A Splitting Primal-dual Proximity Algorithm for Solving Composite Optimization Problems

Yu Chao TANG Chuan Xi ZHU
Department of Mathematics, Nanchang University, Nanchang 330031, P. R. China
E-mail: hhaa0001331@aliyun.com chuanzixu@126.com

Meng WEN Ji Gen PENG
School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, P. R. China
and
Beijing Center for Mathematics and Information Interdisciplinary Sciences,
Beijing 100048, P. R. China
E-mail: wenmeng171987@stu.xjtu.edu.cn jgpeng@xjtu.edu.cn

Abstract Our work considers the optimization of the sum of a non-smooth convex function and a finite family of composite convex functions, each one of which is composed of a convex function and a bounded linear operator. This type of problem is associated with many interesting challenges encountered in the image restoration and image reconstruction fields. We developed a splitting primal-dual proximity algorithm to solve this problem. Furthermore, we propose a preconditioned method, of which the iterative parameters are obtained without the need to know some particular operator norm in advance. Theoretical convergence theorems are presented. We then apply the proposed methods to solve a total variation regularization model, in which the $L^2$ data error function is added to the $L^1$ data error function. The main advantageous feature of this model is its capability to combine different loss functions. The numerical results obtained for computed tomography (CT) image reconstruction demonstrated the ability of the proposed algorithm to reconstruct an image with few and sparse projection views while maintaining the image quality.

Keywords Sparse optimization, proximity operator, saddle-point problem, CT image reconstruction

MR(2010) Subject Classification 90C25, 65K10

1 Introduction

In this paper, we consider solving the following convex optimization problem:

$$\min_{x \in X} \sum_{i=1}^{l} F_i(K_i x) + G(x),$$

where $l$ is an integer, $X$ and $\{Y_i\}_{i=1}^{l}$ are Hilbert spaces, the functions $\{F_i\}_{i=1}^{l}$ and $G$ belong in $\Gamma_0(Y_i)$ and $\Gamma_0(X)$, respectively, and $K_i : X \to Y_i$ is a continuous linear operator for $i = 1, \ldots, l$. The iterative parameters are obtained without the need to know some particular operator norm in advance. Theoretical convergence theorems are presented. We then apply the proposed methods to solve a total variation regularization model, in which the $L^2$ data error function is added to the $L^1$ data error function. The main advantageous feature of this model is its capability to combine different loss functions. The numerical results obtained for computed tomography (CT) image reconstruction demonstrated the ability of the proposed algorithm to reconstruct an image with few and sparse projection views while maintaining the image quality.
Here and in what follows, for a real Hilbert space \( H \), \( \Gamma_0(H) \) denotes the collection of all proper lower semi-continuous (LSC) convex functions from \( H \) to \( (-\infty, +\infty] \). Based on the assumptions of problem (1.1), the functions \((F_i \cdot K_i)_{1 \leq i \leq l}\) may be used to model the data fidelity term, including smooth and non-smooth measures, and \( G \) could be the indicator function of a convex set or \( \ell_1 \)-norm, etc. Therefore, the optimization model (1.1) would be able to accommodate a combination of different data error functions.

In particular, if \( l = 1 \), then problem (1.1) is reduced to the following

\[
\min_{x \in X} F(Kx) + G(x),
\]

where \( F \in \Gamma_0(Y) \), \( G \in \Gamma_0(X) \), and \( K : X \to Y \) is a continuous linear operator. Under the assumption that the proximity operator of \( F^* \) and \( G \) are easy to compute (i.e., it either has a closed-form solution or can be efficiently computed with high precision), Chambolle and Pock \([2]\) proposed a primal-dual proximity algorithm to solve problem (1.2). They proved the convergence of the proposed iterative algorithm in finite dimensional Hilbert spaces. They also pointed out the relationship between the primal-dual proximity algorithm and other existing algorithms, such as extrapolational gradient methods \([13]\), the Douglas–Rachford splitting algorithm \([5]\), and the alternating direction method of multipliers \([1]\). Furthermore, in \([14]\), they introduced a precondition technique to compute the step size of the algorithm automatically. Numerical experiments showed that the preconditioned primal-dual proximity algorithm outperforms the primal-dual proximity algorithm in \([2]\). He and Yuan \([11]\) studied the convergence of the primal-dual proximity algorithm by presenting this algorithm of Chambolle and Pock \([2]\) in the form of a proximal point algorithm in infinite-dimensional Hilbert spaces. Condat \([8]\) also obtained the convergence of the primal-dual proximity algorithm but from a different point of view, namely by studying the following optimization problem:

\[
\min_{x} P(x) + G(x) + F(Kx),
\]

where \( P : X \to \mathbb{R} \) is convex, differentiable, and its gradient is Lipschitz continuous, and \( G, F \), and \( K \) are the same as in Problem (1.2). If \( P(x) = 0 \), then Problem (1.3) reduces to Problem (1.2). Condat \([8]\) proposed an efficient iterative algorithm for solving (1.3) and also proved its convergence based on Krassnoselskii–Mann iteration methods. The primal-dual-proximity algorithm is a special case of Condat’s algorithm by letting \( P(x) = 0 \). Furthermore, he proved the convergence of the primal-dual-proximity algorithm in finite dimensional spaces where the parameters were relaxed from \( \sigma \tau \|K\|^2 < 1 \) to \( \sigma \tau \|K\|^2 \leq 1 \). These are very useful results because it becomes possible to fix one parameter in the algorithm, allowing the other parameter to be tuned in practice.

If we let \( F_0(x) = G(x), K_0 = I \), then the problem (1.1) can also be formulated as follows,

\[
\min_{x} \sum_{i=0}^{l} F_i(K_ix).
\]

Setzer et al. \([17]\) proposed to use an alternating split Bregman method \([9]\) to solve the problem (1.4) and proved \([16, 18]\) that this method coincided with the alternating direction method of multipliers, which can be interpreted as a Douglas–Rachford splitting algorithm applied to the dual problem. However, this iterative algorithm always incorporates linear equations, which
are required to be solved either explicitly or approximately. Condat [8] considered the following general composite optimization problem,

$$\min_x \sum_{i=1}^{l} F_i(K_i x) + G(x) + Q(x),$$

where the linear operators \( \{K_i\}_{i=1}^{l} \), the functions \( \{F_i\}_{i=1}^{l} \), and \( G \) are the same as in Problem (1.1), apart from the fact that the function \( Q(x) \) is also convex, differentiable, and displays a Lipschitz continuous gradient. He obtained an iterative algorithm to solve Problem (1.5) by recasting it as Problem (1.3) using the product spaces method. The iterative parameters in the algorithm introduced in [8] rely on the estimation of the operator norm \( \| \sum_{i=1}^{l} K_i^* K_i \| \), which may affect its practical use. To overcome this disadvantage, we propose a preconditioned iterative algorithm to solve Problem (1.1), where the iterative parameters are calculated self-adaptively.

If the function \( Q(x) \) is equal to the least-squares loss function, i.e., \( Q(x) = \frac{1}{2} \| Ax - b \|_2^2 \), then Problem (1.5) could be viewed as a special case of Problem (1.1).

