Dynamics of Entanglement in Qubit-Qutrit with $x$-Component of DM Interaction

Kapil K. Sharma* and S.N. Pandey†

Department of Physics, Motilal Nehru National Institute of Technology, Allahabad 211004, India

(Received August 27, 2015; revised manuscript received November 23, 2015)

Abstract In this present paper, we study the entanglement dynamics in qubit $A$-qutrit $B$ pair under $x$ component of Dzyaloshinskii–Moriya interaction ($D_x$) by taking an auxiliary qubit $C$. Here, we consider an entangled qubit-qutrit pair initially prepared in two parameter qubit-qutrit states and one auxiliary qubit prepared in pure state interacts with the qutrit of the pair through DM interaction. We trace away the auxiliary qubit and calculate the reduced dynamics in qubit $A$-qutrit $B$ pair to study the influence of the state of auxiliary qubit $C$ and $D_x$ on entanglement. We find that the state (probability amplitude) of auxiliary qubit does not influence the entanglement, only $D_x$ influences the same. The phenomenon of entanglement sudden death (ESD) induced by $D_x$ has also been observed. We also present the affected and unaffected two parameter qubit-qutrit states by $D_x$.

PACS numbers: 03.65.Yz, 03.67.-a, 03.67.Lx

Key words: entanglement dynamics, qubit-qutrit, DM interaction, ESD

1 Introduction

Entanglement\cite{1−3} is the most fascinating phenomenon in quantum information, which has no classical counterpart. Many future quantum technologies are based on the same. However, various mathematical investigations are on the way to explore the inherited properties of entanglement. The phenomenon of entanglement is very much fragile with respect to environmental interactions. Quantum measurements also lead to decoherence which destroys the entanglement in quantum systems. If the entanglement is destroyed for finite interval of time then this phenomenon is called entanglement sudden death (ESD).\cite{4−6} So the study of influence of interactions on entanglement in varieties of quantum systems is important for practical quantum information processing. Zhang et al. have studied the entanglement dynamics of qubit $A$-qubit $B$ by taking a third qubit $C$ which interacts with the qubit $B$ under $x$-component of DM interaction.\cite{7−9} They have shown that DM interaction amplify and periodically kills the entanglement between two qubits ($A$ and $B$) and by adjusting the state (probably amplitude) of third qubit $C$ and DM interaction strength, one can manipulate the entanglement and control the entanglement sudden death (ESD). They have also studied the same system by taking the $x$-component of the DM interaction ($D_x$).\cite{10} Here, we mention that this kind of study is useful not only in qubit-qubit systems but it is also useful for hybrid quantum systems. Recently, we have studied the dynamics in hybrid qubit-qutrit systems by taking the DM interaction strength in $z$ direction.\cite{11−13} DM interaction in different directions has different impact on varieties of quantum states. So, it is interesting to study the dynamics of entanglement with $x$-component of DM interaction.

Motivated, with the above studies we study the entanglement dynamics in hybrid qubit-qutrit systems with $D_x$. In the present paper we have considered a closed system of qubit $A$-qutrit $B$, which has been prepared in two parameter states, and an auxiliary qubit $C$, prepared in pure state, interacts with the qutrit $B$ of the closed system. The auxiliary quantum systems play an important role in quantum information processing. Many auxiliary quantum systems have been investigated which assist in improving the entanglement in quantum systems.\cite{14−17} We trace away the auxiliary qubit $C$ to calculate the reduced system dynamics. Further, we study the influence of the state of auxiliary qubit $C$ and $D_x$ on the entanglement between qubit $A$-qutrit $B$ pair. We find that the state (probability amplitude) of third qubit $C$ does not influence the entanglement in two parameter qubit-qutrit states, only DM interaction influences the same. The $D_x$ is geometry dependent, so by changing the geometrical arrangement between interacting qutrit $B$ and qubit $C$, one can manipulate the entanglement between qubit-qutrit pair. Dzyaoshinshkii–Moriya interaction\cite{18−20} is an anisotropic antisymmetric interaction investigated by taking into consideration the relativistic effects to describe the ferromagnetism of anti ferromagnetic crystals. Quantum spin chains are the important building blocks to execute the practical quantum information processing.\cite{21} Many spin chains with external magnetic filed and DM interaction have been investigated in varieties of configurations even at thermal conditions.\cite{22−29} Spin chains have also been studied by taking as a bath in the form

