Single photon detection generally consists of several stages: the photon has to interact with one or more charged particles, its excitation energy will be converted into other forms of energy, and amplification to a macroscopic signal must occur, thus leading to a “click.” We focus here on the part of the detection process before amplification (which we have studied in a separate publication). We discuss how networks consisting of coupled discrete quantum states provide generic models for that first part of the detection process. The input to the network is a continuum (the continuum of single-photon states), the output is again a continuum describing the next irreversible step. The process of a single photon entering the network, its energy propagating through that network and finally exiting into another output continuum of modes can be described by a single complex transmission function, $T(\omega)$. We discuss how to obtain from $T(\omega)$ the photo detection efficiency, how to find sets of parameters that maximize this efficiency, as well as expressions for frequency-dependent group delays and spectral bandwidth. We then study a large variety of networks.

I. INTRODUCTION

The development of single photon detectors is a state of the art research area [1–3]. Fundamental limits to single photon detector (SPD) performance have yet to be uncovered. That is, even though there is a large amount of theory for each type of photo detector, fundamental limits, independent of platform and architecture, should be derived from a general fully quantum-mechanical model of the whole photo detection process, from the initial physical contact the photon makes with the detector to the final “click.” Device-specific theories, however, are often at least in part phenomenological in nature. In order to develop a useful fully quantum-mechanical theory we cannot be completely general; or, rather, if we are completely general, then the only statements on fundamental limits we can make are likely going to be merely examples of Heisenberg’s uncertainty relations. So we will make three restrictive but—we think—reasonable assumptions about our quantum theory of photo detection.

First, we focus on single-photon detection. The main reason is that number-resolved photo detection is possible using arrays of SPDs where each “pixel” receives at most one photon as in Fig. 1 (also see [4], or [5] for the time-reversed process of creating a single photon on demand). So we focus on an individual pixel here.

Second, although a general state of a single photon is a function of four quantum numbers, one related to the spectral degree of freedom, two related to the two transverse spatial degrees of freedom, and one related to the polarization or helicity degree of freedom, we will restrict ourselves to the spectral (or, equivalently, the temporal) degree of freedom. That is, the input state can be defined in terms of frequency-dependent creation operators $\hat{a}^+(\omega)$ acting on the vacuum. The reason is that the other three degrees of freedom can, in principle, if not in practice, be sorted before detection. For example, if one wishes to distinguish between horizontally and vertically polarized photons, one may use a polarizing beam splitter and put two detectors behind each of the two output ports. Similarly, efficient sorting of photons by their orbital angular momentum quantum number is also possible [6]. It is easier to consider sorting as part of the pre-detection process, rather than a task for the detector itself. On the other hand, the spectral response of a detector cannot be eliminated; it is intrinsic to the resonance-structure of the photo detecting device.

Third, we are going to assume that each pixel’s operation is passive. That is, apart from being turned on at some point, and being turned off at some later point, it operates in a time-independent manner. Thus an incoming photon will interact with a time-independent quantum system. As we will see, active filtering is not needed for perfect detection (if the photo detector has no internal losses).

We can now describe the interaction of a single photon with an arbitrary quantum system as follows. The system may be naturally decomposed into several subsystems, each of which may have discrete and/or contin-
uous energy eigenstates. (For example, the photon may be absorbed by a molecule or atom or quantum dot or any structure with a discrete transition that is almost resonant with the incoming photon.) The continua will in general be structured (for example, containing bands and band gaps in between), but structured continua can be equivalently described as structureless (flat) continua coupled to (fictitious) discrete states [7,10]. And so an arbitrary quantum system may be described by a network of discrete states (some physical, some fictitious), coupled to flat continua. The latter coupling makes the time evolution irreversible. Of course, an actual detector is indeed irreversible. In particular, the amplification process (converting the microscopic input signal into a classical macroscopic output signal) is intrinsically irreversible.

What we do here is give a general description of the photon entering some network of discrete states [indicated by the black box in Fig. 2] up to and including the first irreversible step in the process. We analyzed the (irreversible) amplification step in another paper [11] and found the fundamental limits on added noise arising from amplification are so mild that fundamental tradeoffs of a detector are in essence determined by the pre-amplification process, which is the process we analyze here.

![Diagram of a two-port black box quantum network](image)

**FIG. 2:** Input and output fields coupled to a two-port black box quantum network. The input and output operators are continuous mode (as functions of frequency $\omega$) annihilation operators, satisfying canonical commutation relations [12]. Here the mode $a_{in}(\omega)$ carries the single-photon input state, the reflected mode is described by $b_{out}(\omega)$, and the output mode $a_{out}(\omega)$ contains the excitation energy if the input energy successfully traversed the network. $b_{in}(\omega)$ may contain thermal excitations, but is never occupied by the photon we wish to detect.

As we will show below, this first part of the process can then be fully described in terms of a complex transmission amplitude $T(\omega)$, which is simply the probability amplitude for the frequency component $\omega$ to be detected. (If there are no losses downstream, it would equal the probability of detection.) We are thus particularly interested in identifying quantum systems for which there is at least one frequency $\omega_0$ for which $|T(\omega_0)| = 1$.

Second, an upper bound to the total detectable frequency range is then given by the spectral bandwidth, defined as

$$\tilde{\Gamma} = \frac{1}{\pi} \int d\omega |T(\omega)|^2. \quad (1)$$

Also, the inverse of this quantity is a measure of the time the photon spends in the detector and gives a bound on the timing information obtained about a detected photon.

Third, we can also define a delay (latency) using the polar decomposition of $T(\omega)$

$$T(\omega) = |T(\omega)| \exp(i\phi(\omega)) \quad (2)$$

and using the standard definition of group delay as

$$\tau_g(\omega) = -\frac{d\phi(\omega)}{d\omega}. \quad (3)$$

Finding key conditions that change the transmission function $T(\omega)$ and frequency-dependent group delay $\tau(\omega)$ are important for the design of coupled-resonator optical waveguide (CROW) networks [13] for delay-lines [14] and spectral filtering [15], where the transmission efficiency, frequency-dependent group delay, and spectral bandwidth will all affect performance. These are the three quantities we focus on in the rest of the paper. We'll start with the simplest quantum network to illustrate how we calculate $T(\omega)$ and what our three quantities of interest behave like as functions of $\omega$. After that we'll tackle more complicated networks.

Critically, knowing $T(\omega)$ for a specific photo detector also allows one to construct the positive-operator valued measure (POVM), from which all standard figures of merit can be obtained [16]. The POVM element corresponding to a click after the photo detector has been left on for a very long time (in particular, long compared to the bandwidth $\tau \gg \tilde{\Gamma}^{-1}$; see Appendix A for a detailed POVM construction) has the particularly simple form

$$\Pi = \int_{-\infty}^{\infty} d\omega |T(\omega)|^2 \langle \omega | \langle \omega \rangle. \quad (4)$$

$\Pi$ is defined such that the probability of a photon in a state $\rho$ being detected is given by the Born rule $Pr = Tr(\Pi \rho)$. For example, any photon state $\rho = \sum_i \lambda_i | \omega_i \rangle \langle \omega_i |$ where $|T(\omega_i)|^2 = 1$ and $\sum_i \lambda_i = 1$ will be detected with unit probability. (The states the photo detector can detect perfectly include both pure states [when only one $\lambda_i$ is non-zero] and mixed states comprised entirely of frequencies where $|T(\omega_i)|^2 = 1$.) Of course, no photon is truly monochromatic (or discretely polychromatic, but it be effectively so if the wave-packet envelope is long compared to the inverse spectral bandwidth $\tilde{\Gamma}^{-1}$.
FIG. 3: A single discrete state described by an operator $c(\omega)$ and coupled to input and output continua $a_{\text{in}}(\omega)$ and $a_{\text{out}}(\omega)$ at rates $\gamma_i$ and $\Gamma_i$, respectively.

