FUNDAMENTAL PROPERTIES OF CAUCHY–SZEGÖ PROJECTION ON QUATERNIONIC SIEGEL UPPER HALF SPACE AND APPLICATIONS

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Abstract. We investigate the Cauchy–Szegő projection for quaternionic Siegel upper half space to obtain the pointwise (higher order) regularity estimates for Cauchy–Szegő kernel and prove that the Cauchy–Szegő kernel is non-zero everywhere, which further yields a non-degenerated pointwise lower bound. As applications, we prove the uniform boundedness of Cauchy–Szegő projection on every atom on the quaternionic Heisenberg group, which is used to give an atomic decomposition of regular Hardy space $H^p$ on quaternionic Siegel upper half space for $2/3 < p \leq 1$. Moreover, we establish the characterisation of singular values of the commutator of Cauchy–Szegő projection based on the kernel estimates and on the recent new approach by Fan–Lacey–Li. The quaternionic structure (lack of commutativity) is encoded in the symmetry groups of regular functions and the associated partial differential equations.

1. Introduction

1.1. Background. The theory of slice regular functions of a quaternionic variable has been studied intensively (cf. e.g. [2, 23, 27, 34, 46] and references therein) and applied successfully to the study of quaternionic closed operators, quaternionic function spaces and operators on them, e.g. quaternionic slice Hardy space, quaternionic de Branges space and quaternionic Hankel operator etc. (cf. e.g. [10, 11, 9, 23, 49] and references therein). Meanwhile, quaternionic analysis of several variables has been developed substantially in the last three decades. The quaternionic counterpart of the $\partial$-complex and the $k$-Cauchy–Fueter complex is known explicitly now. And the analysis of these complexes and their operators plays an important role in studying quaternionic regular functions (cf. e.g. [1, 12, 16, 17, 21, 57, 59] and references therein). There are also important results along this direction on quaternionic Monge–Ampère equations, quaternionic pluripotential theories and quaternionic Calabi–Yau problem (cf. e.g. [3, 4, 5, 6, 7, 8, 56] and references therein). The latter one has its origin in supersymmetric string theory.

In this paper, we obtain a higher order regularity estimate and a sharp pointwise lower bound for the Cauchy–Szegő kernel on quaternionic Siegel upper half space. The Cauchy–Szegő projection plays an important role in complex analysis [29, 43, 41]. It is also a key tool connecting the regular function in quaternionic Siegel upper half space to its boundary value. Based on this, we further establish an atomic decomposition of quaternionic regular Hardy space over the quaternionic Siegel upper half space and the characterisation of singular values of the commutator of Cauchy–Szegő projection. Because of lack of commutativity in quaternionic case, most of the known methods from complex analysis are not available. In fact, in many cases even the statements of analogous results are different. Nevertheless, we treat a quaternionic regular function as a vector-valued function satisfying certain partial differential equations, hence certain techniques in harmonic analysis and PDEs can be applied. From this point of view, the quaternionic structure is encoded in the symmetry groups of regular functions, the spaces and PDEs, which are different from the complex ones.

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Let $\mathbb{H}^n$ be the $n$-dimensional quaternion space, which is the collection of $n$-tuples $(q_1, \ldots, q_n), q_i \in \mathbb{H}$. We write $q_l = x_{4l-3} + x_{4l-2}j + x_{4l-1} + x_{4l}k$, $l = 1, \ldots, n$. The quaternionic Siegel upper half space is $U := \{ q = (q_1, \cdots, q_n) \in \mathbb{H}^n \mid \text{Re} \, q_1 > |q|^2 \}$, whose boundary

$$\partial U := \{ (q_1, q') \in \mathbb{H}^n \mid \rho := \text{Re} \, q_1 - |q'|^2 = 0 \}$$

is a quadratic hypersurface. A $C^1$-smooth function $f = f_1 + if_2 + jf_3 + kf_4 : U \to \mathbb{H}$ is called \textit{(left) regular} on $U$ if it satisfies the Cauchy–Fueter equations $\overline{\partial}_q f(q) = 0$, $l = 1, \ldots, n$, for any $q \in U$, where

$$\overline{\partial}_q := \partial_{x_{4l-3}} + \partial_{x_{4l-2}}j + \partial_{x_{4l-1}} + \partial_{x_{4l}}k.$$

The Hardy space $H^p(U)$ consists of all regular functions $F$ on $U$, for which

$$\|F\|_{H^p(U)} := \left( \sup_{\varepsilon > 0} \int_{\partial U} |F(\varepsilon, q)|^p d\beta(q) \right)^{\frac{1}{p}} < \infty,$$

where $F_\varepsilon$ is for its “vertical translate”, i.e. $F_\varepsilon(q) = F(q + \varepsilon e_1)$, where $e_1 = (1, 0, 0, \ldots, 0)$. The Cauchy–Szegő projection is the operator from $L^2(\partial U)$ to $H^2(U)$ satisfying the following reproducing formula:

$$F(q) = \int_{\partial U} S(q, p) F^b(p) d\beta(p), \quad q \in U,$$

whenever $F \in H^2(U)$ with the boundary value $F^b$ on $\partial U$, where $S(q, p)$ is the Cauchy–Szegő kernel:

$$S(q, p) = s \left( q_1 + \overline{p}_1 - 2 \sum_{k=2}^{n} \overline{p}_k q_k \right)$$

for $p = (p_1, \cdots, p_n) \in U$, $q = (q_1, \cdots, q_n) \in U$, and

$$s(\sigma) = c_{n-1}^{-1} \frac{2^{(n-1)}}{|\sigma|^4}, \quad \sigma = x_1 + x_2 j + x_3 k \in \mathbb{H}$$

with the real constant $c_{n-1}$ depending only on $n$ ([20, Theorem A]).

Note that the boundary $\partial U$ can be identified with the quaternionic Heisenberg group $H^{n-1}$, and the Siegel upper half space $U$ can be identified with $\mathcal{U} := \mathbb{R}_+ \times H^{n-1}$ by a quadratic diffeomorphism (2.6). A regular function on $\mathcal{U}$ can be characterized by Proposition 2.1. Details of these notations will be introduced in Section 2. For $0 < p < \infty$, the Hardy space $H^p(\mathcal{U})$ consists of all regular functions $F$ on $\mathcal{U}$ with

$$\|F\|_{H^p(\mathcal{U})} := \left( \sup_{\varepsilon > 0} \int_{\mathbb{R}^{4n-1}} |F(\varepsilon, g)|^p d\sigma(g) \right)^{\frac{1}{p}} < \infty,$$

where $d\sigma$ is the Lebesgue measure on $\mathbb{R}^{4n-1}$, which is an invariant measure on $H^{n-1}$. The Cauchy–Szegő projection integral operator $P : L^2(H^{n-1}) \to H^2(\mathcal{U})$ is given by

$$(P f)(t, g) = \int_{H^{n-1}} K((t, g), g') f(g') d\sigma(g'), \quad (t, g) \in \mathcal{U},$$

for $f \in L^2(H^{n-1})$, satisfying the following reproducing formula:

$$(1.6) \quad f = P f^b,$$

whenever $f \in H^2(\mathcal{U})$ and $f^b$ its boundary value on $H^{n-1}$, where the reproducing kernel $K((t, g), g')$ is induced from $S(q, p)$ in (1.3) (cf. (2.15)).

Just recently, in [19] we have further obtained an explicit formula for this Cauchy–Szegő kernel and then proved that it is a Calderón–Zygmund kernel when restricted to $H^{n-1}$. Moreover, a suitable version of pointwise lower bound is provided: “there exist a large positive constant $r_0$ and a positive constant $C$ such that for every $g \in H^{n-1}$, there exists a ‘twisted truncated sector’ $\Omega_g \subset H^{n-1}$ such that $\inf_{x \in \Omega_g} d(g, g') = r_0$ and for every $g_1 \in B(g, 1)$ and $g_2 \in \Omega_g$ we have $|S(g_1, g_2)| \geq d(g_1, g_2)^{-Q}$. Moreover,
the sector $\Omega_g$ is regular in the sense that $|\Omega_g| = \infty$ and for every $R_2 > R_1 > 2r_0 \cap (B(g, R_2) \setminus B(g, R_1))$ with the implicit constants independent of $R_1, R_2$ and $g$.

Note that this lower bound yields the characterisations of boundedness and compactness of commutators $[b, P]$ for $b \in BMO$ space and VMO space respectively.

1.2. **Statement of main results.** In this paper we investigate several fundamental results for the Cauchy–Szegő projection, including the pointwise regularity (higher order) and the non-degenerated property for the Cauchy–Szegő kernel. To be more explicit, we obtain the following results.

Part 1. The pointwise regularity of the Cauchy–Szegő kernel.

**Theorem 1.1.** Suppose $g \in \mathcal{H}^{n-1} \setminus \{0\}$,

$$|Y^I K((1,0),g)| \lesssim \frac{1}{\langle g \rangle + 1} \langle \Omega \rangle,$$

where $Q = 4n + 2$ is the homogeneous dimension of $\mathcal{H}^{n-1}$, $I = (\alpha_1, \ldots, \alpha_{4n-1}) \in \mathbb{N}^{4n-1}$ is a multi-index, $Y^I, d(I)$ and $I$ are defined in (2.5).

Based on this property, we further have the following result on Cauchy–Szegő projection.

**Proposition 1.1.** Suppose $0 < p < \infty$. For any $(p, \infty, \alpha)$-atom $a$ on the quaternionic Heisenberg group $\mathcal{H}^{n-1}$ (Definition 3.1), $\mathcal{P}(a)$ is an element in $H^p(\mathcal{U})$ with

$$\|\mathcal{P}(a)\|_{H^p(\mathcal{U})} \leq C_{p,n,\alpha}$$

for some constant depending only on $p, n$ and $\alpha$.

This proposition implies that the regular Hardy space $H^p(\mathcal{U})$ is nontrivial, since there are lots of $(p, \infty, \alpha)$-atoms on the quaternionic Heisenberg group $\mathcal{H}^{n-1}$.

Part 2. Pointwise lower bound of the Cauchy–Szegő kernel.

Letting $t \to 0$ in (1.5), we obtain a convolution operator on the quaternionic Heisenberg group, which is also denoted by $\mathcal{P}$ by abuse of notations,

$$(\mathcal{P} f)(g) = p.v. \int_{\mathcal{H}^{n-1}} K(g,h)f(h)dh = p.v. \int_{\mathcal{H}^{n-1}} K(h^{-1}, g)f(h)dh,$$

where the kernel $K(g,h) := \lim_{t\to0} K((t,g),h)$ for $g \neq h$ and $K(g) := K(g,0)$. Note that (1.7) holds whenever $f$ is an $L^2$ function supported in a compact set and $g \notin \text{supp } f$.

**Theorem 1.2.** $K(g) \neq 0$, for all $g \in \mathcal{H}^{n-1} \setminus \{0\}$.

Based on this result, we further have the non-degenerated pointwise lower bound. To begin with, we adapt the work of \cite{[54]} on self-similar tilings to obtain a “nice” decomposition of $\mathcal{H}^{n-1}$, denoted by $\Sigma$, analogous to the decomposition of $\mathbb{R}^n$ into dyadic cubes in classical harmonic analysis, and describe an analogue of a lemma of Journé \cite{[38]}. We recall the detail of tiles in Section 3.

The following pointwise lower bound is a refinement of the result in \cite{[19]} in the sense that for each given tile $T$, we know exactly the location of the existing tile $\hat{T}$.

**Theorem 1.3.** Let $K = K_1 + K_2 i + K_3 j + K_4 k$. There exists a positive integer $a_0$ such that:

1. for any $T \in \Sigma_j$, there is a unique $T_{a_0} \in \Sigma_{j+a_0}$ such that $T \subset T_{a_0}$,

2. there exist positive constants $3 \leq a_1 \leq a_2$ and $C > 0$ such that for any tile $T \in \Sigma_j$, there exists a tile $\hat{T} \in \Sigma_j$ satisfying:

   (a) $\hat{T} \subset T_{a_0}$,

   (b) $a_1 2^j \leq d(\text{cent}(T), \text{cent}(\hat{T})) \leq a_2 2^j$;
(c) for all \((g, \tilde{g}) \in T \times \hat{T}\), there exists \(i \in \{1, 2, 3, 4\}\) such that \(|K_i(g, \tilde{g})| \geq C 2^{-(4n+2)(j)}\) and \(K_i(g, \tilde{g})\) does not change sign.

1.3. Applications of our main results.

1.3.1. Regular Hardy space on the quaternionic Siegel upper half space. The Beltrami–Laplace operator associated to a Kähler metric is a fundamental tool in the study of holomorphic \(H^p\) functions, since it annihilates holomorphic functions [52, chapter III] [35]. On the quaternionic Siegel upper half space, direct calculation shows that regular functions do not satisfy the Beltrami–Laplace equation associated to the quaternionic hyperbolic metric (some modification may work). But by identifying the Siegel upper half space with \(\mathcal{U} = \mathbb{R}^+ \times \mathbb{H}^{n-1}\), we observed that a regular function on \(\mathcal{U}\) satisfies a more simple equation, the heat equation associated with the sub-Laplacian on the quaternionic Heisenberg group.

**Theorem 1.4.** An \(\mathbb{H}\)-valued function \(f(t, g)\) regular on \(\mathcal{U} = \mathbb{R}^+ \times \mathbb{H}^{n-1}\) satisfies

\[
\left(\partial_t + \frac{1}{8(n-1)} \Delta_H\right) f = 0,
\]

where \(\Delta_H\) is the sub-Laplacian on \(\mathbb{H}^{n-1}\).

By using parabolic maximum principle, we show that an \(H^1(\mathcal{U})\) function \(f\) is the heat kernel integral of its boundary value as follows:

\[
f(t, g) = \int_{\mathbb{H}^{n-1}} h_t(g^{-1} \cdot g) f^b(g') dg',
\]

where \(h_t(g)\) is the heat kernel \(e^{-\frac{t}{8(n-1)} \Delta_H}\) and \(f^b\) is the radial limit of \(f\). Consequently, \(f^b\) belongs to boundary Hardy space \(H^p(\mathbb{H}^{n-1})\). This gives us a formula reproducing regular functions directly, as an approximation to the identity, whose kernel is controlled by the geometry of the quaternionic Heisenberg group nicely.

We point out that our argument can be applied to holomorphic functions on Siegel upper half space in \(\mathbb{C}^n\) to simplify corresponding part of Geller’s proof in [35].

**Definition 1.1.** A regular function \(A\) on \(\mathcal{U}\) is a regular \(p\)-atom if there is a \((p, \infty, \alpha)\)-atom \(a\) on \(\mathbb{H}^{n-1}\) such that \(A = \mathcal{P}(a) \in H^p(\mathcal{U})\).

**Definition 1.2.** The atomic Hardy space \(H_{at}^p(\mathcal{U})\) is the set of all regular functions of the form

\[
\sum_{j=1}^{\infty} A_j \lambda_j \quad \text{with} \quad \lambda_j \in \mathbb{H}, \quad \sum_{j=1}^{\infty} |\lambda_j|^p < +\infty,
\]

where each \(A_j\) is a regular \(p\)-atom, such a series converges to a regular function by Lemma 4.1. Moreover, the norm (quasinorm) of \(f \in H_{at}^p(\mathcal{U})\) is the infimum of \(\left(\sum_{j=1}^{\infty} |\lambda_j|^p\right)^{\frac{1}{p}}\) taken over all possible decomposition of \(f\) in terms of \(\sum_{j=1}^{\infty} A_j \lambda_j\).

The atomic Hardy space \(H_{at}^p(\mathcal{U})\) is a right quaternionic vector space (cf. Remark 2.1). Next, we have the following characterisation of the regular Hardy space \(H^p(\mathcal{U})\).

**Theorem 1.5.** For \(\frac{2}{3} < p \leq 1\), \(H_{at}^p(\mathcal{U}) = H^p(\mathcal{U})\) and they have equivalent quasi-norms.

We would like to point out that the restriction of \(p\) comes from the subharmonicity of \(|f|^p\) for \(f \in H^p(\mathcal{U})\) which may not hold for \(p < \frac{2}{3}\). Note that the class of holomorphic functions is unique in the sense that \(|f|^p\) is subharmonic for any holomorphic function \(f\) and any \(p > 0\). Since the Cauchy–Fueter operator in one quaternionic variable is a kind of generalized Cauchy–Riemann operator, \(|f|^p\) is not always subharmonic.
for $p < \frac{2}{n}$, and so regular $H^p(\mathcal{H})$ space for such a $p$ will be completely different from regular $H^p(\mathcal{H})$ space for $\frac{2}{n} < p \leq 1$.

For $p < 1$, it does not make sense to say that the boundary value of a regular $H^p$ function on $\mathcal{H}$ belongs to the boundary Hardy space $H^p(\mathcal{H}^{n-1})$, because an element of $H^p(\mathcal{H}^{n-1})$ may be a distribution. To prove Theorem 1.5, we show that for a regular $H^p$ function $f$ with $\frac{2}{n} < p \leq 1$, $f(\varepsilon, \cdot) \in H^p_{at}(\mathcal{H}^{n-1})$ for any $\varepsilon > 0$ and their $H^p_{at}(\mathcal{H}^{n-1})$ quasi-norms are uniformly bounded. Then there exists a subsequence $\{f(\varepsilon_k, \cdot)\}$ weakly convergent to some $f^b$ in $H^p_{at}(\mathcal{H}^{n-1})$. On the other hand, for $f \in L^1(\mathcal{H}^{n-1}) \cap L^\infty(\mathcal{H}^{n-1})$ and fixed $(t, g) \in \mathcal{H}$, we can show that

$$S_{(t, g)}(f) := \mathcal{P}(f)(t, g)$$

is well defined and can be extended to a continuous linear functional on $H^p_{at}(\mathcal{H}^{n-1})$. The regular atomic decomposition will follow from $S_{(t, g)}(f^b) = \lim_{k \to \infty} S_{(t, g)}f(\varepsilon_k, \cdot)$ and substituting atomic decomposition of $f^b$ on the boundary.

We note that for domains in $\mathbb{C}^n$, the method of obtaining atomic decomposition for the holomorphic Hardy space via the atomic decomposition on the boundary was first used by Krantz–Li [39], but our techniques here are different from [39] in many aspects.

1.3.2. *Singular value estimates for commutators* $[b, \mathcal{P}]$. Recall that in [19] we have obtained the characterisations of boundedness and compactness of commutators $[b, \mathcal{P}]$ for $b \in BMO$ space and $VMO$ space respectively. Based on our Theorem 1.3, we provide a deeper study on the singular value estimates commutators $[b, \mathcal{P}]$. Recall that in the classical setting, characterisations of singular values and the Schmidt decomposition of commutator of Hilbert transform and Riesz transforms were first studied by Peller in $\mathbb{R}$ [44] (see also [45]), then developed by Janson–Wolff in $\mathbb{R}^n$, $n \geq 2$ [37], and later on by Rochberg–Semmes [48, 47], via the Schatten classes and the Besov spaces. We summarise the known results as follows. Let $H$ denote the Hilbert transform and let $R_\ell$ denote the $\ell$-th Riesz transform on $\mathbb{R}^n$.

1. If $n = 1$ and $0 < p < \infty$, then $[b, H]$ is in Schatten class $S^p$ if and only if the symbol $b$ is in the Besov space $B^{1/p}_{p,p}(\mathbb{R})$ [44, 45].

2. Suppose $n \geq 2$ and $b \in L^1_{loc}(\mathbb{R}^n)$. When $p > n$, $[b, R_\ell] \in S^p$ if and only if $b \in B^{n/p}_{p,p}(\mathbb{R}^n)$; when $0 < p \leq n$, $[b, R_\ell] \in S^p$ if and only if $b$ is a constant [37, 47].

The Janson–Wolff inequality has close connection to a quantised derivative of Alain Connes introduced in [24, IV] in terms of the (weak) Schatten norm of the commutator [42]. A similar Janson–Wolff phenomenon has been demonstrated by Feldman and Rochberg [32] for the Cauchy–Szegő projection on the unit ball and Heisenberg group via the Hankel operators. Here we establish the analogous result of Feldman and Rochberg in our setting, based on our main result, Theorem 1.3, and on the recent breakthrough of Fan, Lacey and Li [28], since the theory of Fourier analysis and Hankel operator is not as effective as in the Euclidean setting or the classical Heisenberg group setting.

