Quantum Field Theory in de Sitter Universe: 
Ambient Space Formalism

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The quantum field theory in the 4-dimensional de Sitter space-time is reformulated in the ambient space formalism in a rigorous mathematical framework. This work is based on the group representation theory and the analyticity of the complexified pseudo-Riemannian manifold. The unitary irreducible representations of de Sitter group and corresponding Hilbert spaces were reformulated in the ambient space formalism. Defining the creation and annihilation operators on these Hilbert spaces, quantum field operators, for various massive fields with spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ and massless fields with $s = 0, \frac{1}{2}$ have been constructed and their corresponding analytic two-point functions have been presented. The quantum massless minimally coupled scalar field operator is presented in ambient space formalism which is also analytic. We show that the massless field with $s \leq 2$ can only propagate in de Sitter ambient space formalism. It is well-known that a gauge invariance exists in the massless field theories with higher spin $s \geq 1$. The massless quantum gauge fields with $s = 1, \frac{3}{2}, 2$ are studied by using the gauge principle. The conformal quantum spin-2 field, based on the gauge gravity model, is constructed in this formalism. The gauge spin-$\frac{3}{2}$ fields satisfy the Grassmannian algebra, and hence, naturally provoke one to couple them with the gauge spin-2 field. Then the super-algebra are automatically appeared. We conclude that the gravitational field may be constructed by three parts, namely, the de Sitter background, the gauge spin-2 field and the gauge spin-$\frac{3}{2}$ field.

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I. INTRODUCTION

The quantum field theory in 4-dimensional de Sitter space-time along the logical lines from the first principles is presented. The following principles are used as the axioms in our construction:

- (A) As it is indicated by the observation, our universe can be well approximated by the de Sitter space-time with its symmetrical group $SO(1, 4)$.
- (B) The quantum fields actually are fundamental objects and their corresponding free field operators must be transformed by the unitary irreducible representation of the de Sitter group.
- (C) The interaction between these fields are governed by the gauge principle (gauge theory).
- (D) The conformal symmetry is preserved in the early universe.

The de Sitter (dS) space-time can be considered as a 4-dimensional hyperboloid embedded in a 5-dimensional Minkowski space. The ambient space formalism is used in this paper. This formalism allows us to reformulate the quantum field theory (QFT) in a rigorous mathematical framework, based on the analyticity in the complexified pseudo-Riemannian manifold and the group representation theory. The unitary irreducible representations (UIR) of dS group was completed and finalized by Takahashi [1–4]. The analyticity in complexified dS space-time had been studied by Bros et al [5–8]. In what follows, we combine these two subjects to construct the quantum field operators, the quantum states and the two point functions in dS space-time for various spin fields.

Similar to the QFT in the Minkowski space-time, the construction of QFT in dS space-time, free field quantization, interaction fields or gauge theory, super-symmetry and super-gravity are necessary for gaining a better understanding of the evolution of our universe and the quantum effects of the gravitational field on other fields. In this paper the free field quantization and the gauge theory are reformulated in the ambient space formalism.

The unitary irreducible representations of the dS group and corresponding Hilbert spaces were reformulated in ambient space formalism in the previous paper [9]. The UIRs of dS group are classified by the eigenvalues of the Casimir operators of the dS group as its counterpart in flat space. The eigenvalues are written in terms of the two parameters $j$ and $p$, respectively playing the role of the spin $s$ and mass $m$ in Poincaré group or Minkowskian space-time in the null curvature limit. The dS group has three different types of representations: principal, complementary and discrete series representations [1–4, 10, 11].

The quantum free fields are divided here into the three distinguishable types: massive fields, massless fields and auxiliary fields. A field is called massive, when it propagates inside the dS light-cone and corresponds to the massive Poincaré fields in the null curvature limit. The massive field operators transform by the principal series representation of dS group [3, 12–14]. On the other hand, the massless fields propagate on the dS light-cone and consequently, they possess an additional symmetry, namely, the conformal symmetry and they correspond to the massless Poincaré field at $H = 0$ [15, 16]. The auxiliary fields do not have any counterpart in the null curvature limit, but they appear in the indecomposable representations of the massless fields and also in the conformal invariance of the massless fields [15].

The free field operators, which correspond to the principal and complementary series representations and discrete series with $j \neq p$ can be constructed by using the principle (B). The field operators in these cases transform by the UIR of the dS group, and by defining the creation and annihilation operators on the corresponding Hilbert spaces, the quantum free field operators can be explicitly calculated similar to the Minkowskian space-time, presented by Weinberg [17]. The UIR of dS group can exactly address the quantum states or the vectors in Hilbert space and then the creation operator and vacuum states (up to the state normalization) are defined from this quantum states. For obtaining a well-defined field operator, it must be constructed on the complex dS space-time [5, 6]. Then the
analytic two-point function can be calculated directly from the complex dS plane wave and the vacuum states, up to the normalization constant. The normalization constant can be fixed by imposing the local Hadamard condition \[6\]. The two-point function or the probability amplitude, which is the building blocks of the quantum mechanics, is the boundary value of this analytic two-point function \[6\].

The massless conformally coupled scalar fields correspond to the complementary series representation with \(j = 0\) and \(p = 0\) or \(p = 1\) \[6, 8\], where these two values of \(p\) are unitary equivalent \[4\]. The massless spinor fields correspond to the discrete series representation with \(j = p = \frac{1}{2}\) which is conformally invariant \[4, 8, 18\]. The procedure of defining the field operators for these fields are also similar to the massive case and for all of them, the fields operators are defined as a map on the Fock space:

\[
\text{Field Operators : } \mathcal{F}(\mathcal{H}) \longrightarrow \mathcal{F}(\mathcal{H}),
\]

which is constructed by the corresponding Hilbert spaces.

The massless minimally coupled scalar fields correspond to the value \(j = 0\) and \(p = 2\) which can not correspond with an UIR of dS group. Previously we constructed the minimally coupled scalar field operator in Krein space which transform by an indecomposable representation of dS group \[19\]. But it break the analyticity. Here using the ambient space formalism we obtain the field operator which transform by an indecomposable representation of dS group and satisfy the analyticity properties. It can be written in terms of the massless conformally coupled scalar fields.

The other free massless fields correspond to \(j = p \geq 1\) \[8, 15, 16\]. In these cases, the Hilbert spaces and consequently the quantum states cannot be defined uniquely and there appear a gauge invariance. For the quantum states to be properly obtained, one must fix the gauge. Then the action of the creation operator on the Hilbert spaces results in states which are out of the Hilbert spaces. Therefore, the field operators act on vector spaces which are constructed by an indecomposable representations of dS group. For such fields the massless quantum states can be divided into the three parts:

\[
\mathcal{M} = V_1 \oplus V_2 \oplus V_3,
\]

where \(V_1\) and \(V_3\) are the space of the gauge dependent states and the pure gauge states, respectively. The space \(V_2/V_3 \equiv \mathcal{H}\) is a vector space containing the physical states which are the Hilbert spaces constructed by the corresponding UIR of dS group. This is known as the Gupta-Bleuler triplets \[19–21\]. In these cases the massless field operators are defined as a map on the Fock space which are constructed on the vector space \(\mathcal{M}\):

\[
\text{Massless Field Operators : } \mathcal{F}(\mathcal{M}) \longrightarrow \mathcal{F}(\mathcal{M}).
\]

The structure of unphysical states \(V_3\) and \(V_1\) are obtained by using the gauge invariant transformation and the gauge fixing field equation, respectively. The gauge invariant field equation and the gauge invariant transformation are obtained by using the gauge principle. The physical states (correspond to the Hilbert spaces) are associated with the UIR of the discrete series representation of dS group with \(1 \leq j = p \leq 2\). The vector field \((j = p = 1)\) was considered in a previously published paper \[15\]. The structure of the indecomposable representation of the vector-spinor field \((j = p = \frac{3}{2})\) and linear conformal quantum gravity \((j = p = 2)\) will be considered in forthcoming papers \[22, 23\]. The interactions between the various fields are also written utilizing the gauge principle.

In dS ambient space formalism the tensor or spinor fields are homogeneous functions of degree \(\lambda\) which can be written in terms of a polarization tensor (or spinor) and the dS plane wave \[8, 12, 15, 24\],

\[
\Psi_{\alpha_1...\alpha_l}(x) = D_{\alpha_1...\alpha_l}(x, \partial, \lambda)(x.\xi)^\lambda.
\]
The coordinate system $x^\alpha$ is a five-vector in ambient space notation whereas the $\xi^\alpha$ is a null five-vector which in the null curvature limit becomes the energy-momentum four-vector $k^\mu = (k^t, \vec{k})$. The homogeneous degree $\lambda$ in the null curvature limit has a relation with the mass in Minkowski spacetime. For the discrete series with $p > 2$ the homogeneous degree is positive ($\lambda \geq 0$), therefore, the dS plane wave cannot be defined properly since the plane wave solution has singularity in $x \rightarrow \infty$ or the infra-red divergence appears. In ambient space formalism only the massless fields with $j = p \leq 2$ can propagate and the dS plane wave for the massless fields with $j = p > 2$ are infinite for large $x$. Consequently, the massless fields are allowed to be existed only for the value $j = p \leq 2$. The mass which associates to the $p > 2$ becomes imaginary. Using the principle (D), one can deduce that at the early universe only the massless fields can exist and after the conformal symmetry breaking, such fields get mass and therefore, in the dS universe only the massive and massless field with $j \leq 2$ are present. Hence, in this paper we consider only the various spin fields with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$. Only three types of massless gauge fields or gauge potential exist; spin-1 vector fields ($K_\alpha$, $j = p = 1$), spin-$\frac{3}{2}$ vector-spinor fields ($\Psi_\alpha$, $j = p = \frac{3}{2}$) and spin-2 field ($\lambda_\alpha$, $j = p = 2$). For a spin-2 field there are two possibilities for construction of the field operator: a rank-2 symmetric tensor field ($\mathcal{H}_\alpha\beta$) or a rank-3 mix-symmetric tensor field ($\mathcal{K}_\alpha\beta\gamma^M$).

The massless gauge vector fields can be associated to the electromagnetic, weak and strong nuclear forces in the framework of the abelian and non-abelian gauge theory which will be reformulated in the ambient space formalism. The massless spin-2 field with a rank-2 symmetric tensor field in Dirac six-cone or dS ambient space formalisms breaks the conformal transformation [25–27] and therefore, the ambient space formalism. The massless spin-2 field must be represented by a rank-3 mix-symmetric tensor field ($\mathcal{K}_\alpha\beta\gamma^M$). This part can be quantized and preserves the covariant principle under the action of the de Sitter and the conformal groups simultaneously. Thus $\mathcal{K}_\mu$ describes the (helicity $\pm 2$, massless) radiated quanta, while $\theta$ provides the geodesics for matter to move on. This separation of waves and background could turn out to be an important feature for description of gravitational waves.

It is interesting to note that there exist another massless vector-spinor fields or gauge potentials which corresponds to the discrete series representation with $j = p = \frac{3}{2}$. This gauge potential is spinor and also a Grassmannian function, satisfying the anti-commutation relations. On the other hand, the infinitesimal generators of this gauge group $(Q_i)$ must be spinorial or Grassmannian functions and they must satisfy the anti-commutation relations. The third principle (C) leads us to the super-algebra, where in this case the multiplication of two spinor-generators or the Grassmanian functions become the usual function or algebra and therefore, the spinor-generators cannot satisfy a closed super-algebra. We need the dS group generators $L_{\alpha\beta}$ for obtaining a closed super-algebra [28],

$$\{Q_i, Q_j\} = \left(S_{\alpha\beta}^{\gamma\lambda} \gamma^\lambda \gamma^\gamma\right)_{ij} L_{\alpha\beta}.$$  

The dS group generators, $L_{\alpha\beta}$, in the gauge gravity model are coupled with the gauge potential $\mathcal{K}_{\alpha\beta\gamma}(x)$. In the framework of the gauge theory, the vector-spinor gauge field $\Psi_\alpha$ may be considered as a potential of a new force in the nature but this gauge field must couple with the the gauge potential $\mathcal{K}_{\alpha\beta\gamma}$ and consequently, $\Psi_\alpha$ may be considered as a new part of the gravitational field. It means that the gravitational field may be composed of three parts, namely; the background $\theta_{\alpha\beta}$, the gravitational waves $\mathcal{K}_{\alpha\beta\gamma}^M$ and $\Psi_\alpha$.

The content of this paper is organized as follows. In the next section, the applied mathematical notation of the paper has been introduced, including the definition of the dS group, two independent Casimir operators and the UIR of the dS group. Section III is devoted to defining the field equations for various spin fields in the dS ambient space formalism in two different spaces: $x$-space and $\xi$-space,
which play the similar role of space-time and energy-momentum in Minkowski space-time. Utilizing the plane wave solution in dS ambient space formalism we calculate the homogeneous degrees of various spin fields. This process is analogous to the first quantization in the framework of Minkowski space-time.

The massive quantum field operators (or second quantization) are presented in section IV, where the massive free field operators and the corresponding two-point functions for spin $0, \frac{1}{2}, 1, \frac{3}{2}, 2$ are introduced. A unique vacuum state, Bunch-Davies or Hawking-Ellis vacuum state, is defined for these fields by using the local Hadamard condition. In section V, the quantum field operators for discrete series are studied. This section is divided in two parts for $j \neq p$ and $j = p$. For $j \neq p$ the procedure of defining the field operator is exactly similar to the massive fields. These fields are the auxiliary since they appear in the indecomposable representations of the quantum massless fields. For $j = p \geq 1$, we discuss that one cannot define the creation operators for the quantum states in the Hilbert space because of the gauge invariant, since the quantum state cannot be defined uniquely. For the various massless spin fields $(j = 1, \frac{3}{2}, 2)$ and based on the definition of the gauge-covariant derivative in ambient space formalism, the gauge invariant Lagrangian is obtained for three different types of gauge potentials in section VI. Then the gauge fixing terms are presented and the gauge fixing field equations are calculated.

The massless quantum fields are introduced in section VII. The field operators of massless conformally coupled scalar fields and corresponding two point functions are constructed on the Bunch-Davies vacuum state. We introduce a magic identity in ambient space formalism, in section VII-B, which permits us to write the massless minimally coupled scalar field in terms of the massless conformally coupled scalar field. Therefore we define the field operator and the two-point function of a minimally coupled scalar field in terms of a conformally coupled scalar field and its two-point function. It means that a unique vacuum state, Bunch-Davies vacuum state, is used for construction of the minimally coupled scalar field or equivalently for the linear gravity in dS space. Then the problem of the infra-red divergence of the linear gravity in dS ambient space formalism is completely solved in the Bunch-Davies vacuum state.

The massless spinor field can be defined similar to the massive field. In this case a new type of invariance has been introduced. The massless fields, with $j = 1, \frac{3}{2}, 2$ are also constructed in section VII. Since the massless spin-2 rank-2 symmetric tensor field breaks the conformal invariance, the massless spin-2 rank-3 mix-symmetric tensor field is also studied. It has been shown that the two-point functions of all the massless fields can be written in terms of a polarization tensor (or spinor) and the two-point functions of the massless conformally coupled scalar field.

In section VIII, a brief review of the massless conformal field theory in dS ambient space is presented. First the Dirac 6-cone formalism has been discussed which can be simply mapped to the dS ambient space notation, then discrete series representation of the conformal group $SO(2, 4)$ is represented. The conformal gauge gravity based on the massless spin-2 rank-3 mixed-symmetric tensor fields has been discussed. Finally in section IX the relation between our construction (ambient space formalism) with the intrinsic coordinates is studied and the structure of the maximally symmetric bi-tensors are introduced. The conclusions and an outlook for further investigation have been presented in section X.

II. NOTATION

In this section we briefly recall the notation and conventions which are used in this paper. The dS space-time can be identified with a 4-dimensional hyperboloid embedded in 5-dimensional Minkowskian space-time:

$$M_H = \{x \in \mathbb{R}^5 \mid x \cdot x = \eta_{\alpha\beta} x^\alpha x^\beta = -H^{-2}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4, \quad (II.1)$$
where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$ and $H$ is Hubble parameter. The dS metrics is
\[
d s^2 = \eta_{\alpha\beta} d x^\alpha d x^\beta|_{x^2 = -H^{-2}} = g_{\mu\nu}^{\text{dS}} d x^\mu d X^\nu, \quad \mu = 0, 1, 2, 3, \tag{II.2}
\]
with $X^\mu$ as a 4 space-time intrinsic coordinates on dS hyperboloid. In this paper we use the 5-dimensional Minkowskian space-time $x^\alpha$ with the condition (II.1) which is called the ambient space formalism. For simplicity reasons, $H$ has set to be unity in some equations and been inserted again, whenever it was needed.

### A. de Sitter group

The action of dS group, $SO(1, 4)$, on the intrinsic coordinate $X^\mu$ is non-linear, but its action on the ambient space coordinate $x^\alpha$ is linear:
\[
x'^\alpha = \Lambda^{\alpha}_{\beta} x^\beta, \quad \Lambda \in SO(1, 4) \Rightarrow x \cdot x = x' \cdot x' = -H^{-2},
\]
where $SO(1, 4) = \{ \Lambda \in \text{GL}(5, \mathbb{R}) | \det \Lambda = 1, \ \Lambda \eta \Lambda^t = \eta \}$. $\Lambda^t$ is the transpose of $\Lambda$. The ambient space coordinate $x^\alpha$ can be defined by a $4 \times 4$ matrix $X$:
\[
X = \begin{pmatrix} x^0 I & x \\ x^\dagger & x^0 I \end{pmatrix}, \tag{II.3}
\]
where $I$ is $2 \times 2$ unit matrix and
\[
x = \begin{pmatrix} x^4 + i x^3 & i x^1 - x^2 \\ i x^1 + x^2 & x^4 - i x^3 \end{pmatrix}. \tag{II.4}
\]
x can be defined as a quaternion $x \equiv (x^4, \vec{x})$ with $\bar{x} \equiv (x^4, -\vec{x})$ as its quaternion conjugate \[4, 8\]. The quaternion norm is defined by
\[
|x| = (x \bar{x})^{\frac{1}{2}} = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}.
\]
In the matrices notation we have $\bar{x} = x^\dagger$ and the norm can be written as $|x|^2 = \frac{1}{4} \text{tr}(xx^\dagger)$. The matrix $X$ may be introduced in an alternative form, which is convenient for our considerations in this paper:
\[
\delta x = \eta_{\alpha\beta} \gamma^\alpha x^\beta = X \gamma^0 = \begin{pmatrix} x^0 I & -x \\ \bar{x} & -x^0 I \end{pmatrix}, \quad \delta x \cdot \delta x = x \cdot x \ II, \quad \frac{1}{4} \text{tr}(\delta x \cdot \delta x) = x \cdot x, \tag{II.5}
\]
where $I$ is a $4 \times 4$ unit matrix and the five matrices $\gamma^\alpha$ satisfy the following conditions \[8, 18, 29\]:
\[
\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2 \eta^{\alpha\beta}, \quad \gamma^{\alpha \dagger} = \gamma^0 \gamma^\alpha \gamma^0.
\]
The following representation for $\gamma$ matrices is used in this paper which is appropriate for our formalism \[8, 18, 29\]:
\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, & \gamma^1 &= \begin{pmatrix} 0 & I \\ i \sigma^1 & 0 \end{pmatrix}, & \gamma^2 &= \begin{pmatrix} 0 & -i \sigma^2 \\ -i \sigma^2 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & i \sigma^3 \\ i \sigma^3 & 0 \end{pmatrix},
\end{align*} \tag{II.6}
\]
where $\sigma^i \ (i = 1, 2, 3)$ are the Pauli matrices. In this representation for $\gamma$ matrices, the matrix $\beta$ under the group $Sp(2, 2)$ transform as [29]:

$$\beta' = g \beta g^{-1}, \ g \in Sp(2, 2), \ \text{tr} (\beta, \beta') = \text{tr} (\beta', \beta') \implies x \cdot x = x' \cdot x' = -H^{-2},$$

where the group $Sp(2, 2)$ is defined by:

$$Sp(2, 2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det g = 1, \ \gamma^0 g^0 \gamma^0 = g^{-1} \right\}. \quad (\text{II.7})$$

The elements $a, b, c$ and $d$ are quaternions and $Sp(2, 2)$ is the universal covering group of $SO(1, 4)$ [18, 29]:

$$SO_0(1, 4) \approx Sp(2, 2)/\mathbb{Z}_2, \ \Lambda_{\alpha} \gamma^\alpha = g \gamma^\beta g^{-1}. \quad (\text{II.8})$$

For the $4 \times 4$ matrix $g$, we have $g^\dagger = g^\dagger$. The ambient space coordinate $x^\alpha$ under the action of the group $Sp(2, 2)$ transforms as [31]:

$$\begin{cases} x_0' = (|a|^2 + |b|^2)x_0 + axb + bxa, \\
x_\alpha' = (a\tilde{c} + b\tilde{d})x_0 + b\tilde{x}c + ax\tilde{d}. \end{cases} \quad (\text{II.9})$$

In this paper two different types of homogeneous spaces are used for construction of the UIR of the dS group: the quaternion $u \equiv (u^4, i\bar{u})$ with norm $|u| = 1$ which is called three-sphere $S^3$ (or u-space), and the quaternion $q \equiv (q^4, \bar{q})$ with norm $|q| < 1$ which is called the closed unit ball or for simplicity the unit ball $B$ (or q-space). Their transformations under the group $Sp(2, 2)$ are [31]:

$$u' = g \cdot u = (au + b)(cu + d)^{-1}, \ |u'| = 1,$$

$$q' = g \cdot q = (aq + b)(cq + d)^{-1}, \ |q'| < 1. \quad (\text{II.10})$$

These two homogeneous spaces can be considered as the subspaces of the positive cone $C^+$, which is defined by $C^+ = \{ \xi \in \mathbb{R}^4 | \xi \cdot \xi = 0, \ \xi^0 > 0 \}$. Then in a unique way, the null five vector $\xi^0 = (\xi^0, \tilde{\xi}^4) \in C^+$ can be written as:

$$\xi^0_u \equiv (\xi^0, \xi^0 u), \ |u| = 1; \quad \xi^0_B \equiv (\xi^0, \xi^0 \coth \kappa q), \ |q| = |\tanh \kappa| = r < 1. \quad (\text{II.11})$$

Since $\xi \cdot \xi = 0$, the $\xi^0$ is completely arbitrary from the mathematical point of view, i.e. it is scale invariant and transforms under the dS group as [30]

$$\xi^0 = \xi^0 |cu + d|^2. \quad (\text{II.12})$$

If we chose $q = ru$ with $r < 1$, we obtain

$$\xi^0_B = (\xi^0, \xi^0 \coth \kappa q) \equiv (\xi^0, \xi^0 u) = \xi^0_u.$$

With this choice the unit ball may be considered as the compactified Minkowski space as a group manifold of the unity group $U(2)$ [31, 33]. It may be considered also as the Shilov boundary of the bounded homogeneous complex domain $SU(2, 2)/S(U(2) \times U(2))$ [32, 33].

In this notation $x \cdot \xi$ can be written in the following form:

$$\beta \xi + \xi \beta = 2 x \cdot \xi \mathbb{I} \implies x \cdot \xi = \frac{1}{4} \text{tr} \beta \xi, \quad (\text{II.13})$$
and under the action of the dS group it is a scalar:

$$x' \cdot \xi' = x \cdot \xi, \quad \text{tr } \beta' \xi' = \text{tr } \beta \xi.$$  \hspace{1cm} (II.14)

The $\xi$-space plays the role of the energy-momentum $k^\mu$ in Minkowski space-time and it can be chosen for massive field as \[5, 8\]

$$\xi^a_\alpha = \xi^0 \left(1, \frac{k}{k^0}, \frac{H\nu}{k^0}\right), \quad k^0 \neq 0, \ \nu \neq 0, \quad (k^0)^2 - \vec{k} \cdot \vec{k} = (H\nu)^2,$$

where $\nu$ is introduced in the next subsection. For massless field $\xi$ can be defined by:

$$\xi^a_\alpha = \xi^0 \left(1, \frac{q}{\tau}, \frac{H}{k^0}\right), \quad k^0 \neq 0, \quad (k^0)^2 - \vec{k} \cdot \vec{k} = H^2.$$

In the null curvature limit we obtained exactly the massive and the massless energy momentum, $(k^0)^2 - \vec{k} \cdot \vec{k} = m^2$ and $(k^0)^2 - \vec{k} \cdot \vec{k} = 0$ respectively.

### B. Casimir operators

The dS group has two Casimir operators

$$Q^{(1)} = -\frac{1}{2} L_{\alpha\beta} L^{\alpha\beta}, \quad \alpha, \beta = 0, 1, 2, 3, 4,$$

$$Q^{(2)} = -W_{\alpha} W^{\alpha}, \quad W_{\alpha} = \frac{1}{8} \epsilon_{\alpha\beta\gamma\delta} L^{\beta\gamma} L^{\delta\eta},$$

where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual anti-symmetric tensor in $\mathbb{R}^5$ and $L_{\alpha\beta}$ are the infinitesimal generators of dS group which can be written as $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$. In the ambient space formalism, the orbital part, $M_{\alpha\beta}$, is

$$M_{\alpha\beta} = -i(x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}) = -i(x_{\alpha} \partial_{\beta} - x_{\beta} \partial_{\alpha}),$$

$$\text{where } \partial_{\beta} = \theta_{\beta}^\alpha \partial_{\alpha} \text{ and } \theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_{\alpha} x_{\beta} \text{ is the projection tensor on dS hyperboloid. In order to pin-point the action of the spinorial part, } S_{\alpha\beta}, \text{ on a tensor field or tensor-spinor field, one must treat the integer and half-integer cases, separately. Integer spin fields can be represented by a symmetric tensor field of rank } l, \ K_{\gamma_1...\gamma_4(l)}, \text{ and the spinorial action reads as } [8, 12, 13, 24]:$$

$$S^{(l)}_{\alpha\beta} K_{\gamma_1...\gamma_4(l)} = -i \sum_{i=1}^{l} (\eta_{\alpha\gamma_i} K_{\gamma_1...(\gamma_i \rightarrow \beta)...\gamma_4} - \eta_{\beta\gamma_i} K_{\gamma_1...(\gamma_i \rightarrow \alpha)...\gamma_4}),$$

where $(\gamma_i \rightarrow \beta)$ means $\gamma_i$ index, replaced by $\beta$. Half-integer spin fields with spin $s = l + \frac{1}{2}$ are represented by four component symmetric tensor-spinor field $[\Psi_{\gamma_1...\gamma_4(l)}]$ with the spinor index $i = 1, 2, 3, 4$. In this case, the spinorial part is

$$S^{(s)}_{\alpha\beta} = S^{(l)}_{\alpha\beta} + S^{(\frac{3}{2})}_{\alpha\beta} \quad \text{with} \quad S^{(\frac{3}{2})}_{\alpha\beta} = \frac{i}{4} [\gamma_{\alpha}, \gamma_{\beta}].$$

The $S^{(\frac{3}{2})}_{\alpha\beta}$ acts only on the spinor index $i$.

For $j = l = \text{integer}$, the action of the Casimir operator $Q^{(1)}_j$ on a rank-$l$ symmetric tensor filed $K_{\alpha_1\alpha_2...\alpha_l}(x) \equiv K(x)$ is obtained as \[8, 13\]:

$$Q^{(1)}_l K(x) = Q^{(1)}_0 K(x) - 2\Sigma_1 \partial x \cdot K(x) + 2\Sigma_1 x \partial \cdot K(x) + 2\Sigma_2 \eta K'(x) - l(l + 1) K(x),$$

(II.19)
where

\[ Q^{(1)}_0 = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = -H^{-2} \frac{\partial^T \cdot \partial^T}{2} \equiv -H^{-2} \Box_H. \]  

(II.20)

\( \Box_H \) is the Laplace-Beltrami operator on dS hyperboloid. \( K' \) is the trace of the rank-\( l \) tensor field \( K(x) \) and \( \Sigma_p \) is the non-normalized symmetrization operator:

\[ K'_{\alpha_1...\alpha_{l-2}} = \eta^{\alpha_{l-1}\alpha_l} K_{\alpha_1...\alpha_{l-2}\alpha_{l-1}\alpha_l}, \]  

(II.21)

\[ (\Sigma_p AB)_{\alpha_1...\alpha_l} = \sum_{i_1 < i_2 < ... < i_p} A_{\alpha_1...\alpha_{i_1}} B_{\alpha_{i_1}...\alpha_{i_2}} ... B_{\alpha_{i_p}...\alpha_l}. \]  

(II.22)

The tensor field \( K(x) \) on the dS hyperboloid is a homogeneous function of the variables \( x^\alpha \) with degree \( \lambda \) [36]

\[ x \cdot \partial K(x) = \lambda K(x), \ \text{or} \ K(lx) = l^\lambda K(x). \]

For half-integer case \( j = l + \frac{1}{2} \), the Casimir operator becomes

\[ Q^{(1)}_j = -\frac{1}{2} \left( M_{\alpha\beta} + S^{(l)}_{\alpha\beta} + S^{(l)}_{\frac{1}{2}\beta} \right) \left( M^{\alpha\beta} + S^{\alpha\beta(l)} + S^{\alpha\beta(\frac{1}{2})} \right). \]

Using the identity

\[ S^{(l)}_{\alpha\beta} S^{\alpha\beta(l)} \Psi(x) = l\Psi(x) - \Sigma_1 \gamma(\gamma \cdot \Psi(x)), \]

the action of the Casimir operator \( Q^{(1)}_j \) on a rank-\( l \) symmetric tensor-spinor field \( \Psi_{\alpha_1\alpha_2...\alpha_l}(x) \equiv \Psi(x) \) is [8, 13]:

\[ Q^{(1)}_j \Psi(x) = \left( -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} + \frac{i}{2} \gamma_{\alpha\beta} M^{\alpha\beta} - l(l + 2) - \frac{5}{2} \right) \Psi(x) \]

\[ - 2 \Sigma_1 \partial x \cdot \Psi(x) + 2 \Sigma_1 x \partial \cdot \Psi(x) + 2 \eta \Psi'(x) + \Sigma_1 \gamma(\gamma \cdot \Psi(x)) . \]  

(II.23)

C. Unitary irreducible representation

The Casimir operators commute with the action of the group generators and, as a consequence, they are constant on each UIR of the dS group. The UIR of the dS group are classified by the eigenvalue of the Casimir operators [1, 3]:

\[ Q^{(1)}_{j,p} = (-j(j + 1) - (p + 1)(p - 2)) I_d \equiv Q^{(1)}_j \]  

(II.24)

\[ Q^{(2)}_{j,p} = (-j(j + 1)p(p - 1)) I_d, \]  

(II.25)

where \( I_d \) is the identity operator. Three types of representations, corresponding to the different value of the parameters \( j \) and \( p \), exist:

- Principal series representation

\[ j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, ..., \quad p = \frac{1}{2} + i \nu, \quad \left\{ \begin{array}{l} \nu \geq 0 \ \text{for} \ j = 0, 1, 2, ..., \\ \nu > 0 \ \text{for} \ j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ... \end{array} \right. \]  

(II.26)
• Discrete series representation

\[ j \geq p \geq 1 \quad \text{or} \quad \frac{1}{2}, \quad j - p = \text{integer number}, \]

\[ j = 1, 2, 3, \ldots, \quad p = 0. \quad \text{(II.27)} \]

• Complementary series representation

\[ j = 0, \quad -2 < p - p^2 < \frac{1}{4}, \]

\[ j = 1, 2, 3, \ldots, \quad 0 < p - p^2 < \frac{1}{4}. \quad \text{(II.28)} \]

The principal series representation of dS group was constructed on the compact homogeneous space three-sphere, \( S^3 \) or \( u \)-space [4]:

\[
U^{(j,p)}(g) |u, m_j; j, p \rangle = |cu + d|^{-2(1+p)} \sum_{m'_j} D^{(j)}_{m_m',m'_j} \left( \frac{(cu + d)^{-1}}{|cu + d|} \right) |g^{-1} \cdot u, m'_j; j, p \rangle,
\]

where \( p = \frac{1}{2} + i \nu \) and \( g^{-1} \cdot u = (au + b)(cu + d)^{-1} \) with \( g^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in Sp(2,2) \). The action of the group element on quaternion \( u \) is defined in [11, 10]. \( D^{(j)}_{m_m',m'_j} \) furnish a certain representation of \( SU(2) \) group in a \((2j + 1)\)-dimensional Hilbert space \( V^j \) which is defined explicitly in [11, 34, 37]:

\[
D^{(j)}_{m_m',m'_j}(u) = \sum_n \frac{(j + m')!}{n!(j + m' - n)!} \frac{(j - m')!}{(j + m - n)!(n - m - m')!} u_{11}^{j + m - n} u_{12}^{j + m' - n} u_{21}^{n - m - m'}.
\]

The representation \( D^{(j)} \) can be also defined in terms of the representation of the little group \( SO(3) \), which preserves [17]

\[ \xi^\alpha_u(0) = \xi^0 (1, 0, 0, 0, 1), \quad u(0) = (0, 0, 0, 1), \quad \text{(II.31)} \]

and one can use the "Lorentz boost" to obtain \( \xi^\alpha_u \equiv \xi^0 \left( 1, \frac{k}{k^2}, H_{uv} \right) \). The representation \( U^{(j,p)} \) acts on an infinite dimensional Hilbert space \( \mathcal{H}_{u}^{(j,p)} \):

\[ |u, m_j; j, p \rangle \in \mathcal{H}_{u}^{(j,p)}, \quad \xi_u = (\xi^0, \xi^0 u), \quad \xi^0 > 0, \quad |u| = 1, \quad -j \leq m_j \leq j. \]

The two representations with \( \sigma_1 = 1 + p = \frac{1}{2} + i \nu \) and \( \sigma_2 = 2 - p = \frac{1}{2} - i \nu = \sigma^*_1 \) which satisfy \( \sigma_1 + \sigma_2 = 3 \) (or \( p \to 1 - p \)), are unitary equivalent (Théoréme 2.1. page 389 in [4]). It means that there exist a unitary operator \( S \) \((SS^\dagger = 1)\), which one can write:

\[ S |u, m_j; j, p \rangle = |u, m_j; j, 1 - p \rangle, \quad SU^{(j,p)}(g)S^\dagger = U^{(j,1-p)}(g), \quad \text{(II.32)} \]

where

\[ \langle u', m'_j; j, p |u, m_j; j, p \rangle = \langle u', m'_j; j, 1 - p |u, m_j; j, 1 - p \rangle. \]

For the principal series, one has \( 1 - p = p^* \). These two unitary equivalent representations are needed for construction of the quantum field operator in terms of creation and annihilation operators, which create and annihilate the quantum states.

