Analysis of right-handed Majorana neutrino mass 
in an $SU(4) \times SU(2)_L \times SU(2)_R$ Pati–Salam model 
with democratic texture

Masaki J. S. Yang

Department of Physics, Saitama University,
Shimo-okubo, Sakura-ku, Saitama, 338-8570, Japan

Abstract

In this paper, we attempt to build a unified model with the democratic texture, 
that has some unification between up-type Yukawa interactions $Y_\nu$ and $Y_u$. Since 
the $S_3L \times S_3R$ flavor symmetry is chiral, the unified gauge group is assumed to be 
Pati-Salam type $SU(4)_c \times SU(2)_L \times SU(2)_R$. The breaking scheme of the flavor 
symmetry is considered to be $S_3L \times S_3R \to S_2L \times S_2R \to 0$. In this picture, the four-zero texture is desirable for realistic masses and mixings. This texture is realized by 
a specific representation for the second breaking of the $S_3L \times S_3R$ flavor symmetry.

Assuming only renormalizable Yukawa interactions, type-I seesaw mechanism, 
and neglecting $CP$ phases for simplicity, the right-handed neutrino mass matrix $M_R$ 
can be reconstructed from low energy input values. Numerical analysis shows that 
the texture of $M_R$ basically behaves like the “waterfall texture”. Since $M_R$ tends to 
be the “cascade texture” in the democratic texture approach, a model with type-I seesaw and up-type Yukawa unification $Y_\nu \simeq Y_u$ basically requires fine-tunings 
between parameters. Therefore, it seems to be more realistic to consider universal 
waterfall textures for both $Y_f$ and $M_R$, e.g., by the radiative mass generation or the 
Froggatt–Nielsen mechanism.

Moreover, analysis of eigenvalues shows that the lightest mass eigenvalue $M_{R1}$ 
is too light to achieve successful thermal leptogenesis. Although the resonant leptogenesis might be possible, it also requires fine-tunings of parameters.
1 Introduction

The flavor puzzle is one of the most stringent problems in the current particle physics. In particular, the fermion mixing matrices $U_{\text{CKM}}$ [1,2] and $U_{\text{PMNS}}$ [3,4] are curiously different. Various models and ideas have been considered to explain the underlying flavor dynamics of the standard model (SM). Typical approaches treat the flavor symmetries [5], and/or specific flavor textures [6, 7]. In the latter approach, many researchers have studied the democratic texture [8–25]. In this approach, Yukawa interactions are assumed to have the “democratic matrix” [6], which is realized by $S_3 \times S_3$ symmetry.

In order to explore a more fundamental understanding of flavor, building some unified model is a standard method. The grand unified theory (GUT) with the democratic texture is only discussed in [26, 27], as far as the author knows. However, since these papers assumed a degenerated neutrino Yukawa matrix $Y_\nu$, unification between $Y_\nu$ and other $Y_f$ is difficult. In this paper, we attempt to build another unified model with the democratic texture, which has some unification between up-type Yukawa interactions $Y_\nu$ and $Y_u$. Since the $S_3 \times S_3$ flavor symmetry is chiral, the unified gauge group is assumed to be Pati–Salam (PS) type $SU(4)_c \times SU(2)_L \times SU(2)_R$ ($G_{422}$) [28]. The breaking scheme of the flavor symmetry is considered to be $S_3 \times S_3 \rightarrow S_2 \times S_2 \rightarrow 0$. In this picture, the four-zero texture [29–32] is desirable for realistic masses and mixings. This texture is realized by a specific representation for the second breaking of the $S_3 \times S_3$ flavor symmetry [33–35].

Assuming only renormalizable Yukawa interactions, type-I seesaw mechanism [36], and neglecting $CP$ phases for simplicity, the right-handed neutrino mass matrix $M_R$ can be reconstructed from low energy input values. Numerical analysis shows that the texture of $M_R$ basically behaves like the “waterfall texture” in Table 1. Since $M_R$ tends to be the “cascade texture” in the democratic texture approach, a model with type-I seesaw and up-type Yukawa unification $Y_\nu \simeq Y_u$ basically requires fine-tunings between parameters (including its $CP$ phases, errors of the input parameters, and schemes of gauge symmetry breaking). If we realize the breaking scheme $S_3 \times S_3 \rightarrow S_2 \times S_2 \rightarrow 0$ by some mechanism, the sector of $\nu_R$ might be too complicated to obtain cascade $Y_f$ and waterfall $M_R$ in a unified picture. Therefore, it seems to be more realistic to consider universal waterfall textures for both $Y_f$ and $M_R$, e.g., by the radiative mass generation [37] or the Froggatt–Nielsen mechanism [38].

