DISCRETE SPECTRUM DISTRIBUTION OF THE LANDAU OPERATOR PERTURBED BY AN EXPANDING ELECTRIC POTENTIAL

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Dedicated to our teacher Professor Mikhail Shlemovich Birman on the occasion of his 80-th birthday

Abstract. Under a perturbation by a decreasing potential, the Landau Hamiltonian acquires some discrete eigenvalues between the Landau levels. We study the perturbation by an “expanding” electric potential \( V(t^{-1} x), t > 0 \), and derive a quasi-classical formula for the counting function of the discrete spectrum as \( t \to \infty \).

1. Introduction and main result

The two-dimensional Landau Hamiltonian \( H_0 = (-i \nabla - a)^2 \) describing a charged quantum particle moving in the plane in a constant magnetic field \( B = \text{curl} a \) is one of the earliest explicitly solvable models of Quantum Mechanics. Its spectrum consists of infinitely degenerate eigenvalues (Landau levels) \( \Lambda_q = B(2q + 1) \), \( q = 0, 1, \ldots \) (see, e.g., [10]); we put \( \Lambda_{-1} = -\infty \) for reference convenience. Under a perturbation by an electric or magnetic field decaying at infinity, the Landau levels split, forming clusters (generically, infinite) of eigenvalues with Landau levels being their limit points. Various asymptotic properties of these clusters have been extensively studied in the literature. For instance, [4], [11], [13], [14] studied the rate of convergence of the eigenvalues to their limit points for rapidly decaying potential perturbations. It was found that for a compactly supported electric potential the eigenvalues converge to Landau levels superexponentially fast. A similar effect was observed for the perturbation by a compactly supported magnetic field [15] or impenetrable compact obstacle [12]. Another natural problem is to analyze the eigenvalue behavior as the coupling constant in front of the perturbation becomes large. In this case the eigenvalue asymptotics is described by semi-classical formulas, as shown in [7], [8]. We refer to the above references for further bibliography.

Our objective is to study the discrete spectrum of the Landau Hamiltonian \( H_0 \) perturbed by an expanding potential \( V(t(x) = V(t^{-1} x), t > 0 \). Under the condition \( V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \) the operator \( H = H(t) = H_0 + V(t) \), is properly defined as an operator sum in \( L^2(\mathbb{R}^2) \), and \( V(t) \) is \( H_0 \)-compact. Our aim is to investigate the number \( N(\lambda_1, \lambda_2; H(t)) \) of the eigenvalues of \( H(t) \) on the interval \( (\lambda_1, \lambda_2) \equiv (\Lambda_\nu, \Lambda_{\nu+1}) \) with some \( \nu = -1, 0, 1, \ldots \), as \( t \to \infty \). If \( \lambda_1 = -\infty \), we write \( N(\lambda_2; H(t)) \).

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The behavior of \( N(\lambda_1, \lambda_2; H^{(t)}) \) is determined by the potential \( V \) as follows. For any \( V \in L^1(\mathbb{R}^2) \) and \( -\infty \leq \lambda < \mu \leq \infty \), we define

\[
A(\lambda, \mu; V) = |\{ x : \lambda < V(x) < \mu \}|, 
A^{(0)}(\lambda; V) = |\{ x : V(x) = \lambda \}|.
\]

The coefficients \( A(\lambda, \mu; V) \) are finite if \( \lambda \mu > 0 \). Say, for \( \mu > \lambda > 0 \), we obtain by the Chebyshev inequality

\[
A(\lambda, \mu; V) \leq A(\lambda, \infty; V) < \frac{1}{\lambda}\| V \|_1 < \infty.
\]

The coefficient \( A \) is monotone in \( \lambda, \mu \), so that the limits \( A(\lambda \pm, 0; V) \) are well defined with various combinations of signs \( \pm \). We will call the number \( \lambda \) a *generic value* for \( V \) if \( A^{(0)}(\lambda; V) = 0 \); otherwise, this value is called *exceptional*. For a given function \( V \), there are at most countably many exceptional values. For a generic \( \lambda \), \( A(\lambda - 0, \mu; V) = A^{(\pm)}(\lambda, \mu; V) \), otherwise, \( A(\lambda - 0, \mu; V) = A(\lambda, \mu; V) + A^{(0)}(\lambda; V) \), and similarly for \( \mu \).

For any real \( \lambda_1 < \lambda_2 \), such that \( [\lambda_1, \lambda_2] \) does not contain any of \( \Lambda_q \), we define

\[
\mathcal{A}(\lambda_1, \lambda_2; V) = \frac{B}{2\pi} \sum_{q=0}^{\infty} A(\lambda_1 - \Lambda_q, \lambda_2 - \Lambda_q; V).
\]

Since the sets \( \{ x : \Lambda_q - \lambda_2 < |V(x)| < \Lambda_q - \lambda_1 \} \) are disjoint for different \( q \)'s, this series converges for \( V \in L^1(\mathbb{R}^2) \), and

\[
\mathcal{A}(\lambda_1, \lambda_2; V) \leq A(\min\{ |\Lambda_q - \lambda_2|, |\Lambda_q - \lambda_1| \}, \infty; |V|) < \infty.
\]

The main result of the paper is contained in the following Theorem:

**Theorem 1.1.** Let \( (\lambda_1, \lambda_2) \in (\Lambda_\nu, \Lambda_{\nu+1}) \) with some \( \nu = -1, 0, 1, \ldots \). Suppose that \( V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \). Then

\[
\mathcal{A}(\lambda_1, \lambda_2; V) \leq \liminf_{t \to \infty} t^{-2} N(\lambda_1, \lambda_2; H^{(t)})
\]

\[
\leq \limsup_{t \to \infty} t^{-2} N(\lambda_1, \lambda_2; H^{(t)}) \leq \mathcal{A}(\lambda_1 - 0, \lambda_2 + 0; V).
\]

If \( \lambda_1 - \Lambda_q \) and \( \lambda_2 - \Lambda_q \) are generic for \( V \) for all \( q = 0, 1, \ldots \), then

\[
\lim_{t \to \infty} t^{-2} N(\lambda_1, \lambda_2; H^{(t)}) = \mathcal{A}(\lambda_1, \lambda_2; V).
\]

Note that the right hand side of the asymptotic formula (1.3) coincides with the natural quasi-classical expression for the counting function of the magnetic Schrödinger operator, see e.g. [10]. On the other hand one might juxtapose this result with the classical Szegö Theorem deriving the canonical distribution for the Toeplitz type operators of Fourier type, see [6], Theorem 8.6(c). This comparison is even more appropriate since the proof of Theorem 1.1 relies on the spectral analysis of the Toeplitz type operator \( T^{(t)} = P_q V^{(t)} P_q \), where \( P_q \) is the projection on the spectral subspace associated with the Landau level \( \Lambda_q \). Remembering that the non-zero spectra of the operators \( AB \) and \( BA \) (\( A, B \) being both compact) coincide, instead of \( T^{(t)} \) it is often more convenient to study the operator \( S^{(t)} = WP_q W \) with a function \( W \). If \( W \) is radially symmetric, then this operator splits into an orthogonal sum of one-dimensional operators, whose eigenvalues (and their asymptotics) are computed using the explicit formula for the integral kernel of \( P_q \) (see [23]). To handle the general case we apply a method which is based on the approach put forward by M. Birman and M. Solomyak to study weakly polar
integral operators, see [1]. Precisely, we partition the plane $\mathbb{R}^2$ into disjoint annular sectors, i.e. domains of the form

$$\Omega_{m,l} = \left\{ (\rho, \phi) : (m-1)d < \rho \leq md, \quad \frac{2\pi}{N}(l-1) < \phi \leq \frac{2\pi}{N}l \right\}, \quad m \in \mathbb{N}, l = 1, 2, \ldots, N,$$

with a fixed $d > 0$ and natural $N$. Choosing appropriate $d$ and $N$, we approximate $W$ by a function which is constant on each $\Omega_{m,l}$. This reduces the operator $S^{(t)}$ to a block-matrix form. The crucial point is that the off-diagonal entries do not contribute to the asymptotics, which implies the "additivity" of the asymptotics in the function $W$. This property allows one to reduce the problem to the radially symmetric case, for which the eigenvalues are found in the closed form.

The above Toeplitz operators are linked with the initial Schrödinger operator using the elementary formula

$$(1.4) \quad N(\lambda_1, \lambda_2; H^{(t)}) = N(-\infty, b^2; L^{(t)}), L^{(t)} = (H_0 + V^{(t)} - a)^2,$$

where $a = (\lambda_1 + \lambda_2)/2$, $b = (\lambda_2 - \lambda_1)/2$, which was used previously in [9], [11] in similar circumstances. We represent $L^{(t)}$ in the block-matrix form with the entries of the form $P^\dagger L^{(t)} P$. The off-diagonal terms do not affect the asymptotics, and the diagonal ones are directly expressed via the Toeplitz operators of the form $T^{(t)}$.

The paper is organized as follows. Section 2 is devoted to some abstract operator theory. For families of semi-bounded operators depending on a parameter $t \in (0, \infty)$ we introduce the asymptotic coefficients describing the asymptotic distribution as $t \to \infty$ of eigenvalues in a given interval, and establish their general properties which are used throughout the paper in different concrete environments. In Sections 3 and 4 we prove the crucial asymptotic formulas for the Toeplitz operators $T^{(t)}$. The reduction of the initial problem to the Toeplitz operators is implemented in Sections 5 and 6. The Appendix contains some elementary analytic properties of level sets, needed for our proofs.

