Periodic solution for the magnetohydrodynamic equations with inhomogeneous boundary condition

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Abstract

We show, using the spectral Galerkin method together with compactness arguments, existence and uniqueness of periodic strong solutions for the magnetohydrodynamics’s type equations with inhomogeneous boundary conditions. In particular, when the magnetic field $h(x,t)$ is zero, we obtain existence and uniqueness of strong solutions to the Navier-Stokes equations with inhomogeneous boundary conditions.

1 Introduction

In several situations the motion of incompressible electrical conducting fluid can be modelled by the magnetohydrodynamic (MHD) equations, which correspond to the Navier-Stokes (NS) equations coupled to the Maxwell equations. This system of equations plays an important role in various applications, for example in phenomenons related to the plasma behavior [1] and stochastic dynamics [26]. In the case when the MHD equations have periodic boundary conditions these equations play an important role in MHD generators [15]. Also, these boundary conditions can be considered in the tasks related with processes of the cooling nuclear reactors.

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In presence of a free motion of heavy ions (see Schluter [22], [23] and Pikelner [19]), the MHD equation may be reduced to

\[
\frac{\partial \mathbf{u}}{\partial t} - \frac{\eta}{\rho} \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \frac{\mu}{\rho} \mathbf{h} \cdot \nabla \mathbf{h} = \mathbf{f} - \frac{1}{\rho} \nabla \left( p^* + \frac{\mu}{2} h^2 \right) \\
\frac{\partial \mathbf{h}}{\partial t} - \frac{1}{\mu \sigma} \Delta \mathbf{h} + \mathbf{u} \cdot \nabla \mathbf{h} - \mathbf{h} \cdot \nabla \mathbf{u} = -\nabla \mathbf{w}
\]

(1)

with

\[
\mathbf{u} \mid_{\partial \Omega} = \beta_1(x,t), \quad \mathbf{h} \mid_{\partial \Omega} = \beta_2(x,t).
\]

(2)

Here, \( \mathbf{u} \) and \( \mathbf{h} \) are unknown velocity and magnetic field, respectively; \( p^* \) is an unknown hydrostatic pressure; \( w \) is an unknown function related to the heavy ions (in such way that the density of electric current, \( j_0 \), generated by this motion satisfies the relation \( \text{rot} j_0 = -\sigma \nabla w \)); \( \rho \) is the density of mass of the fluid (assumed to be a positive constant); \( \mu > 0 \) is a constant magnetic permeability of the medium; \( \sigma > 0 \) is a constant electric conductivity; \( \eta > 0 \) is a constant viscosity of the fluid; \( \mathbf{f} \) is a given external force field.

The initial value problem associated to the system (1) has been studied by several authors. Lassner [14], by using the semigroup results of Kato and Fujita [9], proved the existence and uniqueness of strong solutions. Boldrini and Rojas-Medar [5], [21] improved this result to global strong solutions by using the spectral Galerkin method. Damázio and Rojas-Medar [8] studied the regularity of weak solutions, Notte-Cuello and Rojas-Medar [17] used an iterative approach to show the existence and uniqueness of the strong solutions. The initial value problem in time dependent domains was studied by Rojas-Medar and Beltrán-Barrios [20] and by Berselli and Ferreira [4].

The periodic problem for the classical Navier-Stokes equations was studied by Serrin [24] using the perturbation method and subsequently by Kato [12] using the spectral Galerkin method. Following the methodology used by Kato, Notte-Cuello and Rojas-Medar [18] studied the existence and uniqueness of periodic strong solutions with homogeneous boundary conditions for the MDH type equations. In this work it is considered the periodic problem for the MHD equations with inhomogeneous boundary conditions. We prove the existence and the uniqueness of the strong solutions to this system of equations, following the methodology used by Morimoto [16], who presented results of existence and uniqueness of weak solutions to the Navier-Stokes equations and to the Boussinesq equations.

2 Preliminaries and Results

We begin by recalling certain definitions and facts to be used later in this paper.
The $L^2(\Omega)$-product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $| \cdot |$, respectively; the $L^p(\Omega)$-norm by $| \cdot |_{L^p}$, $1 \leq p \leq \infty$; the $H^m(\Omega)$-norm is denoted by $\| \cdot \|_{H^m}$ and the $W^{k,p}(\Omega)$-norm by $| \cdot |_{W^{k,p}}$.

Here $H^m(\Omega) = W^{m,2}(\Omega)$ and $W^{k,p}(\Omega)$ are usual Sobolev spaces, $H^1_0(\Omega)$ is the closure of $C_\infty^0(\Omega)$ in the $H^1$-norm.

If $B$ is a Banach space, we denote $L^q(0,T;B)$ the Banach space of the $B$-valued functions defined in the interval $(0,T)$ that are $L^q$-integrable in the sense of Bochner.

Let $C_\infty^0,\sigma(\Omega) = \{ v \in (C_\infty^0(\Omega))^n; \text{div} v = 0 \}$, $H = \text{closure of } C_\infty^0,\sigma(\Omega)$ in $(L^2(\Omega))^n$, $V = \text{closure of } C_\infty^0(\Omega)$ in $(H^1_0(\Omega))^n$, $H^1_\sigma(\Omega) = \{ u \in (H^1(\Omega))^n : \text{div} u = 0 \}$ and $C^1([0,T];B) = \{ u \in C^1([0,T];B) : u(\cdot,0) = u(\cdot,T) \}$.

Let $P$ be the orthogonal projection from $(L^2(\Omega))^n$ onto $H$ obtained by the usual Helmholtz decomposition. Then, the operator $A : H \to H$ given by $A = -P \Delta$ with domain $D(A) = (H^2(\Omega))^n \cap V$ is called the Stokes operator.

In order to obtain regularity properties of the Stokes operator we will assume that $\Omega$ is of class $C^{1,1}$ [3]. This assumption implies, in particular, that when $Au \in L^2(\Omega)$, then $u \in H^2(\Omega)$ and $\|u\|_{H^2}$ and $|Au|$ are equivalent norms.

Now, let us introduce some functions spaces consisting of $\tau$-periodic functions. For $k \geq 0$, $k \in \mathbb{N}$, we denote by

$$C^k(\tau;B) = \{ f : \mathbb{R} \to B / f \text{ is } \tau\text{-periodic and } D^i_f \in C(\mathbb{R};B) \text{ for any } i \leq k \}.$$ Then, let us define the norm

$$\|f\|_{C^k(\tau;B)} = \sup_{0 \leq t \leq \tau} \sum_{i=1}^{k} \|D^i_f(t)\|_B.$$

We denote for $1 \leq p \leq \infty$, the spaces

$$L^p(\tau;B) = \{ f : \mathbb{R} \to B / f \text{ is measurable, } \tau\text{-periodic and } \|f\|_{L^p(\tau;B)} < \infty \},$$

where

$$\|f\|_{L^p(\tau;B)} = \left( \int_0^\tau \|f(t)\|_B^p \right)^{\frac{1}{p}} \text{ for } 1 \leq p < \infty$$

and

$$\|f\|_{L^\infty(\tau;B)} = \sup_{0 \leq t \leq \tau} \|f(t)\|_B.$$ Similarly, we denote by

$$W^{k,p}(\tau;B) = \{ f \in L^p(\tau;B) / D^i_f \in L^p(\tau;B) \text{ for any } i \leq k \}.$$ In particular, $H^k(\tau;B) = W^{k,2}(\tau;B)$, when $B$ is a Hilbert space.
The problem we consider is as follows: Let the given external force \( f \) be periodic in \( t \) with some periodic \( \tau \). Then we try to prove the existence and uniqueness of periodic strong solutions \((u, h)\) of the magnetohydrodynamic equations (1)-(2) with some periodic \( \tau \):

\[
\begin{align*}
u(x, t + \tau) &= u(x, t); & \quad h(x, t + \tau) &= h(x, t). \tag{3}
\end{align*}
\]

Now, according to the Gauss theorem, the boundary value \( \beta_i \) \( i = 1, 2 \), should satisfy the so-called general outflow condition (GOC)

\[
(G.O.C) \quad \int_{\partial \Omega} \beta_i \cdot nd\sigma = \sum_{k=0}^{N} \int_{\Gamma_k} \beta_i \cdot nd\sigma = 0.
\]

If \( N > 1 \), the stringent outflow condition (S.O.C)

\[
(S.O.C) \quad \int_{\Gamma_k} \beta_i \cdot nd\sigma = 0, \quad (k = 0, 1, ..., N;)
\]

is stronger than G.O.C.

In this work the following assumptions and results are considered,

A0 \( \Omega \subseteq \mathbb{R}^n \quad n = 2, 3 \) bounded domain and \( \partial \Omega \) is smooth and consists of \( N + 1 \) connected components \( \Gamma_0, \Gamma_1, ..., \Gamma_N \) and \( \Omega \) being inside of \( \Gamma_0 \) \( (N \geq 1) \).

