Homogeneous Isotropic Fluid Turbulence
Simulated with the Lattice Vortex Tube Model

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Fully developed turbulence is analyzed with the lattice model employing vortex tube representation which is introduced recently by the authors. Several characteristic features observed in experiments and direct numeric integrations are reproduced. Not only Kolmogorov’s inertial range is observed, but also several local probability distribution functions are obtained as well. Those of the local velocities are close to the Gaussian and exponential-like distributions appear in local vorticity, relative velocities and local velocity consisting of only higher wave number components. Coherent structure of vortex tubes is seen, too. Moreover required cpu-time and memory-size are very little comparing with the conventional pseudo-spectral method.

keywords: fluid turbulence, vortex tube, lattice model, numerical technic.

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I. INTRODUCTION

Fully-developed turbulence is one of the important topics in fluid dynamics. Although it has been studied over a century, we have not yet gotten any clear concept of what turbulence is. Even if we consider only the simplest case of 'Homogeneous Isotropic Turbulence', we do not know what the essential point is. This is mainly because there has been no powerful tool to investigate it on either experiment or theory. In the experiments, the data including most information is no more than flow lines of trace-particles on the two dimensional subsection, which is far from enough to investigate the fluid in details. Theorists have no general method to deal highly nonlinear systems with many degrees of freedom which the turbulence belongs to. Even using numerical methods, it is very hard to get the solution of fully developed turbulence because of 'fractal property' of it. If a system has fractal nature, we cannot find any characteristic length. In such an occasion, we must calculate on a large system to include many kinds of scale length. This requires us to use large lattice sizes (or many number of wave numbers) which needs a long cpu times on computer. Even the largest and fastest computers cannot give us the numerical solution of fully developed turbulence, although recently some part of characteristic features can be reproducible on computer.

In this paper, we propose to give a lattice model which can reproduce some of characteristic feature of statistical property of turbulence, ex. scaling properties etc. If we succeed in constructing a simple model showing some behaviors which are similar to the fluid, we can say that the origin of them is in the common parts. Therefore it will help us to know what the essence of turbulence is.

The paper will be organized as followings. In Sec. II we will denote on which view point we stand and the description of fluid on 'lattice' will be defined. In Sec. III, we will define a dynamics (time-marching) of our model and show it is equivalent to the Navier-Stokes equation. The results and discussion will be presented in Sec. IV and Sec. V will provide us a summary and a conclusion.

II. MODELING FLUID ON THE LATTICE

A. Concept

The concept which we employ in this paper is a vortex tube representation of fluid turbulence. This is because it can easily provide us a lattice model to simulate turbulence.

Recently, in the field of statistical physics, simple lattice models have been shown to be good tools to reproduce qualitative features of non-linear systems even if they were essentially continuous ones. For example, the diffusion limited aggregation on lattices has turned out to be a good model to reproduce fractal dimension of the Laplacian growth systems which are essentially continuous. K. Kaneko has proposed coupled map lattices in order to study chaos in the many degrees of freedom systems. And Y. Oono has proposed the cell dynamical system on lattices and succeeded in getting good results on the phase separation process, which has been believed to be described by the Time-Dependent- Gintzburg-Landau eq.; a sort of partial differential equation. And one of the author (H.T.) has also proposed many discretized models and obtained results consistent with the experimental
data. These lattice models are more suitable to be treated on computer than conventional method like partial differential equations. Following this line we purpose to construct a lattice model which can reproduce qualitative features of turbulence in this paper. To construct these lattice models, we need some physical quantities whose local dynamics are well known. And introducing interactions between them, we can get a complete system to represent a real system. For this purpose, the vortex tube representation is very convenient. The vortex lines are believed to move with the fluid because of the conservation law of circulation. In the following, we will construct the lattice model so as to preserve these essential features. The first step of this process is to answer the question; 'How can we map the fluid on the lattice?'

