Torsions and Curvatures on Jet Fibre Bundle \( J^1(T, M) \)

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Abstract

The aim of this paper is twofold. On the one hand, to study the local representations of d-connections, d-torsions, and d-curvatures with respect to an adapted basis on the jet fibre bundle of order one. On the other hand, to open the problem of prolongations of tensors and connections from a product of two manifolds to 1-jet fibre bundle associated to these manifolds. Section 1 defines the notion of \( \Gamma \)-linear connection on the jet fibre bundle of order one and determines its nine local components. Section 2 studies the main twelve components of torsion d-tensor field, and Section 3 describes the eighteen components of curvature d-tensor field. Via the Ricci and Bianchi identities, Section 4 emphasizes the non-independence of the torsion and curvature d-tensors. Section 5 studies the problem of prolongation of vector fields from \( T \times M \) to 1-jet space \( J^1(T, M) \).

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Key words: 1-jet fibre bundle, nonlinear connection, \( \Gamma \)-linear connection, torsion d-tensor field, curvature d-tensor field.

1 Components of \( \Gamma \)-linear connections

Let us consider \( T \) (resp. \( M \)) a "temporal" (resp. "spatial") manifold of dimension \( p \) (resp. \( n \)), coordinated by \( (t^\alpha)_{\alpha=1}^p \) (resp. \( (x^i)_{i=1}^n \)). Let \( J^1(T, M) \rightarrow T \times M \) be the jet fibre bundle of order one associated to these manifolds.

The bundle of configuration \( J^1(T, M) \) is coordinated by \( (t^\alpha, x^i, x^i_\alpha) \), where \( \alpha = 1, p \) and \( i = 1, n \). Note that, throughout this paper, the indices \( \alpha, \beta, \gamma, \ldots \) run from 1 to \( p \) and the indices \( i, j, k \ldots \) run from 1 to \( n \).

On \( E = J^1(T, M) \), we fix a nonlinear connection \( \Gamma \) defined by the temporal components \( M^{(i)}_{(\alpha)\beta} \) and the spatial components \( N^{(i)}_{(\alpha)j} \). We recall that the transformation rules of the local components of the nonlinear connection \( \Gamma \) are expressed by

\[
\begin{align*}
\tilde{M}^{(j)}_{(\beta)\mu} \frac{\partial t^\mu}{\partial t^\alpha} &= M^{(k)}_{(\gamma)\alpha} \frac{\partial \tilde{x}^i_\gamma}{\partial t^\beta} - \frac{\partial \tilde{x}^i_\beta}{\partial t^\gamma} \\
\tilde{N}^{(j)}_{(\beta)k} \frac{\partial \tilde{x}^k}{\partial x^i} &= N^{(k)}_{(\gamma)i} \frac{\partial \tilde{x}^i_\gamma}{\partial x^k} - \frac{\partial \tilde{x}^i_\beta}{\partial x^k}.
\end{align*}
\]

Let \( \left\{ \frac{\delta}{\delta t^\alpha}, \frac{\delta}{\delta x^i}, \frac{\partial}{\partial x^i_\alpha} \right\} \subset \mathcal{X}(E) \) and \( \left\{ dt^\alpha, dx^i, \delta x^i_\alpha \right\} \subset \mathcal{X}^*(E) \) be the dual adapted bases.
associated to the nonlinear connection $\Gamma = (M^{(i)}_{(\alpha)\beta}, N^{(i)}_{(\alpha)j})$, by the formulas

$$
\begin{align*}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} - M^{(j)}_{(\beta)\alpha} \frac{\partial}{\partial x^j_{\beta}}, \\
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - N^{(j)}_{(\beta)i} \frac{\partial}{\partial x^j_{\beta}}, \\
\delta x^i_\alpha &= dx^i_\alpha + M^{(i)}_{(\alpha)j} dt^\beta + N^{(i)}_{(\alpha)j} dx^j.
\end{align*}
$$

These bases will be used in the description of geometrical objects on $E$, because their transformation laws are very simple [10]:

$$
\begin{align*}
\frac{\delta}{\delta t^\alpha} &= \frac{\partial}{\partial t^\alpha} = \frac{\partial}{\partial \tilde{t}^\beta} \frac{\delta}{\delta \tilde{t}^\beta}, \\
\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} = \frac{\partial}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{x}^j}, \\
\frac{\partial}{\partial x^i_{\alpha}} &= \frac{\partial}{\partial \tilde{x}^j} \frac{\partial}{\partial \tilde{t}^\beta} \frac{\partial}{\partial \tilde{x}^j_{\beta}}.
\end{align*}
$$

In order to develop the theory of $\Gamma$-linear connections on the 1-jet space $E$, we need the following

**Proposition 1.1**  

i) The Lie algebra $\mathcal{X}(E)$ of vector fields decomposes as

$$
\mathcal{X}(E) = \mathcal{X}(H_T) \oplus \mathcal{X}(H_M) \oplus \mathcal{X}(V),
$$

where

$$
\mathcal{X}(H_T) = \text{Span} \left\{ \frac{\delta}{\delta t^\alpha} \right\}, \quad \mathcal{X}(H_M) = \text{Span} \left\{ \frac{\delta}{\delta x^i} \right\}, \quad \mathcal{X}(V) = \text{Span} \left\{ \frac{\partial}{\partial x^i_{\alpha}} \right\}.
$$

ii) The Lie algebra $\mathcal{X}^*(E)$ of covector fields decomposes as

$$
\mathcal{X}^*(E) = \mathcal{X}^*(H_T) \oplus \mathcal{X}^*(H_M) \oplus \mathcal{X}^*(V),
$$

where

$$
\mathcal{X}^*(H_T) = \text{Span} \{dt^\alpha\}, \quad \mathcal{X}^*(H_M) = \text{Span} \{dx^i\}, \quad \mathcal{X}^*(V) = \text{Span} \{\delta x^i_{\alpha}\}.
$$

Let us consider $h_T$, $h_M$ (horizontal) and $v$ (vertical) as the canonical projections of the above decompositions. In this context, we have

