Latin Polytopes

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Abstract

Latin squares are well studied combinatorial objects. In this paper we generalize the concept and propose new objects like Latin triangles, free Latin squares, Latin tetrahedra, free Latin cubes, etc. We start with a classic definition of Latin squares followed by one based on the concept of latinized board. A Latin square appears then as a combinatorial design whose points are geometric and whose lines (the rows and columns) are invariant under the symmetries of the square. The generalization that follows proceeds by 1. broadening this geometric symmetry 2. considering more general configurations of points and 3. admitting lines that intersect more freely. The resulting concept is the Latin board. Finally, we particularize Latin boards to define Latin polytopes, Latin polygons and Latin polyhedra.

Keywords: asterism, biregular polygon, board, Board Symmetrization Problem, class-symmetric board, combinatorial design, hypergraph coloring, kaleidoscopic source, labeling, latinization, Latin board, Latin polyhedron, Latin polytope, Latin puzzle, Latin square, source of symmetry, symmetric board, symmetric Latin board, warp class, weft class, woven board.

1 Introduction

A Latin square of order \(n\) is an \(n \times n\) array filled with \(n\) different labels, each occurring exactly once in each row and exactly once in each column [17]. An example of order three with labels 1, 2 and 3 is shown in Fig. 1.

Latin squares have a long and rich history. They are mentioned in relation to magic squares as early as the 17th century, although they seem to have been used much earlier in amulets [8]. They are called “Latin” because the 18th century Swiss mathematician Leonhard Euler used latin letters in his paper On Magic Squares [12].

![Figure 1. Latin square](image)

\[ \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array} \]

1The set of labels is usually \(\{1, 2, \ldots, n\}\) because they double as indices for the rows and columns in the square.
In the 19th century the connection between Latin squares and finite projective geometries was studied, as well as their link with Algebra\(^2\) and Combinatorics. Already in the 20th century, applications to experimental design were formalized, especially in agricultural engineering [8].

More recently, the advent of computers and digital networks has brought codes based on Latin squares for error detection and correction in data telecommunications\(^3\).

**Geometric Symmetry as a Source of New Objects**

Looking at the Latin square in Fig. 1, it is easy to see that the result of a reflection across its vertical symmetry axis results in another Latin square. Other geometric transformations share this property: for example a 90° counter-clockwise rotation around its center, or a reflection across any of the two diagonals.

In fact any symmetry transformation of the square produces another Latin square. Geometric symmetry alone produces then new Latin squares from a given one. The examples that follow show other objects for which symmetry is also the source of new similar objects.

**Example 1.1.** Fig. 2 shows a square mesh of cells in which each pair of rows with the same letter holds all numbers from 1 to 16, something that also applies to pairs of columns similarly labeled.

This property still holds if, keeping the letters in place, we apply as before any symmetry of the square, like a reflection across the horizontal symmetry axis or a 180° counter-clockwise rotation around its center. In this paper we call objects like this one free Latin squares.

**Example 1.2.** Fig. 3 shows a triangular mesh of cells in which each pair of stripes of triangles pointed to by the same letter has all numbers from 1 to 12.

This condition still holds if, keeping the letters in place, we apply any symmetry transformation of the equilateral triangle, like a 120° counter-clockwise rotation around its center. This is an example of Latin triangle.

**Example 1.3.** Fig. 4 shows a triangular mesh in which each pair of stripes of triangles pointed to by the same letter contains all numbers from 1 to 18.

Here, the property also holds after any symmetry transformation of the regular hexagon, for example a 60° counter-clockwise rotation around its center. We call objects like this one Latin hexagons.

In this paper we generalize these objects by first defining boards as a pair of sets: one containing geometric points, the other subsets of the points. Next, we define Latin boards as specific labelings of the subsets. In both cases geometric symmetry is not taken into account.

Next we define symmetric boards as boards that are invariant under the action of a symmetry group. We finally introduce Latin polytopes, a particularization of Latin boards that, just like the examples mentioned, produce new ones under symmetry transformations.

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\(^2\)Latin squares are multiplication tables of quasi-groups [17].

\(^3\)For additional historical facts about Latin squares see [1, 25, 26].
Figure 2. Free Latin square

Figure 3. Latin triangle

Figure 4. Latin hexagon
2 Boards

In this section and the next, a generalization of Latin squares that does not involve geometric symmetry is proposed.

2.1 Definition of Board

Definition 2.1. A board is a pair \((P, C)\) where \(P\) is a finite set of geometric points and \(C\) is a set of subsets of \(P\) whose union is \(P\).

It follows from the definition that intersection among subsets is allowed. The set \(C\) is the constellation of the board. Each of its subsets is an asterism\(^4\).

Example 2.2. Fig. 5 (left) shows a set of seven \(\mathbb{R}^2\) points. On the right we have formed seven asterisms, each one with three points connected by two segments of the same color. The resulting board is a geometric representation of the Fano Plane\(^5\), a combinatorial structure with nice properties of balance: it has seven points and seven asterisms, with three points in each asterism and three asterisms sharing each point; furthermore, every two points define an asterism and every two asterisms share a point.

Example 2.3. Let \(P\) be the set of nine euclidean points in Fig. 6. Let’s make an asterism out of each set of three points connected by segments of the same color (right), and let \(C\) be the constellation formed by all asterisms. As every point is in \(C\) via an asterism we have the board \(B = (P, C)\).

2.1.1 Points and Cells in Boards

We will often represent a board with a grid of cells. To conciliate this with Def. 2.1 we will adopt the following convention: “cell” will mean either centroid of the cell or point, depending on the context. For example “write a label in the cell” means a centroid, whereas “write a label on a point” means a point.

\(^4\)We have borrowed these terms from Astronomy. They seem less ambiguous in this context than others used in related objects (see Sect. 2.2).
cell” will mean label the centroid of the cell, whereas “a board with n cells” will be a board with n points.

Example 2.4. If we remove the numbers inside the cells in Fig. 1, the resulting object is a board whose points are the (centroids of the) cells. Each row and column of (centroids of the) cells is an asterism of this board.

2.2 Incidence Structures Related to Boards

An abstract system consisting of two types of objects and a single relation between these types of objects is called an incidence structure [8]. Following Def. 2.1, boards are incidence structures in which the two types of objects are the board points and the asterisms. The incidence relation here is the membership of points in asterisms. We see next how boards relate to other existing incidence structures.

2.2.1 Families of Sets

Definition 2.5. A collection $C$ of subsets of a given set $P$ is called a family of subsets of $S$, or a family of sets over $S$.

Identifying respectively the points and the constellation of a board with the elements and the collection of subsets, a board is a family of sets in which no set is repeated in the family.

2.2.2 Finite Geometries

Finite geometries are geometric systems with a finite number of points and a collection of subsets of the points called lines. The incidence relation here is geometric incidence. The most studied finite geometries are finite projective geometries (in which, among other conditions, any two lines have a point in common) and finite affine geometries (in which a necessary condition is: if a point $p$ is not on a line $l_1$ then there is a unique line $l_2$ through $p$ having no points in common with $l_1$). As a finite geometry, the Fano Plane in Fig. 5 (right) is the smallest finite projective plane.

Identifying respectively the points and lines of the geometry with the points and asterisms of a board we see that certain boards are also finite geometries.

2.2.3 Simple Combinatorial Designs

Definition 2.6. A simple combinatorial design is a pair $(P, L)$ where $P$ is a set of points and $L$ is a set of subsets of $P$ called lines [24].

Boards are then simple combinatorial designs whose points are geometric and contained inside lines.

2.2.4 Hypergraphs

Definition 2.7. A hypergraph is a pair $(V, E)$ where $V$ is a finite set of elements called vertices and $E$ is a family of non-empty subsets of $V$ called hyperedges whose union is $V$.

In this specific case, the centers of the small squares.
Hypergraphs become graphs when hyperedges have just one or two vertices [6]. When the vertices of a hypergraph are geometric points we have a geometric hypergraph. If we call the points and asterisks of a board vertices and hyperedges respectively, boards are also geometric hypergraphs.

Example 2.8. Let $V$ be the nine euclidean points in Fig. 6 (left). Let’s make a hyperedge out of the three points connected by segments of the same color (right). Then the pair $(V, E)$ is a geometric hypergraph.

Hypergraphs, finite geometries, combinatorial designs and families of sets are then similar structures, the differences lie in the focus of interest. In hypergraphs the focus is on graph theoretic questions, like connectivity and colorability [6]. In finite geometries it is on geometric aspects like parallelism, collinearity and incidence. The incidence relation here is also much stricter than in other incidence structures. In combinatorial design the interest lies on design theoretic issues, like combinatorial symmetry and balance. In families of sets the focus is on set theoretic questions, like Sperner theory for example (see Sect. 4.8.2).

2.3 Board Automorphisms

Definition 2.9. Two boards $B = (P, C)$ and $B’ = (P’, C’)$ are isomorphic if and only if there is a bijection from $P$ to $P’$ that preserves the constellations. Such bijection is called an isomorphism.

Definition 2.10. An automorphism of a board $B = (P, C)$ is an isomorphism from $B$ to itself.

As the next example shows, isomorphisms are specific permutations of the set of points.

Example 2.11. If we number the points in the board in Fig. 6 as indicated in Fig. 7, we have

$$P = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$C = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}, \{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$$

Using two-line notation for permutations, in which points at the bottom are destinations for those at the top, we have for example that:

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 1 & 3 & 5 & 4 & 6 & 8 & 7 & 9 \\
\end{pmatrix}
$$

swaps horizontally the points in the first and second columns, thus leaving $C$ invariant: the permutation is then an automorphism of the board.
The automorphisms of a board form a group:

**Definition 2.12.** A group obtained under composition of automorphisms is an automorphism group. The group of all automorphisms of a board $B$ is the full automorphism group of $B$, denoted by $\text{Aut}(B)$.

The set of all possible permutations of the elements of a set also forms a group:

**Definition 2.13.** The symmetric group on a finite set $F$, $\text{Sym}(F)$, is the group whose elements are all bijective functions from $F$ to $F$ with function composition as the group operation. A subgroup of $\text{Sym}(F)$ is called a permutation group.

