Fractional Calculus: A Commutative Method on Real Analytic Functions

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Abstract

The traditional first approach to fractional calculus is via the Riemann-Liouville differintegral $aD^k_x$ [1]. The intent of this paper will be to create a space $K$, pair of maps $g : C^\omega(\mathbb{R}) \to K$ and $g' : K \to C^\omega(\mathbb{R})$, and operator $D^k : K \to K$ such that the operator $D^k$ commutes with itself, the map $g$ embeds $C^\omega(\mathbb{R})$ isomorphically into $K$, and the following diagram commutes;

\[
\begin{array}{ccc}
C^\omega(\mathbb{R}) & \xrightarrow{g} & K \\
\downarrow{aD^k_x} & & \downarrow{D^k} \\
C^\omega(\mathbb{R}) & \xrightarrow{g'} & K \\
\end{array}
\]

This implies the following diagram commutes, for analytic $f$ such that $aD^j_x f = 0$ (i.e, if $f = \sum_{i \in I} b_i (x-a)^i$, where $\{b_i\} \subset \mathbb{R}$, and $I \subseteq \{j - 1, ..., j - \lfloor j \rfloor\}$);

\[
\begin{array}{ccc}
f & \xrightarrow{g} & g(f) \\
\downarrow{aD^j_x} & & \downarrow{D^j} \\
0 & \xrightarrow{g'} & D^j g(f) \\
\downarrow{aD^{j+k}_x} & & \downarrow{D^k} \\
aD^{j+k}_x f & \xrightarrow{g'} & D^k D^j g(f) \\
\end{array}
\]

Convention

Henceforth, unless otherwise noted we assume all functions are real-analytic, thus equal to their Taylor series on some interval of $\mathbb{R}$. When a base point for a Taylor series is not given, we assume it converges on $\mathbb{R}$ or the function has been analytically continued. We let $C^\omega(\mathbb{R})$ denote the space of real analytic functions.
The Space $\mathbb{Z}_\omega(a)$

From basic real analysis, for any $f \in C^\omega(\mathbb{R})$, the Taylor series of $f$ (henceforth denoted $T(f)$), equal to $f$ on some open interval in $\mathbb{R}$, is defined by $T(f) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$ for some $a \in \mathbb{R}$. Note the collection $\{f^{(i)}(a) : i \in \mathbb{Z}_{\geq 0}\}$ together with the point $a$ uniquely define $f$ within $C^\omega(\mathbb{R})$. Then, for fixed $a \in \mathbb{R}$, there is a natural bijection between functions $\sigma : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ such that $\sum_{i=0}^{\infty} \frac{\sigma^{(i)}(a)}{i!}(x-a)^i$ converges on some interval about $a$, and functions $f \in C^\omega(\mathbb{R})$ equal to their Taylor series on some interval about $a$.

Define $\mathbb{Z}_\omega(a)$ to be the set of all functions $\sigma : \mathbb{Z} \to \mathbb{R}$ such that $\sum_{i=0}^{\infty} \frac{\sigma^{(i)}(a)}{i!}(x-a)^i$ converges on some interval about $a$. When the point $a$ is understood, or not central to the argument but assumed to be fixed, we may omit it and just write $\mathbb{Z}_\omega$. Note $\mathbb{Z}_\omega$ is non-empty, since the function $f : i \mapsto 0$ (all $i \in \mathbb{Z}$) is an element of $\mathbb{Z}_\omega$. Moreover, $\mathbb{Z}_\omega$ is a vector space with identity $1_{\omega} : i \mapsto 0$; if $\sigma, \rho \in \mathbb{Z}_\omega$ and $k \in \mathbb{R}$, $\sum_{i=0}^{\infty} \frac{(\sigma + \rho)^{(i)}(x-a)}{i!}(x-a)^i = \sum_{i=0}^{\infty} \frac{\sigma^{(i)}(x-a)}{i!}(x-a)^i + \sum_{i=0}^{\infty} \frac{\rho^{(i)}(x-a)}{i!}(x-a)^i \in C^\omega(\mathbb{R})$, and $\sum_{i=0}^{\infty} \frac{k\sigma^{(i)}(x-a)}{i!}(x-a)^i = k \sum_{i=0}^{\infty} \frac{\sigma^{(i)}(x-a)}{i!}(x-a)^i \in C^\omega(\mathbb{R})$.

