On solutions of the Pauli equation in non-static de Sitter metrics

E. M. Ovsiyuk, K. V. Kazmerchuk
Mozyr State Pedagogical University named after I.P. Shamyakin
e.ovsiyuk@mail.ru, kristinash2@mail.ru

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Abstract
A particle with spin 1/2 is investigated both in expanding and oscillating cosmological de Sitter models. It is shown that these space-time geometries admit existence of the non-relativistic limit in the covariant Dirac equation. Procedure for transition to the Pauli approximation is conducted in the equations in the variables \((t, r)\), obtained after separating the angular dependence of \((\theta, \phi)\) from the wave function. The non-relativistic systems of equations in the variables \((t, r)\) is solved exactly in both models. The constructed wave functions do not represent stationary states with fixed energy, however the corresponding probability density does not depend on the time.

1 Introduction
De Sitter and anti de Sitter models take attraction during long time in the context of development of the quantum theory in curved space-time. In particular, long history has the task of exact solving the wave equations for fields with different spins [1]–[12]; the main attention was focused of relativistic wave equations.

In the present paper we will examine the nonrelativistic Pauli approximation for spin 1/2 particle in non-static metrics of de Sitter models: we demonstrate how transition to Pauli approximation can be performed in non-static de Sitter’s metrics, and we will construct exact solutions for these non-relativistic systems in both de Sitter models: expanding and oscillating ones.

2 Pauli equation in expanding de Sitter Universe
Generally covariant Dirac equation in orthogonal coordinates [13]

\[
[i\gamma^k ( e^{\alpha}_{(k)} \partial_{\alpha} + B_k ) - m] \Psi = 0 ,
\]

\[
B_k(x) = \frac{1}{2} e^{\alpha}_{(k)\alpha}(x) = \frac{1}{2} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} e^{\alpha}_{(k)}
\]

being specified for the non-static de Sitter metric

\[
x^\alpha = (t, r, \theta, \phi), \quad dS^2 = dt^2 - \cosh^2 t [dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\phi^2)]
\]

and the corresponding tetrad

\[
e^{\alpha}_{(0)} = (1, 0, 0, 0) , \quad e^{\alpha}_{(3)} = \left(0, \frac{1}{\cosh t}, 0, 0\right) ,
\]

\[
e^{\alpha}_{(2)} = \left(0, 0, 0, \frac{1}{\cosh t \sin r \sin \theta}\right) , \quad e^{\alpha}_{(1)} = \left(0, 0, \frac{1}{\cosh t \sin r}, 0\right)
\]
takes the form
\[
\left[ i \gamma^0 \cosh t \left( \partial_t + \frac{3}{2} \tanh t \right) + \frac{1}{\sin r} \Sigma_{\theta \phi} + i \gamma^3 \left( \partial_r + \frac{1}{\tan r} \right) - m \cosh t \right] \Psi = 0 ,
\]
where
\[
\Sigma_{\theta \phi} = i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + i \sigma^{12} \cos \theta}{\sin \theta}.
\]
Separating a simple factor in the wave function
\[
\Psi(x) = \frac{1}{\sin r \cosh^{3/2} t} \varphi(x) ,
\]
we obtain a simpler equation
\[
\left[ i \gamma^0 \cosh t \frac{\partial}{\partial t} + i \gamma^3 \frac{\partial}{\partial r} + \frac{1}{\sin r} \Sigma_{\theta \phi} - m \cosh t \right] \varphi = 0 .
\]

To diagonalize the total angular moment on the wave functions, we are to use the following substitution [14] (Wigner’s functions [15] are referred as \( D_{j}^{\mp m, \sigma}(\phi, \theta, 0) \equiv D_{\sigma} \)):
\[
\varphi_{jm}(x) = \left| \begin{array}{c}
 f_1(t, r) D_{-1/2}^{-1/2} \\
 f_2(t, r) D_{1/2}^{1/2} \\
 f_3(t, r) D_{-1/2}^{-1/2} \\
 f_4(t, r) D_{1/2}^{1/2}
\end{array} \right| . \tag{2.1}
\]