**Definition 1.3.** Let $f \in L^1_{loc}(\mathcal{H}^{n-1})$. Then we say that $f$ belongs to Besov space $B^{4n+2}_{p,p}(\mathcal{H}^{n-1})$ if

$$\int_{\mathcal{H}^{n-1}} \frac{\|f(g') - f(\cdot)\|_{L^p(\mathcal{H}^{n-1})}}{\rho(g)^{8n+4}} dg < \infty.$$ 

We also recall the definition of the Schatten class $S^p$. Note that if $T$ is any compact operator on $L^2(\mathcal{H}^{n-1})$, then $T^*T$ is compact, positive and therefore diagonalizable. For $0 < p < \infty$, we say that $T \in S^p$ if $\{\lambda_n\} \in \ell^p$, where $\{\lambda_n\}$ is the sequence of square roots of eigenvalues of $T^*T$ (counted according to multiplicity).

Our main theorem on Schatten class commutators is the following.

**Theorem 1.6.** Suppose that $0 < p < \infty$ and $b \in L^1_{loc}(\mathcal{H}^{n-1})$. Then
(1) for $p > 4n + 2$, $[b, P] \in S^p$ if and only if $b \in B_{p, \frac{4n+2}{p}}(H^{n-1})$;
(2) for $0 < p \leq 4n + 2$, $[b, P] \in S^p$ if and only if $b$ is a constant.

This paper is organised as follows. In Section 2 we provide the necessary preliminaries on quaternionic Siegel upper half space and the quaternionic Heisenberg group. In Section 3 we give proofs of our main results Theorems 1.1—1.3 and Proposition 1.1. The application of regular Hardy space on the quaternionic Siegel upper half space will be provided in Section 4 and 5. The application on singular value estimates of the commutator of Cauchy–Szegő projection will be given in Section 6.

2. The flat model of the quaternionic Siegel upper half space

The boundary $\partial U$ can be identified with the quaternionic Heisenberg group $H^{n-1}$, which is the space $\mathbb{R}^{4n-1}$ equipped with the multiplication given by

$$
(t, y) \cdot (t', y') = \left( t + t' + B(y, y'), y + y' \right),
$$

where $t = (t_1, t_2, t_3), \quad t' = (t'_1, t'_2, t'_3) \in \mathbb{R}^3$, $y = (y_1, y_2, \ldots, y_{4n-4}), \quad y' = (y'_1, y'_2, \ldots, y'_{4n-4}) \in \mathbb{R}^{4n-4}$, $B(y, y') = (B_1(y, y'), B_2(y, y'), B_3(y, y'))$, and

$$
B_{\alpha}(y, y') = 2 \sum_{l=0}^{n-2} \sum_{j,k=1}^{4} b_{kj}^{\alpha} y_{4l+k} y_{4l+j}', \quad \alpha = 1, 2, 3,
$$

with

$$
b_1 := \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad b_2 := \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad b_3 := \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}.
$$

The identity element of $H^{n-1}$ is $0 \in \mathbb{R}^{4n-1}$, and the inverse element of $(t, y)$ is $(-t, -y)$. For any $g = (t, y) \in H^{n-1}$, the homogeneous norm of $g$ is defined by

$$
\|g\| := (|y|^4 + |t|^2)^{\frac{1}{2}}.
$$

According to [25], for any $g, g' \in H^{n-1}$, $d(g, g') := \|g^{-1} \cdot g\|$ is a distance. We define balls on $H^{n-1}$ by $B(g, r) := \{g' \mid d(g, g') < r\}$.

The quaternionic Heisenberg group $H^{n-1}$ is a homogeneous group with dilations

$$
\delta_r(t, y) = (r^2 t, r y), \quad r > 0.
$$

Then for any measurable set $E \subset H^{n-1}$,

$$
|\delta_r(E)| = r^Q |E|,
$$

where $Q = 4n + 2$ is the homogeneous dimension of $H^{n-1}$. Denote by $\tau_h$ the left translation by $h$, i.e.

$$
\tau_h(g) = h \cdot g.
$$

The following $4n-1$ vector fields are left invariant on $H^{n-1}$:

$$
Y_{4l+j} = \frac{\partial}{\partial y_{4l+j}} + 2 \sum_{k=1}^{3} \sum_{k=1}^{4} b_{kj}^{\alpha} y_{4l+k} \frac{\partial}{\partial t_{\alpha}}, \quad l = 0, \ldots, n-2, \quad j = 1, \ldots, 4,
$$

$$
T_{\alpha} = \frac{\partial}{\partial t_{\alpha}}, \quad \alpha = 1, 2, 3.
$$
They form a basis for the Lie algebra of left-invariant vector field on $\mathcal{M}^{n-1}$. The only nontrivial commutator relations are
\begin{equation}
[Y_{4l+k}, Y_{4l'+j}] = 4\delta_{l'l} \sum_{\alpha=1}^{3} b_{\alpha}^{k,j} \frac{\partial}{\partial t_{\alpha}}, \quad l, l' = 0, \ldots, n-2; \quad j, k = 1, \ldots, 4.
\end{equation}

For convenience, we set $Y_{4n-4+\alpha} := T_{\alpha}, \quad y_{4n-4+\alpha} := t_{\alpha}, \quad \alpha = 1, 2, 3$. The standard sub-Laplacian on $\mathcal{M}^{n-1}$ is defined by $\Delta_{H} = \sum_{j=1}^{4n-4} Y_{j}^{2}$. For any multi-index $I = (\alpha_{1}, \ldots, \alpha_{4n-1}) \in \mathbb{N}^{4n-1}$, we set $Y^{I} := Y_{1}^{\alpha_{1}} \cdots Y_{4n-1}^{\alpha_{4n-1}}$, and further set
\begin{equation}
d(I) := \sum_{j=1}^{4n-4} \alpha_{j} + \sum_{k=4n-3}^{4n-1} 2\alpha_{k}, \quad |I| := \sum_{j=1}^{4n-1} \alpha_{j},
\end{equation}
which are called the homogeneous degree and topology degree of the differential $Y^{I}$, respectively.

We can identify the Siegel upper half space with $\mathcal{H} = \mathbb{R}_{+} \times \mathcal{M}^{n-1}$ by the following quadratic diffeomorphism:
\begin{equation}
\pi : \mathcal{U} \rightarrow \mathcal{H},
\end{equation}
\begin{equation}
(q_{1}, q') \mapsto (q_{1} - |q'|^{2}, q').
\end{equation}
The diffeomorphism $\pi$ in (2.6) induces an isomorphism of Hardy spaces
\begin{equation}
H^{p}(\mathcal{H}) \rightarrow H^{p}(\mathcal{U}),
\end{equation}
\begin{equation}
f \mapsto F(q_{1}, q') = f(q_{1} - |q'|^{2}, q'),
\end{equation}
with $\| \cdot \|$ preserved.

**Proposition 2.1.** An $\mathcal{H}$-valued function $f$ is regular on $\mathcal{U}$ if and only if $F := f \circ \pi^{-1}$ satisfies
\begin{equation}
\overline{\mathcal{O}}_{m} F = 0, \quad m = 0, \ldots, n-1,
\end{equation}
on $\mathcal{H}$, where
\begin{align*}
\overline{\mathcal{O}}_{0} & := \partial_{t} + i\partial_{t_{1}} + j\partial_{t_{2}} + k\partial_{t_{3}}, \\
\overline{\mathcal{O}}_{l+1} & := Y_{dl+1} + jY_{dl+2} + kY_{dl+3} + Y_{dl+4}, \quad l = 0, \ldots, n-2.
\end{align*}

**Proof.** The proof is similar to [50, Proposition 5.1]. The inverse of $\pi$ is
\begin{equation}
\pi^{-1} : \mathcal{H} \rightarrow \mathcal{U},
\end{equation}
\begin{equation}
(t, t, y) \mapsto (x_{1}, \ldots, x_{4n}) = (t + |y|^{2}, t, y),
\end{equation}
where $t = (t_{1}, t_{2}, t_{3})$. Note that $\overline{\mathcal{O}}_{l+1} + 2q_{l+1} \overline{\mathcal{O}}_{q_{l}}$ is a vector field tangential to the hypersurface $\partial \mathcal{U}$, since by definition, we have
\begin{equation}
(\overline{\mathcal{O}}_{l+1} + 2q_{l+1} \overline{\mathcal{O}}_{q_{l}}) \rho = 0,
\end{equation}
l = 1, \ldots, n-1, \quad \text{where } \rho \text{ is the defining function (1.1) of } \partial \mathcal{U} \text{.}

Denote $\overline{q}_{l+1} := y_{4l-3} + y_{4l-2}i + y_{4l-1}j + y_{4l}k$, \quad l = 1, \ldots, n-1, \text{ and } \overline{\mathcal{O}}_{l+1} \text{ is defined as in (1.2). We claim that}
\begin{equation}
\pi_{*}^{-1} (\overline{\mathcal{O}}_{l+1} + 2\overline{q}_{l+1} \overline{\mathcal{O}}_{q_{l}}) = \overline{\mathcal{O}}_{l+1} + 2q_{l+1} \overline{\mathcal{O}}_{q_{l}},
\end{equation}
where $\overline{\mathcal{O}}_{l} := i\partial_{t_{l}} + j\partial_{t_{l}} + k\partial_{t_{l}}$, and
\begin{equation}
\overline{\mathcal{O}}_{l+1} + 2\overline{q}_{l+1} \overline{\mathcal{O}}_{l} = \overline{\mathcal{O}}_{l},
\end{equation}
l = 1, \ldots, n-1. Namely, we have
\begin{equation}
\pi_{*}^{-1} \overline{\mathcal{O}}_{l} = \overline{\mathcal{O}}_{l+1} + 2q_{l+1} \overline{\mathcal{O}}_{q_{l}}.
\end{equation}
It follows from the definition of $\pi^{-1}$ in (2.9) that
\[
\pi^{-1}_x \partial_{\beta} = \partial_{x_1 + \beta}, \quad \beta = 1, 2, 3,
\]
\[
\pi^{-1}_x \partial_{q_{4(l-1)+j}} = \partial_{x_{4l+j}} + 2x_{4l+j} \cdot \partial_{x_1}, \quad j = 1, \ldots, 4, \quad l = 1, \ldots, n-1,
\]
since $x_{4l+j}$ is the $j$th component of $x_{4l}$.

Then
\[
\pi^{-1}_x (\overline{q}_{l+1} + 2q_{l+1} \overline{\mathbf{t}}_l) = \sum_{j=1}^4 \mathbf{i}_j \left( \partial_{x_{4l+j}} + 2x_{4l+j} \partial_{x_1} \right) + 2q_{l+1} (i\partial_{x_2} + j\partial_{x_3} + k\partial_{x_4})
\]
\[
= \overline{q}_{l+1} + 2q_{l+1} \overline{\mathbf{t}}_l,
\]
if we denote $i_1 := 1, i_2 := i, i_3 := j, i_4 := k$. (2.10) is proved.

To show (2.11), consider right multiplication by $i_3$. Noting that
\[
(x_1 + x_2i + x_3j + x_4k)i_3 = -(b^3\beta_1) - (b^3\beta_2)i - (b^3\beta_3)j - (b^3\beta_4)k = \sum_{j=1}^4 (b^3\beta_j) i_j,
\]
where $b^3\beta_j$s are given by (2.2), and $x$ is the column vector $(x_1, x_2, x_3, x_4)^t$. Then we have
\[
\overline{q}_{l+1} \overline{\mathbf{t}}_l = \overline{(y_{4(l-1)+1} + iy_{4(l-1)+2} + jy_{4(l-1)+3} + ky_{4(l-1)+4})} \left( i\partial_{t_1} + j\partial_{t_2} + k\partial_{t_3} \right)
\]
\[
= \sum_{\beta=1}^3 \sum_{j,k=1}^4 b^3_{jk} y_{4(l-1)+k}\partial_{t_3},
\]
and so
\[
\overline{q}_{l+1} + 2q_{l+1} \overline{\mathbf{t}}_l = \sum_{j=1}^4 \mathbf{i}_j \partial_{q_{4(l-1)+j}} + \sum_{j=1}^4 \mathbf{i}_j \sum_{\beta=1}^3 \sum_{k=1}^4 b^3_{jk} y_{4(l-1)+k}\partial_{t_3}
\]
\[
= Y_{4(l-1)+1} + iY_{4(l-1)+2} + jY_{4(l-1)+3} + kY_{4(l-1)+4},
\]
by the antisymmetry of $b^3\beta_j$s. It is also direct to see that
\[
\pi^{-1}_x \left( \partial_{t_1} + i\partial_{t_1} + j\partial_{t_2} + k\partial_{t_3} \right) = \overline{q}_l.
\]

Now the equivalence follows from
\[
\overline{U}_l (f \circ \pi^{-1}) \big|_{\pi(q)} = (\pi^{-1}_x \overline{U}_l \overline{q}_l \mathbf{f}) \big|_q = (\overline{\mathbf{t}}_{l+1} + 2q_{l+1} \overline{\mathbf{t}}_l) \mathbf{f} \big|_q = 0,
\]
by (2.12), and $(\partial_{t_1} + i\partial_{t_1} + j\partial_{t_2} + k\partial_{t_3}) (f \circ \pi^{-1}) \big|_{\pi(q)} = \overline{q}_l \mathbf{f} \big|_q = 0$ similarly.

\[\square\]

Remark 2.1. If $f$ is regular, then $f\lambda$ for any $\lambda \in \mathbb{H}$ is also regular. This is because $\overline{U}_l (f \lambda) = \overline{U}_l (f) \lambda = 0$ by the associativity of quaternions. Thus the Hardy space $H^p(\mathcal{U})$ is a right quaternionic vector space.

By applying the reproducing formula (1.1) to $F(t + |x|^2, t, x) = f(t, t, x)$, we get the reproducing formula (1.6) on $\mathcal{W} = \mathbb{R}^+ \times \mathcal{H}^{n-1}$ with the reproducing kernel given by
\[
K((t, g), (t', g')) := S((t + |x|^2 + t, x), (|y|^2 + s, y)),
\]
where $g = (t, x), g' = (s, y) \in \mathcal{H}^{n-1}$.

The invariance of the Cauchy–Szegő kernel [20, Proposition 5.1] has the following form.
Proposition 2.2. The Cauchy–Szegő kernel has following invariance properties.

\[
K((t, t_\kappa(g)), t_\kappa(g')) = K((t, g), g'),
K((t, a g), a g') = K((t, g), g'),
K((r^2 t, \delta_r(g)), \delta_r(g')) = K((t, g), g'),
\]
for \(g, g' \in \mathcal{H}^{n-1}\), where \(h \in \mathcal{H}^{n-1}\), \(a \in \text{Sp}(n-1)\) and \(t, r > 0\).

3. Proofs of main results

3.1. Proof of Theorem 1.1. Recall that a function \(P\) on a homogeneous Lie group \(G\) is called a polynomial if \(P \circ \exp\) is a polynomial on its Lie algebra \(\mathfrak{g}\), where \(\exp : \mathfrak{g} \to G\) is the exponential mapping. On the quaternionic Heisenberg group \(\mathcal{H}^{n-1}\), this definition of polynomial coincides with usual one, i.e. polynomials on the Euclidean space \(\mathbb{R}^{4n-1}\). This is because the exponential mapping is easily seen to be the identity mapping in this case as follows. The exponential map \(\exp : \mathbb{R}^{4n-1} \to \mathcal{H}^{n-1}\) is

\[
u \mapsto \exp \left( \sum_{j=1}^{4n-1} u_j Y_j \right)(0).
\]

\(\exp u\) is given by \(\phi^u(1)\), where \(\phi^u\) is the integral curve of the vector field \(\sum_{j=1}^{4n-1} u_j Y_j\) starting from the origin. If we write \(Y_j := \sum_{m=1}^{4n-1} B_{mj}(y) \frac{\partial}{\partial y_m}\), then

\[
\sum_{j=1}^{4n-1} u_j Y_j |_{\phi^u(t)} = \sum_{j,m=1}^{4n-1} B_{mj}(\phi^u(t)) u_j \frac{\partial}{\partial y_m}
\]
and \(\phi^u = (\phi^u_1, \ldots, \phi^u_{4n-1})\) is the solution to the Cauchy problem

\[
\begin{cases}
\phi^u_m(t) = \sum_{j=1}^{4n-1} B_{mj}(\phi^u(t)) u_j, \quad m = 1, \ldots, 4n-1, \\
\phi^u(0) = 0
\end{cases}
\]

(cf. [15, §1.3.4] for the exponential map of a homogeneous Lie group). By the expression of \(Y_j\)'s in (2.3), we see that

\[
\begin{cases}
\phi^u_1(t) = u_1, \\
\vdots \\
\phi^u_{4n-4}(t) = u_{4n-4}, \\
\phi^u_{4n-4+\alpha}(t) = u_{4n-4+\alpha} + 2 \sum_{j=0}^{n-2} \sum_{k=1}^{4} b_{kj}^u \phi^u_{4j+k}(t) u_{4j+k}, \quad \alpha = 1, 2, 3,
\end{cases}
\]
The unique solution is \(\phi^u_j(t) = u_j t, j = 1, \ldots, 4n-1\), since \((b_{kj}^u)\) are skew symmetric. Namely, \(\exp(u) = u\) and the exp is trivial.

Now every polynomial on \(\mathcal{H}^{n-1}\) can be written uniquely as

\[
P = \sum a_I \xi^I, \quad (\xi^I = y_1^{i_1} \cdots y_{2n-4}^{i_{2n-4}} t_1^{i_1} \cdots t_{2n-4}^{i_{2n-4}}),
\]
where all but finitely many of coefficients \(a_I\) vanish. We call a function \(\varphi\) nonisotropic homogeneous of degree \(m\) if

\[
\varphi(\delta_r(t, y)) = r^m \varphi(t, y),
\]
for any \((t, y) \in \mathcal{H}^{n-1}\) and \(r > 0\), where \(\delta_r(t, y) = (r^2 t, r y)\). Then \(\xi^I\) is of homogeneous degree \(d(I) = \sum_{j=1}^{4n-4} i_j + 2 \sum_{k=4n-3}^{4n-1} i_k\). The homogeneous degree of \(P\) is defined to be \(\max\{d(I) : a_I \neq 0\}\).
Proof of Theorem 1.1. Note that \( K((1, 0), g) = s(|y|^2 + 1 - t) \) for \( g = (t, y) \in \mathcal{H}^{n-1} \setminus \{0\} \). We claim that

\[
s(\sigma) = \frac{P_{2n-1}(x)}{|\sigma|^{4n}}, \quad \sigma = x_1 + x_2i + x_3j + x_4k, \tag{3.1}
\]

where \( P_{2n-1}(x) = P_{2n-1}(x_1, x_2, x_3, x_4) \) is some usual homogeneous polynomial of degree \( 2n - 1 \) on \( \mathbb{R}^4 \).

In fact, by induction, we can see

\[
\frac{\partial^l}{\partial x_1^l} \left( \frac{1}{|\sigma|^4} \right) = \frac{P_l(x)}{|\sigma|^{4+l}},
\]

where \( P_l \) is some quaternionic homogeneous polynomial of degree \( l \) on \( \mathbb{R}^4 \). Therefore, by (1.4), we have

\[
s(\sigma) = c_{n-1} \frac{\partial^{2(n-1)}}{\partial x_1^{2(n-1)}} \left( \frac{1}{|\sigma|^2} \right) \sigma + (2n-2)c_{n-1} \frac{\partial^{2n-3}}{\partial x_1^{2n-3}} \left( \frac{1}{|\sigma|^2} \right) \sigma
\]

\[
= \frac{P_{2n-2}(x)}{|\sigma|^{4(n-1)+4}} \sigma + \frac{P_{2n-3}(x)}{|\sigma|^{2(2n-3)+4}} \sigma
\]

\[
= \frac{P_{2n-1}(x)}{|\sigma|^{4n}}.
\]

Next, we claim that for any \( g = (t, y) \in \mathcal{H}^{n-1}, I = (\alpha_1, \cdots, \alpha_{4n-1}) \in \mathbb{N}^{4n-1} \), we have

\[
Y^I s(|y|^2 + 1 + t) = \frac{H_{4n-2+4|I|-d(I)}(t, y)}{(|y|^2 + 1)^2 + |t|^2}^{2n+|I|}, \tag{3.2}
\]

for some nonisotropic polynomial \( H_{4n-2+4|I|-d(I)} \) with homogeneous degree \( 4n - 2 + 4|I| - d(I) \).