The primal-dual algorithm is a very flexible method to solve the optimization Problem (1.2), which has wide potential application in image restoration and image reconstruction, for example [3, 4, 7, 20–23]. Sidky et al. [19] applied the primal-dual proximity algorithm introduced by Chambolle and Pock [2, 14] to solve various convex optimization problems. For example,

$$\min_x \frac{1}{2} \| Ax - b \|_2^2;$$

$$\min_x \frac{1}{2} \| Ax - b \|_2^2, \quad \text{s.t. } x \geq 0;$$

$$\min_x \frac{1}{2} \| Ax - b \|_2^2 + \lambda \| x \|_{TV};$$

$$\min_x \| Ax - b \|_1 + \lambda \| x \|_{TV};$$

$$\min_x KL(Ax, b) + \lambda \| x \|_{TV},$$

where \( \| \cdot \|_1 \) represents the \( \ell_1 \)-norm, \( \| \cdot \|_2 \) represents the \( \ell_2 \)-norm, \( \| \cdot \|_{TV} \) denotes the total variation semi-norm, \( KL(\cdot, \cdot) \) denotes the Kullback–Leibler (KL) divergence, and \( \lambda > 0 \) is the regularization parameter balancing the data error term and the regularization term. The least-squares data error term is used widely in computed tomography (CT) image reconstruction. It is modeled by adding Gaussian noise to the collected data and the \( L1 \) data loss function has the advantage of reducing the impact of image sampling with large outliers. They studied the application of this convex optimization problem in CT image reconstruction to demonstrate the performance of these different models under appropriate levels of noise. The numerical results showed that the primal-dual proximity algorithm can efficiently solve these problems and it exhibited very good performance in terms of reconstructing simulated breast CT data. The work of Sidky et al. [19] motivated us to introduce a general composite optimization problem for image reconstruction. Then, the above optimization problem (1.8) and (1.9) would be a special case of our proposed optimization problem.

The purpose of this paper is to introduce a splitting primal-dual proximity algorithm for solving problem (1.1) and to propose a preconditioning technique to improve the performance of this algorithm. In addition, theoretical convergence theorems are also provided. We then
A Splitting Primal-dual Proximity Algorithm

871

demonstrate the performance of our proposed algorithms by applying them to solve a com-
posite optimization problem, which has wide application in the image restoration and image
reconstruction fields.

The rest of this paper is organized as follows. In Section 2, we provide selected background
information on convex analysis. In Section 3, we briefly review the primal-dual proximal al-
gorithm, together with one of its preconditioned techniques. These iterative algorithms are
employed to develop a splitting primal-dual proximal algorithm for solving problem (1.1) and
the results are presented in Section 4. In Section 5, we apply the proposed iterative algorithm to
solve a particular convex optimization model, which is relevant to the CT image reconstruction
problem. We use numerical results to illustrate the capabilities of our proposed algorithm in
Section 6. Finally, we offer some conclusions.

2 Preliminaries

In this section, we introduce some definitions and notations. Let $H$ be a real Hilbert space,
with its inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$. We denote by $\Gamma_0(H)$ the set of proper
lower semicontinuous (LSC), convex functions from $H$ to $(-\infty, +\infty]$.

Definition 2.1 Let $f$ be a real-valued convex function on $H$, for which the proximity operator
$\text{prox}_f$ is defined by

$$
\text{prox}_f : H \rightarrow H
$$

$$
x \mapsto \arg \min_{y \in H} f(y) + \frac{1}{2}\|x - y\|_2^2. \tag{2.1}
$$

Let $C$ be a nonempty closed convex set of $H$. The indicator function of $C$ is defined on $H$ as

$$
\iota_C(x) = \begin{cases} 
0, & \text{if } x \in C, \\
+\infty, & \text{otherwise}. 
\end{cases} \tag{2.2}
$$

It is easy to see that the proximity operator of the indicator function is the projection operator
onto $C$. That is, $\text{prox}_{\iota_C}(x) = P_C(x)$, where $P_C$ represents the projection operator onto $C$.

For some simple functions, there is a closed-form solution of their proximity functions and
we provide several examples. For other examples of proximity operators with closed-form
expression, we refer the readers to [6] for details.

Example 2.2 Let $\lambda > 0$ and $u \in \mathbb{R}^N$. Then

$$
[\text{prox}_{\lambda \| \cdot \|_1}(u)]_i = \max(|u_i| - \lambda, 0)\text{sign}(u_i).
$$

The proximity operator of $\ell_1$-norm $\| \cdot \|_1$ is often referred to as a Soft-thresholding operator,
and denoted by $\text{Soft}(u, \lambda)$, i.e., $\text{Soft}(u, \lambda) = \text{prox}_{\lambda \| \cdot \|_1}(u)$.

Example 2.3 Let $\lambda > 0$ and $u \in \mathbb{R}^N$. Then

$$
\text{prox}_{\lambda \| \cdot \|_2}(u) = \max(\|u\|_2 - \lambda, 0) \frac{u}{\|u\|_2}.
$$

Example 2.4 Let $u \in \mathbb{R}^{2N}$. Then the norm $\|u\|_{1,2}$ is defined by

$$
\|u\|_{1,2} = \sum_{i=1}^N \sqrt{u_i^2 + u_{N+i}^2}.
$$
Let $\lambda > 0$, $x \in \mathbb{R}^N$ and $\|x\|_2 = \sqrt{x_i^2 + x_{N+i}^2}$. Then $\text{prox}_{\lambda \| \cdot \|_1,1}(x)$ can be expressed as

$$
\text{prox}_{\lambda \| \cdot \|_1,1}(x) = \begin{cases} 
\max \{\|x_i\|_2 - \lambda, 0\} \frac{x_i}{\|x_i\|_2}; \\
\max \{\|x_i\|_2 - \lambda, 0\} \frac{x_{N+i}}{\|x_i\|_2}, \quad i = 1, \ldots, N.
\end{cases}
$$

(2.3)

We also prove some proximity functions which will be used in the following sections.

**Lemma 2.5** For any $u \in \mathbb{R}^N$ and $b \in \mathbb{R}^N$, define the function $f(x) = \|x - b\|_1$. Then the proximity operator of $\text{prox}_{\lambda f}(u)$ is given by

$$
\text{prox}_{\lambda f}(u) = b + \text{Soft}(u - b, \lambda).
$$

(2.4)

**Proof** By the definition of the proximity operator, we know that

$$
\text{prox}_{\lambda f}(u) = \arg \min_x \left\{ \frac{1}{2} \|x - u\|_2^2 + \lambda \|x - b\|_1 \right\}.
$$

Let $x - b = y$. Then the above minimization problem reduces to

$$
\arg \min_y \left\{ \frac{1}{2} \|y + b - u\|_2^2 + \lambda \|y\|_1 \right\} = \arg \min_y \left\{ \frac{1}{2} \|y - (u - b)\|_2^2 + \lambda \|y\|_1 \right\} = \text{Soft}(u - b, \lambda).
$$

Then, $\text{prox}_{\lambda f}(u) = b + \text{Soft}(u - b, \lambda)$.

**Lemma 2.6** For any $u \in \mathbb{R}^N$ and $b \in \mathbb{R}^N$, define the function $f(x) = \frac{1}{2} \|x - b\|_2^2$; then, the proximity operator of $\text{prox}_{\lambda f}(u)$ is given by

$$
\text{prox}_{\lambda f}(u) = \frac{u + \lambda b}{1 + \lambda}.
$$

(2.5)

**Proof** By the definition of the proximity operator, we know that

$$
\text{prox}_{\lambda f}(u) = \arg \min_x \left\{ \frac{1}{2} \|x - u\|_2^2 + \frac{1}{2} \lambda \|x - b\|_2^2 \right\}.
$$

(2.6)

The first-order optimality condition of (2.6) reduces to

$$
0 = (x - u) + \lambda(x - b).
$$

Then $x = \frac{u + \lambda b}{1 + \lambda}$. That is,

$$
\text{prox}_{\lambda f}(u) = \frac{u + \lambda b}{1 + \lambda}.
$$

Similarly, by Example 2.3, we can deduce the proximity operator of function $f(x) = \lambda \|x - b\|_2$ that

$$
\text{prox}_{\lambda \| \cdot \|_2}(u) = \arg \min_x \left\{ \frac{1}{2} \|x - u\|_2^2 + \lambda \|x - b\|_2 \right\}
$$

$$
= \max \{\|u - b\|_2 - \lambda, 0\} \frac{u - b}{\|u - b\|_2} + b.
$$

(2.7)

