A NOTE ON L-SERIES AND HODGE SPECTRUM OF POLYNOMIALS

RICARDO GARCÍA LÓPEZ

(Communicated by Barry Mazur)

Abstract. We compare on the one hand the combinatorial procedure described in [1] which gives a lower bound for the Newton polygon of the L-function attached to a commode, non-degenerate polynomial with coefficients in a finite field and on the other hand the procedure which gives the Hodge theoretical spectrum at infinity of a polynomial with complex coefficients and with the same Newton polyhedron. The outcome is that they are essentially the same, thus providing a Hodge theoretical interpretation of the Adolphson-Sperber lower bound which was conjectured in [1].

1. Introduction

Let $p$ be a prime number, put $q = p^a$. For any $i \geq 1$, denote by $\mathbb{F}_q$ the finite field with $q^i$ elements (for $i = 1$ we write $\mathbb{F}_q = k$). Let $\text{tr}_i : \mathbb{F}_{q^i} \to k$ denote the trace map. To a polynomial $f \in k[x_1, \ldots, x_n]$ and a non-trivial additive character $\psi : k \to \mathbb{C}^*$ one attaches the sequence of exponential sums

$$S_i(f, \psi) = \sum_{x \in (\mathbb{F}_{q^i})^n} \psi(\text{tr}_i(f(x))) \quad (i \geq 1).$$

The generating function

$$L(f, \psi)(t) = \exp \left( \sum_{i \geq 1} S_i(\psi, f) \frac{t^i}{i} \right)$$

is an Artin L-series which encodes much of the information about the arithmetic properties of the polynomial $f$ (see e.g. [6]). If $f$ is commode and non-degenerate (precise definitions are recalled below), it was proven in [1] and [2] that $L(f, \psi)^{(-1)^{n+1}}$ is a polynomial in one variable whose roots are algebraic over $\mathbb{Q}$. It was also shown in [1] that the Newton polygon of $L(f, \psi)^{(-1)^{n+1}}$ is bounded below by a polygonal line that we denote $\text{LB}(f)$ and which can be described in a purely combinatorial way from the Newton polyhedron of $f$. This result gives non-trivial information about the $p$-adic absolute values of the roots of $L(f, \psi)^{(-1)^{n+1}}$.

On the other hand, if $f$ is a polynomial with complex coefficients, one can attach to it a variation of mixed Hodge structures and also a limit mixed Hodge structure
at infinity. Similar to the case of germs of holomorphic functions, the action of the semi-simple part of the monodromy (about infinity) on the limit Hodge filtration gives rise to a set of rational numbers $\text{Spec}_{\mathcal{H}}(f)$, called the spectrum at infinity of the polynomial $f$. If the polynomial is commode and non-degenerate, C. Sabbah proved in [10], [11] that the spectrum at infinity can also be computed in a combinatorial way.

Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a commode, non-degenerate polynomial. Then, for all but a finite number of prime numbers $p$, the reduction $\overline{f}$ of $f$ modulo $p$ will also be commode and non-degenerate, with the same Newton polyhedron as $f$. In this note we compare the combinatorial formulas giving the Hodge spectrum of $f$ and those which give the lower bound of the Newton polygon of $L(\overline{f}, \psi)^{(-1)^{n+1}}$. The outcome is that the datum of $\text{Spec}_{\mathcal{H}}(f)$ is equivalent to that of the lower bound $\text{LB}(\overline{f})$, what provides a Hodge-theoretical interpretation of the later. This interpretation was conjectured in the introduction to [1].

A natural question is whether the relation between the Hodge spectrum of $f$ and the $p$-adic absolute values of the roots of $L(\overline{f}, \psi)^{(-1)^{n+1}}$ is a special feature of the commode, non-degenerate case, or if there is a deeper relation, holding in greater generality. We have no results about this matter, but at the end of this note we present some examples against which the question might be tested. In a different setting, connections between the $p$-adic valuations of the eigenvalues of Frobenius acting in De Rham cohomology and Hodge theory appear already in [8].

