ON THE SKEWED FRACTIONAL DIFFUSION
ADVECTION REACTION EQUATION ON THE INTERVAL

Yulong Li

Abstract
This article provides techniques of raising the regularity of fractional order equations and resolves fundamental questions on the one-dimensional homogeneous boundary-value problem of skewed (double-sided) fractional diffusion advection reaction equation (FDARE) with variable coefficients on the bounded interval. The existence of the true (classical) solution together with norm estimation is established and the precise regularity bound is found; also, the structure of the solution is unraveled, capturing the essence of regularity, singularity, and other features of the solution.

The key analysis lies in exploring the properties of Gauss hypergeometric functions, solving coupled Abel integral equations and dominant singular integral equations, and connecting the functions from fractional Sobolev spaces to the ones from Hölderian spaces that admit integrable singularities at the endpoints.

MSC 2010: Primary 26A33; Secondary 34A08, 46N20, 45E99

Key Words and Phrases: Riemann-Liouville, fractional diffusion, generalized Abel equation, advection, reaction, regularity, double-sided, skewed, integral equations.
1. Main results

Under conditions
\[
\begin{aligned}
0 < \alpha, \beta < 1, & \quad \alpha + \beta = 1, 0 < \mu < 1, \\
f(x) \in H^*(\Omega), & \quad p(x), q(x), k(x) \in C^1(\Omega), \\
p'(x), k'(x) \in H(\Omega), & \quad k(x) > 0 \text{ on } \Omega, \\
0 < \alpha, \beta < 1, & \quad \alpha + \beta = 1, \\
f(x) \in H^*(\Omega), & \quad p(x), q(x), k(x) \in C^1(\Omega), \\
p'(x), k'(x) \in H(\Omega), & \quad k(x) > 0 \text{ on } \Omega, \\
\pi(1 - \mu^2) \cot((1 + \mu)\pi/2) + 4(b - a)\|k'\|_{L^\infty(\Omega)} < 0, \\
\end{aligned}
\]

we consider the problem
\[
\begin{aligned}
[L(u)](x) &= f(x), \quad x \in \Omega = (a, b), \\
u(a) &= u(b) = 0, \\
[L(u)](x) := -Dk(x)(\alpha aD_x^{-(1-\mu)} + \beta bD_b^{-(1-\mu)})Du \\
&+ p(x)Du + q(x)u(x). \\
\end{aligned}
\]

We say a function \(u(x)\) is a true (classical) solution to \((1.2)\) if \(u \in AC(\Omega)\), \(u(a) = u(b) = 0\), \(Du \in C(\Omega)\), \((\alpha aD_x^{-(1-\mu)} + \beta bD_b^{-(1-\mu)})Du \in C^1(\Omega)\) and \([L(u)](x) = f(x) \forall x \in \Omega\).

Our main result is the following:

**Theorem 1.1.** Let conditions \((1.1)\) be satisfied. Then in the Sobolev space \(\hat{H}^{(1+\mu)/2}_0(\Omega)\), there exists a unique true solution \(u(x)\) to \((1.2)\) (up to the equivalence of functions). And it is representable by
\[
u(x) = \alpha D_x^{-t}J_t, \quad x \in \Omega,
\]
\(J_t(x) \in H^*(\Omega)\) (depending on \(t\)) provided that \(t < 1 + \mu\).

We will also show that \(1 + \mu\) is optimal by giving a counter example, namely, the representation \(\nu(x) = \alpha D_x^{-t}J_t\) can fail for any \(t > 1 + \mu\), therefore, Theorem \([1.1]\) is sharp up to the endpoint.

As a byproduct, Corollary \([1.1.1]\) clarifies that the homogeneous boundary-value problems \((1.2)\) and \((11.199)\) that involve different types of fractional derivatives always have the same classical solution; Corollary \([1.2]\) illustrates the behaviour of \(\mu\)-th order derivative and first derivative of the true solution \((i.e., \alpha D_x^\mu u, x D_b^\mu u, Du)\) near the boundary points, more precisely, under suitable conditions, \(\alpha D_x^\mu u\) and \(x D_b^\mu u\) always vanish at \(x = a\) and \(x = b\) respectively; meanwhile, \(Du\) either vanishes or blows up at \(x = a, b\) meaning that it does not admit non-zero values at the endpoints, which suggests difference from integer-order diffusion equations.
2. Introduction

The boundary-value problem of fractional diffusion advection reaction equation (FDARE) is one of the fundamental problems in the subject of fractional-order differential equations. In the last decade, the FDARE has been widely discussed in journals of numerical analysis, applied analysis and applied physics, and its applications have been found in different scientific areas. Compared to integer-order differential equations, fractional-order differential equations exhibit new features and open many opportunities in modelling various phenomena in physics. However, it turns out that the theoretical analysis of fractional-order equations can be very challenging and that many fundamental questions still remain open. Therefore, more additional attention is deserved towards the development of this subject.

We will focus on the homogeneous boundary-value problem of FDARE in this article, as a part of our goal of systematic investigation of FDARE, this work is a continuation of our previous work [4], in which the skewed FDARE was investigated on the whole real axis. In the whole real line $\mathbb{R}$, the properties of the solution to FDARE behave regularly and similarly to the ones of the solution to integer-order diffusion equations, however, this is not the case on the bounded interval. Among extensive papers on the skewed FDARE, pioneering work and excellent discussion that are most related to our work appeared in [3] (2006), [2] (2018) and [5] (2019). Therefore this paper is regarded as a development of these fundamental work from a unified point of view.

During the study of FDARE and in order to obtain a satisfying discussion, the following deserves attention:

1. The results of FDARE on the interval $(0, 1)$ cannot be directly generalized to that on an arbitrary interval $(a, b)$ by a simple transformation of the variable. Scaling of the length of the domain is actually coupled with the variable coefficients. In this work, we are interested in studying FDARE on the general interval $(a, b)$.

2. The variational formulation of FDARE needs to be constructed with caution such that it does not intertwine with raising the regularity at the current stage, which is a simple idea but an important start to the whole analysis.

3. The smoothness of coefficients and the source function of FDARE usually can not guarantee the smoothness of weak solution, what are the appropriate functional spaces to raise the regularity of weak solution and to seek the true solution needs to be considered.
3. Strategy of the proof

The strategy for the whole proof of Theorem 1.1 is two big steps. First, we intend to establish the weak solution for an appropriate variational formulation. Secondly, we try to raise the regularity of weak solution in a certain sense to recover the classical solution. More precisely,

1. due to the presence of the advection term and variable coefficients, we will adopt a suitably modified variational formulation such that we are able to establish the existence of a weak solution and without a need to raise the regularity at this stage. This is where we get started, and later, by picking up some regularity for the weak solution we convert this modified variational formulation equation back to original FDARE pointwisely, which automatically implies that this weak solution is a classical solution (true solution).

2. the main body of the work, which is also the most challenging part, is to convert this weak solution to the true solution and the challenging is attributed to two aspects. First, the solution \( u(x) \) to FDARE usually dramatically lack regularity at the boundary regardless of the smoothness of coefficients and right-hand side function of FDARE, which suggests that the true solution \( u(x) \) should be sought in functional spaces that admit singularities at the endpoints; Second, only limited regularity of the solution can be picked up in the context of fractional Sobolev spaces, which is usually not enough for a pointwise restoration of FDARE and results in difficulty in carrying the mathematical techniques performed in integer-order elliptic equations to the analysis of FDARE (for example, difference quotient technique is not any more applicable, etc.), new framework has to be developed.

Taking into account above philosophy, the analysis is led to: we establish the existence of weak solution \( u(x) \) in the usual fractional Sobolev space \( \tilde{H}^{(1+\mu)/2}_0(\Omega) \), however, raise the regularity of \( aD^\mu_x u \) (therefore raising the regularity of \( u(x) \)) in the Hölderian space that admits integrable singularities at the boundary, namely \( H^*(\Omega) \), to better spaces \( H^*_t(\Omega) (0 < t < 1) \); and in the middle, we need to connect this two spaces \( \tilde{H}^{(1+\mu)/2}_0(\Omega) \) and \( H^*(\Omega) \) by showing that \( aD^\mu_x u \) which is originally from \( \tilde{H}^{(1-\mu)/2}_0(\Omega) \) is actually located in \( H^*(\Omega) \). Roughly speaking, we have essentially “three” steps:

\[
\begin{align*}
    u &\in \tilde{H}^{(1+\mu)/2}_0(\Omega) \rightarrow aD^\mu_x u \in H^*(\Omega) \rightarrow aD^\mu_x u \in \bigcap_{0<t<1} H^*_t(\Omega).
\end{align*}
\]

After integrating \( aD^\mu_x u \) by \( aD^{-\mu}_x \), we have

\[
u(x) = aD^{-(\mu+t)}_x J_t, J_t \in H^*(\Omega), 0 < t < 1,
\]
which will yield the representation in Theorem 1.1 by adding the case \( t \leq 0 \).

4. Organization of the work

Readers can perform quick search either through section titles or Definition, Property, Lemma, Theorem and Corollary numbers.

- Section 5 is about convention and notation.
- Section 6 gathers and lists necessary definitions and associated properties, most of which have been given specific citation information for readers’ convenience. They will be extensively invoked during subsequent proofs.
- Section 7 is to set up a suitable variational formulation and establish the existence of weak solution and norm estimation.
- In Section 8, three lemmas on coupled Abel integral equations will be established, which will serve the counter example at the end of Section 10 and Corollary 11.2 in Section 11.
- In Section 9, another three lemmas on raising the regularity will be established, which are key steps towards the whole proof of Theorem 1.1.
- Section 10 provides the whole proof of Theorem 1.1 and an example.
- Section 11 gives two corollaries and proposes a question.

5. Convention and notation

Convention

- \( \Omega = (a, b), \overline{\Omega} = [a, b] \) and \(-\infty < a, b < \infty\), whenever they appear throughout the material.
- We shall often not distinguish “ = ” at every point from “ = ” almost every point in those equations when there is no chance of misunderstanding. On some occasions, the notation \( a.e. \) will be used for emphasizing the validity for “almost everywhere” to draw the reader’s attention for those cases that are of importance.
- Whenever we deal with a function \( f(x) \) belonging to Sobolev or \( L^p \) spaces, it is implicitly assumed that \( f \) denotes a suitable representative of the equivalence classes, unless otherwise specified.
- All the functions are default to be real-valued, and all the constants that will appear in different contexts will be assumed to be real constants.

Notation

- \( H^\lambda(\overline{\Omega}) \): Hölderian space (\( \lambda > 0 \)).
\( H(\Omega) := \bigcup_{\lambda > 0} H^{\lambda}(\Omega) \).

\( H^{\lambda}(\Omega) := \{ f : f(x) \in H^{\lambda}(\Omega), f(a) = f(b) = 0 \} \).

\( H^{\lambda}(\rho, \Omega) \): weighted Hölderian space, \( (\rho(x) = \prod_{k=1}^{n} |x-x_k|^\mu_k, x_k, x \in \Omega, n \text{ is a positive integer}) \).

\( H^{\lambda}_0(\rho, \Omega) := \{ f : f(x) \in H^{\lambda}(\rho, \Omega), \rho(x_k) = 0, k = 1, \ldots, n \} \).

\( H^{\lambda}_0(\epsilon_1, \epsilon_2) := \{ f : f(x) = g(x)\frac{g(x)}{(x-a)^{1+\epsilon_1}(b-x)^{1+\epsilon_2}}, g(x) \in H^{\lambda}_0(\Omega) \} \).

\( H^{\lambda}_0(\epsilon_1, \epsilon_2) := \bigcup_{0<\epsilon_1, \epsilon_2} H^{\lambda}_0(\epsilon_1, \epsilon_2) \).

\( aD^{-\sigma}_x, bD^{-\sigma}_b, D^{-\sigma} \) and \( D^{-\sigma} \) represent fractional integrals if \( \sigma > 0 \), identity operators if \( \sigma = 0 \) and fractional derivatives if \( \sigma < 0 \) (see specific definitions in Section 6).

\( AC(\Omega) \): the set of absolutely continuous functions on \( \Omega \).

\( C(G) \): the set of all continuous functions on a set \( G \).

\( C^n(G) := \{ f : f^{(n)}(x) \in C(G) \} \).

Both \( Df \) and \( f' \) represent the usual derivative of a function \( f \).

\((f, g)_{\Omega} \) and \((f, g)_{\mathbb{R}} \) denote the integrals \( \int_{\Omega} fg \) and \( \int_{\mathbb{R}} fg \), respectively.

\( C_0^{\infty}(\Omega) \) consists of all the infinitely differentiable functions on \( \Omega \) and with compact support in \( \Omega \).

6. Prerequisite Knowledge

Necessary preliminaries that will be of use are presented first, including definitions and associated properties. These definitions and properties are known and most of them are quoted from the literature directly. Properties given in this section may not be in the strongest forms, but they are adequate for our purpose.

6.1. Riemann-Liouville integrals and their properties.

**Definition 6.1.** Let \( w : (c, d) \to \mathbb{R}, (c, d) \subset \mathbb{R} \) and \( \sigma > 0 \). The left and right Riemann-Liouville fractional integrals of order \( \sigma \) are, formally
respectively, defined as
\[
(aD_x^{-\sigma} w)(x) := \frac{1}{\Gamma(\sigma)} \int_a^x (x - s)^{\sigma - 1} w(s) \, ds, \quad (6.3)
\]
\[
(xD_b^{-\sigma} w)(x) := \frac{1}{\Gamma(\sigma)} \int_x^b (s - x)^{\sigma - 1} w(s) \, ds, \quad (6.4)
\]
where \(\Gamma(\sigma)\) is Gamma function. For convenience, when \(c = -\infty\) or \(d = \infty\) we set
\[
(D_x^{-\sigma} w)(x) := -\infty D_x^{-\sigma} w \quad \text{and} \quad (D^{-\sigma} w)(x) := xD^{-\sigma} w. \quad (6.5)
\]

In particular, if \(\sigma = 0\), \(aD_x^{-\sigma}, xD_b^{-\sigma}, D_x^{-\sigma}\) and \(D^{-\sigma}\) are regarded as identity operators.

**Property 6.1** ([6], eq. (2.72), (2.73), p. 48). Given \(\sigma > 0\), fractional operators \(aD_x^{-\sigma}\) and \(xD_b^{-\sigma}\) are bounded in \(L^p(\Omega)\) (\(p \geq 1\)):
\[
\| aD_x^{-\sigma} \psi \|_{L^p(\Omega)} \leq K \| \psi \|_{L^p(\Omega)}, \quad \| xD_b^{-\sigma} \psi \|_{L^p(\Omega)} \leq K \| \psi \|_{L^p(\Omega)}, \quad K = \frac{(b - a)^\sigma}{\sigma \Gamma(\sigma)}. \quad (6.6)
\]

**Property 6.2** ([6], eq. (2.19), p. 34). Let \(\sigma > 0\) and \((Qf)(x) = f(a + b - x)\), then the following operators are reflective:
\[
Q aD_x^{-\sigma} Q = xD_b^{-\sigma}. \quad (6.7)
\]

**Property 6.3.** If \(0 < \sigma < 1\), \(1 < q < 1/\sigma\), then the fractional operators \(aD_x^{-\sigma}, xD_b^{-\sigma}\) are bounded from \(L^p(\Omega)\) into \(L^q(\Omega)\) with \(q = \frac{p}{1 - \sigma p}\).

**Property 6.4.** If \(0 < \frac{1}{p} < \sigma < 1 + \frac{1}{p}\), fractional operators \(aD_x^{-\sigma}\) and \(xD_b^{-\sigma}\) map the space \(L^p(\Omega)\) into the Hölderian space \(H^{\sigma - 1/p}(\Omega)\).

**Remark 6.1.** Property 6.3 is a combination of Theorem 3.5 ([6], p. 66) and Property 6.2, and Property 6.4 is a combination of Corollary of Theorem 3.6 ([6], p. 69) and Property 6.2.

### 6.2. Riemann-Liouville derivatives and their properties.

**Definition 6.2.** Let \(w : (c, d) \to \mathbb{R}, (c, d) \subset \mathbb{R}\) and \(\sigma > 0\). Assume \(n\) is the smallest integer greater than \(\sigma\) (i.e., \(n - 1 \leq \sigma < n\)). The left and right Riemann-Liouville fractional derivatives of order \(\sigma\) are, formally
respectively, defined as
\[(aD_x w)(x) := \frac{d^n}{dx^n} aD_x^\sigma w \quad \text{and} \quad (xD_0^\sigma w)(x) := (-1)^n \frac{d^n}{dx^n} D_0^\sigma w.\]

For ease of notation, when \( c = -\infty \) or \( d = \infty \) we set
\[(D_\sigma^c w)(x) = -\infty \quad \text{and} \quad (D_\sigma^\infty w)(x) = x D_\infty^\mu w.\]  \[(6.8)\]

**Property 6.5** ([4], Theorem 4.1). For \( v, w \in C_0^\infty(\Omega) \) and \( \sigma \geq 0 \), it is true that
\[(D_\sigma^v, D_\sigma^w)_R = (D_\sigma^w, D_\sigma^v)_R = (2\pi)^2 \int_\Omega |\xi|^{2\sigma} \hat{v}(\xi) \overline{\hat{w}(\xi)} \, d\xi,\]
\[(D_\sigma^v, D_\sigma^w)_R + (D_\sigma^w, D_\sigma^v)_R = 2 \cos(\sigma\pi) (D_\sigma^v, D_\sigma^w)_R.\]  \[(6.9)\]

**Property 6.6** ([4], Property 2.4). Let \( 0 < \sigma \) and \( w \in C_0^\infty(\mathbb{R}) \), then \( D_\sigma^v, D_\sigma^w \in L^p(\mathbb{R}) \) for any \( 1 \leq p < \infty \).

6.3. **Hypergeometric function and properties.**

**Definition 6.3** ([1], pp. 64, 65). For \( |x| < 1 \), the hypergeometric function is defined by the series
\[2F_1(\sigma_1, \sigma_2, \sigma_3; x) = \sum_{n=0}^\infty \frac{(\sigma_1)_n (\sigma_2)_n}{(\sigma_3)_n n!} x^n,\]  \[(6.10)\]
and by analytic continuation elsewhere. One of such analytic continuation is given by
\[2F_1(\sigma_1, \sigma_2, \sigma_3; x) = \frac{\Gamma(\sigma_3)}{\Gamma(\sigma_2)\Gamma(\sigma_3-\sigma_2)} \int_0^1 t^{\sigma_2-1} (1-t)^{\sigma_3-\sigma_2-1} (1-xt)^{-\sigma_1} \, dt\]  \[(6.11)\]
if \( 0 < \sigma_2 < \sigma_3 \).

The Pochhammer symbol \((z)_n\) in above definition with integer \( n \) means
\[(z)_n = z(z+1) \cdots (z+n-1), \quad n = 1, 2, \cdots, (z)_0 = 1.\]  \[(6.12)\]
We put down only some of the properties that will be needed.