Recall that the Fenchel conjugate of a given function $f$ is defined as $f^*(x) = \sup_u \{\langle x, u \rangle - f(u)\}$. The proximity operator of a function $f$ and its Fenchel conjugate $f^*$ are connected by the celebrated Moreau’s identity [12]:

$$
x = \text{prox}_{\lambda f}(x) + \lambda \text{prox}_{\lambda f^*}(\frac{x}{\lambda}).
$$

(2.8)
The well-known Rudin–Osher–Fatemi (ROF) [15] total variation model is one of the most popular image denoising models. The ROF model is given by
\[
    \arg \min_x \left\{ \frac{1}{2} \| x - u \|_2^2 + \lambda \| x \|_{TV} \right\},
\]
where \( u \in \mathbb{R}^d \) denotes the noisy image and \( \| x \|_{TV} \) is the total variation of \( x \). Because total variation regularization can preserve the edges of images, it has been widely used in the image restoration and image reconstruction fields. The total variation norm \( \| x \|_{TV} \) can be viewed as the combination of a convex function with a linear transformation. In fact, let \( B \) denote an \( N \times N \) matrix defined by the following:
\[
    B := \begin{pmatrix}
        -1 & 1 \\
        \ddots & \ddots \\
        -1 & 1 \\
        0
    \end{pmatrix},
\]
and define matrix \( D \) to be \( 2N^2 \times N^2 \), which could be seen as a finite difference discretization of an image from horizontal and vertical,
\[
    D := \begin{pmatrix}
        I \otimes B \\
        B \otimes I
    \end{pmatrix},
\]
where \( I \) is the \( N \times N \) identity matrix and the notation \( P \otimes Q \) denotes the Kronecker product of matrices \( P \) and \( Q \).

Let \( x \) be an image in \( \mathbb{R}^{N^2} \). Two definitions of total variation have appeared in the literature. The first is referred to as anisotropic total variation (ATV) and is defined by the formula
\[
    \| x \|_{TV} := \varphi(Dx) = \| Dx \|_1,
\]
where \( \varphi(z) := \| z \|_1, z \in \mathbb{R}^{2N^2} \), whereas the second definition of total variation is known as isotropic total variation (ITV) and is defined by the equation
\[
    \| x \|_{TV} = \varphi(Dx) = \| Dx \|_{1,2} = \sum_{i=1}^{n} \left\| \begin{pmatrix}
        (Dx)_i \\
        (Dx)_{n+i}
    \end{pmatrix} \right\|_2,
\]
where \( \varphi: \mathbb{R}^{2N^2} \to \mathbb{R} \) as
\[
    \varphi(z) := \sum_{i=1}^{N^2} \left\| \begin{pmatrix}
        z_i \\
        z_{N^2+i}
    \end{pmatrix} \right\|_2, \quad z \in \mathbb{R}^{2N^2}.
\]

3 A Primal-dual Proximity Algorithm for Solving (1.2)

In this section, we recall selected primal-dual proximity algorithms for solving problem (1.2). First, the corresponding dual optimization problem of (1.2) is
\[
    \max_y -F^*(y) - G^*(-K^*y).
\]
Here, \( F^* \) and \( G^* \) represent the Fenchel conjugate of \( F \) and \( G \), respectively. Combining the primal problem (1.2) and dual problem (3.1) leads to the following saddle-point problem:
\[
    \min_x \max_y \langle Kx, y \rangle + G(x) - F^*(y).
\]
Let problem (3.2) have a solution \((\tilde{x}, \tilde{y})\). Then it satisfies the following variational inclusion

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} \in \begin{pmatrix}
K^* \tilde{y} + \partial G(\tilde{x}), \\
-\tilde{x} + \partial F^* \tilde{y},
\end{pmatrix}
\]

(3.3)

where \(\partial F^*\) and \(\partial G\) are the subdifferential of the convex functions \(F^*\) and \(G\).

Chambolle and Pock [2] proposed a primal-dual proximity algorithm for solving (3.2). Choosing \((x^0, y^0) \in X \times Y\) and \(\mathbf{v}^0 = x^0\), the iterative sequences \(\{x^k\}\) and \(\{y^k\}\) are given by

\[
\begin{align*}
x^{k+1} &= \text{prox}_{\sigma F^*}(y^k + \sigma K^* x^k), \\
y^{k+1} &= \text{prox}_{\tau G}(x^k - \tau K^* y^{k+1}), \\
y^{k+1} &= x^{k+1} + \theta(x^{k+1} - x^k),
\end{align*}
\]

(3.4)

where \(\sigma, \tau > 0\), and \(\theta \in [0, 1]\). They proved its convergence with the requirement of \(\theta = 1\) and \(\sigma \tau < 1/\|K\|^2\) in finite-dimensional spaces.

Define \(y^1 = \text{prox}_{\sigma F^*}(y^0 + \sigma K x^0)\). Then the iterative sequence (3.4) can be rewritten as

\[
\begin{align*}
x^{k+1} &= \text{prox}_{\tau G}(x^k - \tau K^* y^{k+1}), \\
y^{k+2} &= \text{prox}_{\sigma F^*}(y^{k+1} + \sigma K(x^{k+1} + \theta(x^{k+1} - x^k))).
\end{align*}
\]

(3.5)

Letting \(y^{k+1} = y^k\), we can simply rewrite the iterative sequence (3.5) as follows:

\[
\begin{align*}
x^{k+1} &= \text{prox}_{\tau G}(x^k - \tau K^* y^k), \\
y^{k+1} &= \text{prox}_{\sigma F^*}(y^k + \sigma K(x^{k+1} + \theta(x^{k+1} - x^k))).
\end{align*}
\]

(3.6)

The only difference between iterative sequences (3.4) and (3.6) is the initial value of \(y^0\). Because these iterative algorithms do not depend on the initial value of \(x^0\) and \(y^0\), they are actually equivalent. Therefore, the details of the primal-dual proximity algorithm introduced by Chambolle and Pock [2] are actually those provided in Algorithm 1.

**Algorithm 1** Primal-dual proximity algorithm for solving (1.2)

**Initialization:** Give \(\tau, \sigma > 0, \theta \in [0, 1]\) and choose \((x^0, y^0) \in X \times Y\);

For \(k = 0, 1, 2, \ldots\) do

1. \(x^{k+1} = \text{prox}_{\tau G}(x^k - \tau K^* y^k)\),
2. \(y^{k+1} = \text{prox}_{\sigma F^*}(y^k + \sigma K(x^{k+1} + \theta(x^{k+1} - x^k)))\).

end for when some stopping criterion is satisfied

**Theorem 3.1** ([2]) Let \(\theta = 1\) and the parameters \(\sigma, \tau\) satisfy \(\sigma \tau \|K\|^2 < 1\). Then, the sequence \((x^k, y^k)\) generated by Algorithm 1 converges weakly to an optimal solution \((x^*, y^*)\) of the saddle-point problem (3.2).

**Remark 3.2** Condat [8] proved that the condition \(\sigma \tau \|K\|^2 < 1\) in Theorem 3.1 could be relaxed to \(\sigma \tau \|K\|^2 \leq 1\) in finite-dimensional spaces.

The convergence of Algorithm 1 relies on the operator norm \(\|K\|\), which is not easy to estimate. Pock and Chambolle [14] attempted to address this shortcoming by proposing a
precondition technique for Algorithm 1 where the step sizes $\tau$ and $\sigma$ are replaced by two symmetric and positive definite matrices, respectively. They also suggested a practical approach for obtaining the matrices $T$ and $\Sigma$, thereby satisfying the convergence requirement of Theorem 3.3.

### Algorithm 2

Preconditioned primal-dual proximity algorithm for solving (1.2)

**Initialization:** Choose symmetric and positive definite matrices $T$ and $\Sigma$, $\theta \in [0, 1]$, $(x^0, y^0) \in X \times Y$.