In fact, by [19, (2.8)] and (3.1), we can see, for \( g = (t, y) \in \mathcal{H}^{n-1} \),

\[
s(|y|^2 + 1 + t) = \frac{P_{2n-1}(|y|^2 + 1, t_1, t_2, t_3)}{(|y|^2 + 1)^2 + |t|^2}^{2n} = \frac{\sum_{|\gamma|=2n-1} C_{\gamma}(|y|^2 + 1)^{\gamma_1}t_1^{\gamma_2}t_2^{\gamma_3}t_3^{\gamma_4}}{(|y|^2 + 1)^2 + |t|^2}^{2n} = \frac{\sum_{|\gamma|=2n-1} \sum_{k=0}^{\gamma_1} A_{\gamma}(\gamma_k)|y|^{2k}t_1^{\gamma_2}t_2^{\gamma_3}t_3^{\gamma_4}}{(|y|^2 + 1)^2 + |t|^2}^{2n} =: \frac{H_{4n-2}(t, y)}{(|y|^2 + 1)^2 + |t|^2}^{2n},
\]

for some nonisotropic polynomial \( H_{4n-2} \) with homogeneous degree \( 4n - 2 \). Then

\[
Y_{4l+j} s(|y|^2 + 1 + t) = \frac{\frac{\partial}{\partial y_{4l+j}} H_{4n-2}(t, y) + 2 \sum_{\alpha=1}^{3} \sum_{k=1}^{4} b_{kj} y_{4l+k} \frac{\partial}{\partial y_{4l}} H_{4n-2}(t, y)}{(|y|^2 + 1)^2 + |t|^2}^{2n} - \frac{2n H_{4n-2}(t, y)}{(|y|^2 + 1)^2 + |t|^2}^{2n+1} \left( 4|y|^2 y_{4l+j} + 4 \sum_{\alpha=1}^{3} \sum_{k=1}^{4} b_{kj} y_{4l+k} t_{\alpha} \right) =: \frac{H_{4n+1}(t, y)}{(|y|^2 + 1)^2 + |t|^2}^{2n+1}.
\]
for some nonisotropic polynomial $H_{4n+1}$ with homogeneous degree $4n + 1$. And

$$T_{\alpha}\cdot s(|y|^2 + 1 + t) = \frac{\partial}{\partial t_{\alpha}} \left\{ \frac{H_{4n-2}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n}} \right\}$$

$$= \frac{\partial}{\partial t_{\alpha}} H_{4n-2}(t, y) \left[ \frac{4nt_{\alpha} H_{4n-2}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+1}} \right]$$

$$= \frac{H_{4n}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+1}}$$

for some nonisotropic polynomial $H_{4n}$ with homogeneous degree $4n$.

Now assume that

$$Y^I\cdot s(|y|^2 + 1 + t) = \frac{H_{4n-2+4|I|-d(I)}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+|I|}}$$

for some nonisotropic polynomial $H_{4n-2+4|I|-d(I)}$ of degree $4n - 2 + 4|I| - d(I)$. Then

$$Y_{4l+j} Y^I\cdot s(|y|^2 + 1 + t) = \frac{\partial}{\partial t_{4l+j}} H_{4n-2+4|I|-d(I)}(t, y) + 2 \sum_{\alpha=1}^{3} \sum_{k=1}^{4} b_{kj} \frac{\partial}{\partial t_{\alpha}} H_{4n-2+4|I|-d(I)}(t, y)$$

$$= \frac{2nH_{4n-2+4|I|-d(I)}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+|I|}} \left( 4|y|^2 y_{4l+j} + 4 \sum_{\alpha=1}^{3} \sum_{k=1}^{4} b_{kj} y_{4l+k t_{\alpha}} \right)$$

$$= \frac{H_{4n+4|I|-d(I)}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+|I|+1}}$$

where $l = 0, \cdots, n-2$, $j = 1, \cdots, 4$. And

$$T_{\alpha} Y^I\cdot s(|y|^2 + 1 + t) = \frac{\partial}{\partial t_{\alpha}} \left\{ \frac{H_{4n-2+4|I|-d(I)}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+|I|}} \right\}$$

$$= \frac{\partial}{\partial t_{\alpha}} H_{4n-2+4|I|-d(I)}(t, y) \left[ \frac{2(2n + |I|) t_{\alpha} H_{4n-2+4|I|-d(I)}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+|I|+1}} \right]$$

$$= \frac{H_{4n+4|I|-d(I)}(t, y)}{(|y|^2 + 1 + |t|^2)^{2n+|I|+1}}$$

The claim is proved by induction.

By (3.2) we can see

$$|Y^I\cdot s(|y|^2 + 1 + t)| = \frac{|H_{4n-2+4|I|-d(I)}(t, y)|}{(|y|^2 + 1 + |t|^2)^{2n+|I|}} \leq C \sum_{n=0}^{4n-2+4|I|} \frac{||t, y||^m}{(|y|^2 + 1 + |t|^2)^{2n+|I|}} \leq C \frac{1}{(|y|^2 + 1 + |t|^2)^{n+\frac{1}{2} + \frac{1}{4}|I|}}$$
\[
\leq C \frac{1}{(|g| + 1)^{4n+2+d(I)}}.
\]

The proof of Theorem 1.1 is complete. \hfill \Box

3.2. Proof of Proposition 1.1. To begin with, we first recall the atoms and atomic Hardy space on \(\mathcal{H}^{n-1}\), which is a special case in [31].

**Definition 3.1.** We say a quaternion valued function \(f\) on \(\mathcal{H}^{n-1}\) is a \((p, \infty, \alpha)\)-atom if

- (i) there exists a ball \(B(g_0, r)\) such that \(f \in L^\infty(B(g_0, r))\) and \(\text{supp} f \subseteq B(g_0, r)\);
- (ii) \(\|f\|_{L^\infty(\mathcal{H}^{n-1})} \leq |B(g_0, r)|^{-\frac{d}{p}}\);
- (iii) \(\int_{\mathcal{H}^{n-1}} P(g) f dg = 0\),

for any quaternion polynomial \(P\) of homogeneous degree \(\leq \alpha\), where \(\alpha \geq \lfloor Q(\frac{1}{p} - 1)\rfloor\), \(Q = 4n + 2\) is the homogeneous dimension of \(\mathcal{H}^{n-1}\) (it is equivalent to the vanishing for any scalar polynomials \(P\) of homogeneous degree \(\leq \alpha\)).

**Definition 3.2.** The atomic Hardy space \(H^P_{at}(\mathcal{H}^{n-1})\) is the set of all quaternion valued tempered distributions of the form

\[
\sum_{j=1}^\infty f_j \lambda_j \quad \text{with} \quad \lambda_j \in \mathbb{H}, \quad \sum_{j=1}^\infty |\lambda_j|^p < +\infty,
\]

(the sum converges in the topology of \(S'\)), where each \(f_j\) is a \((p, \infty, \alpha)\)-atom. Moreover, the norm (quasi-norm) of \(f \in H^P_{at}(\mathcal{H}^{n-1})\) is the infimum of \(\left(\sum_{j=1}^\infty |\lambda_j|^p\right)^\frac{1}{p}\) taken over all possible decomposition of \(f\) in the form \(f = \sum_{j=1}^\infty f_j \lambda_j\).

The atomic Hardy space \(H^P_{at}(\mathcal{H}^{n-1})\) is a right quaternionic vector space.

**Proposition 3.1.** For each positive integer \(k\), there exist \(C_k > 0\) and \(b > 0\) such that for all quaternion valued function \(f \in C^{k+1}(\mathcal{H}^{n-1})\) and all \(g, g' \in \mathcal{H}^{n-1}\),

\[
|f(g \cdot g') - P(g')| \leq C_k \|g'\|^{k+1} \sup_{\|h\| \leq \|g'\|, \; d(I) = k+1} |Y^I f(g \cdot h)|,
\]

where \(P\) is the Taylor polynomial of \(f\) at \(g\) of homogeneous degree \(k\).

**Proof.** Recall that for a real function \(f\) on \(\mathcal{H}^{n-1}\), the left Taylor polynomial of \(f\) at \(g\) with homogeneous degree \(k\) is the unique polynomial \(P\) of \(f\) with homogeneous degree \(k\) such that \(Y^I P(0) = Y^I f(g)\) for all \(d(I) \leq k\). Write

\[
f(g \cdot g') = f_1(g \cdot g') + f_2(g \cdot g')i + f_3(g \cdot g')j + f_4(g \cdot g')k.
\]

Then, by [31, Corollary 1.44], there exist \(C_k > 0\) and \(b > 0\) such that for each \(f_l, l = 1, 2, 3, 4\),

\[
|f_l(g \cdot g') - P^l(g')| \leq C_k \|g'\|^{k+1} \sup_{\|h\| \leq \|g'\|, \; d(I) = k+1} |Y^I f_l(g \cdot h)|, \quad l = 1, 2, 3, 4,
\]

where \(P^l\) is the left Taylor polynomial of \(f_l\) at \(g\) with homogeneous degree \(k\). Let

\[
P(g') = P^1(g') + P^2(g')i + P^3(g')j + P^4(g')k.
\]

Then we have

\[
|f(g \cdot g') - P(g')| = \left(\sum_{l=1}^4 |f_l(g \cdot g') - P^l(g')|^2\right)^\frac{1}{2} \leq C_k \sum_{l=1}^4 \|g'\|^{k+1} \sup_{\|h\| \leq \|g'\|, \; d(I) = k+1} |Y^I f_l(g \cdot h)|
\]

\[
\leq 4C_k \|g'\|^{k+1} \sup_{\|h\| \leq \|g'\|, \; d(I) = k+1} |Y^I f(g \cdot h)|.
\]
The proof of Proposition 3.1 is complete.

**Lemma 3.1.** $\mathcal{P}(a)(1, \cdot)$ is uniformly bounded on $L^p(\mathcal{H}^{n-1})$, $0 < p \leq 1$, for any $(p, \infty, \alpha)$-atom $a$.

**Proof.** Let $a$ be any $(p, \infty, \alpha)$-atom with support in some ball $B = B(g_0, r) \subset \mathcal{H}^{n-1}$, $r > 0$, and $\lambda := \max\{2b^{\alpha+1}, 2\}$, where $b$ is the constant in Proposition 3.1.

Write

\[
\int_{\mathcal{H}^{n-1}} |\mathcal{P}(a)(1, g)|^p \, dg = \int_{B(g_0, \lambda r)} |\mathcal{P}(a)(1, g)|^p \, dg + \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} |\mathcal{P}(a)(1, g)|^p \, dg := I_1 + I_2.
\]

For the term $I_1$, by Hölder’s inequality, we have

\[
I_1 = \int_{B(g_0, \lambda r)} |\mathcal{P}(a)(1, g)|^p \, dg \leq |B(g_0, \lambda r)|^{1 - \frac{2}{p}} \left( \int_{B(g_0, \lambda r)} |\mathcal{P}(a)(1, g)|^2 \, dg \right)^\frac{p}{2} \leq C |B(g_0, \lambda r)|^{1 - \frac{2}{p}} \left( |B(g_0, r)|^{- \frac{2}{p} + 1} \right)^\frac{p}{2} \lesssim 1,
\]

where the second inequality follows from the fact that $\mathcal{P}$ is an orthogonal projection operator from $L^2(\mathcal{H}^{n-1})$ to $H^2(\mathcal{H})$.

For the term $I_2$, by Proposition 3.1 and the vanishing moment condition of $(p, \infty, \alpha)$-atom, we have

\[
I_2 = \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} |\mathcal{P}(a)(1, g)|^p \, dg
\]

\[
= \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left| \int_{\mathcal{H}^{n-1}} K((1, g), g') a(g') \, dg' \right|^p \, dg
\]

\[
= \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left| \int_{B(g_0, r)} K((1, g), g') a(g') \, dg' \right|^p \, dg
\]

\[
= \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left| \int_{B(0, r)} K((1, g_0 \cdot g''), a(0 \cdot g'') \, dg'' \right|^p \, dg
\]

\[
= \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left| \int_{B(0, r)} [K((1, g_0 \cdot g''), -P(g'')) a(g_0 \cdot g'') \, dg'' + \int_{B(0, r)} P(g'') a(g_0 \cdot g'') \, dg'' \right|^p \, dg
\]

\[
\leq \|a\|_{L^\infty(B(g_0, r))} \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left| \int_{B(0, r)} [K((1, g_0 \cdot g''), P(g'')) a(g_0 \cdot g'') \, dg'' \right|^p \, dg
\]

\[
\lesssim \|a\|_{L^\infty(B(g_0, r))} \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left( \int_{B(0, r)} \sup_{\|h\| \leq \|s^{\alpha+1}\| \|g''\|, d(l) = \alpha + 1} |Y^l K((1, g_0 \cdot h)) \, dg'' \right)^p \, dg
\]

\[
\lesssim |B(g_0, r)|^{-1} r^{(\alpha+1)p} \int_{\mathcal{H}^{n-1}\setminus B(g_0, \lambda r)} \left( \int_{B(0, r)} \sup_{\|h\| \leq \|s^{\alpha+1}\| \|g''\|, d(l) = \alpha + 1} |Y^l K((1, g_0 \cdot h)) \, dg'' \right)^p \, dg,
\]

where $P$ is the Taylor polynomial of $K((1, g_0), \cdot)$ at $g_0$ with homogeneous degree $\alpha$.
Therefore, \( \tilde{Y}(3.5) \) with \((t', y') = g^{-1} \cdot g'\), by left invariance, we have 
\[ Y_{(g')}^l [K ((1, g), g')] = Y_{(g')}^l [K ((1, 0), g^{-1} \cdot g')] = Y_{(t', y')} [s (|y'|^2 + 1 - t')] \mid_{(t', y') = g^{-1} \cdot g'}. \]
Then, by Theorem 1.1, we have 
\[ \left| Y_{(g')}^l [K ((1, g), g')] \right| \leq Cd(0, g^{-1} \cdot g')^{-Q - d(1)}. \]
For any \( g \in \mathcal{A}^{n-1} \setminus B(y_0, \lambda r) \) and \( g' \in B(y_0, b^{n+1} r) \),
\[ d(0, g^{-1} \cdot g') = d(g, g') \geq d(g, g_0) - d(g', g_0) \geq \frac{1}{2} d(g, g_0). \]
Consequently,
\[ I_2 \lesssim |B(y_0, r)|^{-1} r^{(p+1)\alpha} \int_{\mathcal{A}^{n-1} \setminus B(y_0, \lambda r)} \left( \sup_{h, |h| \leq \lambda \cdot |y'|} d(0, g^{-1} (g_0 \cdot h))^{-4n - 3 - \alpha} \right)^p \, dg \]
\[ \lesssim |B(y_0, r)|^{-1} r^{(p+1)\alpha} |B(0, r)|^p \int_{\mathcal{A}^{n-1} \setminus B(y_0, \lambda r)} d(g, g_0)^{-4n - 3 - \alpha} \, dg \]
\[ \lesssim r^{Q(p-1) + (p+1)\alpha} \int_{\mathcal{A}^{n-1} \setminus B(y_0, \lambda r)} d(0, g^{-1} (g_0 \cdot h))^{-4n - 3 - \alpha} \, dg \]
\[ \lesssim 1, \]
where \( Q = 4n + 2 \). This completes the proof. \( \square \)

**Proof of Proposition 1.1.** Let \( a \) be any \((p, \infty, \alpha)\)-atom with support in some ball \( B = B(y_0, r) \subset \mathcal{A}^{n-1} \) and \( r > 0 \). By the translation invariance of Cauchy–Szegő kernel in Proposition 2.2, we have
\[ \int_{\mathcal{A}^{n-1}} |P(a)(x, h)|^p \, dh = \int_{\mathcal{A}^{n-1}} |P(a)(x, y_0 \cdot g)|^p \, dg \]
\[ = \int_{\mathcal{A}^{n-1}} \int_{\mathcal{A}^{n-1}} K((x, y_0 \cdot g), g')a(g') \, dg \, dg' \]
\[ = \int_{\mathcal{A}^{n-1}} \int_{\mathcal{A}^{n-1}} K((x, y_0 \cdot g), y_0 \cdot g')a(y_0 \cdot g') \, dg \, dg' \]
\[ = \int_{\mathcal{A}^{n-1}} \int_{\mathcal{A}^{n-1}} K((x, g), g')a(g') \, dg \, dg' \]
\[ = \int_{\mathcal{A}^{n-1}} \int_{\mathcal{A}^{n-1}} K((x, g), \tilde{g}')\tilde{a}(\tilde{g}') \, d\tilde{g}' \, dg \]
where \( \tilde{a}(\tilde{g}') := a(y_0 \cdot g') \) is supported on \( B(0, r) \), and it is clear that
\[ ||\tilde{a}||_{\infty} = ||a||_{\infty} \leq |B(y_0, r)|^{-\frac{1}{p}} = |B(0, r)|^{-\frac{1}{p}}. \]
For any quaternion polynomial \( P \) of homogeneous degree \( \alpha \), since \( P(g_0) \) is also a polynomial of the same order, we have
\[ (3.5) \int_{\mathcal{A}^{n-1}} P(g)\tilde{a}(g) \, dg = \int_{\mathcal{A}^{n-1}} P(g)a(g_0 \cdot g) \, dg = \int_{\mathcal{A}^{n-1}} P(g_0^{-1} \cdot g')a(g') \, dg' = 0. \]
Therefore, \( \tilde{a}(\tilde{g}') \) is a \((p, \infty, \alpha)\)-atom centered at 0. So it’s sufficient to prove the result for \((p, \infty, \alpha)\)-atom supported on \( B(0, r) \). Now by dilation invariance of Cauchy–Szegő kernel in Proposition 2.2, we have
\[ \int_{\mathcal{A}^{n-1}} |P(a)(x, h)|^p \, dh = \int_{\mathcal{A}^{n-1}} \int_{\mathcal{A}^{n-1}} K((x, \delta_{\sqrt{2}} g), \delta_{\sqrt{2}} \tilde{g}')\tilde{a}(\delta_{\sqrt{2}} \tilde{g}') \, d\tilde{g}' \, dg' \]
\[ = \int_{\mathcal{H}^{n-1}} \left| \int_{\mathcal{H}^{n-1}} K((1, \bar{g}), g'') \hat{\alpha}(\delta, \gamma g'') e^{\frac{\bar{g}}{\sqrt{r}}} dg' \right|^p d\bar{g} \]
\[ = \int_{\mathcal{H}^{n-1}} \left| \int_{\mathcal{H}^{n-1}} K((1, \bar{g}), g'') \hat{\alpha}(g'') dg'' \right|^p d\bar{g}, \]
where \( \hat{\alpha}(g'') = e^{\frac{\bar{g}}{\sqrt{r}}} \hat{\alpha}(\delta, \gamma g'') \) is supported on \( B(0, \frac{r}{\sqrt{e}}) \) and
\[ |\hat{\alpha}(g'')| \leq e^{\frac{\bar{g}}{\sqrt{r}}} |B(0, r)|^{-\frac{1}{p}} = |B(0, \frac{r}{\sqrt{e}})|^{-\frac{1}{p}}. \]