**Corollary 1.2**  

i) Any vector field $X$ can be written in the form

$$
X = h_T X + h_M X + v X.
$$

ii) Any covector field $\omega$ can be written in the form

$$
\omega = h_T \omega + h_M \omega + v \omega.
$$
**Theorem 1.3**

In order to describe in local terms a $\Gamma$-linear connection $\nabla$ on $E$, we need nine unique local components,

\[
\nabla = (G^\alpha_{\beta \gamma}, G^k_{\beta}, G^{(i)(\beta)}_{(\alpha)(j) \gamma}, L^\alpha_{\beta}, I^k_{\gamma}, L^{(i)(\beta)}_{(\alpha)(j)k}, C^\alpha_{\beta(k)}, C^j_{i(k)}, C^{(i)(\beta)(\gamma)}_{(\alpha)(j)(k)}),
\]

which are locally defined by the relations

\[
\begin{aligned}
\text{(h)} & \quad \nabla \frac{\delta}{\delta t^a} = C^\alpha_{\beta(j)} \frac{\delta}{\delta x^\alpha} = \nabla \frac{\delta}{\delta x^\alpha}, \\
\text{(h)} & \quad \nabla \frac{\delta}{\delta x^\beta} = L^\alpha_{\beta} \frac{\delta}{\delta t^a} = \nabla \frac{\delta}{\delta t^a}, \\
\text{(v)} & \quad \nabla \frac{\partial}{\partial t^a} = C^{(i)(\beta)_{(\alpha)(j)}} \frac{\partial}{\partial x^\alpha} = \nabla \frac{\partial}{\partial x^\alpha}.
\end{aligned}
\]

The transformation laws of the elements $\left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^\alpha} \right\}$ together with the properties of the $\Gamma$-linear connection $\nabla$, imply

**Theorem 1.3**

i) The components of the $\Gamma$-linear connection $\nabla$ modify by the rules

\[
\begin{aligned}
\text{h} & \quad G^\alpha_{\beta \gamma} = \frac{\partial \tilde{\xi}^\alpha}{\partial \xi^\beta} \frac{\partial \xi^\beta}{\partial t^\gamma} + \frac{\partial^2 \tilde{\xi}^\alpha}{\partial t^\alpha \partial t^\beta}, \\
\text{h} & \quad G^k_{\beta} = \frac{\partial k}{\partial t^m} \frac{\partial k}{\partial t^m} + \frac{\partial k}{\partial t^m} \frac{\partial k}{\partial t^m}, \\
\text{h} & \quad G^{(i)(\beta)}_{(\alpha)(j) \gamma} = \frac{\partial k}{\partial t^m} \frac{\partial k}{\partial t^m} + \frac{\partial k}{\partial t^m} \frac{\partial k}{\partial t^m} + \frac{\partial k}{\partial t^m} \frac{\partial k}{\partial t^m} + \frac{\partial k}{\partial t^m} \frac{\partial k}{\partial t^m}.
\end{aligned}
\]

ii) Conversely, to give a $\Gamma$-linear connection $\nabla$ on the 1-jet space $E$ is equivalent to give a set of nine local components $\left\{ \frac{\delta}{\delta t^a}, \frac{\delta}{\delta x^\alpha}, \frac{\partial}{\partial x^\alpha} \right\}$, whose local transformations laws are described in i.
Theorem 1.3 allows us to construct on the 1-jet space $E$ a natural example of $\Gamma$-linear connection.

**Example 1.1** Suppose that $h_{\alpha\beta}(t)$ (resp. $\varphi_{ij}(x)$) is a pseudo-Riemannian metric on $T$ (resp. $M$). We denote $H^{i\alpha\beta}$ (resp. $\gamma_{ij}^{k}$) the Christoffel symbols of the metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$). The canonical nonlinear connection $\Gamma_{0}$ associated to these metrics is defined by the local components.

\[(1.5)\]
\[
M_{(\alpha)\beta}^{(i)} = -H^{\gamma}_{\alpha\beta}x^{\gamma}_{i}, \quad N_{(\alpha)\beta}^{(i)} = \gamma_{m\alpha}^{i}x^{m}_{\beta}.
\]

In this context, using the well known local transformation rules of the Christoffel symbols $H^{i\alpha\beta}$ and $\gamma_{ij}^{k}$ and setting

\[(1.6)\]
\[
G^{i\alpha\beta} = H^{i\alpha\beta}, \quad G^{(k)(\beta)}_{(\gamma)(\alpha)\alpha} = -\delta_{i}^{k}H^{\alpha\gamma}, \quad L_{ij}^{k} = \gamma_{ij}^{k}, \quad L_{(\gamma)(\alpha)\alpha}^{(k)(\beta)} = \delta_{\gamma}^{\alpha}\gamma_{ij}^{k},
\]

we conclude that the set of local components

\[
BG_{0} = (H^{\gamma}_{\alpha\beta}, \text{0}, G^{(k)(\beta)}_{(\gamma)(\alpha)\alpha}, \text{0}, \gamma_{ij}^{k}, L_{(\gamma)(\alpha)\alpha}^{(k)(\beta)}, \text{0}, \text{0}, \text{0})
\]

defines a $\Gamma_{0}$-linear connection. This is called the *Berwald connection attached to the metrics pair* $(h_{\alpha\beta}, \varphi_{ij})$.

**Remark 1.1** In the particular case $(T, h) = (R, \delta)$, the Berwald connection reduces to that naturally induced by the canonical spray $2G^{i} = \gamma_{jk}^{i}y^{j}y^{k}$ of the classical theory of Lagrange spaces. For more details, see [3], [10].