* E-mail: scienceglobal@gmail.com
† E-mail: snp@mnit.ac.in
© 2016 Chinese Physical Society and IOP Publishing Ltd

http://www.iopscience.iop.org/ctp  http://ctp.ita.ac.cn
of spin chains with DM interaction. It has been observed that in some cases the DM interaction enhance the entanglement present in quantum spin chains. So, DM interaction is a useful resource in quantum information processing. Recently, we also have shown the efficacy of DM interaction to free the bound entangled states in quantum information processing. In the present study, we find that DM interaction does not kill the entanglement in some two parameter qubit-qutrit states, so the affected and unaffected states have also been reported here. The two parameter qubit-qutrit states in $2 \otimes n$ quantum systems have been investigated by Pyo Chi and its generalization in higher dimensions ($m \otimes n, n \geq m \geq 3$) quantum systems by Di. Entanglement dynamics in two-parameter qubit-qutrit states under various channels has also been studied by Wei, et al. and Hao and Fu.

The plan of the paper is as follows. In Sec. 2, we present the Hamiltonian of the system considered. Two parameter qubit-qutrit states its separable and non-separable regions and entanglement measure have been given in Sec. 3. Section 4 is devoted to the entanglement components are Gell–Mann matrices and $\vec{\sigma}$ and $\vec{\sigma}$ are Pauli matrices of qubit $C$ respectively. The above Hamiltonian is a matrix having $6 \times 6$ dimension and is easy to diagonalize by using the method of eigendecomposition. The unitary time evolution operator is easily commutable as

$$U(t) = e^{-iHt},$$

which is also a $6 \times 6$ matrix. This matrix has been used to obtain the time evolution of density matrix of the system.

3 Two Parameter Class of States and Entanglement Measure

We describe the two parameter class of states of qubit-qutrit and entanglement measure (negativity).

The class of states in qubit-qutrit ($2 \otimes 3$) dimensional quantum system is given as

$$\rho_{AB} = \alpha |02\rangle\langle 02| + |12\rangle\langle 12| + \beta (|\phi^+\rangle\langle \phi^+| + |\phi^-\rangle\langle \phi^-| + |\psi^+\rangle\langle \psi^+| + |\psi^-\rangle\langle \psi^-|),$$

where $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$, $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$, and real parameters $\alpha, \beta, \gamma$ satisfy the following condition,

$$2\alpha + 3\beta + \gamma = 1,$$

which is obtained by taking the trace of the density matrix given in Eq. (5).

The negativity $N(\rho^{AB})$ of the two parameter class of states for $2 \otimes 3$ dimension given in (5) can be obtained by using the formula

$$N(\rho^{AB}) = \frac{1}{2} \left( \|\rho^{TA}\| - 1 \right),$$

where $\rho^{TA}$ is the partial transpose of the reduced density matrix and $\|\cdots\|_1$ denotes the trace norm. The negativity is obtained as

$$N(\rho^{AB}) = \max \{0, 2(\alpha + \gamma) - 1\}.$$

The two parameter class of state is separable within the limit $0 \leq (\alpha + \gamma) \leq 1/2$ and non-separable within the limit $1/2 < (\alpha + \gamma) \leq 1$. There are two triangular regions: separable and non-separable as depicted in Fig. 2. And the separable region is bounded with the vertices $(\gamma = 0, \alpha = 0)$, $(\gamma = 0, \alpha = 1/2)$ and $(\gamma = 1/2, \alpha = 0)$. The non-separable region is bounded with the vertices $(\gamma = 1/2, \alpha = 0)$, $(\gamma = 0, \alpha = 1/2)$ and $(\gamma = 1, \alpha = 0)$. The vertices $(\gamma = 0, \alpha = 1/2)$ and $(\gamma = 1/2, \alpha = 0)$ are the common vertices of both separable and non-separable regions. The line joining above mentioned vertices forms the boundaries of separable and non-separable regions. The separable region is bounded by the boundaries $BC$, $BA$ and $AC$, and non-separable region is bounded by the boundaries $CD$, $AC$ and $AD$. The boundary $AC$ is the common boundary of both the regions and states falling on this boundary belong to the separable region. By using
Eq. (10), we plot the initial entanglement in two parameter qubit-qutrit states depicted in Fig. 3. This figure shows that there is no initial entanglement in separable region while the initial entanglement in non-separable region lies between the limits 0 to 0.4.

