THE STRATIFIED SPACES OF REAL POLYNOMIALS &
TRAJECTORY SPACES OF TRAVERSING FLOWS

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Abstract. This paper is the third in a series that researches the Morse Theory, gradient flows, concavity and complexity on smooth compact manifolds with boundary. Employing the local analytic models from [K2], for traversally generic flows on \((n + 1)\)-manifolds \(X\), we embark on a detailed and somewhat tedious study of universal combinatorics of their tangency patterns with respect to the boundary \(\partial X\). This combinatorics is captured by a universal poset \(\Omega_{\bullet}^{n}\) which depends only on the dimension of \(X\). It is intimately linked with the combinatorial patterns of real divisors of real polynomials in one variable of degrees which do not exceed \(2(n + 1)\). Such patterns are elements of another natural poset \(\Omega_{2n+2}\) that describes the ways in which the real roots merge, divide, appear, and disappear under deformations of real polynomials. The space of real degree \(d\) polynomials \(P_d\) is stratified so that its pure strata are cells, labelled by the elements of the poset \(\Omega_{d}\). This cellular structure in \(P_d\) is interesting on its own right (see Theorem 4.1 and Theorem 4.2). Moreover, it helps to understand the localized structure of the trajectory spaces \(T(v)\) for traversally generic fields \(v\), the main subject of Theorem 5.2 and Theorem 5.3.

1. Basics Facts about Boundary Generic and Traversally Generic Vector Fields

For the reader convenience, we start with a short review of few key definitions and lemmas from [K1] and [K2].

Let \(v\) be a vector field on a smooth compact \((n + 1)\)-manifold \(X\) with boundary \(\partial X\). To achieve some uniformity in our notations, let \(\partial_0 X := X\) and \(\partial_1 X := \partial X\).

The vector field \(v\) gives rise to a partition \(\partial_1^+ X \cup \partial_1^- X\) of the boundary \(\partial_1 X\) into two sets: the locus \(\partial_1^+ X\), where the field is directed inward of \(X\), and \(\partial_1^- X\), where it is directed outwards. We assume that \(v\), viewed as a section of the quotient line bundle \(T(X)/T(\partial X)\) over \(\partial X\), is transversal to its zero section. This assumption implies that both sets \(\partial_1^+ X\) are compact manifolds which share a common boundary \(\partial_2 X := \partial(\partial_1^+ X) = \partial(\partial_1^- X)\). Evidently, \(\partial_2 X\) is the locus where \(v\) is tangent to the boundary \(\partial_1 X\).

Morse has noticed that, for a generic vector field \(v\), the tangent locus \(\partial_2 X\) inherits a similar structure in connection to \(\partial_1^+ X\), as \(\partial_1 X\) has in connection to \(X\) (see [Mo]). That is, \(v\) gives rise to a partition \(\partial_2^+ X \cup \partial_2^- X\) of \(\partial_2 X\) into two sets: the locus \(\partial_2^+ X\), where the field is directed inward of \(\partial_1^+ X\), and \(\partial_2^- X\), where it is directed outward of \(\partial_1^+ X\). Again, let us assume that \(v\), viewed as a section of the quotient line bundle \(T(\partial_1 X)/T(\partial_2 X)\) over \(\partial_2 X\), is transversal to its zero section.
For generic fields, this structure replicates itself: the cuspidal locus $\partial_3 X$ is defined as the locus where $v$ is tangent to $\partial_2 X$; $\partial_3 X$ is divided into two manifolds, $\partial_{3}^{+}X$ and $\partial_{3}^{-}X$. In $\partial_{3}^{+}X$, the field is directed inward of $\partial_{2}^{+}X$, in $\partial_{3}^{-}X$, outward of $\partial_{2}^{+}X$. We can repeat this construction until we reach the zero-dimensional stratum $\partial_{n+1}X = \partial_{n+1}^{+}X \cup \partial_{n+1}^{-}X$.

These considerations motivate

**Definition 1.1.** We say that a smooth field $v$ on $X$ is boundary generic if:

- $v|_{\partial X} \neq 0$,
- $v$, viewed as a section of the tangent bundle $T(X)$, is transversal to its zero section,
- for each $j = 1, \ldots, n + 1$, the $v$-generated stratum $\partial_{j}X$ is a smooth submanifold of $\partial_{j-1}X$,
- the field $v$, viewed as section of the quotient 1-bundle
  \[
  T_{j}^{v} := T(\partial_{j-1}X)/T(\partial_{j}X) \to \partial_{j}X,
  \]
  is transversal to the zero section of $T_{j}^{v}$ for all $j > 0$.

We denote by $\mathcal{V}^{\dagger}(X)$ the space of all boundary generic fields on $X$.

Thus a boundary generic vector field $v$ on $X$ gives rise to two Morse stratifications:

\[
\partial X := \partial_{1}X \supset \partial_{2}X \supset \cdots \supset \partial_{n+1}X,
X := \partial_{0}^{+}X \supset \partial_{1}^{+}X \supset \partial_{2}^{+}X \supset \cdots \supset \partial_{n+1}^{+}X
\]

, the first one by closed submanifolds, the second one—by compact ones. Here $\dim(\partial_{j}X) = \dim(\partial_{j}^{\dagger}X) = n + 1 - j$. For simplicity, the notations "$\partial_{j}X$" do not reflect the dependence of these strata on the vector field $v$. When the field varies, we use a more accurate notation "$\partial_{j}X(v)$".

When $v$ is nonsingular on $\partial_{1}X$, we can extend it into a larger manifold $\hat{X}$ so that $\hat{X}$ properly contains $X$ and the extension $\hat{v}$ remains nonsingular in the vicinity of $\partial_{1}X \subset \hat{X}$. Throughout this text, we treat the pair $(\hat{X}, \hat{v})$ as a germ which extends $(X, v)$.

At each point $x \in \partial_{1}X$, the $(-\hat{v})$-flow defines the germ of the projection $p_{x} : \hat{X} \to S_{x}$, where $S_{x}$ is a local section of the $\hat{v}$-flow which is transversal to it. The projection is considered at each point of $\partial_{1}X \subset \hat{X}$. When $\hat{v}$ is a gradient-like field for a function $f : \hat{X} \to \mathbb{R}$, we can choose the germ of the hypersurface $f^{-1}(f(x))$ for the role of $S_{x}$.

For boundary generic vector field $v$, we associate an ordered sequence of multiplicities with each trajectory $\gamma$, such that $\gamma \cap \partial_{1}X$ is a finite set. In fact, for any $v \in \mathcal{V}^{\dagger}(X)$, the intersection $\{a_{i}\} := \gamma \cap \partial_{1}X$ is automatically a finite set. For traversing (see Definition 4.6 from [K1]) generic fields, the points $\{a_{i}\}$ of the intersection $\gamma \cap \partial_{1}X$ are ordered by the field-oriented trajectory $\gamma$, and the index $i$ reflects this ordering.

**Definition 1.2.** Let $v \in \mathcal{V}^{\dagger}(X)$ be a generic field. Let $\gamma$ be a $v$-trajectory which intersects the boundary $\partial_{1}X$ at a finite number of points $\{a_{i}\}$. Each point $a_{i}$ belongs to a unique pure stratum $\partial_{j_{i}}X^{v}$.
The multiplicity $m(\gamma)$ of $\gamma$ is defined by the formula

$$m(\gamma) = \sum_i j_i$$

(1.2)

The reduced multiplicity $m'(\gamma)$ of $\gamma$ is defined by the formula

$$m'(\gamma) = \sum_i (j_i - 1)$$

(1.3)

and the virtual multiplicity $\mu(\gamma)$ of $\gamma$ is defined by

$$\mu(\gamma) = \sum_i \left\lceil \frac{j_i}{2} \right\rceil$$

(1.4)

where $\lceil \sim \rceil$ denotes the integral part function.

For an open and dense subspace $V^\dag(X)$ of $V^\ddagger(X)$, one can interpret $\mu(\gamma)$ as the maximal number of tangency points that any trajectory $\gamma'$ in the vicinity of $\gamma$ may have (see Theorem 3.4 from [K2]).

When $v \in V^\ddagger(X)$, each set $\partial_j X(v)$ is a manifold.

Let $\partial_j X(v)^\circ := \partial_j X(v) \setminus \partial_{j+1} X(v)$ denotes the pure Morse stratum.

The following lemma (see Lemma 3.1 from [K2] and [Morin]) provides us with an analytic description of the Morse strata.

**Lemma 1.1.** Assume that $v$ is a boundary generic field. Denote by $\gamma_a$ the $\hat{v}$-trajectory through a point $a \in X$. If $a \in \partial_k X(v)^\circ$ then in the vicinity of $a$ in $\hat{X}$, there exists a coordinate system $(u, x)$, where $u \in \mathbb{R}$ and $x \in \mathbb{R}^n$, so that:

- each $v$-trajectory $\gamma$ is defined by an equation $\{x = \text{const}\},$
- the boundary $\partial_1 X$ is defined by the equation

$$u^k + \sum_{j=0}^{k-2} x_j u^j = 0$$

(1.5)

- each $v$-trajectory $\gamma$ hits only some strata $\{\partial_j X(v)^\circ\}_{j \in J(a)}$ in such a way that

$$\sum_{j \in J(a)} j \leq k, \quad \text{and} \quad \sum_{j \in J(a)} j \equiv k \ (2)$$

(2)

The introduction of the next class $V^\ddagger(X)$ of vector fields is inspired by the singularity theory of so called Boardman maps with normal crossings (see [Bo], [GG]).

Consider the collection of tangent spaces $\{T_a(\partial_j X^\circ)\}_{i}$ to the pure strata $\{\partial_j X^\circ\}_{i}$ that have a non-empty intersection with a given trajectory $\gamma$. By Lemma 1.1, each space $T_a(\partial_j X^\circ)$ is transversal to the curve $\gamma$.

Let $S$ be a local section of the $\hat{v}$-flow at some point $a_* \in \gamma$ and let $T_*$ be the space tangent to $S$ at $a_*$. Each space $T_a(\partial_j X^\circ)$, with the help of the $\hat{v}$-flow, determines a vector
subspace $T_i = T_i(\gamma)$ in $T_\star$. It is the image of the tangent space $T_{a_i}(\partial_1X)$ under the composition of two maps: (1) the differential of the flow-generated diffeomorphism that maps $a_i$ to $a_\star$ and (2) the linear projection $T_{a_\star}(X) \to T_\star$ whose kernel is generated by $v(a_\star)$.

For a traversing $v$ and a majority of trajectories, we can choose the space $T_{a_\star}(\partial_1^+X)$ for the role of $T_\star$, where $a_\star$ is the lowest point of $\gamma \cap \partial_1X$.

The configuration $\{T_i\}$ of affine subspaces $T_i \subset T_\star$ is called generic (or stable) when all the multiple intersections of spaces from the configuration have the least possible dimensions, consistent with the dimensions of $\{T_i\}$. In other words,

$$\text{codim}(\bigcap_s T_{i_s}, T_\star) = \sum_s \text{codim}(T_{i_s}, T_\star)$$

for any subcollection $\{T_{i_s}\}$ of spaces from the list $\{T_i\}$.

Consider the case when $\{T_i\}$ are vector subspaces of $T_\star$. If we interpret each $T_i$ as the kernel of a linear epimorphism $\Phi_i : T_\star \to \mathbb{R}^{n_i}$, then the property of $\{T_i\}$ being generic can be reformulated as the property of the direct product map $\prod_i \Phi_i : T_\star \to \prod_i \mathbb{R}^{n_i}$ being an epimorphism. In particular, for a generic configuration of affine subspaces, if a point belongs to several $T_i$’s, then the sum of their codimensions $n_i$ does not exceed the dimension of the ambient space $T_\star$.

The definition below resembles the “Normal Crossing Condition” imposed on Boardman maps between smooth manifolds (see [GG], page 157, for the relevant definitions). In fact, for traversing generic fields $v$, the $v$-flow delivers germs of Boardman maps $p(v, \gamma) : \partial_1X \to \mathbb{R}^n$, available in the vicinity of every trajectory $\gamma$.

**Definition 1.3.** We say that a traversing field $v$ on $X$ is traversally generic if:

- the field is boundary generic in the sense of Definition 1.1,
- for each $v$-trajectory $\gamma \subset X$ (not a singleton), the collection of subspaces $\{T_i(\gamma)\}_i$ is generic in $T_\star$: that is, the obvious quotient map $T_\star \to \prod_i (T_\star / T_i(\gamma))$ is surjective.

We denote by $V^\dagger(X)$ the space of all traversally generic fields on $X$. □

**Remark 1.1.** In particular, the second bullet of the definition implies the inequality

$$\sum_i \text{codim}(T_i(\gamma), T_\star) \leq \dim(T_\star) = n.$$ 

In other words, for traversally generic fields, the reduced multiplicity of each trajectory $\gamma$ satisfies the inequality

$$(1.6) \quad m'(\gamma) = \sum_i (j_i - 1) \leq n.$$ 

The following key lemma (see Lemma 3.4 from [K2]) provides us a a semi-local analytic description of the traversally generic fields.
Lemma 1.2. Let \( v \) be a traversing generic field on \( X \) and \( \gamma \) its trajectory such that the intersection \( \gamma \cap \partial_1 X \) is a union of several points \( a_i \in \partial_2 X(v) \).

Then \( \gamma \) has a \( \hat{v} \)-adjusted neighborhood \( V \) with a special system of coordinates
\[
(u, x_{i0}, \ldots, x_{ij_1-2}, \ldots, x_{i0}, \ldots, x_{ij_2-2}, \ldots, x_{ip0}, \ldots, x_{pj_1-2}, y_1, \ldots, y_{n-m(\gamma)})
\]
such that:
- \( \{u = \text{const}\} \) defines a transversal section of the \( \hat{v} \)-flow,
- each \( \hat{v} \)-trajectory in \( V \) is produced by fixing all the coordinates \( \{x_i\} \) and \( \{y_k\} \),
- there is \( \epsilon > 0 \) such that \( V \cap \partial_1 X \subset \bigcap_i V_i \), where \( V_i := u^{-1}((\alpha_i - \epsilon, \alpha_i + \epsilon)) \cap V \), and \( \alpha_i = u(a_i) \),
- the intersection \( V \cap \partial_1 X \) is given by the equation
\[
\prod_i \left((u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i-2} x_{il}(u - \alpha_i)^l\right) = 0.
\]
\( \square \)

2. Bifurcations of the Real Polynomial Divisors — the Combinatorics of Tangency for Traversally Generic Flows

In [K2], we have seen evidence that, for smooth traversally generic fields \( v \) on a compact manifold \( X \), the combinatorial patterns of tangency along \( v \)-trajectories resemble the combinatorics of divisors in \( \mathbb{R} \), produced by real polynomials of an even degree \( d \leq 2 \text{dim}(X) \) (see Lemma 3.4, Theorems 3.1, and Theorem 3.5 from [K2]). Now we are going to devote time to somewhat involved investigations of this combinatorics.

For any polynomial \( P(z) \) with real coefficients, we denote by \( D_\mathbb{R}(P) \) its divisor in \( \mathbb{R} \), and by \( D_\mathbb{C}(P) \) its complex conjugation-invariant divisor in \( \mathbb{C} \). Points in \( \text{sup}(D_\mathbb{R}(P)) \), the support of \( D_\mathbb{R}(P) \), inherit the natural order from \( \mathbb{R} \). The set \( \text{sup}(D_\mathbb{C}(P)) \subset \mathbb{C} \) is invariant under the complex conjugation.

Let \( D_d \) denote the space of all divisors \( D \) in \( \mathbb{R} \) of degree \( |D| = d \). The space \( D_d \) can be identified with the \( d \)-th symmetric power \( \text{Sym}^d(\mathbb{R}) \) of the real number line \( \mathbb{R} \). Alternatively, \( D_d \) can be introduced as the domain \( \Pi_d \subset \mathbb{R}^d \) given by the inequalities \( x_1 \leq x_2 \leq \cdots \leq x_d \) imposed on the coordinates \( (x_1, x_2, \ldots, x_d) \).

The divisors \( D \in D_d \) have combinatorial models represented by maps
\( \omega_D : \{1, 2, 3, \ldots\} \to \{0, 1, 2, 3, \ldots\} \)
where \( \omega_D(i) \geq 1 \) is the multiplicity of the \( i \)-th point in \( \text{sup}(D) \), so that \( \sum_i \omega_D(i) = d \).

Let \( \mathbb{N} \) be the set of all natural numbers and \( \mathbb{Z}_+ \) the set of all non-negative integers. Consider the set \( \Omega \) of all maps \( \omega : \mathbb{N} \to \mathbb{Z}_+ \) with finite support and such that \( \omega(i) \neq 0 \) implies \( \omega(j) \neq 0 \) for all \( j < i \).

Consider a real polynomial \( P \) of degree \( d \) and its complex divisor \( D_\mathbb{C}(P) \). There is \( \epsilon > 0 \) such that the \( \epsilon \)-neighborhood \( U_\epsilon \) of the support \( \text{sup}(D_\mathbb{C}(P)) \subset \mathbb{C} \) is a union of disjoint
open disks. Then any real $d$-polynomial $Q$, sufficiently close to $P$, will have all its roots residing in $U_\varepsilon$.

The radial contraction of each disk from $U_\varepsilon$ to its center commutes with the complex conjugation in $C$ and defines a conjugation-equivariant deformation of the divisor $D_C(Q)$ into the divisor $D_C(P)$. This produces a deformation of $Q$ to $P$, a curve $Q_s$, $s \in [0, 1]$, in the space of real polynomials of degree $d$. At all stages of this deformation, but the last one ($s = 1$), the divisor $D_C(Q_s)$ has the same multiplicity pattern as the one of $D_C(Q)$; moreover, the divisor $D_R(Q_s)$ has the same $R$-ordered multiplicity pattern as the one of $D_R(Q)$.

Now imagine this deformation process from the viewpoint of an observer residing in $R$, so that morphings of all the roots residing in $C \setminus R$ are “invisible”.

Deformations of $P$ within the space of real polynomials of degree $\deg(P)$ change its real divisor $D_R(P)$ by sequences of two elementary operations and their inverses:

1. merging of two adjacent points from the support $\sup(D_R(P))$ (their multiplicities add up);

2. inserting a point of multiplicity 2 to the set $R \setminus \sup(D_R(P))$.

The second elementary operation corresponds to a pair of simple complex-conjugate roots merging at a point of $R \subset C$.

These operations have combinatorial analogues. We define the elementary merge operation $M_j : \Omega \to \Omega$ by the formula

\begin{align*}
M_j(\omega)(i) &= \omega(i) \quad \text{for all } i < j, \\
M_j(\omega)(j) &= \omega(j) + \omega(j + 1), \\
M_j(\omega)(i) &= \omega(i + 1) \quad \text{for all } i > j + 1,
\end{align*}

where $\omega \in \Omega$ and $1 \leq j \leq |\sup(\omega)|$.