The unitary irreducible representation of the dS group for discrete series is constructed on the unit ball homogeneous space or \( q \)-space [4]:
The discrete series representations

\[ T^{(j_1,j_2,p)}(g) |q, m_{j_1}, m_{j_2}; j_1, j_2, p) = |cq + d|^{-2(1+p)} \times \]

\[
\sum_{m'_{j_1}, m'_{j_2}} D^{(j_1)}_{m'_{j_1}, m_{j_1}} \left( \frac{(a + bq)^{-1}}{|cq + d|} \right) D^{(j_2)}_{m_{j_2}, m'_{j_2}} \left( \frac{(cq + d)^{-1}}{|cq + d|} \right) |g^{-1} \cdot q, m'_{j_1}, m'_{j_2}; j_1, j_2, p) . \quad (II.33)\]

In this case, one has an infinite dimensional Hilbert space \( \mathcal{H}^{(j_1,j_2,p)}_q \),

\[ |q, m_{j_1}, m_{j_2}; j_1, j_2, p) \in \mathcal{H}^{(j_1,j_2,p)}_q, \quad q = ru \in \mathbb{R}^4, |q| = r < 1, \quad |u| = 1, \quad -j \leq m_j \leq j. \]

The discrete series representations \( T^{(j,0,p)} \) and \( T^{(0,j,p)} \) are proportional to the two representations \( \Pi^+_{j,p} \) and \( \Pi^-_{j,p} \) in the Dixmier notation \[3\];

\[ T^{(0,j,p)}(g) |q, m_j; j, p) = |cq + d|^{-2p-2} \sum_{m'_{j}} D^{(j)}_{m_{j}, m'_{j}} \left( \frac{(cq + d)^{-1}}{|cq + d|} \right) |g^{-1} \cdot q, m'_{j}; j, p) , \quad (II.34)\]

and

\[ T^{(j,0,p)}(g) |q, m_j; j, p) = |cq + d|^{-2p-2} \sum_{m'_{j}} D^{(j)}_{m'_{j}, m_{j}} \left( \frac{(a + bq)^{-1}}{|cq + d|} \right) |g^{-1} \cdot q, m'_{j}; j, p) . \quad (II.35)\]

The representations \( \Pi^\pm_{j,j} \) in the null curvature limit correspond to the Poincaré massless field with helicity \( \pm j \) \[17, 38, 39\]. For a massless field, \( D^{(j)} \) furnish a certain representation of the little group \( ISO(2) \) \[17\] and we have \( m_j = m'_j = j \) for \( \Pi^+_{j,j} \) and \( m_j = m'_j = -j \) for \( \Pi^-_{j,j} \) and the other cases of \( m_j \) and \( m'_j \) vanish. In this case the little group \( ISO(2) \) preserve the five-vector

\[ \xi^0_B(0) ,\]

and then, one can use the 'Lorentz boost' to obtain \( \xi^0_B \equiv \xi^0 \left( 1, \frac{k}{\sqrt{2}}, \frac{\bar{k}}{\sqrt{2}} \right) \). These representations were constructed previously \[17, 31, 40\].

For the representations \[11.34\] and \[11.35\], similar to the principal series, one can show that the two representations with the values \( p_1 \) and \( p_2 \) which satisfy \( p_1 + p_2 = 1 \), are unitary equivalent \[4\]:

\[ S |q, m_j; j, p) = |q, m_j; j, 1-p) , \quad S^2 T^{(0,j,p)}(g) S^2 = T^{(0,j;1-p)}(g) , \quad (II.37)\]

with \( SS^2 = 1 \). It means that the representation \( T^{(0,j,p)}(g) \) for the values \( p = \frac{1}{2}, 1, \frac{3}{2}, 2, ... \) are unitary equivalent with \( p = \frac{1}{2}, 0, -\frac{1}{2}, -1, ... \). The eigenvalues of the Casimir operators for these two set of value or under the transformation \( p \rightarrow 1 - p \), do not change. The equations \[11.32\] and \[11.37\] are used to define the quantum field operators with various spin in sections IV and V.

The complementary series representations can be only associated with the tensor fields with \( j = 0, 1, 2, ... \). These representation are constructed on three-sphere \( S^3 \) or \( u^- \)– space \[3, 4\]. Among these representation the scalar representation with \( j = p = 0 \) has only corresponding Poincaré group representation in the null curvature limit. This representation associate with the massless conformally coupled scalar field and it will be considered in this paper \[4\]. The massless conformally coupled scalar field is the building block of the massless fields in dS space, since the other massless spin fields can be constructed by this field. The complementary series representations with \( j = p = 0 \) is defined as \[4\]:

\[ U^{(0,0)}(g) |u, 0; 0, 0) = |cu + d|^{-2} |g^{-1} \cdot u, 0; 0, 0) , \quad (II.38)\]
where \( g^{-1} \cdot u = (au + b)(cu + d)^{-1} \) with \( g^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in Sp(2,2) \). The scalar product in these Hilbert spaces are presented by a function \( W(u, u') \) which is defined on \( S^3 \times S^3 \).

The unitary equivalent representation \([II.38]\) is correspond to \( j = 0, p = 1 \) and it is:

\[
U^{(0,1)}(g) |u, 0; 0, 1\rangle = |cu + d|^{-4} |g^{-1} \cdot u, 0; 0, 1\rangle.
\]

Using an unitary operator \( S \), the following relations between these two representations and corresponding states can be defined:

\[
S|u, 0; 0, 1\rangle = |u, 0; 0, 1\rangle, \quad SU^{(0,0)}(g)S^l = U^{(0,1)}(g).
\]

The most important result of this formalism is that the total volume of the homogeneous spaces which the UIR of the dS group constructed on, \( i.e. \) \( q \)-space, \( u \)-space or \( \xi \)-space are finite

\[
\int_{S^3} d\mu(u) = 1, \quad \int_B d\mu(q) = \frac{\pi^2}{2},
\]

where \( d\mu(u) \) is the Haar measure \([4]\) or \( SO(4) \)-invariant normalized volume on three-sphere \( S^3 \) and \( d\mu(q) = 2\pi^2 r^3 dr d\mu(u) \) is the Euclidean measure on the unit ball \( B \). We have used the definition \( q = ru \in B \) with \( u \in S^3, r < 1 \). In the following section we discus that the two \( x \)- and \( \xi \)-spaces play a similar role of the space-time and the energy-momentum spaces in Minkowskian space-time. Since a maximum length for an observable (or an even horizon in dS space or \( x \)-space) exist then a minimum size in the \( \xi \)-space (or the parameters in Hilbert space) can be defined by using the Heisenberg uncertainty principle. Each point in \( \xi \)-space represents a vector in Hilbert space and the number of points is infinite mathematically. Since the total volume of \( \xi \)-space is finite and a minimum length in \( \xi \)-space exists from uncertainty principle, therefore the total number of points become finite physically. It means that the total number of quantum states in these Hilbert spaces is finite.

As an example, we consider the compact space \( S^1 \), where the total volume of space is finite \((2\pi R)\) but the total number of points is infinite. If a minimum length such as Planck length \( l_p \) exist, the total number of points becomes finite \( N = 2\pi R(l_p)^{-1} \). Therefore the total number of quantum states depends to the scale of energy in the system or to the minimum length.

Although one can mathematically define an infinite dimensional Hilbert space, by accepting a minimum length in \( \xi \)-space, the total number of quantum states is physically finite. For principal series \( \mathcal{H}_{u}^{(j,p)} \) we have \([3]\)

\[
N_{\mathcal{H}_{u}^{(j,p)}} = (2j + 1) \int_{S^3} d\mu(\xi_u) = f(H, j, \nu, \xi^0) = \text{finite value}, \tag{II.41}
\]

and for discrete series \( \mathcal{H}_{q}^{(j_1,j_2,p)} \), it is

\[
N_{\mathcal{H}_{q}^{(j_1,j_2,p)}} = (2j_1 + 1)(2j_2 + 1) \int_B d\mu(\xi_B) = \text{finite value}. \tag{II.42}
\]

The total number of quantum states is a function of \( H, p, j \) and \( \xi^0 \). This result is due to the existence of a minimum length and the compactness of the homogeneous spaces which the Hilbert spaces (or the UIR) are constructed on. Then the total number of quantum states becomes finite \([3]\).

D. Physical conditions

The tensor or tensor-spinor fields on dS hyperboloid in the ambient space formalism which are associated with the UIR’s of dS group, must satisfy the following physical conditions:
i) The field equations are:
\[
\left( Q^{(1)}_l - \left< Q^{(1)}_{l,p} \right> \right) K(x) = 0, \quad \left( Q^{(1)}_j - \left< Q^{(1)}_{j,p} \right> \right) \Psi(x) = 0.
\] (II.43)

ii) The tensor or tensor-spinor fields are homogeneous functions with degree \( \lambda \):
\[
x \cdot \partial K(x) = \lambda K(x), \quad x \cdot \partial \Psi(x) = \lambda \Psi(x).
\]

iii) The transversality condition:
\[
x \cdot K(x) = 0, \quad x \cdot \Psi(x) = 0.
\]

iv) The divergencelessness condition:
\[
\partial_l^\top \cdot K(x) = 0, \quad \partial_l^\top \cdot \Psi(x) = 0.
\] (II.44)

\((\partial_l^\top \cdot \cdot \cdot)\) is defined as a generalized divergence on the dS hyperboloid, which acts on a tensor or a tensor-spinor field of rank-\( l \) and results in a transverse tensor or a tensor-spinor field of rank-\( l - 1 \).

v) The tracelessness condition:
\[
K'(x) = \eta \cdot K(x) = 0, \quad \Psi'(x) = \eta \cdot \Psi(x) = 0.
\]

vi) The index symmetrization condition:
\[
K_{\ldots \alpha \beta \ldots} = K_{\ldots \beta \alpha \ldots}, \quad \Psi_{\ldots \alpha \beta \ldots} = \Psi_{\ldots \beta \alpha \ldots}.
\]

vii) For the tensor-spinor fields we have
\[
\gamma \cdot \Psi(x) = 0.
\]

These conditions preserve the dS invariance and present the simplest solutions for the wave equations in the dS space. By the principle (B) we assume that the field operators, which satisfy the above conditions must transform as an UIR of the dS group and correspond to the elementary systems [8]. In the next section we present the field equations and the homogeneous degrees of the tensor (-spinor) fields in dS hyperboloid.

### III. FIELD EQUATIONS

In Minkowski space-time one can define the Fourier transformation between two variables \( X^\mu \) and \( k^\mu \), when the four-vector \( X^\mu \) represents space-time coordinates in Minkowski space and the transformed variable \( k^\mu \) represents energy-momentum four-vector. In general, one cannot define the Fourier transformation on a curved manifold, but in the dS space-time, due to the maximally symmetric properties of dS hyperboloid, one can define a similar transformation, namely, the Fourier-Helgason-type transformation \([41, 42]\) or the Bros-Fourier transformation \([7]\). Then, corresponding to any space-time variable \( x^\alpha \), defined in the ambient space formalism, another variable or the transform variable, \( \xi^\alpha \), exists which is defined in the positive cone \( C^+ \). In what follows, these spaces are called \( x \)-space (dS hyperboloid \( M_H \)) and \( \xi \)-space (positive cone \( C^+ \)). The \( \xi \)-space can be written in terms of two subspaces: the three-sphere \( S^3(\xi_u) \) and the closed unit ball \( B(\xi_B) \). Then the necessary relations concern to these three different homogeneous spaces are presented \([41, 42]\).
The dS invariant volume element in the $x$-space on the dS hyperboloid is \( d\mu(x) = \frac{dx^{(0)} \wedge dx^{(1)} \wedge dx^{(2)} \wedge dx^{(3)} \wedge dx^{(4)}}{d(x.x + H^{-2})} \bigg|_{M_H} = d\mu(x'). \) (III.1)

The total volumes of these three spaces are:

\[
\int_{dS} d\mu(x) = \infty, \quad \int_{S^3} d\mu(u) = 1, \quad \int_{B} d\mu(q) = \frac{\pi^2}{2}.
\]

The following integral is used for defining the scalar product in the Hilbert space of the discrete series:

\[
\int_{B} (1 - |q|^2)^{2p-2} d\mu(q) = \frac{\pi^2}{2p(2p-1)}.
\]

The dS invariant volume element on three-sphere is \( d\mu(u) \) \( \frac{d\mu(u')}{1 + u'^6}, \) \( d\mu(u') = \frac{d\mu(u)}{|cu + d|^6}, \) (III.2)

where \( u' = g^{-1} \cdot u, \) \( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( |1 + u'| = \frac{|1 + u|}{|cu + d|}. \) Also the dS invariant volume element on the unit ball \( B \) is \( d\mu(q) = \frac{d\mu(q')}{(1 - |q|^2)^4}, \) \( d\mu(q') = \frac{d\mu(q)}{|cq + d|^8}, \) (III.3)

where \( q' = g^{-1} \cdot q \) and \( 1 - |q'|^2 = \frac{1 - |q|^2}{|cq + d|^2}. \) These invariant volume elements are used for defining the transformation of the quantum field operators in dS ambient space formalism.

By using the volume element on theses spaces, the Dirac delta function can be defined as \( 43: \)

\[
\int_{dS} \delta_{dS}(x - x') d\mu(x) = 1,
\]

\[
\int_{S^3} \delta_{S^3}(u - u') d\mu(u) = 1, \quad \int_{B} \delta_{B}(q - q') d\mu(q) = 1.
\]

Using the above Dirac delta function, one can define the orthogonal basis for corresponding Hilbert spaces \( 44, 45: \)

\[
\langle x|x' \rangle \equiv N(x)\delta_{dS}(x - x'), \quad \langle u|u' \rangle \equiv N(u)\delta_{S^3}(u - u'), \quad \langle q|q' \rangle \equiv N(q)\delta_{B}(q - q'),
\]

(III.4)

where \( N(x), \) \( N(q) \) and \( N(u) \) are the normalization constants which can be fixed by imposing the physical conditions such as the local Hadamard condition.

### A. Field equation in $x$-space

The tensor field \( K_{\alpha_1...\alpha_l}(x) \) or tensor-spinor fields \( \Psi_{\alpha_1...\alpha_l}(x) \) which can be associated with the UIR of the dS group must satisfy the conditions (i-vii) in section II-D. The homogeneity condition allows us to write the tensor or tensor-spinor fields in the following forms \( 3, 12, 14, 24, 46: \)

\[
K_{\alpha_1...\alpha_l}(x) = D_{\alpha_1...\alpha_l}(x, \partial; \lambda) (x \cdot \xi)^{\lambda} = U_{\alpha_1...\alpha_l}(x, \xi; \lambda) (x \cdot \xi)^{\lambda},
\]
\( \Psi_{\alpha_1...\alpha_l}(x) = D_{\alpha_1...\alpha_l}(x, \partial; \lambda) (x \cdot \xi)^\lambda = U_{\alpha_1...\alpha_l}(x, \xi; \lambda) (x \cdot \xi)^\lambda, \)

where \( \xi \in C^+ \) and the homogeneous degree of \( \mathcal{U} \) and \( U \) is zero:

\[ x \cdot \partial \mathcal{U} = 0 = x \cdot \partial U, \ x \cdot \partial (x \cdot \xi)^\lambda = \lambda (x \cdot \xi)^\lambda. \]

Utilizing the equations (II.19), (II.24), (II.43) and the conditions (i-vii) in section II-D, the field equation for the tensor field \( K_{\alpha_1...\alpha_l}(x) \) \((j = l)\) reads as:

\[
\left[ Q_0^{(1)} + (p+1)(p-2) \right] K_{\alpha_1...\alpha_l}(x) = 0. \tag{III.5}
\]

This equation can be written in the following form:

\[
\left[ \Box_H - H^2(p + 1)(p - 2) \right] K_{\alpha_1...\alpha_l}(x) = 0, \tag{III.6}
\]

and the associated mass parameter to the tensor field or boson field is

\[
m_{b,p}^2 = -H^2(p + 1)(p - 2). \tag{III.7}
\]

For principal series \((p = \frac{1}{2} + i\nu)\) we have \(m_{b,\nu}^2 = H^2(\frac{9}{4} + \nu^2)\) and for complementary and discrete ones \((p = 0, 1, 2, 3, ...)\) we have \(m_{b,0}^2 = 2H^2, \ m_{b,1}^2 = 2H^2, \ m_{b,2}^2 = 0\) and for \(p > 2\) the mass square is negative (mass is imaginary). These fields may be called the Tackyon fields.

Since the tensor field \( K \) is a homogeneous function with degree of \( \lambda \) we obtain [5, 8]:

\[
Q_0^{(1)} (x \cdot \xi)^\lambda = -\lambda(\lambda + 3) (x \cdot \xi)^\lambda,
\]

then the homogeneous degree \( \lambda \) must satisfy the following equation:

\[-\lambda(\lambda + 3) + (p + 1)(p - 2) = 0,
\]

in which, will have two solutions as follows:

\[
\lambda_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + (p + 1)(p - 2)}. \tag{III.8}
\]

These two solutions can be correspond to the two unitary equivalent representations of dS group (II.32), (II.37) and (II.40). In terms of mass parameter, it can be written as

\[
\lambda_{1,2} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{m_{b,p}^2}{H^2}}.
\]

For principal series, we have:

\[
p = \frac{1}{2} + i\nu : \ \lambda_1 = -\frac{3}{2} + i\nu, \ \lambda_2 = -\frac{3}{2} \quad \text{therefore}, \ \ m_{b,\nu}^2 = H^2 \left( \frac{9}{4} + \nu^2 \right).
\]

In the null curvature limit one obtains the mass in the Minkowskian space [3, 6, 8]:

\[
\lim_{H \to 0, \nu \to \infty} m_{b,\nu}^2 = \lim_{H \to 0, \nu \to \infty} \left( \frac{9}{4}H^2 + H^2\nu^2 \right) = \lim_{H \to 0, \nu \to \infty} (H\nu)^2 = m^2,
\]

and for complementary \((p = 0)\) and discrete series \((p \leq l = j = 1, 2, 3, ...)\) obtains:

\[
p = 0 : \ \lambda_1 = -2, \ \lambda_2 = -1, \ m_{b,1}^2 = 2H^2,
\]
\[ p = 1 : \quad \lambda_1 = -2, \quad \lambda_2 = -1, \quad m_{b,1}^2 = 2H^2, \]

\[ p = 2 : \quad \lambda_1 = -3, \quad \lambda_2 = 0, \quad m_{b,2}^2 = 0, \]

\[ p = 3 : \quad \lambda_1 = -4, \quad \lambda_2 = 1, \quad m_{b,3}^2 = -4H^2, \ldots \quad \text{(III.10)} \]

The minimally coupled scalar field correspond to the value \( j = 0 \) and \( p = 2 \) or \( p = -1 \), therefore, \( \lambda_1 = -3, \quad \lambda_2 = 0 \) and \( m_{b,1}^2 = 0 \), but these are not acceptable values for \( p \) in the UIR of the dS group. The constant solution \( (\lambda = 0) \) poses the zero mode problem and for the definition of a covariant quantum field one must use the Krein space quantization and the field operators transform according to the indecomposable representation of the dS group [19]. This problem is also appears for the rank-2 symmetric tensor field with \( p = j = 2 \) (linear gravity) [46]. In this paper we present another method for solving this problem in section VII-B.

For \( p > 2 \), one of the homogeneous degrees of tensor field becomes positive and the plan wave is \( (x, \xi)^n, n > 0 \). This plane wave is infinite for large \( x \):

\[ \lim_{x \to \infty} (x \cdot \xi)^n \to \infty. \quad \text{(III.11)} \]

Consequently, one cannot define a massless particle with spin \( j = p > 2 \). The class of functions of the type \( (x \cdot \xi)^n, n > 0 \) are not infinitely differentiable. They are not the space of all infinitely differentiable functions that are rapidly decreasing at infinity along with all partial derivatives. One cannot defined the field operators for these cases in a distributions sense since there are not exist a tempered distribution on these functional spaces [6, 17].

Using the equations (II.23), (II.24), (II.43) and the conditions (i-vii) in section II-D, the field equation for tensor-spinor field \( \Psi_{\alpha_1 \ldots \alpha_l}(x) \ (j = l + \frac{1}{2}) \) reads as:

\[ Q_0^{(1)} + i \frac{1}{2} \gamma_\alpha \gamma_\beta M^{\alpha_\beta} + (p + 1)(p - 2) - \frac{7}{4} \] \[ \Psi_{\alpha_1 \ldots \alpha_l}(x) = 0. \quad \text{(III.12)} \]

The following identity:

\[ \left( i \frac{1}{2} \gamma_\alpha \gamma_\beta M^{\alpha_\beta} - 2 \right)^2 = -Q_0^{(1)} - i \frac{1}{2} \gamma_\alpha \gamma_\beta M^{\alpha_\beta} + 4, \quad \text{(III.13)} \]

allows us to obtain the first-order field equations in the form:

\[ i \frac{1}{2} \gamma_\alpha \gamma_\beta M^{\alpha_\beta} - 2 \pm \sqrt{\frac{9}{4} + (p + 1)(p - 2)} \] \[ \Psi_{\alpha_1 \ldots \alpha_l}(x) = 0. \quad \text{(III.14)} \]

By replacing the above equation in the equation (III.12), the second-order field equation may be rewritten as:

\[ Q_0^{(1)} + \frac{1}{4} + (p + 1)(p - 2) \] \[ \pm \sqrt{\frac{9}{4} + (p + 1)(p - 2)} \] \[ \Psi_{\alpha_1 \ldots \alpha_l}(x) = 0. \quad \text{(III.15)} \]

The meaning of the \( \pm \) signs, in the first-order spinor field equations (III.14) may be interpreted by the dS Dirac equation and its dual equation [8, 18], but the interpretation of these signs in the second-order equation (III.15) is completely different. We obtain two second-order field equations with two different mass parameters associated with the two spinor fields. This problem will be discussed in section VII-E.

The equation (III.15) can be written in the following form:

\[ \square_H - H^2 \left( \frac{1}{4} + (p + 1)(p - 2) \right) \] \[ \Psi_{\alpha_1 \ldots \alpha_l}(x) = 0, \quad \text{(III.16)} \]
then the mass parameter, associated with the tensor-spinor or fermion fields, is:

\[ m_{f,p}^2 = -H^2 \left( \frac{1}{4} + (p + 1)(p - 2) \mp \sqrt{\frac{9}{4} + (p + 1)(p - 2)} \right). \]  

(III.17)

For principal series \((p = \frac{3}{2} + i\nu)\) we have \(m_{f,\nu}^2 = H^2(2 + \nu^2 \mp \sqrt{-\nu^2}) = H^2(2 + \nu^2 \pm i\nu)\). The mass square has two parts: a real part \((2H^2 + H^2\nu^2)\) and an imaginary part \((\pm H^2\nu)\). In the null curvature limit we obtain the mass in the Minkowski space \([5, 6, 8]\):

\[
\lim_{H \to 0, \nu \to \infty} m_{f,\nu}^2 = \lim_{H \to 0, \nu \to \infty} (2H^2 + H^2\nu^2 \pm iHH\nu) = m^2.
\]

It is known that the mass is a well-defined quantity in Minkowski space contrary to the dS space, where it is only a parameter, which turns to be real mass in the null curvature limit. Although the mass parameter is imaginary in the dS space for spinor fields, the probability amplitude can be defined properly for these fields. In this case, two fields equations, with two different mass parameters, are equivalent.

There is no complementary series representation for the spinor fields, whereas, for the discrete series \((p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots)\) we have:

\[ m_{f,\frac{1}{2}}^2 = 2H^2, \quad m_{f,\frac{3}{2}}^2 = \begin{cases} 2H^2 & , \quad m_{f,\frac{5}{2}}^2 = \begin{cases} 0 & , \quad m_{f,\frac{7}{2}}^2 = \begin{cases} -4H^2 & . \end{cases} \end{cases} \end{cases} \]  

(III.18)

There are two different tensor-spinor fields with two different mass parameters and different field equations for each value of \(p \geq \frac{3}{2}\). Beginning with the spinor field equation (III.12), one arrives to two different spinor field equations (III.15). The relations between these spinor fields with the UIR’s of the dS group will be discussed in the section VII-E.

One defines two field equations for a tensor-spinor field. \(\Psi\) is a homogeneous function with degree \(\lambda\) and we have:

\[-\lambda(\lambda + 3) + \frac{1}{4} + (p + 1)(p - 2) \mp \sqrt{\frac{9}{4} + (p + 1)(p - 2)} = 0,\]

which contains two separate equations with two solutions for each one;

\[
\lambda_{1,2}^{(+)} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{1}{4} + (p + 1)(p - 2) + \sqrt{\frac{9}{4} + (p + 1)(p - 2)}},
\]

\[
\lambda_{1,2}^{(-)} = -\frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{1}{4} + (p + 1)(p - 2) - \sqrt{\frac{9}{4} + (p + 1)(p - 2)}}. \]  

(III.19)

For principal series \((p = \frac{1}{2} + i\nu)\), one obtains:

\[
\lambda_{1}^{(-)} = -2 + i\nu, \quad \lambda_{2}^{(-)} = -1 - i\nu, \quad m_{f,\nu}^2 = H^2(2 + \nu^2 \pm i\nu),
\]

\[
\lambda_{1}^{(+)} = -2 - i\nu, \quad \lambda_{2}^{(+)} = -1 + i\nu, \quad m_{f,\nu}^2 = H^2(2 + \nu^2 \pm i\nu), \]  

(III.20)

with the same mass parameter for both cases. For discrete series \((p = \frac{1}{2}, \frac{3}{2}, \ldots)\), we have:

\[
p = \frac{1}{2}, \quad \lambda^{(+)} = \lambda^{(-)} = -2, \quad -1, \quad m_{f,\frac{1}{2}}^2 = 2H^2.
\]
Note that $\lambda_1 + \lambda_2 = -3$. The only possible values of the homogeneous degree for discrete series are $\lambda = -1$, $-2$ and $\lambda = 0$, $-3$. The first two values are corresponding to the massless conformally coupled scalar fields and the second ones to the minimally coupled scalar fields which can be written in therms of the conformally coupled scalar field (VII-B). For the other value of the homogeneous degree, an infra-red divergence appears [III.11] which can not be eliminated by the Krein space quantization [19] or the other method. Using the identity

$$\frac{i}{2} \gamma_\alpha \gamma_\beta M^{\alpha \beta} = \not{\! \partial} \not{\! \partial}^\top,$$

the spinor field equation can be rewritten in the following form:

$$\left[ \not{\! \partial} \not{\! \partial}^\top - 2 \pm \sqrt{\frac{9}{4} + (p + 1)(p - 2)} \right] \Psi_{\alpha_1 \ldots \alpha_l}(x) = 0,$$

where for principal series we obtain:

$$\left[ \not{\! \partial} \not{\! \partial}^\top - 2 \pm i \nu \right] \Psi_{\alpha_1 \ldots \alpha_l}(x) = 0.$$

The field equation for $\Psi^\dagger$ is:

$$x_\alpha \partial_\beta \Psi^\dagger_{\alpha_1 \ldots \alpha_l}(x) \gamma^0 \gamma^\alpha \gamma_\beta + (-2 \mp i \nu) \Psi^\dagger_{\alpha_1 \ldots \alpha_l}(x) \gamma^0 = 0.$$  

This equation can be rewritten in the following compact form:

$$\Psi^\dagger_{\alpha_1 \ldots \alpha_l}(x) \gamma^0 \left[ \not{\! \partial}^\top - 2 \pm i \nu \right] = 0.$$  

One can obtain proper solutions for these field equations, using the conditions (i-vii) in section II-D. Over the years in a series of papers, the solutions of various tensor or tensor-spinor fields have been studied [5, 13, 15, 18, 19, 46]. The dS plane-wave formalism for discrete series representations with $p > 2$ is not a suitable solution, since it becomes infinite for large values of $x$ [III.11].

A field equation can be derived through a variational procedure from the action $S$ which is defined in terms of the Lagrangian density $\mathcal{L}$:

$$S = \int d\mu(x) \mathcal{L},$$

where $d\mu(x)$ is the dS-invariant volume element [III.11]. The Lagrangian density for free boson fields ($j = l$ integer) in principal series and discrete series with $j = l \neq p$ can be defined as:

$$S[\mathcal{K}, j, p] = \int d\mu(x) \mathcal{L}(\mathcal{K}, j, p) = \int d\mu(x) \mathcal{K}_.. \left( Q_1^{(1)} + (p + 1)(p - 2) \right) \mathcal{K},$$

where .. is a shortened notation for total contraction. For complex free fermion fields ($j$ half-integer) in principal series and discrete series with $j \neq p$ and $j = p = \frac{1}{2}$, the Lagrangian density is given by:

$$S[\Psi, \not{\! \Psi}; j, p] = \int d\mu(x) \mathcal{L}(\Psi, \not{\! \Psi}; j, p) = \int d\mu(x) \Psi^\dagger \gamma^0 \left[ \not{\! \partial} \not{\! \partial}^\top - 2 \pm \sqrt{\frac{9}{4} + (p + 1)(p - 2)} \right] \Psi.$$

For $j = p \geq 1$ one cannot solve the field equations to obtain the field solutions, since a gauge invariance appears in the field equations. For these fields, the gauge invariant Lagrangian must be first defined by using the gauge principle, after which, the gauge fixing parameter will be introduced and then, one can obtain the field equations from the gauge fixing Lagrangian. This procedure will be considered in section VI.
B. Field equation in $\xi$-space

There are three different Hilbert spaces, namely, $\mathcal{H}_q^{(j,p)}$, $\mathcal{H}_q^{(j,0,p)}$ and $\mathcal{H}_q^{(0,j,p)}$ corresponding to the three equations (11.29), (11.31), and (11.33) respectively. Here, we only consider one in a sense that the procedure for others is utterly the same. The action of the Casimir operator $Q_j^{(1)}$ on the Hilbert space $\mathcal{H}_q^{(0,j,p)}$ correspond to the equation (11.31) is:

$$Q_j^{(1)}|q; j,p\rangle = [-j(j+1) - (p+1)(p-2)]|q; j,p\rangle,$$

where $|q; j,p\rangle \in \mathcal{H}_q^{(0,j,p)}$ is a $(2j+1)$-component vector with the component $|q, m; j,p\rangle$. Using the definition $q = ru$ with $|u| = 1$ and $|q| = r$, one can simply obtain the $u$-space equation. Here, only the $q$-space equation is has been discussed. The differential form of the Casimir operator $Q_j^{(1)}$ on the Hilbert space $\mathcal{H}_q^{(0,j,p)}$ is [4]:

$$-Q_j^{(1)}|q; j,p\rangle = \left(\frac{1}{2}(1 - |q|^2)\right)^2 \Delta - p(1 - |q|^2)D - \frac{1}{2}(1 - |q|^2)(A_j D_1 + B_j D_2 + C_j D_3)$$

$$+(p(p+1) - j(j+1))|q|^2 + 2j(j+1) - (p+1)]|q; j,p\rangle,$$

where the differential operators $\Delta, D_1, D_2$ and $D_3$ are defined explicitly in $q$-space by Takahashi [4]:

$$\Delta = \frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2} + \frac{\partial^2}{\partial q_4^2} + \frac{\partial}{\partial r} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{S^3},$$

$$D = q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3} + q_4 \frac{\partial}{\partial q_4} = \frac{\partial}{\partial r} + u \cdot \frac{\partial}{\partial u},$$

$$D_1 = q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1} + q_4 \frac{\partial}{\partial q_3} - q_3 \frac{\partial}{\partial q_4} = u_1 \frac{\partial}{\partial u_2} - u_2 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_4},$$

$$D_2 = q_1 \frac{\partial}{\partial q_3} - q_3 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_4} - q_4 \frac{\partial}{\partial q_2} = u_1 \frac{\partial}{\partial u_3} - u_3 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial u_2},$$

$$D_3 = q_1 \frac{\partial}{\partial q_4} - q_4 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} - q_3 \frac{\partial}{\partial q_3} = u_1 \frac{\partial}{\partial u_4} - u_4 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} - u_3 \frac{\partial}{\partial u_3}. \quad (\text{III.29})$$

$\Delta_{S^3}$ is the Laplace operator on the hypersphere $S^3$. The $A_j$, $B_j$ and $C_j$ are $(2j+1) \times (2j+1)$-matrix representations of the generators of the $SU(2)$ group [4]. By using the equations (III.27) and (III.28), the following partial differential equation for the quantum state $|q; j,p\rangle$ is obtain:

$$\left[\frac{1}{4}(1 - |q|^2)\Delta - pD - \frac{1}{2}(A_j D_1 + B_j D_2 + C_j D_3) + (j(j+1) - p(p+1))\right]|q; j,p\rangle = 0. \quad (\text{III.30})$$

By solving this equation, the state $|q; j,p\rangle$ is determined. This partial differential equation is a $(2j+1) \times (2j+1)$-matrix equation which has a complicated solutions [4]. Takahashi has proven that this equation has a non-trivial solution in the following form, (page 399, Proposition 3.1. in [4]):

$$|q, m; j,p\rangle = F(p - j, j + p + 1; \gamma^2) \sum_{m'} C_{mm'} |u, m'; j,p\rangle. \quad (\text{III.31})$$
\[ F \] is a hyper-geometric function, which is a polynomial function of degree \( j - p \). \(|u, m'; j, p\rangle\) is a base-vector of a \((2j + 1)\)-dimensional Hilbert space \(V'\).

Multiplying above equation with bra \(< x|\), and stating the result in terms of \(< x|\mathbf{q}: j, p \rangle = f(x; \mathbf{q}; j, p)\), one can show that this function has a specific relation with a tensor field \(\mathcal{K}\) or tensor-spinor field \(\Psi\) which are presented in the previous subsection. This process is similar to the first quantization in the usual quantum mechanics, which is not relevant here. In the next section, the second quantization or the quantization of these fields is discussed in the ambient space formalism.

