A ZERO-DIMENSIONAL APPROACH TO HERMITIAN CODES

EDOARDO BALLICO

Department of Mathematics, University of Trento
Via Sommarive 14, 38123 Povo (TN), Italy

ALBERTO RAVAGNANI

Department of Mathematics, University of Neuchâtel
Rue Emile-Argand 11, CH-2000 Neuchâtel, Switzerland

ABSTRACT. We study the algebraic geometry of a family of evaluation codes from plane smooth curves defined over any field. In particular, we provide a cohomological characterization of their dual minimum distance. After having discussed some general results on zero-dimensional subschemes of the plane, we focus on the interesting case of Hermitian s-point codes, describing the geometry of their dual minimum-weight codewords.

1. INTRODUCTION

Let $F$ be any finite field and let $n \geq 1$ be an integer. A linear code of length $n$ and dimension $k$ over $F$ is a $k$-dimensional vector subspace $C \subseteq F^n$. The elements of $C$ are called codewords. For any $v, w \in C$ define the distance between $v$ and $w$ by $d(v, w) := |\{1 \leq i \leq n : v_i \neq w_i\}|$. The weight of a codeword $v \in C$ is defined as $wt(v) := d(v, 0)$. The minimum distance of a code $C \subseteq F^n$ of at least two elements is the positive integer $d(C) = \min_{v \neq w \in C} d(v, w)$.

A code of minimum distance $d$ corrects $\lfloor (d - 1)/2 \rfloor$ errors: the higher is the minimum distance, the higher is the correction capability. Define the component-wise product in $F^n$ by $v \cdot w := \sum_{i=1}^{n} v_i w_i$. The dual code of $C$ is the $(n - k)$-dimensional code $C^\perp := \{u \in F^n : u \cdot v = 0 \text{ for any } v \in C\}$.

Definition 1. We say that codes $C, D \subseteq F^n$ are strongly isometric if the codewords of $C$ are obtained multiplying component-wise the codewords of $D$ by a vector of $F^n$ whose components are all nonzero.

Remark 2. The strong isometry is an equivalence relation on the set of codes in $F^n$. Strongly isometric codes have the same dimension and the same minimum distance. Moreover, a strong isometry preserves the support of the codewords and, in particular, the number of minimum-weight codewords of a code. Codes $C$ and $D$ are strongly isometric if and only if their dual codes $C^\perp$ and $D^\perp$ are strongly isometric.

Let $\mathbb{P}^r$ be the projective $r$-dimensional space over $F$, and let $C \subseteq \mathbb{P}^r$ be a connected smooth curve defined over $F$. Assume that $C$ is a complete intersection. Choose any subset $B \subseteq C(F)$ of $F$-rational points of $C$ and
an integer \( d > 0 \). Finally, consider the linear map

\[ \text{ev} : H^0(C, \mathcal{O}_C(d)) \to \mathbb{F}^{|B|} \]

("\( |B| \) denotes the cardinality of \( B \)) which evaluates a degree \( d \) homogeneous form on \( C \) at the points appearing in \( B \). Being a vector subspace of \( \mathbb{F}^{|B|} \), the image of \( \text{ev} \) say \( \mathcal{C} \), is a linear code of length \( |B| \) over the finite field \( \mathbb{F} \).

Recently, A. Couvreur showed in [5] that a lower bound on the minimum distance of \( \mathcal{C}^\perp \) can be expressed in terms of \( d \) and the projective geometry of \( B \) (for instance, the existence in \( B \) of \( d + 2 \) collinear points). Codes arising from geometric constructions are known to have good parameters for applications and a wide literature on the topic is available (see in particular [18] and [19]).

In this paper we focus on the case \( r = 2 \) of the described approach (i.e., on the case of plane smooth curves) and provide an improvement of Couvreur’s method in this specific context. More precisely, we introduce zero-dimensional schemes in the setup, and study codes obtained evaluating vector spaces of the more general form \( H^0(C, \mathcal{O}_C(d)(-E)) \), where \( E \subseteq \mathbb{P}^2 \) is a zero-dimensional scheme whose support avoids the set \( B \) in the notation above. This class of codes includes many classical Goppa codes (see Remark 14 at page 5). Then we apply the results for arbitrary curves to the special case of codes from the Hermitian curve, providing a geometric characterization of the dual minimum-weight codewords of many Hermitian \( s \)-point codes (see [18], Chapter 10 for the definitions).

1.1. Layout of the paper. The paper is organised in three main parts. In Section 2 we characterize the dual minimum distance of codes arising from smooth plane curves, and establish a key lemma to control zero-dimensional plane schemes from a cohomological point of view. In Section 3 we describe some geometric properties of Hermitian \( s \)-point codes. In Section 4 we prove our main results on Hermitian \( s \)-point codes.

1.2. Main references. One-point codes from the Hermitian curve are well-studied, and efficient methods to decode them are known ([19], [20] and [21]). The minimum distance of Hermitian two-point codes has been first determined by M. Homma and S. J. Kim ([9], [10], [11], [12]) and more recently S. Park gave explicit formulas for the dual minimum distance of such codes (see [17]) using different techniques. The second and the third Hamming weight of one-point codes on the Hermitian curve are studied in [15], [21] and [16]. The second Hamming weight of Hermitian two-point codes is treated in [13].

2. Results on arbitrary plane smooth curves

In this section we study the algebraic geometry of codes obtained evaluating vector spaces of homogeneous forms at some prescribed points of a plane smooth curve. The general results of this section will be applied to interesting examples in later sections.

**Remark 3.** The definition of linear code (see Section 1) can be given over any field \( \mathbb{F} \) (and not only over finite fields). The minimum distance is defined in the same way. We use this extended definition in this section.

**Remark 4.** Fix any field \( \mathbb{F} \), a projective scheme \( T \) and a coherent sheaf \( \mathcal{F} \) on \( T \) defined over \( \mathbb{F} \). Let \( \overline{\mathbb{F}} \) be the algebraic closure of \( \mathbb{F} \). Consider the scheme \( T_{\overline{\mathbb{F}}} \) and the coherent sheaf \( \mathcal{F}_{\overline{\mathbb{F}}} \) obtained by the extension of scalars \( \mathbb{F} \supseteq \overline{\mathbb{F}} \). The definition and main properties of cohomology groups can be found in [7], Chapter II and III, for arbitrary schemes and over an arbitrary field. Since every extension of fields \( \mathbb{F} \supseteq \mathbb{F} \) is flat, for each integer \( i \geq 0 \) the \( \mathbb{F} \)-vector space \( H^i(T, \mathcal{F}) \) and the \( \overline{\mathbb{F}} \)-vector space \( H^i(T_{\overline{\mathbb{F}}}, \mathcal{F}_{\overline{\mathbb{F}}}) \) have the same dimension (see [7], Proposition III.9.3).

**Lemma 5.** Let \( \mathbb{F} \) be any field and let \( \mathbb{P}^2 \) denote the projective plane on \( \mathbb{F} \). Let \( C \subseteq \mathbb{P}^2 \) be a smooth plane curve. Fix an integer \( d > 0 \), a zero-dimensional scheme \( E \subseteq C \) and a finite subset \( B \subseteq C \) such that \( B \cap \text{red} E = \emptyset \). Denote by \( \mathcal{C} \) the code obtained evaluating the vector space \( H^0(C, \mathcal{O}_C(d)(-E)) \) at the points of \( B \). Set \( c := \deg(C), n := |B| \) and assume \( |B| > dc - \deg(E) \). The following facts hold.

---

2 Here \( E_{\text{red}} \) denotes the reduction of the scheme \( E \).
(1) The dimension of $H^0(C, \mathcal{O}_C(d))$ is given by the formulas
\[
h^0(C, \mathcal{O}_C(d)) = \begin{cases} \frac{(d+2)2}{2} & \text{if } d < c, \\
\frac{(d-c+2)2}{2} & \text{if } d \geq c. \end{cases}
\]

(2) The code $\mathcal{C}$ has length $n$ and dimension $k := h^0(C, \mathcal{O}_C(d)) - \deg(E) + h^1(\mathbb{P}^2, \mathcal{I}_E(d))$.

(3) The minimum distance of $\mathcal{C}^\perp$ is the minimal cardinality, say $z$, of a subset of $S \subseteq B$ such that
\[
h^1(\mathbb{P}^2, \mathcal{I}_{S\cup E}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d)).
\]

(4) A codeword of $\mathcal{C}^\perp$ has weight $z$ if and only if it is supported by a subset $S \subseteq B$ such that
(a) $|S| = z$,
(b) $h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d))$,
(c) $h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S'}(d))$ for any $S' \supseteq S$.

