Odd-flavored QCD$^3$ and Random Matrix Theory

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Abstract

We consider QCD$^3$ with an odd number of flavors in the mesoscopic scaling region where the field theory finite-volume partition function is equivalent to a random matrix theory partition function. We argue that the theory is parity invariant at the classical level if an odd number of masses are zero. By introducing so-called pseudo-orthogonal polynomials we are able to relate the kernel to the kernel of the chiral unitary ensemble with $\beta = 2$ in the sector of topological charge $\nu = \frac{1}{2}$. We prove universality and are able to write the kernel in the microscopic limit in terms of field theory finite-volume partition functions.

1 Introduction

In 1993 Shuryak and Verbaarschot realized [1] that a certain chiral random matrix theory (RMT) is equivalent to the low-energy limit of the QCD partition function [2] and thus RMT, a tool that had shown merit in such diverse areas of physics as atomic physics, solid state physics, and nuclear physics, entered the realm of quantum chromodynamics (see [3] for an extensive review of the many applications of RMT and [4] for a review of the application of RMT to QCD). There is an amount of arbitrariness in the RMT of QCD since the potential $V$ that enters in the RMT partition function is arbitrary. It was shown [1] that when $V \sim \lambda^2$ then RMT and QCD are equivalent in the mesoscopic scaling region, also called the (double-)microscopic limit. Thus, if we want RMT to provide us with information about QCD in the microscopic limit we have to consider spectral correlators that are independent of $V$ in this limit. If this is the case the correlators are said to be universal. Clearly, the question of universality is crucial if one wants to extract information about QCD from RMT.

Many papers have been written about the connection between RMT and 4-dimensional QCD (QCD$_4$) with an arbitrary number of flavors as well as 3-dimensional QCD (QCD$^3$) with an even number of flavors (see references in [1]). However, for an odd number $N_f = 2\alpha + 1$ of flavors the situation is less understood and only treated in [1]. The reasons for this lack of balance are many. The RMT partition function is

$$\tilde{Z} = \int dT e^{-NV(T^2)} \prod_{f=1}^{N_f} \det(T + im_f),$$

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where we integrate over $N \times N$ hermitian matrices $T$ with the Haar measure. First of all, the effective field theory partition function, equivalent to \( (1) \) when $V \sim T^2$, depends on whether $N$ is even or odd [3]:

\[
Z_n^{(2\alpha+1)} = \int dU \cosh \left[ \text{Tr}(NMUT_5U^\dagger) \right] \quad (N \text{ even})
\]

\[
Z_n^{(2\alpha+1)} = \int dU \sinh \left[ \text{Tr}(NMUT_5U^\dagger) \right] \quad (N \text{ odd})
\]

Both integrals are over $\text{SU}(N_f)$, $M$ is the mass matrix, and $\Gamma_5 = \text{diag}(1,\ldots,1,-1,\ldots,-1)$. The $N$-dependence is most peculiar. Furthermore, the RMT spectral density $\rho(\lambda)$ obeys the relation [5]

\[
\rho(-\lambda) = (-)^{NN_f} \rho(\lambda)
\]

which implies that for odd $N$ the spectral density is an odd function and thus not positive definite. Finally, the standard method for calculating spectral information in random matrix theories, the method of orthogonal polynomials, is not available for an odd number of flavors due to the fact that it is impossible to define orthogonal polynomials with respect to an odd measure on an interval symmetric about the origin.

In this paper we concentrate on the case of even $N$ and an odd number of flavors $N_f = 2\alpha + 1$. In section 2 we propose a parity transformation for an odd number of flavors that leaves the QCD$_3$-Lagrangian invariant. This is an essential point since the RMT of QCD$_3$ has parity symmetry as an input, and the conclusion is that an odd number of masses have to be zero for the Lagrangian to be invariant (at the classical level). In section 3 we introduce pseudo-orthogonal polynomials and calculate the spectral correlation functions in the microscopic limit for even $N$. It turns out that for an odd number of flavors the correlators can be expressed using the kernel from the chiral unitary ensemble, which is equivalent to four-dimensional QCD, when we equate the number of chiral fermions to $\alpha$ and by analytic continuation work in the sector of topological charge $\nu = 1/2$. In section 4 we use pseudo-orthogonal polynomials to express the RMT kernel in terms of field theory finite-volume partition functions. By looking at the result from [3] for the finite-volume partition function we arrive at the conclusion that when one mass is zero, as is necessary for symmetry reasons, the $N$-odd field theory partition function is zero. The lesson is that the odd-$N$ partition function should not be regarded as defining a theory in itself, but only as a function that is necessary in order to extract spectral information in the even-$N$ theory. This is consistent with the fact that the spectral density does not make sense when $N$ is odd due to (4).

