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Affine connections on complex compact surfaces and Riccati distributions

Received: 24 March 2022 / Accepted: 26 September 2022 / Published online: 12 October 2022

Abstract. Let $M$ be a complex surface. We show that there is a one-to-one correspondence between torsion-free affine connections on $M$ and Riccati distributions on $\mathbb{P}(TM)$. Furthermore, if $M$ is compact, then this correspondence induces a one-to-one correspondence between affine structures on $M$ and Riccati foliations on $\mathbb{P}(TM)$. As applications of this result, we classify the regular $k$-webs on compact complex surfaces for $k \geq 3$, and we also get a new proof of the classification of regular pencils of foliations on compact complex surfaces.

1. Introduction

There are numerous papers on compact complex surfaces that admit holomorphic affine connections (see [4, 6, 8, 10, 15, 20, 21]). In Inoue, Kobayashi and Ochiai [6] give a complete list of all compact complex (connected) surfaces admitting affine holomorphic connections which are not necessarily flat. These surfaces are shown to be biholomorphic (up to a finite covering) to complex tori, primary Kodaira surfaces, affine Hopf surfaces, Inoue surfaces or elliptic surfaces over Riemann surfaces of genus $g \geq 2$ with odd first Betti number. Moreover, it is proved in [6] that all these surfaces have torsion-free flat affine connections that are equivalent to holomorphic affine structures, that is, to atlas with values in open subsets of $\mathbb{C}^2$ whose change of coordinate maps are locally constant mappings in the affine group $GL(2, \mathbb{C}) \rtimes \mathbb{C}^2$. These surfaces are said to be affine. It is also known that they are quotients of a domain in $\mathbb{C}^2$ by a group consisting of affine transformations.

In Klingler [10] classifies holomorphic affine structures and holomorphic projective structures on compact complex surfaces. Also, in Dumitrescu [4] classifies torsion-free holomorphic affine connections and shows that any normal holomorphic projective connection on a compact complex surface has zero curvature. Finally, in his thesis Zhao [22] has studied affine structures and birational structures.

The aim of this paper is to study the correspondence between affine connections on complex surfaces and Riccati distributions. We list our main results.

Theorem A. Let $M$ be a complex surface. There is a one-to-one correspondence between torsion-free affine connections on $M$ and Riccati distributions on $\mathbb{P}(TM)$. 

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Mathematics Subject Classification: 53B05 · 37F75 · 14C21

https://doi.org/10.1007/s00229-022-01435-6
Furthermore, we have a one-to-one correspondence between affine structures on \( M \) and parallelizable Riccati foliations on \( \mathbb{P}(TM) \).

By a Riccati distribution on \( \mathbb{P}(TM) \) we mean a codimension one regular distribution on \( \mathbb{P}(TM) \) that is transversal to every fiber of \( \mathbb{P}^1 \)-bundle \( \pi : \mathbb{P}(TM) \rightarrow M \).

**Theorem B.** If \( M \) is a compact complex surface, then a Riccati distribution on \( \mathbb{P}(TM) \) is parallelizable. In particular, we have a one-to-one correspondence between affine structures on \( M \) and Riccati foliations on \( \mathbb{P}(TM) \).

Since important examples of Riccati foliations are induced by pencils and \( k \)-webs, we can use the previous theorems to study regular pencils and \( k \)-webs on complex surfaces. A regular pencil of foliations \( P = \{ F_t \}_{t \in \mathbb{P}^1} \) on \( M \) is locally defined by a pencil of 1-forms \( F_t = \{ \omega_t = 0 \}_{t \in \mathbb{P}^1} \) with \( \omega_t = \omega_0 + t\omega_\infty \) and \( \omega_0 \land \omega_\infty \neq 0 \), that is, \( F_0 \) and \( F_\infty \) are transversal (see [14] or [18]). We are able to give a new proof of the classification of regular pencils on compact complex surfaces proved by Puchuri [18].

**Theorem C.** If \( M \) is a compact complex surface and \( P \) is a regular pencil on \( M \), then \( M \) is either

1. a complex torus \( M = \mathbb{C}^2 / \Gamma \), where \( \Gamma \) is a lattice in \( \mathbb{C}^2 \), or
2. a Hopf surface \( M \), with universal covering space \( \mathbb{C}^2 \setminus \{(0,0)\} \). Its fundamental group is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} / l\mathbb{Z} \), for some integer \( l \geq 1 \); it is generated by a diagonal automorphism \( (x, y) \mapsto (ax, ay) \) with \( |a| < 1 \), and a diagonal automorphism \( (x, y) \mapsto (\varepsilon x, \varepsilon y) \) where \( \varepsilon \) is a primitive \( l \)-th root of 1.

Moreover, \( P \) is the only regular pencil on \( M \), and it is generated by \( \{ dx + tdy \}_{t \in \mathbb{P}^1} \) on \( \mathbb{C}^2 \) (in the first case) or on \( \mathbb{C}^2 \setminus \{(0,0)\} \) (in the second case).

A regular \( k \)-web \( W = \mathcal{F}_1 \FDots \mathcal{F}_l \) on \( M \) is given locally by \( k \) regular foliations \( \mathcal{F}_t \) in general position, that is, pairwise transversal (see [17] or [19]). When \( M \) is compact and \( k \geq 3 \), we can classify all the regular \( k \)-webs on it.

**Theorem D.** If \( M \) is a compact complex surface and \( W \) is a regular \( k \)-web on \( M \), \( k \geq 3 \), then \( M \) is either

1. a complex torus \( \mathbb{C}^2 / \Gamma \), where \( \Gamma \) is a lattice in \( \mathbb{C}^2 \), or
2. a hyperelliptic surface \( (E \times F) / G \), where \( E \) and \( F \) are elliptic curves, and \( G \) is a finite abelian subgroup of \( E \) (acting on \( E \) by translations), or
3. a Hopf surface \( M \), with universal covering space \( \mathbb{C}^2 \setminus \{(0,0)\} \). Its fundamental group is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} / l\mathbb{Z} \), for some integer \( l \geq 1 \); it is generated by a diagonal automorphism \( (x, y) \mapsto (ax, \xi ay) \) with \( |a| < 1 \) and \( \xi \) is primitive \( j \)-th root of 1, for some integer \( j \geq 1 \), and a diagonal automorphism \( (x, y) \mapsto (\varepsilon_1 x, \varepsilon_2 y) \) where \( \varepsilon_1, \varepsilon_2 \) are primitive \( l \)-th roots of 1.

Moreover, up to a finite cover, the \( k \)-web \( W \) is contained \( P \), where \( P \) denotes the only pencil on \( M \) from Theorem C.
In Sect. 2 we introduce the notion of a Riccati connection on a complex manifold of dimension \( n \geq 2 \). We also define the trace and the curvature of a Riccati connection. The main result in this section guarantees that the existence of a Riccati connection on a compact Kähler manifold yields a relationship between its Chern classes (see Theorem 2.3 and Proposition 2.5). We shall see that in the case of dimension 2, the Kähler hypothesis is not necessary.

In Sect. 3 given a complex surface \( M \) we establish a natural equivalence between reduced Riccati connections on \( M \) and Riccati distributions on \( \mathbb{P}(TM) \). We also introduce the notion of the curvature of a Riccati distribution and show that if \( M \) is compact, then a Riccati distribution is parallelizable, that is, it has zero curvature. Theorem A and Theorem B are proved in this section. By using the geometric description of all the affine compact complex surfaces (see the computations in [10, Théorème 1.2] with a fixed compact complex surface \( M \), where the set of complex affine structures on \( M \) compatible with its analytical structure are determined), we were able to calculate the monodromy of the Riccati foliations on \( \mathbb{P}(TM) \).

In Sect. 4 we apply the results of Sect. 3 to prove Theorem C and Theorem D.

### 2. Riccati connections

The following definition was motivated by R. Molzon and K. Pinney Mortensen [16]. Let \( M \) be a complex manifold of dimension \( n \geq 2 \). A Riccati connection on \( M \) is a \( \mathbb{C} \)-bilinear map

\[
\mathcal{D}R : TM \times TM \rightarrow TM,
\]
satisfying

1. \( \mathcal{D}R_{fX}Y = f \mathcal{D}R_{X}Y - \frac{1}{n} \mathcal{R}(f)Y \),
2. \( \mathcal{D}R_{X}fY = f \mathcal{D}R_{X}Y + \mathcal{R}(f)Y \),

for any local holomorphic functions \( f \) and any local vector fields \( X, Y \).

