the space-time—each integral functional
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Jacobi’s identity for to a separate paper which is
theory’s key element, the proof of Theorem 4.(iii) with
definitions of identity.
understand the mechanism for validity of the Jacobi
vision of the geometry of iterated variations and
threefolds.
the BV-quantization of Chern–Simons models over
deformation quantization, which is not restricted to
zero-curvature geometry for the inverse scattering via

2 Let all functionals that take field configurations to number be
2 In fact, all these BV-, Poisson, or IST models are examples of varia-
tions (see [2]); no ad hoc regularizations occur anywhere in this theory.
3 Test shift of the fields brings its own copy of the domain
of integration into the setup; the locality of couplings
between (co)vectors attached at the domains’ points
ensures a restriction to diagonals in the accumulated
products of bundles, whereas the operational definitions
of Δ and [ , ] are on-the-diagonal reconfigurations
of such couplings. We expect that the reader is familiar
with the concept and notation from §1–2.4 in
[2]. In particular, we let the notation for total deriva-
tives which stem from integrations by parts keep track
of the variations’ arguments, so that (δs /δy(y))∂L(x, [q], [q′])/∂qτ at y = x becomes δs(y).
4 It is readily seen from the proof below that composite objects
such as brackets of functionals retain a kind of memory of the
way how they were produced; in effect, variational derivatives
detect the traces of original objects’ own geometries, whence
a variation within one of them does not mar any of the others.
5 In this note we let the arrow over a variational derivative indi-
cate the direction along which all derivatives act—but not the
opposite direction along which the test shifts were transported
prior to any integration by parts (cf. [2]); we thus have ∂s /∂y(y)∂L(x, [q], [q′])/∂qτ on that diagonal, see
Example 2.4 on pp. 34–36 of [2]. Similarly, the vari-
tional derivatives with respect to (anti)fields q or q′
keep track of the test shifts which those variations
come from: e.g., the formula above yields a term in
deformations, is developed in [2]. We reserved that
theory’s key element, the proof of Theorem 4.(iii) with
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this note. Referring to [2] for detail and discussion, let
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1 The article is published in the original.
2 In fact, all these BV-, Poisson, or IST models are examples of vari-
tional Lie algebroids [4] and their encoding by Q-faces, the con-
struction of gauge automorphisms for the Q-cohomology deter-
mines the next generation of such structures, with new deformation
quantization parameters beyond the Planck constant.
3 Let all functionals that take field configurations to number be
integral in this note; formal (sums of) products of functionals
such as exp\{i \hbar S\} are dealt with by using the Leibniz rule, see
[2, §2.5].

The Jacobi Identity for Graded–Commutative Variational Schouten Bracket Revisited

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Abstract—This short note contains an explicit proof of the Jacobi identity for variational Schouten bracket in Z2-graded commutative setup; an extension of the reasoning and assertion to the noncommutative geometry of cyclic words (see [1]) is immediate. The reasoning refers to the product bundle geometry of iterated variations (see [2]); no ad hoc regularizations occur anywhere in this theory.

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INTRODUCTION

The Jacobi identity for variational Schouten bracket [1,1] is its key property in several cohomologi-
cal theories. For example, one infers that the BV-Laplacian Δ or quantum BV-operator \( \Omega^h = i\hbar \Delta + \{[\hbar^\delta , \cdot \} \) , are differentials in the Batalin–Vilkovisky formalism (available literature is immense; let us refer to
[2] and [3]) or one deduces that \( \partial \partial = \{[\hbar^\partial , \cdot \} \) , yields the Poisson–Lichnerowicz complex for every variational Poisson bi-vector \( \partial \) , see [1]. Likewise, a realization of zero-curvature geometry for the inverse scattering via
the classical master-equation \( [S, S] = 0 \) opens a way for deformation quantization, which is not restricted to
the BV-quantization of Chern–Simons models over threefolds. Therefore, it is mandatory to have a clear
vision of the geometry of iterated variations and understand the mechanism for validity of the Jacobi identity.

A self-regularized calculus of variations, including the
definitions of Δ and [ , ] and a rigorous proof of their
interrelations, is developed in [2]. We reserved that
theory’s key element, the proof of Theorem 4.(iii) with
Jacobi’s identity for [ , ] to a separate paper which is
this note. Referring to [2] for detail and discussion, let
us recall that—in a theory of variations for fields over
the space-time—each integral functional or every

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Theorem. Let $F$, $G$, and $H$ be $\mathbb{Z}_2$-parity homogeneous functionals; denote by $\cdot$ the grading so that $(-)^{1}$ is the parity. The variational Schouten bracket satisfies the shifted-graded Jacobi identity (cf. Eq. (28) in Theorem 4 (iii) on p. 30 versus Eq. (36) on p. 37 in [2]),

$$
\mathcal{S} = \{F, \mathcal{S} [G, H]\} = \{[F, G], H\} + (-)^{|F| - 1} (-)^{|G| - 1} \{G, [F, H]\}. 
$$

The operator $\{F, \cdot \}$ is a graded derivation of $\mathcal{S}$: identity (1) is the Leibniz rule for it.

