Distance Configurations of Points in a Plane
with a Galois group that is not Soluble

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Abstract

We have conjectured that the constraint equations defined by a generic
Laman graph are not soluble by radicals when the graph is 3-connected.
We prove that this conjecture follows from the following simplier conjecture: the constraint equations defined by a generic Laman graph are not soluble by radicals if the graph does not contain a proper subgraph which is itself a Laman graph.

1 Introduction

We consider the algebraic equations that result from specifying the relative
distances between points in a plane. This gives a system of quadratic equations
of the form

\[(x_i - x_j)^2 + (y_i - y_j)^2 = d_{ij},\]  

where there is a pair of variables \((x_i, y_i)\) corresponding to each point \(i\) and an
equation for each of the specified (squared) distances \(d_{ij}\) between points \(i\) and
\(j\). The distances are taken as input parameters.

This is a simplified model of the more general problem that occurs in Me-
chanical Computer Aided Design (MCAD), namely the determination of the
location of various types of geometric elements (points, line segments and arcs)
when the designer specifies the relative distances between them. A similar sys-
tem of equations arises in the study of rigid structures [1], in molecular structure
[2] and in robotics [3]. The configurations that are of special interest are what
engineering designers call properly dimensioned or minimally rigid which means
that the number of distances is just sufficient to ensure discrete solutions for
the points (subject to the obvious rigid body motions in the Euclidean plane).

The general problem of geometric constraint solving is of considerable prac-
tical interest and over the past decade almost all MCAD products have included
some form of geometric constraint solving. The majority of them use algorithms
which are based on the mathematics described here [4].

The sets of equations which are best suited to interactive or real-time so-
lution are those that can be solved without recourse to numerical iteration
methods. Examples include a system of linear equations (which can be solved using Gaussian elimination) or a system of non-linear polynomial equations which can be transformed into triangular form involving at most the solutions of quadratic equations (quadratically soluble, QS) or equations which can be solved by extracting higher roots (radically soluble, RS). As well as being of practical importance, this classification leads to a nice connection with classical mathematics. QS equations correspond to geometric configurations that can be constructed using only a ruler and compass and the zeros of RS equations define an extension field with a Galois group that is a soluble group.

The equations (1) above are invariant under the continuous group of rotations and translations. This invariance can be removed by specifying as additional input parameters the coordinates of any pair of points which are joined by a distance \( \delta \). All such systems are equivalent by an affine transformation (over input parameters) and so the Galois group of a solution to the equations does not depend upon the pair of points which are chosen. We will simply assume that some convenient pair have been chosen.

The determination of the Galois group for an arbitrary geometric constraint problem seems to be very difficult. There are certainly many quite apparently complex configurations that are QS \( [4] \) and there are also some quite simple ones (involving for example just six points in a plane) whose Galois group is not soluble and whose equations are neither QS nor RS \( [5] \) and Section 3 below.

Although any instance of constraint solving will typically assign a set of rational values to the distances, we are commonly interested in finding solution methods which are independent of the particular values used for the distances. This leads us to consider solution methods when the distances are assigned values which are algebraically independent (over the rationals) which we will also call generic. We have shown in \( [5] \) that the form of equations (1) means that for generic distances all of the (finitely many) solutions have the same Galois group.

An abstract graph can be defined to represent a set of points with generic relative distances by corresponding the points with vertices of the graph and the generic distances with edges of the graph. We consider a configuration of \( N \) points in a plane with \( E \) relative distances. The corresponding abstract graph has \( N \) vertices and \( E \) edges. The graph is described as independent if every subgraph with \( n \) vertices and \( e \) edges satisfies \( 2n - e \geq 3 \) and maximally independent if additionally we have \( 2N - E = 3 \). It is a well-known result of Laman \( [6] \) that maximal independence for a graph is equivalent to infinitesimal rigidity for the corresponding generic framework and that these are also equivalent to the rigidity of the framework as a bar-joint mechanism \( [7] \). For this reason, a graph which is maximally independent is also known as a Laman graph and we will sometimes adopt this shorthand terminology. We note that in the usual definition of a generic framework the coordinates of the points (or joints) are taken as the algebraic independents. However, we have shown in \( [5] \) that this is equivalent to considering the distances as algebraic independents. In view of this equivalence we may simply refer to a Laman graph as generic if it represents a system of constraint equations with generic distances and we may also refer to...
the Galois group of a generic Laman graph as a shorthand for the Galois group of the corresponding equations with generic distances.

Some years ago one of us proposed that for generic distances there is also an abstract graph characterisation of whether a configuration has solutions that are QS [8]. We showed that any Laman graph has a recursive decomposition into subgraphs which are also maximally independent (after the possible introduction of some virtual edges) and are either 3-cycles (also called triangles) or are vertex 3-connected. We showed that the graph is QS (and thus RS) if these decomposition subgraphs are all 3-cycles, and proposed that otherwise the graph is not-QS. We have extended this to conjecture that the graph is also not-RS [8]. In other words, we have proved a sufficient condition for any generic graph to be RS and we propose that this condition is also sufficient. This sufficient condition would follow from the truth of the following conjecture:

**Conjecture A:** If a generic Laman graph is 3-connected then it is not solvable by radicals (not-RS).

We still do not have a complete proof of this conjecture. We have proved the conjecture for the (infinite) class of graphs that have a planar embedding (that is, the graph can be drawn on a plane with no edge crossings) [8]. Although our initial focus on planar graphs was made in order to identify a class of graphs for which we could complete a proof, we have subsequently found that planar Laman graphs are of specific interest in robotics problems [8] and so even this partial proof has some merit. Nevertheless the introduction of planar graphs does not seem to be intrinsic to the problem.

