Some remarks on the Balog–Wooley decomposition theorem and
quantities $D^+, D^\times$ *

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Annotation.

In the paper we study two characteristics $D^+(A), D^\times(A)$ of a set $A \subset \mathbb{R}$ which play important role in recent results concerning sum–product phenomenon. Also we obtain several variants and improvements of the Balog–Wooley decomposition theorem. In particular, we prove that any finite subset of $\mathbb{R}$ can be split into two sets with small quantities $D^+$ and $D^\times$.

1 Introduction

Let $A, B \subset \mathbb{R}$ be finite sets. Define the sum set, the product set and quotient set of $A$ and $B$ as

$$A + B := \{a + b : a \in A, b \in B\},$$
$$AB := \{ab : a \in A, b \in B\},$$
and

$$A/B := \{a/b : a \in A, b \in B, b \neq 0\},$$
correspondingly. The Erdős–Szemerédi conjecture [2] says that for any $\epsilon > 0$ one has

$$\max\{|A + A|, |AA|\} \gg |A|^{2-\epsilon}.$$ 

Roughly speaking, it asserts that an arbitrary subset of real numbers (or integers) cannot has good additive and multiplicative structure, simultaneously. Modern bounds concerning the conjecture can be found in [12], [4], [5].

Define

$$E^+(A, B) := |\{a + b = a' + b' : a, a' \in A, b, b' \in B\}|,$$
and

$$E^\times(A, B) := |\{ab = a'b' : a, a' \in A, b, b' \in B\}|$$
be the additive and the multiplicative common energies of $A$ and $B$, correspondingly. Numbers $E^+(A, A), E^\times(A, A)$ are another measures to control the additivity and the multiplicativity of a set.

In [1] the following decomposition theorem was proved.

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**Theorem 1** Let $A \subset \mathbb{R}$ be a finite set and $\delta = 2/33$. Then there are two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and
\[
\max\{E^+(B, B), E^x(C, C)\} \ll |A|^{3-\delta} (\log |A|)^{1-\delta}
\]
and
\[
\max\{E^+(B, C), E^x(B, C)\} \ll |A|^{3-\delta/2} (\log |A|)^{(1-\delta)/2}.
\]

Here and below we suppose that $|A| \geq 2$. All logarithms are base 2. Signs $\ll$ and $\gg$ are the usual Vinogradov’s symbols. We will write $a \lesssim b$ or $b \gtrsim a$ if $a = O(b \cdot \log^c |A|)$, $c > 0$. If $a \lesssim b$ and $b \lesssim a$ then we write $a \sim b$.

The results of such type are useful, see e.g. [3] and will find further applications by the author’s opinion. Also it was proved in [1] that one cannot take $\delta$ greater than $2/3$.

In [5] a different method was applied and it allows to obtain an improvement.

**Theorem 2** Let $A \subset \mathbb{R}$ be a finite set and $\delta = 1/5$. Then there are two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and
\[
\max\{E^+(B, B), E^x(C, C)\} \ll |A|^{3-\delta}.
\]

Actually, a more stronger result takes place. Write
\[
E^+_3(A) = \{|a_1 - a'_1 = a_2 - a'_2 = a_3 - a'_3 : a_1, a'_1, a_2, a'_2, a_3, a'_3 \in A\} \]
and similar for $E^x_3(A)$. Using the fact that $(E^+_3(f))^{1/6}$, $(E^x_3(f))^{1/6}$ are norms of a real function $f$ (more precisely see section [1]), we obtain

**Theorem 3** Let $A \subset \mathbb{R}$ be a finite set and $\delta_1 = 2/5$. Then there are two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and
\[
\max\{E^+_3(B), E^x_3(C)\} \ll |A|^{4-\delta_1},
\]

Besides, inequality (3) cannot holds with $\delta_1$ greater than $3/4$.

Using the Hölder inequality
\[
(E^+(A, A))^2 \leq E^+_3(A)|A|^2, \quad (E^x(A, A))^2 \leq E^x_3(A)|A|^2,
\]
it is easy to see that Theorem 3 implies Theorem 2.

Actually, our proof allows to say much more about the sets $B, C$ than is written in Theorem 3. We consider two quantities $D^+, D^x$ (see the definitions in section 3) and prove the strongest decomposition result.
Theorem 4 Let $A \subset \mathbb{R}$ be a finite set and $\delta_2 = 2/5$. Then there are two disjoint subsets $B$ and $C$ of $A$ such that $A = B \sqcup C$ and
\[
\max\{ D^+(B), D^-(C) \} \lesssim |A|^{1-\delta_2}.
\]
Besides, inequality (4) cannot holds with $\delta_2$ greater than $3/4$.

It is easy to check that Theorem 4 implies both Theorem 2, 3 (see discussion in section 5).

The quantities $D^+, D^-$ play an important role in additive combinatorics, see e.g. [10], [4], [5]. For example, studying the characteristics of a set allows us to improve the famous Solymosi 4/3 result, see [12].

Also, in section 5 we obtain several other forms of the Balog–Wooley Theorem, study quantities $D^+(A), D^-(A)$ and find some applications to sum–product questions, see e.g. Theorem 23 below.

We are going to obtain similar results in $\mathbb{F}_p$ in a forthcoming paper.

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2 Notation

Let $G$ be an abelian group. If $G$ is finite then denote by $N$ the cardinality of $G$. It is well-known [7] that the dual group $\hat{G}$ is isomorphic to $G$ in the case. Let $f$ be a function from $G$ to $\mathbb{C}$. We denote the Fourier transform of $f$ by $\hat{f}$,
\[
\hat{f}(\xi) = \sum_{x \in G} f(x)e(-\xi \cdot x),
\]
where $e(x) = e^{2\pi ix}$ and $\xi$ is a homomorphism from $\hat{G}$ to $\mathbb{R}/\mathbb{Z}$ acting as $\xi : x \to \xi \cdot x$. We rely on the following basic identities
\[
\sum_{x \in G} \left| f(x) \right|^2 = \frac{1}{N} \sum_{\xi \in \hat{G}} \left| \hat{f}(\xi) \right|^2,
\]
\[
\sum_{y \in G} \left| \sum_{x \in G} f(x)g(y-x) \right|^2 = \frac{1}{N} \sum_{\xi \in \hat{G}} \left| \hat{f}(\xi) \right|^2 \left| \hat{g}(\xi) \right|^2,
\]
and
\[
f(x) = \frac{1}{N} \sum_{\xi \in \hat{G}} \hat{f}(\xi)e(\xi \cdot x).
\]
If
\[
(f * g)(x) := \sum_{y \in G} f(y)g(x-y) \quad \text{and} \quad (f \circ g)(x) := \sum_{y \in G} f(y)g(y+x)
\]
then
\[
\hat{f} * \hat{g} = \hat{f} \hat{g} \quad \text{and} \quad \hat{f} \circ \hat{g} = \hat{f} \ast \hat{g} = \hat{f} \hat{g},
\]
where for a function $f : G \to \mathbb{C}$ we put $f^c(x) := f(-x)$. Clearly, $(f * g)(x) = (g * f)(x)$ and $(f \circ g)(x) = (g \circ f)(-x)$, $x \in G$. The $k$–fold convolution, $k \in \mathbb{N}$ we denote by $*^k$, so $^k := (*_{k-1})$. 


