Spectrum of the Dirichlet Laplacian in waveguides with parallel cross-sections

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Abstract

Let $\Omega \subset \mathbb{R}^3$ be a waveguide which is obtained by translating a cross-section in a constant direction along an unbounded spatial curve. Consider $-\Delta^D_\Omega$ the Dirichlet Laplacian operator in $\Omega$. Under the condition that the tangent vector of the reference curve admits a finite limit at infinity, we find the essential spectrum of $-\Delta^D_\Omega$. Then, we state sufficient conditions that give rise to a non-empty discrete spectrum for $-\Delta^D_\Omega$; in particular, we show that the number of discrete eigenvalues can be arbitrarily large since the waveguide is thin enough.

1 Introduction

Let $\Omega$ be an unbounded quantum waveguide in $\mathbb{R}^n$, $n = 2, 3$, and denote by $-\Delta^D_\Omega$ the Dirichlet Laplacian operator in $\Omega$. The spectrum of this operator has been extensively studied in the last years [1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 17, 19, 20, 21, 23, 24, 25, 26, 27]. In fact, the subject is non-trivial since the results depend on the geometry of $\Omega$ [4, 8, 12, 17, 23, 24]. In the particular case where $\Omega$ is a straight waveguide, it is known that its spectrum is purely absolutely continuous and there are no discrete eigenvalues. However, several works show that the situation changes according to the geometry of $\Omega$; for example, curved waveguides or those with deformation on its boundary can give rise to a non-empty discrete spectrum for $-\Delta^D_\Omega$ [16, 17, 20, 23]. In the next paragraphs, we recall some papers and results in the literature; after that, we present the goal of this work.

Let $\Gamma : \mathbb{R} \to \mathbb{R}^2$ be a $C^\infty$ plane curve parameterized by its arc-length $x$ and denote by $k(x)$ its curvature at the point $\Gamma(x)$. Consider a strip $\Omega$ constructed by moving a bounded segment $(a, b) \subset \mathbb{R}$ along $\Gamma$ with respect to its normal vector field; $\Omega$ is delimited by two parallel curves. In the pioneering paper [17], the authors studied the spectral problem of $-\Delta^D_\Omega$. In particular, they proved the existence of discrete spectrum for the operator under the conditions that $k(x) \neq 0$, for some $x \in \mathbb{R}$, and that $k(x)$ decays fast enough at infinity; in addition, the authors had assumed some regularity for $\Gamma$ which were relaxed latter [20, 26, 29].

Now, denote by $\Gamma : \mathbb{R} \to \mathbb{R}^3$ a $C^3$ spatial curve parameterized by its arc-length $x$ which possesses an appropriate Frenet frame; $k(x)$ and $\tau(x)$ denote its curvature and torsion at the point $\Gamma(x)$, respectively. Let $\omega$ be a bounded open connected set in $\mathbb{R}^2$. Consider the case where $\Omega$ is an unbounded waveguide obtained by moving the cross-section $\omega$ along the curve $\Gamma$ according to the Frenet referential. At each point of $\Gamma$ the region $\omega$ also may present a (continuously differentiable) rotation angle $\alpha(x)$. In this situation, several aspects of the spectral problem for $-\Delta^D_\Omega$ were studied in [3, 6, 8, 9, 12, 13, 15, 16, 17, 20, 23]. However, we emphasize the paper [23] where the authors presented, together with new proofs and results, a review of the geometric effects of $\Omega$ on $\sigma(-\Delta^D_\Omega)$. There was discussed how
the spectrum depends on two independent geometric deformations: bending \((k \neq 0)\) and twisting \((w\) is not rotationally invariant with respect to the origin and \(\tau - \alpha' \neq 0\). In particular, the assumptions \(\tau + \alpha' = 0, k \neq 0\) and \(k(x) \to 0, |x| \to \infty\), imply \(\sigma_{\text{dis}}(-\Delta^D_{\Omega}) \neq \emptyset\). Some types of deformation in straight waveguides can also give rise to a non-empty discrete spectrum for \(-\Delta^D_{\Omega}\). For example, let \(\zeta\) be a bounded function such that \(\sup \zeta \subset [-x_0, x_0]\), for some \(x_0 > 0\). Consider the case where \(w\) is a non-circular cross-section and the angle rotation governed by the function \(\alpha(x)\) satisfies \(\alpha'(x) = \alpha_0 - \zeta(x)\), \(\alpha_0 \in \mathbb{R}\). This situation was considered in \([5]\). One of the results of the authors is that the assumption \(\int_{-x_0}^{x_0} (\alpha'(x)^2 - \alpha_0^2)dx < 0\) ensures the existence of discrete eigenvalues for \(-\Delta^D_{\Omega}\).

In this paper we consider the Dirichlet Laplacian operator restricted to a three-dimensional waveguide whose geometry is inspired by a two-dimensional model. Namely, in the recent paper \([4]\) the authors introduced a new model of strip to study the spectral problem of \(-\Delta^D_{\Omega}\). In that work, \(\Omega\) is a strip in \(\mathbb{R}^2\) which is built by translating a segment oriented in a constant direction along an unbounded curve in the plane. More precisely, let \(h : \mathbb{R} \to \mathbb{R}\) be a locally Lipschitz continuous function. Take a positive number \(d > 0\) and define \(\Omega = \{(x,y) \in \mathbb{R}^2 : h(x) < y < h(x) + d\}\). Assume that \(h(x)\) is differentiable almost everywhere and the derivative \(h'(x)\) admits a limit at infinity: \(h'(x) \to \beta \in \mathbb{R} \cup \{\infty\}, |x| \to \infty\).

The spectrum of the operator \(-\Delta^D_{\Omega}\) was carefully studied and the model covers different effects: purely essential spectrum, discrete spectrum or a combination of both. In particular, if \(h \in L_{loc}^\infty(\mathbb{R})\), then \(\sigma_{\text{ess}}(-\Delta^D_{\Omega}) = \{(1 + \beta^2)\pi^2/d^2, \infty\}\). In addition, if \(h'(x)^2 - \beta^2 \in L^1(\mathbb{R})\) and \(\int_{\mathbb{R}} (h'(x)^2 - \beta^2)dx < 0\), then \(\inf \sigma(-\Delta^D_{\Omega}) < (1 + \beta^2)\pi^2/d^2\), i.e., \(\sigma_{\text{dis}}(-\Delta^D_{\Omega}) \neq \emptyset\). In the next paragraphs we present the formal construction of the waveguide which will be considered in this work and we give more details of the problem.

