THE SPECIAL LINEAR GROUP FOR RINGS

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Abstract. We generalise a definition of the special linear group due to Baez to arbitrary rings. At the infinitesimal level we get a Lie ring. We give a description of these special linear rings over some large classes of rings, including all associative rings and all algebras over a field.

1. Introduction

The real and complex special linear groups $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$ are double covers of the groups $SO_0(2,1)$ and $SO_0(3,1)$ of isometries of the hyperbolic plane and hyperbolic 3-space, respectively. The pattern continues with the quaternions, $\mathbb{H}$, as shown by Kugo and Townsend in [KuT], and Sudbery deals with the final normed division algebra, the octonions $\mathbb{O}$, in his 1984 paper [Su]. Unfortunately the way Sudbery defines the special linear group over $\mathbb{O}$ only works in dimensions two and three.

More recently, Baez gives a definition of the special linear group for $\mathbb{O}$ that makes sense in any number of dimensions, and shows that it agrees with Sudbery’s definition in two dimensions $[B]$. He does not treat the general case, and it seems that until now no further investigation has been made.

In Section 2, after covering the necessary background, we reformulate Baez’s definition of the special linear group and algebra in a natural way. This reformulation permits its extension to arbitrary rings, and we determine the corresponding special linear ring (we do not necessarily get an algebra structure) for associative rings. In Section 3 we extend Sudbery’s analysis of the two dimensional case to all real composition algebras, re-estabishing several of the spin isomorphisms (also known as the accidental, or exceptional isomorphisms) and uncovering a new one. Finally, in Section 4 we give a characterisation of the special linear ring, in dimensions greater than two, for a large class of algebras. From this we can compute Baez’s group in dimension greater than two.

In computing Baez’s groups we find that in three dimensions his definition disagrees with Sudbery’s, which gives a real form of the exceptional Lie group $E_6$. It is also worth mentioning that Hitchin has recently proposed an alternative for the two dimensional group over $\mathbb{O}$, in [Hi]. His definition is motivated by a dimension argument, but has the disadvantage that it does not give a Lie group.

The main idea for our definition is the following. Over fields, the set of matrices is naturally identified by left multiplication with the group of linear operators, and in this setting we think of the special linear group as being the set of transformations that preserve signed volume. Over general nonassociative rings, this identification fails to make the appropriate matrices into a subgroup, but if we take the same geometric viewpoint then it is natural to define the special linear group to be the group generated by such operators, and this is essentially what we do.

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2. Preliminaries

2.1. The real octonion algebra $\mathbb{O}$ is an $\mathbb{R}$-algebra with unit 1. It has an orthonormal basis $1, e_1, \ldots, e_7$ with multiplication given by:

$$1e_i = e_i1 = e_i, \quad e_i^2 = -1$$
The nucleus of a nonassociative algebra $A$ is the set of elements that associate with all other pairs of elements. The nucleus of $\mathbb{O}$ is $\mathbb{R} = \mathbb{R}1$, and this is also its centre.

The algebra $\mathbb{O}$ has an anti-involution, conjugation, which we denote by a bar: if $z = z_0 + z_1 e_1 + \cdots + z_7 e_7$ is an octonion then its conjugate is $\bar{z} = z_0 - z_1 e_1 - \cdots - z_7 e_7$. We use this to define a positive definite quadratic form $| \cdot |^2 : z \mapsto |z|^2 = z \bar{z} = \sum_{i=0}^7 z_i^2$, and this form is multiplicative: $|zw|^2 = |z|^2 |w|^2$. It also enables the construction of multiplicative inverses, making $\mathbb{O}$ a division algebra.

More generally, a composition algebra is a unital but not necessarily associative algebra $C$ over a field $F$, together with a nondegenerate quadratic form $| \cdot |^2$ that is multiplicative. Such algebras are necessarily alternative and come with a conjugation anti-involution, again denoted by a bar, which satisfies:

\begin{align*}
(1) \quad z(wz) &= ((zw)z)u \\
zw(zw) &= ((uw)z)z
\end{align*}

We call $z_0$ the real part of $z$, and write $\text{Re}(z) = z_0$, even when the base field is not $\mathbb{R}$.