II. SIMPLE EXAMPLE

The simplest quantum network consists of a single two-level system with a ground state $|g\rangle$ and an excited state $|e\rangle$, with the incoming photon coupling these states. The two-level system is described by fermionic raising and lowering operators $\sigma^+ = |e\rangle \langle g|$ and $\sigma^- = |g\rangle \langle e|$ (this is also the simplest model of a photo detector, see [17]). In the Heisenberg picture, the evolution of these operators will determine whether a photon makes it from one side of the network to the other. By focusing our analysis on cases where at most a single photon is in the network, we can use the equivalence between the two-level system and the simple harmonic oscillator to simplify our problem ab initio: we replace the fermionic raising and lowering operators with bosonic creation and annihilation operators $c$ and $c^\dagger$. We follow standard input-output theory here [12]. After formally solving the Heisenberg evolution equations for two input continuum mode annihilation operators $b_{\text{in}}$ and $a_{\text{in}}$ [12], we write the effective system Hamiltonian that governs the evolution of system operators such as $c$ and $c^\dagger$:

$$H = -\hbar\omega_0 c^\dagger c - \sqrt{\gamma}(c^\dagger b_{\text{in}} + cb_{\text{in}}) - \sqrt{\Gamma}(c^\dagger a_{\text{in}} + ca_{\text{in}})$$

(5)

where we have identified $\omega_0$ as the resonance frequency, and $\gamma$ and $\Gamma$ as the left and right side couplings to two continua (Fig. 3) respectively [13]. In the Heisenberg picture, the time evolution of the discrete state annihilation operator is given by

$$\dot{c} = -\omega_0 c - \frac{\gamma + \Gamma}{2} c - \sqrt{\gamma} a_{\text{in}} - \sqrt{\Gamma} b_{\text{in}}.$$  

(6)

The input mode operators $a_{\text{in}}$ and $b_{\text{in}}$ and output mode operators $a_{\text{out}}$ and $b_{\text{out}}$ are determined by discrete state evolution of the operator $c$ and the two boundary conditions

$$b_{\text{out}} - a_{\text{in}} = -\sqrt{\gamma} c$$

$$a_{\text{out}} - b_{\text{in}} = -\sqrt{\Gamma} c.$$  

(7)

It is easiest to solve the equations by taking the Fourier transform. Unitarity implies the existence of a transfer matrix relating in and out fields in the spectral domain

$$\begin{bmatrix} a_{\text{out}}(\omega) \\ b_{\text{out}}(\omega) \end{bmatrix} = \begin{bmatrix} T(\omega) & R(\omega) \\ R(\omega) & T(\omega) \end{bmatrix} \begin{bmatrix} a_{\text{in}}(\omega) \\ b_{\text{in}}(\omega) \end{bmatrix}$$

(8)

where $|T(\omega)|^2 + |R(\omega)|^2 = 1$ (resulting from our assumption there are no internal losses) [19]. Defining a detuning $\Delta = \omega - \omega_0$, we can easily solve in frequency space

$$c(\omega) = -\sqrt{\gamma} a_{\text{in}}(\omega) - \sqrt{\Gamma} b_{\text{in}}(\omega)$$

(9)

yielding a transmission function

$$T(\omega) = \frac{\sqrt{\gamma} \Gamma}{\frac{\gamma + \Gamma}{2} - i\Delta}.$$  

(10)

We can see from (11) that perfect transmission ($|T(\omega)|^2 = 1$) occurs only when $\gamma = \Gamma$ and $\Delta = 0$. These are the well-known conditions of balanced mirrors and on-resonance required for perfect transmission through a Fabry-Perot cavity [20].

We can also calculate the frequency dependent group delay from (11)

$$\tau_g(\omega) = \frac{\Gamma^+ + \gamma}{\left(\frac{\Gamma^+ + \gamma}{\gamma^2}\right)^2 + \Delta^2}.$$  

(11)

We see that like $T(\omega)$, the group delay is also a Lorentzian with width $\frac{\Gamma^+ + \gamma}{\gamma}$, and that frequencies close to resonance spend the most time in the network with a maximum group delay of $\frac{\gamma}{\Gamma^+ + \gamma}$.

We can also use $T(\omega)$ to define a spectral bandwidth (not to be confused with the channel bandwidth discussed in [10])

$$\Delta \omega = \frac{1}{\pi} \int_{-\infty}^{\infty} d\omega |T(\omega)|^2$$

(12)

$$= \frac{2\Gamma\gamma}{\Gamma^+ + \gamma}$$  

(13)

which is a measure of the number of frequencies that can be efficiently detected. For this simple case, we note that $\tau_g(\omega) = \Gamma^{-1}|T(\omega)|^2$ and thus $\int d\omega \tau_g(\omega) = \pi$. (This will not be true for a general network, as we shall see shortly.)

III. QUANTUM NETWORKS

We now set up the general problem of an arbitrary network of discrete states connecting two continua. The Hamiltonian is a straightforward generalization of (5)

$$H = -\sum_i \hbar \omega_i c_i^\dagger c_i - \sum_{ij} g_{ij} (c_i^\dagger c_j + c_j^\dagger c_i)$$

$$- \sum_{ij} \sqrt{\gamma_i} (c_i b_i + c_i^\dagger b_i^\dagger) - \sqrt{\Gamma_j} (c_i^\dagger a_i + c_i a_i^\dagger)$$

(14)
where we’ve now defined a real coherent coupling between discrete states $g_{ij}$ (we define $g_{ii} = 0$ for each state). Some states may not be coupled to one (or both) continuum, in which case either $\sqrt{\gamma_i}$ or $\sqrt{\Gamma_i}$ (or both) will be zero.

$$-i\Delta_i c_i(\omega) = -\sum_j \left( \frac{i\gamma_i + \sqrt{\Gamma_i} g_{ij}}{2} + ig_{ij} \right) c_j(\omega) - \sqrt{\gamma_i} b_{in}(\omega) - \sqrt{\Gamma_i} a_{in}(\omega).$$  \tag{15}$$

Similarly to (7), we can write boundary conditions for the two continua with an arbitrary network

$$b_{out}(\omega) - a_{in}(\omega) = -\sum_i \sqrt{\gamma_i} c_i(\omega)$$

$$a_{out}(\omega) - b_{in}(\omega) = -\sum_i \sqrt{\Gamma_i} c_i(\omega).$$  \tag{16}$$

Now going from (16) to the transfer matrix (8) involves solving $N$ systems of $N$ coupled first-order differential equations [21]. This complication occurs because each discrete state $c_i(\omega)$ depends on every other discrete state [22]. One can also use numerical techniques to diagonalize the systems of equations and find the transmission function numerically [23]. But this rapidly gets harder with large systems, and masks the analytic conditions for perfect transmission we are interested in identifying. Here, we will instead identify large classes of systems that can be solved exactly with arbitrary couplings.

**Parallel**: each discrete state is directly coupled to both continua ($\gamma_i \neq 0$ and $\Gamma_i \neq 0 \forall i$) but not to each other. Thus there are multiple parallel paths to the same final state and hence we’ll get interference.

**Series**: each of the two continua are coupled to their own discrete state, which are in turn coherently coupled by a chain of intermediate single discrete states. There is just one path from the input continuum to the output continuum.

**Hybrid**: a combination of the above for example, two parallel paths of three steps each.

In all cases, we will solve the systems of equations for $R(\omega)$ directly and make use of the identities $|T(\omega)|^2 = 1 - |R(\omega)|^2$ and $T^2(\omega) = -R^2(\omega)\frac{|T(\omega)|^2}{|R(\omega)|^2}$ when we calculate the transmission efficiency, spectral bandwidth, and group delay.

A salient feature of Eq. (17) is the decay terms produce purely virtual coupling between discrete states coupled...
to the same continua. These cannot be thought of as coupling mediated by a continua, as the continua are flat (Markovian) and perfectly dissipative, and is instead a purely information-theoretic phenomena 23.

