ON ABSOLUTELY CONTINUOUS INVARIANT MEASURES AND KRIEGER-TYPE OF MARKOV SUBSHIFTS

By

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Abstract. It is shown that for a nonsingular conservative shift on a topologically mixing Markov subshift with the Doeblin condition the only possible absolutely continuous shift-invariant measure is a Markov measure. Moreover, if it is not equivalent to a homogeneous Markov measure then the shift is of Krieger-type III 1. A criteria for equivalence of Markov measures is included.

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1 Introduction and Main Theorems

In recent years some general results were obtained about the classification of the Bernoulli shift according to its Krieger-type. The basic problem is classical and goes back to Halmos [18]: for a given sigma-finite Borel measure $\mu$ on a standard Borel space $X$ and a nonsingular Borel transformation $T : (X, \mu) \to (X, \mu)$, determine whether there exists a sigma-finite Borel measure $\nu$ which is both absolutely continuous with respect to $\mu$ and invariant to $T$. Such measure $\nu$ is abbreviated as \textit{a.c.i.m.} (absolutely continuous invariant measure) for $\mu$ and $T$. Hamachi [19] showed that there is a Bernoulli shift without a.c.i.m. but he did not determine its Krieger-type. It was an open question, famously attributed to Krengel and Weiss [28, 25, 13] (see also the MathSciNet review of Krengel on [20]), whether Krieger-types $\text{II}_\infty$ and $\text{III}_\lambda$ ($0 \leq \lambda \leq 1$) can appear in the nonsingular conservative shift. More details on the history of the problem can be found in the survey of Danilenko and Silva [13].

Only in the last few years some general results were discovered on the Bernoulli shift. First, Kosloff [25] showed that in the half-stationary Bernoulli shift on 2 states space, when the distribution on all the negative coordinates is $(1/2, 1/2)$, if the shift is nonsingular and conservative then it is either equivalent to a corresponding stationary Bernoulli measure, and then it is of Krieger-type $\text{II}_1$, or that there is no a.c.i.m. Moreover, in the latter case it is of Krieger-type $\text{III}_1$. This result was later extended by Danilenko and Lemańczyk [12] when the distribution of the negative coordinates is $(p, 1 - p)$ for some $0 < p < 1$.

Recently, a significant progress has been achieved for Bernoulli actions of countable groups. Vaes and Wahl [39] formulated a characterization of a countable group to admit a Krieger-type $\text{III}_1$ Bernoulli action in terms of the first $\ell^2$-cohomology of the group, and proved this characterization for a large family of groups. Björklund and Kosloff [2] showed that every countable amenable group admits a Krieger-type $\text{III}_1$ Bernoulli action on two states. The recent result of Björklund, Kosloff and Vaes [3] confirms the conjecture of Vaes and Wahl, showing that every countable group which is either amenable or has non-trivial first $\ell^2$-Betti number admits a Bernoulli action of Krieger-type $\text{III}_1$.

In contrast to the Bernoulli shift, very little is known about the Markov shift. The ergodicity of a nonsingular conservative Markov subshift was studied by Kosloff [27] and Danilenko [11] (see Theorem 1). The Golden Mean Markov subshift model was used by Kosloff to construct examples of conservative Anosov diffeomorphisms of the torus $\mathbb{T}^2$ without a Lebesgue a.c.i.m. [26, 24]. A special case of a half-stationary Markov shift was studied by Danilenko and Lemańczyk [12],
and they asked about a general half-stationary Markov shift on two states (see [12, Problem (1)]). Here we solve the Markov case to a relatively large extent under the Doeblin condition and we remove the restrictive assumption of half-stationarity.

We now introduce our general setting. Let \( X = \mathbb{S}^\mathbb{Z} \) for a finite state space \( \mathbb{S} \) and consider the left-shift \( T : X \to X \) defined by \((Tx)_n = x_{n+1}\) for every \( n \in \mathbb{Z} \), where \( x_n \) denotes the \( n \)th coordinate of \( x \). For a \{0, 1\}-valued \(|\mathbb{S}| \times |\mathbb{S}|\)-matrix \( A \) let the subshift of finite type (SFT) associated to \( A \) be the shift-invariant space

\[
X_A = \{ x \in X : A(x_n, x_{n+1}) = 1 \ \forall n \in \mathbb{Z} \}.
\]

We call \( A \) the adjacency matrix of \( X_A \). An SFT \( X_A \) is called topologically-mixing if \( A \) is a primitive matrix, that is there exists a positive integer \( M \geq 1 \) such that all the entries of \( A^M \) are positive. Let \((X_n : n \in \mathbb{Z})\) be the coordinates random variables of \( X_A \), defined by \( X_n(x) = x_n \). A Markov measure \( \mu \) on an SFT \( X_A \) is a probability measure defined as follows. Take a sequence \((P_n : n \in \mathbb{Z})\) of transition matrices, which are stochastic \(|\mathbb{S}| \times |\mathbb{S}|\)-matrices with the property that \( P_n(s, t) = 0 \) whenever \( A(s, t) = 0 \) for all \( n \in \mathbb{Z} \) and \( s, t \in \mathbb{S} \). By stochastic matrix we mean a matrix whose entries are non-negative and each of its rows is summed up to 1. Take further a sequence \((\pi_n : n \in \mathbb{Z})\) of probability distributions on \( \mathbb{S} \) which, when relating them as row vectors, satisfy the identities

\[
(1.0.1) \quad \pi_n P_n = \pi_{n+1} \quad \text{for all } n \in \mathbb{Z}.
\]

This defines \( \mu \) on cylinders via

\[
\mu(X_{k+1} = s_1, \ldots, X_{k+m} = s_m) = \pi_{k+1}(s_1)P_{k+1}(s_1, s_2) \cdots P_{k+m-1}(s_{m-1}, s_m),
\]

for all \( k \in \mathbb{Z}, m \in \mathbb{N} \) and \( s_1, \ldots, s_m \in \mathbb{S} \). The consistency conditions (1.0.1) ensures that this definition extends uniquely to a Borel measure \( \mu \) on \( X_A \) such that

\[
\pi_n(s) = \mu(X_n = s), \quad n \in \mathbb{Z}, s \in \mathbb{S},
\]

and

\[
P_n(s, t) = \mu(X_{n+1} = t \mid X_n = s, X_{n-1} = s_1, \ldots, X_{n-k} = s_k),
\]

for all \( t, s, s_1, \ldots, s_k \in \mathbb{S} \) and \( n \in \mathbb{Z} \). This last property is the usual Markov property. We write \( \mu(P_n : n \in \mathbb{Z}) \) for a Markov measure whose sequence of transition matrices is \((P_n : n \in \mathbb{Z})\). Let us denote the reverse transition matrices of a Markov measure \( \mu(P_n : n \in \mathbb{Z}) \) by

\[
(1.0.2) \quad \widehat{P}_n(s, t) = \mu(X_{n-1} = t \mid X_n = s) = \frac{\pi_{n-1}(t)}{\pi_n(s)}P_{n-1}(t, s),
\]
for all $n \in \mathbb{Z}$ and $s, t \in S$ with $\pi_n(s) > 0$. When for some $n \in \mathbb{Z}$ and $s \in S$ we have $\pi_n(s) = 0$, we let $\hat{P}_n(s, t) = 0$ for all $t \in S$. We extend the notation $\hat{Q}$ also for an $S \times S$-stochastic matrix $Q$, by relating it as a constant sequence of transition matrices, which together with the distribution $\lambda$ on $S$ satisfying $\lambda Q = \lambda$ defines a Markov measure on $S^\mathbb{Z}$.

The following condition of a Markov measure is fundamental to our work. We call it the Doeblin condition after various conditions of this type formulated by W. Döblin [9]. Let $\mu = \mu(P_n; n \in \mathbb{Z})$ be a Markov measure on an SFT $X_A$. We say that $\mu$ satisfies the Doeblin condition if

$$\exists \delta > 0, P_n(s, t) \geq \delta \iff A(s, t) = 1 \text{ for all } s, t \in S \text{ and } n \in \mathbb{Z}.$$  

The following result was proved in [27, Proposition 2.2, Theorem 3.4].

**Theorem 1** (Kosloff). Let $X_A \subset S^\mathbb{Z}$ be a topologically-mixing SFT of $S^\mathbb{Z}$ and $\mu$ be a Markov measure on $X_A$ with the Doeblin condition (D). Suppose that the shift is nonsingular with respect to $\mu$. Then if the shift is conservative it is ergodic.

Danilenko [11] strengthened this result and showed that in this case the shift is further weakly-mixing, in the sense that its product with every ergodic probability measure preserving transformation is again ergodic.

Our work goes further into the classification of the shift acting on a Markov subshift of finite type into its Krieger-type. Note that by the Poincaré recurrence theorem when the shift is not conservative it can not admit a shift-invariant probability measure. Then in the following Theorems 2, 3, 4 and 5 we assume that the shift is nonsingular and conservative with respect to the subject measure. See the exact definitions below.

In the case that we call the divergent scenario the following theorem fully answers the question of possible Krieger-type of the shift.

**Theorem 2.** Let $X_A \subset S^\mathbb{Z}$ be a topologically-mixing SFT and $\mu = \mu(P_n; n \in \mathbb{Z})$ be a Markov measure on $X_A$ with the Doeblin condition (D). If the shift is nonsingular and conservative with respect to $\mu$ and the limit $\lim_{|n| \to \infty} P_n$ does not exist, the shift is of Krieger-type III$_1$.

By relating to the limit $\lim_{|n| \to \infty} P_n$ we mean that it exists if, and only if, the limits $\lim_{n \to \infty} P_n$ and $\lim_{n \to -\infty} P_n$ both exist entrywise and are equal.

The other case that we call the convergent scenario is more subtle. We first give a necessary criteria for the conservativeness of the shift. This condition was established in [12, Lemma 8.6] in the special case of the half-stationary bistochastic 2 states case, and here we establish the general case of topologically-mixing Markov SFT with the Doeblin condition.
**Theorem 3.** Let $X_A \subset \mathbb{Z}^Z$ be a topologically-mixing SFT and $\mu = \mu_{(P_n, n \in \mathbb{Z})}$ be a Markov measure on $X_A$ with the Doeblin condition (D). If the shift is nonsingular and conservative with respect to $\mu$ and both limits $\lim_{n \to \infty} P_n$ and $\lim_{n \to \infty} P_{-n}$ exist, then they are equal. That is, $\lim_{|n| \to \infty} P_n$ exists.

Then we can determine the Krieger-type of the shift as follows.

**Theorem 4.** Consider the state space $S = \{0, 1\}$. Let $X_A \subset \mathbb{Z}^Z$ be a topologically-mixing SFT and $\mu = \mu_{(P_n, n \in \mathbb{Z})}$ be a Markov measure on $X_A$ with the Doeblin condition (D). If the shift is nonsingular and conservative with respect to $\mu$, the Krieger-type of the shift is either $\text{II}_1$ or $\text{III}_1$.

Moreover, the shift is of Krieger-type $\text{II}_1$ if, and only if, there exists a stochastic matrix $Q$ such that $Q \equiv \lim_{|n| \to \infty} P_n$ and

$$\sum_{n \geq 1} \sum_{s, u, t \in S} \left( \sqrt{P_{-n}(u, s)P_n(v, t)} - \sqrt{Q(u, s)Q(v, t)} \right)^2 < \infty.$$ 

In this case, the absolutely continuous invariant measure for the shift is the Markov measure defined by $Q$ and the distribution $\lambda$ on $S$ satisfying $\lambda Q = \lambda$.

Consider the Golden Mean SFT $X_G \subset \{0, 1, 2\}^Z$ that is defined by the primitive adjacency matrix

$$G = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

**Theorem 5.** Let $X_G \subset \{0, 1, 2\}^Z$ be the Golden Mean SFT and $\mu = \mu_{(P_n, n \in \mathbb{Z})}$ be a Markov measure on $X_G$ with the Doeblin condition (D). If the shift is nonsingular and conservative with respect to $\mu$, the Krieger-type of the shift is either $\text{II}_1$ or $\text{III}_1$. These alternatives are determined by the same test of Theorem 4.

**1.1 About the proof.** In the first works [25, 12] the authors proved the case of a half-stationary Bernoulli shift by computing the ratio set of the shift using the appropriate cocycle. However, this method relies on the Bernoullicity and the half-stationarity of the shift, and in the general Markov case we found the computation of the essential values of this cocycle to be more involved. In later works [27, 11, 3] it has been found useful to study the ergodicity of the shift by the action of the permutations that change only finitely many coordinates. This approach was applied by Björklund, Kosloff and Vaes [3] for amenable groups, using a ratio ergodic theorem by Danilenko [11], to replace the computation of the ratio set of the Bernoulli shift by the computation of the ratio set of the finite
permutations action. However, also in this approach the Bernoullicity plays a crucial role in two aspects. The first is that the cocycle of the shift satisfies a special identity with respect to finite permutations (see [3, Lemma 3.1]) and this identity no longer holds in the Markov case. The second is that the finite permutations action is ergodic with respect to Bernoulli measures. This is far from being true in general and in the Markov case it is not true even when the shift is measure-preserving. See Example 3 of Blackwell–Freedman [4]. In particular, the action of the finite permutations when not ergodic does not fall under the Krieger classification.

Here we place the above approach for amenable groups in a more general context. We develop a notion of Renormalization Full-Group (Definition 4.1) of one action of a countable group with respect to another action of a countable group, where the latter satisfies a metric property with respect to the former. This metric property is the Maharam extension-version of the notion of equivalence underlying the well-known Hopf Argument. We then establish a version of the Hopf Argument for the Maharam extension (Theorem 4.3), which allows one to study the ratio set of the first action by the ratio set of the corresponding renormalization full-group action. Our use of this renormalization process can be viewed, in a sense, as replacing the computation of the ratio set of groups with a notion of past and future, like the shift, with the computation of the ratio set of some symmetry group.

