Tutte Polynomial of Pseudofractal Scale-Free Web

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Abstract The Tutte polynomial of a graph is a 2-variable polynomial which is quite important in both Combinatorics and Statistical physics. It contains various numerical invariants and polynomial invariants, such as the number of spanning trees, the number of spanning forests, the number of acyclic orientations, the reliability polynomial, chromatic polynomial and flow polynomial. In this paper, we study and obtain a recursive formula for the Tutte polynomial of pseudofractal scale-free web (PSFW), and thus logarithmic complexity algorithm to calculate the Tutte polynomial of the PSFW is obtained, although it is NP-hard for general graph. By solving the recurrence relations derived from the Tutte polynomial, the rigorous solution for the number of spanning trees of the PSFW is obtained. Therefore, an alternative approach to determine explicitly the number of spanning trees of the PSFW is given. Furthermore, we analyze the all-terminal reliability of the PSFW and compare the results with those of the Sierpinski gasket which has the same number of nodes and edges as the PSFW. In contrast with the well-known conclusion that inhomogeneous networks (e.g., scale-free networks) are more robust than homogeneous networks (i.e., networks in which each node has approximately the same number of links) with respect to random deletion of nodes, the Sierpinski gasket (which

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is a homogeneous network), as our results show, is more robust than the PSFW (which is an inhomogeneous network) with respect to random edge failures.

Keywords  Pseudofractal scale-free web · Tutte polynomial · Spanning trees · Reliability polynomial

1 Introduction

The Tutte polynomial is a two-variable polynomial which can be associated with a graph, a matrix, or more generally, with a matroid. This polynomial was introduced by Tutte [1–3] and has many interesting applications in several areas of sciences such as Combinatorics, Probability, Statistical mechanics and Computer sciences. It is quite interesting since several combinatorial, enumerative and algebraic properties of the graph can be investigated by considering special evaluations of it [4]. For instance, one gets information about the number of spanning trees [5,6], spanning connected subgraphs [7], spanning forests [8] and acyclic orientations [9] of the graph by evaluating the Tutte polynomial at particular points \((x, y)\). Moreover, the Tutte polynomial contains several other polynomial invariants, such as flow polynomial [10], reliability polynomial [11] and chromatic polynomial [2,12,13]. It has also many interesting connections with statistical mechanical models such as the Potts model [14,15] and the percolation [16,17]. Despite its ubiquity, there are no widely-available effective tools to compute the Tutte polynomial of a general graph with reasonable size. It is shown that, many of the relevant coefficients do not even have efficient approximation schemes and various decision problems based on the coefficients are NP-hard [18,19]. Although it is hard to compute the Tutte polynomial, considerable attention has been paid to the study of the Tutte polynomials for different graphs such as polygon chain graphs [15], Sierpinski gaskets [20] and strips of lattices [21–24]. As for the Tutte polynomials for networks with scale-free [25] and small-world [26] properties, related research was rarely reported.

The pseudofractal scale-free web (PSFW) we studied is a deterministically growing network introduced by Dorogovtsev [27]. It was introduced to model scale-free network with small-world effect. In the past several years, much effort has been devoted to the study of its properties, such as degree distribution, degree correlation, clustering coefficient [27,28], diameter [28], average path length [29], the number of spanning trees [30] and mean first-passage time [31], the Tutte polynomial and reliability polynomial of the PSFW are still unresolved.

In this paper, we analyze and obtain a recursive formula for the Tutte polynomial of the PSFW. The analytic method is based on its recursive construction and self-similar structure. Recursive formulas for various invariants of the Tutte polynomial can also be obtained by analyzing their connections with the Tutte polynomial. We also obtain the rigorous solution for the number of spanning trees by solving the recurrence relations derived from the Tutte polynomial. The result is consistent with that obtained in Ref. [30]. Thus we give an alternative approach to determine explicitly the number of spanning trees of the PSFW. Also, the correctness of our method is verified. Furthermore, we analyze the all-terminal reliability of the PSFW and compare the results with those of the Sierpinski gasket, which has the same number of nodes and edges as the PSFW. In contrast with the well-known conclusion that inhomogeneous networks (e.g., scale-free networks) are more robust than homogeneous networks (i.e., networks in which each node has approximately the same number of links) against random node removal [32–34], the Sierpinski gasket (which is a homogeneous net-
work), as we show, is more robust than the PSFW (which is an inhomogeneous network) against random edge failures.

2 Structure of the PSFW

The PSFW we studied is a deterministically growing network which can be constructed iteratively [27]. Let $G(n)$ denote the PSFW of generation $n$ ($n \geq 0$). For $n = 0$, $G(0)$ is a triangle. For $n \geq 1$, at each step which $G(n)$ is obtained from $G(n - 1)$, to every edge of $G(n - 1)$, a new node is added, which is attached to both the end nodes of the edge. The constructions of the first three generations are shown in Fig. 1. The PSFW exhibits some typical properties of real networks. Its degree distribution $P(k)$ obeys a power law $P(k) \sim k^{-(1+\ln 3/\ln 2)}$ [27], the average path length scales logarithmically with network order [29]. The network also has an equivalent construction method [30,35] as shown in Fig. 2: $G(n + 1)$ is composed of three copies of $G(n)$ denoted by $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, the three hub nodes of which are represented by $A$, $B$, $C$ in the corresponding triangle (i.e., the left side of the Fig. 2). In the merging process, hub node $C$ of $\Gamma_1$ and hub node $A$ of $\Gamma_3$, hub node $B$ of $\Gamma_3$ and hub node $C$ of $\Gamma_2$, hub node $A$ of $\Gamma_2$ and hub node $B$ of $\Gamma_1$ are identified as a hub node $A$, $B$, $C$ of $G(n + 1)$ respectively (i.e., the right side of Fig. 2). Then, at each step, the total number of edges in the network increases by a factor of 3. Thus, the total number of edges for $G(n)$ is $E_n = 3^n + 1$. We can also find that the total number of nodes for $G(n)$ is $V_n = \frac{3^{n+1}+3}{2}$.

3 Tutte Polynomial of the PSFW

As usual, $G = (V(G), E(G))$ denotes a graph with vertex set $V(G)$ and edge set $E(G)$; we will often write $V$ and $E$, when there is no risk of confusion, and so $G = (V, E)$. A subgraph $H = (V(H), E(H))$ of a graph $G = (V(G), E(G))$ is said spanning if the condition

![Fig. 1](image1.png) Growth process for the PSFW from $n=0$ to $n=2$

![Fig. 2](image2.png) Alternative construction method of the PSFW which highlights self-similar
\( V(H) = V(G) \) is satisfied. In particular, a spanning tree of \( G \) is a spanning subgraph of \( G \) which is a tree. Let \( H \) be a spanning subgraph of \( G \) and \( k(H) \) be the number of connected components of \( H \). Then the rank \( r(H) \) and the nullity \( n(H) \) of \( H \) are defined as \([4]\)

\[
\begin{align*}
   r(H) &= |V(H)| - k(H), \\
   n(H) &= |E(H)| - r(H) = |E(H)| - |V(H)| + k(H),
\end{align*}
\]

where \(|S|\) denotes the number of objects in set \( S \).