The main new result which we present in this paper is that the Conjecture A follows from a simpler conjecture which refers to a smaller class of graphs, namely:

**Conjecture B:** If a generic Laman graph does not contain a proper maximally independent subgraph then it is not solvable by radicals (not-RS).

In order to appreciate the significance of Conjecture B we introduce the notion of a basic Laman graph as follows:

**Definition:** A basic Laman graph has $2N - E = 3$ with a corresponding strict inequality $2n - e > 3$ in every proper subgraph with $n > 2$.

Conjecture B can now be restated in the intuitively appealing form:

**Conjecture B:** Basic Laman graphs are not generically solvable by radicals (not-RS).

It is easy to show (see corollary to Lemma 1 below) that every basic Laman graph is 3-connected and non-planar. The converse is not true. This is why Conjecture B (even when it is taken to refer only to non-planar graphs) refers to a smaller class of graphs than Conjecture A. There is one basic Laman graph with $N = 6$, there are none with $N = 7$ and two with $N = 8$. These are shown in Figure 1. Figure 2 shows a non-planar graph with $N = 7$ which is 3-connected but not basic.
Section 2 of the paper is used to prove Lemma 1 and the following main theorem.

**Theorem 1:** Conjecture B implies Conjecture A.

In section 3 we show by explicit calculation using Maple that the smallest basic Laman graph is not-RS. This adds weight to our conjecture that they are in fact all not-RS.

## 2 Proof of The Main Theorem

We use the same definitions as in [5] for the concepts which we adopt from graph theory and algebraic geometry. The reader is referred to that paper for complete definitions. For the sake of readability we reproduce our main graph theory definitions here.

A graph $G$ is a set of vertices $x$ and a set of edges $(xy)$ such that the edge $(xy)$ is in $G$ only if the vertices $x$ and $y$ are in $G$. The order $|G|$ is the number of vertices in $G$. A vertex $x$ is described as incident to the edge $(xy)$. Vertices $x$ and $y$ are adjacent in $G$ if $G$ contains the edge $(xy)$. The union and intersection of graphs $G$ and $H$ is the union and intersection of the sets (with duplicates removed). A subgraph $S$ of $G$ is described as proper if it is contained in $G$ and not equal to $G$ and has at least three vertices. A subgraph $S$ of $G$ is vertex induced if, whenever $S$ contains $x$ and $y$ and $G$ contains $(xy)$ then $S$ also contains $(xy)$. The graph $G\setminus S$ is the graph induced by the vertices of $G$ which are not in $S$. The vertices of attachment of $S$ in $G$ are the vertices of $S$ which are adjacent in $G$ to vertices in $G\setminus S$. An internal vertex of $S$ is a vertex of $S$ which is not a vertex of attachment. A sub-set $V$ of vertices of $G$ is said to separate $G$ if there exists a pair of vertices $\{a,b\}$ in $G\setminus V$ such that every connected path of edges which joins $a$ and $b$ always includes a member of $V$. A graph is $m$-connected if $|G| > m$ and $G$ has no separation sets of order $m - 1$. The disconnected components $\{C_i\}$ of the graph induced by the vertices of $G\setminus V$ are the separation components of $G$ with respect to $V$ [9]. The subgraphs of $G$ induced by the vertices of $C_i$ and $V$ are the separation blocks $\{S_i\}$ of $G$ with respect to $V$. Thus if a graph is $m$-connected, but not $m + 1$-connected, there is a set of at least two separation blocks $\{S_i\}$ such that $|S_i| > m$, the vertices of $S_i \cap S_j$ are exactly the vertices of $V$ for any $i$ and $j$ and $G = \cup_i S_i$.

If a graph $G$ has $n$ vertices and $e$ edges then the freedom number, $\text{free}(G)$ is $2n - e - 3$.

A graph $G$ is independent if $G$ and every proper subgraph of $G$ have a non-negative freedom number. $G$ is a maximally independent graph or equivalently a Laman graph if in addition $\text{free}(G) = 0$. A Laman graph is basic if it has no proper subgraph which is maximally independent.

The properties of a basic Laman graph which we have described above are easily obtained from the following:

**Lemma 1:** Let $G$ be a Laman graph with a vertex separation pair $\{a,b\}$ which separates $G$ into at least two separation blocks $S_i$ such that $G = \cup_i S_i$ and the
subgraphs $S_i \cap S_j$ have vertex set \{a, b\}. If $G$ contains \((ab)\) then \(\text{free}(S_i) = 0\), for all $i$, otherwise there is a block $S_1$ such that \(\text{free}(S_1) = 0\) and \(\text{free}(S_j) = 1\) all $j > 1$.