Denote by \(\{e_1, e_2, e_3\}\) the canonical basis of \(\mathbb{R}^3\). Pick \(S \neq \emptyset\); an open, bounded and connected subset of \(\mathbb{R}^2\) with \(C^2\)-boundary. Let \(f, g : \mathbb{R} \to \mathbb{R}\) be differentiable functions so that \(f', g' \in L^\infty(\mathbb{R})\), and \(r : \mathbb{R} \to \mathbb{R}^3\) the spatial curve given by

\[
 r(x) = (x, f(x), g(x)), \quad x \in \mathbb{R}. \tag{1}
\]

Define the mapping

\[
 \mathcal{L} : \mathbb{R} \times S \quad \to \mathbb{R}^3
\]

\[
 (x, y_1, y_2) \quad \to \quad r(x) + y_1 e_2 + y_2 e_3. \tag{2}
\]

We define the waveguide

\[
 \Omega := \mathcal{L}(\mathbb{R} \times S).
\]

Roughly speaking, \(\Omega\) is obtained by translating the region \(S\) along the curve \(r(x)\) so that, at each point of it, \(S\) is parallel to the plane generated by \(\{e_2, e_3\}\).

Let \(-\Delta^D_{\Omega}\) be the Dirichlet Laplacian operator in \(\Omega\). In other words, \(-\Delta^D_{\Omega}\) is the self-adjoint operator associated with the quadratic form

\[
 a(\varphi) := \int_{\Omega} |
\nabla \varphi|^2 ds, \quad \text{dom} \ a = H^1_0(\Omega). \tag{3}
\]

The goal of this paper is to study the spectral problem of \(-\Delta^D_{\Omega}\). Inspired by \([4]\), we assume that

\[
 \lim_{|x| \to \infty} f'(x) =: \beta_1, \quad \lim_{|x| \to \infty} g'(x) =: \beta_2, \quad \beta_1, \beta_2 \in \mathbb{R} \tag{4}
\]

We find the essential spectrum of \(-\Delta^D_{\Omega}\) and we discuss conditions that ensure the existence of discrete spectrum for this operator.
Denote by \( y := (y_1, y_2) \) a point of \( S \), \( \partial_{y_1} := \partial / \partial y_1 \), \( \partial_{y_2} := \partial / \partial y_2 \). Consider the two-dimensional operator

\[
H_{\beta_1, \beta_2}(0) := - (\beta_1 \partial_{y_1} + \beta_2 \partial_{y_2})^2 - \partial_{y_1}^2 - \partial_{y_2}^2,
\]

\( \text{dom} \ H_{\beta_1, \beta_2}(0) = H^2(S) \cap H_0^1(S) \). Denote by \( E_1(0) \) the first eigenvalue of \( H_{\beta_1, \beta_2}(0) \) and by \( v_1 \) the corresponding eigenfunction. Since \( H_{\beta_1, \beta_2}(0) \) is an elliptic operator with real coefficients, \( E_1(0) \) is simple and \( v_1 \) can be chosen to be real and positive in \( S \); see, e.g., Chapter 6 of [14]. Our first result states

**Theorem 1.** Suppose that the conditions in (4) hold. Then,

\[
\sigma_{\text{ess}}(-\Delta_\Omega^D) = [E_1(0), \infty).
\]

The proof of Theorem 1 is presented in Section 3.

The next step is to analyze the existence of discrete eigenvalues for \(-\Delta_\Omega^D\). Define the constants

\[
A := \int_S \left| \frac{\partial v_1}{\partial y_1} \right|^2 \, dy, \quad B := \int_S \frac{\partial v_1}{\partial y_1} \frac{\partial v_2}{\partial y_2} \, dy, \quad C := \int_S \left| \frac{\partial v_2}{\partial y_1} \right|^2 \, dy,
\]

and the function

\[
V(x) := A(f'(x)^2 - \beta_1^2) + 2B f'(x) g'(x) + C(g'(x)^2 - \beta_2^2), \quad x \in \mathbb{R}.
\]

**Theorem 2.** Suppose that the conditions in (4) hold and \( V(x) \in L^1(\mathbb{R}) \). If \( \int_\mathbb{R} V(x) \, dx < 0 \), then

\[
\inf \sigma(-\Delta_\Omega^D) < E_1(0),
\]

i.e., \( \sigma_{\text{dis}}(-\Delta_\Omega^D) \neq \emptyset \).

The proof of this result is presented in Section 4. The condition \( \int_\mathbb{R} V(x) \, dx < 0 \) implies the existence of discrete eigenvalues for \(-\Delta_\Omega^D\). However, it is not a necessary condition for this to happen. For example,

**Theorem 3.** Suppose that the conditions in (4) hold with \( g' = 0 \). If \( f'(x)^2 - \beta_1^2 \in L^1(\mathbb{R}) \), \( f''(x) \in L^1_{\text{loc}}(\mathbb{R}) \), \( \int_\mathbb{R} (f'(x)^2 - \beta_1^2) \, dx = 0 \) and \( f' \) is not constant, then

\[
\inf \sigma(-\Delta_\Omega^D) < E_1(0),
\]

i.e., \( \sigma_{\text{dis}}(-\Delta_\Omega^D) \neq \emptyset \).

The proof of this result is also presented in Section 4. Note that the condition \( g' = 0 \) implies that the reference curve (1) belongs to a plane. A similar result can be found if \( f' = 0 \), \( g'(x)^2 - \beta_2^2 \in L^1(\mathbb{R}) \), \( g''(x) \in L^1_{\text{loc}}(\mathbb{R}) \), \( \int_\mathbb{R} (g'(x)^2 - \beta_2^2) \, dx = 0 \) and \( g' \) is not constant.

Now, we are going to show that it is possible to find additional information about \( \sigma_{\text{dis}}(-\Delta_\Omega^D) \) provided that \( \Omega \) is thin enough. For that, we add a small parameter multiplying the points of the cross-section \( S \). More precisely, for \( \varepsilon > 0 \) small enough, we consider the mapping

\[
\mathcal{L}_\varepsilon : \mathbb{R} \times S \to \mathbb{R}^3 \quad (x, y_1, y_2) \mapsto r(x) + \varepsilon y_1 e_2 + \varepsilon y_2 e_3,
\]

and we define the thin waveguide

\[
\Omega_\varepsilon := \mathcal{L}_\varepsilon(\mathbb{R} \times S).
\]
Let $-\Delta^0_{\Omega_\varepsilon}$ be the Dirichlet Laplacian operator in $\Omega_\varepsilon$, i.e., the self-adjoint operator associated with

$$a_\varepsilon(\varphi) := \int_{\Omega_\varepsilon} |\nabla \varphi|^2 \, ds, \quad \text{dom } a_\varepsilon = H^1_0(\Omega_\varepsilon). \quad (7)$$

For simplicity, we denote $-\Delta^D := -\Delta^0_{\Omega_\varepsilon}$. In this case, we have

**Theorem 4.** Suppose that the conditions in [4] hold and $V(x) \in L^1(\mathbb{R})$. If $V(x)$ assumes a negative value, then

$$\sigma_{\text{dis}}(-\Delta^D) \neq \emptyset,$$

for all $\varepsilon > 0$ small enough. Furthermore, given $n \in \mathbb{N}$, there exists $\varepsilon_n > 0$ so that the spectrum of $-\Delta^D_{\varepsilon_n}$ contains at least $n$ discrete eigenvalues, counting multiplicity.

