SYMMETRY AND ASYMPTOTIC BEHAVIOR OF GROUND STATE SOLUTIONS FOR SCHRÖDINGER SYSTEMS WITH LINEAR INTERACTION

Zhitao Zhang* and Haijun Luo

HLM, CEMS, Academy of Mathematics and Systems Science, the Chinese Academy of Sciences, Beijing 100190
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China

(Communicated by Shouchuan Hu)

Abstract. We study symmetry and asymptotic behavior of ground state solutions for the doubly coupled nonlinear Schrödinger elliptic system

\[
\begin{align*}
-\Delta u + \lambda_1 u + \kappa v &= \mu_1 u^3 + \beta uv^2 \quad \text{in } \Omega, \\
-\Delta v + \lambda_2 v + \kappa u &= \mu_2 v^3 + \beta u^2v \quad \text{in } \Omega, \\
u = v = 0 &\text{ on } \partial \Omega \quad \text{(or } u,v \in H^1(\mathbb{R}^N) \text{ as } \Omega = \mathbb{R}^N),
\end{align*}
\]

where \( N \leq 3, \Omega \subseteq \mathbb{R}^N \) is a smooth domain. First we establish the symmetry of ground state solutions, that is, when \( \Omega \) is radially symmetric, we show that ground state solution is foliated Schwarz symmetric with respect to the same point. Moreover, ground state solutions must be radially symmetric under the condition that \( \Omega \) is a ball or the whole space \( \mathbb{R}^N \). Next we investigate the asymptotic behavior of positive ground state solution as \( \kappa \to 0^- \), which shows that the limiting profile is exactly a minimizer for \( c_0 \) (the minimized energy constrained on Nehari manifold corresponds to the limit systems). Finally for a system with three equations, we prove the existence of ground state solutions whose all components are nonzero.

1. Introduction. In this paper, we are mainly concerned with the doubly coupled nonlinear Schrödinger system (see [10])

\[
\begin{align*}
-i \frac{\partial \Phi}{\partial t} &= \frac{1}{2} \Delta \Phi + \mu_1 |\Phi|^2 \Phi + \beta |\Psi|^2 \Phi - \kappa \Phi \quad \text{for } t > 0, x \in \Omega, \\
-i \frac{\partial \Psi}{\partial t} &= \frac{1}{2} \Delta \Psi + \mu_2 |\Psi|^2 \Psi + \beta |\Phi|^2 \Psi - \kappa \Phi \quad \text{for } t > 0, x \in \Omega, \\
\Phi = \Phi(t, x) \in \mathbb{C}, \Psi = \Psi(t, x) \in \mathbb{C}, t > 0, x \in \Omega,
\end{align*}
\]

where \( N \leq 3, \Omega \subseteq \mathbb{R}^N \) is a smooth domain, \( \mu_1, \mu_2 \) are positive constants, \( \kappa, \beta \) are linear and nonlinear coupling constants respectively. System (1) models naturally many physical problems, especially in nonlinear optics. Physically, the solutions \( \Phi \) and \( \Psi \) denote the first and second component of the beam in Kerr-like photo-refractive media (cf.[1]). The positive constant \( \mu_j \) is for self-focusing in the \( j \)-th...
The component of the beam, \( j = 1, 2 \). The nonlinear coupling constant \( \beta \) is the interaction between the two components of the beam. As \( \beta > 0 \), the interaction is attractive, but the interaction is repulsive if \( \beta < 0 \). The linear coupling is generated either by a twist applied to the fiber in the case of circular polarization, or by an elliptic deformation of the fibers core in the case of circular polarizations. Problem (1) also arises in the Hartree-Fock theory for Bose-Einstein condensates, in which all the above parameters have the specific physical meaning, for more details we refer the reader to [10, 13, 19, 21, 25].

To obtain solitary wave solutions, we set \( \Phi = e^{i\lambda_1 t}u(x) \) and \( \Psi = e^{i\lambda_2 t}v(x) \), system (1) is reduced to the following elliptic system for \( u, v \) with Dirichlet boundary conditions:

\[
\begin{cases}
-\Delta u + \lambda_1 u + \kappa v = \mu_1 u^3 + \beta uv^2 & \text{in } \Omega, \\
-\Delta v + \lambda_2 v + \kappa u = \mu_2 v^3 + \beta u^2 v & \text{in } \Omega, \\
u = v = 0 & \text{on } \partial \Omega \quad \text{(or } u, v \in H^1(\mathbb{R}^N) \text{ as } \Omega = \mathbb{R}^N).}
\end{cases}
\]

(2)

Here the coefficients \( \mu_1, \mu_2, \beta \) and \( \kappa \) are each twice the corresponding coefficients in (1). Besides, we assume that \( \lambda_1, \lambda_2 > 0, \beta \in \mathbb{R} \).

The nonlinear elliptic system (2) has attracted considerable attention in the last ten years. When \( \kappa = 0 \), there are many interesting works devoting to study the quantitative and qualitative properties of solutions to the system (2); see [2, 3, 4, 5, 16, 17, 18, 22] for the existence of ground state or bound state solutions, [23, 27, 28] for the symmetry of least energy solutions or bound state solutions, and [20] about the limits of bound state solutions as \( \beta \to -\infty \). However, when \( \kappa \neq 0, \beta \neq 0 \), that is, linear coupling terms and nonlinear coupling terms both exist, only a few interesting results have been obtained in [6, 11, 15, 24]. To be more specific, the existence of bound state and ground state solutions have been investigated by the topological and variational methods in [6, 15], while the authors in [11, 24] study the bifurcation of synchronized solutions with parameter \( \kappa \) and \( \beta \) respectively. With regard to the other properties (such as symmetry, asymptotic limit and so on) of solutions to system (2), there are no results. This may led us to characterize the symmetry and asymptotic behavior of ground state solutions for (2). In addition, as it has been seen in the case that the linear coupling term doesn’t exist, there is a sharking contract between two components and more than two components for system (2). Taking the fact into account, we will also study the existence of ground state solutions to a doubly coupled system with three components (for simplicity), which can be seen as an extension result compared with that of [15].

We begin with some notations. In the following we always assume that \( |\kappa| < \sqrt{\lambda_1 \lambda_2} \).

Set

\[
\mathcal{H} := H^1_0(\Omega) \times H^1_0(\Omega) \quad \text{as } \Omega \text{ is a smooth bounded domain in } \mathbb{R}^N,
\]

or

\[
\mathcal{H} := H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \quad \text{as } \Omega = \mathbb{R}^N.
\]

Define the inner product on \( \mathcal{H} \) as follows:

\[
((u_1,v_1),(u_2,v_2)) = \int_\Omega \left( \nabla u_1 \cdot \nabla u_2 + \lambda_1 u_1 u_2 + \nabla v_1 \cdot \nabla v_2 + \lambda_2 v_1 v_2 + \kappa u_1 v_1 + \kappa u_2 v_2 \right)
\]

for \((u_1,v_1),(u_2,v_2) \in \mathcal{H}\). Since \( |\kappa| < \sqrt{\lambda_1 \lambda_2} \), it is easy to see that \( \| (u,v) \|_\kappa := (\| (u,v) \|_\kappa)^2 \) is the corresponding norm, which is equivalent to the standard product norm on the product space \( \mathcal{H} \).
We observe that the solutions of (2) correspond to critical points of the energy functional
\[ I_\kappa(u, v) = \frac{1}{2} \|(u, v)\|^2_\kappa - \frac{1}{4} F(u, v), \] (3)
where
\[ F(u, v) := \mu_1 \int_\Omega u^4 + \mu_2 \int_\Omega v^4 + 2\beta \int_\Omega u^2 v^2. \] (4)
Since we suppose that \( N \leq 3 \), the Sobolev embedding implies that \( I_\kappa(u, v) \) is well-defined and of class \( C^2 \).

Introduce the Nehari manifold by
\[ \mathcal{N}_\kappa = \{ (u, v) \in \mathcal{H} : (u, v) \neq 0, I'_\kappa(u, v)(u, v) = 0 \}, \]
then we consider the following minimizing problem
\[ c_\kappa = \inf_{(u, v) \in \mathcal{N}_\kappa} I_\kappa(u, v). \] (5)

**Remark 1.** For \( (u, v) \in \mathcal{N}_\kappa \), we have \( I_\kappa(u, v) = \frac{1}{4} \|(u, v)\|^2_\kappa \), thus an equivalent characterization of \( c_\kappa \) is as follows:
\[ c_\kappa = \inf_{(u, v) \in \mathcal{N}_\kappa} \frac{1}{4} \|(u, v)\|^2_\kappa. \]

**Definition 1.1.** We call a solution \( (u, v) \in \mathcal{H} \) a ground state solution of (2) if \( (u, v) \) achieves \( \inf \{ I_\kappa(u, v) : (u, v) \in \mathcal{N}_\kappa \} \), and call a solution \( (u, v) \in \mathcal{H} \) a least energy positive solution of (2) if \( (u, v) \) achieves \( \inf \{ I_\kappa(u, v) : I'_\kappa(u, v)(u, v) = 0, u > 0, v > 0 \} \).

First, as to the existence of ground state solutions to system (2), let us recall from [15] the following theorem.

**Theorem 1.2 ([15]).** Suppose that \( \lambda_1, \lambda_2, \mu_1, \mu_2 > 0, \beta \in \mathbb{R}, \kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0) \cup (0, \sqrt{\lambda_1 \lambda_2}) \), and that \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) or \( \Omega = \mathbb{R}^N, N \leq 3 \). Then system (2) has a ground state solution \( (u, v) \). Moreover, \( u > 0, v > 0 \) as \( \kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0) \); \( u > 0, v < 0 \) or \( u < 0, v > 0 \) as \( \kappa \in (0, \sqrt{\lambda_1 \lambda_2}) \).

**Remark 2.** (1) In fact, in Theorem 1.2, as \( \kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0) \), there exists a ground state solution \( (u, v) \) with \( u < 0, v < 0 \). Thus, we can get the third nontrivial solution by mountain pass theorem constrained on the Nehari manifold \( \mathcal{N}_\kappa \).

