SINGLE PROJECTION KACZMARZ EXTENDED ALGORITHMS

STEFANIA PETRA, CONSTANTIN POPA

ABSTRACT. To find the least squares solution of a very large and inconsistent system of equations, one can employ the extended Kaczmarz algorithm. This method simultaneously removes the error term, such that a consistent system is asymptotically obtained, and applies Kaczmarz iterations for the current approximation of this system. For random corrections of the right hand side and Kaczmarz updates selected at random, convergence to the least squares solution has been shown. We consider the deterministic control strategies, and show convergence to a least squares solution when row and column updates are chosen according to the almost-cyclic or maximal-residual choice.

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1. INTRODUCTION

The Kaczmarz algorithm \cite{Kac37} for solving linear systems of the form
\[ Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m \] (1.1)
in the least-squares sense is a prototypical instance of so-called iterative row-action methods \cite{Cen81} that can be applied to very large systems of equations. Typical applications include image reconstruction from tomographic projections \cite{GBH70} – see \cite{CZ97} for an overview and further examples. The Kaczmarz algorithm has recently gained some renewed interest through the work \cite{SV09} where an expected exponential convergence rate was shown for a randomized control scheme, used to define the sequence of Kaczmarz iterations.

In view of practical applications where measurements define the vector \( b \) in (1.1), the inconsistent case
\[ b \notin \mathcal{R}(A) \] (1.2)

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is significant due to measurement errors and noise that most likely take \( b \) outside the range \( \mathcal{R}(A) \) of \( A \). Needell [Nee10] extended to this case the analysis of [SV09] and showed a similar rate of convergence to a ball around the solution to the consistent system whose radius depends on the condition number of \( A \) and the perturbation of \( b \). Throughout this paper we consider the inconsistent system

\[ Ax = \hat{b} \quad (1.3) \]

after an error vector \( r \) is added to the “clean” right side \( \hat{b} \).

Popa [Pop95b] introduced the extended Kaczmarz iteration so as to achieve convergence to a least-squares solution in the inconsistent case (1.2). The basic idea is to interleave “row-actions” on \( x \) with “columns-actions” on \( \hat{b} \). The latter iteratively remove the spurious component of \( \hat{b} \) orthogonal to \( \mathcal{R}(A) \)

\[ r := P_{\mathcal{R}(A)^\perp} (\hat{b}). \quad (1.4) \]

In a very recent paper [ZF13] a theoretical bound of the expected convergence rate was established for a randomized version of the extended Kaczmarz iteration.

This line of research focusing on the convergence rate of randomized (extended) Kaczmarz iterations also connects to earlier work on establishing convergence of the deterministic Kaczmarz iteration when applied to inconsistent linear systems. The issue of cyclic convergence in this connection was recognized early [GPR67, Tan71, CEG83] but not resolved, as discussed next.

**Contribution.** The present paper has the following objective: we establish convergence of the extended Kaczmarz iteration for a particular control scheme – henceforth called maximal-residual control scheme – used to define the sequence of iterates: at each iterative step the largest residual with respect to \( x \) and \( \hat{b} \) determines the row- and column action to be performed as subsequent iterative step. It is evident that this scheme most aggressively aims to achieve convergence based on additional computational costs encountered when determining the maximal residuals. Convergence however was neither established in [Pop95b] nor somewhere else in the literature, to our knowledge. This also holds for the application of the almost cyclic control scheme [CZ97] to inconsistent linear systems. Our present work also fills this gap in the literature.

**Organization.** We recall the classical Kaczmarz algorithm in Section 2. We specify in Section 3 different iterative schemes based on the Kaczmarz algorithm and its deterministic and randomized extensions discussed above. This section also includes preparatory Lemmata for the convergence analysis of the maximal-residual control scheme, and the almost cyclic control scheme, established in Section 4. We conclude and indicate further directions of research in Section 5.

**Notation.** We set \([n] = \{1, \ldots, n\}\) for \( n \in \mathbb{N} \). \( \langle \cdot, \cdot \rangle \) denotes the Euclidean inner product and \( \| \cdot \|_2 = \langle \cdot, \cdot \rangle^{1/2} \) the corresponding norm. For an \( m \times n \) real matrix \( A \), \( A^\top \) will be its transpose and \( \mathcal{R}(A), \mathcal{N}(A) \) its range and null space, respectively. \( S^\perp \) will denote the orthogonal complement of some vector subspace \( S \subset \mathbb{R}^q \), and \( P_C \) the orthogonal projector onto some closed convex set \( C \). For given \( \hat{b} \in \mathbb{R}^m \) and \( A \in \mathbb{R}^{m \times n} \), we define the orthogonal decomposition

\[ \hat{b} = b + r, \quad b \in \mathcal{R}(A), \quad r \in \mathcal{R}(A)^\perp = \mathcal{N}(A^\top). \quad (1.5) \]

The set of least-squares solution to problem (1.1) is denoted by

\[ LSS(A; \hat{b}) = \{ x \in \mathbb{R}^n : x = x_{LS} + \mathcal{N}(A), \quad Ax_{LS} = P_{\mathcal{R}(A)}(\hat{b}) = b \} \quad (1.6) \]

The probability simplex in \( \mathbb{R}^n \) is

\[ \Delta_n = \{ x \in \mathbb{R}^n : x \geq 0, \sum_{i \in [n]} x_i = 1 \}. \quad (1.7) \]
∥A∥_2 denotes the spectral norm of a linear mapping \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \) defined by
\[
∥A∥_2 = \sup_{x \neq 0} \frac{∥Ax∥_2}{∥x∥},
\]
and \( ∥A∥_F = \left( \sum_{i \in [m], j \in [n]} A_{ij}^2 \right)^{1/2} \) the Frobenius norm. The Moore-Penrose pseudoinverse is denoted by \( A^+ \). Vectors are enumerated with superscripts \( x \) and vector and matrix components with subscripts \( x_i, A_{ij} \). Specifically, matrix rows and columns are denoted by \( A_i \) (row \( i \)) and \( A^j \) (column \( j \)) (1.8)

\[ E[\cdot] \] denotes the expectation operation applied to a random variable. \( ℓ_1 \) denotes the space of all absolutely summable sequences \( (x_k)_{k \in \mathbb{N}} \) satisfying \( \sum_{k \in \mathbb{N}} |x_k| < \infty \), while \( ℓ_+ \) will denote nonnegative sequences. The space of convergent sequences is denoted by \( ℓ_c \), and \( ℓ_{c_0} \) denotes the space of sequences converging to zero.