## 2 Discrete Symmetries of QCD$_3$

Our starting point is the QCD$_3$-Lagrangian with $N_f$ flavors $\psi_f$:

\[
\mathcal{L} = -\frac{1}{4} \text{Tr} F^2 + \sum_{f=1}^{N_f} \bar{\psi}_f D \psi_f + \sum_{f=1}^{N_f} m_f \bar{\psi}_f \psi_f ,
\]

where $F$ is the gluonic field strength and $m_f$ are the eigenvalues of the mass matrix. In three dimensions we do not have chirality at our disposal because $\gamma^0 \gamma^1 \gamma^2 \propto 1$. The lowest dimensional
representation of the $\gamma$-matrices is the Pauli-matrices and the corresponding fields are two-spinors. This has the disadvantage that the mass terms are parity $P: (x^0, x^1, x^2) \mapsto (x^0, -x^1, -x^2)$ odd. If all masses are zero then $\mathcal{L}$ is invariant under $P$ since the Dirac-term in the Lagrangean respects the parity transformation, but in general the masses are non-zero and $\mathcal{L}$ is not invariant. For reasons that will become apparent later we distinguish between an even and an odd number of flavors.

In [7, 8, 9] it is argued that for an even number $N_f = 2\alpha$ of flavors we can achieve parity invariance by using a four-dimensional representation of the $\gamma$-matrices. The representation employed (Minkowski space) is

$$
\gamma^0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & -i\sigma_1 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}.
$$

We see that for a four-spinor $\psi = (\phi \; \chi)^T$ the mass term is

$$
\mathcal{L}_{\text{mass}} = m\bar{\psi}\psi = m\bar{\phi}\gamma^0\phi - m\bar{\chi}\gamma^0\chi.
$$

The last expression is identical to $m\bar{\phi}\phi - m\bar{\chi}\chi$ in a two-dimensional representation of the $\gamma$-matrices with $\gamma_0 = \sigma_3$. If the mass matrix $M$ is chosen to be

$$
M = \text{diag}(m_1, \ldots, m_\alpha, -m_1, \ldots, -m_\alpha)
$$

it follows that the Lagrangean (3) is invariant under parity when the transformation is defined as

$$
P: \psi_i \mapsto \sigma_1\psi_{\alpha+i}, \quad P: \psi_{\alpha+i} \mapsto \sigma_1\psi_i, \quad i = 1, \ldots, \alpha.
$$

For an odd number $N_f = 2\alpha + 1$ of flavors we write the Lagrangean as

$$
\mathcal{L} = -\frac{1}{4} \text{Tr} F^2 + \mathcal{L}_{\text{Dirac}}^{(2\alpha)} + \mathcal{L}_{\text{Dirac}}^{(2\alpha)} + \mathcal{L}_{\text{mass}}^{(1)} + \mathcal{L}_{\text{mass}}^{(1)}
$$

$$
\mathcal{L}_{\text{Dirac}}^{(2\alpha)} = \sum_{f=1}^{2\alpha} \bar{\psi}_f \slashed{D} \psi_f, \quad \mathcal{L}_{\text{Dirac}}^{(1)} = \bar{\psi} \slashed{D} \psi
$$

$$
\mathcal{L}_{\text{mass}}^{(2\alpha)} = \sum_{f=1}^{2\alpha} m_f \bar{\psi}_f \psi_f, \quad \mathcal{L}_{\text{mass}}^{(1)} = m\bar{\psi}\psi.
$$

The $\gamma$-matrices are represented by the Pauli-matrices:

$$
\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_2.
$$

Above we used the grouping of the spinors $\psi_1, \ldots, \psi_{2\alpha}$ into four-spinors and the $4 \times 4$-matrices $\{\gamma_\mu\}$ as $\gamma$-matrices to obtain a parity invariant system when assigning masses of opposite sign to the constituents of the four-spinors. Here we use that if $\{\gamma_\mu\}$ is a representation of the Dirac algebra then $\{ -\gamma_\mu \}$ is an inequivalent representation of the algebra $\{\gamma_\mu\}$. This allows us to use negative masses also for an odd number of flavors, where we cannot group the spinors as in (3). As far

