Verification of the Identifiability Conditions in Some Nonlinear Time Series Models

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Abstract

This paper investigates the identifiability conditions of $M$-estimation for some nonlinear time series models. We present primitive conditions for the identifiability in smooth transition GARCH, nonlinear Poisson autoregressive, and multiple regime smooth transition AR models. As an unexpected result, the smooth transition GARCH model turns out not to require the familiar common root condition for the identifiability. The method for verification is anticipated to be applicable to other nonlinear models.

Key words and phrases: Identifiability, nonlinear time series models, GARCH-type models, smooth transition GARCH models, Poisson autoregressive models, smooth transition AR models.

Abbreviated title: Identifiability in nonlinear time series models

1 Introduction

Verifying the identifiability conditions is a fundamental task to construct consistent parameter estimators and to ensure the positive definiteness of their asymptotic variance matrices. Its verification often appears to be nontrivial in dealing with a class of generalized autoregressive conditional heteroscedasticity (GARCH) models. This issue has a long history and there exist a variety of relevant articles. For instance, we can refer to Rothenberg (1971), who introduced the global and local identification concept and verified that the local identifiability is equivalent to the non-singularity of the information matrix; Phillips (1989), who derived asymptotic theories in partially identified models; Hansen (1996) and Francq et al. (2010), who proposed tests of the hypothesis under which nuisance parameters are unidentifiable; Komunjer (2012), who provided primitive conditions for global identification in moment restriction models. In many cases, the identifiability condition is inherent to given statistical models: for example, the multiple linear regression model becomes unidentifiable when exact multicollinearity exists. Verifying the identifiability becomes more complicated in nonlinear time series models such as the threshold AR and smooth transition GARCH models: see, for instance, Chan (1993) and Meitz & Saikkonen (2011). In this study, we prove the identifiability conditions and derive primitive conditions for some widely adopted nonlinear time series especially with threshold and smooth transition effects.
This paper studies the identifiability conditions within the framework of the $M$-estimation problem. As an illustration, let us consider the nonlinear least squares (NLS) estimation for a sample from a strictly stationary ergodic process \{$(Y_t, Z_t)$\} with $E(Y_t|Z_t) = f(Z_t, \beta^o)$ for some known function $f$. The limit of the random objective functions is uniquely minimized at $\beta^o$ if the following identifiability condition holds:

$$f(Z_1, \beta) = f(Z_1, \beta^o) \text{ a.s. implies } \beta = \beta^o. \quad (1)$$

In most $M$-estimation procedures, the corresponding identifiability conditions are given as a form in (1), where $f$ can be either a conditional mean, variance, or quantile function: see Hayashi (2000, page 463) and Lee & Noh (2013). In view of Wu (1981), to ensure the positive definiteness of the asymptotic variance of the above NLS estimator, it is required to verify that $\lambda^T \partial f(Z_1, \beta^o)/\partial \beta = 0$ a.s. implies $\lambda = 0$. The method described in this paper is also useful to verify the positive definiteness of a given asymptotic variance matrix.

Regarding the verification of the identifiability in nonlinear time series, an early work has been done by Chan & Tong (1986), who studied the asymptotic theory of NLS estimators for the smooth transition AR (STAR) models and verified the positive definiteness of the asymptotic variance matrix. In recent studies of the quasi-maximum likelihood estimators (QMLE) for GARCH-type models, verifying the identifiability conditions emerged as non-trivial and complicated tasks. Straumann & Mikosch (2006) studied the QMLE in general conditionally heteroscedastic models, and proved the identifiability conditions for exponential and asymmetric GARCH($p, q$) models. Medeiros & Veiga (2009) proposed the flexible coefficient GARCH(1,1) model nesting the smooth transition GARCH(1,1) (STGARCH), and verified the identifiability of the proposed model. Kristensen & Rahbek (2009) studied general nonlinear ARCH models and proved the identifiability for various ARCH($q$) models. Meitz & Saikkonen (2011) developed the asymptotic results for nonlinear AR($p$) models with nonlinear GARCH(1,1) errors, and verified the identifiability condition for the STAR($p$)-STGARCH(1,1) model. Lee & Lee (2012) studied the Box-Cox transformed threshold GARCH($p, q$) models in which the identifiability condition was also confirmed. While the uniqueness of linear filter representation has been utilized for the identifiability of linear time series, the above authors had to develop additional tools to confirm the condition (1) for a given nonlinear time series. Further, it can be seen that technical difficulties increase as the nonlinearity of models becomes complicated.

This paper focuses on verifying the identifiability conditions in some widely adopted nonlinear models, in which the identifiability has not yet been confirmed. We deal with STGARCH($p, q$), general Poisson autoregressive, and multiple regime STAR($p$) models, where we present the primitive conditions for the identifiability and their justification. These models were selected to study the identifiability in various situations including discrete conditional distributions and multiple regime structures. We also provide a useful tool to deal with the identifiability of GARCH-type models, and a lemma to handle models with
multiple regime structures: see Lemma 3.1.1 in the Appendix. As an unexpected result, the STGARCH\((p, q)\) model turns out not to require the familiar common root condition for the identifiability.

The paper is organized as follows. Section 2 describes the method of verification through examples. Section 3 verifies the identifiability conditions in the above mentioned nonlinear time series models. Section 4 discusses the identifiability of those models. Proofs are given in the Appendix. Throughout the paper, we denote by \(\{X_t\}\) the data generating process and \(\mathcal{F}_t\) the \(\sigma\)-field generated by \(\{X_s : s \leq t\}\).

## 2 Heuristic examples

In this section, we recall the methods of verifying the identifiability for the STAR and asymmetric GARCH (AGARCH) model in the literature. We also describe our approach to deal with the STGARCH model, which is expected to be useful for various GARCH-type models.

Let us consider the STAR model with two regimes defined by

\[
X_t = m(X_{t-1}, \ldots, X_{t-p}; \theta^o) + \varepsilon_t,
\]

\[m(X_{t-1}, \ldots, X_{t-p}; \theta^o) = \beta_0^T X_{t-1} + \beta_1^T X_{t-1} F \left( \frac{X_{t-d} - \varepsilon^o}{\sigma^o} \right),\]

where \(\{\varepsilon_t\}\) are i.i.d. random variables, \(\theta^o T = (\beta_0^o, \beta_1^o, \varepsilon^o, \rho^o)\), \(X_{t-1} = (1, X_{t-1}, \ldots, X_{t-p})^T\), and \(F(\cdot)\) denotes a smooth distribution function. Chan & Tong (1986) showed the positive definiteness of \(E[\dot{m}_t(\theta^o) \dot{m}_t(\theta^o)^T]\), where \(\dot{m}_t(\theta^o) = \dot{m}(X_{t-1}, \ldots, X_{t-p}; \theta^o)\) and \(\dot{m}(x; \theta^o)\) denotes the gradient of \(m(x; \theta)\) at \(\theta^o\). To do so, it is shown that for a given \(\lambda \neq 0\), \((\lambda^T m(x; \theta^o))^2 > 0\) on \(x \in S \subset \mathbb{R}^p\) where the event \(\{(X_{t-1}, \ldots, X_{t-p}) \in S\}\) has positive probability. Finding such a set \(S\) is implemented by examining the function \(x \mapsto \lambda^T \dot{m}(x; \theta^o)\). Meitz & Saikkonen (2011) considered the above model with \(\{\varepsilon_t\}\) following a STGARCH\((1,1)\) process. They verified that \(m(X_{t-1}, \ldots, X_{t-p}; \theta) = m(X_{t-1}, \ldots, X_{t-p}; \theta^o)\) a.s. implies \(\theta = \theta^o\) by similar arguments.

