ON MORPHISMS OF COMPACT KÄHLER MANIFOLDS WITH SEMI-POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

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Abstract. In this paper, with the aim of establishing a structure theorem for a compact Kähler manifold $X$ with semi-positive holomorphic sectional curvature, we study a morphism $\phi : X \to Y$ to a compact Kähler manifold $Y$ with pseudo-effective canonical bundle. We prove that the morphism $\phi$ is always smooth (that is, a submersion), the image $Y$ admits a finite étale cover $T \to Y$ by a complex torus $T$, and further that all the fibers are isomorphic when $X$ is projective. Moreover, by applying a modified method to maximal rationally connected fibrations, we show that $X$ is rationally connected, if $X$ is projective and $X$ has no truly flat tangent vectors at some point (which is satisfied when the holomorphic sectional curvature is quasi-positive). This result gives a generalization of Yau’s conjecture. As a further application, we obtain a uniformization theorem for compact Kähler surfaces with semi-positive holomorphic sectional curvature.

1. Introduction

One of the fundamental and important problems in differential geometry is to establish structure theorems or classifications for varieties satisfying certain curvature conditions. The Frankel conjecture, which has been solved by Siu-Yau in [SY80] and by Mori in [Mor79], states that a smooth projective variety with positive bisectional curvature is isomorphic to the projective space (see [Mor79] for the Hartshorne conjecture). As one of the extensions of the Frankel conjecture, it is a significant problem to consider the geometry of semi-positive bisectional curvature (or more generally the geometry of nef tangent bundles). In their paper [HSW81], Howard-Smyth-Wu studied a structure theorem for a compact Kähler manifold $M$ with semi-positive bisectional curvature and they showed that $M$ can be decomposed into a “flat” manifold $B$ and a “quasi-positively curved” manifold $M'$ (see also [CG71] and [CG72]). Precisely speaking, they proved that $M$ admits a locally trivial morphism $f : M \to B$ to a flat Kähler...
manifold $B$ such that the fiber $M'$ of $f$ is a smooth projective variety with quasi-positive bisectional curvature. The flat manifold $B$ is a complex torus up to finite étale covers, and thus the geometry of $M$ can be reduced to the smooth projective variety $M'$ with quasi-positive bisectional curvature, thanks to their structure theorem. Further, it can also be proven that the universal cover of $M$ is isomorphic to the product $M' \times \mathbb{C}^m$ (see [DPS94] for compact Kähler manifolds with nef tangent bundle). After the work of Howard-Smyth-Wu, in his paper [Mok88], Mok studied smooth projective varieties with quasi-positive bisectional curvature and he showed that the fiber $M'$ of $f$ is isomorphic to the product of projective spaces and compact Hermitian symmetric manifolds (see [CP91] for the Campana-Peternell conjecture).

This paper is devoted to studies of compact Kähler manifolds whose holomorphic sectional curvature is semi-positive or quasi-positive, motivated by generalizing Howard-Smyth-Wu’s structure theorem for manifolds with semi-positive bisectional curvature and Mok’s result for the geometry of quasi-positive bisectional curvature.

The first contribution of this paper is concerned with the solution and its generalization of the following conjecture posed by Yau in [Yau82], which gives a relation between the “strict” positivity of holomorphic sectional curvature and the geometry of $X$ (rationally connectedness). Yau’s conjecture can be seen as an analogy of Mok’s result in the studies of holomorphic sectional curvature and it corresponds to the geometry of the fiber $M'$ appearing in Howard-Smyth-Wu’s structure theorem.

**Conjecture 1.1** (Yau’s conjecture for projective varieties). *If a smooth projective variety $X$ admits a Kähler metric with positive holomorphic sectional curvature, then $X$ is rationally connected (that is, two arbitrary points can be connected by a rational curve).*

In their paper [HW15], Heier-Wong considered Yau’s conjecture for projective varieties under the weaker assumption that the holomorphic sectional curvature is quasi-positive (that is, it is semi-positive everywhere and positive at some point). We emphasize that it is essentially important to consider Yau’s conjecture for quasi-positive holomorphic sectional curvature from the viewpoint of structure theorems, since the bisectional curvature of the fiber $M'$ is quasi-positive, but not necessarily positive everywhere. Yang affirmatively solved Yau’s conjecture even for the case of compact Kähler manifolds by introducing the notion of RC positivity in [Yan18a] (see [Yan18c] and references therein for recent progress of RC positivity), but it seems to be quite difficult to apply his method to the case of quasi-positive holomorphic sectional curvature.

In this paper, we obtain a generalization of Yau’s conjecture, Heier-Wong’s result, and Yang’s result (see Theorem 1.2), by using an idea in [HW15] and by developing techniques for a partial positivity and certain flatness. This theorem can be seen as a version of Mok’s result for holomorphic sectional curvature, and further it gives a
more precise relation between the positivity of holomorphic sectional curvature and the
dimension of images of maximal rationally connected (MRC for short) fibrations (which
measures how far $X$ is from rationally connectedness). See [Cam92] and [KoMM92] for
MRC fibrations.

**Theorem 1.2.** Let $(X, g)$ be a compact Kähler manifold such that $X$ is projective and
the holomorphic sectional curvature is semi-positive. Let $\phi : X \to Y$ be a MRC
fibration of $X$. Then we have

$$\dim X - \dim Y \geq n_{tf}(X, g).$$

In particular, the manifold $X$ is rationally connected if $n_{tf}(X, g) = \dim X$ (which is
satisfied if the holomorphic sectional curvature is quasi-positive).

Here the invariant $n_{tf}(X, g)$ is defined by

$$n_{tf}(X, g) := \dim X - \inf_{p \in X} \dim V_{flat, p},$$

where $V_{flat, p}$ is the subspace of the tangent space $T_{X, p}$ at $p$ consisting of all the truly flat
tangent vectors (see subsection 2.2 for the precise definition). The invariant $n_{tf}(X, g)$
can be seen as an analogue of the numerical Kodaira dimension in terms of truly flat
tangent vectors introduced in [HLWZ17]. The condition of $n_{tf}(X, g) = \dim X$ (that
is, there is no truly flat tangent vectors at some point) is a weaker assumption than
the quasi-positivity, but it works in a more flexible manner from the viewpoint of our
argument.

The second contribution of this paper is a partial answer for the following structure
conjecture. The following conjecture, which is a revised version of [Mat18, Conjecture
1.1], asks a structure theorem for compact Kähler manifolds with semi-positive holo-
morphic sectional curvature and it can be seen as a natural generalization of Howard-
Smyth-Wu’s structure theorem.

In this paper, we affirmatively solve Conjecture 1.3 under the assumption that a MRC
fibration of a smooth projective variety $X$ can be chosen to be a morphism without
indeterminacy locus (see Theorem 1.4). This assumption is satisfied when $X$ has the
nef anti-canonical bundle by the deep result of [CH17] (see Corollary 3.15). Moreover
we solve Conjecture 1.3 for compact Kähler surfaces without any assumptions (see
Corollary 1.5).

**Conjecture 1.3** (cf. [HSW81] and [Mat18, Conjecture 1.1]). Let $X$ be a compact
Kähler manifold with semi-positive holomorphic sectional curvature.

1. Then there exists a smooth morphism $X \to Y$ with the following properties:
   - The morphism $X \to Y$ is locally trivial (that it, all the fibers $F$ are isomorphic).
   - The fiber $F$ is projective and rationally connected.
• $Y$ is a compact Kähler manifold with flat metric.

In particular, there exist a complex torus $T$ and a finite étale cover $T \to Y$ such that the fiber product $X^* := X \times_Y T$ admits a locally trivial morphism $X^* = X \times_Y T \to T$ to the complex torus $T$ with the rationally connected fiber $F$ and that it satisfies the following commutative diagram:

$$
\begin{array}{c}
X^* = X \times_Y T \\
\downarrow \\
X \\
\downarrow \\
Y.
\end{array}
$$

(2) Moreover we have the decomposition

$$X_{\text{univ}} \cong \mathbb{C}^m \times F,$$

where $X_{\text{univ}}$ is the universal cover of $X$ and $F$ is the rationally connect fiber.

In particular, the fundamental group of $X$ is an extension of a finite group by $\mathbb{Z}^{2m}$.

When we approach to the above conjecture in the case of $X$ being projective, it seems to be the right direction to study a MRC fibration $\phi : X \dashrightarrow Y$ of $X$, based on the strategy explained in [Mat18]. We remark that MRC fibrations are almost holomorphic maps (that is, dominant rational maps with compact general fibers) and they are uniquely determined up to birational equivalence. It can be seen that we can always choose a MRC fibration $\phi : X \dashrightarrow Y$ such that the image $Y$ is smooth by taking a resolution of singularities and that the image $Y$ has the pseudo-effective canonical bundle by [GHS03, Theorem 1.1] and [BDPP13].

The following theorem, which is one of the main results of this paper, reveals a detailed structure of morphisms whose domain has semi-positive holomorphic sectional curvature. Theorem 1.4 is formulated for MRC fibrations of projective varieties and Albanese maps of Kähler manifolds. By applying Theorem 1.4 to MRC fibrations, we can affirmatively solve Conjecture 1.3 for compact Kähler surfaces and (1) of Conjecture 1.3 in the case where a MRC fibration can be chosen to be a morphism (see Corollary 1.5). Further, by applying Theorem 1.4 to Albanese maps, we can obtain a vanishing theorem for the global holomorphic 1-forms (see Corollary 1.6). This vanishing theorem is an extension of [Yan18a, Theorem 1.7].

**Theorem 1.4.** Let $(X, g)$ be a compact Kähler manifold with semi-positive holomorphic sectional curvature and let $Y$ be a compact Kähler manifold with pseudo-effective canonical bundle. Further let $\phi : X \to Y$ be a morphism from $X$ to $Y$. Then the following statements hold:

(1) $\phi$ is a smooth morphism (that is, a submersion).
(2) The standard exact sequence of vector bundles obtained from (1)
\[ 0 \to T_{X/Y} := \text{Ker} \, d\phi_* \to T_X \xrightarrow{d\phi_*} \phi^*T_Y \to 0 \]
splits. Moreover its holomorphic splitting
\[ T_X = T_{X/Y} \oplus \phi^*T_Y \]
coincides with the orthogonal decomposition of \( T_X \) with respect to \( g \). Here \( T_X \) (resp. \( T_Y \)) is the (holomorphic) tangent bundle of \( X \) (resp. \( Y \)).

(3) Let \( g_Q \) be the hermitian metric on \( \phi^*T_Y \) induced by the above exact sequence and the given metric \( g \). Then there exists a hermitian metric \( g_Y \) on \( T_Y \) with the following properties:
- \( g_Q \) is obtained from the pull-back of \( g_Y \) (namely, \( g_Q = \phi^*g_Y \)).
- The holomorphic sectional curvature of \((Y, g_Y)\) is identically zero. In particular, the image \( Y \) is flat and it admits a finite étale cover \( T \to Y \) by a complex torus \( T \).

(4) \( \phi \) is locally trivial if we further assume that \( X \) is projective.

**Corollary 1.5.** Let \( X \) be a compact Kähler manifold with semi-positive holomorphic sectional curvature. Then the followings hold:

- All the statements of Conjecture 1.3 hold in the case of \( X \) being a surface.
- The statement (1) of Conjecture 1.3 holds if \( X \) is projective and a MRC fibration of \( X \) can be chosen to be a morphism.