\footnote{Notice that $(x^0, x^1, x^2) \mapsto (x^0, -x^1, -x^2)$ is a rotation. We choose (somewhat arbitrarily) to let $P$ denote inversion around the 1-axis.}
as $L_{\text{mass}}^{(2\alpha)}$ and $L_{\text{Dirac}}^{(2\alpha)}$ are concerned, we use the transformation defined in (11). For the remaining two-spinor $\psi$ we employ the transformation

$$P: \psi \mapsto \sigma_1 \psi.$$  

(14)

Since a mass term changes sign under this transformation the overall symmetry is retained only when $m = 0$. This is also the case if we had used the representation (13) with opposite signs on all the matrices. Provided that we want to study an invariant Lagrangean we are therefore confined to consider an *odd* number of massless fermions for odd $N_f$. Notice, that if we want to have an optimal amount of discrete symmetry in the theory, with as many non-zero mass terms as possible, there is no alternative to the choice

$$M = \text{diag}(m, m_1, \ldots, m_\alpha, -m_1, \ldots, -m_\alpha)$$  

(15)

(with $m = 0$) of the mass matrix. We have an odd number of two-spinors that have to transform among each other. This means that the transformed fields are permutations of the original fields (multiplied by a Pauli matrix such that mass terms change signs). Unless the fields swap positions two and two their masses have to be zero in order to have invariance, and when two fields interchange positions the $\pm$ assignment of the masses make sure that the mass terms are invariant. This automatically leaves one field that has to transform into itself, thus giving rise to a symmetry breaking mass term unless $m = 0$.

Having discussed the choice of mass matrix we now turn to the RMT of odd-flavored QCD$_3$.

### 3 Universality of Odd-flavored QCD$_3$ in the Microscopic Limit

The QCD$_3$ RMT partition function for an odd number of flavors in the eigenvalue representation of the unitary ensemble is

$$\tilde{Z}(M) = \int_{-\infty}^{+\infty} \prod_{k=1}^{N} d\lambda_k e^{-\frac{N}{2} \sum_{k=1}^{N} (\lambda_k + im)^2} \prod_{f=1}^{\alpha} (\lambda_k^2 + m_f^2) e^{-\frac{N}{2} \sum_{f=1}^{\alpha} (\lambda_k^2 + m_f^2)} \Delta(\{\lambda_i\})^2$$  

(16)

where

$$\Delta(\{\lambda_i\}) = \prod_{i<j}^{N} (\lambda_i - \lambda_j)$$  

(17)

is the Vandermonde determinant. Notice, that it is precisely the choice $m = 0$, which we now make, that makes the partition function real. With

$$\rho_N(\lambda_1, \ldots, \lambda_N) = \frac{1}{\tilde{Z}(M)} \prod_{k=1}^{N} \lambda_k \prod_{f=1}^{\alpha} (\lambda_k^2 + m_f^2) e^{-\frac{N}{2} \sum_{f=1}^{\alpha} (\lambda_k^2 + m_f^2)} \Delta(\{\lambda_i\})^2$$  

(18)

the $k$-point correlation function is given by

$$\rho_k(\lambda_1, \ldots, \lambda_k) = \int_{-\infty}^{+\infty} \prod_{i=k+1}^{N} d\lambda_i \rho_N(\lambda_1, \ldots, \lambda_N).$$  

(19)
The orthodox way to calculate these correlators would be to find orthogonal polynomials \( \{P_n\} \) with respect to the weight function

\[
w(\lambda) = \lambda \prod_{f=1}^{\alpha} (\lambda^2 + m_f^2)e^{-NV(\lambda^2)},
\]

introduce the kernel

\[
K_N(\lambda, \lambda') = \sqrt{w(\lambda)w(\lambda')} \sum_{k=0}^{N-1} \frac{P_k(\lambda)P_k(\lambda')}{h_k}
\]

where \( h_k \) is the norm of \( P_k \), and then use the expression

\[
\rho_k(\lambda_1, \ldots, \lambda_k) = \frac{(N-k)!}{k!} \det_{1 \leq a, b \leq k} K_N(\lambda_a, \lambda_b)
\]

to determine the \( k \)-point correlation functions. This method cannot be used here since \( w(\lambda) \) is odd and the integration interval is even, and thus orthogonal polynomials are not defined.