**Property 6.7.**
\[2F_1(\sigma_1, \sigma_2, \sigma_3; x) = 2F_1(\sigma_2, \sigma_1, \sigma_3; x).\]  \[(6.13)\]
Let $\sigma_2 > 0$, $\sigma_1, \sigma_3 \in \mathbb{R}$, and $\psi(x) = (x - a)^{\sigma_2 - 1}(b - x)^{\sigma_3 - 1}$, then for $x \in \Omega$

$$a^{-\sigma_1}D^\sigma_x = \frac{(b - a)^{\sigma_3 - 1}\Gamma(\sigma_2)}{\Gamma(\sigma_1 + \sigma_2)}(x - a)^{\sigma_1 + \sigma_2 - 1}2F_1(1 - \sigma_3, \sigma_2, \sigma_1 + \sigma_2; \frac{x - a}{b - a}).$$

(6.14)

**Property 6.9** ([1], Theorem 2.3.2, p. 78).

$$2F_1(\sigma_1, \sigma_2, \sigma_1 + \sigma_2 + 1 - \sigma_3; 1 - x) = A \cdot 2F_1(\sigma_1, \sigma_2, \sigma_3; x) + B \cdot x^{1 - \sigma_3}2F_1(1 + \sigma_1 - \sigma_3, 1 + \sigma_2 - \sigma_3, 2 - \sigma_3; x),$$

where

$$A = \frac{\Gamma(\sigma_1 + \sigma_2 + 1 - \sigma_3)\Gamma(1 - \sigma_3)}{\Gamma(\sigma_1 + 1 - \sigma_3)\Gamma(\sigma_2 + 1 - \sigma_3)}; \quad B = \frac{\Gamma(\sigma_3 - 1)\Gamma(\sigma_1 + \sigma_2 + 1 - \sigma_3)}{\Gamma(\sigma_1)\Gamma(\sigma_2)}.$$  

(6.15)

**Property 6.10** ([1], Theorem 2.2.5, p. 68).

$$2F_1(\sigma_1, \sigma_2, \sigma_3; x) = (1 - x)^{\sigma_3 - \sigma_1 - \sigma_2}2F_1(\sigma_3 - \sigma_1, \sigma_3 - \sigma_2, \sigma_3; x).$$

(6.16)

6.4. **Functional spaces** $H^*(\Omega)$, $H_0^*(\Omega)$ **and properties.** We list several mapping properties related to $H^*(\Omega)$ and $H_0^*(\Omega)$, which play an important role in connecting the whole analysis of this work.

**Property 6.11** ([6], Lemma 30.2, p. 618). Let $0 < \sigma < 1$, the weighted singular operator

$$(S_{\nu_a, \nu_b} f)(x) = \frac{1}{\pi} \int_a^b \left( \frac{x - a}{t - a} \right)^{\nu_a} \left( \frac{b - x}{b - t} \right)^{\nu_b} \frac{f(t) dt}{t - x}$$

maps the space $H^*_\sigma(\Omega)$ into itself provided that $\sigma - 1 < \nu_a \leq \sigma$ and $\sigma - 1 < \nu_b \leq \sigma$.

**Property 6.12** ([6], Theorem 13.14, p. 248). Let $0 < \sigma < 1$, the fractional integration operators $aD_x^{-\sigma}$ and $xD_b^{-\sigma}$ map the space $H^*(\Omega)$ one-to-one and onto the space $H^*_\sigma(\Omega)$, respectively. Consequently, $aD_x^{-\sigma}(H^*(\Omega)) = xD_b^{-\sigma}(H^*(\Omega))$.

**Property 6.13**. Let $0 < \sigma < 1$, $\gamma_1, \gamma_2 > 0$, $\gamma_1 aD_x^{-\sigma} + \gamma_2 xD_b^{-\sigma}$ maps $H^*(\Omega)$ one-to-one and onto the space $H^*_\sigma(\Omega)$.
Remark 6.2. Property 6.13 is a combination of Theorem 30.7 ([6], p. 626) and Property 6.12.

6.5. Dominant singular integral and properties.

Definition 6.4. The singular integral operator $S$ is formally defined as

$$(S\psi)(x) = \frac{1}{\pi} \int_a^b \frac{\psi(t)}{t-x} dt, x \in \Omega,$$

the convergence being understood in the principal value sense.

Property 6.14 ([6], Corollary 2, p. 208). Denote $r_a(x) = x-a$, $r_b(x) = b-x, x \in \Omega$.

$$x D_d^{-\lambda} (r_b^{-\lambda} S(r_b^\lambda \psi)) = r_a^\lambda S(r_a^{-\lambda} x D_b^{-\lambda} \psi)$$

is valid for $0 < \lambda < 1, \psi \in L^p(\Omega), p > 1$.

Property 6.15 ([6], Theorem 11.1, p. 200). Let $n$ be a positive integer, $0 < \lambda < 1$, then the operator $S$ is bounded in the space $H^\lambda_0(\Omega)$, and in the weighted space $H^\lambda_0(p,\Omega), \rho(x) = \prod_{k=1}^n |x-x_k|^{\mu_k}, x_k, x \in \Omega$, provided that $\lambda < \mu_k < \lambda + 1 (k = 1, \cdots, n)$.

Property 6.16. Let $c_1, c_2$ be constants and $c_1^2 + c_2^2 \neq 0$. Denote $\frac{c_1 x + c_2}{c_1 + c_2} = e^{i\theta}$ and choose the value of $\theta$ so that $0 \leq \theta < 2\pi$. Further denote spaces $X_1 = H^*(\Omega), X_2 = H^*(\Omega) \cap C((a,b]), X_3 = H^*(\Omega) \cap C([a,b])$ and $X_4 = H^*(\Omega) \cap C([a,b]),$ and define $n_a$ and $n_b$ as follows:

$$n_a(X_1) = n_a(X_2) = 1, n_a(X_3) = n_a(X_4) = 0;$$
$$n_b(X_1) = n_b(X_3) = 1, n_b(X_2) = n_b(X_4) = 0.$$

Consider the problem

$$\begin{cases}
c_1 \psi(x) + \frac{c_2}{\pi} \int_a^b \frac{\psi(t)}{t-x} dt = f(x), x \in \Omega, \\
where f(x) = \frac{f_a(x)}{(x-a)^{1-\nu_a}}, f_b(x) \in H(\Omega), \nu_a, \nu_b \in \mathbb{R}.
\end{cases} (6.19)$$

Then each of the following holds:

(1) If

$$1 - n_a(X_i) - \frac{\theta}{2\pi} < \nu_a, \frac{\theta}{2\pi} - n_b(X_i) < \nu_b,$$
then (6.19) is unconditionally solvable in $X_i$ ($i = 1, 2, 3$) and its general solution $\psi_i(x)$ in $X_i$ ($i = 1, 2, 3$) is given by

$$\psi_i(x) = C(x - a)^{1 - n_0(X_i) - \frac{\phi}{2\pi}(b - x)^{\frac{\phi}{2\pi} - n_0(X_i)} + \frac{c_1 f(x)}{c_1^2 + c_2^2} - \frac{c_2}{\pi(c_1^2 + c_2^2)} \int_a^b \left( \frac{x - a}{t - a} \right)^{1 - n_0(X_i) - \frac{\phi}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\phi}{2\pi} - n_0(X_i)} \frac{f(t)}{t - x} dt,$$

(6.20)

where

$C = 0$ for $i = 2, 3$, and $C$ is an arbitrary constant for $i = 1$.

(2) If (6.19) is solvable in $X_4$, then the solution $\psi_4(x)$ is unique and is given by

$$\psi_4(x) = \frac{c_1 f(x)}{c_1^2 + c_2^2} - \frac{c_2}{\pi(c_1^2 + c_2^2)} \int_a^b \left( \frac{x - a}{t - a} \right)^{1 - n_0(X_4) - \frac{\phi}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\phi}{2\pi} - n_0(X_4)} \frac{f(t)}{t - x} dt.$$

(6.21)

**Remark 6.3.** According to the statement of Property 6.16, it is worth noting that if a solution of (6.19) belongs to the space $X_4$, then it also lies in $X_i$ ($i = 1, 2, 3$) and therefore it has four equivalent representations which differ only in form. This property is a special case of Theorem 30.2 on page 609 in [6] by letting $Z_0(x) = 1$, $a_1(x) = c_1$ and $a_2(x) = c_2$ in our case. And in part (2) we omitted the sufficient and necessary condition for (6.19) to be solvable in $X_4$ since we shall not need it.

6.6. Fractional Sobolev spaces and properties. It is known that there are various ways to define fractional Sobolev spaces, which are essentially equivalent but serve as convenient tools for deriving various properties under different contexts. On the whole real axis, one way is in terms of the Fourier transform as follows.

**Definition 6.5.** Given $0 \leq s$, let

$$H^s(\mathbb{R}) := \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1 + |2\pi\xi|^{2s})|\widehat{w}(\xi)|^2 d\xi < \infty \right\}. \quad (6.22)$$

It is endowed with semi-norm and norm

$$|w|_{H^s(\mathbb{R})} := \| (2\pi\xi)^s \widehat{w} \|_{L^2(\mathbb{R})}, \|w\|_{H^s(\mathbb{R})} := \left( \|w\|^2_{L^2(\mathbb{R})} + |w|^2_{H^s(\mathbb{R})} \right)^{1/2}.$$
Another equivalent definition is achieved with the aid of left or right fractional-order weak derivative, which is a generalization of integer-order weak derivative:

**Definition 6.6.** ([4], Section 3) Given $0 \leq s$ and assume $u(x) \in L^2(\mathbb{R})$, then $u(x) \in \hat{H}^s(\mathbb{R})$ if and only if there exists a unique $\psi_1(x) \in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} u \cdot D^s \psi = \int_{\mathbb{R}} \psi_1 \cdot \psi$$

(6.23)

for any $\psi \in C_0^\infty(\mathbb{R})$.

Similarly, $u(x) \in \hat{H}^s(\mathbb{R})$ if and only if there exists a unique $\psi_2(x) \in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} u \cdot D^{s*} \psi = \int_{\mathbb{R}} \psi_2 \cdot \psi$$

(6.24)

for any $\psi \in C_0^\infty(\mathbb{R})$.

With above definitions, the following property can be deduced, which guarantees the existence of fractional derivatives and provides equivalent semi-norm and norm.

**Property 6.17.** ([4], Section 3) Assume $u \in \hat{H}^s(\mathbb{R})$, $s \geq 0$, then $D^s u, D^{s*} u$ exist a.e. and

$$|u|_{\hat{H}^s(\mathbb{R})} = \|D^s u\|_{L^2(\mathbb{R})} = \|D^{s*} u\|_{L^2(\mathbb{R})}.$$  

(6.25)

Since in this work, we will mainly care about the fractional equation in finite domain, by restricting to the bounded interval we can define the following analogue.

**Definition 6.7.** Given $0 \leq s$.

$$\hat{H}^s_0(\Omega) := \{\text{Closure of } u \in C_0^\infty(\Omega) \text{ with respect to norm } \|\tilde{u}\|_{\hat{H}^s(\mathbb{R})}\},$$

(6.26)

where notation $\tilde{u}$ denotes the extension of $u(x)$ by 0 outside $\Omega$. It is endowed with semi-norm and norm

$$|u|_{\hat{H}^s_0(\Omega)} := |\tilde{u}|_{\hat{H}^s(\mathbb{R})}, \|u\|_{\hat{H}^s_0(\Omega)} := \|\tilde{u}\|_{\hat{H}^s(\mathbb{R})}.$$  

It is well-known that $\hat{H}^s(\mathbb{R})$ is a Hilbert space and so is $\hat{H}^s_0(\Omega)$. We shall also utilize another two useful facts:
Property 6.18. Given \( \frac{1}{2} < s < 1 \), then \( u \in \widetilde{H}_0^s(\Omega) \) can be represented as
\[
u(x) = aD_x^{-s}\psi_1 = xD_b^{-s}\psi_2, \quad (6.27)
\]
for certain \( \psi_1, \psi_2 \in L^2(\Omega) \). As a consequence, \( aD_x^s u \) and \( xD_b^s u \) exist a.e. and coincide with \( \psi_1, \psi_2 \), respectively.

Property 6.19. Given \( 1/2 < s < 1 \), \( g(x) \in C^1(\overline{\Omega}) \), then there exists a positive constant \( C \) such that
\[
g\hat{u} \in \widetilde{H}^s(\mathbb{R}) \quad \text{and} \quad \|g\hat{u}\|_{\widetilde{H}^s(\mathbb{R})} \leq C\|\hat{u}\|_{\widetilde{H}^s(\mathbb{R})} \quad (6.28)
\]
for any \( u(x) \in \widetilde{H}_0^s(\Omega) \). (Notation \( \hat{\cdot} \) denotes the extension by zero outside \( \Omega \).)

7. Variational Formulation

Recall the conditions
\[
\begin{align*}
0 < \alpha, \beta < 1, & \quad \alpha + \beta = 1, 0 < \mu < 1, \\
f(x) & \in H^s(\Omega), p(x), q(x), k(x) \in C^1(\overline{\Omega}), \\
p'(x), k'(x) & \in H^1(\overline{\Omega}), \\
k(x) > 0 & \quad \text{on } \mathbb{R}, \frac{q}{k} - \frac{1}{2}(k')^2 \geq 0 \quad \text{on } \overline{\Omega}, \\
\pi(1 - \mu^2) & \cot((1 + \mu)\pi/2) + 4(b - a)\|k'||_{L^\infty(\Omega)} < 0,
\end{align*}
\]
and the problem
\[
\begin{align*}
[L(u)](x) = & \quad f(x), \quad x \in \Omega = (a, b), \\
u(a) = & \quad u(b) = 0, \\
[L(u)](x) := & \quad -Dk(x)(\alpha_aD_x^{-(1-\mu)} + \beta_xD_b^{-(1-\mu)})Du \\
& \quad +p(x)Du + q(x)u(x). \quad (7.30)
\end{align*}
\]

This section is to set up a proper variational formulation for problem \((7.30)\), so that we can establish the existence of a weak solution and do not require raising any regularity at this stage. To do so, we begin with considering the operator \( \tilde{L} \):
\[
[\tilde{L}(u)](x) := -Dk(x)(\alpha_aD_x^{-(1-\mu)} + \beta_xD_b^{-(1-\mu)})u + p(x)Du + q(x)u(x). \quad (7.31)
\]

(note the difference between operators \( L \) and \( \tilde{L} \)) and construct the suitable bilinear form in Definition \((7.11)\) which is obtained from the left-hand side of
\[
\int_\Omega \frac{[L(u)](t)}{k(t)}v(t) \, dt = \int_\Omega \frac{f(t)}{k(t)}v(t) \, dt, \quad v(t) \in C_0^\infty(\Omega). \quad (7.32)
\]
Only later, we take care to raise the regularity to convert the weak solution to the true solution and operator \( \tilde{L} \) will also automatically become \( L \).

Denote \( s = (1 + \mu) / 2, s' = (1 - \mu) / 2 \) throughout this section (Section 7).

**Definition 7.1.** Define the bilinear form \( B_2[\cdot, \cdot] \) on the space \( \tilde{H}_0^s(\Omega) \) as

\[
B_2[u, v] := -\alpha(aD_x^s u, xD_b^s v)_{\Omega} - \beta(xD_b^s u, aD_x^s v)_{\Omega}
- \alpha((k')_{aD_x^s u, v})_{\Omega} + \beta((k')_{xD_b^s u, v})_{\Omega}
+ (aD_x^{s'} u, xD_b^{s'} (\frac{p}{k} v))_{\Omega} + (\frac{q}{k} u, v)_{\Omega},
\]

for \( u, v \in \tilde{H}_0^s(\Omega) \).

Each term is well-defined under conditions (7.29) by paying an attention to that \( xD_b^s (\frac{p}{k} v) \) exists a.e. and belongs to \( L^2(\Omega) \) by Property 6.19 and Property 6.17.

**Lemma 7.1.** Under conditions (7.29), there exist positive constants \( Q_1, Q_2 \) such that

\[
|B_2[u, v]| \leq Q_1 \|u\|_{\tilde{H}_0^s(\Omega)} \|v\|_{\tilde{H}_0^s(\Omega)}
\]

and

\[
B_2[u, u] \geq Q_2 \|u\|_{\tilde{H}_0^s(\Omega)}^2
\]

for all \( u, v \in \tilde{H}_0^s(\Omega) \).

**Proof.** Let us first prove (7.34) and (7.35) for all \( u, v \in C_0^\infty(\Omega) \).

Assume \( u, v \in C_0^\infty(\Omega) \).

From Definition 7.1, we readily check

\[
|B_2[u, v]| \leq \alpha \|aD_x^s u\|_{L^2(\Omega)} \|xD_b^s v\|_{L^2(\Omega)} + \beta \|xD_b^s u\|_{L^2(\Omega)} \|aD_x^s v\|_{L^2(\Omega)}
+ \alpha \|k'_{L^\infty(\Omega)}\|_{aD_x^s u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \beta \|k'_{L^\infty(\Omega)}\|_{xD_b^s u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}}
+ |(aD_x^{s'} u, xD_b^{s'} (\frac{p}{k} v))_{\Omega}| + \|\frac{q}{k} u\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}.
\]

(7.36)
Now we examine little pieces above separately. 
\[
\begin{align*}
\alpha \|a D_x^s u\|_{L^2(\Omega)} \|x D_b^s v\|_{L^2(\Omega)} &+ \beta \|x D_b^s u\|_{L^2(\Omega)} \|a D_x^s v\|_{L^2(\Omega)} \\
&\leq \alpha \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)} + \beta \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)} \\
&= \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)},
\end{align*}
\]
(7.37) and 
\[
\begin{align*}
\|a D_x^s u, x D_b^s (\frac{p}{k} v)\|_{\Omega} + \|\frac{q}{k} \|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&= \|a D_x^{-\mu} a D_x^s u, x D_b^s (\frac{p}{k} v)\|_{\Omega} + \|\frac{q}{k} \|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \|a D_x^{-\mu} a D_x^s u\|_{L^2(\Omega)} \|a D_b^s (\frac{p}{k} v)\|_{L^2(\Omega)} + \|\frac{q}{k} \|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq C \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)} + \|\frac{q}{k} \|_{L^\infty(\Omega)} \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)} \\
&\leq (C + \|\frac{q}{k} \|_{L^\infty(\Omega)} \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)}).
\end{align*}
\]
(7.38)
(7.39)

(Putting them together we obtain 
\[
|B_2[u, v]| \leq Q_1 \|u\|_{\tilde{H}^\beta(\Omega)} \|v\|_{\tilde{H}^\beta(\Omega)},
\]
(7.40)
for some appropriate positive constant \(Q_1\).