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2 Preliminaries

In this work all the measurable spaces are standard Borel spaces and all the measures are Borel and sigma-finite. Two measures \( \nu \) and \( \mu \) on a standard Borel space \( X \) are called equivalent if each of \( \nu \) and \( \mu \) is absolutely continuous with respect to the other. An automorphism of a measurable space \( (X, \mu) \) is a bi-measurable invertible transformation \( V \) of \( X \) onto \( X \), which is nonsingular with respect to \( \mu \); that is, \( \mu \) and \( \mu \circ V^{-1} \) are equivalent measures. The automorphisms group of \( (X, \mu) \)
is denoted by \( \text{Aut}(X, \mu) \). When there is no confusion we write

\[
V'(x) = \frac{d\mu \circ V}{d\mu}(x) \in L^1(X, \mu), \quad V \in \text{Aut}(X, \mu).
\]

Let \( \Gamma \) be a countable group. We write \( \Gamma \curvearrowright (X, \mu) \) for a group homomorphism \( T : \Gamma \to \text{Aut}(X, \mu) \). When there is no confusion we write \( \gamma x \) for \( T(\gamma)(x) \). Such an action is called \textbf{ergodic} if for every Borel set \( E \subset X \), if \( \gamma E \subset E \) for all \( \gamma \in \Gamma \) then either \( \mu(E) = 0 \) or \( \mu(X\setminus E) = 0 \). It is called \textbf{conservative} if for every Borel set \( E \subset X \) with \( \mu(E) > 0 \) there exists \( \gamma \in \Gamma \) not the identity with \( \mu(E \cap \gamma E) > 0 \). Note that for a non-atomic measure, ergodicity is stronger than conservativeness. Also note that nonsingularity, ergodicity and conservativeness are invariant properties under equivalence of measures.

Let \( (X, \mu) \) be a non-atomic standard measure space and \( \Gamma \curvearrowright (X, \mu) \) be a nonsingular ergodic action. Suppose that there exists a measure \( \nu \) on \( X \) which is both absolutely continuous with respect to \( \mu \) and invariant under the action \( \Gamma \curvearrowright (X, \nu) \). Such measure \( \nu \) is called \textbf{a.c.i.m.} (absolutely continuous invariant measure) for \( \Gamma \curvearrowright (X, \mu) \). In that case the action is said to be of Krieger-type \( \Pi_1 \) or of Krieger-type \( \Pi_{\infty} \), depending on whether its a.c.i.m. is finite or infinite (this does not depend on the choice of the a.c.i.m. by the ergodicity). If the action does not admit an a.c.i.m. it is said to be of Krieger-type III.

\textbf{The full-group, orbital cocycles and essential values.} A \textbf{Borel equivalence relation} \( \mathcal{R} \) is a Borel subset of \( X \times X \) for which \( x \sim y \iff (x, y) \in \mathcal{R} \) is an equivalence relation. For a Borel set \( E \subset X \) we write \( \mathcal{R}(E) \) for the \( \mathcal{R} \)-saturation \( \{ y \in X : \exists x \in E, (x, y) \in \mathcal{R} \} \) of \( E \). For \( x \in X \) write \( \mathcal{R}(x) \) for \( \mathcal{R}([x]) \).

Such \( \mathcal{R} \) is called \textbf{countable} if \( \mathcal{R}(x) \) is a countable set for \( \mu \)-almost every \( x \in X \). It is called \textbf{nonsingular} if \( \mu(\mathcal{R}(E)) = 0 \) whenever \( \mu(E) = 0 \). A fundamental type of Borel countable equivalence relation is the \textbf{orbital equivalence relation} \( \mathcal{O}_\Gamma \) of a countable group action \( \Gamma \curvearrowright (X, \mu) \). This equivalence relation consists of all \( (x, \gamma x) \) for \( x \in X \) and \( \gamma \in \Gamma \). By the Feldman–Moore Theorem [15] every nonsingular countable Borel equivalence relation \( \mathcal{R} \) is the orbital equivalence relation of some (non-unique) countable group of automorphisms \( \text{FM}(\mathcal{R}) \curvearrowright (X, \mu) \).

The \textbf{full-group} \([\mathcal{R}]\) of \( \mathcal{R} \) consists of all \( V \in \text{Aut}(X, \mu) \) such that \( (x, Vx) \in \mathcal{R} \) for \( \mu \)-almost every \( x \in X \). The \textbf{pseudo full-group} \([\mathcal{R}]\) of \( \mathcal{R} \) consists of all nonsingular one-to-one Borel transformations \( V : D \to V(D) \) for some Borel domain \( D \subset X \), such that \( (x, Vx) \in \mathcal{R} \) for \( \mu \)-almost every \( x \in D \). We write \([\Gamma]\) and \([\Gamma]\) for \([\mathcal{O}_\Gamma]\) and \([\mathcal{O}_\Gamma]\), respectively. An \textbf{orbital cocycle}, or simply \textbf{cocycle}, for a Borel equivalence relation \( \mathcal{R} \) is a function \( \varphi : \mathcal{R} \to \mathbb{R} \) for which
there exists $X_0 \subset X$ of $\mu$-full measure such that for all $(x, y), (y, z) \in (X_0 \times X_0) \cap \mathcal{R}$ we have that
\[ \varphi(x, z) = \varphi(x, y) + \varphi(y, z). \]

We write $\varphi_V(x) = \varphi(x, Vx)$ for every $V \in [[\mathcal{R}]]$ and $x \in X_0$. For a nonsingular Borel equivalence relation $\mathcal{R}$ on $(X, \mu)$ there is a fundamental orbital cocycle called the (log) Radon–Nikodym cocycle. This can be defined for every choice of $\Gamma = \text{FM}(\mathbb{R})$ by
\[ \varphi_\gamma(x) = \log \frac{d\mu \circ \gamma}{d\mu}(x) \in L^1(X, \mu), \quad \gamma \in \Gamma. \]

This definition does not depend on the choice of $\text{FM}(\mathbb{R})$ up to a $\mu$-null set.

A number $r \in \mathbb{R}$ is called an essential value for $\hat{\mathcal{W}} \rtimes (X, \mu)$, if for every Borel set $E$ with $\mu(E) > 0$ and every $\epsilon > 0$ there exists $V \in [[\Gamma]]$ such that
\[ \mu(E \cap V^{-1}E \cap \{|\varphi_V - r| < \epsilon\}) > 0. \]

The following lemma is useful to compute essential values. It can be found in several formulations in [7, Lemma 2.1], [12, Lemma 1.1], [26, Lemma 7].

**Lemma 2.1.** Let $\Gamma \rtimes (X, \mu)$ be a countable group of automorphisms and let $\varphi$ be its Radon–Nikodym cocycle. Let $\mathcal{C}$ be a $\mu$-dense countable algebra in the Borel sigma-algebra. Then a number $r \in \mathbb{R}$ is an essential value for $\Gamma \rtimes (X, \mu)$ if there exists $\eta > 0$ depending only on $r$, such that the following condition holds:

For every $\epsilon > 0$ and every $C \in \mathcal{C}$ with $\mu(C) > 0$ there exists $F \subset C$ and $V \in [[\Gamma]]$, such that $V : F \to V(F) \subset C$ and $\mu(F) \geq \eta \mu(C)$ and $|\varphi_V(x) - r| < \epsilon$ for all $x \in F$.

**Krieger’s ratio set and the Maharam extension** The collection of all essential values for the Radon–Nikodym cocycle of $\Gamma \rtimes (X, \mu)$ is called the Krieger ratio set following [29] (see also Schmidt’s monograph [34, Chapter 3]), or simply the ratio set, and is denoted by $\text{e}(\Gamma, \mu)$. When there is no confusion we write $\text{e}(\Gamma)$ for $\text{e}(\Gamma, \mu)$. Observe that $\text{e}(\Gamma, \nu) = \text{e}(\Gamma, \mu)$ whenever $\nu$ and $\mu$ are equivalent measures. It is well-known that the ratio set is not empty if, and only if, the action is conservative, and that the ratio set is a closed additive subgroup of $\mathbb{R}$. Hence, the ratio set of a conservative action is one of the following:

\[ \{0\}, \mathbb{R}, \text{ or } \{n \log \lambda : n \in \mathbb{Z}\} \text{ for some } 0 < \lambda < 1. \]

The ratio set has been defined by Krieger in order to classify nonsingular ergodic actions of type III into types $\text{III}_\lambda$, $0 \leq \lambda \leq 1$ as follows: Type $\text{III}_0$ corresponds
to the ratio set that contains, in an appropriate sense, infinite values, and we do not deal with this here; type $\mathrm{III}_1$ corresponds to the ratio set $e(\Gamma, \varphi) = \mathbb{R}$; and type $\mathrm{III}_\lambda$ for $0 < \lambda < 1$ corresponds to the ratio set $e(\Gamma, \varphi) = \{ n \log \lambda : n \in \mathbb{Z} \}$ for $0 < \lambda < 1$, respectively. For more information on the ratio set and its role as an invariant of orbital equivalence we refer to [21, 19, 23].

Let $\Gamma \curvearrowright (X, \mu)$ be a countable group of automorphisms. Consider the space $\tilde{X} = X \times \mathbb{R}$ with the measure $d\tilde{\mu}(x, t) = d\mu(x)\exp(t)dt$. The Maharam extension of $\Gamma \curvearrowright (X, \mu)$ is the action of $\Gamma$ on $(\tilde{X}, \tilde{\mu})$ defined by

$$
\tilde{\gamma}(x, t) := \left( \gamma x, t - \log \frac{d\mu \circ \gamma}{d\mu}(x) \right), \quad \gamma \in \Gamma.
$$

The Maharam extension is an infinite sigma-finite measure-preserving action and we denote this action by $\tilde{\Gamma} \curvearrowright (\tilde{X}, \tilde{\mu})$. By a well-known theorem of Maharam (for transformations) [32], [1, Chapter 3.4] and Schmidt (for general countable groups) [34, Theorem 5.5], the Maharam extension of a conservative action is conservative. The Maharam extension of an ergodic countable group of automorphisms $\Gamma \curvearrowright (X, \mu)$ is itself ergodic if, and only if, $\Gamma \curvearrowright (X, \mu)$ is of type $\mathrm{III}_1$ [34, Corollary 5.4], [1, Corollary 8.2.5].

3 Notations and asymptotic symbols

We use the following common notations and abbreviations. The function $\operatorname{sign}(x)$ is $+1$ if $x$ is a non-negative number and $-1$ if $x$ is a negative number. For a random variable $Y$ with distribution $\mu$ we write $E_\mu(Y)$ for its mean and $V_\mu(Y)$ for its variance. We abbreviate the mean by $E(Y)$ and the variance by $V(Y)$ when there is no confusion. For an SFT $X_A$ and a set $I \subset \mathbb{Z}$ we write $\sigma(X_n : n \in I)$ for the sigma-algebra generated by cylinders supported on the coordinates of $I$. The operation $*$ will be used for concatenation of finite sequences as follows. For finite sequences $B = (b_1, \ldots, b_L)$ and $B' = (b'_1, \ldots, b'_{L'})$ we let $B * B'$ be the finite sequence $(b_1, \ldots, b_L, b'_1, \ldots, b'_{L'})$.

For a sequence $(a_n : n \geq 1)$ in a metric space $\mathbb{M}$, we denote by $\mathcal{L}(a_n : n \geq 1)$ the set of all partial limits of $(a_n : n \geq 1)$ in $\mathbb{M}$.

We use asymptotic symbols similar to the Vinogradov notations as follows. For sequences $(a_n : n \geq 1)$ and $(b_n : n \geq 1)$ of numbers write

$$
a_n \ll b_n \iff \exists C > 0 \text{ with } |a_n| \leq C|b_n| \text{ for all } n \geq 1.
$$

Write also

$$
a_n \asymp b_n \iff a_n \ll b_n \text{ and } b_n \ll a_n.
$$
This defines an equivalence relation on sequences of numbers.

We use extensively the basic approximation

$$\frac{a - b}{a} < \log(a/b) < \frac{a - b}{b}$$

for all $a, b > 0$.

Restricting ourselves to numbers in an interval $[c, C]$ for $0 < c < C < \infty$, one can derive that for sequences $(a_n : n \geq 1)$ and $(b_n : n \geq 1)$ contained in $[c, C]$,

(3.0.1) \[ \log(a_n/b_n) \approx a_n - b_n. \]

In particular, if for all $n \geq 1$ we have $c \leq b_n \leq a_n \leq C$ then

$$\sum_{n \geq 1} \log(a_n/b_n) = \infty \iff \sum_{n \geq 1} (a_n - b_n) = \infty.$$

## 4 Renormalization and The Hopf argument

Let $(X, \mu)$ be a standard measure space, $G$ be a countable group and $T_G \bowtie (X, \mu)$ be an action of $G$ by automorphisms. Fix a metric $d$ on $X$ that induces its standard Borel structure. For a Borel countable equivalence relation $R \subset X \times X$ we say that an element $(x, y) \in R$ is an asymptotic pair for $T_G$ if

$$d(T_g(x), T_g(y)) \to_\infty 0.$$ 

The collection $\mathcal{R}(T_G) \subset R$ of all asymptotic pairs for $T_G$ is a Borel sub-equivalence relation of $R$ and in particular it is countable. Note that this is the notion of equivalence underlying the well-known Hopf Argument [10]. The term “asymptotic pair” is the common name for the analogous notion in topological dynamics [5, 8].

Let $\Gamma \bowtie (X, \mu)$ be another countable group of automorphisms. We say that $\Gamma$ is asymptotic for $T_G$ if $(x, \gamma x) \in O_\Gamma$ is an asymptotic pair for $T_G$ for $\mu$-a.e. $x \in X$ and every $\gamma \in \Gamma$.

**Definition 4.1** (Renormalization Full-Group). Let $T_G$ and $\Gamma$ be actions of countable groups of automorphisms and suppose that $\Gamma$ is asymptotic for $T_G$. Consider the Maharam extensions $\tilde{T}_G \bowtie (\tilde{X}, \tilde{\mu})$ and $\tilde{\Gamma} \bowtie (\tilde{X}, \tilde{\mu})$. Define the renormalization full-group of $T_G$ with respect to $\Gamma$ to be

$$\mathcal{R}(T_G; \Gamma) := \{ V \in [\Gamma] : ((x, t), \tilde{V}(x, t)) \in O_{\tilde{\Gamma}}(\tilde{T}_G) \text{ for } \tilde{\mu}\text{-a.e. } (x, t) \in \tilde{X} \},$$

where we take the metric on $\tilde{X}$ to be the product of the chosen metric $d$ on $X$ with the standard distance on $\mathbb{R}$. In a similar manner, we define the renormalization pseudo full-group to be the set of all elements in $[[\Gamma]]$ satisfying the same condition as in the renormalization full-group. We will abbreviate the renormalization pseudo full-group by $\mathcal{R}(T_G; [[\Gamma]])$. 
There is a simple description of this object. As $\Gamma$ is asymptotic for $T_G$ the equivalence relation $\mathcal{O}_T(T_G) \subset \mathcal{O}_\Gamma$ consists of all $((x, t), (\tilde{V}(x, t)) \in \mathcal{O}_\Gamma$ for some $V \in [\Gamma]$ such that

$$(t - \log(T_g)'(x)) - (t - \log V(x)) - \log(T_g)'(Vx))$$

$$= \log \frac{(T_g \circ V)'(x)}{(T_g)'(x)} \xrightarrow{g \to \infty} 0.$$}

The occurrence of this condition does not depend on the second variable $t$. We then see that an element $V \in [\hat{\mathcal{W}}]$ belongs to $\mathcal{R}(T_G; \hat{\mathcal{W}})$ if, and only if,

$$\frac{(T_g \circ V)'(x)}{(T_g)'(x)} \xrightarrow{g \to \infty} 1 \quad \text{for } \mu\text{-a.e. } x \in X.$$}

To see that $\mathcal{R}(T_G; \Gamma)$ is a subgroup, recall that by the chain rule

$$\frac{(T_g \circ V^{-1})'(x)}{(T_g)'(x)} = \frac{(T_g)'(V^{-1}x)}{(T_g \circ V)'(V^{-1}x)}, \quad g \in G, V \in [\Gamma],$$

and

$$\frac{(T_g \circ V \circ W)'(x)}{(T_g)'(x)} = \frac{(T_g \circ V)'(Wx)}{(T_g)'(Wx)} \frac{(T_g \circ W)'(x)}{(T_g)'(x)}, \quad g \in G, V, W \in [\Gamma].$$

This also shows that the renormalization pseudo full-group is a pseudo group.