As $\text{Sym}(F)$ is the group of all permutations of the elements of $F$, its order is $|F|!$. This applies to the set of points $P$ of a board $B$. Since every automorphism of the board is a permutation of $P$, it also belongs to $\text{Sym}(P)$, so $\text{Aut}(B)$ is a permutation group:

$$\text{Aut}(B) \subseteq \text{Sym}(P) \quad (2.2)$$

### 2.4 Combinatorial Properties

Borrowing from the theory of Combinatorial Design [8], we define now some concepts for boards. They will be used later in the sections dealing with symmetric boards (Sects. 4 and 5), where we will see that geometric symmetry also induces combinatorial symmetry and balance.

**Definition 2.14.** A board is $k$-uniform if and only if all asterisms have size $k$.

**Example 2.15.** As all six asterisms in the board in Fig. 6 have size three, the board is 3-uniform. The Fano Plane (see Fig. 5) is also 3-uniform.

**Definition 2.16.** A parallel class or resolution class in a board $B = (P, C)$ is a set of asterisms that partitions $P$.

**Definition 2.17.** A parallel class is $k$-uniform if and only if all its asterisms have size $k$.

A $k$-uniform parallel class has then $\frac{|P|}{k}$ asterisms and is a subset of the board constellation. A board can thus have a $k$-uniform parallel class only if $k$ divides $|P|$, as the next example shows.

**Example 2.18.** In the board in Fig. 6, we see that each set $H$ and $V$, with

$$H = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}, \quad V = \{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}$$

partitions the set of points and have asterisms of size three: each is thus a 3-uniform parallel class. It is immediate to see that the Fano Plane (see Fig. 5) has no parallel classes.

**Definition 2.19.** In a board, two parallel classes $PC_1$, $PC_2$ are said to be orthogonal if and only if for any two asterisms $a_1$, $a_2$, with $a_1 \in PC_1$ and $a_2 \in PC_2$, $|a_1 \cap a_2| = 1$.

**Example 2.20.** As every asterism in $H$ in Example 2.18 intersect at a single point every other in $V$, both parallel classes are orthogonal.
Definition 2.21. A resolution of a board is a set of parallel classes that partition the constellation. A board with a resolution is said to be resolvable.

Definition 2.22. A resolution is $k$-uniform if and only if each of its parallel classes is $k$-uniform.

Example 2.23. The board in Fig. 6 is resolvable, as it has a 3-uniform resolution with two orthogonal parallel classes.

In $\mathbb{R}^2$ orthogonality among geometric lines requires both intersection and equality of angles. Parallelism and orthogonality are then special cases of a concept more appropriate to boards:

Definition 2.24. Given a board $B = (P, C)$ we call $\{|a \cap a' | : a, a' \in C, a \neq a'\}$ its set of intersection numbers or SIN.

It is trivial to form one or more parallel classes from a general set of points, but not so if we demand a particular SIN in the resulting board.

Example 2.25. In the board in Fig. 6, the tiled square provides an interesting intersection pattern from the outset: parallelism (within each parallel class) and orthogonality (among classes), i.e. a pattern with a SIN $= \{0, 1\}$. As any two asterisms in the Fano Plane either have one point in common or none at all (see Fig. 5), its SIN is also $\{0, 1\}$.

Example 2.26. Fig. 8 shows a board with 81 points and 27 asterisms, each with points lying on each

- row (9 asterisms that we group in set $H_S$)
- column (9 asterisms that we group in set $V_S$)
- $3 \times 3$ sub-square (9 asterisms that we group in set $Q_S$)

As $H_S$, $V_S$ and $Q_S$ partitions each the set of points, each is a 9-uniform parallel class and the board is resolvable. $H_S$ and $V_S$ are orthogonal, whereas each asterism in $Q_S$ intersects in three points three asterisms in $H_S$ and three in $V_S$, having an empty intersection with the rest.
3 Latin Boards

In this section we assign labels in a particular way to the points of a board to create a new type of object.

3.1 Definition of Latin Board

Definition 3.1. Let $B = (P, C)$ be a board, $L$ a $k$-multiset of labels, $n_i$, the size of asterism $a_i$, and $q_i, r_i$ the respective quotient and remainder of $\frac{n_i}{k}$. A Latin board is a tuple $(B, L, F)$ where $F$ is a function $P \rightarrow L'$, with $L' \subseteq \{l, l \in L\}$, such that the points of every asterism $a_i$ in $B$ are labeled with $q_i$ whole instances of $L$ plus any $r_i$ elements of it.

In simpler terms: each asterism has as many whole copies of the multiset as it can hold, plus some elements of the multiset to pad the remaining cells. We also say that a Latin board is the result of latinizing or labeling a board with a multiset.

According to this definition, and in contrast with Latin squares (see Def. 1), Latin boards have no constraints on topology, number of asterisms, equality of asterism sizes, uniqueness of labels or number of labels relative to asterism sizes. The leeway provided by the definition in the labeling of the remainder points in every asterism is illustrated by the following example. This flexibility can be used to define types of Latin boards, for example by specifying the remaining labels on a per asterism basis.

Example 3.2. Let $\{P, E, A, C, E\}$ be our multiset of labels and let $3, 5, 7, 10, 11$ be the respective number of points in the asterisms of a given board. For the purpose of the example we will order the points on each asterism in a sequence. Then the following sequences, or their distinct permutations, are some of the respective possible latinizations of each asterism:

- $[P, E, A], [A, E, E]$ (any 3 labels of the multiset)
- $[A, C, E, E, P]$ (the full multiset)
- $[P, E, A, C, E, E, C], [E, P, A, C, E, E, E]$ (one copy of the multiset plus any two of its labels)
- $[P, E, A, C, E, P, E, A, C, E]$ (two copies of the multiset)
- $[P, E, A, C, E, P, E, A, C, E, P], [P, E, A, C, E, E, C, A, E, P, E]$ (two copies of the multiset plus any of its labels)

Example 3.3. Fig. 9 (left) shows a board with seven points and a constellation with three asterisms, each one formed by points joined by segments of the same color. The board is not uniform because not all asterisms have the same size: the “green” asterism has seven points, the orange three and the blue four.

If we want to latinize this board with the multiset $\{1, 1, 2, 2\}$ we have to label the green asterism with one whole multiset plus any three labels (since $\frac{7}{2} = 1 + \frac{3}{2}$), for example with multiset $\{1, 1, 2, 2, 1, 2, 2\}$ or $\{1, 1, 2, 2, 1, 1, 2\}$. Similarly, we have to label the blue asterism with just one whole multiset $\frac{4}{2} = \frac{2}{2} = 1$.
1 + \frac{0}{4}); and the orange one with any three labels (since \(\frac{3}{4} = 0 + \frac{3}{4}\)), for example with \(\{1, 2, 2\}\) or \(\{1, 1, 2\}\). Fig. 9 (center and right) shows two example Latin boards.

3.2 Incidence Structures Related to Latin Boards

We see here how Latin boards relate to the incidence structures mentioned in Sect. 2.2.

3.2.1 Families of Sets

In Sect. 2.2.1 we identified the elements and subsets in a family of sets with the points and asterisms of a board. Latinization can be viewed then as the problem of finding an additional set of subsets that partitions the elements of the family, with the property that each one intersects each initial subset in the family at exactly the same number of elements; with the number of elements in the intersection being the count of the corresponding label in the multiset.

3.2.2 Finite Geometries

Latinizing a board that is also a finite geometry means to find subsets of points that meet the requirement imposed by the multiset. These new subsets will not be lines in the geometry in general.

3.2.3 Simple Combinatorial Designs

Once the correspondence between boards and simple combinatorial designs have been established as described in Sect. 2.2.3, latinizing a board is akin to finding specific parallel classes (see Def. 2.16) in the design.

3.2.4 Hypergraphs

Vertices on an hypergraph may be assigned labels called colors. Erdős and Hajnal defined classic hypergraph coloring to generalize graph coloring [13]:

**Definition 3.4.** Let \(\{1, 2, \ldots, n\}\) be a set of colors. A proper \(n\)-coloring of a hypergraph is a labeling of its vertices with the colors in such a way that every hyperedge with a size of at least two has at least two vertices colored differently.

If we choose \(\{1, 2, 3\}\) as a set of colors, then the Latin square in Fig. 1 is a valid classic coloring of the hypergraph in Fig. 6 (right).\(^7\) We also notice that

\(^7\)It is immediately seen that not any coloring is a Latin square though.
if the multiset used in a particular Latin board has at least two different labels, then latinization is a more general and demanding task than classic hypergraph coloring.

A more general view of hypergraph coloring admits any possible partition of the set of vertices into classes of the same color [28]. As Latin boards are geometric hypergraphs, and since latinization induces a partition of the points into classes holding points with the same label, latinization is an instance of this general hypergraph coloring.

### 3.3 Unifit Latin Boards

Unless otherwise stated, we will only deal here with boards of a specific type:

**Definition 3.5.** Let \( B = (P, C) \) be a \( k \)-uniform board and \( L \) a \( k \)-multiset of labels. A Latin board is a tuple \( (B, L, F) \) where \( F \) is a bijective function \( P \rightarrow L \).

This is a restricted version of Def. 3.1. An equivalent simpler definition is:

**Definition 3.6.** A \( k \)-unifit Latin board is a \( k \)-uniform board latinized with a \( k \)-multiset.

**Example 3.7.** Fig. 10 (left) shows a 3-unifit Latin board resulting from the latinization of the board in Fig. 6 with \{1, 2, 3\}. This is identical to the Latin square in Fig. 1, so this particular Latin square is a Latin board. In fact it is simple to prove that any Latin square is a Latin board.

Other latinizations of the same board are possible: if we write any permutation of the labels in the top row, then shift them by one cell to the left in successive rows, we obtain Latin boards that are also Latin squares (see Fig. 10 center). Fig. 10 (right) shows a Latin board with multiset \{1, 1, 2\}. This proves that not all latinizations of the board in Fig. 6 are Latin squares.