With the use of recurrent relations [15]
\[
\begin{align*}
\partial_\theta D_{+1/2} &= a D_{-1/2} - b D_{+3/2} , \\
-m - \frac{1}{2} \cos \theta \frac{\sin \theta}{D_{+1/2}} &= -a D_{-1/2} - b D_{+3/2} , \\
\partial_\theta D_{-1/2} &= b D_{-3/2} - a D_{+1/2} , \\
-m + \frac{1}{2} \cos \theta \frac{\sin \theta}{D_{-1/2}} &= -b D_{-3/2} - a D_{+1/2} ,
\end{align*}
\]
where
\[
a = \frac{j + 1/2}{2} , \quad b = \frac{1}{2} \sqrt{(j - 1/2)(j + 3/2)} ,
\]
we find the action of the angular operator \( \Sigma_{\theta \phi} \)
\[
\Sigma_{\theta \phi} \varphi_{jm}(x) = i \nu \left| \begin{array}{c}
 -f_4(t, r) D_{-1/2}^{-1/2} \\
 +f_3(t, r) D_{1/2}^{1/2} \\
 +f_2(t, r) D_{-1/2}^{-1/2} \\
 -f_1(t, r) D_{1/2}^{1/2}
\end{array} \right| , \tag{2.2}
\]
where \( \nu = j + 1/2 \). Then produce four equations in variables \((t, r)\):
\[
\begin{align*}
 i \cosh t \frac{\partial}{\partial t} f_3 - i \frac{\partial}{\partial r} f_3 - i \frac{\nu}{\sin r} f_4 - m \cosh t f_1 &= 0 , \\
 i \cosh t \frac{\partial}{\partial t} f_4 + i \frac{\partial}{\partial r} f_4 + i \frac{\nu}{\sin r} f_3 - m \cosh t f_2 &= 0 , \\
 i \cosh t \frac{\partial}{\partial t} f_1 + i \frac{\partial}{\partial r} f_1 + i \frac{\nu}{\sin r} f_2 - m \cosh t f_3 &= 0 , \\
 i \cosh t \frac{\partial}{\partial t} f_2 - i \frac{\partial}{\partial r} f_2 - i \frac{\nu}{\sin r} f_1 - m \cosh t f_4 &= 0 . \tag{2.3}
\end{align*}
\]
Usual $P$-reflection operator in Cartesian basis of the tetrad $\hat{\Pi}_C = i\gamma^0 \otimes \hat{P}$

$$\hat{\Pi}_C = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix} \otimes \hat{P}, \quad \hat{P}(\theta, \phi) = (\pi - \theta, \phi + \pi)$$

after translating it to spherical tetrad will take the form

$$\hat{\Pi}_{sph} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} \otimes \hat{P}.$$ (2.4)

From eigenvalue equation $\hat{\Pi}_{sph}\Psi_{jm} = \Pi \Psi_{jm}$ with the us of the known identity [15]

$$\hat{P} D^j_{-m,\sigma} (\phi, \theta, 0) = (-1)^j D^j_{-m,\sigma} (\phi, \theta, 0)$$

we derive linear restrictions of the functions:

$$\Pi = \delta (-1)^{j+1}, \quad \delta = \pm 1, \quad f_4 = \delta f_1, \quad f_3 = \delta f_2.$$ 

Thus, the wave function for states with fixed parity takes the form

$$\varphi(x)_{jm\delta} = \begin{vmatrix} f_1(t, r) D_{-1/2} \\ f_2(t, r) D_{+1/2} \\ \delta f_2(t, r) D_{-1/2} \\ \delta f_1(t, r) D_{+1/2} \end{vmatrix}.$$ (2.5)

Allowing for (2.5), from (2.4) we get more simple equations

$$\left(\frac{\partial}{\partial r} + \frac{\nu}{\sin r}\right)f + \left(i \cosh t \frac{\partial}{\partial t} + \delta m \cosh t\right)g = 0,$$

$$\left(\frac{\partial}{\partial r} - \frac{\nu}{\sin r}\right)g - \left(i \cosh t \frac{\partial}{\partial t} - \delta m \cosh t\right)f = 0,$$ (2.6)

where instead of $f_1(t, r)$ and $f_2(t, r)$ their linear combinations are used:

$$f(t, r) = \frac{f_1 + f_2}{\sqrt{2}}, \quad g(t, r) = \frac{f_1 - f_2}{i\sqrt{2}}.$$