Fig. 2 Separable and non-separable regions in two parameter qubit-qutrit states.

4 Entanglement Dynamics under the $x$-Component of DM Interaction ($D_x$)

To begin the study of entanglement dynamics, we consider that the auxiliary qubit is prepared in pure state as given below

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle,$$

with the normalization condition

$$c_0^2 + c_1^2 = 1.$$

To study the entanglement dynamics with the $D_x$, we need to calculate the reduced density matrix of the system. First we calculate the time evolution composite density matrix as $\rho(t) = U(t)\rho(0)U^\dagger(t)$, where $U(t)$ is the unitary time evolution operator given in Eq. (4) and $\rho(0)$ is the initial composite density matrix of qubit $A$, qutrit $B$ and qubit $C$, which can be written as $\rho(0) = \rho^{AB} \otimes \rho^{C}$. Here $\rho^{C}$ is the density matrix of the auxiliary qubit which is prepared in pure state. Further, we obtain the reduced density matrix by tracing away the auxiliary qubit $C$. The reduced density matrix is obtained for further calculations which is complicated to write here. We get the term $p = (c_0^2 + c_1^2)$ with every element of the matrix, so by imposing the normalization condition given in Eq. (12), the factor disappears from the matrix and hence the probability amplitude of auxiliary qubit does not play any role in entanglement dynamics. Next, we calculate the entanglement by using reduced density matrix with the help of negativity given in Eq. (9) along the boundaries $BC$, $BA$, $AC$, $CD$, $AD$ and the separable and non-separable regions. The study has been discussed in the subsequent sub-sections.

4.1 Entanglement Dynamics along the Boundary BC

In this section, we present the study of entanglement dynamics along the boundary $BC$ which lies in separable region as depicted in Fig. 2. For $D_x = 0$ there is no entanglement present in the quantum states lying along the boundary $BC$ as all the states are separable. We plot the entanglement for $D_x = 0.2, 0.4,$ and $0.6$ with associated contour plots in Figs. 4 and 5.

Fig. 4 Entanglement plot along the boundary $BC$. 
First, we plot the entanglement evolution for $D_x = 0.2$ with $0 \leq \gamma \leq 0.5$, which is shown in Fig. 4(a) and its corresponding contour plot is depicted in Fig. 5(a). The ESD is observed in contour plot as shown in Fig. 5(a). The ESD zone lies within the range $0.11 \leq \gamma \leq 0.50$ and within the time interval $3.44 \leq t \leq 4.31$. Further, we plot the entanglement for $D_x = 0.4$ in Fig. 4(b) and its corresponding contour plot in Fig. 5(b). By observing both the figures, we conclude that the maximum amplitude of the entanglement remains same but the fluctuations of entanglement increases. Next, we plot the results for $D_x = 0.6$ in Figs. 4(c) and 5(c). From the contour plots, we observe that increasing DM interaction strength produces more ESD zones and previous ESD zones shrink. We conclude that increasing strength of DM interaction produces the entanglement in the states lying along the boundary $BC$ and maximum amplitude achieved by the entanglement is 0.1.