It is interesting to note that the tensor fields on dS hyperboloid in terms of the five-variable \(x\) or \(\xi\) (\(f(x)\) or \(g(\xi)\)) are homogeneous functions:

\[
\langle f(lx) = l^\lambda f(x), \quad g(l\xi) = l^\sigma g(\xi),
\]

where \(\lambda\) and \(\sigma\) are the homogeneous degrees. The homogeneous degree \(\lambda\) was obtained by using the field equation in \(x\)-space in this section and the homogeneous degree \(\sigma\) for each tensor (or spinor) field may be fixed by using the UIR of dS group which will be discussed seperately for various spin fields in the following. For tensor fields one obtain \(\lambda = \sigma\) and for tensor-spinor fields \(\lambda = \sigma + \frac{1}{2}\).

IV. MASSIVE QUANTUM FREE FIELD OPERATORS

The massive fields in the dS space-time correspond to the principal series representation of the dS group \((p = \frac{1}{2} + i\nu)\). For constructing the quantum field operator, one must first define the creations and annihilations operators on the Hilbert space \(\mathcal{H}_{u}^{(j,p)}\). A creation operators \(a^\dagger(u, m_j; j, p)\) is defined as an operator that simply adds a state with quantum numbers \((u, m_j; j, p)\) to the vacuum state \(|\Omega\rangle\)

\[
a^\dagger(u, m_j; j, p) |\Omega\rangle \equiv |u, m_j; j, p\rangle,
\]

where the vacuum state \(|\Omega\rangle\) is invariant under the action of the UIR of the dS group:

\[
U^{(j,p)}(g) |\Omega\rangle = |\Omega\rangle.
\]

Here, the norm of the vacuum state can be fixed as \(|\Omega\rangle |\Omega\rangle = 1\), contrary to the norms of the other states which are fixed by the local Hadamard behavior \([6]\). In what follows, it has been shown that, the vacuum state \(|\Omega\rangle\) can be identified with the Bunch-Davies or Hawking-Ellis vacuum state.

The adjoint operator \(a^\dagger(u, m_j; j, p)\) is \(a(u, m_j; j, p)\) and it can be defined from the equation \((IV.1)\):

\[
\langle \Omega | a^\dagger(u, m_j; j, p) \rangle = \langle u, m_j; j, p | \langle \Omega |\).
\]

where

\[
a^\dagger(u, m_j; j, p) \equiv a(\bar{u}, m_j; j, p^* \dagger) = a(\bar{u}, m_j; j, 1 - p).
\]

Using the orthogonality condition on the Hilbert space \(\mathcal{H}_{u}^{(j,p)}\), defined explicitly by Takahashi \([4]\):

\[
\langle u, m_j; j, p | u', m_j'; j, p \rangle = N(u, m_j) \delta_{\mathcal{S}^3}(u - u') \delta_{m_j, m_j'},
\]

where \(N(u, m_j)\) is the states normalization, one can show that the operator \(a(u, m_j; j, \nu)\) removes a state from any state in which it acts on:

\[
\langle \Omega | a(\bar{u}, m_j; j, 1 - p) | u', m_j'; j, p \rangle = N(u, m_j) \delta_{\mathcal{S}^3}(u - u') \delta_{m_j, m_j'} \langle \Omega |\langle \Omega |\).
\]

\(a(u, m_j; j, p)\) is called annihilation operator and it annihilates the vacuum state:

\[
a(u, m_j; j, p) |\Omega\rangle = 0.\]

\(IV.6)\)
Using the equations (II.29), (IV.1) and (IV.2), the creation and annihilation operators under the action of the dS group transform as:

\[
U^{(j,p)}(g)a^{\dagger}(u, m_j; j, p) \left[U^{(j,p)}(g)\right]^\dagger = |cu + d|^{-2(1+p)}
\]

\[
\times \sum_{m_j'} D_{m_j m_j'}^{(j)} \left(\frac{|cu + d|^{-1}}{|cu + d|}\right) a^{\dagger}(g^{-1}.u, m_j'; j, p), \quad (IV.7)
\]

\[
U^{(j,p)}(g)\alpha(u, m_j; j, 1-p) \left[U^{(j,p)}(g)\right]^\dagger = |cu + d|^{-2(2-p)}
\]

\[
\times \sum_{m_j'} \left[D_{m_j m_j'}^{(j)} \left(\frac{|cu + d|^{-1}}{|cu + d|}\right)\right]^* a(g^{-1}.\alpha, m_j'; j, 1-p). \quad (IV.8)
\]

Using the equation (II.32), one can see that the annihilation operator \(a(u, m_j; j, p)\) transforms by the unitary equivalent representation of principal series:

\[
U^{(j,1-p)}(g)a(u, m_j; j, p) \left[U^{(j,1-p)}(g)\right]^\dagger = |cu + d|^{-2(2-p)}
\]

\[
\times \sum_{m_j'} \left[D_{m_j m_j'}^{(j)} \left(\frac{|cu + d|^{-1}}{|cu + d|}\right)\right]^* a(g^{-1}.\alpha, m_j'; j, p). \quad (IV.9)
\]

The quantum state \(|u, m_j; j, p\rangle\) is called “one-particle” state. The N-particle state can be obtained by acting on the vacuum with N creation operators. For identical particles, under the action of a permutation operator, N-particle state is either symmetric or anti-symmetric (bosonic or fermionic respectively). Therefore, similar to the Minkowsky space, one can prove that the following relation is hold:

\[
a(\alpha', m_j'; j, 1-p)a^{\dagger}(u, m_j; j, p) \pm a^{\dagger}(u, m_j; j, p)a(\alpha', m_j'; j, 1-p) = N(u, m_j)\delta_{\alpha \alpha'}(u' - u)\delta_{m_j m_j'}. \quad (IV.10)
\]

with the + and − signs, corresponding to fermionic or bosonic states, respectively.

The quantum field operators for the integer cases \(j = l, 0, 1, 2, \ldots\) can be defined in terms of creation and annihilation operators as follows:

\[
K_{\alpha_1 \ldots \alpha_l}(x) \equiv \sum_{m_j} \int_{S^3} d\mu(u) \left[a(\alpha, m_j; j, 1-p)U_{\alpha_1 \ldots \alpha_l}(x; u, m_j; j, \nu) + a^{\dagger}(u, m_j; j, p)\bar{V}_{\alpha_1 \ldots \alpha_l}(x; u, m_j; j, \nu)\right], \quad (IV.11)
\]

and for the half-integer cases \(j = l + \frac{1}{2} \in \{\frac{3}{2}, \frac{5}{2}, \ldots\}\) one has:

\[
\Psi_{\alpha_1 \ldots \alpha_l}(x) \equiv \sum_{m_j} \int_{S^3} d\mu(u) \left[a(\alpha, m_j; j, 1-p)U_{\alpha_1 \ldots \alpha_l}(x; u, m_j; j, \nu) + a^{\dagger}(u, m_j; j, p)V_{\alpha_1 \ldots \alpha_l}(x; u, m_j; j, \nu)\right], \quad (IV.12)
\]

where \(\Psi_{\alpha_1 \ldots \alpha_l}, U_{\alpha_1 \ldots \alpha_l} \) and \(V_{\alpha_1 \ldots \alpha_l}\) are four-component spinors. The particles that are annihilated and created by these fields may be carry non-zero values of one or more conserved quantum numbers like the electric charge. For these cases the quantum field operator can be written in terms of creation and annihilation operators as:

\[
\Psi_{\alpha_1 \ldots \alpha_l}(x) = \sum_{m_j} \int_{S^3} d\mu(u) \left[a(\alpha, m_j; j, 1-p)U_{\alpha_1 \ldots \alpha_l}(x; u, m_j; j, \nu) + a^{\dagger}(u, m_j; j, p)V_{\alpha_1 \ldots \alpha_l}(x; u, m_j; j, \nu)\right], \quad (IV.13)
\]
where the label 'c' denotes the "charge conjugate". \( a^\dagger \) creates a particle, whereas \( a^\dagger \) creates an anti-particle with the opposite electric charge.

The coefficients \( U_{\alpha_1..\alpha_l} \), \( V_{\alpha_1..\alpha_l} \), \( U_{\alpha_1..\alpha_l} \) and \( V_{\alpha_1..\alpha_l} \) are chosen so that under the dS transformations the field operators transform by the UIR of dS group (principle B) \[8, \[12, \[18\]:

\[
U^{(j,p)}(g)K_{\alpha_1..\alpha_l}(x)\left[U^{(j,p)}(g)\right]^{\dagger} = \Lambda_{\alpha_1..\alpha_l}^{a_1..a_l}K_{\alpha_1'..\alpha_l'}(\Lambda x),
\]

\[
U^{(j,p)}(g)\Psi_{\alpha_1..\alpha_l}(x)\left[U^{(j,p)}(g)\right]^{\dagger} = \Lambda_{\alpha_1..\alpha_l}^{a_1..a_l}g^{-1}\Psi_{\alpha_1'..\alpha_l'}(\Lambda x),
\]

where \( \Lambda \in SO(1,4) \) and \( g \in Sp(2,2) \). To obtain the Minkowskian spinor field in the null curvature limit \((H=0)\), in the ambient space formalism, the adjoint spinor must be defined as follows \[8, \[18, \[29\]:

\[
\Psi(x) \equiv \Psi^\dagger(x)\gamma^0\gamma^4.
\]

Therefore it transforms by the following relation \[8, \[13, \[18\]:

\[
U^{(j,p)}(g)\Psi_{\alpha_1..\alpha_l}(x)\left[U^{(j,p)}(g)\right]^{\dagger} = \Lambda_{\alpha_1..\alpha_l}^{a_1..a_l}\Psi_{\alpha_1'..\alpha_l'}(\Lambda x)\left(-\gamma^4\gamma^4\right).
\]

Using the equations \((IV.7), (IV.8), (IV.11), (IV.14)\) and the dS invariant volume elements on three-sphere \((III.2)\) the coefficients \( U_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu) \) and \( V_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu) \) must satisfy the following relations:

\[
|cu + d|^{-2(1+p)+6}\sum_{m_j} D^{(j)}_{m_j,m_j'} \left(\frac{(cu + d)^{-1}}{|cu + d|}\right) V_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu)
\]

\[
= \Lambda_{\alpha_1..\alpha_l}^{a_1..a_l}V_{\alpha_1'..\alpha_l'}(\Lambda x;g^{-1}u,m_j';j,\nu),
\]

\[
|cu + d|^{-2(2-p)+6}\sum_{m_j} \left[D^{(j)}_{m_j,m_j'} \left(\frac{(cu + d)^{-1}}{|cu + d|}\right)\right]^* U_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu)
\]

\[
= \Lambda_{\alpha_1..\alpha_l}^{a_1..a_l}U_{\alpha_1'..\alpha_l'}(\Lambda x;g^{-1}u,m_j';j,\nu).
\]

These equations are important for the calculation of the coefficients \( U_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu) \) and \( V_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu) \), using the group representation theory.

After making use of the equations \((IV.7), (IV.8), (IV.13), (IV.15)\) and the dS invariant volume elements on three-sphere \((III.2)\) turns out that the coefficients \( U_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu) \) and \( V_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu) \) will satisfy the following relations:

\[
|cu + d|^{-2(1+p)+6}\sum_{m_j} D^{(j)}_{m_j,m_j'} \left(\frac{(cu + d)^{-1}}{|cu + d|}\right) V_{\alpha_1..\alpha_l}(x;u,m_j;j,\nu)
\]

\[
= \Lambda_{\alpha_1..\alpha_l}^{a_1..a_l}g^{-1}V_{\alpha_1'..\alpha_l'}(\Lambda x;g^{-1}u,m_j';j,\nu),
\]
Nevertheless, one can use the complexified dS manifold to obtain a globally well-defined solution \[6\] to the ambiguity of the phase factor \[11\], and also for \(\Re \lambda < 0\), which is singular on \(x \cdot \xi_u = 0\). Nevertheless, one can use the complexified dS manifold to obtain a globally well-defined solution \[6\].

\[
|cu + d|^{-2(2-p)+6} \sum_{m_j} D_{m_j m'_j}^{(j)} \left( \frac{(cu + d)^{-1}}{|cu + d|} \right)^* U_{\alpha_1 \cdots \alpha_{j}}(x; u, m_j; j, \nu) = \Lambda_{\alpha_1 \cdots \alpha_{j}}^{\alpha_1' \cdots \alpha_j'} g^{-1} U_{\alpha_1' \cdots \alpha_j'}(Ax; g^{-1} u, m'_j; j, \nu). \tag{IV.20}
\]

The equations \[(IV.17), (IV.18), (IV.19)\] and \[(IV.20)\] and the conditions (i-vii) in section II-D allow us to calculate the functions \(U, V, V\) and \(U\) explicitly, up to a normalization constant. By using the homogeneity condition, one can write these coefficients in the following form:

\[
U_{\alpha_1 \cdots \alpha_{j}}(x; u, m_j; j, \nu) = u_{\alpha_1 \cdots \alpha_{j}}(x, u, m_j; j, \nu)(x \cdot \xi_u)^{\lambda}, \tag{IV.21}
\]

where \(\xi_u^0 = \xi^0(1, u)\) and \(\lambda\) is given by equation \[(III.9)\] for tensor fields and equation \[(III.20)\] for tensor-spinor fields. Utilizing the equations \[(II.12), (II.14), (IV.18)\] and \[(IV.21)\], one can obtain the transformation law of \(u_{\alpha_1 \cdots \alpha_{j}}(x, u, m_j; j, \nu)\).

Similar to the Minkowski space, knowing the functions \(U, V, V\) and \(U\) for the values \(\xi_u^0(0) = (1, 0, 0, 0, 1)\) and \(x_0^0 = (0, 0, 0, 0, H^{-1})\), one can calculate these functions for the arbitrary values of \(\xi_u^0\) and \(x^0\), in terms of \(\xi_u^0(0)\) and \(x_0^0\), using equations \[(IV.17), (IV.18), (IV.19)\] and \[(IV.20)\]. To do so, first one must determine the transformations \(\Lambda^R\) and \(g^R\) which make \(\xi_u^0(0)\) invariant (\(SO(3)\) little group):

\[
(\Lambda^R)^\beta_\alpha \xi_u^0(0) = \xi_u^0(0), \quad (\Lambda^R)^\alpha_\beta \gamma^\beta = g^R \gamma^\alpha (g^R)^{-1},
\]
or

\[
u'(0) = g^R \nu(0) = (a^R \nu(0) + b^R)(c^R \nu(0) + d^R)^{-1} = I,
\]

where \(\nu(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\). Afterwards, by replacing \(\Lambda^R\) and \(g^R\) in the equations \[(IV.17), (IV.18), (IV.19)\] and \[(IV.20)\], one obtains:

\[
|c^R + d^R|^{-2(1+p)+6} \sum_{m_j} D_{m_j m'_j}^{(j)} \left( \frac{(c^R + d^R)^{-1}}{|c^R + d^R|} \right) V_{\alpha_1 \cdots \alpha_{j}}(x_0; u(0), m_j; j, \nu) = \left(\Lambda^R\right)^{\alpha'_1 \cdots \alpha'_j}_{\alpha_1 \cdots \alpha_j} V_{\alpha'_1 \cdots \alpha'_j}(x_0; u(0), m'_j; j, \nu). \tag{IV.22}
\]

In this matrix equation the only unknown function is \(V_{\alpha_1 \cdots \alpha_{j}}(x_0; u(0), m_j; j, \nu)\). Solving this equation and by the help of the conditions (i-vii) of section II-D, \(V_{\alpha_1 \cdots \alpha_{j}}(x_0; u(0), m_j; j, \nu)\) can be determined up to a normalization constant.

Similar to the Poincaré group in the Minkowski space \[(14)\], one can obtain the coefficients or polarization tensors \(U, V\), and polarization tensor-spinors \(U\) and \(V\), using the above equations and conditions and by considering their transformations under the discrete symmetries in the dS space \[(48)\], in a direct way in which, unfortunately, includes some tedious calculations. This calculations are irrelevant to our work and will be considered in a forthcoming paper.

For the time being, utilizing the representation theory we define the quantum states and the quantum field operators in terms of the dS plane-waves \((x \cdot \xi)^{\lambda}\) and the polarization tensors (or spinors) \(u_{\alpha_1 \cdots \alpha_{j}}(x, u, m_j; j, \nu)\). But these solutions are not globally defined on the dS hyperboloid due to the ambiguity of the phase factor \[(14)\], and also for \(\Re \lambda < 0\), which is singular on \(x \cdot \xi_u = 0\). Nevertheless, one can use the complexified dS manifold to obtain a globally well-defined solution \[(6)\].
In order to obtain a well-defined function one must suggest a proper definition for the $\xi^\alpha_u$ and the complex dS space-time $z^\alpha$, it was called $i\epsilon$ prescription $[3, 4]$. To fulfill such purpose, one considers the solution in a complex dS space-time $M_{H}^{(c)}$:

$$M_{H}^{(c)} = \{ z = x + iy \in \mathbb{C}^5; \eta_{\alpha\beta}z^\alpha z^\beta = (z^0)^2 - \vec{z} \cdot \vec{z} - (z^4)^2 = -H^{-2} \}$$

which will be called forward and backward tubes of the complex dS space $X_{H}^{(c)}$. Finally the “tuboid” above $M_{H}^{(c)} \times M_{H}^{(c)}$ is defined by

$$T_{12} = \{ (z, z') : z \in T^+, z' \in T^- \}. \quad \text{(IV.26)}$$

Details are given in $[3]$. When $z$ varies in $T^+$ (or $T^-$) and $\xi$ lies in the positive cone $C^+$

$$\xi \in C^+ = \{ \xi \in C; \xi^0 > 0 \},$$

the plane wave solutions are globally defined because the imaginary part of $(z, \xi)$ has a fixed sign. The phase is chosen such that

$$\text{boundary value of } (z, \xi)^\lambda |_{x, \xi > 0} > 0. \quad \text{(IV.27)}$$

In the following subsections, we briefly present the quantum field operators and the Wightman two-point functions for the massive spin $j \leq 2$ fields which have been calculated previously in the $x$-space $[, 8, 12, 13, 15]$.

A. Massive scalar field

A massive scalar field associates with the principal series representation (in fact is a simple case with $j = 0$ and $p = \frac{1}{2} + i\nu$, $\nu \geq 0$). The eigenvalue of the Casimir operator for this field is

$$< Q_{0,\nu}^{(1)} > = \nu^2 + \frac{9}{4},$$

with the corresponding mass $m_{b,\nu}^2 = H^2(\nu^2 + \frac{9}{4})$ and the field equation is:

$$\left[ Q_{0}^{(1)} - \left( \nu^2 + \frac{9}{4} \right) \right] \phi(x) = 0, \quad \text{or} \quad \left[ \Box_H + H^2 \left( \nu^2 + \frac{9}{4} \right) \right] \phi(x) = 0.$$

The field operator is defined by:

$$\phi(x) = \int_{S^3} d\mu(\mathbf{u}) \left[ a(\mathbf{u}, 0; 0, 1 - p)\mathcal{U}(x; \mathbf{u}, 0; 0, \nu) + a^\dagger(\mathbf{u}, 0; 0, p)\mathcal{V}(x; \mathbf{u}, 0; 0, \nu) \right], \quad \text{(IV.28)}$$

and the equations $\text{(IV.17)}$ and $\text{(IV.18)}$ become:

$$\mathcal{U}(x; \mathbf{u}; \nu) = |c\mathbf{u} + d|^{2(2-p) - 6} \mathcal{U}(Ax; g^{-1} \cdot u; \nu),$$

$$\mathcal{V}(x; \mathbf{u}; \nu) = |c\mathbf{u} + d| \mathcal{V}(Ax; g^{-1} \cdot u; \nu).$$

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and the equations $\text{(IV.17)}$ and $\text{(IV.18)}$ become:

$$\mathcal{U}(x; \mathbf{u}; \nu) = |c\mathbf{u} + d|^{2(2-p) - 6} \mathcal{U}(Ax; g^{-1} \cdot u; \nu),$$

$$\mathcal{V}(x; \mathbf{u}; \nu) = |c\mathbf{u} + d| \mathcal{V}(Ax; g^{-1} \cdot u; \nu).$$
\[ \mathcal{V}(x; \mathbf{u}; \nu) = |\mathbf{c} \mathbf{u} + \mathbf{d}|^{2(1+p)-6} \mathcal{V}(\Lambda x; g^{-1} \cdot \mathbf{u}; \nu). \]

Using the transformation of \( \xi_u^a \) in our notation (11.12):
\[ \xi_u^a \rightarrow (1, \mathbf{u}) \rightarrow \xi_u^a \equiv |\mathbf{c} \mathbf{u} + \mathbf{d}|^2(1, \mathbf{u}'), \]
and the equation (IV.21), the coefficient \( U \) and \( \mathcal{V} \) can be rewritten in terms of \( \xi_u^a \) as follows:
\[ U(x; \mathbf{u}; \nu) = c_1 (x \cdot \xi_u)^{-\frac{3}{2} - i\nu}, \]
\[ \mathcal{V}(x; \mathbf{u}; \nu) = c_2 (x \cdot \xi_u)^{-\frac{3}{2} + i\nu}, \]
where \( c_1 \) and \( c_2 \) are arbitrary constants. One can see that the homogeneous degree of massive scalar field is \( \lambda = -\frac{3}{2} \pm i\nu \) (11.13). One can simply show that this field operator satisfies the following transformation rule under the dS group:
\[ U(0,p)(g)x \left[ U(0,p)(g) \right]^\dagger = \phi(\Lambda x). \]

The well-defined field operator is obtained by taking the boundary value of the field operator in the complex dS space-time [3, 4]:
\[ \phi(x) = \lim_{y \to 0} \Phi(z) = \lim_{y \to 0} \Phi(x + iy), \]
where
\[ \Phi(z) = \int_{S^3} d\mu(\mathbf{u}) \left[ a(\hat{\mathbf{u}}, 0; 1 - p) U(z; \mathbf{u}, 0; 0, \nu) + a^\dagger(\mathbf{u}, 0; 0, p) V(z; \mathbf{u}, 0; 0, \nu) \right]. \]
Therefore the quantum field operator in this notation can be written in the following form [3, 4]
\[ \phi(x) = \int_{S^3} d\mu(\mathbf{u}) \left\{ c_1 a(\mathbf{u}, 0; 0, 1 - p) [(x \cdot \xi_u)^{-\frac{3}{2} - i\nu} + e^{-i\pi(\frac{3}{2} - i\nu)} (x \cdot \xi_u)^{-\frac{3}{2} + i\nu}] + c_2 a^\dagger(\mathbf{u}, 0; 0, p) [(x \cdot \xi_u)^{-\frac{3}{2} + i\nu} + e^{i\pi(\frac{3}{2} + i\nu)} (x \cdot \xi_u)^{-\frac{3}{2} + i\nu}] \right\}, \quad (IV.29) \]
where [43]:
\[ (x \cdot \xi)_+^+ = \begin{cases} 0 & \text{for } x \cdot \xi \leq 0 \\ (x \cdot \xi) & \text{for } x \cdot \xi > 0 \end{cases}. \]
For a real scalar field we have \( c_1 = c_2 = \sqrt{c_{0,p}} \).

A two point function \( W(x, x') \) is the boundary value (in the distributional sense) of an analytic function \( W(z, z') \). \( W(z, z') \) is maximally analytic, i.e., can be analytically continued to the cut domain [3, 4]
\[ \Delta = \{ (z, z') \in M_H^{(c)} \times M_H^{(c)} : (z - z')^2 \leq 0 \}. \]

The two-point Wightman function \( W(x, x') \) is the boundary value of \( W(z, z') \) from \( T_{12} \) and the “permutated Wightman function” \( W(x', x) \) is the boundary value of \( W(z, z') \) from the domain \( T_{21} = \{ (z, z') : z' \in T^+, z \in T^- \} \).
Therefore the analytic function $W$ is:

$$W(z, z') = \langle \Omega | \Phi(z) \Phi(z') | \Omega \rangle = \int_{S^3} d\mu(u_1) \int_{S^3} d\mu(u_2) U(z, u_1, 0; 0, \nu) V(z', u_2, 0; 0, \nu) < u_1; \nu | u_2; \nu >$$

$$= \int_{S^3} d\mu(u) U(z, u, 0; 0, \nu) V(z', u, 0; 0, \nu) = c_{0, \nu} \int_{S^3} d\mu(u)(z, \zeta_u)^{-\frac{3}{2} - i\nu} (z', \zeta_{u'})^{-\frac{3}{2} + i\nu}. \quad (IV.30)$$

This integral can be calculated in terms of a generalized Legendre function of the first kind $W(z, z') = 2\pi e^{\pi\nu} H^3 c_{0, \nu} P^{(5)}_{d+1}(H^2 z, z')$, \quad (IV.31)

where $c_{0, \nu}$ is fixed by imposing the Hadamard condition \quad (IV.32):

$$c_{0, \nu} = \frac{e^{-\pi\nu} \Gamma(\frac{3}{2} + i\nu) \Gamma(\frac{3}{2} - i\nu)}{2^\nu \pi^4 H}.$$  \quad (IV.32)

The phenomenon of non-uniqueness of the vacuum state in a general curved space-time appears here in the normalization constant $c_{0, \nu}$. In case of the dS space-time, it has been discovered that the Hadamard condition selects a unique vacuum state for the quantum fields operators. The Hadamard condition postulates that the short distance behaviour of the two point function of the field operator on a curved space-time should be the same as of the corresponding minkowskian two point function. In the dS case the preferred vacuum state coincides with the so called Euclidean or Bunch-Davies vacuum state, and singles out one vacuum in the two-parameter family of quantizations constructed by Allen [49].

The generalized Legendre function of the first kind $P^{(d+1)}_{\lambda}(z)$ is defined by the two following integral representations [18]

$$P^{(d+1)}_{\lambda}(\cos \theta) = \frac{2\omega_{d-1}}{\omega_d}(\sin \theta)^{2-d} \int_0^\theta \cos \left( \lambda + \frac{d-1}{2} \right) t \left[ 2(\cos \tau - \cos \theta) \right]^{\frac{d-1}{2}} d\tau, \quad (IV.33)$$

and

$$P^{(d+1)}_{\lambda}(z) = \frac{\Gamma(d/2)}{\sqrt{\pi \Gamma(d-1)}} \int_0^\pi [z + (z^2 - 1)^{1/2} \cos t]^\lambda (\sin t)^{d-2} dt, \quad (IV.34)$$

which are valid on the cut complex plane $\mathbb{C} \setminus (-\infty, -1]$ with $\omega_d = \frac{2\omega_{d/2}}{\Gamma(d/2)}$. These functions are proportional to Gegenbauer functions of the first kind, defined by:

$$C^n_{\lambda}(z) = \frac{\Gamma(\lambda + 2\kappa)}{\Gamma(\lambda + 1)\Gamma(2\kappa)} F \left( \lambda + 2\kappa, -\lambda; \kappa; \frac{1 - z}{2} \right). \quad (IV.35)$$

where $F$ is the hyper-geometric function. Then we have the following relation

$$P^{(d+1)}_{\lambda}(z) = \frac{\Gamma(d-1)\Gamma(\lambda + 1)}{\Gamma(\lambda + d - 1)} C^{d+1}_{\lambda}(z) = F \left( \lambda + d - 1, -\lambda; \frac{d}{2}; \frac{1 - z}{2} \right). \quad (IV.36)$$

This process can be simply generalized to the other massive spin fields.
B. Massive spinor field

A massive spinor field associates with the principal series representation with \( j = \frac{1}{2} \) and \( p = \frac{1}{2} + i\nu \), \( \nu > 0 \). The eigenvalue of the Casimir operator and the corresponding mass parameter are \( Q^{(1)}_{\frac{1}{2}} > = \nu^2 + \frac{3}{2} \) and \( m^2_{\nu} = H^2(\nu^2 + 2 \pm i\nu) \), respectively. The massive spinor field satisfies the following dS-Dirac field equation:

\[
(\not{x} \cdot \beta^\top - 2 \pm i\nu) \psi(x) = 0, \quad \left[ \Box_H + H^2 \left( 2 + \nu^2 \pm i\nu \right) \right] \psi(x) = 0. \tag{IV.37}
\]

Using the following identity in dS ambient space formalism:

\[
(\not{x} \cdot \beta^\top - 2 + i\nu) (\not{x} \cdot \beta^\top - 1 - i\nu) = \Box_H + 2 + \nu^2 + i\nu,
\]

one can write the spinor field in terms of a scalar field

\[
\psi(x) = (\not{x} \cdot \beta^\top - 1 - i\nu) \phi(x), \quad \left[ \Box_H + H^2 \left( 2 + \nu^2 + i\nu \right) \right] \phi(x) = 0. \tag{IV.38}
\]

One can simply replace \( \nu \) with \(-\nu\) and obtaining the other equation for spinor field. The charged spinor field operator is:

\[
\psi(x) = \int_{S^3} d\mu(u) \sum_m \left[ a(\bar{u}, m; \frac{1}{2}, 1 - p) U(x; u, m; \frac{1}{2}, \nu) + \phi^\dagger(u, m; \frac{1}{2}, p) V(x; u, m; \frac{1}{2}, \nu) \right], \quad \tag{IV.39}
\]

where \( m_\pm \equiv m = \frac{1}{2}, \frac{3}{2} \) and \( \psi, U \) and \( V \) are four-components spinors. Afterwards, the equations \( \text{[IV.19]} \) and \( \text{[IV.20]} \) become:

\[
|c u + d|^{3 - 2i\nu} \sum_m D^{(\frac{1}{2})}_{mm'} \left( \frac{(c u + d)^{-1}}{|c u + d|} \right) V(x; u, m; \frac{1}{2}, \nu) = g^{-1} V(\Lambda x; g^{-1} \cdot u, m'; \frac{1}{2}, \nu), \quad \tag{IV.40}
\]

\[
|c u + d|^{3 + 2i\nu} \sum_m \left[ D^{(\frac{1}{2})}_{mm'} \left( \frac{(c u + d)^{-1}}{|c u + d|} \right) \right] U(x; u, m; \frac{1}{2}, \nu) = g^{-1} U(\Lambda x; g^{-1} \cdot u, m'; \frac{1}{2}, \nu). \quad \tag{IV.41}
\]

There are two possibilities for the homogeneous degrees of spinor field: \( \lambda^+ = -2 - i\nu, -1 + i\nu \) and \( \lambda^- = 2 + i\nu, -1 - i\nu \) \( \text{[IV.20]} \). Similar to the case of the scalar fields, the above equations, \( \text{[IV.40]} \) and \( \text{[IV.41]} \), can completely find the homogeneity degree of the spinor fields and yield to the following relations for the spinors \( U \) and \( V \):

\[
U(x; u, m; \frac{1}{2}, \nu) = (x, \xi_u)^{-2 - i\nu} u(x, u, m; \frac{1}{2}, \nu),
\]

\[
V(x; u, m; \frac{1}{2}, \nu) = (x, \xi_u)^{-1 + i\nu} v(x, u, m; \frac{1}{2}, \nu).
\]

Therefore, only one of these two possibilities, \( \lambda^+ \) and \( \lambda^- \) can transform according to the UIR \( U^{(\frac{1}{2} \nu)}(g) \) of the dS group. The acceptable homogeneous degree is \( \lambda^+ \) and consequently, one of
the second order field equations and mass parameters are also suitable in this case. For detailed information about the explicit forms of \( U \) and \( V \) functions, the reader may look into a previously published paper [18]. Following our survey, the equation [IV.15] becomes:

\[
U(t^\mu \tau^\nu)(g)\psi(x) \left[ U(t^\mu \tau^\nu)(g) \right]^\dagger = g^{-1}\psi(\Lambda x).
\]

In this formalism, the charge conjugation spinor is defined as [48]:

\[
\psi^c = \eta_c C \left( \gamma^4 \right)^\dagger \left( \psi \right)^\dagger,
\]

where \( \eta_c \) is an arbitrary unobservable phase value, generally chosen to be a unity, and \( C = \gamma^2\gamma^4 \) in the \( \gamma \) representation [11.e].

The two-point function for the spinor field has been calculated in maximally symmetric space in [50] in addition, the analytic two-point function in the dS ambient space formalism has been also calculated in [18]:

\[
S_{i_1i_2}(z_1, z_2) = \langle \Omega | \psi_{i_1}(z_1) \bar{\psi}_{i_2}(z_2) | \Omega \rangle
\]

\[
= \int_{S^3} d\mu(u)(z_1, \xi_u)^{-2-i\nu}(z_2, \xi_u)^{-2+i\nu} \sum_m u_{i_1}(z_1, u, m; \frac{1}{2}, \nu) \bar{u}_{i_2}(z_2, u, m; \frac{1}{2}, \nu),
\]

and therefore, the "permuted" two-point function \( S' \) is given by:

\[
S'_{i_1i_2}(z_1, z_2) = - \langle \Omega | \bar{\psi}_{i_2}(z_2) \psi_{i_1}(z_1) | \Omega \rangle
\]

\[
= \int_{S^3} d\mu(u)(z_1, \xi_u)^{-1-i\nu}(z_2, \xi_u)^{-1+i\nu} \sum_m \bar{v}_{i_2}(z_2, u, m; \frac{1}{2}, \nu) v_{i_1}(z_1, u, m; \frac{1}{2}, \nu),
\]

where [8, 18, 29]

\[
\sum_m u_{i_1}(z_1, u, m; \frac{1}{2}, \nu) \bar{u}_{i_2}(z_2, u, m; \frac{1}{2}, \nu) = \frac{c_{1, \nu}}{2} \left( \xi_u \gamma^4 \right)_{ij},
\]

\[
\sum_m \bar{v}_{i_2}(z_2, u, m; \frac{1}{2}, \nu) v_{i_1}(z_1, u, m; \frac{1}{2}, \nu) = -\frac{c_{1, \nu}}{2} \left( \xi_{z_1, \xi_u} \bar{\xi}_{z_1, \xi_u} \gamma^4 \right)_{ij}.
\]

\( c_{1, \nu} \) is the normalization constant. In terms of the generalized Legendre function of the first kind, the analytic spinor two-point function can be written as [18]:

\[
S(z_1, z_2) = A_\nu \left[ (2 - i\nu)\nu P_{-2-i\nu}^4(H^2 z_1, z_2) \bar{\xi}_1 - (2 + i\nu)P_{-2+i\nu}^4(H^2 z_1, z_2) \bar{\xi}_2 \right] \gamma^4,
\]

where \( A_\nu = 2i\pi^2e^{\nu\pi} c_{1, \nu} \). The values of the constants \( c_{1, \nu} \) and \( A_\nu \) are fixed by imposing the local Hadamard condition: the short-distance behaviour of \( S(z_1, z_2) \) should coincide with the leading behaviour of the corresponding Minkowskian two-point function [18]:

\[
c_{1, \nu} = \frac{\nu(\nu^2 + 1)}{(2\pi)^3(e^{2\pi\nu} - 1)}, \quad A_\nu = \frac{i \nu(i + \nu^2)}{64\pi \sinh(\pi\nu)}.
\]

The 'permuted' two-point function \( S' \) is [18]:

\[
S'(z_1, z_2) = -\bar{\xi}_1 S(z_1, z_2) \gamma^4 \bar{\xi}_2 \gamma^4.
\]
The mass square associated with the spinor field is a complex number. Bearing in mind that the mass is not a well-defined physical quantity in the dS space-time and it is only a parameter, one finds out that this parameter actually has a relation with the homogeneous degree of the spinor field. In contrary, one obtains a real mass in the null curvature limit (\( H = 0 \)). A massive spinor field, which is a causal field, propagates in the dS light cone. A spinor field in the dS space in the null curvature limit becomes exactly equivalent to its Minkowskian counterpart \([18]\). The homogeneous degree of tensor or spinor field play the role of the mass for causal propagation in the dS space and it must to satisfy \(|\Re \lambda| < 0\).