2. The Kaczmarz Algorithm

The Kaczmarz Algorithm was first published [Kac37]. In it’s simplest form the Kaczmarz iteration proceeds as follows:

**Algorithm 1 Kaczmarz (K)**

**Require:** \( A \in \mathbb{R}^{m \times n}, \hat{b} \in \mathbb{R}^m, k_{\text{max}} \in \mathbb{N} \)

**return** Approximation to \( x_{LS} \) at bounded distance to \( x_{LS} \) (proportional to noise and condition number)

**Initialization** \( x^0 \in \mathbb{R}^n, k_{\text{max}} \)

for \( k = 1, \ldots, k_{\text{max}} \) do

for \( i_k = 1, \ldots, m \) do

Set

\[
x^k = x^{k-1} - \frac{\langle x^{k-1}, A_{i_k} \rangle - \hat{b}_{i_k}}{∥A_{i_k}∥^2} A_{i_k}.
\]

end for

end for

In the field of image reconstruction it is known as ART (Algebraic Reconstruction Technique) and independently rediscovered in [GBH70]. The algorithm is a particular Projection Onto Convex Sets (POCS) algorithm [BB96], and can also be viewed as a special instance of Bregman’s balancing method [Bre65], which for each \( i := (k \mod m) + 1 \) finds

\[
x^{k+1} = x^k + ω_k (P_{\hat{H}_i}(x^k) - x^k),
\]

where \( P_{\hat{H}_i}(x^k) \) is the orthogonal projection of \( x^k \) on the \( i \)-th hyperplane \( \hat{H}_i = \{ x \in \mathbb{R}^n, \langle A_i, x \rangle = \hat{b}_i \} \).

This sequential POCS method converges in the consistent case to a point in the intersection of the convex sets, see [GPR67, Th. 1]. However, in the inconsistent case it does not converge, but convergence of the cyclic subsequences, called cyclic convergence, can be shown [GPR67, Th. 2].

For the Kaczmarz algorithm (without relaxation), Kaczmarz [Kac37] proved convergence to the unique solution of the system, provided \( A \) is square and invertible. Herman et al. showed in [HLL78] that ART with relaxation converges in the consistent case. The case in which no (see also [PZ04]) solution exists has been considered by Tanabe [Tan71], who proved convergence to a limit cycle of vectors. If, the relaxation parameter \( ω_k \) goes to zero, the element of the limit cycle approach the same
vector. This has been considered by Censor et al. [CEG83], who show that the limiting single vector is the least squares solution that is unique provided $A$ has full rank.

However, in both consistent and inconsistent case no convergence rates existed in terms of matrix characteristics like e.g. the matrix condition number. By considering a random row selection strategy a first important step was made in [SV09] for the consistent, full rank case, and expected convergence rates where obtained in terms of linear algebraic characteristics of $A$. The Randomized Kaczmarz algorithm [SV09] triggered a series of recent publications [Nee10, EN11, NT14]. The convergence of the the Randomized Kaczmarz algorithm was analyzed in [Nee10]. The expected convergence to a ball of fixed radius centered at the least squares solution was shown, [Nee10, Thm. 2.1]. This radius is proportional to the norm of the additive noise scaled by the condition number, and equals at most

$$\hat{k}(A) \max_{i \in [m]} \frac{|r_i|}{\|A_i\|},$$

where $\hat{k}(A) = \|A^+\|_2\|A\|_F$.

The bound (2.2) shows that the randomized Kaczmarz method performs well when the noise in inconsistent systems is small. The Kaczmarz method will not converge to the least squares solution of an inconsistent system, since its iterates always lie in a single solution space given by a single row of the matrix $A$.

In order to overcome this problem and converge to a least squares solution we consider an approach first introduced by the second author in [Pop95a], which a employs a iteratively modified right-hand side vector to deal with the inconsistent case. We show next that this strategy breaks the radius barrier of the standard method also for deterministic row and column selection strategies, as shown before in [ZF13] for the random choice.

3. SINGLE PROJECTION KACZMARZ EXTENDED (KE) ALGORITHMS

Algorithm 2 extends Algorithm 1 to inconsistent systems (1.1) due to perturbations $\hat{b} = b + r$ of the right-hand side.

**Algorithm 2** Single Projection Extended Kaczmarz (EK)

**Require:** $A \in \mathbb{R}^{m \times n}, \hat{b} \in \mathbb{R}^m, k_{max} \in \mathbb{N}$

**return** Approximative least-squares solution

**Initialization** $x^0 \in \mathbb{R}^n, y^0 = \hat{b}, \alpha, \omega \in (0, 2)$;

**for** $k = 1, \ldots, k_{max}$ **do**

Select the index $j_k \in [n]$ and set

$$y^k = y^{k-1} - \alpha \langle y^{k-1}, A^{j_k} \rangle A^{j_k}. \quad (3.1)$$

Update the right hand side as

$$\hat{b}^k = \hat{b} - y^k. \quad (3.2)$$

Select the index $i_k \in [m]$ and set

$$x^k = x^{k-1} - \omega \frac{\langle x^{k-1}, A_{i_k} \rangle - \hat{b}^k_{i_k}}{\|A_{i_k}\|^2} A_{i_k}. \quad (3.3)$$

**end for**

The following Lemma examines how the correction step in (3.2) affects the perturbed hyperplanes

$$\hat{H}_{i_k} = \{x : \langle A_{i_k}, x \rangle = \hat{b}^k_{i_k} = b_{i_k} + r_{i_k} - y^k_{i_k} \}$$

