Weight structures and motives; comotives, coniveau and Chow-weight spectral sequences, and mixed complexes of sheaves: a survey

M.V. Bondarko, St. Petersburg State University *

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1 Introduction

This is a short survey of author’s results on Voevodsky’s motives and weight structures; yet it is supplied with detailed references. Weight structures are natural counterparts of $t$-structures (for triangulated categories) introduced by the author in [Bon07] (and also independently by D. Paukstello in [Pau08]). They allow to to construct weight complexes, weight filtrations, and weight spectral sequences. Partial cases of the latter are: 'classical' weight spectral sequences (for singular and étale cohomology), coniveau spectral sequences, and Atiyah-Hirzebruch spectral sequences (we mention all of these below). The details, proofs, and several more results could be found in [Bon07], [Bon10a], and [Bon09] (we also mention certain results of [Bon10b], [Heb10], and [Bon10p]). We describe more motivation for the theory of weight structures, and define weight structures in §4.

Though our ‘main’ weight structures will be defined on certain ‘motivic’ categories, the author tried to make this survey accessible to readers that are rather interested in general triangulated categories (or possibly, the stable homotopy category in topology). Those readers may freely ignore all definitions and results that are related with algebraic geometry (and motives). On the other hand, the main motivic results (see §3) could be understood without knowing anything about weight structures (after §3 a ‘motivic’ reader may proceed directly to §9 to find some more motivation to study weight structures). Alternatively, it is quite possible for any reader to read section §3 only after studying the general theory of weight structures (§§4–8).

The author chose not to pay (much) attention to the differential graded approach to motives in this text; yet it is described in detail in [Bon09] and in §6 of [Bon07] (see also [BeV08]).

This text is based on the talks presented by the author at the conferences "Finiteness for motives and motivic cohomology" (Regensburg, 9–13th of February, 2009) and "Motivic homotopy theory" (Münster, 27–31st of July, 2009); yet some more recent topics are added. The author is deeply grateful to prof. Uwe Jannsen, prof. Eric Friedlander, and to other organizers of these conferences for their efforts.

2 Categoric notation; definitions of Voevodsky

For a category $C$, $A, B \in \text{Obj} C$, we denote by $C(A, B)$ the set of $C$-morphisms from $A$ to $B$.

Below $B$ will be some additive category; $K^b(B) \subset K(B)$ will denote the homotopy category of (bounded) $B$-complexes.

$\mathcal{C}$ and $\mathcal{D}$ will be triangulated categories; for $f \in \mathcal{C}(X, Y)$, $X, Y \in \text{Obj} \mathcal{C}$, we will denote the third vertex of (any) distinguished triangle $X \to Y \to Z$ by $\text{Cone}(f)$.

For $D, E \subset \text{Obj} \mathcal{C}$ we will write $D \perp E$ if $\mathcal{C}(X, Y) = \{0\}$ for all $X \in D$, $Y \in E$. 

2
A will be an abelian category, $D(A)$ is its derived category; $H: C \to A$ will usually be a cohomological functor (i.e. it is contravariant, and converts distinguished triangles into long exact sequences).

Kar($B$) for any $B$ will denote the Karoubization of $B$ i.e. the category of 'formal images' of idempotents in $B$ (so $B$ is embedded into an idempotent complete category).

A full subcategory $C \subset B$ is called Karoubi-closed in $B$ if $C$ contains all $B$-retracts of its objects; $Kar_B C$ will denote the smallest Karoubi-closed subcategory of $B$ that contains $C$ (i.e. its objects are all retracts of objects of $C$ that belong to $B$).

$Ab$ is the category of abelian groups.

Now we introduce our ‘motivic’ definitions; they could be especially interesting to readers that are aware of ‘classical’ motives but do not know much about Voevodsky’s ones.

$k$ is our perfect base field. We will often have to assume that either char $k = 0$ or that we consider (co)motives and cohomology with rational coefficients.

$SmPrVar \subset SmVar \subset Var$ are the sets of (smooth projective) varieties over $k$.

The definition of Voevodsky’s motives starts from smooth correspondences (see [Voe00]):

$ObjSmCor = SmVar; SmCor(X, Y) = \mathbb{Z}^{|U|}; U \subset X \times Y$ is closed reduced, finite dominant over a component of $X$. Compositions of morphisms are given by a natural algebraic analogue of composition of multi-valued functions.

Remark 2.1.

1. So, in contrast to the ‘classical’ definition, we consider only those primitive correspondences (i.e. closed subvarieties of $X \times Y$ of a certain dimension) that are finite over $X$. Note here that any ‘classical’ correspondence is rationally equivalent to some finite one. The advantage of finite correspondences is that the composition is well-defined without factorizing modulo an equivalence relation. This is very important!

2. For any commutative associative ring with a unit $R$ instead of $SmCor$ one can consider a certain category $SmCor_R$; in order to define it one should just replace $\mathbb{Z}^{|U|}$ by $R^{|U|}$ in the definition of $SmCor(X, Y)$. This allows to construct a reasonable theory of Voevodsky’s motives with $R$-coefficients; see [MVW06].

Usually one takes $R = \mathbb{Z}$ or $R = \mathbb{Q}$. In [Bon10b] the author also considers certain intermediate coefficients rings. The case $r = \mathbb{Z}/(n)$ (for $n > 1$) is also interesting.

Cartesian product of varieties yields tensor structure for $SmCor$ (as well as for $K^b(SmCor)$).

One can define (homological) Chow motives in terms of $SmCor$. One starts from the category of rational correspondences: $ObjCorr_{rat} = SmPrVar; Corr_{rat}(X, Y) = SmCor(X, Y)/$ rational equivalence.

Now, one has $Chow^{eff} = Kar(Corr_{rat})$ (this yields a category that is isomorphic to the ‘classical’ effective Chow motives). Formal tensor inversion of $\mathbb{Z}(1)[2]$ (the Lefschetz motif i.e. the ‘complement’ of a point to the projective line) yields the whole category Chow.

$DM^{eff}_{gm}$ is defined as the Karoubization of a certain localization of $K^b(SmCor)$ (so it is triangulated). Similarly, tensor inversion of $\mathbb{Z}(1)[2]$ yields $DM_{gm}$.

We denote by $M$ the composition $SmVar \to SmCor \to K^b(SmCor) \to DM^{eff}_{gm}$; this defines motives of smooth varieties. If char $k = 0$, in $DM^{eff}_{gm}$ there also exist motives and certain motives with compact support for arbitrary varieties.

Voevodsky constructed the following diagram of functors:
Here all arrows are full embeddings of additive categories.

In §3.1 of [Voe00] Voevodsky also defined a certain triangulated category $DM_{gm}^{eff} \supset DM_{gm}^{eff}$.

3 Main motivic results

We list our main results. Assertions 1–6 require $\text{char } k = 0$ (yet see Remark 3.2(5) below).

**Theorem 3.1.** 1. In §3 of [Bon09] $DM_{gm}^{eff}$ was described 'explicitly' in terms of twisted complexes over a certain differential graded category $J$ (see §2.4 of ibid.); the objects of $J$ are cubical Suslin complexes of smooth projective varieties.

2. This description is somewhat similar to (yet 'more convenient' than) those of Hanamura’s motives (see [Han04]). This allowed to compare Voevodsky’s motives with Hanamura’s ones: in §4 of [Bon09] it was proved $DM_{gm}^{eff}$ is anti-isomorphic to Hanamura’s motives.

3. 'Killing all arrows of negative degrees' in the 'description' of $DM_{gm}^{eff}$ immediately yields an exact weight complex functor $t : DM_{gm}^{eff} \rightarrow K^b(Chow^{eff})$; it could also be extended to $t_{gm} : DM_{gm} \rightarrow K^b(Chow)$. In §6 of [Bon09] it was also proved that these functors are conservative (i.e. $t_{gm}(X) = 0 \implies X = 0$). A generalization was described in §3 of [Bon07].

The term 'weight complex' was proposed by Gillet and Soulé in [GiS96]. Their functor was essentially the restriction of $t$ to motives with compact support of varieties (see §6.6 of [Bon09]). In [GNA02] also a functor that is essentially $t \circ M$ was defined.

4. $t$ gives $K_0(DM_{gm}^{eff}) \cong K_0(Chow^{eff})$ and $K_0(DM_{gm}) \cong K_0(Chow)$ (see §6.4 of [Bon09]); a generalization and certain variations of this results are described in §§5.3–5.5 of [Bon07].

Recall that the generators of $K_0(Chow^{eff})$ are $[X], X \in \text{Obj}Chow^{eff}$; the relations are $[X \oplus Y] = [X] + [Y]$ (for in $X, Y \in \text{Obj}Chow^{eff}$). The definition of $K_0(Chow)$ is similar.

