Robust OFDM integrated radar and communications waveform design

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Abstract—An integrated radar and communications system (IRCS) is considered where the radar target response and communications channel response are generally frequency selective but the corresponding frequency responses functions are not exactly known. In particular, these frequency responses are only known to lie in an uncertainty class. To ensure the IRCS simultaneously provides acceptable target classification performance and communications rate, a robust orthogonal frequency division multiplexing (OFDM) integrated radar and communications waveform design method is proposed. The approach finds a waveform that provides a sufficiently large weighted sum of the communications data information rate (DIR) and the conditional mutual information (MI) between the observed signal and the radar target over the entire uncertainty class. First, the conditional MI and DIR based on the integrated OFDM waveform are derived. Then, a robust OFDM integrated radar and communications waveform optimization problem based on the minimax design philosophy is developed such that closed-form solution is derived. Finally, several numerical results are presented to demonstrate the effectiveness of the proposed method.

Index Terms—Robust waveform design, integrated radar and communications, orthogonal frequency division multiplexing, conditional mutual information, data information rate

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I INTRODUCTION

With the increased acceptance of communications and radar systems for both commercial and defense applications, the study of integrated systems has attracted significant attention in the signal processing community [1]-[8]. These systems have the advantages in reducing the hardware cost and improving the spectrum usage. For example, [9] describes how radar, communications, and electronic warfare functionality can be integrated into the same platform with array antennas, signal processing, and display hardware shared. The work in [9] uses the advanced multifunction radio frequency concept (AMRFC) in order to decrease the system size, weight, and electromagnetic interference while performing multiple functions. However, different functions are carried out by using different waveforms, independently. One important application involves intelligent transportation systems (ITS) [10]-[12]. The research in [10]-[12] investigates the integration of radar and communications in an intelligent vehicle system, where the radar can sense collisions and traffic while the communications device connects the car to other cars and information sources or collection points. Some different levels of system integration for such systems is surveyed in [4].

The integration of radar and communications hardware is a topic of great interest [13], [14]. It seems crucial to explore the best approaches to the integration of radar and communications functions as suggested in [13], [14]. The promising approaches that have already been suggested can be classified into two major categories: multiplexing-waveform and identical-waveform. The multiplexing-waveform approaches employ multiplexing techniques, such as space division multiplexing (SDM) [5]-[8], time division multiplexing (TDM) [10]-[12], [15], frequency division multiplexing (FDM) [16]-[18], and code division multiplexing (CDM) [19]-[21]. This approach allows one to easily separate the communications and radar signals so they will not interfere with each other.
The identical-waveform approach picks a single waveform which may be similar to the traditional radar waveforms [22]-[26] or the conventional communications waveforms [27], [28]. A popular approach uses an orthogonal frequency division multiplexing (OFDM) waveform [29]-[41], a waveform that has been widely applied in communications [42]-[44] and recently suggested for radar applications [45]-[59]. In fact, OFDM waveforms have been proposed for car-to-car (C2C) and car-to-infrastructure (C2I) communication [60]-[65] which is an important application for integrated radar and communications systems. OFDM waveforms are employed in this paper.

To effectively allocate the limited total power in an integrated radar and communications system (IRCS), many design criteria have been proposed. The research in [66] splits the total power between the data symbols that perform the information transmission and the training symbols which accomplish the radar function. The study in [67] and [68] splits the total bandwidth into two subbands, one for communications only and the other for both radar and communications. By employing different allocations of the total power between these two subbands, the joint radar and communications system performance with respect to the data information rate (DIR) and estimation rate is explored. To simultaneously improve the radar target parameter estimation performance and the communications channel capacity, optimal assignment of the subcarrier power profile for an integrated OFDM waveform is proposed in [69] by using a multiobjective design criteria. Similarly, to improve the detection performance and channel capacity of the IRCS, an adaptive transmit power allocation for the subcarriers of the integrated OFDM waveform is proposed in [70]. In fact, there is not a uniform criterion to guide integrated waveform design.

In this paper, our focus is on a radar used for target classification or estimation. Based on [71], it seems more appropriate to design the integrated waveform based on information theory as suggested in
Information theory provides much of the foundation of communications theory [72], and various communications waveform design methods are proposed to maximize the DIR by properly assigning the total power according to the channel state information (CSI) [73]-[76]. More recently, the maximization of the conditional MI between the observed signal and the radar target return has emerged as a popular criterion for radar waveform design [56], [71], [77], [78]. For example, the pioneering work in [71] proposed a design criterion for radar waveform design, which maximizes the conditional MI between a random extended target and the received signal echoes. Hence, the research in [79] investigates the conditional mutual information (MI) between the target impulse response and target reflected returns for radar performance while considering channel capacity for communications performance, but waveform optimization is not considered. Based on information theory, the study in [80] explores the adaptive power allocation over the subcarriers of an OFDM integrated radar and communications waveform. However, the proposed method needs to know the precise frequency responses of the extended target and communications channel, which is difficult to attain in practice. In addition, errors in the assumed frequency responses can considerably deteriorate the performance of the IRCS. Hence, it is imperative to explore robust integrated waveform design method for cases where the frequency responses of the target and communications channel may not be known exactly.

In fact, robust waveform design approaches have been proposed in both radar [81]-[85] and communications [86]-[90] applications. However, robust integrated waveform design is still an open problem. In this paper, we employ the minimax robust design criterion from [87] to devise a robust integrated radar and communications waveform. We assume that the target and communications channel frequency responses are not exactly known, but they lie some uncertainty classes with known upper and lower bounds. A robust OFDM integrated radar and communications waveform is developed.
according to the principle of minimax design that optimizes the worst-case performance of the objective function. The approach devises a waveform that ensures acceptable communications DIR and conditional MI between the observed signal and the radar target return regardless of the actual target and channel impulse responses.

The rest of this paper is organized as follows. In Section II, the radar and communications metrics are formulated. In Section III, the minimax robust waveform design method is proposed. In Section IV, several numerical simulations are presented. Finally, conclusions are drawn in Section V.

II PROBLEM DESCRIPTION AND MODELING

In this section, we first present the integrated OFDM signal model of the IRCS. Then the conditional MI and DIR of the integrated radar and communications waveform are derived.

A. Integrated Signal Model

The integrated radar and communications waveform employed in this paper is shown in Fig. 1(a). It is a combination of the OFDM radar waveform and the communications waveform depicted in Fig. 1(b) and Fig. 1(c), respectively. In general, each pulse of the traditional OFDM radar waveform consists of one OFDM symbol without carrying communications information and a cyclic prefix (CP) or guard interval (GI), while the conventional continuous OFDM communications waveform includes several OFDM symbols with GI and communications information.

Fig. 1 The transmitted OFDM waveform in (a) OFDM IRCS, (b) OFDM radar, and (c) OFDM communications system.
Following the integrated radar and communications waveforms shown in Fig. 1(a), the transmitted pulse including $N_c$ consecutive OFDM symbols can be formulated as

$$s(t) = e^{j2\pi f_c t} \sum_{n=0}^{N_c-1} a_n c_{m,n} e^{j2\pi \Delta f \left(\frac{t-nT_s}{T_s}\right)} \text{rect}\left(\frac{t-nT_s}{T_s}\right)$$

(1)

where $f_c$ is carrier frequency, $N_c$ is the number of subcarriers, $\Delta f$ is the subcarrier interval and it satisfies $\Delta f = 1/T$, where $T$ is the elementary length of OFDM symbol, and the duration of each completed OFDM symbol is $T$, which is equal to $T + T_g$ where $T_g$ is the duration of GI. For simplicity, the OFDM symbol will refers to the completed OFDM symbol in the following unless otherwise mentioned. Moreover, $a_n$ is the transmitted weight over the $m$-th subcarrier, $c_{m,n}$ is the transmitted communications code of the $m$-th subcarrier and $n$-th OFDM symbol to transfer the communications information, and $\text{rect}\left(\frac{t}{T_s}\right)$ is the rectangle function, which is equal to one for $0 \leq t \leq T$, and zeros, otherwise.