In particular, Theorem 4 ensures that the number of discrete eigenvalues can be arbitrarily large since the waveguide is thin enough. Its proof is presented in Section 4. In this introduction, we give an alternative proof for the first statement of the theorem.

At first, we remember some spectral properties of a self-adjoint operator. Let $T$ be a self-adjoint operator that is bounded from below and denote by $t(\psi)$ its associated quadratic form. By Min-Max Principle,

$$\lambda_1(T) := \inf\{ t(\psi)/\|\psi\|^2 : 0 \neq \psi \in \text{dom } t \} \quad (8)$$

is either a discrete eigenvalue or the bottom of the essential spectrum of $T$. In the conditions of Theorem 4 one has $\sigma_{\text{ess}}(-\Delta^D) = [E_1(0)/\varepsilon^2, \infty)$. Thus, the strategy is to study the quantity $\lambda_1(-\Delta^D)$ in order to discuss the existence of discrete eigenvalues for $-\Delta^D$ provided that $\varepsilon > 0$ is small enough. This will be done using some estimates for the quadratic form $a_\varepsilon(\varphi)$.

Let $\Lambda := \mathbb{R} \times S$. According to the change of coordinates described in Section 2, $a_\varepsilon(\varphi)$ can be identified with

$$b_\varepsilon(\psi) := \int_{\Lambda} \left( \left| \psi' - \frac{f'(x)}{\varepsilon} \frac{\partial \psi}{\partial y_1} - \frac{g'(x)}{\varepsilon} \frac{\partial \psi}{\partial y_2} \right|^2 + \frac{|\nabla_y \psi|^2}{\varepsilon^2} \right) \, dx \, dy, \quad (9)$$

$\text{dom } b_\varepsilon = H^1_0(\Lambda)$. Recall $y = (y_1, y_2)$ that denotes a point of $S$. Furthermore, $\psi' := \partial \psi / \partial x$, and $\nabla_y \psi := (\partial_{y_1} \psi, \partial_{y_2} \psi)$.

Consider the closed subspace $\mathcal{W} := \{ w v_1 : w \in L^2(\mathbb{R}) \}$ of the Hilbert space $L^2(\Lambda)$. Define the one-dimensional quadratic form

$$s_\varepsilon(w) := b_\varepsilon(w v_1) - \frac{E_1(0)}{\varepsilon^2} \|w v_1\|^2$$

$$= \int_{\mathbb{R}} \left( |w'|^2 + \left( A \frac{f'(x)^2 - \beta_1^2}{\varepsilon^2} |w|^2 + 2B \frac{f'(x)g'(x)}{\varepsilon^2} + C \frac{g'(x)^2 - \beta_2^2}{\varepsilon^2} \right) |w|^2 \right) \, dx,$$

$$= \int_{\mathbb{R}} \left( |w'|^2 + \frac{V(x)}{\varepsilon^2} |w|^2 \right) \, dx,$$

$\text{dom } s_\varepsilon := H^1(\mathbb{R})$. Note that $s_\varepsilon(w)$ is the quadratic form $b_\varepsilon(\psi) - (E_1(0)/\varepsilon^2) \|\psi\|^2$ restricted to the subspace $H^1_0(\Lambda) \cap \mathcal{W}$ which is identified with $H^1(\mathbb{R})$. The self-adjoint operator associated with $s_\varepsilon(w)$ is

$$w \mapsto \left( -\Delta + \frac{V(x)}{\varepsilon^2} \mathbf{1} \right) w, \quad w \in H^2(\mathbb{R}),$$

where $\mathbf{1}$ is the identity operator.
where \(-\Delta_\mathbb{R}\) denotes the one-dimensional Laplacian operator on \(\mathbb{R}\) and \(\mathbf{1}\) is the identity operator on \(L^2(\mathbb{R})\).

The next step is to compare the values \(\lambda_1(-\Delta_\varepsilon^D)\) and \(\lambda_1(-\Delta_\mathbb{R} + (V(x)/\varepsilon^2)\mathbf{1})\). Since \(W \subset H^1_0(\Lambda)\), by (8), one has

\[
\lambda_1(-\Delta_\varepsilon^D) - \frac{E_1(0)}{\varepsilon^2} \leq \lambda_1 \left( -\Delta_\mathbb{R} + \frac{V(x)}{\varepsilon^2} \mathbf{1} \right).
\]  

(10)

However, by Theorem 5 of Appendix B,

\[
\lambda_1 \left( -\Delta_\mathbb{R} + \frac{V(x)}{\varepsilon^2} \mathbf{1} \right) = \frac{1}{\varepsilon^2} \inf_{x \in \mathbb{R}} \{V(x)\} + o(\varepsilon^{-2}),
\]

(11)

for all \(\varepsilon > 0\) small enough. As a consequence of (10) and (11), if \(V(x)\) assumes a negative value, then 

\[
\lambda_1(-\Delta_\varepsilon^D) < \frac{E_1(0)}{\varepsilon^2},
\]

\(\sigma_{\text{dis}}(-\Delta_\varepsilon^D) \neq \emptyset\), for all \(\varepsilon > 0\) small enough.

This work is organized as follows. In Section 2 we describe the usual changes of coordinates to work in the Hilbert space \(L^2(\Lambda)\) with the usual metric. In Section 3 we study the essential spectrum of \(-\Delta_\varepsilon^D\) as well as the proof of Theorem 1. Section 4 is dedicated to study the discrete spectrum of \(-\Delta_\varepsilon^D\). In that section, we present the proofs of Theorems 2, 3 and 4. In Appendices A and B we present useful results used in this work.

2 Change of coordinates

Recall the mapping \(\mathcal{L}\) and the quadratic form \(a(\varphi)\) given by (2) and (3), respectively, in the Introduction. Several arguments to prove the results of this work are based on the study of the quadratic form \(a(\varphi)\) which acts in \(L^2(\Omega)\); again, recall that \(\Omega = \mathcal{L}(\Lambda)\) where \(\Lambda = \mathbb{R} \times S\). Then, in this section we perform a change of coordinates such that \(a(\varphi)\) starts to act in the Hilbert space \(L^2(\Lambda)\) instead of \(L^2(\Omega)\).

