A NOTE ON THE SECOND MOMENT OF AUTOMORPHIC L-FUNCTIONS

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Abstract. We obtain the formula for the twisted harmonic second moment of the L-functions associated with primitive Hecke eigenforms of weight 2. A consequence of our mean value theorem is reminiscent of recent results of Conrey and Young on the reciprocity formula for the twisted second moment of Dirichlet L-functions.

1. Introduction

In this paper, we study the twisted second moment of the family of L-functions arising from $S_2^\ast(q)$, the set of primitive Hecke eigenforms of weight 2, lever $q$ ($q$ prime). For $f(z) \in S_2^\ast(q)$, $f$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{1/2} \lambda_f(n) e(nz),$$

where the normalization is such that $\lambda_f(1) = 1$. The L-function associated to $f$ has an Euler product

$$L(f, s) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \left(1 - \frac{\lambda_f(q)}{q^s}\right)^{-1} \prod_{h \text{ prime} \neq q} \left(1 - \frac{\lambda_f(h)}{h^s} + \frac{1}{h^2s}\right)^{-1}.$$

The series is absolutely convergent when $\Re s > 1$, and admits analytic continuation to all of $\mathbb{C}$. The functional equation for $L(f, s)$ is

$$\Lambda(f, s) := \left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s + \frac{1}{2}) L(f, s) = \varepsilon_f \Lambda(f, 1 - s),$$

where $\varepsilon_f = -q^{1/2} \lambda_f(q) = \pm 1$. We define the harmonic average as

$$\sum_f^h A_f := \sum_{f \in S_2^\ast(q)} \frac{A_f}{4\pi(f, f)},$$

where $(f, g)$ is the Petersson inner product on the space $\Gamma_0(q) \backslash \mathbb{H}$.

We are interested in the twisted second moment of this family of L-functions. We define

$$S(p, q) = \sum_{f \in S_2^\ast(q)}^h L(f, \frac{1}{2})^2 \lambda_f(p).$$

Our main theorem is

**Theorem 1.** Suppose $q$ is prime and $0 < p \leq Cq$, for some fixed $C < 1$. Then we have

$$S(p, q) = \frac{d(p)}{\sqrt{p}} \log \frac{q}{4\pi^2p} + O(p^{1/2}q^{-1+\varepsilon}).$$
Remark 1. The twisted harmonic fourth moment has been considered by Kowalski, Michel and VanderKam [6], where they gave an asymptotic formula for the fourth power mean value provided that \( p \ll q^{1/9 - \varepsilon} \).

Remark 2. In a similar setting, Iwaniec and Sarnak [3] have given the exact formula for the twisted second moment of the automorphic \( L \)-functions arising from \( \mathcal{H}_k(1) \), the set of newforms in \( \mathcal{S}_k(1) \), where \( \mathcal{S}_k(1) \) is the linear space of holomorphic cusp forms of weight \( k \). Precisely, they showed that for \( k > 2, k \equiv 0(\text{mod } 2) \), and for any \( m \geq 1 \), we have

\[
\frac{12}{k-1} \sum_{f \in \mathcal{H}_k(1)} w_f L(f, \frac{1}{2})^2 \lambda_f(m) = 2(1 + i^k) \frac{d(m)}{\sqrt{m}} \left( \sum_{0 \leq t \leq k/2} \frac{1}{t} - \log 2\pi \sqrt{m} \right) - 2\pi^k \sum_{h \neq m} d(h) d(h - m) p_k \left( \frac{h}{m} \right) + 2\pi^k \sum_{h} d(h) d(h + m) q_k \left( \frac{h}{m} \right),
\]

where \( p_k(x) \) and \( q_k(x) \) are Hankel transforms of Bessel functions

\[
p_k(x) = \int_0^\infty Y_0(y \sqrt{x}) J_{k-1}(y) dy, \quad \text{and} \quad q_k(x) = \frac{2}{\pi} \int_0^\infty K_0(y \sqrt{x}) J_{k-1}(y) dy.
\]