2. For (7.35), we consider \(B_2[u, u]\).

Simplifying the term \( (a D_x^s u, x D_b^s (\frac{p}{k} u))_{\Omega} \),
\[
(a D_x^s u, x D_b^s (\frac{p}{k} u))_{\Omega} = (u', \frac{p}{k} u)_{\Omega} = -\frac{1}{2} \langle (\frac{p}{k})' u, u \rangle_{\Omega},
\]
and we find that
\[
B_2[u, u] = -\alpha (a D_x^s u, x D_b^s u)_{\Omega} - \beta (x D_b^s u, a D_x^s u)_{\Omega} \\
- \alpha (\frac{k'}{k} a D_x^s u, u)_{\Omega} + \beta (\frac{k'}{k} x D_b^s u, u)_{\Omega} \\
+ ((\frac{q}{k} - \frac{1}{2} (\frac{p}{k})') u, u)_{\Omega}.
\]
(7.41)
Again, we examine little pieces above separately.
First, using the second identity in Property 6.5 we have
\[
- \alpha(a D_s^x u, x D_b^s u)_\Omega - \beta(x D_b^s u, a D_s^x u)_\Omega \\
= -\alpha(D^s \tilde{u}, D^s \tilde{u})_\mathbb{R} - \beta(D^s \tilde{u}, D^s \tilde{u})_\mathbb{R}
\]
(\(\tilde{u}(x)\) is the extension of \(u(x)\) by 0 outside \(\Omega\))
\[
= -\alpha \cos(s\pi)(D^s u, D^s u)_\mathbb{R} - \beta \cos(s\pi)(D^s u, D^s u)_\mathbb{R}
\]
\[
= -\cos(s\pi)|u|^2_{\tilde{H}_0^\delta(\Omega)}.
\]
(7.42)

Second, \[
| - \alpha(k' k a D^\mu_x u, u)_\Omega + \beta(k' k x D^\mu_b u, u)_\Omega | \leq \|k' k\|_{L^\infty(\Omega)}\|\alpha a D^\mu_x u - \beta x D^\mu_b u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}.
\]
(7.43)

By using Minkowsky inequality and Property 6.1 we deduce
\[
\|\alpha a D^\mu_x u - \beta x D^\mu_b u\|_{L^2(\Omega)} \\
\leq \|\alpha a D^\mu_x u\|_{L^2(\Omega)} + \|\beta x D^\mu_b u\|_{L^2(\Omega)} \\
= \|\alpha a D_x^{-s'}(a D_x^s u)\|_{L^2(\Omega)} + \|\beta x D_b^{-s'}(x D_b^s u)\|_{L^2(\Omega)} \\
\leq \frac{\alpha(b - a)^{s'}}{s' \Gamma(s')} \|a D^s_x u\|_{L^2(\Omega)} + \frac{\beta(b - a)^{s'}}{s' \Gamma(s')} \|x D^s_b u\|_{L^2(\Omega)} \\
\leq \frac{(b - a)^{s'}}{s' \Gamma(s')} |u|_{\tilde{H}_0^\delta(\Omega)},
\]
(7.44)

and
\[
\|u\|_{L^2(\Omega)} = \|a D_x^{-s}(a D_x^s u)\|_{L^2(\Omega)} \\
\leq \frac{(b - a)^s}{s \Gamma(s)} \|a D_x^s u\|_{L^2(\Omega)} \\
\leq \frac{(b - a)^s}{s \Gamma(s)} |u|_{\tilde{H}_0^\delta(\Omega)}.
\]
(7.45)

Thus, inequality (7.43) further becomes
\[
| - \alpha(k' k a D^\mu_x u, u)_\Omega + \beta(k' k x D^\mu_b u, u)_\Omega | \leq \frac{b - a}{s' s \Gamma(s') \Gamma(s)} |u|^2_{\tilde{H}_0^\delta(\Omega)}.
\]
(7.46)
Utilizing this inequality, (7.41) now becomes

\[
B_2[u, u] \geq \left( -\cos(s\pi) - \| \frac{k'}{k} \|_{L^\infty(\Omega)} \frac{b - a}{s' s} \Gamma(s') \Gamma(s) \right) |u|^2_{\tilde{H}_0^s(\Omega)} \\
\quad + \left( \langle \frac{q}{k} - \frac{1}{2} \langle p \rangle' \rangle u, u \rangle\Omega \right) \\
\quad = \left( -\cos(s\pi) - \| \frac{k'}{k} \|_{L^\infty(\Omega)} \frac{(b - a) \sin(s\pi)}{\pi s' s} \right) |u|^2_{\tilde{H}_0^s(\Omega)} \\
\quad + \left( \langle \frac{q}{k} - \frac{1}{2} \langle p \rangle' \rangle u, u \rangle\Omega \right)
\]

(7.47)

(apply the formula \( \Gamma(z) \Gamma(1 - z) = \frac{\pi}{\sin(z\pi)} \), \( z \) is not an integer).

In view of the last two conditions in (7.29) and the fact that the norm \( \| \cdot \|_{\tilde{H}_0^s(\Omega)} \) and the semi-norm \( | \cdot |_{\tilde{H}_0^s(\Omega)} \) are equivalent, we obtain

\[
B_2[u, u] \geq Q_2 |u|^2_{\tilde{H}_0^s(\Omega)},
\]

(7.48)

for some appropriate positive constant \( Q_2 \).

3. Last, let us consider the general case for \( u, v \in \tilde{H}_0^s(\Omega) \).

we claim that for any \( u, v \in \tilde{H}_0^s(\Omega) \) there exist Cauchy sequences \( \{u_n\} \), \( \{v_n\} \subset C_0^\infty(\Omega) \) in \( \tilde{H}_0^s(\Omega) \) such that

\[
B_2[u, v] = \lim_{n \to \infty} B_2[u_n, v_n].
\]

(7.49)

To see this, assume \( u, v \in \tilde{H}_0^s(\Omega) \), since \( C_0^\infty(\Omega) \) is dense in \( \tilde{H}_0^s(\Omega) \), there exist Cauchy sequences \( \{u_n\}, \{v_n\} \subset C_0^\infty(\Omega) \) such that

\[
\lim_{n \to \infty} u_n = u, \lim_{n \to \infty} v_n = v
\]

with respect to \( \| \cdot \|_{\tilde{H}_0^s(\Omega)} \).

From Definition (7.41) it is readily verified that

\[
\lim_{n \to \infty} -\alpha(a D_x u_n, x D_x^s v_n)_{\Omega} = -\alpha(a D_x u, x D_x^s v)_{\Omega},
\]

\[
\lim_{n \to \infty} -\beta(x D_x^s u_n, a D_x v_n)_{\Omega} = -\beta(x D_x^s u, a D_x v)_{\Omega},
\]

\[
\lim_{n \to \infty} -\alpha(\frac{k'}{k} a D_x^s u_n, v_n)_{\Omega} = -\alpha(\frac{k'}{k} a D_x^s u, v)_{\Omega},
\]

\[
\lim_{n \to \infty} \beta(\frac{k'}{k} x D_x^s u_n, v_n)_{\Omega} = \beta(\frac{k'}{k} x D_x^s u, v)_{\Omega},
\]

\[
\lim_{n \to \infty} \langle \frac{q}{k} u_n, v_n \rangle_{\Omega} = \langle \frac{q}{k} u, v \rangle_{\Omega}.
\]
For the term \((aD_x^{s'}u, xD_y^s(P_kv))\),
\[
| (aD_x^{s'}u_n, xD_y^s(P_kv_n)) - (aD_x^{s'}u, xD_y^s(P_kv)) | \\
= | (aD_x^{s'}(u_n - u), xD_y^s(P_kv_n)) - (aD_x^{s'}u, xD_y^s(P_kv_n) - P_kv) | \\
\leq \|aD_x^{s'}(u_n - u)\|_{L^2(\Omega)} \|xD_y^s(P_kv_n)\|_{L^2(\Omega)} + \|aD_x^{s'}u\|_{L^2(\Omega)} \|xD_y^s(P_kv_n) - P_kv\|_{L^2(\Omega)} \\
\leq C_1 \|u_n - u\|_{\tilde{H}_0^s(\Omega)} \|v_n\|_{\tilde{H}_0^s(\Omega)} + C_2 \|u\|_{\tilde{H}_0^s(\Omega)} \|v_n - v\|_{\tilde{H}_0^s(\Omega)}
\]
(\text{using Property 6.19 and noting } s' < s),
for some positive constants \(C_1, C_2\) not depending on \(n\), therefore
\[
\lim_{n \to \infty} (aD_x^{s'}u_n, xD_y^s(P_kv_n)) = (aD_x^{s'}u, xD_y^s(P_kv)) \Omega,
\]
and we see (7.49).

As a result, it follows from (7.40), (7.48) and (7.49) that (7.34) and (7.35) hold for all \(u, v \in \tilde{H}_0^s(\Omega)\). This completes the whole proof. \(\blacksquare\)

Once we have shown that \(B_2[u,v]\) is bounded and coercive on \(\tilde{H}_0^s(\Omega)\), applying the Lax-Milgram theorem gives the following existence of weak solution.

\textbf{Lemma 7.2.} Under conditions (7.29), there exists a unique element \(u \in \tilde{H}_0^s(\Omega)\) such that
\[
B_2[u,v] = (\frac{f}{k}, v)\Omega
\]
(7.50)
for all \(v \in \tilde{H}_0^s(\Omega)\), and there is a positive constant \(C\) such that
\[
\|u\|_{\tilde{H}_0^s(\Omega)} \leq C \|aD_x^{-s}\frac{f}{k}\|_{L^2(\Omega)}.
\]
(7.51)

\textbf{Proof.} 1. Since \(f \in H^s(\Omega)\) and \(k \in C^1(\Omega)\), there exists a certain \(p > 1\) such that \(\frac{f}{k} \in L^p(\Omega)\). Hence \(aD_x^{-s}\frac{f}{k} \in L^2(\Omega)\) is guaranteed by Property 6.3.

2. Define the linear functional \(F\): \(\tilde{H}_0^s(\Omega) \to \mathbb{R}\) as:
\[
F(v) = (aD_x^{-s}\frac{f}{k}, xD_y^s v)\Omega.
\]

On one hand,
\[
(aD_x^{-s}\frac{f}{k}, xD_y^s v)\Omega = (\frac{f}{k}, xD_x^{-s}x D_y^s v)\Omega = (\frac{f}{k}, v)\Omega
\]
by fractional integration by parts (eq. (2.20), p. 34, [6]) and Property 6.18.
On the other hand,\[ |F(v)| \leq \|a D_{x}^{-s} \frac{f}{k}\|_{L^2(\Omega)} \|D_{b}^{s}v\|_{L^2(\Omega)} \leq \|a D_{x}^{-s} \frac{f}{k}\|_{L^2(\Omega)} \|v\|_{\tilde{H}_{0}^{s}(\Omega)} \] (7.52)
by H"older inequality. Thereby, \( F(\cdot) \) is bounded on \( \tilde{H}_{0}^{s}(\Omega) \).

3. (7.50) follows immediately from the Lax-Milgram theorem and Lemma 7.1, and the estimation (7.51) follows from (7.52) and (7.35).

8. Relationship between generalized Abel equations with constant coefficients

In this section, we investigate the relationship between solutions of generalized Abel equations with constant coefficients. More precisely, we will establish in Lemma 8.3 the relationship between \( u \) and \( v \), where \( u, v \) solve equations of the form
\[ \gamma_{1} a D_{x}^{-t} u + \gamma_{2} D_{b}^{-t} u = f, \quad \gamma_{1} a D_{x}^{-t} v + \gamma_{2} D_{b}^{-t} v = Df. \] (8.53)

Before that, we shall need Lemma 8.1 and 8.2.

**Lemma 8.1.** Let \( 0 < t < 1, 0 < \gamma_{1}, \gamma_{2} \). There exists a unique solution \( u(x) \in H^{s}(\Omega) \) to the equation
\[ \gamma_{1} a D_{x}^{-t} u + \gamma_{2} D_{b}^{-t} u = 1, x \in \Omega, \]
and \( u(x) = c \cdot (x - a)^{p} (b - x)^{q} \), where
\[ c = \Gamma(-q) \left( \gamma_{1} (b - a)^{1+p+q} \Gamma(p+1) \int_{a}^{b} (x - a)^{-q-1} (b - x)^{-p-1} dx \right)^{-1} \]
and \( p, q \) are uniquely determined by conditions
\[ p + q = -t \quad \text{and} \quad \gamma_{1} \sin(q\pi) = \gamma_{2} \sin(p\pi). \] (8.54)

**Proof.** 1. We first verify that \( u(x) = c \cdot (x - a)^{p} (b - x)^{q} \) is a solution and will prove the uniqueness in the last step. Before proceeding, we simply see that our assumptions are valid in the lemma, namely, (8.54) uniquely determines \( p \) and \( q \) and \( c \) is well-defined. The first is straightforward to be checked and the later is confirmed by observing that
\[ 0 < \int_{a}^{b} (x - a)^{-q-1} (b - x)^{-p-1} dx < \infty \]
since \((x - a)^{-q-1} (b - x)^{-p-1}\) is strictly positive for \( x \in \Omega \) and is integrable over \( \Omega \).
2. Let \( t_1 = 1 + t + p + q \) and \( \tilde{u}(x) = c_1(x - a)^{-q-1}(b - x)^{-p-1} \), where
\[
c_1 = \left( \int_a^b (x - a)^{-q-1}(b - x)^{-p-1} \, dx \right)^{-1}.
\]

Note from conditions (8.54) that \(-1 < p, q < 0, -1 < -q - 1\). This allows us to apply Property 6.8 to \( \gamma_{1a}D_x^{-t_1}u \) and \( aD_x^{-t_1}\tilde{u} \) separately to obtain
\[
\gamma_{1a}D_x^{-t}u = \gamma_{1c} \frac{(b - a)^q \Gamma(p + 1)}{\Gamma(t + p + 1)} (x - a)^{t+p+1} 2F_1(-q, p + 1; t + p + 1; \frac{x - a}{b-a}),
\]
and
\[
a D_x^{-t_1}\tilde{u} = c_1 \frac{(b - a)^{-p-1} \Gamma(-q)}{\Gamma(t + p + 1)} (x - a)^{t+p+1} 2F_1(p + 1, -q, t + p + 1; \frac{x - a}{b-a}).
\]

Recall now Property 6.7, we know
\[
2F_1(-q, p + 1; t + p + 1; \frac{x - a}{b-a}) = 2F_1(p + 1, -q, t + p + 1; \frac{x - a}{b-a}).
\]

Comparing (8.55) and (8.56), we see that \( \gamma_{1a}D_x^{-t}u = aD_x^{-t_1}\tilde{u} \).

3. Similarly, we show \( \gamma_{2a}D_x^{-t}u = a D_x^{-t_1}\tilde{u} \).

According to Property 6.2 we know \( xD_x^{-t}u = Q_aD_x^{-t}Qu \) and \( x D_x^{-t_1}\tilde{u} = Q_aD_x^{-t_1}Q\tilde{u} \), where the operator \((Qf)(x) := f(a + b - x)\). By direct calculation with Property 6.8, we obtain
\[
\gamma_{2a}QaD_x^{-t}Qu = c_2 \frac{(b - a)^p \Gamma(q + 1)}{\Gamma(t + q + 1)} (b - x)^{t+q+1} 2F_1(-p, q + 1; t + q + 1; \frac{b-x}{b-a}),
\]
and
\[
QaD_x^{-t_1}Q\tilde{u} = c_1 \frac{(b - a)^{-q-1} \Gamma(-p)}{\Gamma(t + q + 1)} (b - x)^{t+q+1} 2F_1(q + 1, -p, t + q + 1; \frac{b-x}{b-a}).
\]

Notice again by Property 6.7 that
\[
2F_1(-p, q + 1; t + q + 1; \frac{b-x}{b-a}) = 2F_1(q + 1, -p, t + q + 1; \frac{b-x}{b-a}).
\]

Now compare (8.57) and (8.58), we readily check that \( \gamma_{2a}x D_x^{-t}u = x D_x^{-t_1}\tilde{u} \) in view of the second piece of conditions (8.54) and the fact that \( \Gamma(z)\Gamma(1-z) = \pi/\sin(z\pi) \), \( z \) is not an integer.

4. Consequently,
\[
\gamma_{1a}D_x^{-t}u + \gamma_{2a}xD_x^{-t}u = a D_x^{-t_1}\tilde{u} + x D_x^{-t_1}\tilde{u}.
\]
Since actually \( t_1 = 1 \) by utilizing the first piece of conditions (8.54),
\[
aD_x^{-t_1} \tilde{u} + xD_b^{-t_1} \tilde{u} = \int_a^x \tilde{u} + \int_x^b \tilde{u} = \int_a^b \tilde{u} = 1.
\]
Therefore, \( \gamma_1 aD_x^{-t} u + \gamma_2 xD_b^{-t} u = 1 \), which confirms that \( u(x) \) is a solution.

5. The uniqueness of the solution to \( \gamma_1 aD_x^{-t} u + \gamma_2 xD_b^{-t} u = 1 \) in the space \( H^*(\Omega) \) is a direct consequence of Property 6.13, provided that \( 1 \in H^*_t(\Omega) \). To see this, and in order to conveniently check the definition of \( H^*_t(\Omega) \) (see Section 5), we simply rewrite 1 in the form of
\[
1 = \frac{(x-a)^{t+\epsilon}(b-x)^{t+\epsilon}}{(x-a)^{1-(1-t-\epsilon)}(b-x)^{1-(1-t-\epsilon)}}, \tag{8.59}
\]
where \( \epsilon \) is chosen such that
\[
0 < \epsilon < 1 - t.
\]
In the numerator of right-hand side of (8.59), it can be verified that
\[
(x-a)^{t+\epsilon}, (b-x)^{t+\epsilon} \in H^{t+\epsilon}(\overline{\Omega}) \tag{8.60}
\]
with the aid of the well-known auxiliary inequality
\[
\frac{|y_1^y - y_2^y|}{|y_1 - y_2|^y} \leq 1, \quad (0 \leq y \leq 1, 0 < y_1, 0 < y_2, y_1 \neq y_2). \tag{8.61}
\]
Therefore, their product
\[
(x-a)^{t+\epsilon}(b-x)^{t+\epsilon} \in H^{t+\epsilon}_0(\Omega) \quad \text{since it vanishes at} \quad x = a, b.
\]
Hence, by the definition of \( H^*_t(\Omega) \), we see \( 1 \in H^*_t(\Omega) \), which completes the whole proof. \( \square \)

Analogously, we have the following.

