Quantum corrections to the effective neutrino mass operator in 5D MSSM

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We discuss in detail a five-dimensional Minimal Supersymmetric Standard Model compactified on $S^1/Z_2$ extended by the effective Majorana neutrino mass operator. We study the evolution of neutrino masses and mixings. Masses and angles, in particular the atmospheric mixing angle $\theta_{23}$, can be significantly lowered at high energies with respect to their value at low energy.

I. INTRODUCTION

Within the Standard Model (SM), the masses of the quarks and charged leptons are determined via Yukawa couplings to the Higgs boson. The origin of their structure (masses and mixing angles between flavours) has no explanation in the context of the SM and it is one of the major challenges for physics beyond the SM. Neutrinos being massless in the SM, the experimental evidence for nonzero neutrino masses gives an important indication for physics beyond the SM. In any case, neutrino masses are many orders of magnitude smaller than those of quarks and charged leptons. Moreover, the experimental results indicate that the lepton mixing matrix has two large mixing angles unlike the quark mixing matrix. All this shows that the neutrino sector plays a special role in understanding the flavour structure of the SM and its possible extensions.

Phenomenological implications of quantum corrections to the neutrino mass and mixing parameters have been investigated intensively in the literature (see e.g. the review [1]). The main reason is that large effects can provide interesting hints for model building. From a theoretical point of view, many models are available proposing different explanations for the particularities of the neutrino sector. In order to study qualitative features in a model independent way, an attractive and simple possibility is to stick to a low-energy effective theory formulation. This means that one organises the effects of additional particles and symmetries present at higher energies within a systematic low-energy expansion. We will assume here that the heavy states arising from physics beyond the standard model completely decouple at low energies. In that case, the degree of suppression of an operator in the low energy effective Lagrangian is characterised by its mass dimension (d). The only operator appearing at dimension $d = 5$ is the lepton-number violating operator [2]

$$-rac{\kappa_{ij}}{4M}(L_i^{ab}\epsilon \phi_b)(L_j^{cd}\phi_d) + h.c,
$$

where $L$ and $\phi$ represent the lepton and the Higgs doublet fields, respectively. $M$ is an energy scale characteristic for the range of validity of the low-energy effective theory description. An operator of this type can be generated, for instance, by the usual seesaw mechanism [3]. In that case the scale $M$ can be identified with the mass of the heavy right-handed neutrino. After spontaneous breakdown of the electroweak symmetry, the Higgs acquires a vacuum expectation value (vev) and the operator in Eq. (1) then represents a Majorana mass term for the neutrinos.

In the context of the Minimal Supersymmetric Standard Model (MSSM), it can be written in the form :

$$-rac{\kappa_{ij}}{4M}(L_i^{ab}\epsilon H_b^{(u)})(L_j^{cd}H_d^{(u)}) + h.c,
$$

where $L$ and $H^{(u)}$ now stand for the lepton and up-type Higgs doublet chiral superfield, respectively. This dimension five operator provides a very efficient way to study neutrino masses and mixings. Renormalisation group equations for this effective operator have been derived in the context of the four-dimensional SM and MSSM in Refs. [1, 4, 5].

Scenarios with compactified extra-dimensions offer many possibilities for model building. For example, there are new ways to generate electroweak symmetry breaking or supersymmetry breaking simply by choosing appropriate boundary conditions (for a review on extra-dimensions and their phenomenology, see [6]). In addition, for flat extra-dimensions the presence of towers of excited Kaluza-Klein states induces the power-law enhancement of the gauge couplings, leading to a possible low-scale unification [7, 8]. This effect can be applied to other couplings such as Yukawa couplings, too, giving an original way to generate mass hierarchies [5, 9]. For the same reasons, extra-dimensions can also provide a possible explanation of the observed pattern of neutrino masses and mixings.

The aim of this paper is to study these effects explicitly in the case of one extra-dimension within a supersymmetric model supplemented by the effective neutrino mass operator, Eq. (4). In the following we shall consider the effects of
renormalisation at one loop in order to test the behaviour of the extra-dimensional model. We shall focus on a five dimensional $\mathcal{N} = 1$ supersymmetric model compactified on the orbifold $S^1/Z_2$ as a simple test ground for the effects of the extra dimension.

The experimental results will be used as a starting point for the evolution of the masses and couplings in order to test the evolution at higher energies and the effects induced by the presence of the extra dimension. Due to the power-law running of the (gauge) couplings, there are of course restrictions on the range of validity of the present model which consequently put limits on the present investigation.

Instead of starting from the observed masses and mixing parameters at low energies, we could take the renormalisation group equations provided here, to constrain parameters of some specific model at high energies by studying the evolution to low energies and comparing the predictions with data. Since we are mainly interested in the qualitative features of a five-dimensional MSSM compactified on the orbifold $S^1/Z_2$ and discuss its low-energy spectrum. The details of the Lagrangian and its Feynman rules are given in appendix 1. The third section is devoted to a discussion of the beta functions for the Yukawa couplings and the effective neutrino mass operator. Numerical results for the evolution of neutrino masses and mixings are given in section 4. In the last section we summarise and discuss the physical implications.

II. 5D MSSM

A. Five-dimensional $\mathcal{N} = 1$ supersymmetry

The beta functions can be most elegantly derived in the superfield formalism. We will therefore begin with briefly discussing $\mathcal{N} = 1$ supersymmetry in a five-dimensional Minkowski space and its description in terms of 4D superfields. More details can be found in Refs. [10, 11, 12]. Space-time coordinates will be denoted by $(x^\mu, y)$.

1. Gauge sector

The gauge sector will be described by a 5D $\mathcal{N} = 1$ vector supermultiplet which consists (on-shell) of a 5D vector field $A^M$, a real scalar $S$ and two gauginos, $\lambda$ and $\lambda'$.

$$S_g = \int d^5x \frac{1}{2k g^2} \text{Tr} \left[ -\frac{1}{2} F^{MN} F_{MN} - D^M S D_M S - i \bar{\lambda} \Gamma^M D_M \lambda 
+ -i \bar{\lambda}' \Gamma^M D_M \lambda' + (\bar{\lambda} + \bar{\lambda}') [S, \lambda + \lambda'] \right]$$

with $D_M = \partial_M + i A_M$, $\Gamma^5 = i \gamma^5$ and $F^{MN} = \frac{2i}{g}[D^M, D^N]$. $k$ normalises the trace over the generators of the gauge groups. Equivalently, one can rearrange these fields in terms of a $\mathcal{N} = 2$, $D = 4$ vector supermultiplet, $\Omega = (V, \chi)$, where:

- $V$ is a $\mathcal{N} = 1$ vector supermultiplet containing $A^\mu$ and $\lambda$;
- $\chi$ is a chiral $\mathcal{N} = 1$ supermultiplet containing $\lambda'$ and $S' = S + i A^5$.

This follows from the decomposition of the 5D supercharge (which is a Dirac spinor) into two Majorana-type supercharges which constitute a $\mathcal{N} = 2$ superalgebra in 4D. Both $V$ and $\chi$ (and their component fields) are in the adjoint representation of the gauge group $G$. Using these supermultiplets, one can rewrite the original 5D $\mathcal{N} = 1$ supersymmetric action (3) only in terms of $\mathcal{N} = 1$ $D = 4$ superfields and the covariant derivative in the $y$ direction [11]:

$$S_g = \int d^5x d^2\theta d^2\bar{\theta} \frac{1}{4k g^2} \text{Tr} \left[ \frac{1}{4} (W^\alpha W_\alpha \delta(\theta^2) + h.c) + (e^{-2g V} \nabla_y e^{2g V})^2 \right]$$ (3)

with $W^\alpha = -\frac{1}{4} \nabla^5 e^{-2g V} D_\alpha e^{2g V}$. $D_\alpha$ is the covariant derivative in the 4D $\mathcal{N} = 1$ superspace (see the textbooks [13, 14]) and $\nabla_y = \partial_y + \chi$. This action can be expanded and quantised to find the Feynman rules to a given order in the gauge coupling $g$. 

2. Matter sector and its coupling to the gauge sector

The $\mathcal{N} = 1$ supersymmetric and 5D Lorentz invariant action describing a free chiral supermultiplet is:

$$S = \int d^5x \left( (\partial_M \phi_i)^\dagger (\partial^M \phi^i) - \bar{\psi} (i\Gamma^M \partial_M + m) \psi \right)$$

(4)

The two complex scalars $\phi^{1,2}$ form a doublet under a ‘$SU(2)_R$’ symmetry and $\psi$ is a $SU(2)_R$-singlet Dirac spinor. Together, they can form two 4D $\mathcal{N} = 1$ chiral supermultiplets, $\Phi$ and $\Phi^c$. Adding the couplings to the gauge sector we obtain for the action in terms of the 4D superfields:

$$S = \int d^5x d^2\theta d^2\bar{\theta} \left( \Phi e^{2gV} \Phi + \Phi^c e^{-2gV} \Phi^c + (\Phi^c (\nabla_5 + m) \Phi) \delta(\bar{\theta}^2) + h.c \right)$$

(5)

This way of writing the action has the disadvantage that the 5D Lorentz invariance and the underlying supersymmetry relating $\Phi$ to $\Phi^c$ and $V$ to $\chi$ are not manifest, but it simplifies considerably the computation of quantum corrections.

