Abstract. In this article, we study the pseudo-isomorphism class of the dual fine Selmer group $X$ attached to a $p$-adic Galois deformation whose deformation ring $\Lambda$ is isomorphic to the ring of formal power series. By using the “Kolyvagin system” arising from a given Euler system $c$, we shall construct a collection $\{C_i(c)\}_{i \geq 0}$ of ideals of $\Lambda$, and prove that the ideals $C_i(c)$ approximate the higher Fitting ideals of $X$ under suitable hypothesis. In particular, we shall prove that the ideals $C_i(c)$ arising from the Euler system of Beilinson–Kato elements determine the pseudo-isomorphism classes of the dual fine Selmer groups attached to ordinary and nearly ordinary Hida deformations satisfying certain conditions.

1. Introduction

1.1. Setting and main results. Let $p$ be an odd prime number. For any $n \in \mathbb{Z}_{>0}$, we denote by $\mu_n$ the group roots of unity in $\mathbb{Q}$, and put $\mu_{p^\infty} := \bigcup_{m \geq 0} \mu_{p^m}$. We fix a finite set $\Sigma$ of prime numbers containing $p$. We denote by $\mathbb{Q}_\Sigma$ the maximal Galois extension field of $\mathbb{Q}$ unramified outside $\Sigma$, and put $G_{\mathbb{Q},\Sigma} := \text{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$. For a topological $G_{\mathbb{Q},\Sigma}$-module $M$ and any $i \in \mathbb{Z}_{\geq 0}$, let

$$H^i(M) := H^i(\mathbb{Q}, M),$$
$$H^i_{\Sigma}(M) := H^i(\mathbb{Q}_\Sigma/\mathbb{Q}, M)$$

be the $i$-th continuous Galois cohomology groups. We denote the inertia subgroup of $G_{\mathbb{Q},\ell} := \text{Gal}(\mathbb{Q}_{\ell}/\mathbb{Q}_{\ell})$ by $I_{\ell}$ for any prime number $\ell$.

Let $F$ be a finite extension field of $\mathbb{Q}_p$, and $\mathcal{O} := \mathcal{O}_F$ the ring of integers of $F$. We fix a uniformizer $\varpi$ of $\mathcal{O}$, and put $k := \mathcal{O}/\varpi \mathcal{O}$. Let $r \in \mathbb{Z}_{>0}$ be a positive integer, and $\Lambda := \Lambda^{(r)} := \mathcal{O}[[x_1, \ldots, x_r]]$ the ring of formal power series. (We write $\Lambda^{(0)} := \mathcal{O}$.) We denote the maximal ideal of $\Lambda$ by $m := m_\Lambda$.

We consider a free $\Lambda$-module of finite rank $d$ equipped with a continuous Galois action

$$\rho_T : \text{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}) \rightarrow \text{Aut}_\Lambda(T) \simeq \text{GL}_d(\Lambda),$$

and put $\mathcal{A}^* := \text{Hom}_{\text{cont}}(T, \mu_{p^\infty})$. (In this article, our main interest is the case when $T$ is an ordinary or nearly ordinary Hida deformation of elliptic modular forms. For
We shall study the higher Fitting ideals of \( \Lambda \)-modules \( X \). We define an ideal \( \text{Ind} \) of units and Beilinson–Kato elements) can be extended to cyclotomic direction. (See §2.)

In this article, we assume the following conditions.

(A1) The representation \( T \) of \( G_{\mathbb{Q}, \Sigma} \) is absolutely irreducible over \( k \).
(A2) There exists an element \( \tau \in G_{\mathbb{Q}(\mu_p^\infty)} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p^\infty)) \) which makes the \( \Lambda \)-module \( T/(\tau - 1)T \) free of rank one.
(A3) We have \( H^0(\mathbb{Q}, T) = H^0(\mathbb{Q}, \Lambda^*[m]) = 0 \).
(A4) At least one of the following is satisfied:
   (a) \( p \geq 5 \), or
   (b) \( \text{Hom}_{\mathbb{F}_p[G_{\mathbb{Q}, \Sigma}])(\overline{T}, \Lambda^*[m]) = 0 \).
(A5) Let \( \Omega \) be the maximal subfield of \( \mathbb{Q} \) which is fixed by \( \ker(\rho_T|_{G_{\mathbb{Q}(\mu_p^\infty)}}) \). Then, we have
   \[
   H^1(\Omega/\mathbb{Q}, \overline{T}) = H^1(\Omega/\mathbb{Q}, \Lambda^*[m]) = 0.
   \]
(A6) For any \( \ell \in \Sigma \setminus \{p\} \), we have \( H^0(I_\ell, \Lambda^*[m]) = 0 \).
(A7) We have \( H^0(\mathbb{Q}_p, \Lambda^*[m]) = 0 \).
(A8) Let \( T^- \) be the maximal \( \Lambda \)-submodule of \( T \) on which the complex conjugate acts via \(-1\). Then, the \( \Lambda \)-module \( T^- \) is free of rank one.

Note that we need the assumptions (A1)–(A5) and (A8) in order to apply the theory of Kolyvagin systems established by Mazur and Rubin in [MR]. The assumptions (A6) and (A7) are technical ones which simplify the local conditions at bad primes.

We define a \( \Lambda \)-module \( \text{Sel}_p(\mathbb{Q}, \Lambda^*) \) by
\[
\text{Sel}_p(\mathbb{Q}, \Lambda^*) := \ker\left( H^1_\Sigma(\Lambda^*) \to H^1(\mathbb{Q}_p, \Lambda^*) \times \prod_{\ell \neq p \in \Sigma} H^1(I_\ell, \Lambda^*) \right),
\]
and define a \( \Lambda \)-module \( X := X(T) \) by
\[
X := \text{Hom}_{\text{cont}}(\text{Sel}_p(\mathbb{Q}, \Lambda^*), \mathbb{Q}_p/\mathbb{Z}_p).
\]
For any \( i \in \mathbb{Z}_{\geq 0} \), the \( \Lambda \)-module \( H^i_\Sigma(\Lambda^*) \) (resp. \( H^i_\Sigma(T) \)) is cofinitely generated (resp. finitely generated). In particular, \( X \) is a finitely generated \( \Lambda \)-module. In this article, we are interested in the pseudo-isomorphism class of the \( \Lambda \)-module \( X \). Note that the pseudo-isomorphism class of a finitely generated \( \Lambda \)-module \( M \) is determined by higher Fitting ideals \( \{\text{Fitt}_{\Lambda,\ell}(M_p)\}_{i \in \mathbb{Z}_{\geq 0}} \) of localizations \( M_p \) at height one primes \( p \).

We shall study the higher Fitting ideals of (localizations of) \( X \).

In order to study higher Fitting ideals, we assume the existence of an Euler system \( \mathbf{c} := \{c(n) \in H^1(\mathbb{Q}_\Sigma/\mathbb{Q}(\mu_n), T)\}_n \) of \( T \). Moreover, we assume that the Euler system \( \mathbf{c} \) can be extended to cyclotomic direction. (For details, see [3.1])

Note that practical Euler systems (like circular units and Beilinson–Kato elements) can be extended to cyclotomic direction. (See Lemma [3.7]) We define an ideal \( \text{Ind}_\Lambda(\mathbf{c}) \) of \( \Lambda \) defined by
\[
\text{Ind}_\Lambda(\mathbf{c}) := \{f(c(1)) \mid f \in \text{Hom}_\Lambda(H^1_\Sigma(T), \Lambda)\}.
\]
Here, we assume the following “non-vanishing” conditions on \( \mathbf{c} \).
The element \( c(1) \in H^1_\Sigma(T) \) is not \( \Lambda \)-torsion.

Note that if \( c \) is the Euler system of circular units or Beilinson–Kato elements corresponding to a certain \( p \)-adic \( L \)-function via the Coleman map, then the property (NV) for \( c \) follows from the non-vanishing of the \( p \)-adic \( L \)-function. In [Oc2], Ochiai proved that under the assumption (NV), the \( \Lambda \)-module \( X \) is torsion, and satisfies

\[
\text{char}_\Lambda(X) \supseteq \text{Ind}_R(c),
\]

where \( \text{char}_\Lambda(X) \) is the characteristic ideal of the \( \Lambda \)-module \( X \). In order to state some pieces of our results, we also consider the following condition (MC) on the pair \((T, c)\).

\[
\text{(MC)} \quad \text{The Euler system } c \text{ satisfies (NV), and it holds that } \text{char}_\Lambda(X) = \text{Ind}_\Lambda(c).
\]

If \( c \) is the Euler system of circular units or Beilinson–Kato elements corresponding to a certain \( p \)-adic \( L \)-function via the Coleman map, then the property (MC) for \( c \) is equivalent to the Iwasawa main conjecture for \( T \).

In §4.1, by using the “universal Kolyvagin system” corresponding to \( c \), we shall construct an ideal \( \mathcal{C}_i(c) \) of \( \Lambda \), which is an analogue of Kurihara’s higher Stickelberger ideal \( \Theta^i \) in [Ku], for any \( i \in \mathbb{Z}_{\geq 0} \).

The following is our first main theorem which does not need the assumption (MC).

**Theorem 1.1.** Let \( T \) be a free \( \Lambda \)-module of finite rank with a continuous \( \Lambda \)-linear action of \( \text{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q}) \). We assume that \( T \) satisfies the assumptions (A1)–(A8). Let \( c \) be an Euler system for \( T \) satisfying the condition (NV) which can be extended to cyclotomic direction. Then, for any height one prime ideal \( p \) of \( \Lambda \) and for any \( i \in \mathbb{Z}_{\geq 0} \), we have

\[
\text{Fitt}_{\Lambda_p,i}(X_p) \supseteq \mathcal{C}_i(c)\Lambda_p.
\]

We shall state stronger results under the assumption (MC). Here, we treat the cases when \( T \) is a one variable deformation, or the cases when \( T \) is the cyclotomic deformation of a one variable deformation. In such cases, we can deduce finer results than Theorem 1.2 and Theorem 1.3 under the assumption (MC).

The results for one variable cases are as follows.

**Theorem 1.2.** Suppose \( \Lambda = \mathcal{O}[[x_1]] \). Let \( (T, c) \) be as in Theorem 1.1. We assume that the Euler system \( c \) for \( T \) satisfies the condition (MC). Let \( p \) be a height one prime ideal of \( \Lambda \). Then, for any \( i \in \mathbb{Z}_{\geq 0} \), we have

\[
\text{Fitt}_{\Lambda_p,i}(X_p) = \mathcal{C}_i(c)\Lambda_p.
\]

In particular, the pseudo-isomorphism class of the \( \Lambda \)-module \( X \) is determined by the collection \( \{\mathcal{C}_i(c)\}_{i \in \mathbb{Z}_{\geq 0}} \) of ideals of \( \Lambda \).

Now, let us consider the case when \( T \) is the cyclotomic deformation of a one variable deformation. We put \( \Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \). Let \( \chi_{\text{cyc}} : \Gamma \to \mathbb{Z}_p^\times \) be the cyclotomic character, and \( \gamma \in \Gamma \) the topological generator given by \( \chi_{\text{cyc}}(\gamma) = 1 + p \). We put \( \Lambda_0 = \Lambda^{(1)} \). From now on, we assume that \( r = 2 \), and that we have

\[
\Lambda = \Lambda_0[[\Gamma]] = \Lambda^{(2)},
\]
where the variable $x_2$ is identified with $\gamma - 1$. Let $T_0$ be a free $\Lambda_0$-module of finite rank $d$ equipped with a continuous $\Lambda_0$-linear action $\rho_{T_0}$ of $\text{Gal}(Q_\Sigma/Q)$ satisfying the conditions (A1)–(A8). Here, we assume

$$\rho_T = (T^\text{cyc}, \rho_T^\text{cyc}) := (T \otimes_{\Lambda_0} \Lambda, \rho_T \otimes \chi_{\text{taut}}),$$

where $\chi_{\text{taut}} : \text{Gal}(Q_\Sigma/Q) \to \Gamma \subseteq \Lambda^\times$ is the tautological character. Note that the assumptions (A1)–(A8) for $T_0$ imply that $T$ satisfies (A1)–(A8). (Moreover, we can easily show the converse: the conditions (A1)–(A8) for $T$ also imply that the free positive integer satisfying $\#\text{Prime}(\kappa)$ images of (modified) Kolyvagin derivatives $\kappa^\text{univ}_n(c)$, where $n$ runs through square-free positive integer satisfying $\#\text{Prime}(n) \leq i$ and contained in a certain set $\mathcal{N}(T, I)$, is identified with $\gamma - 1$.)

Let $\mathfrak{c}$ be an Euler system of $T$. Note that $\mathfrak{c}$ can be extended to cyclotomic direction automatically. (See Lemma 3.7)

**Theorem 1.3.** Let $(T, \mathfrak{c})$ be as above. Namely, we set $\Lambda = \Lambda_0[[\Gamma]] = \Lambda^{(2)}$ and $(T, \rho_T) = (T^\text{cyc}, \rho_T^\text{cyc})$. We assume that the Euler system $\mathfrak{c}$ for $T$ satisfies the condition (MC). Let $\mathfrak{p}$ be a height one prime ideal of $\Lambda$. Then, for any $i \in \mathbb{Z}_{\geq 0}$, we have

$$\text{Fitt}_{\Lambda, i}(X_\mathfrak{p}) = \mathfrak{c}_i(\mathfrak{c})\Lambda_\mathfrak{p}.$$

In particular, the pseudo-isomorphism class of the $\Lambda$-module $X$ is determined by the collection $\{\mathfrak{c}_i(\mathfrak{c})\}_{i \in \mathbb{Z}_{\geq 0}}$ of ideals of $\Lambda$.

1.2. **Strategy.** Here, we introduce the strategy of the proof of our main results.

Theorem 1.1 is proved by the induction on the number $r$ of variables in $\Lambda = \Lambda^{(r)}$. When $r = 0$, namely the case when $\Lambda$ is a DVR, the assertion like Theorem 1.1 follows from the theory of Kolyvagin systems established by Mazur and Rubin in [MR]. When $r = 1$, by using the method developed in [MR] §5.3, we can reduce the proof to the non-variable cases. Namely, for each hight one prime ideal $\mathfrak{p}$ of $\Lambda$, we take a sequence $\{\mathfrak{p}_n\}_{n \geq 0}$ which is a “perturbation” of the prime ideal $\mathfrak{p}$, and observe asymptotic behavior of the reduction by $\mathfrak{p}_n$. For $r \geq 2$, we use the method developed by Ochiai in [Oc2]. We take a suitable “linear element” $g \in \Lambda$ of $\Lambda$, and reduce the proof of Theorem 1.1 for the pair $(T, \mathfrak{c})$ over $\Lambda^{(r)}$ to the proof of that for $(\pi_g^\text{cyc} T, \pi_g^\text{cyc} \mathfrak{c})$ over $\Lambda^{(r-1)}$, where $\pi_g : \Lambda \to \Lambda/g\Lambda = \Lambda^{(r-1)}$ denotes the reduction map. In the induction arguments, a property called “a weak specialization compatibility” which says that the image of the ideal $\mathfrak{c}_i(\mathfrak{c})$ for $(T, \mathfrak{c})$ by the reduction map $\pi_g$ is contained in the ideal $\mathfrak{c}_i(\pi_g^\text{cyc} \mathfrak{c})$. (See Theorem 5.3 and Theorem 5.8.) The proof of the strong specialization compatibility is the most technical part in our article. The difficulty to prove the strong specialization compatibility is as follows. For a positive integer $n$, we denote by $\text{Prime}(n)$ the set of prime divisors of $n$. Roughly speaking, the ideal $\mathfrak{c}_i(\pi_g^\text{cyc} \mathfrak{c})$ is a projective limit of certain ideals of quotient rings $\Lambda/I$ generated by the images of (modified) Kolyvagin derivatives $\kappa^\text{univ}_n(c)$, where $n$ runs through square-free positive integer satisfying $\#\text{Prime}(n) \leq i$ and contained in a certain set $\mathcal{N}(T, I)$. (For the definition of $\mathfrak{c}_i(\pi_g^\text{cyc} \mathfrak{c})$, see Definition 4.5, and for the definition of $\mathcal{N}(T, I)$,
By the definition of $N(T, I)$, a prime divisor $\ell$ of an element $n \in N(T, I)$ makes $(T / (\text{Frob}_T - 1) T) \otimes_{\Lambda} \Lambda / I$ be a free $\Lambda / I$-module of rank one. This implies that the set $N(T, I)$ is smaller than $N(\pi^*_T, \pi(I))$. So, the ideal $\pi_g(\mathcal{C}_i(c))$ may be smaller than $\mathcal{C}_i(\pi^*_c)$. We can overcome this difficulty in the situations of Theorem 1.2 and Theorem 1.3 as follows.

- When $\Lambda = \Lambda^{(1)}$, we can embed $\Lambda / I$ into a certain quotient ring of DVR. By using the theory of Kolyvagin systems over DVR, we can deduce that $\pi_g(\mathcal{C}_i(c))$ is not small. (For details, see the proof of See Theorem 5.3 in [5.3])

- When $T$ is the cyclotomic deformation of a Galois deformation $T_0$ over $\Lambda^{(1)}$, the proof of the strong specialization compatibility for $T$ can be reduced to that for $T_0$, namely Theorem 1.2 (For details, see the proof of Theorem 1.3 in 4.8)

The contents of our article is as follows. In 2, we introduce some basic notion and preliminary results. In 3, we review the theory of Euler systems for Galois deformations and Kolyvagin systems over DVR. In 4, we define the ideal $\mathcal{C}_i(c)$, and prove their basic properties, that is, independence of the choice of a certain system of parameters of $\Lambda$ (Proposition 4.12) and the stability under scalar extensions (Proposition 4.12). In 5, we prove the weak/strong specialization compatibility of $\mathcal{C}_i(c)$. In 6, we prove our main results. In 7, we apply our results to ordinary and nearly ordinary Hida deformation.

**Notation**

We put $\overline{N} := \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $\overline{N}_{>0} := \mathbb{Z}_{>0} \cup \{\infty\}$.

Let $n \in \mathbb{Z}_{>0}$ be any positive integer. We put $H_n := \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$. We define $\Lambda/I[n] := \Lambda/I[H_n]$, and denote by $\pi_I[n]: \Lambda \to \Lambda/I[n]$ the natural ring homomorphism. For simplicity, we write $\pi_I := \pi_I[1]: \Lambda \to \Lambda/I$. We also write $\Lambda[n] := \Lambda[0][n] = \Lambda[H_n]$ and $\pi[n] := \pi[n]$. The tautological action of $H_n$ on $\Lambda/I[n]$ induces the action of $H_n$ on $H^1(\pi_I[1], \mathbb{T})$. By Shapiro’s lemma, we have a natural $H_n$-equivariant homomorphism

$$H^1(\mathbb{Q}(\mu_n), \pi^*_I) \simeq H^1(\pi^*_I[n], \mathbb{T})$$

if $n$ is prime to $\Sigma$.

Let $R$ be a DVR, and $v_R: R \to \mathbb{Z} \cup \{\infty\}$ the additive valuation on $R$. Then, for any ideal $I$ of $R$ generated by an element $a \in R$, we define $v_R(I) := v_R(a)$.

Let $S$ be a commutative ring, and $I$ an ideal of $S$. Let $a$ be an element of $S$, and $C$ a subset of $\Lambda$. Then, we denote by $a_I$ (resp. $C_I$) the image of $a$ (resp. $C$) in $S/I$.

Let $A$ be a set, and $a := (a_0, \ldots, a_r) \in A^{r+1}$ any element. For any $i \in \mathbb{Z}$ with $0 \leq i \leq r$, we define truncated systems $a_{<i}$ and $a_{\geq i}$ by

$$h_{<i} := (a_0, \ldots, a_i) \in A^{i+1},$$
$$h_{\geq i} := (a_i, \ldots, a_r) \in A^{r-i+1}.$$
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2. Preliminaries

Here, we recall some basic notion and preliminary results. In §2.1, we recall structure theorem of \( \Lambda \)-modules, and we also recall the definition and basic properties of Fitting ideals. In §2.2, we define the notion of monic parameter systems which is a system of parameters of \( \Lambda \) consisting of “monic polynomials” in certain sense. The quotient of \( \Lambda \) by ideals generated by powers of elements in a fixed monic parameter system have some useful properties. For instance, such rings become 0-dimensional Gorenstein rings. (See Lemma 2.9.) In §2.3, we recall some control theorems for Galois cohomology groups.

We keep the notation introduced in §1.1. In particular, we fix an odd prime number \( p \), and we put \( \Lambda := \mathcal{O}[[x_1, \ldots, x_r]] \). Let \( T \) be a free \( \Lambda \)-module of finite rank with a continuous \( \Lambda \)-linear action of \( G_{\mathbb{Q}, \Sigma} \) satisfies the conditions (A1)–(A8).

2.1. Higher Fitting ideals and structure theorem of \( \Lambda \)-modules. First, let us recall the definition and basic properties of higher Fitting ideals.

Definition 2.1. Let \( R \) be a commutative ring, and \( M \) a finitely presented \( R \)-module. Suppose that we have an exact sequence

\[
\begin{array}{ccc}
R^n & \xrightarrow{A} & R^n \\
& \rightarrow & M \\
& & \rightarrow 0
\end{array}
\]

of \( R \)-modules. Then, for any \( i \in \mathbb{Z} \), we denote by \( \text{Fitt}_{R,i}(M) \) the ideal of \( R \) generated by all \( (n-i) \times (n-i) \)-minors of \( A \). Note that if \( n-i > m \) (resp. \( n-i \leq 0 \)), then we define \( \text{Fitt}_{R,i}(M) := \{0\} \) (resp. \( \text{Fitt}_{R,i}(M) = R \)). We call \( \text{Fitt}_{R,i}(M) \) the \( i \)-th Fitting ideal of \( R \). Not that the ideals \( \text{Fitt}_{R,i}(M) \) is independent of the choice of the exact sequence (1).

We shall review some basic properties on higher Fitting ideals briefly. Let \( R \) and \( M \) be as in Definition 2.1. Then, by definition, we can verify the following properties easily.

(i) Higher Fitting ideals \( \{\text{Fitt}_{R,i}(M)\} \) forms an ascending filtration of \( R \).
(ii) For any ring homomorphism \( f: R \rightarrow R' \), we have \( \text{Fitt}_{R,i}(f^*M) = f(\text{Fitt}_{R,i}(M))R' \). Namely, the higher Fitting ideals are compatible with base change.
(iii) Let \( \text{ann}_R(M) \) be the annihilator ideal of the \( R \)-module \( M \). Then, we have \( \text{Fitt}_{R,i}(M) \subseteq \text{ann}_R(M) \). (This is a remarkable property of Fitting ideals though we do not use it in this article.)

Now let us introduce some important examples for higher Fitting ideals.
Example 2.2. Let $R$ be a PID, and $M$ a finitely generated $R$-module. Then, by the structure theorem, we have an isomorphism

$$M \simeq \bigoplus_{i=1}^{s} R/d_i R,$$

where $\{d_i\}_i = 1^s$ are sequence contained in $R \setminus R^\times$ satisfying $d_i \mid d_{i+1}$ for any $i$. Hence by the definition of higher Fitting ideals, we have

$$\text{Fitt}_{R,i}(M) = \begin{cases} \left( \prod_{j=1}^{s-i} d_j \right) R & \text{if } i < s \\ R & \text{if } i \geq s. \end{cases}$$

This implies that the isomorphism class of the $R$-module $M$ is determined by the higher Fitting ideals $\{\text{Fitt}_{R,i}(M)\}_{i \geq 0}$.

Example 2.3. Let $R$ be a local ring with the maximal ideal $m_R$, and $M$ the finitely generated $R$-module. By the base change property of Fitting ideals, we have

$$\min \{ i \in \mathbb{Z}_{\geq 0} \mid \text{Fitt}_{R,i}(M) = R \} = \dim_{R/m_R} M \otimes_R R/m_R.$$

Namely, the minimal number of generators of $M$ is determined by higher Fitting ideals $\{\text{Fitt}_{R,i}(M)\}$. In particular, we can easily verify that the $R$-module $M$ is free of rank one if and only if $\text{Fitt}_{R,1}(M) = 0$ and $\text{Fitt}_{R,1}(M) = R$.

Now, let us consider the ring $\Lambda := \Lambda_{\mathbb{Q}_f} = \mathcal{O}[[x_1, \ldots, x_r]]$, where $\mathcal{O}$ is the integer ring of a finite extension field $F$ of $\mathbb{Q}_p$. Note that the ring $\Lambda$ is Noetherian UFD.

First, we recall the notion of pseudo-null modules and pseudo-isomorphisms. Let $f : M \longrightarrow N$ be a homomorphism of finitely generated $\Lambda$-modules. We say that the $\Lambda$-module $M$ is pseudo-null if and only if $M_p = 0$ for any height one prime $p$ of $\Lambda$.

Recall that we have the following structure theorem of finitely generated torsion $\Lambda$-modules.

Proposition 2.4. Let $M$ be finitely generated torsion $\Lambda$-module. Then, we have a pseudo-isomorphism $\iota_M : M \longrightarrow \bigoplus_{i=1}^{s} \Lambda/d_i \Lambda$ of $\Lambda$-modules, where the follow hold.