The basic idea for these previous results can be described as follows. Suppose that \(g(X_{t-1}, \ldots, X_{t-p}; \theta, \theta^o) = 0\) a.s. and the function \(x \mapsto g(x; \theta, \theta^o)\) is continuous. Here, \(g(x; \theta, \theta^o)\) may be \((\theta - \theta^o)^T \dot{m}(x; \theta^o)\) in Chan & Tong (1986), or \(m(x; \theta) - m(x; \theta^o)\) in Meitz & Saikkonen (2011). Then, it can be easily seen that \(g(x; \theta, \theta^o) = 0\) for all \(x \in \text{supp}(X_{t-1}, \ldots, X_{t-p})\), where \(\text{supp}(Y)\) denotes the support of the distribution of a random vector \(Y\). By examining this equation, we can deduce suitable conditions for \(\theta = \theta^o\). Recall that the support of a probability measure on \(\mathbb{R}^d\) is defined as the set of all points \(x\) in \(\mathbb{R}^d\) for which every open neighborhood of \(x\) has a positive measure. It can be noted that Chan & Tong (1986) and Meitz & Saikkonen (2011) verified that the support of the stationary distribution of \((X_{t-1}, \ldots, X_{t-p})\) equals to \(\mathbb{R}^p\) in each model. This paper utilizes the
same approach to verify the identifiability conditions for nonlinear Poisson autoregressive and multiple regime STAR models in Section 3.

In order to deal with GARCH-type models, we introduce a generalized technique compared with articles mentioned in Section 1. In particular, while the observation or conditional volatility is needed to belong to an open interval with positive probability in Kristensen & Rahbek (2009), Meitz & Saikkonen (2011) and Lee & Lee (2012), we do not need to use such properties.

Kristensen & Rahbek (2009) and Straumann & Mikosch (2006) proved the identifiability conditions for the asymmetric power ARCH and AGARCH models, respectively. As an illustration, let us consider the AGARCH(1,1) model with power equal to 2 given by

\[ X_t = \sigma_t \eta_t, \quad \sigma_t^2 = \omega^0 + \alpha^0 (|X_{t-1}| - \gamma^0 X_{t-1})^2 + \beta^0 \sigma_{t-1}^2, \]  

where \{\eta_t\} is a sequence of i.i.d. random variables with \( E\eta_t = 0 \) and \( E\eta_t^2 = 1 \). We denote by \( \theta^0 = (\omega^0, \alpha^0, \beta^0, \gamma^0)^T \) and \( \Theta = (0, \infty) \times [0, \infty) \times [0, 1) \times [-1, 1] \). Assume that \( \alpha^0 > 0 \) and Model (2) has a strictly stationary solution: a necessary and sufficient condition for this is provided by Straumann & Mikosch (2006). Given the solution \{X_t\} and a parameter vector \( \theta \in \Theta \), we can define a strictly stationary process \{\sigma_t^2(\theta)\} as the solution of

\[ \sigma_t^2(\theta) = \omega + \alpha (|X_{t-1}| - \gamma X_{t-1})^2 + \beta \sigma_{t-1}^2(\theta), \quad \forall t \in \mathbb{Z}. \]  

Note that \( \sigma_t^2 = \sigma_t^2(\theta^0) \). To establish the consistency of the QMLE, it is necessary to prove the identifiability condition, that is, if \( \sigma_t^2 = \sigma_t^2(\theta) \) a.s. for some \( t \in \mathbb{Z} \) and \( \theta \in \Theta \), then it should hold that \( \theta = \theta^0 \).

In the verification, we first notice that \( \sigma_t^2 = \sigma_t^2(\theta) \) a.s. for all \( t \), since \( \{\sigma_t^2 - \sigma_t^2(\theta)\} \) is stationary. Then, we can obtain

\[ \omega^0 - \omega + \sigma_{t-1}^2 Y_{t-1} = 0 \quad \text{a.s.,} \]  

where \( Y_{t-1} = \alpha^0 (|\eta_{t-1}| - \gamma^0 \eta_{t-1})^2 - \alpha (|\eta_{t-1}| - \gamma \eta_{t-1})^2 + \beta^0 - \beta \). As given in Lemma 5.3 of Straumann & Mikosch (2006), \( Y_{t-1} \) is \( \mathcal{F}_{t-2} \)-measurable due to (1), but at the same time, it is also independent of \( \mathcal{F}_{t-2} \) since the solution to (2) is causal. Then, \( \theta = \theta^0 \) can be deduced from the degeneracy of \( Y_{t-1} \) and a certain condition on the distribution of \( \eta_{t-1} \). However, it is noteworthy that this idea can not be directly applied to the STGARCH(1,1) model: see (E.5) of Meitz & Saikkonen (2011).

To deal with more complicated models, it is useful to interpret (1) as an equation with respect to \( \eta_{t-1} \) and \( \sigma_{t-1} \). Denote by \( g(\eta_{t-1}, \sigma_{t-1}) \) the left-hand side of (1). Since \( g(\cdot) \) is continuous, it holds that \( g(y) = 0 \) for all \( y \in \supp(\eta_{t-1}, \sigma_{t-1}) \), as mentioned before. Further, the independence of \( \eta_{t-1} \) and \( \mathcal{F}_{t-2} \) implies that \( \supp(\eta_{t-1}, \sigma_{t-1}) = \supp(\eta_{t-1}) \times \supp(\sigma_{t-1}) \). Consequently, we have that

\[ P(g(x, \sigma_{t-1}) = 0 \text{ for all } x \in \supp(\eta_{t-1})) = 1. \]  

(5)
Since $E_{\eta_t} = 0$ and $E_{\eta_t^2} = 1$, $\text{supp}(\eta_{t-1})$ contains both positive and negative values. If the distribution of $\eta_{t-1}$ is not concentrated at two points as assumed in Lemma 5.3 of Straumann & Mikosch (2006), one can deduce from (5) that $\theta = \theta^o$. Further, if $\text{supp}(\eta_{t-1}) = \mathbb{R}$ as usual, (5) implies $\omega = \omega^o$ and the equation:

$$\alpha^o (|x| - \gamma^o x)^2 - \alpha (|x| - \gamma x)^2 + \beta^o - \beta = 0, \quad \forall x \in \mathbb{R},$$

from which $\theta = \theta^o$ can be easily deduced.

It can be seen that the method described in this section simplifies the previous proofs of the identifiability and positive definiteness in Francq & Zakoïan (2004), Straumann & Mikosch (2006) and Lee & Lee (2012). In the next section, we investigate the identifiability conditions in various nonlinear time series models making use of the above argument. It is anticipated that the method will be applicable to other various models.