Inspired by [5] we will now introduce a method based on what we will call pseudo-orthogonal polynomials that can be applied to any odd measure on an even interval as long as \( N \) is even. Consider for simplicity the massless case where \( w(\lambda) = \lambda^{2\alpha+1}e^{-NV(\lambda^2)} \). For the chiral unitary ensemble (chUE, with \( \beta = 2 \)) the weight function for \( N_{f}^{ch} \) massless fermions in the sector of topological charge \( \nu \) is

\[
w_{ch}(\lambda) = \lambda^{N_{f}^{ch}+\nu}e^{-NV(\lambda^2)}, \quad \lambda \in \mathbb{R}_+
\]

and we denote the associated orthogonal polynomials \( \{P_n(\lambda)\} \) with norm \( h_n \). Now put \( N_{f}^{ch} = \alpha \), \( \nu = \frac{1}{2} \), and define the pseudo-orthogonal polynomials by

\[
Q_{2n}(\lambda) \equiv (1 + \lambda)P_n(\lambda^2)
\]
\[
Q_{2n+1}(\lambda) \equiv (1 - \lambda)P_n(\lambda^2).
\]

These new polynomials are pseudo-orthogonal with respect to the weight function \( w(\lambda) = \lambda^{2\alpha+1}e^{-NV(\lambda^2)} \) on the real line in the sense that

\[
\langle 2k|2n \rangle = 2h_n\delta_{kn} \equiv r_{2n}\delta_{kn}
\]
\[
\langle 2k+1|2n+1 \rangle = -2h_n\delta_{kn} \equiv r_{2n+1}\delta_{kn}
\]
\[
\langle 2k|2n+1 \rangle = 0
\]

where the brackets indicate integration. The proof is straightforward. The chUE-measure for \( N_{f}^{ch} \) massive flavors is

\[
w_{ch}(\lambda) = \prod_{f=1}^{N_{f}^{ch}} (\lambda + m_f^2)e^{-NV(\lambda)}, \quad \lambda \in \mathbb{R}_+.
\]

The measure is valid in the sector of vanishing topological charge. In the sector of charge \( \nu \neq 0 \) we obtain the correct measure by adding \( |\nu| \) massive flavors and then letting the masses vanish. We see that in the sector where \( \nu = \frac{1}{2} \) and \( N_{f}^{ch} = \alpha \) the measure is

\[
w_{ch}(\lambda) = \sqrt{\lambda} \prod_{f=1}^{\alpha} (\lambda + m_f^2)e^{-NV(\lambda)}, \quad \lambda \in \mathbb{R}_+.
\]
Simple calculations shows that the pseudo-orthogonal polynomials with respect to \( f_\nu \) are given by the polynomials orthogonal with respect to \( f_\nu \) by the definitions (24) and (25). Because the QCD masses are pairwise assigned with opposite sign we may identify the \( f \)’th chiral mass with the positive mass of the \( f \)’th QCD fermion. Notice that in the matrix representation of chUE half-integer values of \( \nu \) are not permitted since the matrices integrated over are \( N \times (N + |\nu|) \) [3]. In the eigenvalue representation, however, we can without problems make an analytic continuation such that \( \nu = \frac{1}{2} \) makes sense, at least from a formal point of view.

We are now ready to calculate the microscopic spectral correlators of odd-flavored QCD, i.e., the correlation functions taken in the limit \( N \to \infty, \lambda \to 0, m_f \to 0 \) where
\[
\zeta \equiv \pi \rho(0)N\lambda, \quad \zeta' \equiv \pi \rho(0)N\lambda', \quad \mu_f \equiv \pi \rho(0)Nm_f
\]
are kept constant. We always assume that \( N \) is even. \( \Delta(\{\lambda_k\}) \) defined in [7] can be expressed as the determinant of an \( N \times N \) matrix:
\[
\Delta(\{\lambda_k\}) = \det \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_N \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1}
\end{pmatrix}.
\]

Consider the \( N \times N \) matrix
\[
\mathcal{C}(\{\lambda_k\}) = \begin{pmatrix}
1 + \lambda_1 & 1 + \lambda_2 & \cdots & 1 + \lambda_N \\
1 - \lambda_1 & 1 - \lambda_2 & \cdots & 1 - \lambda_N \\
\lambda_1^2 + \lambda_2^2 & \lambda_2^2 + \lambda_3^2 & \cdots & \lambda_N^2 + \lambda_N^2 \\
\lambda_1^2 - \lambda_2^2 & \lambda_2^2 - \lambda_3^2 & \cdots & \lambda_N^2 - \lambda_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{N-2} + \lambda_2^{N-1} & \lambda_2^{N-2} + \lambda_3^{N-1} & \cdots & \lambda_N^{N-2} + \lambda_N^{N-1} \\
\lambda_1^{N-2} - \lambda_2^{N-1} & \lambda_2^{N-2} - \lambda_3^{N-1} & \cdots & \lambda_N^{N-2} - \lambda_N^{N-1}
\end{pmatrix}.
\]

