LONG TIME BEHAVIOR FOR A SEMILINEAR HYPERBOLIC EQUATION  
WITH ASYMPTOTICALLY VANISHING DAMPING TERM AND CONVEX  
POTENTIAL

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Abstract Recently, A. Cabot and P. Frankel studied the long time behavior of solutions to the following semilinear hyperbolic equation:

\[
\frac{d^2 u}{dt^2}(t) + \gamma(t) \frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0, 
\]

where \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \), the damping term, is a decreasing function, \( f \) is the gradient of a given convex function defined on an a real Hilbert space \( V \), and \( A : V \to V' \) is a linear and continuous operator assumed to be symmetric, monotone and semi-coercive. They proved that if the damping term \( \gamma(t) \) behaves like \( \frac{K}{t^\alpha} \) as \( t \to +\infty \), for some \( K > 0 \) and \( \alpha \in [0, 1[ \), then every bounded solution \( u \) to the equation (E) (i.e. \( u \in L^\infty(0, +\infty; V) \)) converges weakly in \( V \) as \( t \to +\infty \) toward a solution to the stationary equation \( Av + f(v) = 0 \). They left open the question: Does convergence still hold without assuming the boundedness of the solution? In this paper, we give a positive answer to this question. Our approach relies on precise estimates on the decay rates for the energy function along trajectories of (E).

keywords: Dissipative hyperbolic equation, asymptotically small dissipation, asymptotic behavior, energy function, convex function.

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1. Introduction and statement of results

Throughout this paper, we follow the same notations as in the paper \[5\]. Let \( H \) be a real Hilbert space with inner product and norm respectively denoted by \( \langle ., . \rangle \) and \( |.| \). Let \( V \) be a real Hilbert space such that \( V \hookrightarrow H \hookrightarrow V' \) with continuous and dense injections, where \( V' \) is the dual space of \( V \). Let \( \gamma : \mathbb{R}^+ \to \mathbb{R}^+ \) be a decreasing function which belongs to the space \( W^{1,1}_{\text{loc}}(\mathbb{R}^+; \mathbb{R}^+) \). Let \( A : V \to V' \) be a linear and continuous operator such that the associated
bilinear form $a : V \times V \to \mathbb{R}$ $(u, v) \mapsto \langle Au, v \rangle_{V', V}$ is symmetric, positive and satisfies the following property:

\[(1.1) \quad \exists \lambda \geq 0, \mu > 0 : \forall v \in V, a(v, v) + \lambda |v|^2 \geq \mu \|v\|^2._V.\]

Let $f : V \to V'$ be a continuous function deriving from a convex potential i.e, there exists a $C^1$ convex function $F : V \to \mathbb{R}$ such that:

$$F'(u)(v) = \langle f(u), v \rangle_{V', V}.$$  

It is clear that the function $\phi : V \to \mathbb{R}$ defined by:

$$\phi(v) = \frac{1}{2}a(v, v) + F(v)$$  

is $C^1$, convex and satisfies the following property:

$$\forall u, v \in V, \phi'(u)(v) = \langle Au + f(u), v \rangle_{V', V}.$$  

We assume moreover that the function $\phi$ is bounded from below and that the set

$$\arg \min \phi = \{v \in V : \phi(v) = \min \phi\}$$  

is not empty. Notice that, since $\phi$ is convex, $\arg \min \phi$ coincides with the set $S = \{v \in V : Av + f(v) = 0\}$ of critical points of $\phi$.

In this paper, our purpose is to investigate the asymptotic behavior of the semilinear hyperbolic equation:

\((E)\)

$$\frac{d^2u}{dt^2}(t) + \gamma(t)\frac{du}{dt}(t) + Au(t) + f(u(t)) = 0, \quad t \geq 0.$$  

This equation and its ODE version (called the heavy ball with friction) have been studied by many authors under various conditions on the damping and potential terms, see for instance, [1], [2], [3], [4], [5], [6], [7], [10], and references there in.

By a solution of (E) we mean a function $u : \mathbb{R}^+ \to H$ which belongs to the class

$$W^{1,1}_{loc}(\mathbb{R}^+, V) \cap W^{2,1}_{loc}(\mathbb{R}^+, H)$$  

and satisfies the equation (E) for almost every $t \geq 0$. A solution $u$ to (E) is said to be bounded if it belongs moreover to the space $L^\infty(0, +\infty; H)$.