**Proof.** We will make implicit use of Remark 4 throughout the proof. The computation of $h^0(C, \mathcal{O}_C(d))$ is well-known. The technical condition $|B| > d_c - \deg(E)$ assures that $h^0(C, \mathcal{O}_C(d)(-E - B)) = 0$. Hence $\mathcal{C}$ has length $n$ and in fact dimension $k$. In the case $E = \emptyset$ the computation of the minimum distance of $\mathcal{C}^\perp$ is just the planar case of [5], Proposition 3.1. In the general case notice that $\mathcal{C}$ is obtained evaluating a family of homogeneous degree $d$ polynomials on the curve $C$ (the ones vanishing on the scheme $E$) at the points of $B$. Since $C$ is projectively normal (it is a plane smooth curve), the restriction map $\rho_d : H^0(\mathbb{P}^2, \mathcal{I}_{E\cup z}(d)) \to H^0(C, \mathcal{O}_C(d))$ is surjective. Hence the restriction map $\rho_{d,E} : H^0(\mathbb{P}^2, \mathcal{I}_E(d)) \to H^0(C, \mathcal{O}_C(d)(-E))$ is surjective. Hence a finite subset $S \subseteq C \setminus E_{\text{red}}$ imposes independent condition to $H^0(C, \mathcal{O}_C(d)(-E))$ if and only if $S$ imposes independent conditions to $H^0(\mathbb{P}^2, \mathcal{I}_E(d))$. Moreover, $S$ imposes independent conditions to $H^0(\mathbb{P}^2, \mathcal{I}_E(d))$ if and only if $h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S}(d)) = h^1(\mathbb{P}^2, \mathcal{I}_E(d))$ (here we use again that $S \cap E_{\text{red}} = \emptyset$). To get the existence of a non-zero codeword of $\mathcal{C}^\perp$ whose support is $S$ (and not only with support contained in $S$) we need that the submatrix $M_S$ of the generator matrix of $\mathcal{C}$ obtained by considering the columns associated to the points appearing in $S$ has the property that each of its submatrices obtained deleting one column have the same rank of $M_S$ (each such column is associated to some $P \in S$ and we require that the codeword has support containing $P$). This is equivalent to the last claim in the statement. \hfill \Box

**Remark 6.** Notice that we may drop the assumption
\[
h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S'}(d))
\]
in the statement of Lemma 5 if there is no subset $A \subseteq B$ with the properties $|A| < z$ and $h^1(\mathbb{P}^2, \mathcal{I}_{E\cup A}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d))$. This is the case when $z$ is smaller or equal then the Hamming distance of $\mathcal{C}^\perp$.

**Remark 7.** Take the set-up of the proof of Lemma 5. Since both the restriction maps $\rho_d$ and $\rho_{d,E}$ are surjective, $h^1(\mathbb{P}^2, \mathcal{I}_{E\cup S}(d)) > h^1(\mathbb{P}^2, \mathcal{I}_E(d))$ is equivalent to $h^0(C, \mathcal{O}_C(d)(-E \cup S)) > h^0(C, \mathcal{O}_C(d)(-E)) - |S|$ or, equivalently (Riemann-Roch theorem), to $h^1(C, \mathcal{O}_C(d)(-E \cup S)) > h^1(C, \mathcal{O}_C(d)(-E))$. In the applications we will usually have $d \leq \deg(C) - 3$ and so $h^1(C, \mathcal{O}_C(d)) > 0$.

**Notation 8.** Let $F$ be any field and $\mathbb{P}^2$ the projective plane over $F$. Let $Z \subseteq \mathbb{P}^2$ be any zero-dimensional scheme. Fix a curve $T \subseteq \mathbb{P}^2$ and set $t := \deg(T)$ (here we do not assume that $T$ is reduced, it may even have multiple components). The residual scheme $\text{Res}_T(Z)$ of $Z$ with respect to the divisor $T$ is defined to be the closed subscheme of $\mathbb{P}^2$ with $\mathcal{I}_Z : \mathcal{I}_T$ as its ideal sheaf. From the general theory of ideal sheaves we have $\text{Res}_T(Z) \subseteq T$ and $\deg(Z) = \deg(T \cap Z) + \text{Res}_T(Z)$. If $Z$ is reduced, i.e. if $Z$ is a finite set, then $\text{Res}_T(Z) = Z \setminus Z \cap T$.

The following Lemma 9 is a key point in the improvement of the method of [5] in the case of plane curves. More precisely, we provide a cohomological control over the zero-dimensional scheme $E$ introduced in Section 1. The lemma is in fact a schematic version of [6], Corollaire 2. Parts (a) and (b) also follow in an arbitrary projective space from [4], Lemma 34. Parts (b), (c), and part (d) are just [6], Remarques at page 116.

**Lemma 9.** Let $F$ be an algebraically closed field and $\mathbb{P}^2$ the projective plane over it. Choose an integer $d > 0$ and a zero dimensional scheme $Z \subseteq \mathbb{P}^2$. Set $z := \deg(Z)$. The following facts hold.
(a) If \( z \leq d + 1 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0 \).

(b) If \( d + 2 \leq z \leq 2d + 1 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if there exists a line \( T_1 \) such that \( \deg(T_1 \cap Z) \geq d + 2 \).

(c) If \( 2d + 2 \leq z \leq 3d - 1 \) and \( d \geq 2 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if either there exists a line \( T_1 \) such that \( \deg(T_1 \cap Z) \geq d + 2 \), or there exists a conic \( T_2 \) such that \( \deg(T_2 \cap Z) \geq 2d + 2 \).

(d) Assume \( z = 3d \) and \( d \geq 3 \). Then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if either there exists a line \( T_1 \) such that \( \deg(T_1 \cap Z) \geq d + 2 \), or there exists a conic \( T_3 \) such that \( \deg(T_3 \cap Z) \geq 2d + 2 \), or there exists a plane cubic \( T_3 \) such that \( Z \) is the complete intersection of \( T_3 \) and a plane curve of degree \( d \). In the latter case, if \( d \geq 3 \) then \( T_3 \) is unique and we may find a plane curve \( C_d \) with \( Z = T_3 \cap C_d \).

(e) Assume \( z \leq 4d - 5 \) and \( d \geq 4 \). Then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if either there exists a line \( T_1 \) such that \( \deg(T_1 \cap Z) \geq d + 2 \), or there exists a conic \( T_2 \) such that \( \deg(T_2 \cap Z) \geq 2d + 2 \), or there exist a subscheme \( W \subseteq Z \) with \( \deg(W) = 3d \) and plane cubic \( T_3 \) such that \( W \) is the complete intersection of \( T_3 \) and a plane curve of degree \( d \), or there is a plane cubic \( C_d \) such that \( \deg(C_d \cap Z) \geq 3d + 1 \).

**Proof.** Since \( Z \) is a zero-dimensional scheme, for every \( d \in \mathbb{Z} \) and any closed subscheme \( W \subseteq Z \) we get \( h^1(Z, \mathcal{I}_W(d)) = 0 \). Hence the restriction map \( h^0(Z, \mathcal{I}_W(d)) \to h^0(W, \mathcal{I}_W(d)) \) is surjective. As a consequence, if \( h^1(\mathbb{P}^2, \mathcal{I}_W(d)) > 0 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \). Take any integer \( y \in \{1, \ldots, d - 1\} \) and any degree \( y \) plane curve \( D_y \) (we allow \( D_y \) to have even multiple components). Set \( W := D_y \cap Z \). From the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}^2}(d - y) \to \mathcal{O}_{\mathbb{P}^2}(d) \to \mathcal{O}_{D_y}(d) \to 0
\]

we get \( h^0(D_y, \mathcal{O}_{D_y}(d)) = (\frac{d + 2}{2} - \frac{d + 3}{2} + y) \) and that the restriction map \( \rho : h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \to h^0(D_y, \mathcal{O}_{D_y}(d)) \) is surjective. Hence if \( h^0(D_y, \mathcal{I}_W(d)) > (\frac{d + 2}{2} - \frac{d + 3}{2} + y) - \deg(W) \), then \( h^1(\mathbb{P}^2, \mathcal{I}_W(d)) > 0 \) and hence \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \). Since \( h^0(D_y, \mathcal{I}_W(d)) \geq 0 \), we will have \( h^1(\mathbb{P}^2, \mathcal{I}_W(d)) > 0 \) if \( \deg(W) > (\frac{d + 2}{2} - \frac{d + 3}{2} + y) \). For \( y = 1, 2 \) it is sufficient to assume \( \deg(W) \geq yd + 2 \). For \( y = 3 \) it is sufficient to assume \( \deg(W) \geq 3d + 1 \). Now take \( y = 3 \) and \( \deg(W) = 3d \). We have \( h^0(D_3, \mathcal{I}_W(d)) > (\frac{d + 2}{2} - \frac{d + 3}{2} + 3) - \deg(W) \) if and only if \( h^0(D_3, \mathcal{I}_W(d)) > 0 \), i.e. (by the surjectivity of \( \rho \)) if and only if there exists a degree \( d \) plane curve \( C_3 \) such that \( W = D_3 \cap C_3 \). Hence in parts (b), (c), (d) and (e) we proved the “if” part. In the remaining part of the proof we check part (a) and the “only if” part of (b), (c), (d) and (e). Let \( \tau \) be the maximal integer such that \( h^1(\mathbb{P}^2, \mathcal{I}_Z(\tau)) > 0 \) (such a \( \tau \) of course exists, because \( h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = 0 \) for \( t \gg 0 \), by a famous theorem of Serre). By assumption we have \( \tau \geq d \). Fix a positive integer \( s \in \{1, 2, 3, 4\} \) and assume \( \tau \geq s - 3 + z/s \). By [4], Corollaire 2, either \( \tau = s - 3 + z/s \) and \( Z \) is the complete intersection of a degree \( s \) plane curve and a degree \( \tau \) curve, or there are \( W \subseteq Z \) and an integer \( t \in \{1, \ldots, \tau - 1\} \) such that \( \deg(W) \geq t(\tau - t + 3) \) and \( W \) is contained in a plane curve of degree \( t \).