A1 \( \beta_i (x, t) \in C^1([0, T], H^{1/2}(\partial \Omega)) \) and satisfies \( (S.O.C) \), \( i = 1, 2 \).

Lemma 1 [(16), p.636] Suppose \( \beta \in C^1(\mathbb{R}, H^{1/2}(\partial \Omega)) \) is \( T \)-periodic and satisfies \( (SOC) \). Then for every \( \varepsilon > 0 \), there exists a solenoidal and \( T \)-periodic function \( v \in C^1(\mathbb{R}, H^1(\Omega)) \) such that

\[
|((u \cdot \nabla)v, u)| \leq \varepsilon |\nabla u|^2, \quad \forall u \in V, \forall t \in \mathbb{R}
\]

Proposition 2 (Giga and Miyakawa [10]). If \( 0 \leq \delta < \frac{1}{2} + \frac{n}{4} \), the following estimate is valid with a constant \( C_1 = C_1(\delta, \theta, \rho) \),

\[
|A^{-\delta} Pu \cdot \nabla v| \leq C_1 |A^\theta u||A^\rho v| \text{ for any } u \in D(A^\theta) \text{ and } v \in D(A^\rho), \tag{4}
\]

with \( \delta + \theta + \rho \geq \frac{n}{4} + \frac{1}{2}, \quad \rho + \delta > \frac{1}{2}, \text{ and } \theta, \rho > 0 \).

Also, we consider the Sobolev inequality [10],

\[
|u|_{L^r(\Omega)} \leq C_2 |u|_{H^{\beta}}, \quad \text{if } \frac{1}{r} \geq \frac{1}{2} - \frac{\beta}{n} > 0,
\]

and the inequality due to Giga and Miyakawa [10]

\[
|u|_{L^r(\Omega)} \leq C_3 |A^\gamma u|, \quad \text{if } \frac{1}{r} \geq \frac{1}{2} - \frac{2\gamma}{n} > 0. \tag{5}
\]

Here, we note that if \( r = n \) in (5) it follows

\[
|u|_{L^n(\Omega)} \leq C_3 |A^\gamma u|, \quad \text{with } \gamma = \frac{n}{4} - \frac{1}{2}.
\]
Lemma 3 If \( u \in D(A^\theta) \) and \( 0 \leq \theta < \beta \), then

\[
|A^\theta u(x)| \leq \mu^{\theta - \beta} |A^\beta u(x)|
\]

where \( \mu = \min \lambda_j > 0 \), where \( \{\lambda_j\}_{j=1}^\infty \) are the eigenvalues of the Stokes operator.

Lemma 4 (Simon [25]) Let \( X, B \) and \( Y \) Banach spaces such that \( X \hookrightarrow B \hookrightarrow Y \), where the first embedding is compact and the second is continuous. Then, if \( T > 0 \) is finite, we have that the following embedding is compact

\[
L^\infty(0,T;X) \cap \{ \phi : \phi_t \in L^r(0,T;Y) \} \hookrightarrow C(0,T;B), \text{ if } 1 < r \leq \infty.
\]

Our results are the following.

Theorem 5 (Existence) Suppose that \( \Omega, \beta \) satisfying the assumption A. Then, there exists a constant \( K_0 = K_0(n) > 0 \) such that if

\[
M = \sup_{0 \leq t \leq \tau} |f|_{L^\nu/2(\Omega)} \leq K_0
\]

the problem (1)-(3) has a \( \tau \)-periodic strong solution \( (\tilde{u}(t), \tilde{h}(t)) \) satisfying

\[
(\tilde{u}, \tilde{h}) \in (H^2(\tau;H))^2 \cap (H^1(\tau;D(A)))^2 \cap (L^\infty(\tau;D(A)))^2 \cap (W^{1,\infty}(\tau;V))^2.
\]

such that

\[
\tilde{u} = u - B_1 \in L^2(0,T;V) \cap L^\infty(0,T;H) \\
\tilde{h} = h - B_2 \in L^2(0,T;V) \cap L^\infty(0,T;H)
\]

for some \( \tau \)-periodic extension \( B_1 \) and \( B_2 \) of the boundary values \( \beta_1 \) and \( \beta_2 \) respectively and \( (u, h) \) satisfying the problem (1)-(2)-(3).

Theorem 6 (Uniqueness) The solution for (1)-(3) given in the above theorem is unique.

The idea of the proof is to use the spectral Galerkin method together with compactness arguments. The principal problem is to obtain the uniform boundedness of certain norms of \( u^k(t) \) and \( h^k(t) \) at some point \( t^* \). This difficulty was early treated by Heywood [11] to prove the regularity of the classical solutions for Navier-Stokes equations.
2.1 Approximate Problem and a priori estimates

We have \((\vec{u} + B_1, \vec{h} + B_2)\) satisfying the following equation:

\[
\begin{align*}
\alpha \frac{\partial}{\partial t} (\vec{u} + B_1) - \nu \Delta (\vec{u} + B_1) + \alpha (\vec{u} + B_1) \cdot \nabla (\vec{u} + B_1) - (\vec{h} + B_2) \cdot \nabla (\vec{h} + B_2) &= \alpha f - \frac{1}{\mu} \nabla \left( p^* + \frac{\mu}{2} (\vec{h} + B_2)^2 \right) \\
\frac{\partial}{\partial t} (\vec{h} + B_2) - \chi \Delta (\vec{h} + B_2) + (\vec{u} + B_1) \cdot \nabla (\vec{h} + B_2) - (\vec{h} + B_2) \cdot \nabla (\vec{u} + B_1) &= -\nabla \cdot w.
\end{align*}
\]

(6)

By putting \(\vec{u} = u\) and \(\vec{h} = h\) and rearranging terms, we obtain

\[
\begin{align*}
\alpha \frac{\partial u}{\partial t} - \nu \Delta u + \alpha u \cdot \nabla u - h \cdot \nabla h + \alpha \frac{\partial B_1}{\partial t} - \nu \Delta B_1 + \alpha B_1 \cdot \nabla B_1 + u \cdot \nabla B_1 \\
+ B_1 \cdot \nabla u - B_2 \cdot \nabla h - h \cdot \nabla B_2 - B_2 \cdot \nabla B_2 &= \alpha f - \frac{1}{\mu} \nabla \left( p^* + \frac{\mu}{2} (h + B_2)^2 \right),
\end{align*}
\]

(7)

By using the operator \(P\), the periodic problem (1)-(3) is formulated as follows

\[
\begin{align*}
\alpha \frac{\partial u(t)}{\partial t} + \nu A u(t) + \alpha P(\nabla u(t)) - P(h(t) \cdot \nabla h(t)) + \alpha \frac{\partial B_1}{\partial t} - \nu AB_1(t) \\
+ \alpha P(B_1(t) \cdot \nabla B_1(t)) + P(\nabla B_1(t)) + P(B_1(t) \cdot \nabla u(t)) \\
- P(B_2(t) \cdot \nabla h(t)) - P(h(t) \cdot \nabla B_2(t)) - P(B_2(t) \cdot \nabla B_2(t)) &= \alpha P f(t),
\end{align*}
\]

(8)

\[
\begin{align*}
\frac{\partial h}{\partial t} + \chi A h + P(\nabla h(t)) - P(h(t) \cdot \nabla u(t)) + \frac{\partial B_2}{\partial t} - \chi AB_2 \\
+ P(B_1(t) \cdot \nabla h(t)) - P(h(t) \cdot \nabla B_1(t)) + P(u(t) \cdot \nabla B_2(t)) \\
- P(B_2(t) \cdot \nabla B_1(t)) - P(B_2(t) \cdot \nabla u(t)) + P(B_1(t) \cdot \nabla B_2(t)) &= 0,
\end{align*}
\]

\[
\begin{align*}
u(x, t + \tau) = u(x, t); & \quad h(x, t + \tau) = h(x, t).
\end{align*}
\]