For any quaternion polynomial \( P \) of homogeneous degree \( \alpha \), since \( P(\delta, \cdot) \) is also a polynomial of the same order, we have
\[ (3.6) \quad \int_{\mathcal{H}^{n-1}} P(g) \hat{\alpha}(g) dg = \int_{\mathcal{H}^{n-1}} P(g) e^{\frac{\bar{g}}{\sqrt{r}}} \hat{\alpha}(\delta, \gamma g) dg = e^{\frac{\bar{g}}{\sqrt{r}}} \int_{\mathcal{H}^{n-1}} P(\delta, \gamma g') \hat{\alpha}(g') dg' = 0. \]
That is to say \( \hat{\alpha} \) is a \( (p, \infty, \alpha) \)-atom supported on \( B(0, \frac{r}{\sqrt{e}}) \), and
\[ (3.7) \quad \int_{\mathcal{H}^{n-1}} |\mathcal{P}(a)(\varepsilon, h)|^p dh = \int_{\mathcal{H}^{n-1}} |\mathcal{P}(\hat{\alpha})(1, \bar{g})|^p d\bar{g}. \]
Therefore, it’s sufficient to prove the result for \( \varepsilon = 1 \) and \((p, \infty, \alpha)\)-atom centered at 0. Then, by Lemma 3.1, we can see that
\[ \int_{\mathcal{H}^{n-1}} |\mathcal{P}(a)(\varepsilon, h)|^p dh \leq C_{p,n,\alpha}. \]
Therefore,
\[ \|\mathcal{P}(a)\|_{H^{p/2}(\mathcal{H})} = \sup_{\varepsilon > 0} \int_{\mathcal{H}^{n-1}} |\mathcal{P}(a)(\varepsilon, h)|^p dh \leq C_{p,n,\alpha}. \]

The proof of Proposition 1.1 is complete. \( \square \)

3.3. Proof of Theorem 1.2. Note that by direct calculation, we can see that
\[ \sigma (x_1 + x_2 i) \sigma = x_1 + x_2 \left[ (2y_2^2 - 1) i + 2y_2y_4 j + 2y_2y_3 k \right], \]
if \( \sigma = y_2i + y_4j + y_3k \) with \( |\sigma| = 1 \) (cf. [20, (5.13)]). Therefore, for given \( \xi = \xi_1 + \xi_2 i + \xi_3 j + \xi_4 k \in \mathbb{H}_+ \)
with \( \xi_3 \xi_4 \neq 0 \) (otherwise, we already have \( \xi = \xi_1 + \xi_2 i \)), if we choose \( x_1 = \xi_1, \ x_2 = -|\text{Im} \xi|, \) where
\[ |\text{Im} \xi| := (\xi_3^2 + \xi_4^2 + \xi_1^2)^{\frac{1}{2}}, \]
and
\[ \begin{align*}
  y_2 &= \frac{1}{2} \left( 1 - \frac{\xi_2}{|\text{Im} \xi|} \right)^{\frac{1}{2}}, \\
  y_3 &= -\frac{\xi_3}{2 |\text{Im} \xi|} \left[ \frac{1}{2} \left( 1 - \frac{\xi_2}{|\text{Im} \xi|} \right) \right]^{-\frac{1}{2}}, \\
  y_4 &= -\frac{\xi_4}{2 |\text{Im} \xi|} \left[ \frac{1}{2} \left( 1 - \frac{\xi_2}{|\text{Im} \xi|} \right) \right]^{-\frac{1}{2}},
\end{align*} \]
which satisfy \( y_2 \neq 0 \) and \( y_3^2 + y_4^2 + y_2^2 = 1 \), then we have \( \sigma (x_1 + x_2 i) \sigma = \xi, \) i.e.
\[ \sigma \xi \bar{\sigma} = x_1 + x_2 i = \xi_1 - |\text{Im} \xi| i. \]
Hence, by \( s(\sigma \xi \bar{\sigma}) = s(\xi) s(\bar{\sigma}) \) for any \( \sigma \in \mathbb{H} \) with \( |\sigma| = 1 \) [20, (5.11)], we have
\[ s(\xi) = s(\xi_1 - |\text{Im} \xi| i) s(\bar{\sigma}). \]
Thus for any \( g = (t, y) \in \mathcal{H}^{n-1} \), we have
\[ (3.8) \quad K(g) = s(|y|^2 + t) = \sigma s(|y|^2 - |t|i) \sigma. \]
On the other hand, by [19, (3.6)], for any \( \alpha = x_1 + x_2i \) with \( x_1 > 0 \), we have

\[
(3.9) \quad s(\alpha) = c_{n-1} \sum_{k=0}^{2n-2} \frac{(2n-2)!}{(2n-k-1)(k+1)} \frac{\bar{\alpha}}{z^{2n-k+2}}
\]

\[-c_{n-1} \sum_{k=0}^{2n-3} \frac{(2n-2)!}{(2n-k-2)(k+1)} \frac{1}{z^{2n-k-1}z^{k+2}},
\]

where \( z = x_1 + |x_2|i \). Now when \( x_2 < 0 \), write \( \alpha = re^{i\theta} \) with \( \theta \in (\frac{3\pi}{2}, 2\pi) \), \( r > 0 \). Then,

\[
(3.10) \quad s(\alpha) = s(re^{i\theta}) = \frac{c_{n-1}(2n-2)!}{r^{2n+1}} \sum_{k=0}^{2n-2} \frac{(2n-k-1)(k+1)e^{i(2n-2k-3)\theta}}{r^{2n+1}}
\]

\[-c_{n-1}(2n-2)! \sum_{k=0}^{2n-2} \frac{(2n-k-2)(k+1)e^{i(2n-2k-3)\theta}}{r^{2n+1}}
\]

\[= c_{n-1}(2n-2)! \sum_{k=0}^{2n-2} (k+1)e^{-2k\theta}
\]

\[= c_{n-1}(2n-2)! e^{i(2n-3)\theta} \sum_{k=0}^{2n-2} (k+1)e^{-2ki\theta}
\]

by using the identity

\[\sum_{k=0}^{2n-2} (k+1)t^k = \frac{(2n-1)t^{2n} - 2nt^{2n-1} + 1}{(1-t)^2}\]

with \( t = e^{-2i\theta} \). Note that for \( t^{2n} \neq 1 \),

\[|(2n-1)t^{2n} - 2nt^{2n-1} + 1| \geq 2n - |(2n-1)t^{2n} + 1| > 2n - 2n = 0,
\]

by \( |t| = 1 \), using the triangle inequality and

\[|(2n-1)(\cos \omega + i \sin \omega) + 1| = |(2n-1)^2 + 1 + 2(2n-1) \cos \omega| \frac{1}{2} < 2n\]

when \( \omega \neq 2k\pi \). But if \( t^{2n} = 1 \),

\[(2n-1)t^{2n} - 2nt^{2n-1} + 1 = (2n-1) - 2nt^{-1} + 1 = 2n(1 - e^{i2\theta}) \neq 0,
\]

for \( \theta \in (\frac{3\pi}{2}, 2\pi) \). Thus \( s(x_1 + x_2i) \neq 0 \) for \( x_1 > 0, x_2 < 0 \). Consequently, by (3.8), \( K(g) \neq 0 \), for \( g \in \mathcal{H}^{n-1} \setminus \{0\} \).

3.4. Proof of Theorem 1.3. We use the work of [54] on self-similar tilings to find a “nice” decomposition of \( \mathcal{H}^{n-1} \), analogous to the decomposition of \( \mathbb{R}^n \) into dyadic cubes in classical harmonic analysis. For \( y = (y_1, \ldots, y_{4n-4}) \), we use \( |y|_\infty \) to denote \( \max\{|y_1|, |y_2|, \ldots, |y_{4n-4}|\} \), while \( \mathcal{H}^{n-1}_Z \) denotes the subgroup \( \{(a, b) \in \mathcal{H}^{n-1} | a \in \mathbb{Z}^{4n-4}, b \in \mathbb{Z}^3\} \).

Denote by \( A \) the basic tile

\[
(3.11) \quad A := \{(t, y) \in \mathcal{H}^{n-1} | y \in [0, 1]^{4n-4}, F_j(y) \leq t_j < F_j(y) + 1, j = 1, 2, 3\},
\]

where

\[F_j(y) = \sum_{m=1}^{\infty} \frac{1}{4^m} B_j([2^m y] \mod 2, \langle 2^m y \rangle),
\]

here \([ , ]\) and \(\langle , \rangle\) denote the integer and fractional part functions, interpreted componentwise (\([y]_j = [y_j]\), etc.), and \(B_j(y, y')\) is defined in (2.1), which is bounded on \([0, 1]^{4n-4} \times [0, 1]^{4n-4}\).

If \( \gamma = (a, b) \in \mathcal{H}^{n-1}_Z \) then the image \( \tau_\gamma(A) \) of \( A \) under left translation by \( \gamma \) is

\[\tau_\gamma(A) = \{(t, y) | y - a \in [0, 1]^{4n-4}, 0 \leq t_j - b_j - B_j(a, y - a) - F_j(y - a) < 1, j = 1, 2, 3\}.
\]
According to \cite{54}, \(\mathcal{H}^{n-1} = \bigcup_{\gamma \in \mathcal{H}^{n-1}_Z} \tau_\gamma(A)\) is a (disjoint) tiling of \(\mathcal{H}^{n-1}\) and it is self-similar, i.e.

\begin{equation}
(3.12) \quad \delta_2(A) = \bigcup_{\gamma \in \Gamma_0} \tau_\gamma(A), \quad \text{or} \quad A = \bigcup_{\gamma \in \Gamma_0} \delta_2 \circ \tau_\gamma(A),
\end{equation}

where \(\Gamma_0 = \{ (a, b) : a_j = 0 \text{ or } 1, b_j = 0, 1, 2, \text{ or } 3 \}\).

**Lemma 3.2.** The tile \(A\) has inner points.

**Proof.** Fix a positive integer \(n_0\) such that \(n_0 > \lceil \log_4 M \rceil\). If \(y \in \mathbb{R}^{4n-4}\) satisfies \(0 < |y|_\infty < 2^{-n_0}\), then \(0 < 2^n y_l < 1, l = 1, \ldots, 4n - 4\), and \([2^n y]\) = 0 for \(n < n_0\).

Let \(M = \max_{y,y'} \in [0,1]^{4n-4} \{ |B_j(y, y')|, j = 1, 2, 3 \}\), then we have

\[|F_j(y)| \leq \sum_{n=n_0}^{\infty} \frac{M}{4^n} < \frac{M}{4^{n_0-1}} < \frac{1}{4}, \quad j = 1, 2, 3.\]

This implies that

\[F_j(y) < \frac{1}{4} \quad \text{and} \quad F_j(y) + 1 > 1 - \frac{1}{4} = \frac{3}{4}, \quad j = 1, 2, 3.\]

Therefore, from the definition of \(A\) as in (3.11), we see that

\[(0, 2^{-n_0})^{4n-4} \times \left(\frac{1}{4}, \frac{3}{4}\right) \subset A.\]

The lemma is proved. \(\square\)

In what follows, we denote \(g_0 := (2^{-n_0-1}, \ldots, 2^{-n_0-1}, \frac{1}{2})\) as the “center” of \(A\).

**Definition 3.3.** We define

\[\mathcal{I}_0 := \{ \tau_\gamma(A) : g \in \mathcal{H}_{Z}^{n-1}\}, \quad \mathcal{I}_j := \delta_2 \mathcal{I}_0 \quad \text{and} \quad \mathcal{I} := \bigcup_{j \in \mathbb{Z}} \mathcal{I}_j.\]

We call the sets \(T \in \mathcal{I}\) tiles. If \(j \in \mathbb{Z}\) and \(g \in \mathcal{H}_{Z}^{n-1}\) and \(T = \delta_2 \circ \tau_\gamma(A)\), then \(T = \tau_{\delta_2(g)}(\delta_2(A))\), and we further define

\[\text{cent}(T) := \delta_2 \circ \tau_\gamma(g_0), \quad \text{width}(T) := 2^j.\]

**Lemma 3.3.** Let \(\mathcal{I}_j\) and \(\mathcal{I}\) be defined as above. Then the following hold:

1. for each \(j \in \mathbb{Z}\), \(\mathcal{I}_j\) is a partition of \(\mathcal{H}_{Z}^{n-1}\), that is, \(\mathcal{H}_{Z}^{n-1} = \bigcup_{T \in \mathcal{I}_j} T\);
2. \(\mathcal{I}\) is nested, that is, if \(T, T' \in \mathcal{I}\), then either \(T\) and \(T'\) are disjoint or one is a subset of the other;
3. for each \(j \in \mathbb{Z}\) and \(T \in \mathcal{I}_j\), \(T\) is a union of \(2^{kn+2}\) disjoint congruent subtiles in \(\mathcal{I}_{j-1}\);
4. there exists \(g \in T\), such that \(B(g, C_1q) \subseteq T \subseteq B(g, C_2q)\), where \(q = \text{width}(T)\) for each \(T \in \mathcal{I}\); the constants \(C_1\) and \(C_2\) depend only on \(n\);
5. if \(T \in \mathcal{I}_j\), then \(\tau_\gamma(T) \in \mathcal{I}_j\) for all \(g \in \delta_2(\mathcal{H}_{Z}^{n-1})\), and \(\delta_2(T) \in \mathcal{I}_{j+k}\) for all \(k \in \mathbb{Z}\).

**Proof.** It is clear that (1) holds. (2) (3) (5) follow from (3.12). (4) can be implied by Lemma 3.2. \(\square\)

Every tile is a dilate and translate of the basic tile \(A\), so all have similar geometry. Hence each tile in \(\mathcal{I}_j\) has fractal boundary and is “approximately” a quaternionic Heisenberg ball of radius \(2^j\). If two tiles in \(\mathcal{I}_j\) are “horizontal neighbours”, then the distance between their centres is \(2^j\), while if they are “vertical neighbours”, then the distance is \(2^{2j}\).

**Proof of Theorem 1.3.** Based on the construction of tiles, we see that for every \(T \in \mathcal{I}_j\) and for each fixed \(N \in \mathbb{N}\), there exists a unique \(T_{N+a_0} \in \mathcal{I}_{N+j+a_0}\) such that \(T \subset T_{N+a_0}\). Here \(a_0\) is a positive integer to be determined later.

We now fix \(N \in \mathbb{N}\) and choose an arbitrary \(T \in \mathcal{I}_j\). From Theorem 1.2, we get that

\[K(g) \neq 0, \quad \forall g \in S^n,\]
where $\mathbb{S}^n = \{ g \in \mathcal{H}^{n-1} : \| g \| = 1 \}$ is the unit sphere in $\mathcal{H}^{n-1}$.

Since $K$ is a $C^\infty$ function in $\mathcal{H}^{n-1}\setminus\{0\}$, there exists $g_0$ in $\mathcal{H}^{n-1}$ with $\rho(g_0) = 1$ such that

$$|K(g_0)| = \min_{g \in \mathbb{S}^n} |K(g)| > 0.$$ 

Hence, there exists $0 < \varepsilon_1 \ll 1$ such that

$$|K(g)| > \frac{1}{2} |K(g_0)|$$

for all $g \in U(\mathbb{S}^n, 4\varepsilon_1) = \{ g \in \mathcal{H}^{n-1} : \exists \hat{g} \in \mathbb{S}^n \text{ such that } d(g, \hat{g}) < 4\varepsilon_1 \}.$

To continue, we first point out that for the chosen $T \in \mathcal{T}_j$ and that unique tile $T_{N+\alpha_0} \in \mathcal{T}_{N+j+\alpha_0}$ with $T \subset T_{N+\alpha_0}$, there exist $\hat{h} \in T_{N+\alpha_0}$ with $d(\hat{h}, \hat{h}) = \mathcal{C}2^{N+j+\alpha_0}$ and $d(\hat{h}, T_{N+\alpha_0}) > 10C_22^j$, then $\hat{g}_0 := (\delta_{\varepsilon-\varepsilon_1-j-\alpha_0}(h^{-1} \cdot \hat{h}))^{-1} \in \mathbb{S}^n$. Without lost of generality, for

$$K(g_0) = K_1(g_0) + K_2(g_0) + K_3(g_0) + K_4(g_0)\mathbf{k},$$

we assume that $|K_1(g_0)| \geq \frac{1}{2}|K(g_0)|(|\frac{1}{4}|K_1(g_0)|)$ and that $K_1(g_0)$ is positive. Then, there exists $0 < \varepsilon_o < \varepsilon_1$ such that

$$K_1(g) > \frac{1}{4}|K(g_0)|, \quad g \in B(\hat{g}_0, 4\varepsilon_o).$$

From the definition of $\hat{g}_0$ we see that

$$\hat{h} = h \cdot \delta_{\varepsilon-\varepsilon_1-j-\alpha_0}(\hat{g}_0^{-1}).$$

Next, we choose the integer $\alpha_0$ so that $2^{N+\alpha_0} > 5C_2\mathcal{C}^{-1}\varepsilon_o^{-1}$. Then fix some $\eta \in (0, 2\varepsilon_o)$ such that the two balls $B(\hat{h}, \eta r)$ and $B(\hat{h}, \eta r)$ with $r = \mathcal{C}2^{N+j+\alpha_0}$ satisfy the following condition:

$$5C_22^j \eta r < 10C_22^j.$$

Then we can deduce that $T \subset B(\hat{h}, \eta r)$ and $B(\hat{h}, \eta r) \subset T_{N+\alpha_0}$.

It is direct that for every $g \in B(\hat{h}, \eta r)$, $\hat{g} \in B(\hat{h}, \eta r)$, we can write

$$g = h \cdot \delta_r(g'_1), \quad \hat{g} = \hat{h} \cdot \delta_r(g'_2),$$

where $g'_1 \in B(0, \eta)$, $g'_2 \in B(0, \eta)$.

As a consequence, we have

$$K(g, \hat{g}) = K(h \cdot \delta_r(g'_1), \hat{h} \cdot \delta_r(g'_2)) = K(h \cdot \delta_r(g'_1), h \cdot \delta_r(\hat{g}_0^{-1}) \cdot \delta_r(g'_2))$$

$$= K(\delta_r(g'_1), \delta_r(\hat{g}_0^{-1}) \cdot \delta_r(g'_2))$$

$$= K(\delta_r(g'_1), \delta_r(g'_2))$$

$$= r^{-Q}K(g'_1, \hat{g}_0^{-1} \cdot g'_2)$$

$$= r^{-Q}K((g'_2)^{-1} \cdot \hat{g}_0 \cdot g'_1),$$

where the second equality comes from (3.15) and the third comes from the property of the left-invariance.

Next, we note that

$$d((g'_2)^{-1} \cdot \hat{g}_0 \cdot g'_1, \hat{g}_0) = d(\hat{g}_0 \cdot g'_1, g'_2 \cdot \hat{g}_0) \leq \left[ d(\hat{g}_0 \cdot g'_1, \hat{g}_0) + d(\hat{g}_0, g'_2 \cdot \hat{g}_0) \right]$$

$$= \left[ d(g'_1, 0) + d(0, g'_2) \right]$$

$$\leq 2\eta$$

$$< 4\varepsilon_o,$$

which shows that $(g'_2)^{-1} \cdot \hat{g}_0 \cdot g'_1$ is contained in the ball $B(\hat{g}_0, 4\varepsilon_o)$ for all $g'_1 \in B(0, \eta)$ and for all $g'_2 \in B(0, \eta)$. 

Thus, from (3.13) and (3.14), we obtain that

\begin{equation}
|K((g'_2)^{-1} \cdot \tilde{g}_0 \cdot g'_1)| > \frac{1}{2} |K(g_0)| \quad \text{with} \quad K_1((g'_2)^{-1} \cdot \tilde{g}_0 \cdot g'_1) > \frac{1}{4} |K(g_0)| > 0,
\end{equation}

for all $g'_1 \in B(0, \eta)$ and for all $g'_2 \in B(0, \eta)$.

Now combining the equality (3.16) and inequality (3.17) above, we obtain that

\begin{equation}
|K(g, \tilde{g})| > \frac{1}{2} r^{-Q} |K(g_0)| \quad \text{with} \quad K_1(g, \tilde{g}) > \frac{1}{4} r^{-Q} |K(g_0)|
\end{equation}

for every $g \in B(h, \eta r)$ and for every $\tilde{g} \in B(\tilde{h}, \eta r)$, where $K_1(g, \tilde{g})$ and $K_1(\tilde{g}_0)$ have the same sign. Here $K(\tilde{g}_0)$ is a fixed constant independent of $\eta, r, h, g_1$ and $g_2$. We denote

$$C(n) = \frac{1}{4} |K(\tilde{g}_0)|.$$

From the lower bound (3.18) above, we further obtain that for the suitable $\eta \in (0, \varepsilon_0)$,

\begin{equation}
|K(g, \tilde{g})| > C(n) r^{-Q} \quad \text{with} \quad K_1(g, \tilde{g}) > \frac{1}{2} C(n) r^{-Q}
\end{equation}

for every $g \in B(h, \eta r)$ and for every $\tilde{g} \in B(\tilde{h}, \eta r)$.