Define the elementary insert operation $I_j : \Omega \to \Omega$ by the formula

\begin{align*}
I_j(\omega)(i) &= \omega(i) \quad \text{for all } i < j, \\
I_j(\omega)(j) &= 2, \\
I_j(\omega)(i) &= \omega(i - 1) \quad \text{for all } i > j \geq 1,
\end{align*}

and

\begin{align*}
I_0(\omega)(1) &= 2, \\
I_0(\omega)(i) &= \omega(i - 1), \text{ for all } i > 1.
\end{align*}

We also allow for the merge of several adjacent points in $\sup(D_R(P))$ and for the merge of groups of conjugate roots (of any multiplicity) at a point(s) of $R$. However, we tend to view these morphings as compositions of the elementary operations $\{M_j\}$ and $\{I_j\}$.

Note that the parity of the degree $\deg(D_R(P))$ is preserved under the merge and insert operations.

Guided by Definition 1.2, we introduce combinatorial analogues of the quantities from that definition:
• the $l_1$-norm $|\omega|$ of $\omega$ by the formula $\sum_i \omega(i)$
• the reduced norm $|\omega'|$ of $\omega$, by the formula $\sum_i (\omega(i) - 1)$,
• the virtual multiplicity $\mu(\omega)$ of $\omega$, by the formula $\sum_i [\omega(i)/2]$, where $[\sim]$ denotes the integral part of a real number.

Now we are in position to define, via the merge and insert operations, a partial order 
"$\succ$" in the set $\Omega$:

**Definition 2.1.** For $\omega_1, \omega_2 \in \Omega$, we write $\omega_1 \succ \omega_2$ if $\omega_2$ can be obtained from $\omega_1$ by a sequence of merge operations $\{M_j\}$ and insert operations $\{I_j\}$ as in (2.1)-(2.3).

Let $\Omega_d \subset \Omega$ denote the set of all $\omega$'s such that $|\omega| = d$. Let $\Omega_d \subset \Omega$ denote the set of all $\omega$'s such that $|\omega| \leq d$ and $|\omega| \equiv d$ (2). The set $\Omega_d$ inherits its partial order from the ambient poset $(\Omega, \succ)$.

For all our applications we will need only the case of an even $d$.

We can give an interpretation to the poset $(\Omega, \succ)$ in the spirit of category theory. In this interpretation, the elements of $\Omega$ become objects of some category $\Omega$ so that the relation $\omega_1 \succ \omega_2$ is transformed into the property $\text{Mor}(\omega_1, \omega_2) \neq \emptyset$.

The following definition mimics the map of conjugation-invariant divisors $D_C(Q) \to D_C(P)$, induced by the radial retraction of the $\epsilon$-neighborhood $U_\epsilon(\text{sup}(D_C(P)))$ to its core, $\text{sup}(D_C(P))$, as observed from within $\mathbb{R} \subset \mathbb{C}$.

**Definition 2.2.** For any two elements $\omega_1, \omega_2 \in \Omega$ we define the set $\text{Mor}(\omega_1, \omega_2)$ as the set of maps $\alpha : \text{sup}(\omega_1) \to \text{sup}(\omega_2)$ such that:

1. For each pair $i < i'$ in $\text{sup}(\omega_1)$, $\alpha(i) \leq \alpha(i')$,
2. $\sum_{j \in \alpha^{-1}(i)} \omega_1(j) \leq \omega_2(i)$ for all $i \in \text{sup}(\omega_2)$,
3. $\sum_{j \in \alpha^{-1}(i)} \omega_1(j) \equiv \omega_2(i) \mod (2)$ for all $i \in \text{sup}(\omega_2)$.

For example, $\text{Mor}((121), (11) = \emptyset$, while $\text{Mor}((11), (121))$ consists a single injective map.

Examining properties (1)-(3) above, we see that there is a natural pairing

$$\text{Mor}(\omega_1, \omega_2) \times \text{Mor}(\omega_2, \omega_3) \to \text{Mor}(\omega_1, \omega_3)$$

defined by the composition of maps, provided $\text{Mor}(\omega_1, \omega_2) \neq \emptyset$ and $\text{Mor}(\omega_2, \omega_3) \neq \emptyset$.

**Lemma 2.1.** In the category $\Omega$, the set $\text{Mor}(\omega_1, \omega_2) \neq \emptyset$, if and only if, $\omega_1 \succeq \omega_2$ in the poset $(\Omega, \succ)$.

**Proof.** Any elementary operation $M_j$ in (2.1) gives rise to an element $\mu_j \in \text{Mor}(\omega, M_j(\omega))$ which maps the pair $j, j+1 \in \text{sup}(\omega)$ to the single element $j \in \text{sup}(M_j(\omega))$; the rest of the elements in $\text{sup}(\omega)$ are mapped bijectively by $\mu_j$. Evidently, the properties (1)-(3) in Definition 2.2 are satisfied. Similarly, any elementary operation $I_j$ in (2.2) gives rise to an

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1This number represents $\deg(D_\delta(P))$.
2This restriction is vacuous when $\alpha^{-1}(i) = \emptyset$.
3In particular, if $\alpha^{-1}(i) = \emptyset$, then $\omega_2(i) \equiv 0 \mod (2)$. 
element \( \nu_j \in Mor(\omega, I_j(\omega)) \) which maps \( \text{sup}(\omega) \) bijectively and in a monotone fashion to \( \text{sup}(I_j(\omega)) \) so that \( j \in \text{sup}(I_j(\omega)) \) is the only element that is not in the image of \( \nu_j \). Again, the properties (1)-(3) in Definition 2.2 are satisfied.

Therefore if two elements, \( \omega_1 \) and \( \omega_2 \), are linked by a sequence of elementary operations of the types \( M_j \) and \( I_k \), then there exists an element \( \alpha \in Mor(\omega_1, \omega_2) \) which is obtained by composing the chain of corresponding maps \( \mu_j \) and \( \nu_k \). Hence, \( Mor(\omega_1, \omega_2) \neq \emptyset \).

On the other hand, if \( Mor(\omega_1, \omega_2) \neq \emptyset \), then any \( \alpha \in Mor(\omega_1, \omega_2) \) can be obtained in such a way via elementary operations. Indeed, for each \( i \in \text{sup}(\omega_2) \), consider the set \( \alpha^{-1}(i) \). Put

\[
\alpha_i = \omega_2(i) - \sum_{j \in \alpha^{-1}(i)} \omega_1(j).
\]

By Definition 2.2, \( \alpha_i \geq 0 \) and \( \alpha_i \equiv 0 \mod (2) \). We apply a sequence of \( \alpha_i/2 \) insert operations \( I_k \) to \( \omega_1 \), localized to the set \( \alpha^{-1}(i) \). They will add \( \alpha_i/2 \) copies of 2’s to the sequence \( \omega_1 \). The location of these 2’s relative to the elements of \( \alpha^{-1}(i) \) is unimportant. The resulting \( \omega'_1 \) now has the property \( Mor(\omega_1, \omega'_1) \neq \emptyset \). These insertions \( \{I_k\} \) define a map \( \gamma \in Mor(\omega_1, \omega'_1) \). Next we merge all the original elements of the set \( \alpha^{-1}(i) \) and the locations of newly inserted 2’s together into a singleton \( i_* \) by a sequence of elementary merges \( \{M_j\} \). Again, the order in which the elementary merges are performed is unimportant. The resulting \( \omega''_1 \) is such that there exists \( \delta \in Mor(\omega_1, \omega''_1) \) with the property

\[
\sum_{j \in \delta^{-1}(i_*)} \omega_1(j) = \omega''_1(i_*) = \omega_2(i).
\]

Finally, we apply this procedure to every element \( i \in \text{sup}(\omega_2) \). The result of these operations transforms \( \omega_1 \) into \( \omega_2 \) by a sequence of elementary operations. Therefore, \( \omega_1 \succ \omega_2 \) in the poset \( \Omega \). \( \square \)

The polynomial inequality \( P(z) \leq 0 \) splits the support \( \text{sup}(D_R(P)) \) into a number of disjoint sets: each set is formed by the maximal string of consecutive roots so that, in the closed interval bounded by the maximal and the minimal root from the string, the inequality \( P(z) \leq 0 \) is valid. For a polynomial of an even degree, each maximal string of roots either is a singleton whose multiplicity is even, or a sequence whose maximal and minimal elements have odd multiplicities, while the rest of roots have even multiplicities. These observations motivate the following combinatorial models.

**Definition 2.3.** Let \( \Omega^* \) denote the set of maps \( \omega \in \Omega \) that satisfy the following properties:

- either
  - (1) \( \omega(1), \omega(q) \) are odd numbers, where \( q = |\text{sup}(\omega)| \), and
  - (2) \( \omega(i) \) is even for \( 1 < i < q \);
- or \( \text{sup}(\omega) = \{1\} \), and \( \omega(1) \equiv 0 \mod (2) \).

A priori \( q \), the cardinality of the support of \( \omega \in \Omega^* \), is not fixed.

**Definition 2.4.** Let \( \Omega^*_{d} \) denote the set of maps \( \omega \in \Omega^* \) such that \( |\omega'| := \sum_i (\omega(i) - 1) = d \).
Let
\[ \Omega^\bullet_{[k,d]} := \prod_{k \leq j \leq d} \Omega^\bullet_j. \]

We also will use the shorter notation “\( \Omega^\bullet_{[d]} \)” for the set \( \Omega^\bullet_{[0,d]} \).

Recall that on a \((n+1)\)-manifold \( X \), any trajectory \( \gamma \) of a boundary generic field \( v \in \mathcal{V}(X) \) gives rise to an element \( \omega_\gamma \in \Omega^\bullet \) according to the rule: \( \omega_\gamma(i) = m(a_i) \), the multiplicity of the \( i \)-th point \( a_i \) in the \( v \)-ordered set \( \gamma \cap \partial_1 X \). By Theorem 3.5 from [K2], for any traversally generic field \( v \in \mathcal{V}(X) \), we get \( |\omega_\gamma'| \leq n = \dim(\partial_1 X) \), so that \( \omega_\gamma \in \Omega^\bullet_{[n]} \).

In the end of the proof of Theorem 3.5 from [K2], we have established the following proposition:

**Lemma 2.2.** For \( \omega \in \Omega^\bullet_{[n]} \), we have \( |\omega| := \sum_i \omega(i) \leq 2n + 2 \).□

We are going to introduce a partial order among the elements of the set \( \Omega^\bullet_{[d]} \), which will match the changing geometry of orbits for traversally generic fields. Crudely, the order is induced from the ambient poset \( \Omega \supset \Omega^\bullet_{[d]} \), but then enhanced. For a more accurate description, we turn to few auxiliary combinatorial constructions.

Given any \( \omega \in \Omega \), we would like to “chop it” into a number of “strings” and “atoms”: each string belongs to some \( \Omega^\bullet_i \) as in the first bullet of the Definition 2.3, while each atom has a singleton for its support and takes there an even value (see Fig. 1). Prior to defining this canonical deconstruction \( \Xi(\omega) \) of \( \omega \), we need to introduce some notations.

Let \( \omega \in \Omega \) be such that \( |\omega| \equiv 0 \) (2). Denote by \( \text{sup}_{\text{odd}}(\omega) \) the points \( l \in \mathbb{N} \) in the support of \( \omega \) such that \( \omega(l) \equiv 1 \) (2) and by \( \text{sup}_{\text{ev}}(\omega) \) the points \( l \) in the support of \( \omega \) such that \( \omega(l) \equiv 0 \) (2). In the geometrical context of traversing flows, the number \( |\text{sup}_{\text{odd}}(\omega)| \) is even. We count the elements of \( \text{sup}_{\text{odd}}(\omega) \) as they appear in the list; some of them acquire odd numerals, others acquire even ones. A string in \( A \subset \text{sup}(\omega) \) is formed by all the elements that are bounded on the left by an element of \( i \in \text{sup}_{\text{odd}}(\omega) \) with an odd numeral and on the right by the next element \( j \in \text{sup}_{\text{odd}}(\omega) \) (it has an even numeral attached to it). In other words, a string \( A := [i,j] \) is formed by such \( i,j \) and all the elements from \( \text{sup}_{\text{ev}}(\omega) \) that lie in-between. The points from \( \text{sup}_{\text{ev}}(\omega) \) that do not belong to any string are called “atoms”. We can compute the values of \( \omega \) at the elements of each string \( A \) as well as at each atomic support. This gives rise to a unique ordered sequence \( \Xi(\omega) \) of strings, interrupted by a number of atoms.

**Definition 2.5.** We define the partial order “\( \prec \)” in \( \Omega^\bullet \) as follows: for \( \omega_1, \omega_2 \in \Omega^\bullet \), we write “\( \omega_1 \prec \omega_2 \)” if \( \omega_2 \) occurs as a string or an atom from the list \( \Xi(\omega) \) for some \( \omega \succ \omega_1 \), where \( \omega \in \Omega \) and the last ordering is considered in the poset \( (\Omega, \succ) \).□

**Example 2.1.** Consider the poset \( (\Omega^\bullet_{[0,3]}, \prec) \) shown in Fig. 2. Then \( (1,4,1) \prec (3,1) \) since \( (3,1) \) is present as a string in \( (1,1,3,1) \succ (1,4,1) \), the order “\( \succ \)” being the one from the poset \( \Omega_{[0,8]} \).□

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4that is, \( a_i \in \partial_{m(a_i)} X(v) \).
Figure 1. An example of the deconstruction \( \omega \Rightarrow \Xi(\omega) \). The strings are boxed, the atoms are circled.

Figure 2. The poset \( \langle \Omega^{*}_{[0,3]}, \succ \rangle \) with the "height" function \( m'(\omega) := |\omega|' \). The bold dark lines indicate the insert operations, the lighter bold lines the merge operations. The dotted lines represent the order relations, as in Definition 2.5 that are not directly induced from the poset \( \Omega_{[0,8]} \supset \Omega^{*}_{[0,3]} \). The upper index next to \( \omega \) shows the value of the norm \( |\omega| \).

Lemma 2.3. For any \( d \), the sub-poset of \( \Omega^{*}_{(d]} \), defined by the constraint \( |\omega|' < d \), is canonically isomorphic to the poset \( \Omega^{*}_{(d-1]} \). As a result, the direct limit

\[
\lim_{d \to +\infty} \Omega^{*}_{(d]} = \Omega^{*}
\]

as posets.

Proof. Let \( \omega_1 \succ \omega_2 \) in \( \Omega^{*}_{(d]} \). This implies that \( \omega_1 \) is string or an atom in some \( \omega \succ \omega_2 \) in \( (\Omega, \succ) \). If \( |\omega_2|' < d \), then \( |\omega|' < d \) as well. In turn, \( |\omega|' < d \) implies that \( |\omega_1|' < d \). Therefore \( \omega_1 \succ \omega_2 \) in \( \Omega^{*}_{(d-1]} \) as well. \( \square \)

Similar to the poset \( (\Omega, \succ) \), the poset \( (\Omega^{*}, \succ) \) allows for an interpretation in terms of a category theory \( \Omega^{*} \) whose objects are elements of \( \Omega^{*} \).

Definition 2.6. For any two elements \( \omega_1, \omega_2 \in \Omega^{*} \) we define \( \text{Mor}^{*}(\omega_1, \omega_2) \) as the set of maps \( \alpha : \sup(\omega_1) \to \sup(\omega_2) \) such that:

1. for each pair \( i < i' \) in \( \sup(\omega_1) \), \( \alpha(i) \leq \alpha(i') \),
\[ (2) \sum_{j \in \alpha^{-1}(i)} \omega_1(j) \leq \omega_2(i) \text{ for all } i \in \sup(\omega_2). \] 

\[ \Box \]

**Lemma 2.4.** In the category \( \Omega^* \), the set \( \text{Mor}^*(\omega_1, \omega_2) \neq \emptyset \), if and only if, \( \omega_1 \succ \omega_2 \) in the poset \( (\Omega_*, \succ) \).

**Proof.** If \( \omega_1 \succ \omega_2 \), then by the definition of the order \( \succ \), there exists an element \( \tilde{\omega}_1 \in \Omega \) such that \( \tilde{\omega}_1 \succ \omega_2 \), and \( \omega_1 \) is a string or an atom in \( \tilde{\omega}_1 \). By Lemma 2.1, \( \text{Mor}(\tilde{\omega}_1, \omega_2) \neq \emptyset \).

When we restrict a map \( \tilde{\alpha} \in \text{Mor}(\tilde{\omega}_1, \omega_2) \) to the subset \( \sup(\tilde{\omega}_1) \subset \sup(\tilde{\omega}_1) \) we produce a map \( \alpha : \sup(\omega_1) \to \sup(\omega_2) \). Under this restriction, the properties (1) and (2) from Definition 2.6. hold, we are evidently preserved for its restriction \( \alpha \): the support of a string consists of consecutive indices, and \( \tilde{\alpha}^{-1}(i) \subset \alpha^{-1}(i) \) for all \( i \in \sup(\omega_2) \), while the property (3) could be violated. Thus \( \alpha \in \text{Mor}^*(\omega_1, \omega_2) \), a nonempty set.

On the other hand, if \( \text{Mor}^*(\omega_1, \omega_2) \neq \emptyset \), then it is possible to construct \( \tilde{\omega}_1 \in \Omega \) such that \( \text{Mor}(\tilde{\omega}_1, \omega_2) \neq \emptyset \). Indeed, consider some \( \alpha : \sup(\omega_1) \to \sup(\omega_2) \) having properties (1) and (2) from Definition 2.6. If, in addition, property (3) from Definition 2.2 holds, we are done. Note that this parity property can only be violated if one or both ends \( \{1\}, \{q\} \) in the support of the string \( \omega_1 \) are mapped by \( \alpha \) to the location \( \alpha(1) \) or \( \alpha(q) \) such that \( \omega_2(\alpha(1)) \equiv 0 \mod (2), \) or \( \omega_2(\alpha(q)) \equiv 0 \mod (2) \).

Let us consider the case \( \omega_2(\alpha(1)) \equiv 0 \mod (2) \). The monotonicity of \( \alpha \) implies that the minimum element \( \{i_\ast\} \) in the support of \( \omega_2 \) is not in the image of \( \alpha \). We can append a string \( \omega' = (1, 1) \) below (to the left) of the string \( \omega_1 \) to form a new element \( \tilde{\omega}_1 \in \Omega \) (not a string!) so that \( \sup(\tilde{\omega}_1) = \sup((1, 1)) \) and \( \sup(\omega) \). Then we extend the map \( \alpha \) to a map \( \tilde{\alpha} : \sup((1, 1)) \to \sup(\omega_2) \) by sending the first element of the extended support to the minimal element \( \{i_\ast\} \in \sup(\omega_2) \), and the second and the third elements both to \( \alpha(1) \) (note that the third element comes from the element \( \{1\} \in \sup(\omega_1) \)). This will repair the parity defect of the original \( \alpha \). A similar treatment applies when \( \omega_2(\alpha(q)) \equiv 0 \mod (2) \). Thus, \( \text{Mor}^*(\omega_1, \omega_2) \neq \emptyset \) implies that \( \text{Mor}(\tilde{\omega}_1, \omega_2) \neq \emptyset \).