C. Massive vector field

A massive vector field associates with the principal series representation of the dS group with \( j = 1 \) and \( p = \frac{1}{2} + i \nu \), \( \nu \geq 0 \). The eigenvalue of the Casimir operator and the corresponding mass parameter are \( \langle Q^{(1)}_{1,\nu} \rangle = \nu^2 + \frac{1}{4} \), and \( m^2_{h,\nu} = H^2(\nu^2 + \frac{9}{4}) \), respectively. The field equation is:

\[
\left[ Q^{(1)}_1 - \left( \nu^2 + \frac{1}{4} \right) \right] K_\alpha(x) = 0, \quad \text{or} \quad \left[ \Box_H + H^2 \left( \nu^2 + \frac{9}{4} \right) \right] K_\alpha(x) = 0.
\]

The massive vector field can be written in terms of the massive scalar field \([IV.28]\) \([12]\):

\[
K_\alpha(x) = \left[ Z^\top + \frac{1}{\nu^2 + \frac{1}{4}} D_{1a} \left( Z \cdot \partial^\top + 2 H^2 x \cdot Z \right) \right] \phi(x),
\]

where \( Z \) is a constant five-vector. The massive vector field operator can be written in the following form:

\[
K_\alpha(x) = \int_{S^3} d\mu(u) \sum_m \left[ a(\tilde{u}, m; 1, 1 - p) U_\alpha(x; u, m; 1, 1, p) + a^\dagger(u, m; 1, \nu) V_\alpha(x; u, m; 1, \nu) \right], \quad (IV.45)
\]

where \( m_1 = m = -1, 0, 1 \). The equations \([IV.17]\) and \([IV.18]\) can indeed fix the homogeneous degrees of the vector field and one obtains:

\[
U_\alpha(x; u, m; 1, \nu) = (x \cdot \xi_u)^{-\frac{1}{2} - i\nu} u_\alpha(x; u, m; 1, \nu),
\]

\[
V_\alpha(x; u, m; 1, \nu) = (x \cdot \xi_u)^{-\frac{1}{2} + i\nu} v_\alpha(x; u, m; 1, \nu).
\]

The explicit form of \( u_\alpha \) and \( v_\alpha \) are not important for us here, but in the previous paper \([12]\) we have calculated these functions by solving the field equation, considering the conditions (i-vii) of section II-D. Their explicit forms also can be calculated, using the equations \([IV.17]\) and \([IV.18]\). The massive vector field operator satisfies the following relation:

\[
U^{(1,p)}(g) K_\alpha(x) \left[ U^{(1,p)}(g) \right]^\dagger = \Lambda_{\alpha}^{\alpha'} K_{\alpha'}(\Lambda x).
\]

The analytic two-point function for the massive vector field is defined by:

\[
W_{\alpha\alpha'}(z, z') = \langle \Omega | K_\alpha(z) K_{\alpha'}(z') | \Omega \rangle
\]

\[
= \int_{S^3} (z \cdot \xi_u)^{-\frac{3}{2} - i\nu} (z' \cdot \xi_u)^{-\frac{3}{2} + i\nu} \sum_m u_\alpha(z; u, m; 1, \nu) v_{\alpha'}(z', u, m; 1, \nu), \quad (IV.46)
\]
where we have [24]:

\[
\sum_{m} u_{\alpha}(z, u, m; 1, \nu) v_{\alpha'}(z', u, m; 1, \nu) =
\]

\[
c_{1,\nu} \left[ -\theta_{\alpha} \cdot \theta_{\alpha'} + \frac{(\theta_{\alpha} \cdot z')(\xi_{\alpha} z - \xi_{\alpha'} z')}{z' \cdot \xi} + \frac{(\theta_{\alpha'} \cdot z)(\xi_{\alpha} z - \xi_{\alpha'} z')}{H^2(z' \cdot \xi)(z' \cdot \xi)} \right].
\] (IV.47)

\(c_{1,\nu}\) is the normalization constant and \(\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 z_{\alpha} z_{\beta}'\). In this formalism, the analytic two-point function can be easily written in terms of the scalar analytic two-point function [12]:

\[
W_{\alpha\alpha'}(z, z') = \left[ \theta_{\alpha} \cdot \theta_{\alpha'} + \frac{1}{H^2(\nu^2 + \frac{4}{9})} \partial_{\alpha}^\top \left( \theta_{\alpha'} \cdot \partial^\top + 2H^2 \theta_{\alpha'} \cdot z \right) \right] W(z, z'),
\] (IV.48)

where \(W(z, z')\) is given by (IV.31).

For the complementary series, (\(< Q^{(1)}_{1,1} >= p - p^2, 0 < p - p^2 < \frac{1}{9}\)), and the discrete series, (\(< Q^{(1)}_{1,1} >= 0\)), one can replace \(\nu\) respectively with \(\pm \sqrt{p - p^2 - \frac{4}{9}}\) and \(\pm \frac{4}{9}\). In the case of the complementary series the associated mass is \(m_{\nu}^2 = H^2(p - p^2)\), but there is no physically meaningful representation, belonging to the Poincaré group, that can be interpreted as a contraction limit \(H \rightarrow 0\) for these representations. Therefore, the physical meaning of a complementary vector field is not clear.

The associated mass of the vector field \((j = 1)\) which correspond to the discrete series representations \(\Pi_{1,1}^\pm\), is \(m_{\nu}^2 = 2H^2\). These representations \((\Pi_{1,1}^\pm)\) have the physically meaningful Poincaré limit. It is a massless vector field so the parameter \(\nu\) must be replaced by \(\pm \frac{4}{9}\) in the equations (IV.45) and (IV.48). Note that the generalized polarization vector \(u_{\alpha}(x, u, m; 1, \nu)\) or the two-point function (IV.48) diverge at this limit. This type of singularity is actually due to the divergencelessness condition imposed on the massless vector field in order to associate this type of field with the representations \(\Pi_{1,1}^\pm\). To overcome this problem, the divergencelessness condition must be dropped out. Then, the massless vector field associates with an indecomposable representation of the dS group and hence a gauge invariance appears. This field will be discussed in the sections VII.

**D. Massive spin-\(\frac{3}{2}\) field**

A massive vector-spinor field is associated with the principal series representation of the dS group with \(j = \frac{3}{2}\) and \(p = \frac{1}{2} + i\nu\), \(\nu > 0\). The corresponding mass parameter and the eigenvalue of the Casimir operator are \(m_{\nu}^2 = H^2(\nu^2 + 2 \pm i\nu)\) and \(< Q^{(1)}_{\frac{3}{2},\nu} >= \nu^2 - \frac{3}{2}\), respectively. This field satisfies the following field equation

\[
\left( \xi^\top \partial - 2 \pm i\nu \right) \Psi_{\alpha}(x) = 0, \quad \left[ \Box H + H^2 \left( 2 + \nu^2 \pm \nu \right) \right] \Psi_{\alpha}(x) = 0.
\]

The spin-\(\frac{3}{2}\) field can be written in terms of the spinor field (IV.39) [13]:

\[
\Psi_{\alpha}(x) = \left[ Z_{\alpha}^\top - \frac{1}{4} \gamma_{\alpha}^\top Z^\top - \frac{1}{\nu^2 + 1} \left( -\frac{1}{4} (1 - i\nu) \gamma_{\alpha}^\top Z^\top + 2 \theta_{\alpha}^\top x.Z \right) \right.
\]

\[
\left. + \frac{2}{3} \theta_{\alpha}^\top Z \partial^\top + \frac{1}{3} (i\nu + 1) \theta_{\alpha}^\top Z^\top x. - \frac{1}{3} (i\nu + 1) \gamma_{\alpha}^\top Z.x \right. \left. x. - \frac{2}{3} \gamma_{\alpha}^\top Z \partial^\top \right.
\]

\[
\left. - \frac{1}{6} \gamma_{\alpha}^\top Z \partial^\top - \frac{1}{6} i\nu \gamma_{\alpha}^\top \right. \left. x.Z \partial^\top - \frac{1}{12} (i\nu + 1) \gamma_{\alpha}^\top \right. \left. Z^\top x. - \frac{2}{3} \gamma_{\alpha}^\top Z \partial^\top \right. \left. \right] \right] \psi(x),
\]
where \( Z \) is a constant five-vector. The vector-spinor field may also be given in terms of a scalar field by using the equation [IV.38]. The charged vector-spinor field operator can be defined as:

\[
\Psi_\alpha(x) = \int_{\mathbb{S}^3} d\mu(u) \sum_m \left[ a(u, m; \frac{3}{2}, 1 - p)U_\alpha(x; u, m; \frac{3}{2}, \nu) + a^{\dagger}(u, m; \frac{3}{2}, p)V_\alpha(x; u, m; \frac{3}{2}, \nu) \right], \quad (IV.49)
\]

where \( m = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \) and \( \Psi_\alpha, U_\alpha \) and \( V_\alpha \) are four-components spinors. Similar to the case of the spinor field, only the homogeneity degree \( \lambda^+ = -2 - iv, -1 + iv \) is allowed:

\[
U_\alpha(x; u, m; \frac{3}{2}, \nu) = (x \cdot \xi_u)^{-2 - iv}u_\alpha(x, u, m; \frac{3}{2}, \nu),
\]

\[
V_\alpha(x; u, m; \frac{3}{2}, \nu) = (x \cdot \xi_u)^{-1 + iv}v_\alpha(x, u, m; \frac{3}{2}, \nu).
\]

Following our previous discussions, we are not interested in the explicit forms of \( u_\alpha \) and \( v_\alpha \), because they are already established in [12]. It is easy to show that the vector-spinor field operator [IV.49] satisfies the following transformation law:

\[
U(\tilde{\xi}, p)(g)\Psi_\alpha(x) \left[ U(\tilde{\xi}, p)(g) \right]^\dagger = \Lambda_\alpha^{\alpha'} g^{-1}\Psi_{\alpha'}(\Lambda x).
\]

Alike to the case of the spinor field, a adjoint vector-spinor field is defined by

\[
\tilde{\Psi}_\alpha(x) \equiv \Psi_\alpha(x)^\dagger \gamma^0 \gamma^4,
\]

which transforms as:

\[
U(\tilde{\xi}, p)(g)\Psi_\alpha(x) \left[ U(\tilde{\xi}, p)(g) \right]^\dagger = \Lambda_\alpha^{\alpha'} \tilde{\Psi}_{\alpha'}(\Lambda x) \left( -\gamma^4 g^4 \right).
\]

The analytic two-point function for the massive vector-spinor field is:

\[
S^\nu_{\alpha \alpha'}(z, z') = \langle \Omega | \Psi_\alpha(z)\Psi_{\alpha'}^\dagger(z') | \Omega \rangle
\]

\[
= \int_{\mathbb{S}^3} d\mu(u)(z \cdot \xi_u)^{-2 - iv}(z' \cdot \xi_u)^{-2 + iv} \sum_m u_\alpha^\dagger(z, u, m; \frac{3}{2}, \nu)u_{\alpha'}^\dagger(z', u, m; \frac{3}{2}, \nu). \quad (IV.50)
\]

Note that the equation [IV.50] can be written in terms of the analytic spinor two-point function as [13]:

\[
S_{\alpha \alpha'}(z, z') = D_{\alpha \alpha'}(z, \partial^\dagger; z', \partial'^\dagger)S(z, z'), \quad (IV.51)
\]

where

\[
D_{\alpha \alpha'} = \theta_\alpha \cdot \theta_\alpha' - \frac{1}{4} \gamma_\alpha \gamma^\top \cdot \theta_\alpha' - \frac{1}{\nu^2 + 1} \frac{(iv - 1)}{4} \gamma_\alpha \gamma^\top \cdot \theta_\alpha' + 2 \partial^\alpha \cdot \theta_\alpha',
\]

\[
+ \frac{2}{3} \partial^\alpha \cdot \partial^\top - \frac{2}{3} \gamma_\alpha \gamma^\top \cdot z \partial^\top + \frac{(iv + 1)}{3} \partial_\alpha \gamma^\top \cdot \theta_\alpha' \partial \cdot \partial - \frac{(iv + 1)}{3} \gamma_\alpha \gamma^\top \cdot \theta_\alpha' \partial \cdot \partial,
\]

\[
- \frac{1}{6} \gamma_\alpha \gamma^\top \cdot \theta_\alpha' \partial \cdot \partial - \frac{1}{6} \gamma_\alpha \gamma^\top \cdot \theta_\alpha' \partial \cdot \partial - \frac{(iv + 1)}{12} \gamma_\alpha \gamma^\top \cdot \theta_\alpha' \partial \cdot \partial, \quad (IV.52)
\]

with \( S(z, z') \) as the analytic spinor two-point function [IV.14].
A vector-spinor field \( (j = \frac{3}{2}) \) which associate with the discrete series representation are correspond to \( p = \frac{1}{2} \) and \( p = \frac{3}{2} \). Their corresponding eigenvalues of the Casimir operators are:

\[
< Q^{(1)}_{\frac{1}{2} \frac{1}{2}} > = \frac{3}{2}, \quad < Q^{(1)}_{\frac{3}{2} \frac{3}{2}} > = \frac{5}{2}.
\]

The case \( p = \frac{1}{2} \) which corresponds to the representations \( \Pi^\pm_{\frac{1}{2} \frac{1}{2}} \), does not have a corresponding representation in Poincaré group at the null curvature limit. Therefore, the corresponding field is called an auxiliary vector-spinor field. Then one can replace \( \nu \) by \( \nu = 0 \) in the equation (IV.49) and (IV.51) for the massive vector-spinor field in this case. This field will be considered in the section V-A.

The second case \( (p = \frac{3}{2}) \) which corresponds to the representations \( \Pi^\pm_{\frac{3}{2} \frac{3}{2}} \), has a physically meaningful Poincaré limit. This is precisely the massless vector-spinor field and hence a gauge invariance appears. This field will be considered in the section VII.

E. Massive spin-2 rank-2 symmetric tensor field

A massive spin-2 rank-2 symmetric tensor field associates with the principal series representation of the dS group with \( j = 2 \) and \( p = \frac{1}{2} + i \nu \), \( \nu \geq 0 \). The eigenvalue of the Casimir operator and its corresponding mass parameter are \( < Q^{(1)}_{2\nu} > = \nu^2 - \frac{15}{4} \) and \( m_{b,\nu}^2 = H^2(\nu^2 + \frac{9}{4}) \), respectively. The field equation is:

\[
\left[ Q^{(1)}_{2} - \left( \nu^2 - \frac{15}{4} \right) \right] K_{\alpha \beta}(x) = 0, \quad \text{or} \quad \left[ \Box H + H^2 \left( \nu^2 + \frac{9}{4} \right) \right] K_{\alpha \beta}(x) = 0.
\]

The massive spin-2 field can be written in terms of a polarization tensor and a massive scalar field (IV.38):

\[
K_{\alpha \beta}(x) = D_{\alpha \beta}(x, \partial; Z_1, Z_2, \nu) \phi(x),
\]

where \( Z_1 \) and \( Z_2 \) are two constant five-vectors and the explicit form of \( D \) was given in [13]. The field operator is defined by:

\[
K_{\alpha \beta}(x) = \sum_m \int_{S^3} d\mu(\mathbf{u}) \left[ a(\mathbf{u}, m; 2, 1 - p) U_{\alpha \beta}(x; \mathbf{u}, m; 2, \nu) + a^\dagger(\mathbf{u}, m; 2, p) V_{\alpha \beta}(x; \mathbf{u}, m; 2, \nu) \right], \quad (IV.53)
\]

where \( m = -2, -1, 0, 1, 2 \). The homogeneous degrees of \( U \) and \( V \) are:

\[
U_{\alpha \beta}(x; \mathbf{u}, m; 2, \nu) = (x \cdot \xi_u)^{-\frac{1}{2} - i\nu} u_{\alpha \beta}(x, \mathbf{u}, m; 2, \nu),
\]

\[
V_{\alpha \beta}(x; \mathbf{u}, m; 2, \nu) = (x \cdot \xi_u)^{-\frac{1}{2} + i\nu} v_{\alpha \beta}(x, \mathbf{u}, m; 2, \nu).
\]

The coefficients \( u \) and \( v \) satisfy the conditions:

\[
v_{\alpha \beta}(x, \mathbf{u}, m; 2, \nu) = v_{\beta \alpha}(x, \mathbf{u}, m; 2, \nu), \quad v^\alpha_{\alpha}(x, \mathbf{u}, m; 2, \nu) = 0.
\]
Again, the explicit forms of $u_{\alpha\beta}$ and $v_{\alpha\beta}$ are given in [24]. They can also be calculated by using the equations (IV.17) and (IV.18). The field operator satisfies the following transformation role:

$$U^{(2,p)}(g)K_{\alpha\beta}(x) [U^{(2,p)}(g)]^\dagger = \Lambda_\alpha^\alpha' \Lambda_\beta^\beta' K_{\alpha'\beta'}(Ax).$$

The analytic two-point function for this field is

$$W_{\alpha'\beta'\alpha\beta}(z, z') = \langle \Omega | K_{\alpha\beta}(z) K_{\alpha'\beta'}(z') | \Omega \rangle,$$

which for the current formalism can be written in terms of the analytic scalar two-point function [24]:

$$W_{\alpha'\beta'\alpha\beta}(z, z') = D_{\alpha'\beta'\alpha\beta}(z, z', \theta, \theta') W(z, z'),$$

with $W(z, z')$ being the analytic scalar two-point function [IV.31]. The polarization tensor is:

$$D_{\alpha'\beta'\alpha\beta}(z, z', \theta, \theta') = \frac{4u^2 + \frac{9}{4}}{3\nu^2 + \frac{9}{4}} \left[ -\Sigma_1 \Sigma_1' \theta \cdot \theta' + \frac{H^2 \Sigma_1(\theta \cdot z') D_1^\dagger}{\lambda^* - 1} + \frac{H^2 \Sigma_1'(\theta' \cdot z) D_2^\dagger}{\lambda - 1} + \frac{z H^2 D_1^\dagger D_2^\dagger}{(\lambda - 1)(\lambda^* - 1)} \right] \times \left[ \frac{\nu^2 + \frac{9}{4}}{\nu^2 + \frac{1}{4}} \left( -\theta_\alpha \cdot \theta_\alpha' + \frac{H^2 \lambda(\theta \cdot z') D_1^\dagger}{\nu^2 + \frac{9}{4}} + \frac{H^2 \lambda^*(\theta' \cdot z) D_1^\dagger}{\nu^2 + \frac{9}{4}} + \frac{H^2 z D_1^\dagger D_1^\dagger}{\nu^2 + \frac{9}{4}} \right) \right],$$

where $\lambda = -\frac{3}{2} + i\nu$, $D_1^\dagger = \partial^\dagger$, $D_2^\dagger = \Sigma_1(\partial^\dagger - x)$ and $\Sigma_1$ is the index symmetrizer [II.22].

For a spin-2 rank-2 symmetric tensor field, which associate with the complementary series representation with $j = 2$ and $0 < p - p^2 < \frac{1}{4}$, the eigenvalue of the Casimir operator and its corresponds mass parameter are $< Q_{2,p}^{(1)} > = p - p^2 - 4$ and $m_{b,p}^2 = H^2(p - p^2 + 2)$, respectively. Although the associated mass is strictly positive, the physical meaning of its associated field remains unclear. These representations in the complementary series at the $H = 0$ limit, do not correspond to any physical representations of the Poincaré group. The field operator and the corresponding two-point function are obtainable simply by replacing the parameter $\nu$ with $\nu = i(p - \frac{1}{4})$.

A spin-2 fields ($j = 2$) which associate with the discrete series representation are correspond to $p = 1$ ( $\Pi_{2,1}^\pm$ ) and $p = 2$ ( $\Pi_{2,2}^\pm$ ) and their corresponding eigenvalues of the Casimir operators are respectively:

$$< Q_{2,1}^{(1)} > = -4, \quad < Q_{2,2}^{(1)} > = -6.$$

The representations $\Pi_{2,1}^\pm$ at the null curvature limit $H = 0$, do not correspond to any physical representations of the Poincaré group. The corresponding mass is $m_{b,1}^2 = 2H^2$. We call it the auxiliary spin-2 fields and it will be considered in the section V-A.

The second case ($p = 2$) which corresponds to the representation $\Pi_{2,2}^\pm$ has a physically meaningful Poincaré limit with its associated mass being $m_{b,2}^2 = 0$. Hence $\Pi_{2,2}^\pm$ corresponds to the massless spin-2 rank-2 symmetric tensor fields (linear quantum gravity in the dS space). In this case $\nu$ should be replaced by $\pm \frac{\nu}{2}$ in the field operators and then in their corresponding two-point functions. The projection operator $D_{\alpha'\beta'\alpha\beta}$ on the classical level [IV.55] and the normalization constant $c_{0,\nu}$ on the
quantum level \([IV, 32]\), become singular in this limit. The first singularity is actually due to the divergencelessness condition in which necessarily associates the rank-2 symmetric tensor field, belonging to the discrete series representation \(\Pi_{2,2}^\pm\). In order to remove this singularity, the divergencelessness condition must be dropped out. Then the field equation becomes gauge invariant, and the field operator must transform under an indecomposable representation of the dS group. By fixing the gauge, the field can eventually be quantized where unfortunately, a second type of singularity appears, which is in fact due to the so-called zero mode problem of the Laplace-Beltrami operator on the dS space inherited from the minimally coupled scalar field \([19]\). Nevertheless, using the Krein space quantization \([19]\), this singularity can be successfully removed and one can defined the quantum massless spin-2 rank-2 symmetric tensor fields in the dS space-time \([16, 51]\) but the theory is not analytic. In the section VII, another method for solving this problem (analyticity) is presented. In this method the tensor field can be written in terms of a polarization tensor and a conformally coupled scalar field. Then we present a linear quantum gravity based on a rank-2 symmetric tensor field which are analytic and free of any infra-red divergence. But this tensor field also break the conformal invariant and then it is not a real massless field in the dS space-time. For solving the problem of conformal invariance, one is obligated to use a rank-3 mixed-symmetric tensor fields for massless spin-2 field which will be considered in section VII.

V. QUANTUM FIELD OPERATORS FOR DISCRETE SERIES

The construction of quantum field operators in terms of the creation and annihilation operators for principal series can be directly generalized to the discrete series representation with \(j \neq p\) and \(p < 2\). For the values \(j = p \geq 1\), the quantum state \(|q; j, j\rangle\) cannot be defined uniquely from the differential equation \((III, 30)\), since a constant vector in \(V^j\) is an arbitrary solution in this case:

\[
|q; j, j\rangle \Rightarrow |q; j, j\rangle_g = |q; j, j\rangle + |q_0; j, j\rangle,
\]

where \(q_0\) is a constant. Because of this arbitrariness, one cannot construct the quantum field operators from the creation and annihilation operators on this state. Neglecting this non-uniqueness, at least one of the functions \(U_l\) or \(V_l\) blows up for this values, in addition the two-point function becomes singular in the limit of \(j = p \geq 1\). A massless field on the dS space associates with the value \(j = p\) and in the null curvature limit corresponds to a massless field of the Poincaré group. In fact, the creation and annihilation operators for discrete series with \(j = p \geq 1\) cannot be used to construct all of the irreducible quantum fields. Therefore, one may remove some of the physical conditions (iv-vii) of the section II-D and consequently, the quantum field must transform with an indecomposable representations of the dS group. Applying this peculiar limitation on different types of the fields naturally leads to the introduction of gauge invariance or gauge principle.

In the present section, the cases with \(j \neq p\), which called auxiliary fields, are being considered and afterwards, two auxiliary quantum fields charactenized with the sets \((j = \frac{3}{2}, \ p = \frac{1}{2})\) and \((j = 2, \ p = 1)\) are being constructed. They appear in the indecomposable representations of the massless fields \([52, 54]\). Next, the important case of \(j = p \geq 1\) and the appearance of the gauge invariance will be discussed.

A. The case \(j \neq p\)

In order to construct the quantum field operators, one must first define the creation and annihilation operators on the proposed Hilbert spaces \(\mathcal{H}_{q, (j,p)}^{(0,j,p)}\) and \(\mathcal{H}_{q, (j,0,p)}^{(j,0,p)}\). For discrete series, similar to the principal series, the creation operator \(a^\dagger(q, m_j; j, p)\) is defined as the operator that simply adds a state with quantum numbers \((q, m_j; j, p)\)

\[
a^\dagger(q, m_j; j, p) |\Omega\rangle \equiv |q, m_j; j, p\rangle,
\]

(V.1)
with \(|\Omega\rangle\) being the vacuum state which is invariant under the action of the discrete UIR of dS group:

\[
T^{(0,j;p)}(g)|\Omega\rangle = |\Omega\rangle.
\]  
(V.2)

In what follows, it has been shown that, this vacuum state \(|\Omega\rangle\) can be identified with the vacuum state in the principal series representation case. Due to the similarity of the cases UIR \(T^{(0,j;p)}\) and \(T^{(j,0;p)}\), only the former will be considered here.

\(a(q,m_j;j,p)\) is the adjoint operator of the creation operator \(a^\dagger(q,m_j;j,p)\) which can be defined using the equation (V.1):

\[
<\Omega| a^\dagger(q, m_j; j, p) |\Omega\rangle = q, m_j; j, p.
\]

For the principal series \(p = \frac{1}{2} + iv\) and one has \(p^* = 1 - p\) but for discrete series \(p\) is real. Since the two representations \(T^{(0,j;p)}\) and \(T^{(0,j;1-p)}\) are unitary equivalent (I.37), then the annihilation operator may be defined similar to the principal series (IV.4) as:

\[
[a^\dagger(q, m_j; j, p)]^\dagger \equiv a(\tilde{q}, m_j; j, 1 - p).
\]  
(V.3)

By the help of the orthogonality condition on the Hilbert space \(H_q^{(0,j;p)}\) (defined explicitly by Takahashi [4]), one shows that the operator \(a(q, m_j; j, p)\) removes a state from any state in which it acts on, similar to the massive case (I.4) and so, appropriately, it is called the annihilation operator and it annihilates the vacuum state:

\[
a(q, m_j; j, p) |\Omega\rangle = 0.
\]  
(V.4)

From (I.34), (V.1) and (V.2), one obtains:

\[
T^{(0,j;p)}(g)a^\dagger(q, m_j; j, p) [T^{(0,j;p)}(g)]^\dagger = |cq + d|^{-2(p+1)}
\]

\[
\times \sum_{m_j'} D^{(j)}_{m_j' m_j} \left( \frac{(cq + d)^{-1}}{|cq + d|} \right) a^\dagger (g^{-1} \cdot q, m_j'; j, p).
\]  
(V.5)

Similar to the principal series, for the annihilation operator one obtain:

\[
T^{(0,j;p)}(g)a(\tilde{q}, m_j; j, 1 - p) [T^{(0,j;p)}(g)]^\dagger = |cq + d|^{-2(2-p)}
\]

\[
\times \sum_{m_j'} \left[ D^{(j)}_{m_j m_j'} \left( \frac{(cq + d)^{-1}}{|cq + d|} \right)^* \right] a (g^{-1} \cdot \tilde{q}, m_j'; j, 1 - p).
\]  
(V.6)

Using the equation (I.37), one can see the annihilation operator \(a(q, m_j; j, p)\) transforms by the unitary equivalent representation of discrete series:

\[
T^{(0,j;1-p)}(g)a(q, m_j; j, p) [T^{(0,j;1-p)}(g)]^\dagger = |cq + d|^{-2(2-p)}
\]

\[
\times \sum_{m_j} \left[ D^{(j)}_{m_j m_j} \left( \frac{(cq + d)^{-1}}{|cq + d|} \right)^* \right] a (g^{-1} \cdot q, m_j'; j, p).
\]  
(V.7)
Similar to the principal series, one can prove that the following relation will hold:

\[ a(\mathbf{q}', m'_j; j, 1 - p)a^\dagger(\mathbf{q}, m_j; j, p) \pm a^\dagger(\mathbf{q}, m_j; j, p)a(\mathbf{q}', m'_j; j, 1 - p) = N(\mathbf{q}, m_j)\delta_{S^3}(\mathbf{q}' - \mathbf{q})\delta_{m, m'_j}, \]  

(V.8)

with + or − signs for fermionic or bosonic states, respectively.

The quantum field operator in terms of creation and annihilation operators can be written as the following forms, for tensor field \( j = l \),

\[ \mathcal{K}_{\alpha_1...\alpha_l}(x) = \int_B \mu(\mathbf{q}) \sum_{m_j} [a(\mathbf{q}, m_j; j, 1 - p)U_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) + a^\dagger(\mathbf{q}, m_j; j, p)V_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p)], \]  

(V.9)

and for tensor-spinor field \( j = l + \frac{1}{2} \),

\[ \Psi_{\alpha_1...\alpha_l}(x) = \int_B \mu(\mathbf{q}) \sum_{m_j} [a(\mathbf{q}, m_j; j, 1 - p)U_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) + a^\dagger(\mathbf{q}, m_j; j, p)V_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p)]. \]  

(V.10)

The coefficients \( U \), \( V \) are chosen so that under the dS transformations the field operators transform by the UIR of the dS group (principle B). For tensor fields \( (j = l) \) one writes:

\[ T^{(0,j;p)}(g)\mathcal{K}_{\alpha_1...\alpha_l}(x) [T^{(0,j;p)}(g)]^\dagger = \Lambda_{\alpha_1'...\alpha_l'}\mathcal{K}_{\alpha_1'...\alpha_l'}(\Lambda x), \]  

(V.11)

which has an equivalent for the tensor-spinor fields \( (j = l + \frac{1}{2}) \), by the form [3, 12, 18]:

\[ T^{(0,j;p)}(g)\Psi_{\alpha_1...\alpha_l}(x) [T^{(0,j;p)}(g)]^\dagger = \Lambda_{\alpha_1'...\alpha_l'}g^{-1}\Psi_{\alpha_1'...\alpha_l'}(\Lambda x), \]  

(V.12)

with \( \Lambda \in SO(1,4) \) and \( g \in Sp(2,2) \). After making use of \([\text{V.5}], [\text{V.6}], [\text{V.9}], [\text{V.11}]\) and the dS invariant volume element on the unit ball \( B \) \([\text{III.3}]\), it is easy to show that the coefficients \( U_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) \) and \( V_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) \) satisfy the following equations:

\[ |\mathbf{c}q + \mathbf{d}|^{-2p+6} \sum_{m_j} D^{(j)}_{m_j,m_j} \left( \frac{(\mathbf{c}q + \mathbf{d})^{-1}}{|\mathbf{c}q + \mathbf{d}|} \right)^2 \mathcal{V}_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) \]

\[ = \Lambda_{\alpha_1'...\alpha_l'}\mathcal{V}_{\alpha_1'...\alpha_l'}(\Lambda x; g^{-1} \cdot \mathbf{q}, m'_j; j, p), \]  

(V.13)

\[ |\mathbf{c}q + \mathbf{d}|^{-2p+4} \sum_{m_j} \left[ D^{(j)}_{m_j,m_j} \left( \frac{(\mathbf{c}q + \mathbf{d})^{-1}}{|\mathbf{c}q + \mathbf{d}|} \right)^2 \right] \mathcal{U}_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) \]

\[ = \Lambda_{\alpha_1'...\alpha_l'}\mathcal{U}_{\alpha_1'...\alpha_l'}(\Lambda x; g^{-1} \cdot \mathbf{q}, m'_j; j, p). \]  

(V.14)

These equations and the conditions (i-vii) in section II-D will allow us to calculate the functions \( \mathcal{U} \) and \( \mathcal{V} \) explicitly. One can write similar equations for a tensor-spinor field. The homogeneity condition results to:

\[ \mathcal{V}_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p) = (x \cdot \xi_B)^\lambda \mathcal{V}_{\alpha_1...\alpha_l}(x; \mathbf{q}, m_j; j, p), \]

where \( \xi_B^\alpha = \xi^0(1, \coth \kappa \mathbf{q}) \). For these fields we can choose \( \xi_B^\alpha(0) = (1, 0, 0, 0, 1) \) similar to the massive case where the little group is \( SO(3) \). The null curvature limit of these field operators does not exist \( i.e. \) these fields do not have any counterparts in the null curvature limit.
Similar to the principal series, one can explicitly calculate the quantum field operator and its corresponding two-point function for these fields only for \( p < 2 \). As it is already known, one of the homogeneous degrees, \( \lambda \), is positive for \( p > 2 \) and as also, the plane wave becomes infinite for the large values of \( x \) (III.11), therefore the quantum field operator is not an infinitely differentiable function. Consequently, one cannot define the field operator in a sense of a distribution function for these cases \( (p > 2) \). The mass associated with these fields is also imaginary or \( m^2 < 0 \) in (III.7) and (III.17).

In these cases \( (j \neq p) \), two auxiliary fields are important for the construction of the massless conformal quantum fields in the dS space, one of which being the auxiliary spin- \( \frac{3}{2} \) field, defined by \( j = \frac{3}{2} \) and \( p = \frac{1}{2} \) and the other one, by \( j = 2 \) and \( p = 1 \). In the following subsection, these two auxiliary field have been described.