We consider \(V_k = \text{span} \{\omega_1(x), \omega_2(x), \ldots, \omega_k(x)\}\) and the approximations \(u^k(t) = \sum_{j=1}^{k} c_{jk}(t)\omega_j(x)\) and \(h^k(t) = \sum_{j=1}^{k} d_{jk}(t)\omega_j(x)\), of \(u\) and \(h\), respectively, satisfying the following system of ordinary differential equations,
\[(\alpha u^k_t + \nu Au^k + \alpha P(u^k \cdot \nabla u^k) - P(h^k \cdot \nabla h^k) + \alpha (B_1)_t, \omega) + (\nu AB_1 + \alpha P(B_1 \cdot \nabla B_1) + P(u^k \cdot \nabla B_1) + P(B_1 \cdot \nabla u^k), \omega) - (P(B_2 \cdot \nabla h^k) + P(h^k \cdot \nabla B_2) + P(B_2 \cdot \nabla B_2), \omega) = (\alpha f(t), \omega), \]

\[(h^k_t + \chi Ah^k + P(u^k \cdot \nabla h^k) - P(h^k \cdot \nabla u^k) + (B_2)_t + \chi AB_2, \omega) + (P(B_1 \cdot \nabla h^k) - P(h^k \cdot \nabla B_1^k) + P(u^k \cdot \nabla B_2) - P(B_2 \cdot \nabla B_1), \omega) - (P(B_2 \cdot \nabla u^k) - P(B_1 \cdot \nabla B_2), \omega) = 0, \]

\[u^k(x, t + \tau) = u^k(x, t); \quad h^k(x, t + \tau) = h^k(x, t). \tag{9}\]

To show that system (9) has an unique \(\tau\)-periodic solution, we consider the following linearized problem:

\[\begin{align*}
(\alpha u^k_t + \nu Au^k, \omega) &= (\alpha f, \omega) - (L_1(v^k, b^k), \omega) - (L_2(v^k, b^k), \omega) \\
(L^k_t + \chi Av^k, \omega) &= -(L_3(v^k, b^k), \omega) - (L_4(v^k, b^k), \omega), \tag{10}
\end{align*}\]

where

\[L_1(v^k, b^k) = \alpha P(v^k \cdot \nabla v^k) - P(b^k \cdot \nabla b^k), \]

\[L_2(v^k, b^k) = \alpha (B_1)_t + \nu AB_1 + \alpha P(B_1 \cdot \nabla B_1) + P(v^k \cdot \nabla B_1) + P(B_1 \cdot \nabla v^k) - P(B_2 \cdot \nabla b^k) - P(b^k \cdot \nabla B_2) - P(B_2 \cdot \nabla B_2), \]

\[L_3(v^k, b^k) = P(v^k \cdot \nabla b^k) - P(b^k \cdot \nabla v^k), \]

\[L_4(v^k, b^k) = (B_2)_t - \chi AB_2 - P(B_1 \cdot \nabla b^k) + P(b^k \cdot \nabla B_1) + P(v^k \cdot \nabla B_2) - P(B_2 \cdot \nabla B_1) - P(B_2 \cdot \nabla v^k) + P(B_1 \cdot \nabla B_2), \]

where \(v^k(t) = \sum_{j=1}^k e_{jk}(t)\omega_j(x)\) and \(b^k(t) = \sum_{j=1}^k g_{jk}(t)\omega_j(x)\) are functions given in \(C^1(\tau; \mathbf{V}_k)\).

It is well know that the linearized system (10) has an unique \(\tau\)-periodic solution \((u^k(t), h^k(t)) \in (C^1(\tau; \mathbf{V}_k))^2\) (see for instance, [2], [6]). Consider the map: \(\Phi : (v^k, b^k) \rightarrow (u^k, h^k)\) in the space \(C^0(\tau; \mathbf{V}_k) \times C^0(\tau; \mathbf{V}_k)\). We shall show that \(\Phi\) has a fixed point by Leray-Schauder Theorem.

We prove that for every \((u^k, h^k)\) and \(\lambda \in [0, 1]\) satisfying \(\lambda \Phi(u^k, h^k) = (u^k, h^k)\),

\[\sup_{0 \leq t \leq \tau} |u^k(t)| \leq C \quad \text{and} \quad \sup_{0 \leq t \leq \tau} |h^k(t)| \leq C \tag{11}\]

where \(C\) is a positive constant independent of \(\lambda\).
For \( \lambda = 0, (u^k, h^k) = (0, 0) \). Let \( \lambda > 0 \) and assume that \( \lambda \Phi(u^k, h^k) = (u^k, h^k) \). Then, from (10), we obtain
\[
\frac{1}{2} \frac{d}{dt} \alpha |u^k|^2 + \nu |\nabla u^k|^2 = \lambda (\alpha f, u^k) - \lambda (L_1(u^k, h^k), u^k) - \lambda (L_2(u^k, h^k), u^k),
\]
\[
\frac{1}{2} \frac{d}{dt} |h^k|^2 + \chi |\nabla h^k|^2 = -\lambda (L_3(u^k, h^k), h^k) - \lambda (L_4(u^k, h^k), h^k).
\]
Summing the above equalities, we obtain
\[
\frac{1}{2} \frac{d}{dt} (\alpha |u^k|^2 + |h^k|^2) + \nu |\nabla u^k|^2 + \chi |\nabla h^k|^2
\]
\[
= \lambda (\alpha f, u^k) - \lambda (L_1(u^k, h^k), u^k) - \lambda (L_2(u^k, h^k), u^k)
- \lambda (L_3(u^k, h^k), h^k) - \lambda (L_4(u^k, h^k), h^k)
+ \lambda \nu (\nabla B_1, \nabla u^k) + \lambda (\nabla B_2, \nabla h^k)
+ \lambda (\nabla B_1, \nabla h^k, B_2) - \lambda (\nabla B_1, \nabla h^k, B_2)
+ \lambda (h^k \cdot \nabla B_2, u^k) + \lambda (h^k \cdot \nabla B_1, h^k) - \lambda (u^k \cdot \nabla B_2, h^k).
\]
We observe that, since \( \lambda \leq 1 \), we obtain
\[
\lambda (\alpha f, u^k) \leq C_\alpha |f||\nabla u^k|,
- \lambda \alpha ((B_1)_t, u^k) \leq C_\alpha |(B_1)_t||\nabla u^k|,
- \lambda ((B_2)_t, h^k) \leq C|(B_2)_t||\nabla h^k|,
\lambda \nu (\nabla B_1, \nabla u^k) \leq \nu |\nabla B_1| |\nabla u^k|,
\lambda \chi (\nabla B_2, \nabla h^k) \leq \chi |\nabla B_2| |\nabla h^k|.
\]
Also, by using the Hölder inequality, we have
\[
\lambda \alpha (B_1 \cdot \nabla u^k, B_1) \leq C_\alpha \|B_1\|_{L^4}^2 |\nabla u^k|,
- \lambda (B_2 \cdot \nabla h^k, B_1) \leq C \|B_1\|_{L^4} |B_2| |\nabla h^k|,
\lambda (B_1 \cdot \nabla h^k, B_2) \leq C \|B_1\|_{L^4} |B_2| |\nabla h^k|,
- \lambda (B_2 \cdot \nabla u^k, B_2) \leq C \|B_2\|_{L^4} |\nabla u^k|.
\]
Now, we use the Lemma 1, to obtain
\[
- \lambda (u^k \cdot \nabla B_1, u^k) \leq \varepsilon_1 |\nabla u^k|^2
\lambda (h^k \cdot \nabla B_2, u^k) \leq \varepsilon_2 |\nabla u^k||\nabla h^k|
\lambda (h^k \cdot \nabla B_1, h^k) \leq \varepsilon_3 |\nabla h^k|^2
- \lambda (u^k \cdot \nabla B_2, h^k) \leq \varepsilon_4 |\nabla u^k||\nabla h^k|.
\]
Using the Young inequality, summing the estimates (14), (15) and (16) together with the equality (13), we have
\[
\frac{1}{2} \frac{d}{dt} (|f|^2 + |h|^2) + |\nabla f|^2 + \chi |\nabla h|^2 \\
\leq C |f|^2 + C |(B_1)_t|^2 + C |(B_2)_t|^2 \\
+ C \|B_1\|_{L^4}^4 + C \|B_1\|_{L^4}^2 C \|B_2\|_{L^4}^2 + C \|B_2\|_{L^4}^2.
\]
(17)

Integrating in \(t\) and using the periodicity of \((u^k, h^k)\) we have
\[
\int_0^T \left( \nu |\nabla u|^2 + \chi |\nabla h|^2 \right) dt \leq \int_0^T (C_1(t) + C_2(t)) dt \leq d_0 M^2,
\]
where \(C_1(t) = C |f(t)|^2\) and \(C_2(t) = C |(B_1)_t|^2 + C |(B_2)_t|^2 + C \|B_1\|_{L^4}^4 + C \|B_1\|_{L^4}^2 C \|B_2\|_{L^4}^2 + C \|B_2\|_{L^4}^2\) belong to \(L^1(0, \tau)\) and are independent of \(k\). Whence by the Mean Value Theorem for integrals, there exists \(t^* \in [0, \tau]\) such that
\[
\nu |\nabla u^k(t^*)|^2 + \chi |\nabla h^k(t^*)|^2 \leq d_1 M^2.
\]
(18)

On the other hand, by using the Lemma 3, with \(\theta = 0\) and \(\beta = 1/2\),
\[
|u^k(t^*)| \leq \mu^{-1/2} |\nabla u^k(t^*)|
\]
and consequently
\[
|u^k(t^*)|^2 \leq \mu^{-1} |\nabla u^k(t^*)|^2 \leq d_0 M^2 / v_1,
\]
(19)

analogously
\[
|h^k(t^*)|^2 \leq \mu^{-1} |\nabla h^k(t^*)|^2 \leq d_0 M^2 / \chi_1.
\]
(20)

Finally, by integrating again (17) from \(t^*\) to \(t + \tau\), with \(t \in [0, \tau]\), we obtain (11). As the map \(\Phi\) is continuous and compact in \(C^0(\tau; V_k)\) we conclude the existence of a fixed point \((u^k, h^k)\) for \(\Phi\). Observe that (11) holds for this \((u^k, h^k)\).