By Lemma 2.1 we get that \( \tilde{\omega}_1 \succ \omega_2 \) in \( (\Omega, \succ) \). Therefore, \( \text{Mor}^*(\omega_1, \omega_2) \neq \emptyset \) implies that \( \omega_1 \succ \omega_2 \). \[ \Box \]

The following lemma can be viewed as an a posteriori justification for introducing the poset \( (\Omega^*, \succ) \). Here, for a boundary generic vector field \( v \in V^\dagger(X) \) and its trajectory \( \gamma \), we view the set \( \gamma \cap \partial_1 X \), together with the multiplicities attached to its points, as a divisor \( D_\gamma \) on \( \gamma \).

**Lemma 2.5.** Let \( v \in V^\dagger(X) \). If a sequence \( \{x_k \in X\}_k \) converges to a point \( x \), then

\[ |D_{\gamma(x)}|' \geq \lim_{k \to +\infty} |D_{\gamma(x_k)}|' \]

, i.e. the reduced norm \( |D_{\gamma(x)}|' \) is a upper semi-continuous function on \( X \). Moreover, if the divisors \( \{D_{\gamma(x_k)}\}_k \) all share a combinatorial type \( \omega \), and \( \omega_1 \neq \omega \) is the combinatorial type of \( D_{\gamma(x)} \), then we get: \( \omega \succ \omega_1 \), \( |\omega_1| \geq |\omega| \), and \( |\omega_1|' > |\omega|' \).

\[ ^5 \text{This restriction is vacuous when } \alpha^{-1}(i) = \emptyset. \]
Proof. The trust of the argument is that any merge or insert operations (see (2.1)-(2.3)) that change the combinatorics of \( D_\gamma \) cannot decrease the multiplicity \( m(\gamma) \), as well as the reduced multiplicity \( m'(\gamma) \) (defined in (1.2)-(1.4)). In fact, any such operation does increase \( m'(g) \).

In the argument below, we use our understanding of local models (1.5) (see Lemma 1.1) of fields \( v \in \mathcal{V}(X) \) in the vicinity of \( \partial_1 X \). They imply that, as we vary \( \gamma \), its divisor \( D_{\gamma} \) (with the support \( \gamma \cap \partial_1 X \)) can only change locally as the real divisors of the family of \( u \)-polynomials in (1.5) do: that is, via resolutions that either preserve their degree, or drop it by an even number. By the same token, as \( \{x_k\} \) converge to \( x \), in the vicinity of \( x \), the divisor \( D_{\gamma(x)} \) is obtained from \( \{D_{\gamma(x_k)}\} \) only via: (1) merging adjacent distinct points from \( \sup(D_{\gamma(x_k)}) \), (2) by inserting new points of even multiplicity, or (3) by increasing the odd multiplicity of the end point from \( \sup(D_{\gamma(x_k)}) \) by an odd number (this happens when the ends of two distinct trajectories merge at \( x \)).

Let us take a closer look at these mechanisms.

By Lemma 1.1, for a boundary generic \( v \in \mathcal{V}(X) \), the multiplicity of each point from \( \gamma \cap \partial_1 X \) does not exceed \( n + 1 \) for all \( \gamma \)'s. Moreover, by the same lemma and a compactness argument, no trajectory \( \gamma \) has an infinite set \( \gamma \cap \partial_1 X \). Thus the multiplicity \( m(\gamma) < \infty \) for each \( \gamma \).

Consider any infinite set of trajectories \( \{\gamma_k\}_k \) such that the set of multiplicities \( \{m(\gamma_k)\}_k \) is unbounded. Since the multiplicity of each point of tangency is uniformly bounded, this implies that the set

\[ A := \bigcup_{1 \leq k < \infty} (\gamma_k \cap \partial_1 X) \]

is infinite and \( \lim_{k \to \infty} |\gamma_k \cap \partial_1 X| = \infty \). Let \( y \) be a limit point for \( A \). Consider the trajectory \( \gamma_y \) through \( y \). Using the compactness of \( \gamma_y \) and the local models from Lemma 1.1, \( \gamma_y \) has a neighborhood in which any trajectory \( \gamma \) has a uniformly bounded cardinality of the set \( \gamma \cap \partial_1 X \). This contradicts to the assumption that \( \lim_{k \to \infty} |\gamma_k \cap \partial_1 X| = \infty \). Therefore there is a number \( d \) so that, for every trajectory \( \gamma \) in \( X \), its combinatorial type belongs to the set \( \Omega(d) \).

Consider the combinatorial types of \( \{D_{\gamma(x_k)}\}_k \), a finite set. At least one of them, say \( \omega \), must occur infinitely many times in a subsequence \( \{D_{\gamma(x_k)}\}_i \) of \( \{D_{\gamma(x_k)}\}_k \). For each \( j \in \sup(\omega) \), consider the sequence of points \( \{y_{ji} \in \gamma(x_k_i)\} \) which occupy the \( j \)-th position in \( \gamma(x_k_i) \cap \partial_1 X \). By its definition, \( y_{ji} \in \partial_\omega(j) X^\circ \). Since \( \partial_\omega(j) X \) is compact, there exists a subsequence \( \{y_{ji}\}_i \) that converges to a point \( y_j \in \partial_\omega(j) X \). Its multiplicity, \( m(y_j) \), is \( \omega(j) \) at least.

Consider the segments \( \gamma_{ij+1,i} \) of \( \gamma(x_k_i) \) bounded by the adjacent pairs \( (y_{ji}, y_{ji+1}) \). The interiors of the segments do not intersect the boundary \( \partial_1 X \). By the continuous dependence of \( v \)-integral curves on their end points, \( \{\gamma_{ij+1,i}\} \) converge to a trajectory segment \( \gamma_{ij} \) that connects \( y_j \) with \( y_{ij+1} \) (that segment may contain ”new” points from \( \partial_1 X \) in its interior). As a result, all the \( y_{ji} \)'s belong to the same trajectory \( \gamma \). Since each \( x_{k_i} \) belongs to some segment \( \gamma_{ij+1,i} \), and \( \lim_{i \to +\infty} x_{k_i} = x \), we get \( \gamma(x) = \gamma \).

By the argument above, \( \sup(D_{\gamma(x)}) \) may contain new points that are not in the limit of \( \bigcup_i \sup(D_{\gamma(x_k_i)}) \). This can only boost the reduced multiplicity \( |D_{\gamma(x)}|' \) in comparison
to $|D_{γ(x_k)}|'$. On the other hand, some points in $sup(D_{γ(x)})$ are the result of merging of two or several consecutive points $\{y_{j,i}\}_j$ from $γ(x_{k_i})$. In the process, thanks to the local models \[\text{L.5}], their multiplicities add; so again, the reduced multiplicity of the limiting point exceeds the sum of the reduced multiplicities of the corresponding merging points. Therefore,

$$m'(γ(x)) ≥ \lim_{k→+∞} m'(γ(x_k)).$$

Moreover, if the combinatorial type $ω$ of $γ(x_k) ∩ \partial_1 X$ differs from the combinatorial type $ω'$ of $γ(x) ∩ \partial_1 X$, then $m'(γ(x)) ≥ \lim_{k→+∞} m'(γ(x_k))$. Similarly, $m(γ(x)) ≥ \lim_{k→+∞} m(γ(x_k)).$

The same same argument shows that the relation between the types $ω$ and $ω'$ is exactly the order relation $ω ≻ ω'$ in $Ω^*$, as introduced in Definition \[\text{L.5}].\[13]

3. On Stratified Spaces

Let $(S, ≻)$ be a poset (a partially ordered set). Given a subset $Θ ⊂ S$ in the poset $(S, ≻)$, we denote by $Θ_≥$ the set

$$\{a ∈ S | b ≳ a \text{ for some } b ∈ Θ\}.$$

Let $Θ_≤ := Θ_≥ \setminus Θ$. Similarly, we introduce

$$Θ_≤ := \{a ∈ S | a ≳ b \text{ for some } b ∈ Θ\} \text{ and } Θ_< := Θ_≤ \setminus Θ.$$

In particular, for any $ω ∈ Ω^*$, we will consider routinely the finite posets $ω_≥, \omega_≤, \omega_≤$, and, for any $ω ∈ Ω$, the finite posets $ω_≥, \omega_≤$.

**Definition 3.1.** A $S$-filtration of a topological space $X$ is a collection of closed topological subspaces $\{X_{≤ ω}\}_{ω ∈ S}$ such that $ω ≺ ω'$ implies the inclusion $X_{≤ ω} ⊂ X_{≤ ω'}$. Moreover, we require that each point $x ∈ X$ which belongs to some stratum $X_{≤ ω}$, $ω ∈ S$, belongs to a unique smallest stratum $X_{≤ ω(x)}$ labelled by a minimal element $ω(x)$ from the poset $\{≤ ω\} ⊂ S$.

When $X$ itself is a member of the collection $\{X_{≤ ω}\}_{ω ∈ S}$, we assume that $S$ has a unique maximal element $ω_*$ and that $X_{≤ ω_*} = X$.

For an $S$-filtered space $X$, we define

$$X_{≤ ω} := \bigcup_{ω' ≺ ω} X_{≤ ω'} \text{ and } X_ω := X_{≤ ω} \setminus X_{≤ ω'}.$$

In general, for any subset $Θ ⊂ S$, put $X_{≤ Θ} := \bigcup_{ω ∈ Θ} X_{≤ ω}$.

**Definition 3.2.** An $S$-filtered map $f : X → Y$ between $S$-filtered spaces $X$ and $Y$ is a continuous map such that $f(X_{≤ ω}) ⊂ Y_{≤ ω}$ for each $ω ∈ S$.

A $S$-filtered homotopy between $S$-filtered maps $f_0 : X → Y$ and $f_1 : X → Y$ is an $S$-filtered map $F : X × [0,1] → Y$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x ∈ X$.\[14]
4. Spaces of Real Polynomials, Stratified by the Combinatorial Types of their Real Divisors

Next, we would like to investigate carefully one natural stratification in the coefficient space $\mathbb{R}^d_{\text{coef}}$ of real monic polynomials $P(z)$ of a given degree $d$. This stratification is generated by different combinatorial patterns of zero divisors $D_R(P)$ in $\mathbb{R}$. Its importance for our program is justified by the local models for traversally generic fields that have been described in Lemmas 1.1 and 1.2.

Let $\tau : \mathbb{C} \to \mathbb{C}$ be the complex conjugation. Via the Viète Map $V$, the coefficient space $\mathbb{R}^d_{\text{coef}}$ can be identified with the space $(\text{Sym}^d \mathbb{C})^\tau$, the space of $\tau$-invariant divisors in $\mathbb{C}$ of degree $d$.

Each function $\omega : \mathbb{N} \to \mathbb{Z}_+$ of $l_1$-norm $|\omega| \leq d$ and such that $|\omega| \equiv d$ (2) defines a pure stratum $(\text{Sym}^d \mathbb{C})^\tau_{\omega}$ in $(\text{Sym}^d \mathbb{C})^\tau$ and therefore, in $\mathbb{R}^d_{\text{coef}}$. We denote this stratum $V((\text{Sym}^d \mathbb{C})^\tau_{\omega})$ by $\mathbb{R}^d_{\omega}$. In $\mathbb{R}^d_{\text{coef}}$, it represents all monic real polynomial $P$ such that the combinatorics of $D_R(P)$ is prescribed by $\omega$.

The depressed form of the polynomial in (1.5) and (1.7) calls for the introduction of similar spaces built out of, so called, balanced divisors.

Let $\alpha \in \mathbb{R}$. A divisor $D = \oplus_i m(i)z_i$ with $z_i \in \mathbb{C}$ is called $\alpha$-balanced, if

$$\sum_i m(i)z_i = \alpha \cdot \sum_i m(i)$$

in $\mathbb{C}$. In other words, $\alpha$ is the center of gravity of the configuration of points-particles in $\mathbb{C}$ representing $D$. In the root space $\text{Sym}^d \mathbb{C} = \{z_1, \ldots, z_d\}$ (where $d = |\omega|$), such divisors are characterized by the equation $\sum_{k=1}^d z_k = \alpha \cdot d$.

The conjugation-invariant $\alpha$-balanced divisors from $(\text{Sym}^d(\mathbb{C}))^\tau$ are described by the equation

$$\sum_{k=1}^d \text{Re}(z_k) = d \cdot \alpha.$$

They form a real hypersurface $(\text{Sym}^d_\alpha(\mathbb{C}))^\tau$ in $(\text{Sym}^d(\mathbb{C}))^\tau$. Zero-balanced divisors (i.e., $\alpha = 0$) are simply called balanced.

The image of $(\text{Sym}^d_\alpha(\mathbb{C}))^\tau$ under the Viète map $V$ consists of real monic polynomials whose $z^{d-1}$-coefficient is $\alpha$. In a similar way, we introduce the stratification

$$\{R_{\alpha,\omega} := V((\text{Sym}^d_\alpha(\mathbb{C}))^\tau_{\omega})\}_{\omega}$$

in the $(d - 1)$-dimensional affine space $\mathbb{R}^d_{\text{coef, } \alpha}$ of such polynomials.

Let us stress that here we use $\omega$’s that attach multiplicities only to real roots (their norms $|\omega|$ do not exceed $d$ and $|\omega| \equiv d$ (2))!

Consider the upper half plane

$$\mathbb{H} := \{z \in \mathbb{C} | \text{Re}(z) \geq 0\}$$

, and let

$$\mathbb{H}^\circ := \{z \in \mathbb{C} | \text{Re}(z) > 0\}.$$
Figure 3. The Swallow Tail singularity is linked to the Whitney projection of the hypersurface \( \{ z^4 + x_2 z^2 + x_1 z + x_0 = 0 \} \) onto the space \( \mathbb{R}_{\text{coef},0}^3 \) with the coordinates \((x_0, x_1, x_2)\). The strata in \( \mathbb{R}_{\text{coef},0}^3 \) are indexed by elements \( \omega \in \Omega_{(4)} \). They divide the target space into three 3-cells, four 2-cells, three 1-cells, and one 0-cell.

Let \( B^2 \subset \mathbb{C} \) be the unit ball centered on the origin. Put \( B^2_+ = B^2 \cap \mathbb{H} \). Similarly, for any \( \alpha \in \mathbb{R} \), let \( B^2(\alpha) \subset \mathbb{C} \) be the unit ball centered on \( \alpha \), and \( B^2_+(\alpha) := B^2(\alpha) \cap \mathbb{H} \).

**Lemma 4.1.** The symmetric product \( \operatorname{Sym}^m \mathbb{H} \) is homeomorphic to the half-space \( \mathbb{R}^m_+ \), bounded by a hyperplane in \( \mathbb{R}^{2m} \).

Let \( B^2 \) denotes a closed 2-ball. The symmetric product \( \operatorname{Sym}^m B^2 \) is homeomorphic to a closed \( 2m \)-ball \( B^{2m} \).

**Proof.** First we validate the second claim of the lemma.

The origin-centered dilation \( t : \mathbb{C} \rightarrow \mathbb{C} \) induces an action \( \psi_t \) on the “root space” \( \operatorname{Sym}^m \mathbb{C} \); under this action, the support of each divisor is scaled by the factor \( t > 0 \). With the help of the Viète homeomorphism \( V : \operatorname{Sym}^m \mathbb{C} \rightarrow \mathbb{C}^m_{\text{coef}} \), this \( \psi_t \)-action induces a \( \Psi_t \)-action on the space \( \mathbb{C}^m_{\text{coef}} \).

The boundary of \( \operatorname{Sym}^m B^2 \) consists of divisors \( D \) whose support is contained in \( B^2 \) and has a non-empty intersection with the circle \( \partial B^2 \). We notice that each \( \psi_t \)-trajectory through a point \( D \in \operatorname{Sym}^m B^2 \), distinct from the divisor \( m\{0\} \), has a unique point of intersection with the boundary \( \partial (\operatorname{Sym}^m B^2) \). Indeed, for each configuration \( \Delta \neq \{0\} \) of points in \( B^2 \), there is a unique origin-centered dilation \( t : \mathbb{C} \rightarrow \mathbb{C} \) such that \( t(\Delta) \cap \partial B^2 \neq \emptyset \) and \( t(\Delta) \subset B^2 \). Since the Viète map \( V : \operatorname{Sym}^m \mathbb{C} \rightarrow \mathbb{C}^m_{\text{coef}} \) is a smooth homeomorphism, \( V(\operatorname{Sym}^m B^2) \subset \mathbb{C}^m_{\text{coef}} \) is homeomorphic to \( \operatorname{Sym}^m B^2 \). Therefore each trajectory \( \Psi_t \)-trajectory, except for the trivial trajectory through \( m\{0\} \), hits \( \partial V(\operatorname{Sym}^m B^2) = V(\partial (\operatorname{Sym}^m B^2)) \) transversally at a singleton. The same property is shared by any \( (2m - 1) \)-sphere in \( \mathbb{C}^m_{\text{coef}} \) that is centered on the origin.
0. Therefore, with the help of $\Psi_t$, $\partial V(\text{Sym}^m B^2)$ is homeomorphic to a $(2m - 1)$-sphere, and $V(\text{Sym}^m B^2)$ to a closed ball $B^{2m}$. Hence $\text{Sym}^m B^2 \approx B^{2m}$ topologically.

A similar argument helps to analyze the topology of $\text{Sym}^m(\mathbb{H})$.

The boundary of the closed half-ball $B^2_+ := B^2 \cap \mathbb{H}$ consists of the segment $I = [-1,1] \subset \mathbb{R}$ and the arc $A := \{ z \in \mathbb{C} \mid |z| = 1, \text{Im}(z) \geq 0 \}$.

Consider the set $S$ of divisors $D \in \text{Sym}^m \mathbb{H}$ such that their support $\Delta \subset B^2_+$ and $\Delta \cap A \neq \emptyset$. Again, there is a unique origin-centered dilation $t : \mathbb{C} \to \mathbb{C}$ such that $t(\Delta) \cap A \neq \emptyset$ and $t(\Delta) \subset B^2_+$, provided that $D \neq m\{0\}$. Therefore each $\psi_t$-trajectory through a point $D \in \text{Sym}^m \mathbb{H}$, except for the trivial trajectory through $m\{0\}$, has a unique point of intersection with $S$.

The boundary $\partial(\text{Sym}^m \mathbb{H})$ in $\text{Sym}^m \mathbb{C}$ consists of divisors $D$ such that their support $\Delta$ has a nonempty intersection with the real line $\mathbb{R} \subset \mathbb{H}$. Evidently, $\partial(\text{Sym}^m \mathbb{H})$ is invariant under $\psi_t$-flow. Thus, $\partial(\text{Sym}^m \mathbb{H})$ topologically is a cone over $\partial(\text{Sym}^m \mathbb{H}) \cap S$, the set of divisors whose support $\Delta \subset B^2_+$ and such that $\Delta \cap I \neq \emptyset$.