One remark is in order here: due to the additional $SU(2)_R$ symmetry of the 5D matter sector, Yukawa-type couplings between $\phi$ and $\psi$ or trilinear couplings between $\Phi$s are forbidden in the bulk.

B. The orbifolding and the low-energy spectrum

If we want to recover the MSSM at low energy, we need chiral zero modes for fermions. To realize this, we will compactify the fifth dimension on the orbifold $S^1/Z_2$. The orbifold construction is crucial in order to obtain chiral zero modes from a vector-like 5D theory.

The $Z_2$ symmetry identifies $y \rightarrow -y$, and reduces the physical interval to $[0, \pi R]$ where $R$ is the radius of the circle (see Fig. 1). We have two orbifold fixed points invariant under the $Z_2$ transformation, namely $y = 0 = -y$ and $y = \pi R = -\pi R + 2\pi R = -y$. These fixed points, called branes (cf. Fig. 2), break the translational invariance in the fifth dimension and therefore the momentum conservation along $y$ and thus part of the 5D supersymmetry. We will choose the transformation properties of the fields and the interactions such that 4D Lorentz symmetry, the $Z_2$ orbifold symmetry and one 4D $\mathcal{N} = 1$ supersymmetry are preserved. The $\chi$-field should then be odd under the $Z_2$ symmetry because it appears together with a derivative $\partial_y$, whereas $V$ is even. For the two matter superfields, we choose $\Phi$ to be even and the conjugate $\Phi^c$ to be odd. This is a pure convention. Note that only the even fields have

FIG. 1: The orbifold projection ($S^1/Z_2$). The physical interval is $0 \leq y \leq \pi R$. 
zero modes. The Fourier decomposition of the fields reads:

\[
\begin{align*}
V(x, y) &= \frac{1}{\sqrt{\pi R}} \left[ V^{(0)}(x) + \sqrt{2} \sum_{n \geq 1} V^{(n)}(x) \cos \left( \frac{ny}{R} \right) \right], \\
\chi(x, y) &= \sqrt{2} \frac{\pi R}{\sqrt{\pi R}} \sum_{n \geq 1} \chi^{(n)}(x) \sin \left( \frac{ny}{R} \right), \\
\Phi(x, y) &= \frac{1}{\sqrt{\pi R}} \left[ \Phi^{(0)}(x) + \sqrt{2} \sum_{n \geq 1} \Phi^{(n)}(x) \cos \left( \frac{ny}{R} \right) \right], \\
\Phi^c(x, y) &= \sqrt{2} \frac{\pi R}{\sqrt{\pi R}} \sum_{n \geq 1} \Phi^{c(n)} \sin \left( \frac{ny}{R} \right),
\end{align*}
\]

where we normalized the massive KK states to have canonical kinetic terms. At energies well below the scale \( R^{-1} \), where the massive Kaluza-Klein states decouple, only the zero modes remain in the spectrum and we assume that physics is described by the usual MSSM. Thus, the matter superfields (and Higgs superfields) of the MSSM will be identified with a \( \Phi^{(0)} \) superfield, and the gauge fields with a \( V^{(0)} \) mode. From the above decomposition it becomes obvious that the orbifold prescription breaks the original \( N = 2 \) supersymmetry which relates \( \Phi \) to \( \Phi^c \) and \( V \) to \( \chi \).

Compactifying on \( S^1/Z_2 \) we have to introduce two Higgs hypermultiplets in the 4D \( N = 2 \) language if we want to have zero modes corresponding to the two Higgs superfields of the MSSM, \( H_u \) and \( H_d \). Introducing an additional \( Z'_2 \) symmetry it is possible to obtain two zero mode Higgs superfields starting from one Higgs hypermultiplet \[15\]. For simplicity we will stay with only one \( Z_2 \) symmetry here.

### C. Flavour physics on the brane

As stated above, Yukawa couplings in the bulk are forbidden by the 5D \( N = 1 \) supersymmetry. However, they can be introduced \textit{on the branes}, which are 4D subspaces with reduced supersymmetry. We will write the following interaction terms, called \textit{brane interactions}, containing Yukawa-type couplings.

\[
S_{\text{brane}} = \int d^8 z dy \delta(y) \left\{ \left( \frac{1}{6} \lambda_{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{4} \tilde{\kappa}_{ij} L_i H_u L_j H_u \right) \delta(\bar{\theta}) + \text{h.c.} \right\},
\]

where \( \Phi_i \) represents a matter superfield, \( L \) is the lepton doublet superfield, and \( H_u \) one Higgs superfield. The last term corresponds to the effective neutrino mass operator, with dimensionful coupling \( \tilde{\kappa}_{ij} \).

Note that we do not write any interactions with conjugate superfields \( \Phi^c \) on the brane which would be allowed by gauge interactions. But, as \( \Phi^c \) vanishes on the branes, these interactions simply vanish. Moreover, we could have
introduced independent interactions on the \( \pi R \) brane. Below we will briefly comment on this possibility, but for our numerical analysis we restrict ourselves to (6).

### III. BETA FUNCTIONS

In this section we will derive the beta functions for the Yukawa couplings and the coupling of the neutrino mass operator assuming that no other operators (generated in the evolution) affect this behaviour. We will begin with recalling briefly the 4D result, mainly in order to set up notations and to explain the method.

#### A. Usual 4D result

Due to the non-renormalisation theorem \[16\], the beta functions for the couplings of the operators in the super-potential are governed by the wave function renormalisation constants \( Z_{ij} = 1 + \delta Z_{ij} \). These relate the bare to the renormalised superfields,

\[
\Phi^{(i)}_B = \sum_{j=1}^{N_Φ} Z_{ij}^{1/2} \Phi^{(j)}_R .
\]

The sum runs here over all \( N_Φ \) chiral superfields of the model.

In a generic 4D super-Yang-Mills theory, the result for the wave function renormalisation constant at one loop and in \( d = 4 - \epsilon \) dimension reads \[13\]:

\[
-\delta Z_{ij} = \frac{1}{16\pi^2 \epsilon} \left( \sum_{k,l=1}^{N_Φ} \frac{1}{2} \lambda_{ikl} \lambda_{jkl} - 2 \sum_{n=1}^{N_χ} g_n^2 C_2(R_n^{(i)}) \delta_{ij} \right) .
\]

We have written the interaction as \( \lambda_{ijk} \Phi^{(i)} \Phi^{(j)} \Phi^{(k)} \) and the sum over \( n \) runs over all gauge groups of the theory. The group-theoretical constants \( C_2(R) \) are defined as

\[
C_2(R) \delta_{ab} = \sum_A (T^A T^A)_{ab} ,
\]

where \( T^A \) are matrix representations of the generators of the gauge group corresponding to the irreducible representation \( R \) under which the fields \( \Phi_i \) transform. The result at two-loops can be found in \[17\].

For future reference we recall here the result at one-loop for the the beta function of \( \kappa' \):

\[
(16\pi^2) \beta_κ = \left( 6 \text{Tr}(Y_u^T Y_u) - \frac{6}{5} g_1^2 - 6g_2^2 \right) \kappa' + \left[ Y_e^T Y_e \right] \kappa' + \left[ Y_e Y_e^T \right] \kappa' + \left[ Y_e^T Y_e \right] \kappa' + \left[ Y_e Y_e^T \right] \kappa'
\]

The two-loop result can be found in Ref. \[18\]. The expression for the Yukawa couplings can be derived analogously.

#### B. 5D result

We will now derive the results in the case of the five-dimensional MSSM discussed above. To deal with the running in extra-dimensional theories, intrinsically non-renormalisable, we briefly remind the point of view introduced in Ref. \[3\]. The theory is treated as a chain of effective field theories where we decouple all the excitations whose mass exceeds the energy \( \mu \) we are interested in. Hence the greater the energy, the larger the number of states considered, which creates to a very good approximation a power law running of the couplings (under the condition that the energy scale of interest is at least about an order of magnitude larger than \( R^{-1} \)). See appendix \[C\] for more details.