Composition algebras have famously been classified for characteristic not two by Jacobson \cite{Jacobson}, and all can be obtained from the following construction, known as the Cayley-Dickson process.

Starting with $F$, which we give trivial conjugation (even if $F = \mathbb{C}$, for example), define a pair of algebras $F_+$ and $F_-$ by setting $F_{\pm} = F \oplus Fi$ as vector spaces, and defining multiplication by

\begin{equation}
(a + bi)(c + di) = (ac - bd) + (da + bc)i
\end{equation}

for $a, b, c, d \in F$. This makes $F_+$ and $F_-$ into $F$-algebras, and defining

$$
\bar{a + bi} = \bar{a} - bi
$$

makes them into composition algebras using (2). Repeating this process with $F_{\pm}$ in place of $F$ yields four composition algebras $F_{\pm \pm}$, and for each of these we can obtain two more. One can continue the process, but beyond this point the algebras obtained are not composition algebras, as one can show that they have zero divisors.

We are mainly interested in the case $F = \mathbb{R}$, and we now state without proof Jacobson’s classification in this case.

**Theorem 2.1** (Jacobson). The real composition algebras are exactly:

(i) $\mathbb{R}$

(ii) $\mathbb{R}_- = \mathbb{C}$

(iii) $\mathbb{R}_+ = \mathbb{R}^2$, with $|a, b|^2 = ab$

(iv) $\mathbb{R}_- = \mathbb{H}$

(v) $M_2(\mathbb{R})$, the $2 \times 2$ matrices over $\mathbb{R}$, with $| \cdot |^2 = \text{det}$

(vi) $\mathbb{R}_- = \mathbb{O}$

(vii) $\mathbb{O}'$, the split octonions

Despite not being a division algebra, the split octonions share many properties with the octonions. In particular they are alternative and satisfy the Moufang identities.
We denote by \( \mathbb{R}^{p,q} \) the vector space \( \mathbb{R}^{p+q} \) with the standard quadratic form of signature 
\((p,q)\). If a space with a quadratic form is isometric to \( \mathbb{R}^{p,q} \) with \( p = q \) then the space is said to be split. All the algebras in the right hand column of Theorem 2.1 are split.

Just as for \( \mathbb{O} \), we write \( 1, e_1, \ldots, e_{d-1} \) for the standard orthonormal basis of a composition algebra \( C \) of dimension \( d \). Then all of these basis elements square to \( \pm 1 \), so since \( C \) is alternative, if \( L_a : C \to C \) is the left multiplication map \( x \mapsto ax \) we have
\[
L_e_i L_e_j = L_{e_i e_j} = e_i^2 = \pm 1
\]

From this, since \( C \) satisfies the first Moufang identity \([1]\), and since every pair of distinct imaginary basis elements anticommutes (as can be seen from (3)), we have for \( i \neq j \):
\[
L_{e_i}L_{e_j} = e_i^2e_j^2(L_{e_i}(L_{e_i}L_{e_i})L_{e_j})(L_{e_i}L_{e_j}) = e_i^2e_j^2(L_{e_i}L_{e_i})(L_{e_i}L_{e_i})(L_{e_i}L_{e_i}) = \delta_{ij}
\]
\[
= e_i^2e_j^2(L_{e_i}L_{e_i})(L_{e_i}L_{e_i}) = e_i^2e_j^2(L_{e_i}L_{e_i})(L_{e_i}L_{e_i}) = \delta_{ij}
\]
\[
= -e_i^2(L_{e_i}L_{e_i})(e_j^2) = -L_{e_i}L_{e_i}
\]

2.2. For a ring \( R \) we write \( M_m(R) \) to mean the space of \( m \times m \) matrices with entries in \( R \), and \( E_{ij} \) is an element of the standard basis. The trace of a matrix \( x \) is written \( \text{tr} x \), and the left multiplication map \( R^m \to R^m, v \mapsto xv \) is denoted \( L_x \). The definition of the octonionic special linear group and algebra given by Baez is as follows \([3] \ p.28\).

