Nonlinear Operator Superalgebras and BFV–BRST Operators for Lagrangian Description of Mixed-symmetry HS Fields in AdS Spaces

A.A. Reshetnyak

Institute of Strength Physics and Materials Science of SB of Russian Academy of Sciences, 634021 Tomsk, Russia

We study the properties of nonlinear superalgebras \( \mathcal{A} \) and algebras \( \mathcal{A}_b \) arising from a one-to-one correspondence between the sets of relations that extract AdS-group irreducible representations \( D(E_0, s_1, s_2) \) in AdS\(_d\)-spaces and the sets of operators that form \( \mathcal{A} \) and \( \mathcal{A}_b \), respectively, for fermionic, \( s_i = n_i + \frac{1}{2} \), and bosonic, \( s_i = n_i \), \( n_i \in \mathbb{N}_0 \), \( i = 1, 2 \), HS fields characterized by a Young tableaux with two rows. We consider a method of constructing the Verma modules \( V_\mathcal{A}, V_{\mathcal{A}_b} \) for \( \mathcal{A}, \mathcal{A}_b \) and establish a possibility of their Fock-space realizations in terms of formal power series in oscillator operators which serve to realize an additive conversion of the above (super)algebra \( \mathcal{A} \mathcal{A}_b \), containing a set of 2nd-class constraints, into a converted (super)algebra \( \mathcal{A}_{bc} = \mathcal{A}_b + \mathcal{A}'_b \) \( \mathcal{A}_c = \mathcal{A} + \mathcal{A}' \), containing a set of 1st-class constraints only. For the algebra \( \mathcal{A}_{bc} \), we construct an exact nilpotent BFV–BRST operator \( Q' \) having nonvanishing terms of 3rd degree in the powers of ghost coordinates and use \( Q' \) to construct a gauge-invariant Lagrangian formulation (LF) for HS fields with a given mass \( m \) (energy \( E_0(m) \)) and generalized spin \( s = (s_1, s_2) \). LFs with off-shell algebraic constraints are examined as well.

PACS Numbers: 11.10.Ef, 11.10.Kk, 11.15.-q, 03.65.Fd, 04.20.Fy, 03.70.+k

1. INTRODUCTION

The growing interest in field-theoretical models of higher dimensions is due to the
problem of a unified description of the known interactions and the variety of elementary
particles, which becomes especially prominent at high energies (partially accessible to the
LHC), thus stimulating the present-day development of a mixed-symmetry higher-spin
(HS) field theory in view of its close relation to superstring theory in constant-curvature
spaces [1, 2], which operates with an infinite set of bosonic and fermionic HS fields (that
correspond to arbitrary tensor representations of the Wigner little algebra) subject to a
multi-row Young tableaux (YT) \( Y(s_1, ..., s_k) \), \( k \geq 1 \), whose study has been initiated by [3]
and continued in [4]; for a review on HS field theory, see [5, 6]. The theory of free and
interacting mixed-symmetry HS fields has been developed in the framework of various
approaches, which may be classified as the light-cone formalism [7], Vasiliev’s frame-like
formalism [8–10] using the unfolded approach [11], and Fronsdal’s [12], both constrained
[13] and unconstrained [14], metric-like formalism. While the results of constructing a
Lagrangian formulation (LF) for free bosonic mixed-symmetry HS fields in the flat space
are well-known within all of these approaches, see for instance [15–18], the corresponding
results for the AdS\(_d\)-space have been developed in the light-cone, for the AdS\(_5\)-space [19],
and in the frame-like, for an integer spin [20, 21], formulations and remain unknown in a
more involved case of half-integer spins with a YT \( Y(s_1, ..., s_k) \), \( s_i = n_i + \frac{1}{2}, n_i \in \mathbb{N}_0 \).

The present article is devoted to solving this problem for free integer and half-integer
HS fields in the AdS\(_d\)-space that are subject to a YT with two rows, in unconstrained and
constrained metric-like formulations, on a basis of the (initially elaborated for a Hamiltonian
quantization of gauge theories, and being universal for all of the above constructions)
BFV–BRST formalism [22,23]; see the review [24] as well. The basic idea here consists in a
solution of a problem inverse to that of the method [22], just as in string field theory [25],
in the sense of constructing a gauge LF with respect to a nilpotent BFV–BRST operator
\( Q \), constructed, in turn, from a system \( \{ O_\alpha \} \) of 1st-class constraints that include a special
nonlinear non-gauge operator symmetry (super)algebra \((\mathcal{A}_c), \mathcal{A}_{bc} \) for (half-)integer HS
fields \( \{ O_I \} \): \( \{ O_I \} \supset \{ O_\alpha \} \). These quantities \( \{ O_I \} \) correspond to the initial AdS\(_d\)-group
irreducible representation (irrep) relations extracting the spin-tensors of a definite mass \( m \)
(including \( m = 0 \)) and spin (except for such algebraic conditions as the gamma- and trace-
less conditions in the case of a constrained description) and realized as operator constraints
for a vector of a special Fock space whose coefficients are (spin-)tensors related to the spin
of the basic HS field. As a result, the final action and the sequence of reducible gauge
transformations are reproduced by means of the simplest operations of decomposing the
resulting gauge vectors of the Hilbert space that contain the initial HS (spin-)tensors and
the gauge parameters with respect to the initial oscillator and ghost variables, subject to
the spin and ghost number conditions, and also by means of calculating the corresponding
scalar products, first realized in [26,27]. Due to the required presence of auxiliary (spin-)
tensors with a lesser spin, in order to have a closed LF for the basic (spin-)tensor with a
given spin (mentioned in the pioneering works of Fierz–Pauli [28] and Singh–Hagen [29] as a crucial part in the definition of a correct number of physical degrees of freedom), there arises a necessity of converting the sub(super)algebra of the total HS symmetry (super)algebra corresponding to the subset of 2nd-class constraints into that of 1st-class constraints. This conversion procedure is realized as an additive version of [30, 31], by means of constructing the Verma modules [32] for Lie (super)algebras corresponding to HS fields in the flat case and for specially deformed nonlinear (super)algebras \((\mathcal{A}')_{\mathcal{A}'b}\) in the AdS\(_d\)-space for HS fields with \(Y(s_1)\); see [33, 34]. A transition to mixed-symmetry HS fields with \(Y(s_1, \ldots, s_k), k \geq 2\) meets a significant obstacle to an application of a Cartan-like decomposition for \((\mathcal{A}')_{\mathcal{A}'b}\), which is one of the goals of the present article.

Another aspect concerns the structure of the BFV–BRST operator \(Q\), being more involved in the case of a converted nonlinear (super)algebra for HS fields subject to \(Y(s_1, s_2), (\mathcal{A}_c)_{\mathcal{A}bc}\), in view of the presence of nonvanishing terms of 3rd order in the powers of ghosts, because of a nontrivial character of the Jacobi identity for \(O_I\), in comparison, first, with \(Q\) for (half-)integer HS fields in the flat space [16,27], second, with \(Q\) for totally-symmetric (half-)integer HS fields in the AdS\(_d\)-space [33,34], and, third, with \(Q\) for special classical quadratic (super)algebras investigated in [36,37], because of a partially nonsupercommuting character of the operators \(O_I\).