At first, note that \(\Omega\) can be identified with the Riemannian manifold \((\Lambda, G)\), where \(G = (G_{ij})\) is the metric induced by \(\mathcal{L}\), i.e.,

\[
G_{ij} = \langle \mathcal{G}_i, \mathcal{G}_j \rangle = \mathcal{G}_{ji}, \quad i,j = 1,2,
\]

where

\[
\mathcal{G}_1 = \frac{\partial \mathcal{L}}{\partial x}, \quad \mathcal{G}_2 = \frac{\partial \mathcal{L}}{\partial y_1}, \quad \mathcal{G}_3 = \frac{\partial \mathcal{L}}{\partial y_2}.
\]

More precisely,

\[
G = \nabla \mathcal{L} \cdot (\nabla \mathcal{L})^t = \begin{pmatrix} 1 + (f'(x))^2 + (g'(x))^2 & f'(x) & g'(x) \\ f'(x) & 1 & 0 \\ g'(x) & 0 & 1 \end{pmatrix}, \quad \det G = 1.
\]

Since \(\Omega\) is homeomorphic to the straight waveguide \(\Lambda\), we consider the unitary operator

\[
\mathcal{U} : L^2(\Omega) \rightarrow L^2(\Lambda) \quad \psi \mapsto \psi \circ \mathcal{L},
\]

(12)

and, we define

\[
b_{f',g'}(\psi) := a(\mathcal{U}^{-1}\psi) = \int_{\Lambda} \langle \nabla \psi, G^{-1}\nabla \psi \rangle \sqrt{\det G} \, dx dy
\]

\[
= \int_{\Lambda} \left( |\psi' - f'(x) \frac{\partial \psi}{\partial y_1} - g'(x) \frac{\partial \psi}{\partial y_2}|^2 + |\nabla_y \psi|^2 \right) dx dy,
\]

(13)
dom \( b_{f',g'} := \mathcal{U}(H^1_0(\Omega)) = H^1_0(\Lambda) \); in fact, \( f', g' \in L^\infty(\mathbb{R}) \). Recall that \( y = (y_1, y_2) \) denotes a point of \( S \), \( \psi' = \partial\psi/\partial x \), and \( \nabla_y \psi = (\partial_{y_1} \psi, \partial_{y_2} \psi) \). Denote by \( H_{f',g'} \) the self-adjoint operator associated with the quadratic form \( b_{f',g'}(\psi) \).

**Remark 1.** We can perform a similar change of coordinates for the quadratic form \( a_0(\varphi) \) given \([7]\) in the Introduction. Recall the mapping \( L_\varepsilon \) and the quadratic form \( b_\varepsilon(\psi) \) given by \([5]\) and \([6]\), respectively. In this case, consider the unitary transformation \( \mathcal{U}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Lambda), \mathcal{U}_\varepsilon \psi := \varepsilon^2 \psi \circ L_\varepsilon \). Then, some calculations show that \( b_\varepsilon(\psi) = a_\varepsilon(\mathcal{U}_\varepsilon^{-1} \psi), \) dom \( b_\varepsilon = \mathcal{U}_\varepsilon(H^1_0(\Omega_\varepsilon)) = H^1_0(\Lambda) \). Furthermore, if we compare this quadratic form with that in \([13]\), we can note that \( b_\varepsilon = b_{f'/\varepsilon,g'/\varepsilon} \).

## 3 Essential spectrum

This section is dedicated to the proof of Theorem \( \square \). We start with the following result.

**Proposition 1.** Suppose that the conditions in \([4]\) hold. Then, \( \sigma_{\text{ess}}(H_{f',g'}) = \sigma_{\text{ess}}(H_{\beta_1,\beta_2}) \).

The proof of this proposition is discussed in Appendix \( \square \). As a consequence, we are going to study the essential spectrum of the operator \( H_{\beta_1,\beta_2} \) instead of \( H_{f',g'} \).

Let \( \mathcal{F}_x : L^2(\Lambda) \to L^2(\Lambda) \) be the partial Fourier transform in the longitudinal variable \( x \). \( \mathcal{F}_x \) is a unitary operator and, for functions \( \psi \in L^1(\Lambda) \), the explicit expression for this transformation is given by

\[
(\mathcal{F}_x \psi)(p, y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ipx} \psi(x, y) dx.
\]

We consider the operator \( \hat{H}_{\beta_1,\beta_2} := \mathcal{F}_x H_{\beta_1,\beta_2} \mathcal{F}_x^{-1} \) which admits a direct integral decomposition

\[
\hat{H}_{\beta_1,\beta_2} = \int_{\mathbb{R}} \hat{H}_{\beta_1,\beta_2}(p) \, dp,
\]

where, for each \( p \in \mathbb{R} \), \( H_{\beta_1,\beta_2}(p) \) is the self-adjoint operator associated with the quadratic form

\[
h_{\beta_1,\beta_2}(p)(\psi) = \int_S \left( |ip\psi - \beta_1 \partial_{y_1} \psi - \beta_2 \partial_{y_2} \psi|^2 + |\nabla_y \psi|^2 \right) dy, \quad \text{dom } h_{\beta_1,\beta_2}(p) = H^1_0(S).
\]

Since \( S \) is a bounded domain with \( C^2 \)-boundary, one has

\[
H_{\beta_1,\beta_2}(p) = -(ip - \beta_1 \partial_{y_1} - \beta_2 \partial_{y_2})^2 - \partial_{y_1}^2 - \partial_{y_2}^2, \quad \text{dom } H_{\beta_1,\beta_2}(p) = H^2(S) \cap H^1_0(S);
\]

if \( p = 0 \), we obtain the operator given by \([5]\) in the Introduction.

By the compactness of the embedding \( H^1_0(S) \hookrightarrow L^2(S) \), each \( H_{\beta_1,\beta_2}(p) \) has purely discrete spectrum. Denote by \( \{E_n(p)\}_{n \in \mathbb{N}} \) the sequence of the eigenvalues of \( H_{\beta_1,\beta_2}(p) \) and by \( \{v_n(p)\}_{n \in \mathbb{N}} \) the sequence of the corresponding normalized eigenfunctions, i.e.,

\[
H_{\beta_1,\beta_2}(p)v_n(p) = E_n(p)v_n(p), \quad n \in \mathbb{N}, \quad p \in \mathbb{R}.
\]

As already done in the Introduction, for simplicity, we denote \( v_1 := v_1(0) \).