3 Identifiability in nonlinear time series

3.1 Smooth transition GARCH models

González-Rivera (1998) suggested the STGARCH($p, q, d$) model defined by

$$X_t = \sigma_t \eta_t,$$

$$\sigma_t^2 = \omega^o + \sum_{i=1}^{q} \alpha^o_{i1} X_{t-i}^2 + \left( \sum_{j=1}^{q} \alpha^o_{2i} X_{t-i}^2 \right) F(X_{t-d}, \gamma^o) + \sum_{j=1}^{p} \beta^o_j \sigma_{t-j}^2,$$

(6)

where $\{\eta_t\}$ is the same as that defined in Model (2) and

$$F(X_{t-d}, \gamma^o) = \frac{1}{1 + e^{\gamma^o X_{t-d}}} - \frac{1}{2}.$$

In the above transition function, $d \in \{1, \ldots, q\}$ is pre-specified and $\gamma^o > 0$ is the smoothness parameter determining the speed of transition. It is noteworthy that when $\gamma^o \to \infty$, the STGARCH($1, 1, 1$) model becomes the GJR-GARCH($1, 1$) model proposed by Glosten et al. (1993), which is equivalent to Model (2). Denote by $\theta^o = (\gamma^o, \omega^o, \alpha^o_{11}, \ldots, \alpha^o_{1q}, \alpha^o_{21}, \ldots, \alpha^o_{2q}, \beta^o_1, \ldots, \beta^o_p)^T$ the true parameter vector. Let the parameter space $\Theta = [0, \infty) \times (0, \infty) \times A \times B$, where

$$A = \left\{ (\alpha_{11}, \ldots, \alpha_{1q}, \alpha_{21}, \ldots, \alpha_{2q}) \in \mathbb{R}^{2q} : \alpha_{1i} \geq 0, \ |\alpha_{2i}| \leq 2\alpha_{1i}, \forall i \right\},$$

$$B = \left\{ (\beta_1, \ldots, \beta_p) \in [0, 1)^p : \sum_{j=1}^{p} \beta_j < 1 \right\}. \quad (7)$$

Assume $\theta^o \in \Theta$ so that the positive conditional variance is guaranteed.

Though specific conditions under which Model (6) admits a stationary solution are not found in the literature, Straumann & Mikosch (2006) suggest such conditions for general stochastic recurrence equations. In the case of the STGARCH($1, 1, d$) model, the conditions
are provided by Medeiros & Veiga (2009) who suggest the flexible coefficient GARCH model nesting the STGARCH(1,1,d). In this case, notice that the region of parameters for the existence of a stationary process is a proper subset of Θ.

Given the stationary solution \{X_t\} and a parameter vector \( \theta \in \Theta \), we define

\[
c_t(\alpha) = \omega + \sum_{i=1}^{q} \alpha_{1i} X_{t-i}^2 + \left( \sum_{i=1}^{q} \alpha_{2i} X_{t-i}^2 \right) F(X_{t-d}, \gamma),
\]

where \( \alpha = (\gamma, \omega, \alpha_{11}, \ldots, \alpha_{1q}, \alpha_{21}, \ldots, \alpha_{2q}) \). Note that the polynomial \( \beta(z) = 1 - \sum_{j=1}^{p} \beta_j z^j \) has all its zeros outside the unit disc due to (7). Define \( \sigma_t^2(\theta) = \beta(B)^{-1} c_t(\alpha) \), where \( B \) is the backshift operator.

**Theorem 1.** Let \( \{X_t\} \) be a stationary process satisfying (6). Assume that

(a) \( \alpha_{2i}^0 \neq 0 \) for some \( 1 \leq i \leq q \) and \( \gamma^0 > 0 \);

(b) The support of the distribution of \( \eta_1 \) is \( \mathbb{R} \).

Then if \( \sigma_t^2 = \sigma_t^2(\theta) \) a.s. for some \( t \in \mathbb{Z} \) and \( \theta \in \Theta \), we have \( \theta = \theta^0 \).

**Remark 1.** It is remarkable that the identifiability in STGARCH models need no restriction concerning orders \( p \) and \( q \). Our result shows that a STGARCH\((p, q, d)\) model satisfying the above conditions can be consistently estimated by fitting any STGARCH\((p^*, q^*, d)\) model with \( p^* \geq p \) and \( q^* \geq q \). This is different from the results in GARCH and AGARCH models, where the identifiability conditions such as (c) in Theorem 2 below are necessary: see Francq & Zakoïan (2004) and Straumann & Mikosch (2006).

**Remark 2.** As in the case of the AGARCH model in Section 2, the support need not to be \( \mathbb{R} \): for example, \( \text{supp}(\eta_1) = \mathbb{Z} \) is sufficient.

The condition (a) in Theorem 1 means that there exists a smooth-transition mechanism, that is, conditional variances asymmetrically respond to positive and negative news. When it fails, the STGARCH model becomes the standard GARCH model. The following theorem shows that the parameters in Model (6) are partially identified when there is no smooth-transition mechanism.

**Theorem 2.** Let \( \{X_t\} \) be a stationary process satisfying (6) with \( \gamma^0 = 0 \) or \( \alpha_{2i}^0 = 0 \), \( i = 1, \ldots, q \). Suppose that condition (b) in Theorem 1 holds and the following condition holds:

(c) \( \alpha_{1i}^0 > 0 \) for some \( 1 \leq i \leq q \), \( (\alpha_{1q}, \beta_0^0) \neq (0, 0) \), and the polynomials \( \alpha_1^i(z) = \sum_{i=1}^{q} \alpha_{1i}^2 z^i \) and \( \beta^0(z) = 1 - \sum_{j=1}^{p} \beta_j^0 z^j \) have no common zeros.

Then if \( \sigma_t^2 = \sigma_t^2(\theta) \) a.s. for some \( t \in \mathbb{Z} \) and \( \theta \in \Theta \), it holds that \( \omega = \omega^0 \), \( \alpha_{1i} = \alpha_{1i}^0 \), \( \beta_j = \beta_j^0 \) for \( 1 \leq i \leq q, 1 \leq j \leq p \), and either \( \gamma = 0 \) or \( \alpha_{2i} = 0 \), \( 1 \leq i \leq q \) holds.
Remark 3. The hypothesis testing whether the smoothness mechanism exists or not has been studied by González-Rivera (1998). This is one of testing problems when nuisance parameters are not identifiable under the null hypothesis. Inference under similar situations have been studied by Hansen (1996) and Francq et al. (2010).

Remark 4. Condition (c) allows that either $p$ or $q$ is over-specified: see Remark 2.4 of Francq & Zakoïan (2004) in the case of GARCH models. Further, if we consider smooth transition ARCH$(q,d)$ models, the condition (c) is unnecessary.

3.2 Integer-valued threshold GARCH models

Poisson autoregressive models are popular for time series of counts, for example, daily epileptic seizure counts for a patient and the number of transactions per minute of some stock; see Fokianos et al. (2009) and the references therein. The models are also known as integer-valued GARCH models.