The paper is organized as follows. In Section 2, we examine the initial operator (super)algebra \((\mathcal{A})_{\mathcal{A}b}\). In Section 3, we consider Proposition, which determines a way to obtain algebraic relations for the (super)algebras of the parts \((\mathcal{A}')_{\mathcal{A}'b}\) additional to those for a specially modified (super)algebra \((\mathcal{A}_{mod})_{\mathcal{A}bmod}\), and examine a construction of Verma modules that realize the highest-weight representation of \((\mathcal{A}')_{\mathcal{A}'b}\) and their realization in an auxiliary Fock space. An exact BFV–BRST operator for a converted (super)algebra \((\mathcal{A}_c)_{\mathcal{A}bc}\) is obtained in Section 4, on the basis of a solution of the Jacobi identity, due to the absence of non-trivial higher-order relations for \((\mathcal{A}_c)_{\mathcal{A}bc}\). The action and the sequence of reducible gauge transformations, mainly for bosonic HS fields of a fixed spin \(s = (s_1, s_2)\), are deduced in Section 5. In the conclusion, we summarize the results of this article and discuss some open problems.

We mainly use the conventions of Refs. [16,34].

2. NONLINEAR (SUPER)ALGEBRA FOR MIXED-SYMMETRY HS FIELDS IN ADS SPACE-TIME

A massive spin \(s = (s_1, s_2), s_i = n_i + \frac{1}{2}, n_1 \geq n_2\), representation of the AdS group in an AdS\(_d\) space is realized in a space of mixed-symmetry spin-tensors with a suppressed
Dirac index, and is characterized by \(Y(s_1, s_2)\),

\[
\Phi_{(\mu)_{n_1}, (\nu)_{n_2}} \equiv \Phi_{\mu_1...\mu_{n_1}, \nu_1...\nu_{n_2}}(x) = \\
\begin{array}{cccccccc}
\mu_1 & \mu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \mu_{n_1} \\
\nu_1 & \nu_2 & \cdots & \cdots & \cdots & \cdots & \cdots & \nu_{n_2}
\end{array},
\]

subject to the following equations \((\beta = (2; 3) \iff (n_1 > n_2; n_1 = n_2))\); \(r\) being the inverse squared AdS\(_d\) radius, and Dirac’s matrices satisfying the relation \(\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}(x)\):

\[
\left(i\gamma^\mu \nabla_\mu - r\frac{\bar{\gamma}}{2}(n_1 + \frac{d}{2} - \beta) - m\right)\Phi_{(\mu)_{n_1}, (\nu)_{n_2}} = \Phi_{(\mu)_{n_1}, (\nu)_{n_2}} = 0.
\]

For a simultaneous description of all half-integer HS fields, one introduces a Fock space \(\mathcal{H}\), generated by 2 pairs of creation \(a^i_\mu(x)\) and annihilation \(a^{i+}_\mu(x)\) operators, \(i, j = 1, 2, \mu, \nu = 0, 1, \ldots, d - 1; [a^{i}_\mu, a^j_{\mu}^{\dagger}] = -g_{\mu\nu}\delta_{ij}\), and a set of constraints for an arbitrary string-like vector \(|\Phi\rangle \in \mathcal{H}\),

\[
\begin{align*}
\langle \tau^i_0|\Phi\rangle &= -i\bar{\gamma}^\mu D_\mu + \bar{\gamma}(m + \sqrt{r}(g_0^d - \beta))|\Phi\rangle = 0, \\
\langle t^i, t^j|\Phi\rangle &= (\tilde{\gamma}^\mu a^{i+}_\mu, a^{j+}_\mu)|\Phi\rangle = 0, \\
|\Phi\rangle &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{n_1} \Phi_{(\mu)_{n_1}, (\nu)_{n_2}}(x) a^{i+}_{\mu_1} \cdots a^{i+}_{\mu_{n_1}} a^{j+}_{\nu_1} \cdots a^{j+}_{\nu_{n_2}} |0\rangle,
\end{align*}
\]

given in terms of the operator \(D_\mu = \partial_\mu - \omega^{ab}_\mu(x)(\sum_i a^i_{\mu a} a_{ib} - \frac{1}{8} \bar{\gamma}[\bar{\gamma} g_{0}\bar{\gamma}]\}, a^{+}\mu(x) = \epsilon^d_\mu(x)a^{+}\mu, equivalent to the covariant derivative \(\nabla_\mu\) in its action in \(\mathcal{H}\), with a vielbein \(e^\mu_a\), a spin connection \(\omega^{ab}_\mu\), and tangent indices \(a, b = 0, 1, \ldots, d - 1\). The scalar fermionic operators \(\tilde{\tau}^i_0, t^i\) are defined with the help of an extended set \(\tilde{\gamma}^\mu, \bar{\gamma}\) of Grassmann-odd gamma-matrix-like objects \([34], \{\tilde{\gamma}^\mu, \bar{\gamma}\} = 2g^{\mu\nu}, \{\tilde{\gamma}^\mu, \bar{\gamma}\} = 0, \bar{\gamma}^2 = -1\), related to the conventional gamma-matrices by an odd non-degenerate transformation: \(\gamma^\mu = \tilde{\gamma}^\mu\bar{\gamma}\). The validity of relations \([3], [4]\) is equivalent to a simultaneous fulfilment of Eqs. \([2]\) for all the spin-tensors \(\Phi_{(\mu)_{n_1}, (\nu)_{n_2}}\).

The construction of a Hermitian BFV–BRST charge \(Q\), whose special cohomology in the zero-ghost-number subspace of a total Hilbert space \(\mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}^\prime \otimes \mathcal{H}_{gh}\) will coincide with the space of solutions of Eqs. \([2]\), implies constructing a set of 1st-class quantities \(O_I, \{O_a\} \subset \{O_I\}\), closed under the operations of a) Hermitian conjugation with respect to an odd scalar product, \(\langle \Psi|\Phi\rangle_1 \equiv \langle \bar{\Psi}|\Phi\rangle_1 [16]\), with a measure \(d^Gx\sqrt{-\text{det}g}\), and b) supercommutator multiplication \([, ]\). As a result, the final massive (massless for \(m = 0\)) half-integer HS symmetry superalgebra in a space \(AdS_d\) with \(Y(s_1, s_2), \mathcal{A} = \{O_I\} = \{\tilde{t}^i_0, t^i, t^{i+}, t^i, l^{i+}, l^i, g_0^d, t^i_0\}, i \leq j; i, j = 1, 2;\)

\[
(t^{i+}; g_0^d; t^{i}; l^{i+}; l^i; l^{i+}; l^i; g_0^d; t^i_0), \quad i \leq j; i, j = 1, 2,
\]

\[
\begin{align*}
(t^{i+}; g_0^d; t^{i}; l^{i+}; l^i; l^{i+}; l^i; g_0^d; t^i_0) &= (\tilde{\gamma}^\mu a^{i+}_\mu; -a^{i+}_\mu a^\mu + \frac{d}{2}; a^{i+}_\mu a^\mu; -i(a^{\nu+}; a^{\mu+})D_\mu; \frac{1}{2} a^{i+}_\mu a_{\mu j}), \\
\tilde{t}^i_0 &= g^{\mu\nu}(D_\nu D_\mu - \Gamma^\sigma_{\mu\nu} D_\sigma) - r\left(\sum_i (g_0^d + t^{i+} t^i) + \frac{d(d-5)}{4}\right) + (m + \sqrt{r}(g_0^d - \beta))^2,
\end{align*}
\]
Table 1: The superalgebra of the modified initial operators.

