Towards fault-tolerant quantum computing with trapped ions

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Today ion traps are among the most promising physical systems for constructing a quantum device harnessing the computing power inherent in the laws of quantum physics\textsuperscript{1,2}. The standard circuit model of quantum computing requires a universal set of quantum logic gates for the implementation of arbitrary quantum operations. As in classical models of computation, quantum error correction techniques\textsuperscript{3,4} enable rectification of small imperfections in gate operations, thus allowing for perfect computation in the presence of noise. For fault-tolerant computation\textsuperscript{5}, it is commonly believed that error thresholds ranging between $10^{-4}$ and $10^{-2}$\textsuperscript{6–8} will be required depending on the noise model and the computational overhead for realizing the quantum gates. Up to now, all experimental implementations have fallen short of these requirements. Here, we report on a Mølmer-Sørensen\textsuperscript{9,10} type gate operation entangling ions with a fidelity of 99.3(1)\% which together with single-qubit operations forms a universal set of quantum gates. The gate operation is performed on a pair of qubits encoded in two trapped calcium ions using a single amplitude-modulated laser beam interacting with
both ions at the same time. A robust gate operation, mapping separable states onto max-
mally entangled states is achieved by adiabatically switching the laser-ion coupling on and
off. We analyse the performance of a single gate and concatenations of up to 21 gate opera-
tions. The gate mechanism holds great promise not only for two-qubit but also for multi-qubit
operations.

For ion traps, all building blocks necessary for the construction of a universal quantum
computer\(^1\) have been demonstrated over the last decade. Currently, the most important challenges
consist in scaling up the present systems to higher number of qubits and in raising the fidelity
of gate operations up to the point where quantum error correction techniques can be success-
fully applied. While single-qubit gates are easily performed with high quality, the realisation of
high-fidelity entangling two-qubit gates\(^{11-16}\) is much more demanding since the inter-ion distance
is orders of magnitude bigger than the characteristic length scale of any state-dependent ion-ion
interaction. Apart from quantum gates of the Cirac-Zoller type\(^2,12\) where a laser couples a single
qubit with a vibrational mode of the ion string at a time, most other gate realizations entangling ions
have relied on collective interactions of the qubits with the laser control fields\(^{11,13-15}\). These gate
operations entangle transiently the collective pseudo-spin of the qubits with the vibrational mode
and produce either a conditional phase shift\(^{17}\) or a collective spin flip\(^{9,10,18}\) of the qubits. While the
highest fidelity \(F = 97\%\) reported to date\(^{13}\) has been achieved with a conditional phase gate acting
on a pair of hyperfine qubits in \(^9\text{Be}^+\), spin flip gates have been limited so far to \(F \approx 85\%\)\(^{11,14}\).
All of these experiments have used qubits encoded in hyperfine or Zeeman ground states and a Ra-
man transition mediated by an electric-dipole transition for coupling the qubits. While spontaneous
scattering from the mediating short-lived levels degrades the gate fidelity due to the limited amount of laser power available in current experiments, this source of decoherence does not occur for optical qubits, i.e. qubits encoded in a ground state and a metastable electronic state of an ion. In the experiment presented in this paper where the qubit is comprised of the states $|S\rangle \equiv S_{1/2}(m = 1/2)$ and $|D\rangle \equiv D_{5/2}(m = 3/2)$ of the isotope $^{40}$Ca$^+$, spontaneous decay of the metastable state reduces the gate fidelity by less than $5 \cdot 10^{-5}$.