For $k = 0, 1, 2, \ldots$ do

1. $x^{k+1} = \text{prox}_T G(x^k - TK^*y^k)$,
2. $y^{k+1} = \text{prox}_\Sigma F^*(y^k + \Sigma K(x^{k+1} + \theta(x^{k+1} - x^k)))$.

end for when some stopping criterion is satisfied

#### Theorem 3.3 ([14])

Let $\theta = 1$ and let $T, \Sigma$ be symmetric and positive definite matrices such that $\|\Sigma^\frac{1}{2}KT^\frac{1}{2}\| < 1$. Then, the sequence $(x^k, y^k)$ generated by Algorithm 2 converges weakly to an optimal solution $(x^*, y^*)$ of the saddle-point problem (3.2).

As mentioned in [14], the matrices $\Sigma$ and $T$ could be any symmetric and positive matrices. However, it is a prior requirement of Algorithm 2 that the proximity operators are simple. Thus, they proposed to choose $\Sigma$ and $T$ with some diagonal matrices which satisfy all these requirements and guarantee the convergence of the algorithm.

#### Lemma 3.4 ([14])

Let $T = \text{diag}(\tau)$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ and $\Sigma = \text{diag}(\sigma)$, where $\sigma = (\sigma_1, \ldots, \sigma_m)$. In particular,

$$\tau_j = \frac{1}{\sum_{i=1}^m |K_{i,j}|^{2-\alpha}}, \quad \sigma_i = \frac{1}{\sum_{j=1}^n |K_{i,j}|^{\alpha}},$$

then for any $\alpha \in [0, 2]$,

$$\|\Sigma^\frac{1}{2}KT^\frac{1}{2}\|^2 = \sup_{x \in X, x \neq 0} \frac{\|T^\frac{1}{2}K\Sigma^\frac{1}{2}x\|^2}{\|x\|^2} \leq 1.$$  

In the next section, we shall see how to judiciously use the primal-dual proximity algorithms, including Algorithm 1 and Algorithm 2, to derive a variety of flexible convex optimization algorithms for the proposed problem (1.1).

### 4 A Splitting Primal-dual Proximity Algorithm for Solving (1.1)

In comparison with the well-known forward-backward splitting algorithm and the alternating direction method of multipliers for solving Problem (1.2), the forward-backward splitting algorithm needs one of the functions $F$ or $G$ to satisfy the differential and requires a Lipschitz continuous gradient, and the alternating direction method of multipliers always involves a system of linear equations as its subproblem. In contrast, every subproblem of the primal-dual proximity algorithm is easy to solve and does not require any inner iteration numbers. This motivated us to extend the primal-dual proximity algorithm to solve the general optimization Problem (1.1).

First, we present the main iterative algorithm to solve Problem (1.1) and prove its convergence as follows.
Algorithm 3  A splitting primal-dual proximity algorithm for solving (1.1)

Initialization: Give $\tau, \sigma > 0$ such that $\tau \sigma < 1/\|\sum_{i=1}^{l} K_i^* K_i\|$, choose \((x^0, y_1^0, y_2^0, \ldots, y_l^0) \in X \times Y_1 \times Y_2 \times \cdots \times Y_l;\)
For $k = 0, 1, 2, \ldots$ do
1. $x^{k+1} = \text{prox}_{\tau G}(x^k - \tau \sum_{i=1}^{l} K_i^* y_i^k),$
2. $y_i^{k+1} = \text{prox}_{\sigma F_i^*}(y_i^k + \sigma K_i(2x^{k+1} - x^k)), \text{ for } i = 1, 2, \ldots, l.$
end for when some stopping criterion is satisfied

The dual problem of (1.1) is
$$
\max_{y_1, \ldots, y_l} -G^* \left( -\sum_{i=1}^{l} K_i^* y_i \right) - \sum_{i=1}^{l} F_i^*(y_i), \quad (4.1)
$$
and the saddle-point problem is
$$
\min_x \max_{y_1, \ldots, y_l} \sum_{i=1}^{l} \langle K_i x, y_i \rangle + G(x) - \sum_{i=1}^{l} F_i^*(y_i). \quad (4.2)
$$

Theorem 4.1  Let $\sigma > 0$ and $\tau > 0$ be the parameters of Algorithm 3. Then the iterative sequence \((x^k, y_1^k, \ldots, y_l^k)\) converges weakly to an optimal solution \((x^*, y_1^*, \ldots, y_l^*)\) of the saddle-point problem (4.2).

Proof  First, we transfer the optimization problem (1.1) into the form of problem (1.2) by using a product spaces technique. For this purpose, we introduce the notation $y := (y_1, \ldots, y_l)$ for an element of the Hilbert space $Y := Y_1 \times \cdots \times Y_l$, equipped with the inner product $\langle y, z \rangle = \sum_{i=1}^{l} \langle y_i, z_i \rangle$. For any $y \in Y$, we define the function $F \in \Gamma_0(Y)$ by $\tilde{F}(y) = \sum_{i=1}^{l} F_i(y_i)$ and the linear function $\tilde{K} : X \to Y$ by $\tilde{K} x := (K_1 x, \ldots, K_l x)$, i.e.,

$$
\tilde{K} = \begin{pmatrix} K_1 \\
K_2 \\
\vdots \\
K_l \end{pmatrix}.
$$

Then, we know that $(\tilde{F} \circ \tilde{K})(x) = F_1(K_1 x) + F_2(K_2 x) + \cdots + F_l(K_l x)$. Therefore, the optimization problem (1.1) can be reformulated as the following
$$
\min_x (\tilde{F} \circ \tilde{K})(x) + G(x),
$$
which is the exact optimization problem (1.2). Taking $\theta = 1$ in Algorithm 1, we obtain the iterative sequence for solving (1.1).

$$
\begin{cases}
x^{k+1} = \text{prox}_{\tau G}(x^k - \tau \tilde{K}^* y^k), \\
y_i^{k+1} = \text{prox}_{\sigma F_i^*}(y_i^k + \sigma \tilde{K}(2x^{k+1} - x^k)).
\end{cases} \quad (4.3)
$$

By Theorem 3.1, we can conclude that the iterative sequence \((x^k, y_1^k, \ldots, y_l^k)\) converges weakly to an optimal solution \((x^*, y_1^*, \ldots, y_l^*)\) of the saddle-point Problem (4.2). Furthermore, as the function $\tilde{F}$ is separable with variables, the Fenchel conjugate of $\tilde{F}^*(u) = F_1^*(u_1) + F_2^*(u_2) + \cdots + F_l^*(u_l)$
where $x \in Y$. Then, the proximity operator $\text{prox}_{\sigma \Phi^*}$ can be calculated independently, i.e.,

$$\text{prox}_{\sigma \Phi^*}(u) = (\text{prox}_{\sigma F^*_1}(u_1), \ldots, \text{prox}_{\sigma F^*_l}(u_l)).$$

Therefore, we can split the iterative sequence (4.3) and obtain the corresponding Algorithm 3 as stated before.

**Remark 4.2** Based on the results of Condat [8], the parameters can be relaxed to

$$\tau \sigma \leq 1\left\| \sum_{i=1}^{l} K_i^T K_i \right\|$$

in a finite-dimensional Hilbert space.

**Algorithm 4** A preconditioned splitting primal-dual proximity algorithm for solving (1.1)

**Initialization:** Choose symmetric and positive definite matrices $T$ and $\Sigma_i$, for $i = 1, 2, \ldots, l$, $(x^0_1, y^0_1, y^0_2, \ldots, y^0_l) \in X \times Y_1 \times Y_2 \times \cdots \times Y_l$;

For $k = 0, 1, 2, \ldots$ do
1. $x^{k+1} = \text{prox}_{T \Theta}(x^k - T \sum_{i=1}^{l} K_i^* y^k_i)$,
2. $y^{k+1}_i = \text{prox}_{\sigma_i F_i^*}(y^k_i + \Sigma_i K_i (2x^{k+1} - x^k))$, for $i = 1, 2, \ldots, l$.

end for when some stopping criterion is satisfied

Based on Lemma 3.4, we are able to suggest a practical way to choose the matrices $T$ and $(\Sigma_k)_{k=1}^l$, respectively.