Franke et al. (2012) considered the conditional LS estimation in the following integer-valued threshold GARCH (INTGARCH) model. Let $\{X_t : t \geq 0\}$ be a time series of counts and $\{\lambda_t : t \geq 0\}$ be an associated intensity process. Denote by $F_{0,t}$ the $\sigma$-field generated by $\{\lambda_0, X_0, \ldots, X_t\}$. The INTGARCH model is defined by

$$X_t|F_{0,t-1} \sim \text{Poisson}(\lambda_t),$$

$$\lambda_t = \omega + \alpha_1^0 X_{t-1} + (\alpha_2^0 - \alpha_1^0)(X_{t-1} - l^0)^+ + \beta^0 \lambda_{t-1},$$

for $t \geq 1$ where $\alpha^+$ denotes $\max\{0, \alpha\}$. Assume that the true parameter vector $\theta^0 = (\omega^0, \alpha_1^0, \alpha_2^0, \beta^0, l^0)$ belongs to a parameter space $\Theta = (0, \infty) \times [0, 1)^3 \times \mathbb{N}$. Theorem 2.1 of Neumann (2011) verifies that if $\beta^0 + \max\{\alpha_1^0, \alpha_2^0\} < 1$, there exists a unique stationary bivariate process $\{(X_t, \lambda_t) : t \geq 0\}$ satisfying (8). Then, the time domain can be extended from $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ to $\mathbb{Z}$.

Given the stationary process $\{X_t : t \in \mathbb{Z}\}$ and a parameter vector $\theta \in \Theta$, define a stationary process $\{\lambda_t(\theta)\}$ as the solution of

$$\lambda_t(\theta) = \omega + \alpha_1 X_{t-1} + (\alpha_2 - \alpha_1)(X_{t-1} - l)^+ + \beta \lambda_{t-1}(\theta), \quad t \in \mathbb{Z}.$$

**Theorem 3.** Suppose that $\{X_t : t \in \mathbb{Z}\}$ is a stationary process satisfying (8) and $\alpha_1^0 \neq \alpha_2^0$. Then if $\lambda_t = \lambda_t(\theta)$ a.s. for some $t \in \mathbb{Z}$ and $\theta \in \Theta$, we have $\theta = \theta^0$.

**Remark 5.** When $\alpha_2^0 = \alpha_2 > 0$, Model (8) becomes an integer-valued GARCH$(1,1)$ model. Under such overspecification, parameters except for the threshold parameter $l$ are equal to the true parameters and $l$ is allowed to be arbitrary.

3.3 General Poisson autoregressive models

Neumann (2011) considered the following general class of nonlinear Poisson autoregressive models. Let $\{X_t : t \in \mathbb{Z}\}$ be a time series of counts accompanied by an intensity process
\{\lambda_t : t \in \mathbb{Z}\} satisfying:

\[ X_t|\mathcal{F}_{t-1} \sim \text{Poisson}(\lambda_t), \quad \lambda_t = f(\lambda_{t-1}, X_{t-1}, \theta^o) \tag{9} \]

for some known function \( f : [0, \infty) \times \mathbb{N}_0 \times \Theta \rightarrow [0, \infty) \). According to Theorems 2.1 and 3.1 of [Neumann (2011)], if \( f(\cdot, \theta^o) \) satisfies the following contractive condition:

\[ |f(\lambda, y, \theta^o) - f(\lambda', y', \theta^o)| \leq \kappa_1|\lambda - \lambda'| + \kappa_2|y - y'|, \quad \forall \lambda, \lambda' \geq 0, \forall y, y' \in \mathbb{N}_0, \]

where \( \kappa_1, \kappa_2 \geq 0 \) and \( \kappa_1 + \kappa_2 < 1 \), then there exists a stationary process \( \{(X_t, \lambda_t)\} \) with \( \lambda_t \in \mathcal{F}_{t-1} \) satisfying (9).

In view of Theorem 3.1 of [Neumann (2011)], we can define a stationary process \( \{\lambda_t(\theta)\} \) satisfying

\[ \lambda_t(\theta) = f(\lambda_{t-1}(\theta), X_{t-1}, \theta), \quad \forall t \in \mathbb{Z} \]

for a given stationary process \( \{X_t\} \) and a parameter vector \( \theta \in \Theta \). The ML estimation for nonlinear Poisson autoregressive models is studied by [Fokianos & Tjøstheim (2012)].

The following theorem presents mild requirements of \( f \) for their identifiability assumptions: its proof is straightforward in view of the proof of Theorem 3.

**Theorem 4.** Let \( \{(X_t, \lambda_t)\} \) be a stationary process satisfying (9). Assume that

(a) For each \( \theta \in \Theta \), \( f(\cdot, \theta) \) is continuous on \( \text{supp}(\lambda_1) \times \mathbb{N}_0 \);

(b) \( f(\lambda, y, \theta) = f(\lambda, y, \theta^o), \forall \lambda \in \text{supp}(\lambda_1), \forall y \in \mathbb{N}_0 \) implies \( \theta = \theta^o \).

Then if \( \lambda_t = \lambda_t(\theta) \) a.s. for some \( t \in \mathbb{Z} \) and \( \theta \in \Theta \), \( \theta = \theta^o \).

### 3.4 Multiple regime smooth transition autoregressive models

Regime-switching models for financial data have been received considerable attention. [Teräsvirta (1994)] discussed inference for two-regime STAR models. Recently, [McAleer & Medeiros (2008)] and [Li & Ling (2012)] studied multiple-regime smooth transition and threshold AR models, respectively. This subsection considers the nonlinear LS estimation in multiple-regime STAR model with heteroscedastic errors suggested by [McAleer & Medeiros (2008)].

A time series \( \{X_t\} \) is said to follow a multiple-regime STAR model of order \( p \) with \( M+1 \) limiting regimes if

\[ X_t = \beta_0^o X_{t-1} + \sum_{i=1}^{M} \beta_i^o X_{t-1} G(X_{t-d^o}; \gamma_i^o, c_i^o) + \varepsilon_t, \tag{10} \]

where \( \{\varepsilon_t\} \) is a white noise, \( \beta_i^o = (\phi_{i0}^o, \phi_{i1}^o, \ldots, \phi_{ip}^o)^T \), for \( 0 \leq i \leq M \), \( X_{t-1} = (1, X_{t-1}, \ldots, X_{t-p})^T \) and \( G(X_{t-d^o}; \gamma_i^o, c_i^o) \) is a logistic transition function given by

\[ G(X_{t-d^o}; \gamma_i^o, c_i^o) = \frac{1}{1 + e^{-\gamma_i^o(\lambda_{t-d^o} - c_i^o)}}. \tag{11} \]
The regimes switch from one to another according to the value of the transition variable, \( X_{t-d} \). The parameter \( d^o \in \{1, \ldots, p\} \) is called the delay parameter and assumed to be unknown. The threshold parameters are \(-\infty < c^o_i < \cdots < c^o_M < \infty\) and the smoothness parameters are \( \gamma^o_i > 0, \ i = 1, \ldots, M \). When \( \gamma^o_i \)'s are very large, Model (10) is barely distinguishable from the threshold model studied by Li & Ling (2012). Denote by \( \theta = (\beta^T_0, \beta^T_1, \ldots, \beta^T_M; \gamma_1, \ldots, \gamma_M, c_1, \ldots, c_M, d)^T \) a parameter vector belonging to a parameter space \( \Theta \subset \mathbb{R}^{(M+1)(p+1)+2M} \times \{1, \ldots, p\} \). Define

\[
m(X_{t-1}, \ldots, X_{t-p}, \theta) = \beta^{T}_0 X_{t-1} + \sum_{i=1}^{M} \beta^{T}_i X_{t-1} G(X_{t-d}; \gamma_i, c_i).
\]