The crucial observation is that \( \det \mathcal{C}(\{\lambda_k\}) \) is related to \( \Delta(\{\lambda_k\}) \). Performing on \( \mathcal{C} \) the row operations
\[
\text{row}(2k-1) \to \text{row}(2k-1) + \text{row}(2k), \quad k = 1, \ldots, N/2
\]
followed by
\[
\text{row}(2k) \to \text{row}(2k) - \frac{1}{2} \text{row}(2k-1), \quad k = 1, \ldots, N/2
\]
we end up with
\[
\hat{\mathcal{C}}(\{\lambda_k\}) = \begin{pmatrix}
2 & 2 & \cdots & 2 \\
-\frac{1}{2}\lambda_1 & -\frac{1}{2}\lambda_2 & \cdots & -\frac{1}{2}\lambda_N \\
\vdots & \vdots & \ddots & \vdots \\
2\lambda_1^{N-2} & 2\lambda_2^{N-2} & \cdots & 2\lambda_N^{N-2} \\
-\frac{1}{2}\lambda_1^{N-1} & -\frac{1}{2}\lambda_2^{N-1} & \cdots & -\frac{1}{2}\lambda_N^{N-1}
\end{pmatrix},
\]
where we have discarded a factor coming from the row operations. The determinant of an \( N \times N \) matrix \( M \) is defined as
\[
\det M = \sum_{\pi \in S_N} (-)^\pi M_{1\pi(1)} \cdots M_{N\pi(N)}
\]
(37)

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from which it follows that \( \det C(\{\lambda_k\}) = (-)^{N/2} \Delta(\{\lambda_k\}) \), or
\[
(-)^{N/2} \det C(\{\lambda_k\}) = (-)^N \Delta(\{\lambda_k\}) = \Delta(\{\lambda_k\}) .
\] (38)

Now notice that

\[
C(\{\lambda_k\}) = \begin{pmatrix}
(1 + \lambda_1)1 & (1 + \lambda_2)1 & \cdots & (1 + \lambda_N)1 \\
(1 - \lambda_1)1 & (1 - \lambda_2)1 & \cdots & (1 - \lambda_N)1 \\
(1 + \lambda_1)\lambda_1^2 & (1 + \lambda_2)\lambda_2^2 & \cdots & (1 + \lambda_1)\lambda_N^2 \\
(1 - \lambda_1)\lambda_1^2 & (1 - \lambda_2)\lambda_2^2 & \cdots & (1 - \lambda_1)\lambda_N^2 \\
\vdots & \vdots & \ddots & \vdots \\
(1 + \lambda_1)\lambda_1^{N-2} & (1 + \lambda_1)\lambda_2^{N-2} & \cdots & (1 + \lambda_1)\lambda_N^{N-2} \\
(1 - \lambda_1)\lambda_1^{N-2} & (1 - \lambda_1)\lambda_2^{N-2} & \cdots & (1 - \lambda_1)\lambda_N^{N-2}
\end{pmatrix}.
\] (39)

Looking at (39) we see that we are in a position where we can substitute the entries in \( C(\{\lambda_k\}) \) with the pseudo-orthogonal polynomials (properly normalized). Because of pseudo-orthogonality we may then write, as in the case of orthogonal polynomials,
\[
K_N(\lambda, \lambda') = \sqrt{\lambda \lambda'} \prod_{f=1}^{\alpha} \sqrt{\lambda^2 + m_f^2} \sqrt{\lambda'^2 + m_f^2} e^{-\frac{N}{2} (V(\lambda^2) + V(\lambda'^2))} \frac{1}{N} \sum_{j=0}^{N-1} \frac{Q_j(\lambda)Q_j(\lambda')}{r_j} ,
\] (40)

