GENERALIZED STRICHARTZ ESTIMATES ON PERTURBED WAVE EQUATION AND APPLICATIONS ON STRAUSS CONJECTURE

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1. Introduction and Main Result.

The purpose of this paper is to show a general Strichartz estimate for certain perturbed wave equation under known local energy decay estimates, and as application, to get the Strauss conjecture for several convex obstacles in \( n = 3, 4 \). Our results improve on earlier work in Hidano, Metcalfe, Smith, Sogge and Zhou [14]. First, and most important, we can drop the nontrapping hypothesis and handle trapping obstacles with some loss of derivatives for data in the local energy decay estimates (see (1.2) below). This hypothesis is fulfilled in many cases in the non-trapping case when there is local decay of energy with no loss of derivatives (see [32], [21], [30], [3], [23]). (1.2) is also known to hold in several examples involving hyperbolic trapped rays (see [15], [16], [9]). In addition to improving the hypotheses on the obstacles, we give the obstacle version of sharp life span for semilinear wave equations when \( n = 3, \ p < p_c \), by using a real interpolation between KSS estimate and endpoint Trace Lemma, and by getting a corresponding finite time Strichartz estimates (see section 3). Lastly, we are able to use the general Strichartz estimates we have gained to get the Strauss conjecture for some perturbed semilinear wave equations with trapped rays when \( n = 3, 4 \) (see Section 4).

We consider wave equations on an exterior domain \( \Omega \subset \mathbb{R}^n \):

\[
\begin{cases}
(\partial_t^2 - \Delta_g)u = F(t, x) \quad \text{on } \mathbb{R}_+ \times \Omega \\
 u|_{t=0} = f, \\
 \partial_t u|_{t=0} = g, \\
 (Bu)(t, x) = 0, \quad \text{on } \mathbb{R}_+ \times \partial \Omega
\end{cases}
\]

(1.1)

where for simplicity we take \( B \) to either be the identity operator or the inward pointing normal derivative \( \partial_n \). The operator \( \Delta_g \) is the Laplace-Beltrami operator associated with a smooth, time independent Riemannian metric \( g_{jk}(x) \) which we assume equals the Euclidean metric \( \delta_{jk} \) for \( |x| \geq R \), some \( R \). The set \( \Omega \) is assumed to be either all of \( \mathbb{R}^n \), or else is a subset of \( |x| < R \) with smooth boundary. Note that here we do not require that \( \mathbb{R}^n / \Omega \) is nontrapping.

We will make the following assumption:

**Hypothesis B'**. Fix the boundary operator \( B \) and the exterior domain \( \Omega \subset \mathbb{R}^n \) as above.
We then assume that given \( R_0 > 0 \)

\[
\int_0^S \left( \| u(t, \cdot) \|_{H^1(|x| < R_0)}^2 + \| \partial_t u(t, \cdot) \|_{L^2(|x| < R_0)}^2 \right) dt 
\lesssim \| f \|_{H^{1+\varepsilon}}^2 + \| g \|_{H^{1+\varepsilon}}^2 + \int_0^S \| F(s, \cdot) \|_{H^\varepsilon}^2 ds,
\]

where \( u \) is a solution of (1.1) with data \((f, g)\) and forcing term \( F \) that both vanish for \(|x| > R_0\).

**Remark 1.1.** We assume \( S \) to be finite time \( T \) or \( \infty \), and although \( \varepsilon \) could be any real number without affecting our techniques much, here we assume \( \varepsilon \geq 0 \) is an arbitrarily small number (which is all we need for now) throughout the paper for clarity of explanation. Note that when \( \varepsilon = 0 \) and \( T = \infty \), it is just the case in [14]. More specifically, when the obstacle is nontrapping, and \( \Delta \) is the standard Euclidean Laplacian, we will have that local energy decays exponentially in odd dimension \( n \geq 3 \) and polynomially in even dimensions except \( n = 2 \) ([20]). For \( n = 2 \), local energy decays like \( O((\log(2+t))^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}}) \) ([32]). These results imply (1.2). When \( \Delta_g \) is a time-independent variable coefficient compact perturbation of \( \Delta \), one also has that (1.2) is valid for the Dirichlet-wave equation for \( n \geq 3 \) as well for \( n = 2 \) if \( \partial \Omega \neq \emptyset \) ([20], [3]). On the other hand, when there are trapped rays, it is known that a uniform decay rate is generally not possible ([24]), but we can get some local energy decay by trading some derivatives in the initial data. Ikawa, for example, gets the following exponential decay when \( n = 3 \) and there are several convex obstacles which are far apart (see [16]),

\[
\| u'(t, x) \|_{L^2(|x| < 1)} \lesssim e^{-at} \| u'(0, x) \|_{H^2(|x| < 1)},
\]

where \( a \) is a constant.

By an interpolation of this estimate and standard energy estimates, it is easy to get

\[
\| u'(t, x) \|_{L^2(|x| < 1)} \lesssim e^{-ct} \| f \|_{H^{1+\varepsilon}(|x| < 1)},
\]

which implies our Hypothesis B'. When there is only one hyperbolic trapped ray, Christianson ([9]) also showed that for all odd dimensions \( n \geq 3 \) we have the local energy decay

\[
\| u'(t, x) \|_{L^2(|x| < 1)} \lesssim e^{-t^{\frac{1}{2}}/C} \| u'(0, x) \|_{H^{1+\varepsilon}(|x| < 1)}
\]

which gives Hypothesis B' as well. Further work in this direction can be seen in [4], [5], [8], [11].

Now we will introduce a revised homogenous Sobolev norm:

**Definition 1.2.** Define \( \tilde{H}^\gamma_\varepsilon(\mathbb{R}^n) \) and \( \tilde{H}^\gamma_\varepsilon(\Omega) \) to be the space with norm defined by

\[
\| h \|_{\tilde{H}^\gamma_\varepsilon} = \left\| D^\gamma (1 - \Delta)^{\frac{1}{2}} \tilde{h} \right\|_{L^2} = \int_{\mathbb{R}^n} \| \xi |^\gamma (1 + |\xi|^2)^{\frac{1}{2}} \tilde{h}(\xi) \|^2 d\xi.
\]

Notice that if \( 0 \leq \varepsilon_1 < \varepsilon_2 \), then

\[
\tilde{H}^\gamma_{\varepsilon_2} \subset \tilde{H}^\gamma_{\varepsilon_1}.
\]
We also notice that, when $\varepsilon \geq 0$ the above norm is equivalent to the following useful form:

\begin{equation}
\|h\|_{\tilde{H}_\gamma^\varepsilon} \simeq \|h\|_{\dot{H}^{\gamma}(\{|x|<1\})} + \|h\|_{\dot{H}^{\gamma+\varepsilon}(\{|x|>1\})} \approx \|h\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|h\|_{\dot{H}^{\gamma+\varepsilon}(\mathbb{R}^n)}.
\end{equation}

The norm on manifold $\Omega$ is defined like in [12] and [26]. Roughly speaking,

\[ \|f\|_{\tilde{H}_\gamma^\varepsilon(\Omega)} = \|\beta f\|_{\tilde{H}_\gamma^\varepsilon(\Omega')} + \|(1-\beta)f\|_{\tilde{H}_\gamma^\varepsilon(\mathbb{R}^n)} \]

where $\Omega'$ is the embedding of $\Omega \cap \{|x|<2R\}$ into the torus obtained by periodic extension of $\Omega \cap [-2R,2R]^n$, so that $\partial \Omega' = \partial \Omega$. And the spaces $\tilde{H}_\gamma^\varepsilon(\Omega')$ are defined by a spectral decomposition of $\Delta_{\mathbb{S}}|_{\Omega'}$ subject to the boundary condition $B$.

We also redefine ”admissible” as follows:

**Definition 1.3.** We say that $(X, \gamma, \eta, p)$ is almost admissible if it satisfies

1. Minkowski almost Strichartz estimates

\begin{equation}
\|u\|_{L_t^p X([0,T] \times \mathbb{R}^n)} \lesssim A(S) \|u(0,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\partial_t u(0,\cdot)\|_{\dot{H}^{\gamma-1}(\mathbb{R}^n)}
\end{equation}

where $A(S)$ is a just constant when $S = \infty$.

2. Local almost Strichartz estimates for $\Omega$

\begin{equation}
\|u\|_{L_t^p X([0,1] \times \Omega)} \lesssim \|u(0,\cdot)\|_{\tilde{H}_\gamma^\varepsilon(\Omega)} + \|\partial_t u(0,\cdot)\|_{\tilde{H}_\gamma^{\varepsilon-1}(\Omega)}
\end{equation}

Notice that here we assume a weaker local Strichartz estimates by losing some derivatives in the regularity of initial data, which probably will happen when there are broken rays in the manifold. We also assume $\eta \geq 0$ is an arbitrarily small number in our theorems, and actually in our application we only need the case when $\eta = 0$.