**Lemma 8.2.** Let \( 0 < t < 1 \) and \( 0 < \gamma_1, \gamma_2 \). Then one of the solutions to the equation
\[
D(\gamma_1 aD_x^{-t} u + \gamma_2 xD_b^{-t} u) = 1, \quad x \in \Omega
\]
is \( u(x) = c \cdot D(x-a)^{p+1}(b-x)^{q+1} \), where
\[
c = \frac{(-p-t)(-p-t+1)\Gamma(t)}{(1-t)(2-t)\gamma_2 \Gamma(t+p+1)\Gamma(q+2)}, \tag{8.62}
\]
and \( p, q \) are uniquely determined by
\[
p + q = -t \quad \text{and} \quad \gamma_1 \sin(q\pi) = \gamma_2 \sin(p\pi). \tag{8.63}
\]
Proof. 1. We begin with calculating
\[ \gamma_1 D_x^{-t}((x-a)^{p+1}(b-x)^{q+1}) \text{ and } \gamma_2 D_b^{-t}((x-a)^{p+1}(b-x)^{q+1}). \]

2. In view of Property 6.8 and the condition \( p + q = -t \),
\[ \gamma_1 D_x^{-t}((x-a)^{p+1}(b-x)^{q+1}) \]
\[ = A \cdot \left( \frac{x-a}{b-a} \right)^{t+p+1} 2F_1(-q-1, p+2, t+p+2; \frac{x-a}{b-a}), \quad (8.64) \]
where
\[ A = \frac{\gamma_1(b-a)^2\Gamma(p+2)}{\Gamma(t+p+2)}. \quad (8.65) \]

3. We continue to compute with the condition \( p + q = -t \)
\[ \gamma_2 D_b^{-t}((x-a)^{p+1}(b-x)^{q+1}) \]
\[ = \gamma_2 Q_a D_x^{-t}Q((x-a)^{p+1}(b-x)^{q+1}) \text{ (recall } (Qf)(x) := f(a+b-x)) \]
\[ = B \cdot \left( \frac{b-x}{b-a} \right)^{t+q+1} 2F_1(-p-1, q+2, t+q+2; \frac{b-x}{b-a}), \quad (8.66) \]
where
\[ B = \frac{\gamma_2(b-a)^2\Gamma(q+2)}{\Gamma(t+q+2)}. \]

4. Now we take a closer look at the term \( 2F_1(-p-1, q+2, t+q+2; \frac{b-x}{b-a}) \). Applying Property 6.9 and the condition \( p + q = -t \) yields
\[ 2F_1(-p-1, q+2, t+q+2; \frac{b-x}{b-a}) \]
\[ = C \cdot 2F_1(-p-1, q+2, -p-t; \frac{x-a}{b-a}) \quad (8.67) \]
\[ + D \cdot \left( \frac{x-a}{b-a} \right)^{p+t+1} 2F_1(t, 3, t+p+2; \frac{x-a}{b-a}), \]
where
\[ C = \frac{\Gamma(t+q+2)\Gamma(t+p+1)}{\Gamma(t)\Gamma(3)}, \quad D = \frac{\Gamma(-t-p-1)\Gamma(t+q+2)}{\Gamma(-p-1)\Gamma(q+2)}. \]

5. Inserting (8.67) into (8.66) and taking \( p + q = -t \) into account, we thus arrive at
\[ \gamma_2 D_b^{-t}((x-a)^{p+1}(b-x)^{q+1}) \]
\[ = E \cdot \left( \frac{b-x}{b-a} \right)^{t+q+1} 2F_1(-p-1, q+2, -p-t; \frac{x-a}{b-a}) \quad (8.68) \]
\[ + F \cdot \left( \frac{x-a}{b-a} \right)^{p+t+1} \left( \frac{b-x}{b-a} \right)^{t+q+1} 2F_1(t, 3, t+p+2; \frac{x-a}{b-a}), \]
where
\[ E = \frac{\gamma_2(b-a)^2\Gamma(t+p+1)\Gamma(q+2)}{\Gamma(t)\Gamma(3)}, \quad F = \frac{\gamma_2(b-a)^2\Gamma(-t-p-1)}{\Gamma(-p-1)}. \]

Notice \( t + q + 1 = 1 - p \) and we apply Property 6.10 to (8.68) to obtain
\[ \gamma_2 x D^{-t}_b ((x-a)^{p+1}(b-x)^{q+1}) 
= E \cdot 2F_1(1-t,-2,-p-t;\frac{x-a}{b-a}) 
+ F \cdot (\frac{x-a}{b-a})^{p+t+1} 2F_1(p+2,t+p-1,t+p+2;\frac{x-a}{b-a}). \quad (8.69) \]

Since
\[ 2F_1(p+2,t+p-1,t+p+2;\frac{x-a}{b-a}) = 2F_1(-q-1,p+2,t+p+2;\frac{x-a}{b-a}) \]
(by Property 6.7), (8.69) further becomes
\[ \gamma_2 x D^{-t}_b ((x-a)^{p+1}(b-x)^{q+1}) 
= E \cdot 2F_1(1-t,-2,-p-t;\frac{x-a}{b-a}) 
+ F \cdot (\frac{x-a}{b-a})^{p+t+1} 2F_1(-q-1,p+2,t+p+2;\frac{x-a}{b-a}). \quad (8.70) \]

Compare the last term with (8.64), we therefore see that
\[ \gamma_1 a D^{-t}_x ((x-a)^{p+1}(b-x)^{q+1}) 
= E \cdot 2F_1(1-t,-2,-p-t;\frac{x-a}{b-a}) 
+ F \cdot A^{-1} \cdot \gamma_1 a D^{-t}_x ((x-a)^{p+1}(b-x)^{q+1}). \quad (8.71) \]

6. Consequently, summing up (8.64) and (8.71) and simplifying by using the definition of Hypergeometric function (Definition 6.3) give
\[ \gamma_1 a D^{-t}_x ((x-a)^{p+1}(b-x)^{q+1}) + \gamma_2 x D^{-t}_b ((x-a)^{p+1}(b-x)^{q+1}) 
= E \cdot 2F_1(1-t,-2,-p-t;\frac{x-a}{b-a}) 
+ (1 + F \cdot A^{-1}) \gamma_1 a D^{-t}_x ((x-a)^{p+1}(b-x)^{q+1}) 
= E \cdot \sum_{n=0}^{2} \frac{(1-t)_n(-2)_n}{(-p-t)_n n!} \frac{x-a}{b-a}^n \quad \text{(every term is zero after } n = 2) 
+ (1 + F \cdot A^{-1}) \gamma_1 a D^{-t}_x ((x-a)^{p+1}(b-x)^{q+1}). \quad (8.72) \]
7. It suffices to show that $1 + F \cdot A^{-1} = 0$ by taking advantage of the formula $\Gamma(z)\Gamma(1-z) = \pi / \sin(z\pi)$, $(z$ is not an integer). Namely, we check

$$F \cdot A^{-1} = \frac{\gamma_2 \Gamma(-1-t-p)\Gamma(t+p+2)}{\gamma_1 \Gamma(-p-1)\Gamma(p+2)}$$

$$= \frac{\gamma_2}{\gamma_1} \cdot \frac{\sin(p\pi)}{\sin((t+p)\pi)}$$

$$= -\frac{\gamma_2}{\gamma_1} \cdot \frac{\sin(p\pi)}{\sin(q\pi)},$$

thus $1 + F \cdot A^{-1} = 0$ in view of the second piece of conditions $[8.63]$. It follows that

$$\gamma_1 a D_x^{-t}((x-a)^{p+1}(b-x)^{q+1}) + \gamma_2 x D_b^{-t}((x-a)^{p+1}(b-x)^{q+1})$$

$$= E \cdot \sum_{n=0}^{2} \frac{(1-t)_n(-2)_n x - a}{(-p-t)_n n!} (b - a)^n. \quad (8.74)$$

8. Last, on one hand, differentiating $(8.74)$ twice at both sides gives

$$DD(\gamma_1 a D_x^{-t} + \gamma_2 x D_b^{-t})((x-a)^{p+1}(b-x)^{q+1})$$

$$= \frac{(1-t)(2-t)\gamma_2 \Gamma(t+p+1)\Gamma(q+2)}{(-p-t)(-p-t+1)\Gamma(t)}.$$

On the other hand, interchanging the order of differentiation and fractional integrations is permitted ([6], Theorem 2.2, p. 39) and results in

$$DD(\gamma_1 a D_x^{-t} + \gamma_2 x D_b^{-t})((x-a)^{p+1}(b-x)^{q+1})$$

$$= D(\gamma_1 a D_x^{-t} + \gamma_2 x D_b^{-t})D((x-a)^{p+1}(b-x)^{q+1}).$$

Thereby,

$$D(\gamma_1 a D_x^{-t} + \gamma_2 x D_b^{-t})D((x-a)^{p+1}(b-x)^{q+1})$$

$$= \frac{(1-t)(2-t)\gamma_2 \Gamma(t+p+1)\Gamma(q+2)}{(-p-t)(-p-t+1)\Gamma(t)}.$$

Dividing both sides of $(8.75)$ by the right-hand side concludes

$$D(\gamma_1 a D_x^{-t} + \gamma_2 x D_b^{-t})u = 1,$$

as desired, which completes the whole proof. \hfill \Box

Based on Lemma $[8.1]$ and $[8.2]$ we arrive at the following:

**Lemma 8.3.** Let $0 < \sigma < 1$ and $0 < \gamma_1, \gamma_2$. Assume that $f \in H^*_s(\Omega)$, $Df$ exists in $\Omega$ and $Df \in H^*_s(\Omega)$. If $u(x), v(x) \in H^*(\Omega)$ and satisfy

$$(\gamma_1 a D_x^{-\sigma} + \gamma_2 x D_b^{-\sigma})u = f \text{ and } (\gamma_1 a D_x^{-\sigma} + \gamma_2 x D_b^{-\sigma})v = Df, \text{ } x \in \Omega,$$

$(8.77)$
then
\[ D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})(u - Y) = 0, \ x \in \Omega, \]  
(8.78)

where
\[ Y = \int_a^x v(t) \, dt - \frac{cS}{S_1} \int_a^x (t-a)^p (b-t)^q \, dt + \frac{c_1 S}{S_1} D((x-a)^{p+1}(b-x)^q+1), \]
\[ S = \int_a^b v(t) \, dt, \]
\[ c = \Gamma(-q) \left( \gamma_1 (b-a)^{1+p+q} \Gamma(p+1) \int_a^b (t-a)^{-q-1}(b-t)^{-p-1} \, dt \right)^{-1}, \]
\[ S_1 = \frac{\Gamma(-q) \int_a^b (t-a)^p (b-t)^q \, dt}{\gamma_1 (b-a)^{1+p+q} \Gamma(p+1) \int_a^b (t-a)^{-q-1}(b-t)^{-p-1} \, dt}, \]
\[ c_1 = \frac{(-p-\sigma)(-p-\sigma+1) \Gamma(\sigma)}{(1-\sigma)(2-\sigma) \gamma_2 \Gamma(\sigma + p + 1) \Gamma(q+2)}, \]  
(8.79)

and \( p, q \) are uniquely determined by
\[ p + q = -\sigma \quad \text{and} \quad \gamma_1 \sin(q\pi) = \gamma_2 \sin(p\pi). \]  
(8.80)

**Proof.** 1. According to the assumption in the lemma we first have
\[ D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})u - (\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})v = 0, \ x \in \Omega. \]  
(8.81)

We examine the second term
\[ (\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})v \]
\[ = \gamma_1 D_x^{-\sigma} D_x^{-1} v - \gamma_2 D_b^{-\sigma} D_b^{-1} v \]
\[ = \gamma_1 D_a D_x^{-\sigma} a D_x^{-1} v - \gamma_2 D x D_b^{-\sigma} b D_b^{-1} v \]
(interchanging the order of operators by Theorem 2.2, p. 39, [6])
\[ = \gamma_1 D_a D_x^{-\sigma} a D_x^{-1} v - \gamma_2 D x D_b^{-\sigma} b D_b^{-1} v (S - a D_x^{-1} v) \]  
(let \( S = \int_a^b v(t) \, dt \))
\[ = D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})a D_x^{-1} v + \frac{\gamma_2 S}{\Gamma(\sigma)}(b-x)^{\sigma-1}. \]  
(8.82)

Substituting this into (8.81) gives
\[ D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})(u - a D_x^{-1} v) = \frac{\gamma_2 S}{\Gamma(\sigma)}(b-x)^{\sigma-1}. \]  
(8.83)

2. We claim
\[ D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})(c \int_a^x (t-a)^p (b-t)^q \, dt) = 1 - \frac{\gamma_2 S_1}{\Gamma(\sigma)}(b-x)^{\sigma-1}, \]  
(8.84)
where
\[
c = \Gamma(-q) \left( \gamma_1(b-a)^{1+p+q} \Gamma(p+1) \int_a^b (t-a)^{-q-1}(b-t)^{-p-1} \, dt \right)^{-1},
\]
\[
S_1 = c \int_a^b (t-a)^p(b-t)^q \, dt,
\]
and \(p, q\) satisfy\[p + q = -\sigma \quad \text{and} \quad \gamma_1 \sin(q\pi) = \gamma_2 \sin(p\pi).\]

To see this,
\[
D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})(c \int_a^x (t-a)^p(b-t)^q \, dt) = \gamma_1 D_x^{-\sigma}(c \int_a^x (t-a)^p(b-t)^q \, dt) + \gamma_2 D_b^{-\sigma}(S_1 - c \int_a^b (t-a)^p(b-t)^q \, dt)
\]
\[
= \gamma_1 D_x^{-\sigma} D(c \int_a^x (t-a)^p(b-t)^q \, dt) + \gamma_2 D_b^{-\sigma} D(-c \int_a^b (t-a)^p(b-t)^q \, dt) + \gamma_2 D_x D_b^{-\sigma} S_1
\]
(interchanging the order of operators)
\[
= (\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})(c(x-a)^p(b-x)^q) - \frac{\gamma_2 S_1}{\Gamma(\sigma)(b-x)^{\sigma-1}}
\]
\[
= 1 - \frac{\gamma_2 S_1}{\Gamma(\sigma)(b-x)^{\sigma-1}} \quad \text{(applying Lemma 8.1)}.
\]

3. Before going further, let us simply see that \(c\) is well-defined, \(c \neq 0\) and \(S_1 \neq 0\) by observing that
\[
0 < \int_a^b (t-a)^{-q-1}(b-t)^{-p-1} \, dt, \quad \int_a^b (t-a)^p(b-t)^q \, dt < \infty.
\]

Adding (8.83) to (8.84) multiplied by \(\frac{S}{S_1}\), we have
\[
D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma})(u - a D_x^{-\sigma} v + \frac{cS}{S_1} \int_a^x (t-a)^p(b-t)^q \, dt) = \frac{S}{S_1},
\]
(8.85)

Invoking Lemma 8.2 we know
\[
D(\gamma_1 D_x^{-\sigma} + \gamma_2 D_b^{-\sigma}) \left( -\frac{c_1 S}{S_1} D(x-a)^{p+1}(b-x)^{q+1} \right) = -\frac{S}{S_1},
\]
(8.86)
where
\[ c_1 = \frac{(-p - \sigma)(-p - \sigma + 1)\Gamma(\sigma)}{(1 - \sigma)(2 - \sigma)\gamma_2 \Gamma(\sigma + p + 1)\Gamma(q + 2)}. \]

Summing up (8.85) and (8.86) produces
\[ D(\gamma_1 D_x^\sigma + \gamma_2 D_y^{-\sigma})(u - Y) = 0, \quad (8.87) \]
where
\[ Y = \int_a^x v(t) \, dt - \frac{cS}{S_1} \int_a^x (t - a)^p (b - t)^q \, dt + \frac{c_1 S}{S_1} D(x - a)^{p+1}(b - x)^{q+1}, \]
which is the desired result. We will keep the coefficients this way since it is convenient to be used later. \(\square\)

9. Raising the regularity

In this section, we will establish three lemmas, which are the key analysis of the whole work and crucial steps towards the proof of Theorem 1.1. Also, as explained in Section 3, this three lemmas will help us connect the weak solution from the Sobolev space \(H_0^{1+\mu/2}(\Omega)\) to \(H^*(\Omega)\) and further raise the regularity from the space \(H^*(\Omega)\) to better spaces \(H^*_\sigma(\Omega)\).

Let \(0 < \mu, \alpha, \beta < 1, \alpha + \beta = 1\) throughout this section (Section 9).

**Lemma 9.1.** Let \(A = \alpha - \beta \cos(\mu \pi), B = \beta \sin(\mu \pi)\) and \(f(x) \in H(\Omega)\). Denote \(r_t(x) = b - x, x \in \Omega, F(x) = (b - x)^\mu f(x)\) and \(A = \frac{A - iB}{A + iB} = e^{i\theta}\) with the value of \(\theta\) chosen so that \(0 \leq \theta < 2\pi\). Consider the problem
\[ A\psi(x) + B \pi \int_a^b \frac{\psi(t)}{t - x} \, dt = F(x), x \in \Omega. \quad (9.88) \]
Then each of the following is valid:

(1) \(9.88\) is solvable in spaces \(X_2 = H^*(\Omega) \cap C((a, b))\) and \(X_3 = H^*(\Omega) \cap C([a, b])\) respectively, and its according solution \(\psi_i(i = 2, 3)\) is unique and is represented as
\[ \psi_i(x) = \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \left( \frac{x - a}{t - a} \right)^{1 - n_a(X_i) - \frac{\mu}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\mu}{2\pi} - n_b(X_i)} F(t) \frac{t - x}{t - x} \, dt, \quad (9.89) \]
where
\[ n_a(X_2) = 1, n_a(X_3) = 0; \]
\[ n_b(X_2) = 0, n_b(X_3) = 1. \]
(2) $\theta$ satisfies
\[ \mu < \frac{\theta}{2\pi} < 1. \] (9.90)

(3) The solution $\psi_i(x)$ in part (1) satisfies
\[ \frac{\psi_i(x)}{(b - x)^\mu} \in H^s(\Omega), \quad (i = 2, 3). \] (9.91)

(4) The solution $\psi_2(x)$ in part (1) satisfies
\[ aD_x^\mu \frac{\psi_2}{r_b^p} \in L^p(I_b), \] (9.92)
where $p = \frac{1}{1 - \frac{2\pi}{\theta}}$, $I_b = (\frac{b - a}{2}, b)$.

(5) If the solution $\psi_i(x)$ in part (1) satisfies $\psi_2(x) = \psi_3(x)$, $x \in \Omega$, then $\psi_2$ (or $\psi_3$, which is the same) has four equivalent representations:

\[
\psi_2(x) = \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \frac{(x - a)}{(t - a)^{\frac{\theta}{2\pi}}} \frac{(b - x)}{(b - t)^{\frac{\theta}{2\pi}}} \frac{F(t)}{t - x} dt,
\]
\[
= \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \frac{(x - a)}{(t - a)^{\frac{\theta}{2\pi}}} \frac{(b - x)}{(b - t)^{\frac{\theta}{2\pi} - 1}} \frac{F(t)}{t - x} dt,
\]
\[
= \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \frac{(x - a)}{(t - a)^{\frac{\theta}{2\pi}}} \frac{(b - x)}{(b - t)^{\frac{\theta}{2\pi} - 1}} \frac{F(t)}{t - x} dt.
\] (9.93)

**Proof.** 1. Proof for part (1).

Let us see that part (1) is just a direct consequence of the first part of Property 6.16, we only need to justify the applicability of Property 6.16.

We shall need to check three aspects.

Firstly, $A^2 + B^2 \neq 0$ by recalling $0 < \mu, \alpha, \beta < 1, \alpha + \beta = 1$ (see the beginning of Section 9).

Secondly, the function $F(x)$ can be equivalently rewritten as
\[ F(x) = (b - x)^\mu f(x) = \frac{(b - x)^\mu f(x)}{(x - a)^{1-1}(b - x)^{1-1}}. \] (9.94)

In the numerator, since $f \in H(\Omega)$, there exists a $0 < \lambda_0 < \mu$ so that
\[ f \in H^{\lambda_0}(\Omega), \] (9.95)
and we directly check that
\[ (b - x)^\mu \in H^{\mu}(\Omega) \]
with the assistance of the well-known inequality
\[ \frac{|y_1^y - y_2^y|}{|y_1^y - y_2^y|} \leq 1, \quad (0 \leq y \leq 1, 0 < y_1, 0 < y_2, y_1 \neq y_2). \] (9.96)

Therefore, their product \((b - x)^{\mu} f(x) \in H^{\lambda} \Omega \) follows.

Lastly, observe that \( B \neq 0 \), which implies that \( \theta \neq 0 \). Hence,
\[ 1 - n_a(X_i) - \frac{\theta}{2\pi} < 1, \quad \frac{\theta}{2\pi} - n_b(X_i) < 1 \]
hold for \( i = 2, 3 \).