Now, let us perform nonrelativistic approximation in the system (2.6) (here we will adhere the method exposed in [13, 14]). First, we separate the rest energy by the formal change

$$i \frac{\partial}{\partial t} \implies M + i \frac{\partial}{\partial t}.$$

The cases $\delta = \pm 1$ should be considered separately:

$\delta = +1,$

$$\frac{1}{\cosh t} \left(\frac{\partial}{\partial r} + \frac{\nu}{\sin r}\right)f + \left(M + i \frac{\partial}{\partial t} + M\right)g = 0,$$

$$\frac{1}{\cosh t} \left(\frac{\partial}{\partial r} - \frac{\nu}{\sin r}\right)g - \left(M + i \frac{\partial}{\partial t} - M\right)f = 0;$$ (2.7)
\[ \delta = -1, \]
\[ \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) f + \left( M + i \frac{\partial}{\partial t} - M \right) g = 0, \]
\[ \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) g - \left( M + i \frac{\partial}{\partial t} + M \right) f = 0; \] (2.8)

the term \( i \partial_t \) should be neglected in comparing with \( 2M \). Equations (2.7), (2.8) give respectively \( \delta = +1, \)
\[ \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) f + 2Mg = 0, \]
\[ \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) g - i \frac{\partial}{\partial t} f = 0; \]
\[ \delta = -1, \]
\[ \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) f + i \frac{\partial}{\partial t} g = 0, \]
\[ \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) g - 2Mf = 0. \]

In each case, we derive an equation for the big component (the concomitant equation permits to express the small component via the big one)
\[ \delta = +1, \]
\[ i \frac{\partial}{\partial t} f = -\frac{1}{2M \cosh^2 t} \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) f, \] (2.9)
\[ g = -\frac{1}{2M \cosh t} \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) f; \]
\[ \delta = -1, \]
\[ i \frac{\partial}{\partial t} g = -\frac{1}{2M \cosh t} \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sin r} \right) \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) g, \] (2.10)
\[ f = \frac{1}{2M \cosh^2 t} \frac{1}{\cosh t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sin r} \right) g. \]

Recall that the Pauli wave function is constructed from relativistic one in accordance with
\[ \phi_{jm\delta}(x) = \begin{vmatrix} f_1 D_{-1/2} \\ f_2 D_{+1/2} \\ \delta f_2 D_{-1/2} \\ \delta f_1 D_{+1/2} \end{vmatrix} \implies \Psi_{jm\delta}^{Pauli}(x) = \begin{vmatrix} f_1 D_{-1/2} + \delta f_2 D_{-1/2} \\ f_2 D_{+1/2} + \delta f_1 D_{+1/2} / \end{vmatrix}. \] (2.11)

For different parities, eq. (2.11) gives respectively \( \delta = +1, \)
\[ \Psi_{jm+1}^{Pauli}(t, r, \theta, \phi) = \begin{vmatrix} f_1 D_{-1/2} + f_2 D_{-1/2} \\ f_2 D_{+1/2} f_1 D_{+1/2} / \end{vmatrix} = \begin{vmatrix} f(t, r) D_{-1/2} \\ f(t, r) D_{+1/2} / \end{vmatrix}; \]
\[ \delta = -1, \]
\[ \Psi^{\text{Pauli}}_{jm,-1}(t, r, \theta, \phi) = \begin{vmatrix} f_1 D_{-1/2} - f_2 D_{-1/2} \\ f_2 D_{1/2} - f_1 D_{1/2} \end{vmatrix} = -ig(t, r)D_{-1/2} - ig(t, r)D_{1/2}. \]

