The shift $\Delta E$ of a maxon-roton $^4$He excitation spectrum $E$ via the $^3$He admixture is considered theoretically. The Bogoliubov formula for $E(\rho, g)$ as a function of Bose-condensate density $\rho$ and of $^4$He/$^4$He interaction $g$ is used as an element of partition function formalism of the whole system, presented in the form of a path (functional) integral. The dependence of $\rho$ on $g$, $^4$He/$^3$He interaction $\lambda$ as well as on the pressure $P$, induced by the $^3$He admixture, is taken into account. The new model of $g$, suitable for analytical calculations, is used for a description of a change of $P$ within the constant volume. The cases of positive and negative shifts are found as the result of competition between interactions $g$ and $\lambda$.

Keywords: Bose-Einstein condensation, $^4$He $\leftrightarrow ^3$He energy transfer, pressure, spectrum of non-condensate excitations.

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1 Introduction

This is a continuation of the paper [1] over the same title. The main idea of these papers is to look for the theoretical key to understand the $^4\text{He}$ maxon-roton excitation curve shift, caused by $^3\text{He}$ admixture. The sections 1-4 of [1] are shortly repeated here in the sections 2-3. The phenomenological interpretation of section 5 in [1] is replaced by a new theoretical investigation, represented in the sections 4,5,6 of the present paper. This investigation classifies the types of dispersion shifts as a result of competition between the $^4\text{He}/^3\text{He}$ and $^4\text{He}/^4\text{He}$ interactions, accompanying by effects of pressure, induced by the $^3\text{He}$ admixture in the $V = \text{const}$ volume.

The motivation for our theory, in general, was provided by experiments [2,3] on this subject. We tried to take into account some of experimental preconditions, but hope to accomplish the accordance with them later, after the more detailed discussion the problem with specialists.

2 Inelastic and elastic scattering

The Hamiltonian of the whole system $H$ is written as a sum of three terms: $H_B$, $h$ and $V_1$. $H_B$ is the Bogoliubov Hamiltonian for a superfluid $^4\text{He}$

$$H_B = g_0 \frac{|a|^2}{2V} + \sum_{k \neq 0} \left[ \Omega_k b_k^+ b_k + \frac{g_k}{2V} (b_k^+ b_k) a^2 + b_k b_{-k} (a^*)^2 + 2b_k^+ b_k |a|^2 \right] + \frac{g_0}{V} b_k^+ b_k |a|^2 , \quad (1)$$

where $a$, $a^*$—classic amplitudes of condensate bosons, and $b_k, b_k^+$—noncondensate operators with $k \neq 0$, $[b_k, b_{k'}^+] = \delta_{kk'}$, $\Omega_k = k^2/2m_b$, $m_b$—is the boson mass.

Then, $h$ is the Hamiltonian of $^3\text{He}$ atoms $h = \sum \varepsilon_k c_k^+ c_k$ with energy $\varepsilon_k = k^2/2m$. The $V_1$ term describes the interaction between bosons and fermions and is represented as a sum of four parts, describing the elastic scattering of $^3\text{He}$ on $^4\text{He}$ condensate (I), elastic scattering on the noncondensate bosons (II), inelastic excitation of condensate (III) and inelastic scattering on the noncondensate bosons (IV):

$$V_1 = \frac{1}{V} \sum_{q_1+q_2=q_3+q_4} \lambda c_{q_1}^+ c_{q_1} b_{q_2}^+ b_{q_4} = I + II + III + IV =$$

$$= \sum_q c_q^+ c_q |a|^2 \frac{\lambda(q)}{V} + \sum_q c_q^+ c_q \sum_{k \neq 0} \frac{b_k^+ b_k \lambda(k - q)}{V} +$$

elastic scattering of $^3\text{He}$ on condensate

elastic scattering on noncondensate bosons
\[ + \sum_{k,q} c^+_k c_{q-k} a^* + b^+_k a \frac{\lambda(k-q)}{V} \]  
\[ + \sum_{k,q} c^+_q c_k b^+_k b_{k+q-k} \frac{\lambda(k-q)}{V}. \]  
\(\text{inelastic excitation of condensate}\)
\(\text{inelastic scattering of } ^3\text{He on the noncondensate bosons}\)