4.2 Entanglement Dynamics along the Boundary BA

The entanglement dynamics along the boundary $BC$ has been discussed in the previous section. In this section, we study the entanglement dynamics through DM interaction along the boundary $BA$. All the states lying along this boundary are separable. We plot the entanglement evolution lying along this boundary with $0 \leq \alpha \leq 0.5$. The results are plotted in Figs. 6 and 7.
First we plot the result for $D_x = 0.2$ in Fig. 6(a) and its corresponding contour plot in Fig. 7(a). We observe that for initial values of $\alpha$ the entanglement is produced but as the value of $\alpha$ increases the entanglement dies out with the advancement of time. We observe the ESD zone in contour plot as shown in Fig. 7(a). The maximum amplitude of entanglement achieved is 0.2. For $D_x = 0.4$, we show the evolution of entanglement in Fig. 6(b) and its corresponding contour plot in Fig. 7(b). Here, we observe that the maximum value of entanglement is 0.3. The ESD is also seen in Fig. 7(b). Next, we plot the entanglement with $D_x = 0.6$ in Fig. 6(c) and its corresponding contour plot in Fig. 7(c). We observe that the maximum amplitude of entanglement is now stabilized with the increasing value of DM interaction strength. In the corresponding contour plot we again observe ESD zone.

4.3 Entanglement Dynamics inside the Region $ABC$ including the Boundary $AC$

In this section, we discuss the entanglement evolution inside the region $ABC$ including the boundary $AC$. The region satisfies the condition $0 \leq (\alpha + \gamma) \leq 0.5$. It is very complicated to obtain the results by varying the parameters $\alpha$, $\gamma$, $D_x$ and $t$ all together so we plot the entanglement evolution for different values of the parameter $D_xt$. The three-dimensional plots are shown in Fig. 8. We observe that with the increasing value of $D_xt$, the entanglement is produced in the region with ESD effects. The ESD zones periodically shrink and expand. By observing all the figures we find that the maximum amplitude of entanglement is 0.3 (which is obtained for $D_xt = 6$). Further, we find that the maximum ESD zone is produced with $D_xt = 11$ with the amplitude 0.02 of entanglement. This result is also shown separately in Fig. 10(a). We also plot the corresponding contour plots of Fig. 8 in Fig. 9 to depict the ESD zones.

![Fig. 8 Entanglement plot inside the region $ABC$.](image-url)
4.4 Entanglement Dynamics inside the Region ACD

In this section, we describe the results obtained for entanglement evolution for the region ACD. The result has been shown in Fig. 10(b). We plot the entanglement evolution for different values of $D_x t$ and observe that as the value of $D_x t$ increases the amplitude of entanglement increases and achieves the maximum limit as 0.4. And it is important to note that there is no ESD found in this region ACD. We here recall the result obtained with the $z$-component of DM interaction of earlier study.\[1] In both the cases we observe that the states lying in the region ACD do not suffer from ESD.

![Entanglement in ACD](image)

Fig. 9 Contour Plot of Entanglement inside the region ABC.

![Entanglement plots](image)

Fig. 10 (a) Maximum entanglement plot with $D_x t = 11$; (b) Entanglement plot inside the region ACD.
5 Conclusion

In this paper, we present the study of entanglement dynamics in two parameter qubit-qutrit states with the $x$-component of DM interaction by taking an auxiliary qubit. We studied the entanglement dynamics in separable and non-separable regions and along their associated boundaries. First, we observe that as the value of DM interaction strength increases, the entanglement is produced in the separable region with ESD. The maximum amount of entanglement achieved in this region is 0.3. Next, we present the dynamics along the boundary $BC$ of the separable region. We observe that the states lying along this boundary go under ESD as time advances. As the value of the parameter value $D_zt = 11$. We also present the entanglement dynamics inside the non-separable region and plot it for different values of the parameter $D_zt$. We find that the increasing strength of DM interaction increases the amplitude of the entanglement present in this region but do not produce the ESD zones.

We get maximum amount of entanglement as 0.15 due to $z$-component of DM interaction, $x$-component of DM interaction is producing maximum entanglement as 0.30 in separable region $ABC$. By comparing the maximum entanglement produced by $x$ and $z$ components of DM interaction in separable region $ABC$, we conclude that $x$-component of DM interaction is producing more amount of maximum entanglement. However in the region $ACD$, the $x$ and $z$ components are producing same amount of entanglement.