**Lemma 4.3** Let $T = \text{diag}(\tau)$, where $\tau = (\tau_1, \tau_2, \ldots, \tau_n)$ and $\Sigma_k = \text{diag}(\sigma^k)$, where $\sigma^k = (\sigma^k_1, \ldots, \sigma^k_m)$ for $k = 1, 2, \ldots, l$. In particular,

$$\tau_j = \frac{1}{\sum_{k=1}^{l} \sum_{i=1}^{m} |K_k(i,j)|^{2-\alpha}}, \quad \sigma^k_i = \frac{1}{\sum_{j=1}^{n} |K_k(i,j)|^{\alpha}},$$

then for any $\alpha \in [0, 2],

$$\| T^\frac{1}{2} \tilde{K} \tilde{\Sigma}^\frac{1}{2} x \| = \sup_{x \in X, x \neq 0} \frac{\| T^\frac{1}{2} \tilde{K} \tilde{\Sigma}^\frac{1}{2} x \|}{\| x \|} \leq 1,$$

where $\tilde{K} = (K_1; K_2; \ldots; K_l)$ and $\tilde{\Sigma} = (\Sigma_1; \Sigma_2; \ldots; \Sigma_l)$.

**Remark 4.4** The advantages of our approach are the following: (i) There are limited assumptions for the functions $\{F_i\}_{i=1}^l$ and $G$; (ii) There is no inner iteration involved in the main process; (iii) The iterative parameters are easy to select.

**5 Applications**

In this section, we consider solving the following constrained composite optimization problem,

$$\min_{x} \frac{1}{2} w_1 \| Ax - b \|^2_2 + w_2 \| Ax - b \|_1 + \lambda \| x \|_{TV},$$

s.t. $x \in C$,

$$w_1 + w_2 = 1,$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C$ is a closed convex set, $w_1, w_2 \in [0, 1]$ satisfying

$$w_1 + w_2 = 1.$$
\( \lambda \) is the regularization parameter, and \( \| x \|_{TV} \) denotes the total variation (TV) norm. It is easy to see that problem (5.1) includes the well-known \( L2 + TV \) (1.8) and \( L1 + TV \) (1.9) problem as its special case. If \( w_2 = 0 \), then it reduces to the constrained \( L2 + TV \) problem, and if \( w_1 = 0 \), then it reduces to the constrained \( L1 + TV \) problem, respectively.

In the following, we show that the optimization problem (5.1) is a special case of problem (1.1). The flexibility of problem (1.1) lies in the case with which constraints can be incorporated into this problem. It is observed from the definition of the total variation semi-norm (2.11) and (2.12) that \( \| x \|_{TV} = (\varphi \circ D)(x) \), with \( \varphi \) a convex lower semicontinuous function and \( D \) a real matrix. Then, the optimization problem (5.1) can be reformulated as follows.

\[
\min_x \frac{1}{2} w_1 \| Ax - b \|_2^2 + w_2 \| Ax - b \|_1 + \lambda \varphi(Dx) + \iota_C(x),
\]

where \( \iota_C \) is the indicator function of the closed convex set \( C \).

To match the formulation (1.1) with the problem at hand (5.2), we follow two approaches to obtain its solution.

**Method I** Let \( G(x) = 0 \), \( F_1(v) = \frac{1}{2} w_1 \| v - b \|_2^2 \), \( K_1 = A \), \( F_2(v) = w_2 \| v - b \|_1 \), \( K_2 = A \), \( F_3(v) = \lambda \varphi(v) \), \( K_3 = D \), \( F_4(v) = \iota_C(v) \), and \( K_4 = I \). Then, we can apply Algorithm 3 to solve the problem (5.2). The detailed structure of the algorithm is presented as follows.

**Algorithm 5** A first class of splitting primal-dual proximity algorithm for solving problem (5.1)

**Initialization:** Give \( \tau, \sigma > 0 \) such that \( \tau \sigma \leq 1/\| 2A^T A + D^T D + I \| \), choose \((x_0, y_1^0, y_2^0, y_3^0, y_4^0)\) \( \in X \times Y_1 \times Y_2 \times Y_3 \times Y_4 \);

for \( k = 0, 1, 2, \ldots \) do

1. \( x^{k+1} = x^k - \tau (A^T y_1^k + A^T y_2^k + D^T y_3^k + y_4^k) \),
2. \( y_1^{k+1} = \text{prox}_{\sigma F_1}(y_1^k + \sigma A(2x^{k+1} - x^k)) \),
3. \( y_2^{k+1} = \text{prox}_{\sigma F_2}(y_2^k + \sigma A(2x^{k+1} - x^k)) \),
4. \( y_3^{k+1} = \text{prox}_{\sigma F_3}(y_3^k + \sigma D(2x^{k+1} - x^k)) \),
5. \( y_4^{k+1} = \text{prox}_{\sigma F_4}(y_4^k + \sigma (2x^{k+1} - x^k)) \).

end for when some stopping criterion is satisfied

In the following, we explain that every subproblem of Algorithm 5 can be calculated explicitly. In fact, the proximal operator of \( F^* \) is determined via one of the functions \( F \) obtained by using Moreau’s identity (2.8).

First, according to Moreau’s identity (2.8) and Lemma 2.6, we have

\[
y_1^{k+1} = \text{prox}_{\sigma F_1} \left( \sigma \left( \frac{1}{\sigma} y_1^k + A(2x^{k+1} - x^k) \right) \right) \\
= \sigma (I - \text{prox}_{\frac{1}{\sigma} F_1}) \left( \frac{1}{\sigma} y_1^k + A(2x^{k+1} - x^k) \right) \\
= \left( y_1^k + \sigma A(2x^{k+1} - x^k) \right) - \frac{\sigma}{w_1 + \sigma} \left( y_1^k + \sigma A(2x^{k+1} - x^k) + w_1 b \right) \\
= \frac{w_1}{w_1 + \sigma} \left( y_1^k + \sigma A(2x^{k+1} - x^k) - \sigma b \right).
\]

(5.3)
Second, by Lemma 2.5, we can obtain the proximity operator of function $\sigma F_2^\ast$. That is,
\[
y_2^{k+1} = \text{prox}_{\sigma F_2^\ast} \left( \sigma \left( \frac{1}{\sigma} y_2^k + A(2x^{k+1} - x^k) \right) \right)
= \sigma(I - \text{prox}_{\frac{1}{\sigma} F_2}) \left( \frac{1}{\sigma} y_2^k + A(2x^{k+1} - x^k) \right)
= (y_2^k + \sigma A(2x^{k+1} - x^k)) - \sigma \left( b + \text{Soft} \left( \frac{1}{\sigma} y_2^k + A(2x^{k+1} - x^k) - b, \frac{w_2}{\sigma} \right) \right).
\]
(5.4)

Third, by taking into account the definition of the TV norm, the function $F_3(v)$ is equal to $\|v\|_1$ or $\|v\|_{1,2}$, respectively. Then, the proximity of $\sigma F_3^\ast$ can also be calculated by
\[
y_3^{k+1} = \text{prox}_{\sigma F_3^\ast} \left( \sigma \left( \frac{1}{\sigma} y_3^k + D(2x^{k+1} - x^k) \right) \right)
= \sigma(I - \text{prox}_{\frac{1}{\sigma} F_3}) \left( \frac{1}{\sigma} y_3^k + D(2x^{k+1} - x^k) \right).
\]
(5.5)

Then, for the anisotropic TV (ATV), we have
\[
y_3^{k+1} = (y_3^k + \sigma D(2x^{k+1} - x^k)) - \sigma \text{Soft} \left( \frac{1}{\sigma} y_3^k + D(2x^{k+1} - x^k), \frac{\lambda}{\sigma} \right).
\]
(5.6)
and for the isotropic TV (ITV), we also have a closed-form solution due to (2.3).