In [5], Cabot and Frankel proved the following interesting convergence result:

**Theorem 1.1** (A. Cabot and P. Frankel). *Assume that there exist $\alpha \in]0, 1[\, and $K_1, K_2 > 0$ such that for every $t \geq 0$, $\frac{K_1}{(t + 1)^{1+\alpha}} \leq \gamma(t) \leq \frac{K_2}{(t + 1)^{1+\alpha}}$. Let $u$ be a bounded solution to (E). Then there exists $u_\infty \in \arg \min \phi$ such that $u(t)$ converges weakly in $V$ to $u_\infty$ as $t \to +\infty$.*
An open question left in the paper [5] was whether the condition $u \in L^\infty(0, +\infty; H)$ is really necessary in the previous theorem (see Remark 3.15 in [5]). In the present paper, we will show, without assuming the boundedness of the solution, that the weak convergence result still holds in the case $\alpha \in \left[0, \frac{1}{2}\right]$ and in the case $\alpha \in \left]\frac{1}{2}, 1\right]$ up to a supplementary assumption on the derivative of the damping term $\gamma$. Such assumption is satisfied, for instance, by functions of the form $\frac{K}{(1+t)^\alpha}$ where $K > 0$. Moreover, in each case, we will establish an estimate on the rate of the decay for the energy function on the trajectories of (E). More precisely, we will prove the following theorems:

**Theorem 1.2.** Assume that there exist $\alpha \in [0, \frac{1}{2}]$ and $K > 0$ such that for every $t \geq 0$, $\gamma(t) \geq \frac{K}{(1+t)^\alpha}$. Then for every solution $u$ to (E) there exists $u_\infty \in \arg \min \phi$ such that $u(t)$ converges weakly in $V$ to $u_\infty$ as $t \to +\infty$. Moreover,

$$\frac{1}{2} \left|\frac{du}{dt}(t)\right|^2 + \phi(u(t)) - \min \phi = o\left(\frac{1}{t}\right) \text{ as } t \to +\infty.$$

**Theorem 1.3.** Assume that there exist $\alpha \in [0, 1]$, $K > 0$ and $t_0 \geq 0$ such that $\gamma(t) \geq \frac{K}{(1+t)^\alpha}$ for every $t \geq 0$ and $\gamma'(t) \leq -\alpha \frac{\gamma(t)}{1+t}$ for almost every $t \geq t_0$. Let $u$ be a solution to (E), then $u(t)$ converges weakly in $V$ as $t \to +\infty$ toward some $u_\infty \in \arg \min \phi$. Moreover, for every $\bar{\alpha} < \alpha$,

$$\frac{1}{2} \left|\frac{du}{dt}(t)\right|^2 + \phi(u(t)) - \min \phi = o\left(\frac{1}{t^{1+\bar{\alpha}}}\right) \text{ as } t \to +\infty.$$

2. **Proof of Theorem 1.2 and Theorem 1.3**

We will first prove some preliminary results under the following general hypothesis on the damping term $\gamma$:

$$\exists K > 0 \text{ and } \alpha \in [0, 1]; \forall t \geq 0, \ \gamma(t) \geq \frac{K}{(1+t)^\alpha}. \tag{2.1}$$

These results will be useful in the proofs of Theorem 1.2 and Theorem 1.3. Let $u$ be a solution to the equation (E). Define the energy function

$$\mathcal{E}(t) = \frac{1}{2} \left|\frac{du}{dt}(t)\right|^2 + \phi(u(t)) - \min \phi, \ t \geq 0. \tag{2.2}$$

A simple computation yields

$$\frac{d\mathcal{E}}{dt}(t) = -\gamma(t) \left|\frac{du}{dt}(t)\right|^2, \ a.e. \ t \geq 0.$$

Thus the function $\mathcal{E}$ is decreasing and converges as $t \to +\infty$ to some real number $\mathcal{E}_\infty$ which will be identified later. Moreover

$$\int_0^{+\infty} \gamma(t) \left|\frac{du}{dt}(t)\right|^2 dt < \infty \tag{2.3}$$
\(\forall t \geq 0, \mathcal{E}(t) - \mathcal{E}_\infty = \int_t^{+\infty} \gamma(s) \left| \frac{du}{dt}(s) \right|^2 ds.\)

