A family of exact solutions for unpolarized Gowdy models

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ABSTRACT

Unpolarized Gowdy models are inhomogeneous cosmological models that depend on time and one spatial variable with complicated nonlinear equations of motion. There are two topologies associated with these models, $T^3$ (three-torus) and $S^1 \times S^2$. The $T^3$ models have been used for numerical studies because it seems to be difficult to find analytic solutions to their nonlinear Einstein equations. The $S^1 \times S^2$ models have even more complicated equations, but at least one family of analytic solutions can be given as a reinterpretation of known solutions. Various properties of this family of solutions are studied.
I. Introduction

In 1971 Gowdy proposed a set of inhomogeneous cosmological metrics with the idea of constructing vacuum models in which inhomogeneity appeared in a simple way [1]. He wanted his models to have compact topology and two commuting Killing vectors constructed so that the model depended only on time and one of the spatial variables. Models of this sort have the metric structure of axisymmetric or cylindrically symmetric static or stationary metrics with the radial coordinate replaced by $t$ and the metric independent of the new “radial” coordinate instead of being independent of time. Such metrics share many of the properties of axisymmetric metrics and a number of well-known results for these metrics can be applied to the Gowdy models almost without change.

There are two topologies for the $t =$ constant three surfaces of these metrics that were proposed by Gowdy. One was a $T^3$ (three-torus) topology and another was an $S^1 \times S^2$ topology. For some reason the $T^3$ topology metrics have been used extensively for a number of studies of different types, while the models with $S^1 \times S^2$ topology have been almost forgotten.

In the notation of Berger and Garfinkle [2] the original $T^3$ Gowdy model takes the form

$$ds^2 = e^{-\lambda/2}e^{\tau/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}[e^Pd\sigma^2 + e^{-P}d\delta^2],$$

(1.1)

where $\lambda$ and $P$ are functions of $\theta$ and $\tau$. The model is compactified by requiring $0 \leq \theta, \sigma, \delta \leq 2\pi$, and the metric functions must be periodic in $\theta$. It is often stated that this form of the metric can be achieved by coordinate transformations alone, but (see for example, [3]) the form of the coefficient of $d\delta^2$ comes from the fact that the determinant of the two-metric $g_2$ formed by the $d\sigma^2$ and $d\delta^2$ terms, which in this case is $e^{-2\tau}$, is a solution of a certain combination of the Einstein equations. If we assume that $g_2$ is $g_2(\tau, \theta)$, and define $T \equiv e^{-\tau}$, then

$$R_\delta^2 + R_\sigma^2 = 0 \Rightarrow \frac{(g_2)_{TT}}{g_2} - \frac{1}{2} \left( \frac{(g_2)_T}{g_2} \right)^2 - \left[ \frac{(g_2)_{\theta\theta}}{g_2} - \frac{1}{2} \left( \frac{(g_2)_\theta}{g_2} \right)^2 \right] = 0.$$  

(1.2)

This expression is valid for any two-metric analogous to the $d\sigma^2$, $d\delta^2$ part of (1.1) of the form $A(\theta, \tau)d\sigma^2 + B(\theta, \tau)d\sigma d\delta + C(\theta, \tau)d\delta^2$, not just the special case given in (1.1), a fact that we will use below. In the case of (1.1) $g_2 = T^2 = e^{-2\tau}$ is a solution of (1.2), which allows us to write the $d\delta^2$ term of (1.1) in the form given above.

The Einstein equations for (1.1) become a linear equation for $P$ and a set of first-order equations for $\lambda$ which can be integrated to give $\lambda$ given any solution to the equation for $P$. These Einstein equations are easily solved in terms of ordinary Bessel functions, and several particular solutions were given by Gowdy [1].

Gowdy [4] also proposed “unpolarized” models (the name comes from the fact that for small values of $P$ the metric [1.1] becomes that of polarized linearized gravitational waves, and the unpolarized models become that of unpolarized waves), which, in the notation of [2] become, for the $T^3$ models,

$$ds^2 = e^{-\lambda/2}e^{\tau/2}(-e^{-2\tau}d\tau^2 + d\theta^2) + e^{-\tau}(e^Pd\sigma^2 + 2e^PQd\sigma d\delta + [e^{-P} + e^PQ^2]d\delta^2),$$

(1.3)

where the form of the $d\delta^2$ term comes from taking $g_2 = e^{-2\tau}$ as above, and $\lambda$, $P$, $Q$ are functions of $\theta$ and $\tau$. The Einstein equations for (1.3) are a set of coupled nonlinear
equations for $P$ and $Q$, and, again, first-order equations for $\lambda$ that can be integrated once $P$ and $Q$ are known.