**Lemma 7** Let \((u^k(t), h^k(t))\) be the solution of (9). Suppose that
\[
M < \min \left\{ \left( \frac{v}{P_1} \right)^2, \left( \frac{\chi}{P_2} \right)^2, 1 \right\}
\]
where
\[
P_1 = z \left( \frac{d_0}{\bar{v}} \right)^{1/2} \mu^{-\gamma} + C_1 \alpha \left( \frac{d_0}{\bar{v}} \right)^{1/2} \mu^{\gamma-1/2} + d_5 + d_4 \bar{C}
\]
\[+ 2C_2 \left( \frac{d_0}{\bar{v}} \right)^{1/2} \mu^{\gamma-1/2} \bar{C},
\]
\[P_2 = d_3 \left( \frac{d_0}{\bar{v}} \right)^{1/2} \mu^{-\gamma} + C_9 \left( \frac{d_0}{\bar{v}} \right)^{1/2} \mu^{\gamma-1/2} + d_6 + d_3 M^{1/2} \bar{C}
\]
\[+ 2C_1 \left( \frac{d_0}{\bar{v}} \right)^{1/2} \mu^{\gamma-1/2} \bar{C}.
\]
then, we have

$$|A^\gamma u^k(t)|^2 + |A^\gamma h^k(t)|^2 \leq E\mu^{2\gamma-1} M.$$  

**Proof:** Taking $A^{2\gamma} u^k$ and $A^{2\gamma} h^k$ as test functions in (9), we get

$$\frac{\alpha}{2} \frac{d}{dt} |A^\gamma u^k|^2 + \nu |A^{(1+2\gamma)/2} u^k|^2 =$$

$$(\alpha f(t) - \alpha P(u^k \cdot \nabla u^k) + P(h^k \cdot \nabla h^k) - \alpha (B_1)_t - \nu AB_1, A^{2\gamma} u^k)$$

$$- (\alpha P(B_1 \cdot \nabla B_1) + P(u^k \cdot \nabla B_1) - P(B_1 \cdot \nabla u^k) + P(B_2 \cdot \nabla h^k), A^{2\gamma} u^k)$$

$$+ (P(h^k \cdot \nabla B_2) + P(B_2 \cdot \nabla B_2), A^{2\gamma} u^k),$$

$$\frac{1}{2} \frac{d}{dt} |A^\gamma h^k|^2 + \chi |A^{(1+2\gamma)/2} h^k|^2 =$$

$$-(P(u^k \cdot \nabla h^k) + P(h^k \cdot \nabla u^k) - (B_2)_t - \chi AB_2 - P(B_1 \cdot \nabla h^k), A^{2\gamma} h^k)$$

$$+(P(h^k \cdot \nabla B^k_1) - P(u^k \cdot \nabla B_2) - P(B_2 \cdot \nabla B_1), A^{2\gamma} h^k)$$

$$-(P(B_2 \cdot \nabla u^k) - P(B_1 \cdot \nabla B_2), A^{2\gamma} h^k).$$

Now, we estimate the right hand side of the above equalities as follows:

$$|(\alpha f(t), A^{2\gamma} u^k)| \leq \alpha |f|_{L^{n/2}} |A^{2\gamma} u^k|_{L^{n/(n-2)}} \leq \alpha \tilde{C} M |A^{(1+2\gamma)/2} u^k|,$$

here we use the Hölder’s inequality

$$|(Pv \cdot b, A^{2\gamma} \phi)| = |(A^{2\gamma-1} Pv \cdot b, A^{2\gamma+1} \phi)|$$

$$\leq C |A^{2\gamma} v| |A^{(1+2\gamma)/2} b| |A^{(1+2\gamma)/2} \phi|,$$

where we use the Giga-Miyakawa estimate with $\theta = \gamma$ and $\rho = (1 + 2\gamma)/2$. Now, we must estimate the $L_1$ and $L_2$ terms, where

$$L_1(A^{2\gamma} u^k) = (\alpha f(t) - \alpha (B_1)_t - \nu AB_1 - \alpha P(B_1 \cdot \nabla B_1), A^{2\gamma} u^k)$$

$$+ (P(u^k \cdot \nabla B_1) - P(B_1 \cdot \nabla u^k) + P(B_2 \cdot \nabla h^k), A^{2\gamma} u^k)$$

$$+ (P(h^k \cdot \nabla B_2) + P(B_2 \cdot \nabla B_2), A^{2\gamma} u^k),$$

$$L_2(A^{2\gamma} h^k) = -(B_2)_t - \chi AB_2 - P(B_1 \cdot \nabla h^k) + (P(h^k \cdot \nabla B_1^k), A^{2\gamma} h^k)$$

$$- (P(u^k \cdot \nabla B_2) - P(B_2 \cdot \nabla B_1) - P(B_2 \cdot \nabla u^k), A^{2\gamma} h^k)$$

$$- (P(B_1 \cdot \nabla B_2), A^{2\gamma} h^k),$$

now, we note that

$$|(\alpha (B_1)_t, A^{2\gamma} u^k)| \leq \alpha C_2 |A^{(2\gamma-1)/2} (B_1)_t| |A^{(2\gamma+1)/2} u^k|$$

$$\leq \alpha C_2 \|(B_1)_t\|_{L^{n/2}} |A^{(2\gamma+1)/2} u^k|.$$
or taking into account that \( \|A^{2\gamma}u\| \leq C\|A^{(2\gamma+1)/2}u\| \) we have

\[
|\langle \alpha(B_1)_t, A^{2\gamma}u^k \rangle| \leq \alpha C_2 \|B_1\| \|A^{(2\gamma+1)/2}u^k\|,
\]

\[
|\langle \nu AB_1, A^{2\gamma}u^k \rangle| \leq |\langle \nu A^{2\gamma+1}B_1, A^{2\gamma+1}u^k \rangle| \\
\leq \nu C_3 A^{(2\gamma+1)/2}B_1 \|A^{(2\gamma+1)/2}u^k\| \\
\leq \nu \bar{c}_C \|AB_1\| A^{(2\gamma+1)/2}u^k,
\]

similarly

\[
|\langle \alpha P(B_1 \cdot \nabla B_1), A^{2\gamma}u^k \rangle| \leq \alpha C_4 A\gamma B_1 \|A^{(2\gamma+1)/2}B_1\| A^{(2\gamma+1)/2}u^k, \\
|\langle P(u^k \cdot \nabla u^k), A^{2\gamma}u^k \rangle| \leq C_5 A\gamma/2 B_1 \|A^{(2\gamma+1)/2}u^k\|^2, \\
|\langle P(B_1 \cdot \nabla u^k), A^{2\gamma}u^k \rangle| \leq C_6 A\gamma B_1 \|A^{(2\gamma+1)/2}u^k\|^2, \\
|\langle P(B_2 \cdot \nabla h^k), A^{2\gamma}u^k \rangle| \leq C_7 A\gamma B_2 \|A^{(2\gamma+1)/2}h^k\| A^{(2\gamma+1)/2}u^k, \\
|\langle P(h^k \cdot \nabla B_2), A^{2\gamma}u^k \rangle| \leq C_8 A\gamma/2 B_2 \|A^{(2\gamma+1)/2}h^k\| A^{(2\gamma+1)/2}u^k, \\
|\langle P(B_2 \cdot \nabla B_2), A^{2\gamma}u^k \rangle| \leq C_9 A\gamma B_2 \|A^{(2\gamma+1)/2}B_2\| A^{(2\gamma+1)/2}u^k.
\]

