TOPOLOGY OF PARETO SETS OF STRONGLY CONVEX PROBLEMS

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Dedicated to Professor Takashi Nishimura on the occasion of his 60th birthday.

Abstract. A multiobjective optimization problem is simplicial if the Pareto set and front are homeomorphic to a simplex and, under the homeomorphisms, each face of the simplex corresponds to the Pareto set and front of a subproblem. In this paper, we show that strongly convex problems are simplicial under a mild assumption on the ranks of differentials of the objective mapping. We further prove that one can make any strongly convex problem satisfy the assumption by a generic linear perturbation, provided that the dimension of the source is sufficiently larger than that of the target. We also show that several examples of practical problems can be reduced to strongly convex ones via transformations preserving the Pareto ordering and the topology.

1. Introduction

Multiobjective optimization is widely used in science and engineering. Encouraged by the rapid growth of computation power, practitioners now get trying to find the entire Pareto set (rather than a single Pareto solution), and extract the underlying trade-off between conflicting objective functions [1].

In general it becomes easier to obtain the entire Pareto set once we can find that it has a simple topological structure. For example, an algorithm to extend a given Pareto point to the entire connected Pareto set is given in [4]. Furthermore, we can efficiently obtain the entire Pareto set, provided that the problem is simplicial [3, 7].

The reader can refer to the precise definition of simplicial problems in section 2.2. Figure 1 describes an example of a simplicial problem with three objective functions $f_1, f_2, f_3$. As is seen in the figure, the Pareto set of a simplicial problem is homeomorphic to a simplex, and its faces are the Pareto sets of subproblems.

Simplicial problems appear in practical situations. In 1967, Kuhn [8] showed that the Pareto set of a facility location problem under the Euclidean norm is the convex hull of demand points, which becomes a simplex when the demand points are in general position. Moreover, many other problems seem to be simplicial: Shoal et al. [12] proposed a multiobjective model for phenotypic divergence of species in evolutionary biology and indicated that its Pareto set is a curved simplex. Smale [13] asserted (without proof) that the Pareto set of a pure exchange economy (under some conditions on utility functions) is homeomorphic to a simplex.

As shown in the previous paragraph, there are plenty of practical problems which are presumably simplicial, and this observation will be indeed justified by the main theorem below:
Theorem 1.1. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a strongly convex $C^r$–mapping ($2 \leq r \leq \infty$). The multi-objective optimization problem minimizing $f$ is $C^{r-1}$–weakly simplicial. Furthermore, this problem is $C^{r-1}$–simplicial if the corank of the differential $df_x$ is equal to 1 for any $x \in X^*(f)$.

For the definitions of strong convexity and weak simpliciality, see section 2. Note that some of the problems in the previous paragraph become strongly convex after suitable transformations of the target space preserving the Pareto ordering (see section 5).

The paper is organized as follows: The proof of the main theorem will be given in section 3. In section 4 we will show that any strongly convex problem becomes simplicial after a generic linear perturbation, provided that the dimension of the source is sufficiently larger than that of the target. Section 5 will be devoted to discussing practical problems. Although these problems are not strongly convex, we will observe that one can make them strongly convex by applying suitable homeomorphisms preserving the Pareto ordering.

2. Preliminaries

We introduce the definition of strongly convex problems and their properties and define $C^r$–(weakly) simplicial problems. Throughout the paper, we denote the index set $\{1, \ldots, m\}$ by $M$.

2.1. Multiobjective optimization. A multiobjective optimization problem is a problem minimizing objective functions $f_1, \ldots, f_m : X \rightarrow \mathbb{R}$ over a domain $X \subseteq \mathbb{R}^n$:

\[
\begin{align*}
\text{minimize } & \quad f(x) = (f_1(x), \ldots, f_m(x)) \\
\text{subject to } & \quad x \in X(\subseteq \mathbb{R}^n).
\end{align*}
\]

According to the Pareto ordering, i.e.,

\[
f(x) < f(y) \overset{\text{def}}{=} \forall i \in M, f_i(x) \leq f_i(y) \land \exists j \in M, \text{ s.t. } f_j(x) < f_j(y),
\]

we basically would like to obtain the Pareto set

\[
X^*(f) = \{ x \in X \mid \forall y \in X, f(y) \neq f(x) \}.
\]
and the Pareto front
\[ f(\mathbf{x}^*(f)) = \{ f(x) \in \mathbb{R}^m \mid x \in X^*(f) \}. \]

2.2. Simplicial problems. Here, we explain the definition of \( C^r \)- (weakly) simplicial problems for \( 0 \leq r \leq \infty \). For \( \varepsilon \geq 0 \), we define the subset \( \Delta_{\varepsilon}^{m-1} \subseteq \mathbb{R}^m \) as follows:
\[
\Delta_{\varepsilon}^{m-1} = \left\{ (w_1, \ldots, w_m) \in \mathbb{R}^m \mid \sum_{i=1}^m w_i = 1, \ w_i > -\varepsilon \right\} \subsetneq \mathbb{R}^m.
\]

Note that the closure \( \overline{\Delta_{\varepsilon}^{m-1}} \) is the standard simplex, which we will denote by
\[
\Delta^{m-1} = \left\{ (w_1, \ldots, w_m) \in \mathbb{R}^m \mid \sum_{i=1}^m w_i = 1, \ w_i \geq 0 \right\} \subsetneq \mathbb{R}^m.
\]
We also denote a face of \( \Delta^{m-1} \) for \( I \subseteq M \) by
\[
\Delta_I = \left\{ (w_1, \ldots, w_m) \in \Delta^{m-1} \mid w_i = 0 \ (i \notin I) \right\} \subsetneq \mathbb{R}^m.
\]

For a subset \( U \subseteq \mathbb{R}^m \), a continuous mapping \( f : \Delta^{m-1} \to U \) is a \( C^r \)-mapping if there exist \( \varepsilon > 0 \) and a \( C^r \)-mapping \( f : \Delta_{\varepsilon}^{m-1} \to \mathbb{R}^m \) satisfying \( f|_{\Delta_{\varepsilon}^{m-1}} = f \). A subspace \( X \subseteq \mathbb{R}^m \) is \( C^r \)-diffeomorphic to the simplex \( \Delta^{m-1} \) if there exist \( \varepsilon > 0 \) and a \( C^r \)-immersion \( \phi : \Delta^{m-1} \to \mathbb{R}^m \) such that \( \phi|_{\Delta^{m-1}} : \Delta^{m-1} \to X \) is a homeomorphism. The reader can refer to [9, §2] for more general definition of diffeomorphisms between manifolds with corners.

