The relativistic correction of the quarkonium melting temperature with a holographic potential

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The relativistic correction of the AdS/CFT implied heavy quark potential is examined within the framework of the potential model. For the typical range of the coupling strength appropriate to heavy-ion collisions, we find the correction is significant in size and lowers the dissociation temperature of quarkonia.

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1. INTRODUCTION

The phase structure of quantum chromodynamics (QCD) remains an active field of researches. At sufficiently high temperature, hadronic matter will evolve into a quark-gluon plasma which has been explored experimentally by relativistic heavy ion collisions (RHIC). The quarkonium dissociation is an important signal of this transition [1].

The quarkonium are bound states composed of a heavy quark $q$ and its antiparticle $\bar{q}$. It is found that, the ground states and the excitation levels of quarkonium are very much smaller than the normal hadrons, and that they are very tightly bounded [2]. Theoretically, there are two approaches to study the quarkonium: Lattice QCD and potential models [3]. From Lattice QCD, we can calculate the spectral function numerically via the quarkonium correlators and identify the quarkonium states with the resonance peaks [4–8]. The potential model relies on the small velocity ($v \ll 1$) of the constituent quarks. By solving a non-relativistic Schrödinger equation with a temperature dependent effective potential, we can determine the energy levels and thereby the threshold temperature when the bound state dissolves [9–14]. The potential model will be applied in the present work.

AdS/CFT correspondence is a powerful tool to explore the strongly coupled $\mathcal{N} = 4$ super Yang-Mills plasma. The equation of state and viscosity ratio, etc. extracted from the AdS/CFT show remarkable agreement with the Lattice QCD or experimental data from the quark-gluon plasma (QGP) created via Relativistic Heavy Ion Collisions (RHIC). It would be interesting to extend the comparison to a wide range of other quantities, for instance heavy quark dissociation, which are calculable in both ways to assess whether the super Yang-Mills serves an important reference model of the QGP phase of QCD. This is the primary motivation of this paper.

In a previous work [15], we examined the heavy quarkonium dissociation within the potential model with the AdS/CFT implied potential function (holographic potential). We found that the holographic potential can be approximated by a truncated Coulomb potential to a great accuracy. With the typical values of the ’t Hooft coupling constant, $\lambda \equiv \sqrt{N_c g_{YM}^2}$ considered in the literature [17],

$$5.5 < \lambda < 6\pi, \quad (1.1)$$

our dissociation temperatures are systematically lower, though not far from the lattice prediction. [20] On the other hand, an estimate of the velocity of the constituent quarks inside the bound state indicates that the non-relativistic approximation may be marginal, especially for $J/\Psi$. This motivates us to examine the relativistic corrections of the holographic potential with the aid of a two-body Dirac equation (TBDE).

While the holographic potential alone is sufficient in the nonrelativistic limit, it does not provide all information necessary for the relativistic corrections even to the order $v^4$ term. Except for the correction brought about by the relativistic kinetic energy, the spin-orbital coupling and the Darwin term depend on how the holographic potential is introduced in the two-body Dirac equation. In addition, gravity dual of spin dependent forces is not available in the literature. Therefore our result remains incomplete at this stage. We would like to comment that the same issues exists for the relativistic corrections of the heavy quark potential extracted from the lattice QCD simulations.

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In the next section, the work reported in [15] will be reviewed and the dissociation temperature beyond the truncated Coulomb approximation is presented. The corrections to the dissociation temperature are computed in the section 3 through a Foldy-Wouthuysen (F.W.) transformation of a two-body Dirac Hamiltonian. The kinetic energy contribution and the contribution from the Darwin and the spin-orbit coupling are calculated separately with the latter obtained simply by replacing the perturbative Coulomb potential in the two-body Dirac Hamiltonian with the holographic potential. The section 4 conclude the paper.

2. THE HOLOGRAPHIC POTENTIAL MODEL

In the conventional potential model of QCD, the non-relativistic wave function of a heavy quarkonium satisfies the Schroedinger equation

\[
[-\frac{1}{2\mu}\nabla^2 + U(r, T)]\psi = -E(T)\psi,
\]

where \(E(T)\) is the binding energy and \(U(r, T)\) is identified with the internal energy of a pair a static \(q\) and \(\bar{q}\) in QGP and is related to the free energy \(F(r, T)\) via

\[
U(r, T) = -T^2\left[\frac{\partial}{\partial T}\left(\frac{F(r, T)}{T}\right)\right]_r.
\]

The free energy \(F(r, T)\) can be extracted from the expectation of a pair of Wilson loops operator according to:

\[
e^{-\frac{1}{T}F(r, T)} = \frac{\text{tr} < W^+(L_+) W(L_-) >}{\text{tr} < W^+(L_+)> < W(L_-)>}
\]

where \(L_\pm\) stands for the Wilson loop running in Euclidean time direction at spatial coordinates \((0,0,\pm 1/2 r)\) and is closed with the periodicity \(\beta = \frac{1}{T}\) and

\[
W(L_\pm) = Pe^{-\frac{i}{\beta} \oint_{L_\pm} dx^a A_a(x)}.
\]

The spatial coordinates of \(L_\pm\) are \((0,0,\pm 1/2 r)\). The lattice QCD simulation of the expectation value \(\langle 2.3 \rangle\) can be found if Ref. [10, 18].