B. The representation of the fluid on the lattice

To map the fluid on the lattice, we must take special care for some basic values on the vector analysis. For example, the divergence of velocity \( v \), \( \text{div} v \), must be easy to calculate, because incompressibility condition requires \( \text{div} v = 0 \). And \( \text{rot} v \) also should be calculated easily because of the definition of the vorticity \( \vec{\omega} \); \( \text{rot} v = \vec{\omega} \). Although there is no uniqueness on how to map the fluid on the lattice to satisfy these requirements, we employ the following description.

First, we map the velocity field \( v(r) \) on each bond on the simple cubic lattice (we will call it as the \( v \)-lattice), such that \( x \) bonds must have the only \( x \) component of the velocity. \( y \) and \( z \) bonds also should have the only component along the bond direction. On the other hand, the vorticity field \( \vec{\omega}(r) \) is assigned on each bond on the dual lattice (we call it \( \vec{\omega} \)-lattice). Again each bond has only one component. This means, on the \( v \)-lattice, the vorticity is located on the center of each face and its direction is perpendicular to the face (See Fig. 1). For example, the position vector of the bond between the lattice points \((x_0, y_0, z_0)\) and \((x_0 + 1, y_0, z_0)\) is written as \((x_0 + 1/2, y_0, z_0)\). In this description, the vorticity field \( \vec{\omega}(r) \) is related to the velocity \( v(r) \) by the equation

\[
\omega_i(r) = v_k(r + \frac{1}{2}\hat{j}) + v_j(r - \frac{1}{2}\hat{k}) - v_k(r - \frac{1}{2}\hat{j}) - v_j(r + \frac{1}{2}\hat{k}),
\]

(1)

where \( \hat{i}, \hat{j}, \hat{k} \) are unit lattice vector of \( x, y, z \) directions, which can be exchanged their order cyclically. Therefore \( \vec{\omega} \) is circulation rather than the vorticity. However we are not interested in the difference between them, because they share the same basic property mentioned in the previous subsection, for example folding etc. Independent of whether our \( \vec{\omega}(r) \) is the vorticity or the circulation, we get the same description.

Now, we have everything necessary to construct the map to the lattice model. For example, \( \text{div} v(r) \) is given as

\[
\text{div} v(r) = \sum_{i=x,y,z} \pm v(r \pm \frac{1}{2}\hat{i}),
\]

(2)

where \( \hat{x}, \hat{y} \) and \( \hat{z} \) are the unit lattice vectors of \( x, y \) and \( z \) direction respectively. However in this description, we have a problem. If we give \( v(r) \) first, then we can calculate \( \vec{\omega}(r) \) without any problems. But if \( \vec{\omega}(r) \) is given first, we can not obtain \( v(r) \) uniquely. As will be
shown, this is necessary for time evolution of our model. In addition to this, we require the incompressibility \( \text{div} \mathbf{v}(\mathbf{r}) = 0 \). To get \( \mathbf{v}(\mathbf{r}) \) satisfying this condition seems not to be trivial.

The basic procedure to calculate \( \mathbf{v}(\mathbf{r}) \) from \( \mathbf{\omega}(\mathbf{r}) \) is essentially solving a set of linear equations given by (1). One may think that it is a similar task to solving the Laplace equation. However, this is much more difficult because we cannot use the relaxation method. The method which we employ is the cg method which is explained briefly in Appendix A. In principle, this method can give us accurate numerical solutions even for large matrices. However, if we assume the periodic boundary condition for the lattice, we have another problem. Because of the periodicity, one of a set of linear equations mentioned above is not independent to the rest equations. Although we cannot prove this point analytically, the rank of coefficient matrix of the set of linear equations is always less than the number of components by unity. (The total number of the components is equivalent to the total number of independent components on the \( \mathbf{v} \) lattice. This is \( 3 \times L^3 \) if the linear dimension of the lattice is \( L \).) Therefore we have one degree of freedom on ambiguity of values of \( \mathbf{v}(\mathbf{r}) \). Fortunately this problem turns out to be solved by requiring a discretized representation of \( \text{div} \mathbf{v} = 0 \) at an arbitrary point \( \mathbf{r}_0 \). This gives us the coefficient matrix whose dimension is equal to its rank. The velocity field \( \mathbf{v}(\mathbf{r}) \) which this matrix gives turns out to satisfy \( \text{div} \mathbf{v} = 0 \) and \( \text{rot} \mathbf{v} = \mathbf{\omega} \). Therefore we succeed in mapping fluid on the lattice.