**Definition 1** Let \( G = (V, E) \) be a graph. The Tutte polynomial \( T(G; x, y) \) of \( G \) is defined as \([4]\)

\[
T(G; x, y) = \sum_H (x - 1)^{r(G) - r(H)} (y - 1)^{n(H)},
\]

where the sum runs over all the spanning subgraphs \( H \) of \( G \).

Let \( T_n(x, y) \) denotes the Tutte polynomial of the PSFW \( G(n) \) and \( D_n \) denotes the set of spanning subgraphs of \( G(n) \), and for any subgraph \( H \in D_n \),

\[
\Phi_n(H) \equiv (x - 1)^{r(G(n)) - r(H)} (y - 1)^{n(H)}.
\]

Then

\[
T_n(x, y) = \sum_{H \in D_n} \Phi_n(H). \tag{4}
\]

But \( T_n(x, y) \) doesn’t satisfy any recurrence relation directly. We split it into five separate polynomials which satisfy recurrence relation. The split policy is to divide \( D_n \) into five different subsets and define corresponding polynomials on them. The five different subsets of \( D_n \) are defined as:

- \( D_{1,n} \) denotes the set of spanning subgraphs of \( G(n) \), where the three hub nodes \( A, B, C \) belong to the same connected component;
- \( D_{2,n}^C \) denotes the set of spanning subgraphs of \( G(n) \), where the hub nodes \( A \) and \( B \) belong to the same connected component, but \( C \) belongs to a different component. Similarly, \( D_{2,n}^A \) (or \( D_{2,n}^B \)) denotes the set of spanning subgraphs of \( G(n) \), where \( A \) (or \( B \)) does not belong to the connected component which contains the other two hub nodes;
- \( D_{3,n} \) denotes the set of spanning subgraphs of \( G(n) \), where the three hub nodes belong to three different connected components.

Thus, for any \( n \geq 0 \),

\[
D_n = D_{1,n} \cup D_{2,n}^A \cup D_{2,n}^B \cup D_{2,n}^C \cup D_{3,n}. \tag{5}
\]

Correspondingly, the five polynomials are defined by:

\[
\begin{align*}
   T_{1,n}(x, y) &= \sum_{H \in D_{1,n}} \Phi_n(H), & T_{2,n}^A(x, y) &= \sum_{H \in D_{2,n}^A} \Phi_n(H), \\
   T_{2,n}^B(x, y) &= \sum_{H \in D_{2,n}^B} \Phi_n(H), & T_{2,n}^C(x, y) &= \sum_{H \in D_{2,n}^C} \Phi_n(H), \\
   T_{3,n}(x, y) &= \sum_{H \in D_{3,n}} \Phi_n(H).
\end{align*}
\]
By symmetry, we have
\[ T_{2,n}^A(x, y) = T_{2,n}^B(x, y) = T_{2,n}^C(x, y). \]
Therefore we can simply use \( T_{2,n}(x, y) \) to denote one of the three polynomials.
As proved in the Appendix 1, we obtain the following result.

**Theorem 1** For \( n \geq 0 \), the Tutte polynomial \( T_n(x, y) \) of \( G(n) \) is given by
\[ T_n(x, y) = T_{1,n}(x, y) + 3T_{2,n}(x, y) + T_{3,n}(x, y), \] (6)
where \( T_{1,n}(x, y), T_{2,n}(x, y), T_{3,n}(x, y) \) satisfy the following recurrence relation:
\[
\begin{align*}
T_{1,n+1}(x, y) &= (y - 1)T_{1,n}^3 + 3(y - 1)T_{1,n}^2T_{2,n} + 3(y - 1)T_{1,n}T_{2,n}^2 \\
&\quad + (y - 1)T_{2,n}^3 + \frac{1}{x - 1}(3T_{1,n}^2T_{3,n} + 6T_{1,n}^2T_{2,n} \\
&\quad + 12T_{1,n}T_{2,n}^2 + 6T_{2,n}^3 + 3T_{3,n}T_{2,n}^2 + 6T_{1,n}T_{2,n}T_{3,n}) \\
T_{2,n+1}(x, y) &= \frac{1}{x - 1}(4T_{1,n}T_{2,n}^2 + 4T_{1,n}T_{2,n}T_{3,n} + 4T_{3,n}^2 \\
&\quad + 4T_{2,n}^2T_{3,n} + T_{1,n}T_{3,n}^2 + T_{2,n}T_{3,n}^2) \\
T_{3,n+1}(x, y) &= \frac{1}{x - 1}(8T_{2,n}^3 + 12T_{2,n}^2T_{3,n} + 6T_{2,n}T_{3,n}^2 + T_{3,n}^3)
\end{align*}
\] (7)
with initial conditions
\[ T_{1,0}(x, y) = y + 2, \quad T_{2,0}(x, y) = x - 1, \quad T_{3,0}(x, y) = (x - 1)^2, \]
while \( T_{1,n}, T_{2,n}, T_{3,n} \) are the abbreviations of \( T_{1,n}(x, y), T_{2,n}(x, y), T_{3,n}(x, y) \) respectively.

**Lemma 1** For any \( n \geq 0 \), \( x - 1 \) is a factor of the polynomial \( T_{2,n}(x, y) \) and \( (x - 1)^2 \) is a factor of the polynomial \( T_{3,n}(x, y) \). As a consequence, we can write
\[
\begin{align*}
T_{2,n}(x, y) &= (x - 1)P_n(x, y) \\
T_{3,n}(x, y) &= (x - 1)^2Q_n(x, y)
\end{align*}
\] (8)
where \( P_n(x, y), Q_n(x, y) \) are polynomials of \( x \) and \( y \).

The lemma is proved in Appendix 2.

Substituting \( T_{2,n}(x, y), T_{3,n}(x, y) \) from Eq. (8) in Eq. (7), we obtain the following Theorem.