Proof. The logic is straightforward as soon as the matching of (co) vectors and reconfigurations of couplings are understood in [2, § 1–2]. We consider first the l.-h.s. of (1). By construction, we have that

$$
\mathcal{S} \{G, \mathcal{S} [F, H]\} = \{[G, F], H\} - (F(x_j) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(F(x_j))) = - (F(x_j) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(F(x_j))).
$$

Now expanding

$$
\{[F, G], H\} = \mathcal{S} [F, \mathcal{S} [G, H]] = \{G, [F, H]\},
$$

we obtain the sum of eight enumerated terms:

- $F(x_j) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(F(x_j))$
- $- (-)^{|F| - 1} F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$
- $- (-)^{|F| - 1} F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$
- $- F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$
- $- F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$
- $- F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$
- $- F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$
- $- F(x_j) \mathcal{S} q(z_{j_2}) G(x_{j_2}) \mathcal{S} q(y_j) - \mathcal{S} q(y_j)(G(x_{j_2}) \mathcal{S} q(z_{j_2}))$

Arguing as above, we see that the term $\{\mathcal{S} [F, G], H\}$ in the r.-h.s. of (1) is

$$
\{[F, G], H\} = \mathcal{S} [F, \mathcal{S} [G, H]] = \{G, [F, H]\},
$$

which cancel out in the two r.-h.s. summands in Jacobi’s identity.
In the same way, we obtain the term $\|G, [F, H]\|$ not yet multiplied by the extra sign factor:

$\begin{align*}
(1) & \quad G(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(2) & \quad + (-)^{|F|}G(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(3) & \quad - G(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot \overrightarrow{\delta}/\delta q(z_3, F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(4) & \quad - (-)^{|F|}G(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(5) & \quad - G(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot \overrightarrow{\delta}/\delta q(z_3, F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(6) & \quad - G(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot \overrightarrow{\delta}/\delta q(z_3, F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(7) & \quad + (x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot \overrightarrow{\delta}/\delta q(z_3, F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3) \\
(8) & \quad + (x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot F(x_3)\overleftarrow{\delta}/\delta q(y_1) \cdot \overrightarrow{\delta}/\delta q(z_3, x_3)\overleftarrow{\delta}/\delta q(y_1, x_3)H(x_3).
\end{align*}$

Terms (1)–(8) are present in the r.h.s. of (1) and terms (9)–(12) cancel out; it is only the indices (3) and (12) which require special attention. Consider (3) in $\|F, G, H\|$; by relabelling the integration variables, $y \mapsto z$ (i.e., by swapping the test shifts), we obtain

$\begin{align*}
F(x_3)\overleftarrow{\delta}/\delta q(z_3) \cdot \overleftarrow{\delta}/\delta q(z_3, G(x_3)) \\
\times \overrightarrow{\delta}/\delta q(y_1, x_3)H(x_3).
\end{align*}$

The variation’s argument in parentheses has grading $|G| - 1$, which yields the sign factor $(-)^{|G| - 1}$ when the left-acting parity-odd variation $\overrightarrow{\delta}/\delta q(y_1)$ is brought to the other side of its argument, becoming $\overrightarrow{\delta}/\delta q(y_1)$.

Hence $(-)^{|G| - 2} \overrightarrow{\delta}/\delta q(y_1)(\overrightarrow{\delta}/\delta q(z_3, G(x_3)))^{(i)} = (-)^{|G| - 1} \overrightarrow{\delta}/\delta q(y_1)(\overrightarrow{\delta}/\delta q(z_3, G(x_3)))^{(ii)}$.

We do the same with (12). Consider such term in $(-)^{|F| - 1} |G, [F, H]|$: clearly, the factor $(-)^{|G|}$ is irrelevant because it is present also near (12) in $\|F, G, H\|$. Transporting the parity-odd variation $\overleftarrow{\delta}/\delta q(z_3)$ around the object of grading $|F| - 1$ in

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10For each term labelled by (1)–(8) in $\|G, [F, H]\|$, let us calculate the product of three signs: one which was written near the respective summand, the other which comes from the reorderings to $F \lesssim G$, and thirdly, $(-)^{|F| - 1}|G| - 1$; here is the list:

(1): $(-)^{|F| - 1}|G| - 1$; (2): $(-)^{|F| - 1}|G| - 1$; (3): $(-)^{|F| - 2}G - |G| - 1(G) - 1 = (-)^{|F| - 1}|G| - 1$; (4): $(-)^{|F| - 1}|G| - 1$; (5): $(-)^{|F| - 1}|G| - 1|G| - 1 = (-)^{|F| - 1}|G| - 1$; (6): $(-)^{|F| - 1}|G| - 1$; (7): $(-)^{|F| - 1}|G| - 1|G| - 1 = (-)^{|F| - 1}|G| - 1$; (8): $(-)^{|F| - 1}|G| - 1|G| - 1 = (-)^{|F| - 1}|G| - 1$. 
parentheses, we gain the factor $(-)^{|F| - 2}$, which cancels out with $(-)^{|F|}$. Next, relabel $y \rightarrow z$, which gives

$$F(x_1)\frac{\delta}{\delta q^4} (z_{13})$$

$$\times \frac{\delta}{\delta q^1} (y_1) \cdot G(x_2)\frac{\delta}{\delta q^2} (y_2) \cdot \frac{\delta}{\delta q^3} (z_{13}) H(x_3).$$

The parity-odd variations follow in the order which is reverse with respect to that in $\langle \{ F, G, H \} \rangle$, hence these terms cancel out. The proof is complete.

CONCLUSIONS

Variations $\delta \sigma$ act via graded Leibniz rule on products of integral functionals, e.g., $F \cdot \{ G, H \}$; within composite objects like $\{ F, G, H \}$, they act also by derivation w.r.t. own geometries of the blocks $G, H$; variations are graded-permutable in each block. Neither $\Delta$ nor $\delta$ depend on a choice of normalized test shift $\delta \sigma$. This yields (1) and $\Delta^2 (F \cdot G \cdot H) = 0$.

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