We will assume $1 - \frac{n}{2} < \gamma < \frac{n}{2}$ throughout, so that $(\dot{H}_\gamma, \dot{H}_\gamma^\varepsilon)$ and $(\dot{H}_{1-\gamma}, \dot{H}_{1-\gamma}^\varepsilon)$ are comparable pairs for functions supported in a ball. Besides, for $\beta \in C^\infty_0(\mathbb{R}^n)$, with $\beta = 1$ on a neighborhood of $\mathbb{R}^n \setminus \Omega$, we assume that

\[ \|(1-\beta)f\|_{X(\Omega)} \approx \|(1-\beta)f\|_{X(\mathbb{R}^n)} \]

Now we will state our main Strichartz estimates:

**Theorem 1.4.** Let $n > 2$ and assume that $(X, \gamma, \eta, p)$ is almost admissible with

\begin{equation}
p > 2 \text{ and } \gamma \in \left[-\frac{n-3}{2}, -\frac{n-1}{2}\right)
\end{equation}

Then if Hypothesis $B'$ is valid and if $u$ solves (1.1) with forcing term $F = 0$, we have the abstract Strichartz estimates

\begin{equation}
\|u\|_{L_t^p X([0,T] \times \Omega)} \lesssim A(S)(\|f\|_{\tilde{H}_\gamma^{\varepsilon+\eta}(\Omega)} + \|g\|_{\tilde{H}_\gamma^{\varepsilon-1}(\Omega)}).
\end{equation}

**Remark 1.5.** We need $-\frac{n-3}{2} \leq \gamma < \frac{n-1}{2}$ since we will use Lemma 2.11 which requires $\gamma + \varepsilon + \eta \leq \frac{n}{2} - 1$ for any positive number $\varepsilon$, thus precisely $\gamma \in \left[-\frac{n-3}{2}, \frac{n-1}{2} - \varepsilon - \eta\right]$. On the other hand, when $\varepsilon$ is allowed to take large values, which depends on our local energy estimates, we can easily adapt our arguments to show

\[ \|u\|_{L_t^p X([0,T] \times \Omega)} \lesssim A(S)(\|f\|_{\tilde{H}_\gamma^{\varepsilon+\eta}(\Omega)} + \|g\|_{\tilde{H}_\gamma^{\varepsilon-1}(\Omega)}) \]

under the assumption $\gamma \in \left[-\frac{n-3}{2}, \frac{n-1}{2}\right]$. 

Next we will see two corollaries that involve adding forcing term to the equation.

**Corollary 1.6.** Assume that \((X, \gamma, \eta, p)\) and \((Y, 1 - \gamma, \eta, r)\) are almost admissible and that Hypothesis B' is valid. Also assume that \([1.9]\) holds for \((X, \gamma, \eta, p)\) and \((Y, 1 - \gamma, \eta, r)\).

Then we have the following global abstract Strichartz estimates for the solution of \([1.11]\)

\[
\|u\|_{L^p_t(X([0,T]\times\Omega))} \lesssim A(S)(\|f\|_{\dot{H}^{\gamma-\kappa}(\Omega)} + \|g\|_{\dot{H}^{\gamma^{-\kappa}}(\Omega)}) + A^2(S)\|\Lambda^{2(\varepsilon+\eta)}F\|_{L^p_t Y'(\Omega)}
\]

where \(\Lambda = (1 - \Delta)^{\frac{1}{4}}\), \(r'\) denotes the conjugate exponent to \(r\) and \(\|\cdot\|_{Y'}\) is the dual norm to \(\|\cdot\|_Y\).

**Proof.** Since

\[
\|h\|_{\dot{H}^{\gamma}} = \|\|D\|\Lambda^{\varepsilon}h\|_{L^2} \quad \varepsilon \geq 0
\]

it is easy to see that the dual norm is

\[
\|h\|_{(\dot{H}^{-\gamma})'} = \|h\|_{\dot{H}^{-\gamma}} = \|\|D\|\Lambda^{-\varepsilon}h\|_{L^2}
\]

To prove \([1.11]\), we may assume by \([1.10]\) that the initial data vanishes. If \(|D| = \sqrt{-\Delta_g}\) is the square root of minus the Laplacian (with the boundary conditions \(B\)), then we need to show

\[
(1.12) \quad \left\| \int_0^t e^{-i(t-s)|D|} |D|^{-1} F(s, \cdot) \, ds \right\|_{L^p_t X([0,T]\times\Omega)} \lesssim A^2(S)\|\Lambda^{2(\varepsilon+\eta)}F\|_{L^p_t Y'([0,T]\times\Omega)}.
\]

Since \(p > r'\), an application of the Christ-Kiselev Lemma (cf. \([10]\)) and \([1.10]\) show that it suffices to prove the estimate

\[
(1.13) \quad \left\| \int_0^S e^{-is|D|} |D|^{-1} F(s, \cdot) \, ds \right\|_{\dot{H}^{\gamma}(\Omega)} = \left\| \int_0^S e^{-is|D|} |D|^{-1+\gamma} \Lambda^{\varepsilon+\eta} F(s, \cdot) \, ds \right\|_{L^2(\Omega)} \lesssim A(S)\|\Lambda^{2(\varepsilon+\eta)}F\|_{L^p_t Y'([0,T]\times\Omega)}.
\]

But by duality of \([1.10]\) for \((Y, 1 - \gamma, \eta, r)\) gives

\[
\left\| \int_0^S e^{-is|D|} F(s, \cdot) \, ds \right\|_{\dot{H}^{-\gamma-\kappa}(\Omega)} \lesssim A(S)\|F\|_{L^p_t Y'(\Omega)}.
\]

i.e.

\[
(1.14) \quad \left\| \int_0^S e^{-is|D|} |D|^{\gamma-1} \Lambda^{-\varepsilon-\eta} F(s, \cdot) \, ds \right\|_{L^2(\Omega)} \lesssim A(S)\|F\|_{L^p_t Y'([0,S]\times\Omega)}.
\]

Now \([1.13]\) follows from \([1.14]\).

\(\square\)

As a special case of \([1.11]\) when the spaces \(X\) and \(Y\) are the standard Lebesgue spaces, we have the following
Corollary 1.7. Suppose that \( n \geq 3 \) and that Hypothesis B’ is valid. Suppose that \( p, r > 2, \ q, s \geq 2 \) and that
\[
\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{r} + \frac{n}{s} - 2
\]
and
\[
\frac{2}{p} + \frac{n - 1}{q} \cdot \frac{2}{r} + \frac{n - 1}{s} \leq \frac{n - 1}{2}.
\]
Then if the local Strichartz estimate (1.8) holds respectively for \((L^q(\Omega), \gamma, \eta, p)\) and \((L^s(\Omega), 1 - \gamma, \eta, r)\), it follows that when \( u \) solves (1.1)
\[
\|u\|_{L^p_t L^q_x(\mathbb{R}^+ \times \Omega)} \lesssim \|f\|_{\dot{H}^{\gamma}_{\mathcal{E}_+\eta}(\Omega)} + \|g\|_{\dot{H}^{\gamma-1}_{\mathcal{E}_+\eta}(\Omega)} + \|(1 - \Delta)^{\epsilon + \eta} F\|_{L^r_t L^s_x(\mathbb{R}^+ \times \Omega)}.
\]

These estimates of course are the obstacle versions of the mixed-norm estimates for \( R^n \) and \( \Delta_\mathcal{E} = \Delta \). For nontrapping obstacle and \( B = I \), these results were proved in odd dimensions by Smith and Sogge [26] and then by Burq [3] and Metcalfe [22] for even dimensions. The Neumann case was not treated, but it follows from the same proof. Unfortunately, the known techniques don’t apply to the case of \( n = 2 \) due to (1.9), and Hypothesis B’ seems also to require \( B = Id \) and \( \partial \Omega \neq \emptyset \) in this case. Also, at present, the knowledge of the local Strichartz estimates (1.8) when \( X = L^r(\Omega) \) is limited. When \( \Omega \) is the exterior of a geodesically convex obstacle, they were obtained by Smith and Sogge [25], and their results apply to the case when there are finitely many convex obstacles by finite propagation of speed. Recently, there has been work on proving local Strichartz estimates when \( X = L^r(\Omega) \) for more general exterior domains ([6], [7], [2], [27]), but only partial results for a more restrictive range of exponents than the ones described in Corollary 1.7 have been obtained.

2. Proof of Theorem 1.4

Lemma 2.1. Fix \( \beta \in C_0^\infty(\mathbb{R}^n) \) and assume that \( \gamma \leq \frac{n-1}{2} \). Then
\[
\int_{-\infty}^{\infty} \left\| \beta(\cdot)(e^{it|D|}f)(t, \cdot) \right\|_{\dot{H}^\gamma(\mathbb{R}^n)}^2 \, dt \lesssim \|f\|_{\dot{H}^\gamma(\mathbb{R}^n)}^2,
\]
if \( |D| = \sqrt{-\Delta} \).

Proof. Refer to Lemma 2.2 in [26].