Here the weight \( w_f = \zeta(2) L(\text{sym}^2(f), 1)^{-1} \), where the symmetric square \( L \)-function \( L(\text{sym}^2(f), s) \) corresponding to \( f \) is defined by

\[
L(\text{sym}^2(f), s) = \zeta(2s) \sum_{n=1}^\infty \lambda_f(n^2) n^{-s}.
\]

In the context of Dirichlet \( L \)-functions, consider

\[
M(p, q) = \frac{1}{\varphi^*(q)} \sum_{\chi \equiv \chi(q)}^* |L(\frac{1}{2}, \chi)|^2 \chi(p),
\]

where \( \sum^* \) denotes summation over all primitive characters \( \chi \equiv \chi(q) \), and \( \varphi^*(q) \) is the number of primitive characters. This is the twisted second moment of Dirichlet \( L \)-functions. In a recent paper, Conrey [11] proved that there is a kind of reciprocity formula relating \( M(p, q) \) and \( M(-q, p) \) when \( p \) and \( q \) are distinct prime integers. Precisely, Conrey showed that

\[
M(p, q) = \sqrt{\frac{q}{p}} M(-q, p) + \frac{1}{\sqrt{p}} \left( \log \frac{q}{p} + A \right) + \frac{B}{2\sqrt{q}} + O \left( \frac{p}{q} + \frac{\log q}{q} + \frac{\log pq}{\sqrt{pq}} \right),
\]

where \( A \) and \( B \) are some explicit constants. This provides an asymptotic formula for \( M(p, q) - \sqrt{p/q} M(-q, p) \) under the condition that \( p \ll q^{2/3 - \varepsilon} \). The error term above was improved by Young [7] so that the asymptotic formula holds for \( p \ll q^{1-\varepsilon} \).

We now take \( p \) to be prime and, similarly as before, \( S(q, p) \) is defined as the harmonic second moment, twisted by \( \lambda_q(q) \), of the family of \( L \)-functions arising from \( g(z) \in \mathcal{S}_2^*(p) \). We note that as \( q \) is prime, the Ramanujan bound \( |\lambda_f(n)| \leq d(n) \) [2] yields

\[
S(q, p) \ll \sum_{g \in \mathcal{S}_2^*(p)} h L(g, \frac{1}{2})^2 \ll \log p.
\]

Thus as a trivial consequence of Theorem 1, for \( p < q \) we have

\[
S(p, q) - \sqrt{p/q} S(q, p) = \frac{2}{\sqrt{p}} \log \frac{q}{4\pi^2 p} + O(p^{1/2+\varepsilon} q^{-1/2}).
\]
This leads to an asymptotic formula for $S(p, q) - \sqrt{p/q}S(q, p)$, at least for $p$ as large as $q^{1/2-\varepsilon}$. The results in the Dirichlet $L$-functions case \cite{17} suggest that the asymptotic formula should hold for $p \ll q^\theta$, for any $\theta < 1$. However, our technique fails to extend the range to any power $\theta > 1/2$. For that purpose, we need more refined estimates for the off-diagonal terms of $S(p, q)$ and $S(q, p)$. The intricate calculations seem to suggest that there is a large cancellation between these two expressions. The nature of this is not well-understood.

2. Preliminary lemmas

We require some lemmas. We begin with Hecke’s formula for primitive forms.

**Lemma 1.** For $m, n \geq 1$,

$$
\lambda_f(m)\lambda_f(n) = \sum_{d \mid (m, n)} \lambda_f \left( \frac{mn}{d^2} \right).
$$

The next lemma is a particular case of Petersson’s trace formula.