Higgs superfields and gauge superfields will always propagate into the fifth dimension. Different possibilities of localisation for the matter superfields will be studied by taking the two limiting cases of superfields containing the SM fermions in the bulk or all superfields containing SM fermions restricted to the brane, respectively. We will begin with the case where all matter fields propagate in the bulk.
1. All matter superfields propagate in the bulk

If all matter chiral superfields of the MSSM are allowed to propagate in the fifth dimension, we find the wave function renormalisation constant of a matter (chiral) superfield:

\[ -(16\pi^2) \delta Z^D = \left( -8 \sum_{n=1}^{N_u} g_n^2 C_2(R_n^{(i)}) \delta_{ij} \right) \Lambda R + \left( 2\pi \sum_{k,l=1}^{N_k} \lambda^*_{ijkl} \lambda_{ijkl} \right) \Lambda^2 R^2. \]  \hspace{1cm} (11)

Here \( R \) corresponds to the radius of the compactified fifth dimension and \( \Lambda \) is a cutoff parameter. We only retained the contributions which diverge in the limit \( \Lambda \to \infty \), see appendix \([\) where we discuss the evaluation of the sums over KK states. The same result is obtained for the Higgs superfield. A collection of explicit expressions for the different wave function renormalisation constants can be found in appendix \([D4]\).

As in the previous section, the beta functions can be directly calculated from the above expression for the wave function renormalisation constants. Within the model discussed in Section \([II] \) we obtain for the beta function of \( \kappa = \kappa / (\pi R)^2 \) at one loop:

\[ (16\pi^2)\beta_\kappa = \left( -\frac{12}{5} g_1^2 - 12 g_2^2 \right) \Lambda R + 24\pi \text{Tr}(Y_u^\dagger Y_u) \Lambda^2 R^2 \kappa + \left( [Y^T Y^*] + \kappa [Y_e^T Y_e] \right) 4\pi \Lambda^2 R^2. \]  \hspace{1cm} (12)

The beta functions for Yukawa couplings are given by:

\[ (16\pi^2)\beta_{Y_d} = Y_d (3 \text{Tr}(Y^\dagger_d Y_d) + \text{Tr}(Y^\dagger_e Y_e) + 3 Y^\dagger_d Y_d + Y^\dagger_u Y_u) 4\pi \Lambda^2 R^2 \]
\[ \hspace{1cm} - Y_d \left( \frac{14}{15} g_1^2 + 6 g_2^2 + \frac{32}{3} g_3^2 \right) \Lambda R \]  \hspace{1cm} (13)

\[ (16\pi^2)\beta_{Y_u} = Y_u (3 \text{Tr}(Y^\dagger_u Y_u) + 3 Y^\dagger_d Y_d + Y^\dagger_y Y_y) 4\pi \Lambda^2 R^2 \]
\[ \hspace{1cm} - Y_u \left( \frac{26}{15} g_1^2 + 6 g_2^2 + \frac{32}{3} g_3^2 \right) \Lambda R \]  \hspace{1cm} (14)

\[ (16\pi^2)\beta_{Y_e} = Y_e (3 \text{Tr}(Y^\dagger_d Y_d) + \text{Tr}(Y^\dagger_e Y_e) + 3 Y^\dagger_e Y_e) 4\pi \Lambda^2 R^2 \]
\[ \hspace{1cm} - Y_e \left( \frac{18}{5} g_1^2 + 6 g_2^2 \right) \Lambda R \]  \hspace{1cm} (15)

Note that all beta functions contain a term quadratic in \( \Lambda R \). This term will dominate the evolution of the Yukawa couplings and of \( \kappa \). They will evolve much more rapidly than the gauge couplings, which only contain a linear term. This linear term arises from the sum over the tower of KK states. Since the Yukawa interactions and the effective neutrino mass operator are localised on the brane, we have to sum over two towers of KK excitations giving rise to the quadratic term. This effect has already been noticed in Ref. \([15] \), where limitations of the model due to the quadratic running have been mentioned, too. For the same reason at higher orders in the loop expansion higher powers of \( \Lambda R \) appear which limit the validity of the present approach. The top Yukawa coupling can become non-perturbative before the gauge couplings and at rather low energy thus limiting considerably the range of validity of the present model. This will become evident from the discussion of the numerical results in Section \([IV] \). We could have introduced another independent interaction on the \( \pi R \) brane. This would not change the general problem since it is not possible to mutually compensate the quadratic terms. Thus, without allowing for Yukawa interactions in the bulk, which would break the supersymmetry, it is not possible to avoid this quadratic running if there are matter fields in the bulk.

2. Matter superfields on the brane

In this section we will discuss the results for the beta function in the case where all matter superfields are constrained to live on the 4D brane at \( y = 0 \), i.e. there are no KK excitations for the matter superfields. We thus expect that the quadratic evolution due to the sum over two KK towers will become milder. The renormalisation constant of a matter (chiral) superfield becomes:

\[ -(16\pi^2) \delta Z^D = \left( -16 \sum_{n=1}^{N_u} g_n^2 C_2(R_n^{(i)}) \delta_{ij} + \sum_{k,l=1}^{N_k} \lambda^*_{ijkl} \lambda_{ijkl} \right) \Lambda R \]  \hspace{1cm} (16)
Since the Higgs-type superfields can propagate in the bulk, in contrast to the matter superfields, the wave function renormalisation has no longer the same structure. For example, the contribution containing the vector superfields contains a sum over KK states for the Higgs superfield, whereas only the zero mode enters in the case of the matter superfields. For the Higgs superfield we obtain

\[- (16\pi^2) \delta Z_H = \left( -8 \sum_{n=1}^{N_u} g_n^2 C_2 (R_n^{(i)}) \delta_{ij} \right) \Lambda R + \left( \sum_{k,l=1}^{N_\Phi} \Lambda_{kl}^{\ast} \Lambda_{kl} \right) \log \Lambda R \]  

Again, explicit expressions for the wave function renormalisation constants can be found in appendix D.2.

As can be seen from the above equations neither $\kappa$ nor the Yukawa couplings contain a term quadratic in $\Lambda$ any more, since there are no KK excitations for the matter superfields. This means that the range of validity of this model will be wider since the couplings, in particular the top Yukawa coupling, will not become non-perturbative at low energies (a few $R^{-1}$). This will be discussed in more detail in the next section where numerical results will be presented.

IV. NUMERICAL RESULTS

Within this section we will apply the beta functions derived in the previous section to study the evolution of the different couplings. We are particularly interested in the evolution of the neutrino mass parameters. To that end we employ the MATHEMATICA package REAP [19] in a slightly modified version including our models. The SUSY threshold is taken at 1 TeV. Throughout the discussion we will identify the cutoff $\Lambda$ with the energy scale $\mu$ for the evolution.

Before coming to a detailed discussion of the numerical results, let us mention that the discussion of the fixed point structure for the mixing angles follows closely the MSSM one [20, 21]. The evolution equations (given in detail in Appendix A) have the same structure as in MSSM. We will come back to this point in section IVB.

A. Matter superfields in the bulk

Let us begin with the model described in Sec. III B 1. The corresponding beta functions have been given in Eqs. 12-15. In this case, as already anticipated, the evolution is dominated by the quadratic terms. This puts stringent limits on the model. As soon as the energy scale $\mu$ becomes of the order of $R^{-1}$, i.e. the couplings start to feel the effect of the extra dimension, $y_t$ starts to increase rapidly and it diverges for $\mu R \sim 2 - 3$. Moreover cubic and higher divergences arise at higher orders in the loop expansion which can strongly affect the results. This is due to the presence of sums on the KK excitations in the loops which are not restricted by any conservation of KK numbers at the vertices.
Throughout the calculation of the beta functions, we have assumed that $\mu_R \gg 1$. How reliable is this approximation if we reach the perturbative limit of the model already at $\mu_R \sim 2 - 3$? We checked the validity of the approximation by integrating the beta functions without using the power law approximation. This can be done numerically, adding at every threshold the contributions from the new degrees of freedom. The results do not change considerably. For instance, $y_t$ diverges for a slightly larger value of $\mu R \sim 4 - 5$ (see Fig. 3).

![Graphs of $y_t$ and $g_i$ vs. $\mu$](image)

**FIG. 3:** Evolution of the top Yukawa coupling (left panel) and the gauge couplings (right panel) $g_1$, $g_2$ and $g_3$ (from top to bottom) in the case $R^{-1} \sim 10^{10}$ GeV and matter fields in the bulk. We clearly see that $y_t$ diverges before any perturbative unification is possible.

However, a unification of the gauge couplings before the non-perturbativity of the top Yukawa coupling is possible at a scale $R^{-1} \geq 2.10^{14}$ GeV. This scale is of the same order as the standard 4D unification scale, such that the extra dimensional scenario loses much of its interest. In addition, the rapid divergence of the top Yukawa coupling prevents us from making any reliable prediction on the effects of the extra dimension on the neutrino sector. We will thus not show any results for the neutrino parameters within this model.