The renormalization full-group may be uncountable, but its orbital equivalence relation $\mathcal{O}_{\mathcal{R}(T_G; \Gamma)}$ is countable as a sub relation of $\mathcal{O}_\Gamma$. It is also Borel since

$$(4.1.1) \quad \mathcal{O}_{\mathcal{R}(T_G; \Gamma)} = \bigcup_{\gamma \in \Gamma} O_\gamma,$$

where $O_\gamma$ for $\gamma \in \Gamma$ is the set of all $(x, \gamma x) \in X \times X$ for which

$$\frac{(T_g \circ \gamma)'(x)}{(T_g)'(x)} \xrightarrow{g \to \infty} 1.$$}

Then by the Feldman–Moore Theorem we can consider the ratio set $e(\mathcal{R}(T_G; \Gamma))$.

**Example 4.2** (Bernoulli Shift). Let $X = S^Z$ for some finite set $S$ and suppose that $G = \mathbb{Z}$ acting by the shift $T$ and that $\Gamma = \Pi$ is the group of all permutations of $\mathbb{Z}$ that change only finitely many elements. This group acts naturally on $(X, \mu)$ by letting $\pi \in \Pi$ be the automorphism defined by $(\pi x)_n = x_{\pi(n)}$ for all $n \in \mathbb{Z}$ and $x \in X$. It is clear that $\Gamma$ is asymptotic for $T$ for the metric

$$d(x, y) = 2^{-\inf \{n \in \mathbb{N} : x_n \neq y_n\}}.$$
Consider a product measure $\mu = \prod_{n \in \mathbb{Z}} \mu_n$ on $X$ and suppose that $\mu$ satisfies the Doeblin condition (D). We claim that if the shift is nonsingular with respect to $\mu$ then the renormalization full-group $\mathcal{R}(T; \Pi)$ of the shift with respect to the finite permutations is $[\Pi]$ itself. First note that the shift satisfies $(T^n)\prime(x) = \prod_{k \in \mathbb{Z}} \frac{\mu_{k-n}(x_k)}{\mu_k(x_k)}, \quad n \in \mathbb{Z}.$

Let $V := V_{a,b} \in [\Pi]$ for some $a, b \in \mathbb{Z}$ be the transposition defined by $(Vx)_a = x_b$, $(Vx)_b = x_a$ and $(Vx)_k = x_k$ for any other $k \in \mathbb{Z}$. Then we have the formula $V'(x) = \frac{\mu_a(x_b)\mu_b(x_a)}{\mu_a(x_a)\mu_b(x_b)}$, so one can see that $(T^n \circ V)\prime(x) = (T^n)\prime(Vx) = \frac{\mu_{a-n}(x_b)\mu_{b-n}(x_a)}{\mu_{a-n}(x_a)\mu_{b-n}(x_b)}.$

Assuming that the shift is nonsingular with respect to $\mu$, we later see in Corollary 5.3 that it satisfies $\mu_n(s) - \mu_{n-1}(s) \xrightarrow{|n| \to \infty} 0$ for all $s \in S$.

Using the Doeblin condition we conclude that $\mu_n(s)/\mu_{n-1}(s) \xrightarrow{|n| \to \infty} 1$ for all $s \in S$ which implies that $\mu_{a-n}(s)/\mu_{b-n}(s) \xrightarrow{|n| \to \infty} 1 \quad \text{for fixed } a, b \in \mathbb{Z}.$

This shows that $V \in \mathcal{R}(T; \Pi)$.

**Theorem 4.3** (The Hopf Argument for the Maharam Extension). Let $(X, \mu)$ be a standard Borel probability space. Let $G$ be a countable amenable group and let $T_G \curvearrowright (X, \mu)$ be a conservatives action of $G$ by automorphisms. Let $\hat{W} \curvearrowright (X, \mu)$ be a countable group of automorphisms and assume that $\hat{W}$ is asymptotic for $T_G$.

If the renormalization full-group satisfies $e(\mathcal{R}(T_G; \hat{W})) = \mathbb{R}$ then also $e(T_G) = \mathbb{R}$.

In particular, if $T_G$ is ergodic and $e(\mathcal{R}(T_G; \hat{W})) = \mathbb{R}$ then $T_G$ is of type $\text{III}_1$.

**Remark 4.4.** The renormalization process can be described in terms of equivalence relations using (4.1.1) in a similar way introduced by Danilenko in [11]. There can be found a translation of the following Lemma 4.5 to [11, Theorem 2.3] under appropriate assumptions, by passing to an equivalent probability measure. For the sake of completeness we present here a self-contained proof.

We formulate two lemmas that together imply Theorem 4.3. The first lemma is a refinement of [3, Lemma 3.1].
Lemma 4.5. Let \( T_G \) and \( \Gamma \) be as in Theorem 4.3. Then every \( \hat{T}_G \)-invariant \( L^1(\hat{X}, \hat{\mu}) \)-function is also \( \hat{\mathcal{R}}(T_G; \Gamma) \)-invariant.

The second lemma is a refinement of the well-known fact that an ergodic action is of type III₁ if, and only if, its Maharam extension is ergodic.

Lemma 4.6. Let \( \Gamma \sim (X, \mu) \) be a countable group of automorphisms on a standard Borel probability space. Then \( e(\Gamma) = \mathbb{R} \) if, and only if, every \( \hat{\Gamma} \)-invariant function \( F \in L^1(\hat{X}, \hat{\mu}) \) is of the form \( F(x, t) = f(x) \) for \( \hat{\mu} \)-a.e. \( (x, t) \in \hat{X} \).

Of course, when \( \Gamma \) is ergodic then the only such functions are the constant functions.

Proof of Theorem 4.3 assuming Lemmas 4.5 and 4.6. By Lemma 4.6 we need to show that if every \( \hat{\Gamma} \)-invariant function \( F : \hat{X} \to \mathbb{R} \) is of the form \( F(x, t) = f(x) \) in \( L^1(\hat{\mu}) \), then the same holds for every \( \hat{T}_G \)-invariant function. This is straightforward from Lemma 4.5. \( \square \)

Proof of Lemma 4.5. The main idea of the proof is similar to that of [3, Lemma 3.1]. Denote by \( \tilde{\mathcal{J}} \) the sigma-algebra of Borel \( \hat{T}_G \)-invariant sets. We show that for every \( V \in \tilde{\mathcal{R}}(T_G; \Gamma) \) there exists a positive function \( v(x) \), such that for every \( F \in L^1(\tilde{X}, \tilde{\mu}) \) we have that

\[
(4.6.1) \quad v(x)^{-1}E(F(x, t) \mid \tilde{J}) \leq E(F \mid \tilde{J}) \circ \hat{V}(x, t) \leq v(x)E(F(x, t) \mid \tilde{J})
\]

for \( \hat{\mu} \)-a.e. \( (x, t) \in \hat{X} \). It will follow that if \( A \in \tilde{\mathcal{J}} \) so that the function \( F(x, t) = 1_A(x, t) \) is \( \hat{T}_G \)-invariant, then for every \( V \in \tilde{\mathcal{R}}(T_G; \Gamma) \) we have

\[
v(x)^{-1}F(x, t) \leq F(\hat{V}(x, t)) \leq v(x)F(x, t)
\]

for \( \hat{\mu} \)-a.e. \( (x, t) \in \hat{X} \). Since \( F \) is taking only the values 0 and 1 it will follow that \( F(\hat{V}(x, t)) = F(x, t) \) so that \( A \) is \( \hat{V} \)-invariant. Now every \( \hat{T}_G \)-invariant \( L^1(\tilde{X}, \tilde{\mu}) \)-function is an \( L^1(\tilde{X}, \tilde{\mu}) \)-limit and hence \( \hat{\mu} \)-a.e. limit of a sequence of \( \hat{T}_G \)-invariant simple functions, so proving (4.6.1) will finish the proof of Lemma 4.5.

We prove (4.6.1). Instead of the usual infinite measure \( \hat{\mu} \) on \( \hat{X} \) we consider the equivalent probability measure \( \hat{\nu} \) on \( \hat{X} \) defined by

\[
d\hat{\nu}(x, t) = d\mu(x)e(t)dt, \quad \text{where } e(t) := \exp(-|t|)/2.
\]

Then the Maharam extension \( \hat{T}_G \) is nonsingular and conservative with respect to \( \hat{\nu} \) as well. We also have that \( L^1(\hat{X}, \hat{\mu}) \subset L^1(\hat{X}, \hat{\nu}) \), so it is enough to prove (4.6.1) for \( L^1(\hat{X}, \hat{\nu}) \). Consider the class \( \mathcal{L} \subset L^1(\hat{X}, \hat{\nu}) \) of functions of the form \( L(x, t) = \phi(x)\varphi(t) \), where \( \phi \in L^1(X, \mu) \) and \( \varphi \in L^1(\mathbb{R}, e(t)dt) \) are both uniformly
continuous bounded functions satisfying \( \inf_{x \in X} \phi(x) > 0 \) and \( \inf_{t \in \mathbb{R}} \varphi(t) > 0 \). Then the linear space generated by \( \mathcal{L} \) is dense in \( L^1(\tilde{X}, \tilde{\mu}) \). Passing to subsequences converging \( \tilde{\mu} \)-a.e. and using the continuity of the conditional expectation, it is enough to establish (4.6.1) for the functions of \( \mathcal{L} \).

Given any \( L(x, t) = \phi(x) \varphi(t) \in \mathcal{L} \), by the ratio ergodic theorem of [11, Theorem 0.4] for the Maharam extension \( \tilde{T}_g \), there exists for \( L \) an increasing sequence of finite sets \( G_1 \subset G_2 \subset \cdots \) whose union is \( G \), such that

\[
\text{(4.6.2)} \quad E(L(x, t) \mid \tilde{\mathcal{G}}) = \lim_{N \to \infty} \frac{\sum_{g \in G_N} \tilde{T}_g'(x, t)L(\tilde{T}_g(x, t))}{\sum_{g \in G_N} \tilde{T}_g'(x, t)}
\]

and

\[
\text{(4.6.3)} \quad E(L \mid \tilde{\mathcal{G}}) \circ \tilde{V}(x, t) = \lim_{N \to \infty} \frac{\sum_{g \in G_N} \tilde{T}_g'(\tilde{V}(x, t))L(\tilde{T}_g(\tilde{V}(x, t)))}{\sum_{g \in G_N} \tilde{T}_g'(\tilde{V}(x, t))}
\]

for \( \tilde{\mu} \)-a.e. \((x, t) \in \tilde{X}\), where here and in the rest of this proof the notation \( \tilde{T}_g' \) refers to the Radon–Nikodym derivative with respect to \( \tilde{\mu} \).

We first claim that

\[
\text{(4.6.4)} \quad \frac{L(\tilde{T}_g(\tilde{V}(x, t)))}{L(\tilde{T}_g(x, t))} = \frac{\phi(\tilde{T}_g(Vx))}{\phi(\tilde{T}_g(x))} \frac{\varphi(t - \log(\tilde{T}_g \circ \tilde{V}(x)))}{\varphi(t - \log \tilde{T}_g'(x))} \xrightarrow{g \to \infty} 1
\]

for \( \tilde{\mu} \)-a.e. \((x, t) \in \tilde{X}\). The first factor converges to 1 as \( g \to \infty \) for \( \mu \)-a.e. \( x \in X \), since \((x, Vx)\) is an asymptotic pair and by the choice of \( \phi \). The second factor also converges to 1 as \( g \to \infty \) for \( \tilde{\mu} \)-a.e. \((x, t) \in \tilde{X}\), since \( V \in \mathcal{R}(T_G; \Gamma) \) and by the choice of \( \varphi \).