**Corollary 1.6.** Let \((X, g)\) be a compact Kähler manifold with semi-positive holomorphic sectional curvature. Then we have
\[ h^0(X, \Omega_X) \leq \dim X - n_{\text{tf}}(X, g). \]
In particular, we obtain \( h^0(X, \Omega_X) = 0 \) if \( n_{\text{tf}}(X, g) = \dim X \) (which is satisfied if the holomorphic sectional curvature is quasi-positive).

For the proof of Theorem 1.4, we will carefully observe the curvature current and its integration of an induced “singular” hermitian metric on \( \phi^*K_Y \), and further we investigate the scalar curvature of the Kähler form \( g \), based on the idea in [HW15]. The main difficulty of Theorem 1.4 is that the given metric \( g \) has no a priori relation with the morphism \( \phi \). To overcome this difficulty, we will show that all the tangent vector in the horizontal direction of \( \phi : X \to Y \) are truly flat, which produces a relation (for example, the statements (2) and (3)) between the metric \( g \) and the morphism \( \phi \).

The key point here is to construct a suitably chosen orthonormal basis of \( T_X \) by using an argument on a partial positivity developed in [Mat18].
By modifying the above techniques for a general MRC fibration $\phi : X \to Y$ (which is not necessarily a morphism), we can prove that the numerical dimension of the image $Y$ is equal to zero. Moreover we can obtain the same conclusions as in (1), (2), (3) of Theorem 1.4 over the smooth locus of $\phi$.

**Theorem 1.7.** Let $(X, g)$ be a compact Kähler manifold with semi-positive holomorphic sectional curvature and $Y$ be a compact Kähler manifold with pseudo-effective canonical bundle. Let $\phi : X \to Y$ be an almost holomorphic map from $X$ to $Y$.

Then the numerical dimension $\nu(Y)$ of $Y$ is equal to zero. Moreover, the same conclusions as in (1), (2), (3) of Theorem 1.4 hold if we replace $X$ and $Y$ in the statements of Theorem 1.4 with $X_1$ and $Y_1$ (see Theorem 3.10 for the precise statement). Here $X_1$ and $Y_1$ are Zariski open sets such that $\phi : X_1 := \phi^{-1}(Y_1) \to Y_1$ is a morphism.

In Section 2, we will recall some basic results on curvature and truly flat tangent vectors. In Section 3, we will prove all the theorems and corollaries. In Section 4, we will discuss open problems related to the geometry of semi-positive holomorphic sectional curvature.

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2. Preliminaries

For reader’s convenience, we summarize some formulas and properties of curvature tensors, holomorphic sectional curvature, and truly flat tangent vectors in this section.

2.1. Curvature and exact sequences of vector bundles. In this subsection, we recall several formulas of curvature of induced hermitian metrics and properties of exact sequences of vector bundles.

Let $(E, g)$ be a (holomorphic) vector bundle on a complex manifold $X$ equipped with a (smooth) hermitian metric $g$. The Chern curvature of $(E, g)$

$$\sqrt{-1}\Theta_g := \sqrt{-1}\Theta_g(E) \in C^\infty(X, \Lambda^{1,1} \otimes \text{End}(E)),$$

defines the curvature tensor

$$R_g := R_{(E, g)} \in C^\infty(X, \Lambda^{1,1} \otimes E^\vee \otimes \bar{E}^\vee)$$

to be

$$R_g(v, \bar{w}, e, \bar{f}) := \langle \sqrt{-1}\Theta_g(v, \bar{w})(e), f \rangle_g$$
for tangent vectors $v, w \in T_X$ and vectors $e, f \in E$. We denote the dual vector bundle of $E$ by the notation $E'$ and the inner product with respect to $g$ by the notation $(\cdot, \cdot)_g$ throughout this paper. The metric $g$ induces the hermitian metric $\Lambda^m g$ on the vector bundle $\Lambda^m E$ of the $m$-th exterior product. Then it follows that

$$
\langle \sqrt{-1} \Theta_{\Lambda^m g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \rangle_{\Lambda^m g} = \sum_{k=1}^m \langle \sqrt{-1} \Theta_g(v, \bar{v})(e_k), e_k \rangle_g
$$

for a tangent vector $v \in T_X$ and vectors $\{e_k\}_{k=1}^m$ in $E$ with $\langle e_i, e_j \rangle_g = \delta_{ij}$ since the curvature $\sqrt{-1} \Theta_{\Lambda^m g}$ associated to $\Lambda^m g$ satisfies that

$$
\sqrt{-1} \Theta_{\Lambda^m g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m) = \sum_{k=1}^m e_1 \wedge \cdots \wedge e_{k-1} \wedge \sqrt{-1} \Theta_g(v, \bar{v})(e_k) \wedge e_{k+1} \wedge \cdots \wedge e_m.
$$

In particular, the curvature $\sqrt{-1} \Theta_{\det g}$ of the determinant bundle $\det E := \Lambda^{rkE} E$ with the induced metric $\det g := \Lambda^{rkE} g$ satisfies that

$$
\sqrt{-1} \Theta_{\det g}(v, \bar{v}) = \langle \sqrt{-1} \Theta_{\det g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_{rkE}), e_1 \wedge e_2 \wedge \cdots \wedge e_{rkE} \rangle_{\det g} = \sum_{k=1}^{rkE} \langle \sqrt{-1} \Theta_g(v, \bar{v})(e_k), e_k \rangle_g
$$

for an orthonormal basis $\{e_k\}_{k=1}^{rkE}$ of $E$.

For a subbundle $S$ of $E$ and its quotient vector bundle $Q := E/S$, we consider the hermitian metric $g_S$ (resp. $g_Q$) on $S$ (resp. $Q$) induced by the metric $g$ and the exact sequence

$$
0 \longrightarrow (S, g_S) \longrightarrow (E, g) \longrightarrow (Q, g_Q) \longrightarrow 0.
$$

The quotient bundle $Q$ is isomorphic to the orthogonal complement $S^\perp$ of $S$ in $(E, g)$ as $C^\infty$-vector bundles. By this isomorphism, the quotient bundle $Q$ can be identified with the $C^\infty$-vector bundle $S^\perp$ and the metric $g_Q$ can be regarded as the hermitian metric on $S^\perp$. Also, it can be proven that there exist smooth sections (which are called fundamental forms)

$$
A \in C^\infty(X, \Lambda^{1,0} \otimes \text{Hom}(S, S^\perp)) \quad \text{and} \quad B \in C^\infty(X, \Lambda^{0,1} \otimes \text{Hom}(S^\perp, S))
$$

satisfying that

$$
\langle \sqrt{-1} \Theta_g(v, \bar{v})(e), e \rangle_g + \langle B_v(e), B_v(e) \rangle_{g_S} = \langle \sqrt{-1} \Theta_{g_Q}(v, \bar{v})(e), e \rangle_{g_Q},
$$

$$
\langle \sqrt{-1} \Theta_g(v, \bar{v})(f), f \rangle_g - \langle A_v(f), A_v(f) \rangle_{g_Q} = \langle \sqrt{-1} \Theta_{g_S}(v, \bar{v})(f), f \rangle_{g_S},
$$

$$
\langle A_v(f), e \rangle_{g_Q} + \langle f, B_v(e) \rangle_{g_S} = 0
$$
for a tangent vector $v \in T_X$, a vector $e \in S^1$, and a vector $f \in S$. In particular, we have
\begin{equation}
\langle \sqrt{-1} \Theta_g(v, \bar{v})(e), e \rangle_g \leq \langle \sqrt{-1} \Theta_{gQ}(v, \bar{v})(e), e \rangle_{gQ},
\end{equation}
\begin{equation}
\langle \sqrt{-1} \Theta_g(v, \bar{v})(f), f \rangle_g \geq \langle \sqrt{-1} \Theta_{gs}(v, \bar{v})(f), f \rangle_{gs}.
\end{equation}

Moreover it can be shown that the above exact sequence determines the holomorphic orthogonal decomposition $E = S \oplus Q$ (that is, $S^\perp$ is a holomorphic vector bundle and it is isomorphic to $Q$) if and only if $A$ (equivalently $B$) is identically zero.

In the rest of this subsection, we summarize the notion of singular hermitian metrics on a line bundle $L$ (see [Dem] for more details). A hermitian metric $h$ on $L$ is said to be a \textit{singular hermitian metric}, if $\log |e|_h$ is an $L_{1 \text{loc}}^1$-function for any local frame $e$ of $L$. Then the curvature current $\sqrt{-1} \Theta_h$ of $(L, h)$ is defined by
\begin{equation}
\sqrt{-1} \Theta_h := \sqrt{-1} \Theta_h(L) := -\sqrt{-1} \partial \bar{\partial} \log |e|_h^2
\end{equation}
in the sense of distributions. The singular hermitian metric $h$ is said to have neat \textit{analytic singularities}, if there exists an ideal sheaf $I \subset \mathcal{O}_X$ such that the function $-\log |e|_h^2$ can be locally written as
\begin{equation}
-\log |e|_h^2 = c \log (|f_1|^2 + |f_2|^2 + \cdots + |f_k|^2) + \text{smooth function},
\end{equation}
where $c$ is a positive real number and $f_1, \ldots, f_k$ are local generators of $I$. We say that $h$ has \textit{divisorial singularities} when the ideal sheaf $I$ is defined by an effective divisor. The dual singular hermitian metric $h^\vee$ on the dual line bundle $L^\vee$ can be defined to be $|e^\vee|_{h^\vee} := |e|_{h}^{-1}$ for the dual local frame $e^\vee$. Further, for a morphism $f : Z \to X$, the singular hermitian metric $f^*h$ on the pull-back $f^*L$ can also be defined to be $|f^*e|_{f^*h} := f^*(|e|_h)$ for the local frame $f^*e$ of $f^*L$. Then we have
\begin{equation}
\sqrt{-1} \Theta_h = -\sqrt{-1} \Theta_{h^\vee} = \sqrt{-1} \partial \bar{\partial} \log |e^\vee|_{h^\vee}^2 \quad \text{and} \quad f^* \sqrt{-1} \Theta_h := \sqrt{-1} \Theta_{f^*h}.
\end{equation}

\subsection*{2.2. Holomorphic sectional curvature and truly flat tangent vectors.}

In this subsection, we summarize some properties of holomorphic sectional curvature and truly flat tangent vectors. For a hermitian metric $g$ on the (holomorphic) tangent bundle $T_X$, the holomorphic sectional curvature $H_g$ is defined to be
\begin{equation}
H_g([v]) := \frac{R_g(v, \bar{v}, v, \bar{v})}{|v|^4_g} = \frac{\langle \sqrt{-1} \Theta_g(v, \bar{v})(v), v \rangle_g}{|v|^4_g}
\end{equation}
for a non-zero tangent vector $v \in T_X$. The holomorphic sectional curvature $H_g$ is said to be \textit{positive} (resp. \textit{semi-positive}) if $H_g([v]) > 0$ (resp. $H_g([v]) \geq 0$) holds for any non-zero tangent vector $v \in T_X$. Also $H_g$ is said to be \textit{quasi-positive} if it is semi-positive everywhere and it is positive at some point in $X$. 
In this paper, we handle only the case of $g$ being a Kähler metric (that is, the associated $(1,1)$-form $\omega_g$ is $d$-closed). In this case, the following symmetries hold:
\[
R_g(e_i, \bar{e}_j, e_k, \bar{e}_\ell) = R_g(e_k, \bar{e}_\ell, e_i, \bar{e}_j) = R_g(e_k, \bar{e}_j, e_i, \bar{e}_\ell).
\]
The above symmetries lead to the following lemmas.