**B. Matter superfields on the brane**

If we restrict the matter fields to the brane, we no longer face this problem. First of all, there is no divergent quadratic term at 1-loop. In addition, thanks to the large negative contribution of the gauge couplings to $\beta_{y_t}$, $y_t$ decreases. This allows for a perturbative unification of the gauge sector at a value $\mu \sim 40 R^{-1} = M_{GUT}$ for $R^{-1} = 10^3$ TeV. We have also checked explicitly that 2-loop terms are at most quadratic in the cut-off. These terms are further suppressed by an extra factor of $16\pi^2$ so that the results are presumably not modified within the range of validity of the effective theory established at 1-loop.

This model is thus much more promising and worthwhile studying the evolution of the neutrino mass parameters. We discuss our conventions for the mixing matrix and masses together with the explicit renormalisation group equations (RGE) in appendix A.

Here a few general comments are in order. The RGE for the neutrino masses and mixings are similar to the 4D case since the beta functions have a similar structure. Consequently the relevant parameters will be essentially the same as in the 4D case. $\tan \beta$ plays an important role as all the mixing angles and phases depend on $y_t$. In addition, $m_1$ is important since the running of $\theta_{12}$ will be stronger if $m_1 \sim m_2$. This is the reason why we generally choose $\tan \beta \sim 50$ and $m_1 \sim 0.1$ eV to explore the effects of the extra-dimension.

In Fig. 4 we show the influence of $\tan \beta$ on the evolution of $\theta_{12}$, all other parameters are kept fixed. Depending on the value of $\tan \beta$, $\theta_{12}$ can assume almost any value at the scale where gauge coupling unification is achieved within our model. It indicates the scale where new physics will come into play. This sensitivity of $\theta_{12}$ to $\tan \beta$ has already been noticed in 4D (cf for instance Ref. [22]).
FIG. 4: Running of $\theta_{12}$ for the values $\tan\beta = \{10, 30, 50\}$ (from top to bottom), $R^{-1} = 10^4$ GeV, $m_1 = 0.1$ eV, $\theta_{13} = 0$ and all phases vanish at $M_Z$.

1. Masses

Still, the 5D case is intrinsically different from the 4D one and has some particularities that are quite independent of any choice of low energy parameters. This is the case for the evolution of the masses (Fig. 5). As can be seen in Fig. 5 the evolution has exactly the same form for the three masses and is much sharper above $\mu \sim R^{-1}$, i.e. where the masses start to feel the effect of the fifth dimension. This leads to a reduction of up to a factor of the order of five for the masses in the UV with respect to the values at low energies. This prediction is extremely stable and can be explained as follows.

The evolution of the masses is governed by the following equation

$$\frac{d m_i}{dt = \ln \mu R} = \dot{m}_i = \frac{1}{16\pi^2} [\alpha + 4\mu Rg_i^2a_i] m_i . \quad (22)$$

where the parameters $a_i$ induce a priori a non-universal behaviour and the parameter $\alpha$ is detailed in appendix A in 4D and 5D. In contrast to the MSSM, the evolution in our case is completely dominated by the universal part. The essential point is that in the MSSM the positive contribution to $\alpha$, approximately $6g_t^2$, is of the same order as the negative contribution from the gauge part, leaving an $\mathcal{O}(1)$ factor. In the setup we are interested in, however, the situation is completely different: on the one hand $y_t$ decreases very rapidly and on the other hand the contribution from the gauge part to $\alpha$ is multiplied by $12\mu R$ with respect to the MSSM which makes it completely dominant compared to any other contribution. We can therefore write:

$$\dot{m}_i \sim \frac{1}{4\pi^2} \mu R \left(-\frac{18}{5}g_1^2 - 18g_2^2\right) m_i . \quad (23)$$

This equation is universal. From this approximation we immediately see that all masses decrease with increasing energy and eventually become zero.

This discussion can be extended to $\Delta m^2$, too, except in the case where $m_1$ is not small compared with $m_2$ and $\tan\beta$ is large. In the latter case we can check from the analytic formula,

$$\frac{d}{dt} \Delta m^2_{sol} = \frac{1}{8\pi^2} [\alpha \Delta m^2_{sol} + 4\mu Rg_t^2 [2s_{23}^2(m_2^2c_{12}^2 - m_1^2s_{12}^2) + \mathcal{O}(\theta_{13})]] \quad (24)$$

that for $\sqrt{\Delta m^2_{sol}} \ll m_1$ the first term does not necessarily dominate. Still, because all the masses decrease rapidly, $\Delta m^2_{sol}$ has to stop growing at some point. This is indeed the case, see Fig. 6.
FIG. 5: Running of the three masses for the values $R^{-1} = 10^{4}$ GeV, $m_1 = 0.1$ eV, $\theta_{13} = 0$ and all phases vanish at $M_Z$.

FIG. 6: Running of $\Delta m^2_{\text{sol}}$ and $\Delta m^2_{\text{atm}}$ for the values $\tan \beta = 50$, $R^{-1} = 10^{4}$ GeV, $m_1 = 0.1$ eV and all phases vanish at $M_Z$.

2. Mixing angles

Note also that, assuming a hierarchical spectrum ($m_1 = 0.001$ eV) and $\tan \beta = 10$, $\theta_{12}$ will vary twice more than in the MSSM between $M_Z$ and $M_{\text{GUT}}$, provided $\theta_{13}$ is rather large ($\simeq 9$ degrees).

Another interesting observation is that $\theta_{23}$ can become much smaller than in the 4D case. This allows for a unification of $\theta_{12}$ and $\theta_{23}$ at a value below ten degrees. This can be seen in Fig. 7 where we display the three mixing angles for $\tan \beta = 50$ and $m_1 = 0.1$ eV.

In Figs. 8 and 9 we show results for $\theta_{23}$ at the cut-off scale in our 5D model and in the 4D MSSM. Fig. 8 corresponds to an inverse hierarchy whereas Fig. 9 corresponds to normal hierarchy. For the results shown we have $0.01$ eV $< m_1 < 0.1$ eV. For smaller masses the spread is reduced. In general, compared with the MSSM we obtain new interesting points where $\theta_{23}$ is small, ($\leq 20$ degrees) for quite large $\tan \beta \sim 30 - 50$ and $m_1 \sim 0.1$ eV. Note that it is even easier to get such small $\theta_{23}$ with an inverted hierarchy. In this case we can also find a portion of the parameter space where $\theta_{13}$ can be rather large, see Fig. 10. This is in sharp contrast with the 4D MSSM, thus suggesting the possibility that $\theta_{13} \simeq \theta_{12}$ at the cut-off scale with values around roughly 30 degrees. There are even some points with larger values, up to 45 degrees.

Let us note that, although previous studies have investigated the possibility of having CKM-like values for the PMNS angles at high energies, they required even larger values $0.1$ eV $< m_1 < 0.6$ eV creating some tension with cosmological bounds. Here a value $m_1 \leq 0.1$ eV is sufficient, which is new compared with the 4D case. This opens the interesting possibility to find a set of parameters which generate a maximal mixing from a small CKM-like mixing at some new physics scale or even allow for unification of the three mixing angles at high energy.

Despite these interesting features we note that the region in parameter space most studied here is the region where
\[ \mu \text{ (GeV)} \]

FIG. 7: Running of \( \theta_{12} \) (red/grey), \( \theta_{13} \) (blue/dark grey) and \( \theta_{23} \) (green/light grey) for the values \( \tan \beta = 50, R^{-1} = 10^4 \text{GeV}, \ m_1 = 0.1 \text{eV} \) and no phases at \( M_Z \).

\[ m_1 \simeq m_2. \] In this case the low energy values of the mixing angles (in particular \( \theta_{12} \)) are very sensitive to the precise value of \( m_1 \) at the high energy scale (at the percent level).

As mentioned above, we have in principle the same fixed points for the angles as in the 4D MSSM. In the case where \( \theta_{13} = 0 \) it can be seen immediately from the equations that \( \theta_{23} = \theta_{12} = 0 \) are fixed points. For nonzero \( \theta_{13} \) the fixed points are less obvious, but as the structure of the equations does not change with respect to the MSSM, the result is in principle the same. It leads to fixed points predicting the relation \[ 20, 21 \]

\[ \sin^2 2\theta_{12} = \frac{s_{13}^2 \sin^2 2\theta_{23}}{(s_{23}^2 c_{13}^2 + s_{13}^2)^2} \quad \text{with} \quad s_{23}^2 = \frac{1}{2} \left( 1 \pm \sqrt{1 - \sin^2 2\theta_{23}} \right) \] (25)

for different patterns for the coefficients. The coefficients change with respect to the 4D MSSM and therefore the way we can approach the fixed points. Two essential differences between the 5D and 4D MSSM case should be noted in that context. First, the coefficient \( C \) governing the evolution, see appendix \[ \Box \] is \( C = 1 \) in the 4D MSSM whereas it is \( 4\mu R \) in the 5D case. Second, the numerical coefficients entering the equations for the angles depend on the masses and become almost constant if we enter the universal regime for the running of the masses discussed above. This can
be understood as follows. Neglecting the phases, these coefficients can be written as \((m_i + m_j)/(m_i - m_j)\) such that the indirect dependence on \(\mu\) via the masses cancels out within the universal regime. From our numerical analysis we find that in general the fixed points are not reached within the range of validity of the present model. Increasing \(\tan \beta\) accelerates the evolution such that the angles come closer to the fixed points at the cut-off scale. For very large values, \(\tan \beta > 50\), we reach in some cases a fixed point.