In the paper we use the same letter to denote a set \( S \subseteq G \) and its characteristic function \( S : G \to \{0, 1\} \). By \(|S|\) denote the cardinality of \( S \). For a positive integer \( n \), we set \([n] = \{1, \ldots, n\}\).

Put \( E^+(A, B) \) for the additive energy of two sets \( A, B \subseteq G \) (see e.g. [13]), that is

\[
E^+(A, B) = \{|a_1 + b_1 = a_2 + b_2 : a_1, a_2 \in A, b_1, b_2 \in B\|.
\]

If \( A = B \) we simply write \( E^+(A) \) instead of \( E^+(A, A) \). Clearly,

\[
E^+(A, B) = \sum_x (A * B)(x)^2 = \sum_x (A \circ B)(x)^2 = \sum_x (A \circ A)(x)(B \circ B)(x).
\]

Note also that

\[
E^+(A, B) \leq \min\{|A|^2|B|, |B|^2|A|, |A|^{3/2}|B|^{3/2}\}, \tag{11}
\]

and by the Cauchy–Schwarz inequality

\[
E^+(A, B) \geq \frac{|A|^2|B|^2}{|A + B|}. \tag{12}
\]

In the same way define the multiplicative energy of two sets \( A, B \subseteq G \)

\[
E^\times(A, B) = \{|a_1 b_1 = a_2 b_2 : a_1, a_2 \in A, b_1, b_2 \in B\|.
\]

Certainly, multiplicative energy \( E^\times(A, B) \) can be expressed in terms of multiplicative convolutions, similar to (9). Let also

\[
\sigma_k(A) := (A *_k A)(0) = \{|a_1 + \cdots + a_k = 0 : a_1, \ldots, a_k \in A\|.
\]

Now more generally, let

\[
E_k(A) = \sum_{x \in G} (A \circ A)(x)^k = \sum_{x \in G} |A_x|^k = \{|a_1 - a'_1 = \cdots = a_k - a'_k : a_j, a'_j \in A\|; \tag{13}
\]

and

\[
E_k(A, B) = \sum_{x \in G} (A \circ B)^k(x) = \{|a_1 - b_1 = \cdots = a_k - b_k : a_j \in A, b_j \in B\| \tag{14}
\]

be the higher energies of \( A \) and \( B \), see [8]. Here \( A_x = A \cap (A - x), \ x \in A - A \). Similarly, we write \( E_k(f, g) = \sum_x (\mathcal{J} \circ g)^k(x) \) for any complex functions \( f, g \) and even more generally, we consider

\[
E_k(f_1, \ldots, f_k) = \sum_x (\mathcal{J}_1 \circ f_1)(x) \cdots (\mathcal{J}_k \circ f_k)(x).
\]

There is a simple connection between \( E_k \) and \( E_2 \) energies. Indeed,

\[
E_{k+1}(A) = E(\Delta^k(A), A^k), \tag{15}
\]

where

\[
\Delta(A) = \Delta_k(A) := \{(a, a, \ldots, a) \in A^k\}.
\]
Identity (15) allows us to use lower bound (12) to estimate higher energies $E_k^+$, $E_k^-$. In particular,

$$|A|^6 \leq E_3^+(A)|A^2 \pm \Delta(A)|$$  \hspace{1cm} (16)

and similar for $E_3^-(A)$.

Quantities $E_k(A, B)$ can be written in terms of generalized convolutions. Let $f_1, \ldots, f_{k+1} : G \to \mathbb{C}$ be functions and put $F = (f_1, \ldots, f_{k+1}) : G^{k+1} \to \mathbb{C}$. Denote by

$$C_k(F)(x) = C_{k+1}(f_1, \ldots, f_{k+1})(x_1, \ldots, x_k)$$

the function

$$C_k(F)(x) = C_{k+1}(f_1, \ldots, f_{k+1})(x_1, \ldots, x_k) = \sum_x f_1(z)f_2(z + x_1) \cdots f_{k+1}(z + x_k).$$

Thus, $C_2(f_1, f_2)(x) = (f_1 \circ f_2)(x)$. If $f_1 = \cdots = f_{k+1} = f$ then write $C_{k+1}(f)(x_1, \ldots, x_k)$ for $C_{k+1}(f, \ldots, f)(x_1, \ldots, x_k)$.

Then (see Lemma 5 below) the following holds

$$E_{k+1}(A, B) = \sum_{x_1, \ldots, x_k} C_{k+1}(A)(x_1, \ldots, x_k)C_{k+1}(B)(x_1, \ldots, x_k).$$

### 3 Preliminaries

We begin with a lemma from [9] concerning ”commutativity” of generalized convolution.

**Lemma 5** For any functions $f_i, g_j : G \to \mathbb{C}$ the following holds

$$\sum_{x_1, \ldots, x_{l-1}} C_l(f_0, \ldots, f_{l-1})(x_1, \ldots, x_{l-1}) C_l(g_0, \ldots, g_{l-1})(x_1, \ldots, x_{l-1}) =$$

$$= \sum_z (f_0 \circ g_0)(z) \cdots (f_{l-1} \circ g_{l-1})(z) \hspace{1cm} \text{(scalar product),}$$  \hspace{1cm} (17)

moreover

$$\sum_{x_1, \ldots, x_{l-1}} C_l(f_0)(x_1, \ldots, x_{l-1}) \cdots C_l(f_{k-1})(x_1, \ldots, x_{l-1}) =$$

$$= \sum_{y_1, \ldots, y_{k-1}} C_{k-1}(f_0, \ldots, f_{k-1})(y_1, \ldots, y_{k-1}) \hspace{1cm} \text{(multi–scalar product),}$$  \hspace{1cm} (18)

and

$$\sum_{x_1, \ldots, x_{l-1}} C_l(f_0)(x_1, \ldots, x_{l-1}) (C_l(f_1) \circ \cdots \circ C_l(f_{k-1}))(x_1, \ldots, x_{l-1}) =$$

$$= \sum_z (f_0 \circ \cdots \circ f_{k-1})(z) \hspace{1cm} (\sigma_k \text{ for } C_l).$$  \hspace{1cm} (19)
The next lemma shows that "higher sum sets" can be expressed in terms of ordinary sums, see e.g. [8].

**Lemma 6** Let \( A \subseteq G \) be a set. Then

\[
|A^2 \pm \Delta(A)| = \sum_{x \in A-A} |A \pm Ax|.
\]

The main objects of the paper are two quantities \( D^+(A), D^\times(A) \). Let us recall the definitions.

**Definition 7** A finite set \( A \subset \mathbb{R} \) is said to be of additive Szemerédi–Trotter type with a parameter \( D^+(A) > 0 \) if the inequality

\[
\left| \{ s \in A - B \mid |A \cap (B + s)| \geq \tau \} \right| \leq \frac{D^+(A)|A||B|^2}{\tau^3},
\]

holds for every finite set \( B \subset \mathbb{R} \) and every real number \( \tau \geq 1 \).