A Mølmer-Sørensen gate inducing collective spin flips is achieved with a bichromatic laser field with frequencies $\omega_\pm = \omega_0 \pm \delta$, with $\omega_0$ the qubit transition frequency and $\delta$ close to the vibrational mode frequency $\nu$ (see Fig. 1). For optical qubits, the bichromatic field can be a pair of co-propagating lasers which is equivalent to a single laser beam resonant with the qubit transition and amplitude-modulated with frequency $\delta$. For a gate mediated by the axial centre-of-mass (COM) mode, the Hamiltonian describing the laser-qubit interaction is given by $\hat{H} = \hbar \Omega e^{-i\phi} \sigma_+ (e^{-i(\delta t + \zeta)} + e^{i(\delta t + \zeta)}) e^{i\eta(\hat{a}e^{-i\nu t} + \hat{a}^\dagger e^{i\nu t})} + \text{h.c.}$. Here, $\sigma_j = \sigma_j^{(1)} + \sigma_j^{(2)}$, $j \in \{+, -, x, y, z\}$, denotes a collective atomic operator, and $\sigma_+^{(i)} \langle S \rangle_{i} = \langle D \rangle_{i}$. The operators $a, a^\dagger$ annihilate and create phonons of the COM mode with Lamb-Dicke factor $\eta$. The optical phase of the laser field is labeled $\phi$, and the phase $\zeta$ accounts for a time difference between the start of the gate operation and the maximum of the amplitude modulation of the laser beam. In the Lamb-Dicke regime, and for $\phi = 0$, the gate operation is very well described by the propagator

$$U(t) = e^{-iF(t)S_0} \hat{D}(\alpha(t)S_{y,\psi}) \exp(-i(\lambda t + \chi \sin(\nu - \delta) t) S_{y,\psi}^2).$$

Here, the operator to the right describes collective spin flips induced by the operator $S_{y,\psi} = S_y \cos \psi + S_z \sin \psi$, $\psi = \frac{4\Omega}{\delta} \cos \zeta$, and $\lambda \approx \eta^2 \Omega^2 / (\nu - \delta)$, $\chi \approx \eta^2 \Omega^2 / (\nu - \delta)^2$. With $\alpha(t) =$
\( \alpha_0(e^{i(\nu-\delta)t} - 1) \), the displacement operator \( \hat{D}(\beta) = e^{\beta a^\dagger - \beta^* a} \) accounting for the transient entanglement between the qubits and the harmonic oscillator becomes equal to the identity after the gate time \( \tau_{\text{gate}} = \frac{2\pi}{|\nu - \delta|} \). The operator \( e^{-iF(t)S_z} \) with \( F(t) = (2\Omega/\delta)(\sin(\delta t + \phi) - \sin \phi) \) describes fast non-resonant excitations of the carrier transition that occur in the limit of short gates when \( \Omega \ll \delta \) no longer strictly holds. Non-resonant excitations are suppressed by intensity-shaping the laser pulse so that the Rabi frequency \( \Omega(t) \) is switched on and off smoothly. Moreover, adiabatic switching makes the collective spin flip operator independent of \( \zeta \) as \( S_{y,\psi} \rightarrow S_y \) for \( \Omega \rightarrow 0 \).

To achieve adiabatic following, it turns out to be sufficient to switch on the laser within 2.5 trap cycles. In order to realize an entangling gate of duration \( \tau_{\text{gate}} \) described by the unitary operator \( U_{\text{gate}} = \exp(-i\frac{\pi}{8}S_y^2) \), the laser intensity needs to be set such that \( \eta \Omega \approx |\delta - \nu|/4 \).

Two \(^{40}\text{Ca}^+\) ions are confined in a linear trap \(^{20}\) with axial and radial frequencies of \( \nu_{\text{axial}}/2\pi = 1.23 \text{ MHz} \) and \( \nu_{\text{radial}}/2\pi = 4 \text{ MHz} \), respectively. After Doppler cooling and frequency-resolved optical pumping \(^{21}\), the two axial modes are cooled close to the motional ground state (\( \bar{n}_{\text{com}}, \bar{n}_{\text{stretch}} < 0.05(5) \)). Both ions are now initialized to \( |SS\rangle \) with a probability of more than 99.8%. Then, the gate operation is performed, followed by an optional carrier pulse for analysis. Finally, we measure the probability \( p_k \) of finding \( k \) ions in the \( |S\rangle \) state by detecting light scattered on the \( S_{1/2} \leftrightarrow P_{1/2} \) dipole transition with a photomultiplier for 3 ms. The error in state detection due to spontaneous emission is estimated to be less than 0.15%. Each experimental cycle is synchronised with the frequency of the AC-power line and repeated 50-200 times. The laser beam performing the entangling operation is controlled by a double-pass acousto-optic modulator (AOM) which allows setting the frequency \( \omega_L \) and phase \( \phi \) of the beam. By means of a variable gain amplifier, we
control the radio-frequency (r.f.) input power and hence the intensity profile of each laser pulse. To generate a bichromatic light field, the beam is passed through another AOM in single-pass configuration that is driven simultaneously by two r.f. signals with difference frequency $\delta/\pi$. Phase coherence of the laser frequencies is maintained by phase-locking all r.f. sources to an ultra-stable quartz oscillator. We use 1.8 mW of light focused down to a spot size of 14 $\mu$m Gaussian beam waist illuminating both ions from an angle of 45° with equal intensity to achieve the Rabi frequencies $\Omega/(2\pi) \approx 110$ kHz required for performing a gate operation with $(\nu - \delta)/(2\pi) = 20$ kHz and $\eta = 0.044$. To make the bichromatic laser pulses independent of the phase $\zeta$, the pulse is switched on and off by using pulse slopes of duration $\tau_r = 2 \mu s$.