The models (1.3) have been used for numerical studies [5], since it is usually assumed that there is little chance of finding analytic solutions for such a complicated system of equations. One aim of these studies has been to show that, in general, the solutions to these models become asymptotically velocity term dominated (AVTD) as $\tau \to +\infty$, that is, near the singularity of the model. The idea of AVTD models, originally due to Belinskii, Khalatnikov and Lifshitz [6] (BKL), is that near a singularity in the ADM action,

$$I = \frac{1}{16\pi} \int [\pi^{ij} g_{ij} - N H_\perp - N_i H^i] dt d^3 x,$$

$$H_\perp = \frac{1}{\sqrt{g}} [\pi^{ij} \pi_{ij} - \frac{1}{2} (\pi^k_k)^2] - \sqrt{g^3} R],$$

$$H^i = -2 \pi^{ij} \left| \partial_j \right|,$$

where $^3R$ is the curvature on $t = \text{const.}$ surfaces and $\left| \right|$ is a covariant derivative on these surfaces, the $H^i = 0$ terms will imply that the curvature (“potential”) term in the $H_\perp$ can, at best, be ignored completely, or at worst become algebraic in $g_{ij}$, so that the ADM equations can be solved as if the metric were homogeneous (with the $g_{ij}$ completely independent functions of time at each point $x$). The conjecture of BKL is that this behavior is generic, an idea which is still being debated. Numerical studies of the models (1.3) (see [2] and references therein) have shown that these models do indeed become AVTD except at a set of isolated points ($\theta = \theta_i$), where there are “spikes” in the solution. The origin of these spikes is studied in Ref. [2].

While the $T^3$ Gowdy models have been used extensively in classical and minisuperspace quantum gravity [7], the $S^1 \times S^2$ topology seems to have been almost ignored. The polarized form of these models can be written, in the notation of [2], as

$$ds^2 = e^{-\lambda/2} e^{\tau/2} (-e^{-2\tau} d\tau^2 + d\theta^2) + \sin(e^{-\tau}) [e^P d\delta^2 + e^{-P} \sin^2 \theta d\phi^2],$$

(1.6)

where $g_2 = \sin^2 T \sin^2 \theta$ is a solution of (1.2). These models have Einstein equations that consist, as above, of a linear equation for $P$ and first-order equations for $\lambda$ that can be directly integrated. The linear equation can be solved by separation of variables, and several particular solutions were given by Gowdy [1].

The unpolarized form of these models,

$$ds^2 = e^{-\lambda/2} e^{\tau/2} (-e^{-2\tau} d\tau^2 + d\theta^2) + \sin(e^{-\tau}) [e^P d\delta^2 + 2 e^P Q d\delta d\phi + (e^P Q^2 + e^{-P} \sin^2 \theta) d\phi^2],$$

(1.7)

(where, once again, $g_2 = \sin^2 e^{-\tau} \sin^2 \theta$ gives the form of the $d\phi^2$ term), has Einstein equations similar to those of (1.3), though quite a bit more complicated. These models seem to have been ignored, and there are, as far as we know, no numerical or analytic solutions given in the literature.

One could argue that the fact that the Einstein equations for (1.7) are even more complicated than those for (1.3) would mean that only numerical solutions could be found. However, we will show that at least one family of particular analytic solutions is readily
available as a reinterpretation of the portion of the Kerr metric [7] between the inner, 
\( r_- = M - \sqrt{M^2 - a^2} \), and the outer, \( r_+ = M + \sqrt{M^2 - a^2} \) horizons (where the “radial” coordinate is timelike). We will study some of the properties of this family of solutions and compare some of the results with those of numerical solutions of (1.3). One important point is that these solutions are not AVTD. Of course, since such particular solutions form a set of measure zero in the space of all solutions, it is not clear what the generic behavior of these models near the singularities is. Another point is that the “singularities” of these models are not true curvature singularities, since they represent points where the solution crosses a lightlike surface into regions where the metric is the usual stationary axisymmetric black hole metric. We will discuss possible generalizations briefly.

The paper is organized as follows: in Sec. II we will give the Einstein equations for all the metrics listed above, and show that the Kerr metric between its horizons is a family of exact solutions for the unpolarized \( S^1 \times S^2 \) model. Sec. III discusses the properties of the solutions (including possible AVTD behavior). Sec. IV consists of conclusions and some suggestions for extending the results of this article.
II. Exact Solutions

Equations of Motion

The equations of motion for the polarized Gowdy models were given in the original works of Gowdy [1, 4]. In our parametrization the equations for the $T^3$ cosmologies become, as we have mentioned above, a linear equation for $P$,

$$P_{,\tau\tau} - e^{-2\tau} P_{,\theta\theta} = 0,$$  \hfill (2.1)

and a set of first-order equations for $\lambda$,

$$\lambda_{,\tau} = P_{,\tau}^2 + e^{-2\tau} P_{,\theta}^2,$$  \hfill (2.2a)

$$\lambda_{,\theta} = 2P_{,\theta}P_{,\tau},$$  \hfill (2.2b)

which, in principle can be integrated directly to give $\lambda$ if they obey an integrability condition which is just (2.1). Eq. (2.1) is exactly soluble in terms of trigonometric functions of $\theta$ and ordinary Bessel functions of $\tau$, and there are even tabulated integrals that give $\lambda$ explicitly in certain cases [8]. Some of these solutions were given by Gowdy in his original papers.