The quantity \( D^+(A) \) can be considered as the infimum of numbers \( D \) such that (21) takes place for any \( B \) and \( \tau \geq 1 \) but, of course, the definition is applicable just for sets \( A \) with small quantity \( D^+(A) \). It is easy to see that \( D^+(A) \leq |A| \), so \( |A| \) can be considered as a trivial upper bound for the quantity. Note also that \( D^+(A) \geq 1 \) (just take \( B \) equals any one–element set and substitute \( \tau = 1 \) into formula (21)).

Any SzT–type set has small number of solutions of a wide class of linear equations, see e.g. Corollary 8 from [4] (where nevertheless another quantity \( D^+(A) \) was used) and Lemmas 7, 8 from [10], say.

**Lemma 8** Let \( A_1, A_2 \subset \mathbb{R} \) be any finite sets. Then

\[
E^+(A_1, A_2) \ll (D^+(A_1))^{1/2}|A_1||A_2|^{3/2},
\]

and

\[
E^+(A_1, A_2) \ll D^+(A_1)|A_1||A_2|^{2} \cdot \log(\min\{|A_1|, |A_2|\}).
\]

Similarly, consider a dual characteristic of a set of real numbers.

**Definition 9** A finite set \( A \subset \mathbb{R} \) is said to be of multiplicative Szemerédi–Trotter type with a parameter \( D^\times(A) > 0 \) if the inequality

\[
\left| \{ s \in A/B \mid |A \cap sB| \geq \tau \} \right| \leq \frac{D^\times(A)|A||B|^2}{\tau^3},
\]

holds for every finite set \( B \subset \mathbb{R} \) and every real number \( \tau \geq 1 \).
Of course a multiplicative analog of Lemma 8 takes place.

**Lemma 10** Let $A_1, A_2 \subset \mathbb{R}$ be any finite sets. Then

$$E^x(A_1, A_2) \ll (D^x(A_1))^{1/2} |A_1| |A_2|^{3/2},$$

and

$$E^y(A_1, A_2) \ll D^y(A_1)|A_1| |A_2|^{2} \cdot \log(\min\{|A_1|, |A_2|\}).$$

### 4 Norms $E_k$

For any function $f : G \to \mathbb{C}$ and an arbitrary integer $k \geq 1$ put

$$\|f\|_{E_k} := (E_k(f))^{1/2k} = \left(\frac{1}{|G|} \sum_{x} (f \circ f)(x)^k\right)^{1/2k}. \quad (23)$$

By formula (10), we get

$$\|f\|_{E_k}^{2k} = N^{-(k-1)} \sum_{x_1 + \cdots + x_k = 0} |\hat{f}(x_1)|^2 \cdots |\hat{f}(x_k)|^2 \quad (24)$$

and hence the expression is nonnegative. Another way is to think about $\|f\|_{E_k}$ is to note that by formula (17) of Lemma 5, we have

$$\|f\|_{E_k}^{2k} = \sum_{x_1, \ldots, x_{k-1}} |C_k(f)(x_1, \ldots, x_{k-1})|^2. \quad (25)$$

Note that there are nonzero functions $f$ with $\|f\|_{E_k} = 0$, e.g. $G = \mathbb{F}_p$, $p$ is a prime number, $k < p$ and $f(x) = e^{2\pi ix/p}$. If we restrict ourselves to consider just real functions then again it is possible to find nonzero functions $f$ with $\|f\|_{E_k} = 0$.

**Example 11** Let $G = \mathbb{F}_2^n$ and $f(x) = f(x_1, \ldots, x_n) = (-1)^{x_1 + \cdots + x_n}$. Then $\hat{f}(r) = 0$ for any $r \neq (1, \ldots, 1)$. Thus by formula (24), we have $\|f\|_{E_k} = 0$ for all odd $k$.

If $k \geq 2$ is even then the last situation is not possible.

**Lemma 12** Let $f : G \to \mathbb{R}$ be a function and $k \geq 2$ be an even number. Then $\|f\|_{E_k} = 0$ iff $f \equiv 0$.

**Proof.** We give even three proofs. The first one uses Fourier analysis and another two do not. Applying formula (24) we see that

$$|\hat{f}(x_1)|^2 \cdots |\hat{f}(x_k)|^2 = 0 \quad (26)$$

for all $x_1, \ldots, x_k \in G$ such that $x_1 + \cdots + x_k = 0$. By the assumption $f$ is a real function, thus $\hat{f}(-x) = \hat{f}(x)$. Using the fact and substitute variables $x_1 = x$, $x_2 = -x$, $x_3 = x$, $x_4 = -x$, $\ldots$, $x_{k-1} = x$, $x_k = x$ into formula (26), we obtain $|\hat{f}(x)|^{2k} = 0$ for every $x$ and hence $f \equiv 0$. 


Lemma 13 For any \( k \geq 1 \) and arbitrary functions \( f_1, f_1', \ldots, f_k, f_k' : \mathbb{C} \rightarrow \mathbb{C} \) the following holds
\[
\left| \sum_x (f_1 \circ f_1')(x) \ldots (f_k \circ f_k')(x) \right| \leq \prod_{j=1}^k ||f_j||_{E_k} ||f_j'||_{E_k}.
\] (27)

If \( k \geq 2 \) is even and all functions are real then one can remove modulus from formula (27).

Proof. By formula (17) of Lemma 13 we have
\[
\sigma := \sum_x (f_1 \circ f_1')(x) \ldots (f_k \circ f_k')(x) = \sum_{x_1, \ldots, x_k} C_{k+1}(f_1, \ldots, f_k)(x_1, \ldots, x_k)C_{k+1}(f_1', \ldots, f_k')(x_1, \ldots, x_k).
\]
Using the Cauchy–Schwarz inequality and formula (17) of Lemma 13 again, we obtain
\[
|\sigma|^2 \leq \sum_{x_1, \ldots, x_k} |C_{k+1}(f_1, \ldots, f_k)(x_1, \ldots, x_k)|^2 \cdot \sum_{x_1, \ldots, x_k} |C_{k+1}(f_1', \ldots, f_k')(x_1, \ldots, x_k)|^2
\]
\[
= \sum_x (f_1 \circ f_1')(x) \ldots (f_k \circ f_k')(x) \cdot \sum_x (f_1' \circ f_1')(x) \ldots (f_k' \circ f_k')(x).
\]
By the Hölder inequality it is sufficient to prove that
\[
|\sigma| = \sum_x |(f \circ f')(x)|^k \leq \sum_x (|f| \circ |f|)(x)^k = ||f||_{E_k}^{2k}
\]
for any function \( f : \mathbb{C} \rightarrow \mathbb{C} \). If \( k \) is even and \( f \) is a real function then we need to check that
\[
\sigma = \sum_x (f \circ f(x))^k = ||f||_{E_k}^{2k}.
\]
The last two formulas coincide with the definition of the norm \( E_k \). This completes the proof. \( \square \)