III. DYNAMICS OF THE LATTICE MODEL

In the previous section, we have finished introducing basic elements to the lattice model; \( \mathbf{\omega}(\mathbf{r}) \) and its ‘interaction’ \( \mathbf{v}(\mathbf{r}) \). Actually, our model corresponds to a set of the vortex tubes moving with the velocity induced by the Biot-Savier interaction. In this section, we define a concrete process of dynamics.

In our model, vortex tubes are represented as a sequence of bonds having non-zero vorticity. It can be folded and have branches but no ends. This character must be held even while the vortex tubes are transferred by the flow of the fluid. The simplest dynamics to satisfy these condition is to move the vorticity on some \( i \)-bond to the neighboring \( i \)-bond along \( j \)-direction by the amount of \( \omega_i(\mathbf{r})v_j(\mathbf{r}) \Delta t \) and to add some vorticity to neighboring \( j \)-bonds in order to have no ends. However in stead of that, to move half of the net flow forward and the rest half backwards is more favorable in order to recover isotropy (See the Appendix B). If we consider \( i = j \) makes no change, the time development is defined as;

\[
\omega_z(\mathbf{r}_0, t + \Delta t) = \omega_z(\mathbf{r}_0, t) + \Delta t [ \pm J_{zy}(\mathbf{r}_0 + \hat{y}/2, t) \mp J_{zy}(\mathbf{r}_0 - \hat{y}/2, t) ] + J_{xz}(\mathbf{r}_0 \mp \hat{x}, t) \frac{\Delta}{2} \nonumber
\]

\[
= \omega_z(\mathbf{r}_0, t) + J_{xz}(\mathbf{r}_0 - \hat{z}/2 + \hat{x}/2, t) \mp J_{xz}(\mathbf{r}_0 + \hat{z}/2 - \hat{x}/2, t) \frac{\Delta}{2},
\]

where \( J_{ij}(\mathbf{r}, t) = \tilde{v}_i(\mathbf{r}, t)\omega_j(\mathbf{r}, t) \) and \( \tilde{v}_i(\mathbf{r}) \) is the \( i \)-th component of the velocity on the \( \mathbf{\omega} \)-lattice and is defined as the average over neighboring bonds in the \( \mathbf{v} \)-lattice. For example, \( \tilde{\mathbf{v}} \) on \( z \)-bond is defined as...
\[ \vec{v}(r_0) = \left( \frac{v_x(r_0 + \hat{y}/2) + v_x(r_0 - \hat{y}/2)}{2}, \frac{v_y(r_0 + \hat{x}/2) + v_y(r_0 - \hat{x}/2)}{2}, 0 \right) \]

The y and z components have similar time developments. \( \Delta t \) is the time interval and chosen so as to \( \max_r |J_i(r, t)\Delta t| = \alpha \). This makes the change of the strength of \( \vec{\omega}(r) \) less than \( \alpha \) where \( \alpha \) is a small positive constant. In addition to the above, we add the renormalization of \( \vec{v}(r) \) and \( \vec{\omega}(r) \) at each step. The above procedure does not guarantee the conservation of energy. Although the increasing of energy at each step is small, it may cause some influence for long time behavior. For safety, we renormalize \( \vec{v}(r) \) and \( \vec{\omega}(r) \) just after the vortex dynamics given by (3).

Next we introduce the viscosity. The viscosity is introduced as a diffusion process of the vorticity. And it is easily represented as:

\[ \omega_z(r_0, t + \Delta t) = (1 - \nu \Delta t)\omega_z(r_0, t) + \sum_{i=x, y, z} \omega_z(r_0 \pm \hat{i}, t)\nu \Delta t/6 \] (4)

where \( \nu \) is the viscosity. y and z components follow similar equations. This is the conventional and the simplest discretization of the diffusion equation.