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2. Asymptotic coefficients

Let $L$ be a self-adjoint operator, semi-bounded from below, with

$$\eta_0 = \eta_0(L) = \inf \sigma_{\text{ess}}(L).$$

Denote by $N(\eta; L)$, $\eta < \eta_0$ the number of discrete eigenvalues of $L$ strictly below $\eta$. If $K$ is a compact operator, then $\sigma_{\text{ess}}(K) = \{0\}$ and we use the traditional notation

$$n_-(\lambda; K) = N(-\lambda; K), \quad n_+(\lambda; K) = N(-\lambda; -K), \quad \lambda > 0,$$

for the counting functions of the negative and positive eigenvalues respectively. For an arbitrary compact operator $K$, not necessarily self-adjoint, introduce also the counting function of its singular values:

$$n(\lambda; K) = n_+(\lambda^2; K^* K).$$
If there exists a linear set \( N \) satisfying (2.2), let the domain of the quadratic form \( l[u] \) of the operator \( L \). Then \( \eta < \eta_0 \) if and only if there exists a linear set \( \mathcal{L} \subset \mathfrak{d}(L) \) of finite codimension, satisfying the property
\[
l[u] \geq \eta \|u\|^2 \quad \text{for all } u \in \mathcal{L}.
\]
Moreover,
\[
N(\eta; L) = \min \operatorname{codim} \mathcal{L},
\]
where the minimum is taken over all linear sets \( \mathcal{L} \subset \mathfrak{d}(L) \), satisfying (2.1).

First, recall the version of the min-max principle for the counting function, which is usually referred to as Glazman’s Lemma:

**Lemma 2.1.** Let \( L \) be a semi-bounded operator with \( \eta_0 = \inf \sigma_{\min}(L) \), and let \( \mathfrak{d}(L) \) be the domain of the quadratic form \( l[u] \) of the operator \( L \). Then \( \eta < \eta_0 \) if and only if there exists a linear set \( \mathcal{L} \subset \mathfrak{d}(L) \) of finite codimension, satisfying the property (2.1)
\[
l[u] \geq \eta \|u\|^2 \quad \text{for all } u \in \mathcal{L}.
\]
Moreover,
\[
N(\eta; L) = \min \operatorname{codim} \mathcal{L},
\]
where the minimum is taken over all linear sets \( \mathcal{L} \subset \mathfrak{d}(L) \), satisfying (2.1).

In this form Glazman’s Lemma appeared in [5] (Ch. 1, Theorems 12, 12bis); equivalent formulations (however in terms of eigenvalues, and not the distribution function), are given in many books on spectral theory.

The most general form of the eigenvalue distribution function inequality we need is the following.

**Lemma 2.2.** Let \( L_1, L_2 \) be semi-bounded (from below) self-adjoint operators such that \( \mathfrak{d}(L_1) \subset \mathfrak{d}(L_2) \) and \( L_2 \) is \( L_1 \)-form bounded with a bound strictly less than 1. Then \( \eta_0(L_1 + L_2) \geq \eta_0(L_1) + \eta_0(L_2) \) and
\[
N(\eta_1 + \eta_2; L_1 + L_2) \leq N(\eta_1; L_1) + N(\eta_2; L_2)
\]
for any \( \eta_j < \eta_0(L_j) \), \( j = 1, 2 \), and
\[
N(\eta_1 - \eta_2; L_1 - L_2) \geq N(\eta_1; L_1) - N(\eta_2; L_2),
\]
for any \( \eta_1, \eta_2 \) such that \( \eta_1 - \eta_2 < \eta_0(L_1 - L_2) \), \( \eta_2 < \eta_0(L_2) \).

**Proof.** Denote \( N_j = N(\eta_j; L_j) \), and let \( \mathcal{L}_j \subset \mathfrak{d}(L_j) \) be a subspace of codimension \( N_j \) on which \( (L_j u, u) \geq \eta_j \|u\|^2 \), \( j = 1, 2 \). Then the subspace \( \mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2 \subset \mathfrak{d}(L_1 + L_2) \) has codimension not greater than \( N_1 + N_2 \), and for \( u \in \mathcal{L} \),
\[
((L_1 + L_2)u, u) \geq (\eta_1 + \eta_2)\|u\|^2,
\]
so (2.3) follows from Glazman’s lemma. The inequality (2.1) follows from (2.3) by an obvious change of notation. \( \square \)

We also need a simpler version of the above inequalities. Let \( L \) be a semi-bounded operator, and let \( K \) be compact and self-adjoint. Clearly, \( \eta_0(L + K) = \eta_0(L) \), and by Lemma 2.2
\[
N(\eta - \lambda; L + K) \leq N(\eta; L) + n_-(\lambda; K)
\]
for any \( \eta < \eta_0 \), \( \lambda > 0 \). It is useful to write a similar inequality for a pair of compact self-adjoint operators \( K_1, K_2 \):
\[
n_\pm(\lambda_1 + \lambda_2; K_1 + K_2) \leq n_\pm(\lambda_1; K_1) + n_\pm(\lambda_2; K_2),
\]
1Subspace, which is not necessarily closed
for any \( \lambda_1, \lambda_2 > 0 \). For a pair of compact operators (not necessarily self-adjoint) a similar inequality holds:

\[
(2.7) \quad n(\lambda_1 + \lambda_2; K_1 + K_2) \leq n(\lambda_1; K_1) + n(\lambda_2; K_2)
\]

for any \( \lambda_1, \lambda_2 > 0 \), see [3], Section 9.2, Theorem 9.

For a family of semi-bounded operators \( L = L(t) \), depending on a parameter \( t > 0 \), with the value \( \eta_0 = \eta_0(L(t)) \) independent of \( t \), we introduce the asymptotic coefficients

\[
\mathcal{B}(\eta; L) = \limsup_{t \to \infty} t^{-2} N(\eta; L(t)), \quad b(\eta; L) = \liminf_{t \to \infty} t^{-2} N(\eta; L(t)), \quad \eta < \eta_0.
\]

Clearly, one can introduce similar asymptotic coefficients, with \( t^{-2} \) replaced by \( t^{-\gamma} \) with any \( \gamma > 0 \). Although such characteristics of operator families may prove to be useful for some other eigenvalue counting problems, the case \( \gamma = 2 \) is sufficient for our purposes. General properties of such coefficients are the same as for \( \gamma = 2 \). The asymptotic coefficients, just introduced, are not necessarily continuous in \( \eta \), but they are monotone. We systematically use naturally defined limits such as \( \mathcal{B}(\eta \pm 0; L) \).

In order to keep in line with the traditional definition of counting functions \( n_\pm \) for compact operators, for a compact self-adjoint family \( K = K(t) \) we denote

\[
\mathfrak{R}(\pm)(\lambda; K) = \mathcal{B}(-\lambda; \mp K), \quad \mathfrak{v}(\pm)(\lambda; K) = b(-\lambda; \mp K),
\]

and for arbitrary compact family introduce also

\[
\mathfrak{R}(\lambda; K) = \mathfrak{R}(+)(\lambda^2; K^* K), \quad \mathfrak{v}(\lambda; K) = \mathfrak{v}(+)(\lambda^2; K^* K).
\]

The bounds \([2.4]\), \([2.5]\), \([2.6]\) and \([2.6]\) imply similar bounds for the functionals \( \mathcal{B}, b \). In particular, for a semi-bounded \( L \) and compact \( K \) it follows from \([2.6]\) that

\[
(2.8) \quad \mathcal{B}(\eta - \lambda; L + K) \leq \mathcal{B}(\eta; L) + \mathfrak{R}(-)(\lambda; K), \quad b(\eta - \lambda; L + K) \leq b(\eta; L) + \mathfrak{R}(-)(\lambda; K), \quad \eta < \eta_0, \ \lambda > 0.
\]

For compact operators, the inequality \([2.6]\) produces the bounds

\[
(2.9) \quad \mathfrak{R}(\lambda_1 + \lambda_2; K_1 + K_2) \leq \mathfrak{R}(\lambda_1; K_1) + \mathfrak{R}(\lambda_2; K_2), \quad \mathfrak{v}(\lambda_1 + \lambda_2; K_1 + K_2) \leq \mathfrak{v}(\lambda_1; K_1) + \mathfrak{v}(\lambda_2; K_2),
\]

for any \( \lambda_1, \lambda_2 > 0 \) and similar bounds hold for the functionals \( \mathfrak{R}(\pm), \mathfrak{v}(\pm) \).

We systematically use analogues of Birman-Solomyak asymptotic perturbation lemma for the eigenvalues, see [2].

**Lemma 2.3.** Let \( L = L(t), t > 0 \), be a family of self-adjoint semi-bounded from below operators with a value of \( \eta_0 = \eta_0(L(t)) \) independent of \( t \). Suppose that for any \( \delta > 0 \) the family \( L \) can be represented as \( L = L_\delta + Y_\delta^1 + Y_\delta^\prime \) with some \( L(t) \)-form-compact self-adjoint \( Y_\delta^1 = Y_\delta^{(t)} \) and \( Y_\delta^\prime = Y_\delta^{(t)} \), such that

\[
(2.10) \quad \lim_{\delta \to 0} \mathcal{B}(\tau; L + MY_\delta^\prime) = 0
\]
for any \( \tau < \eta_0 \) and any \( M \in \mathbb{R} \). Then for any \( \eta < \eta_0 \)

\[
\lim_{\epsilon \downarrow 0} \lim_{M \downarrow 1} \lim_{\delta \to 0} \mathfrak{B}(\eta - \epsilon; L_\delta + MY_\delta') \\
\leq \mathfrak{B}(\eta; L) \leq \lim_{\epsilon \downarrow 0} \lim_{M \downarrow 1} \lim_{\delta \to 0} \mathfrak{B}(\eta + \epsilon; L_\delta + MY_\delta') ;
\]

(2.11) \[
\lim_{\epsilon \downarrow 0} \lim_{M \downarrow 1} \lim_{\delta \to 0} \mathfrak{b}(\eta - \epsilon; L_\delta + MY_\delta') \\
\leq \mathfrak{b}(\eta; L) \leq \lim_{\epsilon \downarrow 0} \lim_{M \downarrow 1} \lim_{\delta \to 0} \mathfrak{b}(\eta + \epsilon; L_\delta + MY_\delta').
\]

Proof. It suffices to prove the Lemma for \( \eta_0 = 0 \). Using (2.3) with \( L_1 = (1 - \mu)L + Y_\delta' \) and \( L_2 = \mu L + Y_\delta'' \), we obtain

\[
\mathfrak{b}(\eta; L) \leq \mathfrak{b}(\eta + \epsilon; (1 - \mu)L_\delta + Y_\delta') + \mathfrak{B}(-\epsilon; \mu L_\delta + Y_\delta''),
\]

for any \( \eta < 0, \mu \in (0, 1) \) and any \( 0 < \epsilon < |\eta| \). Using (2.10), and passing to the limit as \( \delta \to 0 \), we get

\[
\mathfrak{b}(\eta; L) \leq \lim_{\delta \to 0} \mathfrak{b}(\eta + \epsilon; L_\delta + (1 - \mu)^{-1}Y_\delta').
\]

Passing to the limit as \( \mu \downarrow 0 \) and \( \epsilon \downarrow 0 \), we get the proclaimed upper bound for \( \mathfrak{b}(\eta; L) \), where \( M = (1 - \mu)^{-1} \). Similarly for \( \mathfrak{B}(\eta; L) \).