(3.4)
in view of the unperturbed hyperplanes

\[ H_{ik} = \{ x : \langle A_{ik}, x \rangle = b_{ik} \}. \]  \tag{3.5}

**Lemma 3.1.** Consider \( \hat{H}_{ik} \) and \( H_{ik} \) defined by (3.4) and (3.5). Then

\[ \hat{H}_{ik} = \{ x + \gamma_{ik} : x \in H_{ik} \} \quad \text{where} \quad \gamma_{ik} = \delta_{ik} A_{ik}, \quad \delta_{ik} = \frac{r_{ik} - y_{ik}^k}{\| A_{ik} \|^2}. \]  \tag{3.6}

**Proof.** Denote \( i := i_k \) for simplicity. For \( x \in H_i \), we have \( \langle A_i, x + \delta_i A_i \rangle = \langle A_i, x \rangle + \delta_i \| A_i \|^2 = b_i + r_i - y_i^k = \hat{b}_i^k \). Thus, \( x + \delta_i A_i \in \hat{H}_i \). Conversely, choose \( \hat{x} \in \hat{H}_{ik} \) arbitrary and set \( x = \hat{x} - \delta_i A_i \). Then \( \langle A_i, x \rangle = \langle A_i, \hat{x} \rangle - r_i + y_i^k = b_i + r_i - y_i^k - r_i + y_i^k = b_i \) holds. Consequently \( x \in H_i \). \( \square \)

**Remark 3.1.** We observe that due to the initialization \( y^0 = \hat{b} \) of Algorithm 2, the decomposition (1.5) and the update rule (3.1), it always holds that

\[ y^k - r \in \mathcal{R}(A), \quad \forall k \in \mathbb{N}. \]  \tag{3.7}

### 3.1. Control Sequences

We will consider the following basic deterministic control sequences, cf. [Cen81], besides randomized control sequences [SV09 ZF13].

- **Cyclic control:** Set \( i_k = k \text{ mod } m + 1 \), \( j_k = k \text{ mod } n + 1 \).
- **Almost cyclic control:** Select \( i_k \in [m] \), \( j_k \in [n] \), such there exist integers \( m_0, n_0 \) with
  \[ [m] \subseteq \{ i_{k+1}, \ldots, i_{k+m_0} \} \]  \tag{3.8}
  \[ [n] \subseteq \{ j_{k+1}, \ldots, j_{k+n_0} \}. \]  \tag{3.9}
  for every \( k \in \mathbb{N} \).
- **Set-based control:** Select \( j_k \in [n] \) and \( i_k \in [m] \) such that
  \[ j_k = \arg \max_{j \in [n]} |\langle A^j, y^{k-1} \rangle|, \]  \tag{3.10}
  \[ i_k = \arg \max_{i \in [m]} |\langle A_i, x^{k-1} \rangle - \hat{b}_i^k |. \]  \tag{3.11}

Note that these sequences depend on each other through (3.1)–(3.3). Sequence \( (j_k)_{k \in \mathbb{N}} \) relates to largest component \( |P_{\mathcal{R}(A)}(y^k)| \) of \( y^k \) weighted by \( \| A^j \| \), whereas the sequence \( (i_k)_{k \in \mathbb{N}} \) relates to the largest distance of \( x^k \), weighted by \( \| A_i \| \), to the hyperplane defined by some row \( A_i \) and the right-hand side \( \hat{b}_i^k \), that is updated due to (3.2).

- **Random control:** Define the discrete distributions
  \[ p \in \Delta_m, \quad p_i = \frac{\| A_i \|^2}{\| A \|^2}, \quad i \in [m], \quad q \in \Delta_n, \quad q_j = \frac{\| A^j \|^2}{\| A \|^2}, \quad j \in [n], \]  \tag{3.12}
  and sample in each step \( k \) of the iteration (3.1)
  \[ j_k \sim q \]  \tag{3.13}
  and each step \( k \) of the iteration (3.3)
  \[ i_k \sim p. \]  \tag{3.14}

**Remark 3.2.** We note that the cyclic control is a special case of the almost cyclic control. The maximal residual choice is also known as remote set control [Cen81].
Algorithm 3 Randomized Extended Kaczmarz Algorithm (REK)

Require: $A \in \mathbb{R}^{m \times n}, \hat{b} \in \mathbb{R}^m, k_{\text{max}} \in \mathbb{N}$ return Approximative least-squares solution

Initialization $x^0 \in \mathbb{R}^n, y^0 = \hat{b}, \alpha, \omega \in [0, 2]$

for $k = 1, \ldots, k_{\text{max}}$ do
    Select the index $j_k \in [n]$ randomly according to (3.13) and set
    $$y^k = y^{k-1} - \alpha \langle y^{k-1}, A^{j_k} \rangle A^{j_k}. \tag{3.15}$$
    Update the right hand side as
    $$\hat{b}^k = \hat{b} - y^k. \tag{3.16}$$
    Select the index $i_k \in [m]$ randomly according to (3.14) and set
    $$x^k = x^{k-1} - \omega \frac{\langle x^{k-1}, A^{i_k} \rangle - \hat{b}_{i_k}^k}{\| A_{i_k} \|^2} A_{i_k}. \tag{3.17}$$
end for

3.2. The Randomized Extended Kaczmarz Algorithm. In the recent paper [ZF13], authors considered Algorithm 3 with a random selection of the indices $j_k$ and $i_k$ and $\alpha = \omega = 1$. They proved the following convergence result along with a convergence rate.

Theorem 3.2. For any $A$, $\hat{b}$, and $x^0 = 0$, the sequence $(x^k)_{k \in \mathbb{N}}$ generated by REK Algorithm 3 with $\alpha = \omega = 1$ converges in expectation to the minimal norm solution $x_{LS}$ of (1.3), with the asymptotic error reduction factor

$$\mathbb{E}[\|x^k - x_{LS}\|] \leq \left(1 - \frac{1}{k^2(\hat{A})}\right)^{|k/2|} (1 + 2k^2(\hat{A}))\|x_{LS}\|^2,$$

where $\hat{k}(A) = \|A^+\|_2\|A\|_F$ and $k(A) = \sigma_1/\sigma_r$, where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ are the nonzero singular values of $A$ and $r = \text{rank}(A)$.