**Lemma 2.** For $m, n \geq 1$, we have

$$
\sum_{f \in S^*_2(q)} h \lambda_f(m)\lambda_f(n) = \delta_{m,n} - J_q(m, n),
$$

where $\delta_{m,n}$ is the Kronecker symbol and

$$
J_q(m, n) = 2\pi \sum_{c=1}^\infty \frac{S(m, n; cq)}{cq} J_1 \left( \frac{4\pi \sqrt{mn}}{cq} \right).
$$

Here $J_1(x)$ is the Bessel function of order 1, and $S(m, n; c)$ is the Kloosterman sum

$$
S(m, n; c) = \sum_{\alpha \mod c}^* e \left( \frac{ma + n\alpha}{c} \right).
$$

Moreover we have

$$
J_q(m, n) \ll (m, n, q)^{1/2}(mn)^{1/2+\varepsilon}q^{-3/2}.
$$

The above estimate follows easily from the bound $J_1(x) \ll x$ and Weil’s bound on Kloosterman sums.

We mention a result of Jutila \cite{4} (cf. Theorem 1.7), which is an extension of the Voronoi summation formula.

**Lemma 3.** Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a $C^\infty$ function which vanishes in the neighbourhood of 0 and is rapidly decreasing at infinity. Then for $c \geq 1$ and $(a, c) = 1$,

$$
c \sum_{m=1}^\infty d(m) e \left( \frac{am}{c} \right) f(m) = 2 \int_0^\infty \left( \log \frac{\sqrt{x}}{c} + \gamma \right) f(x) dx
$$

$$
-2\pi \sum_{m=1}^\infty d(m) e \left( \frac{-\overline{am}}{c} \right) \int_0^\infty Y_0 \left( \frac{4\pi \sqrt{mx}}{c} \right) f(x) dx
$$

$$
+ 4 \sum_{m=1}^\infty d(m) e \left( \frac{am}{c} \right) \int_0^\infty K_0 \left( \frac{4\pi \sqrt{mx}}{c} \right) f(x) dx.
$$
The next lemma concerns the approximate functional equation for $L$-functions.

**Lemma 4.** Let $G(s)$ be an even entire function satisfying $G(0) = 1$ and $G$ has a double zero at each $s \in \mathbb{Z}$. Furthermore let assume that $G(s) \ll_{A,B} (1 + |s|)^{-A}$ for any $A > 0$ in any strip $-B \leq \Re s \leq B$. Then for $f \in \mathcal{S}_2^*(q),$

$$L(f, \frac{1}{2})^2 = 2 \sum_{n=1}^{\infty} \frac{d(n) \lambda_f(n)}{\sqrt{n}} W_q\left(\frac{4\pi^2 n}{q}\right),$$

where

$$W_q(x) = \frac{1}{2\pi i} \int_{(1)} G(s) \Gamma(s + 1)^2 \zeta_q(2s + 1)x^{-s} \frac{ds}{s}.$$

Here $\zeta_q(s)$ is defined by

$$\zeta_q(s) = \sum_{n=1 \atop (n,q)=1}^{\infty} n^{-s} \quad (\sigma > 1).$$

**Proof.** From Lemma 1 we first note that

$$L(f, s)^2 = \zeta_q(2s) \sum_{n=1}^{\infty} \frac{d(n) \lambda_f(n)}{n^s} \quad (\sigma > 1).$$

Consider

$$A(f) := \frac{1}{2\pi i} \int_{(1)} G(s) \Lambda(f, s + \frac{1}{2})^2 ds \frac{ds}{\sqrt{s}}.$$

Moving the line of integration to $\Re s = -1$, and applying Cauchy’s theorem and the functional equation, we derive that $A(f) = L(f, \frac{1}{2})^2 - A(f)$. Expanding $\Lambda(f, s + \frac{1}{2})^2$ in a Dirichlet series and integrating termwise we obtain the lemma. □

For our purpose, $W_q$ is basically a “cut-off” function. Indeed, we have the following.