The $R$ Operator

For $\sigma \in \mathbb{Z}_\omega(a)$, define the operator $R : \mathbb{Z}_\omega(a) \to C^\omega(\mathbb{R})$ by $R\sigma = \sum_{i=0}^{\infty} \frac{\sigma^{(i)}(a)}{i!}(x-a)^i$. Clearly $R$ is surjective, with kernel $\{\sigma \in \mathbb{Z}_\omega : \sigma(i) = 0, i \geq 0\}$. For some pair $(f, a) \in C^\omega(\mathbb{R}) \times \mathbb{R}$, we also define the operator $R^{-1} : C^\omega(\mathbb{R}) \times \mathbb{R} \to \mathbb{Z}_\omega(a)$ by

$$R^{-1}(f, a)(i) = \begin{cases} 0 & : i < 0 \\ f^{(i)}(a) & : i \geq 0 \end{cases}$$

This allows the following identities;

(R1) $RR^{-1}(f, a) = (f, a)$
(R1') $RR^{-1}f = f$
(R2) $R^{-1}R\sigma(i) = \begin{cases} \sigma(i) & : i \geq 0 \\ 0 & : i < 0 \end{cases}$ By definition of the maps $R$ and $R^{-1}$, it follows they are both homomorphisms, where $R^{-1}$ is injective with image $\mathbb{Z}_\omega/\ker(R)$, and $R$ is surjective.
The $D$-Operator and $\Gamma$ Function

Given $\sigma \in \mathbb{Z}_\omega(a)$, we define $D^k \sigma(i) = \sigma(i + k)$ for all $k \in \mathbb{Z}$. From the definition of the operator $D$, we immediately have the identities

(D1) $D^a D^b = D^b D^a$
(D2) $D^a D^b = D^{a+b}$
(D3) $D^a D^{-a} = D^{-a} D^a = D^0$
(D4) $D^a(\sigma + \rho) = D^a \sigma + D^a \rho$
(D5) $D^a(k\sigma) = kD^a \sigma$ for all $k \in \mathbb{R}$

Relating the $D$ Operator to Differentiation

Induction on the power rule provides the identity $\frac{d^a}{dx^a} x^k = \frac{k!}{(k-a)!} x^{k-a}$, and the relation $n! = \Gamma(n+1)$ provides the identity $\frac{d^a}{dx^a} x^k = \frac{\Gamma(k+1)}{\Gamma(k+1-a)} x^{k-a}$.

Applying these to Taylor series, we obtain the identity

$$T(f) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!} (x-a)^i = \sum_{i=0}^{\infty} \frac{f^i(a)}{\Gamma(i+1)} (x-a)^i$$

while the power rule allows for the identities

$$\frac{d^j}{dx^j} T(f) = T\left(\frac{d^j}{dx^j} f\right) = \sum_{i=0}^{\infty} \frac{i!}{(i-j)!} f^{(i)}(a)(x-a)^{i-j} = \sum_{i=0}^{\infty} \frac{\Gamma(i+1)}{\Gamma(i+1-j)} f^{(i)}(a)(x-a)^i$$

For $f \in C^\omega(\mathbb{R})$ and $\sigma \in \mathbb{Z}_\omega$ such that $R\sigma = f$, straightforward calculation yields the following identities for all $a \in \mathbb{Z}$;

(D6) $RD^a R^{-1} f = \frac{d^a}{dx^a} f = f^{(a)}$

(D7) $R^{-1} \frac{d^a}{dx^a} R\sigma(i) = \begin{cases} 
\sigma(i + a) & : i \geq -a \\
0 & : i < -a
\end{cases}$

(D8) $\frac{d^a}{dx^a} RD^{-a} \sigma = f$

Together, (D1) - (D8), along with (R1’) and (R2) will form the core of our arguments for the rest of the paper.