The equations for models with $S^1 \times S^2$ topology are only slightly more complicated, again we have a linear equation for $P$,

$$P_{,\tau\tau} - \frac{e^{-2\tau}}{\sin \theta} (\sin \theta P_{,\theta})_{,\theta} - e^{-2\tau} - [e^{-\tau} \cot(e^{-\tau}) - 1] P_{,\tau} = 0,$$  \hfill (2.3)

this equation can be separated in terms of Legendre polynomials and a function of $\tau$. Some exact solutions to this equation were given by Gowdy [1] (in another coordinate system). The equations for $\lambda$ become

$$\cot(e^{-\tau}) \lambda_{,\theta} - 2e^{\tau} (P_{,\tau}P_{,\theta}) + \cot \theta [-e^{\tau} \lambda_{,\tau} + 2e^{\tau} P_{,\tau} + e^{\tau} + 2\cot(e^{-\tau})] = 0,$$  \hfill (2.4a)

$$\cot(e^{-\tau})(\lambda_{,\tau} - 1) - e^{\tau}[(P_{,\tau})^2 + e^{-2\tau}(P_{,\theta})^2]$$

$$+ e^{-\tau} [\cot^2(e^{-\tau}) + 4] + e^{-\tau} (-\cot \theta \lambda_{,\theta} + 2 \cot \theta P_{,\theta}) = 0.$$  \hfill (2.4b)

The unpolarized models have somewhat more complicated equations. The equations for $P$ become nonlinear because nonlinear terms in $Q$ are added, and an equation for $Q$ appears which has nonlinear terms in both $Q$ and $P$. The $T^3$ model has been used extensively for numerical studies of such concepts as AVTD models [2, 5]. Its equations for $P$ and $Q$ are

$$P_{,\tau\tau} - e^{-2\tau} P_{,\theta\theta} - e^{2\tau} (Q_{,\tau}^2 - e^{-2\tau} Q_{,\theta}^2) = 0,$$  \hfill (2.5a)

$$Q_{,\tau\tau} - e^{-2\tau} Q_{,\theta\theta} + 2(P_{,\tau}Q_{,\tau} - e^{-2\tau} P_{,\theta}Q_{,\theta}) = 0,$$  \hfill (2.5b)

and once these are solved one can integrate the equations for $\lambda$,

$$\lambda_{,\tau} - [P_{,\tau}^2 + e^{-2\tau} P_{,\theta}^2 + e^{2\tau} (Q_{,\tau}^2 + e^{-2\tau} Q_{,\theta}^2)] = 0,$$  \hfill (2.6a)

$$\lambda_{,\theta} - 2(P_{,\theta}P_{,\tau} + e^{2\tau} Q_{,\theta}Q_{,\tau}) = 0.$$  \hfill (2.6b)
One of the reasons usually given for solving these equations numerically is that there is little chance of finding analytic solutions to such a complicated system of nonlinear equations.

The unpolarized $S^1 \times S^2$ models have even more complicated dynamical equations for $P$ and $Q$,

$$
P_{,\tau\tau} - e^{-2\tau} \frac{(\sin \theta P_{,\theta})_{,\theta}}{\sin \theta} - e^{-2\tau} - \frac{e^{2P}}{\sin^2 \theta} \{(Q_{,\tau})^2 - e^{-2\tau} (Q_{,\theta})^2\} - \left[ e^\tau \cot(e^{-\tau}) - 1 \right] P_{,\tau} = 0,
$$

$$
Q_{,\tau\tau} - e^{-2\tau} Q_{,\theta \theta} + e^{-2\tau} \cot \theta Q_{,\theta} + 2(P_{,\tau} Q_{,\tau} - e^{-2\tau} P_{,\theta} Q_{,\theta}) - \left[ e^\tau \cot(e^{-\tau}) - 1 \right] Q_{,\tau} = 0.
$$

(2.7a)

There are two coupled first-order equations for $\lambda$ (which, of course, can be reduced to separate equations for $\lambda_{,\tau}$ and $\lambda_{,\theta}$, but give equations that are more complicated than the originals). They are

$$\cot(e^{-\tau}) \lambda_{,\theta} - 2e^\tau (P_{,\tau} P_{,\theta} + \frac{e^{2P} Q_{,\tau} Q_{,\theta}}{\sin^2 \theta})$$

$$+ \cot \theta [-e^\tau \lambda_{,\tau} + 2e^\tau P_{,\tau} + e^\tau + 2 \cot(e^{-\tau})] = 0,$$

(2.8a)

$$\cot(e^{-\tau})(\lambda_{,\tau} - 1) - e^\tau [(P_{,\tau})^2 + e^{-2\tau} (P_{,\theta})^2] - e^\tau \frac{e^{2P}}{\sin^2 \theta} [(Q_{,\tau})^2 + e^{-2\tau} (Q_{,\theta})^2]$$

$$+ e^{-\tau} [\cot^2(e^{-\tau}) + 4] + e^{-\tau} (-\cot \theta \lambda_{,\theta} + 2 \cot \theta P_{,\theta}) = 0.$$

(2.8b)

One could expect that it might be difficult to find exact solutions to this system of equations, but one is at hand without any work.