In case of super Yang-Mills, the holographic principle places the Wilson lines \(L_\pm\) on the boundary \((y \to \infty)\) of the 5D AdS-Schwarzschild metric [19]:

\[
d s^2 = \pi^2 T^2 y^2 (f dt^2 + d\vec{x}^2) + \frac{1}{\pi^2 T^2 y^2 f} dy^2
\]

where \(f = 1 - \frac{1}{y^2}\), \(d\vec{x}^2 = dx_1^2 + dx_2^2 + dx_3^2\) with the ansatz \(x_1 = x_2 = 0\) and \(x_3\) a function of \(y\).

The free energy \(F(r, T)\) of the corresponding super Yang-Mills at large \(N_c\) and large ’t Hooft coupling is proportional to the minimum area of the worldsheet in the AdS bulk bounded by \(L_+\) and \(L_-\), and is given parametrically by [15, 20]:

\[
\begin{align*}
F(r, T) &= T \min(I, 0) \\
r &= \frac{2q}{\pi T} \int_{y_c}^{\infty} \frac{dy}{\sqrt{(y^4 - 1)(y^4 - y_c^4)}}
\end{align*}
\]

where

\[
I = \sqrt{\lambda} \int_{y_c}^{\infty} dy \left(\sqrt{\frac{y^4 - 1}{y^4 - y_c^4}} - 1 + y_c\right)
\]

and the parameter \(y_c \in (1, \infty)\). Eliminating \(y_c\) between \(\langle 2.6 \rangle\) and \(\langle 2.7 \rangle\), we find that

\[
F(r, T) = -\frac{\alpha}{r} \Phi(\rho) \theta(\rho_0 - \rho)
\]

where \(\alpha \approx 0.2285\sqrt{\lambda}, \rho = \pi Tr, \rho_0 = 0.7541\) and \(\Phi(\rho)\) is the screening factor. The corresponding internal energy is

\[
U(r, T) = -\frac{\alpha}{r} [\Phi(\rho) - \rho \frac{d\Phi}{d\rho}(y_c) - \rho \frac{d\Phi}{dy_c}(\rho_0)] \theta(\rho_0 - \rho)
\]
\[ \Phi(\rho) = 1 - \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3} \rho + \frac{3\Gamma^8\left(\frac{1}{8}\right)}{64\pi^6} \rho^4 + O(\rho^8). \] (2.10)

Within the screening radius \( \rho_0 \), the first two terms of the series (2.10) approximate the exact \( \Phi \) well as is shown in Fig.1. If we keep only the first two terms, the screening radius \( \rho_0 \approx 0.7359 \) and \( U(r,T) \) becomes a truncated Coulomb potential

\[ U = -\frac{\alpha}{\rho} \theta(\rho_0 - \rho) \] (2.11)

under the approximation.

We define the dissociation temperature \( T_d \) as the temperature when the binding energy falls to zero, i.e. \( E(T_d) = 0 \), and the corresponding radial Schrödinger equation reads [15]:

\[ \frac{d^2 R}{d\rho^2} + \frac{2}{\rho} \frac{dR}{d\rho} - \left[ \frac{l(l+1)}{\rho^2} + V \right] R = 0 \] (2.12)

where the reduced potential, \( V = \frac{mU}{\pi^2 T^2} \) is dimensionless.

The truncated Coulomb potential approximation was employed in [15] and the dissociation temperature of the bound state of \( l \)-th partial wave and \( n \)-th radial quantum number is given by

\[ T_d = 4\alpha x_{nl} m \] (2.13)

with \( x_{nl} \) the \( n \)-th nonzero root (ascending order) of the Bessel function \( J_{2l}(x) \). The corresponding radial wave function reads

\[ R(r) = \frac{1}{\sqrt{\rho}} J_{2l+1} \left( x_{nl} \sqrt{\frac{\rho}{\rho_0}} \right), \] (2.14)

for \( \rho \leq \rho_0 \) and \( R(r) = \text{const.} / r^{l+1} \) for \( \rho > \rho_0 \).

In this work, we have calculated the dissociation temperature with the exact holographic potential [2.9]. The comparison with that obtained from the truncated Coulomb potential in [15] for \( J/\Psi \) is shown in Table I, where we choose \( m = 1.65 \) GeV for the mass of \( c \) quarks. From the comparison of these two results we confirmed that the truncated Coulomb approximation is a good approximation and we shall stay with the truncated Coulomb approximation for the rest of this paper.