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| \( \begin{array}{ccc} \epsilon & \epsilon & \epsilon \\ \epsilon & \delta & \delta \\ \epsilon & \delta & 1 \end{array} \) | \( \begin{array}{ccc} \epsilon^2 & \epsilon \delta & \epsilon \\ \epsilon \delta & \delta^2 & \delta \\ \epsilon & \delta & 1 \end{array} \) |
|---|---|
| Cascade | Waterfall |

Table 1: The cascade and waterfall texture, with $1 \gg \delta \gg \epsilon$ [39].

Moreover, analysis of eigenvalues shows that the lightest mass eigenvalue $M_{R1}$ is too light to achieve successful thermal leptogenesis [40]. Although the resonant leptogenesis
might be possible, it also requires fine-tunings of parameters.

In this study, we assume only renormalizable Yukawa interactions. However, this strong tendency to the waterfall texture originates from the seesaw relation $M_R \sim Y_u^T Y_u$. Therefore, it would be rather robust for non-renormalizable Yukawa interactions, as far as the type-I seesaw mechanism is assumed.

This paper is organized as follows. The next section is a review of the Yukawa matrices with the democratic texture. In Sect. 3, we construct a unified model with the $S_{3L} \times S_{3R}$ flavor symmetry. Section 4 is a numerical analysis of mass matrix $M_R$ in this model. Section 5 is devoted to conclusions.

2 The four-zero texture from the democratic matrix approach

The democratic matrix is defined as

$$Y_f^0 = \frac{K_f}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{K_f}{3} D,$$

which is invariant under $S_{3L} \times S_{3R}$, the permutation symmetry between rows and columns. It is diagonalized by the unitary matrix $U_{DC}$

$$U_{DC} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix},$$

and eigenvalues are given by $Y_{f1}^0 = \text{diag}(0, 0, K_f)$. Then, the democratic matrix produces mass only for the third generation. In order to provide masses for the first and second generations, the breaking scheme of the flavor symmetry is chosen as $S_{3L} \times S_{3R} \rightarrow S_{2L} \times S_{2R} \rightarrow 0$. Then, Yukawa matrices are represented as

$$Y_f = \frac{K_f}{3} D + \delta_f Y_f^\delta + \epsilon_f Y_f^\epsilon,$$

where $Y_f^\delta, Y_f^\epsilon$ breaks $S_{3L} \times S_{3R}$ and $S_{2L} \times S_{2R}$ respectively. This breaking scheme is discussed in several papers [35, 43–47]. The origin and specific realization of this breaking scheme have not been discussed by the authors who proposed it. For example, the radiatively generated light fermion masses by broken $S_3$ symmetry [37] could explain this breaking scheme. In Ref. [37], $S_3$ breaking effects induce departures from the democratic texture only radiatively, and light fermion masses are suppressed by typical loop factors $[1/(16\pi^2)]^{1/2}$. It naturally predicts the hierarchical relation

$$K_f \gg \delta_f \gg \epsilon_f,$$
which is required from realistic masses and mixings. A pedagogical explanation is also found in the review [48]. The following discussion is equivalent to Ref. [35].

The term $\delta_f Y_f^\delta$ is invariant under $S_{2L} \times S_{2R}$ between first and second indices, in order to provide mass only for the second generation. The most general form of the $S_{2L} \times S_{2R}$ invariant symmetric $Y_f^\delta$ is

$$Y_f^\delta = \begin{pmatrix} a & a & b \\ a & a & b \\ b & b & c \end{pmatrix}.$$  \hfill (5)

For later convenience, we parametrize $\delta_f Y_f^\delta$ as follows:

$$\delta_f Y_f^\delta = \delta_f \begin{pmatrix} \frac{\sqrt{2}}{3} + \frac{r}{6} & \frac{\sqrt{2}}{3} + \frac{r}{6} & -\frac{r}{3\sqrt{2}} - \frac{1}{3} \\ -\frac{r}{3\sqrt{2}} - \frac{1}{3} & \frac{\sqrt{2}}{3} + \frac{r}{6} & -\frac{r}{3\sqrt{2}} - \frac{1}{3} \\ \frac{2}{3} - \frac{2\sqrt{2}}{3} & \frac{2}{3} - \frac{2\sqrt{2}}{3} & 0 \end{pmatrix}.$$  \hfill (6)

In Eq. (6), there are only two free parameters $r, \delta_f$. However, it does not lose generality, because one of the parameters in Eq. (5) can be absorbed by the redefinition of $K_f$. Similarly, $\epsilon_f Y_f^\epsilon$ provide mass for the first generations. Refs. [34, 35] proposed that $\epsilon_f Y_f^\epsilon$ may be the doublet complex tensorial representation of the $S_3(L+R)$ diagonal subgroup:

$$\epsilon_f Y_f^\epsilon = \begin{pmatrix} \epsilon_1 & i\epsilon_2 & -\epsilon_1 - i\epsilon_2 \\ -i\epsilon_2 & -\epsilon_1 & \epsilon_1 + i\epsilon_2 \\ -\epsilon_1 + i\epsilon_2 & \epsilon_1 - i\epsilon_2 & 0 \end{pmatrix}.$$  \hfill (7)