We also claim that there are positive functions \( v_0(x) \) and \( v_1(x) \) depending only on \( V \), such that

\[
\text{(4.6.5)} \quad v_0(x) \leq \lim_{g \to \infty} \frac{\tilde{T}_g'(\tilde{V}(x, t))}{\tilde{T}_g'(x, t)} \leq v_1(x)
\]

for \( \tilde{\mu} \)-a.e. \((x, t) \in \tilde{X}\). The reason for that is the following. It is straightforward to verify that the Radon–Nikodym derivatives with respect to \( \tilde{\mu} \) take the form

\[
\frac{\tilde{T}_g'(\tilde{V}(x, t))}{\tilde{T}_g'(x, t)} = \frac{T_g'(Vx) e(t - \log T_g'(Vx))}{T_g'(x) e(t - \log T_g'(x))} \quad \text{for} \quad \tilde{\mu} \text{-a.e. } (x, t) \in \tilde{X}.
\]

Since \( V \in \mathcal{R}(T_G; \Gamma) \) we have that

\[
\frac{T_g'(Vx)}{T_g'(x)} \xrightarrow{g \to \infty} (V'(x))^{-1} \quad \text{for } \mu \text{-a.e. } x \in X.
\]
Using the bound $\exp(|s|)^{-1} \leq e(t+s)/e(t) \leq \exp(|s|)$ we have that
\[
\frac{e(t - \log T_g'(Vx))}{e(t - \log T_g'(x))} \leq \exp \left( \left| \log \frac{T_g'(Vx)}{T_g'(x)} \right| \right) \xrightarrow{g \to \infty} \exp(| \log V'(x)|),
\]
and that
\[
\frac{e(t - \log T_g'(Vx))}{e(t - \log T_g'(x))} \geq \exp \left( \left| \log \frac{T_g'(Vx)}{T_g'(x)} \right| \right)^{-1} \xrightarrow{g \to \infty} \exp(| \log V'(x)|)^{-1}.
\]
Then (4.6.5) holds for
\[
v_0(x) := (V'(x))^{-1} \exp(| \log V'(x)|)^{-1} \quad \text{and} \quad v_1(x) := (V'(x))^{-1} \exp(| \log V'(x)|).
\]
Recall that by the conservativeness of $\tilde{T}_G$ we have
\[
\lim_{N \to \infty} \sum_{g \in G_N} \tilde{T}_g(x, t) = \infty \quad \text{for } \tilde{\mu}\text{-a.e. } (x, t) \in \tilde{X},
\]
so that plugging (4.6.4) and (4.6.5) into (4.6.3), we obtain that (4.6.1) holds for the positive function $v(x) = v_1(x)/v_0(x)$. $\square$

Before we prove Lemma 4.6, let us mention some basic facts about the ergodic decomposition and its relation to the Maharam extension. Let $\Gamma \curvearrowright (X, \mu)$ be a countable group of automorphisms. As described by Bowen following Zimmer [6], the ergodic decomposition of this action is the standard Borel space $(E, \nu)$, where $E$ is the space of all $\tilde{\Gamma}$-nonsingular $\tilde{\Gamma}$-ergodic probability measures on $X$ and $\nu$ is a Borel measure on $E$ that satisfies
\[
\mu(\cdot) = \int \kappa(\cdot) d\nu(\kappa).
\]
Moreover, there exists a regular choice of Radon–Nikodym cocycle with respect to the ergodic decomposition in the following sense. There exists a Borel function $\varphi_{\gamma}(x) : \Gamma \times X \to \mathbb{R}$ that satisfies the cocycle identity
\[
(4.6.6) \quad \varphi_{\gamma_1}(x) + \varphi_{\gamma_2}(\gamma_1 x) = \varphi_{\gamma_1 \gamma_2}(x), \quad \forall \gamma_1, \gamma_2 \in \Gamma, \forall x \in X,
\]
such that
\[
(4.6.7) \quad \varphi_{\gamma}(x) = \log \frac{d\kappa \circ \gamma}{d\kappa}(x), \quad \forall \gamma \in \Gamma, \nu\text{-a.e. } \kappa \in \mathcal{E}, \kappa\text{-a.e. } x \in X.
\]
Fix such a cocycle and a corresponding $\nu$-full measure set $\mathcal{E}_0 \subset \mathcal{E}$. Consider the Maharam extension $\tilde{\Gamma} \curvearrowright (\tilde{X}, \tilde{\mu})$ and the Maharam extensions $\tilde{\Gamma} \curvearrowright (\tilde{X}, \tilde{\kappa})$ for the ergodic components $\kappa \in \mathcal{E}_0$. Note that the property
\[
(4.6.8) \quad \tilde{\mu}(\cdot) = \int \tilde{\kappa}(\cdot) d\nu(\kappa)
\]
can be easily verified on sets in $\mathcal{B}(X) \times \mathcal{B}(\mathbb{R})$ using the Fubini theorem, thus it holds for all Borel sets of $\tilde{X}$.

**Proof of Lemma 4.6.** We prove the Lemma for indicator functions, showing that $e(\Gamma) = \mathbb{R}$ if, and only if, every $\tilde{\Gamma} \sim (\tilde{X}, \tilde{\mu})$-invariant set $E \subset \tilde{X}$ with $\tilde{\mu}(E) > 0$ is of the form $E = E' \times \mathbb{R}$ mod $\tilde{\mu}$ for some $E' \subset X$.

One implication is standard: Suppose that $r \in \mathbb{R} \setminus e(\Gamma, \mu)$. Then there exist $\epsilon > 0$ and some $E_0 \subset X$ with $\mu(E_0) > 0$, such that $-\log \gamma'(x) \notin (r, r + \epsilon)$ for $\mu$-a.e. $x \in E_0$ for every $\gamma \in \Gamma$. Consider the $\tilde{\Gamma} \sim (\tilde{X}, \tilde{\mu})$-invariant set

$$E := \bigcup_{\gamma \in \Gamma} \tilde{\gamma}(E_0 \times (0, \epsilon/2)) \subset \tilde{X}.$$ 

Note that for $\tilde{\mu}$-a.e. $(x, t) \in E_0 \times (0, \epsilon/2)$ and every $\gamma \in \Gamma$,

$$\text{proj}_{\mathbb{R}}(\tilde{\gamma}(x, t)) = t - \log \gamma'(x) \notin (r + \epsilon/2, r + \epsilon).$$

Hence $\text{proj}_{\mathbb{R}}(E) \cap (r + \epsilon/2, r + \epsilon) = \emptyset$ so that $E$ can not be of the form $E = E' \times \mathbb{R}$.

For the other implication, assume that $e(\Gamma, \mu) = \mathbb{R}$ and let $E \subset \tilde{X}$ be a $\tilde{\Gamma} \sim (\tilde{X}, \tilde{\mu})$-invariant Borel set. For $x \in X$ let $E_x = \{t \in \mathbb{R} : (x, t) \in E\}$. Let $\lambda$ be the Lebesgue measure on $\mathbb{R}$. Fix a cocycle $\phi$ satisfying (4.6.6) and (4.6.7) and a corresponding $\nu$-full measure set of ergodic components $\mathcal{E}_0 \subset \mathcal{E}$. Note that since $E$ is $\tilde{\Gamma} \sim (\tilde{X}, \tilde{\mu})$-invariant, using formula (4.6.8) we have that

$$0 = \tilde{\mu}(E \triangle \tilde{\gamma}^{-1}E) = \int_{\mathcal{E}_0} \tilde{\kappa}(E \triangle \tilde{\gamma}^{-1}E)d\nu(\kappa), \quad \gamma \in \Gamma,$$

so for every $\gamma \in \Gamma$ there is a $\nu$-full measure set $\mathcal{E}_\gamma \subset \mathcal{E}_0$ such that $\tilde{\kappa}(E \triangle \tilde{\gamma}^{-1}E) = 0$ for every $\kappa \in \mathcal{E}_\gamma$. Considering the $\nu$-full measure set $\mathcal{E}_1 := \bigcap_{\gamma \in \Gamma} \mathcal{E}_\gamma \subset \mathcal{E}_0$, we get that $E$ is $\tilde{\Gamma} \sim (\tilde{X}, \tilde{\kappa})$-invariant for every $\kappa \in \mathcal{E}_1$.

By Bowen’s theorem [6, Theorem 2.1] and our assumption, the ratio set of $\nu$-a.e. $\kappa \in \mathcal{E}$ satisfies $e(\Gamma, \kappa) = e(\Gamma, \mu) = \mathbb{R}$. Let $\mathcal{E}_2 \subset \mathcal{E}_1$ be a $\nu$-full measure set satisfying this property. By Schmidt’s theorem [34, Theorem 5.2], [1, Theorem 8.2.4], the ratio set of an ergodic action is the same as its *periods set*, which means that in our case for every $\kappa \in \mathcal{E}_2$,

$$\mathbb{R} = e(\Gamma, \kappa) = \{r \in \mathbb{R} : S_rF = F \mod \tilde{\kappa}, \forall \tilde{\Gamma} \sim (\tilde{X}, \tilde{\kappa})\text{-invariant set } F \subset \tilde{X}\},$$

where for $r \in \mathbb{R}$, $S_r(x, t) = (x, t + r)$. Then for every $\kappa \in \mathcal{E}_2$ we have that $S_rE = E \mod \tilde{\kappa}$ for every $r \in \mathbb{R}$, hence $E_x = E_x - r \mod \lambda$ for every $r \in \mathbb{R}$. As the Lebesgue measure $\lambda$ is translation-invariant, it has the property that every pair of positive Lebesgue measure sets $A_0$ and $A_1$ admits a positive length interval $I$.
such that $\lambda(A_0 \cap (A_1 - r)) > 0$ for every $r \in I$, so it is impossible that both $\lambda(E_\kappa) > 0$ and $\lambda(\mathbb{R} \setminus E_\kappa) > 0$. That is, $E_\kappa \in \{\emptyset, \mathbb{R}\} \mod \lambda$ for $\kappa$-a.e. $x \in X$ for every $\kappa \in \mathcal{F}_2$ which is a $\nu$-full measure set. By the Fubini theorem and the above ergodic decomposition, this is equivalent to $E = E' \times \mathbb{R} \mod \tilde{\mu}$ where $E'$ is the set of all $x \in X$ for which $E_x = \mathbb{R} \mod \lambda$, so the proof is complete. \qed

5 Markov subshift of finite type (MSFT)

A Markov Subshift of Finite Type (Markov SFT or MSFT) is a measure space $(X, A, \mu)$ where $X$ is an SFT on a finite state space $S$ with adjacency matrix $A$, and $\mu$ is a Markov measure on $X$ which is compatible with $A$ in the sense that if the transition matrices of $\mu$ are $(P_n : n \in \mathbb{Z})$ then

$P_n(s, t) > 0 \iff A(s, t) = 1, \quad s, t \in S, n \in \mathbb{Z}.$

For a pair of integers $n < m$ denote

$P^{(n,m)} = P_n \cdots \cdots P_m,$

which is a row-stochastic $|S| \times |S|$-matrix that has the interpretation

$P^{(n,m)}(s, t) = \mu(X_{m+1} = t \mid X_n = s), \quad s, t \in S.$

Note that if $(X, \mu)$ is a topologically-mixing MSFT with $A^M > 0$ then

$P^{(n,n+M-1)}(s, t) > 0, \quad n \in \mathbb{Z}, s, t \in S.$

The following proposition will be used constantly in our work and we include its proof in Appendix A.

**Proposition 5.1.** Let $(X, \mu)$ be a topologically-mixing MSFT with $A^M > 0$, that satisfies the Doeblin condition (D) for $\delta > 0$. Let $(P_n : n \in \mathbb{Z})$ be the transition matrices of $\mu$ and let $(\pi_n : n \in \mathbb{Z})$ be the coordinate distributions of $\mu$. Then the following properties hold:

1. For every $n \in \mathbb{Z}$,

$\delta^M \leq \pi_n(s) \leq 1 - \delta^M, \quad s \in S.$

2. For every integer $N \geq M$ and $n \in \mathbb{Z}$,

$\delta^M \leq P^{(n,n+N)}(s, t) \leq 1 - \delta^M, \quad s, t \in S.$

3. There exists a constant $C(\delta, M) \in (0, 1)$ depending only on $\delta$ and $M$, such that for every $n, m \in \mathbb{Z}$ and every pair of Borel sets $E \in \sigma(\ldots, X_{n-1}, X_n)$ and $F \in \sigma(X_m, X_{m+1}, \ldots)$, if $m - n \geq M$ then

$C(\delta, M) \mu(E) \mu(F) \leq \mu(E \cap F) \leq C(\delta, M)^{-1} \mu(E) \mu(F).$
Here we establish a deterministic criteria for equivalence of Markov measures which will be fundamental to our work. The Hahn–Lebesgue decomposition of one Markov measure with respect to another is known but, unlike the Kakutani dichotomy in product measures, in general it is not a 0-1 event and there is no deterministic criterion to distinguish between the alternatives. For one-sided Markov chains some authors assumed tail triviality as well as other regularity assumptions to establish such deterministic criteria; see for instance [31, 30, 12]. Here we take a different approach, which under the assumption of the Doeblin condition makes no reference to tail triviality. We provide the proof of Theorem 5.2 as well as a detailed background in Appendix B.

**Theorem 5.2.** Let $\nu = \nu((P_n, n \in \mathbb{Z}))$ and $\mu = \mu((Q_n, n \in \mathbb{Z}))$ be Markov measures on a topologically-mixing SFT $X_A$, both satisfying the Doeblin condition (D). Then $\nu \ll \mu$ if, and only if,

$$\sum_{n \geq 1} \sum_{s, u, v, t \in S} d_n^2[\nu, \mu](s, u, v, t) < \infty,$$

where for $n \geq 1$ and $s, t, u, v \in S$ we denote the numbers

$$d_n^2[\nu, \mu](s, u, v, t) := \left( \sqrt{\hat{P}_n(u, s)P_n(v, t)} - \sqrt{\hat{Q}_n(u, s)Q_n(v, t)} \right)^2.$$

In particular, since

$$d_n^2[\nu, \mu](s, u, v, t) = d_n^2[\mu, \nu](s, u, v, t)$$

for all $n \geq 1$ and $s, u, v, t \in S$, it follows that

$$\nu \ll \mu \iff \mu \ll \nu$$

so that every such two measures are either equivalent or that none of them is absolutely continuous with respect to the other.

**Corollary 5.3.** Let $(X_A, \mu)$ be a topologically-mixing MSFT that satisfies the Doeblin condition. The coefficients for the nonsingularity of the shift $T$ are

$$d_n^2[\mu, \mu \circ T^{-1}](s, u, v, t) = \left( \sqrt{\hat{P}_n(u, s)P_n(v, t)} - \sqrt{\hat{P}_{-(n+1)}(u, s)P_{n+1}(v, t)} \right)^2.$$

Thus, when the shift is nonsingular, using the stochasticity of the matrices $P_n$ and $\hat{P}_n$ for all $n \in \mathbb{Z}$ we get by Theorem 5.2 that

$$P_{n-1}(v, t) - P_n(v, t) \xrightarrow{n \to \infty} 0, \quad v, t \in S$$

and

$$\hat{P}_{-n}(u, s) - \hat{P}_{-(n+1)}(u, s) \xrightarrow{n \to \infty} 0, \quad s, u \in S.$$
Corollary 5.4. Let \((X_A, \mu)\) be a MSFT and let \((P_n : n \in \mathbb{Z})\) be the sequence of transition matrices of \(\mu\). Then a necessary condition for \(\mu\) to be equivalent to a homogeneous Markov measure \(\nu\) defined by a matrix \(Q\) is that
\[
\lim_{|n| \to \infty} P_n = Q.
\]
If this holds, then \(\mu\) is equivalent to \(\nu\) if, and only if,
\[
\sum_{n \geq 1} \sum_{s,u,v,t} \left( \sqrt{P_n(u,s)P_n(v,t)} - \sqrt{Q(u,s)Q(v,t)} \right)^2 < \infty.
\]

5.1 Renormalization in MSFT. Let \((X_A, \mu)\) be a topologically-mixing MSFT that satisfies the Doeblin condition. Consider the action of \(G = \mathbb{Z}\) by the shift \(T : (X_A, \mu) \to (X_A, \mu)\). Let \(\Pi\) be the group of all permutations of \(\mathbb{Z}\) that change only finitely many coordinates, and consider the equivalence relation that consists of all \((x, y) \in X_A \times X_A\) for which \(y = \pi x\) for some \(\pi \in \Pi\). This is a Borel countable equivalence relation, so by the Feldman–Moore Theorem it is the orbital equivalence relation of a countable group \(\Pi_A\) of nonsingular automorphisms of \(X_A\).