**B. Conditional Mutual Information**

For radar target identification and classification, the target impulse response is normally employed. Hence, it is imperative for the radar to obtain a precise estimation of the target impulse response. This can be guaranteed if there is a sufficiently large conditional MI between the observed signal and the target impulse response [70], called conditional MI in this paper for short. In fact, under some conditions [77] minimizing the mean square error (MMSE) in estimating the target impulse response can be equivalent to maximizing the conditional MI. In this section, this conditional MI that the IRCS can obtain with the integrated OFDM waveform is derived.

Assume that the impulse response of the propagation channel from transmitter to target and target to receiver is $h(t)$, called the radar channel, and that an extended target is illuminated which has an impulse response $g(t)$ which is a Gaussian random process. Therefore, due to the transmitted signal
The received signal of the IRCS is described as

$$y(t) = s(t) * h_t(t) * g(t) + n(t)$$  \( (2) \)

where $*$ denotes the convolution operator and $n(t)$ is complex additive white Gaussian noise (AWGN) with zero mean and power spectral density $N(f)$.

Suppose that for any frequency $f$ in $\Delta_n = [f_m, f_{m+1}]$, $S(f) \approx S(f_m)$, $H_r(f) \approx H_r(f_m)$, $G(f) \approx G(f_m)$, $N(f) \approx N(f_m)$, where $S(f)$, $H_r(f)$ and $G(f)$ are the Fourier transforms of $s(t)$, $h_t(t)$ and $g(t)$, respectively, and $f_m = f_c + m \Delta f$ is the $m$-th subcarrier frequency.

Following the guideline in [70], the conditional MI between the target impulse response and the received echoes can be formulated as

$$I(y(t); g(t)|s(t), h_t(t)) = \frac{\Delta T_p}{2} \sum_{m=0}^{N-1} \log_2 \left( 1 + \frac{|S(f_m)|^2 |H_r(f_m)|^2 |G(f_m)|^2}{N(f_m) T_p} \right)$$  \( (3) \)

where $T_p = N T_s$ is the pulse duration.

To evaluate $I(y(t); g(t)|s(t), h_t(t))$, the $|S(f_m)|^2$ must be calculated. First, the Fourier transform of $s(t)$ is expressed as

$$S(f) = T_s \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} a_m c_n e^{j \pi n T_s} s_n \left( \pi (f - f_m) T_s \right) e^{-j 2 \pi (f - f_m) T_s}$$  \( (4) \)

where $s_n(t) = \sin(\pi t / T)$.

Hence, $U(f) = |S(f)|^2$ can be described as

$$U(f) = T_s^2 \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} a_m a_m^* c_n c_{n'}^* e^{-j 2 \pi (f - f_m + f_{m'}) T_s} e^{-j 2 \pi (n - n') T_s} s_n \left( \pi (f - f_m) T_s \right) s_{n'} \left( \pi (f - f_{m'}) T_s \right)$$  \( (5) \)

where $(\cdot)^*$ denotes the complex conjugation.

If the communications symbol $c_{m,n}$ is normalized to have unit magnitude and is random with zero mean for any fixed $m$ and $n$ while any code symbols for distinct $(m,n)$ are statistically independent then

$$E[c_{m,n} c_{m',n'}^*] = \begin{cases} 1, & m = m', n = n' \\ 0, & \text{else} \end{cases}$$  \( (6) \)

where $E[\cdot]$ indicates the expectation operator.
In practice, phase shift keying modulation is widely used in communications, and (6) can be attained through precoding [72], although the communications symbol \( c_{m,n} \) is determined by the conveyed information. Using (6), the following can be obtained

\[
E[U(f)] = T_s^2 N_c \sum_{n=0}^{N_c-1} |a_n|^2 \left( \frac{s_n}{2} \right)^2
\]  

(7)

If the number of subcarriers \( N_c \) is sufficient large, \( U(f) \) will approach \( E[U(f)] \). In practice, the number of subcarriers can be much larger than one hundred, and the following reasonable approximation can be achieved

\[
U(f) \approx T_s^2 N_c \sum_{n=0}^{N_c-1} |a_n|^2 \left( \frac{s_n}{2} \right)^2
\]  

(8)

![Fig. 2](image)

**Fig. 2** The variation of relative error with the number of subcarriers.

To illustrate the effectiveness of the approximation, the dependence of relative error between \( U(f) \) and \( E[U(f)] \) on the number of subcarriers is depicted in Fig. 2. The relative error is defined as

\[
Re = \frac{\int_{-\infty}^{\infty} |U(f) - E[U(f)]| df}{\int_{-\infty}^{\infty} U(f) df}
\]  

(9)

In Fig. 2, the number of OFDM symbols is 4, and the subcarrier interval is 0.25MHz and a standard Monte Carlo technique with 1000 independent trials is employed. Obviously, the relative error is initially decreasing with the increase of the number of subcarrier until it reaches to a small value which is less than 4%. The relative error is less than 3%, when the number of subcarriers is larger than 100, which is typical in practice.
At the frequency \( f = f_n \), (8) can also be rewritten as

\[
U(f_n) \approx \frac{N}{N-1} \sum_{m=0}^{N-1} |a_m|^2 \left[ s_m \left( \pi (f_m - f_n) T_s \right) \right]^2
\]

\[
= \frac{T_s^2 N_s}{N} \left| a_m \right|^2 + \frac{T_s^2 N_s}{N} \sum_{m=0, m \neq m'}^{N-1} \left| a_m \right|^2 \left[ s_m \left( \pi (f_m - f_n) T_s \right) \right]^2
\]

(10)

If the length of GI \( T_g \) is zero, \( s_m \left( \pi (f_m - f_n) T_s \right) \) will be zero, for \( m' \neq m \). However, in practice, the length of GI \( T_g \) must larger than the maximum time delay of communications channel. The typical value of \( T_g \) is \( T/4 \), \( T/8 \), or \( T/16 \). Hence, \( s_m \left( \pi (f_m - f_n) T_s \right) \) is not zero, but its value is comparable with the sidelobes of \( s_m(t) \), which is far less than one. Thus, the following approximation is reasonable

\[
\frac{T_g^2 N_s}{N} \sum_{m=0, m \neq m'}^{N-1} \left| a_m \right|^2 \left[ s_m \left( \pi (f_m - f_n) T_s \right) \right]^2 \approx 0
\]

(11)

From (10) the following expression can be achieved

\[
U(f_n) \approx \frac{T_s^2 N_s}{N} \left| a_m \right|^2
\]

(12)

To demonstrate the effectiveness of the approximation in (12), the relative approximation error of \( U(f_n) \) is shown in Fig. 3. The relative approximation error is defined as

\[
Rae = \frac{\sum_{m=0}^{N-1} \left| U(f_m) - \frac{T_s^2 N_s}{N} \left| a_m \right|^2 \right|}{\sum_{m=0}^{N-1} U(f_m)}
\]

(13)

![Graph](image)

**Fig. 3** The dependence of relative approximation error on the number of subcarriers.

In Fig. 3, the simulation settings are same as that in Fig. 2. As expected, the relative approximation error is decreasing with an increase in the number of subcarriers and the relative approximation is less
than 4%, when the number of subcarriers is greater than 100. In practice, the number of subcarriers is usually larger than 100. Hence, the approximation in (12) is reasonable.