3.3. The MREK Algorithm. In this subsection we will show that $\|\gamma_{i_k}\|$ from (3.6) decays geometrically for the maximal residual choice $i_k$ from (3.11) and, in particular, that the error norms are absolutely summable. These results will be in turn used to establish convergence of the MREK algorithm in Section 4. We first collect some facts and state a basic assumption. For any invertible matrix $D \in \mathbb{R}^{n \times n}$ we have (cf. (1.6))

$$x \in LSS(A; \hat{b}) \iff D^{-1}x \in LSS(AD; \hat{b}). \tag{3.18}$$

As a consequence, by choosing $D = \text{Diag}(\|A^1\|^{-1}, \ldots, \|A^n\|^{-1})$, we may assume w.l.o.g. that

$$\|A^j\| = 1, \quad j \in [n]. \tag{3.19}$$

First we need a preparatory result, which can be easily proved, see e.g. [Ans84].

Lemma 3.3. Let $\alpha \in (0, 2)$, $\delta \geq 0$ be defined by

$$\delta = \inf\{\| A^\top \zeta \|, \zeta \in N(A^\top) = \mathcal{R}(A), \| \zeta \| = 1\}, \tag{3.20}$$

and let $A = U\Sigma V^\top$, $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$ be a singular value decomposition of $A$, where $r = \text{rank}(A)$. Then

$$0 < \delta = \sigma_r \leq \sigma_1. \tag{3.21}$$
Remark 3.3. Since $\sigma_i^2$, $i \in [r]$, are the eigenvalues of $A^\top A$ and $AA^\top$, respectively, scaling of $A \leftarrow \frac{1}{c} A$ by some factor $c > 0$ scales the singular values $\sigma_i \leftarrow \sigma_i/c$ as well. Thus, by scaling the linear system (3.11),
\[
\| Ax - \hat{b} \| = \min! \iff \| \frac{\sqrt{n}}{\sigma_r} Ax - \frac{\sqrt{n}}{\sigma_r} \hat{b} \| = \min!,
\]
we may assume w.l.o.g. that $\delta \sqrt{\alpha (2 - \alpha)} \leq \sqrt{n}$, hence
\[
1 - \frac{\delta^2 \alpha (2 - \alpha)}{n} \in [0, 1).
\] (3.22)

Lemma 3.4. Let $\alpha \in (0, 2)$, $k \in \mathbb{N}$ denote an arbitrary fixed number of iterations of Algorithm 4, with $i_k$ selected according to the maximal residual choice (3.11), and let $\delta_i \in \mathbb{R}$ and $\gamma_i \in \mathbb{R}^n$ be given by (3.6). Then,

(i) there exist $M \geq 0$ and $\gamma \in [0, 1)$, independent on $k$, such that
\[
\| \gamma_i \| \leq M \gamma_k,
\] (3.23)

(ii) $(\| \gamma_i \|)^2 k \in \mathbb{N} \in \ell_+ \cap \ell_1$,

(iii) $y^k \to r$ for $k \to \infty$, with $r$ given by (3.5).

Proof. (i) Update rule (3.28) yields
\[
y^k - r = (y^{k-1} - r) - \alpha \langle y^{k-1}, A_i^k \rangle A_i^k.
\]
Using $y^k - r \in \mathcal{R}(A)$, $\forall k$, and $\mathcal{R}(A)^\perp \ni r \perp A_i^j$, $j \in [n]$, we compute
\[
\| y^k - r \|^2 = \| y^{k-1} - r \|^2 - \alpha (2 - \alpha) \langle y^{k-1}, A_i^j \rangle^2.
\] (3.24)

Based on property (3.27) defining the index $j_k \in [n]$, we upper bound
\[
\| y^k - r \|^2 \leq \| y^{k-1} - r \|^2 - \frac{\alpha (2 - \alpha)}{n} \sum_{j \in [n]} \langle y^{k-1}, A_i^j \rangle^2.
\]

Exploiting again $r \perp A_i^j$, $j \in [n]$, and Lemma 3.3, we obtain
\[
\| y^k - r \|^2 \leq \| y^{k-1} - r \|^2 - \frac{\alpha (2 - \alpha)}{n} \sum_{j \in [n]} \langle y^{k-1} - r, A_i^j \rangle^2
\]
\[
= \| y^{k-1} - r \|^2 \left( 1 - \frac{\alpha (2 - \alpha)}{n} \| A_i^\top y^{k-1} - r \| \| y^{k-1} - r \| \right) \leq \left( 1 - \frac{\delta^2 \alpha (2 - \alpha)}{n} \right) \| y^{k-1} - r \|^2
\]
\[
\leq \left( 1 - \frac{\delta^2 \alpha (2 - \alpha)}{n} \right)^k \| y^0 - r \|^2.
\]

Thus, with $y^0 - r = b$,
\[
\| \gamma_i \| = \frac{1}{\| A_i^k \|} |(r - y^k)_i| \leq \left( 1 - \frac{\delta^2 \alpha (2 - \alpha)}{n} \right)^{k/2} \frac{\| b \|}{\min_{i \in [n]} \| A_i \|} =: \gamma^k M,
\] (3.25)
\[
\gamma = \left( 1 - \frac{\delta^2 \alpha (2 - \alpha)}{n} \right)^{1/2},
\] (3.26)
and $\gamma \in [0, 1)$ due to (3.22).