- We have $s \in \mathbb{Z}_{>0}$.
- For each $i \in \mathbb{Z}$ with $1 \leq i \leq s$, we have $d_i \in \Lambda \setminus (\Lambda^\times \cup \{0\})$.
- For any $i, j \in \mathbb{Z}$ with $1 \leq i < j \leq s$, we have $d_j \in d_i \Lambda$.

For the finitely generated torsion $\Lambda$-module $M$ in Proposition 2.4, we define the characteristic ideal $\text{char}_\Lambda(M)$ of the $\Lambda$-module $M$ by

$$\text{char}_\Lambda(M) = \left( \prod_{i=1}^{s} d_i \right) \Lambda.$$

Note that the higher Fitting ideals are independent of the choice of the pseudo-isomorphism $\iota_M$ in Proposition 2.4.
Example 2.5. Let $M$ be a finitely generated torsion $\Lambda$-module, and $\{d_i\}_{i \geq 0}$ the sequence in $\Lambda \setminus \Lambda^\times$ as in Proposition 2.4. Then, by the base change property of higher Fitting ideals, for any any $i \in \mathbb{Z}_{\geq 0}$ and for any prime ideal $p$ of $\Lambda$, we have

$$
\text{Fitt}_{\Lambda,p,i}(M_p) = \begin{cases} 
\prod_{j=1}^{s-i} f_j & \text{if } i < s \\
\Lambda_p & \text{if } i \geq s.
\end{cases}
$$

Namely, for any $i \in \mathbb{Z}_{\geq 0}$, there exists an ideal $\mathfrak{a}_i$ of height at least two satisfying

$$
\text{Fitt}_{\Lambda,i}(M) = \begin{cases} 
\prod_{j=1}^{s-i} f_j & \text{if } i < s \\
\mathfrak{a}_i & \text{if } i \geq s.
\end{cases}
$$

In particular, the characteristic ideal $\text{char}_\Lambda(M)$ is the minimal principal ideal of $\Lambda$ containing $\text{Fitt}_{\Lambda,0}(M)$. The pseudo-isomorphism class of $M$ is determined by the higher Fitting ideals $\{\text{Fitt}_{\Lambda,i}(M)\}_{i \in \mathbb{Z}_{\geq 0}}$.

2.2. Monic parameter systems. Here, we use the notation introduced in §Notation, namely, in the end of §1. For instance, for each $n \in \mathbb{Z}_{>0}$ and each ideal $I$ of $\Lambda$, we denote by $\pi_{I,[n]} : \Lambda \to \Lambda/I[H_n]$ the natural map.

Definition 2.6. Recall that we put $N := \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Let $h := (h_0, \ldots, h_r) \in (m_\Lambda)^{r+1}$ and $m := (m_0, \ldots, m_r)$, $m' := (m_0', \ldots, m_r') \in (N_{>0})^{r+1}$ be arbitrary elements.

(i) We write $m' \geq m$ if we have $m_i' \geq m_i$ for any $i$.
(ii) We put $h^m := (h_0^{m_0}, \ldots, h_r^{m_{r+1}}) \in (m_\Lambda)^r$.

(Here, we define $h_0^\infty := 0$)

(iii) We regard $N_{>0}$ as a subset of $(N_{>0})^{r+1}$ via the diagonal embedding. In particular, for any integer $N$, we write $h^N := h^{(N, \ldots, N)}$. Note that we write $N \geq m$ if and only if $N \geq m_i$ for any $i$.

(iv) We denote by $I(h)$ the ideal of $\Lambda$ generated by $\{h_i \mid 0 \leq i \leq r\}$.

Now let us introduce a notion monic parameter systems, which plays an important role in our article.

Definition 2.7. An element $h := (h_0, \ldots, h_r) \in (m_\Lambda)^{r+1}$ is called a monic parameter system of $\Lambda$ if it satisfies the following conditions (1)–(3):

(i) The 0-th component $h_0$ is a non-zero element contained in $m_\Omega = \omega \mathcal{O}$.
(ii) For any $i \geq 1$, the $i$-th component $h_i$ is a monic polynomial in the variable $x_i$ whose coefficients of lower terms are contained in the maximal ideal of the ring $\Lambda^{(i-1)} = \mathcal{O}[x_1, \ldots, x_{i-1}]$.

Note that $x := (\omega, x_1, \ldots, x_r)$ forms a monic parameter system of $\Lambda$. We call $x$ the standard monic parameter system of $\Lambda$.

Here, let us observe some basic properties of monic parameter systems.
Lemma 2.8. Let \( h \) be a monic parameter system of \( \Lambda \). Then, the following hold.

(i) The \((r + 1)\)-ple \( h \) forms a system of parameters for the maximal ideal \( \mathfrak{m}_\Lambda \). In particular, the order of \( \Lambda/I(h^m) \) is finite.

(ii) For any \( m := (m_0, \ldots, m_r) \in (\mathbb{Z}_{>0})^{r+1} \), the \((r + 1)\)-ple \( h^m \) is a monic parameter system.

(iii) Let \( i \in \mathbb{Z} \) be any integer satisfying \( 0 \leq i \leq r \), and put \( h(i) := (h_{i-1}, 0, h_{i+1}) \).

Then, there exists an exact sequence

\[ 0 \rightarrow \pi^*_I(h(i)) \mathbb{T} \overset{\times h_i}{\longrightarrow} \pi^*_I(h(i)) \mathbb{T} \rightarrow \pi^*_I(h) \mathbb{T} \rightarrow 0 \]

of \( \Lambda \)-modules.

**Proof.** The assertions (i) and (ii) immediately follows from the definition of monic parameter systems. Let \( i \in \mathbb{Z}_{>0} \) be any element. Since \( h = (h_i)_i \) is a monic parameter system, the \( \Lambda^{(i)}/I(h_{i-1}) \)-module \( \pi_I(h(i)) \mathbb{T} \) is free of finite rank. Moreover, since \( h \) is a monic parameter system, the image of \( h_i \) in \( \Lambda^{(i)}/I(h_{i-1}) \) is not a zero divisor. This implies that the scalar multiplication map \( \times h_i : \pi^*_I(h(i)) \mathbb{T} \rightarrow \pi^*_I(h(i)) \mathbb{T} \) is injective. Hence the assertion (iii) follows. \( \square \)

Let \( h \) be a monic parameter system of \( \Lambda \), and \( m = (m_0, \ldots, m_r) \in (\mathbb{Z}_{>0})^{r+1} \) any element. Take any element \( n \in N_{\Sigma} \). Then, by definition, the ring \( \Lambda/I(h^m)_{[n]} \) becomes locally complete intersection. In particular, the ring \( \Lambda/I(h^m)_{[n]} \) is Gorenstein. Since \( \Lambda/I(h^m)_{[n]} \) is a finite direct product of 0-dimensional local rings, we obtain the following lemma which becomes a key of some ring theoretic arguments in our article.

Lemma 2.9. The \( \Lambda/I(h^m)_{[n]} \)-module \( \Lambda/I(h^m)_{[n]} \) is injective.

**Remark 2.10.** It is convenient to consider the notion of “monic parameter systems” to construct systems of parameters for the local ring \( \Lambda \) systematically. Note that Lemma 2.8 (iii) and Lemma 2.9 are benefits of the fact that a monic parameter systems forms systems of parameters for \( \Lambda \). For instance, if we use a power of the maximal ideal of \( \Lambda \) instead of \( I(h^m) \), similar assertion to Lemma 2.9 does not hold.

2.3. Control theorems of Galois cohomology groups. Here, we introduce some preliminary results which are related to Iwasawa theoretic reduction arguments. Let us consider the \( \Lambda[\text{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})] \)-module \( \mathbb{T} \) satisfying all the conditions in Theorem 1.1. First, we give a description of our Selmer group \( X = X(\mathbb{T}) \) in terms of the Galois cohomology group of the second degree.

Lemma 2.11. We have a natural isomorphism \( X \simeq H^2_\Sigma(\mathbb{T}) \) of \( \Lambda \)-modules.

**Proof.** By the assumption (A6), we have

\[ \text{Sel}_p(\mathbb{Q}, \mathcal{A}^*) = \ker \left( H^1_\Sigma(\mathcal{A}^*) \rightarrow \prod_{\ell \in \Sigma} H^1(\mathbb{Q}_\ell, \mathcal{A}^*) \right). \]
So, by (the projective limit of) the Poitou–Tate exact sequences, we obtain
\[ X \simeq \ker \left( H^2_\Sigma(T) \to \prod_{\ell \in \Sigma} H^2(Q_\ell, T) \right). \]

By the local duality theorem of Galois cohomology groups and the assumptions (A6) and (A7) imply \( H^2(Q_\ell, T) = 0 \) for any \( \ell \in \Sigma \). Hence we obtain our proposition. \( \square \)

Let \( h \in \Lambda^{r+1} \) be a monic parameter system, and \( m \in \mathbb{Z}_{>0} \) any element. We put \( I := I(h^m) \). By the similar arguments to that in the proof of \( \text{[Ne]} \) (8.4.8.1) Proposition, we obtain the following proposition.

**Proposition 2.12.** We have a spectral sequence
\[ E_{2}^{p,q} := \text{Tor}^\Lambda_p(H^q_\Sigma(T), \Lambda/I) \Rightarrow H^{p+q}_\Sigma(\pi^*_T \mathbb{T}). \]

**Remark 2.13.** In \( \text{[Ne]} \) (8.4.8.1) Proposition, Nekovár obtained the spectral sequence in the case when \( \Lambda \) is an Iwasawa algebra of a Galois group \( \Gamma \simeq \mathbb{Z}_p^r \), and \( I \) is the augmentation ideal of \( \Lambda \). But, by Lemma 2.8 (iii), similar arguments work in our situation.

By Lemma 2.11 and Proposition 2.12, we immediately obtain the following corollary.

**Corollary 2.14** (Control Theorem). The following hold.

(i) We have an exact sequence
\[ \text{Tor}^\Lambda_2(H^2_\Sigma(T), \Lambda/I) \to H^1_\Sigma(\pi^*_T \mathbb{T}) \otimes_\Lambda \Lambda/I \to H^1_\Sigma(\pi^*_T \mathbb{T}) \to \text{Tor}^\Lambda_1(H^2_\Sigma(T), \Lambda/I). \]

(ii) We have a natural homomorphism
\[ X(T) \otimes_\Lambda \Lambda/I \simeq X(\pi^*_T \mathbb{T}). \]

3. **Euler systems**

Let \( T \) be as in Theorem 1.1. In this section, we recall the definition and some basic preliminary results on Euler systems and Kolyvagin systems. In §3.1 we introduce the definition of Euler systems for Galois deformations. In §3.2 we recall the definition and basic properties of Kolyvagin derivatives. In §3.3 we recall the theory of Kolyvagin systems over DVR established by Mazur and Rubin in \[ \text{[MR]} \]. In §3.4 we introduce a notion of universal Kolyvagin system, which is a system of linear combinations of Kolyvagin derivatives whose specialization to DVR forms a Kolyvagin system.
3.1. Definition. Here, we recall the notion of Euler systems for Galois deformations, which is a generalization of the Euler systems in the sense of [MR]. (We adopt the axiom of Euler systems slightly different from Ochiai’s one in [Oc2].) For any \( \sigma \in G_{Q, \Sigma} \), we define a polynomial \( P(x; T|\sigma) \) by
\[
P(x; T|\sigma) := \det_{A}(1 - x\rho_{T}(\sigma); T),
\]
where \( \text{Frob}_{\ell} \in \text{Gal}(Q_{\Sigma}/Q) \) is an arithmetic Frobenius element at \( \ell \). The definition of Euler system in this paper is as follows.

**Definition 3.1.** We define the set \( N_{\Sigma} \) by
\[
N_{\Sigma} := \{ n \in \mathbb{Z}_{>0} \mid n \text{ is square free and prime to } \Sigma \}.
\]
We call a collection
\[
c := \{ c(n) \in H^{1}(Q(\mu_{n}), T) \mid n \in N_{\Sigma} \}
\]
of Galois cohomology classes an Euler system for \( T \) if the collection \( c \) satisfies the following conditions.

(i) For any \( n \in \Sigma \), the Galois cohomology class \( c(n) \) is unramified outside \( p \).

(ii) Let \( \ell \) be any prime number not contained in \( \Sigma \), and let \( n \in N_{\Sigma} \) be any integer prime to \( \ell \). Then, we have
\[
\text{Cor}_{Q(\mu_{n})/Q}(c(n\ell)) = P(\text{Frob}_{\ell}^{-1}; T|\text{Frob}_{\ell})c(n),
\]
where \( \text{Cor}_{Q(\mu_{n})/Q}(c(n\ell)) \) is the corestriction map of Galois cohomology.

**Remark 3.2.** Our Euler system axiom is slightly different from that in [Ru] or [Oc2]. Indeed, Euler systems \( z = \{ z(n) \in H^{1}(Q(\mu_{n}), T) \}_{n} \) in the sense of [Oc2] satisfies the following condition (ii)' instead of (ii) in Definition 3.1.

(ii)' Let \( \ell \) be any prime number not contained in \( \Sigma \), and let \( n \in N_{\Sigma} \) be any integer prime to \( \ell \). Then, we have
\[
\text{Cor}_{Q(\mu_{n})/Q}(z(n\ell)) = P(\text{Frob}_{\ell}^{-1}; T^{*}|\text{Frob}_{\ell}^{-1})z(n).
\]
Here, we put \( T^{*} := \text{Hom}_{A}(T, A) \otimes_{\mathbb{Z}_{p}} \lim_{\longleftarrow} \mu_{p^{m}} \). We prefer the axiom (ii) to (ii)' since (ii)' is useful to apply the theory of Kolyvagin systems. By Proposition 3.3 below, we can construct an Euler system in our sense by a canonical way when an Euler system in the sense of [Oc2] is given.

**Proposition 3.3** ([Ru] Lemma 9.6.1 and Corollary 9.6.4). Let \( z = \{ z(n) \in H^{1}(Q(\mu_{n}), T) \}_{n} \in N_{\Sigma} \) be a collection of continuous Galois cohomology classes satisfying (i) in Definition 3.1 and (ii)' in Remark 3.2. For each \( n \in N_{\Sigma} \) and each divisor \( d \) of \( n \) satisfying \( d > 1 \), we define an element \( A(n, d; T) \in \Lambda[Gal(Q(\mu_{n})/Q)] \) by
\[
A(n, d; T) := \prod_{\ell \in \text{Prime}(n/d)} \frac{P(\text{Frob}_{\ell}^{-1}; T|\text{Frob}_{\ell}) - P(\text{Frob}_{\ell}^{-1}; T^{*}|\text{Frob}_{\ell}^{-1})}{\varphi(n/d)},
\]
where \( \varphi \) denotes Euler’s \( \varphi \) function. Then, the system
\[
c := \left\{ c(n) := z(n) + \sum_{1 < d | n} A(n, d; T)z(d) \in H^{1}(Q(\mu_{n}), T) \mid n \in N_{\Sigma} \right\}
\]
forms an Euler system for $\mathbb{T}$ in the sense of Definition 3.1.

**Definition 3.4.** Let $\mathcal{R}$ be a topological $\Lambda$-algebra with a structure map $\pi : \Lambda \rightarrow \mathcal{R}$.

(i) If a collection
\[ c := \{ c(n) \in H^1(\mathbb{Q}(\mu_n), \mathbb{T}) \mid n \in \mathbb{N}_\Sigma \} \]
satisfies conditions (1) and (2) in Definition 3.1, we call $c$ an Euler system for $\pi^* \mathbb{T}$.

(ii) For any Euler system $c$ for $\mathbb{T}$, we can define an Euler system $\tilde{c}$ for $\pi^* \mathbb{T}$ by
\[ \pi^* c := \{ c(n) \in H^1(\mathbb{Q}(\mu_n), \pi^* \mathbb{T}) \mid n \in \mathbb{N}_\Sigma \} . \]

We consider the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty / \mathbb{Q}$, and put $\Gamma := \text{Gal}(\mathbb{Q}_\infty / \mathbb{Q})$. Let $\chi_{\text{taut}} : \text{Gal}(\mathbb{Q}_\Sigma / \mathbb{Q}) \rightarrow \Gamma \subseteq \Lambda^\times$ be the tautological character. We define a $\Lambda[[\Gamma]]$-module $\mathbb{T}^\text{cyc} := \mathbb{T} \otimes_{\Lambda} \Lambda[[\Gamma]]$ on which $\text{Gal}(\mathbb{Q}_\Sigma / \mathbb{Q})$ acts via $\rho_T \otimes \chi_{\text{taut}}$. The augmentation $\Lambda[[\Gamma]] \rightarrow \Lambda$ induces a $\text{Gal}(\mathbb{Q}_\Sigma / \mathbb{Q})$-equivariant map
\[ \text{aug}_T : \mathbb{T}^\text{cyc} \rightarrow \mathbb{T} . \]

**Definition 3.5.** In this article, we say that an Euler system $c := \{ c(n) \}_n$ for $\mathbb{T}$ can be extended to the cyclotomic direction if there exists an Euler system $\tilde{c} := \{ \tilde{c}(n) \}_n$ for $\mathbb{T}^\text{cyc}$ such that $\text{aug}_T(\tilde{c}) := \{ \text{aug}_T(\tilde{c}(n)) \}_n$ coincides with $c$. The Euler system $\tilde{c}$ is called an extension of $c$ to the cyclotomic direction.

**Remark 3.6.** Let $\mathcal{R}$ be a topological $\Lambda$-algebra. If an Euler system $c$ for $\mathcal{R} \otimes \Lambda \mathbb{T}$ satisfies similar conditions to that in Definition 3.5, we say that $c$ can be extended to the cyclotomic direction.

**Lemma 3.7.** Let $c$ be an Euler system for $\mathbb{T}$.

(i) Let $\mathcal{R}$ be any topological $\Lambda$-algebra, and $\pi : \Lambda \rightarrow \mathcal{R}$ the structure map of the $\Lambda$-algebra $\mathcal{R}$. If an Euler system $c$ for $\mathbb{T}$ can be extended to the cyclotomic direction, then $\pi^* c$ can be extended to the cyclotomic direction.

(ii) Let $\tilde{c}$ be an Euler system for $\mathbb{T}$ satisfying $\text{aug}_T(\tilde{c}) = c$. Then, $\tilde{c}$ can be extended to the cyclotomic direction.

**Proof.** First, Let us show the first assertion. The map $\pi$ induces a continuous homomorphism $\tilde{\pi} : \Lambda[[\Gamma]] \rightarrow \mathcal{R}[[\Gamma]]$. Note that we have a natural isomorphism $\tilde{\pi}^* \mathbb{T}^\text{cyc} \simeq (\pi^* \mathbb{T})^\text{cyc}$. By definition, the Euler system $\tilde{\pi}^*(\tilde{c})$ for $(\pi^* \mathbb{T})^\text{cyc}$ satisfies
\[ \text{arg}_{\pi^* T}(\tilde{\pi}^*(\tilde{c})) = \pi^* c . \]

So $\pi^* c$ can be extended to the cyclotomic direction.

Next, Let us show the second assertion. The diagonal embedding $\Delta : \Gamma \rightarrow \Gamma \times \Gamma$ induces a continuous homomorphism
\[ \Delta : \Lambda[[\Gamma]] \rightarrow \Lambda[[\Gamma \times \Gamma]] . \]
Galois deformation and the pseudo-isomorphism class. We define an isomorphism \( \iota : \Lambda[[\Gamma \times \Gamma]] \xrightarrow{\cong} \Lambda[[\Gamma]] \) of topological rings by composite

\[
\Lambda[[\Gamma \times \Gamma]] \cong \varprojlim_{n_1, n_2} \Lambda[\Gamma/\Gamma_{p^{n_1}} \times \Gamma/\Gamma_{p^{n_2}}] \\
\cong \varprojlim_{n_2} \left( \varprojlim_{n_1} \Lambda[\Gamma/\Gamma_{p^{n_1}}] \right)[\Gamma/\Gamma_{p^{n_2}}] \\
\cong (\Lambda[[\Gamma]])[\Gamma].
\]

Then we have a commutative diagram

\[
\begin{array}{ccc}
\Lambda[[\Gamma \times \Gamma]] & \xrightarrow{\Delta} & \Lambda[[\Gamma]] \\
\downarrow{\text{pr}_1} & & \downarrow{\text{aug}} \\
\Lambda[[\Gamma]] & \xrightarrow{\iota} & \Lambda[[\Gamma]]
\end{array}
\]

where \( \text{aug} \) is the augmentation map, and \( \text{pr}_1 \) is the ring homomorphism induced by the first projection map \( \Gamma \times \Gamma \to \Gamma \). By the commutative diagram (2), we can immediately verify that the Euler system \( \tilde{c} := (\iota \circ \Delta)^* (\tilde{c}) \) for \((\mathbb{T}^\text{cyc})^\text{cyc} \) satisfies \( \arg_{\mathbb{T}^\text{cyc}} (\tilde{c}) = \tilde{c} \).

This completes the proof of Lemma 3.7. \( \square \)

### 3.2. Kolyvagin derivatives

Here, we fix an Euler system \( \mathbf{c} = \{c(n)\}_n \) for \( \mathbb{T} \). Let us recall the construction of Kolyvagin derivative arising from the Euler system \( \mathbf{c} \).

First, we introduce some notation in general setting. Let \( R \) be any pro-finite commutative ring, and \( T \) a free \( R \)-module of finite rank with continuous \( R \)-linear \( G_{\mathbb{Q}, \Sigma} \)-action \( \rho_T \). Let \( I \) be an ideal of \( R \) of finite index. For any prime number \( \ell \notin \Sigma \), we set the ideal \( \mathcal{I}_{T, \ell} \) of \( R \) by \( \mathcal{I}_{T, \ell} := (1 - x\rho_T(\text{Frob}_\ell); T) \).

For any \( n \in \mathcal{N}_\Sigma \), we define the ideal \( \mathcal{I}_{T, n} \) of \( \Lambda \) by

\[
\mathcal{I}_{T, n} := \sum_{\ell \in \text{Prime}(n)} \mathcal{I}_{T, \ell}.
\]

We define a set \( \mathcal{P}(T; I) \) of prime numbers by

\[
\mathcal{P}(T; I) := \left\{ \ell \mid \ell \notin \Sigma, \mathcal{I}_{T, \ell} \subseteq I, \text{ and the } \Lambda/I\text{-module } T/(IT + (\text{Frob}_\ell - 1)T) \text{ is free of rank one} \right\},
\]

and define a set \( \mathcal{N}(T, I) \) of positive integers by

\[
\mathcal{N}(T, I) := \left\{ n \in \mathbb{Z}_{>0} \mid n \text{ is a square free integer, and all prime divisors of } n \text{ are contained in } \mathcal{P}(T, I) \right\}.
\]

For simplicity, when \( (R, T) = (\Lambda, \mathbb{T}) \), we write \( \mathcal{I}_n := \mathcal{I}_{T, n} \).
Remark 3.8. Let $I$ be any ideal of $\Lambda$ of finite index. Here, we give some remarks on the sets $\mathcal{P}(T, I)$ and $\mathcal{N}(T, I)$.

(i) The assumption (A2) and the Chebotarev density theorem imply that the sets $\mathcal{P}(T, I)$ and $\mathcal{N}(T, I)$ are not empty.

(ii) Let $\ell$ be an integer not contained in $\Sigma$. Suppose that $(\ell - 1) \in I$, and the $\Lambda/I$-module $T/(IT + (\text{Frob}_\ell - 1)T)$ is free of rank one. Then, we have $\ell \in \mathcal{P}(T, I)$.

(iii) We note that $1 \in \mathcal{N}(T, I)$.

Let $\ell \in \mathcal{P}(T, I)$ be any element. and fix a generator $\sigma_\ell$ of $H_\ell \simeq (\mathbb{Z}/\ell\mathbb{Z})^\times$. We define an element $D_\ell \in \mathbb{Z}[H_n]$ by
\[
D_\ell := \ell - 2 \sum_{\nu=1}^{\ell-2} \nu \sigma_\ell^\nu.
\]

Let $n \in \mathcal{N}(T, I)$ be any element. We denote the set of prime numbers dividing $n$ by $\text{Prime}(n)$. Then, we have $H_n = \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \simeq \prod_{\ell \in \text{Prime}(n)} H_\ell$. We put
\[
H_n^\otimes := \bigotimes_{\ell \in \text{Prime}(n)} H_\ell,
\]
where tensor products are taken over $\mathbb{Z}$. We define
\[
D_n := \prod_{\ell \in \text{Prime}(n)} D_\ell \in \mathbb{Z}[H_n].
\]

By the similar arguments to the proof of [Ru] Lemma 4.4.2, we deduce that
\[(1 - \sigma)D_n c(n) \in I H^1(\pi_{I,[n]}^* T)\]
for any $\sigma \in H_n$. So, we obtain
\[
D_n c(n)_I \in H^1(\pi_{I,[n]}^* T)^{H_n},
\]
where $c(n)_I$ is the image of $c(n)$ in $H^1(\pi_{I,[n]}^* T)$. By the assumptions (A1) and (A3), the restriction map
\[
H^1(\pi_{I,[n]}^* T) \rightarrow H^1(\pi_{I,[n]}^* T)^{H_n}
\]
becomes an isomorphism. Hence we obtain the following definition.