Substituting (12) into (3) yields

\[
I(y(t); g(t) | s(t), h(t)) = \frac{1}{2} \Delta f T_p \sum_{n=0}^{N-1} \log_2 \left( 1 + \frac{T_s^2 N_s |p_n|^2 |G(f_n)|^2 |H_r(f_n)|^2}{N(f_n) T_p} \right)
\]

\[
= \frac{1}{2} \Delta f T_p \sum_{n=0}^{N-1} \log_2 (1+p_n \nu_m)
\]

where \( p_n = |p_n|^2 \), and \( \nu_m = T_r^2 N_s |G(f_n)|^2 |H_r(f_n)|^2 / N(f_n) T_p \) is the channel to noise ratio (CNR) for the \( m \)-th subchannel. For simplicity, the conditional MI is referred to as MI in the rest of this paper.

C. Data Information Rate

In the IRCS, both the radar and communications performance is employed to assess the overall utility. Since the DIR is an accepted metric to evaluate the communications performance, the DIR of the IRCS will be calculated in this subsection.

\[h(f)\]

\[h_1, h_2, \ldots\]

\[B = N_s \Delta f\]

Fig. 4 The frequency response of communications channel.

Without loss of generality, we suppose that the communications channel exhibits slowly varying frequency selective fading. The frequency response of the frequency selective communications channel \( h(f) \) is shown in Fig. 4 for a given time. Following the guidelines in [72], the combined DIR from all subchannels can be formulated as

\[C_r = \sum_{n=0}^{N-1} \Delta f \log_2 \left( 1 + |p_n|^2 |\sigma_n|^2 / \sigma_n^2 \right)\]

\[= \sum_{n=0}^{N-1} \Delta f \log_2 (1 + p_n \sigma_n)\]
where \( h_m = h(f_m) \) represents the channel frequency response of the \( m \)-th subchannel, \( \sigma_c^2 \) is the noise power in the communications channel, \( p_m \) effectively indicates the transmit power of the \( m \)-th subchannel, and \( \sigma_m = \|h_m\|^2/\sigma_c^2 \) can be regarded as the CNR for the \( m \)-th communications subchannel. From (15), we can see that the higher the CNR is (i.e., the better the communications subchannel is), the larger the DIR.

### III Minimax Robust Waveform Design

In this section, we focus on the minimax robust waveform design for the IRCS to optimally allocate the limited transmit power. First, two individual optimal waveform design criteria, i.e., optimal radar waveform design and optimal communications waveform design under known CNR, are discussed.

#### A. Radar Waveform and Communications Waveform Design

1) Optimal radar waveform for known CNR

In order to improve the performance of radar for target identification and classification, it is imperative to maximize the MI. Hence, with a constraint on total power, the optimization problem can be formulated as

\[
\mathbf{p}_r = \arg \max_{\mathbf{p} \in \mathbb{R}^N} \log \left( 1 + \frac{\Delta f_p \sum_{n=0}^{N-1} \log_2 (1 + p_m \nu_n) }{ \sigma_c^2 } \right)
\]

subject to \( \mathbf{1}_N^T \mathbf{p} \leq 1 \), \( p_m \geq 0 \), \( m = 0, 1, \cdots, N_c - 1 \)

where \( \mathbf{1}_N \) indicates an \( N_c \times 1 \) vector of ones, \( \mathbf{p} = [p_0 \; p_1 \; \cdots \; p_{N_c-1}]^T \) is an \( N_c \times 1 \) vector containing the subcarrier transmit powers, and \( \mathbf{p}_r = [p_{r,0} \; p_{r,1} \; \cdots \; p_{r,N_c-1}]^T \) is the optimal power allocation for the optimal radar waveform.

The optimization problem is convex, since the object function is concave, and the inequality constraint is convex [91]. The optimal solution can be achieved by utilizing a convex optimization
toolbox, such as SeDuMi [92] or cvx [93]. In addition, one can obtain a closed-form expression for the solution by using the Karush-Kuhn-Tucker (KKT) conditions [91], and the following optimal solution can be achieved:

\[ p_{c,m} = \left[ \lambda_c - 1/\sigma_m \right]^+ , m = 0,1,\ldots,N_c - 1 \]  

(17)

where \( \left[ x \right]^+ = \max \{ x, 0 \} \), \( \lambda_c \) is the water-level, which satisfies that

\[ \sum_{m=0}^{N_c-1} \left[ \lambda_c - 1/\sigma_m \right]^+ = 1 \]  

(18)

Utilizing the algorithm in [76], the optimal solution in (17) can be specified, and it indicates that more transmit power will be assigned to the subchannels with larger CNR, given sufficient power is available.

2) Optimal communications waveform for known CNR

For communications, the DIR is an important evaluation criterion and the DIR can be improved through reasonably assignment of the limited transmit power. Under the total transmit power constraint, the optimization problem can be formulated as follows:

\[
p_c = \arg \max_{p \in \mathbb{R}^N} \sum_{n=0}^{N_c-1} \Delta f \log_2 \left( 1 + p_n \sigma_n \right)
\]

subject to \( \mathbf{1}^T_p \leq 1, \quad p_n \geq 0, m = 0,1,\ldots,N_c - 1 \)

(19)

where \( p_c = \left[ p_{c,0} \quad p_{c,1} \quad \cdots \quad p_{c,N_c-1} \right]^T \) is an \( N_c \times 1 \) vector.

The optimization problem in (19) is also convex, since the object function is concave, and the inequality constraint is convex. The following optimal solution can be achieved by using the KKT condition

\[ p_{c,m} = \left[ \lambda_c - 1/\sigma_m \right]^+ , m = 0,1,\ldots,N_c - 1 \]  

(20)

where \( \lambda_c \) is the water-level, which satisfies

\[ \sum_{m=0}^{N_c-1} \left[ \lambda_c - 1/\sigma_m \right]^+ = 1 \]  

(21)
The optimal solution in (20) can be determined by using the algorithm in [76]. It shows that when sufficient power is available, the greater the CNR is in a given subchannel, the more power will be assigned to that subchannel, which is consistent with the optimal solution in (17) for the optimal radar waveform design.