V. CONCLUSIONS

The aim of this paper was to perform a study of the neutrino sector in a simple supersymmetric extra-dimensional framework. We have presented beta functions for Yukawa couplings and \(\kappa\), the coupling of the effective neutrino mass operator, within two distinct setups in the context of a five-dimensional MSSM on \(S^1/Z_2\). In the first case, all fields associated with the SM fermions are allowed to propagate in the fifth dimension whereas in the second they are restricted to the brane. Due to the 5D \(\mathcal{N} = 1\) supersymmetry, Yukawa interactions are forbidden in the bulk and must be introduced on the branes. Within the first model, this induces a quadratic running for Yukawa couplings and \(\kappa\). \(y_t\) then becomes non-perturbative already at rather low energies and even before the gauge couplings. This strongly limits the range of validity of the model. Within the second scenario the dependence on the energy scale is only linear and Yukawa couplings remain perturbative until gauge coupling unification. The evolution of neutrino masses and mixings shows interesting possibilities to explain the observed values at low energies from some specific scenario at high energies. As a generic prediction, neutrino masses are reduced up to a factor of five at the unification scale with respect to their values at low energies. From a top-down point of view, one can radiatively generate a large mixing pattern at low energy starting with small
values of $\theta_{23}$ and $\theta_{12}$ at a high energy scale with values for $m_1$ consistent with cosmological bounds. It is also possible to generate a small $\theta_{13}$ at $M_Z$ from large values at the cut-off scale.

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APPENDIX A: CONVENTIONS FOR NEUTRINO MASSES AND MIXING PARAMETERS

Within this section we would like to stress our conventions for the mixing angles and phases and briefly discuss different scenarios for neutrino masses. The mixing matrix which relates gauge and mass eigenstates is defined to diagonalise the neutrino mass matrix in the basis where the charged lepton mass matrix is diagonal. It is usually parameterised as follows [24]:

$$U = \text{diag}(e^{i\delta_1}, e^{i\delta_2}, e^{i\delta_3}) U_{\text{MNS}},$$

$$U_{\text{MNS}} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{23}s_\alpha & s_{13}e^{-i\delta}
    
    -s_{12}c_{23} - c_{12}s_{23}s_\alpha e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_\alpha e^{i\delta} & s_{13}e^{-i\delta}
    
    s_{12}s_{23} - c_{12}c_{23}s_\alpha e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_\alpha e^{i\delta} & c_{13}c_{23}
\end{pmatrix} \text{diag}(e^{-i\phi_1/2}, e^{-i\phi_2/2}, 1).$$

(A1)

where $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$. We follow the conventions of Ref. [22] to extract mixing parameters from the $MNS$ matrix.

Experimental information on neutrino mixing parameters and masses is obtained mainly from oscillation experiments. The most simple interpretation of these oscillation data is in terms of massive neutrinos. There are essentially three types of experiments providing us with data: solar neutrino experiments (Kamiokande and Superkamiokande), looking for a deficit of $\nu_e$ from the sun, atmospheric neutrino experiments looking for a deficit of $\nu_\mu$ and $\bar{\nu}_\mu$ produced by cosmic rays in the earth’s atmosphere and reactor experiments looking for the neutrino flux from a reactor. Present data are compatible with oscillations between the three known neutrino flavours. In general $\Delta m^2_{\text{atm}}$ is assigned to a mass difference between $\nu_3$ and $\nu_2$ whereas $\Delta m^2_{\text{sol}}$ to a mass difference between $\nu_2$ and $\nu_1$. Current values [25] are summarised in Table A. Data indicate that $\Delta m^2_{\text{sol}} \ll \Delta m^2_{\text{atm}}$, but the masses themselves are not determined. Either they follow the hierarchical scenario with $\Delta m^2_{\text{atm}} \approx m^2_{\nu_3} \gg m^2_{\nu_2(1)} \gg m^2_{\nu_1(2)}$ or (partly) degenerate scenarios with masses approximately equal.

| Parameter | Value (90% CL) |
|-----------|----------------|
| $\sin^2(2\theta_{12})$ | $0.86(0.03)$ |
| $\sin^2(2\theta_{23})$ | $> 0.92$ |
| $\sin^2(2\theta_{13})$ | $< 0.19$ |
| $\Delta m^2_{\text{sol}}$ | $(8.0 \pm 0.4) \times 10^{-5} eV^2$ |
| $\Delta m^2_{\text{atm}}$ | $1.9$ to $3.0 \times 10^{-3} eV^2$ |

With these conventions we can derive the approximate equations for the mixing angles and masses. The derivation is similar to the one in ref. [22], and with the same notations we write the result:
\[ \dot{m}_1 = \frac{1}{16\pi^2} [\alpha + C y_1^2 (2 s_{12}^2 s_{23} + O(\theta_{13}))] m_1 \]
\[ \dot{m}_2 = \frac{1}{16\pi^2} [\alpha + C y_2^2 (2 c_{12}^2 s_{23} + O(\theta_{13}))] m_2 \]
\[ \dot{m}_3 = \frac{1}{16\pi^2} [\alpha + C y_2^2 c_{12}^2] m_3 \]
\[ \dot{\theta}_{12} = -\frac{C y_1^2}{32\pi^2} \sin(2\theta_{12}) s_{23}^2 \frac{|m_1 e^{i\phi_1} + m_2 e^{i\phi_2}|^2}{\Delta m_{sol}^2} + O(\theta_{13}) \]
\[ \dot{\theta}_{13} = \frac{C y_1^2}{32\pi^2} \sin(2\theta_{12}) \sin(2\theta_{23}) \frac{m_3}{\Delta m_{atm}^2 (1 + \zeta)} \]
\[ \times [m_1 \cos(\phi_1 - \delta) - (1 + \zeta)m_2 \cos(\phi_2 - \delta) - \zeta m_3 \cos \delta + O(\theta_{13})] \]
\[ \dot{\theta}_{23} = -\frac{C y_2^2}{32\pi^2} \sin(2\theta_{23}) \frac{1}{\Delta m_{atm}^2} \left[ c_{12}^2 |m_2 e^{i\phi_2} + m_3|^2 + s_{12}^2 |m_1 e^{i\phi_1} + m_3|^2 \right] + O(\theta_{13}) \]

where \( \Delta m_{sol}^2 = m_2^2 - m_1^2 \), \( \Delta m_{atm}^2 = m_3^2 - m_2^2 \), and \( \zeta = \Delta m_{sol}^2 / \Delta m_{atm}^2 \). The main difference here lies in the expressions of \( \alpha \) and \( C \), which are in the model with matter superfields on the brane:

\[
\alpha = \left[ \left( -\frac{18}{5} g_1^2 - 18 g_2^2 \right) \mu R + 6 \text{Tr}(Y_u^3 Y_u) \right] \quad C = C(\mu) = 4 \mu R \quad (A2)
\]

This has to be compared with the MSSM coefficients:

\[
\alpha = \left( 6 \text{Tr}(Y_u^3 Y_u) - \frac{6}{5} g_1^2 - 6 g_2^2 \right) \quad C = 1 \quad (A3)
\]

**APPENDIX B: COMPLETE LAGRANGIAN AND FEYNMAN RULES**

Here we write the complete 5D action of the model where all fields can propagate in the bulk [10]:

\[
S_{\text{gauge}} = \frac{\text{Tr}}{C_2(G)} \int d^8 z d\gamma \left\{ -V \Box (P_T - \frac{1}{\xi} (P_1 + P_2)) V - V \partial_6^2 V - \frac{\xi}{2} \partial_6^2 \chi + \frac{1}{2} \bar{\chi} \chi + \frac{\bar{g}}{4} \left( \bar{D}^2 D^6 V \right) \left[ V, D_6 V \right] + \bar{g} \left( \partial_6 V \left[ V, \chi + \bar{\chi} \right] - \left( \chi + \bar{\chi} \right) \left[ V, \chi + \bar{\chi} \right] \right) \right. \\
+ O(g^2) + \text{ghosts} \right\}
\]

\[
S_{\text{matter}} = \int d^8 z d\gamma \left\{ \bar{\Phi}_i \Phi_i + \bar{\Phi}_i \bar{\Phi}_i + \bar{\Phi}_i \partial_6 \Phi_i \delta(\bar{\theta}) - \bar{\Phi}_i \partial_6 \bar{\Phi}_i \delta(\bar{\theta}) \right. \\
+ \bar{g} \left( 2 \bar{\Phi}_i V \Phi_i - 2 \Phi_i V \bar{\Phi}_i + \bar{\Phi}_i \chi \Phi_i \delta(\tilde{\theta}) + \Phi_i \bar{\chi} \bar{\Phi}_i \delta(\tilde{\theta}) \right) \}
\]

\[
S_{\text{brane}} = \int d^8 z d\gamma \delta(y) \left\{ \left( \frac{1}{6} \bar{\lambda}_{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{4} \bar{\kappa}_{ij} L_i H_u L_j H_u \right) \delta(\bar{\theta}) + \text{h.c.} \right\}
\]

where we separated the pure gauge, the coupling of matter to gauge and the Yukawa sector. The latter is localised on the brane in order to respect 5D SUSY. \( \xi \) is the gauge fixing parameter.