(2) According to the proof of theorem 1.2(cf.[15]), any minimizer \((u_\kappa, v_\kappa)\) for \( c_\kappa \) is a solution of system (2). Note that (2) has no semi-trivial solution which assumes the form of \( (u, 0) \) with \( u \neq 0 \) or \( (0, v) \) with \( v \neq 0 \), thus any minimizer for \( c_\kappa \) is a ground state solution. However, there may exist semi-trivial solutions for a system with more than two equations, which shows that to find its ground state solutions(cf. Definition 1.7) may be more involved.

We are in position to give our main results.

### 1.1. Symmetry results.
When \( \beta \geq 0, \kappa = 0 \) and the underlying domain \( \Omega \) is radially symmetric, the authors in [27] proved the partial symmetry of the least energy solutions to system (2). For \( \kappa \neq 0 \), we can obtain a similar result.

Before the following theorem, let us recall that a positive function \( u \) defined on a radially symmetric domain \( \Omega \) is said to be foliated Schwarz symmetric with respect to \( e \in \mathbb{S}^{N-1} \) if \( u \) depends only on \((r, \theta) = (|x|, \arccos(x \cdot e)/|x|)\) and is non-increasing in \( \theta \). Thus for a negative function \( u \), we can call \( u \) foliated Schwarz symmetric with
respect to $e \in S^{N-1}$ if $u(x)$ depends only on $(r, \theta) = (|x|, \arccos(x \cdot e)/|x|)$ and is non-decreasing in $\theta$. Let $B_r(0) := \{x \in \mathbb{R}^N, |x| < r\}, r > 0$.

Now we state our theorems.

**Theorem 1.3.** Suppose that $N = 2$ or 3, $\beta \geq 0, \kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0) \cup (0, \sqrt{\lambda_1 \lambda_2})$, and that $\Omega$ is a bounded radial domain in $\mathbb{R}^N$ (i.e., $\Omega = B_r(0), r > 0$ or $\Omega = B_r(0) \setminus \overline{B}_{r_2}(0), r_1 > r_2 > 0$) or the whole space $\mathbb{R}^N$, let $(u, v)$ be a ground state solution to system (2). Then there exists $e \in S^{N-1}$ such that $u$ and $v$ are foliated Schwarz symmetric with respect to $e$.

Moreover, when $\Omega$ is a ball or the whole space $\mathbb{R}^N$, we can give a further result on the symmetry of ground state solutions.

**Theorem 1.4.** Suppose that $N = 2$ or 3, $\beta \geq 0, \kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0) \cup (0, \sqrt{\lambda_1 \lambda_2})$, and that $\Omega$ is a ball $B_r(0), r > 0$ or the whole space $\mathbb{R}^N$. Then any ground state solution of system (2) must be radially symmetric.

1.2. **Asymptotic behavior as $\kappa \to 0^-$.** If $\kappa = 0$, due to the nonexistence result in [4], system (2) may have no positive solution for some ranges of parameter $\beta$, whereas if $\kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0)$, there is a positive ground state solution to system (2) for any $\beta \in \mathbb{R}$ by Theorem 1.2. In the following, we denote the positive ground state solution to (2) by $(u_\kappa, v_\kappa)$ which is a minimizer for $c_\kappa$. Based on the above fact, we investigate the asymptotic behavior of $(u_\kappa, v_\kappa)$ as $\kappa \to 0^-$.

Specifically, we deal with the case $\lambda_1 = \lambda_2 > 0$ and may assume $\lambda_1 = \lambda_2 = 1$, besides, we only consider the domain $\Omega = \mathbb{R}^N$ and the parameter $\kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0)$. Thus we have

$$
\begin{cases}
-\Delta u + u + \kappa v = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\
-\Delta v + v + \kappa u = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N).
\end{cases}
$$

Fixing $\mu_1, \mu_2 > 0$ we may assume without loss of generality that $\mu_1 \leq \mu_2$.

Consider the limiting system of (6)

$$
\begin{cases}
-\Delta u + u = \mu_1 u^3 + \beta uv^2 & \text{in } \mathbb{R}^N, \\
-\Delta v + v = \mu_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N).
\end{cases}
$$

Let

$$
I_0(u, v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2 + |\nabla v|^2 + v^2)
- \frac{1}{4} \left( \mu_1 \int_{\mathbb{R}^N} u^4 + \mu_2 \int_{\mathbb{R}^N} v^4 + 2\beta \int_{\mathbb{R}^N} u^2 v^2 \right)
= \frac{1}{2} ||(u, v)||_0^2 - \frac{1}{4} F(u, v),
$$

$U_i(x) = \sqrt{\mu_i} w(x)$ ($i = 1, 2$), where $w$ is the unique positive solution of

$$
-\Delta w + w = w^3 \text{ in } \mathbb{R}^N, \quad w(0) = \max_{x \in \mathbb{R}^N} w(x), \quad w(x) \to 0 \text{ as } |x| \to \infty.
$$

By direct computations, we have

$$
I_0(U_1, 0) = \frac{1}{\mu_1} I_0(w, 0), \quad I_0(0, U_2) = \frac{1}{\mu_2} I_0(0, w).
$$

(9)
Since $\mu_1 \leq \mu_2$, then $I_0(w,0) \geq I_0(0,w)$, whence
\begin{equation}
I_0(U_1,0) \geq I_0(0,U_2).
\end{equation}

Besides, if $\mu_1 < \mu_2$, system (7) has a synchronized solution of the form
\begin{equation}
(u^\beta, v^\beta) = \left(\sqrt{\frac{\beta - \mu_2}{\beta^2 - \mu_1 \mu_2}} w(x), \sqrt{\frac{\beta - \mu_1}{\beta^2 - \mu_1 \mu_2}} w(x)\right),
\end{equation}
for $\beta \in (-\sqrt{\mu_1 \mu_2}, \mu_1) \cup (\mu_2, +\infty)$.

But if $\mu_1 = \mu_2 = \mu$, system (7) has a synchronized solution of the form
\begin{equation}
(u^\beta, v^\beta) = \left(\sqrt{\frac{1}{\beta + \mu}} w(x), \sqrt{\frac{1}{\beta + \mu}} w(x)\right),
\end{equation}
for $\beta \in (-\mu, \mu) \cup (\mu, +\infty)$, while for $\beta = \mu$,
\begin{equation}
(\bar{u}_0, \bar{v}_0) = \left(\sqrt{\frac{1}{\beta + \mu}} w(x) \cos \theta, \sqrt{\frac{1}{\beta + \mu}} w(x) \sin \theta\right), \quad \theta \in (0, \pi/2)
\end{equation}
consist of a family of solutions to system (7).

With regard to the asymptotic behavior of positive ground state solutions to (6), we obtain the following two theorems.

**Theorem 1.5.** Suppose that $\Omega = \mathbb{R}^N, N = 2$ or 3, $\lambda_1 = \lambda_2 = 1$, $\mu_1 < \mu_2, \kappa \in (-1,0)$. Let $(u_\kappa, v_\kappa)$ be a positive ground state solution to system (6), then for any sequence $\kappa_n \to 0^-$,

(i) for $\beta < 0$, there exist a subsequence (still denoted by $\kappa_n$) and $\{x_n\} \subseteq \mathbb{R}^N$ such that
\begin{equation}
(u_{\kappa_n}(\cdot + x_n), v_{\kappa_n}(\cdot + x_n)) \to (0, U_2) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N);
\end{equation}

(ii) for $\beta \in [0, \mu_2]$, there exists a subsequence (still denoted by $\kappa_n$) such that
\begin{equation}
(u_{\kappa_n}, v_{\kappa_n}) \to (0, U_2) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N);
\end{equation}

(iii) for $\beta \in (\mu_2, +\infty)$, there exists a subsequence (still denoted by $\kappa_n$) such that
\begin{equation}
(u_{\kappa_n}, v_{\kappa_n}) \to (u^\beta, v^\beta) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).
\end{equation}

**Theorem 1.6.** Suppose that $\Omega = \mathbb{R}^N, N = 2$ or 3, $\lambda_1 = \lambda_2 = 1$, $\mu_1 = \mu_2 = \mu, \kappa \in (-1,0)$. Let $(u_\kappa, v_\kappa)$ be a positive ground state solution to system (6), then for any sequence $\kappa_n \to 0^-$,

(i) for $\beta < 0$, there exist a subsequence (still denoted by $\kappa_n$) and $\{x_n\} \subseteq \mathbb{R}^N$ such that
\begin{equation}
(u_{\kappa_n}(\cdot + x_n), v_{\kappa_n}(\cdot + x_n)) \to (0, U_2) \text{ or } (U_1, 0) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N);
\end{equation}

(ii) for $\beta \in [0, \mu]$, there exists a subsequence (still denoted by $\kappa_n$) such that
\begin{equation}
(u_{\kappa_n}, v_{\kappa_n}) \to (0, U_2) \text{ or } (U_1, 0) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N);
\end{equation}

(iii) for $\beta = \mu$, there exists a subsequence (still denoted by $\kappa_n$) such that
\begin{equation}
(u_{\kappa_n}, v_{\kappa_n}) \to (\bar{u}_{\pi/4}, \bar{v}_{\pi/4}) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N);
\end{equation}

(iv) for $\beta \in (\mu, +\infty)$, there exists a subsequence (still denoted by $\kappa_n$) such that
\begin{equation}
(u_{\kappa_n}, v_{\kappa_n}) \to (u^\beta, v^\beta) \quad \text{strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).
\end{equation}

**Remark 3.** (1) Compared with Theorem 1.5, the reason why the limit profile of $(u_\kappa, v_\kappa)$ for $\beta < \mu$ in Theorem 1.6 assumes the form of $(0, U_2)$ or $(U_1, 0)$ is because of the invariance $\sigma(u, v) = (v, u)$ for the solutions to (6) when $\mu_1 = \mu_2$. 
(2) When $\Omega$ is a ball in $\mathbb{R}^N$, $N = 2$ or $N = 3$, we have similar results which may be easier to prove. For the sake of simplicity, we omit it here.