Note that according to (2.9) and (2.10), non-relativistic functions \( f(t, r) \) and \( g(t, r) \) obey similar but slightly different equations:

\[ \delta = +1, \]
\[ i \frac{\partial f}{\partial t} = \frac{1}{2M} \frac{1}{\cosh^2 t} \left( \frac{\partial^2}{\partial t^2} - \frac{\nu^2 + \nu \cos r}{\sin^2 r} \right) f; \] (2.12)

\[ \delta = -1, \]
\[ i \frac{\partial g}{\partial t} = \frac{1}{2M} \frac{1}{\cosh^2 t} \left( \frac{\partial^2}{\partial t^2} - \frac{\nu^2 - \nu \cos r}{\sin^2 r} \right) g. \] (2.13)

In (2.12), (2.13), the variables are separated by the substitutions

\[ f(t, r) = f(t) f(r), \quad g(t, r) = g(t) g(r); \]

in this way we get

\[ \delta = +1, \]
\[ i \cosh^2 t \frac{1}{f(t)} \frac{d}{dt} f(t) = -\frac{1}{2M} \frac{1}{f(r)} \left( \frac{d^2}{dr^2} - \frac{\nu^2 - \nu \cos r}{\sin^2 r} \right) f(r) = E; \]

\[ \delta = -1, \]
\[ i \cosh^2 t \frac{1}{g(t)} \frac{d}{dt} g(t) = -\frac{1}{2M} \frac{1}{g(r)} \left( \frac{d^2}{dr^2} - \frac{\nu^2 + \nu \cos r}{\sin^2 r} \right) g(r) = E. \]

The final separated equations are

\[ \delta = +1, \]
\[ i \cosh^2 t \frac{1}{f(t)} \frac{d}{dt} f(t) = E \quad \Rightarrow \quad f(t) = e^{-iE \tanh t}, \]
\[ -\frac{1}{2M} \frac{1}{f(r)} \left( \frac{d^2}{dr^2} - \frac{\nu^2 + \nu \cos r}{\sin^2 r} \right) f(r) = E; \]

\[ \delta = -1, \]
\[ i \cosh^2 t \frac{1}{g(t)} \frac{d}{dt} g(t) = E \quad \Rightarrow \quad g(t) = e^{-iE \tanh t}, \]
\[ -\frac{1}{2M} \frac{1}{g(r)} \left( \frac{d^2}{dr^2} - \frac{\nu^2 - \nu \cos r}{\sin^2 r} \right) g(r) = E. \]

Note the symmetry between equations for \( f(r) \) and \( g(r) \): \( \nu \Rightarrow -\nu \); therefore it suffices to examine in detail only one case and then to employ the formal change. For definiteness, we will consider the variant \( \delta = +1 \):

\[ \left( \frac{d^2}{dr^2} - \frac{\nu^2 + \nu \cos r}{\sin^2 r} + 2ME \right) f(r) = 0. \] (2.14)
Because the 3-subspace in de Sitter space-time coincides with the compact spherical Riemann model, the motion if the variable \( r \) must be quantized; besides, we must assume that \( 2ME > 0 \). In eq. (2.14), it is convenient to use a new variable, \( z = \cos r \), \( z \in (-1, +1) \):

\[
(1 - z^2) \frac{d^2 f}{dz^2} - z \frac{df}{dz} - \left( \nu - \frac{z}{1 - z^2} + \frac{\nu^2}{1 - z^2} - 2ME \right) f = 0,
\]

and then to introduce another variable

\[
y = \frac{1 - z}{2} = \frac{1 - \cos r}{2},
\]

With the substitution \( f = y^A (1 - y)^B F \), we get

\[
y(1 - y) \frac{d^2 F}{dy^2} + \left( \frac{1}{2} - y \right) \frac{df}{dy} - \left[ \nu(\nu + 1) + \frac{\nu(\nu - 1)}{4(1 - y)} - 2ME \right] f = 0.
\]