Fourth, because the proximity of indicator function $\iota_C$ is equal to the projection operator onto the set $C$, we obtain
\[
y_4^{k+1} = \text{prox}_{\sigma F_4^\ast} \left( \sigma \left( \frac{1}{\sigma} y_4^k + (2x^{k+1} - x^k) \right) \right)
= \sigma(I - \text{prox}_{\frac{1}{\sigma} F_4}) \left( \frac{1}{\sigma} y_4^k + (2x^{k+1} - x^k) \right)
= (y_4^k + \sigma(2x^{k+1} - x^k)) - \sigma P_C \left( \frac{1}{\sigma} y_4^k + (2x^{k+1} - x^k) \right).
\]
(5.7)

Therefore, the original problem (5.1) is decomposed into an iterative sequence consisting of subproblems which are much easier to solve, each one with a closed-form solution.

Next, we follow another approach to solve problem (5.2).

**Method II** Let $G(x) = \iota_C(x)$, $F_1(v) = \frac{1}{2} w_1 \|v - b\|_2^2$, $K_1 = A$, $F_2(v) = w_2 \|v - b\|_1$, $K_2 = A$, $F_3(v) = \varphi(v)$, and $K_3 = D$. Then, we can apply Algorithm 3 to solve the problem (5.2).

**Algorithm 6** A second class of splitting primal-dual proximity algorithm for solving problem (5.1)

**Initialization:** Give $\tau, \sigma > 0$ such that $\tau \sigma \leq \|2A^T A + D^T D\|$, choose $(x^0, y_1^0, y_2^0, y_3^0, y_4^0) \in X \times Y_1 \times Y_2 \times Y_3 \times Y_4$;

For $k = 0, 1, 2, \ldots$ do
1. $x^{k+1} = P_C(x^k - \tau (A^T y_1^k + A^T y_2^k + D^T y_3^k),$
2. $y_1^{k+1} = \text{prox}_{\sigma F_1^\ast} (y_1^k + \sigma A(2x^{k+1} - x^k)),$
3. $y_2^{k+1} = \text{prox}_{\sigma F_2^\ast} (y_2^k + \sigma A(2x^{k+1} - x^k)),$
4. $y_3^{k+1} = \text{prox}_{\sigma F_3^\ast} (y_3^k + \sigma D(2x^{k+1} - x^k)),$

end for when some stopping criterion is satisfied
To demonstrate the performance of these proposed algorithms, we apply them to the test problem (5.1). In Section 5, we derived an instance of the proposed splitting primal-dual proximity algorithms.

6 Numerical Experiments

Based on the preconditioned splitting primal-dual proximity algorithm (Algorithm 4), we obtain the corresponding preconditioned Algorithm 5 and Algorithm 6, respectively.

Algorithm 7 A first class of preconditioned splitting primal-dual proximity algorithm for solving problem (5.1)

Initialization: Follow the lemma to define the matrices $T$ and $\Sigma_i$, $i = 1, 2, 3, 4$; Choose $(x^0, y^0_1, y^0_2, y^0_3, y^0_4) \in X \times Y_1 \times Y_2 \times Y_3 \times Y_4$.

For $k = 0, 1, 2, \ldots$, do

1. $x^{k+1} = x^k - T(A^T y^k_1 + A^T y^k_2 + D^T y^k_3 + y^k_4)$,
2. $y^{k+1}_1 = \text{prox}_{\Sigma_1 F_1}(y^k_1 + \Sigma_1 A(2x^{k+1} - x^k))$,
3. $y^{k+1}_2 = \text{prox}_{\Sigma_2 F_2}(y^k_2 + \Sigma_2 A(2x^{k+1} - x^k))$,
4. $y^{k+1}_3 = \text{prox}_{\Sigma_3 F_3}(y^k_3 + \Sigma_3 D(2x^{k+1} - x^k))$,
5. $y^{k+1}_4 = \text{prox}_{\Sigma_4 F_4}(y^k_4 + \Sigma_4 (2x^{k+1} - x^k))$.

end for when some stopping criterion is satisfied

Similarly, we can provide preconditioned Algorithm 6 as follows.

Algorithm 8 A second class of preconditioned splitting primal-dual proximity algorithm for solving problem (5.1)

Initialization: Follow the lemma to define the matrices $T$ and $\Sigma_i$, $i = 1, 2, 3$; Choose $(x^0, y^0_1, y^0_2, y^0_3) \in X \times Y_1 \times Y_2 \times Y_3$.

For $k = 0, 1, 2, \ldots$, do

1. $x^{k+1} = P_C(x^k - T(A^T y^k_1 + A^T y^k_2 + D^T y^k_3))$,
2. $y^{k+1}_1 = \text{prox}_{\Sigma_1 F_1}(y^k_1 + \Sigma_1 A(2x^{k+1} - x^k))$,
3. $y^{k+1}_2 = \text{prox}_{\Sigma_2 F_2}(y^k_2 + \Sigma_2 A(2x^{k+1} - x^k))$,
4. $y^{k+1}_3 = \text{prox}_{\Sigma_3 F_3}(y^k_3 + \Sigma_3 D(2x^{k+1} - x^k))$.

end for when some stopping criterion is satisfied

Remark 5.2 (1) For the unconstrained optimization problem (5.1), i.e., $C := \mathbb{R}^n$, Algorithm 5 and Algorithm 6 are equivalent, as are Algorithm 7 and Algorithm 8. (2) In comparison with Algorithms 5 and 6, Algorithms 7 and 8 can be used to obtain the iterative parameters self-adaptively without the need to know the respective matrix norm.

6 Numerical Experiments

In Section 5, we derived an instance of the proposed splitting primal-dual proximity algorithms. To demonstrate the performance of these proposed algorithms, we apply them to the test
problems described in Section 5. All experiments were performed using MATLAB on a Lenovo
Thinkstation running Windows 7 with an Intel Core 2 CPU and 4 GB of RAM.

Two-dimensional tomography test problems were created by using AIRTools [10], which
is a MATLAB software package for tomographic reconstruction that was developed by Prof.
Perchristian Hansen and his collaborators. The package includes two core functions “fanbeamtomo”
and “paralleltomo”, which were used to generate the simulation data. For example, the
function “paralleltomo” creates a 2D tomography test problem using parallel beams.

\[ [A, b, x] = \text{paralleltomo}(N, \theta, p), \]

where the input variables are as follows: \( N \) is a scalar denoting the number of discretization
intervals in each dimension such that the domain consists of \( N^2 \) cells, \( \theta \) is a vector containing
the angles in degrees (default: \( \theta = 0 : 1 : 179 \)), and \( p \) is the number of parallel rays for each
angle (default: \( p = \text{round}(\sqrt{2N}) \)). The output variables are the following: \( A \) is a coefficient
matrix with \( N^2 \) columns and length \((\theta) \times p \) rows, \( b \) is a vector containing the projection data,
and \( x \) is a vector containing the exact solution with elements between 0 and 1. We refer the
reader to the AIRTools manual for further details. The test image is the standard benchmark
Shepp–Logan phantom (see Figure 1) with size \( 256 \times 256 \) and pixels are assigned values varying
from 0 to 1.

![Figure 1 Original Shepp–Logan phantom](image)

We measured the quality of recovered images by using the criterion signal-to-noise ratio
(SNR),

\[
\text{SNR} = \log_{10} \left( \frac{\|x_{\text{true}}\|_2^2}{\|x_{\text{true}} - x_{\text{rec}}\|_2^2} \right),
\]

where \( x_{\text{true}} \) is the original image, \( x_{\text{rec}} \) denotes the reconstructed image obtained by using the
iterative algorithms. The iterative process is stopped when the relative error

\[
\frac{\|x^{k+1} - x^k\|_2}{\|x^k\|_2} \leq \epsilon,
\]

where \( \epsilon \) is a given small real number.