For the lower bound we use (2.4) with \( L_1 = (1 + \mu)L + Y_\delta' \) and \( L_2 = \mu L - Y_\delta'' \), which gives

\[
\mathfrak{b}(\eta; L) \geq \mathfrak{b}(\eta - \epsilon; (1 + \mu)L_\delta + Y_\delta') - \mathfrak{B}(-\epsilon; \mu L_\delta - Y_\delta'');
\]

for any \( \eta < 0, \mu \in (0, 1) \) and \( \epsilon > 0 \). Using (2.10) again, and passing to the limit as \( \delta \to 0 \), we get

\[
\mathfrak{b}(\eta; L) \geq \lim_{\delta \to 0} \mathfrak{b}(\eta - \epsilon; L_\delta + (1 + \mu)^{-1}Y_\delta').
\]

Passing to the limit as \( \mu \downarrow 0 \) and \( \epsilon \downarrow 0 \), we get the proclaimed lower bound for \( \mathfrak{b}(\eta; L) \), where \( M = (1 + \mu)^{-1} \). Similarly for \( \mathfrak{B}(\eta; L) \). \( \square \)

Lemma 2.4. Let \( L = L^{(t)}, t > 0 \), be a family of self-adjoint semi-bounded from below operators with a value of \( \eta_0 = \eta_0(L^{(t)}) \) independent of \( t \). Suppose that for any \( \delta > 0 \) the family \( L \) can be represented as \( L = L_\delta + K_\delta' \) with a compact self-adjoint \( K_\delta' = K^{(t)}_\delta \), such that

\[
\lim_{\delta \to 0} \mathfrak{A}(\epsilon; K_\delta') = 0
\]

for any \( \epsilon > 0 \). Then for any \( \eta < \eta_0 \)

\[
\lim_{\epsilon \downarrow 0} \lim_{\delta \to 0} \mathfrak{B}(\eta - \epsilon; L_\delta) \leq \mathfrak{B}(\eta; L) \leq \lim_{\epsilon \downarrow 0} \lim_{\delta \to 0} \mathfrak{B}(\eta + \epsilon; L_\delta);
\]

(2.13) \[
\lim_{\epsilon \downarrow 0} \lim_{\delta \to 0} \mathfrak{b}(\eta - \epsilon; L_\delta) \leq \mathfrak{b}(\eta; L) \leq \lim_{\epsilon \downarrow 0} \lim_{\delta \to 0} \mathfrak{b}(\eta + \epsilon; L_\delta).
\]

Proof. For \( \delta \) fixed, write (2.8) for families \( L_\delta, K_\delta' \) and pass to lim sup as \( \delta \to 0 \) and then as \( \epsilon \to 0 \). This proves the upper bounds in (2.13). The lower bounds are proved similarly. \( \square \)

The next result is a direct consequence of this lemma applied to compact operators:
Lemma 2.5. Let $K = K^{(t)}$ be a family of compact operators. Suppose that for any $\delta > 0$ the family $K$ can be represented as a sum $K = K_\delta + K_\delta'$ such that for any $\varepsilon > 0$ the condition (2.12) is satisfied. Then for any $\lambda > 0$

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \Re(\lambda + \varepsilon, K_\delta) \leq \Re(\lambda, K) \leq \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \Re(\lambda - \varepsilon, K_\delta);$$

(2.14)

$$\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \Im(\lambda + \varepsilon, K_\delta) \leq \Im(\lambda, K) \leq \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \Im(\lambda - \varepsilon, K_\delta).$$

If, moreover, the families $K_\delta, K_\delta'$ are self-adjoint, then the relations (2.14) hold with $\Re, \Im$ replaced respectively by $\Re^{(\pm)}, \Im^{(\pm)}$.

3. Eigenvalue bounds for Toeplitz operators

3.1. Eigenvalue bounds for auxiliary integral operators. Here we obtain spectral estimates for integral operators involving the projections $P_q$ on the spectral subspaces (Landau subspaces) $L_q$ associated with the landau Levels $\Lambda_q$, $q = 0, 1, 2, \ldots$. Choosing the gauge $a = (-\frac{B}{2}x_2, \frac{B}{2}x_1)$ for the magnetic potential, one can write the orthonormal basis of the subspace $L_q$ using the generalized Laguerre polynomials

$$L_q^{(\alpha)}(\xi) = \sum_{m=0}^{q} \frac{(q + \alpha)}{q - m} \frac{(-\xi)^m}{m!}, \quad \xi \geq 0,$$

as follows:

(3.1) $\psi_{q, \alpha}(x) = \sqrt{\frac{q!}{(q + \alpha)!}} \left[ \sqrt{\frac{B}{2}}(x_1 + i x_2) \right]^\alpha L_q^{(\alpha)} \left( \frac{B|x|^2}{2} \right) \sqrt{\frac{B}{2\pi}} \exp \left( -\frac{B|x|^2}{4} \right)$

for $\alpha = -q, -q + 1, \ldots$. The orthonormality follows from the standard relation

(3.2) $\int_0^\infty \xi^\alpha e^{-\xi} L_q^{(\alpha)}(\xi)L_{q'}^{(\alpha)}(\xi)d\xi = \frac{\Gamma(\alpha + q + 1)}{q!} \delta_{q,q'}$.

The integral kernel of the projection $P_q$ is

(3.3) $P_q(x, y) = \frac{B}{2\pi} L_q^{(0)} \left( \frac{B|x - y|^2}{2} \right) \exp \left( -\frac{B}{4} |x - y|^2 + 2ix \wedge y \right)$.

The following important estimate for the Laguerre polynomials can be found in [14].

Lemma 3.1. Let $k \in \mathbb{Z}_+$. Then

(3.4) $|L_k^{(\alpha)}(\xi)| \leq (\alpha + k)^k e^{-\frac{\xi}{2}}$

for all $\xi \geq 0$ and $\alpha \geq 1 - k$.

For $t > 0$ and any function $f = f(x)$ we denote $f^{(t)}(x) = f(t^{-1}x)$. We consider the operator families of the form

$$S^{(t)}(W_1, W_2) = S_q^{(t)}(W_1, W_2) = W_1^{(t)} P_q W_2^{(t)}, t > 0,$$

where $W_1, W_2$ are some complex-valued functions. Along with $S^{(t)}$ we also consider

$$T_{q,q'}^{(t)}(V) = P_q V P_{q'}, \quad T_q^{(t)}(V) = T_{q,q}^{(t)}(V),$$

with some complex-valued function $V$; these are Toeplitz type operators for $q' = q$ and Hankel type operators for $q' \neq q$. The labels $q, q' = 0, 1, 2, \ldots$ are fixed and as a rule, they are not reflected in the notation of the operators. The superscript $(t)$ is sometimes omitted as well. The functions $W_j, V$ will be referred to as weight...
Using (3.3) and (3.2), we find
\[ S^{(t)}(W_1, W_2) = Z^{(t)}(W_1)(Z^{(t)}(W_2))^*, \]
\[ T^{(t)}(V) = (Z^{(t)}(V_1))^*(Z^{(t)}(V_2)), \]
where \( V_1 = \sqrt{|V|}, \ V_2 = V|V|^{-1/2}. \)

Under mild assumptions on \( W_1, W_2, V \) the above operators are compact.

**Lemma 3.2.** If \( W, W_1, W_2 \in \mathcal{L}^2(\mathbb{R}^2), \ V \in \mathcal{L}^1(\mathbb{R}^2), \) then \( Z^{(t)}(W) \in \mathcal{S}, \ S^{(t)}(W_1,W_2), \ T^{(t)}(V) \in \mathcal{S}, \) and
\[
\|Z^{(t)}(W)\|_{\mathcal{S}} = t\sqrt{\frac{B}{2\pi}} \|W\|_2,
\]
\[
\|S^{(t)}(W_1,W_2)\|_{\mathcal{S}} \leq t^2 \frac{B}{2\pi} \|W_1\|_2 \|W_2\|_2, \quad \|T^{(t)}(V)\|_{\mathcal{S}} \leq t^2 \frac{B}{2\pi} \|V\|_1.
\]

**Proof.** It suffices to prove the equality for the Hilbert-Schmidt norm of \( Z^{(t)}(W). \)
Using (3.3) and (3.2), we find
\[
\|Z^{(t)}(W)\|^2_{\mathcal{S}} = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(t^{-1}x)|^2 |P_q(x,y)|^2 \ dx \ dy
\]
\[
= \frac{B^2}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |W(t^{-1}x)|^2 \left( L_q^{(0)} \left( \frac{B|y|^2}{2} \right) \right)^2 \exp \left( -\frac{B^2|y|^2}{2} \right) \ dx \ dy
\]
\[
= t^2 \|W\|^2_2 \frac{B}{2\pi} \int_0^\infty (L_q^{(0)}(s))^2 e^{-s} \ ds = t^2 \|W\|^2_2 \frac{B}{2\pi},
\]
as required. \( \square \)

Using the notations introduced in Section 2, we set
\[
(3.5) \quad \mathcal{M}(\lambda; W_1, W_2) = \mathcal{R}(\lambda; S^{(t)}(W_1, W_2)); \ \ m(\lambda; W_1, W_2) = r(\lambda; S^{(t)}(W_1, W_2)).
\]

When \( W_1 = W_2 = W, \) we write \( \mathcal{M}(\lambda; W) \) and \( m(\lambda; W). \) In this case the operator \( S(W_1, W_2) \) is self-adjoint, so that we can also define the functionals \( \mathcal{M}^{(+)}(W). \)
Clearly, \( \mathcal{M}^{(-)}(W) = 0 \) and \( \mathcal{M}^{(+)}(W) = \mathcal{M}(W). \)