**Example 3.8.** The triangular and hexagonal objects in Figs. 3 and 4 are unifit Latin boards with respective sets \{1, 2, 3, ..., 12\} and \{1, 2, 3, ..., 18\}. They are examples of what will be called Latin triangles and Latin hexagons later. As indicated in Sect. 2.1.1, the points here are the centroids of the cells. The superimposed graphical structure—the grids of squares and triangles—is just a way to rendering the board points and asterisms apparent.

In what follows we will usually drop the adjective “unifit”.

|   |   | 3 |
|---|---|---|
| 1 | 2 | 3 |
| 2 | 3 | 1 |
| 3 | 1 | 2 |

|   |   | 1 |
|---|---|---|
| 3 | 2 | 1 |
| 2 | 1 | 3 |
| 1 | 3 | 2 |

|   |   | 1 |
|---|---|---|
| 1 | 2 | 1 |
| 2 | 1 | 1 |
| 1 | 1 | 2 |

*Figure 10. Different latinizations of the same board*
3.4 Latinization Techniques

Latinizations in Fig. 10 were easily found by inspection, but latinizing a generic board is not trivial. Several methods are available [21]; the author used constraint programming techniques [3] to generate the examples that follow.

Example 3.9. Fig. 11 (right) shows a Latin square of order seven.

Example 3.10. The next figures show three different latinizations of the board in Fig. 8: Fig. 12 (right), labeled with the set \{1, 2, 3, 4, 5, 6, 7, 8, 9\}; Fig. 13 (right), latinized with the multiset \{1, 1, 1, 2, 2, 2, 3, 3, 3\} and Fig. 14 (right) with the multiset \{N, E, O, S, U, D, O, K, U\}.

3.4.1 Latinizing the Fano Plane

Definition 3.11. In a trivial latinization all labels in the board are equal.

Theorem 3.12. The only latinization with a 3-multiset admitted by the Fano Plane is the trivial one.

Proof. As the asterisms in the Fano Plane have size three, there are only two types of candidate 3-multisets for latinization: one with at least one unique label –this one includes the case in which all labels are different– and another with all labels equal. Let \( u \) be the unique label in a multiset of the first type. Fig. 15 shows the Fano Plane with one of its points labeled with this label. The
### Figure 13. Sudoku Ripeto puzzle and solution

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 3 | 2 | 1 | 3 | 1 |
| 3 | 2 | 1 | 2 | 3 |
| 2 | 3 | 1 | 2 | 2 |
|   | 1 | 2 |   | 2 |
|   | 3 | 3 | 3 | 2 |
| 1 | 1 | 2 | 3 | 2 |
|   | 1 | 3 | 1 | 2 |
| 1 | 1 | 2 | 2 | 1 |
| 2 | 2 | 2 | 3 | 1 |

### Figure 14. Custom Sudoku puzzle and solution

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| K | S | D | O |
| O | N | U | O | U |
| U | O | K | O |
| N | E | O |   |
| O | S | U | D | E |
| E | O | K | U | N | D |
| U | S | K | D | U |
| O | U | S | O |
| D | U |
| K | U | S | D | O | U | O | N |
| O | N | D | U | O | K | U | E |
| E | O | U | S | N | K | D | O |
| D | K | U | N | E | O | S | O |
| U | O | N | S | U | D | O | K |
| S | E | O | O | K | U | N | U | D |
| U | S | O | K | N | E | D | U | O |
| O | U | E | D | U | S | O | N | K |
| N | D | K | O | O | U | E | U | S | U |
“light green” asterism should have also a point labeled $u$, but that point cannot be its leftmost one because it is being shared with the dark blue line, already holding the label. The same applies to the two other points in the light green line, due to the light blue and dark green lines. As any point in the Fano Plane has the same connectivity pattern, the argument is valid for any point, so the only possibility is to use a multiset with all labels equal, i.e.: one that produces the trivial latinization.

\[ \square \]

### 3.5 Latin Puzzles

Latin squares are redundant structures: they contain more information than the strictly necessary to describe them, as the next example shows.

**Example 3.13.** If we remove all labels in the top row of the Latin square in Fig. 1 the result is completable to just the original square.

Squares with missing labels having the property of being completable to a single Latin square are called *uniquely completable partial Latin squares* [17] and *Latin square puzzles* in [21].

**Example 3.14.** Fig. 11 shows a Latin square (right) and an associated Latin square puzzle (left).

The corresponding objects for Latin boards are called *Latin puzzles*, and are defined likewise [21]. Methods to find Latin puzzles are intimately related to those used to find Latin boards.

**Example 3.15.** The following examples were found by the author using constraint programming techniques [21]: Fig. 12 (left) shows a *Sudoku* puzzle [23]. Fig. 13 (left) is an example of *Sudoku Ripeto* puzzle [22]. Fig. 14 (left) shows a *Custom Sudoku* puzzle [22]. More Latin puzzles may be found in [21, 22].

### 3.6 Equivalence Classes

Latin boards may be easily derived from a given one in some cases. We illustrate this with examples involving Latin squares, *Sudokus* and Latin boards in general.
3.6.1 Equivalence Classes of Latin Squares

It is easily seen that, if we permute the labels, the rows or the columns of a Latin square we obtain another Latin square. Both squares are said to be isotopic. Isotopism is an equivalence relation that partitions the set of Latin squares of a particular order into isotopy classes [17].

On the other hand, a Latin square of order $n$ with labels $\{1, \ldots, n\}$ may be represented with an $n^2$-set of 3-tuples $(i, j, k)$, one for each cell, where $i$ is the index of the row, $j$ that of the column and $k$ the label in the cell. This set, called the orthogonal array representation of the square [17] makes apparent that any pair of components in the tuple may act as row and column indices in yet another Latin square, with the remaining one becoming the corresponding cell content. As there are six possible ways to do this, from one Latin square we can obtain a maximum of five others$^8$ called conjugates of the original one.

Both operations can be combined: two Latin squares are said to be paratopic if one of them is isotopic to a conjugate of the other. This is an equivalence relation that creates paratopy classes, each one containing up to six isotopy classes$^9$.

3.6.2 Equivalence Classes of Sudokus

Similarly, from a Sudoku (see Fig. 12 right) we can easily obtain new ones by any combination of the following operations:

- a permutation of the labels
- a permutation of rows within a 3-row block
- a permutation of the 3 row blocks
- a permutation of columns within a 3-column block
- a permutation of the 3 column blocks
- a reflection across the vertical or horizontal symmetry axis
- a $90^\circ$, $180^\circ$ or $270^\circ$ rotation around the center of the board

We can likewise say that two Sudokus are equivalent if we can transform one into the other using a combination of the mentioned operations$^{10}$. We can divide then the set of Sudoku boards into several equivalence classes$^{11}$.

3.6.3 Equivalence Classes of Latin Boards

If we can transform a Latin board into another one with a set of operations we can define a notion of equivalence among boards. Each set of operations will generate different equivalence classes. The more elements in the set, the less equivalence classes there will be and the more boards per class will result.

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$^8$Or less, since some permutations may result in the same Latin square.

$^9$For the number of paratopy classes of Latin squares as a function of the order see [8, 20].

$^{10}$In fact, not all of them are necessary to generate all equivalent boards, just a subset of the operations.

$^{11}$Enumerating the classes is a somewhat involved process, see [14].
Equivalence classes are important when enumerating Latin boards. If several boards are found with a particular method [21], they will be more valuable if they belong to different equivalence classes, since if they are not, from one of them we can easily find the others using the available operations. To avoid unnecessary calculations, algorithms that enumerate or count Latin boards can eliminate the need to look for boards in the same class of those already found [8, 21], or eliminate dead ends in the search for that matter. This needs as a key element the set of operations that generates the classes, that should be as large as possible for the reasons just explained.

As we will see in Sect. 5.5, when a Latin board is symmetric and there are no other equivalence operations available, those linked to its symmetry group can be used to generate equivalence classes.

3.7 Woven Boards

Given a Latin board, we can naturally extend the underlying board by adding to it an asterism for each set of points with the same label.

Definition 3.16. Let $B = (\{P, C\}, L, M)$ be a Latin board. Let $C' = \{a_l, l \in L\}$ be a constellation each of whose asterisms contains all points with the same label. Then the board $W = (P, C \cup C')$ is the woven board of $B$.

Latin boards with different permutations of the labels have then the same woven board. In this context, the asterisms in $C$ are called weft asterisms and $C$ proper, the weft constellation. The asterisms in $C'$ are the warp asterisms, while $C'$ proper is the warp constellation. Like in a loom, in a woven board the weft constellation is interwoven with the warp one.

Example 3.17. Fig. 16 (left) shows a latinization of the board in Fig. 6 with $\{1, 2, 3\}$. On the right, its woven board. Solid colored segments indicate weft asterisms; warp asterisms are represented by dashed segments. Woven boards obtained from Latin squares are combinatorial designs known as 3-nets [8].

Lemma 3.18. The warp constellation of a woven board is a parallel class.

Proof. As all points in the originating Latin board are labeled, the warp asterisms partition them, so they constitute a parallel class.

Lemma 3.19. Let $B = (P, C)$ be a unifit Latin board. If the weft constellation in its woven board contains a $k$-uniform parallel class, then the size of each warp asterism is $\frac{|P|}{k} c$, where $c$ is the count, in the multiset of labels, of the label the asterism was derived from.
As the parallel class partitions $P$, the number of weft asterisms in the class is $|P|_k$. As the Latin board is unifit, a label with count $c$ in the multiset will be present in $c$ points on each weft asterism, so the warp asterism corresponding to this label has $|P|_k c$ points.

**Theorem 3.20.** If a unifit Latin board with $|P|$ points has a $k$-uniform parallel class, and each label in the multiset has count $k^2 |P|_k$, then its woven board is also $k$-uniform.

**Proof.** By Lemma 3.19 the size of each warp asterism is $|P|_k c = |P|_k k^2 |P|_k = k$. As weft asterisms are also $k$-uniform, the woven board is also $k$-uniform.

The repetition pattern of labels in the multiset serves then to fine-tune the combinatorial properties of the resulting woven board, like the number and size of the warp asterisms and its SIN (set of intersection numbers, see Def. 2.24).