Multiple application of the bichromatic pulse of duration $\tau_{gate}$ ideally maps the state $|SS\rangle$ to

$$
|SS\rangle \xrightarrow{\tau_{gate}} |SS\rangle + i|DD\rangle \xrightarrow{\tau_{gate}} |DD\rangle \xrightarrow{\tau_{gate}} |DD\rangle + i|SS\rangle \xrightarrow{\tau_{gate}} |SS\rangle \xrightarrow{\tau_{gate}} ...
$$

up to global phases. Maximally entangled states occur at instances $\tau_m = m \cdot \tau_{gate}$ ($m = 1, 3, \ldots$). A similar mapping of product states onto Bell states and vice versa also occurs when starting from state $|SD\rangle$. In order to assess the fidelity of the gate operation, we adapt the strategy first applied in refs. $^{11,13}$ consisting in measuring the fidelity of Bell states created by a single application of the gate to the state $|SS\rangle$ (see Fig. 2 (a)). The fidelity $F = \langle \Psi_1 | \rho^{exp} | \Psi_1 \rangle = (\rho^{exp}_{SS,SS} + \rho^{exp}_{DD,DD})/2 + \text{Im} \rho^{exp}_{DD,SS}$, with the density matrix $\rho^{exp}$ describing the experimentally produced qubits’ state, is inferred from measurements on a set of 42,400 Bell states continuously produced within a measurement time of 35 minutes. Fluorescence measurements on 13,000 Bell states reveal that $\rho^{exp}_{SS,SS} + \rho^{exp}_{DD,DD} = p_2 + p_0 = 0.9965(4)$. The off-diagonal element $\rho^{exp}_{DD,SS}$ is determined by measuring $P(\phi) = \langle \sigma_{z}(1) \sigma_{y}(2) \rangle$ for different values of $\phi$, where $\sigma_{\phi} = \sigma_x \cos \phi + \sigma_y \sin \phi$.
by applying $(\frac{\pi}{2})_\phi$-pulses to the remaining 29,400 states and measuring $p_0 + p_2 - p_1$ to obtain the parity $\langle \sigma_1^{(1)} \sigma_2^{(2)} \rangle$. The resulting parity oscillation $P(\phi)$ shown in Fig. 2(b) is fitted with a function $P_{fit}(\phi) = A \sin(2\phi + \phi_0)$ that yields $A = 2|p_{DD,SS}^{exp}| = 0.990(1)$. Combining the two measurements, we obtain the fidelity $F = 99.3(1)\%$ for the Bell state $\Psi_1$.

A wealth of further information is obtained by studying the state dynamics under the action of the gate Hamiltonian. Starting from state $|SS\rangle$, Fig. 3 depicts the time evolution of the state populations for pulse lengths equivalent up to 17 gate times. The ions are entangled and disentangled consecutively up to nine times, the populations closely following the predicted unitary evolution of the propagator (1) for $\zeta = 0$ shown in Fig. 3 as a solid line.