1.3. **Ground states to the system with three equations.** In this subsection, we briefly discuss the systems with more than two equations. To simplify notation, we focus on a system of three equations like

$$
\begin{cases}
-\Delta u_1 + \lambda_1 u_1 + \kappa_{12} u_2 + \kappa_{13} u_3 = \mu_1 u_1^3 + \beta_{12} u_1 u_2^2 + \beta_{13} u_1 u_3^2 & \text{in } \Omega, \\
-\Delta u_2 + \lambda_2 u_2 + \kappa_{21} u_1 + \kappa_{23} u_3 = \mu_2 u_2^3 + \beta_{21} u_1^2 u_2 + \beta_{23} u_2 u_3^2 & \text{in } \Omega, \\
-\Delta u_3 + \lambda_3 u_3 + \kappa_{31} u_1 + \kappa_{32} u_2 = \mu_3 u_3^3 + \beta_{31} u_1^2 u_3 + \beta_{32} u_2^2 u_3 & \text{in } \Omega,
\end{cases}
$$

where $\Omega \subseteq \mathbb{R}^N$ is a smooth domain, $N \leq 3, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3$ are positive constants, $\kappa_{ij} = \kappa_{ji}, \beta_{ij} = \beta_{ji}, 1 \leq i, j \leq 3, i \neq j$.

Let $\overline{H} := [H^1_0(\Omega)]^3$ as $\Omega$ is bounded in $\mathbb{R}^N$, or $\overline{H} := [H^1(\mathbb{R}^N)]^3$ as $\Omega = \mathbb{R}^N$. Set matrix

$$
K := 
\begin{bmatrix}
\lambda_1 & \kappa_{12} & \kappa_{13} \\
\kappa_{21} & \lambda_2 & \kappa_{23} \\
\kappa_{31} & \kappa_{32} & \lambda_3
\end{bmatrix}
$$

We assume that $K$ is positive definite, then we can define an inner product on $\overline{H}$ as follows.

For any $\vec{u} = (u_1, u_2, u_3), \vec{v} = (v_1, v_2, v_3) \in \overline{H}$,

$$
((u_1, u_2, u_3), (v_1, v_2, v_3)) = \sum_{i=1}^3 \int_\Omega \nabla u_i \nabla v_i + \int_\Omega (u_1, u_2, u_3)K(v_1, v_2, v_3)^T,
$$

where $\top$ represents for vector transpose. Thus,

$$
\|((u_1, u_2, u_3))_K := ((u_1, u_2, u_3), (u_1, u_2, u_3))^{1/2}
$$

is the corresponding norm.

**Remark 4.** Since $K$ is positive definite, denoting the smallest and biggest eigenvalues by $\sigma_{\min}$ and $\sigma_{\max}$ respectively, we have $\sigma_{\max} \geq \sigma_{\min} > 0$. This implies that $\|((u_1, u_2, u_3))_K$ is equivalent to the standard product norm on the product space $\overline{H}$.

For convenience, set $\beta_{ii} = \mu_i$ ($1 \leq i \leq 3$). Then the corresponding energy functional is

$$
I_K(\vec{u}) = \frac{1}{2} \|(u_1, u_2, u_3)\|^2_K - \frac{1}{4} \sum_{i,j=1}^3 \int \beta_{ij} u_i^2 u_j^2,
$$

and the Nehari manifold is

$$
\mathcal{N}_K = \{\vec{u} \in \overline{H} : \vec{u} \neq 0, Q(\vec{u}) = 0\},
$$

where

$$
Q(\vec{u}) = Q((u_1, u_2, u_3)) := \int (u_1, u_2, u_3)\nu(\vec{u})d\vec{u}.
$$

We consider the following minimizing problem

$$
c_K = \inf_{\vec{u} \in \mathcal{N}_K} I_K(\vec{u}).
$$

**Definition 1.7.** We call a solution $\vec{u} = (u_1, u_2, u_3) \in \overline{H}$ a ground state solution of (11) if $\vec{u}$ achieves $\inf\{I_K(\vec{u})/c_K(\vec{u}) = 0, u_i \neq 0, i = 1, 2, 3\}$.

The following theorem gives the existence of ground state solutions to system (11).
Theorem 1.8. Suppose that \( \lambda_i > 0, \mu_i > 0, i = 1, 2, 3; \beta_{ij} \in \mathbb{R}, \beta_{ij} = \beta_{ji}, 1 \leq i \neq j \leq 3, K \) is positive definite with \( \kappa_{12}\kappa_{13}\kappa_{23} < 0 \), and that \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^N \) or the whole space \( \mathbb{R}^N \), \( N \leq 3 \). Then there is a ground state solution \( \bar{u} = (u_1, u_2, u_3) \) to system (11).

Moreover, when \( K \) is positive definite and \( \kappa_{12} < 0, \kappa_{13} < 0, \kappa_{23} < 0 \), we have \( u_i > 0, i = 1, 2, 3 \) or \( u_i < 0, i = 1, 2, 3 \); when \( K \) is positive definite with \( \kappa_{ij} < 0, \kappa_{ik} > 0, \kappa_{kj} > 0 (i \neq j \neq k) \), then \( u_i > 0, u_j > 0, u_k > 0 \) or \( u_i < 0, u_j < 0, u_k < 0 \).

This paper is organized as follows. In Section 2, we devote to state some preliminaries. The symmetry results Theorems 1.3-1.4 will be proved in Section 3. In Section 4, we give a detailed analysis on the asymptotic behavior of the positive ground state solutions to (2). The proof of existence of ground state solution to system with three equations will be done in the final section.

2. Preliminaries. In this section, some preliminaries are given by the following lemmata.

We define the sets
\[
\mathcal{H}_0 = \{ H \subset \mathbb{R}^N : H \text{ is a closed half-space in } \mathbb{R}^N \text{ and } 0 \in \partial H \}
\]
and, for \( p \neq 0 \),
\[
\mathcal{H}_0(p) = \{ H \in \mathcal{H}_0 : p \in \text{int}(H) \}.
\]
For each \( H \in \mathcal{H}_0 \) we denote by \( \sigma_H : \mathbb{R}^N \to \mathbb{R}^N \) the reflection in \( \mathbb{R}^N \) with respect to the hyperplane \( \partial H \), and define the polarization of a function \( u : \Omega \to \mathbb{R} \) with respect to \( H \) by
\[
\sigma_H(x) = \begin{cases} 
\max\{u(x), u(\sigma_H(x))\} & x \in H \cap \Omega, \\
\min\{u(x), u(\sigma_H(x))\} & x \in \Omega \setminus H. 
\end{cases}
\]

Definition 2.1 (cf. [30]). We call a half space \( H \in \mathcal{H}_0 \) dominant for \( u \) if \( u(x) \geq u(\sigma_H(x)) \) for all \( x \in \Omega \cap H \) with \( \sigma_H(x) \in \Omega \).

Lemma 2.2. (cf. [30, Propostion 2.4]). Let \( u : \Omega \to \mathbb{R} \) be a continuous function. Then \( u \) is foliated Schwarz symmetric with respect to \( p \in S^{N-1} \) if and only if every half space \( H \in \mathcal{H}_0(p) \) is dominant for \( u \).

Lemma 2.3 (cf. [7]). Let \( f, g \) be nonnegative and in \( \mathcal{L}^2(\mathbb{R}^N) \). Suppose that \( f^* \) and \( g^* \) are the Schwarz symmetrized function of \( f \) and \( g \) respectively, then
\[
\int_{\mathbb{R}^N} f(x)g(x)dx \leq \int_{\mathbb{R}^N} f^*(x)g^*(x)dx.
\]

Lemma 2.4 (cf. [31]). Let \( r > 0 \) and \( 2 \leq q < 2^* \) \((2^* = \frac{2N}{N-2} \text{ if } N \geq 3 \text{ and } 2^* = \infty \text{ if } N = 1, 2)\). If \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \) and if
\[
\liminf_{n \to \infty} \sup_{x \in \mathbb{R}^N} \int_{B_r(x)} |u_n|^q = 0,
\]
then \( u_n \to 0 \) in \( L^p(\mathbb{R}^N) \) for \( 2 < p < 2^* \).

3. Proof of Theorems 1.3-1.4. This section is devoted to the symmetry results of ground state solutions to system (2). Firstly, we give a proof concerned with a partial symmetry result.
Proof of Theorem 1.3. First, it can be easily seen that system (2) is invariant under the following transformation:
\[ \sigma : \mathbb{R} \times \mathbb{R} \times \mathcal{H} \to \mathbb{R} \times \mathbb{R} \times \mathcal{H}, \sigma(\kappa, \beta, u, v) = (-\kappa, \beta, u, -v). \]

Using the \( \sigma \)-invariance of system (2), we can see that \((u, v)\) is a ground state solution of the system (2) with coupling coefficients \(\kappa\) and \(\beta\) if and only if \((u, -v)\) is a ground state solution of the system (2) with coupling coefficients \(-\kappa\) and \(\beta\). Thus, we only suffice to deal with the case \(\beta \geq 0, \kappa \in (-\sqrt{\lambda_1 \lambda_2}, 0)\). By Theorem 1.2, we know that \(u > 0, v > 0\) or \(u < 0, v < 0\). Up to a sign, we may assume that \(u > 0, v > 0\).

Next, let us divide the proof into three steps.

Step 1: We claim that \((u_H, v_H)\) (defined as (13)) is also a minimizer for \(c_\kappa\).

Since \(u > 0, v > 0\), we have

\[ (u_H)^p = (u^p)_H, (v_H)^p = (v^p)_H, \quad p \geq 1. \]

By Proposition 31.7 of [32], we have

\[ \int_\Omega uvdx \leq \int_\Omega u_H v_H dx, \int_\Omega u^2 v^2 dx \leq \int_\Omega (u_H^2)(v_H^2) dx. \]  

(14)

Furthermore, in virtue of Lemma 3.1 in [30], we have

\[ \int_\Omega |\nabla f_H|^2 dx = \int_\Omega |\nabla f|^2 dx, \int_\Omega (f_H)^q dx = \int_\Omega f^q dx. \]  

(15)

where \(f = u\) or \(v, q = 2\) or \(4\).