### 3. The Relativistic Correction of the Holographic Potential

As it is mentioned in the introduction, the velocity of the heavy quarks is not low enough so the relativistic correction may be significant, especially for \( J/\Psi \). To explore this correction, one has to go beyond the Schrödinger equation (2.1) and switch to the two body Dirac equation [21, 24]:

\[ i\frac{\partial \Psi}{\partial t} = H \Psi \] (3.1)
In the center-of-mass frame, the Hamiltonian of the two body Dirac equation is:

\[ H = \alpha_1 \cdot \vec{p} + \beta_1 \cdot m - \alpha_2 \cdot \vec{p} + \beta_2 \cdot m + U \]  

(3.2)

where, \( \alpha_1 = \vec{\alpha} \otimes I, \alpha_2 = I \otimes \vec{\alpha} \), \( \beta_1 = I \otimes \beta, \beta_2 = \beta \otimes I \). \( \beta_1 \), \( \beta_2 \), \( \vec{\alpha} \), \( \vec{\alpha} \) are usual 4 \times 4 Dirac matrices, \( \vec{p} = -i \vec{\nabla} \) and \( U \) is the interaction potential between the two particles. The Hamiltonian \( H \) is a 16 \times 16 matrix. A quarkonium state corresponds to a bound state of \( H \) with the eigenvalue \( 2m - E(T) \), which goes to \( 2m \) at the dissociation temperature, i.e. \( E(T) = 0 \). Since we are interested in the leading order relativistic correction of the dissociation temperature for the quarkonium, we have to expand the Hamiltonian to the order \( v^4 \). The sorting of the order in \( v \) follows from the rules that \( \frac{\vec{p}^2}{m} \sim U \sim v^2 \) and \( \vec{\nabla} \sim \frac{1}{m} \sim m v \). Also the expectation values of \( \alpha_1 \) and \( \alpha_2 \) are of the order \( v \).

In analogous to the one body Foldy-Wouthuysen transformation \cite{25}, we introduce the unitary operator

\[ U = e^{i S'_1 e^i S'_2 e^i S_1} \]

(3.3)

where

\[ S_1 = -i \frac{1}{2m} \beta_1 \cdot O_1 \]

(3.4)

\[ S_2 = -i \frac{1}{2m} \beta_2 \cdot (-O_2) \]

(3.5)

\[ S'_1 = -i \frac{1}{2m} \beta_1 \cdot O'_1 \]

(3.6)

\[ S'_2 = -i \frac{1}{2m} \beta_2 \cdot (-O'_2) \]

(3.7)

with

\[ O_1 = \vec{\alpha}_1 \cdot \vec{p}, \quad O_2 = \vec{\alpha}_2 \cdot \vec{p}. \]

The transformed Hamiltonian reads

\[ H_{FW} = U H U^\dagger \]

\[ = (\beta_1 + \beta_2) (m + \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3}) + U + \frac{1}{4m^2} \nabla^2 U + \frac{1}{4m^2r} \frac{dU}{dr} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{L} \]

(3.8)

where, higher order terms in \( v \) have been dropped. The details of the transformation are deferred to the appendix. The non-relativistic wave function, \( \Psi = \Psi_{s_1 s_2} (r_1 - r_2) \) with subscript \( s_1, s_2 \) labelling the spin components of the two quarks, corresponding to the sector with \( \beta_1 = \beta_2 = 1 \) (This wave function can be expanded in the series of products of the orbital wave functions of the preceding section and the spin wave functions). We may stay within this sector for the 1st order perturbation of the \( v^4 \) term of \( \Psi \) with the effective Hamiltonian \( H_{eff.} = H_0 + H_1 \), where

\[ H_0 = 2m + \frac{\vec{p}^2}{m} + U \]

corresponds to the non-relativistic part, and

\[ H_1 = -\frac{\vec{p}^4}{4m^3} + \frac{1}{4m^2} \nabla^2 U + \frac{1}{4m^2r} \frac{dU}{dr} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{L} \]

\[ = -\frac{\vec{p}^4}{4m^3} + \frac{1}{4m^2} \nabla^2 U + \frac{1}{4m^2r} \frac{dU}{dr} (J^2 - L^2 - S^2), \]

(3.9)

is a relativistic correction. We have introduced the total spin \( \vec{S} = \frac{1}{2} (\vec{\sigma}_1 + \vec{\sigma}_2) \) and the total angular momentum \( \vec{J} = \vec{L} + \vec{S} \) in the last step. The contribution of \( H_1 \) is somewhat like the that to the fine structure of the hydrogen atom, and we can see that the first term is the first order of the kinetic energy correction; the second term is Drawin term; and the third term is the spin orbit coupling which can be decomposed into the spin singlet and the spin triplet channels.