We compare the performance of Algorithms 5 and 6, as well as that of Algorithms 7 and 8.
The anisotropic TV (ATV) and isotropic TV (ITV) perform similarly; therefore, we use the
(ATV) regularization term in the following test. The initial values of the variables are set
to zero in all iterative algorithms. In Algorithms 5 and 6, the induced norm of the operator
\( \|2A^T A + D^T D + I\| \) and \( \|2A^T A + D^T D\| \) are estimated using the standard power iteration
algorithm. For the preconditioned algorithm, the parameter \( \alpha \) was set to one in Lemma 4.3.
The projection angles in (6.1) are set as \( \theta = 0 : 10 : 179 \). A total of 18 angles were used in the simulation test. The \( p \) value is set as default. Then, the system matrix \( A \) is \( 6516 \times 65536 \), which is an under-determined matrix. Both Gaussian and impulsive noise are added to the projection data vector \( b \). The performance of Algorithms 5, 6, 7, and 8 are listed in Table 1 and Table 2, respectively. The “-” entries indicate that the algorithm failed to reduce the error below the given tolerance \( \epsilon \) within a maximum number of 40000 iterations.

**Table 1** Comparison of the performance of Algorithms 5, 6, 7, and 8 in terms of SNR and iteration numbers with non-negativity constraints, i.e., \( C = \{ x | x \geq 0 \} \)

| Regularization Parameter | Methods | \( \epsilon = 10^{-3} \) | \( \epsilon = 10^{-4} \) | \( \epsilon = 10^{-5} \) | \( \epsilon = 10^{-6} \) |
|---------------------------|---------|----------------|----------------|----------------|----------------|
| \( \lambda = 0.6 \)       | Algorithm 5 | 18.66/3253 | 24.31/15804 | 25.25/36598 | 25.34/− |
|                           | Algorithm 7 | 24.40/430 | 25.75/1236 | 26.17/5852 | 26.18/21265 |
|                           | Algorithm 6 | 19.08/1882 | 25.65/14659 | 26.11/30009 | 26.15/− |
|                           | Algorithm 8 | 24.52/378 | 25.86/1154 | 26.17/4634 | 26.18/16433 |
| \( \lambda = 0.8 \)       | Algorithm 5 | 18.73/3399 | 25.49/17359 | 26.73/38831 | 26.78/− |
|                           | Algorithm 7 | 25.33/465 | 27.11/1346 | 27.67/5805 | 27.73/22503 |
|                           | Algorithm 6 | 19.01/2158 | 26.84/17613 | 27.53/36010 | 27.55/− |
|                           | Algorithm 8 | 25.43/404 | 27.27/1286 | 27.67/4559 | 27.74/16967 |
| \( \lambda = 1.2 \)       | Algorithm 5 | 18.67/3754 | 27.41/19697 | 28.87/− | 28.87/− |
|                           | Algorithm 7 | 26.57/495 | 29.10/1473 | 29.86/4462 | 30.09/21411 |
|                           | Algorithm 6 | 18.31/2593 | 28.29/20614 | 29.47/− | 29.47/− |
|                           | Algorithm 8 | 26.65/445 | 29.13/1412 | 29.87/3930 | 30.11/19898 |
| \( \lambda = 1.6 \)       | Algorithm 5 | 18.37/3921 | 28.31/21132 | 29.78/− | 29.78/− |
|                           | Algorithm 7 | 26.89/502 | 30.00/1521 | 30.99/4223 | 31.43/23334 |
|                           | Algorithm 6 | 17.78/2791 | 28.79/21704 | 30.32/− | 30.32/− |
|                           | Algorithm 8 | 27.00/471 | 30.05/1476 | 30.98/3751 | 31.44/20318 |
| \( \lambda = 1.8 \)       | Algorithm 5 | 18.26/4006 | 28.10/21545 | 29.80/− | 29.80/− |
|                           | Algorithm 7 | 26.74/504 | 29.92/1518 | 31.06/4278 | 31.59/20884 |
|                           | Algorithm 6 | 17.73/2920 | 28.52/21850 | 30.25/− | 30.25/− |
|                           | Algorithm 8 | 26.80/478 | 29.98/1490 | 31.02/3889 | 31.64/21790 |
| \( \lambda = 2 \)         | Algorithm 5 | 17.92/3942 | 27.73/21777 | 29.40/− | 29.40/− |
|                           | Algorithm 7 | 26.30/506 | 29.53/1529 | 30.76/4410 | 31.33/20260 |
|                           | Algorithm 6 | 17.62/3073 | 27.94/22010 | 29.79/− | 29.79/− |
|                           | Algorithm 8 | 26.32/487 | 29.72/1552 | 30.73/4134 | 31.35/19732 |
The results in Tables 1 and 2 indicate that preconditioned iterative Algorithm 7 and Algorithm 8 converge faster than iterative Algorithm 5 and Algorithm 6, respectively. The second class of splitting primal-dual proximity algorithms (Algorithm 6 and Algorithm 8) achieve higher SNR values than the first class (Algorithm 5 and Algorithm 7), respectively.

In addition, the results in Tables 1 and 2 show that when the error tolerance decreases, the SNR value increases accordingly; however, this requires a larger number of iterations and is more time consuming. The regularization parameter also has an impact on the performance of these iterative algorithms. A large regularization parameter means that the total variation term is strongly penalized. We found the SNR value to increase as we increased the regularization parameter; however, the SNR value was observed to decrease when the regularization parameter exceeded the value of 2.

A comparison between Tables 1 and 2 revealed that the SNR values of the reconstructed images are very similar for the given regularization parameter level. The reconstructed images are shown in Figure 2 and Figure 3, where the regularization parameter $\lambda = 1.8$ and the tolerance $\epsilon = 10^{-6}$. 

---

**Table 2** Comparison of the performance of Algorithms 5, 6, 7, and 8 in terms of SNR and iteration numbers with box constraints, i.e., $C = \{x|0 \leq x \leq 1\}$