1. Auxiliary spin- \( \frac{3}{2} \) field

An auxiliary vector-spinor field associates with the discrete series representation with \( j = \frac{3}{2} \), \( p = \frac{1}{2} \) and its corresponding mass is \( m^2 \frac{1}{2} = 2H^2 \). It satisfies the conditions (i-vii) of the section II-D and its relevant eigenvalue of the Casimir operator is \( \langle Q^{(1)}_{p,q} \rangle = -\frac{3}{2} \). This field satisfies the following field equation

\[
\left(\kappa \, \partial^\top - 2\right) \Psi_\alpha(x) = 0, \quad \left(\Box_H + 2H^2\right) \Psi_\alpha(x) = 0.
\]

The field operator of the auxiliary vector-spinor field is defined by:

\[
\Psi_\alpha(x) = \int_B d\mu(q) \sum_m \left[ a(q, m; \frac{3}{2}, \frac{1}{2}) U_\alpha(x; q, m; \frac{3}{2}, \frac{1}{2}) + a^\dagger(q, m; \frac{3}{2}, \nu) V_\alpha(x; q, m; \frac{3}{2}, \frac{1}{2}) \right], \quad \text{(V.15)}
\]

where \( m = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \) and \( \Psi_\alpha, U_\alpha \) and \( V_\alpha \) are four-component spinors. From the equation (III.21), it is well-known that the homogeneous degrees of spinor-vector field with \( p = \frac{1}{2} \) are \( \lambda = -1 \) and \( -2 \):

\[
U_\alpha(x; q, m; \frac{3}{2}, \frac{1}{2}) = (x \cdot \xi_B)^{-2} u_\alpha(x, q, m; \frac{3}{2}, \frac{1}{2}),
\]

\[
V_\alpha(x; q, m; \frac{3}{2}, \frac{1}{2}) = (x \cdot \xi_B)^{-1} u_\alpha(x, q, m; \frac{3}{2}, \frac{1}{2}).
\]

Through defining \( q = ru \), one obtains \( \xi_B = (1, \coth \kappa q) = (1, \coth \kappa r u) = (1, \pm u) \). Then the field operator and the four-components spinors \( U \) and \( V \) and also the two-point function can be obtained simply from the principal series counterpart by replacing \( \nu = 0 \) in their corresponding relations.

The analytic two-point function for auxiliary vector-spinor field is:

\[
S^{\alpha\beta'}_{\alpha\alpha'}(z, z') = \langle \Omega | \Psi_\alpha^\dagger(z) \tilde{\Psi}_{\alpha'}^\dagger(z') | \Omega \rangle
\]

\[
= \int_B d\mu(q) (z \cdot \xi_B)^{-2}(z' \cdot \xi_B)^{-2} \sum_m u_\alpha(z, q, m; \frac{3}{2}, \frac{1}{2}) \bar{u}_{\alpha'}(z', q, m; \frac{3}{2}, \frac{1}{2}). \quad \text{(V.16)}
\]

This analytic two-point function can be obtained from the principal counterpart (IV.51) by replacing \( \nu = 0 \):

\[
S_{\alpha\alpha'}(z, z') = D_{\alpha\alpha'}(z, \partial^\top z', \partial'^\top) S_{c}(z, z'), \quad \text{(V.17)}
\]
where
\[ D_{\alpha\alpha'}(z, \partial^T; z', \partial'^T) = \theta_\alpha \cdot \theta_{\alpha'} - \frac{1}{4} \gamma^\alpha \gamma^\top \cdot \theta_{\alpha'} - \left(-\frac{1}{4} \gamma^\top \cdot \theta_{\alpha'} + 2 \theta^T z \cdot \theta_{\alpha'} \right) + \frac{2}{3} \partial_a \theta_{\alpha'} \cdot \partial^T - \frac{2}{3} \gamma^\top \gamma^\top \cdot \theta_{\alpha'} \cdot z \cdot \rho^T + \frac{1}{3} \gamma^\top \gamma^\top \cdot \theta_{\alpha'} \cdot \rho - \frac{1}{3} \gamma^\top \theta_{\alpha'} \cdot \rho^T \right]. \] (V.18)

\( S_c(z, z') \) is the analytic two-point function of conformally massless spinor field \[ \text{which will be considered in the section VII-C:} \]
\[ S_c(z, z') = \frac{i}{2\pi^2} \left[ \frac{(\rho - \rho')\gamma^4}{(z - z')^2} \right]. \] (V.19)

One can see that the two-point function becomes singular for the massless spin-\( \frac{3}{2} \) field \( j = 2 \) in the limit \( \nu = i \) and consequently, one cannot define the two-point function or the field operator in this limit. Such feature can be interpreted as the disability of the massless spin-\( \frac{3}{2} \) field to transform as an UIR of the dS group which is equivalent to the appearance of a gauge invariance.

2. Auxiliary spin-2 rank-2 symmetric tensor field

An auxiliary spin-2 rank-2 symmetric traceless tensor field corresponds to the discrete series representation with \( j = 2 \) and \( p = 1 \). The eigenvalue of the Casimir operator and its corresponding mass are \( Q(\alpha, \beta) = -4 \) and \( m_{\alpha, \beta}^2 = 2H^2 \). The field equation is:
\[ \left( Q(\alpha, \beta) + 4 \right) K_{\alpha\beta}(x) = 0, \quad \left( \Box_H + 2H^2 \right) K_{\alpha\beta}(x) = 0. \]
The tensor field \( K_{\alpha\beta} \) can be written in terms of a massless conformally coupled scalar field \( \phi_c \):
\[ K_{\alpha\beta}(x) = \left( -\frac{2}{3} \theta Z_1 + \Sigma_1 Z_1^\top + \frac{1}{3} D_2 \left[ \frac{1}{9} D_1 Z_1 + x \cdot Z_1 \right] \right) \left( Z_2^\top + D_1 \left( x \cdot Z_2 \right) \right) \phi_c, \] (V.20)
where \( Z_1 \) and \( Z_2 \) are constant five-vectors and \( \Sigma_1 \) was defined in equation (II.22). \( \phi_c \) will be considered in section VII.A. The field operator can be written in the following form:
\[ K_{\alpha\beta}(x) = \sum_m \int_B d\mu(q) \left[ a(q, m; 2, 0) U_{\alpha\beta}(x; q, m; 2, 1) + a^\top(q, m; 2, 1) V_{\alpha\beta}(x; q, m; 2, 1) \right], \] (V.21)
with \( m = 2, 1, 0, -1, -2 \). The homogeneous degrees of such field are \( \lambda = -1 \) and \( -2 \): \[ U_{\alpha\beta}(x; q, m; 2, 1) = (x \cdot \xi_B)^{-2} u_{\alpha\beta}(x, q, m; 2, 1), \]
\[ V_{\alpha\beta}(x; q, m; 2, 1) = (x \cdot \xi_B)^{-1} v_{\alpha\beta}(x, q, m; 2, 1). \]
The coefficients \( u \) and \( v \) satisfy the conditions (v-vi) of section II-D:
\[ u_{\alpha\beta}(x, q, m; 2, 1) = v_{\beta\alpha}(x, q, m; 2, 1), \quad v_{\alpha\beta}(x, q, m; 2, 1) = 0, \]
and, the field operator satisfies the following transformation role:
\[ T(0,2;1)(g) K_{\alpha\beta}(x) \left[T(0,2;1)(g) \right]^\top = \Lambda_{\alpha}^{\alpha'} \Lambda_{\beta}^{\beta'} K_{\alpha';\beta'}(\Lambda x). \]
where defined by this equation, and also, it is invariant under the following transformation:

\[ \Delta \]

Since \( r_1; 2; (V.24) \) and relations (III.29):

\[ j \]

Here, we only consider three sets of values, namely \( \mathbb{D} \) with existing equivalent massless fields in the Minkowski space-time, in the null curvature limit.

and \( W_\epsilon(z, z') \) cannot define proper plane waves. Even for these cases, one has trouble with the construction of the field operators, which transform as an UIR of the dS group.

and \( W_\epsilon(z, z') \) is the analytic two-point function of massless conformally coupled scalar field \( r, g, c \) which will be considered in the section VII-A:

\[ W_\epsilon(z, z') = \frac{-iH^2}{24\pi^2} P^{(5)}_\epsilon (H^2z,z') = \frac{H^2}{8\pi^2} \frac{-1}{1 - Z(z, z')} \]  
(\text{V.23})

\[ Z(z, z') = -H^2z \cdot z' \] is the geodesic distance between two points on the complex dS hyperboloid.

B. The case \( j = p \geq 1 \)

The analytic two-point function is defined by:

\[ W_{\alpha\beta'\beta'}(z, z') = \langle \Omega | K_{\alpha\beta}(z) K_{\alpha'\beta'}(z') | \Omega \rangle \]

\[ = \int_B d\mu(q)(z \cdot \xi_B)^{-2}(z' \cdot \xi_B)^{-1} \sum_{m} u_{\alpha\beta}(z, q, m; 2, 1) v_{\alpha'\beta'}(z', q, m; 2, 1). \]  
(\text{V.22})

This analytic two-point function has been calculated in the previous paper \( \rangle \): \( W_{\alpha\beta'\beta'}(z, z') = D_{\alpha\beta'\beta'}(z, \partial; z', \partial'; 1)W_\epsilon(z, z'), \)

where

\[ D_{\alpha\beta'\beta'}(z, \partial; z', \partial'; 1) = -\frac{2}{3} \sum' \theta' \left( \theta' + D'_1(\theta'.z) \right) \]

\[ + \sum_1 \sum' \theta' \left( \theta' + D'_1(\theta'.z) \right) + \frac{1}{3} D_2^1 \sum_1 \left( \frac{1}{9} D_1^3 \theta' + z' \theta' \right) \left( \theta' + D'_1(\theta'.z) \right), \]

and \( W_\epsilon(z, z') \) is the analytic two-point function of massless conformally coupled scalar field \( \rangle \) which will be considered in the section VII-A:

\[ W_\epsilon(z, z') = \frac{-iH^2}{24\pi^2} P^{(5)}_\epsilon (H^2z,z') = \frac{H^2}{8\pi^2} \frac{-1}{1 - Z(z, z')} \]  
(\text{V.23})

\[ Z(z, z') = -H^2z \cdot z' \] is the geodesic distance between two points on the complex dS hyperboloid.

The case \( j = p \geq 1 \) is of particular interest, since it corresponds to three massless fields in the dS space with existing equivalent massless fields in the Minkowski space-time, in the null curvature limit. Here, we only consider three sets of values, namely \( j = p = 1, 2, 3 \), and for other values one cannot define proper plane waves. Even for these cases, one has trouble with the construction of the field operators, which transform as an UIR of the dS group.

The field equation in \( \xi \)-space \( \langle \text{III.30} \rangle \) for these cases become:

\[ \left[ \frac{1}{4} (1 - |q|^2) \Delta - jD - \frac{1}{2} (A_j D_1 + B_j D_2 + C_j D_3) \right] |q; j, j\rangle = 0. \]  
(\text{V.24})

Since \( \Delta, D, D_1, D_2 \) and \( D_3 \) are differential operators, the quantum state \( |q; j, j\rangle \) is not uniquely defined by this equation, and also, it is invariant under the following transformation:

\[ |q; j, j\rangle \Rightarrow |q; j, j\rangle^g = |q; j, j\rangle + |q_0; j, j\rangle, \]  
(\text{V.25})

where \( |q_0; j, j\rangle \in V^j \) and \( q_0 \) is constant.

If we assume \( q = ru \) with \( |u| = 1 \) and \( |q| = r \), then the \( |q; j, p\rangle \) can be written as \( \langle \text{III.31} \rangle \):

\[ |q, m; j, p\rangle = F(p - j, j + p + 1; r^2) \sum_{m'} C_{mm'} |u, m'; j, p\rangle, \]

whit \( |u; j, p\rangle \in V^j \) and \( F(p - j, j + p + 1; r^2) \) being the hyper-geometric function. \( F(p - j, j + p + 1; r^2) \) satisfies the following differential equation \( \rangle \) which can be simply obtained, using equation \( \langle \text{V.24} \rangle \) and relations \( \langle \text{III.29} \rangle \):

\[ \left[ \frac{1 - r^2}{4} \left( \frac{d^2}{dr^2} + 3 \frac{d}{dr} \right) \right] F(p - j, j + p + 1; r^2) = 0. \]  
(\text{V.26})
One of the solutions of the above equation for \( j = p \) is constant and therefore the hyper-geometric function becomes constant:

\[
F(0, 2j + 1; 2; r^2) = \text{constant}.
\]

This is precisely alike to a problem, previously occurred in the establishment of the minimally coupled scalar fields in the dS space, where one needed to introduce an indecomposable representation of the dS group to define field operators in an indefinite inner product space (Krein space quantization) \[8, 19, 53\]. It means that the creation operator \( a^\dagger(q, m_j; j, j) \) cannot be defined properly whenever it is being transformed by an UIR \( T^{(0;j:j)}(g) \) of the dS group.

Similar to the previous section, by ignoring this difficulty, one can define a field operator which suffers the lack of the capability of being transformed under an UIR of the dS group \[15, 19, 51\]. For such a case, one has:

\[
T^{(0;j:j)}(g)K_{\alpha_1...\alpha_l}(x) \big[ T^{(0;j:j)}(g) \big]^\dagger = \Lambda_{\alpha_1}^{\alpha'_1}...\Lambda_{\alpha_l}^{\alpha'_l}K_{\alpha'_1...\alpha'_l}(\Lambda x) + \left[ D_l^\top \Omega \right]_{\alpha_1...\alpha_l}, \tag{V.27}
\]

where \( D_l^\top \) is a generalized-gradient which preserves the trasversality and symmetric type of tensor field \( K \) on the dS hyperboloid. \( \Omega(x) \) is a linear combination of annihilation and creation operators whose precise form is non of our concern here. It is sufficient to know that \( \Omega(x) \) is a rank-(\( l-1 \)) tensor field with some constituents. There are two types of tensor field \( \Omega \): the first type satisfies the following field equation:

\[
\left[ \partial_l^\top \cdot D_l^\top \Omega_3(x) \right]_{\alpha_1...\alpha_{l-1}} = 0,
\]

where \( (\partial_l^\top \cdot) \) is the generalized divergence in the ambient space formalism on the dS hyperboloid. It is called the pure gauge state. The second type is

\[
\left[ \partial_l^\top \cdot D_l^\top \Omega_1(x) \right]_{\alpha_1...\alpha_{l-1}} \neq 0,
\]

which satisfies some conditions (ii-vii) of section II-D. It is the gauge-dependent state. The explicit form of \( \Omega_3 \) and \( \Omega_1 \) can be obtained form the gauge invariant transformation and gauge fixing field equation which are being discussed in the next section. These two type of un-physical states with the physical state construct the Gupta-Bluler triplet states of the massless fields in the dS space-time.

The field operators act on a vector space which is constructed on an indecomposable representations of the dS group. For these fields a massless quantum state can be divided into three parts:

\[
\mathcal{M} = V_1 \oplus V_2 \oplus V_3,
\]

where \( V_1 \) is the space of the gauge dependent states and \( V_3 \) is the pure gauge states. The physical states appear in \( V_2/V_3 \equiv H, \) which are being constructed on the UIR of discrete series with \( j = p \geq 1, \)

\( i.e. \ H_q^{(j,0;j)} \) and \( H_q^{(0;j:j)} : \)

\[
| \text{physical states} > = H_q^{(j,0;j)} \oplus H_q^{(0;j:j)}.
\]

The precise forms of \( V_3 \) and \( V_1 \) spaces can be defined by using the gauge invariant transformation and the gauge fixing field equation in \( x \)-space.

The gauge invariance in the \( x \)-space appear in a different form in comparison with the \( q \)-space. When \( j = p \), the coefficient \( v_{\alpha_1...\alpha_2}(x, q, m_j; j, j) \) can not be defined uniquely and it becomes singular. From the viewpoint of the quantum field theory, the coefficient \( v_{\alpha_1...\alpha_2}(x, q, m_j; j, j) \) has infinite normalization \[13, 14\] and with the conditions (i-vii) of section II-D one cannot obtain it \[14, 15, 24, 46\]. This problem appears due to the divergencelessness condition which is necessary for associated each
field with an UIR of dS group. After introducing the Lagrangian formalism and fixing the gauge, one can derive the field equation off the Lagrangian and hence, define the quantum field operators which are associated with an indecomposable representation of the dS group.

Up to this point, based of the gauge theory approach, the three cases of vector fields, vector-spinor fields and spin-2 fields have been discussed. The structure of the indecomposable representation will not be the subject of attention in this paper but the vector field was also discussed previously in [13].

VI. GAUGE INVARIANT FIELD EQUATION

Establishment of the gauge invariant equation (V.24) and the gauge invariant transformation (V.25) of the massless fields with \( j = p \geq 1 \) in the \( \xi \)-space was the subject of the previous section. Additionally, the gauge invariant equations of the three important cases, namely the vector fields, the vector-spinor fields and the spin-2 rank-2 symmetric tensor fields in the \( x \)-space have been presented in the previously published papers [8, 15, 16, 51, 52]. In this section the gauge invariant equations are obtained in \( x \)-space by the use of the gauge principle.

The interactions between the elementary systems in the universe are governed by the gauge principle. An interaction is defined through the gauge-covariant derivative which is defined as quantity that preserves the gauge invariant transformation of the Lagrangian. For an instructive review of gauge transformation and gauge potentials the reader can see [56]. The following includes only the three gauge fields, namely the vector fields \( (j = p = 1) \), vector-spinor field \( (j = p = \frac{3}{2}) \) and spin-2 field \( (j = p = 2) \). These gauge fields, in the language of the gauge theory, are the sources or potentials of the various forces. Such fact can be considered as a connection in the gauge group manifold. One can associate these gauge transformations with the local symmetrical groups, noting that these gauge fields transform according to these symmetrical groups.

The gauge vector fields \( (j = p = 1) \), \( K^a_\alpha \) with \( a = 1, 2, \ldots, n^2 - 1 \), may be associated with the gauge group \( SU(n) \). The same association may be engaged between the gauge spin-2 fields \( (j = p = 2) \) in the gauge gravity framework and the gauge group \( SO(1, 4) \), or equivalently, the gauge group \( SO(2, 4) \) [57]. Imposing some physical conditions on the gauge field or on the connection, such case can be described as a spin-2 rank-3 mix-symetric tensor field \( K^{M}_{\alpha\beta\gamma} \). The vector-spinor gauge fields \( (j = p = \frac{3}{2}) \), \( \Psi^a_\alpha \) with \( a = 1, 2, \ldots, N \), are the spinor fields and consequently, their corresponding gauge group must have spinorial generators to justify a set of well defined gauge-covariant derivative. Therefore, a set of anti-commutative generators satisfy a super-algebra. Nevertheless, such algebra would not be closed, since its constituent generators are Grassmanian functions which will have usual functions as their multiplication products \( i.e. \) the anti-commutation of two spinor generators become a tensor generator. In this case, for obtaining a closed super-algebra, the Grassmanian generators must be coupled with the generators of the dS group. Additionally, in the language of the gauge theory, one may describe a vector-spinor field as a real force which must be coupled with a spin-2 gauge potential, and consequently, the gauge potentials \( K^{M}_{\alpha\beta\gamma} \) and \( \Psi^a_\alpha \) can be interpreted as a unique force. These two gauge fields may be described the gravitational field. The gauge group in this case is a super-group.

A. Vector gauge theory

This subsection is a direct generalization of the abelian gauge theory in the dS ambient space formalism [58]. Since such formalism in the ambient space notation is utterly similar to its Minkowskian counterpart, only the differences and necessary considerations are presented here. The vector gauge potential \( K^a_\alpha \) with \( a = 1, 2, \ldots, n^2 - 1 \), can be associated with the gauge group \( SU(n) \). One can
assume that \( t^a \) is the generator of \( SU(n) \) group, satisfying the following commutation relation
\[
[t_a, t_b] = C_{ab}^c t_c. \tag{VI.1}
\]

The notation of local gauge symmetry with its space-time-dependent transformation can be used to generate the gauge interaction. For obtaining a local gauge invariant Lagrangian, it is necessary to replace the transverse-covariant derivative \( \nabla^\top_\beta \) with the gauge-covariant derivative \( D^K_\beta \) which is defined by
\[
D^K_\beta = \nabla^\top_\beta + iK_a^\beta t_a, \tag{VI.2}
\]
where the gauge potential or connection \( K_a^\beta \) is a vector field. The transverse-covariant derivative which acts on tensor field is [27]:
\[
\nabla_\beta T_{\alpha_1...\alpha_l} = \partial_\beta T_{\alpha_1...\alpha_l} - H^2 \sum_{n=1}^l x_{\alpha_n} T_{\alpha_1...\alpha_{n-1}\beta\alpha_{n+1}...\alpha_l}. \tag{VI.3}
\]

Considering a local infinitesimal gauge transformation generated by \( \epsilon^a(x) t_a \), one has
\[
\delta \epsilon K_a^\alpha = [D^K_\alpha, \epsilon^a(x) t_a] = (D^K_\alpha \epsilon^a) t_a,
\]
so that
\[
\delta \epsilon K_a^\alpha = D^K_\alpha \epsilon^a = \partial_\alpha \epsilon^a + C_{cb}^a K_c^\alpha \epsilon^b.
\]
The curvature \( F \), for the Lie algebra of the gauge group \( SU(N) \), is defined by:
\[
F(D^K_\alpha, D^K_\beta) = -[D^K_\alpha, D^K_\beta] = F_{\alpha\beta}^a t_a,
\]
where
\[
F_{\alpha\beta}^a = \nabla^\top_\alpha K_{\beta}^a - \nabla^\top_\beta K_{\alpha}^a + K_b^\beta K_c^\alpha C_{bc}^a, \quad x^a F_{\alpha\beta} = 0 = x^\beta F_{\alpha\beta}.
\]

The \( SU(N) \) gauge invariant action or Lagrangian in the dS background for the gauge field \( K_{\alpha}^a \) is:
\[
S_1[K] = -\frac{1}{2} \int d\mu(x) \text{tr} \left( F_{\alpha\beta} F^{\alpha\beta} \right) = -\frac{1}{2} \int d\mu(x) F_{\alpha\beta}^a F^{\alpha\beta}_{ab} \text{tr} (t_a t_b),
\]
with \( F_{\alpha\beta} = t^a F_{\alpha\beta}^a \) and summing over the repeated indices. The normalization of the structure constant is usually fixed by requiring that, in the fundamental representation, the corresponding matrices of the generators \( t_a \) are normalised such as [56]
\[
\text{tr} (t_a t_b) = \frac{1}{2} \delta_{ab}.
\]

Then the action becomes
\[
S[K] = -\frac{1}{4} \int d\mu(x) F_{\alpha\beta}^a F^{\alpha\beta}_{ab}.
\]
The field equation for this action is [58]:
\[
\theta^\gamma \gamma^\top \nabla^\top_\gamma \left( \nabla^\top_\alpha K_{\beta}^a - \nabla^\top_\beta K_{\alpha}^a \right) + O([K]^2) = 0,
\]
\[
\partial^\top \partial^\top K_{\alpha}^a + 2K_{\alpha}^a - 2x_a \partial^\top K^a - \partial_\alpha \partial \cdot K^a + O([K]^2) = 0, \tag{VI.4}
\]
where $O[(K)^2]$ is the second order of $K$. In terms of Casimir operator (II.19), such field equation can be rewritten in the following form [13, 58]:

$$Q_1^{(1)} K^\alpha_a + \partial^\top_\alpha \partial \cdot K^a + O[(K)^2] = 0,$$

with the non-linear terms being defined as the interaction between the gauge potentials $K^a_{\alpha}$. The linear field equation is invariant under the following linear gauge transformation:

$$K^a_{\alpha} \longrightarrow (K^g)^{\alpha}_a = K^a_{\alpha} + \partial^\top_\alpha \phi^a,$$

where $\phi^a$ is an arbitrary scalar fields. The linear gauge fixing field equation is

$$Q_1^{(1)} K^a_{\alpha} + c\partial^\top_\alpha \partial \cdot K^a = 0,$$  \hspace{1cm} (VI.5)

where $c$ is gauge fixing parameter. Similar to the Minkowski space, the gauge fixing equation can be obtained from the new Lagrangian with the additional gauge fixing terms:

$$S_1[K] = \int d\mu(x) \left[ -\frac{1}{4} F^a_{\alpha\beta} F_a^{\alpha\beta} - \frac{c'}{2} \partial \cdot K^a \partial \cdot K^a \right],$$

with summing over index $a$ and $c'$ being the gauge fixing parameter. The Faddeev-Popov ghost fields can also be added in the quantization procedure in an exact similar way as in the Minkowski space-time.

### B. Spin-2 gauge theory

Utiyama was the first who proposed that the general relativity can be seen as a gauge theory based on the local Lorentz group [59]. Kibble comprehensively extended the Utiyama’s gauge theory of gravitation by showing that the local Poincaré symmetry can generate a space with torsion as well as curvature [60]. It is well-known that the gauge gravity model, which is already built for the Poincaré group, becomes the Einstein general relativity in the case of the metric compatible and torsion free condition [61, 62]. For obtaining a similar construction with the Yang-Mills theory (or previous sub-section) one uses the Fronsdal paper for spin-two, conformal gauge theory on the Dirac 6-cone formalism [57], which can simply be mapped on the dS ambient space formalism.

In what follows, a brief introduction of the formalism of the gauge gravity model in the dS ambient space for gauge groups $SO(1,4)$ will be presented. The conformal consideration or $SO(2,4)$ gauge gravity will be discussed in section VIII-C. In this formalism, imposing some physical conditions on the gauge potentials or the gauge vector fields (10 vector fields for $SO(1,4)$ and 15 vector fields for $SO(2,4)$), one can describe these gauge vector fields using the higher rank tensor fields on the dS hyperboloid. This field may be described the gravitational waves which propagate on the dS light cone, and in the quantum level, the field operator transforms by an indecomposable representation of dS group.

Let us generalize the previous discussions on the gauge theory in dS ambient space formalism to the gauge dS group $SO(1,4)$ by presenting a brief review of the procedure of obtaining the linear gauge invariant field equation and the gauge transformation and focusing only on the main results. The complete considerations will be revealed in a forthcoming paper.

For simplicity, we change the notation, as in a previous subsection, and defined the group generators by:

$$L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta} \equiv X_A, \hspace{0.5cm} A = 1, 2, ..., 10,$$
afterwards, the results will be presented in the suitable notation, i.e. ambient space formalism. Let us write the commutation relation as:

$$[X_A, X_B] = f_{BA}^C X_C.$$ 

In this case we have 10 gauge vector fields $K^A_\alpha \equiv K^\beta_\alpha \gamma = -K^\gamma_\alpha \beta$. Since these gauge vector fields exist on the dS hyperboloid, they must be transverse with respect to the first index $x \cdot K^A_\alpha \equiv x^\alpha K^A_\alpha = 0$. Additionally, one counts 40 degrees of freedom for these gauge vector fields and recognizes them as the connection coefficients in the general relativity framework. In the ambient space notation the gauge-covariant derivative can be defined as

$$D^K_\beta = \nabla^\top_\beta + iK^B_\alpha X^A_\alpha,$$

where $\nabla^\top_\beta$ is defined in equation (VI.3). The gauge-covariant derivative must be a map from spinor or tensor fields of rank-$l$ to a spinor or tensor fields of rank-$l + 1$ on the dS hyperboloid in the ambient space formalism. It means that the gauge-covariant derivative of a tensor field must satisfy the transversality condition:

$$x^\alpha D^K_\beta T^\alpha_{\alpha_1 \ldots \alpha_l} = 0, \quad x^\alpha T^\alpha_{\alpha_1 \ldots \alpha_l} = 0.$$

The first part of the gauge-covariant derivative $D^K_\beta$ satisfies the transversality condition, but the second part is not transverse:

$$x^\alpha K^\beta_\alpha X^A_\alpha T^\alpha_{\alpha_1 \ldots \alpha_l} \neq 0,$$

since $x^\alpha M_{\alpha \beta} \neq M_{\alpha \beta} x^\alpha$. Imposing the following subsidiary condition on the gauge vector fields $K^A_\alpha$,

$$x^\alpha K^\beta_\alpha = 0, \quad x^\gamma K^\beta_\alpha = 0,$$

one obtains

$$x^\alpha K^\beta_\alpha X^A_\alpha T^\alpha_{\alpha_1 \ldots \alpha_l} = 0.$$

These conditions (VI.6) preserve the transversality condition for the gauge-covariant derivative. In this case, the combination of the 10 gauge vector fields $K^A_\alpha$ can be considered as a tensor field of rank-3 with $K^\beta_\alpha \gamma = -K^\gamma_\alpha \beta$ on the dS hyperboloid, with 24 degrees of freedom in the ambient space formalism.

Now, one can repeat the construction of the gauge gravity in its canonical manner. Under a local infinitesimal gauge transformation generated by $\epsilon^A(x)X_A$, one has

$$\delta_\epsilon D^K_A = [D^K_A, \epsilon^A(x)X_A] = (D^K_A \epsilon^A)X_A,$$

so that

$$\delta_\epsilon K^A_\alpha = D^K_A \epsilon^A = \nabla^\top_\alpha \epsilon^A + f_{CB}^A K^C_\alpha \epsilon^B.$$  

(VI.7)

The curvature $\mathcal{R}$, with values in the Lie algebra of the dS group, is defined by:

$$\mathcal{R}(D^K_A, D^K_B) = -[D^K_A, D^K_B] = R^A_{\alpha \beta} X_A,$$

where

$$R^A_{\alpha \beta} = \nabla^\top_\alpha K^A_\beta - \nabla^\top_\beta K^A_\alpha + K^B_\beta K^C_\alpha f_{BC}^A.$$  

(VI.8)
The $SO(1, 4)$ gauge invariant action or Lagrangian in the dS background for the gauge potential $K_{\alpha\beta\gamma} = -K_{\alpha\gamma\beta}$ with the condition (VI.6), can be written in the following form [62–64]:

$$S_2[K] = \int d^4x \alpha_{\alpha\beta\gamma\delta}(R_{\alpha\beta}^A R_{\gamma\delta}^B)Q_{AB},$$

where $Q_{AB}$ is a numerical constant and therefore, for the dS group, one can chose as $Q_{AB} \equiv \epsilon_{\alpha\beta\gamma\delta}$ [62–64]. The action can be written in the following form:

$$S_2[K] = \int d\mu(x) \left[ \theta^\alpha \nabla^\top_{\alpha} \left( \nabla^\tau_{\gamma} K_{\delta}^B - \nabla^\tau_{\delta} K_{\gamma}^B \right) + \theta^\alpha \left( \nabla^\top_{\alpha} K_{\beta\gamma} + \theta^\alpha \nabla^\top_{\alpha} \right) \right] Q_{AB} + O[\mathcal{K}^3], \quad (VI.9)$$

with $O[(\mathcal{K})^3]$ being the third order of $\mathcal{K}$. The field equation is:

$$\theta^\alpha \nabla^\top_{\alpha} \left( \nabla^\tau_{\gamma} K_{\delta}^B - \nabla^\tau_{\delta} K_{\gamma}^B \right) Q_{AB} + O[(\mathcal{K})^2] = 0, \quad (VI.10)$$

with $O[(\mathcal{K})^2]$ being the second order of $\mathcal{K}$. Simplifying the relation between the gauge potential $K_{\alpha\beta\gamma}$ and the UIR of the dS group, one must write this equation in terms of the Casimir operators of the dS group.

The effect of the Casimir operator on the totally symmetric tensor and tensor-spinor fields of rank-$l$ has been presented in the section II-B. Using the equations (II.15), (II.17) and (II.18) the effect of the Casimir operator on the totally symmetric tensor field and tensor-spinor fields of rank-$l$ can be calculated. One obtain:

$$Q^{(1)}_3 K_{\alpha\beta\gamma} = Q_0 K_{\alpha\beta\gamma} - 4K_{\alpha\beta\gamma} - 2K_{\alpha\beta\gamma} - 2K_{\alpha\beta\gamma}$$

$$+ 2\delta_{\alpha\beta\gamma} K_{\alpha\beta\gamma} + 2\delta_{\alpha\beta\gamma} K_{\alpha\beta\gamma} + 2\delta_{\alpha\beta\gamma} K_{\alpha\beta\gamma} + 2\delta_{\alpha\beta\gamma} K_{\alpha\beta\gamma} + (VI.11)$$

where $K_{\alpha\beta\gamma} = K_{\beta\alpha\gamma}$ is traversed over the first and the second indices. Now in terms of the above Casimir operator (VI.11), the field equation (VI.10), with the condition $K_{\alpha\beta\gamma} = 0$, can be rewritten in the following form [22]:

$$\left( Q_3^{(1)} + 6 \right) K_{\alpha\beta\gamma} + \nabla^\top_{\alpha} \nabla_{\alpha} K_{\alpha\beta\gamma} + O[(\mathcal{K})^2] = 0, \quad (VI.12)$$

with $\partial^\top_{\alpha} \cdot$ being the generalized divergence:

$$\partial^\top_{\alpha} \cdot K_{\alpha\beta\gamma} = \partial^\top \cdot K_{\alpha\beta\gamma} - x_{\beta} K_{\alpha\gamma} - x_{\gamma} K_{\alpha\beta}.$$

$\partial^\top \cdot K_{\alpha\beta\gamma} \equiv \Lambda_{\alpha\beta\gamma}$ is a rank-2 anti-symmetric tensor field and its generalized gradient is (VI.3):

$$\nabla^\top_{\alpha} \Lambda_{\alpha\beta\gamma} = \partial^\top_{\alpha} \Lambda_{\alpha\beta\gamma} - x_{\beta} \Lambda_{\alpha\gamma} - x_{\gamma} \Lambda_{\alpha\beta}.$$

The field equation (VI.12) is invariant under the gauge transformation:

$$K_{\alpha\beta\gamma} \rightarrow K_{\alpha\beta\gamma}^g = K_{\alpha\beta\gamma} + \nabla^\top_{\alpha} \Lambda_{\alpha\beta\gamma} + O[\mathcal{K}]. \quad (VI.13)$$

The gauge invariance can be checked simply by using the two following identities:

$$Q_3^{(1)} \nabla^\top_{\alpha} \Lambda_{\beta\gamma} = \nabla^\top_{\alpha} Q_3^{(1)} \Lambda_{\beta\gamma}, \quad \partial^\top_{\beta} \nabla^\top_{\beta} \Lambda_{\beta\gamma} = - \left( Q_3^{(1)} + 6 \right) \Lambda_{\beta\gamma},$$

where $Q_3^{(1)}$ is the Casimir operator on the rank-2 anti-symmetric tensor field:

$$Q_3^{(1)} \Lambda_{\beta\gamma} = Q_0 \Lambda_{\beta\gamma} - 4\Lambda_{\beta\gamma} + x_{\gamma} \partial^\top \cdot \Lambda_{\beta\gamma} + x_{\beta} \partial^\top \cdot \Lambda_{\beta\gamma}.$$
Similar to the Minkowski space, the gauge fixing terms can be added to the Lagrangian for obtaining the gauge fixing field equation:

\[
(Q^{(1)}_3 + 6) K_{\alpha\beta\gamma} + c \nabla_\alpha^T \partial_3^T \cdot K_{\beta\gamma} + O((K)^2) = 0, \tag{VI.14}
\]

where \(c\) is gauge fixing parameter. In the section VII-G we prove that this tensor field cannot transform by the UIR of the dS group. This gauge potential can be written in the sum of two field with definite symmetry, a totally antisymmetric part \(K_{A\alpha_1\alpha_2\alpha_3}\) and a mixed symmetry part \(K_{M\alpha_1\alpha_2\alpha_3}\). The mixed symmetric part can be associated with the tensor field with helicity \(\pm 2\). This part can propagate on the dS light cone and it must be conformal invariant.