**Definition 2.2.** Baez’s octonionic special linear algebra, which we temporarily denote \( \mathfrak{sl}_m(\mathbb{O})_B \), is the Lie algebra of linear operators generated under commutators by the set \( \{ L_x : x \in M_m(\mathbb{O}), \text{tr} x = 0 \} \). Baez’s octonionic special linear group, \( \text{SL}_m(\mathbb{O})_B \), is the Lie group generated by exponentiating \( \mathfrak{sl}_m(\mathbb{O})_B \).

This definition is not well suited to computation, and we prefer to use a simpler, more general one:

**Definition 2.3.** Let \( R \) be a not necessarily associative or unital ring. We define \( \mathfrak{sl}_m(R) \), the special linear ring of \( R \), to be the ring generated by \( \{ L_{aE_{ij}} : a \in R, i \neq j \} \) under commutators. Similarly, \( \text{SL}_m(R) \), the special linear group of \( R \), is the group generated by \( \{ L_{1+aE_{ij}} : a \in R, i \neq j \} \) under composition.

**Proposition 2.4.** The groups \( \text{SL}_m(\mathbb{O})_B \) and \( \text{SL}_m(\mathbb{O}) \) are equal.

**Proof.** Let \( T \) be the set of tangent vectors at the identity of \( \text{SL}_m(\mathbb{O}) \) that have the form
\[
\frac{d}{dt}|_{t=0} L_{(1+taE_{ij})} = L_{aE_{ij}} \quad i \neq j
\]
Note that \( [L_{aE_{ij}}, L_{E_{ji}}] = L_{-aE_{ij}} \), so since the set of traceless matrices is \( R \)-spanned by matrices \( aE_{ij} \) and \( a(E_{ii} - E_{jj}) \), the Lie algebra generated by \( T \) contains \( \mathfrak{sl}_m(\mathbb{O})_B \). Then because \( \text{SL}_m(\mathbb{O})_B \) is generated by exponentiation, \( \text{SL}_m(\mathbb{O})_B \subset \text{SL}_m(\mathbb{O}) \). The other containment is clear because \( \exp(aE_{ij}) = I + aE_{ij} \).\(\square\)

Straight from the definition we can obtain a nice description of the special linear algebra of an associative ring.

**Theorem 2.5.** Let \( R \) be an associative (not necessarily unital) ring. Then there is an isomorphism \( \mathfrak{sl}_m(R) \cong \{ x \in M_m(R) : \text{tr} x \in [R, R] \} \).\(\square\)

**Proof.** Since \( R \) is associative, we have \( L_aL_b = L_{ab} \) for all \( a, b \in R \). This allows us to identify \( L_{aE_{ij}} \) with the matrix \( aE_{ij} \), and we henceforth think of \( \mathfrak{sl}_m(R) \) as consisting of matrices, rather than multiplication maps. Now \( \mathfrak{sl}_m(R) \) contains all matrices with all diagonal entries zero, as these form the linear span of the generators. Furthermore, the commutator of two generators is
\[
[aE_{ij}, bE_{kl}] = \delta_{jk}abE_{il} - \delta_{il}baE_{kj}
\]
where \( \delta \) denotes the Kronecker delta. If \( \delta_{jk} \) and \( \delta_{il} \) are not both 1 then we get either zero or a generator. If both are 1 then we get \( abE_{ii} - baE_{jj} \). Clearly this has trace lying in


$[R, R]$, and by varying $a$ and $b$ we can get the whole of $[R, R]$. Then varying $i$ and $j$ gives the right hand side of the result. Note that the commutator of such a diagonal matrix with a generator is traceless, so all further commutators have trace in $[R, R]$, and we are done. □

Theorem 2.3 shows that our definition is a generalisation of the usual special linear algebra, for if $R$ is a field then $[R, R] = 0$, and if $R = \mathbb{H}$ then $[R, R] = \{ z \in \mathbb{H} : \text{Re} z = 0 \}$, which gives the standard definition of $\mathfrak{s}l_m(\mathbb{H})$ (see [HM, p.52], for instance).