B. Robust Integrated Waveform Design

The individual optimal radar and communications waveforms design have been discussed. For the IRCS, both the radar and communications performance deserve to be considered. In [80], we have proposed the optimal integrated radar and communications waveform design method. However, the devised optimal waveforms are designed for their specific CNRs which are assumed known. Under such designs the performance may degrade seriously from predicted performance for different CNRs. Furthermore, in practice, it is difficult to know the precise frequency responses of the extended target and communications channels and they may change over time. Hence, it is more reasonable to suppose that the frequency responses of the overall radar channel/target and the communications channel are not known exactly, but they lie in the uncertainty classes

$$\Xi_{gh} = \left\{ \rho_{gh} : 0 < l_{gh,m} \leq \rho_{gh} \left( f_m \right) \leq l_{gh,m} \leq u_{gh,m}, \forall m = 0,1,\cdots,N_c - 1 \right\}$$

$$\Xi_h = \left\{ \rho_h : 0 < l_{h,m} \leq \rho_h \left( f_m \right) \leq l_{h,m} \leq u_{h,m}, \forall m = 0,1,\cdots,N_c - 1 \right\}$$

for given $l_{gh,m}$, $u_{gh,m}$, $l_{h,m}$, $u_{h,m}$, $\forall m = 0,1,\cdots,N_c - 1$ which might be determined by field measurement or propagation modeling by considering the best and worst cases. In this paper, we suppose that $l_{gh,m}$, $u_{gh,m}$, $l_{h,m}$, $u_{h,m}$, $\forall m = 0,1,\cdots,N_c - 1$, called the upper and lower bounds, are known.

According to the minimax robust waveform design criterion, we wish to solve
\[
\max_{p \in \mathcal{R}_N^c} \left\{ \min_{\rho \in \mathcal{R}_N^c} I_{MD} \left( p, \rho \right) \right\}
\]  
(24)

where \( \mathcal{R}_N^c \) represents the space \( \mathcal{R}_N^c = \{ x \in \mathcal{R}_N^c : x_m \geq 0, \forall m = 0, 1, \ldots, N-1 \} \), and

\[
I_{MD} \left( p, \rho \right) = \frac{W}{2F_r} \Delta f \sum_{m=0}^{N-1} \log_2 \left( 1 + p \nu_m \left( \rho \left( f_m \right) \right) \right) + \frac{W}{F_c} \sum_{m=0}^{N-1} \Delta f \log_2 \left( 1 + p \sigma_m \left( \rho \left( f_m \right) \right) \right)
\]  
(25)

is the joint performance criterion of the radar and communications system, in which \( w_i \) and \( w_c \) are weighting factors which satisfy \( w_i + w_c = 1 \), \( \nu_m \left( \rho \left( f_m \right) \right) = N_i T \rho \left( f_m \right) / \left( N \left( f_m \right) T \right) \), and \( \sigma_m \left( \rho \left( f_m \right) \right) = \rho \left( f_m \right) / \sigma_i^2 \). For simplicity, \( \nu_m \) and \( \sigma_m \) represent \( \nu_m \left( \rho \left( f_m \right) \right) \) and \( \sigma_m \left( \rho \left( f_m \right) \right) \), respectively. It is an interesting problem to choose the weighting factors, which is related to the relative importance of the radar and communications performance of the IRCS. \( F_r \) and \( F_c \) are the optimal values (i.e., maximum MI and maximum DIR) in (16) and (19) with the frequency responses of the overall radar channel/target and communications channel being the upper bounds. They are normalization factors such that the two performance criteria are approximately within the same range and are of similar magnitudes. The solution to (24) is called the robust waveform design.

To solve (24), we need to obtain the saddle point \([86], [87] \) \( p_s, \rho_{gh,s}, \rho_{hs} \) that satisfies

\[
I_{MD} \left( p, \rho_{gh,s}, \rho_{hs} \right) \left|_{U_{pcl}} \right. \leq I_{MD} \left( p_s, \rho_{gh,s}, \rho_{hs} \right) \left|_{U_{pcl}} \right. \leq I_{MD} \left( p, \rho_{gh,s}, \rho_{hs} \right) \left|_{U_{pcl}} \right.
\]  
(26)

Since \( I_{MD} \left( p, \rho_{gh}, \rho_{hs} \right) \) is monotonically increasing in \( \rho_{gh} \left( f_m \right) \) and \( \rho_{hs} \left( f_m \right) \), the minimum value of \( I_{MD} \left( p, \rho_{gh}, \rho_{hs} \right) \) for \( \rho_{gh} \in \Xi_{gh} \), \( \rho_{hs} \in \Xi_{hs} \) is \( I_{MD} \left( p, \mathbf{1}_{gh}, \mathbf{1}_{hs} \right) \), i.e.,

\[
\min_{\rho_{gh} \in \Xi_{gh}, \rho_{hs} \in \Xi_{hs}} I_{MD} \left( p, \rho_{gh}, \rho_{hs} \right) \left|_{U_{pcl}} \right. = I_{MD} \left( p, \mathbf{1}_{gh}, \mathbf{1}_{hs} \right) \left|_{U_{pcl}} \right.
\]  
(27)

where \( \mathbf{1}_{gh} = \left[ I_{gh,0} \ I_{gh,1} \ \cdots \ I_{gh,N-1} \right]^T \), and \( \mathbf{1}_{hs} = \left[ I_{hs,0} \ I_{hs,1} \ \cdots \ I_{hs,N-1} \right]^T \). Hence, the right most inequality in (26) is satisfied by this choice of \( \rho_{gh,s} = \mathbf{1}_{gh}, \rho_{hs,s} = \mathbf{1}_{hs} \) for any \( p = p_s \). The left most inequality in (26) requires the solution of \( \mathbf{p} = p_s \) of

\[
\max_{p \in \mathcal{R}_N^c} \left\{ I_{MD} \left( p, \mathbf{1}_{gh}, \mathbf{1}_{hs} \right) \left|_{U_{pcl}} \right. \right\}
\]  
(28)
\[ p_k = p_m = \arg \max_{p \in \mathbb{R}^n} \frac{w_{\gamma}}{2F} \sum_{n=0}^{N-1} \log_2 \left( 1 + p_n \nu_{i,n} \right) + \frac{w_{\gamma}}{F_c} \sum_{n=0}^{N-1} \log_2 \left( 1 + p_n \sigma_{r,n} \right) \]

subject to \( \mathbf{1}_N^T p \leq 1, \quad p_n \geq 0, \quad m = 0, 1, \cdots, N_c - 1 \)

where \( \nu_{i,n} = \nu_m \left( l_{i,n,m} \right) = N_r f_{i,m}^T \delta f / \left( N \left( f_m \right) f_p^T \right) \), and \( \sigma_{r,n} = \sigma_n \left( l_{i,n,m} \right) = \sigma_n / \sigma_c^2 \) are determined by the lower bounds. Thus these \( p_k, \rho_{BH,S}, \rho_{BS} \) satisfies the condition of saddle point conditions.