C. Vector-spinor gauge theory

Now we consider the vector-spinor gauge field \(\Psi_\alpha(x)\) (\(j = p = \frac{3}{2}\)). According to the principle of the gauge invariance (principle C), the interactions between different fields with a specific gauge field should be defined through the definition of the gauge-covariant derivatives. One may ask: Which force can be associated with such gauge potential? What kind of symmetry is involved? The gauge potential in the present case is a spinor field which satisfies the Grassmann algebra. Correspondingly, the involved symmetry group includes spinorial generators (generators with the anti-commutation relations). Assuming that there are \(N\) gauge vector-spinor fields (\(\Psi^A_\alpha\), with \(A = 1, \ldots, N\)), one defines the gauge-covariant derivative, similar to the other previous cases, as

\[
D^\Psi_\beta = \nabla_\beta^T + i \left(\Psi^A_\beta\right)^\dagger \gamma^0 Q_A,
\]

where \(\nabla_\beta^T\) is the transverse-covariant derivative. It is already defined when acts on a tensor field (VI.3). The transverse-covariant derivative which acts on tensor-spinor field is defined by the following relation

\[
\nabla_\beta^T \Psi_{\alpha_1 \ldots \alpha_l} \equiv D^T_{\frac{1}{2}\alpha} \Psi_{\alpha_1 \ldots \alpha_l} - H^2 \sum_{n=1}^l x_{\alpha_n} \Psi_{\alpha_1 \ldots \alpha_{n-1} \beta \alpha_{n+1} \ldots \alpha_l}, \tag{VI.15}
\]

The transverse derivative of spinor field, \(D^T_{\frac{1}{2}\alpha}\), is defined as

\[
D^T_{\frac{1}{2}\alpha} = H^{-2} \partial_{\frac{1}{2}\alpha} + \gamma_{\frac{1}{2}\alpha} \mathbf{f}, \tag{VI.16}
\]

with \(Q^{(1)}_\frac{1}{2} D^T_{\frac{1}{2}\alpha} \psi = D^T_{\frac{1}{2}\alpha} Q^{(1)}_\frac{1}{2} \psi\), where \(\psi\) is a spinor field. The generators \(Q_A\) are spinor-like, satisfying some anti-commutation relations. It has been seen that the super-algebra in the dS ambient space formalism will naturally appear. A brief discussion of the simple case \(N = 1\), would be utterly instructive. The gauge-covariant derivative:

\[
D^\Psi_\beta = \nabla_\beta^T + i \left(\Psi_\beta\right)^\dagger \gamma^0 Q = \nabla_\beta^T + i \left(-\Psi_\beta \gamma^4\right) \gamma^i Q_i,
\]

where \(i = 1, \ldots, 4\) is the spinorial index and \(\Psi_\beta \gamma^0 \gamma^4 = -\bar{\Psi}_\beta \gamma^4\) which is defined in this form, solely for simplicity. In order to acquire a rank-1 tensor field for the covariant derivative, one needs to add a spinor generator \(Q\). The difference between this case and the other cases (\(SU(N), SO(1,4), SO(2,4)\)) is that the super-algebra between the Grassmanian generators is not closed, since the product of two Grassmanian numbers become a usual number. So, for obtaining a closed super-algebra, one must couple these generators with the dS group generators \(L_{\alpha\beta}\). The \(N = 1\) super-algebra in the dS ambient space formalism is already calculated [28]:

\[
\{Q_i, Q_j\} = \left(S^{(\frac{1}{2})}_{\alpha\beta} \gamma^4\gamma^2\right)_{ij} L_{\alpha\beta}, \tag{VI.17}
\]
\[ [Q_i, L_{\alpha \beta}] = \left(S_{\alpha \beta}^{(i)} Q\right)_i, \quad [\tilde{Q}_i, L_{\alpha \beta}] = -\left(\tilde{Q} S_{\alpha \beta}^{(i)}\right)_i, \quad (VI.18) \]

\[ [L_{\alpha \beta}, L_{\alpha \delta}] = -i(\eta_{\alpha \gamma} L_{\beta \delta} + \eta_{\alpha \delta} L_{\beta \gamma} - \eta_{\beta \delta} L_{\alpha \gamma} - \eta_{\beta \gamma} L_{\alpha \delta}), \quad (VI.19) \]

where \( \tilde{Q}_i = (Q^t \gamma^4 C)_i = \tilde{Q}_i \), and \( Q^t \) is the transpose of \( Q \). The charge conjugation \( C \) is defined in section IV-B. It can be shown that \( \tilde{Q} \gamma^4 Q \) is a scalar field under the dS transformation [48]. Then for defining the gauge-covariant derivative one must use the above \( N = 1 \) super-algebra. Consequently, the gauge fields are \( H^A_{\alpha} = (K^{\beta \gamma}_{\alpha} \Psi^i_{\alpha}) \), along with the generators which are \( Z_A \equiv (L_{\alpha \beta}, Q_i) \):

\[ D^H_{\beta} = \nabla_{\beta} + iH^A Z_A. \quad (VI.20) \]

One can rewrite the above \( N = 1 \) super-algebra as the following form:

\[ [Z_A, Z_B] = C_{BA} C Z_C, \]

where \( [Z_A, Z_B] \) is a commutation or an anti-commutation relation. Under a local infinitesimal gauge transformation generated by \( \epsilon^A(x)Z_A \) one has

\[ \delta_{\epsilon} H^A_{\beta} = D^H_{\beta} \epsilon^A = \nabla_{\beta}^T \epsilon^A + C_{BC} A^A H^B_{\gamma} C_{\beta \gamma}. \]

The curvature \( \mathcal{R} \), with values in this super-algebra is defined by:

\[ \mathcal{R}(D^H_{\alpha}, D^H_{\beta}) = -[D^H_{\alpha}, D^H_{\beta}] = R_{\alpha \beta}^A Z_A, \]

with

\[ R_{\alpha \beta}^A = \nabla_{\alpha}^T H_{\beta}^A - \nabla_{\beta}^T H_{\alpha}^A + H_{\beta}^B H_{\alpha}^C C_{\beta \gamma}^A, \quad \nabla_{\alpha}^T R_{\alpha \beta}^A = 0 = \nabla_{\beta}^T R_{\alpha \beta}^A. \]

For the torsorial part this curvature becomes exactly the curvature for gauge dS group (VI.8) and for spinorial part the curvature is:

\[ R_{\alpha \beta}^i = \nabla_{\alpha}^T \Psi^i_{\beta} - \nabla_{\beta}^T \Psi^i_{\alpha} + H_{\beta}^B H_{\alpha}^C C_{\beta \gamma}^i, \]

where the gauge potential \( K^{\beta \gamma}_{\alpha} \) appears in the non-linear terms and the dS covariant derivative becomes (VI.15):

\[ \nabla_{\alpha}^T \Psi_{\beta} = \partial_{\alpha}^T \Psi_{\beta} + \gamma_{\alpha}^\tau A^\tau \Psi_{\beta} - x_{\beta} \Psi_{\alpha}. \]

For defining the Lagrangian, the curvature and transverse covariant derivative for the spinorial part are defined by \( \mathcal{R} = R^i \gamma^0 \gamma^4 \) and \( \nabla^T = \left(\nabla^T\right)^\dagger \gamma^0 \gamma^4 \) respectively.

The super-gauge invariant action or Lagrangian in the dS background for these gauge fields, \( \Psi^i_{\alpha} \) and \( K^{\beta \gamma}_{\alpha} \), is

\[ S_g[\Psi, K] = \int d^4 x x^\alpha [Q_{\alpha \beta}^A Q_{\beta \gamma}^B R_{\gamma \delta}^B], \]

with \( Q_{AB} \) as the numerical constants. Establishing the maximal irreducibility of the gauge multiplets \( \Psi^i_{\alpha} \) and \( K^{\beta \gamma}_{\alpha} \), \( Q_{AB} \) can be chosen as [62]

\[ Q_{(\alpha \beta)(\gamma \delta)} = \epsilon_{\alpha \beta \gamma \delta}, \quad Q_{ij} = \left(\gamma^4\right)_{ij}, \quad Q_{(\alpha \beta)i} = 0 = Q_{i(\alpha \beta)}. \]

Utilizing above relations, one obtains

\[ S_g[\Psi, K] = \int d\mu(x) \mathcal{L}(\Psi, K) \equiv S_2 + S_2, \]
where the action is separated into two parts, the first part is the action of spin-2 field $S_2$, which is exactly the same as (VI.9). The vector-spinor fields have no contribution in $S_2$, since the gauge group $SO(1,4)$ is a closed algebra. The second part is the action of spin-$\frac{3}{2}$ field, which depends on the gauge potential $K$,

$$S_{\frac{3}{2}}[\Psi, K] = \int d\mu(x) \theta^{\alpha\gamma} \theta^{\beta\delta} \left( \bar{R}_{\alpha\beta} \gamma^4 R_{\gamma\delta} \right)$$

$$= \int d\mu(x) \theta^{\alpha\gamma} \theta^{\beta\delta} \left( \nabla_{\gamma}^{\top} \Psi_{\beta} - \nabla_{\beta}^{\top} \Psi_{\gamma} \right) \bar{\gamma}^0 \left( \nabla_{\gamma}^{\top} \Psi_{\delta} - \nabla_{\delta}^{\top} \Psi_{\gamma} \right) + O([(\Psi, K)]) .$$

Here $O([(\Psi, K)])$ is the simultaneous second order of $\Psi$ and the product of $\Psi$ and $K$. The field equation of the vector-spinor field in the linear approximation is:

$$\theta^{\gamma\beta} \nabla_{\gamma}^{\top} \left( \nabla_{\alpha}^{\top} \Psi_{\beta} - \nabla_{\beta}^{\top} \Psi_{\alpha} \right) + O([(\Psi, K)]) = 0.$$

(VI.22)

Using the condition $\gamma \cdot \Psi = 0$, the Casimir operator (II.23) and the first order field equation (VII.29) the linear vector-spinor field equation (VI.22) can be rewritten in the following form [23] :

$$\left( Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \Psi_{\alpha} + D_{\alpha}^{\top} \partial^{\top} \cdot \Psi = 0.$$

(VI.23)

The equation (VI.23) is invariant under the following gauge transformation [52, 53]

$$\Psi_{\alpha} \longrightarrow \Psi_{\alpha}^{\prime} = \Psi_{\alpha} + D_{\alpha}^{\top} \psi,$$

where $\psi$ is an arbitrary spinor field. The non-linear terms define the interaction between the gauge potentials. The gauge fixing terms are added to the Lagrangian and in the linear approximation the gauge fixing field equation becomes:

$$\left( Q_{\frac{1}{2}}^{(1)} + \frac{5}{2} \right) \Psi_{\alpha} + cD_{\alpha}^{\top} \partial^{\top} \cdot \Psi = 0,$$

(VI.24)

where $c$ is the gauge fixing parameter. One sees that one cannot establish the gauge-covariant Lagrangian only with a vector-spinor field $\Psi_{\alpha}$ equation (VI.22), since the spinor generator $Q$ does not have a closed super-algebra and one must couple the vector-spinor field with a rank-3 tensor field $K_{\alpha\beta\gamma}$. Consequently, such gauge potential cannot be considered as a new forces but it may be considered as a part of the gravitational field.

Now we have the gauge transformations and the gauge invariant field equations for massless fields with spin 1, $\frac{3}{2}$ and 2 in the linear approximation. In the proceeding section, our survey continues with the study of the massless quantum field theory and the calculation of its two-point functions.

### VII. MASSLESS QUANTUM FIELD THEORY

In this section the massless quantum field operators are studied. There are two conditions for defining a massless field in the dS space:

- (1) the massless field operator (for $j = p = 0$ and $j = p = \frac{1}{2}$) must transform by the UIR of the dS group and corresponds to the massless field of the Minkowskian space in the null curvature limit,

- (1)' the massless field operator (for $j = p = 1, \frac{3}{2}$ and 2) must transform as an indecomposable representation of dS group, in such, its central part imitates the UIR of the discrete series $(\Pi_{k}^{+} \oplus \Pi_{k}^{-})$ and, in the null curvature limit, this massless field operator corresponds to the massless field of the Minkowskian space-time,
\begin{itemize}
  \item (2) the massless field must propagate on the dS light cone or, in other words, the propagator must be conformal invariant, implying the existence of an extension of the UIR of the dS group to the conformal group $SO(2,4)$.
\end{itemize}

For scalar field ($j = 0$), one recognizes two important possibilities: the massless conformally coupled scalar field $p = 0$, and the massless minimally coupled scalar field $p = 2$. The first case satisfies the two above conditions (1) and (2) [8], where the second case does not satisfy the above conditions so it does not transform as an UIR of the dS group and therefore, validating the covariant quantization procedure, one needs to introduce an indecomposable representation of dS group. Previously, the quantization in the Krein space was presented [24]. The minimally coupled scalar field is an auxiliary field, which appears in the indecomposable representation of the vector field and spin-2 field.

The spinor field $j = p = \frac{1}{2}$ satisfies the above conditions (1) and (2) [52, 65]. For $j = p = 2$, a rank-2 symmetric tensor field can be quantized, during which, the conformal invariance will be broken. Then, preserving the dS and the conformal covariants simultaneously, one must use for the case of $j = p = 2$, a rank-3 mixed symmetry tensor field [27]. This field can also be quantized in the Koren space which is free of any infrared divergence and preserves the dS invariance [16]. In what follows, a brief review of quantization of these massless fields will be presented.

\section*{A. Massless conformally coupled scalar field}

The various spin massless fields and the auxiliary fields in the discrete series can be constructed in terms of the massless conformally coupled scalar field, then the field operator, the quantum states and the two-point function of conformally coupled scalar field are explicitly constructed in this subsection. The massless conformally coupled scalar field satisfies the following field equation [6, 8]:

$$
\left(Q_0^{(1)} - 2\right) \phi_c(x) = 0, \quad \left(\Box_H + 2H^2\right) \phi_c(x) = 0.
$$

This field corresponds to the complementary series representation of the dS group with $j = p = 0$ [II.38] which is unitary equivalent with the representation $j = 0, \quad p = 1$ [II.39]. This representation was constructed on $u-$space or the three-sphere $S^3$, utterly similar to the principal series representation for the scalar case $j = 0, \quad p = \frac{1}{2} + i\nu$ [2, 8]. The associated mass is $m_{5,1}^2 = 2H^2$ and the homogeneous degrees of this field are $\lambda = -1, -2$. This field satisfies the conditions (1) and (2) for the massless fields. This field can be obtained by replacing the parameter $\nu$ in the principal series representation for massive scalar field by $\nu = \frac{1}{2}$ and $\nu = -\frac{1}{2}$ in $U(0,\frac{1}{2}+i\nu)$ and $U(0,\frac{1}{2}-i\nu)$ respectively. In this limit, one can simply write the quantum field operator and the two-point function. Therefore the massless minimally coupled scalar field operator is:

$$
\phi_c(x) = \int_{S^3} d\mu(u) \left[a(\mathbf{u}, 0; 0, 1) U(x; \mathbf{u}, 0; 0, 0) + a^\dagger(\mathbf{u}, 0; 0, 0) \mathcal{V}(x; \mathbf{u}, 0; 0, 0)\right],
$$

(VII.1)

where the coefficients $U$ and $\mathcal{V}$ can be written in terms of $\xi_u$:

$$
U(x; \mathbf{u}, 0; 0, 0) = \sqrt{c_0(x \cdot \xi_u)^{-2}}, \quad \mathcal{V}(x; \mathbf{u}, 0; 0, 0) = \sqrt{c_0(x \cdot \xi_u)^{-1}}.
$$

(VII.2)
\[ c_0 \] is the normalization constant. The field operator in the complex dS spacetime \( X_H^{(c)} \) can be written as follows:

\[
\Phi_c(z) = \sqrt{c_0} \int_{S^3} d\mu(u) \left[ a(\tilde{u}, 0; 0, 1) \mathcal{U}(z; u, 0; 0, 0) + a^\dagger(\tilde{u}, 0; 0, 0) \mathcal{V}(z; u, 0; 0, 0) \right],
\]

and consequently, the quantum field operator in this notation is given by:

\[
\phi_c(x) = \sqrt{c_0} \int_{S^3} d\mu(u) \left\{ a(\tilde{u}, 0; 0, 1) [(x \cdot \xi_u)^{-2} + e^{-i\pi(-2)}(x \cdot \xi_u)^{-2}] + a^\dagger(\tilde{u}, 0; 0, 0) [(x \cdot \xi_u)^{-1} + e^{i\pi(-1)}(x \cdot \xi_u)^{-1}] \right\}.
\]

(VII.3)

It is interesting to note that one can construct the field operator on the closed unit ball \( B \) or \( \mathbf{q} \)-space:

\[
\phi_c(x) = \sqrt{c_0} \int_B d\mu(\mathbf{q}) \left\{ a(\tilde{\mathbf{q}}, 0; 0, 1) [(x \cdot \xi_B)^{-2} + e^{-i\pi(-2)}(x \cdot \xi_B)^{-2}] + a^\dagger(\tilde{\mathbf{q}}, 0; 0, 0) [(x \cdot \xi_B)^{-1} + e^{i\pi(-1)}(x \cdot \xi_B)^{-1}] \right\},
\]

(VII.4)

since \( \xi_B \equiv \xi_u \) and \( d\mu(\mathbf{q}) = 2\pi^2 r^3 dr d\mu(u) \). These two field operators are equivalent up to a normalization constant which can be fixed by the Hadamard conditions. The field operator (VII.4) can be used for construction the other massless fields in the dS ambient space formalism.

The analytic two-point function for scalar field is \([3, 4]\):

\[
W_c(z_1, z_2) = \langle \Omega | \Phi(z_1) \Phi(z_2) | \Omega \rangle = c_0 \int_{S^3} d\mu(u)(z_1 \cdot \xi_u)^{-2}(z_2 \cdot \xi_u)^{-1},
\]

(VII.5)

and \( c_0 = c_{0,\nu} \) is obtain by replacing \( \nu = \pm \frac{i}{2} \) in [IV.32]. One can easily calculate (VII.5) in terms of the generalized Legendre function (V.23):

\[
W_c(z_1, z_2) = -\frac{iH^2}{2^4\pi^2} \mathbb{P}^{(3)}(H^2 z_1 \cdot z_2) = \frac{H^2}{8\pi^2} \frac{1}{1 - \mathcal{Z}(z_1, z_2)}.
\]

The vacuum state in this case is equivalent with the Bunch-Davies vacuum state. The Wightman two-point function \( \mathcal{W}_c(x_1, x_2) \) is the boundary value (in the sense of its interpretation as a distribution function, according to the theorem A.2 in [1]) of the function \( W_c(z_1, z_2) \) which is analytic in the domain \( \mathcal{T}_2 \) of \( M_H^{(c)} \times M_H^{(c)} \) (IV.28). The boundary value is defined for \( z_1 = x_1 + iy_1 \in \mathcal{T}^- \) and \( z_2 = x_2 + y_2 \in \mathcal{T}^+ \)

\[
\mathcal{Z}(z_1, z_2) = \mathcal{Z}(x_1, x_2) - i\tau \epsilon(x_1^0, x_2^0),
\]

where \( y_1 = (-\tau, 0, 0, 0, 0) \in \mathcal{V}^- \), \( y_2 = (\tau, 0, 0, 0, 0) \in \mathcal{V}^+ \) (IV.24) and \( \tau \to 0 \). Then, one obtains [4, 8, 63]:

\[
\mathcal{W}_c(x_1, x_2) = \frac{-H^2}{8\pi^2} \lim_{\tau \to 0} \frac{1}{1 - \mathcal{Z}(x_1, x_2) + i\tau \epsilon(x_1^0, x_2^0)}
\]

\[
= \frac{-H^2}{8\pi^2} \left[ \mathcal{P} \frac{1}{1 - \mathcal{Z}(x_1, x_2)} - i\tau \epsilon(x_1^0, x_2^0) \delta(1 - \mathcal{Z}(x_1, x_2)) \right],
\]

(VII.6)
where the symbol $P$ is the principal part and $Z$ is the geodesic distance between two points $x$ and $y$ on the dS hyperboloid:

$$Z(x_1, x_2) = -H^2 x_1 \cdot x_2 = 1 + \frac{H^2}{2} (x_1 - x_2)^2,$$

and

$$
\epsilon(x_1^0 - x_2^0) = \begin{cases} 
1 & x_1^0 > x_2^0 \\
0 & x_1^0 = x_2^0 \\
-1 & x_1^0 < x_2^0.
\end{cases}
$$

Finally one can show that the two solutions (VII.2) are equivalent in the intrinsic coordinate system. In the SO(4)-orbital basis $(\xi_\alpha^\mu = (\xi^0, \xi^0 u))$, the relation between the dS plane waves and the partial waves (intrinsic coordinate solution) is given by [8, 9, 15, 30]:

$$
(x \cdot \xi_u)^\lambda = 2\pi^2 (\xi^0)^\lambda \sum_{Llm} \Phi^\lambda_{Llm}(X) \mathcal{Y}^\ast_{Llm}(u),
$$

(VII.8)

where $\Re \lambda < 0$ and

$$
\Phi^\lambda_{Llm}(X) = i^{L-\lambda} e^{-i(L+\lambda+3)\rho} (2\cos \rho)^{\lambda+3} \frac{\Gamma(L-\lambda)}{(L+1)!\Gamma(-\lambda)}
\times 2F_1(\lambda + 2, L + \lambda + 3; L + 2; -e^{-2i\rho}) \mathcal{Y}_{Llm}(v),
$$

(VII.9)

$\mathcal{Y}_{Llm}$ stands for the hyperspherical harmonics. The conformal global coordinates

$$
x^\alpha = (H^{-\tan \rho}, (H \cos \rho)^{-1} \nu)
$$

are being used, where $\nu = (\nu^4, \bar{\nu})$ is a quaternion with the norm of 1. Using the following relation

$$
2F_1(a, b; c; z) = (1 - z)^{-c-a-b} 2F_1(c - a, c - b; c; z)
$$

one can show that the two solutions $(x \cdot \xi_u)^{-1}$ and $(x \cdot \xi_u)^{-2}$ in the intrinsic global coordinate are equal up to a normalisation constant. These two solutions can be correspond to the two unitary equivalent representations $U^{(0,0)}(g)$ (III.38) and $U^{(0,1)}(g)$ (III.39).

B. Massless minimally coupled scalar field

The values $j = 0$ and $p = 2$, which are characteristics of the massless minimally coupled scalar field, are not permissible, according to the classification of the UIR of the dS group. The corresponding mass parameter is $m_{2}^2 = 0$. The field equation can be written in the following form:

$$Q_0^{(1)} \phi_m(x) = 0, \quad \text{or} \quad \Box H \phi_m(x) = 0.$$

This field equation is similar to the $\xi$-space field equation (V.26) for $j = p$, and is invariant under the transformation

$$\phi'_m(x) = \phi_m(x) + \text{const.}.$$

The homogeneous degrees of this field are: $\lambda = 0, -3$. The constant solution $(\lambda_1 = 0, (x \cdot \xi)^0 = \text{constant})$, poses the zero mode problem. With just one solution $(\lambda_2 = -3, (x \cdot \xi)^{-3})$, one cannot to establish a proper covariant quantum field operator on the Hilbert space-constructed on an
UIR of the dS group \([19, 49, 67]\). One cannot obtain a massless minimally coupled scalar field and its two-point function, by replacing the parameter \(\nu\) of a massive field in the principal series representation with the \(\nu = \pm \frac{3i}{2}\), since such replacement would cause a singularity in the determination of the normalization constant \(c_{0, \nu}\), in the sense that such field operator is not in a correspondence with an UIR of the dS group. Nevertheless, one can associate a massless minimally coupled scalar field with an indecomposable representation of the dS group and conclude the relevant field operator as follows.

Using these two followings identities

\[
Q_0^{(1)\alpha} \phi(x) - \partial^\alpha Q_0^{(1)\alpha} \phi(x) = 2 \partial^\alpha \phi(x) + 2x_\alpha Q_0^{(1)\alpha} \phi(x),
\]

\[
Q_0^{(1)\alpha} x_\alpha \phi(x) - x_\alpha Q_0^{(1)\alpha} \phi(x) = -2 \partial^\alpha \phi(x) - 4x_\alpha \phi(x),
\]

with \(\phi\) as an arbitrary scalar field, one can proved the existence of a magic relation between the minimally coupled and the conformally coupled scalar fields in the dS ambient space formalism:

\[
\phi_m(x) = N \left[ Z \cdot \partial^\alpha + 2Z \cdot x \right] \phi_c(x). \tag{VII.10}
\]

\(N\) is a normalization constant which can be fixed, using the local Hadamard condition, and \(Z^\alpha\) is a constant polarization five-vector, defined as the by-product of the relation between the minimally coupled scalar field with an indecomposable representation of the dS group. The quantum field operator is defined by:

\[
U(g)\Phi_m(z, Z)U(g)^{-1} = \Phi_m(\Lambda z, \Lambda Z),
\]

where \(U(g)\) is an indecomposable representation of the dS group. Such representation can be constructed as the product of two representations of the dS group: the scalar complementary series representation \(j = 0\) and \(p = 0\), and a five-dimensional trivial representation in respect to \(Z_{\alpha}^{(l)}\) \([68]\).

For a thorough investigation regarding the five existing polarization states \(l = 1, 2, 3, 4, 5\), the reader may refer to \([68]\). This subject will not be pursued here.

Concerning the polarization five-vector \(Z_{\alpha}^{(l)}\), the quantum field operator can be defined properly from the quantum field operator of conformally coupled scalar field:
\[ \Phi_m(z) = \sqrt{c_0} N \sum_{l=1}^{5} \left[ Z^{(l)} \cdot \partial^\top + 2Z^{(l)} \cdot z \right] \int_{S^3} d\mu(u) \left\{ a(\tilde{u}, 0; 0, 1)(z \cdot \xi_u)^{-2} + a^l(u, 0; 0, 0)(z \cdot \xi_u)^{-1} \right\} \]

\[ = \sqrt{c_0} N \sum_{l=1}^{5} \int_{S^3} d\mu(u) \left\{ a(\tilde{u}, 0; 0, 1) \left[ -2(Z^{(l)} \cdot \xi_u)^{-3} + 2(Z^{(l)} \cdot z)(z \cdot \xi_u)^{-2} \right] + a^l(u, 0; 0, 0) \left[ -(Z^{(l)} \cdot \xi_u)^{-2} + 2(Z^{(l)} \cdot z)(z \cdot \xi_u)^{-1} \right] \right\} \equiv \Phi_m(z, Z). \quad (VII.11) \]

The analytic two-point function can be defined on the vacuum state of the conformally coupled scalar field or Bunch-Davies vacuum:

\[ W_m^H(z, z') = \langle \Omega | \Phi_m(z) \Phi_m(z') | \Omega \rangle \]

\[ = \sum_{l=1}^{5} \sum_{l'=1}^{5} \left[ Z^{(l)} \cdot \partial^\top + 2Z^{(l)} \cdot z \right] \left[ Z^{(l')} \cdot \partial'^\top + 2Z^{(l')} \cdot z' \right] W_c(z, z'). \]

The explicit form of this function depends on the representation \( U(g) \). As a simple case, one can chose:

\[ \sum_{l=1}^{5} \sum_{l'=1}^{5} Z^{(l)} \cdot Z^{(l')} = \eta_{\alpha\beta}, \]

which will be concluded to the following analytic two-point function:

\[ W_m^H(z, z') \equiv \left[ \partial^\top \cdot \partial'^\top + 2z \cdot \partial'^\top + 2z' \cdot \partial^\top + 4z \cdot z' \right] W_c(z, z'), \quad (VII.12) \]

with \( W_c \) being the analytic two-point function of conformally coupled scalar field \((V.23)\). By using the following relations

\[ Z = -H^2 z \cdot z', \quad \frac{\partial}{\partial \alpha} = -H^2 z'_\alpha \frac{d}{dZ}, \quad \partial^\top_\alpha = (z_\alpha Z - z'_\alpha) \frac{d}{dZ}, \]

\[ \partial^\top \cdot \partial'^\top = \left( 1 - Z^2 \right) \left[ \frac{d}{dZ} + Z \frac{d^2}{dZ^2} \right], \quad z' \cdot \partial^\top = \left( 1 - Z^2 \right) \frac{d}{dZ}, \]

one can show that this two-point function also satisfies the minimally coupled scalar field equation for the variables \( z \) and \( z' \). In conclusion, the analytic two-point function \((VII.12)\) is free of any infrared divergences. The two-point function in the real dS space is the boundary value of the analytic two-point function \( W_m^H(z, z') \) \((VII.12)\):

\[ \mathcal{W}_m^H(x, x') = -\frac{H^2}{8\pi^2} \left[ \partial^\top \cdot \partial'^\top + 2x \cdot \partial'^\top + 2\partial^\top \cdot x' - 4Z(x, x') \right] \times \left[ \frac{1}{1 - Z(x, x') - i\pi \epsilon(x^0 - x'^0)} \delta(1 - Z(x, x')) \right]. \quad (VII.13) \]

For the sole purpose of a covariant quantizing such fields, an entirely new method of quantization, 'the Krein space quantization' was developed, based on the definition of the field operator in the
intrinsic coordinates, preserving the dS invariance and commencing transformations as an a specific indecomposable representation of the dS group [19]. Nevertheless, this method has been ignored here, since it breaks the analyticity and uses the intrinsic coordinates contrary to the present formalism which has been established in the ambient space. The reader is encouraged to refer to the previously published papers for a detailed study [8, 19]. Nevertheless, for the sake of an interesting comparison, the two-point function of the massless minimally coupled scalar field in the Krein space quantization is presented [69]:

\[ W^K_m(x, x') = \frac{iH^2}{8\pi} \epsilon(x^0 - x'^0)\left[\delta(1 - Z(x, x')) - \theta(\mathcal{Z}(x, x') - 1)\right], \quad (VII.14) \]

where \( \theta \) is the Heaviside step function. This expression is dS invariant, i.e. coordinate independent and also free of any infra-red divergence. Unfortunately, since the propagation is placed inside the dS light-cone, the appearance of constant term in the two-point function (Heaviside step function), breaks the conformal invariance.

C. Massless spinor field

The massless spinor field is defined by \( j = p = \frac{1}{2} \) and transform as the UIR of the discrete series \( T^{0,1/2} \) and \( T^{\pm,1/2} \) [4] (or \( \Pi_{1/2}^L \) in Dixmier notation [3]). Its corresponding mass and eigenvalue of the Casimir operator are \( m^2 = 2H^2 \) and \( <Q^{(1)}_{1/2,1/2}> = \frac{3}{2} \), respectively. This field satisfies the conditions (1) and (2) as well as following field equations:

\[ (Q_0^{(1)} - 2) \psi(x) = 0, \quad \text{and} \quad (\not{x} \not{\partial}^T - 2) \psi(x) = 0. \]

(VII.15)

The equations (VII.15) are invariant under the following transformation

\[ \psi \rightarrow \psi' = H \not{x} \psi. \]

(VII.16)

Through definition of the spinor fields: [70, 71]

\[ \psi_L(x) = \frac{1 + H \not{x}}{2} \psi(x), \quad \psi_R(x) = \frac{1 - H \not{x}}{2} \psi(x), \]

which are also independent solutions for the field equations (VII.15), one obtains \( H \not{x} \psi_L(x) = \psi_R(x) \). The massless spinor field can be simply extracted, replacing the parameter \( \nu \) in the quantum field operator and the two-point function for a massive spinor field in the principal series representation with the value of \( \nu = 0 \).