Perturbatively, we may write \( E(T) = E_0(T) + \delta E(T) \) and \( T = T_0 + \delta T \), where \( E_0(T) \) is the non-relativistic binding energy in \( \Psi \) and \( T_0 \) are the dissociation temperature given by \( \Psi \), and \( \delta E(T) \) and \( \delta T \) are the \( v^4 \) corrections.
We have \( E_0(T_0) = 0 \). Expanding the dissociation condition \( E(T) = 0 \) to the order \( v^4 \), we obtain the formula for \( \delta T \), i.e.

\[
\delta T = \delta_1 T + \delta_2 T = - \frac{\delta_1 E(T_0) + \delta_2 E(T_0)}{(\frac{\partial E}{\partial T})_{T_0}}. \tag{3.10}
\]

where,

\[
\delta_1 E(T_0) = - \left< \frac{\hat{p}^4}{4m^3} \right>
\]

\[
\delta_2 E(T_0) = \left[ \left< \frac{1}{4m^2} \nabla^2 U \right> + \left< \frac{1}{4m^2} \frac{dU}{dr} (J^2 - L^2 - S^2) \right> \right]
\]

and

\[
\left( \frac{\partial E_0}{\partial T} \right)_{T=T_0} = \left< \frac{\partial H_0}{\partial T} \right> = \left< \frac{\partial U}{\partial T} \right>. \tag{3.12}
\]

The average

\[
\left< O \right> = \frac{\int d^3\hat{r} \psi^*(\hat{r}) O(\hat{r}) \psi(\hat{r})}{\int d^3\hat{r} \psi^*(\hat{r}) \psi(\hat{r})}.
\]

with \( \psi(\hat{r}) \) the non-relativistic wave function. The reason for our separating the contribution from \( p^4 \), \( \delta_1 T \) and that from the Darwin and spin-orbital terms, \( \delta_2 T \) is the uncertainty in the representation of the holographic potential in (3.2), which does not impact on the \( p^4 \) correction. We will come to this point in the next section. In the limit of zero binding energy, we find \( \left< \frac{\hat{p}^4}{4m^3} \right> = \left< \frac{1}{4m^2} U \right> \). For the truncated Coulomb potential,

\[
\nabla^2 U = 4\pi \alpha \pi^3 T^3 \delta^3(\vec{r}) \theta(\rho_0 - \rho) + \frac{\alpha \pi^3 T^3}{\rho} \delta'(\rho - \rho_0). \tag{3.14}
\]

In terms of the radial wave function \( R_l(r) \) of \( \psi(\hat{r}) \),

\[
\left< \frac{\hat{p}^4}{4m^3} \right> = - \frac{\alpha^2}{4\pi m T} \int_0^{\rho_0} R_l(\rho)^2 d\rho
\]

\[
\left< \frac{1}{4m^2} \nabla^2 U \right> = \frac{\alpha}{4m^2} \{ |R_l(0)|^2 - 2R_l(\rho_0)R_l'(\rho_0)\rho_0 - R_l^2(\rho_0) \}
\]

\[
\left< \frac{1}{4m^2} \frac{dU}{dr} \right> = \frac{\alpha}{4m^2} \int_0^{\rho_0} \frac{d\rho}{\rho} R_l^2(\rho) + \frac{\alpha}{4m^2} R_l^2(\rho_0)
\]

\[
\left< \frac{\partial U}{\partial T} \right> = \frac{\alpha \pi}{\pi^3 T^3} R_l^2(\rho_0) \rho_0^2
\]

(3.15)

For the \( ns \) state, we find that

\[
\delta_1 T = \frac{\pi \alpha T_0^2}{4m_0} \left[ \frac{1}{J_1^2(x_{n0})} - \frac{J_0^2(x_{n0})}{J_1^2(x_{n0})} - 1 \right]
\]

\[
\delta_1 T + \delta_2 T = - \frac{\pi \alpha T_0^2}{4m_0} \left[ \frac{J_0^2(x_{n0})}{J_1^2(x_{n0})} + 1 + \frac{2}{x_{n0}^2} \right]. \tag{3.16}
\]

For the \( np \) state, we can also get analytical expressions, which are more lengthy.

The numerical values of the corrected temperature \( T_0 + \delta_1 T \) and \( T_0 + \delta_1 T + \delta_2 T \) in MeV’s for 1s, 2s and 1p states are listed in the Table II below.

4. DISCUSSIONS

In summary, we have explored the leading relativistic correction to the dissociation temperature of heavy quarkonium state through a F.W.-like transformation of the two body Dirac Hamiltonian with the AdS/CFT implied
|       | $c\bar{c}$ |       | $b\bar{b}$ |
|-------|------------|-------|------------|
|       | $\lambda = 5.5$ | $\lambda = 6\pi$ | $\lambda = 5.5$ | $\lambda = 6\pi$ |
| $1s$  | 162.54     | 387.54 | 478.76     | 1139.11    |
| $2s$  | 29.15      | 62.75  | 85.67      | 184.44     |
| $1p$  | 32.04      | 62.14  | 94.18      | 182.66     |

TABLE II: This lists the final results of the corrected temperature in MeV’s, where the upper one corresponds to the results of $T_0 + \delta_1 T$ and the lower one corresponds to $T_0 + \delta_1 T + \delta_2 T$. For the lower one, we wrote the states as $nL^{\pm1}$ in the first column, where $n$ is the main quantum number, $L$ is the orbit angular momentum quantum number, $S$ is the spin quantum number and $J$ is the total angular momentum quantum number. Since there are no spin-orbit coupling term for $ns$ states, the spin singlet and spin triplet are degenerate. However, for $p$ state, adding the coupling term, there will have energy level splitting, so we can see the different results for different total angular momentum $J$.

