Generalizations of singular value decomposition to dual-numbered matrices

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ABSTRACT
We present two generalizations of Singular Value Decomposition from real-numbered matrices to dual-numbered matrices. We prove that every dual-numbered matrix has both types of SVD. Both of our generalizations are motivated by applications, either to geometry or to mechanics.

1. Introduction
In this paper, we consider two possible generalizations of Singular Value Decomposition (T-SVD and ∗-SVD) to matrices over the ring of dual numbers. We prove that both generalizations always exist. Both types of SVD are motivated by applications.

A dual number is a number of the form \( a + b \epsilon \) where \( a, b \in \mathbb{R} \) and \( \epsilon^2 = 0 \). The dual numbers form a commutative, associative and unital algebra over the real numbers. Let \( \mathbb{D} \) denote the dual numbers, and \( M_n(\mathbb{D}) \) denote the ring of \( n \times n \) dual-numbered matrices.

Dual numbers have applications in automatic differentiation [1], mechanics (via screw theory, see [2]), computer graphics (via the dual quaternion algebra [3]), and geometry [4]. Our paper is motivated by the applications in geometry described in Section 1.1 and in mechanics given in Section 1.2.

Our main results are the existence of the T-SVD and the existence of the ∗-SVD. T-SVD is a generalization of Singular Value Decomposition that resembles that over real numbers, while ∗-SVD is a generalization of SVD that resembles that over complex numbers. We prove the following:

**Theorem 1.1 (Dual T-SVD):** Given a square dual number matrix \( M \in M_n(\mathbb{D}) \), we can decompose the matrix as:

\[
M = U \Sigma V^T
\]

where \( U^T U = V^T V = I \), and \( \Sigma \) is a diagonal matrix.
Theorem 1.2 (Dual $*$-SVD): Every square dual-numbered matrix \( M \in M_n(\mathbb{D}) \) can be decomposed as

\[
M = U \Sigma V^T
\]

where \( U^T U = V V^T = I \), and \( \Sigma \) is a block-diagonal matrix where each block is of one of the forms \( (\sigma_i) \), \( (\sigma_i, -\epsilon \sigma_i') \), or \( (\epsilon \sigma_i') \), where each \( \sigma_i \) and \( \sigma_i' \) is real, and \( \sigma_i' \neq 0 \).

1.1. Dual $*$-SVD

Our study of $*$-SVD is motivated by Yaglom’s 1968 book *Complex numbers in geometry* [4]. Yaglom considers the group of Laguerre transformations, which is analogous to the group of Moebius transformations over the complex numbers. The Laguerre transformations are the group of functions of the form \( z \mapsto az + b/cz + d \) where \( a, b, c, d \) are elements of \( \mathbb{D} \), \( z \) is a variable over \( \mathbb{D} \), and \( ad - bc \) is not a zero divisor. It can easily be seen that every Laguerre transformation \( z \mapsto \frac{az + b}{cz + d} \) can be represented as the \( 2 \times 2 \) matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

Yaglom classifies the elements of this group in a geometric way. We restate the classification in the language of matrices. Yaglom argues that every invertible \( 2 \times 2 \) matrix over the dual numbers can be expressed in exactly one of the following two ways:

- \( U \Sigma V^T \) where \( U U^T = V V^T = I \) and \( \Sigma \) is a diagonal matrix with real-valued entries.
- \( U \Sigma \) where \( U U^T = I \), \( \Sigma = \begin{pmatrix} \sigma & -\epsilon \sigma' \\ \epsilon \sigma' & \sigma \end{pmatrix} \), and both \( \sigma \) and \( \sigma' \) are real (and \( \sigma' \) is non-zero).

The first of these forms resembles Singular Value Decomposition. What we propose in this paper is a generalization of Singular Value Decomposition to square dual-numbered matrices which includes both forms as special cases.

1.2. Dual $T$-SVD

Our study of $T$-SVD is motivated by recent research [5–8] in mechanics, where the authors consider either a form of SVD that is essentially $T$-SVD, or a form of Polar Decomposition that is essentially $T$-Polar Decomposition. For completeness, we describe $T$-Polar Decomposition below.