To study sources of gate imperfections we measured the fidelity of Bell states obtained after a pulse length $\tau_m$ for up to $m = 21$ gate operations. The sum of the populations $p_0(t) + p_2(t)$ does not return perfectly to one at times $\tau_m$ as shown in Fig. 4 but decreases by about 0.0022(1) per gate. This linear decrease could be explained by resonant spin flip processes caused by spectral components of the qubit laser that are far outside the laser’s linewidth of 20 Hz (see Methods). The figure also shows the amplitude of parity oscillation scans at odd integer multiples of $\tau_{gate}$ similar to the one in Fig. 2(b). The Gaussian shape of the amplitude decay is consistent with low-frequency noise of the magnetic field and the laser frequency as the source of imperfections (see Methods).

The observed Bell state infidelity of $7 \cdot 10^{-3}$ suggests that the gate operation has an infidelity below the error threshold required by some models of fault-tolerant quantum computation. How-
ever, further experimental advances will be needed before fault-tolerant computation will become a reality as the overhead implied by these models is considerable. Nevertheless, in addition to making the implementation of quantum algorithms with tens of entangling operations look realistic, the gate presented here also opens interesting perspectives for generating multi-particle entanglement by a single laser interacting with more than two qubits at once. For the generation of N-qubit GHZ states, there exist no constraints on the positioning of ions in the bichromatic beam that otherwise makes generation of GHZ states beyond N=6 so difficult in the case of hyperfine qubits. While the bichromatic force lacks a strong spatial modulation that would enable tailoring of the gate interaction by choosing particular ion spacings, more complex multi-qubit interactions could be engineered by interleaving entangling laser pulses addressing all qubits with a focussed laser inducing phase shifts in single qubits. Akin to nuclear magnetic resonance techniques, this method should allow for refocussing of unwanted qubit-qubit interactions and open the door to a wide variety of entangling multi-qubit interactions.
Methods

**AC-Stark-shift compensation** The red- and the blue-detuned frequency components $\omega_\pm$ of the bichromatic light field cause dynamic (ac-) Stark shifts by non-resonant excitation on the carrier and the first-order sidebands that exactly cancel each other if the corresponding laser intensities $I_\pm$ are equal. The remaining ac-Stark shift due to other Zeeman transitions and far-detuned dipole transitions amounts to 7 kHz for a gate time $\tau_{\text{gate}} = 50 \mu s$. These shifts could be compensated by using an additional far-detuned light field\(^\text{27}\) or by properly setting the intensity ratio $I_+/I_-$. We utilize the latter technique which makes the coupling strengths $\Omega_{SS\rightarrow DD} \propto 2\sqrt{I_+I_-}$, $\Omega_{SD\rightarrow DS} \propto I_+ + I_-$ slightly unequal. However, the error is insignificant as $\frac{\Omega_{SD\rightarrow DS}}{\Omega_{SS\rightarrow DD}} - 1 = 4 \cdot 10^{-3}$ in our experiments.

**Sources of gate infidelity** Spin flips induced by incoherent off-resonant light of the bichromatic laser field reduce the gate fidelity. A beat frequency measurement between the gate laser and a similar independent laser system that was spectrally filtered indicates that a fraction $\gamma$ of about $2 \cdot 10^{-7}$ of the total laser power is contained in a 20 kHz bandwidth $B$ around the carrier transition when the laser is tuned close to a motional sideband. A simple model predicts spin flips to cause a gate error with probability $p_{\text{flip}} = (\pi\gamma|\nu - \delta|)/(2\eta^2 B)$. This would correspond to a probability $p_{\text{flip}} = 8 \cdot 10^{-4}$ whereas the measured state populations shown in Fig.4 would be consistent with $p_{\text{flip}} = 2 \cdot 10^{-3}$. Spin flip errors could be further reduced by two orders of magnitude by spectrally filtering the laser light and increasing the trap frequency $\nu$ to above 2 MHz where noise caused by the laser frequency stabilization is much reduced. Variations in the coupling strength $\Omega$ induced by low-frequency laser intensity noise and thermally occupied radial modes can be independently
measured by recording the amplitude decay of carrier oscillations. For $\delta \Omega / \Omega$, we find a variation of $7 \cdot 10^{-3}$. The Gaussian decay of the parity contrast shown in Fig. 4 is attributed to low frequency noise randomly shifting the laser frequency with respect to the atomic transition frequency $\omega_0$ and having a Gaussian distribution with full-width at half maximum of $\Delta \omega / (2\pi) = 180$ Hz. This value is consistent with Ramsey measurements on a single ion predicting $\Delta \omega / (2\pi) = 160$ Hz. For a single gate, the frequency uncertainty gives rise to a fidelity loss of 0.1%.