At restrictions

\[
2A = \nu + 1, -\nu; \quad 2B = -\nu + 1, +\nu,
\]

the above equation simplifies

\[
y(1 - y) \frac{d^2 F}{dy^2} + \left[ 2A + \frac{1}{2} - (2A + 2B + 1)y \right] \frac{df}{dy} - [(A + B)^2 - 2ME] F = 0,
\]

and coincides with the equation of hypergeometric type

\[
y(1 - y) F'' + [(c - (a + b + 1)y] F' - ab F = 0,
\]

where

\[
c = 2A + \frac{1}{2}, \quad a + b = 2A + 2B, \quad ab = (A + B)^2 - 2ME;
\]

from whence it follows

\[
a = A + B - \sqrt{2ME}, \quad b = A + B + \sqrt{2ME}.
\]

From physical considerations, we should construct solutions in polynomials and they should be associated with discrete spectrum of energy \( 2ME > 0 \). Appropriate solutions are possible at positive \( A \) and \( B \):

\[
2A = +\nu + 1 = j + 3/2, \quad 2B = +\nu = j + 1/2,
\]

polynomial condition provides us with the quantization rule

\[
a = -n, \quad n = 0, 1, 2, ... \quad 2ME = (j + 1 + n)^2;
\]

corresponding radial functions are

\[
f(y) = y^{(\nu + 1)/2} (1 - y)^{\nu/2} F(a, b, c, y),
\]

\[
b = 2(j + 1) + n, \quad c = j + 2 \quad (\nu = j + 1/2).
\]
With the use of the mentioned symmetry between \( f(r) \) and \( g(r) \), one gets description of states with other parity

\[
g(y) = y^A (1 - y)^B G(y),
\]

\[
2A' = -\nu + 1, \nu, \quad 2B' = \nu + 1, -\nu;
\]

at positive \( A' \) and \( B' \):

\[
2A' = \nu = j + 1/2, \quad c' = j + 1 = c - 1,
\]

\[
2B' = +\nu + 1 = j + 3/2, \quad A' + B' = j + 1,
\]

\[
a' = -n = a, \quad b' = 2(j + 1) + n = b,
\]

\[
g(y) = y^{\nu/2} (1 - y)^{\nu + 1/2} F(a, b, c - 1, y).
\]

Thus, solutions of the Pauli equation in non-static de Sitter metrics have been constructed \( \delta = +1 \),

\[
\Psi^{Pauli}_{Ejm,1}(t, r, \theta, \phi) = e^{-iE \tanh t} f(r) \begin{vmatrix} D_{j\cdot-1/2}^{j\cdot-1/2}(\phi, \theta, 0) \\ D_{j\cdot+1/2}^{j\cdot+1/2}(\phi, \theta, 0) \end{vmatrix},
\]

\[
f(r) = (\sin \frac{r}{2})^{j+3/2} \left( \cos \frac{r}{2} \right)^{j+1/2} F(-n, n + 2j + 2, j + 2, \sin^2 \frac{r}{2});
\]

\[
\delta = -1,
\]

\[
\Psi^{Pauli}_{Ejm,-1}(t, r, \theta, \phi) = e^{-iE \tanh t} g(r) \begin{vmatrix} D_{j\cdot-1/2}^{j\cdot-1/2}(\phi, \theta, 0) \\ D_{j\cdot+1/2}^{j\cdot+1/2}(\phi, \theta, 0) \end{vmatrix},
\]

\[
g(r) = (\sin \frac{r}{2})^{j+1/2} \left( \cos \frac{r}{2} \right)^{j+3/2} F(-n, n + 2j + 2, j + 1, \sin^2 \frac{r}{2}).
\]

Spectral parameter \( E \) (it does not represent the energy of stationary states) is quantized according to the rule (the same for state with different parities)

\[
2ME = (j + 1 + n)^2.
\]