Now, we bound the terms of \( L_2(A^{2\gamma}h^k) \),

\[
|\langle (B_2)_t, A^{2\gamma}h^k \rangle| \leq \|A^{(2\gamma-1)/2}(B_2)_t, A^{(2\gamma+1)/2}h^k\| \\
\leq \bar{c}_1 \|(B_2)_t\| A^{(2\gamma+1)/2}h^k,
\]

\[
|\chi(AB_2, A^{2\gamma}h^k)| \leq |\chi(A^{(2\gamma-1)/2}AB_2, A^{(2\gamma+1)/2}h^k)| \\
\leq \bar{c}_2 \|A^{(2\gamma+1)/2}B_2\| A^{(2\gamma+1)/2}h^k, \\
|\langle P(B_1 \cdot \nabla h^k), A^{2\gamma}h^k \rangle| = |\langle A^{(2\gamma-1)/2}P(B_1 \cdot \nabla h^k), A^{(2\gamma+1)/2}h^k \rangle| \\
\leq C \|A^{(2\gamma-1)/2}P(B_1 \cdot \nabla h^k)\| A^{(2\gamma+1)/2}h^k| \\
\leq \bar{c}_3 \|A\gamma B_1\| A^{(2\gamma+1)/2}h^k|^2, \\
|\langle P(h^k \cdot \nabla B_1), A^{2\gamma}h^k \rangle| = |\langle A^{(2\gamma-1)/2}P(h^k \cdot \nabla B_1), A^{(2\gamma+1)/2}h^k \rangle| \\
\leq C \|A^{(2\gamma+1)/2}h^k\| A^{3\gamma/2}B_1 \|A^{(2\gamma+1)/2}h^k\| \\
\leq \bar{c}_4 \|A^{3\gamma/2}B_1\| A^{(2\gamma+1)/2}h^k|^2,
\]
here we use $\theta = \frac{2^{\gamma+1}}{2}$ and $\rho = \frac{3^{\gamma}}{2}$ in Giga-Miyakawa estimate,

$$
\|(P(u^k \cdot \nabla B_2), A^{2\gamma} h^k)\| \leq \tilde{C}_5 |A^{(2\gamma+1)/2} u_k||A^{3/2} B_2||A^{(2\gamma+1)/2} h^k|,
$$

$$
\|(P(B_2 \cdot \nabla B_1), A^{2\gamma} h^k)\| = \|(A^{\frac{2^{\gamma}-1}{2}} P(B_2 \cdot \nabla B_1), A^{(2\gamma+1)/2} h^k)\|
\leq \tilde{C}_6 |A^{\gamma} B_2||A^{(2\gamma+1)/2} B_2||A^{(2\gamma+1)/2} h^k|,
$$

$$
\|(P(B_2 \cdot \nabla u^k), A^{2\gamma} h^k)\| \leq \tilde{C}_7 |A^{\gamma} B_2||A^{(2\gamma+1)/2} u^k||A^{(2\gamma+1)/2} h^k|,
$$

$$
\|(P(B_1 \cdot \nabla B_2), A^{2\gamma} h^k)\| \leq \tilde{C}_8 |A^{\gamma} B_1||A^{(2\gamma+1)/2} B_2||A^{(2\gamma+1)/2} h^k|.
$$

Now, summing the above estimates, we get

$$
\begin{align*}
\frac{\alpha}{2} \frac{d}{dt} |A^{\gamma} u^k|^2 + \frac{1}{2} \frac{d}{dt} |A^{\gamma} h^k|^2 + \nu |A^{\frac{1+2\gamma}{2}} u^k|^2 + \chi |A^{\frac{1+2\gamma}{2}} h^k|^2 \\
\leq z M |A^{\frac{1+2\gamma}{2}} u^k| + M |A^{(2\gamma+1)/2} h^k| + 2C_1 |A^{\gamma} h^k||A^{(2\gamma+1)/2} h^k||A^{\frac{2\gamma+1}{2}} u^k| \\
+ M |A^{(2\gamma+1)/2} h^k||A^{\frac{2\gamma+1}{2}} u^k| + C_1 \alpha |A^{\gamma} u^k||A^{\frac{2\gamma+1}{2}} u^k|^2 + M |A^{\frac{2\gamma+1}{2}} u^k|^2 + \tilde{C}_9 |A^{\gamma} u^k||A^{(2\gamma+1)/2} h^k|^2 M |A^{(2\gamma+1)/2} h^k|^2,
\end{align*}
$$

(21)

where we put

$$
daC_2 |(B_1) t| + \nu \tilde{C}_3 |A^{\gamma} B_1| + \alpha C_4 |A^{2\gamma} B_1||A^{(2\gamma+1)/2} B_1| \\
= d_2 \leq M,
$$

$$
\tilde{C}_1 |(B_2) t| + \tilde{C}_2 |A^{(2\gamma+1)/2} B_2| + \tilde{C}_6 |A^{\gamma} B_2||A^{(2\gamma+1)/2} B_1| \\
+ \tilde{C}_8 |A^{\gamma} B_1||A^{(2\gamma+1)/2} B_2| = d_3 \leq M,
$$

and

$$
C_7 |A^{\gamma} B_2| + C_8 |A^{3\gamma/2} B_2| + \tilde{C}_5 |A^{3\gamma/2} B_2| + \tilde{C}_7 |A^{\gamma} B_2| = d_4 \leq M,
$$

$$
C_5 |A^{3\gamma/2} B_1| + C_6 |A^{\gamma} B_1| = d_5 \leq M,
$$

$$
\tilde{C}_3 |A^{\gamma} B_1| + \tilde{C}_4 |A^{3\gamma/2} B_1| = d_6 \leq M, \\
z = \alpha \tilde{C} + 1.
$$

Now, by using the Lemma 3, with $\theta = 0$ and $\beta = 1/2$ we have

$$
|u^k(t^*)| \leq \mu^{-1/2} |\nabla u^k(t^*)|
$$

and consequently

$$
|u^k(t^*)|^2 \leq \mu^{-1} |\nabla u^k(t^*)|^2
$$
and from (19) and (20), we have

\[ |A^\gamma u^k(t^*)| \leq \mu^{\gamma-1/2} |\nabla u^k(t^*)| \leq \left( \frac{d_0}{v_1} \right)^{1/2} \mu^{\gamma-1/2} M, \]

\[ |A^\gamma h^k(t^*)| \leq \mu^{\gamma-1/2} |\nabla h^k(t^*)| \leq \left( \frac{d_0}{\chi_1} \right)^{1/2} \mu^{\gamma-1/2} M. \]

Thus, if suppose that \( M < 1 \), we obtain at \( t = t^* \),

\[ |A^\gamma u^k(t^*)| \leq \left( \frac{d_0}{v_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2} \quad \text{and} \quad |A^\gamma h^k(t^*)| \leq \left( \frac{d_0}{\chi_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}. \quad (22) \]

Then, we can write

\[ |A^\gamma u^k(t^*)|^2 + |A^\gamma h^k(t^*)|^2 \leq \left( \frac{d_0}{v_1} + \frac{d_0}{\chi_1} \right) \mu^{2\gamma-1} M = E \mu^{2\gamma-1} M. \]

Let \( T^* = \sup \left\{ \frac{T}{|A^\gamma u^k (t^*)|^2 + |A^\gamma h^k (t^*)|^2} \leq E \mu^{2\gamma-1} M, \quad t \in [t^*, T) \right\} \). We will prove by contradiction that \( T^* = \infty \). In fact, it \( T^* \) is finite it should follow that \( \forall t \in [t^*, T^*) \).

\[ |A^\gamma u^k(t^*)| \leq \left( \frac{d_0}{v_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}, \]

\[ |A^\gamma h^k(t^*)| \leq \left( \frac{d_0}{\chi_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}, \]

and

\[ |A^\gamma u^k(T^*)| = \left( \frac{d_0}{v_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}, \quad (23) \]

\[ |A^\gamma h^k(T^*)| = \left( \frac{d_0}{\chi_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}. \]

From above inequations, we can obtain

\[ |A^\gamma u^k(t^*)|^2 + |A^\gamma h^k(t^*)|^2 \leq E \mu^{2\gamma-1} M, \quad t \in [t^*, T). \]

and

\[ |A^\gamma u^k(T^*)|^2 + |A^\gamma h^k(T^*)|^2 = E \mu^{2\gamma-1} M, \]

where \( E = d_0/\chi_1 + d_0/v_1 \).