Here \(\lambda(k,q)\) is the interaction between bosons and \(^3\text{He} \) atoms. It is clear that the first term I of the right-hand side of the formula \(2\) gives the small contribution to the boson excitation spectrum. The contribution II of the elastic scattering due to the noncondensate bosons is received as the limit of the fourth term of \(2\) with a small transfer momentum \(p = k - q\). The fourth term with a big \(p\) presents a change of the noncondensate state and we don’t consider it. Only the third term III is analyzed here, and the corresponding part of \(h\) and \(V_1\) may be written in a form:

\[
H_h = \sum_{k,q} \left( \frac{\varepsilon_k}{2} - \mu_f/2 \right) \left( \frac{\lambda}{2V} \gamma_{k-q} - \varepsilon_q / 2 - \mu_f/2, \right) (c_k/c_q) = \]
\[= \sum_q (c^*_k c_q \phi_{kq}) (c_k/c_q), \quad \gamma_{k-q} = b_{k-q} a^* + b^*_{q-k} a.\]

The partition function of the system \(H\) is taken in a form of a path-integral over trajectories of bosons and fermions with periodic and antiperiodic boundary conditions respectively

\[Q = \text{Sp} e^{-\beta(H - \mu_b n - \mu_f n_f)}, \quad H = H_B + H_h,\]  
\[Q = \frac{1}{\sqrt{2\pi}} \int d\rho \prod_{k \neq 0} \int Db_k^* Db_k \exp \{S_B + \mu_b n\}.\]
\[\cdot \prod_{k,q} \int Dc^*_k Dc_q \exp \{S_{H_h} + \mu_f n_f\}, \quad \rho = |a|^2 / V, \quad n = |a|^2 + \sum_{k \neq 0} b^*_k b_k.\]

Here \(n\) is a "quasiclassical" (see Appendix) integral of motion for bosons, \(n_f\) – the conserved number of fermions, \(\rho\) is the density of the Bose-condensate; \(S_B\) and \(S_{H_h}\) are the actions for Hamiltonians \(H_B\) and \(H_h\) respectively. So, the most attention in \(3\) is paid to the deep inelastic interaction term III, as far as it is responsible for condensate-noncondensate transitions of \(^4\text{He} \) atoms, and, therefore, for the excitation spectrum.

### 3 \(^3\text{He} \leftrightarrow ^4\text{He} \) energy transfer

After the integration over fermions the partition function is represented as an integral over the boson condensate and non-condensate variables. The term \(H_h\) gives a contribution to two processes. In one of them atoms of \(^3\text{He} \) with initial energy \(\varepsilon_k/2\) interact with Bose-condensate and escape with the momentum \(q\), and equivalent process goes with respect to the reversing of time, so the transfer momentum \(p = k - q\) appears:
Eigenvalues $f_1, f_2$ of matrix $H_h$ are found from the following equation:

$$\det (\phi_{kq} - f) = 0,$$

$$f_{1,2} = \frac{\varepsilon_q + \varepsilon_k - 2\mu_f}{4} \pm \sqrt{\left(\frac{\varepsilon_k - \varepsilon_q}{4}\right)^2 + \frac{\lambda^2}{4V^2} \gamma_{k-q}\gamma_{q-k}}. \quad (4)$$

Therefore the partition function looks like

$$Q = \int D\mu_B e^{S_B} \prod_{k,q} \left[ 1 + e^{-\beta(f_1 + f_2)} + \exp \left( \int_0^\beta f_1 dt \right) + \exp \left( \int_0^\beta f_2 dt \right) \right] \quad (5)$$

an integral over condensate and noncondensate variables. The integral over momentums of fermions in formula for $^3\text{He}$ spectrum is calculated in assumption of ideal-gas Fermi distribution function of pure $^3\text{He}$ atoms.