Let \(v\) be a fixed point in \(\arg \min \phi\) and define the function \(p(t) = \frac{1}{2} |u(t) - v|^2, t \geq 0.\) Proceeding as in the proof of Proposition 3.5 in [5], one can easily prove that for almost every \(t\) in \(\mathbb{R}^+\) we have

\[\dot{p}(t) + \gamma(t)p(t) \leq \frac{3}{2} \left| \frac{du}{dt}(t) \right|^2 - \mathcal{E}(t).\]

Multiplying the last inequality by \(\lambda_r(t) = (1 + t)^r, r \in \mathbb{R},\) and integrating by parts over the interval \([0, T], T > 0,\) we easily obtain after simplification

\[\int_0^T \lambda_r(t)\mathcal{E}(t) dt \leq \frac{3}{2} \int_0^T \lambda_r(t) \left| \frac{du}{dt}(t) \right|^2 dt - \lambda_r(T)\dot{p}(T) + \left[ \lambda_r' - (\gamma \lambda_r) \right](T)p(T)
\]

\[+ \int_0^T \left[ (\lambda_r \gamma)' - \lambda_r'' \right] (t)p(t)dt + C_r\]

(2.5)

where \(C_r = \dot{p}(0) + (\gamma(0) - r)p(0).\)

Since \(\gamma\) satisfies (2.1), \(\lambda_r'(T) = \circ [(\gamma \lambda_r)(T)]\) as \(T \to +\infty.\) Thus, there exists \(T_r \geq 0\) such

\(\forall t \geq T_r, \lambda_r'(T) - (\gamma \lambda_r)(T) \leq -\frac{1}{2} (\gamma \lambda_r)(T).\)

On the other hand, thanks to Cauchy-Schwarz inequality, we have

\[|\dot{p}(T)| \leq \left| \frac{du}{dt}(t) \right| |u(t) - v|\]

\[\leq 2 \sqrt{\mathcal{E}(T)} \sqrt{p(T)}\]

(2.7)

Inserting estimates (2.6) and (2.7) into (2.5) and using hypothesis (2.1) and the following elementary inequality

\[\forall a > 0 \forall x, b \in \mathbb{R}, \ bx - ax^2 \leq \frac{b^2}{4a},\]

with \(x = \sqrt{p(T)},\) we deduce that for every \(T \geq T_r\) we have

\[\int_0^T \lambda_r(t)\mathcal{E}(t) dt \leq \frac{3}{2} \int_0^T \lambda_r(t) \left| \frac{du}{dt}(t) \right|^2 dt + \frac{2}{K} \lambda_{r+\alpha}(T)\mathcal{E}(T)
\]

\[+ \int_0^T \left[ (\lambda_r \gamma)'(t) - \lambda_r''(t) \right] p(t)dt + C_r.\]

(2.8)

Let us notice that if \(r \leq 0, (\lambda_r \gamma)'(t) - \lambda_r''(t) \leq 0\) a.e. on \(\mathbb{R}^+\) (since the function \(\lambda_r \gamma\) is decreasing and the function \(\lambda_r\) is convex); then, in the case where \(r \leq 0, (2.8)\) becomes

\[\forall T \geq T_r, \int_0^T \lambda_r(t)\mathcal{E}(t) dt \leq \frac{3}{2} \int_0^T \lambda_r(t) \left| \frac{du}{dt}(t) \right|^2 dt + \frac{2}{K} \lambda_{r+\alpha}(T)\mathcal{E}(T) + C_r.\]

(2.9)
Proof. Let us now prove the following crucial lemma: we get
\[ \forall T \geq T_{-\alpha}, \int_0^T \lambda_{-\alpha}(t) \mathcal{E}(t) dt \leq \frac{3}{2K} \int_0^{+\infty} \gamma(t) \left| \frac{du}{dt}(t) \right|^2 dt + \frac{2}{K} \mathcal{E}(0) + C_{-\alpha}, \]
which implies that
\[ \int_0^{+\infty} \lambda_{-\alpha}(t) \mathcal{E}(t) dt < \infty. \]
Recalling that \( \alpha < 1 \), we then deduce that the limit \( \mathcal{E}_\infty \) of \( \mathcal{E}(t) \) as \( t \to +\infty \) is equal to zero.