3. The Two Dimensional Case

For an $m \times m$ matrix $x = (x_{ij})$ with entries in a composition algebra $C$, the hermitian conjugate of $x$ is defined to be $x^* = (\overline{x_{ji}})$. If $x^* = x$ then $x$ is said to be hermitian, and the set of such matrices is denoted $\mathfrak{h}_m(C)$. Note that all diagonal entries lie in the base field, for only this is fixed by conjugation in $C$. We henceforth restrict our attention to the case $F = \mathbb{R}$ and $m = 2$, in which case the alternativity of $C$ ensures that the determinant map is a well defined quadratic form on $\mathfrak{h}_2(C)$.

Thus if $C$ has dimension $d$ and signature $(p, q)$, then writing $z = z_0 + z_1 e_1 + \cdots + z_{d-1} e_{d-1}$ for an element of $C$, we have that $\mathfrak{h}_2(C)$ is isometric (possibly after reordering coordinates) to $\mathbb{R}^{q+1,p+1}$ via the map

$$\begin{pmatrix} r & z \\ \bar{z} & s \end{pmatrix} \mapsto \begin{pmatrix} r + s & r - s \\ \frac{r - s}{2} & \frac{r + s}{2}, z_0, z_1, \ldots, z_{d-1} \end{pmatrix}$$

We now define a representation of $SL_2(C)$ on $\mathfrak{h}_2(C)$. Let $y = L_{I + aE_{ij}}$ be a generator of $SL_2(C)$, and for $x = (x_{ij}) \in \mathfrak{h}_2(C)$ set $y \cdot x = (I + aE_{ij})x(I + aE_{ij})$. This product is well defined because $C$ is alternative, and we extend to the rest of $SL_2(C)$ in the obvious way.

**Lemma 3.1.** The action of $SL_2(C)$ on $\mathfrak{h}_2(C)$ is by isometries. That is, if $x \in \mathfrak{h}_2(C)$ and $y \in SL_2(C)$ then $\det(y \cdot x) = \det x$.

**Proof.** It suffices to show that this holds for the generators of $SL_2(C)$, and the two cases are similar, so we just do $y = L_{I + aE_2}$. Let $x = \begin{pmatrix} r & z \\ \bar{z} & s \end{pmatrix}$, recalling that $r, s \in \mathbb{R}$. Equation (2) allows us to choose the order in which we multiply off-diagonal terms when calculating the determinant, so

$$\det(y \cdot x) = \det \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} r & z \\ \bar{z} & s \end{pmatrix} \begin{pmatrix} 1 & \bar{a} \\ 0 & 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} r & ra + \bar{z} \\ ra + \bar{z} & r\bar{a} + az + s \end{pmatrix}$$

$$= r(ra\bar{a} + az + s) - (ra + \bar{z})(r\bar{a} + z)$$

$$= rs - \bar{z}z = \det x$$

□

We have an action by isometries of $SL_2(C)$ on a space isometric to $\mathbb{R}^{q+1,p+1}$, so there is a corresponding homomorphism of connected Lie groups

$$\psi : SL_2(C) \to SO_0(q + 1, p + 1)$$

In order to analyse this homomorphism we describe a basis of $\mathfrak{sl}_2(C)$.

**Lemma 3.2.** Let $1, e_1, \ldots, e_{d-1}$ be an orthonormal basis for $C$. Then $\mathfrak{sl}_2(C)$ is based by

$$\{ L_{E_{12}}, L_{E_{21}}, [L_{E_{12}}, L_{E_{21}}], \alpha_i = L_{e_i E_{12}}, \beta_i = L_{e_i E_{21}}, \gamma_i = [L_{E_{12}}, \beta_i], \varepsilon_{ij} = [\alpha_i, \beta_j] : i < j \}.$$

In particular, $\dim SL_2(C) = 3 + 3(d - 1) + \frac{(d - 1)(d - 2)}{2} = \frac{(d+1)(d+2)}{2}$. 


Proof. It follows from identities (4) and (5) that the set in question bases the subspace spanned by products of length two, so it suffices to show that this is the whole of $\mathfrak{s}l_2(C)$. Products of length three in $\mathfrak{s}l_2(C)$ are spanned by generators and elements $\delta$ and $\delta^T$, where

$$
\delta = \begin{pmatrix}
0 & L_{e_i} & L_{e_j} & L_{e_k} & L_{e_j}L_{e_k} & L_{e_k}L_{e_i}
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

There are three cases for $\delta$, depending on the choice of $e_i, e_j,$ and $e_k$.