The "quasiclassical" momentum quantisation is used for the transfer momentum $p$ separation:

$$-\beta \sum_k f_{1,2} \to -\beta f_0 \pm \frac{\beta Y}{8m}, \quad f_0 = c \int_0^{k_f} (\varepsilon_q + \varepsilon_k - 2\mu_f) k^2 dk, \quad c = V(\pi^2\hbar^3)^{-1},$$

$$Y = cV \int_0^{k_f} k^2 \sqrt{X} dk = Y_0 \frac{\lambda^2}{V^2} m^2 \gamma_{-p}\gamma_p, \quad Y_0 = \left[ 2k_f \left( \frac{k_f}{p} - 1 \right) + p \ln \left( 1 + \frac{2k_f}{p} \right) \right],$$

$$X = p^4 + 4kp^3 + 4k^2p^2 + 16 \frac{\lambda^2}{V^2} m^2 \gamma_{-p}\gamma_p.$$ 

For the discussed experimental situation the estimation $k_f \simeq 0.3A^{-1}$ and a maxon-roton interval of an experimental transfer momentum $p$ 0.5$A^{-1} < p < 2A^{-1}$ were presented [3], so that $k_f < p$. It is seen, that if the $^3\text{He}$ admixture is absent, $k_f = 0$ and $Y_0 = 0$. Taking into account the equation $k_f = (3\pi^2)^{1/3} \rho_f^{1/3}\hbar$, we get

$$Y_0 \sim \frac{8k^3}{3p^2} \sim 8\pi^2 \rho_f \hbar^3.$$
and the series decomposition of $Y_0(k_f^r)$ is confined by the first order in $\rho_f$. This approximation is valid for $T < 1$ K, as the Fermi energy of $^3$He atoms is $k_f^2/2m \sim 1.5$ K. Finally, the $^3$He density in the mixture is $\rho_f \sim 1.1 \cdot 10^{21}$ atoms/cc, while the pure liquid $^3$He density is $1.6 \cdot 10^{22}$ atoms/cc [6], and pure liquid $^4$He density is $2.17 \cdot 10^{22}$ atoms/cc (superfluid state [7]).

4 Effective action for Bose-condensate

All the terms in the brackets (5) give the contribution to the boson partition function. The path integral over $\rho$ and non-condensate bosons $b_p, b_p^*$ looks like

$$Q = \prod_{\rho \neq 0} \int d\rho \left[ \left( 1 + e^{-2\beta f_0} \right) J_B + e^{-\beta f_0} (J_+ + J_-) \right],$$

where $J_B, J_+$ and $J_-$ are three path integrals of the same type

$$J_{B,\pm} = \frac{1}{\sqrt{2\pi}} \int Db_p^* Db_p \exp \left[ \int_0^\beta (b_p^* b_{-p}) K_B,\pm \left( b_p, b_p^* \right) dt \right],$$

$$K_B,\pm = \begin{pmatrix} -\frac{d}{dt} - \frac{\alpha_p}{2} & -\zeta_p \\ -\zeta_p^* & \frac{d}{dt} - \frac{\alpha_p}{2} \end{pmatrix}_{B,\pm}.$$

The coefficients in these matrices are

$$\alpha|_B(p) = \Omega(p) + (g_0 + g_p)\rho,$$

$$\alpha|_\pm = \alpha|_B \pm \alpha_\lambda, \quad \alpha_\lambda = Y_0 \frac{\lambda^2}{V} cm\rho,$$

$$\zeta_B = \frac{1}{2} g(p)\rho, \quad \zeta_\pm = \zeta_B \pm \alpha_\lambda.$$