In this case, the Yukawa matrices are approximately diagonalized as

$$U^\dagger_{DC} Y_f U_{DC} = U^\dagger_{DC} \left[ \frac{K_f}{3} D + \delta_f Y_f^\delta + \epsilon_f Y_f^\epsilon \right] U_{DC} = \begin{pmatrix} 0 & \epsilon_f e^{i\phi_f} & 0 \\ \epsilon_f e^{-i\phi_f} & \delta_f & r\delta_f \\ 0 & r\delta_f & K_f \end{pmatrix},$$  \hfill (8)

where $\epsilon_f e^{i\phi_f} = \sqrt{3} (\epsilon_1 + i\epsilon_2)$. Then, these Yukawa matrices lead to the “four-zero texture” or the “modified Fritzsch texture” [23–25]. This relationship between the democratic texture and the four-zero texture is studied by several authors [33–35]. In Eq. (8), $r \sim O(1)$ is required to obtain the successful Cabibbo-Kobayashi-Maskawa (CKM) matrix. This is a natural condition because $S_{3L} \times S_{3R}$ breaking would produce a relation $Y_{22} \sim Y_{23}$.

For simplicity, we neglect all $CP$ phases of the Yukawa matrices (cf. $\phi_f = 0$ in Eq. (8)). The effect of $CP$ phases is discussed later. However, the qualitative result is considered to be rather robust with finite $CP$ phases.

For the real Yukawa matrices, Eq. (8) is perturbatively diagonalized as

$$B_f^\dagger U^\dagger_{DC} Y_f U_{DC} B_f = \text{diag}(y_{1f}, y_{2f}, y_{3f}),$$  \hfill (9)

where

$$y_{1f} \approx -\frac{\epsilon_2}{\delta_f} - \frac{\epsilon_2^2}{K_f}, \quad y_{2f} \approx \delta_f + \frac{\epsilon_2}{\delta_f} - \frac{\epsilon_2^2}{K_f}, \quad y_{3f} \approx K_f + \frac{\epsilon_2^2}{K_f}.$$  \hfill (10)
The unitary matrix $B_f$ at leading order is found to be

$$B_f \simeq \begin{pmatrix} 1 & -\frac{\epsilon_f}{\delta_f} & 0 \\ \frac{\epsilon_f}{\delta_f} & 1 & r \frac{\delta_f}{K_f} \\ -r \frac{\epsilon_f}{K_f} & -r \frac{\delta_f}{K_f} & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -\sqrt{\frac{y_{f1}}{y_{f2}}} & 0 \\ \sqrt{\frac{y_{f1}}{y_{f2}}} & 1 & r \frac{y_{f2}}{y_{f3}} \\ -r \frac{y_{f2}}{y_{f3}} & -r \frac{y_{f2}}{y_{f3}} & 1 \end{pmatrix}. \quad (11)$$

Note that $y_{f1}/y_{f2} \simeq -\epsilon_f^2/\delta_f^2$ is always negative.

Therefore, the CKM matrix $V_{\text{CKM}} = B_u^* B_d$ (without complex phase) is calculated as

$$V_{\text{CKM}} \simeq \begin{pmatrix} 1 & -r \sqrt{\frac{m_u}{m_c}} & 0 \\ -r \sqrt{\frac{m_u}{m_c}} & 1 & r \frac{m_s}{m_b} \\ 0 & r \frac{m_s}{m_b} & 1 \end{pmatrix} \begin{pmatrix} 1 & -r \sqrt{\frac{m_d}{m_s}} & 0 \\ -r \sqrt{\frac{m_d}{m_s}} & 1 & r \frac{m_s}{m_b} \\ 0 & r \frac{m_s}{m_b} & 1 \end{pmatrix} \begin{pmatrix} 1 & -r \left[ \sqrt{\frac{m_u}{m_c}} - \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] r \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] \\ -r \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] r \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] \\ \left[ \sqrt{\frac{m_u}{m_c}} \right] - \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] r \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] \left[ \sqrt{\frac{m_u}{m_c}} \right] \end{pmatrix} \quad (12)$$

Here, we omit the minus sign in the square root ($\sqrt{-m_u/m_c} \rightarrow \sqrt{m_u/m_c}$). It predicts $V_{cb}$ and $V_{ts}$ at leading order as follows

$$V_{cb} \simeq -V_{ts} \simeq r \left[ \frac{m_s}{m_b} - \frac{m_c}{m_t} \right]. \quad (14)$$