Definition 3.9. We denote by $\kappa(c; n)_I$ the unique element of $H^1(\pi_I^* T) \otimes \mathbb{Z} H_n^\otimes$ whose image by the restriction map
\[(5)\]
\[
H^1(\pi_I^* T) \otimes \mathbb{Z} H_n^\otimes \rightarrow H^1(\pi_{I,[n]}^* T)^{H_n} \otimes \mathbb{Z} H_n^\otimes
\]
coincides with $D_n c(n)_I \otimes \bigotimes_{\ell \in \text{Prime}(n)} \sigma_\ell$. Note that the element $\kappa(c; n)_I$ is independent of the choice of generators $\sigma_\ell \in H_\ell$. The element $\kappa(c; n)_I$ is called the Kolyvagin derivative of the Euler system $c$. 
3.3. Mazur-Rubin theory over DVR. Let us recall the theory of Kolyvagin systems over DVR, namely the case when $\Lambda = \Lambda^{(0)}$, established by Mazur and Rubin in the book [MR].

First, let us recall basic settings in the book [MR]. Let $R$ be the integer ring of a finite extension field of $\mathbb{Q}_p$. We denote the maximal ideal of $R$ by $m_R$. Let $T$ be a free $R$ module of finite rank with continuous $R$-linear $G_{\mathbb{Q},\Sigma}$-action. We put $A^* := \text{Hom}_{\mathbb{Z}_p}(T,\mu_{p^\infty})$. Let $\mathcal{F}_{\text{can}}$ be the canonical local condition for $T$ in the sense of [MR], namely

$$H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_v,T) = \begin{cases} H^1(\mathbb{Q}_v,T) & (v \in \{p,\infty\}) \\ H^1_{\mathfrak{p}}(\mathbb{Q}_v,T) & (v \notin \{p,\infty\}) \end{cases}$$

in our setting, where $H^1_{\mathfrak{p}}(\mathbb{Q}_v,T)$ denotes Bloch–Kato’s finite local condition. We denote by $\mathcal{F}_{\text{can}}^*$ the dual local condition for $\mathcal{F}_{\text{can}}$ in the sense of [MR] Definition 2.3.1. Let $\mathcal{P}$ be a set of prime numbers contained in $\mathcal{P}(T,m_R)$. Here, the set $\mathcal{P}(T,m_R)$ is defined by similar manner to [B]. We denote by $\mathcal{N}(\mathcal{P})$ the set of positive integers which are square free products of several prime numbers in $\mathcal{P}$. We assume that the triple $(T,\mathcal{F}_{\text{can}},\mathcal{P})$ satisfies the hypotheses (H1)–(H6) in [MR]. Mazur and Rubin introduced the notion of the core rank $\chi(T) := \chi(T,\mathcal{F}_{\text{can}},\mathcal{P}) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ of the triple $(T,\mathcal{F}_{\text{can}},\mathcal{P})$. (See [MR] Definition 4.1.11 and Definition 5.2.4.) In our situation, the core rank $\chi(T)$ is given by the formula

$$\chi(T) = \text{rank}T^- + \text{corank}H^0(\mathbb{Q}_p,A^*),$$

where $T^-$ is the maximal $R$-submodule of $T$ where the complex conjugate acts via the scalar multiplication by $-1$. (See [MR] Theorem 5.2.15.)

Example 3.10. Let $\mathcal{P}$ be as in Theorem 1.1. In particular, we assume that $\mathcal{T}$ satisfies conditions (A1)–(A8). Let $f : \Lambda \longrightarrow R$ be any continuous ring homomorphism. Here, we put $T := f^*T$. Take any $n \in \mathbb{Z}_{>0}$, and put $\mathcal{P} := \mathcal{P}(T,m_R^n)$. Then, we can easily verify that the triple $(T,\mathcal{F}_{\text{can}},\mathcal{P})$ satisfies hypotheses (H1)–(H6), and $\chi(T) = 1$. Moreover, we have a canonical isomorphism

$$\text{Hom}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}_v,A^*),\mathbb{Q}_p/\mathbb{Z}_p) \simeq X(T).$$

Now let us recall the definition of Kolyvagin systems. Let $(T,\mathcal{F}_{\text{can}},\mathcal{P})$ be as above. For each $n \in \mathcal{P}$, We define a new local condition $\mathcal{F}_{\text{can}}(n)$ by

$$H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}_v,T/I_nT) = \begin{cases} H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_v,T/I_nT) & (\text{if } \ell \nmid n) \\ \text{Ker}(H^1_{\mathcal{F}_{\text{can}}}(\mathbb{Q}_v,T/I_nT) \rightarrow H^1(\mathbb{Q}_v(\mu_\ell^\infty),T/I_nT)) & (\text{if } \ell | n). \end{cases}$$

Definition 3.11 ([MR] Definition 3.1.3). A collection

$$\kappa = \{\kappa_n \in H^1(T/I_nT) \otimes \mathbb{Z} \overset{\mathcal{H}_n^{(0)}}{\longrightarrow} \}_{n \in \mathcal{N}(\mathcal{P})}$$

is called a Kolyvagin system for the triple $(T,\mathcal{F}_{\text{can}},\mathcal{P})$ if it satisfies the following two conditions:

(i) For any $n \in \mathcal{P}$, we have $\kappa_n \in H^1_{\mathcal{F}_{\text{can}}(n)}(\mathbb{Q}_v,T/I_nT)$. 

(ii) Take any \( n \in \mathcal{N}(\mathcal{P}) \), and let \( \ell \in \mathcal{P} \) be an element prime to \( n \). We put
\[
H^1_\ell(\mathbb{Q}_\ell, T/I_nT) := H^1(\mathbb{Q}_\ell, T/I_nT)/H^1_\ell(\mathbb{Q}_\ell, T/I_nT).
\]
Then, we have
\[
\psi_{n,\ell}^{n,\ell}(\kappa_{n\ell}) = \psi_n^{n,\ell}(\kappa_n) \in H^1_\ell(\mathbb{Q}_\ell, T/I_nT) \otimes \mathbb{Z} H^\oplus_{n\ell}
\]
where
\[
\psi_{n,\ell}^{n,\ell} : H^1_\ell(\mathcal{F}_{\text{can}}(n\ell))(\mathbb{Q}, T/I_{n\ell}T) \otimes \mathbb{Z} H^\oplus_{n\ell} \to H^1(\mathbb{Q}_\ell, T/I_{n\ell}T) \otimes \mathbb{Z} H^\oplus_{n\ell}
\]
is a composite of the localization (namely, restriction) map and the natural surjection, and
\[
\psi_n^{n,\ell} : H^1_\ell(\mathcal{F}_{\text{can}}(n))(\mathbb{Q}, T/I_nT) \otimes \mathbb{Z} H^\oplus_n \to H^1(\mathbb{Q}_\ell, T/I_nT) \otimes \mathbb{Z} H^\oplus_n
\]
is the composite of the localization map and the finite singular comparison map \( \text{(MR Definition 1.2.2)} \).

We denote the set of Kolyvagin systems for \( (T, \mathcal{F}_{\text{can}}, \mathcal{P}) \) by \( \mathcal{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \). The set \( \mathcal{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \) has a natural \( R \)-module structure.

The following are main results on Kolyvagin systems (of rank one) over DVR proved in \( \text{MR} \) §5.2.

**Theorem 3.12** (\( \text{MR} \) Theorem 5.2.10). Suppose that \( \chi(T) = 1 \). Then, the \( R \)-module \( \mathcal{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \) is free of rank one.

Let \( \kappa = \{\kappa_n\}_n \in \mathcal{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \) be any Kolyvagin system. For each \( n \in \mathcal{N}(\mathcal{P}) \), we define
\[
\partial(\kappa; n) := \max\{j \in \mathbb{Z}_{\geq 0} \mid \kappa_n \in m_R^j H^1_\ell(\mathcal{F}_{\text{can}}(n))(\mathbb{Q}, T/I_nT)\}.
\]
Let \( t \in \mathbb{Z}_{>0} \), and put \( \mathcal{P}(t) := \{\ell \in \mathcal{P} \mid I_t \subseteq m_R^t\} \). Then, for each \( i \in \mathbb{Z}_{\geq 0} \), we put
\[
\partial_i(\kappa) := \min\{\partial(\kappa; n) \mid \#\text{Prime}(n) = i, \, \text{Prime}(n) \subseteq \mathcal{P}(t)\}.
\]
In particular, we put \( \partial_0(\kappa) := \partial(\kappa; 1) \).

**Theorem 3.13** (\( \text{MR} \) Theorem 5.2.12). Let \( t \in \mathbb{Z}_{>0} \) be any positive integer. Suppose that \( \chi(T) = 1 \). Let \( \kappa = \{\kappa_n\}_n \) be a generator of the \( R \)-module \( \mathcal{KS}(T, \mathcal{F}_{\text{can}}, \mathcal{P}) \). Then, for any \( i \in \mathbb{Z}_{\geq 0} \), we have
\[
\text{Fitt}_{R,t}(\text{Hom}_{\mathbb{Z}_p}(H^1_{\mathcal{F}_{\text{can}}^*}(\mathbb{Q}, A^*), Q_p/\mathbb{Z}_p)) = m_{R}^{\partial_i(\kappa)}.
\]

### 3.4. The universal Kolyvagin system.

Here, by using Kolyvagin derivatives of a fixed Euler system \( \mathbf{c} \), we shall construct a collection of Galois cohomology classes called a “universal Kolyvagin system”, whose specializations to DVRs become Kolyvagin systems. From now on, we assume that the Euler system \( \mathbf{c} \) extends to cyclotomic direction. Fix an Euler system \( \mathbf{c} = \{c(n)\}_{n \in \mathbb{N}_2} \) on \( \mathbb{T}^{\text{cyc}} \) which is an extension of \( \mathbf{c} \) to the cyclotomic direction.
We use the following notation. Recall that we put $\Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$. We denote by $\mathfrak{a}_I$ the augmentation ideal of the completed group ring $\Lambda[[\Gamma]]$. Let $I$ be an ideal of $\Lambda[[\Gamma]]$, and $n \in \mathcal{N}_\Sigma$ any element. Then, we put $\Lambda[[\Gamma]]_I := (\Lambda[[\Gamma]]/I)[H_\ell]$, and let $\tilde{\pi}_I : \Lambda[[\Gamma]] \longrightarrow \Lambda[[\Gamma]]/I$ be the natural projection. We write $\mathcal{I}_n := \mathcal{I}_{\text{cycl},n}$. If the ideal $I$ contains $\mathcal{I}_n$, then as in Definition 3.9 we can define the Kolyvagin derivative

$$\kappa(\tilde{c}; n)_I \in H^1(\tilde{\pi}_I^* T^\text{cycl}) \otimes_{\mathbb{Z}} H^\circ_n$$

arising from the Euler system $\tilde{c}$ on $\widetilde{T}$.

Let $n \in \mathcal{N}_\Sigma$ be any element. We denote by $\mathfrak{S}_{\text{Prime}(n)}$ the set of Permutations on $\text{Prime}(n)$. Let $\alpha \in \mathfrak{S}_{\text{Prime}(n)}$ be any element. the sign of $\alpha$ is denoted by $\text{sign}(\alpha)$, and the set of elements fixed by $\alpha$ is denoted by $\text{Prime}(n)^\alpha$. We put

$$d_\alpha := \prod_{\ell \in \text{Prime}(n)^\alpha} \ell \in \mathcal{N}_\Sigma.$$

Let $\ell$ be a prime number not contained in $\Sigma$, and $I$ an ideal of $\Lambda[[\Gamma]]$. We denote by $\mathfrak{a}_{I,H_\ell}$ the augmentation ideal of the group ring $\Lambda[[\Gamma]]/_{\tilde{\pi}_I^*} = (\Lambda[[\Gamma]]/I)[H_\ell]$ on the group $H_\ell$. Namely, we put

$$\mathfrak{a}_{I,H_\ell} := \text{Ker}(\text{aug} : \Lambda[[\Gamma]]/_{\tilde{\pi}_I^*} \longrightarrow \Lambda[[\Gamma]]/I).$$

We have an isomorphism

$$e_{I,H_\ell} : \mathfrak{a}_{I,H_\ell} \otimes \mathfrak{a}_{I,H_\ell}^2 \longrightarrow (\Lambda[[\Gamma]]/I) \otimes_{\mathbb{Z}} H_\ell$$

doing $\Lambda[[\Gamma]]/I$-modules defined by $e_{I,H_\ell}(\sigma - 1) = 1 \otimes \sigma$ for any $\sigma \in H_\ell$. Similarly, for any ideal $I$ of $\Lambda$ of finite index, we denote the augmentation ideal of $\Lambda/I_{\ell}[\ell] = (\Lambda/I)[H_\ell]$ by $\mathfrak{a}_{I,\ell}$, and define the isomorphism

$$e_{I,\ell} : \mathfrak{a}_{I,\ell} \otimes \mathfrak{a}_{I,\ell}^2 \longrightarrow (\Lambda/I) \otimes_{\mathbb{Z}} H_\ell$$

doing $\Lambda/I$-modules.

**Definition 3.14.** Let $n \in \mathcal{N}_\Sigma$ be any element.

(i) For any ideal $\tilde{I}$ of $\Lambda[[\Gamma]]$ of finite index containing $\mathcal{I}_n$, we define an element $\kappa_{n}^{\text{univ}}(\tilde{c})_I$ of

$$H^1(\tilde{\pi}_I^* T^\text{cycl}) \otimes_{\mathbb{Z}} H^\circ_n = H^1(\tilde{\pi}_I^* T^\text{cycl}) \otimes_{\Lambda[[\Gamma]]/I} \bigotimes_{\ell|n} \left( (\Lambda[[\Gamma]]/I) \otimes_{\mathbb{Z}} H_\ell \right)$$

by

$$\kappa_{n}^{\text{univ}}(\tilde{c})_I := \sum_{\alpha \in \mathfrak{S}_{\text{Prime}(n)}} \text{sign}(\alpha) \kappa(\tilde{c}; d_\alpha)_I \otimes_{\ell|(n/d_\alpha)} e_{I,H_\ell}(P_\ell(\text{Frob}_{\alpha(\ell)}; T^\text{cycl})).$$

(ii) Similarly to (i), for any ideal $I$ of $\Lambda$ of finite index containing $\mathcal{I}_n$, we define an element $\kappa_{n}^{\text{univ}}(c)_I \in H^1(\pi_I^* T) \otimes_{\mathbb{Z}} H^\circ_n$ by

$$\kappa_{n}^{\text{univ}}(c)_I := \sum_{\alpha \in \mathfrak{S}_{\text{Prime}(n)}} \text{sign}(\alpha) \kappa(c; d_\alpha)_I \otimes_{\ell|(n/d_\alpha)} e_{I,H_\ell}(P_\ell(\text{Frob}_{\alpha(\ell)}; T)).$$
Remark 3.15. Let \( I \) containing \( \mathcal{I}_n \) from the theory of Kolyvagin systems over DVRs.) (In our article, we can omit to check it since our main results follow
We have not proved that our “universal Kolyvagin systems” satisfies the axioms re-
slightly different from that in \[Bu\]. On the one hand, in \[Bu\], a “universal Kolyvagin
An axiomatic framework of Kolyvagin systems for Galois deforma-
Remark 3.16. Let \( n \in \mathcal{N}_\Sigma \) be any element, and \( I \) any ideal of \( \Lambda \) of finite index containing \( \mathcal{I}_n \). Then, by definition, the following hold.
(i) When \( n = 1 \), we have
\[
d_1^\text{univ}(c)_I = \kappa_1^\text{univ}(c)_I = c(1)_I \in H^1(\pi^*_1 \mathbb{T}).
\]
(ii) The image of \( d_n^\text{univ}(c)_I \otimes \otimes_{\ell \in \text{Prime}(n)} \sigma_\ell \) by the inverse map of the restriction map
\[
H^1(\pi^*_1 \mathbb{T}) \otimes_{\mathbb{Z}} H^\otimes_n \rightarrow H^1(\pi^*_1 \mathbb{T}) \otimes_{\mathbb{Z}} H^\otimes_n
\]
coincides with \( \kappa_n^\text{univ}(c)_I \).
(iii) We identify the topological \( (\Lambda/I)[G_{\mathbb{Q}, \Sigma}] \)-module \( \hat{\pi}^*_1 \mathbb{T} \) with \( \pi^*_1 \mathbb{T} \) via the
natural isomorphism. Then, we have
\[
\kappa_n^\text{univ}(\mathcal{C})_{I+I\Lambda[\mathbb{F}]} = \kappa_n^\text{univ}(c)_I \in H^1(\pi^*_1 \mathbb{T}) \otimes_{\mathbb{Z}} H^\otimes_n.
\]
Remark 3.16. An axiomatic framework of Kolyvagin systems for Galois deformations are studied in \[Bu\]. Note that our notion of “universal Kolyvagin system” is
slightly different from that in \[Bu\]. On the one hand, in \[Bu\], a “universal Kolyvagin
system in \[Bu\] is a system of Galois cohomology classes satisfying certain axioms.
We have not proved that our “universal Kolyvagin systems” satisfies the axioms required
in \[Bu\]. (In our article, we can omit to check it since our main results follow
from the theory of Kolyvagin systems over DVRs.)

Recall that we have fixed an Euler system \( \mathcal{C} = \{ \mathcal{C}(n) \}_{n \in \mathcal{N}_\Sigma} \) on the cyclotomic
deformation \( \mathbb{T} \). Let \( \chi \in \text{Hom}_{\text{cont}}(\Gamma, \mathbb{Z}_p^\times) \simeq \mathbb{Z}_p \) be any element. The character \( \chi \)
induces a continuous \( \Lambda \)-algebra homomorphism \( e_\chi : \Lambda[\mathbb{F}] \rightarrow \Lambda \) given by \( e_\chi(g) = \chi(g) \) for each \( g \in \Gamma \). We put \( \mathbb{T} \otimes \chi := e_\chi \mathbb{T} \), and consider the Euler system \( c \otimes \chi := \mathcal{C} \) on \( \mathbb{T} \otimes \chi \). Roughly speaking, the following proposition implies that a
specialization of the universal Kolyvagin systems becomes Kolyvagin systems.

Proposition 3.17. Let \( \mathcal{O}' \) be the ring of integers of a finite extension field \( F' \) of \( F \), and \( f : \Lambda \rightarrow \mathcal{O}' \) any continuous ring homomorphism. Then, for any open subgroup
\( U \) of \( \text{Hom}_{\text{cont}}(\Gamma, \mathbb{Z}_p) \), there exists a character \( \chi \in U \) such that the system
\[
f^* \kappa_n^\text{univ}(c \otimes \chi) = \left\{ f^* \kappa_n^\text{univ}(c \otimes \chi) \in H^1(f^*_1 \mathbb{T} \otimes_{\mathbb{Z}} (\mathbb{T} \otimes \chi)) \otimes_{\mathbb{Z}} H^\otimes_n \right\}
\]
forms a Kolyvagin system for the Selmer triple \( (f^*(\mathbb{T} \otimes \chi), \mathcal{F}_\text{can}, \mathcal{P}(\mathbb{T} \otimes \mathfrak{m}_\Lambda)) \) over \( \mathcal{O}' \),
where \( f^* \kappa_n^\text{univ}(c \otimes \chi) \) is the image of \( \kappa_n^\text{univ}(\mathcal{C})_{I+I\Lambda[\mathbb{F}]} \) by the map
\[
H^1(\mathcal{C}_{I+I\Lambda[\mathbb{F}]}) \otimes_{\mathbb{Z}} H^\otimes_n \rightarrow H^1(f^*_1 \mathbb{T} \otimes_{\mathbb{Z}} (\mathbb{T} \otimes \chi)) \otimes_{\mathbb{Z}} H^\otimes_n
\]
induced by the continuous ring homomorphism \( f \otimes e_\chi : \Lambda[[\Gamma]] \rightarrow \mathcal{O}' \), and
\[
\hat{f}_{\mathbb{T} \otimes X,n} : \Lambda \rightarrow \mathcal{O}' / f(I_n) \mathcal{O}'
\]
denotes the continuous ring homomorphism induced by \( f \).

**Proof.** For each \( \ell \in \mathcal{P}(\mathbb{T}, m_\Lambda) \) and \( N \in \mathbb{Z}_{>0} \), we denote by \( \Xi_{\ell,N} \) the subset of \( \text{Hom}_{\text{cont}}(\Gamma, \mathbb{Z}_p^\times) \) which consists of characters \( \psi \in \text{Hom}_{\text{cont}}(\Gamma, \mathbb{Z}_p^\times) \) such that the operator \( f^* (\rho \otimes \psi) (\text{Frob}_\ell)^N - \text{id} \) does not acts on \( f^*(\mathbb{T} \otimes \chi) \) invertively. Note that the set \( \Xi_{\ell,N} \) is finite. We put
\[
\Xi := \bigcup_{\ell \in \mathcal{P}(\mathbb{T}, m_\Lambda)} \bigcup_{N>0} \Xi_{\ell,N}.
\]
Since \( \Xi \) is a countable set, the set \( \mathcal{U} \setminus \Xi \) is not empty.

Take any character \( \chi \in \mathcal{U} \setminus \Xi \). Then, the pair \( (f^*(\mathbb{T} \otimes \chi), \mathcal{P}(\mathbb{T}, m_\Lambda)) \) satisfies the assumptions in [MR, Theorem 3.2.4]. So by the construction of the Kolyvagin system \( \kappa \) in Theorem 3.2.4, which is explained in [MR, Appendix A], the system \( f^* \kappa_{\text{univ}}(c \otimes \chi) \) becomes a Kolyvagin system for the Selmer triple \( (f^*(\mathbb{T} \otimes \chi), \mathcal{F}_{\text{can}}, \mathcal{P}(\mathbb{T}, m_\Lambda)) \).