In addition to the above, if we would like to maintain steady state, we should introduce an external force. The time development of \( \vec{\omega}(r) \) due to the force is very simple,

\[ \omega_z(r_0, t + \Delta t) = \omega_z(r_0, t) + \text{rot}f(r_0)\Delta t \] (5)

where \( f \) is the external force.

The whole process of our algorithm is summarized as follows;

1. Calculating \( \sum_r \vec{v}(r)^2/2 \)
2. Deciding \( \Delta t \)
3. Time development corresponding to (3)
4. Renormalizing \( \vec{v}(r), \vec{\omega}(r) \) so as to keep \( \sum_r \vec{v}(r)^2/2 \) constant.
5. Time development corresponding to the diffusion process. (See (4))
6. Time development corresponding to the external force. (See (5))
7. Return to process 1.

IV. THE NUMERICAL RESULTS

In this section, we show some numerical results. In the numerical simulation, we use the lattice with \( L = 24 \) and set \( \alpha = 0.1 \). External force \( f(r) \) is given as

\[ f_x(r) = 4 \sin(k(x + \frac{1}{2})) \cos(ky) \cos(kz) \]
\[ + 2(\cos(k(x + \frac{1}{2})) \cos(ky) + \cos(ky) \cos(kz) + \sin(k(x + \frac{1}{2})) \sin(kz)) \]
\[ f_y(r) = -4 \cos(kx) \sin(k(y + \frac{1}{2})) \cos(kz) \]
\[ + 2(\sin(k(y + \frac{1}{2})) \sin(kx) + \cos(kz) \cos(k(y + \frac{1}{2})) + \cos(kz) \cos(kx)) \]
\[ f_z(r) = 2(\cos(k(z + \frac{1}{2})) \cos(kx) + \cos(kx) \cos(ky) + \sin(ky) \sin(k(z + \frac{1}{2}))) \]
where \( k = 2\pi/L \). Actually, these correspond to the Taylor-Green vortex accompanied with the components of wave number \( 2\sqrt{2}\pi/L \). This choice is not unique but, this is the case which contains the fewest number of kinds of wave number among those which recover isotropy in the numerical simulations.

Remaining parameter is the viscosity \( \nu \). We investigate four values of \( \nu \): 5, 8, 10, \( 20 \times 10^{-3} \). If we define microscopic Reynolds number \( R_\lambda = \sqrt{10/3\langle v^2/2 \rangle/\nu \langle \vec{\omega}^2/2 \rangle} \), \( R_\lambda \approx 5 \sim 20 \) (\( \langle \cdots \rangle \) means average over the whole lattice point and time). However we are not sure whether absolute values have meanings or not, because in our model the unit cubic has already been coarse grained.

First we show the time development of the total \( \sum_r v(r)^2 \) and \( \sum_r \vec{\omega}(r)^2 \) in Fig 3. These behaviors look like those of fully developed turbulence gotten with the conventional direct integration (For example, Ref. [12]). Every results shown in below are averaged over nine snap shots apart from each other by the same intervals after the state reaches steady state (Actually speaking, \( 600 < t < 3000 \)). One can see total square velocities are not influenced by the change of \( \nu \). On the other hand, the total square vorticities are influenced by it strongly.

Before going forward, we show snapshot configurations at a instance. Fig. 4 shows a velocity field snapshot. Nothing abnormal is observed in this figure. In Fig. 5, the bonds on which strong vorticities exist are plotted. The number of plotted bonds are 527 which is much less than total number of bonds : 41472. In spite of small ratio of number of plotted bonds, we can see large number of connected bonds. This means our velocity field surely succeeds in presenting coherent structure of vorticities which is observed in direct integration [4].

For each calculations with four different \( \nu \), we need less than nine hours on CRAY-XMP (3000 steps each). This is surprisingly short, if obtained velocity fields really correspond to fully developed turbulence.