Example 14 Let \( \mathbb{G} = \mathbb{F}_2^n \), \( f_1(x) = f_\lambda(x) = f_\lambda(x_1, \ldots, x_n) = (-1)^{\lambda_1 x_1 + \cdots + \lambda_n x_n} \), \( \lambda = (\lambda_1, \ldots, \lambda_n) \) and similar \( f_2(x) = f_\lambda'(x) \), \( f_3(x) = f_\lambda''(x) \). Take \( \lambda' \), \( \lambda'' \), \( \lambda''' \) be three nonzero vectors such that \( \lambda' + \lambda'' + \lambda''' = 0 \). Then by simple calculations, we get
\[
\sum_x (f_1 \circ f_1)(x)(f_2 \circ f_2)(x)(f_3 \circ f_3)(x) = 2^{4n}
\]
but for any \( j = 1, 2, 3 \) one has \( \sum_x (f_j \circ f_j)^3(x) = 0 \). Thus (27) does not hold without modulus in the case of odd \( k \) and \( f_j \) are real functions.
We need a combinatorial lemma. Let \( l \) be a positive integer and let \( \Omega_l = \{0,1\}^l \). For any \( \varepsilon \in \Omega_l \) put \( \text{wt}(\varepsilon) \) equals the number of ones in \( \varepsilon \). Finally, given numbers \( k_1, \ldots, k_s \) such that \( k_1 + \cdots + k_s = k \) write \( \binom{k}{k_1, \ldots, k_s} \) for \( \frac{k!}{k_1! \cdots k_s!} \).

**Lemma 15** Let \( n, k, l \) be positive integers, \( n \leq lk \). Then
\[
\sum_{\varepsilon \in \Omega_l} \sum_{n_\varepsilon = k, \sum \text{wt}(\varepsilon)n_{\varepsilon} = n} \frac{k!}{n!(lk-n)!} = \frac{(lk)!}{n!(lk-n)!}.
\]

In particular,
\[
\sum_{n_1 + n_2 + n_3 + n_4 = k, 2n_1 + n_2 + n_3 = n} \frac{k!}{n_1!n_2!n_3!n_4!} = \frac{(2k)!}{n!(2k-n)!}.
\]

**Proof.** One has
\[
\sum_n \binom{lk}{n} x^n = \left(1+x\right)^{lk} = \left(\sum_{i=0}^l \binom{l}{i} x^i\right)^k = \sum_{m_0+m_1+\cdots+m_l = k} \sum_{r=0}^l \frac{k!}{m_0!m_1!\cdots m_l!} \prod_{r=0}^l \binom{l}{r}^{m_r}.
\]

It follows that
\[
\binom{lk}{n} = \sum_{m_0+m_1+\cdots+m_l = k} \sum_{r=0}^l \frac{k!}{m_0!m_1!\cdots m_l!} \prod_{r=0}^l \binom{l}{r}^{m_r} = \sum_{m_0+m_1+\cdots+m_l = k} \sum_{r=0}^l \frac{k!}{m_0!m_1!\cdots m_l!} \prod_{r=0}^l \binom{l}{r}^{m_r}.
\]

Fix \( r \in \{0, \ldots, l\} \) and redenote \( \binom{l}{r} \) variables \( s^{(i)}_r \) by \( n_{\varepsilon}, \varepsilon \in \Omega_l \) such that \( \text{wt}(\varepsilon) = r \). Hence we have redenoted all \( 2^l \) variables \( s^{(i)}_r \), \( r = 0, \ldots, l \) as \( n_{\varepsilon}, \varepsilon \in \Omega_l \). Note that \( \sum_{\varepsilon \in \Omega_l} n_{\varepsilon} = m_0 + m_1 + \cdots + m_l = k \). Further, by our choice of enumeration of \( n_{\varepsilon} \), we get \( \sum_{\varepsilon \in \Omega_l} \text{wt}(\varepsilon)n_{\varepsilon} = \sum_{j=1}^l jm_j = n \).

Thus, we obtain
\[
\binom{lk}{n} = \sum_{\varepsilon \in \Omega_l, n_{\varepsilon} = k, \sum_{\varepsilon \in \Omega_l} \text{wt}(\varepsilon)n_{\varepsilon} = n} \frac{k!}{n!(lk-n)!},
\]

as required. \( \square \)

Using the lemmas above we are ready to prove the main result of the section.

**Proposition 16** Let \( k \geq 2 \) be an integer. Then for any pair of functions \( f, g : G \to C \) the following holds
\[
\|f + g\|_{E_k} \leq \|f\|_{E_k} + \|g\|_{E_k}.
\]

If we consider just real functions and \( k \) is even then \( \| \cdot \|_{E_k} \) is a norm.
Proof. We have
\[ \sigma := \|f + g\|_{E_k}^{2k} = \sum_x ((\overline{f} \circ f)(x) + (\overline{f} \circ g)(x) + (\overline{g} \circ f)(x) + (\overline{g} \circ g)(x))^k = \]
\[ = \sum_{n_1+n_2+n_3+n_4=k} \binom{k}{n_1,n_2,n_3,n_4} \sum_x (f \circ f)^{n_1}(x)(f \circ g)^{n_2}(x)(g \circ f)^{n_3}(x)(g \circ g)^{n_4}(x). \]
Using Lemma [13] and Lemma [15] in the case \( l = 2 \), we get
\[ \sigma \leq \sum n_1+n_2+n_3+n_4=k \binom{k}{n_1,n_2,n_3,n_4} (\|f\|_{E_k}^{2n_1+n_2+n_3}\|g\|_{E_k}^{n_2+n_3+2n_4} = \]
\[ = \sum_{n+m=2k} \frac{(2k)!}{n!m!} \|f\|_{E_k}^n \|g\|_{E_k}^m = (\|f\|_{E_k} + \|g\|_{E_k})^{2k} \]
as required.

If \( f, g \) are real functions and \( k \) is even then we apply the second part of Lemma [13] and obtain \( \|f + g\|_{E_k} \leq \|f\|_{E_k} + \|g\|_{E_k} \). Also Lemma [12] says that \( \|f\|_{E_k} = 0 \) iff \( f \equiv 0 \) in the case. This completes the proof.

\[ \square \]

5 The proof of Theorem 4

In the next two sections we prove Theorems [3] [4]. We begin with the stronger Theorem [4] and after that use similar arguments, combining with the results of section [4] to get Theorem [3].

First of all express quantities \( D^+(A), D^\times(A) \) in terms of the energies \( E^+(A, B), E^\times(A, B) \). Consider the case of \( D^+(A) \) the second variant is similar. For any finite set \( A \subseteq \mathbb{R} \) put
\[ q^+(A) := \max_{B \neq \emptyset} \frac{E^+(A, B)}{|A||B|^2}. \]  
(30)

It is easy to see that the maximum in (30) is attained. Indeed, shifting we can suppose that \( 0 \in B \), further the size of \( B \) is bounded in terms of \( |A| \) and by Lemma [5] one has
\[ E^+(A, B) = \sum_{(x,y) \in A^2 - \Delta(A)} C_3(B)(x,y)C_3(A)(x,y). \]  
(31)

Whence by induction we show that \( B \subseteq k(A - A) \), where \( k \) is bounded in terms of \( |A| \). Thus the maximum in (30) is attained.