If the parameters $K_f, \delta_f, \epsilon_f$ have CP phases, each CKM matrix element obtains overall phases and relative phases, such as $\sqrt{\frac{m_u}{m_c}} - \sqrt{\frac{m_u}{m_c}} \rightarrow e^{i\phi} \left[ \sqrt{\frac{m_u}{m_c}} - e^{i\eta} \sqrt{\frac{m_u}{m_c}} \right]$. In particular, the best value of $\chi^2$ fit $r = \sqrt{81/32} \simeq 1.59$ gives excellent agreement between the prediction and the observation of absolute values of the CKM matrix elements.

### 3 $SU(4)_c \times SU(2)_L \times SU(2)_R$ model with democratic texture

In order to explore a more fundamental understanding of flavor, building some unified model is a standard method. The grand unified theory (GUT) with the democratic
texture is only discussed in [26, 27], as far as the author knows. However, since these papers assumed degenerated $Y_\nu$, unification between $Y_\nu$ and other $Y_f$ is difficult. In this paper, we attempt to build another unified model with the democratic texture, which has some unification between $Y_\nu$ and $Y_u$. Since the $S_{3L} \times S_{3R}$ flavor symmetry is chiral [1], the unified gauge group is assumed to be Pati–Salam (PS) type $SU(4)_c \times SU(2)_L \times SU(2)_R$ ($G_{422}$) [28].

To produce realistic fermion masses, we consider the minimal contents of Higgs fields with the following representations under the $G_{422}$ group:

$$
\Phi : (1, 2, 2), \quad \Sigma : (15, 2, 2), \quad \Delta_R : (10, 1, 3).
$$

(15)

Although other representations are also possible, such as (4,1,2) in [13, 50], we consider only renormalizable interactions to control Yukawa interactions.

The field contents of the unified model are in Table 2. These Higgs contents are sufficient to break the PS gauge group $G_{224}$ to the SM gauge group $G_{SM}$. For example, a breaking scheme of the gauge symmetry with these Higgs contents is discussed in the context of the noncommutative geometry [51, 52]. We do not discuss the energy scales and order of the symmetry breakings. However, the final result is considered to be rather independent from breaking schemes.

|            | $SU(4)_c$ | $SU(2)_L$ | $SU(2)_R$ | $S_{3L}$   | $S_{3R}$   |
|------------|-----------|-----------|-----------|------------|------------|
| $\Psi_Li$  | $q_{Li}^\alpha$, $l_{Li}$ | 4         | 2         | 1          | $1_L + 2_L$ | $1_R$      |
| $\Psi_Ri$  | $q_{Ri}^\alpha$, $l_{Ri}$ | 4         | 1         | 2          | $1_L$      | $1_R + 2_R$ |
| $\Phi$     |           | 1         | 2         | 2          | $1_L$      | $1_R$      |
| $\Sigma$   |           | 15        | 2         | 2          | $1_L$      | $1_R$      |
| $\Delta_R$ |           | 10        | 1         | 3          | $1_L$      | $1_R$      |

Table 2: The charge assignments of the SM fermions and Higgs fields under the gauge and the flavor symmetries.

The renormalizable Yukawa interactions invariant under $G_{422}$ are found to be

$$
\mathcal{L}_{\text{Yukawa}} = \bar{\Psi}_{Ri}(Y_1^{ij}\Phi + Y_{15}^{ij}\Sigma)\Psi_{Lj} + \text{H.c.}.
$$

(16)

Note that Yukawa matrices $Y_1, Y_{15}$ become symmetric matrices if we impose the left-right symmetry between $\Psi_L \leftrightarrow \Psi_R$. These $Y_1, Y_{15}$ are divided into $S_{3L} \times S_{3R}$ preserving and breaking parts respectively:

$$
Y_1 = K_1D + \delta Y_1, \quad Y_{15} = K_{15}D + \delta Y_{15}.
$$

(17)

In order to obtain the desirable masses and mixings, we assume $K_{15} = 0$ and $\delta Y_1$ does not have $S_{3L} \times S_{3R}$ breaking elements $\delta f$. Then $Y_{15}$ is treated as a perturbation, as in the

\[1\) In the $SO(10)$ GUT, the flavor symmetry should be single $S_3$, and the condition $c_f = 0$ similar to Eq. (31) should be assumed.
previous study \[26\]. Vacuum expectation values of these Higgs fields are taken to be

\[
\langle \Phi \rangle = \text{Diag}(1, 1, 1, 1) \times \begin{pmatrix} v_u^1 & 0 \\ 0 & v_d^1 \end{pmatrix}, \quad \langle \Sigma \rangle = \text{Diag}(1, 1, 1, -3) \times \begin{pmatrix} v_u^{15} & 0 \\ 0 & v_d^{15} \end{pmatrix},
\] (18)

in the representation space of $\Psi_{L,R} = (q_{1,L,R}, q_{2,L,R}, q_{3,L,R}, l_{L,R})$.