Proposition 2.2. Let \( w \) solve the inhomogeneous wave equation
\[
\begin{cases}
(\partial_t^2 - \Delta)u = F(t, x) & \text{on } \mathbb{R}^+ \times \mathbb{R}^n \\
u|_{t=0} = 0, & \partial_t u|_{t=0} = 0
\end{cases}
\]
Assume that (1.7) is valid whenever \( u \) is a solution of the homogeneous wave equation
\[
\begin{cases}
(\partial_t^2 - \Delta)u = 0 & \text{on } \mathbb{R}^+ \times \mathbb{R}^n \\
u|_{t=0} = f, & \partial_t u|_{t=0} = g
\end{cases}
\]

\[
\|u\|_{L_t^p L_x^q(\mathbb{R}^+ \times \mathbb{R}^n)} \lesssim \|f\|_{\dot{H}^{\gamma\epsilon_n}(\Omega)} + \|g\|_{\dot{H}^{\gamma-1\epsilon_n}(\Omega)} + \|(1 - \Delta)^{\epsilon + \eta} F\|_{L_r^t L_s^x(\mathbb{R}^+ \times \Omega)}.
\]
Assume further that $p > 2$, $\gamma \geq \frac{3p - 3}{2}$. Then, if
\[
F(t, x) = 0 \quad \text{if } |x| > 2R
\]
we have
\[
\|u\|_{L^p_tL^\gamma_x([0, S] \times \mathbb{R}^n)} \lesssim A(S) \|F\|_{L^p_tH^{\gamma - 1}_{x}([0, S] \times \mathbb{R}^n)}.
\]

Proof. When $S = \infty$, this is just Proposition 2.1 in [14]. And their argument is easy to be modified to give the proof when $S = T$ is finite.

\[\Box\]

Lemma 2.3. Let $u$ solves (1.1) and assume that Hypothesis B’ holds, let $\beta \in C_0^\infty(\mathbb{R}^n)$ equal 1 on a neighborhood of $\mathbb{R}^n \setminus \Omega$, then we have the following estimates:

i), if $f, g, F$ are supported in $|x| < 2R$,
\[
(2.5) \quad \|\beta u\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)} + \|\beta \partial_t u\|_{L^\infty_tH^{\gamma - 1}_{x}([0, S] \times \Omega)} \lesssim \|f\|_{H^{\gamma}(\Omega)} + \|g\|_{H^{\gamma - 1}(\Omega)} + \|F\|_{L^\infty_tH^{\gamma + 1}_{x}([0, S] \times \Omega)}.
\]

ii), if $F$ is supported in $|x| < R$, $\gamma < \frac{3p - 3}{2}$,
\[
(2.6) \quad \|u\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)} + \|\partial_t u\|_{L^\infty_tH^{\gamma - 1}_{x}([0, S] \times \Omega)} \lesssim \|f\|_{H^{\gamma}(\Omega)} + \|g\|_{H^{\gamma - 1}(\Omega)} + \|F\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)}.
\]

Proof. i), Since $f, g, F$ are supported in a ball, by (1.2) and elliptic regularity arguments for $\gamma \in \mathbb{Z}$, and by interpolation for the remaining $\gamma \in \mathbb{R}$, we get
\[
(2.7) \quad \|\beta u\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)} + \|\beta \partial_t u\|_{L^\infty_tH^{\gamma - 1}_{x}([0, S] \times \Omega)} \lesssim \|f\|_{H^\gamma(\Omega)} + \|g\|_{H^{\gamma - 1}(\Omega)} + \|F\|_{L^\infty_tH^{\gamma + 1}_{x}([0, S] \times \Omega)}.
\]

By Duhamel’s principle, the inhomogeneous solution $v$ satisfies
\[
\|\beta v\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)} + \|\beta \partial_t v\|_{L^\infty_tH^{\gamma - 1}_{x}([0, S] \times \Omega)} \lesssim \|F\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)}.
\]

Now by duality of the above estimate, energy estimates and elliptic regularity, we get
\[
(2.8) \quad \|u\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)} + \|\partial_t u\|_{L^\infty_tH^{\gamma - 1}_{x}([0, S] \times \Omega)} \lesssim \|f\|_{H^\gamma(\Omega)} + \|g\|_{H^{\gamma - 1}(\Omega)} + \|F\|_{L^\infty_tH^\gamma_x([0, S] \times \Omega)}.
\]

Now (2.5) is a result of (2.7) and (2.8).

ii), For the homogeneous solution $v$, we can assume $f = g = 0$ for $|x| \leq \frac{3p}{2}R$ by i). Decompose $v = (1 - \eta)v_0 + \hat{v}$, with $\eta \in C_0^\infty(\mathbb{R}^n)$ equals 1 for $|x| < R$ and vanishes for $|x| > \frac{3p}{2}R$.

$(1 - \eta)v_0$ solves Cauchy problem for the Minkowski space wave equation with initial data $((1 - \eta)f, (1 - \eta)g)$ and forcing term $G$; $\hat{v}$ solves wave equation with initial data $(0, 0)$ and
forcing term $-G$. Notice that $G$ is supported in $R < |x| < 2R$, we get the $L^2_t$ bounds for $(1 - \eta)v_0$ by Lemma 2.1 and i), and $L^2_t$ bounds for $\hat{v}$ by i) and Lemma 2.1. $L^2_t$ bounds for the inhomogeneous solution $w$ follows from i) since $F$ is still compactly supported. Similarly to i), the $L^\infty_t$ bounds for $u$ follows also from energy estimates, elliptic regularity and duality.

\[ \tag{2.10} \]

**Proposition 2.4.** Let $u$ solves (1.1) and assume that

\[ f(x) = g(x) = F(t, x) = 0, \quad \text{when } |x| > 2R, \]

if $(X, \gamma, \eta, p)$ is almost admissible with $p > 2, \gamma \geq -\frac{2-\eta}{2}$, and hypothesis $B'$ holds, then we have

\[ \|u\|_{L^t_x([0, S] \times \Omega)} \lesssim A(S)(\|f\|_{\dot{H}^{\gamma+\epsilon+\eta}} + \|g\|_{\dot{H}^{\gamma+\epsilon+\eta}} + \|F\|_{L^2_t\dot{H}^{\gamma+\epsilon+\eta-1}}). \]

**Proof.** Fix $\beta \in C^\infty_0(\mathbb{R}^n)$ satisfying $\beta(x) = 1, \ |x| \leq 3R$ and write $u = v + w, \ \text{where } v = \beta u, \ w = (1 - \beta)u.$

Then $w$ solves the free wave equation

\[
\begin{align*}
(\partial_t^2 - \Delta)w &= [\beta, \Delta]u \\
\left. w \right|_{t=0} &= \partial_t w \big|_{t=0} = 0.
\end{align*}
\]

Notice that $[\beta, \Delta]u$ is compactly supported, an application of Proposition 2.2 shows that $\|w\|_{L^t_x}$ is dominated by $A(S)\|\rho u\|_{L^2_t\dot{H}^{\gamma+\epsilon+\eta}}$ if $\rho \in C^\infty_0$ equals one on the support of $\beta$. Therefore, by (2.5), $\|w\|_{L^t_x}$ is dominated by the right side of (2.10).

As a result, we are left with showing that if $v = \beta u$ then

\[ \tag{2.11} \|v\|_{L^t_x([0, S] \times \Omega)} \lesssim A(S)(\|f\|_{\dot{H}^{\gamma+\epsilon+\eta}} + \|g\|_{\dot{H}^{\gamma+\epsilon+\eta}} + \|F\|_{L^2_t\dot{H}^{\gamma+\epsilon+\eta-1}}) \]

assuming, as above, that (2.9) holds. To do this, fix $\varphi \in C^\infty_0((-1, 1))$ satisfying $\sum_{j=-\infty}^{\infty} \varphi(t - j) = 1$. For a given $j \in \mathbb{N}$, let $v_j = \varphi(t - j)v.$ Then $v_j$ solves

\[
\begin{cases}
(\partial_t^2 - \Delta_k)v_j = -\varphi(t - j)[\Delta, \beta]u + [\partial_t^2, \varphi(t - j)]\beta u + \varphi(t - j)F \\
Bv_j(t, x) = 0, \quad x \in \partial\Omega \\
v_j(0, \cdot) = \partial_t v_j(0, \cdot) = 0,
\end{cases}
\]

while $v_0 = v - \sum_{j=1}^{\infty} v_j$ solves

\[
\begin{cases}
(\partial_t^2 - \Delta_k)v_0 = -\tilde{\varphi}[\Delta, \beta]u + [\partial_t^2, \tilde{\varphi}]\beta u + \tilde{\varphi}F \\
Bv_0(t, x) = 0, \quad x \in \partial\Omega \\
v_0|_{t=0} = f, \quad \partial_t v_0|_{t=0} = g,
\end{cases}
\]

if $\tilde{\varphi} = 1 - \sum_{j=1}^{\infty} \varphi(t - j)$ if $t \geq 0$ and 0 otherwise. If we then let $G_j = (\partial_t^2 - \Delta_k)v_j$ be the forcing term for $v_j, \ j = 0, 1, 2, \ldots, \ \text{then, by the local Strichartz estimates} \ 1.8\ \text{and DuHamel, we get for } j = 1, 2, \ldots,$

\[
\|v_j\|_{L^t_x([0, S] \times \Omega)} \leq \int_0^S \|G_j(s, \cdot)\|_{\dot{H}^{\gamma-1}} \, ds \lesssim \|G_j\|_{L^2_t\dot{H}^{\gamma+\epsilon+\eta-1}},
\]
using Schwarz’s inequality and the support properties of the $G_j$ in the last step. Similarly,

$$
\|v_0\|_{L^p_x(\mathbb{R}_+ \times \Omega)} \lesssim \|f\|_{\dot{H}^{\gamma}} + \|g\|_{\dot{H}^{\gamma-1}} + \|G_0\|_{L^2_t\dot{H}^{\gamma+n-1}_x}.
$$