The theorems from [1] and [10] quoted in this note are not stated in their most general form. In particular, some of the theorems we recall which concern commode and non-degenerate polynomials hold also for Laurent polynomials, under suitable hypotheses. For the convenience of the reader, we have tried to keep the same notation as in [1] and [3].

2. $L$-series attached to CND polynomials

The class of commode polynomials, non-degenerate with respect to their Newton boundary, was defined by A. Kouchnirenko in [7]. In order to recall his definition, we introduce some notation: Let $k$ be any field, let $\overline{k}$ denote an algebraic closure of $k$, let $f \in \overline{k}[x_1, \ldots, x_n]$ be a polynomial. Write

$$f(x_1, \ldots, x_n) = \sum_I c_I x^I$$

where $I = (i_1, \ldots, i_n) \in \mathbb{N}^n$, $x^I := x_1^{i_1} \cdots x_n^{i_n}$ and $c_I \in \overline{k}$. The Newton polyhedron of $f$, denoted $\Delta(f)$, is the convex envelope in $\mathbb{R}^n$ of the set

$$\{I \in \mathbb{N}^n \mid c_I \neq 0\} \cup \{(0, \ldots, 0)\}.$$

If $\tau$ is a face of $\Delta(f)$, put $f_\tau = \sum_{I \in \tau} c_I x^I$.

**Definition 1.** The polynomial $f$ is said to be non-degenerate if for any face $\tau$ of $\Delta(f)$ not containing $(0, \ldots, 0)$, there are no solutions $(x_1, \ldots, x_n) \in \overline{k}$ of the system of equations

$$\frac{\partial f_\tau}{\partial x_i}(x_1, \ldots, x_n) = 0 \quad (1 \leq i \leq n)$$

with $x_i \neq 0$ for all $i \in \{1, \ldots, n\}$. This condition is Zariski generic (cf. [7, 6.1]).
The polynomial $f$ is said to be commode (or convenient in some articles) if $\Delta(f)$ intersects all coordinate axes in points different from $(0, \ldots, 0)$. If $f$ is simultaneously commode and non-degenerate we will say that $f$ is a CND polynomial.

Let $k$ be a finite field, $f \in k[x_1, \ldots, x_n]$ a CND polynomial, $\psi : k \rightarrow \mathbb{C}^*$ a non-trivial additive character. The following is proved in [1] and [2]:

i) $L(f, \psi)^{(-1)^{n+1}}$ is a polynomial and its roots are algebraic over $\mathbb{Q}$.

ii) For any embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, the complex absolute value of the roots of $L(f, \psi)^{(-1)^{n+1}}$ is $q^{n/2}$.

iii) The roots of $L(f, \psi)^{(-1)^{n+1}}$ are $\ell$-adic units for any $\ell \neq p$ and any embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$.

The results of Adolphson and Sperber in [1] give information about the possible $p$-adic absolute values of the roots of $L(f, \psi)^{(-1)^{n+1}}$. Their theorem is stated in terms of Newton polygons for polynomials in one variable (these Newton polygons have nothing to do with the Newton polyhedra introduced before, but this is the standard terminology and we will also use it). We recall the definition:

**Definition 2.** Let $\Omega_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Denote by “ord” the additive valuation of $\Omega_p$ normalized by the condition $\text{ord}(q) = 1$. Given $h(t) = 1 + a_1 t + \ldots + a_k t^k \in \Omega_p[t]$, the Newton polygon of $h(t)$, denoted $\mathcal{N}(h(t))$, is the lower boundary of the convex envelope in $\mathbb{R}^2$ of the set

$\{(0,0) \cup \{(i, \text{ord}(a_i))\}_{i=1,\ldots,k}\}$.

It is well-known that $\mathcal{N}(h(t))$ has a side of slope $s$ and horizontal projection of length $l$ if and only if $h(t)$ has exactly $l$ roots $\alpha \in \Omega_p$ (counting multiplicities) with $\text{ord}(\alpha) = -s$. Thus, the knowledge of $\mathcal{N}(p(t))$ is equivalent to that of the $p$-adic valuations of the roots of $p(t)$.