Case 1: $e_j$ is equal to either $e_i$ or $e_k$. Then $\delta$ is a generator by identity (4).

Case 2: $e_i = e_k \neq e_j$. Using both identities (4) and (5) we see that $\delta$ is a generator.

Case 3: $e_i, e_j, e_k$ are distinct. Then $\delta = 0$ by identity (5).

Thus products of length three are spanned by generators, so $\mathfrak{s}l_2(C)$ is spanned by the given set. \qed

Remark 3.3. What we have done so far is valid over any field of characteristic not two, and in particular if $F = C$ then we get a homomorphism $\text{SL}_2(C) \to \text{SO}(d + 2, C)$, because the latter group is connected.

We are now ready for the main result of this section.

Theorem 3.4. If $C$ is a real composition algebra of dimension $d$ and signature $(p, q)$ then $\text{SL}_2(C) \cong \text{Spin}(p + 1, q + 1)$.

Proof. Because $\text{Spin}(p+1, q+1) \cong \text{Spin}(q+1, p+1)$ it suffices to show that $\psi$ is onto and has two-point kernel. By Lemma 3.2 we have $\dim \text{SL}_2(C) = \frac{(d+1)(d+2)}{2} = \dim \text{SO}(q + 1, p + 1)$, so if $\ker d\psi = 0$ then $d\psi$ is onto, and

$$
\psi(\text{SL}_2(C)) = \exp(d\psi(\mathfrak{s}l_2(C))) = \exp(\mathfrak{s}o(q + 1, p + 1)) = \text{SO}(q + 1, p + 1)
$$

Claim 1: $\ker d\psi = 0$

Proof: The action of $\text{SL}_2(C)$ on $\mathfrak{h}_2(C)$ induces an action of $\mathfrak{s}l_2(C)$: if $x \in \mathfrak{h}_2(C)$ and $y$ is a generator of $\mathfrak{s}l_2(C)$ then $y \cdot x = yx + xy^*$. By definition, any element of $\ker d\psi$ acts trivially, so we calculate the action of the basis of Lemma 3.2 on an arbitrary $x = \begin{pmatrix} r & z \\ \bar{z} & \bar{w} \end{pmatrix} \in \mathfrak{h}_2(C)$.

Here $r, s \in \mathbb{R}$, $z = z_0 + \sum_{i=1}^{d-1} z_i e_i \in C$, and below the $\lambda_i$ and $\kappa_{ij}$ are real. In several places we find it convenient to write $w = \lambda_0 + \sum_{i=1}^{d-1} \lambda_i e_i$.

(a) : \( \begin{pmatrix} \lambda_0 L_{E_{12}} + \sum_{i=1}^{d-1} \lambda_i \alpha_i \end{pmatrix} \cdot x = L_{wE_{12}} \cdot x = (wE_{12})x + x(\bar{w}E_{21}) = \begin{pmatrix} w\bar{z} & ws \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} z\bar{w} & 0 \\ s\bar{w} & 0 \end{pmatrix} = \begin{pmatrix} 2\text{Re}(w\bar{z}) & sw \\ sw & 0 \end{pmatrix} \)

(b) : \( \begin{pmatrix} \lambda_0 L_{E_{21}} + \sum_{i=1}^{d-1} \lambda_i \beta_i \end{pmatrix} \cdot x = L_{wE_{21}} \cdot x = \begin{pmatrix} 0 & r\bar{w} \\ rw & 2\text{Re}(wz) \end{pmatrix} \)