**Definition 2.1.** Let \( X \) be a subset of \( \mathbb{R}^n \) and \( f = (f_1, \ldots, f_m) \) be a mapping from \( X \) to \( \mathbb{R}^m \). For \( I = \{i_1, \ldots, i_k\} \subseteq M \) such that \( i_1 < \cdots < i_k \), we put \( f_I = (f_{i_1}, \ldots, f_{i_k}) \). The problem minimizing \( f \) is \( C^r \)-simplicial if there exists a \( C^r \)-mapping \( \Phi : \Delta^{m-1} \to X^*(f) \) such that both of the restrictions \( \Phi|_{\Delta_I} : \Delta_I \to X^*(f_I) \) and \( f|_{X^*(f_I)} \) are \( C^r \)-diffeomorphisms for any \( I \subseteq M \). The problem minimizing \( f \) is \( C^r \)-weakly simplicial if there exists a \( C^r \)-mapping \( \phi : \Delta^{m-1} \to f(X^*(f)) \) satisfying \( \phi(\Delta_I) = f(X^*(f_I)) \) for any \( I \subseteq M \).

2.3. Pareto optimal solutions of strongly convex mappings. In section 2.3, a characterization of Pareto optimal solutions of strongly convex \( C^1 \)-mappings is given (see proposition 2.5). We begin this subsection with quickly reviewing the definition of (strong) convexity. A subset \( X \) of \( \mathbb{R}^n \) is convex if \( (1-t)x + ty \in X \) for all \( x, y \in X \) and all \( t \in [0, 1] \). Let \( X \) be a convex set in \( \mathbb{R}^n \). A function \( f : X \to \mathbb{R} \) is convex if
\[
(f(tx) + (1-t)y) - f(x) \leq f((1-t)x + ty)
\]
for all \( x, y \in X \) and all \( t \in [0, 1] \). A function \( f : X \to \mathbb{R} \) is strongly convex if there exists \( \alpha > 0 \) such that
\[
f(tx + (1-t)y) - f(x) - (1-t)f(y) \leq \frac{1}{2} \alpha(t^2 - 1) ||x - y||^2
\]
for all \( x, y \in X \) and all \( t \in [0, 1] \), where \( ||x - y|| \) denotes the Euclidian norm of \( x - y \). The constant \( \alpha \) is called a convexity parameter of the function \( f \). A mapping \( f = (f_1, \ldots, f_m) : X \to \mathbb{R}^m \) is (strongly) convex if every \( f_i \) is (strongly) convex. A problem minimizing a strongly convex mapping is called a strongly convex problem.

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The usual definition of a \( C^r \)-mapping on a manifold with corners is slightly different from that given here: the latter is stronger (as a condition) than the former.
The followings are basic properties of (strongly) convex mappings which will be needed later on:

**Lemma 2.2** ([11, Theorem 2.1.2 (p. 54)]). Let \( X \subseteq \mathbb{R}^n \) be a convex open subset. A \( C^1 \)-function \( f : X \to \mathbb{R} \) is convex if and only if \( f(x) + df_x \cdot (y - x) \leq f(y) \) for any \( x, y \in X \).

**Lemma 2.3** ([11, Theorem 2.1.11 (p. 65)]). Let \( X \subseteq \mathbb{R}^n \) be a convex open subset. A \( C^2 \)-function \( f : X \to \mathbb{R} \) is strongly convex if and only if there exists \( \beta > 0 \) such that \( m(f)_x \geq \beta \) for any \( x \in X \), where \( m(f)_x \) is the minimal eigenvalue of the Hessian matrix of \( f \) at \( x \).

**Lemma 2.4** ([11, Theorem 2.2.6 (p. 85)]). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a strongly convex \( C^1 \)-function. Then, there exists a unique point such that the function \( f \) is minimized.

In the rest of this subsection we will prove the following proposition:

**Proposition 2.5.** Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a strongly convex \( C^1 \)-mapping. Then, there exists a unique point such that the function \( f \) is minimized.

For the proof, we prepare some lemmas.

**Lemma 2.6** ([10, Theorem 3.1.3 (p. 79)]). Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a mapping and let \((w_1, \ldots, w_m) \in \Delta^{m-1}\) be an element. If \( x \in \mathbb{R}^n \) is a unique element such that the function \( \sum_{i=1}^{m} w_i f_i \) is minimized, then \( x \in X^*(f) \).

The following is a special case of the Karush-Kuhn-Tucker necessary condition for Pareto optimality.

**Lemma 2.7** ([10, Theorem 3.1.5 (p. 39)]). Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a \( C^1 \)-mapping. If \( x \in X^*(f) \), then there exists an element \((w_1, \ldots, w_m) \in \Delta^{m-1}\) satisfying \( \sum_{i=1}^{m} w_i (df_{i})_x = 0 \).

**Lemma 2.8.** Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a convex \( C^1 \)-mapping. Let \((w_1, \ldots, w_m) \in \Delta^{m-1}\) be an element. Then, the following conditions for \( x \in \mathbb{R}^n \) are equivalent.

1. \( \sum_{i=1}^{m} w_i (df_{i})_x = 0 \).
2. The function \( \sum_{i=1}^{m} w_i f_i : \mathbb{R}^n \to \mathbb{R} \) attains its minimum at \( x \).
Proof of lemma 2.8. Set $g = \sum_{i=1}^{m} w_i f_i$. Then, for any $x \in \mathbb{R}^n$, we have

\begin{equation}
\sum_{i=1}^{m} w_i (df_i)_x = dg_x.
\end{equation}

Since $g$ is convex, we can deduce from Lemma 2.2 that the following inequality holds for any $y \in \mathbb{R}^n$:

\begin{equation}
g(x) + dg_x : (y - x) \leq g(y).
\end{equation}

Suppose that $\sum_{i=1}^{m} w_i (df_i)_x = 0$. We can easily deduce the assertion 2 from (1) and (2). Suppose that $\sum_{i=1}^{m} w_i f_i : \mathbb{R}^n \to \mathbb{R}$ attains its minimum at $x$. Since $dg_x$ is equal to 0 and the equality (1) holds, we have the assertion 1.

Proof of proposition 2.5. Suppose that $x \in X^* (f)$. Using lemma 2.7, we can verify that there exists an element $(w_1, \ldots, w_m) \in \Delta^{m-1}$ satisfying $\sum_{i=1}^{m} w_i (df_i)_x = 0$. From lemma 2.8, the point $x \in \mathbb{R}^n$ is an element such that $\sum_{i=1}^{m} w_i f_i$ is minimized. Since $\sum_{i=1}^{m} w_i f_i$ is strongly convex $C^1$–function, by lemma 2.4, we have the assertion 2. Finally, suppose the assertion 2. Then, from lemma 2.6, we get $x \in X^* (f)$.

