Evaluation of the Pion-Nucleon Sigma Term from CHAOS Data

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Abstract

We have reanalyzed the \( \pi^\pm p \) scattering data at low energy in the Coulomb-nuclear interference region as measured by the CHAOS group at TRIUMF with the aim to determine the pion-nucleon sigma term. The resulting value \( \sigma = (44 \pm 12) \text{MeV} \), while in agreement with lattice QCD calculations and compatible with other recent analyses, is significantly lower than that from the GWU-TRIUMF analysis of 2002.

Keywords: pion-proton scattering, derive sigma term

1. Introduction

During the last decade it became known [1],[2],[3],[4],[5] that the (spin-independent) cross section for elastic scattering of supersymmetric cold dark matter particles on nucleons depends strongly on the value of the pion-nucleon sigma term \( \sigma_{\pi N} \). This is but one example of the role of the sigma term, a concept that was introduced in chiral perturbation theory (ChPT) to measure the explicit breaking of chiral symmetry due to non-zero masses of light quarks [6]. The sigma term represents the contribution from the finite quark masses to the mass of the proton. Its value is related to the strange quark content of the nucleon. The pion-nucleon sigma term \( \sigma_{\pi N} \) is also related to the value of the pion-nucleon invariant amplitude at the unphysical Cheng-Dashen point where \( s - u = 0, ~ t = 2m^2_\pi \) (Here, \( s, ~ t, ~ u \)

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are the Mandelstam variables). Consequently, its determination is mostly attempted through pion-nucleon scattering experiments at low energies and by spectroscopy of the hadronic shift and width in the ground state of pionic hydrogen and deuterium [7].

Ellis et al. [2] pleaded strongly for an experimental campaign to better determine the pion nucleon sigma term. Indeed, due to the difficulty of experiments with low energy pions, which are notoriously plagued by pion decay and the emerging muon background, the resulting cross sections from different $\pi N$ scattering experiments often do not agree with each other and the derived values of $\sigma_{\pi N}$ differ substantially, often by more than the quoted systematic errors. Ironically, however, the experimental campaign requested by the authors of [2] already existed and had been published [8] after a strong and long running effort with the CHAOS detector at TRIUMF [9], which was a dedicated detector system developed to cope with the peculiarities of low-energy pion scattering. The experiment provided differential cross sections for elastic scattering of positively and negatively charged pions off hydrogen at five energies between 19.9 and 43.3 MeV in fine angular steps ranging from the Coulomb-nuclear interference region to nearly 180°.

What was really missing in ref.[8] was a theoretical analysis leading to a value of $\sigma_{\pi N}$. The authors of ref.[8] merely considered the isospin-even forward scattering amplitudes $ReD^+(T_\pi)$ obtained directly from $\pi^+p$ and $\pi^-p$ differential cross sections and extrapolated the values to threshold ($T_\pi = 0$) using the functional forms given by the KH80 phase shifts [10] or, alternatively, from the more recent SAID FA02 phase shifts [11]. In both cases the threshold values and the related isospin-even scattering lengths turned out to be smaller than in the previous analyses. From this observation it was qualitatively concluded that the CHAOS data favour values of $\sigma_{\pi N}$ that are smaller than recently claimed [12].

It is the purpose of this paper to complete the analysis quantitatively and to derive the value of the $\pi N$ sigma term using the CHAOS cross sections as much as possible. This approach differs substantially from the method favored by the GWU-TRIUMF group [12] who uses the huge $\pi N$ data base of SAID hoping that the errors of partially contradictory measurements are averaged out. We, instead, prefer to rely as much as possible on the results taken from the most advanced low-energy pion spectrometer to date. Furthermore, by adopting the analysis methods and the notation of the Karlsruhe group [16], we use constraints that warrant analyticity and unitarity and we exploit forward dispersions relations of the $\pi N$-scattering amplitudes which ensures,
in a sense, consistency with the existing $\pi N$ data base. For a criticism of the VPI-GWU methods see [17].

2. Formalism

The $\pi N$ $\sigma$ term is defined as a matrix element of the quark mass term

$$\sigma_{\pi N} = \frac{m_u + m_d}{4m} \langle p | \bar{u}u + \bar{d}d | p \rangle,$$

where $m_u$ and $m_d$ are up and down quark masses, and $m$ is the proton mass. It is an empirical measure of the chiral asymmetry generated by the up and down quark masses. The $\sigma$ term may be written in the form

$$\sigma_{\pi N} = \frac{m_u + m_d}{4m} \frac{\langle p | \bar{u}u + \bar{d}d - 2\bar{s}s | p \rangle}{1 - y},$$

where the parameter $y$, the strange quark content of the proton, is defined as

$$y = 2 \frac{\langle p | \bar{s}s | p \rangle}{\langle p | \bar{u}u + \bar{d}d | p \rangle}.$$