Let \( \mathcal{O}' \) be the ring of integers of a finite extension field \( E' \) of \( F \), and \( f : \Lambda \rightarrow \mathcal{O}' \) any continuous ring homomorphism. For each \( n \in N_{\mathcal{U}} \), we define
\[
\partial(c, f ; n) := \max \{ j \in \mathbb{Z}_{\geq 0} \mid f^* \kappa_{\text{univ}}(c) \in \mathcal{H}^1_{\text{can}}(\mathbb{Q}, f^* \mathbb{T} / I_{\mathbb{T} \otimes X,n} f^* \mathbb{T}) \}.
\]
For each \( t \in \mathbb{Z}_{>0} \) and \( i \in \mathbb{Z}_{\geq 0} \), we put
\[
\partial(t, c, f) := \min \{ \partial(c, f ; n) \mid \# \text{Prime}(n) = i, \text{Prime}(n) \subseteq \mathcal{P}(t) \}.
\]

**Corollary 3.18.** Let \( f : \Lambda \rightarrow \mathcal{O}' \) be as in Proposition 3.17 and \( m_{\mathcal{O}' \mathcal{V}} \) the maximal ideal of \( \mathcal{O}' \). Assume that the order of \( X(f^* \mathbb{T}) \) is finite, and we have
\[
\text{Fitt}_{\mathcal{O}' \mathcal{V},0}(X(f^* \mathbb{T})) = m_{\mathcal{O}' \mathcal{V}}^{N+\partial(c, f ; 1)}
\]
for some \( N \in \mathbb{Z}_{\geq 0} \). Let \( t \in \mathbb{Z}_{>0} \) be a positive integer satisfying \( t > \text{length}_{\mathcal{O}'}(X(f^* \mathbb{T})) \). Then, for any \( i \in \mathbb{Z}_{\geq 0} \), we have
\[
m_{\mathcal{O}' \mathcal{V}}^{N} \cdot \text{Fitt}_{\mathcal{O}' \mathcal{V},i}(X(f^* \mathbb{T})) = m_{\mathcal{O}' \mathcal{V}}^{\partial(t, c, f ; i)}.
\]

**Proof.** Recall we have fixed a topological generator \( \gamma \in \Gamma \). By proposition 3.17, there exists a continuous homomorphism \( \chi : \Gamma \rightarrow \mathbb{Z}_p^\times \) satisfying \( \chi(\gamma) - 1 \in m_{\mathcal{O}' \mathcal{V}} \) such that the system
\[
f^* \kappa_{\text{univ}}(c \otimes \chi) = \left\{ f^* \kappa_n(c \otimes \chi) \in H^1_{\mathcal{O}' \mathcal{V},n} \mathbb{T} \otimes \chi \right\}_{n \in N_{\Gamma}(\mathbb{T}, m_\Lambda)}
\]
becomes a Kolyvagin system for the Selmer triple \( (f^*(\mathbb{T} \otimes \chi), \mathcal{F}_{\text{can}}, \mathcal{P}(\mathbb{T}, m_\Lambda)) \). By Theorem 3.12 and Theorem 3.13 there exists \( N \in \mathbb{Z}_{>0} \) such that
\[
\mathcal{O}' : f^* \kappa_{\text{univ}}(c \otimes \chi) = m_{\mathcal{O}' \mathcal{V}}^{N} \cdot \text{KS}(f^*(\mathbb{T} \otimes \chi), \mathcal{F}_{\text{can}}, \mathcal{P}(\mathbb{T}, m_\Lambda)).
\]

Fix a uniformizer \( \varpi' \in \mathcal{O}' \). Since \( X(f^* \mathbb{T}) \) is annihilated by \( m_{\mathcal{O}' \mathcal{V}} \), the cohomological exact sequence arising from the short exact sequence
\[
0 \rightarrow f^* \mathbb{T} \xrightarrow{x \varpi''} f^* \mathbb{T} \rightarrow f^* \mathbb{T} / m_{\mathcal{O}' \mathcal{V}} f^* \mathbb{T} \rightarrow 0
\]
implies that we have a natural isomorphism

\[ X(f^*T) \simeq H^2_\Sigma(f^*T) \simeq H^2_\Sigma(f^*T/m'_Ov,f^*T). \]

We also have a natural isomorphism

\[ X(f^*(T \otimes \chi))/m'_Ov,X(f^*(T \otimes \chi)) \simeq H^2_\Sigma(f^*(T \otimes \chi)/m'_Ov,f^*(T \otimes \chi)). \]

Since we have a natural \( \mathcal{O}'[\mathcal{G}_{\mathcal{L},\Sigma}] \)-equivariant isomorphism

\[ f^*(T \otimes \chi)/m'_Ov,f^*(T \otimes \chi) \simeq f^*T/m'_Ov,f^*T, \]

we obtain the isomorphism

\[ X(f^*(T \otimes \chi))/m'_Ov,X(f^*(T \otimes \chi)) \simeq X(f^*T). \]

Hence by (0) and Theorem 3.13, for any \( i \in \mathbb{Z}_{\geq 0} \), we have

\[
\begin{align*}
m_N^Ov,Fitt_{\mathcal{O}'(i)}(X(f^*T)) &= m_N^Ov,Fitt_{\mathcal{O}'(i)}(X(f^*(T \otimes \chi))/m'_Ov,X(f^*(T \otimes \chi))) \\
&= m_N^Ov \cdot (Fitt_{\mathcal{O}'(i)}(X(f^*(T \otimes \chi))) + m'_Ov) \\
&= m_N^Ov + \partial_i(f^*s^\text{univ}(e \otimes \chi))t \\
&= m_N^Ov + \partial_i(f^*s^\text{univ}(e \otimes \chi))t.
\end{align*}
\]

This completes the proof. \(\square\)

4. THE IDEAL \( \mathcal{C}_i(c) \)

Let \((T, c)\) be as in Theorem 1.1. In this section, we shall construct the ideals \( \mathcal{C}_i(c) \), and prove their basic properties. In \( \S 4.1 \) we fix a monic parameter system \( h \), and construct the ideals \( \mathcal{C}_i(c; h) \). (Note that we shall define \( \mathcal{C}_i(c):= \mathcal{C}_i(c;x) \), where \( x \) denotes the standard monic parameter system.) In \( \S 4.2 \) we vary the monic parameter system \( \mathcal{C}_i(c; h) \), and prove the independence of the ideal \( \mathcal{C}_i(c; h) \) of the choice of the monic parameter system \( h \). In \( \S 4.3 \) we prove a basic property of the ideals \( \mathcal{C}_i(c; h) \), namely the stability under scalar extensions (Proposition 4.12). This property plays an important role in the reduction arguments in \( \S 6 \) based on Ochiai’s work \([\text{Oc}2]\). In \( \S 4.2 \) we show another basic property of of the ideals \( \mathcal{C}_i(c; h) \), that is, the stability under affine transformations (Proposition 4.14). This stability will not be used in the proof of our main results, but it seems to be important ingredient to deal with concrete problems since this stability implies that in some sense, the definition of \( \mathcal{C}_i(c; h) \) does not depend on parameters \( x_1, \ldots, x_r \) of \( \Lambda \).

4.1. The construction. In this subsection, we fix a monic parameter system \( h \) of \( \Lambda \). We shall construct an ideal \( \mathcal{C}_i(c; h) \) of \( \Lambda \) for any \( i \in \mathbb{Z}_{\geq 0} \).

Definition 4.1. Let \( I \) and \( I' \) be ideals of \( \Lambda \) satisfying \( I \subseteq I' \), and \( n \in \mathcal{N}_\Sigma \) any element.

(i) We define the ideal \( \mathcal{K}_I(c; I; n) \) of \( \Lambda_{I'/[n]} \) by

\[
\mathcal{K}_I(c; I'; n) := \left\{ f(d_{n}^\text{univ}(c))_{I'} \mid f \in \text{Hom}_{\Lambda_{I'/[n]}}(H^1(\pi_{I'/[n]}^*T),\Lambda_{I'/[n]}) \right\}.
\]
Remark 4.2. Let \( \mathcal{I}(c; I; n) \) the ideal of Kolyvagin images.

(i) We denote by \( \mathcal{I}(c; I'; n) \) the image of \( \mathcal{I}(c; I; n) \) in \( \Lambda/I, n \).

(ii) Assume that \( n \in \mathcal{N}(\mathbb{T}, I) \), and then, by (3.2) in §3.2, the set \( \mathcal{I}(c; I; n) \) is fixed by the action of \( H_n \). We define the ideal \( \mathcal{E}(c; I'; n) \) of \( \Lambda/I \) by the inverse image of \( \mathcal{I}(c; I'; n) \) by the isomorphism

\[
\Lambda/I \xrightarrow{\cong} (\Lambda/I, n)^H_n = N_{H_n} \Lambda/I, n; \quad x \mapsto N_{H_n} x,
\]

where we put

\[
N_{H_n} := \sum_{\sigma \in H_n} \sigma.
\]

(iv) If \( I = I' \), then we put \( \mathcal{E}(c; I; n) := \mathcal{E}(c; I; n) \).

Remark 4.2. Let \( h \) be a monic parameter system of \( \Lambda \), and \( m, m' \in (\mathbb{Z}_{>0})^r+1 \) any element. We put \( I := I(h^m) \). Let \( n \in \mathcal{N}(\mathbb{T}, I) \) be any element. Then, by the isomorphism (3.3) in §3.3 and the injectivity of the \( \Lambda/I, n \)-module \( \Lambda/I, n \) (see Lemma 2.9), we have

\[
\mathcal{E}(c; I; n) = \left\{ f(\kappa_n^\text{inv}(c)) \mid f \in \text{Hom}_{\Lambda/I}(H^1(\pi^{-1}_I \mathbb{T}) \otimes H^\circ_{n}, \Lambda/I) \right\}.
\]

Note that the ideal \( \mathcal{E}(c; I; n) \) is independent of the choice of generators \( \sigma \) and elements \( A(n; \alpha, \ell) \in \Lambda \).

In order to define the ideals \( \mathcal{E}(c; I; n) \) the following lemma becomes a key.

Lemma 4.3. Let \( h \) be a monic parameter system of \( \Lambda \), and \( m, m' \in (\mathbb{Z}_{>0})^r+1 \) elements satisfying \( m' \geq m \). We put \( I := I(h^m) \) and \( I' := I(h^{m'}) \). Let \( n \in \mathcal{N}_\Sigma \) be any element.

(i) For any \( f' \in \text{Hom}_{\Lambda/I, n}(H^1(\pi^{-1}_{I', [n]} \mathbb{T}), \Lambda/I, [n]) \), there exists a homomorphism \( f \in \text{Hom}_{\Lambda/I, n}(H^1(\pi^{-1}_{I, [n]} \mathbb{T}), \Lambda/I, [n]) \) which makes the diagram

\[
\begin{array}{ccc}
H^1(\pi^{-1}_{I', [n]} \mathbb{T}) & \xrightarrow{f'} & \Lambda/I, [n] \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
H^1(\pi^{-1}_{I, [n]} \mathbb{T}) & \xrightarrow{-} & \Lambda/I, [n]
\end{array}
\]

commute, where \( \pi_1 \) and \( \pi_2 \) are natural maps.

(ii) For any \( f \in \text{Hom}_{\Lambda/I, [n]}(H^1(\pi^{-1}_{I, [n]} \mathbb{T}), \Lambda/I, [n]) \), there exists a homomorphism \( f' \in \text{Hom}_{\Lambda/I, [n]}(H^1(\pi^{-1}_{I', [n]} \mathbb{T}), \Lambda/I, [n]) \) which makes the diagram

\[
\begin{array}{ccc}
H^1(\pi^{-1}_{I, [n]} \mathbb{T}) & \xrightarrow{f} & \Lambda/I, [n] \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
H^1(\pi^{-1}_{I', [n]} \mathbb{T}) & \xrightarrow{-} & \Lambda/I, [n]
\end{array}
\]

commute.
**Proof.** We put \( h = (h_0, \ldots, h_r) \) and \( \mathbf{h} = (m_0, \ldots, m_r) \). In order to show our lemma, we may and do assume that \( \mathbf{m}' \) is written in the form \( \mathbf{m}' = (m_{i-1}, m_i + 1, m_{i+1}) \) for some integer \( i \) with \( 0 \leq i \leq r \). Then, we have an isomorphism
\[
\nu_2: \Lambda_{/I,[n]} \cong \Lambda_{/I',[n]}[I] \xrightarrow{h_i} \Lambda_{/I',[n]}; \ x \mapsto h_i x.
\]
We denote by
\[
\nu_1: H^1(\pi^*_{I,[n]} \mathbb{T}) \longrightarrow H^1(\pi^*_{I',[n]} \mathbb{T})
\]
the map induced by \( \nu_2 \otimes \text{id}_{\mathbb{T}}: \pi^*_{I,[n]} \mathbb{T} \longrightarrow \pi^*_{I',[n]} \mathbb{T} \). By definition, the maps \( \nu_1 \circ \pi_1 \) and \( \nu_2 \circ \pi_2 \) are endomorphisms defined by the scalar multiplication by \( h_i \) respectively. Since the Galois representation \( \mathbb{T} \) is unramified at each prime divisors of \( n \), the assumption (A3) and the short exact sequence
\[
0 \longrightarrow \pi^*_{I,[n]} \mathbb{T} \xrightarrow{\times h_i} \pi^*_{I',[n]} \mathbb{T} \longrightarrow \pi^*_{I(m_{i-1}, m_{i+1})[n]} \mathbb{T} \longrightarrow 0
\]
implies that the map \( \nu_1 \) is injective.

Let us prove the first assertion. Note that the image of \( f' \circ \nu_1 \) is annihilated by \( I \). We define a homomorphism
\[
f: H^1(\pi^*_{I,[n]} \mathbb{T}) \longrightarrow \Lambda_{/I,[n]}
\]
of \( \Lambda_{/I,[n]} \)-modules by \( f := \nu_2^{-1} \circ f' \circ \nu_1 \). Then, we have
\[
f \circ \pi_1 = \nu_2^{-1} \circ f' \circ \nu_1 \circ \pi_1 = \nu_2^{-1} \circ h_i f' = \nu_2^{-1} \circ \nu_2 \circ \pi_2 \circ f' = \pi_2 \circ f',
\]
so the map \( f \) satisfies the required properties.

Next, let us show the second assertion. Since the map \( \nu_1 \) is an injection, we can define the \( \Lambda_{/I',[n]} \)-linear map
\[
f'_0: \text{Im} \nu_1 \longrightarrow \Lambda_{/I',[n]}
\]
by \( f'_0 := \nu_2 \circ f \circ \nu_1^{-1} \). Recall that by lemma 2.2, the \( \Lambda_{/I',[n]} \)-module \( \Lambda_{/I',[n]} \) is injective. So we have a homomorphism \( f': H^1(\pi^*_{I,[n]} \mathbb{T}) \longrightarrow \Lambda_{/I,[n]} \) whose restriction to \( \text{Im} \nu_1 \) coincides with \( f'_0 \). By definitions of \( f'_0 \) and \( f' \), it holds that
\[
\nu_2 \circ \pi_2 \circ f' = h_i f' = f' \circ \nu_1 \circ \pi_1 = \nu_2 \circ f \circ \pi_1.
\]
Since \( \nu_2 \) is an injection, we obtain \( \pi_2 \circ f' = f \circ \pi_1 \). This completes the proof. \( \square \)

By Lemma 4.3, we obtain the following corollary, which is plays an important role in §5.

**Corollary 4.4.** Let \( \mathbf{h} \) be a monic parameter system of \( \Lambda \), and \( \mathbf{m}, \mathbf{m}' \in (\mathbb{Z}_{>0})^{r+1} \) elements satisfying \( \mathbf{m}' \geq \mathbf{m} \). We put \( I := I(\mathbf{h}^m) \) and \( I' := I(\mathbf{h}^{m'}) \). Let \( n \in \mathcal{N}_\Sigma \) be any element. Then, we have
\[
\mathcal{K}(c; I'; n) = \mathcal{K}(c; I; n)
\]
In particular, if \( n \in \mathcal{N}(\mathbb{T}, I) \), then we have
\[
\mathcal{C}(c; I'; n) = \mathcal{C}(c; I; n).
\]
Let \( h \) be a monic parameter system of \( \Lambda \), and fix a collection of non-empty subsets
\[ \mathcal{N} := \{ N_m \}_{m \in (\mathbb{Z}_{>0})^r} \]
satisfying \( N_{m'} \subseteq N_m \subseteq N(\mathbb{T}, I(h^m)) \) for any \( m, m' \in (\mathbb{Z}_{>0})^{r+1} \) with \( m' \geq m \). We put
\[ \mathcal{N}(\mathbb{T}; h) := \{ N(\mathbb{T}, I(h^m)) \}_{m \in (\mathbb{Z}_{>0})^r} \]
Let \( m \in (\mathbb{Z}_{>0})^{r+1} \) and \( i \in \mathbb{Z}_{\geq 0} \) be arbitrary elements. We define
\[ N_m(i) := \{ n \in N_m | \# \text{Prime}(n) \leq i \} . \]
Then, we denote by \( \mathcal{C}_i(c; I(h^m); \mathcal{N}) \) the ideal of \( \Lambda/I(h^m) \) generated by
\[ \bigcup_{n \in N_m(i)} \mathcal{C}(c; I(h^m); n) . \]

Let \( m, m' \in (\mathbb{Z}_{>0})^{r+1} \) be elements satisfying \( m' \geq m \). Put \( I := I(h^m) \), and \( I' := I(h^{m'}) \). The image of \( \mathcal{C}_i(c; I'; \mathcal{N}) \) in \( \Lambda/I \) is denoted by \( \mathcal{C}_i(c; I'; \mathcal{N})_I \). We write \( \mathcal{C}_i(c; I') := \mathcal{C}_i(c; I'; \mathcal{N}(\mathbb{T}; h)) \) and \( \mathcal{C}_i(c; I')_I := \mathcal{C}_i(c; I'; \mathcal{N}(\mathbb{T}; h))_I \). By Corollary 4.4, we have
\[ \mathcal{C}_i(c; I'; \mathcal{N})_I \subseteq \mathcal{C}_i(c; I; \mathcal{N}) . \]
Hence we obtain a projective system \( \{ \mathcal{C}_i(c; I(h^m); \mathcal{N}) \}_{m \in (\mathbb{Z}_{>0})^{r+1}} \).

**Definition 4.5.** Let \( h \) be a monic parameter system of \( \Lambda \). Then, we define an ideal \( \mathcal{C}_i(c; h) \) of the ring \( \Lambda \) by
\[ \mathcal{C}_i(c; h; \mathcal{N}) := \varprojlim_m \mathcal{C}_i(c; I(h^m); \mathcal{N}) \subseteq \varprojlim_m \Lambda/I(h^m) \simeq \Lambda . \]
Especially, we put \( \mathcal{C}_i(c; h) := \mathcal{C}_i(c; h; \mathcal{N}(\mathbb{T}; h)) \) and \( \mathcal{C}_i(c) := \mathcal{C}_i(c; x; \mathcal{N}(\mathbb{T}; x)) \), where \( x \) denotes the standard monic parameter system.

4.2. **Varying monic parameter systems.** In the previous subsection, we have constructed the collection \( \{ \mathcal{C}_i(c; h) \}_{i \geq 0} \) of ideals of \( \Lambda \) by using the “universal Kolyvagin system” arising from the Euler system \( c \). Note that a priori, the definition of the ideals \( \mathcal{C}_i(c; h) \) depends on the choice of the monic parameter system \( h \). Here, we shall prove Proposition 4.8, which assert that the ideals \( \mathcal{C}_i(c; h) \) are independent of \( h \). This independence becomes a key of the induction arguments in the proof of our main results.

**Lemma 4.6.** Let \( h \) be a monic parameter system of \( \Lambda \), and \( i \in \mathbb{Z} \) an integer satisfying \( 0 \leq i \leq r \). We assume that \( h_{\leq i-1} = x_{\leq i-1} \) if \( i \geq 1 \). We put
\[ \tilde{h} := (x_{\leq i}, h_{\geq i+1}) \in (\Lambda)^{r+1} . \]
Then, for any \( m, m', m'' \in (\mathbb{Z}_{>0})^{r+1} \) satisfying \( I(h^{m''}) \subseteq I(h^{m'}) \subseteq I(h^m) \), we have
\[ K\mathcal{I}(c; I(h^{m''}); n)_{I(h^{m'})} = K\mathcal{I}(c; I(h^{m'}); n) \]
and
\[ K\mathcal{I}(c; I(h^{m'}); n)_{I(h^m)} = K\mathcal{I}(c; I(h^m); n) . \]
**Proof.** First, suppose that \( i = 0 \). Let \( v_\mathcal{O} \) be the additive valuation on \( \mathcal{O} \) normalized by \( v_\mathcal{O}(\pi) = 1 \), and put \( \tilde{m}'' = (m''_0, v_\mathcal{O}(h_0), m''_{\geq 1}) \). Then we have \( I(\tilde{h}''') = I(h'''') \). This implies that we have

\[
\kappa I(c; I(\tilde{h}'''); n) = \kappa I(c; I(h''''); n).
\]

So by Corollary 4.4, we deduce the assertion of Lemma 4.6 for \( i = 0 \).

Next, let us assume that \( i \geq 1 \). By the definition of monic parameter systems, there exist a positive integer \( \delta \) and elements \( g_0, g_1, \ldots, g_{\delta-1} \in m_{A(i-1)} \) satisfying

\[
h_i = x_i^\delta + \sum_{j=0}^{\delta-1} g_j x_i^j.
\]

Here, \( m_{A(i-1)} \) denotes the maximal ideal of \( A^{(i-1)} = \mathcal{O}[[x_1, \ldots, x_{i-1}]] \). Let \( M \) be a positive integer satisfying \( p^M \geq m'', \delta p^M > m' \) and \( m_{A(i-1)}^M \subseteq I(x_{\leq i-1})^M \). Then, we have

\[
h_i^{p^M} \equiv x_i^{\delta p^M} \mod m_{A(i-1)}^M.
\]

Define elements \( \nu, \tilde{\nu} \in (\mathbb{Z}_{>0})^{r+1} \) by

\[
\nu = (m''_{\leq i-1}, p^M, m''_{\geq i+1}),
\]

\[
\tilde{\nu} = (m''_{\leq i-1}, \delta p^M, m''_{\geq i+1}).
\]

By the congruence (8), we obtain \( I(h^{\nu}) = I(h''') \) and

\[
\kappa I(c; I(h^{\nu}); n) = \kappa I(c; I(h''''); n).
\]

Hence the assertion of Lemma 4.6 follows from Corollary 4.4.

\( \square \)

**Lemma 4.7.** Let \( h \) be a monic parameter system of \( A \). Then, for any \( m, m', m'' \in (\mathbb{Z}_{>0})^{r+1} \) satisfying \( I(h^{m''}) \subseteq I(x^{m'}) \subseteq I(h^{m}) \), we have

\[
\kappa I(c; I(h^{m''}); n)_{I(x^{m'})} = \kappa I(c; I(h^{m''}); n)
\]

and

\[
\kappa I(c; I(x^{m'}); n)_{I(h^{m})} = \kappa I(c; I(h^{m}); n)
\]

**Proof.** For each \( i \in \mathbb{Z} \) with \( 0 \leq i \leq r \), we define \( h^{(i)} := (x_{\leq r-i}, h_{r-i+1}) \). Note that in particular, we have \( h^{(0)} := x \) and \( h^{(r)} := h \). Let \( \{N_i\}_{i=0}^r \) be a sequence of integers satisfying \( I(h^{m''}) \supseteq I(h^{N_i}) \) and \( I((h^{(i)})^{N_i}) \supseteq I((h^{(i)})^{N_i}) \) for any integer \( i \) with \( 1 \leq i \leq r \). Then, by using Lemma 4.3, we deduce, via induction on \( i \), that

\[
\kappa I(c; I(x^{N_0}); n)_{I(h^{N_1})} = \kappa I(c; I((h^{(i)})^{N_i}); n)
\]

for any \( 0 \leq i \leq r \). Hence, by Corollary 4.4, the equality (9) for \( i = r \) implies the assertion of our lemma.

\( \square \)

**Proposition 4.8.** For any monic parameter system \( h \) of \( A \) and any \( i \in \mathbb{Z}_{\geq 0} \), we have \( \mathcal{C}_i(c; h) = \mathcal{C}_i(c) \). Namely, the ideal \( \mathcal{C}_i(c; h) \) is independent of the choice of \( h \).
**Proof.** Let \( \{N_\nu\}_{\nu > 0} \) be an increasing sequence of positive integers which satisfies \( I(x^{N_\nu}) \supseteq I(h^{N_\nu}) \supseteq I(x^{N_{\nu+1}}) \) for any \( j \in \mathbb{Z}_{\geq 0} \). We put \( I_{2j-1} := I(x^{N_{2j-1}}) \) and \( I_{2j} := I(h^{N_{2j}}) \) for any \( j \in \mathbb{Z}_{>0} \). Then, Lemma 4.7 implies that \( \{\mathcal{C}_i(c; I_\nu)\}_{\nu > 0} \) forms a projective system. Hence we obtain

\[
\mathcal{C}_i(c; h) = \lim_{j \to \infty} \mathcal{C}_i(c; I_{2j}) = \lim_{j \to \infty} \mathcal{C}_i(c; I_{2j-1}) = \mathcal{C}_i(c; x).
\]

This completes the proof of Proposition 4.8.

### 4.3. Extension of Scalars.

Here, we observe the behavior of the ideals \( \mathcal{C}_i(c) \) along extensions of the rings of constants.

Let \( F' \) be a finite extension field of \( F \), and \( \mathcal{O}' \) the ring of integers of \( F' \). We denote the ramification index of \( F'/F \) by \( e \). Fix a uniformizer \( \pi \in \mathcal{O}' \). We put \( \Lambda_{\mathcal{O}'} := \Lambda \otimes_{\mathcal{O}} \mathcal{O}' = \mathcal{O}'[\{x_1, \ldots, x_r\}] \), and \( \Lambda_{\mathcal{O}'} := \Lambda \otimes_{\mathcal{O}} \mathcal{O}' \). We define the new “standard parameter system”

\[
x' = (x'_0, \ldots, x'_r) := (\pi', x_1, \ldots, x_r).
\]

We define \( T_{\mathcal{O}'} := T \otimes_{\mathcal{O}} \mathcal{O}' \). The Euler system \( c \) induces an Euler system \( c' := c \otimes 1 \) for \( T_{\mathcal{O}'} \).

Let \( m = (m_0, \ldots, m_r) \in (\mathbb{Z}_{>0})^{r+1} \) be any element, and put \( m' := (em_0, m_{>1}) \). and denote by \( I_{\mathcal{O}'}(x^{m'}) \) the ideal of \( \Lambda \) generated by \( \{x_i^{m_i} \mid 0 \leq i \leq r\} \). Then, clearly we have

\[
I_{\mathcal{O}'}(x^{m'}) = I(x^{m})\Lambda_{\mathcal{O}'} = I(x^{m})\mathcal{O}'.
\]

Put \( I := I(x^m) \) and \( I' := I_{\mathcal{O}'}(x^{m'}) \). Since \( \mathcal{O}' \) is a free \( \mathcal{O} \)-module of finite rank, we have an isomorphism

\[
H^i(\pi^* T_{\mathcal{O}'}) \simeq H^i(\pi^* T \otimes_{\mathcal{O}} \mathcal{O}') \simeq H^i(\pi^* T) \otimes_{\mathcal{O}} \mathcal{O}'
\]

of \( \Lambda \otimes_{\mathcal{O}} \mathcal{O}' \)-modules for any \( i \in \mathbb{Z}_{\geq 0} \). Let \( \ell \) be a prime number not contained in \( \Sigma \). Since \( \mathcal{O}' \) is faithfully flat over \( \mathcal{O} \), the \( \mathcal{O}' \)-module

\[
T_{\mathcal{O}'}/(I' T_{\mathcal{O}'}) \simeq (T/(I T + (\text{Frob}_{\ell} - 1) T)) \otimes_{\mathcal{O}} \mathcal{O}'
\]

is free of rank one if and only if the \( \mathcal{O} \)-module \( T/(I T + (\text{Frob}_{\ell} - 1) T) \otimes_{\mathcal{O}} \mathcal{O}' \) is free of rank one. So, we obtain

\[
\mathcal{N}(T, I) = \mathcal{N}(T_{\mathcal{O}'}, I').
\]

For any \( n \in \mathcal{N}(T, I) \), we denote by \( \mathcal{C}(c'; I'; n)_{\mathcal{O}'} \) the ideal of \( \Lambda_{\mathcal{O}'} / I' \) constructed in Definition 4.1 (iv) for the new data

\[
(\Lambda_{\mathcal{O}'}, \Lambda_{\mathcal{O}'}, T_{\mathcal{O}'}, c \otimes 1, x').
\]

Similarly, we define \( \mathcal{C}_i(c'; I')_{\mathcal{O}'} \) and \( \mathcal{C}_i(c)_{\mathcal{O}'} \). We can easily check that the new data satisfies all conditions required in Theorem 4.1 if the old data \( (\Lambda, T, c) \) satisfies them.