(i) Parts (a) and (b) are just the plane case of [4], Lemma 34.

(ii) Now assume \( d + 2 \leq z \leq 2d + 1 \). Since \( \tau \geq d \), we have \( \tau \geq d + 2 \). Take \( s = 1 \). Since \( \tau \geq d \), we have \( \tau \geq 1 + 3 + z \). Hence we may apply [4], Corollaire 2, and get the existence of \( W \subseteq Z \) and a line \( T_1 \) such that \( \deg(W) \geq d + 2 \) and \( W \subseteq T_1 \).

(iii) Now assume \( 2d + 2 \leq z \leq 3d \). Since \( \tau \geq d \geq 3 - 3 + z/3 \), we can apply [4], Corollaire 2, with the integer \( s := 3 \) and get parts (c) and (d).

(iv) Finally assume \( 3d + 1 \leq z \leq 4d - 5 \). Since \( \tau \geq d \geq 4 - 3 + z/4 \), by [4], Corollaire 2, we get part (e).

\[ \square \]

In the next section we study the algebraic geometry of codes arising from a Hermitian curve \( X \) defined over a finite field \( \mathbb{F}_{q^2} \) (briefly, of Hermitian codes). In particular, we will employ the well-known characterization of the tangent lines to \( X \) in order to reveal a precise cohomological structure in the dual minimum-weight codewords of such codes.

### 3. The Hermitian curve

In this section \( q \) denotes a prime power and \( \mathbb{F}_{q^2} \) is the finite field with \( q^2 \) elements. We denote by \( \mathbb{P}^2 \) the projective plane over the field \( \mathbb{F}_{q^2} \) of projective coordinates \((x : y : z)\). The Hermitian curve \( X \subseteq \mathbb{P}^2 \) is defined
to be the zero locus of the polynomial \( y + y^2 = x^d + 1 \) (as an affine equation). This curve is clearly defined over \( \mathbb{F}_q \). Its function field is studied in [19], Example 6.3.6. From a geometric point of view, \( X \) is known to be a smooth curve of genus \( g(X) = q(q - 1)/2 \) given by the genus-degree formula. The projective geometry of tangent lines to \( X \) is completely known and summarized in the following two results.

**Lemma 10.** Let \( X \) be the Hermitian curve. Every line \( L \) of \( \mathbb{P}^2 \) either intersects \( X \) in \( q + 1 \) distinct \( \mathbb{F}_q \)-rational points, or \( L \) is tangent to \( X \) at a point \( P \) (with contact order \( q + 1 \)). In the latter case \( L \) does not intersect \( X \) in any other point different from \( P \).

**Proof.** See [8], part (i) of Lemma 7.3.2, at page 247. □

**Lemma 11.** Fix an integer \( e \in \{2, \ldots, q + 1\} \) and \( P \in X(\mathbb{F}_q) \). Let \( E \subset X \) be the divisor \( eP \), seen as a closed degree \( e \) subscheme of \( \mathbb{P}^2 \). Let \( T \subset \mathbb{P}^2 \) any effective divisor (i.e. a plane curve, possibly with multiple components) of degree smaller or equal than \( e - 1 \) and such that the contact order of \( X \) and \( T \) at \( P \) is at least \( e \). Then \( L_{X,P} \subset T \), i.e. \( L_{X,P} \) is one of the components of \( T \).

**Proof.** Since \( L_{X,P} \) has order of contact \( q + 1 \geq e \) with \( X \) at \( P \), we have \( E \subset L_{X,P} \). Since \( \deg(E) > \deg(T) \) and \( E \subset T \cap L_{X,P} \), Bezout theorem implies \( L_{X,P} \subset T \). □

The Hermitian curve carries \( |X(\mathbb{F}_q)| = q^3 + 1 \) rational points ([19], p. 250, or [8]). It follows that \( X \) is a maximal curve, in the sense of the Hasse-Weil bound (see [18], Chapter 6).

**Notation 12.** For any point \( P \in X(\mathbb{F}_q) \) we denote by \( L_{X,P} \subset \mathbb{P}^2 \) the tangent line to \( X \) at \( P \). Clearly, \( L_{X,P} \) is a line defined over \( \mathbb{F}_q \).

**Lemma 13.** Fix integers \( d, s \) such that \( d \geq s \geq 1 \). Choose \( s \) distinct points \( P_1, \ldots, P_s \in X(\mathbb{F}_q) \) and \( s \) integers \( b_1, \ldots, b_s \) such that \( 0 \leq b_i \leq d - 2 - i \) and \( b_i \leq q + 1 \) for any \( i \in \{1, \ldots, s\} \). Let \( E := \sum_{i=1}^s b_i P_i \) be a degree \( b_1 + \cdots + b_s \) effective divisor of \( X \), seen also as a degree \( b_1 + \cdots + b_s \) zero-dimensional subscheme of \( \mathbb{P}^2 \). Then we have \( h^1(\mathbb{P}^2, \mathcal{I}_E(t)) = 0 \).

**Proof.** For any integer \( j \in \{1, \ldots, s\} \) set \( E[j] := \sum_{i=1}^j b_i P_i \) and \( E_i := b_i P_i \). Hence \( E[1] = E \) and \( E[i] = \cap_{i \leq j \leq s} E[j] \).

We can see each \( E[i] \) as a degree \( b_1 + \cdots + b_i \) zero-dimensional subscheme of \( \mathbb{P}^2 \). Since \( L_{X,P} \) has order of contact \( q + 1 \) with \( X \) at \( P_i \) and \( b_i \leq q + 1 \), we have \( E_i \subset L_{X,P} \). Hence \( E[i+1] = E[i] \cap L_{X,P} \) for any \( i = 1, \ldots, s - 1 \). See \( E[i] \) and \( E[i+1] \) as zero-dimensional subschemes of \( \mathbb{P}^2 \) and \( L_{X,P} \) as a degree 1 curve of \( \mathbb{P}^2 \). Then for any \( t \in \mathbb{Z} \) and any \( i \in \{1, \ldots, s\} \) we get the following exact sequence of coherent sheaves on \( \mathbb{P}^2 \):

\[
0 \rightarrow \mathcal{I}_{E[i+1]}(t-1) \rightarrow \mathcal{I}_{E[i]}(t) \rightarrow \mathcal{I}_{E_i,L_{X,P}}(t) \rightarrow 0
\]

in which we see \( E_i \) as a degree \( b_i \) divisors of \( L_{X,P} \cong \mathbb{P}^1 \). Hence \( h^1(L_{X,P}, \mathcal{I}_{E_i,L_{X,P}}(t)) = 0 \) for any \( t \geq b_i + 1 \). Taking \( t = d \) and \( i = 1 \) in (2) we get \( h^1(\mathbb{P}^2, \mathcal{I}_E(d)) \leq h^1(\mathbb{P}^2, \mathcal{O}_{E[2]}(d-1)) \). If \( s = 1 \) then we are done, because \( E[2] = 0 \) in this case. In the general case we use induction on \( s \). Notice that we may apply the inductive assumption to \( E[2] \) with respect to the integer \( d' := d - 1 \). Hence the inductive assumption gives \( h^1(\mathbb{P}^2, \mathcal{I}_{E[2]}(d-1)) = 0 \). Conclude by using the long cohomology exact sequence of (2) in the case \( t = d \) and \( i = 1 \). □

**Remark 14.** Fix an integer \( s \geq 2 \) and take \( s \) distinct points \( P_1, \ldots, P_s \in X(\mathbb{F}_q) \). Choose integers \( a_1, \ldots, a_s \) and set \( E := \sum_{i=1}^s a_i P_i \), both viewed as a divisor on \( X \) and as a zero-dimensional subscheme \( E \subset \mathbb{P}^2 \). For any \( \mathbb{F}_q \)-rational point \( P \in X(\mathbb{F}_q) \) we have an isomorphism of sheaves \( \mathcal{O}_X((q + 1)P) \cong \mathcal{O}_X(1) \). It follows that there exists a rational function \( f_P \) such that \( (f_P) = (q + 1)P - (q + 1)Q \). Set \( G := d(q + 1)P_1 - E \) and denote by \( \mathcal{L}(G) \) the Riemann-Roch space associated to the divisor \( G \). The codes obtained evaluating at the points of \( B := X(\mathbb{F}_q) \setminus \{P_1, \ldots, P_s\} \) the vector spaces

\[
H^0(X, \mathcal{O}_X(d)(-E)) \quad \text{and} \quad \mathcal{L}(G)
\]

are in fact strongly isometric (see Definition[8]). Notice that codes obtained evaluating a Riemann-Roch space \( \mathcal{L}(G) \) at the rational points of a curve avoiding the support of \( G \) are the famous Goppa codes (see [18] for further details). We will denote by \( \mathcal{C}(B, d, -E) \) the code obtained evaluating the vector space \( H^0(X, \mathcal{O}_X(d)(-E)) \)
on the set \( B \) defined above and by \( \mathcal{C}^{-1}(B, d, -E) \) its dual code. This remark shows that the class of \( \mathcal{C}(B, d, -E) \) codes, here studied, includes many codes of classical interest in geometric coding theory.