Based on the fact that $B(h, \eta r) \subset T_{N+a_0}$ and $\eta r > 5C_22^j$, there must be some tile $\tilde{T} \subset H_j$ such that $\tilde{T} \subset B(h, \eta r)$. Also note that $T \subset B(h, \eta r)$. Hence we obtain that $a_1 2^{N+j} \leq d(\text{cent}(T), \text{cent}(\tilde{T})) \leq a_2 2^{N+j}$, where $a_1$ and $a_2$ depend only on $a_0$ and $C$. Moreover, we see that for all $(g, \tilde{g}) \in T \times \tilde{T}$, $K_1(g, \tilde{g})$ does not change sign and that for all $(g, \tilde{g}) \in T \times \tilde{T}$, $|K_1(g, \tilde{g})| \geq 2^{-Q(N+j)}$, where the implicit constant depends on $C(n)$ and $a_0$.

The proof of Theorem 1.3 is complete.

\end{proof}

4. Regular functions and heat kernel integrals on the quaternionic Heisenberg group

4.1. Subharmonicity. In the early attempts to generalize $H^p$ to several variables, Stein and Weiss [53] considered a $\mathbb{R}^n$-valued function $u = (u_1, u_2, ..., u_n)$ on $\mathbb{R}^n_+$ that satisfies the generalized Cauchy–Riemann equations

\begin{equation}
\frac{\partial u_j}{\partial x_k} = \frac{\partial u_k}{\partial x_j} \quad \text{and} \quad \sum_{k=1}^{n} \frac{\partial u_k}{\partial x_k} = 0,
\end{equation}

which implies that $u$ is harmonic and $|u(x)|^p$ is subharmonic for $p \geq \frac{n-2}{n-1}$. Recall that for a domain $\Omega \subset \mathbb{R}^n$, an upper semicontinuous function $v : \Omega \to \mathbb{R}$ is subharmonic if $\Delta v \geq 0$ in the sense of distributions, where $\Delta$ is the Laplace operator. Based on the subharmonicity, they built an $H^p$ theory for the generalized Cauchy–Riemann systems. It is natural to try to get below $p = \frac{n-2}{n-1}$, and this can be done by studying higher gradients of harmonic functions in place of (4.1) by Calderón-Zygmund [18]. So there is no restriction for real variable theory of the $H^p$ spaces, but such restriction is natural for the $H^p$ space of functions satisfying the generalized Cauchy–Riemann equations.

The subharmonicity of solutions to the Cauchy–Fueter equation on $\mathbb{R}^4$ was mentioned in [53, p.164] without proof. Let us recall Stein-Weiss general results [53]. Let $U, V$ be two finite dimensional complex vector spaces which are irreducible representations of $\text{Spin}(n)$, the covering group $\text{SO}(n)$. $V$ is $n$-dimensional with a distinguished real $n$-dimensional subspace $V_0$, identified with $\mathbb{R}^n$. For $u \in C^1(\Omega, U)$ for a domain $\Omega$ in $\mathbb{R}^n$, the gradient of $u$, $\nabla u(x)$ for each $x \in \Omega$, is valued in $U \otimes V$. $U \otimes V$ is irreducible and can be decomposed as

$$U \otimes V = U \otimes V \oplus (U \otimes V)^{\perp}$$
where $U \boxtimes V$ is the Cartan composition, and $(U \boxtimes V)^\perp$ is its orthogonal complement. Then, we can decompose
\[
\nabla u(x) = \partial u(x) + \overline{\nabla} u(x)
\]
orthogonally, and $\overline{\nabla}$ is called the generalized Cauchy–Riemann operator.

**Theorem 4.1.** [53, Theorem 1] If $u$ is a solution of $\overline{\nabla} u = 0$, then $u$ is harmonic and $|u(x)|^p$ is subharmonic for $p \geq \frac{4}{n-2}$.

**Corollary 4.1.** For any solution $F$ to the Cauchy–Fueter equation on a domain in $\mathbb{R}^4$, $|F(x)|^p$ is subharmonic for $p \geq \frac{4}{3}$.

**Proof.** Let us check that the Cauchy–Fueter operator on $\mathbb{R}^4$ is the generalized Cauchy–Riemann operator for $U = \mathbb{C}^2$, a spin representation of Spin(4). In [53, Section 8], Stein-Weiss discussed all such operators on $\mathbb{R}^4$. Then $\frac{4}{3}$ is best one for the Cauchy–Fueter operator by [53, Theorem 4].

Assume a unitary representation $\zeta \rightarrow R_\zeta$ of Spin($n$) acts on $U$. Suppose that $\{f_\alpha\}$ is an orthonormal basis of $U$ and $\{e_j\}$ is an orthonormal basis of $V_0$. We can write $u(x) = \sum u_\alpha(x) f_\alpha$. Then $U \otimes V$ has a basis $\{f_\alpha \otimes e_j\}$, and
\[
\nabla u(x) = \sum \frac{\partial u_\alpha}{\partial x_j}(x) f_\alpha \otimes e_j.
\]
Recall [53, Section 2] that if $v(x) := R_\zeta[u(\rho_\zeta^{-1}(x))]$, where $\zeta \rightarrow \rho_\zeta$ is the rotation representation on $V_0$, then
\[
(4.2) \quad \nabla v(0) := (R_\zeta \otimes \rho_\zeta)(\nabla u)(0).
\]

Now let $V$ be $\mathbb{C}^4$ with a distinguished real 4-dimensional subspace given by the embedding of $\mathbb{H}$ into the space of complex $2 \times 2$ matrices $\mathbb{C}^{2 \times 2} \simeq \mathbb{C}^4$

\[
q = x_1 + x_2 i + x_3 j + x_4 k \mapsto \tau(q) = \begin{pmatrix} x_1 + i x_4 & -x_2 - i x_3 \\ x_2 - i x_3 & x_1 - i x_4 \end{pmatrix}.
\]
\[
\tau \text{ is a representation, i.e. } (\tau(q_1) \tau(q_2) = \tau(q_1) \tau(q_2) \text{ (cf. [58])}. \text{ We can identify } SU(2) \text{ with unit quaternions. It is well known Spin}(4) \cong SU(2) \times SU(2), \text{ which acts on } V_0 \text{ as } \rho(q_1, q_2) \tau(q) = \tau(q_1) \tau(q) \tau(q_2). \text{ We denote } \text{Spin}(4) = SU(2)_L \times SU(2)_R, \text{ where } SU(2)_L \text{ and } SU(2)_R \text{ acts on } \mathbb{C}^2 \text{ naturally, denoted by } C^2_L \text{ and } C^2_R, \text{ as 2-dimensional column and row vectors, respectively. They are two spin representations of Spin}(4). \mathbb{C}^4 \text{ as the space of } 2 \times 2 \text{ matrices means}
\mathbb{C}^4 \cong C^2_L \otimes C^2_R.
\]

For $n = 4$ and $U = \mathbb{C}^2_R$, we see that $(\frac{\partial u_\alpha}{\partial x_j}(x))$ is in the representation $C^2_R \otimes \mathbb{R}^4$ by (4.2). Under action $\tau$ in (4.3), we see that $\tau((\frac{\partial u_\alpha}{\partial x_j}(x))) \in C^2_R \otimes C^2_L \otimes C^2_R \cong C^2_R \otimes C^2_R \otimes C^2_L$ given by
\[
\nabla u(x) = \sum_{\alpha, \beta, \mu = 0, 1} \partial_{\mu, \beta} u_\alpha(x) f_\alpha \otimes f_\beta \otimes g_\mu
\]
if $\{f_\alpha\}$ is an orthonormal basis of $C^2_R$ and $\{g_\mu\}$ is an orthonormal basis of $C^2_L$, and we denote
\[
(4.4) \quad (\partial_{\mu, \beta}) := \begin{pmatrix} \partial_{x_1} + i \partial_{x_2} & -\partial_{x_3} - i \partial_{x_4} \\ \partial_{x_3} - i \partial_{x_4} & \partial_{x_2} - i \partial_{x_1} \end{pmatrix}.
\]

$C^2_R \otimes \mathbb{C}^4$ decomposes into irreducible ones as
\[
C^2_R \otimes \mathbb{C}^4 \cong C^2_R \otimes C^2_L \otimes C^2_R \cong C^2_R \otimes C^2_R \otimes C^2_L \cong (\mathbb{C}^2 \otimes \Lambda^2 \mathbb{C}^2_R) \otimes \mathbb{C}^2_L \cong (\mathbb{C}^2 \otimes \mathbb{C}^2_R) \otimes \mathbb{C}^2_L.
\]

This symmetric and antisymmetric parts decomposition is realized as
\[
f_\alpha \otimes f_\beta = f_\alpha \otimes f_\beta + f_\alpha \wedge f_\beta.
\]
where, \( f_\alpha \otimes f_\beta := (f_\alpha \otimes f_\beta + f_\beta \otimes f_\alpha) / 2 \), \( f_\alpha \wedge f_\beta := (f_\alpha \otimes f_\beta - f_\beta \otimes f_\alpha) / 2 \). Thus, the projection of \( \nabla u(x) \) to \( \mathbb{C}_L^2 \) is simply
\[
\mathcal{D} u(x) = \sum_{\mu = 0, 1} (\partial_{\mu 1} u_0(x) - \partial_{\mu 0} u_1(x)) f_0 \wedge f_1 \otimes g_\mu.
\]
As a result, the generalized Cauchy–Riemann \( \overline{\mathcal{D}} u(x) = 0 \) for \( U = \mathbb{C}_L^2 \) is equivalent to
\[(4.5)\]
\[
\partial_{\mu 1} u_0(x) - \partial_{\mu 0} u_1(x) = 0, \quad \mu = 0, 1.
\]
On the other hand, apply the representation \( \tau (4.3) \) to the Cauchy–Fueter \( \overline{\mathcal{D}} q F = 0 \) to get
\[
(4.6) \quad \begin{pmatrix} \partial_{00} & \partial_{01} \\ \partial_{10} & \partial_{11} \end{pmatrix} \begin{pmatrix} \overline{F}_0 - \overline{F}_1 \\ \overline{F}_1 \end{pmatrix} = 0,
\]
where \( \overline{F}_0 = F_1 + iF_2, \overline{F}_1 = F_3 - iF_4 \). From the second column of this matrices equation, we see that \( u_1 = \overline{F}_1 \) and \( u_0 = \overline{F}_0 \) satisfy the generalized Cauchy–Riemann (4.5) for \( U = \mathbb{C}_L^2 \). Thus, by Theorem 4.1, \(|u|^p = |F|^p\) is subharmonic for \( p \geq \frac{2}{3} \).

**Proposition 4.1.** There is a positive constant \( C \) such that for all \( \frac{2}{3} \leq p \leq 1 \) and \( (\varepsilon, \xi) \in \mathcal{U} \), we have
\[(4.7) \quad |f(\varepsilon, \xi)| \leq C \| f \|_{H^p(\mathcal{U})} \varepsilon^{-\frac{2p+1}{3}}
\]
for any \( f \in H^p(\mathcal{U}) \).

**Proof.** Note that if \( f \in H^p(\mathcal{U}) \), then \( \overline{f}(\varepsilon, \xi) = f(\varepsilon, g \cdot \xi) \) (for a fixed \( g \)) is also regular, because
\[
\overline{\mathcal{D}} q \overline{f}(\varepsilon, \xi) = \overline{\mathcal{D}} q f(\varepsilon, g \cdot \xi) = 0
\]
due to the left invariance of \( Y_j \)’s and the fact that \( f \) is regular. Moreover, we have
\[
\int_{\mathbb{H}^{n-1}} |\overline{f}(\varepsilon, \xi')|^p \, d\varepsilon' = \int_{\mathbb{H}^{n-1}} |f(\varepsilon, \xi')|^p \, d\varepsilon'
\]
by the invariance of the measure. We get \( \overline{f} \in H^p(\mathcal{U}) \). So it is sufficient to prove (4.7) for \( g = 0 \).

Recall that \( F(q_1, q_2) = f(q_1 - |q_2|^2, q_2) \) is an element of \( H^p(\mathcal{U}) \) with the same norm by (2.7). By Corollary 4.1, \( |F|^p \) is subharmonic on \( \mathcal{U} \) for each quaternionic variable. Consequently, the submean value inequality holds for each quaternionic variable. By the subharmonicity of \( |F|^p(q_1, 0, \ldots, 0) \) in variable \( q_2 \) to get
\[
|F|^p(\varepsilon, 0) \leq \frac{1}{|B_{\mathbb{H}}(\varepsilon, \xi/2)| \cdot B_{\mathbb{H}}(\varepsilon, \xi/2)} \int_{B_{\mathbb{H}}(\varepsilon, \xi/2)} |F|^p(q_1, 0, \ldots, 0) \, dV(q_1),
\]
where \( B_{\mathbb{H}}(x, r) \) is a ball in \( \mathbb{H} \) with radius \( r \) and center \( x \). Then apply the submean value inequality to subharmonic function \( |F|^p(q_1, q_2, 0, \ldots, 0) \) in variable \( q_2 \) to get
\[
|F|^p(\varepsilon, 0) \leq \frac{1}{B_{\mathbb{H}}(\varepsilon, \xi/2) \times B_{\mathbb{H}}(0, \sqrt{\varepsilon}/2)} \int_{B_{\mathbb{H}}(\varepsilon, \xi/2)} \int_{B_{\mathbb{H}}(0, \sqrt{\varepsilon}/2)} |F|^p(q_1, q_2, 0, \ldots, 0) \, dV(q_1) \, dV(q_2).
\]
Repeating this procedure, we finally get
\[
|F|^p(\varepsilon, 0) \leq \frac{1}{|D|} \int_D |F|^p(q) \, dV(q),
\]
where \( D = B_{\mathbb{H}}(\varepsilon, \xi/2) \times B_{\mathbb{H}}(0, \sqrt{\varepsilon}/2) \times \cdots \times B_{\mathbb{H}}(0, \sqrt{\varepsilon}/2) \). Noting that \( D \subset \{ q \in \mathcal{U} : \varepsilon/4 < \Re q_1 - |q_2|^2 < 3\varepsilon/2 \} \), we have
\[
|F|^p(\varepsilon, 0) \leq \frac{1}{|D|} \int_{\{ \varepsilon/4 < \Re q_1 - |q_2|^2 < 3\varepsilon/2 \}} |F(x_1, x_2, \ldots, x_{4n})|^p \, dx_1 \, dx_2 \cdots \, dx_{4n}
\]
\[
\leq \left( \frac{2^{4n}}{\varepsilon^{2n+2}} \right)^{\frac{4n}{2n+2}} \int_{\left( \frac{\varepsilon}{2} \right)^{4n} \times \mathbb{R}^{4n-1}} \left| F \left( x_1 + \sum_{j=5}^{4n} |x_j|^2, x_2, \ldots, x_{4n} \right) \right|^p \, dx_1 \, dx_2 \cdots \, dx_{4n}
\]
choose a Kähler metric matching C-C geometry on the boundary. Geller [18] wrote down the Beltrami–Laplace equation. This solution formula reproduces holomorphic functions, playing the role of Poisson integral, and can be used to prove the boundary value of a holomorphic $H^1$ function on the Siegel upper half space belongs to the boundary Hardy space $H^1$ over the Heisenberg group [35].

4.2. The heat equation. Recall that it is a fundamental fact in $H^1$ theory that an $H^1$ harmonic function on $\mathbb{R}^n$ can be expressed as Poisson integral of its boundary value. To develop $H^1$ theory for holomorphic functions on a strongly pseudoconvex domain $D$, there is a natural candidate, the Poisson-Szegő kernel, which is the function $\mathcal{P}(z, w)$ on $D \times \partial D$ defined by

$$\mathcal{P}(z, w) = \frac{|S(z, w)|^2}{S(z, z)}, \quad z \in D, w \in \partial D,$$

if $S(z, z) \neq 0$. Here $S(z, w)$ is the Cauchy–Szegő kernel reproducing $H^2(D)$ functions, which is holomorphic in $z$ and anti-holomorphic in $w$. $P$ reproduces $H^2(D)$ functions, because, for $f \in H^2(D)$,

$$\int_D \mathcal{P}(z, w)f(w)d\beta(w) = \frac{1}{S(z, z)} \int_{\partial D} S(z, w)S(z, w)f(w)d\beta(w) = f(z)$$

by Cauchy–Szegő kernel reproducing $H^2(D)$ function $S(z, \cdot)f(\cdot)$ for fixed $z$ and $S(z, z)$ real. We have $\int_{\partial D} \mathcal{P}(z, w)d\beta(w) = 1$. Note that Cauchy–Szegő kernel restricted to the boundary is a singular integral, and so it is not an approximation to the identity. The Poisson-Szegő reproducing formula was used by Koranyi [40] and Garnett-Latter [33] to study holomorphic $H^1$ space over the unit ball in $\mathbb{C}^n$ and its atomic decomposition. This construction does not work in the noncommutative case, because $S(z, \cdot)\tilde{f}(\cdot)$ is not regular in general although $S(z, \cdot)$ and $f(\cdot)$ are both regular. For example $x_1 + ix_2$ and $x_2 + jx_4$ are both annihilated by the Cauchy–Fueter operator $\partial_q := \partial_{x_1} + i\partial_{x_2} + j\partial_{x_3} + k\partial_{x_4}$, but their product $(x_1 + ix_2)(x_2 + jx_4)$ is not.

In early 1970s, little information of Cauchy–Szegő kernel for a strongly pseudoconvex domain in $\mathbb{C}^n$ was known. Because real and imaginary parts of a holomorphic function are both harmonic, Stein [52] used the Euclidean Poisson integral in a clever way to prove the admissible convergence almost everywhere of a bounded holomorphic function with approach regions defined in terms of Carnot–Carathéodory (C-C) distance on the boundary. This technique was used to control the (non)tangential maximal function of an $H^p$ function on some bounded pseudoconvex domains in $\mathbb{C}^n$ by Krantz–Li [39] and Dafni [26]. But in the unbounded case, Poisson integral is an integral with respect to the surface measure of the boundary, which blows up at infinity with respect to the Lebesgue measure on the Heisenberg group. Moreover, since the Lipschitz constant of the boundary is unbounded, the detail of Poisson integral is not known directly.

The key point is to find a reproducing formula that matches the geometry of the boundary.

On the other hand, since a holomorphic function is annihilated by the Beltrami–Laplace operator associated to a Kähler metric, potential theory associated to this operator (the Dirichlet problem and its solution, etc.) gives us information of holomorphic function on the domain. In general, we need to choose a Kähler metric matching C-C geometry on the boundary. Geller [35] wrote down the Beltrami–Laplace operator associated to the complex hyperbolic metric on the Siegel upper half space, and found the solution to the corresponding Dirichlet problem explicitly. It was generalized by Graham [36] to some modifications of the Beltrami–Laplace equation. This solution formula reproduces holomorphic functions, playing the role of Poisson integral, and can be used to prove the boundary value of a holomorphic $H^1$ function on the Siegel upper half space belongs to the boundary Hardy space $H^1$ over the Heisenberg group [35].
Proof of Theorem 4.4. By (2.2)—(2.4), we have
\[ Q_{t+1}Q_{t+1} = (Y_{4t+1} - iY_{4t+2} - jY_{4t+3} - kY_{4t+4})(Y_{4t+1} + iY_{4t+2} + jY_{4t+3} + kY_{4t+4}) \]
\[ = \sum_{j=1}^{4} Y_{4t+j}^2 + i[(Y_{4t+1}, Y_{4t+2}) - [Y_{4t+3}, Y_{4t+4}]] + j[(Y_{4t+1}, Y_{4t+3}) + [Y_{4t+2}, Y_{4t+4}]] \]
\[ + k([Y_{4t+1}, Y_{4t+4}] - [Y_{4t+2}, Y_{4t+3}]) \]
\[ = \sum_{j=1}^{4} Y_{4t+j}^2 + 4 \sum_{\alpha=1}^{3} (\delta_{j2}^\alpha - b_{3j}^\alpha) \partial_{\alpha} + 4j \sum_{\alpha=1}^{3} (b_{13}^\alpha + b_{24}^\alpha) \partial_{\alpha} + 4k \sum_{\alpha=1}^{3} (b_{14}^\alpha - b_{23}^\alpha) \partial_{\alpha} \]
\[ = \sum_{j=1}^{4} Y_{4t+j}^2 + 4 (i\partial_{t_1} + j\partial_{t_2} + k\partial_{t_3}), \]
for \( l = 0, \cdots, n - 2 \). Hence if \( f \) is regular, we have
\[ (\triangle_H + 8(n-1)\partial_t)f = \left( - \sum_{l=0}^{n-2} Q_{l+1}Q_{l+1} + 8(n-1)Q_0 \right) f = 0 \]
by \( Q_j f = 0 \) by Proposition 2.1. \( \square \)

We can use heat semigroup \( e^{-\frac{t}{4}} \) to define the Littlewood–Paley function and the Lusin area integral to study boundary behavior of regular functions on \( \mathcal{H} \).