**Lemma 4.9.** The following hold.

(i) For any \( n \in \mathcal{N}(T, I) \), we have \( \mathcal{C}(c'; I'; n)_{\mathcal{O}'} = \mathcal{C}(c; I; n)_{\mathcal{O}'} \).

(ii) For any \( i \in \mathbb{Z}_{\geq 0} \), we have \( \mathcal{C}_i(c'; I')_{\mathcal{O}'} = \mathcal{C}_i(c; I)_{\mathcal{O}'} \).
Proof. By the equality \((11)\), the assertion (i) implies the assertion (ii). So, it suffices to show the assertion (i). We have
\[
\text{Hom}_{\Lambda_{\mathcal{O}'}/I'}(H^1(\pi_{I'}^*T_{\mathcal{O}'}, \Lambda_{\mathcal{O}'}/I'), \Lambda_{\mathcal{O}'}/I') = \text{Hom}_{\Lambda/I}(H^1(\pi_{I}^*T), \Lambda_{\mathcal{O}'}/I')
\]
\[
= \text{Hom}_{\Lambda/I}(H^1(\pi_{I}^*T), \Lambda/I) \otimes_{\mathcal{O}} \mathcal{O}'.
\]
Indeed, the first equality follows from \(10\) for \(i = 1\), and the second equality holds since \(\Lambda_{\mathcal{O}'}/I' = \Lambda/I \otimes_{\mathcal{O}} \mathcal{O}'\) is a free \(\Lambda/I\)-module. Hence by the definition of the ideals \(\mathcal{E}(c; I; n)\) and \(\mathcal{E}(c'; I'; n)_{\mathcal{O}'}\), we obtain the assertion (i). \(\square\)

Here, let us prove Proposition \(4.12\) below, which states that the ideals \(\mathcal{C}_i(c)\) is compatible with base change and descent arguments along extension of the coefficient rings. We need some lemmas.

Lemma 4.10. Let \(J\) be an ideal of \(\Lambda/I\). Then, as a subset of \(\Lambda_{\mathcal{O}'}/I'\) we have \(J = J\Lambda_{\mathcal{O}'}/I' \cap \Lambda/I\).

Proof. Put \(\tilde{J} := J\Lambda_{\mathcal{O}'}/I' \cap \Lambda/I\). Clearly, we have \(J \subseteq \tilde{J}\), and \(\tilde{J}\Lambda_{\mathcal{O}'}/I' = J\Lambda_{\mathcal{O}'}/I'\). Since \(\mathcal{O}'\) is flat over \(\mathcal{O}\), we have
\[J\Lambda_{\mathcal{O}'}/I' = J \otimes_{\mathcal{O}} \mathcal{O}' \subseteq \Lambda_{\mathcal{O}'}/I' = \Lambda/I \otimes_{\mathcal{O}} \mathcal{O}'\]
Similarly, we have \(J\Lambda_{\mathcal{O}'}/I' = \tilde{J} \otimes_{\mathcal{O}} \mathcal{O}'\), so we obtain \(J \otimes_{\mathcal{O}} \mathcal{O}' = \tilde{J} \otimes_{\mathcal{O}} \mathcal{O}'\). The ring \(\mathcal{O}'\) is faithfully flat over \(\mathcal{O}\), so we obtain \(J = \tilde{J}\). \(\square\)

By Lemma \(4.9\) and Lemma \(4.10\) we immediately obtain the following lemma.

Lemma 4.11. The following hold.

(i) For any \(n \in N(T, I)\), we have \(\mathcal{E}(c; I; n) = \mathcal{E}(c'; I'; n)_{\mathcal{O}'} \cap \Lambda/I\).
(ii) For any \(i \in \mathbb{Z}_{\geq 0}\), we have \(\mathcal{C}_i(c; I)_{\mathcal{O}} = \mathcal{C}_i(c'; I')_{\mathcal{O}'}\).

By the above arguments, we deduce the following.

Proposition 4.12. We have \(\mathcal{C}_i(c')_{\mathcal{O}'} = \mathcal{C}_i(c)_{\mathcal{O}'}\) and \(\mathcal{C}_i(c) = \mathcal{C}_i(c')_{\mathcal{O}'} \cap \Lambda'\) for any \(i \in \mathbb{Z}_{\geq 0}\).

In the end of this section, we give a remark on the behavior of the Selmer group side along the extension of the rings of coefficients. The following holds.

Proposition 4.13. We have
\[
\text{Fitt}_{\Lambda_{\mathcal{O}'}, \mathcal{O}'}(X(T_{\mathcal{O}'})) = \text{Fitt}_{\Lambda, \mathcal{O}}(X(T))\Lambda_{\mathcal{O}'}.
\]

Proof. Since \(\mathcal{O}'\) is a free \(\mathcal{O}\)-module of finite rank, we have an isomorphism
\[
H^i(\pi_{I, [n]}^*T \otimes_{\mathcal{O}} \mathcal{O}') \simeq H^i(\pi_{I, [n]}^*T) \otimes_{\mathcal{O}} \mathcal{O}'
\]
of \(\Lambda_{\mathcal{O}'}\)-modules for any \(i \in \mathbb{Z}_{\geq 0}\), any \(n \in N_{\Sigma}\) and any ideal \(I\) of \(\Lambda\). So, by definition of the ideal \(\mathcal{C}_i(c)\) and the base change property of Fitting ideals, we obtain the assertion of Proposition \(4.13\). \(\square\)
4.4. **Affine transformations.** Here, we introduce affine transformations on the ring $\Lambda = \Lambda^{(r)}$, and show that the ideals $\xi_i(e)$ are stable under affine transformations. (Note that we do not use this property in the proof of our main results.)

Let us define affine transformations. For any $A \in \text{GL}_r(\mathcal{O})$ and any $v \in (\mathcal{O})^{\oplus r}$, we define an automorphism

$$T(A, v): \Lambda^{(r)} \xrightarrow{\cong} \Lambda^{(r)}; \ f(x) \mapsto f(Ax + v),$$

where we regard $x = (x_i)_{i=1}^r$ and $v$ as column vectors. We call this automorphism an **affine transformation** on $\Lambda = \Lambda^{(r)}$.

Now, we introduce certain special affine transformations called elementary affine transformations. First, let us recall the definition of elementary matrices.

(i) For any $u \in \mathcal{O}^\times$ and any $\nu \in \mathbb{Z}$ with $1 \leq \nu \leq r$, we define a matrix $P_\nu(u) = (c_{ij})_{i,j} \in \text{GL}_r(\mathcal{O})$ by

$$c_{ij} := \begin{cases} u & \text{if } i = j = \nu \\ 1 & \text{if } i = j \neq \nu \\ 0 & \text{if } i \neq j. \end{cases}$$

(ii) Let $\nu, \mu \in \mathbb{Z}$ be distinct integers with $1 \leq \nu, \mu \leq r$. Then, we define a matrix $Q_{\mu,\nu} = (c_{ij})_{i,j} \in \text{GL}_r(\mathcal{O})$ by

$$c_{ij} := \begin{cases} 1 & \text{if } i = j, \text{ and if } i \notin \{\mu, \nu\} \\ 1 & \text{if } (i, j) = (\mu, \nu), (\nu, \mu) \\ 0 & \text{otherwise}. \end{cases}$$

(iii) Let $\nu, \mu \in \mathbb{Z}$ be integers with $1 \leq \nu, \mu \leq r$ and $\mu > \nu$. For any $a \in \mathcal{O}$, we define a matrix $R_{\mu,\nu}(a) = (c_{ij})_{i,j} \in \text{GL}_r(\mathcal{O})$ by

$$c_{ij} := \begin{cases} 1 & \text{if } i = j \\ a & \text{if } (i, j) = (\mu, \nu) \\ 0 & \text{otherwise}. \end{cases}$$

The matrices of the form $P_\nu(u), Q_{\mu,\nu}$ or $R_{\mu,\nu}(a)$ are called **elementary matrices**. Note that since $\mathcal{O}$ is a local ring, any element $A \in \text{GL}_r(\mathcal{O})$ is decomposed into a product of elementary matrices. For any $a \in \mathcal{O}$ and any $\nu \in \mathbb{Z}$ with $1 \leq \nu \leq r$, we define an element $\delta_\nu(a) = (v_i)_{i=1}^r \in (\mathcal{O})^{\oplus r}$ by $v_\nu := a$ and $v_j := 0$ if $j \neq \nu$. An affine transformation $T(A, v)$ is called an **elementary affine transformation** if the pair $(A, v)$ is one of the following:

- The matrix $A$ is elementary, and $v = 0$
- The matrix $A$ is the identity matrix, and $v = \delta_i(a)$ for some $a \in \mathcal{O}$ and $\nu \in \mathbb{Z}$ with $1 \leq \nu \leq r$. 

Note that any affine transformation is a composite of finitely many elementary affine transformations.
Proposition 4.14. Let $A \in \text{GL}_r(O)$ and $v = (v_i)^{i=1}_{i=r} \in (\varpi O)^{\oplus r}$ be arbitrary elements. We put $$y = (y_i)^{i=1}_{i=r} := Ax + v \in (\mathfrak{m}_A)^{\oplus r}.$$ Then, for any $i \in \mathbb{Z}_{\geq 0}$, we have $$T(A, v)(\mathcal{C}_i(c; x)) = \mathcal{C}_i(c; y) = \mathcal{C}_i(c; x).$$

Proof. By the definition of the ideal $\mathcal{C}_i(c; x)$, the first equality $T(A, v)(\mathcal{C}_i(c; x)) = \mathcal{C}_i(c; y)$ is clear. Let us prove the second equality. Since any affine transformation is decomposed into a composite of elementary affine transformations, we may assume that $T(A, v)$ is elementary.

Let us consider the case when $v = 0$. First, we assume that $A$ is equal to $P_{\mu}(u)$ or $Q_{\mu, \nu}$. In this case, we have $I(x^N) = I(y^N)$ for any $N \in \mathbb{Z}_{\geq 1}$. So, by definition, we have $\mathcal{C}_i(c; I(x^N)) = \mathcal{C}_i(c; I(y^N))$. This implies that $\mathcal{C}_i(c; y) = \mathcal{C}_i(c; x)$. Next, let us assume that $A = R_{\mu, \nu}(a)$. Then, the system $$y = (\varpi, x_1, \ldots, x_{\mu-1}, x_\mu + ax_\nu, x_{\mu+1}, \ldots, x_r)$$ becomes a monic parameter system. Hence by Proposition 4.8 we obtain $\mathcal{C}_i(c; y) = \mathcal{C}_i(c; x)$.

Let us consider the case when $A = 1$. Put $v = \delta_\nu(a)$. In this case, the system $$y = (\varpi, x_1, \ldots, x_{\nu-1}, x_\nu + a, x_{\nu+1}, \ldots, x_r)$$ forms a monic parameter system, so Lemma 4.8 implies that $\mathcal{C}_i(c; y) = \mathcal{C}_i(c; x)$. □

5. Reduction of $\mathcal{C}_i(c)$

Let $\Lambda = \Lambda^{(r)}$ and $(\mathbb{T}, c)$ be as in Theorem 1.1. In particular, we assume that $c$ satisfies (NV). In the proof of our main results, namely Theorem 1.1, Theorem 1.2 and Theorem 1.3, a certain properties of the ideals $\mathcal{C}_i(c)$ related to the specialization of the coefficient ring called the weak/strong specialization compatibility become a key. Roughly speaking, the specialization weak (resp. strong) compatibility says that if the reduction map $\pi_I: \Lambda \to \Lambda/I \simeq \Lambda^{(r-1)}$ for a certain ideal $I$ is given, then for any $i \in \mathbb{Z}_{\geq 0}$, the image of $\mathcal{C}_i(c)$ by $\pi_I$ is contained in (resp. coincides with) the ideal $\mathcal{C}_i(\pi_I c)$. $\mathcal{C}_i(\pi_I c)$ defined by the data $(\pi_I \mathbb{T}, \pi_I c, \mathfrak{N}(\pi_I \mathbb{T}; x_{<r-1}))$. In this section, we shall study the specialization compatibilities of ideals $\mathcal{C}_i(c)$.

In §5.1, we shall prove Proposition 5.2 that is, the weak specialization compatibility for general multi-variable cases. In §5.2 we will show the strong compatibility in one variable cases, namely Theorem 5.3. Note that the proof of Theorem 5.3 is the most technical part of this article. In §5.3, we shall prove Theorem 5.8 which asserts that the strong compatibilities hold in the case when $\mathbb{T}$ is a cyclotomic deformation of a one variable deformation.
5.1. Weak specialization compatibility. Here, let us study the weak specialization compatibilities. We need the notion of linear elements in the sense of Ochiai’s article [Oc2].

**Definition 5.1.** A linear element \( g \) in \( \Lambda = \Lambda^{(r)} \) is a polynomial written in a form

\[
g = a_0 + \sum_{i=1}^{r} a_i x_i \in \Lambda,
\]

where \( a_0 \in \varpi \mathcal{O}, \) and \((a_1, \ldots, a_r) \in \mathcal{O}^{\oplus r} \setminus (\varpi \mathcal{O})^{\oplus r}.\)

As in §Notation, namely, the end of [I], for each ideal \( J \) of \( \Lambda \), we denote by \( \mathcal{C}_i(c)_J \) the image of \( \mathcal{C}_i(c) \) in \( \Lambda/J \).

**Proposition 5.2.** Let \( h = (x_{\leq r-1}, h) \) be a monic parameter system of \( \Lambda \) such that \( h \) is a linear element. We put \( I := h\Lambda = I(h^{(\infty, \ldots, \infty, 1)}) \). Let \( \pi_I : \Lambda \rightarrow \Lambda/I \simeq \Lambda^{(r-1)} \) be the reduction map. Then, We have

\[
\mathcal{C}_i(c)_I \subseteq \mathcal{C}_i(\pi_I^*c) := \mathcal{C}_i(\pi_I^*c; x_{\leq r-1}; \mathcal{M}(\pi_I^*T; x_{\leq r-1}))
\]

**Proof.** Note that by definition, we have

\[
\mathcal{M}(\pi_I^*T; x_{\leq r-1}) = \{ \mathcal{M}(I(h^{(m,1)})) \}_{m=(m_0, \ldots, m_{r-1}) \in (\mathbb{Z}_{>0})^r}.
\]

For any \( N \in \mathbb{Z}_{>0} \), we put \( m(N) := (N, \ldots, N, 1) \in (\mathbb{Z}_{>0})^{r+1} \). Then, by definition, we have

\[
\mathcal{C}_i(\pi_I^*c; x; \pi_I^*\mathcal{M}(T; h)) = \lim_{N} \mathcal{C}_i(c; I(h^{m(N)}); \mathcal{M}(T; h)).
\]

So, Proposition 4.8 and Corollary 4.4 imply our proposition. \( \square \)

5.2. Strong compatibility for one variable cases. Here, we set \( \Lambda = \Lambda^{(1)} = \mathcal{O}[[x_1]]. \) Let prove the following theorem, which says that in one variable cases, the strong specialization compatibilities hold.

**Theorem 5.3.** Let \( a \in \mathfrak{m}_\mathcal{O} = \varpi \mathcal{O} \) be any element. Then, for any \( i \in \mathbb{Z}_{\geq 0} \), the image of \( \mathcal{C}_i(c) \) in \( \Lambda/I(0, x_1 - a) \) coincides with

\[
\mathcal{C}_i(\pi_{I(0,x_1-a)}^*c) := \mathcal{C}_i(\pi_{I(0,x_1-a)}^*c; x; \mathcal{N}(\pi_{I(0,x_1-a)}^*T)).
\]

Let \( i \in \mathbb{Z}_{\geq 0} \) be any element. Recall that we assume that \( c \) satisfies (NV). So, we have a non-negative integer \( c \) such that

\[
\mathcal{C}_i(\pi_{I(0,x_1-a)}^*c) \supseteq \mathcal{C}_i(\pi_{I(0,x_1-a)}^*c) = \varpi^c \mathcal{O}.
\]

By in §1.1 in order to prove Theorem 5.3, it suffices to show the following Proposition.

**Proposition 5.4.** Fix \( a \in \mathfrak{m}_\mathcal{O} \). Let \( m_0 \in \mathbb{Z}_{>0} \) and \( m' = (m'_0, m'_1) \in \mathbb{Z}_{>0}^2 \) be elements satisfying \( m'_0 \geq m_0 > c \). We put \( I := I(\varpi^{m_0}, x_1 - a) \) and \( I' := I(\varpi^{m'_0}, (x_1 - a)^{m'_1}) \). Then, we have

\[
\mathcal{C}_i(c; I')_I \supseteq \mathcal{C}_i(c; I).
\]
\textbf{Proof.} By (7), we may replace \( m' \) with suitable larger one, and assume that \( m'_1 \) is prime to \( p \), and that \( m'_0 = m_0 m'_1 \). Moreover, by Lemma 4.11 we may assume that \( F \) contains a primitive \( m'_1 \)-th root of unity. We also assume that \( \text{Fix a generator } \bar{b} \text{ of the cyclic group } k^*, \) and let \( b \in \mathcal{O} \) be a lift of \( \bar{b} \). Then, we fix a \( m'_1 \)-th root \( \beta \in \overline{F} \) of \( b \), and put \( F' := F(\beta) \). The ring of integers of \( F' \) is denoted by \( \mathcal{O}' \). Note that \( F'/F \) is an unramified extension with \([ F': F ] = m'_1 \). We define a homomorphism
\[
e: \Lambda \rightarrow \mathcal{O}', \quad x_1 \mapsto a + \varpi^m \beta
\]
of \( \mathcal{O} \)-algebras. The kernel of \( e \) is generated by the irreducible distinguished polynomial
\[
g := (x_1 - a)^{m'_1} - \varpi^{m_0} b \in \mathcal{O}[x_1].
\]
By definition, we have \( I(\varpi^{m_0}, g) = I' \), and \( e \) induces an injection
\[
\bar{e}_{m'_0}: \Lambda/I' \hookrightarrow \mathcal{O}'/\varpi^{m_0} \mathcal{O}'; \quad x_1 \mod I' \mapsto a + \varpi^m \beta \mod \varpi^{m_0} \mathcal{O}'.
\]
Similarly, we have an injection \( \bar{e}_{m_0}: \Lambda/I \hookrightarrow \mathcal{O}'/\varpi^{m_0} \mathcal{O}' \) given by
\[
\bar{e}_{m_0}(x_1 \mod I) = a \mod \varpi^m \mathcal{O}' = a + \varpi^m \beta \mod \varpi^{m_0} \mathcal{O}'.
\]
Then, by definition, we obtain a commutative diagram
\[
\begin{array}{ccc}
\Lambda/I'\Lambda & \xrightarrow{\bar{e}_{m'_0}} & (\mathcal{O}'/\varpi^{m_0} \mathcal{O}') \\
\pi_1 & & \pi_2 \\
\Lambda/I\Lambda & \xrightarrow{\bar{e}_{m_0}} & (\mathcal{O}'/\varpi^{m_0} \mathcal{O}'),
\end{array}
\]
where \( \pi_1 \) and \( \pi_2 \) are natural surjections.

We put \( \mathcal{N}(m) := \mathcal{N}(e^* \mathcal{T}; \varpi^m \mathcal{O}') \) for any \( m \in \mathbb{Z}_{>0} \). Then, the following lemma holds.

\textbf{Lemma 5.5.} We have \( \mathcal{N}(m_0) = \mathcal{N}(\mathcal{T}; I) \) and \( \mathcal{N}(m'_0) = \mathcal{N}(\mathcal{T}; I') \).

\textbf{Proof.} First, let us show first equality \( \mathcal{N}(m_0) = \mathcal{N}(\mathcal{T}; I) \). Let \( \ell \) be a prime number not contained in \( \Sigma \), and put
\[
\mathcal{T}_\ell := \mathcal{T}/(\text{Frob}_\ell - 1) \mathcal{T}.
\]
Since the ring \( \mathcal{O}'/\varpi^{m_0} \mathcal{O}' \) is faithfully flat over \( \Lambda/I \simeq \mathcal{O}/\varpi^{m_0} \mathcal{O} \), the \( \Lambda/I \)-module \( \mathcal{T}_\ell/\mathcal{T}_\ell \) is free of rank one if and only if the \( \mathcal{O}'/\varpi^{m_0} \mathcal{O}' \)-module
\[
e^* \mathcal{T}_\ell/\varpi^m \mathcal{E}^* \mathcal{T}_\ell \simeq e^*(\mathcal{T}_\ell/\mathcal{T}_\ell)
\]
is free of rank one. Hence the first equality follows from the definition of \( \mathcal{N}(m_0) \) and \( \mathcal{N}(\mathcal{T}; I) \).

Let us show the second equality, namely \( \mathcal{N}(m'_0) = \mathcal{N}(\mathcal{T}; I') \). We denote the composite
\[
\bar{e}_{m'_0} \circ \pi_{I'}: \Lambda \rightarrow \Lambda/I' \rightarrow \mathcal{O}'/(\varpi^{m'_0})
\]
by \( e_{m'_0} \). Let \( \ell \) be a prime number not contained in \( \Sigma \). In order to show the second equality of our lemma, it suffices to show that the \( \Lambda/I \)-module \( \pi_{I'}^* \mathcal{T}_\ell \) is free of rank one if and only if the \( \mathcal{O}'/(\varpi^{m'_0}) \)-module \( e_{m'_0}^* \mathcal{T}_\ell \) is free of rank one. Note that by Example 2.3, the \( \Lambda/I' \)-module \( \pi_{I'}^* \mathcal{T}_\ell \) is free of rank one if and only if \( \text{Fitt}_{\Lambda/I', 0}(\pi_{I'}^* \mathcal{T}_\ell) = \{0\} \).
and $\text{Fitt}_{\mathcal{A}/I'}(\pi^*_I T_I) = \Lambda/I'$. Similarly, the $O'/(\varpi^{m_0})$-module $e^{*}_{m_0}T_I$ is free of rank one if and only if $\text{Fitt}_{O'/(\varpi^{m_0})}(e^{*}_{m_0}T_I) = \{0\}$ and $\text{Fitt}_{O'/(\varpi^{m_0})}(e^{*}_{m_0}T_I) = O'/(\varpi^{m_0})$. Since $e^{*}_{m_0}T_I = \overline{e}^{*}_{m_0}(\pi^*_I T_I)$, we have

$$\text{Fitt}_{O'/(\varpi^{m_0})}(e^{*}_{m_0}T_I) = \overline{e}^{*}_{m_0}(\text{Fitt}_{\mathcal{A}/I'}(\pi^*_I T_I)) O'/(\varpi^{m_0})$$

for any $i \in \mathbb{Z}_{\geq 0}$. Note that since $\overline{e}^{*}_{m_0}$ is injective, we have $\text{Fitt}_{\mathcal{A}/I'}(\pi^*_I T_I) = \{0\}$ if and only if $\text{Fitt}_{O'/(\varpi^{m_0})}(e^{*}_{m_0}T_I) = \{0\}$. Moreover, since $\overline{e}^{*}_{m_0}$ is a homomorphism of local rings, we have $\text{Fitt}_{\mathcal{A}/I'}(\pi^*_I T_I) = \Lambda/I'$ if and only if $\text{Fitt}_{O'/(\varpi^{m_0})}(e^{*}_{m_0}T_I) = O'/(\varpi^{m_0})$.

Hence we deduce that the $\Lambda/I$-module $\pi^*_I T_I$ is free of rank one if and only if the $O'/(\varpi^{m_0})$-module $e^{*}_{m_0}T_I$ is free of rank one. This implies that $\mathcal{N}(m_0') = \mathcal{N}(T; I')$.

We need the following lemma and its corollary.

**Lemma 5.6.** Let $n \in \mathcal{N}(m_0')$ be any element. Then, the following hold.

1. $\mathcal{C}(e^{*} c; \varpi^{m_0} O'; n) = \overline{e}^{*}_{m_0}(\mathcal{C}(c; I'; n)) \cdot O'/(\varpi^{m_0}) O'$.
2. $\mathcal{C}(e^{*} c; \varpi^{m_0} O'; n) = \overline{e}^{*}_{m_0}(\mathcal{C}(c; I'; n)) \cdot O'/(\varpi^{m_0}) O'$.
3. $\mathcal{C}(c; I'; n) = \overline{e}^{*}_{m_0}(\mathcal{C}(e^{*} c; \varpi^{m_0} O'; n))$.
4. $\mathcal{C}(c; I'; n) = \overline{e}^{*}_{m_0}(\mathcal{C}(e^{*} c; \varpi^{m_0} O'; n))$.