Potential. Among the contributions we considered, the $p^4$ correction of the kinetic energy, being negative, enhances the binding but the Darwin term does the opposite and dominates. Consequently, the dissociation temperature of s-state is lowered, leaving the corrected values further below the lattice result. This disagreement can be attributed to the short screening length $r_0 = \frac{\rho_0}{\pi T}$, about 0.25fm at $T = 200$ MeV, of the AdS/CFT potential and the sharp cutoff nature of the screening. In case of $J/\psi$, the magnitude of the correction ranges from 8% for $\lambda = 5.5$ to 30% for $\lambda = 6\pi$, indicating significant relativistic towards the high end of the domain (2.13).

The potential model, though physically more transparent than spectral function approach, does not provide complete $v^4$ corrections with the holographic potential extracted from the Wilson loop alone. The same deficiency applies the relativistic correction based on the lattice heavy quark potential alone. As the Wilson loop for a non-Abelian theory involves multi-gluon exchanges, its form in the two-body Dirac Hamiltonian (3.2) may not be adequate unless the single gluon exchange serves a reasonable approximation. Among the four sectors $\beta_1 = \pm 1$ and $\beta_2 = \pm 1$, only
two of them $\beta_1 = \beta_2 = \pm 1$ correspond to $q\bar{q}$ interacting via the holographic potential. The other two sectors with $\beta_1$ and $\beta_2$ correspond to $qq$ or $q\bar{q}$ and the interacting potential is unknown. A more general form of the interaction in (3.2) without violating the charge conjugation symmetry is to replace $U$ by

$$(\Lambda_{++} + \Lambda_{--})U + (\Lambda_{+-} + \Lambda_{-+})U' = U_+ + \beta_1 \beta_2 U_-$$

(4.1)

where the projection operator $\Lambda_{\pm 1, \pm 1} \equiv \frac{1 \pm \beta_1 \pm \beta_2}{2}$, $U'$ the potential between $qq$ or $q\bar{q}$ and $U_\pm = \frac{U + U'}{2}$. $U' = U$ in the previous section. Repeating the steps of FW transformation in the appendix, we find that the perturbing Hamiltonian (3.9) is replaced by

$$H_1 = -\frac{\vec{p}^4}{4m^3} + \frac{1}{4m^2} \nabla^2 U_+ + \frac{1}{4m^2r} \frac{dU_+}{dr} (\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{L} + \frac{1}{8m^2} \{\{\vec{\sigma}_1 \cdot \vec{\nabla}, \{\vec{\sigma}_1 \cdot \vec{\nabla}, U_+\}\} + \{\vec{\sigma}_2 \cdot \vec{\nabla}, \{\vec{\sigma}_2 \cdot \vec{\nabla}, U_-\}\}$$

(4.2)

with $\{...\}$ an anticommutator. This will modify the potential part of (3.9). Only the correction from the kinetic energy, $p^4$ term of (3.9) is robust, which raise the dissociation temperature.

In addition to the holographic potential considered in this work, there should be spin dependent ones that splits degeneracies between spin singlets and spin triplets (e.g. between $\eta_c$ and $J/\psi$). The gravity dual of the latter are unknown in the literature. A first principle derivation of the spin-dependent forces associate them to the expectation value of Wilson loops with operator insertions, $< \text{tr} W_{\mu\nu}(L_+) W_{\rho\lambda}(L_-) >$, where

$$W_{\mu\nu}(L) = PF_{\mu\nu}(x) e^{-i \int_L dx^\nu A_\nu}$$

(4.3)

with $F_{\mu\nu}$ the Yang-Mills field strength and $x$ a point along $L$, $W_{\mu\nu}(L)$ is obtained from the Eqs. (2.4) by small distortion of $L$ at $x$. Within the AdS/CFT framework, it corresponds to the perturbation of the Nambu-Goto action of the world shee underlying the holographic potential under a small distortion of its boundary. It is a challenging boundary value problem and we hope to report our progress along this line in future.

Finally, we would like to comment on a phenomenological formulation of the two-body Dirac equation [27], which has been applied recently to the same problem addressed in this work [28]. It amounts to divide the heavy quark potential of Cornell type into a linearly confining term and a single gluon Coulomb term and to generate all spin-dependent terms of (4.2) without violating the charge conjugation symmetry is to replace

$$\{\vec{\sigma}_1 \cdot \vec{\nabla}, \{\vec{\sigma}_1 \cdot \vec{\nabla}, U_+\}\}$$

with \{\{\vec{\sigma}_1 \cdot \vec{\nabla}, \vec{\sigma}_1 \cdot \vec{\nabla}, U_+\}\} + \{\vec{\sigma}_2 \cdot \vec{\nabla}, \vec{\sigma}_2 \cdot \vec{\nabla}, U_-\}\}