Corollary 1.1 (Dual $T$-Polar Decomposition): Every square dual-numbered matrix \( M \) can be expressed in the form \( M = UP \) where \( U^T U = I \) and \( P \) is symmetric.

One of the main applications of the $T$-SVD is in finding the Moore-Penrose generalized inverse of a dual matrix (whenever it exists). This has applications in kinematic synthesis (see [7]).

Very recently, Udwadia et al. [9] studied the existence of the Moore–Penrose generalized inverse for dual-numbered matrices and showed that unlike in the case of real and complex matrices, not all dual-numbered matrices have such inverses. In contrast to their result, we prove that the $T$-SVD does exist for all dual-numbered matrices. Our result is new as none of the authors above have proved that the $T$-SVD exists in general.
1.3. Structure of the paper

Section 2 contains preliminaries. In Section 3, we will prove the */T-spectral theorems. In Section 4, we will show that every square dual-number matrix has a */T-SVD.

2. Preliminaries

2.1. Dual numbers

The dual numbers are the ring \( \mathbb{R}[\epsilon]/(\epsilon^2) \). In other words the dual numbers are pairs of real numbers, usually written as \( a + b\epsilon \), with the following operations defined on them:

- \((a + b\epsilon) + (c + d\epsilon) = (a + c) + (b + d)\epsilon \).
- \((a + b\epsilon)(c + d\epsilon) = ac + (ad + bc)\epsilon \).
- \((a + b\epsilon)^{-1} = a^{-1} - \epsilon ba^{-2} \).

Given a polynomial with real coefficients \( F(X) \in \mathbb{R}[X] \), evaluating \( F \) on a dual number gives \( F(a + b\epsilon) = F(a) + b\epsilon F'(a) \) where \( F' \) denotes the derivative of \( F \). Multiples of the dual number \( \epsilon \) are sometimes referred to as ‘infinitesimal’ with the intuition being that \( \epsilon \) is ‘so small’ that it squares to 0.

The dual numbers contrast with the complex numbers. The complex numbers are defined as the ring \( \mathbb{R}[i]/(i^2 + 1) \). Like the dual numbers, the complex numbers are pairs of real numbers with certain operations defined on them. The difference is that while the complex numbers are defined by adjoining an element \( i \) such that \( i^2 = -1 \), the dual numbers are defined by adjoining an element \( \epsilon \) such that \( \epsilon^2 = 0 \).

Unlike the complex numbers, the dual numbers do not form a field. As such, instead of talking about vector spaces over dual numbers, one must talk about modules over dual numbers. Modules over dual numbers don’t necessarily have bases: For instance, the module \( \epsilon \mathbb{D} \) (that is, multiples of the dual number \( \epsilon \)) doesn’t have a basis. For this reason, one cannot in general talk about the ‘rank’ of a matrix over the dual numbers. A module over the dual numbers (or generally any ring) that does have a basis is referred to as a free module.

2.2. Spectral theorem over symmetric real matrices

The spectral theorem over symmetric matrices states that a symmetric matrix \( A \) over the real numbers can be orthogonally diagonalized. In other words, if \( A \) is a real matrix such that \( A = A^T \), then there exists a matrix \( P \) such that \( P^T P = I \) and \( A = PDP^T \) for some diagonal matrix \( D \).

2.3. Spectral theorem over skew-symmetric real matrices

The spectral theorem over skew-symmetric matrices states that a skew-symmetric matrix \( A \) over the real number can be orthogonally block-diagonalized, where every block is a skew-symmetric \( 2 \times 2 \) block except possibly for one block, which has dimensions \( 1 \times 1 \) and whose only entry equals 0.
2.4. Notation and terms

By \( a + b\epsilon \), we mean \( a - b\epsilon \). Given a dual number \( a + b\epsilon \), we call \( a \) the standard part and \( b \) the infinitesimal part. We sometimes denote the standard and infinitesimal parts of a dual number \( z \) by \( \text{st}(z) \) and \( \Im(z) \), respectively. We also use the term infinitesimal to describe a dual number whose standard part is zero, and appreciable to describe a dual number whose standard part is non-zero. We sometimes call a dual number whose infinitesimal part is zero as a real number. All the above terms generalize to matrices and vectors over dual numbers in the obvious way. Likewise, the operations infinitesimal part and standard part generalize to sets of dual-numbered vectors by applying these operations to each element of the set.