In a one loop calculation of chiral perturbation theory (ChPT) Gasser et al. [13] obtained

$$\sigma_{\pi N} = \frac{35.5 \pm 5}{1 - y} \text{ MeV.}$$

This relation provides a simple way to calculate the strange quark content of the proton from the known value of the $\sigma$ term. The low energy theorem [14], [15] relates the so called experimental $\pi N$ sigma term $\Sigma$ to the isospin-even scattering amplitude $D^+$ at the Cheng-Dashen point $\nu = 0$, $t = 2m^2_{\pi}$

$$\Sigma = F^2_{\pi} \bar{D}^+(\nu = 0, t = 2m^2_{\pi}).$$

Here $F_\pi = 92.4$ MeV is the pion decay constant. The pion mass is denoted by $m_\pi$, the nucleon mass by $m$, $s$, $u$, $t$ are Mandelstam variables and $\nu = (s - u)/4m$. The amplitude $\bar{D}^+$ is defined in terms of the $\pi N$ invariant amplitudes $A$ and $B$: $D^+ = A^+ + \nu B^+$. $\bar{D}^+$ denotes the $D^+$ amplitude from
which the pseudo vector Born term is subtracted: $\bar{D}^+ = D^+ - D_{N\mu}^+$. Within the framework of ChPT Gasser et al. [6] obtained

$$\sigma_{\pi N} = \Sigma - \Delta_\sigma - \Delta_R,$$

where

$$\Delta_\sigma = (15.2 \pm 0.4)\text{MeV}, \quad \Delta_R = 0.35\text{MeV}. \quad (3)$$

More recent determinations of $\Delta_R$ will be given below.

Since the Cheng-Dashen point is outside the physical region, one has to perform an analytic continuation of the $\bar{D}^+$ amplitude to this point. Mandelstam analyticity, unitarity and crossing symmetry of the $\pi N$ invariant amplitudes are very strong constraints when analyzing experimental data or when performing an analytic continuation of the invariant amplitudes outside the physical region. The most frequently used method for that purpose is the application of dispersion relations along different curves in the Mandelstam plane. It is important to stress that use of dispersion relations as a method of analytic continuation outside the physical region assumes input from the whole energy region in the physical $\pi N$ channel. Using results from phase-shift analyses in dispersion relations implies the use of a whole body of pion-nucleon data. Two different methods, both based on dispersion relations, are used for the calculation of the $\bar{D}^+$ amplitude at the Cheng-Dashen point. In the first approach, the dispersion curves pass through the Cheng-Dashen point which allows calculating the amplitude $D$ directly [21, 23, 24]. In the second approach [12, 20], one determines the coefficients in the Taylor expansion of $\bar{D}^+$ around the center of the Mandelstam triangle (also known as a sub-threshold expansion). Invariant amplitudes are real inside the Mandelstam triangle. The $\bar{D}^+$ amplitude is crossing symmetric and is a function of $\nu^2$

$$\bar{D}^+(\nu, t) = \sum_{m,n} \bar{d}^+_{mn} \nu^{2m} t^n.$$

At the Cheng-Dashen point one has:

$$\bar{D}^+(0, 2m_\pi^2) = \bar{d}^+_{00} + \bar{d}^+_{01} \cdot 2m_\pi^2 + \ldots$$

$$\Sigma = F^2_\pi \cdot \bar{D}^+(0, 2m_\pi^2) = F^2_\pi \cdot (\bar{d}^+_{00} + \bar{d}^+_{01} \cdot 2m_\pi^2) + \Delta_d$$
with

\[ \Sigma = \Sigma_d + \Delta_d \]

\[ \Sigma_d = F_\pi^2 \cdot (\bar{d}^{+\nu}_{00} + \bar{d}^{+\nu}_{01} \cdot 2m_\pi^2). \]  \hspace{1cm} (4)  

\( \Sigma_d \) stands for the leading contribution, linear in \( t \). The term \( \Delta_d \), a curvature term \( [6] \), describes contributions of higher order in \( t \). Calculations show \([6, 23]\) that the curvature term is determined mainly by contributions from the \( t \)-channel, and is considered as a known quantity. The final relation that we use to calculate \( \sigma_{\pi N} \) reads:

\[ \sigma_{\pi N} = \Sigma_d + \Delta_d - \Delta_\sigma - \Delta_R. \]  \hspace{1cm} (5)  

In our calculation we use the value from \([6]\) \( \Delta_d = 11.9 \) MeV. With the values from (3) this yields \( \Delta = \Delta_d - \Delta_\sigma = (-3.3 \pm 0.4) \) MeV. Other, recent evaluations obtained \( \Delta = (-1.8 \pm 0.2) \) MeV \([18]\) and \( |\Delta_R| \leq 2 \) MeV \([19]\) instead. But with regard to the large final uncertainties (see eq.15 below) their application would not change the result significantly.