The renormalization full-group \(R(T; \Pi_A)\) is usually a proper subgroup of \([\Pi_A]\). We write \(R_A\) and \([R_A]\) for the renormalization full-group \(R(T; \Pi_A)\) and for the renormalization pseudo full-group \(\hat{R}(T; [\Pi_A])\), respectively.

Here we identify a collection of elements of \([\Pi_A]\) inside \(\hat{R}(T; \Pi_A)\).

A block \(B\) in an SFT \(X_A\) is a finite sequence \(B = [b_1, \ldots, b_L]\) of symbols from \(S\), such that \(A(b_l, b_{l+1}) = 1\) for \(1 \leq l \leq L - 1\). For such \(B\) we write \(L = \text{Length}(B)\).

For a block \(B\) in \(X_A\) with \(L = \text{Length}(B)\) and for \(i \in \mathbb{Z}\), we have the corresponding cylinder
\[
B(i) := \{ x \in X_A : [x_i, \ldots, x_{i+L-1}] = B \} \subset X_A.
\]

A pair \((B, B')\) of two blocks in \(X_A\) is called an admissible pair in \(X_A\) with length \(L \geq 1\), and we write \(\text{Length}(B, B') = L\), if it satisfies the following properties.

(1) \(\text{Length}(B) = \text{Length}(B') = L\).
(2) \(B\) and \(B'\) have the same first symbol.
(3) \(B\) and \(B'\) have the same last symbol.

To avoid trivialities we always assume that \(B \neq B'\). In particular we always have \(\text{Length}(B, B') \geq 3\). An example for admissible pair \((B, B')\) in the Golden Mean SFT is the one of length \(L = 4\) defined by \(B = [0, 0, 0, 0]\) and \(B' = [0, 2, 1, 0]\).
Definition 5.5. Let $X_A$ be a topologically-mixing SFT with $A^M > 0$ for some $M \geq 1$. An **admissible configuration** in $X_A$ is a sequence $(B_k(i_k), B'_k(j_k)), k \geq 1$, built out of the following ingredients:

- A sequence $(B_k, B'_k), k \geq 1$, of admissible pairs in $X_A$ with some $L \geq 1$ such that

  $\text{Length}(B_k, B'_k) \leq L$ for all $k \geq 1$.

- A sequence $(j_k : k \geq 1)$ of positive integers with

  $j_{k+1} - j_k \geq L + M, \quad k \geq 1$.

- A sequence $(i_k : k \geq 1)$ of negative integers with

  $i_k - i_{k+1} \geq L + M, \quad k \geq 1$.

Definition 5.6. Let $\mu$ be a Markov measure on $X_A$ defined by $(P_n : n \in \mathbb{Z})$. Let $(B, B')$ be an admissible pair in $X_A$ and $i, j \in \mathbb{Z}$ with $|i - j| \geq \text{Length}(B, B')$. Denote

$E_{ij} := B(i) \cap B'(j)$ and $E'_{ij} := B'(i) \cap B(j)$.

We define two types of elements of the pseudo full-group $[\Gamma_A]$.  
- The corresponding **asymmetric admissible permutation** is of the form

  $V : E_{ij} \rightarrow E'_{ij}$

  and is defined to exchange the block $B$ in the coordinates $\{i, \ldots, i+L-1\}$ with the block $B'$ in the coordinates $\{j, \ldots, j + L - 1\}$. We write such element by

  $V : B(i) \rightleftharpoons B'(j) \in [\Gamma_A]$.

- The corresponding **symmetric admissible permutation** is of the form

  $V : E_{ij} \cup E'_{ij} \rightarrow E_{ij} \cup E'_{ij}$

  and is defined on $E_{ij}$ by $V : B(i) \rightleftharpoons B'(j)$ and on $E'_{ij}$ by $V : B'(i) \rightleftharpoons B(j)$. We write such element by

  $V : B(i) \circledast B'(j) \in [\Gamma_A]$.

Note that if we define admissible permutations to be the identity mappings outside of their domains, then symmetric admissible permutations remain one-to-one while asymmetric admissible permutations are no longer one-to-one.

Let us establish a notation. Given a Markov measure $\mu$ on $X_A$ defined by $(P_n : n \in \mathbb{Z})$, for a block $B = [b_1, b_2, \ldots, b_L]$ and $i \in \mathbb{Z}$ we write

$P_i(B) := P_i(b_1, b_2) \cdots P_{i+L-1}(b_{L-1}, b_L)$. 

The following formula is a direct computation using the properties of admissible permutations and of the Radon–Nikodym derivative.

**Claim 5.7.** For an asymmetric admissible permutation $V : B(i) \rightleftharpoons B'(j)$ we have

$$V'(x) = \frac{P_i(B')P_j(B)}{P_i(B)P_j(B')} \quad \text{for } x \in B(i) \cap B'(j).$$

In particular, $V'$ is taking exactly one value on $B(i) \cap B'(j)$ and this value depends only on the coordinates of its admissible pair.

Similarly, for a symmetric admissible permutation $V : B(i) \cap B'(j)$ we have

$$V'(x) = \begin{cases} \frac{P_i(B')P_j(B)}{P_i(B)P_j(B')} & \text{for } x \in B(i) \cap B'(j), \\ \frac{P_i(B)P_j(B')}{P_i(B)P_j(B')} & \text{for } x \in B'(i) \cap B(j). \end{cases}$$

For an admissible configuration $(B_k(i_k), B'_k(j_k)), k \geq 1$, we denote by $(D_k : k \geq 1)$ the sequence of numbers

$$D_k := \log \left( \frac{P_i(B'_k)P_j(B_k)}{P_i(B_k)P_j(B'_k)} \right) = P_i(B'_k)P_j(B_k) - P_i(B_k)P_j(B'_k),$$

where the approximation is by the approximation of the logarithm in (3.0.1).

**Lemma 5.8.** Let $(X_A, \mu)$ be a topologically-mixing MSFT that satisfies the Doeblin condition (D) and suppose that the shift is nonsingular with respect to $\mu$. Taking the metric on $X_A$ as in Example 4.2, we have that every symmetric admissible permutation belongs to $\mathcal{R}_A$. Similarly, every asymmetric admissible permutation belongs to $[[\mathcal{R}_A]]$.

**Proof.** Since any symmetric admissible permutation is defined by two asymmetric admissible permutations on disjoint domains, it is enough to consider only the asymmetric case. Let $V : B(i) \rightleftharpoons B'(j)$,

when $B = [b_1, b_2, \ldots, b_L]$ and $B' = [b'_1, b'_2, \ldots, b'_L]$ with $b_1 = b'_1$ and $b_L = b'_L$.

Note that for every $n \in \mathbb{Z}$ and $x \in B(i) \cap B'(j)$, $T^n(x)$ and $T^n(Vx)$ differ only in the coordinates $\{j-n, \ldots, j-n+L-1\}$ and $\{i-n, \ldots, i-n+L-1\}$, so since $b_1 = b'_1$ and $b_L = b'_L$ we have

$$\frac{(T^n \circ V)'(x)}{(T^n)'(x)} = \frac{d\mu \circ T^n \circ V}{d\mu \circ T^n}(x) = \prod_{l=1}^{L} \frac{P_{j-n+l-1}(b_l, b_{l+1})}{P_{j-n+l-1}(b'_l, b'_{l+1})} \cdot \frac{P_{i-n+l-1}(b'_l, b'_{l+1})}{P_{i-n+l-1}(b_l, b_{l+1})}. $$
By the Doeblin condition and Corollary 5.3, for every $s, t \in S$ with $A(s, t) = 1$ and every $1 \leq l \leq L$,
\[
\frac{P_{j-n+l-1}(s, t)}{P_{i-n+l-1}(s, t)} \to 1 + \frac{P_{j-n+l-1}(s, t) - P_{i-n+l-1}(s, t)}{[n] \to \infty}.
\]
As the length of the product is bounded by $L$ uniformly in $n$, this shows that
\[
\left( \frac{T^n \circ V'(x)}{T^n(x)} \right) \to 1 \quad \text{for } \mu\text{-a.e. } x \in X_A.
\]

6 Proof of the Divergent Scenario

Here we prove Theorem 2.

**Lemma 6.1.** For every admissible configuration $(B(i_k), B'(j_k)), k \geq 1$, the set $\mathcal{L}(D_k : k \geq 1)$ of partial limits of $(D_k : k \geq 1)$ that was defined in (5.7.1) is contained in the ratio set $e(\mathcal{R}_A)$. In particular, if this partial limits set contains a positive length interval, or at least two numbers independent over the rationals, then $e(\mathcal{R}_A) = \mathbb{R}$.

**Proof.** Let $r \in \mathcal{L}(D_k : k \geq 1)$. Let $0 < \epsilon < \min(\{|r|, \delta\})$, where $\delta > 0$ is the constant of the Doeblin condition. Let $E \in \sigma(X_k : |k| \leq N)$ for some $N \geq 1$. Fix some large $K \geq 1$ such that
\[
j_K > N + M, \quad i_K < -N - M \quad \text{and} \quad |D_k - r| < \epsilon,
\]
where $M$ is such that $A^M > 0$. Let
\[
F := B_K(i_K) \cap B'_K(j_K) \cap E \subset E
\]
and consider the asymmetric admissible permutation $V : B_K(i_K) \equiv B'_K(j_k)$. Then $V$ is a mapping of the form $V : F \to E$ and by Claim 5.7 it satisfies
\[
\log V'(x) = D_K \in (r - \epsilon, r + \epsilon) \quad \text{for } x \in F.
\]
Finally, since $j_K - N > M$ and $i_K + N < M$, we apply Proposition 5.1 twice to get
\[
\mu(F) \geq C(\delta, M)^2 \mu(B_K(i_K)) \mu(B'_K(j_K)) \mu(E)
\]
\[
\geq C(\delta, M)^2 \delta^{2(M+L)} \mu(E),
\]
where we used that in general, for every admissible block $B = [b_1, \ldots, b_L]$ and every $i \in \mathbb{Z}$, by Proposition 5.1 we have
\[
(6.1.1) \quad \mu(B(i)) = \pi_i(b_1)P_i(b_1, b_2) \cdots P_{i+L-1}(b_{L-1}, b_L) \geq \delta^{M+L}.
\]
This shows that the condition for extending $r$ to be an essential value of $e(\mathcal{R}_A)$ as in Lemma 2.1 is fulfilled for $\eta := C(\delta, M)^2 \delta^{2(M+L)} > 0$. \qed
Lemma 6.2. Let $S' \subset S \times S$ be some set with cardinality $d'$. Consider the set
$L(P'_n : n \geq 1)$ of partial limits of the sequence
\[ P'_n := (P_n(s, t) : (s, t) \in S') \in [\delta, 1 - \delta]^{d'}, \quad n \geq 1. \]
Then the image of every continuous real-valued function on $L(P'_n : n \geq 1)$ is a compact, connected set.

Proof. A continuous real-valued function $f$ on $L(P'_n : n \geq 1)$ satisfies
\[ f(L(P'_n : n \geq 1)) = L(f(P'_n) : n \geq 1). \]
By Corollary 5.3 we have
\[ d(P'_n, P'_{n-1}) \xrightarrow[n \to \infty]{} 0 \]
for some Euclidean metric $d$ on $[\delta, 1 - \delta]^{d'}$, and since $f$ is uniformly continuous we have
\[ f(P'_n) - f(P'_{n-1}) \xrightarrow[n \to \infty]{} 0. \]
Then the lemma follows from the following elementary fact. The partial limits set of a sequence $(p_n : n \geq 1)$ of numbers with the property $p_n - p_{n-1} \xrightarrow[n \to \infty]{} 0$ is a compact, connected set. □

Lemma 6.3. Let $S$ be a finite set. Let $P$ and $Q$ be a pair of different irreducible and aperiodic stochastic $|S| \times |S|$-matrices such that
\[ P(s, t) = 0 \iff Q(s, t) = 0, \quad s, t \in S. \]
Then there is $L \geq 1$ and a pair of elements $\alpha$ and $\beta$ in $S$, as well as a pair of finite paths $[b_1, \ldots, b_L]$ and $[b'_1, \ldots, b'_L]$ in $S$ that are admissible for $P$ (and $Q$), such that
\[ \frac{P(a, b_1) \cdots P(b_L, \beta)}{P(a, b'_1) \cdots P(b'_L, \beta)} \neq \frac{Q(a, b_1) \cdots Q(b_L, \beta)}{Q(a, b'_1) \cdots Q(b'_L, \beta)}. \]

Before the proof we establish some notations. For a matrix $P$ and a block $B = [b_1, \ldots, b_L]$ we write
\[ P(B) = P(b_1, b_2) \cdots P(b_{L-1}, b_L). \]
For a stochastic matrix $P$, consider the topologically-mixing SFT $X_A$ of $S^\mathbb{Z}$ where $A$ is the $\{0, 1\}$-valued $|S| \times |S|$-matrix defined by
\[ A(s, t) = 1 \iff P(s, t) > 0, \quad s, t \in S. \]
For such $A$ and $P$ denote by $\mu_P$ the homogeneous Markov measure on $X_A$ defined by $P$ and its stationary distribution. For every $\alpha$ and $\beta$ in $\mathcal{S}$ and every integer $n < m$, denote by $\mathbb{B}_A^{[n,m]}(\alpha, \beta)$ the finite collection of all $A$-admissible blocks on the coordinates $\{n, \ldots, m\}$, which take the form $[\alpha, s_{n+1}, \ldots, s_{m-1}, \beta]$ for some $s_{n+1}, \ldots, s_{m-1}$ in $\mathcal{S}$.

**Proof of Lemma 6.3.** Suppose toward a contradiction that the assertion in the lemma is false. This means that

\[(6.3.1) \quad P(B)Q(B') = Q(B)P(B') \quad \text{for every admissible pair } (B, B') \text{ in } X_A,\]

where $A$ is the adjacency matrix corresponding to $P$ (and $Q$). Consider the space $X_A \times X_A \subset S^Z \times S^Z$ with the Borel sigma-algebra $\mathcal{B}(X_A \times X_A)$ and let

$\mathfrak{A} = \{ (B(n) \times B'(n) : (B, B') \text{ is an admissible pair in } X_A \text{ and } n \in \mathbb{Z} \}$.

We consider the two trivial ways to define joining on $X_A \times X_A$:

$\mu_{(P,Q)} := \mu_P \otimes \mu_Q$ and $\mu_{(Q,P)} := \mu_Q \otimes \mu_P$.