Remark 4.1. (1) In [35], Geller developed the \( H^1 \) theory for holomorphic functions on the Siegel upper half space \( \mathcal{D} := \{(z', z_{n+1}) \in \mathbb{C}^n \times \mathbb{C}^1 \mid \rho = \text{Im} z_{n+1} - |z'|^2 \} \), by using the Laplace–Beltrami operator
\[ \triangle_B = 4 \rho \left( \sum_{j=1}^{n} Z_j \overline{Z}_j + 4 \rho Z_{n+1} \overline{Z}_{n+1} + 2b_1 \overline{Z}_{n+1} \right) = 4 \rho \left( \frac{1}{4} \sum_{j=1}^{n} (X_j^2 + Y_j^2) + \partial_\rho^2 + \partial_s^2 - n\partial_\rho \right) \]
with respect to the Bergman metric (cf. [36, (2.2)]), which annihilates holomorphic functions \( (\overline{Z}_j f = 0) \), where \( Z_j = \frac{1}{2}(X_j - iY_j) \) with \( X_j = \partial_{x_j} + 2y_j \partial_y \) and \( Y_j = \partial_{y_j} - 2x_j \partial_x \) and \( Z_{n+1} = \frac{1}{2}(\partial_s - i\partial_{\rho}) \).

(2) If we throw out the term \( 4 \rho Z_{n+1} \overline{Z}_{n+1} \) in (4.9), holomorphic functions are annihilated by the heat operator \( \sum_{j=1}^{n} (X_j^2 + Y_j^2) - 4\partial_\rho \). This fact seems to not have been noticed before in literatures, and can be used to simplify proofs in [35] and may be generalized to other domains.

4.3. The heat kernel integral. Denote
\[ \mathcal{L} := \frac{1}{8(n-1)} \sum_{j=1}^{4n-4} Y_j^2 - \partial_t. \]
We need the following maximum principle for heat equation. Although there is a maximum principle for heat equation on any homogeneous group for smooth functions in [31, Proposition 8.1], we give its proof here since we need the proof for nonsmooth functions later.

Proposition 4.2. (Maximum principle) Let \( D \) be a bounded domain in \( \mathcal{H}^{n-1} \) and \( \Omega = (0, T) \times D \) for \( T > 0 \). Suppose that \( v \in C^2(\overline{\Omega}), v|_{[0,T) \times \partial D} \leq 0, v|_{(0)} \times D \leq 0 \) and \( \mathcal{L} v \geq 0 \) in \( \Omega \). Then \( v \leq 0 \) in \( \Omega \).

Proof. If replace \( v \) by \( v - \kappa_1 t - \kappa_2 \) with \( \kappa_1, \kappa_2 > 0 \), we may assume \( v|_{[0,T) \times \partial \Omega} < 0, v|_{\{0\}} \times \Omega < 0 \) and \( \mathcal{L} v > 0 \).

Suppose that \( v > 0 \) somewhere in \( \Omega \). Let \( t^* := \inf \{ t \mid v(t, g) > 0 \text{ for some } g \in \overline{\Omega} \} \). By continuity and \( v \) negative on the boundary \([0, T) \times \partial \Omega \cup \{0\} \times \Omega \), we see that there exists \((t^*, g^*) \in \Omega \) such that \( v(t^*, g^*) = 0 \),
and $t^* > 0$. We must have $v(t, g) < 0$ for $0 < t < t^*$, $g \in \Omega$, and so $\partial_t v(t^*, g^*) \geq 0$. On the other hand, $v(t^*, \cdot)$ attains its maximum at $g^*$, which implies that $Y_j v(t^*, g^*) = 0$ and

$$Y_j^2 v(t^*, g^*) = \left. \frac{d^2}{ds^2} v(t^*, g^*(\ldots, 0, s, 0, \ldots)) \right|_{s=0} \leq 0,$$

$j = 1, \ldots, 4n-4$, where $s$ appears in the $j$-th entry. Consequently, we get $Lv(t^*, g^*) \leq 0$, which contradicts to $Lv > 0$ in $\Omega$. Thus $v - \kappa_1 t - \kappa_2 \leq 0$. Now letting $\kappa_1, \kappa_2 \to 0+$, we get the result. \hfill $\square$

**Proposition 4.3.** Suppose $f \in H^1(\mathcal{U})$ and $h_t(g)$ is the heat kernel $e^{-\frac{(g-g_0)^2}{4t}}$. Then (1.9) holds.

**Proof.** Since $f_k(t, g) := f(t + 1/k, g)$ is smooth and satisfies the heat equation (1.8) for $t \geq 0, g \in \mathcal{H}^{n-1}$, and $f(1/k, \cdot) \in L^1(\mathcal{H}^{n-1}),$ let

$$\hat{f}_{(k)}(t, g) = \int_{\mathcal{H}^{n-1}} h_t(g^{-1} \cdot g) f\left(\frac{1}{k}, g\right) dg',$n$

which is smooth in $\mathcal{U}$ by the heat kernel estimate [55, Theorem IV 4.2], and satisfies $L\hat{f}_{(k)} = 0$. To apply maximum principle to each components, write $f_k = f_{(k):1} + if_{(k):2} + jf_{(k):3} + kf_{(k):4}$, $\hat{f}_{(k)} = \hat{f}_{(k):1} + i\hat{f}_{(k):2} + j\hat{f}_{(k):3} + k\hat{f}_{(k):4}$, then

$$L\left(f_{(k):j} - \hat{f}_{(k):j}\right) = 0 \quad \text{and} \quad \left(f_{(k):j} - \hat{f}_{(k):j}\right)|_{\{0\} \times \mathcal{H}^{n-1}} = 0,$$

$j = 1, 2, 3, 4$. We claim that for given $T > 0$ and $\varepsilon > 0$, there exists $R > 0$ such that

$$(4.10) \quad |f_{(k):j} - \hat{f}_{(k):j}| \leq \varepsilon \quad \text{on} \quad |0, T) \times \partial B(0, R).$$

Then we can apply the maximum principle (Proposition 4.2) to $f_{(k):j} - \hat{f}_{(k):j} - \varepsilon$ to get $f_{(k):j} - \hat{f}_{(k):j} \leq \varepsilon$. Now let $R \to \infty$ and $T \to \infty$, we get $f_{(k):j} \leq \hat{f}_{(k):j}$ on $\mathcal{U}$. The same argument gives us $f_{(k):j} \leq \hat{f}_{(k):j}$ on $\mathcal{U}$. Hence $f_k = \hat{f}_{(k)}$ on $\mathcal{U}$, i.e.

$$f\left(t + \frac{1}{k}, g\right) = \int_{\mathcal{H}^{n-1}} h_t(g^{-1} \cdot g) f\left(\frac{1}{k}, g\right) dg'.$n$$

Then letting $k \to +\infty$, we get (1.9).

To prove the claim (4.10), note that

$$(4.11) \quad \left|\int_{\mathcal{H}^{n-1}} h_t(g^{-1} \cdot g) f\left(\frac{1}{k}, g\right) dg\right| \leq \frac{\varepsilon}{2},$$

by the heat kernel estimate again, for $\|g\| \geq R$ and $t > \frac{1}{T}$ for sufficiently large $R$, since $f\left(\frac{1}{k}, \cdot\right) \in L^1(\mathcal{H}^{n-1})$. For the estimate for $f_k$, we need to use the mean value formula for $\pi^* f$, which is regular on $\mathcal{U}$ and

$$\tau^*_{\pi^{-1}(g_0)} \circ \pi^* f = \pi^* \circ \tau^*_0 f$$

by definition of $\pi$ in (2.6), where $\tau^*_{\pi^{-1}(g_0)}$ is the corresponding translation of $\mathcal{U}$ with $\pi^{-1}(g_0) \in \partial \mathcal{U}$. Then given $\varepsilon > 0$, if $R$ is sufficiently large, we have

$$(4.12) \quad \int_{\|g\| \geq R} |f(t, g)| dt dg \leq \int_{\|h\| \geq R, t \in \left(\frac{1}{2T}, T + \frac{1}{2}\right)} |f(t, g)| dt dg \leq \varepsilon,$$n

for $\|g_0\| > R + 1, t_0 \in \left(\frac{1}{k}, T + \frac{1}{k}\right)$, since $f \in L^1(\left(\frac{1}{k}, T + \frac{1}{k}\right) \times \mathcal{H}^{n-1})$ by $f \in H^1(\mathcal{U})$. Note that $\pi^* \circ \tau^*_0 f$ is regular in the product $\Omega_k := B_{\Omega_k}(0, \delta) \times B_{\Omega_k}(0, \delta) \times \ldots \times B_{\Omega_k}(0, \delta) \subset \mathcal{U}$, for some $\delta > 0$ only depending on $k$, satisfying

$$\pi(\Omega_k) \subset \left\{(t, g) : \|g\| < \frac{1}{k}, |t - t_0| < \frac{1}{2k}\right\} \subset \mathcal{U},$$
where $B_{\mathbb{H}}(q,r)$ is a ball in $\mathbb{H}$. Hence,

\begin{equation}
|f(t_0, g_0)| = |\pi^* \tau_{g_0}^* f(t_0, 0)| = \frac{1}{|\Omega_k|} \left| \int_{\Omega_k} \pi^* \tau_{g_0}^* f(t, g) dtdg \right|
\end{equation}

\begin{align*}
&\leq \frac{1}{|\Omega_k|} \left| \int_{\|g\| < \frac{1}{2} |t-t_0| < \frac{1}{10}} \tau_{g_0}^* f(t, g) dtdg \right| \\
&\leq \frac{1}{|\Omega_k|} \left| \int_{\|g^{-1} - g\| < \frac{1}{2} |t-t_0| < \frac{1}{10}} f(t, g) dtdg \right| \\
&\leq \frac{\varepsilon}{|\Omega_k|},
\end{align*}

for $\|g_0\| > R + 1, t_0 \in \left( \frac{1}{8}, T + \frac{1}{5} \right)$, by (4.12). The claim follows from estimates (4.11) and (4.13). □

**Proposition 4.4.** Suppose $f \in H^p(\mathcal{U})$ for $\frac{2}{3} < p \leq 1$ and continuous on $\overline{\mathcal{U}}$. Then for $\frac{2}{3} < q \leq p$, we have

\begin{equation}
|f(t,g)|^q \leq \int_{\mathcal{U}^{n-1}} h_1(g^{-1} \cdot g)|f(0,g')|^q dg'.
\end{equation}

We need the following parabolic version of subharmonicity of $|u|^p$ in the Euclidean case [51, section 3.2.1 in chapter 7].

**Proposition 4.5.** Suppose $f$ is regular on $\mathcal{U}$. Then for any $\frac{2}{3} \leq p \leq 1$, we have

\begin{equation}
\mathcal{L}|f|^p(t,g) \geq 0,
\end{equation}

for $(t,g) \in \mathcal{U}$ with $f(t,g) \neq 0$.

**Proof.** Since $\tau_{g_0}^* f$ is also a regular function for fixed $g_0 \in \mathcal{H}^{n-1}$ and $\mathcal{L}$ is invariant under translations, we only need to show (4.15) at point $(t,0)$. Let $F(q) = \pi^* f(q)$ be the pull back function on $\mathcal{U}$. Recall that the Cauchy–Fueter equation $\overline{\partial}_q F = 0$ is equivalent to the generalized Cauchy–Riemann $\overline{\partial}u(x) = 0$, where $u = (u_0, u_1)$ is a $\mathbb{C}^2$-valued function with

\begin{equation}
\begin{aligned}
\tau_{g_0}^* u &= F_1 - iF_2, \\
\tau_{g_0}^* u_1 &= F_3 + iF_4
\end{aligned}
\end{equation}

in (4.5)-(4.6). Thus $\pi^* f$ is also a regular function, and by Corollary 4.1

\[ |u|^p = |\pi^* f|^p, \]

for $\frac{2}{3} < p \leq 1$ is subharmonic with respect to each quaternionic variable; i.e.

\begin{equation}
\sum_{j=l+1}^{4l+4} \partial_j^2 |u|^p \geq 0
\end{equation}

for $l = 0, \ldots, n - 1$.

On the other hand, we have

\begin{equation}
\partial_l |f|^p = \frac{p}{2} (f, f)^{\frac{p}{2} - 1} ((\partial_l f, f) + (f, \partial_l f)),
\end{equation}

and

\begin{equation}
Y_j |f|^p = \frac{p}{2} (f, f)^{\frac{p}{2} - 1} ((Y_j f, f) + (f, Y_j f)),
\end{equation}
where $(\cdot, \cdot)$ is the quaternionic inner product on $\mathbb{H}$ given by $(q, q') = \overline{q}q'$ for $q, q' \in \mathbb{H}$. Note that $(Y_j f, f) + (f, Y_j f) = 2\text{Re}(Y_j f, f)$. Then, we have

$$
\sum_{j=1}^{4n-4} Y_j^2|f|^p = \frac{p}{2} \left( \frac{p}{2} - 1 \right) (f, f) \frac{2}{p-2} \sum_{j=1}^{4n-4} \left( \text{Re}(Y_j f, f) \right)^2 
+ \frac{p}{2} (f, f) \frac{2}{p-2} \left( \sum_{j=1}^{4n-4} Y_j^2 f, f \right) + \left( f, \sum_{j=1}^{4n-4} Y_j^2 f \right) + 2 \sum_{j=1}^{4n-4} |Y_j f|^2. 
$$

(4.19)

Then (4.19) minus (4.18) multiplied by $8(n-1)$ gives us

$$
8(n - 1)\mathcal{L}|f|^p = p(p - 2) (f, f) \frac{2}{p-2} \sum_{j=1}^{4n-4} \left( \text{Re}(Y_j f, f) \right)^2 + p(f, f) \frac{2}{p-2} \sum_{j=1}^{4n-4} |Y_j f|^2
$$

(4.20)

by Theorem 1.4.

Following the above calculation, the subharmonicity (4.17) implies

$$
0 \leq \sum_{j=1}^{4n-4} \partial_{x_j}^2 |u|^p = p(p - 2) (u, u) \frac{2}{p-2} \sum_{j=1}^{4n-4} \left( \text{Re}(\partial_{x_j} u, u) \right)^2 + p(u, u) \frac{2}{p-2} \sum_{j=1}^{4n-4} |\partial_{x_j} u|^2
$$

(4.21)

$$
\quad\quad\quad\quad\quad\quad\quad\quad\quad= p(p - 2) (F, F) \frac{2}{p-2} \sum_{j=1}^{4n-4} \left( \text{Re}(\partial_{x_j} F, F) \right)^2 + p(F, F) \frac{2}{p-2} \sum_{j=1}^{4n-4} |\partial_{x_j} F|^2
$$

by using (4.16). Here, by abuse of notations, $(u, u)$ or $(\partial_{x_j} u, u)$ is complex inner product on $\mathbb{C}$. Apply (4.21) to (4.20) to get $\mathcal{L}|f|^p(t, 0) \geq 0$ by $Y_j f(t, 0) = \partial_{x_j} f(t, 0)$ by $\pi, \partial_{x_j}|_{(t, 0)} = Y_j|_{(t, 0)}$.

The proof of Proposition 4.5 is complete. 

Proof of Proposition 4.4. The maximum principle can not be applied to $|f|^q$ directly because it is not smooth on $|f| = 0$. It is not easy to construct an auxiliary regular function as in [51, section 3.2.1 in chapter 7] to overcome this difficulty. But the argument of the proof of maximum principle can be applied in this case as follows. Denote $v(t, g) = |f|^q(t, g) - \kappa t$ for $\kappa > 0$ and

$$
\tilde{v}(t, g) := \int_{\mathbb{H}^n-1} h_t(g' \cdot g') v(0, g) dg'.
$$

Here $v(0, \cdot) = |f|^q(0, \cdot)$ is in $L^r(\mathbb{H}^n-1)$ with $r = p/q$. Thus, $\tilde{v}(t, g)$ is smooth and

$$
\mathcal{L} \tilde{v} = 0.
$$

As (4.11)-(4.13) in the proof of Proposition 4.3, we can show that for given $T > 0$ and $\varepsilon > 0$, there exists $R > 0$ such that $|v - \tilde{v}| \leq \varepsilon$ on $[0, T) \times \partial B(0, R)$ with the mean value formula replaced by the submean value inequality for $|\pi^* \circ \tau^*_g f|^q$ in (4.13). Then,

$$
\mathcal{L}(v - \tilde{v} - 2\varepsilon)(t, g) > 0
$$

(4.22)

when $|v(t, g)| \neq 0$, by Proposition 4.5 and $\mathcal{L}(-\kappa t) = \kappa > 0$. Moreover, $v - \tilde{v} - 2\varepsilon$ is negative on $[0, T) \times \partial B(0, R) \cup \{0\} \times B(0, R)$.

Now suppose that $(v - \tilde{v} - 2\varepsilon)(t, g) \geq 0$ at some point in $(0, T) \times B(0, R)$. Then, as in the proof of Proposition 4.2, we can find $(t^*, g^*) \in (0, T) \times B(0, R)$ such that $(v - \tilde{v} - 2\varepsilon)(t^*, g^*) = 0$ and $(v - \tilde{v} - 2\varepsilon)(t, g) < 0$ for $0 < t < t^*$, $g \in B(0, R)$. Note that we must have $|v(t^*, g^*)| \neq 0$. Otherwise, we have $(v - \tilde{v} - 2\varepsilon)(t^*, g^*) < 0$, since $\tilde{v}(t, g) = \int_{\mathbb{H}^n-1} h_t(g' \cdot g') |f|^q(0, \cdot) dg' > 0$ on $\mathbb{H}$ by positivity of the heat kernel [55, Theorem IV 4.3]. Consequently, $v - \tilde{v} - 2\varepsilon$ is smooth at $(t^*, g^*)$ and so we have $\mathcal{L}(v - \tilde{v} - 2\varepsilon)(t^*, g^*) \leq 0$, which contradicts (4.22). Thus $(v - \tilde{v} - 2\varepsilon)(t, g) < 0$ on $(0, T) \times B(0, R)$ for any fixed $\varepsilon, T > 0$ and sufficiently large $R$. The result follows.

The proof of Proposition 4.4 is complete. 

