Four-Fermion Theory and the Conformal Bootstrap

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Abstract

We employ the conformal bootstrap to re-examine the problem of finding the critical behavior of four-Fermion theory at its strong coupling fixed point. Existence of a solution of the bootstrap equations indicates self-consistency of the assumption that, in space-time dimensions less than four, the renormalization group flow of the coupling constant of a four-Fermion interaction has a nontrivial fixed point which is generally out of the perturbative regime. We exploit the hypothesis of conformal invariance at this fixed point to reduce the set of the Schwinger-Dyson bootstrap equations for four-Fermion theory to three equations which determine the scale dimension of the Fermion field $\psi$, the scale dimension of the composite field $\bar{\psi}\psi$ and the critical value of the Yukawa coupling constant. We solve the equations assuming this critical value to be small. We show that this solution recovers the fixed point for the four-fermion interaction with $N$-component fermions in the limit of large $N$ at (Euclidean) dimensions $d$ between two and four. We perform a detailed analysis of the $1/N$-expansion in $d = 3$ and demonstrate full agreement with the conformal bootstrap. We argue that this is a useful starting point for more sophisticated computations of the critical indices.
1 Introduction

Conformal symmetry governs the critical behavior of a quantum field theory near the fixed points of its renormalization group flow [1, 2, 3]. It is most powerful in two spacetime dimensions where the requirement of conformal invariance of correlation functions leads to a solution of a quantum field theory in terms of a few parameters [4]. In this Paper we shall show that conformal symmetry is also useful in higher dimensions. In any spacetime dimensions, it determines the form of the two and three-point Green functions up to a few constants, the scaling dimension of the field operators and the value of the coupling constant at the fixed point. When this information is combined with Schwinger-Dyson equations, it can provide a powerful tool for the analysis of field theories. We shall apply the conformal bootstrap technique to four-Fermion interactions in dimensions $2 < d < 4$. An application of this technique to $\lambda \phi^4$ theory [5] and some preliminary work for spinors with four-Fermion interactions [6] between 2 and 4 dimensions have been given previously by Yu. Makeenko.

Fermionic field theories with four-fermion interactions have a long history. A classic example is the Nambu-Jona-Lasinio model which was proposed some thirty years ago by Nambu and Jona-Lasinio [7] (see also [8]) as a quantum field theory which exhibits dynamical chiral symmetry breaking for sufficiently strong attractive fermion-anti-fermion interactions in four (space-time) dimensions. The study of related models has recently seen a revival with speculations that the symmetry breaking mechanisms which it describes could be used in the standard model to describe the Higgs particle as a top quark condensate [9, 10].

In the conventional perturbative expansion in powers of the coupling constant, four-fermion interactions are not renormalizable (or are irrelevant operators) in space-time dimensions greater than two. This can be seen by simple power-counting. Consider the Euclidean action in d dimensions

$$S = \int d^d x \left( \bar{\psi} \hat{\partial} \psi - \frac{G}{2} (\bar{\psi} \psi)^2 \right),$$

where $\hat{\partial} \equiv \gamma \cdot \partial$ and $G$ is the (bare) coupling constant. The coefficient of the four-
fermion coupling must have engineering dimension \(2 - d\):

\[
G = \frac{g^2}{\Lambda^{d-2}}. \tag{1.2}
\]

Here, we have defined the dimensionless coupling constant \(g^2\) and have included powers of the ultraviolet cutoff \(\Lambda\) to give the vertex its correct dimension. In \(d = 2\) the coupling constant \(G\) is dimensionless and the model is renormalizable in the conventional sense. It was shown by Gross and Neveu \[1\] that it is asymptotically free, has nontrivial scaling and contains fermion-anti-fermion bound states.

In \(d > 2\), the four-fermion interaction is irrelevant in weak coupling perturbation theory. Because of the powers of cutoff \(1/\Lambda^{d-2}\) in the bare coupling, perturbative contributions to the full four-fermion vertex vanish as the cutoff is sent to infinity. For example, the contribution of each of the bubble diagrams in Fig. 1 (the corresponding Feynman integral has ultraviolet divergence \(\sim \Lambda^{d-2}\)) goes to zero like \(1/\Lambda^{d-2}\) if the external momentum is kept fixed at a finite value and \(\Lambda \to \infty\). Thus, the four-fermion operator is irrelevant at weak coupling: its contributions to four-fermion scattering vanish and its contribution to relevant vertices can be absorbed by local renormalizable counterterms.

On the other hand, it was conjectured long ago \[2\] that the four-fermion interaction could perhaps be relevant if, instead of a simple perturbative expansion, one considered some nontrivial resummation of infinite numbers of diagrams. For example, if we consider bubble graphs of the kind depicted in Fig. 1, the term with \(n\) bubbles is

\[
n\text{-bubble graph} \sim g^{2(n+1)}\Lambda^{(n+1)(2-d)} (\Gamma(\Lambda, p))^n, \tag{1.3}
\]

where \(\Gamma(\Lambda, p) \sim \Lambda^{d-2} + \ldots\) is the contribution from a single bubble. The geometric sum over all bubble graphs of the kind depicted in Fig. 1 has the form

\[
\sum \sim \frac{g^2}{\Lambda^{d-2} - g^2 \Gamma(\Lambda, p)} + \text{crossed}. \tag{1.4}
\]

The interaction can be non-zero if

\[
g^2 = \frac{\Lambda^{d-2}}{\Gamma(\Lambda, 0)} + \mathcal{O}\left(\frac{\text{finite mass scales}/\Lambda}{d-2}\right). \tag{1.5}
\]
Thus, in order to produce non-zero interactions, the coupling constant must be precisely tuned to this strong coupling (ultraviolet stable) fixed point, within errors proportional to \( (\text{finite mass scales}/\Lambda)^{d-2} \). At the critical point, (1.4) behaves like \( \text{sum} \sim 1/p^{d-2} \) so that the effective scale dimension of the composite operator \( \bar{\psi} \psi \) is 1 (in momentum units) for any \( d \). When \( d \neq 2 \) this differs from its classical dimension, \( (d - 1) \). Moreover, the dimension of the four-fermion vertex is approximately two, rather than its classical dimension \( 2(d - 1) \). It is this large anomalous dimension of composite operators which makes the four-fermion interaction relevant.

The latter idea appeared in the work of Wilson [13] who observed that the sum of bubble diagrams in (1.4) is the leading order in the large \( N \) expansion where \( N \) is the number of fermion species. He argued that, in dimensions \( 2 < d < 4 \), as well as the infrared stable fixed point at zero coupling, the four-fermion coupling has an ultraviolet stable fixed point at a non-zero value of the coupling constant \( g_0^2 > 0 \). As \( d \to 2 \) this fixed point moves to the origin, \( g_0^2 \to 0 \) and gives the asymptotic freedom of the Gross-Neveu model [11]. He also observed that at \( d = 4 \) the four-fermion theory is trivial and is described by \( N \) non-interacting spinor fields and a free scalar field.

This led to the conjecture that, in \( 2 < d < 4 \) the systematic large \( N \) expansion of four-fermion theory is renormalizable in the conventional sense that, in order to remove the cutoff dependence of physical quantities as they are computed order by order in \( 1/N \), it is necessary to add local counterterms which have the same form as these already occurring in the action [14][15][16]. That this is indeed the case in \( d = 3 \) has only recently been proven using the renormalization group technique of constructive field theory [17].

There have also been numerous recent efforts, both analytical [18] and numerical [19], to understand the critical behavior of the four-fermion model around its fixed points. The numerical data in \( d=3 \) is in good agreement with analytical calculations in the large \( N \) expansion. The ultraviolet fixed point has the interesting feature that the critical indices there are much different from those at the Gaussian infrared fixed point.