We use (8) to find \( R(\omega) \) by considering an input on only one side of the network (thus setting the expectation value of the other input field operator to zero). This yields an expression \( b_{\text{out}}(\omega) = R(\omega) b_{\text{in}}(\omega) \) (or \( a_{\text{out}}(\omega) = R(\omega) a_{\text{in}}(\omega) \)), from which we reconstruct \( T(\omega) \). We analytically find the general form of \( R(\omega) \)

\[
R(\omega) = \frac{\prod_i \left( \frac{\Gamma_i - \gamma_i}{2} - i\Delta_i \right) - X(N)}{\prod_i \left( \frac{\Gamma_i + \gamma_i}{2} - i\Delta_i \right) - X(N)}
\]  

where \( X(N) \) is a polynomial of order \( N - 2 \) in the detunings \( \Delta_i \). (We trivially find \( X(1) = 0 \).) For \( N = 2 \), we can see that \( X(2) \) is symmetric (anti-symmetric) between \( \Gamma_i \) and \( \gamma_i \)

\[
X(2) = \left( \frac{\sqrt{\Gamma_1 \Gamma_2} \pm \sqrt{\gamma_1 \gamma_2}}{2} \right)^2.
\]

This (anti-)symmetry is also present in higher-\( N \) coefficients. From (18) we can determine a key feature of parallel quantum networks: if some subset of the discrete states have balanced decay rates such that \( \gamma_i = \Gamma_i \), for large spacings between discrete states compared to the other decay rates \( |\omega_j - \omega_k| \gg \gamma_j, \Gamma_j \), we find \( R(\omega_i) = 0 \); input monochromatic photons with frequencies on resonance with those discrete states are transmitted perfectly through the network 26.

In general, finding the specific form of \( X(N) \) is a numerical task, and we will focus on a simpler case where we can utilize another salient feature of parallel networks: the purely virtual coupling present in (17). Before we consider a network that is uniformly coupled (all decays are the same), we can consider a network with couplings that are inhomogeneous but uniformly unbalanced such that \( \Gamma_i = k \gamma_i \) \( \forall i \). We can then write (17) in a simplified form

\[
\frac{i \Delta_i}{\sqrt{\gamma_i}} c_i(\omega) = \sum_j \frac{\sqrt{\gamma_j}(1 + k)}{2} c_j(\omega) + b_{\text{in}}(\omega) + \sqrt{k} a_{\text{in}}(\omega)
\]

so that we observe strong correlations between discrete state amplitudes

\[
\frac{i \Delta_i}{\sqrt{\gamma_i}} c_i(\omega) = \frac{i \Delta_i}{\sqrt{\gamma_i}} c_i(\omega).
\]

For non-degenerate states, Eq. (21) means that a photon that is resonant with one state will only excite that state. (For infinitely-narrow discrete states, these correlations are satisfied trivially of course; \( c_i(\omega) \) is only non-zero at \( \omega_i \) - Eq. (21) also indicates that the relative phase of discrete state amplitudes is frequency dependent; when \( \omega_i < \omega < \omega_j \), there is a relative phase of \( \pi \) between \( c_i \) and \( c_j \). This is a possible alternative explanation for the destructive interference present in atomic chains on a fiber 27,28 and multi-mode Fabry-Perot cavities 29.

We can see destructive interference directly from the form of the reflection coefficient

\[
R(\omega) = \frac{i - (k - 1) \sum \gamma_i}{i - (k + 1) \sum \gamma_i} T(\omega)^2
\]

from which we can determine \( |T(\omega)|^2 \) and the other quantities of interest.

We can further specialize to the case of homogenous coupling where \( \gamma_i = \gamma \) and \( \Gamma_i = \Gamma \). This case is of interest for several reasons: most generally, this assumption directly follows from the first Markov approximation if the spacing between discrete states is small compared to the decays. Assuming homogenous coupling is also appropriate for modeling a variety of systems (i.e. atomic chains on a fiber 27,28 and parallel Fabry-Perot micro-cavities coupled to the same input and monitored continua). It also simplifies the form of the correlations between discrete states \( \Delta_i c_i(\omega) = \Delta_i c_i(\omega) \) 30, as is the form of the reflection coefficient

![FIG. 5: Transmission function for parallel networks with \( N = 4 \) equally spaced discrete states, each with a decay rate to the input continuum \( \gamma_i = \gamma_i (1.4)^{i-1} \). The decays to the monitored continuum are uniformly unbalanced (\( \frac{\Gamma_i}{\gamma_i} = k \forall i \)), with \( k = .5 \) in blue and \( k = 1 \) in red. Frequency is measured w.r.t. the average resonance frequency. We observe the four resonance frequencies \( \omega_i \) have maximum transmission probability \( |T(\omega_i)|^2 = \frac{4k}{(i+1)\pi} \). The three frequencies of perfect refection correspond to solutions of \( \sum_i \frac{\Delta_i}{\Delta_i i} = 0 \).]
FIG. 6: Transmission function amplitude, group delay, and transmission function phase for a parallel network with $N = 5$ equally spaced discrete states with balanced decay rates to both continua ($\gamma = \Gamma$). Frequency is measured w.r.t. the average resonance frequency. We observe the five resonance frequencies $\omega_i$ each correspond to a perfectly transmitted frequency. The four frequencies of perfect refraction correspond to solutions of $\sum_i \frac{1}{\Delta_i} = 0$. The group delay is always longest for the highest and lowest frequency resonances (except in the large spacing limit, where they are of equal magnitude). The total change in phase of the transmission function is proportional to the number of discrete states.

Perfect transmission. — From (22) and (23) we can see that in both cases there are $N$ frequencies of maximum transmission corresponding to each resonant frequency $\Delta_i = 0$ with transmission probability $|T(\omega_i)|^2 = \frac{4k}{(k+1)^2}$ and $|T(\omega_i)|^2 = \frac{4\gamma_i\Gamma}{(\gamma_i+\Gamma)^2}$, respectively. We also see $N - 1$ frequencies of destructive interference corresponding to the $N - 1$ solutions of $\sum_i \frac{1}{\Delta_i} = 0$ (Fig. 5) determined solely by the decays and resonances (and notably not by $k$). We similarly observe for the case of a network with homogeneous decay rates $N - 1$ frequencies of destructive interference corresponding to the $N - 1$ solutions of $\sum_i \frac{1}{\gamma_i} = 0$ (Fig. 5). One might think that, in principle, a resonant frequency could coincide with a frequency of perfect reflection when $N > 2$, in which they can annihilate. However, this only occurs when two discrete states are energetically degenerate, and since they couple to the same 1D continuum, this is forbidden by unitarity; as we decrease the spacing between states, we see a resonant frequency and a frequency of destructive interference annihilate as a discrete state is forced to decouple from the system as the degeneracy becomes exact.

In general, the condition for perfect transmission through a parallel network is that all the couplings be balanced ($\gamma = \Gamma$ or in the inhomogeneous uniformly unbalanced case, $k = 1$). As we saw in the case for a completely arbitrary parallel network, we see that perfect transmission at some discrete state frequency $\omega_i$ not only requires balanced coupling $\gamma_i = \Gamma_i$, but also that all the other discrete states either be far away in frequency compared to the their decay rates ($|\omega_i - \omega_j| \gg \gamma_j, \Gamma_j$), or also be balanced ($\gamma_j = \Gamma_j$), or a mix of the two.

Spectral Bandwidth. — Once we have the reflection coefficient, we can calculate the three quantities of interest. For all three cases we’ve discussed, we find that the spectral bandwidth is purely additive $\tilde{\Gamma} = \sum_j \frac{2\gamma_j\Gamma_j}{\gamma_j+\Gamma_j}$ and is completely independent of the spacing between discrete states

Group Delay. — We note that, unlike the simple model, the sharp peaks in the group delay (Fig. 6b) correspond to frequencies of destructive interference and are greatest for the outermost frequencies of destructive interference despite all the frequencies of note in (Fig. 6a) being completely destructive or constructive. We also observe that the relationship between the three quantities of interest discussed for the simple model is not present: here $|T(\omega_i)|^2$ for each resonant frequency. Considering the phase itself, we find that it increases by $2\pi$ with each resonant frequency (Fig. 6c). This provides a potentially novel application for single-photon interferometry for resolving tightly-structured resonance structures. It also serves to explain why, whereas the spectral bandwidth $\tilde{\Gamma}$ is independent of discrete state spacing, we find that the group delay increases with close spacing; the same change in phase is occurring in a smaller spectral range so $\tau_g(\omega) = -\frac{d\phi(\omega)}{d\omega}$ increases.