### 3.8 Sequences of Latin Boards

#### 3.8.1 Sequences of Latin Squares

If we latinize the woven board in Fig. 16 (right) with $\{1, 2, 3\}$ we can obtain the Latin Square in Fig. 17. Weaving in turn this second Latin square we obtain the woven board in Fig. 18, where the three new warp asterisms are represented with even more spaced dashed lines. This board features four parallel classes, each with three asterisms of three points each. If we identify the points as in Fig. 7, the classes are:

\[
\begin{align*}
\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\} \\
\{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\} \\
\{\{1, 6, 8\}, \{2, 4, 9\}, \{3, 5, 7\}\} \\
\{\{1, 5, 9\}, \{2, 6, 7\}, \{3, 4, 8\}\}
\end{align*}
\]

These classes have an interesting property: any two asterisms, each from a different class, intersect at exactly one point. The two intervening Latin squares in the sequence constitute a set of mutually orthogonal Latin squares (MOLS) \[8\]. Can we extend the just constructed Latin sequence? It is known that the maximum size of a set of MOLS of order $n$ is $n - 1$ \[8\], so no further extensions of the sequence are possible.
3.8.2 Sequences of Free Latin Squares

The same process can be applied to more general Latin boards. In the next example we construct a Latin sequence that illustrates the generalization of orthogonality to *uniform intersection* among parallel classes. Let $P$ be the set of points in Fig. 19. Let’s make an asterism out of points indexed by the same letter; each asterism has then the points on either two rows or two columns. The result is a board with SIN $\{0, 4\}$ and parallel classes: $\{a, b, c, d\}$ and $\{e, f, g, h\}$.

Using the multiset $\{1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4\}$ we have obtained the three Latin boards\(^\text{12}\) shown in Fig. 20. The derived woven boards have also SIN $\{0, 4\}$. Let’s consider the following parallel classes, each with four asterisms:

- $H$: $\{a, b, c, d\}$
- $V$: $\{e, f, g, h\}$
- $S_1$: containing one asterism per set of points with the same symbol in the first Latin board in the figure
- $S_2$: *idem* in the second one
- $S_3$: *idem* in the third one

\(^{12}\)As we will see in Sect. 5.3.3, these Latin boards are called *free Latin squares*. 

---

\[\text{Figure 18. Second woven board}\]

\[\text{Figure 19. Board with SIN \{0, 4\}}\]
It is easily verified by inspection that any two asterisms, each from a different class and not necessarily from the same board, intersect at exactly four points: the three Latin boards constitute then a Latin sequence that features uniform intersection among parallel classes.

4 Symmetric Boards

In this section we study boards with geometric symmetry and how this translates into combinatorial and algebraic symmetries.

4.1 Symmetry Groups

Definition 4.1. The (full) symmetry group $T$ of a geometric object embedded in a space $S$ is the group of all transformations under which the object remains invariant, with composition of transformations as the group operation.

If $S$ is a metric space, $T$ is a subgroup of the isometry group of $S$ [7].

Example 4.2. Consider the following eight $\mathbb{R}^2$ transformations of the square:

- $0^\circ$, $90^\circ$, $180^\circ$ and $270^\circ$ counter-clockwise rotations
- 2 reflections across the two diagonals
- 2 reflections, one across the vertical symmetry axis and another across the horizontal one

Each of them leaves the square invariant. Together with the operation of composition of transformations, they form the dihedral group $D_4$ [10] – a subgroup of Euclidean group in two dimensions – the algebraic structure that captures the symmetry of the square.

Example 4.3. The symmetry group of the equilateral triangle is the dihedral group $D_3$, that comprises three counter-clockwise rotations ($0^\circ$, $120^\circ$ and $240^\circ$) and three reflections across the axes that go through the vertices to the midpoint of the edges each.
4.1.1 Symmetry Groups from Regular Polytopes

Natural candidates for $T$ in Def. 4.1 are the symmetry groups of regular polytopes (see Sect. 4.7.1). Examples in $\mathbb{R}^2$ are the symmetry groups of the regular polygons (the dihedral groups $D_n$). In $\mathbb{R}^3$ we have the symmetry groups of the finite real convex regular polyhedra (see Table 4.2). For symmetry groups in higher dimensions see [7, 10].

4.2 Actions of Symmetry Groups

The effect of a symmetry group on a finite set of points is an instance of the more general notion of action of a generic group on a generic finite set:

**Definition 4.4.** If $T$ is a group and $P$ is a finite set, a *group action on $P$* is a group homomorphism—called the *action homomorphism*—from $T$ to $\text{Sym}(P)$ that satisfies two axioms:

1. the identity element of $T$ is mapped to the identity permutation of $P$
2. a composition of two elements of $T$ is mapped to the composition of the corresponding permutations

We say in this case that the group acts on the set, and also that the set is acted upon by the group. There is a natural action of symmetry groups on finite sets of geometric points, as the next example shows.

**Example 4.5.** Let $P$ be the set of vertices of an equilateral triangle. If we map the identity transformation in $D_3$ to the identity permutation in $\text{Sym}(P)$, and the other transformations in $D_3$ to the permutation of the vertices that result after the transformation is applied to the triangle, then it is easy to prove that this map is a group homomorphism between $D_3$ and $\text{Sym}(P)$ that fulfills the axioms in Def. 4.4, i.e.: the map is a group action of $D_3$ on $P$. We say that $D_3$ acts on the vertices of the triangle. By a similar argument we can prove that $D_3$ also acts on the sides of the triangle.

4.3 Definition of Symmetric Board

A symmetry group that acts on the points of a board may also act on its asterisms. We define:

**Definition 4.6.** A *symmetric board* is one whose asterisms are acted upon by a symmetry group.

In a symmetric board the elements of the group take asterisms to asterisms, i.e.: it preserves the points, the constellation and hence the board itself.

**Example 4.7.** The asterisms in the board in Fig. 6 are acted upon by $D_4$, so the board is symmetric.

4.3.1 Class-symmetric Boards

**Definition 4.8.** The action of a group $T$ on a set $P$ is *transitive* if $P$ is non-empty, and if for any $p_1, p_2 \in P$ there exists a $t \in T$ such that the action of $t$ on $p_1$ is $p_2$. 

20
So any element in $P$ can be reached from any other if the action is transitive.

**Example 4.9.** $D_4$ does not act transitively on the asterisms of the board in Fig. 6, as there is no element in $D_4$ that brings the top horizontal asterism to the middle horizontal one. For the same reasons, $D_3$ does not act transitively on the Fano Plane asterisms (see Fig. 45 right).

If the symmetric board is resolvable (see Def. 2.21) then additional symmetry may be present at another level in the board:

**Definition 4.10.** Let $B$ be a resolvable symmetric board with symmetry group $T$. If $T$ acts transitively on the parallel classes, then $B$ is a class-symmetric board.

In a class-symmetric board the elements of the group not only take classes to classes: there is always a sequence of group elements that take any class in the board to any other too. A symmetric board may not be class-symmetric for several reasons:

- there are no parallel classes (the case of the Fano Plane)
- there are parallel classes but the group does not act upon them
- there are parallel classes acted upon by the group, but the action is not transitive (the case of the board in Fig. 8)

**Example 4.11.** $D_4$ acts on the set of parallel classes of the board in Fig. 6 (it takes the set of rows to the set of columns and vice versa), so the board is class-symmetric. This shows that a board may be class-symmetric even if the group does not act transitively on its asterisms.

### 4.4 Permutations linked to Actions of Symmetry Groups

The permutations linked to the action of a symmetry group form themselves a group:

**Lemma 4.12.** Let $B = (P, C)$ be a symmetric board and $T$ the symmetry group acting on the asterisms in $C$. Then the permutations of $P$ linked to the action form a group with composition of permutations as the group operation.

**Proof.** Let $h$ be the action homomorphism between $T$ and $\text{Sym}(P)$, with $P$ being a set of points with symmetry $T$. Since $T$ acts on $C$ then, as each asterism is a subset of the points and the union of all asterisms is $P$, $T$ also acts on $P$. Since a group homomorphism preserves the subgroups, and as $T$ is an improper subgroup of itself, then $h(T)$ is a subgroup of $\text{Sym}(P)$, hence a group.

### 4.5 Board Automorphisms from Geometric Symmetry

Board automorphisms are point permutations that preserve the board (see Sect. 2.3). We see next that some automorphisms come from the geometric symmetry of the board, i.e.: how geometric symmetry entails algebraic geometry.
Lemma 4.13. Let $B = (P, C)$ be a symmetric board and let $T$ be the symmetry group acting on the asterisms in $C$. Let $A_T(B)$ be the group of permutations of $P$ linked to the action. Then $A_T(B) \subseteq \text{Aut}(B)$.

Proof. As $T$ acts on the asterisms in $C$, then the elements in their image $A_T(B)$ under the action homomorphism are isomorphisms of $B$, hence $A_T(B) \subseteq \text{Aut}(B)$.

Example 4.14. Consider the eight transformations in $D_4$ (see Example 4.2) and the board in Fig. 6. Each transformation is linked to a point permutation that leaves the asterisms invariant (i.e.: to automorphisms). For example a $90^\circ$ counter-clockwise rotation induces the permutation (see Fig. 7):

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
2 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3
\end{pmatrix}
$$

which is an automorphism. As is the permutation linked to a reflection across the vertical axis:

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
3 & 2 & 1 & 6 & 5 & 4 & 9 & 8 & 7
\end{pmatrix}
$$

Other automorphisms of the board have no correspondence in $D_4$. An example is the automorphism linked to permutation 2.1: it swaps the two leftmost columns in the resulting Latin squares, something that no element of $D_4$ can perform.

4.6 Group Hierarchy

The relation among symmetry groups, automorphism groups and symmetric groups (see Sect. 2.3) is given by Lemma 4.13 and Relation (2.2) in Sect. 2.3:

$$A_T(B) \subseteq \text{Aut}(B) \subseteq \text{Sym}(P)$$

This shows that if we want interesting boards with large $\text{Aut}(B)$, a symmetric board whose asterisms are acted upon by a symmetry group $T$ provides at least $A_T(B)$ upfront.

