On Errors Generated by Unitary Dynamics of Bipartite Quantum Systems

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Abstract—Given a quantum channel it is possible to define the non-commutative operator graph whose properties determine a possibility of error-free transmission of information via this channel. The corresponding graph has a straight definition through Kraus operators determining quantum errors. We are discussing the opposite problem of a proper definition of errors that some graph corresponds to. Taking into account that any graph is generated by some POVM we give a solution to such a problem by means of the Naimark dilatation theorem. Using our approach we construct errors corresponding to the graphs generated by unitary dynamics of bipartite quantum systems. The cases of POVMs on the circle group \( \mathbb{Z}_n \) and the additive group \( \mathbb{R} \) are discussed. As an example we construct the graph corresponding to the errors generated by dynamics of two mode quantum oscillator.

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1. INTRODUCTION

Since the famous work by Peter Shor [1] it were developed many approaches to construction of quantum error-correcting codes. All these approaches require to have some conjectures on how noise acts on the quantum states which we want to preserve. Mathematically the action of a noise is described by a set of completely positive maps, called errors, on the set of states of a quantum system. The choice of this set is a separate theoretical problem. There are various types of conjectures, for example, Shor code [1] corrects an arbitrary error in one physical qubit of a 9-qubit cluster in which we encode one logical qubit. Or in the case of stabilizer codes [2] noise is given by a subset of \( n \)-fold Pauli group.

In the general setting of that problem [3] for any set of errors, it is possible to define a unique non-commutative operator graph [4], the knowledge of this graph is enough to determine the existence of quantum error-correcting code and define all possible codes for given errors. Such a correspondence between sets of errors and non-commutative operator graphs is not one-to-one, but each non-commutative operator graph describes codes for some set of errors in that sense [5, 6]. So it could be meaningful to explore the problem of quantum error-correcting codes for graphs with parametrizations given in some separate way from mentioned error-correction formalism, in that description each conjecture on a graph also is the conjecture on a possible noise. It is shown [5–7], that any non-commutative operator graph is the closure of the linear envelope of some positive operator valued measure (POVM). So the problem of quantum error correction could be viewed in terms of POVMs. In particular, if a non-commutative operator graph is linearly generated by a POVM covariant with respect to the action of a unitary-represented group it is possible to give the sufficient conditions of error-correcting code existence. Several examples of such graphs were constructed [8–11].
Suppose that we have a solution to the error-correction problem for the graph which we define through some suitable parametrization, then it immediately appears a back problem on how to define errors generating this graph. In the present article we give partial solutions to this problem for graphs generated by POVMs. In Section 2 we give a detailed description of the problem and describe the set of errors corresponding to the graph generated by POVM on the locally compact group $G$ by means of the Naimark dilation theorem. In Section 3 we consider the special case of the cyclic group $G = \mathbb{Z}_n$. Section 4 is devoted to the group $G = \mathbb{R}$. The example of two-mode quantum oscillator is studied in Section 5. The last section is for conclusions.

2. ERRORS ASSOCIATED WITH GRAPHS GENERATED BY POVM

Denote $S_+(H)$ the set of positive nuclear operators in a Hilbert space $H$. Together with $S_+(H)$ we shall use the convex set of quantum states $S(H)$ consisting of unit trace operators from $S_+(H)$. Any $V$ belonging to the algebra of all bounded operators $B(H)$ in $H$ defines a linear completely positive map on $S_+(H)$ by the formula

$$\Phi_V : \rho \to V \rho V^*, \quad \rho \in S_+,$$

which can be considered as an error appearing under the information transmission. In the case, a state $\rho \in S(H)$ is mapped to $\Phi_V(\rho) = \frac{\text{Tr}(V^* \rho)}{\text{Tr}(\rho)}$. Let us pick up the set of errors $\Phi_V$ having the form (1) and define a linear space consisting of linear bounded operators $V = \text{span}(V^*_jV_k)$.

Following to the general theory of error correcting codes [3] if there exists the orthogonal projection $P$ with the property

$$PV^*_kV_kP = c_{jk}P$$

(2)

for some $c_{jk} \in \mathbb{C}$, then any error of the form (1) for $V \in V$ can be corrected if $\text{supp} \rho \in PH$ in the sense that there is a completely positive map $\Psi$ such that

$$\Psi \circ \Phi_V(\rho) = d_\rho \rho,$$

(3)

where $d_\rho > 0$.