**Theorem 2** For \( n \geq 0 \), the Tutte polynomial \( T_n(x, y) \) of \( G(n) \) is given by
\[ T_n(x, y) = T_{1,n}(x, y) + 3(x - 1)P_n(x, y) + (x - 1)^2Q_n(x, y), \] (9)
where \( T_{1,n}(x, y), P_n(x, y), Q_n(x, y) \) satisfy the following recurrence relation:
\[
\begin{align*}
T_{1,n+1} &= (y - 1)T_{1,n}^3 + 3(y - 1)(x - 1)T_{1,n}^2P_n \\
&\quad + 3(x - 1)^2(y - 1)T_{1,n}P_n^2 + (x - 1)^3(y - 1)P_n^3 \\
&\quad + 3(x - 1)T_{1,n}^2Q_n + 6T_{1,n}^2P_n + 12(x - 1)T_{1,n}P_n^2 \\
&\quad + 6(x - 1)^2P_n^3 + 3(x - 1)^3P_n^2Q_n + 6(x - 1)^2T_{1,n}P_nQ_n + 4(x - 1)^2P_n^3 \\
&\quad + 4(x - 1)^2P_n^2Q_n + (x - 1)^2T_{1,n}Q_n^2 + (x - 1)^3P_nQ_n^2 \\
Q_{n+1} &= 8P_n^3 + 12(x - 1)P_n^2Q_n + 6(x - 1)^2P_nQ_n^2 + (x - 1)^3Q_n^3
\end{align*}
\] (10)

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with initial conditions
\[ T_{1,0}(x, y) = y + 2, \quad P_0(x, y) = Q_0(x, y) = 1, \]

while \( T_{1,n}, P_n, Q_n \) are the abbreviations of \( T_{1,n}(x, y), P_n(x, y), Q_n(x, y) \) respectively.

**Remark** Although it is NP-hard to calculate the Tutte polynomial for general graph, the recurrence relations we obtain show that we can calculate the Tutte polynomial of the PSFW with time complexity \( O(n) = O(\log(V_n)) \). Thus, we have obtained logarithmic complexity algorithm to calculate the Tutte polynomial of the PSFW.

We can also obtain the recursive formulas for various invariants of the Tutte polynomial by analyzing their connections with the Tutte polynomial, such as the number of spanning trees, the number of connected spanning subgraphs, the number of spanning forests, the number of acyclic orientations, the reliability polynomial and the chromatic polynomial. In this paper, we only analyze the number of spanning trees and the reliability polynomial.

### 4 Exact Result for the Number of Spanning Trees of the PSFW

Let \( N_{ST}(n) \) denotes the number of spanning trees of the PSFW \( G(n) \). Then [4]
\[ N_{ST}(n) = T_n(1, 1). \]

Let \( x = 1 \) and \( y = 1 \) in Eq. (9), we obtain
\[ N_{ST}(n) = T_{1,n}(1, 1). \]

Let \( x = 1 \) and \( y = 1 \) in Eq. (10), the following recurrence relations hold for any \( n \geq 0 \).
\[
\begin{cases}
N_{ST}(n + 1) = 6[N_{ST}(n)]^2 P_n \\
P_{n+1} = 4N_{ST}(n) P_n^2
\end{cases}
\tag{11}
\]

where \( P_n \) is abbreviation of \( P_n(1, 1) \), and the initial conditions are
\[ N_{ST}(0) = T_{1,0}(1, 1) = 3, \quad P_0 = 1. \]

Using Eq. (11) recursively for \( k (n \geq k \geq 1) \) times, we find that \( N_{ST}(n) \) can be expressed as
\[ N_{ST}(n) = 6^{a_k} 4^{b_k} [N_{ST}(n - k)]^{c_k} P_{n-k}^{d_k}, \tag{12} \]

with
\[ a_1 = 1, \quad b_1 = 0, \quad c_1 = 2, \quad d_1 = 1. \]

We can also find that
\[
N_{ST}(n) = 6^{a_k-1} 4^{b_k-1} [N_{ST}(n - k + 1)]^{c_k-1} P_{n-k+1}^{d_k-1} \\
= 6^{a_k-1} 4^{b_k-1} [6[N_{ST}(n - k)]^2 P_{n-k}^2]^{c_k-1} [4N_{ST}(n - k) P_{n-k}^2]^{d_k-1}.
\]
\[ \begin{align*}
&= 6^{a_{k-1}} 4^{b_{k-1}} [N_{ST}(n-k)]^{2c_{k-1}+d_{k-1}} P^{e_{k-1}+2d_{k-1}}_{n-k} \\
&= 6^{a_k} 4^{b_k} [N_{ST}(n-k)]^{c_k} P^{d_k}_{n-k},
\end{align*} \]

where the second line of Eq. (13) is obtained by replacing \( N_{ST}(n-k+1) \) and \( P_{n-k+1} \) from Eq. (11). Comparing the last two lines of Eq. (13), we find that \( a_k, b_k, c_k, d_k (k \geq 1) \) satisfy the following recurrence relations

\[
\begin{cases}
    a_k = a_{k-1} + c_{k-1} \\
    b_k = b_{k-1} + d_{k-1} \\
    c_k = 2c_{k-1} + d_{k-1} \\
    d_k = c_{k-1} + 2d_{k-1}
\end{cases}
\]

Thus

\[
\begin{cases}
    c_k + d_k = 3(c_{k-1} + d_{k-1}) = 3^{k-1}(c_1 + d_1) = 3^k \\
    c_k - d_k = c_{k-1} - d_{k-1} = c_1 - d_1 = 1
\end{cases}
\]

Therefore

\[
c_k = \frac{3^k + 1}{2},
\]

and

\[
d_k = \frac{3^k - 1}{2}.
\]

Note that the first two equations of Eq. (14) can be rewritten by

\[
\begin{align*}
ak - a_{k-1} &= c_{k-1}, \\
b_k - b_{k-1} &= d_{k-1}.
\end{align*}
\]

Therefore

\[
\begin{align*}
&= \sum_{i=1}^{k-1} (a_{i+1} - a_i) = \sum_{i=1}^{k-1} c_i = \frac{k - 1}{2} + \frac{3^k - 3}{4}, \\
&= \sum_{i=1}^{k-1} (b_{i+1} - b_i) = \sum_{i=1}^{k-1} d_i = -\frac{k - 1}{2} + \frac{3^k - 3}{4}.
\end{align*}
\]

Hence

\[
\begin{align*}
ak &= \frac{k + 1}{2} + \frac{3^k - 3}{4}, \\
b_k &= -\frac{k - 1}{2} + \frac{3^k - 3}{4}.
\end{align*}
\]

Thus, for any \( n \geq 0 \), the number of spanning trees of the PSFW \( G(n) \) is given by

\[
N_{ST}(n) = 6^{a_n} 4^{b_n} [N_{ST}(0)]^{c_n} P^{d_n}_{0} = 2^{\frac{3n+1-2n-3}{4}} 3^{\frac{3^{n+1}+4n+1}{4}} \frac{3^{n+1}+4n+1}{2} \frac{3^{n-1}}{4} \frac{3^{n-1}}{2} \frac{3^{n-1}}{4}.
\]
The result is consistent with that obtained in Ref. [30], which is smaller than the number of spanning trees for the Sierpinski gasket [36]. Thus we give an alternative approach to determine explicitly the number of spanning trees for the PSFW. The result also verifies the correctness of our result.