In our approach to extract information from the low energy data, we use for convenience the Lorentz invariant amplitude \( C^+(\nu, t) = A^+(\nu, t) + \frac{4m_\nu^2}{4m_\nu^2 - t} B^+(\nu, t) \). It is useful to recall \([16]\) that for \( t = 0 \): \( \bar{C}^+(\nu, t = 0) = \bar{D}^+(\nu, t = 0) \) and \( \bar{d}_{00}^+ = \bar{c}_{00}^+, \bar{d}_{01}^+ = \bar{c}_{01}^+ \).

Published values of the \( \sigma \) term \([21, 20, 22, 12, 23, 24, 25, 40]\) range from about 50 MeV to about 75 MeV. The various results depend on the method and technique used to extrapolate to the Cheng-Dashen point and on the data input used. The values at the upper end of this range were criticised as corresponding, in the framework of ChPT, to unrealistically large values of the strange quark content of the nucleon. Therefore, it is worthwhile to mention recent work \([26]\) where it is shown in the framework of covariant heavy baryon perturbation theory that a large value of the sigma term does not necessarily entail an implausibly large strange quark content of the nucleon.

3. Method

Whichever of the methods mentioned above is used, data from low energy \( \pi^+ p \) scattering assure stable and reliable extrapolations of the \( \bar{C}^+ \) amplitude
to the Cheng-Dashen point. The goal of the CHAOS [8] experiment was to obtain high quality data needed for that purpose. In the CHAOS experiment $\pi^p$ differential cross sections were measured at low energies and at particularly small angles in the Coulomb-nuclear interference region, where the known Coulomb non-spin-flip amplitude $G_C$ [10, 27] interferes with the nuclear amplitude $G^+$ [16]. The real part of forward nuclear amplitude $C^+$ is obtained from the experimental scattering data using the relation

$$ReC^+(q^2, t = 0) = \frac{4\pi\sqrt{s}}{m}ReG^+(q^2, t = 0) =$$

$$= \lim_{t \to 0} \frac{4\pi\sqrt{s}}{m} \left[ \frac{d\sigma_{\pi^+p}}{d\Omega} - \frac{d\sigma_{\pi^-p}}{d\Omega} \right] = \lim_{t \to 0} \Delta^+(t),$$

where $G_C$ is the known Coulomb non-spin-flip amplitude, $\frac{d\sigma_{\pi^+p}}{d\Omega}$ are $\pi^p$ differential cross sections, and $q^2$ is momentum squared in the center of mass frame. To adapt our notation to the kinematical variables used in the experiment, the momentum squared is used as an argument of the invariant amplitude $C^+$. Measurements of differential cross sections for $\pi^p$ scattering in the Coulomb-nuclear interference region allow a determination of the real part of the forward amplitude $C^+$ at five energies covered by the CHAOS experiment in the low energy region ranging from $T_\pi = 19.9$ MeV to 43.3 MeV.

In the first step of our analysis the values $\Delta^+$, derived from the CHAOS data at a given energy, are fitted to a polynomial of order $N$ in $t$ and extrapolated to the forward point $t = 0$. The coefficients $a_n$ in a polynomial fit to $\Delta^+$ are determined using a robust convergence test function method [28] by minimizing

$$\chi^2 = \chi^2_{\text{data}} + \Phi,$$

$$\chi^2_{\text{data}} = \sum_{i=1}^{N_D} \frac{(\Delta_i^+ - P_N(t_i))^2}{\sigma_i^2}.$$
the form \( \Phi = \lambda \sum_{n=0}^{N} a_n^2(n + 1)^3 \), where \( \lambda \) is a smoothing parameter which is determined in the method. The convergence test function method was used in the Karlsruhe-Helsinki phase shift analysis \([10]\). There, the invariant amplitudes were fitted by polynomials of order 50 to 60 whereas in our fit to \( \Delta^+ \) the order does not exceed \( N = 5 \) (see section 4).