For the operator \( T^{(t)} = T^{(t)}_{q,q'} \) we introduce the related quantities:
\[
\mathcal{N}(\lambda; V) = \mathcal{R}(\lambda; T^{(t)}(V)), \quad n(\lambda; V) = r(\lambda; T^{(t)}(V)),
\]
and in case when \( V \) is real-valued and \( q = q', \) we introduce the natural notation \( \mathcal{N}^{(+)}(V) \) and \( n^{(+)}(V) \) as well. Since the nonzero eigenvalues of \( S_q^{(t)}(W) \) and \( T_q^{(t)}(|W|^2) \) coincide, we have
\[
(3.6) \quad \mathcal{N}^{(+)}(\lambda; |W|^2) = \mathcal{M}(\lambda; W), \quad n^{(+)}(\lambda; |W|^2) = m(\lambda; W).
\]
If necessary, we reflect the dependence on \( q, q' \) in the notation of the above asymptotic coefficients; for instance, we may write \( m_q(W_1, W_2) \) and \( \mathcal{N}_q^{(+)}(V), \ \mathcal{M}_{q,q'}(V). \)

Now Lemma 3.2 leads to the following result.
Lemma 3.3. If $W, W_1, W_2 \in L^2(\mathbb{R}^2)$ and $V \in L^1(\mathbb{R}^2)$, then for any $\lambda > 0$ and $q, q' = 0, 1, 2, \ldots$, we have
\[
\Re(\lambda; Z_q^{(i)}(W)) \leq \frac{B}{2\pi \lambda^2} \|W\|_2^2.
\]
\[
\mathfrak{R}_q(\lambda; W_1, W_2) \leq \frac{B}{2\pi \lambda^2} \|W_1\|_2 \|W_2\|_2, \quad \Re_q(\lambda; V) \leq \frac{B}{2\pi \lambda^2} \|V\|_1.
\]

Proof. For any operator $T \in \mathfrak{S}_p$ we have $n(\lambda; T) \leq \lambda^{-p}\|T\|_{\mathfrak{S}_p}$. It remains to use Lemma 3.2. \hfill \Box

3.2. Localization. In our study of the eigenvalue behavior, we systematically represent the operators as block-matrices associated with certain orthogonal decompositions. The results of this subsection help to show that off-diagonal terms do not contribute to the asymptotic coefficients.

We consider an auxiliary integral operator $K^{(i)}(W_1, W_2, f)$ having the kernel
\[
K^{(i)}_q(x, y; W_1, W_2, f) = W_1^{(i)}(x)\mathcal{P}_q(x, y)f^{(i)}(x - y)W_2^{(i)}(y),
\]
with some functions $W_1, W_2$ and $f$.

Lemma 3.4. Let $W_1 \in L^p, W_2 \in L^s, f \in L^\infty$ with arbitrary $2 \leq p, s \leq \infty$ such that $\frac{1}{p} + \frac{1}{s} \geq \frac{1}{2}$. Suppose that for some $\delta > 0$
\[
f(z) = 0, \quad \text{for } |z| \leq \delta.
\]
Then for any $\lambda > 0$
\[
\Re(\lambda; K^{(i)}(W_1, W_2, f)) = 0.
\]

Proof. By assumption $K^{(i)}(x, y; W_1, W_2, f) = 0$ if $|x - y| \leq \delta t$. For $|x - y| > \delta t$ we use (3.3):
\[
|\mathcal{P}_q(x, y)| \leq \frac{B}{2\pi} \left| L_q^{(0)} \left( \frac{B|x-y|^2}{2} \right) \right| e^{-\frac{B}{4}\delta^2t^2} e^{-\frac{B}{4}|x-y|^2}.
\]
Let $r \in [2, \infty]$ be defined by $p^{-1} + s^{-1} + r^{-1} = 1$. By the Hölder and Young inequalities
\[
\|K^{(i)}(W_1, W_2, f)\|_2^2 \leq t^{\frac{1}{p} + \frac{1}{2}} \|W_1\|_p^2 \|W_2\|_2 \|f\|_r^2 \leq \left( \frac{B^2}{(2\pi)^2} e^{-\frac{B^2}{4}\delta^2t^2} \right) \left( \int_{\mathbb{R}^2} \left| L_q^{(0)} \left( \frac{B|x|^2}{2} \right) \right|^r e^{-\frac{B}{4}|x|^2} e^{-\frac{B}{4}\delta^2t^2} \right)^{\frac{2}{r}} \leq \left( \int_{\mathbb{R}^2} \left| L_q^{(0)} \left( \frac{B|x|^2}{2} \right) \right|^r e^{-\frac{B}{4}|x|^2} e^{-\frac{B}{4}\delta^2t^2} \right)^{\frac{2}{r}} \leq Ct^{\frac{1}{p} + \frac{1}{2}} e^{-\frac{B}{4}\delta^2t^2}.
\]
Due to the presence of the exponentially decaying factor, for sufficiently large $t$ the operator $K^{(i)}(W_1, W_2, f)$ has no singular values above $\lambda$, whence (3.7). \hfill \Box

The above Lemma has a few useful corollaries.

Corollary 3.5. Let $V \in L^2(\mathbb{R}^2)$ and $R^{(i)}(V) = R_q^{(i)}(V) = [P_q, V^{(i)}]$. Then
\[
\Re(\lambda; R^{(i)}(V)) = 0.
\]
Moreover, if \( V \in L^1(\mathbb{R}^2) + L^2(\mathbb{R}^2) \) and \( q \neq q' \) then also
\[
\mathfrak{N}_{q,q'}(\lambda; V) = 0,
\]
for all \( \lambda > 0 \).

**Proof.** Fix a \( \delta > 0 \) and find \( \tilde{V} \in C_0(\mathbb{R}^2) \) such that \( \| V - \tilde{V} \|_2 < \delta \). We have
\[
[P_q, V^{(t)}] = [P_q, \tilde{V}^{(t)}] + P_q(V^{(t)} - \tilde{V}^{(t)}) - (V^{(t)} - \tilde{V}^{(t)})P_q.
\]
To two last terms in (3.10) we can apply Lemma 3.3, which gives
\[
\Re(\epsilon; P(V^{(t)} - \tilde{V}^{(t)}) - (V^{(t)} - \tilde{V}^{(t)})P) \leq C\delta^2/\epsilon^2,
\]
for any \( \epsilon > 0 \). Since \( \delta \) is arbitrarily small, it suffices to prove that \( \Re(\lambda; [\tilde{V}^{(t)}, P]) = 0 \) for any \( \lambda > 0 \) and \( \tilde{V} \in C_0(\mathbb{R}^2) \), and then apply Lemma 2.14 The integral kernel of
\[
[\tilde{V}^{(t)}, P_q]
\]
is
\[
(\tilde{V}(t^{-1}x) - \tilde{V}(t^{-1}y))P_q(x, y).
\]
Denoting \( f(z) = \chi(|z| \leq \epsilon) \), we rewrite this kernel as follows:
\begin{equation}
(\tilde{V}(t^{-1}x) - \tilde{V}(t^{-1}y))P_q(x, y)f^{(t)}(x - y)
\quad + \mathcal{X}^{(t)}(x; y; \tilde{V}, 1, 1 - f) - \mathcal{X}^{(t)}(x; y; 1, \tilde{V}, 1 - f).
\end{equation}
The operators, corresponding to the last two terms satisfy (3.7). For the first term in (3.11), we use that \( \tilde{V} \) has a compact support and so
\[
|\tilde{V}(t^{-1}x) - \tilde{V}(t^{-1}y)|f^{(t)}(x - y) \leq t^{-1} \max_z |\nabla \tilde{V}(z)| |x - y| \leq C\epsilon.
\]
Thus the norm of the operator corresponding to the first term in (3.12) is bounded by
\[
C\epsilon \max_x \int_{\mathbb{R}^2} |P_q(x, y)|dy \leq C'\epsilon,
\]
and hence it can be made arbitrarily small, which proves (3.8).

If \( V \in L^2(\mathbb{R}^2) \), then (3.9) follows immediately from (3.8) in view of the identity
\[
T_{q,q}(V) = R_{q,q}(V)P_q.
\]
If \( V \in L^1(\mathbb{R}^2) \), then for arbitrary \( \delta > 0 \) we approximate \( V \) with a function \( \tilde{V} \in C_0(\mathbb{R}^2) \), such that \( \| V - \tilde{V} \|_1 < \delta \), and use again Lemmas 3.3 and the formula (3.8). \( \square \)

**Corollary 3.6.** Let \( W_1 \in L^2(\mathbb{R}^2) \) and \( W_2 \in L^\infty(\mathbb{R}^2) \) be such that \( W_1W_2 = 0 \). Then
\[
\Re(\lambda; W_1, W_2) = 0 \quad \text{for all } \lambda > 0.
\]

**Proof.** Rewrite: \( S^{(t)}(W_1, W_2) = -R^{(t)}(W_1)W_2^{(t)} \). Consequently,
\[
\Re(\lambda; W_1, W_2) \leq \Re(\lambda\|W_2\|^{-1}; R^{(t)}(W_1)) = 0,
\]
by Corollary 3.5 \( \square \)
4. Eigenvalues of Toeplitz operators

4.1. Additivity of asymptotic coefficients. Further on, we will approximate weight functions by piece-wise constant ones. To describe these approximations we cut the plane $\mathbb{R}^2$ in the following way. For fixed $N \in \mathbb{N}$, $d > 0$ we tile $\mathbb{R}^2$ by disjoint annular sectors

\[ \Omega_{m,t} = \left\{ x = (\rho, \theta) : (m-1)d < \rho \leq md, \frac{2\pi}{N}(l-1) < \theta \leq \frac{2\pi}{N}l \right\}, \quad m \in \mathbb{N}, \quad l = 1, 2, \ldots, N. \]

For any set $\Omega$ we denote by $\chi(x \in \Omega)$ its characteristic function. Let $X_{m,l} = \chi(x \in \Omega_{m,l})$.