This setup leads to the following mass matrices \[53–55\]

\[
M_u = v_u^1(K_1 D + \delta Y_1) + v_u^{15}\delta Y_{15} = v_u^1 K_1 D + v_u^1 \delta Y_1 + v_u^{15} \delta Y_{15},
\] (19)

\[
M_\nu^D = v_u^1(K_1 D + \delta Y_1) - 3v_u^{15}\delta Y_{15} = v_u^1 K_1 D + v_u^1 \delta Y_1 - 3v_u^{15} \delta Y_{15},
\] (20)

\[
M_d = v_d^1(K_1 D + \delta Y_1) + v_d^{15}\delta Y_{15} = v_d^1 K_1 D + v_d^1 \delta Y_1 + v_d^{15} \delta Y_{15},
\] (21)

\[
M_e = v_d^1(K_1 D + \delta Y_1) - 3v_d^{15}\delta Y_{15} = v_d^1 K_1 D + v_d^1 \delta Y_1 - 3v_d^{15} \delta Y_{15}.
\] (22)

In particular, effective Yukawa matrices are explicitly written as

\[
Y_u = \begin{pmatrix} 0 & \epsilon_u & 0 \\ \epsilon_u & \delta_u & r\delta_u \\ 0 & r\delta_u & K_u \end{pmatrix}, \quad Y_d = \begin{pmatrix} 0 & \epsilon_d & 0 \\ \epsilon_d & \delta_d & r\delta_d \\ 0 & r\delta_d & K_d \end{pmatrix},
\] (23)

\[
Y_\nu = \begin{pmatrix} 0 & \epsilon_\nu & 0 \\ \epsilon_\nu & \delta_\nu & r\delta_\nu \\ 0 & r\delta_\nu & K_\nu \end{pmatrix}, \quad Y_e = \begin{pmatrix} 0 & \epsilon_e & 0 \\ \epsilon_e & \delta_e & r\delta_e \\ 0 & r\delta_e & K_e \end{pmatrix},
\] (24)

with

\[
K_u,d = K_{u,e}, \quad \delta_{u,d} = -\frac{1}{3}\delta_{u,e}, \quad \epsilon_{u,d} = \epsilon_{u,e}.
\] (25)

These conditions lead to the famous Georgi–Jarlskog relation \[56\]

\[
m_d = 3m_e, \quad m_s = \frac{1}{3}m_u, \quad m_b = m_\tau,
\] (26)

and similar formulae hold for up-type fermions.

### 4 Analysis of the right-handed Majorana neutrino mass matrix

In this section, we analyze the right-handed neutrino mass matrix $M_R$ in the PS model with the four-zero Yukawa textures. Many papers have studied this kind of model, such as SO(10) GUT with the four-zero texture \[30, 31, 57, 58\]. However, the purpose of this paper is to analyze texture of $M_R$ quantitatively in a united model with the democratic texture.

$M_R$ emerges from the following interaction

\[
\mathcal{L}_{\text{Majorana}} = \bar{\Psi}_{Ri}^c Y_{10}^{ij} \Delta_R \Psi_{Rj} + \text{H.c.},
\] (27)
when $\Delta_R$ obtain a vacuum expectation value

$$\langle \Delta_R \rangle = \text{Diag}(0, 0, 0, 1) \times \begin{pmatrix} 0 & 0 \\ v_R & 0 \end{pmatrix}.$$  \hspace{1cm} (28)

Because $Y^{10}$ is transformed as $(1_R + 2_R) \times (1_R + 2_R)$, it has two $S_{3R}$ invariant terms

$$Y^{10} = K_{10} D + c_{10} 1_3 + \delta Y_{10}.$$  \hspace{1cm} (29)

where $1_3$ is the $3 \times 3$ identity matrix.

To obtain the observed light neutrino masses, we assume the type-I seesaw mechanism [36]

$$m_\nu = \frac{v^2}{2} Y_{\nu}^T M_R^{-1} Y_{\nu}.$$  \hspace{1cm} (30)

In this case,

$$\delta Y_{10} \gg c_{10} \simeq 0,$$  \hspace{1cm} (31)

is required by phenomenological reason. The numerical analysis shown later reveals that $Y_{\nu}$ with a large $c_{10} \gg \delta Y_{10}$ are incompatible to obtain the observed large neutrino mixings.