In the next step, the obtained values of \( \text{Re}C^+(q^2, 0) \) are fitted to a polynomial of second order in \( q^2 \)

\[
\text{Re}C^+(q^2, 0) = c_0 + c_1 q^2 + c_2 q^4, \quad (6)
\]

and extrapolated to the threshold \((q^2 = 0, t = 0)\). It is important to make a clear distinction between the coefficients in expansions (4) and (6). The first two coefficients \(c_0\) and \(c_1\) in (6) determine \( \text{Re}C^+(q^2, t = 0) \) and its derivative \( \frac{\partial \text{Re}C^+(q^2, t=0)}{\partial q^2} \) at threshold, while \( \bar{c}^+_{00} \) \((d^+_{00})\) and \( \bar{c}^+_{01} \) \((d^+_{01})\) in (4) are related to \( \bar{C}^+ \) and its forward slope \( \frac{\partial \bar{C}^+(\nu=0,t)}{\partial t} \) at the center of Mandelstam triangle \((t = 0, \nu = 0)\). The basic idea behind our method is just to connect the coefficients in eq. (6), obtained from the analysis of the CHAOS data, to the coefficients in (4) needed to calculate the pion-nucleon \( \sigma \) term. To this aim we exploit the analyticity of the pion-nucleon invariant amplitudes through applications of forward dispersion relations for the amplitude \( C^+ \) and its forward slope \([16]\).

Using the partial wave decomposition of \( \text{Re}C^+ \) and the effective range formula for partial waves \( f_{l\pm} = \text{Re} \left( \frac{T_{l\pm}}{q} \right) = q^{2l}(a_{l\pm} + b_{l\pm} q^2) \) close to threshold, one obtains the following approximation of the \( C^+ \) amplitude (formula A.3.60 in ref [16])

\[
\left(1 - \frac{t}{4m^2}\right) \frac{\text{Re}C^+(q^2, t)}{4\pi(1 + x)} = a_{0^+}^+ + \frac{1}{2} \left(2a_{1^+}^+ + a_{1^-}^- - \frac{a_{0^+}^+}{4m^2}\right) \cdot t
\]

\[
+ \left(b_{0^+}^+ + 2a_{1^+}^+ + a_{1^-}^- + \frac{a_{0^+}^+}{2m \cdot m_\pi}\right) \cdot q^2
\]

\[
+ \text{higher order terms in } t \text{ and } q^2.
\]

Here, \( a_{l\pm}^+ \) are isoscalar \( s \)-and \( p \)-wave scattering lengths, \( b_{0^+}^+ \) is the \( s \)-wave effective range parameter \([16]\) and \( x = \frac{m_\pi}{m} \). According to its definition the coefficient \( c_0 \) in expansion (6) is connected with \( a_{0^+}^+ \) by

\[
c_0 = 4\pi(1 + x)a_{0^+}^+.
\]
The third term on the right hand side is proportional to the first derivative of the $C^+$ amplitude with respect to $q^2$ at threshold:

\[
\frac{1}{4\pi(1+x)} \left. \frac{\partial Re C^+(q^2,0)}{\partial q^2} \right|_{q^2=0} = b_{0+}^+ + 2a_{1+}^+ + a_{1-}^+ + \frac{a_{0+}^+}{2m \cdot m_{\pi}}.
\]

\[
2a_{1+}^+ + a_{1-}^+ = \frac{c_1}{4\pi(1+x)} - b_{0+}^+ - \frac{a_{0+}^+}{2m \cdot m_{\pi}}.
\]

Taking the derivative of $Re C^+$ with respect to $t$ (forward slope), one obtains:

\[
\frac{1}{4\pi(1+x)} \left. \frac{\partial Re C^+(0,t)}{\partial t} \right|_{t=0} = \frac{1}{2} \left( 2a_{1+}^+ + a_{1-}^+ + \frac{a_{0+}^+}{4m^2} \right) \approx \frac{1}{2}(2a_{1+}^+ + a_{1-}^+). (8)
\]

From eqs. (7) and (8) one may derive an expression relating $\left. \frac{\partial Re C^+(q^2,t)}{\partial q^2} \right|_{q^2=0}$ to the forward slope $\left. \frac{\partial Re C^+(q^2,0)}{\partial t} \right|_{t=0}$:

\[
\left. \frac{\partial Re C^+(0,t)}{\partial t} \right|_{t=0} = \frac{c_1}{2} - 2\pi(1+x) \left( b_{0+}^+ + \frac{a_{0+}^+}{2m \cdot m_{\pi}} \right).
\]

The corresponding forward slope of the amplitude $\bar{C}^+$ is obtained by subtracting the pseudovector Born term:

\[
\left. \frac{\partial Re \bar{C}^+(0,t)}{\partial t} \right|_{t=0} = \left. \frac{\partial Re C^+(0,t)}{\partial t} \right|_{t=0} - \left. \frac{\partial C^+_{N\nu}(0,t)}{\partial t} \right|_{t=0}.
\]