2.1. Assume now that $k$ is an arbitrary field and $f \in k[x_1, \ldots, x_n]$ is a CND polynomial, put $R = k[x_1, \ldots, x_n]$. For any face $\sigma$ of $\Delta(f)$ denote by $L_\sigma$ the unique linear form with coefficients in $\mathbb{Q}$ such that $L_\sigma \equiv 1$ on $\sigma$. Given $g \in R$, put $w_\sigma(g) := \max_j L_\sigma(J)$ where $J$ runs over the multiindexes corresponding to exponents of monomials appearing in $g$ with a non-zero coefficient. Consider the weight function $w(g) = \max_\sigma w_\sigma(g)$. Let $M$ be an integer such that $w(R) \subset M^{-1}\mathbb{N}$. Define an increasing filtration on $R$ by:

$$R_{i/M} = \{g \in R \mid w(g) \leq i/M\}$$

(cf. [7], [1]). Let $\overline{R} = \oplus_{i \in \mathbb{N}} \overline{R}^{(i/M)}$ denote the associated graded algebra, where $\overline{R}^{(i/M)} = R_{i/M}/R_{(i-1)/M}$. Adolphson and Sperber consider in [1] also the graded, non-unitary algebra $\overline{R}_S = x_1 \ldots x_n \overline{R}$. It is easy to check that $x_i \partial f / \partial x_i \in R_1$ for all $1 \leq i \leq n$; let $\overline{f}_i \in \overline{R}^{(1)}$ denote the class of $x_i \partial f / \partial x_i$.

The ideal $T_S = \langle \{x_1 \ldots \hat{x}_i \ldots x_n \overline{f}_1\}_{1 \leq i \leq n} \rangle \subset \overline{R}_S$ is homogeneous and the quotient $\overline{R}_S/T_S$ is also a graded algebra, namely $\overline{R}_S/T_S = \oplus_{i \in \mathbb{N}}(\overline{R}_S/T_S)^{(i/M)}$ with

$$\left(\overline{R}_S/T_S^{(i/M)}\right) = \frac{\overline{R}_S^{(i/M)}}{\overline{R}_S^{(i/M)} \cap T_S};$$

put $b_i := \dim_k((\overline{R}_S/T_S^{(i/M)})$. It can be shown that $b_i < \infty$ for every $i \in \mathbb{N}$ and $b_i = 0$ for $i > nM$ ([7, 2.9]). It is plain from the definition of $w$ that if $g \in R_{i/M}$,
then all monomials appearing in $g$ are also elements of $R_{i/M}$. It follows that the $k$-vector spaces $(\overline{R_S}/I_S)^{(i/M)}$ have a basis formed by monomials, and so the numbers $b_i \in \mathbb{N}$ are independent of the base field $k$. The following theorem is proved in [1, 3.11]:

**Theorem.** If $k$ is a finite field and $f \in k[x_1, \ldots, x_n]$ is a CND polynomial, then the Newton polygon of $L(f, \psi)^{(-1)^{n+1}}$ lies on or above the Newton polygon of the polynomial $\prod_{i=0}^{nM}(1 - q^{i/M}t)^{b_i}$.

**Remarks.**

i) In [1] it is proven also that the endpoints of both Newton polygons coincide.

ii) In [1] the filtration considered above is defined (for more general polynomials than CND’s) on a certain subring $R(f)$ of $R$. For commode polynomials, $R(f) = R$.

iii) It is easy to see that the Newton polygon of $\prod_{i=0}^{nM}(1 - q^{i/M}t)^{b_i}$ is the polygonal line $LB(f)$ obtained by juxtaposition of segments of slope $i/M$ and with horizontal projection of length $b_i$ $(0 \leq i \leq nM)$, starting at the origin in $\mathbb{R}^2$.

**Definition 3.** We denote by $\text{Spec}_{AS}(f)$ the set of rational numbers with multiplicities defined as follows: The number of times $\frac{i}{M}$ appears in $\text{Spec}_{AS}(f)$ equals $b_i \in \mathbb{N}$.