If the flavor symmetry breaking $S_{3L} \times S_{3R} \rightarrow S_{2L} \times S_{2R} \rightarrow 0$ also controls the structure of $M_R$, and if there is no fine-tuning between the parameters, the form of $M_R$ should be the following cascade texture in Table 1:

$$M_R \sim v_R \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \delta & \delta \\ \epsilon & \delta & 1 \end{pmatrix}.$$  \hspace{1cm} (32)

The light neutrino mass, Eq. (30), is diagonalized by

$$m_\nu \equiv V_{\nu}^* m_\nu^{\text{diag}} V_{\nu}^\dagger,$$  \hspace{1cm} (33)

where $m_\nu^{\text{diag}} = \text{diag}(m_1, m_2, m_3)$. This mass matrix is rewritten as

$$m_\nu = B_e^* U_{\text{PMNS}}^* m_\nu^{\text{diag}} U_{\text{PMNS}}^T B_e^\dagger,$$  \hspace{1cm} (34)

with the neutrino mixing matrix $U_{\text{PMNS}} = B_e^\dagger V_{\nu}$ and $B_e$ [11] for the charged leptons.

Ignoring all of the complex phases for simplicity, we can reconstruct $M_R$ by the seesaw formula:

$$M_R = \frac{v^2}{2} Y_{\nu}^T m_\nu^{-1} Y_{\nu}$$  \hspace{1cm} (35)

$$= \frac{v^2}{2} Y_{\nu}^T B_e U_{\text{PMNS}} (m_\nu^{\text{diag}})^{-1} U_{\text{PMNS}}^T B_e^T Y_{\nu}.$$  \hspace{1cm} (36)
As a benchmark, \( M_R(\Lambda_{\text{GUT}}) = Y_\nu(\Lambda_{\text{GUT}})^T m_\nu(\Lambda_{\text{GUT}}) Y_\nu(\Lambda_{\text{GUT}}) \) at the GUT scale \( \Lambda_{\text{GUT}} = 2 \times 10^{16} \text{GeV} \) can be evaluated as

\[
\frac{M_R(\Lambda_{\text{GUT}})}{[\text{GeV}]} \sim \frac{[\text{meV}]}{m_1} \begin{pmatrix} 1.876 \times 10^7 & -3.623 \times 10^8 & -1.009 \times 10^{11} \\ -3.623 \times 10^8 & 6.996 \times 10^9 & 1.948 \times 10^{12} \\ -1.009 \times 10^{11} & 1.948 \times 10^{12} & 5.424 \times 10^{14} \end{pmatrix}
\]

(37)

\[
+ \frac{[\text{meV}]}{m_2} \begin{pmatrix} 3.302 \times 10^7 & -2.173 \times 10^9 & -2.849 \times 10^{11} \\ -2.173 \times 10^9 & 1.429 \times 10^{11} & 1.874 \times 10^{13} \\ -2.849 \times 10^{11} & 1.874 \times 10^{13} & 2.457 \times 10^{15} \end{pmatrix}
\]

(38)

\[
+ \frac{[\text{meV}]}{m_3} \begin{pmatrix} 6.255 \times 10^7 & 1.012 \times 10^{10} & 3.975 \times 10^{11} \\ 1.012 \times 10^{10} & 1.637 \times 10^{12} & 6.431 \times 10^{13} \\ 3.975 \times 10^{11} & 6.431 \times 10^{13} & 2.526 \times 10^{15} \end{pmatrix}.
\]

(39)

The parameters used here are summarized in Table 3. The fermion masses at the GUT scale \( m_f(\Lambda_{\text{GUT}}) \) are taken from [60]. In most cases of this model, the order of light neutrino masses \( m_i \) becomes the normal hierarchy. The inverted hierarchy \( m_1 \approx m_2 \gg m_3 \) is unnatural because the hierarchy of \( M_R \) should overcome the ratio \( m_t^2/m_c^2 \). The renormalization of the neutrino mass can be neglected for the normal hierarchy case [61, 62]. Then, neutrino mixing angles and mass square differences are taken from the latest global fit [63], without renormalization running. A similar parameter set is used in [59].