**Lemma 5.** The function $W_q$ satisfies

$$W_q^{(j)}(x) \ll_{j,N} x^{-N} \text{ for } x \geq 1 \text{ and all } j, N \geq 0,$$

$$x^i W_q^{(j)}(x) \ll_{i,j} |\log x| \text{ for } 0 < x < 1 \text{ and all } i \geq j \geq 0,$$

and

$$W_q(x) = -\left(1 - \frac{1}{q}\right) \frac{\log x}{2} + \frac{\log q}{q} + O_N(x^N) \text{ for } 0 < x < 1 \text{ and all } N \geq 0.$$

The implicit constants are independent of $q$.

**Proof.** The first estimate is a direct consequence of Stirling’s formula after differentiating under the integral sign and shifting the line of integration to $\Re s = N$. The only difference in the other two estimates is that one has to move the line of integration to $\Re s = -N$. □
3. Proof of Theorem 1

Our argument in this section follows closely \[5\]. From Lemma 4 and Lemma 2 we obtain

\[ S(p, q) = 2 \frac{d(p)}{\sqrt{p}} W_q \left( \frac{4\pi^2 p}{q} \right) - 2R(p, q), \]

where

\[ R(p, q) = \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} J_q(n, p) W_q \left( \frac{4\pi^2 n}{q} \right). \]

Using Lemma 5, the first term is

\[ \frac{d(p)}{\sqrt{p}} \log \frac{q}{4\pi^2 p} + O(p^{-1/2} q^{-1+\epsilon} + p^{1/2+\epsilon} q^{-1}). \]

Thus, we are left to consider \( R(p, q) \). We have

\[ R(p, q) = 2\pi \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} \sum_{c=1}^{\infty} S(n, p; cq) \frac{J_1 \left( \frac{4\pi \sqrt{np}}{cq} \right)}{cq} W_q \left( \frac{4\pi^2 n}{q} \right). \]

Using Weil’s bound for Kloosterman sums and \( J_1(x) \ll x \), the contribution from the terms \( c \geq q \) is

\[ \ll p^{1/2} q^{-3/2} \sum_{n=1}^{\infty} (n, p)^{1/2} d(n) W_q \left( \frac{4\pi^2 n}{q} \right) \sum_{c \geq q} \frac{d(c)}{c^{3/2}} \ll p^{1/2} q^{-1+\epsilon}. \]

Thus we need to study

\[ \frac{2\pi}{q} \sum_{c<q} \frac{1}{c} \sum_{a \equiv 0 \mod cq}^{\ast} e \left( \frac{ap}{cq} \right) \sum_{n=1}^{\infty} d(n) e \left( \frac{an}{cq} \right) \frac{J_1 \left( \frac{4\pi \sqrt{np}}{cq} \right)}{\sqrt{n}}. \]

We fix a \( C^\infty \) function \( \xi : \mathbb{R}^+ \to [0, 1] \), which satisfies \( \xi(x) = 0 \) for \( 0 \leq x \leq 1/2 \) and \( \xi(x) = 1 \) for \( x \geq 1 \), and attach the weight \( \xi(n) \) to the innermost sum. Using Lemma 3, this is equal to

\[ \frac{4\pi}{q^2} \sum_{c<q} \frac{1}{c^2} S(0, p; cq) \int_0^{\infty} \left( \log \frac{\sqrt{t}}{cq} + \gamma \right) J_1 \left( \frac{4\pi \sqrt{tp}}{cq} \right) W_q \left( \frac{4\pi^2 t}{q} \right) \xi(t) \frac{dt}{\sqrt{t}} - Y + K, \]

where

\[ Y = \frac{4\pi^2}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p - n; cq) \]

\[ \int_0^{\infty} Y_0 \left( \frac{4\pi \sqrt{nt}}{cq} \right) J_1 \left( \frac{4\pi \sqrt{tp}}{cq} \right) W_q \left( \frac{4\pi^2 t}{q} \right) \xi(t) \frac{dt}{\sqrt{t}}, \quad (1) \]

and

\[ K = \frac{8\pi}{q^2} \sum_{c<q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p + n; cq) \]