2.4. Fold singularities. In this subsection we will briefly review the definition and basic properties of fold singularities (for details, see [2]). For $0 \leq k \leq \min \{ n, m \}$, we define a subset $S_k \subsetneq J^1(\mathbb{R}^n, \mathbb{R}^m)$ as follows:

$$S_k = \left\{ j^1 g(x) \in J^1(\mathbb{R}^n, \mathbb{R}^m) \mid x \in \mathbb{R}^n, g : \mathbb{R}^n \to \mathbb{R}^m : C^2 \text{–mapping}, \min \{ n, m \} - \text{rank} (dg_x) = k \right\},$$

where $j^1 g : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R}^m)$ is the 1–jet extension of $g$. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be a $C^2$–mapping, $S \subset J^1(\mathbb{R}^n, \mathbb{R}^m)$ be a submanifold, and $x \in \mathbb{R}^n$. The mapping $j^1 g$ is transverse to $S$ at $x$ if either of the following conditions holds:

- $j^1 g(x)$ is not contained in $S$,
- $j^1 g(x) \in S$ and $d(j^1 g)_x (T_x \mathbb{R}^n) + T_{j^1 g(x)} S = T_{j^1 g(x)} J^1(\mathbb{R}^n, \mathbb{R}^m)$.

The mapping $j^1 g$ is transverse to $S$ if it is transverse to $S$ at any point in $\mathbb{R}^n$.

Suppose that $n$ is greater than or equal to $m$. For a $C^2$–mapping $f : \mathbb{R}^n \to \mathbb{R}^m$, we denote the critical point set of $f$ by $\text{Crit}(f) \subseteq \mathbb{R}^n$. A point $x \in \text{Crit}(f)$ is called a fold if the following conditions hold:

1. $j^1 f$ is transverse to $S_1$ at $x_0$.
2. $T_{x_0} S_1 (f) \oplus \ker df_{x_0} = T_{x_0} \mathbb{R}^n$, where $S_1 (f) := (j^1 f)^{-1} (S_1)$.

Note that we can easily deduce from the condition 2 that the restriction $f|_{\text{Crit}(f)}$ is an immersion around a fold.
Remark 2.9. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a \( C^\infty \)–mapping and \( x \in \text{Crit}(f) \) be a fold. One can take coordinate neighborhoods \((U, \varphi)\) and \((V, \psi)\) at \( x \) and \( f(x) \), respectively, so that they satisfy:

\[
\psi \circ f \circ \varphi^{-1}(x_1, \ldots, x_n) = \left( x_1, \ldots, x_{m-1}, \sum_{k=m}^{n} \pm x_k^2 \right).
\]

In what follows we will give a useful criterion for detecting fold singularities. Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a \( C^2 \)–mapping and \( x_0 \in \text{Crit}(f) \). Suppose that the corank\(^2\) of \( df_{x_0} \) is 1 and the matrix \( \left( \frac{\partial f_i}{\partial x_j}(x_0) \right)_{1 \leq i, j \leq m-1} \) is regular. We define the function \( \lambda_f : \mathbb{R}^n \to \mathbb{R}^{n-m+1} \) as follows:

\[
\lambda_f(x) = (J_1(x), \ldots, J_{n-m+1}(x)),
\]

where

\[
J_i(x) = \det \left( \begin{array}{ccc}
\frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_1}(x) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_1}{\partial x_{m-1}}(x) & \cdots & \frac{\partial f_m}{\partial x_{m-1}}(x) \\
\frac{\partial f_1}{\partial x_{m-1+i}}(x) & \cdots & \frac{\partial f_m}{\partial x_{m-1+i}}(x)
\end{array} \right).
\]

Lemma 2.10. Under the situation above, \( x_0 \) is a fold if and only if the following conditions hold:

1. the differential \( (d\lambda_f)_{x_0} \) has rank \( n - m + 1 \),
2. \( \ker(d\lambda_f)_{x_0} \oplus \ker df_{x_0} = T_{x_0} \mathbb{R}^n \).

Remark 2.11. The two conditions in lemma 2.10 are equivalent to those in the original definition above. Indeed, the first condition is equivalent to the condition that \( j^1 f \) is transverse to \( S_1 \) at \( x_0 \), and \( T_{x_0} S_1(f) \) is equal to \( \ker(d\lambda_f)_{x_0} \).

3. PROOF OF THE MAIN RESULT

In this section we will show that strongly convex problems are simplicial. Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m \) be a strongly convex \( C^r \)–mapping \((2 \leq r \leq \infty)\). Since \( \sum_{i=1}^{m} w_i f_i \) is strongly convex for any \((w_1, \ldots, w_m) \in \Delta^{m-1}\), there exists a unique point \( x \in \mathbb{R}^n \) such that \( \sum_{i=1}^{m} w_i f_i \) is minimized (see lemma 2.4). We denote this minimizing point by \( \arg \min_{x \in \mathbb{R}^n} \left( \sum_{i=1}^{m} w_i f_i(x) \right) \in \mathbb{R}^n \), which is contained in \( X^*(f) \) by lemma 2.6. We can thus define a mapping \( x^* : \Delta^{m-1} \to X^*(f) \) as follows:

\[
x^*(w) = \arg \min_{x \in \mathbb{R}^n} \left( \sum_{i=1}^{m} w_i f_i(x) \right).
\]

\(^2\)For a linear mapping \( \varphi : V \to W \) the non-negative number \( \dim W - \text{rank}(\varphi) \) is called the corank of \( \varphi \).
Theorem 3.1. Let \( f = (f_1, \ldots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a strongly convex \( C^r \)-mapping \((2 \leq r \leq \infty)\).

1. The mapping \( x^* : \Delta^{m-1} \rightarrow X^*(f) \) (and thus \( f \circ x^* : \Delta^{m-1} \rightarrow f(X^*(f)) \)) is a surjective mapping of class \( C^{r-1} \).

2. Suppose that the corank of \( df_x \) is equal to 1 for any \( x \in X^*(f) \).
   
   A. The mapping \( x^* : \Delta^{m-1} \rightarrow X^*(f) \) is a \( C^{r-1} \)-diffeomorphism.
   
   B. The restriction \( f|_{X^*(f)} : X^*(f) \rightarrow \mathbb{R}^m \) is a \( C^{r-1} \)-embedding.

Note that this theorem obviously holds for \( m = 1 \). For this reason, in the rest of this section we will assume \( m \geq 2 \).

Theorem 1.1 follows from this theorem as follows: It is easy to see that any subproblem of a strongly convex problem is again strongly convex. In particular, by applying theorem 3.1 to each subproblem, we can show that the image of the restriction \( x^* \) on \( \Delta_I \) is equal to \( X^*(f_I) \) for any \( I \subseteq M \). Thus a strongly convex problem is weakly simplicial. We can further deduce from 2 of theorem 3.1 that a strongly convex problem is simplicial under the assumption on the coranks of differentials.

Remark 3.2. The corank assumption in 2 implies that \( n \) is greater than or equal to \( m - 1 \). As we will show in the proof, under this assumption any point in \( X^*(f) \) for a mapping \( f \) is a fold if \( n \geq m \).