**Black Holes inside their Horizons.**

Kantowski and Sachs [9] studied cosmological models with four-dimensional groups of motion which had three-dimensional subgroups, and one special case was a cosmological model of the form

$$ds^2 = -N^2(t) dt^2 + e^{2\sqrt{3}\beta(t)} dr^2 + e^{-2\sqrt{3}\beta(t)} e^{-2\sqrt{3}\Omega(t)} (d\theta^2 + \sin^2 \theta d\varphi^2).$$

(2.9)

They gave a one parameter family of solutions of the form

$$N^2 = \frac{1}{\frac{\alpha}{t} - 1}, \quad e^{2\sqrt{3}\beta} = \frac{\alpha}{t} - 1, \quad e^{-2\sqrt{3}\beta(t)} e^{-2\sqrt{3}\Omega(t)} = t^2.$$

(2.10)

They note that this family is nothing more than the Schwarzschild model inside the horizon. That is, if we write the usual Schwarzschild metric in Schwarzschild coordinates, we have

$$ds^2 = -(1 - \frac{2m}{r}) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

(2.11)

and for $r < 2m$ we can make the coordinate transformation $t \leftrightarrow r$ and we find (2.9) with the functions given by (2.10).

One advantage of looking at the Kantowski-Sachs model as Schwarzschild is that we can see the character of the singular points of the metric. The singularity at $t = 0$ is the true singular point of Schwarzschild ($r = 0$), where the curvature is infinite, but the singularity at $t = \alpha$ is just a lightlike surface where the curvature is regular and we pass
from the Kantowski-Sachs region to the Schwarzschild region. Something similar happens with the vacuum Taub model [10], where there are two singular points, but both of them are lightlike surfaces where we pass into the NUT [11, 12] region.

The Gowdy models have two commuting Killing vectors, and in the \( T^3 \) and \( S^1 \times S^2 \) topologies are axisymmetric. If we were able to find a static or stationary axisymmetric metric with a horizon of the Schwarzschild type for which the region inside of the horizon had the proper character, then one could find a solution to a Gowdy model in the same way that one finds the Kantowski-Sachs solution from Schwarzschild. One difference is that in the Schwarzschild case the Birkhoff theorem says that the Kantowski-Sachs model is the general solution to the Einstein equations for this model. In the Gowdy case we can probably expect no more than a particular solution.

A metric that generalizes Schwarzschild and has two commuting Killing vectors is the Kerr metric. In Boyer-Lindquist coordinates the Kerr metric is

\[
ds^2 = -\frac{r^2 - 2Mr + a^2}{r^2 + a^2 \cos^2 \theta} [dt - a \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{r^2 + a^2 \cos^2 \theta} [(r^2 + a^2) d\phi - adt]^2
\]

\[+ \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2Mr + a^2} dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2. \tag{2.11}\]

For this metric the outer horizon, \( r_+ = M + \sqrt{M^2 - a^2} \) is a lightlike surface where the light cones tip over to the point where \( \partial/\partial r \) is a timelike vector, so that a coordinate change of the \( r \leftrightarrow t \) is possible. Unfortunately, at the inner horizon, \( r_- = M^2 - \sqrt{M^2 - a^2} \), another lightlike surface, the light cones tilt back to the point where \( \partial/\partial r \) becomes spacelike again. In the region \( r_- < r < r_+ \) we can make the transformation \( r \leftrightarrow t \) and the Kerr metric becomes a cosmological model,

\[
ds^2 = \frac{2Mt - t^2 - a^2}{t^2 + a^2 \cos^2 \theta} [dr - a^2 \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{t^2 + a^2 \cos^2 \theta} [(t^2 + a^2) d\phi - adr]^2
\]

\[+ \frac{t^2 + a^2 \cos^2 \theta}{2Mt - t^2 - a^2} dt^2 + (t^2 + a^2 \cos^2 \theta) d\theta^2. \tag{2.12}\]

This metric, with the simple transformation

\[t = \alpha [\sqrt{1 - \beta^2 \cos(e^{-\tau})} + 1], \tag{2.13}\]

\[\alpha = M, \beta = a/M, \text{ makes } (2.12) \text{ into } (1.7) \text{ with}
\]

\[\lambda = \tau - 2 \ln(\alpha^2 [\sqrt{1 - \beta^2 \cos(e^{-\tau})} + 1]^2 + \beta^2 \cos^2 \theta), \tag{2.14a}\]

\[P = \ln[(1 - \beta^2) \sin^2(e^{-\tau}) + \beta^2 \sin^2 \theta] - \ln[\alpha \sqrt{1 - \beta^2 \sin(e^{-\tau})}]
\]

\[- \ln([\sqrt{1 - \beta^2 \cos(e^{-\tau})} + 1]^2 + \beta^2 \cos^2 \theta) \tag{2.14b}\]

\[Q = -\frac{2\alpha \beta \sin^2 \theta [\sqrt{1 - \beta^2 \cos(e^{-\tau})} + 1]}{(1 - \beta^2) \sin^2(e^{-\tau}) + \beta^2 \sin^2 \theta} \tag{2.14c}\]