Let us give some simple bounds for \( q^+(A) \). In view of (11), (13), we have, clearly,
\[ 1 \leq \frac{E^+(A)}{|A|^3} \leq q^+(A) \leq |A|. \]  
(32)

More precisely, by formulas (11), (31) and the Cauchy–Schwarz inequality, one has
\[ q^+(A) \leq \frac{(E^+(A))^{1/2}}{|A|} \leq |A|. \]  
(33)

Now we are ready to show that \( D^+(A) \) is proportional to \( q^+(A) \) up to logarithms.
Lemma 17 Let $A \subset \mathbb{R}$ be a finite set. Then
\[
q^+(A) \lesssim D^+(A) \leq q^+(A).
\] (34)

**Proof.** The lower bound in (34) immediately follows from Lemma 8. Now suppose that for some $B \subset \mathbb{R}$ and \(\tau \geq 1\) the following holds
\[
D^+(A) |A||B|^2 = \tau^3 |S_{\tau}(A, B)|,
\]
where
\[
S_{\tau}(A, B) := \{ s \in A - B : |A \cap (B + s)| \geq \tau \}.
\]
Hence
\[
E^+_3(A, B) = \sum_s (A \circ B)^3(s) \geq \sum_{s \in S_{\tau}(A, B)} (A \circ B)^3(s) \geq \tau^3 |S_{\tau}(A, B)| = |A||B|^2 D^+(A).
\]
The last formula implies that \(q^+(A) \geq D^+(A)\). This completes the proof. \(\square\)

Lemma above, combining with Lemmas 8, 10 and inequality (33), implies that there is a close connection between quantities \(D^+(A), D^\times(A)\) and \(E^+_3(A), E^\times_3(A)\), correspondingly.

Corollary 18 Let $A \subset \mathbb{R}$ be a finite set. Then
\[
|A|^2 (D^+(A))^2 \leq E^+_3(A) \lesssim D^+(A)|A|^3.
\]
The same holds for $D^\times(A)$.

Secondly, let us note that the quantities $D^+$, $D^\times$ have some kind of "subadditive" property. Actually, our results hold in a general abelian group $G$.

Lemma 19 Let $A_1, \ldots, A_k \subset \mathbb{R}$ be finite sets. Then
\[
D(A_1 \cup \cdots \cup A_k) \lesssim k^3 \cdot \frac{1}{|A|} \sum_{j=1}^k D(A_j)|A_j|.
\]
where $D = D^+$ or $D^\times$.

**Proof.** Let us consider the case of $D^\times$, the situation with $D^+$ is similar. We give even two proofs. Put $A = A_1 \cup \cdots \cup A_k$. Take any set $G \subset \mathbb{R}$ and a real number \(\tau \geq 1\). We need to estimate the size of the set
\[
S_\tau = S_\tau(A, G) := \{ s \in A/G : |A \cap sG| \geq \tau \}.
\]
For any $s \in S_\tau$ one has
\[
\tau \leq |A \cap sG| \leq \sum_{j=1}^k |A_j \cap sG| \lesssim \sum_{j \in \Omega_\Delta(s)} |A_j \cap sG| \leq 2\Delta(s)|\Omega_\Delta(s)|,
\] (35)
where we write $\Delta(s)$ for some number of the form $2^j$, $j \geq 0$ such that

$$
\Omega_\Delta(s) = \{j : \Delta(s) < |A_j \cap sG| \leq 2\Delta(s)\} \subseteq [k]
$$

and $\tau k^{-1} \lesssim \Delta(s)$. The existence of the number $\Delta(s)$ and the set $\Omega_\Delta(s)$ follows from the pigeonhole principle. Using the pigeonhole principle once more, we find a set $S' \subseteq S_\tau$ such that $|S'| \approx |S_\tau|$, further, the number $\Delta$ with

$$
\Omega_\Delta(s) = \{j : \Delta < |A_j \cap sG| \leq 2\Delta\} \subseteq [k],
$$

and such that for any $s \in S'$, we have an analog of (5), namely,

$$
\tau \leq |A \cap sG| \lesssim \Delta |\Omega_\Delta(s)|.
$$

By our construction, we have $S' \subseteq \bigcup_{j=1}^{k} S_\Delta(A_j, G)$. In view of (36), we obtain

$$
|S_\tau| \approx |S'| \leq \sum_{j=1}^{k} |S_\Delta(A_j, G)| \leq \frac{|G|^2}{A^3} \sum_{j=1}^{k} D(A_j)|A_j| \lesssim \frac{|G|^2 k^3}{A^3} \sum_{j=1}^{k} D(A_j)|A_j| =
$$

$$
= \frac{|G|^2 |A| k^3}{A^3} \cdot \frac{1}{|A|} \sum_{j=1}^{k} D(A_j)|A_j|.
$$

By the definition it means that $D(A) \lesssim \frac{k^3}{|A|} \sum_{j=1}^{k} D(A_j)|A_j|$ as required.

Now let us give another proof via quantity $q$. Applying the Hölder inequality and Lemmas 8, 10, we have

$$
E_3(\bigcup_{i=1}^{k} A_i, B) \leq \sum_{x} \left(\sum_{i=1}^{k} (A_i \circ B)^3(x)\right)^{1/3} =
$$

$$
= \sum_{i_1, i_2, i_3=1}^{k} \sum_{x} (A_{i_1} \circ B)(x)(A_{i_2} \circ B)(x)(A_{i_3} \circ B)(x) \leq \left(\sum_{i=1}^{k} \left(\sum_{x} (A_i \circ B)^3(x)\right)^{1/3}\right)^{3} \leq
$$

$$
\leq k^3 \sum_{i=1}^{k} \sum_{x} (A_i \circ B)^3(x) \lesssim k^3 |A||B|^2 \cdot \frac{1}{|A|} \sum_{j=1}^{k} D(A_j)|A_j|.
$$

Thus, by Lemma 17, we obtain $D(A) \leq q(A) \lesssim \frac{k^3}{|A|} \sum_{j=1}^{k} D(A_j)|A_j|$, where $q$ equals $q^+$ or $q^\times$, correspondingly to $D$. This completes the proof. \hfill \Box

In [5] we considered two another characteristics of $A$. Put

$$
\text{Sym}^x_t(Q, R) = \{x : |Q \cap xR^{-1}| \geq t\},
$$
and
\[
d^+(A) = \min_{t > 0} \min_{\emptyset \neq Q, R \subset \mathbb{R}\{0\}} \frac{|Q|^2|R|^2}{|A|^3},
\]
where the second minimum in (37) is taken over any \(Q, R\) such that \(A \subseteq \text{Sym}_t^+(Q, R)\) and \(\max\{|Q|, |R|\} \geq |A|\).

Similarly, for any sets \(Q, R\) and a real number \(t > 0\) put
\[
\text{Sym}_t^+(Q, R) := \{x : |Q \cap (x - R)| \geq t\}
\]
and consider the following quantity
\[
d^\times(A) := \min_{t > 0} \min_{\emptyset \neq Q, R \subset \mathbb{R}\{0\}} \frac{|Q|^2|R|^2}{|A|^3},
\]
where the second minimum in (38) is taken over any \(Q, R\) such that \(A \subseteq \text{Sym}_t^+(Q, R)\) and \(\max\{|Q|, |R|\} \geq |A|\). It is easy to see [5] that \(1 \leq d^+(A), d^\times(A) \leq |A|\).