So, all the hypotheses are met for applying the first part of Property 6.16, the part (1) of Lemma 9.1 follows.

2. Now we prove the part (2).

Since \( A - iB \)
\[ \frac{A - iB}{A + iB} = e^{i\theta} \quad \text{and} \quad 0 \leq \theta < 2\pi, \]
(9.97)

it is clear that
\[ \frac{\theta}{2\pi} < 1. \]

Substituting for \( A, B \) and simplifying (9.97), we have
\[ \frac{\alpha - \beta \cos(\mu\pi) - i\beta \sin(\mu\pi)}{\alpha - \beta \cos(\mu\pi) + i\beta \sin(\mu\pi)} = \frac{\alpha - \beta e^{i\mu\pi}}{\alpha - \beta e^{-i\mu\pi}} = \frac{\alpha - e^{i\mu\pi}}{\alpha - e^{-i\mu\pi}} = e^\theta. \]

Solving the last equality for \( \frac{\alpha}{\beta} \) gives
\[ \frac{\alpha}{\beta} = \frac{e^{i(\theta - 2\mu\pi)} - 1}{e^{i\theta} - 1} e^{i\mu\pi}. \]

Taking the fact \( e^{iz} - 1 = 2ie^{iz/2} \sin(z/2), \ z \in \mathbb{C} \) into account, we arrive at
\[ \frac{\alpha}{\beta} = \frac{\sin((\theta - 2\mu\pi)/2)}{\sin(\theta/2)}. \]

Again, recalling \( 0 < \mu, \alpha, \beta < 1, \alpha + \beta = 1 \) we derive
\[ 0 < (\theta - 2\mu\pi)/2 < \pi. \]

Hence, we see the assertion
\[ \mu < \frac{\theta}{2\pi} < 1. \]

This fact will be used in the rest of proof and in the proof of subsequent lemmas.

3. To prove the part (3), namely \( \frac{\psi_i(x)}{(b - x)^{\mu}} \in H^* \Omega \) \((i = 2, 3)\), we discuss the two cases separately in this step and the next step.
For $i = 2$, substituting for $\psi_2$ into $\frac{\psi_2(x)}{(b-x)^\mu}$ by using the representation in part (1) of the lemma and simplifying, we obtain

$$\frac{\psi_2(x)}{(b-x)^\mu} = \frac{Af(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right) - \frac{\theta}{\pi} \left( \frac{b-x}{b-t} \right) \frac{f(t)}{t-x} dt$$

$$= \frac{A}{A^2 + B^2} \Pi_1 - \frac{B}{\pi(A^2 + B^2)} \Pi_2. \tag{9.98}$$

Let us look at $\Pi_1$ and $\Pi_2$.

$\Pi_1$, namely $f(x)$ (recall $f \in H^{\lambda_0}(\Omega)$ in (9.95)), is relatively easily seen to belong to $H^*(\Omega)$ by manipulating as follows:

$$\Pi_1 = \frac{(x-a)^\epsilon f(x)(b-x)^\epsilon}{(x-a)^{(1-\epsilon)}(b-x)^{(1-\epsilon)}}, \tag{9.99}$$

where $\epsilon$ is chosen so that $0 < \epsilon < \lambda_0$.

In the numerator, it can be directly verified with (9.96) that

$$(x-a)^\epsilon, (b-x)^\epsilon \in H^\epsilon(\Omega). \tag{9.100}$$

Therefore, the product

$$(x-a)^\epsilon f(x)(b-x)^\epsilon \in H^\epsilon_0(\Omega) \tag{9.101}$$

by taking into account the boundary and $f \in H^{\lambda_0}(\Omega)$.

In the denominator of (9.99), simply observe that $1 - \epsilon > 0$.

Hence, $\Pi_1 \in H^*(\Omega)$ follows by the definition of $H^*(\Omega)$ (see notation in Section 5).

Similarly, to see $\Pi_2 \in H^*(\Omega)$, we rewrite

$$\Pi_2 = \int_a^b \left( \frac{x-a}{t-a} \right)^\epsilon \left( \frac{b-x}{b-t} \right) \frac{f(t)}{t-x} dt$$

$$= \int_a^b \left( \frac{x-a}{t-a} \right)^\epsilon \left( \frac{b-x}{b-t} \right) \frac{\theta^{/(2\pi)} - \mu + \epsilon}{t-x} \frac{(t-a)^{\theta^{/(2\pi)} + \epsilon} f(t) (b-t)^\epsilon}{t-x} dt$$

$$= \frac{(x-a)^{(1-\theta^{/(2\pi)} - \epsilon)} (b-x)^{(1-\epsilon)}}{(x-a)^{(1-\epsilon)} (b-x)^{(1-\epsilon)}}, \tag{9.102}$$

where $\epsilon$ is chosen so that $0 < \epsilon < \min\{\lambda_0, 1 - \frac{\theta}{2\pi}\}$ (it should be clear this $\epsilon$ is different from the one in (9.99) and we will use notation $\epsilon$ this way several times in the rest of proof).

In the denominator, $1 - \theta^{/(2\pi)} - \epsilon > 0$, $1 - \epsilon > 0$.

According to the definition of $H^*(\Omega)$, $\Pi_2$ belongs to $H^*(\Omega)$ is ensured provided that the numerator is in $H^{\theta^{/(2\pi)}}(\Omega)$, namely

$$\int_a^b \left( \frac{x-a}{t-a} \right)^\epsilon \left( \frac{b-x}{b-t} \right) \frac{\theta^{/(2\pi)} - \mu + \epsilon}{t-x} \frac{(t-a)^{\theta^{/(2\pi)} + \epsilon} f(t) (b-t)^\epsilon}{t-x} dt \in H^{\theta^{/(2\pi)}}_0(\Omega). \tag{9.103}$$
is verified by a direct application of Property 6.15 by checking that
\[(t - a)^{\theta/(2\pi)} + \epsilon f(t)(b - t)^\epsilon \in H^{\epsilon/2}_0(\Omega),\]
which can be justified analogously to (9.101), and that
\[\epsilon/2 < \epsilon < 1 + \epsilon/2, \quad \epsilon/2 < \theta/(2\pi) - \mu + \epsilon < 1 + \epsilon/2,
\]
where the fact that \(\theta/(2\pi) - \mu > 0\) from (9.90) was used in the second piece.

Hence, \(\Pi_2 \in H^*(\Omega),\)
and therefore,
\[\frac{\psi_2(x)}{(b - x)^\mu} \in H^*(\Omega).
\]

4. We continue to consider the case \(i = 3.\)

Substituting for \(\psi_3(x)\) into \(\frac{\psi_3(x)}{(b - x)^\mu}\) by using the representation in part (1) of the lemma and simplifying, we have
\[
\psi_3(x) = Af(x) - \frac{B}{\pi(A^2 + B^2)} \int_a^b \frac{1}{(t - a)^{\theta/(2\pi)} + \epsilon} f(t)(b - t)^\epsilon \frac{(b - x)(t - a)^{\theta/(2\pi)} - \mu - \epsilon}{t - x} dt
\]
\[= \frac{A}{A^2 + B^2} \Sigma_1 - \frac{B}{\pi(A^2 + B^2)} \Sigma_2.
\]

For the first term, \(\Sigma_1 = f(x),\) which is the same as \(\Pi_1\) in the previous step, thus, \(\Sigma_1 \in H^*(\Omega).\)

For \(\Sigma_2,\)
\[
\Sigma_2 = \int_a^b \frac{(x - a)^{1-\theta/(2\pi) + \epsilon}}{(t - a)^{1-\theta/(2\pi) - \mu - \epsilon}} \frac{(b - x)(t - a)^{\theta/(2\pi)} - \mu - \epsilon}{t - x} \frac{1}{(x - a)^{1-(1-\epsilon)(b - x)^{1-(\theta/(2\pi) - \mu - \epsilon)}}} dt
\]
\[(9.106)
\]
where \(\epsilon\) is chosen so that \(0 < \epsilon < \min\{\lambda_0, \frac{\theta}{2\pi} - \mu\}.
\nAgain, observe that in the denominator \(1 - \epsilon > 0\) and \(\frac{\theta}{2\pi} - \mu - \epsilon > 0.
\nThen by the definition of \(H^*(\Omega),\) \(\Sigma_2\) belongs to \(H^*(\Omega)\) is guaranteed provided that the numerator is in \(H^{\epsilon/2}_0(\Omega),\) namely
\[
\int_a^b \frac{(x - a)^{1-\theta/(2\pi) + \epsilon}}{(t - a)^{1-\theta/(2\pi) - \mu - \epsilon}} \frac{(b - x)(t - a)^{\theta/(2\pi)} - \mu - \epsilon}{t - x} \frac{1}{(x - a)^{1-(1-\epsilon)(b - x)^{1-(\theta/(2\pi) - \mu - \epsilon)}}} dt \in H^{\epsilon/2}_0(\Omega).
\]
\[(9.107)\]
(9.107) is guaranteed by Property 6.15 and can be justified similarly to (9.103) without essential difference. Hence, \( \Sigma_2 \in H^*(\Omega) \), and therefore,

\[
\psi_3(x) = (b-x)\mu \in H^*(\Omega).
\]

This completes the proof for part (3).

5. Proof for part (4).

Using (9.98) and integrating both sides by \( aD_x^{-\mu} \),

\[
aD_x^{-\mu} \psi_2 \sim \frac{A}{A^2 + B^2} aD_x^{-\mu} \Pi_1 - \frac{B}{\pi(A^2 + B^2)} aD_x^{-\mu} \Pi_2. \tag{9.108}
\]

It is clear that \( aD_x^{-\mu} \Pi_1 \in L^p(I_b), p = \frac{1}{1 - \frac{\theta}{2\pi}}, I_b = (\frac{b-a}{2}, b) \), since \( \Pi_1 = f(x) \in H^{\lambda_0}(\Omega) \). In order to show \( aD_x^{-\mu} \psi_2 \in L^p(I_b) \), we only need to show \( aD_x^{-\mu} \Pi_2 \in L^p(I_b) \).

Recall from (9.104) that \( \Pi_2 \in H^*(\Omega) \), which implies that \( \Pi_2 \in L^z(\Omega) \) for some \( z > 1 \). This allows us to apply the fact (eq. (11.17), p. 206, [6]) that

\[
aD_x^{-\mu} g = \cos(\mu \pi) aD_x^{-\mu} \sin(\mu \pi) aD_x^{-\mu} (r_b^{-\mu} S(r_b^\mu g)), \quad \text{for } g(x) \in L^p(\Omega), p > 1
\]

to \( aD_x^{-\mu} \Pi_2 \) to obtain

\[
aD_x^{-\mu} \Pi_2 = \cos(\mu \pi) aD_x^{-\mu} \sin(\mu \pi) aD_x^{-\mu} (r_b^{-\mu} S(r_b^\mu \Pi_2)). \tag{9.109}
\]

Let us examine \( xD_b^{-\mu} \Pi_2 \) first. Indeed,

\[
\Pi_2 \in L^1(I_b), \quad \text{for any } t > 0, \tag{9.110}
\]

by seeing that in (9.102) the \( \epsilon \) can be chosen as small as possible. Certainly,

\[
xD_b^{-\mu} \Pi_2 \in L^p(I_b), \quad p = \frac{1}{1 - \frac{\theta}{2\pi}}. \tag{9.111}
\]

Secondly, we look \( xD_b^{-\mu} (r_b^{-\mu} S(r_b^\mu \Pi_2)) \). Utilizing the second line of (9.102), we obtain

\[
r_b^{-\mu} \Pi_2 = \int_a^b \frac{(x-a)^{\epsilon}}{t-a} \frac{(b-t)^{\theta/(2\pi)} + \epsilon (t-a)^{\theta/(2\pi)} + f(t)(b-t)^{\epsilon + \mu}}{t-x} dt, \tag{9.112}
\]

where \( 0 < \epsilon < \min\{\lambda_0, 1 - \frac{\theta}{2\pi}\} \).
On the right-hand side of (9.112), the numerator belongs to \( H^{\epsilon/2}_0(\Omega) \), which is justified analogously to (9.103). Therefore, by virtue of Property 6.15,

\[
\begin{align*}
\frac{b^{-\mu}}{r_b^{-\mu} S(r_b^\mu \Pi_2)} &= \frac{h(x)}{(x-a)^{\theta/(2\pi)} + \epsilon(b-x)^{e+\mu}}, \\
\end{align*}
\]

(9.113)

for a certain \( h(x) \in H^{\epsilon/2}_0(\Omega) \) and \( \epsilon \) is the same as in (9.112). It is straightforward to check

\[
\frac{b^{-\mu}}{r_b^{-\mu} S(r_b^\mu \Pi_2)} \in L^{1/(\mu+2\epsilon)}(I_b).
\]

Remembering that the \( \epsilon \) can be chosen as small as possible in (9.113) and by another use of Property 6.3 to (9.113) over the interval \((\frac{a}{2}, b)\), we derive

\[
\begin{align*}
xD_{b}^{-\mu}(r_b^{-\mu} S(r_b^\mu \Pi_2)) \in L^\nu(I_b), 
\end{align*}
\]

(9.114)

In particular,

\[
\begin{align*}
xD_{b}^{-\mu}(r_b^{-\mu} S(r_b^\mu \Pi_2)) \in L^p(I_b), 
\end{align*}
\]

(9.115)

Combining (9.114) and (9.115) gives the desired result

\[
\begin{align*}
\frac{\psi_2}{r_b^\mu} \in L^p(I_b),
\end{align*}
\]

(9.116)

where

\[
\begin{align*}
p = \frac{1}{1 - \frac{\theta}{2\pi}}, 
I_b = \left(\frac{b-a}{2}, b\right).
\end{align*}
\]

6. Proof for part (5).

Since we assume the solutions satisfy \( \psi_2(x) = \psi_3(x) \), \( \psi_2 \) (or \( \psi_3 \)) belongs to \( H^*(\Omega) \cap C([a, b]) \) due to \( \psi_2 \in H^*(\Omega) \cap C([a, b]) \) and \( \psi_3 \in H^*(\Omega) \cap C([a, b]) \). This means the problem (9.88) is solvable in the following four spaces:

\[
\begin{align*}
H^*(\Omega), H^*(\Omega) \cap C([a, b]), H^*(\Omega) \cap C([a, b]), H^*(\Omega) \cap C([a, b]).
\end{align*}
\]

(9.117)
From Property 6.16 we know that, as a solution of (9.88), \( \psi_2 \) totally has four representations in these four spaces, namely:

\[
\psi_2(x) = \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right)^{1-\frac{\theta}{2\pi}} \left( \frac{b-x}{b-t} \right)^{\frac{\theta}{2\pi}} F(t) \frac{dt}{t-x},
\]

\( \in H^*(\Omega) \cap C([a,b]) \),

\[
\psi_2(x) = \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right)^{1-\frac{\theta}{2\pi}} \left( \frac{b-x}{b-t} \right)^{\frac{\theta}{2\pi}-1} F(t) \frac{dt}{t-x},
\]

\( \in H^*(\Omega) \cap C([a,b]) \),

\[
\psi_2(x) = \frac{AF(x)}{A^2 + B^2} - \frac{B}{\pi(A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right)^{-\frac{\theta}{2\pi}} \left( \frac{b-x}{b-t} \right)^{\frac{\theta}{2\pi}} F(t) \frac{dt}{t-x},
\]

\( + C(x-a)^{-\frac{\theta}{2\pi}}(b-x)^{\frac{\theta}{2\pi}-1} \)

\( \in H^*(\Omega) \). (9.118)

This completes the proof provided that \( C = 0 \) in the forth equation.

To see this, first we calculate

\[
C = \left( \psi_2(x) - \frac{AF(x)}{A^2 + B^2} \right)(x-a)^{\frac{\theta}{2\pi}}(b-x)^{1-\frac{\theta}{2\pi}}
\]

\[
+ \frac{B}{\pi(A^2 + B^2)} \int_a^b \left( t-a \right)^{\theta/(2\pi)} \frac{f(t)(b-t)^{1-(\theta/(2\pi)) - \mu}}{t-x} dt, x \in \Omega.
\] (9.119)

The first term equals to 0 at the boundary point \( x = b \).

In the second term, we can directly verify, analogously to (9.101), that

\[
(t-a)^{\theta/(2\pi)} f(t)(b-t)^{1-(\theta/(2\pi)) - \mu} \in H_0^{\epsilon_0}(\Omega), \epsilon_0 = \min\{\lambda_0, \frac{\theta}{2\pi}, 1 - (\frac{\theta}{2\pi} - \mu)\}.
\]

Thus, in light of Property 6.15 we see

\[
\frac{1}{\pi} \int_a^b \left( t-a \right)^{\theta/(2\pi)} \frac{f(t)(b-t)^{1-(\theta/(2\pi)) - \mu}}{t-x} dt \in H_0^{\epsilon_0}(\Omega).
\]
Consequently,

\[
C = \lim_{x \to b^-} \left( \psi(x) - \frac{AF(x)}{A^2 + B^2} \right) (x - a) \frac{\theta}{\pi} (b - x)^{1 - \frac{\theta}{\pi}} \\
+ \lim_{x \to b^-} \frac{B}{\pi(A^2 + B^2)} \int_{a}^{b} \left( \frac{(t - a)^{\theta/(2\pi)} f(t)(b - t)^{1 - \theta/(2\pi) - \mu}}{t - x} \right) dt 
(9.120)
= 0 + 0 \\
= 0.
\]

This completes the proof of part (5), and the whole proof of Lemma 9.1 is completed. \qed

The following lemma provides a bridge connecting the solutions of coupled Abel integral equations in the Sobolev space to the solutions in the space \( H^*(\Omega) \). By which we mean that, if a solution \( \psi(x) \) of the coupled Abel integral equation is located in \( \hat{H}^{(1+\mu)/2}(\Omega) \) (see equation (9.121)), then its \( \mu \)-th order derivative \( aD_x^{-\mu}\psi \) actually has a representative belonging to \( H^*(\Omega) \) (which is equivalent to what equation (9.122) says). This is an important connection and preparation for us to continue to raise the regularity of \( aD_x^{-\mu}\psi \) to better spaces \( H^*_\sigma(\Omega) \) from \( H^*(\Omega) \) in subsequent Lemma 9.3, thereby raising the regularity of \( \psi(x) \).

As we will see later, this \( \psi \) in (9.121) actually represents the weak solution of our problem (1.2); Lemma 9.2 and 9.3 are two key intermediate steps towards converting the weak solution to the classical solution.

**Lemma 9.2.** Let \( c \) be a constant, \( \psi(x) \in \hat{H}^{(1+\mu)/2}(\Omega) \) and \( f(x) \in H(\Omega) \). If

\[
\alpha_aD_x^{-(1-\mu)}\psi + \beta_xD_b^{-(1-\mu)}\psi a.e. = aD_x^{-1}f + c, \ x \in \Omega, 
(9.121)
\]

then the solution \( \psi(x) \) has a representation

\[
\psi(x) = aD_x^{-\mu}J, 
(9.122)
\]
where \( J(x) \in H^*(\Omega) \) and \( J(x) \) has four equivalent representations:

\[
J(x) = \frac{Af(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right) \left( \frac{b-x}{b-t} \right) f(t) \, dt,
\]

\[
J(x) = \frac{Af(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right) \left( \frac{b-x}{b-t} \right) \frac{\mu}{2\pi} f(t) \, dt,
\]

\[
J(x) = \frac{Af(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right) \left( \frac{b-x}{b-t} \right) \left( \frac{\mu}{2\pi} - \frac{\mu-1}{2\pi} \right) f(t) \, dt,
\]

\[
J(x) = \frac{Af(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_a^b \left( \frac{x-a}{t-a} \right) \left( \frac{b-x}{b-t} \right) \left( \frac{\mu}{2\pi} - \frac{\mu-1}{2\pi} \right) f(t) \, dt.
\]

(9.123)

where \( A = \alpha - \beta \cos(\mu \pi) \), \( B = \beta \sin(\mu \pi) \).