**Proof.** Let us show the assertion (i) of Lemma 5.6. First, we shall prove

$$\mathcal{C}(e^{*} c; \varpi^{m_0} O'; n) \supseteq \overline{e}^{*}_{m_0}(\mathcal{C}(c; I'; n)) \cdot O'/(\varpi^{m_0}) O'$.

The map $\overline{e}^{*}_{m_0}$ induces an exact sequence

$$0 \rightarrow \pi^*_I T \xrightarrow{\overline{e}^{*}_{m_0} \circ \pi^*_I T \xrightarrow{\overline{e}^{*}_{m_0}} \pi^*_I T \xrightarrow{\text{Coker}(\overline{e}^{*}_{m_0})} 0,$$

where $\overline{e}^{*}_{m_0} := \overline{e}^{*}_{m_0} \otimes \text{id}_T$. The assumption (A3) implies that we have

$$H^0(\mathcal{Q}, \mathcal{T} \otimes \text{Coker}(\overline{e}^{*}_{m_0})) = 0.$$

So, the map $H^1(\overline{e}^{*}_{m_0}, T) : H^1(\pi^*_I T) \rightarrow H^1(\overline{e}^{*}_{m_0}, T)$ induced by $\overline{e}^{*}_{m_0}$ is injective. Note that by construction, we have

$$H^1(\overline{e}^{*}_{m_0}, T)(\kappa^{\text{univ}}_{\mathcal{C}} c) = \kappa^{\text{univ}}_{\mathcal{C}}(\overline{e}^{*} c; \varpi^{m_0} O').$$

We regard $H^1(\pi^*_I T)$ as an $\Lambda/I'$-submodule of $H^1(\overline{e}^{*}_{m_0}, T)$ via the injection $H^1(\overline{e}^{*}_{m_0}, T)$, and identify $\kappa^{\text{univ}}_{\mathcal{C}}(\overline{e}^{*} c; \varpi^{m_0} O')$ with $\kappa^{\text{univ}}_{\mathcal{C}}(c)$. Let $f \in \text{Hom}_{\Lambda/I'}(H^1(\pi^*_I T), \Lambda/I')$ be any element. Since $\overline{e}^{*}_{m_0} \circ f$ is an $\mathcal{O}$-linear map, and since $F'/F$ is unramified, we can define an $\mathcal{O}'/\varpi^{m_0} \mathcal{O}'$-linear map

$$f' : (\mathcal{O}'/\varpi^{m_0} \mathcal{O}')_{k^{\text{univ}}_{\mathcal{C}}(c)} \rightarrow \mathcal{O}'/\varpi^{m_0} \mathcal{O}' ; x \cdot (\kappa^{\text{univ}}_{\mathcal{C}}(c)) \mapsto x \cdot (\overline{e}^{*}_{m_0} \circ f)(\kappa^{\text{univ}}_{\mathcal{C}}(c)),$$

where $(\mathcal{O}'/\varpi^{m_0} \mathcal{O}')_{k^{\text{univ}}_{\mathcal{C}}(c)}$ is the $\mathcal{O}'/\varpi^{m_0} \mathcal{O}'$-submodule of $H^1(\overline{e}^{*}_{m_0}, T)$ generated by $k^{\text{univ}}_{\mathcal{C}}(c)$. Since $\mathcal{O}'/\varpi^{m_0} \mathcal{O}'$ is an injective $\mathcal{O}'/\varpi^{m_0} \mathcal{O}'$-module, the map $f'$ can be extended to a homomorphism defined on $H^1(\overline{e}^{*}_{m_0}, T)$. So, we have

$$\overline{e}^{*}_{m_0} \circ f(k^{\text{univ}}_{\mathcal{C}}(c)) \in \mathcal{C}(e^{*} c; \varpi^{m_0} O'; n).$$
Hence we obtain (13).

Next, let us prove

\[ \mathfrak{C}(\epsilon^* c; \mathcal{O}'^0; n) \subseteq \overline{\epsilon_m}(\mathfrak{C}(c; I'; n)) \cdot \mathcal{O}'^0/\mathcal{O}'^0. \]

Since the ring \( \mathcal{O}'^0/\mathcal{O}'^0 \) is a quotient of a DVR, there exists an element \( \tilde{f}' \in \text{Hom}_{\mathcal{O}}(H^1(\overline{\epsilon_m^0} \pi^0_1 \mathcal{T}), \mathcal{O}'^0/\mathcal{O}'^0) \) such that \( \tilde{f}'(\kappa_{n}^\text{univ}(c)_{I'}) \) generates the ideal \( \mathfrak{C}(\epsilon^* c; \mathcal{O}'^0; n) \).

Since \( F'/F \) is unramified, we may assume that

\[ \tilde{f}'(\kappa_{n}^\text{univ}(c)_{I'}) \in \mathcal{O}/\mathcal{O}'^0. \]

We denote by \( f_0 \) the restriction of \( \tilde{f}' \) to \((\Lambda/I')\kappa_{n}^\text{univ}(c)_{I'}\). Then, we have

\[ f_0 \in \text{Hom}_{\Lambda/I'} ((\Lambda/I')\kappa_{n}^\text{univ}(c)_{I'}, \Lambda/I'). \]

By Lemma 2.19, the \( \Lambda/I' \)-module \( \Lambda/I' \) is injective, so we have a homomorphism

\[ f \in \text{Hom}_{\Lambda/I'}(\pi^0_1 \mathcal{T}, \Lambda/I') \]

which is an extension of \( f_0 \). Then, we have

\[ \mathfrak{C}(\epsilon^* c; \mathcal{O}'^0; n) = \overline{\epsilon_m}(\mathfrak{C}(c; I'; n)) \cdot \mathcal{O}'^0/\mathcal{O}'^0 \]

\[ \subseteq \overline{\epsilon_m}(\mathfrak{C}(c; I'; n)) \cdot \mathcal{O}'^0/\mathcal{O}'^0. \]

Hence we obtain the assertion (i). The assertion (ii) follows similarly.

We shall show the assertion (iii). By the assertion (i), we have

\[ \mathfrak{C}(c; I'; n) \subseteq \overline{\epsilon_m}(\mathfrak{C}(\epsilon^* c; \mathcal{O}'^0; n)). \]

Let \( y \in \overline{\epsilon_m}(\mathfrak{C}(\epsilon^* c; \mathcal{O}'^0; n)) \) be any element. Then, we have an element

\[ \tilde{f}' \in \text{Hom}_{\mathcal{O}^0}(H^1(\overline{\epsilon_m^0} \pi^0_1 \mathcal{T}), \mathcal{O}'^0/\mathcal{O}'^0) \]

such that \( \tilde{f}'(\kappa_{n}^\text{univ}(c)_{I'}) = \overline{\epsilon_m}(y) \). Since the map \( \epsilon_m \) is an injection, we obtain an element \( f \in \text{Hom}_{\Lambda/I'}(\pi^0_1 \mathcal{T}, \Lambda/I') \) satisfying \( f(\kappa_{n}^\text{univ}(c)_{I'}) = y \) by similar arguments to those in the previous paragraph. So, we have \( y \in \mathfrak{C}(c; I'; n) \). This implies that the assertion (iii) holds. We also obtain the assertion (iv) by similar manner.

\[ \square \]

**Corollary 5.7.** For any \( i \in \mathbb{Z}_{\geq 0} \), the following hold.

- (i) \( \mathfrak{C}_i(\epsilon^* c; \mathcal{O}'^0) = \overline{\epsilon_m}(\mathfrak{C}_i(c; I')) \cdot \mathcal{O}'^0/\mathcal{O}'^0. \)
- (ii) \( \mathfrak{C}_i(\epsilon^* c; \mathcal{O}'^0) = \overline{\epsilon_m}(\mathfrak{C}_i(c; I)) \cdot \mathcal{O}'^0/\mathcal{O}'^0. \)
- (iii) \( \mathfrak{C}_i(c; I') = \overline{\epsilon_m}(\mathfrak{C}_i(\epsilon^* c; \mathcal{O}'^0)). \)
- (iv) \( \mathfrak{C}_i(c; I) = \overline{\epsilon_m}(\mathfrak{C}_i(\epsilon^* c; \mathcal{O}'^0)). \)

**Proof.** The assertion (i) (resp. (ii)) of Corollary 5.7 immediately follows from Lemma 5.5 and Lemma 5.6 (i) (resp. Lemma 5.5 and Lemma 5.6 (ii)).

Let us show the assertion (iii). By the assertion (i), we have

\[ \mathfrak{C}_i(c; I') \subseteq \overline{\epsilon_m}(\mathfrak{C}_i(\epsilon^* c; \mathcal{O}'^0)). \]
Since $O'/\varpi^{m_0}O'$ is a quotient of a DVR, there exists an element $n \in N(m_0)$ satisfying $\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O') = \bar{e}_{m_0}^{-1}(\mathcal{C}(\epsilon^*; \varpi^{m_0}O'; n))O'$. So, Lemma 5.6 (iii) implies

$$\bar{e}_{m_0}^{-1}(\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O')) = \bar{e}_{m_0}^{-1}(\mathcal{C}(\epsilon^*; \varpi^{m_0}O'; n)) = \mathcal{C}(\epsilon; I'; n) \subseteq \mathcal{C}_i(\epsilon; I)$$. Hence we obtain the assertion (iii). Similarly, the assertion (iv) follows.

Let us finish the proof of Proposition 5.4. Recall that we have the commutative diagram (12). By Corollary 3.18, we have

$$\pi_2(\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O')) = \mathcal{C}_i(\epsilon^*; \varpi^{m_0}O')$$.

Since $O'/\varpi^{m_0}O'$ is a quotient of a DVR, and since we assume that $m_0 \geq m_0 > c$, we have

$$(14) \quad \mathcal{C}_i(\epsilon^*; \varpi^{m_0}O') = \pi_2^{-1}(\pi_2(\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O')))$$.

So, by Corollary 5.7 (iii) and (iv), we obtain

$$\mathcal{C}_i(\epsilon; I')_I = \pi_1(\bar{e}_{m_0}^{-1}(\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O'))) = \bar{e}_{m_0}^{-1}\left(\pi_2\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O')\right) = \bar{e}_{m_0}^{-1}\mathcal{C}_i(\epsilon^*; \varpi^{m_0}O') = \mathcal{C}_i(\epsilon; I)$$. Note that the second equality follows from the equality (14), the commutativity of (12), and the surjectivity of the homomorphism $\pi_1 : \Lambda/I' \to \Lambda/I$. This completes the proof of Proposition 5.4.

5.3. **Strong compatibility for cyclotomic two variable deformations.** Here, we study a special case of $r = 2$, that is, the cyclotomic deformation of one variable deformations. The goal of this section is Theorem 5.8, which is an analogous result to Theorem 5.3.

In this section, we put $\Lambda_0 := \mathbb{Z}_p[[x]]$. Let $\mathbb{T}_0$ be the $\Lambda_0[G_{\mathbb{Q}}]$-module studied in the previous subsection, which is denoted by $\mathbb{T}$ there. Recall that we put $\Gamma := \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and $\Lambda := \Lambda_0[\Gamma]$. We fix a topological generator $\gamma \in \Gamma$. Here, we use similar notation to that in Definition 3.5. In this subsection, we put $\mathbb{T} := \mathbb{T}_0^{\text{cyc}}$. Via the isomorphism $O[[x_1, x_2]] \simeq \Lambda$ defined by $x_2 \mapsto \gamma - 1$, and regard $\Lambda$ as the ring of two-variable formal power series. Note that $\mathbb{T}$ is a $\Lambda$-module on which $\text{Gal}(\mathbb{Q}_S/\mathbb{Q})$ acts via $\rho_{\pi_0} \otimes \chi_{\text{taut}}$.

Let $h \in \Lambda$ an element of the form $h = a x_1 + 2 x_2 + b$, where $a \in O$ and $b \in \varpi O$. We put $\overline{\Lambda} := \Lambda/(h)$, and let $\pi := \pi_{h\Lambda} : \Lambda \to \overline{\Lambda}$ be the natural projection. Note that $\overline{\Lambda}$ is naturally isomorphic to $\Lambda_0$ as a $\Lambda_0$-algebra. Now we can state the main result in 5.3.

**Theorem 5.8.** For any $i \in \mathbb{Z}_{\geq 0}$, we have

$$\pi(\mathcal{C}_i(\epsilon)) = \mathcal{C}_i(\pi^* \epsilon)$$,

where $\mathcal{C}_i(\pi^* \epsilon)$ is the ideal of $\overline{\Lambda}$ defined in Definition 4.5 for the data

$$\langle \Lambda_0, \pi^* \mathbb{T}, \pi^* \epsilon, x_1 \rangle$$. 

Proof. For any positive integer \( M \), we define a set \( \mathcal{P}_M \) of prime numbers by
\[
\mathcal{P}_M := \{ \ell \notin \Sigma \mid \ell \equiv 1 \mod \varpi^M \mathcal{O} \}.
\]
Let \( m \) be any positive integer. We put \( I := I(m) := I(\varpi^m, x_1^m, h) \) and \( I' := I'(m) := I(\varpi^m, x_1^m, h^m) \). Since \( \chi_{\text{taut}} \) factors through the group \( \Gamma \simeq 1 + p\mathbb{Z}_p \), the continuity of \( \chi_{\text{taut}} \) implies that there is a positive integer \( N = N(m) \) with \( N \geq m \) such that for any \( \ell \in \mathcal{P}_N \), it holds that
\[
\chi_{\text{taut}}(\text{Frob}_\ell) \equiv 1 \mod I'.
\]
We put \( I'' := I''(m) := I(\varpi^N, x_1^m, h) \).

Let \( \ell \in \mathcal{P}_N \) be any element, and \( J \in \{ I, I' \} \). We denote by \( J_0 \) the ideal of \( \Lambda_0 \) generated by \( \varpi^m \) and \( x_1^m \). We put \( T_0,\ell := T_0 / (\text{Frob}_\ell - 1)T_0 \) and \( T_\ell := T / (\text{Frob}_\ell - 1)T \).

By the congruence (15), we have
\[
T_\ell \otimes_\Lambda \Lambda / J = (T_0,\ell \otimes_{\Lambda_0} \Lambda_0 / J_0) \otimes_{\Lambda_0 / J_0} \Lambda / J.
\]
Since \( \Lambda / J \) is faithfully flat over \( \Lambda_0 / J_0 \), the \( \Lambda / J \)-module \( T_\ell \otimes_\Lambda \Lambda / J \) is free of rank one if and only if the \( \Lambda_0 / J_0 \)-module \( T_0,\ell \otimes_{\Lambda_0} \Lambda_0 / J_0 \) is free of rank one. So we obtain
\[
\mathcal{N}(T, I', I) \cap \mathcal{P}_N = \mathcal{N}(T, I) \cap \mathcal{P}_N \supseteq \mathcal{N}(T, I'') \cap \mathcal{P}_N.
\]
Hence by Corollary 4.4 and (7) in §4.1, we have
\[
\mathcal{C}_i(c; I'')_I \subseteq \mathcal{C}_i(c; I')_I \subseteq \mathcal{C}_i(c; I),
\]
and obtain
\[
\mathcal{C}_i(\pi^* c) = \lim_{m} \mathcal{C}_i(c; I''(m))_{I(m)} \subseteq \pi(\mathcal{C}_i(c)) = \lim_{m} \mathcal{C}_i(c; I'(m))_{I(m)} \subseteq \mathcal{C}_i(\pi^* c) = \lim_{m} \mathcal{C}_i(c; I(m))_{I(m)}.
\]
This completes the proof of Theorem 5.8 (i). \( \square \)

6. PROOF OF MAIN RESULTS

Let \((T, c)\) be as in Theorem 1.1. In this section, we prove our main results, namely Theorem 1.1, Theorem 1.2 and Theorem 1.3. In §6.1 we review Mazur–Rubin’s observations in [MR] §5.3 which become important ingredients to study the one-variable deformations, that is, the modules over \( \Lambda^{(1)} \). In §6.2 we prove the results for one-variable cases, namely Theorem 1.1 for \( \Lambda = \Lambda^{(1)} \) and Theorem 1.2. In §6.3, we prove Theorem 1.3 in general setting. In §6.4 we prove Theorem 1.3.

6.1. Asymptotic behavior. In this and the next subsections, we set \( \Lambda = \Lambda^{(1)} = O[[x_1]] \). Here, let us recall some observations by Mazur and Rubin in [MR] §5.3 which reduce studies on the pseudo-isomorphism class of a \( \Lambda \)-module to that on the asymptotic behaviors of their specializations.
Let $f$ be $\varpi$ or a linear element of $\Lambda = \mathcal{O}[[x]]$, and put $p := f\Lambda$. For any $N \in \mathbb{Z}_{>0}$, we define an element $f_N \in \Lambda$ by

$$f_N := \begin{cases} f + \varpi^N & (f \neq \varpi) \\ \varpi + x_1^N & (f = \varpi). \end{cases}$$

We also write $f_\infty := f$. We put $p_N := f_N\Lambda$ for each $N \in \mathbb{Z}_{>0}$. Note that $p_N$ is a prime ideal of $\Lambda$, and the residue ring $\mathcal{O}_N := \Lambda/p_N$ becomes a DVR which is finite flat over $\mathcal{O}$. Indeed, by definition, we have $\mathcal{O}_N = \mathcal{O}[\varpi^{1/N}]$ (resp. $\mathcal{O}_N = \mathcal{O}$) if $f = \varpi$ (resp. $f \neq \varpi$). Let $\pi_N : \Lambda \rightarrow \mathcal{O}_N$ be the modulo $p_N$ reduction map.

We adopt the following notation.

**Definition 6.1.** Let $\{\alpha_N\}_{N \in \mathbb{Z}_{>0}}$ and $\{\beta_N\}_{N \in \mathbb{Z}_{>0}}$ be sequences of real numbers. We write $\alpha_N \prec \beta_N$ if and only if we have $\liminf_{N \rightarrow \infty}(\beta_N - \alpha_N) > -\infty$. Moreover if we have $\alpha_N \prec \beta_N$ and $\beta_N \prec \alpha_N$, we write $\alpha_N \sim \beta_N$. Namely, we write $\alpha_N \sim \beta_N$ if and only if the sequence $\{\alpha_N - \beta_N\}_{N \in \mathbb{Z}_{>0}}$ is bounded.

The following elementary lemma becomes a key.

**Lemma 6.2.** Let $g \in \Lambda$ be a prime element.

1. If $g$ is prime to $f$, then there exist a positive integer $N$ such that for any $n \in \mathbb{Z}$ with $n > N$, the length $\text{length}_{\mathcal{O}}(\Lambda/(g, f_n))$ of the $\mathcal{O}$-module $\Lambda/(g, f_n)$ is finite. Moreover, the sequence $\{\text{length}_{\mathcal{O}}(\Lambda/(g, f_n))\}_{n > N}$ is bounded.
2. Let $e \in \mathbb{Z}_{>0}$. If $g = f^e$, then for any $N$, the length of $\Lambda/(g, f_N)$ as $\mathcal{O}_N$-module is finite. Moreover, we have $\text{length}_{\mathcal{O}_N}(\Lambda/(g, f_N)) \sim eN$.

Let $M$ be a finitely generated $\Lambda$-module, and fix an integer $i \in \mathbb{Z}_{>0}$. We define an integer $\alpha$ by

$$\text{Fitt}_{\alpha,i}(M) = p^\alpha \Lambda_p.$$

For any $N \in \mathbb{Z}_{>0}$, we define $\tilde{\alpha}(N) \in \mathbb{N}$ by

$$\text{Fitt}_{\alpha,i}(\pi_N M) = \varpi_{N}^{\tilde{\alpha}(N)} \mathcal{O}_N,$$

where $\varpi_N$ is a prime element of $\mathcal{O}_N$. Then, by Lemma 6.2 and the structure theorem of finitely generated torsion $\Lambda$-modules, we obtain the following corollary.

**Corollary 6.3.** There exist a positive integer $N(M, p)$ such that $\tilde{\alpha}(N) < \infty$ for any $N \in \mathbb{Z}$ with $N > N(M, p)$. Moreover, we have

$$\tilde{\alpha}(N) \sim \alpha \cdot N.$$

6.2. Proof of results on one variable cases. Here, we prove Theorem 1.1 for $\Lambda = \Lambda^{(1)}$ and Theorem 1.2. Let $f \in \Lambda$ be a prime element of $\Lambda$ and put $p := f\Lambda$. Fix any $i \in \mathbb{Z}_{>0}$. We define integers $\alpha_i, \beta_i \in \mathbb{N}_{>0}$ by

$$\text{Fitt}_{\alpha_i, i}(X(\mathcal{T}p)) = f^{\alpha_i} \Lambda_p,$$

$$\mathcal{C}_i(c) \Lambda_p = f^{\beta_i} \Lambda_p.$$
In order to show Theorem 1.1 for the case when $\Lambda = \Lambda^{(1)}$ (resp. Theorem 1.2), it suffices to show $\alpha \leq \beta$ (resp. $\alpha = \beta$ under the assumption (MC)).

If $f = \varpi$, then we put $\mathcal{O}' := \mathcal{O}$, and $\tilde{f} := f \cdot \mathcal{O}$ If $f$ is a distinguished polynomial, namely if $f \neq \varpi$ then we define the ring $\mathcal{O}'$ and a prime element $\tilde{f} \in \mathcal{O}'[[x_1]]$ as follows.

Let $F'$ be a finite extension field of $F$ such that in $F'[x_1]$, the polynomial $f(x_1)$ is decomposed into the product of linear factors. We denote by $\mathcal{O}'$ the ring of integers of $F'$, and fix a root $a \in \mathcal{O}'$ of $f$. We put $\tilde{f} := x_1 - a \in \mathcal{O}'[[x_1]]$.

We put $\Lambda' := \mathcal{O}'[[x_1]]$, and $\tilde{p} := \tilde{f}\mathcal{O}'[[x_1]]$. Let $e$ be the ramification index of the extension $\Lambda'_p/\Lambda_p$ of DVRs. By Proposition 4.12 and Proposition 4.13, we have

$$Fitt_{\Lambda_p}(X(\mathbb{T} \otimes \mathcal{O}')_p) = \tilde{f}e_{\alpha}, \Lambda'_p,$$

$$\mathcal{C}_i(\mathcal{C} \otimes \mathcal{O})\Lambda'_p = \tilde{f}e_{\beta}, \Lambda'_p.$$

So, in order to show Theorem 1.1 for the case when $\Lambda = \Lambda^{(1)}$ and resp. Theorem 1.2, we may replace $(\mathcal{O}, \mathbb{T}, \mathcal{C})$ with $(\mathcal{O}', \mathbb{T} \otimes \mathcal{O}', \mathcal{C} \otimes \mathcal{O}')$, and assume that $f = f' = x_1 - a$.

Let us apply the observation in §6.1 to our situation. For any $N \in \mathbb{N}_{>0}$, we put

$$f_N := \begin{cases} f + a^N = x_1 - a + a^N & (f \neq \varpi), \\ \varpi + x_1^N & (f = \varpi). \end{cases}$$

We set $p_N := f_N\Lambda$, and $\mathcal{O}_N := \Lambda/p_N$. Let $\pi_N : \Lambda \rightarrow \mathcal{O}_N$ be the modulo $p_N$ reduction map. For each $N \in \mathbb{Z}_{>0}$, we fix a uniformizer $\varpi_N$ of the DVR $\mathcal{O}_N$, and define $\bar{\alpha}_i(N), \bar{\beta}_i(N) \in \mathbb{N}$ by

$$Fitt_{\mathcal{O}_N}(X(\pi_N^* \mathbb{T})) = \varpi_N^{\bar{\alpha}_i(N)} \mathcal{O}_N,$$

$$\mathcal{C}_i(\pi_N^* \mathcal{C}) = \varpi_N^{\bar{\beta}_i(N)} \mathcal{O}_N.$$

**Proof of Theorem 1.1 for the case when $\Lambda = \Lambda^{(1)}$.** By Corollary 2.14 Theorem 5.3 and Corollary 6.3, we obtain

\begin{align}
(16) & \quad \bar{\alpha}_i(N) \sim \alpha_i \cdot N, \\
(17) & \quad \bar{\beta}_i(N) \prec \beta_i \cdot N.
\end{align}

Since Theorem 3.13 implies $\bar{\alpha}_i(N) \leq \bar{\beta}_i(N)$, we have

$$\alpha_i \cdot N \sim \bar{\alpha}_i(N) \leq \bar{\beta}_i(N) \prec \beta_i \cdot N.$$ 

Hence we obtain $\alpha_i \leq \beta_i$. \qed

For the proof of Theorem 1.2 we need a bit more careful arguments. Assume that (MC) for $(\mathbb{T}, \mathcal{C})$ holds. Let us show $\alpha_j = \beta_j$. By (16) and (17), it suffices to show $\bar{\beta}_j(N) \prec \bar{\alpha}_j(N)$. We need the following lemma which describes a relation between the ideals $\text{Ind}((\mathcal{C})$ and $\mathcal{C}_0(\mathcal{C})$. 