where the \( Q_j \)'s are pseudo-orthogonal. With \( n \equiv N/2 \) we obtain
\[
\sum_{k=0}^{N-1} \frac{Q_k(\lambda)Q_k(\lambda')}{r_k} = \sum_{i=0}^{n-1} \frac{1}{r_{2i}} (1 + \lambda)(1 + \lambda')P_i(\lambda^2)P_i(\lambda'^2) + \sum_{j=0}^{n-1} \frac{1}{r_{2j+1}} (1 - \lambda)(1 - \lambda')P_j(\lambda^2)P_j(\lambda'^2)
\]
\[
= \sum_{i=0}^{n-1} \frac{1}{r_{2i}} (1 + \lambda)(1 + \lambda')P_i(\lambda^2)P_i(\lambda'^2) - \sum_{j=0}^{n-1} \frac{1}{r_{2j}} (1 - \lambda)(1 - \lambda')P_j(\lambda^2)P_j(\lambda'^2)
\]
\[
= (\lambda + \lambda') \sum_{k=0}^{n-1} \frac{P_k(\lambda^2)P_k(\lambda'^2)}{h_k} ,
\] (41)

We have discarded irrelevant overall factors. The introduction of pseudo-orthogonal polynomials have led us to an expression for the kernel that involves orthogonal polynomials for which the microscopic limit is universal \cite{11, 12}:
\[
K_N(\lambda, \lambda') = (\lambda + \lambda') \sqrt{\lambda \lambda'} \prod_{f=1}^{\alpha} \sqrt{\lambda^2 + m_f^2} \sqrt{\lambda'^2 + m_f^2} e^{-\frac{N}{2} (V(\lambda^2) + V(\lambda'^2))} \frac{1}{N} \sum_{j=0}^{n-1} \frac{P_j(\lambda^2)P_j(\lambda'^2)}{h_j} .
\] (42)

By the Christoffel-Darboux formula the sum over the polynomials is proportional to \([P_{n-1}(\lambda)P_n(\lambda') - P_n(\lambda)P_{n-1}(\lambda')]/(\lambda^2 - \lambda'^2)\) and since \( n = N/2 \) the \( N \to \infty \) limit corresponds to the \( n \to \infty \) limit. Now compare \cite{12} with the expression for the chUE kernel \cite{11}:
\[
K_{\text{chUE}, N}(z_1, z_2) = \sqrt{|z_1z_2|} \prod_{f=1}^{N_{ch}} \sqrt{z_1^2 + m_f^2} \sqrt{z_2^2 + m_f^2} e^{-\frac{N}{2} (V(z_1^2) + V(z_2^2))}
\]
\[
\times \frac{P_{N-1}(z_1^2)P_N(z_2^2) - P_N(z_1^2)P_{N-1}(z_2^2)}{z_1^2 - z_2^2}.
\] (43)
in the sector of zero topological charge. To go to the sector of charge \( \nu \) we can, by flavor-topology
duality, take the \( \nu = 0 \) kernel with \( N_f^{ch} + |\nu| \) massive flavors and then let \(|\nu|\) masses vanish. The
sector with \( \nu = \frac{1}{2} \) is then by continuation given by \(13\) multiplied by a factor \( \sqrt{z_1 z_2} \). We arrive at the
following master formula for odd-flavored QCD3:

\[
K_s^{(N_f=2\alpha+1)}(0, \mu_1, \ldots, \mu_\alpha, -\mu_1, \ldots, -\mu_\alpha, \zeta_1, \zeta_2) = \frac{\zeta_1 + \zeta_2}{\sqrt{\zeta_1 \zeta_2}} K_{chUE,s}^{(N_f^{ch}=\alpha, \nu=1/2)}(\mu_1, \ldots, \mu_\alpha, \zeta_1, \zeta_2). \quad (44)
\]

The formula implies that all properties of the chUE that are universal similarly are universal in
odd-flavored QCD3 and especially that all microscopic correlators are universal. We see that in the
case of the microscopic spectral density, where we have to evaluate \( K_s(\zeta, \zeta) \), the odd-flavored QCD3
kernel and the chUE kernel agree up to a proportionality constant. They do not, however, agree for
higher-order correlators. The chUE kernel is given by \(8, 12\):

\[
K_{chUE,s}(\mu_1, \ldots, \mu_\alpha, \zeta_1, \zeta_2) = C_2 \frac{(-)^{\nu+[N_f^{ch}/2]+1} \sqrt{\zeta_1 \zeta_2}}{(\zeta_1^2 - \zeta_2^2) \prod_f \sqrt{(\zeta_1^2 + \mu_f^2)(\zeta_2^2 + \mu_f^2)}} \det B \det A, \quad (45)
\]