In order to implement the large \( N \) expansion it is convenient to introduce an
auxiliary scalar field, so that the action takes the form

\[ S = \int d^d x \left( \bar{\psi}_i \hat{\partial} \psi_i + \frac{1}{\sqrt{N}} \phi \bar{\psi}_i \psi_i + \frac{1}{2G} \phi^2 \right), \tag{1.6} \]

where \( i = 1, \ldots, N \). The four-fermion interaction in (1.1) is recovered by solving the equation of motion for \( \phi \) or, equivalently, performing the gaussian functional integral over \( \phi \) in the partition function

\[ Z = \int d\psi d\bar{\psi} d\phi \exp(-S). \tag{1.7} \]

The introduction of \( \phi \) results in a Yukawa interaction which, as we shall see, is essential to the method we shall introduce in this paper. In the past this trick has been used to discuss \[20\] the equivalence of the four-Fermi theory and a fermion-scalar field theory coupled through Yukawa interactions when the wavefunction renormalization constant for the \( \phi \) field vanishes. This is known as the compositeness condition and, as has been suggested in some recent literature \[10\], can be used to reduce the number of parameters in the fermion-scalar field theory, thereby giving the model more predictive power. Indeed, one could consider a generalization of (1.6) which contains a kinetic term for the scalar field \( \frac{1}{2} \phi \Box \phi \) and possibly higher polynomials – in fact all of the relevant operators of the scalar fields theory. Then if one considers the renormalized kinetic term, \( \frac{1}{2} Z \phi_R \Box \phi_R \) the compositeness condition can be achieved if as a function of \( g^2 \) and other parameters, the equation \( Z \phi(g^2, \ldots) = 0 \) has a solution. If, at this solution, correlators of the field theory remain finite, the interacting fermion theory which would result from eliminating the scalar fields is also finite. (If the scalar field action has terms of higher than quadratic order, the resulting fermion theory is rather complicated, possibly non-polynomial.)

The action (1.6) is invariant under the discrete chiral transformation

\[ \psi_i \rightarrow \gamma_5 \psi_i, \quad \bar{\psi}_i \rightarrow -\bar{\psi}_i \gamma_5, \quad \phi \rightarrow -\phi. \tag{1.8} \]

where we consider the appropriate definition of \( \gamma^5 \) in less than 4 dimensions. (We use spinors with \( 2^{[d/2]} \) components, where \([d/2]\) is the smallest integer greater than or equal to \( d/2 \). In any integer dimensions such spinors have an analog of \( \gamma^5 \) which is the product of the gamma matrices, \( \prod_\mu \gamma^\mu \).) This chiral symmetry prevents the
fermion from acquiring a mass. When the four-fermion interaction is attractive and strong enough, there is a phase transition and the fermion acquires mass by spontaneous chiral symmetry breaking. The corresponding critical coupling constant is just the ultraviolet fixed point. A natural order parameter for the spontaneous chiral symmetry breaking is the expectation value of the auxiliary field $\langle \phi \rangle$ which, in our convention, has the dimension of mass. The critical point where the phase transition occurs can be approached from either the symmetric or the broken phase. Assuming the phase transition here is of second order, as strongly suggested by numerical Monte-Carlo simulation [19], the critical indices calculated in one phase must be equivalent to those calculated in the other.

In the present paper, we shall use the conformal bootstrap to examine four-fermion interactions in $2 < d < 4$ at the critical point. The conformal bootstrap was invented by Polyakov [1], Migdal [2], Parisi [3] and many others in the early seventies.

The basic idea is as follows. As the transition point is approached, various quantities either vanish (such as masses) or diverge and long-range correlations appear. Moreover, just at the critical point, the correlation function of fluctuations takes the form

$$\langle \delta s(x) \delta s(0) \rangle \mid x \mid \rightarrow \infty \sim |x|^{-(d-2-\eta)}.$$  \hfill (1.9)

This is exactly the form of two-point correlation functions which is implied by scale invariance, where the effective scale dimension of the operator $\delta s(x)$ is $\frac{d-2-\eta}{2}$.

In fact, as was first observed in [21], in addition to scale invariance, at the phase transition point the system is invariant under the full group of special conformal transformations. To see this, consider the conformal current $K_\mu^a(g^2, x)$ where $g^2$ is the effective coupling constant. Then the divergence of the conformal current is related to the beta function of $g^2$ through the trace of the energy-momentum tensor:

$$\partial_\mu K_\mu^a(g^2, x) \propto x^a \theta_{\mu\mu}(g^2, x),$$  \hfill (1.10)

which in turn is the divergence of the dilatation current $D_\mu(g^2, x)$ and, therefore, proportional to the beta function:

$$\theta_{\mu\mu}(g^2, x) = \partial_\mu D_\mu(g^2, x) \propto \beta(g^2).$$  \hfill (1.11)
As usual, the last equation is understood in the sense of matrix elements. If more than one coupling constant is involved, the right hand side of Eq. (1.11) would be a linear combination of all the corresponding beta functions.

As well as the two-point function, which is actually fixed (up to the critical exponent) by scale invariance, conformal symmetry also fixes the form of three-point functions up to multiplicative constants. The multiplicative constants are the fixed point values of the renormalized coupling constants — by definition — the zero point of their beta function.

To determine the unknown exponents and multiplicative constant (fixed point), one invokes the Schwinger-Dyson equations in bootstrap form – the homogeneous Schwinger-Dyson equations. The inhomogeneous term arising from bare couplings in the Schwinger-Dyson equation for the three-point vertex vanishes since it is multiplied by the inverse of an infinite renormalization constant. Since the bare vertex does not have the correct form to satisfy the conformal ansatz, this cancellation is in fact necessary for consistency of the assumption of conformal symmetry. This is discussed in detail in [2]. The bootstrap equations for the three-vertex are closed and have a conformal solution. They are free of ultraviolet and infrared divergences, as a result of renormalization and are sufficient to determine the scale dimension of the fields and the value of the coupling constant at the fixed point. In the present model, these are the critical coupling constant and the scaling dimensions of the fields $\psi$ and $\phi$. If the hyperscaling hypothesis [22] is correct, all other critical exponents can be determined.

This paper is organized as follows. In Section 2 we analyze the $1/N$ expansion of four-fermion theory at $d = 3$ to order $1/N$, demonstrate renormalizability and review the calculation of the anomalous dimensions of $\psi$ and $\phi$ to that order. In Section 3 we assume conformal invariance of the four-fermion theory at fixed point to derive the set of bootstrap equations which determines the critical indices. In Section 4 we consider the three-vertex approximation which is legitimate when the index of the Yukawa vertex is small, calculate the critical indices and show that results agree with the $1/N$ expansion. In the Appendix we present a derivation of the Parisi procedure for the case of the small index of Yukawa vertex.
2 \ 1/N expansion

In this section, we conduct a perturbative expansion of the theory in 1/N with emphasis on the conformal structure. For simplicity, we restrict our computations to \( d = 3 \), the generalization to any dimensionality in \( 2 < d < 4 \) is straightforward.

2.1 Bubble graphs

Our convention is that each fermion has two components and the Dirac matrices are Hermitean, namely, \( \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2 \delta_{\mu\nu} 1 \) and \( \text{Tr} \, 1 = 2^{d/2} = 2 \). In the symmetric phase, fermions have no dynamical mass. The Feynman rules in the momentum space use the fermion propagator and three-point vertex

\[
S_0(p) = \frac{1}{i\hat{p}} \quad \text{and} \quad \Gamma_0 = -\frac{1}{\sqrt{N}},
\]

respectively.

The scalar, \( \phi \), is not a dynamical but an auxiliary field. However, there formally exists a bare \( \phi \) ‘propagator’, \( G \), corresponding to the last term in the action (1.6). \( G \), the only free parameter of the theory, has dimension of mass\(^{-1}\).