For triangulated categories one imposes more relations: $K_0(DM_{gm}^{eff})$ is generated by $[M], M \in \text{Obj}DM_{gm}^{eff}$; if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished triangle then $[B] = [A] + [C]$. $K_0(DM_{gm})$ is defined similarly.

5. Motivically functorial weight spectral sequences for any cohomology theory $H : DM_{gm}^{eff} \rightarrow A$ (generalizing Deligne’s ones for étale and singular cohomology of varieties) were constructed (see §6.6 and Remark 2.4.3 of [Bon09]; they were called Chow-weight spectral sequences since they correspond to the Chow weight structure; see below).

6. All triangulated subcategories and localizations of $DM_{gm}^{eff}$ were 'described' (see §8.1–8.2 of [Bon09]). In particular, one obtains 'reasonable' descriptions of Tate motives and of the (triangulated) category of birational motives (i.e. of the localization of $DM_{gm}^{eff}$ by $DM_{gm}^{eff}(1)$; see [KaS02]) this way.
7. A certain category \( \mathcal{D} \) (of comotives) that contains 'nice homotopy limits' of Voevodsky’s motives was constructed (see §3.1 and §5 of [Bon10a]). In particular, it contains certain (co)motives for all function fields over \( k \).

Some of the properties of \( \mathcal{D} \) are dual (in a certain sense) to the corresponding properties of the 'usual large motivic' categories. In particular, though we have a covariant embedding \( \mathcal{D} \) \( \rightarrow \) \( \mathcal{D}_{\text{eff}} \), it yields a family of cocompact cogenerators for \( \mathcal{D} \). This is why we call the objects of \( \mathcal{D} \) comotives.

Comotives allow to prove the following results.

8. Motivically functorial coniveau spectral sequences for cohomology of motives were constructed (in §4.2 of [Bon10a]; cf. also §7.4 of [Bon07]).

For \( H \) represented by a motivic complex (i.e. an object of \( \mathcal{D}_{\text{eff}} \)) we prove that these spectral sequences can be described in terms of the homotopy t-truncations of \( H \). This vastly extends seminal results of Bloch and Ogus (see [BOg94]).

9. Let \( k \) be countable.

Then the cohomology of any smooth semi-local scheme (over \( k \)) is a direct summand of the cohomology of its generic point; the cohomology of function fields contain twisted cohomology of their residue fields (for all geometric valuations) as direct summands.

Remark 3.2. 1. Parts 3–5 of the Theorem will be vastly generalized below (to triangulated categories endowed with weight structures).

They follow from the existence of a certain Chow weight structure for \( \mathcal{D}_{\text{eff}} \); whereas assertions 8–9 follow from the existence of a certain Gersten weight structure for a certain triangulated \( \mathcal{D} \), such that \( \mathcal{D}_{\text{eff}} \subset \mathcal{D} \subset \mathcal{D}^* \).

2. Recently (independently in [Heb10] and in [Bon10p]) it was also proved that the Chow weight structure could be defined for the category of Voevodsky’s motives with rational coefficients over any 'reasonable' base scheme \( S \) (in [CiD10] where the basic properties of \( S \)-motives were established, they were called Beilinson’s motives; one could either consider the ‘large’ category of \( S \)-motives or its subcategory of constructible i.e. ‘geometric’ objects here). The heart of this weight structure is ‘generated’ by the motives of regular schemes that are projective over \( S \) (tensored by \( \mathbb{Q}(n)[2n] \) for all \( n \in \mathbb{Z} \)). So, we obtain certain analogues of parts 3–5 of the Theorem for \( S \)-motives also.

In [Bon10p] the weights for \( S \)-motives were also related with the ‘classical’ weights of mixed complexes of sheaves. To this end the notion of a relative weight structure was introduced; see below.

3. \( t \) allows to compute \( E_2 \) of 'weight' spectral sequences (see assertion 5). Hence for (rational) singular/étale cohomology of varieties (and motives) it computes the factors of the weight filtration; whence the name.

4. No explicit comparison functor in the ‘description’ of part 1 is known (the two triangulated categories in question are compared by means of a third triangulated category). Note also that the category of twisted complexes considered is a ‘twisted’ analogue of \( K^b(B) \) i.e. one considers morphisms and objects up to (a certain) homotopy equivalence. Hence in order to work with \( \mathcal{D}_{\text{eff}} \) one needs constructions that do not depend on the choices of representatives in these homotopy equivalence classes. Weight structures really help here!

5. All the assertions of the theorem remain valid if we replace motives with integral coefficients by those with rational (or \( \mathbb{Z}/n\mathbb{Z} \)-) ones; see Remark 2.1(2).
Moreover, the requirement char $k = 0$ is only needed to apply the resolution of singularities (that is required to prove some of the statements in \cite{Voe00}, which are necessary to deduce our results). Yet for motives with rational coefficients (we denote them by $\text{Chow}^{\text{eff}} \subset \text{DM}^{\text{eff}} \subset \text{DM}_\text{gm} \subset \text{DM}_\text{gm}^{\text{eff}} \subset \text{DM}_\text{gm}$) it usually suffices to apply de Jong’s alterations. In particular, this allows to prove the ‘rational’ analogues of assertions 3–5 also for any perfect $k$ of characteristic $p$.

Moreover, a recent resolution of singularities result of Gabber (see Theorem 1.3 of \cite{Ill08}) allow also to prove the analogues of assertions 3–5 with $\mathbb{Z}[\frac{1}{p}]$-coefficients (over $k$). Note here: Gabber’s theorem could be called ‘$\mathbb{Z}(l)$-resolution of singularities’ (for all $l \in \mathbb{P} \setminus \{p\}$); yet weight structures allow to prove motivic results with $\mathbb{Z}[\frac{1}{p}]$-coefficients (that is a priori more difficult); see \cite{Bon10b}.

6. In §6.3 of \cite{Bon09} certain length of motives was defined (it is a certain ‘length’of $t(X)$). This is a motivic analogue of the length of the weight filtration for mixed Hodge structures (coming from cohomology of varieties). In particular, the length of a motif of a smooth variety is is not greater than its dimension and not less than the length of the weight filtration for its cohomology.

7. One can prove more than conservativity for $t$. In particular, $X \in \text{ObjDM}_\text{gm}$ is mixed Tate whenever $t_{\text{gm}}(X)$ is (see Corollary 8.2.3 of \cite{Bon09}).

## 4 Weight structures: basics

Now we define weight structures. They are related with stupid truncations of complexes (i.e. of objects of $K(B)$) in a way similar to the relation of $t$-structures with canonical truncations (see \cite{BBD82} for the foundations of the theory of $t$-structures); certainly, the distinctions here are also very significant!

Stupid truncations are not very popular since they are not canonical (whereas canonical truncations are canonical and functorial). Yet we will explain (starting from §5 below) how they do yield functorial cohomological information; these results are new even in the case $C = K(B)$. There are a lot of examples when non-canonical constructions yield important functorial information: projective and injective resolution of objects and complexes over abelian categories allow to define derived functors; nice compactifications and smooth hyper-resolutions of varieties allow to define weight spectral sequences for étale and singular cohomology; skeletal filtration for topological spectra allow to construct Atiyah-Hirzebruch spectral sequences for their cohomology. All of these observations have very natural ‘explanations’ inside the theory of weight structures!

Weight structures have (at least) two distinct incarnations important for Voevodsky’s motives (related to weight and coniveau spectral sequences), and also one that is relevant for the stable homotopy category (in topology). Yet we describe illustrate some basics of the theory on a (more) simple (though quite interesting) example.

For $C = K(B)$ we denote by $C^{w \leq 0}$ the class of complexes, homotopy equivalent to those concentrated in non-positive degrees; we denote by $C^{m \geq 0}$ the class complexes, equivalent to those concentrated in degrees $\geq 0$.

Then the classes of complexes described satisfy the following properties (we write them down in the form that reminds the axioms of $t$-structures; this is very convenient).
Definition 4.1 (Axioms of weight structures). (i) $C_{w\geq 0}, C_{w\leq 0}$ are additive and Karoubi-closed in $C$.

(ii) 'Semi-invariance' with respect to translations. 
$C_{w\geq 0} \subset C_{w\geq 0}[1], C_{w\leq 0} \subset C_{w\leq 0}$.

(iii) Orthogonality. 
$C_{w\geq 0} \perp C_{w\leq 0}$.

(iv) Weight decompositions. 
For any $X \in ObjC$ there exists a distinguished triangle 
\[ B[-1] \to X \to A \to B \] 
\[ 2 \] 
such that $A \in C_{w\leq 0}, B \in C_{w\geq 0}$.