Let \(t_H = \|(u_H, v_H)\|_\kappa/F(u_H, v_H)\) then \((t_H u_H, t_H v_H) \in \mathcal{N}_\kappa\). Besides, observe that \(\kappa < 0\), together with (14) and (15), we can infer that \(t_H \in (0, 1]\). Therefore, we have

\[ c_\kappa \leq I_\kappa(t_H u_H, t_H v_H) = \frac{(t_H)^2}{4} \|(u_H, v_H)\|^2 \leq \frac{1}{4} \|(u_H, v_H)\|^2 \leq \frac{1}{4} \|(u, v)\|^2 = c_\kappa, \]

where the last equality follows from Remark 1 and the fact that \((u, v)\) is a minimizer for \(c_\kappa\). This means that all the previous inequalities are indeed equalities, and in particular, which implies that \(t_H = 1\) and \(I_\kappa(u_H, v_H) = c_\kappa\). Thus, \((u_H, v_H) \in \mathcal{N}_\kappa\) is also a minimizer for \(c_\kappa\).

Choose \(\bar{r} > 0\) such that the sphere \(\{|x| = \bar{r}\} \subseteq \Omega\), then we have

\[ u(y_1) = \max_{|x| = \bar{r}} u(x), \quad v(y_2) = \max_{|x| = \bar{r}} v(x) \]

for some \(y_1, y_2 \in \Omega\), and set \(\epsilon_i = y_i/\bar{r}, i = 1, 2\).

Step 2: Let us show that \(u\) and \(v\) are foliated Schwarz symmetric with respect to \(c_1\) and \(c_2\) respectively.

By Step 1, \((u_H, v_H)\) is a minimizer for \(c_\kappa\) and therefore a solution of system (2)/(cf. (2) of Remark 2). Hence \((u_H, v_H)\) satisfies the following system

\[ \begin{cases} -\Delta u_H + \lambda_1 u_H + \kappa v_H = \mu_1 u_H^3 + \beta u_H v_H^2 & \text{in } \Omega, \\ -\Delta v_H + \lambda_2 v_H + \kappa u_H = \mu_2 v_H^2 + \beta u_H v_H & \text{in } \Omega, \\ u_H = v_H = 0 & \text{on } \partial \Omega. \end{cases} \]

Then

\[ -\Delta(u_H - u) + \lambda_1(u_H - u) = \mu_1(u_H^3 - u^3) + \beta(u_H v_H^2 - u v^2) - \kappa(v_H - v). \]  

(16)

Note that \(\kappa < 0, \beta \geq 0\) and \(u_H \geq u, v_H \geq v\) in \(H \cap \Omega\), together with (16), we get

\[ -\Delta(u_H - u) + \lambda_1(u_H - u) \geq 0, \quad x \in H \cap \Omega, \]

\[ u_H - u = 0, \quad x \in \partial(H \cap \Omega). \]
Since $u_H - u \geq 0$ in $H \cap \Omega$, by the maximum principle, we have

$$u_H(x) = u(x) \quad \text{or} \quad u_H(x) > u(x), \quad x \in H \cap \Omega.$$  

For $H \in \mathcal{H}_0(e_1)$, we have $u_H(x) = u(x)$ for all $x \in H \cap \Omega$ because $u_H(y_1) = u(y_1)$. By Lemma 2.2, $u$ is foliated Schwarz symmetric with respect to $e_1$.

According to the analogous argument as above, we can also show that $v$ is foliated Schwarz symmetric with respect to $e_2$.

**Step 3:** $v(y_2) = v(y_1)$.

Due to the very definition of $y_2$, we have $v(y_2) \geq v(y_1)$. Thus we can suppose, by contradiction, that $v(y_2) > v(y_1)$. Choose $H \in \mathcal{H}_0(e_1)$ such that $H \notin \mathcal{H}_0(e_2)$ and $\sigma_H(y_1) = y_2$. Note that $u_H(x) = u(x)$ for all $x \in H \cap \Omega$, combining with (16), we get

$$\beta u(v_H^2 - v^2) - \kappa (v_H - v) = 0, \quad x \in H \cap \Omega$$

This gives $v_H(x) = v(x)$ for all $x \in H \cap \Omega$, recall the definition of $v_H$, choose $y_1 \in H \cap \Omega$ given by Step 1, we obtain

$$v(y_1) = v_H(y_1) \geq v(\sigma_H(y_1)) = v(y_2) > v(y_1),$$

this leads to a contradiction.

If $y_2 = y_1$, i.e. $e_2 = e_1$, set $e = e_2 = e_1$, then $u$ and $v$ are foliated Schwarz symmetric with respect to the same point $e$. If $y_2 \neq y_1$, by Step 2, we can imply that $v$ is foliated Schwarz symmetric with respect to both $e_1$ and $e_2$. Therefore, let $e = e_1$, then the desired result also follows.

If the underlying domain $\Omega$ is a ball or the whole space $\mathbb{R}^N$, we can improve the above result, as is said in Theorem 1.4. Next, we commerce with the proof of the theorem.

**Proof of Theorem 1.4.** We divide the proof into two cases according to the range of the parameter $\kappa$.

**Case 1:** $\kappa \in (\sqrt{1 \over \lambda_1 \lambda_2}, 0)$. For this case, we know that up to a sign the ground state solutions are positive. In addition, system (2) can also be written into the following form

$$\begin{align*}
-\Delta u + \lambda_1 u &= \mu_1 u^3 + \beta u v^2 - \kappa v &\text{in } \Omega, \\
-\Delta v + \lambda_2 v &= \mu_2 v^3 + \beta u^2 v - \kappa u &\text{in } \Omega, \\
u = v &= 0 &\text{on } \partial \Omega \text{ (or } u, v \in H^1(\mathbb{R}^N) \text{ as } \Omega = \mathbb{R}^N).}
\end{align*}$$

Since $\beta \geq 0, \kappa < 0$, we can infer that this is a cooperative system. The positive solutions must be radially symmetric and strictly decreasing following from [26] as $\Omega$ is a ball $B_r(0), r > 0$, while from [9] as $\Omega = \mathbb{R}^N$.

**Case 2:** $\kappa \in (0, \sqrt{1 \over \lambda_1 \lambda_2})$. From Theorem 1.2, system (2) has a ground state solution $(u, v)$ with $u > 0, v < 0$ or $u < 0, v > 0$. Without loss of generality, we may assume that $u > 0, v < 0$.

Let $u^*$ be the Schwarz symmetrized function of $u$, we regard $u \in H^1_0(\Omega)$ as $u \in H^1(\mathbb{R}^N)$ by setting $u = 0$ on $\mathbb{R}^N \setminus \Omega$ when $\Omega$ is a ball. Note that if $u \in H^1_0(\Omega)$, then $u^* \in H^1_0(\Omega)$. Besides, for $v \leq 0$, we make $v^* = -(v)^*$. By Sobolev embedding theorems, $u \in L^2(\Omega) \cap L^4(\Omega)$. By Lemma 2.3, we get

$$\int_{\Omega} u(-v)dx \leq \int_{\Omega} u^*(-v)^*dx, \int_{\Omega} u^2(-v)^2dx \leq \int_{\Omega} (u^*)^2((-v)^*)^2dx. \quad (17)$$
Moreover, we have
\[ \int_{\Omega} |\nabla u^*|^2 \, dx \leq \int_{\Omega} |\nabla u|^2 \, dx, \int_{\Omega} |\nabla (-v)^*|^2 \, dx \leq \int_{\Omega} |\nabla (-v)|^2 \, dx. \]
Note that \( v^* = -(-v)^* \) and \( \kappa > 0 \), together with (17), we get
\[ \kappa \int_{\Omega} uv \, dx \geq \kappa \int_{\Omega} u^*v^* \, dx. \]
Thus, if \((u, v)\) with \( u > 0, v < 0\) is a ground state solution, then
\[ ||(u^*, v^*)||^2 = \int_{\Omega} |\nabla u^*|^2 + \lambda_1 \int_{\Omega} (u^*)^2 + \lambda_2 \int_{\Omega} (v^*)^2 + 2\kappa \int_{\Omega} u^*v^* \]
\[ \leq \int_{\Omega} |\nabla u|^2 + \lambda_1 \int_{\Omega} u^2 + \int_{\Omega} |\nabla v|^2 + \lambda_2 \int_{\Omega} v^2 + 2\kappa \int_{\Omega} uv \]
\[ = ||(u, v)||^2_\kappa \]
\[ = \mu_1 \int_{\Omega} u^4 + \mu_2 \int_{\Omega} v^4 + 2\beta \int_{\Omega} u^2v^2 \]
\[ \leq \mu_1 \int_{\Omega} (u^*)^4 + \mu_2 \int_{\Omega} (v^*)^4 + 2\beta \int_{\Omega} (u^*)^2(v^*)^2 \]
\[ = F(u^*, v^*). \]

Introduce the function \( \varphi(t) := I_\kappa(tu^*, tv^*) \), then by a direct computation, we get that
\[ t_0 = \frac{||u^*, v^*||_\kappa}{F(u^*, v^*)^{1/2}} \in (0, 1] \]
is a maximal point of \( \varphi \), which implies that \((t_0u^*, t_0v^*) \in N_\kappa \). Furthermore, we have
\[ c_\kappa \leq I_\kappa(t_0u^*, t_0v^*) = \frac{t_0^2}{4} ||(u^*, v^*)||^2_\kappa \leq \frac{1}{4} ||u^*, v^*||^2_\kappa \leq \frac{1}{4} ||(u, v)||^2_\kappa = c_\kappa, \]
which means that all the previous inequalities are indeed equalities. Therefore, we have \( I_\kappa(u^*, v^*) = c_\kappa \) and
\[ \int_{\Omega} |\nabla u^*|^2 = \int_{\Omega} |\nabla u|^2, \quad \int_{\Omega} |\nabla v^*|^2 = \int_{\Omega} |\nabla v|^2. \quad (18) \]
Since \((u^*, v^*)\) is also a minimizer for \( c_\kappa \), then we have
\[ \begin{cases} -\Delta u^* + \lambda_1 u^* + \kappa v^* = \mu_1(u^*)^3 + \beta u^*(v^*)^2 & \text{in } \Omega, \\ -\Delta v^* + \lambda_2 v^* + \kappa u^* = \mu_2(v^*)^3 + \beta v^*(u^*)^2 & \text{in } \Omega, \\ u^* = v^* = 0 & \text{on } \partial \Omega. \end{cases} \quad (19) \]
By the elliptic regularity theory, we get that \( u^*, v^* \in C^2(\overline{\Omega}) \). By the strong maximum principle, we have \( u^* > 0, -v^* = (-v)^* > 0 \) in \( \Omega \).