Since $p > 2$, by (2.5) and disjoint support of $G_j$, we have

$$
\|v\|_{L^p_x([0,S] \times \Omega)} \lesssim \sum_{j=0}^{\infty} \|v_j\|_{L^p_x([0,S] \times \Omega)} \\
\lesssim \sum_{j=1}^{\infty} \|G_j\|_{L^2_t\dot{H}^{\gamma+n-1}_x([0,S] \times \Omega)} + \|v_0\|_{L^p_x(\mathbb{R}_+ \times \Omega)} \\
\lesssim \|f\|_{\dot{H}^{\gamma+n}} + \|g\|_{\dot{H}^{\gamma+n-1}} + \|F\|_{L^2_t\dot{H}^{\gamma+n-1}_x},
$$

and so we get

$$
(2.12)
\|v\|_{L^p_x([0,S] \times \Omega)} \lesssim \|f\|_{\dot{H}^{\gamma+n}} + \|g\|_{\dot{H}^{\gamma+n-1}} + \|F\|_{L^2_t\dot{H}^{\gamma+n-1}_x} \\
\lesssim A(S)(\|f\|_{\dot{H}^{\gamma+n}} + \|g\|_{\dot{H}^{\gamma+n-1}} + \|F\|_{L^2_t\dot{H}^{\gamma+n-1}_x})
$$

as desired, which finishes the proof of Proposition 2.4.

\[ \square \]

**Proof of Theorem 1.4:** Recall that we are assuming that $(\partial_x^2 - \Delta_x)u = 0$. By Proposition 2.4, we may also assume that the initial data for $u$ vanishes when $|x| < 3R/2$. We then fix $\beta \in C_0^\infty(\mathbb{R}^n)$ satisfying $\beta(x) = 1$, $|x| \leq R$ and $\beta(x) = 0$, $|x| > 3R/2$ and write

$$
u = u_0 - v = (1 - \beta)u_0 + (\beta u_0 - v),
$$

where $u_0$ solves the Cauchy problem for the Minkowski space wave equation with initial data defined to be $(f,g)$ if $x \in \Omega$ and 0 otherwise. By the free estimate (1.7), Proposition 2.4 and Lemma 2.1,

$$
\|(1 - \beta)u_0\|_{L^p_x([0,S] \times \mathbb{R}^n)} \lesssim A(S)(\|f\|_{\dot{H}^{\gamma}} + \|g\|_{\dot{H}^{\gamma-1}} + \|G\|_{L^2_t\dot{H}^{\gamma}_x}) \\
\lesssim A(S)(\|f\|_{\dot{H}^{\gamma}} + \|g\|_{\dot{H}^{\gamma-1}}).
$$

where $G = \Box((1 - \beta)u) = \Delta \beta \cdot u_0 + 2\nabla \beta \cdot \nabla u_0$ is supported in $R < |x| < 3R/2$.

Now consider $\tilde{u} = \beta u_0 - v$, which has forcing term $-G$ and zero initial data, again by Proposition 2.4 and Lemma 2.1,

$$
\|\beta u_0 - v\|_{L^p_x([0,S] \times \Omega)} \lesssim A(S)(\|G\|_{L^2_t\dot{H}^{\gamma+n}_x}) \\
\lesssim A(S)(\|\rho u_0\|_{L^2_t\dot{H}^{\gamma+n}_x} \quad \text{(here } \rho \text{ is a } C_0^\infty \text{ function.)} \\
\lesssim A(S)(\|f\|_{\dot{H}^{\gamma+n}} + \|g\|_{\dot{H}^{\gamma+n-1}}).
$$

The proof is complete.

\[ \square \]
3. Application 1: Sharp life span bounds for $p < p_c$ when $n = 3$.

First let us describe the wave equation we shall consider:

$$
\begin{cases}
(\partial_t^2 - \Delta_g)u = F_p(u(t, x)) \quad &\text{on } \mathbb{R}^+ \times \Omega \\
u|_{t=0} = f, \\
\partial_t u|_{t=0} = g, \\
(Bu)(t, x) = 0, &\text{on } \mathbb{R}^+ \times \partial\Omega
\end{cases}
$$

(3.1)

with $B$ as above, the set $\Omega$ is assumed to be either all of $\mathbb{R}^3$, or else $\Omega = \mathbb{R}^n \setminus \kappa$ where $\kappa$ is a compact subset of $|x| < R$ with smooth boundary, and $\kappa$ is nontrapping in the sense that any geodesic restricted to $|x| < R$ has bounded length.

We shall assume that the nonlinear term behaves like $|u|^p$ when $u$ is small, and so we assume that

$$
\sum_{0 \leq j \leq 2} |u|^j |\partial_x^j F_p(u)| \lesssim |u|^p
$$

(3.2)

when $u$ is small.

Based on the discussion in the first section, we will assume Hypothesis B’ holds with $\varepsilon = 0, S = T$. Now if we set

$$
\{Z\} = \{\partial_l, x_j \partial_k - x_k \partial_j : 1 \leq l \leq n, 1 \leq j < k \leq n\}
$$

then we have the following existence theorem for (3.1).

**Theorem 3.1.** Let $n = 3$, and fix $\Omega \subset \mathbb{R}^n$ and boundary operator $B$ as above. Assume further that Hypothesis B’ is valid with $\varepsilon = 0$. Then if

$$
2 < p < p_c = 1 + \sqrt{2}, \quad \gamma = \frac{1}{2} - \frac{1}{p},
$$

and if

$$
T_{\varepsilon', p} = c \varepsilon^{\frac{p(1-p)}{2p-1}}
$$

there is an $\varepsilon_0 > 0$ depending on $\Omega, B$ and $p$ so that (3.1) has an almost global solution in $[0, T_{\varepsilon', p}] \times \Omega$, satisfying $(Z^\alpha u(t, \cdot), \partial_t Z^\alpha u(t, \cdot)) \in \dot{H}_{\mu}^\alpha \times \dot{H}_{\mu}^{\alpha - 1}$, $|\alpha| \leq 2, t \in [0, T_{\varepsilon', p}]$, whenever the initial data satisfies the boundary conditions of order 2, and

$$
\sum_{|\alpha| \leq 2} (\|Z^\alpha f\|_{\dot{H}_{\mu}^\alpha(\Omega)} + \|Z^\alpha g\|_{\dot{H}_{\mu}^{\alpha - 1}(\Omega)}) < \varepsilon'
$$

(3.3)

with $0 < \varepsilon' < \varepsilon_0$.

In the case where $\Omega = \mathbb{R}^3$ and $\Delta_g = \Delta$ it is known that $p > p_c$ is necessary for global existence (see John [17]). In this case under a somewhat more restrictive smallness condition global existence was established by John [17] for the case where $n = 3$. For the local existence result, Lindblad [19] handled the case $1 < p < 1 + \sqrt{2}$ in $\mathbb{R}^3$, then Zhou [34] for $p = 1 + \sqrt{2}$. In their works it was also shown that the lifespan estimates given are sharp.

For nontrapping obstacles, Hidano, Metcalfe, Smith, Sogge and Zhou [14] dealt with the global existence part (i.e., $p > p_c$) for (3.1) with $n = 3, 4$. 
On the other hand, when the data is spherically symmetrical and \( n = 3 \), Sogge\[28\] and Hidano\[13\] obtained the sharp local well-posedness theorem for Minkowski wave equation separately by using some radial estimates. It is also shown in \[28\] that the regularity \( \gamma = \frac{1}{2} - \frac{1}{p} \) is sharp for radial data.

Here we will use a real interpolation method to get the local existence theorem for \( (3.1) \) when the perturbation is nontrapping.

Before handling the obstacle problem we will first see an alternative proof for Minkowski space results, which involves an interpolation between the following two estimates.

**Lemma 3.2.** (a variant of KSS estimate) For \( n \geq 3 \), let \( u \) solve the homogeneous wave equation \( (2.3) \) in Minkowski space, then we have

\[
\| x > a e^{it} f \|_{L_t^2 L_x^2([0,T] \times \mathbb{R}^n)} \lesssim B(T) \| f \|_{\dot{H}^0},
\]

where

\[
B(T) = \begin{cases} 
T^\frac{1}{2} + a, & \text{if } -\frac{1}{2} < a \leq 0 \\
(\log((2 + T))^\frac{1}{2}, & \text{if } a = -\frac{1}{2} \\
\text{Constant, } & \text{if } a < -\frac{1}{2}
\end{cases}
\]

In particular, for \(-\frac{1}{2} < a \leq 0\), we have

\[
\| x > a e^{it} f \|_{L_t^2 L_x^2([0,T] \times \mathbb{R}^n)} \lesssim T^\frac{1}{2} + a \| f \|_{\dot{H}^0}.
\]

**Proof.** Actually the cases when \( a \leq -\frac{1}{2} \) have been well set up in Du, Sogge, Zhou \[12\], and can be adapted to handle the case \(-\frac{1}{2} < a \leq 0\). Specifically, considering

\[
u = -i \sum_{j=1}^n \partial_j v_j, v_j = \hat{F}^{-1} \hat{u}(t, \xi) \xi_j \]

it is implied by

\[
\| x > a v' \|_{L_t^2 L_x^2([0,T] \times \mathbb{R}^n)} \lesssim B(T) \| f \|_{\dot{H}^1}, \quad a \leq 0.
\]

But this follows from the following estimates

\[
\| v'(t,x) \|_{L_t^2 L_x^2([0,T] \times |x| < 1)} \lesssim \| v'(0,x) \|_{L_x^2}
\]

and a scaling argument for \( |x| < T \), the energy inequality for \( |x| > T \). For a proof of \( (3.7) \) refer to Keel, Smith and Sogge \[15\].