**Lemma 4.1.** If $(m,l) \neq (m',l')$, then $M(\lambda; X_{m,l}, X_{m',l'}) = 0$ for all $\lambda > 0$.

**Proof.** Immediately follows from Corollary 3.6. □

This result leads to the additivity of the asymptotic coefficients for piece-wise constant functions of the form

\[ W = \sum_{m,l} w_{m,l}X_{m,l}, \]

where $w_{m,l}$ are some complex numbers and the sum is finite.

**Lemma 4.2.** Let $W$ have the form (4.1). Then

\[ \sum_{m,l} m(\lambda + 0; w_{m,l}X_{m,l}) \leq m(\lambda; W) \leq \sum_{m,l} \mathfrak{R}(\lambda - 0; w_{m,l}X_{m,l}), \]

**Proof.** We prove the upper bound only, the lower bound is established in the same way, with obvious changes. Represent the operator $S^{(t)}(W)$ as

\[ S^{(t)}(W) = \sum_{m,l} |w_{m,l}|^2 S^{(t)}(X_{m,l}) + \sum_{(m,l) \neq (m',l')} w_{m,l}\overline{w_{m',l'}} S^{(t)}(X_{m,l}, X_{m',l'}) \]

\[ = S' + S''. \]

The family $S''$ in (4.3) is a finite sum of operators of the form considered in Lemma 4.1, therefore $\mathfrak{R}(\epsilon, S'') = 0$ for any $\epsilon > 0$. Further, the operator $S'$ is a direct sum of the operators $|w_{m,l}|^2 S^{(t)}(X_{m,l})$, therefore its spectrum is the union of spectra of summands, so

\[ \mathfrak{R}(\lambda, S') \leq \sum \mathfrak{R}(\lambda, |w_{m,l}|^2 S^{(t)}(X_{m,l})). \]

Now we can apply Lemma 2.5. □

Let us establish a similar additivity property for the operator $T(V)$ with a real-valued function $V$ of the form

\[ V = \sum_{m,l} v_{m,l}X_{m,l}, \]

where $v_{m,l}$ are real and the sum is finite.

**Lemma 4.3.** Let $V$ be of the form (4.5). Then

\[ \sum_{\pm v_{m,l} > 0} n^{(\pm)}(\lambda + 0; \pm v_{m,l}X_{m,l}) \leq n^{(\pm)}(\lambda; V) \]

\[ \leq \mathfrak{R}^{(\pm)}(\lambda; V) \leq \sum_{\pm v_{m,l} > 0} \mathfrak{R}^{(\pm)}(\lambda - 0; \pm v_{m,l}X_{m,l}). \]
Proof. As in the previous lemma, we prove only the upper bound. Let \( \Omega = \bigcup \Omega_{m,l} \) be the set where \( V(x) \neq 0 \), and let \( \Omega_{0,0} = \mathbb{R}^2 \setminus \Omega \). It is convenient to include the set \( \Omega_{0,0} \) in the family of \( \Omega_{m,l}'s \). Rewrite:

\[
T^{(t)}(V) = \sum X_{m,l} T^{(t)}(v_{n,s} X_{n,s}) X_{m',t'} = T' + T'',
\]

where

\[
T' = \sum T_{m,l}, \quad T_{m,l} = X_{m,l} T^{(t)}(v_{m,l} X_{m,l}) X_{m,l},
\]

and \( T'' \) is the sum in which at least one of the pairs \((m, l), (m', l')\) is distinct from \((n, s)\). Consider, for instance the term with \((m, l) \neq (n, s)\), and rewrite it as follows:

\[
X_{m,l} T^{(t)}(v_{n,s} X_{m',t'}) = v_{n,s} S^{(t)}(X_{m,l}, X_{n,s}) P X_{m',t'},
\]

so that by Corollary 3.3, the value \( \mathfrak{R}(\epsilon; \cdot \cdot) \) for this operator equals zero for any \( \epsilon > 0 \). Consequently, \( \mathfrak{R}(\epsilon; T'') = 0 \) for any \( \epsilon > 0 \). Next, \( T' \) is an orthogonal sum of operators \( T_{m,l} \), therefore

\[
n_{\pm}(\lambda; T') = \sum_{\pm \nu_{m,l}>0} n_{\pm}(\lambda; \pm T_{m,l}), \quad \mathfrak{R}^{(\pm)}(\lambda; T') \leq \sum_{\pm \nu_{m,l}>0} \mathfrak{R}^{(\pm)}(\lambda; \pm T_{m,l}).
\]

So, by Lemma 2.5

\[
\mathfrak{R}^{(\pm)}(\lambda; V) \leq \sum_{\pm \nu_{m,l}>0} \mathfrak{R}^{(\pm)}(\lambda - 0; \pm T_{m,l}).
\]

Now we apply again Corollary 3.6 and Lemma 2.5 to each of operators \( T_{m,l} \), which gives

\[
\mathfrak{R}^{(\pm)}(\lambda - 0; \pm T_{m,l}) \leq n_{\pm}(\lambda; \pm T_{m,l}, X_{m,l}),
\]

and this leads to the required upper bound. \( \square \)

Another kind of additivity holds with respect to the Landau projections. For \( J > 1 \) we denote by \( P^{(J)} \) the projection \( P^{(J)} = \sum_{q \leq J} P_q \). Consider the Toeplitz family \( T^{(t)}_{(J)}(V) = P^{(J)} V^{(t)} P^{(J)} \).

Lemma 4.4. For \( V \in L^1(\mathbb{R}^2) + L^2(\mathbb{R}^2), \lambda > 0 \),

\[
\sum_{q \leq J} n_q^{(\pm)}(\lambda + 0; V) \leq \mathfrak{e}^{(\pm)}(\lambda; T^{(t)}_{(J)}) \leq \mathfrak{R}^{(\pm)}(\lambda; T^{(t)}_{(J)}) \leq \sum_{q \leq J} \mathfrak{R}^{(\pm)}(\lambda - 0; V).
\]

Proof. We split the family \( T^{(t)}_{(J)} \) as follows:

\[
T^{(t)}_{(J)} = \sum_{q} T^{(t)}_{q} + \sum_{q \neq q'} T^{(t)}_{q,q'} = T' + T''.
\]

The operators in \( T' \) act in orthogonal subspaces, so their distribution functions add up. By Corollary 3.3 \( \mathfrak{R}(\epsilon; T^{(t)}_{q,q'}) = 0, \ q \neq q' \) for any \( \epsilon > 0 \), so that \( \mathfrak{R}(\epsilon; T'') = 0 \), and (4.7) follows by Lemma 2.5 \( \square \)
4.2. Model integral operators. In order to pass from the conditional results in Subsection 4.1 to actual calculations, we need at least some operators for which the asymptotic coefficients are known. Here we consider such model operators.

Lemma 4.5. Let $0 \leq d_1 < d_2 < \infty$, and let $W = \chi(d_1 < |x| < d_2)$. Then for any $\lambda > 0$

$$\mathfrak{m}(\lambda; W) = m(\lambda; W) = \begin{cases} \frac{B}{2}(d_2^2 - d_1^2), & \lambda < 1, \\ 0, & \lambda \geq 1. \end{cases}$$

(4.8) Proof. Since the function $W$ is radially symmetric, using (3.1) we can immediately find all eigenvalues of the operator $S^{(t)}_q(W)$ explicitly (see [14], Lemma 3.4):

$$\lambda_j = \lambda^{(t)}_j = \lambda^{(t)}_j(d_1, d_2) = \int_{d_1 t < |x| < d_2 t} |\psi_{q,j}(x)|^2 dx = \frac{q^j}{(j+q)!} \int_{\eta_1}^{\eta_2} \xi^j e^{-\xi} (L^{(j)}_q(\xi))^2 d\xi,$$

where we denote $\eta_k = B(d_k t)^2/2, k = 1, 2$. Note that $\lambda^{(t)}_j$’s are not necessarily labeled in the usual decreasing order. The functions $\psi_{q,j}$ are normalized, so $\lambda_j \leq 1$ and therefore (4.8) holds for $\lambda \geq 1$.

Let now $\lambda < 1$. We will find the asymptotics of $\lambda_j = \lambda^{(t)}_j(0, d)$ as $t \to \infty$. Denote $\eta = B(t d)^2/2$ and fix an $\epsilon > 0$. Suppose first that $j \geq (1+\epsilon)\eta$. Then (3.4) implies

$$\lambda_j \leq (j+q)^{2\epsilon} \frac{q^j}{(j+q)!} \int_0^{\eta} \xi^j \exp\left(-\left(1 - \frac{2}{j+q}\right)\xi\right) d\xi \leq (j+q)^{2\epsilon} \frac{q^j}{(j+q)!} e^{j+1} \int_{\eta}^{\eta + q} \xi^j e^{-\xi} d\xi.$$

The maximum of the integrand is attained at $\xi = j$, it grows for $\xi < j$, thus we can estimate it from above by $\eta^j e^{-\eta}$, which leads to the bound

$$\lambda_j \leq C(j+q)^{2\epsilon} \frac{q^j}{(j+q)!} \eta^{j+1} e^{-\eta}.$$

Using the Stirling formula, we get

$$\lambda_j \leq C \frac{(j+q)^{2\epsilon} q^{j}}{(j+q)^{j+q+1/2}} q^{j+q-\eta} \leq e^{\eta j + (q+1/2) q^{j+q-\eta}} = e^{\eta j + (q+1/2) q^{j+q-\eta}} \tau_j(\eta), \quad \tau_j(\eta) = \left( \frac{\eta}{j} \right)^j \epsilon^{-\eta}.$$

To estimate $\tau_j(\eta)$ we rewrite it as

$$\tau_j(\eta) = \exp \left[ - j \int_{\eta}^{\eta + q} \left( \frac{t}{s} - 1 \right) ds \right].$$