Remark 3.3. In general, the mapping \( x^* \) for a strongly convex problem (without the corank assumption) is not necessarily a diffeomorphism. We will give an explicit example of such a problem with a non-injective \( x^* \) in example 3.4.

Proof of 1 in theorem 3.1. First of all, we can immediately deduce from proposition 2.5 that \( x^* \) is surjective. Let \( d' > 0 \) be a positive number and \( g : \Delta^{m-1} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a \( C^{r-1} \)-mapping defined by \( g(w, x) = \sum_{k=1}^{m} w_k (df_k)_x \). We can easily deduce from lemma 2.8 that \( x^* \) is an implicit function of the equation \( g(w, x) = 0 \) defined on \( \Delta^{m-1} \). Differentiating \( g \), we have

\[
\frac{\partial g_k}{\partial x_j} = \sum_{k=1}^{m} w_k \frac{\partial f_k}{\partial x_i} \frac{\partial f_k}{\partial x_j}.
\]

Thus the matrix \( \left( \frac{\partial g_i}{\partial x_j} \right)_{1 \leq i,j \leq n} \) is equal to \( m \sum_{k=1}^{m} w_k H(f_k)_x \), where \( H(f_k)_x \) is the Hessian matrix of \( f_k \) at \( x \). Since \( f_k \) is strongly convex, the Hessian matrix \( H(f_k)_x \) is positive definite (see lemma 2.3). Thus, the matrix \( \left( \frac{\partial g_i}{\partial x_j} \right)_{1 \leq i,j \leq n} \) is regular on \( \Delta^{m-1} \). By the implicit function theorem, for any \( y \in \Delta^{m-1} \) there exists an open neighborhood \( U_y \subseteq \Delta^{m-1} \) and a (unique) \( C^{r-1} \)-mapping \( x^*_y : U_y \rightarrow \mathbb{R}^n \) such that \( x^*_y(y) = x^*(y) \) and \( g(w, x^*_y(w)) = 0 \) for any \( w \in U_y \). We can further deduce from uniqueness of an implicit function that \( x^*_y \) coincides with \( x^*_{y'} \) on \( U_y \cap U_{y'} \) for distinct \( y, y' \in \Delta^{m-1} \). Since \( U = \bigcup_{y \in \Delta^{m-1}} U_y \) is an open neighborhood of \( \Delta^{m-1} \), one can

\[\text{It is indeed a “definite” fold.}\]
take $\delta < \delta'$ so that $\Delta^{m-1}_\delta$ is contained in $U$\footnote{If not, we can take $x_n \in U^c \cap \Delta^{m-1}_{1/n}$ for any $n \in \mathbb{N}$. Since $\Delta^{m-1}_1$ is compact, $\{x_n\}_{n \in \mathbb{N}}$ has a cluster point $x$, which is contained in $\Delta^{m-1}_1$. However, $x$ is also contained in $U^c$ since it is closed, contradicting the fact that $\Delta^{m-1}_1 \not\subseteq U$.}. We can define $\tilde{x}^* : \Delta^{m-1}_0 \to \mathbb{R}^n$ by $\tilde{x}^*(w) = x^*_y(w)$ for $w \in U_y$. It is easy to see that $\tilde{x}^*$ is a $C^{r-1}$-mapping and is an extension of $x^*$. Thus $x^*$ is a $C^{r-1}$-mapping. \hfill $\square$

**Proof of 2.A in theorem 3.1.** For $\varepsilon \geq 0$, we define a subset $D^{m-1}_\varepsilon \subsetneq \mathbb{R}^{m-1}$ by

$$D^{m-1}_\varepsilon = \left\{(z_1, \ldots, z_{m-1}) \in \mathbb{R}^{m-1} \mid \sum_i z_i < 1 + \varepsilon, \; z_i > -\varepsilon \right\}.$$  

We denote the closure $\overline{D^{m-1}_0}$ by $D^{m-1}$. It is easy to check that the projection $p : \Delta^{m-1}_0 \to D^{m-1}_0$ defined by $p(w_1, \ldots, w_m) = (w_1, \ldots, w_{m-1})$ is a diffeomorphism. In what follows we will identify $\Delta^{m-1}_0$ with $D^{m-1}_0$ by $p$.

Since $x^*$ constructed in the proof above is an implicit function of the equation $g(w, x) = 0$, the following equality holds:

$$0 = \sum_{k=1}^m w_k(df_k)_{\tilde{x}^*(w)} = \sum_{k=1}^{m-1} z_k(df_k)_{\tilde{x}^*(z)} + \left(1 - \sum_{k=1}^{m-1} z_k\right)(df_m)_{\tilde{x}^*(z)},$$  

where $(z_1, \ldots, z_{m-1}) \in D^{m-1}_\varepsilon$ is a point corresponding to $w \in \Delta^{m-1}_0$. Differentiating the both sides of the equation above by $z_j$ ($j = 1, \ldots, m-1$), we obtain the following equality:

$$0 = \sum_{k=1}^{m-1} z_k \left(H(f_k)_{\tilde{x}^*(z)} \frac{\partial x^*}{\partial z_j}\right) + (df_j)_{\tilde{x}^*(z)} + \left(1 - \sum_{k=1}^{m-1} z_k\right) \left(H(f_m)_{\tilde{x}^*(z)} \frac{\partial x^*}{\partial z_j}\right) - (df_m)_{\tilde{x}^*(z)}.$$  

We thus obtain:

$$\frac{\partial x^*}{\partial z_j} = -\left(\sum_{k=1}^{m-1} z_k H(f_k)_{\tilde{x}^*(z)} + \left(1 - \sum_{k=1}^{m-1} z_k\right) H(f_m)_{\tilde{x}^*(z)}\right)^{-1} \left((df_j)_{\tilde{x}^*(z)} - (df_m)_{\tilde{x}^*(z)}\right),$$  

where we denote the matrix $\left(\sum_{k=1}^{m-1} z_k H(f_k)_{\tilde{x}^*(z)} + \left(1 - \sum_{k=1}^{m-1} z_k\right) H(f_m)_{\tilde{x}^*(z)}\right)^{-1}$ by $A(z)$, which is positive definite for $z \in D^{m-1}$. Since $D^{m-1}$ is compact and the corank of $df_{\tilde{x}^*(z)}$ is equal to 1 for any $z \in D^{m-1}$, by retaking $\delta$ if necessary, we can assume that the corank of $df_{\tilde{x}^*(z)}$ is equal to 1 and $A(z)$ is positive definite, and thus regular, for any $z \in D^{m-1}_\delta$. (Note that the condition $\text{corank}(df_x) = 1$ is an open condition in $\text{Crit}(f)$.)