Now, take a point $p \in \partial B^2$ and a small open $\epsilon$-ball $B_\epsilon(p) \subset \mathbb{C}$ with the center at $p$. Since $H := B_\epsilon(p) \cap B^2$ is homeomorphic to the half-plane $\mathbb{H}$, $\text{Sym}^m \mathbb{H} \approx \text{Sym}^m H$.

Consider $m \cdot p$ as a point in $\partial(\text{Sym}^m B^2) \approx \partial B^{2m}$. Then $\text{Sym}^m H$ can be viewed as an open $\epsilon$-neighborhood of $m \cdot p$ in the space $\text{Sym}^m B^2 \approx B^{2m}$, the distance in the closed ball $\text{Sym}^m B^2$ being the Hausdorff distance between $S_m$-orbits in $(B^2)^m \subset \mathbb{C}^m$. At least for a sufficiently small $\epsilon > 0$, that neighborhood $\text{Sym}^m H$, being an $\epsilon$-neighborhood of a point $m \cdot p \in \partial(B^{2m})$, is homeomorphic to $\mathbb{R}^{2m}$, and so is $\text{Sym}^m \mathbb{H}$. \hfill \Box

For $\omega \in \Omega_{\alpha|\alpha}$, consider the space

$$
(4.1) 
\sigma_\omega := \text{Sym}^{\text{sup}(\omega)} | \mathbb{R} \times \text{Sym}^{d - |\omega|} \mathbb{H}.
$$

Denote by $\sigma_\omega$ the subset of $e_\omega$ that consists of divisor pairs $D' \times D''$ such that $\text{sup}(D') \cup \text{sup}(D'') \subset B^2$, but not of its interior.

Let the subset $e_{\alpha, \omega} \subset e_\omega$ be defined by the constraint: the divisor $D' + D'' + \tau(D'')$ is $\alpha$-balanced. Similarly, let $\sigma_{\alpha, \omega} \subset \sigma_\omega$ be defined by the property of $D' + D'' + \tau(D'')$ being $\alpha$-balanced.

In what follows, by a “cell complex” we mean a Hausdorff topological space $Z$ which is a disjoint union of its subsets $\{e_\alpha\}_\alpha$, where each $e_\alpha$ is homeomorphic to an open ball $B^2_\alpha$ of some dimension $d_\alpha$. Let $B_\alpha$ be either a closed ball, or an open ball together with a northern hemisphere in its spherical boundary $\mathbb{H}$. The homeomorphism $B^2_\alpha \to e^\circ_\alpha$ must extend to a continuous map $\phi_\alpha : B_\alpha \to Z$, so that the image $\phi_\alpha(\partial B_\alpha)$ is contained in a union of finitely many cells $e_\beta$ of dimensions lower than $d_\alpha$. By definition, $Y \subset Z$ is closed if, for each $\alpha$, $\phi^{-1}_\alpha(Y \cap \phi_\alpha(B_\alpha))$ is closed in $B_\alpha$. Note that $e_\alpha$, the closure of $e_\alpha$ in $Z$, coincides with $\phi_\alpha(B_\alpha)$. We do not require that $e_\alpha$ will be homeomorphic to a closed ball.

---

6This deviation from the standard definition of CW-complex is due to our need to consider germs of classical CW-complexes.
The next proposition describes one particularly important cellular structure on the space \( \mathcal{P}^d \simeq \mathbb{R}^d_{\text{coeff}} \) of real degree \( d \) monic polynomials, the structure induced by the combinatorial types \( \omega \) of their real divisors.

**Theorem 4.1.** Let \( \omega \in \Omega_{[d]} \). Then the following structures are available:

- each pure stratum \( R_\omega \subset \mathbb{R}^d_{\text{coeff}} \) is homeomorphic to an open ball\(^7\) of codimension \(|\omega|'\).
- the space \( \sigma_\omega \) is homeomorphic to a closed \((d - |\omega|' - 1)\)-ball, and the space \( \sigma_{\alpha, \omega} \) to a closed \((d - |\omega|' - 2)\)-ball. The space \( e_\omega \) is a positive cone over \( \sigma_\omega \), and \( e_{\alpha, \omega} \) is a positive cone over \( \sigma_{\alpha, \omega} \).
- for each \( \tilde{\omega} \in \Omega_{[d]} \), the strata \( \{R_\omega\}_{\omega \leq \tilde{\omega}} \), define the structure of a cell complex on the real affine variety \( R_{\omega^+} \subset \mathbb{R}^d_{\text{coeff}} \). The attaching maps
  \[
  \Phi_\omega^\beta : \partial e_\omega \to R_{\omega^+} \setminus R_\omega
  \]
  for the cells \( e_\omega \) are described in formulas (4.4).
- the space \( \mathbb{R}^d_{\text{coeff}} \) admits a 1-parameter flow \( \Psi_t \), which has a single stationary point \( 0 \) (a source), is transversal to each sphere \( S^d_{\text{coeff}} \subset \mathbb{R}^d_{\text{coeff}} \), centered on \( 0 \), and preserves each stratum \( R_\omega \). Thus, the cellular structure on \( \mathbb{R}^d_{\text{coeff}} \) (described in the third bullet) is a cone over a similar cellular structure on \( S^d_{\text{coeff}} \).

**Proof.** Any divisor in \( \mathbb{R} \) comes with a particular linear order among the points in its support. Let \( P \) be a real polynomial of degree \( d \). Its real divisor \( D_\mathbb{R}(P) \) of the combinatorial type \( \omega \) is determined (with the help of \( \omega \)) by the support \( \sup(D_\mathbb{R}(P)) \), a point the chamber \( \Pi_\omega^p \) of \( \mathbb{R}^p = \{(y_1, \ldots, y_p)\} \), defined by the inequalities \( y_1 < y_2 < \cdots < y_p \), where \( p = |\sup(\omega)| \). In fact, \( \Pi_\omega^p \) is one of the \( 2^p - 1 \) chambers-cells in which the hyperplanes \( \{y_i = y_{i+1}\} \) divide \( \mathbb{R}^p \).

The set \( \Pi_\omega^p \) admits an obvious embedding into the space \( \mathbb{R}^{[\omega]} \) with the coordinates \((x_1, \ldots, x_{|\omega|})\). It is defined there by the system of \( \omega \)-dependent equations and inequalities:

\[
x_1 = x_2 = \cdots = x_{|\omega|} < x_{|\omega|+1} = x_{|\omega|+2} = \cdots = x_{|\omega|+|\omega|+2} < \cdots
\]

We denote by \( \Pi_\omega^{[\sup(\omega)]} \) the solution set of this system. Note that \( \Pi_\omega^{[\sup(\omega)]} \) is homeomorphic to the open chamber \( \Pi_\omega^{[\sup(\omega)]} \subset \mathbb{R}^{[\sup(\omega)]} \). Let \( \Pi_\omega \) be the closure of \( \Pi_\omega^{[\sup(\omega)]} \) in \( \mathbb{R}^{[\omega]} \). That closure is homeomorphic to \( \Pi^{[\sup(\omega)]} \simeq \text{Sym}^{[\sup(\omega)]}\mathbb{R} \). Indeed, each orbit of the natural action of the symmetric group \( \text{Sym}^{[\sup(\omega)]} \) on \( \mathbb{R}^{[\sup(\omega)]} \) intersects the closed chamber \( \Pi^{[\sup(\omega)]} \) at a singleton.

With its real roots being fixed, a polynomial \( P \) is determined by the unordered pairs of its conjugate roots (possibly with multiplicities). Then the conjugate (non-real) pairs can be identified with points of the space \( \text{Sym}^m \mathbb{H}^\circ \), where \( m = \frac{1}{2}(d - |\omega|) \). Therefore, \( \langle \text{Sym}^d \mathbb{C} \rangle_\omega \) is homeomorphic to the space \( \Pi_\omega^p \times \text{Sym}^m \mathbb{H}^\circ \), where \( p := |\sup(\omega)| \). Since \( \mathbb{H}^\circ \) is

\footnote{In general, the intersection of \( R_\omega \) with a ball \( B^d \subset \mathbb{R}^d \), centered at the origin, topologically is not a ball.}

\footnote{See the proof of this theorem for the constructions of the relevant maps.}
homeomorphic to \( \mathbb{C} \) and \( \text{Sym}^n \mathbb{C} \) is homeomorphic to \( \mathbb{C}^m \) via the Viète Map \( V \), we conclude that \( \text{Sym}^m \mathbb{H} \) is homeomorphic to \( \mathbb{C}^m \). Thus,

\[
(\text{Sym}^d \mathbb{C})^0 \approx \Pi^p_0 \times \text{Sym}^m \mathbb{H} \approx \Pi^p_0 \times \mathbb{C}^m
\]

, an open cell of dimension

\[
p + 2m = |\sup(\omega)| + (d - |\omega|) = d - |\omega|'.
\]

Note that, the Viète map \( V \) generates a smooth homeomorphism between the spaces \( (\text{Sym}^d \mathbb{C})^0 \approx \mathbb{R}^{d - |\omega|'} \) and \( R_\omega := (\mathbb{R}^d_{\text{coef}})_\omega \). Thus the claim of the first bullet is validated.

The flow \( \Psi_t : \mathbb{R}^{d}_{\text{coef}} \to \mathbb{R}^{d}_{\text{coef}} \) (see the fourth bullet) is induced by the flow \( \psi_t \) in the root space \( (\text{Sym}^d \mathbb{C})^\tau \) that applies dilatations by real factors \( t > 0 \) to each \( \tau \)-symmetric root configuration in \( \mathbb{C} \). The transplantation of \( \psi_t \) to \( \mathbb{R}^{d}_{\text{coef}} \) is done via the Viète Map \( V \), a smooth homeomorphism \( (\text{Sym}^d \mathbb{C})^\tau \to \mathbb{R}^{d}_{\text{coef}} \). Evidently, each stratum \( (\text{Sym}^d \mathbb{C})^\omega_\tau \) is \( \psi_t \)-invariant; therefore each stratum \( R_\omega \) must be invariant under \( \Psi_t \).

We leave to the reader to verify that, for any polynomial \( P \neq z^d \), its \( \Psi_t \)-trajectory is transversal to the spheres \( S^{d - 1}_{\text{coef}} \subset \mathbb{R}^{d}_{\text{coef}} \), centered on \( 0 \) (the verification is a straightforward computation). This proves the claim in the forth bullet.

Next, we turn our attention to the second bullet of the theorem. Each divisor \( D \in \text{Sym}^m \mathbb{H} \) produces to a unique divisor \( D_{\mathbb{R}} \), the part of \( D \) that is supported in \( \mathbb{R} \). We view \( D_{\mathbb{R}} \) as an element of \( \text{Sym}^l \mathbb{R} \), \( l \leq m \). For the majority of \( D \)'s, \( D_{\mathbb{R}} \) will have an empty support, so we interpret \( \text{Sym}^0 \mathbb{R} \) as an empty set.

Recall that \( e_\omega := \Pi_\omega \times \text{Sym}^m \mathbb{H} \), where \( m = \frac{1}{2}(d - |\omega|) \) (see (4.1)). In the proof of Lemma 4.1, we have seen that \( e_\omega \) is homeomorphic to \( \Pi_\omega \times \mathbb{R}^{d - |\omega|} \) - a \( |\sup(\omega)| \)-dimensional pyramid times a half-space in \( \mathbb{R}^{d - |\omega|} \).

The boundary of the unit half-disk \( B^2_+ := \{ z \in \mathbb{C} : |z| \leq 1, \text{Im}(z) \geq 0 \} \) consists of the segment \( I = [-1, 1] \subset \mathbb{R} \) and the arc

\[
A := \{ z \in \mathbb{C} : |z| = 1, \text{Im}(z) \geq 0 \}.
\]

The multiplicative group \( \mathbb{R}^*_+ \) of positive real numbers acts semi-freely on the space \( \Pi_\omega \times \text{Sym}^m \mathbb{H} \) by the diagonal \( \psi_t \)-action which is induced by the origin-centered dilatations in \( \mathbb{C} \). Its only fixed point is the origin \( 0 := \{0\} \times \{0\} \). This \( \mathbb{R}^*_+ \)-action on \( e_\omega \setminus 0 \) admits a compact section \( \sigma_\omega \) that consists of points \( D' \times D'' \in e_\omega \) such that \( \sup(D') \cup \sup(D'') \) is contained in the unit half-disk \( B^2_+ \subset \mathbb{H} \) (centered on the origin) and has a nonempty intersection with the arc \( A \).

Thus \( \sigma_\omega \) is the set of pairs \( D' \in \text{Sym}^p(I), D'' \in \text{Sym}^m(B^2_+) \) such that either \( \sup(D') \cap \partial I \neq \emptyset \), or \( \sup(D'') \cap A \neq \emptyset \), or both. Here \( p = |\sup(\omega)| \). Note that if \( \sup(D') \cap \partial I \neq \emptyset \), then \( \sup(D') \cap A \neq \emptyset \) since \( \partial A = \partial I \). Therefore, \( \sigma_\omega \) can be also described as a set of pairs \( (D', D'') \) such that \( \sup(D' + D'') \subset B^2_+ \) and \( \sup(D' + D'') \cap A \neq \emptyset \).

Recall that \( \text{Sym}^p I \approx \Delta^p \), a \( p \)-simplex. By Lemma 4.1, \( \text{Sym}^m B^2_+ \approx B^2_+ \), a half-ball. Let \( \delta B^{2m}_+ \subset \partial(B^{2m}_+) \) denote the northern hemisphere in the boundary of the ball \( B^{2m}_+ \cap B^2_+ \).

It corresponds to the divisors \( D'' \) with the property \( \sup(D'') \cap A \neq \emptyset \).
In new notations, the section \( \sigma_\omega \) is the set of pairs \( D' \in \Delta^p, D'' \in B^{2m}_+ \) such that either \( D' \in \partial \Delta^p \), or \( D'' \in \delta B^{2m}_+ \), or both. As a result, we get a homeomorphism

\[
\sigma_\omega \approx (\partial \Delta^p \times B^{2m}_+) \cup_{\partial \Delta^p \times \delta B^{2m}_+} (\Delta^p \times \delta B^{2m}_+)
\]
whose target topologically is a \((d - |\omega'| - 1)\)-ball. Indeed, as the formula above testifies, \( \sigma_\omega \) is obtained from the sold torus \( T := S^{p-1} \times D^{2m} \) by attaching a \( p \)-handle along \( S^{p-1} \times D^{2m-1} \subset \partial T \).

Therefore \( e_\omega \), an infinite positive cone over the ball \( \sigma_\omega \), is homeomorphic to a positive cone in \( \mathbb{R}^{d-|\omega'|} \) with a closed \((d - |\omega'| - 1)\)-ball base.

Now we proceed to describe the attaching maps (see (4.3) and (4.4)) for the cells \( e_\omega \). With this goal in mind, we will “partially compactify” the pure stratum \( R_\omega \) in order to form an “honest” \((d - |\omega'|)\)-cell \( e_\omega \) and will show how to attach its boundary \( \partial e_\omega \) to the strata \( \{ R_\gamma \}_{\gamma \in \omega_\omega} \) of dimensions smaller than \( \dim(R_\omega) \). This cell \( e_\omega \) can be regarded as a resolution of the real variety \( R_{\omega_\omega} \).

Consider the maps

\[
\Theta_\omega : e_\omega := \Pi_\omega \times \text{Sym}^{d-|\omega|} \mathbb{H} \longrightarrow (\text{Sym}^d \mathbb{C})^\tau
\]
defined by the formula \( \Theta_\omega(D' \times D'') = D' + D'' + \tau(D'') \), where \( D' \in \Pi_\omega, D'' \in \text{Sym}^{d-|\omega|} \mathbb{H} \).

\( \tau(D'') \) stands for the complex conjugate of the divisor \( D'' \), and “+” denotes the sum of divisors in \( \mathbb{C} \).

As formula (1.2) testifies, each map \( \Theta_\omega \) doubles the real part \( D''_\mathbb{R} \) of each divisor \( D'' \in \text{Sym}^n \mathbb{H} \) and thus mimics the merging of conjugate pairs of complex roots into the appropriate real roots of even multiplicity.

Note that the restriction of \( \Theta_\omega \) to \( e_\omega^o := \Pi_\omega^o \times \text{Sym}^{d-|\omega|} \mathbb{H}^o \), the interior of \( e_\omega \), is a homeomorphism onto the pure stratum \((\text{Sym}^d \mathbb{C})^\tau_{\omega_\omega} \).

The restriction \( \Theta_\omega^\partial \) of \( \Theta_\omega \) to the boundary \( \partial e_\omega \) provides us with the attaching maps

\[
\{ \Theta_\omega^\partial : \partial e_\omega \rightarrow (\text{Sym}^d \mathbb{C})^\tau \}_{\omega \in \Omega(d)}
\]
By the very construction of \( e_\omega \), the \( \Theta_\omega \)-image of \( \partial e_\omega \) belongs to the union of strata \( \{ (\text{Sym}^d \mathbb{C})^\tau_{\omega'} \}_{\omega' \in \omega_\omega} \), where \( \omega' \in \omega_\omega \). Indeed, if \( D' \times D'' \in \partial e_\omega \), then either \( D' \in \partial \Pi_\omega \), or \( D'' \in \partial (\text{Sym}^{d-|\omega|} \mathbb{H}) \). In the first case, \( D' \) is obtained from some \( D' \in \Pi_\omega^o \) via merge operations; thus \( (D' + D'' + \tau(D''))_\mathbb{R} \) is obtained from \( (D + D'' + \tau(D''))_\mathbb{R} \) by the same merges. In the second case, \( (D' + D'' + \tau(D''))_\mathbb{R} \) can be obtained from \( D' \) by inserting \( (D'' + \tau(D''))_\mathbb{R} \).

Therefore the maps \( \{ \Theta_\omega^\partial \} \) from (4.3) define cellular structures on the real affine variety \((\text{Sym}^d \mathbb{C})^\tau \) and its subvarieties \( \{ (\text{Sym}^d \mathbb{C})^\tau_{\omega_\omega} \}_{\omega_\omega} \).