Given a quantum channel (completely positive trace preserving map) $\Phi : S(H) \to S(K)$ consider the Kraus decomposition

$$\Phi(\rho) = \sum_k V_k \rho V_k^*,$$

where $V_k : H \to K$ are linear bounded operators and

$$\sum_k V^*_kV_k = I.$$

A linear operator subspace $V \subset B(H)$ defined by the formula $V = \text{span}(V^*_jV_k)$ is said to be a non-commutative operator graph associated with $\Phi$. It immediately follows from the definition that

- $I \in V$;
- if $V \in V$ then $V^* \in V$.

Hence $V$ is an operator system in the sense of [12]. Moreover any operator system is a non-commutative operator graph associated with some quantum channel [5–7]. If there is a projection $P$ called a (quantum) anticlique [13] that has the property

$$PV^*_kV_k = \{CP\}$$

then (2) is satisfied and all errors generated by $V \in V$ can be corrected in the sense of (3).
Let $G$ be a locally compact group with the Haar measure $\mu$. Denote $\mathfrak{B}(G)$ the $\sigma$-algebra generated by compact subsets of $G$. The map $B \in \mathfrak{B}(G) \to M(B)$ from $\mathfrak{B}(G)$ to the cone of all positive operators in a Hilbert space $H$ is said to be a normalized positive operator valued measure (POVM) if [14, 15]

$$M(\emptyset) = 0, \quad M(G) = I \quad \text{and} \quad M(\cup_j B_j) = \sum_j M(B_j), \quad \text{for} \quad B_j \cap B_k = \emptyset, \quad j \neq k,$$

where the sum in the last equation converges in weak operator topology. Any non-commutative operator graph $V$ is generated by some POVM $M(B)$ such that

$$V = \overline{\text{span}}(M(B), \quad B \in \mathfrak{B}(G)).$$

Following to the Naimark dilation theorem $H$ can be isometrically embedded into a Hilbert space $K$ such that

$$M(B) = P_H E(B)|_H, \quad B \in \mathfrak{B}(G),$$

where $E(B)$ is an orthogonal projection valued measure and $P_H$ is a projection in $K$ onto $H$. Following to the Naimark construction it is possible to pick up $K$ generated by all $E(B)H$. Consider a set of completely positive maps $\Phi_G : B(H) \to B(K)$ defined by

$$\Phi_G(\rho) = E(B)\rho E(B), \quad \rho \in \mathfrak{S}(H). \quad (4)$$

Denote $V_B = E(B)|_H : H \to K$.

**Proposition 1.** Errors of the form (4) generate the non-commutative graph (2) in the sense

$$V = \overline{\text{span}}(V_B V_B', \quad B, B' \in \mathfrak{B}(G)).$$

**Proof.** Note that $V_B^* = P_H E(B) : K \to H$. Hence

$$V_B^* V_B' = P_H E(B) E(B') = P_H E(B \cap B') = M(B \cap B'),$$

$B, B' \in \mathfrak{B}(G)$.

\[ \square \]

3. THE CYCLIC GROUP $G = \mathbb{Z}_n$

Following to the ideas of [16] let us consider two mutually unbiased bases $(e_j)_{j=1}^{n-1}$ and $(f_j)_{j=0}^{n-1}$ in a finite dimensional Hilbert space $K$, $\dim K = n$. Define an orthogonal projection valued measure $E$ on $\mathbb{Z}_n$ by the formula $E(\{j\}) = |e_j\rangle\langle e_j|$, $j \in \mathbb{Z}_n$ and two unitary representations $\mathbb{Z}_n \ni j \to U_j$ and $\mathbb{Z}_n \ni j \to \hat{U}_j$ by the formulae

$$U_j = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{n} jk} |e_k\rangle\langle e_k|, \quad j \in \mathbb{Z}_n, \quad \hat{U}_j = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{n} jk} |f_k\rangle\langle f_k|, \quad j \in \mathbb{Z}_n.$$

Then,

$$\hat{U}_j E(\{k\}) \hat{U}_j^* = E(\{k + j\}), \quad j, k \in \mathbb{Z}_n,$$

and $E$ is covariant with respect to the action $j \to \hat{U}_j$.