**Lemma 2.1** ([Yan17, Lemma 4.1], [Mat18, Lemma 2.2] cf. [Bre], [BKT13]). Let $g$ be a Kähler metric of $X$ and $V$ be a subspace of $T_{X,p}$ at a point $p \in X$. If a unit vector $v \in V$ is a minimizer of the holomorphic sectional curvature $H_g$ on $V$, that is, it satisfies
\[
H_g([v]) = \min \{H_g([x]) \mid 0 \neq x \in V\},
\]
then we have
\[
2R_g(v, \bar{v}, x, \bar{x}) \geq (1 + |\langle v, x \rangle_g|^2)R_g(v, \bar{v}, v, \bar{v})
\]
for any unit vector $x \in V$. In particular, if the holomorphic sectional curvature $H_g$ is semi-positive, a minimizer $v$ of $H_g$ on $V$ satisfies that
\[
R_g(v, \bar{v}, x, \bar{x}) \geq 0
\]
for any tangent vector $x \in V$.

The above lemma was proved in [Yan17, Lemma 4.1] when the subspace $V$ in Lemma 2.1 coincides with the tangent space $T_{X,p}$. It is easy to see that the same argument as in [Yan17, Lemma 4.1] works even in the case of $V$ being a subspace of $T_{X,p}$, and thus we omit the proof of Lemma 2.1. Note that we can always take the minimizer of $H_g$ on a given subspace $V$ of $T_{X,p}$ at a point $p \in X$, since the holomorphic sectional curvature can be regarded as a smooth function on the projective space bundle $\mathbb{P}(T_X^\vee)$ (that is, the set of all complex lines $[v]$ in $T_X$) and $\mathbb{P}(V^\vee) \subset \mathbb{P}(T_X^\vee)$ is compact.

Now we define truly flat tangent vectors and the invariant $n_{tf}(X, g)$ introduced in [HLWZ17]. We remark that the invariant $n_{tf}(X, g)$ was denoted by the different notation $r_{-tf}^-$ in [HLWZ17].

**Definition 2.2** (Truly flat tangent vectors and the invariant $n_{tf}(X, g)$). Let $(X, g)$ be a Kähler manifold.

- A tangent vector $v \in T_X$ at $p$ is said to be **truly flat** with respect to $g$ if $v$ satisfies that
\[
R_g(v, \bar{x}, y, \bar{z}) = 0 \text{ for any tangent vectors } x, y, z \in T_{X,p}.
\]
- We define the subspace $V_{\text{flat},p}$ of $T_{X,p}$ at $p$ by
\[
V_{\text{flat},p} := \{v \in T_{X,p} \mid v \text{ is a truly flat tangent vector in } T_{X,p}\}.
\]
- We define the invariants $n_{tf}(X, g)_p$ and $n_{tf}(X, g)$ by
\[
n_{tf}(X, g)_p := \dim X - \dim V_{\text{flat},p} \quad \text{and} \quad n_{tf}(X, g) := \dim X - \inf_{p \in X} \dim V_{\text{flat},p}.
\]
It is easy to see that the invariant \( n_{tf}(X, g)_p \) is lower semi-continuous with respect to \( p \in X \) in the classical topology. In particular, if we have the equality \( n_{tf}(X, g)_p = n_{tf}(X, g) \) at \( p \), the same equality holds on a neighborhood of \( p \). The following lemma gives a characterization of truly flat tangent vectors in terms of holomorphic sectional curvature and bisectional curvature.

**Lemma 2.3** (cf. [HLWZ17]). Let \( g \) be a Kähler metric of \( X \) with semi-positive holomorphic sectional curvature and \( V \) be a subspace of \( T_{X,p} \) at a point \( p \in X \). If a tangent vector \( v \in T_X \) satisfies that

\[
H_g([v]) = 0 \quad \text{and} \quad R_g(v, \bar{v}, w, \bar{w}) = 0 \quad \text{for any tangent vector } w \in V,
\]

then \( v \) satisfies that

\[
R_g(v, \bar{x}, y, \bar{z}) = 0 \quad \text{for any tangent vectors } x, y, z \in V.
\]

In particular, if \( v \) satisfies the above assumptions for any tangent vector \( w \in T_{X,p} \), then \( v \) is a truly flat tangent vector at \( p \).

**Proof.** When the holomorphic sectional curvature is semi-negative and the subspace \( V \) coincides with the tangent space \( T_{X,p} \), the same conclusion was proved in [HLWZ17, Lemma 2.1]. For reader’s convenience, we will give a sketch of the proof.

For an arbitrary complex number \( re^{\sqrt{-1}\theta} \), we obtain that

\[
0 \leq H([v + re^{\sqrt{-1}\theta}w]) |v + re^{\sqrt{-1}\theta}w|_g^4
\]

\[
= 2\Re(e^{\sqrt{-1}\theta} R_g(v, \bar{v}, w, \bar{w})) r^3 + 2\Re(e^{\sqrt{-1}\theta} R_g(v, \bar{w}, w, \bar{w})) r + R_g(w, \bar{w}, w, \bar{w})
\]

from the assumptions \( R_g(v, \bar{v}, w, \bar{w}) = 0 \) and \( R_g(v, \bar{v}, w, \bar{w}) = 0 \). Here we used the symmetries obtained from Kähler metrics. If \( R_g(v, \bar{v}, w, \bar{w}) \) is not zero, we have a contradiction by suitably choosing \( \theta \) such that \( \Re(e^{\sqrt{-1}\theta} R_g(v, \bar{v}, w, \bar{w})) < 0 \) and by taking a sufficiently large \( r > 0 \). Hence we obtain \( R_g(v, \bar{v}, w, \bar{w}) = 0 \). By repeating the same argument as above for \( e^{\sqrt{-1}\theta} R_g(v, \bar{w}, w, \bar{w}) \), we can see that \( R_g(v, \bar{w}, w, \bar{w}) = 0 \) for any tangent vector \( w \in T_X \). Then we can easily check the desired equality by the standard polarization argument. \( \square \)

### 3. Proof of the results

**3.1. Proof of Theorem 1.4.** In this subsection, we give a proof of Theorem 1.4. The arguments in this subsection will be modified to handle almost holomorphic maps in the proof of Theorem 1.7. This subsection is the core of this paper.

**Theorem 3.1** (=Theorem 1.4). Let \( (X, g) \) be a compact Kähler manifold with semi-positive holomorphic sectional curvature and let \( Y \) be a compact Kähler manifold with
pseudo-effective canonical bundle. Further let $\phi : X \to Y$ be a morphism from $X$ to $Y$. Then the following statements hold:

1. $\phi$ is a smooth morphism (that is, a submersion).
2. The standard exact sequence of vector bundles obtained from (1)
   $0 \longrightarrow T_{X/Y} := \text{Ker } d\phi^* \longrightarrow T_X \xrightarrow{d\phi^*} \phi^* T_Y \longrightarrow 0$
splits. Moreover its holomorphic splitting
   
   
   $T_X = T_{X/Y} \oplus \phi^* T_Y$

   coincides with the orthogonal decomposition of $T_X$ with respect to $g$. Here $T_X$ (resp. $T_Y$) is the (holomorphic) tangent bundle of $X$ (resp. $Y$).
3. Let $g_Q$ be the hermitian metric on $\phi^* T_Y$ induced by the above exact sequence and the given metric $g$. Then there exists a hermitian metric $g_Y$ on $T_Y$ with the following properties:
   - $g_Q$ is obtained from the pull-back of $g_Y$ (namely, $g_Q = \phi^* g_Y$).
   - The holomorphic sectional curvature of $(Y, g_Y)$ is identically zero. In particular, the image $Y$ is flat and it admits a finite étale cover $T \to Y$ by a complex torus $T$.
4. $\phi$ is locally trivial if we further assume that $X$ is projective.

Proof. Throughout this proof, let $(X, g)$ be a compact Kähler manifold with the semi-positive holomorphic sectional curvature $H_g$ and let $\phi : X \to Y$ be a morphism (that is, a surjective holomorphic map) to a compact Kähler manifold $Y$ with the pseudo-effective canonical bundle $K_Y$. For simplicity, we put $n := \text{dim } X$ and $m := \text{dim } Y$. We will divide the proof into five steps to refer later.

**Step 1** (Singularities of induced singular hermitian metrics). Our first purpose is to prove that $\phi$ is actually a smooth morphism. For this purpose, in this step, we first construct a possibly “singular” hermitian metric $G$ on the line bundle $\phi^* K_Y$ from the given Kähler metric $g$ of $X$ such that the singularities of $G$ corresponds to the non-smooth locus of $\phi$. This enables us to reduce our first purpose to observe the singularities of $G$. Moreover, in this step, we show that the pull-back $\pi^* G$ has divisorial singularities and its curvature current can be decomposed into a smooth part and a divisorial part, after we take a suitable modification $\pi : \bar{X} \to X$.

Now we have the injective sheaf morphism

$$
(\phi^* K_Y, H) \xrightarrow{d\phi^*} (\Lambda^m \Omega_X, \Lambda^m \Omega)
$$

between the vector bundle $\Lambda^m \Omega_X := \Lambda^m T_X^\vee$ of the $m$-th exterior product and the line bundle $\phi^* K_Y := \phi^* \Lambda^m \Omega_Y$. We interchangeably use the words “line bundles” and
\textit{invertible sheaves} (also \textit{vector bundles} and \textit{locally free sheaves}) throughout this paper. Note that the above morphism is not a bundle morphism since the rank drops on the non-smooth locus of \( \phi \), but it is an injective morphism as sheaf morphisms. Let \( h \) be the dual hermitian metric of \( g \) on the (holomorphic) cotangent bundle \( \Omega_X = T_X^\vee \) and let \( \Lambda^m h \) be the induced metric on \( \Lambda^m \Omega_X \). Then, from the above morphism, we can construct a possibly singular hermitian metric \( H \) on \( \phi^* K_Y \) to be

\[ |e_H| := |d\phi^*(e)|_{\Lambda^m h} \]

for a local frame \( e \) of \( \phi^* K_Y \).

From now on, we mainly consider the dual singular hermitian metric \( G := H^\vee = H^{-1} \) on \( \phi^* K_Y^\vee \). For a local coordinate \((t_1, t_2, \ldots, t_m)\) of \( Y \), the \( m \)-form \( dt := dt_1 \wedge dt_2 \wedge \cdots \wedge dt_m \) naturally determines the local frame of \( \phi^* K_Y \), which we denote by the same notation \( dt \). By the definitions of the curvature and the dual metric, the curvature current of \((\phi^* K_Y^\vee, G)\) can be locally written as

\[ \sqrt{-1} \Theta_G := \sqrt{-1} \Theta_G(\phi^* K_Y^\vee) = \sqrt{-1} \partial \bar{\partial} \log |\phi^* dt|_{\Lambda^m h}^2, \]

where \( \phi^* dt \) is the pull-back of the \( m \)-form \( dt \) by \( \phi \). We remark that the pull-back \( \phi^* dt \) coincides with the image \( d\phi^*(dt) \) of the section \( dt \) of \( \phi^* K_Y \) by \( d\phi^* \).