As for \( (3.6) \), we just need to take care of the case when \( |x| < 1 \), but that is just a direct result from Lemma \[2.1\] and a scaling argument for a partition of \( \{ x : 0 < |x| < 1 \} \), see details in \[13\]. \( \square \)

In the following we will employ \( (3.6) \) to do the interpolations for simplicity, while we remark that the weaker estimate \( (3.4) \) can actually lead us to the same conclusion \( (3.11) \) as well by the same argument.

The next estimate is a result from a complex interpolation between \( (3.6) \) and endpoint Trace Lemma.

**Proposition 3.3.** For \( n = 3 \), let \( u \) solve the homogeneous wave equation \( (2.3) \) in Minkowski space, then we have

\[
\| x > \frac{14a}{3} e^{it} f \|_{L_t^2 L_x^2([0,T] \times \mathbb{R}^n)} \lesssim T^{\frac{4}{5} + a} \| f \|_{\dot{H}^\frac{1}{2}}.
\]
for \(-\frac{1}{2} < a \leq 0\).

Here, and in what follows, we are using the mixed-norm notation with respect to the volume element
\[
\|h\|_{L^q_t L^r_x} = \left( \int_0^\infty \left( \int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \right)^{q/p} r^{n-1} \, dr \right)^{1/q}
\]
for finite exponents and
\[
\|h\|_{L^\infty_t L^r_x} = \sup_{r > 0} \left( \int_{S^{n-1}} |h(r\omega)|^p \, d\sigma(\omega) \right)^{1/p}.
\]

**Proof.** Recall that we have the endpoint Trace Lemma (see [33]):
\[
\|x^{\frac{n-1}{2}} e^{it|D|} f\|_{L^\infty_t L^\infty_x([0,T] \times \mathbb{R}^n)} \lesssim \|f\|_{\dot{B}^{\frac{3}{2}}_{2,1}}.
\]
Now if we use a complex interpolation between this estimate and (3.6) for \(n = 3\), and set \(\theta = \frac{1}{3}\), use the fact that for complex interpolation between besov spaces we have
\((\dot{B}^{0}_{2,2}, \dot{B}^{\frac{3}{2}}_{2,1})[\theta] = \dot{B}^{\frac{3}{2}}_{2,2}\) (see Section 6.4 in [1]), we get the desired estimate (3.8) for \(-\frac{1}{2} < a \leq 0\). \(\square\)

Now we will cite some notations and results in [1] and [31]. Let \(A_0, A_1\) be Banach spaces, define the real interpolation space \((A_0, A_1)_{\theta, q}\) for \(0 < \theta < 1, 1 \leq q \leq \infty\) via the norm:
\[
\|a\|_{(A_0, A_1)_{\theta, q}} = \|a\|_{(A_0, A_1)_{\theta, 1, \infty}} = \left( \int_0^\infty (t^{-\theta} K(t, a))^q \, dt \right)^{1/q},
\]
where
\[
K(t, a) = \inf_{a = a_0 + a_1} \{\|a_0\|_{A_0} + \|a_1\|_{A_1}\}.
\]
Now if we let
\[
A_0 = \dot{B}^{0}_{2,2}, \quad A_1 = \dot{B}^{\frac{3}{2}}_{2,2},
\]
\[
B_0 = L^1_t L^2_w([0, T] \times [0, \infty) \times S^2, r^{2+2a} \, dt \, dr \, dw)
\]
\[
B_1 = L^1_t L^2_w([0, T] \times [0, \infty) \times S^2, r^{3+2a} \, dt \, dr \, dw),
\]
then by (3.6) and (3.8), we have
\[
(3.9) \quad T f = e^{it|D|} f : \tilde{A} \to \tilde{B}, \quad \text{where} \quad \tilde{A} = (A_0, A_1), \ \tilde{B} = (B_0, B_1)
\]
and
\[
(3.10) \quad M_0 \lesssim T^{\frac{1}{2} + a}, \quad M_1 \lesssim T^{\frac{1}{2} (\frac{1}{2} + a)}, \quad \text{where} \quad M_i = \|T\|_{A_j \to B_j}, \quad j = 0, 1
\]

Now we can state the main weighted Strichartz estimates as follows:

**Proposition 3.4.** For \(n = 3\), let \(u\) solve the homogeneous wave equation \((\ref{wave})\) in Minkowski space, then we have
\[
(3.11) \quad \|x^{\frac{1}{2} - \frac{1}{p}} u\|_{L^2_t L^2_x ([0,T] \times \mathbb{R}^n)} \lesssim T^{\frac{1}{p} + \frac{1}{2} + a} \left( \|f\|_{\dot{H}^{\gamma}} + \|g\|_{\dot{H}^{\gamma-1}} \right),
\]
where \(\gamma = \frac{1}{2} - \frac{1}{p}\) and \(2 < p < 1 + \sqrt{3}\).
Proof. Since $K_{\theta,q}$ is an exact interpolation functor of exponent $\theta$ (Theorem 3.1.2 in [1]), from (3.9) and (3.10) we get

$$\| T f \|_{B_{\theta,2}} \leq M^{1-\theta} M^\theta \| f \|_{A_{\theta,2}}$$

$$\lesssim T^{(1-\frac{1}{\theta})\left(\frac{1}{p}+\alpha\right)} \| f \|_{A_{\theta,2}}$$

(3.12)

if $-\frac{1}{p} < a \leq 0$. But from Theorem 6.4.5 in [1] we have

$$\left(B^w_{p_0}, B^w_{p_1}\right)_{\theta,r} = B^w_{p_1^*}, \text{ if } s_0 < s_1, 0 < \theta < 1, r, q_0, q_1 \geq 1, s^* = (1-\theta)s_0 + \theta s_1.$$ 

Set $a = \frac{-p+\frac{1}{p}+1}{2}, \theta = 3 - \frac{2}{p}$, then we have $0 < \theta < 1$ and $-\frac{1}{p} < a \leq 0$ satisfied since $2 < p < 1 + \sqrt{2}$, thus

$$\text{RHS of (3.12)} \lesssim T^{-\frac{p+\frac{1}{p}+2}{p}} \| f \|_{B_{2,2}^\theta} = T^{-\frac{p+\frac{1}{p}+2}{p}} \| f \|_{H^\theta_r}.$$

On the other hand, we can use the fact (Theorem 3.4.1(b) in [1])

$$A_{\theta,q} \subset A_{\theta,r}, \text{ if } q \leq r$$

and bilinear weighted interpolation (Section 1.18.5 in [31])

$$\left(\left(L^p_{t,r}, L^{w_0}_{q} \right)_{\theta, r} dtdrdw, \left(L^p_{t,r}, L^{w_1}_{q} \right)_{\theta, r} dtdrdw\right)_{\theta, p} = \left(L^p_{t,r}, L^2_{w} \right)_{\theta, p} dtdrdw$$

if $\frac{1}{p} = \frac{1-\theta}{s_0} + \frac{\theta}{p_1}, w(r) = w_0^{\frac{p(1-\theta)}{s_0}} w_1^{\frac{\theta}{p_1}}.$

And since $p > 2$, we have

$$\text{LHS of (3.12)} \gtrsim \| T f \|_{B_{\theta,p}} = \| T f \|_{\left(L^p_{t,r}, L^2_{w} \right)_{\theta, p} dtdrdw}$$

$$= \left\| \left|x - \frac{\theta}{p} \right| T f \right\|_{L^p_{t,r}, L^2_{w} \left([0,T] \times \mathbb{R}^n\right)}$$

(3.14)

Now (3.11) is just the result of (3.13) and (3.14). \hfill \square

As a result, by the arguments to follow, (3.11) is strong enough to show the local existence of solution as described in Theorem 3.1 in Minkowski space case.

To prove the obstacle version of this result, we define $X = X_{\gamma,q}(\mathbb{R}^n)$ to be the space with norm defined by

$$\| h \|_{X_{\gamma,q}} = \| h \|_{L^{s\gamma}_{\gamma}([|x| < 2R])} + (A(T))^{-1} \left\| \left|x - \frac{\theta}{p} \right| h \right\|_{L^p_{t,r}, L^2_{w}([|x| > 2R])},$$

with $A(T) = T^{-\frac{p+\frac{1}{p}+2}{p}}$ and $s_\gamma = \frac{2n}{n-2\gamma}$.

Now we can prove the following estimate provided $\gamma = \frac{1}{2} - \frac{1}{p}$ and $p \geq 2$:

$$\| u \|_{L^p_{t} X([0,T] \times \mathbb{R}^n)} \lesssim \| f \|_{H^\gamma} + \| g \|_{H^{\gamma-1}},$$

(3.16)

if $\Omega u = 0$.

Indeed, the contribution of the second part of the norm in (3.15) is controlled by (3.11), and the contribution of the first term is due to Sobolev estimates and an interpolation between (2.6) (Note that $\varepsilon = 0$ in our case).
Furthermore, by finite propagation speed of the wave equation, Sobolev estimates and interpolation in (2.6), we have the local estimate for solutions of (1.1) with $F = 0$:

$$\|u\|_{L^p_t L^\gamma_x([0,1] \times \Omega)} \lesssim (\|f\|_{H^\gamma} + \|g\|_{H^{\gamma-1}})$$

where $p \geq 2$.