### 3. Complex polynomials

Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial. There exists a finite subset $\Sigma \subset \mathbb{C}$ such that the restriction of $f$

$$f : \mathbb{C}^n - f^{-1}(\Sigma) \longrightarrow \mathbb{C} - \Sigma$$

is a locally trivial smooth fibration (see e.g. [9]). This is not straightforward due to the fact that $f$ is not proper), and the smallest set $\Sigma_f$ verifying this condition is called the set of bifurcation values of $f$. It follows that we can speak of the topological type of the generic fiber of $f$, and it can be proven that for polynomials which are CND, this generic fiber has the homotopy type of a bouquet of $(n-1)$-dimensional spheres ([7, Théorème V]).

The cohomology spaces $H^{n-1}(f^{-1}(t), \mathbb{Q})$ carry mixed Hodge structures and the fibration considered above defines a variation of mixed Hodge structures over $\mathbb{C} - \Sigma_f$. In this situation, one can define a limit mixed Hodge structure at infinity (by the results in [13], cf. [5]) and this allows one to define a spectrum, the so-called spectrum at infinity. More precisely, let $F^\bullet$ be the limit Hodge filtration and $T^\infty$ the semisimple part of the monodromy at infinity. The Hodge spectrum is the finite set of rational numbers with multiplicities, denoted $\text{Spec}_H(f)$, defined by the following condition: The number of times that $a \in \mathbb{Q}$ appears in the spectrum is the dimension of the space of eigenvectors of $T^\infty$ of eigenvalue $\exp(-2\pi ia)$, where $T^\infty$ is regarded as an endomorphism of the quotient $F^{[n-a+1]}/F^{[n-a+1]+1}$ and where $[\cdot]$ denotes integer part (cf. [12, (2.1)] in the local case, notice that we have introduced a shifting of +1 with respect to Steenbrink’s definition).

### 3.1. In [10], [11] C. Sabbah introduced the class of cohomologically tame polynomials (a large class which includes the CND polynomials). For these polynomials, he gives a $\mathcal{D}$-module theoretic interpretation of the Hodge spectrum, and he proves
that if $f$ is a CND polynomial, the spectrum can be computed from the Newton polyedron of $f$. The combinatorial procedure is described in [3] (we will use slightly different notations as those in [3]): Put $R = \mathbb{C}[x_1, \ldots, x_n]$. Let $w$ be the same weight function in $R$ defined in section 2, define $w^*: R \to \mathbb{Z}$ by $w^*(g) := w(x_1 \ldots x_n g)$. For $\alpha \in \mathbb{Q}$, put

$$
V^\alpha R = \{ g \in R \mid w^*(g) \leq \alpha \},
$$

$$
\text{gr}_V^\alpha R = V^\alpha R/\cup_{\beta < \alpha} V^\beta R,
$$

$$
\text{gr}_V^\bullet R = \oplus_{\alpha \in \mathbb{Q}} \text{gr}_V^\alpha R.
$$

Given $g \in R$, denote $\text{in}(g)$ the class of $g$ in $\text{gr}_V^{w^*(g)} R$, let $J(f) = \langle \{ \partial f/\partial x_i \}_{1 \leq i \leq n} \rangle$ denote the Jacobian ideal of $f$, put $\text{in}(J(f)) = \langle \{ \text{in}(g) \mid g \in J(f) \} \rangle$. The quotient $\text{gr}_V^\bullet R/\text{in}(J(f))$ is a graded algebra and the Hodge spectrum of $f$ is given by the following rule: The number of times $\alpha \in \mathbb{Q}$ appears in $\text{Spec}_{\mathbb{H}}(f)$ is the dimension of the $\alpha$-graded piece of the graded ring

$$(1) \quad \frac{\text{gr}_V^\bullet R}{\text{in}(J(f))} = \bigoplus_{\alpha} \frac{\text{gr}_V^\alpha R}{\text{in}(J(f)) \cap \text{gr}_V^\alpha R}.
$$