To check this point, three dimensional energy spectrums are shown in Fig 6. One must take care of that drastic decreasing of energy spectrums on high wave number region is due to not only the energy dissipation but also the decreasing of number of mode having wave number of \( k \). So we should consider only \( |k| \leq \pi \). In order to see whether they exhibit inertial range, that is, \( k^{-5/3} \) law, the averaged spectrum is shown in Fig. 7. For \( \nu = 10 \) and \( 20 \times 10^{-3} \), we can see inertial ranges. However, for other two with smaller viscosity, humps are seen in higher wave number region. This means, for later two cases, dynamics in small scale are damaged and they cannot be regarded as a good representation as highly developed turbulence.

One may think these spectrum do not have any dissipation ranges, because decreasing of energy spectrum may be caused by only insufficient number of modes as pointed out in above. For these four cases, we also calculate one dimensional energy spectrum \( E_z = \langle | \int v_z(x, y, z) \exp(-ikx)dx |^2 \rangle_{y, z} \), where \( \langle \cdots \rangle_{y, z} \) stands for the average over \( y, z \). We can see some scaling region obeying Kolmogorov’s scaling region and short inertial subrange. Here we can see clear dissipation range. This time decreasing of energy spectrum at the higher wave number is surely caused by the energy dissipation because shortness of number of modes does not occur in one dimension.

Therefore, we think we can think cases with two larger viscosities can be investigated as representing fully developed turbulence, but not for other two cases. However, we show
results for all four cases for comparison.

In Figs. 9, the probability distribution functions (PDF) of $v_x$ are shown. They look like almost Gaussian which is also observed in experiments and numerical calculation [2]. In order to see the dependence upon $\nu$, semi-logarithmic plot of them is shown in Fig. 10. Those having smaller $\nu$ are closer to Gaussian than others with larger $\nu$. This means even if higher wave number components are damaged, lower component can properly behave and become closer to fully developed turbulence. This also reveals that our model works well to produce developed turbulence.

Next several PDFs which are close to exponential-like distributions are shown. Figs. 11 to 14 show PDF of $\omega_x$, $\partial v_x/\partial x$, $\partial v_x/\partial y$ and $v_x$ having only higher wave number components $k \geq 2\pi/3$. All of them are normalized so as to have standard deviations of unity. Roughly speaking all of them look like exponential distribution. They agree with the results obtained before [2, 3, 4, 5, 13, 14, 15, 16] well. Especially, asymmetry of PDF of $\partial v_x/\partial x$ is properly reproduced. It is extremely surprising because these PDFs reflect the property in the high wave number region which is damaged for lower viscosities. The fact means that our model can catch the essential feature of turbulence which is hardly destroyed even by the insufficient resolution. It enables us to get some aspects on what the essence of turbulence is [17].

These results clearly demonstrate our model works so well. It is clear that our model can work very well to represent fully developed turbulence although it requires small memory-sizes and short cpu-time.

V. CONCLUSION AND SUMMARY

From the above results, our model works remarkably well. For example, since our model has succeeded in reproducing some characteristic features of fully developed turbulence; the Kolmogorov’s scaling of the energy spectrum, the development of energy and enstrophy, coherent structure of vortex tubes, the qualitative feature of the distribution functions of the velocity and its differential we can conclude that we have achieved our purpose; to construct the simple model which can reproduce some qualitative features of turbulence. We think it proves that the lattice model is also valid in investigating the turbulence. In contrast to the conventional direct integration of Navier-Stokes equation, our modeling is very tough to reproduce the qualitative features of turbulence. Our model is not damaged even with the crude procedures like artificial rescaling in order to conserve energy and rough discretization of the field, which cause severe catastrophe to the direct integration.

Moreover we can expect many things for our model. Since our model is so simple, we can find some essential reason about the appearance of the non-Gaussian PDFs, which will be reported elsewhere [17]. And even some rigorous calculation like estimating upper bounds of some value may be possible, because many such calculation have been done on the lattice systems. In addition to this, our model is connected with other lattice models in statistical physics. This will allow us to investigate several phenomena like chemical reaction and phase ordering process under the existence of turbulent flow.
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APPENDIX A: THE CG METHOD

In this appendix, we give a brief explanation of the cg (conjugate gradient) method. For details, see references [18,19].