We sometimes write \( M^* \) for \( M^T \).

We use \( (u, v) \) to denote \( u^T v \), and \( \langle u, v \rangle \) to denote \( u^T v \). Two vectors are considered T-orthogonal if \( (u, v) = 0 \), and \( * \)-orthogonal if \( \langle u, v \rangle = 0 \).

A dual matrix \( U \) which satisfies \( UU^T = \bar{U}^T U = I \) is called unitary. Similarly, one which satisfies \( U^T U = UU^T = I \) is called T-orthogonal.

A dual matrix \( A \) such that \( A = \bar{A}^T \) is called Hermitian. Similarly, one which satisfies \( A = A^T \) is called symmetric.

A vector \( v \) is called an eigenvector of a dual-numbered matrix \( A \) if \( v \) is appreciable and \( Av = \lambda v \) for some \( \lambda \in \mathbb{D} \).

3. Dual-number \( * / T \)-spectral theorem

Theorem 3.1 (Dual \( * \)-spectral): Given a Hermitian dual number matrix \( M \in \mathbb{M}_n(\mathbb{D}) \), we can decompose the matrix as:

\[
M = V \Sigma V^*
\]

where \( V \) is unitary, and \( \Sigma \) is a block-diagonal matrix where each block is either of the form:

- \( (\sigma_i) \)
- \( \begin{pmatrix} \sigma_i & -\epsilon \sigma_i' \\ \epsilon \sigma_i' & \sigma_i \end{pmatrix} \)

where each \( \sigma_i \) and \( \sigma_i' \) is real, and \( \sigma_i' \neq 0 \).

Proof: Let \( M \) be a Hermitian matrix. We find a real matrix \( S \) such that \( S \text{st}(A)S^T \) is diagonal. We let \( M' = SMST \), which we write as a block matrix

\[
\begin{pmatrix}
\lambda_1 I + \epsilon B_{11} & \epsilon B_{12} & \cdots & \epsilon B_{1n} \\
-\epsilon B_{12}^T & \lambda_2 I + \epsilon B_{22} & \cdots & \\
& \ddots & \ddots & \epsilon B_{n-1,n} \\
-\epsilon B_{1n}^T & \cdots & -\epsilon B_{n-1,n}^T & \lambda_n I + \epsilon B_{nn}
\end{pmatrix}
\]

where \( \epsilon B_{ij} \) are matrices of appropriate dimensions.
where each $B_{ii}$ is skew-symmetric. We let

$$P = \begin{pmatrix} I & \frac{\epsilon B_{12}}{\lambda_1 - \lambda_2} & \cdots & \frac{\epsilon B_{1n}}{\lambda_1 - \lambda_n} \\ \frac{\epsilon B_{12}^T}{\lambda_1 - \lambda_2} & I & \cdots & \vdots \\ \vdots & \ddots & \ddots & \frac{\epsilon B_{n-1,n}}{\lambda_{n-1} - \lambda_n} \\ \frac{\epsilon B_{1n}^T}{\lambda_1 - \lambda_n} & \cdots & \frac{\epsilon B_{n-1,n}^T}{\lambda_{n-1} - \lambda_n} & I \end{pmatrix},$$

and let $M'' = PM'P^*$. We end up with $M''$ being equal to a direct sum of matrices: $M'' = (\lambda_1 I + \epsilon B_{11}) \oplus (\lambda_2 I + \epsilon B_{22}) \oplus \cdots \oplus (\lambda_n I + \epsilon B_{nn})$. We finally use the spectral theorem for skew-symmetric matrices to find matrices $Q_i$ such that $Q_iB_{ii}Q_i^T$ is equal to a direct sum of skew-symmetric 2 $\times$ 2 blocks except for potentially one zero block. We thus get that $(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n)M''(Q_1 \oplus Q_2 \oplus \cdots \oplus Q_n)^T$ is a block-diagonal matrix in the desired form.

\[\blacksquare\]