Proof. Suppose that disjoint graphs $H$ and $K$ are joined to create $H \cup K$ by identifying a pair of vertices \{a, b\} from each and removing repetition of edge \((ab)\) if both contain \((ab)\). If neither graph has edge \((ab)\) then \(\text{free}(H \cup K) = \text{free}(H) + \text{free}(K) - 1\). It follows (by repeated joining) that if $G$ does not contain \((ab)\) then \(\text{free}(G)\) is the sum of the freedom numbers of the components minus $k - 1$ where $k$ is the number of components. Since the components are independent and so have non-negative freedom number, the second half of the Lemma follows. When $G$ contains \((ab)\) there is a similar argument. \qed

Corollary: Every basic Laman graph $G$ with $|G| > 3$ is 3-connected and non-planar.

Proof. Every Laman graph $G$ is 2-connected so if $G$ is not 3-connected with $|G| > 3$ it has a separation pair and thus a proper subgraph $S_1$ with \(\text{free}(S_1) = 0\). Every planar Laman graph $G$ contains a 3-cycle $C$, by Corollary 4.10 of [5], for which \(\text{free}(C) = 0\), and $C$ is a proper subgraph for $|G| > 3$. \qed

Our strategy to prove Theorem 1 is to show that if there is a 3-connected, non-basic Laman graph with $N > 6$ vertices which is radically soluble (RS), then there is a 3-connected Laman graph with fewer than $N$ vertices which is also RS. This defines a sequence of 3-connected Laman graphs such that each successor graph is RS if its ancestor is RS. The sequence terminates on either a non-basic Laman graph with $N = 6$ or on a basic Laman graph. The only non-basic Laman graph with $N = 6$ is the doublet, which we have proved to be not-RS in [5] and so Conjecture A is proved true if Conjecture B is true.

The main graph reduction step which we use is edge contraction [9]. This is illustrated in figure 3. If $G$ is a graph with an edge \((xy)\) which joins vertices $x$ and $y$ then the edge contracted graph $G/(xy)$ is obtained from $G$ by deleting the edge \((xy)\), identifying the vertices $x$ and $y$ and deleting any duplicate edges which may result. The fundamental result which we need from algebraic geometry is proved in Theorem 6.1 of [5], namely:

If both $G$ and $G/e$ are maximally independent graphs then $G/e$ is generically soluble by radicals (RS) if $G$ is generically soluble by radicals (RS).

In order to use this graph reduction we must find edge contractions which maintain both maximal independence and 3-connectivity. It is easy to see that a necessary condition to preserve maximal independence when an edge in a graph $G$ is contracted is that the edge should be in exactly one 3-cycle of $G$. In [5] we focused on planar graphs in order to ensure that $G$ contains a 3-cycle. We say that an edge $e$ of a maximally independent graph $G$ is contractible if $G/e$ is maximally independent.

To prove Theorem 1 when $G$ may not contain a 3-cycle we first perform some surgery on $G$ in order to generate a graph which does have a 3-cycle. Since $G$
can be assumed to have a subgraph \( R \) which is maximally independent we can replace this subgraph in \( G \) with a subgraph which is a simple triangulation of the vertices of attachment of \( R \) in \( G \). This surgery is illustrated in Figure 4 and defined precisely in Lemma 4 below.

This subgraph replacement is a special case of a more general replacement scheme in which a maximally independent subgraph \( R \) is replaced with another maximally independent graph \( R' \) using the same vertices of attachment. This preserves many important properties of \( G \) as shown by the following.

**Lemma 2.** Let \( G \) be a graph with a maximally independent proper vertex induced subgraph \( R \) which has vertices of attachment \( \{c_1, \ldots, c_m\} \) in \( G \). Let \( R' \) be any maximally independent graph with \( n \geq m \) vertices such that \( m \) of its vertices are identified with the vertices \( \{c_1, \ldots, c_m\} \) in \( G \) and the remaining vertices are called internal vertices of \( R' \). A graph \( G' \) is obtained from \( G \) by deleting all of the edges and all of the internal vertices of \( R \) and replacing them with the edges and internal vertices of \( R' \). Then (i) \( \text{free}(G') = \text{free}(G) \). (ii) If \( G \) is independent then \( G' \) is independent. (iii) If \( G \) is maximally independent and both \( G \) and \( R' \) are generically RS, then \( G' \) is generically RS.

**Proof.** Suppose that \( G, G', R \) and \( R' \) have \( N, N', n \) and \( n' \) vertices and \( E, E', e \) and \( e' \) edges. Then \( N' = N - n + n' \), \( E' = E - e + e' \) and \( \text{free}(G') = 2N' - E' - 3 = 2N - E + 2n' - e' - (2n - e) - 3 = 2N - E - 3 = \text{free}(G) \).

To prove (ii) we show that every subgraph \( S' \) of \( G' \) has \( \text{free}(S') \geq 0 \).

Suppose to the contrary that there is a subgraph \( S' \) of \( G' \) with \( \text{free}(S') < 0 \). Then \( S' \) contains at least 2 of the vertices \( c_i \) or else it would also be a subgraph of \( G \). Then \( \text{free}(S' \cup R') = \text{free}(S') + \text{free}(R') - \text{free}(S' \cap R') \) and \( \text{free}(S' \cap R') \geq 0 \) because \( (S' \cap R') \) is a subgraph of \( R \), \( \text{free}(R') = 0 \) and \( \text{free}(S') \) is negative. This implies \( \text{free}(S' \cup R') < 0 \).