Now, let $\nabla$ be a $\Gamma$-linear connection on $E$, locally defined by [4]. The linear connection $\nabla$ induces a natural linear connection on the d-tensors set of the jet fibre bundle $E = J^{1}(T, M)$, in the following fashion: starting with a vector field $X$ and a d-tensor field $D$ locally expressed by

\[
X = X^{\alpha}_{\sigma} \frac{\delta}{\delta t^{\alpha}} + X^{m}_{\sigma} \frac{\delta}{\delta x^{m}} + X^{(m)}_{\sigma} \frac{\partial}{\partial x^{m}},
\]
\[
D = D^{\alpha(i)(\delta)\ldots}_{\gamma k(\beta)(l)\ldots} \frac{\delta}{\delta t^{\alpha}} \otimes \frac{\delta}{\delta x^{i}} \otimes \frac{\delta}{\delta x^{j}} \otimes dt^{\gamma} \otimes dx^{k} \otimes dx^{l} \ldots,
\]

we introduce the covariant derivative

\[
\nabla_{X}D = X^{\varepsilon} \nabla_{\frac{\delta}{\delta t^{\varepsilon}}} D + X^{p} \nabla_{\frac{\delta}{\delta x^{p}}} D + X^{(p)}_{(\varepsilon)} \nabla_{\frac{\partial}{\partial x^{p}}} D = \left\{ X^{\varepsilon}D^{\alpha(i)(\delta)\ldots}_{\gamma k(\beta)(l)\ldots} / \varepsilon + X^{p}
\right\} \frac{\delta}{\delta t^{\alpha}} \otimes \frac{\delta}{\delta x^{i}} \otimes \frac{\delta}{\delta x^{j}} \otimes dt^{\gamma} \otimes dx^{k} \otimes dx^{l} \ldots,
\]

where

\[
(h_{\Gamma})
\]
\[
\begin{align*}
D^{ai(j)(\ldots)}_{\gamma k(\beta)(l)\ldots} / \varepsilon &= \delta D^{ai(j)(\ldots)}_{\gamma k(\beta)(l)\ldots} / \varepsilon + D^{ai(j)(\ldots)}_{\gamma k(\beta)(l)\ldots} G^{\varepsilon}_{\beta m} + \\
D^{ai(m)(\ldots)}_{\gamma k(\beta)(l)\ldots} G^{\beta}_{m} + D^{ai(m)(\ldots)}_{\gamma k(\beta)(l)\ldots} G^{\beta}_{m} + \ldots - \\
- D^{ai(j)(\ldots)}_{\gamma m(\beta)(l)\ldots} G^{\varepsilon}_{m} - D^{ai(j)(\ldots)}_{\gamma m(\beta)(l)\ldots} G^{\varepsilon}_{m} - D^{ai(m)(\ldots)}_{\gamma k(\beta)(l)\ldots} G^{\varepsilon}_{m} - D^{ai(m)(\ldots)}_{\gamma k(\beta)(l)\ldots} G^{\varepsilon}_{m} - \ldots .
\end{align*}
\]
i) In the particular case of a function $f(t^\gamma, x^k, x_\gamma^k)$ on $J^1(T,M)$, the above covariant derivatives reduce to

\[
\begin{align*}
D_{\gamma k(l)}^{\alpha i}(j)(\delta)\ldots l_{\mu p} & = \frac{\delta D_{\gamma k(l)}^{\alpha i}(j)(\delta)\ldots l_{\mu p}}{\delta x^p} + D_{\gamma k(l)}^{j(\mu)(\delta)\ldots l_{\mu p}} + D_{\gamma k(l)}^{i(\delta)\ldots l_{\mu p}} + \ldots, \\
D_{\alpha m(\gamma)}^{\alpha i}(j)(\delta)\ldots l_{\gamma p} & = D_{\gamma k(l)}^{\alpha i}(j)(\delta)\ldots l_{\gamma p} + D_{\gamma k(l)}^{i(\delta)\ldots l_{\gamma p}} + \ldots,
\end{align*}
\]

\[
\begin{align*}
D_{\mu k(l)}^{\alpha i}(j)(\delta)\ldots l_{\nu p} & = D_{\gamma k(l)}^{\alpha i}(j)(\delta)\ldots l_{\nu p} - D_{\gamma m(\beta)}^{\alpha i}(j)(\mu)(\delta)\ldots l_{\nu p} + \ldots,
\end{align*}
\]

The local operators "/" \(l\) and "\(\gamma\)" are called the $T$-horizontal covariant derivative, $M$-horizontal covariant derivative and vertical covariant derivative of the $\Gamma$-connection $\nabla$.

**Remarks 1.2** i) In the particular case of a function $f(t^\gamma, x^k, x_\gamma^k)$ on $J^1(T,M)$, the above covariant derivatives reduce to

\[
\begin{align*}
f_{/\varepsilon} & = \frac{\delta f}{\delta \varepsilon} - M_{(\gamma)\varepsilon}^{(k)} \frac{\partial f}{\partial x^\gamma_k}, \\
f_{p/} & = \frac{\delta f}{\delta x^p} - N_{(\gamma)\varepsilon}^{(k)} \frac{\partial f}{\partial x^\gamma_k}, \\
f_{(\varepsilon)/p} & = \frac{\delta f}{\delta \varepsilon}. 
\end{align*}
\]

ii) Particularly, starting with a d-vector field $X$ on $J^1(T,M)$, locally expressed by

\[
X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} + X^{(i)} \frac{\partial}{\partial x^i},
\]

the following expressions of above covariant derivatives hold good:

\[
\begin{align*}
X_{/\varepsilon}^\alpha & = \frac{\delta X^\alpha}{\delta \varepsilon} + X^\mu \tilde{G}_\mu \varepsilon, \\
X^i_{/\varepsilon} & = \frac{\delta X^i}{\delta \varepsilon} + X^m \tilde{G}_m \varepsilon, \\
X^{(i)}_{(\alpha)/\varepsilon} & = \frac{\delta X^{(i)}}{\delta \varepsilon} + X_{(\mu)} \tilde{G}^{(i)}(\mu)_{(\beta)(\varepsilon)}, \\
X_{/p}^\alpha & = \frac{\delta X^\alpha}{\delta x^p} + X^\mu \tilde{L}_\mu p, \\
X^i_{/p} & = \frac{\delta X^i}{\delta x^p} + X^m L^i_{mp}, \\
X^{(i)}_{(\alpha)/p} & = \frac{\delta X^{(i)}}{\delta x^p} + X_{(\mu)} L^{(i)}(\mu)_{(\alpha)(p)}. 
\end{align*}
\]
Proposition 2.1

\[
\begin{align*}
X^{\alpha}(\varepsilon)_{\mu} &= \frac{\partial X^{\alpha}}{\partial \varepsilon_{\mu}} + X^{\mu}C_{\mu}^{\alpha}(\varepsilon) \\
X^{i}(\varepsilon)_{\mu} &= \frac{\partial X^{i}}{\partial x_{\mu}} + X^{\mu}C_{\mu}^{i}(\varepsilon) \\
X^{ij}(\varepsilon)_{\mu} &= \frac{\partial X^{ij}}{\partial x_{\mu}} + X^{\mu}C_{\mu}^{ij}(\varepsilon).
\end{align*}
\]

iii) The local covariant derivatives associated to the Berwald $\Gamma_0$-linear connection, will be denoted by $"/\varepsilon","/\mu$" and $"(\varepsilon)_{\mu}"$.