For any triangulated category $C$ we will say that some $(C_{w\leq 0}, C_{w\geq 0})$ yield a weight structure if they satisfy the properties listed.

Remark 4.2. 1. For $C = K(B)$ weight decompositions come from 'stupid truncations': 
\[ \ldots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to X^2 \to \ldots \] 
\[ \downarrow \alpha \] 
\[ \ldots \to X^{-2} \to X^{-1} \to X^0 \to 0 \to 0 \to \ldots \] 
\[ \downarrow \beta \] 
\[ \ldots \to 0 \to 0 \to X^1 \to X^2 \to X^3 \to \ldots \]

2. In this partial case we also have an opposite orthogonality property; yet this additional orthogonality is not important, and does not generalize to other (more interesting) examples.

3. For $t$-structures the orthogonality axiom is opposite; also, the arrows in $t$-decompositions 'go in the converse direction'. These distinctions result in a drastic difference between the properties of these two types of structures. Note that dualization does not change anything here (since the axiomatics of $t$-structures is self-dual, as well as the one of weight structures).

4. We demand (in (i)) $C_{w\geq 0}$ and $C_{w\leq 0}$ to be Karoubi-closed; this is a technical condition that is not really important. The corresponding condition for $t$-structures is also true (though in contrast to the weight structure case, one does not have to include it in the axioms).

We also define the heart $H_w$ of $w$ (similarly to hearts of $t$-structures): $Obj H_w = C_{w=0} = C_{w\geq 0} \cap C_{w\leq 0}$, $H_w(X,Y) = C(X,Y)$ for $X, Y \in C_{w=0}$.

Now we list some very basic properties of weight structures (and their hearts).

Theorem 4.3. 1. The axiomatics of weight structures is self-dual: if $D = C_{w}^{\text{op}}$ (so $ObjC = ObjD$) then one can define the (opposite) weight structure $w'$ on $D$ by taking $D_{w\leq 0} = C_{w\geq 0}$ and $D_{w\geq 0} = C_{w\leq 0}$.

2. $C_{w\leq 0}, C_{w\geq 0},$ and $C_{w=0}$ are extension-stable i.e. for a distinguished triangle $A \to B \to C$ if $A, C$ belong to $C_{w\leq 0}$ (resp. to $C_{w\geq 0}$, resp. to $C_{w=0}$) then $B$ belongs to the corresponding class also.

3. If $A \to B \to C \to A[1]$ is a distinguished triangle and $A, C \in C_{w=0}$, then $B \cong A \oplus C$. 7
4. \( Hw \) is negative i.e. \( Hw \perp \bigcup_{i > 0} Hw[i] \).

5. Conversely, for a triangulated \( C \) let an additive \( D \subset \text{Obj}C \) be negative; suppose that the smallest triangulated subcategory of \( C \) containing \( D \) is \( C \) itself. Then there exists a unique weight structure \( w \) for \( C \) such that \( D \subset {_Cw}0 \), for it we have \( Hw = \text{Kax}_{\text{C}}D \) (see Theorem 4.3.2 of [Bon07]).

One can construct all bounded weight structures (i.e. those ones that satisfy \( \cap_{i \in \mathbb{Z}} {_{Cw}i} = \{ 0 \} \) this way.

Remark 4.4. 1. Examples

Assertion 5 allows to construct the 'stupid' weight structure for \( K^b(B) \) mentioned above (note: as for \( t \)-structures, a single \( C \) may support more than one distinct weight structures).

Besides, in the stable homotopy category there are no morphisms of positive degrees between coproducts of the sphere spectrum \( S^0 \). Hence assertion [5] allows to construct a certain weight structure for the subcategory of finite spectra. In §4 of [Bon07] several other existence of weight structures results (for unbounded weight structures) were proved. In particular, they allow to construct a certain \( w_{S^0} \) for the whole \( SH \) (see §4.6 of ibid.). The corresponding weight decompositions correspond to cellular filtration of spectra; one gets Atiyah-Hirzebruch spectral sequences this way (as weight spectral sequences; see below)!

Lastly, \( \text{Chow}^{eff} \) is negative inside \( DM^{eff}_{gm} \subset DM^{eff} \); \( \text{Chow} \) is negative inside \( DM_{gm} \) (see (11)). This allows to construct certain \( \text{Chow} \) weight structures for all of these categories. We denote all of them by \( w_{\text{Chow}} \), since they are compatible; see §§6.5-6.6 of [Bon07], and also Remark 4.3.4 above.

2. The obvious analogue of assertion [5] for \( t \)-structures (i.e. we want to construct a \( t \)-structure such that a positive \( D \subset C \) lies in its heart) is very far from being true. So, negative subcategories of triangulated categories are much more valuable than positive ones! Besides, weight structures 'are more likely to exist for small triangulated categories' (than \( t \)-structures); see Remark 4.3.4 of [Bon07].

3. Yet another distinction of weight structures from \( t \)-structures is demonstrated by assertion 3: distinguished triangles in \( C \) do not yield non-trivial extensions in \( Hw \).

In fact, one may say that the notion of the heart of a weight structure is a 'triangulated analogue' of the category of projective (or injective) objects of an abelian category \( A \). Note here: we have \( D(\mathbb{A})(P,Q[i]) = \{ 0 \} \) if \( i \neq 0 \) and \( P,Q \) are both projective (or injective) objects of \( A \); this allows to construct resolutions of objects of \( A \) (and hyperresolutions of complexes) that are functorial up to homotopy equivalence. The theory of weight structures demonstrates that one mostly needs \( D(\mathbb{A})(P,Q[i]) = \{ 0 \} \) if \( i > 0 \); the absence of 'positive extensions' is sufficient to prove certain functoriality of the corresponding 'resolutions' (i.e. Postnikov towers); see below. So, weight structures yield a vast generalization of projective and injective hyperresolutions!

5  Functoriality of weight decompositions; truncations for cohomology

Now we discuss to what extent weight decompositions are functorial, and how this allows to define nice canonical 'truncations' and filtration for cohomology.
Weight decompositions (as in (2)) are (almost) never unique. Still we will denote any pair of \((A, B)\) as in (2) by \(X^{w \leq 0}\) and \(X^{w \geq 1}\). \(X^{w \leq i}\) (resp. \(X^{w \geq i}\)) will denote \((X[i])^{w \leq 0}\) (resp. \((X[i-1])^{w \geq 1}\)).

Note: though one can prove this statement easily without weight structures, now we observe that weight decompositions are 'weakly functorial'.

**Proposition 5.1.** 1. Any \(g \in C(X, Y)\) could be completed (non-uniquely) to a morphism weight decompositions.

2. Moreover, for any \(i \in \mathbb{Z}, j > 0\), \(g\) extends to a diagram

\[
\begin{array}{ccc}
  w_{\geq i+1} X & \rightarrow & X \\
  \downarrow & & \downarrow g \\
  w_{\geq i+1+j} Y & \rightarrow & Y \\
\end{array}
\]

in a unique way if we fix the corresponding weight decompositions.

**Remark 5.2.** 1. A nice illustration for assertion 1 is: for \(C = DM_{gm}^{eff}\), \(w = w_{Chow}\), it implies (in particular) that any morphism of smooth varieties (coming from \(SmVar, SmCor\), or \(DM^{eff}_{gm}\)) could be completed in \(DM_{gm}^{eff}\) to a morphism of (any choices of) their smooth compactifications. Note: though one can prove this statement easily without weight structures, yet it is somewhat 'counterintuitive'.

2. For \(C = K(B)\) assertion 2 means: if we fix the choice of weight decompositions, then the diagram

\[
\begin{array}{ccccccc}
  \ldots & \rightarrow & X^{-2} & \rightarrow & X^{-1} & \rightarrow & X^0 \\
  \downarrow g^{-2} & & \downarrow g^{-1} & & \downarrow g^0 & & \downarrow g^1 \\
  \ldots & \rightarrow & Y^{-2} & \rightarrow & Y^{-1} & \rightarrow & Y^0 \\
\end{array}
\]

is compatible with a unique choice of the following diagram

\[
\begin{array}{ccccccc}
  \ldots & \rightarrow & X^{-2} & \rightarrow & X^{-1} & \rightarrow & X^0 \\
  \downarrow g^{-2} & & \downarrow g^{-1} & & \downarrow g^0 & & \downarrow g^1 \\
  \ldots & \rightarrow & Y^{-2} & \rightarrow & Y^{-1} \\
\end{array}
\]

in \(C\) (i.e. if we consider all morphisms up to homotopy equivalence).