Now we fix $\epsilon_1 \in (0, \epsilon)$ and obtain:

$$\tau_j(\eta) \leq \exp \left[ - j \int_{\eta}^{\eta + q} \left( \frac{t}{s} - 1 \right) ds \right] \leq \exp \left[ - j \epsilon_1 \int_{\eta}^{\eta + q} ds \right] \leq \exp \left( - j \epsilon_1 \frac{\epsilon - \epsilon_1}{(1+\epsilon)(1+\epsilon_1)} \right).$$
This shows that \( \lambda_j \) tends to zero very fast as \( \eta \to \infty \) and \( j \geq (1 + \epsilon)\eta \). Assume now that \( j \leq (1 - \epsilon)\eta \). Since the functions \( \nu_{q,j} \) are normalized, we have

\[
\lambda_j = 1 - \mu_j, \quad \mu_j = \frac{q!}{(j + q)!} \int_\eta^\infty \xi^j e^{-\xi} (L_q^j(\xi))^2 d\xi > 0.
\]

Then, using (3.4) again, we obtain

\[
\mu_j \leq (j + q)^2 \frac{q!}{(j + q)!} \int_\eta^\infty \xi^j \exp \left( -\left(1 - \frac{2}{j + q}\right)\xi \right) d\xi
\]

For an arbitrary \( \eta_1 \in (0, 1) \) rewrite the integrand as follows:

\[
\left[ \xi^j e^{-(1 - \epsilon_1)\xi} \right] \exp \left( -\left(1 - \frac{2}{j + q}\right)\xi \right)
\]

The maximum of the term in brackets is attained at \( j(1 - \epsilon_1)^{-1} \). For \( \epsilon_1 > \epsilon \), so that \( j(1 - \epsilon_1)^{-1} < \eta \), we conclude that on the interval \([\eta, \infty)\) the integrand does not exceed

\[
\left[ \eta^j e^{-(1 - \epsilon_1)\eta} \right] \exp \left( -\left(1 - \frac{2}{j + q}\right)\xi \right).
\]

Integrating, we get

\[
\mu_j \leq (j + q)^2 \frac{q!}{(j + q)!} \frac{1}{\epsilon_1 - 2(j + q)^{-1}} \eta^j e^{-(1 - 2(j + q)^{-1})\eta}.
\]

As at the first step of the proof, by the Stirling formula we obtain

\[
\mu_j \leq C(j + q)^{\eta_1 - 1/2} \eta e^{\eta_1 - 2(j + q)^{-1}} e^{2(j + q)^{-1}\eta} \tau_j(\eta),
\]

with the function \( \tau_j(\eta) \) defined in (4.10). To estimate it, we rewrite

\[
\tau_j(\eta) = \exp \left[ -\int_j^\eta \left(1 - \frac{j}{s} \right) ds \right].
\]

Choose an \( \epsilon_1 \in (0, \epsilon) \) and estimate:

\[
\tau_j(\eta) \leq \exp \left[ -\int_j^\eta \left(1 - \frac{j}{s} \right) ds \right] \leq \exp \left[ -\epsilon_1 \int_j^\eta ds \right] \leq \exp \left( -\epsilon_1 \frac{\eta - \epsilon_1}{1 - \epsilon_1} \right).
\]

This shows that \( \mu_j \) tends to zero very fast as \( \eta \to \infty \) and \( j \leq (1 - \epsilon)\eta \), that is \( \lambda_j \to 1 \) as \( t \to \infty \), uniformly for \( j \leq (1 - \epsilon)\eta \). Summarizing the above calculations, we see that for sufficiently large \( t \) the following inequalities hold:

\[
\lambda_j(d_1, d_2) \equiv \lambda_j(0, d_2) - \lambda_j(0, d_1) \begin{cases} < \lambda, & j < (1 - \epsilon)\eta_1; \\
> \lambda, & (1 + \epsilon)\eta_1 < j < (1 - \epsilon)\eta_2; \\
< \lambda, & j > (1 + \epsilon)\eta_2.
\end{cases}
\]

Consequently, \( (1 - \epsilon)\eta_2 - (1 + \epsilon)\eta_1 \leq n(\lambda, S(t)(W)) \leq (1 + \epsilon)\eta_2 - (1 - \epsilon)\eta_1 \) for sufficiently large \( t \). Passing to the limit, we obtain

\[
m(\lambda; W) \geq (1 - \epsilon) \frac{Bd_2^2}{2} - (1 + \epsilon) \frac{Bd_1^2}{2}, \quad \Re(\lambda; W) \leq (1 + \epsilon) \frac{Bd_2^2}{2} - (1 - \epsilon) \frac{Bd_1^2}{2}.
\]

Since \( \epsilon \in (0, 1) \) is arbitrary, this entails (4.8). \( \square \)

Now, using the additivity and the calculations for the model operator we can find \( m(\lambda; w_{m,t} X_{m,t}) \) and \( \Re(\lambda; w_{m,t} X_{m,t}) \) which turn out to be equal.
Lemma 4.6. If \( \lambda \geq |w_{m,l}|^2 \), then \( \mathfrak{M}(\lambda, w_{m,l}X_{m,l}) = 0 \). For any \( \lambda \in (0, |w_{m,l}|^2) \)
\[
\mathfrak{m}(\lambda; w_{m,l}X_{m,l}) = 2\mathfrak{M}(\lambda; w_{m,l}X_{m,l}) = \frac{B}{2\pi}|\Omega_{m,l}|.
\]

Proof. Since \( n(\lambda, S^{(t)}(w_{m,l}X_{m,l})) = n(\lambda|w_{m,l}|^{-2}, S^{(t)}(X_{m,l})) \), it suffices to consider
the case \( w_{m,l} = 1 \). The norm of the operator \( S^{(t)}(X_{m,l}) \) is not greater than 1, and
this takes care of the case \( \lambda \geq 1 \). Next consider \( X_m = \sum_{l=1}^{N} X_{m,l} \). It is clear
that the asymptotic coefficients are the same for all sectors \( X_{m,l}, l = 1, \ldots, N \).
Therefore, by Lemma 4.2,
\[
(4.11) \quad \mathfrak{m}(\lambda + 0; X_m) \leq N\mathfrak{m}(\lambda; X_{m,l}) \leq N\mathfrak{M}(\lambda; X_{m,l}) \leq \mathfrak{M}(\lambda - 0; X_m).
\]
Now (4.8), produces the required formula. \( \square \)

Before we proceed to treating more general functions \( W \), we introduce, similarly
to (1.1), the appropriate asymptotic coefficients. For \( V \in L^1(\mathbb{R}^2) \) and \( \lambda > 0 \) we
define \( sup- \) and \( sub- \) measures of \( V \):
\[
(4.12) \quad A^{(\pm)}(\lambda; V) = A(\lambda, \infty; \pm V) = |\{ x : \pm V(x) > \lambda \}|,
\]

With this notation, the result of Lemma 4.6 reads
\[
(4.13) \quad \mathfrak{m}(\lambda; w_{m,l}X_{m,l}) = 2\mathfrak{M}(\lambda; w_{m,l}X_{m,l}) = \frac{B}{2\pi} A^{(+)}(\lambda; |w_{m,l}|^2 X_{m,l}),
\]
for all \( \lambda > 0 \).

Corollary 4.7. Let \( v_{m,l} \) be real for some \( m, l \). Then
\[
\mathfrak{n}^{(\pm)}(\lambda; v_{m,l}X_{m,l}) = \mathfrak{N}^{(\pm)}(\lambda; v_{m,l}X_{m,l}) = \frac{B}{2\pi} A^{(\pm)}(\lambda; v_{m,l}X_{m,l}),
\]
for all \( \lambda > 0 \).

Proof. Denote \( V = v_{m,l}X_{m,l} \). If \( \pm v_{m,l} > 0 \), then obviously \( \mathfrak{N}^{(\pm)}(\lambda; V) = 0 \), and
\( \mathfrak{N}^{(\pm)}(\lambda; V) = \mathfrak{N}^{(\pm)}(\lambda; \pm V) = \mathfrak{M}(\lambda; \sqrt{\pm V}) \). It remains to use formula (4.13). \( \square \)

In order to proceed we need to establish the "continuity" of the coefficients
\( A^{(\pm)}(\lambda; V) \) in the function \( V \):

Lemma 4.8. Let \( W_\delta = V + V_\delta^\prime, \ \delta \neq 0, \) where \( V, V_\delta^\prime \in L^1(\mathbb{R}^2), \ V \) does not depend
on \( \delta \) and \( \|V_\delta^\prime\|_1 \rightarrow 0 \) as \( \delta \rightarrow 0 \). Then
\[
(4.14) \quad A^{(\pm)}(\lambda; V) \leq \liminf_{\delta \rightarrow 0} A^{(\pm)}(\lambda; W_\delta)
\leq \limsup_{\delta \rightarrow 0} A^{(\pm)}(\lambda; W_\delta) \leq A^{(\pm)}(\lambda - 0; V).
\]

Proof. It suffices to consider the sign "+". By definition, for any \( \epsilon \in (0, \lambda) \),
\[
A^{(+)}(\lambda; W_\delta) \leq |\{ x : W_\delta(x) > \lambda \} \cap \{ x : |V_\delta^\prime(x)| \leq \epsilon \}| + |\{ x : |V_\delta^\prime(x)| > \epsilon \}|
\leq A^{(+)}(\lambda - \epsilon; V) + \epsilon^{-1}\|V_\delta^\prime\|_1.
\]
For the last estimate we have used the Chebyshev inequality. Passing to the limit
as \( \delta \rightarrow 0 \) and \( \epsilon \downarrow 0 \), we get the upper bound in (4.14). Similarly, write
\[
A^{(+)}(\lambda; W_\delta) \geq A^{(+)}(\lambda + \epsilon; V) - \epsilon^{-1}\|V_\delta^\prime\|_1.
\]
Passing again to the limit as \( \delta \rightarrow 0 \) and \( \epsilon \downarrow 0 \), we get the lower bound in (4.14). \( \square \)
4.3. Eigenvalue asymptotics for Toeplitz operators. Corollary \[4.3\] enables us to establish the spectral asymptotics for the Toeplitz operator \(T^{(t)}(V)\) with a piece-wise constant function \(V\).

**Lemma 4.9.** Let \(V = \sum v_{ml}X_{ml}\), where the sum is finite and \(v_{ml}\) are real-valued. Then for any \(q \geq 0\)

\[
(4.15) \quad \frac{B}{2\pi} \lambda(\pm; V) \leq n^{(\pm)}(\lambda; V) \leq \mathcal{N}(\pm)(\lambda; V) \leq \frac{B}{2\pi} \lambda(\pm; V - 0; V),
\]

for all \(\lambda > 0\).