Denoting by $"A"$ one of the covariant derivatives $"/\varepsilon","/\mu$" or $"(\varepsilon)_{\mu}"$, one easily proves the following

**Proposition 1.4** If $D_{\cdots}^\mu$ and $F_{\cdots}^\mu$ are two d-tensor fields on $E$, then the following statements hold good.

i) $D_{\cdots}^\mu A$ are the components of a new d-tensor field,

ii) $(D_{\cdots}^\mu + F_{\cdots}^\mu)A = D_{\cdots}^\mu A + F_{\cdots}^\mu A$,

iii) $(D_{\cdots}^\mu \otimes F_{\cdots}^\mu)A = D_{\cdots}^\mu \otimes A + D_{\cdots}^\mu \otimes F_{\cdots}^\mu A$,

iv) The operator $"A"$ commutes with the operation of contraction.

## 2 Components of torsion d-tensor field

Let us start with a fixed $\Gamma$-linear connection $\nabla$ on $E = J^1(T, M)$, defined by the local components $[\mathcal{I}]$. The torsion d-tensor field associated to $\nabla$ is

$$T : \mathcal{X}(E) \times \mathcal{X}(E) \to \mathcal{X}(E), \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad \forall X, Y \in \mathcal{X}(E).$$

To characterize locally the torsion d-tensor $T$ of the connection $\nabla$, we need the next

**Proposition 2.1** The following bracket identities are true,

\[
\begin{align*}
\begin{bmatrix} \delta \\ \delta \varepsilon^\alpha \delta \varepsilon^\beta \end{bmatrix} &= R_{(\mu)\alpha\beta}^{(m)} \frac{\partial}{\partial x^\mu}, \quad \begin{bmatrix} \delta \\ \delta \varepsilon^\alpha \delta \varepsilon^\beta \end{bmatrix} = R_{(\mu)\alpha\beta}^{(m)} \frac{\partial}{\partial x^\mu}, \\
\begin{bmatrix} \delta \varepsilon^\alpha \delta \varepsilon^\beta \end{bmatrix} &= \frac{\partial M_{(\mu)\alpha}^{(m)}}{\partial x^\alpha}, \quad \begin{bmatrix} \delta \varepsilon^\alpha \delta \varepsilon^\beta \end{bmatrix} = \frac{\partial M_{(\mu)\alpha}^{(m)}}{\partial x^\alpha}, \\
\begin{bmatrix} \delta \varepsilon^\alpha \delta \varepsilon^\beta \end{bmatrix} &= \frac{\partial N_{(\mu)\alpha}^{(m)}}{\partial x^\beta}, \quad \begin{bmatrix} \delta \varepsilon^\alpha \delta \varepsilon^\beta \end{bmatrix} = \frac{\partial N_{(\mu)\alpha}^{(m)}}{\partial x^\beta},
\end{align*}
\]

where $M_{(\mu)\alpha}^{(m)}$ and $N_{(\mu)\alpha}^{(m)}$ are the local components of the nonlinear connection $\Gamma$ while the components $R_{(\mu)\alpha\beta}^{(m)}$, $R_{(\mu)\alpha\beta}^{(m)}$, $R_{(\mu)\alpha\beta}^{(m)}$ are d-tensors expressed by

\[
\begin{align*}
R_{(\mu)\alpha\beta}^{(m)} &= \frac{\delta M_{(\mu)\alpha}^{(m)}}{\delta \varepsilon^\beta} - \frac{\delta M_{(\mu)\beta}^{(m)}}{\delta \varepsilon^\alpha}, \quad R_{(\mu)\alpha\beta}^{(m)} = \frac{\delta M_{(\mu)\beta}^{(m)}}{\delta \varepsilon^\beta} - \frac{\delta M_{(\mu)\alpha}^{(m)}}{\delta \varepsilon^\beta}, \quad R_{(\mu)\alpha\beta}^{(m)} = \frac{\delta N_{(\mu)\alpha}^{(m)}}{\delta \varepsilon^\beta} - \frac{\delta N_{(\mu)\beta}^{(m)}}{\delta \varepsilon^\beta}.
\end{align*}
\]
Consequently, the torsion d-tensor field $T$ of the $\Gamma$-linear connection $\nabla$ can be described locally by

**Theorem 2.2** The torsion d-tensor $T$ of the $\Gamma$-linear connection $\nabla$ is determined by the following local expressions:

$$h_T T \left( \frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha} \right) = T_{\alpha\beta}^\mu \frac{\partial}{\partial t^\mu}, \quad h_M T \left( \frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha} \right) = 0,$$

$$v_T \left( \frac{\delta}{\delta t^\beta}, \frac{\delta}{\delta t^\alpha} \right) = R^{(m)}_{(\mu)\alpha\beta}(x), \quad h_T T \left( \frac{\delta}{\delta x^\beta}, \frac{\delta}{\delta x^\alpha} \right) = T^{m}_{\alpha\beta} \frac{\delta}{\delta x^m},$$

$$h_T T \left( \frac{\partial}{\partial x^\beta}, \frac{\delta}{\delta t^\alpha} \right) = 0, \quad h_M T \left( \frac{\partial}{\partial x^\beta}, \frac{\delta}{\delta t^\alpha} \right) = 0,$$

$$v_T \left( \frac{\partial}{\partial x^\beta}, \frac{\delta}{\delta t^\alpha} \right) = P^{(m)}_{(\mu)\alpha\beta}(x), \quad h_T T \left( \frac{\partial}{\partial x^\beta}, \frac{\delta}{\delta x^\alpha} \right) = 0,$$

$$v_T \left( \frac{\partial}{\partial x^\beta}, \frac{\delta}{\delta x^\alpha} \right) = P^{(m)}_{(\mu)\beta}(x), \quad h_T T \left( \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\alpha} \right) = 0,$$