(ii) Using (3.25), $\gamma \in [0, 1)$ and convergence of geometric series, we get
\[
\sum_{k \in \mathbb{N}} \| \gamma_i \|^2 \leq \sum_{k \in \mathbb{N}} M^2 \gamma^{2k} = \frac{M^2}{1 - \gamma^2} < \infty.
\]
(iii) The derivation of (i) shows that relation (3.25) is valid for every $i \in [m]$. Hence, since $\gamma \in [0, 1)$, 
$$\|r - y^k\|_\infty \leq \text{const. } \gamma^k \to 0 \quad \text{for} \quad k \to \infty.$$ 
\hfill \Box

**Algorithm 4** Algorithm Maximal Residual Extended Kaczmarz (MREK)

**Require:** $A \in \mathbb{R}^{m \times n}, \hat{b} \in \mathbb{R}^m, k_{\text{max}} \in \mathbb{N}$

**return** Approximative least-squares solution

**Initialization** $x^0 \in \mathbb{R}^n, y^0 = \hat{b}; \alpha, \omega \in [0, 2]$

for $k = 1, \ldots, k_{\text{max}}$ do

Select the index $j_k \in [n]$ such that

$$|\langle A_{j_k}, y^{k-1} \rangle| \geq |\langle A_j, y^{k-1} \rangle|, \forall j \in [n],$$

and set

$$y^k = y^{k-1} - \alpha \langle y^{k-1}, A_{j_k} \rangle A_{j_k}.$$  

(3.27)

(3.28)

Update the right hand side as

$$\hat{b}^k = \hat{b} - y^k.$$  

(3.29)

Select the index $i_k \in [m]$ such that

$$|\langle A_{i_k}, x^{k-1} \rangle - \hat{b}_i^k| \geq |\langle A_i, x^{k-1} \rangle - \hat{b}_i^k|, \forall i \in [m],$$

and set

$$x^k = x^{k-1} - \omega \frac{\langle x^{k-1}, A_{i_k} \rangle - \hat{b}_i^k}{\| A_{i_k} \|^2} A_{i_k}.$$  

(3.30)

(3.31)

end for

3.4. The ACEK Algorithm. In this section we will establish a result analogous to Lemma 3.4 for Algorithm 5 that corresponds to Algorithm 2 in the case of the almost cyclic index selection scheme. First of all, related to (3.52) we introduce the notations

$$\varphi_j(y) = y - \frac{\langle y, A_j \rangle}{\| A_j \|^2} A_j, \quad \varphi_j^\alpha(y) = y - \alpha \frac{\langle y, A_j \rangle}{\| A_j \|^2} A_j,$$

and observe that the application $\varphi_j^\alpha$ is no more a projection and we have the equalities

$$\varphi_j^\alpha(y) = (1 - \alpha)I + \alpha \varphi_j(y).$$  

(3.33)

We will replay below Lemma 21 from [Pop95a] (see also [Nat86]) with respect to the above applications.

**Lemma 3.5.** For any $\alpha \in (0, 2), y \in \mathbb{R}^m, j = 1, \ldots, n$ the following are true

$$\| \varphi_j^\alpha \| \leq 1,$$

$$\| \varphi_j^\alpha y \|^2 - \| y \|^2 = (2 - \alpha)\alpha(\| \varphi_j y \|^2 - \| y \|^2).$$  

(3.34)

(3.35)

We can now state the result analogous to Lemma 3.4.
Lemma 3.6. Let \( k \geq n_0 \in \mathbb{N} \) denote an arbitrary fixed number of iterations of Algorithm A with \( n_0 \) defined by (3.9), and with \( i_k, j_k \) selected according to the almost cyclic choice (3.8) and (3.9), respectively. Let \( \delta_{i_k} \in \mathbb{R} \) and \( \gamma_{i_k} \in \mathbb{R}^n \) be given by (3.6). Then,

(i) there exist \( M \geq 0 \) and \( \gamma \in [0, 1) \), independent on \( k \), such that

\[
\| \gamma_{i_k} \| \leq M \gamma^n,
\]

(3.36)

with \( k = n \cdot n_0 + l_0, \; n \in \mathbb{N}, \; n_0 \geq l_0 \in \mathbb{N}_0 \).

(ii) \( \langle \gamma_{i_k} \|^2 \rangle_{k \in \mathbb{N}} \in \ell^1 \).

(iii) \( y^k \rightarrow r \) for \( k \rightarrow \infty \), with \( r \) given by (1.5).

Proof. Step 1. Let \( k \geq 0 \) be an arbitrary fixed iteration of the algorithm ACEK, \( J = \{1, \ldots, n\} \), \( J_k = \{j_{k+1}, \ldots, j_k \} \) and (see (3.52))

\[
y^{k+\Gamma} = \varphi_{j_{k+\Gamma}}^\alpha \circ \cdots \circ \varphi_{j_{k+1}}^\alpha y^k.
\]

(3.37)

We will first show that it exists \( \hat{\gamma} \in [0, 1) \) such that

\[
\| y^{k+\Gamma} - P_{N(A \Gamma)}(y^k) \| \leq \hat{\gamma} \| y^k - P_{N(A \Gamma)}(y^k) \|.
\]

(3.38)

From (3.37) we get

\[
y^{k+\Gamma} = \Phi_k^\alpha y^k, \text{ where } \Phi_k^\alpha = \varphi_{j_{k+\Gamma}}^\alpha \circ \cdots \circ \varphi_{j_{k+1}}^\alpha : \mathbb{R}^m \rightarrow \mathbb{R}^m.
\]

Let \( A(k) \) be the \( n \times \Gamma \) matrix defined by

\[
A(k) = \text{col}(A^{j_{k+1}}, \ldots, A^{j_{k+\Gamma}}).
\]

(3.40)

Because the additional \( \Gamma - n \) columns of \( A(k) \) are among the columns of the initial matrix \( A \) (see (3.9)), we have

\[
N(A \Gamma(k)) = N(A \Gamma), \text{ thus } P_{N(A \Gamma(k))} = P_{N(A \Gamma)}.
\]

(3.41)

If we define \( \tilde{\Phi}_k^\alpha = \Phi_k^\alpha P_{\mathcal{R}(A)} \) we know that (see e.g. [Pop95a])

\[
\tilde{\Phi}_k^\alpha = \Phi_k^\alpha + P_{N(A \Gamma)}(y^k) \tilde{\Phi}_k^\alpha = 0, \| \tilde{\Phi}_k^\alpha \| < 1.
\]

(3.42)

Then \( y^{k+\Gamma} = \Phi_k^\alpha (y^k) = \tilde{\Phi}_k^\alpha (y^k) + P_{N(A \Gamma)}(y^k) \), thus

\[
\| y^{k+\Gamma} - P_{N(A \Gamma)}(y^k) \| = \| \tilde{\Phi}_k^\alpha (y^k) \| = \| \tilde{\Phi}_k^\alpha \left( y^k - P_{N(A \Gamma)}(y^k) \right) \| \leq \| \tilde{\Phi}_k^\alpha \| \cdot \| y^k - P_{N(A \Gamma)}(y^k) \|.
\]

The set \( J_k \setminus J \) has at most \( \Gamma - n \) elements which are among the indices from \( J \). It results that there are finitely many matrices \( A(k) \), thus finitely many applications \( \tilde{\Phi}_k^\alpha \), i.e.

\[
\hat{\gamma} = \max_{k \geq 0} \| \tilde{\Phi}_k^\alpha \| \text{ belongs to } [0, 1),
\]

(3.43)

which gives us (3.38).