In coordinates \((x_1, \ldots, x_n) \in U \subset \mathbb{C}^n\), a trivialization of \( TU \) is given by the basis \((\partial x_1, \ldots, \partial x_n)\) and the Riccati connection is given by

\[
\mathcal{D}R_{X}(Y) = d(Y) + \theta Y - \frac{1}{n} \text{div}(X)Y,
\]
where \( \theta \) is the matrix of 1-forms associated with the Riccati connection \( \mathcal{D}R \) and \( \text{div} \) represents the divergence operator.

Letting \((\varphi_\alpha, U_\alpha), (\varphi_\beta, U_\beta)\) be two local systems of coordinates on \( M \) with \( U_\alpha \cap U_\beta \neq \emptyset \), and denoting the corresponding change of coordinates by \( \varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} \), we have

\[
\theta_\alpha = g_{\alpha\beta} \theta_\beta g_{\alpha\beta}^{-1} - d g_{\alpha\beta} g_{\alpha\beta}^{-1} + \frac{1}{n} \text{Tr}(dg_{\alpha\beta} g_{\alpha\beta}^{-1})I
\]
(1)

where \( g_{\alpha\beta} \) represents the Jacobian matrix of \( \varphi_{\alpha\beta} \) and \( \text{Tr} \) the trace operator.

Note that the trace of \( \theta \) represents a 1-form on \( M \) due to equation (1).

Now we will describe two invariants defined from a Riccati connection.
2.1. Trace and curvature of a Riccati connection

**Definition 2.1.** The *trace* of a Riccati connection $\mathcal{D}R$, denoted by $\text{Tr}(\mathcal{D}R)$, is the 1-form defined as the trace of a matrix of 1-forms on $M$ associated with $\mathcal{D}R$. We say that $\mathcal{D}R$ is a *reduced Riccati connection* if its trace is zero, i.e., $\text{Tr}(\mathcal{D}R) = 0$.

**Remark 2.2.** Every Riccati connection $\mathcal{D}R$ on $M$ determines a reduced Riccati connection $\tilde{\mathcal{D}}R$ on $M$ as follows

$$\tilde{\mathcal{D}}R_{X Y} = \mathcal{D}R_{X Y} - \frac{\text{Tr}(\mathcal{D}R)(X)}{n} Y.$$

Two Riccati connections $\mathcal{D}R$ and $\hat{\mathcal{D}}R$ on $M$ determine the same reduced Riccati connection if and only if there is a 1-form $\gamma$ on $M$ such that $\hat{\mathcal{D}}R_{X Y} - \mathcal{D}R_{X Y} = \gamma(X)Y$.

We introduce the notion of curvature of a Riccati connection following ideas similar to those of Kato [9]. Using the same notation as in (1), we define

$$W_\alpha := d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha. \quad (2)$$

The matrices of 2-forms $W_\alpha$ is called the *curvature* of the Riccati connection $\mathcal{D}R$. Using equation (1) we can verify that $W_\alpha = g_{\alpha\beta} W_\beta g_{\alpha\beta}^{-1}$, i.e., $W \in H^0(M, \Lambda^2 T^* M \otimes \text{End}(TM))$.

Let $t$ be an indeterminate and $A$ an $n \times n$ matrix. Define the elementary symmetric polynomials $C_k$ by

$$\det(tI + A) = \sum_{k=0}^{n} C_{n-k}(A)t^k.$$

We put $R_k(M) = C_k(\frac{i}{2\pi} W_\alpha)$, for $k = 0, 1, \ldots, n$.

**Theorem 2.3.** Let $M$ be a complex manifold of dimension $n \geq 2$. If $M$ admits a Riccati connection, then

$$c_k(M) = \sum_{j=0}^{k} \binom{n-j}{k-j} R_j(M) \left( \frac{c_1(M)}{n} \right)^{k-j},$$

where $c_k(M)$ is the $k$-th Chern form.

**Proof.** We know that there exists a Riccati connection $\mathcal{D}R$, for which equation (1) holds. Let $\nabla^0$ be a (smooth) connection on $\det(TM)$. That is, there exist $(1,0)$-forms $\eta_\alpha$ on $U_\alpha$ satisfying

$$\eta_\beta - \eta_\alpha = \text{Tr}(dg_{\alpha\beta} g_{\alpha\beta}^{-1}). \quad (3)$$

Note that the connection $\nabla^0$ has curvature form $K_{\nabla^0}^{\alpha} = d\eta_\alpha$ and $c_1(M) = c_1(\det TM) = \frac{i}{2\pi} d\eta_\alpha$. 

Define a matrix-valued smooth \((1,0)\)-form \(\Theta_\alpha\) on \(U_\alpha\) by
\[
\Theta_\alpha = \theta_\alpha + \frac{1}{n} \eta_\alpha,
\]
Using equations (1) and (3) we have
\[
\Theta_\alpha = g_{\alpha\beta} \Theta_{\beta} g_{\alpha\beta}^{-1} - d g_{\alpha\beta} g_{\alpha\beta}^{-1}
\]
This shows that \(\{\Theta_\alpha\}_\alpha\) is a (smooth) connection \(\nabla\) on \(TM\). Let \(K^\alpha_\nabla\) be the form curvature of \(\nabla\). We can see that
\[
K^\alpha_\nabla = W_\alpha + \frac{1}{n} K^\alpha_\nabla 0.
\]
Set \(A = \frac{i}{2\pi} K^\alpha_\nabla\), \(B = \frac{i}{2\pi} W_\alpha\) and \(\eta = \frac{i}{2\pi} K^\alpha_\nabla 0\). Then
\[
\det(tI + A) = \sum_{j=0}^{n} C_{n-j}(A)t^j = \sum_{j=0}^{n} C_{n-j}(M)t^j,
\]
The other side we can rewrite
\[
\det(tI + A) = \det(tI + B + \frac{1}{n} \eta I) = \det \left( (t + \frac{1}{n} \eta)I + B \right),
\]
\[
= \sum_{l=0}^{n} C_{n-l}(B)(t + \frac{1}{n} \eta)^l = \sum_{l=0}^{n} R_{n-l}(M)(t + \frac{1}{n} c_1(M))^l,
\]
\[
= \sum_{l=0}^{n} R_{n-l}(M) \sum_{s=0}^{l} \binom{l}{s} t^s \left( \frac{c_1(M)}{n} \right)^{l-s},
\]
\[
= \sum_{j=0}^{n} \left( \sum_{l=j}^{n} \binom{l}{j} R_{n-l}(M) \left( \frac{c_1(M)}{n} \right)^{l-j} \right) t^j.
\]
Comparing the terms of \(t^j\) of these two equalities, we get
\[
C_{n-j} = \sum_{l=j}^{n} \binom{l}{j} R_{n-l}(M) \left( \frac{c_1(M)}{n} \right)^{l-j}, \quad j = 0, \ldots, n
\]
Exchanging the indices \(n - j\) and \(k\) we complete the proof of the theorem. \(\square\)

Using Theorem 2.3 we have \(R_0(M) = 1\), \(R_1(M) = 0\),
\[
R_2(M) = c_2(M) - \frac{(n-1)}{2n} c_1^2(M), \ldots
\]
Thus we obtain the following corollary.

**Corollary 2.4.** The \(R_k(M)\) forms are \(d\)-closed. The de Rham cohomology classes of the \(R_k(M)\) forms are real cohomology classes and independent of the choice of Riccati connections.
**Proposition 2.5.** Let $M$ be a compact complex manifold of dimension $n \geq 2$ admitting a Riccati connection. Then

$$R_k(M) = 0, \text{ for } k \geq n/2.$$  

If $M$ is a complex surface, then

$$4c_2(M) = c_1^2(M).$$

If $M$ is Kähler, then the classes $R_k(M)$, $k \geq 1$, are zero and

$$c_k(M) = n^{-k} \binom{n}{k} c_1^k(M).$$

**Proof.** Since $R_k(M)$ is a holomorphic $2k$-form, if $2k > n$ then $R_k(M) = 0$.

Since every $c_k(M)$ is represented by a real $(k, k)$-form, using induction we can show that $R_k(M)$ is represented by a real $(k, k)$-form.

For $k = 2n$, we put $\gamma = R_k(M)$ is represented by a real $(k, k)$-form $\eta$. Since $\eta$ is real and cohomologous to $\gamma$, it is cohomologous to $\overline{\gamma}$. Hence $\gamma$ and $\overline{\gamma}$ are cohomologous to each other. Then

$$\int_M \gamma \wedge \overline{\gamma} = \int_M \gamma \wedge \gamma = 0.$$  

Thus, we get $\gamma = 0$.