$$v_T \left( \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\alpha} \right) = S^{(m)(\alpha)(\beta)}_{(\mu)(\nu)(\rho)}(x), \quad v_T \left( \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\alpha} \right) = 0,$$

where $R^{(m)}_{(\mu)\alpha\beta}$, $R^{(m)}_{(\mu)\alpha\gamma}$, $R^{(m)}_{(\mu)\beta}$ are the d-tensors constructed above, and

$$T_{\alpha\beta}^\mu = C_{\alpha\beta}^\mu - G_{\beta\alpha}^\mu, \quad T_{\alpha\gamma}^\mu = L_{\alpha\gamma}^\mu, \quad P^{(m)(\beta)}_{(\mu)\alpha\beta} = C_{\alpha\beta}^\mu, \quad T_{\alpha\beta}^m = -G_{\beta\alpha}^m,$$

$$P^{(m)(\beta)}_{(\mu)\alpha\beta} = \frac{\partial M^{(m)}_{(\mu)\alpha\beta}}{\partial x^\beta} - G^{(m)(\beta)}_{(\mu)\alpha\beta}, \quad P^{(m)(\beta)}_{(\mu)\alpha\gamma} = \frac{\partial N^{(m)}_{(\mu)\alpha\beta}}{\partial x^\beta} - L^{(m)(\beta)}_{(\mu)\alpha\gamma}.$$
Corollary 2.3 The torsion $T$ of the $\Gamma$-linear connection $\nabla$ is determined by twelve effective $d$-tensor fields, arranged in the following table:

|       | $h_T$ | $h_M$ | $\nu$ |
|-------|-------|-------|-------|
| $h_T h_T$ | $T^\mu_{\alpha \beta}$ | 0 | $R^{(m)}_{(\mu)\alpha \beta}$ |
| $h_M h_T$ | $T^m_{\alpha j}$ | $T^m_{\alpha j}$ | $R^{(m)}_{(\mu)\alpha j}$ |
| $h_M h_M$ | 0 | $T^m_{ij}$ | $R^{(m)}_{(\mu)ij}$ |
| $v h_T$ | $T^m_{\mu (\beta)}$ | 0 | $P^{(m)}_{(\mu)\beta i j}$ |
| $v h_M$ | 0 | $P^{m (\beta)}_{i j}$ | $P^{(m)}_{(\mu)ij}$ |
| $\nu$ | 0 | 0 | $S^{(m)}_{(\mu)ij}$ |

(2.1)

Remark 2.1 In the particular case of the Berwald $\Gamma_0$-linear connection associated to the metrics $h_{\alpha \beta}$ and $\varphi_{ij}$, all torsion $d$-tensors vanish, except

$$R^{(m)}_{(\mu)\alpha \beta} = -H^\gamma_{\mu \alpha \beta} x^m, \quad R^{(m)}_{(\mu)ij} = r^{m}_{ij} l^\mu,$$

where $H^\gamma_{\mu \beta \gamma}$ (resp. $r^{m}_{ij}$) are the curvature tensors of the metric $h_{\alpha \beta}$ (resp. $\varphi_{ij}$).

3 Components of curvature $d$-tensor field

From the general theory of linear connections, we recall that the curvature $d$-tensor field associated to the $\Gamma$-linear connection $\nabla$ is

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad \forall X, Y, Z \in \mathcal{X}(E).$$

Using an adapted basis and the properties of the $\Gamma$-linear connection $\nabla$, one easily prove the following

Theorem 3.1 The curvature $d$-tensor $R$ of the $\Gamma$-linear connection $\nabla$ is determined by the following eighteen local expressions:

$$R \left( \frac{\delta}{\delta t^\gamma}, \frac{\delta}{\delta t^\beta} \right) \frac{\delta}{\delta x^\alpha} = R^\delta_{\alpha \beta \gamma} \frac{\delta}{\delta t^\delta}, \quad R \left( \frac{\delta}{\delta x^k}, \frac{\delta}{\delta t^\beta} \right) \frac{\delta}{\delta x^i} = R^l_{i \beta k} \frac{\delta}{\delta x^l},$$

$$R \left( \frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^\alpha} \right) \frac{\delta}{\delta x^i} = R^l_{i \beta k} \frac{\delta}{\delta x^l}, \quad R \left( \frac{\delta}{\delta x^k}, \frac{\delta}{\delta t^\beta} \right) \frac{\delta}{\delta x^i} = R^l_{i \beta k} \frac{\delta}{\delta x^l}.$$
which we arrange in the following table

\[ \begin{array}{|c|c|c|c|}
\hline
 & h_T & h_M & \nu \\
\hline
h_T h_T & R^\delta_{\alpha \beta \gamma} & R^i_{ijk} & R^{(1)(\alpha)(\beta)(\gamma)} \\
\hline
h_M h_T & R^\delta_{\alpha \beta \gamma} & R^i_{ijk} & R^{(1)(\alpha)(\beta)(\gamma)} \\
\hline
h_M h_M & R^\delta_{\alpha \beta \gamma} & R^i_{ijk} & R^{(1)(\alpha)(\beta)(\gamma)} \\
\hline
\nu h_T & \delta \tilde{C}^\delta_{\alpha \beta \gamma} & \delta \tilde{C}^i_{\alpha \beta \gamma} & \delta \tilde{C}^{(1)(\alpha)(\beta)(\gamma)} \\
\hline
\nu h_M & \delta \tilde{C}^\delta_{\alpha \beta \gamma} & \delta \tilde{C}^i_{\alpha \beta \gamma} & \delta \tilde{C}^{(1)(\alpha)(\beta)(\gamma)} \\
\hline
\nu \nu & \delta \tilde{C}^{(1)(\alpha)(\beta)(\gamma)} & \delta \tilde{C}^{(1)(\alpha)(\beta)(\gamma)} & \delta \tilde{C}^{(1)(\alpha)(\beta)(\gamma)} \\
\hline
\end{array} \]

(3.1)