Finally,

\[
\sigma(H_{\beta_1,\beta_2}) = \cup_{p \in \mathbb{R}} \sigma(H_{\beta_1,\beta_2}(p)) = \cup_{n \in \mathbb{N}} \{E_n(p) : p \in \mathbb{R}\}. \tag{14}
\]

**Lemma 1.** Each \( E_n(\cdot) \), \( n \in \mathbb{N} \), is a real-analytic function of \( p \) and

\[
\lim_{p \to \pm \infty} E_n(p) = \infty. \tag{15}
\]
Thus, we obtain the pointwise limit $H_{\beta_1, \beta_2}(p)$ coincides with $H_{\beta_1, \beta_2}(0)$, and we write
$$H_{\beta_1, \beta_2}(p) = H_{\beta_1, \beta_2}(0) + p^2 + 2ip(\beta_1 \partial y_1 + \beta_2 \partial y_2).$$

Now, take $z \in \mathbb{C}$ with $\text{img } z \neq 0$, and denote $R_z := (H_{\beta_1, \beta_2}(0) - z1)^{-1}$. We have the estimate
$$\|2ip(\beta_1 \partial y_1 + \beta_2 \partial y_2)\|^2 \leq 4p^2 \langle \psi, H_{\beta_1, \beta_2}(0)\psi \rangle$$
$$\leq 4p^2 \langle R_z(H_{\beta_1, \beta_2}(0) - z1)\psi, H_{\beta_1, \beta_2}(0)\psi \rangle$$
$$\leq 4p^2 \langle R_zH_{\beta_1, \beta_2}(0)\psi, H_{\beta_1, \beta_2}(0)\psi \rangle + |z|\|\psi, R_zH_{\beta_1, \beta_2}(0)\psi\|$$
$$\leq 4p^2 \|R_zH_{\beta_1, \beta_2}(0)\psi\| \|H_{\beta_1, \beta_2}(0)\psi\| + |z|\|\psi, (1 + R_z)\psi\|$$
$$\leq 4p^2 \left(\|R_z\|\|H_{\beta_1, \beta_2}(0)\psi\| + (|z| + |z|^2\|R_z\|) \|\psi\|^2\right),$$
for all $\psi \in \text{dom } H_{\beta_1, \beta_2}(0)$ and all $p \in \mathbb{R}$. Since $\|R_z\| \to 0$, as $\text{img } z \to \infty$, the operator $2ip(\beta_1 \partial y_1 + \beta_2 \partial y_2)$ is $H_{\beta_1, \beta_2}(0)$-bounded with zero relative bound. Consequently, since $p^2$ is clearly analytic, $\{H_{\beta_1, \beta_2}(p) : p \in \mathbb{R}\}$ is a type A analytic family. As a consequence, Theorem 3.9 of [22] implies that all the $E_n(\cdot)$ are real-analytic functions of $p$.

Now, fix $\delta > 0$. For each $\psi \in \text{dom } h_{\beta_1, \beta_2}(p)$, one has
$$(1 + \delta)|\beta_1 \partial y_1 \psi + \beta_2 \partial y_2 \psi|^2 + \frac{p^2}{1 + \delta}|\psi|^2 \geq 2 \text{Re}(i\psi(\beta_1 \partial y_1 \overline{\psi} + \beta_2 \partial y_2 \overline{\psi})).$$

Consequently,
$$h_{\beta_1, \beta_2}(p)(\psi) = \int_S \left(p^2|\psi|^2 - 2 \text{Re}(i\psi(\beta_1 \partial y_1 \overline{\psi} + \beta_2 \partial y_2 \overline{\psi})) + |\beta_1 \partial y_1 \psi + \beta_2 \partial y_2 \psi|^2\right) dy$$
$$\geq \int_S \left(p^2 + \frac{\delta}{1 + \delta}|\psi|^2 - |\beta_1 \partial y_1 \psi + \beta_2 \partial y_2 \psi|^2\right) dy.$$

Thus, we obtain the pointwise limit $h_{\beta_1, \beta_2}(p)(\psi) \to \infty$, as $p \to \pm \infty$. In particular, for each $n \in \mathbb{N}, h_{\beta_1, \beta_2}(p)(v_n(p)) = E_n(p) \to \infty$, as $p \to \pm \infty$. Then, (15) is proven.

Remark 2. Lemma [1] ensures that the functions $E_n(p)$ are nonconstant and analytic in $\mathbb{R}$. Consequently, the spectrum of the operator $H_{\beta_1, \beta_2}$ is purely absolutely continuous; see Theorem XIII.86 of [23].

Proposition 2. One has $\sigma(H_{\beta_1, \beta_2}) = \{E_1(0), \infty\}$.

Proof. Due to decomposition in (14) and Lemma [1] we have $[E_1(0), \infty) \subset \sigma(H_{\beta_1, \beta_2})$. It remains to show that $(\infty, E_1(0)) \cap \sigma(H_{\beta_1, \beta_2}) = \emptyset$. Since $C^0_0(S)$ is a core of $h_{\beta_1, \beta_2}(p)$, by Min-Max Principle,
$$\inf \sigma(H_{\beta_1, \beta_2}(p)) = \inf_{0 \neq \psi \in C^0_0(S)} \left\{\int_S \left(\|i\psi - \beta_1 \partial y_1 \psi - \beta_2 \partial y_2 \psi\|^2 + |\nabla y|^2\psi|^2\right) dy / \int_S |\psi|^2 dy\right\}.$$

Now, we perform the change of variable $\psi = \varphi v_1$, $\varphi \in C^0_0(S)$; note that $v^{-1}_1 \psi \in C^0_0(S)$.
if $\psi \in C_0^\infty(S)$. Some straightforward computations show that

$$
\int_S \left( |ip\psi - \beta_1 \partial_{y_1}\psi - \beta_2 \partial_{y_2}\psi|^2 + |\nabla_y \psi|^2 \right) dy
= \int_S \left( |ip\varphi - \beta_1 \partial_{y_1}\varphi - \beta_2 \partial_{y_2}\varphi|^2 + |\nabla_y \varphi|^2 \right) |v_1|^2 dy
+ \int \left( -(1 + \beta_1^2) \frac{\partial^2 v_1}{\partial y_1^2} - (1 + \beta_2^2) \frac{\partial^2 v_1}{\partial y_2^2} - 2\beta_1\beta_2 \frac{\partial^2 v_1}{\partial y_1 \partial y_2} \right) v_1 |\varphi|^2 dy
= \int_S \left( |ip\varphi - \beta_1 \partial_{y_1}\varphi - \beta_2 \partial_{y_2}\varphi|^2 + |\nabla_y \varphi|^2 \right) |v_1|^2 dy
+ E_1(0) \int_S |\psi|^2 dy.
$$