By the above expression, it can be shown that the singular locus of \( G \) (that is, the polar set of the quasi-psh function \( \log |\phi^* dt|_{\Lambda^m h} \)) coincides with the non-smooth locus of \( \phi \), since the zero locus of the section \( \phi^* dt \) of \( \Lambda^m \Omega_X \) is equal to the non-smooth locus of \( \phi \). Therefore it is sufficient for our first purpose (that is, the proof of the smoothness of \( \phi \)) to prove that \( G \) is actually a smooth hermitian metric.

We take a (log) resolution \( \pi : \bar{X} \to X \) of the degenerate ideal \( \mathcal{I} \) of the above sheaf morphism. The degenerate ideal \( \mathcal{I} \) is the ideal sheaf generated by the coefficients of \( \phi^* dt \) with respect to local frames of \( \Lambda^m \Omega_X \). Then we obtain the following claim:

**Claim 3.2.** Let \( Z \) be the non-smooth locus of \( \phi \). Then the following statements hold:

- \( \pi^{-1}(Z) \) has codimension one.
- \( \pi : \bar{X} \setminus \pi^{-1}(Z) \cong X \setminus Z \).
- \( \phi^* G \) has divisorial singularities along \( \pi^{-1}(Z) \).

More precisely, the pull-back \( \pi^* \sqrt{-1} \Theta_G \) of the curvature current \( \sqrt{-1} \Theta_G \) can be written as

\[ \pi^* \sqrt{-1} \Theta_G := \sqrt{-1} \partial \bar{\partial} \log \pi^* (|\phi^* dt|_{\Lambda^m h}^2) = \gamma + [E], \]

where \( \gamma \) is a smooth \((1,1)\)-form on \( \bar{X} \) and \([E]\) is the integration current defined by an effective divisor \( E \).

**Proof of Claim 3.2.** The subvariety \( Z \) coincides with the support of the cokernel \( \mathcal{O}_X/\mathcal{I} \), and thus the first and second statements are obvious by the choice of \( \pi \). However the
third statement seems to be a subtle problem, since we do not know whether or not the metric $G$ itself has neat analytic singularities (see Remark 3.3 for more details).

To check the third statement, we fix an arbitrary point $p \in \bar{X}$. When $p$ is outside $\pi^{-1}(Z)$, the metric $G$ is smooth on a neighborhood of $\pi(p)$ since $Z$ also coincides with the zero locus of $\phi^*dt$. The third statement is obvious in this case, and thus we may assume that $p \in \pi^{-1}(Z)$.

We take a local frame $\{s_i\}_{i=1}^N$ of $\Lambda^m\Omega_X$ on a neighborhood of $\pi(p)$. Here we put $N := \binom{n}{m}$ for simplicity. The holomorphic $m$-form $\phi^*dt$ can be locally written as

$$\phi^*dt = \sum_{i=1}^N f_i s_i$$

for some holomorphic functions $\{f_i\}_{i=1}^N$. The degenerate ideal $\mathcal{I}$ is generated by $\{f_i\}_{i=1}^N$ and $\pi^{-1}\mathcal{I} = \mathcal{I} \cdot \mathcal{O}_X$ is the ideal sheaf associated to an effective divisor $E$. Let $t$ be a local holomorphic function such that $t$ determines the effective divisor $E$. Then it follows that $g_i := \pi^*f_i/t$ is a holomorphic function and the common zero locus $\cap_{i=1}^N g_i^{-1}(0)$ is empty from the choice of $\pi$. Therefore a simple computation yields

$$\log \pi^*(|\phi^*dt|^2_{\Lambda^m h}) = \log |t|^2 + \log \sum_{i,j=1}^N g_i g_j \pi^*\langle s_i, s_j \rangle_{\Lambda^m h}.$$

It can be proven that the Levi form of the first term is equal to the integration current $[E]$ by the Poincaré-Lelong formula. On the other hand, it follows that the Levi form of the second term determines a smooth $(1,1)$-form $\gamma$, since it is easy to see that the function

$$\sum_{i,j=1}^N g_i g_j \pi^*\langle s_i, s_j \rangle_{\Lambda^m h}$$

is a non-vanishing smooth function. \hfill \square

**Remark 3.3.**

- It follows that the smooth form $\gamma$ can be identified with the curvature $\sqrt{-1}\Theta_G$ under the isomorphism $\pi : \bar{X} \setminus \pi^{-1}(Z) \cong X \setminus Z$ from the second and third properties.
- The metric $G$ itself may not have neat analytic singularities although the pull-back $\pi^*G$ by $\pi$ has divisorial singularities. For example, in the case of $n = 2$ and $m = 1$, we consider the following situation:

$$\phi^*dt = z_1 s_1 + z_2 s_2$$

and $h = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with respect to a local frame $(s_1, s_2)$ of $\Omega_X$. 
Here \((z_1, z_2)\) is a local coordinate of \(X\). Then we can see that the function
\[
\frac{\vert \phi^* dt \vert_h^2}{\vert z_1 \vert^2 + \vert z_2 \vert^2} = \frac{2 \vert z_1 \vert^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2 + 2 \vert z_2 \vert^2}{\vert z_1 \vert^2 + \vert z_2 \vert^2}
\]
can not be extended to a smooth function defined at the origin. Of course, when we take a resolution of the degenerate ideal (which is just one point blow-up in this case), we can easily check that the pull-back of the above function is a non-vanishing smooth function.

**Step 2** (Construction of orthonormal basis in the horizontal direction). In this step, by using the argument in [Mat18, Lemma 3.5], we will choose a suitable orthonormal basis of \(T_X\) at a smooth point \(p\) of \(\phi\), in order to obtain a partial positivity of \(\sqrt{-1} \Theta_G\) and \(\gamma\) in the horizontal direction.

For a given point \(p \in X\) at which \(\phi\) is smooth, we consider the standard exact sequence
\[
0 \longrightarrow T_{X/Y} := \text{Ker} \ d\phi^* \longrightarrow T_X \xrightarrow{d\phi^*} \phi^* T_Y \longrightarrow 0 \text{ at } p.
\]
In the proof, we say that a tangent vector \(v \in T_X\) is in the horizontal direction (resp. in the vertical direction) in the case of \(v \in (T_{X/Y})^\perp\) (resp. \(v \in T_{X/Y}\)). Here \((T_{X/Y})^\perp\) is the orthogonal complement of \(T_{X/Y}\) in \(T_X\) with respect to \(g\) and it is identified with \(\phi^* T_Y\) at \(p\). Then we obtain the following claim:

**Claim 3.4.** For a smooth point \(p\) of \(\phi\), there exists an orthonormal basis \(\{e_k\}_{k=1}^n\) of \(T_X\) at \(p\) with the following properties:

- \(\{e_i\}_{i=1}^m\) is an orthonormal basis of \((T_{X/Y})^\perp\) at \(p\).
- \(R_g(e_i, \bar{e}_i, e_j, \bar{e}_j) \geq 0\) for any \(1 \leq i, j \leq m\).
- \(\sqrt{-1} \Theta_G(e_i, \bar{e}_i) \geq 0\) for any \(i = 1, 2, \ldots, m\).

**Proof of Claim 3.4.** We first take an arbitrary orthonormal basis \(\{e_k\}_{k=1}^m\) of \(T_X\) at \(p\) such that
\[
(T_{X/Y})^\perp = \text{Spn} \langle \{e_i\}_{i=1}^m \rangle \quad \text{and} \quad T_{X/Y} = \text{Spn} \langle \{e_j\}_{j=m+1}^n \rangle.
\]
By choosing a new orthonormal basis \(\{e_k\}_{k=1}^m\) of \((T_{X/Y})^\perp\), we may assume that \(e_1\) is the minimizer of \(H_g\) on \((T_{X/Y})^\perp = \text{Spn} \langle \{e_k\}_{k=1}^m \rangle\), that is, the unit tangent vector \(e_1\) satisfies that
\[
H_g([e_1]) = \min \{H_g([v]) \mid 0 \neq v \in \text{Spn} \langle \{e_k\}_{k=1}^m \rangle\}.
\]
After we fix the tangent vector \(e_1\) chosen as above, we choose an orthonormal basis \(\{e_i\}_{i=2}^m\) of \((T_{X/Y} \oplus \text{Spn} \langle e_1 \rangle)^\perp = \text{Spn} \langle \{e_k\}_{k=2}^m \rangle\)
such that $e_2$ is the minimizer of $H_g$ on $\text{Spn}\langle \{e_k\}_{k=2}^m \rangle$. By repeating this process, we can construct an orthonormal basis $\{e_i\}_{i=1}^m$ of $(T_X/Y)^\perp$ satisfying that

$$H_g([e_i]) = \min \{H_g([v]) \mid 0 \neq v \in \text{Spn}\langle \{e_k\}_{k=i}^m \rangle \}.$$ 

for any $i = 1, 2, \ldots, m$.

Then, for this orthonormal basis, we can prove that

$$R_g(e_i, \bar{e}_i, e_j, \bar{e}_j) \geq 0 \text{ for any } 1 \leq i, j \leq m.$$ 

Indeed, we may assume that $i \leq j$ by the symmetry $R_g(e_i, \bar{e}_i, e_j, \bar{e}_j) = R_g(e_j, \bar{e}_j, e_i, \bar{e}_i)$. Further, for $i \leq j$, the tangent vector $e_i$ is the minimizer of $H_g$ on the subspace $\text{Spn}\langle \{e_k\}_{k=i}^m \rangle$ which contains $e_j$. Therefore it follows that $R_g(e_i, \bar{e}_i, e_j, \bar{e}_j)$ is non-negative from Lemma 2.1.

By applying the formulas (2.1) and (2.3) to the exact sequence

$$0 \longrightarrow \text{Ker} \, d\phi_* \longrightarrow (\Lambda^m T_X, \Lambda^m g) \xrightarrow{d\phi_*} (\phi^* K_Y^\vee, G) \longrightarrow 0$$

on a neighborhood of $p$, we obtain that

$$\sum_{k=1}^m R_g(v, \bar{v}, e_k, \bar{e}_k) = \langle \sqrt{-1} \Theta_{\Lambda^m g}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \rangle_{\Lambda^m g}$$

$$\leq \langle \sqrt{-1} \Theta_G(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), e_1 \wedge e_2 \wedge \cdots \wedge e_m \rangle_G$$

$$= \sqrt{-1} \Theta_G(v, \bar{v}) |e_1 \wedge e_2 \wedge \cdots \wedge e_m|^2_G$$

$$= \sqrt{-1} \Theta_G(v, \bar{v})$$

for any tangent vector $v \in T_X$. Note that $G$ (defined by the dual metric of $H$) is equal to the quotient metric induced by $\Lambda^m g$ since $p$ is a smooth point of $\phi$. When we consider the above formula in the case of $v = e_i$, we can see that the left hand side is non-negative by the second statement in Claim 3.4. Therefore we can conclude that $\sqrt{-1} \Theta_G(e_i, \bar{e}_i)$ is non-negative for any $i = 1, 2, \ldots, m$. \hfill \Box

Remark 3.5. At the end of the proof, we can conclude that all the tangent vectors in the horizontal direction are actually truly flat and that the curvature $\sqrt{-1} \Theta_G$ is flat, but a further argument is needed for these conclusions.