On the other hand, at a fixed point all dimensionful parameters must cancel from expressions for correlation functions. Here, the cancellation of the dimensional coupling constant \( G \) occurs through the the regularization and renormalization procedure. If the theory is regularized by introducing a cutoff \( \Lambda \) which has the dimension of mass, one can choose \( 1/G \) so that it exactly cancels the terms in the scalar field self-energy that are linearly divergent, \( \sim \Lambda \). Or, if the theory is regularized by a method such as dimensional regularization where linear divergences are absent, \( 1/G \) set to zero. We can regard both of these as fine tuning of the bare coupling constant \( 1/G \) to the ultraviolet stable fixed point.

In this Section, we shall introduce a naive cutoff \( \Lambda \) as a regulator. In this case, we can define a dimensionless coupling constant \( g \) according to Eq. (1.2):

\[
G = \frac{g^2}{\Lambda}.
\]

(In this Section, \( d=3 \).)
The bare $\phi$-propagator is a momentum independent and is proportional to the coupling constant, $G$. In the $1/N$ expansion, the fermion bubble chains (Fig. 2) are of the same 0-th order as the bare $\phi$ propagator. Therefore, the correct $\phi$ propagator in the Feynman rules should be an effective one that sums over all one-fermion-loop chains. It is referred to as the dressed $\phi$ propagator and its inverse takes the form

$$
\Delta_0^{-1}(p, \Lambda) = \frac{1}{G} - \int^\Lambda \frac{d^3k}{(2\pi)^3} \text{Tr} \frac{\hat{k}(\hat{k} + \hat{p})}{k^2(k + p)^2} = \frac{\Lambda}{g^2} + \frac{\Lambda}{\pi^2} + \frac{1}{8} |p|.
$$

(2.3)

Setting $g = g_c$ so that to cancel the linear in $\Lambda$ term in Eq. (2.3), the $h1/G - \Lambda/\pi^2 = 0$, we have the critical coupling constant at this order

$$
g_c = \pi,
$$

(2.4)

and the dressed $\phi$ propagator

$$
\Delta_0(p) = \frac{8}{|p|}.
$$

(2.5)

Now we see that, at the order $1/N^0$, the two- and three-vertices given by Eqs.(2.1) and (2.5) have indeed a conformal structure, as if the fields $\psi$ and $\phi$ both have dimension 1 and the Yukawa interaction is dimensionless. While this value of the dimension of the fermion field $\psi$ coincides with the canonical one at $d = 3$, in zeroth order in $1/N$ the scale dimension $pg \phi$ is 1, different from the value $3/2$ which is implied by the bare propagator. This is related to the shift of the scale dimension of the composite operator $\bar{\psi}\psi$ to 1 after summing the bubble graphs and tuning the bare coupling to the critical value (2.4) which was discussed in Section 1. Notice that the scale dimension 1 coincides with the standard mass dimension of the field $\phi$, so that there is no need of a mass parameter in Eq. (2.5).

2.2 Primary divergences at $O(1/N)$

At the order $O(1/N)$, we need to consider the diagrams in Figs. 3 – 6. We shall see that the primary divergences are these in the fermion self-energy and in the Yukawa

\footnote{Since, for large momentum $|p|$, the momentum-space propagator of a field with the scale dimension $l$ is $\sim |p|^l - d/2$ in a $d$-dimensional space, the canonical scale dimension of the auxiliary field is $d/2$ while that of a conventional dynamical scalar field is $d/2 - 1.$}
vertex. A straightforward calculation of Fig. 3a gives the regulated fermion two-point vertex

\[
S^{-1}(p, \Lambda) = i\hat{p} - \frac{8i}{N} \int_{\Lambda} \frac{d^3 k}{(2\pi)^3} \frac{\hat{k} + \hat{p}}{|k|(k + p)^2}
\]

\[
= i\hat{p}[1 + \frac{2}{3\pi^2 N} \ln(\frac{\Lambda^2}{p^2}) + \text{finite}]
\]  

(2.6)

and of Fig. 4a the regulated three-point vertex

\[
-\Gamma(p_1, p_2, \Lambda) = -\frac{1}{\sqrt{N}} + \frac{8}{\sqrt{N^3}} \int_{\Lambda} \frac{d^3 k}{(2\pi)^3} \frac{(\hat{k} + \hat{p}_1)(\hat{k} + \hat{p}_2)}{|k|(k + p_1)^2(k + p_2)^2}
\]

\[
= -\frac{1}{\sqrt{N}}[1 - \frac{2}{\pi^2 N} \ln(\frac{\Lambda^2}{p_{\text{max}}^2}) + \text{finite}]
\]  

(2.7)

where \(p_{\text{max}}^2\) is the largest of \(p_1^2\) and \(p_2^2\), with \(p_1\) and \(p_2\) the incoming and outgoing fermion momenta, respectively.

The diagrams in Fig. 5, which have a subdiagram with a fermion loop and three boson legs. The subdiagram vanishes because of parity symmetry. We shall see in the next section that this form of the vertex function is in fact prescribed by conformal invariance.

To render the above two- and three-point vertices finite, we need to introduce local counterterms, Figs. 3a’ and 4a’,

\[
\text{Fig.3a’} = -\frac{2}{3\pi^2 N} \ln(\frac{\Lambda^2}{\mu^2})\bar{\psi}\psi
\]  

(2.8)

\[
\text{Fig.4a’} = -\frac{2}{\pi^2 \sqrt{N^3}} \ln(\frac{\Lambda^2}{\mu^2})\phi\bar{\psi}\psi
\]  

(2.9)

respectively, where a reference mass scale \(\mu\) has been introduced. The corresponding renormalization constants, using minimal subtraction, are

\[
Z_\psi(\Lambda) = 1 - \frac{2}{3\pi^2 N} \ln(\frac{\Lambda^2}{\mu^2})
\]  

(2.10)

\[
Z_1(\Lambda) = 1 + \frac{2}{\pi^2 N} \ln(\frac{\Lambda^2}{\mu^2})
\]  

(2.11)

The anomalous dimension of the fermion field is

\[
\gamma_\psi = \frac{2}{3\pi^2 N}
\]  

(2.12)
From (2.7), we see that the three-point vertex, while dimensionless at the tree level, develops an anomalous dimension, $-2\gamma$, beyond the tree level. We shall call $\gamma$ the index of the Yukawa vertex. At the order $1/N$,

$$\gamma = -\frac{2}{\pi^2 N}.$$  \hspace{1cm} (2.13)

### 2.3 A demonstration of renormalizability

The renormalization of the operator $\phi^2$ is crucial. By power counting, the inverse of the two-point function $\langle \phi \phi \rangle$ has engineering dimension one, instead of two, which would be the case for a dynamical scalar field. The self energy for the scalar is linearly divergent, with the divergent terms occurring in Feynman diagrams cancelled order by order in the $1/N$ expansion by adjusting the value of the critical coupling constant. The remainder of the scalar self-energy can (and generally does) contain individual integrals with non-local divergent terms like $|p| \ln(\Lambda^2/p^2)$. It should be rendered finite by the insertion of the fermion wave-function renormalization and vertex renormalization counterterms into the internal lines of lower order diagrams. Otherwise, non-local counterterms would be necessary to cancel ultraviolet divergences and the $1/N$ expansion of the theory would be non-renormalizable in the conventional sense. In this Section, we shall demonstrate that, to order $1/N$, these nonlocal divergences are indeed cancelled by local counterterms.