5 Reliability Analysis of the PSFW

5.1 All-Terminal Reliability of the PSFW

In this section, we look upon the PSFW $G(n)$ as a probabilistic graph. Each edge of $G(n)$ has a known probability $p$ ($0 < p < 1$) of being operational, otherwise it is failed. Operations of different edges are statistically independent, while the nodes of $G(n)$ never fail. The all-terminal reliability $R(G(n), p)$ of $G(n)$ is defined as the probability that there is a path of operational edges between any pair of vertices of $G(n)$. It is a polynomial of $p$. The connection between the Tutte polynomial and all-terminal reliability is given by [4]

$$R(G(n), p) = p^{V_n - 1}(1 - p)^{E_n - V_n + 1}T_n \left(1, \frac{1}{1 - p}\right),$$  

(21)

where $V_n, E_n$ denote the total number of nodes and edges for $G(n)$ respectively. It is easy to know from Eq. (9) that

$$T_n \left(1, \frac{1}{1 - p}\right) = T_{1,n} \left(1, \frac{1}{1 - p}\right).$$

Thus

$$R(G(n), p) = p^{V_n - 1}(1 - p)^{E_n - V_n + 1}T_{1,n} \left(1, \frac{1}{1 - p}\right).$$  

(22)

Let us simply denote by $T_{1,n}, P_n$, the expression $T_{1,n}(1, \frac{1}{1 - p}), P_n \left(1, \frac{1}{1 - p}\right)$ respectively. For $n \geq 0$, inserting $x = 1, y = \frac{1}{1 - p}$ into Eq. (10), we obtain the following recurrence relations:

$$\begin{align*}
T_{1,n+1} \left(1, \frac{1}{1 - p}\right) &= \frac{p}{1 - p} T_{1,n}^3 + 6T_{1,n}^2 P_n, \\
P_{n+1} \left(1, \frac{1}{1 - p}\right) &= 4T_{1,n} P_n^2
\end{align*}$$  

(23)

with initial conditions

$$T_{1,0} \left(1, \frac{1}{1 - p}\right) = \frac{3 - 2p}{1 - p}, \quad P_0 \left(1, \frac{1}{1 - p}\right) = 1.$$  

For $n \geq 0$, let

$$B(n) = p^{(V_n - 2)}(1 - p)^{(E_n - V_n + 2)} P_n.$$  

(24)
We obtain the following recurrence relation from Eqs. (22), (23), (24):

\[
\begin{align*}
R(n + 1) &= p^{V_n+1} - 1(1 - p)^{E_n+1} - V_n+1 + 1 T_{1,n+1} \\
&= p^{3V_n - 4} - 1(1 - p)^{3E_n - 3V_n + 4}(\frac{p}{1-p}) T_{1,n}^3 + 6T_{1,n} P_n \\
&= p^{3(V_n - 1)} - 1(1 - p)^3(E_n - V_n + 1) T_{1,n}^3 \\
&+ 6p^2(V_n - 1)(1 - p)^2(E_n - V_n + 1) T_{1,n}^2 B_n(V_n - 2)(1 - p)(E_n - V_n + 2) P_n \\
&= R^3(n) + 6R^2(n) B(n), \\
B(n + 1) &= p^{(V_n+1 - 2)}(1 - p)^{(E_n+1 - V_n+1 + 2)} P_{n+1} \\
&= 4p^{3(V_n - 5)}(1 - p)^{(3E_n - 3V_n + 5)} T_{1,n}^2 P_n^2 \\
&= 4p^{V_n - 1}(1 - p)^{E_n - V_n + 1} T_{1,n} [p^{(V_n - 2)}(1 - p)(E_n - V_n + 2) P_n]^2 \\
&= 4R(n) B^2(n),
\end{align*}
\]

where \( R(n) \) is the abbreviation of reliability polynomial \( R(G(n), p) \) and the initial conditions satisfy

\[
R(0) = p^2(3 - 2p), \quad B(0) = p(1 - p)^2.
\]

Note that \( R(n) \gg B(n) \) while \( n \to \infty \), ones get

\[
\begin{align*}
R(n + 1) + 2B(n + 1) &= R(n)^3 + 6R(n)^2 B(n) + 8R(n) B(n)^2 \\
&= (R(n) + 2B(n))^3 - 4(R(n) + 2B(n)) B(n)^2 \\
&\approx (R(n) + 2B(n))^3.
\end{align*}
\]

Thus

\[
R(n) \approx R(n) + 2B(n) \approx (R(n - 1) + 2B(n - 1))^3 \\
\approx (R(0) + 2B(0))^3 = [p^2(3 - 2p)]^3 \approx [p^2(3 - 2p)]^\frac{3}{2} V_n,
\]

which shows that all terminal reliability decreases approximately as an exponential function of network order \( V_n \). The reason we don’t use \( R^3(n - 1) \) as a approximation of \( R(n) \) is that it has larger relative error than \( (R(n - 1) + 2B(n - 1))^3 \).

**Remark** If we denote by \( \mathcal{A} \) the collection of edges sets for spanning tree of \( G(n) \) whose edge have a probability \( 1 - p \) of being failed, \( R(G(n), p) \) is just the percolation probability for percolation model on \( \mathcal{A} \) [17]. Hence, the percolation threshold for the percolation model is \( P_c = 1 \) because \( \lim_{n \to \infty} R(G(n), p) = 0 \). But more significant problems, such as percolation threshold, percolation probability, average cluster size, etc, for site percolation and bond percolation on the PSFW, are still unsolved.

5.2 Comparison of All-Terminal Reliability Between the Sierpinski Gasket and the PSFW

The Sierpinski gasket is a famous fractal which attracted considerable attention [37]. If we look upon the Sierpinski gasket as a network, it is a deterministically growing network which has the same starting point as the PSFW. But the method of iteration is different from the PSFW [38]. The construction process of the first three generation is shown in Fig. 3. We denote the Sierpinski gasket of generation \( n (n \geq 0) \) by \( SG(n) \). It has the same number of nodes and edges as the PSFW for any \( n \geq 0 \), but the structure is quite different. For the Sierpinski gasket, except the 3 outmost nodes whose degrees are 2, the degrees of all other vertices are 4. Therefore \( SG(n) \) is almost 4-regular and it is a homogeneous network (i.e.,
networks in which each node has approximately the same number of links) [32]. But the PSFW is an inhomogeneous network (scale free network) whose degree distribution obeys power law. Thus, the Sierpinski gasket and the PSFW are typical examples of homogeneous network and inhomogeneous network which have the same number of nodes and edges.