B. Series Networks

We cannot in general diagonalize an arbitrary network in terms of parallel modes with purely virtual couplingthere may be a causal relationship built embedded in the network structure. A class of such systems are series
quantum networks, where only one state is coupled to each continuum, with the other states forming a chain between the two outer ones ($\gamma_{i=1} = 0$, $\Gamma_{i=N} = 0$, and $g_{ij} = 0$ for $j \neq i \pm 1$). This provides an especially natural (and simple) model for energy transport (i.e. along an organic compound) as well as repeated spectral filtering (a series of Fabry-Perot cavities). Furthermore, we can analytically determine the transmission function for arbitrary series networks, as we will now proceed to do.

Since each state is only coupled to the two adjacent states, the system of $N$ equations of the form of [13], describing evolution of the system is solvable in a stepladder-type approach. Setting the expectation value of the second input continuum to zero, we solve for the $N$th state in terms of the $N-1$th. As we step up the ladder, we arrive at an expression for the reflection coefficient with the form of a generalized continued fraction

$$R(\omega) = 1 - \frac{\gamma}{\frac{\gamma}{2} - i\Delta_1} + \frac{g_{12}^2}{-i\Delta_2 + \frac{g_{23}^2}{\frac{g_{N-1,N}^2}{2} - i\Delta_N}}$$

(we have dropped the subscripts on $\gamma_1$ and $\Gamma_N$) which is easily analyzable using the Wallis-Euler recursion relations for continued fractions:

$$A_n = b_n A_{n-1} + a_n A_{n-2} \quad (n \geq 1)$$

$$B_n = b_n B_{n-1} + a_n B_{n-2} \quad (n \geq 1)$$

$$B_{-1} = 0, \quad B_0 = 1, \quad A_{-1} = 1, \quad A_0 = b_0.$$  

Perfect transmission. — We begin by considering the unphysical but illuminating case of an infinite series of identical discrete states ($g_{i,i+1} = g \forall i$ and $\Delta_i = \Delta \forall i$) analytically. We find that the limit $\lim_{N \to \infty} R(\omega)$ only converges on-resonance for the special “critical” case

$$g = \frac{\sqrt{\gamma \Gamma}}{2}.$$ \hspace{1cm} (28)

Furthermore, the limit only converges to zero on resonance (perfect transmission) when we also have $\gamma = \Gamma$. This first condition corresponds to a series of discrete states coupled through their decays (e.g. Fabry-Perot cavities coupled through their evanescent fields), and the second condition corresponds to the same requirement of balanced decays we saw for parallel quantum networks.

While infinite series networks of discrete states are not realistic, these two conditions play different but important roles in all infinite series networks where the discrete states are identical (though introducing detuning will change the critical values of $g$ and $\Gamma/\gamma$, as we will see). That there are two conditions can be explained thus; since $R(\omega)$ is in general complex, the condition $R(\omega) = 0$ gives two constraint equations on the real and imaginary parts of $R(\omega)$. When $N$ is even [odd], the real part is an $N$-$[N-1]$ polynomial in the detunings $\Delta_i$, while the imaginary part is order $N-1$ $[N]$. In general, this means there are conditions for a minimum of $N-1$ frequencies of perfect transmission and we may find $N$ frequencies of perfect transmission only if the lower-order equation is satisfied trivially for all frequencies. When considering finite series networks of identical discrete states, these same two conditions appear in the constraint equations for perfect transmission [32].

We now explore in detail the effects of these two conditions on series networks of identical discrete states with uniform coupling; first, consider fixing the coupling $g$ to be critical ($g = \frac{\sqrt{\gamma \Gamma}}{2}$). For odd $N$, we find that when the decays are balanced ($\gamma = \Gamma$), this ensures $N$ frequencies of perfect transmission (Fig. 8a), with the resonance frequency at a local maxima of unity, and when the decays are unbalanced ($\gamma \neq \Gamma$), $N-1$ frequencies of perfect transmission (Fig. 8b) with the resonant frequency at a local minima. For even $N$, letting $g = \frac{\sqrt{\gamma \Gamma}}{2}$ always results in $N-1$ frequencies of perfect transmission (Fig. 8c) with the on-resonant frequency at a local maxima. Here, having balanced decays broadens the the on-resonance maxima (Fig. 8d), which will be desirable for detection of non-monochromatic photons (wave packets).

Now we instead consider fixing the decays to be balanced ($\gamma = \Gamma$) and observe a switch in the on-resonance behavior; whereas above we had found for even $N$ the critical coupling condition was sufficient for on-resonance transmission to be at a local maxima, we now find that, for odd $N$ that the balanced decay condition results in $N$ peaks of unity transmission (Fig. 10a) with on-resonance transmission is at a local maxima. Here three peaks become degenerate to give $N - 2$ frequencies of
FIG. 8: Transmission functions for series networks with an odd (top) and even (bottom) number of discrete states \( N \) with no internal detuning, for the special case of homogenous critical coupling \( g_{ij} \rightarrow g = \sqrt{\gamma \Gamma} \). Networks with both balanced decays (top left and bottom right, \( \gamma = \Gamma \)) and unbalanced decays (top right and bottom left, \( \frac{\Gamma}{\gamma} = 2 \)) are plotted. While it is not required for perfect transmission at some frequency, meeting both the balanced decay and critical coupling conditions ensures on-resonance transmission is both unity and maximally broadened.

FIG. 9: A series network of three coherently coupled discrete states, each described by an operator \( c_i(\omega) \) and coupled to each other at rates \( g_{ij} \). The first and last states decay at rates \( \gamma \) and \( \Gamma \) to input and output continua \( a_{in}(\omega) \) and \( a_{out}(\omega) \) respectively.

perfect transmission when \( g \lesssim \frac{\sqrt{\gamma \Gamma}}{2} \) (Fig. 10b). (This inequality rapidly becomes exact \( g < \frac{\sqrt{\gamma \Gamma}}{2} \) with increasing \( N \).) Similarly, we find that for even \( N \), the behavior of \( T(\omega) \) depends strongly on the coupling, flipping between \( N \) frequencies of perfect transmission with on-resonance transmission at a local minima for \( g > \frac{\sqrt{\gamma \Gamma}}{2} \) (Fig. 10c) and \( N - 2 \) frequencies of perfect transmission with on-resonance transmission at a non-unity local maxima for \( g < \frac{\sqrt{\gamma \Gamma}}{2} \) (Fig. 10d).

For both even and odd \( N \), the width is increasingly determined by \( g \) with increasing \( N \), with the half-width asymptotically approaching \( 2g \). In the large-\( N \) limit, we observe that the transmission function is asymptotically bounded between a circle and a square when both conditions for perfect transmission are met (Fig. 11).
FIG. 10: Transmission functions for series networks with an odd (top) and even (bottom) number of discrete states \( N \) with no internal detuning, for the special case of homogenous balanced decays \( \gamma = \Gamma \) for all states. Networks that are both over-coupled (left, \( g > \sqrt{\frac{\gamma}{\Gamma}} \)) and under-coupled (right, \( g < \sqrt{\frac{\gamma}{\Gamma}} \)) are plotted. A balanced over-coupled network will always have more peaks of perfect transmission than an under-coupled or critically-coupled one, though on-resonance transmission may not be at a maxima.

also observe that increasing \( g \) past \( \sqrt{\frac{\gamma}{\Gamma}} \) while maintaining \( \gamma = \Gamma \) induces Rabi splitting, with the \( N \) or \( N-1 \) frequencies of perfect transmission spreading outwards (for odd and even \( N \), respectively). In the high-\( g \) limit, we start seeing a frequency-comb structure emerge (Fig. 12), with dips that approach perfect reflection. This is because in the strong-coupling limit, the causal ordering of the discrete states asymptotically disappears: the system becomes approximately diagonalizable as parallel modes with purely virtual coupling and the \( N-1 \) frequencies of perfect reflection from (22) manifest.