To see this, we consider the diagrams at the order $\mathcal{O}(1/N)$, Fig. 6. Figs. 6a and 6b each has a subdiagram, the one-loop fermion self-energy, that is logarithmically divergent, as calculated above. Therefore accompanying Figs. 6a and 6b are Fig. 6a’ and 6b’ containing the counterterm (2.8). Similarly, Fig. 6c has a subdiagram, the one-loop vertex correction, that is also logarithmically divergent, and therefore we need to calculate Figs. 6c’ and 6c’’, together with Fig. 6c. Once these counterterm diagrams have been taken into account all logarithmically divergent terms cancel out between Figs. 6a and 6a’; Figs. 6b and 6b’; and among Figs. 6c, 6c’ and 6c’’. Consequently, no momentum dependent counterterms are necessary.

The linearly divergent terms in the $\phi$ self-energy can be cancelled by adjusting the critical coupling constant $g_c^2$, which has been determined to leading order in
the previous subsection. By this analysis, we have confirmed the renormalizability of \( (1.6) \) at the order \( 1/N \). In the language of multiplicative renormalization, introducing counterterms implies introducing renormalization constants for the vertices appearing in the action. In the the present case, \( Z_\phi \) for \( \phi \) is not independent but takes the form
\[
Z_\phi = \frac{Z^2_\phi}{Z^2_{\psi}} = [1 + \frac{16}{3\pi^2 N} \ln(\frac{\Lambda^2}{\mu^2})]. \tag{2.14}
\]

The following explicit calculation confirms \( (2.14) \):

By using \( (2.3) \), we have
\[
\Delta^{-1}(p) = \frac{1}{g^2} \left[ \frac{\Lambda}{\pi^2} \right] - \frac{2}{3} \pi \ln(\frac{\Lambda^2}{p^2}) + \text{const} \times p. \tag{2.16}
\]
Choosing \( g = g_c \) so that \( \Lambda/g^2_c - (1 - 2/N)\Lambda/\pi^2 = 0 \), we have the critical coupling constant at the order \( 1/N \)
\[
g_c = \pi(1 + \frac{1}{N}) \cdot g^2_c = \frac{1}{\pi^2}(1 - \frac{2}{N}). \tag{2.17}
\]

The renormalized scalar two-point function is
\[
\Delta^{-1}(p) = Z_\phi(\Lambda)\Delta^{-1}(p, \Lambda) = \frac{p}{8} [1 - \frac{16}{3\pi^2 N} \ln(\frac{\mu^2}{p^2})], \tag{2.18}
\]
where we have used the \( \phi \) wavefunction renormalization constant \( (2.14) \). The anomalous dimension of \( \phi \) is
\[
\gamma_\phi = -\frac{16}{3\pi^2 N}. \tag{2.19}
\]

From \( (2.12), (2.13) \) and \( (2.19) \), it can be checked that the index of Yukawa interaction and the anomalous dimensions of \( \psi \) and \( \phi \) at the order \( \mathcal{O}(1/N) \) satisfy a relation.
\[ \gamma = \gamma_\psi + \frac{1}{2} \gamma_\phi. \] (2.20)

As we shall show in the next section, this is not an accident but a result of conformal symmetry. The point is that the renormalization constants in the conformal theory are equal to their fixed point values:

\[ Z_{\psi}(\Lambda) = \left( \frac{\Lambda^2}{\mu^2} \right)^{\gamma_\psi}, \] (2.21)
\[ Z_1(\Lambda) = \left( \frac{\Lambda^2}{\mu^2} \right)^\gamma, \] (2.22)
\[ Z_{\phi}(\Lambda) = \left( \frac{\Lambda^2}{\mu^2} \right)^{\gamma_\phi}. \] (2.23)

The expressions (2.10), (2.11) and (2.14) which are explicitly calculated at the order \( \mathcal{O}(1/N) \) can be viewed therefore as the \( \mathcal{O}(1/N) \) terms of the expansion of (2.21), (2.22) and (2.23) in \( 1/N \) with \( \gamma_\psi, \gamma \) and \( \gamma_\phi \) given by Eqs. (2.12), (2.13) and (2.19), respectively.

## 3 Conformal bootstrap

In this section, we derive the bootstrap equations for the system with four-fermion interactions in \( d \) dimensions. The result is three equations which determine two anomalous dimensions and the value of the critical coupling constant. We start with a systematic analysis of the conformal structure of a general theory involving fermions and bosons with Yukawa interaction(s).

### 3.1 Conformal structure

Let \( l \) and \( b \) be the scaling dimensions of the primary spinor and scalar fields, respectively. Under a scaling transformation of the coordinates, \( x \to \rho x \), the transformation of the fields are defined as

\[ \psi'\!(\rho x) = \rho^{-l}\psi(x), \quad \bar{\psi}'\!(\rho x) = \rho^{-l}\bar{\psi}(x), \quad \phi'\!(\rho x) = \rho^{-b}\phi(x). \] (3.1)
Moreover, the special conformal transformation of the fields are defined as:

\[
\begin{align*}
\psi(x) & \rightarrow \psi'(x') = \sigma_x^{-1/2}(1 + \hat{t}\hat{x})\psi(x), \\
\bar{\psi}(x) & \rightarrow \bar{\psi}'(x') = \sigma_x^{-1/2}\bar{\psi}(x)(1 + \hat{x}\hat{t}), \\
\phi(x) & \rightarrow \phi'(x') = \sigma_x^b\phi(x),
\end{align*}
\]

where \(\sigma_x \equiv \sigma_t(x) = \left| \frac{\partial x}{\partial x'} \right|^{1/d} = 1 + 2t \cdot x + t^2 x^2\), under a special conformal transformation of coordinates, parameterized with a constant vector \(t_\mu\),

\[
x_\mu \rightarrow x'_\mu = \frac{x_\mu + t_\mu x^2}{1 + 2t \cdot x + t^2 x^2}.
\]

Scaling and special conformal transformations together with translations and Lorentz rotations comprise the conformal group which has \((d + 1)(d + 2)/2\) generators in \(d\) dimensions.

The transformations of the primary fields determine the transformations of the Green functions. For instance, under the conformal transformation (3.5), the two- and three-point functions transform as

\[
\begin{align*}
G'(x' - y') &= \langle T\psi'(x')\bar{\psi}'(y') \rangle = (\sigma_x\sigma_y)^{l-1/2}(1 + \hat{t}\hat{x})G(x - y)(1 + \hat{y}\hat{t}) , \\
D'(x' - y') &= \langle T\phi'(x')\phi'(y') \rangle = (\sigma_x\sigma_y)^bD(x - y), \\
G_3(x', y'; z') &= \langle T\phi'(z')\psi'(x')\bar{\psi}'(y') \rangle = \sigma_{x}^{b}(\sigma_{x}\sigma_{y})^{l-1/2}(1 + \hat{t}\hat{x})G_{3}(x, y, z)(1 + \hat{y}\hat{t}).
\end{align*}
\]

The unique (modulo normalization) solution to the special conformal transformations (3.6) – (3.8) is

\[
\begin{align*}
G(x - y) &= \frac{1}{4^{h-l} \pi^{h-i}N(l)} \left| (x - y)^2 \right|^{l+1/2} , \\
D(x - y) &= \frac{1}{4^{h-b} \pi^{h-i}N(b)} \left| (x - y)^2 \right|^{b} , \\
G_3(x, y; z) &= \frac{C}{\left| (x - z)^2 \right|^{b/2+1/2} \left| (z - y)^2 \right|^{b/2+1/2} \left| (x - y)^2 \right|^{l-b/2}} ,
\end{align*}
\]

where \(h = d/2\), and

\[
\tilde{N}(\tau) = \frac{\Gamma(h - \tau + 1/2)}{\Gamma(\tau + 1/2)}, \quad N(\tau) = \frac{\Gamma(h - \tau)}{\Gamma(\tau)}.
\]
Note that, unlike the case of \( d=2 \) where the conformal group has an infinite number of generators, the finite dimensional conformal symmetry in \( d>2 \) restricts but does not fix the functional forms of higher point correlation functions.