To get the final expression for the coefficient $\bar{c}_{01}^+$, we switch to the Mandelstam variable $\nu$ as an argument of the $C^+$ amplitude. In order to determine the coefficient $\bar{d}_{01}^+ = \frac{c_{01}^+}{c_1} = \left. \left( \frac{\partial \bar{C}^+_{(\nu=0,t)}}{\partial t} \right) \right|_{t=0}$ one adds and subtracts the forward slope at threshold

\[
\left. \frac{\partial Re \bar{C}^+(0,t)}{\partial t} \right|_{t=0} = \left. \frac{\partial Re C^+(\nu_{th},t)}{\partial t} \right|_{t=0} - \left. \frac{\partial C^+_{N\nu}(\nu_{th},t)}{\partial t} \right|_{t=0} + \left. \left( \frac{\partial Re \bar{C}^+(0,t)}{\partial t} \right) \right|_{t=0} - \left. \left( \frac{\partial Re \bar{C}^+(\nu_{th},t)}{\partial t} \right) \right|_{t=0}
\]

\[
\bar{c}_{01}^+ = \frac{c_1}{2} - 2\pi(1+x) \left( b_{0+}^+ + \frac{a_{0+}^+}{2m \cdot m_{\pi}} \right) + \Delta_1,
\]

(9)
where

$$\Delta_1 = - \frac{\partial C^+_{Npv}(\nu_{th}, t)}{\partial t} \bigg|_{t=0} + \left( \frac{\partial \text{Re}C^+(0, t)}{\partial t} \bigg|_{t=0} - \frac{\partial \text{Re}C^+(\nu_{th}, t)}{\partial t} \bigg|_{t=0} \right).$$

The Born term $C^+_{Npv}$ and its derivative are explicitly known (formula A.8.1 in [16])

$$C^+_{Npv}(\nu, t) = - \frac{g^2}{4m^3} \left( \frac{m^2_\pi - \frac{1}{4m^2}}{\nu^2 - \nu_B^2} \right) (1 - \frac{t}{4m^2}),$$

$$C^+_{N}(\nu, t) = C^+_{Npv}(\nu, t) - \frac{g^2}{m},$$

$$\left. \frac{\partial C^+_{Npv}(\nu, t)}{\partial t} \right|_{t=0} = - \frac{g^2}{4m^3} \frac{\omega}{\omega^2 - \omega_B^2} \left( \omega - \frac{m^2_\pi}{\omega + \omega_B} \right).$$

$\omega$ is the total energy of the pion in the laboratory frame related to variable $\nu$ by $\nu = \omega + \frac{t}{4m}$, $\nu_{th} = m_\pi + \frac{t}{4m}$, $\omega_B = -\frac{m^2_\pi}{2m}$, ( for $t = 0$: $\nu = \omega$, $\nu_{th} = m_\pi$). $C^+_{N}(\nu, t)$ is a pseudoscalar Born term which appears in dispersion relations and $g$ is the $\pi N$ pseudoscalar coupling constant, which is related to the pseudovector coupling constant $f$ by $\frac{g^2}{4\pi} = \frac{4m^2_\pi}{m^2_\pi} f^2$. In our calculations we use $f^2=0.076$ [37].

From dispersion relations for the forward slope of $C^+$ [16] one obtains the following expression:

$$\Delta_1 = - \frac{\partial \text{Re}C^+_{Npv}(\nu_{th}, t)}{\partial t} \bigg|_{t=0}$$

$$- \frac{2m^2_\pi}{\pi} \int_{m_\pi}^{\infty} \frac{d\omega'}{\omega' (\omega'^2 - m^2_\pi)} \left. \frac{\partial C^+ (\omega' + \frac{t}{4m}, t)}{\partial t} \right|_{t=0}$$

$$- \frac{m_\pi}{2m^2_\pi} \int_{m_\pi}^{\infty} \frac{d\omega'}{\omega'^2 (\omega' + m_\pi)^2} \text{Im} C^+(\omega', 0)$$

$$\Delta_1 = - \left. \frac{\partial C^+_{Npv}(\nu_{th}, t)}{\partial t} \right|_{t=0} - I_1 - I_2,$$  \hspace{1cm} (10)
where $I_1$ and $I_2$ are short for the first and the second integral, respectively. Both integrals are fast converging and may be accurately evaluated using results from existing phase shift analyses or, by virtue of the optical theorem, from existing data for total cross sections.

Forward dispersion relations relate the amplitude $\bar{C}^+(0)$ at the center of the Mandelstam triangle and at the threshold $\bar{C}^+(m_\pi)$ by the dispersion integral [16] (with the variable $t$ omitted as an variable):

$$\Delta_2 = \bar{C}^+(0) - \bar{C}^+(m_\pi) = -\frac{2m^2_\pi}{\pi} \int_{m_\pi}^\infty \frac{ImC^+(\omega')}{\omega'(\omega'^2 - m^2_\pi)} d\omega',$$

which leads to:

$$\bar{C}^+(0) = \bar{C}^+(m_\pi) + \Delta_2 = C^+(m_\pi) + 1.88f^2 + \Delta_2,$$

and finally:

$$\bar{c}^{+}_{00} = c_0 + 1.88f^2 + \Delta_2.$$  (12)

Eqs. (9) and (12) establish the required relations between the coefficients in expansions (6) and (4).