Then, $E_1(0) \leq \inf \sigma(H_{\beta_1,\beta_2}(p))$, for all $p \in \mathbb{R}$. Thus, $(-\infty, E_1(0)) \cap \sigma(H_{\beta_1,\beta_2}) = \emptyset$. □

**Proof of Theorem 1** It just to apply Propositions 1 and 2.

**Remark 3.** If $\beta_1 \beta_2 = 0$, we can give an alternative proof to find $\sigma_{ess}(-\Delta^D_{\Omega})$. In fact, suppose $\beta_2 = 0$. Define

$$
\gamma := \frac{p\beta_1}{(1 + \beta_1^2)},
$$

and consider the multiplication operators $e^{i\gamma y_1}$ and $e^{-i\gamma y_1}$. Then, for each $p \in \mathbb{R}$,

$$
-(ip - \beta_1 \partial_{y_1})^2 - \partial_{y_1}^2 - \partial_{y_2}^2 = -(1 + \beta_1^2) \left( \partial_{y_1} - \frac{ip\beta_1}{1 + \beta_1^2} \right)^2 - \partial_{y_2}^2 + \frac{p^2}{1 + \beta_1^2}
= e^{i\gamma y_1} \left[ -(1 + \beta_1^2) \partial_{y_1}^2 - \partial_{y_2}^2 + \frac{p^2}{1 + \beta_1^2} \right] e^{-i\gamma y_1}.
$$

Consequently,

$$
\sigma(H_{\beta_1,0}(p)) = \left\{ E_n(0) + \frac{p^2}{1 + \beta_1^2} \right\}_{n=1}^\infty, \quad \text{and} \quad \sigma(H_{\beta_1,0}) = [E_1(0), \infty).
$$

### 4 Discrete spectrum

This section is dedicated to prove Theorems 2, 3 and 4 stated in the Introduction. For simplicity, we denote $b := b_{f',g'}$.

**Proof of Theorem 2.** For each $n \in \mathbb{N}$, consider a linear function $\varphi_n : \mathbb{R} \to \mathbb{R}$ in $C^\infty(\mathbb{R})$ satisfying the following conditions:

$$
||\varphi_n|| \leq 1, \quad \varphi_n \equiv 1 \text{ on } [-n, n], \quad \varphi_n \equiv 0 \text{ on } \mathbb{R} \setminus (-2n, 2n).
$$

Define

$$
\psi_n(x,y) := \varphi_n(x)v_1(y), \quad n \in \mathbb{N},
$$

and note that $\psi_n \in \text{dom } b$, for all $n \in \mathbb{N}$. Some calculations show that

$$
b(\psi_n) - E_1(0)||\psi_n||^2 = \int_{\mathbb{R}} |\varphi_n'|^2 dx + \int_{\mathbb{R}} \left( A(f'(x)^2 - \beta_1^2) + 2B f'(x) g'(x) + C(g'(x)^2 - \beta_2^2) \right) |\varphi_n|^2 dx
= \int_{\mathbb{R}} |\varphi_n'|^2 dx + \int_{\mathbb{R}} V(x)|\varphi_n|^2 dx.
$$
Since $\|\varphi'_n\|^2 = 2/n$, for all $n \in \mathbb{N}$, by dominated convergence theorem,

$$b(\psi_n) - E_1(0)\|\psi_n\|^2 \to \int_{\mathbb{R}} V(x)dx, \quad n \to \infty.$$ 

Thus, there exists $n$ such that $b(\psi_n) - E_1(0)\|\psi_n\|^2 < 0$.

**Proof of Theorem 3.** For each $n \in \mathbb{N}$, consider $\varphi_n$ and $\psi_n$ as in the proof of Theorem 2. However, we add a small perturbation and we define

$$\psi_{n,\delta}(x, y) := \psi_n(x, y) + \delta \xi(x) y_1 v_1(y), \quad n \in \mathbb{N},$$

where $\delta$ is a real number and $\xi \in C_0^\infty(\mathbb{R})$. In this case, we have

$$\lim_{n \to \infty} \left( b(\psi_{n,\delta}) - E_1(0)\|\psi_{n,\delta}\|^2 \right)$$

$$= \lim_{n \to \infty} \left[ b(\psi_n) - E_1(0)\|\psi_n\|^2 + 2\delta \left( b(\psi_n, \xi y_1 v_1) - E_1(0) \int_{\Lambda} \psi_n \xi y_1 v_1 dx dy \right) 
+ \delta^2 \left( b(\xi y_1 v_1) - E_1(0)\|\xi y_1 v_1\|^2 \right) \right]$$

$$= \lim_{n \to \infty} \left[ 2\delta \left( b(\psi_n, \xi y_1 v_1) - E_1(0) \int_{\Lambda} \psi_n \xi y_1 v_1 dx dy \right) + \delta^2 \left( b(\xi y_1 v_1) - E_1(0)\|\xi y_1 v_1\|^2 \right) \right].$$

In the last equality was used the assumption $\int_{\mathbb{R}} (f'(x)^2 - \beta_1^2)dx = 0$. Now, we need to show that there exists a function $\xi$ satisfying

$$\lim_{n \to \infty} \left( b(\psi_n, \xi y_1 v_1) - E_1(0)\|\psi_n, \xi y_1 v_1\|^2 \right) \neq 0. \quad (16)$$

In fact, if (16) holds true, it is enough to choose $\delta$ such that $\left( b(\psi_{n,\delta}) - E_1(0)\|\psi_{n,\delta}\|^2 \right) < 0$, for some $n$ large enough.

Define the constant

$$\tilde{A} := \int_{S} y_1 \left( \frac{\partial v_1}{\partial y_1} \right)^2 dy.$$ 