If $M$ is Kähler, then $R_k(M) = 0$, since $R_k(M)$ is represented by a real $(k, k)$-form.

As an immediate consequence of the proposition above and [6, (2.2) Corollary], we have:

**Corollary 2.6.** A compact Kähler manifold $M$ with $c_1(M) = 0$ admits a Riccati connection if and only if it is covered by a complex torus.

**Remark 2.7.** In Molzon and Mortensen [16] introduced the concept of a projective connection on $M$, which is a $\mathbb{C}$-bilinear map

$$\Pi : TM \times TM \to TM$$

satisfying

1. $\Pi fX Y = f \Pi X Y - \frac{1}{n+1} X(f)Y$, for $f \in \mathcal{O}(M)$,
2. $\Pi X f Y = f \Pi X Y + X(f)Y - \frac{1}{n+1} Y(f)X$, for $f \in \mathcal{O}(M)$,
3. $\Pi X Y - \Pi Y X = [X, Y]$.

In the next section, we will see that a Riccati connection on a complex surface $M$ induces a projective connection on $M$.

From now on, unless stated otherwise, $M$ will denote an arbitrary complex surface (i.e., not necessarily compact).
3. Riccati distributions

Consider the total space $S = \mathbb{P}(TM)$ of the $\mathbb{P}^1$-bundle $\pi : \mathbb{P}(TM) \to M$. The three dimensional variety $S$ is called the contact variety.

For each point $q = (p, [v]) \in S$, i.e. $p \in M$ and $v \in T_p M$, one has the plane $\mathcal{D}_q := (d\pi(q))^{-1}(\mathbb{C}v)$. We obtain in this way a two dimensional distribution $\mathcal{D}$ on $S$, namely the so called contact distribution.

A codimension one holomorphic regular distribution $\mathcal{H}$ on $S$ is called a Riccati distribution if every fibre of $\pi$ is transverse of $\mathcal{H}$.

In local coordinates $(x, y) : U \to \mathbb{C}^2$, vectors $z_1 \partial_x + z_2 \partial_y$ in the fiber of $TM$ are replaced by homogeneous coordinates $(z_1 : z_2) = (1 : z)$ with $z \in \mathbb{P}^1$ in $\mathbb{P}(TM)$. Therefore, the $\mathbb{P}^1$-bundle writes

$\pi : \mathbb{P}^1 \times U \to U$

$((1 : z), (x, y)) \mapsto (x, y),$

over the set $U$, we know that $\mathcal{D}$ is given by $dy - zdx$ and the contact structure $dy = zdx$, and hence the Riccati distribution $\mathcal{H}$ is given by a 1-form of the type:

$$\omega = dz + \gamma + \delta z + \eta z^2,$$

with $\gamma$, $\delta$, and $\eta$ 1-forms on $U$.

The corresponding differential equation $dz = -\gamma - \delta z - \eta z^2$ is called the Riccati equation.

We write

$$\theta = \begin{pmatrix} -\delta & -\eta \\ \gamma & \delta \end{pmatrix}.$$

This matrix of 1-form represents a reduced Riccati connection $\mathcal{D}\mathcal{R}$. In fact, taking two local systems of coordinates $(\varphi_\alpha, U_\alpha)$ and $(\varphi_\beta, U_\beta)$ on $M$ with $U_\alpha \cap U_\beta \neq \emptyset$ and letting $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ denote the corresponding change of coordinates, we have

$$\omega_\alpha = h_{\alpha\beta} \omega_\beta, \quad h_{\alpha\beta} \in \mathcal{O}^*(\pi^{-1}U_\alpha \cap \pi^{-1}U_\beta),$$

and $\mathcal{H}_{|\pi^{-1}U_\alpha} = \{\omega_\alpha = 0\}$. Using equation (5) we get

$$\theta_\alpha = g_{\alpha\beta} \theta_\beta g_{\alpha\beta}^{-1} - d g_{\alpha\beta} g_{\alpha\beta}^{-1} + \frac{1}{2} \text{Tr}(d g_{\alpha\beta} g_{\alpha\beta}^{-1}) I.$$

Furthermore, the following conditions are equivalent

- $\mathcal{D}\mathcal{R}$ has zero curvature: $\theta \wedge \theta + d\theta = 0$;
- $\mathcal{H}$ is integrable (Frobenius): $\omega \wedge d\omega = 0$.

Therefore, we obtain the following.
Proposition 3.1. We have a one-to-one correspondence between reduced Riccati connections on $M$ and Riccati distributions on $\mathbb{P}(TM)$.

Furthermore, this correspondence induces a one-to-one correspondence between reduced Riccati connections with zero curvature on $M$ and Riccati foliations on $\mathbb{P}(TM)$, i.e. Frobenius integrable Riccati distributions.

In particular, finding Riccati foliations on $\mathbb{P}(TM)$ is equivalent to finding $2 \times 2$ holomorphic matrices of 1-forms $\theta_\alpha$ on $U_\alpha$ that satisfy the structure equations

$$d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = 0,$$

$$\text{Tr}(\theta_\alpha) = 0,$$

and whose changes of coordinates satisfy (6).

Corollary 3.2. If $M$ admits a Riccati connection, then it admits a projective connection.

Proof. By Proposition 3.1 there is $\mathcal{H}$ a Riccati distribution on $\mathbb{P}(TM)$. By intersecting $\mathcal{H}$ with the contact distribution $\mathcal{D}$ we obtain a geodesic foliation $\mathcal{G}$ that induces a projective connection on $M$. See [7, Section 2] for more details. \qed

3.1. Curvature of a Riccati distribution

Using the same notation as in (4), we write

$$\begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \delta_1 \\ \eta_1 \end{pmatrix} dx + \begin{pmatrix} \gamma_2 \\ \delta_2 \\ \eta_2 \end{pmatrix} dy, \quad \gamma_i, \delta_i, \eta_i \in \mathcal{O}(U).$$

We define the following 1-form

$$\kappa = \left( \frac{\delta_1}{2} - \gamma_2 \right) dx + \left( \eta_1 - \frac{\delta_2}{2} \right) dy,$$

which we call the connection form. Now, we will see that $\kappa$ determines the so-called holomorphic connection on $\det(TM)$.

Proposition 3.3. If $((x, y), U_\alpha)$, $((\tilde{x}, \tilde{y}), U_\beta)$ are two local systems of coordinates on $M$, with $U_\alpha \cap U_\beta \neq \emptyset$, then

$$\kappa_\alpha - \kappa_\beta = \frac{1}{2} \text{Tr}(dg_{\alpha \beta}g^{-1}_{\alpha \beta}) = \frac{1}{2} d \log(\det(g_{\alpha \beta})).$$

In particular, $\{-2\kappa_\alpha\}_\alpha$ defines a holomorphic connection on the canonical bundle $K_M$.

Proof. We denote by $\kappa = \kappa_\alpha = \kappa_1 dx + \kappa_2 dy$, $\tilde{\kappa} = \kappa_\beta = \tilde{\kappa}_1 d\tilde{x} + \tilde{\kappa}_2 d\tilde{y}$ and $g = g_{\alpha \beta} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$. Let $\mathcal{D} \mathcal{R}$ be the Riccati connection induced by $\mathcal{H}$, and we define $T(X, Y) = \mathcal{D} \mathcal{R}_X Y - \mathcal{D} \mathcal{R}_Y X - [X, Y]$. We see that $T(\partial_x, \partial_y) = -\kappa_2 \partial_x + \kappa_1 \partial_y$. 


We can verify the following properties: $T$ is $\mathbb{C}$-bilinear and

$$T(fX, gY) = fgT(X, Y) + \frac{1}{2}(fY(g)X - gX(f)Y),$$

where $f, g \in \mathcal{O}_M$, $X, Y \in TM$. Using these properties and the fact that $\partial_\tilde{x} = g_{11}\partial_x + g_{21}\partial_y$ and $\partial_\tilde{y} = g_{12}\partial_x + g_{22}\partial_y$, we get

$$T(\partial_\tilde{x}, \partial_\tilde{y}) = (-(\det g)\kappa_2 + \frac{1}{2}h_1)\partial_x + ((\det g)\kappa_1 + \frac{1}{2}h_2)\partial_y, \quad (9)$$

where,

$$h_1 = g_{11}(\partial_x(g_{12}) + \partial_y(g_{22})) - g_{12}(\partial_x(g_{11}) + \partial_y(g_{21})),
$$

$$h_2 = g_{21}(\partial_x(g_{12}) + \partial_y(g_{22})) - g_{22}(\partial_x(g_{11}) + \partial_y(g_{21})).$$

On the other hand, we have

$$T(\partial_\tilde{x}, \partial_\tilde{y}) = (-\tilde{\kappa}_2g_{11} + \tilde{\kappa}_1g_{12})\partial_x + (-\tilde{\kappa}_2g_{21} + \tilde{\kappa}_1g_{22})\partial_y. \quad (10)$$

Comparing (9) and (10), we get

$$(\det g)\kappa = (\det g)\tilde{\kappa} + \frac{1}{2}((h_1g_{21} - h_2g_{11})dx + (h_1g_{22} - h_2g_{12})d\tilde{y}).$$

It is not difficult to verify that

$$(\det g)\text{Tr}(dg \cdot g^{-1}) = (h_1g_{21} - h_2g_{11})d\tilde{x} + (h_1g_{22} - h_2g_{12})d\tilde{y}. \quad (\Box)$$

This completes the proof of the proposition.