4. Results

The CHAOS data [8] consist of differential cross sections grouped in two angular regions. The first region is in the forward scattering hemisphere and is defined by values of the Mandelstam variable $t \geq -0.01$ GeV$^2$. The second group, defined by $t < -0.01$ GeV$^2$, comprises all larger angles. We have performed two separate analyses of the CHAOS data. In the first analysis we have included data at forward angles only. Values of $\Delta^+(t)$ obtained from the experimental data have been fitted to a polynomial of third order in $t$. In the second analysis, all data were included and fitted to a polynomial of order five. In such a way two sets of values of $ReC^+$ were obtained from the CHAOS data and averaged. This way we deliberately increased the weight of the small angle data, keeping in mind the design features of the CHAOS detector. The resulting values of $ReC^+$ are nuclear. Hadronic values were calculated applying electromagnetic corrections according to the Nordita procedure from ref. [27] and used as an input for determination of the coefficients $\bar{c}^{+}_{00}$ and $\bar{c}^{+}_{01}$. 
Forward dispersion relations (FDR) are an essential part of our extrapolation of the $C^+$ amplitude to the threshold. In the first step, values of $\text{Re}C^+$, obtained from the CHAOS data, combined with the once subtracted dispersion relations for the $C^+$ amplitude, are used to calculate its value at threshold. The FDR for the amplitude $C^+$, subtracted at the threshold, read [16]:

$$C^+(m_\pi) = \text{Re}C^+(\omega) - C^+_N(\omega) + C^+_N(m_\pi) - I^+$$  \hspace{1cm} (13)

with

$$I^+ = \frac{2k^2}{\pi} \int_{m_\pi}^{\infty} \frac{\text{Im}C^+(\omega')}{(\omega'^2 - \omega^2)(\omega'^2 - m^2_\pi)} \omega' d\omega',$$

where $k$ is the pion lab momentum and $C^+_N$ the nucleon Born term introduced above. For each energy where $\text{Re}C^+$ is determined from the data, one calculates the dispersion integral and, using equation (13), finds a corresponding subtraction constant $C^+(m_\pi)$. The average value of the five subtraction constants that we obtain this way is $C^+(m_\pi) = (-0.139 \pm 0.019)m_\pi^{-1}$. We stress that the obtained average value is not our final result for $c_0$ as defined in (6). Rather, it is used as part of the input (square in Fig.1) when calculating the final values of the amplitude and its forward slope at threshold. Partial wave analyses do not give errors of partial waves so that the error of the integral in (13) may not be given and errors quoted in our results are due only to the numerical procedure. Our estimates show, however, that errors of the subtraction constant due to the uncertainty of dispersion integrals are negligible compared to the experimental errors.

The dispersion integrals in (10), (11), and (13) may be accurately evaluated using available scattering data. In the energy region where results from phase shift analyses exist ($k \leq k_{\text{max}}$) imaginary parts are calculated from partial waves (GWU/VPI: $k_{\text{max}} = 2.6 \text{ GeV/c}$; Ka84: $k_{\text{max}} = 6.0 \text{ GeV/c}$). Tables of total $\pi^\pm p$ cross sections, needed to calculate $\text{Im}C^+$, are available up to lab momenta of $k = 340 \text{ GeV/c}$ [30]. Parametrization of the total cross sections at high energies are also available [36]. The integrals in (11) and (13) are fast converging. More than 97% of their values are due to contributions below $k = 2.6 \text{ GeV/c}$, so that uncertainties from the high energy parts of the integrals may be neglected. Using the GWU/VPI solution Fa08 [37] one obtains a value of $\Delta_2 = -1.381m_\pi^{-1}$. $\Delta_1$ has been calculated using the same
input. Our evaluations, using several solutions (Ka84, Fa02, GW06, Fa08), show that the integral $I_1$ is saturated already at $\omega_{max}$ which corresponds to the highest lab momentum in the GWU/VPI partial wave solution. Using the discrepancy method we found that contributions from higher energies to the integral $I_1$ do not exceed one percent of its value. Hence, uncertainties from high energies may be neglected. Using results from the GWU/VPI solution Fa08 and tables for total cross sections at high energies, a value of $\Delta_1 = -133.8 \text{ GeV}^{-3} = -0.364 \text{ } m_\pi^{-3}$ was obtained.