where \( C_2 \) is a normalization constant, and where \( B \) is an \((N_f^{ch} + 2) \times (N_f^{ch} + 2)\) matrix and \( A \) is an
\(N_f^{ch} \times N_f^{ch}\) matrix defined as

\[
A_{ij} = \mu_i^{j-1} I_{\nu+j-1}(\mu_i) \quad (46)
\]

\[
B_{ij} = (\zeta_i)^{j-1} J_{\nu+j-1}(\zeta_i), \quad i = 1, 2 \quad (47)
\]

\[
B_{ij} = (-\mu_i^*)^{j-1} I_{\nu+j-1}(\mu_i), \quad i = 3, \ldots, N_f^{ch} + 2. \quad (48)
\]

In QCD3 the hitherto only known microscopic spectral correlator is the massless microscopic spectral
density \(8\)

\[
\rho_s^{(2\alpha+1)}(\zeta) = \frac{\zeta}{2} \left[ J_{\alpha+\frac{1}{2}}(\zeta) - J_{\alpha+\frac{3}{2}}(\zeta) \right] \quad (49)
\]

which coincides with the massless chUE microscopic spectral density

\[
\rho_{chUE,s}^{(N_f^{ch}, \nu)}(\zeta) = \frac{\zeta}{2} \left[ J_{N_f^{ch}+\nu}(\zeta) - J_{N_f^{ch}+\nu+1}(\zeta) \right] \quad (50)
\]

precisely when \( N_f^{ch} = \alpha \) and \( \nu = \frac{1}{2} \).

From \(14\) it directly follows that the microscopic spectral density obeys a series of decoupling
relations \(12\):

\[
\lim_{\mu_k \to \infty} \rho_s^{(2k+1)}(0, \mu_1, \ldots, \mu_k, \zeta) = \rho_s^{(2k-1)}(0, \mu_1, \ldots, \mu_{k-1}, \zeta), \quad k = 1, \ldots, \alpha. \quad (51)
\]

These relations tell us that when we make two quarks with equal masses (up to a sign) infinitely
heavy the spectral density of the new system becomes that of the system consisting of the remaining
quarks. This is illustrated in figure \(1\) which shows the massless microscopic spectral density (msd)
for one and three massless flavors, and for one massless and two massive flavors. The massive msd is
calculated using the master formula \(14\). When the masses vanish the massive msd will become
that of three massless fermions while the act of making the masses infinitely heavy will make the
massive msd become that of one massless flavor.
Figure 1: The microscopic spectral density in three different cases: $\rho_{s}^{(1)}(0, \zeta)$ (solid curve), $\rho_{s}^{(3)}(0,0,0,\zeta)$ (dashed curve), and $\rho_{s}^{(3)}(0,\mu,-\mu,\zeta)$ with $\mu = 10$ (dotted curve). In the $\mu \to \infty$ limit the dotted curve will approach the solid curve, cf. the decoupling relations (51). In the $\mu \to 0$ limit the dotted curve will converge to the dashed curve as the massive spectral density becomes massless.

4 The Kernel in Terms of Finite-volume Partition Functions

The newly acquired universality of the microscopic spectral correlators will now allow us to express the odd-flavored RMT kernel for even $N$ in terms of the field theory finite-volume partition functions of $\text{QCD}_3$, as it was done in [6, 13] for the other ensembles. Having done this we go on to simplify the expression for the finite-volume partition functions considerably when we take the conditions for parity invariance into account.

The main ingredient in the proof is a result from [14] that in our case says that

$$K_{N}^{(N_{f})}(\lambda, \lambda') = \frac{1}{\tilde{Z}_{N}} e^{-\frac{N}{2}[V(\lambda^{2})+V'(\lambda')^{2}]} \sqrt{\lambda \lambda'} \prod_{f=1}^{\alpha} \sqrt{(\lambda^{2} + m_{f}^{2})(\lambda'^{2} + m_{f}^{2})}$$

$$\times \int d^{(N-1)^{2}} M e^{-N\text{tr}V(M^{2})} \det(M) \prod_{f=1}^{\alpha} \det(M^{2} + m_{f}^{2}) \det(\lambda - M) \det(\lambda' - M). \quad (52)$$