Then assumption (6.3.1) means that $\mu_{(P,Q)} = \mu_{(Q,P)}$ on $\mathfrak{A}$.

**Claim 6.4.** *The product of the shifts*

$T \times T : X_A \times X_A \to X_A \times X_A$

*is ergodic with respect to* $\mu_{(P,Q)}$.

**Proof.** It is well-known [33, Corollary 1.1] that for an irreducible and aperiodic stochastic matrix $P$ the shift is (strongly-)mixing with respect to the Markov measure $\mu_P$. In our case, since the shift $T$ is mixing with respect to both $\mu_P$ and $\mu_Q$ it follows that $T \times T$ is ergodic with respect to $\mu_{(P,Q)}$. \qed

**Claim 6.5.** $\mathfrak{A}$ is generating $\mathcal{B}(X_A \times X_A)$ up to $\mu_{(P,Q)}$-null sets.

**Proof.** Consider a basic cylinder $C_0 \times C_1 \in \mathcal{B}(X_A) \times \mathcal{B}(X_A)$ supported on the coordinates $\{-N, \ldots, N\} \times \{-N, \ldots, N\}$ for some $N \geq 1$. Define stopping times $\tau_+$ and $\tau_-$ on $X_A \times X_A$ by

$\tau_+(x, y) = \inf\{ n > N : x_n = y_n \}$ and $\tau_-(x, y) = \inf\{ n > N : x_{-n} = y_{-n} \}$.

By Claim 6.4 $T \times T$ is ergodic with respect to $\mu_{(P,Q)}$, so by the pointwise ergodic theorem we have that

$\lim_{K\to\infty} \frac{1}{K} \sum_{k=0}^{K-1} 1_{[x_k=y_k]} = \mu_{(P,Q)}(x_0 = y_0) > 0$ for $\mu_{(P,Q)}$-a.e. $(x, y) \in X_A \times X_A$. 
This shows that $\tau_+ < \infty$, $\mu_{(P,Q)}$-a.e. and similarly, by the ergodicity of $T^{-1} \times T^{-1}$, also $\tau_- < \infty$, $\mu_{(P,Q)}$-a.e. Observe that for every $s$ and $t$ in $\mathcal{S}$ and every choice of $B_0$ and $B_1$ in $\mathcal{B}_{A,\{x_{\tau_0},t\}}(s,t)$ it holds that

$$(C_0 \cap B_0) \times (C_1 \cap B_1) \in \mathcal{A} \quad \text{for } \mu_{(P,Q)}\text{-a.e. } (x,y) \in X_A \times X_A.$$ 

This shows that

$$C_0 \times C_1 = \bigcup_{s,t \in \mathcal{S}} \bigcup_{-N,n \leq m} (C_0 \cap B_0) \times (C_1 \cap B_1)$$

up to a $\mu_{(P,Q)}$-null set. \hfill $\square$

**Claim 6.6.** Every finite intersection of elements in $\mathcal{A}$ is a disjoint union of finitely many elements of $\mathcal{A}$.

**Proof.** For some integers $n_1 \leq n_2$, let

$$B_1 := B_1(n_1) \times B_1'(n_1) \text{ of length } L_1$$

and

$$B_2 := B_2(n_2) \times B_2'(n_2) \text{ of length } L_2$$

be elements in $\mathcal{A}$ such that $B_1 \cap B_2$ is non-empty. If the sets of coordinates

$$\{n_1, \ldots, n_1 + L_1\} \text{ and } \{n_2, \ldots, n_2 + L_2\}$$

are not disjoint and $B_1 \cap B_2$ is non-empty, then simply $B_1 \cap B_2 \in \mathcal{A}$. The same holds also if $n_1 + L_1 = n_2 - 1$. Assume then that $n_1 + L_1 < n_2 - 1$. In this case we can write

$$B_1(n_1) \cap B_2(n_2) = \bigcup_{s,t \in \mathcal{S}} \bigcup_{B \in \mathcal{B}_{A,\{n_1 + L_1, n_2 - 1\}}(s,t)} B_1(n_1) \cdot B \cdot B_2(n_2),$$

where some of the concatenated blocks may be empty. Of course, we can write $B_1'(n_1) \cap B_2'(n_2)$ in a similar way. Observe that

$$(B_1(n_1) \cdot B \cdot B_2(n_2), B_1'(n_1) \cdot B' \cdot B_2'(n_2))$$

is an admissible pair on the coordinates $\{n_1, \ldots, n_2 + L_2\}$ for every choice of $B$ and $B'$ in $\mathcal{B}_{A,\{n_1 + L_1, n_2 - 1\}}(s,t)$. Thus, as

$$B_1 \cap B_2 = (B_1(n_1) \cap B_2(n_2)) \times (B_1'(n_1) \cap B_2'(n_2)),$$

we conclude that it is a disjoint union of finitely many elements of $\mathcal{A}$ and the proof of Claim 6.6 is complete. \hfill $\square$
We will now prove that \( \mu_{(P,Q)} = \mu_{(Q,P)} \) as measures on \( \mathcal{B}(X_A \times X_A) \). Our argument is based on Dynkin’s \( \pi \)-\( \lambda \) Theorem and we will follow the terminology of [37, Chapter II, §2]. Let

\[
\mathcal{F} = \{ E \in \mathcal{B}(X_A \times X_A) : \mu_{(P,Q)}(E) = \mu_{(Q,P)}(E) \}.
\]

Then \( \mathcal{F} \) is a \( \mathfrak{d} \)-system containing \( \mathfrak{A} \). By Claim 6.6, if \( B_1 \) and \( B_2 \) are in \( \mathfrak{A} \) then \( B_1 \cap B_2 \in \mathcal{F} \). Thus, the \( \pi \)-system \( \pi(\mathfrak{A}) \) generated by \( \mathfrak{A} \), which consists of all finite intersections of elements of \( \mathfrak{A} \), is also contained in \( \mathcal{F} \). Then by Dynkin’s \( \pi \)-\( \lambda \) Theorem the sigma-algebra generated by \( \pi(\mathfrak{A}) \) is contained in \( \mathcal{F} \). By Claim 6.5 this sigma-algebra is \( \mathcal{B}(X_A \times X_A) \) so that \( \mathcal{B}(X_A \times X_A) = \mathcal{F} \).

Finally, to complete the proof of Lemma 6.3, we get a contradiction by showing that \( P = Q \). For every \( s \in S \) let

\[
B = \bigcup_{t \in S} B_t \text{ for } B_t = \{ (x, y) \in X_A \times X_A : x_0 = s, y_0 = t \}, t \in S.
\]

Then \( \mu_{(P,Q)}(B) = \pi_P(s) \) and \( \mu_{(Q,P)}(B) = \pi_Q(s) \) so that \( \pi_P(s) = \pi_Q(s) \) for all \( s \in S \). Next, for every \( s, t \in S \) let

\[
B = \bigcup_{u \in S} B_u \text{ for } B_u = \{ (x, y) \in X_A \times X_A : (x_0, y_0) = (s, t), (x_1, y_1) = (s, u) \}, u \in S.
\]

Then

\[
\mu_{(P,Q)}(B) = \pi_P(s)P(s, t)\pi_Q(s) \quad \text{and} \quad \mu_{(Q,P)}(B) = \pi_Q(s)Q(s, t)\pi_P(s).
\]

As \( \pi_P(s) = \pi_Q(s) \) we see that \( P(s, t) = Q(s, t) \). \( \square \)

We are now in a position to prove Theorem 2.

**Proof of Theorem 2.** By Theorem 1 we know that under the conditions of Theorem 2 the shift on \( (X_A, \mu) \) is ergodic. Thus, by our Hopf Argument (Theorem 4.3), if we show that \( e(\mathcal{R}_A) = \mathbb{R} \) for the renormalization full-group \( \mathcal{R}_A := \mathcal{R}(T; \Pi_A) \) it will follow that the shift is of type III\(_1\).

We consider the case where \( (P_n : n \geq 1) \) does not converge regardless of the convergence of \( (P_{-n} : n \geq 1) \), and the other case is similar. For the rest of the proof we fix an arbitrary sequence \( i_k \xrightarrow{k \to \infty} -\infty \) of coordinates that satisfies \( i_k - i_{k+1} \xrightarrow{k \to \infty} \infty \), such that \( P_{i_k} \xrightarrow{k \to \infty} R \) for some arbitrary stochastic matrix \( R \). For every partial limit \( P_{j_k} \xrightarrow{k \to \infty} P \) for some \( j_k \xrightarrow{k \to \infty} \infty \) and for every admissible pair of the form

\[
(6.6.1) \quad (B, B') \quad \text{for } B = [b_1, b_2, \ldots, b_{L-1}, b_L], \quad B' = [b_1', b_2', \ldots, b'_{L-1}, b_L],
\]
assuming without loss of generality that \( j_k - j_{k-1} > L + M \) for all \( k \geq 1 \), the sequence of admissible permutations \( V_k : B(i_k) \to B'(j_k) \) satisfies

\[
V_k(x) = \frac{P_k(B'_k)}{P_k(B_k)} \xrightarrow{k \to \infty} \frac{R(B')}{R(B)} \cdot \frac{P(B)}{P(B')}, \quad x \in B(i_k) \cap B'(j_k),
\]

where in this convergence we used the nonsingularity of the shift and Corollary 5.3 to see that for every fixed \( l \in \mathbb{Z} \) we have that

\[
\lim_{k \to \infty} \frac{P_{j_k+l}(s, t)}{P_{j_k}(s, t)} = 1 \quad \text{and} \quad \lim_{k \to \infty} \frac{P_{j_k+l}(s, t)}{P_{j_k}(s, t)} = 1 \quad \text{for all} \quad s, t \in \mathcal{B}.
\]

Letting \( c := \log(R(B')/R(B)) \) we see by Lemma 6.1 that

\[
c + \log \frac{P(B)}{P(B')} \in \mathfrak{e}(\mathcal{A}(T; \Pi_A)).
\]

Note that by the nonsingularity of the shift and Lemma 6.2, for every admissible pair \((B, B')\) of the form of (6.6.1) the set of partial limits

\[
\mathcal{L}(P_n(b_1, b_2), \ldots, P_{n+L-2}(b_{L-1}, b_L), P_n(b_1, b'_2), \ldots, P_{n+L-2}(b'_{L-1}, b_L)) : n \geq 1
\]

is a compact, connected subset of \([\delta, 1 - \delta]^{2(L-1)}\). Denoting this partial limits set by \( \mathcal{L}(B, B') \), we see that the image \( F(\mathcal{L}(B, B')) \) of the set \( \mathcal{L}(B, B') \) under the continuous function \( F : [\delta, 1 - \delta]^{2(L-1)} \to \mathbb{R} \) defined by

\[
F : (r_1, \ldots, r_{L-1}, r'_1, \ldots, r'_{L-1}) \mapsto \log \frac{r_1 \cdots r_{L-1}}{r'_1 \cdots r'_{L-1}}
\]

is a compact, connected set of \( \mathbb{R} \), which is simply a compact interval. By the above argument we have that \( c + F(\mathcal{L}(B, B')) \subset e(\mathfrak{A}) \). We then only need to show that there can be found some \( L \geq 3 \) and an admissible pair \((B, B')\) of length \( L \) such that \( F(\mathcal{L}(B, B')) \) is an interval of positive length or, equivalently, that \( F \) is not constant on \( \mathcal{L}(B, B') \). This is straightforward from Lemma 6.3.

\[\square\]

7 Proof of the necessary condition for conservativeness

Here we prove Theorem 3. First let us establish a general simple necessary condition for conservativeness. Let \((X, \mathcal{B}, \mu)\) be a standard probability space and \( T : X \to X \) an invertible bi-measurable transformation. Denote the ergodic sums of a function \( f \) on \( X \) by

\[
S_N^f(x) = \sum_{n=0}^{N-1} f(T^n x) \quad \text{and} \quad S_N^f(x) = \sum_{n=0}^{N-1} f(T^{-n} x) \quad \text{for} \quad N \geq 1.
\]
Lemma 7.1. In the above setting, if there is a real-valued function $f$ on $X$ such that for some numbers $a < b$ we have that

$$
\limsup_{N \to \infty} \frac{1}{N} S_N^- f \leq a < b \leq \liminf_{N \to \infty} \frac{1}{N} S_N^+ f
$$
on a set of $\mu$-positive measure, then $T$ is not conservative.

**Proof.** Let $\epsilon = (b - a)/3$ and fix $N_0$ large enough such that the set

$$
E_0 := \bigcap_{N \geq N_0} \left\{ \frac{1}{N} S_N^- f \leq a + \epsilon < b - \epsilon \leq \frac{1}{N} S_N^+ f \right\}
$$is of $\mu$-positive measure. Then for every $x \in E_0$ and every $N \geq N_0$,

$$
\frac{1}{N} S_N^+ f(T^{N-1}x) = \frac{1}{N} S_N^+ f(x) \geq b - \epsilon > a + \epsilon,
$$showing that $T^{N}x \notin E_0$ for all but at most finitely many positive integers $N$. Then by Halmos’ Recurrence Theorem [1, Chapter 1.1] $E_0$ is a $\mu$-positive measure set which is not in the conservative part of the shift. \qed

**Theorem 7.2** (Wen–Weiguo [40, 41]). Let $(X_n : n \geq 0)$ be a non-homogeneous one-sided Markov chain and $(f_n : n \geq 0)$ be a bounded sequence of functions on $S \times S$. Then

$$
\frac{1}{N} \sum_{n=0}^{N-1} (f_n(X_n, X_{n+1}) - \mathbb{E}(f_n(X_n, X_{n+1}) \mid X_n)) \quad \text{a.e.} \quad N \to \infty \to 0.
$$

Once we observe that

$$
\xi_n := f_n(X_n, X_{n+1}) - \mathbb{E}(f_n(X_n, X_{n+1}) \mid X_n), \quad n \geq 1,
$$is a sequence of martingale differences for the natural filtration, Theorem 7.2 follows from the Law of Large Numbers for martingales [17, Theorem 2.19].

Applying Theorem 7.2 to the functions $1_{\{X_{n+1} = t_0\}}, t_0 \in S$ and to the functions $1_{\{(X_n, X_{n+1}) = (s_0, t_0)\}}, s_0, t_0 \in S$ we get the following.