Proposition 5.1 immediately allows to construct some functorial filtration and 'truncations' for cohomology (i.e. for some contravariant \(H : C \rightarrow A\) that will usually be cohomological).

**Proposition 5.3.** 1. For any contravariant \(H : C^{op} \rightarrow A, j > 0\), Proposition 5.1(1) yields that the weight filtration \(W^iH(X) = \text{Im}(H(w_{\leq i} X) \rightarrow H(X))\) of \(H(X)\) is \(C\)-functorial in \(X\).

2. Applying both parts of the proposition we obtain that \(H^i : X \mapsto \text{Im}(H(w_{\leq i} X) \rightarrow H(w_{\leq i+j} X))\) also defines a functor.

3. If \(H\) is cohomological, \(j = 1\), \(H^1\) is cohomological also.
4. \( H^i_2 = \text{Im}(H(w_{\geq i} X) \to H(w_{\geq i+1} X)) \) is also functorial and cohomological (if \( H \) is); there is a long exact sequence of functors (i.e. it becomes a long exact sequence in \( A \) when applied to any object of \( C \))

\[ \cdots \to H^i_2 \circ [1] \to H^i_1 \to H^i \to H^i_2 \circ [1] \to \cdots \]

We call \( H^i_1 \) and \( H^i_2 \) virtual t-truncations of \( H \). The reason for this is that they 'behave as' if \( H \) is 'represented' by an object of some triangulated category \( D \), and the truncations are 'represented' by its actual t-truncations with respect to some t-structure of \( D \). We will observe that it is often the case in the next section; yet note that virtual t-truncations can be defined (and have nice properties) without specifying any \( D \) and any t-structure for it (in fact, it is far from being obvious that such \( D \) and t exist always; even if they do, \( D \) is definitely not determined by \( C \) in a functorial way)!

Virtual t-truncations are studied (in detail) in §2.5 of [Bon07] (the theory is also developed for covariant functors; certainly, the difference is quite formal) and in §§2.3–2.5 of [Bon10a]. Also, \( \tilde{W}_t H_{BM}^n (-, -) \) in Definition 5.8 of [FrH04] are essentially (restrictions to varieties of) virtual t-truncations of Borel-Moore homology with respect to \( w_{\text{Chow}} \).

6 Dualities of triangulated categories; orthogonal and adjacent weight and t-structures

Let \( D \) also be a triangulated category.

**Definition 6.1.** 1. We will call a (covariant) bi-functor \( \Phi : C^\text{op} \times D \to A \) a duality if it is bi-additive, homological with respect to both arguments; and is equipped with a (bi)natural transformation \( \Phi(X, Y) \cong \Phi(X[1], Y[1]) \).

2. Suppose now that \( C \) is endowed with a weight structure \( w \), \( D \) is endowed with a t-structure \( t \). Then we will say that \( w \) is (left) orthogonal to \( t \) with respect to \( \Phi \) if the following orthogonality condition is fulfilled:

\[ \Phi(X, Y) = 0 \text{ if: } X \in C^{w \leq 0} \text{ and } Y \in D^{t \geq 1}, \text{ or } X \in C^{w \geq 0} \text{ and } Y \in D^{t \leq -1}. \]  \hspace{1cm} (4)

**Remark 6.2.** 1. If \( t \) is orthogonal to \( w \), then: for any \( X \in C^{w=0} \) the functor \( Y \mapsto \Phi(X, Y) \) is exact when restricted to \( H_t \).

Virtual t-truncations of \( \Phi(-, Y) \) are 'represented' by t-truncations of \( Y \): for example, \( \Phi(X, Y^{t \geq [j]}) \cong \text{Im}(\Phi([X^{w \geq -j}, Y[i]) \to \Phi([X^{w \geq -1-j}, Y[i-1]])) \).

2. **Adjacent structures**

A very important example of a duality is: \( D = C \), \( \Phi(X, Y) = C(X, Y) \). This duality is also nice (see Definition 2.5.1 of [Bon10a]); niceness is a technical condition needed for spectral sequence calculations (see below).

In this situation, we call orthogonal \( w \) and t adjacent structures; \( w \) is (left) adjacent to \( t \) whenever \( C^{w \leq 0} = C^{t \leq 0} ; \text{ see §4.4 of } [Bon07] \).

3. **Weight-exact functors; relation with adjoint functors.**

Recall now that if an exact functor \( C \to C' \) is t-exact with respect to some t-structures on these categories, its (left or right) adjoint is usually not t-exact (it is only left or right
t-exact, respectively). This situation can be described much more precisely if there exist adjacent weight structures for these t-structures (see Proposition 4.4.5 of ibid.).

Suppose that $C$ is endowed with a weight structure $w$ and its left adjacent t-structure $t$; $C'$ is endowed with a weight structure $w$ and its left adjacent t-structure $t'$; $F : C 	o C'$ is exact, $G : C' 	o C$ is its left adjoint.

We will say that $G$ is left (resp. right) weight-exact if $G(C_{w \leq 0}) \subset C_{w \leq 0}$ (resp. $G(C_{w \geq 0}) \subset C_{w \geq 0}$).

Then: $G$ is left (resp. right) weight-exact whenever $F$ is right (resp. left) t-exact (in the well-known and similarly defined sense).

4. Examples.

A simple example of adjacent structures is: if Proj $A \subset A$ denotes the full subcategory of projective objects, $D^i(A)$ (i.e. some version of $D(A)$) is isomorphic to the corresponding $K^i$ (Proj $A$), then for $C = D^i(A)$ the canonical t-structure for $C$ is orthogonal to the ‘stupid’ weight structure for $C \cong K^i$ (Proj $A$) (mentioned above). Note that this example allows to compute extension functors in $A$ (and hyperextensions i.e. morphisms in $D^i(A)$)!! Besides, the spherical weight structure ($w_{\geq 0}$ for SH mentioned above) is adjacent to the Postnikov t-structure $t_{post}$ (for SH).

Moreover, a process similar to the construction of Eilenberg-MacLane spectra allows to construct a Chow t-structure for $DM^{eff}$ such that $H_{\text{chow}} \cong \text{AddFun}(\text{Chow}^{eff}, Ab)$ (see §7.1 of [Bon07]). $t_{\text{chow}}$ is adjacent to the Chow weight structure for $DM^{eff}$; it is related with unramified cohomology (see §7.6 of ibid.). Other related calculations of hearts of orthogonal structures were made in §§4.4–4.6 of ibid. and in §6.2 of [Bon10a].

Lastly, there also exists a nice duality $\mathcal{D}^{pop} \times DM^{eff} \to Ab$ (see §4.5 of [Bon10a]). If (the base field) $k$ is countable, there also exists a triangulated category $\mathcal{D}$ (such that $DM^{eff}_{gm} \subset \mathcal{D}_{s} \subset \mathcal{D}$) endowed with a Gersten weight structure (see §4.1 of ibid.), that is orthogonal to the homotopy t-structure for $DM^{eff}$ (defined in [Voe00]). So, the objects of its heart induce exact covariant functors from $H^i$ (i.e. the category of homotopy invariant sheaves with transfers) to $Ab$. It is no surprise that this heart is ’generated’ by comotives of (spectra of) function fields (over $k$).

Note that in this case $\mathcal{C} \neq \mathcal{D}$.

5. Hence (the recently proved) Beilinson-Lichtenbaum conjecture implies that the homotopy t-truncations of complexes of sheaves that represent $\mathbb{Z}/n\mathbb{Z}$-étale cohomology yield $\mathbb{Z}/n\mathbb{Z}$-motivic cohomology. Hence one can express torsion motivic cohomology (of smooth varieties, motives, and comotives) in terms of virtual t-truncations of torsion étale cohomology with respect to the Gersten weight structure. This allows to obtain some new formulae for motivic cohomology; cf. §§7.4–7.5 of [Bon07] and Remark 4.5.2 of [Bon10a].

7 Weight spectral sequences

Applying $H$ to (shifted) weight decompositions of $X$ one obtains an exact couple with: $D^1_{pq} = H(X^{w \leq -p}[-q])$, $E^1_{pq} = H(X^{-p}[-q])$.

Here $X^i \in \text{Obj} \mathcal{H}_w$ are the terms of the weight complex of $X$; the latter coincides with $X$ for $\mathcal{C} = K(B)$, was mentioned in Theorem [3.1.3] for $\mathcal{C} = DM^{eff}_{gm}$ or $DM_{gm}$, and will be considered in §6.2 in the general case. We will call the corresponding spectral sequence
a weight spectral sequence and denote it by $T_w(H, X)$ (we will often omit $w$ in this notation). It always weakly converges to $E^p_{s}T(H, X) = H(X[-p-q])$. Under certain (quite weak) boundedness conditions the convergence is strong. Note here: it is natural to denote $H(X[-i])$ by $H^i(X)$; see §2.3–2.4 of [Bon07] for more detail.