The objective function in (29) is concave, since it is the affine combination of two concave functions. In addition, the inequality constraint in (29) is convex. Hence, the optimization problem in (29) is convex [90], and it is solvable by using the KKT conditions [90]. Introduce the Lagrange multipliers \( \mu \) and \( \mu_m, \quad m = 0, 1, \cdots, N_c - 1 \), for the constraints in (29). Taking the gradient of the objective function in (29) with respect to \( p = [p_0 \quad p_1 \quad \cdots \quad p_{N_c-1}]^T \), setting each component to zero and adding the conditions on \( \mu \) and \( \mu_m \), for \( m = 0, 1, \cdots, N_c - 1 \), from the KKT conditions [90] provides the requirements on the solution to (29) as that \( p = [p_0 \quad p_1 \quad \cdots \quad p_{N_c-1}]^T, \mu \) and \( \mu_m \), for \( m = 0, 1, \cdots, N_c - 1 \) satisfying

\[
\mu - \mu_m = w_{\gamma} \nu_m \Delta f / \left[ 2 \ln 2 F \left( 1 + p_n \nu_{i,n} \right) \right] + w_{\gamma} \sigma_n \Delta f / \left[ \ln 2 F \left( 1 + p_n \sigma_{r,n} \right) \right], \quad m = 0, 1, \cdots, N_c - 1 \quad (30a)
\]

\[
\mu \left( \sum_{m=0}^{N-1} p_m - 1 \right) = 0 \quad (30b)
\]

\[
\mu_m p_m = 0, \quad m = 0, 1, \cdots, N_c - 1 \quad (30c)
\]

\[
\mu \geq 0, \quad \mu_m \geq 0, \quad m = 0, 1, \cdots, N_c - 1. \quad (30d)
\]

The solution to (30a)-(30d) is

\[
p_{c,m} = \frac{1}{2} \left[ \mu \left( \alpha' + \beta' \right) - \left( \nu_{r,m} + \sigma_{r,m} \right) + \sqrt{\left( \nu_{r,m} - \nu_{i,m} \right)^2 + \mu \left( \alpha' - \beta' \right) + 4 \mu^2 \alpha' \beta'} \right]
\]

where \( \alpha' = w_{\gamma} \Delta f / \left( 2 \ln 2 F \right), \quad \beta' = w_{\gamma} \Delta f / \left( \ln 2 F \right), \quad \nu_{r,m} = \nu_{i,m}, \quad \sigma_{r,m} = \sigma_{i,m}, \) and \( \mu' = 1/\mu \) is chosen to solve

\[
\sum_{m=0}^{N-1} p_{c,m} - 1 = 0 \quad (32)
\]
The positive Lagrange multiplier $\mu'$ can be obtained by a simple bisection search over the interval

$$0 < \mu' \leq \frac{1}{\min_{n} \left\{ \alpha'/\left(\nu_{m}^{n} + 1\right) + \beta'/\left(\sigma_{n}^{m} + 1\right) \right\}} \quad \text{for} \quad m = 0, 1, \cdots, N_c - 1,$$

where $\min_{n} \{x_n\}$, for $m = 0, 1, \cdots, N_c - 1$, indicates the minimum value in the set $\{x_{s, \cdots, s_{N_c}}\}$. Once $\mu'$ is obtained, the optimal power assignment will be determined using (31). See Appendix for the detailed derivations of the optimal solution.

C. Performance Analysis

In this section, the performance of the designed robust waveform will be analyzed. The following Theorems will be useful for this purpose. Let $|G(f_m)||H_i(f_m)|$ define the magnitude of the overall radar channel/target frequency response and $|H(f_m)|$ define the magnitude of the communications channel frequency response, as described in (22) and (23).

**THEOREM 1:** For a given power allocation, the best radar and communications performance will occur when the true frequency responses are exactly the upper bounds of the uncertainty classes.

**PROOF:** From (25), $I_{\text{MD}}(p, \rho_{\phi}, \rho_{s})$ is monotonically increasing in $\nu_n\left(\rho_{\phi}(f_m)\right)$ and $\sigma_n\left(\rho_{s}(f_m)\right)$, for $m = 0, 1, \cdots, N_c - 1$ when the power allocation $p$ is given. When $\rho_{\phi}(f_m)$ and $\rho_{s}(f_m)$ are exactly the upper bounds $u_{\phi,m}$, and $u_{s,m}$, for $m = 0, 1, \cdots, N_c - 1$, respectively, both $\nu_n\left(\rho_{\phi}(f_m)\right)$ and $\sigma_n\left(\rho_{s}(f_m)\right)$ will be maximum.

**THEOREM 2** For a given power allocation, the worst radar and communications performance will occur when the true frequency responses are exactly the lower bounds of the uncertainty classes.

**Proof** is similar to the Proof for Theorem 1.

**THEOREM 3:** Suppose that the power allocation is designed to maximize any weighted sum of the DIR and MI assuming some given overall radar channel/target frequency response and some communications channel frequency response from the uncertainty classes in (22) and (23). If the
overall radar channel/target frequency response and the communications channel frequency response both have their minimum values at the same subcarrier and the upper and lower bounds of the uncertainty classes in (22) and (23) are sufficiently separated with respect to the power available, then the worst power allocation puts all the power in this minimizing subcarrier:

**Proof** Without loss of generality, assume that $N_c = 2$, $\mathbf{p}_{\text{min}} = [P_1, 0]^T$, $\mathbf{p} = [p_1, p_2]^T$, $\rho_{\Phi}(f_1) \leq \rho_{\Phi}(f_2)$ and $\rho_{\chi}(f_1) \leq \rho_{\chi}(f_2)$, i.e., $\nu_1 \leq \nu_2$ and $\sigma_1 \leq \sigma_2$, where $p_1 + p_2 = P_1$. We first prove that $I_{\text{MD}}(\mathbf{p}_{\text{min}}, \rho_{\Phi}, \rho_{\chi}) \leq I_{\text{MD}}(\mathbf{p}, \rho_{\Phi}, \rho_{\chi})$.

From (25),

$$I_{\text{MD}}(\mathbf{p}_{\text{min}}, \rho_{\Phi}, \rho_{\chi}) = \alpha \log_2 (1+p_1 \nu_1) + \beta \log_2 (1+p_1 \sigma_1)$$

$$= \alpha \log_2 (1+p_1 \nu_1 + p_2 \nu_2) + \beta \log_2 (1+p_1 \sigma_1 + p_2 \sigma_2)$$

(33)

where $\alpha = \omega_1 \Delta f_T / (2 \nu_1)$ and $\beta = \omega_1 \Delta f / F_c$. Now consider a general power weighting

$$I_{\text{MD}}(\mathbf{p}, \rho_{\Phi}, \rho_{\chi}) = \alpha \left( \log_2 (1+p_1 \nu_1) + \log_2 (1+p_2 \nu_2) \right) + \beta \left( \log_2 (1+p_1 \sigma_1) + \log_2 (1+p_2 \sigma_2) \right)$$

$$= \alpha \log_2 (1+p_1 \nu_1) (1+p_2 \nu_2) + \beta \log_2 (1+p_1 \sigma_1) (1+p_2 \sigma_2)$$

$$= \alpha \log_2 (1+p_1 \nu_1 + p_2 \nu_2 + p_1 p_2 \nu_1 \nu_2) + \beta \log_2 (1+p_1 \sigma_1 + p_2 \sigma_2 + p_1 p_2 \sigma_1 \sigma_2)$$

(34)

Since $p_1, p_2, \nu_1, \nu_2 \geq 0$ and $p_1, p_2, \sigma_1, \sigma_2 \geq 0$

$$I_{\text{MD}}(\mathbf{p}_{\text{min}}, \rho_{\Phi}, \rho_{\chi}) \leq I_{\text{MD}}(\mathbf{p}, \rho_{\Phi}, \rho_{\chi})$$

(36)

For $N_c > 2$, we can obtain the same result. Here we have proved that the worst power allocation puts all the power in the minimizing subcarrier at which both the overall radar channel/target frequency response and the communications channel frequency response have their minimum values.

Next, we prove that under some conditions the worst power allocation $\mathbf{p}_{\text{min}}$ also maximizes some weighted sum of the DIR and MI for some given overall radar channel/target frequency response and some communications channel frequency response from the uncertainty classes in (22) and (23). These conditions on the frequency responses of overall radar channel/target and communications channel will
be derived in the following.