In the following result the geometry of tangent lines to \( X \) is explicitly involved in proving that we may restrict, in studying Hermitian codes, to a very particular subclass of them.

**Lemma 15.** Fix integers \( d > 0 \) and \( s \geq 1 \). Choose \( s \) integers \( a_1, \ldots, a_s \in \{1, \ldots, q+1\} \) with \( a_1 \leq \cdots \leq a_s \) and denote by \( r \) be the maximal integer \( i \leq s \) with the property \( a_i \leq d - s + i \). Set \( d' := d - s + r \) and assume \( d' > 0 \). Assume \( d' = d \), \( E' = E \) and so there is noting to prove. Assume \( r < s \), i.e. \( a_r > d \). Take any \( f \in H^0(X, \mathcal{O}_X(d)(-E)) \), which is a degree \( d \) homogeneous polynomial vanishing on the zero-dimensional scheme \( E \subseteq \mathbb{P}^2 \). Fix any \( i \in \{r+1, \ldots, s\} \). Let \( \mathcal{I}_{E_i} \) denote the scheme \( E_i \) and with \( \deg E_i = \deg E_i + 0 \) if and only if there exists a line \( T_1 \) such that \( \deg(T_1 \cap Z) \geq d + 2 \).

Since a tangent line to \( X \) at a rational point \( P \) does not intersect \( X \) at any other rational point, the codes \( \mathcal{C}(B, d, -E) \) and \( \mathcal{C}(B, d', -E') \) are in fact strongly isometric. Conclude by Remark 1.

We notice that Lemma 15 works over an algebraically closed field. The following result improves Lemma 9 for the case of the Hermitian curve over finite fields.

**Lemma 16.** Let \( \mathbb{F}_{q^2} \) be the finite field with \( q^2 \) elements \( (q \text{ a prime power}) \) and denote by \( \mathbb{P}^2 \) the projective plane over the field \( \mathbb{F}_{q^2} \). Let \( X \subseteq \mathbb{P}^2 \) be the Hermitian curve. Choose an integer \( d > 0 \) and a zero-dimensional scheme \( Z \subseteq X(\mathbb{F}_{q^2}) \) of degree \( z > 0 \). The following facts hold.

(a) If \( z \leq d + 1 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) = 0 \).
(b) If \( 2d + 2 \leq z \leq 2d + 1 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if there exists a line \( T_1 \) such that \( \deg(T_1 \cap Z) \geq d + 2 \).
(c) If \( 2d + 2 \leq z \leq 3d - 1 \) and \( d \geq 2 \), then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if there exists a plane cubic \( T_3 \) such that \( \deg(T_3 \cap Z) \geq d + 2 \) or there exists a plane cubic \( T_3 \) such that \( \deg(T_3 \cap Z) \geq 2d + 2 \).
(d) Assume \( z = 3d \) and \( d \geq 3 \). Then \( h^1(\mathbb{P}^2, \mathcal{I}_Z(d)) > 0 \) if and only if there exists a plane cubic \( T_3 \) such that \( \deg(T_3 \cap Z) \geq d + 2 \) or there exists a plane cubic \( T_3 \) such that \( \deg(T_3 \cap Z) \geq 2d + 2 \). Hence (a) holds for \( Z \) over \( \mathbb{F}_{q^2} \), while in the other case we need to inquire whether the curves of low degree claimed in the statement are defined over \( \mathbb{F}_{q^2} \), or not. First assume \( z = 3d \) and that \( Z \) is the complete intersection of a plane curve and a degree \( d \) curve. By Remark 4 the homogeneous ideal of \( Z \) in \( \mathbb{P}^2 \)
is generated by forms defined over $\mathbb{F}_{q^2}$. Hence $Z$ is the complete intersection of a plane cubic defined over $\mathbb{F}_{q^2}$ and a degree $d$ curve defined over $\mathbb{F}_{q^2}$. The other cases are in general more complicated, but in the applications to the Hermitian curve we know more about $Z$: not only $Z$ is defined over $\mathbb{F}_{q^2}$, but each connected component of it is defined over $\mathbb{F}_{q^2}$ and hence $Z_{\text{red}} \subseteq \mathbb{P}_x^2$; moreover, we also know that the lines $D$ with $\deg(D \cap Z) \geq 2$ are defined over $\mathbb{F}_{q^2}$. Hence it is sufficient to notice the following facts.

1. A line of $\mathbb{P}_x^2$ containing two points of $\mathbb{P}_x^2$ is defined over $\mathbb{F}_{q^2}$.
2. A conic of $\mathbb{P}_x^2$ containing 5 points of $\mathbb{P}_x^2$, no 4 of them on a line, is defined over $\mathbb{F}_{q^2}$.
3. Let $C$ be a plane cubic defined over $\mathbb{F}_{q^2}$ and containing a set $S \subseteq \mathbb{P}_x^2$ of at least 12 points. Assume that no 5 of the points of $S$ are contained in a line and no 8 of the points of $S$ are contained in a conic. Then $C$ is defined over $\mathbb{F}_{q^2}$.

\[ \square \]

4. HERMITIAN CODES

In this last section we apply our results and describe the dual minimum distance of many s-point codes on the Hermitian curve (with $s \geq 2$). For $s \geq 3$ this parameter was unknown, except in some particular cases covered by [5]. Until the end of the paper we work over the finite field $\mathbb{F}_{q^2}$ with $q^2$ elements ($q$ fixed) and we will denote by $X \subseteq \mathbb{P}_x^2$ the Hermitian curve defined in Section 5. The following Theorem and Theorem describe Hermitian three-point codes, while Theorem 23 and Theorem 24 deal with the more complicated case of s-point codes with $s \geq 2$ arbitrary.

**Theorem 17.** Fix any three distinct points $P_1, P_2, P_3 \in X(\mathbb{F}_{q^2})$ and assume $P_1, P_2$ and $P_3$ to be not collinear. Set $B := X(\mathbb{F}_{q^2}) \setminus \{P_1, P_2, P_3\}$. Fix an integer $d \geq 5$ such that $1 \leq d \leq q - 1$ and integers $a_1, a_2, a_3 \in \{1, \ldots, d\}$ such that $a_1 + a_2 + a_3 \leq 3d - 5$ and $a_i = d$ for at most one index $i \in \{1, 2, 3, 4\}$. Set $E := a_1P_1 + a_2P_2 + a_3P_3$. Let $\mathcal{C} := \mathcal{C}(B, d, -E)$ be the code obtained evaluating the vector space $H^0(X, \mathcal{O}_X(d)(-E))$ on the set $B$. Then $\mathcal{C}$ is a code of length $n := |B| = q^3 - 2$ and dimension $k := \left(\frac{d^2 + 2}{2}\right) - a_1 - a_2 - a_3$. For any $i \in \{1, 2, 3\}$ let $L_i$ denote the line spanned by $P_i$ and $P_3$ with $i, j, h \in \{1, 2, 3\}$. Then $\mathcal{C}^\perp$ has minimum distance $d$ and its minimum-weight codewords are exactly the ones whose support is formed by $d$ points of $B \cap L_i$ for some $i \in \{1, 2, 3\}$.

**Proof.** The length of $\mathcal{C}$ is obviously $n = |B| = q^3 - 2$. Since $d \leq q < \deg(X)$, we have $h^0(X, \mathcal{O}_X(d)) = \left(\frac{d^2 + 2}{2}\right)$. If, say, $a_1 \geq a_2 \geq a_3$, the assumptions $a_1 \leq d$ and $a_1 + a_2 + a_3 \leq 3d - 5$ give $a_i \leq d - 2 + i$ for all $i$. Hence Lemma 13 implies $h^1(\mathbb{P}_x^2, \mathcal{F}_x(d)) = 0$ and so $h^0(X, \mathcal{O}_X(d)(-E)) = \left(\frac{d^2 + 2}{2}\right) - a_1 - a_2 - a_3 = k$. Since $|B| > d \cdot \deg(X)$, there is not a non-zero element of $H^0(X, \mathcal{O}_X(d))$ vanishes at all the points of $B$. Hence $\mathcal{C}$ has dimension $k$. By Lemma 5 it is sufficient to prove the following two facts.

