Detect genuine multipartite entanglement in the one-dimensional transverse-field Ising model

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Recently Seevinck and Uffink argued that genuine multipartite entanglement (GME) had not been established in the experiments designed to confirm GME. In this paper, we use the Bell-type inequalities introduced by Seevinck and Svetlichny [Phys. Rev. Lett. 89, 060401 (2002)] to investigate the GME problem in the one-dimensional transverse-field Ising model. We show explicitly that the ground states of this model violate the inequality when the external transverse magnetic field is weak, which indicate that the ground states in this model with weak magnetic field are fully entangled. Since this model can be simulated with nuclear magnetic resonance, our results provide a fresh approach to experimental test of GME.

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Quantum entanglement is central to the foundations of quantum mechanics and has been playing a vital role in quantum communication, information and computation processing, such as quantum teleportation [1, 2], quantum cryptographic schemes [3], quantum parallelism [4], dense coding [5, 6], decoherence in quantum computers, and the evaluation of quantum cryptographic schemes [7]. While two-particle entanglement is well-known, multipartite entanglement is far from comprehensively understanding not only because the classification of different types of this form of entanglement is still an open problem [8], but also it requires different conditions for actual experimental confirmation. Extensive efforts related to this issue have been made both theoretically and experimentally [9, 10, 11, 12]. In fact, recent experiments using photons and atom-cavity techniques [9] have claimed experimental conformation of genuine multipartite entanglement (GME). Nevertheless, Seevinck and Uffink argued that GME had not been established in these experiments by pointing out a possible loophole problem [13]. Here, we use the Bell-type inequalities introduced by Seevinck and Svetlichny [10, 11] to investigate the GME problem in this model based on the Bell-type inequalities introduced by Seevinck and Svetlichny [10]. We restrict our study on the ground states of this model and find that these states violate the inequalities when the external transverse magnetic field is weak. We also find that the violations might decrease as the external magnetic field gets stronger and stronger. The more particles involved in the model, the faster the violations decrease.

To begin with, let first briefly review the the N-particle Bell-type inequalities presented by Seevinck and Svetlichny [10], which are useful for detecting GME. Consider an experimental situation involving N particles in which two measurements $X^{[j]}_1$ and $X^{[j]}_2$ ($j = 1, \cdots, N$) can be performed on each particle. Each of the measurement has two possible outcomes: $X^{[j]}_{k_j} = \pm 1$ ($k_j = 1, 2$).

In a specific run of the experiment the correlations between all N observations can be represented by the product $\Pi_{j=1}^{N} X^{[j]}_{k_j}$, then the correlation function is the average over many runs of the experiment:

$$Q_{k_1\cdots k_N} = \left\langle \Pi_{j=1}^{N} X^{[j]}_{k_j} \right\rangle$$

Based on the partial separability [10, 15], or more generally speaking, on the so called hybrid local-nonlocal hidden variables (HLNHV) models [11, 18], Seevinck and Svetlichny introduced the following Bell-type inequali-
ties:
\[
I^N = \frac{1}{2^{N-1}} \sum_{k_1, \ldots, k_N} V^\pm_{\kappa(K)} Q_{k_1 \cdots k_N} \leq 1, \tag{2}
\]

where \( K = (k_1, k_2, \ldots, k_N) \) and \( \kappa(K) \) is the number of times of index 2 appears in \( K \); \( V^\pm_{\kappa(K)} \) be a sequence of signs given by \( V^\pm_{\kappa(K)} = (-1)^{\kappa(\pm 1)/2} \). As analyzed in Ref. [10], the two inequalities of Eq. (2) are equivalent for even \( N \) since they are interchanged by a global change of labels 1 and 2. While for odd \( N \) this is not the case and should be considered a priori independent. Moreover, the inequalities of Eq. (2) are invariant under a permutation of the \( N \) particles. Although the inequalities of Eq. (2) are derived from the HLNHV model, they also hold for quantum states that are partially entangled. The violations of these inequalities are sufficient conditions for GME. Actually, in quantum mechanics, the observables could be spin projections onto unit vectors \( X_{k_j}^{(j)} = n_{k_j} \cdot \bar{\sigma}_j \). Here \( n_{k_j} \) is a unit vector and \( \bar{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z) \) is a vector of local Pauli operators for the \( j \)-th observer. For any quantum state \( \rho \), the quantum expression of the correlation function reads: \( Q_{k_1 \cdots k_N} = \text{Tr}[\rho X_{k_1}^{[1]} \otimes \cdots \otimes X_{k_N}^{[N]}] \). Then it is obviously that if the quantum state \( \rho \) is partially entangled, the quantum expression of the correlation function might be factorizable. Consequently, from the construction of the inequalities (2), any partially entangled quantum state might obey these inequalities. For detailed analysis, please see Ref. [10].