Step 2. We will now show that it exists \( \hat{M} \geq 0 \), independent on \( k \) such that

\[
\| \gamma_k \| \leq \hat{M} \hat{\gamma}^{k-k\text{mod} \Gamma},
\]

(3.44)

with \( \hat{\gamma} \) from (3.43). From (1.5) and (3.52) it results that \( y^k \in \mathcal{R}(A), \forall k \geq 0 \), i.e. \( P_{N(A \Gamma)}(y^k) = r, \forall k \geq 0 \). Thus,

\[
\| y^{k+\Gamma} - r \| \leq \hat{\gamma} \| y^k - r \|, \forall k \geq 0,
\]

(3.45)

and recursively

\[
\| y^{\mu \Gamma} - r \| \leq \hat{\gamma} \| y^{(\mu-1)\Gamma} - r \|, \forall \mu \geq 1.
\]

(3.46)
For the arbitrary fixed index $k \geq 0$, let $\mu$ be the integer given by

$$\mu = \frac{k - k \pmod{\Gamma}}{\Gamma}, \text{ i.e.}$$

$$k = \mu \Gamma + q, \text{ for some } q \in \{0, 1, \ldots, \Gamma - 1\}.$$ (3.47)
(3.48)

If we define $\widehat{M}$ as

$$\widehat{M} = \max\{\|y^{\Gamma - 1} - r\|, \ldots, \|y^0 - r\|\},$$

from (3.45) - (3.49) we get for any $\mu \geq 1$

$$\|y^k - r\| = \|y^{\mu \Gamma + q} - r\| \leq \gamma \|y^{(\mu - 1) \Gamma + q} - r\| \leq \ldots \leq \gamma^\mu \|y^q - r\| \leq \widehat{M} \gamma^\mu.$$ (3.50)

Hence

$$\|\gamma_{ik}\| = \frac{|r^k_{ik} - \hat{b}_{ik}|}{\|A^k_{ik}\|} \leq \frac{\|y^k - r\|}{\min_{i=1,\ldots,m}\{\|A_i\|\}} \leq \gamma^\mu \frac{\widehat{M}}{\min_{i=1,\ldots,m}\{\|A_i\|\}}.$$ (3.51)

which is exactly (3.44), with $\widehat{M} = \min_{i=1,\ldots,m}\{\|A_i\|\}$.

**step 3.** Then, relation (3.36) holds directly from (3.44) and gives us also the conclusion (ii). Conclusion (iii) holds from (3.38) and the proof is complete. □

**Algorithm 5** Algorithm Almost Cyclic Extended Kaczmarz (ACEK)

Require: $A \in \mathbb{R}^{m \times n}$, $\hat{b} \in \mathbb{R}^m$, $k_{\text{max}} \in \mathbb{N}$, $\alpha \neq 0$, $\omega \neq 0$

return Approximative least-squares solution

**Initialization**

$x^0 \in \mathbb{R}^n$, $y^0 = \hat{b}$;

for $k = 1, \ldots, k_{\text{max}}$ do

Select the index $j_k \in [n]$ in an almost cyclic way according to (3.9)

and set

$$y^k = y^{k-1} - \alpha \frac{\langle y^{k-1}, A_{jk} \rangle}{\|A_{jk}\|^2} A_{jk}. \quad (3.52)$$

Update the right hand side as

$$\hat{b}^k = \hat{b} - y^k.$$ (3.53)

Select the index $i_k \in [m]$ in an almost cyclic way according to (3.8)

and set

$$x^k = x^{k-1} - \omega \frac{\langle x^{k-1}, A_{ik} \rangle - \hat{b}_{ik}^k}{\|A_{ik}\|^2} A_{ik}. \quad (3.54)$$

end for

**4. Convergence Analysis**

In order to prove the convergence of the two algorithms MREK\[4] and ACEK\[5] we next examine how the distance to any fixed least-squares solution changes.

To this end, we denote by $x_k^* = P_{H_{ik}}(x_k^{k-1})$, where $H_{ik}$ is the unperturbed hyperplane from (3.5), given by

$$x_k^* = x_k^{k-1} - \omega \frac{\langle A_{ik}, x_k^{k-1} \rangle - b_{ik}}{\|A_{ik}\|^2} A_{ik}, \quad (4.1)$$

**Proposition 4.1.** For any $x \in \text{LSS}(A; \hat{b})$ and for all $k \in \mathbb{N}$, we have for every iterate $x^k$ generated by the algorithm MREK\[4] or ACEK\[5] respectively and for any $i_k \in [m]$. ...
(i) $$\| x^k - x \|^2 = \| x^k - x \|^2 + \| \gamma_{i_k} \|^2$$,  

(4.2)

(ii) $$\| x^k - x \|^2 = \| x^{k-1} - x \|^2 - \omega(2 - \omega) \left( \frac{(A_{i_k}, x^{k-1}) - b_{i_k}}{\| A_{i_k} \|^2} \right)^2 + \omega \| \gamma_{i_k} \|^2$$,  

(4.3)

(iii) $$\| x^k - x \|^2 \leq \| x^{k-1} - x \|^2 + \omega \| \gamma_{i_k} \|^2$$,  

(4.4)

with $$\gamma_{i_k}$$ from (3.6).