Finally, we slightly redefine the operator $R$ based on properties of the $\Gamma$ function. By definition, $R\sigma = \sum_{i=0}^{\infty} \frac{\sigma(i)}{\Gamma(i+1)}(x-a)^i$. However, for $i \leq 0$, $\frac{\sigma(i)}{\Gamma(i+1)} = 0$ so $\frac{\sigma(i)}{\Gamma(i+1)}(x-a)^i = 0$ and $\sum_{i=-\infty}^{\infty} \frac{\sigma(i)}{\Gamma(i+1)}(x-a)^i = \sum_{i=0}^{\infty} \frac{\sigma(i)}{\Gamma(i+1)}(x-a)^i = R\sigma$, so from this point on we will define
\[ R\sigma = \sum_{i=-\infty}^{\infty} \frac{\sigma(i)}{\Gamma(i+1)} (x-a)^i \]

Clearly, properties (R1), (R1'), (R2) and (D1) - (D8) still hold.

**Mapping \( C^\omega(\mathbb{R}) \) To \( \mathbb{R}_\omega \) And Back**

For a power function \( b(x-s)^\alpha \), the Riemann-Liouville derivative \( _aD^k_x \) is given by [1]

\[ _sD^k_x b(x-s)^\alpha = \frac{b}{\Gamma(-k)} \int_s^x (t-s)^\alpha (x-t)^{-k-1} dt = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} b(x-s)^{\alpha-k} \]

If, and only if, \( k \notin \mathbb{R} \) and \( \alpha + 1 - k \in \mathbb{Z}_{\leq 0} \), then the numerator of the fraction \( \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-k)} \) is finite while the denominator goes to \( \pm\infty \), and in the limit we see \( _sD^k_x f = 0 \). This shows that, when restricted to \( C^\omega(\mathbb{R}) \), \( \ker(_sD^k_x) = \{b(x-s)^\alpha : b, \alpha \in \mathbb{R}, \alpha + 1 - k \in \mathbb{Z}_{\leq 0}\} \). If we wish to preserve the identities (D1) - (D8), (R1') and (R2') when generalizing \( D^k \) to all real \( k \), we must define a new operator with a significantly smaller kernel. Note that (D1) - (D8) and (R1'), (R2) are only consistent if \( \ker(D^k) = \{0\} \), the zero function in \( \mathbb{Z}_\omega \).

To summarize the situation, then, on the one hand we have the (commutative) \( D \) operator on elements of \( \mathbb{Z}_\omega \) which, when coupled with the \( R \) operator, allows identities (D1) - (D8), and on the other we have the Riemann-Liouville derivative, which is commutative for analytic functions when the degrees of differentiation under consideration never sum to a nonpositive integer [2].

We now create maps \( f, f' \) and a generalization of the \( D \) operator to a space \( \mathbb{R}_\omega \) (to be defined) such that the following diagram commutes;

\[
\begin{array}{ccc}
C^\omega(\mathbb{R}) & \xrightarrow{f} & \mathbb{R}_\omega \\
_sD^k_x & \downarrow & D^k \\
C^\omega(\mathbb{R}) & \xrightarrow{f'} & \mathbb{R}_\omega
\end{array}
\]

It will, however, be more convenient to express the map \( f \) as a composition of maps \( C^\omega(\mathbb{R}) \xrightarrow{R^{-1}} \mathbb{Z}_\omega \xrightarrow{\iota} \mathbb{R}_\omega \) and extending the domain of the operator \( R \) to \( \mathbb{R}_\omega \) so \( f = \iota \circ R^{-1} \) and \( f' = R \). Our goal, then, will be to define the maps and spaces which make the following diagram commute, while maintaining analogs of (R1') and (R2);
Define $\mathbb{R}_\omega(a) = \{ \rho : \mathbb{R} \rightarrow \mathbb{R} : \sum_{i=-\infty}^{\infty} \frac{\Gamma(i+1-k)}{\Gamma(i+1)} \rho(i)(x-a)^{i-k} \in C^\omega(\mathbb{R}) \forall k \in \mathbb{R} \}$. By definition, for any $\rho \in \mathbb{R}_\omega$, $\rho|_z \in \mathbb{Z}_\omega$. In fact, for any $\rho = \rho(x) \in \mathbb{R}_\omega$, $\rho(x-k)|_z \in \mathbb{Z}_\omega$. Observing $D$ is merely a shift operator on $\mathbb{Z}_\omega(\mathbb{R})$, we naturally extend $D$ to $\mathbb{R}_\omega$ by setting $D_k \rho(i) = \rho(i-k)$ for all $k \in \mathbb{R}, \rho \in \mathbb{R}_\omega$. By definition of $\mathbb{R}_\omega$, $(D_k \rho)|_z \in \mathbb{Z}_\omega$ for all $k$. This leads to the natural extension of $R$ to $\mathbb{R}_\omega$ by $R \rho = R(\rho|_z) = \sum_{i=-\infty}^{\infty} \frac{\rho(i)}{\Gamma(i+1)} (x-a)^i$.