Assume we have a set of linear equations. In matrix form, \( A\hat{x} = b \), where \( A \) is the coefficient matrix, \( \hat{x} \) is a vector consisting of variables and \( b \) is the constant vector having the same length as that of \( \hat{x} \). The following procedures can be applied to the case that \( A \) is not symmetric neither positive definite.

The cg method is one of those which are suitable to search extreme values. If one would like to search the extreme values of the function \( f(\mathbf{r}) \), one can follow along the flow \( \nabla f(\mathbf{r}) \) iteratively. These are applicable to find the solution \( A\hat{x} = b \), if we define \( f(\mathbf{r}) \) as \( \mathbf{r}^2 \) with \( \mathbf{r} = A\hat{x} - b \). To accelerate this process, in each iteration of the cg method, the direction of flow is tuned so as to be orthogonal to the flow directions in all the previous iterations. Therefore the cg method should converge within the \( n \) iterations, where \( n \) is the number of components; that is, the total number of dimension in the variable space. Skipping detailed points, the actual algorithm of cg method is

1. For given \( k \)-th approximate solution \( \hat{x}_k \) and \( k \)-th residue \( \mathbf{r}_k \), prepare a store array \( \mathbf{p}_k \). For \( k=1 \), \( \mathbf{p}_1 = \mathbf{r}_1 = b - A\hat{x}_1 \)

2. Calculate \( \alpha_k = \frac{(A\mathbf{p}_k, \mathbf{r}_k)}{(A\mathbf{p}_k, A\mathbf{p}_k)} = \frac{(A^T\mathbf{r}_k, A^T\mathbf{r}_k)}{(A\mathbf{p}_k, A\mathbf{p}_k)} \)

3. Calculate the \( k + 1 \)-th approximate solution \( \hat{x}_{k+1} = \hat{x}_k + \alpha_k \mathbf{p}_k \)

4. Calculate the \( k + 1 \)-th residue \( \mathbf{r}_{k+1} = b - A\hat{x}_{k+1} = \mathbf{r}_k - \alpha_k A\mathbf{p}_k \)

5. Calculate \( \beta_k = -\frac{(A\mathbf{p}_k, A^T\mathbf{r}_{k+1})}{(A\mathbf{p}_k, A\mathbf{p}_k)} = \frac{(A^T\mathbf{r}_{k+1}, A^T\mathbf{r}_{k+1})}{(A^T\mathbf{r}_k, A^T\mathbf{r}_k)} \)

6. Calculate \( \mathbf{p}_{k+1} = A^T\mathbf{r}_{k+1} + \beta_k \mathbf{p}_k \)

For the convenience of numerical simulation, one can do as followings using store vectors \( \mathbf{g}, \mathbf{q} \). For \( k = 1 \), \( \mathbf{r}_1 = b - A\hat{x}_1 \), \( \mathbf{p}_0 = 0 \), \( c_0 = 1 \), and iterate like

\[
\begin{align*}
g_k &= A^T\mathbf{r}_k \\
c_k &= (\mathbf{g}_k, \mathbf{g}_k) \\
\beta_k &= c_k/c_{k-1} \\
\mathbf{p}_k &= \mathbf{g}_k + \beta_k \mathbf{p}_{k-1} \\
\mathbf{q}_k &= A\mathbf{p}_k \\
d_k &= (\mathbf{q}_k, \mathbf{q}_k) \\
\alpha_k &= c_k/d_k \\
\hat{x}_{k+1} &= \hat{x}_k + \alpha_k \mathbf{p}_k \\
\mathbf{r}_{k+1} &= \mathbf{r}_k - \alpha_k \mathbf{q}_k
\end{align*}
\]
In our case, $\mathbf{x}_k$ consists of each component of velocity $v_i(\mathbf{r})$. Therefore if the linear dimension of the lattice is $L$, the total number of the component $n$ is $3 \times L^3$. $A$ is coefficient matrix of (1) and (2). $b$ is the vector containing all the components of $\mathbf{\omega}(\mathbf{r})$.