Therefore, for such a value \( t = T^* \), from the estimates of the right hand side of (21) and from (23) we obtain

\[ zM |A^{(1+2\gamma)/2} u^k| \leq z \left( \frac{v_1}{d_0} \right)^{1/2} \mu^{1/2 - \gamma} |A^\gamma u^k|M^{1/2} |A^{(1+2\gamma)/2} u^k| \]

\[ \leq z \left( \frac{v_1}{d_0} \right)^{1/2} \mu^{-\gamma} M^{1/2} |A^{(1+2\gamma)/2} u^k|^2 \]
where we use the inequality $|A^\gamma u^k| \leq \mu^{-1/2}|A^{(1+2\gamma)/2}u^k|$. Similarly,

$$d_3M|A^{(1+2\gamma)/2}h^k| \leq d_3 \left( \frac{d\alpha}{\alpha} \right)^{1/2} \mu^{-\gamma} M^{1/2}|A^{(1+2\gamma)/2}h^k|^2,$$

$$C_1\alpha |A^\gamma u^k||A^{(1+2\gamma)/2}u^k|^2 \leq C_1\alpha \left( \frac{dn}{m} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}|A^{(1+2\gamma)/2}u^k|^2,$$

$$d_5M|A^{(1+2\gamma)/2}u^k|^2 \leq d_5M^{1/2}|A^{(1+2\gamma)/2}u^k|^2,$$

$$\tilde{C}_9|A^\gamma u^k||A^{(1+2\gamma)/2}h^k|^2 \leq \tilde{C}_9 \left( \frac{d\alpha}{v_1} \right)^{1/2} \mu^{\gamma-1/2} M^{1/2}|A^{(1+2\gamma)/2}h^k|^2,$$

$$d_6M|A^{(1+2\gamma)/2}h^k|^2 \leq d_6M^{1/2}|A^{(1+2\gamma)/2}h^k|^2,$$

and

$$d_4M|A^{(1+2\gamma)/2}h^k||A^{(1+2\gamma)/2}u^k| \leq d_4M^{1/2}\tilde{C} \left\{ |A^{(1+2\gamma)/2}h^k|^2 + |A^{(1+2\gamma)/2}u^k|^2 \right\} \leq 2C_1^{1/2}M^{1/2}\tilde{C} \left\{ |A^{(1+2\gamma)/2}h^k|^2 + |A^{(1+2\gamma)/2}u^k|^2 \right\}.$$  

Consequently, the above estimate and (21) imply

$$\alpha d 2dt|A^\gamma u^k|^2 + \frac{1}{2} \frac{d}{dt}|A^\gamma h^k|^2 + \nu|A^{1+2\gamma}u^k|^2 + \chi|A^{1+2\gamma}h^k|^2$$

$$\leq P_1M^{1/2}|A^{(1+2\gamma)/2}u^k|^2 + P_2M^{1/2}|A^{(1+2\gamma)/2}h^k|^2,$$

where

$$P_1 = z \left( \frac{v_1}{d_0} \right)^{1/2} \mu^{-\gamma} + C_1\alpha \left( \frac{d_0}{v_1} \right)^{1/2} \mu^{-1/2} + d_5 + d_4\tilde{C} + 2C_1 \left( \frac{d_0}{\alpha} \right)^{1/2} \mu^{\gamma-1/2}\tilde{C}$$

and

$$P_2 = d_3 \left( \frac{\chi_1}{d_0} \right)^{1/2} \mu^{-\gamma} + \tilde{C}_9 \left( \frac{d_0}{v_1} \right)^{1/2} \mu^{-1/2} + d_6 + d_4M^{1/2}\tilde{C} + 2C_1 \left( \frac{d_0}{\alpha} \right)^{1/2} \mu^{\gamma-1/2}\tilde{C}.$$  

Then, if $M < \min \left\{ \left( \frac{d_0}{\alpha} \right)^2, \left( \frac{d_0}{\chi^2} \right)^2, 1 \right\}$, we have

$$\alpha d 2dt|A^\gamma u^k|^2 + \frac{1}{2} \frac{d}{dt}|A^\gamma h^k|^2 < 0, \quad \text{at } t = T^*.$$  

Thus, in a neighborhood of $t = T^*$ it follows

$$|A^\gamma u^k(t)|^2 + |A^\gamma h^k(t)|^2 \leq E\mu^{2\gamma-1}M \quad \text{for any } t \in [T^*, T^* + \delta).$$  

which implies $T^* = \infty$. Then, we have

$$|A^\gamma u^k(t)|^2 \leq E\mu^{2\gamma-1}M \quad \text{for any } t \in (-\infty, \infty)$$

$$|A^\gamma h^k(t)|^2 \leq E\mu^{2\gamma-1}M \quad \text{for any } t \in (-\infty, \infty)$$

since $u^k(t)$ and $h^k(t)$ are periodical.
3 Estimates of derivatives of higher order

To show the convergence of the approximate solutions we shall derive estimates of derivatives of higher order. By Lemma 7, if $M$ is sufficiently small the approximate solutions satisfy

$$\sup_t |A^\gamma u^k(t)| \leq C(M), \sup_t |A^\gamma h^k(t)| \leq C(M)$$

with $\gamma = \frac{N}{4} - \frac{1}{2}$, where $C(M)$ denotes a constant depending on $M$ and on norm involving the border function $\beta_i(x,t)$ and independent of $k$.

**Lemma 8** Let $(u^k(t), h^k(t))$ be the solution of (9) given above. Set

$$M_0 = \left( \int_0^\tau |f|^2 \, dt \right)^{\frac{1}{2}}, M_1 = \left( \int_0^\tau |f_t|^2 \, dt \right)^{\frac{1}{2}}.$$

Then, we have

$$\sup_{0 \leq t \leq \tau} |\nabla u^k(t)|^2 \leq C(M_0, M, L_{1B}, L_{2B}), \quad \sup_{0 \leq t \leq \tau} |\nabla h^k(t)|^2 \leq C(M_0, M, L_{1B}, L_{2B}),$$

and

$$\sup_t (\alpha |u^k(t)|^2 + |h^k_t(t)|^2) \leq C(M_0, M_1, M, L_{1B}, L_{2B}),$$

where $C(M_0, M, L_{1B}, L_{2B})$ and $C(M_0, M_1, M, L_{1B}, L_{2B})$ denote constants depending on $M_0, M_1, M, L_{1B}, L_{2B}$ and independent of $n$. Here, $L_{1B}, L_{2B}$ are constants depending on the norm of $\beta_i(x,t)$.

**Proof.** Taking $Au^k$ and $Ah^k$ as test functions in (9), we get

$$(\alpha u^k_t + \nu Au^k, Au^k) =$$

$$(\alpha f(t) - \alpha P(u^k \cdot \nabla u^k) + P(h^k \cdot \nabla h^k) - \alpha(B_1)_t - \nu AB_1, Au^k)$$

$$- (\alpha P(B_1 \cdot \nabla B_1) + P(u^k \cdot \nabla B_1) - P(B_1 \cdot \nabla u^k) + P(B_2 \cdot \nabla h^k), Au^k)$$

$$- (P(h^k \cdot \nabla B_2) + P(B_2 \cdot \nabla B_2), Au^k),$$

$$(h^k_t + \chi Ah^k, Ah^k) =$$

$$(-P(u^k \cdot \nabla h^k) + P(h^k \cdot \nabla u^k) - (B_2)_t - \chi AB_2 - P(B_1 \cdot \nabla h^k), Ah^k)$$

$$+ (P(h^k \cdot \nabla B_2^k) - P(u^k \cdot \nabla B_2) - P(B_2 \cdot \nabla B_1), Ah^k)$$

$$- (P(B_2 \cdot \nabla u^k) - P(B_1 \cdot \nabla B_2), Ah^k).$$
Now, we set

\[ L_1(Au^k) = (\alpha \mathbf{f}(t) - \alpha(B_1) - \nu AB_1 - \alpha P(B_1 \cdot \nabla B_1) + P(u^k \cdot \nabla B_1, Au^k) +\]

\[-P(B_1 \cdot \nabla u^k) + P(B_2 \cdot \nabla h^k) + P(h^k \cdot \nabla B_2) + P(B_2 \cdot \nabla B_2, Au^k),\]

\[ L_2(Ah^k) = (-B_2 \cdot \nabla B_1 + \chi AB_2 - P(B_1 \cdot \nabla h^k) + P(h^k \cdot \nabla B_1) - P(u^k \cdot \nabla B_2, Ah^k) +\]

\[ (B_2 \cdot \nabla B_1) + P(B_2 \cdot \nabla u^k) - P(B_1 \cdot \nabla B_2, Ah^k),\]

and the system above can be written as

\[
\frac{\alpha}{2} \frac{d}{dt} |\nabla u^k|^2 + \nu |Au^k|^2 \leq |(P(h^k \cdot \nabla h^k), Au^k)| + |(P(u^k \cdot \nabla B_1), Au^k)| + |(P(B_2 \cdot \nabla h^k), Au^k)| + |(P(h^k \cdot \nabla B_2), Au^k)| + |L_1(Au^k)|,
\]

\[
\frac{1}{2} \frac{d}{dt} |\nabla h^k|^2 + \chi |Ah^k|^2 \leq |(P(h^k \cdot \nabla u^k), Ah^k)| + |(P(u^k \cdot \nabla B_2), Ah^k)| + |(P(B_2 \cdot \nabla u^k), Ah^k)| + |(P(h^k \cdot \nabla B_1), Ah^k)| + |L_2(Ah^k)|.
\]