Suppose that the worst power allocation $\mathbf{p}_{\text{min}}$ also maximizes some weighted sum of the DIR and MI for some given overall radar channel/target frequency response and some communications channel frequency response from the uncertainty classes in (22) and (23), and that the $m_i$-th element of $\mathbf{p}_{\text{min}}$ is 1 and any other element is zero. Hence, $\mathbf{p}_{\text{min}}$ satisfies the optimization solution in (31), i.e.,

$$p_{v_{c,m}} = \frac{1}{2} \left[ \mu' (\alpha' + \beta') - (\nu'_{m} + \sigma'_{m}) + \sqrt{\left( \sigma'_{m} - \nu'_{m} \right) + \mu' (\alpha' - \beta')} \right] + 4 \mu'^2 \alpha' \beta' = P_i = 1 \quad (37)$$

$$p_{v_{c,m}} = \frac{1}{2} \left[ \mu' (\alpha' + \beta') - (\nu'_{m} + \sigma'_{m}) + \sqrt{\left( \sigma'_{m} - \nu'_{m} \right) + \mu' (\alpha' - \beta')} \right] + 4 \mu'^2 \alpha' \beta' \leq 0,$$

$$m_n = 0, 1, \ldots, N_c - 1, \ m_s \neq m_i \quad (38)$$

where

$$\nu'_{m} = \frac{1}{\nu_{m}} \left( \rho_{\phi} \left( f_{m} \right) \right) = \left( N \left( f_{m} \right) T_i \right) / N T \rho_{\phi} \left( f_{m} \right)$$

and

$$\sigma'_{m} = \frac{1}{\sigma_{m}} \left( \rho_{\phi} \left( f_{m} \right) \right) = \sigma'_{m} / \rho_{\phi} \left( f_{m} \right), \ for \ m_n = 0, 1, \ldots, N_c - 1.$$

Simplifying (37), the following can be obtained

$$\mu = 1/\mu' = \frac{\alpha' + \sigma'_{m} \alpha' + \beta' + \nu'_{m} \beta'}{1 + \nu'_{m} \sigma'_{m} + \nu'_{m} + \sigma'_{m}} \quad (39)$$

Simplifying (38) we can get

$$\left( \frac{\alpha'}{\nu'_{m}} + \frac{\beta'}{\sigma'_{m}} \right) \leq \frac{1}{\mu} = \mu, \ m_n = 0, 1, \ldots, N_c - 1, \ m_s \neq m_i \quad (40)$$

Substituting (39) into (40) yields

$$\max_{m_n} \left\{ \frac{\alpha'}{\nu'_{m}} + \frac{\beta'}{\sigma'_{m}} \right\} \leq \frac{\alpha'}{1 + \nu'_{m} \sigma'_{m} + \nu'_{m} + \sigma'_{m}}, \ m_n = 0, \ldots, m_i - 1, m_i + 1, \ldots, N_c - 1$$

where $\max_{n} \{ x_n \}, \ m = 0, 1, \ldots, N_c - 1$, indicates the maximum value in the set $\{ x_0, x_1, \ldots, x_{N_c - 1} \}$.

Eq. (41) can be rewritten as

$$\max_{m_n} \left\{ \frac{\alpha'}{\nu'_{m}} + \frac{\beta'}{\sigma'_{m}} \right\} \leq \frac{\alpha'}{1 + \nu'_{m} \sigma'_{m} + \nu'_{m} + \sigma'_{m}}, \ m_n = 0, \ldots, m_i - 1, m_i + 1, \ldots, N_c - 1$$

If $\nu'_{m}$ and $\sigma'_{m}$ are sufficiently small, and $\nu'_{m}$ and $\sigma'_{m}$, for $m_n = 0, \ldots, m_i - 1, m_i + 1, \ldots, N_c - 1$, are
sufficient large, the inequality in (42) will hold. If the upper and lower bounds are sufficiently separated, some \( \nu_m \) and \( \sigma_m \), for \( m = 0,1,\cdots,N_c -1 \) will satisfy the condition in (42), since \( \nu_m \) and \( \sigma_m \), for \( m = 0,1,\cdots,N_c -1 \) are limited by the lower and upper bounds of the uncertainty classes.

IV SIMULATION

In this section, we present several numerical examples to demonstrate the effectiveness of the proposed robust waveform design method. In the examples, the noise is complex AWGN, and the frequency response of the propagation channel for radar is flat so that \( G(\bullet) \) describes the overall (combined) radar channel/target frequency response. The magnitude of the overall radar channel/target frequency response and the magnitude of the communications channel frequency response are shown in Fig. 5. Other simulation parameters are shown in Table 1.

| Simulation parameters | Value | Parameter | Value |
|-----------------------|-------|-----------|-------|
| Duration of GI        | 1 us  | Number of subcarriers | 128   |
| Subcarrier spacing    | 0.25 MHz | Number of OFDM symbols | 16    |

In Fig. 5, the lower bound of the magnitude of the overall radar channel/target frequency response is

\[
\left| G_L (f_m) \right| = e^{-\left[ 2(m-N_c/2) \right]^2 / N_c}, \quad m = 0,1,\cdots,N_c -1, \quad (43)
\]

the upper bound of the magnitude of the overall radar channel/target frequency response is

\[
\left| G_U (f_m) \right| = 2 + e^{-\left[ 2(m-N_c/2) \right]^2 / N_c}, \quad m = 0,1,\cdots,N_c -1, \quad (44)
\]

the lower bound of the magnitude of the communications channel frequency response is

\[
\left| h_L (f_m) \right| = e^{-\left[ 3(m-N_c/2-10) \right]^2 / N_c}, \quad m = 0,1,\cdots,N_c -1, \quad (45)
\]

and the upper bound of the magnitude of the communications channel frequency response is

\[
\left| h_U (f_m) \right| = 1.5 + e^{-\left[ 3(m-N_c/2-10) \right]^2 / N_c}, \quad m = 0,1,\cdots,N_c -1. \quad (46)
\]
In the following, the optimal waveform is designed based on the specific frequency responses of overall radar channel/target and communications channel as shown in figures. The robust waveform is obtained by solving (24). In the following figures, when the actual overall radar channel/target frequency response and communications channel frequency response correspond to the corresponding uncertainty class lower bounds (UCLBs), the performance of the optimal waveform and that of the robust waveform are labeled as ‘OptW, AFR=UCLB’ and ‘RobW, AFR=UCLB’, respectively. Similarly, when the actual overall radar channel/target frequency response and communications channel frequency response correspond to the corresponding uncertainty class upper bounds (UCUBs), the performance of the optimal waveform and that of the robust waveform are labeled as ‘OptW, AFR=UCUB’ and ‘RobW, AFR=UCUB’, respectively.

A. Communications Performance

In this subsection, we will examine the DIR of communications with different waveforms. In Fig. 6, the variation of DIR with the SNR is depicted. The weighting factor for communications is 0.5. The magnitude of the overall radar channel/target frequency response and that of the communications channel frequency response are shown in Fig. 5. In Fig. 6, the DIR is enhanced with an increase in the
SNR. When the actual overall radar channel/target frequency response and communications frequency response correspond to the UCLBs, the robust waveform outperforms the optimal waveform for the specific frequency responses in Fig. 5, which implies that the worst performance, as described in Theorem 2, of the robust waveform is better than that of the optimal waveform, since the robust waveform optimizes the worst case performance over the uncertainty class and thus guarantees performance is always better than this quantity. However, the robust waveform is not always superior to the optimal waveform when the actual overall radar channel/target frequency response and communications frequency response correspond to the UCUBs, which indicates that the best performance, as described in Theorem 1, of the robust waveform is not always better than that of the optimal waveform. As expected, the robust waveform can improve the worst performance of the IRCS.

