Remarks on a paper by Cordero and Nicola on Feichtinger’s Wiener amalgam spaces and the Schrödinger equation

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Abstract

We derive some consequences of very recent results of Cordero and Nicola on the metaplectic representation, the Wiener amalgam spaces, (whose definition is due to Feichtinger), and their applications to the regularity of the solutions of Schrödinger equation with quadratic Weyl symbol. We do not however discuss the validity of Cordero and Nicola’s claims.

1 Introduction

There are very few results in the literature about the regularity of the solutions of Schrödinger’s equation in terms of the Wiener amalgam spaces introduced by Feichtinger in the early 1980s. Very recently, Cordero and Nicola [1] have proposed such a study for the quantum oscillator (their study actually applies to larger classes of quadratic Hamiltonians as well); their study is based on properties of the action of the metaplectic group on Feichtinger’s spaces they prove.

The aim of this short Note is to derive a few consequences of Cordero and Nicola’s results by using the complete properties of the metaplectic operators they invoke. We do not, however, discuss the validity of their results.

2 The metaplectic group

Let Sp(n) be the symplectic group of the space $\mathbb{R}^{2n}$ equipped with the standard symplectic structure $\sigma = dp \wedge dx$. It is a connected classical Lie
group, contractible to the unitary group $U(n, \mathbb{C})$. We thus have $\pi_1[\text{Sp}(n)] = (\mathbb{Z}, +)$ and $\text{Sp}(n)$ therefore has covering groups of all orders. It turns out that the double cover $\text{Sp}_2(n)$ can be faithfully represented by a group of unitary operators on $L^2(\mathbb{R}^n)$. That group is the metaplectic group $\text{Mp}(n)$. Since the projection $\pi : \text{Mp}(n) \to \text{Sp}(n)$ is two-to-one, one associates to each $\mathcal{A} \in \text{Sp}(n)$ two operators $\pm \mu(\mathcal{A}) \in \text{Mp}(n)$. In particular, if

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \det B \neq 0$$

then

$$\pm \mu(\mathcal{A}) f(x) = (2\pi \hbar)^{-n/2} m^{-n/2} |\det B|^{-1/2} \int e^{i W(x,x')} f(x') d^n x'$$

where $m$ ("the Maslov index [6, 12]") corresponds to a choice of $\arg \det B$ and

$$W(x,x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'.$$

It turns out that each $\mu(\mathcal{A})$ can be written as a product of exactly two operators of the type above. In fact, writing $\mu_{W,m}(\mathcal{A})$ for the operator (2) one the Maslov index $m$ has been chosen:

**Proposition 1** For every operator $\mu(\mathcal{A}) \in \text{Mp}(n)$ there exist quadratic forms $W$, $W'$ and integers (modulo 4) $m$ and $m'$ such that

$$\mu(\mathcal{A}) = \mu_{W,m}(\mathcal{A}) \mu_{W',m'}(\mathcal{A}).$$

(This factorization is of course not unique).

**Proof.** See de Gosson [4, 5, 6, 7, 9]. ■

**Remark 2** A similar result holds on the matrix level: every $\mathcal{A} \in \text{Sp}(n)$ can be written as a product of two symplectic matrices [7].

### 3 On Cordero and Nicola’s estimate

In [1], Cordero and Nicola prove the following continuity result for metaplectic operators on Wiener amalgam spaces (Theorem 4.1, formula (25), p. 17):

**Proposition 3** Let $\mathcal{A} \in \text{Sp}(n) be as in (1). Then, for $1 \leq p, q \leq \infty$,

$$||\mu(\mathcal{A}) f||_{W(F_{L^p,L^q})} \leq \alpha(\mathcal{A}, p, q) ||f||_{W(F_{L^q,L^p})}.$$  (3)

where $\alpha(\mathcal{A}, p, q) > 0$
We claim that:

**Proposition 4** Let $A \in \text{Sp}(n)$. Then, for $1 \leq p, q \leq \infty$,

$$||\mu(A)f||_{W(F^p, L^q)} \leq C_{A, p, q}||f||_{W(F^p, L^q)}$$  \hspace{1cm} (4)

where $C_{A, p, q} > 0$ only depends on $A, p, q$.

**Proof.** It immediately follows from (3) using Proposition 1. Thus:

**Corollary 5** Metaplectic operators are continuous operators on the Wiener amalgam spaces $W(F^p, L^q)$, $1 \leq p, q \leq \infty$.

**Remark 6** This result can actually be obtained in a simpler way, using different methods (the Weyl representation of metaplectic operators studied in de Gosson [8]).

## 4 Schrödinger’s equation

Consider the time-dependent Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} f(x, t) = H_{\text{Weyl}} f(x, t) \hspace{1cm} f(x, 0) = f_0(x)$$  \hspace{1cm} (5)

where the operator $H_{\text{Weyl}}$ is the partial differential operator with Weyl symbol

$$H(x, p) = \frac{1}{2} (x, p)M(x, p)^T$$

where $M$ is a real symmetric $2n \times 2n$ matrix. The corresponding Hamilton equations of motion

$$\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial x}$$

are then linear; the Hamiltonian flow determined by these equations is thus a one-parameter subgroup $(A_t)$ of $\text{Sp}(n)$. In view of the lifting theorem of algebraic topology (see e.g. [11]), there exists a unique one-parameter group $(\mu(A_t))$ of $\text{Mp}(n)$ covering $(A_t)$.

**Proposition 7** Assume $f_0 \in L^2(\mathbb{R}^n)$ and set $f(x, t) = \mu(A_t)f_0(x)$. The function $f$ is the (unique) solution of Schrödinger’s equation (5).

**Proof.** See [11] [9]. Thus:

An immediate consequence of this classical result is:
Corollary 8 Assume that $f_0 \in W(\mathcal{F}L^p, L^q)$. Then $f(\cdot, t) \in W(\mathcal{F}L^p, L^q)$ for all $t$.

Proof. It immediately follows from Proposition 7 using the estimate (4) in Proposition 4.

5 Conclusion

The usefulness of Feichtinger’s Wiener amalgam spaces in quantum mechanics is obvious, as is the usefulness of Feichtinger’s algebra [3], which deserves to be investigated separately. We will come back in a forthcoming paper [10] to a precise study of then regularity of the solutions to Schrödinger’s equation for quite arbitrary Hamiltonians (i.e. not necessarily associated to a quadratic Hamiltonian) in terms of these spaces.

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