In this paper, we restrict ourselves to the case of \( V^-_{\kappa(K)} \) and the case of \( V^+_{\kappa(K)} \) can be analyzed similarly. For the case of \( V^-_{\kappa(K)} \), the inequality for three-qubit system reads:
\[
I^3 = \frac{1}{4} \left( Q_{111} + Q_{112} + Q_{121} + Q_{211} - Q_{122} - Q_{212} + Q_{221} - Q_{222} \right) \leq 1. \tag{3}
\]

Inequality (3) was first introduced by Svetlichny in 1987 to distinguish three-particle from two-particle entanglement [15]. Here we apply it in the one-dimensional transverse-field Ising model.

The Hamiltonian of the one-dimensional transverse-field Ising model with periodic boundary conditions reads
\[
H_N = -\sum_{i=1}^N (\sigma_i^x \sigma_{i+1}^x + h \sigma_i^z), \quad \sigma_{N+1}^x = \sigma_1^x, \tag{5}
\]

where \( h \) is the transverse field and \( N \) is the number of spins involved in the model. As inferred from the model’s name, the Hamiltonian describes a chain of spins with the nearest neighboring Ising interaction along \( x \)-direction, and all spins are subject to a transverse magnetic field \( h \) along the \( z \)-direction. If \( h \to \infty \), the Ising interaction is negligible, and all spins are fully polarized along \( z \)-direction. It is easy to prove that the ground state for a finite system in the whole region \( h > 0 \) is nondegenerate. An illustration of this model is shown in Fig. 1.

To detect GME in this model, let first focus on the three-qubit case, namely \( N = 3 \). In this case, the Hamiltonian of this model in the matrix form reads:
\[
H_3 = \begin{pmatrix}
-3h & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\
0 & -h & -1 & 0 & -1 & 0 & 0 & -1 \\
0 & -1 & -h & 0 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & h & 0 & -1 & 0 & -1 \\
0 & -1 & -1 & 0 & -h & 0 & 0 & -1 \\
-1 & 0 & 0 & -1 & 0 & h & -1 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & h & 0 \\
0 & -1 & -1 & 0 & -1 & 0 & 0 & 3h
\end{pmatrix}. \tag{6}
\]

Solve the Hamiltonian \( H_3 \) directly, one can easily find that the ground state of this Hamiltonian is:
\[
|\psi_3\rangle_g = (\gamma_1 |000\rangle + |011\rangle + |101\rangle + |110\rangle)/\sqrt{N_1}, \tag{7}
\]

where \( \gamma_1 = -1 + 2h + 2\sqrt{1 - h + h^2} \) and \( N_1 = 3 + \gamma_1^2 \) is the normalization constant. Our numerical results show that, for small \( h \), the ground state \( |\psi_3\rangle_g \) violates the Bell-type inequality (3), indicating that the state is a genuine three-particle entangled state. For instance, denote the unit vector on to which the spin projected by \( n_{k_j}^{[j]} = (\sin \theta_{k_j}^{[j]} \cos \phi_{k_j}^{[j]}, \sin \theta_{k_j}^{[j]} \sin \phi_{k_j}^{[j]}, \cos \phi_{k_j}^{[j]}) \). The maximal violation of the inequality (3) occurs at \( h = 0 \) and the violation is \( \sqrt{2} \). To obtain the maximal violation, we can set \( \theta_{[1]} = \theta_{[2]} = \phi_{[1]} = \phi_{[2]} = 0 \). The violation is \( \sqrt{2} \) for the GHZ form \( |\Psi_3\rangle_{\text{max}} = 1/\sqrt{2}(|000\rangle + |111\rangle) \) due to local unitary transformations [19]. The violations decrease as the external magnetic field gets increased. At

![FIG. 1: (Color online) A sketch of the one-dimensional transverse-field Ising model with periodic boundary conditions. Here only eighteen spins are considered. In this model, all spins are subject to an external field \( h \) along \( z \)-direction and two arbitrary neighboring spins interact with each other by the Ising interaction \( \sigma_j^x \sigma_{j+1}^x \).](Image 327x605 to 394x726)
the critical point of a quantum phase transition in this model [19, 20], namely \( h = 1 \), the quantum violation decreases to 1.08866 and the angles for this violation are as follows: \( \theta_2^{[1]} = \theta_2^{[3]} = -\theta_1^{[1]} = -\theta_1^{[2]} = -\phi_1^{[3]} = 1.9\pi \), \( \phi_1^{[1]} = \phi_1^{[2]} = -\phi_1^{[3]} = \phi_1^{[3]} = \phi_2^{[3]} = \pi/2 \) and \( \theta_2^{[2]} = 7\pi/4 \). The threshold value of \( h \) for the ground state \( |\psi_3\rangle_g \) to violate the inequality (3) is 1.375, indicating that the ground state with \( h > 1.375 \) in this model might not be fully entangled. Consequently, if one wants to detect GME in the model, the external magnetic field should not surpass this threshold value. It is worthwhile to clarify that the violations of the inequalities (2) are only sufficient but not necessary conditions for GME, namely, some states that do not violate these inequalities may also be genuinely full entangled. That is the reason why we use the term “might not be fully entangled” to describe the entanglement of the ground state with \( N = 3 \).