We claim that
\[ \{x \in \Omega : |\nabla u^*| = 0\} \cap u^{-1}(0, ||u^*||_{\infty}) = 0, \]
\[ \{x \in \Omega : |\nabla v^*| = 0\} \cap (-v)^{-1}(0, ||v^*||_{\infty}) = 0, \]
where \(| \cdot |\) denotes the \( N \)-dimension Lebesgue measure.

In fact, by the very definition of \( u^* \), we know that \( u^* \) is a non increasing radially symmetric function in \( \Omega \), thus \((u^*)'(s) \leq 0 \) in \((0, r)\) as \( \Omega = B_r(0) \) or in \((0, \infty)\) as
\[ \Omega = \mathbb{R}^N. \]

Note that \( u^* \) is \( C^2 \), we can suppose, in views of contradiction, that there exist \( 0 < a < b \) such that

\[
(u^*)'(s) = 0, \quad \forall s \in [a, b] \subseteq (0, r) \text{ as } \Omega = B_r(0) \text{ or } s \in [a, b] \subseteq (0, \infty) \text{ as } \Omega = \mathbb{R}^N.
\]

This implies that \( u^*(x) \equiv C > 0 \) for some constant \( C \) in the annulus \( a < |x| < b \). 

Nextly we show that the situation doesn’t occur and hence obtain a contradiction. 

Note that \( u^*(x) = 0, |x| = r \text{ as } \Omega = B_r(0) \) or \( \lim_{|x| \to \infty} u^*(x) = 0 \text{ as } \Omega = \mathbb{R}^N \), we can infer that \( b < r \) or \( b < +\infty \).

If \( \Omega = B_r(0) \), the function \( C - u^* \) satisfies the following equations in the annulus \( b < |x| < r \) :

\[
\begin{cases}
-\Delta (C - u^*) + \lambda_1 (C - u^*) = \kappa v^* - \mu_1 (u^*)^3 - \beta (u^*)^2 + \lambda_1 C & \text{in } b < |x| < r, \\
C - u^* \geq 0 & \text{in } b < |x| < r,
\end{cases}
\]

\[
C - u^* = 0 \text{ on } |x| = b, \quad C - u^* = C > 0 \text{ on } |x| = r.
\]

For \( a < |x| < b \), \( u^*(x) \equiv C \), together with the system (19), we have

\[
\lambda_1 C + \kappa v^*(x) = \mu_1 C^3 + \beta C (v^*)^2(x), \forall a < |x| < b.
\]

In particular, take \( x = x_0 \) with \( |x_0| = (a + b)/2 \), we obtain

\[
\lambda_1 C = -\kappa v^*(x_0) + \mu_1 C^3 + \beta C (v^*)^2(x_0).
\]

Since \( u^*, -v^* \) are positive radially non-increasing functions in \( B_r(0) \), we can imply that

\[
\kappa v^*(x) - \mu_1 (u^*)^3(x) - \beta (u^*)^2(x) + \lambda_1 C = \kappa v^*(x) - \mu_1 (u^*)^3(x) - \beta (u^*)^2(x) - k v^*(x_0) + \mu_1 C^3 + \beta (v^*)^2(x_0) \geq 0
\]

for every \( b < |x| < r \). Since \( C - u^* \neq 0 \), by the Hopf’s lemma (see [12, Lemma 3.4]), we obtain that \( \frac{\partial u^*}{\partial n}(x) = -\frac{\partial (C - u^*)}{\partial n}(x) > 0 \) for \( |x| = b \), where \( n \) denotes the unit outward normal to \( \partial \{ b < |x| < r \} \). However, observe that \( u^* \) is \( C^2 (B_r(0)) \) and identically equals to the positive constant \( C \) in \( a < |x| < b \), this leads to a contradiction.

If \( \Omega = \mathbb{R}^N \), choose a constant \( R > b \) such that \( u^*(x) = C/2 \) on \( |x| = R \). By a similar argument in \( b < |x| < R \), we can also obtain a contradiction.

In this way, we prove that

\[
\left| \{ x \in \Omega : |\nabla u^*| = 0 \} \cap u^*^{-1}(0, ||u^*||_{\infty}) \right| = 0.
\]

Analogous argument holds for \( v^* \).

According to the above claim, together with (18) and by Theorem 1.1 of Brothers and Ziemer [8], we obtain

\[
u^* = u, v^* = v \quad \text{a.e. in } \Omega.
\]

By the continuity of \( u, v, u^*, v^* \), we have \( u(x) = u^*(x), v(x) = v^*(x), \forall x \in \Omega \). Thus, \((u, v)\) is a radially symmetric ground state solution.

**Remark 5.** For the second case in the proof of Theorem 1.4, in virtue of the \( \sigma \)-invariance of system (2) in the proof of Theorem 1.3, we can infer that the ground state solutions to system (2) are radially symmetric following from the first case. However, the method to prove the second case has its own interest.
4. **Asymptotic behavior as $\kappa \to 0^-$**. We study the following elliptic systems:

\[
\begin{aligned}
-\Delta u + u + \kappa v &= \mu_1 u^3 + \beta uv^2 \\
-\Delta v + v + \kappa u &= \mu_2 v^3 + \beta u^2 v
\end{aligned}
\] in $\mathbb{R}^N$, \quad (20)

where $N = 2, 3, \kappa \in (-1, 0), \mu_1, \mu_2 > 0, \beta \in \mathbb{R}$. Fixing $\mu_1, \mu_2 > 0$, we may assume without loss of generality that $\mu_1 \leq \mu_2$.

By Theorem 1.2, there exists a positive ground state solution to system (20), denoted by $(u_\kappa, v_\kappa)$ to emphasize the dependence on parameter $\kappa$. To investigate the asymptotic behavior of $(u_\kappa, v_\kappa)$, we firstly deal with the limit equations:

\[
\begin{aligned}
-\Delta u + u &= \mu_1 u^3 + \beta u v^2 \\
-\Delta v + v &= \mu_2 v^3 + \beta u^2 v
\end{aligned}
\] in $\mathbb{R}^N$, \quad (21)

When the underlying domain $\Omega$ is the whole space $\mathbb{R}^N$, recall that $\mathcal{H}$ is defined by $\mathcal{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Set

$$\mathcal{H}_r := \{(u, v) \in \mathcal{H} : u, v \text{ are radially symmetric}\},$$

$$\mathcal{M}_0 = \{(u, v) \in \mathcal{H}_r : u \neq 0 \neq v, \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 = \int_{\mathbb{R}^N} \mu_1 u^4 + \beta u^2 v^2, \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 = \int_{\mathbb{R}^N} \mu_2 v^4 + \beta u^2 v^2\},$$

$$\mathcal{M}_0' = \{(u, v) \in \mathcal{H}_r : u \neq 0 \neq v, \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 = \int_{\mathbb{R}^N} \mu_1 u^4 + \beta u^2 v^2, \int_{\mathbb{R}^N} |\nabla v|^2 + v^2 = \int_{\mathbb{R}^N} \mu_2 v^4 + \beta u^2 v^2\},$$

$$\mathcal{N}_0 = \{(u, v) \in \mathcal{H} : (u, v) \neq 0, I_0(u, v) = 0\},$$

$$\mathcal{N}_0' = \{(u, v) \in \mathcal{H}_r : (u, v) \neq 0, I_0(u, v) = 0\}.$$ We consider the following minimizing problems:

\[
c_0 = \inf_{(u, v) \in \mathcal{N}_0} I_0(u, v), c_0' = \inf_{(u, v) \in \mathcal{N}_0'} I_0(u, v), \]

\[
A_0 = \inf_{(u, v) \in \mathcal{M}_0} I_0(u, v), A_0' = \inf_{(u, v) \in \mathcal{M}_0'} I_0(u, v). \quad (22)
\]

In this section, we emphasize that we only deal with the dimension $N = 2, 3$.

**Definition 4.1.** We call a solution $(u, v)$ nontrivial if at least one component of $(u, v)$ is not zero.

First, we give some results on the existence and quantitative properties of bound state or ground state solutions to (21), please refer to [2, 3, 4, 14, 16, 22].

**Lemma 4.2.** According to the range of parameter $\beta$, we have

(a) (see [16, 22]) for $\beta < 0$, $A_0$ can’t be achieved, that is, (21) has no least energy positive solution. However, $A_0'$ can be obtained;

(b) (see [14]) for $\beta \in (0, \mu_1)$, $(u^3, v^3)$ is a least energy positive solution to system (21) with

$$I_0(u^3, v^3) \geq \max\{I_0(U_1, 0), I_0(0, U_2)\};$$
(c) (see [4]) if \( \mu_1 < \mu_2 \), for \( \beta \in [\mu_1, \mu_2] \), (21) has no positive solution, if \( \mu_1 = \mu_2 \), (21) has at least a positive solution for \( \beta > 0 \);

(d) (see [2, 3]) if \( \mu_1 \leq \mu_2 \), for \( \beta \in (\mu_2, \infty) \), (21) has a positive radial ground state \((u, v)\) with

\[
I_0(u, v) < \min\{I_0(U_1, 0), I_0(0, U_2)\}.
\]

Next, some uniqueness results of positive solutions to (21) have been offered via the following lemma, please refer to [11, 29].

**Lemma 4.3** (see [11, 29]). There exists a positive constant \( \beta_0 \) such that \((u^\beta, v^\beta)\) is the unique positive solution to system (21) for \( \beta \in (0, \beta_0) \cup (\mu_2, \infty) \). Moreover, if \( \beta = \mu_1 = \mu_2 \), all the positive solutions of system (21) have the following form

\[
(u_\theta, v_\theta) = \left(\sqrt{\frac{1 + \kappa}{\beta + \mu}} w(\sqrt{1 + \kappa} x), \sqrt{\frac{1 + \kappa}{\beta + \mu}} w(\sqrt{1 + \kappa} x)\right), \quad \theta \in (0, \pi/2).
\]

Before analysing the asymptotic behavior of \((u_\kappa, v_\kappa)\), we finally provide a uniqueness result on system (20).