The dependence of the DIR on the width of the uncertainty range is shown in Fig. 7. In Fig. 7(a), the lower bounds of the uncertainty classes are fixed, but the upper bounds change with the width of the uncertainty range. The fixed lower bounds of the magnitude of the overall radar channel/target frequency response and that of the magnitude of the communications channel frequency response are same as those in Fig. 5. In contrast, in Fig. 7(b), the upper bounds of the uncertainty classes are fixed, but the lower bounds change with the width of the uncertainty range. The fixed upper bounds of the
magnitude of the overall radar channel/target frequency response is
\[ G_u(f_m) = 5.1 + e^{\left[ \frac{(N_c/2)}{N_u} \right]} , m = 0, 1, \cdots, N_c - 1 \]. The fixed upper bounds of the magnitude of the communications channel frequency response is \[ h_u(f_m) = 5.1 + e^{\left[ \frac{(N_c/2)}{N_u} \right]} , m = 0, 1, \cdots, N_c - 1 \]. The specific frequency responses of communications channel and overall radar channel/target are depicted in Fig. 8(a) for fixed lower bounds, and in Fig. 8(b) for fixed upper bounds. The weighting factor for communications is 0.5 and the SNR is 5 dB in Fig. 7.

![Figure 7](image_url)

**Fig. 7** The variation of data information rate with the width of uncertainty range. (a) The lower bounds are fixed. (b) The upper bounds are fixed.

In Fig. 7(a), when the true frequency responses of overall radar channel/target and communications channel correspond to the UCUBs, the performance of the robust waveform and the optimal waveform is improved as the uncertainty classes become wider with fixed lower bounds. However, when the true frequency responses of overall radar channel/target and communications channel correspond to the UCLBs, the performance of the robust waveform and optimal waveform is unchanged as the uncertainty classes become wider, since lower bounds of the uncertainty classes are fixed. Similar to the results in Fig. 6, the performance of the robust waveform is superior to that of the optimal waveform when the true frequency responses of overall radar channel/target and communications channel correspond to the UCLBs, although the robust waveform is not always better than the optimal waveform when the true frequency responses of overall radar channel/target and communications
channel correspond to the UCUBs.

When the true frequency responses of overall radar channel/target and communications channel correspond to the UCLBs, as shown in Fig. 7(b), an increase in the width of uncertainty range causes the performance of the robust waveform and optimal waveform to be deteriorated, since the lower bounds of the uncertainty classes are decreasing. In contrast, when the true frequency responses of overall radar channel/target and communications channel correspond to the UCUBs, the performance of the optimal waveform is unchanged since the upper bounds of the uncertainty classes are fixed, while the performance of the robust waveform is changed with the decrease of UCLBs, since the robust waveform is determined by the UCLBs under fixed total power. Similar to the results in Fig. 7(a), with the increase of the width of the uncertainty range, the performance of the optimal waveform eventually becomes worse than that of the robust waveform when the actual frequency responses of the overall radar channel/target and communications channel correspond to the UCUBs. As expected, the robust waveform can improve the worst performance of the IRCS.

![Fig. 8](image)

**Fig. 8** The magnitude of the specific frequency responses of overall radar channel/target and communications channel. (a) The lower bounds are fixed. (b) The upper bounds are fixed.

**B. Radar Performance**

In this subsection, the radar performance of the IRCS is evaluated. The dependence of MI on SNR is
shown in Fig. 9. The simulation conditions are same as those in Fig. 6. As expected the MI increases with the increase of the SNR. Similar to the communications performance in Fig. 6, the performance of the robust waveform is better than that of the optimal waveform when the actual frequency responses of overall radar channel/target and communications channel equal UCLBs. In this case, both the robust waveform and the optimal waveform achieve the worst performance as described in Theorem 2. This means that the robust waveform can achieve a favorable performance over the entire uncertainty class, even under the worst possible situation. However, when the true frequency responses of overall radar channel/target and communications channel are exactly UCUBs, the performance of the robust waveform is not always better than that of the optimal waveform.

The variation of the MI with the width of uncertainty range is depicted in Fig. 10. The parameters are same as those in Fig. 7. In Fig. 10(a), the performance of the optimal waveform and the robust waveform is improved with the increase of the width of uncertainty range when the actual frequency responses of overall radar channel/target and communications channel are the UCUBs. At this situation, both the optimal waveform and robust waveform achieve the best performance as described in Theorem 1. Moreover, the best performance of the robust waveform is not always superior to that of the optimal waveform. In contrast, when the actual frequency responses of overall radar channel/target
and communications channel are the UCLBs, the performance of the optimal waveform and the robust waveform does not change, since the lower bounds of the uncertainty classes stay constant in the example. As expected, the potential worst case performance of the IRCS is improved by using the robust waveform.

Fig. 10  The variation of mutual information with the width of uncertainty range. (a) The lower bounds are fixed. (b) The upper bounds are fixed.

In Fig. 10(b), when the true frequency responses of overall radar channel/target and communications channel are the UCLBs, the performance of the optimal waveform and the robust waveform is deteriorated with the increase of the width of uncertainty range, since the lower bounds of the uncertainty classes are decreasing in the example. Moreover, in this case, the performance of the optimal waveform eventually becomes worse than that of the robust waveform. Similar to the results in Fig. 7(b), when the true frequency responses of overall radar channel/target and communications channel are the UCUBs, with the increase of the width of uncertainty range the performance of the optimal waveform stays constant while that of the robust waveform is gradually deteriorated.

The previous simulation results show that the robust waveform has favorable radar and communications performance. Moreover, the robust waveform can simultaneously improve the worst case performance of the radar and communications, although it is not as good as the optimal waveform when the true frequency responses of overall radar channel/target and communications channel are
exactly the UCUBs.

C. Trade-off Curve

For the robust integrated waveform design, the weighting factors $w_t$ and $w_c$ are required to be specified. In this section we consider how the weighting factors will impact the radar and communications performance of the IRCS by showing all possible solutions for all weighting factors in Fig. 11. In the simulation, the SNR is 15 dB. Fig. 12 shows the lower and upper bounds of the frequency responses of the uncertainty classes along with some specific frequency responses which lie in the uncertainty classes.

![Fig. 11 The optimal trade-off curve of the IRCS.](image)

![Fig. 12 The uncertainty classes of frequency response. (a) The magnitude of the overall radar channel/target frequency response. (b) The magnitude of the communications channel frequency response.](image)

In Fig. 11, in the arrow direction, the weighting factor $w_c$ for communications increases from 0 to
1 in increments of 0.1. As expected, with the increase of the weighting factor for communications, the DIR is improved and the MI is decreased. According to the Theorem 1, the trade-off curves of the robust waveform and the optimal waveform show the variations of the best performance of these waveforms with the increase of the weighting factor for communications, when the true frequency responses of overall radar channel/target and communications channel correspond to the UCBs. In this case, the trade-off curves show the best trade-off. In contrast, according to the Theorem 2, when the true frequency responses of overall radar channel/target and communications channel correspond to the UCLBs, the trade-off curves of the robust waveform and the optimal waveform show the variations of the worst performance of these waveforms with the increase of the weighting factor for communications. In this case, the trade-off curves show the worst trade-off. For any overall radar channel/target and communications channel frequency responses which lie in the uncertainty classes, the performance of the robust waveform will located in the region between the best trade-off curves and the worst trade-off curves. The performance of the optimal waveform has similar results. Hence, in practice using the trade-off curves one can select the weighting factors to meet the demands for MI and DIR. Moreover, the worst performance of the robust waveform are superior to that of the optimal waveform although the best performance of the robust waveform is not as good as that of the optimal waveform due to the fact that the robust waveform can ensure the worst possible performance over the whole uncertainty class is optimal while cannot insure the best possible performance is optimal. The optimal trade-off curves also reveal the inherent compromise between the radar and communications performance, and the trade-off between the robustness and the best possible performance in the integrated waveform design.