(3.9) and (3.10) are normalized so that their Fourier transformations take a simple form:

\[
G(p) = \frac{1}{ip^2} (p^2)^{l-h+1/2}, \quad \text{(3.13)}
\]
\[
D(p) = \frac{1}{p^2} (p^2)^{b-h+1}, \quad \text{(3.14)}
\]

Amputating the external legs from the three-point function (3.11), we obtain the conformal three-point vertex (depicted in Fig. 7)

\[
\Gamma(x, y; z) = g_* \frac{\Gamma(h)}{4^\gamma \pi^d} N(\gamma) \tilde{N}^2 (b/2) N(l-b/2) \int \frac{d^d k}{\pi^h} \frac{\hat{k} + \hat{p}_1}{[(k + p_1)^2]^{b/2+l+1/2}} \frac{\hat{k} + \hat{p}_2}{[(k + p_2)^2]^{b/2+l+1/2}} \frac{1}{[(x - z)^2]^{l-b/2}} , \quad \text{(3.15)}
\]

which is normalized so that the Fourier transformation of (3.15) is

\[
\Gamma(p_1, p_2) = g_* \Gamma(h) N(\gamma) \int d^d k \frac{\hat{k} + \hat{p}_1}{\pi^h} \frac{\hat{k} + \hat{p}_2}{\pi^h} \frac{1}{[(k + p_1)^2]^{b/2+l+1/2}} \frac{1}{[(k + p_2)^2]^{b/2+l+1/2}} \frac{1}{[(k^2)]^{l-b/2}} , \quad \text{(3.16)}
\]

where \( g_* \) is the unknown dimensionless fixed-point coupling constant and

\[
\gamma = l + b/2 - h , \quad \text{(3.17)}
\]

is defined as the index of the vertex. By dimensional analysis, it is easy to check that the dimension of the vertex is \(-2\gamma\).

In the previous section, we have introduced in \( d=3 \) the anomalous dimensions \( \gamma_\psi \) and \( \gamma_\phi \) by Eqs. (2.21) and (2.23). They are related to the scale dimensions \( l \) and \( b \) as follows:

\[
l = h - \frac{1}{2} + \gamma_\psi , \quad \text{(3.18)}
\]
\[
b = 1 + \gamma_\phi . \quad \text{(3.19)}
\]

\[2\]This amputation in conformal theories replaces the dimensions by ‘shadow’ dimensions \( l^* = d-l, \)
\( b^* = d-b. \)
When these formulas are substituted into Eq. (3.17), it recovers Eq. (2.20) which is obtained in the previous section by explicit calculations at the order $O(1/N)$. It is worth also noting that when (3.18) and (3.19) with $\gamma_\psi = \gamma_\phi = 0$ are substituted into (3.16) the result coincides in $d = 3$ with the vertex function (2.7) which explains its conformal origin.

### 3.2 Bootstrap equations

The critical coupling $g_*$ and the indices $b$ and $l$ are determined by the set of homogeneous Schwinger-Dyson equations, which we shall now derive. To simplify notations, we introduce operators $S\Gamma$ and $S'\Gamma$ as

\[
\langle p_1 | S\Gamma | p_2 \rangle = G(p_1)\Gamma(p_1, p_2)G(p_2), \quad \langle p_1 | S'\Gamma | p_2 \rangle = G(p_1)\Gamma(p_1, p_2)D(p_2 - p_1),
\]

where the matrix multiplication over the spinor indices is implied. Here, $S$ and $S'$ are the operation of attaching two fermion or a fermion and boson leg to the vertex function, respectively. Then the set of Schwinger-Dyson equations reads

\[
\Gamma = g_0 + \Gamma S K, \quad (3.21)
\]

\[
\Sigma \equiv G_0^{-1} - G^{-1} = g_0 S' \Gamma, \quad (3.22)
\]

\[
\Pi \equiv D_0^{-1} - D^{-1} = -g_0 N \text{Tr} (S\Gamma), \quad (3.23)
\]

where $\text{Tr}$ is the trace over the spinor indices and $K$ stands for the standard Bethe–Salpeter kernel.

The bare coupling $g_0$ in Eq. (3.21) must vanish so that this equation is consistent with the conformal ansatz. This is realized by renormalization [2]. Moreover, since the three vertex is fixed by conformal invariance, both sides of (3.21) must have the same dependence on the space-time coordinates. Therefore the vertex bootstrap equation takes a simple form

\[
1 = g_0^2 f(l, l, b; g_*), \quad (3.24)
\]

where $f$, named the vertex function, is a function solely of the scaling dimensions carried by the legs and of the critical coupling. The vertex function, to be explored to
some extent in the next section, plays a key role in the conformal bootstrap approach.
calculated below. lines is shown $l$ or $b$).

However, one cannot immediately set the bare coupling $g_0$ to zero in Eqs. (3.22) and (3.23). The reason is that the quantities that $g_0$ multiplies in (3.22) and (3.23) are divergent. Therefore the right hand side of (3.22) and (3.23) has the form $0 \cdot \infty$.

To deal with this uncertainty, we differentiate both sides of Eqs. (3.22) and (3.23) with respect to the external momentum $k$. Denoting $\frac{\partial \Sigma}{\partial k} \equiv \Sigma_\alpha$ and $\frac{\partial \Pi}{\partial k} \equiv \Pi_\alpha$ and replacing $g_0$ with $\Gamma - \Gamma SK$ and $\Gamma - \Gamma S'K'$, we have

$$\Sigma_\alpha = (\Gamma - \Gamma S'K')S'_\alpha \Gamma + (\Gamma - \Gamma S'K')S'\Gamma_\alpha ,$$  
$$\Pi_\alpha = -N \text{Tr}[(\Gamma - \Gamma SK)S_\alpha \Gamma + (\Gamma - \Gamma SK)S\Gamma_\alpha] ,$$

where $K'$ denotes the boson-fermion Bethe-Salpeter kernel. The skeleton expansions of $K$ and $K'$ are depicted in Fig. 8. The vertex bootstrap equation now is

$$\Gamma = \Gamma SK \equiv \Gamma S'K' .$$

Now we see that the first term on the right hand side of Eqs. (3.23) and (3.26) vanishes as the bracket is zero, according to (3.27), and $S'_\alpha \Gamma$ and $S_\alpha \Gamma$ are finite. However, the second term of Eqs. (3.25) and (3.26) is still the form $0 \cdot \infty$.

### 3.3 Parigi approach

To remove the remaining uncertainty in (3.23) and (3.26), we use a regularization proposed by Parigi [3]. The idea is to shift slightly the dimension of one of the operators in the Green functions:

$$l \rightarrow l' = l + \epsilon , \text{ or } b \rightarrow b' = b + \epsilon .$$

The corresponding change of the expression on the right hand side of the vertex bootstrap equation is

$$\Gamma^\epsilon S'K' = g^2 f(l + \epsilon, l, b; g_*) \Gamma^\epsilon ,$$  
$$\Gamma^\epsilon SK = g^2 f(l, l, b + \epsilon; g_*) \Gamma^\epsilon ,$$

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where \( f \) is defined similarly to the right hand side of Eq. (3.24). Using Eq. (3.24) we obtain

\[
\Sigma_\alpha = - g^2 \frac{\partial f(l', l, b; g_s)}{\partial l'} \bigg|_{l'=l} \epsilon \Gamma^\epsilon S' \Gamma_\alpha , \tag{3.31}
\]

\[
\Pi_\alpha = - N g^2 \frac{\partial f(l, l, b'; g_s)}{\partial b'} \bigg|_{b'=b} \epsilon \text{ Tr } \Gamma^\epsilon S \Gamma_\alpha . \tag{3.32}
\]

Notice that, in this procedure, the dimension of only one external line is analytically continued in the equation.