\( (S' \cup R') \) is a graph which contains the maximally independent subgraph \( R' \). If \( R' \) is replaced by the maximally independent graph \( R \) in the manner described in the Lemma statement above then the resulting graph has the form \( (S \cup R) \) for some subgraph \( S \) of \( G \) and by the first assertion of the lemma \( \text{free}(S \cup R) = \text{free}(S' \cup R') < 0 \). This contradicts the requirement that \( G \) is independent since \( (S \cup R) \) is a subgraph of \( G \).

To prove (iii) we appeal to some standard results of algebraic geometry [10].

Let the (squared) distances in \( R \) be \( \{d_e\} \), distances in \( R' \) be \( \{d_s\} \), distances in \( G \setminus R \) and in \( G \setminus R' \) be \( \{d_t\} \), and the distances in \( G \) be \( \{d_g\} \). Each set of generic distances is thus an algebraically independent set of indeterminates and in particular the (transcendental) field extension \( \mathbb{Q}(\{d_g\}) \) is a base field over which we can form the algebraic closure. It follows from the maximal independence of \( R \) and \( G \) that their equation sets determine algebraic varieties \( V(R) \) and \( V(G) \) (over the appropriate algebraic closures) which consist of finitely many points \([4], [16]\), and so (by definition) are zero dimensional varieties.

The equation set \( R \) is contained in the equation set \( G \) and so any point of \( V(G) \) gives a point of \( V(R) \) simply by restricting to the variables \( R \). It will be convenient to use the notation \( G|_w \) to indicate the equation set \( G \) with (some of) its variables evaluated at the point \( w \) and \( \pi_{G|_w}(V(G)) \) to represent the
points of the variety \( V(G) \) projected onto the subset of points represented by the variables in the equation set \( G \setminus R \).

We have
\[
\pi_{G \setminus R}(V(G)) = \bigcup_{w \in V(R)} (V(G|w)) = \bigcup_{w \in V(R)} (V((G \setminus R)|w)).
\]

Now \( V(G) \) is radical over \( \mathbb{Q}(\{d_r\}, \{d_t\}) \) and so \( \pi_{G \setminus R}(V(G)) \) is radical over \( \mathbb{Q}(\{d_r\}, \{d_t\}) \) and for any zero \( w \) in \( V(R) \) we have that \( V((G \setminus R)|w) \) is radical over \( \mathbb{Q}(\{d_r\}, \{d_t\}) \). Let \((x_1, \ldots, x_{2k}) (k \geq m)\) be a zero of \( V(R) \) where \( x_{2i}, x_{2i+1} \) are the coordinates of \( c_i \). Then \( \mathbb{Q}(\{d_r\}) \) is contained in \( \mathbb{Q}(\{x_1, \ldots, x_{2k}\}) \) so \( V((G \setminus R)|w) \) is radical over \( \mathbb{Q}(\{x_1, \ldots, x_{2k}\}, \{d_t\}) \). But \( \{x_1, \ldots, x_{2k}\} \) are algebraically independent over \( \mathbb{Q} \) and so \( V((G \setminus R)|(x_1, \ldots, x_{2k})) \) is radical over \( \mathbb{Q}(\{x_1, \ldots, x_{2k}\}, \{d_t\}) \). This holds for any algebraically independent set \( \{x_1, \ldots, x_{2k}\} \).

To construct a radical zero of \( G' \), we select the vertices \( c_1 \) and \( c_2 \) as the base vertices \( \mathbb{Q} \) and obtain the coordinates for the vertices \( c_3, \ldots, c_m \) by solving for the coordinates of the vertices \( c_3, c_4 \) etc in sequential order where the coordinates for the vertex \( c_j \) is determined from the equations represented by the edges \((c_j, c_{j-1}) \) and \((c_j, c_1) \). This can be done for each vertex in turn by solving a single quadratic equation \( \mathbb{Q} \). Thus each of the coordinates of \( c_1, \ldots, c_m \) are in a radical extension of \( \mathbb{Q}(\{d_r\}) \) and these coordinates are algebraically independent because \( R' \) is also a generic Laman graph. The coordinates of the remaining points of \( G' \) are determined as zeros of the equations of \( G' \setminus R' \) which are the same as the zeros of the equations of \( G \setminus R \) and therefore lie in a radical extension of \( \mathbb{Q}(\{x_1, \ldots, x_{2m}\}, \{d_t\}) \) which is in turn in a radical extension of \( \mathbb{Q}(\{d_r\}, \{d_t\}) \).

If this surgery is performed on an arbitrary Laman subgraph then the resulting graph may not be 3-connected. This is illustrated in Figure 5, where the graph \( G \) is 3-connected but the graph \( G \# R \) is not. Notice however that the subgraph \( R \) can be expanded to include the three vertices at the bottom of the graph. The resulting subgraph \( T \) is maximally independent and has 3 vertices of connection in \( G \). The graph \( G \# T \) is the doublet which is 3-connected. These considerations lead us to restrict attention to subgraphs which are maximal according to the following definition:

A proper subgraph \( R \) of a graph \( G \) is maximal with respect to property \( P \) if there is no proper subgraph \( S \) of \( G \) with the property \( P \) such that \( R \) is a proper subgraph of \( S \).

In order to use this property we need the following lemma:

**Lemma 3:** Let \( G' \) be the graph which is obtained from a maximally independent graph \( G \) by the replacement of a maximally independent proper subgraph \( R \) with a maximally independent graph \( R' \), in the manner described in Lemma 2. If \( R \) is maximal in \( G \) (with respect to the property of maximal independence) then \( R' \) is maximal in \( G' \).