For the four-qubit case, i.e., \( N = 4 \), the ground states of the Hamiltonian \( H_4 \) reads:

\[
|\psi_4\rangle_g = \frac{1}{\sqrt{N_2}} \left[ \begin{array}{c} \gamma_2 - 2\sqrt{2}h(1 - \gamma_3^2) \\ \gamma_3 \end{array} \right] |0000\rangle + \left( h + \frac{\gamma_3}{\sqrt{2}} \right) (0011) + \frac{4h + 2\sqrt{2}\gamma_3}{2\sqrt{2}\gamma_3} (0101) + \left( h + \frac{\gamma_3}{\sqrt{2}} \right) (0110) + \frac{4h + 2\sqrt{2}\gamma_3}{2\sqrt{2}\gamma_3} (1010) + \left( h + \frac{\gamma_3}{\sqrt{2}} \right) (1100) + (1111),
\]

where \( \gamma_2 = -1 + 2h^2 + 2\sqrt{1 + h^2}, \gamma_3 = \sqrt{1 + h^2 + \sqrt{1 + h^2}} \) and \( N_2 = 1 + 4(h + \frac{\gamma_3}{\sqrt{2}})^2 + \frac{(4h + 2\sqrt{2}\gamma_3)^2}{4\gamma_3^2} + (\gamma_2 - 2\gamma_3^2 + 2\sqrt{2}h\gamma_3)^2 \) is the normalization constant. A similar analysis shows that the ground state \( |\psi_4\rangle_g \) violates the inequalities (2) with small values of \( h \) and the maximal violation is also \( \sqrt{2} \) and occurs at \( h = 0 \). A notable difference between four-qubit case and three-qubit case is that the violations for four-qubit ground state decrease faster than that for three-qubit. The threshold value of \( h \) for the ground state \( |\psi_4\rangle_g \) to violate the inequalities (2) is 0.935, which is smaller than 1.375, the corresponding threshold values of \( h \) for three-qubit case. The detailed quantum violations as a function of \( h \) for the cases \( N = 3, N = 4 \), and \( N = 5 \) are illustrated in Fig. 2. From Fig. 2, we can see that the quantum violations decrease faster and faster as the system size increase. At the critical point of a quantum phase transition, i.e., \( h = 1 \), the corresponding ground states for \( N = 4 \) and \( N = 5 \) do not violate the inequalities (2), which is different from the \( N = 3 \) case.

For larger systems, it is very difficult to calculate the quantum violations since there are so many parameters in the maximizing procedure. From the numerical results above, we believe that GME does exist in the one-dimensional transverse-field Ising model, but the more particles involved in the model, the small threshold value of \( h \) for the ground states to preserve GME.

It is worthwhile to note that the Bell-type inequalities (2) originate from the HLNHV models, thus these inequalities are also useful for the study of genuine multiparticle nonlocality (GMNL), which is quite different from the concept GME (for details, see Ref. [18]). Actually, Tóth and Acín have presented a concrete example in Ref. [21] illustrating that a certain family of genuine three-qubit entangled states preserve local hidden-variable models. Any violation of inequalities (2) is a sufficient condition of GMNL. Consequently, the quantum violations of these inequalities in the one-dimensional transverse-field Ising model also confirm GMNL in this model, implying that the model can not be described by HLNHV theories.

Recent experiments designed to detect GME are based on the photons and atom-cavity techniques [9]. It has been argued that their claims of experimental confirmation of three- and four-particle entanglement are questionable [13]. Fortunately, the rapid development in the field of Nuclear Magnetic Resonance Quantum Information Processing (NMR-QIP) has shown that NMR is a valuable and feasible testing tool for the new ideas in quantum information science (for recent reviews see Ref. [22] and references there in). More than fifty years of development has put NMR in an unique position to perform complex experiments and many physical models can be simulated using the NMR technologies [23, 24]. Because of the fundamental importance of the GME for large scale quantum information processing, further experimental tests of GME might be widely welcomed. Based on the numerical results analyzed above, experiments utilizing NMR technologies to confirm GME in the one-dimensional transverse-field Ising model are nice.
alternatives. Actually, the significance of such experiments is at least two-folded: On the one hand, these experiments might close the loophole problem in the recent experiments mentioned above and lead to a more comprehensive understanding of entanglement in the model; on the other hand, this approach can also address the GMNL problems and might provide experimental evidences concerning the contradictions between quantum mechanics and HLNHV theories.

In summary, we studied the GME problem in the one-dimensional transverse-field Ising model based on the Bell-type inequalities. By numerically investigating the quantum violations of the ground states of this model, we showed evidently that these ground states are genuinely full entangled when the external transverse magnetic field is weak. The violations decrease as the external magnetic field gets stronger and stronger and the more particles involved in the model, the faster the violations decrease. Our approach also addressed the GMNL problem. The quantum violations of the Bell-type inequalities (2) also confirm GMNL in this model, indicating that the model can not be described by HLNHV theories. Based on the numerical analysis, we suggest experiments using the rapid developing NMR technologies to simulate the one-dimensional transverse-field Ising model be carried out to detect GME and GMNL in the model.

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