Example 4.15. Let $B = (P, C)$ be a board with $P = \{v_i, i = 1, 2, 3\}$ being the set of vertices of an equilateral triangle, with $C = \{\{v_i\}, i = 1, 2, 3\}$ (i.e.: each asterism has just a single point). As the dihedral group $D_3$ acts on the asterisms in $C$, $B$ is a symmetric board. We have in this case $A_{D_3}(B) = \text{Aut}(B) = \text{Sym}(P)$, all with order 6.

Example 4.16. Let $B = (P, C)$ be a board with $P = \{v_i, i = 1, 2, 3, 4\}$ being the set of vertices of the square, with $C = \{\{v_i\}, i = 1, 2, 3, 4\}$. As $D_4$ acts on the asterisms in $C$, $B$ is a symmetric board. We now have $A_{D_4}(B) = \text{Aut}(B)$ and $\text{Aut}(B) \subseteq \text{Sym}(P)$. $\text{Sym}(P)$ has order $4! = 24$, while $A_{D_4}(B)$ and $\text{Aut}(B)$ have both order 8.
4.7 Symmetric Boards from Sources of Symmetry

A way to find symmetric boards is to start with a source of symmetry (or source for short): a geometric object with a symmetry group $T$ (see Sect. 4.1.1). Board points can then be picked in the source and asterisms formed in such a way that $T$ acts on them. If we want the board to be class-symmetric, classes can be chosen so that $T$ acts transitively on them. The resulting board $B$ is assured to have at least $A_T(B)$ as an automorphism group following Lemma (4.13).

Sources are usually drawn along with the boards to render the symmetry apparent. If the source is intricate or has many dimensions it may be given analytically instead of graphically.

We describe next how to obtain different sources, and give example boards based on them. To highlight the effect of geometric symmetry on combinatorial symmetry (see Sect. 2.4), the examples include properties such as the pattern of intersection of the asterisms (SIN), existence of parallel classes, orthogonality, resolvability, etc.

4.7.1 Sources from Regular Polytopes

A polytope is a geometrical figure bounded by portions of lines, planes and hyperplanes in a particular space [10]. Polygons and polyhedra are polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Regular polytopes are polytopes with special properties of symmetry that make them ideal sources. Among the finite real regular polytopes we have\(^\text{13}\):

- in $\mathbb{R}$ the closed intervals
- in $\mathbb{R}^2$ the regular polygons (convex) and the regular star-polygons (non-convex)
- in $\mathbb{R}^3$ the five platonic polyhedra (convex) and the four Kepler-Poinsot polyhedra (non-convex)
- in $\mathbb{R}^4$ the six convex regular 4-polytopes and the ten non-convex Schläfli-Hess 4-polytopes
- in $\mathbb{R}^n$ ($n > 4$) the simplexes, cubes and cross-polytopes

Example 4.17. The source in Fig. 21 (left) has the symmetry of the regular dodecahedron (dihedral symmetry $D_{12}$). Let’s take as board points the intersections between segments (thirty six intersections in all) and let’s define a 6-uniform asterism out of each 6-set of points lying on the same segment (twelve asterisms in all). The board is shown on the right. The asterisms are acted upon by $D_{12}$ and hence we have a symmetric board. The board is not class-symmetric because there are no parallel classes here.

4.7.2 Sources from Projections

A way to obtain sources in a space with a given dimension is to properly project onto it sources from higher dimensions. We can obtain bidimensional sources with dihedral symmetry as proper projections of the regular polyhedra [10].

\(^{13}\)For a formal definition and a comprehensive list of regular real polytopes –including apeirotopes, the infinite ones– see [10]. For the complex regular polytopes see [9]; for abstract polytopes -a purely combinatorial generalization- see [19].
4.7.3 Sources from Kaleidoscopic Sets

Sources can also be built from scratch in any geometric space using symmetry groups (see Sect. 4.1.1), group actions (see Def. 4.4) and ideas from [10].

Definition 4.18. Let $T$ be a group acting on a set of points $P$. The orbit of a point $p$ is the set of points in $P$ to which $p$ is moved by the elements of $T$.

Definition 4.19. Let $T$ be a group acting on a space $S$. A fundamental region is a subset of $S$ containing exactly one point from the orbit of every point in $S$.

Definition 4.20. Let $T$ be a symmetry group acting on the points of a space $S$. Let $F$ be a set of points in a fundamental region of $S$. Then $K = \{\text{orbit of } p, p \in F\}$ is the kaleidoscopic set of $F$ by $T$.

A kaleidoscopic set so constructed is evidently a source. The elements in the definition have clear counterparts in a kaleidoscope: points in $F$ are the colored beads inside the tube, while the fundamental region is the chamber in which they are confined; the set of images of a bead is the orbit of a point; $S$ is the plane in which the beads move; the kaleidoscopic set is the image seen for a certain rotation of the tube, while $T$ generators are the inside mirrors.

Example 4.21. Let $S$ be $\mathbb{R}^2$ and $T$ be $D_3$. Fig. 22 (left) shows the fundamental regions, six in this case. $F$ is in the middle and $K$ on the right. Now let’s choose as board points its fifteen vertices and make an asterism out of each pair of vertices on the same segment (thirty asterisms in all). The board is shown on Fig. 23. As $D_3$ acts on the asterisms we have a symmetric board.

Example 4.22. Let $S$ be $\mathbb{R}^2$ and $T$ be $D_4$. The eight fundamental regions are shown in Fig. 24 (left), $F$ is in the center and $K$ on the right. If we choose the centers of the squares as board points and make asterisms out of every row and every column of points we obtain the board Latin squares are based on.

4.7.4 Enriching Sources

Sources with rich structure give more options to define board points and asterisms. We can obtain a richer source from a given one with a combination of techniques that add structure without breaking the symmetry, like the ones described next.
Figure 22. Kaleidoscopic set by $D_3$

Figure 23. $D_3$ symmetric board

Figure 24. Kaleidoscopic set by $D_4$
4.7.4.1 Composition
We can combine several sources with the same symmetry to obtain a richer one. *Compound polytopes* [10]—sets of regular polytopes with a common center—are an example.

4.7.4.2 Truncation
A source properly truncated may gain new vertices, edges, faces, etc. depending on the dimension of the source. See [10] for truncated versions of the regular polytopes.

4.7.4.3 Kaleidoscopic Sets
If we have a source with symmetry group $T$ we can enrich it while preserving its symmetry by adding a kaleidoscopic set generated by $T$ (see Sect. 4.7.3). We can add in this way more than one point per edge, face or vertex.

4.7.4.4 Extra Regular Tilings. Biregular Polytopes
We define *biregular polytopes* as regular polytopes tiled with other regular polytopes. In two dimensions for example we have *biregular polygons*: regular polygons tiled with regular polygons. In the context of sources biregular polytopes thus enrich regular polytopes.

The number of tiling polygon sides on each side of the tiled polygon is the order of the biregular polygon. The tiling polygons are called *faces* or *cells*. Since there are only three regular tessellations in $\mathbb{R}^2$ [10] it is easy to prove that in this space there are only three families of biregular polygons (see Fig. 25):

- equilateral triangles tiled with equilateral triangles (*biregular triangles*)
- squares tiled with squares (*biregular squares*)
- regular hexagons tiled with equilateral triangles (*biregular hexagons*)

As biregular polygons feature dihedral symmetry, they are eligible sources for both symmetric and class-symmetric boards (see Sect. 4.3). They also feature structure and scalability while keeping the symmetry, so entire families of symmetric boards can be defined in a very compact way. Table 4.1 shows relevant parameters of each family. We can enrich candidate sources by transforming parts of them that are regular polytopes:

- polygons into biregular polygons (see Sects. 4.7.4.4.1, 4.7.4.4.2, 4.7.4.4.3 and 4.7.4.4.4)
- polyhedra into biregular polytopes
- $n$-polytopes into biregular polytopes

4.7.4.4.1 Biregular Triangles
Example 4.23. Fig. 26 shows a biregular triangle of order six where face barycenters (red dots) have been chosen as board points. An asterism here is the set of points pointed to by the same letter. Each asterism has twelve points and belongs to one of three parallel classes: $\{a, b, c\}$, $\{d, e, f\}$ or $\{g, h, i\}$. We
Figure 25. Biregular polygons

| biregular polygon | order  | tiling polygon | tiled polygon | symmetry group | #vertices | #edges | #faces |
|-------------------|--------|----------------|--------------|----------------|-----------|--------|--------|
| triangle          | \(n\)  | triangle       | triangle     | \(D_4\)       | \(\frac{(n+1)(n+2)}{2}\) | \(\frac{3n(n+1)}{2}\) | \(n^2\) |
| square            | \(n\)  | square         | square       | \(D_4\)       | \((n+1)^2\) | \(2n(n+1)\) | \(n^2\) |
| hexagon           | \(n\)  | triangle       | hexagon      | \(D_6\)       | \(3n^2+3n+1\) | \(3n(3n+1)\) | \(6n^2\) |

Table 4.1 Parameters of biregular polygons

have then a uniform resolution. Furthermore, any two asterisms from different classes intersect in four points. To see this, take the smaller, three-row upper triangle, and turn it clockwise around the center of the larger triangle’s right side until both halves of this side coincide. We see that every asterism in the first class intersects in four points any other in the second. The symmetry of the board allows us to conclude that this applies to any two asterisms from different classes, so the SIN of the board is \(\{0, 4\}\). It is simple to verify that \(D_3\) acts transitively on the classes, so we have a class-symmetric board (see Sect. 4.3.1). In fact, provided that the order \(n\) of the biregular triangle is even, we have a whole family of class-symmetric boards likewise built, each with \(n^2\) points and three uniform parallel classes, each with \(\frac{n^2}{2}\) asterisms of size \(2n\) each.

Example 4.24. Fig. 27 shows a biregular triangle of order seven with vertices as board points. An asterism here is the set of points pointed to by the same letter. Each has nine points and belongs to one of three parallel classes:

Figure 26. Triangular board with points at the barycenters
Figure 27. Triangular board with points at the vertices

Figure 28. Triangular board with points at the edge midpoints

We have then a board with uniform resolution. It is simple to verify that $D_3$ acts transitively on the classes. In fact, provided that the order $n$ of the biregular triangle is odd, we have a whole family of class-symmetric boards, each with \((n+1)(n+2)/2\) points and three uniform parallel classes, each with \((n+1)/2\) asterisms of size $n+2$ each.