(a) $h^1(\mathbb{P}_x^2, \mathcal{I}_{E\cup A}(d)) = 0$ for all $A \subseteq B$ such that $|A| \leq d - 1$.
(b) For any $S \subseteq B$ such that $|S| = d$ we have $h^1(\mathbb{P}_x^2, \mathcal{I}_{E\cup S}(d)) > 0$ if and only if $S \subseteq L_i$ for some $i \in \{1, 2, 3\}$. Each line $L_i$ contains $q - 1$ points of $B$, while $\deg(E \cap L_i) = 2$. Hence for any $S \subseteq L_i \cap B$ with $|S| = d$ we have $h^1(\mathbb{P}_x^2, \mathcal{I}_{E\cup S}(d)) > 0$ (see Lemma 16). Let $E_i := a_iP_i$, view as a divisor on $X$. We have $E = E_1 \cup E_2 \cup E_3$. Fix a set $S \subseteq B$ such that $|S| \leq d$ and assume $h^1(\mathbb{P}_x^2, \mathcal{I}_{E\cup S}(d)) > 0$. We have $S \cap \{P_1, P_2, P_3\} = \emptyset$ and $\deg(E \cup S) = a_1 + a_2 + a_3 + |S|$. Since $a_1 + a_2 + a_3 + |S| \leq 4d - 5$, we may apply Lemma 16 to the scheme $E \cup S$. Let $T \subseteq \mathbb{P}_x^2$ be the curve arising from the statement of the lemma. Set $x := \deg(T) \in \{1, 2, 3\}$ and $e_i := \deg(T \cap E_i)$ for $i \in \{1, 2, 3\}$. We have $0 \leq e_i \leq a_i$. If $e_i \geq x + 1$ then Lemma 11 gives $L_{x, P_i} \subseteq T$. Assume $e_i \leq x$ for all $i \in \{1, 2, 3\}$. For $x = 2$ we get $\deg(T \cap (E \cup S)) \leq d + 6 \leq 2d + 1$. For $x = 3$ we get $\deg(T \cap (E \cup S)) \leq d + 9 \leq 3d - 1$. Finally, for $x = 1$ we may have $e_1 > 0$ only for at most two indices, say $i = 1, 2$. Since $|S| \leq d$, we get $|S| + e_1 + e_2 \geq d + 2$ and $|S| + e_1 + e_2 = d + 2$ if and only if $T = L_3, S \subseteq L_3 \cap B$ and $|S| = d$. Now assume that $T$ contains one of the lines $L_{x, P_i}$, say $L_{x, P_1}$. Let $T'$ be the curve whose equation is obtained dividing an equation of $T$ by an equation of $L_{x, P_1}$. We have $\deg(T') = x - 1, T' + L_{x, P_1} = T$ (as divisors of $\mathbb{P}_x^2$) and $T = L_{x, P_1} \cup T'$ (as sets). Since $L_{x, P_1} \cap B = \emptyset$, we have $T \cap S = T' \cap S$ and $\deg(T \cap (E \cup S)) = \deg(T' \cap (E \cup S))$. 

7
Let us compare the results of Theorem 17 with the classical Goppa bound for codes obtained on a line.

Assume $x = 2$. The curve $T'$ must be a line such that $\deg(T' \cap (E_2 \cup E_3 \cup S)) \geq 2d + 2 - a_1$. If either $T' = L_{X,P_1}$ or $T' = L_{X,P_2}$, we get $T' \cap S = \emptyset$ and $\deg(T' \cap (E_2 \cup E_3 \cup S)) \leq \max\{e_2, e_3\} \leq d$, a contradiction. If neither $T' = L_{X,P_1}$ nor $T' = L_{X,P_2}$, then $\deg(T' \cap E_2) \leq 1$, $\deg(T' \cap E_3) \leq 1$ and $\deg(T' \cap (E_2 \cup E_3)) = 2$ if and only if $T' = L_1$. Since $|S| \leq d$ we deduce $\deg(T' \cap (E \cup S)) \leq a_1 + 2 + |S|$. Moreover, the equality holds if and only if $T' = L_1$ and $S \subseteq L_1$. Since $\deg(T' \cap (E \cup S)) \geq 2d + 2$ by assumption, $|S| = d$ and $S \subseteq L_1$, as claimed.

Now assume $x = 3$. We get $\deg(T' \cap (E_2 \cup E_3 \cup S)) \geq 3d - a_1$ and $T'$ is a conic. If neither $L_{X,P_1}$, nor $L_{X,P_2}$, is a component of $T$ then Lemma 11 gives $e_2 \leq 2$ and $e_3 \leq 2$ and so $|T' \cap S| \geq 3d - 4 - a_1 \geq 2d - 4 > d$. If, say, $T'$ contains $L_{X,P_1}$ and $T''$ is the line with $T' = T'' + L_{X,P_2}$, then we get $|(S \cup E_3) \cap T''| \geq 3d - a_1 - a_2$. Since $a_1 + a_2 \leq 2d - 1$ we deduce $\deg(T'' \cap (E_3 \cup S)) \geq d + 1$. Since $\deg(T'' \cap E_3) \leq 1$, we get $a_1 + a_2 = 2d - 1$, say $a_1 = d$, $a_2 = d - 1$ and that $S$ is formed by $d$ points on a line $T''$ through $P_3$. If either $T'' = L_1$ or $T'' = L_3$, then we are done. In any case it is sufficient to prove that the case $x = 3$ of Lemma 16 does not apply, i.e., that $E_1 \cup E_2 \cup \{P_3\} \cup S$ is not the complete intersection of $T = L_{X,P_1} \cup L_{X,P_2} \cup T''$ and a degree $d$ curve, say $C_d$. Since $a_2 = d - 1$, $E_2$ is not the complete intersection of $L_{X,P_1}$ and $C_d$, while $L_{X,P_2} \cap (\{P_3\} \cup S) = \emptyset$, a contradiction.

\section*{Remark 18.} Let us compare the results of Theorem 17 with the classical Goppa bound for codes obtained from curves. Choose $B$, $d$, $P_1$, $P_2$, $P_3$, $a_1$, $a_2$, $a_3$ as in the theorem. Denote by $P_\infty := (0 : 1 : 0)$ the point at infinity of the Hermitian curve $X$. As in Remark 14 we have

\[
H^0(X, \mathcal{O}_X(d)) = \mathcal{L}(d(q + 1)P_\infty).
\]

As a consequence, $\mathcal{C} := \mathcal{C}(B, d, -E)$ is the code obtained evaluating the Riemann-Roch space $\mathcal{L}(G)$ at the points of $B$, where $G := d(q + 1)P_\infty - a_1P_1 - a_2P_2 - a_3P_3$. The dual minimum distance of $\mathcal{C}$ is lower-bounded by $\deg(G) - (2g - 2)$ (see [19], Theorem 2.2.7). Since $d \leq q - 1$, we have

\[
\deg(G) - (2g - 2) = d(q + 1) - a_1 - a_2 - a_3 - q(q - 1) + 2 \leq q + 1 - a_1 - a_2 - a_3.
\]

As a consequence, if $d > q + 1 - a_1 - a_2 - a_3$ then the dual minimum distance of $\mathcal{C}$ is higher than the designed distance $\deg(G) - (2g - 2)$.

\section*{Remark 19.} Take the set-up of Theorem 17 and assume $d \geq 6$, $a_1 = a_2 = d$ and $1 \leq a_3 \leq d - 5$. Take any line $L$ through $P_3$ with $L \neq L_{X,P_1}$ and any $S \subseteq L \cap B$ such that $|S| = d$. Then $S$ is the support of a code-word of $\mathcal{C}^\perp$ of weight $d$. Indeed, Lemma 13 gives $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$. Hence it is enough to prove that $h^1(\mathbb{P}^2, \mathcal{I}_E \cup E_2 \cup \{P_3\} \cup S(d)) > 0$. Since $\deg(E_1 \cup E_2 \cup \{P_3\} \cup S) = 3d + 1$ and $E_1 \cup E_2 \cup \{P_3\} \cup S$ is contained into the degree 3 curve $L_{X,P_1} \cup L_{X,P_2} \cup L_2$, we can simply apply the “if” part of Lemma 16 (part (e)).

In the following result we modify the set $B$ in the statement of Theorem 17 in order to obtain (under certain assumptions) $\mathcal{C}^\perp$ codes of improved parameters. To be precise, we will exhibit codes whose dual minimum distance is $d + 1$ instead of $d$.

\section*{Theorem 20.} Fix any three distinct points $P_1, P_2, P_3 \in X(\mathbb{F}_q^2)$ and assume $P_1$, $P_2$ and $P_3$ to be not collinear. Fix an integer $d \geq 5$ such that $1 \leq d \leq q - 1$ and integers $a_1, a_2, a_3 \in \{1, \ldots, d\}$ such that $a_1 + a_2 + a_3 \leq 3d - 5$ and $a_1 + a_2 \leq 2d - 2$. Set $E := a_1P_1 + a_2P_2 + a_3P_3$ and $B' := X(\mathbb{F}_q^2) \setminus (X(\mathbb{F}_q^2) \cap (L_1 \cup L_2 \cup L_3))$, where $L_1, L_2, L_3$ are the three lines spanned by $P_1, P_2, P_3$. Let $\mathcal{C} := \mathcal{C}(B', d, -E)$ be the code obtained evaluating the vector space $H^0(X, \mathcal{O}_X(d) (-E))$ on the set $B'$. Then $\mathcal{C}$ has length $n = q^2 - 3q + 1$ and dimension $k := \left(\frac{d+2}{2}\right) - a_1 - a_2 - a_3$. Let $\mathcal{J}$ denote be the set of all lines in $\mathbb{P}^2$ defined over $\mathbb{F}_q$ through one of the points $P_1, P_2, P_3$, but different from $L_1, L_2, L_3$ and from the tangent lines $L_{X,P_i}$, $i = 1, 2, 3$, to $X$ at $P_i$. Let $\mathcal{J}(d + 1)$ denote the set of all the subsets $S \subseteq B$ such that $|S| = d + 1$ and contained in some line $L \in \mathcal{J}$. We have $|\mathcal{J}(d + 1)| = 3(q^2 - 2)(\frac{q}{d+1})$. The minimum distance of $\mathcal{C}^\perp$ is $d + 1$ and for each $S \in \mathcal{J}(d + 1)$ there exists a minimum-weight codeword (unique up to a scalar multiplication) with $S$ as its support. If $d \geq 6$ and $a_1 + a_2 + a_3 \leq 3d - 6$ then all the minimum-weight codewords of $\mathcal{C}^\perp$ arise in this way from a unique $S \in \mathcal{J}(d + 1)$. 