(c) : \( \begin{pmatrix} \lambda_0 [L_{E_{12}}, L_{E_{21}}] + \sum_{i=1}^{d-1} \lambda_i \gamma_i \end{pmatrix} \cdot x = [L_{E_{12}}, L_{wE_{21}}] \cdot x = \begin{pmatrix} 2\text{Re}(rw) & 2\text{Re}(wz) \\ 2\text{Re}(wz) & 2\text{Re}(w) \end{pmatrix} - \begin{pmatrix} 0 & 2\text{Re}(\bar{z})\bar{w} \\ 2\text{Re}(\bar{z})\bar{w} & 0 \end{pmatrix} = \begin{pmatrix} r\lambda_0 & wz - z\bar{w} \\ \bar{z}\bar{w} - w\bar{z} & -s\lambda_0 \end{pmatrix} \)
As in the proof of Theorem 2.5 we can consider elements of $\text{SL}_2$ that above (with respective sets of coefficients $\lambda_i$, $\mu_i$, $\nu_i$, and $\kappa_{ij}$), consider the upper left entry of $y \cdot x$. We see that

$$2\text{Re}\left((\lambda_0 + \sum_{i=1}^{d-1} \lambda_i e_i)\bar{z}\right) + 2rv_0 = 0$$

for all $r \in \mathbb{R}$ and $z \in C$. Taking $z = 0$, $r = 1$ gives $v_0 = 0$, and then cycling $z$ through the $e_i$ gives $\lambda_i = 0$. Similarly, considering the lower right entry gives

$$2\text{Re}\left((\mu_0 + \sum_{i=1}^{d-1} \mu_i e_i)z\right) - 2sv_0 = 0$$

We already have $v_0 = 0$, and cycling $z$ through the $e_i$ gives $\mu_i = 0$. Finally, considering the top right entry, we are left with

$$\sum_{i=1}^{d-1} 2(\nu_i e_i z - 2\nu_i \bar{e}_i) + \sum_{0<i<j<d} 2\kappa_{ij}(e_j^2 z_j e_i - e_i^2 z_i e_j) = 0$$

Taking $z = 1$ gives $\sum_{i=1}^{d-1} 4\nu_i e_i = 0$, so all $\nu_i$ are zero. Then successively considering $z = e_1, z = e_2, \ldots$ we find that all $\kappa_{ij}, \kappa_{ji}, \ldots$ are zero. Thus, since any element of $\ker d\psi$ acts trivially, $\ker d\psi = 0$.

As remarked we now have that $\psi$ is onto, and it remains to show that $\psi$ has two-point kernel. Consider the real matrices

$$a = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

and the linear map $\iota = L_a L_b L_c L_b L_a \in \text{SL}_2(C)$, which acts on $(v_1, v_2)^T \in C^2$ as follows:

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \xrightarrow{L_b L_a} \begin{pmatrix} v_1 - v_2 \\ v_1 \end{pmatrix} \xrightarrow{L_a} \begin{pmatrix} -v_1 - v_2 \\ v_1 \end{pmatrix} \xrightarrow{L_a L_b} \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix}$$

Because $a, b, c \in M_2(\mathbb{R})$, the expression for $\iota \cdot x$ associates, so $\iota \cdot x = (-I)x(-I) = x$, and $\iota \in \ker \psi$. Hence ker $\psi$ consists of at least two elements.

Claim 2: If $C$ is associative then ker $\psi = \{1, \iota\}$.

**Proof:** As in the proof of Theorem 2.5 we can consider elements of $\text{SL}_2(C)$ to be matrices. Then $\iota = -I$. Also, if $y = (y_{ij}) \in \text{SL}_2(C)$ acts trivially on $b_2(C)$ we have

$$\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = y \cdot \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} ry_{11} y_{11}^{*} + sy_{12} y_{12}^{*} \\ ry_{21} y_{11}^{*} + sy_{22} y_{12}^{*} \end{pmatrix} \begin{pmatrix} ry_{11} y_{11}^{*} + sy_{12} y_{12}^{*} \\ ry_{21} y_{11}^{*} + sy_{22} y_{12}^{*} \end{pmatrix}$$