This exact couple (and so the whole spectral sequence) is functorial in $H$ (in the obvious sense). Also, it is easily seen that any $g \in \mathbb{C}(X, X')$ could be extended to some morphism of exact couples. Still, this extension is almost never unique.

Yet this problem vanishes completely if one passes to the derived exact couple! It is easily seen that $D_2$-terms are virtual $t$-truncations of $H$ (defined in [4] above); $E_2$ are certain 'truncations from both sides'; so both are given by cohomological functors $\mathbb{C} \to \mathbb{A}$ (see loc.cit. and §2.4 of [Bon10a]). Hence $T(H, X)$ is $\mathbb{C}$-functorial (in $X$) starting from $E_2$.

Besides, the relation between virtual $t$-truncations and truncations with respect to an orthogonal $t$-structure (described above) yields: for a nice duality $\Phi$, $H = \Phi(-, Y)$, $Y \in \text{ObjC}$, one has a functorial description of $T(H, -)$ (starting from $E_2$) in terms of $t$-truncations of $Y$; see Theorem 2.6.1 of [Bon10a]. This is a powerful tool for comparing spectral sequences (in this situation); it does not require constructing any complexes (and filtrations for them) in contrast to the method of [Par96] (probably, originating from Deligne).

Remark 7.1 (Examples; change of weight structures). 1. Weight spectral sequences generalize Deligne’s weight spectral sequences, coniveau, and Atiyah-Hirzebruch spectral sequences.

Weight spectral sequences corresponding to $w_{\text{Chow}}$ (we call them Chow-weight spectral sequences since they relate cohomology of Voevodsky’s motives with those of Chow motives) essentially generalize Deligne’s weight spectral sequences; see Remark 2.4.3 and §6 of [Bon07]. For $H$ being étale or singular cohomology (of motives) this yields motivic functoriality of $T_{w_{\text{Chow}}}(H, -)$ for integral (or torsion) coefficients. Note that the 'classical' way of proving uniqueness of these spectral sequences uses Deligne’s weights for sheaves, and so requires rational coefficients (one also uses heavily the fact that in this particular case weight spectral sequences degenerate at $E_2$).

One could also take the motivic cohomology theory for $H$. This yields completely new spectral sequences (yet see Remark 2.4.3(2) of ibid.). This $T_{w_{\text{Chow}}}(H, -)$ does not degenerate at any fixed level (even with rational coefficients, in general), and so its functoriality definitely cannot be proved by ‘classical’ methods.

2. Let $F : \mathbb{C} \to \mathbb{C}'$ be an exact functor that is right weight-exact with respect to $w$ for $\mathbb{C}$ and $w'$ for $\mathbb{C}'$ (see Remark 6.2.3)); let $H : \mathbb{C}' \to \mathbb{A}$ be cohomological. Then in §2.7 of [Bon10a] it was proved: for any $X \in \text{ObjC}$ there exists some comparison morphism of weight spectral sequences $M : T_{w'}(H, F(X)) \to T_w(H \circ F, X)$. Moreover, this morphism is unique and additively functorial starting from $E_2$. The proof uses a natural (and easy) generalization of [3].

In particular, this yields comparison functors from coniveau spectral sequences to Chow-weight ones (see [9] below for more detail).

If $F$ is left weight-exact, there exists a comparison transformation $N$ in the inverse direction. We call both $M$ and $N$ ‘change of weight structures’ transformations.

3. Using the Gersten weight structure (for $\mathcal{D}_s$, see above) one can extend coniveau spectral sequences to $\mathcal{D}_s \supset DM^{eff}_y$ in a natural way (for an arbitrary cohomology theory $H$ defined on $DM^{eff}_y$, such that $\mathbb{A}$ satisfies AB5). This also yields motivic functoriality of coniveau spectral sequences (which is far from being obvious from their definition; see
Remark 4.4.2 of [Bon10a]). Note also that we obtain this functoriality for a not necessarily countable $k$, since one can always define the coniveau spectral sequence for $(H, X)$ over $k$ as the limit of the related coniveau spectral sequences over countable perfect fields of definition of $X$ (see §4.6 of ibid.). Here we use the 'change of weight structure' transformations (that we denoted by $N$ above).

The orthogonality of the Gersten weight structure with the homotopy $t$-structure (for $\text{DM}^{eff}$; see the previous section) yields that the coniveau spectral sequence for $H$ represented by some $Y \in \text{ObjDM}^{eff}$ could be described in terms of the homotopy $t$-truncations of $H$. This extends vastly the coniveau spectral sequence calculations of Bloch&Ogus (in [BOg94]; see §4.5 of [Bon10a]).

4. Since $t_{\text{Post}}$ and $w_{\text{Sh}}$ are adjacent, we obtain the well-known fact: the Atiyah-Hirzebruch spectral sequence converging to $[X, Y]$ for $X, Y \in \text{ObjSH}$ could be expressed either in terms of the $t_{\text{Post}}$-truncations of $Y$ or in terms of $w_{\text{Sh}}$-truncations of $X$ (i.e., in terms of cellular filtration of $X$).

8 More on weight structures

8.1 'Functoriality' of weight structures: localizations and gluing

Weight structures could be carried over to localizations and also 'glued' similarly to $t$-structures.

If $w$ (for $\mathcal{C}$) induces a weight structure also on some triangulated $\mathcal{D} \subset \mathcal{C}$ then it also induces a compatible weight structure on the Verdier quotient $\mathcal{C}/\mathcal{D}$; its heart could be easily calculated (see §8.1 of [Bon07]).

Moreover, one can glue weight structures (i.e. recover a weight structure for $\mathcal{C}$ from those for $\mathcal{D}$ and $\mathcal{C}/\mathcal{D}$ when certain adjoint functors exist) in a way that is just slightly different from those for $t$-structures (see §8.2 of ibid.). We discuss an interesting example of such a gluing in §8.3 below.

This statement was also used in [Bon10p] in (one of the methods of) the construction of the Chow weight structure for motives over $S$.

8.2 The weight complex functor

There are two ways to construct the weight complex functor for a general $(\mathcal{C}, w)$ (that generalizes the exact conservative functor $t : \text{DM}^{eff}_{gm} \to K^b(\text{Chow}^{eff})$ mentioned in Theorem 3.1).

First we describe the 'rigid' method. Suppose that $\mathcal{C}$ has a 'description' in terms of twisted complexes over a negative differential graded category (i.e. a differential graded enhancement; see §2 of [Bon09] or §6 of [Bon07]). Suppose also that $w$ is compatible with this enhancement (i.e. that $w$ coincides with the weight structure given by Proposition 6.2.1 of ibid.). Then there exists an exact weight complex functor $t : \mathcal{C} \to K(Hw)$; see §6.3 of ibid. (actually, in loc.cit. only bounded twisted complexes are considered, so the target of $t$ is $K^b(Hw)$).

The main disadvantage of this method is that it requires some extra information on $\mathcal{C}$. A differential graded enhancement does not have to exist at all (for a general $\mathcal{C}$; for example,
\(SH\) does not have a differential graded enhancement; an exact functor does not have to extend to enhancements (and if such an extension exists, it is not necessarily unique).

Luckily, in [Bon07] another method was developed; it always works and does not depend on any external structures. There is a construction that associates a certain complex to each \(X \in \text{Obj}_{\mathbb{C}}\) for any \(\mathbb{C}\) and depends only on \(w\). It is closely related with the definition of a weight Postnikov tower for \(X\) (see Definitions 1.1.5 and 2.1.2 of [Bon10a]). The terms of the (weight) complex \(t(X)\) are \(X^i = \text{Cone}(w_{\leq i - 1}X \to w_{\leq i}X)[i] \cong \text{Cone}(w_{\geq i}X \to w_{\geq i+1}X)[i-1]\) (see Remark 2.1.3 of loc.cit.); the corresponding triangles yield some boundary morphisms \(X^i \to X^{i+1}\) (see §2.2 of [Bon10a]). It is easily seen that any \(g \in \mathbb{C}(X, X')\) is compatible with some \(t(g) : t(X) \to t(X')\). This method has the following serious disadvantage: in general, \(t(g)\) is only well-defined up to morphisms of the form \(df + gd\) (i.e. modulo an equivalence relation that is more coarse than homotopy equivalence of morphisms of complexes). Still, this equivalence relation has certain nice properties: equivalent morphisms yield the same map on the cohomology of complexes; the homotopy equivalence class of \(t(X)\) does not depend on the choices mentioned. So, we obtain a certain weakly exact functor \(\mathbb{C} \to K_w(Hw)\) (see Definition 3.1.5 of loc.cit.). For any \(H\) one has \(E_1^{pq}T(H, X) = H(X^{-p}[-q])\); hence \(E_2^wT(H, X)\) can be described in terms of \(t(X)\) (in a functorial way); see Remark 3.1.7 of loc.cit.