Take a unit vector $f = \frac{1}{\sqrt{n-1}} \sum_{k=0}^{n-1} e_k$ and define the subspace $H$ by the condition $h \in H$ iff $(h, f) = 0$. It immediately follows from Proposition 1 that the statement below holds true.

**Corollary 1.** The graph

$$\mathcal{V} = \text{span}(P_H U_j|_H, \quad j \in \mathbb{Z}_n)$$

is generated by the errors $\rho \to U_j \rho U_j^*$, $\rho \in \mathfrak{S}(H)$. On the other hand, (3) is generated by the POVM

$$M(\{j\}) = P_H E(\{j\})|_H, \quad j \in \mathbb{Z}_n,$$

in $H$. 

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 41 No. 12 2020
Let $\mathbb{T} = [0, 2\pi)$ be the circle group with the operation $+/2\pi$. Following to [10] put $H = \mathbb{C}^n \otimes \mathbb{C}^n$ for $n \geq 2$ and denote $|j,k\rangle$, $j, k \in \mathbb{Z}_n$ the orthonormal basis in $H$. Let us consider the following reducible unitary representation of $\mathbb{T}$

$$\hat{U}_\varphi|j,k\rangle = e^{i\varphi j}|j,k\rangle, \quad \varphi \in \mathbb{T}.$$ 

We also need the orthonormal basis $|\eta_j^k\rangle$, $j, k \in \mathbb{Z}_n$ consisting of the generalized Bell states [17]

$$|\eta_j^k\rangle = \frac{1}{\sqrt{n}} \sum_{s \in \mathbb{Z}_n} e^{\frac{2\pi i}{n}js}|s,s-k\rangle.$$ 

So

$$\hat{U}_\varphi|\eta_j^k\rangle = \sum_{s \in \mathbb{Z}_n} e^{i(\varphi + \frac{2\pi i}{n})s}|s,s-k\rangle.$$ 

Consider the orthogonal projections

$$Q_j = \sum_{k \in \mathbb{Z}_n} |j - k j\rangle \langle j - k|, \quad j \in \mathbb{G}.$$ 

Then [11], the non-commutative operator graphs $\mathcal{V}_j = \text{span}(\hat{U}_\varphi Q_j \hat{U}_\varphi^*, \varphi \in \mathbb{T})$ coincide and the graph $\mathcal{V} \equiv \mathcal{V}_j$ has the following unitary generators

$$U_j = \sum_{k,l \in \mathbb{Z}_n} |\eta_j^k\rangle \langle \eta_l^k|, \quad j \in \mathbb{Z}_n.$$ (5)

Now Proposition 1 gives rise to the following.

**Corollary 2.** Formula (5) determines a unitary representation of $\mathbb{Z}_n$ in $H$. The graph $\mathcal{V} = \text{span}(U_j, j \in \mathbb{Z}_n)$ is generated by the errors $\rho \rightarrow U_j \rho U_j^*$, $\rho \in \mathcal{S}(H)$, $j \in \mathbb{Z}_n$.

4. THE CASE $G = \mathbb{R}$

Suppose that $H$ is isometrically embedded into $K = H \otimes H_E$ by means of the rule

$$f \rightarrow f \otimes e, \quad f \in H,$$ (6)

where $e$ is a fixed unit vector in a Hilbert space $H_E$.

Let $U_t : H \otimes H_E \rightarrow H \otimes H_E$, $t \in \mathbb{R}$, be a one-parameter unitary group describing the interaction between the system $H$ and its environment $H_E$. Denote $\mathfrak{B}(\mathbb{R})$ the $\sigma$-algebra of Borel sets on the real line $\mathbb{R}$. The Stone theorem reads

$$U_t = \int_{\mathbb{R}} e^{i t x} E(dx), \quad t \in \mathbb{R},$$

where $E$ is an orthogonal projection valued measure on $\mathbb{R}$. Our goal is to protect information encoded by states belonging to $\mathcal{S}(H)$ against errors having the form (4). Denote $\mathfrak{A}$ the commutative algebra generated by the projections $E(B)$, $B \in \mathfrak{B}(\mathbb{R})$. We consider linear completely positive maps from $\mathfrak{S}_+(H)$ to $\mathfrak{S}_+(H \otimes H_E)$ defined by the formula

$$\Phi_A(\rho) = A(\rho \otimes \rho_e)A^*, \quad \rho \in \mathfrak{S}(H), \ A \in \mathfrak{A}, \ \rho_e = |e\rangle \langle e| \in \mathfrak{S}(H_E)$$ (7)

as errors that can occur under the information transmission. Given $A \in \mathfrak{A}$ denote $V_A : H \rightarrow H \otimes H_E$ the linear operator defined by the formula $V_A f = A(f \otimes e)$, $f \in H$, then the adjoint operator $V_A^* : H \otimes H_E \rightarrow H$. Consider the linear operator space

$$\mathcal{V} = \text{span}(V_A^* V_{A'}, A, A' \in \mathfrak{A}).$$ (8)