Prop. 1. Differentiating both sides of equation (9.121) is valid by the assumption \( \psi \in \tilde{H}_0^{(1+\mu)/2} (\Omega) \) and Property 6.18 from which we have

\[
D(a_0 D_x^{-(1-\mu)} \psi + \beta_x D_b^{-(1-\mu)} \psi) \overset{a.e.}{=} f.
\]

(9.124)

Distributing the differentiation operator \( D \) is permitted and gives

\[
\alpha_a D_x^{\mu} \psi - \beta_x D_b^{\mu} \psi \overset{a.e.}{=} f.
\]

(9.125)

Noting \((1+\mu)/2 > \mu \) and recalling the knowledge of embedding \( \tilde{H}_0^{(1+\mu)/2} (\Omega) \subset \tilde{H}_0^\mu (\Omega) \), we see from Property 6.18 that \( \psi(x) \) can be represented as

\[
\psi(x) = a D_x^{\mu} \psi \quad \text{and} \quad a D_x^{\mu} \psi \in L^2(\Omega).
\]

Doing back substitution for \( \psi \) into (9.125), the left-hand side becomes

\[
\alpha_a D_x^{\mu} \psi - \beta_x D_b^{\mu} \psi = \alpha_a D_x^{\mu} \psi - \beta_x D_x^{\mu} a D_x^{\mu} \psi.
\]

(9.126)

For notation simplicity, we denote \( r_a(x) = x - a \), \( r_b(x) = b - x \), \( x \in \Omega \).

Applying the fact (eq. (11.17), p. 206, [6]) that

\[
a D_x^{\mu} g = \cos(\mu\pi) x D_b^{\mu} g - \sin(\mu\pi) x D_b^{\mu} (r_b^{\mu} S(r_b^{\mu} g)), \quad g(x) \in L^p(\Omega), \mu > 1
\]

to the right-hand side of (9.126), we have

\[
\alpha_a D_x^{\mu} \psi - \beta \cos(\mu\pi) a D_x^{\mu} \psi + \beta \sin(\mu\pi) r_b^{\mu} S(r_b^{\mu} a D_x^{\mu} \psi)
\]

\[
= (\alpha - \beta \cos(\mu\pi)) a D_x^{\mu} \psi + \beta \sin(\mu\pi) r_b^{\mu} S(r_b^{\mu} a D_x^{\mu} \psi).
\]

(9.127)

Inserting this back into equation (9.125), multiplying both sides by \( r_b^{\mu} \) and denoting

\[
A = \alpha - \beta \cos(\mu\pi), \quad B = \beta \sin(\mu\pi), \quad \Psi_1(x) = r_b^{\mu} a D_x^{\mu} \psi, \quad F(x) = r_b^{\mu} f(x),
\]

(9.128)
we arrive at
\[ A\Psi_1(x) + \frac{B}{\pi} \int_a^b \frac{\Psi_1(t)}{t-x} dt \overset{a.e.}{=} F(x), x \in \Omega. \tag{9.128} \]

2. On the other hand, by Lemma 9.1, we already know that there exist solutions \( \Psi_2 \in H^s(\Omega) \cap C((a, b]) \) and \( \Psi_3 \in H^s(\Omega) \cap C([a, b)) \) satisfying
\[ A\Psi_i(x) + \frac{B}{\pi} \int_a^b \frac{\Psi_i(t)}{t-x} dt = F(x), x \in \Omega, (i = 2, 3), \tag{9.129} \]
and that
\[ \frac{\Psi_2}{(b-x)^\mu}, \frac{\Psi_3}{(b-x)^\mu} \in H^s(\Omega), \quad \mu < \frac{\theta}{2\pi} < 1. \tag{9.130} \]

Notice that the distinction between (9.128) and (9.129) is that (9.128) holds \( a.e. \) and (9.129) holds for every \( x \in \Omega \) and that \( \Psi_2 \) and \( \Psi_3 \) belong to \( H^s(\Omega) \) but \( \Psi_1 \) is not immediately clear yet for now.

In the following, our strategy is to intend to show that
\[ \Psi_1(x) = \Psi_2(x) = \Psi_3(x), \tag{9.131} \]
which produces (9.122) in the lemma, and after this, (9.123) will be obtained shortly, the whole lemma hence will be eventually completed.

3. Now let us continue.
Subtracting (9.129) from (9.128) gives
\[ A\tilde{\Psi}(x) + \frac{B}{\pi} \int_a^b \frac{\tilde{\Psi}(t)}{t-x} dt \overset{a.e.}{=} 0, x \in \Omega, \tag{9.132} \]
where
\[ \tilde{\Psi}(x) = \Psi_1(x) - \Psi_i(x), (i = 2, 3). \]
(Evidently, \( \tilde{\Psi}(x) \) depends on \( i \) and it should not be confused. We denote \( \tilde{\Psi}(x) \) this way simply because it is more convenient to discuss this two cases \( i = 2, 3 \) together in the rest of proof rather than separately.)

Dividing both sides of (9.132) by \( r_b^\mu \), we arrive at
\[ A\tilde{\Psi}(x) + \frac{B}{\pi} \frac{1}{(b-x)^\mu} \int_a^b \frac{(b-t)^\mu \tilde{\Psi}(t)}{t-x} dt \overset{a.e.}{=} 0, x \in \Omega, \tag{9.133} \]
where
\[ \tilde{\Psi}(x) = \frac{\Psi(x)}{(b-x)^\mu} = \frac{\Psi_1(x)}{(b-x)^\mu} - \frac{\Psi_i(x)}{(b-x)^\mu}, (i = 2, 3). \]

4. Let us examine which functional space \( \tilde{\Psi}(x) \) belongs to (discussing the two cases \( i = 2, 3 \) together).
First,
\[
\frac{\Psi_1(x)}{(b-x)\mu} = aD_x^\mu \psi \in L^2(\Omega), \text{ since } \psi \in \hat{H}_0^{(1+\mu)/2}(\Omega).
\]

Secondly, since
\[
\frac{\Psi_i}{(b-x)\mu} \in H^*(\Omega),
\]
we can always find a certain \( p > 1 \) that does not depend on \( i \) such that
\[
\frac{\Psi_i}{(b-x)\mu} \in L^p(\Omega), (i = 2, 3),
\]
by taking into account the definition of \( H^*(\Omega) \).

Combining these together, we conclude that
\[
\tilde{\Psi}(x) \in L^p(\Omega), \text{ for a certain } p > 1.
\]

5. The establishment of (9.136) allows us to be able to apply Property 6.14 to equation (9.133). To do so, integrating both sides of (9.133) by \( xD_b^{-\mu} \), we obtain
\[
A_xD_b^{-\mu}\tilde{\Psi} + \frac{B}{\pi}(x-a)^\mu \int_a^b \frac{tD_b^{-\mu}\tilde{\Psi}}{(t-a)^\mu(t-x)} dt \overset{a.e.}{=} 0, \ x \in \Omega.
\]

Dividing both sides by \( (x-a)^\mu \), we arrive at
\[
\frac{\tilde{\Psi}}{\pi} + \frac{B}{\pi} \int_a^b \frac{\tilde{\Psi}(t)}{t-x} dt \overset{a.e.}{=} 0, \ x \in \Omega,
\]
where
\[
\tilde{\Psi}(x) = \frac{xD_b^{-\mu}\tilde{\Psi}}{(x-a)^\mu}.
\]

Before going further, let us call attention to that (9.138) holds \( a.e. \) at this stage since \( \tilde{\Psi}(x) \) is essentially in terms of \( \psi(x) \), and in the next two steps we intend to show that \( \tilde{\Psi}(x) \) actually admits a good representative such that (9.138) holds for every \( x \in \Omega \).

6. Now we assert that \( \tilde{\Psi}(x) \in H^*(\Omega) \) (for both \( i = 2, 3 \)). (It should be clear that \( \tilde{\Psi}(x) \in H^*(\Omega) \) means there is a representative in the equivalence classes of \( \tilde{\Psi}(x) \) such that it belongs to \( H^*(\Omega) \)).
To see this, substituting for $\tilde{\Psi}$ into $\tilde{\Psi}(x)$, we have

$$\tilde{\Psi}(x) = \frac{1}{(x - a)^{\mu}} x D_{b}^{-\mu} \left( \frac{1}{r_b^\mu} \Psi_1 - \frac{1}{r_b^\mu} \Psi_i \right) = \frac{1}{(x - a)^{\mu}} x D_{b}^{-\mu} a D_x^\mu \Psi_i - \frac{1}{(x - a)^{\mu}} x D_{b}^{-\mu} \frac{\Psi_i}{r_b^\mu} \quad (9.139)$$

It suffices to investigate $M_1$ and $M_2$, respectively.

Consider $M_1$ first and examine the piece $x D_{b}^{-\mu} a D_x^\mu \Psi_i$. By Property 6.18, there exists a function $\psi_1(x) \in L^2(\Omega)$ such that

$$a D_x^\mu \psi_1 = a D_x^\mu (1 + \mu)/2 \psi_1 = a D_x^{(1-\mu)/2} \psi_1.$$ 

On the other hand, there exists a function $\psi_2(x) \in L^2(\Omega)$ such that

$$a D_x^{(1-\mu)/2} \psi_1 = x D_{b}^{(-1+\mu)/2} \psi_2,$$

(Corollary 1, p. 208, [6]). Thus,

$$x D_{b}^{-\mu} a D_x^\mu \psi = x D_{b}^{-(1+\mu)/2} \psi_2 \in H^{\mu/2}(\Omega)$$

is guaranteed by Property 6.4.

If we equivalently rewrite $M_1$ as

$$M_1 = \frac{(x - a)^{\epsilon} (x D_{b}^{-\mu} a D_x^\mu \psi)(b - x)^{\epsilon}}{(x - a)^{1-(1-\mu-\epsilon)}(b - x)^{1-(1-\epsilon)}}, \quad (9.140)$$

where $\epsilon$ is chosen so that $0 < \epsilon < \min\{1 - \mu, \mu/2\}$, then by simple steps as we justified for $\Pi_1$ in the step 3 of the proof of Lemma 9.1, we see

$$M_1 \in H^*(\Omega),$$

and not repeated here.

For $M_2$, indeed, by virtue of Property 6.12 and recalling (9.130), $x D_{b}^{-\mu} \Psi_i$ can be represented as

$$x D_{b}^{-\mu} \Psi_i = \frac{g_i(x)}{(x - a)^{1-\epsilon_1}(b - x)^{1-\epsilon_2}}, \quad (9.141)$$

for certain functions $g_i(x)$ and real numbers $k, \epsilon_1, \epsilon_2$ satisfying

$$g_i(x) \in H_0^k(\Omega), \quad \mu < k, 0 < \epsilon_1, 0 < \epsilon_2 \quad (k, \epsilon_1, \epsilon_2 \text{ depend on } i). \quad (9.142)$$

Taking into account the useful fact that

$$\frac{g_i(x)}{(x - a)^{\mu}} \in H_0^l(\Omega), \quad \text{for any } 0 < l < k - \mu, \quad (9.143)$$

(the value of $\frac{g_i(x)}{(x - a)^{\mu}}$ at $x = a$ is understood in the limiting sense) and the definition of $H^*(\Omega)$, we see
\[ M_2 = \frac{1}{(x-a)^\mu} D_b^{-\mu} \frac{\Psi_i}{\tau_b} = \frac{g_i(x)}{(x-a)^{\mu(r_1(b-a))^{1-\epsilon_2}}} \in H^*(\Omega), (i = 2, 3). \]

(9.144)

Combing \( M_1 \) and \( M_2 \) yields the assertion

\[ \tilde{\Psi}(x) \in H^*(\Omega) \text{ for both } i = 2, 3. \]

7. Once we have the above, it now is notable that equation (9.138) becomes valid for every point in \( \Omega \), namely

\[ A \tilde{\Psi}(x) + \frac{B}{\pi} \int_a^b \tilde{\Psi}(t) \frac{dt}{t-x} = 0, \text{ for each } x \in \Omega. \]

(9.145)

(Putting it another way, there is a representative for the equivalence classes of \( \tilde{\Psi}(x) \) such that it belongs to \( H^*(\Omega) \) and makes (9.138) hold for every \( x \in \Omega \).

On the other hand, remember that, in \( H^*(\Omega) \), (9.145) is unconditionally solvable according to the part (1) of Property 6.16. Hence by utilizing (6.20), we derive that \( \tilde{\Psi}(x) \) can be represented as a constant multiple of \( (x-a)^{1-n_a-\frac{\mu}{2\pi}} (b-x)^{\frac{\mu}{2\pi}-n_b} \) with choosing \( n_a = 1, n_b = 1 \), namely

\[ \tilde{\Psi}(x) = C_i \cdot (x-a)^{-\frac{\mu}{2\pi}} (b-x)^{\frac{\mu}{2\pi}-1}, \]

(9.146)

where \( C_i \) depends on the \( i \) in \( \tilde{\Psi}(x) (i=2,3) \).

In the next two steps, we will show \( C_2 \) and \( C_3 \) have to be zero, separately.

8. Using the second line of expression (9.139), we solve equation (9.146) for \( aD^\mu_x \psi \) to obtain

\[ aD^\mu_x \psi = \frac{\Psi_i(x)}{(b-x)^\mu} + C_i x D^\mu_b ((x-a)^{-\frac{\mu}{2\pi}} (b-x)\frac{\mu}{2\pi}-1), \text{ (i = 2, 3).} \]

(9.147)

Integrating both sides by \( aD^{-\mu}_x \), which is valid, and also noting \( \psi(x) = aD^{-\mu}_x aD^\mu_x \psi \), we have

\[ \psi(x) = aD^{-\mu}_x \frac{\Psi_i}{\tau_b} + C_i aD^{-\mu}_x x D^\mu_b ((x-a)^{-\frac{\mu}{2\pi}} (b-x)\frac{\mu}{2\pi}-1), \text{ (i = 2, 3).} \]

(9.148)
Calculating the second term on the right-hand side (using eq. (11.4) and (11.19), [6]),

\[
aD_x^{-\mu}D_b^{\mu}((x-a)^{-\frac{\theta}{2\pi}}+\mu(b-x)^{\frac{\theta}{2\pi}-1}) = \left(\cos(\mu\pi) + \sin(\mu\pi)\cot(\pi - \frac{\theta}{2})\right)(x-a)^{-\frac{\theta}{2\pi}+\mu(b-x)^{\frac{\theta}{2\pi}}-1} (9.149)
\]

where

\[
\tilde{C} = \cos(\mu\pi) + \sin(\mu\pi)\cot(\pi - \frac{\theta}{2}).
\]

Equation (9.148) further becomes

\[
\psi(x) = aD_x^{-\mu}\frac{\Psi_i}{r_b^i} + C_i\tilde{C} \cdot (x-a)^{-\frac{\theta}{2\pi}+\mu(b-x)^{\frac{\theta}{2\pi}}-1}, \ (i = 2, 3). \quad (9.150)
\]

Consider the case \(i = 3\) now. On one hand, recall that \(\mu < \frac{\theta}{2\pi} < 1\) from (9.130). This implies that the coefficient \(\tilde{C}\) is non-zero and hence that \(\tilde{C} \cdot (x-a)^{-\frac{\theta}{2\pi}+\mu(b-x)^{\frac{\theta}{2\pi}}-1}\) is unbounded at the boundary point \(x = a\). On the other hand, both \(\psi(x)\) (belonging to \(\tilde{H}_0(1+\mu)/2(\Omega)\)) and \(aD_x^{-\mu}\frac{\Psi_3}{r_b^3}\) (remember \(\Psi_3 \in H^*(\Omega) \cap C([a,b])\)) are bounded at \(x = a\), which contradicts the unboundedness of \(C_3\tilde{C} \cdot (x-a)^{-\frac{\theta}{2\pi}+\mu(b-x)^{\frac{\theta}{2\pi}}-1}\) unless \(C_3 = 0\). Thereby, we arrive at

\[
\psi(x) = aD_x^{-\mu}\frac{\Psi_3}{r_b^3}. \quad (9.151)
\]

So, we have proved (9.122) in the lemma by letting \(J(x) = \frac{\Psi_3}{r_b^3}\) and noting \(J(x) \in H^*(\Omega)\) from (9.130). We remain to show that \(\frac{\Psi_3}{r_b^3}\) has four equivalent representations, namely (9.123), which follows from the next step.

9. In this last step we show that the constant \(C_2\) in (9.150) for the case \(i = 2\) has to be zero as well, namely, \(C_2 = 0\).

Using equation (9.150) and doing subtraction with each other for \(i = 2, 3\),

\[
aD_x^{-\mu}\frac{\Psi_3}{r_b^3} - aD_x^{-\mu}\frac{\Psi_2}{r_b^2} = C_2\tilde{C} \cdot (x-a)^{-\frac{\theta}{2\pi}+\mu(b-x)^{\frac{\theta}{2\pi}}-1}. \quad (9.152)
\]

Let us take care about each term above at the boundary point \(x = b\) to obtain a contradiction.
For the left-hand side, using (9.151) and invoking the part (4) of Lemma 9.1, we know
\[ aD_x^{-\mu} \frac{\Psi_3}{r_b^\mu} (x) = \psi(x) \in \hat{H}_0^{(1+\mu)/2}(\Omega), \quad aD_x^{-\mu} \frac{\Psi_2}{r_b^\mu} \in L^p(I_b), \] (9.153)
where \( p = \frac{1}{1+\theta}, \) \( I_b = \left( \frac{b-a}{2}, b \right). \) However, in the right-hand side of (9.152), it is clear that
\[ \tilde{C} \cdot (x-a)^{-\theta \pi+\mu}(b-x)^{\theta \pi-1} \notin L^p(I_b). \] (9.154)
This means that \( C_2 \) has to be zero in (9.152), from which it follows that
\[ \psi(x) = aD_x^{-\mu} \frac{\Psi_2}{r_b^\mu}. \] (9.155)
Comparing (9.151) and (9.155), we see
\[ aD_x^{-\mu} \left( \frac{\Psi_3}{r_b^\mu} - \frac{\Psi_2}{r_b^\mu} \right) = 0. \] (9.156)
Only trivial solution is allowed by seeing \( \frac{\Psi_2}{r_b^\mu}, \frac{\Psi_3}{r_b^\mu} \in H^*(\Omega) \) from (9.130), and thus
\[ \Psi_2(x) = \Psi_3(x), \quad x \in \Omega. \] (9.157)
Once we have equality (9.157), by virtue of the part (5) of Lemma 9.1 (9.123), namely the four desired representations for \( \frac{\Psi_3}{r_b^\mu} \) (or \( \frac{\Psi_2}{r_b^\mu} \)) in the lemma, follows immediately after dividing by \( r_b^\mu. \)
This finally completes the whole proof. \( \square \)

Now we go one step further from above lemma by showing that \( aD_x^{\mu} \psi \) can actually go to better spaces \( H^*(\Omega) \) from \( H^* (\Omega) \) provided that \( f \) lies in \( H^*_\sigma (\Omega) \) (which is the same as what (9.159) means in the following).