**Theorem 3.2 (Dual T-spectral):** Every symmetric matrix can be orthogonally diagonalized.

**Proof:** The proof is the same as for the $*$-spectral case except that

$$M' = \begin{pmatrix} \lambda_1 I + \epsilon B_{11} & \epsilon B_{12} & \cdots & \epsilon B_{1n} \\ \epsilon B_{12}^T & \lambda_2 I + \epsilon B_{22} & \cdots & \vdots \\ \vdots & \ddots & \ddots & \epsilon B_{n-1,n} \\ \epsilon B_{1n}^T & \cdots & \epsilon B_{n-1,n}^T & \lambda_n I + \epsilon B_{nn} \end{pmatrix},$$

and

$$P = \begin{pmatrix} I & \frac{\epsilon B_{12}}{\lambda_1 - \lambda_2} & \cdots & \frac{\epsilon B_{1n}}{\lambda_1 - \lambda_n} \\ -\frac{\epsilon B_{12}^T}{\lambda_1 - \lambda_2} & I & \cdots & \vdots \\ \vdots & \ddots & \ddots & \frac{\epsilon B_{n-1,n}}{\lambda_{n-1} - \lambda_n} \\ -\frac{\epsilon B_{1n}^T}{\lambda_1 - \lambda_n} & \cdots & \frac{\epsilon B_{n-1,n}^T}{\lambda_{n-1} - \lambda_n} & I \end{pmatrix},$$

and $M'' = PMP^T$. And instead of using the spectral theorem for skew-symmetric matrices, we use the one for symmetric matrices. \[\blacksquare\]

The following theorem can be used to show that the $*/T$-spectral decompositions are unique. Note that in order to show that the $*$-spectral decomposition is unique, one must first fully diagonalize the matrix $\begin{pmatrix} \sigma & -\epsilon \sigma' \\ \epsilon \sigma' & \sigma \end{pmatrix}$ to $\begin{pmatrix} \sigma & i\epsilon \sigma' \\ -i\epsilon \sigma' & \sigma \end{pmatrix}$ by working over $\mathbb{C} \otimes \mathbb{D}$.

**Theorem 3.3 (Uniqueness of eigenvalues):** An eigenbasis of a linear endomorphism $T : \mathbb{D}^n \to \mathbb{D}^n$ corresponds to a unique multi-set of eigenvalues.
Proof: Let $e_1, e_2, \ldots, e_n$ be a basis made up of eigenvectors of $T$. Let the corresponding eigenvalues be $\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively.

Claim 1: Let $e$ be an eigenvector of $T$ of eigenvalue $\lambda$. We claim that $\lambda = \lambda_i$ for some $i$.

Proof of claim 1.: Suppose not. We then have that $e = \sum_{i=1}^{n} \alpha_i e_i$ for some $(\alpha_i)$. Thus on the one hand we have $T(e) = \lambda e = \lambda (\sum_{i=1}^{n} \alpha_i e_i)$, and on the other hand we have $T(e) = \sum_{i=1}^{n} \alpha_i T(e_i) = \sum_{i=1}^{n} \alpha_i \lambda_i e_i$. This implies that $\alpha_i (\lambda_i - \lambda) = 0$ for all $i$. Since $\lambda_i - \lambda$ is never zero, we get that $\alpha_i$ must be infinitesimal for all $i$. But then the vector $e$ is infinitesimal, and is therefore not an eigenvector. We have a contradiction.

Next, we assume that $e_1, e_2, \ldots, e_k$ all have some eigenvalue $\lambda$, and no other $e_i$ has eigenvalue $\lambda$.

Claim 2: $\text{st}(e_1), \text{st}(e_2), \ldots, \text{st}(e_k)$ form a spanning set of $\text{st}(V_\lambda)$ (the standard part of the eigenspace corresponding to $\lambda$).

Proof of claim 2.: Let $\text{st}(e) \in \text{st}(V_\lambda)$. We can express $e$ as $e = \sum_{i=1}^{n} \alpha_i e_i$. On the one hand we have $T(e) = \lambda e = \lambda (\sum_{i=1}^{n} \alpha_i e_i)$, and on the other hand we have $T(e) = \sum_{i=1}^{n} \alpha_i T(e_i) = \sum_{i=1}^{n} \alpha_i \lambda_i e_i$. This implies that $\alpha_i (\lambda_i - \lambda) = 0$ for all $i$. If $\lambda \neq \lambda_i$ then $\alpha_i$ is infinitesimal. We thus get that $\text{st}(e) = \sum_{i=1}^{k} \text{st}(\alpha_i) \text{st}(e_i)$.