3 Pauli equation in the oscillating anti de Sitter Universe

In the non-static anti de Sitter metric

\[
x^\alpha = (t, r, \theta, \phi), \quad dS^2 = dt^2 - \cos^2 t [dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2)]
\]

and the corresponding diagonal tetrad

\[
e^\alpha_{(0)} = (1, 0, 0, 0), \quad e^\alpha_{(3)} = \left( 0, \frac{1}{\cos t}, 0, 0 \right),
\]

\[
e^\alpha_{(2)} = \left( 0, 0, \frac{1}{\cos t \sinh r \sin \theta} \right), \quad e^\alpha_{(1)} = \left( 0, 0, \frac{1}{\cos t \sinh r}, 0 \right),
\]

generally covariant Dirac equation takes the form

\[
\left[ i\gamma^0 \cos t \left( \partial_t - \frac{3}{2} \tan t \right) + \frac{1}{\sinh r} \Sigma_{\theta \phi} + i\gamma^3 (\partial_r + \frac{1}{\tanh r}) - m \cos t \right] \Psi = 0.
\]
Separating a simper multiplier in the wave function

\[ \Psi(x) = \frac{1}{\sinh r} \frac{1}{\cos^{3/2}t} \varphi(x), \]

we obtain a more simple form

\[ \left[ i_\gamma^0 \cos t \frac{\partial}{\partial t} + i_\gamma^3 \frac{\partial}{\partial r} + \frac{1}{\sinh r} \Sigma_{\theta \phi} - m \cos t \right] \varphi = 0. \]

Using the technique of D-Wigner functions (2.1)–(2.2), we obtain the radial equation

\[ \begin{align*}
i \cos t \frac{\partial}{\partial t} f_3 - i \frac{\partial}{\partial r} f_3 - i \frac{\nu}{\sinh r} f_4 - m \cos t f_1 &= 0, \\
i \cos t \frac{\partial}{\partial t} f_4 + i \frac{\partial}{\partial r} f_4 + i \frac{\nu}{\sinh r} f_3 - m \cos t f_2 &= 0, \\
i \cos t \frac{\partial}{\partial t} f_1 + i \frac{\partial}{\partial r} f_1 + i \frac{\nu}{\sinh r} f_2 - m \cos t f_3 &= 0, \\
i \cos t \frac{\partial}{\partial t} f_2 - i \frac{\partial}{\partial r} f_2 - i \frac{\nu}{\sinh r} f_1 - m \cos t f_4 &= 0.
\end{align*} \]

Using the operator of spatial parity (2.4)–(2.5), we get more simple equations

\[ \begin{align*}
\left( \frac{\partial}{\partial r} + \frac{\nu}{\sinh r} \right) f + \left( i \cos t \frac{\partial}{\partial t} + \delta m \cos t \right) g &= 0, \\
\left( \frac{\partial}{\partial r} - \frac{\nu}{\sinh r} \right) g - \left( i \cos t \frac{\partial}{\partial t} - \delta m \cos t \right) f &= 0.
\end{align*} \] (3.1)

Now, let us perform the nonrelativistic approximation in the system (3.1). First, we separate the rest energy. The cases \( \delta = \pm 1 \) should be considered separately:

\( \delta = +1, \)

\[ \begin{align*}
\frac{1}{\cos t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sinh r} \right) f + \left( M + i \frac{\partial}{\partial t} + M \right) g &= 0, \\
\frac{1}{\cos t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sinh r} \right) g - \left( M + i \frac{\partial}{\partial t} - M \right) f &= 0;
\end{align*} \]

\( \delta = -1, \)

\[ \begin{align*}
\frac{1}{\cos t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sinh r} \right) f + \left( M + i \frac{\partial}{\partial t} - M \right) g &= 0, \\
\frac{1}{\cos t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sinh r} \right) g - \left( M + i \frac{\partial}{\partial t} + M \right) f &= 0.
\end{align*} \]

These system give respectively (the term \( i \partial_t \) should be neglected in comparing with \( 2M \))

\( \delta = +1, \)

\[ \begin{align*}
\frac{1}{\cos t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sinh r} \right) f + 2M g &= 0, \\
\frac{1}{\cos t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sinh r} \right) g - i \frac{\partial}{\partial t} f &= 0;
\end{align*} \]
\[ \delta = -1, \]
\[ \frac{1}{\cos t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sinh r} \right) f + i \frac{\partial}{\partial t} g = 0, \]
\[ \frac{1}{\cos t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sinh r} \right) g - 2Mf = 0. \]