Even though it is obvious that Eqs. (2.14) are a solution of the vacuum Einstein equations (since the Kerr metric is a solution), we have, using a symbolic manipulation program, inserted (2.14) into (2.7) and (2.8) and showed explicitly that these expressions are solutions. In the next section we will discuss the properties of this family of solutions.
III. Properties of the Solutions

The general behavior of this family of solutions is given in Figs. 1 and 2, which show

\[ P + \ln[\alpha \sqrt{1 - \beta^2 \sin(e^{-\tau})}] \]

and \( Q \) as functions of \( \theta \) for various values of \( \tau \). The functions are relatively flat at \( \tau = -\ln \pi \), which is the “big bang”, then grow a spike in the region of \( \theta = \pi/2 \) which disappears as \( \tau \to +\infty \), so the functions become flat again at the “big crunch”. As \( \tau \) goes either to \( -\ln \pi \) or \( +\infty \), \( P \) becomes infinite, but the combination

\[ P + \ln[\alpha \sqrt{1 - \beta^2 \sin(e^{-\tau})}] \]

(i.e. adding a \( \theta \)-independent term) is finite for all \( \theta \) and \( \tau \) except at the extremes of \( \theta \), \( \theta = 0 \) and \( \theta = \pi \), where, for \( \tau = -\ln \pi \) and \( \tau = +\infty \), it becomes \( -\infty \) \( (e^P = 0) \).

At first glance the “spiky” behavior of these solutions seems to have something to do with similar spikes that occur in numerical solutions. While we will show that there is a remote connection between the two situations, the spikes in our solution show up at different parts of the evolution from the numerical spikes.

The simplest manner of showing the difference between the numerical and analytic solutions is to study the Hamiltonian formulation of the Einstein equations for the \( S^1 \times S^2 \) topology, which is similar to that of the \( T^3 \) topology given in Refs. [2, 13]. Equations (2.7) can be modified by defining \( e^{P'} = e^P / \sin \theta \) and collecting the first and last terms of each equation. They become

\[ e^{-\tau} \csc(e^{-\tau})(e^\tau \sin(e^{-\tau})P'_{\tau},_\tau) - e^{-2\tau} \frac{(\sin \theta P'_{\theta},_\theta)}{\sin \theta} = -e^{P'} \{Q^2_{,\tau} + e^{-2\tau} Q^2_{,\theta}\} = 0, \]  \hspace{1cm} (3.1a)

\[ e^{-\tau} \csc(e^{-\tau})(e^\tau \sin(e^{-\tau})Q_{,\tau},_\tau) - e^{-2\tau} \frac{(\sin \theta Q_{,\theta},_\theta)}{\sin \theta} + 2(P'_{,\tau} Q_{,\tau} - e^{-2\tau} P'_{,\theta} Q_{,\theta}) = 0. \]  \hspace{1cm} (3.1b)

If we define a new time variable \( w \) by means of \( dw = e^{-\tau} \csc(e^{-\tau})d\tau \), or

\[ w = -\ln \left[ \tan \left( \frac{e^{-\tau}}{2} \right) \right], \]  \hspace{1cm} (3.2)

then, using \( \sin(e^{-\tau}) = \text{sech} \ w \), and multiplying through by \( e^{2\tau} \sin^2(e^{-\tau}) \), Eqs. (3.1) reduce to

\[ P'_{,ww} - \text{sech}^2 w \frac{(\sin \theta P'_{\theta},_\theta)}{\sin \theta} - e^{2P'} \{Q^2_{,w} + \text{sech}^2 w Q^2_{,\theta}\} = 0, \]  \hspace{1cm} (3.3a)

\[ Q'_{,ww} - \text{sech}^2 w \frac{(\sin \theta Q_{,\theta},_\theta)}{\sin \theta} + 2(P'_{,w} Q_{,w} - \text{sech}^2 w P'_{,\theta} Q_{,\theta}) = 0. \]  \hspace{1cm} (3.3b)

These equations are similar to those of the \( T^3 \) topology [2], with \( P_{,\theta \theta} \) and \( Q_{,\theta \theta} \) replaced by \( (\sin \theta P'_{\theta},_\theta)/\sin \theta \), \( (\sin \theta Q_{,\theta},_\theta)/\sin \theta \) and \( e^{-2\tau} \) replaced by \( \text{sech}^2 w \). If we look at (3.2), we see that near the big bang, \( w \to -\infty \), and near the big crunch, \( w \to +\infty \), \( \text{sech}^2 w \to e^{-2|w|} \).