**Lemma 9.3.** Given \( 0 < \sigma < 1 \), let \( c \) be a constant, \( \psi(x) \in \hat{H}_0^{(1+\mu)/2}(\Omega) \) and \( f(x) \in H(\Omega) \cap H^*_\sigma (\Omega) \). If
\[ \alpha aD_x^{-(1-\mu)} \psi + \beta bD_x^{-(1-\mu)} \psi = aD_x^{-1} f + c, \quad x \in \Omega, \] (9.158)
then the solution \( \psi(x) \) has a representation
\[ \psi(x) = aD_x^{-(\mu+\sigma)} K_\sigma, \] (9.159)
where function \( K_\sigma(x) \) belongs to \( H^*(\Omega) \) (\( K_\sigma \) depends on \( \sigma \)).

**Proof.** Notice that, compared to Lemma 9.2, only one more condition is imposed in this lemma, namely \( f(x) \in H^*_\sigma (\Omega), 0 < \sigma < 1. \)
1. As a direct consequence of Lemma 9.2, we already know that the function \( \psi(x) \) must have a representation
\[
\psi(x) = a D_{x}^{\mu} J,
\] (9.160)
where \( J(x) \in H^{\sigma}(\Omega) \) and \( J(x) \) has four equivalent representations:
\[
J(x) = \frac{A f(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_{a}^{b} \left( \frac{x - a}{t - a} \right)^{1-\frac{\theta}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\theta}{2\pi} - \mu} f(t) \frac{dt}{t - x};
\]
\[
J(x) = \frac{A f(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_{a}^{b} \left( \frac{x - a}{t - a} \right)^{1-\frac{\theta}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\theta}{2\pi} - \mu - 1} f(t) \frac{dt}{t - x};
\]
\[
J(x) = \frac{A f(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_{a}^{b} \left( \frac{x - a}{t - a} \right)^{\frac{\theta}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\theta}{2\pi} - \mu} f(t) \frac{dt}{t - x};
\]
\[
J(x) = \frac{A f(x)}{A^2 + B^2} - \frac{B}{\pi (A^2 + B^2)} \int_{a}^{b} \left( \frac{x - a}{t - a} \right)^{\frac{\theta}{2\pi}} \left( \frac{b - x}{b - t} \right)^{\frac{\theta}{2\pi} - \mu - 1} f(t) \frac{dt}{t - x}.
\] (9.161)

where \( A = \alpha - \beta \cos(\mu \pi) \), \( B = \beta \sin(\mu \pi) \).

For ease of notation, we denote \((9.161)\) as
\[
J(x) = \text{Expression 1};
\]
\[
J(x) = \text{Expression 2};
\]
\[
J(x) = \text{Expression 3};
\]
\[
J(x) = \text{Expression 4}.
\] (9.162)

2. For any given \( 0 < \sigma < 1 \), \( \sigma \) must satisfy one of the following inequality cases:

Case&1 : \( \sigma + \frac{\theta}{2\pi} \geq 1, \sigma + \mu \geq \frac{\theta}{2\pi}; \)

Case&2 : \( \sigma + \frac{\theta}{2\pi} \geq 1, \sigma + \mu < \frac{\theta}{2\pi}; \)

Case&3 : \( \sigma + \frac{\theta}{2\pi} < 1, \sigma + \mu \geq \frac{\theta}{2\pi}; \)

Case&4 : \( \sigma + \frac{\theta}{2\pi} < 1, \sigma + \mu < \frac{\theta}{2\pi}. \) (9.163)

3. We associate different cases to different expressions as follows:

Case&1 with \( \text{Expression 1}; \)

Case&2 with \( \text{Expression 2}; \)

Case&3 with \( \text{Expression 3}; \)

Case&4 with \( \text{Expression 4}. \) (9.164)
For each of (9.164), applying Property 6.11 to the according expression of $J(x)$ is valid by checking the two inequality conditions in Property 6.11 and yields that
\[ J(x) \in H^s_0(\Omega). \] (9.165)

4. Utilizing Property 6.12, we know that there exists a certain function $K_\sigma(x) \in H^s_0(\Omega)$ such that
\[ J(x) = aD^{-\sigma}_x K_\sigma, \] (9.166)
and therefore, by inserting (9.166) back into (9.160),
\[ \psi(x) = aD^{-(\mu+\sigma)}_x K_\sigma \] (9.167)
follows from the semigroup property of R-L integral operators.

This completes the whole proof. \qed

10. Proof of Theorem 1.1

We intend to show that the weak solution of problem (1.2) associated with conditions (1.1) is actually the true solution to (1.2) by picking up regularity and inverting the variational formulation back to original problem (1.2) pointwisely. The whole proof will be completed by invoking the lemmas that were established in previous sections; the majority of proof is about the existence of $u(x)$ satisfying $u(x) = aD^{-\tau}_x J_t$, $t < 1 + \mu$, from which the confirmation of true solution follows. The uniqueness will be proved at the very end.

Proof. 1. For symbol convenience, denote $s = (1 + \mu)/2$ throughout the proof. From Lemma 7.2, we know that there exists a unique $u \in \hat{H}^s_0(\Omega)$ such that
\[ B_2[u, \psi] = (f_k, \psi)_\Omega, \] (10.168)
for any $\psi \in C_0^\infty(\Omega)$.

Utilizing the expression of $B_2[\cdot, \cdot]$ in Definition 7.1, simplifying and operating both sides of (10.168), we obtain
\[ (L_1(u), D\psi)_\Omega = (xD^{-1}_b \frac{f}{k}, D\psi)_\Omega, \]
where
\[ L_1(u) = \alpha_a D^\mu_x u - \beta_x D^\mu_b u \]
\[ - \alpha_x D^{-1}_b \left( \frac{k'}{k} aD^\mu_x u \right) + \beta_x D^{-1}_b \left( \frac{k'}{k} xD^\mu_b u \right) \]
\[ - \frac{p}{k} u - xD^{-1}_b \left( \frac{p}{k} \right) u + xD^{-1}_b \left( \frac{q}{k} \right) u. \] (10.169)
Namely,

\[(L_1(u) - \int_0 f \, D^\psi \Omega = 0, \forall \psi \in C_0^\infty(\Omega). \] (10.170)

2. Observe that \(L_1(u)\) is a summable function, and so is \(\int_0 f\). From (10.170), there exists a constant \(C_1\) such that

\[L_1(u) - \int_0 f \, a.e. = C_1, x \in \Omega. \] (10.171)

Isolating the first two terms of \(L_1(u)\) gives

\[\alpha a D^\mu x u - \beta b D^\mu u \, a.e. = L_2(u) + \frac{p}{k} u, \] (10.172)

where

\[L_2(u) = \alpha a D^\mu \left(\frac{k'}{k}a D^\mu u\right) - \beta b D^\mu \left(\frac{k'}{k}b D^\mu u\right) + \int_0 f \, a.e. = C_1. \] (10.173)

Namely,

\[D(\alpha a D_x^{-1} u + \beta b D_b^{-1} u) \, a.e. = L_2(u) + \frac{p}{k} u. \] (10.174)

3. In order to get rid of the differentiation operator \(D\) in the left-hand side above, we would like to integrate both sides by \(\int_0 f\), which is, however, in general valid for absolutely continuous functions on \(\Omega\).

To see \(\alpha a D_x^{-1} u + \beta b D_b^{-1} u \in AC(\Omega)\) (it should be clear that it means there exists a representative of equivalence classes of \(\alpha a D_x^{-1} u + \beta b D_b^{-1} u\) belonging to \(AC(\Omega)\), similarly in the following steps), notice that \(u \in \tilde{H}_0^s(\Omega)\), which implies that (Property 6.18)

\[u = a D_x^{-s} \theta_1 = x D_b^{-s} \theta_2, \] (10.175)

for certain functions \(\theta_1, \theta_2 \in L^2(\Omega)\). It follows that

\[\alpha a D_x^{-1} u = a D_x^{-1} \tilde{\theta}_1 \text{ and } \beta b D_b^{-1} u = x D_b^{-1} \tilde{\theta}_2, \] (10.176)

for certain \(\tilde{\theta}_1, \tilde{\theta}_2 \in L^2(\Omega)\).

This amounts to saying that both \(\alpha a D_x^{-1} u\) and \(\beta b D_b^{-1} u\) are absolutely continuous on \(\Omega\), and so is \(\alpha a D_x^{-1} u + \beta b D_b^{-1} u\).

So, now taking the integration by \(\int_0 f\) at both sides of (10.174) yields

\[\alpha a D_x^{-1} u + \beta b D_b^{-1} u \, a.e. = a D_x^{-1} \left(\int_0 f \, u\right) + C, \] (10.177)

\(C\) is another certain constant.
Before going any further, it is worthwhile to observe that the right-hand side of (10.177) is “better” than the left-hand side in the sense that \(a D_x^{-1}(L_2(u) + \frac{p}{k} u)\) has at least integration of order 1, plus a constant \(C\), meanwhile, the left-hand side has only fractional integrations of order \(1 - \mu\). Since this is an identity, the two sides should behave the “same”, which seemingly suggests that \(u\) in the left-hand side is supposed to possess at least fractional integration of order \(\mu\) to balance the right-hand side. However, note that both sides are essentially in terms of \(u\) except the constant \(C\), if \(u\) is of fractional integration of order \(\mu\) on the left, then \(a D_x^{-1}(L_2(u) + \frac{p}{k} u)\) on the right is of fractional integration of order \(1 + \mu\), plus \(C\), which is always “better” than the left-hand side by order \(\mu\). This way, we can keep continuing to raise the regularity of \(u(x)\) until some threshold (if any) is reached.

Let us come back to (10.177) and put this idea into action in the next two steps.

4. We assert that \(u(x)\) in (10.177) can be represented as

\[
    u(x) = a D_x^\mu J, \quad \text{where } J(x) \in H^*(\Omega). \tag{10.178}
\]

To see this, we start with showing that, in the right-hand side of (10.177), \(L_2(u) + \frac{p}{k} u \in H(\Omega)\).

Check \(\frac{p}{k} u\) first, in view of (10.175) and Property 6.4,

\[
    u \in H^{\mu/2}(\Omega).
\]

Hence,

\[
    \frac{p}{k} u \in H^{\mu/2}(\Omega) \quad \text{since } \frac{p}{k} \in C^1(\Omega).
\]

Similarly, for each term of \(L_2(u)\) by recalling the expression (10.173), it can be directly checked that

\[
    \alpha_x D_b^{-1} \left( \frac{k'}{k} a D_x^\mu u \right), \beta_x D_b^{-1} \left( \frac{k'}{k} D^\mu_b u \right), \gamma_x D_b^{-1} \left( \frac{k'}{k} \beta' \right) u, \delta_x D_b^{-1} \left( \frac{k'}{k} \beta \right) u \in H^{1/2}(\Omega),
\]

(we can go beyond 1/2 in the exponent, but it is enough already.)

\[
    \xi D_b^{-1} \frac{f}{k} \in H(\Omega), \quad C_1 \in H^1(\Omega),
\]

by using (10.175), Property 6.4, conditions (E.1) and the fact that the quotient of two Hölderian functions is still Hölderian provided the denominator does not vanish.

Thus

\[
    L_2(u) + \frac{p}{k} u \in H(\Omega). \tag{10.179}
\]
The establishment of (10.179) permits us to apply Lemma 9.2 to (10.177), which concludes our assertion that \( u \) admits a representation of the form
\[
 u(x) = a D_x^{-\mu} J, \quad \text{where } J(x) \in H^s(\Omega). \quad (10.180)
\]

According to Property 6.12, \( u(x) \) can also be represented by
\[
 u(x) = x D_b^{-\mu} \tilde{J}, \quad \text{where } \tilde{J}(x) \in H^s(\Omega). \quad (10.180)
\]

It follows that
\[
a D_x^\mu u, x D_b^\mu u \in H^s(\Omega). \quad (10.181)
\]

5. We claim that, for any given \( 0 < \sigma < 1 \), the solution \( u \) in (10.177) can be represented as
\[
 u(x) = a D_x^{-(\mu+\sigma)} K_\sigma, \quad (10.182)
\]
where function \( K_\sigma(x) \) belongs to \( H^s(\Omega) \) and depends on \( \sigma \).

Substituting (10.181), (10.180) into \( L_2(u) \) and (10.180) into \( \frac{p}{k} u \), we deduce that
\[
 L_2(u) + \frac{p}{k} u \in H^s_\mu(\Omega). \quad (10.183)
\]

Indeed, it is directly checked that
\[
 L_2(u) \in H_t^s(\Omega), \forall 0 < t < 1, \text{ i.e., } L_2(u) \in \bigcap_{0 < t < 1} H_t^s(\Omega) \quad (10.184)
\]
and that
\[
 \frac{p}{k} u \in H^s_\mu(\Omega),
\]
by using conditions (1.1), (10.150), (10.181), Property 6.12 and the fact that the product of two Hölderian functions is still Hölderian.

(10.183) gives us the permission to apply Lemma 9.3 to equation (10.177), from which we raise the fractional integration order of \( u \) from \( \mu \) to \( \mu + \nu \), namely,
\[
 u(x) = a D_x^{-(\mu+\mu)} K_\nu = a D_x^{-2\mu} K_\nu, \text{ for a certain } K_\nu \in H^s(\Omega). \quad (10.185)
\]

At this stage, if \( \mu \geq \sigma \), the claim (10.182) has already been achieved due to the knowledge that \( H^s_{y_2}(\Omega) \subseteq H^s_{y_1}(\Omega) \) for \( 0 < y_2 \leq y_1 < 1 \).

Otherwise, by a second substitution of (10.185) into \( \frac{p}{k} u \) and noting (10.184), we obtain
\[
 L_2(u) + \frac{p}{k} u \in H^s_\nu(\Omega), \quad \nu = \min\{2\mu, \sigma\}. \quad (10.186)
\]
And then by a second application of Lemma 9.3 to (10.177), we have
\[
 u(x) = a D_x^{-(\mu+\nu)} K_\nu, \text{ for a certain } K_\nu \in H^s(\Omega). \quad (10.187)
\]

\[
 FRACTIONAL DIFFUSION ADVECTION REACTION 47
\]
Again, if $2\mu \geq \sigma$, the claim (10.182) has been confirmed. Otherwise, repeating, from now on, this procedure \( n \) times, where \( n \) is the smallest integer satisfying \( (2 + n)\mu \geq \sigma \), we obtain

\[
L_2(u) + \frac{p}{k} u \in H^*_\sigma(\Omega), \text{ and } u(x) = aD_{x}^{-(\mu+\sigma)}K_\sigma, K_\sigma \in H^*(\Omega). \tag{10.188}
\]

(Notice: it is not concluded that \( L_2(u) + \frac{p}{k} u \in H^*_\sigma(\Omega) \) and \( u(x) = aD_x^{-(\mu+\sigma)}K_\sigma \) since \( (2 + n)\mu \) can be equal or beyond 1, which can not guarantee the applicability of Lemma 9.3, this is the essential difficulty that stops us reaching the extreme value \( 1 + \mu \) in Theorem 1.1.)

Thus, we have concluded our claim (10.182), which amounts to saying that for any given \( 0 < t < 1 + \mu \), \( u(x) \) can be represented as

\[
u(x) = aD_x^{-t}J_t, \text{ for a certain } J_t(x) \in H^*(\Omega).
\]

6. For \( t \leq 0 \), \( aD_x^{-t} \) denotes the fractional derivatives or identity operators (see notation in Section 5) and

\[
u(x) = aD_x^{-t} (aD_x^t u), aD_x^t u \in H^*(\Omega), x \in \Omega
\]
is always true.

Therefore, for any \( t < 1 + \mu \), \( u(x) \) is representable by

\[
u(x) = aD_x^{-t}J_t, \text{ for a certain } J_t(x) \in H^*(\Omega). \tag{10.189}
\]

7. It remains to prove that \( u(x) \) in (10.189) is a true solution to problem (1.2) and is unique.

To see \( u(x) \) is a true solution, we need to show that \( u \in C(\Omega), u(a) = u(b) = 0, Du \in C(\Omega), (\alpha aD_x^{-(1-\mu)} + \beta D_b^{-(1-\mu)})Du \in C^1(\Omega) \) and \( (L(u))(x) = f(x) \forall x \in \Omega \).

First, from (10.189), \( u(x) \in AC(\Omega) \) and \( Du \in C(\Omega) \) are satisfied. Second, since \( u(x) \in \bar{H}^{(1+\mu)/2}(\Omega), u(a) = u(b) = 0 \) is ensured.

This allows us to interchange the order of differentiation and fractional integrations to obtain

\[
(\alpha aD_x^{-(1-\mu)} + \beta D_b^{-(1-\mu)})Du = D(\alpha aD_x^{-(1-\mu)} + \beta D_b^{-(1-\mu)})u,
\]

by invoking (10.174) we hence see

\[
\alpha aD_x^{-(1-\mu)}Du + \beta D_b^{-(1-\mu)}Du = L_2(u) + \frac{p}{k} u \in C^1(\Omega). \tag{10.190}
\]

Differentiating both sides of (10.190), substituting for \( L_2(u) \), then multiplying both sides by \( k(x) \) and simplifying, we recover FDARE pointwisely, namely

\[
(L(u))(x) = f(x), \text{ for every point } x \in \Omega.
\]
Hence, \( u(x) \) in (10.189) is a true solution (classical solution) to problem (1.2).

8. Lastly, we prove the uniqueness. Suppose that there is another function \( v(x) \in \hat{H}_0^s(\Omega) \) such that it has a representative which is also a true solution to problem (1.2), we show \( v \) has to coincide with \( u \).

Multiplying both sides of
\[
L(v) = \frac{f}{k}
\]
by an arbitrary \( \psi(x) \in C^\infty_0(\Omega) \) and integrating over \( \Omega \), we have
\[
\int_{\Omega} \frac{[L(v)](t)}{k(t)} \psi(t) \, dt = \int_{\Omega} \frac{f}{k(t)} \psi(t) \, dt.
\]
(10.191)

Since \( v(x) \) is a true solution, by definition, \( v \in AC(\Omega) \), \( v(a) = v(b) = 0 \), which implies
\[
D(\alpha a D_x^{(1-\mu)} + \beta_x D_b^{(1-\mu)}) u = (\alpha a D_x^{(1-\mu)} + \beta_x D_b^{(1-\mu)}) Du.
\]
(10.192)

Simplifying and manipulating the left-hand side of (10.191) by taking advantage of (10.192) and the assumption \( v(x) \in \hat{H}_0^s(\Omega) \) yields
\[
B_2[v, \psi] = (\frac{f}{k}, \psi)_\Omega, \psi \in C^\infty_0(\Omega).
\]

Therefore, \( v(x) \) is also a weak solution to (7.50), which implies \( u(x) = v(x) \) a.e. by the uniqueness of weak solution in \( \hat{H}_0^s(\Omega) \).