**Step 3** (Positivity of scalar curvature and its integration). In this step, we will consider the scalar curvature of $g$ and its integration, based on the idea in [HW15]. Let $\omega$ be the Kähler form associated to the Kähler metric $g$. The first Chern class of $\pi^* \phi^* K_Y^\vee$ can be represented by the curvature current $\pi^* \sqrt{-1} \Theta_G/2\pi$. Hence, by taking the wedge
product of the equality in Claim 3.2 with \( \pi^* \omega^{n-1} \) and by considering the integration over \( \bar{X} \), we obtain\\n\\n\[
(3.2) \quad 2\pi \int_{\bar{X}} c_1(\pi^* K_Y) \wedge \pi^* \omega^{n-1} = \int_{\bar{X}} \gamma \wedge \pi^* \omega^{n-1} + \int_E \pi^* \omega^{n-1}.
\]

The purpose of this step is to prove that the first term of the right hand side is non-negative. If it is proven, all the terms can be shown to be zero. Indeed, the left hand side is non-positive since \( K_Y \) is pseudo-effective by the assumption and the second term of the right hand side is non-negative. We will show that this observation leads to a certain flatness of \( K_Y \) and the smoothness of \( \phi \) in Step 4.

We first decompose the first term into the vertical part and the horizontal part. The integration of \( \gamma \wedge \pi^* \omega^{n-1} \) on \( \bar{X} \) is equal to the integration on a Zariski open set since \( \gamma \) and \( \pi^* \omega \) are smooth differential forms. Further \( \bar{X} \setminus \pi^{-1}(Z) \) is isomorphic to \( X \setminus Z \) by the morphism \( \pi \) and the equality \( \gamma = \pi^* \sqrt{-1} \Theta_G \) holds on the Zariski open set \( \bar{X} \setminus \pi^{-1}(Z) \) (cf. Remark 3.3). Therefore we can obtain that\\n\\n\[
\int_{\bar{X}} \gamma \wedge \pi^* \omega^{n-1} = \int_{\bar{X} \setminus \pi^{-1}(Z)} \gamma \wedge \pi^* \omega^{n-1} = \int_{\bar{X} \setminus \pi^{-1}(Z)} \pi^* (\sqrt{-1} \Theta_G \wedge \omega^{n-1}) = \int_{X \setminus Z} \sqrt{-1} \Theta_G \wedge \omega^{n-1} = \int_{X_0} \sqrt{-1} \Theta_G \wedge \omega^{n-1}.
\]

Here \( X_0 \) is the Zariski open set defined by \( X_0 := \phi^{-1}(Y_0) \) and \( Y_0 \) is the maximal Zariski open set of \( Y \) such that the restriction \( \phi : X_0 = \phi^{-1}(Y_0) \to Y_0 \) is a smooth morphism over \( Y_0 \).

On the other hand, for a given point \( p \in X_0 \), we take an orthonormal basis \( \{e_k\}_{k=1}^n \) of \( T_p X \) satisfying the properties in Claim 3.4. Then we have \( \omega = (\sqrt{-1}/2) \sum_{k=1}^n e_k^\vee \wedge \bar{e}_k^\vee \) at \( p \), and thus we obtain\\n\\n\[
(3.3) \quad \frac{n}{2} \int_{X_0} \sqrt{-1} \Theta_G \wedge \omega^{n-1} = \int_{X_0} \sum_{i=1}^m \sqrt{-1} \Theta_G(e_i, \bar{e}_i) \omega^n + \int_{X_0} \sum_{j=m+1}^n \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \omega^n
\]

from straightforward computations of the scalar curvature. The integrand of the first term (which measures positivity of the scalar curvature in the horizontal direction) is non-negative by Claim 3.4. We will show that the second term (that is, the vertical
part) is also non-negative by using Stokes’s theorem and Fubini’s theorem (see Claim 3.6). Note that the integrand of the second term can be shown to be non-negative later (cf. Remark 3.5). However it seems to be quite difficult to directly check this fact. For this reason, we need to handle the integration instead of the integrand.

Claim 3.6. Under the above situation, the second term is non-negative, namely, we have

\[
\int_{X_0} \sum_{j=m+1}^n \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \omega^n \geq 0.
\]

Proof of Claim 3.6. Let \( \omega_Y \) be a Kähler form on \( Y \). Then, for a given local coordinate \((t_1, t_2, \ldots, t_m)\) of \( Y_0 \), there exists a smooth positive function \( f \) defined on an open set in \( Y_0 \) such that

\[
\omega^n = \frac{1}{\phi^* f \cdot |\phi^* dt|_{\Lambda^m h}} \phi^* \omega_Y^m \wedge \omega^{n-m},
\]

where \( dt := dt_1 \wedge dt_2 \wedge \cdots \wedge dt_m \). We remark that \( \phi^* f \) and \( |\phi^* dt|_{\Lambda^m h} \) depend on the choice of local coordinates, but the product is independent of the coordinates and it is globally defined on \( X_0 \). Indeed, it can be seen that

\[
\langle \phi^* dt_\ell, e_j' \rangle_h = \langle \phi^* dt_\ell, e_j \rangle_{\text{pairing}} = \langle dt_\ell, \phi_* e_j \rangle_{\text{pairing}} = 0
\]

for any \( j = m + 1, \ldots, n \) since \( e_j \) is in the kernel of \( d\phi_* \). Therefore we obtain

\[
\phi^* dt_\ell = \sum_{k=1}^n \langle \phi^* dt_\ell, e_k' \rangle_h e_k' = \sum_{i=1}^m \langle \phi^* dt_\ell, e_i' \rangle_h e_i'.
\]

Further we obtain

\[
\phi^* dt = \det[\langle \phi^* dt_\ell, e_i' \rangle_h] e_1' \wedge e_2' \wedge \cdots \wedge e_m' \quad \text{and} \quad |\phi^* dt|_{\Lambda^m h}^2 = |\det[\langle \phi^* dt_\ell, e_i' \rangle_h]|^2
\]

by straightforward computations. On the other hand, the Kähler form \( \omega_Y \) can be locally written as

\[
\omega_Y = \sqrt{-1} \sum_{i,j=1}^m f_{ij} dt_i \wedge d\bar{t}_j
\]

in terms of the given local coordinate \((t_1, t_2, \ldots, t_m)\). From this local expression, we can easily show that

\[
\phi^* \omega_Y^m \wedge \omega^{n-m} = c_{n,m} \phi^* (\det[f_{ij}]) \phi^* (dt \wedge d\bar{t}) \wedge \omega^{n-m} = d_{n,m} \phi^* (\det[f_{ij}]) |\det[\langle \phi^* dt_\ell, e_i' \rangle_h]|^2 \omega^n,
\]

where \( c_{n,m} \) and \( d_{n,m} \) are the universal constants depending only on \( n \) and \( m \). Therefore it can be seen that \( f := d_{n,m} \det[f_{ij}] \) satisfies the desired equality.
By Fubini’s theorem, we have
\[
\int_{X_0} \sum_{j=m+1}^{n} \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \omega^n = \int_{Y_0} \frac{1}{f} \omega_{y}^m \int_{X_y} \frac{1}{\phi^* dt|_{\Lambda_{m}^{m_h}}} \sum_{j=m+1}^{n} \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \omega^{n-m}
\]
\[
= \frac{n-m}{2} \int_{Y_0} \frac{1}{f} \omega_{y}^m \int_{X_y} \frac{1}{\phi^* dt|_{\Lambda_{m}^{m_h}}} \sqrt{-1} \Theta_G \wedge \omega^{n-m-1},
\]
where \(X_y\) is the fiber of \(\phi\) at \(y \in Y_0\). Here we used the equality
\[
\sum_{j=m+1}^{n} \sqrt{-1} \Theta_G(e_j, \bar{e}_j) \omega^{n-m} = \frac{n-m}{2} \sqrt{-1} \Theta_G \wedge \omega^{n-m-1}
\]
of the scalar curvature on the fiber \(X_y\). We finally prove that the fiber integral in the above equality is non-negative. For simplicity, we put \(F := |\phi^* dt|_{\Lambda_{m}^{m_h}}\). Then, by the definition of the curvature \(\sqrt{-1} \Theta_G\), we can show that
\[
\int_{X_y} \frac{1}{F} \sqrt{-1} \partial \bar{\partial} \log F \wedge \omega^{n-m-1}
\]
\[
= \int_{X_y} \frac{1}{F} \sqrt{-1} \partial \bar{\partial} \log F \wedge \omega^{n-m-1}
\]
\[
= \sqrt{-1} \int_{X_y} \partial \left( \frac{1}{F} \bar{\partial} \log F \wedge \omega^{n-m-1} \right) - \sqrt{-1} \int_{X_y} \bar{\partial} \left( \frac{1}{F} \right) \wedge \partial \log F \wedge \omega^{n-m-1}
\]
\[
= \int_{X_y} \frac{1}{F^2} \sqrt{-1} \partial F \wedge \bar{\partial} F \wedge \omega^{n-m-1}.
\]
The last equality follows from Stokes’s theorem. The integrand of the right hand side is non-negative, and thus the desired inequality can be obtained. \(\square\)

**Step 4** (Curvature of the canonical bundle \(K_Y\)). In this step, from the assumption that \(K_Y\) is pseudo-effective, we will show that the curvature \(\sqrt{-1} \Theta_G\) is flat and \(\phi\) is a smooth morphism. The key point here is the observation on the flatness of curvature in the horizontal direction obtained from Claim 3.4 and Claim 3.6.

**Claim 3.7.** The following statements hold:

- The canonical bundle \(K_Y\) is numerically zero (that is, \(c_1(K_Y) = 0\)).
- \(H_g([e_i]) = R_g(e_i, \bar{e}_i, e_i, \bar{e}_i) = 0\) for any \(i = 1, 2, \ldots, m\).
- \(R_g(v, \bar{v}, e_i, \bar{e}_i) \geq 0\) for any tangent vector \(v \in T_X\) and any \(i = 1, 2, \ldots, m\).
- The curvature \(\sqrt{-1} \Theta_G\) is flat on \(X\). In particular, the effective divisor \(E\) is actually the zero divisor, and thus the morphism \(\phi\) is smooth.
Proof of Claim 3.7. The left hand side of the equality (3.2) is non-positive since $\pi^*\phi^*K_Y$ is pseudo-effective by the assumption, and further each term of the right hand side is non-negative by Claim 3.6. Hence we obtain

$$\int_X c_1(\phi^*K_Y) \wedge \omega^{n-1} = \int_X c_1(\pi^*\phi^*K_Y) \wedge \pi^*\omega^{n-1} = 0.$$ 

In general, if a pseudo-effective line bundle $L$ satisfies $c_1(L) \cdot \{\omega^{n-1}\} = 0$ for some Kähler form $\omega$, then $L$ is numerically zero (for example see [Mat13]). Indeed, for an arbitrary $d$-closed $(n-1, n-1)$ form $\eta$, we can take a positive constant $C$ such that

$$\frac{1}{C} \omega^{n-1} \leq \eta \leq C \omega^{n-1}.$$ 

Then we obtain $c_1(L) \cdot \{\eta\} = 0$ by the assumption $c_1(L) \cdot \{\omega^{n-1}\} = 0$. This leads to $c_1(L) = 0$ by the duality. Therefore it can be seen that $\phi^*K_Y$ is numerically zero.