**Proposition 2.** Suppose that $\rho_e = |e\rangle \langle e| \in \mathfrak{S}(H_E)$ is a fixed pure state of the environment. Then, the maps $\rho \rightarrow U_t(\rho \otimes \rho_e)U_t^*$ have the form (7). Moreover, the set of errors $\rho \rightarrow U_t \rho U_t^*$, $t \in \mathbb{R}$ generates the same operator space as (8).
shown that if the graph is linearly generated by POVM, then the Naimark dilatation operators determine the formula

\[ U_t^{(B_j)} = \sum_{j=1}^{n} e^{i t s_j} E_j, \quad t \in \mathbb{R}, \]

where \( s_j \in B_j \). Then, \( U_t^{(B_j)} \in \mathcal{A} \). Hence, \( U_t \in \mathcal{A} \) also because \( \mathcal{A} \) is closed. To finish the proof it is sufficiently to take into account that \( \text{span}(U_t, t \in \mathbb{R}) = \mathcal{A} \). \( \square \)

5. EXAMPLE: TWO MODE OSCILLATOR

Here we study the explicit example described in [8] in view of Section 4. For a Hamiltonian of a two-mode quantum oscillator

\[ H = -\frac{1}{2} \frac{\partial^2}{\partial x^2} - \frac{1}{2} \frac{\partial^2}{\partial y^2} + \frac{(x - y)^2}{2}, \]

acting in the bipartite quantum system \( K = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \), it is possible to give formulae for the unitary group \( U_t = e^{-itH} \) in terms of the products of coherent states. Given a complex number \( \alpha \in \mathbb{C} \), the coherent state is an eigenvector of the annihilation operator corresponding to eigenvalue \( \alpha \). The wave function of a coherent state equals

\[ \xi_{\alpha}(x) = \frac{1}{\pi^{1/4}} \exp \left( -\frac{|\alpha|^2}{2} \right) \exp \left( -\frac{x^2 - 2\sqrt{2}\alpha x + \alpha^2}{2} \right), \quad \alpha \in \mathbb{C}. \]

Given two complex numbers \( \alpha \) and \( \beta \) let us consider the following products

\[ \psi_{\alpha\beta}(x, y) = \frac{1}{\sqrt{2}} \xi_{\alpha} \left( \frac{x + y}{\sqrt{2}} \right) \xi_{\beta} \left( \frac{x - y}{\sqrt{2}} \right). \]

The group \( U_t \) can be defined by its action on the overcomplete system \( \psi_{\alpha\beta} \) such that

\[ (U_t \psi_{\alpha\beta})(x, y) = \frac{e^{-it\alpha \beta}}{\sqrt{2} \sqrt{1 + 2ti}} \xi_{\alpha} \left( \frac{x + y}{\sqrt{2}} \sqrt{1 + 2ti} \right) \xi_{\beta} \left( \frac{x - y}{\sqrt{2}} \right). \]

Take new variables \( \tilde{x} = \frac{x + y}{\sqrt{2}}, \tilde{y} = \frac{x - y}{\sqrt{2}} \) and denote \( \langle z | \alpha \rangle = \frac{1}{\sqrt{2}} \xi_{\alpha}(z), z \in \mathbb{C} \). Then, (5) goes to

\[ \langle \tilde{x}, \tilde{y} | U_t | \alpha \beta \rangle = \frac{e^{-it\alpha \beta}}{\sqrt{1 + 2ti}} \langle \tilde{x} | \alpha \rangle \langle \tilde{y} | e^{-i2t\beta} \rangle. \]

Following to (6) let us introduce the isometrical embedding of \( H = L^2(\mathbb{R}) \) into \( K = L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \) by the formula \( |\alpha\rangle \rightarrow |\alpha\rangle \otimes |\beta\rangle \equiv |\alpha\beta\rangle \), where \( \beta \) is a fixed complex number, while \( \alpha \) runs over \( \mathbb{C} \).