The integral appearing in the second line is the RMT partition function in the case where the square hermitian matrices are of odd dimensions, and with two extra imaginary masses $i\lambda$ and $i\lambda'$. The proof goes as in [14]:

$$\tilde{Z}_{N-1}(M, i\lambda, i\lambda') = \int d^{(N-1)^{2}} M \det(\lambda - M) \det(\lambda' - M) W(M) \quad (53)$$

where $W(M) = \tilde{Z}_{N}^{-1} e^{-N\text{tr}V(M^{2})} \det(M) \prod_{f=1}^{\alpha} \det(M^{2} + m_{f}^{2})$ is the RMT measure. Rewriting in
such that the rows $\alpha$ the interchange of two rows amounts to an overall sign change of the determinant we see that

$$\mu_{\text{Formula (59) can be simplified considerably when interchange}}$$

In the expression $\Delta(\mathbf{M})$ terms of eigenvalue variables we get

$$\mu_{\text{in formula (58) now has the microscopic masses as entries, and the partition functions have been labelled with $N$ and $N-1$ implicitly through $\eta \equiv (-)^{N}$. Notice that we have already taken the double-microscopic limit - the label is important because the partition functions depend on $N$ in a crucial manner. We have}}$$

$$\mu_{\text{In the expression $\Delta(\mathbf{M})$ is a Vandermonde determinant of the masses and $\mathbf{D}(\mu, \{\mu_k\})$ is an $N_f \times N_f$ matrix of the form ($\mu_0 \equiv \mu$, $\mu_{\alpha+i} = -\mu_i$ for $i = 1, \ldots, \alpha)$}}$$

$$\mu_{\text{Formula (59) can be simplified considerably when $\mu = 0$. Take the matrix $\mathbf{D}(-\mu, \{-\mu_k\})$ and interchange}}$$
because of the ± symmetry of the masses and the block structure of (60)-(61). Thus

\[ Z^{(N_f)}(\eta; \mu = 0, \{\mu_k\}) = \frac{1}{\Delta(M)} \left[ \det D(\mu = 0, \{\mu_k\}) + (-)^{N+2\alpha} \det D(\mu = 0, \{\mu_k\}) \right] \]

\[ = \begin{cases} 2 \det D(\mu = 0, \{\mu_k\}) & \text{if } N \text{ is even} \\ 0 & \text{if } N \text{ is odd} \end{cases} \]  

(64)

Besides the simplification this expression tells us that we should not look upon the odd-\(N\) partition function as defining a theory in itself, but only as a function that enters in the expressions of the even-\(N\) theory.

Note that when \(\nu = \frac{1}{2}\) the chUE kernel (45) becomes complex and thus the normalization coefficient

\[ C_2 = (-)^{\nu+|N_{\text{ch}}/2|} \]  

(65)

becomes complex also. We have explicitly checked that (58) gives the expected results in a few cases, and in each case the normalization constants had to be chosen complex. Finally, we mention that the decoupling relations (51) can be explicitly derived from (58).

There are two major points to take away from this section. First, the fact that for even \(N\) we are able to rewrite the RMT kernel in terms of finite-volume partition functions while the proof cannot be applied to the case of odd \(N\). Second, the fact the finite-volume partition function for odd \(N\) vanishes when the only input is the mass assignment that guarantees parity invariance, a crucial point when the RMT is constructed. From this we conclude that when extracting spectral information about odd-flavored QCD\(_3\) from RMT we should look upon the even-\(N\) theory as defining the theory while the odd-\(N\) finite-volume partition function should be looked upon merely as a function that enters in the expression for the even-\(N\) kernel.

## 5 Summary

We have considered the RMT formulation of QCD\(_3\) with an odd number of flavors in the mesoscopic scaling region and we have calculated the microscopic kernel from which all microscopic spectral correlation functions can be found. We have shown the universality of the kernel by the discovery of a deep connection between the kernel of odd-flavored QCD\(_3\) with \(N_f = 2\alpha + 1\) flavors and QCD\(_4\) (\(\beta = 2\)) with \(N_{\text{ch}}^f = \alpha\) flavors in the sector of topological charge \(\nu = \frac{1}{2}\), brought about by the introduction of pseudo-orthogonal polynomials. The result has the consequence that all properties of the four-dimensional theory that are universal similarly are universal in three dimensions. We end by noting that odd-flavored QCD\(_3\) now finally has caught up with its older and more mature sisters QCD\(_4\) and even-flavored QCD\(_3\) as far as universality is concerned.

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