Proof. (i) Choose $$x \in LSS(A; \hat{b})$$ arbitrarily. Then $$Ax = b$$ and, in particular, $$x \in H_{i_k}$$. Since $$x_{i_k}^k \in H_{i_k}$$, Lemma 3.1 (see also (3.54)) asserts $$x^k = x_{i_k}^k + \omega \gamma_{i_k}$$. The orthogonality relation $$\gamma_{i_k} \perp (x_{i_k}^k - x) \in H_{i_k}$$ due to $$\gamma_{i_k} = \delta_{i_k} A_{i_k} \gamma_{i_k}$$, immediately gives $$\| x^k - x \|^2 = \| x_{i_k}^k - x \|^2 + \omega \| \gamma_{i_k} \|^2$$.

(ii) We will denote by $$P_{H_{i_k}}^*$$ the right hand side of (4.1), i.e.

$$x_{i_k}^k = P_{H_{i_k}}^* (x^{k-1}) = x^{k-1} - \omega \frac{(A_{i_k}, x^{k-1}) - b_{i_k}}{\| A_{i_k} \|^2} A_{i_k}$$,  

(4.5)

If $$S_{i_k} = \{ x : (A_{i_k}, x) = 0 \}$$ denotes the corresponding vector subspace (see (3.5)), and because $$b_{i_k} = (A_{i_k}, x)$$ then the application $$P_{S_{i_k}}^* (z) = z - \omega \frac{(A_{i_k}, z)}{\| A_{i_k} \|^2} A_{i_k}$$, which satisfies

$$x_{i_k}^k - x = P_{S_{i_k}}^* (x^{k-1} - x)$$,  

(4.6)

has similar properties with $$\varphi_j^S$$ from (3.32). Let also $$P_{S_{i_k}}^* (z) = z - \omega \frac{(A_{i_k}, z)}{\| A_{i_k} \|^2} A_{i_k}$$. Then, from Lemma 3.5 (3.35) applied to $$P_{S_{i_k}}^*$$ and $$P_{S_{i_k}}$$ we get (by also using the fact that the projection $$P_{S_{i_k}}$$ is an idempotent operator)

$$\| x_{i_k}^k - x \|^2 = \| P_{S_{i_k}}^* (x^{k-1} - x) \|^2 = \omega(2 - \omega) \left( \| P_{S_{i_k}} (x^{k-1} - x) \|^2 - \| x^{k-1} - x \|^2 \right) + \| x^{k-1} - x \|^2 = \omega(2 - \omega) \langle (A_{i_k}, x^{k-1} - x), x^{k-1} - x \rangle + (1 - \omega(2 - \omega)) \| x^{k-1} - x \|^2 = \| x^{k-1} - x \|^2 - \omega(2 - \omega) \frac{(A_{i_k}, x^{k-1} - x)^2}{\| A_{i_k} \|^2}$$.

(4.7)

Then, equation (4.3) follows from (4.2) and (4.7).

(iii) It results directly from (4.3) and the proof is complete.

Remark 4.1. Proposition 4.1 (iii), together with Lemmata 3.3 (ii) and 3.6 (ii) shows that the sequence $$(x^k)_{k \in \mathbb{N}}$$ generated by the MREK [4] or the ACEK algorithm [5] is quasi-Féjer of Type II, see [Com01] Def. 1.1.

The next Lemma is a special case of Lemma 3.1 in [Com01]. The corresponding simplified proof is included for completeness.

Lemma 4.2. Let $$(\alpha_k)_{k \in \mathbb{N}} \in \ell_+$$ and $$(\beta_k)_{k \in \mathbb{N}} \in \ell_+$$ be two nonnegative sequences, and $$(\varepsilon_k)_{k \in \mathbb{N}} \in \ell_+ \cap \ell_1$$ satisfying

$$\alpha_{k+1} = \alpha_k - \beta_k + \varepsilon_k$$.

(4.8)

Then the following statements hold true.

(i) $$(\beta_k)_{k \in \mathbb{N}} \in \ell^1$$. In particular $$(\beta_k)_{k \in \mathbb{N}} \in \ell_{\varepsilon_0}$$.

(ii) $$(\alpha_k)_{k \in \mathbb{N}}$$ converges.
Proof. (i) From (4.8), we have \( \beta_k = \alpha_k - \alpha_{k+1} + \varepsilon_k \). Furthermore, 
\[
\sum_{k=0}^{n} \beta_k = \sum_{k=0}^{n} (\alpha_k - \alpha_{k+1}) + \sum_{k=0}^{n} \varepsilon_k = \alpha_0 - \alpha_{n+1} + \sum_{k=0}^{n} \varepsilon_k < \alpha_0 + \sum_{k=0}^{n} \varepsilon_k,
\]
which yields \( \sum_{k \in \mathbb{N}} \beta_k < \alpha_0 + \sum_{k \in \mathbb{N}} \varepsilon_k < +\infty \). Hence \( (\beta_k)_{k \in \mathbb{N}} \in \ell^1 \). Now \( \ell^1 \subset \ell_+ \), shows (i).

(ii) Now, both \( (\varepsilon_k)_{k \in \mathbb{N}} \in \ell_{\alpha_0} \) and \( (\beta_k)_{k \in \mathbb{N}} \in \ell_{\alpha_0} \). By (4.8), 
\[
|\alpha_{k+1} - \alpha_k| = |\varepsilon_k - \beta_k| \leq |\varepsilon_k| + |\beta_k| = \varepsilon_k + \beta_k,
\]
with \( (\varepsilon_k + \beta_k)_{k \in \mathbb{N}} \in \ell^1 \). This shows that \( (\alpha_k)_{k \in \mathbb{N}} \) is a Cauchy sequence. Since \( (\alpha_k)_{k \in \mathbb{N}} \in \ell_+ \subset \mathbb{R} \) it also converges.

We are now ready to prove convergence of MREK, Algorithm 4.

Theorem 4.3. Let \( \alpha, \omega \in (0, 2) \). The sequence \( (x^k)_{k \in \mathbb{N}} \) generated by the MREK, Algorithm 4 converges to a least-squares solution in \( LSS(A; b) \), for any starting vector \( x^0 \in \mathbb{R}^n \).