**Proposition 1.** Assume \( \lambda_1 = \lambda_2 = 1, \mu_1 = \mu_2 = \mu, \beta \geq \mu, \kappa \in (-1, 0) \), then \((u^\kappa_\mu, v^\kappa_\mu)\) is the unique positive solutions to system (20), where \((u^\kappa_\mu, v^\kappa_\mu)\) is defined as follows:

\[
(u^\kappa_\mu, v^\kappa_\mu) = \left(\sqrt{\frac{1 + \kappa}{\beta + \mu}} w(\sqrt{1 + \kappa} x), \sqrt{\frac{1 + \kappa}{\beta + \mu}} w(\sqrt{1 + \kappa} x)\right),
\]

where \(w\) is given by (8).

**Proof.** Since the proof of the above proposition can be carried out line by line following from [29, Theorem 4.2], for the sake of simplicity, we omit it here. \(\square\)

With the above preparations, we now commence to study the asymptotic behavior of positive ground state solution \((u_\kappa, v_\kappa)\) to system (20).

**Proof of Theorem 1.5 and 1.6.** For clarity, we divide our proof into three steps.

**Step 1:** Energy estimate.

By proposition 3.7 of [15], we know that \(c_\kappa < \min\{I_0(U_1, 0), I_0(0, U_2)\}\) when \(\kappa \in (-1, 0)\), combining with (10), we have

\[
c_\kappa < I_0(0, U_2) \leq I_0(U_1, 0).
\]

On the other hand, since we only consider \(\kappa \to 0^-\), we may assume that \(\kappa \in (-1/2, 0)\), then

\[
\frac{1}{2} \|(u, v)\|_0^2 \leq \|(u, v)\|_\kappa^2 \leq \frac{3}{2} \|(u, v)\|_0^2.
\]

**Step 2:** (see [11, 29]) if \( \kappa \rightarrow 0^- \), then the following holds

\[
c_\kappa = \frac{1}{4} \int_{\mathbb{R}^N} \left(\nabla u_\kappa \right)^2 + u_\kappa^2 + |\nabla v_\kappa|^2 + v_\kappa^2 + 2\kappa u_\kappa v_\kappa = \frac{1}{4} \|(u_\kappa, v_\kappa)\|_\kappa^2,
\]

we have

\[
\|(u_\kappa, v_\kappa)\|_\kappa^2 = \left(\mu_1 \int_{\mathbb{R}^N} u_\kappa^4 + \mu_2 \int_{\mathbb{R}^N} v_\kappa^4 + 2\beta \int_{\mathbb{R}^N} u_\kappa^2 v_\kappa^2\right)
\]

\[
\leq C(\mu_1, \mu_2, \beta)\left(\int_{\mathbb{R}^N} u_\kappa^4 + \int_{\mathbb{R}^N} v_\kappa^4\right)
\]

\[
\leq C(N, \mu_1, \mu_2, \beta)\|(u_\kappa, v_\kappa)\|_0^4.
\]
By virtue of (25), we obtain
\[ \|(u_\kappa, v_\kappa)\|_0^2 \leq C(N, \mu_1, \mu_2, \beta) \|(u_\kappa, v_\kappa)\|_0^4. \]
Therefore, there exists a positive constant \( \delta_0 \) independent of \( \kappa \) such that
\[ c_\kappa \geq \delta_0, \quad \text{for all } \kappa \in (-1/2, 0). \] (28)

**Step 2:** The limit profile is nontrivial.

Combining (24), (25), (26) and (28), then for any sequence \( \kappa_n \to 0 \), we can get that \( \{(u_{\kappa_n}, v_{\kappa_n})\} \) are bounded in \( H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \). When \( \beta \geq 0 \), up to a subsequence (still denoted by \( \kappa_n \)), there exists some \( (u_0, v_0) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \) such that
\[ (u_{\kappa_n}, v_{\kappa_n}) \rightharpoonup (u_0, v_0) \quad \text{in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N). \]

By Theorem 1.4, \( \{(u_{\kappa_n}, v_{\kappa_n})\} \) are radially symmetric. Together with (26), (27) and (28), there exists a constant \( \delta' \) independent of \( \kappa \) such that
\[ \int_{\mathbb{R}^N} (u_{\kappa_n}^4 + v_{\kappa_n}^4) \, dx \geq \delta' > 0, \quad \forall \kappa \in (-1/2, 0). \]

Since \( H^1(\mathbb{R}^N) \) is compactly embedded into \( L^4(\mathbb{R}^N) \), we can infer that \((u_0, v_0) \neq 0\).

For the case \( \beta < 0 \), if
\[ \liminf_{n \to \infty} \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} u_{\kappa_n}^2(y) \, dy = 0 \quad \text{and} \quad \liminf_{n \to \infty} \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} v_{\kappa_n}^2(y) \, dy = 0, \]
then Lion’s compactness principle (cf. Lemma 2.4) gives
\[ (u_{\kappa_n}, v_{\kappa_n}) \to (0, 0) \quad \text{in } L^4(\mathbb{R}^N) \times L^4(\mathbb{R}^N). \]

Therefore,
\[ \|(u_\kappa, v_\kappa)\|_0^2 = \left( \mu_1 \int_{\mathbb{R}^N} u_\kappa^4 + \mu_2 \int_{\mathbb{R}^N} v_\kappa^4 + 2\beta \int_{\mathbb{R}^N} u_\kappa^2 v_\kappa^2 \right) \to 0, \]
which contradicts to (28). Hence, without loss of generality, we assume that
\[ \liminf_{n \to \infty} \sup_{x \in \mathbb{R}^N} \int_{B_1(x)} v_{\kappa_n}^2(y) \, dy = \alpha \]
for some constant \( \alpha > 0 \). Going if necessary to a subsequence, we may assume the existence of \( \{x_n\} \subseteq \mathbb{R}^N \) such that
\[ \int_{B_1(x_n)} v_{\kappa_n}^2(y) \, dy > \frac{\alpha}{2}. \] (29)

Note that
\[ \|(u_\kappa \cdot + x_n, v_\kappa \cdot + x_n)\|_0^2 = \|(u_\kappa, v_\kappa)\|_0^2, \]
and hence \( \{(u_{\kappa_n} \cdot + x_n, v_{\kappa_n} \cdot + x_n)\} \) is bounded. Set
\[ \overline{u}_{\kappa_n}(x) = u_{\kappa_n}(x + x_n), \quad \overline{v}_{\kappa_n}(x) = v_{\kappa_n}(x + x_n), \]
then up to a subsequence, for some \((u_0, v_0) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\), we have
\[ (\overline{u}_{\kappa_n}, \overline{v}_{\kappa_n}) \rightharpoonup (u_0, v_0) \quad \text{in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \]
and
\[ (\overline{u}_{\kappa_n}, \overline{v}_{\kappa_n}) \to (u_0, v_0) \quad \text{in } L^2_{loc}(\mathbb{R}^N) \times L^2_{loc}(\mathbb{R}^N). \]

Hence (29) gives
\[ \int_{B_1(0)} v_0^2(y) \, dy > \frac{\alpha}{2}, \]
which implies
\[ v_0 \neq 0. \]
Thus, we get \((u_0, v_0) \neq 0\). At the same time, we see that \((u_0, v_0)\) is nonnegative and solves the limit equations (21).

**Step 3:** Identity \((u_0, v_0)\).

By step 2, we know that \((u_0, v_0)\) is a nonnegative nontrivial solution to system (21). Firstly, we point out if \((u_0, v_0)\) has one component with zero, then \((u_0, v_0) = (0, U_2)\) or \((U_1, 0)\), which follows from the uniqueness of positive solution of scalar equation. Furthermore, we have for \(\beta \geq 0\)
\[
I_0(u_0, v_0) = \frac{1}{4} \|(u_0, v_0)\|_0^2 \leq \liminf_{n \to +\infty} \frac{1}{4} \|(u_{\kappa_n}, v_{\kappa_n})\|_{\kappa_n}^2 = \liminf_{n \to +\infty} c_{\kappa_n},
\]
while for \(\beta < 0\)
\[
I_0(u_0, v_0) = \frac{1}{4} \|(u_0, v_0)\|_0^2 \leq \liminf_{n \to +\infty} \frac{1}{4} \|(\tilde{u}_{\kappa_n}, \tilde{v}_{\kappa_n})\|_{\kappa_n}^2 = \liminf_{n \to +\infty} c_{\kappa_n}.
\]
Note that (24), we have
\[
I_0(u_0, v_0) \leq I_0(0, U_2) \leq I_0(U_1, 0).
\]
Nextly, we take account of several cases by the range of parameter \(\beta\).

**Case 1:** \(\beta \in (-\infty, 0)\).

Recall that \(c_0 = \inf_{(u, v) \in \mathcal{N}_0} I_0(u, v)\), by a standard argument and Lion’s compactness principle, we can obtain a nonzero nonnegative minimizer \((\hat{u}, \hat{v})\). By (a) of Lemma 4.2, \((\hat{u}, \hat{v})\) must have one zero component, otherwise \(A_0\) can be achieved by \((\hat{u}, \hat{v})\), this is a contradiction. Thus, we have \((\hat{u}, \hat{v}) = (0, U_2)\) or \((U_1, 0)\). If \(\mu_1 < \mu_2\), then we have
\[
I_0(\hat{u}, \hat{v}) = c_0 \leq I_0(0, U_2) < I_0(U_1, 0).
\]
It implies that \(c_0 = I_0(0, U_2)\), hence the minimizer \((\hat{u}, \hat{v})\) is \((0, U_2)\). Again by (a) of Lemma 4.2, we infer that \(I_0(u, v) > I_0(0, U_2)\) for all positive solution \((u, v)\) to system (21). Combining with (32), we know that \((u_0, v_0)\) is not a positive solution, hence a semi-trivial solution. Moreover, \((u_0, v_0) = (0, U_2)\). In virtue of (31), we have
\[
\lim_{n \to +\infty} \|(\tilde{u}_{\kappa_n}, \tilde{v}_{\kappa_n})\|_{\kappa_n}^2 = \|(u_0, v_0)\|_0^2,
\]

Thus \((\tilde{u}_{\kappa_n}, \tilde{v}_{\kappa_n}) \to (0, U_2)\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).