In the case \(\mathbb{C} = \mathbb{SH}\) we have \(K_w(Hw) = K(Hw)\); so \(t\) is actually an exact functor (see Remark 3.3.4 of ibid.). Moreover, this (weak) weight complex functor is compatible with the (strong) one given by the differential graded approach; see §6.3 of ibid. It is conservative if \(w\) is bounded (i.e. if \(\cap_{i \in \mathbb{Z}} \mathbb{C}^{w \leq 0}[i] = \cap_{i \in \mathbb{Z}} \mathbb{C}^{w \geq 0}[i] = \{0\}\); see Theorem 3.3.1 of ibid. for the proof of this fact and of several other nice properties of \(t\).

### 8.3 Certain \(K_0\)-calculations

Suppose that \(w\) is bounded, \(Hw\) is idempotent complete. Then \(\mathbb{C}\) is idempotent complete also; see Lemma 5.2.1 of ibid. In particular, this allows to prove that \(DM^{eff}_{gm}\) is generated by \(\text{Chow}^{eff}\) (i.e. the only strict full triangulated subcategory of \(DM^{eff}_{gm}\) containing \(\text{Chow}^{eff}\) is \(DM^{eff}_{gm}\) itself); it seems that §3.5 of [Voe00] does not contain a complete proof of this statement.

Besides, we have \(K_0(\mathbb{C}) \cong K_0(Hw)\). Recall that the generators of \(K_0(\mathbb{C})\) (resp. \(K_0(Hw)\)) are \([X], X \in \text{Obj}\mathbb{C}\) (\(X \in \text{Obj}_{Hw}\)), and the relations are: \([B] = [A] + [C]\) if \(A \to B \to C\) is a distinguished triangle (resp. \(B \cong A \bigoplus C\)).

In particular, we obtain Theorem 8.3.14 this way.

### 8.4 A generalization: relative weight structures

Now we describe a formalism that generalizes those of weight structures. It is actual since in the (derived) category of mixed complexes of sheaves over a variety \(X_0\) defined over a finite field \(\mathbb{F}_q\) the subcategories of objects of non-positive and non-negative weights do not quite satisfy the orthogonality axiom (iii) of Definition 1.1. So, we adjust this axiom in order make it compatible with Proposition 5.1.15 of [BBDS2]. Note here: in our notation the roles of \(\mathbb{C}^{w \leq 0}\) and \(\mathbb{C}^{w \geq 0}\) are permuted with respect to the notation of ibid.
Definition 8.1. Let \( F : \mathcal{C} \to \mathcal{D} \) be an exact functor (of triangulated categories).

A pair of extension-stable (see Theorem 4.3(2)) Karoubi-closed subclasses \( \mathcal{C}^{w \leq 0}, \mathcal{C}^{w \geq 0} \subset \text{ObjC} \) for a triangulated category \( \mathcal{C} \) will be said to define a relative weight structure \( w \) for \( \mathcal{C} \) with respect to \( F \) (or just \( F \)-weight structure) if they satisfy conditions (ii) and (iv) of Definition 4.1 as well as the following orthogonality assumptions:

\[
\mathcal{C}^{w \geq 0} \perp \mathcal{C}^{w \leq 0}[2]; \quad F \text{ kills all morphisms between } \mathcal{C}^{w \geq 0} \text{ and } \mathcal{C}^{w \leq 0}[1].
\]

Relative weight structures satisfy several properties similar to those of 'absolute' weight structures (note: an 'absolute' weight structure is the same thing as an \( \text{idC} \)-weight structure); see below.

9 'Motivic’ weight structures; comotives; gluing Chow and Gersten structures from 'birational slices’

We briefly summarize how weight structures help in the proof of Theorem 3.1 (this information could be found above, yet it is somewhat scattered). We also make several other remarks.

Weight structures yield a mighty instrument for constructing and studying certain functorial spectral sequences for cohomology functors (defined on a triangulated category \( \mathcal{C} \)); so they also yield certain functorial ('weight') filtration. They also describe how objects of \( \mathcal{C} \) could be 'constructed from' objects of a 'more simple' additive \( \mathcal{H} \subset \mathcal{C} \).

We have two main 'motivic' weight structures. They correspond to (Chow)-weight and coniveau spectral sequences, respectively. Note that both of these spectral sequences were 'classically' defined only for cohomology of varieties; still our approach allows to define them for arbitrary Voevodsky’s motives, and also yields their motivic functoriality (which is very far from being obvious).

9.1 Chow weight structure(s); relation with mixed motivic \( t \)-structure and weight filtration

Our first ('motivic') weight structure (being more precise, we have a system of compatible weight structures on distinct 'motivic' categories) is \( w_{\text{Chow}} \); it is defined on \( DM^{eff}_{gm} \subset DM_{gm} \), its heart is \( \text{Chow}^{eff} \subset \text{Chow} \); \( w_{\text{Chow}} \) can also be extended to \( DM^{eff} \) and \( \mathcal{Q} \).

So, it closely relates \( DM^{eff}_{gm} \) with \( \text{Chow}^{eff} \) (in particular, the weight complex functor \( DM^{eff}_{gm} \to K^b(\text{Chow}^{eff}) \) is conservative; note that \( DM^{eff}_{gm} \) is very far from being isomorphic to \( K^b(\text{Chow}^{eff}) \)). So, the cohomology of Voevodsky’s motives can be 'functionally related' with the cohomology of Chow ones; one obtains a vast generalization of Deligne’s weight spectral sequences.

Besides, there exists a \( \text{Chow} t \)-structure for \( \mathcal{M}^{eff} \) such that \( H^t_{\text{Chow}} \cong \text{AddFun}(\text{Chow}^{eff}, \text{Ab}) \); \( t_{\text{Chow}} \) is adjacent to the Chow weight structure for \( \mathcal{M}^{eff} \).

Now we relate \( w_{\text{Chow}} \) with the 'usual expectations for weights of motives'; see §8.6 of [Bon07] for more detail.

Conjecturally, \( DM^{eff}_{gm} \mathbb{Q} \) (and \( DM_{gm} \mathbb{Q} \)) should support a mixed motivic \( t \)-structure \((t_{MM}, \text{whose heart is the abelian category } MM \text{ of mixed motives}) \) and a weight filtration.
(by certain triangulated subcategories); the latter one comes from certain weight filtration functors $MM \to MM$ (compatible via cohomology with the weight filtration of mixed Hodge structures and of mixed Galois modules; these functors are idempotent). So, there should be three important filtrations for $DM_{gm}^{eff} \subset DM_{gm}Q$ altogether.

Now, one can easily verify that the (widely believed to be true) conjectural properties of the two conjectural filtrations mentioned yield: for a subcategory of objects that are ‘pure of some fixed weight $i$’ with respect to one of these three filtrations, the filtrations induced by two remaining structures differ only by a shift of indices (that depends on $i$). In particular, $t_{MM}$ ‘should split’ Chow motives into components that are ‘pure with respect to the weight filtration’. $w_{Chow}$-weight decompositions induce the (conjectural!) weight filtration for mixed motives. Note here: though weight decompositions (of objects of triangulated categories) are (usually) highly non-unique, for any $i \in \mathbb{Z}$, $X \in \text{Obj}MM \subset \text{Obj}DM_{gm}^{eff}Q$, there ‘should exist’ a unique weight decomposition of $X[i]$ such that $w_{\leq i}X, w_{\geq i+1}X \in MM$; this choice of $w_{\geq i+1}X$ is what one expects to be the corresponding level of the weight filtration of $X$ in $MM$.

In [Wil09at] this (conjectural) picture was justified in the case when $k$ is a number field for the triangulated category $DAT \subset DM_{gm}^{eff}Q$ (of so-called Artin-Tate motives; this is the triangulated subcategory of $DM_{gm}^{eff}Q$ generated by Tate twists of motives of spectra of finite extensions of $k$). It was also shown that the restriction of $w_{Chow}$ to $DAT$ can be completely characterized in terms of weights of singular homology. Actually, this corresponds to the fact that the triangulated category $DHS$ of mixed Hodge complexes has a weight filtration (by triangulated subcategories) and could be endowed with a weight structure; these filtrations and the ‘canonical’ $t$-structure for $DHS$ are connected by the same relations as those that ‘should connect’ the corresponding filtrations of $DM_{gm}^{eff}Q \subset DM_{gm}Q$. It could be easily seen that singular (co)homology ‘respects’ weight structures; it should also ‘strictly respect’ them (and this was essentially proved in [Wil09at] for Artin-Tate motives).