References

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47 (1935) 777.
[2] M.A. Nielsen and I.L. Chuang, Quantum computation and quantum information. Cambridge University Press, Cambridge (2000).
[3] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. 80 (2008) 517.
[4] T. Yu and J.H. Eberly, Phys. Rev. Lett. 93 (2004) 140404.
[5] T. Yu and J.H. Eberly, Opt. Commun. 264 (2006) 393.
[6] T. Yu and J.H. Eberly, Science 30 (2009) 598.
[7] Q. Zheng, X.P. Zhang, Q.J. Zhi, and Z.Z. Ren, Chin. Phys. B 18 (2009) 3210.
[8] Q. Zheng, P. Sun, X.P. Zhang, and Z.Z. Ren, Chin. Phys. C 34 (2010) 1583.
[9] Q. Zheng, Q.J. Zhi, X.P. Zhang, and Z.Z. Ren, Chin. Phys. C 35 (2011) 135.
[10] Q. Zheng, P. Sun, X.P. Zhang, and Z.Z. Ren, Commun. Theor. Phys. 54 (2010) 417.
[11] K.K. Sharma, S.K. Awasthi, and S.N. Pandey, Quantum Info. Proc. 12 (2013) 3437.
[12] K.K. Sharma and S.N. Pandey, Quantum Info. Proc. 13 (2014) 1361.
[13] K.K. Sharma and S.N. Pandey, Quantum Info. Proc. 14 (2015) 1361.
[14] W. Dr, G. Vidal, J.I. Cirac, N. Linden, and S. Popescu, Phys. Rev. Lett. 87 (2001) 137901.
[15] B. Kraus and J.I. Cirac, Phys. Rev. A 63 (2001) 062309.
[16] W. Dr, G. Vidal, and J.I. Cirac, Phys. Rev. Lett. 89 (2002) 057901.
[17] B.L. Hu and Y.M. Di, Commun. Theor. Phys. 47 (2007) 1029.
[18] I. Dzyaloshinsky, J. Phys. Chem. Solids 4 (1958) 241.
[19] T. Moriya, Phys. Rev. Lett. 4 (1960) 228.
[20] T. Moriya, Phys. Rev. Lett. 120 (1960) 91.
[21] R.I. Nepomechie, Int. J. Mod. Phys. A 13 (1999) 2973.
[22] E.A. Ivanchenko, Int. J. Quantum Inf. 10 (2012) 1250068.
[23] R. Jafari and A. Langari, Int. J. Quantum Inf. 9 (2011) 1057.
[24] Y.Y. Yan, L.G. Qin, and L.J. Tian, arXiv:1109.5458v1.
[25] M.L. Hu, Phys. Lett. A 374 (2010) 3520.
[26] M. Tursun, A. Abliz, R. Mamtimin, et al., arXiv:1207.-0277v4.
[27] Z.N. Gurkan and O.K. Pashaev, arXiv:0804.0710v2.
[28] X.Z. Yuan, Hsi-Sheng Goan, and K.D. Zhu, Phy. Rev. B 75 (2007) 045331.
[29] W. You and Y. Dong, Eur. Phys. J. D 57 (2010) 439.
[30] H.P. Breuer, D. Burgarth, and F. Petruione, arXiv:quant-ph/041051v2.
[31] K.K. Sharma and S.N. Pandey, Quant. Info. Proc. (2016), DOI 10.1007/s11128-015-1234-3, arXiv:1501.00942.
[32] K.K. Sharma and S.N. Pandey, arXiv:1412.4883v2.
[33] D. Pyo Chi and S. Lee, J. Phys A: Math. Gen. 36 (2003) 11503.
[34] Y.M. Di, S.P. Liu, and D.D. Liu, Science Chin. Phys. Mechanics and Astronomy 53 (2010) 1868.
[35] H.R. Wei, B.C. Ren, T. Li, et al., Commun. Theor. Phys. 57 (2012) 983.
[36] Y. Hao and W.L. Fu, Chin. Phys. B 22 (2013) 050303.