As we all know, inhomogeneous networks, such as scale-free networks, are more robust than homogeneous networks with respect to random deletion of nodes [32, 39, 40]. Thus, the PSFW is more robust than the Sierpinski gasket again random node failures. But the robustness with respect to node removal does not implies robustness with respect to link removal [41].

Noticing that all-terminal reliability is a good measure of robustness with respect to random deletion of links, we compare all-terminal reliability between the PSFW and the Sierpinski gasket. For $n \geq 0$, let us simply denote by $R_s(n)$, $R(n)$ the all-terminal reliability of the Sierpinski gasket and the PSFW respectively. As derived in Appendix 3,

$$R_s(0) = R(0), \quad R_s(1) = R(1),$$

and for $n \geq 2$

$$R_s(n) > R(n). \quad (29)$$

We have also checked our analytic result against numerical calculations for different $p \in (0, 1)$ and $n$. we find that $R_s(n)$ and $R(n)$ converge to 0 quickly with $n \to \infty$ and Eq. (29) holds for $n \geq 2$. The results for $n = 6$ are shown in Fig. 4. In contrast with the well-known conclusion that inhomogeneous networks (e.g., scale-free networks) are more robust than homogeneous networks with respect to random node removal , the Sierpinski gasket (which is a homogeneous network), as we show, is more robust than the PSFW (which is an inhomogeneous network) with respect to random edge failures. Thus, we obtain an example which shows that homogeneous network (e.g. , the Sierpinski gasket) is more robust than inhomogeneous network (e.g. , the PSFW) with respect to random edge failures.

6 Conclusion

In this paper, we study and obtain recursive formula for the Tutte polynomial of the PSFW, which implies that recursive formulas for various invariants of Tutte polynomial can also be obtained by analyzing their connections with the Tutte polynomial. We also obtain the rigorous solution for the number of spanning trees of the PSFW by solving the recurrence relation derived from Tutte polynomial, and an alternative approach to determine explicitly the number of spanning trees of the PSFW is given. Furthermore, we analyze the all-terminal reliability of the PSFW based on the the recurrence relations derived from Tutte polynomial and compare the results with those of the Sierpinski gasket. In contrast with the well-known

![Fig. 3 Growth process for the Sierpinski gasket $SG(n)$ from $n=0$ to $n=2$](image-url)
conclusion that inhomogeneous networks are more robust than homogeneous networks with respect to random deletion of nodes, the Sierpinski gasket (which is a homogeneous network), as our results show, is more robust than the PSFW (which is an inhomogeneous network) against random edge failures.

Having the Tutte polynomial of the PSFW, some further works might be the percolation problems on the PSFW, such as percolation threshold, percolation probability and average cluster size, etc.

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Appendix 1: A Proof of Theorem 1

In order to derive recursive formula of $T_n(x, y)$, we must analyze the relation of spanning subgraphs between $G(n + 1)$ and its three subgraphs $\Gamma_i (i = 1, 2, 3)$ which are a copy of $G(n)$ respectively.

For any $H \in D_{n+1}$, it is a spanning subgraphs of $G(n + 1)$. Let

$$ E_i = E(H) \cap E(\Gamma_i), V_i = V(H) \cap V(\Gamma_i), i = 1, 2, 3. \quad (30) $$

We find that $H_i = (V_i, E_i)$ is a spanning subgraphs of $\Gamma_i$ which is a copy of $G(n)$, thus $H_i \in D_n$, for $i = 1, 2, 3$.

On the contrary, given three spanning subgraphs $H_1, H_2$ and $H_3$ of $\Gamma_1, \Gamma_2$ and $\Gamma_3$. Let

$$ E = E(H_1) \cup E(H_2) \cup E(H_3), V = V(H_1) \cup V(H_2) \cup V(H_3). \quad (31) $$

We find that $H = (V, E)$ is a spanning subgraphs of $G(n + 1)$ and there exists a bijection between spanning subgraphs of $G(n + 1)$ and spanning subgraphs of $\Gamma_1, \Gamma_2, \Gamma_3$ inside $G(n + 1)$. Therefore

$$ T_{n+1}(x, y) = \sum_{H \in D_{n+1}} \Phi_{n+1}(H) = \sum_{H_i \in D_n, i=1,2,3} \Phi_{n+1}(H_i). \quad (32) $$
where the relation between \( H = (V, E) \) and \( H_i = (V_i, E_i) \) \( (i = 1, 2, 3) \) are shown in Eqs. (30) and (31). We denote this relation always holds when they appear in the following paragraphs.

We first derive the relations between \( \Phi_{n+1}(H) \) and \( \Phi_n(H_i) \) \( (i = 1, 2, 3) \), and then the recursive formula of \( T_n(x, y) \).

Relations Between \( \Phi_{n+1}(H) \) and \( \Phi_n(H_i) \)

Now, we first derive recursive formula of \( r(G(n)) \), and the relations between \( r(H) \) and \( r(H_i) \), \( n(H) \) and \( n(H_i) \). Then we can obtain relations between \( \Phi_{n+1}(H) \) and \( \Phi_n(H_i) \).

Because the PSFW is a connected graph, therefore \( k(G(n)) = 1 \), for any \( n \geq 0 \). Note that \( G(n + 1) \) is composed of three copies of \( G(n) \) denoted as \( \Gamma_1, \Gamma_2, \Gamma_3 \), we have

\[
|V(G(n + 1))| = |V(\Gamma_1)| + |V(\Gamma_2)| + |V(\Gamma_3)| - 3 = 3|V(G(n))| - 3.
\]

Thus

\[
r(G(n + 1)) = |V(G(n + 1))| - k(G(n + 1))
= (3|V(G(n))| - 3) - 1
= 3(|V(G(n))| - 1) - 1
= 3r(G(n)) - 1. \tag{33}
\]

According to the second construction algorithm of the PSFW, we have

\[
|V(H)| = |V(H_1)| + |V(H_2)| + |V(H_3)| - 3, \tag{34}
\]

\[
|E(H)| = |E(H_1)| + |E(H_2)| + |E(H_3)|. \tag{35}
\]

As for the relation between \( r(H) \) and \( r(H_i) \), \( n(H) \) and \( n(H_i) \), they can be divided into two cases. Thus relations between summation term of \( T_{n+1}(x, y) \) and that of \( T_n(x, y) \) can also be divided into two cases.