![Figure 1: Data points: Real parts of the forward $C^+$ amplitude from the CHAOS experiment [8] (solid circles) and their average subtraction constant $C^+(m_\pi)$ (solid square) as described in the text. Solid curve: Best fit parabola to the six points.](image)

Fitting a parabola to the five data points for $ReC^+$ and the average subtraction constant $C^+(m_\pi)$, obtained as described above, yields the first two coefficients in the expansion (6):

\[
\begin{align*}
    c_0 &= (-0.140 \pm 0.013) m_\pi^{-1}, \\
    c_1 &= (2.146 \pm 0.187) m_\pi^{-3},
\end{align*}
\]

which represent essential steps in our analysis. As Fig.1 suggests both coefficients $c_0$ and $c_1$ are well defined as a result of the small errors of $ReC^+$. 

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The only parameter left to be determined is the $s$-wave effective range parameter $b_{0+}^+$. One may follow the authors of ref. [31] and use the Karlsruhe value $b_{0+}^+ = (-44 \pm 7) \cdot 10^{-3} m_{\pi}^{-3}$ from ref. [16]. The very same value was obtained from partial wave relations derived from the fixed $t$ dispersion relations [32]. Due to the fact that partial waves from partial wave relations are strictly consistent with analyticity, the method allows a reliable determination of threshold parameters. Unfortunately, like some other programs from the Karlsruhe group, Koch’s program for evaluation of partial wave relations was lost. To our knowledge there are no recent determinations of the $s$-wave effective range parameters with such a degree of sophistication. In [33] a system of Roy-Steiner equations for $\pi N$ scattering was derived. Numerical evaluation of these equations would greatly improve our knowledge about the $\pi N$ low energy parameters.

Until results from a new evaluation of partial wave relations become available, we rely on a simple extrapolation of the $s$-wave to threshold. Applying the effective range approximation and using the GWU/VPI partial wave solution FA08 up to $k = 80.0$ MeV/c, we obtain $b_{0+}^+ = (-52 \pm 4) \cdot 10^{-3} m_{\pi}^{-3}$. The value $b_{0+}^+ = (-50 \pm 4) \cdot 10^{-3} m_{\pi}^{-3}$, which we use in the present analysis, is a weighted average of the Karlsruhe value and a value obtained from the FA08 partial wave solution.

Inserting the values of $c_0$ and $\Delta_2$ into (12), we obtain $\tilde{c}_{00}^+ = (-1.378 \pm 0.013)m_{\pi}^{-1}$. The values of $\Delta_1$ and $b_{0+}^+$ inserted into (9) lead to $\tilde{c}_{01}^+ = (1.075 \pm 0.098)m_{\pi}^{-3}$. With these results for $\tilde{c}_{00}^+$, $\tilde{c}_{01}^+$, and $\tilde{d}_{00}^+$, eq. (4) yields $\Sigma_d = (47.2 \pm 12.2)$ MeV. Adding the curvature term, we obtain our final result for the $\pi N \sigma$-term as derived from the CHAOS data:

$$\Sigma = (59 \pm 12) \text{ MeV} \quad \sigma_{\pi N} = (44 \pm 12) \text{ MeV}. \quad (15)$$

As stated before, the quoted errors are mainly due to errors having their origin in the experimental data. Compared to these, errors due to uncertainties of dispersion integrals, as shown by our calculations, are negligible.

5. Discussion

The result obtained for the coefficient $c_0$ corresponds to a $s$-wave scattering length $a_{0+}^+ = (-9.7 \pm 0.9) \cdot 10^{-3} m_{\pi}^{-1}$ which is comparable to the old Karlsruhe result [16]. It differs, however, significantly (and even in sign) from the results based on the PSI precision experiments on pionic hydrogen and
deuterium. These yielded values of the scattering length that developed over the years from \((-0.1^{+0.9}_{-2.1}) \cdot 10^{-3}m_\pi^{-1}\) \([41]\) to \((7.6 \pm 3.1) \cdot 10^{-3}m_\pi^{-1}\) \([34]\). The latter value, however, should not be compared to our result which was obtained from data in charged pion channels alone. Instead, applying corrections for isospin violation as may be obtained from Table 6 of ref. \([35]\), a value of \((-1.0 \pm 0.9) \cdot 10^{-3}m_\pi^{-1}\) is suggested \([42]\) for comparison with our result.

The situation appears much clearer for p-waves. The combination \(2a_1^+ + a_1^-\) has been stable during the last three decades. Our result \(2a_1^+ + a_1^- = (199 \pm 14) \cdot 10^{-3}m_\pi^{-3}\), obtained from (7) using our values for coefficient \(c_1\), effective range \(b_{0+}\) and scattering length \(a_{0+}\), is in good agreement with those obtained in partial wave analyses performed during the last several years \([37, 38, 39]\), e.g. \(2a_1^+ + a_1^- = (203.9 \pm 1.9) \cdot 10^{-3}m_\pi^{-1}\) \([39]\).