Lemma 20 Let \(A \subset \mathbb{R}\) be a finite set. Then \(A\) is of additive Szemerédi–Trotter type with \(O(d^+(A))\) and \(A\) is of multiplicative Szemerédi–Trotter type with \(O(d^\times(A))\).

Now we can formulate a new result, which implies Theorem 2 with \(\delta = 1/5\) if one combines Theorem 21 below with Lemmas 8, 10.

Theorem 21 Let \(A \subset \mathbb{R}\) be a finite set and \(\delta = 2/5\). Then there are two disjoint subsets \(B\) and \(C\) of \(A\) such that \(A = B \cup C\) and
\[
\max\{D^+(B), D^\times(C)\} \lesssim |A|^{1-\delta}.
\]

Proof. Let \(1 \leq M \leq |A|\) be a parameter which we will choose later. Our arguments is a sort of an algorithm. We construct a decreasing sequence of sets \(C_1 = A \supseteq C_2 \supseteq \cdots \supseteq C_k\) and an increasing sequence of sets \(B_0 = \emptyset \subseteq B_1 \subseteq \cdots \subseteq B_{k-1} \subseteq A\) such that for any \(j = 1, 2, \ldots, k\) the sets \(C_j\) and \(B_{j-1}\) are disjoint and moreover \(A = C_j \cup B_{j-1}\). If at some step \(j\) we have \(D^\times(C_j) \leq |A|/M\) then we stop our algorithm putting \(C = C_j\), \(B = B_{j-1}\), and \(k = j - 1\). Consider the opposite situation, that is \(D^\times(C_j) > |A|/M\). Put \(C' = C_j\). By the definition there exists a number \(\tau \geq 1\) and a finite set \(G = G_j \subset \mathbb{R}\) such that the set
\[
S_\tau = S_\tau(C_j, C') := \{s \in C'/G : |C' \cap sG| \geq \tau\}
\]
has size at least \(\frac{|C'|^2|G|^2}{M^\tau}\). We have
\[
\tau |S_\tau| \leq \sum_{s \in S_\tau} |C' \cap sG| = \sum_{a \in C'} |S_\tau \cap aG^{-1}|.
\]
By the pigeonholing principle there is a set $A' \subseteq C'$ and a number $q$ such that $|A'|q \sim \tau|S_\tau|$ and $q < |S_\tau \cap aG^{-1}| \leq 2q$ for any $a \in A'$. In other words, $A' \subseteq \Sym^\tau_1(S_\tau, G)$. Applying Lemma \ref{lemma:notation} with $P = S_\tau$, $Q = G$, we get

$$D^+(A') \ll d^+(A') \leq \frac{|S_\tau|^2|G|^2}{q^3|A'|}. \quad (39)$$

Further, we know that $|A'|q \sim \tau|S_\tau|$ and $|S_\tau| > \frac{|C'|^2|G|^2}{M^{3/2}}$. Combining these inequality with bound \ref{equation:39}, we obtain

$$D^+(A') \ll d^+(A') \leq \frac{|A'|^2|G|^2}{\tau^3|S_\tau|} < \frac{|A'|^2M}{|C'|^2}. \quad (40)$$

Since $1 \leq d^+(A')$, we have $|A'| \gtrsim |C'|M^{-1/2}$. Trivially, $A' \subseteq C'$ and hence $D^+(A') \lesssim M$.

After that we put $D_j = A', C_j+1 = C_j \setminus D_j$, $B_j = B_{j-1} \cup D_j$ and repeat the procedure. Clearly, at step $k$ one has $B_k = \bigcup_{j=1}^k D_j$ and because of $|D_j| \gtrsim |C_j|M^{-1/2}$, we have after some calculations that $k \gtrsim M^{1/2}$, so $k$ is finite. It remains to estimate $D^+(B_k) = D^+(D_1 \cup \cdots \cup D_k)$. Put $U_j = \{i \in [k] : 2^{j-1} \leq D^+(D_i) \leq 2^j\}$, $j \in [t]$ and $k_j = |U_j|$. Since for any $i$ one has $D^+(D_i) \lesssim M$ it follows that the number of sets $U_j$ is $t \lesssim 1$. Let also $B_k^{(j)} = \bigcup_{i \in U_j} D_i$. Applying Lemma \ref{lemma:notation} we get

$$D^+(B_k) \lesssim \frac{1}{|B_k|} \sum_{j=1}^t D^+(B_k^{(j)})|B_k^{(j)}|. \quad (41)$$

Now fixing $j \in [t]$, we see that $k_j \gtrsim M^{1/2}2^{-j/2}$. Using Lemma \ref{lemma:notation} once more, we obtain

$$D^+(B_k^{(j)}) \lesssim \frac{k_j^3}{|B_k^{(j)}|} \sum_{i \in U_j} D^+(D_i)|D_i| \lesssim M^{3/2}2^{-j/2} \frac{1}{|B_k^{(j)}|} \sum_{i \in U_j} |D_i| = M^{3/2}2^{-j/2}. \quad (42)$$

Substituting the last bound into \ref{equation:41}, we find

$$D^+(B_k) \lesssim \frac{M^{3/2}}{|B_k|} \sum_{j=1}^t 2^{-j/2}|B_k^{(j)}| \ll M^{3/2}. \quad (43)$$

Optimizing over $M$, that is solving the equation $M^{3/2} = |A|/M$ and choosing $M = |A|^{2/5}$, we obtain the result. This completes the proof. \hfill \square

As for lower bounds in Theorems \ref{theorem:bounds} \ref{theorem:bounds}, we use small modification of the construction from \cite{1}. A counterexample is so-called $(H + \Lambda)$–sets, see \cite{2}.

**Theorem 22** For any positive integer $N$ there exists a set $A \subseteq \mathbb{N}$, $|A| = N$ such that for an arbitrary $B \subseteq A$ with $|B| \geq |A|/2$ one has $\mathcal{E}_3(B)$, $\mathcal{E}_5(B) \gg |A|^{13/4}$ and $D^+(B), D^\times(B) \gtrsim |A|^{1/4}$.

**Proof.** Take an integer parameter $1 \leq K \ll N$, which we will choose later, $t = \lceil N/K \rceil$ and put $G = \{2^i\}_{i=1}^K$, $P = \{3 = p_1 < p_2 \cdots < p_t\}$ be $t$ consecutive odd primes. Finally, put $A = PG$, $|A| = tK = N + \theta K$, where $|\theta| \leq 1$. Thus, redefining $N$ if needed one can think that $|A| = N$. 