Eqs. (37) - (39) shows that the right-handed neutrino mass matrix \( M_R \sim Y_u^T Y_u \) rather tends to be the waterfall texture in Table 1,

\[
M_R \sim v_R \begin{pmatrix} \epsilon^2 & \epsilon \delta & \epsilon \\ \epsilon \delta & \delta^2 & \delta \\ \epsilon & \delta & 1 \end{pmatrix},
\]

(40)

for each small mass eigenvalue \( m_i \). Then, it seems to be difficult to explain this texture by the breaking scheme \( S_{3L} \times S_{3R} \rightarrow S_{2L} \times S_{2R} \rightarrow 0 \). Hereafter we precisely check the form of the \( M_R \) by numerical analysis.
4.1 Numerical results

Using the mass difference values $\Delta m^2_{31}$ in Table 3,

$$
\begin{align*}
    m_3 &= \pm \sqrt{m^2_1 + 2457} \, [\text{meV}], \\
    m_2 &= \pm \sqrt{m^2_1 + 75} \, [\text{meV}], \\
\end{align*}
$$

(41)

the mass matrix $M_R$ (37) - (39) is expressed as a function of $m_1$, $M_R(\Lambda_{\text{GUT}}) = M_R(m_1)$.

\[m_2 < 0, m_3 > 0\]

\[m_2 > 0, m_3 > 0\]

\[m_2 < 0, m_3 < 0\]

\[m_2 > 0, m_3 < 0\]

Figure 1: Lighter matrix elements $(M_R)_{11}, (M_R)_{12}, (M_R)_{13},$ and $(M_R)_{22}$ of the $M_R(m_1)$ at the GUT scale $\Lambda_{\text{GUT}} = 2 \times 10^{16} \, [\text{GeV}]$, as a function of $m_1$. The signatures of $m_2$ and $m_3$ are taken as the top of the figures.

Figure 1 shows lighter matrix elements $(M_R)_{11}, (M_R)_{12}, (M_R)_{13},$ and $(M_R)_{22}$ of the $M_R(m_1)$ at the GUT scale $\Lambda_{\text{GUT}} = 2 \times 10^{16} \, [\text{GeV}]$, as a function of $m_1$. The signatures of $m_2$ and $m_3$ are taken as the top of the figures. From Fig. 1, we can see the hierarchical structure of the $M_R$. These matrix elements basically behave like the waterfall texture $(M_R)_{22} \sim (M_R)_{13} \gg (M_R)_{22} \gg (M_R)_{11}$. Several changes of sign are due to cancellations among Eqs. (37) - (39).
This behavior shows that the cascade texture \((M_R)_{22} \gg (M_R)_{13} \sim (M_R)_{22} \sim (M_R)_{11}\) cannot be realized without fine-tunings of parameters in this model. In particular, the four-zero texture for \(M_R\) (equivalent to \((M_R)_{11} = (M_R)_{13} = 0\)), is also difficult to realize without fine-tuning. However, in this analysis, approximate four-zero texture \((M_R)_{12} \gg (M_R)_{13} \sim (M_R)_{11}\) is realized around \(m_1 \sim 4\) meV with \(m_{2,3} < 0\).

So far, the parameters of the model have been assumed to be real. Here we will discuss the effect of \(CP\) phases shortly. Figure 2 shows lighter matrix elements \((M_R)_{11}, (M_R)_{12}, (M_R)_{13}\), and \((M_R)_{22}\) of the \(M_R(m_1)\), with finite dirac \(CP\) phase \(\delta_{CP} = \pi/2\) of the \(\) PMNS matrix. Other parameters are taken to be the same as Fig. 1 (for \(m_2\), only negative sign \(m_2 < 0\) is presented).

![Figure 2: Lighter matrix elements \((M_R)_{11}, (M_R)_{12}, (M_R)_{13}\), and \((M_R)_{22}\) of the \(M_R(m_1)\), with finite dirac \(CP\) phase \(\delta_{CP} = \pi/2\) of the \(\) PMNS matrix. Other parameters are taken to be the same as Fig. 1 (for \(m_2\), only negative sign \(m_2 < 0\) is presented).](image)

and \((M_R)_{22}\) of the \(M_R(m_1)\), with finite dirac \(CP\) phase \(\delta_{CP} = \pi/2\) of the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix. Other parameters are taken to be the same as Fig. 1 (for \(m_2\), only negative sign \(m_2 < 0\) is presented). In Fig. 2, the cancellations of \((M_R)_{ij}\) found in Fig. 1 vanish by the finite \(CP\) phases, and the cascade texture is evidently impossible with this parameter set. By assuming finite \(CP\) phases for other parameters, we found that the cancellations are basically smoothed or vanished. It is plausible that \(M_R\) is strongly tend to be the waterfall texture [11]. Therefore, in this democratic matrix approach, a model with type-I seesaw and up-type Yukawa unification \(Y_\nu \simeq Y_u\) basically requires fine-tunings between parameters (including its \(CP\) phases, errors of the input parameters, and gauge symmetry breaking schemes). If we realize the breaking scheme \(S_{3L} \times S_{3R} \rightarrow S_{2L} \times S_{2R} \rightarrow 0\) by some mechanism, the sector of \(\nu_R\) might be too complicated to obtain cascade \(Y_f\) and waterfall \(M_R\) in a unified picture. Therefore, it seems to be more realistic to consider universal waterfall textures for both \(Y_f\) and \(M_R\), e.g., by the radiative mass generation [37] or the Froggatt–Nielsen mechanism [38].
4.2 Mass eigenvalues and thermal leptogenesis