Claim 3: $\text{st}(e_1), \text{st}(e_2), \ldots, \text{st}(e_k)$ forms a linearly independent set.

Proof of claim 3.: Assume that there exist $(\alpha_i)$ (where each $\alpha_i$ is real) such that $\sum_{i=1}^{k} \alpha_i \text{st}(e_i) = 0$. We then have that $\text{st}(\sum_{i=1}^{k} \alpha_i e_i) = 0$. If $\alpha_j \neq 0$ for some $j$ then we get that $\sum_{i=1}^{k} \epsilon \alpha_i e_i = 0$, which contradicts the linear independence of $e_1, e_2, \ldots, e_n$. Therefore all $\alpha_i$ equal 0.

Since $\text{st}(e_1), \text{st}(e_2), \ldots, \text{st}(e_k)$ form a basis of $\text{st}(V_\lambda)$, we have that $k = \dim(\text{st}(V_\lambda))$. Thus any eigenbasis of $T$ must have $\dim(\text{st}(V_\lambda))$ many vectors of eigenvalue $\lambda$.

4. Dual-number $\ast$-Singular value decomposition

These are the theorems we will prove in this section.

Theorem 4.1 (Dual $\ast$-SVD): Given a square dual number matrix $M \in M_n(\mathbb{D})$, we can decompose the matrix as:

$$M = U \Sigma V^\ast$$

where $U$ and $V$ are unitary, and $\Sigma$ is a block-diagonal matrix where each block is either of the form $(\sigma_i, \begin{pmatrix} \sigma_i & -\epsilon \sigma_i' \\ \epsilon \sigma_i' & \sigma_i \end{pmatrix})$ or $(\epsilon \sigma_i')$, where each $\sigma_i$ and $\sigma_i'$ is real, and $\sigma_i' \neq 0$.

Theorem 4.2 (Dual T-SVD): Given a square dual number matrix $M \in M_n(\mathbb{D})$, we can decompose the matrix as:

$$M = U \Sigma V^T$$

where $U$ and $V$ are $T$-orthogonal, and $\Sigma$ is a diagonal matrix.
We will start by proving these theorems for invertible matrices.

**Theorem 4.3 (SVD for invertible matrices):** Every invertible matrix has a $T/\star$-SVD.

**Proof:** We shall prove this for the $T$-SVD, but the argument is the same for the $\star$-SVD.

Let $M$ be an arbitrary dual matrix. Observe that $M^T M$ is a symmetric matrix. As such, by the $T$-spectral theorem, we have that $M^T M = V \Sigma V^T$ for some orthogonal matrix $V$ and diagonal $\Sigma$. Also observe that the standard part of $M^T M$ is positive-definite. From this, it follows that the standard part of $\Sigma$ is positive. It follows that $\sqrt{M^T M}$ exists and is equal to $V \sqrt{\Sigma} V^T$. Observe also that $M(\sqrt{M^T M})^{-1}$ is an orthogonal matrix which we shall call $U$. Finally, we have that $(UV) \sqrt{\Sigma} V^T$ is the $T$-SVD of $M$. ■

**Theorem 4.4 (Dual $\star$-SVD):** Given a square dual number matrix $M \in M_n(\mathbb{D})$, we can decompose the matrix as:

$$M = U \Sigma W^\star$$

where $U$ and $W$ are unitary, and $\Sigma$ is a block-diagonal matrix where each block is either of the form $(\sigma_i, (-\epsilon \sigma'_i, \epsilon \sigma'_i))$ or $(\epsilon \sigma'_i)$, and each $\sigma_i$ and $\sigma'_i$ is real.