If \(\mu_1 = \mu_2\), we get
\[
I_0(\hat{u}, \hat{v}) = c_0 \leq I_0(0, U_2) = I_0(U_1, 0).
\]
It means that \(c_0 = I_0(0, U_2) = I_0(U_1, 0)\), hence the minimizer \((\hat{u}, \hat{v})\) is \((0, U_2)\) or \((U_1, 0)\). Then by the same argument described above, we obtain
\[
(\tilde{u}_{\kappa_n}, \tilde{v}_{\kappa_n}) \to (0, U_2)\) or \((U_1, 0)\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).

**Case 2:** \(\beta \in [0, \mu_1)\).

By (b) of Lemma 4.2, (21) has a least energy positive solution \((u, v)\) with \(I_0(u, v) > \max\{I_0(U_1, 0), I_0(0, U_2)\}\) Combining with (32), we get that \((u_0, v_0)\) is not a positive solution. If \(\mu_1 < \mu_2\), we have \((u_0, v_0) = (0, U_2)\) and
\[
I_0(u_0, v_0) = \frac{1}{4} \|(u_0, v_0)\|_0^2 \leq \liminf_{n \to +\infty} \frac{1}{4} \|(u_{\kappa_n}, v_{\kappa_n})\|_{\kappa_n}^2 = \liminf_{n \to +\infty} c_{\kappa_n} \leq I_0(0, U_2),
\]
whence we get
\[
(u_{\kappa_n}, v_{\kappa_n}) \to (0, U_2)\) in \(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).
Thus, if \( \mu_1 = \mu_2 \), argue as above, we have \((u_0, v_0) = (0, U_2)\) or \((U_1, 0)\) and 
\[
(u_{n_0}, v_{n_0}) \to (0, U_2) \text{ or } (U_1, 0) \text{ in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).
\]

**Case 3:** If \( \mu_1 < \mu_2 \), consider the case \( \beta \in [\mu_1, \mu_2] \).

By (c) of Lemma 4.2, system (21) has no positive solution. Since \((u_0, v_0)\) is nonnegative and nontrivial, and note that (32), we have \((u_0, v_0) = (0, U_2)\). By virtue of (30), we get
\[
\lim_{n \to +\infty} \|(u_{n_0}, v_{n_0})\|^2 = \|(u_0, v_0)\|^2,
\]

and
\[
\mu = 1 = 1 = \mu\]

therefore, we know
\[
(u_{n_0}, v_{n_0}) \to (0, U_2) \text{ in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).
\]

If \( \mu_1 = \mu_2 \), for the case \( \beta = \mu_1 = \mu_2 \), by a uniqueness result of Lemma 4.3, we know that
\[
(u_{n_0}, v_{n_0}) = (u_\beta_{n_0}, v_\beta_{n_0}) = \left(\sqrt{\frac{1 + \kappa_n}{2\beta + \mu}} w(\sqrt{1 + \kappa_n} x), \sqrt{\frac{1 + \kappa_n}{\beta + \mu}} w(\sqrt{1 + \kappa_n} x)\right).
\]

Let \( \kappa \to 0 \), by a direct computation, we have
\[
(u_0, v_0) = \left(\sqrt{\frac{1}{2\beta}} w(x), \sqrt{\frac{1}{2\beta}} w(x)\right) = (\bar{u}_{\pi/4}, \bar{v}_{\pi/4})
\]
and
\[
(u_{n_0}, v_{n_0}) \to (\bar{u}_{\pi/4}, \bar{v}_{\pi/4}) \text{ strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N).
\]

**Case 4:** \( \beta \in (\mu_2, \infty) \)

For this case, we have \((t_n u^\beta, t_n v^\beta) \in \mathcal{N}_{\kappa_n}\) with
\[
t_n = \frac{\|(u^\beta, v^\beta)\|_{\kappa_n}}{F(u^\beta, v^\beta)^{\frac{1}{2}}}
\]
Thus
\[
c_0 \leq I_0(u_0, v_0) = \frac{1}{4} \|(u_0, v_0)\|^2 \leq \liminf_{n \to +\infty} \frac{1}{4} \|(u_{n_0}, v_{n_0})\|_{\kappa_n}^2 = \liminf_{n \to +\infty} c_{\kappa_n}
\]
and
\[
\leq \liminf_{n \to +\infty} \frac{1}{4} \|(t_n u^\beta, t_n v^\beta)\|_{\kappa_n}^2 = \liminf_{n \to +\infty} \frac{1}{4} \left(\frac{\|(u^\beta, v^\beta)\|_{\kappa_n}^4}{F(u^\beta, v^\beta)}\right)^{\frac{1}{2}}.
\]
Note that
\[
F(u^\beta, v^\beta) = \|(u^\beta, v^\beta)\|_0^2
\]
and
\[
\|(u^\beta, v^\beta)\|_{\kappa_n}^2 = \|(u^\beta, v^\beta)\|_0^2 + 2\kappa_n \int_{\mathbb{R}^N} u^\beta v^\beta,
\]
then we know
\[
\frac{1}{4} \frac{\|(u^\beta, v^\beta)\|_{\kappa_n}^4}{F(u^\beta, v^\beta)} = \frac{1}{4} \frac{\|(u^\beta, v^\beta)\|_{\kappa_n}^4}{\|(u^\beta, v^\beta)\|_0^2}
\]
\[
= \frac{1}{4} \{(u^\beta, v^\beta)\}^2_0 + \kappa_n \int_{\mathbb{R}^N} u^\beta v^\beta + \frac{\left(\kappa_n \int_{\mathbb{R}^N} u^\beta v^\beta\right)^2}{\|(u^\beta, v^\beta)\|_0^2}
\]
\[
= c_0 + \kappa_n \int_{\mathbb{R}^N} u^\beta v^\beta + \frac{\left(\kappa_n \int_{\mathbb{R}^N} u^\beta v^\beta\right)^2}{\|(u^\beta, v^\beta)\|_0^2}.
\]
Thus, we have
\[
c_0 = I_0(u_0, v_0) \quad \text{and} \quad \|(u_0, v_0)\|_0^2 = \lim_{n \to +\infty} \|u_{k_n}, v_{k_n}\|_{\kappa_n}^2.
By Lemma 4.3, the ground state solution to system (21) is unique. Therefore, 
\((u_0, v_0) = (u^\beta, v^\beta)\).

In addition, we have 
\((u_{\kappa_n}, v_{\kappa_n}) \to (u^\beta, v^\beta) \text{ in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)\).

\[\square\]

**Remark 6.** Via the above method, we think that the similar results hold for the spatial dimension \(N = 1\).

5. **System with three equations.** In this section, we study the existence of ground state solutions to system (11). As we point out in Remark 2, compared to system (2), the situation seems to be more complicated because of the fact that system (2) has no semi-trivial solution while system (11) may have semi-trivial solutions which have one zero component. Thus, the key is to how to distinguish the ground state solutions and the semi-trivial solutions to (11). The following proof provides a method based on energy to distinguish them.

**Proof of Theorem 1.8.** To start with, by an analogous argument as the proof of Theorem 1.1 of [15], we can obtain a nontrivial solution \(\vec{u} = (u_1, u_2, u_3)\) to system (11).

Next, we show that \(\vec{u}\) is a ground state solution. Since \(K\) is positive definite, then its every principal minor is positive definite. Thus, we have 
\[|\kappa_{ij}| < \sqrt{\lambda_i \lambda_j}, \quad 1 \leq i < j \leq 3.\]

Note that system (11) has no semi-trivial solution with two zero components, so we only suffice to distinguish between the solution and those semi-trivial solutions with one zero component.

Define 
\[c_{ij} = \inf_{N_k \cap \{u_k = 0\} \cap \{u_i = 0\} \cap \{u_j = 0\}} I_K(\vec{u}) = \inf_{(u_i, u_j) \in N_{\kappa_{ij}}} I_{\kappa_{ij}}(u_i, u_j), \quad 1 \leq i \neq j \neq k \leq 3.\]

According to Theorem 1.2, \(c_{ij}\) can be achieved by a solution \((u_i, u_j)\) with \(u_i u_j \neq 0\) to system (2). Moreover, we have

\[
\begin{align*}
\text{If } \kappa_{ij} > 0, & \quad \text{then } u_i > 0, u_j < 0 \text{ or } u_i < 0, u_j > 0; \\
\text{If } \kappa_{ij} < 0, & \quad \text{then } u_i > 0, u_j > 0 \text{ or } u_i < 0, u_j < 0.
\end{align*}
\]

(33)

We claim that \(c_K < \min\{c_{12}, c_{13}, c_{23}\}\) and we only prove that \(c_K < c_{12}\), the remaining cases follow from the similar argument.

Assume \((u_1, u_2)\) is a minimizer for \(c_{12}\), then according to (33), we always choose \((\tilde{u}_2, u_3)\) as a minimizer for \(c_{23}\) such that \(\text{sign}(\tilde{u}_2) = \text{sign}(u_2)\). Recall that \(Q(\vec{u})\) is defined by

\[Q(\vec{u}) = Q((u_1, u_2, u_3)) := I_K'(\vec{u})\vec{u},\]

introduce the function \(G(t, s) := Q(tu_1, tu_2, tsu_3)\), since \((u_1, u_2)\) is a minimizer for \(c_{12}\), we have

\[\|(u_1, u_2)\|_{c_{12}} = \sum_{i,j=1}^{2} \int_{\Omega} \beta_{ij} u_i^2 u_j^2,
\]