We will show that the matrix

$$(df_1)_{\tilde{x}^*(z)} - (df_m)_{\tilde{x}^*(z)} \quad \cdots \quad (df_{m-1})_{\tilde{x}^*(z)} - (df_m)_{\tilde{x}^*(z)}$$

$$\left(\sum_{k=1}^{m-1} z_k H(f_k)_{\tilde{x}^*(z)} + \left(1 - \sum_{k=1}^{m-1} z_k\right) H(f_m)_{\tilde{x}^*(z)}\right)^{-1}$$

is a $C^{r-1}$-mapping and is an extension of $x^*$. Thus $x^*$ is a $C^{r-1}$-mapping.
has rank $m - 1$ for any $z \in D^{m - 1}$. If not so, there exists $a_i \in \mathbb{R}$ with $(a_1, \ldots, a_{m - 1}) \neq 0$ such that

$$\sum_{i=1}^{m-1} a_i \left( (df_i)^\ast_{x^\ast}(z) - (df_m)^\ast_{x^\ast}(z) \right) = 0.$$ 

On the other hand, by the definition of $x^\ast$, we obtain

$$\sum_{j=1}^{m-1} z_j (df_j)^\ast_{x^\ast}(z) + \left( 1 - \sum_{i=1}^{m-1} z_i \right) (df_m)^\ast_{x^\ast}(z) = 0.$$ 

Thus, both of the vectors $\left( z_1, \ldots, z_{m-1}, 1 - \sum_{i=1}^{m-1} z_i \right)$ and $(a_1, \ldots, a_{m-1}, -\sum_{i=1}^{m-1} a_i)$ are contained in $\ker df_{x^\ast}(z)$. However, these are linearly independent and contradict the assumption $\text{corank} \left( df_{x^\ast}(z) \right) = 1$. Therefore, the matrix in (3) has rank $m - 1$ for any $z \in D^{m - 1}$. Since the condition that the matrix in (3) has rank $m - 1$ is an open condition for $z$, we can assume that this condition holds for any $D^{m - 1}$ by making $\delta$ sufficiently small. The differential

$$dx^\ast z = -A(z) \left( (df_1)^\ast_{x^\ast}(z) - (df_m)^\ast_{x^\ast}(z), \ldots, (df_{m-1})^\ast_{x^\ast}(z) - (df_m)^\ast_{x^\ast}(z) \right)$$

also has rank $m - 1$ since $A(z)$ is regular for any $z \in D^{m - 1}$.

We next show that the mapping $x^\ast$ is injective. Assume that $x^\ast(w)$ is equal to $x^\ast(w')$ for $w, w' \in \Delta^{m - 1}$. Since the corank of $df_{x^\ast(w)}$ is equal to 1 and $\sum_{j=1}^{m} w_j (df_j)_{x^\ast(w)} = 0$, we can obtain $\text{Im}(df_{x^\ast(w)}) = \langle w \rangle^\perp$. In the same way, we can also prove that $\text{Im}(df_{x^\ast(w')})$ is equal to $\langle w' \rangle^\perp$. From the assumption, we can deduce that $\langle w \rangle^\perp$ is equal to $\langle w' \rangle^\perp$ and thus $w = w'$.

We have shown that $x^\ast$ is an injective immersion. Since $\Delta^{m - 1}$ is compact, $x^\ast$ is a homeomorphism and thus a diffeomorphism to its image, which is equal to $X^\ast(f)$. \hfill \Box

**Proof of 2.B in theorem 3.1** We first prove that $f|_{X^\ast(f)}$ is injective. Let $w, z \in \Delta^{m - 1}$, $x = x^\ast(w)$ and $y = x^\ast(z)$. Suppose that $f(x)$ is equal to $f(y)$. Then

$$\left( \sum_{i=1}^{m} w_i f_i \right) (x)$$

is also equal to

$$\left( \sum_{i=1}^{m} w_i f_i \right) (y).$$

Since the function $\sum_{i=1}^{m} w_i f_i$ is strongly convex, the point minimizing $\sum_{i=1}^{m} w_i f_i$ is unique (see lemma 2.4). Thus, $x$ is equal to $y$.

As we noted in remark 3.2, the corank assumption implies that $n$ is greater than or equal to $m - 1$. If $n = m - 1$, this assumption further implies that $f$ is an immersion at any point in $X^\ast(f)$. The restriction $f|_{X^\ast(f)}$ is thus an embedding since any injective immersion on a compact manifold is an embedding. In what follows we will assume $n \geq m$.

We next show that any point $x \in X^\ast(f)$ is a fold of $f$. The following transformations preserve strong convexity of $f$:

- $(f_1, \ldots, f_m) \mapsto (f_{\sigma(1)}, \ldots, f_{\sigma(m)}) \quad (\sigma \in S_m)$,
- $(f_1, \ldots, f_m) \mapsto (f_1, \ldots, f_m + \alpha f_i) \quad (\alpha > 0, i = 1, \ldots, m - 1)$, and
• linear transformations of the source of $f$,

where $\mathcal{S}_m$ is the symmetric group of degree $m$. By applying these transformations if necessary, we can assume the followings:

$$(df_m)_x = 0, \quad \left(\frac{\partial f_i}{\partial x_j}(x)\right)_{1 \leq i, j \leq m-1} = I_{m-1} \quad \text{and} \quad \frac{\partial f_i}{\partial x_j}(x) = 0 \quad (i = 1, \ldots, m, \quad j = m, \ldots, n).$$

Let $\lambda_f : \mathbb{R}^n \rightarrow \mathbb{R}^{n-m+1}$ be the mapping defined in section 2.4. By lemma 2.10 it suffices to show the followings:

(A) the rank of $(d\lambda_f)_x$ is $n-m+1$,

(B) $\ker(d\lambda_f)_x \oplus \ker df_x = T_x\mathbb{R}^n$.