**Example 4.25.** Fig. 28 shows a biregular triangle of order five on which we have chosen edge centers as board points. As before, an asterism is specified as the set of points pointed to by the same letter. Each asterism has ten points and belongs to one of three parallel classes: \{a, b, c\}, \{d, e, f\} or \{g, h, i\}. We have again a uniform resolution. It is simple to verify that $D_3$ acts transitively on the classes, so this is a class-symmetric board.

### 4.7.4.4.2 Biregular Squares

**Example 4.26.** Fig. 19 shows a biregular square of order eight (see Sect. 4.7.4.4) with face centers as board points. A board asterism is made up of sets of points pointed to by the same letter. Each asterism has sixteen points and belongs to parallel class \{a, b, c, d\} or \{e, f, g, h\}. We have then a 16-uniform
resolution in which every pair of asterisms not in the same class share four points, so the board has SIN is \{0, 4\}. It is straightforward to verify that $D_4$ acts transitively on the classes, so the board is class-symmetric. We can build symmetric boards on biregular squares of any order $n$ with $n^2$ points. If the order is prime we can only have symmetric boards that will originate conventional Latin squares. For composite orders we have several options to group rows and columns into board asterisms. If $n$ is even and asterisms come in two pieces as in Example 4.26, the board will have two uniform parallel classes, each with $2n$ asterisms of size $2n$ each. The pairing chosen will determine if the board is class-symmetric, just symmetric or non-symmetric.

Example 4.27. Fig. 29 shows the board in Fig. 19 with an extra parallel class whose asterisms are the four $4 \times 4$ highlighted sub-squares. Its SIN is also \{0, 4\}. The board is evidently symmetric but not class-symmetric, as there is no way to take the extra class to one of the others with the elements of $D_4$.

Example 4.28. Fig. 30 shows an arrangement of points with $D_4$ symmetry. They come from the vertices and tile centers –with one missing– of a biregular square of order four. They have been organized into asterisms, each one formed by points pointed to by the same letter. The asterisms have eight points each and are acted upon by $D_4$ –asterisms $b$ and $h$ for example are swapped by a vertical reflection. This is thus a 8-uniform $D_4$ board that is also class-symmetric. Its SIN is \{0, 2\}, as any two asterisms either intersect at two points or do not intersect at all.

4.7.4.4.3 Biregular Hexagons

Example 4.29. Fig. 31 shows a biregular hexagon of order three on which we have chosen face centers as board points. An asterism is made up of the eighteen points pointed to by the same letter, and belongs to one of three parallel classes: \{a, b, c\}, \{d, e, f\} or \{g, h, i\}. We have then a board with a 18-uniform resolution. It is easy to verify that $D_6$ acts on the asterisms and transitively on the classes, so the board is class-symmetric\(^\text{14}\). It is also easy to see that its SIN is \{0, 6, 8\}. We can have symmetric boards like this one for any order $n$ of the biregular hexagon ($6n^2$ points) with three uniform parallel classes, each with $n$ asterisms of size $6n$ each.

\(^{14}\) $D_6$ also acts on the asterisms here but we choose to focus on the largest symmetry group.
Figure 30. Square board with SIN \{0, 2\}

Figure 31. Hexagonal board with points at the barycenters
Example 4.30. Fig. 32 shows a biregular hexagon of order three on which we have chosen edge centers as board points. An asterism is made up of sets of points pointed to by the same letter. Each asterism has fourteen points and belongs to one of three parallel classes: \{a, b, c\}, \{d, e, f\} or \{g, h, i\}. The board has a 14-uniform resolution. It is simple to verify that $D_6$ acts transitively on the classes, so the board is class-symmetric.

4.7.4.4.4 Regular Polyhedra with Biregular Faces

There are five finite real convex regular polyhedra, and three polyhedral full symmetry groups (see Table 4.2). Each of the following examples consists of a polyhedral source of symmetry (see sect. 4.7) and a board based on it. The corresponding pictures are meant to be folded along the inner black lines and glued along the outer ones. With the exception of those in the example dodecahedral board, every asterism in the boards is made up of points enclosed by two lines of the same color. The asterism becomes a closed band –so to speak– when the board is folded.

Example 4.31. Tetrahedral Board. Fig. 33 (left) shows a tetrahedral source. On the right we have selected face centroids (triangle orthocenters) as board points. As indicated above, asterisms are made up of points contained between lines of the same color. The board has SIN $\{0, 4\}$ and is resolvable. However, when unfolded, the board is also a Latin triangle like the one in Fig. 47, which has SIN $\{0, 4\}$.

---

**Table 4.2** Polyhedral symmetry groups of the platonic solids

| group   | regular polyhedron                  | order |
|---------|------------------------------------|-------|
| tetrahedral | tetrahedron                        | 24    |
| octahedral  | cube, octahedron                    | 48    |
| icosahedral | dodecahedron, icosahedron           | 120   |

Figure 32. Hexagonal board with points at the edge midpoints
It is also class-symmetric, since there are three parallel classes that are acted upon transitively by the tetrahedral group.

**Example 4.32. Cubical Board.** The picture in Fig. 34 is a cubical source. In Fig. 35 we have chosen the face centroids as board points. The asterisms, defined as before as the points between lines of the same color, are acted upon by the octahedral group. The board has SIN \( \{0, 8\} \) and no parallel classes, so it is not resolvable.

**Example 4.33. Octahedral Board.** The picture in Fig. 36 is an octahedral source. In Fig. 37 we have chosen face orthocenters as board points and asterisms as indicated before. The board is symmetric since the asterisms are acted upon by the octahedral group. The board has SIN \( \{0, 4\} \) and no parallel classes, and hence it is not resolvable.

**Example 4.34. Icosahedral Board.** The picture in Fig. 38 is an icosahedral source. In Fig. 39 we have chosen face orthocenters as board points. The asterisms, defined as above, are acted upon by the icosahedral group, so the board is symmetric. The board has SIN \( \{0, 4\} \) and no parallel classes.

**Example 4.35. Dodecahedral Board.** The picture in Fig. 40 is a dodecahedral source. Its pentagonal faces are trivial biregular pentagons. In Fig. 41 we have chosen the edge centers as board points. Each asterism is made up of points lying on a drawn line of a particular color. Once folded, each asterism becomes a closed polyline on the dodecahedron surface. The board is symmetric because the asterisms so defined are acted upon by the icosahedral group. It has SIN \( \{0, 2\} \) and no parallel classes, and so it is not resolvable.

### 4.7.4.5 Extra General Tilings

We can also enrich sources by tiling some of its elements (edges, faces, hyperplanes) with tiles that are not necessarily regular.

**Example 4.36.** The picture in Fig. 42 is meant to be folded along the inner black lines and glued along the outer ones. The resulting dodecahedron, with faces tiled with non-equilateral triangles, is a source with icosahedral symmetry (see Table 4.2).
Figure 34. Cubical source

Figure 35. Cubical board
Figure 36. Octahedral source

Figure 37. Octahedral board
Figure 38. Icosahedral source

Figure 39. Icosahedral board
Figure 40. Dodecahedral source

Figure 41. Dodecahedral board

Figure 42. Source with dodecahedral symmetry
Figure 43. Points and asterism on a dodecahedral board

In Fig. 43 we have a board whose points are the orthocenters of the triangles inside the pentagons. Each of its asterisms is made up of points lying on lines like the one shown in yellow. The asterisms, each indicated by a colored dashed line, are shown in Fig. 44. Once the board folded, the portions of each asterism become one on the polyhedron surface. The board has no parallel classes and thus it is not resolvable, but it is symmetric because the asterisms are acted upon by the icosahedral group.

4.8 Symmetric Boards from Asymmetric Boards

In Sect. 4.7 we have obtained symmetric boards for sources of symmetry. We discuss next how to obtain symmetric boards from asymmetric ones.

4.8.1 Board Symmetrization Problem (BSP)

**Definition 4.37.** Given an asymmetric board, find symmetric boards isomorphic to it. We call this problem the **Board Symmetrization Problem** (BSP).

**Example 4.38.** Fig. 45 (left) shows the Fano Plane as an asymmetric board. We would like to find boards like the one in the center, which is both isomorphic to the first –this is easily verified by inspection– and symmetric –$D_3$ acts on the
asterisms$^{16}$.

4.8.2 Symmetric Boards with unknown Automorphism Group

Let $B = (P, C)$ be our asymmetric board and let’s suppose that $Aut(B)$ is unknown. We can approach the BSP geometrically by first establishing a bijection between points in $P$ and other points in $P'$, that may reside in the same space or in a different one. We define an asterism as the set of points in $P'$ that are linked by the bijection to points in an asterism in $P$. If we group all resulting asterisms in constellation $C'$ we obtain the isomorphic board $B' = (P', C')$.

$^{16}$The boards derived from this one by the action of $D_3$ are also solutions to the problem.
Figure 44. Points and asterisms on a dodecahedral board
Now we consider a set $S_{SG}$ of candidate symmetry groups (see Sect. 4.1). For each of them we arrange the points in $P'$ so that the group acts on them, then repeatedly permute the points to see if the group could also act on the asterisms in $C'$. When this is achieved the resulting board is a solution to the BSP for $B$.

**Example 4.39.** By taking $D_3$ as a candidate symmetry group for the Fano Plane in Fig. 45 (left) and arranging its points as described we have obtained the solution to the BSP shown in the center. We could try now $D_6$. Fig. 45 (right) shows a failed attempt to rearrange the points in which $D_6$ acts on them but not on the asterisms (a $60^\circ$ counter-clockwise rotation for example does not take the red asterism to any other).

As we saw in sect. 4.5, a symmetric board $B'$ isomorphic to another one $B$ provides a subgroup of $\text{Aut}(B)$. So if $\text{Aut}(B)$ is unknown, finding solutions to the BSP provides at least some previously unknown automorphisms.