(In [3] the weight function used is $-\omega$, which makes the spectral numbers have the opposite sign of those we consider). Let $\overline{R}_S$, $\overline{T}_S$ be defined as in section 2, taking as field of coefficients the field $\mathbb{C}$. Multiplication by the monomial $x_1 \ldots x_n$ induces an isomorphism of graded vector spaces

$$(2) \quad \text{gr}_V^\bullet \left( \frac{R}{J(f)} \right) \cong \overline{R}_S/\overline{T}_S
$$

where the graded ring on the right hand side was defined in section 2 (see [3, pg. 321, Lemme]). On the other hand, it is a standard fact that

$$(3) \quad \frac{\text{gr}_V^\bullet R}{\text{in}(J(f))} \cong \text{gr}_V^\bullet \left( \frac{R}{J(f)} \right)
$$

as graded rings, where in $R/J(f)$ one considers the graduation attached to the filtration induced by $V^\bullet R$.

**Proposition.** Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial. Denote $\overline{f}$ the reduction of $f$ modulo $p$ and assume that both $\overline{f} \in \mathbb{F}_p[x_1, \ldots, x_n]$ and $f \in \mathbb{C}[x_1, \ldots, x_n]$ are CND polynomials with the same Newton polyedron. Then

$$\text{Spec}_{\mathbb{H}}(f) = \text{Spec}_{\mathbb{AS}}(\overline{f}).$$

In particular, the polygonal line $\text{LB}(\overline{f})$ is determined by the Hodge spectrum of $f$.

**Proof.** By the theorem of Sabbah recalled in (3.1), the multiplicity of $\alpha \in \mathbb{Q}$ in $\text{Spec}_{\mathbb{H}}(f)$ is

$$\dim_{\mathbb{C}} \frac{\text{gr}_V^\alpha R}{\text{in}(J(f)) \cap \text{gr}_V^\alpha R}.$$ 

By (1) and isomorphisms (2), (3), this multiplicity coincides with the dimension of the $\alpha$-th graded piece of the graded ring $\overline{R}_S/\overline{T}_S$, defined as in (2.1) with $k = \mathbb{C}$. As noted in (2.1), this dimension is independent of the coefficient field. Since $f$ and $\overline{f}$ are CND polynomials with the same Newton polyedron, it coincides with the multiplicity of $\alpha$ in $\text{Spec}_{\mathbb{AS}}(\overline{f})$ (See definition 3). It follows that $\text{Spec}_{\mathbb{H}}(f) = \text{Spec}_{\mathbb{AS}}(\overline{f})$. The last statement follows from the equality of spectra and the definition of the lower bound $\text{LB}(\overline{f})$ in remark iii) above. \qed
Remark. There are polynomials $f \in \mathbb{Z}[x_1, \ldots, x_n]$ which are either non-commode or degenerated, but such that:

i) Their spectrum at infinity can be explicitly computed.

ii) For all but a finite number of primes $p$ and for all non-trivial additive characters $\psi$, if $\overline{f}$ denotes the reduction of $f$ modulo $p$, then the $L$-series $L(\overline{f}, \psi)^{(-1)^{n+1}}$ is a polynomial.

As far as we know, it is an open question whether in these cases the Hodge spectrum provides a lower bound for the Newton polygon of $L(\overline{f}, \psi)^{(-1)^{n+1}}$.

Example. Let $f \in \mathbb{Z}[x, y]$ be a polynomial of degree $d$, write it in the form $f = f_d + f_{d-1} + \ldots$ where $f_i$ is a homogeneous form of degree $i$. Write

$$f_d = \prod_{j=1}^{m} l_j^{\alpha_j}$$

where $l_j \in \mathbb{C}[x, y]$ are distinct linear forms and $d = \sum \alpha_j$. Assume that $\alpha_j \mid d$ for all $1 \leq j \leq m$ and that, whenever $\alpha_j > 1$, the linear form $l_j$ does not divide $f_{d-1}$ in $\mathbb{C}[x, y]$. Given $\beta \in \mathbb{Q} \cap [0, 1]$, let $m_{\beta}$ be the multiplicity of $\beta$ in $\text{Spec}_H(f)$. The Hodge spectrum of $f$ is given by the following formulas:

i) If $\beta = 1$, then $m_1 = m - 1$

ii) If $\beta = \frac{s}{d}$ with $s \in \mathbb{N}$, $0 < s < d$, then

$$m_\beta = m - d + s - 1 + \sum_j \left\lfloor \frac{(d-s)\alpha_j}{d} \right\rfloor$$

iii) If $d \cdot \beta \notin \mathbb{Z}$, $(d-1)\beta \notin \mathbb{Z}$, and $d(1-\beta) - [(d-1)(1-\beta)] < 1$, then

$$m_\beta = \# \{ j \mid (d-1)\alpha_j \beta \in \mathbb{Z} \}$$

iv) Otherwise, $m_\beta = 0$.