**Case 1** The hub nodes of \( G(n + 1) \) belong to the same connected component in the spanning subgraph \( H_i \) of \( G_i \), for any \( i = 1, 2, 3 \) (i.e., hub nodes \( A, C \) of \( G(n + 1) \) belong to the same connected component in \( H_1 \), hub nodes \( B, C \) of \( G(n + 1) \) belong to the same connected component in \( H_2 \), hub nodes \( A, B \) of \( G(n + 1) \) belong to the same connected component in \( H_3 \)).

According to the second construction algorithm of the PSFW, we have

\[
k(H) = k(H_1) + k(H_2) + k(H_3) - 2. \tag{36}
\]

Substituting \( |V(H)|, |k(H)| \) from Eqs. (34) and (36) respectively in Eq. (1),

\[
r(H) = |V(H)| - k(H)
= |V(H_1)| - k(H_1) + |V(H_2)| - k(H_2) + |V(H_3)| - k(H_3) - 1
= r(H_1) + r(H_2) + r(H_3) - 1. \tag{37}
\]

Substituting \( |E(H)|, |r(H)| \) from Eqs. (35) and (37) respectively in Eq. (2),

\[
n(H) = |E(H)| - r(H) = n(H_1) + n(H_2) + n(H_3) + 1.
\]

Let Eq. (33) minus Eq. (37), we obtain

\[
r(G(n + 1)) - r(H) = \sum_{i=1}^{3}(r(G(n)) - r(H_i)).
\]
Hence
\[
\Phi_{n+1}(H) = (x - 1)^{r(G(n+1)) - r(H)}(y - 1)^{n(H)} - r(H)) \prod_{i=1}^{3} (x - 1)^{r(G(n)) - r(H_i)} \Phi_n(H_i).
\]

**Case 2** The conditions for case 1 are not satisfied (i.e., for certain \( i = 1, 2, 3 \), the hub nodes of \( G(n + 1) \) do not belong to the same connected component in the spanning subgraph \( H_i \)), we have
\[
k(H) = k(H_1) + k(H_2) + k(H_3) - 3.
\]

Substituting \(|V(H)|, |k(H)|\) from Eqs. (34) and (39) respectively in Eq. (1), we have
\[
r(H) = |V(H)| - k(H) = |V(H_1)| + |V(H_2)| + |V(H_3)| - k(H_1) - r(H_2) - r(H_3).
\]

Thus
\[
n(H) = |E(H)| - r(H) = n(H_1) + n(H_2) + n(H_3),
\]

and
\[
r(G(n + 1)) - r(H) = \sum_{i=1}^{3} (r(G(n)) - r(H_i)) - 1.
\]

Hence
\[
\Phi_{n+1}(H) = (x - 1)^{r(G(n+1)) - r(H)}(y - 1)^{n(H)} - r(H)) \prod_{i=1}^{3} (x - 1)^{r(G(n)) - r(H_i)} \Phi_n(H_i).
\]

Recursive Formulas of \( T_{1,n}(x, y), T_{2,n}(x, y) \) and \( T_{3,n}(x, y) \)

Here, we proof \( T_{1,n}(x, y), T_{2,n}(x, y), T_{3,n}(x, y) \) satisfy the following recurrence relations:
\[
\begin{align*}
T_{1,n+1} &= (y - 1)T_{1,n}^3 + 3(y - 1)T_{2,n}^2 + 3(y - 1)T_{1,n}T_{2,n}^2 \\
&\quad + (y - 1)T_{2,n}^3 + \left(3T_{1,n}^2T_{3,n} + 6T_{1,n}T_{2,n}ight)T_{2,n}^2 \\
&\quad + 12T_{1,n}T_{2,n}T_{3,n} + 3T_{3,n}T_{2,n}^2 + 6T_{1,n}T_{2,n}T_{3,n}, \\
T_{2,n+1} &= \frac{1}{x-1} \left(4T_{1,n}T_{2,n}^2 + 4T_{1,n}T_{2,n}T_{3,n} + 4T_{2,n}^3 \\
&\quad + 4T_{2,n}^2T_{3,n} + T_{1,n}^2T_{3,n} + T_{2,n}T_{3,n}^2ight), \\
T_{3,n+1} &= \frac{1}{x-1} \left(8T_{2,n}^3 + 12T_{2,n}^2T_{3,n} + 6T_{2,n}T_{3,n}^2 + T_{3,n}^3\right),
\end{align*}
\]
with initial conditions

\[ T_{1,0}(x, y) = y + 2, \quad T_{2,0}(x, y) = x - 1, \quad T_{3,0}(x, y) = (x - 1)^2, \]

where \( T_{1,n}, T_{2,n}, T_{3,n} \) are the abbreviations of \( T_{1,n}(x, y), T_{2,n}(x, y), T_{3,n}(x, y) \) respectively.

The initial conditions are easy to be verified according to the definition of \( T_{1,n}(x, y), T_{2,n}(x, y) \) and \( T_{3,n}(x, y) \). As for the recurrence relations, the strategy of proof is to study all the possible structure of spanning subgraphs \( H \) of \( G(n + 1) \) and analyze which kind of contribution they give to \( T_{1,n+1}(x, y), T_{2,n+1}(x, y) \) and \( T_{3,n+1}(x, y) \).

As defined in Sect. 3, the spanning subgraphs of \( G(n) \) has five different structures (i.e., \( D_1, n, D_2, n, D_2, B, D_2, C, D_3, n \)), thus \( H_i \) also has five different structures, for any \( i = 1, 2, 3 \), and \( H \) has \( 5^3 \) kinds of structure.

In order to depict the relations between the structure of \( H_i \) and \( H \) clearly, we introduce five corresponding notations for the five different structures of \( H_i \) which are shown in Fig. 5. Each structure is denoted by a triangle whose three vertices denote the three hub nodes of \( G(n) \). The hub nodes which are in the same connected component are connected by a solid line, and the hub nodes which are not in the same connected component connected by a dotted line.

Now, we will study all the \( 5^3 \) kinds of structures of \( H \), and analyze which kind of contributions they give to \( T_{1,n+1}(x, y), T_{2,n+1}(x, y) \) and \( T_{3,n+1}(x, y) \).