**Proof.** Use Lemma \[4.3\] and Corollary \[4.7\].\[4.7\]

We are now in position to treat the general case.

**Theorem 4.10.** For any \(V \in L^1(\mathbb{R}^2)\)

\[
(4.16) \quad \frac{B}{2\pi} \lambda(\pm; V) \leq n^{(\pm)}(\lambda; V) \leq \mathcal{N}(\pm)(\lambda; V) \leq \frac{B}{2\pi} \lambda(\pm; V - 0; V).
\]

Moreover, if \(\lambda\) is a generic value for \(\pm V\), we have the asymptotics

\[
(4.17) \quad n^{(\pm)}(\lambda; V) = \mathcal{N}(\pm)(\lambda; V) = \frac{B}{2\pi} \lambda(\pm; \lambda; V).
\]

**Proof.** Since \(n_-(\lambda; T^{(t)}(V)) = n_+(\lambda; T^{(t)}(-V))\), it suffices to consider the sign \(\pm\) only.

For a positive \(\delta\) we find a sufficiently fine tiling of the plane by annular sectors \(\Omega_{m,t}\) and a piecewise constant function \(V_\delta\) represented in the form \[4.5\] with a finite sum, such that \(\|V - V_\delta\|_1 < \delta\). Hence by Lemma \[3.3\] \(\mathcal{N}(\epsilon; V - V_\delta) \leq B\delta(2\pi\epsilon)^{-1}\). Furthermore, by Lemma \[4.8\]

\[
(4.18) \quad \mathcal{N}(\pm)(\lambda; V_\delta) \leq \frac{B}{2\pi} \lambda(\pm; V_\delta).
\]

Thus by Lemma \[2.5\]

\[
\mathcal{N}(\pm)(\lambda; V) \leq \frac{B}{2\pi} \limsup_{\epsilon \to 0} \limsup_{\delta \to 0} \lambda(\pm; \epsilon; V_\delta).
\]

By virtue of Lemma \[1.8\] the right hand side does not exceed \(B(2\pi)^{-1}\lambda(\pm; V - 0; V)\), as required.

The corresponding lower bound for \(n^{(\pm)}(\lambda; V)\) is established similarly.\[4.7\]

5. Reduction to Toeplitz operators

Let \((\lambda_1, \lambda_2) \in (\Lambda_\nu, \Lambda_{\nu+1})\) with some \(\nu = -1, 0, \ldots\), as in Theorem \[1.1\]. We denote

\[
a = (\lambda_1 + \lambda_2)/2, \quad b = (\lambda_2 - \lambda_1)/2, \quad H_0a = H_0 - a, \quad H_a = H_a(V^{(t)}) = H + V^{(t)} - a.
\]

Our analysis of the counting function \(N(\lambda_1, \lambda_2; H)\) is based upon the obvious relations (cf. \[1.4\]):

\[
(5.1) \quad N(\lambda_1, \lambda_2; H) = N(\lambda_1 - a, \lambda_2 - a; H - a) = N(-b, b; H - a) = N(b^2; (H - a)^2),
\]

where, recall, \(N(\eta; L)\) stands for the number of eigenvalues of a semi-bounded operator \(L\) below \(\eta\). For methodological purposes we need to consider an operator
having somewhat more general form. Namely, with a fixed, for real-valued functions \( V \in L^1(\mathbb{R}^2), Z \in L^1(\mathbb{R}^2) \) we set

\[
L^{(t)}(V, Z) = H^2_{0a} + V^{(t)} H_{0a} + H_{0a} V^{(t)} + Z^{(t)},
\]

so that \((H - a)^2 = L^{(t)}(V, V^2)\). Under these conditions the operator is well defined via the quadratic form

\[
\|H_{0a} u\|^2 + 2 \text{Re}(H_{0a} u, V^{(t)} u) + (Z^{(t)} u, u)
\]

for \( u \in \text{Dom}(H_0) \). Using the diamagnetic inequality, it is easy to check that each term of the perturbation \( V^{(t)} H_{0a} + H_{0a} V^{(t)} + Z^{(t)} \) is \( H^2_{0a} \)-form-compact. The lowest point of the essential spectrum of \( L^{(t)}(V, Z) \) is

\[
\eta_0 = \min_{q \geq 0} (A_q - a)^2 > 0.
\]

We are going to study the discrete spectrum of the operator \( L^{(t)} \), independently of its connection with \( H^{(t)} \). The following important lemma establishes asymptotic relations of the spectrum of \( L^{(t)}(V, Z) \) below \( \eta_0 \) and the spectra of certain Toeplitz type operators.

**Lemma 5.1.** Suppose that \( V, Z \in C_0^\infty(\mathbb{R}^2) \), and let \( \eta < \eta_0 \). Then for any \( \epsilon > 0 \) there exists an \( \tilde{\epsilon} \downarrow 0 \) as \( \epsilon \downarrow 0 \), and an integer \( J_0 = J_0(\epsilon) \) such that for all \( J \geq J_0 \) and the projection

\[
P^{(J)} = \sum_{q \leq J} P_q
\]

we have

\[
b_0(\eta; L(V, Z)) \geq b_0(\eta; P^{(J)} (1 - \epsilon) L(1 - \epsilon V, (1 - \epsilon) Z) P^{(J)}),
\]

\[
\mathfrak{B}_0(\eta; L(V, Z)) \leq \mathfrak{B}_0(\eta + \tilde{\epsilon}; P^{(J)} (1 + \epsilon) V, (1 + \epsilon) Z) P^{(J)}).
\]

**Proof.** We denote \( P = P^{(J)}, Q = I - P \). Let \( v \) be a constant such that

\[
\sup_{x} (|V(x)| + |\nabla V(x)| + |\Delta V(x)| + |Z(x)|) \leq v.
\]

Our first aim is to bound the counting function \( N(\eta, L^{(t)}) \) from above and from below by similar counting functions for \( P L^{(t)} P \) and \( Q L^{(t)} Q \).

Recalling that \( H_0 = \Pi_1^2 + \Pi_2^2 \) with \( \Pi_k = -i \partial_k - a_k \), \( k = 1, 2 \), and denoting \( \tilde{V}_k(x) = -i(\partial_k V)(x) \), we conclude that

\[
[H_0, V^{(t)}] = \sum_{k=1}^{2} (\Pi_k [\Pi_k, V^{(t)}] + [\Pi_k, V^{(t)}] \Pi_k)
\]

\[
= t^{-1} \sum_{k=1}^{2} (\Pi_k \tilde{V}_k^{(t)} + \tilde{V}_k^{(t)} \Pi_k) = t^{-1} \sum_{k=1}^{2} \Pi_k \tilde{V}_k^{(t)} + t^{-2} (\Delta V)^{(t)}.
\]
Consequently,

\[
|\langle P[H_0, V(t)]Qu, u \rangle| \leq t^{-1} \sum_{k=1}^{2} |\langle \tilde{V}_k(t)Qu, \Pi_k Pu \rangle + v t^{-2} \|u\|^2 |
\]

\[
\leq (2t^2)^{-1} (H_0Pu, Pu) + 2^{-1}v^2 \|Qu\|^2 + vt^{-2} \|u\|^2
\]

\[
\leq (4t^2)^{-1} (H_{0a}Pu, Pu) + 2^{-1}v^2 \|Qu\|^2 + (v + a + 1)t^{-2} \|u\|^2.
\]

Now it follows from (5.2) that for any \( \delta > 0 \)

\[
|\langle PL(t)Qu, u \rangle| \leq 2|\langle V(t)Qu, H_{0a}Pu \rangle| + |\langle P[H_0, V(t)]Qu, u \rangle| + |\langle Z(t)Qu, Pu \rangle|
\]

\[
\leq \delta \|H_{0a}Pu\|^2 + \frac{v}{\delta} \|Qu\|^2 + (4t^2)^{-1}(H_{0a}^2Pu, Pu)
\]

\[
+ 2^{-1}v^2 \|Qu\|^2 + (v + a + 1)t^{-2} \|u\|^2 + \frac{\delta}{2} \|Pu\|^2 + \frac{v^2}{29} \|Qu\|^2
\]

\[
\leq (\delta + (4t^2)^{-1})(H_{0a}^2Pu, Pu) + \frac{\delta}{2} \|Pu\|^2
\]

\[
+ 2^{-1}v^2 (1 + \delta^{-1}) \|Qu\|^2 + (v + a + 1)t^{-2} \|u\|^2.
\]

Therefore,

\[
(L(t)u, u) \leq (L(t)Pu, Pu) + 2(\delta + (4t^2)^{-1})(H_{0a}Pu, Pu) + \delta \|Pu\|^2
\]

\[
+ v^2(1 + \delta^{-1}) \|Qu\|^2 + 2(v + a + 1)t^{-2} \|u\|^2 + (QL(t)Qu, u),
\]

and

\[
(L(t)u, u) \geq (L(t)Pu, Pu) - 2(\delta + (4t^2)^{-1})(H_{0a}^2Pu, Pu) - \delta \|Pu\|^2
\]

\[
- v^2(1 + \delta^{-1}) \|Qu\|^2 - 2(v + a + 1)t^{-2} \|u\|^2 + (QL(t)Qu, u).
\]

Denote

\[
\epsilon_1 = 2(\delta + (4t^2)^{-1}), \quad \epsilon_2 = \delta + 2(v + a + 1)t^{-2}, M = v^2(1 + \delta^{-1}) + 2(v + a + 1)t^{-2},
\]

so that

(5.6)

\[
(L(t)u, u) \leq (L(t)Pu, Pu) + \epsilon_1(H_{0a}^2Pu, Pu) + \epsilon_2\|Pu\|^2 + M \|Qu\|^2 + (QL(t)Qu, u),
\]

\[
(L(t)u, u) \geq (L(t)Pu, Pu) - \epsilon_1(H_{0a}^2Pu, Pu) - \epsilon_2\|Pu\|^2 - M \|Qu\|^2 + (QL(t)Qu, u).
\]

Thus we have bounded \( L(t) \) from above and from below by orthogonal sums of operators acting in ranges of \( P \) and \( Q \). We show now that for sufficiently large \( J \) the operators containing \( Q \) do not contribute to the \( t \)-asymptotics of the counting function. We have

\[
(QL(t)Qu, u) = \|H_{0a}Qu\|^2 + (H_{0a}Qu, V(t)Qu) + (V(t)Qu, H_{0a}Qu) + (Z(t)Qu, Qu)
\]

\[
\geq \|H_{0a}Qu\|^2 - \frac{1}{2} \|H_{0a}Qu\|^2 - 2\|V(t)Qu\|^2 - v \|Qu\|^2
\]

\[
\geq \frac{1}{2} \|H_{0a}Qu\|^2 - (2v^2 + v)\|Qu\|^2 \geq \frac{1}{2} \|(A_J - a)Qu\|^2 - (2v^2 + v)\|Qu\|^2.
\]
So, for sufficiently large $J$ (depending on $M$ and $\delta$), we have $QL^{(t)}Q - MQ \geq \eta Q$ and therefore the operator $QL^{(t)}Q - MQ$ acting in the range of $Q$ has no spectrum below $\eta$.