Taking $r = 1, s = 0$ in this gives $y_{11} y_{11}^{*} = 1$ and $y_{21} = 0$. Similarly, taking $r = 0, s = 1$ gives $y_{22} y_{22}^{*} = 1$ and $y_{12} = 0$. Now we have

$$\begin{pmatrix} 0 & \bar{z} \\ \bar{z} & 0 \end{pmatrix} = \begin{pmatrix} y_{11} & 0 \\ 0 & y_{22} \end{pmatrix} \cdot \begin{pmatrix} 0 & \bar{z} \\ \bar{z} & 0 \end{pmatrix} = \begin{pmatrix} 0 & y_{11} \bar{z} + y_{22}^{*} \bar{z} \\ y_{22} \bar{z} + y_{11} \bar{z} & 0 \end{pmatrix}$$

Taking $z = 1$ gives $y_{11} y_{22} = 1$. But $y_{11} y_{11}^{*} = 1$, so $y_{22} = y_{11}$. From this we get $z = y_{11} \bar{z} y_{11}^{*}$, so $zy_{11} = y_{11} z$ for all $z \in C$. Hence $y_{11} = y_{22} = \pm 1$, and thus $y \in \{1, \iota\}$. ■

Claim 2 gives us the result when $C$ is associative. Next we consider the case $C = \mathbb{O}$. In this case the corresponding special orthogonal group $SO(9,1)$ has maximal compact subgroup $SO(9) \times SO(1) \cong SO(9)$ [K p.73], and thus has fundamental group $\pi_1(SO(9,1)) = \mathbb{Z}$. **HARRY PETYT**
\[ \pi_1(\text{SO}(9)) = \mathbb{Z}_2 \] [H, p.377]. Hence any connected group that covers \( \text{SO}_0(9,1) \) nontrivially is a double cover, so we also have the result for \( C = \mathbb{O} \).

This just leaves the case \( C = \mathbb{O}' \), which is more difficult because \( \pi_1(\text{SO}(5,5)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \). However, since \(-1\) is square in \( \mathbb{C} \), the complexification \( \mathbb{C} \otimes \mathbb{O}' \) of the split-octonions is isomorphic to the bioctonions \( \mathbb{C} \otimes \mathbb{O} \), so \( \text{SL}_2(\mathbb{O}') \) is a real form of \( \text{SL}_2(\mathbb{C} \otimes \mathbb{O}) \). Moreover, the bioctonions can be obtained from the Cayley-Dickson construction by starting with \( \mathbb{R} \) as the base field instead of \( \mathbb{R} \), so by Remark 3.3 \( \text{SL}_2(\mathbb{C} \otimes \mathbb{O}) \) covers \( \text{SO}(10, \mathbb{C}) \), which has fundamental group \( \mathbb{Z}_2 \) [K, p.73]. We still have \( \iota \in \text{SL}_2(\mathbb{C} \otimes \mathbb{O}) \), so the covering is 2:1. We thus have a commutative diagram

\[
\begin{array}{ccc}
\text{SL}_2(\mathbb{O}') & \text{complexify} & \text{SL}_2(\mathbb{C} \otimes \mathbb{O}) \\
\downarrow \iota & & \downarrow 2:1 \\
\text{SO}_0(5,5) & \text{complexify} & \text{SO}(10, \mathbb{C})
\end{array}
\]

which gives the result in the final case, \( C = \mathbb{O}' \). \( \square \)

As well as recovering Sudbery’s result for the octonions, a small amount of manipulation in the split case shows that Theorem 3.4 re-establishes the spin isomorphisms

\[ \text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) \cong \text{Spin}(2,2) \quad \text{SL}_4(\mathbb{R}) \cong \text{Spin}(3,3) \]

4. THE GENERAL CASE

In this section \( R \) is a commutative, associative, unital ring, and \( A \) is a finite dimensional \( R \)-algebra that is free as an \( R \)-module, with basis \( \{e_1, \ldots, e_n\} \). Note that every finite dimensional algebra over a field satisfies these conditions. The choice of a basis gives an identification \( A \cong R^n \) as \( R \)-modules, and under this identification \( L_{e_i} \in \text{M}_n(R) \).

**Definition 4.1.** The (left) multiplication algebra of \( A \), denoted \( \mathfrak{M}_A \), is the \( R \)-algebra generated by the set \( \{L_{re_i} : r \in R\} \subset \text{M}_n(R) \) of left multiplications by scaled basis elements under composition.