Let us now prove the following crucial lemma:

Lemma 2.1. Let \( r \in \mathbb{R} \backslash \{-1\} \). If \( \int_0^{+\infty} \lambda_r(t) \mathcal{E}(t) dt < \infty \) then \( \mathcal{E}(t) = o(1/t^{1+r}) \) as \( t \to +\infty \) and
\[ \int_0^{+\infty} \lambda_{r+1-\alpha}(t) \left| \frac{du}{dt}(t) \right|^2 dt < \infty. \]

Proof. Since the energy function \( \mathcal{E} \) is decreasing, we have
\[ (2.11) \quad \mathcal{E}(t) \int_{\frac{1}{2}}^t (1 + s)^r ds \leq \int_{\frac{1}{2}}^{+\infty} \lambda_r(s) \mathcal{E}(s) ds. \]
A simple computation yields \( \int_{\frac{1}{2}}^t (1+s)^r ds \simeq M_r t^{r+1} \) for \( t \) large enough where \( M_r \) is a nonnegative constant depending only on \( r \). Inserting this last estimate into (2.11), we get \( \lim_{t \to +\infty} t^{1+r} \mathcal{E}(t) = 0 \). On the other hand, by using equality (2.4), the fact that \( \mathcal{E}_\infty = 0 \), and Fubini Theorem, we obtain
\[ \int_0^{+\infty} \lambda_r(t) \mathcal{E}(t) dt = \frac{1}{1+r} \int_0^{+\infty} \gamma(s) \left[ (1 + s)^{r+1} - 1 \right] \left| \frac{du}{dt}(s) \right|^2 ds, \]
which clearly implies that \( \int_0^{+\infty} \lambda_{r+1-\alpha}(t) \left| \frac{du}{dt}(t) \right|^2 dt < \infty \) since \( \int_0^{+\infty} \gamma(s) \left| \frac{du}{dt}(s) \right|^2 ds < \infty \) and \( \gamma(s) \geq \frac{K}{(1+s)^r} \).

Proof of Theorem 1.2. In view of (2.10), Lemma 2.1 implies \( \mathcal{E}(t) = o(t^{\alpha-1}) \) as \( t \to +\infty \) and \( \int_0^{+\infty} \lambda_{1-2\alpha}(t) \left| \frac{du}{dt}(t) \right|^2 dt < \infty \). Hence by letting \( r = 0 \) in (2.9), we get, for \( T \) large enough,
\[ \int_0^T \mathcal{E}(t) dt \leq \frac{3}{2} \int_0^T \left| \frac{du}{dt}(t) \right|^2 dt + o(T^{2\alpha-1}) + C_0. \]
Therefore, by letting \( T \to +\infty \) and using the assumption \( \alpha \leq \frac{1}{2} \), we get
\[ \int_0^{+\infty} \mathcal{E}(t) dt \leq \frac{3}{2} \int_0^{+\infty} \lambda_{1-2\alpha}(t) \left| \frac{du}{dt}(t) \right|^2 dt + C_0, \]
Hence, by using once again Lemma 2.1, we deduce that \( \mathcal{E}(t) = o(1/t) \) as \( t \to +\infty \) and that \( \int_0^{+\infty} \lambda_{1-\alpha}(t) \left| \frac{du}{dt}(t) \right|^2 dt < \infty \) which implies, since \( \alpha \leq \frac{1}{2} \), that \( \int_0^{+\infty} (1 + t)\alpha \left| \frac{du}{dt}(t) \right|^2 dt < \infty \). Therefore we deduce the weak convergence of \( u(t) \) in \( V \) as \( t \to +\infty \) from the following lemma which is implicitly proved in [3] (see the proofs of Theorem 3.7 and Theorem 3.13) by adapting a classical arguments originated by F. Alvarez [1] based on the famous Opial’s lemma [9].
Lemma 2.2. Assume (2.7). Let $u$ be a solution to (E). If $\int_0^\infty (1 + t)^\alpha |\frac{du}{dt}(t)|^2 dt < \infty$ then $u(t)$ converges weakly in $V$ as $t \to +\infty$ to some $u_\infty \in \text{arg min } \phi$.

Now we are going to prove our second main theorem. Hence, hereafter, we assume that the function $\gamma$ satisfies (2.1) and the hypothesis on its derivative given in Theorem 1.3. First we will prove the following key lemma:

Lemma 2.3. If $\nu < 2\alpha - 1$ and $\int_0^\infty \lambda_\nu(t)\mathcal{E}(t)dt < +\infty$ then $\int_0^\infty \lambda_{\nu+1}(t)\mathcal{E}(t)dt < +\infty$.