**4.8.3 Symmetric Boards with known Automorphism Group**

On the other hand, if $\text{Aut}(B)$ is known, we can further simplify the BSP: Lemma 4.13 tells us that we need to focus only in symmetry groups whose actions are linked to permutation groups that are subgroups of $\text{Aut}(B)$\(^{17}\).

**Example 4.40.** The automorphism group of the Fano Plane $\text{Aut}(FP)$ has 168 elements [8]. If we number the points as indicated in Fig. 46 (left) we have:

$$P = \{0, 1, 2, 3, 4, 5, 6\}$$

$$C = \{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$$

It is easy to verify, just by making the corresponding changes in the asterisms in $C$, that any of the following permutations takes any asterism in $FP = (P, C)$

\(^{17}\)This leaves us of course with the additional problem of finding the subgroups of a group.
to another one, i.e.: that it is an automorphism of $FP$:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 0 & 1 & 4 & 5 & 3 & 6 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 0 & 5 & 3 & 4 & 6 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 0 & 1 & 4 & 5 & 3 & 6 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 0 & 5 & 3 & 4 & 6 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 0 & 5 & 4 & 3 & 6 \\
\end{pmatrix}
\]

It is also easy to see that the six permutations form a group, with composition of permutations as the group operation. We notice how the group only leaves invariant point 6. We wonder if we could link this group to the action of a symmetry group. The obvious candidate is $D_3$, because it has six elements (three rotations and three reflections) and fixes a single point.

We could try now to rearrange the points so that they exhibit $D_3$ symmetry, and permute them until $D_3$ acts on the new asterisms as prescribed. Fig. 46 (right) shows that this is possible. The action homomorphism here maps the identity element in $D_3$ to the identity permutation above; the $120^\circ$ counter-clockwise rotation to the top center permutation; the $240^\circ$ counter-clockwise rotation to the top right permutation; and each of the 3 reflections to one of the bottom permutations. For the Fano Plane we have then (see Sect. 4.6):

\[A_{D_3}(FP) \subset Aut(FP) \subset Sym(P)\]

with respective orders 6, 168 and $7! = 5040$. An example of an automorphism not linked to $D_3$ is:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 5 & 4 & 3 & 6 & 1 & 0 \\
\end{pmatrix}
\]

4.8.4 Sperner Families of Boards

Let’s suppose that $S_{SB}$ is the set of symmetric boards that are solutions of the BSP for a given asymmetric board. If we put their symmetry groups in a set $S_{SG}$ we can remove duplicate symmetry groups. To further reduce redundancy in $S_{SB}$, we can retain instead a subset $S'_{SB}$ of the former such that $S_{SG}$, the set of its symmetry groups, is the Sperner family [2] of $S_{SG}$\footnote{Besides the identity, groups in the Sperner family may have other common elements.}, i.e: a set with two properties:

- no group in $S'_{SG}$ is a proper subgroup of another group in $S'_{SG}$
- any group in $S_{SG}$ not in $S'_{SG}$ is a subgroup of at least one member of $S'_{SG}$

4.8.5 Computational Complexity of the BSP

There are methods based on geometric symmetry to find graph automorphisms\footnote{I.e.: permutations of the vertices that leave the edges invariant.} [11, 16], a problem known to be in the class NP of computational complexity [18]. Since graphs are a subset of hypergraphs, and since boards are geometric hypergraphs, the BSP complexity class is also at least NP. For algebraic methods to find automorphisms of hypergraphs see [4].
5 Latin Polytopes

In Sect. 3 we defined Latin boards and described some of their properties. In this section we focus on a subset of Latin boards with special properties of symmetry.

5.1 Symmetric Latin Boards

Definition 5.1. A Latin board is symmetric under a geometric transformation if it is transformed by this one into another Latin board.

Example 5.2. A Latin board is trivially symmetric under the identity transformation. A Latin square is symmetric under a reflection across its vertical symmetry axis.

Example 5.3. The board in Fig. 45 (center) is symmetric under a 120° counter-clockwise rotation. The board on its left is not symmetric under the same transformation.

5.2 Definition of Latin Polytope

The transformations under which a Latin board is symmetric may form a group. When the group is the symmetry group of a finite regular polytope we have the following objects:

Definition 5.4. A Latin finite regular polytope is a Latin board with the symmetry of a finite regular polytope.

We will normally drop the adjectives and use simply “Latin polytopes” when there is no risk of ambiguity. Latin squares for example are symmetric under any of the symmetries of the square –a finite regular polytope with $D_4$ symmetry—hence Latin squares are Latin polytopes. The Latin boards in Figs. 3 and 4 are also Latin polytopes, since they are symmetric under $D_3$ and $D_6$ respectively.

5.3 Latin Polygons

Definition 5.5. A Latin polygon is a Latin polytope with the symmetry of a finite regular polygon.

The symmetry groups of the regular polygons are the dihedral groups $D_n$, with order $2n$ ($n$ reflections and $n$ rotations, see Sect. 4.1.1). Some dihedral groups properly contain others, for example $D_3 \subset D_6$. When a Latin polygon has more than one symmetry group we will name it after the polygon with the group of highest order.

Example 5.6. Conventional Latin squares have $D_4$ symmetry, so they are Latin polygons. Sudoku (see Fig. 12 right) are also Latin polygons since they feature the same symmetry.
5.3.1 About the Example Latin Polygons

The next section defines some Latin polygons. The accompanying examples are latinizations (see Sect. 3.3) of the boards in Sect. 4.7.4.4. The multiset used in each latinization defines the balance and intersection pattern in the resulting Latin polygon and its woven board (see Sect. 3.7). The examples were created by the author using the technique described in [21].

5.3.2 Latin Triangles

Definition 5.7. A Latin triangle is a Latin polytope with $D_3$ symmetry.

Example 5.8. If we latinize the board in Fig. 26 with numbers 1 to 12 we obtain Latin triangles like the one in Fig. 47 (left). To derive its woven board we make up a warp asterism out of the points with the same label. Each warp asterism intersects in one point each weft asterism. Warp asterisms constitute then an orthogonal class: the resulting woven board has SIN $\{0,1,4\}$.

Example 5.9. Fig. 47 (right) shows a Latin triangle with the board in Fig. 26 and the multiset \{1,1,1,2,2,2,3,3,3\}. The SIN in the derived woven board is \{0,4\}, as every warp asterism intersects in four points every weft one. Comparing these results with those in Example 5.8 we notice the role of the multiset in the balance and intersection pattern of the resulting boards.

Example 5.10. Fig. 48 (left) shows a Latin triangle with the board in Fig. 27 latinized with numbers 1 to 9, i.e.: with an orthogonal warp class in the derived woven board.

Example 5.11. Fig. 48 (right) shows a Latin triangle resulting from the latinization of the board in Fig. 27 with the multiset \{1,1,1,2,2,2,3,3,3\}. Each warp asterism in the derived woven board intersects in four points every weft asterism.

Example 5.12. Fig. 49 (left) shows a Latin triangle resulting from the latinization of the board in Fig. 28 with numbers 0 to 9, i.e.: with an orthogonal warp class in its woven board.

Example 5.13. Fig. 49 (right) shows a Latin triangle with the board in Fig. 28 and the multiset \{0,0,1,1,2,2,3,3,3,4,4\}. Here each warp asterism in the woven board intersects in four points every weft one.

5.3.3 Free Latin Squares

Definition 5.14. A free Latin square is a Latin polytope with $D_4$ symmetry.

Example 5.15. Fig. 2 shows a free Latin square resulting from the latinization of the board in Fig. 19 with all numbers from 1 to 16. Its derived woven board has SIN \{0,1,4\} because its warp asterisms constitute an orthogonal class.

---

20This example corresponds to the notion of Latin triangle defined in [15], which is a more restricted concept than the one proposed here. See Fig. 56 for a larger Latin triangle with points in the barycenters, like this one.
Figure 47. Latin triangles with points at the barycenters

Figure 48. Latin triangles with points at the vertices

Figure 49. Latin triangles with points at the edge midpoints
Example 5.16. Fig. 20 shows three free Latin square with the board in Fig. 19 and the multiset \( \{1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4\} \). Their derived woven boards have also \( \text{SIN} \{0, 4\} \) because every warp asterism intersects in four points every weft asterism.

Example 5.17. Fig. 50 shows a free Latin square with the board in Fig. 29 and numbers 1 to 16. The warp class in the derived woven board is orthogonal to every weft class, so its \( \text{SIN} \) is \( \{0, 1, 4\} \).

Example 5.18. The base board for Sudoku (see Fig. 8) is clearly \( D_4 \)-symmetric although not class-symmetric, because there is no way to take the rows or columns to the \( 3 \times 3 \) sub-square asterisms with an element of the group. Latinizations of this board with different multisets, like the ones shown on the right in Figs. 12, 13 and 14 are also free Latin squares.

Example 5.19. A conventional Latin square is a Latin polygon having

1. a biregular square (see Sect. 4.7.4.4.2) as a source
2. face centers as board points
3. a class-symmetric board with \( D_4 \) symmetry and \( \text{SIN} \{0, 1\} \)
4. a derived woven board with \( \text{SIN} \{0, 1\} \) too

This description does not univocally determines conventional Latin squares, as we see next. Fig. 51 (left) shows a biregular square of order seven whose face centers have been chosen as board points. Fourteen asterisms of seven points each have been formed, each one coming from either one or two slanted sets of points pointed to by the same letter (for example asterism \( c \) in green and asterism \( n \) in brown). There are two parallel classes here: \( \{a, b, c, d, e, f, g\} \) and \( \{h, i, j, k, l, m, n\} \). As \( D_4 \) acts transitively on the classes the board is class-symmetric, and has \( \text{SIN} \{0, 1\} \). Fig. 51 (right) shows a free Latin square with this board labeled with numbers 1 to 7. Its derived woven board has \( \text{SIN} \{0, 1\} \) because the warp asterisms constitute an orthogonal class. This free Latin square has then the main features of conventional Latin squares without being itself one of them.
Example 5.20. Fig. 52 shows a free Latin square with the board in Fig. 30 latinized with numbers 1 to 8. The SIN in the derived woven board is \{0, 1, 2\} because each warp asterism is orthogonal to every weft one. This example shows that the number of points in a free Latin square need not be a perfect square.