We define the curvature of $\mathcal{H}$ as

$$K(\mathcal{H}) = d\kappa \in \Omega^2(M),$$

We say that $\mathcal{H}$ is parallelizable if $K(\mathcal{H}) = 0$.

**Theorem 3.4.** Let $\mathcal{H}$ be a Riccati distribution on $\mathbb{P}(TM)$ and $\{-2\kappa_\alpha\}_\alpha$ be the holomorphic connection on the canonical bundle $K_M$ induced by $\mathcal{H}$. Then $\{-2\kappa_\alpha\}_\alpha$ is flat if and only if $\mathcal{H}$ is parallelizable.

If $M$ is a compact complex surface, then $\mathcal{H}$ is parallelizable.

**Proof.** The first part follows directly from the definition.

For the second part, by Proposition 3.3, the 1-forms $\{2\kappa_\alpha\}_\alpha$ define a holomorphic connection on $\det(TM)$. The curvature of this connection is $2d\kappa$, which represents $c_1(\det TM) = c_1(M)$. We conclude by following the same ideas as Proposition 2.5. $\Box$
3.2. Riccati distributions and affine connections

An affine connection on $M$ is a (linear) holomorphic connection on the tangent bundle $TM$, i.e., a $C$-bilinear map $\nabla : TM \times TM \to TM$ satisfying the Leibnitz rule $\nabla_X(f \cdot Z) = f \cdot \nabla_X(Z) + df(X)Z$ and $\nabla f(X)(Z) = f \nabla_X(Z)$, for any holomorphic function $f$ and any vector fields $X, Z$. The connection $\nabla$ is torsion free when the torsion vanishes, that is $\nabla_X Z - \nabla_Z X - [X, Z] = 0$, for all vector fields $X, Z$, and the curvature of $\nabla$ is denoted by $K_\nabla = \nabla \cdot \nabla$. The connection $\nabla$ is flat when the curvature vanishes, that is $K_\nabla = \nabla \cdot \nabla = 0$.

We describe a map from the set of connections to into the set of distributions as follows: In coordinates $(x, y) \in U \subset \mathbb{C}^2$, a trivialization of $TU$ is given by the basis $(\partial_x, \partial_y)$ and the affine connection is given by

$$\nabla(Z) = d(Z) + \theta Z, \quad \theta = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix},$$

where $Z = z_1 \partial_x + z_2 \partial_y$ and $\theta_{ij} \in \Omega^1(U)$. On the projectivized bundle $\mathbb{P}(TU)$, with trivializing coordinate $z = z_2/z_1$, the equation $\nabla = 0$ induces a Riccati distribution $\mathcal{H}_\nabla$ that is locally given by

$$\omega = dz + \theta_{21} + (\theta_{22} - \theta_{11})z - \theta_{12}z^2. \quad (11)$$

**Theorem 3.5.** This map induces a one-to-one correspondence between torsion free affine connections on $M$ and Riccati distributions on $\mathbb{P}(TM)$.

Furthermore, there are one-to-one correspondences between:

1. Torsion free affine connections $\nabla$ on $M$ with zero curvature of the trace of connection $\text{Tr}(\nabla)$, i.e., $K_{\text{Tr}(\nabla)} = 0$, and parallelizable distributions on $\mathbb{P}(TM)$.
2. Torsion free affine connections $\nabla$ on $M$ with $K_\nabla = \frac{1}{2}K_{\text{Tr}(\nabla)}I$ and Riccati foliations on $\mathbb{P}(TM)$.
3. Affine structures on $M$ and parallelizable Riccati foliations on $\mathbb{P}(TM)$.

**Proof.** To verify that we have a bijection we will describe the inverse mapping as follows: Let $\mathcal{H}$ be a Riccati distribution on $\mathbb{P}(TM)$. By Proposition 3.1 we have a reduced Riccati connection $\mathcal{DR}$ induced by $\mathcal{H}$ that verifies $\theta_\alpha = g_{\alpha\beta}\theta_\beta g_{\alpha\beta}^{-1} - d g_{\alpha\beta}g_{\alpha\beta}^{-1} + \frac{1}{2}\text{Tr}(dg_{\alpha\beta}g_{\alpha\beta}^{-1})I$, where $\theta_\alpha$ is the matrix of 1-forms of $\mathcal{DR}$ in $U_\alpha$. By Proposition 3.3 we have $\frac{1}{2}\text{Tr}(dg_{\alpha\beta}g_{\alpha\beta}^{-1}) = \kappa_\alpha - \kappa_\beta$, where $\kappa_\alpha \in \Omega^1(U_\alpha)$ is the connection form. Define $\tilde{\theta}_\alpha = \theta_\alpha - \kappa_\alpha I$, so that $\tilde{\theta}_\alpha = g_{\alpha\beta}\tilde{\theta}_\beta g_{\alpha\beta}^{-1} - d g_{\alpha\beta}g_{\alpha\beta}^{-1}$. Then the 1-forms $\{\tilde{\theta}_\alpha\}_\alpha$ define an affine connection $\nabla^{\mathcal{H}}$ on $M$.

To verify items 1., 2. and 3. it is sufficient to see that $K_{\text{Tr}(\nabla)} = -2K(\mathcal{H}^{\nabla})$ and $K_\nabla = W - K(\mathcal{H}^{\nabla})I$, where $W$ is the curvature of the Riccati connection induced by $\mathcal{H}_\nabla$. The theorem is proved. \qed

**Corollary 3.6.** The following assertions are equivalent

- $M$ admits an affine connection,
- $M$ admits a Riccati connection,
- $\mathbb{P}(TM)$ admits a Riccati distribution.
It is worth pointing out that by using Corollary 3.2 together with the classification of (normal) projective connections \([11,12]\) and Proposition 2.5 one can obtain another proof of Corollary 3.6 in the case of a compact complex surface.

**Corollary 3.7.** If \(M\) is compact, then we have a one-to-one correspondence between Riccati foliations on \(\mathbb{P}(TM)\) and affine structures on \(M\).

**Proof.** This follows from Theorem 3.4 and Theorem 3.5. \qed

### 3.3. Monodromy of a Riccati foliation

The **monodromy representation** of the Riccati foliation \(\mathcal{H}\) on \(S = \mathbb{P}(TM)\) is the representation

\[
\rho_\mathcal{H} : \pi_1(M) \to \text{Aut}(\mathbb{P}^1)
\]

defined by lifting paths on \(M\) to the leaves of \(\mathcal{H}\). The image of \(\rho_\mathcal{H}\) in \(\text{Aut}(\mathbb{P}^1)\) is, by definition, the **monodromy group** of \(\mathcal{H}\), denoted by \(\text{Mon}(\mathcal{H})\). Let’s see some particular cases.

#### 3.3.1. Pencils of foliations and Riccati foliations

A regular pencil of foliations \(\mathcal{P}\) on \(U\) is a one-parameter family of foliations \(\{\mathcal{F}_t\}_{t \in \mathbb{P}^1}\) defined by \(\mathcal{F}_t = [\omega_t = 0]\) for a pencil of 1-forms \(\{\omega_t = \omega_0 + t\omega_\infty\}_{t \in \mathbb{P}^1}\) with \(\omega_0, \omega_\infty \in \Omega^1(U)\) and \(\omega_0 \wedge \omega_\infty \neq 0\) on \(U\). The pencil of 1-forms defining \(\{\mathcal{F}_t\}_{t \in \mathbb{P}^1}\) is unique up to multiplication by a non vanishing function: \(\tilde{\omega}_t = f \omega_t\) for all \(t \in \mathbb{P}^1\) and \(f \in \mathcal{O}(U)\). In fact, the parametrization by \(t \in \mathbb{P}^1\) is not intrinsic; we will say that \(\{\mathcal{F}_t\}_{t \in \mathbb{P}^1}\) and \(\{\tilde{\mathcal{F}}_t\}_{t \in \mathbb{P}^1}\) define the same pencil on \(U\) if there is a Möbius transformation \(\varphi \in \text{Aut}(\mathbb{P}^1)\) such that \(\tilde{\mathcal{F}}_t = \mathcal{F}_{\varphi(t)}\) for all \(t \in \mathbb{P}^1\).