We notice that all the maps \( \Theta_\omega \) are equivariant under the \( \mathbb{R}_+^* \)-actions \( \psi_t \) and \( \Psi_t \) in the source and target spaces, respectively, so that the attaching maps are consistent with the cone structures of \( e_\omega \) and of the strata \( \{ (\text{Sym}^d \mathbb{C})^\tau_{\omega'} \} \). Moreover, the sections \( \sigma_\omega \) are mapped by \( \Theta_\omega \) to some sections \( S_\omega \subset R_\omega \) of the flow \( \Psi_t \). The space \( S_\omega \) is defined as the set of real polynomials with all their roots residing in the ball \( B^2 \subset \mathbb{C} \), but not in its interior, and with the combinatorics of the real roots being prescribed by the poset \( \omega_\omega \subset \Omega(d) \).
With the help of the Viète homeomorphism $V$, the maps

\begin{equation}
\Phi_\omega : e_\omega \mapsto (\text{Sym}^d \mathbb{C})^\tau \to R_{\omega,\preceq} \subset \mathbb{R}^d_{\text{coef}} \}
\omega \in \Omega_{(d)}
\end{equation}

\begin{equation}
\Phi_\omega^\partial : \partial e_\omega \mapsto (\text{Sym}^d \mathbb{C})^\tau \to R_{\omega,\succeq} \subset \mathbb{R}^d_{\text{coef}} \}
\omega \in \Omega_{(d)}
\end{equation}

define cellular structures in $\mathbb{R}^d_{\text{coef}}$ and its subvarieties $\{R_{\omega,\preceq}\}_{\omega}$. Again, the attaching maps $\Phi_\omega, \Phi_\omega^\partial$ are $\mathbb{R}^d_{\text{coef}}$-equivariant. They consistent with the cone structures in the strata $e_\omega$ and $R_{\omega,\succeq}$. In particular, we get the maps:

\begin{equation}
\Psi_\omega : \sigma_\omega \mapsto (\text{Sym}^d \mathbb{C})^\tau \to S_{\omega,\preceq} \subset S^d_{\text{coef}} \}
\omega \in \Omega_{(d)}
\end{equation}

\begin{equation}
\Phi_\omega^\partial : \partial \sigma_\omega \mapsto (\text{Sym}^d \mathbb{C})^\tau \to S_{\omega,\succeq} \subset S^d_{\text{coef}} \}
\omega \in \Omega_{(d)}
\end{equation}

Here $S^d_{\text{coef}}$ denotes the space of degree $d$ real monic polynomials whose roots are in $B^2$, but not in its interior. With the help of $\Psi_t$, the space $S^d_{\text{coef}}$ is diffeomorphic to the standard sphere $S^1$. This completes the proof of the third bullet.

Results, similar to the ones described in Theorem 4.1 are valid for the space of real degree $d$ monic polynomials with a fixed coefficient $d \cdot \alpha$ of $z^{d-1}$.

**Theorem 4.2.** Let $\omega \in \Omega_{(d)}$ and $\alpha \in \mathbb{R}$. We denote by $\omega_* : 1 \to d$ the minimal element of the poset $\Omega_{(d)}$. The following properties hold:

- the stratum $R_{\alpha,\omega} \subset \mathbb{R}^d_{\text{coef},\alpha}$ is an open ball of codimension $|\omega|^\tau$,
- for each $\alpha \in \Omega_{(d)}$, the strata $R_{\alpha,\omega}$ give rise to a cellular structure on the affine variety $\mathbb{R}^d_{\text{coef},\alpha}$. The attaching maps

$$
\Phi_{\alpha,\omega} : e_{\alpha,\omega} \to R_{\alpha,\omega,\preceq} \subset \mathbb{R}^d_{\text{coef},\alpha}
$$

for the cells $e_{\alpha,\omega}$ are described by formulas similar to formulas \[4.4\] and \[4.5\],
- the space $\mathbb{R}^d_{\text{coef},\alpha}$ admits a 1-parameter flow $\Psi_t$ which has a single stationary point $O_\alpha$ (a source), is transversal to each sphere $S^d_{\text{coef},\alpha}$, centered on $O_\alpha$, and preserves each stratum $R_{\alpha,\omega}$. Thus, the $\Omega_{(d)}$-labeled cellular structure on $\mathbb{R}^d_{\text{coef},\alpha}$ is a cone over a similar $(\Omega_{(d)} \setminus \omega_*)$-labelled cellular structure on the sphere $S^d_{\text{coef},\alpha}$.

**Proof.** We turn to the $\alpha$-balanced divisors, which tell a similar story.

Consider a flow $\phi_t : \{z_1, \ldots, z_d\} \to \{z_1 - \alpha t, \ldots, z_d - \alpha t\}$ in $(\text{Sym}^d \mathbb{C})^\tau$. For $t = 1$, it maps $\{z_1, \ldots, z_d\}$ to $\{z_1 - \alpha, \ldots, z_d - \alpha\}$, a $\alpha$-balanced configuration. Note that $\phi_t$ preserves the $\omega$-stratification in $(\text{Sym}^d \mathbb{C})^\tau$. Hence, with the help of the Viète map $V$, $\phi_t$ gives rise to a retraction of the stratum $R_{\omega,\preceq}$ onto the stratum $R_{\alpha,\omega}$. Therefore, $R_{\alpha,\omega}$ is an open ball of dimension $d - |\omega|^\tau - 1$ and codimension $|\omega|^\tau$ in $\mathbb{R}^d_{\text{coef},\alpha}$.

Next, consider the flow $\psi_t : \mathbb{R}^d_{\text{coef},\alpha} \to \mathbb{R}^d_{\text{coef},\alpha}$ induced with the help of the Viète map $V$ by a flow $\{\psi_t\}_{t>0}$ in the root space $(\text{Sym}^d \mathbb{C})^\tau$. The flow $\psi_t$ applies $t$-dilatations that are centered on the point $c_\alpha := \alpha/d \in \mathbb{R}$ to each $\tau$-symmetric and $\alpha$-ballanced divisor $D$ in $\mathbb{C}$. The dilatation $\psi_t^\alpha$ keeps the center of gravity of the weighted configuration $\psi_t^\alpha(D)$ at the
point \( o_\alpha \in \mathbb{C} \), so that \( \psi_\alpha^o(D) \) is \( \alpha \)-balanced for all \( t \). Evidently, these dilatations preserve the \( \omega \)-type of each \( \alpha \)-balanced configuration in \( \mathbb{C} \). As a result, each \( R_{\alpha, \omega} \) is \( \Psi^o_\alpha \)-invariant. The exception is the trivial trajectory through \( O_\alpha \).

We denote by \( e_{\alpha, \omega} \) the subset of \( e_\omega \) (see (4.1)) that consists of pairs \((D', D'')\), where \( D' \in \Pi_\omega, D'' \in \text{Sym}^m \mathbb{H} \), and such that the divisor 

\[
D' + D'' + \tau(D'')
\]

is \( \alpha \)-balanced. In other words, if \( D' = \sum_i m_i x_i \) and \( D'' = \sum_k m_k z_k \), then \( e_{\alpha, \omega} \) is the space of a fibration over the base \( \text{Sym}^m \mathbb{H} \approx \mathbb{R}^{2m} \) with the cell-like fiber \( F_{D''} \subset \Pi_\omega \) (over the point \( D'' \)) that is defined by the constraint

\[
\sum_i m_i x_i = d \cdot \alpha - \sum_k 2m_k \text{Re}(z_k).
\]

Denote by \( \sigma_{\alpha, \omega} \) the subset of \( e_{\alpha, \omega} \) that consists of \( \alpha \)-balanced pairs \((D', D'')\) such that \( \sup(D') \cup \sup(D'') \) is contained in the \( \alpha \)-centered unit ball \( B_2^2(\alpha) \), but not in its interior. As in the “unbalanced” case, \( \sigma_{\alpha, \omega} \) is a section of the \( \psi^o_\alpha \)-flow in \( e_{\alpha, \omega} \). This results in \( e_{\alpha, \omega} \) being a positive cone over the base \( \sigma_{\alpha, \omega} \). By an argument as in the proof of Theorem 4.1, \( \sigma_{\alpha, \omega} \) is homeomorphic to a closed \((d - 2 - |\omega'|)\)-ball.

For any real \( \alpha \), the maps \( \Theta_{\alpha, \omega} \) in (4.2), being restricted to \( e_{\alpha, \omega} \subset e_\omega \), produce the maps \( \Theta_{\alpha, \omega} : e_{\alpha, \omega} \to (\text{Sym}^d \mathbb{C})^r \) which, with the help of \( V \), give rise to the maps

\[
\Phi_{\alpha, \omega} : e_{\alpha, \omega} \to R_{\alpha, \omega^-} \subset \mathbb{R}_\text{coef,}\alpha^{d-1}
\]

, already familiar from the “unbalanced” formulas (4.4). They define a cellular structure on the real variety \( \mathbb{R}_\text{coef,}\alpha^{d-1} \) and its subvarieties \( R_{\alpha, \omega^-} \). Analogously, the maps

\[
\Phi_{\alpha, \omega} : \sigma_{\alpha, \omega} \to S_{\alpha, \omega^-} \subset \mathbb{S}_\text{coef,}\alpha^{d-2}
\]

define a cellular structure on the sphere \( S_{\text{coef,}\alpha}^{d-2} \) and its strata \( S_{\alpha, \omega^-} \). Here \( S_{\text{coef,}\alpha}^{d-2} \) is the space of real monic polynomials \( P(z) \) of the form

\[
z^d + \alpha z^{d-1} + \ldots
\]

such that the \( P \)-roots are contained in \( B_2^2(\alpha) \), but not in its interior. Note that \( P(z) \neq (z - \frac{\alpha}{2})^d \) — the apex of the cone. Similarly, \( S_{\alpha, \omega^-} \subset S_{\text{coef,}\alpha}^{d-2} \), is the set of such polynomials \( P(z) \) whose real roots have the combinatorics that is prescribed by the poset \( \omega^- \). \[\square\]

**Example 4.1.** Let \( \omega^\star \) be the minimal element of the poset \( \Omega_[d] \). Theorem 4.1 claims that the sphere \( S^{d-1} \) admits a cellular structure whose cells \( \{S_\omega\} \) of codimension \( |\omega'| \) are indexed by the elements of the poset \((\Omega_{[d]} \setminus \omega^\star, \succ)\). Moreover, \( S_\omega^\omega \subset S_{\text{coef,}\alpha}^{d-2} \) if and only if \( \omega'' \succeq \omega' \). In other words, the poset \((\Omega_{[d]} \setminus \omega^\star, \succ)\) provides a complete set of instructions for assembling \( S^{d-1} \)! You may glance at Fig. 3 to examine how the assembly works for \( S^2 \).
Note that this cellular structure is not a regular one: the closures $S_{\omega, \hat{\omega}}$ of cells $S_{\omega}$ are not necessarily closed balls.

The next lemma describes the attaching maps $\Phi_{\omega} : e_{\omega} \to R^d_{\text{coef}}$ for the cells $e_{\omega}$ as being finitely ramified over their images. The degrees of ramification are described by combinatorics-flavored numbers $\{o(\omega, \hat{\omega})\}$ whose exact Definition 4.1 will be provided below.

**Lemma 4.2.** Each map $\Phi_{\omega} : e_{\omega} \to R^d_{\text{coef}}$ from (4.4) has finite fibers, and is a bijection on the interior of the cell $e_{\omega}$.

For any point-polynomial $Q \in \Phi_{\omega}(e_{\omega})$, the cardinality of the fiber $\Phi_{\omega}^{-1}(Q)$ is equal to the number $o(\omega, \hat{\omega})$ introduced in Definition 4.1 below. Here $\hat{\omega}$ denotes the combinatorial pattern of the real divisor $D_\mathbb{R}(Q)$.

**Proof.** Let us fix a degree $d$ conjugation-invariant divisor

$$D = \Theta_{\omega}((D', D'')) := D' + D'' + \tau(D'')$$

, where $D' \in \Pi_{\omega}$, $D'' \in \text{Sym}^m \mathbb{H}$, and $m := (d - |\omega|)/2$. We denote by $\hat{\omega}$ the combinatorial type of $D_\mathbb{R}$. Then there exist only finitely many pairs $(D', D'')$ that deliver $D$. Indeed, since $\sup(D') \subset \mathbb{R}$, the divisor $D'' - D'^{\flat}_\mathbb{R}$ with the support in $\mathbb{H}$ is uniquely determined by $D$. Note that $D_\mathbb{R}$ and $D'$ (with the support in $\mathbb{R}$) differ by $2D'^{\flat}_\mathbb{R}$. This leaves only a finite set of choices for $D' = D_\mathbb{R} - 2D'^{\flat}_\mathbb{R}$, whose support must be contained in $\sup(D_\mathbb{R})$ and whose degree is bounded by $\deg(D_\mathbb{R})$.

If $(D', D'') \in \partial e_{\omega}$, then either $D' \in \partial \Pi_{\omega}$, or $D'' \in \partial (\text{Sym}^m \mathbb{H})$ (i.e. $\sup(D'') \cap \mathbb{R} \neq \emptyset$), or both. In the first case, $\omega(D')$, the combinatorial type of $D'$, is obtained from $\omega$ by a sequence of merge operations $\{M_j\}$. In the second case, $\hat{\omega} := \omega(D_\mathbb{R})$, the combinatorial type of the $\Theta_{\omega}$-image of $(D', D'')$, is obtained from $\omega$ by a sequence of insert operations $\{I_k\}$. In the third mixed case, one applies both types of operations.

As a result, the combinatorics of reconstructing $(D', D'')$ from $D = D' + D'' + \tau(D'')$ can be described as follows. If $\hat{\omega}$ is the combinatorial pattern of $D_\mathbb{R}$, then $\omega'$, the combinatorial pattern of $D' = D'_\mathbb{R}$ can be obtained by: (1) subtracting from $\hat{\omega}$ a non-negative function $2\omega''$, such that $\hat{\omega} - 2\omega''$ is again a non-negative function ($\omega''$ represents the divisor $D''_\mathbb{R}$), and (2) deleting all the positions $i$ where $(\hat{\omega} - 2\omega'')(i) = 0$.

Let us denote by $K$ this “repackaging” operator from (2); it takes any histogram and deletes from it all the columns of zero height.

Since $D' \in \partial \Pi_{\omega}$, we conclude that $\omega' = K(\hat{\omega} - 2\omega'')$ should be obtainable from $\omega$ via a sequence of merge operations alone.

For each $\omega \in \Omega$, let us denote by $\omega_{\geq M}$ the subset of $\Omega$ that consists of elements that can be obtained from $\omega$ by the merge operations $\{M_j\}$ alone.

In these notations, we get that $K(\hat{\omega} - 2\omega'') \in \omega_{\geq M}$.

**Definition 4.1.** For any pair $\omega \geq \hat{\omega}$ in $\Omega$, consider the subset $O(\omega, \hat{\omega}) \subset \Omega$ such that, for any $\omega'' \in O(\omega, \hat{\omega})$, the following properties hold:

- the function $\hat{\omega} - 2\omega'' \geq 0$
- $K(\hat{\omega} - 2\omega'') \in \omega_{\geq M}$
Let \( o(\omega, \tilde{\omega}) \) denote the cardinality of \( O(\omega, \tilde{\omega}) \).

Recall that the Viète map \( V : (\text{Sym}^d\mathbb{C})^r \to \mathbb{R}^{d}_{\text{coef}} \) is a stratification-preserving homeomorphism of the \( \Omega_2 \)-stratified spaces; in particular, it is bijective. Therefore the cardinality of each fiber \( \Phi^{-1}_i(Q) \), where \( Q \in \mathbb{R}^{d}_{\text{coef}} \), is equal to the cardinality of the fiber \( \Theta^{-1}_i(V^{-1}(Q)) \).

Pick \( Q \in \Phi_i(\partial \omega) \) and let \( \tilde{\omega} \) denote the combinatorial type of the divisor \( D_R(Q) \). Then, by the considerations above, we have proved that \( |\Phi_i^{-1}(Q)| = |\Theta_i^{-1}(V^{-1}(Q))| = o(\omega, \tilde{\omega}). \)

For each \( \omega \in \Omega_{[d]} \), let \( \omega \xrightarrow{\sim} k \) be the set of elements \( \tilde{\omega} \in \Omega_{[d]} \) that can produce \( \omega \) as a result of a sequence of \( k \) elementary operations \( \{M_j, I_j\} \) being applied to \( \tilde{\omega} \) (each such \( \tilde{\omega} \) is the maximal element in a chain of \( \omega \)-predecessors in \( \Omega_{[d]} \) of length \( k \)). Similar, let \( \omega \xrightarrow{\sim} k \) be the set of elements in \( \Omega_{[d]} \) that can produce \( \omega \) by a sequence of \( k \) elementary operations at most.

Next lemma is an extension of Theorem 4.1 and Theorem 4.2. It spells out the local arrangement of cells in the direction normal to a typical pure stratum \( R_\omega \) or \( R_{\alpha, \omega} \) in \( \mathbb{R}^{d}_{\text{coef}} \) or in \( \mathbb{R}^{d-1}_{\text{coef}, \alpha} \), respectively.

**Lemma 4.3.** Let \( \omega \in \Omega_{[d]} \) and \( P \in R_\omega \). We denote by \( St_\nu(P) \) the star, normal in \( \mathbb{R}^{d}_{\text{coef}} \) to the pure stratum \( R_\omega \) at the point \( P \). This normal star is homeomorphic to the space \( \mathbb{R}^{d-1}_{\text{coef}, \alpha} \) and inherits the structure of a cell complex from \( \mathbb{R}^{d}_{\text{coef}} \). The cells \( f_\omega := St_\nu(P) \cap R_\omega \) of dimension \( |\omega'| - |\tilde{\omega}'| \) are indexed by the elements \( \tilde{\omega} \sim \omega \).

The incidence of cells \( \{f_\omega\} \) in \( St_\nu(P) \) is prescribed by the partial order in the poset \( \omega \xrightarrow{\sim} \): specifically, any element \( \tilde{\omega} \in \omega \xrightarrow{\sim} k \) gives rise to a cell \( f_\omega \subset St_\nu(P) \) of dimension \( k \). It is contained in exactly \( |\tilde{\omega}'| + \#(\omega^{-1}(2)) \) cells \( \{f_\nu\} \) of dimension \( k + 1 \). The total number of cells \( f_\omega \subset St_\nu(P) \) of dimension \( k \) is the cardinality of the set \( \omega \xrightarrow{\sim} k \).

Similar properties hold for the strata \( \{R_{\nu, \omega} \cap R_{\alpha, \omega} \} \) in the \( \alpha \)-balanced polynomial space \( \mathbb{R}^{d}_{\text{coef}, \alpha} \).

**Proof.** Let \( St(R_\omega) \) be the union of all cells \( \{R_\omega\} \) in \( \mathbb{R}^{d}_{\text{coef}} \) such that \( R_\omega \supset \omega \). In contrast with the standard definition of the star, here we ignore the cells \( R_\omega \) such that \( R_\omega \cap R_\omega \neq \emptyset \), but \( R_\omega \) does not contain \( R_\omega \). Recall that any elementary operation \( M_j \) or \( I_j \) from (2.1)-(2.3), being applied to an element \( \omega \in \Omega \), lowers its reduced norm \( |\omega'| \) by 1. Thus each \( R_\omega \) from \( St(R_\omega) \) has \( \text{codim}(R_\omega, R_{\omega}) = k \), if and only if, \( \tilde{\omega} \in \omega \xrightarrow{\sim} k \), that is, if \( \tilde{\omega} \) can be obtained from \( \omega \) by a sequence of \( k \) elementary resolutions \( \{M_j^{-1}\} \) and elementary reductions by two \( \{I_j^{-1}\} \).