\[ \int_0^{\infty} K_0 \left( \frac{4\pi \sqrt{nt}}{cq} \right) J_1 \left( \frac{4\pi \sqrt{tp}}{cq} \right) W_q \left( \frac{4\pi^2 t}{q} \right) \xi(t) \frac{dt}{\sqrt{t}}. \quad (2) \]
We will deal with $Y$ and $K$ in the next three lemmas. For the first sum, since $S(0, p; cq) = \mu(q)S(0, p; c)$ and $J_t(\alpha) \ll \alpha$, this is
\[ \ll p^{1/2}q^{-3} \sum_{\epsilon < q} \frac{1}{c^2} \int_{1/2}^{\infty} W_q \left( \frac{4\pi^2 t}{q} \right) (\log t) \frac{\xi(t)}{t} dt \ll p^{1/2}q^{-2+\varepsilon}. \]

**Lemma 6.** For $K$ defined as in (2), we have
\[ K \ll p^{1/2}q^\varepsilon(q - p)^{-1+\varepsilon}. \]
And hence $K \ll p^{1/2}q^{-1+\varepsilon}$, given that $p \leq Cq$ for some fixed $C < 1$.

**Remark 3.** This is the only place where the condition $p \leq Cq$ for some constant $C < 1$ is used.

**Proof.** The integral involving $K_0$, using $K_0(y) \ll y^{-1/2}e^{-y}$, is
\[ \int_0^\infty K_0 \left( \frac{4\pi \sqrt{nL}}{cq} \right) J_1 \left( \frac{4\pi \sqrt{tp}}{cq} \right) W_q \left( \frac{4\pi^2 t}{q} \right) \frac{\xi(t)}{t} dt \ll \frac{cq}{2\pi \sqrt{n}} \int_0^\infty K_0(y)J_1 \left( \frac{\sqrt{p}}{n}y \right) W_q \left( \frac{c^2qy^2}{4n} \right) \xi \left( \frac{c^2qy^2}{16\pi^2n} \right) dy \ll \frac{cp^{1/2}q^{1+\varepsilon}}{n} \int_{\sqrt{n/cq}}^\infty y^{1/2}e^{-y}dy \ll \frac{cp^{1/2}q^{1+\varepsilon}}{n} e^{-cq}. \]

Thus, as $S(0, p+n; cq) = S(0, (p+n)c; c)S(0, p+n; q)$ and $|S(0, (p+n)c; c)| \leq \sum_{l|(p+n, c)} l$,\[ K \ll p^{1/2}q^{-1+\varepsilon} \sum_{n=1}^{\infty} \frac{d(n)}{n} e^{-\sqrt{n}/2q^2} |S(0, p+n; q)| \sum_{l|(p+n, c)} l \ll p^{1/2}q^{-1+\varepsilon} \sum_{n=1}^{\infty} \frac{d(n)\lambda(p+n)}{n} e^{-\sqrt{n}/2q^2} |S(0, p+n; q)|. \]

We break the sum over $n$ according to whether $q|(p+n)$ or $q \nmid (p+n)$. The contribution of the latter is $O(p^{1/2}q^{-1+\varepsilon})$. That of the former is
\[ \ll p^{1/2}q^\varepsilon \sum_{l=1}^{\infty} \frac{d(l)d(ql - p)}{ql - p} e^{-\sqrt{ql - p}/2q^2} \ll p^{1/2}q^\varepsilon(q - p)^{-1+\varepsilon} + p^{1/2}q^{-1+\varepsilon}. \]

The lemma follows. \[ \square \]

The case of $Y$ is more complicated as $Y_0$ is an oscillating function. For that we need the following standard lemma (for example, see [5]).