Figure 3 shows three mass eigenvalues $M_{Ri}$ of the $M_R(m_1)$ at the GUT scale $\Lambda_{\text{GUT}} = 2 \times 10^{16}$ [GeV], as a function of $m_1$. The parameters are taken to be the same as Figure 1 (for $m_2$, only negative sign $m_2 < 0$ is presented). Basically the eigenvalues $M_{Ri}$ are strongly hierarchical, because $M_R$ has large hierarchy such as $M_R \sim Y_T Y_u$. The largest eigenvalue $M_{R3}$ changes its sign around $m_1 \sim 2$ meV. This is due to cancellation for the 33 element of $M_R$, between Eq. (37) and Eq. (38) around the region $m_2 \sim 5m_1$. Similarly, the cancellation for $(M_R)_{11}$ induces the change of sign for two smaller eigenvalues, $M_{R1}$ and $M_{R2}$.

These figures exhibit that the lightest mass eigenvalue tends to be rather small $M_{R1} \lesssim 10^5$ GeV, except the cancellation regions. The successful thermal leptogenesis [40] requires $M_{R1} > 4.9 \times 10^8$ GeV for the hierarchical $M_{Ri}$ [54, 55]. Then, it is nearly impossible to explain the observed baryon asymmetry by the thermal leptogenesis in this model. The resonant leptogenesis [41, 42] would be possible in the cancellation region with $M_{R1} \simeq M_{R2}$ ($m_3 < 0, m_1 \simeq 3$ meV). Similar results for SO(10) are found in Ref. [58]. However, this cancellation region can be easily vanished by finite CP phases. Therefore, successful leptogenesis also requires fine-tunings of the parameters in this model.

In this study, we assume only renormalizable Yukawa interactions. However, this strong tendency to the waterfall texture originates from the seesaw relation $M_R \sim Y_T Y_u$. Therefore, it would be rather robust for non-renormalizable Yukawa interactions, as far as the type-I seesaw mechanism is assumed.
5 Conclusions

In this paper, we attempt to build a unified model with the democratic texture, which has some unification between up-type Yukawa interactions $Y_\nu$ and $Y_u$. Since the $S_{3L} \times S_{3R}$ flavor symmetry is chiral, the unified gauge group is assumed to be Pati-Salam (PS) type $SU(4)_C \times SU(2)_L \times SU(2)_R$ ($G_{422}$). The breaking scheme of the flavor symmetry is considered to be $S_{3L} \times S_{3R} \rightarrow S_{2L} \times S_{2R} \rightarrow 0$. In this picture, the four-zero texture is desirable for realistic mass and mixings. This texture is realized by a specific representation for the second breaking of the $S_{3L} \times S_{3R}$ flavor symmetry.

Assuming only renormalizable Yukawa interactions, type-I seesaw mechanism, and neglecting $CP$ phases for simplicity, the right-handed neutrino mass matrix $M_R$ can be reconstructed from low energy input values. Numerical analysis shows that the texture of $M_R$ basically behaves like the waterfall texture in Table 1. Since $M_R$ tends to be the cascade texture in the democratic texture approach, a model with type-I seesaw and up-type Yukawa unification $Y_\nu \approx Y_u$ basically requires fine-tunings between parameters (including its $CP$ phases, errors of the input parameters, and schemes of gauge symmetry breaking). If we realize the breaking scheme $S_{3L} \times S_{3R} \rightarrow S_{2L} \times S_{2R} \rightarrow 0$ by some mechanism, the sector of $\nu_R$ might be too complicated to obtain cascade $Y_f$ and waterfall $M_R$ in a unified picture. Therefore, it seems to be more realistic to consider universal waterfall textures for both $Y_f$ and $M_R$, e.g., by the radiative mass generation or the Froggatt–Nielsen mechanism.

Moreover, analysis of eigenvalues shows that the lightest mass eigenvalue $M_{R1}$ is too light to account the baryon asymmetry of the universe by the thermal leptogenesis. Although the resonant leptogenesis might be possible, it also requires fine-tunings of parameters.

In this study, we assume only renormalizable Yukawa interactions. However, this strong tendency to the waterfall texture originates from the seesaw relation $M_R \sim Y_u^T Y_u$. Therefore, it would be rather robust for non-renormalizable Yukawa interactions, as far as the type-I seesaw mechanism is assumed.

Acknowledgement

This study is financially supported by the Iwanami Fujukai Foundation, and the Sasakawa Scientific Research Grant from The Japan Science Society, No. 28-214.

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