Put \( \rho_e = |\beta\rangle \langle \beta| \) and denote \( \tilde{H} = \text{span}(f \otimes |\beta\rangle, f \in H) \). The following Corollary is due to Proposition 2.

**Corollary 3.** The set of errors \( \rho \rightarrow U_t (\rho \otimes \rho_e) U_t^\dagger, t \in \mathbb{R}, \) generates the non-commutative operator graph

\[ \mathcal{V} = \text{span}(T_t = P_{\tilde{H}} U_t |_{\tilde{H}}, \quad t \in \mathbb{R}), \]

where

\[ \langle \tilde{x}, \tilde{y} | T_t | \alpha\beta \rangle = \exp \left( e^{-i2t|\beta|^2} - \frac{it}{\sqrt{2}} \right) \left( 1 + \sqrt{2ti} \right)^{-\frac{1}{2}} \langle \tilde{x} | \alpha \rangle \langle \tilde{y} | \beta \rangle. \]

6. CONCLUSION

We discussed the problem of searching for a quantum noise corresponding to the operator graph. It is shown that if the graph is linearly generated by POVM, then the Naimark dilatation operators determine errors for our graph. The cases of the graphs generated by covariant POVMs on the cyclic group \( \mathbb{Z}_n \) and on the additive group of reals \( \mathbb{R} \) were discussed separately. Also we introduced the explicit example originated from the unitary dynamics of the two-mode quantum oscillator.
ON ERRORS GENERATED BY UNITARY DYNAMICS

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REFERENCES

1. P. Shor, “Scheme for reducing decoherence in quantum memory,” Phys. Rev. A 52, 2493 (1995).
2. D. Gottesman, “Stabilizer codes and quantum error correction,” Ph. D. Thesis (Caltech, 1997); arXiv: quant-ph/9705052.
3. E. Knill and R. Laflamme, “Theory of error-correction codes,” Phys. Rev. A 55, 900 (1997).
4. R. Duan, S. Severini, and A. Winter, “Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovasz theta function,” IEEE Trans. Inform. Theory 59, 1164-1174 (2013).
5. R. Duan, “Superactivation of zero-error capacity of noisy quantum channels,” arXiv:0906.2527 (2009).
6. M. E. Shirokov and T. Shulman, “On superactivation of zero-error capacities and reversibility of a quantum channel,” Commun. Math. Phys. 335, 1159–1179 (2015).
7. V. I. Yashin, “Properties of operator systems, corresponding to channels (2020),” arXiv: 2004.13661 (2020).
8. G. G. Amosov, A. S. Mokeev, and A. N. Pechen, “Non-commutative graphs and quantum error correction for a two-mode quantum oscillator,” Quantum Inform. Process. 19, 95 (2020).
9. G. G. Amosov and A. S. Mokeev, “On non-commutative operator graphs generated by reducible unitary representation of the Heisenberg–Weyl group,” Int. J. Theor. Phys. (2018). https://doi.org/10.1007/s10773-018-3963-4
10. G. G. Amosov and A. S. Mokeev, “On non-commutative operator graphs generated by covariant resolutions of identity,” Quantum Inform. Process. 17, 325 (2018).
11. G. G. Amosov and A. S. Mokeev, “On linear structure of non-commutative operator graphs,” Lobachevskii J. Math. 40(10), 1440–1443 (2019).
12. M. D. Choi and E. G. Effros, “Injectivity and operator spaces,” J. Funct. Anal. 24, 156–209 (1977).
13. N. Weaver, “A “quantum” Ramsey theorem for operator systems,” Proc. Am. Math. Soc. 145, 4595-4605 (2017).
14. A. S. Holevo, Probabilistic and Statistical Aspects of Quantum Theory (Edizioni della Normale, 2011).
15. A. S. Holevo, Quantum System, Channels, Information (De Gruyter, Berlin, 2012).
16. G. G. Amosov, “On operator systems generated by reducible projective unitary representations of compact groups,” Turk. J. Math. 43, 2366–2370 (2019).
17. C. H. Bennett, G. Brassard, R. Jozsa, C. Crepeau, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels,” Phys. Rev. Lett. 70, 1895 (1993).