The calculations of \( \Gamma^\epsilon S' \Gamma_\alpha \) and \( \Gamma^\epsilon S \Gamma_\alpha \) are performed similarly to those of the scalar \( \Phi^3 \) theory [3] and of \( \Phi^4 \) theory [5]. In the fermion model, a useful formula is [24]

\[
\int \frac{dz}{\pi^h} \frac{(\hat{x}_1 - \hat{z})(\hat{z} - \hat{x}_2)}{[x_1 - z)^2]^{\delta_1+\frac{1}{2}}[(x_2 - z)^2]^{\delta_2+\frac{1}{2}}[(x_3 - z)^2]^{\delta_3}} \]

\[
= \frac{\hat{x}_{13}}{[x_{13}^2]^{h-\delta_2+\frac{1}{2}}} \frac{\hat{x}_{23}}{[x_{23}^2]^{h-\delta_1+\frac{1}{2}}} \frac{N(\delta_1)N(\delta_2)N(\delta_3)}{[x_{12}^2]^{h-\delta_3}} . \tag{3.33}
\]

It is valid providing

\[
\delta_1 + \delta_2 + \delta_3 = d . \tag{3.34}
\]

The meaning of these formulas is very simple: if one performs the conformal transformation of \( x \)'s, the left hand side of Eq. (3.33) obeys Eq. (3.8) providing Eq. (3.34) for \( \delta \)'s is fulfilled. Therefore, the integral is determined (modulo an \( x \)-independent factor) by the conformal invariance to be the right hand side of Eq. (3.33).

Using Eq. (3.33) in calculating \( \Gamma^\epsilon S' \Gamma_\alpha \) and \( \Gamma^\epsilon S \Gamma_\alpha \) for each of the integrals (the all are of the type (3.33)) and picking up the terms \( \sim \epsilon^{-1} \), we rewrite Eqs. (3.31) and (3.32) as [3]

\[
1 = - \frac{g^4}{(4\pi)^h} \Gamma(h)N(\gamma)N(d-l)(l-b/2) \frac{\partial f(l', l, b; g_s)}{\partial l'/2} \bigg|_{l'=l} . \tag{3.35}
\]

\[
1 = N \Tr 1 \frac{g^4}{(4\pi)^h} \Gamma(h)N(\gamma)N(d-b)N(l-b/2)(l-b/2) \frac{\partial f(l, l', b; g_s)}{\partial b'/2} \bigg|_{b'=b} . \tag{3.36}
\]

Eqs. (3.35) and (3.36) together with Eq. (3.24) determine the critical indices. All the three equations depend on the vertex function \( f(l,l,b) \), which we shall discuss in the next section.
4 Three-point vertex function

The skeleton expansion of the Bethe-Salpeter kernel $K$ (or $K'$) in the vertex bootstrap equation (3.27) contains an infinite number of terms. If the critical coupling is reasonably small, the three vertex approximation makes sense. Then the vertex equation has a solution with small $g^*$ for $\gamma \ll 1$. The point is that the integral in the vertex equation is slowly convergent if $\gamma \ll 1$ and is therefore large ($\sim 1/\gamma$). This enables it to cancel the small factor of $g^*_2$ in the bootstrap equation.

4.1 Momentum-space analysis

It is convenient to perform calculation in the momentum space where the term $\sim 1/\gamma$ comes from the region with large integration momentum. The Fourier image of the conformal vertex (3.15) has been given in (3.16). The integration on the right hand side of (3.16) has an expression of the Appel functions $F_1$ in general, and takes a simple form for $\gamma \ll 1$. We shall derive the corresponding formula directly analyzing the integral on the right hand side of Eq. (3.16).

Let us first note that the coefficient in front of the integral is $N(\gamma) \approx \gamma$ for small $\gamma$, so that one has to peak up the term $\sim 1/\gamma$ for the vertex to be of order 1. It is easy to see that this term comes from the region of integration with $|k| \gtrsim \max\{|p_1|, |p_2|\}$. (Recall that $|p_1 - p_2| \lesssim \max\{|p_1|, |p_2|\}$ in an Euclidean domain.) One gets

$$
\int \frac{d^dk}{\pi^h} \frac{\hat{k} + \hat{p}_1}{[(k + p_1)^2]^{b/2+1/2} [(\hat{k} + \hat{p}_2)^2]^{b/2+1/2} [k^2]^{l-b/2}} = \int_{\max\{p_1^2, p_2^2\}}^{\infty} \frac{dk^2}{[k^2]^{1+\gamma}} = \frac{1}{\gamma (\max\{p_1^2, p_2^2\})^\gamma}. \quad (4.1)
$$

and

$$
\Gamma(p_1, p_2) = g^* \frac{1}{(\max\{p_1^2, p_2^2\})^\gamma}. \quad (4.2)
$$

This dependence of the three-point vertex solely on the largest momentum is typical for logarithmic theories in the ultraviolet region where one can put, say, $p_1 = 0$ without changing the integral with logarithmic accuracy. This is valid, however, if the integral is fast convergent in infrared regions. For our case this means that the dimensions $l$ and $b$ should be far away from the values at which the integral on the
left hand side of Eq. (4.1) is infrared divergent. For instance if \( l - b/2 \approx h \), the term of the form

\[
\text{infrared term} = \frac{1}{l - b/2 - h} \left[ \max \{ p_1^2, p_2^2 \} \right]^{b/2 + 1/2} \left[ \min \{ p_1^2, p_2^2 \} \right]^{l+1/2 - h},
\]

were appear in the integral which would depend therefore both on the maximal and on the minimal momenta.

In particular, for \( \gamma = -2/(N\pi^2) \) Eq. (4.2) reproduces to order \( O(1/N) \) Eq.(2.??) which was obtained by direct calculations. For this reason our analysis of the case of small \( \gamma \) is very similar to that of Section 2.

Let us explicitly write down the dependence of the vertex function \( f \) on \( g^* \):

\[
f(l, l, b; g^*) = g^2 f_3(l, l, b) + g^2 f_5(l, l, b) + \ldots,
\]

where \( f_3(l, l, b) \) is associated with the three-point vertex contribution on the right hand side of the vertex bootstrap equation (3.27), \( f_5(l, l, b) \) with the five-vertex terms and so on. If one restricts oneself to the three-point vertex term, the resulting equation is depicted in Fig. 9. This truncation is referred as the three-vertex approximation.

It is not difficult to calculate \( f_3 \) if \( \gamma \ll 1 \). In order to pick up the term \( \sim 1/\gamma \) one performs quite similar calculations to the logarithmic case. Let \( p_2^2 \gtrsim p_1^2 \). Then the three vertex equation reads

\[
\frac{1}{[p_2^2]^{\gamma}} = -g^2 \int d^d k \frac{1}{(2\pi)^d \left[ \max \{ p_1^2, k^2 \} \right]^\gamma} \frac{k + \hat{p}_1}{\left[ \max \{ p_2^2, k^2 \} \right]^\gamma} \left[ \left( k + p_1 \right)^2 \right]^{h-l+1/2} \left[ \left( k^2 \right)^{h-b} \right] \cdot \frac{1}{\left[ \min \{ p_2^2, k^2 \} \right]^\gamma} \left[ \left( k + p_2 \right)^2 \right]^{h-l+1/2} \left[ \left( k^2 \right)^{h-b} \right] \int d^d k_2 \frac{1}{\left[ k_2^2 \right]^{1+\gamma}}.
\]

(4.5)

where the momenta are depicted in Fig. 9.