For both cases, we derive equations
\[ \delta = +1, \]
\[ i \frac{\partial}{\partial t} f = -\frac{1}{2M} \frac{1}{\cos^2 t} \left( \frac{\partial^2}{\partial r^2} - \nu^2 + \nu \cosh r \right) f, \]
\[ g = -\frac{1}{2M} \frac{1}{\cos t} \left( \frac{\partial}{\partial r} + \frac{\nu}{\sinh r} \right) f; \]
\[ \delta = -1, \]
\[ i \frac{\partial}{\partial t} g = -\frac{1}{2M} \frac{1}{\cos^2 t} \left( \frac{\partial^2}{\partial r^2} - \nu^2 - \nu \cosh r \right) g, \]
\[ f = \frac{1}{2M} \frac{1}{\cos t} \left( \frac{\partial}{\partial r} - \frac{\nu}{\sinh r} \right) g. \]

Nonrelativistic functions \( f(t, r) \) and \( g(t, r) \) obey similar equations
\[ \delta = +1, \]
\[ i \cos^2 t \frac{1}{f(t)} \frac{d}{dt} f(t) = -\frac{1}{2M} \frac{1}{f(r)} \left( \frac{d^2}{dr^2} - \nu^2 + \nu \cosh r \right) f(r) = E; \]
\[ \delta = -1, \]
\[ i \cos^2 t \frac{1}{g(t)} \frac{d}{dt} g(t) = -\frac{1}{2M} \frac{1}{g(r)} \left( \frac{d^2}{dr^2} - \nu^2 - \nu \cosh r \right) g(r) = E. \]

In (3.2), (3.3) the variables are separated by the substitutions
\[ f(t, r) = f(t)f(r), \quad g(t, r) = g(t)g(r), \]
in this way we get
\[ \delta = +1, \]
\[ i \cos^2 t \frac{1}{f(t)} \frac{d}{dt} f(t) = -\frac{1}{2M} \frac{1}{f(r)} \left( \frac{d^2}{dr^2} - \nu^2 + \nu \cosh r \right) f(r) = E; \]
\[ \delta = -1, \]
\[ i \cos^2 t \frac{1}{g(t)} \frac{d}{dt} g(t) = -\frac{1}{2M} \frac{1}{g(r)} \left( \frac{d^2}{dr^2} - \nu^2 - \nu \cosh r \right) g(r) = E. \]

The final separated equations are
\[ \delta = +1, \]
\[ i \cos^2 t \frac{1}{f(t)} \frac{d}{dt} f(t) = E \quad \implies \quad f(t) = e^{-iE \tan t}, \]
\[ -\frac{1}{2M} \frac{1}{f(r)} \left( \frac{d^2}{dr^2} - \frac{\nu^2 + \nu \cosh r}{\sinh^2 r} \right) f(r) = E; \]
\[ \delta = -1, \]

\[ i \cos^2 t \frac{1}{g(t)} \frac{d}{dt} g(t) = E \quad \implies \quad g(t) = e^{-iEt \tan t}, \]

\[ - \frac{1}{2M} \frac{1}{g(r)} \left( \frac{d^2}{dr^2} - \frac{\nu^2 - \nu \cosh r}{\sinh^2 r} \right) g(r) = E. \]

Due to the symmetry between equations for \( f(r) \) and \( g(r) \): \( \nu \mapsto -\nu \); it suffices to examine in detail only one case and then to employ the formal change. For definiteness, we will consider the variant \( \delta = +1 \):

\[ \left( \frac{d^2}{dr^2} - \frac{\nu^2 + \nu \cosh r}{\sinh^2 r} + 2ME \right) f(r) = 0. \]  \hspace{1cm} (3.6)

In eq. (3.6), it is convenient to use the new variable \( z = \cosh r, z \in (1, +\infty) \):

\[ (1 - z^2) \frac{d^2 f}{dz^2} - z \frac{df}{dz} - \left( \nu \frac{z}{1 - z^2} + \frac{\nu^2}{1 - z^2} + 2ME \right) f = 0, \]

and then to introduce another variable

\[ y = \frac{1 - z^2}{2} = \frac{1 - \cosh r}{2}, \]

\[ y (1 - y) \frac{d^2 f}{dy^2} + \left( \frac{1}{2} - y \right) \frac{df}{dy} - \left[ \frac{\nu (\nu + 1)}{4y} + \frac{\nu (\nu - 1)}{4(1 - y)} + 2ME \right] f = 0. \]