The graphs \(S_t\) of the foliations \(\mathcal{F}_t\) are disjoint sections (since foliations are pairwise transversal) and form a codimension one foliation \(\mathcal{H}\) on \(\mathbb{P}(TU)\) transversal to the projection \(\pi : \mathbb{P}(TU) \to U\). The foliation \(\mathcal{H}\) is a Riccati foliation, i.e. a Frobenius integrable Riccati distribution:

\[
\mathcal{H} : [\omega = 0], \quad \omega = dz + y + \delta z + \eta z^2, \quad \omega \wedge d\omega = 0.
\]

In local coordinates \((x, y)\) such that \(\mathcal{F}_0\) and \(\mathcal{F}_\infty\) are defined by \(dx = 0\) and \(dy = 0\) respectively, we can assume the pencil is generated by \(\omega_0 = dx\) and \(\omega_\infty = u(x, y)dy\) (we have normalized \(\omega_0\) with \(u(0, 0) \neq 0\). Then the graph of each foliation \(\mathcal{F}_t\) is given by the section \(S_t = \{z = -t/u(x, y)\} \subset \mathbb{P}(TU)\). These sections are the leaves of the Riccati foliation \(\mathcal{H} : [dz + du/du \cdot z = 0]\).

We know that the curvature \(K(\mathcal{P})\) of a regular pencil \(\mathcal{P} = \{\mathcal{F}_t\}_{t \in \mathbb{P}^1}\) is a 2-form (see [14]). For instance, if \(\mathcal{P}\) is in normal form \(\mathcal{P} = dx + tu(x, y)dy\), then the curvature is given by

\[
K(\mathcal{P}) = -(\ln u)_{xy} dx \wedge dy.
\]

We recall that a pencil \(\mathcal{P}\) is flat when its curvature is zero, that is \(K(\mathcal{P}) = 0\).
On the other hand, \( \mathcal{P} \) induces a Riccati foliation \( \mathcal{H} : [dz + \frac{du}{u^2}z = 0] \) on \( \mathbb{P}(TU) \). So we have \( \kappa = \frac{u_x}{2u} dx - \frac{u_y}{2u} dy \). Thus the curvature of \( K(\mathcal{H}) = K(\mathcal{P}) = d\kappa \), this means that the definition of the curvature of a Riccati distribution extends the definition of the curvature of a pencil of foliations. So, we have the following proposition.

**Proposition 3.8.** The following data are equivalent:

- a regular pencil of foliations \( \{\mathcal{F}_t\}_{t \in \mathbb{P}^1} \) on \( M \),
- a Riccati foliation \( \mathcal{H} \) on \( \mathbb{P}(TM) \) with trivial monodromy, i.e., \( \text{Mon}(\mathcal{H}) = \{id\} \).

Furthermore, if \( M \) is compact and \( \mathcal{P} \) is a regular pencil on \( M \), then \( \mathcal{P} \) is flat.

**Proof.** It follows from the previous construction and the fact that the leaves of the foliation \( \mathcal{H} \) are defined by the graphs \( S_t \) of the foliations \( \mathcal{F}_t \) that the monodromy is trivial. \( \square \)

The following lemma exhibits the normal form of a flat pencil that represents an affine structure.

**Lemma 3.9.** (\cite{14, Lemma 2.1.4}). Let \( \mathcal{P} \) be a flat regular pencil defined in a neighborhood of the origin \( 0 \in \mathbb{C}^2 \). Then, there is a change of local coordinates \( \varphi(x, y) \) sending \( \mathcal{P} \) to the pencil defined by \( dx + tdy, t \in \mathbb{P}^1 \).

### 3.3.2. Webs and pencils of foliations

Let \( \mathcal{W} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \boxtimes \mathcal{F}_3 \boxtimes \mathcal{F}_4 \) be a regular 4-web on \( (\mathbb{C}^2, 0) \)

\[
\mathcal{F}_i = [X_i = \partial x + e_i(x, y) \partial y] = [\eta_i = e_i dx - dy], \quad i = 1, 2, 3, 4.
\]

The cross-ratio

\[
(\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4) := \frac{(e_1 - e_3)(e_2 - e_4)}{(e_2 - e_3)(e_1 - e_4)}
\]

is a holomorphic function on \( (\mathbb{C}^2, 0) \) intrinsically defined by \( \mathcal{W} \). Then, we have:

**Proposition 3.10.** If \( \mathcal{W} = \mathcal{F}_0 \boxtimes \mathcal{F}_1 \boxtimes \mathcal{F}_\infty \) is a regular 3-web on \( (\mathbb{C}^2, 0) \), then there is a unique pencil \( \{\mathcal{F}_t\}_{t \in \mathbb{P}^1} \) that contains \( \mathcal{F}_0, \mathcal{F}_1 \) and \( \mathcal{F}_\infty \) as its elements. More precisely, \( \mathcal{F}_i \) is the only foliation such that

\[
(\mathcal{F}_i, \mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_\infty) = t.
\]

Conversely, any Riccati foliation comes from a 3-web: it suffices to choose 3 elements of a pencil. In particular, any 4 elements of a pencil \( \{\mathcal{F}_t\}_{t \in \mathbb{P}^1} \) have constant cross-ratio.

In general, we can define a cross-ratio for a \( k \)-web, \( k > 3 \), on a compact complex surface \( M \). Indeed, in a finite cover of \( M \) any four foliations define a constant cross-ratio, and therefore they define a cross-ratio on \( M \), thus giving a Riccati foliation on \( M \).

The proposition below follows from the previous considerations.

**Proposition 3.11.** If \( M \) is compact and \( \mathcal{W} \) is a regular \( k \)-web on \( M, k \geq 3 \), then there is a Riccati foliation \( \mathcal{H} \) on \( \mathbb{P}(TM) \) induced by \( \mathcal{W} \) such that

\[
\text{Mon}(\mathcal{H}) \subset \text{Sym}(1, \ldots, k).
\]

See \cite{17} and \cite{7} for more details.
3.4. Calculations of the monodromy of a Riccati foliation

What we will do is the following: calculate the Riccati connections on $M$ of zero curvature and zero trace, and hence the space of Riccati foliations on $\mathbb{P}(TM)$. If $M$ has at least one affine structure, the Riccati connections are the elements of $H^0(M, V)$, where $V = \text{End}(TM) \otimes T^*M$ (vector bundle, Higgs field) = Higgs bundle. This follows from (6) since $g_{\alpha\beta}$, can be chosen constant so that $dg_{\alpha\beta} = 0$.

Now pull back $V$ via the universal cover $\pi : \tilde{M} \to M$ of $M$, and compute the holomorphic sections of $\pi^*(V)$; those invariant by $\pi_1(M)$ will be the holomorphic sections of $V$.

The universal cover $\tilde{M}$ of each of these surfaces is a subdomain of $\mathbb{C}^2$ and the covering transformations are affine so that the standard flat holomorphic linear connection on $\mathbb{C}^2$ restricted to $\tilde{M} \subset \mathbb{C}^2$ can be “pulled down” to $M$. In this case, $M$ has a natural or usual affine connection whose corresponding affine coordinates are those of $\tilde{M} \subset \mathbb{C}^2$ defined locally on $M$. Since $\pi^*(V) = \tilde{M} \times \mathbb{C}^8$, the Riccati connections on $\tilde{M}$ are of the form $\theta = A_1(x, y)dx + A_2(x, y)dy$ where $A_k(x, y)$ is a holomorphic $2 \times 2$ matrix and $(x, y)$ are the global coordinates on $\tilde{M} \subset \mathbb{C}^2$. The Riccati connections on $M$ are the 2-forms $\theta$ that are invariant by $\pi_1(M)$. Using the structure equations (7) and (8), the zero curvature condition becomes

$$0 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} + [A_1, A_2],$$

and the relation $\text{Tr}(\theta) = 0$ turns into $\text{Tr}(A_1) = \text{Tr}(A_2) = 0$.