By the assumptions above, we can calculate $(d\lambda_f)_x$ as follows:

$$(d\lambda_f)_x = \pm \left(\frac{\partial^2 f_m}{\partial x_j \partial x_{m-1+i}}(x)\right)_{1 \leq i \leq n-m+1, 1 \leq j \leq n} = \pm \left(0_{(n-m+1) \times (m-1)} \bigg| I_{n-m+1}\right) H(f_m)_x.$$

Since $f_m$ is strongly convex, the Hessian matrix $H(f_m)_x$ is positive definite, in particular regular by lemma 2.3. Thus, the condition (A) holds. The above calculation also implies the following equality:

$$\ker(d\lambda_f)_x = \langle H(f_m)_x^{-1} e_1, \ldots, H(f_m)_x^{-1} e_{m-1}\rangle.$$ 

Let $v \in \ker(d\lambda_f)_x \cap \ker(df)_x$. From the equality above, we can find $w \in \mathbb{R}^{m-1} \times \{0\} \subseteq \mathbb{R}^m$ such that $v = H(f_m)_x^{-1} w$. Thus the following holds:

$$0 = df_x(v) = \left(I_{m-1} \begin{array}{c} 0 \end{array} \right) H(f_m)_x^{-1} w.$$

We can deduce the following from this equality:

$$^{t}w H(f_m)_x^{-1} w = 0.$$ 

Since $H(f_m)$ is positive definite by lemma 2.3, $w$ is equal to 0. Since the corank of $df_x$ is equal to 1, we can deduce the following from the condition (A):

$$\dim \ker(d\lambda_f)_x + \dim \ker(df_x) = n.$$

Thus the condition (B) also holds. We can eventually conclude that $f|_{X^*(f)}$ is an immersion. \hfill \Box

3.1. Examples. One of the most simple and representative instances of strongly convex problem is the multi-objective facility location problem under the Euclidian norm. It is well known that the Pareto set (resp. the Pareto front) of this problem is a convex hull of minimizing points (resp. their values) of individual objective functions [8]. Thus, if these minimizing points are in general position, then the convex hull becomes a simplex and this problem is a $C^0$-simplicial problem.

In this section we will show that in the strongly convex case, the condition that minimizing points are in general position is no longer necessary nor sufficient to ensure $C^0$-simpliciality, and the corank assumption is still essential to determine the topology of the Pareto set and the Pareto front. To this end, we will give two examples of strongly convex mappings from $\mathbb{R}^3$ to $\mathbb{R}^3$, and discuss the configurations of Pareto sets of them. As we mentioned in the beginning of section 3.1 for any strongly convex mapping $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ we can define a mapping $x^* : \Delta^2 \rightarrow X^*(f)$. The first example (given in example 3.4) has a corank 2 differential at a point in the Pareto
set, and the corresponding $x^*$ is not a diffeomorphism (despite the fact that the minimizing points of the three component functions are in general position). This example implies that we cannot drop the corank assumption in 2 of theorem 3.1. The second example (given in example 3.5) satisfies the corank assumption, and thus the corresponding $x^*$ is a diffeomorphism (although the minimizing points of the three component functions are not in general position).

**Example 3.4** (general position with corank 2). We define a mapping $f = (f_1, f_2, f_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as follows:

$$f_1(x, y, z) = x^2 + y^2 + z^2,$$
$$f_2(x, y, z) = x + y + x^2 + y^2 + z^2,$$
$$f_3(x, y, z) = -(x + y) + x^2 + 2y^2 + z^2.$$

The mapping $f$ is strongly convex. We will check that $X^*(f)$ contains a singularity of corank 2, and is not diffeomorphic to $\Delta^2$. The differentials at $p = (x, y, z) \in \mathbb{R}^3$ are

$$df_{1,p} = (2x, 2y, 2z)$$
$$df_{2,p} = (1 + 2x, 1 + 2y, 2z)$$
$$df_{3,p} = (-1 + 2x, -1 + 4y, 2z),$$

and thus the corank of $df_0$ is 2. Since $f$ is strongly convex, the mapping $x^* : \Delta^2 \rightarrow X^*(f)$ is surjective by 1 of theorem 3.1. Regarding $\Delta^2$ as $D^2 = \{(w_2, w_3) | w_2, w_3 \geq 0, w_2 + w_3 \leq 1\}$, we obtain

$$x^*(w_2, w_3) = \left(-\frac{w_2 - w_3}{2}, -\frac{w_2 - w_3}{2(1 + w_3)} - 0\right).$$

Obviously $x^*$ maps the line defined by $w_2 - w_3 = 0$ in $\Delta^2 = D^2$ into single point (the origin), while it is injective at points outside the above line. Thus $X^*(f) (= x^*(\Delta^2))$ is not diffeomorphic to $\Delta^2$. Figure 2 describes the Pareto set of $f$, together with the contours of the functions $f_1$ (red), $f_2$ (blue) and $f_3$ (green).

![Figure 2. The Pareto set of $f$. The union of two domains colored with orange and blue is the Pareto set.](image-url)
For $\varepsilon \in \mathbb{R} - \{0\}$, we define another mapping $f_{\varepsilon} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as follows:

$$f_{\varepsilon}(x, y, z) = \left( f_1(x, y, z) + \varepsilon z, f_2(x, y, z), f_3(x, y, z) \right).$$

Note that the mapping $f_{\varepsilon}$ is a linear perturbation of $f$. It is easy to verify that the mapping $f_{\varepsilon}$ is strongly convex and never has corank 2 critical points. Thus the problem minimizing $f_{\varepsilon}$ is simplicial. We will see in section 4 that in general any strongly convex problem becomes simplicial after a generic linear perturbation (see theorem 4.1).

**Example 3.5** (non-general position without corank 2). We define a mapping $f = (f_1, f_2, f_3) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ as follows:

$$f_1(x, y, z) = x^2 + (-y + x)^2 + z^2,$$
$$f_2(x, y, z) = 2(x - 1)^2 + (-y + x - 1)^2 + z^2,$$
$$f_3(x, y, z) = (x - 2)^2 + (y + x - 2)^2 + z^2.$$

The mapping $f$ is strongly convex. We will check that $f$ satisfies the assumption in 2 of theorem 3.1 (although the minimizing points of $f_1, f_2, f_3$ are not in general position). Let $p = (x, y, z)$ be a point in $X^*(f)$. It is easy to see that $z$ is equal to 0. Thus the differentials at $p$ are calculated as follows:

$$(df_1)_p = (4x - 2y, -2x + 2y, 0),$$
$$(df_2)_p = (6x - 2y - 6, -2x + 2y + 2),$$
$$(df_3)_p = (4x + 2y - 8, 2x + 2y - 4).$$

Suppose that the corank of $df_p$ is greater than or equal to 2. Then the following equalities hold:

$$0 = \det \left( \begin{array}{cc} 4x - 2y & -2x + 2y \\ 6x - 2y - 6 & -2x + 2y + 2 \end{array} \right) = 4 \left( x - \frac{y}{2} - \frac{1}{2} \right)^2 - (y - 3)^2 + 8,$$

$$0 = \det \left( \begin{array}{cc} 4x - 2y & -2x + 2y \\ 4x + 2y - 8 & 2x + 2y - 4 \end{array} \right) = 2 \left( 8(x - 1)^2 - 4 \left( y - \frac{3}{2} \right)^2 + 1 \right).$$

These equalities give rise to two hyperbolas given in Figure 3(a) (the red hyperbola is defined by [4], while the blue one is defined by [5]). As shown in Figure 3(a), the two hyperbolas intersect at two points. One is the origin $0$ and let $q = (x', y')$ be the other. Since the rank of $((df_1)_0, (df_2)_0, (df_3)_0)$ is 2, $(x, y)$ is not equal to $(0, 0)$. Thus we obtain $(x, y) = (x', y')$. However, since $y' > x'$ and $x', y' > 1$, all of the three values $-2x' + 2y', -2x' + 2y' + 2$ and $2x' + 2y' - 4$ are greater than 0, contradicting proposition 2.3. Hence we can conclude that there is no point $p \in X^*(f)$ with $\text{corank}(df_p) \geq 2$. Figure 3(b) describes the Pareto set of $f$, together with the contours of the functions $f_1$ (red), $f_2$ (blue) and $f_3$ (green).