**Proof.** Suppose to the contrary that there is a proper subgraph \( S' \) of \( G' \) which is maximally independent and which contains \( R' \) as a proper subgraph. This
means there is a vertex \( w \) in \( G' \setminus S' \), so \( w \) is also in \( G \setminus R' \). The vertices and edges of \( S'\setminus R' \) are vertices and edges of \( G \) and so replacing the proper subgraph \( R' \) in \( S' \) by the graph \( R \) gives a graph \( S \) which is a proper subgraph of \( G \) (proper, because it does not contain \( w \)) and which contains \( R \) as a proper subgraph. \( S \) is maximally independent by Lemma 2 which contradicts the fact that \( R \) is maximal.

With these preliminaries we can now prove the following useful properties of the proposed graph surgery.

**Lemma 4:** Let \( G \) be a 3-connected maximally independent graph which has a maximally independent proper subgraph \( R \) with \( m > 2 \) vertices of attachment \( c_1, \ldots, c_m \). Define \( G \# R \) to be the graph obtained from \( G \) by deleting all the edges of \( R \) and all of the internal vertices of \( R \), adding the cycle of edges \((c_1c_2), (c_2c_3), \ldots, (c_m c_1)\) and, for \( m > 3 \), the edges \((c_1c_3), (c_1c_4), \ldots, (c_m c_{m-3}), (c_1c_{m-2})\). Then

(i) \( G \# R \) is maximally independent.

If \( R \) is maximal (with respect to the property of maximal independence) then

(ii) \( G \# R \) is 3-connected.

(iii) Each of the edges \((c_ic_{i+1})\) is contractible.

(iv) Any proper maximally independent subgraph of \( G \# R \) with an internal vertex is also a proper maximally independent subgraph of \( G \) with an internal vertex.

**Proof.** Point (i) follows immediately from Lemma 2.

Let \( R' \) be the subgraph of \( G \# R \) induced by the vertices \( c_1, \ldots, c_m \). \( R' \) is maximally independent by construction and is maximal in \( G \# R \) by Lemma 3.

To prove (ii), (iii) and (iv) we first show that if \( S' \) is a proper maximally independent subgraph of \( G \# R \) not contained in \( R' \), then \( |R' \cap S'| < 2 \). For if \( |R' \cap S'| > 1 \) then by Lemma 4.2 of [5] \( R' \cup S' \) is maximally independent and thus \( G = R' \cup S' \) because \( R' \) is maximal. Each vertex \( c_i \) is incident to an edge which is not in \( R' \) so this edge and thus each \( c_i \) is in \( S' \) which implies that \( S' \) contains \( R' \) and thus \( S' = G' \) which contradicts the requirement that \( S' \) is proper.

If \( G \# R \) is not 3-connected then it has a separation pair \( \{a, b\} \) and both \( a \) and \( b \) are in \( R' \) or else \( \{a, b\} \) would also be a separation pair for \( G \). By Lemma 1 there is a proper subgraph \( S_1 \) with \( \text{free}(S_1) = 0 \) which has exactly \( a \) and \( b \) as vertices of attachment in \( G \). Thus \( |(R' \cap S_1)| > 1 \) and \( S_1 \) is contained in \( R' \) which is impossible because every vertex of \( R' \) is incident to a vertex of \( G \setminus R \) and would be a vertex of attachment of \( S_1 \).

The edge \((c_ic_{i+1})\) is in the 3-cycle \( \{c_i, c_{i+1}, c_1\} \) of \( R' \). By the point proved above there is no maximally independent subgraph which is not contained in \( R' \) and which also contains \((c_ic_{i+1})\). All maximally independent subgraphs in \( R' \) contain \( c_1 \). Thus by Lemma 4.5 of [5] the edge \((c_ic_{i+1})\) is contractible.

If \( S' \) is a proper maximally independent subgraph of \( G \# R \) with an internal vertex then \( S' \) is not contained in \( R' \) so \( R' \cap S' \) has a most one vertex and no edges. Thus \( S' \) is also a proper subgraph of \( G \).
In order to complete the reduction required to prove Theorem 1, we have to allow for the fact that even though \( G \neq R \) has contractible edges, the result of a contraction may not be 3-connected. We have shown in Lemma 1 that if a Laman graph \( G \) has a vertex separation pair \( \{a, b\} \), then \( G \) is the union of subgraphs \( S_i \), all with free \( (S_i) = 0 \) or 1, and that at least one of these, say \( S_1 \), has free \( (S_1) = 0 \). If a new (virtual) edge \((ab)\) is added to every \( S_i \) for which \( S_i = 1 \) then each graph \( S'_i \), defined as \( S_i \) with \((ab)\) added if free \( (S_i) = 1 \), is a Laman graph.

We have shown in \[1\] that the graph \( G \) is generically RS if and only if each of the \( S'_i \) is generically RS. This observation forms the basis of our recursive method for generating the zeros of the generic varieties of Laman graphs which are RS.