First, we analyze the possible structures of \( H \) which give contribution to \( T_{1,n+1}(x, y) \). According to the definition of \( T_{1,n+1}(x, y) \), the condition \( H \in D_1, n+1 \) holds. We find it has ten kinds of possible structures. The first three of them are shown in Fig. 6, the rest seven kinds of structures are shown in Fig. 7. For the first structure (left one of Fig. 6), \( H_i \in D_1, n \) for any \( i = 1, 2, 3 \), the hub nodes \( A, B \) of \( G_{n+1} \) are connected by a solid line in \( H_3 \), the hub nodes \( A, C \) of \( G_{n+1} \) are connected by a solid line in \( H_1 \), the hub nodes \( B, C \) of \( G_{n+1} \) are connected by a solid line in \( H_2 \). Thus the conditions for the case 1 of Appendix 1 are satisfied. According to Eq. (38), we have

\[
\sum_{H_i \in D_1, n, i = 1, 2, 3} \Phi_{n+1}(H) = \sum_{H_i \in D_1, n, i = 1, 2, 3} \left[ (y - 1) \prod_{i=1}^{3} \Phi_n(H_i) \right] = (y - 1) \prod_{i=1}^{3} \Phi_n(H_i) = (y - 1) T_{1,n}^3.
\]

Thus, it contributes to \( T_{1,n+1}(x, y) \) by a term \( (y - 1) T_{1,n}^3 \).
Fig. 6 Three kinds of structures which satisfy $H \in D_{1,n+1}$. Each one of them corresponds to a kind of spanning subgraph $H$ of $G(n + 1)$. The hub nodes $A, B, C$ of $G(n + 1)$ are connected by a path of solid line which shows they are in the same connected component.

![Fig. 6](image)

Fig. 7 Seven kinds of structures which satisfy $H \in D_{1,n+1}$. The hub nodes $A, B, C$ are connected by a path of solid line which shows they are in the same connected component.

![Fig. 7](image)

Fig. 8 All possible structures which satisfy $H \in D^C_{2,n+1}$. Each one of them corresponds to a kind of spanning subgraph $H$ of $G(n + 1)$. The hub nodes $A, B$ are connected by solid line, but $C$ are not connected with $A, B$ by a path of solid line which shows $C$ are not in the same connected component with $A, B$.

![Fig. 8](image)

For the second structure (center one of Fig. 6), $H_1 \in D_{1,n}, H_2 \in D_{1,n}$ and $H_3 \in D^C_{2,n}$, the hub nodes $A, B$ of $G_{n+1}$ are connected by a solid line in $H_3$, the hub nodes $A, C$ of $G_{n+1}$ are connected by a solid line in $H_1$, the hub nodes $B, C$ of $G_{n+1}$ are connected by a solid line in $H_2$. Thus the conditions for the case 1 of Appendix 1 are also satisfied. According to Eq. (38), we have

$$
\sum_{H_1 \in D_{1,n}, H_2 \in D_{1,n}, H_3 \in D^C_{2,n}} \Phi_{n+1}(H)
= (y - 1) \prod_{i=1}^{2} \sum_{H_i \in D_{1,n}} \Phi_n(H_i) \cdot \sum_{H_3 \in D^C_{2,n}} \Phi_n(H_3)
= (y - 1)T_{1,n}^2 T_{2,n}.
$$

Noticing the possible rotations, we find it has 3 equivalent structures. Thus this kind of spanning subgraphs contributes to $T_{2,n+1}(x, y)$ by a term $3(y - 1)T_{1,n}^2 T_{2,n}$. Computing the contributions of all possible structures and adding them together, we obtain Eq. (42).

Next, we analyze the possible structures of $H$ which give contribution to $T_{2,n+1}(x, y)$. By symmetry, we only study $T^C_{2,n+1}(x, y)$. Then the condition $H \in D^C_{2,n+1}$ holds. We find it has 6 possible structures which are shown in Fig. 8. For the first structure, $H_3 \in D_{1,n}, H_1 \in D^B_{2,n}$ and $H_2 \in D^A_{2,n}$. Therefore the hub nodes $A, C$ of $G_{n+1}$ are connected by a dotted line in $H_1$ which shows that they are not in the same connected component. Thus the conditions for case 2 are satisfied. According to Eq. (41), it contributes to $T^C_{2,n+1}(x, y)$ by a term $\frac{4}{y-1}T_{1,n}T_{2,n}^2$ (we have consider 4 equivalent structures: $H_1 \in D^C_{2,n}$ (or $D^B_{2,n}$), $H_2 \in D^A_{2,n}$ (or $D^C_{2,n}$)).
Computing the contributions of all the 6 possible structures and adding them together, we obtain Eq. (43).

Finally, we analyze the possible structures of $H$ which give contribution to $T_{3,n+1}(x, y)$. Then the condition $H \in D_{3,n+1}$ holds. We find it has 4 possible structures which is shown in Fig. 9. For the first structure, $H_1 \in H_{3,n+1}$ holds. We find it has 4 possible structures which is shown in Fig. 9. For the first structure, $H_1 \in H_{3,n+1}$ holds. We find it has 4 possible structures which is shown

Fig. 9 All possible structures which satisfy $H \in D_{3,n+1}$. The hub nodes $A$, $B$, $C$ are not connected by a path of solid line which shows $A$, $B$, $C$ belong to three different connected components

Appendix 2: Proof of Lemma 1

In this appendix, we prove the following result by mathematical induction.

For any $n \geq 0$, $T_{2,n}(x, y)$, $T_{3,n}(x, y)$ can be factored as

$$
\begin{align*}
T_{2,n}(x, y) &= (x - 1)P_n(x, y), \\
T_{3,n}(x, y) &= (x - 1)^2Q_n(x, y),
\end{align*}
$$

(47)

where $P_n(x, y)$, $Q_n(x, y)$ are polynomials of $x$ and $y$.

Initial step: for $n = 0$, $T_{2,0}(x, y) = x - 1$, $T_{3,0}(x, y) = (x - 1)^2$, let $P_0(x, y) = Q_0(x, y) = 1$, we know Eq. (47) holds.

Inductive step: assuming Eq. (47) holds for certain $n \geq 0$, we prove Eq. (47) holds for $n + 1$ as follows.