**Theorem 5.** Let \( \{X_t\} \) be a stationary process satisfying (17). Assume that

(a) For each \( i = 1, \ldots, M, \ \beta^o_i \neq (0, \ldots, 0)^T \in \mathbb{R}^{p+1}; \)

(b) The support of the stationary distribution of \( (X_p, \ldots, X_1) \) is \( \mathbb{R}^p; \)

(c) The parameter space \( \Theta \) satisfies that \( \gamma_i > 0, \ i = 1, \ldots, M \) and \(-\infty < c_1 < \cdots < c_M < \infty.\)

Then if \( m(X_{t-1}, \ldots, X_{t-p}, \theta^o) = m(X_{t-1}, \ldots, X_{t-p}, \theta) \) a.s. for some \( t \in \mathbb{Z} \) and \( \theta \in \Theta \), we have \( \theta = \theta^o. \)

**Remark 6.** Theorem 5 is closely related to the identifiability of finite mixture of logistic distributions: see Lemma A.1 in the Appendix. Although the restriction on threshold parameters has natural interpretation, it is not indispensable. If we assume \((\gamma^o_i, c^o_i), i = 1, \ldots, M\) are distinct instead of the condition \( c^o_i < c^o_{i+1}\), Model (10) is weakly identifiable in the sense of Redner & Walker (1984).

## 4 Identifiability of the model

The identifiability conditions in Sections 2–3 are stated in terms of certain estimation problem and ensure that the limit of random criterion functions is uniquely maximized or minimized at the true parameter value. On the other hand, one might raise the question of whether the same data generating process can be obtained by different parameter values. This section shows that Theorems 1 and 3–5 also ensure the identifiability of the corresponding models.

Here, we reconsider the AGARCH(1,1) model discussed in Section 2. Suppose that a sequence of i.i.d. random variables \( \{\eta_t\} \) is defined on a probability space \((\Omega, F, P)\). Let \( \{X_t\} \) be defined by the equation (2) and \( \{X^*_t\} \) follow another AGARCH(1,1) model with a parameter \( \theta^* = (\omega^*, \alpha^*, \beta^*, \gamma^*)^T \in \Theta:\)

\[
X^*_t = \sigma^*_t \eta_t, \quad \sigma^2_t = \omega^* + \alpha^* (|X^*_{t-1}| - \gamma^* X^*_{t-1})^2 + \beta^* \sigma^2_{t-1}.
\]
Suppose that $X_t = X_t^* \ a.s.$ for all $t \in \mathbb{Z}$. Then, it follows that $\sigma_t = \sigma_t^* \ a.s.$ for all $t$. Note that

$$
\sigma_t^2(\theta^*) = (1 - \beta^*)^{-1} \omega^* + \alpha^* \sum_{j=0}^{\infty} \beta^*^j \left( |X_{t-1-j}^*| - \gamma X_{t-1-j}^* \right)^2 = \sigma_t^{*2},
$$

where $\sigma_t^2(\theta^*)$ is the one defined by (3). Hence, $\theta^* = \theta^o$ follows from the identifiability condition in Section 2. By similar arguments, Theorems 1 and 3–5 entail that any two different parameters can not produce the same data generating process in the corresponding models. In Poisson autoregressive models, $\lambda_t = E[X_t|\mathcal{F}_{t-1}]$ can be used instead of a specific innovation process.

Appendix

Proof of Theorem 1. We prove the theorem when $d = 1$. The proof for the other case is completely analogous. By the stationarity, it holds that $\sigma_t^2 = \sigma_t^2(\theta) \ a.s.$ for any $t \in \mathbb{Z}$. Since $\beta^o(z) \neq 0$ for $|z| \leq 1$ and $\sigma_t^2 = \beta^o(B)^{-1} c_t(\alpha^o)$, we can have

$$
c_t(\alpha) = \beta(B)\sigma_t^2(\theta) = \beta(B)\beta^o(B)^{-1} c_t(\alpha^o) = c_t(\alpha^o) + \sum_{j=1}^{\infty} b_j c_{t-j}(\alpha^o), \quad \text{(A.1)}
$$

where $1 + \sum_{j=1}^{\infty} b_j z^j = \beta(z)/\beta^o(z)$ for $|z| \leq 1$. As discussed in Section 2, we can express (A.1) as a function of $\eta_{t-1}$ and $\mathcal{F}_{t-2}$-measurable random variables given by:

$$
g_1(\eta_{t-1}, \sigma_{t-1}, A_{t,2}, B_{t,2}, A^o_{t,2}, B^o_{t,2}, D_{t,2}) \quad := \quad (\alpha_1 - \alpha_1^o)\sigma_{t-1}^2 \eta_{t-1}^2 + A_{t,2} - A^o_{t,2} + \left( \alpha_2 \sigma_{t-1}^2 \eta_{t-1}^2 + B_{t,2} \right) F(\sigma_{t-1} \eta_{t-1}, \gamma)
$$

$$
\quad \quad - \left( \alpha_2^o \sigma_{t-1}^2 \eta_{t-1}^2 + B^o_{t,2} \right) F(\sigma_{t-1} \eta_{t-1}, \gamma^o) - D_{t,2}
$$

$$
= 0 \quad \text{a.s.,}
$$

where for $2 \leq i^* \leq q$ and $2 \leq k$,

$$
A_{t,i^*} = \omega + \sum_{i=i^*}^{q} \alpha_{1i} X_{t-i}^2, \quad B_{t,i^*} = \sum_{i=i^*}^{q} \alpha_{2i} X_{t-i}^2, \quad D_{t,k} = \sum_{j=k-1}^{\infty} b_j c_{t-j}(\alpha^o),
$$

$$
A^o_{t,i^*} = \omega^o + \sum_{i=i^*}^{q} \alpha^o_{1i} X_{t-i}^2, \quad B^o_{t,i^*} = \sum_{i=i^*}^{q} \alpha^o_{2i} X_{t-i}^2.
$$