From (3.16) and (3.17), we know $(X, \gamma, 0, p)$ is admissible, by Theorem 1.4, we have the following Proposition:

**Proposition 3.5.** For $n = 3$, let $u$ be a solution of (1.1) with $F = 0$, $\Omega$ is as described in this section, and also assume

$$\gamma = \frac{1}{2} - \frac{1}{p}, \quad p \in (2, 1 + \sqrt{2}),$$

then

$$\|u\|_{L^p_t L^\gamma_x([0,T] \times \Omega)} \lesssim \|f\|_{H^\gamma} + \|g\|_{H^{\gamma-1}}.$$ 

**Corollary 3.6.** For $n = 3$, let $u$ be a solution of (1.1), $\Omega$ is as described in this section, and assume condition (3.18) is satisfied, then

$$\|u\|_{L^p_t L^\gamma_x([0,T] \times |x| < 2R)} + (A(T))^{-1} \left\| \left| x \right|^{-\frac{\gamma}{2} - \frac{1}{p}} u \right\|_{L^p_t L^\gamma_x([0,T] \times |x| < 2R)} \lesssim \|f\|_{H^\gamma} + \|g\|_{H^{\gamma-1}} + \|F\|_{L^1_t L^{\gamma'}_x ([0,T] \times |x| < 2R)} + \left\| \left| x \right|^{-\frac{1}{2} - \gamma} F \right\|_{L^1_t L^{\gamma'}_x ([0,T] \times |x| > 2R)}$$

**Proof.** By Duhamel’s principle, Sobolev estimates and the following estimate (refer to (3.7) in [14]):

$$\|\varphi\|_{H^{\gamma-1}} \lesssim \left\| \left| x \right|^{-\frac{1}{2} + 1 - \gamma} \varphi \right\|_{L^1_t L^\infty_x}, \quad \text{if } \frac{1}{2} < 1 - \gamma < \frac{n}{2}.$$ 

Here the condition $\frac{1}{2} < 1 - \gamma < \frac{n}{2}$ is satisfied due to (3.18).

If we set $\Gamma = \{\partial_t, Z\}$, then we can easily adapt the argument as in [14] to get the following two estimates:

$$\sum_{|\alpha| \leq 2} (\|\Gamma^\alpha u\|_{L^p_t L^\gamma_x ([0,T] \times |x| < 2R)} + A(T)^{-1} \left\| \left| x \right|^{-\frac{1}{2} + \gamma} \Gamma^\alpha u \right\|_{L^p_t L^\gamma_x ([0,T] \times |x| < 2R)}) \lesssim \sum_{|\alpha| \leq 2} \left( \|Z^\alpha f\|_{H^\gamma} + \|Z^\alpha g\|_{H^{\gamma-1}} \right) + \sum_{|\alpha| \leq 2} \left( \|\Gamma^\alpha F\|_{L^1_t L^{\gamma'}_x ([0,T] \times |x| < 2R)} + \left\| \left| x \right|^{-\frac{1}{2} - \gamma} \Gamma^\alpha F \right\|_{L^1_t L^{\gamma'}_x ([0,T] \times |x| > 2R)} \right).$$
complex and real interpolation method as above, we will get Proposition 3.4 with

\[ u \]

the nonlinear term behaves like

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\[ u \]

exponential decay estimates with a loss of 2 derivatives of data, to assure this we need
critical power when

\[ 4.1 \]

Now if we set

\[ I kawa [16] managed to show that solutions of (4.1) with \]

\[ n \]

We will consider wave equations of the form

\[ A(T) \]

Note. if we use the KSS estimate for \( a = -\frac{1}{3} \) instead of \(- \frac{1}{2} < a \leq 0 \), and the same complex and real interpolation method as above, we will get Proposition 3.4 with \( p = 1 + \sqrt{2} \), \( a = -\frac{1}{2} \) and \( A(T) = (\log(2+T))^{\frac{1}{2}} \), and more further get the local-wellposedness for critical power \( p = 1 + \sqrt{2} \) with \( T_\varepsilon = \exp(C\varepsilon^{-p-1}) \), but this life span is Not optimal. (The optimal one should be \( T_\varepsilon = \exp(C\varepsilon^{-p-1}) \)).

4. Application2: Strauss conjecture on semilinear wave equations with finitely many obstacles.

We will consider wave equations of the form

\[ (\partial_t^2 - \Delta_g)u(t,x) = F_p(u(t,x)), \quad (t,x) \in \mathbb{R}_+ \times \Omega \]

\[ Bu = 0, \quad \partial_\nu u = g(x), \quad x \in \Omega, \]

with \( B \) described as in the first section. \( \Omega = \mathbb{R}^n \setminus \bigcup_{i=1}^m \kappa_i \) where \( \kappa_i(i = 1, 2, \cdot, m) \) are disjoint compact convex subsets of \(|x| < R\) with smooth boundary. We shall assume that the nonlinear term behaves like \(|u|^p\) when \( u \) is small, and so we assume that

\[ \sum_{0 \leq j \leq 2} |u|^j \cdot |\partial_u^j F_p(u)| \leq |u|^p, \]

when \( u \) is small.

Ikawa [16] managed to show that solutions of (4.1) with \( n = 3, \Delta_g = \Delta, B = I \) have exponential decay estimates with a loss of 2 derivatives of data, to assure this we need
some technical assumptions, which we will assume are satisfied here. Now by using an interpolation of this estimate and energy estimate we get an estimate of the form:

\[ \|u'(t, x)\|_{L^2_\infty(|x|<1)} \lesssim e^{-ct} \|u'(0, x)\|_{H^s(|x|<1)}, \quad \text{for any positive number } \varepsilon. \]

Therefore we will assume Hypothesis B' holds for (4.1).

In the next theorem we are abusing the Hypothesis B' a little by assuming it is true for \( n = 4 \). Actually there has been no polynomially local energy decay set up for even dimensions when there are trapped rays, which could be expected though. And Burq did show that local energy decays at least logarithmically with some loss in derivatives(\[4\]).

**Theorem 4.1.** Let \( n = 3 \) or \( 4 \), and fix \( \Omega \subset \mathbb{R}^n \) and boundary operator \( B \) as above. Assume further that Hypothesis B' is valid.

Let \( p = p_c \) be the positive root of

\[ (n - 1)p^2 - (n + 1)p - 2 = 0. \]

Then if

\[ p_c < p < (n + 3)/(n - 1), \quad \gamma = \frac{2}{p - 1}, \]

there is an \( \varepsilon_0 > 0 \) depending on \( \Omega, B \) and \( p \) and \( \varepsilon > 0 \) arbitrarily small so that (4.1) has a global solution satisfying \( (Z^n u(t, \cdot), \partial_t Z^n u(t, \cdot)) \in \dot{H}^{\gamma}_B \times \dot{H}^{\gamma - 1}_B \), \(|\alpha| \leq 2, t \in \mathbb{R}_+, \)

whenever the initial data satisfies the boundary conditions of order 2, and

\[ \sum_{|\alpha| \leq 2} \left( \|Z^n f\|_{\dot{H}^{2\gamma}_\infty(\Omega)} + \|Z^n g\|_{\dot{H}^{2\gamma - 1}_\infty(\Omega)} \right) < \varepsilon' \]

with \( 0 < \varepsilon' < \varepsilon_0 \).

On the other hand, if

\[ n = 3, \gamma = \frac{1}{2} - \frac{1}{p}, p \in (2, 1 + \sqrt{2}). \]

and

\[ T_{\varepsilon'} = c \varepsilon'^{-\frac{2(p-1)}{2p-1}}, \quad 2 < p < 1 + \sqrt{2} \]

then there is a unique solution in \([0, T_{\varepsilon'}) \times \Omega\) such that \((Z^n u(t, \cdot), \partial_t Z^n u(t, \cdot)) \in \dot{H}^{\gamma}_B \times \dot{H}^{\gamma - 1}_B \) under condition (4.5).

Before we turn to the proof of this existence theorem, we will first use the Strichartz estimates to get important estimates that will be used.

Define \( X = X_{\gamma,p}(\mathbb{R}^n) \) to be the space with norm defined by

\[ \|h\|_{X_{\gamma,p}} = \|h\|_{L^{\infty}(|x|<2R)} + (A(S))^{-1} \left\| \frac{|x|^{\frac{2}{p} + 1 - \gamma}}{p} h \right\|_{L^p_{\gamma}L^p_\infty(|x|>2R)}, \]

where \( s_\gamma = \frac{2n}{n-2\gamma}, \ S = T \) and \( A(T) \) is as defined in last section for \( n = 3, p < p_c, \gamma = \frac{1}{2} - \frac{1}{p}; \)

\( S = \infty \) for \( n = 3, 4, p > p_c, \gamma = \frac{2}{p - 1}. \)

Now by using (5.11), a known result (3.6) in [14] and energy estimates, we can adapt the argument in section 3 to get the following Proposition:
Proposition 4.2. For \( n = 3 \) or 4, let \( u \) be a solution of (4.1) with \( F = 0 \), and assume condition (4.4) or (4.6) is satisfied, then

\[
\|u\|_{L_2^s L_2^\gamma([0,T] \times \Omega)} \lesssim \|f\|_{\tilde{H}_2^s} + \|g\|_{\tilde{H}_2^{-1}}.
\]

(4.9)

Corollary 4.3. For \( n = 3, 4 \), let \( u \) be a solution of (4.1), and assume condition (4.4) or (4.6) is satisfied, then

\[
\|u\|_{L_1^s L_2^s([0,T] \times \Omega)} \lesssim \|f\|_{\tilde{H}_1^s} + \|g\|_{\tilde{H}_1^{-1}}.
\]

(4.10)

Proof. By (4.9), we can assume \( f = g = 0 \), and by Duhamel’s principle we have

\[
\|u\|_{L_2^s L_2^\gamma([0,T] \times \Omega)} \lesssim \|f\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} + \|g\|_{L_2^{1-\gamma}(\mathbb{R} \times \Omega)}.
\]

Recall the following estimate (refer to (3.16) [14]):

\[
\|F\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} \lesssim \|f\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} + \|g\|_{L_2^{1-\gamma}(\mathbb{R} \times \Omega)},
\]

(4.11)

Here the condition \( \frac{1}{2} < 1 - \gamma < \frac{3}{2} \) is satisfied due to (4.4) or (4.6).