On the other hand, by the equalities (3.2) and (3.3), we have

$$\int_{X_0} \sum_{i=1}^m \sqrt{-1} \Omega_G(e_i, \bar{e}_i) \omega^n = 0.$$ 

It follows that $\sqrt{-1} \Omega_G(e_i, \bar{e}_i) = 0$ for any $i = 1, 2, \ldots, m$ at a point $p \in X_0$ since the integrand $\sqrt{-1} \Omega_G(e_i, \bar{e}_i)$ is non-negative by Claim 3.4. By applying the formula (3.1) to the case of $v = e_i$, we obtain

$$0 \leq \sum_{k=1}^m R_g(e_i, \bar{e}_i, e_k, \bar{e}_k) \leq \sqrt{-1} \Omega_G(e_i, \bar{e}_i) = 0.$$ 

The left inequality follows from Claim 3.4. In particular, we can see that

$$H_g([e_i]) = R_g(e_i, \bar{e}_i, e_i, \bar{e}_i) = 0 \text{ for any } i = 1, 2, \ldots, m.$$ 

This implies that $e_i$ is the minimizer of the semi-positive holomorphic sectional curvature $H_g$ on $T_X$, and thus it can be shown that $R_g(v, \bar{v}, e_i, \bar{e}_i)$ is non-negative for any tangent vector $v \in T_X$ by Lemma 2.1. By applying the formula (3.1) to an arbitrary tangent vector $v \in T_X$ again, we obtain

$$0 \leq \sum_{i=1}^m R_g(v, \bar{v}, e_i, \bar{e}_i) \leq \sqrt{-1} \Omega_G(v, \bar{v}).$$ 

This means that the curvature $\sqrt{-1} \Omega_G$ is semi-positive on $X_0$. We can see that $\gamma \geq 0$ holds on the Zariski open set $\pi^{-1}(X_0)$, since $\gamma = \pi^*\sqrt{-1} \Omega_G$ holds on $\tilde{X} \setminus \pi^{-1}(Z)$ and we have $X_0 \subset X \setminus Z$. Hence it follows that $\gamma \geq 0$ on the ambient space $\tilde{X}$ since $\gamma$ is a smooth form. By the above arguments, the first Chern class $c_1(\pi^*\phi^*K_Y^\vee)$ (which is numerically zero) is represented by the sum of the semi-positive form $\gamma$ and the positive
Therefore we can conclude that $\gamma = 0$ and $E = 0$ (namely, $\sqrt{-1}\Theta_G = 0$). In particular, we can see that $G$ is a smooth metric (that is, $\phi$ is a smooth morphism). □

**Step 5 (Truly flatness in the horizontal direction and its applications).** In this step, we first show that the statement (2) in Theorem 1.4 holds and all the tangent vectors in the horizontal direction are truly flat. We will prove the statement (3) as an application of the truly flatness. Further we finally obtain the statement (4) from the theory of foliations.

Now we have the exact sequence of vector bundles

\[
0 \longrightarrow (T_{X/Y}, g_S) \longrightarrow (T_X, g) \xrightarrow{d\phi} (\phi^*T_Y, g_Q) \longrightarrow 0
\]

on the ambient space $X$ since $\phi$ is a smooth morphism by Claim 3.7. Let $g_Q$ (resp. $g_S$) be the induced hermitian metric on $\phi^*T_Y$ (resp. $T_{X/Y}$). Then we prove the following claim:

**Claim 3.8.** The following statements hold:

- $e_i$ is a truly flat tangent vector for any $i = 1, 2, \ldots, m$.
- The exact sequence (3.4) splits, and its splitting $T_X = T_{X/Y} \oplus \phi^*T_Y$ coincides with the orthogonal decomposition of $(T_X, g)$.
- There exists a hermitian metric $g_Y$ on $T_Y$ such that $g_Q = \phi^*g_Y$ and $H_{g_Y} \equiv 0$. In particular, the image $Y$ admits a finite étale cover $T \rightarrow Y$ by a complex torus $T$.

**Proof of Claim 3.8.** For any $i = 1, 2, \ldots, m$, the tangent vector $e_i$ is the minimizer of the holomorphic sectional curvature $H_g$, and further $\langle \sqrt{-1}\Theta_g(v, \bar{v})(e_i), e_i \rangle_g$ is non-negative for any tangent vector $v \in T_X$ by Claim 3.7. By the formula (2.2), we obtain that

\[
0 \leq \langle \sqrt{-1}\Theta_g(v, \bar{v})(e_i), e_i \rangle_g + \langle B_{v}(e_i), B_{\bar{v}}(e_i) \rangle_{g_S} = \langle \sqrt{-1}\Theta_{g_Q}(v, \bar{v})(e_i), e_i \rangle_{g_Q}
\]

for a tangent vector $v \in T_X$.

On the other hand, the induced metric $\det g_Q$ on $\phi^*K_Y^\vee = \det \phi^*T_Y$ coincides with the metric $G$ constructed in Step 1 and the curvature of $\det g_Q = G$ is flat by Claim 3.7. Therefore we obtain that

\[
\sum_{i=1}^{m} \langle \sqrt{-1}\Theta_{g_Q}(v, \bar{v})(e_i), e_i \rangle_{g_Q} = \langle \sqrt{-1}\Theta_{\det g_Q}(v, \bar{v})(e_1 \wedge e_2 \wedge \cdots \wedge e_m), (e_1 \wedge e_2 \wedge \cdots \wedge e_m) \rangle_{\det g_Q} = 0
\]

by the equality (2.1). By combining with the inequality (3.5), we can obtain that

$$R_g(v, \bar{v}, e_i, e_i) = \langle \sqrt{-1}\Theta_g(v, \bar{v})(e_i), e_i \rangle_g = 0 \quad \text{and} \quad \langle B_{v}(e_i), B_{\bar{v}}(e_i) \rangle_{g_S} = 0$$
for any tangent vector $v \in T_X$ and $i = 1, 2, \ldots, m$. Here we used the fact that $\langle B_v(e_i), B_v(e_i) \rangle_{g_S}$ is non-negative. Then, by Lemma 2.3, we can see that $e_i$ is truly flat since $e_i$ satisfies that $R_g(v, \bar{v}, e_i, \bar{e}_i) = 0$ and $H_g(e_i) = 0$. Further it follows that $B \in C^\infty(X, \Lambda^{0,1} \otimes \text{Hom}(S^\perp, S))$ is identically zero since $\langle B_v(\cdot), B_v(\cdot) \rangle_{g_S}$ is a semi-positive definite quadratic form on $S^\perp$ and its trace $\sum_{i=1}^m \langle B_v(e_i), B_v(e_i) \rangle_{g_S}$ is zero by the above argument. Hence we obtain the holomorphic orthogonal decomposition $T_X = T_{X/Y} \oplus \phi^*T_Y$ (see subsection 2.1).

Now we prove the last statement. For a (local) vector field $v$ of $T_Y$ defined on an open set $U$ in $Y$, we consider the section $\phi^*v \in H^0(\phi^{-1}(U), \phi^*T_Y)$ defined by

$$\phi^{-1}(U) \ni p \mapsto v(\phi(p)) \in T_{Y, \phi(p)} = (\phi^*T_Y)_p,$$

which we will denote by the notation $\phi^*v$. If the function $|\phi^*v|_{g_Q}$ is a constant on the fiber $X_y$, we can define the hermitian metric $g_Y$ of $Y$ by $|v|_{g_Y} := |\phi^*v|_{g_Q}$. Then we have $g = \phi^*g_Y$ by the definition.

If we can show that the restriction of $\sqrt{-1}\partial\bar{\partial} \log |\phi^*v|_{g_Q}$ to the fiber is a semi-positive $(1,1)$-form, it should be a constant by the maximal principle, since $|\phi^*v|_{g_Q}$ is a psh function globally defined on the compact fiber. For this purpose, we consider the subline bundle $L$ of $\phi^*T_Y$

$$(L := \text{Spn}\langle \phi^*v \rangle, g_L) \subset (\phi^*T_Y, g_Q)$$

spanned by $\phi^*v$. Let $g_L$ be the induced metric on $L$. By the definition of the curvature and the induced metric, we obtain

$$\sqrt{-1}\Theta_{gL} := \sqrt{-1}\Theta_{gL}(L) = -\sqrt{-1}\partial\bar{\partial} \log |\phi^*v|_{gL}^2 = -\sqrt{-1}\partial\bar{\partial} \log |\phi^*v|_{g_Q}^2.$$

By applying the formula (2.4) to the above injective bundle morphism, we obtain that

$$\sqrt{-1}\Theta_{gL}(w, \bar{w})|\phi^*v|_{gL}^2 = \langle \sqrt{-1}\Theta_{gL}(w, \bar{w})(\phi^*v), \phi^*v \rangle_{gL} \leq \langle \sqrt{-1}\Theta_{gQ}(w, \bar{w})(\phi^*v), \phi^*v \rangle_{gQ}$$

for a tangent vector $w \in T_X$. We have already shown that the tangent vectors $\{e_i\}_{i=1}^m$ are truly flat by the above argument. The vector $\phi^*v$ can be written as a linear combination of $\{e_i\}_{i=1}^m$, and thus it is also truly flat. On the other hand, by the holomorphic orthogonal decomposition $T_X = T_{X/Y} \oplus \phi^*T_Y$, the section $\phi^*v$ of $\phi^*T_Y$ determines the section of $T_X$, which we denote by the same notation $\phi^*v$. Then we obtain

$$\langle \sqrt{-1}\Theta_{gQ}(w, \bar{w})(\phi^*v), \phi^*v \rangle_{gQ} = \langle \sqrt{-1}\Theta_{g}(w, \bar{w})(\phi^*v), \phi^*v \rangle_{g} = 0.$$

The right equality follows from the truly flatness of $\phi^*v$. Therefore we can see that $\sqrt{-1}\Theta_{gL}$ is semi-negative (in particular $|\phi^*v|_{g_Q}$ is a constant).

We finally check that the holomorphic sectional curvature of $g_Y$ is identically zero. Note that, in general, a compact Kähler manifold is a complex torus up to finite étale covers when the holomorphic sectional curvature is identically zero (see [Igu54],...
[HLW16, Proposition 2.2], [Ber66], [Igu54]). For a given tangent vector $v \in T_Y$, the
tensor $\varphi^*v$ satisfies $d\varphi^*(\varphi^*v) = v$. Hence we can easily see that
\[
0 = \langle \sqrt{-1}\Theta_g(\varphi^*v, \bar{\varphi}^*v), \varphi^*v \rangle_g
= \langle \sqrt{-1}\Theta_g^Q(\varphi^*v, \bar{\varphi}^*v), \varphi^*v \rangle_{g^Q}
= \langle \sqrt{-1}\Theta_{g^Y}(v, \bar{v})(v), v \rangle_{g^Y}
\]
by $d\varphi^*(\varphi^*v) = v$, $g_Q = \varphi^*g_Y$, and the truly flatness of $\varphi^*v$.

We check the statement (4) in Theorem 1.4. When $X$ is projective, it can be shown
that the morphism $\phi : X \to Y$ is a (holomorphic) fiber bundle (in particular, all the
fibers are isomorphic) by the classical Ehresmann theorem and [Hör07, Lemma 3.19].

We finish the proof of Theorem 1.4.