3.4.1. Complex torus surfaces

In this case $M$ is the quotient of $\mathbb{C}^2$ by some lattice $\Gamma = \bigoplus_{i=1}^4 (k_i, l_i)\mathbb{Z}$. Then $\theta = A_1dx + A_2dy$, where $A_1, A_2$ are complex $2 \times 2$ matrices such that

$$A_1A_2 = A_2A_1 \text{ and } \text{Tr}(A_1) = \text{Tr}(A_2) = 0. \quad (12)$$

The pairs of matrices $(A_1, A_2)$ are divided into conjugacy classes: $(A_1, A_2)$ is said to be conjugate to $(B_1, B_2)$ iff there exists $C \in \text{GL}(2, \mathbb{C})$ such that $CB_1C^{-1} = A_1$ and $CB_2C^{-1} = A_2$. Clearly, if we can find one member $(B_1, B_2)$ of a conjugacy class satisfying (12) then the other members $(A_1, A_2)$ are easily obtained. So we may assume that $B_1, B_2$ are in Jordan normal form, and in this case what we obtain is given in the listed below:

| Type | $B_1 =$ | $B_2 =$ | Monodromy |
|------|----------|----------|------------|
| 1    | $\begin{pmatrix} -\frac{a}{2} & 0 \\ 0 & \frac{a}{2} \end{pmatrix}$ | $\begin{pmatrix} -\frac{b}{2} & 0 \\ 0 & \frac{b}{2} \end{pmatrix}$ | $z \mapsto z \exp(ak_i + bl_i)$ |
| 2    | $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$ | $z \mapsto z - (k_i + cl_i)$ |
| 3    | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ | $z \mapsto z - l_i$ |
where \( a, b, c \in \mathbb{C}, i = 1, 2, 3, 4 \) and the corresponding monodromy (seen as a Möbius transformation on one complex variable \( z \)) is also included.

We write \( C = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \). Then, in general, for \( A_1 \) and \( A_2 \) we have:

| Type | Riccati foliation \( \mathcal{H} \) induced by \( \theta \) | Monodromy |
|------|-------------------------------------------------|------------|
| 1    | \( dz + \frac{1}{\det C}(g - ez)(-h + fz)(ax + by) \) | \( z \mapsto z \exp(ak_i + bl_i) \) |
| 2    | \( dz + \frac{1}{\det C}(g - ez)(-g + ez)(dx + cdy) \) | \( z \mapsto \begin{cases} 
    z - \frac{g}{f}(k_i + cl_i), & \text{if } e = 0, \\
    z - e(k_i + cl_i), & \text{if } e \neq 0.
  \end{cases} \) |
| 3    | \( dz + \frac{1}{\det C}(g - ez)(-g + ez)dy \) | \( z \mapsto \begin{cases} 
    z - \frac{g}{f}l_i, & \text{if } e = 0, \\
    z - el_i, & \text{if } e \neq 0.
  \end{cases} \) |

where \( a, b, c \in \mathbb{C} \) and \( i = 1, 2, 3, 4 \).

From the list above we deduce the following lemma.

**Lemma 3.12.** Let \( \mathcal{H} \) be a Riccati foliation on \( \mathbb{P}(T(\mathbb{C}^2 / \Gamma)) \) induced by \( \theta \). Then \( \text{Mon}(\mathcal{H}) \) is trivial if and only if \( \theta = 0 \), i.e., it is of type 1 with \( a = b = 0 \). Furthermore, if \( \text{Mon}(\mathcal{H}) \) is finite, then \( \text{Mon}(\mathcal{H}) \) is trivial.

**Proof.** It is enough to verify that if the monodromy is finite, then \( \theta = 0 \). In fact, we can assume that \( (k_1, l_1) = (1, 0), (k_4, l_4) = (0, 1) \) and \( \det(\text{im } \tau) \neq 0 \), where \( \tau = \begin{pmatrix} k_1 & k_2 \\ l_1 & l_2 \end{pmatrix} \). So, \( \mathcal{H} \) is of type 1 and \( \text{Mon}(\mathcal{H}) \) is generated by \( h_j(z) = z \exp(ak_j + bl_j) \) such that \( h_j^n(z) = z \), for all \( j = 1, 2, 3, 4 \) and for some \( n \in \mathbb{N} \). Thus we see \( na = 2\pi ir \) and \( nb = 2\pi is, r, s \in \mathbb{Z} \). Also, we have \( rk_1 + sl_1 \in \mathbb{Z} \) and \( rk_2 + sl_2 \in \mathbb{Z} \). Looking at the imaginary part of these numbers we deduce \( \det(\text{im } \tau T \begin{pmatrix} r \\ s \end{pmatrix}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), so we get \( r = s = 0 \) and therefore \( a = b = 0 \). \( \square \)

### 3.4.2. Hyperelliptic surfaces

A hyperelliptic surface, or bi-elliptic surface, is a surface that can be written as the quotient of a product of two elliptic curves by a finite abelian group. According to [3, Theorem 4], there are seven types of hyperelliptic surfaces that can written as \( M = E \times F / G \), where \( E, F \) elliptic curves and \( G \) is finite abelian subgroup of \( E \). Therefore we can also calculate the monodromy of Riccati foliations on \( \mathbb{P}(TM) \), and these are given in the following table.

where \( \nu \) is a primitive cube root of 1. The following lemma is a straightforward consequence of the above table.

**Lemma 3.13.** Let \( \mathcal{H} \) be a Riccati foliation on \( \mathbb{P}(T(E \times F / G)) \). Then \( \text{Mon}(\mathcal{H}) \) is non-trivial, finite and cyclic. In particular, there are no regular pencils on \( E \times F / G \) and there exists regular k-webs on \( E \times F / G \), for \( k \geq 3 \).
3.4.3. Primary Kodaira surfaces  A primary Kodaira surface $K$ is the quotient of $\mathbb{C}^2$ by some group $G$ generated by

\[
g_1(x, y) = (x, y + 1), \quad g_2(x, y) = (x, y + \tau_1), \quad g_3(x, y) = (x + 1, ax + y), \quad g_4(x, y) = (x + \tau_2, bx + y),
\]

where $\tau_1, \tau_2$ are generators of the fundamental domain of the modular group and $a, b \in \mathbb{C}$ satisfy $a\tau_2 - b = m\tau_1$ for some positive integer $m$. See more details in [21].

A Riccati connection on $K$ has the form

\[
\theta = \begin{pmatrix} e & 0 \\ c & h \end{pmatrix} dx + \begin{pmatrix} 0 & 0 \\ e & h \end{pmatrix} dy,
\]

for some constants $e, c, h$. Therefore the Riccati foliations on $\mathbb{P}(TK)$ are of the form $dz + cdx, c \in \mathbb{C}$. So, the monodromy group Mon is generated by $f_1(z) = z - a - c$, and $f_2(z) = z - b - c\tau_2$. From this we can conclude that there are no regular pencils $\mathcal{P}$ and regular $k$-webs $\mathcal{W}$ on a primary Kodaira surface, for $k \geq 3$.

3.4.4. Hopf surfaces  A compact complex surface $M$ whose universal covering space is biholomorphic to $\mathbb{C}^2 - \{0\}$ is called a Hopf surface.

1. Primary Hopf surfaces. We consider first the primary Hopf surfaces, which are quotients of $\mathbb{C}^2 - \{0\}$ by the infinite cyclic group generated by an automorphism $g$ of $\mathbb{C}^2 - \{0\}$. According to [13, Part II], 10, $g$ has the form:

\[
g(x, y) = (ax + \lambda y^m, by),
\]

for some positive integer $m$ and some complex numbers $a, b, \lambda$ with $0 < |a| \leq |b| < 1$ and $(a - b^m)\lambda = 0$.

By using [6, (7.5) Theorem] we have:

| Condition | Riccati foliation $\mathcal{H}$ induced by $\theta$ | Monodromy |
|-----------|-----------------------------------------------|------------|
| $\lambda \neq 0, m = 1$ | $dz$ | $z \mapsto \frac{z b + \lambda}{a}$ |
| $\lambda = 0, a \neq b^2$ | $dz$ | $z \mapsto \frac{z b}{a}$ |
| $\lambda = 0, a = b^2$ | $dz - cz^2 dy$ | $z \mapsto \frac{z}{b}$ |
where \( c \in \mathbb{C} \).

From the above list we deduce directly the following.

**Lemma 3.14.** Let \( M \) be a primary Hopf surface and \( \mathcal{H} \) be a Riccati foliation on \( \mathbb{P}(TM) \).