The whole proof of Theorem 1.1 is completed. \( \square \)

10.1. **An example.** We provide an example to show that the value \( 1 + \mu \) in Theorem 1.1 is optimal, namely, for any given \( t > 1 + \mu \), there exists a true solution \( u(x) \) to (1.2) under conditions (1.1) such that \( u(x) \) is not representable by \( a D_x^{-t} J_t \), namely, \( u(x) \neq a D_x^{-t} J_t \) for any \( J_t(x) \in H^s(\Omega) \).

Let us consider
\[
\begin{aligned}
|L(u)|(x) &= -1, \ x \in \Omega, \\
u(a) &= u(b) = 0, \\
|L(u)|(x) &:= -D(\alpha a D_x^{(1-\mu)} + \beta_x D_b^{(1-\mu)}) Du,
\end{aligned}
\]
(10.193)

it is clear that conditions in (1.1) are satisfied. From Lemma 8.2, we know the true solution is \( u(x) = c \cdot (x - a)^{p+1} (b - x)^{q+1} \), where
\[
c = \frac{(-p - 1 + \mu)(-p + \mu)\Gamma(1 - \mu)}{\mu(1 + \mu) \beta \Gamma(2 - \mu + p) \Gamma(q + 2)}.
\]
(10.194)

and \( p, q \) are uniquely determined by
\[
p + q = -(1 - \mu) \quad \text{and} \quad \alpha \sin(q\pi) = \beta \sin(p\pi).
\]
(10.195)
Using conditions (10.195), we carry out
\[ \frac{\beta}{\alpha} = \cot(-p\pi) \sin((1-\mu)\pi) - \cos((1-\mu)\pi). \]
(10.196)

In fact, if \( \frac{\beta}{\alpha} \) (similarly for \( \frac{\alpha}{\beta} \)) is small enough, for example, let \( \frac{\beta}{\alpha} \) satisfies
\[ 0 < \frac{\beta}{\alpha} < \cot((2-t_0)\pi) \sin((1-\mu)\pi) - \cos((1-\mu)\pi), \]
where \( 1+\mu < t_0 < \min\{2,t\} \), then (10.195), (10.196) and (10.197) produce
\[ p + 1 - t < -1. \]
(10.198)

It is directly evaluated by using Property 6.8 that
\[ aD^p_x u = C \cdot (x-a)^{p+1-t} {}_2F_1(-1-q,p+2,p+2-t;x-\frac{a}{b-a}), a < x < b, \]
where \( C \) is a non-zero constant.

We see that \( aD^p_x u \notin L^1(\Omega) \) due to (10.198), which implies that \( u(x) \) cannot be represented as \( aD^{\frac{p}{2}+t} I_t \) for any \( J_t(x) \in H^*(\Omega) \).

11. Applications

We conclude this paper by giving two corollaries from a practical point of view.

11.1. FDARE with Riemann-Liouville derivative. The FDARE with Riemann-Liouville derivative has also attracted the attention of many authors in modelling, namely,
\[ \begin{cases} [\hat{L}(u)](x) = f(x), x \in \Omega, \\ u(a) = u(b) = 0, \\ [\hat{L}(u)](x) := -Dk(x)D(\alpha aD^{-(1-\mu)}_x + \beta xD^{-(1-\mu)}_b)u \\ \quad +p(x)Du + q(x)u(x). \end{cases} \]
(11.199)

We say a function \( u(x) \) is a true (classical) solution to (11.199) if \( u \in AC(\Omega), u(a) = u(b) = 0, Du \in C(\Omega), D(\alpha aD^{-(1-\mu)}_x + \beta xD^{-(1-\mu)}_b)u \in C^1(\Omega) \) and \( [\hat{L}(u)](x) = f(x) \forall x \in \Omega \).

The following corollary clarifies that the FDARE with either Riemann-Liouville derivative or Caputo derivative always has the same solution.

**Corollary 11.1.** Let conditions (1.1) be satisfied. Then in the Sobolev space \( \widehat{H}_0^{(1+\mu)/2}(\Omega) \), problem (1.2) and (11.199) have the same true solution.
Proof. First, according to the definition of true solution, we readily verify that \( u(x) \) is a true solution of (1.2) if and only if \( u(x) \) is a true solution of (11.199) by noting that
\[
D(\alpha_a D_x^{(1-\mu)} + \beta_x D_b^{-(1-\mu)}) u = (\alpha_a D_x^{(1-\mu)} + \beta_x D_b^{-(1-\mu)}) Du
\]
if \( u \in AC(\Omega), u(a) = u(b) = 0. \)

Second, the existence of such a true solution directly follows from Theorem 1.1. \( \square \)

11.2. More structure information on the solution. In practice, one is also interested in knowing what happens at the boundary points of the \( \mu \)-th-order derivative and the first-order derivative of the solution, namely, \( a D_x^{\mu} u, x D_b^{\mu} u \) and \( Du \), which is essentially related to the well-posedness problems of FDARE associated with other types of conditions. The following corollary gives refined structures of the solution by imposing a little bit stronger conditions, namely,
\[
\begin{align*}
0 < \alpha, \beta < 1, & \quad \alpha + \beta = 1, 1/2 < \mu < 1, \\
f \in H^{1-\mu}_1(\Omega), & \quad p(x), q(x), k(x) \in C^2(\Omega), \\
k(x) > 0 \text{ on } \Omega, & \quad \frac{q}{k} - \frac{1}{2} \frac{(k')'}{k} \geq 0 \text{ on } \Omega, \\
\pi(1 - \mu^2) \cot((1 + \mu)\pi/2) + 4(b - a)\|\frac{k'}{k}\|_{L^\infty(\Omega)} < 0,
\end{align*}
\]
(11.200)
or
\[
\begin{align*}
0 < \alpha, \beta < 1, & \quad \alpha + \beta = 1, 0 < \mu < 1, \\
f \in H^{1-\mu}_1(\Omega), & \quad q(x), k(x) \in C^2(\Omega), \\
p(x) = 0, x \in \overline{\Omega}, & \quad k(x) > 0 \text{ on } \Omega, \\
\frac{q}{k} - \frac{1}{2} \frac{(k')'}{k} \geq 0 \text{ on } \Omega, \\
\pi(1 - \mu^2) \cot((1 + \mu)\pi/2) + 4(b - a)\|\frac{k'}{k}\|_{L^\infty(\Omega)} < 0.
\end{align*}
\]
(11.201)

Corollary 11.2. Let conditions (11.200) or (11.201) (or both!) be satisfied. Then in \( H^{(1+\mu)/2}_0(\Omega) \), there exists a unique true solution \( u(x) \) (up to the equivalence classes) to problem (1.2) and it satisfies the following:

1. \( a D_x^{\mu} u, x D_b^{\mu} u \in C(\Omega) \) and
\[
a D_x^{\mu} u|_{x=a} = x D_b^{\mu} u|_{x=b} = 0.
\]
(11.202)

2. \( Du \in C(\Omega) \) and precisely one of the following holds:
   (a) \( \lim_{x \to a^+} Du = 0, \lim_{x \to b^-} Du = 0 \).
(11.203)
(b) \[ \lim_{x \to a^+} Du = 0, \quad \lim_{x \to b^-} |Du| = +\infty, \] (11.204)

(c) \[ \lim_{x \to a^+} |Du| = +\infty, \quad \lim_{x \to b^-} Du = 0, \] (11.205)

(d) \[ \lim_{x \to a^+} |Du| = +\infty, \quad \lim_{x \to b^-} |Du| = +\infty. \] (11.206)

Proof. Let us see that, compared to Theorem 1.1, stronger conditions are imposed in Corollary 11.2, namely, either (11.200) or (11.201) ensures (1.1); therefore, all the justifications in the proof of Theorem 1.1 remain valid to Corollary 11.2.

1. The existence and uniqueness of the true solution \( u(x) \) is an immediate consequence of Theorem 1.1. And it is representable by

\[ u(x) = a D^{-\tau} J_t, \] (11.207)

\( J_t(x) \in H^*(\Omega) \) (depending on \( t \)) provided that \( t < 1 + \mu \).

Taking into account \( u(a) = u(b) = 0 \) and Property 6.12 we also have

\[ u(x) = b D^{-\tau} \tilde{J}_t, \] (11.208)

\( \tilde{J}_t(x) \in H^*(\Omega) \) (depending on \( t \)) provided that \( t < 1 + \mu \).

It remains to show part (1) and part (2).

2. Let us invoke the identity in (10.190), namely,

\[ \alpha a D^{-\tau} u + \beta x D^{-\tau} b u = L_2(u) + \frac{p}{k} u. \] (11.209)

Recalling the expression of \( L_2(u) \) and differentiating the right-hand side give

\[ D(L_2(u) + \frac{p}{k} u) = -\alpha \frac{k'}{k} a D^\mu_x u + \beta \frac{k'}{k} x D^\mu_b u + \frac{q}{k} u - \frac{f}{k} + \frac{p}{k} Du. \] (11.210)

From (11.207) and (11.208), we see

\[ a D^\mu_x u, x D^\mu_b u \in H^*_{\sigma}(\Omega) \quad \forall 0 < \sigma < 1. \] (11.211)

Substituting (11.211) into (11.210) and checking each term by utilizing either conditions (11.200) or conditions (11.201), it is guaranteed that

\[ D(L_2(u) + \frac{p}{k} u) \in H^*_{1-\mu}(\Omega). \]

In view of Property 6.13 we know there exists a function \( v(x) \in H^*(\Omega) \) such that

\[ \alpha a D^{-\tau} (1-\mu) u + \beta x D^{-\tau} b (1-\mu) u = D(L_2(u) + \frac{p}{k} u). \] (11.212)
Once we have (11.209) and (11.212), according to Lemma 8.3, we know
\[ D(\alpha_a D_x^{-(1-\mu)} + \beta_x D_b^{-(1-\mu)})(Du - Y) = 0, \quad x \in \Omega, \quad (11.213) \]
where
\[
Y = \int_a^x v(t) \, dt - \frac{cS}{S_1} \int_a^x (t-a)^p(b-t)^q \, dt + \frac{c_1S}{S_1} D((x-a)^{p+1}(b-x)^{q+1}),
\]
\[
S = \int_a^b v(t) \, dt,
\]
\[
c = \Gamma(-q) \left( \alpha(b-a)^{1+p+q} \Gamma(p+1) \int_a^b (t-a)^{-q-1}(b-t)^{-p-1} \, dt \right)^{-1},
\]
\[
S_1 = \frac{\Gamma(-q) \int_a^b (t-a)^p(b-t)^q \, dt}{\alpha(b-a)^{1+p+q} \Gamma(p+1) \int_a^b (t-a)^{-q-1}(b-t)^{-p-1} \, dt},
\]
\[
c_1 = \frac{(-p + \mu)(-p + \mu)\Gamma(1-\mu)}{\mu(1+\mu)\beta \Gamma(2-\mu+p)\Gamma(q+2)}, \quad (11.214)
\]
and \( p, q \) are uniquely determined by
\[
p + q = -(1 - \mu) \quad \text{and} \quad \alpha \sin(q\pi) = \beta \sin(p\pi). \quad (11.215)
\]

3. We would like to get rid of the derivative operator \( D \) at the left-hand side of \( (11.213) \). Observe that
\[
(\alpha_a D_x^{-(1-\mu)} + \beta_x D_b^{-(1-\mu)})(Du - Y) \in C^1(\Omega),
\]
as a result,
\[
(\alpha_a D_x^{-(1-\mu)} + \beta_x D_b^{-(1-\mu)})(Du - Y) = c_2, \quad (11.216)
\]
for a certain constant \( c_2 \).

On the other hand, taking Lemma 8.1 into account, we know
\[
(\alpha_a D_x^{-(1-\mu)} + \beta_x D_b^{-(1-\mu)})(c_3 \cdot (x-a)^p(b-x)^q) = c_2, \quad (11.217)
\]
for a suitable constant \( c_3 \).

(11.216) subtracting (11.217) yields
\[
(\alpha_a D_x^{-(1-\mu)} + \beta_x D_b^{-(1-\mu)})(Du - Y_1) = 0, \quad (11.218)
\]
where
\[
Y_1 = Y + c_3 \cdot (x-a)^p(b-x)^q. \quad (11.219)
\]
Since \( Du - Y_1 \in H^*(\Omega) \), we conclude by Property 6.13 that
\[
Du - Y_1 = 0, \quad (11.220)
\]
namely,
\[
Du(x) = \Pi_1(x) + \Pi_2(x), \quad x \in \Omega, \quad (11.221)
\]
where
\[
\Pi_1(x) = \int_a^x v(t) \, dt - \frac{cS}{S_1} \int_a^x (t-a)^p(b-t)^q \, dt,
\]
\[
\Pi_2(x) = \frac{c_1 S}{S_1} D(x-a)^{p+1}(b-x)^{q+1} + c_3 \cdot (x-a)^p(b-x)^q.
\]  
(11.222)

4. Let us prove part (1) now.
First we claim \(Du \in H^*_\mu(\Omega)\).
To see this, we just need to show \(\Pi_1(x), \Pi_2(x) \in H^*_\mu(\Omega)\).
\(\Pi_1(x) \in H^*_\mu(\Omega)\) is clear. To see \(\Pi_2(x) \in H^*_\mu(\Omega)\), expanding
\[
\Pi_2(x) = \frac{c_1(p+1) S}{S_1} (x-a)^p(b-x)^{q+1} - \frac{c_1(q+1) S}{S_1} (x-a)^{p+1}(b-x)^q
\]
\[+ c_3 \cdot (x-a)^p(b-x)^q.\]  
(11.223)

For the first term, manipulating \((x-a)^p(b-x)^{q+1}\) as
\[
(x-a)^p(b-x)^{q+1} = \frac{(x-a)^{p+1+q+\epsilon}(b-x)^{q+1}}{(x-a)^{1+q+\epsilon}}
\]
\[= \frac{(x-a)^{\mu+\epsilon}(b-x)^{-p+\mu}}{(x-a)^{1-(q-\epsilon)}}
\]
(\text{using the condition } p + q = -(1-\mu)),
where \(\epsilon\) is chosen so that \(0 < \epsilon < \min \{ -q, -p \} \). In the numerator, we see
\[(x-a)^{\mu+\epsilon}(b-x)^{q+1} \in H^\mu_0(\Omega)\]
with the auxiliary inequality
\[
\frac{|y_1^p - y_2^p|}{|y_1 - y_2|^q} \leq 1, \ (0 \leq y \leq 1, 0 < y_1, 0 < y_2, y_1 \neq y_2).
\]  
(11.224)

Therefore, by the definition of \(H^*_\mu(\Omega)\), \((x-a)^p(b-x)^{q+1} \in H^*_\mu(\Omega)\).
As regards the second and third term in (11.223), it can be analogously verified, respectively. This concludes our claim \(Du \in H^*_\mu(\Omega)\), which in turn implies, in view of Property [6.12] that
\[
Du = aD^{-\mu}_x J_1 = aD^{-\mu}_b (-J_2) , \text{ for certain } J_1(x), J_2(x) \in H^*(\Omega).
\]
Taking \(u(a) = u(b) = 0\) into account, we obtain
\[
u(x) = aD^{-1}_x Du = -aD^{-1}_b Du = aD^{-(1+\mu)}_x J_1 = aD^{-(1+\mu)}_b J_2.
\]
So,
\[
aD^\mu_x u = \int_a^x J_1(t) \, dt, \quad xD^\mu_b u = \int_x^b J_2(t) \, dt,
\]
from which we see \( aD_x^\mu u, xD_b^\mu u \in C(\overline{\Omega}) \) and
\[ aD_x^\mu u|_{x=a} = xD_b^\mu u|_{x=b} = 0. \]

5. Last, we prove part (2).
Let us take a closer look at \( \Pi_1 \) and \( \Pi_2 \) in (11.222).

Clearly, \( Du \in C(\Omega) \) and \( \Pi_1(a) = 0 \). A direct calculation by substituting
\( c, S, S_1 \) into \( \Pi_1(x) \) from (11.214) yields \( \Pi_1(b) = 0 \).

If we can show that \( \Pi_2(x) \) either vanishes or blows up at the boundary
point \( x = a \) and either vanishes or blows up at the boundary point \( x = b \),
then part (2) is proved.

Let us first show that \( \Pi_2(x) \) either vanishes or blows up at \( x = a \).
To see this, according to the expression of \( \Pi_2(x) \) in (11.222), if \( \frac{c_3S}{S_1} \neq 0 \), using the other expression in (11.223) and operating with
the condition \( p + q = -(1 - \mu) \), we have
\[ \Pi_2(x) = \frac{c_1(p + 1)S(b - x)^q}{S_1} \left( b - a + \frac{c_3S_1}{c_1(p + 1)S} \right) (x - a)^p \]
\[ - \frac{c_1(1 + \mu)S}{S_1} (x - a)^{p+1}(b - x)^q, \]
and again we see
\[ \text{either } \lim_{x \to a^+} \Pi_2(x) = 0 \text{ or } \lim_{x \to a^+} |\Pi_2(x)| = +\infty, \]
by noting \( p < 0 \) and \( p + 1 > 0 \), depending on whether \( b - a + \frac{c_3S_1}{c_1(p + 1)S} \)
disappears.

In a similar fashion, it can be shown the same at \( x = b \). Namely,
\[ \text{either } \lim_{x \to b^-} \Pi_2(x) = 0 \text{ or } \lim_{x \to b^-} |\Pi_2(x)| = +\infty. \]

Hence, part (2) follows.
The whole proof is completed. \( \square \)

In Corollary 11.2 two key conditions have been imposed, i.e., \( 1/2 < \mu < 1 \) and \( p(x) = 0, x \in \Omega \), which corresponds to the diffusion order and
the advection term, respectively. We want to know if they are intrinsically
related to physical phenomena or not, or equivalently, from a mathematical
point of view, whether these constrains can be removed. It turns out that
it is connected to the extreme value \( 1 + \mu \) in Theorem 1.1 in a certain sense
and therefore we propose the following question:

Q: Under conditions (1.1), can each true solution of problem (1.2) be
represented as \( u(x) = aD_x^{-(1+\mu)} J, J(x) \in H^*(\Omega)? \)
References

[1] G. E. Andrews, R. Askey, and R. Roy, Special functions, vol. 71 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1999.
[2] V. J. Ervin, N. Heuer, and J. P. Roop, Regularity of the solution to 1-D fractional order diffusion equations, Math. Comp., 87 (2018), pp. 2273–2294.
[3] V. J. Ervin and J. P. Roop, Variational formulation for the stationary fractional advection dispersion equation, Numer. Methods Partial Differ. Equ., 22 (2006), pp. 558–576.
[4] V. Ginting and Y. Li, On the fractional diffusion-advection-reaction equation in $\mathbb{R}$, Fract. Calc. Appl. Anal., 22 (2019), pp. 1039–1062.
[5] Y. Li, On Fractional Differential Equations and Related Questions, ProQuest LLC, Ann Arbor, MI, 2019. Thesis (Ph.D.)–University of Wyoming.
[6] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional integrals and derivatives, Gordon and Breach Science Publishers, Yverdon, 1993.

Science and Mathematics Cluster
Singapore University of Technology and Design
8 Somapah Road Singapore 487372, Singapore
e-mail: liyulong0807101@gmail.com