\textbf{Remark 3.9.} If the foliation $\varphi^*T_Y \subset T_X$ obtained from Theorem 1.4 is integrable (that
is, it is closed under the Lie bracket), then it can be shown that $\phi$ is locally trivial and
we have the decomposition $X_{\text{univ}} \cong Y_{\text{univ}} \times F_{\text{univ}} = \mathbb{C}^m \times F_{\text{univ}}$
by the Ehresmann theorem (for example see [Hör07, Theorem 3.17]). The integrability
of $\varphi^*T_Y \subset T_X$ is satisfied when the dimension of $Y$ is one. In this case, the image $Y$ is
automatically an elliptic curve by the statement (3) in Theorem 1.4. See [Hör07] and
references therein for more details.

\subsection{Proof of Theorem 1.7.}
In this subsection, we will prove Theorem 1.7 by modifying the arguments in the proof of Theorem 1.4 for almost holomorphic maps.

\textbf{Theorem 3.10} (=Theorem 1.7). \textit{Let $(X, g)$ be a compact Kähler manifold with semi-positive holomorphic sectional curvature and $\phi : X \to Y$ be an almost holomorphic map to a compact Kähler manifold $Y$ with pseudo-effective canonical bundle. Let $X_1$ and $Y_1$ be Zariski open sets such that $\phi : X_1 := \phi^{-1}(Y_1) \to Y_1$ is a morphism. Then we have the followings:}

\begin{enumerate}
\item[(0)] The numerical dimension $\nu(Y)$ of $Y$ is equal to zero.
\item[(1)] $\phi$ is a smooth morphism on $X_1$.
\item[(2)] The standard exact sequence of vector bundles on $X_1$
\[
0 \longrightarrow T_{X_1/Y_1} := \text{Ker } d\phi_* \longrightarrow T_{X_1} \xrightarrow{d\phi_*} \varphi^*T_{Y_1} \longrightarrow 0
\]
gives the holomorphic orthogonal decomposition
\[
T_{X_1} = T_{X_1/Y_1} \oplus \varphi^*T_{Y_1}.
\]
Moreover, all the tangent vectors in $\varphi^*T_{Y_1} \subset T_{X_1}$ are truly flat.
\end{enumerate}
(3) Let $g_Q$ be the hermitian metric on $\phi^*T_{Y_1}$ induced by the above exact sequence and the given metric $g$. Then there exists a hermitian metric $g_Y$ on $T_{Y_1}$ with the following properties:
- $g_Q$ is obtained from the pull-back of $g_Y$ (namely, $g_Q = \phi^*g_Y$).
- The holomorphic sectional curvature of $(Y_1, g_Y)$ is identically zero.

Proof of Theorem 1.7. The strategy of the proof is essentially the same as that of Theorem 1.4. We will only explain how to revise the proof of Theorem 1.4 to avoid repeating the same arguments. We use the same notations as in the proof of Theorem 1.4.

For an almost holomorphic map $\phi : X \to Y$, we take a resolution $\tau : \Gamma \to X$ of the indeterminacy locus of $\phi$. We denote, by the notation $\bar{\phi} : \Gamma \to Y$, the morphism with the following commutative diagram:

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{\bar{\phi}} & Y \\
\downarrow{\tau} & & \downarrow{\phi} \\
X & \xrightarrow{-} & Y
\end{array}
\]

Then we have the injective sheaf morphism

\[
\bar{\phi}^*K_Y \xrightarrow{d\bar{\phi}^*} \Lambda^m\Omega_{\Gamma}.
\]

By taking the push-forward by the modification $\tau$, we obtain the injective sheaf morphism

\[
(L := \tau_*\bar{\phi}^*K_Y, H) \xrightarrow{f} (\Lambda^m\Omega_X, \Lambda^m h).
\]

Here we used the formula $\tau_*\Lambda^m\Omega_{\Gamma} = \Lambda^m\Omega_X$. For simplicity, we denote the line bundle $\tau_*\bar{\phi}^*K_Y$ by the notation $L$ and the above sheaf morphism by the notation $f$. We can take a (non-empty) Zariski open set $Y_1$ such that the restriction $\phi : X_1 := \phi^{-1}(Y_1) \to Y_1$ is a morphism without the indeterminacy locus since $\phi$ is an almost holomorphic map. Also, we take the maximal Zariski open set $Y_0 \subset Y_1$ such that $\phi : X_0 := \phi^{-1}(Y_0) \to Y_0$ is smooth. One of our purposes is to prove that $Y_0 = Y_1$.

From now on, we will check that the same arguments as in each step in the proof of Theorem 1.4 work by replacing $\phi^*K_Y$ with $L$ and $\phi : X \to Y$ with $\phi : X_1 \to Y_1$.

By the same way as in Step 1, we can construct a singular hermitian metric $H$ on $L$ and its dual metric $G$ on $L'$. The line bundle $L = \tau_*\bar{\phi}^*K_Y$ coincides with the usual pull-back $\phi^*K_Y$ on $X_1$ (that is, $L$ can be seen as the extension of the pull-back $\phi^*K_Y$ defined on $X_1$ to $X$). Let $\mathcal{I}$ be the degenerate ideal of $f$ and let $Z'$ be the support of the cokernel $\mathcal{O}_X/\mathcal{I}$. We do not know whether $Z'$ coincides with the non-smooth locus $Z$ of $\phi$, but we have $Z' \cap X_1 = Z \cap X_1$ since $f$ is just the morphism $d\phi^*$ of the pull-back on $X_1$. For a resolution $\pi : X \to X$ of the degenerate ideal $\mathcal{I}$, we can easily check the
same statements as in Claim 3.2 by replacing $Z$ with $Z'$. In particular, we have

$$2\pi c_1(\pi^*L') \supseteq \pi^*\sqrt{-1}\Theta_G = \gamma + [E],$$

for some smooth $(1,1)$-form $\gamma$ and the integration current $[E]$ of an effective divisor $E$.

In Step 2, we only considered tangent vectors at a smooth point of $\phi$. Hence there is no difficulty to obtain Claim 3.4.

In Step 3, we essentially discussed local problems in $Y_1$. Therefore we can obtain the equality (3.2), the equality (3.3), and Claim 3.6 by replacing $\phi^*K_Y$ with $L$.

In Step 4, we used the global condition that $\phi^*K_Y$ is pseudo-effective. However we can see that the line bundle $L$ is pseudo-effective by the definition, and thus we can repeat the same argument as in Claim 3.7. As a result, we can conclude that $L$ is numerically zero and $\sqrt{-1}\Theta_G$ is flat on $X$. This implies that $\bar{\phi}^*K_Y$ is numerically equivalent to some exceptional divisor by the definition $L = \tau_*\phi^*K_Y$. Hence the numerical dimension of $K_Y$ is zero. Further the morphism $f$ is an injective bundle morphism since the effective divisor $E$ is the zero divisor by $\sqrt{-1}\Theta_G$ is flat. In particular, since the morphism $f$ is equal to the morphism $d\phi^*$ of the pull-back over $Y_1$, the morphism $\phi$ is smooth over $Y_1$ (namely, $Y_1 = Y_0$).

The rest arguments in Step 5 are local in $Y_1 = Y_0$, and thus we can easily check the same conclusions as in Theorem 1.4 over $Y_0$ by replacing $X$ and $Y$ with $X_0$ and $Y_0$. □

3.3. Proof of Theorem 1.2 and Corollary 1.6. In this subsection, we will prove the following theorem. Theorem 1.2 and Corollary 1.6 can be directly obtained from the following theorem.

**Theorem 3.11.** Let $(X, g)$ be a compact Kähler manifold with semi-positive holomorphic sectional curvature and $Y$ be a compact Kähler manifold with pseudo-effective canonical bundle. Let $\phi : X \dashrightarrow Y$ be an almost holomorphic map from $X$ to $Y$.

Then we have

$$\dim X - \dim Y \geq n_{tf}(X, g).$$

**Proof of Theorem 3.11.** For the proof, we will use the arguments in the proof of Theorem 1.4 and Theorem 1.7. For simplicity, we put $k := n_{tf}(X, g)$. We take a Zariski open set $Y_1$ in $Y$ such that $\phi : X_1 = \phi^{-1}(Y_1) \to Y_1$ is a morphism (over which $\phi$ is actually smooth by Theorem 1.7). The invariant $n_{tf}(X, g)_p$ is lower semi-continuous with respect to $p \in X$ in the classical topology (see Definition 2.2). In particular, the condition $n_{tf}(X, g)_p = k$ is an open condition. Hence we can find a point $p$ such that

$$n_{tf}(X, g)_p = k = \max_{p \in X} n_{tf}(X, g)_p \quad \text{and} \quad p \in X_1.$$  

It follows that tangent vectors $\{e_i\}_{i=1}^m$ in the horizontal direction are truly flat by (2) in Theorem 3.10 (see also Claim 3.8). In particular, the vector space $\phi^*T_Y = \text{Span}\{e_i\}_{i=1}^m$ at $p$ is contained in $V_{\text{flat}, p}$. Therefore we obtain the desired inequality $m \leq n - k$. □
In the rest of this subsection, we will check Theorem 1.2 and Corollary 1.6.

**Theorem 3.12 (=Theorem 1.2).** Let \((X, g)\) be a compact Kähler manifold such that \(X\) is projective and the holomorphic sectional curvature is semi-positive. Let \(\phi : X \rightarrow Y\) be a MRC fibration of \(X\). Then we have
\[
\dim X - \dim Y \geq \text{ntf}(X, g).
\]
In particular, the manifold \(X\) is rationally connected if \(\text{ntf}(X, g) = \dim X\) (which is satisfied if the holomorphic sectional curvature is quasi-positive).

**Proof.** We obtain the desired inequality by applying Theorem 3.11 to MRC fibrations. The latter conclusion is obvious. Indeed, when \(\text{ntf}(X, g) = \dim X\), the image \(Y\) should be one point. This implies that \(X\) is rationally connected. \(\square\)

**Corollary 3.13 (=Corollary 1.6).** Let \((X, g)\) be a compact Kähler manifold with semi-positive holomorphic sectional curvature. Then we have
\[
h^0(X, \Omega_X) \leq \dim X - \text{ntf}(X, g).
\]
In particular, we obtain \(h^0(X, \Omega_X) = 0\) if \(\text{ntf}(X, g) = \dim X\) (which is satisfied if the holomorphic sectional curvature is quasi-positive).

**Proof.** We consider the Albanese map \(\alpha : X \rightarrow \text{Alb}(X)\) of \(X\). Then the canonical bundle \(K_{\text{Alb}(X)}\) is trivial, and thus the assumptions in Theorem 3.11 are satisfied. We obtain the desired conclusion by \(\dim Y = \dim \text{Alb}(X) = h^0(X, \Omega_X)\). \(\square\)

### 3.4. Proof of Corollary 1.5

In this subsection, we will obtain Corollary 1.5 as an application of Theorem 1.4.

**Corollary 3.14 (=Corollary 1.5).** Let \(X\) be a compact Kähler manifold with semi-positive holomorphic sectional curvature. Then the followings hold:

- All the statements of Conjecture 1.3 hold in the case of \(X\) being a surface.
- The statement (1) of Conjecture 1.3 holds if \(X\) is projective and a MRC fibration of \(X\) can be chosen to be a morphism.