1. Mon\((\mathcal{H})\) is trivial if and only if \( \theta = 0 \), i.e., \( \lambda = 0 \) and \( a = b \).
2. \( \mathcal{H} \) is induced by a regular \( k \)-web, \( k \geq 3 \), if and only if \( \lambda = 0 \) and \( a = \xi b \), where \( \xi \) is a primitive \( j \)-th root of 1, for some integer \( j \geq 1 \).

2. **Secondary Hopf surfaces.** A secondary Hopf surface \( M \) is the quotient of \( \mathbb{C}^2 - \{0\} \) by the free action of a group \( \Gamma \) containing a central, finite index subgroup generated by an automorphism \( g \) of the above type. The primary Hopf surface \( N = \mathbb{C}^2 - \{0\}/g \) is a finite étale cover of \( M \) and the corresponding finite subgroup is generated by \( e(x, y) = (\epsilon_1 x, \epsilon_2 y) \), where \( \epsilon_1 \) and \( \epsilon_2 \) are primitive \( l \)-th roots of unity and \( (\epsilon_1 - \epsilon_2^m)\lambda = 0 \). Then we have:

| Condition                  | Riccati foliation \( \mathcal{H} \) induced by \( \theta \)                                      | Monodromy          |
|---------------------------|-------------------------------------------------------------------------------------------------|--------------------|
| \( \lambda \neq 0, m = 1 \) | \( dz \)                                                                                         | \( z \mapsto \frac{\epsilon_2}{\epsilon_1} \) |
| \( \lambda = 0, \epsilon_1 \neq \epsilon_2^2 \) | \( dz \)                                                                                         | \( z \mapsto \frac{\epsilon_2}{\epsilon_1} \) |
| \( \lambda = 0, \epsilon_1 = \epsilon_2^2 \) | \( dz - cz^2 dy \)                                                                               | \( z \mapsto \frac{\epsilon_2}{\epsilon_1} \) |

where \( c \in \mathbb{C} \).

Hopf surfaces with fundamental group isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}/l\mathbb{Z} \), where \( l \geq 1 \) is an integer, are generated by two diagonal automorphisms \( g(x, y) = (ax, ay) \) with \( 0 < |a| < 1 \), and \( e(x, y) = (\epsilon x, \epsilon y) \) where \( \epsilon \) is a primitive \( l \)-th root of 1. We denote them by \( H_{a,l} \). From these two previous lists we conclude the following.

**Lemma 3.15.** Let \( M \) be a Hopf surface and \( \mathcal{H} \) be a Riccati foliation on \( \mathbb{P}(TM) \).

1. Mon\((\mathcal{H})\) is trivial if and only if \( M = H_{a,l} \).
2. \( \mathcal{H} \) is induced by a regular \( k \)-web, \( k \geq 3 \), if and only if \( H_{a,l} \) is a finite cover of \( M \).
3. If Mon\((\mathcal{H})\) is finite, then Mon\((\mathcal{H})\) is cyclic.

### 3.4.5. Inoue surfaces

Inoue surfaces are compact complex surfaces of type \( VII_0 \), see [1] for more details. In Inoue [5] shows that these surfaces are obtained as quotients of \( \mathbb{H} \times \mathbb{C} \) by a group \( \Gamma \) of affine transformations of \( \mathbb{C}^2 \) preserving the open set \( \mathbb{H} \) (\( \mathbb{H} \) being the Poincaré upper half-plane). In particular, each Inoue surface inherits an affine structure induced by the canonical affine structure of \( \mathbb{C}^2 \), which is unique by [10, Lemma 4.3]. Up to a double unramified cover, Inoue surfaces are obtained by one of the following two procedures [5].

1. **Surfaces \( S_M \).** Consider a matrix \( M \in \text{SL}(3, \mathbb{Z}) \) with eigenvalues \( \alpha, \beta, \bar{\beta} \) such that \( \alpha > 1 \) and \( \beta \neq \bar{\beta} \). Choose a real eigenvector \((a_1, a_2, a_3)\) associated with
the eigenvalue \( \alpha \) and an eigenvector \((b_1, b_2, b_3)\) associated with the eigenvalue \( \beta \). Consider also the group \( \Gamma \) of (affines) transformations of \( \mathbb{C}^2 \) generated by:

\[
\gamma_0(x, y) = (\alpha x, \beta y), \\
\gamma_i(x, y) = (x + a_i, y + b_i), \quad \text{with } i = 1, 2, 3.
\]

The action of \( \Gamma \) on \( \mathbb{C}^2 \) preserves \( \mathbb{H} \times \mathbb{C} \) and the quotient is the compact complex surface \( S \).

In this case, there is only one Riccati foliation on \( \mathbb{P}(TS_M) \) and it has the form \( dz \). Thus, the monodromy group is generated by \( f(z) = \frac{\beta}{\alpha}z \).

2. Surfaces \( S^+_{N, p, q, r, t} \). Let \( N = (n_{ij}) \in \text{SL}(2, \mathbb{Z}) \) be a diagonalizable matrix on \( \mathbb{R} \) with eigenvalues \( \alpha > 1 \) and \( \alpha^{-1} \) and eigenvectors \((a_1, a_2)\) and \((b_1, b_2)\) respectively. Choose \( r \in \mathbb{Z}^*, \ p, q \in \mathbb{Z}, \ t \in \mathbb{C} \) and real solutions \( c_1, c_2 \) of the equation

\[
(c_1, c_2) = (c_1, c_2)N^t + (e_1, e_2) + \frac{1}{r}(b_1a_2 - b_2a_1)(p, q),
\]

where \( e_i = \frac{1}{2}n_{1i}(n_{1i} - 1)a_1b_1 + \frac{1}{2}n_{2i}(n_{2i} - 1)a_2b_2 + n_{1i}n_{2i}b_1a_2 \) and \( N^t \) denotes the transpose of \( N \).

In this case \( \Gamma \) is generated by the transformations

\[
\gamma_0(x, y) = (\alpha x, y + t), \\
\gamma_i(x, y) = (x + a_i, y + b_1x + c_i), \quad (i = 1, 2), \\
\gamma_3(x, y) = (x, y + r^{-1}(b_1a_2 - b_2a_1)).
\]

This group is discrete and acts properly and discontinuously on \( \mathbb{H} \times \mathbb{C} \) and the quotient we obtain is the compact complex surface \( S^+_{N, p, q, r, t} \).

Also in this case, there is only one Riccati foliation on \( \mathbb{P}(TS^+_{N, p, q, r, t}) \) which has the form \( dz \). Moreover, the monodromy group is generated by \( f_0(z) = \frac{\beta}{\alpha}z \) and \( f_i(z) = \frac{z}{1+b_1z}, \ i = 1, 2 \).

Thus, we conclude that there are no regular pencils \( \mathcal{P} \) nor regular \( k \)-webs \( \mathcal{W} \) on an Inoue surface, for \( k \geq 3 \).

3.4.6. Elliptic surfaces over a Riemann surface of genus \( g \geq 2 \), with odd first Betti number. The existence of a holomorphic affine structure on an elliptic surface over a Riemann surface of genus \( g \geq 2 \), of odd first Betti number, is a result due to Maehara [15]. The global geometry of affine holomorphic structures and holomorphic affine connections without torsion on these surfaces are studied in [10] and [4] respectively.

Up to a finite covering and a finite quotient, this surface \( M \) is constructed as follows. Let \( \Gamma \) be a discrete torsion-free subgroup of \( \text{PSL}(2, \mathbb{R}) \) such that \( \Sigma = \Gamma \backslash \mathbb{H} \), where \( \mathbb{H} \) denotes the Poincaré half-plane. If we think of \( \Gamma \) as a subgroup of \( \text{SL}(2, \mathbb{R}) \), then the action of an element \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \) of \( \Gamma \) on \( \mathbb{C} \times \mathbb{H} \) is given by the following formula:\( \gamma(x, y) = (x + \log(cy + d), \gamma y) \), for all \( (x, y) \in \mathbb{C} \times \mathbb{H} \).
where \( \log \) denotes a branch of the logarithm function and the action of \( \gamma \) on \( \mathbb{H} \) comes from the canonical action of \( \text{SL}(2, \mathbb{R}) \) on \( \mathbb{H} \). See [10] for more details.

The quotient of \( \mathbb{C} \times \mathbb{H} \) by the action of \( \Gamma \) is the complex surface compact \( M \).