In order to consider the 4D effective theory we compactify this action by expanding the fields as in section [11] and we keep only the terms that will be of interest to renormalise the Yukawa beta functions:
\[ S_{\text{gauge}} = \int d^8 z \left\{ -V_a^{(0)} \Box (P_T - \frac{1}{\xi} (P_1 + P_2)) V_a^{(0)} - \sum_{n \geq 1} V_a^{(n)} \Box (P_T - \frac{1}{\xi} (P_1 + P_2)) V_a^{(n)} \right. \\
+ \sum_{n \geq 1} \frac{n^2}{R^2} V_a^{(n)} V_a^{(n)} + \frac{1}{3} \sum_{n \geq 1} \chi_a^{(n)} \chi_a^{(n)} + \left. \frac{1}{2} \sum_{n \geq 1} \chi_a^{(n)} \chi_a^{(n)} \right\} \]

\[ S_{\text{matter}} = \int d^8 z \left\{ \Phi^{(0)} \Phi^{(0)} + \sum_{n \geq 1} (\Phi^{(n)} \Phi^{(n)} + \Phi^{(n)} \Phi^{(n)}) - \sum_{n \geq 1} \frac{n}{R^2} (\Phi^{(n)} \Phi^{(n)} \delta(\tilde{\theta}) + \Phi^{(n)} \Phi^{(n)} \delta(\tilde{\theta})) \right. \\
+ g \left[ 2 \Phi^{(0)} V^{(0)} \Phi^{(0)} + 2 \sum_{n \geq 1} (\Phi^{(n)} V^{(n)} \Phi^{(n)} + \Phi^{(n)} V^{(n)} \Phi^{(0)}) + \Phi^{(0)} V^{(0)} \Phi^{(0)} \right] \\
- 2 \sum_{n \geq 1} \Phi^{(n)} V^{(n)} \Phi^{(n)} + \sum_{n \geq 1} (\Phi^{(n)} \chi^{(n)} \Phi^{(n)} \delta(\tilde{\theta}) + \Phi^{(n)} \chi^{(n)} \Phi^{(n)} \delta(\tilde{\theta})) \right] \\
+ g \left[ \sqrt{2} \sum_{m,n \geq 1} \Phi^{(m)} V^{(n)} (\Phi^{(m+n)} + \Phi^{(|m-n|)}) + \sqrt{2} \sum_{m,n \geq 1} \Phi^{(m)} (V^{(m+n)} - V^{(|m-n|)}) \Phi^{(n)} \right. \\
- \frac{1}{\sqrt{2}} \sum_{m,n \geq 1} (\Phi^{(m)} \chi^{(n)} (\Phi^{(m+n)} - \Phi^{(|m-n|)}) \delta(\tilde{\theta}) + \Phi^{(m)} \chi^{(n)} (\Phi^{(m+n)} - \Phi^{(|m-n|)}) \delta(\tilde{\theta}) \right) \left. \right\} \]

\[ S_{\text{brane}} = \int d^8 z \left\{ \frac{\lambda_{ij}}{6} \left[ \Phi_i^{(0)} \Phi_j^{(0)} \Phi_k^{(0)} + 3 \sqrt{2} \sum_{n \geq 1} \Phi_i^{(n)} \Phi_j^{(0)} \Phi_k^{(0)} + 6 \sum_{m,n \geq 1} \Phi_i^{(0)} \Phi_j^{(m)} \Phi_k^{(n)} \right. \\
+ 2 \sqrt{2} \sum_{m,n \geq 1} \left. \Phi_i^{(m)} \Phi_j^{(n)} \Phi_k^{(p)} \right] \delta(\tilde{\theta}) + \text{h.c.} \right. \\
+ \frac{\kappa_{ij}}{4} \left[ L_i^{(0)} H_u^{(0)} L_j^{(0)} H_u^{(0)} + 2 \sqrt{2} \sum_{n \geq 1} (L_i^{(n)} H_u^{(0)} L_j^{(0)} H_u^{(0)} + L_i^{(0)} H_u^{(n)} L_j^{(0)} H_u^{(0)}) \right. \\
+ 4 \sum_{m,n \geq 1} (L_i^{(m)} H_u^{(n)} L_j^{(0)} H_u^{(0)} + L_i^{(m)} H_u^{(0)} L_j^{(n)} H_u^{(0)}) \right. \\
+ 4 \sqrt{2} \sum_{m,n \geq 1} (L_i^{(0)} H_u^{(m)} L_j^{(n)} H_u^{(p)} + L_i^{(m)} H_u^{(0)} L_j^{(n)} H_u^{(p)}) \right. \\
+ 4 \sum_{m,n,p,q \geq 1} L_i^{(m)} H_u^{(n)} L_j^{(p)} H_u^{(q)} \left. \right] \delta(\tilde{\theta}) + \text{h.c.} \right\} . \]

We kept the interaction terms involving only excited KK modes although they are not relevant for the one-loop beta functions, since they can be relevant to the renormalisation of the Kaluza-Klein masses and to the mixing of the excited states. We defined the 4D effective couplings: 
\[ g = \frac{\tilde{g}}{\sqrt{\pi R}}, \quad \lambda = \frac{\tilde{\lambda}}{(\sqrt{\pi R})^3}, \quad \kappa = \frac{\tilde{\kappa}}{(\pi R)^2} . \]

The propagators are read off from the action (we choose the gauge \( \xi = -1 \)): 
the vertices that are relevant for our one loop calculation:

\[
\begin{align*}
\Phi_i^{(c)(n)}(-p, \theta) & \xrightarrow{p} \Phi_j^{(c)(m)}(p, \theta') = \frac{-i}{p^2 + \frac{n^2}{R^2} - i\epsilon} \delta_{ij} \delta_{mn} \delta^4(\theta - \theta') \\
\Phi_i^{(n)}(-p, \theta) & \xrightarrow{p} \Phi_j^{(c)(m)}(p, \theta') = \frac{D^2(p)}{4} \frac{-i\gamma}{p^2 (p^2 + \frac{n^2}{R^2}) - i\epsilon} \delta_{ij} \delta_{mn} \delta^4(\theta - \theta') \\
\Phi_i^{(c)(n)}(-p, \theta) & \xrightarrow{p} \Phi_j^{(m)}(p, \theta') = \frac{-i\gamma}{p^2 (p^2 + \frac{n^2}{R^2}) - i\epsilon} \delta_{ij} \delta_{mn} \delta^4(\theta - \theta') \\
V_{a(n)}(-p, \theta) & \xrightarrow{p} V_{b}^{(m)}(p, \theta') = \frac{i}{p^2 + \frac{n^2}{R^2} - i\epsilon} \delta_{ab} \delta_{mn} \delta^4(\theta - \theta') \\
\chi_a^{(n)}(-p, \theta) & \xrightarrow{p} \chi_b^{(m)}(p, \theta') = \frac{2i}{p^2 + \frac{n^2}{R^2} - i\epsilon} \delta_{ab} \delta_{mn} \delta^4(\theta - \theta')
\end{align*}
\]

and we derive

\[
\begin{align*}
\Phi^{(0)} & \xrightarrow{V^{a(0,n)}} 2i g T^a \\
\chi^{a(n)}(\Phi^{(0), n}) & \xrightarrow{i g T^a} \Phi^{(0)} \\
\Phi_k^{(0)} & \xrightarrow{i \frac{1}{6} \lambda_{ijk}} \Phi_k^{(n)} \\
\Phi_i^{(0)} & \xrightarrow{i \frac{1}{2} \lambda_{ijk}} \Phi_i^{(m)} \\
\Phi_j^{(0)} & \xrightarrow{i \lambda_{ijk}} \Phi_j^{(n)}
\end{align*}
\]