□
5. Hardy space $H^p(\mathcal{U})$ and the regular atomic decomposition

We need some results about the Hardy space $H^p$ over homogeneous groups in FoSt, of which the quaternionic Heisenberg group is a special case. Recall that if $f \in S'(\mathcal{H}^{n-1})$ and $\phi \in \mathcal{S}$, the nontangential maximal function $M_\phi f$ is defined as

$$M_\phi f(g) := \sup_{\|g^{-1}h\| < t} |f * \phi_t(g')|.$$ 

If $N \in \mathbb{N}$, $f \in S'(\mathcal{H}^{n-1})$, we define the nontangential grand maximal function $M_{(N)}f$ by

$$M_{(N)}f(g) := \sup_{\|f\|_{(N)} \leq 1} M_\phi f(g),$$

where

$$\|\phi\|_{(N)} := \sup_{|t| \leq N} (1 + |g||t|^{Q}) |X^1 \phi(g)|,$$

$Q = 4n + 2$. If $0 < p \leq 1$, define the (boundary) Hardy space $H^p(\mathcal{H}^{n-1})$ over the quaternionic Heisenberg group to be

$$H^p(\mathcal{H}^{n-1}) := \{ f \in S'(\mathcal{H}^{n-1}) \mid M_{(N)}f \in L^p(\mathcal{H}^{n-1}) \},$$

where $N_p := [Q(\frac{1}{p} - 1)] + 1$.

If $u$ is a continuous function on $\mathcal{U}$, define the maximal function $u^*$ on $\mathcal{H}^{n-1}$ by

$$(5.1) \quad u^*(g) := \sup_{\|g^{-1}h\| < t} |u(t, g')|.$$

**Proposition 5.1.** [31, Proposition 8.4] Suppose that $0 < p \leq 1$, and $f \in S'(\mathcal{H}^{n-1})$. Then $f \in H^p(\mathcal{H}^{n-1})$ if and only if $u^* \in L^p(\mathcal{H}^{n-1})$, where $u(g) = f * h_t(g)$.

In Proposition 5.1, $h_t f(g)$ is well defined since the heat kernel $h_t \in S(\mathcal{H}^{n-1})$ [31, Proposition 1.74].

**Theorem 5.1.** [31] Suppose $0 < p \leq 1$, then $H^p(\mathcal{H}^{n-1}) = H^p(\mathcal{H}^{n-1})$ and $\|\cdot\|_{H^p(\mathcal{H}^{n-1})} \approx \|\cdot\|_{H^p(\mathcal{H}^{n-1})}$.

**Proposition 5.2.** If $f \in H^p(\mathcal{U})$ for $\frac{2}{3} < p \leq 1$, then $f(\varepsilon, \cdot) \in H^p(\mathcal{H}^{n-1})$ for any $\varepsilon > 0$ and

$$\|f(\varepsilon, \cdot)\|_{H^p(\mathcal{H}^{n-1})} \leq C \|f\|_{H^p(\mathcal{U})}.$$

**Proof.** Note that $f$ is smooth on $\mathcal{U}$ since $\pi^* f$ is harmonic. For fixed $\varepsilon > 0$, $f(\varepsilon + \cdot, \cdot) \in H^p(\mathcal{U})$ by definition and is continuous on $\mathcal{U}$. Apply Proposition 4.4 to it to get

$$|f(t + \varepsilon, g)|^{q} \leq \int_{\mathcal{H}^{n-1}} h_t(g^{-1} \cdot g') |f(\varepsilon, g')|^{q} dg'$$

if we choose $\frac{2}{3} < q < p$. Then, $r = \frac{2}{q} > 1$ and $|f(\varepsilon, \cdot)|^{q} \in L^r(\mathcal{H}^{n-1})$ and

$$\sup_{|g^{-1}h| < t} |f(t + \varepsilon, g)|^{q} \leq \sup_{|g^{-1}h| < t} \int_{\mathcal{H}^{n-1}} h_t(g^{-1} \cdot g') |f(\varepsilon, g')|^{q} dg' \leq C_0 M(\|f(\varepsilon, \cdot)|^{q})(h),$$

where $M$ is the Hardy–Littlewood maximal function on $\mathcal{H}^{n-1}$.

On the other hand, $f(t + \varepsilon, g) = e^{\frac{-1}{(2n-1)^{m}}\Delta^h} f(\varepsilon, g)$ by Proposition 4.3 since we can see that $f(\varepsilon + \cdot, \cdot) \in H^1(\mathcal{U})$ by its boundedness on $\mathcal{U}$ by Lemma 4.1. Hence,

$$\left\| \sup_{|g^{-1}h| < t} e^{\frac{-1}{(2n-1)^m}\Delta^h} f(\varepsilon, g) \right\|_{L^p(\mathcal{H}^{n-1})} \leq C_0 \|M(\|f(\varepsilon, \cdot)|^{q})(h)\|_{L^p(\mathcal{H}^{n-1})} \leq C C_0 \|f(\varepsilon, \cdot)\|_{L^p(\mathcal{H}^{n-1})} \leq C C_0 \|f\|_{H^p(\mathcal{U})}.$$ 

Thus, $f(\varepsilon, \cdot) \in H^p(\mathcal{H}^{n-1})$ by Proposition 5.1 and their $H^p(\mathcal{H}^{n-1})$ quasinorms are uniformly bounded.

The proof of Proposition 5.2 is complete. □
Recall that a quasinorm $\| \cdot \|_A$ of vector space $A$ is called $c$-norm if $\| a + b \|_A \leq c(\| a \|_A + \| b \|_A)$.

**Lemma 5.1** ([13], [Lemma 3.10.1]). For a $c$-norm on vector space $A$, there exists a norm $\| \cdot \|^*_A$ on $A$ such that

\begin{equation}
\| \cdot \|^*_A \leq \| \cdot \|^*_A \leq 2 \| \cdot \|^*_A,
\end{equation}

with $(2c)^p = 2$.

We now turn to the proof of Theorem 1.5.

**Proof of Theorem 1.5.** (1) Suppose $f \in H^p(\mathcal{H})$. We now prove that $f \in H^p_{\alpha}(\mathcal{H})$.

To begin with, for $f \in L^1(\mathcal{H}^{n-1}) \cap L^\infty(\mathcal{H}^{n-1})$ and fixed $(t, g) \in \mathcal{H}$, let

\[ S_{(t,g)}(f) := P(f)(t,g) \]

be defined as the integral of the product of the bounded function $K((t,g), \cdot)$ and $f$. Thus, $S_{(t,g)}$ is a linear functional on $H^p(\mathcal{H}^{n-1}) \cap L^1(\mathcal{H}^{n-1}) \cap L^\infty(\mathcal{H}^{n-1})$, a dense subspace of the quasi-Banach space $H^p(\mathcal{H}^{n-1})$ by [31, Theorem 3.33]. We claim that $S_{(t,g)}$ is continuous on $H^p(\mathcal{H}^{n-1}) \cap L^1(\mathcal{H}^{n-1}) \cap L^\infty(\mathcal{H}^{n-1})$ with respect to the $H^p(\mathcal{H}^{n-1})$ norm. Then it can be naturally extended to a continuous linear functional on $H^p(\mathcal{H}^{n-1})$.

To prove the claim for $f = \sum_{i=1}^\infty f_i a_i \in H^p(\mathcal{H}^{n-1}) \cap L^1(\mathcal{H}^{n-1}) \cap L^\infty(\mathcal{H}^{n-1})$ (for $i_1, i_2, i_3, i_4 = k$), note that scalar functions $f_i \in H^p(\mathcal{H}^{n-1}) \cap L^1(\mathcal{H}^{n-1}) \cap L^\infty(\mathcal{H}^{n-1})$ has the Calderón–Zygmund decomposition ([31, Section B in chapter 3])

\[ f_i = g_i^k + \sum_{j} b_{ij}^k \]

of degree $a$ and height $2^k$, where $b_{ij}^k$ is supported on $B(g_j, 2r_j)$ with

\[ \cup B(g_j, r_j) = \Omega^k \approx \{ g \in \mathcal{H}^{n-1} : M_{\{g\}} f_i (g) > 2^k \}, \quad B(g_j, 2r_j) \cap \Omega^k = \emptyset \]

and no point of $\Omega^k$ lies in more than $L$ of the balls $B(g_j, 2r_j)$.

Note also that by [31, Theorem 3.17], $g_i^k \to f_i$ in $H^p(\mathcal{H}^{n-1})$ as $k \to +\infty$ and by [31, Theorem 3.20] $g_i^k \to 0$ uniformly as $k \to -\infty$. Hence, $f_i$ has the following decomposition [31, Section B in chapter 3]

\begin{equation}
(f_i = \sum_{k=-\infty}^{\infty} (g_i^{k+1} - g_i^k)).
\end{equation}

We point out that the summation (5.3) is only taken over $k \leq N_\alpha$ for some $N_\alpha$ depending on $\| f_i \|_{L^\infty(\mathcal{H}^{n-1})}$, because $g_i^{k+1} - g_i^k = 0$ for $k > N_\alpha$ by $\Omega_\alpha^k = \emptyset$ for such $k$. That is,

\begin{equation}
(f_i = \sum_{k=-\infty}^{N_\alpha} (g_i^{k+1} - g_i^k)).
\end{equation}

Moreover, we have that

\begin{equation}
g_i^{k+1} - g_i^k = \sum_{j} \lambda_j a_{ij}^k
\end{equation}

with $a_{ij}^k$ being a $(p, \infty, \alpha)$-atom supported on $B(g_i, 2r_i)$, where $\lambda_j \in \mathbb{R}$ and

\begin{equation}
\| \lambda_j a_{ij}^k \|_{L^\infty(\mathcal{H}^{n-1})} \leq C_2 2^k, \quad \sum_j \sum_k |\lambda_j a_{ij}^k|^p \leq C_4 \| f_i \|^p_{L^p(\mathcal{H}^{n-1})},
\end{equation}

for some absolute constants $C_2, C_4 > 0$. 


Note that
\[(5.7) \quad \sum_{k=-\infty}^{N_n} \sum_{i} \|\lambda_{\alpha,i}a_{\alpha,i}^k\|_{L^2(\mathcal{H}^n)}^2 \leq \sum_{k=-\infty}^{N_n} \sum_{i} C_2 2^{2k} |B(g_i, T_2r_i)| \leq C_2 L \sum_{k=-\infty}^{N_n} 2^{2k} |\Omega_{\alpha}^k| \]
\[\leq 2N_n(2-p)C_2 L \sum_{k=-\infty}^{N_n} 2^{k(p-1)} |\Omega_{\alpha}^k| 2^{k-1} \]
\[\leq 2N_n(2-p)C_2 L \int_{0}^{+\infty} s^{p-1} \{\{ g \mid M_{(N)} f_\alpha(g) > s \}\mid ds \]
\[= 2N_n(2-p)C_2 L \|M_{(N)} f_\alpha\|_{L^p(\mathcal{H}^n)}^p \]
\[\approx 2N_n(2-p)C_2 L \|f_\alpha\|_{H^p(\mathcal{H}^n)}^p \]
\[< \infty, \]
by \(f_\alpha \in H^p(\mathcal{H}^n) \cap L^1(\mathcal{H}^n) \cap L^\infty(\mathcal{H}^n).\) Thus, we have
\[\sum_{\alpha=1}^{4} \sum_{k=-\infty}^{N_n} \sum_{i} a_{\alpha,i}^k \lambda_{\alpha,i}^k i_\alpha = f \]
in \(L^2(\mathcal{H}^n),\) and by the continuity of \(S_{(t,g)}\) on \(L^2(\mathcal{H}^n),\) we find that
\[S_{(t,g)}(f) = \sum_{\alpha} \sum_{k=-\infty}^{N_n} \sum_{i} S_{(t,g)}(a_{\alpha,i}^k) \lambda_{\alpha,i}^k i_\alpha \]
for \(f \in H^p(\mathcal{H}^n) \cap L^1(\mathcal{H}^n) \cap L^\infty(\mathcal{H}^n).\) Then we can apply this identity to get
\[|S_{(t,g)}(f)| \leq C \sum_{\alpha} \sum_{k=-\infty}^{N_n} \sum_{i} |P(a_{\alpha,i}^k)(t, g) \lambda_{\alpha,i}^k| \leq C \sum_{\alpha} \sum_{k=-\infty}^{N_n} \sum_{i} \|P(a_{\alpha,i}^k)\|_{H^p(\mathcal{H}^n)} t^{-\frac{2n+1}{p}} |\lambda_{\alpha,i}^k| \]
\[\leq CC_{p,n,\alpha} t^{-\frac{2n+1}{p}} \left( \sum_{\alpha} \sum_{k=-\infty}^{N_n} \sum_{i} |\lambda_{\alpha,i}^k|^p \right)^{\frac{1}{2}} \]
\[\approx CC_{p,n,\alpha} t^{-\frac{2n+1}{p}} \|f\|_{H^p(\mathcal{H}^n)},\]
by using Lemma 4.1, Proposition 1.1 and (5.6). Thus \(S_{(t,g)}\) is bounded on \(H^p(\mathcal{H}^n) \cap L^1(\mathcal{H}^n) \cap L^\infty(\mathcal{H}^n)\) with respect to the \(H^p(\mathcal{H}^n)\) norm.

We now apply Lemma 5.1 to the quasi-Banach space \(H^p(\mathcal{H}^n)\) to get a norm \(\| \cdot \|_{H^p(\mathcal{H}^n)}\). Then by (5.2), it is obvious that \((H^p(\mathcal{H}^n), \| \cdot \|_{H^p(\mathcal{H}^n)})\) is complete. Namely, \((H^p(\mathcal{H}^n), \| \cdot \|_{H^p(\mathcal{H}^n)})\) is a Banach space.

By Proposition 5.2, we see that \(\{f(\varepsilon, \cdot)\}_{\varepsilon > 0}\) is a bounded set in the quasi-Banach space \(H^p(\mathcal{H}^n)\) if \(f \in H^p(\mathcal{U})\) for \(\frac{4}{3} < p \leq 1\). It is also bounded in \(H^p(\mathcal{H}^n)\) under the norm \(\| \cdot \|_{H^p(\mathcal{H}^n)}\). It follows from Banach–Alaoglu theorem that there exists a subsequence \(\{f(\varepsilon_k, \cdot)\}\) weakly convergent to some \(f^b\) in \((H^p(\mathcal{H}^n), \| \cdot \|_{H^p(\mathcal{H}^n)})\). Then by (5.2) we can obtain that \(\{f(\varepsilon_k, \cdot)\}\) is also weakly convergent to the same \(f^b\) in the quasi-Banach space \(H^p(\mathcal{H}^n)\), since every continuous linear functional on the quasi-Banach space \(H^p(\mathcal{H}^n)\) must be continuous under the norm \(\| \cdot \|_{H^p(\mathcal{H}^n)}\) by (5.2).

Note that \(f(\varepsilon_k + t, g)\) is uniformly bounded on \(\mathcal{U}\) by Lemma 4.1, and so it is in \(H^2(\mathcal{U})\). Thus we have \((Pf(\varepsilon_k, \cdot))(t, g) = f(\varepsilon_k + t, g)\) by the reproducing formula. Now apply \(S_{(t,g)}\) to \(\{f(\varepsilon_k, \cdot)\}\) to get
\[S_{(t,g)}(f^b) = \lim_{k \to \infty} S_{(t,g)}f(\varepsilon_k, \cdot) = \lim_{k \to \infty} Pf(\varepsilon_k, \cdot)(t, g) = \lim_{k \to \infty} f(\varepsilon_k + t, g) = f(t, g).\]
On the other hand, since \( f^b = \sum_k a_k \lambda_k \) by Theorem 5.1, and
\[
S_{(t,g)}(f^b) = \sum_k P(a_k)(t,g)\lambda_k,
\]
by continuity of \( S_{(t,g)} \) on \( H^p(S^{n-1}) \). Consequently, \( f(t,g) = \sum_k P(a_k)(t,g)\lambda_k \) for each point \((t,g) \in \mathcal{U}\), i.e., \( f \) is in \( H^p(S^{n-1}) \).

(2) Suppose \( u \in H^p_{\text{tr}}(\mathcal{U}) \). We now prove that \( u \in H^p(\mathcal{U}) \).

Let \( u = \sum_{j=1}^{\infty} A_j \lambda_j \in H^p_{\text{tr}}(\mathcal{U}) \) such that \( \sum_{j=1}^{\infty} |\lambda_j|^p \approx \| u \|_{H^p_{\text{tr}}(\mathcal{U})} \), where \( A_j \)'s are regular \( p \)-atoms; i.e. there exist \((p, \infty, \alpha)\)-atoms \( a_j \) on \( S^{n-1} \) such that \( A_j = P(a_j) \). By Proposition 1.1, we see that \( A_j \in H^p(\mathcal{U}) \) with \( \| A_j \|_{H^p(\mathcal{U})} \leq C_{p,n,\alpha} \) for all \( j \). Apply Proposition 4.1 to \( A_j \) to see that \( |A_j(\varepsilon, g)| \leq CC_{p,n,\alpha} \frac{\varepsilon^{2n+1}}{r^p} \), and so \( \sum_{j=1}^{\infty} A_j \lambda_j \) converges uniformly on compact subset of \( \mathcal{U} \) and defines a regular function on \( \mathcal{U} \) and
\[
\left| \sum_{j=1}^{\infty} A_j(\varepsilon, g)\lambda_j \right| \leq CC_{p,n,\alpha} \frac{\varepsilon^{2n+1}}{r^p} \sum_{j=1}^{\infty} |\lambda_j|.
\]
Moreover,
\[
\int_{S^{n-1}} \left| \sum_{j=1}^{\infty} A_j(\varepsilon, g)\lambda_j \right|^p \, dg \leq \int_{S^{n-1}} \sum_{j=1}^{\infty} |\lambda_j|^p |A_j(\varepsilon, g)|^p \, dg \leq \sum_{j=1}^{\infty} |\lambda_j|^p \int_{S^{n-1}} |A_j(\varepsilon, g)|^p \, dg
\]
\[
\leq C_{p,n,\alpha} \sum_{j=1}^{\infty} |\lambda_j|^p
\]
\[
\approx C_{p,n,\alpha} \| u \|_{H^p_{\text{tr}}(\mathcal{U})}.
\]
Namely, \( u = \sum_{j=1}^{\infty} A_j \lambda_j \in H^p(\mathcal{U}) \) with \( \| u \|_{H^p(\mathcal{U})} \leq C_p \| u \|_{H^p_{\text{tr}}(\mathcal{U})} \).

The proof of Theorem 1.5 is complete. \( \square \)

6. SINGULAR VALUE ESTIMATES OF THE COMMUTATOR \([b, P]\): PROOF OF THEOREM 1.6

Based on Theorems 1.1 and 1.3, and the recent result in [28], we see that (1) in Theorem 1.6 holds.

We now prove (2) in Theorem 1.6. The sufficient condition is obvious, since \([b, P]\) is 0 when \( b \) is a constant. We now prove necessary condition. To show this, it suffices to consider the critical case \( p = 4n + 2 \), by the inclusion \( S^p \subset S^{2n+2} \) for \( p < 2n + 2 \).

**Lemma 6.1.** There exists a positive integer \( b \) such that for any tile \( T \in \mathcal{T}_k \) and \( a_j = \pm 1 \) \( (j = 1, 2, \ldots, 4n - 4) \), there are tiles \( T' \in \mathcal{T}_{k-b} \), \( T'' \in \mathcal{T}_{k-b} \) such that \( T' \subset T, T'' \subset T \) and if \( g = (g_1, \ldots, g_{4n-4}, t) \in T'' \),
\[
h = (h_1, \ldots, h_{4n-4}, t') \in T', \text{ then } a_j(g_j - h_j) \geq \text{width}(T) \] \( (j = 1, 2, \ldots, 4n - 4) \).