By the same argument obtaining (5) and condition (b), it holds with probability 1 that $g_1(x, \sigma_{t-1}, A_{t,2}, B_{t,2}, A^o_{t,2}, B^o_{t,2}, D_{t,2}) = 0$ for all $x \in \mathbb{R}$. In particular, this implies that

$$
g_1(0, \sigma_{t-1}, A_{t,2}, B_{t,2}, A^o_{t,2}, B^o_{t,2}, D_{t,2}) = A_{t,2} - A^o_{t,2} - D_{t,2} = 0 \quad \text{a.s.} \quad \text{(A.2)}
$$
Expressing (A.2) as a function of $\eta_{t-2}$ and $\mathcal{F}_{t-3}$-measurable random variables, we have

$$
g_2(\eta_{t-2}, \sigma_{t-2}, A_{t,3}, A_{t,3}^o, A_{t-1,2}^o, B_{t-1,2}^o; D_{t,3})
$$

$$
:= (\alpha_{12} - \alpha_{12}^o)\sigma_{t-2}^2\eta_{t-2}^2 + A_{t,3} - A_{t,3}^o - b_1\epsilon_{t-1}(\alpha^o) - D_{t,3}
$$

$$
= (\alpha_{12} - \alpha_{12}^o)\sigma_{t-2}^2\eta_{t-2}^2 + A_{t,3} - A_{t,3}^o - D_{t,3}
$$

$$
- b_1 \left( \alpha_{11}^o\sigma_{t-2}^2\eta_{t-2}^2 + A_{t-1,2}^o + (\alpha_{21}^o\sigma_{t-2}^2\eta_{t-2}^2 + B_{t-1,2}^o) \cdot F(\sigma_{t-2}\eta_{t-2}, \gamma^o) \right)
$$

$$
= 0 \quad \text{a.s.}
$$

Then, it follows that

$$
P \left( g_2(x, \sigma_{t-2}, A_{t,3}, A_{t,3}^o, A_{t-1,2}^o, B_{t-1,2}^o; D_{t,3}) = 0, \forall x \in \mathbb{R} \right) = 1
$$

(A.4)

Consider an identity: $\forall x \in \mathbb{R},$

$$
f(x) = ax^2 + b + (cx^2 + d)F(x, \gamma^o) = 0,
$$

(A.5)

where $a, b, c, d, \sigma > 0, \gamma^o > 0$ are real constants. Since $\lim_{x \to \pm \infty} x^{-2}f(x) = 0$, we have $a = c = 0$, then it follows from $\lim_{x \to \pm \infty} f(x) = 0$ that $b = d = 0$. Hence, (A.4) implies in particular that $b_1\alpha_{21}^o = 0$ and $b_1B_{t-1,2}^o = 0$ a.s. It can be easily seen that $B_{t-1,2}^o = 0$ a.s. if and only if $\alpha_{22}^o$ is a.s. 0. From condition (a) and (A.3), we obtain that $b_1 = 0$ and $A_{t,3} - A_{t,3}^o - D_{t,3} = 0$ a.s. By repeating similar arguments, we can have that $b_k = 0$, $k \geq 2$ and $A_{t,k+2} - A_{t,k+2}^o - D_{t,k+2} = 0$ a.s. for $2 \leq k \leq q - 2$, and $\omega - \omega^o - D_{t,k+2} = 0$ a.s. for $q - 1 \leq k$. Hence, we obtain that $\beta(\cdot) \equiv \beta^o(\cdot), \omega = \omega^o,$ and $A_{t,2} = A_{t,2}^o, \ldots, A_{t,q} = A_{t,q}^o$ almost surely, from which it follows that $\alpha_{1q} = \alpha_{1q}^o, \ldots, \alpha_{12} = \alpha_{12}^o$.

Now, the relation (A.1) is reduced to

$$
h_1(\eta_{t-1}, \sigma_{t-1}, B_{t,2}, B_{t,2}^o) := (\alpha_{11} - \alpha_{11}^o)\sigma_{t-1}^2\eta_{t-1}^2 + (\alpha_{21}^o\sigma_{t-1}^2\eta_{t-1}^2 + B_{t,2}^o) \cdot F(\sigma_{t-1}\eta_{t-1}, \gamma)
$$

$$
- (\alpha_{21}^o\sigma_{t-1}^2\eta_{t-1}^2 + B_{t,2}^o) \cdot F(\sigma_{t-1}\eta_{t-1}, \gamma^o)
$$

$$
= 0 \quad \text{a.s.}
$$

(A.6)

Suppose that $\gamma = 0$, thus $F(X_{t-1}, \gamma) \equiv 0$. Using the identity (A.3) again, it follows from (A.6) that $\alpha_{21}^o = 0$ and $B_{t,2}^o = 0$ a.s. Since this is a contradiction to condition (a), $\gamma$ should be positive. Then, it follows from (A.6) that

$$
\lim_{x \to \infty} x^{-2}h_1(x, \sigma_{t-1}, B_{t,2}, B_{t,2}^o) = \alpha_{11}^o\sigma_{t-1}^2\eta_{t-1}^2 \cdot \{ \alpha_{11} - \alpha_{11}^o - 2^{-1}(\alpha_{21} - \alpha_{21}^o) \} = 0 \quad \text{a.s.}
$$

Also by taking the limit $x \to -\infty$, we obtain $\alpha_{11} = \alpha_{11}^o$ and $\alpha_{21} = \alpha_{21}^o$. Using this, we can have that

$$
\lim_{x \to \infty} h_1(x, \sigma_{t-1}, B_{t,2}, B_{t,2}^o) = -2^{-1}B_{t,2} + 2^{-1}B_{t,2}^o = 0 \quad \text{a.s.},
$$

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from which it can be easily seen that \( \alpha_{2i} = \alpha_{2i}^0, \ 2 \leq i \leq q \). Now, \( (A.6) \) is reduced to
\[
h_2(\eta_{t-1}, \sigma_{t-1}, B_{t,2}^0) := (\alpha_{21}^0 \sigma_{t-1}^2 \eta_{t-1}^2 + B_{t,2}^0) \left( \frac{1}{1 + e^{\gamma \sigma_{t-1} \eta_{t-1}}} - \frac{1}{1 + e^{\gamma^0 \sigma_{t-1} \eta_{t-1}}} \right) = 0 \quad \text{a.s.}
\]
Suppose that \( \gamma < \gamma^0 \). In the case of \( \alpha_{21}^0 \neq 0 \), it follows that
\[
\lim_{x \to \infty} x^{-2} e^{\gamma \sigma_{t-1} x} h_2(x, \sigma_{t-1}, B_{t,2}^0) = \alpha_{21}^0 \sigma_{t-1}^2 = 0 \quad \text{a.s.,}
\]
which is a contradiction. When \( \alpha_{21}^0 = 0 \), we have \( \lim_{x \to \infty} e^{\gamma \sigma_{t-1} x} h_2(x, \sigma_{t-1}, B_{t,2}^0) = B_{t,2}^0 = 0 \)
a.s., which leads to a contradiction to condition (a). Since \( \gamma > \gamma^0 \) also contradicts, we conclude that \( \gamma = \gamma^0 \), which completes the proof.