(4.2)

\[ \begin{align*}
\Lambda_{++} & = \frac{1 + \beta_1 \beta_2}{2} \\
\Lambda_{--} & = \frac{1 - \beta_1 \beta_2}{2} \\
\Lambda_{+-} & = \frac{1 + \beta_1 - \beta_2}{2} \\
\Lambda_{-+} & = \frac{1 - \beta_1 + \beta_2}{2}
\end{align*} \]

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\Lambda_{++} & = \frac{1 + \beta_1 \beta_2}{2} \\
\Lambda_{--} & = \frac{1 - \beta_1 \beta_2}{2} \\
\Lambda_{+-} & = \frac{1 + \beta_1 - \beta_2}{2} \\
\Lambda_{-+} & = \frac{1 - \beta_1 + \beta_2}{2}
\end{align*} \]

(4.1)

with $F_{\mu\nu}$ the Yang-Mills field strength and $x$ a point along $L$, $W_{\mu\nu}(L)$ is obtained from the Eqs. (2.4) by small distortion of $L$ at $x$. Within the AdS/CFT framework, it corresponds to the perturbation of the Nambu-Goto action of the world shee underlying the holographic potential under a small distortion of its boundary. It is a challenging boundary value problem and we hope to report our progress along this line in future.

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Appendix A

In this appendix, we shall fill in the details of the F.W. transformation for the two body Dirac equation which we have done in section 4.

Let us recall the Foldy-Wouthuysen transformation of the one body Dirac Hamiltonian, $H = \vec{\sigma} \cdot \vec{p} + \beta m + V$. For a 4-component spinor with velocity $v \ll 1$ and positive energy, one can work in a representation where the upper two components correspond to the non-relativistic limit, referred to as the large components, while the lower two components are suppressed by a power of $v$, referred to as the small components. An operator is even(odd) if it is diagonal(off-diagonal) with respect to large and small components. For example, $\beta$ is even and $\bar{\alpha}$ is odd. The Foldy-Wouthuysen transformation amounts to successive unitary transformations that push the odd operators to higher orders in $v$.

The Foldy-Wouthuysen transformation can be easily generalized to the two body case, the Hilbert space of which is spanned by the direct products of two one body spinors. To the leading order relativistic correction, we stop at the $v^4$ terms ignoring all the higher order term. The Hamiltonian of the two body Dirac equation is:

$$H = \alpha_1 \bar{p} + \beta_1 m - \alpha_2 \bar{p} + \beta_2 m + V$$

(A.1)

let:

$$\alpha_1 \bar{p} = O_1$$

$$\alpha_2 \bar{p} = O_2$$
so,\[
H = \beta_1 m + O_1 + \beta_2 m - O_2 + V
= H_1 + H_2 + V
\] (A.2)
where \( H_1 \) corresponds to the first under-brace, and \( H_2 \) corresponds to the second under-brace. And it is easy to proof the commutation relation that: \([O_1, O_2] = \beta_1, \beta_2 = [O_1, \beta_2] = [O_2, \beta_1] = 0\), and \( O_1 \sim O_2 \sim v \).

We do the first transformation:

so, the transformed Hamiltonian turns into: \( H' = e^{iS_2} e^{iS_1} H e^{-iS_1} e^{-iS_2} \). We calculate the mid over-brace first.

\[
e^{iS_1} H e^{-iS_1} = e^{iS_1} H_1 e^{-iS_1} + e^{iS_1} H_2 e^{-iS_1} + e^{iS_1} V e^{-iS_1}
= \overline{H}_1 + \overline{H}_2 + V'
\] (A.3)

also, \( \overline{H}_1 \) corresponds to the first under-brace, \( \overline{H}_2 \) corresponds to the second, and \( V' \) corresponds to the third.

\[
\overline{H}_1 = H_1 + [iS_1, H_1] + \frac{1}{2!} [iS_1, [iS_1, H_1]] + \frac{1}{3!} [iS_1, [iS_1, [iS_1, H_1]]] + \frac{1}{4!} [iS_1, [iS_1, [iS_1, [iS_1, H_1]]]] + \cdots \quad \text{(A.4)}
\]

\[
[iS_1, H_1] = -O_1 + \frac{1}{m} \beta_1 O_1^2
\]
\[
\frac{1}{2!} [iS_1, [iS_1, H_1]] = -\frac{1}{2m} \beta_1 O_1^2 - \frac{1}{m^2} O_1^3
\]
\[
\frac{1}{3!} [iS_1, [iS_1, [iS_1, H_1]]] = \frac{1}{6m^2} O_1^3 - \frac{1}{6m^3} \beta_1 O_1^4
\]
\[
\frac{1}{4!} [iS_1, [iS_1, [iS_1, [iS_1, H_1]]]] \sim \frac{1}{24m^3} \beta_1 O_1^4
\]

where \( v \) is the velocity of the particles.