In accordance with formula [8]

$$\frac{Q}{Q_0} = \frac{\det K_0}{\det K} = \exp \left[ - \int_0^\zeta \text{Sp} \left( \frac{1}{K} \frac{\partial K}{\beta x} \right) dx \right]$$

the Gaussian integrals $J_{B,\pm}$ over the trajectories $b_p, b_p^*$ for noncondensate variables are calculated

$$J(p)|_{B,\pm} = \frac{e^{\beta \alpha_p/2}}{4 \text{sh}^2 \frac{\beta E_p}{2}}, \quad E_p = \sqrt{\frac{\alpha^2(p)}{4} - |\zeta(p)|^2},$$

$$\alpha = \{\alpha_B, \alpha_+, \alpha_-\}, \quad \zeta = \{\zeta_B, \zeta_+, \zeta_-\}.$$
Finally, looking for the small shifts of $E(p)$, we found

$$Q = \prod_{p \neq 0} \int d\rho J_R \left[ (1 + e^{-2\beta f_0}) + 2e^{-\beta f_0}ch \left( \frac{\beta \alpha \lambda}{2} \right) \right].$$

(7)

The effective action $S_{ef}(\rho)$ in $Q = \int d\rho \exp S_{ef}$ is represented in the form

$$S_{ef} = -\beta V \left( g_0 \frac{\rho^2}{2} - \mu \rho + \mu R \right) +$$

$$cV \int_0^{\infty} p^2 dp \left\{ (\omega_p - \mu) \frac{\beta}{2} - 2 \ln sh \left( \frac{\beta E_p}{4} \right) + \ln \left[ 1 + e^{-2\beta f_0} + 2ch \left( \frac{\beta \alpha \lambda}{2} \right) \cdot e^{-\beta f_0} \right] \right\}.$$

Taking into account the equation $\mu_b = \rho g_0$, promoting the gap-less maxon-roton excitations, we get

$$\omega_p - \mu = \Omega_p + g_p \rho$$

with the Bogoliubov spectrum

$$E_p = \sqrt{\Omega_p^2 + 2\Omega_p g(p)}.$$

In the view of the quasiclassical approach for bosons in (1,3) (see Appendix) the variational equation

$$\delta S_{ef} \beta V \delta \rho = 0$$

determine the extremal value of $\rho$. In the case $\rho < R/2$ this equation approximately looks like

$$\delta S_{ef} \beta V \delta \rho \approx g_0 (\rho - R) + c \int_0^{\infty} p^2 dp g_p \left( 1 - \frac{\Omega_p}{E_p} \right) + K_\lambda = 0,$$

(8)

$$K_\lambda = c \int_0^{\infty} p^2 dp \frac{sh(\alpha \beta/2) \alpha_0}{ch(\beta f_0) + ch(\beta \alpha \lambda/2)}, \quad \alpha_0 = \frac{\partial \alpha \lambda}{\partial \rho} = Y_0 \frac{\lambda^2}{V} cm,$$

$R$ is a total atomic density. The thermodinamical definition of a pressure

$$P = - \left( \frac{\partial F}{\partial V} \right)_T,$$

$$F = -kT \ln Q, \quad P = \frac{1}{\beta Q} \int d\rho \left( \frac{\partial}{\partial V} \exp S_{ef} \right),$$

$$P = \frac{1}{Q} \int d\rho \exp S_{ef} \left( \frac{S_{ef}}{V \beta} - K_\lambda \rho \right) = \left( \frac{S_{ef}}{V \beta} - K_\lambda \rho \right).$$

The last equation gives an exact connection between the pressure $P$ and a condensate atoms density $\rho$. Still the interaction potentials between the atoms take part in all the formulas, so a progress will follow, if we introduce a more detailed information on $g$ and $\lambda$.

\[\text{Note, that all the terms of the partition function (6) are taken into account in (7) regardless their dependence (or not) on the boson variables. It was not so with the formula (18) in [1], where some terms were illegally omitted.}\]
5 Interatomic $^4$He potential