**Proof of Theorem 2.** By following the same lines in the proof of Theorem 1, we can obtain similarly to \( (A.3) \) that
\[
g'_2(\eta_{t-2}, \sigma_{t-2}, A_{t,3}, A_{t,1}^0, A_{t-1,2}^0, D_{t,3})
:= (\alpha_{22} - \alpha_{12}^0 - b_1(\alpha_{11}^0)\sigma_{t-2}^2 \eta_{t-2}^2 + A_{t,3} - A_{t,1}^0 - b_1 A_{t-1,2}^0 - D_{t,3})
= 0 \quad \text{a.s.}
\]
As in \( (A.2) \), this entails that \( A_{t,3} - A_{t,1}^0 - b_1 A_{t-1,2}^0 - D_{t,3} = 0 \) a.s. By repeating similar arguments, we have that \( \omega - \omega^0 - b_1 A_{t-1,q} - b_2 A_{t-2,q-1} - \cdots - b_{q-1} A_{t-q+1,2} - D_{t,q+1} = 0 \) a.s. Then, it holds with probability 1 that for all \( x \in \mathbb{R} \),
\[
g(x) := (\omega - \omega^0) - b_1 (\alpha_{1q}^0 \sigma_{t-q-1}^2 x^2 + \omega^0) - b_2 (\alpha_{1q-1}^0 \sigma_{t-q-1}^2 x^2 + A_{t-2,q}^0) - \cdots - b_{q-1} (\alpha_{12}^0 \sigma_{t-q-1}^2 x^2 + A_{t-q+1,3}) - b_q (\alpha_{11}^0 \sigma_{t-q-1}^2 x^2 + A_{t-q,2}) - D_{t,q+2}
= 0.
\]
This identity implies that
\[
P \left( \lim_{x \to \infty} \frac{-g(x)}{\sigma_{t-q-1}^2 x^2} = b_1 \alpha_{1q}^0 + \cdots + b_q \alpha_{11}^0 = 0 \right) = 1.
\]
Furthermore, we can repeat the above arguments and obtain relations between \( \eta_{t-q-k} \) and \( F_{t-q-k-1} \)-measurable random variables for \( k \geq 2 \). Consequently, it follows that \( b_k \alpha_{1q}^0 + \cdots + b_{k+q-1} \alpha_{11}^0 = 0 \) for all \( k \geq 1 \), that is, \( \beta(z) \beta^0(z)^{-1} \alpha_{11}^0 \) is a polynomial of degree at most \( q \). By condition (c) and standard arguments (for instance, those in Straumann and Mikosch (2006), p.2481), we conclude that \( \beta'(\cdot) \equiv \beta^0(\cdot) \). Hence, \( b_j = 0 \) for \( j \geq 1 \). In view of \( (A.2) \) and \( (A.7) \), it follows that \( A_{t,2} = A_{t,3}, \ldots, A_{t,q} = A_{t,q}^0 \) a.s. and \( \omega = \omega^0 \), which also imply that \( \alpha_{1q} = \alpha_{1q}^0, \ldots, \alpha_{12} = \alpha_{12}^0 \).

Now the relation \( (A.1) \) is reduced to
\[
h_2'(\eta_{t-1}, \sigma_{t-1}, B_{t,2}) := (\alpha_{11} - \alpha_{11}^0) \sigma_{t-1}^2 \eta_{t-1}^2 + (\alpha_{21} \sigma_{t-1}^2 \eta_{t-1}^2 + B_{t,2}) F(\sigma_{t-1} \eta_{t-1}, \gamma)
= 0 \quad \text{a.s.}
\]
Irrespective of $\gamma = 0$ or not, we can easily obtain $\alpha_{11} = \alpha_{11}^0$. Suppose $\gamma > 0$. By using the identity (A.5) again, it follows that $\alpha_{21} = \cdots = \alpha_{2q} = 0$. This completes the proof. 

**Proof of Theorem 3.** We first show that the support of the stationary distribution of $(X_1, \lambda_1)$ equals to the Cartesian product of $\mathbb{N}_0$ and $\text{supp}(\lambda_1)$. If not, there exists $(m', \lambda') \in \mathbb{N}_0 \times \text{supp}(\lambda_1)$ such that $(m', \lambda') \notin \text{supp}(X_1, \lambda_1)$. Then, we can take a positive number $r$ such that

$$0 = P(X_1 = m', \lambda_1 \in (\lambda' - r, \lambda' + r)) = \int_{\lambda' - r}^{\lambda' + r} (m')^{-1} e^{-u} m' dF_{\lambda_1}(u),$$

where $F_{\lambda_1}$ is the distribution function of $\lambda_1$. Since the integrand is positive, the above equality implies $P(\lambda_1 \in (\lambda' - r, \lambda' + r)) = 0$, which is a contradiction to $\lambda' \in \text{supp}(\lambda_1)$.

By the stationarity, it holds that for all $t \in \mathbb{Z}$,

$$g(X_{t-1}, \lambda_{t-1}) = (\omega - \omega^o) + (\alpha_1 - \alpha_1^o) X_{t-1} + (\alpha_2 - \alpha_1)(X_{t-1} - l)^+ + (\alpha_2 - \alpha_1^o)(X_{t-1} - l^o)^+ + (\beta - \beta^o) \lambda_{t-1} = 0 \text{ a.s.}$$

Since $g(\cdot)$ is continuous and $\text{supp}(X_1, \lambda_1) = \mathbb{N}_0 \times \text{supp}(\lambda_1)$, we have

$$g(m, \lambda) = 0 \text{ for all } m \in \mathbb{N}_0 \text{ and } \lambda \in \text{supp}(\lambda_1). \tag{A.8}$$

In particular, $g(0, \lambda) = (\omega - \omega^o) + (\beta - \beta^o) \lambda = 0$ for any $\lambda \in \text{supp}(\lambda_1)$. Note that $\lambda_t$ is not degenerate under the condition $\alpha_1^o \neq \alpha_2^o$; otherwise $X_{t-1}$ should be degenerate. Thus, we conclude $\omega = \omega^o$ and $\beta = \beta^o$. Then, we have that $g(1, \lambda) = \alpha_1 - \alpha_1^o = 0$. It follows from (A.8) that $\lim_{m \to \infty} m^{-1} g(m, \lambda) = \alpha_2 - \alpha_2^o = 0$. Using the fact that $g(l, \lambda) = g(l^o, \lambda) = 0$ and the condition $\alpha_1^o \neq \alpha_2^o$, we obtain $l = l^o$, which completes the proof. 

**Proof of Theorem 5.** From condition (b) and the continuity of $m(\cdot, \theta)$, it follows that

$$m(x_1, \ldots, x_p, \theta^o) = m(x_1, \ldots, x_p, \theta), \quad \forall x_j \in \mathbb{R}, \quad 1 \leq j \leq p. \tag{A.9}$$

Without loss of generality, we may assume that $d^o = 1$. Suppose that $d \neq 1$. Express (A.9) as an identity with respect to $x_1$ given by

$$m(x_1, \ldots, x_p, \theta^o) - m(x_1, \ldots, x_p, \theta)$$

$$= \left\{ f_0^o(x_2) - f_0(x_2) - \sum_{i=1}^M f_i(x_2) G(x_d; \gamma_i, c_i) \right\}$$

$$+ \left\{ \phi_0^o - \phi_0 - \sum_{i=1}^M \phi_1 G(x_d; \gamma_i, c_i) \right\} x_1 + \sum_{i=1}^M (f_i^o(x_2) + \phi_{i1}^o x_1) G(x_1; \gamma_i^o, c_i)$$

$$= 0,$$
Applying Lemma A.1 again, we conclude $\phi_i = 0$ for some $x$ with respect to $x$ and $\phi_i \in \mathcal{G}$ members of $\mathcal{G}$.