A torsion-free flat connection on \( M \) has the form

\[
\theta = \begin{pmatrix} dx & 0 \\ dy & dx \end{pmatrix} + \begin{pmatrix} 0 & f(y)dy \\ 0 & h(y)dy \end{pmatrix},
\]

where \( h(\gamma y) = h(y)(cy+d)^2 \) and \( f(\gamma y) = f(y)(cy+d)^4 + ch(y)(cy+d)^3 \), for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \). Therefore the Riccati foliations on \( \mathbb{P}(TM) \) are of the form

\[
\omega = dz + dy + h(y)dyz - f(y)dyz^2.
\]

(13)

In this case we did not succeed in calculating the monodromy but we prove the following lemma.

**Lemma 3.16.** There are no regular pencils \( \mathcal{P} \) on \( M \). In particular, there are no regular \( k \)-webs \( \mathcal{W} \) on \( M \), for \( k \geq 3 \).

**Proof.** Suppose that there exists a regular pencil \( \mathcal{P} = \{ F_t \}_{t \in \mathbb{P}^1} \) on \( M \). By [18, Lemma 4.3] we may assume that \( \mathcal{F}_\infty \) is the foliation tangent to the elliptic fibration on \( M \). Let \( \pi : \mathbb{C} \times \mathbb{H} \to M \) be the universal covering and \( \mathcal{P} = \pi^*\mathcal{P} = \{ \mathcal{F}_t \}_{t \in \mathbb{P}^1} \) be the pencil on \( \mathbb{C} \times \mathbb{H} \). Since \( \mathcal{F}_0 \) and \( \mathcal{F}_\infty \) are transversal we have \( \mathcal{F}_0 = \{ dx + B(x, y)dy = 0 \} \) and \( \mathcal{F}_\infty = \{ dy = 0 \} \), \( B \in \mathcal{O}(\mathbb{C} \times \mathbb{H}) \). So, \( \mathcal{P} \) is induced by \( \omega_t = dx + B(x, y)dy + tA(x, y)dy \), for some \( A \in \mathcal{O}^*(\mathbb{C} \times \mathbb{H}) \). Thus the Riccati foliation is induced by \( \omega = dz + dy + h(y)dyz - f(y)dyz^2 \), which is in contradiction with the equation (13). \( \square \)

4. Applications

The following theorem was proved in [18, Theorem 4.6] by using foliation techniques.

**Theorem 4.1.** If \( M \) is compact and \( \mathcal{P} \) is a regular pencil on \( M \), then \( M \) is either a complex torus \( \mathbb{C}^2/\Gamma \) or a Hopf surface \( M = H_{a,1} \).

Moreover, \( \mathcal{P} \) is the only regular pencil on \( M \), and it is generated by \( \{ dx + tdy \}_{t \in \mathbb{P}^1} \) on \( \mathbb{C}^2 \) (in the first case) or on \( \mathbb{C}^2 \setminus \{(0,0)\} \) (in the second case).

**Proof.** Let \( \mathcal{H} \) be the Riccati foliation on \( \mathbb{P}(TM) \) induced by \( \mathcal{P} \). By Proposition 3.8 we have that \( \text{Mon}(\mathcal{H}) \) is trivial. Using the calculations of the monodromy from Sect. 3.4, see Lemmas 3.12 and 3.15, we conclude the proof. \( \square \)

**Theorem 4.2.** If \( M \) is compact and \( \mathcal{H} \) is a Riccati foliation on \( \mathbb{P}(TM) \) with finite monodromy, then \( M \) is either a complex torus \( \mathbb{C}^2/\Gamma \), a hyperelliptic surface, or a Hopf surface.

Moreover, in the first case \( \text{Mon}(\mathcal{H}) \) is trivial, while in the last two cases it is cyclic.
Proof. This follows readily from the procedure we employed when computing monodromy in the previous subsection, see Lemma 3.12, Lemma 3.13 and Lemma 3.15.

Theorem 4.3. If $M$ is compact and $\mathcal{W}$ is a regular $k$-web on $M$, $k \geq 3$, then $M$ is either a complex torus, a hyperelliptic surface, or, up to a finite unramified cover, the Hopf surface $H_{a,1}$.

Moreover, $\mathcal{W} \subset \mathcal{P}$ up to a finite unramified cover, where $\mathcal{P}$ is the only pencil on $M$.

Proof. By Proposition 3.11, there is a Riccati foliation $\mathcal{H}$ on $\mathbb{P}(TM)$ induced by $\mathcal{W}$ with finite monodromy. So we can apply Theorem 4.2. Now up to a finite unramified cover we can suppose that the monodromy is trivial and $\mathcal{W} = \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_k$ is a (completely decomposable) regular $k$-web on $M$. To complete the proof we apply Theorem 4.1 and proceed as we did when computing monodromy in Sect. 3.4.

Finally, we point out that the classification of regular 2-webs follows directly from [2, Theorem C].

Acknowledgements The author wishes to express his deepest gratitude to Frank Loray for lots of fruitful discussions about the content of this paper, and also for reading the drafts and suggesting improvements. The author also wishes to thank Cesar Hilario for the suggestions and comments on the manuscript. The author acknowledge financial support from CAPES/COFECUB (Ma 932/19 “Feuilletages holomorphes et intégration avec la géométrie” / process number 88887.356980/2019-00). The author is grateful to the Institut de Recherche en Mathématique de Rennes, IRMAR and the Université de Rennes 1 for their hospitality and support. The author is supported by FAPERJ (Grant number E-26/010.001143/2019).

Declarations

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces. Springer, Berlin (1984)
[2] Beauville, A.: Complex Manifolds with Split Tangent Bundle, Vol. en mémoire de M.Schneider, de Gruyter, à paraître
[3] Bombieri, E., Mumford, D.: Enriques’ Classification of Surfaces in Char, p, II. In: Complex Analysis and Algebraic Geometry, Iwanami Shoten, pp. 23–42 (1977)
[4] Dumitrescu, S.: Connexions affines et projectives sur les surfaces complexes compacts. Math. Zeit. 264, 301–316 (2010)
[5] Inoue, M.: On surfaces of class $VII_0$. Invent. Math. 24, 269–310 (1974)
[6] Inoue, M., Kobayashi, S., Ochiai, T.: Holomorphic affine connections on compact complex surfaces. J. Fac. Sci. Univ. Tokyo 27, 888 (1980)
[7] Falla Luza, M., Loray, F.: Projective structures and neighborhood of rational curves. https://hal.archives-ouvertes.fr/hal-01567347v2
[8] Gunning, R.: Lectures on Riemann Surfaces. Princeton Mathematical Notes, Princeton (1966)
[9] Kato, M.: On characteristic forms on complex manifolds. J. Algebra **138**, 424–439 (1991)
[10] Klingler, B.: Structures affines et projectives sur les surfaces complexes. Ann. Inst. Fourier Grenoble **48**(2), 441–477 (1998)
[11] Kobayashi, S., Ochiai, T.: Holomorphic projective structures on compact complex surfaces. Math. Ann. **249**, 75–94 (1980)
[12] Kobayashi, S., Ochiai, T.: Holomorphic projective structures on compact complex surfaces. II. Math. Ann. **255**, 519–521 (1981)
[13] Kodaira, K.: On the structure of compact complex analytic surfaces I, II, III. Am. J. Math. **86**, 751–798 (1964)
[14] Lins Neto, A.: Curvature of pencil of foliations. Astérisque **296**, 167–190 (2004)
[15] Maehara, K.: On elliptic surfaces whose first Betti numbers are odd. Intl. Symp. Alg. Geom. Kyoto **2**, 565–574 (1977)
[16] Molzon, R., Pinney, M.K.: The Schwarzian derivate for maps between manifolds with complex projective connections. In: Transactions of the American Mathematical Society, vol. 348, no. 8 (1996)
[17] Pereira, J.V., Pirio, L.: An invitation to web geometry. Publicações Matemáticas do IMPA. Instituto Nacional de Matemática Pura e Aplicada (IMPA), Rio de Janeiro (2009)
[18] Puchuri, L.: Classification of flat pencils of foliations on compact complex surfaces. Annales de l’Institut Fourier. Tome **70**(5), 2191–2214 (2020)
[19] Robert, G.: Poincaré maps and Bol’s Theorem. http://park.ite.u-tokyo.ac.jp/MSF/topology/conference/GHC/data/Robert.pdf
[20] Suwa, T.: Compact quotient spaces of $\mathbb{C}^2$ by affine transformation groups. J. Differ. Geom. **10**, 239–252 (1975)
[21] Vitter, A.: Affine structures on compact complex manifolds. Invent. Math. **17**, 231–244 (1972)
[22] Zhao, S.Y.: Groupes kleiniens birationnels en dimension deux. L’Université de Rennes 1 Thesis 2020. https://perso.univ-rennes1.fr/shengyuan.zhao/index.html/paper/main.pdf

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