Using the following identity in dS ambient space formalism:

\[ (\not{x} \not{\partial}^T - 2) \left( - \not{x} \not{\partial}^T + 1 \right) = \left( - \not{x} \not{\partial}^T + 1 \right) \left( \not{x} \not{\partial}^T - 2 \right) = Q^{(1)}_0 - 2, \]

one can write the massless spinor field in terms of the massless conformally coupled scalar field

\[ \psi(x) = \left( - \not{x} \not{\partial}^T + 1 \right) \phi_c(x). \]

(VII.17)

The charged spinor field operator can be defined as:

\[ \psi(x) = \int_B d\mu(q) \sum_m \left[ a(q, m; \frac{1}{2}; \frac{1}{2}) U(x; q, m; \frac{1}{2}; \frac{1}{2}) + a^c(q, m; \frac{1}{2}; \frac{1}{2}) V(x; q, m; \frac{1}{2}; \frac{1}{2}) \right], \quad (VII.18) \]
with \( m = -\frac{1}{2}, \frac{1}{2} \) and \( \psi \), \( U \) and \( V \) being four-components spinors. The homogeneous degrees of the spinor field are \( \lambda = -2, -1 \) \((\text{VII.21)}\):

\[
U(x; q, m; \frac{1}{2}, \frac{1}{2}) = (x \cdot \xi_B)^{-2} u(x, q, m; \frac{1}{2}, \frac{1}{2}),
\]

\[
V(x; q, m; \frac{1}{2}, \frac{1}{2}) = (x \cdot \xi_B)^{-1} v(x, q, m; \frac{1}{2}, \frac{1}{2}).
\]

The analytic two-point function for a spinor field, equation \((\text{V.19)}\), was previously calculated in \([18]\):

\[
S_c(x_1, x_2) = \frac{1}{16\pi^2} D_\frac{1}{2}(z_2) \gamma^4 P_{-1}(H^2 z_1, z_2) = \frac{iH^2 (\gamma_1 - \gamma_2) \gamma^4}{2\pi^2 [(z_1 - z_2)^2]^2},
\]

where \( D_\frac{1}{2}(z_2) = - \gamma_2 \rho_{z_2}^T + 1 \). The Wightman spinor two-point function can be written in terms of the Wightman two-point function for a conformally coupled scalar field \((\text{VII.6)}\):

\[
S_c(x, y) = \frac{H^2}{16\pi^2} D_\frac{1}{2}(y) \gamma^4 \left[ \frac{1}{1 - \mathcal{Z}(x, y)} - i\pi\epsilon(x^0 - y^0)\delta(1 - \mathcal{Z}(x, y)) \right], \quad (\text{VII.19})
\]

### D. Massless vector field

The massless vector field corresponds to \( j = p = 1 \) with the corresponding eigenvalue of Casimir operator being \( < Q^{(1)}_{1,1} > = 0 \). The associated mass is \( m^2_{b,1} = 2H^2 \). This field corresponds to the discrete series representation \( \Pi^\updownarrow_{1,1} \) and the field equation

\[
Q^{(1)}_1 K(x) = 0, \quad \text{or} \quad (Q^{(1)}_0 - 2) K(x) = 0.
\]

Nevertheless, minding the involvement of the conditions (i-vii) from section II-B, one concludes that the above field equation simply cannot be solved and the quantum states \( |q; 1, 1> \) cannot be fixed in the \( q \)-space by the field equation \((\text{VII.21)}\). One needs to drop the divergenceless condition in order for the field equation to become gauge invariant. In the previous section, utilizing the gauge principle, the gauge invariant field equations in the \( x \)-space was obtained \((\text{VII.15)}\):

\[
Q^{(1)}_1 K(x) + cD_1^\top \partial \cdot K(x) = 0, \quad (\text{VII.20})
\]

where \( D_1^\top = H^{-2} \partial^\top \). For \( c = 1 \), this equation is invariant under the gauge transformation:

\[
K(x) \Rightarrow K^\alpha(x) = K_\alpha(x) + D_{1\alpha} \Omega(x), \quad (\text{VII.21})
\]

with \( \Omega(x) \) as an arbitrary scalar field. In this case the field equation \((\text{VII.20)}\) replaces the condition (i) of the section II-D and disables the condition (iv). The gauge invariance can be checked simply by using the two following identities:

\[
Q^{(1)}_1 D_1^\top \Omega(x) = D_1^\top Q^{(1)}_0 \Omega(x), \quad \partial \cdot D_{1\alpha} \Omega(x) = -Q^{(1)}_0 \Omega(x).
\]

By taking the divergence of the equation \((\text{VII.20)}\), one obtains \([13]\):

\[
(1 - c)Q_0 \partial \cdot K = 0.
\]

The \( \partial \cdot K = \Phi \) is a scalar state, in which for the case \( c \neq 1 \) yields \( Q_0 \Phi = 0 \). Similarly the divergence of equation \((\text{VII.21)}\) with \( \partial \cdot K = 0 \), one obtains \( Q_0 \Omega = 0 \). Combining these two states with the physical states results into the Gupta-Bleuler triplet \([13]\). Let us now define the Gupta-Bleuler triplet
Due to the previous calculations, it has been shown that the two-point function for such massless vector field is not interested in their explicit forms. The reader may follow up this subject in [15]. The analytic field operator can be introduced as:

$$K_\alpha(x) = \left[ Z^\top - \frac{c}{2(1-c)} D_{1a} \left( H^2 x \cdot Z + Z \cdot \bar{\partial} \right) + \frac{2 - 3c}{1-c} H^2 D_{1a} \left[ Q_0^{(1)} \right]^{-1} x \cdot Z \right] \phi_c. \quad (VII.22)$$

The field operator can be introduced as:

$$K_\alpha(x) = \int_B d\mu(q) \sum_m \left[ a(q, m; 1, 0) U_\alpha(x; q, m; 1, 1) + a^\dagger(q, m; 1, 1) V_\alpha(x; q, m; 1, 1) \right], \quad (VII.23)$$

where $m = 0, 1, 2, 3$. $K_\alpha(x)$ transforms by an indecomposable representation of the dS group. The homogeneous degrees of this field are $\lambda = -1, -2$ and $U$ and $V$ may be written in the following form:

$$U_\alpha(x; q, m; 1, 1) = (x \cdot \xi_B)^{-2} u_\alpha(x, q, m; 1, \nu),$$

$$V_\alpha(x; q, m; 1, 1) = (x \cdot \xi_B)^{-1} v_\alpha(x, q, m; 1, 1).$$

The coefficients $u_\alpha$ and $v_\alpha$ are obtained by solving the field equation (VII.20) but, at the moment, we are not interested in their explicit forms. The reader may follow up this subject in [15]. The analytic two-point function for such massless vector field is

$$W_{\alpha\alpha'}(z, z') \equiv \langle \Omega | K_\alpha(z) K_{\alpha'}(z') | \Omega \rangle. \quad (VII.24)$$

Due to the previous calculations, it has been shown that:

$$W_{\alpha\alpha'}(z, z') = \theta_\alpha \cdot \theta_{\alpha'} W_c(z, z') - \frac{c}{2(1-c)} H^{-2} \bar{\partial}_\alpha \left[ \theta_{\alpha'} \cdot \bar{\partial} + H^2 z \cdot \theta_{\alpha'} \right] W_c(z, z')$$

$$+ \frac{2 - 3c}{1-c} \bar{\partial}_\alpha Q_0^{(1)} z \cdot \theta_{\alpha'} W_c(z, z'). \quad (VII.25)$$

where $W_c$ is the two-point function of a massless conformally coupled scalar field (V.23). It is proved that the value

$$c_s = \frac{2}{2s + 1}, \quad c = \frac{2}{3}$$

corresponds to the minimal (or optimal) choice, without any logarithmic singularities [8, 15, 16, 68].

**E. Massless vector-spinor field**

The vector-spinor field, associated with the UIR of the discrete series representation $\Pi^\perp_{1,1}$, corresponds to $j = p = \frac{3}{2}$ with the corresponding eigenvalue of Casimir operator being $< Q_0^{(1)} > = -\frac{5}{2}$. This field satisfies the second order field equation:

$$\left( Q_0^{(1)} + \frac{5}{2} \right) \Psi_\alpha(x) = 0, \quad \text{or} \quad \left( Q_0^{(1)} + \frac{i}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} - 3 \right) \Psi_\alpha = 0.$$
There are two possibilities for the relevant first order field equation and by using the identity (III.22), one obtains:

\[
\left( \not{x} \not{D} - 3 \right) \Psi_m^\alpha(x) = 0, \quad Q_0^{(1)} \Psi_m^\alpha(x) = 0,
\]

\[
\left( \not{x} \not{D} - 1 \right) \Psi_c^\alpha(x) = 0, \quad \left( Q_0^{(1)} - 2 \right) \Psi_c^\alpha(x) = 0.
\]

The mass parameter associated to the first one, \( (\Psi_m^\alpha) \), is \( m^2 = 0 \) with its homogeneous degrees being \( \lambda = 0, -3 \). For the second field, \( (\Psi_c^\alpha) \), the associated mass parameter and the homogeneous degrees of this field are \( m^2 = 2H^2 \) and \( \lambda = -1, -2 \), respectively. One can describe \( \Psi_m^\alpha \) in terms of \( \Psi_c^\alpha \), using the identity (VII.10).

By the exclusive use of the conditions (i-vii) from section II-D and because of the appearance of the gauge invariance, one cannot find proper solutions for these field equations. Similar to the case of the vector fields, using the gauge principle which was obtained in the previous section (VI.23), a second order gauge invariant field equation is obtained:

\[
(Q_0^{(1)} + \frac{5}{2}) \Psi(x) + D^\top_{\not{\beta}} \not{D} \cdot \Psi(x) = 0,
\]

(VII.26)

where \( D^\top_{\not{\beta}} = H^{-2} \partial^\top + \gamma^\top /x \). One can show that this equation is invariant under the gauge transformation:

\[
\Psi_\alpha(x) \rightarrow \Psi_\beta^\alpha(x) = \Psi_\alpha(x) + D^\top_{\not{\beta}} \zeta,
\]

(VII.27)

with \( \zeta \) as an arbitrary spinor field. To provide the gauge invariance, the following identities are used:

\[
Q_0^{(1)} D^\top_{\not{\beta}} = D^\top_{\not{\beta}} Q_0^{(1)}, \quad \not{D} \cdot D^\top_{\not{\beta}} \zeta = - \left( Q_0^{(1)} + \frac{5}{2} \right) \zeta.
\]

Let us introduce a gauge fixing parameter \( c \). The wave equation now reads as

\[
\left( Q_0^{(1)} + \frac{5}{2} \right) \Psi(x) + c D^\top_{\not{\beta}} \not{D} \cdot \Psi(x) = 0,
\]

(VII.28)

whereas the role of \( c \) is just to fix the gauge spinor field \( \zeta \).

Up to the first order, there are two gauge invariant field equations. The first one is (52):

\[
(\not{x} \not{D} - 3) \Psi(x) - x_\alpha \not{x} \not{D} \not{\beta} \Psi_\alpha(x) + \partial^\top_{\not{\beta}} \not{x} \not{D} \not{\beta} \Psi_\alpha(x) = 0,
\]

(VII.29)

which is invariant under the following gauge transformation:

\[
\Psi_\alpha(x) \rightarrow \Psi_\beta^\alpha(x) = \Psi_\alpha(x) + D^\top_{\not{\beta}} \zeta.
\]

(VII.30)

The second field equation reads as (52)

\[
(\not{x} \not{D} - 1) \Psi(x) - x_\alpha \not{x} \not{D} \not{\beta} \Psi_\alpha(x) + D^\top_{\not{\beta}} \not{x} \not{D} \not{\beta} \Psi_\alpha(x) = 0.
\]

(VII.31)

This field equation is invariant under the following gauge transformation:

\[
\Psi_\alpha(x) \rightarrow \Psi_\beta^\alpha(x) = \Psi_\alpha(x) + \partial^\top_{\not{\beta}} \zeta.
\]

(VII.32)

Introducing the massless vector-spinor field operator, one can think of two possibilities; the first choice is to take the equations (VII.26) and (VII.29), and the second, taking the equations (VII.26) and (VII.31). The structure of the Gupta-Bleuler triplet for these two cases are different but the
central part or physical states are the same. Similar to the vector field, the pure gauge state \( V_g \) and the gauge dependent state \( V_c \) can be defined for each case. These two states, plus the physical states construct the Gupta-Bleuler triplet. Let us now define the Gupta-Bleuler triplet as

\[
V_g \subset V \subset V_c.
\]

Then the the gauge fixing field equations and the gauge transformation and the UIR of discrete series completely determine the covariant massless vector-spinor field in the dS space.

The vector-spinor field operator can be defined generally as:

\[
\Psi_\alpha(x) = \int_B d\mu(\mathbf{q}) \sum_m \left[ a(\mathbf{q}, m; \frac{3}{2}, \frac{3}{2}) U_\alpha(x; \mathbf{q}, m; \frac{3}{2}, \frac{3}{2}) + a^c(\mathbf{q}, m; \frac{3}{2}, \frac{3}{2}) V_\alpha(x; \mathbf{q}, m; \frac{3}{2}, \frac{3}{2}) \right],
\]

(VII.33)

where \( \Psi_\alpha \), \( U_\alpha \) and \( V_\alpha \) are four-components spinors and we can write:

\[
U_\alpha(x; \mathbf{q}, m; \frac{3}{2}, \frac{3}{2}) = (x \cdot \xi_B)^{-\frac{3}{2}} u_\alpha(x, \mathbf{q}, m; \frac{3}{2}, \frac{3}{2}),
\]

\[
V_\alpha(x; \mathbf{q}, m; \frac{3}{2}, \frac{3}{2}) = (x \cdot \xi_B)^{-1} v_\alpha(x, \mathbf{q}, m; \frac{3}{2}, \frac{3}{2}).
\]

The four-components spinors \( u_\alpha \) and \( v_\alpha \) can be obtained by imposing the condition that the vector-spinor field must satisfy simultaneously the first and second field equations which is not important for us here.

The analytic two-point function for vector-spinor field can be written in the following form:

\[
S_{\alpha\alpha'}(x, x') = D_{\alpha\alpha'}(x, \partial^T; x', \partial'^T; \frac{3}{2}) S_c(x, x'),
\]

(VII.34)

where \( S_c \) is the massless spinor analytic two-point function (V.19) and the explicit form of \( D_{\alpha\alpha'}(x, \partial^T; x', \partial'^T; \frac{3}{2}) \) depends to the structure of the indecomposable representation which is not of our concern here.

F. Massless spin-2 rank-2 symmetric tensor field

The massless spin-2 field \( j = p = 2 \) in the dS ambient space formalism may be presented by a rank-2 symmetric tensor field \( \mathcal{H}_{\alpha\beta} \) or a rank-3 mixed symmetric tensor field \( \mathcal{K}^M_{\alpha\beta\gamma} \) which describes the equivalent free theories \([72]\). It can be possible to define a map between these two schemes \([72]\). The aim of the present subsection is to introduce the rank-2 symmetric tensor field. The rank-3 mixed-symmetric tensor field will be studied in the following subsection.

A massless spin-2 rank-2 symmetric traceless tensor field associates with the discrete series representation \( \Pi_{\frac{5}{2}, 2} \) and satisfies the field equation

\[
\left( Q_{2}^{(1)} + 6 \right) \mathcal{H}_{\alpha\beta} = 0, \quad \text{or} \quad Q_{0}^{(1)} \mathcal{H}_{\alpha\beta} = 0.
\]

The corresponding mass parameter is \( m_{h_{2,2}}^2 = 0 \). One cannot construct this field from the massive spin-2 rank-2 symmetric tensor field, only by imposing the limit \( \nu = \frac{3}{2} i \). The two-point function in this limit becomes singular, due to the appearance of the gauge invariance. The gauge invariant field equation can be obtained from the gauge invariant field equation of \( \mathcal{K}^M_{\alpha\beta\gamma} \) which will be discussed in the next subsection. The gauge invariant field equation was also presented previously \([16]\):

\[
\left( Q_{2}^{(1)} + 6 \right) \mathcal{H} + D_2^T \partial_2^T \cdot \mathcal{H} = 0,
\]

(VII.35)
which is invariant under the following gauge transformation:

$$\mathcal{H}(x) \mapsto \mathcal{H}^\theta(x) = \mathcal{H}(x) + D_2^\top K(x).$$  

(VII.36)

$K_\beta(x)$ is an arbitrary vector field, $D_2^\top = H^{-2} \Sigma_1 (\bar{\partial} - H^2 x)$ is the generalized gradient, $\bar{\partial}_2^\top \cdot \mathcal{H}$ is the generalized divergence and $\Sigma_1$ is the index symmetrizer operator (II.22). The following identities are used to provide the gauge invariance:

$$Q_2^{(1)} D_2^\top K(x) = D_2^\top Q_1^{(1)} K(x), \quad \partial_2^\top \cdot D_2^\top K(x) = - \left( Q_1^{(1)} + 6 \right) K(x).$$

The homogeneous degrees of tensor field $\mathcal{H}_{\alpha\beta}$ are: $\lambda = 0, -3$ (III.10). The solution $\lambda = 0$ poses the zero mode problem as well as the appearance of infrared divergence in the process of quantization, just like the case of the minimally coupled scalar field. Our previously prepared grand solution, the Krein space quantization, will remedy this problem [16, 19]. A massless spin-2 rank-2 symmetric tensor field can be written in terms of a minimally coupled scalar field $\phi_m$ and polarization tensor $\mathcal{D}$ [16]:

$$\mathcal{H}_{\alpha\beta}(x) = \mathcal{D}_{\alpha\beta}(x, \bar{\partial}) \phi_m(x).$$

One may use the Krein space quantization, the ambient space formalism breaks down for must utilize the simultaneous combination of the intrinsic coordinate and ambient space formalism. For calculating the polarization states the ambient space formalism is needed. In present paper, using the identity (VII.10), a covariant quantization of the minimally coupled scalar fields in the ambient space formalism is being introduced in terms of the conformally coupled scalar fields.

The procedure of the construction of the Gupta-Bleuler triplet is similar to the massless vector field. Here we briefly review the construction. The gauge fixing field equation

$$\left( Q_2^{(1)} + 6 \right) \mathcal{H} + c D_2^\top \bar{\partial}_2^\top \cdot \mathcal{H} = 0,$$  

along with the gauge transformation (VII.36) and the UIR of discrete series $\Pi_{2,2}^\pm$ exactly define the covariant massless tensor field in the dS space (see [16, 51]). Applying a divergence operator on the equation (VII.37), one obtains [51]:

$$(1-c)Q_1^{(1)} \partial \cdot \mathcal{H} = 0.$$

The $\partial \cdot \mathcal{H} = K_1$ is a vector state in which for $c \neq 1$ yields $Q_1^{(1)} K_1 = 0$. Imposing the divergenceless condition and taking as $\mathcal{H} = D_2 K_2$, one obtains $Q_1^{(1)} K_2 = 0$. These two states in addition to the physical states construct the Gupta-Bleuler triplet [51]. Let us now define the Gupta-Bleuler triplet $V_g \subset V \subset V_c$ carrying the indecomposable representation of the dS group appearing in our problem. $V_c/V$ is the vector state $K_1$ which is a massless vector field. $V_g$ is the gauge state $K_2$ which is also a massless vector field. The space $V/V_g$ contains the physical states $\Pi_{2,2}^\pm \oplus \Pi_{2,2}^\pm$.

The explicit form of the tensor field depends to the gauge fixing parameter $c$ and for the value $c = \frac{2}{3}$ (VII.25) the tensor field becomes [16]

$$\mathcal{H}_{\alpha\beta}(x) = \mathcal{D}_{\alpha\beta}(x, \bar{\partial}) \phi_m(x),$$

where

$$\mathcal{D}(x, \bar{\partial}, Z_1, Z_2) = \left( -\frac{2}{3} \theta Z_1 \cdot + SZ_1^\top + \frac{1}{9} D_2^\top (H^2 x Z_1 \cdot - Z_1 \cdot \bar{\partial}^\top + \frac{2}{3} H^2 D_1^\top Z_1) \right)$$

$$+ \left( Z_2 - \frac{1}{2} D_1^\top (Z_2 \cdot \bar{\partial}^\top + 2H^2 x \cdot Z_2) \right),$$  

(VII.38)
with $Z_1$ and $Z_2$ as two constant five-vectors. $\phi_\alpha(x)$ is a minimally coupled scalar field. By using the identity (VII.10), the tensor field can be written in terms of the conformally coupled scalar field:

$$\mathcal{H}_{\alpha\beta}(x) = D_{\alpha\beta}(x, \partial) \left[ Z_3 \cdot \partial^\top + 2Z_3 \cdot x \right] \phi_c(x).$$  \hfill (VII.39)

The quantum field operator can be written in the following form:

$$\mathcal{H}_{\alpha\beta}(x) = \int_B d\mu(q) \sum_m [a(q, m; 2, -1)U_{\alpha\beta}(x; q, m; 2, 2) + a(q, m; 2, 2)V_{\alpha\beta}(x; q, m; 2, 2)].$$  \hfill (VII.40)

In terms of the minimally coupled scalar field operator (VII.11), the coefficients $U$ and $V$ become:

$$U_{\alpha\beta}(x; q, m; 2, 2) = (x \cdot \xi_B)^{-2}u_{\alpha\beta}(x, m; 2, 2),$$

$$V_{\alpha\beta}(x; q, m; 2, 2) = (x \cdot \xi_B)^{-1}v_{\alpha\beta}(x, m; 2, 2).$$

The coefficients $u_{\alpha\beta}$ and $v_{\alpha\beta}$ can be calculated explicitly by using the equation (VII.39).

The two-point function can be written in terms of the two-point function of minimally coupled scalar field and a polarization tensor. For $c = \frac{2}{3}$, one has (17, 51, 73)

$$\mathcal{W}_{\alpha\beta\alpha'\beta'}(x, x') = \Delta_{\alpha\beta\alpha'\beta'}(x, x')\mathcal{W}_m(x, x'),$$ \hfill (VII.41)

where

$$\Delta(x, x') = -\frac{2}{3} \Sigma_1 \theta \cdot \theta' \left( \theta \cdot \theta' - \frac{1}{2} D_1^\top [2H^2 x \cdot \theta' + \theta' \cdot \theta] \right)$$

$$+ \Sigma_1 \Sigma_2 \theta \cdot \theta' \left( \theta \cdot \theta' - \frac{1}{2} D_1^\top [2H^2 x \cdot \theta' + \theta' \cdot \theta] \right)$$

$$+ \frac{H^2}{9} \Sigma_1 D_2^\top \left( \frac{2}{3} D_1^\top \theta' \cdot + x \theta' \cdot - H^{-2} \theta' \cdot \theta \right) \left( \theta \cdot \theta' - \frac{1}{2} D_1^\top [2H^2 x \cdot \theta' + \theta' \cdot \theta] \right).$$ \hfill (VII.42)

$\mathcal{W}_m$ is the two-point function of the minimally coupled scalar field. In terms of conformally coupled scalar field it is (VII.12):

$$\mathcal{W}_m^H(z, x') = \frac{-H^2}{8\pi^2} \left[ \partial^\top \cdot \partial' \top + 2x \cdot \partial' \top + 2\partial^\top \cdot x' - 4Z(x, x') \right]$$

$$\times \left[ \frac{1}{1 - Z(x, x')} - i\pi \epsilon(x^0 - x'^0)\delta(1 - Z(x, x')) \right],$$

and in the Krein space quantization it becomes (VII.14):

$$\mathcal{W}_m^K(x, x') = \frac{iH^2}{8\pi} \epsilon(x^0 - x'^0)[\delta(1 - Z(x, x')) - \theta(Z(x, x') - 1)].$$

It is clear that these two-point functions are dS covariant and also free of any infrared divergences. Note that such two-point function breaks the conformal transformation, and hence, in order to preserve the conformal transformation, the spin-2 field must be described by a rank-3 mixed symmetric tensor field.
G. Massless spin-2 rank-3 mixed-symmetric tensor field

The rank-3 tensor field on the dS hyperboloid with the conditions $x \cdot \mathcal{K} = 0$ and $\mathcal{K}_{\alpha_1\alpha_2\alpha_3} = -\mathcal{K}_{\alpha_1\alpha_3\alpha_2}$ satisfies the field equation (VI.12). Associating this field with an UIR of the dS group, the following subsidiary condition must be imposed: $\partial \mathcal{K} = 0$, $\mathcal{K}' = 0$, then the field equation becomes:

$$(Q_3^{(1)} + 6) \mathcal{K}_{\alpha_1\alpha_2\alpha_3} = 0,$$

which, in terms of $Q_0$, is

$$Q_0 \mathcal{K}_{\alpha_1\alpha_2\alpha_3} + 2 \mathcal{K}_{\alpha_1\alpha_2\alpha_3} + 2 \mathcal{K}_{\alpha_2\alpha_3\alpha_1} + 2 \mathcal{K}_{\alpha_3\alpha_1\alpha_2} = 0. \quad (VII.43)$$

Because of the existence of the two last terms in the field equation, the rank-3 tensor field $\mathcal{K}$ cannot associate with the UIR of the dS group $\Pi_{2,2}^\pm$. The tensor field $\mathcal{K}$ can be written in the sum of two fields with definite symmetry [57]:

$$\mathcal{K}_{\alpha_1\alpha_2\alpha_3} = \frac{1}{3} (\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M + \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^A), \quad (VII.44)$$

$$\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M = 2 \mathcal{K}_{\alpha_1\alpha_2\alpha_3} - \mathcal{K}_{\alpha_2\alpha_3\alpha_1} - \mathcal{K}_{\alpha_3\alpha_1\alpha_2},$$

$$\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^A = \mathcal{K}_{\alpha_1\alpha_2\alpha_3} + \mathcal{K}_{\alpha_2\alpha_3\alpha_1} + \mathcal{K}_{\alpha_3\alpha_1\alpha_2},$$

where $\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^A$ is a totally antisymmetric tensor field with 4 degrees of freedom, and $\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M$ is a mixed symmetry tensor field with 20 degrees of freedom:

$$\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M + \mathcal{K}_{\alpha_2\alpha_3\alpha_1}^M + \mathcal{K}_{\alpha_3\alpha_1\alpha_2}^M = 0.$$

The rank-3 mix symmetry tensor field $\mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M$ on the dS hyperboloid can be considered as a spin-2 field in dS space. Such a field may be considered as a part of the gravitational field with its quantum field operator being transformed by an indecomposable representation of the dS group and its physical states correspond to the UIR of the discrete series representations $T^{(0,2;2)}$ and $T^{(2,0;2)}$ (or $\Pi_{2,2}^\pm$ in Dixmier notation).

On the other hand, decomposing a rank-3 tensor field as (VII.44), one can rewrite the field equation (VII.43) in the following form:

$$Q_0 \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M + (Q_0 + 6) \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^A = 0,$$

and consequently, both parts must vanish, separately. The mixed symmetry part can associate to the UIR of the dS group $\Pi_{2,2}^\pm$, since the associated mass parameter is $m_0^2 = 0$ and the homogeneous degrees of this part are $\lambda = 0, -3$

$$Q_0 \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M = 0, \quad \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^M(x) = \mathcal{D}_{\alpha_1\alpha_2\alpha_3}^M(x, \partial)(x \cdot \xi)^\lambda.$$

The totally anti-symmetric part cannot associate to an UIR of the dS group and accompanies an imaginary mass $m_0^2 = -6H^2$, plus one of the homogeneous degrees is positive $\lambda = -\frac{3}{2} \pm \frac{1}{2} \sqrt{33}$,

$$(Q_0 + 6) \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^A = 0, \quad \mathcal{K}_{\alpha_1\alpha_2\alpha_3}^A(x) = \mathcal{D}_{\alpha_1\alpha_2\alpha_3}^A(x, \partial)(x \cdot \xi)^\lambda.$$

This part, which was called "the ghost field" by Frønsdal, can be eliminated by imposing some conditions which preserve the gauge invariance [57].
In the previous section using the $SO(1,4)$ gauge gravity model, the calculation of the gauge invariant field equations for the massless spin-2 rank-3 tensor field in the $x$-space was carried on [VI.12]:

$$\left(Q^1_3 + 6\right) \mathcal{K}_{a_1a_2a_3} + \nabla_{a_1}^T \partial_{a_3} \mathcal{K}_{a_2a_3} = 0.$$ 

The field equation is invariant under the gauge transformation [VI.7] or [VI.13]:

$$\mathcal{K}_{a_1a_2a_3} \rightarrow \mathcal{K}^g_{a_1a_2a_3} = \mathcal{K}_{a_1a_2a_3} + \nabla_{a_1}^T A_{a_2a_3},$$

dropping the non-linear term and introducing $A_{a_1a_2}$ as an arbitrary rank-2 anti-symmetric tensor field on the dS hyperboloid. One can simplify above equations for the mixed symmetric and totally antisymmetric parts by introducing subscripts $M$ and $A$ to represent these parts, respectively. For mixed symmetry parts, the gauge invariant field equation is:

$$\left(Q^1_3 + 6\right) \mathcal{K}^M_{a_1a_2a_3} + \nabla_{a_1}^T \partial_{a_3} \mathcal{K}^M_{a_2a_3} = 0,$$ 

and the gauge transformation become:

$$\mathcal{K}^M_{a_1a_2a_3} \rightarrow \mathcal{K}^g_M = \mathcal{K}^M_{a_1a_2a_3} + \left[\nabla_{a_1}^T A_{a_2a_3}\right]^M,$$ 

where

$$\left[\nabla_{a_1}^T A_{a_2a_3}\right]^M = 2\nabla_{a_1}^T A_{a_2a_3} - \nabla_{a_2}^T A_{a_3a_1} - \nabla_{a_3}^T A_{a_1a_2}.$$ 

The gauge fixing field equation can be obtained similar to the previous cases:

$$\left(Q^1_3 + 6\right) \mathcal{K}^M_{a_1a_2a_3} + c\nabla_{a_1}^T \partial_{a_3} \mathcal{K}^M_{a_2a_3} = 0.$$ 

Since the divergencelessness condition is dropped out, the field operator does not transform by an UIR of the dS group and must transform according to an indecomposable representation of the dS group. For the construction of this representation or the Goupta-Bleuler triplet, one needs the gauge transformation [VII.46], the gauge fixing equation [VII.47] and the UIR’s $T^{(0;2,2)}$ and $T^{(2;0,2)}$, or $\Pi_2^\pm$ in Dixmir notation of the dS group. The physical states, which correspond to $\Pi_2^\pm$, transform simultaneously under the conformal group representations $C^{(\pm3,0,2)}$ and $C^{(3,2,0)}$, which will be considered in the next section.

The gauge fixing field equation [VII.47], the gauge transformation [VII.46] and the UIR of discrete series $\Pi_2^\pm$ are exactly defined the covariant massless spin-2 rank-3 mixed symmetric tensor field in the dS space formalism. Taking the divergence of the equation [VII.47], one gains the field equation of the gauge dependent states $\partial_{a_3}^T \mathcal{K}^M_{a_2a_3} \neq 0$. These states $\partial_{a_3}^T \mathcal{K}^M_{a_2a_3} = (A_1)_{a_2a_3}$ are forming a rank-2 anti-symmetric tensor field which can be fixed if $c \neq 1$. Impose the divergencelessness condition on the tensor field and visualizing it as $\mathcal{K}^M_{a_1a_2a_3} = \left[\nabla_{a_1}^T A_{a_2a_3}\right]^M$, one obtains the field equation, defining this tensor field as a pure gauge state $(A_2)_{a_2a_3}$. These two tensor fields $A_1$ and $A_2$, and the physical states construct the Gupta-Bleuler triplet, with the exact same process, defining the Gupta-Bleuler triplet as $V_q \subset V \subset V_c$, identifying $V_c/V$ as the rank-2 anti-symmetric tensor field state $A_1$, $V_g$ as the gauge state $A_2$. $V/V_g$ contains the physical states $\Pi_2^\pm \oplus \Pi_2^\pm$.

The quantum field operator can be written as the following form:

$$\mathcal{K}_{\alpha\beta\gamma}(x) = \int_B d\mu(q) \sum_m \left[a(q, m; 2, -1) \mathcal{U}_{\alpha\beta\gamma}(x; q, m; 2, 2) + a^\dagger(q, m; 2, 2) \mathcal{V}_{\alpha\beta\gamma}(x; q, m; 2, 2)\right].$$ 

(VII.48)
By using the homogeneity conditions, one obtains:
\[ U_{\alpha\beta\gamma}(x; q, m; 2, 2) = (x \cdot \xi_B)^{-2} u_{\alpha\beta\gamma}(x, q, m; 2, 2), \]
\[ V_{\alpha\beta\gamma}(x; q, m; 2, 2) = (x \cdot \xi_B)^{-1} v_{\alpha\beta\gamma}(x, q, m; 2, 2), \]
and also, solving the field equation and imposing the auxiliary conditions, the coefficients \( u_{\alpha\beta\gamma} \) and \( v_{\alpha\beta\gamma} \) can be established.

The analytic two-point function for the Massless spin-2 rank-3 mixed-symmetric tensor field is
\[ W_{\alpha\beta\gamma\alpha'\beta'\gamma'}(z, z') = \langle \Omega | K_{\alpha\beta\gamma}(z) K_{\alpha'\beta'\gamma'}(z') | \Omega \rangle = \hat{B} \int_B \mu(q) (z \cdot \xi_B)^{-2} (z' \cdot \xi_B)^{-1} \sum_m u_{\alpha\beta\gamma}(z, q, m; 2, 2) v_{\alpha'\beta'\gamma'}(z', q, m; 2, 2). \quad (VII.49) \]

Similar to the other cases, this two-point function can be written in the following form:
\[ W_{\alpha\beta\gamma\alpha'\beta'\gamma'}(z, z') = D_{\alpha\beta\gamma\alpha'\beta'\gamma'}(z, \partial; z', \partial') W_c(z, z'), \]
where \( W_c \) is the analytic two-point function of a massless conformally coupled scalar field. The explicit forms of the polarization tensor are calculated in [22].

VIII. CONFORMAL TRANSFORMATION

A thorough and rather interesting study, regarding the relation between the dS and the conformal groups in the ambient space formalism (such relation often appears in describing the massless fields), would be the subject of this section. Considering the global conformal transformation on the Dirac 6-cone formalism, one notices that the physical states of massless field operators transform according to the UIR of the \( SO(2, 4) \) group. We show that the massless fields in the dS ambient space formalism are exactly the same as the massless fields in the Dirac 6-cone formalism.

Afterwards, the local conformal transformation group, \( SO(2, 4) \), can be considered as one of the basis of the gauge gravity model, with the gauge potential being considered as the conformal gravity in the dS ambient space formalism. Then, one can construct the Hilbert space for massless fields on the closed unit ball \( q \).

A. Dirac 6-cone formalism

Undesirably, the conformal group acts non-linearly on the Minkowski coordinates. Obviating this problem, Dirac proposed a manifestly conformal covariant formulation in which the Minkowski coordinates are replaced by a set of suitable coordinates, in order to provide the possibility of linear action for the conformal group. The result was a theory, established as a 5-dimensional hyper-cone in a 6-dimensional flat space, named Dirac’s six-cone. This approach to conformal symmetry, leading to the best path for exploiting the physical symmetry was first used by Dirac, to construct the manifestly conformal invariant field equations for spinor and vector fields in \( (1 + 3) \)-dimensional space-time [74], and afterwards has been developed by Mack and Salam [75].

Dirac’s six-cone or Dirac’s projection cone is defined by
\[ y^2 \equiv (y^0)^2 - \vec{y}^2 - (y^4)^2 + (y^5)^2 = \eta_{ab} y^a y^b = 0, \quad \eta_{ab} = \text{diag}(1, -1, -1, -1, -1, 1), \quad (VIII.1) \]
where \( y^a \in \mathbb{R}^6; a, b = 0, 1, 2, 3, 4, 5 \) and \( \vec{y} \equiv (y^1, y^2, y^3) \). Reduction to four dimensions is achieved by projection after fixing the degrees of homogeneity of all the fields. Wave equations, subsidiary
conditions, etc., must be expressed in terms of operators that are defined intrinsically on the cone. These are well-defined operators that map tensor fields to tensor fields with the same rank on the cone $y^2 = 0$. So, the out coming equations which are obtained by this method, are conformally invariant.