We now consider the effects of introducing relative detunings between discrete states as, generally, a quantum network will not be comprised of identical states. Still there are sufficient degrees of freedom in (24) such that, by tuning the parameters, perfect transmission at some frequencies is always possible. This is even true when \( \gamma \neq \Gamma \) (for \( N > 1 \)): considering only two discrete states in series, we find that perfect transmission occurs at a frequency \( \omega = \frac{\omega_1 + \omega_2}{2} + \frac{\Gamma \omega_1 - \gamma \omega_2}{2} \) when \( g_{12} = \sqrt{\frac{\gamma}{\Gamma} + \left( \frac{\Gamma \omega_1 - \gamma \omega_2}{\Gamma - \gamma} \right)^2 - \left( \frac{\omega_1 - \omega_2}{2} \right)^2} \) (Fig. 13). For two discrete states in series with balanced decays \( \gamma = \Gamma \), perfect transmission is impossible as the critical value of \( g \) is infinite (the transmission efficiency asymptotically becomes perfect in the strong-coupling limit), but this is a special case and is not true for \( N > 2 \) discrete states. Given a series network of \( N \) discrete states with arbitrary relative detunings, we can always find a set of parameters
FIG. 11: Transmission function for a series network with $N = 70$ identical discrete states ($\omega_i = \omega_j$) with both balanced decays ($\gamma = \Gamma$) and uniform critical coupling ($g = \sqrt{\gamma \Gamma}$) conditions met. This results in a maximally-broadened unity on-resonance transmission. In the large-$N$ limit, this ideal (perfect and broadened) transmission function is bounded below by a circle and above by a square, both of width $2g$.

such that $N - 1$ frequencies are perfectly transmitted.

Spectral Bandwidth.— Once we have the form of the transmission function, we can calculate the spectral bandwidth for these systems. The spectral bandwidth decreases with additional discrete states and is strictly bounded above by the single discrete state bandwidth $\Gamma \leq \frac{2g}{\gamma + \Gamma}$. For discrete states without detuning, equality is reached in the strong-coupling limit of $g \gg \sqrt{\frac{\gamma \Gamma}{\gamma + \Gamma}}$ but independently of whether $\gamma = \Gamma$ (Fig. 14). Introducing detuning between discrete states lowers the bandwidth, but equality with the upper limit still occurs for sufficiently strong coupling (Fig. 15); the strong coupling is able to better mask the discrepancy between discrete state frequencies as the dressed states become the more physical description and as the system more strongly resembles a parallel network. (In this limit, each dressed state is coupled to the continua at reduced decays so that the total bandwidth is still bounded by $\Gamma \leq \frac{2g}{\gamma + \Gamma}$.)

Group Delay.— Lastly, we consider the group-delay for

FIG. 12: Transmission function for a series network with $N = 70$ identical discrete states with balanced decays ($\gamma = \Gamma$) yielding perfect transmission in the strong-and-uniform coupling limit ($g \gg \sqrt{\frac{\gamma \Gamma}{\gamma + \Gamma}}$). The result is a frequency-comb structure with 70 frequencies of unity transmission separated by 69 frequencies of near-perfect reflection.

FIG. 13: Transmission functions for series networks with $N = 2$ discrete states with no relative detuning (blue), and a non-zero relative detuning $\frac{\omega_2 - \omega_1}{\gamma} = .75$ (red). In both cases, the decays are not balanced ($\frac{\Gamma}{\gamma} = 2$) but the couplings are chosen such that perfect transmission at a frequency is achieved.

FIG. 14: Normalized spectral bandwidth $\tilde{\Gamma}/\tilde{\Gamma}_{\text{max}}$ for a series network with $N = 2$ discrete states with no relative detuning. Here the maximum bandwidth, given by that of a single discrete state $\tilde{\Gamma}_{\text{max}} = \frac{2g}{\gamma + \Gamma}$, is attained in the strong coupling limit. The balanced decay and critical coupling conditions do not effect the bandwidth significantly; increasing the decay $\Gamma$ just scales the strong coupling limit regime ($g \gg \sqrt{\frac{\gamma \Gamma}{\gamma + \Gamma}}$).
FIG. 15: Normalized spectral bandwidth $\tilde{\Gamma}/\tilde{\Gamma}_{\text{max}}$ for a series network with $N = 2$ discrete states with relative detuning $\omega_2 - \omega_1$. We still observe $\tilde{\Gamma}/\tilde{\Gamma}_{\text{max}} = 1$, but at a higher coupling strength for greater detunings.

series networks, which increases with both $g$ and $N$ but for different reasons. As $g$ increases, the peaks of the transmission function sharpen so that the phase changes more rapidly. This results in an increased group delay (Fig. 16a-d). As $N$ increase, we observe (as we did for parallel networks) that the peaks in the group delay are not of uniform magnitude, even when the transmission function itself is rather flat (Fig. 17a). Instead, the frequencies of maximum delay are those closest to $\pm 2g$ (Fig. 17c) increase the most with $N$. This is because the oscillations in the transmission function are most dense here, resulting in a rapid change in transmission function phase and thus a larger group delay.

We can also consider the effect of detuning between discrete states on the frequency-dependent group delay. We observe that the group delay can be negative for series networks with detuning (Fig. 17d), with an asymmetric structure that depends on the ordering of the detunings (except in the strongly coupled limit). As before, the magnitude of the group delay depends on the density state resonances.

FIG. 16: Transmission functions (left) and group delays (right) for series networks with $N = 2$ discrete states with critical coupling (top) and over-coupling (bottom), both with balanced decay rates ($\gamma = \Gamma$) and no relative detuning. A small change in the sharpness of the transmission function’s peaks can have a large effect on the magnitude of the group delay.
FIG. 17: Transmission functions and group delays for series networks with $N = 20$ discrete states with balanced decay rates $\gamma = \Gamma$, critical coupling $g = \sqrt{\gamma \Gamma}$, and either no relative detuning (left) or relative detunings of 0.02$\gamma$ between resonances (right). As before the group delay is of the largest magnitude near the edge of the transmission function ($\pm 2g$ in the high $N$ limit). A relative detuning may result in a negative group delay.

C. Hybrid Networks

FIG. 18: A hybrid network of two manifolds of $N_k$ discrete states, each described by an operator $c^{(k)}_i(\omega)$. The discrete states are coherently coupled at rates $g^{(1,2)}_{ij}$ and decay to input and output continua $a_{\text{in}}(\omega)$ and $a_{\text{out}}(\omega)$ at rates $\gamma^{(1)}_i$ and $\Gamma^{(2)}_j$, respectively.

We now begin to approach the case of a general two-sided quantum network. The fully general problem is intractable analytically, but luckily there are several simplifications we can make that correspond to the network representing realistic photo detecting systems. To illustrate, consider the case of two parallel networks in series: each discrete state connected to each discrete state in the other manifold (but not necessarily at the same rate) with each of the two manifolds of purely virtually coupled discrete states coupled to their own continuum. One could imagine generating different networks from this one by removing a coupling $g_{ij}$ between discrete states, permuting which discrete states are disconnected, removing an additional coupling, permuting, and so on. However, this is unphysical: two discrete states in parallel cannot be prevented from coupling to the same discrete state except by selection rules. But the discrete states are also coupled to the same continuum (which has been already made 1D in effect), so they must satisfy the same selection rules. The same argument applies in reverse: no discrete state within a manifold can individually stop being coupled to a continuum without the rest of the discrete states doing so as well. And we can similarly apply it to manifolds embedded in a larger network away from a continuum: the requirement that all states satisfy the same selection rules is strong. This means we can ignore partially connected networks. It also gives us a helpful way to organize discrete states: into manifolds of discrete
states (which are purely virtually coupled to each other after diagonalization) that are all coupled to the same set of discrete or continuum states.