Moreover, using the properties of the d-tensor \( R \) and the expressions of local \( T^\alpha \), \( M \)-horizontal and vertical covariant derivatives attached to the \( \Gamma \)-linear connection \( \nabla \), we derive the following local components of the curvature d-tensor,

\[
\begin{align*}
1. \quad & \delta \tilde{C}^\delta_{\alpha \beta \gamma} = \frac{\delta \tilde{C}^\delta_{\alpha \beta \gamma}}{\delta x^\gamma} - \frac{\delta \tilde{C}^\delta_{\alpha \beta \gamma}}{\delta t^\gamma} + \bar{G}^\mu_{\alpha \beta} \bar{G}^\delta_{\mu \gamma} - \bar{G}^\mu_{\alpha \gamma} \bar{G}^\delta_{\mu \beta} + \bar{C}^{\delta(\mu)}_{\alpha(\beta)} R^{(m)}_{(\mu) \beta \gamma} \\
2. \quad & \delta \tilde{C}^\delta_{\alpha \beta \gamma} = \frac{\delta \tilde{C}^\delta_{\alpha \beta \gamma}}{\delta x^k} - \frac{\delta \tilde{C}^\delta_{\alpha \beta \gamma}}{\delta t^k} + \bar{G}^\mu_{\alpha \beta} \bar{L}^\delta_{\mu \gamma} - \bar{L}^\mu_{\alpha \beta} \bar{L}^\delta_{\mu \gamma} + \bar{C}^{\delta(\mu)}_{\alpha(\beta)} R^{(m)}_{(\mu) \beta \gamma} \\
3. \quad & \delta \tilde{C}^\delta_{\alpha \beta \gamma} = \frac{\delta \tilde{C}^\delta_{\alpha \beta \gamma}}{\delta x^\gamma} - \frac{\delta \tilde{C}^\delta_{\alpha \beta \gamma}}{\delta t^\gamma} + \bar{G}^\mu_{\alpha \beta} \bar{L}^\delta_{\mu \gamma} - \bar{L}^\mu_{\alpha \beta} \bar{L}^\delta_{\mu \gamma} + \bar{C}^{\delta(\mu)}_{\alpha(\beta)} R^{(m)}_{(\mu) \beta \gamma} \\
\end{align*}
\]

\[ \{ h_T \} \]

\[
\begin{align*}
4. \quad & \delta \tilde{C}^{(\gamma)}_{\alpha \beta (k)} = \frac{\delta \tilde{C}^{(\gamma)}_{\alpha \beta (k)}}{\delta x^\gamma} - \bar{G}^{(\gamma)}_{\alpha \beta (k)} + \bar{C}^{(\gamma)}_{\alpha(\beta)} P^{(m)}_{(\mu) \beta (k)} \\
5. \quad & \delta \tilde{C}^{(\gamma)}_{\alpha \beta (k)} = \frac{\delta \tilde{C}^{(\gamma)}_{\alpha \beta (k)}}{\delta x^k} - \bar{G}^{(\gamma)}_{\alpha \beta (k)} + \bar{C}^{(\gamma)}_{\alpha(\beta)} P^{(m)}_{(\mu) \beta (k)} \\
6. \quad & \delta \tilde{C}^{(\gamma)}_{\alpha \beta (k)} = \frac{\delta \tilde{C}^{(\gamma)}_{\alpha \beta (k)}}{\delta x^\gamma} - \bar{G}^{(\gamma)}_{\alpha \beta (k)} + \bar{C}^{(\gamma)}_{\alpha(\beta)} P^{(m)}_{(\mu) \beta (k)} \\
\end{align*}
\]

\[ \{ h_M \} \]

9
Remark 3.1 In the case of the Berwald $\Gamma_0$-linear connection associated to the metrics pair $(h_{\alpha\beta}, \varphi_{ij})$, all curvature d-tensors vanish, except

$$R^l_{i\beta\gamma} = H^l_{i\beta\gamma}, \quad R^l_{i\beta jk} = r^l_{i\beta jk},$$

where $H^l_{i\beta\gamma}$ (resp. $r^l_{i\beta jk}$) are the curvature tensors of the metric $h_{\alpha\beta}$ (resp. $\varphi_{ij}$).
4 Ricci and Bianchi identities

Taking into account the local form of the $T-,M$-horizontal and vertical covariant derivatives defined in Section 1, by a direct calculation one proves

**Theorem 4.1** If $X = X^\alpha \frac{\delta}{\delta x^\alpha} + X^i \frac{\delta}{\delta x^i} + X_{(\alpha)} \frac{\partial}{\partial x^i}$ is an arbitrary vector field on the 1-jet space $E$, then the following Ricci identities hold good:

\[
\begin{align*}
(X_1) & \left\{ X_{\beta\gamma} - X_{\gamma\beta} = X^\mu R_{\mu\beta\gamma} - X^\mu \bar{T}_{\mu\beta\gamma} - X_{(\mu)} R_{(\mu)\beta\gamma}, \\
X_{i\beta} - X_{i\gamma} & = X^m R_{m\beta\gamma} - X^m \bar{T}_{m\beta\gamma} - X_{(m)} R_{(m)\beta\gamma}, \quad X^i = X^m R_{m\gamma}\beta - X^m \bar{T}_{m\gamma}\beta - X_{(m)} R_{(m)\gamma}\beta, \\
X_{ij} - X_{ij} & = X^m R_{m\gamma\beta} - X^m \bar{T}_{m\gamma\beta} - X_{(m)} R_{(m)\gamma\beta}, \\
X_{i\gamma} & = X^m R_{m\beta\gamma} - X^m \bar{T}_{m\beta\gamma} - X_{(m)} R_{(m)\beta\gamma}. \end{align*}
\]