Thus using the Cauchy–Schwarz inequality, we obtain
\[
E^+_\sigma(B_j) \geq \frac{|B_j|^6}{|P + P|^2} \gtrsim \frac{|B_j|^6}{|P|^2}.
\]

Thus using the Cauchy–Schwarz inequality, we obtain
\[
E^+_\sigma(B) \geq \sum_{j=1}^{K} E^+_\sigma(B_j) \gtrsim |P|^{-2} \sum_{j=1}^{K} |B_j|^6 \gtrsim \frac{|B|^6}{|P|^2 K^5} \gg \frac{N^4}{K^3}.
\] (42)

Now let us calculate \( E^\times_\sigma(B) \). By formula (16), we have
\[
|A|^6 \ll |B|^6 \leq E^\times_\sigma(B)|B^2 \cdot \Delta(B)| \leq E^\times_\sigma(B)|A^2 \cdot \Delta(A)|
\] (43)

and thus it is sufficient to estimate the size of the set \( A^2 \cdot \Delta(A) \). Put \( A_x = A \cap xA \). Applying formula (20) of Lemma 6 in its multiplicative form, we obtain
\[
|A^2 \cdot \Delta(A)| = \sum_{x \in A/A} |AA_x| = \sum_{x \in G/G} |AA_x| + \sum_{x \in (A/A) \setminus (G/G)} |AA_x| \leq |G/G||GGP| + \sum_{x \in (A/A) \setminus (G/G)} |AA_x| \ll N^2 + \sigma.
\] (44)

Let us prove that \( \sigma \ll N^3/K \). Put \( x \in (A/A) \setminus (G/G) \). In other words \( x = g_1/g_2 \cdot p_1/p_2 \) and \( p_1 \neq p_2 \). Now taking \( a \in A_x \), we have
\[
a = p'g' = g_1 p_1 p'' g'' / g_2 p_2
\]
or
\[
p' p_2 \cdot g' g_2 = p'' p_1 \cdot g'' g_1,
\]
where \( p', p'' \in P \), \( g', g'' \in G \). Thus \( p' = p_1 \), \( p'' = p_2 \) and \( g' g_2 = g'' g_1 \). Hence \( A_x = p_1 g_1 g_2^{-1} \cdot G \) and \( |AA_x| \leq |GGP| \leq 2N \). It follows that
\[
\sigma \leq 2 |A/A| |N| \ll N |G/G \cdot P/P| \ll N^3/K.
\]

Returning to (43), (44) and recalling that \( K \ll N \), we get
\[
N^6 \ll E^+_\sigma(B) \cdot \frac{N^3}{K}.
\]

In view of (12) the optimal choice of \( K \) is \( K \sim N^{1/4} \).

Finally, inequalities \( D^+(B), D^\times(B) \gtrsim |A|^{1/4} \) immediately follows from the obtained lower bounds for \( E^+_\sigma(B), E^\times_\sigma(B) \), formula (12) and Lemma 17. This completes the proof. \( \square \)

Combining Theorem 22 with Theorem 21, we obtain Theorem 3 as well as Theorem 3. It is easy to see that estimate (11) implies bound (3) of Theorem 1 via the Cauchy–Schwarz inequality and hence we lose \( \delta/2 \). Our method allows to avoid such loses.
6 Another proof of Theorem 3 and further remarks

As was shown in [5] that quantities \( d^+ (A), d^x (A) \) are bounded above by

\[
d^+ (A) := \min_{B \neq \emptyset} \frac{|AB|^2}{|A||B|} \quad \text{and} \quad d^x (A) := \min_{B \neq \emptyset} \frac{|A + B|^2}{|A||B|},
\]

correspondingly, see also [6]. It turns out that there is a sum–product-type result involving just

the quantities \( d^+ (A), d^x (A) \) but not sum sets or product sets (which are hidden in the definitions

df d^+ (A), d^x (A), nevertheless).

**Theorem 23** Let \( A \subset \mathbb{R} \) be a finite set. Then

\[
|A| \lesssim d^+_x (A) \cdot d^x (A).
\] (45)

**Proof.** Applying Lemma 20, Lemma 17 and the Hölder inequality, we obtain for any nonempty
finite set \( B \subset \mathbb{R} \) that

\[
\frac{|A|^2 |B|}{|A + B|^2} \leq \frac{E^+_3 (A, -B)}{|A||B|^2} \leq q^+ (A) \lesssim D^+ (A) \ll d^+ (A) \leq d^+_x (A).
\]

In other words, for any such \( B \) one has

\[
|A| \lesssim d^+_x (A) \cdot \frac{|A + B|^2}{|A||B|}
\]
as required. \( \square \)

Of course bound (45) is optimal up to logarithms. Actually, we have proved in Theorem 23
that \( |A| \lesssim D^+ (A) \cdot d^x (A) \) and \( |A| \lesssim D^x (A) \cdot d^+ (A) \).

**Theorem 21** combining with Lemmas 8, 10 gives us an analog of Theorem 2 (or one can
repeat the arguments from [5] directly, we left this for the interested reader).

**Corollary 24** Let \( A \subset \mathbb{R} \) be a finite set and \( \delta = 1/5 \). Then there are two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \uplus C \) and

\[
\max \{ E^+ (A, B), E^x (A, C) \} \lesssim |A|^{3-\delta}.
\]

Of course Theorem 21 and Lemmas 8, 10 allows to calculate the higher energies of the
splitting sets \( B, C \). We give a sketch of a more direct proof in the case of \( E^+_3, E^x_3 \) energies, using
Proposition 16.
**Theorem 25** Let \( A \subseteq \mathbb{R} \) be a finite set and \( \delta_1 = 2/5 \). Then there exists two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \sqcup C \) and

\[
\max\{E_3^+(B), E_3^+(C)\} \lesssim |A|^{4-\delta_1},
\]

**Proof. (sketch)** Using the arguments of the proof of Theorem 20 from [5] one finds a set \( A_1 \subseteq A \) such that

\[
|A_1|^5|A|^2 \gtrsim E_3^+(A)E_3^+(A_1)
\]

(and, similarly, a dual version). After that applying the notation and the algorithm of the proof of Corollary 21 of the paper or following the proof of Theorem 21 as well as Proposition 16, we obtain

\[
E_3^+(D_j) \lesssim M|A|^{-2}|D_j|^5,
\]

where

\[
|D_j| \gtrsim |A|M^{-1/2},
\]

and with help of the Hölder inequality

\[
(E_3^+(B_k))^{1/6} = (E_3^+(D_1 \cup \cdots \cup D_k))^{1/6} \leq \sum_{j=1}^k (E_3^+(D_j))^{1/6} \lesssim (M|A|^{-2})^{1/6} \sum_{j=1}^k |D_j|^{5/6} \leq (M|A|^{-2})^{1/6}A|^{5/6}k^{1/6} \lesssim M^{1/4}|A|^{1/2}.
\]

The last bound is a consequence of (46), namely, \( k \lesssim M^{1/2} \). Hence

\[
E_3^+(B_k) \lesssim M^{3/2}|A|^3.
\]

Optimizing over \( M \), that is solving the equation \( M^{3/2}|A|^3 = |A|^4/M \) and choosing \( M = |A|^{2/5} \), we obtain the result. \( \square \)

We do not consider the situation of higher energies (although Proposition 16 allows to do it) because they will not so effective. The fact is the Szemerédi–Trotter theorem naturally corresponds to \( E_3^+(A), E_3^+(A) \) energies.

In [11] the author considered a more general context than usual sum–product setting. The method, combining with the arguments of [5], allows to obtain a variant of Theorem 2 in particular.