9.2 Comotives; the Gersten weight structure

Our second ‘motivic’ weight structure is the Gersten weight structure $w$ defined on the category $D_s \supset DM_{gm}^{eff}$ (for a countable $k$). Here $D_s$ is a full triangulated subcategory of a certain category $D$ of comotives (already mentioned in Theorem 3.1).

The idea is that $w$ should be orthogonal to the homotopy $t$-structure on $DM_{gm}^{eff}$ (recall that the latter is the restriction of the canonical $t$-structure of the derived category of Nisnevich sheaves with transfers). So, $Hw$ is ‘generated’ by comotives of function fields over $k$ (note that these are Nisnevich points); in particular, $w$ cannot be defined on $DM_{gm}^{eff}$ (or on $DM_{gm}^{eff}$).

The problem with $DM_{gm}^{eff} \supset DM_{gm}^{eff}$ is that there are no ‘nice’ homotopy limits in it. In order to have them one needs ‘nice’ (small) products; one also needs the objects of $DM_{gm}^{eff}$ to be cocompact (in this ‘category of homotopy limits’). $DM_{gm}^{eff}$ definitely does not satisfy these conditions. Instead in §5 of [Bon10a] a category $D'$ that is opposite to a certain category of differential graded modules (i.e. covariant differential graded functors from the differential graded enhancement of $DM_{gm}^{eff}$ to complexes of abelian groups) was considered; $D$ is its homotopy category (with respect to a certain closed model structure; so it is opposite to the corresponding derived category of differential graded modules). So,
we have a contravariant Yoneda embedding of $DM_{gm}^{eff}$ to the category opposite to $\mathcal{D}$ whose image consists of compact objects; in this category 'nice' homotopy colimits exist. Thus, inverting arrows we obtain a 'nice' category of comotives. Inside $\mathcal{D}$ we define $\mathcal{D}_s$ as its smallest Karoubi-closed triangulated category that contains comotives of functions fields. Note: we need $k$ to be countable since without this the author does not know how to prove that (our candidate for) $Hw$ is negative; still comotives can be defined over any perfect $k$.

The general theory of weight spectral sequence yields those for cohomological functors $\mathcal{D}_s \to \mathcal{A}$, the problem here is that $\mathcal{D}_s$ is a 'large' and rather 'mysterious' category. Yet, any $H : DM_{gm}^{eff} \to \mathcal{A}$ has a 'nice' extension to $\mathcal{D}_s$ (and also to $\mathcal{D} \supset \mathcal{D}_s$) if $\mathcal{A}$ satisfies AB5 (see Proposition 4.3.1 of [Bon10a]). So, we can consider weight spectral sequences $T = T_w(H, X)$ for any such $H$ and any $X \in ObjDM_{gm}^{eff}$ or $X \in Obj\mathcal{D}_s$. It turns out that for $X$ being the motif of a smooth variety, $T$ is isomorphic to the coniveau spectral sequence (corresponding to $H$) starting from $E_2$; see Proposition 4.4.1 of ibid. So, we call $T$ a coniveau spectral sequence for any $X$. As well as for 'classical' coniveau spectral sequences, if $H$ is represented by an object of $DM_{gm}^{eff}$, $T_w(H, X)$ can be described in terms of cohomology of $X$ with the coefficients in the homotopy $t$-truncations of $H$ (see Corollary 4.5.3 of ibid.); this fact extends the related results of Bloch-Ogus and Paranjape (see [BO94] and [Par96]). The latter result follows from the existence of a nice duality $\mathcal{D}^{op} \times DM_{gm}^{eff} \to \text{Ab}$. 

Remark 9.1. $w$ can be restricted to the category $DAT \subset DM_{gm}^{eff}$ of Artin-Tate motives (mentioned above; one may take integral coefficients here; $k$ is any perfect field). Indeed, we don’t need comotives here, since (co)motives of (spectra of) finite extensions of $k$ belong to $ObjDM_{gm}^{eff}$.

We explain this in more detail. $DAT$ is generated by $\mathcal{M}(F)(j)[j]$, where $F$ runs through all (spectra of) finite field extensions of $k$, $j \geq 0$. $D = \{ \oplus_i \mathcal{M}(F_i)(j_i)[j_i] \}$ is a negative (additive) subcategory of $DAT$, so Theorem 4.3(5) implies: there exists a weight structure $w_{DAT}$ with $D \subset Hw_{DAT}$. Since $Hw_{DAT} \subset Hw(\subset \mathcal{D})$, we obtain that $w_{DAT}$ is compatible with $w$ (at least, for a countable $k$).

In particular, this implies that coniveau spectral sequences for cohomology of any $X \in ObjDAT$ have quite 'economical' descriptions (starting from $E_2$).

9.3 Comparison of weight structures; 'gluing from birational slices'

Now, we describe the relation between $T' = T_{w_{Chow}}(H, X)$ and $T = T_w(H, X)$ (for $X$ being a motif or even $X \in Obj\mathcal{D}_s$). Firstly, the 'change of weight structure transformation' (see Remark 4.4.5) yield some morphism $M : T' \to T$ (functorially starting from $E_2$; see §4.8 of [Bon10a]). $M$ is an isomorphism if $H$ is birational i.e. kills $DM_{gm}^{eff}(1)$; here $- \otimes \mathbb{Z}(1)$ is the Tate twist isomorphism of $DM_{gm}^{eff}$ into itself.

Now, $- \otimes \mathbb{Z}(1)$ can be extended from $DM_{gm}^{eff}$ to $\mathcal{D}$ (see §5.4.3 of ibid.); this is also true for $w_{Chow}$ (see §4.7 of ibid.). It is easily seen that $w$ and $w_{Chow}$ induce the same weight structure $w_{bir}^G$ on the category of birational comotives $\mathcal{D}_{bir} = \mathcal{D}/\mathcal{D}(1)$ (the Verdier quotient); the heart of this localization contains images of all (co)motives of all smooth varieties. One obtains that (roughly!) $w$ and $w_{Chow}$ coincide on slices and only differ by the value of a single integral parameter: $w$ is $- \otimes \mathbb{Z}(1)[1]$-stable and $w_{Chow}$ is $- \otimes \mathbb{Z}(1)[2]$-stable!

We try to make this more precise; see §4.9 of ibid. for more details. We consider the localizations $\mathcal{D}/\mathcal{D}(n)$ for all $n > 0$. Though none of them is isomorphic to $\mathcal{D}$, they
'approximate it pretty well'. Also, for all $n$ we have short exact sequences of triangulated categories $\mathcal{D}/\mathcal{D}(n) \xrightarrow{i_*} \mathcal{D}/\mathcal{D}(n+1) \xrightarrow{j^*} \mathcal{D}_{bir}$. Here the notation for functors comes from the 'classical' gluing data setting (cf. §8.2 of [Bon07]); $i_*$ can be given by $- \otimes \mathbb{Z}(1)[s]$ for any $s \in \mathbb{Z}$, $j^*$ is just the localization. Now, if we choose $s = 2$ then both $i_*$ and $j^*$ are weight-exact with respect to weight structures induced by $w_{Chow}$ on the corresponding categories; if we choose $s = 1$ these functors are weight-exact with respect to the weight structures coming from $w$. So, the Chow and Gersten weight structures induce weight structures on the localizations $\mathcal{D}(n)/\mathcal{D}(n+1) \cong \mathcal{D}_{bir}$ (we call these localizations 'slices') that differ only by a shift.

One can show that for any short exact sequence $\mathcal{D} \xrightarrow{i_*} \mathcal{C} \xrightarrow{j^*} \mathcal{E}$ of triangulated categories, if $\mathcal{D}$ and $\mathcal{E}$ are endowed with weight structures such that both $i_*$ and $j^*$ weight-exact. So, if one calls the filtration of $\mathcal{D}$ by $\mathcal{D}(n)$ the slice filtration (this term was already used by A. Huber, B. Kahn, M. Levine, V. Voevodsky, and other authors for other 'motivic categories'), then one may say that the weight structures induced by $w$ and $w_{Chow}$ on all $\mathcal{D}/\mathcal{D}(n)$ 'can be recovered from slices'; the only difference between them is 'how we shift the slices'!