If $\beta \in [1, 2] \cap \mathbb{Q}$, then $m_\beta$ is determined by the symmetry of the spectrum, namely $m_\beta = m_{2-\beta}$. If $\beta \notin [0, 2] \cap \mathbb{Q}$, then $m_\beta = 0$.

These formulas follow from the computation of primitive equivariant Hodge numbers done in [5, (7.1)]. The translation in terms of spectral numbers is easily obtained from (3.2) and the definition of the spectral pairs in [5, pg. 1575].

On the other hand, it is proved in [4] that, for all but a finite number of primes $p$, the series $L(f, \psi)^{-1}$ is a polynomial, where $\overline{f}$ is the reduction of $f$ modulo $p$ and $\psi : \mathbb{F}_p \to \mathbb{C}^*$ is any non-trivial additive character.

References

[1] A. Adolphson and S. Sperber, Exponential sums and newton polyhedra: Cohomology and estimates, Annals of Math., 130 (1989), 367–406. MR 1014928 (91e:11094)

[2] J. Denef and F. Loeser, Weights of exponential sums, intersection cohomology, and Newton polyhedra, Invent. Math., 106 (1991), 275–294. MR 1128216 (93a:14019)

[3] A. Douai, Équations aux différences finies, intégrales de fonctions multiformes et polyédres de Newton, (French) [Finite difference equations, integrals of multivalent functions and Newton polyhedral], Compos. Math., 87 (1993), 311–355. MR 1227450 (94h:32060)

[4] R. García López, Exponential sums and singular hypersurfaces, Manuscripta Math., 97 (1998), 45–58. MR 1642630 (99e:11112)

[5] R. García López and A. Némethi, Hodge numbers attached to a polynomial map, Ann. Inst. Fourier, 49 (1999), 1547–1579. MR 1723826 (2001i:32045)
[6] N. Katz, “Sommes Exponentielles,” (French) [Exponential sums] Course taught at the University of Paris, Orsay, Fall 1979. With a preface by Luc Illusie. Notes written by Gérard Laumon. With an English summary, Astérisque, 79, Société Mathématique de France, Paris, 1980. MR 0617009 (82m:10059)

[7] A. G. Kouchnirenko, Polyèdres de Newton et nombres de Milnor, (French), Invent. Math., 32 (1976), 1–31. MR 0419433 (54 7454)

[8] B. Mazur, Frobenius and the Hodge filtration (estimates), Ann. of Math. (2), 98 (1973), 58–95. MR 0321932 (48 297)

[9] F. Pham, Vanishing homologies and the \( n \) variable saddlepoint method, In Singularities, Part 2 (Arcata, Calif., 1981), 319–333, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, (1983). MR 0713258 (85d:32026)

[10] C. Sabbah, Hypergeometric periods for a tame polynomial, C. R. Acad. Sci. Paris Sér. I Math., 328 (1999), 603–608. MR 1679978 (2000b:32053)

[11] C. Sabbah, Hypergeometric period for a tame polynomial, Portugal. Math., 63 (2006), 173–226. MR 2229875 (2007i:32014)

[12] J. Steenbrink, Semicontinuity of the singularity spectrum, Invent. Math., 79 (1985), 557–565. MR 0782235 (86h:32033)

[13] J. Steenbrink and S. Zucker, Variation of mixed Hodge structure. I, Invent. Math., 80 (1985), 489–542. MR 0791673 (87h:32050a)

Departament d’Algebra i Geometria. Universitat de Barcelona. Gran Via, 585. E-08007, Spain.

E-mail address: ricardogarcia@ub.edu