where \(\beta_{ii} = \mu_i, i = 1, 2\). It implies that \((u_1, u_2, 0) \in N_K\), then \(G(1, 0) = Q(u_1, u_2, 0) = 0\).
By direct computations, we have
\[
\frac{\partial G}{\partial s}
\bigg|_{t=1, s=0}
= 2 \left( \kappa_{13} \int_{\Omega} u_1 u_3 + \kappa_{23} \int_{\Omega} u_2 u_3 \right),
\]
\[
\frac{\partial G}{\partial t}
\bigg|_{t=1, s=0}
= -2 \| (u_1, u_2) \|_{\kappa_{12}} < 0.
\]
By the Implicit Function theorem, there exists an \( s_0 > 0 \) and a function \( t(s) \in C^1(-s_0, s_0) \) such that
\[
t(0) = 1, \quad G(t(s), s) = 0, \quad t'(s) = -\frac{G_s(t, s)}{G_t(t, s)}, \quad \text{for } s \in (-s_0, s_0).
\]
Therefore,
\[
(t(s)u_1, t(s)u_2, t(s)su_3) \in \mathcal{N}_K.
\]
Since \( (u_1, u_2) \) and \( (\tilde{u}_2, u_3) \) are minimizers for \( c_{12} \) and \( c_{23} \) respectively, by suitably choosing the minimizers according to the sign of \( \kappa_{12}, \kappa_{13}, \kappa_{23} \), we always have
\[
\kappa_{13} \int_{\Omega} u_1 u_3 < 0, \quad \kappa_{23} \int_{\Omega} u_2 u_3 < 0.
\]
In fact, note that \( \kappa_{12}\kappa_{13}\kappa_{23} < 0 \), we can proceed as the following way:
\[
\begin{align*}
(1) & \text{ if } \kappa_{12} > 0, \kappa_{13} < 0, \kappa_{23} > 0, \text{ then } u_1 > 0, u_2 < 0, u_3 > 0; \\
(2) & \text{ if } \kappa_{12} > 0, \kappa_{13} > 0, \kappa_{23} > 0, \text{ then } u_1 > 0, u_2 < 0, u_3 < 0; \\
(3) & \text{ if } \kappa_{12} < 0, \kappa_{13} > 0, \kappa_{23} > 0, \text{ then } u_1 < 0, u_2 < 0, u_3 < 0; \\
(4) & \text{ if } \kappa_{12} < 0, \kappa_{13} < 0, \kappa_{23} > 0, \text{ then } u_1 < 0, u_2 > 0, u_3 > 0.
\end{align*}
\]
Set \( \alpha := t'(0) = -G_s(1, 0)/G_t(1, 0) < 0 \), then
\[
t(s) = t(0) + t'(0)s + o(s) = 1 + \alpha s + o(s).
\]
For any \( s \in (-s_0, s_0) \), note that \( (t(s)u_1, t(s)u_2, t(s)su_3) \in \mathcal{N}_K \) and
\[
\alpha = \frac{\left( \kappa_{13} \int_{\Omega} u_1 u_3 + \kappa_{23} \int_{\Omega} u_2 u_3 \right)}{\| (u_1, u_2) \|_{\kappa_{12}}}.
\]
we have
\[
c_K \leq I_K(t(s)u_1, t(s)u_2, t(s)su_3) = \\
= \frac{1}{4} t^2(s) \| (u_1, u_2, su_3) \|_K = \\
= \frac{1}{4} \left( 1 + 2\alpha s(1+o(1)) \right) \left[ \| (u_1, u_2) \|_{\kappa_{12}} + \left( 2\kappa_{13} \int_{\Omega} u_1 u_3 + 2\kappa_{23} \int_{\Omega} u_2 u_3 \right) s + o(s) \right] = c_{12} + \left( \kappa_{13} \int_{\Omega} u_1 u_3 + \kappa_{23} \int_{\Omega} u_2 u_3 \right) s + o(s).
\]
Since \( \kappa_{13} \int_{\Omega} u_1 u_3 < 0, \kappa_{23} \int_{\Omega} u_2 u_3 < 0 \), then we can choose \( s > 0 \) sufficiently small such that \( \alpha \| (u_1, u_2) \|_{\kappa_{12}} + \left( \kappa_{13} \int_{\Omega} u_1 u_3 + \kappa_{23} \int_{\Omega} u_2 u_3 \right) s + o(s) < 0 \), whence \( c_K < c_{12} \). In this way, we have \( c_K < \min \{ c_{12}, c_{13}, c_{23} \} \), hence \( u_i \neq 0, i = 1, 2, 3 \), that is, \( \tilde{u} \) is a ground state solution to system (11).

Finally, if \( K \) is positive definite and \( \kappa_{ij} < 0(1 \leq i < j \leq 3) \), by similar arguments as the proof of Theorem 1.1 of [15], we can get that either \( u_i > 0, i = 1, 2, 3 \) or \( u_i < 0, i = 1, 2, 3 \). Note that system (11) is invariant under the following transformation
\[
\sigma : \mathbb{R}^3 \times \mathcal{H} \rightarrow \mathbb{R}^3 \times \mathcal{H}, \sigma(k_{ij}, k_{ik}, k_{kj}, u_i, u_j, u_k) = (k_{ij}, -k_{ik}, -k_{kj}, u_i, u_j, -u_k)
\]
for \( i \neq j \neq k \). When two of \( \{ \kappa_{ij}, \kappa_{ik}, \kappa_{kj} \} \) are positive, the other is negative, we can also infer the sign of the ground state solution.

For example, \( \kappa_{12} < 0, \kappa_{13} > 0, \kappa_{23} > 0 \), using the \( \sigma \)-invariance of system (11), we can get

\[
 u_1 > 0, u_2 > 0, u_3 < 0 \text{ or } u_1 < 0, u_2 < 0, u_3 > 0.
\]

Acknowledgments. The authors thank the referees for their helpful suggestions.

REFERENCES

[1] N. Akhmediev and A. Ankiewicz, Partially coherent solitons on a finite background, Phys. Rev. Lett., 82 (1999), 2661–2664.
[2] A. Ambrosetti and E. Colorado, Bound and ground states of coupled nonlinear Schrödinger equations, C. R. Math. Acad. Sci. Paris, 342 (2006), 453–458.
[3] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations, J. Lond. Math. Soc., 75 (2007), 67–82.
[4] T. Bartsch and Z. Q. Wang, Note on ground states of nonlinear Schrödinger systems, J. Partial Differential Equations, 19 (2006), 200–207.
[5] T. Bartsch, Z. Q. Wang and J. C. Wei, Bound states for a coupled Schrödinger system, J. Fixed Point Theory Appl., 2 (2007), 353–367.
[6] J. Belmonte-Beitia, V. M. Pérez-García and P. J. Torres, Solitary waves for linearly coupled nonlinear Schrödinger equations with inhomogeneous coefficients, J. Nonlinear Sci., 19 (2009), 437–451.
[7] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations. I. Existence of a ground state, Arch. Rational Mech. Anal., 82 (1983), 313–345.
[8] J. E. Brothers and W. P. Ziemer, Minimal rearrangements of Sobolev functions, Acta Univ. Carolin. Math. Phys., 28 (1987), 13–24.
[9] J. Busca and B. Sirakov, Symmetry results for semilinear elliptic systems in the whole space, J. Differential Equations, 163 (2000), 41–56.
[10] B. Deconinck et al., Linearly coupled Bose-Einstein condensates: From Rabi oscillations and quasiperiodic solutions to oscillating domain walls and spiral waves, Phys. Rev. A, 70 (2004), 705–706.
[11] G. W. Dai, R. S. Tian and Z. T. Zhang, Global bifurcation, priori bounds and uniqueness of positive solutions for coupled nonlinear Schrödinger systems, preprint.
[12] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 ed., Springer-Verlag, Berlin, 2001.
[13] D. S. Hall, M. R. Matthews, J. R. Ensher and C. E. Wieman, Dynamics of component separation in a binary mixture of Bose-Einstein condensates, Phys. Rev. A, 57 (1998), 1539–1542.
[14] N. Ikoma and K. Tanaka, A local mountain pass type result for a system of nonlinear Schrödinger equations, Calc. Var. Partial Differential Equations, 40 (2011), 449–480.
[15] K. Li and Z. T. Zhang, Existence of solutions for a Schrödinger system with linear and nonlinear couplings, J. Math. Phys., 57 (2016), 17 pp.
[16] T. C. Lin and J. C. Wei, Ground state of N coupled nonlinear Schrödinger equations in \( \mathbb{R}^n \), \( n \leq 3 \), Comm. Math. Phys., 255 (2005), 629–653.
[17] Z. L. Liu and Z. Q. Wang, Multiple bound states of nonlinear Schrödinger systems, Comm. Math. Phys., 282 (2008), 721–731.
[18] L. A. Maia, E. Montefusco and B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system, J. Differential Equations, 229 (2006), 743–767.
[19] C. J. Myatt, et al., Production of two overlapping Bose-Einstein condensates by sympathetic cooling, Phys. Rev. Lett., 88 (1997), 586–589.
[20] B. Noris, H. Tavares, S. Terracini and G. Verzini, Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition, Comm. Pure Appl. Math., 63 (2010), 267–302.
[21] Ch. Rüegg, et al., Bose-Einstein condensate of the triplet states in the magnetic insulator TlCuCl3, Nature, 423 (2003), 62–65.
[22] B. Sirakov, Least energy solitary waves for a system of nonlinear Schrödinger equations in \( \mathbb{R}^n \), Comm. Math. Phys., 271 (2007), 199–221.
[23] H. Tavares and T. Weth, Existence and symmetry results for competing variational systems, *NoDEA Nonlinear Differential Equations Appl.*, **20** (2013), 715–740.
[24] R. S. Tian and Z. T. Zhang, Existence and bifurcation of solutions for a double coupled system of Schrödinger equations, *Sci. China Math.*, **58** (2015), 1607–1620.
[25] E. Timmermans, *Phase separation of Bose-Einstein condensates*, *Phys. Rev. Lett.*, **81** (1998), 5718–5721.
[26] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Differential Equations*, **42** (1981), 400–413.
[27] Z. Q. Wang and M. Willem, Partial symmetry of vector solutions for elliptic systems, *J. Anal. Math.*, **122** (2014), 69–85.
[28] J. C. Wei and T. Weth, Nonradial symmetric bound states for a system of coupled Schrödinger equations, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, **18** (2007), 279–293.
[29] J. C. Wei and W. Yao, Uniqueness of positive solutions to some coupled nonlinear Schrödinger equations, *Commun. Pure Appl. Anal.*, **11** (2012), 1003–1011.
[30] T. Weth, Symmetry of solutions to variational problems for nonlinear elliptic equations via reflection methods, *Jahresber. Dtsch. Math.-Ver.*, **112** (2010), 119–158.
[31] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.
[32] M. Willem, *Principles d’analyse fonctionnelle*, Cassini, Paris, 2007.

Received June 2017; revised June 2017.

_E-mail address:_ zzt@math.ac.cn
_E-mail address:_ luohj@amss.ac.cn