Some calculations show that
\[
\lim_{n \to \infty} \left[ b(\psi_n, \xi y_1 v_1) - E_1(0) \int_{\Lambda} \psi_n \xi y_1 v_1 \, dx \right]
\]
\[
= \lim_{n \to \infty} \int_{\Lambda} \left( \varphi'_n v_1 - f'(x) \varphi_n \frac{\partial v_1}{\partial y_1} \right) \left( \xi y_1 v_1 - f'(x) \xi v_1 - f'(x) \xi y_1 \frac{\partial v_1}{\partial y_1} \right) \, dx \, dy
\]
\[
+ \lim_{n \to \infty} \int_{\Lambda} \varphi_n \xi (\nabla_y v_1, \nabla_y (y_1 v_1)) \, dx \, dy - E_1(0) \lim_{n \to \infty} \int_{\Lambda} \varphi_n \xi y_1 v_1^2 \, dx \, dy
\]
\[
= \int_{\Lambda} \left( -f'(x) \frac{\partial v_1}{\partial y_1} \xi y_1 v_1 + (f'(x))^2 \frac{\partial v_1}{\partial y_1} \xi y_1 + (f'(x))^2 \xi y_1 \left( \frac{\partial v_1}{\partial y_1} \right)^2 \right) \, dx \, dy
\]
\[
- \int_{\Lambda} \xi y_1 |\nabla y v_1|^2 \, dx \, dy - E_1(0) \int_{\Lambda} \xi y_1 v_1^2 \, dx \, dy
\]
\[
= \int_{\Lambda} \left( f''(x) \frac{\partial v_1}{\partial y_1} y_1 v_1 + (f'(x))^2 y_1 \left( \frac{\partial v_1}{\partial y_1} \right)^2 \right) \, dx \, dy
\]
\[
- \int_{\Lambda} \xi y_1 |\nabla y v_1|^2 \, dx \, dy - E_1(0) \int_{\Lambda} \xi y_1 v_1^2 \, dx \, dy
\]
\[
= \int_{\Lambda} \left( f''(x) \frac{\partial v_1}{\partial y_1} y_1 v_1 + (f'(x))^2 \beta_1 y_1 \left( \frac{\partial v_1}{\partial y_1} \right)^2 \right) \, dx \, dy
\]
\[
= \int_{\mathbb{R}} \xi \left( -\frac{f''(x)}{2} + \tilde{A} (f'(x)^2 - \beta_1^2) \right) \, dx.
\]

If the last integral is zero for all possible choices of \( \xi \), then the function \( f \) satisfy the differential equation
\[
- \tau_1'(x) + 2 \tilde{A} \tau_1^2(x) + 2 \beta_1 \tau_1(x) = 0,
\]
where \( \tau_1(x) = f'(x) - \beta_1 \). The family of curves
\[
\tau_{1,c}(x) = \frac{2 \beta_1}{ce^{-4A|x|} - 1}, \quad c \in \mathbb{R},
\]
describes all the solutions of the equation \([17]\). Now, note that \( c > 0 \) (resp. \( c < 0 \)) implies \( f'(x)^2 - \beta_1^2 > 0 \) (resp. \( f'(x)^2 - \beta_1^2 < 0 \)); the case \( c = 0 \) admits singularity as \( x = (4A\beta_1)^{-1} \ln c \). The case \( c = 0 \) implies that \( f' \) is constant. Then, the solutions in \([18]\) are not admissible. Thus, there exists a function \( \xi \) satisfying \([16]\).

**Remark 4.** The differential equation \([17]\) is very similar to that find in the proof Theorem 1.2 of \([4]\). Intuitively, we can justify this similarity by the fact that, in the conditions of Theorem 3, the reference curve \([\hat{I}]\) belongs to a plane.

**Proof of Theorem 4.** The case \( n = 0 \) is trivial. Suppose \( n \geq 1 \). We are going to show that there exists \( \epsilon_n > 0 \) so that the spectrum of \(-\Delta_n^D\) contains at least \( n \) discrete eigenvalues, counting multiplicity. Let \( I \subset \mathbb{R} \) be a bounded interval so that \( V(x) < 0 \), for all \( x \in I \). Define \( x_0 := \inf I \) and \( x_i := x_0 + i|I|/n \), for all \( i \in \{1, \ldots, n\} \). Let \( \psi_0 \) a non-zero function from \( H^1(\mathbb{R}) \) so that \( \text{supp} \psi_0 \subset (x_0, x_1) \). For each \( i \in \{1, \ldots, n\} \), define
\[
\Phi := \int_{x_{i-1}}^{x_i} |\psi_0(x_0 + x - x_{i-1})|^2 \, dx, \quad \psi_i(x) := \Phi^{-\frac{1}{2}} \psi_0(x_0 + x - x_{i-1}), \quad x \in \mathbb{R},
\]
and

\[ \varphi_i(x, y) := \psi_i(x) v_0(y), \quad (x, y) \in \Lambda. \]

The subset \( \{ \varphi_i \}_{i=1}^n \) is a basis of a subspace of \( H_0^2(\Lambda) \). Furthermore, for \( i \neq j \), since \( \varphi_i \) and \( \varphi_j \) have disjoint supports, one has \( b_\varepsilon(\varphi_i, \varphi_j) = 0 \). Some calculations show that

\[ b_\varepsilon(\varphi_i, \varphi_i) - \frac{E_i(0)}{\varepsilon^2} \| \varphi_i \|^2 = \int_{\mathbb{R}} \left( |\varphi_i'|^2 + \frac{V(x)}{\varepsilon^2} |\varphi_i|^2 \right) dx < 0, \]

for all \( i \in \{1, \ldots, n\} \), for all \( \varepsilon > 0 \) small enough. Then, by taking \( \varepsilon_n > 0 \) small enough, the result follows by Lemma 4.5.4 of [10] and Theorem 1.

A Appendix

Stability of the essential spectrum

This appendix is dedicated to the proof of Proposition 1. We use the arguments of [4]. In that work, the authors employed a different characterization of the spectrum essential which can be adapted to our problem. In fact,

**Lemma 2.** A real number \( \lambda \) belongs to the essential spectrum of \( H_{f', g'} \) if, and only if, there exists a sequence \( \{ \psi_n \}_{n=1}^\infty \subset \text{dom } b_{f', g'} \) such that the following conditions hold:

(i) \( \| \psi_n \| = 1 \), for all \( n \geq 1 \);

(ii) \( (H_{f', g'} - \lambda I) \psi_n \to 0 \), as \( n \to \infty \), in the norm of the dual space \( (\text{dom } b_{f', g'})^* \);

(iii) \( \text{supp } \psi_n \subset Q \setminus (-n, n) \times S \), for all \( n \geq 1 \).

The proof of Lemma 2 is very similar to the proof of Lemma 5 of [4], it will not presented in this text.