Given the similarity between Eqs. (3.3) and those of the \( T^3 \) case, it is not difficult to construct the Hamiltonian form of the action that gives them [14],

\[ I = \int d\theta dw \left[ \Sigma P'_{,\theta} + \Sigma Q_{,\theta} \right] \]
\[-\frac{1}{2} \left\{ \frac{\Pi_{P'}^2}{\sin \theta} + \frac{e^{-2P'}\Pi_Q^2}{\sin \theta} + \sin \theta \text{sech}^2 w[(P'_\theta)^2 + e^{2P'}Q^2_{\theta}] \right\}. \quad (3.4)\]

Since \(\text{sech}^2 w \rightarrow e^{-2|w|}\) for \(w \rightarrow \pm \infty\), one might expect that the solutions (2.14) would become AVTD at both the big bang and the big crunch. In this case the potential terms would become negligible and the Hamiltonian density would approach

\[\mathcal{H} = \frac{\Pi_{P'}^2}{2 \sin \theta} + \frac{e^{-2P'}\Pi_Q^2}{2 \sin \theta}, \quad (3.5)\]

which is, again, similar to the \(T^3\) expression given in Refs. [2, 13, 14]. This reduced Hamiltonian has an exact solution of the form

\[P' = P_0 + \ln[\cosh vw + \cos \psi \sinh vw], \quad (3.6a)\]

\[Q = Q_0 + \frac{e^{P_0} \sin \psi \tanh vw}{1 + \cos \psi \sinh vw}, \quad (3.6b)\]

\[\Pi_{P'} = \left( \frac{v \sinh vw + v \cos \psi \cosh vw}{\cosh vw + \cos \psi \sinh vw} \right) \sin \theta, \quad (3.6c)\]

\[\Pi_Q = e^{P_0} v \sin \psi \sin \theta, \quad (3.6d)\]

where \(P_0, v, Q_0\) and \(\psi\) are arbitrary functions of \(\theta\).

If the solution (2.14) is to be AVTD at either the big bang or the big crunch, we have to rewrite \(P\) and \(Q\) in terms of \(w\) and see whether \(P'\) and \(Q\) take the form of \(P'\) and \(Q\) in (3.6). Since \(P\) and \(Q\) in (2.14) are functions of \(\sin(e^{-\tau})\) and \(\cos(e^{-\tau})\), and we have \(\sin(e^{-\tau}) = \text{sech} w, \cos(e^{-\tau}) = \tanh w\) and \(P' = P - \ln(\sin \theta)\), we find that

\[P' = \ln \left[ \frac{(1 - \beta^2) \text{sech} w + \beta^2 \sin^2 \theta \cosh w}{\alpha \sqrt{1 - \beta^2 \sin \theta(\sqrt{1 - \beta^2 \tanh w} + 1)^2 + \beta^2 \cos^2 \theta)} \right], \quad (3.7a)\]

\[Q = \frac{-2\alpha \beta \sin^2 \theta[\sqrt{1 - \beta^2 \tanh w} + 1]}{(1 - \beta^2) \text{sech}^2 w + \beta^2 \sin^2 \theta}. \quad (3.7b)\]

There are at least two ways to take the limit as \(w \rightarrow \pm \infty\); one is to assume that \(\text{sech} w \rightarrow 0\) and that

\[P' \rightarrow \ln \left[ \frac{\beta^2 \sin \theta \cosh w}{\alpha \sqrt{1 - \beta^2(2 - \beta^2 \sin^2 \theta + 2\sqrt{1 - \beta^2})}} \right]\]

\[= \ln \left( \frac{\beta^2 \sin \theta \cosh w}{\alpha \sqrt{1 - \beta^2(2 - \beta^2 \sin^2 \theta + 2\sqrt{1 - \beta^2})}} \right) + \ln \cosh w, \quad (3.8a)\]

\[Q \rightarrow \frac{-2\alpha}{\beta} [1 + \sqrt{1 - \beta^2 \tanh w}], \quad (3.8b)\]

which imply \(v = 1, \psi = \pi/2\), \(P_0 = \ln(\beta \sin \theta / \alpha \sqrt{1 - \beta^2(2 - \beta^2 \sin^2 \theta + 2\sqrt{1 - \beta^2})})\), \(Q_0 = -2\alpha / \beta\). However, the coefficient of \(\tanh w\) in (3.8b) is not \(e^{-P_0}\), which it should be from
(3.6b). It is worse if we calculate the momenta. For example, \( \Pi_Q = e^{P'} Q_w \sin \theta \), and using the expression (3.7b) we find that \( \Pi_Q \) calculated in this manner is not the same as (3.6d).

While this is the simplest limit, it is possible to keep terms of order \( e^{-w} \) and still have a limit that has approximately the form of (3.6). Since \( \text{sech} w \to e^{-|w|} \), \( \cosh w \to e^{+|w|} \), and \( e^{\pm |w|} = \cosh |w| \pm \sinh |w| \), and \( \tanh w \to \pm 1 \), the limit of \( P' \) as \( w \to \pm \infty \) is

\[
P' \to -\ln \left[ \frac{\alpha \sqrt{1 - \beta^2} (2 - \beta^2 \sin^2 \theta \pm 2 \sqrt{1 - \beta^2})}{1 - \beta^2 \cos^2 \theta} \right]
\]

\[
+ \ln \left[ \frac{|w| + [1 - \beta^2 (1 + \sin^2 \theta)]}{1 - \beta^2 \cos^2 \theta} \sinh \theta \right],
\]

which implies \( \cos \psi = \pm [1 - \beta^2 (1 + \sin^2 \theta)]/(1 - \beta^2 \cos^2 \theta) \), \( v = 1 \), \( P_0 = (1 - \beta^2 \cos^2 \theta)/(2 \alpha \beta \sqrt{1 - \beta^2} (2 - \beta^2 \sin^2 \theta \pm 2 \sqrt{1 - \beta^2}) \}