In order to use this observation in the proof of Theorem 1, we need to carry the analysis a little further because the components \( S'_i \) may not be 3-connected. This could happen either because \( G \) has more than one vertex separation pair or because some subgraph such as \( S_1 \) which does not have a virtual edge added to it may have a separation pair which is not a separation pair of \( G \). The former problem is managed by separating \( G \) at all of its separation pairs. The latter is managed by adding a new virtual edge \((ab)\) to every subgraph \( S_i \) which does not already contain this edge and defining \( B_i(ab) = S_i \cup (ab) \), for all \( i \). A new virtual edge \((ab)\) in \( B_i(ab) \) is marked as "redundant" if free \( (S_i) = 0 \). The redundant edges are included when we consider the 3-connectivity of \( B_i \) but are excluded when we consider whether \( B_i \) is RS. Each \( B_i \) is 2-connected because every Laman graph is 2-connected \[9\]. It is easy to see that any separation pair of any of the \( B_i(ab) \) is distinct from \( \{a, b\} \) and that it is also a separation pair of \( G \) \[11\]. The converse is also true because \( G \) is a Laman graph. This is shown in the following lemma and leads to a unique decomposition for \( G \) which is a little simpler than that given in \[11\] for a general graph.

**Lemma 5:** Let \( G \) be a Laman graph. Then

(i) If \( \{a, b\} \) and \( \{c, d\} \) are any distinct separation pairs of \( G \) which separate \( G \) respectively into blocks \( B_i(a, b) \) and \( B_j(c, d) \) as described above then \( \{c, d\} \) is also a separation pair of some block \( B_i(a, b) \). If \( B_i(c, d) \) is the corresponding block of the separation pair \( \{c, d\} \) which contains \( \{a, b\} \), then separating \( G \) at \( \{a, b\} \) and \( \{c, d\} \) gives blocks \( B_i(a, b), i > 1, B_j(c, d), j > 1, \) and \( B_1(a, b) \cap B_1(c, d) \). These blocks are independent of the order of the separations.

(ii) \( G \) has a unique decomposition into blocks \( B_i \) where each block is a subgraph of \( G \) plus virtual edges, some of which are marked as redundant. Each \( B_i \) is either (a) a 3-cycle, or (b) a 3-connected graph.

(iii) Each \( B_i \) (ignoring redundant virtual edges) is maximally independent and \( G \) is generically RS if and only if each \( B_i \) is generically RS.

(iv) At least one of the blocks \( B_i \) contains no redundant virtual edges.

**Proof.** If a separation pair \( \{c, d\} \) of \( G \) is not a separation pair of any \( B_i(a, b) \) then the vertices \( c \) and \( d \) are in different blocks \( B_i(a, b) \) and \( B_j(a, b) \). Then \( G \) is the union of four proper subgraphs as follows. \( G = S_1 \cup S_2 \cup S_3 \cup S_4 \).
where \( S_1 \cap S_2 = a, S_2 \cap S_3 = c, S_3 \cap S_4 = b, S_4 \cap S_1 = d \) (see [9], this is the case that produces cycles of length greater than 3 of original and virtual edges in the decomposition described in this reference). Then \( \text{free}(G) = \text{free}(S_1) + \text{free}(S_2) + \text{free}(S_3) + \text{free}(S_4) + 1 \) and this contradicts \( \text{free}(G) = 0 \) because each \( \text{free}(S_i) \geq 0 \). Thus the pair \( \{c, d\} \) is in some block \( B_1(a, b) \) and similarly the pair \( a, b \) is in some block \( B_1(c, d) \). The remaining blocks \( B_i(c, d), i > 1 \) are all contained in \( B_1(a, b) \) because they contain the pair \( \{c, d\} \) and do not contain the pair \( \{a, b\} \). Thus the pair \( \{c, d\} \) separates \( B_1(a, b) \) into \( B_i(c, d) \), all \( i > 1 \) and \( B_1(a, b) \cap B_3(c, d) \). Similarly the pair \( \{a, b\} \) separates \( B_1(c, d) \) into \( B_j(a, b) \), all \( j > 1 \) and \( B_1(a, b) \cap B_3(c, d) \). The two pairs \( \{a, b\} \) and \( \{c, d\} \) separate \( G \) into \( B_i(a, b), i > 1, B_j(c, d), j > 1 \) and \( B_1(a, b) \cap B_1(c, d) \) and this is independent of the order in which \( G \) is separated.

Suppose that \( G \) has \( m \) distinct separation pairs. The decomposition in (ii) is obtained by successively separating \( G \) and the resulting separation blocks and adding virtual edges and marking some of them as redundant as described above. This procedure terminates after \( m \) separations when none of the blocks have any separation pairs. The resulting blocks are independent of the order of the separations by (i). Each of the blocks \( B_i \) is either 3-connected or a 3-cycle of edges since these are the only 2-connected graphs without a separation pair.

Each block \( B_i \) is maximally independent (when redundant virtual edges are ignored) since this property is maintained for every block at every separation. Similarly \( G \) is RS if and only if every \( B_i \) is RS because this property is also maintained at each separation.

At most one virtual edge is marked as redundant for each separation pair of \( G \), so there are at most \( m \) redundant virtual edges. However after \( m \) separations there are at least \( m + 1 \) blocks and so at least one block has no redundant virtual edges.

With these preliminary lemmas complete we now give the main graph reduction lemma which allows us to maintain 3-connectivity following edge contractions.