**Proof of Proposition 1.** Let \( \lambda \in \sigma_{\text{ess}}(H_{\beta_1, \beta_2}) \). By Lemma 2 there exists a sequence \( \{ \psi_n \}_{n=1}^\infty \subset \text{dom } b_{\beta_1, \beta_2} \) such that the conditions (i) – (iii) are satisfied. For simplicity, write

\[ \tau_1(x) := f'(x) - \beta_1, \quad \tau_2(x) := g'(x) - \beta_2. \]

Some calculations show that

\[
\begin{align*}
    b_{f', g'}(\varphi, \psi_n) - \lambda(\varphi, \psi_n) &= b_{\beta_1, \beta_2}(\varphi, \psi_n) - \lambda(\varphi, \psi_n) \\
    &= + \int_{\Lambda} \left( -\tau_1 \frac{\partial \varphi}{\partial y_1} - \tau_2 \frac{\partial \varphi}{\partial y_2} \right) \left( \psi'_n - \beta_1 \frac{\partial \psi_n}{\partial y_1} - \beta_2 \frac{\partial \psi_n}{\partial y_2} \right) dxdy \\
    &\quad + \int_{\Lambda} \left( \varphi' - \tau_1 \frac{\partial \varphi}{\partial y_1} - \tau_2 \frac{\partial \varphi}{\partial y_2} \right) \left( -\tau_1 \frac{\partial \psi_n}{\partial y_1} - \tau_2 \frac{\partial \psi_n}{\partial y_2} \right) dxdy.
\end{align*}
\]

Since \( \| \partial \varphi / \partial y_1 \|^2, \| \partial \varphi / \partial y_2 \|^2 \leq b_{f', g'}(\varphi) = \| \varphi \|_{H_0^2}^2 \), one has

\[
\sup_{0 \neq \varphi \in H_0^2(\Lambda)} \left\{ \int_{\Lambda} \left( -\tau_1 \frac{\partial \varphi}{\partial y_1} - \tau_2 \frac{\partial \varphi}{\partial y_2} \right) \left( \psi'_n - \beta_1 \frac{\partial \psi_n}{\partial y_1} - \beta_2 \frac{\partial \psi_n}{\partial y_2} \right) dxdy / \| \varphi \|_H^2 \right\} \leq \left( \| \tau_1 \|_{L^\infty((-n, n))} + \| \tau_2 \|_{L^\infty((-n, n))} \right) \left( \| \psi_n' \| + \beta_1 \| \psi_n / \partial y_1 \| + \beta_2 \| \psi_n / \partial y_2 \| \right) \to 0,
\]

\( 11 \).
Consider the quadratic form \( s \) satisfying \( M \). Now, we need to obtain the opposite estimate. Consider the multiplication operator \( \lambda \). Since \( \phi \), \( \mu \) and \( N \) are the eigenvalues of the generated operator \( H_{\beta, \gamma} \). The inclusion \( \sigma_{\text{ess}}(H_{\beta, \gamma}) \subset \sigma_{\text{ess}}(H_{\beta, \gamma}) \) can be obtained in a similar way.

**B Appendix**

**Asymptotic behavior of eigenvalues**

Let \( W : \mathbb{R} \rightarrow \mathbb{R} \) be a bounded function and \( \mu \in \mathbb{R} \setminus \{0\} \). Define \( W_{\min} := \inf_{x \in \mathbb{R}} \{W(x)\} \). Consider the quadratic form

\[
\begin{align*}
    n_{\mu}(\phi) &= \int_{\mathbb{R}} |\phi'|^2 \, dx + \mu \int_{\mathbb{R}} W(x)|\phi|^2 \, dx, \quad \text{dom } n_{\mu} = H^1(\mathbb{R}).
\end{align*}
\]

Denote by \( n_{\mu}(\psi, \varphi) \) and \( N_{\mu} \) its sesquilinear form and its self-adjoint operator associated, respectively. Consider the sequence \( \{\lambda_j(N_{\mu})\}_{j \in \mathbb{N}} \) given by Min-Max Principle; see, e.g., Theorem XIII.1 of \cite{28}.

The next result is a simpler version of Theorem 4 of \cite{18}.

**Theorem 5.** For each \( j \in \mathbb{N} \),

\[
\liminf_{\mu \rightarrow +\infty} \frac{\lambda_j(N_{\mu})}{\mu} = W_{\min}.
\]

**Proof.** Since \( N_{\mu} \geq \mu W_{\min} \mathbf{1} \), by Min-Max Principle,

\[
\liminf_{\mu \rightarrow +\infty} \frac{\lambda_j(N_{\mu})}{\mu} \geq W_{\min}.
\]

Now, we need to obtain the opposite estimate. Consider the multiplication operator

\[
\mathcal{M}_W \phi = W \phi, \quad \text{dom } \mathcal{M}_W = \{ \phi \in L^2(\mathbb{R}) : W \phi \in L^2(\mathbb{R}) \}.
\]

The spectrum \( \mathcal{M}_W \) is purely essential and equal to the essential range of the generated function \( W \). In particular, \( W_{\min} \in \sigma(\mathcal{M}_W) \). By the spectral theorem, there exists a sequence \( \{\psi_i\}_{i \in \mathbb{N}} \) orthonormalized in \( L^2(\mathbb{R}) \) so that \( \| (\mathcal{M}_W - W_{\min} \mathbf{1}) \psi_i \| \rightarrow 0 \), as \( i \rightarrow \infty \). Since \( C_0^\infty(\mathbb{R}) \) is dense in dom \( \mathcal{M}_W \), it follows that there is also a sequence \( \{\varphi_i\}_{i \in \mathbb{N}} \) satisfying

\[
\langle \varphi_i, \varphi_j \rangle - \delta_{ij} \rightarrow 0, \quad \langle \varphi_i, (\mathcal{M}_W - W_{\min}) \varphi_j \rangle \rightarrow 0,
\]

as \( i, j \rightarrow \infty \). Now, given \( N \in \mathbb{N} \), take \( k = k(\mathbb{N}) \) sufficiently large so that

\[
A(N) - W_{\min} \mathbf{1} \leq N^{-1} \mathbf{1},
\]

where \( A(N) \) is a symmetric matrix with entries \( \langle \varphi_{i+k}, \varphi_{j+k} \rangle \), for \( i, j \in \{1, \ldots, N\} \). Since the subspace generated by \( \{\varphi_{1+k}, \ldots, \varphi_{N+k}\} \) is a \( N \)-dimensional subspace of dom \( N_{\mu} \), one has \( \lambda_j(N_{\mu}) \leq c_j(N_{\mu}) \), for all \( j \in \{1, \ldots, N\} \), where \( \{c_j(N_{\mu})\}_{j=1}^N \) are the eigenvalues.
(written in increasing order and repeated according to multiplicity) of the matrix $C(N_\mu)$ defined by $C_{ij}(N_\mu) = n_\mu(\varphi_{i+k}, \varphi_{i+j})$. Now, we can see that

$$C(N_\mu) \leq \mu(W_{\text{min}} + N^{-1})1 + d(N)1,$$

where $d(N)$ denotes the maximal eigenvalue of the matrix with entries $(\langle \nabla \varphi_{i+k}, \nabla \varphi_{j+k} \rangle)$. Then, follows that

$$\liminf_{\mu \to +\infty} \frac{\lambda_j(N_\mu)}{\mu} \leq W_{\text{min}} + N^{-1},$$

for $j \in \{1, 2, \ldots, N\}$, with $N$ being arbitrarily large. \hfill \qed

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