Summing the above inequalities, we have

\[
\frac{d}{dt}(\alpha |\nabla u^k|^2 + |\nabla h^k|^2) + 2\nu |Au^k|^2 + 2\chi |Ah^k|^2 \leq 2|(P(h^k \cdot \nabla h^k), Au^k)| + 2|(P(u^k \cdot \nabla B_1), Au^k)| + 2|(P(B_2 \cdot \nabla h^k), Au^k)| + 2|(P(h^k \cdot \nabla B_2), Au^k)| + 2|(P(B_2 \cdot \nabla u^k), Ah^k)| + 2|(P(h^k \cdot \nabla B_1), Ah^k)| + 2|L_1(Au^k)| + 2|L_2(Ah^k)|.
\]

then, by using the inequality (4) we have (with \( \delta = 0, \theta = \gamma \) and \( \rho = 1 \))

\[ 2|(P(h^k \cdot \nabla h^k), Au^k)| \leq 2|A^{-\delta} h^k \cdot \nabla h^k||Au^k| \leq c_1 |A^\theta h^k||A^\theta h^k||Au^k| \leq c_1 |A^\gamma h^k||Ah^k||Au^k|.\]
In the same manner,

\[2|\langle P(u^k \cdot \nabla B_1), Au^k \rangle| \leq c_2|A^\gamma u^k||AB_1||Au^k|,
\]

\[2|\langle P(h^k \cdot \nabla u^k), Ah^k \rangle| \leq c_3|A^\gamma h^k||Au^k||Ah^k|,
\]

\[2|\langle P(u^k \cdot \nabla B_2), Ah^k \rangle| \leq c_4|A^\gamma u^k||AB_2||Ah^k|,
\]

\[2|\langle P(h^k \cdot \nabla B_2), Ah^k \rangle| \leq c_5|A^\gamma h^k||AB_2||Au^k|,
\]

\[2|\langle P(h^k \cdot \nabla B_1), Ah^k \rangle| \leq c_6|A^\gamma h^k||AB_1||Ah^k|,
\]

\[2|\langle P(B_2 \cdot \nabla h^k), Au^k \rangle| \leq c_7|A^\gamma B_2||Ah^k||Au^k|,
\]

\[2|\langle P(B_2 \cdot \nabla u^k), Ah^k \rangle| \leq c_8|A^\gamma B_2||Au^k||Ah^k|.
\]

and

\[2|L_1(Au^k)|
\]

\[\leq \alpha|f||Au^k| + \left[ \alpha c|B_{1l}| + \nu c|AB_1| + \alpha \varepsilon_1^{L_1} |\nabla B_1| + \varepsilon_2^{L_2} |\nabla B_2| \right] |Au^k|
\]

\[\leq \mathcal{M}_1(M, L_{1B})|Au^k|,
\]

\[2|L_2(Au^k)|
\]

\[\leq \left[ c|B_{2l}| + \chi c|AB_2| + \varepsilon_1^{L_2} |\nabla B_1| + \varepsilon_2^{L_2} |\nabla B_2| \right] |Ah^k|
\]

\[\leq \mathcal{M}_2(L_{2B})|Ah^k|,
\]

where

\[
\left[ \alpha c|B_{1l}| + \nu c|AB_1| + \alpha \varepsilon_1^{L_1} |\nabla B_1| + \varepsilon_2^{L_2} |\nabla B_2| \right] \leq L_{1B},
\]

\[
\left[ c|B_{2l}| + \chi c|AB_2| + \varepsilon_1^{L_2} |\nabla B_1| + \varepsilon_2^{L_2} |\nabla B_2| \right] \leq L_{2B}.
\]

Then, from (25) we have

\[
\frac{d}{dt}(\alpha|\nabla u^k|^2 + |\nabla h^k|^2) + 2\nu|Au^k|^2 + 2\chi|Ah^k|^2
\]

\[\leq c_1|A^\gamma h^k||Ah^k||Au^k| + c_2|A^\gamma u^k||AB_1||Au^k| + c_3|A^\gamma h^k||Au^k||Ah^k|
\]

\[+ c_4|A^\gamma u^k||AB_2||Ah^k| + c_5|A^\gamma h^k||AB_2||Au^k| + c_6|A^\gamma h^k||AB_1||Ah^k|
\]

\[+ c_7|A^\gamma B_2||Ah^k||Au^k| + c_8|A^\gamma B_2||Au^k||Ah^k|
\]

\[+ \mathcal{M}_1(M, L_{1B})|Au^k| + \mathcal{M}_2(L_{2B})|Ah^k|.
\]

Now, by the hypothesis we have done, we obtain

\[\sup_t |A^\gamma u^k| \leq C_1(M) \quad \text{and} \quad \sup_t |A^\gamma h^k| \leq C_2(M).
\]
Thus, from the above inequality we can write
\[
\frac{d}{dt}(\alpha |\nabla u^k|^2 + |\nabla h^k|^2) + 2\nu |Au^k|^2 + 2\chi |Ah^k|^2 \\
\leq \overline{M}_0(M)|Au^k||Ah^k| + \overline{M}_1(M, L_{1B})|Au^k| + \overline{M}_2(L_{2B})|Ah^k|
\]
\[\leq N_1(M)|Au^k|^2 + N_2(M)|Ah^k|^2 + \overline{M}_1(M, L_{1B})|Au^k| + \overline{M}_2(L_{2B})|Ah^k|\]
where
\[
[c_1C_2(M) + c_3C_2(M) + c_7 |A^\gamma B_2| + c_8 |A^\gamma B_2|] \leq \overline{M}_0(M),
\]
\[
[c_2C_1(M) |AB_1| + c_5C_2(M) |AB_1| + \overline{M}_1(M, L_{1B})] \leq \overline{M}_1(M, L_{1B}),
\]
\[
[c_4C_1(M) |AB_2| + c_6C_2(M) |AB_1| + \overline{M}_2(M, L_{2B})] \leq \overline{M}_2(M, L_{2B}).
\]
Then, integrating (26) and recalling the periodicity of $\nabla u^k(t)$ and $\nabla h^k(t)$, we have
\[
\int_0^\tau (\nu |Au^k|^2 + \chi |Ah^k|^2) dt \\
\leq N_1(M) \int_0^\tau |Au^k|^2 dt + N_2(M) \int_0^\tau |Ah^k|^2 dt \\
+ M_1(M, L_{1B}) (\int_0^\tau |Au^k|^2)^{1/2} + M_2(L_{2B}) (\int_0^\tau |Ah^k|^2)^{1/2}
\]
or
\[
\int_0^\tau ((\nu - N_1(M))|Au^k|^2 + (\chi - N_2(M))|Ah^k|^2) dt \\
\leq M_1(M, L_{1B}) \left( \int_0^\tau |Au^k|^2 \right)^{1/2} + M_2(L_{2B}) \left( \int_0^\tau |Ah^k|^2 \right)^{1/2}.
\]
Then, if $d = \min \{(\nu - N_1(M)), (\chi - N_2(M))\} > 0$, from (27) we have,
\[
\int_0^\tau (|Au^k|^2 + |Ah^k|^2) dt \\
\leq d^{-1} M_1(M, L_{1B}) (\int_0^\tau |Au^k|^2)^{1/2} + d^{-1} M_2(L_{2B}) (\int_0^\tau |Ah^k|^2)^{1/2} \\
\leq \left( \frac{d^{-1} M_1(M, L_{1B})^2}{2} \right) + \frac{1}{2} \int_0^\tau |Au^k|^2 + \left( \frac{d^{-1} M_2(L_{2B})^2}{2} \right) + \frac{1}{2} \int_0^\tau |Ah^k|^2,
\]
where we have used the Young’s inequality, then
\[
\int_0^\tau (|Au^k|^2 + |Ah^k|^2) dt \leq (d^{-1} M_1(M, L_{1B})^2) + (d^{-1} M_2(L_{2B})^2)
\leq D(M, L_{1B}, L_{2B}).
\]
Finally, applying the Mean Value Theorem for integrals, we have that there exists $t^* \in [0, \tau]$ such that
\[
|Au^k(t^*)|^2 + |Ah^k(t^*)|^2 \leq \tau^{-1} D(M, L_{1B}, L_{2B}).
\]
By using the Lemma 3, with $\theta = \frac{1}{2}$, $\beta = 1$, we have

$$|\nabla u^k(t^*)|^2 \leq \mu^{-1} |Au^k(t^*)|^2 \leq \mu^{-1} \tau^{-1} D(M, L_{1B}, L_{2B})$$

and

$$|\nabla h^k(t^*)|^2 \leq \mu^{-1} |Ah^k(t^*)|^2 \leq \mu^{-1} \tau^{-1} D(M, L_{1B}, L_{2B}).$$

Now, integrating the inequality (26) from $t^*$ to $t + \tau$ ($t \in [0, \tau]$), we deduce easily

$$\sup_t |\nabla u^k(t)| \leq D(M, L_{1B}, L_{2B}), \quad \sup_t |\nabla h^k(t)| \leq D(M, L_{1B}, L_{2B})$$

(28)

where $D(M, L_{1B}, L_{2B})$ is independent of $k$.