Lenard-Jones and Morse potentials may be written as

$$\Phi_{LD}(r) = \frac{A}{r^n} - \frac{B}{r^m}, \quad (n > m > 0), \quad \Phi_M(r) = C \left( \exp[-2\alpha r] - 2 \exp[-\alpha r] \right),$$

and Yukawa potential looks like $\Phi_Y(r) = D \exp(-r)/r$. Ordinary $\Phi_M$ is used for atoms in molecule, $\Phi_{LD}$ is used for atoms in condensed matter and $\Phi_Y$ is used in nuclear physics. Atoms of quantum liquids $^4$He, $^3$He are “light” and has a “small hard core”, so they may be attributed to an intermediate type of interaction. Here we suggest the new interatomic potential for atoms of $^4$He

$$g(r) = \varepsilon \left( \frac{1}{r^2} \exp[-ar] - \frac{1}{rd} \exp[-br] \right), \quad (9)$$

where $a^{-1}$ and $b^{-1}$ are the effective interaction ranges for attraction and repulsion correspondingly, $d$ is the unit length power. Fourier image $g(p)$ of the potential (9) looks like

$$g(p) = \varepsilon \left[ \frac{1}{p} \arctan \left( \frac{p}{a} \right) - \frac{1}{d(p^2 + b^2)} \right],$$

and the negative value of $g(p)$ is responsible for the roton minimum. In order to take the potential shift dependence on $P$ into account we suppose, that the pressure moves the point of potential minimum to the left and down in the axes $g(r)$ and $r$. This process is imitating by changing $a, b$ coefficients: $P_1 \rightarrow P_2$: $a_1 \rightarrow a_2, \ b_1 \rightarrow b_2$, so that

$$\delta g(p) = g_2(p) - g_1(p) = \varepsilon \left( \arctan \left( \frac{p}{a_2} \right) - \arctan \left( \frac{p}{a_1} \right) \right) + \varepsilon \left( \frac{1}{p^2 + b_2^2} - \frac{1}{p^2 + b_1^2} \right), \quad (10)$$

$$\delta g(0) = \varepsilon \left( \frac{1}{a_2} - \frac{1}{db_2^2} - \frac{1}{a_1} + \frac{1}{db_1^2} \right).$$

For the situation $P_1 \rightarrow P_2$, provided by the $^3$He admixture to $^4$He in the same volume $V$, we use Bogoliubov formula for the spectrum with initial and shifted potentials $g(p) \rightarrow g(p) + \delta g(p)$ and the condensate densities $\rho \rightarrow \rho + \Delta \rho$:

$$E_p^2 = \Omega_p^2 + 2\Omega_p \rho g(p), \quad \Omega_p = \frac{p^2}{2m},$$

$$E_p^2|_\lambda = \Omega_p^2 + 2\Omega_p [\rho + \Delta \rho] \cdot [g(p) + \delta g(p)].$$

The equation for $\Delta \rho$ appears after the combination of variational equations for $\lambda = 0$ and $\lambda \neq 0$

$$\Delta \rho \left( g_0 + K_\lambda + c \int_0^\infty \frac{p^2 \rho g dp}{\Omega_p} \right) = -\rho K_\lambda + \left( R - \rho \right) \delta g(0) - \rho \left( \frac{g(p)}{\Omega_p} \right) D_\varepsilon, \quad (11)$$

7
$$D_\varepsilon = \int_0^\infty \delta g(p)dp = \frac{\pi \varepsilon}{2} \left[ \ln \frac{a_1}{a_2} + \frac{1}{d} \left( \frac{1}{b_1} - \frac{1}{b_2} \right) \right].$$

Thus, there are three origins for the shift $\Delta E = E_p|_\lambda - E_p$ of maxon-roton spectrum $E_p$:

a. direct $^4$He/$^3$He interaction, represented by the term $K_\lambda$ in (11),

b. shift (10) of the potential $\delta g(p)$ via the pressure $P$,

c. shift $\Delta \rho$ of a Bose-condensate density due to the factors (a,b).

6 Shifts of maxon-roton excitations

The value of $\Delta \rho$, determined by the equation (11), valid for $\rho < R/2$, depends mainly on the competition between $K_\lambda$ and $g$, which represent the $^4$He/$^3$He and $^4$He/$^4$He interactions correspondingly. We have no apriory information about $g$ and $\lambda$ magnitudes, but we may look for the different versions of the situation with shifts $\Delta E = E_p|_\lambda - E_p$ in general.