**Theorem 26** Let \( A \subseteq \mathbb{R} \) be a finite set, \( \alpha \neq 0 \) be a real number, and \( \delta = 1/5 \). Then there are two disjoint subsets \( B \) and \( C \) of \( A \) such that \( A = B \sqcup C \) and

\[
\max\{E^+(B), E^+(C + \alpha)\} \lesssim |A|^{3-\delta}.
\]

Further, there are disjoint subsets \( B' \) and \( C' \) of \( A \) such that \( A = B' \sqcup C' \) and

\[
\max\{E^+(B'), E^+(\alpha/C')\} \lesssim |A|^{3-\delta}.
\]
Again one can prove a similar result for the energies $E^+_3(B)$, $E^x_3(C + \alpha)$ or for the quantities $D^+(B)$, $D^x(C + \alpha)$ but we left it for the interested reader. Let us derive a consequence of the result above.

**Corollary 27** Let $A \subset \mathbb{R}$ be a finite set. Put

$$R[A] = \left\{ \frac{a_1 - a}{a_2 - a} : a, a_1, a_2 \in A, a_2 \neq a \right\}.$$ 

Then there are two sets $R', R'' \subseteq R[A]$, $|R'|, |R''| \geq |R[A]|/2$ such that $E^x(R') \lesssim |R'|^{3-1/5}$ and $E^+(R'') \lesssim |R''|^{3-1/5}$.

**Proof.** First of all let us prove the existence of the set $R'$. Put $R = R[A]$, $R^* = R \setminus \{0\}$, and $\delta = 1/5$. Using Theorem 26 we find $B, C \subseteq R$ such that $R = B \cup C$ and

$$\max\{E^x(B), E^x(C - 1)\} \lesssim |R|^{3-\delta}.$$ 

If $|B| \geq |R|/2$ then we are done. Suppose not. Then $|C| \geq |R|/2$ and in view of the formula $R = 1 - R$, see [11], we obtain that $C' := 1 - C \subseteq R$, $|C'| = |C| \geq |R|/2$ and

$$E^x(C') = E^x(1 - C) = E^x(C - 1) \lesssim |R|^{3-\delta}.$$ 

So, putting $R'$ equals $B$ or $C'$, we obtain the result.

To find the set $R''$ note that $(R^*)^{-1} = R^*$ and use the second part of Theorem 26 as well as the arguments above. This completes the proof. \(\square\)

In particular, Corollary 27 says that the set $R$ has large $R + R$ and $RR$ (the last fact is known from paper [11] or can be obtained as a direct application of the Szemerédi–Trotter theorem).

The same proof allows us to find a subset $A'_s$ of the set $A_s \cup (A - s) = A_s \cup (A_s - s)$, $A_s = A \cap (A + s)$, $s \in (A - A) \setminus \{0\}$ of cardinality $|A_s|/2$ such that $E^x(A'_s) \lesssim |A'_s|^{3-1/5}$ (similarly one can consider the set $A \cap (s - A)$ and find a subset of size at least $|A \cap (s - A)|/2$ with small multiplicative energy). This question is a dual one which appeared in [4], [5]. The same result holds for some multiplicative analog of the sets $A_s$, namely, $A^*_s = A \cap (s/A)$, $s \in AA \setminus \{0\}$.

**7 Appendix**

We finish the paper discussing some generalizations of norms $E_k$. Because our arguments almost repeat the methods of section 5 we give the sketch of the proofs sometimes.

Take $l \geq 2$, $k \geq 2$ and suppose that either $k$ or $l$ is even. Basically, we restrict ourselves considering the case of real functions. For any such a function $f$ put

$$\|f\|_{E_{k,l}}^{kl} := \sum_{x_1, \ldots, x_{k-1}} C^l_k(f)(x_1, \ldots, x_{k-1}) = \sum_{y_1, \ldots, y_{l-1}} C^k_l(f)(y_1, \ldots, y_{l-1}) \geq 0,$$ \hspace{1cm} (47)

where we have used formula (138) of Lemma 5 to obtain the second identity in (47). Again for even $k$ and $l$, we get $\|f\|_{E_{k,l}} \geq \|f\|_l$, $\|f\|_k$ and hence $\|f\|_{E_{k,l}} = 0$ iff $f \equiv 0$ in the case.

Similarly, we obtain an analog of Lemma 13.
Lemma 28 For any $k, l \geq 2$ and arbitrary functions $\varphi_1, \ldots, \varphi_k : G^l \to \mathbb{C}$, $\varphi_j = (\varphi_j^{(1)}, \ldots, \varphi_j^{(l)})$ the following holds

$$\left| \sum_{x \in G^{l-1}} C_l(\varphi_1)(x) \cdots C_l(\varphi_k)(x) \right| \leq \prod_{j=1}^k \prod_{i=1}^l \|\varphi_j^{(i)}\|_{E_{k,l}}. \quad (48)$$

If $k, l \geq 2$ are even and all functions are real then one can remove modulus from formula (48).

Proof. Let $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(l)})$. By the H"older inequality it is sufficient to have deal with

$$\sum_{x \in G^{l-1}} C_l^k(\varphi)(x) = \sum_{x \in G^{k-1}} C_k(\varphi^{(1)})(x) \cdots C_k(\varphi^{(l)})(x),$$

where we have used formula (18) of Lemma 5. Applying the H"older inequality again we obtain the required result. □

An analog of Proposition [16] is the following.

Proposition 29 Let $k, l \geq 2$ be integers. Then for any pair of functions $f, g : G \to \mathbb{C}$ the following holds

$$\|f + g\|_{E_{k,l}} \leq \|f\|_{E_{k,l}} + \|g\|_{E_{k,l}}.$$

If we consider just real functions and $k, l$ are even numbers then $\| \cdot \|_{E_{k,l}}$ is a norm.

Proof. Recall that $\Omega_l := \{0, 1\}^l$. We have

$$\sigma := \|f + g\|_{E_{k,l}}^k = \sum_{x \in G^{l-1}} C_k^k(f + g)(x) = \sum_{x \in G^{k-1}} \left( \sum_{\varepsilon \in \Omega_l} C_l(\varphi_\varepsilon)(x) \right)^k =$$

$$= \sum_{x \in G^{l-1}} \left( \sum_{n_\varepsilon = k} C_l^{n_\varepsilon}(\varphi_\varepsilon)(x) \right)^k,$$

where for $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l) \in \Omega_l$ we put $\varphi_\varepsilon = (\varphi_{\varepsilon_1}, \ldots, \varphi_{\varepsilon_l})$ and denote $\varphi_1 = f$, $\varphi_0 = g$. Let $n = (n_\varepsilon) \in \mathbb{N}_0^l$ and $q(n) := \sum_{\varepsilon \in \Omega_l} \text{wt}(\varepsilon) n_\varepsilon$. Applying Lemma 28 and Lemma 15 we obtain

$$\sigma \leq \sum_{x \in G^{l-1}} \left( \sum_{n_\varepsilon = k} C_l^{n_\varepsilon}(\varphi_\varepsilon)(x) \right)^k \|f\|_{E_{k,l}}^{q(n)} \|g\|_{E_{k,l}}^{q(n)} = \sum_{i+j = k} \frac{(kl)!}{i!j!} \|f\|_{E_{k,l}}^i \|g\|_{E_{k,l}}^j \leq$$

$$= (\|f\|_{E_{k,l}} + \|g\|_{E_{k,l}})^{kl}$$

as required. □
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