V CONCLUSION
In this paper, a robust OFDM integrated radar and communications waveform design method is proposed. Under a constraint on the total power, the minimax robust waveform design method is employed to design the robust integrated waveform and a closed form solution is derived. The devised robust waveform has acceptable performance in the worst case. Moreover, compared with the optimal waveform the robust waveform has a performance improvement when the true frequency responses of overall radar channel/target and communications channel correspond to the UCLBs although it is not as good as the performance of the optimal waveform when the true frequency responses of overall radar channel/target and communications channel correspond to the UCUBs. It is inevitable for the robust OFDM integrated radar and communications waveform design to make a trade-off between the radar and communications performance as well as the robustness and the best possible performance.

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**APPENDIX**

**DERIVATION OF OPTIMAL SOLUTION TO THE ROBUST WAVEFORM DESIGN**

In this section, we will derive the optimal solution to the optimization problem in (29). Rewritten the KKT conditions:

\[
\mu - \mu_m = w_r \Delta T_r \left[ \ln 2F_r \left( 1 + p_m \nu_{r,m} \right) \right] + w_c \Delta f \left[ \ln 2F_c \left( 1 + p_m \sigma_{c,m} \right) \right], \quad m = 0, 1, \cdots, N_c - 1 \tag{47a}
\]

\[
\mu \left( \sum_{m=0}^{N_c-1} p_m - 1 \right) = 0 \tag{47b}
\]

\[
\mu_m p_m = 0, \quad m = 0, 1, \cdots, N_c - 1 \tag{47c}
\]

\[
\mu \geq 0, \quad \mu_m \geq 0, \quad m = 0, 1, \cdots, N_c - 1 \tag{47d}
\]

where \( \mu \), and \( \mu_m \), for \( m = 0, 1, \cdots, N_c - 1 \) are Lagrange multipliers.

Define \( \lambda_m = \frac{1}{\mu - \mu_m} \), \( \alpha' = w_r \Delta T_r \left( \ln 2F_r \right) \), \( \beta' = w_c \Delta f \left( \ln 2F_c \right) \), and (47a) can be represented as

\[
\frac{1}{\lambda_m} = \alpha' \nu_{r,m} \left( 1 + p_m \nu_{r,m} \right) + \beta' \sigma_{c,m} \left( 1 + p_m \sigma_{c,m} \right) \tag{48}
\]

Furthermore, (48) can be rewritten as

\[
\frac{1}{\lambda_m} = \alpha' \left( \nu_{r,m} + p_m \right) + \beta' \left( \sigma_{c,m} + p_m \right) \tag{49}
\]

where \( \nu_{r,m}' = \frac{1}{\nu_{r,m}} > 0 \), \( \sigma_{c,m}' = \frac{1}{\sigma_{c,m}} > 0 \). Suppose \( p_m > 0 \), by (49) we have

\[
p_m^2 + \left[ \left( \nu_{r,m}' + \sigma_{c,m}' - \lambda_m \left( \alpha' + \beta' \right) \right) p_m + \nu_{r,m}' \sigma_{c,m}' - \lambda_m \left( \alpha' \sigma_{c,m}' + \beta' \nu_{r,m}' \right) \right] = 0 \tag{50}
\]

The following result can be obtained

\[
p_m = \frac{\lambda_m \left( \alpha' + \beta' \right) - \left( \nu_{r,m}' + \sigma_{c,m}' \right) \pm \sqrt{\left( \nu_{r,m}' + \sigma_{c,m}' \right)^2 + 4\lambda_m \left( \alpha' - \beta' \right) \left( \alpha' \sigma_{c,m}' + \beta' \nu_{r,m}' \right)}}{2} \tag{51}
\]
According to (49), we can obtain that

$$\frac{1}{\lambda_m} < \alpha' \nu_{i,m} + \beta' \sigma_{i,m}$$  \hspace{1cm} (52)

Equation (52) is equivalent to

$$\nu_{i,m} \sigma_{i,m} - \lambda_m (\alpha' \sigma_{i,m} + \beta' \nu_{i,m}) < 0$$  \hspace{1cm} (53)

Using (53) and the property of quadratic equation, we can obtain that the negative root in (51) is less than zero, hence

$$p_m = \frac{\lambda_m (\alpha' + \beta') - (\nu_{i,m} + \sigma_{i,m}) + \sqrt{[\sigma_{i,m} - \nu_{i,m} + \lambda_m (\alpha' - \beta')]^2 + 4 \lambda_m^2 \alpha' \beta'}}{2}$$  \hspace{1cm} (54)

If $p_m > 0$, in order to satisfy (47c), we must have $\mu_m = 0$. Therefore, (54) can be rewritten as

$$p_m = \frac{\mu' (\alpha' + \beta') - (\nu_{i,m} + \sigma_{i,m}) + \sqrt{[\sigma_{i,m} - \nu_{i,m} + \mu' (\alpha' - \beta')]^2 + 4 \mu'^2 \alpha' \beta'}}{2}$$  \hspace{1cm} (55)

where $\mu' = 1/\mu$.

If $p_m = 0$, and $\lambda_m = 1/(\mu - \mu_m) \leq 0$, to satisfy (49) we must have

$$p_m < 0$$  \hspace{1cm} (56)

which is contrary to the practical problem.

If $p_m = 0$, and $\lambda_m = 1/(\mu - \mu_m) > 0$, since $\lambda_m = 1/(\mu - \mu_m) \geq \mu' > 0$, to satisfy (49) we have

$$p_m = \frac{\mu' (\alpha' + \beta') - (\nu_{i,m} + \sigma_{i,m}) + \sqrt{[\sigma_{i,m} - \nu_{i,m} + \mu' (\alpha' - \beta')]^2 + 4 \mu'^2 \alpha' \beta'}}{2} < 0$$  \hspace{1cm} (57)

The results in (55) and (57) can be summarized as

$$p_{\text{rc},m} = \frac{1}{2} \left[ \mu' (\alpha' + \beta') - (\nu_{i,m} + \sigma_{i,m}) + \sqrt{[\sigma_{i,m} - \nu_{i,m} + \mu' (\alpha' - \beta')]^2 + 4 \mu'^2 \alpha' \beta'} \right]$$  \hspace{1cm} (58)

where $[x] = \max \{x, 0\}$, and $\mu'$ satisfies that

$$\left\{ \sum_{m=0}^{\nu-1} p_{\text{rc},m} - 1 \right\} = 0$$  \hspace{1cm} (59)

Hence, the optimal solution to the optimization problem in (29) is obtained.