**Lemma 7.** Let $v \geq 0$ and $J$ be a positive integer. If $f$ is a compactly supported $C^\infty$ function on $[Y, 2Y]$, and there exists $\beta > 0$ such that
\[ y^j f^{(j)}(y) \ll_j (1 + \beta Y)^j \]
for $0 \leq j \leq J$, then for any $\alpha > 1$, we have
\[ \int_0^\infty Y_v(\alpha y)f(y)dy \ll \left( \frac{1 + \beta Y}{1 + \alpha Y} \right)^J Y. \]

**Lemma 8.** For $Y$ defined as in (1), we have
\[ Y \ll p^{1/2}q^{-1+\varepsilon}. \]
Proof. We have

\[ Y = \frac{4\pi^2}{q^2} \sum_{c < q} \frac{1}{c^2} \sum_{n=1}^{\infty} d(n) S(0, p - n; cq) y(n), \]  

where

\[ y(n) = \int_0^{\infty} Y_0 \left( \frac{4\pi \sqrt{nt}}{cq} \right) J_1 \left( \frac{4\pi \sqrt{tp}}{cq} \right) W_q \left( \frac{4\pi^2 t}{q} \right) \xi(t) \frac{dt}{\sqrt{t}}. \]  

We make a smooth dyadic partition of unity that \( \xi = \sum_k \xi_k \), where each \( \xi_k \) is a compactly supported \( C^\infty \) function on the dyadic interval \([X_k, 2X_k]\). Moreover, \( \xi_k \) satisfies \( x^j \xi_k^{(j)}(x) \ll 1 \), for all \( j \geq 0 \). We work on each \( \xi_k \) individually, but we write \( \xi \) instead of \( \xi_k \) and, accordingly, \( X \) rather than \( X_k \).

By the change of variable \( x := 2\sqrt{t}/cq \), we have

\[ y(n) = cq \int_0^{\infty} Y_0(2\pi \sqrt{nx}) J_1(2\pi \sqrt{px}) W_q(\pi^2 c^2 q x^2) \xi \left( \frac{c^2 q^2 x^2}{4} \right) dx. \]

We define

\[ f(x) := J_1(2\pi \sqrt{px}) W_q(\pi^2 c^2 q x^2) \xi \left( \frac{c^2 q^2 x^2}{4} \right). \]

This is a \( C^\infty \) function compactly supported on \([\rho, 2\rho]\), where \( \rho = 2\sqrt{X}/cq \).

We first treat the case \( 1/2 \leq X \leq q \). We note that this involves \( O(\log q) \) dyadic intervals. From Lemma 5 we have \( x^j W^{(j)}(x) \ll_j \log q \) for \( 1/q \ll x \ll 1 \). This, together with the recurrence relation \( (x^v J_v(x))' = x^v J_{v-1}(x) \), gives

\[ x^j f^{(j)}(x) \ll_j (1 + \sqrt{px})^j \log q. \]  

We are in a position to apply Lemma 7 to \( f \) with \( \alpha = 2\pi \sqrt{n} \), \( \beta = \sqrt{p} \) and \( Y = \rho = 2\sqrt{X}/cq \). The lemma yields, for any positive integer \( J \),

\[ y(n) \ll cq\rho \left( \frac{1 + \sqrt{p}\rho}{1 + \sqrt{n}\rho} \right)^J \log q. \]  

Later, we will break the sum over \( n \) in (3) in the following way

\[ \sum_{n \geq 1} = \sum_{n \leq \rho^{-\kappa}} + \sum_{n > \rho^{-\kappa}}, \]

where \( \kappa > 2 \) will be chosen later. The estimate (6) will be used for \( n > \rho^{-\kappa} \). We need another estimate for the range \( n \leq \rho^{-\kappa} \). For this we go back to (4), using \( Y_0(x) \ll 1 + |\log x| \) and \( J_1(x) \ll x \), to derive

\[ y(n) \ll \frac{\sqrt{pX}}{cq} (\log q)^2. \]
We denote by $Y_1$ and $Y_2$ the corresponding splitted sums ($Y = Y_1 + Y_2$). For the first sum, using (7), we have