Since \( R \) is in the centre and the nucleus of \( A \) the multiplication algebra is an associative \( R \)-algebra. Before proceeding we check that it is well defined.

**Proposition 4.2.** The multiplication algebra of \( A \) is independent of: (i) The choice of basis. (ii) The choice of ring over which \( A \) is an algebra.

**Proof.** Part (i) is clear because any two bases are linear combinations of one another.

For part (ii), if \( A \) is both a free \( R \)-module and a free \( S \)-module with respective multiplication algebras \( \mathfrak{M}_{A,R} \) and \( \mathfrak{M}_{A,S} \) then assume \( \text{dim}_R A \geq \text{dim}_S A \) and let \( e_1, \ldots, e_s \) be a basis of \( A \) over \( S \). Identifying \( S \) as a subalgebra makes it into a free \( R \)-module with basis \( f_1, \ldots, f_r \). Then \( f_ie_j \) is a basis of \( A \) over \( R \), and in this basis it is immediate that \( \mathfrak{M}_{A,R} \cong \mathfrak{M}_{A,S} \), which is sufficient by part (i). \( \square \)

Because we have identified \( L_{e_i} \in \text{M}_n(R) \) we have an identification \( \mathfrak{M}_A \subset \text{M}_n(R) \). This allows us to state:

**Theorem 4.3.** Under the assumptions stated at the beginning of this section, if \( m \geq 3 \) then \( \mathfrak{sl}_m(A) = \{ x \in \text{M}_m(\mathfrak{M}_A) : \text{tr}x \in [\mathfrak{M}_A, \mathfrak{M}_A] \} \). In particular \( \mathfrak{sl}_m(A) \subset \mathfrak{sl}_m(\mathbb{R}) \).

**Proof.** The nonzero products of two generators are:

\[
[La_{E_i}, Lb_{E_j}] = L_{La_{E_i}b_{E_j}} - L_{b_{E_j}a_{E_i}} = \text{trx} \in [\mathfrak{M}_A, \mathfrak{M}_A]
\]

and from this it follows by taking successive products that \( \mathfrak{sl}_m(A) \) is the span of the set

\[
\left\{ L_{E_{ij}} : \alpha \in \mathfrak{M}_A, \ i \neq j \right\} \bigcup \left\{ L_{E_{ij}} : \alpha, \beta \in \mathfrak{M}_A, \ i \neq j \right\}
\]
Now because $\mathcal{M}_A$ is associative we can identify these multiplication operators with the corresponding matrices. Clearly all such matrices have trace in $[\mathcal{M}_A, \mathcal{M}_A]$, and varying $\alpha$ and $\beta$ gives the whole of $[\mathcal{M}_A, \mathcal{M}_A]$. Varying $i$ and $j$ then gives the result. □

This reduces the problem of determining $\mathfrak{s}\mathfrak{l}_m(A)$ to that of finding the multiplication algebra of $A$.

In the case of the $\mathbb{R}$-algebra $\mathcal{O}$, the group generated by left multiplications by units is $\text{SO}(8)$ (see [CS], p. 92, for a proof). This $\mathbb{R}$-spans the full matrix algebra $M_8(\mathbb{R})$, and it follows that $\mathcal{O}_\mathbb{R} = M_8(\mathbb{R})$. We can thus calculate Baez’s groups:

**Corollary 4.4.** If $m \geq 3$ then $\text{SL}_m(\mathcal{O}) \cong \text{SL}_8(\mathbb{R})$.

**Proof.** Since $[M_8(\mathbb{R}), M_8(\mathbb{R})]$ is the set of traceless matrices, we conclude from Theorem [1.3] that for $m \geq 3$ there is an isomorphism of Lie algebras $\mathfrak{s}\mathfrak{l}_m(\mathcal{O}) \cong \mathfrak{s}\mathfrak{l}_8(\mathbb{R})$. Exponentiating both sides gives the result. □

Together with Theorem [3.4], this describes all the groups Baez defined. In fact, the same argument works for $\mathcal{O}'$, and we similarly obtain $\text{SL}_m(\mathcal{O}') \cong \text{SL}_8(\mathbb{R})$ for $m \geq 3$.

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