For convince, we abbreviate $P_n(x, y)$, $Q_n(x, y)$ as $P_n$ and $Q_n$ respectively. Substituting $T_{2,n}(x, y)$, $T_{3,n}(x, y)$ from Eq. (47) in Eqs. (43) and (44), we have

$$
T_{2,n+1} = \frac{1}{x - 1}(4T_{1,n}T_{2,n}^2 + 4T_{1,n}T_{2,n}T_{3,n} + 4T_{2,n}^3)
$$

and

$$
T_{3,n+1} = \frac{1}{x - 1}(8T_{2,n}^3 + 12T_{2,n}T_{3,n} + 6T_{2,n}T_{3,n}^2 + T_{3,n}^3)
$$
\[ + 6(x - 1)P_n((x - 1)^2 Q_n)^2 + ((x - 1)^2 Q_n)^3 \]
\[ = (x - 1)^2(8P_n^3 + 12(x - 1)P_n^2 Q_n + 6(x - 1)^2 P_n Q_n^2 + (x - 1)^3 Q_n^3). \]

Let
\[
P_{n+1} = 4T_{1,n}P_n^2 + 4(x - 1)T_{1,n}P_nQ_n + 4(x - 1)P_n^3 \\
+ 4(x - 1)^2 P_n^2 Q_n + (x - 1)^2 T_{1,n}Q_n^2 + (x - 1)^3 P_nQ_n^2, \tag{48}\]
\[
Q_{n+1} = 8P_n^3 + 12(x - 1)P_n^2 Q_n + 6(x - 1)^2 P_n Q_n^2 + (x - 1)^3 Q_n^3. \tag{49}\]

We find
\[
\begin{align*}
T_{2,n+1}(x, y) &= (x - 1)P_{n+1}(x, y) \\
T_{3,n+1}(x, y) &= (x - 1)^2 Q_{n+1}(x, y).
\end{align*} \tag{50}\]

Thus Eq. (47) holds for \( n + 1 \) which end the proof.

### Appendix 3: Comparison of \( R_s(n) \) and \( R(n) \)

For the Sierpinski gasket \( SG(n) \) \( (n \geq 0) \), Alfredo Donno [20] found that the all-terminal reliability polynomial is given by

\[ R_s(n) = p^{V_n - 1}(1 - p)^{E_n - V_n + 1}T_{1,n}\left(1, \frac{1}{1 - p}\right). \tag{51}\]

and \( T_{1,n}\left(1, \frac{1}{1 - p}\right) \) satisfy the following recurrence relation

\[
\begin{align*}
T_{1,n+1}\left(1, \frac{1}{1 - p}\right) &= \frac{p}{1 - p}T_{1,n}^3 + 6T_{1,n}^2N_n, \\
N_{n+1}\left(1, \frac{1}{1 - p}\right) &= \frac{p}{1 - p}T_{1,n}^2N_n + T_{1,n}^2M_n + 7T_{1,n}N_n^2, \\
M_{n+1}\left(1, \frac{1}{1 - p}\right) &= \frac{3p}{1 - p}T_{1,n}N_n^2 + 12T_{1,n}N_nM_n + 14N_n^3.
\end{align*} \tag{52}\]

with initial conditions
\[
T_{1,0}\left(1, \frac{1}{1 - p}\right) = \frac{3 - 2p}{1 - p}, \quad N_0\left(1, \frac{1}{1 - p}\right) = M_0\left(1, \frac{1}{1 - p}\right) = 1.
\]

Let
\[
B_s(n) = p^{(V_n - 2)}(1 - p)^{(E_n - V_n + 2)}N_n, \tag{53}\]
\[
C_s(n) = p^{(V_n - 3)}(1 - p)^{(E_n - V_n + 3)}M_n. \tag{54}\]
We obtain the following recurrence relation from Eqs. (51), (52), (53) and (54).

\[
\begin{aligned}
R_s(n+1) &= p^{V_n+1}(1-p)E_{n+1}-V_{n+1}+1T_{1,n+1}(1, 1, 1-p) \\
&= p^{3V_n-4}(1-p)^3E_n-3V_n+1 \left( \frac{p}{1-p}T_n^3 + 6T_{1,n}^2N_n \right) \\
&= R_s^3(n) + 6R_s^2(n)B_s(n), \\
B_s(n+1) &= p^{V_n+1}(1-p)(E_n-1-V_{n+1}+1)N_{n+1} \\
&= p^{3V_n-5}(1-p)^3E_n-3V_n+5 \left( \frac{p}{1-p}T_n^2N_n + T_{1,n}^2M_n + 7T_{1,n}N_n^2 \right) \\
&= R_s^2(n)B_s(n) + R_s^2(n)C_s(n) + 7R_s(n)B_s^2(n), \\
C_s(n+1) &= p^{V_n+1-3}(1-p)(E_n-1-V_{n+1}+3)M_{n+1} \\
&= p^{3V_n-6}(1-p)^3E_n-3V_n+6 \left( \frac{3p}{1-p}T_nN_n^2 + 12T_{1,n}N_nM_n + 14N_n^3 \right) \\
&= 3R_s(n)B_s^2(n) + 12R_s(n)B_s(n)C_s(n) + 14B_s^3(n),
\end{aligned}
\]

with initial conditions

\[
R_s(0) = p^2(3-2p), \quad B_s(0) = p(1-p)^2, \quad C_s(0) = (1-p)^3.
\]

Now, we compare all-terminal reliability between the Sierpinski gasket and the PSFW while \( p \in (0, 1) \).

For \( n = 0 \), the Sierpinski gasket and the PSFW have the same topology (i.e., a triangle). Thus

\[
R_s(0) = R(0), \quad B_s(0) = B(0).
\]

According to Eqs. (25), (55), (26) and (56), we have

\[
R_s(1) = R(1), B_s(1) > B(1)
\]

But for \( n \geq 2 \), we have

\[
\begin{aligned}
R_s(n) - R(n) > 0, \\
B_s(n) - B(n) > 0,
\end{aligned}
\]

which is proved by mathematical induction as follows.

Initial step: for \( n = 2 \),

\[
R_s(2) - R(2) = R_s^3(1) - R^3(1) + 6R_s^2(1)B_s(1) - 6R^2(1)B(1) \\
= 6R^2(1)[B_s(1) - B(1)] > 0,
\]

and

\[
B_s(2) - B(2) = R_s^2(1)B_s(1) + R_s^2(1)C_s(1) + 7R_s(1)B_s^2(1) - 4R(1)B^2(1) \\
\geq R_s^2(1)B_s(1) + R_s^2(1)C_s(1) + 3R_s(1)B_s^2(1) > 0.
\]

Thus Eq. (57) holds for \( n = 2 \).

Inductive step: assuming that Eq. (57) holds for certain \( n \geq 2 \), we prove Eq. (57) holds for \( n + 1 \).
Let Eq. (55) minus Eq. (25) and Eq. (56) minus Eq. (26), we have

\[ R_s(n + 1) - R(n + 1) = R_s^3(n) - R^3(n) + 6R_s^2(n)B_s(n) - 6R^2(n)B(n) > 0, \]

and

\[ B_s(n + 1) - B(n + 1) = R_s^2(n)B_s(n) + R_s^2(n)C_s(n) + 7R_s(n)B_s^2(n) - 4R(n)B^2(n) > 0. \]

Thus Eq. (57) holds for \( n + 1 \) which end the proof.

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