Proof. We split the proof into two parts, showing convergence of \( (x^k)_{k \in \mathbb{N}} \), and convergence to a point in \( LSS(A; \hat{b}) \), respectively.

(i) Choose any \( x \in LSS(A; \hat{b}) \) and set 
\[
\alpha_{k+1} = \|x^k - x\|^2, \quad \beta_k = \omega (2 - \omega) \frac{(\langle A_{ik}, x^{k-1} \rangle - b_{ik})^2}{\|A_{ik}\|^2}, \quad \varepsilon_k = \omega^2 \|\gamma_{ik}\|^2,
\]

The above Lemma (see also (4.7) !!!) asserts convergence of \( (\alpha_k)_{k \in \mathbb{N}} \) and \( (\beta_k)_{k \in \mathbb{N}} \in \ell^1 \), in view of \( \varepsilon_k \in \ell^1 \), due to Lemma 5.4(ii) and Prop. 4.1(ii) respectively. In view of (3.31), we get 
\[
\|x^k - x^{k-1}\|^2 = \omega^2 \left( \frac{(\langle A_{ik}, x^{k-1} \rangle - b_{ik})^2}{\|A_{ik}\|^2} \right) = \frac{2\omega}{2 - \omega} \beta_k + 2\varepsilon_k. \tag{4.9}
\]

Now \( \frac{2\omega}{2 - \omega} \beta_k + 2\varepsilon_k \in \ell^1 \) implies that \( (x^k)_{k \in \mathbb{N}} \) is a Cauchy sequence and converges as well. In particular, using again (3.31), 
\[
\|x^k - x^{k-1}\|^2 = \omega^2 \left( \frac{(\langle A_{ik}, x^{k-1} \rangle - \hat{b}_{ik})^2}{\|A_{ik}\|^2} \right) \rightarrow 0. \tag{4.10}
\]

(ii) Assume that \( x^k \rightarrow \pi \). We show that \( \pi \in LSS(A, \hat{b}) \). Fix any \( i \in [m] \). Due to the particular choice of \( i_k \) in (3.11), we have 
\[
|\langle A_i, x^{k-1} \rangle - b_i| = |r_i - y_i^k| \leq |\langle A_i, x^{k-1} \rangle - b_i - (r_i - y_i^k)|
\]
\[
= |\langle A_i, x^{k-1} \rangle - \hat{b}_{ik}| \leq |\langle A_{ik}, x^{k-1} \rangle - \hat{b}_{ik}|, \tag{4.11}
\]

Thus \( |\langle A_i, x^{k-1} \rangle - b_i| \rightarrow 0 \), due to \( |r_i - y_i^k| \rightarrow 0 \) by Lemma 5.4(iii) and (4.10), respectively. Summarizing, we get \( \lim_{k \rightarrow \infty} \|Ax^{k-1} - b\| = 0 = \|A\pi - b\| \). Thus, \( \pi \in LSS(A, \hat{b}) \).
The main result concerning convergence of ACEK, Algorithm 5, is stated next.

**Theorem 4.4.** The sequence \( (x^k)_{k \in \mathbb{N}} \) generated by ACEK, Algorithm 5, converges to a least-squares solution in \( LSS(A; \hat{b}) \), for any starting vector \( x^0 \in \mathbb{R}^n \).

**Proof.** Choose any \( x \in LSS(A; \hat{b}) \) and set
\[
\alpha_{k+1} = \| x^k - x \|^2, \quad \beta_k = \omega(2 - \omega) \frac{\left( \langle A_{i_k}, x^{k-1} \rangle - b_{i_k} \right)^2}{\| A_{i_k} \|^2}, \quad \varepsilon_k = \omega^2 \| \gamma_{i_k} \|^2.
\]

The proof of convergence \( x^k \to \Pi \) is identically to the first part of the proof of Thm. 4.3 with the only difference that we have \( (\varepsilon_k)_{k \in \mathbb{N}} \in \ell^1 \) due to Lemma 3.6 (ii). Moreover
\[
\langle A_{i_k}, x^{k-1} \rangle - b_{i_k} \to 0 \quad (4.11)
\]
holds. The selection of \( i_k \) in (3.8) ensures \([m] \subset (i_k)_{k \in \mathbb{N}}\). This, together with (4.11), implies \( A\bar{x} = b \) and completes the proof. \( \square \)

5. Conclusions

We consider an inconsistent system of linear equations and our goal is to find the least squares (LS) solution. It is known that the Kaczmarz method does not converge to the LS solution in this case. In its randomized form the Kaczmarz method converges with a radius proportional the magnitude of the largest entry of the noise in the system. Convergence to the LS solution can be achieved if step lengths converging to zero are used. Unfortunately this significantly compromises convergence speed. A different approach is adopted by the extended Kaczmarz (EK) algorithm. In both randomized and deterministic forms, the methods alternates between projections on hyperplanes defined by the rows of the matrix and projections on the subspace orthogonal to the matrix range defined by the matrix columns. By this procedure the method iteratively builds a corrected right hand side which is then simultaneously exploited by Kaczmarz steps applied to a corrected system. The randomized extended Kaczmarz (REK) converges in expectation to the least squares solution and convergence rates can be obtained, as recently shown by Zouzias and Freris. For deterministic control strategies however, the convergence was still open when alternating between row and columns updates. We close this gap by showing convergence to the LS solution.

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(S. Petra) **IMAGE AND PATTERN ANALYSIS GROUP, UNIVERSITY OF HEIDELBERG, SPEYERER STR. 6, 69115 HEIDELBERG, GERMANY**

*E-mail address: {petra}@math.uni-heidelberg.de*

*URL: iwr.ipa.uni-heidelberg.de*

(C. Popa) **FACULTY OF MATHEMATICS AND INFORMATICS, OVIDIUS UNIVERSITY OF CONSTANTA, BLVD. MAI1A 124, 900527 CONSTANTA, ROMANIA**

*E-mail address: {copopa}@univ-ovidius.ro*

*URL: http://math.univ-ovidius.ro/doc/CVpdf/cv_C_Poparo.pdf*