\[
| [1, \rightarrow] | t_0 | t^i | t^{i+} | t | t^+ | l_0 | l^i | l^{i+} | l^{ij} | l^{ij+} | g_0^i \|
\begin{array}{cccccccccccc}
-2l_0 & 2l^i & 2l^{i+} & 0 & 0 & 0 & M^i & -M^{i+} & 0 & 0 & 0 \\
2l^k & 4l^{ki} & A^{ki} & -t^2\delta^{k1} & -t^3\delta^{k2} & 2M^k & 0 & -t_0\delta^{ik} & 0 & B^{k,ij} & t^i\delta^{k1} \\
2l^{k+} & A^{k+} & 4l^{k+i+} & t^1+\delta^{k2} & t^2+\delta^{k1} & -2M^{k+} & t_0\delta^{ik} & 0 & -B^{k,ij+} & 0 & -t^{i+}\delta^{k1} \\
t & 0 & t^2\delta^{i1} & -t^1+\delta^{i2} & 0 & g_0^1 - g_0^2 & 0 & t^2\delta^{i1} & -t^1+\delta^{i2} & D^{ij} & -G^{ij} & F^i \\
t^+ & 0 & t^1+\delta^{i2} & -t^2+\delta^{i1} & g_0^2 - g_0^1 & 0 & 0 & t^1+\delta^{i2} & -t^2+\delta^{i1} & G^{ij} & -D^{ij+} & -F^i \\
l_0 & 0 & -2M^i & 2M^{i+} & 0 & 0 & 0 & -rK_k^{bi+} & rK_k^{bi} & 0 & 0 & 0 \\
l^k & -M^k & 0 & -t_0\delta^{ik} & -t^2\delta^{k1} & -t^3\delta^{k2} & rK_k^{k+b} & W^{ki} & X^{ki} & 0 & J^{k,ij} & t^i\delta^{k1} \\
l^{k+} & M^{k+} & t_0\delta^{ik} & 0 & t^1+\delta^{k2} & t^2+\delta^{k1} & -rK_k^{k+b} & -X^{ik} & -W^{k+i} & -J^{k,ij+} & 0 & -t^{i+}\delta^{k1} \\
l^{kl} & 0 & 0 & B^{i,kl+} & -D^{kl} & -G^{kl} & 0 & 0 & J^{i,kl+} & 0 & L^{kl,ij} & t^{i}\delta^{kl} \\
l^{kl+} & 0 & -B^{i,kl} & 0 & G^{kl+} & D^{kl+} & 0 & -J^{i,kl} & 0 & -L^{i,kl,ij} & 0 & -l^{i}_{(k}\delta_{l)} \\
g_0^k & 0 & -t_k\delta^{ik} & t^k+\delta^{ik} & -F^k & F^{k+} & 0 & -t^k\delta^{ik} & t^k+\delta^{ik} & t^{k}(i\delta)^{k}_{l} & l^{k+}_{(i}\delta_{j)}^{k}_{l} & 0
\end{array}
\]

will contain a central charge \( \tilde{m} = (m - \beta \sqrt{r}) \), a subset of \((4+12)\) differential \( \{l_i, l^{i+}\} \subset \{a_{\alpha}\} \) and algebraic \( \{t_i, t^+_i, t, t^+, l_{ij}, l^{i+}_{ij}\} \subset \{a_{\alpha}\} \) 2-class constraints, as well as some particle-number operators \( g_0^i \), composing, along with \( \tilde{m}^2 \), an invertible supermatrix \( ||[a_{\alpha}, b_{\beta}]|| = \|\Delta_{ab}(g_0^i, \tilde{m})\| + O(a_{\alpha}) \), and obeys some non-linear algebraic relations w.r.t. \[ , \]. To construct an appropriate LF, it is sufficient to have a simpler (so-called modified) superalgebra \( \mathcal{A}_{\text{mod}} \), obtained from \( \mathcal{A} \) by a linear non-degenerate transformation of \( \alpha_I \) to another basis \( \tilde{\alpha}_I = u_I^\dagger \alpha_J, \tilde{\gamma} \notin \{\tilde{\alpha}_I\} \), so that the AdS-mass term \( m_D = (m + \sqrt{r}(n_1 + \frac{d}{2} - \beta)) \) factors out of \( l_0^2, \tilde{l}_0 \), which change only to \( t_0 = -i\tilde{\gamma}^\mu D_\mu, l_0 = -l_0^2 \).

As a result, the operators \( \tilde{\alpha}_I \), given by \([4], [6]\), with the central charge \( \tilde{m} \), satisfy the relations given by Table \[] where the quantities \( A^{ik}, B^{k,ij}, D^{ij}, F^i, G^{ij}, J^{k,ij}, L^{kl,ij} \) are defined as follows:

\[
A^{ik} = -2(g_0^i\delta^{nk} - t^i\delta^{k1} + t^+\delta^{n1}\delta^{k2}), \quad D^{ij} = l^{(i2}\delta^{j1)}, \quad G^{ij} = l^{(i}\delta^{j2)}, \quad (8)
\]

\[
J^{k,ij} = -\frac{1}{2}l^{(i+}\delta^{j)k}, \quad B^{k,ij} = -\frac{1}{2}l^{(i}\delta^{j)k}, \quad F^i = t(\delta^{i2} - \delta^{i1}), \quad (9)
\]
\[ L^{kl,ij} = -L^{kl,ij+} = \frac{1}{4} \left\{ \delta^{ik} \delta^{lj} [2g_0^k \delta^{kl} + g_0^i + g_0^j] - \delta^{ik} \left[ t (\delta^{lj}(\delta^{i1} + \delta^{k1} \delta^{lj}) + \delta^{ik} \delta^{lj}) + t^+ (1 \leftarrow 2) \right] - \delta^{lj} [t (\delta^{ik}(\delta^{i1} + \delta^{k1} \delta^{ij}) + \delta^{ik} \delta^{ij}) + t^+ (1 \leftarrow 2)] \right\}, \]

whereas the nonlinear operators \( M^i, W^{ij}, K^{bi}_i, X^{ij} \) are given by \((\varepsilon^{ij} = -\varepsilon^{ji}, \varepsilon^{12} = 1)\)

\[ M^i = r \left( 2 \sum_k t^k + t^k i - \frac{1}{2} t^i - tt^1 \delta^{i2} - t^+ t^2 \delta^{i1} \right), \]
\[ W^{ij} = W_b^{ij} + \frac{r}{4} t^{ij} t^k = 2 r \varepsilon^{ij} \left[ (g_0^i - g_0^j)t^{12} - t^1 t^{11} + t^1 t^{22} + t^{12} + t^{12} \right], \]
\[ K^{bi}_i = \left( 4 \sum_k l^{ik} + t^i (2g_0^i - 1) - 2 t^2 + t^1 \right), \]
\[ X^{ij} = \left\{ l_0 + r \left( \sum_k \left( g_0^i + t^k + t^k \right) - \frac{5}{2} g_0^i \right) \right\} \delta^{ij} + r \left\{ \frac{1}{2} t^j + t^i - 4 \sum_k l^{ik} + t^k \right\} \]
\[ -(g_0^1 + g_0^2 - \frac{3}{2} t^d \delta^{i1} \delta^{j2} - t^+ (g_0^1 + g_0^2 - \frac{3}{2} t^d \delta^{j2} \delta^{i1} + (g_0^2 - g_0^2) \delta^{i1} \delta^{j1}) \right\}. \]