**Proof.** Consider first \( T = \delta^2(A) \). Based on (4) in Lemma 3.3, there exist \( o = (x, t) \in \mathcal{H}^{n-1} \) such that \( B(o, C_1 2^k) \subset T \). Then one can choose \( g_{o,1} = (\hat{x}, \hat{t}) \in B(o, C_1 2^k) \) such that \( d(g_{o,1}, o) = \frac{3C_1}{4} 2^k \), and
\[
\hat{x}_j - x_j = \frac{3C_1}{4\sqrt{4n - 4}} 2^k, \quad j = 1, \ldots, 4n - 4.
\]
Thus, we can choose \( C'_1 \) sufficiently small such that for each \( g = (x', t') \in B(g_{o,1}, C'_1 2^k) \), we have
\[
x'_j - x_j > \frac{3C_1}{8\sqrt{4n - 4}} 2^k, \quad j = 1, \ldots, 4n - 4.
\]
Take $g_{o,2} = o(o^{-1}g_{o,1})^{-1}$ and write $g_{o,2} = (\hat{x}, \hat{t})$, then $d(g_{o,2}, o) = \frac{3C_1}{4} 2^k$ and
\[ \hat{x}_j - x_j = -\frac{3C_1}{4\sqrt{4n-4}} 2^k, \quad j = 1, \cdots, 4n - 4. \]

For any $\hat{y} = (y, s) \in B(g_{o,1}, C_1^2 2^k)$, we have
\[ y_j - x_j < -\frac{3C_1}{8\sqrt{4n-4}} 2^k, \quad j = 1, \cdots, 4n - 4. \]

As a consequence, if we choose $b$ sufficiently large such that $2^k - \text{width}(A) < C_1^2 2^k$, then there must exist $T' \in \mathcal{T}_{k-b}$ such that $T' \subset B(g_{o,1}, C_1^2 2^k)$ and $T'' \in \mathcal{T}_{k-b}$ such that $T'' \subset B(g_{o,2}, C_1^2 2^k)$. Then it is clear that if $g \in T'$, $h \in T''$, then $g_j - h_j \geq \text{width}(T) \ (j = 1, 2, \cdots, 4n - 4)$. The proof of other cases are similar. This ends the proof of Lemma 6.1.

Recall the following first order Taylor’s inequality on $\mathcal{H}^{n-1}$ from [14].

**Lemma 6.2.** Let $f \in C^\infty(\mathcal{H}^{n-1})$, then for every $g = (x_1, \cdots, x_{4n-4}, t), g_0 = (x_0^1, \cdots, x_0^{4n-4}, t_0) \in \mathcal{H}^{n-1}$, we have
\[ f(g) = f(g_0) + \sum_{k=1}^{4n-4} \frac{Y_k f(g_0)}{k!} (x_k - x_0^k) + R(g, g_0), \]
where the remainder $R(g, g_0)$ satisfies the following inequality:
\[ |R(g, g_0)| \leq C \left( \sum_{k=1}^{2} \frac{c_k^k}{k!} \sum_{i_1, \cdots, i_k \leq 4n-1} \|g_0^{-1} \cdot g\|^{d(I)} \sup_{\|h\| \leq c \|g_0^{-1} \cdot g\|} |Y^I f(g_0z)| \right) \]
for some constant $c > 0$.

In the sequel, for any $T \in \mathcal{T}_k$, let $T'$ be the tile chosen in Lemma 6.1. Also, we denote $\nabla_H$ be the horizontal gradient of $\mathcal{H}^{n-1}$ defined by $\nabla_H f := (Y_1 f, \cdots, Y_{4n-4} f)$. Then we can show a lower bound for a local pseudo-oscillation of the symbol $b$ in the commutator.

**Lemma 6.3.** Let $b \in C^\infty(\mathcal{H}^{n-1})$. Assume that there is a point $g_0 \in \mathcal{H}^{n-1}$ such that $\nabla b(g_0) \neq 0$. Then there exist $C > 0$, $\varepsilon > 0$ and $N > 0$ such that if $k > N$, then for any tile $T \in \mathcal{T}_k$ satisfying $d(\text{cent}(T), g_0) \varepsilon$, one has
\[ \frac{1}{|T|} \int_T |b(g) - (b)_{T'}| \, dg \geq C \text{width}(T) \| \nabla_H b(g_0) \| . \]

**Proof.** Denote $c_T := \text{cent}(T) := \{c_T^1, \cdots, c_T^{4n-4}, t_T\}$ and $g = (g_1, \cdots, g_{4n-4}, t)$, then by Lemma 6.2,
\[ b(g) = b(c_T) + \sum_{j=1}^{4n-4} \frac{Y_j b(c_T)}{j!} (g_j - c_T^j) + R(g, c_T), \]
where the remainder term $R(g, c_T)$ satisfies
\[ |R(g, c_T)| \leq C \left( \sum_{j=1}^{2} \frac{c_j^j}{j!} \sum_{i_1, \cdots, i_j \leq 4n-1} \|c_T^{-1} \cdot g\|^{d(I)} \sup_{\|h\| \leq c \|c_T^{-1} \cdot g\|} |Y^I b(c_T \cdot h)| \right) . \]
Note that the condition \( \|h\| \leq c|c_T^{-1} \cdot g| \) implies that \( d(c_T \cdot h, c_T) = \|h\| \leq c|c_T^{-1} \cdot g| \lesssim \text{width}(T) \) whenever \( g \in T \). Hence, if \( g \in T \), then
\[
|R(g, c_T)| \leq C \text{width}(T)^2 \sum_{j=1}^{2} \sum_{t(i_1), i_2 \leq 4n-1, \text{l} = (i_1, \ldots, i_2), \text{d}(l) \geq 2} \|Y^T b\|_{L^\infty(B(g_0, 1))},
\]
Besides, it follows from Lemma 6.1 that there exist cubes \( T' \in \mathfrak{S}_{k-6}, T'' \in \mathfrak{S}_{k-1} \) such that \( T' \subset T, T'' \subset T \) and \( \text{sgn}(Y_jb(c_T)(g_j - \tilde{g}_j) \geq \text{width}(T) \) \( (j = 1, 2, \ldots, 4n - 4) \). Therefore,
\[
\frac{1}{|T|} \int_T |b(g) - (b)_{T'}| dg \\
\geq \frac{1}{|T||T'|} \int_T \left( \sum_{j=1}^{4n-4} \frac{Y_jb(c_T)}{j!} (g_j - \tilde{g}_j) dh \right) \int_T |R(g, c_T)| dg - \frac{1}{|T|} \int_T |R(g, c_T)| dg \\
\geq \sum_{j=1}^{4n-4} \frac{Y_jb(c_T)}{j!} \text{width}(T) - C \text{width}(T)^2 \sum_{j=1}^{2} \sum_{t(i_1), i_2 \leq 4n-1, \text{l} = (i_1, \ldots, i_2), \text{d}(l) \geq 2} \|Y^T b\|_{L^\infty(B(g_0, 1))}
\]
\[
\geq C \text{width}(T) |\nabla_H b(g_0)|,
\]
where the last inequality holds since we choose \( N \) to be a sufficient large constant such that the remainder term can be absorbed by the first term. \( \square \)

Denote by \( t^h \) the right translation by \( h \).

**Lemma 6.4.** Let \( b \in L^1_{\text{loc}}(\mathcal{H}^{n-1}) \). Suppose that
\[
\sup_{h \in B(0, 1)} \left\| \left( \frac{1}{|T^h|} \int_{T^h T} |b(g) - (b)_{T^h T'}| dg \right)_{T \in \mathfrak{T}} \right\|_{\ell^{n+2}} < +\infty,
\]
then \( b \) is a constant.

**Proof.** Denote \( \psi_\epsilon(g) := \epsilon^{-4n-2} \psi(\delta_{\epsilon-1} g), \) where \( \psi \) is a smooth bump function and \( \epsilon \) is a small positive constant. Note that
\[
\left\| \left( \frac{1}{|T|} \int_T |b \ast \psi_\epsilon(g) - (b \ast \psi_\epsilon)_{T'}| dg \right)_{T \in \mathfrak{T}} \right\|_{\ell^{n+2}} \leq \left\| \left( \frac{1}{|T^h|} \int_{T^h T} |b(g) - (b)_{T^h T'}| dg dh \right)_{T \in \mathfrak{T}} \right\|_{\ell^{n+2}} \leq \sup_{h \in B(0, 1)} \left( \left\| \left( \frac{1}{|T^h|} \int_{T^h T} |b(g) - (b)_{T^h T'}| dg \right)_{T \in \mathfrak{T}} \right\|_{\ell^{n+2}} \right) < +\infty.
\]

Next we will show that for any \( \epsilon > 0, b \ast \psi_\epsilon \) is a constant, which implies that \( b \) is a constant by letting \( \epsilon \to 0 \). If not, then it follows from [15, Proposition 1.5.6] that there exists a point \( g_0 \in \mathcal{H}^{n-1} \) such that \( \nabla_H b \ast \psi_\epsilon(g_0) \neq 0 \). By Lemma 6.3, there exist \( \epsilon > 0 \) and \( N > 0 \) such that if \( k > N \), then for any tile \( T \in \mathfrak{S}_k \) satisfying \( d(\text{cent}(T), g_0) < \epsilon \),
\[
\frac{1}{|T|} \int_{T} |b \ast \psi_\epsilon(g) - (b \ast \psi_\epsilon)_{T'}| dg \geq C \text{width}(T) |\nabla b \ast \psi_\epsilon|.
\]
Note that for \( k > N \), the number of \( T \in \mathfrak{S}_k \) and \( d(\text{cent}(T), g_0) < \epsilon \) is at least \( c2^k(4n+2) \). Therefore,
\[
\left\| \left( \frac{1}{|T|} \int_{T} |b \ast \psi_\epsilon(g) - (b \ast \psi_\epsilon)_{T'}| dg \right)_{T \in \mathfrak{T}} \right\|_{\ell^{n+2}} \geq C \text{width}(T) |\nabla b \ast \psi_\epsilon|.
\]
\[ \begin{aligned}
&\geq \left( \sum_{k=N+1}^{\infty} \sum_{T \in \mathcal{T}_k : d(\text{cent}(T), g_0) < \varepsilon} \text{width}(T)^{4n+2} |\nabla b * \psi_\varepsilon(g_0)|^{4n+2} \right)^{\frac{1}{4n+2}} \\
&\geq |\nabla H b * \psi_\varepsilon(g_0)| \left( \sum_{k=N+1}^{\infty} \sum_{T \in \mathcal{T}_k : d(\text{cent}(T), g_0) < \varepsilon} 2^{k(4n+2)} \right)^{\frac{1}{4n+2}} \\
&= +\infty.
\end{aligned} \]

This is in contradiction with inequality (6.1). Therefore, the proof of Lemma 6.4 is complete. \(\square\)

Based on our fundamental results in Lemmas 6.1—6.4, (2) holds by using the argument in [28]. Hence, the proof of Theorem 1.6 is complete. \(\square\)

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References

[1] Adams, W., Berenstein, C., Loustaunau, P., Sabadini, I. and Struppa, D., Regular functions of several quaternionic variables and the Cauchy–Fueter complex, J. Geom. Anal., 9 (1999) 1–15.
[2] Ahrens, J., Cowling, M., Martini, A. and Müller, D., Quaternionic spherical harmonics and a sharp multiplier theorem on quaternionic spheres, Math. Z., 294 (2020), 1659–1686.
[3] Alesker, S., Non-commutative linear algebra and plurisubharmonic functions of quaternionic variables, Bull. Sci. Math., 127(1) (2003), 1–35.
[4] Alesker, S., Quaternionic Monge-Ampère equations, J. Geom. Anal., 13(2) (2003), 205–238.
[5] Alesker, S., Valuations on convex sets, non-commutative determinants, and pluripotential theory, Adv. Math., 195(2) (2005), 561–595.
[6] Alesker, S. and E. Shelukhin, A uniform estimate for general quaternionic Calabi problem (with an appendix by Daniel Barlet), Adv. Math., 316 (2017), 1–52.
[7] Kolodziej, S. and Sroka, M., Regularity of solutions to the quaternionic Monge-Ampère equation, J. Geom. Anal., 30 (2020), 2852–2864.
[8] Alesker, S. and Verbitsky, M., Plurisubharmonic functions on hypercomplex manifolds and HKT-geometry, J. Geom. Anal., 16 (2006), 375–399.
[9] Alpay, D., Colombo, F. and Sabadini, I., Quaternionic de Branges spaces and characteristic operator function, Springer Briefs in Mathematics, Springer, Cham. (2020).
[10] Alpay, D., Colombo, F., Qian, T. and Sabadini, I., The $H^\infty$ functional calculus based on the $S$-spectrum for quaternionic operators and for $n$-tuples of noncommuting operators, J. Funct. Anal., 271(6) (2016), 1544–1584.
[11] Alpay, D., Colombo, F. and Sabadini, I., On slice hyperholomorphic fractional Hardy spaces, Math. Nachr., 290(17-18) (2017), 2725–2739.
[12] Baston, R., Quaternionic complexes, J. Geom. Phys., 8 (1992), 29–52.
[13] Bergli, J. and Lofstrom, J., Interpolation spaces: an introduction, Grundlehren der mathematischen Wissenschaften 223, 1976.
[14] Bonfiglioli, A., Taylor formula for homogeneous groups and applications, Math. Z., 262 (2009), 255–279.
[15] Bonfiglioli A., Lanconelli E. and Uguzzoni F., Stratified Lie Groups and Potential Theory for Their Sub-Laplacians, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2007.
[16] Bures, J. and V. Soucek, V., Complexes of invariant differential operators in several quaternionic variables, Complex Var. Elliptic Equ., 51(5-6) (2006), 463-487.
[17] Bureš, J., Damiano, A. and Sabadini, I., Explicit resolutions for several Fueter operators, J. Geom. Phys., 57, (2007), 765–775.
[18] Calderón, A.-P. and Zygmund, A., On higher gradients of harmonic functions, *Studia Math.*, 24 (1964), 211–226.

[19] Chang, D.-C., Duong, X. T., Li, J., Wang, W. and Wu, Q. Y., An explicit formula of Cauchy–Szegő kernel for quaternionic Siegel upper half space and applications, to appear in *Indiana Univ. Math. J.*, 2, 3, 5, 10, 16.

[20] Chang, D.-C., Markina, I. and Wang, W., On the Cauchy–Szegő kernel for quaternion Siegel upper half-space, *Complex Anal. Oper. Theory*, 7(5) (2013), 1623–1654.

[21] Colombo, F., Souček, V. and Struppa, D., Invariant resolutions for several Fueter operators, *J. Geom. Phys.*, 56(7) (2006), 1175–1191.

[22] Colombo, F., Gentili, G., Sabadini, I. and Struppa, D., Extension results for slice regular functions of a quaternionic variable, *Adv. Math.*, 222 (2009), 1793–1808.

[23] Colombo, F. and Gantner, J., Quaternionic closed operators, fractional powers and fractional diffusion processes, *Operator Theory: Advances and Applications*, 274. Birkhäuser, Springer, Cham, 2019.

[24] Connes, A., Noncommutative Geometry, Academic Press, Inc., San Diego, CA, 1994.

[25] Cygan, J., Subadditivity of homogeneous norms on certain nilpotent Lie groups, *Proc. Am. Math. Soc.*, 83 (1981), 69–70.

[26] Dafni, G., Hardy spaces on some pseudoconvex domains, *J. Geom. Anal.*, 4 (1994), 273–316.

[27] Dou, X., Ren, G., Sabadini, I. and Yang, T., Weak slice regular functions on the n-dimensional quadratic cone of octonions, *J. Geom. Anal.*, 31 (2021), 11312–11337.

[28] Fan, Z., Lacey, M. and Li, J., Schatten classes and commutators of Riesz transform on Heisenberg group and applications, arXiv:2107.10569.

[29] Fefferman, C., The Bergman kernel and biholomorphic mappings of pseudoconvex domains, *Invent. Math.*, 26 (1974), 1–65.

[30] Fefferman, C. and Stein, E.M., The $H^p$ spaces of several variables, *Acta Math.*, 129 (1972), 137–193.

[31] Folland, G.B. and Stein, E.M., *Hardy Spaces on Homogeneous Groups*, Mathematical Notes, vol. 28, Princeton University Press, Princeton, NJ, 1982.

[32] Feldman, M. and Rochberg, R., Singular value estimates for commutators and Hankel operators on the unit ball and the Heisenberg group. Analysis and partial differential equations, 121–159, Lecture Notes in Pure and Appl. Math., 122, Dekker, New York, 1990.

[33] Garnett, J. and Latter, R., The atomic decomposition for Hardy spaces in several complex variables, *Duke J. Math.*, 45 (1978), 815–845.

[34] Gentili, G. and Struppa, D., A new theory of regular functions of a quaternionic variable, *Adv. Math.*, 216(1) (2007), 279–301.

[35] Geller, D., Some results in $H^p$ theory for the Heisenberg group, *Duke Math. J.*, 47 (1980), 365–390.

[36] Graham, C.R., The Dirichlet problem for the Bergman Laplacian I, *Comm. in P. D. E.*, 8 (1983), 305–317.

[37] Janson, S. and Wolff, T., Schatten classes and commutators of singular integral operators, *Ark. Mat.*, 20 (1982), 301–310.

[38] Journé, J.-L., Calderón–Zygmund operators on product space, *Rev. Mat. Iberoam.*, 1 (1985), 55–92.

[39] Krantz, S.G. and Li, S.-Y., On decomposition theorems for Hardy spaces on domains in $C^n$ and applications, *J. Fourier Anal. Appl.*, 2 (1995), 65–107.

[40] Koranyi, A., A Poisson integral for homogeneous wedge domains, *J. Anal. Math.*, 14 (1965), 275–284.

[41] Lanzani, L. and Stein, E., The Cauchy–Szegő projection for domains in $C^n$ with minimal smoothness, *Duke Math. J.*, 166 (2017), 125–176.

[42] Lord, S., McDonald, E., Sukochev, F. and Zanin, D., Quantum differentiability of essentially bounded functions on Euclidean space, *J. Funct. Anal.*, 273(7) (2017), 2353–2387.

[43] Nagel, A., Rosay, J.-P., Stein, E. M. and Wainger, S., Estimates for the Bergman and Szegő kernels in $C^2$, *Ann. of Math. (2)*, 129 (1989), 113–149.

[44] Peller, V.V., Nuclearity of Hankel operators, *Mat. Sbornik*, 113 (1980), 538–581.

[45] Peller, V.V., Hankel operators and their applications, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[46] Qian, T., Singular integrals on star-shaped Lipschitz surfaces in the quaternionic space, *Math. Ann.*, 310 (1998), 601–630.

[47] Rochberg, R. and Semmes, S., Nearly weakly orthonormal sequences, singular value estimates, and Calderón-Zygmund operators, *J. Funct. Anal.*, 86 (1989), 237–306.

[48] Rochberg, R. and Semmes, S., A decomposition theorem for BMO and applications, *J. Funct. Anal.*, 67 (1986), 228–263.

[49] Sarfatti, G., Quaternionic Hankel operators and approximation by slice regular functions, *Indiana Univ. Math. J.*, 65(5) (2016), 1735–1757.
[50] Shi, Y. and Wang, W., The tangential $k$-Cauchy–Fueter complexes and Hartogs’ phenomenon over the right quaternionic Heisenberg group, *Ann. Mat. Pura Appl.*, **199** (2020), 651–680.

[51] Stein E. M., Singular integrals and differentiability properties of functions, Princeton Mathematical Series **30**, Princeton University Press, Princeton, NJ, 1970.

[52] Stein E. M., Boundary behavior of holomorphic functions of several complex variables, Mathematical Notes, Princeton University Press (1972).

[53] Stein E. M. and Weiss, G., Generalization of the Cauchy–Riemann equations and representations of the rotation group, *Amer. J. Math.*, **90** (1968), 163–196.

[54] Strichartz, R. S., Self-similarity on nilpotent Lie groups. Pages 123–157 in: Generalized convex bodies and generalized envelopes. Contemp. Math. 140. American Mathematical Society, Providence, 1992.

[55] Varopoulos, N., Saloff-Coste, L. and Coulhon, T., Analysis and geometry on groups, Cambridge Tracts in Math. **100**, Cambridge University Press, Cambridge, 1992.

[56] Wan, D. and Wang, W., On quaternionic Monge–Ampère operator, closed positive currents and Lelong–Jensen type formula on the quaternionic space, *Bull. Sci. Math.*, **141**(4) (2017), 267–311.

[57] Wang, W., The $k$-Cauchy–Fueter complex, Penrose transformation and Hartogs’ phenomenon for quaternionic $k$-regular functions, *J. Geom. Phys.*, **60** (2010), 513–530.

[58] Wang, W., The linear algebra in the quaternionic pluripotential theory, *Linear Alg. Appl.*, **562** (1) (2019), 223–241.

[59] Wang, W., The Neumann problem for the $k$-Cauchy–Fueter complex over $k$-pseudoconvex domains in $\mathbb{R}^4$ and the $L^2$ estimate, *J. Geom. Anal.*, **29** (2019), 1233–1258.

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