In particular, there are exactly as many cells \( R_\omega \) of dimension \( \text{dim}(R_\omega) + 1 \) as there are elementary resolutions and reductions of \( \omega \). Each value \( \omega(i) \) can be ”resolved” in \( \omega(i) - 1 \) distinct ways:

\[
(\omega(i) - 1, 1), (\omega(i) - 2, 2), \ldots, (1, \omega(i) - 1)
\]

(the order does matter!), and each \( \omega(i) = 2 \) can be “deleted” or “reduced”. Therefore, the total number of elementary resolutions is \( \sum_i (\omega(i) - 1) = |\omega'| \). The total number of

\[\text{These mimic a bifurcation of a real multiple root in a pair of real roots of the same combined multiplicity.}
\[\text{These mimic the resolution of a real root } \alpha \text{ of multiplicity 2 into two simple complex-conjugate roots.} \]
reductions is \(\#(\omega^{-1}(2))\), the cardinality of \(\omega^{-1}(2)\). All together, there are \(|\omega'| + \#(\omega^{-1}(2))\) elementary operations applicable to \(\omega\). Stated differently, the multiplicity of each cell \(R_\omega\)—the number of \((\dim(R_\omega) + 1)\)-cells containing it—is the codimension of \(R_\omega\) in \(\mathbb{R}_\text{coef}^d\) (or in \(\mathbb{R}_{\text{coef},\alpha}^{d-1}\) plus \(\#(\omega^{-1}(2))\)).

By Theorem 5.1, \(\Gamma : X^m_a\) of multiplicities theorem in the next paper.

Remark 4.1. Note that the combinatorics of these cellular structures is different from the combinatorics of the standard simplex \(\Delta^d\): in a simplex, the multiplicity of each subsimplex is equal to its codimension (to \(|\omega'|\) in our notations); thus the “defect” \(\#(\hat{\omega}^{-1}(2))\) measures the deviation of the \(\Omega_{|d|,\succ}\)-labeled cellular structures in \(\mathbb{R}_\text{coef}^d\) from the standard simplicial one in \(\Delta^d\).

5. On Spaces of Multi-tangent Trajectories

As before, let \(X\) be a compact smooth \((n+1)\)-manifold with boundary. For a boundary generic field \(v\), each \(v\)-trajectory \(\gamma\) intersects the boundary \(\partial_v X\) at a finite number of points \(a\) of multiplicities \(m(a) \leq n+1\). Recall that formulas (1.2)-(1.4) attach the multiplicity \(m(\gamma)\), the reduced multiplicity \(m'(\gamma)\), and the virtual multiplicity \(\mu(\gamma)\) to each trajectory \(\gamma\). For traversally generic fields, \(m(\gamma) \leq 2(n+1), m'(\gamma) \leq n\) for every \(\gamma\) (see Theorem 3.4 from [K2]).

The space of \(v\)-trajectories \(T(v)\) is given the quotient topology, so that the obvious map \(\Gamma : X \to T(v)\) is continuous. Let us stress again that if \(v\) has singularities, the trajectory space \(T(v)\) is quite pathological (non-separable). In contrast, for nonsingular generic gradient fields \(T(v)\) is a decent space, a compact CW-complex! We will prove this theorem in the next paper.

For traversing fields \(v\), all the fibers of \(\Gamma\) are closed intervals or singletons. This property leads to

**Theorem 5.1.** For a traversally generic field \(v\) on \(X\), the map \(\Gamma : X \to T(v)\) is a weak homotopy equivalence\(^{11}\).

For any traversing field \(v\) and any local coefficient system \(A\) on \(X\), the map \(\Gamma\) is a homology equivalence:

\[
\Gamma^* : H^*(T(v); \Gamma^*(A)) \approx H^*(X; A)
\]

for all \(\ast \geq 0\).

\(^{11}\)In the next paper, we will show that \(\Gamma\) actually is a homotopy equivalence for traversally generic fields.
Proof. Suppose there exists an open cover $U = \{U_\alpha\}_\alpha$ of $T(v)$, such that all the maps $C: \Gamma^{-1}(U_\alpha) \to U_\alpha$ are weak homotopy equivalences. Then, by Corollary 1.4 from [May], $\Gamma: X \to T(v)$ is a weak homotopy equivalence, provided that the cover $U$ is closed under finite intersections of its elements.

So we need to construct the appropriate cover of $T(v)$ and to prove that $\Gamma$ is a weak homotopy equivalence just locally.

By Lemma 1.2 for each $v$-trajectory $\gamma$, there exists a $\tilde{v}$-adjusted neighborhood $\tilde{V}_\gamma \subset \tilde{X}$ of $\gamma$ such that, in special coordinates $(u, x, y)$, $X$ is described by the polynomial inequality $\{P(u, x) \leq 0\}$, and the cylindrical neighborhood $\tilde{V}_\gamma$ of $\gamma \subset X$ by the additional inequalities $\|x\| \leq \epsilon$, $\|y\| \leq \epsilon'$.

Being the union of all $\tilde{v}$-trajectories close to $\gamma$, this neighborhood $\tilde{V}_\gamma$ determines a neighborhood $U_{\gamma}$ of the point $\gamma \in T(v)$. Note that $\tilde{V}_\gamma$ may have $\tilde{v}$-trajectories that do not intersect $X$. Let us denote by $V_\gamma$ the subset of $\tilde{V}_\gamma$ built out of $\tilde{v}$-trajectories $\tilde{\gamma}$ with the property $\tilde{\gamma} \cap X \neq \emptyset$. By definition, the restriction $\Gamma: V_\gamma \to U_\gamma$ is surjective.

Let us denote by $\omega_x \in \Omega$ the combinatorial pattern of the divisor $D_\mathbb{R}(P(u, x))$, where $x \in \mathbb{R}^{\lfloor \omega_0 \rfloor'}$ and $\omega_0$ is the combinatorial pattern of the divisor $D_\mathbb{R}(P(u, 0))$. If $x$ is such that $P(u, x) > 0$ for all $u$, then $\omega_x : N_+ \to \mathbb{Z}_+$ is defined to be the zero map. The reduced norm $|\omega_x|'$ of such trivial $\omega_x$ is defined to be $-1$.

For each $k \in [0, \lfloor \omega_0 \rfloor']$, consider the real subvariety $X_k \subset \mathbb{R}^{\lfloor \omega_0 \rfloor'}$ defined by the constraint $|\omega_x|' \geq k$. In other words, $x \in X_k$ if and only if the reduced multiplicity $m'(D_\mathbb{R}(P(u, x))) \geq k$. In view of Theorem 4.1 and Theorem 4.2, $\text{codim}(X_k, \mathbb{R}^{\lfloor \omega_0 \rfloor'}) = k$.

Let $\pi : V_\gamma \to \mathbb{R}^{\lfloor \omega_0 \rfloor'} \times \mathbb{R}^{n-\lfloor \omega_0 \rfloor'}$ denote the projection $(u, x, y) \mapsto (x, y)$.

Put

$$V_{\gamma,k} := \pi^{-1}(X_k \times B'_{\epsilon'}(0))$$

where $B_{\epsilon'}(0)$ is the $\epsilon'$-ball in $\mathbb{R}^{n-\lfloor \omega_0 \rfloor'}$ with the center at 0. Thus $V_{\gamma,0} := V_\gamma$, and $V_{\gamma,\lfloor \omega_0 \rfloor'} = \gamma \times B_{\epsilon'}(0)$.

The map $\pi$ can be viewed as a composition $p \circ q$ of two maps: the quotient surjective map $q : V_\gamma \to U_\gamma \subset T(v)$, whose fibers are closed segments, and a finitely ramified map $p : U_\gamma \to \mathbb{R}^{\lfloor \omega_0 \rfloor'}$. The map $q$ is the restriction of the map $\Gamma : X \to T(v)$ to the neighborhood $V_\gamma$.

Let $U_{\gamma,k} := q(V_{\gamma,k})$. We will argue by induction “$k \Rightarrow k - 1$”. We claim that if $q : V_{\gamma,k} \cap X \to U_{\gamma,k}$ is a weak homotopy equivalence, then so is the map $q : V_{\gamma,k-1} \cap X \to U_{\gamma,k-1}$, provided $k > 0$.

First we will show that the map

$$q_{k-1} : (V_{\gamma,k-1} \cap X)/(V_{\gamma,k} \cap X) \to U_{\gamma,k-1}/U_{\gamma,k}$$

admits a continuous section $\sigma_{k-1}$ such that $\sigma_{k-1}(U_{\gamma,k-1}/U_{\gamma,k})$ is a deformation retract of $(V_{\gamma,k-1} \cap X)/(V_{\gamma,k} \cap X)$.

For each $x \in X_{k-1}$, the set $\{P(u, x) \leq 0\}$ is a disjointed union of closed intervals $\{I_i(x)\}$. Let $u^*_i(x)$ be the center of the interval $I_i(x)$. With the help of $q$, the pair $(x, u^*_i(x))$ determines the point $q(x, u^*_i(x))$ in $T(v)$. Then we define the “protosection”
\[ \sigma_{k-1} \text{ by the formula } \]

\[ \sigma_{k-1}(q(x, u^*_k(x))) := (x, u^*_k(x)). \]

This formula is discontinuous for points \( q(x, u^*_k(x)) \in U_{\gamma, k} \), where some intervals \( \{I_i(x)\}_{x \in \mathcal{X}_{k-1} \setminus \mathcal{X}_k, i} \) merge; however, it produces a continuous section

\[ \sigma_{k-1} : U_{\gamma, k-1}/U_{\gamma, k} \to (V_{\gamma, k-1} \cap X)/(V_{\gamma, k} \cap X) \]

of the quotients. Now \((V_{\gamma, k-1} \cap X)/(V_{\gamma, k} \cap X)\) retracts on \( \sigma_{k-1}(U_{\gamma, k-1}/U_{\gamma, k}) \) by collapsing each interval \( I_i(x) \) on its center \( u^*_k(x) \).

The basis of induction is \( k = |\omega_0| \). In this case, with the help of \( q, V_{\gamma, \omega_0} := \gamma \times B_{\epsilon'}(0) \) is homotopy equivalent to \( U_{\gamma, \omega_0} := B_{\epsilon'}(0) \).

By the inductive assumption, \( q_k : V_{\gamma, k} \cap X \to U_{\gamma, k} \) is a weak homotopy equivalence. We have shown that

\[ \tilde{q}_k : (V_{\gamma, k-1} \cap X)/(V_{\gamma, k} \cap X) \to U_{\gamma, k-1}/U_{\gamma, k} \]

is a homotopy equivalence. Therefore, comparing the exact homotopy sequences of the two triples, we conclude that \( q_{k-1} : V_{\gamma, k-1} \cap X \to U_{\gamma, k-1} \) is a weak homotopy equivalence as well. In particular, it follows that \( q_0 := q : V_{\gamma, 0} \cap X \to U_{\gamma, 0} \) is a weak homotopy equivalence.

By compactness of \( X \), we can pick a finite \( v \)-adjusted cover \( \mathcal{V} := \{V_\gamma\} \) of \( X \subset \hat{X} \) and the corresponding cover \( \mathcal{U} := \{U_\gamma := \Gamma(V_\gamma)\} \) of \( \mathcal{T}(v) \), so that each map \( \Gamma : V_\gamma \to U_\gamma \) is a weak homotopy equivalence. Add to the list \( \mathcal{V} \) all the multiple intersections \( V_{\gamma_1} \cap V_{\gamma_2} \cap \cdots \cap V_{\gamma_r} \) of elements from \( \mathcal{V} \), thus forming a larger lists \( \hat{\mathcal{V}} \) and a new corresponding list \( \hat{\mathcal{U}} \) comprising all the intersections \( U_{\gamma_1} \cap U_{\gamma_2} \cap \cdots \cap U_{\gamma_r} \).

For each \( k \), the locally-defined sets \( \{V_{\gamma_1, k}\} \) have an intrinsic description in terms of the combinatorial patterns of tangency. So they automatically agree on multiple intersections: \( X \cap V_{\gamma_1, k} \cap V_{\gamma_m} \subset X \cap V_{\gamma_m, k} \) for all \( l, m \). Now the same inductive argument in \( k \) works for each map

\[ \Gamma : X \cap V_{\gamma_1} \cap V_{\gamma_2} \cap \cdots \cap V_{\gamma_r} \to U_{\gamma_1} \cap U_{\gamma_2} \cap \cdots \cap U_{\gamma_r} \]

, so that this map is a weak homotopy equivalence as well.

As a result, by Corollary 1.4 \cite{May}, \( \Gamma : X \to \mathcal{T}(v) \) is a week homotopy equivalence.

Now consider a traversing field \( v \) on \( X \). Let \( \mathcal{A} \) be any local coefficient system (a sheaf) on \( X \) with an abelian group \( \mathcal{A} \) for the stock. We denote by \( \Gamma_*(\mathcal{A}) \) its push-forward residing on the trajectory space \( \mathcal{T}(v) \).

Let \( U \) be an open neighborhood of a typical trajectory \( \gamma \) in \( X \). Since \( X \) is compact and a typical \( \Gamma \)-fiber—a trajectory \( \gamma \)—is closed, the canonical homomorphism

\[ \lim \text{ind}(U \ni \gamma) \quad H^*(U; \mathcal{A}(U)) \to H^*(\gamma; \mathcal{A}|_{\gamma}) \]

is an isomorphism (see Theorem 4.11.1 in \cite{God}). Since all \( \gamma \)'s are either segments or singletons, \( H^*(\gamma; \mathcal{A}|_{\gamma}) = 0 \) for all \( * \neq 0 \) and \( H^0(\gamma; \mathcal{A}|_{\gamma}) = \mathcal{A} \). Thus, the Leray spectral sequence

\[ \{E_2^{pq} = H^p(\mathcal{T}(v); H^q(\gamma; \mathcal{A}))\}_{p, q} \]
of the map $\Gamma : X \rightarrow \mathcal{T}(v)$ collapses (see Theorem 4.17.1 in [God]). As a result, we get that the map $\Gamma$ establishes an isomorphism $\Gamma^* : H^*(\mathcal{T}(v); \Gamma_*(A)) \approx H^*(X; A)$.

In particular, for a trivial local system $A$, we get $H^*(X; A) \approx H^*(\mathcal{T}(v); A)$. □

**Remark 5.1.** If a traversing field $v$ is such that $\mathcal{T}(v)$ has a homotopy type of a CW-complex, then by Whitehead Theorem [Wh], $\Gamma : X \rightarrow \mathcal{T}(v)$ is a homotopy equivalence. In the next paper, we will prove that, for a traversally generic field $v$, the trajectory space $\mathcal{T}(v)$ can be given the structure of a compact CW-complex. □

For any sub-poset $\Theta \subset \Omega^*_{[n]}$ and a traversally generic field $v$ on $X$, let us consider the subsets $X(v, \Theta) \subset X$ and $\mathcal{T}(v, \Theta) \subset \mathcal{T}(v)$ comprised of points $x \in X$ or of trajectories $\gamma_x \in \mathcal{T}(v)$ whose divisors $D_{\gamma_x}$ have the combinatorial models prescribed by the elements of $\Theta$.

In particular, we will see that the webs of subspaces

$$\{X(v, \omega_{\leq}^n)\}_{\omega \in \Omega^*_{[n]}} \quad \text{and} \quad \{\mathcal{T}(v, \omega_{\leq}^n)\}_{\omega \in \Omega^*_{[n]}}$$

form a remarkable geometric structure. It will preoccupy us for the rest of this series of articles.

A cruder stratification (filtration) of $X$ and $\mathcal{T}(v)$ is provided by the spaces

$$\{X(v, \Omega^*_{[k,n]})\}_{0 \leq k \leq n} \quad \text{and} \quad \{\mathcal{T}(v, \Omega^*_{[k,n]})\}_{0 \leq k \leq n},$$

respectively.

Lemma 3.4 (see also Lemma 1.2) and Theorem 3.5 from [K2] have an useful implication:

**Corollary 5.1.** Let $X$ be a smooth $(n + 1)$-manifold with boundary. For any traversally generic field $v$, the obvious map $\Gamma : \partial_1 X \rightarrow \mathcal{T}(v)$ is $(n + 2)$-to-1 at most. At the same time, $\Gamma : \partial_2 X \rightarrow \mathcal{T}(v)$ is $n$-to-1 at most. For each $\omega$, the restriction of $\Gamma$ to the subspace $\Gamma^{-1}(\mathcal{T}(v, \omega))$ is $|\sup(\omega)|$-to-1.

When restricted to the $\Gamma$-preimage of the proper stratum $\mathcal{T}(v, \omega)$, $\Gamma$ is a covering map with a trivial monodromy and a fiber of cardinality $|\sup(\omega)|$.

**Proof.** For a traversally generic field, by Theorem 3.5 from [K2], $m(\gamma) \leq 2(n + 1)$. Each trajectory has exactly two points of odd multiplicity, the rest of the points are tangent points of even multiplicity. Their number does not exceed $n$. Thus $\Gamma$ is $(n + 2)$-to-1 at most, and $\Gamma|_{\partial_2 X}$ is $n$-to-1 at most.

The statement dealing with the cardinalities of the fibers of

$$\Gamma : \Gamma^{-1}(\mathcal{T}(v, \omega)) \rightarrow \mathcal{T}(v, \omega)$$

follows instantly from the definitions of the relevant spaces.

Let $\beta$ be a loop in $\mathcal{T}(v, \omega)$, and let $E_\beta := \Gamma^{-1}(\beta) \subset X$. Note that $\Gamma : E_\beta \rightarrow \beta$, thanks to the orientation by $v$, is a cylinder (and not a Möbius band). Consider the intersection $E_\beta \cap \partial_1 X$. Since $\beta$ is contained in the pure stratum $\mathcal{T}(v, \omega)$, $\Gamma : E_\beta \cap \partial_1 X \rightarrow \beta$ is a covering map with a finite fiber. Because its space $E_\beta \cap \partial_1 X$ is contained in the cylinder $E_\beta$, we conclude that $\Gamma : E_\beta \cap \partial_1 X \rightarrow \beta$ must be a trivial covering. □
Assuming that \( v \) is traversally generic, our immediate goal is to describe one particular localized cellular structure of the trajectory space \( \mathcal{T}(v) \). As we mentioned before, it is governed by the combinatorics of the divisors in \( \mathbb{R} \) of real degree \( \leq 2(n + 1) \) polynomials.

First, we would like to understand better the \( \Omega^{\bullet(\omega)} \)-stratified structure of \( \mathcal{T}(v) \), localized to the vicinity of given \( v \)-trajectory \( \gamma \).