---

**Note:** The text contains mathematical expressions and proofs that are typical in algebraic number theory and valuation theory, involving concepts such as DVRs, ramification indices, and ideals. The proofs are structured to show how theorems and corollaries are interrelated through specific calculations and theorems. The use of notation such as $\varpi$, $\alpha$, $\beta$, $\mathcal{O}'$, and $\mathcal{O}_N$ indicates a focus on the ring structures and their properties in these fields.
Lemma 6.4. We denote by $X_{pn}$ the maximal pseudo-null \( \Lambda \)-submodule of \( X \). Then, we have

\[
\text{ann}_\Lambda(X_{pn}) \text{Ind}(c) \subseteq \mathcal{C}_0(c).
\]

Proof of Lemma 6.4. Let \( \psi \in \text{Hom}_\Lambda(H^1_{\Sigma}(T), \Lambda) \) be any element, and \( m \in \mathbb{Z}_{>0} \) any positive integer. In order to prove Lemma 6.4, it suffices to show that for any \( a \in \text{ann}_\Lambda(X_{pn}) \), we have \( \pi_{I(x^m)}(a \psi(c(1))) \in \mathcal{C}_0(\pi_{I(x^m)}^*c) \). Note that we have a projective resolution

\[
0 \longrightarrow \Lambda \xrightarrow{d_2} \Lambda^2 \xrightarrow{d_1} \Lambda \xrightarrow{\pi_{I(x^m)}} \Lambda/I(x^m) \longrightarrow 0
\]

of the \( \Lambda \)-module \( \Lambda/I(x^m) \) where the map \( d_1 \) is defined by \( d_1(z_1, z_2) = w^m z + x^m z_2 \) for each \( (z_1, z_2) \in \Lambda^2 \), and the map \( d_2 \) is defined by \( d_2(z) = (-x^m z, w^m z) \) for each \( z \in \Lambda \). So, we have

\[
\text{Tor}_2(H^1_{\Sigma}(T), \Lambda/I(x^m)) \cong \text{Ker} \left( d_2 \otimes \text{id}_{H^2_{\Sigma}(T)} \right) = H^2_{\Sigma}(T)[I(x^m)] \supseteq X_{pn}.
\]

Let \( a \in \text{ann}_\Lambda(X_{pn}) \) be any element. By Corollary 2.14 the element \( a \) annihilates the kernel of the natural map \( P_m : \pi_{I(x^m)}^* H^1_{\Sigma}(T) \longrightarrow H^1_{\Sigma}(\pi_{I(x^m)}^* T) \). This implies that the map

\[
(a \psi) \otimes \pi_{I(x^m)} : \pi_{I(x^m)}^* H^1_{\Sigma}(T) \longrightarrow \Lambda/I(x^m)
\]

factors through \( \text{Im}(P_m) \). Since the \( \Lambda/I(x^m) \) is an injective \( \Lambda/I(x^m) \)-module, there exists a \( \Lambda/I(x^m) \)-linear map \( \tilde{\psi}_{a,m} : \pi_{I(x^m)}^* H^1_{\Sigma}(\pi_{I(x^m)}^* T) \longrightarrow \Lambda/I(x^m) \) which makes the diagram

\[
\begin{array}{ccc}
\pi_{I(x^m)}^* H^1_{\Sigma}(T) & \xrightarrow{(a \psi) \otimes \pi_{I(x^m)}} & \Lambda/I(x^m) \\
\downarrow P_m & & \downarrow \tilde{\psi}_{a,m} \\
H^1_{\Sigma}(\pi_{I(x^m)}^* T) & & \\
\end{array}
\]

commute. Hence we obtain

\[
\pi_{I(x^m)}(a \psi(c(1))) = \tilde{\psi}_{a,m}(\kappa_{I(x^m)}^{\text{univ}}(c)) \in \mathcal{C}_0(\pi_{I(x^m)}^*c).
\]

This completes the proof of Lemma 6.4. \( \square \)

Proof of Theorem 1.2. Fix any \( i \in \mathbb{Z}_{>0} \). Let us show that \( \tilde{\alpha}_i(N) \sim \tilde{\beta}_i(N) \). By Theorem 1.1 for one-variable cases, it suffices to show that \( \beta_i(N) \prec \alpha_i(N) \). By (MC) for \( (T, c) \) and Lemma 6.4 we have

\[
\text{ann}_\Lambda(X_{pn}) \text{ Fitt}_{\Lambda,0}(X) \subseteq \text{ann}_\Lambda(X_{pn}) \text{ char}_\Lambda(X) = \text{ann}_\Lambda(X_{pn}) \text{Ind}(c) \subseteq \mathcal{C}_0(c).
\]

By Corollary 2.14 and Theorem 5.3 we obtain

\[
\pi_N(\text{ann}_\Lambda(X_{pn})) \cdot \text{Fitt}_{\mathcal{O}_N,0}(X(\pi^*_N T)) \subseteq \mathcal{C}_0(\pi_N^*c)
\]

for any \( N \in \mathbb{Z}_{>0} \). Since the height of the ideal \( \text{ann}_\Lambda(X_{pn}) \) of the ring \( \Lambda \) is at least two, there exists a positive integer \( L_0 \in \mathbb{Z}_{>0} \) such that for any \( N \in \mathbb{Z}_{>0} \), we have \( \pi_N(\text{ann}_\Lambda(X_{pn})) \supseteq \omega_{L_0}^N \mathcal{O}_N \). So we obtain

\[
\omega_{L_0}^N \text{ Fitt}_{\mathcal{O}_N,0}(X(\pi^*_N T)) \subseteq \mathcal{C}_0(\pi_N^*c).
\]

for any \( N \in \mathbb{Z}_{>0} \). By Corollary 3.18 we have

\[
\omega_{L_0}^N \text{ Fitt}_{\mathcal{O}_N,0}(X(\pi^*_N T)) \subseteq \mathcal{C}_0(\pi_N^*c).
\]
for any $N \in \mathbb{Z}_{>0}$. Hence we obtain $\bar{\beta}_i(N) < \bar{\alpha}_i(N)$.

6.3. **Proof of Theorem 1.1.** In §6.2, we have already proved the assertions of Theorem 1.1 for one-variable cases. Here, let us complete the proof of Theorem 1.1 by induction on the number $r$ of the variables in the coefficient ring $\Lambda$. For each $r \in \mathbb{Z}_{\geq 1}$, we consider the following induction hypothesis $(I)_r$, which claims that the assertion of Theorem 1.1 for the $r$-variable cases holds:

$$(I)_r \quad \text{Let } F \text{ be a finite extension field of } \mathbb{Q}_p. \text{ We denote } \mathcal{O} \text{ the ring of integers of } F, \text{ and put } \Lambda^{(r)} := \mathcal{O}[[x_1, \ldots, x_r]]. \text{ Let } T' \text{ be an arbitrary free } \Lambda^{(r)}\text{-module of finite rank equipped a continuous } \Lambda^{(r)}\text{-linear action of } G_{\mathbb{Q}, \Sigma} \text{ satisfying the conditions (A1)--(A8). Let } \mathcal{C}' \text{ be an Euler system for } T' \text{ satisfying the condition (NV) which can be extended to cyclotomic direction. Then, for any height one prime ideal } p \text{ of } \Lambda^{(r)} \text{ and for any } i \in \mathbb{Z}_{\geq 0}, \text{ we have }$$

$$\text{Fitt}_{\Lambda^{(r)}, i}(X(T')_p) \supseteq \mathcal{C}_i(\mathcal{C}')\Lambda_p.$$ \hspace{1cm} (18)

Note that in the assertion of $(I)_r$, we vary the field $F$.

Here, we fix any finite extension field $F$ of $\mathbb{Q}_p$. Let $r \in \mathbb{Z}_{\geq 2}$, and put $\Lambda = \Lambda^{(r)}$. In order to prove Theorem 1.1, it suffices to show the following assertion:

**Proposition 6.5.** Suppose that the hypothesis $(I)_{r-1}$ holds. Let $(T, \mathcal{C})$ be a pair over $\Lambda^{(r)}$ satisfying the assumptions in Theorem 1.1. Then, for any height one prime ideal $p$ of $\Lambda$ and for any $i \in \mathbb{Z}_{\geq 0}$, we have

$$\text{Fitt}_{\Lambda^{(r)}, i}(X(T)_p) \supseteq \mathcal{C}_i(\mathcal{C}')\Lambda_p.$$ \hspace{1cm} (19)

Now let $(T, \mathcal{C})$ be the pair over $\Lambda = \Lambda^{(r)}$ satisfying the assumptions in Theorem 1.1, and fix a height one prime ideal $p$ of $\Lambda$ and an integer $i \in \mathbb{Z}_{\geq 0}$. By the structure theorem, we have pseudo-isomorphisms

$$\iota_X: X := X(T) \rightarrow \bigoplus_{a=1}^{s} \bigoplus_{b=1}^{t_a} \Lambda/p_a^{e_{ab}(X)},$$

and

$$\iota_C: \Lambda/\mathcal{C}_i(\mathcal{C}) \rightarrow \bigoplus_{a=1}^{s} \Lambda/p_a^{e_a(C)},$$

where $p_1, \ldots, p_s$ are distinct height one prime ideals of $\Lambda$, and $\{e_{ab}(X)\}_{b=1}^{t_a}$ (resp. $e_a(C)$) is an increasing sequence of non-negative integers (resp. a non-negative integer) for any $a \in \mathbb{Z}$ with $1 \leq a \leq s$. We may (and do) assume $p_1 = p$, and $e_{11}(X) > 0$. We may also assume that $t_1 > i$. We put $\delta := \sum_{a=1}^{t_1} e_{1a}(X)$. Then, in order to prove the assertion (1) of Proposition 6.5, it suffices to show the inequality

$$\delta \leq e_1(C)$$

under the hypothesis $(I)_{r-1}$.

Here, we shall prove the inequality (18) via the reduction arguments by using principal ideals of $\Lambda$ generated by a linear element, which are developed in the Ochiai’s work [Oc2].
Recall that in Definition 5.1, we have introduced the notion of linear elements in the sense of Ochiai’s article [Oc2]. A principal ideal of $\Lambda$ generated by a linear element is called a linear ideal of $\Lambda$. Ochiai introduced the following sets consisting of certain linear ideals.

**Definition 6.6 ([Oc2] Definition 3.3).** Recall that we put $\Lambda = \Lambda^{(r)} = \mathcal{O}[[x_1, \ldots, x_r]]$.

(i) We denote by $\mathcal{L}_\mathcal{O}^{(r)}$ by the set of all linear ideals of $\Lambda = \mathcal{O}[[x_1, \ldots, x_r]]$.

(ii) For any finite set $\mathfrak{I}$ of ideals of $\Lambda$, we denote by $\mathcal{L}_\mathcal{O}^{(r)}[\mathfrak{I}]$ the set of all linear ideals of $\Lambda$ not contained in any ideal belonging to $\mathfrak{I}$.

(iii) For any finitely generated torsion $\Lambda$-module $M$, we denote by $\mathcal{L}_\mathcal{O}^{(r)}(M)$ the set of all linear ideals $I$ of $\Lambda$ satisfying the following two conditions.

(a) The quotient $\pi_I^* M = M/IM$ is a torsion $\Lambda/I$-module, where $\pi_I \Lambda \to \Lambda/I$ denotes the residue map.

(b) It holds that $\text{char}_{\Lambda/I}(\pi_I^* M) = \pi_I(\text{char}(M))$.

We put $\overline{A} := \Lambda^{(r-1)} = \mathcal{O}[[x_1, \ldots, x_{r-1}]]$ and $\Lambda := \Lambda^{(r)}$. For any linear element $g = a_0 + \sum_{i=1}^{r} a_i x_i \in \Lambda$ with $a_r \in \mathcal{O}^*$, we have a natural isomorphism $\overline{A} \simeq \Lambda/g\Lambda$.

Let $F'$ be a finite extension field of $F$, and $\mathcal{O}'$ the ring of integers. We fix a uniformizer $\varpi'$ of $\mathcal{O}'$, and denote by $k'$ the residue field of $\mathcal{O}'$. We have a bijection $P_{\mathcal{O}'} = (P_{m,\mathcal{O}'}, P_{g,\mathcal{O}'}) : \mathcal{L}_\mathcal{O}^{(r)} \to \varpi'\mathcal{O}' \times \mathbb{F}^{r-1}(\mathcal{O}')$

defined by

$$(a_0 + \sum_{i=1}^{r} a_i x^i) \Lambda \mapsto (a_0 a_0^{-1}, (a_1 : a_2 : \cdots : a_r))$$

where we put $i_0 := \min\{i \in \mathbb{Z} \mid 1 \leq i \leq r, \ a_i \in \mathcal{O}'^{\times}\}$. In the article [Oc2], Ochiai defined a map $\text{Sp}_{\mathcal{O}'} : \mathcal{L}_{\mathcal{O}'} \to \mathbb{F}^{r-1}(k')$ to be the composite of the map $P_{g,\mathcal{O}'}$ and the reduction map $\mathbb{F}^{r-1}(O') \to \mathbb{F}^{r-1}(k')$.

Let us recall the following two lemmas proved in [Oc2].

**Lemma 6.7 ([Oc2] Lemma 3.5).** Let $a$ be a height two prime ideal of $\Lambda$.

(i) The set $\mathcal{L}_\mathcal{O}^{(r)} \setminus \mathcal{L}_\mathcal{O}^{(r)}\{a\}$ is infinite if and only if the ideal $a$ contains at least two elements of $\mathcal{L}_\mathcal{O}^{(r)}$. Moreover, if the set $\mathcal{L}_\mathcal{O}^{(r)} \setminus \mathcal{L}_\mathcal{O}^{(r)}\{a\}$ is infinite, there exist two distinct linear element $g_1, g_2 \in \Lambda$ satisfying $a = (g_1, g_2)$.

(ii) Assume that there exist two distinct element $g_1, g_2 \in \Lambda$ satisfying $a = (g_1, g_2)$. If $\varpi \in a$, then there exists an element $\bar{x} \in \mathbb{F}^{r-1}(k)$ such that the set $\mathcal{L}_\mathcal{O}^{(r)} \setminus \mathcal{L}_\mathcal{O}^{(r)}\{a\}$ is contained in $\text{Sp}_{\mathcal{O}'}^{-1}(\bar{x})$.

(iii) Assume that there exist two distinct element $g_1, g_2 \in \Lambda$ satisfying $a = (g_1, g_2)$. If $\varpi \notin a$, then there exists a section $s : \mathbb{F}^{r-1}(\mathcal{O}) \to \mathcal{L}_\mathcal{O}^{(r)}$ of $P_{g,\mathcal{O}'}$ such that the set $\mathcal{L}_\mathcal{O}^{(r)} \setminus \mathcal{L}_\mathcal{O}^{(r)}\{a\}$ is contained in the image of $s$.

**Lemma 6.8 ([Oc2] Lemma 3.5).** Let $M$ be a finitely generated pseudo-null $\Lambda^{(r)}$-module, and let $\text{Assoc}^2_{\Lambda^{(r)}}(M)$ be the set of all height two associated primes of $M$. 

Then, we have
\[ \mathcal{L}_{\mathcal{O}}^{(r)}(M) = \bigcap_{a \in \text{Assoc}_{\Lambda}^{2}(\Lambda)} \mathcal{L}_{\mathcal{O}}^{(r)}(\Lambda^{(r)}/a) = \bigcap_{a \in \text{Assoc}_{\Lambda}^{2}(\Lambda)} \mathcal{L}_{\mathcal{O}}^{(r)}([a]). \]

**Proof of Theorem 1.1.** We denote by \( \mathcal{J}_{\text{char}} \) the set of height one primes of \( \Lambda \) containing the ideal \( \text{char}_{\Lambda}(X) \text{char}_{\Lambda}(\Lambda / \text{Ind}_C(c)) \). We define a set \( \mathcal{I} \) of prime ideals of \( \Lambda \) by

\[ \mathcal{I} := \mathcal{J}_{\text{char}} \cup \bigcup_{a=2}^{s} \text{Assoc}_{\Lambda}^{2}(\Lambda/(p + p_a)) \]

\[ \cup \text{Assoc}_{\Lambda}^{2}(\text{Ker } \iota_X) \cup \text{Assoc}_{\Lambda}^{2}(\text{Coker } \iota_X) \]

\[ \cup \text{Assoc}_{\Lambda}^{2}(\text{Ker } \iota_C) \cup \text{Assoc}_{\Lambda}^{2}(\text{Coker } \iota_C). \]

By definition, the set \( \mathcal{I} \) is finite. So Lemma 6.7 and Lemma 6.8 imply that after replacing \( F \) by a suitable finite extension field of it, there exists a linear element \( g = a_0 + \sum_{i=1}^{r} a_i x_i \in \Lambda \) which is not contained in any \( a \in \mathcal{I} \). By Proposition 4.14, moreover, Lemma 6.7 implies that we may assume that \( a_r \in \mathcal{O}^{\times} \).

For any pseudo-isomorphism \( \iota : M_1 \rightarrow M_2 \) of \( \Lambda \)-modules, we consider the following condition \( P_t(h) \) for a linear element \( h \in \Lambda \).

\( P_t(h) \) “For any height two prime ideal \( a \in \text{Assoc}_{\Lambda}(\text{Ker } \iota) \cup \text{Assoc}_{\Lambda}(\text{Coker } \iota) \), the ideal \( h\Lambda \) is not contained in \( a \).”

By the definition of \( \mathcal{I} \), the linear element \( g \) satisfies the following properties:

(LE1) The ideal \( g\Lambda \) does not contain the ideal \( \text{char}_{\Lambda}(X) \text{char}_{\Lambda}(\Lambda / \text{Ind}_C(c)) \).

(LE2) The linear element \( g \in \Lambda \) satisfies the conditions \( P(tX) \) and \( P(tC) \).

(LE3) For any \( a \in \mathbb{Z} \) with \( 2 \leq a \leq s \), the height of the ideal \( p + p_a + g\Lambda \) of \( \Lambda \) is three.

Let \( \pi_g : \Lambda \rightarrow \Lambda / g\Lambda = \overline{\Lambda} \) be the reduction map. It follows from the property (LE1) that \( \pi_g(p) \) becomes a non-zero principal ideal of the UFD \( \overline{\Lambda} \). We fix a height one prime \( \mathfrak{p} \) of \( \overline{\Lambda} \) containing \( \pi_g(p) \), and define a positive integer \( m \) by \( \mathfrak{p}^m \overline{\Lambda}_\mathfrak{p} = \pi_g(p) \overline{\Lambda}_\mathfrak{p} \).

Note that by the above property (LE2), the maps

\[ \overline{t}_X = t_X \otimes \pi_g : X(T) \otimes_{\Lambda, \pi_g} \overline{\Lambda} \rightarrow \bigoplus_{a=1}^{s} \bigoplus_{b=1}^{s_a} \overline{\Lambda}/\pi_g(p_a)^{c_{ab}(X)} \]

and

\[ \overline{t}_C = t_C \otimes \pi_g : \overline{\Lambda}/\pi_g(C) \rightarrow \bigoplus_{a=1}^{s} \overline{\Lambda}/\pi_g(p_a)^{c_{a}(C)}, \]

induced by \( \pi_g \) are pseudo-isomorphisms of \( \overline{\Lambda} \)-modules. By Corollary 2.14, we have natural isomorphism \( X(T) \otimes_{\Lambda, \pi_g} \overline{\Lambda} \simeq X(\pi_g^*(T)) \). Since \( g \) satisfies (LE3), and since \( \overline{t}_X \) is a pseudo-isomorphism, we obtain

\[ \text{Fitt}_{\pi_g,X}(X(\pi^*_g(T))_\mathfrak{p}) = \mathfrak{p}^{m\delta(\mathfrak{p})} \overline{\Lambda}_\mathfrak{p} \]
By Proposition 5.2, we have \( \pi_g(\mathcal{C}(c)) \subseteq \mathcal{C}(\pi_g^*c) \). So, the property (LE3) and the pseudo-isomorphism \( \bar{\iota}_C \) imply that
\[
\pi_g(\mathcal{C}(c)) \Lambda_{\bar{p}} = \bar{p}^{me_1(C)} \Lambda_{\bar{p}} \subseteq \mathcal{C}(\pi_g^*c) \Lambda_{\bar{p}}.
\]
Hence the induction hypothesis \( (I_r) \) implies that we have
\[
\bar{p}^{me_1(C)} \Lambda_{\bar{p}} = \bar{p}^{me_1(C)} \Lambda_{\bar{p}} \supseteq \bar{p}^{me_1(C)} \Lambda_{\bar{p}}.
\]
and we obtain the inequality (18), namely \( \delta \leq e_1(C) \).

6.4. Proof of Theorem 1.3. Here, let us prove Theorem 1.3.

First, we recall some notation. We denote the cyclotomic \( \mathbb{Z}_p \)-extension by \( \mathbb{Q}_\infty / \mathbb{Q} \), and put \( \Gamma := \text{Gal}(\mathbb{Q}_\infty / \mathbb{Q}) \). We fix a topological generator \( \gamma \in \Gamma \). We put \( \Lambda_0 := \Lambda^{(1)} = \mathbb{O}[[x]] \) and identify the completed group ring \( \Lambda_0(\Gamma) \) with \( \Lambda := \Lambda^{(2)} = \mathbb{O}[[x_1, x_2]] \) via the isomorphism \( \Lambda \simeq \Lambda(\Gamma) \) of \( \Lambda_0 \)-algebras defined by \( 1 + x_2 \mapsto \gamma \). Let \( T_0 \) be a free \( \Lambda_0 \)-module of finite rank \( d \) with a continuous \( \Lambda_0 \)-linear action of \( G_{\mathbb{Q}, \Sigma} \) satisfying the conditions (A1)–(A8). We denote by \( T \) the cyclotomic deformation of \( T_0 \). Then, the \( \Lambda \)-module \( T \) also satisfies the conditions conditions (A1)–(A8). Let \( c \) be an Euler system for the \( \Lambda \)-module \( T \) satisfying the condition (MC). Note that since \( T \) is a cyclotomic deformation, the Euler system \( c \) can be extended to the cyclotomic direction. (See Lemma 3.7)

Proof of Theorem 1.3. In order to prove Theorem 1.3 it suffices to show that for any \( i \in \mathbb{Z}_{\geq 0} \) and any height one prime \( p \) of \( \Lambda \), we have
\[
\text{Fitt}_{\Lambda,p,i}(X(T)) = \mathcal{C}_i(c).
\]

As in §6.3 we take pseudo-isomorphisms
\[
\iota_X : X := X(T) \longrightarrow \bigoplus_{a=1}^{s} \bigoplus_{b=1}^{s_a} \Lambda/p_a^{e_a(X)}
\]
and
\[
\iota_C : \Lambda/\mathcal{C}_i(c) \longrightarrow \bigoplus_{a=1}^{s} \Lambda/p_a^{e_a(C)}.
\]
We may assume \( p = p_1 \). Put \( \delta := \sum_{a=1}^{s} e_a(X) \). In order to show Theorem 1.3 it suffices to show the equality
\[
(19) \quad \delta = e_1(C).
\]

Let \( \mathcal{I} \) be the finite set of prime ideals of \( \Lambda \) defined in §6.3. Let \( p\text{Ind}_{\Lambda}(c) \) be the minimal prime ideal of \( \Lambda \) containing \( \text{Ind}_{\Lambda}(c) \), and \( \mathfrak{A}_{\text{Ind}} \) the ideal of \( \Lambda \) satisfying
\[
\text{Ind}_{\Lambda}(c) = \mathfrak{A}_{\text{Ind}} \cdot p\text{Ind}_{\Lambda}(c).
\]
Note that since \( \Lambda \) is a UFD, the ideals \( p\text{Ind}_{\Lambda}(c) \) and \( \mathfrak{A}_{\text{Ind}} \) do exist. We denote by \( \mathcal{I}_{\text{Ind}} \) the set of height one prime ideals of \( \Lambda \) containing \( p\text{Ind}_{\Lambda}(c) \), and by \( \mathcal{I}_{\text{Ind}}^2 \) the set
of height two prime ideals of \( \Lambda \) containing \( \mathfrak{a}_{\text{Ind}} \). Let \( X_{pn} \) be the maximal pseudo-null \( \Lambda \)-submodule of \( X(\mathbb{T}) \). We define a finite set \( \tilde{\mathcal{I}} \) of prime ideals of \( \Lambda \) by
\[
\tilde{\mathcal{I}} := \mathcal{I}_{\text{Ind}}^1 \cup \mathcal{I}_{\text{Ind}}^2 \cup \operatorname{Assoc}_\Lambda^2(X_{pn}) \cup \mathcal{I}.
\]
By Lemma 6.7 and Lemma 6.8, after replacing \( F \) by a suitable finite extension field of it, we can take a linear element \( g = a_0 + a_1x + x_2 \in \Lambda \) which is not contained in any \( a \in \mathcal{I} \). Since \( \tilde{\mathcal{I}} \) contains \( \mathcal{I} \), the linear element \( g \) satisfies the properties (LE1)–(LE3) for our \((T, c)\). Let us show that the linear element \( g \) satisfies the following additional property:

(LE4) Let \( \pi_g: \Lambda \to \Lambda/(g) \simeq \Lambda_0 \) be the projection. Then, we have
\[
p\text{Ind}_{\Lambda_0}(\pi_g^*c) = \pi_g(p\text{Ind}_\Lambda(c)).
\]
First, we shall prove
\[
(20) \quad \pi_g(p\text{Ind}_\Lambda(c)) \supseteq p\text{Ind}_\Lambda(\pi_g^*c).
\]
Since the pair \((T, c)\) satisfies (MC), we have
\[
\pi_g(p\text{Ind}_\Lambda(c)) = \pi_g(\text{char}_\Lambda(X(\mathbb{T}))).
\]
By the properties (LE1) and (LE2), the \( \Lambda_0 \)-module \( X(\pi_g^*T) \) is torsion, and we have
\[
\pi_g(\text{char}_\Lambda(X(\mathbb{T}))) = \text{char}_{\Lambda_0}(X(\pi_g^*T)).
\]
By Ochiai’s results on Euler systems for Galois deformations (see [Oc2] Theorem 2.4), it holds that
\[
\text{char}_{\Lambda_0}(X(\pi_g^*T)) \supseteq p\text{Ind}_\Lambda(\pi_g^*c).
\]
Hence we obtain (20). Next, let us show
\[
(21) \quad \pi_g(p\text{Ind}_\Lambda(c)) \subseteq p\text{Ind}_\Lambda(\pi_g^*c).
\]
Let \( \psi \in \text{Hom}_\Lambda(H_1^\Sigma(\mathbb{T}), \Lambda) \) be any homomorphism. We denote by
\[
\tilde{\psi}: \pi_g^*H_1^\Sigma(\mathbb{T}) \to \Lambda/(g) \simeq \Lambda_0
\]
the homomorphism induced by \( \psi \). By the cohomological exact sequence induced by
\[
0 \to T \xrightarrow{\times g} T \to T/gT \to 0,
\]
we deduce that the natural map \( \pi_g^*H_1^\Sigma(\mathbb{T}) \to H^1(\pi_g^*T) \) is injective, and its cokernel is annihilated by \( \text{ann}_{\Lambda_0}(X(\mathbb{T})[g]) \). So, for any \( \bar{h} \in \text{ann}_{\Lambda_0}(X(\mathbb{T})[g]) \), there exists a homomorphism \( \phi_{\bar{h}} \in \text{Hom}_{\Lambda_0}(H_1^\Sigma(\pi_g^*T), \Lambda_0) \) which makes the diagram
\[
\begin{array}{ccc}
\pi_g^*H_1^\Sigma(\mathbb{T}) & \xrightarrow{\bar{h}} & \Lambda_0 \\
\downarrow & & \\
H_1^\Sigma(\pi_g^*T) & \xrightarrow{\phi_{\bar{h}}} & \\
\end{array}
\]
commutes. This implies that we have
\[
\text{ann}_{\Lambda_0}(X(\mathbb{T})[g]) \cdot \pi_g(p\text{Ind}_\Lambda(c)) \subseteq p\text{Ind}_\Lambda(\pi_g^*c).
\]
In order to show (21), it suffices to prove that the height of the ideal \( \text{ann}_{\Lambda_0}(X(\mathbb{T})[g]) \) is at least two. Since \( g \) is prime to \( \text{char}_\Lambda(X(\mathbb{T})) \), the \( \Lambda \)-module \( X(\mathbb{T})[g] \) is a pseudo-null.
Namely, we have $X(T)[g] \subseteq X_{pn}$. Moreover, since $g \not\in \mathfrak{a}$ for any $\mathfrak{a} \in \text{Assc}_{S}^{1}(X_{pn})$, we deduce that $X(T)[g]$ is also pseudo-null as $\Lambda_{0}$-module. So the height of $\text{ann}_{\Lambda_{0}}(X(T)[g])$ is at least two, and we obtain (23). Hence we deduce that the linear element $g$ satisfies the property (LE4).