If we use (4.11), we get

\[
\|F\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} \lesssim \|f\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} + \|g\|_{L_2^{1-\gamma}(\mathbb{R} \times \Omega)} + \|F\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} + \|F\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)}
\]

(4.12)

when \( \varepsilon > 0 \) is small enough, which completes the proof.

\[
\square
\]

Proposition 4.4. (Higher order Strichartz Estimates). Under assumptions in Corollary 4.3 and assume that \( (f, g, F) \) satisfy \( H_2^s \times H_0^1 \times H_0^1 \) boundary conditions, then

\[
\sum_{|\alpha| \leq 2} \left( \|\Gamma^\alpha u\|_{L_2^s \tilde{H}_2^{-1}} + \|\partial_t \Gamma^\alpha u\|_{L_2^s \tilde{H}_2^{-1}} \right) \lesssim \sum_{|\alpha| \leq 2} \left( \|Z^\alpha f\|_{\tilde{H}_2^s} + \|Z^\alpha g\|_{\tilde{H}_2^{-1}} \right)
\]

(4.13)

+ \sum_{|\alpha| \leq 2} \left( \|\Gamma^\alpha F\|_{L_2^s \tilde{H}_2^{-1}(\mathbb{R} \times \Omega)} \right),

(4.14)
assuming that $\beta(2.6)$ and Duhamel’s principle we control $\|\Gamma^\alpha u\|_{L^p_t L^\gamma_x(R^+ \times \{x: |x| > 2R\})} + \|\Gamma^\alpha u\|_{L^p_t L^\gamma_x(R_+ \times \{x: |x| < 2R\})}$

\[ \lesssim \sum_{|\alpha| \leq 2} \left( \|Z^\alpha f\|_{\dot{H}^\gamma_x} + \|Z^\alpha g\|_{\dot{H}^\gamma_x^{-1}} \right) + \sum_{|\alpha| \leq 2} \left( \|\left| x - \frac{\alpha}{2} \right|^{\frac{1}{2} + \gamma} \Gamma^\alpha F\|_{L^1_t L^\gamma_x(R_+ \times \{x: |x| > 2R\})} + \|\Gamma^\alpha F\|_{L^1_t L^\gamma_x(R_+ \times \{x: |x| < 2R\})} \right). \]

**Proof.** We first consider the Cauchy data for $\Gamma^\alpha$. This is clear if $\Gamma^\alpha$ is replaced by $Z^\alpha$. On the other hand, the Cauchy data for $\partial_t u$ is $(g, \Delta_g f + F(0, \cdot))$. We may control

\[ \sum_{|\alpha| \leq 1} \left( \|Z^\alpha g\|_{\dot{H}^\gamma_x} + \|Z^\alpha \Delta_g f\|_{\dot{H}^\gamma_x^{-1}} \right) \leq \sum_{|\alpha| \leq 2} \left( \|Z^\alpha f\|_{\dot{H}^\gamma_x} + \|Z^\alpha g\|_{\dot{H}^\gamma_x^{-1}} \right). \]

Recall that $\gamma \in (0, \frac{1}{2})$, so that $\dot{H}^\gamma_x(\Omega) = \dot{H}^\gamma(\Omega)$. To control the term $F(0, \cdot)$, we recall that $\Gamma = \{\partial_t, Z\}$, and use the bound

\[ \sum_{|\alpha| \leq 1} \|\Gamma^\alpha F\|_{L^\infty_t \dot{H}^\gamma_x^{-1}(R^+ \times \Omega)} \leq \sum_{|\alpha| \leq 2} \|\Gamma^\alpha F\|_{L^1_t \dot{H}^\gamma_x^{-1}(R_+ \times \Omega)}, \]

which by (4.14) is seen to be dominated by the right hand side of (4.13). Similar considerations apply to the Cauchy data for $\partial_t^2 u$.

Let us now give the argument for (4.13). We first fix $\beta_0 \in C^\infty_0$ satisfying $\beta_0 = 1$ for $|x| \leq R$ and vanishes for $\{x: |x| > 2R\}$. Let

\[ \Gamma^\alpha u = (1 - \beta_0)\Gamma^\alpha u + \beta_0\Gamma^\alpha u = v + w. \]

Since the $\Gamma$ commute with $\Box_g$ when $|x| \geq R$, we have

\[ \Box_g v = (1 - \beta_0)\Gamma^\alpha F - [\beta_0, \Delta_g]\Gamma^\alpha u. \]

We can therefore write $(1 - \beta_0)\Gamma^\alpha u$ as $v_1 + v_2$ where $\Box_g v_1 = (1 - \beta_0)\Gamma^\alpha F$ and $v_1$ has initial data $((1 - \beta_0)\Gamma^\alpha u(0, \cdot), \partial_t (1 - \beta_0)\Gamma^\alpha u(0, \cdot))$, while $\Box_g v_2 = -[\beta_0, \Delta_g]\Gamma^\alpha u$ and $v_2$ has vanishing initial data. If we do this, it follows by (4.10) that if for $|\alpha| \leq 2$ we replace the term involving $\Gamma^\alpha u$ by $v_1$ in the left side of (4.13), then the resulting expression is dominated by the right side of (4.13). If we use Proposition 2.2, we find that if we replace $\Gamma^\alpha u$ by $v_2$ then the resulting expression is dominated by

\[ \sum_{|\alpha| \leq 2} \| [\beta_0, \Delta_g]\Gamma^\alpha u\|_{L^2_t \dot{H}^\gamma_x^{-1}} \lesssim \sum_{j \leq 2} \|\beta_j \partial_t^j u\|_{L^2_t \dot{H}^\gamma_x^{-2-j}}, \]

assuming that $\beta_j$ equals one on the support of $\beta_0$ and is supported in $R < |x| < 2R$. By (2.6) and Duhamel’s principle we control $\|\beta_j \partial_t^j u\|_{L^2_t \dot{H}^\gamma_x}$ by the right hand side of (4.13). On the other hand, by Cauchy-Schwarz and Parseval’s Formula,

\[ \|\beta_j \partial_t^j u\|_{L^2_t \dot{H}^\gamma_x^{-1}} \lesssim \|\beta_j \partial_t^2 u\|_{L^2_t \dot{H}^\gamma_x} \|\beta_j \partial_t u\|_{L^2_t \dot{H}^\gamma_x}, \]
So it suffices to dominate $\|\beta_1 u\|_{L^2_t H^{\gamma+2}_B}$. Since $\Delta_x u = \partial_t^2 u - F$, then if $\beta_2$ equals one on support of $\beta_1$ and is supported in the set where $|x| < 2R$, we may use elliptic regularity and the equation to bound
\[
\|\beta_1 u\|_{L^2_t H^{\gamma+2}_B} \lesssim \|\beta_2 \Delta_x u\|_{L^2_t H^{\gamma}_B} + \|\beta_2 u\|_{L^2_t H^{\gamma}_B}
\lesssim \|\beta_2 \partial_t^2 u\|_{L^2_t H^{\gamma}_B} + \|\beta_2 u\|_{L^2_t H^{\gamma}_B} + \|\beta_2 F\|_{L^2_t H^{\gamma}_B}.
\]
The first two terms are dominated as above using (2.6) and Duhamel’s principle. For the first term is dominated as above, and the bounds for the second term is just from (4.12). The inequality involving $v_1$ and the equation to bound by almost the same argument as above we get the desired bound in (4.12).

For $w = \beta_0 \Gamma^\alpha u$, since the coefficients of $\Gamma$ are bounded on support of $\beta_0$, if $\beta_1$ equals one on the support of $\beta_0$ and is supported in $|x| < 2R$, then by Sobolev embedding
\[
\sum_{|\alpha| \leq 2} \|\beta_0 \Gamma^\alpha u\|_{L^p_t L^\gamma_x (\mathbb{R}^n \times \Omega)} \lesssim \sum_{|\alpha| \leq 2} \|\beta_1 \Gamma^\alpha u\|_{L^p_t H^{\gamma}_B}
\lesssim \sum_{|\alpha| \leq 2} \|\beta_1 \partial_t^j u\|_{L^p_t H^{\gamma+2-j}_B} + \|\beta_1 \Gamma^\alpha u\|_{L^\infty_t H^{\gamma}_B}.
\]
The first term is dominated as above, and the bounds for the second term is just from (4.12), so we are done with proof of (4.13).

Now we turn to the proof of (4.12).

As before we first consider the inequality where $\Gamma^\alpha u$ is replaced by $v = v_1 + v_2$ in (4.12). The inequality involving $v_1$ just follows from energy estimates on $\mathbb{R}^n$, Duhamel’s principle and (4.11). For $v_2$ by (2.6) we see that it is controlled by
\[
\sum_{|\alpha| \leq 2} \|\beta_0, \Delta_x \Gamma^\alpha u\|_{L^2_t H^{\gamma+1}_B} \lesssim \sum_{j \leq 2} \|\beta_1 \partial_t^j u\|_{L^2_t H^{\gamma+1}_B}
\]
by almost the same argument as above we get the desired bound in (4.12).