**Corollary 7.3.** Under the conditions of Theorem 7.2, we have

$$
\frac{1}{N} \sum_{n=0}^{N-1} (1_{\{X_{n+1} = t_0\}} - P_n(X_n, t_0)) \quad \text{a.e.} \quad N \to \infty \to 0, \quad t_0 \in S,
$$

and

$$
\frac{1}{N} \sum_{n=0}^{N-1} (1_{\{(X_n, X_{n+1}) = (s_0, t_0)\}} - 1_{\{X_n = s_0\}} P_n(X_n, t_0)) \quad \text{a.e.} \quad N \to \infty \to 0, \quad s_0, t_0 \in S.
$$
In the following discussion it will be useful to use the notation
\[ W_N \approx W_N' \iff W_N - W_N' \xrightarrow{a.c.} N \to \infty 0, \]
which defines an equivalence relation on the collection of all sequences of random variables on a specified probability space.

The following proposition was proved by Wen and Weiguo [41, Theorem 2] in the context of \( m \)th order Markov chains. In the following we provide a simplified version of their proof.

**Proposition 7.4** (Wen–Weiguo). Let \( (X_n : n \geq 0) \) be a Markov chain with the distribution defined by \( (\pi_n, P_n : n \geq 0) \). If \( P_n(s, t) \xrightarrow{n \to \infty} P(s, t) \) for all \( s, t \in S \) for an irreducible and aperiodic stochastic matrix \( P \) with stationary distribution \( \pi \), then

\[(7.4.1) \quad \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_n = t_0\}} \approx \pi(t_0), \quad t_0 \in S\]

and

\[(7.4.2) \quad \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{(X_n, X_{n+1}) = (s_0, t_0)\}} \approx \pi(s_0)P(s_0, t_0), \quad s_0, t_0 \in S.\]

**Proof.** By Corollary 7.3 and the Cesaro convergence for all \( s_0, t_0 \in S \),
\[ \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{(X_n, X_{n+1}) = (s_0, t_0)\}} \approx P(s_0, t_0) \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_n = s_0\}}. \]

Hence (7.4.1) implies (7.4.2). To establish (7.4.1), fix \( t_0 \in S \) and observe that by Corollary 7.3 and the Cesaro convergence we have
\[ \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_n = t_0\}} \approx \sum_{s \in S} P(s, t_0) \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_{n+1} = s\}}. \]

Denoting the \( k \)-fold product of \( P \) by \( P^k \) we get recursively that, for every \( k \geq 1 \),
\[ \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_{n+1} = t_0\}} \approx \sum_{s \in S} P^k(s, t_0) \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_{n+1} = s\}}. \]

Since \( P \) is irreducible and aperiodic, by the convergence theorem for homogeneous Markov chains \( P^k(s, t_0) \xrightarrow{k \to \infty} \pi(t_0) \) for all \( s \in S \) and (7.4.1) follows. \( \square \)
Proof of Theorem 3. Let
\[ P = \lim_{n \to \infty} P_n \quad \text{and} \quad Q = \lim_{n \to \infty} P_n, \]
and denote their stationary distributions by \( \pi \) and \( \lambda \), respectively. Since the SFT \( X_A \) is topologically-mixing so that all the entries of \( A^M \) are positive, and since the matrices \( (P_n : n \in \mathbb{Z}) \) satisfy the Doeblin condition, it is clear that all the entries of \( P^M \) and \( Q^M \) are positive, so that \( P \) and \( Q \) are irreducible and aperiodic. It follows from Proposition 7.4 that
\[ \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_n = s_0\}} \approx \lambda(s_0), \quad s_0 \in \mathbb{S}. \]
Note that the reversed sequence \((X_0, X_{-1}, X_{-2}, \ldots)\) is a Markov chain with the transition matrices \((\pi_n, \hat{P}_n : n \leq 0)\), where
\[ \hat{P}_n(s, t) = \frac{\pi_{n-1}(t)}{\pi_n(s)} P_{n-1}(t, s), \quad s, t \in \mathbb{S}, n \leq 0. \]
This sequence converges to \( \hat{P}(s, t) := \frac{\pi(t)}{\pi(s)} P(t, s) \). Note also that \( P \) and \( \hat{P} \) share the same stationary distribution \( \pi \) so by Proposition 7.4 we have
\[ \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{X_{-n} = s_0\}} \approx \pi(s_0), \quad s_0 \in \mathbb{S}. \]
If \( \pi(s_0) \neq \lambda(s_0) \) for some \( s_0 \in \mathbb{S} \), then applying Lemma 7.1 to the function \( f = 1_{\{X_0 = s_0\}} \) shows that the shift is not conservative. Assume then that \( \pi = \lambda \). Fix \( s_0, t_0 \in \mathbb{S} \) and let \( f = 1_{\{(X_0, X_1) = (s_0, t_0)\}} \). By Proposition 7.4 we have
\[ \frac{1}{N} S^+_N f = \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{(X_n, X_{n+1}) = (s_0, t_0)\}} \approx \lambda(s_0)Q(s_0, t_0) = \pi(s_0)Q(s_0, t_0). \]
By the reasoning mentioned above, we can apply Proposition 7.4 to the reversed chain to get that
\[ \frac{1}{N} S^-_N f = \frac{1}{N} \sum_{n=0}^{N-1} 1_{\{(X_{-n+1}, X_{-n}) = (t_0, s_0)\}} \approx \pi(t_0)\hat{P}(t_0, s_0) = \pi(s_0)P(s_0, t_0). \]
Thus, if \( P(s_0, t_0) \neq Q(s_0, t_0) \) for some \( s_0, t_0 \in \mathbb{S} \), applying Lemma 7.1 to the function \( f = 1_{\{(X_0, X_1) = (s_0, t_0)\}} \) shows that the shift is not conservative. \( \Box \)
8 Proofs of the convergent scenarios

We start with a sufficient condition for the Central Limit Theorem (CLT).

**Theorem 8.1** (Dobrushin). Let \((Y_n : n \geq 1)\) be a non-homogeneous Markov chain that its distribution satisfies the Doeblin condition \((D)\). Let \((f_n : n \geq 1)\) be a uniformly bounded sequence of real-valued functions. If

\[
\sum_{n \geq 1} V(f_n(Y_n)) = \infty,
\]

then the sequence \((f_n(Y_n) : n \geq 1)\) satisfies

\[
\frac{S_N - \mathbb{E}(S_N)}{\sqrt{\mathbb{V}(S_N)}} \xrightarrow{d} \mathcal{N}, \quad N \to \infty,
\]

where \(S_N := \sum_{n=1}^{N} f_n(Y_n)\) for \(N \geq 1\) and \(\mathcal{N}\) is the standard normal distribution.

This formulation is a special case of a sufficient condition for CLT established by Dobrushin [14]. See the formulation and the proof by Sethuraman and Varadhan [35]. In their notations, the constants \(C_n\) are uniformly bounded as \((f_n : n \geq 1)\) is uniformly bounded, and the ergodic coefficients \(\alpha_n\) are all in \([2\delta, 1]\) by the Doeblin condition.

The following lemma is a Markovian version of what is sometimes called the Araki–Woods Lemma [12, Chapter 2].

**Lemma 8.2.** Let \((X_A, \mu)\) be a topologically-mixing MSFT that satisfies the Doeblin condition \((D)\). If there is an admissible configuration \((B_k(i_k), B'_k(j_k))\), \(k \geq 1\), (recall Definition 5.5), such that for the corresponding sequence \((D_k : k \geq 1)\) defined in (5.7.1) we have that

\[
D_k \xrightarrow{k \to \infty} 0 \quad \text{and} \quad \sum_{k \geq 1} D_k^2 = \infty,
\]

then \(e(\mathcal{R}_A) = \mathbb{R}\) for the renormalization full-group \(\mathcal{R}_A := \mathcal{R}(T; \Pi_A)\).

**Proof.** Consider the sequence of symmetric admissible permutations

\[
V_k : B_k(i_k) \cap B'_k(j_k), \quad k \geq 1.
\]

Let the random variables

\[
Y_k(x) := 1_{B_k(i_k) \cap B'_k(j_k)}(x) - 1_{B'_k(i_k) \cap B_k(j_k)}(x), \quad k \geq 1,
\]

so that according to Claim 5.7 and the notation in (5.7.1),

\[
\log V'_k(x) = D_k Y_k(x), \quad k \geq 1.
\]
Claim 8.3. The sequence \((Y_k : k \geq 1)\) defined in (8.2.1) is a one-sided Markov chain on the state space \([-1, 0, 1]\), with respect to the distribution induced from \(\mu\) in the obvious way. Moreover, if \(\mu\) satisfies the Doeblin condition for \(\delta > 0\) then the distribution of \((Y_k : k \geq 1)\) satisfies the Doeblin condition for some \(\delta' > 0\).

Proof. Since the distribution \(\mu\) of \((X_n : n \in \mathbb{Z})\) satisfies the Markov property, it follows from [16, Remark 10.9] that \(\mu\) satisfies the Markov field property, namely

\[
\sigma(X_k : |k| > n) \text{ conditioned on } \sigma(X_{-n}, X_n) \text{ is independent on } \sigma(X_k : |k| < n)
\]

for every \(n \geq 1\) with respect to \(\mu\). This readily implies that the distribution of \((Y_k : k \geq 1)\) satisfies the Markov property.

To see that the Markov chain \((Y_k : k \geq 1)\) satisfies the Doeblin condition we use Proposition 5.1, and that by the construction of an admissible configuration, \(i_k - i_{k+1}\) and \(j_{k+1} - j_k\), as well as \(j_k - i_k\), are all greater than \(L + M\) for every \(k \geq 1\). First recall that for every \(A\)-admissible block \(B\) of length \(L\), for every \(n \in \mathbb{Z}\),

\[
\delta L M \leq \mu(B(n)) \leq (1 - \delta M)^L
\]

holds. We then get that

\[
\mathbb{P}(Y_k = 1) = \mu(B_k(i_k) \cap B'_k(j_k)) \\
\geq C(\delta, M) \mu(B_k(i_k)) \mu(B'_k(j_k)) \\
\geq C(\delta, M) \delta^{2LM},
\]

and similarly \(\mathbb{P}(Y_k = -1) \geq C(\delta, M) \delta^{2LM}\). Also we get that

\[
\mathbb{P}(Y_k = 0) \geq \mu(B_k(i_k)^c \cap B_k(j_k)^c) \\
\geq C(\delta, M) \mu(B_k(i_k)^c) \mu(B_k(j_k)^c) \\
\geq C(\delta, M) (1 - (1 - \delta M)L)^2.
\]

Then let \(0 < \eta \leq 1/2\) such that for all \(k \geq 1\) and \(a \in \{-1, 0, 1\}\) we have that \(\mathbb{P}(Y_k = a) \geq \eta\). Considering the transition probabilities, using the same considerations we see that for all \(a, b \in \{-1, 0, 1\},\)

\[
\mathbb{P}(Y_{k+1} = b, Y_k = a) \geq C(\delta, M)^3 \eta^4,
\]

hence

\[
\mathbb{P}(Y_{k+1} = b \mid Y_k = a) \geq \delta' := \frac{C(\delta, M)^2 \eta^3}{1 - \eta},
\]

which concludes that the Markov chain \((Y_k : k \geq 1)\) satisfies the Doeblin condition for \(\delta' > 0\) depending only on the constants \(\delta, M\) and \(L\). This completes the proof of Claim 8.3. \(\square\)
Claim 8.4. The sequence $\{\log V'_k : k \geq 1\}$ satisfies the central limit theorem.

Proof. As we mentioned, we have the identity $\log V'_k(x) = D_k Y_k(x)$ for all $k \geq 1$, where $(D_k : k \geq 1)$ is a convergent sequence of numbers. It follows by Claim 8.3 that $(\log V'_k : k \geq 1)$ is a Markov chain that satisfies the Doeblin condition. To use Theorem 8.1, note that for every $k \geq 1$ the events $B_k(i_k) \cap B'_k(j_k)$ and $B'_k(i_k) \cap B_k(j_k)$ are disjoint, so by Proposition 5.1 we have that

$$V(Y_k) = \mu(B_k(i_k) \cap B'_k(j_k)) (1 - \mu(B_k(i_k) \cap B'_k(j_k)))$$

$$+ \mu(B'_k(i_k) \cap B_k(j_k)) (1 - \mu(B'_k(i_k) \cap B_k(j_k)))$$

$$+ 2 \mu(B_k(i_k) \cap B'_k(j_k)) \mu(B'_k(i_k) \cap B_k(j_k))$$

$$\geq \mu(B_k(i_k) \cap B'_k(j_k)) \geq C(\delta, M) 2^{2LM},$$

hence

$$V(\log V'_k) = D_k^2 V(Y_k) \ll D_k^2.$$

By the assumption in the Lemma we conclude that $\sum_{k \geq 1} V(\log V'_k) = \infty$, hence the sequence $\{\log V'_k : k \geq 1\}$ satisfies the condition of Theorem 8.1. This completes the proof of Claim 8.4. \hfill $\square$

Claim 8.5. For integers $1 \leq k \leq K$ denote

$$S^K_k(x) = \sum_{i=k}^K \log V'_i(x).$$

Fix some $k_0 \geq 1$. Then for every $r < 0$,

$$\lim_{K \to \infty} \inf \mathbb{P}(S^K_{k_0} < r) \geq \lim_{K \to \infty} \mathbb{P}(S^K_{k_0} < \mathbb{E}(S^K_{k_0})) = 1/2.$$

Proof. The second equality is a straightforward corollary of the CLT as in Claim 8.4. For the first inequality, it is enough to show that $\mathbb{E}(S^K_{k_0}) \xrightarrow[K \to \infty]{} -\infty$. Note that since every $(B_k, B'_k)$ is an admissible pair, if we denote the mutual first symbol by $b_0$ and the mutual last symbol by $b_1$, then we have that

$$\mu(B_k(i_k) \cap B'_k(j_k)) = \pi_{i_k}(b_0) P_{i_k}(B_k) P^{i_k+L-1,j_k}(b_1, b_0) P_{j_k}(B'_k)$$

and

$$\mu(B'_k(i_k) \cap B_k(j_k)) = \pi_{i_k}(b_0) P_{i_k}(B'_k) P^{i_k+L-1,j_k}(b_1, b_0) P_{j_k}(B_k).$$

It then follows that

$$\mathbb{E}(\log V'_k) = D_k E(Y_k)$$

$$= D_k (\mu(B_k(i_k) \cap B'_k(j_k)) - \mu(B'_k(i_k) \cap B_k(j_k)))$$

$$\asymp D_k (P_{i_k}(B_k) P_{j_k}(B'_k) - P_{i_k}(B'_k) P_{j_k}(B_k))$$

$$\asymp -D_k^2,$$
where the first approximation is by the above calculation and Proposition 5.1, and the second approximation was mentioned in (5.7.1). By the assumption in the Lemma we conclude that $E(S^k)_{k \to \infty} \to -\infty$, which completes the proof of Claim 8.5.