We can now focus on a very large class of quantum networks, hybrid systems consisting of manifolds in series. It will be helpful to denote couplings between discrete states \(i\) and \(j\) or manifolds \(k\) and \(\ell\) as \(g_{ij}^{(k,\ell)}\), and denote

\[-i\Delta \epsilon_i^{(k)}(\omega) = -\sum_{j=1}^{N_k} \sqrt{\gamma_i^{(k)}(\omega) + \Gamma_i^{(k)}(\omega)} c_j^{(k)}(\omega) - i \sum_{j=1}^{N_k-1} g_{ij}^{(k-1,k)} c_j^{(k-1)}(\omega) - i \sum_{j=1}^{N_k+1} g_{ij}^{(k,k+1)} c_j^{(k+1)}(\omega) - \sqrt{\gamma_i^{(k)}} b_{in}(\omega) - \sqrt{\Gamma_i^{(k)}} a_{in}(\omega)\]

where the superscripts denote labels of manifolds and we’ve implicitly defined \(g_{ij}^{(0,1)} = g_{ij}^{(N,N+1)} = 0\) \(\forall i, j\). We denote the number of discrete states in each manifold \(N_k\) such that \(\sum_{k=1}^{M} N_k = N\). In general (29) is still only numerically solvable but we can now note two cases that yield analytic solutions.

The first is the case of critically coupled between members of each adjacent manifold and, additionally, uniformly unbalanced decays: we first define an effective decay rate for internal couplings within the system so that \(g_{ij}^{(k,\ell)} = \sqrt{\gamma_i^{(k)}(\omega)}\) (effectively specializing to the critical coupling case for series networks), and then consider the special case of \(\Gamma_i^{(k)}/\gamma_i^{(k)} = k^{(k)} \forall i, k\) (inhomogeneous decays that are uniformly unbalanced within each manifold). This leads to a reflection coefficient of the form

\[R(\omega) = 1 - \frac{2h^{(1)}}{h^{(1)} - i + \frac{k^{(1)}h^{(2)}}{k^{(2)}h^{(3)}} - i + \frac{k^{(N-1)}h^{(N-1)}h^{(N)}}{\sqrt{k^{(N)}}h^{(N)} - i}}\]

where we have defined a new function \(h^{(k)} = \sum_{i=1}^{N_k} \frac{\gamma_i^{(k)}}{2\Delta \epsilon_i^{(k)}}\).

(The appearance of a lone \(\sqrt{k^{(N)}}\) at the end of (30) is due to the final purely virtual coupling to the output continuum, since we’ve absorbed the rest of the decay rate into the function \(h^{(k)}\).) This function encodes the zeroes and singularities we found for parallel networks, which previously gave rise to frequencies of constructive interference and completely destructive interference.

While (30) is tractable, it is of limited applicability to real systems. More relevant is the second solvable case of homogenous coupling and decays within manifolds: \(g_{ij}^{(k,\ell)} = g^{(k,\ell)}\), \(\gamma_i^{(k)} = \gamma^{(k)}\), and \(\Gamma_i^{(k)} = \Gamma^{(k)}\). This leads to a reflection coefficient of the form

\[R(\omega) = 1 - \frac{\gamma f^{(1)}}{2\gamma f^{(1)} - i + \frac{(g^{(1,2)})^2 f^{(1)} f^{(2)}}{(g^{(2,3)})^2 f^{(2)} f^{(3)}} - i + \cdots + \frac{(g^{(N-1,N)})^2 f^{(N-1)} f^{(N)}}{\frac{\gamma}{2} f^{(N)} - i}}\]

where again we have defined a new function \(f^{(k)} = \sum_{i=1}^{N_k} \frac{1}{\Delta \epsilon_i^{(k)}}\). This provides a nice model for multi-mode systems in series (for instance, a linear network of multimode cavities).

We see in both (30) and (31) a combination of the structures we observed in (22) and (23) for parallel networks and (24) for series networks: correlations in amplitudes between discrete states within a given manifold manifest via a function \(h^{(k)}\) or \(f^{(k)}\) with \(N_k\) poles and \(N_k - 1\) zeroes (potentially perfectly transmitted and reflected frequencies, depending on the hybrid network’s resonance structure and couplings), and causal ordering of the manifolds manifests in a continued fraction structure. This latter property makes them easily analyzable using the Wallis-Euler recurrence relations, allowing us to find \(R(\omega)\) and from there \(T(\omega)\) and the other quantities of interest.

Perfect transmission.—Focusing on the case of homogenous coupling, we can use the same trick of examining the convergence of an infinite series of identical manifolds around one of resonant frequencies \(\omega_i\) to find the two critical conditions as we did for series networks, which again are \(\gamma = \Gamma\) and \(g = \frac{1}{\sqrt{2\pi}}\). We also observe that the form of \(T(\omega)\) for hybrid systems exhibits a combination of features of series and parallel networks (Fig. 19a and 20a). This results in layers of structure, with the small dips and peaks corresponding to intra-manifold structure layered on top of the larger dips and peaks of the inter-manifold structure.

When at least one discrete state in each manifold have the same resonance, letting \(\gamma = \Gamma\) and \(g = \frac{1}{\sqrt{2\pi}}\)
ensures perfect transmission at $M$ frequencies. For a general hybrid network, the number of peaks of unity (perfect transmission) of $T(\omega)$ are bounded above by $M \min\{N_k\} \leq N$, with $M$ the number of manifolds each with $N_k$ discrete states. Here the latter equality is reached for networks that are either completely in parallel or completely in series, with critical parameters in either case. \[ \text{Spectral Bandwidth.} \quad \text{For hybrid networks where the first and last manifold have the same number of discrete states } N_1 = N_M, \text{ the spectral bandwidth is bounded above by } M \min\{N_1, M_M\} + X \text{ with } X \text{ a network-dependent number that is always less than } 1/2. \]

**Group Delay.** — We observe the same structural properties of the group delay (Fig. 19b) as we did for other networks; the frequencies with the largest delays are those where oscillations in the transmission function are most dense. We find that networks with non-identical manifolds can give rise to group delays that are not strictly positive (Fig. 20b).
FIG. 21: Transmission function and group delay for a hybrid network consisting of 3 manifolds in series, the first two with 2 discrete states and the third with 3 discrete states, with balanced decays $\gamma = \Gamma$ and uniform coupling $g = \sqrt{\gamma \Gamma}$. Within the first and second manifolds, the discrete states are detuned by $2.5\gamma$. Within the third manifold, the discrete states are detuned by $7\gamma$. Now no frequencies are perfectly transmitted, and finding couplings such that perfect transmission is achieved becomes less trivial.

IV. OTHER NETWORKS

A. General Two-Sided Networks

We now begin to extrapolate from the above analyses to a larger class of two-sided networks. Both the series networks and hybrid networks we’ve discussed have the key property of asymptotic irrelevance of the causal ordering of discrete states in the strong coupling limit. This means that the asymptotically strong-coupling behavior of these networks is entirely determined by the properties of fully parallel networks [18], and we can make several statements that will apply to any network that has the same behavior (that is, a network that resembles [18] in the strong coupling limit). From the structure of [18] alone, we can bound the total number of peaks and troughs of $|T(\omega)|^2$; we find there are at most $2N - 1$ peaks and $2N - 2$ troughs in the strong coupling limit and since the number of maxima never decreases with increasing $g$, we can extrapolate these bounds to weakly coupled systems as well. In transitioning from $2N - 1$ peaks or constructive interference to $N$ peaks of perfect transmission, we observe a pair-wise merging of peaks (with one extra peak leftover when $N$ is odd). We also find that the number of dips of zero transmission (perfect reflection) is at most $\sum_{k=1}^{M} (N_k - 1) \leq N - 1$ where the latter equality is only reached for a completely parallel network or in the strong coupling limit.

We suspect that the above argument for upper bounds applies more generally than just to the specific networks consisting of manifolds in series; that is, there is a large class of networks that reduce to a parallel description in the strong coupling limit [18]. However, we cannot generalize from the above analysis to a fully arbitrary two-sided network as there also are networks with different topologies. For instance, we can consider networks with loops of discrete states that exist outside the main chain of manifolds that connect the two continua. Since it is not necessary for a photon to pass through the loop of discrete states to make it through the network, these networks behave differently in the strong coupling limit. We can make a further distinction between disconnected loops (dead-ends where photons have to back track) and connected loops (chains of manifolds that provide an alternate route to the output continua). It is possible that connected loops behave more like the loop-free networks discussed above in the strong-coupling limit, but with their specific loop-structure encoded in $T(\omega)$ in unexpected ways.