\[
\overline{H}_2 = H_2 + [iS_1, H_2] + \frac{1}{2!} [iS_1, [iS_1, H_2]] + \frac{1}{3!} [iS_1, [iS_1, [iS_1, H_2]]] + \frac{1}{4!} [iS_1, [iS_1, [iS_1, [iS_1, H_2]]]] + \cdots \quad \text{(A.5)}
\]

\[
[iS_1, H_2] = \frac{1}{2!} [iS_1, [iS_1, H_2]] = \frac{1}{3!} [iS_1, [iS_1, [iS_1, H_1]]] = \cdots = 0
\]

\[
\overline{H}_2 = H_2 = \beta_2 m - O_2
\]

\[
V' = V + \frac{1}{2m} \beta_1 [O_1, V] - \frac{1}{8m^2} [O_1, [O_1, V]]
\] (A.7)

so,

\[
e^{iS_1} H e^{-iS_1} = \overline{H}_1 + \overline{H}_2 + V'
= \beta_1 m + \beta_2 m - O_2 + V + \frac{1}{2m} \beta_1 O_1^2 - \frac{1}{8m^2} [O_1, [O_1, V]] - \frac{1}{8m^3} \beta_1 O_1^4 + \frac{1}{2m} \beta_1 [O_1, V] - \frac{1}{3m^2} O_1^3 + O(v^5)
\]

\[
= \overline{H}
\] (A.8)
that we mark $e^{iS_i}He^{-iS_i}$ as $H'$ for convenience. So,

\[
H' = e^{iS_2}H' e^{-iS_2} = H + [iS_2, H] + \frac{1}{3!}[iS_2, [iS_2, H]] + \frac{1}{5!}[iS_2, [iS_2, [iS_2, H]]] + \cdots \tag{A.9}
\]

\[
[iS_2, H] = O_2 + \frac{1}{2m} \beta_2 O_2^2 - \frac{1}{2m^2} \beta_2 [O_2, V] - \frac{1}{4m} \beta_1 \beta_2 [O_2, [O_1, V]]
\]

\[
\frac{1}{2}[iS_2, [iS_2, H]] = -\frac{1}{2m} \beta_2 O_2^2 + \frac{1}{2m^2} O_2^4 - \frac{1}{8m^2} [O_2, [O_2, V]]
\]

\[
\frac{1}{3}[iS_2, [iS_2, [iS_2, H]]] = -\frac{1}{6m^2} O_2^3 - \frac{1}{6m^3} \beta_2 O_2^4
\]

\[
\frac{1}{4}[iS_2, [iS_2, [iS_2, [iS_2, H]]]] = \sim \frac{1}{24m^3} \beta_2 O_2^4
\]

\[
H' = \beta_1 m + \beta_2 m + V
\]

\[
+ \frac{1}{2m} \beta_1 O_1^2 - \frac{1}{8m^2} [O_1, [O_1, V]] - \frac{1}{8m^3} \beta_1 O_1^4 + \frac{1}{2m} \beta_1 [O_1, V] - \frac{1}{3m^3} O_1^3
\]

\[
+ \frac{1}{2m} \beta_2 O_2^2 - \frac{1}{8m^2} [O_2, [O_2, V]] - \frac{1}{8m^3} \beta_2 O_2^4 - \frac{1}{2m} \beta_2 [O_2, V] + \frac{1}{3m^3} O_2^3
\]

\[
- \frac{1}{4m^3} \beta_1 \beta_2 [O_2, [O_1, V]] + O(v^5)
\]

(A.10)

since,

\[
H' = (\beta_1 m + V + \frac{1}{2m} \beta_1 O_1^2 - \frac{1}{8m^2} [O_1, [O_1, V]] - \frac{1}{8m^3} \beta_1 O_1^4 + \frac{1}{2m} \beta_1 [O_1, V] - \frac{1}{3m^3} O_1^3) \Rightarrow \text{mark: } H'_1
\]

\[
+ (\beta_2 m + \frac{1}{2m} \beta_2 O_2^2 - \frac{1}{8m^2} [O_2, [O_2, V]] - \frac{1}{8m^3} \beta_2 O_2^4 - \frac{1}{2m} \beta_2 [O_2, V] + \frac{1}{3m^3} O_2^3) \Rightarrow \text{mark: } H'_2
\]

\[
- \frac{1}{4m^3} \beta_1 \beta_2 [O_2, [O_1, V]]
\]

(A.11)

\[
H'_1 = \beta_1 m + V' + O_1'
\]

(A.12)

\[
H'_2 = \beta_2 m + \frac{1}{2m} \beta_2 O_2^2 - \frac{1}{8m^2} [O_2, [O_2, V]] - \frac{1}{8m^3} \beta_2 O_2^4 - \frac{1}{2m} \beta_2 [O_2, V] - \frac{1}{3m^3} O_2^3
\]

(A.13)

\[
H' = H'_1 + H'_2 - \frac{1}{4m^2} \beta_1 \beta_2 [O_2, [O_1, V]]
\]

(A.14)

the first under-braces upside in the expression of $H'_1(H'_2)$ correspond to $V'_1(V'_2)$, and the second correspond to $O'_1(O'_2)$ for convenience.