**Theorem 4.2** If $X = X^\alpha \frac{\delta}{\delta x^\alpha} + X^i \frac{\delta}{\delta x^i} + X_{(\alpha)} \frac{\partial}{\partial x^i}$ is an arbitrary vector field on the 1-jet space $E$, then the following Bianchi identities hold good:

\[
\begin{align*}
(X_2) & \left\{ X_{\beta\gamma} - X_{\gamma\beta} = X^\mu \bar{T}_{\mu\beta\gamma} - X^\mu \bar{T}_{\mu\gamma\beta} - X_{(\mu)} \bar{T}_{(\mu)\beta\gamma}, \\
X_{i\beta} - X_{i\gamma} & = X^m \bar{T}_{m\beta\gamma} - X^m \bar{T}_{m\gamma\beta} - X_{(m)} \bar{T}_{(m)\beta\gamma}, \\
X_{ij} - X_{ij} & = X^m \bar{T}_{m\gamma\beta} - X^m \bar{T}_{m\beta\gamma} - X_{(m)} \bar{T}_{(m)\gamma\beta}, \\
X_{i\gamma} & = X^m \bar{T}_{m\beta\gamma} - X^m \bar{T}_{m\gamma\beta} - X_{(m)} \bar{T}_{(m)\beta\gamma}. \end{align*}
\]
Remark 4.1 For the arbitrary vector fields $X, Y, Z \in \mathcal{X}(E)$ and the arbitrary 1-form $\omega \in \mathcal{X}^*(E)$ on $J^1(T, M)$, the relations

$$\begin{align*}
\{ & R(X, Y)\omega = -\omega \circ R(X, Y), \\
& R(X, Y)(Z \otimes \omega) = R(X, Y)Z \otimes \omega + Z \otimes R(X, Y)\omega,
\end{align*}$$

(4.1)

are true. These relations allow us to generalize the Ricci identities to the $d$-tensors set of the 1-jet fibre bundle $E$. The generalization is a natural one, but the expressions of Ricci identities become extremely complicated. For that reason, we exemplify this generalization writing just one Ricci identity. For example, if $D = (D_{\alpha j}(\eta)(l))_{\gamma\beta}$ is an arbitrary $d$-tensor field on $E$, then the following Ricci identity

$$\begin{align*}
& D_{\alpha j}(\eta)(l)_{\gamma\beta} - D_{\alpha j}(\eta)(l)_{\gamma\beta} \gamma - \bar{D}_{\alpha j}(\eta)(l)_{\gamma\beta} + D_{\alpha j}(\eta)(l)_{\gamma\beta} = D^{\mu j}(\kappa)(\eta)(l)_{\gamma\beta} + D^{\mu j}(\kappa)(\eta)(l)_{\gamma\beta} + D_{\alpha j}(\eta)(l)_{\gamma\beta}
\end{align*}$$

holds good.

Now, let us consider the Liouville canonical vector field $\mathbf{C} = x^\alpha \frac{\partial}{\partial x^\alpha}$ and the deflection $d$-tensors associated to the $\Gamma$-linear connection $\nabla$, defined by the local components

$$\begin{align*}
\bar{D}^{(i)}_{\alpha j} = x^i_{\alpha j}, \quad D^{(i)}_{\alpha j} = x^i_{\alpha j}, \quad d^{(i)}_{\alpha j} = x^i_{\alpha j}.
\end{align*}$$

By a direct calculation, we find

$$\begin{align*}
\bar{D}^{(i)}_{\alpha j} = -M^{(i)}_{\alpha j} + G^{(i)}_{\alpha j}(\mu)m x^m_{\mu} \\
D^{(i)}_{\alpha j} = -N^{(i)}_{\alpha j} + F^{(i)}_{\alpha j}(\mu)m x^m_{\mu} \\
d^{(i)}_{\alpha j} = \delta^{(1)}_{\alpha j}(\beta) + C^{(i)}(\mu)(\beta)m x^m_{\mu}.
\end{align*}$$

(4.2)

Applying the $v$-set of the Ricci identities to the components of the Liouville vector field, we obtain

**Theorem 4.2** The deflection $d$-tensors, attached to the $\Gamma$-linear connection $\nabla$, satisfy:

$$\begin{align*}
& \bar{D}^{(i)}_{\alpha j}/\gamma - \bar{D}^{(i)}_{\alpha j}/\gamma \beta = x^m_{\mu} R^{(i)}(\alpha)(\mu)m x^m_{\mu} - D^{(i)}_{\alpha j}/\gamma \beta - d^{(i)}(\mu)m R^{(m)}(\mu)_{\gamma\beta} \\
& \bar{D}^{(i)}_{\alpha j} / k - \bar{D}^{(i)}_{\alpha j} / k \beta = x^m_{\mu} R^{(i)}(\alpha)(\mu)m x^m_{\mu} - \bar{D}^{(i)}_{\alpha j}/k \beta - D^{(i)}_{\alpha j}/k \beta - d^{(i)}(\mu)m R^{(m)}(\mu)_{\gamma\beta} \\
& D^{(i)}_{\alpha j} / k - D^{(i)}_{\alpha j} / k \beta = x^m_{\mu} R^{(i)}(\alpha)(\mu)m x^m_{\mu} - D^{(i)}_{\alpha j}/k \beta - d^{(i)}(\mu)m R^{(m)}(\mu)_{\gamma\beta} \\
& \bar{D}^{(i)}_{\alpha j}/(\gamma) - \bar{D}^{(i)}_{\alpha j}/(\alpha)(\gamma) = x^m_{\mu} R^{(i)}(\alpha)(\mu)m x^m_{\mu} - \bar{D}^{(i)}_{\alpha j}/(\gamma) - d^{(i)}(\mu)m R^{(m)}(\mu)_{\gamma\beta} \\
& D^{(i)}_{\alpha j}/(\gamma) - D^{(i)}_{\alpha j}/(\alpha)(\gamma) = x^m_{\mu} R^{(i)}(\alpha)(\mu)m x^m_{\mu} - D^{(i)}_{\alpha j}/(\gamma) - d^{(i)}(\mu)m R^{(m)}(\mu)_{\gamma\beta} \\
& d^{(i)}_{\alpha j}/(\beta) = x^m_{\mu} S^{(i)}(\mu)(\beta)m x^m_{\mu} - d^{(i)}(\mu)m R^{(m)}(\mu)_{\gamma\beta}.
\end{align*}$$
Finally, note that the torsion $T$ and the curvature $R$ of the $\Gamma$-linear connection $\nabla$ are not independent. They verify the general Bianchi identities

\[
\sum_{\{X,Y,Z\}} \{(\nabla_X T)(Y,Z) - R(X,Y)Z + T(T(X,Y),Z)\} = 0, \quad \forall X,Y,Z \in \mathcal{X}(E)
\]
\[
\sum_{\{X,Y,Z\}} \{(\nabla_X R)(U,Y,Z) + R(T(X,Y),Z)U\} = 0, \quad \forall X,Y,Z,U \in \mathcal{X}(E),
\]

where $\{X,Y,Z\}$ means cyclic sum.