**Proof of Corollary 1.5.** We consider a compact Kähler manifold \(X\) with semi-positive sectional curvature. If the holomorphic sectional curvature is identically zero, then \(X\) itself admits a finite étale cover by a complex torus by [Igu54] (see also [HLW16, Proposition 2.2] and [Ber66]). Then there is nothing to prove.

From now on, we consider the case where \(H_g\) is semi-positive, but not identically zero. In this case, we can conclude that the canonical bundle \(K_X\) is not pseudo-effective. Indeed, it follows that the scalar curvature \(S\) of the Kähler metric \(g\) is positive since the scalar curvature \(S\) can be expressed as the integral of the holomorphic sectional curvature...
curvature on the projective space $\mathbb{P}(T^\vee_{X,p})$ (for example, see [Ber66]). Then we can see that the canonical bundle $K_X$ is not pseudo-effective by the formula

$$\int_X c_1(K_X) \wedge \omega^{n-1} = -\frac{1}{n\pi} \int_X S \omega^n < 0,$$

where $\omega$ is the Kähler form associated to $g$.

To check the first statement, we assume that $X$ is a compact Kähler surface. By the classification of compact complex surfaces, it can be seen that a Kähler surface such that $K_X$ is not pseudo-effective is a minimal rational surface or a ruled surface over a curve of genus $\geq 1$. It is sufficient to consider the case of $X$ being a ruled surface over a curve of genus $\geq 1$ since a minimal rational surface is rationally connected. In this case, we can conclude that the ruling $X \to B$ is minimal (that is, a submersion) and the base is elliptic curve, by applying Theorem 1.4. The direct summand $\phi^*T_B$ is integrable since the rank of $\phi^*T_B$ is one (see Remark 3.9). Hence the universal cover can be shown to be the product of $\mathbb{C} \times \mathbb{P}^1$ by the classical Ehresmann theorem.

To check the second statement, we consider a smooth projective variety whose holomorphic sectional curvature is semi-positive, but not identically zero. Then, since $K_Y$ not pseudo-effective by the first half argument, a MRC fibration $\phi : X \to Y$ is non-trivial. Then the image $Y$ is not uniruled by [GHS03, Theorem 1.1] and the canonical bundle $K_Y$ of $Y$ is pseudo-effective by [BDPP13]. Therefore we can directly apply Theorem 1.4 if a MRC fibration can be chosen to be a morphism. Then the statement (1) of Conjecture 1.3 is obvious.

In the rest of subsection, we give a remark on smooth projective varieties with nef anti-canonical bundle. Even if a compact Kähler manifold $X$ has the semi-positive holomorphic sectional curvature, the anti-canonical bundle $K_X^\vee$ is not necessarily nef (for example, see [Yan16, Example 3.6]). However it is worth to mention that we can confirm that Conjecture 1.3 holds when $X$ is projective and $X$ has the nef anti-canonical bundle, by using Theorem 1.4 and the deep structure theorem proved by Cao-Höring in [CH17].

**Corollary 3.15.** Let $(X, g)$ be a compact Kähler manifold such that $X$ is projective and the holomorphic sectional curvature is semi-positive. Further we assume that the anti-canonical bundle $K_X^\vee$ is nef. Then Conjecture 1.3 can be affirmatively solved.

**Proof.** By the structure theorem of [CH17], we can choose a MRC fibration $\phi : X \to Y$ to be a (locally trivial) morphism. Further we have the decomposition $X_{\text{univ}} \cong F \times Y_{\text{univ}}$, where $F$ is the rationally connected fiber of $\phi$. By applying Theorem 1.4 to this MRC fibration $\phi : X \to Y$, we can see that $Y$ admits a finite étale cover $T \to Y$ by an abelian variety $T$. This finishes the proof. \qed
4. OPEN PROBLEMS RELATED TO SEMI-POSITIVE HOLOMORPHIC SECTIONAL CURVATURE

In this section, we discuss open problems related to the geometry of semi-positive sectional curvature.

The first problem is concerned with Conjecture 1.3. If (1) and (2) in the following problem are affirmatively solved, then Conjecture 1.3 for smooth projective varieties can be obtained from Theorem 1.4.

**Problem 4.1.** Let \((X, g)\) be a compact Kähler manifold such that \(X\) is projective and the holomorphic sectional curvature \(H_g\) is semi-positive. Let \(\phi : X \to Y\) be a MRC fibration of \(X\).

1. Can we choose a MRC fibration of \(X\) to be a morphism?
2. Does \(\phi^*T_Y\) determine an integrable foliation?
3. Does the fiber \(F\) admit a Kähler metric \(g_F\) such that \(n_{tf}(F, g_F) = \dim F\)?
4. Does the equality \(\dim X = \dim Y + n_{tf}(X, g)\) hold?

The following problem seems to give a strategy to study Conjecture 1.3 for compact Kähler manifolds. If the following problem can be solved, we can apply Theorem 1.4 and Conjecture 1.3 to MRC fibrations of the fiber \(Z\).

**Problem 4.2.** Let \((X, g)\) be a compact Kähler manifold with the semi-positive holomorphic sectional curvature \(H_g\). After we take a suitable finite étale cover \(X^* \to X\), we consider the Albanese map \(\alpha : X^* \to \text{Alb}(X^*)\).

1. Is the fiber \(Z\) projective?
2. Is the holomorphic sectional curvature \(H_{g_Z}\) of the induced metric \(g_Z\) semi-positive?

When \(X\) admits a Kähler metric with quasi-positive holomorphic sectional curvature, it seems to be natural to expect that \(X\) is automatically projective (cf. [Yan18a, Theorem 1.7]). The following problem, which was posed by Yang in a private discussion, gives a generalization of this expectation. Also, it is an interesting problem to consider rationally connectedness or holomorphic sectional curvature from the viewpoint of (uniform) RC positivity introduced by Yang. See [Yan18b] for vanishing theorems and [Yan18c, Theorem 1.7, Conjecture 1.9] for rationally connectedness.

**Problem 4.3.** Let \((X, g)\) be a compact Kähler manifold (or more generally a hermitian manifold) with the semi-positive holomorphic sectional curvature \(H_g\). Assume that \(X\) has no truly flat vector at some point of \(X\) (or \(H_g\) is quasi-positive).

1. Can we obtain \(h^i(X, \mathcal{O}_X) = 0\) for any \(i > 0\)?
2. Is \(X\) automatically projective and rationally connected?
References

[BDPP13] S. Boucksom, J.-P. Demailly, M. Păun, T. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, J. Algebraic Geom. **22** (2013), no. 2, 201–248.

[Ber66] M. Berger, *Sur les variétés d’Einstein compactes*, Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d’Expression Latine (Namur, 1965) pp. 35–55 Librairie Universitaire, Louvain (1966).

[BKT13] Y. Brunebarbe, B. Klingler, B. Totaro, *Symmetric differentials and the fundamental group*, Duke Math. J. **162** (2013), no. 14, 2797–2813.

[Bre] S. Brendle, *Ricci flow and the sphere theorem*, Graduate Studies in Mathematics, **111**, American Mathematical Society, Providence, RI, 2010. viii+176 pp. ISBN: 978-0-8218-4938-5.

[Cam92] F. Campana, *Connexité rationnelle des variétés de Fano*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 539–545.

[Cam16] J. Cao, *Albanese maps of projective manifolds with nef anticanonical bundles*, to appear in Annales Scientifiques de l’École Normale Supérieure, arXiv:1612.05921v3.

[CH17] J. Cao, A. Höring, *A decomposition theorem for projective manifolds with nef anticanonical bundle*, to appear in Journal of Algebraic Geometry, arXiv:1706.08814v1.

[CG71] J. Cheeger, D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), 119–128.

[CG72] J. Cheeger, D. Gromoll, *On the structure of complete manifolds of nonnegative curvature*, Ann. of Math., **96** (1972), 413–443.

[CP91] F. Campana, T. Peternell, *Projective manifolds whose tangent bundles are numerically effective*, Math. Ann., **289** (1991), 169–187.

[Dem] J.-P. Demailly, *Analytic methods in algebraic geometry*, Surveys of Modern Mathematics, **1**, International Press, Somerville, Higher Education Press, Beijing, (2012).

[DPS94] J.-P. Demailly, T. Peternell, M. Schneider, *Compact complex manifolds with numerically effective tangent bundles*, J. Algebraic Geom., **3**, (1994), no.2, 295–345.

[GHS03] T. Graber, J. Harris, J. Starr, *Families of rationally connected varieties*, J. Amer. Math. Soc. **16** (2003), no. 1, 57–67.

[HLW16] G. Heier, S. S. Y. Lu, B. Wong, *Kähler manifolds of semi-negative holomorphic sectional curvature*, J. Differential Geom. **104** (2016), no. 3, 419–441.

[HLWZ17] G. Heier, S. S. Y. Lu, B. Wong, F. Zheng, *Reduction of manifolds with semi-negative holomorphic sectional curvature*, Preprint, arXiv:1705.00605v1.

[Hör07] A. Höring, *Uniruled varieties with split tangent bundle*, Math. Z., **256** (2007), no.3, 465–479.

[HSW81] A. Howard, B. Smyth, H. Wu, *On compact Kähler manifolds of nonnegative bisectional curvature I and II*, Acta Math. **147** (1981), no. 1-2, 51–70

[HW15] G. Heier, B. Wong, *On projective Kähler manifolds of partially positive curvature and rational connectedness*, Preprint, arXiv:1509.02149v1.

[Igus4] J. Igusa, *On the structure of a certain class of Kaehler varieties*, Amer. J. Math. **76**, (1954), 669–678.

[Mok88] N. Mok, *The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature*, J. Differential Geom. **27** (1988), no. 2, 179–214.

[KoMM92] J. Kollár, Y. Miyaoka, S. Mori, *Rationally connected varieties*, J. Algebraic Geom. **1** (1992), no. 3, 429–448.
[Mat13] S. Matsumura, *Asymptotic cohomology vanishing and a converse to the Andreotti-Grauert theorem on surfaces*, Ann. Inst. Fourier (Grenoble) 63 (2013), no. 6, 2199–2221.

[Mat18] S. Matsumura, *On the image of MRC fibrations of projective manifolds with semi-positive holomorphic sectional curvature*, Preprint, arXiv:1801.09081v1.

[Mor79] S. Mori, *Projective manifolds with ample tangent bundles*, Ann. of Math. (2) 110 (1979), no. 3, 593–606.

[SY80] Y.-T. Siu, S.-T. Yau, *Compact Kähler manifolds of positive bisectional curvature*, Invent. Math. 59 (1980), no. 2, 189–204.

[Yan16] X. Yang, *Hermitian manifolds with semi-positive holomorphic sectional curvature*, Math. Res. Lett. 23 (2016), no. 3, 939–952.

[Yan17] X. Yang, *Big vector bundles and complex manifolds with semi-positive tangent bundles*, Math. Ann. 367 (2017), no. 1-2, 251–282.

[Yan18a] X. Yang, *RC-positivity, rational connectedness and Yau’s conjecture*, Camb. J. Math. 6 (2018), 183–212.

[Yan18b] X. Yang, *RC-positivity, vanishing theorems and rigidity of holomorphic maps*, Preprint, arXiv:1807.02601v2

[Yan18c] X. Yang, *RC-positive metrics on rationally connected manifolds*, Preprint, arXiv:1807.03510v2.

[Yau82] S.-T. Yau, *Problem section*, Seminar on Differential Geometry, 669–706, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, N.J, (1982).

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