Proof. Since $|L_i \cap X(\mathbb{F}_{q^2})| = q + 1$ for any $i \in \{1, 2, 3\}$, we have $|X(\mathbb{F}_{q^2}) \cap (L_1 \cup L_2 \cup L_3)| = 3q$. Hence $|B'| = q^3 - 3q + 1$. Lemma 13 gives $h^0(X, \mathcal{O}_X(d)(-E)) = (d^2 - 2) - a_1 - a_2 - a_3$. Since $E \subseteq X$ and $|B'| + \deg(E) > d \cdot \deg(X)$, there is not non-zero element of $H^0(X, \mathcal{O}_X(d)(-E))$ vanishing at any point of $B'$. Hence $\mathcal{C}$ has dimension $k = (d^2 - 2) - a_1 - a_2 - a_3$. Theorem 17 implies that $\mathcal{C}^+$ has minimum distance at least $d + 1$. The set of all the lines through any $P_i$ has cardinality $q^2 + 1$. One of these lines is the tangent line $L_{X,P_i}$ and two of these lines are in $\{L_1, L_2, L_3\}$. Hence $|\mathcal{S}| = 3(q^2 - 2)$. Since $|B'| \cap L_i = q$ for any $L_i \in \mathcal{S}$, we have $|\mathcal{S}(d + 1)| = 3(q^2 - 2)(d^2 - 1)$. Fix any $S \in \mathcal{S}(d + 1)$. We have $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$ (by Lemma 13). Parts (a) and (b) of Lemma 16 give $h^1(\mathbb{P}^2, \mathcal{I}_{E,S}(d)) > 0$. Hence Lemma 5 tells us that $S$ is the support of a (up to a non-zero scalar) minimum-weight codeword of $\mathcal{C}^+$. Now assume $d \geq 6$ and $a_1 + a_2 + a_3 \leq 3d - 6$. Look at the proof of Theorem 17. Fix any $S \subseteq B'$ such that $|S| = d + 1$. Since $a_1 + a_2 + a_3 + |S| \leq 4d - 5$, we may apply Lemma 16. Let $T$ be a curve arising from the Lemma and set $x := \deg(T) \in \{1, 2, 3\}$ and $e_i := \deg(E_i \cap T)$. First assume $L_{X,P_i} \nsubseteq T$ for any $i \in \{1, 2, 3\}$.

(i) If $x = 1$ then we get $\deg(E \cap T) \leq 1$ and $\deg(E \cap T) > 0$ if and only if $P_i \in T$. Since $S \cap T = \emptyset$, we get $T \in \mathcal{S}$ and $S \in \mathcal{S}(d + 1)$.

(ii) If $x = 2$, then we have $e_1 + e_2 + e_3 \leq 6$. Since $\deg(T \cap (E \cup S)) \geq 2d + 2$, we get $d + 7 \geq 2d + 2$, a contradiction.

(iii) Now assume $x = 3$. Since $\deg(T \cap (E \cup S)) \geq 3d$ and $e_i \leq 3$ for any $i \in \{1, 2, 3\}$, we get $d \leq 5$, a contradiction.

From now on we assume $L_{X,P_i} \subseteq T$ for some $i \in \{1, 2, 3\}$ and write $T = L_{X,P_i} + T'$. In the case $x = 1$ we obviously get a contradiction. In the case $x = 2$ we have $\deg((S \cup E_{j \in \mathcal{S}}) \cap T') \leq |S| + 1$ (for $\{i, j, h\} = \{1, 2, 3\}$), because $L_i \cap B' = \emptyset$. This is a contradiction. Now assume $x = 3$ and that no $L_{X,P_i}, j \neq i$, is contained in $T'$. It follows $e_j + e_h \leq 4$ (for $\{i, j, h\} = \{1, 2, 3\}$). We deduce $a_i + 4 + d \geq 3d$, a contradiction. Now assume $L_{X,P_i} \subseteq T'$, say $T' = R + L_{X,P_i}$. Conclude by using the last part of the proof of Theorem 17 by setting $|S| = d + 1$ and $a_1 + a_2 \leq 2d - 2$.

The following two results are the analogues of Theorem 17 and Theorem 20 for the more complicated case of Hermitian $s$-point codes, with $s \geq 2$ arbitrary.

Remark 21. In [14] G. L. Matthews computed the Weierstrass semigroup of any $s$-collinear points in $X(\mathbb{F}_{q^2})$. In particular, the dimensions of the vector spaces of the form $H^0(X, \mathcal{O}_X(t)(-\sum_{i=1}^s a_i P_i))$, with $s \geq 1$, $P_1, \ldots, P_s \in X(\mathbb{F}_{q^2})$ and $t, a_1, \ldots, a_s \geq 0$ arbitrary, are known.

Lemma 22. Fix integers $s, d$ and such that and $2 \leq s \leq d - 1 \leq q - 2$. Choose integers $a_1, \ldots, a_s$ and fix $s$ distinct collinear points $P_1, \ldots, P_s \in X(\mathbb{F}_{q^2})$. Denote by $R$ the line containing the $s \geq 2$ points $P_1, \ldots, P_s$ and take $B := X(\mathbb{F}_{q^2}) \setminus \{P_1, \ldots, P_s\}$. Set $E := \sum_{i=1}^s a_i P_i$. Let $S \subseteq B$ be any subset. For any integer $t$ we have an exact sequence of coherent sheaves:

$\begin{align*}
0 \to \mathcal{I}_{E,(S \cap R)}(t - 1) \to \mathcal{I}_{E,S}(t) \to \mathcal{I}_{(P_1, \ldots, P_s),(S \cap R)}(t) \to 0
\end{align*}$

in which $\{P_1, \ldots, P_s\} \cup (S \cap R)$ is a set of $s + |S \cap R|$ points of $R$. For each integer $i \geq 0$ we have

$\begin{align*}
h^i(\mathbb{P}^2, \mathcal{I}_{E,S}(t)) \leq h^i(\mathbb{P}^2, \mathcal{I}_{E,(S \cap R)}(t - 1)) + h^i(\mathbb{P}^2, \mathcal{I}_{(P_1, \ldots, P_s),(S \cap R)}(t - 1)).
\end{align*}$

If $t \geq |S \cap R| + s - 1$, then $h^1(\mathbb{P}^2, \mathcal{I}_{E,S}(t)) \leq h^1(\mathbb{P}^2, \mathcal{I}_{E,(S \cap R)}(t - 1))$.

Proof. For any closed subscheme $Z \subseteq \mathbb{P}^2$ the zero-dimensional scheme $\text{Res}_Z(Z)$ is the closed subscheme of $\mathbb{P}^2$ with $\mathcal{I}_Z : \mathcal{I}_R$ as its ideal sheaf (see Notations 3). For any finite set $A \subseteq \mathbb{P}^2$ we have $\text{Res}_A(S) = S \setminus \cap R$. Since $R$ is a degree 1 divisor of $\mathbb{P}^2$ we have the residual sheaf

$\begin{align*}
0 \to \mathcal{I}_{\text{Res}_Z(Z)}(t - 1) \to \mathcal{I}_{Z}(t) \to \mathcal{I}_{Z \cap R,R}(t) \to 0
\end{align*}$

Take $Z := E \cup S$ with $S \subseteq B$. We have $\text{Res}_Z(S) = S \setminus \cap R$. Since $R$ is not tangent to $C$ and $P_i \in R$ for any $i \in \{1, \ldots, s\}$, we have $\text{Res}_E(E) = E'$. Hence $\text{Res}_E(E \cup S) = E' \cup (S \cap R)$. Since $P_i \in R$ for any $i$ and $R$ is transversal to $C$, we have $R \cap E = \{P_1, \ldots, P_s\}$ and hence $R \cap (E \cup S) = \{P_1, \ldots, P_s\} \cup (S \cap R)$. Applying (5) we get (3). The cohomology exact sequence induced by (3) gives (4). Since $R \cong \mathbb{P}^1$ and $\deg(\{P_1, \ldots, P_s\} \cup$
Theorem 23. Fix integers $s$ and $d$ such that $2 \leq s \leq d - 1 \leq q - 2$. Choose integers $a_1, \ldots, a_r$ such that $0 < a_i \leq d + 1 - i$ for any $i \in \{1, \ldots, s\}$ and such that $a_1 + \cdots + a_s \leq 3d - 7 + s$. Fix $s$ distinct collinear points $P_1, \ldots, P_s \in X(\F_q)$. Denote by $R$ the line containing the $s \geq 2$ points $P_1, \ldots, P_s$. Take $B := X(\F_q^{d}) \setminus \{P_1, \ldots, P_s\}$ and set $E := \sum_{i=1}^{s} a_i P_i$. Then the code $\mathcal{C} \subseteq \mathcal{C}(B, d, -E)$ obtained evaluating the vector space $H^0(X, \mathcal{O}_X(d)(-E))$ on the set $B$ has length $n := q^3 + 1 - s$ and dimension $k := (d + 2)^s - \sum_{i=1}^{s} a_i$. The code $\mathcal{C}$ has minimum distance $d + 2 - s$ and its minimum-weight codewords are exactly the ones whose support, $S$, consists of $d + 2 - s$ points of $B \cap R$. Any $S \subseteq R \cap B$ with $|S| = d + 2 - s$ is the support of exactly one (up to multiplication by a non-zero scalar) minimum-weight codeword.