It should be noted that for \( r = 0 \) the superalgebras \( A, A_{\text{mod}} \) are Lie superalgebras \([16]\), \( A^{\text{Lie}}, A^{\text{Lie, mod}} \), which obey the condition (1.2) mentioned for Lie superalgebras in Ref. [38] for \( A^{\text{Lie, mod}} \) and do not obey it for \( A^{\text{Lie}} \). In their turn, the original \( A_0 \) and modified \( A_{\text{mod}} \) nonlinear massive (massless, \( m = 0 \)) integer-spin algebras for bosonic HS fields, i.e., tensors in \( \Pi \), for \( s_i = n_i \), contain only the respective bosonic elements \( o_1t \), with \( \tilde{o}_b \) being Lorentz scalar having no \( \gamma \)-matrices in the definition of \( D_{\mu} \) as compared to the \([2^{(d)}_2 \times 2^{(d)}_2] \)-matrix structure of \( o_1t \), \( o_1t \) (with \([x] \) being fractional part of a number \( x \in \mathbb{R} \)) for the superalgebras \( A, A_{\text{mod}} \), and obey, in the case of \( \tilde{o}_b \), the same algebraic relations as those given by Table 1 without the fermionic operators \( t_0, t_i, t^+_i \). Only some of the nonlinear relations \([12], [14] \) are changed: first, \( t^i \) must be removed from \([12] \), second, Eqs. \([14] \), along with the new definition of \( l_{ob} = (D^2 - r^2 d(d-6)) \in \{ \tilde{o}_b \}, \tilde{l}_{ob} \in \{ o_b \} \), acquire the form

\[ X_b^{ij} = \left\{ l_{ob} + r \left( (g_0^1 - 1 - \delta^{i1})g_0^j - (1 + \delta^{i2})g_0^2 + t^+ t \right) \right\} \delta^{ij} - r \left\{ 4 \sum_k l^{ik} + t^k \right\} \]
\[ + (g_0^1 + g_0^2 - 2 t^d \delta^{i1} \delta^{j2} + t^+ (g_0^1 + g_0^2 - 2 t^d \delta^{j2} \delta^{i1} \right), \]
\[ \tilde{l}_{ob} = l_{ob} + m_{ob}^2 + r ((g_0^1 - 2 \beta - 2 g_0^1 - g_0^2), \quad m_{ob}^2 = m^2 + r \beta (\beta + 1), \]

being a consequence of the AdS-group irreps equations for the tensor \( \Phi_{(\mu)_{s_1}, (\nu)_{s_2}} \), see [19],

\[ \left\{ \nabla^2 + r (s_1 - \beta - 1 + d)(s_1 - \beta) - s_1 - s_2 \right\} \Phi_{(\mu)_{s_1}, (\nu)_{s_2}} = 0, \]
\[ \left( g^{\mu \nu_2}, g^{\nu \nu_2}, g^{\nu_1 \nu_2} \right) \Phi_{(\mu)_{s_1}, (\nu)_{s_2}} = \Phi_{(\mu_{s_1}, \nu_1)_{s_2}} = 0, \]

realized in \( \mathcal{H} \), with a standard scalar product \( \langle | \rangle \), as constraints: \( \langle \tilde{l}_{ob}, l_{ij}, t \rangle \Phi = 0 \).

\(^{\dagger}\text{Indeed, there are anticommutators, } [\tilde{t}_0, t_k] = 2 t^k - \tilde{\tau} r \frac{1}{2} t^j \delta^{k1}, \text{ that violate the requirement } C_{mn} = 0 \text{ if } \varepsilon(i) + \varepsilon(m) + \varepsilon(n) \neq 0 \text{ for the Grassmann parities } \varepsilon(i) = \varepsilon(\chi_i) \text{ of the quantities } \chi_i \text{ in Eq. (1.1): } | \chi_m, \chi_n \rangle = C^{mn}_{\chi_m \chi_m} \text{ in } [38], \text{ because } C^{[\ell]}_{[\chi]} \delta^{\ell^1} = -\tilde{\tau} r \frac{1}{2} \delta^{\ell^1} \text{ for } \delta^2 = -1.\)
3. ADDITIVE CONVERSION FOR NONLINEAR SUPERALEGEBRAS AND VERMA MODULE CONSTRUCTION

To convert additively non-linear superalgebras with a subset of 2nd class constraints, we need the following easily verified

Proposition: If a set of operators \( \{ \tilde{o}_I \} \): \( \mathcal{H} \to \mathcal{H} \) is subject to \( n \)-th order polynomial supercommutator relations (with the Grassmann parities \( \varepsilon_I = \varepsilon(\tilde{o}_I) = 0, 1 \))

\[
[\tilde{o}_I, \tilde{o}_J] = \sum_{m=2}^{n} \tilde{f}_{ij}^{K_1 \cdots K_m} \prod_{l=1}^{m} \tilde{o}_{K_l}, \quad \tilde{f}_{ij}^{K_1 \cdots K_m} = -(-1)^{\varepsilon_I \varepsilon_J} \tilde{f}_{ij}^{K_1 \cdots K_m}, \tag{19}
\]

then due to the requirement of the composition law for a direct sum,

\[
\tilde{o}_J \longrightarrow \tilde{O}_J = \tilde{o}_J + \tilde{o}_J^\prime : \{ \tilde{o}_I, \tilde{o}_J^\prime \} : \mathcal{H} \to \mathcal{H}, \quad [\tilde{o}_I, \tilde{o}_J] = 0, \quad \mathcal{H} \cap \mathcal{H}^\prime = \emptyset,
\]

such that the set of enlarged operators \( \{ \tilde{O}_J \} \) must obey involution relations, \( [\tilde{O}_I, \tilde{O}_J] = \tilde{F}^{(l)}_{IJ}(\tilde{o}_I, \tilde{o}_J)\tilde{O}_K \), the sets \( \{ \tilde{o}_J^\prime \}, \{ \tilde{O}_J \} \) form nonlinear superalgebras \( \mathcal{A}', \mathcal{A}_c \), given in \( \mathcal{H}' \) and \( \mathcal{H} \otimes \mathcal{H}' \) with the corresponding explicit multiplication laws

\[
\{ o'_I, o'_J \} = \sum_{m=2}^{n} (-1)^{l-1+\varepsilon_{K(l)}} \tilde{f}_{ij}^{K_1 \cdots K_l} \prod_{s=1}^{l} o'_{K_s}, \tag{20}
\]

\[
[\tilde{O}_I, \tilde{O}_J] = \left( \sum_{l=2}^{n} F^{(l)}_{IJ} (o'_I, \tilde{O}_J) \right) \tilde{O}_K, \quad \varepsilon_{K(l)} = \sum_{s=1}^{n-1} \varepsilon_K \left( \sum_{l=1}^{n} \varepsilon_{K_I} \right), \tag{21}
\]

\[
F^{(l)}_{IJ} = \sum_{l=2}^{n} \left( \sum_{s=1}^{n-1} \varepsilon_K \right) \tilde{f}_{ij}^{K_1 \cdots K_l} \prod_{s=1}^{l} o'_{K_s}, \tag{22}
\]

where the sum \( \sum_{s=1}^{n-1} \varepsilon_{K(l)} \) contains \( \frac{n!}{s!(n-s)!} \) terms with all the possible ways of ordering the indices \( (K_{s+1}, \ldots, K_l) \) among the indices \( (K_s, \ldots, K_1) \) in \( f_{ij}^{K_s \cdots K_1} K_{s+1} \).