4. Generic linearly perturbed strongly convex mappings

In this section, we will investigate the multiobjective optimization problem minimizing a generic linearly perturbed strongly convex mapping. Let $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be the space consisting of all linear mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$. In what follows we will regard $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ as the Euclidian space $(\mathbb{R}^n)^m$ in the obvious way. The purpose of this section is to show the following theorem:
Theorem 4.1. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) (\( n \geq m \)) be a strongly convex \( C^r \)-mapping (\( 2 \leq r \leq \infty \)). If \( n - 2m + 4 > 0 \), then there exists a subset \( \Sigma \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) - \Sigma \) the mapping \( f + \pi \) never has differential with corank greater than 1 on its Pareto set. In particular, the multiobjective optimization problem minimizing \( f + \pi : \mathbb{R}^n \to \mathbb{R}^m \) is \( C^{r-1} \)-simplicial.

We begin with observing that strong convexity is preserved under linear perturbations.

Lemma 4.2. Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a strongly convex mapping. Then, for any \( \pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \), the mapping \( f + \pi : \mathbb{R}^n \to \mathbb{R}^m \) is also a strongly convex mapping.

Proof of lemma 4.2. Obviously, it is sufficient to show the statement under the assumption that \( f \) is a function (i.e. \( m = 1 \)). For \( x, y \in \mathbb{R}^n \) and \( t \in [0, 1] \), the following holds:

\[
t (f + \pi)(x) + (1 - t) ((f + \pi)(y)) - (f + \pi)(tx + (1 - t)y) \\
= t f(x) + \pi(x) + (1 - t) (f(y) + \pi(y)) - f(tx + (1 - t)y) - \pi(tx + (1 - t)y) \\
= tf(x) + (1 - t) f(y) - f(tx + (1 - t)y),
\]

where the last equality holds since \( \pi \) is linear. Since \( f \) is strongly convex, there exists \( \alpha > 0 \) satisfying the following inequality for any \( x, y \in \mathbb{R}^n \) and \( t \in [0, 1] \):

\[
 tf(x) + (1 - t) f(y) - f(tx + (1 - t)y) \geq \frac{1}{2} \alpha t(1 - t)||x - y||^2.
\]

Hence, the mapping \( f + \pi \) is also strongly convex. \( \square \)

Before proving theorem 4.1, we will briefly review the result in [5] needed here. Let \( S_k \subseteq J^1(\mathbb{R}^n, \mathbb{R}^m) \) be the subset defined in section 2.4. It is known that \( S_k \) is a submanifold of \( J^1(\mathbb{R}^n, \mathbb{R}^m) \) satisfying the following (see [2]):

\[
 \text{codim } S_k = (n - v + k)(m - v + k),
\]
where \( \text{codim } S_k = \dim J^1(\mathbb{R}^n, \mathbb{R}^m) - \dim S_k \) and \( v = \min\{n, m\} \). The following lemma is merely a special case of [3, Theorem 1]:

**Lemma 4.3** (cf. [5]). Let \( f : \mathbb{R}^n \to \mathbb{R}^m \) be a \( C^r \)-mapping. Let \( k \) be an integer satisfying \( 1 \leq k \leq \min\{n, m\} \). If \( r > \max\{n - \text{codim } S_k, 0\} + 1 \), then there exists a subset \( \Sigma \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) with Lebesgue measure zero such that for any \( \pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) - \Sigma \), the mapping \( j^1(f + \pi) : \mathbb{R}^n \to J^1(\mathbb{R}^n, \mathbb{R}^m) \) is transverse to \( S_k \).

**Proof of theorem 4.1.** In the case \( m = 1 \), it is clearly seen that theorem 4.1 holds. Hence, we will consider the case \( m \geq 2 \). Since \( n \geq m \), the codimension of \( S_2 \) is equal to \( 2(n - m + 2) \). By the assumption \( n - 2m + 4 > 0 \), we can obtain the inequality codim \( S_2 > n \). Let \( k \) be an integer with \( 2 \leq k \leq m \). It follows that

\[
n - \text{codim } S_k \leq n - \text{codim } S_2 < 0.
\]

In particular, for a mapping \( g : \mathbb{R}^n \to \mathbb{R}^m \), transversality of \( j^1g \) to \( S_k \) is equivalent to the condition that \( j^1g(\mathbb{R}^n) \cap S_k = \emptyset \), that is, \( g \) has no corank \( k \) critical points (see [2, Ch. II, Proposition 4.2]). Furthermore, the following inequality holds:

\[
r \geq 2 > \max\{n - \text{codim } S_k, 0\} + 1.
\]

We can deduce from lemma 4.3 together with the observations above, that there exists \( \Sigma_k \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \) with Lebesgue measure zero such that the mapping \( f + \pi \) has no corank \( k \) critical points for any \( \pi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) - \Sigma_k \). Set \( \Sigma = \bigcup_{i=2}^{m} \Sigma_i \subseteq \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \), which also has Lebesgue measure zero. Lastly, we can easily verify that \( \Sigma \) satisfies the conditions in theorem 4.1. \( \square \)

### 5. Applications

As we have seen, strongly convex problems have a variety of desirable properties which make their Pareto sets easy to understand. Although lots of practical problems are not necessarily strongly convex, we can apply suitable “structure-preserving” transformations to these problems so that they become strongly convex. In this section, we will give several examples of such problems.

#### 5.1. Facility location problems

One of the most traditional examples is the facility location problem, which requires to find the place \( x \in \mathbb{R}^n \) for a facility so that the weighted sum \( \sum_{i=1}^{m} w_i \|x - p_i\| \) (for given \( w \in \Delta^{m-1} \)) of distances from demand points \( p_1, \ldots, p_m \in \mathbb{R}^n \) is minimized. Its multiobjective version [8] is the following problem:

\[
\begin{align*}
\text{minimize } & f(x) = (f_1(x), \ldots, f_m(x)) \text{ subject to } x \in \mathbb{R}^n \\
& \text{where } f_i(x) = \|x - p_i\| \quad (i = 1, \ldots, m).
\end{align*}
\]

The mapping \( f \) is called a *distance mapping* [6], which is not differentiable. Each \( f_i \) is convex but not strongly convex, and thus so is the problem eq. (6).