Let $\mathfrak{p}$ be a height one prime ideal of $\Lambda_{0}$ containing $\pi_{g}(\mathfrak{p})$, and $m$ an integer satisfying $\mathfrak{p}^{m} = \pi_{g}(\mathfrak{p})\Lambda_{0,\mathfrak{p}}$. It The property (LE3) and Corollary [2.14] imply that we have

$$\text{Fitt}_{\Lambda_{0,\mathfrak{p}}}(X(\pi_{g}^{*}T)) = \text{Fitt}_{\Lambda_{0,\mathfrak{p}}}(\pi_{g}^{*}X(T)) = \mathfrak{p}^{mG}\Lambda_{0,\mathfrak{p}},$$

and Theorem 5.8 implies

$$\pi_{g}(\mathcal{C}(\mathfrak{c}))\Lambda_{0,\mathfrak{p}} = \mathcal{C}(\pi_{g}^{*}\mathfrak{c})\Lambda_{0,\mathfrak{p}} = \mathfrak{p}^{m\mathfrak{c}_{1}(\mathfrak{c})}\Lambda_{0,\mathfrak{p}}.$$ 

By the assumption (MC) for the pair $(T, \mathcal{C})$, we have $\pi_{g}^{*}T, \pi_{g}^{*}\mathfrak{c}$ satisfies (MC). So, we can apply Theorem 1.2 for the pair $(\pi_{g}^{*}T, \pi_{g}^{*}\mathfrak{c})$, and we obtain

$$\mathfrak{p}^{mG}\Lambda_{0,\mathfrak{p}} = \text{Fitt}_{\Lambda_{0,\mathfrak{p}}}(X(\pi_{g}^{*}(T)_{\mathfrak{p}})) = \mathcal{C}(\pi_{g}^{*}\mathfrak{c})\Lambda_{0,\mathfrak{p}} = \mathfrak{p}^{m\mathfrak{c}_{1}(\mathfrak{c})}\Lambda_{0,\mathfrak{p}}.$$ 

Hence we obtain the equality (19), that is, $\delta = \varepsilon_{1}(\mathfrak{c})$. This completes the proof of Theorem 1.3.

7. APPLICATION TO (NEARLY) ORDINARY HIDA DEFORMATIONS

Here, we apply our main results to ordinary and nearly ordinary Hida deformations.

First, let us fix our notation. Here, suppose that $p \geq 5$, and fix an isomorphism $\overline{Q}_{p} \simeq \mathbb{C}$. As in §1 let $F$ be a finite extension field, and $\mathcal{O} := \mathcal{O}_{F}$ the ring of integers of $F$. Fix a positive integer $N$ prime to $p$. We put $\Sigma := \text{Prime}(pN)$, namely the set of all prime divisors of $pN$. Let $D_{\infty}$ be a pro-finite group equipped with an isomorphism $1 + p\mathbb{Z}_{p} \simeq D_{\infty} : a \mapsto \langle a \rangle$. Note that $D_{\infty}$ is regarded as the projective limit of the direct factor $D_{m} \simeq (\mathbb{Z}/p^{m+1}\mathbb{Z})^{x} \otimes_{\mathbb{Z}} \mathbb{Z}_{p}$ of the group naturally isomorphic to $(\mathbb{Z}/Np^{m+1}\mathbb{Z})^{x}$ consisting of the diamond operators acting on the modular curve $Y_{1}(Np^{m+1})$. We denote by $\chi_{D}: D_{\infty} \rightarrow 1 + p\mathbb{Z}_{p}$ the inverse of $\langle \cdot \rangle$. In §7 we set $\Lambda := \mathcal{O}[[D_{\infty}]] \simeq \Lambda^{(1)}$. Let $\mathfrak{m}_{\Lambda}$ be the maximal ideal of $\Lambda$. Then, we have $k := \mathcal{O}/\mathfrak{m}_{\Lambda} \simeq \Lambda/\mathfrak{m}_{\Lambda}$.

Let $\omega: G_{Q, \Sigma} \rightarrow \mu_{p-1} \subseteq \mathbb{Z}_{p}^{x}$ be the Teichmüller character. We regard $\omega$ as a Dirichlet character modulo $pN$ defined by $\omega(\ell) := \omega(\text{Frob}_{\ell}^{-1})$ for each prime number $\ell$ not dividing $p$. Let $\psi = \psi_{0}\omega^{j}$ be an $\mathbb{O}^{x}$-valued Dirichlet character modulo $pN$, where $\psi_{0}$ denotes a character modulo $N$, and $j$ is an integer satisfying $0 \leq j \leq p - 2$. We take a Hida family

$$\mathcal{F} = \sum_{n=1}^{\infty} A(n, \mathcal{F})q^{n} \in \Lambda[[q]]$$

of cuspforms of tame level $N$ with Dirichlet character $\psi$. For any $k \in \mathbb{Z}_{\geq 2}$ and any character $\varepsilon: D_{\infty} \rightarrow \overline{Q}_{p}$ of finite order, the power series

$$\mathcal{F}_{\chi_{D}^{k-2}\varepsilon} := \sum_{n=1}^{\infty} \chi_{D}^{k-2}\varepsilon(A(n, \mathcal{F}))q^{n} \in \overline{Q}_{p}[[q]]$$
becomes the $q$-expansion of a $p$-ordinary Hecke eigen cuspidal form of weight $k$ at the cusp $\infty$. We say that $F_{\chi_{D}^{k-2}\varepsilon}$ is the specialization of $F$ at the arithmetic point $\chi_{D}^{k-2}\varepsilon$. Note that here, we assume that the coefficients $A(n,F)$ are contained in $\Lambda$.

Remark 7.1. Let $k$ be an integer with $k \geq 2$, and $f \in S_k(\Gamma_1(Np),\psi\omega^{2-k};\overline{\mathbb{Q}}_p)$ be a $p$-stabilized newform of tame level $N$. Then, we have a Hida family

$$F' = \sum_{n=1}^{\infty} A(n,F')q^n \in I[[q]]$$

of tame level $N$ with Dirichlet character $\psi$ such that $f$ becomes a specialization of $F'$, where $I$ is a domain finite flat over $\mathbb{Z}_p[[D_\infty]]$. (See [Hi1] Corollary 3.2 and Corollary 3.7.) A sufficient condition on $(p,f')$ to ensure that $F' \in \mathcal{O}'[[D_\infty]][[q]]$ for some DVR $\mathcal{O}'$ finite flat over $\mathbb{Z}_p$ is studied in [Go]. See [Go] Proposition 8 and Corollary 9.

By [Hi2] Theorem 2.1, we have a free $\Lambda$-module $T(F)$ of rank 2 with a continuous $\Lambda$-linear action $\rho_{T(F)}$ of $G_{Q,\Sigma}$ satisfying

$$\det(1-x\text{Frob}_\ell^{-1}|V(F)) = 1 - A(\ell,F)x + \ell \cdot \psi(\ell)\langle \rho(\ell)\rangle \Lambda[x]$$

for any prime number $\ell$ not dividing $pN$, where $\text{Frob}_\ell$ denotes the arithmetic Frobenius at $\ell$, and $pr: \mathbb{Z}_p^\times = \mu_{p-1} \times 1 + p\mathbb{Z}_p \rightarrow 1 + p\mathbb{Z}_p$ denotes the projection. (In this article, for the description of $T(F)$, we use the cohomological convention.) We have an exact sequence

$$0 \rightarrow F^+ T(F) \rightarrow T(F) \rightarrow F^- T(F) \rightarrow 0$$

of $\Lambda[G_{Q,p}]$-modules, where $F^+ T(F)$ and $F^- T(F)$ are free $\Lambda$-modules of rank one, and the $G_{Q,p}$-action on $F^+ T(F)$ is unramified. We put

$$T'_F := \text{Hom}_\Lambda(T(F),\Lambda) \otimes_{\mathbb{Z}_p} \lim_m \mu_{p^m},$$

and denote by $\rho_{T'_F}: G_{Q,\Sigma} \rightarrow \text{Aut}_\Lambda(T'_F) \simeq \text{GL}_2(\Lambda)$ the Galois action on $T'_F$. Note that $T'_F$ satisfies the assumption (A8). Moreover, since we assume that $p \geq 5$, the assumption (A4) for $T'_F$ is satisfied. Here, we assume the following hypothesis:

(Full) The image of $\rho_{T(F)}$ contains the special linear subgroup $\text{SL}_2(\Lambda)$.

Since the commutator subgroup $\text{Aut}_\Lambda(T(F))$ is $\text{SL}_2(\Lambda)$, we have

$$\rho_{T(F)}(\text{Gal}(Q_\Sigma/\mathbb{Q}(\mu_p^{\infty}))) \supseteq \text{SL}_2(\Lambda)$$

if $T(F)$ satisfies (Full). So, under the assumption (Full), The conditions (A1)-(A3) for $T'_F$ are satisfied obviously. Moreover, by Proposition 7.2 below, we can deduce that $T'_F$ also satisfies the condition (A5) under the assumption (Full). (Note that Proposition 7.2 easily follows from the induction on $i$.)

Proposition 7.2. For any $i \in \mathbb{Z}_{>0}$, we have $H^1(\text{SL}_2(\Lambda/m^i_\Lambda),k^2) = 0$, where $k^2$ is regarded as a $\Lambda[\text{SL}_2(\Lambda/m^i_\Lambda)]$-module equipped with the standard matrix action of $\text{SL}_2(\Lambda/m^i_\Lambda)$.

Let $\tilde{T}'_F := T'_F \otimes_{\Lambda} \Lambda[[\Gamma]]$ be the cyclotomic deformation of $T'_F$. We fix a basis $d$ of the free $\Lambda$-module $D := (F^+ T(F) \otimes_{\mathbb{Z}_p} W(\overline{\mathbb{Q}}_p))/\Gamma_{Q,p}$ of rank one, and a basis $b$ of
the free \(\Lambda\)-module \(\mathcal{B}^{(-1)^s}\) of rank one consisting of \(\Lambda\)-adic modular symbols in the sense of \([Ki]\). Then, we have an Euler system \(Z^{Ki}_{b,d} = \{Z^{Ki}_{b,d}(n)\}_{n \in \mathbb{Z}}\) for \(\tilde{T}_F\) such that \(Z^{Ki}_{b,d}(1)\) maps to the two-variable \(p\)-adic \(L\)-function \(L^{Ki}_{p,b}\) attached to \(\mathcal{F}\) constructed by Kitagawa in \([Ki]\) via the generalized Coleman map \(\Xi_d: H^I(Q_p, \tilde{T}_F)/H^I(Q_p, \tilde{T}_F) \to \Lambda[[\Gamma]]\) constructed by Ochiai in \([Oc1]\) Theorem 3.13. (For details on \(Z\), see \([Oc3]\) Theorem 6.11.) Note that \(Z\) is constructed by gluing the Euler system of Beilinson–Kato elements defined in \([Ka]\).) Let \(c_F\) be the Euler system for \(\tilde{T}_F\) corresponding to \(Z^{Ki}_{b,d}\) in the sense of Proposition 7.5. The Euler system \(c_F\) satisfies the condition (NV). Moreover, the pair \((\tilde{T}_F, c_F)\) satisfies the condition (MC) if and only if the two-variable Iwasawa–Greenberg main conjecture for nearly ordinary Hida deformation \(\tilde{T}(\mathcal{F})\) proposed by Greenberg \([Gr]\) holds. (For the precise statement of the two-variable Iwasawa–Greenberg main conjecture for \(\tilde{T}(\mathcal{F})\) also implies \((\tilde{T}_F, \text{aug}^*c_F)\) satisfies the condition (MC). (See \([Oc3]\) Corollary 7.5.)

**Theorem 7.3.** Suppose that \(\tilde{T}_F\) satisfies the hypotheses (A6), (A7) and (Full). We denote by \(\text{aug}_i: \Lambda[[\Gamma]] \to \Lambda\) the augmentation map. Take any \(i \in \mathbb{Z}_{\geq 0}\). Let \(\mathfrak{p}\) be a height one prime ideal of \(\Lambda\). Then, we have \(\text{Fitt}_{\Lambda[[\Gamma]]_{\mathfrak{p},i}}(X(\tilde{T}_F)) \supseteq \mathfrak{c}_i(c_F)\Lambda[[\Gamma]]_{\mathfrak{p}}\), and \(\text{Fitt}_{\Lambda_p,i}(X(\tilde{T}_F)) \supseteq \mathfrak{c}_i(\text{aug}^*c_F)\Lambda_p\). Moreover, if the two-variable Iwasawa–Greenberg main conjecture for nearly ordinary Hida deformation \(\tilde{T}(\mathcal{F})\) holds, we have the equalities \(\text{Fitt}_{\Lambda[[\Gamma]]_{\mathfrak{p},i}}(X(\tilde{T}_F)) = \mathfrak{c}_i(c_F)\Lambda[[\Gamma]]_{\mathfrak{p}}\), and \(\text{Fitt}_{\Lambda_p,i}(X(\tilde{T}_F)) = \mathfrak{c}_i(\text{aug}^*c_F)\Lambda_p\).

**Remark 7.4.** Some sufficient conditions for (Full) are studied, for instance, in \([Bo]\) and \([MW]\). In our setting, the condition (Full) for \(\tilde{T}(\mathcal{F})\) is satisfied if \(\mathcal{O} = \mathbb{Z}_n\), and if the residual representation \(T(\mathcal{F}) \otimes_{\Lambda} k\) contains is \(\text{SL}_2(k)\). (For details, see \([MW]\) §10 Proposition.)

**Remark 7.5.** Here, we give some remarks on the conditions (A6) and (A7) on \(\tilde{T}_F\). Let \(k \in \mathbb{Z}_{\geq 2}\), and suppose that the conductor of \(\psi_0 := \psi_0^{2-k}\) divides \(N\). Let \(f = \sum_{a=1}^{\infty} a(n, f)q^a \in S_k(\Gamma_1(N), \varepsilon; \mathcal{T}_p)\) be a \(p\)-ordinary normalized Hecke eigenform with \(r \in \mathbb{Z}_{\geq 0}\). We denote by \(f^*\) the \(p\)-stabilization ordinary newform of tame level \(N\) attached to \(f\). Suppose that \(f^*\) is a specialization of \(\mathcal{F}\) at an arithmetic point \(\eta\). We denote by \(\mathcal{O}_{\eta}\) the normalization of \(\eta(\Lambda)\). Note that the residue field of \(\mathcal{O}_{\eta}\) is naturally isomorphic to \(k = \Lambda/\mathfrak{m}_\Lambda\). We fix an \(\mathcal{O}_{\eta}\)-lattice \(T(f)\) of the \(p\)-adic Galois representation \(V(f)\) attached to \(f\).

(i) Obviously, the condition (A6) on \(T(F)\) holds if and only if the following property (A6)$_{f,\ell}$ holds for for any prime divisor \(\ell\) of \(N\).

\[(A6)_{f,\ell} \quad \text{We have } H^0(I_\ell, T(f) \otimes_{\mathcal{O}} k) = 0.\]

A sufficient condition for (A6)$_{f,\ell}$ is given in Proposition 7.6 below.

(ii) By Deligne’s unpublished work, the continuous modulo \(p\) Galois representation \((T(f) \otimes_{\mathcal{O}_{\eta}} k, \tilde{p}_T(f) |_{G_{q,p}})\) is given by

\[
\tilde{p}_T(f) |_{G_{q,p}} \simeq \left( \begin{array}{cc} \lambda(\tilde{a}(p, f)) & 0 \\ \lambda(\psi_0(p)/\tilde{a}(p, f)) & \tilde{\chi}^k_{\text{cyc}} \end{array} \right).
\]
where we denote the image of an element \( x \in \mathcal{O}_\eta \) in \( k \) by \( \bar{x} \), and for any \( \bar{a} \in k^\times \), we define an unramified character \( \lambda(\bar{a}) : G_{Q_p} \to k^\times \) by \( \lambda(\bar{a})(\text{Frob}_p^{-1}) := \bar{a} \).

(For the proof of this fact under assumption \( k \leq p+1 \), see [Gro] Proposition 12.1. Note that Galois representations in [Gro] are written in the homological convention.) In particular, if we have \( A(F,p) \not\equiv \psi_0^{-1}(p) \mod m_\Lambda \), then \( T_F \) satisfies (A7).

(iii) Recall that here, we assume that \( p \geq 5 \). For any Hida family \( F \) of ordinary cuspidal Hecke eigennewforms of tame level \( N \) with Dirichlet character \( \psi \), there exists an integer \( i \in \mathbb{Z} \) with \( 0 \leq i \leq p - 2 \) such that the representation \( T(F \otimes \omega^i) \simeq T(F) \otimes \omega^i \) corresponding to the Hida family

\[
F \otimes \omega^i = \sum_{n=1}^{\infty} A(n,F) \omega^i(\ell) q^n \in \Lambda[[q]]
\]

of cuspforms of tame level \( N \) with Dirichlet character \( \psi \omega^{2i} \) satisfies (A7).

**Proposition 7.6.** Let \( f = \sum_{n=1}^{\infty} a(n,f) q^n \in S_k(\Gamma_1(N); \mathcal{O}_\eta) \) be as in Remark 7.5. Suppose that \( \mathcal{O}_\eta \) is unramified over \( \mathbb{Z}_p \). Let \( \ell \) be a prime divisor of \( N \). If we have \( a(\ell,f) \neq 0 \), and if \( \ell^2 - 1 \) is prime to \( \# k^\times \), then \( f \) satisfies (A6)_{f,\ell}.

**Proof.** We denote by \( \pi_f = \bigotimes_v \pi_{f,v} \) the automorphic representation of \( GL_2(\mathbb{A}_{\mathbb{Q}}) \) attached to \( f \). We put \( G_\ell := GL(\mathbb{Q}_\ell) \). Let \( T_\ell \) be the maximal torus of \( G_\ell \) consisting of diagonal matrices, and \( B_\ell \) the Borel subgroup of \( G_\ell \) consisting of upper triangle matrices.

In order to prove Proposition 7.6 we treat the following three cases.

(I) The representation \( \pi_{f,\ell} \) of \( G_\ell \) is principal series, but not special in the sense of [BH] §9.11 Classification Theorem.

(II) The representation \( \pi_{f,\ell} \) of \( G_\ell \) is special.

(III) The representation \( \pi_{f,\ell} \) of \( G_\ell \) is supercuspidal.

First, let us consider the case (I). Since we assume that \( a(\ell,f) = 0 \), the representation \( \pi_{f,\ell} \) of \( G_\ell \) is not \( p \)-primitive in the sense of [AL] p. 236. (See also [LW] Proposition 2.8.) This fact and (the proof of) [BH] §33.3 Theorem imply that there exists two ramified characters \( \chi_1 \) and \( \chi_2 \) of \( G_{\mathbb{Q}_\ell} \) such that we have \( \rho_{T(f)}|_{G_{\mathbb{Q}_\ell}} = \chi_1 \oplus \chi_2 \). This implies that \( f \) satisfies (A6)_{f,\ell}.

Next, let us consider the case (II). The assumption that \( a(\ell,f) \) that is equal to \( 0 \) implies that \( \pi_{f,\ell} \) is a twist of the Steinberg representation by a ramified character. From this fact and [BH] §33.3 Theorem, it follows that \( f \) satisfies (A6)_{f,\ell}.

Let us consider the case (III). In this case, the representation \( \rho_{\nu(f)}|_{G_{\mathbb{Q}_\ell}} \) is irreducible. (See [BH] §33.4 Theorem.) Suppose that \( T(f) \otimes \mathcal{O}_\eta \) has a non-zero fixed vector \( \bar{v} \) under the action of \( I_{\ell} \). We denote by \( I_{\ell}^w \) the wild inertia subgroup of \( G_{\mathbb{Q}_\ell} \), and put \( I_{\ell}^w := I_{\ell}/I_{\ell}^w \simeq \lim_{\longleftarrow} \mathbb{F}_p^\times \). Since the kernel of the reduction map \( GL_2(\mathcal{O}_\eta) \to GL_2(\mathbb{F}_p) \) is pro-\( p \), and since \( I_{\ell}^w \) is a pro-\( \ell \) group with \( \ell \neq p \), we deduce that there exists a lift \( v \in T(f) \) of \( \bar{v} \) fixed by \( I_{\ell}^w \). The group \( I_{\ell} \) acts on \( v \) via a character \( \chi \) of \( I_{\ell}^w \). Since \( I_{\ell} \) fixes \( \bar{v} \), the order of \( \chi \) is a power of \( p \). By the assumption that \( p \geq 5 \), and that \( \mathcal{O}_\eta \) is unramified over \( \mathbb{Z}_p \), we do not have an element of \( GL(\mathcal{O}_\eta) \) whose order is a positive power of \( p \). So, the character \( \chi \) is trivial. This contradicts the assertion that
\( \rho_{V(f)}|_{G_{\ell}} \) is irreducible. Hence \( T(f) \otimes_{\mathcal{O}_\kappa} k \) does not have a non-zero fixed vector under the action of \( I_\ell \). \hfill \Box

**Remark 7.7.** By Skinner and Urban, For the precise statement of the two-variable Iwasawa–Greenberg main conjecture for nearly ordinary deformations are proved in many cases. Indeed, the “three-variable” conjectures which imply two–variable ones (after combined with [Oc3] §2 Theorem 3 under suitable assumptions) are also proved under certain hypotheses. (See [SU] Theorem 3.6.6.) However, in order to apply Skinner–Urban’s work, we need to assume that the tame level \( n \) has a prime divisor whose square does not divide \( d \). Under this assumption, the condition (A6) \( \mathcal{T}^e_F \) does not hold. So, we cannot apply the results in [SU] in our setting.

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