Similarly, taking $u^k_t$ and $h^k_t$ as test functions in (9), we can show that

$$\sup_t |u^k(t)| \leq D(M_0, M, L_{1B}, L_{2B}), \quad \sup_t |h^k(t)| \leq D(M_0, M, L_{1B}, L_{2B}).$$

This completes the proof of lemma.

The proof of the following lemma is omitted, since it is similar to the proofs of the previous lemmas and one can follow the methodology of Lemma 3.2 of [18].

**Lemma 9** Let $(u^k(t), h^k(t))$ be the approximate solution of (9) given above. Then, we have

$$\sup_t |Au^k(t)| \leq C(M_0, M_1, M, L_{1B}, L_{2B}), \quad \sup_t |Ah^k(t)| \leq C(M_0, M_1, M, L_{1B}, L_{2B})$$

$$\int_0^\tau (|Au^k(t)|^2 + |Ah^k(t)|^2) dt \leq C(M_0, M_1, M, L_{1B}, L_{2B}),$$

$$\int_0^\tau (|u^k(t)|^2 + |h^k(t)|^2) dt \leq C(M_0, M_1, M, L_{1B}, L_{2B}).$$

### 4 Proof of Theorems

By the Aubin-Lions theorem, we have from (11) estimates that there are subsequences $u^k(t)$ and $h^k(t)$ such that

$$u^k \to u, \ h^k \to h, \text{ strongly in } L^\infty(\tau; V).$$

Also, by the estimates (24), we have

$$u^k \to u, \ h^k \to h, \ w^* \text{ in } L^\infty(\tau; D(A))$$

and

$$u^k_t \to u_t, \ h^k_t \to h_t, \ w^* \text{ in } L^\infty(\tau; V),$$
and the functions $u(t)$ and $h(t)$ satisfy

$$u, h \in H^2(\tau; H) \cap H^1(\tau; D(A)) \cap L^\infty(\tau; D(A)) \cap W^{1,\infty}(\tau; V).$$

We will show that

$$u^k_t \to u_t, h^k_t \to h_t, \text{ strongly in } L^\infty(\tau; H).$$

Taking $\phi = u_t$ and $\phi = h_t$ in Lemma 4, with $X = V, Y = B = H$, we obtain the desired convergences.

Once these latter convergences are established, it is a standard procedure to take the limit along the previous subsequences in (9), and we conclude that $(u, h)$ is a periodic strong solution of (1)-(3).

To prove the uniqueness, we consider that $(u_1, h_1)$ and $(u_2, h_2)$ are two solutions of problem (1)-(3). By defining differences

$$w = u_1 - u_2, z = h_1 - h_2,$$

from (8) we have

$$\frac{\partial w}{\partial t} + \nu A w + \alpha P w \cdot \nabla u_1 + \alpha P u_2 \cdot \nabla w - P z \cdot \nabla h_1 = 0,$$

$$\frac{\partial z}{\partial t} + \chi A z + P w \cdot \nabla h_1 + P u_2 \cdot \nabla z - P z \cdot \nabla u_1 = 0,$$

from which we obtain,

$$(\alpha w_t + \nu A w, w) = (-\alpha P w \cdot \nabla u_1, w) + (P z \cdot \nabla h_1, w) + (P h_2 \cdot \nabla z, w)$$

$$- (P w \cdot \nabla B_1, w) + (P B_2 \cdot \nabla z, w) + (P z \cdot \nabla B_2, w),$$

$$(z_t + \chi A z, z) = (-P w \cdot \nabla h_1, z) + (P z \cdot \nabla u_1, z) + (P h_2 \cdot \nabla w, z)$$

$$+ (P z \cdot \nabla B_1, z) - (P w \cdot \nabla B_2, z) + (P B_2 \cdot \nabla w, z),$$

or

$$\frac{1}{2} \frac{d}{dt} (|w|^2 + |z|^2) + \nu |\nabla w|^2 + \chi |\nabla z|^2$$

$$= \alpha (P w \cdot \nabla w, u_1) - (P z \cdot \nabla w, h_1) + (P w \cdot \nabla w, B_1) - (P z \cdot \nabla w, B_2)$$

$$+ (P w \cdot \nabla z, h_1) - (P z \cdot \nabla z, u_1) - (P z \cdot \nabla z, B_1) + (P w \cdot \nabla z, B_2).$$
Thus, considering that $D$ above inequality, we have
\[ |\alpha(Pw \cdot \nabla u_i)| = \alpha(A^{-\gamma}Pw \cdot \nabla u_i) \leq \alpha |A^{-\gamma}Pw \cdot \nabla w| A^{-\gamma}u_i \leq C_1 |A^{1/2}w| |A^{1/2}w| A^{-\gamma}u_i \leq C_1 |\nabla w|^2 |A^{-\gamma}u_i| \leq C_1 C(M) |\nabla w|^2, \]

similarly
\[ |(Pz \cdot \nabla h_1)| \leq C_1 |\nabla z| |\nabla w| A^{-\gamma}h_1 \]
\[ \leq C_1 C(M) |\nabla z| |\nabla w| \leq \frac{C_1 C(M)}{2} |\nabla z|^2 + \frac{C_1 C(M)}{2} |\nabla w|^2, \]

Then, by using the estimations above we have
\[ \frac{1}{2} \frac{d}{dt}(\alpha |w|^2 + |z|^2) + \nu |\nabla w|^2 + \chi |\nabla z|^2 \leq D(M)(\nu |\nabla w|^2 + \chi |\nabla z|^2), \]

where $D(M)$ is an appropriate constant depending of $M$. Now, we can write
\[ \frac{d}{dt}(\alpha |w|^2 + |z|^2) \leq 2(D(M) - 1)(\nu |\nabla w|^2 + \chi |\nabla z|^2). \]

Thus, considering that $D(M) < 1$, we conclude that $L = 2(1 - D(M)) > 0$, and then, from the above inequality, we have
\[ \frac{d}{dt}(\alpha |w|^2 + |z|^2) \leq -L(\nu |\nabla w|^2 + \chi |\nabla z|^2). \]  

On the other hand, recall that we can choose the basis \{\omega_i; i = 1, 2, \ldots\} such that the eigenfunctions $\omega_i$ of $A$ are also eigenfunctions of $A^\gamma$ and that we can write
\[ A\omega_i = \mu_i \omega_i, \quad A^\gamma \omega_i = \mu_i^\gamma \omega_i \]

where the $\mu_i$ are eigenvalue of $A$. We obtain that
\[ |\nabla w| \leq \mu^{1/2} |w| \quad \text{and} \quad |\nabla z| \leq \mu^{1/2} |z|, \]
then from (29) we can write
\[
\frac{d}{dt}(\alpha|w|^2 + |z|^2) \leq -L(\nu \mu |w|^2 + \chi \mu |z|^2)
\]
\[
\leq -Q(\alpha|w|^2 + |z|^2),
\]
where \(Q = \mu \min \{\nu, \chi\} \left(\frac{1}{\alpha} + 1\right) > 0\).

Finally,
\[
(\alpha|w(t)|^2 + |z(t)|^2 \leq (\alpha|w(0)|^2 + |z(0)|^2) \exp(-Qt),
\]
for any \(t \in (0, \infty)\).

Since \(w(t)\) and \(z(t)\) are periodic in \(t\), for any \(t \in (-\infty, +\infty)\) there exists a positive integer \(n_0\) such that \(t + n_0 \tau > 0\) and
\[
\alpha|w(t)|^2 + |z(t)|^2 = \alpha|w(t + n_0 \tau)|^2 + |z(t + n_0 \tau)|^2.
\]

Hence, it follows,
\[
\alpha|w(t)|^2 + |z(t)|^2 \leq (\alpha|w(0)|^2 + |z(0)|^2) \exp(-Qnt)
\]

\((n \geq n_0)\), which implies
\[
\alpha|w(t)|^2 + |z(t)|^2 = 0
\]
and finally \(u_1 = u_2\) and \(h_1 = h_2\).

Note that the NS equations
\[
\frac{\partial u}{\partial t} - \frac{\eta}{\rho} \Delta u + u \cdot \nabla u = f - \frac{1}{\rho} \nabla p^* \quad \text{on } \partial \Omega
\]
\[
\text{div } u = 0
\]
are a particular case of the MHD equations when the magnetic field \(h\) is identically zero, in this case when \(h = 0\), we prove existence and uniqueness of periodic strong solutions to the NS equations with inhomogeneous boundary conditions. Also, Morimoto in [16] shows existence and uniqueness of weak solutions with inhomogeneous boundary conditions for the NS equations.
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