Proof. The parameters $n$ and $k$ are obvious by Lemma 13 and the inequality $\deg(E) + |B| > d \cdot \deg(X)$. Lemma 13 gives also $h^1(\mathbb{P}^2, \mathcal{I}_E(d))$. By Lemma 5, it is enough to prove that $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) = 0$ for any $A \subseteq B$ such that $|A| \leq d + 1 - s$ and that for any $S \subseteq B$ such that $|S| = d + 2 - s$ we have $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) > 0$ and only if $S \subseteq R$. If $S \cap B \cap R$ and $|S| = d + 2 - s$ then $\deg((E \cup S) \cap R) = d + 2$. Hence the part (b) of Lemma 16 gives $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) > 0$. So the “if” part of the statement is proved. Now we check the “only if” part. Fix $S \subseteq B$ such that $|S| \leq d + 2 - s$ and $h^1(\mathbb{P}^2, \mathcal{I}_E(d)) > 0$. Since $\deg(E \cup S) \leq 4d - s$, we may apply Lemma 16. Let $T$ be a curve arising from the statement of that lemma and set $x := \deg(T)$. Define for any $i \in \{1, \ldots, s\}$, $E_i := a_i P_i$ and $e_i := \deg(E_i \cap T)$. Set $f := |S \cap T|$. Notice that if $R$ is not a component of $T$ then $e_i > 0$ for at most $x$ indices $i$. Set $E(t) := E \setminus E_i$ and $E' := \sum_{i=1}^{s}(a_i - 1) P_i$. We will see $E'$ both as a positive divisor of $X$ and a zero-dimensional subscheme of $\mathbb{P}^2$. We have $\deg(E') = \deg(E) - s$. Consider the scheme $\text{Res}_R(E) \subseteq E \subseteq \mathbb{P}^2$ and observe that since $R$ is transversal to $X$ at each $P_i$, we have $E' = \text{Res}_R(E)$. Keep on mind Lemma 22 and its proof.

(i) In this step we assume that $T$ contains no one of the tangent lines $L_{X, P_i}$, $1 \leq i \leq s$. Hence $e_i \leq x$ for any $i$ (Lemma 11). Hence $\deg((E \cup S) \cap T) \leq sx + f$. First assume $x = 1$. We get $|\{P_1, \ldots, P_s\} \cap T| + f \geq d + 2$ and hence $S \subseteq T$, $P_i \in T$ for any $i$ (i.e. $T = R$) and $f = d + 2 - s$, as claimed. Now assume $x = 2$ and that $R$ is not a component of $T$. We get $2 \cdot 2 + (d + 2 - s) \geq 2d + 2$, a contradiction. Now assume that $x = 2$ and that $R$ is a component of $T$, say $T = R \cup L'$. If $|S \cap R| \geq d - s - 1$ then $S \cap R \cap B$, as claimed; if $S \cap R \leq d - s$ then $\deg(L' \cap (E' \cup (S \setminus S \cap R))) \geq d - s + 2 = 2d - 2 + 2$. Since $L' \neq L_{X, P_i}$ for any $i$ and $L' \neq R$, we have $\deg(E' \cap L') \leq 1$. Hence $|S| \geq |S \cap L'| \geq d + 1$, contradiction. If $x = 3$ and $R$ is not a component of $T$ then we get $3 \cdot 3 + d + 2 - s \geq 3d - 1$, a contradiction. Now assume $x = 3$ and $T = R \cup L'$. We may assume $f \leq d - 1$. Hence $\deg(T' \cap (E' \cap (S \setminus S \cap R))) \geq 3d - (d - 1 - s) = 2d - 1 + s$. Since $2d - 1 + s \geq 2d + 2$, we get a contradiction as in the case $x = 2$.

(ii) Now assume that $T$ contains one of the tangent lines $L_{X, P_i}$. If $x = 1$, then $L_{X, P_i} = T$ and hence $\deg(T \cap (E \cup S)) = e_i \leq d$, contradiction. Now assume $x \geq 2$ and write $T = T' \cup L_{X, P_i}$. We have $T \cap (E \cup S) = E_i \cup T' \cap (S \cup E(i))$. Assume for the moment that $T'$ contains no tangent line to $X$ at one of the points $P_j$, $j \neq i$. We get $e_j \leq x - 1$ for all $j \neq i$ (Lemma 11). Hence $\deg(T \cap (E \cup S)) = e_i + \deg(T' \cap (S \cup E(i))) \leq e_i + f + (x - 1)(s - 1)$. First assume $x = 2$. Since $e_i \leq d$, we get $f + (s - 1) \geq d + 2$, a contradiction. Now assume $x = 3$, i.e. $\deg(T') = 2$. First assume $R \subseteq T'$, say $T' = R \cup L''$ with $L''$ a line. If $|S \cap R| \geq d - 2 + 2$, then we are done. Hence we may assume $|S \cap R| \leq d - 1 - s$. Hence $\deg(L'' \cap ((S \setminus S \cap R) \cup E)) \geq 3d - e_i - (d - 1 - s) \geq d + 2$. Since $L'' \neq R$ and $L'' \neq L_{X, P_i}$ for any $j$, we have $\deg(E \cap L'') \leq 1$. Hence $|S| \geq d + 1$, a contradiction. Now assume $R \nsubseteq T'$. Since the points $P_1, \ldots, P_r$ are collinear, Bezout theorem gives $\deg(T' \cap \{P_1, \ldots, P_s\}) \leq 2$. So Lemma 11 implies $\deg(E(i) \cap T') \leq 4$. We conclude $f \geq 3d - 1 - a_i - 4 \geq 2d - 5 > d + 2 - s$, a contradiction. Now assume the existence of an index $j \neq i$ such that $T'$ contains the tangent line $L_{X, P_j}$. We have $L_{X, P_j} \cap B = \emptyset$. Hence $f = 0$ if $x = 2$, a contradiction. Assume $x = 3$ and write $T' = L_{X, P_i} \cup L$ with $L$ a line and $|S \cap L| = f$. If $L$ is a tangent line to $X$, say at $P$, then either $\deg((E \cup S) \cap T) = e_i + e_j + e_h$ (i.e. $P = P_h$ with $h \in \{1, \ldots, s\} \setminus \{i, j\}$, or $\deg((E \cup S) \cap T) \leq e_i + e_j + 1$ (i.e. $P \notin \{P_1, \ldots, P_s\}$). In any case we get a contradiction, because $e_i + e_j + e_h \leq 3d - 1$ when $i, j, h$ are distinct. If $L \neq L_{X, P_i}$ for any $h \notin \{i, j\}$,
then $\deg((E \setminus (E_i \cup E_j)) \cup S) \cap L) \leq f + 1$ and equality holds if and only if $P_h \in L$ for some $h \not\in \{i, j\}$. Since $f \leq d + 1 - s$, we get $e_1 + e_j + d + 2 - s \geq 3d$, a contradiction.

\[ \square \]

As in the case of 3-point codes, in the following Theorem 24 we construct $s$-point codes with improved parameters.

**Theorem 24.** Fix integers integers $s$ and $d$ such that $2 \leq s \leq d - 1 \leq q - 2$. Choose integers $a_1, \ldots, a_s$ such that $0 < a_i \leq d + 1 - i$ for any $i \in \{1, \ldots, s\}$, $a_1 + \cdots + a_s \leq 3d - 6$ and $a_i + a_j \leq 2d - 2$ for any $i \neq j$. Fix $s$ distinct collinear points $P_1, \ldots, P_s \in X(F_{q^2})$. Denote by $R$ the line containing the $s \geq 2$ points $P_1, \ldots, P_s$. Take $B' := X(F_{q^2}) \cap R \times X(F_{q^2})$ and set $E := \sum_{i=1}^s a_i P_i$. Then the code $C = C(B', d, -E)$ obtained evaluating the vector space $H^0(X, \mathcal{O}_X(d)(-E))$ on the set $B'$ has length $n := q^2 - q$ and dimension $k := (d^2/2) - \sum_{i=1}^s a_i$. Let $\mathcal{S}$ denote the set of the lines in $\mathbb{P}^2$ different from $R$, containing one of the points $P_1, \ldots, P_s$ and not tangent to $X$. Let $\mathcal{S}(d + 1)$ be the set of all the subsets $S \subseteq B'$ such that $|S| = d + 1$ and $S \subseteq L$ for some $L \in \mathcal{S}$. We have $|\mathcal{S}(d + 1)| = s(q^2 - 1)q$. The code $C^\perp$ has minimum distance $d + 1$ and for any set $S \in \mathcal{S}(d + 1)$ there exists a unique (up to multiplication by a non-zero scalar) minimum-weight codeword with $S$ as its support. Moreover, all the minimum-weight codewords of $C^\perp$ are associated to a unique $S \in \mathcal{S}(d + 1)$.