**Lemma 6:** Let \( G \) be a 3-connected, maximally independent graph \(|G| > 6\). Suppose that \( G \) has no proper maximally independent subgraph with an internal vertex and that \( G \) has a contractible edge \( e \). Then either

(i) \( G \) has a contractible edge \( f \) such the \( G/f \) is 3-connected, or (ii) All separation blocks of \( G/e \) (as defined above) are 3-connected.

**Proof.** If \( e \) is on a 3-cycle of \( G \) which also has a vertex of degree 3 then by Lemma 4.7 of [5] at least one edge of the 3-cycle is contractible to give a 3-connected graph and (i) is satisfied. Otherwise we may assume that \( G/e \) is not 3-connected and that the edge \( e \) is on a 3-cycle all of whose vertices have degree at least 4. Since every vertex of \( G \) has degree at least 3 this implies that every vertex of \( G/e \) has degree at least 3.

Let \( e = (xy) \). Since \( G/e \) is not 3-connected it has a separation pair \((v, w)\) and one of the vertices, \( v \) say, is the combination of the vertices \( x \) and \( y \) from \( G \). This means that the vertex triple \((x, y, w)\) separates \( G \).
We claim that in fact \((x, y, w)\) divides \(G/e\) into exactly two components and that \(G\) does not contain the edge \((xw)\) or the edge \((yw)\). In [5] we proved this property (Lemma 4.8) when \(G\) is a planar graph using Kuratowski’s theorem. In fact this property is also true for non-planar graphs which do not contain a proper maximally independent subgraph. For suppose \((x, y, w)\) divides \(G\) into \(m\) blocks \(S_i\) each with freedom number \(f_i\) and that \(d = 0, 1\) or 2 according to how many of the edges \((xw)\) or \((yw)\) are in \(G\). The freedom number of the subgraph induced by the vertices \((x, y, w)\) is \((2 - d)\). Then \(\sum f_i = \text{free}(G) = \Sigma_i(f_i) - (m - 1)(2 - d)\) and so \(\Sigma_i(f_i - 2 + d) = d - 2\). Each \(f_i > 0\) since otherwise the subgraph \(S_i\) would be maximally independent with 3 vertices of attachment in \(G\) and an internal vertex. Thus \(\Sigma_i(d - 1) \leq d - 2\) and this requires \(d = 0\) and then at least two of the \(f_i = 1\), say \(f_1 = f_2 = 1\). The subgraph \(S_1 \cup S_2\) has freedom number \(f_1 + f_2 - 2 = 0\) and is a proper maximally independent subgraph of \(G\) with an internal vertex unless \(G = S_1 \cup S_2\).

We claim further (compare Lemma 4.17 of [5]) that each of the two separation blocks \(S_1(w)\) and \(S_2(w)\) has at least two internal vertices which are adjacent to the vertex \(w\). For suppose to the contrary that \(w\) has only one neighbour in \(C_i(w)\). Then the proper subgraph induced by the vertices of \(S_i(w)\) has freedom number \(f_i - 1 = 0\) and 3 vertices of attachment \(\{w', x, y\}\). Therefore, it cannot have an internal vertex so it is the 3-cycle of edges \((w'x), (w'y)\) and \((xy)\). This means the vertex \(w'\) has degree 3 and is on a 3-cycle containing \((xy)\) which was excluded at the start of the proof.

Note that there may be several different vertices \(w\) such that \((x, y, w)\) separate \(G\). Let \(w\) and \(w'\) be two vertices such that \((x, y, w)\) and \((x, y, w')\) both separate \(G\) and suppose that \(G\) contains the edge \((ww')\). Then the vertex pair \(\{x, y\}\) does not separate the graph \(G \setminus \{ww'\}\) because this would imply that one component of the separation set \((x, y, w)\) would have \(w\) adjacent to only one internal vertex \(w'\).

Now consider a complete separation of \(G/e\) at all separation pairs as described in Lemma 5. By this lemma every separation block is either a 3-cycle or a 3-connected graph. We will show that in fact each of the separation blocks is a 3-connected graph.

Since \(G\) is 3-connected every separation pair of \(G/e\) consists of the vertex \(v\) which results from combining the vertices \(x\) and \(y\) of \(G\) and a vertex \(w\) as described above. Thus every virtual edge is incident to the vertex \(v\). Since neither \((xw)\) nor \((yw)\) is an edge of \(G\), \((vw)\) is not an edge of \(G/e\) and so every vertex in every separation block which is a vertex of a separation pair of \(G\) gets a virtual edge. Every vertex of \(G/e\) has degree at least 3 and so any vertex in any separation block with degree less than three is incident to a virtual edge. In particular there are no 3-cycles in which there are two original (non-virtual) edges at a vertex. Since all virtual edges are adjacent to the vertex \(v\) the only possible 3-cycle consists of the vertex \(v\), virtual edges \((vw)\) and \((vw')\) and an original edge \((ww')\). This means that the vertex \(v\) separates \((G/e) \setminus \{ww'\}\). This implies that \((x, y, w)\) and \((x, y, w')\) are distinct separation sets for \(G\) and that the vertex pair \(\{x, y\}\) separates \(G \setminus \{ww'\}\). This contradicts the remark above and shows that all separation blocks of \(G/e\) are 3-connected graphs. □
These Lemmas now provide the basis for our proof of the main theorem

**Theorem 1**: Conjecture B implies Conjecture A.

**Proof.** Suppose to the contrary that Conjecture B is true and that G is a 3-connected maximally independent graph which is not basic and which is generically RS. Suppose further that G is a vertex minimal graph with this property. Then |G| > 6 since the only 3-connected maximally independent graph which is not basic with |G| ≤ 6 is the doublet and this is not RS.