As \( w \to \pm \infty \), \( \tanh w \to \pm 1 \), the limit of \( P' \) as \( w \to \pm \infty \) is

\[
\frac{-2\alpha \beta \sin^2 \theta (1 - \beta^2) e^{-|w|}}{(1 - \beta^2 e^{-|w|} + \beta^2 \sin^2 \theta)},
\]

and using, as above, \( e^{\pm |w|} = \cosh |w| \pm \sinh |w| \), we find that

\[
Q \to -\frac{2\alpha \beta \sin^2 \theta (1 + \sqrt{1 - \beta^2} \cosh w \pm (1 + \frac{3}{2} \sqrt{1 - \beta^2} \sinh w)}{(1 - \beta^2 \cos^2 \theta) \cosh w \pm [-1 + \beta^2 (1 + \sin^2 \theta) \sinh w]}
\]

\[
= \frac{-2\alpha \beta \sin^2 \theta (1 + \frac{3}{2} \sqrt{1 - \beta^2}) \left[\frac{1 + \frac{3}{2} \sqrt{1 - \beta^2} \tanh w}{1 - \beta^2 \cos^2 \theta} \right]}{(1 - \beta^2 \cos^2 \theta) \left[1 + \frac{\beta^2 (1 + \sin^2 \theta) - 1}{1 - \beta^2 \cos^2 \theta} \tanh w \right]},
\]

which is of the form of (3.6b) with \( Q_0 = -2\alpha \beta \sin^2 \theta (1 + \frac{1}{2} \sqrt{1 - \beta^2})/(1 - \beta^2 \cos^2 \theta) \) and \( \psi \) as above. However, if we calculate \( \Pi_Q \) as before, it is not the same as (3.6d).

The fact that these solutions do not become AVTD is, in fact, not surprising if we study the velocity \( v \) [2, 14],

\[
v = \frac{1}{\sin \theta} \sqrt{\Pi_{P'}^2 + e^{-2P'} \Pi_Q^2}
\]

that appears in (3.6). This quantity can be calculated for all values of \( w \), and is

\[
v(\theta, w) = \frac{\cosh w}{[(1 - \beta^2) \text{sech}^2 w + \beta^2 \sin^2 \theta]([\sqrt{1 - \beta^2} \tanh w + 1]^2 + \beta^2 \cos^2 \theta)}
\]

\[
\times \left[\sinh w [(1 - \beta^2) \text{sech}^2 w - \beta^2 \sin^2 \theta]([\sqrt{1 - \beta^2} \tanh w + 1]^2 + \beta^2 \cos^2 \theta)\right]
\]
As \( w \to \pm \infty \), \( v \to 1 \) and \( P' \to |w| \).

If we consider Eq. (3.3a), the solution will be AVTD if the terms involving \( \theta \)-derivatives decay much more rapidly than those involving \( w \)-derivatives. The term involving \( Q^2_{,\theta} \),

\[
\text{sech}^2 w e^{2P'_{,\theta}} Q^2_{,\theta} \to e^{-2|w|} e^{2|w|} Q^2_{,\theta}
\]
does not necessarily decay rapidly as \( w \to \pm \infty \). However, calculating \( Q_{,\theta} \), we see that it goes to zero as \( w \to \pm \infty \), so this term does not contribute in the limit. However, calculating \( e^{2P'_{,w}} \) and \( P'_{,ww} \), we see that they decay as \( \text{sech}^2 w \) (which is consistent with the AVTD form from \([3.6]\), where they should decay as \( \text{sech}^2 v \)). However, \( (\sin \theta P'_{,\theta})_{,\theta}/\sin \theta \) becomes constant in \( w \) as \( w \to \pm \infty \), so \( \text{sech}^2 w (\sin \theta P'_{,\theta})_{,\theta}/\sin \theta \) is of the same order as \( P'_{,ww} \)
and \( e^{2P'_{,w}} \), and it is not surprising that the solution (2.14) does not become AVTD as \( w \to \pm \infty \). Of course, if \( v \) were less than one, \( P'_{,ww} \) and \( e^{2P'_{,w}} \) would become AVTD at \( w = 0 \) and \( w = 0 \) sheds some light on the origin of the spike at \( \beta = \pi/2 \).