B. Additional Continua

FIG. 22: Input-output field operators for a black box network with two additional side channels. These will give rise to losses that make perfect transmission impossible, as well as dark counts if the side channels contain excitations (i.e. thermal).

We have thus far only considered quantum networks...
with two continuua, and have shown that perfect transmission through a general network structure is possible. We will briefly analyze more general multi-port quantum networks (Fig. 22) to illustrate how they lead to inefficiencies and dark counts.

Introducing a third and fourth continuum coupled to our network of discrete states at rates $\mu_i$ and $\nu_i$, we rewrite (15) in the form of the full transfer matrix, which now will include functions $D_m(\omega)$ and $D_n(\omega)$ that governs the probability of thermal excitations in the side channels to end up in the monitored continuum.

Of course, the internal mode $b_{in}$ may be occupied by thermal excitations as well, but the contribution to dark counts by these will depend strongly on the amplification mechanism. Furthermore, if only frequencies such that $|T(\omega_i)| = 1$ are amplified, we find that thermal excitations $b_{in}$ do not contribute at all to dark counts as they always leak out of the system, ending up in the reflected $b_{out}$.

\[ -i\Delta_i c_i(\omega) = -\sum_j \left( \sqrt{i}i\gamma_j + \sqrt{i}\Gamma_i\gamma_j + \mu_i\mu_j + ig_{ij} \right) c_j(\omega) - \sqrt{i}\gamma_i b_{in}(\omega) - \sqrt{i}\mu_i a_{in}(\omega) - \sqrt{i}\mu_i m_{in}(\omega) - \sqrt{i}\nu_i n_{in}(\omega), \]

where we have introduced new input field annihilation operators $m_{in}(\omega)$ and $n_{in}(\omega)$ for the additional continua (satisfying the canonical commutation relations). These continua satisfy the same form of input-output relations as in (16)

\[
\begin{align*}
  n_{out}(\omega) &= m_{in}(\omega) + \sum_i \sqrt{i}\mu_i c_i(\omega) \\
  m_{out}(\omega) &= n_{in}(\omega) + \sum_i \sqrt{i}\mu_i c_i(\omega).
\end{align*}
\]

When there are side channels and all decay rates in the system are homogenous ($\mu_i = \mu$, $\nu_i = \nu$, $\gamma_i = \gamma$, and $\Gamma_i = \Gamma$), it is impossible to achieve perfect transmission at any input frequency without adding additional excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and posing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering). To see this, consider taking expectation values and imposing excitations (active filtering).

It is still possible that side channels with inhomogeneous coupling $\mu_i$ and $\nu_i$ could yield a system that perfectly transmit some light at a frequency $\omega'$ satisfying all three conditions: $b_{out}(\omega') = a_{in}(\omega') - \sum_i \sqrt{i}\gamma_i c_i(\omega') = 0$, $\sum_i \sqrt{i}\mu_i c_i(\omega') = 0$, and $\sum_i \sqrt{i}\mu_i c_i(\omega') = 0$. After all, the $c_i(\omega')$ with resonant frequencies above and below $\omega'$ differ in phase by $\pi$ and could in principle cancel out. However, this would require incredible fine-tuning of the system.

For a uniformly inhomogeneous parallel network such as (22), we see that this approach fails even in the low-loss limit on resonance: we find $m_{out}(\omega_j) = \frac{2\mu_j}{\gamma_j + \Gamma_j} b_{in}(\omega_j)$ and $n_{out}(\omega_j) = \frac{2\nu_j}{\gamma_j + \Gamma_j} b_{in}(\omega_j)$. In general, output channels outside your control lead to loss.

Similarly, we can determine from (32) and (33) that, in general, input channels outside your control lead to extra dark counts; photons that are in initially populated side channel can end up in the monitored continuum. If the side channel is at a finite temperature, there will in general be thermal photons that could populate the system. The probability of this vanishes for $k_{B}T \ll \hbar \omega_i \forall i$, but will generally be non-negligible for high-temperature systems. The specific contribution depends on the specific conditions and dark counts.

V. CONCLUSIONS

We have studied the behavior of coherent quantum networks and have found that they provide a diverse structure of transmission functions for modeling the first stage of single photon detectors (transmission of a single excitation from the input continuum, through the system, to a monitored output continuum). Inefficiencies and dark counts can be modeled through the incorporation of additional continua (side channels). While we do not find fundamental limits to transmission efficiency, spectral bandwidth, or frequency-dependent group delay across all the studied networks (series, parallel, and hybrid), we do find that some networks are better suited to certain applications than others, as we will now discuss.

Perfect Transmission.—Perhaps the most important metric for a photo detector is detection efficiency. For a loop and side-channel-free network of discrete states with arbitrary configuration and relative detunings, we are always able to find conditions such that perfect transmission ($|T(\omega_i)| = 1$) is achieved for some frequency $\omega_i$ (or multiple frequencies). These either amount to requiring the network to have balanced decays ($\gamma = \Gamma$) or to be critically coupled ($g = \sqrt{\gamma \Gamma}$), and often either will work (35). That finding conditions such that $|T(\omega)| = 1$ is always possible indicates that perfect photo detection is possible in a wide variety of systems.

If one additionally wants a broadened transmission spectrum at a particular perfectly transmitted frequency—so that a broad range of frequencies is de-
tected almost perfectly—we similarly find a variety of ways to accomplish it. One way is to use a parallel network distributed over a small range of states. However, this will result in a number of dark states which will not be detected at all (spectral hole-burning). Instead, it is better to use a series network that meets both the critical coupling and balanced decay conditions, resulting in a maximally broadened on resonance transmission function as seen in Fig. 11.

Spectral Bandwidth.— We do not find fundamental limits to the minimum or maximum bandwidth of a network $\Gamma$ (or, conversely, to the interaction time between a network and incident light $\Gamma^{-1}$), but that it is generally proportional to the decay rates to both the input and monitored continua ($\Gamma \propto \frac{\gamma_i^2}{\gamma_i + \Gamma_i}$), assumed here to be homogenous across states). We also find that some network structures are more suited to high-bandwidth applications than others: for networks with equivalent decay rates, we generally find that $\Gamma_{\text{series}} \leq \Gamma_{\text{simple}} \leq \Gamma_{\text{parallel}}$. For a series network, equality with the upper limit of $\Gamma_{\text{simple}} = \frac{2\gamma_i \Gamma_i}{\gamma_i + \Gamma_i}$ is reached only in the strong-coupling limit, and the lower limit ($\Gamma = 0$) corresponds to a completely de-coupled or infinitely-detuned network (so that a photon can never pass through). For a parallel network, the bandwidth is always given $\Gamma_{\text{parallel}} = \sum_i = \frac{2\gamma_i^2 \Gamma_i}{\gamma_i + \Gamma_i}$ regardless of detuning, so the lower limit simply corresponds to a single discrete state (reproducing the simple model). Unlike both series and hybrid networks where the spectral bandwidth decreases with detuning, the spectral bandwidth of a parallel network is independent of detuning. This makes parallel networks the better candidate for implementation of broadband single photon detection where the frequencies that need to be detected are distinct (so the spectral hole-burning problem is a non-issue).

Considering hybrid networks, we find that their bandwidths are bounded above and below by parallel and series networks, respectively; given a series network with the same coupling strengths and manifold-number, and a parallel network with the same manifold resonance structure and decay rates, the bandwidth of a Hybrid network is bounded $\Gamma_{\text{series}} \leq \Gamma_{\text{hybrid}} \leq \Gamma_{\text{parallel}}$. The upper limit for a hybrid network is approached only in the limit of strong coupling, and the lower limit is approached when there is only a single discrete state in each manifold.