**APPENDIX B: RELATION WITH THE VORTICITY EQUATION**

One may think that our model lacks mathematical justification and therefore cannot be trust. In principle, we do not think our model needs any justifications, because anyway our model is purposed only to reproduce qualitative results at the moment. If one has such a purpose, one dose not have to make his model being related with the real system exactly. This attitude is justified with the hope of existence of ‘universality class’. The universality class is a set of model or parameters which can give the same results on some property. One of the famous universality class in statistical physics is ‘Ising universality class’ in the magnetic phase transition. Ising model is a set of magnetic moments which takes only values of $\pm 1$, therefore far from the true magnetic materials. However, surprisingly, it can reproduce almost all basic nature of ‘Critical Phenomena’. Although no one cannot prove a true magnetic material is exactly same as the Ising model, one can get many deep insights about the ‘true’ phase transition accompanied with the critical phenomena. And it has helped many people who would like to know what the critical phenomena are.

We stand at the same point on the turbulence. Therefore if our model can reproduce some qualitative feature of turbulence, it is useful enough to use in order to investigate turbulence. In this meaning, we do not need any special justifications other than the results itself. However, we can actually shown some relationship with the true fluid equation as follows.

In order to expand (3) in space, first we expand $\mathbf{v}(\mathbf{r})$ and $\mathbf{\omega}(\mathbf{r})$ itself.

$$F_i(\mathbf{r}_0 + \delta \mathbf{x} + \delta \mathbf{y} + \delta \mathbf{z}) = \sum_{n=0}^{\infty} \frac{1}{n!} (\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z})^n F_i$$

$$= \exp(\delta x \frac{\partial}{\partial x} + \delta y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z}) F_i$$

where $F_i$ is either $v_i$ or $\omega_i$. Considering the definition of $\tilde{v}$, we get from eq.(3),

$$\omega_i(t + \Delta t) = \omega_i(t)$$

$$+ \Delta t [\cosh(\frac{\delta j}{2} \frac{\partial}{\partial j}) \sinh(\frac{\delta k}{2} \frac{\partial}{\partial k}) (-v_k \omega_i + 2 \cosh(\frac{\delta i}{2} \frac{\partial}{\partial i}) v_i \omega_k)]$$

$$+ [\cosh(\frac{\delta k}{2} \frac{\partial}{\partial k}) \sinh(\frac{\delta j}{2} \frac{\partial}{\partial j}) (-v_j \omega_i + 2 \cosh(\frac{\delta i}{2} \frac{\partial}{\partial i}) v_i \omega_j)]$$

where $i, j, k$ is $x, y, z$ respectively and change order cyclical and the partial differentiation with tilde is applied to only the velocity not to the vorticity. And taking into account to the second order of $\delta i, \delta j, \delta k$ and the first order of $\delta t$, we can get

$$\frac{\partial \omega_i(t)}{\partial t} = \delta j \frac{\partial}{\partial j} (v_i \omega_j - v_j \omega_i) + \delta k \frac{\partial}{\partial k} (v_i \omega_k - v_k \omega_i)$$
Using \( \text{div} \mathbf{v} = \text{div} \mathbf{\omega} = 0 \) and setting \( \delta x = \delta y = \delta z = 1 \), this reduces to

\[
\frac{\partial \mathbf{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = (\mathbf{\omega} \cdot \nabla)\mathbf{v}.
\]

This is the vorticity equation without viscosity term itself. And it does not have ‘any second order’ terms. This means we can introduce viscosity term using eq.(4) correctly. Therefore our model reduces to the dynamics of vortex by taking the continuum limit.