In view of the spread among the various isoscalar scattering lengths mentioned above, we performed two tests to study the sensitivity of our determination of \(\sigma_{\pi N}\) to a variation of the scattering length. By the first test we demonstrate that it is a frequent, but naive misconception that a larger scattering length will necessarily result in an increased \(\sigma_{\pi N}\). This is only so if one considers the contribution of \(\tilde{c}_{00}^+\) to \(\sigma_{\pi N}\) alone, as may be easily concluded from eqs. (4) and (12). However, there is also a strong dependence of \(\sigma_{\pi N}\) on the coefficient \(\tilde{c}_{01}^+\) i.e. on the forward slope of \(\tilde{C}^+\) at the center of the Mandelstam triangle, \(t = 0, \nu = 0\).

This is exactly what we observe when we arbitrarily adopt the recommended \([42]\) higher value \(a_{0+}^+ = (-1.0 \pm 0.9) \cdot 10^{-3}m_\pi^{-1}\) and obtain the \(C^+\) amplitude for \(q^2 = 0\) in Fig.1. Combined with the five data points from the CHAOS experiment we obtain best fit parameters \(c_0 = (-0.016 \pm 0.014)m_\pi^{-1}\) and \(c_1 = (1.400 \pm 0.184)m_\pi^{-3}\) in (6). Hence the higher value of \(c_0\) entails a smaller slope parameter \(c_1\) leading eventually to a ridiculously small \(\sigma_{\pi N} = (7 \pm 11)\text{ MeV}\). Needless to say that this procedure is inconsistent with our use of forward dispersion relations as described in section 4. Rather, it serves the purpose to show that it would be a gross oversimplification to suspect the small isoscalar scattering length as the sole origin of our low sigma-term. In addition, the then resulting value of \(2a_1^+ + a_1^- = (149 \pm 13) \cdot 10^{-3}m_\pi^{-3}\) would be in disagreement with recently obtained values \([37, 38, 39]\) unless a large negative value \(b_{0+}^+ \approx -100 \cdot 10^{-3}m_\pi^{-3}\) for the s-wave effective range was allowed.

In a second test we assume that the low value of our scattering length results from a problem with the normalisation of the CHAOS cross sections.
Hence, we retain the shape, notably the slope, of the $C^+$ curve in Fig. 1, but shift all values by the same amount of $+0.013 \, m^{-1}$ in order to get the intercept with the vertical axis at $C^+ = c_0 = (-0.014 \pm 0.013)m^{-1}$, which corresponds to the recommended value \cite{42} of $a_{0+}^+ = (-1.0 \pm 0.9) \cdot 10^{-3} m^{-1}$. As a result we now obtain $c_1 = (2.146 \pm 0.187) m^{-3}$ and $\sigma_{\pi N} = (51 \pm 12) MeV$.

In order to generate the applied upward shift of the CHAOS data points in Fig. 1 one has to decrease the differential cross sections for $\pi^+ p$ scattering and/or increase those for $\pi^- p$ cross sections suitably. (This follows from the definition of $ReC^+$ in terms of $\pi^\pm p$ differential cross sections in Sect. 3). Quantitatively the required modifications are compatible (within 2 standard deviations) with the systematic errors quoted in \cite{8}. Therefore, we note that the errors of the CHAOS experiment do not allow a determination of the s-wave scattering length. However, it is very comforting to observe that the value of $\sigma_{\pi N}$ is robust: it is only by half a standard deviation away from our result in eq.(15).

6. Conclusions

Our value of the $\sigma$-term $\sigma_{\pi N} = (44 \pm 12)$ MeV, based on an analysis of the CHAOS data, is at the lower end of the range of published values. Given the systematic uncertainties of the CHAOS experiment one could not expect a precision determination of its value. Using an analysis which respects the analytic properties of the $\pi N$ amplitudes we were, however, able to confirm quantitatively the conjecture of Denz et al. \cite{8} that the sigma term is small, comparable to the canonical value $\sigma = 49$ MeV of Koch et al. \cite{10, 21} which remarkably is based on data from the pre-meson-factory era. This stability is partly owed to the constraints imposed by our use of dispersion relations.

Our result agrees, within the (combined) 1$\sigma$-errors, with those from recent determinations in the framework of covariant baryon chiral perturbation theory \cite{25, 40, 44} which range from 41 MeV to 59 MeV. Moreover, it is perfectly compatible with recent lattice QCD calculations \cite{5, 43} which yielded $\sigma_{\pi N} = (47 \pm 9)$ MeV. However, our result is about 20 MeV smaller than that from the phenomenological extraction by the GWU-TRIUMF group \cite{12}. With no experimental facility available to improve the $\pi N$ data base in the foreseeable future we considered it important to challenge that frequently adopted large value.
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