Now we are only left with $w = \beta_0 \Gamma^\alpha u$, first notice that the left hand side of (4.12) with $w$ is dominated by $\sum_{j\leq 3} \|\beta_1 \partial_t^j u\|_{L^\infty_t H^{2-j}_B}$. For the case $j = 0, 1$, since
\[
\begin{cases}
\Delta_x (\beta_1 u) = \beta_1 F + [\Delta_x, \beta_2] u \\
(\beta_1 u, \partial_t \beta_1 u)|_{t=0} = (\beta_1 f, \beta_1 g),
\end{cases}
\]
we use (2.6) with the DuHamel formula to bound
\[
\|\beta_1 u\|_{L^\infty_t H^{\gamma+2}_B} + \|\beta_1 \partial_t u\|_{L^\infty_t H^{\gamma+1}_B} \lesssim ||\beta_1 f||_{H^{\gamma+2}} + ||\beta_1 g||_{H^{\gamma+1}} + \|\beta_2 u\|_{L^2_t H^{\gamma+2}_B} + \|\beta_1 F\|_{L^1_t H^{\gamma+1}_B}.
\]
The term on the right involving $u$ was controlled previously; on the other hand, since $F$ satisfies the $H^{\gamma+1}_B$ boundary conditions,

$$
\|\beta_1 F\|_{L^1_t H^{\gamma+1}_B} \lesssim \sum_{|\alpha| \leq 2} \|\partial_x^{\alpha} F\|_{L^1_t L^2_x (1-\gamma)^{\alpha}}.
$$

To handle the terms for $j = 2, 3$ we use the equation to bound

$$
\sum_{j=2}^3 \|\beta_1 \partial_t u\|_{L^j_t H^{\gamma+1-j}_B} \leq \sum_{j=0,1} \left(\|\beta_1 \partial_t \Delta_g u\|_{L^\infty_t H^{\gamma-j}_B} + \|\beta_2 \partial_x^{\alpha} F\|_{L^\infty_t H^{\gamma-j}_B}\right).
$$

The terms involving $\Delta_g u$ are dominated by $\|\beta_2 \partial_x^{\alpha} F\|_{L^j_t H^{\gamma+2-j}}$ with $j = 0, 1$. The terms involving $F$ are controlled for $j = 1$ by (4.14), and for $j = 0$ by observing that (4.16) holds with $L_t^j$ replaced by $L_t^\infty$. This completes the proof of (4.12).

**Proof of Theorem 4.1**

We will adapt the argument from [14]. First, let $u_0$ solve the Cauchy problem (1.1) with $F = 0$. We iteratively define $u_k$, for $k \geq 1$, by solving

$$
\begin{cases}
(\partial_t^2 - \Delta_g) u_k(t, x) = F_p(u_{k-1}(t, x)), & (t, x) \in \mathbb{R}_+ \times \Omega \\
u_k(0, \cdot) = f \\
\partial_t u_k(0, \cdot) = g \\
(Bu_k)(t, x) = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega.
\end{cases}
$$

Our aim is to show that if the constant $\varepsilon' > 0$ in (4.5) is small enough, then so is

$$
M_k = \sum_{|\alpha| \leq 2} \left(\|\Gamma^\alpha u_k\|_{L^\infty_t \dot{H}^{\gamma}_B([0,S] \times \Omega)} + \|\partial_t \Gamma^\alpha u_k\|_{L^\infty_t \dot{H}^{\gamma-1}_B([0,S] \times \Omega)} \right)
$$

$$
+ (A(S))^{-1} \|x|^{-\frac{\gamma + 1}{2} - \frac{\gamma}{p}} \|\Gamma^\alpha u_k\|_{L^p_t L^2_x ([0,S] \times \{ |x| > 2R \})} + \|\Gamma^\alpha u_k\|_{L^\infty_t L^2_x ([0,S] \times \{ x \in \Omega: |x| < 2R \})}
$$

for every $k = 0, 1, 2, \ldots$. For $k = 0$, it follows by (4.12) and (4.13) that $M_0 \leq C_0\varepsilon'$, with $C_0$ a fixed constant. More generally, (4.12) and (4.13) yield that

$$
M_k \leq C_0\varepsilon' + C_0 \sum_{|\alpha| \leq 2} \left(\|\partial_t \Gamma^\alpha F_p(u_{k-1})\|_{L^1_t L^2_x ([\mathbb{R}_+ \times \{ |x| > 2R \})} \right)
$$

$$
+ \|\Gamma^\alpha F_p(u_{k-1})\|_{L^1_t L^2_x (\mathbb{R}_+ \times \{ x \in \Omega: |x| < 2R \})}.
$$

Note that our assumption (4.2) on the nonlinear term $F_p$ implies that for small $v$

$$
\sum_{|\alpha| \leq 2} \|\Gamma^\alpha F_p(v)\| \lesssim |v|^{p-1} \sum_{|\alpha| \leq 2} \|\Gamma^\alpha v\| + |v|^{p-2} \sum_{|\alpha| \leq 2} \|\Gamma^\alpha v\|^2.
$$

Furthermore, since $u_k$ will be locally of regularity $H^{\gamma+2}_B \subset L^\infty$ and $F_p$ vanishes at 0, it follows that $F_p(u_k)$ satisfies the $B$ boundary conditions if $u_k$ does. Since the collection $\Gamma$ contains vectors spanning the tangent space to $S^{n-1}$, by Sobolev embedding for $n = 3, 4$ we have

$$
\|v(r \cdot)\|_{L^\infty} + \sum_{|\alpha| \leq 1} \|\Gamma^\alpha v(r \cdot)\|_{L^2_x} \lesssim \sum_{|\alpha| \leq 2} \|\Gamma^\alpha v(r \cdot)\|_{L^2_x}.
$$
Consequently, for fixed $t, r > 0$
\[
\sum_{|\alpha| \leq 2} \left\| \Gamma^\alpha F_p(u_{k-1}(t, r \cdot)) \right\|_{L_x^2} \lesssim \sum_{|\alpha| \leq 2} \left\| \Gamma^\alpha u_{k-1}(t, r \cdot) \right\|_{L_x^2}^p.
\]
Thus the first summand in the right side of (4.18) is dominated by $C_1 (A(S) M_{k-1})^p$.

We next observe that, since $s, \gamma > 2$ and $n \leq 4$, it follows by Sobolev embedding on \( \Omega \cap \{|x| < 2R\} \) that
\[
\left\| v \right\|_{L^\infty(\{x \in \Omega: |x| < 2R\})} + \sum_{|\alpha| \leq 1} \left\| \Gamma^\alpha v \right\|_{L^4(\{x \in \Omega: |x| < 2R\})} \lesssim \sum_{|\alpha| \leq 2} \left\| \Gamma^\alpha v \right\|_{L^\gamma(\{x \in \Omega: |x| < 2R\})}.
\]

Since $s'_1 - \gamma - 2 \varepsilon < 2$, it holds for each fixed $t$ that
\[
(4.19) \quad \sum_{|\alpha| \leq 2} \left\| \Gamma^\alpha F_p(u_{k-1}(t, \cdot)) \right\|_{L_x^{s'_1 - \gamma - 2 \varepsilon}(\{x \in \Omega: |x| < 2R\})} \lesssim \sum_{|\alpha| \leq 2} \left\| \Gamma^\alpha F_p(u_{k-1}(t, \cdot)) \right\|_{L^2(\{x \in \Omega: |x| < 2R\})} \lesssim \sum_{|\alpha| \leq 2} \left\| \Gamma^\alpha u_{k-1}(t, \cdot) \right\|_{L_x^\gamma(\{x \in \Omega: |x| < 2R\})}^p.
\]

The second summand in the right side of (4.18) is thus dominated by $C_1 M_{k-1}^p$, and we conclude that $M_k \leq C_0 \varepsilon' + 2C_0 C_1 (A(S) M_{k-1})^p$. For $\varepsilon'$ sufficiently small, by the definition of $A(S)$, then
\[
(4.20) \quad M_k \leq 2 C_0 \varepsilon', \quad k = 1, 2, 3, \ldots
\]

To finish the proof of Theorem 4.1 we need to show that $u_k$ converges to a solution of the equation (4.11). For this it suffices to show that
\[
A_k = (A(S))^{-1} \left\| \frac{x^2 + 1}{p} u_k - u_{k-1} \right\|_{L^p_t L^2_x([0, T] \times \{|x| > 2R\})} + \left\| u_k - u_{k-1} \right\|_{L^p_t L^2_x([0, T] \times \{|x| < 2R\})}
\]
tends geometrically to zero as $k \to \infty$. Since $|F_p(v) - F_p(w)| \lesssim |v - w|(|v|^{p-1} + |w|^{p-1})$ when $v$ and $w$ are small, the proof of (4.20) can be adapted to show that, for small $\varepsilon' > 0$,
there is a uniform constant $C$ so that
\[
A_k \leq C (A(S))^p A_{k-1}(M_{k-1} + M_{k-2})^{p-1},
\]
which, by (4.20), implies that $A_k \leq \frac{1}{2} A_{k-1}$ for small $\varepsilon'$. Since $A_1$ is finite, the claim follows, which finishes the proof of Theorem 4.1 \( \square \)

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