The term \( \sim 1/\gamma \) on the right hand side of Eq. (4.5) comes from the domain where the integration momentum \( k^2 \gtrsim p_2^2 \). Then we have finally

\[
f_3(l, l, b) = -\frac{1}{(4\pi)^h \Gamma(h)} \int p_2^2 \frac{dk^2}{[k^2]^{1+\gamma}} = -\frac{1}{(4\pi)^h \Gamma(h) \gamma}.
\]

(4.6)

It has been verified simultaneously that the momentum dependence (4.2) is reproduced by the three vertex equation at small \( \gamma \). This is a consequence of conformal
invariance which fixes as is mentioned above the coordinate (or momentum) dependence of the three-point vertex.

To apply the procedure described in the previous section, we repeat the calculation above with the dimension of one of the external lines shifted and have

\[ f_3(l', l, b) = -\frac{1}{(4\pi)^h \Gamma(h) \gamma'}, \quad \gamma' = \frac{l' + l + b - d}{2}; \quad (4.7) \]

\[ f_3(l, l', b) = -\frac{1}{(4\pi)^h \Gamma(h) \gamma''}, \quad \gamma'' = \frac{l + l' + b - d}{2}. \quad (4.8) \]

Such a modification is prescribed by the dimensional analysis.

Substituting Eqs. (4.7) and (4.8) into Eqs. (3.35) and (3.36) for propagator, we arrive at the algebraic equation set that determines the critical indices in the three vertex approximation:

\[ 1 = -\frac{g^2}{(4\pi)^h \Gamma(h) \gamma'}, \quad (4.9) \]

\[ 1 = \frac{g^2}{(4\pi)^h} \bar{N}(b/2) N(l - b/2) \bar{N}(d - l), \quad (4.10) \]

\[ 1 = -N \text{Tr} \frac{g^2}{(4\pi)^h} \bar{N}^2(b/2) N(d - b). \quad (4.11) \]

### 4.2 The solution at large \(N\)

We look for a solution to the algebraic equations (4.9) – (4.11) for small \(\gamma = l + b/2 - h\).

Let us assume the scaling dimension of \(\psi\) is close to the canonical value, \(l \approx h - 1/2\), i.e. \(\gamma_\psi\) which is defined by Eq. (3.18) is small. In order that \(\gamma = \gamma_\psi + (b - 1)/2 \ll 1\), the scaling dimension of \(\phi\) should be \(b \approx 1\), i.e. \(\gamma_\phi\) which is defined by Eq. (3.19) should also be small.

From Eqs. (4.9) and (4.10) we have Let us first equal the r.h.s.’s of Eqs. (4.11) and (4.10) for \(b \approx 1\) and \(l - h + 1/2 \ll 1\). Solving the equation, we get

\[ l = h - 1/2 + \frac{1}{N \text{Tr} 1} \frac{\Gamma(2h - 1) \sin \left[\pi(h - 1)\right]}{\Gamma(h - 1) \Gamma(h + 1) \pi}, \quad (4.12) \]

and from Eqs. (4.9) and (4.10)

\[ \gamma = -\frac{1}{N \text{Tr} 1} \frac{\Gamma(2h - 1) \sin \left[\pi(h - 1)\right]}{\Gamma^2(h) \pi}, \quad (4.13) \]
which gives

\[ b = 1 - \frac{2}{N \text{Tr} \mathbf{1}} \frac{\Gamma(2h) \sin [\pi(h - 1)]}{\Gamma(h)\Gamma(h + 1)\pi}. \]  

(4.14)

Now we see that the scaling dimensions of \( \psi \) and \( \phi \) are such that \( \gamma_\psi \) and \( \gamma_\phi \) are \( \sim 1/N \) and therefore are indeed small in the large \( N \) limit. Moreover, since \( g_*^2 \sim 1/N \), it is verified that for large \( N \) the three-vertex approximation is a good one.

Substituting \( d = 3 \) and \( \text{Tr} \mathbf{1} = 2 \) into (4.9), (4.12) and (4.14), we obtain

\[ l = 1 + \frac{2}{3\pi^2N}, \]  

(4.15)

\[ b = 1 - \frac{16}{3\pi^2N}. \]  

(4.16)

These values of scaling dimensions coincide with those, given by (2.12) and (2.19), which are calculated in Section 2 by the \( 1/N \) expansion while the (nonuniversal) value of \( g_* \), which is determined by Eq. (4.9), is to be compared to (2.17).

Notice that the value of \( \gamma_\psi \) which is given by the last term on the right hand side of Eq. (4.12) is positive for \( 2 < d < 4 \). In other words, the dimension \( l \) is bigger than the canonical value. This is in agreement with the Lehman theorem which says that the momentum space propagator in an interacting theory without ghosts should grow with momentum faster than the free one. For the auxiliary field \( \phi \) the Lehman theorem is not applicable since there is no pole term in the spectral representation of the propagator, so that the dimension \( b \) can be arbitrary in four-fermion theory. However, if we were consider instead a field theory with dynamical scalars coupled to fermions through Yukawa interaction, it would be described by exactly the same bootstrap equations with the solution given by (4.12) – (4.14). In this case the Lehman theorem for the scalar field would say that \( b \) should be bigger than \( h - 1 \), \( i.e. \) than the canonical value for a dynamical scalar field. We see that Eq. (4.14) obeys this restriction providing \( d < 4 \). Therefore, our solution is applicable to Yukawa theory as well as to four-fermion theory. This confirms that the two theories are equivalent, as was discussed in Section 1, for \( 2 < d < 4 \).
4.3 Estimating high-vertex contributions

Let us stress once more that the system of equations (4.9) – (4.11) is valid provided no infrared divergences emerge in the vertex bootstrap equation. This is indeed the case for the values of \( l, \gamma \) and \( b \) given by Eqs. (4.12) – (4.14) (or (4.15) and (4.16) for \( d = 3 \)).

The three vertex approximation works if five etc. vertex contributions are small. It is easy to estimate the order in \( 1/N \) of these higher-vertex terms which have been disregarded. This can be done as follows.

The index of integrals in the Bethe-Salpeter kernel (which coincides with that for a four-fermion vertex) is

\[
t = 2l - h.
\]

(4.17)

If \( t \gg \gamma \), five etc. vertex contributions are suppressed as

\[
\lambda^2 \frac{t}{t} \sim \frac{\gamma}{t}.
\]

(4.18)

This is true in particular for \( d = 4 - \epsilon \) \((\epsilon \ll 1)\) when \( \gamma \sim \epsilon \) but \( t \sim 1 \). However, if \( t \sim \gamma \), all parquet graphs are of the same order and should be summed up (like four-boson interaction in \( 4 - \epsilon \) \((\epsilon \ll 1)\)). Another example is Gross-Neveu model in \( 2 + \epsilon \) \((\epsilon \ll 1)\) when the s-channel graph is compensated by the u-channel one so that the three vertex approximation works while \( t \sim \gamma \).

5 Conclusions

The method of conformal bootstrap turned out to be very useful for calculating the critical indices for the four-Fermi interaction at the fixed point. From the viewpoint of the \( 1/N \) expansion this method corresponds to a resumming of the perturbative series, so that at each order in \( 1/N \) one deals with manifestly conformal invariant two and three point functions whose dependence on momenta is fixed by conformal invariance. The only unknown quantities: the scale dimensions of \( \psi \) and \( \bar{\psi}\psi \) field as well as the critical value of the coupling constant are determined order by order in \( 1/N \). In \( d = 3 \) we have shown by explicit calculations how the logarithms of perturbative theory are nicely collected into conformally invariant structures. As is
discussed above, the reason behind this is the exponentiation of the log’s into scale-invariant expressions which is a nonperturbative phenomenon from the viewpoint on the $1/N$-expansion.