If we consider \( v(\theta, 0) \), we find that

\[
v(\theta, 0) = \frac{2\sqrt{1 - \beta^2 \cos^2 \theta}}{1 + \beta^2 \cos^2 \theta}
\]

which varies from \( 2\sqrt{1 - \beta^2}/(1 + \beta^2) \) at \( \theta = 0, \pi \) to two at \( \theta = \pi/2 \). The value at \( \theta = 0, \pi \) is always less than two, and for \( \beta \) near one the difference becomes large. Notice that the system becomes formally AVTD at \( w = 0, \theta = 0 \), in that both \( (P'_{,\theta})^2 \) and \( (Q_{,\theta})^2 \) are zero at this point, while \( v = 2 \). If one could assume that this fact was relevant to the time development of the system, it would show that there is an indication that the spiky behavior of Refs. [2, 5], which is tied to \( v > 1 \) at specific points of the manifold, is present here also. However, in our case, “velocity dominated” at one point in \( \theta \) is meaningless, since the \( \theta \)-derivative terms in (3.3) can be shown to be considerable at \( \theta = \pi/2 \). Nevertheless, the existence of these velocity peaks may give us some clue about the true meaning of the spikes.

One last property of our solutions is their behavior as functions of \( \beta \). As \( \beta \to 0 \), the Kerr solution (2.12) becomes Schwarzschild, and our Gowdy model becomes Kantowski-Sachs (compactified in the “radial” coordinate), and one of the singularities \( (w \to -\infty) \) becomes a curvature singularity. For the equivalent of extreme Kerr, \( \beta = 1 \), both \( \lambda \) and \( Q \) seem reasonable, although time independent, but \( P \) is singular. However, the inner and outer horizons become the same, and there is no Gowdy solution. Notice that \( t \) from (2.13) becomes \( \alpha \) for all values of \( \tau \).
IV. Conclusions and Suggestions for Further Research

We have shown that the Kerr metric between the inner and outer horizons can be interpreted as an exact solution to the Gowdy models with $S^1 \times S^2$ topology. The Gowdy solutions generated by this method are relatively flat functions of $\theta$ at the horizons (the "big bang" at the inner horizon and the "big crunch" at the outer horizon) which grow a spike near $\theta = \pi/2$ in both the Gowdy functions $P(\theta, \tau)$ and $Q(\theta, \tau)$.

The singularities in the solutions are just lightlike surfaces where the solution passes from the coordinate patch suitable for the Gowdy interpretation to patches where we have true Kerr (black hole) metrics. One important point is that we have compactified in the $\delta$-direction which becomes the $t$-direction in the Kerr patches, giving solutions with closed timelike lines in the outer regions, a property shared with the Taub-NUT solution. Notice that if one compactifies the Kantowski-Sachs model (2.9) in the $r$-direction, there are also closed timelike lines in the Schwarzschild portion.

The solutions we have found are not AVTD near the singularities, which can be shown directly from the solutions and the Hamiltonian form of the action which generates their equations of motion. The fact that the velocity that would appear in the AVTD solution is one seems to give a convincing argument why this is so.

There are a number of directions in which this analysis can be extended. An obvious idea is to study numerical solutions of Eqs. (3.3), which might shed light on the structure of solutions for these models more general than the particular solution (2.14), and could give reasons for the form that (2.14) takes.

Another possibility [16] would be to create new solutions to the models where the big bang and big crunch "singularities" become true curvature singularities. The technique for doing this is described in Ref [17].

One final way to extend our idea would be to consider other models which have cosmological sectors and "black hole" sectors separated by horizons. There are a number of metrics where one or the other of these sectors is well known, but where the other has never been interpreted as either a "black hole" solution or a cosmological model. The cosmological sector of some of these metrics can be compactified as we did with the metric (1.7) (or they may naturally admit a compact topology), leading to "black hole" sectors with closed timelike lines. The global geometry of some of these manifolds (including the Kantowski-Sachs model compactified in the $r$ coordinate of Eq. (2.9)) has not been investigated in detail.

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FIGURE CAPTIONS

Figure 1a-g. These figures show the evolution of \( P - h, \ h = \ln[\alpha\sqrt{1 - \beta^2}\sin(e^{-\tau})] \), as a function of \( \theta \) for the solution given in Eqs. (2.14) for \( \alpha = 1 \) and \( \beta = 1/2 \). Fig. 1a corresponds to \( \tau = -1.1447 \), Fig. 1b to \( \tau = -1.144 \), Fig. 1c to \( \tau = -1.1 \), Fig. 1d to \( \tau = -0.75 \), Fig. 1e to \( \tau = -\ln(\pi/2) \), Fig. 1f to \( \tau = +5 \), and Fig. 1g to \( \tau = +10 \).

Figure 2a-d. These figures show the evolution of \( Q \) as a function of \( \theta \) for the solution given in Eqs. (2.14) for \( \alpha = 1 \) and \( \beta = 1/2 \). Fig. 2a corresponds to \( \tau = -1.1439 \), Fig. 2b to \( \tau = -1.1 \), Fig. 2c to \( \tau = -\ln(\pi/2) \), and Fig. 2d to \( \tau = +5 \).
(Figure 1f)
(Figure 1g)
(Figure 1d)
(Figure 1b)
(Figure 1a)
(Figure 2a)

\[ \theta \]