$$Y_1 = \frac{4\pi^2}{q^2} \sum_{c < q} \frac{1}{c^2} \sum_{n \leq \rho^{-\kappa}} d(n)S(0, p - n; cq)y(n)$$

$$\ll p^{1/2}Xq^{3+\varepsilon} \sum_{n \leq \rho^{-\kappa}} \frac{d(n)|S(0, p - n; q)|}{n^{1/2}} \sum_{l \mid (p - n, c)} l \sum_{c < q} \frac{1}{c^2} \sum_{l \mid (p - n, c)} l$$

$$\ll p^{1/2}Xq^{3+\varepsilon} \sum_{n \leq \rho^{-\kappa}} \frac{d(n)|S(0, p - n; q)|}{n^{1/2}} \sum_{l \mid (p - n, c)} l \sum_{c < q} \frac{1}{c^2} \sum_{l \mid (p - n, c)} l$$

$$\ll p^{1/2}q^{-1+\varepsilon} \sum_{n \leq \rho^{-\kappa}} \frac{d(n)d(p - n)}{n^{2/\kappa}} |S(0, p - n; q)|$$

$$\ll p^{1/2}q^{-5+\varepsilon}.$$ \hspace{1cm} (8)

For the second sum, we note that $\sqrt{p} \rho \ll 1$ in this range. Using (6), we have

$$y(n) \ll \sqrt{X}(\log q)n^{-1/2}\rho^{-J}.$$ 

Similarly to above, we deduce that

$$Y_2 = \frac{4\pi^2}{q^2} \sum_{c < q} \frac{1}{c^2} \sum_{n \geq \rho^{-\kappa}} d(n)S(0, p - n; cq)y(n)$$

$$\ll \sqrt{X}q^{-2+\varepsilon} \sum_{n \geq \rho^{-\kappa}} \frac{d(n)}{n^{1/2}} |S(0, p - n; q)| \sum_{c \mid (p - n, c)} l \rho^{-J}$$

$$\ll X^{-(J - 1)/2}q^{-J-2} \sum_{n} \frac{d(n)}{n^{1/2}} |S(0, p - n; q)| \sum_{l \mid (p - n)} l^{-1} \sum_{c \mid (p - n)} c^{J-2}$$

$$\ll q^{-1+\varepsilon} \sum_{n} \frac{d(n)d(p - n)}{n^{J-2-(J - 1)/\kappa}} |S(0, p - n; q)|.$$ \hspace{1cm} (9)

To this end, we choose $\kappa = 2 + \varepsilon/2$ and $J$ large enough so that $J/2 - (J - 1)/\kappa > 1$. We hence obtain $Y_1 \ll p^{1/2}q^{-1+\varepsilon}$ and, since the sum over $n$ in (9) converges, $Y_2 \ll q^{-1+\varepsilon}$.

For $X > q$, similarly to (5), using the bound $x^jW^j(x) \ll_j x^{-2}$, we have

$$x^j f^{(j)}(x) \ll_j (1 + \sqrt{p}x)^j q^{-2}(cx)^{-4}.$$ 

Lemma 7 then gives

$$y(n) \ll cq \left(1 + \sqrt{p}\rho \right) \frac{1}{1 + \sqrt{p}\rho} \frac{1}{X^2}.$$ 

For the range $n \leq \rho^{-\kappa}$, a better bound than (7) in this case is

$$y(n) \ll \frac{\sqrt{\pi X}}{cq} (\log q) \frac{q^2}{X^2}.$$ 

Since $q^2/X^2 \ll 1$, all the previous estimates remain valid. The only place where this is not the case is the sum over $n \ll (q^2/2\sqrt{X})^\kappa$ in (8). However, this sum is void for $X > q^4/4$ and the former estimate still works in the larger interval $X \leq q^4/4$. Also, the
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quantity saved $q^2/X^2$ is sufficient to allow the sum over the dyadic values of $X$ involved to converge. The lemma follows.

The proof of the theorem is complete.

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