Here we should mention our dynamics differ from the conventional ones. In the conventional concepts, vortex tube should move with fluid in nonviscous case. In our dynamics, as show in Fig.3 the center of vortex tube does not move. In stead of that, the width of the vortex tube will increase rapidly. We see that ours are better than conventional ones. This is because the conventional ones give the second order term which vanish in our dynamics. The second order terms conflict with the true viscosity term which is also of the second order. We guess this is the reason why the conventional vortex method cannot either get good results nor consider viscosity correctly. Therefore if one follows our dynamics, vortex method can give us physical results.
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[20] For example, the width of the vortex tube on \( j - k \) surface can be defined as

\[
W_i = \left( \frac{\int djd\omega^2_i(j,k) \left((j-j_0)^2 + (k-k_0)^2\right)^{1/2}}{\int djd\omega^2_i(j,k)} \right) \text{ where } i, j \text{ and } k \text{ is } x, y \text{ and } z \text{ and changes the order cyclically.}
\]

\( j_0 \) and \( k_0 \) is the center of the vortex tube, \( j_0 = \left( \frac{\int djd\omega^2_i(j,k)j}{\int dkd\omega^2_i(j,k)} \right) \).
FIGURES

FIG. 1. The relation of a vortex tube (the doubled vector) and velocity vectors (the solid vectors). $\omega$ is equal to $v_{1y} - v_{2y} - v_{2z} + v_{1z}$.

FIG. 2. The actual changes of the distribution of ”vortex tubes” on the $\vec{\omega}$-lattice during the dynamical process. If the part of the vortex tube has the vorticity $\omega$ and the velocities on that part are $\vec{v}_x, \vec{v}_y$ (left), after the dynamics it has branches of vortex tubes shown as narrow arrows(right). The direction of the arrows shows the sign of the value of the vorticity. And absolute values of the branches have two values depending on whether it was induced by $\vec{v}_x$ or $\vec{v}_y$. One must notice the vorticities on the part which has existed before dynamics does not change at all. This shows in our dynamics the center of the vortex tube does not move and the vortex tube itself collapses automatically. In Appendix. B we can see this is better than the conventional picture.

FIG. 3. Time development of $v(r)^2$(thin lines) and $\vec{\omega}(r)^2$(thick lines) (a) $\nu = 5 \times 10^{-3}$ (b) $\nu = 8 \times 10^{-3}$ (c) $\nu = 10 \times 10^{-3}$ (d) $\nu = 20 \times 10^{-3}$

FIG. 4. A velocity snapshot taken from $\nu = 8 \times 10^{-3}$ at $t \sim 1500$. Two dimensional cross section at $z = 4$.

FIG. 5. Coherent structure of vortex tubes at the same instance as that in Fig. 4. Bonds with strong vorticities are plotted. Summation over them is 14 percent of total square vorticities.

FIG. 6. Three dimensional energy spectrums. (a) to (d) correspond to those in Fig. 3. Solid lines indicate $k^{-5/3}$.

FIG. 7. Averaged spectrum corresponding to Figs. 3. Whole range is divided into 50 parts and spectrum is averaged within each band.

FIG. 8. The scaled energy spectrum. $k_0 = (\varepsilon/\nu^3)^{1/4}\pi/12$ is proportional to Kolmogorov wave number. $\varepsilon = \nu \langle \vec{\omega}^2/2 \rangle$ is dissipation energy. Solid line indicates 5/3 law. The symbols represent the cases with different values of the viscosity rate $\nu$; $\times : 5 \times 10^{-3}$, $\square : 8 \times 10^{-3}$, $\Diamond : 10 \times 10^{-3}$, $+: 20 \times 10^{-3}$. Error bars are smaller than the size of character.

FIG. 9. PDF of $v_x$. (a) to (d) correspond to those of Fig. 3. The lines without character represent Gaussian distribution.

FIG. 10. Semi-logarithmic plot of Fig. 9. $\times, \square, \Diamond$ and $+$ correspond to (a),(b),(c) and (d), respectively.

FIG. 11. PDF of $\omega_x$. Characters are the same as those of Fig. 10.
FIG. 12. PDF of $\partial v_x / \partial x$. Characters are the same as those of Fig.10

FIG. 13. PDF of $\partial v_x / \partial y$. Characters are the same as those of Fig.10

FIG. 14. PDF of $v_x$. Their lower wave number components $k \leq 2\pi/3$ are removed. Characters are the same as those of Fig.10