Since G is 3-connected, every proper subgraph of G has more than two vertices of attachment in G. Let \{M\} be the set of maximally independent proper subgraphs of G and note that \{M\} is not empty since G is not basic. If \{M\} contains a subgraph, S which has an internal vertex then choose R from \{M\} such that R contains S and R is maximal. Otherwise simply choose R from \{M\} to be maximal. In the former case, the internal vertex of S is also an internal vertex of R.

Let G#R be the graph derived from G as described in Lemma 4. Then by Lemma 4 G#R is 3-connected and maximally independent and is generically RS. If R has any internal vertices, then |G#R| < |G| and this contradicts the requirement that G is minimal.

Otherwise |G#R| = |G| > 6 and we may assume that G and thus G#R has no maximally independent subgraphs which contain internal vertices. By Lemma 6 either G#R has a contractible edge f such that (G#R)/f is 3-connected or all of the separation blocks of (G#R)/e (including all their virtual edges) are 3-connected. In the first case (G#R)/f is RS by Theorem 6.1 of [5] which is quoted above and |(G#R)/f| = |G| − 1. This contradicts the requirement that G is minimal. In the second case, we conclude from Lemma 5 that at least one of the separation blocks B_1 of (G#R)/e contains no redundant edges and so remains 3-connected when redundant edges are ignored. Then by Lemma 5 G is generically RS only if the 3-connected graph B_1 is generically RS and since |B_1| < |G| this contradicts the requirement that G is minimal. □

3 **K(3,3) is not Soluble by Radicals**

Let us label the vertices of K(3,3) as shown in figure 1 and select the coordinates of vertices 1 and 2 to be (0,0) and (0,1) respectively. Then the polynomials represented by K(3,3) are:

\[ x_3^2 + y_3^2 - d_1, \quad (x_4 - 1)^2 + y_4^2 - d_3, \quad (x_5 - y_5)^2 - d_2, \]
\[ (x_6 - 1)^2 + y_6^2 - d_4, \quad (x_3 - x_4)^2 + (y_3 - y_4)^2 - d_5, \quad (x_4 - x_5)^2 + (y_4 - y_5)^2 - d_6, \]
and
\[ (x_4 - x_6)^2 + (y_4 - y_6)^2 - d_7, \quad (x_3 - x_4)^2 + (y_3 - y_6)^2 - d_8. \]

For each choice of real algebraically independent (squared) distances \(d_1, \ldots, d_8\), these equations determine a zero dimensional complex affine variety.
In order to prove that the zeros of these equations are non-radical for generic distances, we need to show that a polynomial generator for the elimination ideal in one of the variables is a polynomial whose Galois group is not a soluble group. Although this can in principle be done by treating the distances as generic parameters and performing the algebra in the field \( \mathbb{Q}(\{d\}) \), in practice the Maple software package which we use is unable to complete the calculation. We therefore choose specialised rational values for the distances, perform the calculations in \( \mathbb{Q} \) and then use the specialisation Theorem 7.2 of [5] to make a conclusion for generic distances. We select the specialised values for the (squared) distances to be 

\[
d_1 = d_2 = d_3 = d_4 = 1, \quad d_5 = 1/4, \quad d_6 = 4, \quad d_7 = 9/16 \quad \text{and} \quad d_8 = 9/4.
\]

The fifth equation and its three successors admit the squared form

\[
(d_5 - (x_3 - x_4)^2 + y_1^2 + y_4^2)^2 - 4y_3^2y_4^2 = 0,
\]

which in turn yields an equation in \( x_3 \) and \( x_4 \) alone on substituting for \( y_3^2 \) and \( y_4^2 \) from the first two equations. In this way we obtain a system \( \{g\} = \{g_1, g_2, g_3, g_4\} \) of four quartic equations in \( x_3, x_4, x_5, x_6 \) and the squared distances. Computing the resultants of pairs of these equations with respect to \( x_4, x_5 \) and \( x_6 \) gives a polynomial for \( x_3 \) of degree 20 for the specialised distance values given. This polynomial is known to be a specialisation of a polynomial with generic distances in the elimination ideal for \( x_3 \) by Theorem 8.3 of [5]. This polynomial factors into three polynomials, namely (with \( x = x_3 \)), \((x-1)^6\) and the following polynomials of degree 6 and degree 8:

\[
8773791129600x^6 - 280160493061120x^5 + 486601784497152x^4 - 58173137040244x^3 + 396516248769992x^2 - 110509387701405x + 1912924250825
\]

and

\[
19741148184576x^8 - 103544588664832x^7 + 29867097677824x^6 + 440356364853504x^5 - 761674146310464x^4 + 517152016022904x^3 - 215063281430796x^2 + 118596291789193x - 45476733930709
\]

The factor \((x-1)\) does not extend to give a zero of \( K(3,3) \) because this would require \( x_3 = 1, y_3 = 0 \) so that points 2 and 3 are coincident which would require \( d_3 = d_5 \). Maple shows that the degree 6 and degree 8 polynomials have a non-soluble Galois group. Then the specialisation Theorem 7.2 of [5] allows us to conclude that the graph \( K(3,3) \) is generically not radically soluble.

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