Proof of Lemma 2.3. Let $\nu < 2\alpha - 1$ such that $\int_0^\infty \lambda_\nu(t)\mathcal{E}(t)dt < +\infty$. According to Lemma 2.1 we have:

$$\mathcal{E}(t) = o\left(\frac{1}{t^{1+\nu}}\right)\text{ as } t \to +\infty$$  \hspace{1cm} (2.12)

and

$$\int_0^{\infty} \lambda_{1+\nu-\alpha}(t) \left|\frac{du}{dt}(t)\right|^2 dt < \infty.$$  \hspace{1cm} (2.13)

Let $\rho = 1 + \nu - \alpha$. Using the hypothesis on the damping term $\gamma$ and the fact that $\rho < \alpha$, we find that for almost every $t \geq t_0$ we have

$$\left[(\lambda_\rho)^\prime - \lambda_\rho''\right](t) \leq (\rho - \alpha)\lambda_{\rho-1}(t)\gamma(t) - \rho(\rho - 1)\lambda_{\rho-2}(t)$$

$$\leq (\rho - \alpha)K\lambda_{\rho-\alpha-1}(t) - \rho(\rho - 1)\lambda_{\rho-2}(t)$$

$$\simeq (\rho - \alpha)K\lambda_{\rho-\alpha-1}(t) \text{ as } t \to +\infty.$$  \hspace{1cm} (2.14)

The last inequality implies that there exists $\tau_0 \geq \max(T_0, t_0)$ such that for almost every $t \geq \tau_0$ we have $\left[(\lambda_\rho)^\prime - \lambda_\rho''\right](t) \leq 0$. Inserting this last inequality into (2.8) with $r = \rho$, we obtain

$$\int_0^T \lambda_\rho(t)\mathcal{E}(t)dt \leq \frac{3}{2} \int_0^T \lambda_\rho(t) \left|\frac{du}{dt}(t)\right|^2 dt + \frac{2}{K}\lambda_1\lambda_{\nu}(T)\mathcal{E}(T) + A_\rho \text{ for a.e. } T \geq \tau_0,$$  \hspace{1cm} (2.15)

where $A_\rho = C_\rho + \int_0^{\tau_0} \left[(\lambda_\rho)^\prime(t) - \lambda_\rho''(t)\right]p(t)dt$. Hence, by using estimates (2.12), (2.13) and by letting $T \to +\infty$ in (2.15), we deduce that $\int_0^\infty \lambda_\rho(t)\mathcal{E}(t)dt < \infty$.

Now we are in position to prove our second main theorem.

Proof of Theorem 1.3: We will proceed as in the proof of Theorem 1.3 in [8]. Let $A = \{\nu \in \mathbb{R} : \int_0^\infty \lambda_\nu(t)\mathcal{E}(t)dt < +\infty\}$. From (2.8), $-\alpha \in A$, thus $A$ is a non empty interval of $\mathbb{R}$ which is on the forme $A = [-\infty, \alpha_0]$ or $A = (-\infty, \alpha_0]$ where $\alpha_0 = \text{sup } A$. The previous lemma asserts that: if $\nu < \alpha_0$ and $\nu < 2\alpha - 1$ then $\nu + 1 - \alpha \leq \alpha_0$ which means that $\min(\alpha_0, 2\alpha - 1) \leq \alpha_0 + \alpha - 1$. Now since $\alpha - 1 < 0$, the last inequality reads as $2\alpha - 1 \leq \alpha_0 + \alpha - 1$, thus $\alpha \leq \alpha_0$. Therefore, by using the definition of $\alpha_0$ and Lemma 2.1 we infer that for all $\tilde{\alpha} < \alpha$, $\mathcal{E}(t) = o\left(\frac{1}{t^{1+\tilde{\alpha}}}\right)$ as $t \to +\infty$ and $\int_0^{\infty} (1 + t)^{1+\tilde{\alpha}-\alpha} \left|\frac{du}{dt}(t)\right|^2 dt < \infty$. Hence, by taking $\tilde{\alpha}$ closed enough to $\alpha$ and using the fact that $\alpha < 1$, we deduce that $\int_0^\infty (1 + t)^\alpha \left|\frac{du}{dt}(t)\right|^2 dt < \infty$ which completes the proof thanks to Lemma 2.2.
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