As a consequence, for \( n = 2 \) in Proposition, as well as for the algebraic relations given by Table [1] for \( (\mathcal{A})_6 \), we can obtain relations (for the first time deduced in [34] for quadratic superalgebras) for the (super)algebras \( (\mathcal{A}')_6 \) of the additional \( \tilde{o}_I^I \) and for the (super)algebras \( (\mathcal{A}')_6 \) of the converted operators \( \tilde{O}_I \). These relations remain the same for the linear (Lie) part of the superalgebras, with the only respective change \( \tilde{o}_I \to (o'_I, \tilde{O}_I) \), whereas the quadratic ones [11]–[15] take the form (with a preservation of
Table [1] except for the replacement \((\mathcal{K}^{bi}_0, \mathcal{K}^{bi+}_0, M^i, M^{i+}) \rightarrow -(\mathcal{K}^{bi}_0, \mathcal{K}^{bi+}_0, M^i, M^{i+})\), for \(\mathcal{A}'\), \(r\mathcal{K}_1^{bi}, r\mathcal{K}_1^{bi+}, M^i, M^{i+}\) \(\rightarrow -(rV^i_W, -rV^i_W, M^i_W, M^{i+}_W)\) for \(\mathcal{A}_c\).

\[
M^{ni} = -r\left(2\sum_{k} t^{lki}l^{kli} + g_0^it^ni - \frac{1}{2}t'^{ni} - t'l'^1\delta^{i2} - tl'^2\delta^{i1}\right),
\]

(23)

\[
W^{nij} = W'^{nij} - \frac{r}{4}t'^{ij}t^ni = -2r\varepsilon^{ij} \left[(g_0^i - g_0^j)t^l12 - t'^lt'^ni1 + t'^lt'^l22 - \frac{r}{4}t'^{ij}t^ni\right],
\]

(24)

\[
\mathcal{K}^{bi}_1 = \left(4\sum_{k} l^{ki}l^{kni} + t'^{(2g_0^i - 1) - 2t'^{l2} + t'^l1 - 2l'^{i1} + l'^{i2}}\right),
\]

(25)

\[
X^{nij} = \left\{t'^{ni} - r\left(\sum_{k} K^{i1k}_0 + K^{b0i}_0 + \frac{1}{2}K^{i1i}_0 + K^{ij2}_0\right)\delta^{ij} + r\left[4\sum_{k} l^{ki}l^{k2i} - \frac{1}{2}t'^{i1} + t'^{i2}
\right.
\]
\[
\left.\left(\sum_{k} g_0^k - \frac{3}{2}t\right)\delta^{i1}\delta^{i2} + \left\{4\sum_{k} l^{ki}l^{k2i} - \frac{1}{2}t'^{i2} + t'^{i1} + t'^\left(\sum_{k} g_0^k - \frac{3}{2}\right)\right\}\right)\right\},
\]

(26)

\[
X'^{nij} = \left\{t'^{ni} - r\left(K^{i0i}_0 + K^{ij2}_0\right)\delta^{ij} + r\left[4\sum_{k} l^{ki}l^{k2i} + \left(\sum_{k} g_0^k - 2t\right)\delta^{i1}\delta^{i2}
\right.
\]
\[
\left.\left[4\sum_{k} l^{ki}l^{k2i} + t'^\left(\sum_{k} g_0^k - 2t\right)\right\}\right\},
\]

(27)

In their turn, the only modified relations, for instance, in the converted algebra \(\mathcal{A}_{bc}\), have the form (with the choice of Weyl’s ordering of \(\hat{O}_b\) in the r.h.s. of the commutators), which implies, as in [34], an exact expression for the BRST operator,

\[
V_{bW}^{0i} = -r\left(2(L^{ii} + 2t'^{ii})L^i + 2(L^i - 2t'^{i})L^{ii} + (L^{i+} - 2t'^{i})G_0^i + (G_0^i - 2g_0^i)L^{i+}
\right.
\]
\[
\left.\left.+ 2\left(L^{12} + 2t'^{12}\right)L^{i1} + L^i - 2t'^{i}L^{l2} + (L^{i2} - 2t'^{i2})\right)\delta^{i1}(L^{2i} - 2t'^{2i})T
\right.
\]
\[
\left.\left.+ (T - t'^i)\delta^{i1}(L^{2i} - 2t'^{2i}) + (T + t'^i)\delta^{i1}(L^{2i} - 2t'^{2i})\right)\right\},
\]

(28)

\[
W_{bW}^{ij} = r\varepsilon^{ij}\left\{\sum_{k} (\delta^{j1} - 1)^k(G_0^k - 2g_0^k)L^{12} + (L^{12} - 2t'^{12})\sum_{k}(\delta^{j1} - 1)^kG_0^k - [(T - t'^i)\]
\]
\[
\left.\left(L^{i1} - 2t'^{i1}) + (T + t'^i)\right)\delta^{j1}\delta^{i2} + \left[\sum_{k}(G_0^k - 2g_0^k)T^{i1} + (T - t'^i)\right.\right]
\]
\[
\left.\left.\times \left(\delta^{j1} - 1)^kG_0^k - \left(\sum_{k}(G_0^k - 2g_0^k)T^{i1}\right)\delta^{j1}\delta^{i2}\right)\right\},
\]

(29)

In Eqs. (26), (27), we have presented the quantities \(K^{0i}_0, K^{bi}_0 = (g_0^i - 2g_0^j - 4t'^{i1})\), being Casimir operators for the bosonic subalgebras \(so(2,1)\) generated by \(l_{ii}, l'^{i1}, g_0^i\) for each \(i = 1, 2\). The operators \(K^{li}_0, K^{li+}_0 = (g_0^i + t'^i + t')\) extend \(K^{bi}_0\) up to the Casimir operators \(K^{0i}_0, K^{0j}_0 = (K^{0i}_0 + K^{j0}_0)\), of the Lie subsuperalgebras in \(\mathcal{A}_c\) generated by \((t'^i, l'^{i1}, l'^{i1}, g_0^i)\) for each \(i = 1, 2\), and the quantity \(K^{ij2}_0\) extend \(\sum_{i} K^{0i}_0, \sum_{i} K^{ij}_0\) up to the respective Casimir operators \(K_0, K^{bi}_0\),

\[
K_0 = K^{bi}_0 + \sum_{i} (K^{0i}_0 + K^{li}_0) + 2K^{ij2}_0, K^{ij2}_0 = t'^{i1}t' - g_0^{ij} - 4t'^{i2}t'^{ij2},
\]

(31)

of the maximal (in \(A\)) Lie superalgebra \(A^{Lie}\) generated by \((t'^i, l'^{i1}, l'^{i1}, g_0^i, t, t')\), \(i, k = 1, 2\), and of its \(so(3,2)\) subalgebra.
These operators appear to be crucial to realize the operators of the (super)algebra \( (\mathcal{A}')_f \) in terms of the creation and annihilation operators of a new Fock space \( (\mathcal{H}')_b \), whose number of pairs is equal to that of the converted 2nd-class constraints \( o_a \), which allows one to obtain the correct number of physical degrees of freedom describing the basic spin-tensor [\( \Pi \)] in the final LF, after an application of the BFV–BRST procedure to the resulting first-class constraints \( \{\hat{O}_a\} \subset \{\hat{O}_I\} \).