Group Delay.— The maximum magnitude of frequency-dependent group delay $\tau(\omega)$ increases with both the couplings between discrete states and the density of oscillations in the transmission function $T(\omega)$. In particular, the bounded-box structure of a uniformly coupled series networks yields large delays near $\omega = \pm 2g$. We observe that a negative group delay may occur for series networks with relative detuning between the discrete states. (For more on negative group delay, see Ref. [32].) For general networks, the group delay can vary immensely over the range of frequencies where $|T(\omega)|^2$ is non-negligible, with the effect being strongest for hybrid networks (where the resonance structures can be most dense). This means we can expect dispersion effects to be substantial in many photo detection platforms. For applications such as frequency discriminating delay-lines, we may expect hybrid networks to be the best performing candidate as they allow for the finest control of group delay structure.

Tradeoffs.— For an arbitrary series or hybrid network with arbitrary decay rates, perfect transmission requires a particular choice of the couplings. This critical value is generically of order $\sqrt{\frac{\gamma}{2\gamma_i}}$. However, the spectral bandwidth is maximized when $g \gg \sqrt{\frac{\gamma}{2\gamma_i}}$. Furthermore, the magnitude and location of the maximum group delays depends strongly on the coupling $g$ (and especially so in the high-$N$ limit). So for series and hybrid networks, there is a clear tradeoff between efficient transmission and the spectral bandwidth (which saturates in the high-$g$ limit $\sqrt{\gamma}$ with the group delay changing as well. We contrast this with the case of parallel networks where the three quantities are completely independent: perfect detection requires balanced decays ($\gamma_i = \Gamma_i$), the bandwidth can be directly scaled by scaling both decay rates together, and the frequency-dependent group delay $\tau(\omega)$ alone depends on the detuning between discrete states, which is an independent quantity. So for photo detectors where all three quantities must be determinable independently, parallel networks form the clear contenders. The exception to this are situations where a negative group delay is required, which parallel networks never exhibit. Then the use of a hybrid or series network is unavoidable, as are their accompanying tradeoffs.

Final Remarks.— We have studied a variety of quantum networks to uncover tradeoffs and limits fundamental to single photon detection that may arise in the first step of photo detection (filtering). The spectral bandwidth is the only main quantity of interest where there appear to be fundamental limits for particular classes of networks (parallel, series, and hybrid), but even these scale with the relevant decay rates for the networks. Given the freedom to adjust coupling strengths and decay rates, it appears that arbitrary group delay, transmission efficiency, and spectral bandwidth are attainable for any network so long as they are attained individually (together there may be tradeoffs for series and hybrid networks).

We do not find fundamental limitations to dark counts and losses in this analysis: in general uncontrolled input channels lead to dark counts and uncontrolled output channels lead to loss. These can always be mitigated by cooling the system ($k_B T \ll \hbar \omega$) and reducing coupling to side channels ($\mu_i, \nu_i \ll \gamma_i + \Gamma_i$).

Other sources of noise, such as from signal amplification (for more on this specifically, see Ref. [1]), classical parameter fluctuations, as well as the noise inherent in a non-ideal quantum measurement (as described by an arbitrary photo detection POVM) can also be included in our model to give a fully quantum description of the
entire single photon detection process.

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splitting of one peak to two when the number of manifolds $M$ is even.

[34] It’s worth reiterating here that all networks reduce to (18) in the very-weak coupling limit $2g_{ij} \ll \sqrt{\pi \gamma_j + \sqrt{1/T_j}} \forall i,j$.

[35] The only exception being $N = 2$ discrete states in series with non-zero relative detuning and balanced decays ($\gamma = \Gamma$), where no coupling strength can result in perfect transmission as the critical value of $g$ goes to infinity with the difference between the decays.

[36] D. Solli, R. Y. Chiao, and J. M. Hickmann, Physical Review E 66 (2002).

[37] Of course, this assumes a large spectral bandwidth is a desired. There is always an obvious tradeoff between the spectral bandwidth and the amount one learns about the incoming light; the more frequencies that can be detected efficiently, the less a single click tells you about the incoming light. Whether a high bandwidth or a low bandwidth is preferred will ultimately depend on the application of a SPD.

[38] C. Baxter, Journal of Physics B: Atomic, Molecular and Optical Physics 25, L589 (1992).

[39] S. J. van Enk, Physical Review A 96 (2017).

[40] K. Molmer and Y. Castin, Quantum and Semiclassical Optics: Journal of the European Optical Society Part B 8, 49 (1996).

**Appendix A: Photo detection POVM**

So far in this paper, we have not discussed in detail what happens to the excitation after it ends up in the output mode $b_{out}$, except that it gives rise to a macroscopic photo detector click. Following Ref. [39], we can construct the full POVM describing a photo detector from which all standard figures of merit can be obtained [10]. Assume that we finalize the photo detection process by simply ascertaining at some time $t$ whether the excitation is indeed in $b_{out}$ and also that the amplification is ideal and lossless so that, if a photon makes it to the monitored continuum, it will be detected (this assumption is revisited in more depth in [11]). We can then define normalized filtered photon states

$$| T \phi_i \rangle = \frac{1}{\sqrt{\pi \Gamma}} \int_{-\infty}^{\infty} d\omega \, T^* (\omega) \, e^{i \omega t} \, \tilde{a}^\dagger (\omega) \, | \text{vac} \rangle .$$  \hspace{1cm} (A1)

From the quantum jump method [10, 40], we know that a quantum jump from the last manifold of discrete states to the monitored continuum will occur in an infinitesimal time $dt$ with condition probability $\hat{\Gamma} dt$. We can then infer the POVM element for detecting a photon at a particular time $t$ after the photo detector has been on for a time $dt$

$$\hat{\Pi}_t = \frac{\hat{\Gamma}}{2} \, dt \, | T \phi_i \rangle \langle T \phi_i | .$$  \hspace{1cm} (A2)

(To reiterate, here $t$ refers to the time of detection, not the time evolution of the input state.) The probability of getting a click for a normalized input photon $\hat{\rho}$ is $\text{Tr} (\Pi_t \rho) \leq 1$. So the input state that will be detected with maximum probability is $\rho = | T \phi_i \rangle \langle T \phi_i |$, but this yields an infinitesimal probability of detection of $\frac{\hat{\Gamma} dt}{\tau}$. To remedy this, we consider the time-integrated POVM element, where a click occurs at some time before $t = 0$ and after the photo detector was turned on at some time in the past $t = -\tau$. Then the time-integrated POVM element is

$$\Pi_\tau = \int_{-\tau}^{0} dt \, \frac{\hat{\Gamma}}{2} \, | T \phi_i \rangle \langle T \phi_i |$$

$$\approx \int_{-\infty}^{\infty} d\omega \, | T (\omega) |^2 \, | \omega \rangle \langle \omega | \, (\tau \gg \hat{\Gamma}^{-1})$$  \hspace{1cm} (A3)

where, for $\tau \to \infty$, the projectors $| \omega \rangle \langle \omega |$ are truly monochromatic because no timing information is obtained. With the time-integrated form in (A3), we can see that the conditions for perfect transmission discussed above correspond to perfect detection in the limit of $\tau \gg \hat{\Gamma}^{-1}$ as we used previously in (4). Physically, having a non-infinitesimal integration time $\tau$ means that the photo detector has been left on long enough to interact with the system before we check whether there was a click. This is in agreement with our observation that, for the simple model in (4), $\tau_g (\omega_0) = \hat{\Gamma}^{-1}$: we must at least allow enough time for an on-resonance photon to travel through the device to achieve perfect detection. (A3) is also illustrative of what the POVM element represents: the information a detector click reveals about what led up to it.

We can also use (A3) to construct the POVM element for not getting a click in the finite time $\tau$, which is simply $\Pi_0 = 1 - \Pi_\tau$, such that the full POVM $\{ \Pi_0, \Pi_\tau \}$ forms a partition of unity for the relevant Hilbert space; that is, the Hilbert space spanned by single photon states and the vacuum. (See Ref. [16] for inclusion of general photon Fock states in the photo detection POVM.)