| Regularization Parameter | Methods | $\epsilon = 10^{-3}$ | $\epsilon = 10^{-4}$ | $\epsilon = 10^{-5}$ | $\epsilon = 10^{-6}$ |
|--------------------------|---------|---------------------|---------------------|---------------------|---------------------|
|                          |         | SNR(dB)/k           | SNR(dB)/k           | SNR(dB)/k           | SNR(dB)/k           |
| $\lambda = 0.6$          | Algorithm 5 | 19.20/2899         | 24.86/14335         | 26.05/35501         | 26.15/-             |
|                          | Algorithm 7 | 24.65/380          | 26.65/1208          | 27.23/5966          | 27.31/22942         |
|                          | Algorithm 6 | 19.08/655          | 26.55/12137         | 27.18/28035         | 27.24/-             |
|                          | Algorithm 8 | 25.03/343          | 26.78/1123          | 27.25/4450          | 27.30/16060         |
| $\lambda = 0.8$          | Algorithm 5 | 19.21/2969         | 25.94/15457         | 27.40/37330         | 27.48/-             |
|                          | Algorithm 7 | 25.25/384          | 27.83/1273          | 28.65/5385          | 28.77/22840         |
|                          | Algorithm 6 | 19.28/799          | 27.58/14095         | 28.47/32327         | 28.55/-             |
|                          | Algorithm 8 | 25.53/353          | 28.08/1244          | 28.71/14688         | 28.79/18123         |
| $\lambda = 1.2$          | Algorithm 5 | 19.08/3187         | 27.46/17257         | 29.09/-             | 29.09/-             |
|                          | Algorithm 7 | 25.90/396          | 29.23/1342          | 30.22/14588         | 30.48/22166         |
|                          | Algorithm 6 | 19.00/1132         | 28.61/16171         | 29.80/37411         | 29.84/-             |
|                          | Algorithm 8 | 26.13/354          | 29.30/1309          | 30.22/3899          | 30.50/19944         |
| $\lambda = 1.6$          | Algorithm 5 | 18.79/3346         | 28.00/18658         | 29.78/-             | 29.78/-             |
|                          | Algorithm 7 | 26.11/413          | 29.94/1422          | 30.98/4097          | 31.44/23096         |
|                          | Algorithm 6 | 18.52/1478         | 28.96/17626         | 30.38/-             | 30.38/-             |
|                          | Algorithm 8 | 26.41/378          | 29.97/1378          | 31.00/3790          | 31.45/20929         |
| $\lambda = 1.8$          | Algorithm 5 | 18.42/3366         | 27.91/19126         | 29.78/-             | 29.78/-             |
|                          | Algorithm 7 | 25.89/415          | 29.91/1462          | 31.08/4373          | 31.59/21341         |
|                          | Algorithm 6 | 18.38/1645         | 28.78/18234         | 30.29/-             | 30.29/-             |
|                          | Algorithm 8 | 26.20/392          | 29.92/1419          | 31.03/3922          | 31.63/21140         |
| $\lambda = 2$            | Algorithm 5 | 18.04/3432         | 27.65/19592         | 29.44/-             | 29.44/-             |
|                          | Algorithm 7 | 25.53/422          | 29.52/1488          | 30.76/4380          | 31.33/19802         |
|                          | Algorithm 6 | 18.31/1858         | 28.13/18920         | 29.81/-             | 29.81/-             |
|                          | Algorithm 8 | 25.89/408          | 29.56/1464          | 30.73/4070          | 31.35/20100         |
Figure 2 Reconstructed images obtained from Algorithms 5, 6, 7, and 8 with non-negativity constraints, respectively

Figure 3 Reconstructed images obtained from Algorithms 5, 6, 7, and 8 with box constraints, respectively
7 Conclusions
In this paper, we proposed a splitting primal-dual proximity algorithm to solve the general optimization problem (1.1). As its iterative parameters rely on estimating some operator norm, this may affect its practical use. Thus, we introduced a precondition technique to compute the iterative parameters self-adaptively. Under some mild assumptions, we proved the theoretical convergence of both iterative algorithms. The methods proposed in this paper have been applied to the constrained optimization model (5.1), which has wide application in image restoration and image reconstruction problems. We verified the numerical performance of these iterative algorithms by applying them to CT image reconstruction problems. The numerical results were very promising.

Although we have illustrated the use of our proposed methods in the context of a CT image reconstruction problem, the proposed methods can also be used to solve other application problems such as image deblurring and denoising, and statistical learning problems.

Acknowledgements We thank the editors for their time and comments concerned with this paper.

References
[1] Boyd, S., Parikh, N., Chu, E., et al.: Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends in Machine Learning, 3, 1–122 (2010)
[2] Chambolle, A., Pock, T.: A first-order primal-dual algorithm for convex problems with applications to imaging. J. Math. Imag. Vis., 40, 120–145 (2011)
[3] Chen, P. J., Huang, J. G., Zhang, X. Q.: A primal-dual fixed point algorithm based on proximity operator for convex set constrained separable problem. J. Nanjing Normal University (Natural Science Edition), 36, 1–5 (2013)
[4] Chen, P. J., Huang, J. G., Zhang, X. Q.: A primal-dual fixed point algorithm for convex separable minimization with applications to image restoration. Inverse Prob., 29, 025011 (33pp) (2013)
[5] Combettes, P. L., Pesquet, J. C.: A Douglas–Rachford splitting approach to nonsmooth convex variational signal recovery. IEEE J. Sel. Top. Sign. Proces., 1, 564–574 (2007)
[6] Combettes, P. L., Pesquet, J. C.: Fixed-point algorithm for inverse problems in science and engineering. Proximal Splitting Methods in Signal Processing, Bauschke, H., Burachik, R., Combettes, P., Elser, V., Luke, D., and Wolkowicz, H.: Eds. Springer-Verlag, New York, 2010, 185–212
[7] Combettes, P. L., Pesquet, J. C.: Primal-dual splitting algorithm for solving inclusions with mixtures of composite, Lipschitzian, and parallel-sum type monotone operators. Set-Valued Var. Anal., 20, 307–330 (2012)
[8] Condat, L.: A primal-dual splitting method for convex optimization involving Lipschitzian, proximable and linear composite terms. J. Optim. Theory Appl., 158, 460–479 (2013)
[9] Goldstein, T., Osher, S.: The split Bregman method for ℓ1-regularized problems. SIAM J. Imaging Sci., 2, 323–343 (2009)
[10] Hansen, P., Saxild-Hansen, M.: AIR tools—a MATLAB package of algebraic iterative reconstruction methods. J. Comput. Appl. Math., 236, 2167–2178 (2012)
[11] He, B., Yuan, X.: Convergence analysis of primal-dual algorithms for a saddle-point problem: from contraction perspective. SIAM J. Imaging Sci., 5, 119–149 (2012)
[12] Moreau, J.: Fonctions convexes dues et points proximaux dans un espace Hilbertien. C. R. Acad. Sci., Paris Ser. A Math., 255, 2897–2899 (1962)
[13] Popov, L. D.: A modification of the Arrow–Hurwitz method for search of saddle points. Mathematical notes of the Academy of Sciences of the USSR, 28, 845–848 (1980)
[14] Pock, T., Chambolle, A.: Diagonal preconditioning for first order primal-dual algorithms in convex optimization. IEEE International Conference on Computer Vision (ICCV), IEEE, 1762–1769 (2011)
[15] Rudin, L., Osher, S., Fatemi, E.: Nonlinear total variation based noise removal algorithms. *Phys. D*, 60, 259–268 (1992)

[16] Setzer, S.: Split Bregman algorithms, Douglas–Rachford splitting and frame shrinkage. *Scale Space and Variational Methods in Computer Vision*, Tai, X., Morken, K., Lysaker, M., Lie, K.: Eds. Springer-Verlag, 5567, 464–476 (2009)

[17] Setzer, S., Steidl, G., Teuber, T.: Deblurring Poissonian images by split Bregman techniques. *J. Vis. Comm. Image Rep.*, 21, 193–199 (2010)

[18] Setzer, S.: Operator splittings, Bregman methods and frame shrinkage in image processing. *Int. J. Comput. Vis.*, 92, 265–280 (2011)

[19] Sidky, E., Jørgensen, J., Pan, X.: Convex optimization problem prototyping for image reconstruction in computed tomography with the Chambolle–Pock algorithm. *Phys. Med. Biol.*, 57, 3065–3091 (2012)

[20] Tang, Y.: A primal dual fixed point algorithm for constrained optimization problems with applications to image reconstruction. *Proc. SPIE 9413, Medical Imaging 2015: Image Processing*, 94131W, 1–9 (2015)

[21] Tang, Y., Cai, Y., Wang, X., et al.: A primal dual proximal point method of Chambolle–Pock algorithm for total variation image reconstruction. 2013 *IEEE International Conference on Medical Imaging Physics and Engineering* (ICMIPE), IEEE, 6–10 (2013)

[22] Tang, Y., Peng, J., Yue, S., et al.: A primal dual proximal point method of Chambolle–Pock algorithms for $\ell_1$-tv minimization problems in image reconstruction. 2012 5-th *International Conference on Biomedical Engineering and Informatics* (BMEI), IEEE, 12–16 (2012)

[23] Zhang, X., Burger, M., Osher, S.: A unified primal-dual framework based on Bregman iteration. *J. Sci. Comput.*, 46, 20–46 (2011)