Among the two variants of an additive conversion for the non-linear superalgebras [35] of \( \{\alpha_I\} \) into the 1st-class system \( \{O_\alpha\} \) [first, for the total set of \( \{o_a\} \)], resulting in an unconstrained LF, second, for the differential and partly algebraic constraints \( l_i, l_i^+, t, t^+ \), restricting the (super)algebra \( \mathcal{A} \) to the surface \( \{o_a\} = \{o_a\} \setminus \{l_i, l_i^+, t, t^+\} \) at all the stages of the construction, resulting in an LF with off-shell \( \gamma \)-traceless (only for the fermionic HS field \( \Phi(\mu_{\alpha_1},(\nu)_{\alpha_2}) \) and (only) traceless conditions for the fields and gauge parameters), we consider in detail the former case. To find \( o'_I \) explicitly, we need, first, to construct an auxiliary representation, known as the Verma module [32], on the basis of a Cartan-like decomposition, extended from the one for \( \mathcal{A}^{\text{Lie}} \),

\[
\mathcal{A}' = \{t_i^+, t_i^{ij+}, t_i^{t+}; t_i^{l_2^+}\} \oplus \{g_0^i; t_0^i, l_0^i\} \oplus \{t_i', t_i'^{ij}, t_i'; l_i^i\} \equiv \mathcal{E}^- \oplus H \oplus \mathcal{E}^+, \tag{32}
\]

and then to realize the above Verma module as an operator-valued formal power series

\[
\sum_{n \geq 0} \sqrt{r^n} \mathcal{P}_n[(a, a^+)_a]\] in a new Fock space \( \mathcal{H}' \) generated by \( (a, a^+)_a = f_i; f_i^+ b_i, b_i^+, b_{ij}, b_{ij}^+, b, b^+ \) (for a constrained LF, \( \{o'_a\} \leftrightarrow (a, a^+)_a = \{b_i, b_i^+, b, b^+\} \)).

A solution of these problems is more involved than the analysis made for a non-linear \( \mathcal{A}' \), see [34], and for a Lie superalgebra \( \mathcal{A}'^{\text{Lie}} \), see [16], due to a nontrivial entanglement of the triplet of non-commuting negative root vectors \( (l_1^+ t^+ l_2^+) \) and their ordered products \( ((l_1^+)^n(t^+)^n(l_2^+)^n) \), \( n_i, n \in \mathbb{N}_0 \) (composing the non-commuting part of an arbitrary vector of the Verma module \( V_{\mathcal{A}}' \)), as follows from Table II. This task should be effectively solved iteratively, e.g., for an action of the operator \( t' \) on \( (l_2^+)^{n_2} \),

\[
t'((l_2^+)^{n_2}) \rightarrow \sum_{m_j=0} (t_2^+)^{n_2-1-2m_j} t_1^{1+} + \ldots \rightarrow t' \sum_{m'=0} \sum_{m=0} (t_2^+)^{n_2-2-2m-2m'} + \ldots ,
\]

where the remaining summand does not contain any incorrectly ordered terms, thus, extending the known results of Verma module construction [33] and its Fock space realization in \( \mathcal{H}' \). First, note that there is no nontrivial entanglement of the above triplet of negative root vectors, due to a restriction of Table II to the surface determined by the non-converted second-class constraints \( l^{ij}, l^{ij+}, t^i, t^{t+} \), and, therefore, the finding of an operator realization of the restricted \( \mathcal{A}'_r \) follows the known way [34]. Second, within our conversion procedure the enlarged central charge \( \tilde{M} = \tilde{m} + \tilde{m}' \) vanishes, whereas explicit expressions for \( o'_I \) in terms of \( (a, a^+)_a \) and the new constants \( m_0, h^i, (l_0^i, g_0^i) = (m_0^2, h^i) + \ldots \) [they are to be determined later from the condition of reproducing the correct form of Eqs. (3)] are found by partially following [33, 34].
4. BFV-BRST OPERATOR FOR CONVERTED (SUPER)ALGEBRA

To construct a BRST operator for a non-linear non-gauge (super)algebra \((A_c)A_{bc}\), we shall use an operator version of finding a BRST operator, described in Ref. [23], and classically in Ref. [24]. Due to the quadratic algebraic relations \(28\)–\(30\) and their Hermitian conjugates, we must check a nontrivial existence of new structure relations and new structure functions of 3rd order [24], implied by a resolution of the Jacobi identities \((-1)^{\varepsilon \varepsilon K}[\tilde{O}_I, \tilde{O}_J], \tilde{O}_K] + \text{cycl.perm.}(I, J, K) = 0\), for \((A_c)A_{bc}\), \(n = 2\) in [21], [22],

\[
(-1)^{\varepsilon \varepsilon K}(f_{IJ}^M + F_{IJ}^{(2)M})(f_{MK}^P + F_{MK}^{(2)P}) + (-1)^{\varepsilon \varepsilon K}(f_{IJ}^{(2)P}, \tilde{O}_K)
\]

+\text{cycl.perm.}(I, J, K) - \frac{1}{2} F_{IJK}^{RS}(J_{RS}^P + F_{RS}^{(2)P}) = F_{IJK}^{RP} \tilde{O}_R,
\]

\[
F_{IJK}^{(2)K}(\phi', \tilde{O}) = -(f_{IJ}^{MK} + (-1)^{\varepsilon \varepsilon M} f_{IJ}^{KM}) \phi'_M + f_{IJ}^{MK} \tilde{O}_M,
\]

(33)

with the 3rd order structure functions \(F_{IJK}^{RS}(\phi', \tilde{O})\) satisfying the properties of generalized antisymmetry with respect to a permutation of any two of the lower indices \((I, J, K)\) and the upper indices \(R, S\).

If the 4th-, 5th- and 6th-order structure functions \(F_{IJKL}^{PRS}(\phi', \tilde{O})\), \(F_{IJKLM}^{PRST}(\phi', \tilde{O})\), \(F_{IJKLMN}^{PQRSTUVWXYZ}(\phi', \tilde{O})\) are zero, the BRST operator \(Q'\) has the form of the one for a formal 2nd-rank “gauge” theory [24], i.e., it has an exact form for the \((CP)\)-ordering of the ghost coordinates \(C^I\), bosonic, \(q_0, q_i, q'^i_0\), and fermionic, \(\eta_0, \eta^+_i, \eta_\bar{i}, \eta^+_{i\bar{j}}, \eta_{ij}, \eta, \eta^+\), \(\eta_{i\bar{j}}\), and their conjugated momenta operators \(P_I\): \(p_0, p_i, \bar{p}_i, \bar{P}_0, \bar{P}_i, \bar{P}_i, \bar{P}_0\), \(P, P^+, \bar{P}, \bar{P}^+, P_G\), see [16], with the Grassmann parities opposite to those of \(\tilde{O}_I\) and the values of ghost number \(gh(C^I) = -gh(P_I) = 1\),

\[
Q' = C^I[\tilde{O}_I + \frac{1}{2} C^I(f_{IJ}^P + F_{IJ}^{(2)P})P_P(-1)^{\varepsilon I + \varepsilon P} + \frac{1}{12} C^I C^K F_{IKJ}^{RP} P_K P_P(-1)^{\varepsilon I + \varepsilon K + \varepsilon J + \varepsilon R}].
\]

(34)

The requirement of \((CP)\)-ordering for the equations following from the nilpotency condition for \(Q'\) in the \(n\)th-order in \(C^I\), \(n = 3, 4, 5, 6\), leads, for instance, for \(n = 3\) to the necessity or of the separate fulfillment the relations which do not appear in the classical case [24]:

\[
(-1)^{\varepsilon \varepsilon K}(F_{IJ}^{(2)M}, F_{MK}^{(2)P}) + \text{cycl.perm.}(I, J, K) - \frac{1}{2} [F_{IJK}^{RS}, F_{RS}^{(2)P}] = [F_{IJK}^{RP}, \tilde{O}_R],
\]

(35)

or of a corresponding extension of the 3rd structural relations \((33)\) by these terms.