Properties (D1) - (D5) still hold for $D$ on $\mathbb{R}_\omega$, since elements of $\mathbb{R}_\omega$, like those of $\mathbb{Z}_\omega$, are functions. Thus, we are only left to define the map $\iota$ and verify its properties.

Let $\sigma \in \mathbb{Z}_\omega$, then define $\iota(\sigma)(z) = \lim_{k \rightarrow \infty} (aD_z^{x+k}RD^{-k}\sigma)(a)$ whenever the limit exists. We then have the following (equivalent) identities;

(I1) $\iota(\sigma)|_z = \sigma$

(I2) $(D_k \iota(\sigma))|_z = D_k \sigma$ when $k \in \mathbb{Z}$

(I3) $\iota(R^{-1}f)|_z = R^{-1}f$

(I4) $R(\iota(R^{-1}f)|_z) = f$

Identity (I4) is our analog of (R2), and (R1') follows from properties of $R$ and (I1). Finally, we will show Diagram 2 commutes; that is, $aD_z^{x+k} = RD^k \iota(R^{-1}f)$ for all $k \in \mathbb{R}$. Let $f \in C^\omega(\mathbb{R})$, and $f_k : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f_k(z) = (aD_z^{x+k}(RR^{-1}f))(a)$, then

$$RD^k \iota(R^{-1}f) = R\iota(R^{-1}f - k)$$
$$= R(f_k)$$
$$= \sum_{i=0}^{\infty} \frac{1}{\Gamma(i+1)} (aD_z^{-i+k}f)(a)(x-a)^i$$
$$= T(aD_z^k f)$$
$$= aD_z^k f$$
and the diagram commutes. Then (D7) and the following analogs of (D6) and (D8) hold;

\begin{align*}
(D6') \quad RD^k \iota (R^{-1} f) &= a D^k_x f \\
(D8') \quad a D^k_x RD^{-k} \iota (R^{-1} f) &= f.
\end{align*}

**Conclusion**

In conclusion, we have created a space $\mathbb{R}_\omega$, and a collection of maps and operators $R, R^{-1}, D^k, \iota$ such that the operator $D^k$ acts exactly the same as the Riemann-Liouville operator as $s D^k_x$ when applied to an element of $C^\omega (\mathbb{R})$ mapped through $\mathbb{R}_\omega$, and the operator $D^k$ commutes with itself.

That is, if $f \in C^\omega (\mathbb{R})$ is such that $s D^j_x f = 0$ for some $j \in \mathbb{R}$, then for $\sigma = R^{-1} f, \rho = \iota (\sigma)$, and all $k \in \mathbb{R}$, we have the following commutative diagram

\[
\begin{array}{ccccccccc}
f & \xrightarrow{R^{-1}} & \sigma & \xrightarrow{\iota} & \rho \\
\downarrow a D^{j+k}_x & & & & \downarrow a D^j_x \\
0 & \xleftarrow{R} & D^k \rho & \xleftarrow{D^k} & a D^{j+k}_x f
\end{array}
\]

which is equivalent to the second diagram in the abstract. This, together with the second diagram in this section - which is equivalent to the first diagram in the abstract - completes the paper.

**References**

[1] Keith B. Oldham and Jerome Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover, Mineola, New York, 2006.

[2] Kenneth S. Miller, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley-Interscience, 1993.