A bichromatic force with time-dependent $\Omega(t)$ acting on ions prepared in an eigenstate of $S_y$ creates coherent states $\alpha(t)$ following trajectories in phase space that generally do not close$^{19,28}$. For the short rise times used in our experiments, this effect can be made negligibly small by slightly increasing the gate time.
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Addendum

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Figure 1: **Gate mechanism.** (a) A bichromatic laser field with frequencies $\omega_+$, $\omega_-$ satisfying $2\omega_0 = \omega_+ + \omega_-$ is tuned close to the upper and lower motional sideband of the qubit transition. The field couples the qubit states $|SS\rangle \leftrightarrow |DD\rangle$ via the four interfering paths shown in the figure, $n$ denoting the vibrational quantum number of the axial COM mode. Similar processes couple the states $|SD\rangle \leftrightarrow |DS\rangle$. (b) The qubits are encoded in the ground state $S_{1/2}(m = 1/2)$ and the metastable state $D_{5/2}(m = 3/2)$ of $^{40}\text{Ca}$ ions and manipulated by a narrow bandwidth laser emitting at a wavelength of 729 nm.
Figure 2: **High-fidelity gate operation.** (a) State evolution induced by a Mølmer-Sørensen bichromatic pulse of duration \( \tau \). The Rabi frequency \( \Omega(t) \) is smoothly switched on and off within 2 \( \mu s \) and adjusted such that a maximally entangled state is created at \( \tau_{\text{gate}} = 50 \, \mu s \). The dashed lines are calculated for \( \bar{n}_{\text{com}} = 0.05 \) from the propagator \((1)\), neglecting pulse shaping and non-resonant carrier excitation. The solid lines are obtained from numerically solving the Schrödinger equation for time-dependent \( \Omega(t) \) and imbalanced Rabi frequencies \( \Omega_+ / \Omega_- = 1.094 \) (see Methods).

(b) A \( (\frac{\pi}{2})_p \) analysis pulse applied to both ions prepared in \( \Psi_1 \) gives rise to a parity oscillation \( P(\phi) = \sin(2\phi) \) as a function of \( \phi \). A fit with a function \( P_{\text{fit}} = A \sin(2\phi + \phi_0) \) yields the parity oscillation amplitude \( A = 0.990(1) \) and \( \phi_0 / \pi = -1.253(1) \). The precise value of the phase \( \phi_0 \) is without significance. It arises from phase-locking the frequencies \( \omega_0, \omega_+, \omega_- \) and could have been experimentally adjusted to zero.
Figure 3: Entanglement and disentanglement dynamics of the Mølmer-Sørensen interaction.

Starting from state $|SS\rangle$ for a detuning of the bichromatic laser from the sidebands set to $\delta - \nu = -20$ kHz, the figure shows the time evolution of the populations $p_0$, $p_1$, and $p_2$ denoted by the symbols ($\bullet$), ($\diamond$), and ($\triangle$), respectively. The length of the pulse is equivalent to the application of up to 17 gate operations. Maximally entangled states are created whenever $p_0(\tau)$ and $p_2(\tau)$ coincide and $p_1(\tau)$ vanishes.
Figure 4: **Multiple gate operations.** Gate imperfections as a function of the bichromatic pulse length $\tau_m = m \cdot \tau_{gate}$ given in equivalent number of gate operations $m$. The upper curve shows a linear decrease of the state populations $p_0 + p_2$ with a slope of 0.0022(1). The lower curve shows the magnitude of the coherence $2\rho_{DD,SS}$ measured by detecting parity oscillations and fitted by a Gaussian decay function that accounts for low-frequency noise of the laser frequency and the magnetic field. Combining both measurements yields the Bell state fidelity $F_m$ shown as the middle trace. For $m = 21$, the fidelity is still $F_{21} = 80(1)\%$. Similar results are achieved when replacing the entangling pulse of length $\tau_m$ by $m$ amplitude-shaped pulses each of which is realizing an entangling gate operation.