As usual, we operate within an extension germ \((X, \hat{v})\) of \((X, v)\). Let \( \{a_i\}_i := \gamma \cap \partial_i X \) be the \( i \)-ordered finite set of points, where \( a_i \in \partial_i X^\circ \). By Theorem 3.5 from [K2] (see also Lemma 1.2), this tangency pattern \( \omega = (j_1, j_2, \ldots) \) is described by an element \( \omega \in \Omega^{\bullet(\omega)}_\varepsilon \cap \Omega^{\bullet(\omega)}_\varepsilon(\gamma \times \mathbb{R}^n) \).

By Lemma 1.2 and formula (1.7), in special coordinates \((u, x, y)\) on some \( \hat{v} \)-adjusted tube surrounding \( \gamma \), the manifold \( X \) is given by the inequality

\[
(5.1) \quad P(u, x) := \prod_i (u - \alpha_i)^{j_i} + \sum_{l=0}^{j_i-2} x_{i,l}(u - \alpha_i)^l \leq 0
\]

, where \( \alpha_i = u(a_i) \) and \( x = \{x_{i,l}\} \).

Next, in our local analysis, we can pick a canonical model of \( X \) in the vicinity of \( \gamma \) by assuming that each \( \alpha_i = i \).

If \( |\omega| = 0(2) \), for each fixed value of the coordinates \((x, y)\), the solution set of (5.1) is a disjoint union of several closed intervals and singletons residing in the \( u \)-line \( \hat{\gamma}_x \) (see Fig. 1 in [K2]), the union depending on \( x \) alone. Each of these intervals and singletons represent a \( v \)-trajectory suspended over \((x, y)\) (for some \( x, \hat{\gamma}_x \) can be empty!). So to get the space of trajectories \( \mathcal{T}(v) \) in the vicinity of \( \gamma \), we need to collapse each interval to a point-marker that resides in it. Let us formalize the collapsing procedure.

Consider the solution set \( E_\omega \) of (5.1). We say that two points \((u, x, y), (u', x', y') \in E_\omega \) are equivalent (“\( \sim \)”), if \( x = x', y = y' \), and the interval \([u, u'], x, y \in E_\omega \). Now we define the space \( \mathcal{T}_\omega \) as the quotient space \( E_\omega / \sim \).

The space \( \mathcal{T}_\omega \) comes equipped with the map \( \mathcal{T}_\omega \to \mathbb{R}^{|\omega'| \times \mathbb{R}^{n - |\omega'|}} \) induced by the obvious projection \((u, x, y) \to (x, y) \). Since \( X \) is compact, for each \( x \), the polynomial \( P(u, x) \) in (5.1) has finitely many intervals where it is negative, \( p \) is a ramified map with finite fibers.

For any fixed \( x \), the \( u \)-polynomial in \( P(u, x) \) in (5.1) can be viewed also as an element of the space \( \mathcal{P}^{|\omega|} := \mathbb{R}^{|\omega|}_\text{coeff} \) and as such belongs to a unique pure stratum \( R^{|\omega'|}_\omega := R^{|\omega'|}(x) \subset \mathcal{P}^{|\omega|} \), where \( \omega' \geq \omega \in \Omega_{\{\omega\}} \). Therefore, with the help of (5.1), each \( x \in \mathbb{R}^{|\omega'|} \) has a well-defined combinatorial type \( \omega(x) = \omega' \in \Omega_{\{\omega\}} \) associated to it. As a result, \( \mathbb{R}^{|\omega'|} \) is an \( \Omega_{\{\omega\}} \)-stratified space, and so is \( E_\omega \subset \mathbb{R} \times \mathbb{R}^{|\omega'|} \times \mathbb{R}^{n - |\omega'|} \).

In fact, the space \( E_\omega \) admit a “more intrinsic” stratification which is labeled by the elements of the poset \( \Omega^\bullet_{\{\omega\}} \), where \( d = |\omega'| \). In a sense, this stratification is cruder than the \( \Omega^\bullet_{\{\omega\}} \)-stratification of \( \mathbb{R}^{|\omega'|} \). Here is the description of this \( \Omega^\bullet_{\{\omega\}} \)-stratification.

For each point \((u_*, x, y) \in E_\omega \), there is a unique closed interval \( I_{u_*, x} := [a, b] \) such that \( u_* \in [a, b] \), \( P(u, x) \leq 0 \) for all \( u \in [a, b] \), and \( I_{u_*, x} \) is the maximal closed interval
possessing these two properties. Consider the real zero divisor $D_{(u_*, x)}$ of the $u$-polynomial $P(u, x)$ being restricted to the interval $I_{u_*, x}$. Its combinatorial type $\omega(u_*, x) \in \Omega^*$ and is independent on the choice of $u_*$ within the interval $I_{u_*, x}$. Thus, $\omega(u_*, x)$ depends only on the equivalence class of $(u_*, x, y) \in E_\omega$, a point in $T_\omega$.

On the other hand, if $\omega(x) = \omega(x')$ for some $x, x' \in \mathbb{R}[\omega']$, then there exist $$(u_*, x, y), (u'_*, x', y) \in E_\omega$$
such that $\omega(u_*, x) = \omega(u'_*, x')$. Moreover, the orders which the intervals $I_{u_*, x}$ and $I_{u'_*, x'}$ occupy inside the sets $P^{-1}((\infty, 0), x)$ and $P^{-1}((\infty, 0), x')$, respectively, are the same. Stated differently, the combinatorial type $\omega(x) \in \Omega_{[\omega]}$ determines the ordered sequence $\Xi(\omega(x))$ of types $\{\omega(u_*, x)\}_{u_*}$ for points in the fiber $p^{-1}(x)$ (cf. the discussion preceding Fig. 1).

Since the construction of the space $T_\omega$ and its $\Omega^*_{[\omega']}$-stratification depends only on the combinatorial pattern $\omega \in \Omega^*_{[\omega]}$, we get:

**Theorem 5.2.** For any traversally generic field $v$ on a $(n + 1)$-manifold $X$ and any $v$-trajectory $\gamma$ with the tangency multiplicity pattern $\omega \in \Omega^*_{[\omega]}$, the $\Omega^*_{[\omega']}$-stratified topological type of the germ of the trajectory space $T(v)$ at the point $\gamma$ is determined by the combinatorial pattern $\omega$ alone. \qed

**Example 5.1.** For the traversally generic fields $v$ on 4-folds $X$, there are 11 distinct local topological types for $T(v)$. They are labeled by the elements of the poset from Fig. 2. \qed

Next, we will employ similar considerations to describe the germ at $\gamma$ of a cellular structure in $T_\omega$, a structure subordinate to the filtration of $T_\omega$ by spaces which are labeled by the elements of the poset $\Omega^*_{[\omega']}$. Eventually, these investigations will culminate in Theorem 5.3 below.

By choosing an appropriately narrow $\hat{v}$-adjusted neighborhood $U \subset \hat{X}$ of $\gamma$, for any $x$ sufficiently close to the origin, the complex zeros of $P(u, x)$ from (5.1) can be separated into disjointed groups. These groups correspond to the real zeros $\{\alpha_i\}_i$ of the polynomial $P(u, 0)$ (which is a product of linear polynomials over $\mathbb{R}$) and reside in their vicinity. Moreover, each portion of the complex zero divisor of $P(u, x)$, taken within each group, by (5.1), is $\alpha_i$-balanced.

Thus, for any $\hat{v}$-trajectory $\hat{\gamma} \subset U$, the “real” zero divisor $D_{\hat{\gamma}}$ splits into several “real” divisors $D_{\hat{\gamma}, i}$. Their combinatorial types are described by some elements $\hat{\omega}_i \in \Omega_{[\omega(i)]}$ such that $|\hat{\omega}_i| \leq \omega(i)$ and $|\hat{\omega}_i| \equiv \omega(i) \mod (2)$. In other words, $\omega_{\hat{\gamma}}$, the combinatorial type of $D_{\hat{\gamma}}$, is determined by an element of the product $\prod_i \Omega_{[\omega(i)]}$. Evidently, there is a canonical map $$\kappa : \prod_i \Omega_{[\omega(i)]} \rightarrow \Omega_{[\omega]}$$
that places $\{\hat{\omega}_i\}_i$ in a single array $\omega_{\hat{\gamma}} := (\hat{\omega}_1, \hat{\omega}_2, \ldots)$. For each $i$, we denote by $x_i$ the ordered subset $\{x_{i, 1}, x_{i, 2}, \ldots\}$ of the $x$-coordinates, amenable to the formula (5.1).
The decomposition of $P(u, x) = \prod_i P_i(u - \alpha_i, x_i)$ of the left-hand side of (5.1) into a product of monic depressed polynomials $P_i(u - \alpha_i, x_i)$ in $u - \alpha_i$ of degrees $j_i := \omega(i)$ gives rise to a continuous map

$$A_\omega : \mathbb{R}^{\omega(i)} \to \prod_i \mathbb{R}^{\omega(i)-1}_{\text{coeff}, \alpha_i}$$

from the space of $x$-coordinates $\mathbb{R}^{\omega(i)}$, to the product of polynomial spaces $\{\mathbb{R}^{\omega(i)-1}_{\text{coeff}, \alpha_i}\}_{i}$, of equal dimension $|\omega(i)|$. If $x \neq x'$, then at least for one $i$, the vectors $x_i, x'_i$ are distinct. Thus the $(u - \alpha_i)$-polynomials $P_i(u - \alpha_i, x_i)$ and $P_i(u - \alpha_i, x'_i)$ must have distinct coefficients. As a result, $A_\omega$ is a 1-to-1 map. Evidently, by (5.1), $A_\omega$ is surjective. So $A_\omega$ establishes a smooth homeomorphism of the two spaces from (5.2).

With the help of Theorems 4.1 and 4.2 we get the cellular structures

$$\{\Phi_{\alpha_i, \omega_i} : e_{\alpha_i, \omega_i} \to R_{\alpha_i, \hat{\omega}_i} \omega_i \in \Omega_{(\omega(i))}\}$$

for each space $\mathbb{R}^{\omega(i)-1}_{\text{coeff}, \alpha_i}$. By $A_\omega^{-1}$ from (5.2), this product of cellular structures in the target space of (5.2) gives rise to a cellular structure in the space $\mathbb{R}^{\omega(i)}$ of $x$-coordinates. Employing

$$\kappa : \prod_i \Omega_{(\omega(i))} \to \Omega_{(\omega)}$$

, that structure

$$\Psi_\omega := A_\omega^{-1} \circ (\prod_i \Phi_{\alpha_i, \omega_i}) : \prod_i e_{\alpha_i, \hat{\omega}_i} \longrightarrow \mathbb{R}^{\omega(i)}$$

is consistent with the stratification of $\mathbb{R}^{\omega(i)}$, labeled by the elements $\kappa(\{\hat{\omega}_i\}_{i})$ of the poset $\Omega_{(\omega)}$.

In what follows, the cellular structure in the product $\mathbb{R}^{\omega(i)} \times \mathbb{R}^{n-|\omega(i)|}$ is chosen to be this cellular structure in $\mathbb{R}^{\omega(i)}$ given by (5.3), being multiplied by a single open cell $e^{n-|\omega(i)|} \approx \mathbb{R}^{n-|\omega(i)|}$.

Note that, so far, the cells in $\mathbb{R}^{\omega(i)}$ are labelled by the elements of the poset $\omega \preceq \Omega_{(\omega)}$, and not by elements of the poset $\omega \preceq \Omega_{(\omega)}^*$, appropriate for the trajectories in $E_\omega$ (see the discussion preceding Lemma 4.3).

Next, using the ramified map $p : T_\omega \to \mathbb{R}^{\omega(i)} \times \mathbb{R}^{n-|\omega(i)|}$ with finite fibers, we will employ the cellular structure $\Psi_\omega$ in the target space $\mathbb{R}^{\omega(i)}$ to produce a cellular structure in the source space $T_\omega$, so that $p$ will become a cellular map. That task will preoccupy us for a while...

With this goal in mind, we introduce markers, a new combinatorial contraption (see Fig. 1 and Fig. 4).

For each $\omega \in \Omega_{(d)}$, consider the auxiliary $u$-polynomial

$$\varrho_\omega(u) = \prod_{\{i \in \text{sup} (\omega)\}} (u - i)^{\omega(i)}$$

of degree that does not exceed $d$ and shares the same parity with it.
For a given $\omega$, we consider a pair $(\omega, k)$, where the marker $k \in \sup(\omega) \subset \mathbb{N}$ is such that:

- either $\omega(k) \equiv 0 \mod (2)$ and $\varphi_\omega(k - 0.5) > 0$, $\varphi_\omega(k + 0.5) > 0$,
- or $\omega(k) \equiv 1 \mod (2)$ and $\varphi_\omega(k - 0.5) > 0$, $\varphi_\omega(k + 0.5) < 0$.

We denote by $\Omega^\mu_{[d]}$ the set of all marked pairs $(\omega, k)$ as above.

Let us denote by $\Upsilon(\omega) \subset \mathbb{N}$ the set of markers $k$ associated with $\omega$. Each marker $k \in \Upsilon(\omega)$, as an element of linearly ordered set $\Upsilon(\omega)$, acquires its ordinal $p$. We will denote the $p$-th marker in $\Upsilon(\omega)$ by $k_p$. Let $\Upsilon_p(\omega) \subset \mathbb{N}$ be the maximal set of consecutive natural numbers $j \geq k_p$ such that $\varphi_\omega(u) \leq 0$ for all $u \in [k_p, j]$.

It is possible to extend the elementary operations $M_j$ and $I_j$ (merge and insert), introduced in (2.1, 2.3) for the poset $\Omega_{[d]}$, to the elements of the new set $\Omega^\mu_{[d]}$.

The new merge operation $M^\mu_j : \Omega^\mu_{[d]} \rightarrow \Omega^\mu_{[d]}$ is shown in Fig. 4. Let

$$\tag{5.4} M^\mu_j(\omega, k) := (M_j(\omega), \mu_j(k)),$$

where $\mu_j(k) \in \Upsilon(M_j(\omega))$ is defined as follows.

If both $j$ and $j + 1$ belong to the same set $\Upsilon_q(\omega)$, and the marker $k \in \Upsilon_q(\omega)$, $q \neq p$, then $\mu_j(k) = k$ as elements of the set $\Upsilon_q(M_j(\omega)) = \Upsilon_q(\omega)$. If $j$ and $j + 1$ belong to $\Upsilon_p(\omega)$, and the marker $k \in \Upsilon_p(\omega)$, then again $\mu_j(k) = k$ as elements of the set $\Upsilon_p(M_j(\omega)) \subset \Upsilon_p(\omega)$.

At the same time, if $j \in \Upsilon_p(\omega)$, $j + 1 \in \Upsilon_{p+1}(\omega)$, and $k \in \Upsilon_{p+1}(\omega)$, then $\mu_j(k)$ is the unique minimal element in $\Upsilon_p(M_j(\omega))$. When $k \in \Upsilon_q(\omega)$ and $q \neq p + 1$, the marker keeps its minimal position within the subset $\Upsilon_q(\omega)$.

Similarly, the new insert operation

$$\tag{5.5} I^\mu_j(\omega, k) := (I_j(\omega), \mu_j(k))$$

is described as follows. Under $I^\mu_j$, $\omega$ is subjected to the old insert operation $I_j$, while the marker $k \in \Upsilon_p(\omega)$ keeps its minimal position within the set $\Upsilon_p(\omega) \subseteq \Upsilon_q(I_j(\omega))$ (where $q$ is uniquely determined by $p$ and $j$), that is, $\mu_j(k) = k$ as the minimal elements of the appropriate sets.

Here is a slightly more geometrical look at the markers and their behavior, a look motivated by interactions of vector flows with the boundary $\partial_1 X$. Let $\gamma \subset X$ be a typical $\hat{v}$-trajectory in the vicinity of a given trajectory $\gamma \subset X$. The intersection $\hat{\gamma} \cap \partial_1 X = \{b_1\}_{1 \leq l \leq q}$, allows us to shade some of the intervals $(b_l, b_{l+1})$ in which $\partial_1 X$ divides $\hat{\gamma}$: the interval is shaded if it belongs to $X$. Using any auxiliary function $z : X \rightarrow \mathbb{R}$ (as in Lemma 3.1 from [K2]), the shading is defined as the locus where $z|_\gamma \leq 0$\footnote{Recall that $z$ is chosen to possess the following properties: 1) $0$ is a regular $z$-value and $z^{-1}(0) = \partial_1 X$, 2) $z^{-1}((-\infty, 0]) = X$.} Therefore, the ordered sequence of the multiplicities $\omega_\gamma := (m(b_1), m(b_2), \ldots, m(b_q))$ uniquely determines which intervals $(b_l, b_{l+1})$ along $\hat{\gamma}$ are shaded (cf. Fig. 1, where the shading corresponds to "strings" and "atoms"): each shaded interval can be marked with a unique lowest odd-multiplicity point in it; also each atom can be marked.

As in Section 2 (see the discussion preceding Fig. 1), we denote by $\sup_{\text{odd}}(\omega)$ the points $l \in \mathbb{N}$ in the support of $\omega$ such that $\omega(l) \equiv 1 \mod (2)$ and by $\sup_{\text{ext}}(\omega)$ the points $l$ in the
Figure 4. The rules by which the markers evolve under the merge operations (depicted as the V-shaped passages from top to bottom bars in each of the seven diagrams).

Support of \( \omega \) such that \( \omega(l) \equiv 0 \mod (2) \). We can count the elements of \( \text{sup}_{\text{odd}}(\omega) \) and pick only the ones that acquire odd numerals in that count (in this way, we pick half of the elements in \( \text{sup}_{\text{odd}}(\omega) \)) (cf. Fig. 1). We denote this set by \( \text{sup}_{\text{odd}}^{+}(\omega) \) and its complement by \( \text{sup}_{\text{odd}}^{-}(\omega) \). We divide the points of \( \text{sup}_{\text{ev}}(\omega) \) into two complementary sets: the first one, \( \text{sup}_{\text{ev}}(\omega) \), contains points that are bounded from below by a point from \( \text{sup}_{\text{odd}}^{+}(\omega) \) and from above by a point from \( \text{sup}_{\text{odd}}^{-}(\omega) \); the second one, \( \text{sup}_{\text{ev}}^{+}(\omega) \), contains points that are bounded from below by a point from \( \text{sup}_{\text{odd}}^{-}(\omega) \) and from above by a point from \( \text{sup}_{\text{odd}}^{+}(\omega) \). In fact, \( \Upsilon(\omega) = \text{sup}_{\text{odd}}^{+}(\omega) \cup \text{sup}_{\text{ev}}^{+}(\omega) \).