Finally, to estimate the operator containing $P$ in (5.6), we can write

$$L^{(t)}(V, Z) \pm \epsilon_1 H_0 = (1 \pm \epsilon_1) L^{(t)}(V(1 \pm \epsilon_1)^{-1}, Z(1 \pm \epsilon_1)^{-1}).$$

Therefore, for sufficiently large $J$ it follows from (5.6) that

$$N(\eta; L^{(t)}) \leq N(\eta + \epsilon_2; (1 - \epsilon_1) PL^{(t)}(V(1 - \epsilon_1)^{-1}, Z(1 - \epsilon_1)^{-1}) P),$$

$$N(\eta; L^{(t)}) \geq N(\eta - \epsilon_2; (1 + \epsilon_1) PL^{(t)}(V(1 + \epsilon_1)^{-1}, Z(1 + \epsilon_1)^{-1}) P),$$

and (5.5) follows.

Next we are going to derive an asymptotic estimate for the operator $PL^{(t)}P$ as $t \to \infty$. We define the asymptotic coefficient as

$$\mathcal{B}(\eta; V, Z) = \sum_{q=0}^{\infty} \frac{B}{2\pi} A^{(+)}((\Lambda_q - a)^2 - \eta, -2(\Lambda_q - a)V - Z).$$

Under the assumptions $V \in L^2(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$, $Z \in L^1(\mathbb{R}^2)$ the above series is absolutely convergent uniformly in $\eta$ varying on a compact set. Indeed,

$$A^{(+)}((\Lambda_q - a)^2 - \eta, -2(\Lambda_q - a)V - Z) \leq A^{(+)}((\Lambda_q - a)^2 - \eta, |\Lambda_q - a| |V| + |Z|)$$

$$\leq A^{(+)}((\Lambda_q - a)^2 - \eta, \frac{1}{2}(\Lambda_q - a)^2 + 2|V|^2 + |Z|)$$

$$\leq A^{(+)}(\frac{1}{2}(\Lambda_q - a)^2 - \eta, 2|V|^2 + |Z|).$$

For sufficiently large $q$, by the Chebyshev inequality the last term is bounded from above by

$$\left(\frac{1}{2}(\Lambda_q - a)^2 - \eta\right)^{-1}(2||V||_2^2 + ||Z||_1).$$

This shows that indeed the series in (5.7) converges. Each term in (5.7) is a left semi-continuous non-decreasing function of $\eta$, and as a result, $\mathcal{B}(\eta; V, Z)$ is left semi-continuous as well.

Let $P^{(J)}$ be as defined in (5.4).

**Lemma 5.2.** Let $V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, $Z \in L^1(\mathbb{R}^2)$. Then for any $\eta < \eta_0$

$$\liminf_{J \to \infty} b(\eta; P^{(J)}L(V, Z)P^{(J)}) \geq \mathcal{B}(\eta; V, Z),$$

$$\limsup_{J \to \infty} 2b(\eta; P^{(J)}L(V, Z)P^{(J)}) \leq \mathcal{B}(\eta + 0; V, Z).$$

In particular, if $(\Lambda_q - a)^2 - \eta$ are generic values for $-2(\Lambda_q - a)V - Z$ for all $q = 0, 1, \ldots$, then

$$\lim_{J \to \infty} \lim_{t \to \infty} t^{-2} N(\eta; P^{(J)}L^{(t)}(V, Z)P^{(J)}) = \mathcal{B}(\eta; V, Z).$$
\section*{6. Reduction to a smooth potential}

\subsection*{6.1. Further estimates for the operator $L^{(t)}(V, Z)$}

In this section we continue the study of the operator $L^{(t)}(V, Z)$. Our aim now is to extend Corollary \ref{cor5.3} to non-smooth functions $V$ and $Z$.

Recall again the notation

$$L^{(t)} = L^{(t)}(V, Z) = H_{0a}^2 + V^{(t)}H_{0a} + H_{0a}V^{(t)} + Z^{(t)}$$

for some real-valued functions $V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, $Z \in L^1(\mathbb{R}^2)$. We start with an eigenvalue estimate for the operator $L^{(t)}$.

\begin{lemma}
Suppose that $V \in L^2(\mathbb{R}^2)$ and $Z \in L^1(\mathbb{R}^2)$. Then

\begin{equation}
\mathcal{B}(\eta; L(V, Z)) \leq \frac{C}{(\eta_0 - \eta)^2}(\|V\|_2^2 + \|Z\|_1),
\end{equation}

for all $\eta < \eta_0$ and some constant $C$ independent of $\eta$.
\end{lemma}

Proof. Since \(|(V^{(t)}u, H_{0a}u)| \leq \frac{1}{2} \|V^{(t)}u\|^2 + \frac{\gamma}{2} \|H_{0a}u\|^2\), we have for \(z \in (0, 1)\):
\[L^{(t)}(V, Z) \geq (1 - z)H_{0a}^2 - z^{-1}(V^{(t)})^2 - |Z^{(t)}|\].
We chose \(z = \frac{\eta - \eta_0}{2\eta_0}\) so that \((1 - z)H_{0a}^2 - \eta > 0\) and \((1 - z)H_{0a}^2 - \eta \geq c(\eta_0 - \eta)H_{0a}^2 - 1\), and denote \(Y^{(t)}_{a,\eta} = (1 - z)H_{0a}^2 - \eta\). By the Birman-Schwinger principle we have
\[N(\eta; L^{(t)}(V, Z)) \leq n(1; (Y^{(t)}_{a,\eta})^* Y^{(t)}_{a,\eta})\].
By the diamagnetic inequality we have
\[\|Y^{(t)}_{a,\eta}\|_{\mathcal{B}^2} \leq C(\eta_0 - \eta)^{-1} \|Y^{(t)}_{a,0}\|_{\mathcal{B}^2}\]
\[\leq C(\eta_0 - \eta)^{-1} \|(-\Delta + I)^{-1}(z^{-1}(V^{(t)})^2 + |Z^{(t)}|)^{1/2}\|_{\mathcal{B}^2}\]
\[\leq (\eta_0 - \eta)^{-2}t^2 (\|V\|^2 + \|Z\|_1)\].
Therefore \((Y^{(t)}_{a,\eta})^* Y^{(t)}_{a,\eta}\) is a trace class operator and
\[\|((Y^{(t)}_{a,\eta})^* Y^{(t)}_{a,\eta})\|_{\mathcal{B}^1} \leq C(\eta_0 - \eta)^{-2}t^2 (\|V\|^2 + \|Z\|_1)\].
The inequality (6.1) follows now by applying the bound \(n(\lambda; K) \leq \lambda^{-1} \|K\|_{\mathcal{B}^1}\).

**Theorem 6.2.** Let \(V \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)\) and \(Z \in L^1(\mathbb{R}^2)\). Then formula (5.10) holds for any \(\eta < \eta_0\).

**Proof.** The idea is to approximate \(V, Z\) by smooth functions and then use Lemma 5.4 to show that the error does not contribute.

For a fixed \(\delta > 0\) represent \(V, Z\) as \(V = V_{\delta}' + V_{\delta}''\), \(Z = Z_{\delta}' + Z_{\delta}''\) so that \(V_{\delta}', Z_{\delta}' \in C_0^\infty(\mathbb{R}^2)\) and
\[\|V_{\delta}'\|_1 + \|V_{\delta}''\|_2 + \|Z_{\delta}'\|_1 < \delta\].
We apply Lemma 2.3. Clearly,
\[L(V, Z) = L(0, 0) + Y_{\delta}' + Y_{\delta}'',\]
\[Y_{\delta}' = V_{\delta}' H_{0a} + H_{0a} V_{\delta}' + Z_{\delta}', \ Y_{\delta}'' = V_{\delta}'' H_{0a} + H_{0a} V_{\delta}'' + Z_{\delta}''.\]
\[L(V, Z) = L(0, 0)\]. According to (6.1)
\[\mathcal{B}(\tau; L(0, 0) + MY_{\delta}'') \leq C(1 + M^2)\frac{\delta}{(\eta_0 - \tau)^2}\]
for any \(\tau < \eta_0\) and any \(M \in R\), and thus the condition (2.10) is fulfilled. Now it follows from Corollary 5.3 and Lemma 2.3 that
\[\mathcal{B}(\eta; L(V, Z)) \leq \lim_{\eta \to 0} \lim_{M \to 0} \sup_{\delta \to 0} \mathcal{B}(\eta + \epsilon; MV_{\delta}', M Z_{\delta}'')\]
By Lemma 1.3 the right hand side does not exceed \(\mathcal{B}(\eta + 0; V, Z)\), as claimed. Similarly, one obtains the appropriate lower bound for \(b(\eta, L(V, Z))\). □

6.2. The general asymptotic estimate: proof of Theorem 1.1. Recall that the coefficients \(a\) and \(b\) are defined in (1.2) and (5.7) respectively. By (5.1) we have
\[N(\lambda_1, \lambda_2; H^{(t)}) = N(b^2; L^{(t)}(V, V^2))\]. Observe that \(a(\lambda_1, \lambda_2; V) = \mathcal{B}(b^2; V, V^2)\). It remains to apply Theorem 6.2.
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