In the case of a bosonic algebra \(A_{bc}\), there are 3 types of nontrivial Jacobi identities for 6 triplets \((L_1, L_2, L_0), (L^+_1, L^+_2, L_0), (L_i, L^+_i, L_0)\), with the existence of 3rd-order structure functions. For instance, one of the solutions for \((L_i, L^+_i, L_0)\) after a reduction of \(L^+_i\) has

\footnote{Given by Eqs. \((33)\), the resolution of Jacobi identities for a nonlinear superalgebra is more general than the one presented in Ref. [37] for a classical (super)algebra, because we do not examine a more restrictive vanishing of all the coefficients at the 1st, 2nd and 3rd degrees in \(\tilde{O}_I\).}
the form

\[ 2 \left\{ \delta^2 \delta^1 \left[ (L^{22} - 2t^{22})(T^+ - 2t^{\prime+}) + (G^i_0 - 2g^i_0)(L^{12} - 2t^{12}) \right] + r^{-1}(\tilde{W}^{ij}_{\delta^i} - 2W^{\delta^i}_{\delta^i}) - (T - 2t')(L^{11} - 2t^{11}) - (G^i_0 - 2g^i_0)(L^{12} - 2t^{12}) \right] \]

\[ - \varepsilon^{(1j)\delta^2i} \left[ (L^{12} - 2t^{12})(T^+ - 2t^{\prime+}) - (T^+ - 2t^{\prime+})(L^{12} - 2t^{12}) \right] \right\} = \delta^2 \delta^1 \left\{ \{ L_{11}, T \} - \{ L_{22}, T^{+} \} - \{ L_{12}, G^2_0 - G^1_0 \} - 4L^{12} \right\} + 2 \varepsilon^{(1j)\delta^2i} L^{11}. \]  

(36)

As a result, in view of the absence of higher-order structure functions, a nilpotent BRST operator $Q'$ [34] for $A_{bc}$ has an exact form of the maximal 3rd degree in the powers of ghosts $C'$:

\[ Q' = \frac{1}{2} \eta_0 L_0 + \eta_i^+ L^i + \eta_{im} L^{im} + \eta^+ T + \frac{1}{2} \eta_i G_i + \frac{1}{2} \eta_i^+ \eta^+ \mathcal{P}_0 + \frac{1}{2} \eta_i^+ \eta_i^+ \mathcal{P}_G + \frac{1}{2} \eta_i^+ \eta^i \mathcal{P}_0 \\
+ \frac{1}{2} \eta_i^+ \eta^i \mathcal{P}_G + (\eta_i^+ G_i + \eta_i^+ \eta^i) \mathcal{P}_i + 2 \eta_i^+ \eta_i^+ \mathcal{P}_{ii} - \eta_{12}(\eta_1^+ \mathcal{P}_{11}^0 + \eta^2 \mathcal{P}_G^0) \\
- 2 \left[ \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_{12} + \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_G \\
+ \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_{12} + \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_G \\
+ \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_{12} + \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_G \right] \\
+ \left\{ \begin{array}{l}
\frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_{12} + \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_G \\
+ \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_{12} + \frac{1}{2} \sum_k (L^k - 2t^{2k}) \mathcal{P}_G \right\} \right]. \]  

(37)

The property of the BRST operator $Q'$ to be Hermitian is defined by the rule

\[ Q'^+ K = K Q', \quad \text{for} \quad K = \hat{i} \otimes K' \otimes \hat{i}_{gh}, \]  

(38)
with unity operators in \( \mathcal{H} \), \( \mathcal{H}_{gh} \), and the operator \( K' \) providing the Hermiticity of the additional parts \( \sigma'_j \) in \( \mathcal{H}' \), as in Refs. [16,34]. We first note that the BRST operator for the superalgebra \( \mathcal{A}_c \) is treated in the same way, with the above nontrivial Jacobi identities, whose r.h.s. is extended by a fermionic operator. Second, in the case \( i = j = 1 \), \( Q \) yields a BRST operator for the converted algebra \( \mathcal{A}_{bc} \) in the case of totally-symmetric bosonic HS fields [33].

5. LAGRANGIAN FORMULATIONS

A covariant extraction of \( G^i_0 = g^i_0 + g^i_0(h^i) \) from \( \{ \bar{O}_j \} \), in order to pass to that part of the converted 1st-class constraints \( \{ O_a \} \) which corresponds only to equations (17), (18), i.e., the constraints \( \{ L_0, L^i, L^{ij}, T \} \), is based on a special representation for \( \mathcal{H}_{gh} \) in \( \mathcal{H}_{tot} = \mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{gh} \), such that the operators \( \eta_i, \eta_{ij}, \eta, P_0, P_i, P_{ij}, P, P_G \) should annihilate the vacuum \( |0\rangle \), as well as on the elimination from \( Q' \) of the terms proportional to \( P_G^i, \eta_G^i, K^i = (\sigma^i + h^i) \), as in [16,17],

\[
Q' = Q + \eta_0^i K^i + B^i P_G^i, \quad K^i = G^i_0 + (\sum_j (1 + \delta_{ij}) \eta^{ij} + (1 - h^i) \eta^i P + h.c.). \tag{39}
\]

The same applies to a scalar physical vector \( |\chi\rangle \in \mathcal{H}_{tot}, |\chi\rangle = |\Phi\rangle + |\Phi_A\rangle, |\Phi_A\rangle_{(a,a^+)} = c = p = 0 \) = 0, with \( |\Phi\rangle \) given by (5) and with the use of the BFV–BRST equation \( \tilde{Q}'|\chi\rangle = 0 \) that determines physical states,

\[
\tilde{Q}|\chi\rangle = 0, \quad (\sigma^i + h^i)|\chi\rangle = 0, \quad (\varepsilon, gh)(|\chi\rangle) = (0,0). \tag{40}
\]

Notice that the second equations must take place in the entire \( \mathcal{H}_{tot} \), thus determining the spectrum of spin values for \( |\chi\rangle \) and the corresponding proper eigenvectors,

\[
h^i = -(s_i + \frac{d^2 - 5}{2} - 2\delta^2), \quad (s_1, s_2) \in (\mathbb{Z}, \mathbb{N}_0), \quad |\chi\rangle_{(s_1,s_2)}, \tag{41}
\]

whereas the first equation is valid only in the subspace of \( \mathcal{H}_{tot} \) with the zero ghost number.

Because of the commutativity of \( \sigma^i \) with \( Q \), the latter, being subject to the substitution \( h^i \to -(s_i + \frac{d^2 - 5}{2} - 2\delta^2), \) i.e., \( Q \to Q_{(s_1,s_2)} \), is nilpotent in each of the subspaces \( H_{tot(s_1,s_2)} \) whose vectors obey Eqs. (40), (41). Thus, the equations of motion are in a one-to-one correspondence with Eqs. (17), (18), whereas the sequence of reducible gauge transformations has the form

\[
Q_{(s_1,s_2)}|\chi^0\rangle_{(s_1,s_2)} = 0, \quad \delta|\chi^l\rangle_{(s_1,s_2)} = Q_{(s_1,s_2)}|\chi^{1+l}\rangle_{(s_1,s_2)}, \quad l = 0, \ldots, 6, \tag{42}
\]

for \( |\chi^0\rangle \equiv |\chi\rangle \), and can be obtained from the Lagrangian action

\[
S_{n_1,n_2} = \int d\eta_0 (s_1, s_2) \langle \chi^0 | K_{(s_1, s_2)} Q_{(s_1, s_2)} |\chi^0\rangle_{(s_1, s_2)} K_{(s_1, s_2)} = K_{\chi^0} \rightarrow -(s_i + \frac{d^2 - 5}{2} - 2\delta^2), \tag{43}
\]

where the standard \( \varepsilon \)-even scalar product in \( \mathcal{H}_{tot} \) is assumed.
The corresponding LF of a bosonic field with a specific value of spin $s$ subject to $Y(s_1, s_2)$ is an unconstrained reducible gauge theory of $L = 5$-th stage of reducibility.