Let us consider the transformation of the target \( T : [0, \infty)^m \to [0, \infty)^m \) defined by \( T(y_1, \ldots, y_m) = (y_1^2, \ldots, y_m^2) \), which preserves the Pareto ordering of \( [0, \infty)^m \).

We have a transformed problem

\[
\text{minimize } T \circ f(x) \text{ subject to } x \in \mathbb{R}^n.
\]

The mapping \( T \circ f \) (called a *distance-squared mapping* [6]) is differentiable and strongly convex, in particular the problem eq. (7) is strongly convex. Since \( T \)
preserves the Pareto ordering, the Pareto sets of eqs. (6) and (7) are identical and
the Pareto fronts are homeomorphic.

For the original problem eq. (6), there is a weight with which the weighted sum
scalarization has non-unique solutions (e.g. the problem minimizing \( \sum_{i=1}^{m} \frac{1}{m} \| x - p_i \| \)),
in particular we cannot define a mapping \( x^* \) given in section 3. On the other hand,
for the transformed problem eq. (7), every scalarized problem has a unique solution
and the entire Pareto set consists of such elements by 1 of theorem 3.1. It is further
easy to verify that the corank of \( d(T \circ f) \) is 1 for any \( x \in X^*(T \circ f) \), provided
that \( n \geq m - 1 \) and \( p_1, \ldots, p_m \) are in general position. Thus, the statement 2
of theorem 3.1 guarantees that the problem eq. (7) is \((C^\infty-)\)simplicial. Since \( T \)
preserves the Pareto ordering, one can easily see that the problem eq. (6) is also
\((C^0-)\)simplicial.

5.2. Phenotypic divergence model. Another example minimizing distances from
points arises in evolutionary biology. Let \( A_i \) be a symmetric, positive definite ma-
trix of size \( n \) and \( p_i \in \mathbb{R}^n \) (\( i = 1, \ldots, m \)). Shoval et al. [12] provided a model for
describing phenotypic divergence of species, which is an extension of the facility
location problem:

\[
\begin{align*}
\text{minimize} & \quad f(x) = (f_1(x), \ldots, f_m(x)) \quad \text{subject to} \quad x \in \mathbb{R}^n \\
\text{where} & \quad f_i(x) = \| A_i(x - p_i) \| \quad (i = 1, \ldots, m)
\end{align*}
\]

As before, the problem minimizing eq. (8) is convex but not strongly convex. We
can again apply the transformation \( T \) used in the previous subsection and obtain

\[
\begin{align*}
\text{minimize} & \quad T \circ f(x) \quad \text{subject to} \quad x \in \mathbb{R}^n.
\end{align*}
\]

Since affine transformations of the source space preserve strong convexity of a prob-
lem, each component of \( T \circ f \) (and thus the problem eq. (9)) is strongly convex.
Applying 1 of theorem 3.1 we can conclude that both of the problems eqs. (8)
and (9) are weakly simplicial. In order to further show that these problems are
simplicial, we have to check the corank condition in 2 of theorem 3.1 which would
be a hard task, even if the demand points \( p_1, \ldots, p_m \) are in general position. Indeed,
problems appearing in section 3.1 are special cases of the problems we are dealing
with here. As discussed in section 3.1 generality of the configuration of demand
points does not necessarily imply the corank condition.

6. Conclusions

In this paper, we have shown that a \( C^r \)-strongly convex problem is \( C^{r-1} \)-weakly
simplicial. We have further proved that it is \( C^{r-1} \)-simplicial under some mild
assumption on the corank of the objective mapping. We have also shown that one
can always make strongly convex problems satisfy this assumption by generic linear
perturbations, provided that the dimension of the source is sufficiently larger than
that of the target.

While lots of multiobjective optimization problems appearing in practice are
not strongly convex, we have demonstrated that several examples of such problems
can be reduced to strongly convex problems via transformations preserving the
Pareto ordering and the topology. Moreover, we have discussed what information
in original problems can be extracted from transformed ones.
We plan to extend the theorems to those for $C^1$–mappings. To do this, we will require different techniques since one cannot define the Hessian matrices for $C^1$–mappings. Another interesting research project is to classify the singularity types of map germs at Pareto points under the equivalence relation induced by the structure-preserving transformations.

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References

[1] K. Deb. Multi-objective Optimization Using Evolutionary Algorithms. John Wiley & Sons, Inc., New York, NY, USA, 2001.
[2] M. Golubitsky and V. Guillemin. Stable Mappings and Their Singularities, volume 14 of Graduate Texts in Mathematics. Springer New York, 1974.
[3] Naoki Hamada, Yuichi Nagata, Shigenobu Kobayashi, and Isao Ono. Adaptive weighted aggregation: A multiobjective function optimization framework taking account of spread and evenness of approximate solutions. In Proceedings of the 2010 IEEE Congress on Evolutionary Computation (CEC 2010), pages 787–794, 2010.
[4] C. Hillermeier. Nonlinear Multiobjective Optimization: A Generalized Homotopy Approach, volume 25 of International Series of Numerical Mathematics. Birkhäuser Verlag, Basel, Boston, Berlin, 2001.
[5] S. Ichiki. Transversality theorems on generic linearly perturbed mappings. Methods and Applications of Analysis special volume in memory of John Mather. To appear.
[6] Shunsuke Ichiki and Takashi Nishimura. Distance-squared mappings. Topology Appl., 160(8):1005–1016, 2013.
[7] Ken Kobayashi, Naoki Hamada, Akiyoshi Sannai, Akinori Tanaka, Kenichi Bannai, and Masashi Sugiyama. Bézier simplex fitting: Describing Pareto fronts of simplicial problems with small samples in multi-objective optimization. In Proceedings of the Thirty-Third AAAI Conference on Artificial Intelligence (AAAI-19). To appear.
[8] Harold W Kuhn. On a pair of dual nonlinear programs. Nonlinear Programming, 1:38–45, 1967.
[9] P. W. Michor. Manifolds of Differentiable Mappings, volume 3 of Shiva mathematics series. Birkhauser, 1980.
[10] Kaisa M. Miettinen. Nonlinear Multiobjective Optimization, volume 12 of International Series in Operations Research & Management Science. Springer-Verlag, GmbH, 1999.
[11] Yuuri Nesterov. Introductory Lectures on Convex Optimization: A Basic Course. Kluwer Academic Publishers, 2004.
[12] O. Shoval, H. Sheftel, G. Shinar, Y. Hart, O. Ramote, A. Mayo, E. Dekel, K. Kavanagh, and U. Alon. Evolutionary trade-offs, Pareto optimality, and the geometry of phenotype space. Science, 336(6085):1157–1160, 2012.
[13] S. Smale. Global analysis and economics I: Pareto optimum and a generalization of Morse theory. In M. M. Peixoto, editor, Dynamical Systems, pages 531–544. Academic Press, 1973.
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