We can do the same for the case where all superfields containing SM fermions are restricted to the brane. The part of the action involving only gauge and Higgs fields is not modified, whereas the action for the superfields containing the SM fermions splits into:

\[
S_{brane} = \int d^8 z d y \delta(y) \left\{ \bar{\Phi}_i \Phi_i + 2g \bar{\Phi}_i V_i \right\}
= \int d^8 z \left\{ \bar{\Phi}_i \Phi_i + 2g \bar{\Phi}_i V^{(0)} \Phi_i + 2 \sqrt{2} g \sum_{n \geq 1} \bar{\Phi}_i V^{(n)} \Phi_i \right\}
\]

\[
S_{Yukawa} = \int d^8 z d y \delta(y) \left\{ \tilde{Y}_c E^c L H_d + \tilde{Y}_d D^c Q H_d + \tilde{Y}_u U^c Q H_u + \frac{1}{4} \kappa L H_u L H_u + h.c. \right\}
= \int d^8 z \left\{ Y_c E^c L H_d^{(0)} + Y_d D^c Q H_d^{(0)} + Y_u U^c Q H_u^{(0)} + \frac{1}{4} \kappa L H_u^{(0)} L H_u^{(0)} \right. \\
+ \sum_{n \geq 1} \sqrt{2} \left( Y_c E^c L H_d^{(n)} + Y_d D^c Q H_d^{(n)} + Y_u U^c Q H_u^{(n)} + \frac{1}{2} \kappa L H_u^{(n)} L H_u^{(n)} \right) \\
+ \sum_{m,n \geq 1} \frac{1}{2} \kappa L H_u^{(m)} L H_u^{(n)} + h.c. \right\}
\]
where we have written: $Y_i = \hat{Y}_i / \sqrt{\pi R}$ et $\kappa = \hat{\kappa} / \pi R$.

The propagators and Feynman rules can be derived in the same way as before.

APPENDIX C: KALUZA-KLEIN INTEGRALS

We will give here the major steps to the derivation of the relevant Kaluza-Klein integrals and the computation of divergences following [7]. We will not show finite terms, i.e. terms which vanish in the limit $\Lambda \to \infty$. For the calculation of the one-loop contributions to the wave function renormalisation constants we need four types of two-point functions which we will discuss in turn.

1. Two excited KK modes with the same KK number

The first case contains two excited KK modes, but their KK number is restricted to be the same. It is illustrated in Fig. 11 and arises typically from bulk interactions. For example, it enters the contribution from the one-loop correction containing one vector superfield and one chiral superfield (which can be only Higgs for the model discussed in Sect. III B 2 and Higgs or matter superfield for the model in Sec. III B 1).

Let us start from the following general expression

$$K = \sum_{n \geq 1} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2 - n^2/R^2} \frac{1}{k^2 - M_0^2 - n^2/R^2}. \quad (C1)$$

After introducing Feynman parameters and performing a Wick rotation in the standard way, the resulting momentum integration can be re-expressed using a proper-time regularised form. This allows to introduce two cutoff parameters, $t_{IR}$ and $t_{UV}$ to treat infrared and ultraviolet divergences, respectively. Assuming $M_0 \sim m_0 \ll R^{-1}$ the sum over KK states can be evaluated. We obtain

$$(16\pi^2)K = \int_0^1 dx \int_{t_{UV}}^{t_{IR}} \frac{dt}{t} \left[ \frac{1}{2} \theta_3 \left( \frac{it}{\pi R^2} \right) - \frac{1}{2} \right] \quad (C2)$$

The $\theta_3$ function arising from the sum over KK states is defined as:

$$\theta_3(x) = \sum_{n=-\infty}^{n=+\infty} e^{i\pi n^2 x} \quad (C3)$$

As discussed in the appendix of Ref. [7] we can use the approximate form of the $\theta_3$ function,

$$\theta_3 \left( \frac{it}{\pi R^2} \right) \approx R \sqrt{\frac{\pi}{t}} \quad (C4)$$

to evaluate the integral. This form of the $\theta_3$ function can be applied in general if $t_{IR}, t_{UV} \ll R^2$, but it gives also a
very good approximation to the integral if \( t_{\text{IR}} \sim R^2 \). We then obtain:

\[
(16\pi^2) \mathcal{K} \simeq \int_0^1 dx \int_{t_{\text{UV}}}^{t_{\text{IR}}} dt \left( \frac{1}{2} \sqrt{\frac{\pi R^2}{t}} - \frac{1}{2} \right) = \int_{t_{\text{UV}}}^{t_{\text{IR}}} dt \left( \frac{\sqrt{\pi R}}{2t^{3/2}} - \frac{1}{2t} \right) \tag{C5}
\]

With the redefinitions \( t_{\text{UV}} = r\Lambda^{-2} \), \( t_{\text{IR}} = rR^2 \) and \( r = \pi/4 \) the final result reads:

\[
(16\pi^2) \mathcal{K} = -2 + 2\Lambda R - \frac{1}{2} \log \Lambda^2 R^2 \tag{C6}
\]

\[
\simeq 2\Lambda R - \frac{1}{2} \log \Lambda^2 R^2 \tag{C7}
\]

In the last line we have supposed that \( \Lambda R \gg 1 \).

In Ref. [26], the authors discussed another coherent cut-off regularisation scheme which allows for obtaining the preceding result \([C8]\) naturally without any rescaling of the cut-off when the KK tower is truncated at \( \Lambda R \).

### 2. Two KK excitations with different KK numbers

We now perform the integral where two Kaluza-Klein states run in the loop and are not constrained to have the same KK number,

\[
\mathcal{G} = \sum_{n,m \geq 1} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2 - n^2/R^2 k^2 - M_0^2 - m^2/R^2} . \tag{C9}
\]

This type of integral arises only in the model discussed in Sec. III [12], where all matter superfields propagate in the bulk. It appears in connection with the Yukawa interactions restricted to the brane. This type of function is illustrated in Fig. 12.

![Fig. 12: One-loop diagram where two Kaluza-Klein states are not constrained to be equal](image)

Following the same steps as in the first case, we have:

\[
(16\pi^2) \mathcal{G} = \int_0^1 dx \sum_{n,m \geq 1} \int \frac{dt}{t} e^{-\Delta_{nm} t} = \int_0^1 dx \int_{t_{\text{UV}}}^{t_{\text{IR}}} \frac{dt}{t} \left[ \frac{1}{2} \theta_3 \left( \frac{itx}{\pi R^2} \right) - \frac{1}{2} \right] \left[ \frac{1}{2} \theta_3 \left( \frac{it(1-x)}{\pi R^2} \right) - \frac{1}{2} \right] , \tag{C10}
\]

where

\[
\Delta_{nm} = (m_0^2 + n^2/R^2)x + (M_0^2 + m^2/R^2)(1 - x) \simeq n^2 x/R^2 + m^2 (1-x)/R^2 . \tag{C11}
\]

In exactly the same way as in the previous section we obtain in this case

\[
(16\pi^2) \mathcal{G} = \pi \Lambda^2 R^2 - \pi + 4 - 4\Lambda R + \frac{1}{4} \log \Lambda^2 R^2 \tag{C12}
\]

\[
\simeq \pi \Lambda^2 R^2 - 4\Lambda R + \frac{1}{4} \log \Lambda^2 R^2 . \tag{C13}
\]
3. One KK mode running in the loop

The third type of integral contains one zero mode and one excited KK mode:

\[ \mathcal{H} = \sum_{n \geq 1} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2} \frac{1}{k^2 - M_0^2} . \] (C14)

It is illustrated in Fig. 13. The evaluation proceeds in exactly the same way as before. We obtain

\[ (16\pi^2) \mathcal{H} = \int_0^1 dx \sum_{n \geq 1} \int dt \frac{e^{-n^2x/R^4}}{t} = 4\Lambda R - 4 - \frac{1}{2} \log \Lambda^2 R^2 . \] (C15)

which gives upon performing the same approximations as before

\[ (16\pi^2) \mathcal{H} = 4\Lambda R - 4 - \frac{1}{2} \log \Lambda^2 R^2 \] (C16)

\[ \simeq 4\Lambda R - \frac{1}{2} \log \Lambda^2 R^2 . \] (C17)

![FIG. 13: One-loop diagram with only one Kaluza-Klein state in the loop.](image)

4. Only zero modes running in the loop

We recall the result of a loop with only zero modes,

\[ I = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m_0^2} \frac{1}{k^2 - M_0^2} , \] (C18)

which reads

\[ (16\pi^2) I = \int_0^1 dx \int dt e^{-t(m_0^2x + M_0^2(1-x))} \simeq \log \Lambda^2 R^2 . \] (C19)

APPENDIX D: RENORMALISATION CONSTANTS

Within this section we summarise the explicit expressions for all wave function renormalisation constants needed to compute the beta functions. In this section we will display the diagrams entering in the one-loop renormalisation of the Yukawa couplings and their values, in order to deduce the wave functions renormalisation and the beta functions. Extensive use is made of the integrals calculated in the last section.
1. Matter fields propagating in the bulk

We have 5 types of diagrams:

We provide some steps of the calculation for the first diagram and give the result for the four others:

\[
\delta Z^{(1)}_{ij} = -4g^2(T^aT^a)_{rs} \delta_{ij} \sum_{n \geq 0} \int \frac{d^4 k}{(2\pi)^4} d^4 \theta_1 d^4 \theta_2 \frac{i \delta_{12}^4}{2(k^2 + \frac{m^2}{4})} \\
\times \frac{1}{16} D^2 \overline{D}^2 \frac{-i \delta_{12}^4}{(k + p)^2 + \frac{m^2}{4}} \phi_i^{r(0)}(-p, \theta_1) \overline{\phi_j^{s(0)}}(p, \theta_2)
\]

\[
= -i 2g^2 C_2(R) \delta_{ij}^r \left( K + \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2(p + k)^2} \right) \int d^4 \theta_1 \phi_i^{r(0)}(-p, \theta_1) \overline{\phi_j^{s(0)}}(p, \theta_1)
\]

\[
= i \frac{-2g^2 C_2(R) \delta_{ij}^r}{16\pi^2} (2\Lambda R + \log(\Lambda R)) \int d^4 \theta_1 \phi_i^{r(0)}(-p, \theta_1) \overline{\phi_j^{s(0)}}(p, \theta_1)
\]

\[
\delta Z^{(2)}_{ij} = i \frac{-2g^2 C_2(R) \delta_{ij}^r}{16\pi^2} (2\Lambda R - \log(\Lambda R)) \\
\delta Z^{(3)}_{ij} = i \frac{\lambda_{ikl} \lambda_{jkl}^* \delta_{rs}}{16\pi^2} \log(\Lambda R)
\]

\[
\delta Z^{(4)}_{ij} = i \frac{2\lambda_{ikl} \lambda_{jkl}^* \delta_{rs}}{16\pi^2} (4\Lambda R - \log(\Lambda R)) \\
\delta Z^{(5)}_{ij} = i \frac{\lambda_{ikl} \lambda_{jkl}^* \delta_{rs}}{16\pi^2} (2\pi(\Lambda R)^2 - 8\Lambda R + \log(\Lambda R))
\]

We only displayed the integral over the \( \theta \) coordinate for the first contribution, omitting it in the others.

Summing the different contributions, every subdominant divergence disappears for both the gauge and the Yukawa contribution and we obtain:

\[
-(16\pi^2) \delta Z^{(D)}_{\Phi} = \left( -8 \sum_{n=1}^{N_\phi} g^2 C_2(R^{(i)}_n) \delta_{ij} \right) \Lambda R + \left( 2\pi \sum_{k,l=1}^{N_\phi} \lambda_{ikl}^* \lambda_{jkl} \right) \Lambda^2 R^2 . \tag{D1}
\]
Applying it to the matter and Higgs superfields:

\begin{align}
-(16\pi^2) \delta Z_{H_u} &= 12\pi \text{Tr}(Y_u^\dagger Y_u) \Lambda^2 R^2 - \left(\frac{6}{5} g_1^2 + 6g_2^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{H_d} &= 4\pi [3\text{Tr}(Y_d^\dagger Y_d) + \text{Tr}(Y_e^\dagger Y_e)] \Lambda^2 R^2 - \left(\frac{6}{5} g_1^2 + 6g_2^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{L} &= 4\pi (Y_T^\dagger Y_e) \Lambda^2 R^2 - \left(\frac{6}{5} g_1^2 + 6g_2^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{E_C} &= 8\pi (Y_e^\dagger Y_T^\dagger Y_e) \Lambda^2 R^2 - \left(\frac{24}{5} g_1^2 + 6g_2^2 + \frac{32}{3} g_3^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{D_c} &= 8\pi (Y_e^\dagger Y_T^\dagger Y_e) \Lambda^2 R^2 - \left(\frac{16}{5} g_1^2 + \frac{32}{3} g_3^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{Q} &= 4\pi (Y_u^\dagger Y_u + Y_d^\dagger Y_d) \Lambda^2 R^2 - \left(\frac{2}{15} g_1^2 + 6g_2^2 + \frac{32}{3} g_3^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{U^c} &= 8\pi (Y_u^\dagger Y_u) \Lambda^2 R^2 - \left(\frac{32}{15} g_1^2 + \frac{32}{3} g_3^2\right) \Lambda R \\
-(16\pi^2) \delta Z_{U^c} &= 8\pi (Y_u^\dagger Y_u) \Lambda^2 R^2 - \left(\frac{32}{15} g_1^2 + \frac{32}{3} g_3^2\right) \Lambda R
\end{align}

from which we deduce the Yukawa beta functions:

\begin{align}
\beta_{Y_d} &= -\frac{1}{2} \frac{\partial}{\partial \Lambda} (\delta Z_{D_c}^T Y_d + Y_d \delta Z_{Q} + Y_d \delta Z_{H_d}) \\
\beta_{Y_u} &= -\frac{1}{2} \frac{\partial}{\partial \Lambda} (\delta Z_{U^c}^T Y_u + Y_u \delta Z_{Q} + Y_u \delta Z_{H_u}) \\
\beta_{Y_d} &= -\frac{1}{2} \frac{\partial}{\partial \Lambda} (\delta Z_{E_C}^T Y_e + Y_e \delta Z_{L} + Y_e \delta Z_{H_d})
\end{align}

2. Matter fields restricted to the brane

There again we show all 4 diagrams contributing for the 3 generations of flavour:

\begin{align}
&\delta Z_{ij}^{(1)} = i \frac{-4 g^2 C_2(R) \delta_{ij}^T}{16\pi^2} \log(\Lambda R) \\
&\delta Z_{ij}^{(2)} = i \frac{8 g^2 C_2(R) \delta_{ij}^T}{16\pi^2} (4\Lambda R - \log(\Lambda R)) \\
&\delta Z_{ij}^{(3)} = i \frac{\lambda_{ikl} \lambda_{jkl}^T}{16\pi^2} \log(\Lambda R) \\
&\delta Z_{ij}^{(4)} = i \frac{\lambda_{ikl} \lambda_{jkl}^T}{16\pi^2} (4\Lambda R - \log(\Lambda R))
\end{align}

The sum gives:

\begin{align}
\delta Z_{ij}^L &= -\frac{1}{16\pi^2} \left[ -16\Lambda R g^2 C_2(R) \delta_{ij} + 4\Lambda R \lambda_{ikl} \lambda_{jkl}^T \right] \\
\delta Z_{ij}^H &= \frac{1}{16\pi^2} \left[ -8\Lambda R g^2 C_2(R) + 2 \log(\Lambda R) \text{Tr}(Y_i^\dagger Y_i^\dagger) \right]
\end{align}

As for the Higgses, the gauge diagrams are those the previous case and for \(\delta Z\) we collect:

\begin{align}
\delta Z^H &= \frac{1}{16\pi^2} \left[ -8\Lambda R g^2 C_2(R) + 2 \log(\Lambda R) \text{Tr}(Y_i^\dagger Y_i^\dagger) \right]
\end{align}
It is straightforward to deduce the following renormalisation constants for the matter and Higgs superfields:

\[(16\pi^2) \delta Z_{H_u} = 6 \text{Tr}(Y_u^\dagger Y_u) \log \Lambda R - \left(\frac{6}{5} g_1^2 + 6 g_2^2\right) \Lambda R \quad (D11)\]

\[-(16\pi^2) \delta Z_{H_d} = \left[6 \text{Tr}(Y_d^\dagger Y_d) + 2 \text{Tr}(Y_e^\dagger Y_e)\right] \log \Lambda R - \left(\frac{6}{5} g_1^2 + 6 g_2^2\right) \Lambda R \quad (D12)\]

\[-(16\pi^2) \delta Z_L = \left[8(Y_e^\dagger Y_e) - \frac{12}{5} g_1^2 - 12 g_2^2\right] \Lambda R \quad (D13)\]

\[-(16\pi^2) \delta Z_{E^c} = \left[16(Y_e^\dagger Y_e^T) - \frac{48}{5} g_1^2\right] \Lambda R \quad (D14)\]

\[-(16\pi^2) \delta Z_{D^c} = \left[16(Y_d^\dagger Y_d^T) - \frac{16}{15} g_1^2 - \frac{64}{3} g_3^2\right] \Lambda R \quad (D15)\]

\[-(16\pi^2) \delta Z_{Q} = \left[8(Y_1^\dagger Y_1 + Y_d^\dagger Y_d) - \frac{4}{15} g_1^2 - 12 g_2^2 - \frac{64}{3} g_3^2\right] \Lambda R \quad (D16)\]

\[-(16\pi^2) \delta Z_{U^c} = \left[16(Y_u^\dagger Y_u^T) - \frac{64}{15} g_1^2 - \frac{64}{3} g_3^2\right] \Lambda R \quad (D17)\]
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