Let us take the positive shifts $\delta g(0)$ and $D_\varepsilon$, using the initial parameters as $a_1 = 0.2245$, $b_1 = 0.4848$ and the final parameters as $a_2 = 0.22$, $b_2 = 0.4809$. It means, that the ranges in space of attraction and repulsion in (9) are diminishing. This deformation of the potential, corresponding the situation $P_2 > P_1$, is rather small in the scale $r, g(r)$. Still it will lead to a considerable changes of the dispersion (maxon-roton) curves, shown below in the Fig.1.

The curve 1 corresponds the values $a_1$, $b_1$, $K_\lambda = 0$ and arbitrary constant $\rho$. The values $a_2$, $b_2$ for $K_\lambda = 0$ and the same value of $\rho$ lead to the curve 2, that has the single intersection with the curve 1. This is a change of excitation spectrum via the pressure $P_1 \to P_2$ itself, without any admixture of $^3$He.

If $\rho$ will be changed for any given $K_\lambda$, we have two different cases: the curves 3 and 4, both of them have two intersection points with 1. For example, if $\rho \to 0.5 \rho$, $\Delta \rho \sim -0.5 \rho$, the curve 3 appears, and $A$ is the second intersection point between the curves 1 and 3. Then, for the amplitude $\rho \to 1.5 \rho$, $\Delta \rho \sim 0.5 \rho$, we get the curve 4, where $B$ is the second intersection point between the curves 1 and 4. Note, that these two cases appears as a result of competition between interactions $g$ and $\lambda$. Maxon energy and maxon momentum in Fig.1 are supposed to be $\sim 13K$ and $\sim 2A^{-1}$ correspondingly.

We hope, that the arguments of such a kind for the other versions of the parameters $a_1$, $a_2$, $b_1$ and $b_2$ will show more complete correspondence between experimental conditions and theoretical background on this subject.

7 Appendix

Bogoliubov model $H_B$ [9] gives the mixed description of bosons: the quantum operators $b_k^\dagger, b_k$ for non-condensate particles and complex numbers $a^*, a$ for Bose-condensate. The
"quasiclassical" integral of motion $n$ for $H_B$ was introduced, following [4], in Section 2

$$ n = |a|^2 + \sum_{k \neq 0} b_k^+ b_k, \quad \{H_B, |a|^2\} + i[H_B, \sum_{k \neq 0} b_k^+ b_k] = 0 $$

where $\{,\}$ are classical Poisson brackets and $[,]$ is quantum commutator. A question may be asked: what happens with $n$, if the high-power $b_k^+, b_k$ terms $h$ are taken into account for boson-boson interaction?

We may take the exact operator of energy

$$ H = \frac{1}{V} \sum_{k_1+k_2=k_3+k_4} b_{k_1}^+ b_{k_2}^+ b_{k_3} b_{k_4} $$

and use of the ordinary [10] $C$-shift $b_k^+ \rightarrow b_k^+ + \delta_{k,0} a^*, b_k \rightarrow b_k + \delta_{k,0} a$, so that $H \rightarrow H_B + h$,

$$ h = \frac{1}{V^{1/2}} g_0(b_0 a^* a^2 + a^* a^2 b_0^+) + \frac{1}{2V^{3/2}} \sum_{k_1+k_2=k_3} (g_{k_1} + g_{k_2}) (a b_{k_1}^+ b_{k_2}^+ b_{k_3} + a^* b_{k_3}^+ b_{k_2} b_{k_1}) + $$

$$ + \frac{1}{2V} \sum_{k_1+k_2=k_3+k_4} (g_{k_1-k_3} + g_{k_1-k_4}) b_{k_1}^+ b_{k_2}^+ b_{k_3} b_{k_4}. $$

It was shown in [5], that the term $h$ does not disturb the quasiclassical conservation law, as the equation

$$ \{H_B + h, |a|^2\} + i[H_B + h, \sum_{k=0} b_k^+ b_k] = 0 $$

is valid. Therefore, the initial approximation in $H_B$ is not so strong, as it is usually supposed. So the quasiclassical integral of motion $n$ was justly used in a wide context of path-integral calculations (the equation $\mu_b = \rho \sigma_0$ in Section 4 etc, for example).

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