Consider \( \omega' \succeq \omega \), where \( \omega', \omega \in \Omega_{(m)} \), and two elements-markers \( k' \in \Upsilon(\omega') \) and \( k \in \Upsilon(\omega) \). Our next task is to define a relation \( \rightsquigarrow \) between \( k' \) and \( k \). Recall that \( \omega \) can be obtained from \( \omega' \) by a sequence of elementary merges and multiplicity 2 inserts as described in (2.1)-(2.3). Given a marker \( k' \in \Upsilon(\omega') \), we will describe its evolution under these elementary transformations of \( \omega' \). An insertion of an even multiplicity point does not affect any marker below the insertion and shifts the location of the marker above the insertion by one. If two points from \( \text{sup}_{\text{ev}}^{+}(\omega') \) merge and one of them is marked, the marker is placed at the location where the merge took place. If a marked point from \( \text{sup}_{\text{ev}}^{+}(\omega') \) is merging with a point from \( \text{sup}_{\text{odd}}^{-}(\omega') \), the location of the merge is marked. On the other hand, if a marked point from \( \text{sup}_{\text{ev}}^{+}(\omega') \) is merging with a point from \( \text{sup}_{\text{odd}}^{-}(\omega') \), then the marker is placed at the first point of \( \text{sup}_{\text{odd}}^{-}(\omega') \) located below the merge—the
marker “travels down until it reaches a point of \( \text{sup}_+^\omega (\omega') \)” . Finally, if a marked point from \( \text{sup}_+^\omega (\omega') \) merges with the adjacent point from \( \text{sup}_-^\omega (\omega') \), the marker is placed at the first point of \( \text{sup}_+^\omega (\omega') \) located below the merge. If one inserts a point of an even multiplicity, the marker does not change its location.

Therefore, when \( \omega' \supset \omega \), the marker \( k' \) defines a unique marker \( k \in \Upsilon(\omega) \). In such a case, we say that \( k' \) collapses to \( k \) and write “\( k' \leadsto k \)” .

There is a natural and already familiar map \( \Xi : \Omega^\omega \to \Omega^\ast \) which acts from the set of all marked pairs \((\omega, k)\) to the set \( \Omega^\ast \) of strings and atoms (see Fig. 1). By definition, \( \Xi \) takes each pair \((\omega, k)\), \( k \in \Upsilon(\omega) \), to the restriction of \( \omega \) to the unique maximal interval \([k, j(k)]\) of indices in \( \text{sup}(\omega) \) with the property \( \varphi_j(j) \leq 0 \) for all \( j \in [k, j(k)] \). Then, by a shift of indices, one reinterprets \( \omega : [k, j(k)] \to \mathbb{N} \) as a map \( \omega : [1, j(k) - k - 1] \to \mathbb{N} \), an element of \( \Omega^\ast \). In fact, \( \Xi : \Omega^\omega_{[2n+2]} \to \Omega^\ast_{[n]} \) for each \( n \).

In the following theorem, we combine the acquired knowledge about one particular cellular structure of the model spaces \( \{T_\omega\} \) with the local models for traversally generic vector fields (described in Lemma 1.2). This leads to a local purely combinatorial description of trajectory spaces for traversally generic fields. Regrettfully, the formulation of the theorem is lengthy.

Theorem 5.3. Let \( X \) be a compact smooth \( (n+1) \)-manifold with boundary. Let \( v \) be a traversally generic vector field on \( X \) and \( \gamma \) its trajectory of the combinatorial type \( \omega \). Then the following statements hold:

- In the vicinity of \( \gamma \), the trajectory space \( T(v) \) has a structure of the model \( |\omega'| \)-dimensional finite cell complex \( T_\omega \) times \( \mathbb{R}^{n-|\omega'|} \).
- Each open cell \( E_{\omega,k} \subset T_\omega \) (of dimension \( |\omega'| - |\hat{\omega}'| \)) is indexed by an element \( \hat{\omega} \subset \Upsilon(\omega) \) of the poset \( \prod_i \Omega(\omega(i)) \) together with a marker \( k \in \Upsilon(\hat{\omega}) \). Every point in \( E_{\omega,k} \) belongs to the pure stratum \( T(v, \Xi(\kappa(\hat{\omega}), k)) \) of \( T(v) \), labeled by the element \( \Xi(\kappa(\hat{\omega}), k) \in \Omega^\ast(\hat{\omega}) \).
- \( E_{\omega',k'} \subset E_{\omega',k} \) if and only if the following two conditions are satisfied:
  1. \( \hat{\omega} \supset \hat{\omega}' \), and
  2. the markers \( k \) and \( k' \) satisfy the relation “\( k \leadsto k' \)”.
- Employing the attaching maps \( \Psi_{\omega,k} : e_{\hat{\omega}} : \prod_i e_{\omega_i} \to \mathbb{R}^{|\omega'|} \) from (5.3), the space \( T_\omega \) can be assembled from the marked cells \( \{e_{\hat{\omega}}\} \) according to the rules described in Fig. 4. Each closed cell \( E_{\omega',k} \subset T_\omega \) is the image of the cell \( (e_{\hat{\omega}}, k) \) under the attaching map \( \Psi_{\omega,k} : \prod_i e_{\omega_i} \to T_\omega \) based on (5.3).
- Each cell \( E_{\omega',k} \), where \( \kappa(\hat{\omega}) \supset \omega \), topologically is a positive cone with the apex at \( \gamma \in T(v) \) over a compact cell \( S_{\omega',k} \). These cells \( \{S_{\omega',k}\} \) form a link of the point \( \gamma \) in \( T_\omega \). The rules that describe their incidence are similar to the incidence rules for \( \{E_{\omega',k}\} \) in \( T_\omega \).

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\(^{13}\)where the partial order in \( \prod_i \Omega(\omega(i)) \) is the one of the posets product.
$T_\omega$ is equipped with a ramified cellular map $p : T_\omega \to \prod_i \mathbb{R}^{\omega(i)-1}_{\text{coef}, i}$ whose fibers are of the cardinality $|\omega|/2$ at most. The image $E_\omega := p(E_{\omega,k})$ is a product of cells $\{E_{\tilde{\omega}_i}\}_{i}$. The cellular structure $\{E_{\tilde{\omega}_i}\}_{\tilde{\omega}_i \in \Omega(\omega(i))}$ in each space $\mathbb{R}^{\omega(i)-1}_{\text{coef}, i}$ is described in Theorem 4.1 and Lemma 4.3 in terms of the poset $(\Omega(\omega(i)), \succ)$.

**Figure 5.** The Swallow Tail singularity poset $\Omega_{\omega}$ with all possible markers $k$, denoted here by the symbol “△”. The dimensions of the strata are shown on the right. The diagram indicates that the 3-complex $T_{(4)}$ is assembled from three 3-cells, six 2-cells, four 1-cells and one 0-cell.

**Figure 6.** The poset $\Omega_{\omega}$ with the downwards evolution “$\rightsquigarrow$” of a particular marker $\triangle$ attached to $\omega = 1111$. The diagram indicates that the marked 3-cell $e_{1,1,1,1,\triangle}$ in $T_{(4)}$ shares its boundary with three marked 2-cells: $e_{2,1,1,1,\triangle}$, $e_{1,1,2,1,\triangle}$, $e_{1,1,2,1,\triangle}$, and that each of these is attached to two marked 1-cells, and so on...
depicted in Fig. 4. Also glance at Fig. 6 which gives an example of such evolution.

Thus a pair \((\hat{\gamma}, \Delta)\), where \(\hat{\gamma} \in \mathcal{T}(v)\), can be viewed as a point of \(\mathcal{T}(v)\). Here we distinguish between the combinatorial marker \(k\) and its geometrical realization \(\Delta \in \partial_1 X \cap \hat{\gamma}\).

In the vicinity of a given trajectory \(\gamma \subset X\), each trajectory \(\hat{\gamma} \subset \hat{X}\) is determined by the intersection point \(\hat{\gamma} \cap S\), where \(S\) denotes a local section of the \(\hat{v}\)-flow through a point on \(\gamma\) which resides slightly below the first intersection \(a_1 \in \gamma \cap \partial_1 X\). By Theorems [1.1] and [1.2], the combinatorial types \(\hat{\omega} \in \prod_i \Omega(\omega(i))\), or rather their \(\kappa\)-images, label cells \(E_{\omega}\) of codimension \(|\kappa(\hat{\omega})|\) in \(S\). To define the corresponding cell decomposition of \(\mathcal{T}(v)\) in the vicinity of \(\gamma\), we need multiple copies \(\{E_{\omega,k}\}_k\) of each cell \(E_{\omega}\), the copies that are indexed by all admissible markers \(k\). Their number is the cardinality of the set \(\Upsilon(\kappa(\hat{\omega}))\). For each \(\hat{\gamma}\), there is a canonical correspondence between the markers \(k \in \Upsilon(\kappa(\hat{\omega}))\) and the connected components \(\hat{\tau} \in \pi_0(\hat{\gamma} \cap X)\). So we will use elements \(k \in \Upsilon(\kappa(\hat{\omega}))\) and \(\hat{\omega} \in \prod_i \Omega(\omega(i))\) to label all the cells in \(\mathcal{T}(v)\).

We are going to show that, for any \(\hat{\omega} \succ \hat{\omega}'\) in \(\prod_i \Omega(\omega(i))\), a \(k\)-marked cell \(E_{\omega,k}\) is incident to a \(k'\)-marked cell \(E_{\omega',k'}\), where \(k \sim k'\) if and only if \(E_{\omega}\) is incident to \(E_{\omega'}\) in \(S\).

In the vicinity of \(\gamma\), let us employ the special coordinates \((u, x, y)\) as in Lemma [1.2]. Consider a smooth path \(\{w(t)\}_{0 \leq t < 1}\) in the pure stratum \(S(\hat{\omega}) \subset S\) so that \(\lim_{t \to 1} w(t) = w_1 \in S(\hat{\omega}')\). The corresponding trajectories \(\gamma_{w(t)} \subset \hat{X}\) converge to a trajectory \(\gamma_{w_1}\) in such a way that the sets \(\gamma_{w(t)} \cap \partial_1 X\) go through merges of adjacent points or/and insertions of even multiplicity points. Examining the evolution of each element \(\hat{\tau}_t \in \pi_0(\gamma_{w(t)} \cap X)\) into an element \(\hat{\tau}' \in \pi_0(\gamma_{w_1} \cap X)\), we see that it is captured by a marker collapse \(\Delta \sim \Delta'\) as depicted in Fig. 4. Also glance at Fig. 6 which gives an example of such evolution.

With the rigid rules for the collapses \(\tau \sim \tau'\) from Fig. 4 in place, we can reconstruct the germ \(T_\gamma \approx \mathcal{T}_\omega \times \mathbb{R}^n - \{\omega'\}\). Let us describe a more formal construction of \(T_\gamma \approx \mathcal{T}_\omega \times \mathbb{R}^n - \{\omega'\}\).

For \(\hat{\omega} = \{\hat{\omega}_i \in \Omega(\omega(i))\}_i\), let \(e_{\hat{\omega}} = \prod_i e_{\hat{\omega}_i}\), where the hypersurface
\begin{equation}
(5.6) e_{\hat{\omega}_i} \subset \text{Sym}^{|\sup(\hat{\omega}_i)|}(\mathbb{R}) \times \text{Sym}^{\frac{|\omega(i) - \hat{\omega}_i|}{2}}(\mathbb{R})
\end{equation}
consists of divisors \((D'_i, D''_i)\) such that \(D'_i + D''_i + \tau(D''_i)\) is \(i\)-balanced.

Then \(T_\omega\) is the quotient of the space
\begin{equation}
(5.7) Z_\omega := \prod_{\hat{\omega} \in \prod_i \Omega(\omega(i))} (e_{\hat{\omega}}, k)
\end{equation}
by the equivalence relation "\((z, k) \sim (z', k')\)" that is defined as follows:
• $\hat{\omega} \succ \hat{\omega}'$ in the poset $\prod_i (\Omega(\omega(i)), \succ)$,
• $z \in \partial e_{\hat{\omega}}$, $z' \in e_{\hat{\omega}}^0$ are such that $\Psi_{\hat{\omega}}(z) = \Psi_{\hat{\omega}'}(z')$ in the space $\prod_i \mathbb{R}^{\omega(i)-1}$, where the
  $\Psi$-maps are defined by (5.3),
• $k \sim k'$.

Each closed cell $E_{\hat{\omega},k} \subset T_\omega$ is defined as the equivalence class of the set $(e_{\hat{\omega}}, k)$ in the quotient space $Z_\omega/\sim$. Its interior is homeomorphic to an open ball $e_{\hat{\omega}}^0$ of dimension $|\omega'| - |\hat{\omega}'|$. As a result, for a traversally generic $v$-flow, we have precise instructions for assembling the germ of a cell complex $T_\gamma$ associated with each trajectory $\gamma \in T(v)$, or rather, with its combinatorial type $\omega$.

In the vicinity of $\gamma$, the projection $p : T(v) \to S$ is evidently a finitely-ramified cellular map with respect to the cellular structures $\{E_{\hat{\omega},k}\}_{\hat{\omega},k}$ and $\{E_{\hat{\omega}}\}_\omega$.

Note that not every cell $E_{\hat{\omega}} \subset S$ is the $p$-image of some cell from $T(v)$: the cells in $S$ that are pierced by the trajectories $\hat{\gamma} \subset \hat{X}$ with the property $\hat{\gamma} \cap X = \emptyset$ are not in $p(T(v))$. However, each $E_{\hat{\omega}}$ with $\sup(\hat{\omega}) \neq \emptyset$ belongs to $p(T(v))$.

For each $\hat{\omega}$, the $p$-fiber over the pure stratum $S(\kappa(\hat{\omega}))^\circ \subset S$ is of the cardinality $\#[\Omega(\kappa(\hat{\omega}))]$. We would like to get an upper bound in $\#[\Omega(\kappa(\hat{\omega}))]$ in terms of $\omega$. In fact, $m(\gamma)/2 = |\omega|/2$ is such an upper bound. Indeed, the divisor $D_\gamma$ can be resolved in a divisor with $m(\gamma)$ simple roots. Recall that $m(\gamma)$ must be even, and, by Theorem 3.3 and Theorem 3.5 from [K2], any potential resolution of $D_\gamma$ (with the combinatorics described by elements of $\kappa(\prod_i \Omega(\omega(i)))$) is realized in the vicinity of $\gamma$. Therefore $m(\gamma)/2$ shaded intervals do occur in the vicinity of $\gamma$. In fact, for traversally generic $v$, the number $m(\gamma)/2 \leq \dim(X)$ is the maximal possible cardinality of the $p$-fibers in the vicinity of $\gamma$.

Next we turn to describing the positive cone structure of $T(v)$ at $\gamma$, which is consistent with its cellular structure $\{E_{\hat{\omega},k}\}_{\hat{\omega},k}$. In order to construct a base $\sigma_{\hat{\omega}}$ for the cone structure in each cell $e_{\hat{\omega}} = \prod_i e_{\hat{\omega}_i}$ (see [4.5] and (5.3)), consider a set of disjoint closed 2-balls $\{B(i) \subset \mathbb{C}\}_i$ of radius $1/3$ and with the center at $(i,0) \in \mathbb{C}$. We denote by $\sigma_{\hat{\omega}} \subset e_{\hat{\omega}}$ the set of complex conjugation-invariant divisor pairs $(D', D'') = \oplus_i (D'_i, D''_i)$, where the divisor $D'_i + D''_i + \tau(D''_i)$ of degree $\omega(i)$ is $i$-balanced and the sup$(D'_i + D''_i) \subset B(i)$; furthermore, at least for one $i$, sup$(D'_i + D''_i)$ is not contained in the interior of $B(i)$. The group $\mathbb{R}_+^*$ acts semi-freely on $e_{\hat{\omega}} = \prod_i e_{\hat{\omega}_i}$ by the diagonal action that applies a $t$-dilatation centered on the point $(i,0) \in \mathbb{C}$ to each pair $(D'_i, D''_i) \in e_{\hat{\omega}_i}$. Evidently, for each pair $(D', D'')$ there is a single $t \in \mathbb{R}_+^*$ so that $t(D', D'') \in \sigma_{\hat{\omega}}$. This $\mathbb{R}_+^*$-action extends to the space $Z_{\hat{\omega}}$ in (5.3), the action on the markings being trivial.

By Theorem [4.1] and with the help of the Viète map $V$, a similar semi-free action is available on the space $\prod_i \mathbb{R}^{\omega(i)-1}_{\text{coef}, i}$. Since the maps $\prod_i \Phi_i : \hat{\omega}_i$ are $\mathbb{R}_+^*$-equivariant, the quotient space $T_\omega = Z_{\hat{\omega}}/\sim$ inherits a semi-free $\mathbb{R}_+^*$-action for which $\gamma$ is the only fixed point.

Let $S_{\hat{\omega},k}$ be the image of $\sigma_{\hat{\omega}} \times k$ under the obvious map $Z_{\hat{\omega}} \to T_\omega$. Any nontrivial $\mathbb{R}_+^*$-trajectory meets $S_{\hat{\omega},k}$ at a singleton. Therefore $E_{\hat{\omega},k}$ is a positive cone over $S_{\hat{\omega},k}$. By their constructions, these cone structures in the individual cells $E_{\hat{\omega},k}$ are well-correlated and
produce a positive cone structure in \( T_\omega \). As a result, \( \{S_{\tilde{\omega},k}\}_{\tilde{\omega},k} \) define a cellular structure in the link of \( \gamma \) in \( T_\gamma = T_\omega \times \mathbb{R}^{n-|\omega|'} \).

Here is a short summary of what we have established in this paper: a traversally generic \( v \)-flow generates a stratification \( \{T(v, \tilde{\omega})\}_{\tilde{\omega}\in\Omega_{\langle n \rangle}} \) of the trajectory space \( T(v) \), which is consistent (in a subtle way!) with, but cruder than, the \( \gamma \)-local cellular structure

\[
\{E_{\tilde{\omega},k} \times \mathbb{R}^{n-|\omega|'}\}_{\tilde{\omega}\in\prod_i \Omega_{\langle \omega(i) \rangle}, \ k\in\Upsilon(\kappa(\hat{\omega}))}
\]

in the model space \( T_\gamma \approx T_\omega \times \mathbb{R}^{n-|\omega|'} \) that we just have described. Each cell \( E_{\tilde{\omega},k} \times \mathbb{R}^{n-|\omega|'} \) belongs to the stratum \( T(v, \tilde{\omega}) \), where \( \tilde{\omega} := \Xi(\kappa(\hat{\omega})) \in \Omega_{\langle n \rangle} \). In the process, distinct cells could acquire the same label \( \tilde{\omega} \in \Omega_{\langle n \rangle} \).

This understanding of the cellular structure of the spaces of real polynomials of the degree \( 2n+2 \)—the structure which reflect the universal posets \( \Omega_{\langle 2n+2 \rangle} \) and \( \Omega_{\langle n \rangle} \)—will enable us, in the papers to follow, to define new rich characteristic classes of traversally generic flows on \((n+1)\)-manifolds with boundary.

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