The tensor fields $\Psi$ on the Dirac 6-cone are a homogeneous functions of variable $y^2$ and are transverse [74]:

$$y^a \frac{\partial}{\partial y^a} \Psi^{cd..} = \sigma \Psi^{cd..}, \quad y_a \Psi^{ab..} = 0.$$  

In order to project the coordinates on the cone $y^2 = 0$ to the $(4 + 1)$-dimensional dS space, one chooses the following relations:

$$\begin{cases} 
  x^a = (y^5)^{-1} y^a, \\
  x^5 = y^5. 
\end{cases}$$  

(VIII.2)

Note that $x^5$ becomes superfluous when one deals with the projective cone. The choice $x^5 = H^{-1}$, precisely results in the dS hyperboloid structure, with the tensor field, previously defined in the Dirac 6-cone formalism, turning to be the tensor field on the dS space. Six-tensors $\Psi^{ab..}$ are related to complexes of five-tensors by [72]

$$\Psi^{ab..}(y) = \frac{\partial x^\alpha}{\partial y^a} \frac{\partial x^\beta}{\partial y^b} \cdots K_{\alpha\beta..}(x).$$

For example, after performing some algebraic calculation, one can show that a symmetric rank-2 tensor field satisfies the following relation for the limit ($y^5 = H^{-1}$) on the cone:

$$\Psi^{ab}(y) = \left( \Psi^{\alpha\beta}(y) \propto K^{\alpha\beta}(x), \Psi^{a5}(y) \propto K^a(x), \Psi^{55}(y) \propto \phi(x) \right).$$

The same statement holds for tensor fields of other ranks. These fields are conformal invariant, since they are coming from the cone. In the following section, by representing the discrete series of the conformal group to enlighten the subject of the massless fields, the correspondence of these tensor fields with the discrete series representations of the dS group has been illustrated.

B. Discrete UIR of the conformal group

The conformal group $SO(2,4)$ acts on the cone as:

$$y^\alpha = \Lambda^\alpha_\beta y^\beta, \quad \Lambda \in SO(2,4), \quad \det \Lambda = 1, \quad \Lambda \eta \Lambda^T = \eta \implies y \cdot y = 0 = y' \cdot y'.$$

Defining a $4 \times 4$ matrix, $Y$, as:

$$Y = \begin{pmatrix}
  y^0 + iy^5 \\
  P \\
  P^{-1} \\
  y^0 - iy^5
\end{pmatrix},$$

where $p = (y^4, \vec{y})$ is a quaternion, one can show that under a transformation of the conformal group $SU(2,2)$, it transforms as:

$$Y' = g Y g^T, \quad g \in SU(2,2) \implies y \cdot y = 0 = y' \cdot y'.$$

$SU(2,2)$ is the universal covering group of $SO(2,4)$:

$$SO_0(2,4) \approx SU(2,2)/\mathbb{Z}_2,$$  

(VIII.3)
which is defined by

\[ SU(2, 2) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det g = 1, \ Jg^tJ = g^{-1}, J = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix} \right\}, \quad \text{(VIII.4)} \]

where \( a, b, c \) and \( d \) are complex quaternion \( a = q_1 + iq_2 \).

In the previous works \cite{20, 31, 34, 35, 75, 76}, a variety of realizations of the UIR of the conformal group and their corresponding Hilbert spaces have been constructed. Particularly, Graev’s realization of the discrete series is important for our purposes here \cite{34}. The homogeneous space is by definition a complex \( 2 \times 2 \) matrix, \( Z \), in the domain \( D \) which in turn is defined by the constraint of positive definiteness \cite{34, 35}:

\[ \Pi - Z^tZ > 0, \quad \Pi - ZZ^t > 0. \quad \text{(VIII.5)} \]

If one defines \( Z \) as a quaternion:

\[ Z = q = \begin{pmatrix} q^1 + iq^3 \\ iq^1 + q^2 \end{pmatrix}, \]

then the condition \( \Pi - Z^tZ > 0 \) holds when \( |q| < 1 \). It is exactly the homogeneous space \( (\xi_B^0) \) in which the discrete series representation of the dS group has been constructed on. The discrete series representation of the conformal group and its corresponding Hilbert space on this homogeneous space, has been introduced by Ruhl (for value \( E_0 > j_1 + j_2 + 3 \)) \cite{34, 35}:

\[ C^{(E_0,j_1,j_2)}(g) |q, m_{j_1}, m_{j_2}; j_1, j_2, E_0) = [\det(cq + d)]^{-E_0} \times \]

\[ \sum_{m'_{j_1}, m'_{j_2}} D^{(j_1)}_{m'_{j_1} m'_{j_2}} \left( a^1 + cq^1 \right) D^{(j_2)}_{m'_{j_2} m_{j_2}} (cq + d) \left| g^{-1} \cdot q, m'_{j_1}, m'_{j_2}; j_1, j_2, E_0 \right>, \quad \text{(VIII.6)} \]

where \( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2, 2) \) and \( g^{-1} \cdot q = (aq + b)(cq + d)^{-1}. \)

\( D^{(j_1)} \) and \( D^{(j_2)} \) furnish a certain representation of \( SU(2) \) group \cite{11, 30, 31, 33, 34}. The UIR \( C^{(E_0,j_1,j_2)}(g) \) acts on an infinite dimensional Hilbert space \( \mathcal{H}^{(E_0,j_1,j_2)} \). The scalar product was defined in \cite{34}. One can easily verify that this representation on the Hilbert space \( \mathcal{H}^{(E_0,j_1,j_2)} \) satisfies \cite{34, 35}:

\[ C^{(E_0,j_1,j_2)}(g) C^{(E_0,j_1,j_2)}(g') |q, m_{j_1}, m_{j_2}; j_1, j_2, E_0) = C^{(E_0,j_1,j_2)}(gg') |q, m'_{j_1}, m'_{j_2}; j_1, j_2, E_0), \]

\[ C^{(E_0,j_1,j_2)}(g) \left[ C^{(E_0,j_1,j_2)}(g) \right]^\dagger |q, m_{j_1}, m_{j_2}; j_1, j_2, E_0) = |q, m_{j_1}, m_{j_2}; j_1, j_2, E_0). \]

One has an infinite dimensional Hilbert space \( \mathcal{H}^{(E_0,j_1,j_2)} \),

\[ |q, m_{j_1}, m_{j_2}; j_1, j_2, E_0) \in \mathcal{H}^{(E_0,j_1,j_2)}, \quad q \in \mathbb{R}^4, |q| = r < 1, \quad -j \leq m_j \leq j. \]

The total number of quantum states is finite. Although our interest lies only with the values \( j_1 j_2 = 0 \), \( E_0 = j_1 + j_2 + 1 \), with helicity \( j_1 - j_2 \) and \( 2j_1 \) and \( 2j_2 \) as non-negative integers (values associated with massless fields), the reader can find a detailed discussion regarding all relevant values of \( E_0, j_1 \) and \( j_2 \), in correspondence with the UIR of the conformal group, presented by Mack \cite{31}.

The representations \( C^{(j_1,j_0)}(g) \) and \( C^{(j_1+1,0)}(g) \) correspond to the massless representation of the Poincaré massless group \cite{31, 33, 34, 39}. They correspond to two helicities of the massless fields. The massless representation \( D^{(j)} \) furnishes a certain representation of the little group \( ISO(2) \) \cite{17, 31, 31
and therefore, one can write \( m_j = m'_j = j \) for \( C^{(j+1,j,0)}(g) \) and \( m_j = m'_j = -j \) for \( C^{(j+1,0,j)}(g) \) with the vanishing values for \( m_j \) and \( m'_j \) in other cases \([31]\). In this case the little group is defined by

\[
\xi_B^0(0) = \xi^0 \left( 1, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad \text{or} \quad \mathbf{q}(0) = \left( 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).
\]  

(VIII.7)

The massless field equations in the Minkowski space are considered to be conformally invariant, therefore, every massless representation of the Poincaré group has only one corresponding representation in the conformal group \([20, 38]\). In the dS space, for massless fields, only two representations in discrete series \( T^{(0,j;j)} \) and \( T^{(j,0;j)} \) (in Dixmier notation \( \Pi^\pm_{j,j} \)) have Minkowski interpretations. The signs \( \pm \) correspond to the two types of helicity for the massless fields. The representation \( \Pi^+_{j,j} \) has a unique extension to a direct sum of two UIR’s \( C^{(j+1,j,0)} \) and \( C^{(-j-1,j,0)} \) of the conformal group \( SO_0(2, 4) \). Note that \( C^{(j+1,j,0)} \) and \( C^{(-j-1,j,0)} \) correspond to positive and negative energy representations in the conformal group respectively \([20, 38]\). The concept of energy cannot be defined in the dS space. The UIR of the dS group is restricted to the sum of the massless Poincaré UIR’s \( P^{(0,j)}_+ \) and \( P^{(0,j)}_- \) with positive and negative energies respectively. The following diagrams illustrate these connections

\[
\Pi^+_{j,j} \leftrightarrow \bigoplus_{H=0} \bigoplus_{1 \leq \alpha \leq 15} C^{(j+1,j,0)} \leftrightarrow P^{(0,j)}_+,
\]

\[
\Pi^-_{j,j} \leftrightarrow \bigoplus_{H=0} \bigoplus_{1 \leq \alpha \leq 15} C^{(-j-1,j,0)} \leftrightarrow P^{(0,j)}_-,
\]

where the arrows \( \leftrightarrow \) designate unique extension. \( P^{(0,j)}_{\pm} \) are the massless Poincaré UIRs with positive and negative energies and negative helicity.

C. Conformal gauge gravity

This subsection contains a generalization of the dS gauge gravity to the conformal group \( SO(2, 4) \), using of the Dirac 6-cone formalism and introducing the following set of the gauge generators:

\[
L_{ab} = M_{ab} + S_{ab} \equiv Y_A, \quad A = 1, 2, \ldots, 15, \quad a, b = 0, 1, \ldots, 5.
\]

In this notation the commutation relation is rewritten as:

\[
[Y_A, Y_B] = f_{B\overline{A}} \xi_{\overline{C}} Y_C.
\]

Obviously, there are 15 one-form potentials or gauge vector fields \( \mathcal{G}_a^A \equiv G_{a}^{ab} = -G_{a}^{ba} \). These gauge potentials satisfy the transversality condition \( x^a \mathcal{G}_a^A = 0 \) and they have 60 degrees of freedom. In ambient space notation, the conformal gauge-covariant derivative can be defined as

\[
D^A = \nabla^A + \mathcal{G}^{\overline{A}}_\beta Y_\beta.
\]

Similar to the dS gauge gravity this gauge-covariant derivative is not transverse since:

\[
x^a \mathcal{G}_\beta^A Y_A T_{\alpha_1 \cdots \alpha_n} \neq 0, \quad x^{\alpha_n} M_{\alpha \beta} \neq M_{\alpha \beta} x^{\alpha_n}.
\]
Dividing the gauge potential $G_{\alpha}^{ab}$ into two parts, $G_{\alpha}^{\beta\gamma}$ and $G_{\alpha}^{5\gamma}$, and imposing the following subsidiary conditions

$$x_{\beta}G_{\alpha}^{\beta\gamma} = 0 = x_{\gamma}G_{\alpha}^{\beta\gamma}, \quad x_{\gamma}G_{\alpha}^{5\gamma} = 0,$$

(VIII.8)
on the gauge potentials, one obtains:

$$x^{a}G_{\beta}^{\ A}Y_{A}T_{\alpha_{1}...\alpha_{n}} = 0.$$

The 15 gauge vector fields $G_{\alpha}^{ab}$ can be separated into two groups; 10 gauge vector fields $G_{\alpha}^{\beta\gamma}$ (with 40 degrees of freedom) and 5 gauge vector fields $G_{\alpha}^{5\gamma}$ (with 20 degrees of freedom). Noticing the auxiliary conditions (VIII.8), one assigns 24 degrees of freedom to the gauge fields $G_{\alpha}^{\beta\gamma}$ and considers the collection of them as a tensor field of rank-3 (exact equivalent of a tensor field of rank-3 in the dS space, up to a normalization constant). Afterwards, applying the mix-symmetry condition on $G_{\alpha}^{\beta\gamma}$, one decides that this is the exact same gauge field, subjected to the last section $G_{\alpha}^{\beta\gamma} = K_{\alpha}^{\beta\gamma}$. It can be considered as a conformal gravitational field. The gauge vector potential $G_{\alpha}^{5\gamma} = H_{\alpha}^{\gamma}$ is considered as a tensor field of rank-2 with 16 degrees of freedom on dS hyperboloid.

Now we can repeat the conformal gauge gravity construction in its conventional way. Under a local infinitesimal gauge transformation generated by $\epsilon^{A}(x)Y_{A}$ we have

$$\delta_{\epsilon}D_{\alpha}^{G} = [D_{\alpha}^{G}, \epsilon^{A}(x)Y_{A}] = (D_{\alpha}^{G} \epsilon^{A})Y_{A},$$

so that

$$\delta_{\epsilon}G_{\alpha}^{A} = D_{\alpha}^{G} \epsilon^{A} = \nabla_{\alpha}^{\top} \epsilon^{A} + f_{CB}^{A} G_{\alpha}^{C} \epsilon^{B}.$$

The curvature $C_{\alpha}^{A}$, with values in the Lie algebra of conformal group $SO(2, 4)$, is defined by:

$$C(D_{\alpha}^{G}, D_{\beta}^{G}) = -[D_{\alpha}^{G}, D_{\beta}^{G}] = C_{\alpha\beta}^{A} Y_{A},$$

where

$$C_{\alpha\beta}^{A} = \nabla_{\alpha}^{\top} G_{\beta}^{A} - \nabla_{\beta}^{\top} G_{\alpha}^{A} + G_{\beta}^{B} G_{\alpha}^{C} f_{BC}^{A}.$$

The $SO(2, 4)$ gauge invariant action or Lagrangian in dS background for the gauge field $G_{\alpha}^{A}$ is [62, 64]:

$$S_{c}[G] = \int d^{4}X \epsilon^{\alpha\beta\gamma\delta} \left( C_{\alpha\beta}^{A} C_{\gamma\delta}^{B} \right) Q_{AB},$$

(VIII.9)

$$= \int d^{4}X \epsilon^{\alpha\beta\gamma\delta} \left( \nabla_{\alpha}^{\top} G_{\beta}^{A} - \nabla_{\beta}^{\top} G_{\alpha}^{A} \right) \left( \nabla_{\gamma}^{\top} G_{\delta}^{B} - \nabla_{\delta}^{\top} G_{\gamma}^{B} \right) Q_{AB} + O[G^{3}],$$

where $Q_{AB}$ are the numerical constants and $O([G]^{3})$ is the third order of $G$. The gauge invariant action can be rewritten in the following form:

$$S_{c}[G] = \int d\mu(x) \theta^{\alpha\gamma} \theta^{\beta\delta} \left( \nabla_{\alpha}^{\top} G_{\beta}^{A} - \nabla_{\delta}^{\top} G_{\gamma}^{B} \right) \left( \nabla_{\gamma}^{\top} G_{\delta}^{B} - \nabla_{\gamma}^{\top} G_{\beta}^{B} \right) Q_{AB} + O[G^{3}].$$

The field equation can be written in the following form:

$$\theta^{\alpha\gamma} \nabla_{\alpha}^{\top} \left( \nabla_{\gamma}^{\top} G_{\delta}^{B} - \nabla_{\delta}^{\top} G_{\gamma}^{B} \right) Q_{AB} + O([G]^{2}) = 0,$$

(VIII.10)
with $O([\mathcal{G}]^2)$ being the second order of $\mathcal{G}$. One imposes the subsidiary conditions on $\mathcal{Q}_{AB}$ to achieve the maximal irreducibility of the gauge multiplet $\mathcal{G}_\alpha^{\beta\gamma} \equiv \mathcal{K}_\alpha^{\beta\gamma}$ and $\mathcal{G}_\alpha^{5\beta} \equiv H_\alpha^{\beta}$ on the dS space

$$\quad \mathcal{Q}_{(\alpha\beta)(\gamma\delta)} = \epsilon_{\alpha\beta\gamma\delta}, \quad \mathcal{Q}_{(5\alpha)(5\beta)} = \epsilon_{\alpha\beta}, \quad \mathcal{Q}_{(\alpha\beta)(5\gamma)} = 0 = \mathcal{Q}_{(5\gamma)(\alpha\beta)}.$$ 

The effect of these conditions splits the action into two parts:

$$S_c[H,\mathcal{K}] = \int d\mu(x) \mathcal{L}(H,\mathcal{K}) \equiv S_{c1} + S_{c2},$$

where

$$S_{c1}[H,\mathcal{K}] = -\frac{1}{4} \int d\mu(x) \theta^{\alpha\gamma} \theta^{\beta\delta} \left( \nabla_\alpha H_\beta^{\gamma'} - \nabla_\beta H_\alpha^{\gamma'} \right) \left( \nabla_\gamma H_\delta^{\beta'} - \nabla_\delta H_\gamma^{\beta'} \right) \epsilon_{\gamma'\delta'} + O([H,\mathcal{K}]),$$

and

$$S_{c2}[H,\mathcal{K}] = -\frac{1}{4} \int d\mu(x) \theta^{\alpha\gamma} \theta^{\beta\delta} \left( \nabla_\alpha \mathcal{K}_\beta^A - \nabla_\beta \mathcal{K}_\alpha^A \right) \left( \nabla_\gamma \mathcal{K}_\delta^B - \nabla_\delta \mathcal{K}_\gamma^B \right) Q_{AB} + O([H,\mathcal{K}]).$$

$O([H,\mathcal{K}])$ is the non-linear order of $H$ and $\mathcal{K}$. The first part of the action, $S_{c2}$, is in direct correspondence to the first part of the action in the $SO(1,4)$ gauge group (VI.12), resulting in an exact similarity between linear field equation, obtain from the action $S_{c2}$, and the field equation of the spin-2 rank-3 field in the dS space (VI.12). For obtaining the relation between the gauge potential $\mathcal{G}^{\alpha\beta}_\gamma$ and the UIR of dS group, one must write this equation in terms of the Casimir operator of the dS group.

It is interesting to note that the field equation, obtained from the conformal action (VIII.12), is a second order field equation, invariant under the local conformal transformation. Similar to the Minkowski space, the gauge fixing terms can be added to the conformal Lagrangian:

$$S_c[\mathcal{G}] = \int d\mu(x) \left[ \theta^{\alpha\gamma} \theta^{\beta\delta} \mathcal{G}_\alpha^{\gamma\delta} - \frac{c}{2} \partial^A \mathcal{G}_\alpha^{\gamma} \partial^B \mathcal{G}_\alpha^{\delta} \right] Q_{AB}. \quad \text{(VIII.11)}$$

The field equation can be calculated for this action being invariant under the following gauge transformation:

$$\quad \mathcal{G}_{\alpha A}^{\gamma} = \mathcal{G}_\alpha^{A} + \nabla_\alpha^{\gamma} A^A + f_{CB} A^C \theta^B + f_{AC} A^C \theta^B. \quad \text{(VIII.12)}$$

The field equation can be written in terms of Casimir operator of the dS group. Utilizing the gauge fixing field equation (obtained form the action (VIII.11)), the gauge transformation (VIII.12) and the UIR of the conformal group $\mathcal{C}(3,2,0) \oplus \mathcal{C}(-3,2,0)$ and $\mathcal{C}(3,0,2) \oplus \mathcal{C}(-3,0,2)$, one can construct the Gupta-Bleuler triplet for conformal gauge gravity. This rather complicated procedure will be the subject of a forthcoming paper.

**IX. RELATION WITH INTRINSIC COORDINATES**

At this point, in order to compare our results with the work of the other authors in the intrinsic coordinates and the other authors can use our result [77, 78], the relation between these two formalism, i.e. the ambient space formalism and the intrinsic coordinates, is presented. These relation are discussed for the tensor fields, the spinor fields and the two-point functions. The de Sitter metrics is

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = g^{dS}_{\mu\nu} dx^\mu dx^\nu, \quad \mu = 0, 1, 2, 3,$$

where the $X^\mu$’s are the 4-dimensional space-time coordinates in the dS hyperboloid. Different coordinate systems can be chosen.
A. Tensor fields

Here we consider only the spin-2 rank-2 symmetric tensor field and one can simply generalize this method to the other tensor fields. We use the fact that the intrinsic field \( h_{\mu\nu}(X) \) is locally determined by the transverse tensor field \([8, 12, 24]\) \( K_{\alpha\beta}(x) \) through

\[
h_{\mu\nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} K_{\alpha\beta}(x(X)).
\]  

(IX.1)

In the same way one can show that the transverse projector \( \theta \) is the only symmetric and transverse tensor that is linked to the dS metric \( g_{\mu\nu} \):

\[
g_{\mu\nu}^{dS} = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \theta_{\alpha\beta}.
\]

Covariant derivatives acting on a symmetric second rank tensor are transformed according to

\[
\nabla_\rho \nabla_\lambda h_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\rho} \frac{\partial x^\beta}{\partial X^\lambda} \frac{\partial x^\gamma}{\partial X^\mu} \frac{\partial x^\delta}{\partial X^\nu} \text{Trpr} \nabla_\alpha ^\top \nabla_\beta ^\top K_{\gamma\delta}.
\]

(IX.2)

By contraction of the covariant derivatives, i.e. \( \nabla_\rho \nabla^\rho \), the d’Alambertian operator becomes:

\[
\Box_H h_{\mu\nu} = g^{\lambda\rho} \nabla_\lambda \nabla_\rho h_{\mu\nu} = \frac{\partial x^\alpha}{\partial X^\rho} \frac{\partial x^\beta}{\partial X^\lambda} \left[ \frac{\partial_\gamma ^\top}{\partial X^\mu} - 2 \right] \left[ \frac{\partial_\delta ^\top}{\partial X^\nu} - 2 \right] K_{\alpha\beta} - 2 S_{\alpha\beta} (\partial \cdot K) + 2 x_\alpha x_\beta K',
\]

(IX.4)

and one can define the other necessary relations accordingly.

B. Spinor fields

The Dirac equation in the general curved space-time is \([48]\)

\[
(\gamma^\mu(X) \nabla_\mu - m) \Psi(X) = 0 = (\bar{\gamma}^a \nabla_a - m) \Psi(X),
\]

(IX.5)

where

\[
\{\gamma^\mu(X), \gamma^\nu(X)\} = 2g^{\mu\nu}, \quad \{\bar{\gamma}^a, \gamma^b\} = 2\eta^{ab}, \quad \mu, a = 0, 1, 2, 3.
\]

Here \( \nabla_\mu \) is the spinor covariant derivative

\[
\nabla_a \Psi(X) = e^\mu_a (\partial_\mu + \frac{i}{2} \epsilon_c \epsilon^\nu \epsilon^{bc} \nabla_\mu e_b \Sigma_{cb}) \Psi(X),
\]
with $e_\mu^a$ as the local vierbein, $e_\mu^a e_\nu^b \eta_{ab} = g_{\mu\nu}$, and $\Sigma_b = \frac{1}{4} [\gamma_e, \bar{\gamma}_b]$ as the spinor representation of the generators of the Lorentz transformation. At this point, it seems relevant to briefly recall the relation of the dS-Dirac equation in two formalism (IV.37) and (IX.5), extracted by Gürsey and Lee [71]. They have introduced a set of coordinates $\{y^\alpha\} \equiv (y^\mu, y^4 = H^{-1})$ related to the $\{x^\alpha\}$’s by

$$x^\alpha = (Hy^4) f^\alpha(y^0, y^1, y^2, y^3),$$

where arbitrary functions $f^\alpha$ satisfies $f^\alpha f_\alpha = -H^{-2}$. Matrices $\beta^\alpha \equiv \left( \frac{\partial y^\alpha}{\partial x^\beta} \right) \gamma^\beta$, can be introduced through the anticommutation properties

$$\{\beta^\mu, \beta^\nu\} = 2g^{\mu\nu},$$

$$\{\beta^\mu, \beta^4\} = 0$$

with $g^{\mu\nu} = \eta^{\alpha\beta} \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu}$, $\mu, \nu = 0, \ldots, 3$, and $\psi = (1 \pm i\beta^4)\chi$, where $\chi$ satisfies the Gürsey-Lee equation

$$\left( \beta^\mu \frac{\partial}{\partial y^\mu} - 2H \beta^4 - m \right) \chi(y) = 0,$$

(IX.6)

for $m = H\nu$. Through the selection of a local vierbein $e_\mu^a$ and setting the gamma function to be $\gamma^\mu(X) \equiv e_\mu^a \gamma^a$, one can find a transformation $V$ so $\gamma^\mu(X) = V \beta^\mu(y) V^{-1}$. Then, under the transformation $V$, the Gürsey-Lee equation (IX.6) becomes an exact replica of the equation (IV.37) with $\Psi(X) = V\chi(y)$.

It is interesting to note that the matrix $\beta^4 = \gamma a x^a = \not{a}x$ is related to the constant matrix $\gamma^4$ by [71]:

$$\gamma^4 = V\beta^4V^{-1} = V \not{a}xV^{-1}.$$ 

Now we can write the relation between the spinor field in two steps, first introducing

$$\psi(x) = V^{-1} \left[ a \left( 1 + i\gamma^4 \right) + b \left( 1 - i\gamma^4 \right) \right] \Psi(X),$$

where $a$ and $b$ are normalization constants and second by calculating the matrix that transforms the $\psi(x)$ to $\Psi(X)$ as follows:

$$\Psi(X) = \left[ \frac{1}{2a} \left( 1 + i\gamma^4 \right) + \frac{1}{2b} \left( 1 - i\gamma^4 \right) \right] V\psi(x).$$

(IX.7)

One may directly conclude the relation between the tensor-spinor fields in the two formalisms by the equation (IX.7).

C. Two-point function

Following Allen and Jacobson in [96] we will write the two-point functions in de Sitter space (maximally symmetric) in terms of bi-tensors. These are functions of two points $(x, x')$ which behave like tensors under coordinate transformations at either point. As shown in [96], any maximally symmetric bi-tensor can be expressed as a sum of products of three basic tensors. The coefficients in this expansion are functions of the geodesic distance $\sigma(x, x')$, that is the distance along the geodesic connecting the points $x$ and $x'$ (note that $\sigma(x, x')$ can be defined by unique analytic extension also when no geodesic connects $x$ and $x'$). In this sense, these fundamental tensors form a complete set. They can be obtained by differentiating the geodesic distance [24]:

$$n_\mu = \nabla_\mu \sigma(x, x'), \quad n_{\mu'} = \nabla_{\mu'} \sigma(x, x'),$$
and the parallel propagator
\[ g_{\mu\nu} = -c^{-1}(Z)\nabla_\mu n_\nu + n_\mu n_\nu. \]

The basic bi-tensors in ambient space notation are found through
\[ \partial_\alpha^\top \sigma(x, x') , \ \partial_\beta'^\top \sigma(x, x') , \ \theta_\alpha \theta'_\beta', \]
restricted to the hyperboloid by
\[ T_{\mu\nu}(X, X') = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x'^\beta'}{\partial X'^\nu} T_{\alpha\beta'}(x, x'). \]

For \( Z = -H^2 x \cdot x' = \cos(\sigma) \), one can find
\[ n_\mu = \frac{\partial x^\alpha}{\partial X^\mu} \bar{\bar{\theta}}_\alpha \sigma(x, x') = \frac{\partial x^\alpha}{\partial X^\mu} \frac{x \cdot \theta_\alpha}{\sqrt{1 - Z^2}}, \]
\[ n_\nu' = \frac{\partial x'^\beta'}{\partial X'^\nu} \bar{\bar{\theta}}'_{\beta'} \sigma(x, x') = \frac{\partial x'^\beta'}{\partial X'^\nu} \frac{x' \cdot \theta'_{\beta'}}{\sqrt{1 - Z^2}}, \]
\[ \nabla_\mu n_\nu = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x'^\beta'}{\partial X'^\nu} \frac{\partial x \cdot \theta_\alpha}{\sqrt{1 - Z^2}}, \]
\[ \nabla_\nu n_\mu = \frac{\partial x'^\beta'}{\partial X'^\nu} \frac{\partial x \cdot \theta_\alpha}{\sqrt{1 - Z^2}}, \]
with \( c^{-1}(Z) = -\frac{1}{\sqrt{1 - Z^2}} \). In the case \( Z = \cosh(\sigma) \), \( n_\mu \) and \( n_\nu \) are multiplied by \( i \) and \( c(Z) \) becomes \(-i \sqrt{1 - Z^2}\). In both cases we have
\[ g_{\mu\nu} + (Z - 1)n_\mu n_\nu = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x'^\beta'}{\partial X'^\nu} \theta_\alpha \cdot \theta'_{\beta'}. \]
and the two-point functions are related through
\[ Q_{\mu\nu\rho\sigma}(X, X') = \frac{\partial x^\alpha}{\partial X^\mu} \frac{\partial x^\beta}{\partial X^\nu} \frac{\partial x'^\alpha'}{\partial X'^\rho} \frac{\partial x'^\beta'}{\partial X'^\sigma} W_{\alpha\beta\alpha'\beta'}(x, x'). \]

By using these equations, the relation between the two-point functions in the two formalisms can be obtained directly [12, 14, 15, 24, 46, 54].

X. CONCLUSION AND OUTLOOK

In this paper the quantum field theory (including the gauge theory) is reformulated in ambient space formalism. This formalism allows us to construct the quantum field operator in a rigorous mathematical framework based on complexified pseudo-Riemannian manifold and the group representation theory. The linear quantum field theory was studied in this paper and the procedure of defining the Lagrangian for the non-linear terms was presented in ambient space formalism. Some interesting results of this formalism are:

- Since the action of the dS group on the ambient space coordinate \( x^\alpha \) is linear, QFT in this formalism has a very simple form, in comparison with its form in the intrinsic coordinate \( X^\mu \) where the action of the dS group is non-linear.

- Using the dS plane waves in ambient space formalism, the construction of quantum field theory in dS space is very similar to its counterpart in the Minkowski space-time.
• The dS space predicts the existence of a maximum length for an observable (or equivalently, an "event horizon") in the $x$-space. The uncertainty principle results in to the existence of a minimum size in the $\xi$-space. We know that the total volume of this parameter in Hilbert space or $\xi$-space, is finite. As a direct consequence, the total number of quantum states in the Hilbert space turns out to be finite physically \[9\]. This is the one of the most important results of this formalism.

• If we assume that the early universe had conformal symmetry and after the symmetry breaking the fields become massive, then fields with $j > 2$ are banned to exist in the dS universe. It means that the massless fields with $s = j = p > 2$ cannot propagate on the dS space in our formalism. Nonetheless, only fields with spins $j = 0, \frac{1}{2}, 1, \frac{3}{2}$ and 2 have natural presence in our universe.

• There are three types of gauge fields in our formalism, characterized by $j = p = 1, \frac{3}{2}$, and 2 with two of them ($j = p = 1, 2$) being tensor fields. Gauge vector fields $K_{\alpha}(x)$ with $j = p = 1$ can be considered as the potential of the electromagnetic, weak and strong nuclear forces. The spin-2 gauge field $K_{\alpha\beta\gamma}(x)$ in the gauge gravity model can be considered as a part of gravitational field. The case $j = p = \frac{3}{2}$, corresponds to a vector-spinor gauge potential $\Psi_{\alpha}(x)$ with anti-commutating operators and therefore, coupled to spinorial generator that would naturally satisfy some anti-commutation relations. The consequence is the self-appearance of a super-algebra.

• The spin-2 gauge potential $K_{\alpha\beta\gamma}(x)$ can be quantized and their quantum field operators transform according to the indecomposable representations of dS group. The physical states transforms simultaneously as the UIR of dS and conformal groups.

• One can construct the Hilbert space and then the Fock space for the universe including the gauge potentials $K_{\alpha\beta\gamma}(x)$ and $\Psi_{\alpha}(x)$ as gravitational fields. The total number of quantum states for our universe turns out to be finite and hence the Hilbert space for interaction fields can be constructed from the perturbation theory.

• Previously, the Krein space quantization (Hilbert space $\oplus$ anti-Hilbert space) has been used for the extraction of the covariant quantization of minimally coupled scalar fields that suppresses the infrared divergences but breaks the analyticity \[19\]. To remedy this later drawback, we have used the identity (VII.10) in ambient space formalism. This innovative formalism permits one to quantize the massless minimally coupled scalar field in terms of a massless conformally coupled scalar field. It means that the quantum field operator (VII.11) and the two-point function (VII.13) of minimally scalar field can be constructed on the Bunch-Davies vacuum state. Then the problem of infrared divergence of the linear gravity in the dS space is completely solved on the Bunch-Davies vacuum. The theory is also analytic.

• The quantum massless tensor (-spinor) field can be reformulated in terms of a polarization tensor (-spinor) and a massless conformally coupled scalar field. Its analytic two-point function can also be written in terms of the analytic two-point function of conformally coupled scalar field and a polarization tensor which can be obtained when the indecomposable representation becomes fixed.
• In the Dirac six-cone formalism the conformal invariance of the theory becomes manifest. One of the interesting properties of Dirac six-cone formalism is that the tensor and spinor fields on the cone can be simply mapped to the massless fields on de Sitter space-time in the ambient space formalism. It means that there is an exact correspondence between massless conformally fields on the dS ambient space formalism with tensor (or spinor) fields on the Dirac six-cone formalism.

The dS gauge gravity in the null curvature limit becomes the Poincaré gauge gravity. The Poincaré gauge gravity can be reduced to the Einstein general relativity in the torsion free and metric compatible limit. In this paper using the gauge principle we present two perspectives for defining the gravitational field: (1) The gravitational field is constructed of two parts, the dS background $g^{dS}_{\mu\nu}(X)$ or equivalently $\theta_{\alpha\beta}(x)$ in ambient space formalism and the massless spin-2 rank-3 mixed symmetric tensor field $K^{M}_{\alpha\beta\gamma}(x)$. The effect of the first part is in the definitions of the Hilbert space for the free fields. The second part can be considered as a quantum gravitational field, which propagates on the dS light cone. In this perspective the vector-spinor gauge field $\Psi_{\alpha}(x)$ is absent and for unification of the gravitational field with the other forces we need a supplementary principle i.e. the super-gravity principle. (2) The gravitational field is constructed of three parts: the dS background $\theta_{\alpha\beta}$, the spin-2 gauge potential $K^{M}_{\alpha\beta\gamma}$ and the vector-spinor gauge potential $\Psi_{\alpha}$. In this perspective the super-algebra naturally appears. A detailed discussion of these subjects will be discussed in the forthcoming papers.

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