Notice that Lemma A.1 verifies the linear independence of the family of real-valued functions, $\mathcal{G} = \{1, y\} \cup \{G(y; \gamma, c) : \gamma > 0, c \in \mathbb{R}\} \cup \{yG(y; \gamma, c) : \gamma > 0, c \in \mathbb{R}\}$. Thus, any member of the linear span of $\mathcal{G}$ has a unique representation as a linear combination of members of $\mathcal{G}$; see Yakowitz & Spragins (1968). Further, it is possible to choose a vector $x'_2 \in \mathbb{R}^{p-1}$ such that $(f_i(x'_2), \phi_{i1}) \neq (0, 0)$ for all $i = 1, \ldots, M$; otherwise $\phi_{i1} = \cdots = \phi_{ip} = 0$ for some $i$, which contradicts to the condition (a). Hence, from the identity (A.11) with respect to $x_1$ when $x_2 = x'_2$ and the condition (c), we deduce in particular that $\phi_{01} = \phi_{01}$ and $\phi_{i1} = \phi_{i1}$, $\gamma_i = \gamma_i$, $c_i = c_i$ for each $i = 1, \ldots, M$. Then, it follows from (A.11) that for all $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}^{p-1}$,

$$(f_0(x_2) - f_0(x_2)) + \sum_{i=1}^M (f_i(x_2) - f_i(x_2)) G(x_1; \gamma_i, c_i) = 0.$$ 

Applying Lemma A.1 again, we conclude $\phi_{0i} = \phi_{0i}$ and $\phi_{ij} = \phi_{ij}$ for $j = 2, \ldots, p$, $i = 0, 1, \ldots, M$. This completes the proof.

**Lemma A.1.** Let $(\gamma_1, c_1), \ldots, (\gamma_k, c_k)$ be distinct real vectors with $\gamma_i > 0$, $i = 1, \ldots, k$. Suppose that for all $y \in \mathbb{R}$,

$$d_{00} + d_{01}y + \sum_{i=1}^k (d_{i0} + d_{i1}y) \frac{1}{1 + e^{-\gamma_i(y-c_i)}} = 0. \quad (A.12)$$

Then, $d_{00} = d_{01} = 0$ for each $i = 0, 1, \ldots, k$.

**Proof.** Denote by $g(y)$ the left-hand side of (A.12). Note that (A.12) implies $\lim_{y \to -\infty} y^{-1} g(y) = d_{01} = 0$, and thus, $\lim_{y \to -\infty} g(y) = d_{00} = 0$. Consider the two-sided
Laplace transform of a function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( \mathcal{L}\{f(y)\}(s) = \int_{-\infty}^{\infty} e^{-sy} f(y) dy \). The transform of the logistic distribution function is given by

\[
F_0(s; \gamma, c) := \mathcal{L}\{G(y; \gamma, c)\}(s) = \frac{\pi \gamma^{-1} e^{-cs}}{\sin \pi \gamma^{-1} s}, \quad 0 < s < \gamma.
\]

Further, it follows that

\[
F_1(s; \gamma, c) := \mathcal{L}\{y G(y; \gamma, c)\}(s) = \frac{\pi \gamma^{-1} ce^{-cs}}{\sin \pi \gamma^{-1} s} + \frac{\pi^2 \gamma^{-2} e^{-cs} \cos \pi \gamma^{-1} s}{\sin^2 \pi \gamma^{-1} s}, \quad 0 < s < \gamma.
\]

Without loss of generality, assume that \((\gamma_i, c_i)\) \(i = 1, \ldots, k\) satisfy a lexicographical ordering such that \(\gamma_i \leq \gamma_{i+1}\) and \(c_i < c_{i+1}\) when \(\gamma_i = \gamma_{i+1}\). Suppose \(\gamma_1 = \cdots = \gamma_l < \gamma_{l+1} \leq \cdots \leq \gamma_k\), thus \(c_1 < \cdots < c_l\). Then, it follows from \((A.12)\) that for all \(0 < s < \gamma_1\),

\[
\sum_{i=1}^{k} d_{0i} F_0(s; \gamma_i, c_i) + \sum_{i=1}^{k} d_{1i} F_1(s; \gamma_i, c_i) = 0. \tag{A.13}
\]

Since the numerator of the left-hand side of \((A.13)\) is an analytic function on \(\mathbb{R}\), the relation \((A.13)\) is still valid for all \(s \in \mathbb{R} \setminus D\), where \(D = \{s : s = \gamma_{im}, 1 \leq i \leq k, m \in \mathbb{Z}\}\). Multiplying both sides of \((A.13)\) by \(\sin^2 \pi \gamma^{-1} s\), it follows that

\[
\sin \pi \gamma^{-1} s \sum_{i=1}^{l} \left\{ d_{0i} \pi \gamma_{1}^{-1} e^{-c_{i}s} + d_{1i} \pi \gamma_{1}^{-1} c_{i} e^{-c_{i}s} \right\} \\
+ \sin^2 \pi \gamma^{-1} s \sum_{i=l+1}^{k} \left\{ d_{0i} \frac{\pi \gamma_{i}^{-1} e^{-c_{i}s}}{\sin \pi \gamma_{i}^{-1} s} + d_{1i} \frac{\pi \gamma_{i}^{-1} c_{i} e^{-c_{i}s}}{\sin \pi \gamma_{i}^{-1} s} \right\} \\
+ \cos \pi \gamma^{-1} s \sum_{i=1}^{l} d_{1i} \pi^2 \gamma_{1}^{-2} e^{-c_{i}s} + \sin^2 \pi \gamma^{-1} s \sum_{i=l+1}^{k} d_{1i} \frac{\pi^2 \gamma_{i}^{-2} e^{-c_{i}s} \cos \pi \gamma_{i}^{-1} s}{\sin^2 \pi \gamma_{i}^{-1} s} = 0.
\]

Define \(N_1 = \{n \in \mathbb{N} : \gamma_1 n \neq \gamma_{im}, l < i < k, m \in \mathbb{N}\}\), which can be shown to have infinitely many numbers. Fix \(n \in N_1\). Letting \(s \to \gamma_1 n\) through values in \(\mathbb{R} \setminus D\), we obtain that

\[
\sum_{i=1}^{l} d_{1i} e^{-c_{i}\gamma_1 n} = 0. \tag{A.14}
\]

Since \((A.14)\) holds for all \(n \in N_1\), by multiplying both sides by \(e^{c_{i}\gamma_1 n}\) and letting \(n \to \infty\) through values in \(N_1\), we get \(d_{1i} = 0\). Further, it follows from the same arguments that \(d_{21} = \cdots = d_{1l} = 0\). Now, multiplying the both sides of \((A.13)\) by \(\sin \pi \gamma_{1}^{-1} s\) and letting \(s \to \gamma_1 n\), it follows that \(\sum_{i=1}^{l} d_{0i} e^{-c_{i}\gamma_1 n} = 0\) for any \(n \in N_1\). The preceding argument shows that \(d_{10} = \cdots = d_{1l} = 0\). By repeating the above arguments, the lemma is validated. \(\square\)
Remark 7. According to the theorem of [Yakowitz & Spragins (1968)], Lemma A.1 entails the identifiability of logistic mixture distributions, which was verified by [Sussmann (1992)]. [Hwang & Ding (1997)] also proved the linear independence of logistic distribution functions and density functions for identifiability in the artificial neural network problems. However, their results do not imply Lemma A.1. Notice that our proof is simpler and based on the method of Theorem 2 of [Teicher (1963)].

Acknowledgements

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2012R1A2A2A01046092).

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