6. CONCLUSIONS

We have found and studied the properties of nonlinear operator algebras and superalgebras underlying integer and half-integer HS fields in an AdS$_d$ space that are subject to a Young tableaux with two rows. To construct a BFV–BRST operator, whose special cohomology in the corresponding Hilbert space with a vanishing ghost number should coincide with the space of solutions for the equations that determine (spin-)tensors of the AdS-group irreducible representation with a given mass and generalized spin, we formulate a proposition that determines the form of algebraic relations both for the parts additional to the operators of the initial polynomial superalgebra with given relations and for the superalgebra $\mathcal{A}_c$ of additively converted operators. We have briefly shown, first, the solvability of the algorithm of constructing the Verma module for the superalgebra of the additional parts, and, second, have described a way to realize it in terms of a formal power series in the corresponding Fock-space variables, which completes our procedure of converting the subset with 2nd-class constraints into that of converted 1st-class ones. We have obtained an exact BFV–BRST operator having terms of at most 3rd degree in ghost coordinates for a general quadratic operator superalgebra having nontrivial 3rd-order structure functions and relations that resolve the Jacobi identities for $\mathcal{A}_c$ only. Following this prescription, we have found an exact BRST operator for a nonlinear algebra $\mathcal{A}_{bc}$ of converted operators. Finally, on a basis of the resulting BRST operator, we have developed a gauge-invariant approach to an unconstrained LF, given by Eqs. (42), (43), which describes the dynamics of free bosonic HS fields in an AdS$_d$ space with an index symmetry corresponding to a two-row Young tableaux. We must draw the attention of the reader to some problems that have remained not entirely solved: an explicit form of the Verma module for the (super)algebra in question, the deduction of a BRST operator for a converted superalgebra with fermionic HS fields, its application to a construction of the corresponding LF, and a more detailed development of the constrained LF.

The author is grateful to I.A. Batalin for the comments and indication the typos, to P.Yu. Moshin for useful discussions and thanks the organizers of the “XXVII International Colloquium on Group Theoretical Methods in Physics”, Yerevan, Armenia, August 2008, for support and hospitality. The work was supported by an RFBR grant, project No. 08-02-08602.

[1] N. Beisert, M. Bianchi, et al, JHEP 0407 058 (2004).
[2] P.J. Heslop, F. Riccioni, JHEP 0510 060 (2005).
[3] T. Curtright, Phys. Lett. B165 (1985) 304,
[4] J.M.F. Labastida, T.R. Morris, Phys. Lett. B180 101 (1986).
[5] M. Vasiliev, Fortsch.Phys. 52 702 (2004).
[6] D. Sorokin, AIP Conf.Proc. 767 172 (2005).
[7] R.R. Metsaev, Phys. Lett. B354 78 (1995).
[8] M.A. Vasiliev, Sov. J. Nucl. Phys. 32 439 (1980).
[9] K.B. Alkalaev, O.V. Shaynkman, M.A. Vasiliev, Nucl. Phys. B692 (2004) 363.
[10] G. Barnich, M. Grigoriev, JHEP 08 013 (2006).
[11] M.A. Vasiliev, Phys. Lett. B209 491 (1988).
[12] C. Fronsdal, Phys. Rev. D18 3624 (1978).
[13] Yu.M. Zinoviev, *On Massive Mixed Symmetry Tensor Fields in Minkowski space and (A)dS*, arXiv:hep-th/0211233.
[14] D. Francia, A. Sagnotti, Phys. Lett. B543 303 (2002).
[15] X. Bekaert, N. Boulanger, Commun. Math. Phys. 271 723 (2007).
[16] P.Yu. Moshin, A.A. Reshetnyak, JHEP, 10 040 (2007).
[17] I.L. Buchbinder, V.A. Krykhtin, H. Takata, Phys. Lett. B656 253 (2007).
[18] E.D. Skvortsov, Nucl.Phys. B808 569 (2009).
[19] R.R. Metsaev, Class. Quant. Grav. 22 2777 (2005).
[20] K.B. Alkalaev, O.V. Shaynkman, M.A. Vasiliev, JHEP 0508 069 (2005).
[21] Yu.M. Zinoviev, *Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields* arXiv:0809.3287.
[22] E.S. Fradkin, G.A. Vilkovisky, Phys. Lett. B55 224 (1975); I.A. Batalin, G.A. Vilkovisky, Phys. Lett. B69 309 (1977); I.A. Batalin, E.S. Fradkin, Phys. Lett. B128 303 (1983).
[23] I.A. Batalin, E.S. Fradkin, Ann. Inst. H. Poincare, A49 145 (1988).
[24] M. Henneaux, Phys. Rept. 126 1 (1985).
[25] E. Witten, Nucl.Phys B268 253 (1986); N. Seiberg E. Witten, Nucl.Phys. B276 272 (1986).
[26] C.S. Aulakh, I.G. Koh, S. Ouvry, Phys. Lett. 173B 284 (1986); S. Ouvry, J. Stern, Phys. Lett. 177B 335 (1986).
[27] A. Pashnev, M. Tsulaia, Mod. Phys. Lett. A15 281 (2000).
[28] M. Fierz, W. Pauli, Proc.R.Soc.London, Ser. A, 173 211 (1939);
[29] L.P.S. Singh, C.R. Hagen, Phys. Rev. D9 898 (1974); *ibid*, Phys. Rev. D9 910 (1974).
[30] L.D. Faddeev, S.L. Shatashvili, Phys.Lett. B167 225 (1986).
[31] I.A. Batalin, E.S. Fradkin, T.E. Fradkina, Nucl. Phys. B314 158 (1989); I.A. Batalin, I.V. Tyutin, Int.J.Mod.Phys. A6 (1991) 3255.

[32] J. Dixmier, Algebres enveloppantes, Gauthier-Villars, Paris (1974).

[33] C. Burdik, O. Navratil, A. Pashnev, On the Fock Space Realizations of Nonlinear Algebras Describing the High Spin Fields in AdS Spaces, [hep-th/0206027]; I.L. Buchbinder, V.A. Krykhtin, P.M. Lavrov, Nucl.Phys. B762 344 (2007).

[34] I.L. Buchbinder, V.A. Krykhtin, A.A. Reshetnyak, Nucl. Phys. B787 211 (2007).

[35] A.A. Reshetnyak, On Lagrangian formulations for mixed-symmetry HS fields on AdS spaces within BFV-BRST approach, to appers in Proc. of XIII International Conference “Some Modern Problems of Modern Theoretical Physics” (SMTPP08), June 23–27, 2008. Dubna, [arXiv:0809.4815].

[36] K. Schoutens, A. Sevrin, P. van Nieuwenhuizen, Comm. Math. Phys. 124 87 (1989).

[37] I.L. Buchbinder, P.M. Lavrov, Math.Phys. 48 082306 (2007) ;M. Asorey, P.M. Lavrov et all, BRST structure of non-linear superalgebras, [arXiv:0809.3322].

[38] A.P. Isaev, S.O. Krivonos and O.V. Ogievetsky, J. Math.Phys.49 073512 (2008).