We have demonstrated that the calculation of critical indices by the method of conformal bootstrap is an economical one in the sense that the number of skeleton graphs in the vertex bootstrap equation is much less than the total number of graphs of the perturbative expansion to the given order in $1/N$. Moreover, one only needs to calculate the vertex function while the two propagator equations are expressed via it.

It would be very interesting to extend the conformal bootstrap approach to the current-current four-Fermi interaction and to quantum electrodynamics. In these cases one might expect further simplifications due to the conservation of the vector current.

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Appendix A  Parisi procedure at small $\gamma$

Here we derive the conformal bootstrap equations from Schwinger-Dyson equations for the self-energies, for small $\gamma$. According to (3.31) and (3.32), we should extract the $\epsilon^{-1}$ term in $\Gamma^e S \Gamma_\alpha$ and $\Gamma^e S' \Gamma_\alpha$. To the leading order in $\gamma$ we have

$$
\text{Tr} [\Gamma^e S \Gamma_\alpha] = - \text{Tr} \int \frac{dq}{\pi^h} \Gamma^e (q, p + q) G(p + q) \frac{\partial}{\partial p_\alpha} \Gamma(p + q, q) G(q)
$$

$$
= \text{Tr} \int \frac{dq}{\pi^h} \frac{g^2 \gamma \Gamma(h)}{\max\{q^2, p^2\}^{\gamma+\epsilon/2} \left[(q + p)^2\right]^{h-l+1/2} \left[(q + p)^2\right]^{h-l+1/2}} \frac{\hat{q} + \hat{p}}{\hat{k}} \frac{1}{\pi^h \left[(q - k)^2\right]^{1-b/2} \left[k^2\right]^{b/2+1/2} \partial \left[(k - p)^2\right]^{b/2+1/2}} . \tag{A.1}
$$

Since the integral over $dq$ equals $2\pi^h/\epsilon \Gamma(h)$, it turns out to be

$$
\text{Tr} [\Gamma^e S \Gamma_\alpha] = \frac{2}{\epsilon} g^2 \gamma \text{Tr} \frac{\partial}{\partial p_\alpha} \int \frac{dk}{\pi^h} \frac{\hat{k}}{\left[k^2\right]^{b/2+1/2} \left[(k - p)^2\right]^{b/2+1/2}} \frac{\hat{k} - \hat{p}}{N(d - \frac{b}{2}) N\left(\frac{b}{2}\right)} . \tag{A.2}
$$

Similarly, we obtain

$$
\Gamma^e S' \Gamma_\alpha = \frac{2}{\epsilon} g^2 \gamma \frac{\partial}{\partial p_\alpha} [i\hat{p}(p^2)^{h-l-1/2}] \tilde{N}\left(\frac{b}{2}\right) N(d - l) N\left(l - \frac{b}{2}\right) . \tag{A.3}
$$

Substituting Eqs. (A.2), (A.3) into (3.31) and (3.32), we arrive at, in the small $\gamma$-limit, (4.9), (4.10) and (4.11).
References

[1] A.M. Polyakov, JETP Lett. 12 (1970) 381.
[2] A.A. Migdal, Phys. Lett. 37B (1971) 98; 386.
[3] G. Parisi, Lett. Nuovo Cim. 4 (1972) 777.
[4] A. Belavin, A. Polyakov and A. Zamalodchikov, Nucl. Phys. B247, 83 (1984).
[5] Yu.M. Makeenko, *Conformal Bootstrap for $\Phi_4^4$ Interaction*, ITEP-44 (1979).
[6] Yu.M. Makeenko, *Dynamics of Four-fermion Interaction*, Ph.D. thesis, Moscow 1978 (in Russian).
[7] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345.
[8] V.G. Vaks and A.I. Larkin, ZhETF 40 (1961) 282, 1392.
[9] W. A. Bardeen, C. N. Leung and S. T. Love, Phys. Rev. Lett. 56, 1230 (1986);
Y. Nambu, in *New Trends in Physics*, Proc of Kazimierz 1988, Z. Ajduk, S. Pokorski and A. Trautman ed. (World Scientific, Singapore, 1989);
T. Appelquist, T. Takeuchi, M. Einhorn and L. C. R. Wijewardhana, Phys. Lett. B220, 223 (1989);
A. Miranski, M. Tanabashi and K. Yamawaki, Mod. Phys. Lett. A4, 1043 (1989); Phys. Lett. B221, 177 (1989).
[10] W. A. Bardeen, C. T. Hill and M. Lindner, Phys. Rev. D41, 1467 (1990).
[11] D.J. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.
[12] G. Feinberg and A. Pais, Phys. Rev. 131 (1963) 2724, 133 (1964) B477.
[13] K.G. Wilson, Phys. Rev. D7 (1973) 2911.
[14] G. Parisi, Nucl. Phys. B100 (1975) 368; D.J. Gross, in *Methods in Field Theory*, eds. R. Balian and J. Zinn-Justin (North-Holland, 1976).
[15] B. Rosenstein, B.J. Warr and S.H. Park, Phys. Rev. Lett. 62 (1989) 1433, Phys. Rep. 205 (1991) 205; G. Gat, A. Kovner, B. Rosenstein and B.J. Warr, Phys. Lett. B240 (1990) 158.
[16] S.J. Hands, A. Kocic and J.B. Kogut, Phys. Lett. B273 (1991) 111.
[17] C. de Calan, P.A. Faria Da Veiga, J. Magnen and R. Seneor, Phys. Rev. Lett. 66 (1991) 3233.
[18] J. Zinn-Justin, Nucl. Phys. B367 (1991) 105.
[19] S.J. Hands, A. Kocic and J.B. Kogut, *Four-Fermi Theories in Fewer Than Four Dimensions*, ILL-(TH)-92-#19.
[20] D. Lurie and A.J. Macfarlane, Phys. Rev. 136 (1964) B816.
[21] G. Mack and A. Salam, Ann. Phys. 53 (1969) 144; D. Gross and J. Wess, Phys. Rev. D2 (1970) 753.
[22] L.P. Kadanoff, Physics 2 (1966) 263.
[23] M. D’Eramo, L. Peliti and G. Parisi, Lett. Nuovo Cim. 2 (1971) 878.
[24] K. Symanzik, Lett. Nuovo Cim. 3 (1972) 734.
[25] S. Ferrara, G. Gatto, A. Grillo and G. Parisi, Nuovo Cim. 19A (1974) 667.
[26] J.A. Gracey, Int. J. Mod. Phys. A6 (1991) 395, 2755 (E).
Figures

Fig. 1  The four-Fermion scattering in the four-Fermion theories. The solid line is the fermion propagator.

Fig. 2  The summation of infinite series of one-loop Fermion bubble chains gives the dressed scalar propagator, which is of order $\mathcal{O}(N^0)$. The cross line is the dressed scalar propagator, the dashed line is the ‘bare’ scalar propagator.
Fig. 3  The fermion self-energy at $\mathcal{O}(1/N)$. $a'$ is the proper counterterm.

Fig. 4  The vertex correction at order $\mathcal{O}(1/N)$. $a'$ is the proper counterterm.
Fig. 5  The null diagrams in the symmetric phase at order $O(1/N)$.

Fig. 6  The scalar self energy at order $O(1/N)$. $a'$, $b'$, $c'$ and $c''$ are due to the proper counterterms.
Fig. 7 The conformal Yukawa vertex constructed by truncating the external legs of the conformal three-point function $\langle \phi \psi \bar{\psi} \rangle$.

\[ \Gamma = \]

Fig. 8 The skeleton expansions of a) Fermion-Fermion and b) boson-Fermion Bethe–Salpeter kernel. The double solid line is the full Fermion two-point function, the double dashed line is the full scalar two-point function.
Fig. 9  The three-vertex approximation of the vertex bootstrap equation.