Then, we do the second transformation:

We select:

\[
S'_1 = -\frac{i}{2m} \beta_1 O'_1
\]

\[
S'_2 = -\frac{i}{2m} \beta_2 (-O'_2)
\]

and $O'_1 \sim O'_2 \sim v^3$

so,

\[
H'' = e^{iS'_2}He^{-iS'_2}H' e^{-iS'_1} e^{-iS'_2}
\]

\[
= e^{iS'_2}He^{-iS'_1} e^{-iS'_1} e^{-iS'_2} + e^{iS'_2}He^{-iS'_1} e^{-iS'_2} - \frac{1}{4m^2} e^{iS'_2}He^{-iS'_1} e^{-iS'_2} e^{-iS'_2}
\]

\[
= e^{iS'_2}He^{-iS'_2} + e^{iS'_2}He^{-iS'_2} - \frac{1}{4m^2} e^{iS'_2}He^{-iS'_2} e^{-iS'_2}
\]

(A.15)
where, the first under-brace upside correspond to $H_1^t$, and the second correspond to $H_2^t$, and we can directly apply the results of the first transformation

$$H_1^t = H'_1 + [iS_1', H'_1] + O(v^5)$$

$$= \beta_1 m + V'_1 + \frac{1}{2m} \beta_1 [O'_1, V'_1]$$ (A.16)

$$e^{iS_2} H_1^t e^{-iS'_2} = H'_1 + [iS'_2, H'_1] + O(v^5)$$

$$= \beta_1 m + V'_1 + \frac{1}{2m} \beta_1 [O'_1, V'_1] - \frac{1}{2m} \beta_2 [O'_2, V'_1]$$ (A.17)

$$H_2^t = H'_2 + [iS'_1, H'_2] + O(v^5)$$

$$= \beta_2 m + V'_2 - \frac{1}{2m} \beta_2 [O'_2, V'_2] + \frac{1}{2m} \beta_1 [O'_1, V'_2] - \frac{1}{2m} \beta_1 [O'_1, O'_2]$$ (A.18)

$$e^{iS'_2} H_2^t e^{-iS'_2} = H'_2 + [iS'_2, H'_2] + O(v^5)$$

$$= \beta_2 m + V'_2 - \frac{1}{2m} \beta_2 [O'_2, V'_2] + \frac{1}{2m} \beta_1 [O'_1, V'_2] - \frac{1}{2m} \beta_1 [O'_1, O'_2]$$ (A.19)

Then, we see the last term in $H''$: 

$$e^{iS'_1} \beta_1 \beta_2 [O_2, [O_1, V]] e^{-iS'_1} = \frac{1}{4m^2} e^{iS'_2} \beta_1 \beta_2 [O_2, [O_1, V]] e^{-iS'_2}$$

we do the calculation of the over-brace first.

$$e^{iS'_1} \beta_1 \beta_2 [O_2, [O_1, V]] e^{-iS'_1} = \beta_1 \beta_2 [O_2, [O_1, V]] + [iS'_1, \beta_1 \beta_2 [O_2, [O_1, V]]] + O(v^5)$$

$$= (\beta_1 \beta_2 [O_2, [O_1, V]] + \frac{1}{4m^2} [\beta_1 [O_1, V], [O_2, [O_1, V]]]) \implies \text{mark : A}$$ (A.20)

$$e^{iS'_2} A e^{-iS'_2} = A + O(v^5)$$ (A.21)

$$e^{iS'_2} A e^{-iS'_2} = A + O(v^5)$$ (A.22)

considering $O_1, O_2 \sim v, O'_1, O'_2 \sim v^3$

$$H'' = \beta_1 m + V'_1 + \beta_2 m + V'_2 - \frac{1}{4m^2} \beta_1 \beta_2 [O_2, [O_1, V]]$$

$$= \beta_1 m + \beta_2 m + V + \frac{1}{2m} \beta_1 O_1^2 + \frac{1}{2m} \beta_2 O_2^2 - \frac{1}{8m^2} [O_1, [O_1, V]] - \frac{1}{8m^2} [O_2, [O_2, V]]$$

since, $O_1^2 = O_2^2 = \hat{p}^2, O_1 = O_2 = \hat{p}^4$.

$$[O_1, [O_1, V]] = -\nabla^2 V - \frac{2}{r} \frac{\partial V}{\partial r} \Sigma_1 L$$

$$[O_2, [O_2, V]] = -\nabla^2 V - \frac{2}{r} \frac{\partial V}{\partial r} \Sigma_2 L$$

$$[O_2, [O_1, V]] = -\alpha_1 \alpha_2 \nabla_i \nabla_j V$$

$$H'' = \beta_1 (m + \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^2}) + \beta_2 (m + \frac{\hat{p}^2}{2m} - \frac{\hat{p}^4}{8m^2}) + V$$

$$+ \frac{1}{4m^2} \nabla^2 V + \frac{1}{4m^2} \frac{\partial V}{\partial r} (\Sigma_1 + \Sigma_2) L + \frac{1}{4m^2} \beta_1 \beta_2 \alpha_1 \alpha_2 \nabla_i \nabla_j V$$ (A.24)

The last term in (A.24), though of the order of $v^4$, is a direct product of two odd operators and therefore does not contribute to the first order perturbation considered in this paper.

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