Moreover, Theorem 8.2.3 of [Bon07] shows that if both adjoints to both $i_*$ and $j^*$ exist, then one can use this gluing data in order to 'glue' (any pair) of weight structures for $\mathcal{D}$ and $\mathcal{E}$ into a weight structure for $\mathcal{C}$. So, suppose that we have a weight structure $w_{n,s}$ for $\mathcal{D}/\mathcal{D}(n)$ that is $- \otimes (1)[s]$-stable and 'compatible with $w_{bir}$ on all slices'. Then we can also construct $w_{n+1,s}$ satisfying similar properties, since general homological algebra yields that all adjoints needed exist in our situation. So, $w_{n,s}$ exist for all $n > 0$ and all $s \in \mathbb{Z}$. Moreover, there exists a 'large' subcategory of $\mathcal{D}$ (containing $DM^{eff}_{gm}$) that for any $s$ can be endowed with a weight structure $w_s$ compatible with all $w_{n,s}$. Hence Gersten and Chow weight structures (for $\mathcal{D}_s/\mathcal{D}_s(n) \subset \mathcal{D}/\mathcal{D}(n)$) are members of a rather natural family of weight structures indexed by a single integral parameter! It could be interesting to study other members of this family (for example, the one that is $- \otimes \mathbb{Z}(1)$-stable).

### 9.4 Weights for relative motives and mixed sheaves

Let $S$ be a scheme of finite type over some excellent noetherian scheme $S_0$ of dimension $\leq 2$.

As we have already said, on the category $DM^c(S)$ (of constructible i.e. 'geometric' motives with rational coefficients over $S$) there exists a weight structure $w_{Chow}$ whose heart $Chow(S)$ is the idempotent completion of $\{M_S(X)(n)[2n]\}$, for $X$ running through all regular schemes equipped with a projective (or proper) morphism to $S$, $n \in \mathbb{Z}$ (see [Heb10] and [Bon10p]).

The corresponding Chow-weight spectral sequences yield: for any cohomological $H : DM^c(S) \to A$, $X \in \text{Obj}DM^c(S)$, there exists a filtration on $H^*(X)$ (that is $DM^c(S)$-functorial in $X$) whose factors are subfactors of cohomology of some regular projective $S$-schemes.

We also obtain that $K_0(DM^c(S)) \cong K_0(Chow(S))$ (see §3.3). The author hopes that this observation could be useful for motivic integration. Note that we obtain a ring structure on $K_0(Chow(S))$ this way (since $DM^c(S)$ is a tensor triangulated category) in spite of the fact that the tensor product on $DM^c(S)$ does not restrict to $Chow(S)$.

Now denote by $\mathbb{H}$ the étale realization functor $DM^c(S) \to DSH$, where $DSH = DSH(S)$.
is the category $D^b_m(S,\mathbb{Q})$ of mixed complexes of $\mathbb{Q}_l$-étale sheaves as considered in \cite{Hub97} and in \cite{BBDS2}. Then $H$ sends Chow motives over $S$ to pure complexes of sheaves (see Definition 3.3 of \cite{Hub97}). We deduce certain consequences from this fact.

Suppose that $S$ is a finite type scheme over a number ring. We take $H_{per}$ being the perverse étale cohomology theory, i.e. $H^i_{per}(M)$ (for $M \in \text{Obj}_{DM}^c(S)$, $i \in \mathbb{Z}$) is the $i$-th cohomology of $H(M)$ with respect to the perverse $t$-structure of $DSH$ (see Proposition 3.2 of \cite{Hub97}). Then $T_{w_{\text{Chow}}}(H_{per}, M)$ for any $M \in \text{Obj}DM^c(S)$ yields: all $H^i_{per}(M)$ have weight filtrations (defined using Definition 3.3 of loc.cit., for all $i \in \mathbb{Z}$). Note that this is not at all automatic (for perverse sheaves over $S$); see Remark 6.8.4(i) of \cite{Jan90}. Certainly, one can replace perverse sheaves over $S$ here by $\mathbb{Q}_l$-adic representations of the absolute Galois group of the function field of $S$; cf. §6.8 of loc.cit.

Now let $S = X_0$ be a variety over a finite field $\mathbb{F}_q$; let $X$ denote $X_0 \times_{\text{Spec} \mathbb{F}_q} \text{Spec} \mathbb{F}$, where $\mathbb{F}$ is the algebraic closure of $\mathbb{F}_q$. The results of §5 of \cite{BBDS2} (along with some of the results of \cite{Bon10}) yield that the category $DSH(= D^b_m(X_0, \mathbb{Q}))$ can be endowed with an $\mathbb{F}$-weight structure $w_{DSH}$ whose heart is the category of pure complexes of sheaves, for $\mathbb{F}$ being the extension of scalars functor $DSH \to D^b(X, \mathbb{Q})$. In particular, we obtain that any object $M$ of $DSH$ possesses a 'filtration' (a weight Postnikov tower) whose 'factors' belong to $H_{w_{DSH}}$.

Next, our $H$ is a weight-exact functor (i.e. it sends $DM^c(S)^{w_{\text{Chow}} \leq 0}$ to $DSH^{w_{DSH} \leq 0}$ and sends $DM^c(S)^{w_{\text{Chow}} \geq 0}$ to $DSH^{w_{DSH} \geq 0}$). Hence this is no wonder that the weight-exactness properties of motivic base change functors (for $DM^c(-)$, as established in \cite{Bon10}) are parallel to 5.1.14 of \cite{BBDS2}.

Lastly, let $G : D^b(X, \mathbb{Q}) \to A$ be any cohomological functor, $H = G \circ F$, $M \in \text{Obj}DM^c(S)$. Then the weight-exactness of $H$ yields that the (Chow)-weight filtration for $(H \circ H)^*(M)$ is exactly the $w_{DSH}$-weight filtration for $H^*(H(M))$.

Very probably, some analogues of these results are valid for $H$ replaced by a ‘Hodge module’ realization of motives (for $S$ being a complex variety); the problem is that (to the knowledge of the author) no such realization is known at the moment.

10 Possible applications to finite-dimensionality of motives

Recall that $DM_{gm}^{eff} \subset DM_{gm}$, as well as their ‘rational versions’ $DM_{gm}^{eff, \mathbb{Q}} \subset DM_{gm, \mathbb{Q}}$ (see Remark 3.1(2)) are tensor triangulated categories. This allows to define external and symmetric powers of objects in two latter categories, since those are direct summands of tensor powers (for $\mathbb{Q}$-linear motivic categories).

$M \in DM_{gm, \mathbb{Q}}$ is called Kimura-finite (or finite-dimensional) if $M = M_1 \bigoplus M_2$, where some external power of $M_1$ and some symmetric power of $M_2$ is 0. In this case $M_1$ is called evenly finite-dimensional. Now, $t_{\mathbb{Q}} : DM_{gm}^{eff, \mathbb{Q}} \to K^b(Chow^{eff, \mathbb{Q}})$ (the rational version of the weight complex) is a conservative tensor functor; so $X \in \text{Obj}DM_{gm}^{eff}$ (or $DM_{gm}$, or $\text{Obj}DM_{gm, \mathbb{Q}}$) is Kimura-finite whenever $t_{\mathbb{Q}}(X)$ is.

Now we describe a series of motives that ‘should be’ finite-dimensional (very similar objects were considered by A. Beilinson and M. Nori though in somewhat distinct contexts).

Let $X/k$ be smooth affine of dimension $n$, $Y$ be its generic hyperplane section (with respect to some projective embedding). Then for $M = (Y \to X)$ the only non-zero cohomology
is $H^*_c(M_{\kappa^{vir}})$. Hence some external power of $M \otimes \mathbb{Q}[-n]$ should vanish (since a certain external power of its cohomology vanishes). So $M[-n]$ should be evenly finite-dimensional.

We can also pass to $K^b(Chow_{eff}^r(Q))$ here (i.e. consider $t(M)$ instead of $M$) since the rational version of the weight complex functor is a tensor functor.

**Remark 10.1.**
1. If all such $M$ are Kimura-finite at least numerically (i.e. we consider their images in $K^b(Mot_{num})$ obtained via $t$), then one can prove that $Mot_{num}$ is a tannakian category.

2. Widely-believed conservativity of étale cohomology (as a functor on $DM^r_{gm}(\mathbb{Q})$) immediately implies that all such $M$ are Kimura-finite indeed (as mentioned above). Alternatively, it is possible to deduce Kimura-finiteness of $M$ from a certain weak Lefschetz for motivic cohomology. The latter should be true since it easily follows from the (widely believed, yet conjectural!) existence of a reasonable motivic t-structure for $DM^r_{gm}(\mathbb{Q})$.

Unfortunately, the author has no idea how to prove anything here unconditionally.

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