In the adapted basis $(X_A)$, we have sixty-four effective Bianchi identities, obtained by the relations

(4.3) \[
\begin{cases}
\sum_{\{A,B,C\}} \{R^F_{ABC} - T^E_{ABC} - T^G_{AB}T^E_{CG}\} = 0 \\
\sum_{\{A,B,C\}} \{R^F_{DABC} + T^G_{AB}R^F_{DAG}\} = 0,
\end{cases}
\]

where $R(X_A, X_B)X_C = R^D_{CBA}X_D$, $T(X_A, X_B) = T^P_{BA}X_D$ and "$\cdot\cdot\cdot$" represents one of the covariant derivatives "$\partial_{\alpha}$", "$\partial_{i}$" or "$\partial^{(i)}_{\alpha}$". The large number and the complicated form of the above Bianchi identities associated to a $\Gamma$-linear connection determine us to study them, in a subsequent paper \cite{9}, in the more particular case of the $h$-normal $\Gamma$-linear connections. In that case, the number of Bianchi identities reduces to thirty.

## 5 Jet prolongation of vector fields

A general vector field $X^*$ on $J^1(T,M)$ can be written under the form

\[
X^* = X^\alpha \frac{\partial}{\partial t^\alpha} + X^i \frac{\partial}{\partial x^i} + X^{(i)}_{\alpha} \frac{\partial}{\partial x^i_{\alpha}},
\]

where the components $X^\alpha$, $X^i$, $X^{(i)}_{\alpha}$ are functions of $(t^\alpha, x^i, x^i_{\alpha})$.

The prolongation of a vector field $X$ on $T \times M$ to a vector field on the 1-jet bundle $J^1(T,M)$ was solved by Olver \cite{12} in the following sense.

**Definition 5.1** Let $X$ be a vector field on $T \times M$ with corresponding (local) one-parameter group $\exp(\varepsilon X)$. The 1-th prolongation of $X$, denoted by $pr^{(1)}X$, will be a vector field on the 1-jet space $J^1(T,M)$, and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $pr^{(1)}[\exp(\varepsilon X)]$, i.e.,

\[
[pr^{(1)}X](t^\alpha, x^i, x^i_{\alpha}) = \frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} pr^{(1)}[\exp(\varepsilon X)](t^\alpha, x^i, x^i_{\alpha}).
\]

In order to write the components of the prolongation, Olver used the $\alpha$-th total derivative $D_\alpha f$ of an arbitrary function $f(t^\alpha, x^i)$ on $T \times M$, which is defined by the relation

(5.2) \[
D_\alpha f = \frac{\partial f}{\partial t^\alpha} + \frac{\partial f}{\partial x^i} x^i_{\alpha}.
\]

Thus, starting with $X = X^\alpha(t,x) \frac{\partial}{\partial t^\alpha} + X^i(t,x) \frac{\partial}{\partial x^i}$ like a vector field on $T \times M$, Olver introduced the 1-th prolongation of $X$ as the vector field

(5.3) \[
pr^{(1)}X = X + X^{(i)}_{\alpha}(t^\beta, x^j, x^j_{\beta}) \frac{\partial}{\partial x^i_{\alpha}},
\]
where
\[
X^{(i)} = D_\alpha X^i - (D_\alpha X^\beta)x^i_\beta = \frac{\partial X^i}{\partial t^\alpha} + \frac{\partial X^i}{\partial x^j}x^j_\alpha - \left( \frac{\partial X^\beta}{\partial t^\alpha} + \frac{\partial X^\beta}{\partial x^j}x^j_\alpha \right)x^i_\beta.
\]

Let us use a geometrical approach for obtaining jet prolongations of vector fields. If we assume that is given a nonlinear connection \( \Gamma = (M_{(\alpha)}^{(i)} x^j, N_{(\alpha)}^{(i)} x^j) \) on \( J^1(T,M) \), then the \( \alpha \)-th total derivative used by Olver can be written as
\[
(5.4) \quad D_\alpha f = \frac{\delta f}{\delta t^\alpha} + \frac{\delta f}{\delta x^i}x^i_\alpha = f/\alpha + f^i_\alpha x^i_\alpha,
\]
and, consequently, \( D_\alpha f \) represents the local components of a distinguished 1-form on \( J^1(T,M) \), which is expressed by \( Df = (D_\alpha f) dt^\alpha \).

Now, let there be given a vector field \( X \) on \( T \times M \). If we suppose that \( J^1(T,M) \) is endowed at the same time with a \( \Gamma \)-linear connection \( \Gamma_0 \), we can define the geometrical 1-th jet prolongation of \( X \),
\[
(5.5) \quad \operatorname{pr}^{(1)} X = X^\alpha \frac{\delta}{\delta t^\alpha} + X^i \frac{\delta}{\delta x^i} + Y^{(i)}_{(\alpha)}(x^j, x^j_\alpha) \frac{\partial}{\partial x^i_\alpha},
\]
setting
\[
Y^{(i)}_{(\alpha)} = X^{(i)}_{(\alpha)} + M^{(i)}_{(\alpha)\mu} X^\mu + N^{(i)}_{(\alpha)m} X^m.
\]

**Remarks 5.1**

i) Our prolongation coincides with that of Olver. Moreover, we have the relation
\[
(5.6) \quad Y^{(i)}_{(\alpha)} = X^{(i)}_{(\alpha)} + M^{(i)}_{(\alpha)\mu} X^\mu + N^{(i)}_{(\alpha)m} X^m.
\]

ii) In the particular case of the Berwald \( \Gamma_0 \)-linear connection associated to the metrics \( h_{\alpha\beta} \) and \( \varphi_{ij} \), the expression of \( Y^{(i)}_{(\alpha)} \) reduces to
\[
(5.7) \quad Y^{(i)}_{(\alpha)} = X^{(i)}_{(\alpha)} + M^{(i)}_{(\alpha)\mu} X^\mu + N^{(i)}_{(\alpha)m} X^m,
\]
where \( \gamma^{i}_{jk} \) represent the Christoffel symbols of the metric \( \varphi_{ij} \).

**Open problem.**
Study the prolongations of vectors, 1-forms, tensors, \( G \)-structures from \( T \times M \) to \( J^1(T,M) \).

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