With the substitution \( f = y^A (1 - y)^B F \), we get

\[ y(1 - y) \frac{d^2 F}{dy^2} + \left[ \frac{1}{2} + 2A - (1 + 2A + 2B)y \right] \frac{dF}{dy} - \left[ (A + B)^2 - \frac{2A(2A - 1) - \nu (\nu + 1)}{4y} - \frac{2B(2B - 1) - \nu (\nu - 1)}{4(1 - y)} + 2ME \right] F = 0. \]

At additional restriction

\[ A = -\frac{1}{2} \nu, \quad B = \frac{1}{2}, \]

the above equation simplifies

\[ y(1 - y) \frac{d^2 F}{dy^2} + \left[ \frac{1}{2} + 2A - (1 + 2A + 2B)y \right] \frac{dF}{dy} - [(A + B)^2 + 2ME] F = 0, \]

and coincides with the equation of hypergeometric type for \( F(a, b, c, y) \) wuith

\[ c = 2A + \frac{1}{2}, \quad a = A + B - i\sqrt{2ME}, \quad b = A + B + i\sqrt{2ME}. \]

Corresponding solutions are

\[ a = \frac{1}{2} + \nu - i\sqrt{2ME}, \quad b = \frac{1}{2} + \nu + i\sqrt{2ME}, \quad c = \frac{3}{2} + \nu = j + 2, \]

\[ a = 1 + j - i\sqrt{2ME}, \quad b = 1 + j + i\sqrt{2ME}, \quad f(r) = C y^{(\nu+1)/2} (1 - y)^{\nu/2} F(a, b, c, y). \]
Allowing for the mentioned symmetry between \( f(r) \) and \( g(r) \), one gets description of states with other parity:

\[
g = C' y^A(1 - y)^B G, \quad A' = \frac{1}{2} \nu, \quad \frac{1}{2} - \frac{1}{2} \nu, \quad B' = \frac{1}{2} + \frac{1}{2} \nu, \quad -\frac{1}{2} \nu,
\]

\[
a' = 1 + j - i\sqrt{2ME}, \quad b' = 1 + j + i\sqrt{2ME}, \quad c' = j + 1 = c - 1,
\]

\[
g(r) = C' y^{\nu/2}(1 - y)^{(\nu+1)/2} F(a, b, c - 1, y).
\]

Thus, solutions of the Pauli equation in non-static anti de Sitter metrics have been constructed

\[
\delta = +1, \quad \Psi_{Pauli}^{Ejm,1}(t, r, \theta, \phi) = e^{-iEt\tan t} f(r) \begin{vmatrix} D^j_{m,-1/2}(\phi, \theta, 0) \\ D^j_{m+1/2}(\phi, \theta, 0) \end{vmatrix},
\]

\[
f(r) = C \left( \frac{1 - \cosh r}{2} \right)^{(\nu+1)/2} \left( \frac{1 + \cosh r}{2} \right)^{\nu/2} F\left(-n, n + 2\nu + 1, \nu + \frac{3}{2}, \frac{1 - \cosh r}{2}\right);
\]

\[
\delta = -1, \quad \Psi_{Pauli}^{Ejm,-1}(t, r, \theta, \phi) = e^{-iEt\tan t} g(r) \begin{vmatrix} D^j_{m,-1/2}(\phi, \theta, 0) \\ D^j_{m+1/2}(\phi, \theta, 0) \end{vmatrix},
\]

\[
g(r) = C' \left( \frac{1 - \cosh r}{2} \right)^{\nu/2} \left( \frac{1 + \cosh r}{2} \right)^{(\nu+1)/2} F\left(-n, n + 2\nu + 1, \nu + \frac{1}{2}, \frac{1 - \cosh r}{2}\right).
\]

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