5.3.4 Latin Hexagons

Definition 5.21. A Latin hexagon is a Latin polytope with $D_6$ symmetry.

Example 5.22. Fig. 53 (left) shows a Latin hexagon featuring the board in Fig. 31 labeled with numbers 1 to 18. The derived woven board has SIN \{0, 1, 6, 8\} since the warp asterisms constitute an orthogonal class here.

Example 5.23. Fig. 53 (right) shows a Latin hexagon with the board in Fig. 31 and the multiset \{1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3\}. This time the derived woven board has SIN \{0, 6, 8\} because each warp asterism intersects in six points every weft asterism.

Example 5.24. Fig. 54 (left) shows a Latin hexagon with the board in Fig. 32 and all numbers from 1 to 14. The derived woven board has an orthogonal warp class because every warp asterism intersect in one point every weft asterism.
Example 5.25. Fig. 54 (right) shows another Latin hexagon with the board in Fig. 31 latinized this time with multiset \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7\}. Here every warp asterism in the derived woven board intersects in two points every weft asterism.

5.3.5 Latin Dodecagons

Definition 5.26. A Latin dodecagon is a Latin polytope with $D_{12}$ symmetry.

Example 5.27. Fig. 55 shows a Latin dodecagon with the board in Fig. 21 latinized with the set \{H, E, L, I, O, S\}. The SIN of the derived woven board is \{0, 1\} because the warp asterisms constitute a parallel class orthogonal to the weft asterisms.

5.4 Latin Polyhedra

Definition 5.28. A Latin polyhedron is a Latin polytope with the symmetry of a finite regular polyhedron.
There are five finite real convex regular polyhedra, and three polyhedral symmetry groups (see Table 4.2). We may name the resulting Latin polyhedron either after its polyhedral group or after the polyhedron that is closer to the source of symmetry used. In any case the underlying symmetry group should be clear. Hereafter we will adopt the second naming convention.

5.4.1 About the Example Latin Polyhedra

Each of the following examples presents a Latin polyhedron and a corresponding Latin puzzle (see [21]). All pictures are meant to be folded along the inner black lines and glued along the outer ones. With the exception of those in the example Latin dodecahedron, every asterism is made up of points enclosed by two lines of the same color. The asterism becomes a closed band –so to speak– in the polyhedron surface when the picture is folded. The examples were created by the author using the latinization techniques described in [21].

5.4.2 Latin Tetrahedra

Fig. 56 (left) shows the board in Fig. 33 latinized with all numbers from 1 to 16, i.e.: a Latin tetrahedron\(^21\). The derived woven board has \(\text{SIN} \{0, 1, 4\}\), since it has an orthogonal warp class with sixteen asterisms with four points each. On the right there is a corresponding Latin puzzle.

5.4.3 Free Latin Cubes

Fig. 57 shows a free Latin cube\(^22\) with the board in Fig. 35 latinized with all numbers from 1 to 16. The derived woven board has \(\text{SIN} \{0, 1, 8\}\) because it...

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\(^{21}\)If we similarly fold and glue the Latin triangle in Example 4.23 we also obtain a (smaller) Latin tetrahedron.

\(^{22}\)Or Latin octahedron if we follow the first naming criterion in Sect. 5.4. We use “free Latin cube” instead of “Latin cube” because this last name already identifies two different combinatorial objects. One is the natural generalization of Latin squares: an \(n \times n \times n\) array in which each of a set of \(n\) elements occurs once in each row, once in each column and once in each file. The other is a different object used by statisticians. This is why the first meaning cited, the generalization of Latin squares, is usually called “permutation cube” [17].
Figure 56. Latin tetrahedron and Latin tetrahedron puzzle

has an orthogonal warp class with sixteen asterism with six points each. Fig. 58 shows a corresponding puzzle.

5.4.4 Latin Octahedra

Fig. 59 shows a Latin octahedron resulting from the latinization of the board in Fig. 37 with all numbers from 1 to 18. The derived woven board has SIN \{0, 1, 4\} because it has an orthogonal warp class with eighteen asterisms with four points each. Fig. 60 shows a corresponding Latin puzzle.

5.4.5 Latin Icosahedra

Fig. 61 shows a Latin icosahedron resulting from the latinization of the board in Fig. 39 with all numbers from 1 to 20. The derived woven board has SIN \{0, 1, 4\} because it has an orthogonal warp class with twenty asterisms with four points each. Fig. 62 shows a corresponding Latin puzzle.

5.4.6 Latin Dodecahedra

Fig. 63 shows a Latin dodecahedron resulting from the latinization of the board in Fig. 41 with all numbers from 1 to 6. The derived woven board has SIN \{0, 1, 2\} because there is an orthogonal warp class with six asterisms with five points each. Fig. 64 shows a corresponding Latin puzzle.

5.5 Equivalence Classes of Latin Polytopes

We saw in Sect. 3.6 that a set of operations that easily generate Latin boards from a given one serves to define a notion of equivalence among boards that classify them into equivalence classes. We also saw how this was useful when enumerating or counting Latin boards.

On the other hand, asterisms in a Latin polytope are taken to other asterisms by the action of a symmetry group. This not only gives us some automorphisms of the board—the see Relation 4.6—it also moves the labels so as to produce another,

\(^{23}\)Or Latin icosaedron if we follow the first naming criterion in Sect. 5.4.
### Figure 57. Free Latin cube

| 7 | 3 | 9 | 6 | 12 | 8 | 15 | 16 |
|---|---|---|---|----|---|----|----|
| 4 | 11| 15| 8 |
| 1 | 14| 6 | 15 |
| 8 | 12| 5 | 16 |
| 14| 2 | 13| 1 |

### Figure 58. Free Latin cube puzzle

| 3 | 6 | 7 | 9 |
|---|---|---|---|
| 10| 11| 3 | 12 |
| 16| 4 | 13| 10 |
| 5 | 9 | 2 | 7 |
Figure 59. Latin octahedron

Figure 60. Latin octahedron puzzle
Figure 61. Latin icosahedron

Figure 62. Latin icosahedron puzzle

Figure 63. Latin dodecahedron
not necessarily different, Latin polytope\textsuperscript{24}. This means that, in absence of larger groups, geometric symmetry may be used to derive equivalence classes.

As commented in Sect. 3.6, the larger the symmetry group the less equivalence classes there will be and the more boards per class will result\textsuperscript{25}. This is illustrated by the next example. We know that the board of a Latin square has $D_4$ symmetry. Some elements of this group have the same effect as an isotopism – for example a reflection across the vertical symmetry axis – while others produce a conjugate Latin square – for example a reflection across the top-left to bottom-right diagonal, that swaps rows and columns.

The symmetry group of the square-tiled torus of order $n$, whose fundamental polygon features single horizontal and vertical borders (see Fig. 65 for the $n = 3$ case), has $D_4$ as a subset. In a square-tiled torus of order $n$ the group also includes rotations of $\frac{360}{n} \times k$, $k = 1, \ldots, n$, around both its vertical and internal axis. These rotations induce circular permutations in rows and columns; these permutations are a subset of the full set of permutations of rows and columns in the Latin square. The symmetry group of the square-tiled torus of order $n$ will produce then less equivalence classes of Latin squares of order $n$ than $D_4$, but still more than the number of isotopic classes (see Sect. 3.6.1).

6 Conclusions

Taking Latin squares as a departure point, we have introduced the concept of latinization and defined boards, Latin boards, symmetric boards and Latin polytopes. We have also provided examples that prove their existence for given values of the relevant parameters.

These results are accompanied by corresponding generalizations of other concepts related to Latin squares, like the orthogonal intersection property in sets

\textsuperscript{24} Not all board automorphisms produce Latin polytopes though. In the case of the board of a Latin square for example, swapping just the first row and first column of points is a board automorphism that does not produce a Latin square in general.

\textsuperscript{25} The precise number of classes generated by the group can be calculated using Burnside’s Lemma [23].
of mutually orthogonal Latin squares (MOLS). The generalization of uniquely completable partial Latin squares to Latin puzzles has also been mentioned. This topic is studied in the companion paper *Latin Puzzles* [21], where several latinization techniques and their computational complexity are also discussed.

The new defined combinatorial objects offer varied compromises between balance and symmetry. This is especially true in the case of symmetric boards and Latin polytopes, in which geometric symmetry translates naturally into both combinatorial symmetry (resolvable boards with simple intersection patterns) and algebraic symmetry (automorphisms of the board and equivalence classes of Latin polytopes). On a related note, we have seen how the use of multisets—rather than sets—for latinization offers the possibility to fine-tuning these properties in the resulting Latin polytopes and woven boards.

We have proposed several ways to construct symmetric boards from sources of symmetry (orthogonal projections of sources onto lower dimensions, kaleidoscopic sets generated by the action of symmetry groups, etc.). We have also discussed ways to enhance the symmetry of a given source (composition, truncation, tessellation, etc.), with an emphasis on tessellations based on biregular polytopes, a method that can easily generate whole families of sources of symmetry, and hence whole families of symmetric boards and Latin polytopes.

Finally, we have also touched upon the problem of deriving symmetric boards from existing non-symmetric ones.

### 7 Suggestions for Future Work

Future work on Latin polytopes could proceed along the lines mentioned below.

**New Latin Boards and Polytopes**

- based on the different sources of symmetry listed in Sect. 4.7
- in spaces other than $\mathbb{R}^n$ (elliptic, hyperbolic or complex spaces [9, 10])
- Latin abstract polytopes (from sources derived from abstract regular polytopes [10])
- class-symmetric polyhedral boards (in Sect. 4.7.4.4.4 only the tetrahedral board was class-symmetric)
Study of Properties

- classification and enumeration
- notions of isotopism and conjugacy when applicable
- relation with hypergraphs (automorphism group, coloring, etc.)
- relation with combinatorial designs (extension of the notion of orthogonality, resolvability, etc.)
- notions of equivalence and derived equivalence classes
- properties of related partial objects like partial Latin boards [21]

Applications

- design of experiments
- coding
- scheduling
- timetabling
- resource allocation
- mathematical puzzles

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