**Proof:** Let $T : V \rightarrow V$ be a linear endomorphism over a finite-dimensional free $\mathbb{D}$-module.

The operator $T^\star T$ is Hermitian. Thus, apply the $\star$-spectral theorem to it to get an orthonormal basis $B = \{b_1, b_2, \ldots, b_n\}$. Let $V^L$ be the span of the set of vectors in $B$ that get mapped to appreciable vectors by $T$. Let $V^R$ be the span of the set of vectors in $B$ that get mapped to infinitesimal vectors by $T$.

Observe that $T$ is injective over $V^L$. Let $I$ be the image of $V^L$ under $T$. Observe that $I$ has the same dimension as $V^L$. Thus, there exists a unitary operator $U$ such that $U(I) = V^L$. Clearly, $UT(V^L) = V^L$. Observe that $UT(V^R) \subseteq V^R$.

Since $UT|_{V^L}$ is a linear endomorphism, we may take its $\star$-SVD by theorem Dual $\star$-SVD for invertible matrices. We also observe that $UT|_{V^R}$ is an infinitesimal map (meaning that it maps every argument to infinitesimals), and can therefore be expressed as $\epsilon T'$. We can thus take the SVD of $T'$. Taking the direct sum of the $\star$-SVDs of $UT|_{V^L}$ and $UT|_{V^R}$ yields the $\star$-SVD of $UT$.

The block types $(\sigma_i)$ and $(\sigma_i, -\epsilon \sigma'_i)$ come from $UT|_{V^L}$, and the block type $(\epsilon \sigma'_i)$ comes from $UT|_{V^R}$.

Finally, we multiply the SVD of $UT$ on the left by $U^{-1}$, and we are done. ■

**Theorem 4.5 (Dual $T$-SVD):** Given a square dual number matrix $M \in M_n(\mathbb{D})$, we can decompose the matrix as:

$$M = U \Sigma V^T$$

where $U$ and $V$ are $T$-orthogonal, and $\Sigma$ is a diagonal matrix.

**Proof:** Essentially the same as for the $\star$-SVD, except that the role of the $\star$-spectral theorem is changed to the $T$-spectral theorem. ■
5. Discussion

The proofs in this paper may help in designing backwards-stable algorithms for computing the $T^*/-$-SVD. One obstacle for computing the $T^*/-$-SVDs in a numerically stable way is the presence of singularities when eigenvalues get close to one another; this problem can be addressed by setting eigenvalues equal to each other when they are sufficiently close to one another.

Note
1. It so happens that $V^L$ is equal to the image of $T^*T$, and that $V^R$ is equal to both $\ker(T^*T)$ and $\ker(T^tT)$ (where $T^t$ denotes the transpose of $T$). These are easy to prove, and we won’t do so here.

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References
[1] Hoffmann PH. A hitchhiker’s guide to automatic differentiation. Numer Algorithms. July 2016;72(3):775–811.
[2] Fischer IS. Dual-number methods in kinematics, statics and dynamics. Boca Raton, Florida: CRC Press; 1998.
[3] Kenwright B. A beginner’s guide to dual-quaternions: What they are, how they work, and how to use them for 3D character hierarchies. The 20th International Conference on Computer Graphics, Visualization and Computer Vision, WSCG 2012 Communication Proceedings; 2012. p. 1–13.
[4] Yaglom IM. Complex numbers in geometry. Cambridge, Massachusetts: Academic Press; 1968.
[5] Han S, Bauchau OA. On the global interpolation of motion. Comput Methods Appl Mech Eng. 2018;337:352–386.
[6] Han S, Bauchau OA. Spectral collocation methods for the periodic solution of flexible multi-body dynamics. Nonlinear Dyn. 2018;92:1599–1618.
[7] Pennestri E, Valentini PP, de Falco D. The Moore–Penrose dual generalized inverse matrix with application to kinematic synthesis of spatial linkages. J Mech Design. 07 2018;140(10):102303.
[8] Pennestri E, Stefanelli R. Linear algebra and numerical algorithms using dual numbers. Multibody Syst Dyn. Oct 2007;18:323–344.
[9] Udwadia F, Pennestri E, de Falco D. Do all dual matrices have dual Moore-Penrose generalized inverses? Mech Mach Theor. Sept 2020;151:103878.
[10] Meurer A, Smith CP, Paprocki M. Sympy: symbolic computing in Python. PeerJ Computer Science. January 2017;3:e103.