**Proof.** The parameters $n,k$ of $C$ are obvious, because $h^1(\mathbb{P}^2, \mathcal{O}_d(E)) = 0$ by Lemma 13 and the inequality $\deg(E) + |B'| > d \cdot \deg(X)$ holds. If $A \in \mathcal{S}(d + 1)$, then the “if” part of Lemma 16 gives $h^1(\mathbb{P}^2, \mathcal{E}_{\mathcal{S}(d)}(d)) > 0$. At this point it is enough to prove that if $S \subseteq B'$ satisfies $|S| \leq d + 1$ and $h^1(\mathcal{E}_{\mathcal{S}(d)}(d)) > 0$ then $S \in \mathcal{S}(d + 1)$. We will make use of the notations $E_i, E^{(i)}, E'$, and so on, introduced in the proof of Theorem 23. Keep on mind Lemma 22 and its proof. Since $\deg(E \cup S) = a_1 + \cdots + a_s + |S| \leq 4d - 5$, we may apply Lemma 16 and get a curve $T \subseteq \mathbb{P}^2$ arising from the statement of that lemma. Set $x := \deg(T) \in \{1, 2, 3\}$. Define $e_i := \deg(T \cap E_i)$ and $f := |T \cap S|$. Let $L \subseteq \mathbb{P}^2$ be any line. We have $\deg(L \cap (E \cup S)) = s$ if $L = R$; $\deg(L \cap (E \cup S)) = a_i$ if $L = L_{X,P_i}$; $\deg(L \cap E) = 0$ if $L \neq R, L$ is not tangent to $X$ and $L \cap \{P_1, \ldots, P_s\} = \emptyset$; $L \cap E = \emptyset$ if $L \cap \{P_1, \ldots, P_s\} = \emptyset$.

(i) Here we assume that $R$ is not an irreducible component of $T$. First assume that no $L_{X,P_i}$ is a component of $T$. Hence $e_i \leq x$ for all $i$ (Lemma 11). If $x = 1$ we get $S \subseteq T$, $|S| = d + 1$ and $L \cap \{P_1, \ldots, P_s\} = \emptyset$ (i.e. $T \in \mathcal{S}$ and $S \in \mathcal{S}(d + 1)$), because $s = \deg(E \cup S) \cap R) = s < d + 2$. Now assume $x = 2$ (resp. $x = 3$). Since the points $P_1, \ldots, P_s$ are collinear and $R \not\subset T$, Bezout theorem gives $|\{P_1, \ldots, P_s\} \cap T| \leq x$. Hence $2 \cdot 2 + |S| \geq 2d + 2$ (resp. $3 \cdot 3 + |S| \geq 3d$), a contradiction. Now assume, say, $L_{X,P_i} \subseteq T$ and set $T = L_{X,P_i} \cup T'$ with $S \cap T = S \cap T'$ and $\deg(T') = x - 1$. If $x = 1$ then $\deg((E \cup S) \cap T) = a_i < d + 2$, a contradiction. If $x = 2$ then $a_i + (\deg(E \cup S) \cap T') \geq 2d + 2 - a_i - d + 2$. Hence $S \subseteq T'$ and $S \subseteq \mathcal{S}(d + 1)$. Now assume $x = 3$ and that $T'$ does not contain a line $L_{X,P_i}$ for any $j \neq i$. We get $\deg(T' \cap (E \cup S)) \leq \deg(T')^2 = 4$. Since $|S| \leq d + 1$ we conclude $\deg(T \cap (E \cup S)) \leq |S| + 4 + d + 1 < 3d$, a contradiction. Now assume $L_{X,P_i} \subseteq T$ for some $j \neq i$. We still have $a_i + a_j + d + 1 < 3d$ (and hence a contradiction), because we assumed $a_i + a_j \leq 2d - 2$ for all $i \neq j$.

(ii) Now assume that $R$ is an irreducible component with multiplicity $c \geq 1$ of $T$ and write $T = cR + T_1$ with $\deg(T_1) = x - c$. Since $R \cap B' = \emptyset$, we have $f = |S \cap T_1|$. Hence $x > c$. Assume $x = 2$, so that $c = 1$. We get $\deg(T_1 \cap (E \cup S)) \geq 2d + 2 - s \geq d + 3$, a contradiction. Now assume $x = 3$ and $c = 2$. We get $\deg(T_1 \cap (E \cup S)) \geq 3d - 2s \geq d + 2$. Hence $T_1 \in \mathcal{S}$ and $S \in \mathcal{S}(d + 1)$. Assume $x = 3$ and $c = 1$. If $T_1$ contains no tangent line $L_{X,P_i}$, then $\deg(T_1 \cap E) \leq 2 \cdot 2$. Hence $3d \leq \deg(T \cap (E \cup S)) \leq s + \deg(T_1 \cap (E \cup S)) \leq s + 4 + d + 1$, a contradiction. Assume $L_{X,P_i} \subseteq T_1$, say $T_1 = L_{X,P_i} \cup T_2$. Set $E''[i] := \sum_{j \neq i} (a_j - 1)P_j$. We have $E''[i] = \Res_k(E^{(i)})$, and $3d \leq \deg(T \cap (E \cup S)) = (a_i - 1) + s + \deg(T_2 \cap (E \cup E''[i]) \leq s + a_i - 1 + d + 2$, a contradiction.

\[ \square \]

5. Conclusion

We introduce classical geometric tools in the study of geometric codes. The most common methods to investigate Goppa codes are Weierstrass semigroups, Groebner bases and other more computational tools (see the References). Many important results can be established through these techniques. In this paper we show
how a more geometric approach can reveal an interesting bond between the coding-theoretic properties of a Goppa code and the true geometric attributes of the underlying curve, such as the geometry of its tangent lines.

REFERENCES

[1] E. Ballico, A. Ravagnani, *On the duals of geometric Goppa codes from norm-trace curves*. Finite Fields and their Applications, 20 (2013), pp. 30 – 39.

[2] E. Ballico, A. Ravagnani, *On the geometry of Hermitian one-point codes*. Journal of Algebra (to appear).

[3] E. Ballico, A. Ravagnani, *The dual geometry of Hermitian two-point codes*. Discrete Mathematics, 313 (2013), 23, pp. 2687 – 2695.

[4] A. Bernardi, A. Gimigliano, M. Idà, *Computing symmetric rank for symmetric tensors*. Journal of Symbolic Computation, 46 (2011), 1, pp. 34–53.

[5] A. Couvreur: The dual minimum distance of arbitrary dimensional algebraic-geometric codes. Journal of Algebra, 350 (2012), 1, pp. 84–107.

[6] P. Ellia, C. Peskine: Groupes de points de $\mathbb{P}^2$: caractère et position uniforme. Algebraic geometry (L’Aquila, 1988), 111–116, Lecture Notes in Math., 1417, Springer Berlin (1990).

[7] R. Hartshorne, Algebraic Geometry, Springer Berlin, 1977.

[8] J. W. P. Hirschfeld, *Projective Geometries over Finite Fields*, Clarendon Press, Oxford, 1979.

[9] M. Homma, S. J. Kim, *Toward the determination of the minimum distance of two-point codes on a Hermitian curve*. Designs, Codes and Cryptography, 37 (2005), 1, pp. 11–132.

[10] M. Homma, S. J. Kim, *The two-point codes on a Hermitian curve with the designed minimum distance*. Designs, Codes and Cryptography, 38 (2006), 1, pp. 55–81.

[11] M. Homma, S. J. Kim, *The Two-Point Codes with the Designed Distance on a Hermitian Curve in Even Characteristic*. Designs, Codes and Cryptography, 39 (2006), 3, pp. 375–386.

[12] M. Homma, S. J. Kim, *The complete determination of the minimum distance of two-point codes on a Hermitian curve*. Designs, Codes and Cryptography, 40 (2006), 1, pp. 5–24.

[13] M. Homma, S. J. Kim, *The second generalized Hamming weight for two-point codes on a Hermitian curve*. Designs, Codes and Cryptography, 50 (2009), 1, pp. 1–40.

[14] G. L. Matthews, *The Weierstrass semigroup of an m-tuple of collinear points on a Hermitian curve*. Lecture Notes in Computer Science, Finite fields and applications, pp. 12–24, Springer Berlin, 2004.

[15] C. Munuera, *On the generalized Hamming weights of geometric Goppa codes*. IEEE Transactions on Information Theory, 40 (1994), pp. 2092–2099.

[16] C. Munuera, D. Ramirez, *The second and third generalized Hamming weights of Hermitian codes*. IEEE Transactions on Information Theory, 45 (1999), 2, pp. 709–712.

[17] S. Park, *Minimum distance of Hermitian two-point codes*. Designs, Codes and Cryptography, 57 (2010), pp. 195–213.

[18] S. A. Stepanov, *Codes on Algebraic Curves*. Springer, 1999.

[19] H. Stichtenoth, *Algebraic Function Fields and Codes*. Springer Berlin (1993).

[20] K. Yang, P. V. Kumar, *On the true minimum distance of Hermitian codes*. Lecture Notes in Mathematics, Coding theory and algebraic geometry, pp. 99–107, Springer Berlin, 1992.

[21] K. Yang, P. V. Kumar, H. Stichtenoth: On the weight hierarchy of geometric Goppa codes. IEEE Transactions on Information Theory, 40 (1994), pp. 913–920.