We are now ready to establish that $e(\mathcal{A}) = \mathbb{R}$. Since the ratio set is an additive subgroup of $\mathbb{R}$ it is enough to show that it contains every negative number. Let $r < 0$ and $0 < \epsilon < |r|$. Let $E \in \sigma(X_k : |k| \leq N)$ for some $N \geq 1$. Find positive integers $k_0 \leq K_0$ to satisfy the following properties:

1. $i_{k_0} + N \leq -M$ and $j_{k_0} - N \geq M$;
2. $|\log V'_k(x)| < \epsilon$ everywhere for all $k \geq k_0$; and
3. $\mu(S^k_{k_0} < r) \geq 1/4$.

The first property clearly holds for every large $k_0$. The second property holds for every large $k_0$ since $D_k \to 0$. The third property holds for every large $k_0$ and every $K_0$ which is large enough with respect to the choice of $k_0$ by Claim 8.5.

Consider the set $F := E \cap \{S^k_{k_0} < r\} \subset E$. We now define $V \in \mathcal{[\mathcal{A}]}$ of the form $V : F \to E$ with $\log V'(x) \in (r - \epsilon, r + \epsilon)$ for all $x \in F$. For $x \in F$ let

$$k(x) := \inf\{k \geq k_0 : S^k_{k_0}(x) < r\} \leq K_0$$

and

$$K(x) := \{k_0 \leq k \leq k(x) : Y_k(x) \neq 0\} \subset \{k_0, \ldots, k(x)\}.$$ 

Define $V_x$ for $x \in F$ to be the composition of all $V_{k_0}x$ for $k \in K(x)$.

Recall that $j_{k+1} - j_k$ and $i_k - i_{k+1}$ are both greater than $L + M$ for all $k \geq 1$, and in particular the coordinates that are being changed by the $V_k$’s are distinct, so that $V_x$ is a well-defined transformation with domain in $X_A$. Also note that $V_x \in E$ for all $x \in F$, since $i_{k_0} + N \leq -M$ and $j_{k_0} - N \geq M$ while $E \in \sigma(X_k : |k| \leq N)$. We show that $V$ is one-to-one on $F$. Assume that $V_x = V_y$ for $x, y \in F$. If $k(x) < k(y)$, since $V_x = V_y$ implies that $x_k = y_k$ for all $|k| \leq k(x)$, we get that

$$S^k_{k_0}(x) = S^k_{k_0}(y) < r,$$

a contradiction to the definition of $k(y)$. By the symmetric reasoning it is also impossible that $k(x) > k(y)$, hence $k(x) = k(y)$. Then we see that for every $k_0 \leq k \leq k(x) = k(y),$

$$x \in B_k(i_k) \cap B'_k(j_k)$$

$$\iff V_y = V_x \in B_k'(i_k) \cap B_k(j_k)$$

$$\iff y \in B_k(i_k) \cap B'_k(j_k),$$
and similarly
\[ x \in B'_k(i_k) \cap B_k(j_k) \iff y \in B'_k(i_k) \cap B_k(j_k). \]
It follows that \( K(x) = K(y) \). Finally, since each of the \( V_k \)'s is one-to-one and since \( Vx = Vy \) is the composition of all \( V_k \)'s for \( k \in K(x) = K(y) \), we see that \( x = y \) so that \( V \) is one-to-one on \( F \). We also see that for every \( x \in F \),
\[
\log V'(x) = \sum_{k \in K(x)} \log V'_k(x) = \delta^{k(x)}_{k_0}(x) \in (r - \epsilon, r),
\]
by the definition of \( k(x) \) and since \( |\log V'_k(x)| < \epsilon \) for \( k \geq k_0 \). This establishes the condition for \( r \) to be essential value for all sets supported on finitely many coordinates. As this collection forms a countable algebra that is dense in the Borel sigma-algebra, in order to finish we establish the condition of Lemma 2.1. Note that since \( i_{k_0} + N \leq -M \) and \( j_{k_0} - N \geq M \) while \( E \in \sigma(X_k : |k| \leq N) \), by Proposition 5.1 we have that
\[
\mu(F) \geq C(\delta, M) \mu(S^{k_0}_{k_0} < r) \mu(E) \geq \frac{C(\delta, M)^2}{4} \mu(E).
\]
Thus, \( \eta := C(\delta, M)^2/4 \) satisfies the condition of Lemma 2.1 and \( r \in e(\mathcal{R}_A) \). \( \square \)

We are now in a position to prove Theorems 4 and 5. By Theorem 1 we know that under the conditions of Theorems 4 and 5 the shift is ergodic. Thus, by our Hopf Argument (Theorem 4.3) if we show that \( e(\mathcal{R}_A) = \mathbb{R} \) for the renormalization full-group \( \mathcal{R}_A = \mathcal{R}(T; \Pi_A) \) it will follow that the shift is of type III1.

Note that if
\[
\sum_{n \geq 1} \sum_{s, t \in S} \left( \sqrt{P_n(s, t)} - \sqrt{Q(s, t)} \right)^2 < \infty,
\]
then by Theorem 5.2 we can assume that \( P_n = Q \) for all \( n \geq 1 \) without changing the equivalence class of the measure. In a similar way, if
\[
\sum_{n \geq 1} \sum_{s, t \in S} \left( \sqrt{\hat{P}_n(s, t)} - \sqrt{\hat{Q}(s, t)} \right)^2 < \infty
\]
we can assume without loss of generality that \( \hat{P}_n = \hat{Q} \) for \( n \geq 1 \). Then if both of those series are finite, \( \mu \) is equivalent to the homogeneous Markov measure defined by \( Q \) and the shift is of type II1. Thus, if the shift is not of type II1 then one of the above series diverges. We consider the case where the first series diverges,
\[
\sum_{n \geq 1} \sum_{s, t \in S} \left( \sqrt{Q(s, t)} - \sqrt{P_n(s, t)} \right)^2 = \infty,
\]
and the other case is similar.
**Proof of Theorem 4.** Let us consider first the fullshift \( T : (X, \mu) \to (X, \mu) \) where \( X = \{0, 1\}^\mathbb{Z} \) and \( \mu \) is the Markov measure defined by the transition matrices

\[
P_n = \begin{pmatrix} p_n & 1 - p_n \\ p'_n & 1 - p'_n \end{pmatrix}, \quad n \in \mathbb{Z}.
\]

Assume that \( \lim_{|n| \to \infty} P_n = Q \) for

\[
Q = \begin{pmatrix} q & 1 - q \\ q' & 1 - q' \end{pmatrix}.
\]

If the shift is not of type II, we assume without loss of generality that

\[
\sum_{n \geq 1} (\sqrt{p_n} - \sqrt{q})^2 + (\sqrt{1 - p_n} - \sqrt{1 - q})^2 + \sum_{n \geq 1} (\sqrt{p'_n} - \sqrt{q'})^2 + (\sqrt{1 - p'_n} - \sqrt{1 - q'})^2 = \infty.
\]

By the Doeblin condition the square roots do not affect this divergence so that

\[
\sum_{n \geq 1} (p_n - q)^2 + \sum_{n \geq 1} (p'_n - q')^2 = \infty.
\]

We consider the case of \( \sum_{n \geq 1} (p_n - q)^2 = \infty \) regardless of \( \sum_{n \geq 1} (p'_n - q')^2 \). The other case can be treated symmetrically.

We now construct an admissible configuration that is satisfying the conditions of Lemma 8.2. Let

\[
I := \{ n \geq 1 : \text{sign}(p_n - q) = \text{sign}(p_{n+1} - q) \}.
\]

Then one can easily see that \( (p_n - p_{n+1})^2 \geq (p_n - q)^2 \) for all \( n \notin I \) so by the nonsingularity of the shift, using Corollary 5.3 we get that

\[
\sum_{n \notin I} (p_n - q)^2 \leq \sum_{n \notin I} (p_n - p_{n+1})^2 < \infty.
\]

It follows that

\[
\sum_{n \in I} (p_n - q)^2 = \infty.
\]

Then we can find a subsequence \( (j_k : k \geq 1) \subset I \) satisfying

- \( j_k - j_{k-1} \geq 3 \) for all \( k \geq 1 \);
- \( \text{sign}(p_{j_k} - q) = \text{sign}(p_{j_{k+1}} - q) \) is constant for \( k \geq 1 \) and
- \( \sum_{k \geq 1} (p_{j_k} - q)^2 = \infty \).
Write \( s := \text{sign}(p_{jk} - q) \) for any \( k \geq 1 \). Since \( p_n \xrightarrow{n \to -\infty} q \) there is a sequence \( (i_k : k \geq 1) \) of negative integers satisfying

- \( i_k - i_{k+1} \geq 3 \) for all \( k \geq 1 \);
- \( \text{sign}(p_{jk} - p_{ik}) = s \) for all \( k \geq 1 \); and
- \( \sum_{k \geq 1} (p_{jk} - p_{ik})^2 = \infty \).

Consider the admissible pairs

\[
(B_0, B'_0) \text{ for } B_0 = [0, 0, 0], \quad B'_0 = [0, 1, 0],
\]

and

\[
(B_1, B'_1) \text{ for } B_1 = [0, 0, 1], \quad B'_1 = [0, 1, 1].
\]

For every \( k \geq 1 \) we have that

\[
D_{0,k} := \log \left( \frac{P_{ik}(B'_0)P_{jk}(B_0)}{P_{ik}(B_0)P_{jk}(B'_0)} \right) = \log \frac{p_{jk}}{p_{ik}} + \log \frac{1 - p_{ik}}{1 - p_{jk}} + \log \frac{p_{ik+1}}{p_{jk+1}} + \log \frac{p'_{ik+1}}{p'_{jk+1}}.
\]

and that

\[
D_{1,k} := \log \left( \frac{P_{ik}(B'_1)P_{jk}(B_1)}{P_{ik}(B_1)P_{jk}(B'_1)} \right) = \log \frac{p_{jk}}{p_{ik}} + \log \frac{1 - p_{ik}}{1 - p_{jk}} + \log \frac{1 - p_{ik+1}}{1 - p_{jk+1}} + \log \frac{1 - p'_{ik+1}}{1 - p'_{jk+1}}.
\]

Define, for \( k \geq 1 \),

\[
g(k) = \begin{cases} 
0, & \text{sign}(\log \frac{p'_{ik+1}}{1 - p'_{jk+1}}) = s, \\
1, & \text{sign}(\log \frac{p_{ik}}{1 - p_{jk}}) = s.
\end{cases}
\]

Claim 8.6. Let \( D_k := D_{g(k),k} \) for \( k \geq 1 \). Then

\[
D_k \xrightarrow{k \to \infty} 0 \quad \text{and} \quad \sum_{k \geq 1} D_k^2 = \infty.
\]

Proof of Claim 8.6. It is clear that \( D_k \xrightarrow{k \to \infty} 0 \). We prove the second part.

By the approximation in (3.0.1) we have that

\[
\log \frac{p_{jk}}{p_{ik}} \asymp \log \frac{1 - p_{ik}}{1 - p_{jk}} \asymp \log \frac{p_{ik+1}}{p_{jk+1}} \asymp p_{jk} - p_{ik}.
\]

By the definition of \( I \) we have that

\[
\text{sign}\left( \log \frac{1 - p_{ik}}{1 - p_{jk}} \right) = \text{sign}\left( \log \frac{p_{ik+1}}{p_{jk+1}} \right) = s, \quad k \geq 1.
\]
It follows that for $g(k) = 0$ we have

$$D_k = D_{0,k} \ni p_{ik} - p_{jk},$$

so in the case of $\sum_{g(k)=0}(p_{jk} - p_{ik})^2 = \infty$ we have

$$\sum_{k \geq 1} D_k^2 \geq \sum_{g(k)=0} D_k^2 = \infty.$$

In the case of $\sum_{g(k)=0}(p_{jk} - p_{ik})^2 < \infty$ we must have $\sum_{g(k)=1}(p_{jk} - p_{ik})^2 = \infty$. For $g(k) = 1$, the general term of the sequence $D_k = D_{1,k}$ is the sum of the general term of the sequence

$$\log \frac{p_{jk}}{p_{ik}} + \log \frac{1 - p'_{i,k+1}}{1 - p'_{j,k+1}} \nless p_{jk} - p_{ik}$$

and the general term of the sequence

$$\log \frac{1 - p_{ik}}{1 - p_{jk}} + \log \frac{1 - p_{j,k+1}}{1 - p_{i,k+1}} \nless (p_{i,k+1} - p_{ik}) + (p_{jk} - p_{j,k+1}),$$

which is square-summable by the nonsingularity of the shift as in Corollary 5.3. It follows that also in this case we have

$$\sum_{k \geq 1} D_k^2 \geq \sum_{g(k)=1} D_k^2 = \infty,$$

so the proof of Claim 8.6 is complete.

By Claim 8.6 we see that the admissible configuration $(B_{g(k)}(i_k), B'_{g(k)}(j_k)), k \geq 1,$ satisfies the conditions of Lemma 8.2 so that $e(\mathcal{A}) = \mathbb{R}$. This completes the proof of Theorem 4 for the fullshift.

Let us now consider a subshift on two states. The primitive adjacency matrices in this case, except for the fullshift, are

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The treatment in these two subshifts is similar, and we consider the first one. Let the transition matrices

$$P_n = \begin{pmatrix} p_n & 1 - p_n \\ 1 & 0 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

Assume that $\lim_{|n| \to \infty} P_n = Q$ for

$$Q = \begin{pmatrix} q & 1 - q \\ 1 & 0 \end{pmatrix}.$$
If the shift is not of type II$_1$, then without loss of generality $\sum_{n \geq 1} (p_n - q)^2 = \infty$. Choose sequences $(i_k : k \geq 1)$ and $(j_k : k \geq 1)$ in the same way we did in the fullshift and consider the admissible pair 

$$(B, B')$$

for $B = [0, 1, 0]$, $B' = [0, 0, 0]$.

Then $(B(i_k), B'(j_k)), k \geq 1$, is an admissible configuration that satisfies

$$D_k := \log \left( \frac{P_{i_k}(B')P_{j_k}(B)}{P_{i_k}(B)P_{j_k}(B')} \right)$$

$$= \log \frac{p_{i_k}}{p_{j_k}} + \log \frac{1 - p_{i_k}}{1 - p_{j_k}} + \log \frac{p_{i_k+1}}{p_{j_k+1}} \propto p_{i_k} - p_{j_k},$$

as $\text{sign}(p_{i_k} - p_{j_k}) = \text{sign}(p_{i_k+1} - p_{j_k+1})$ is constant for $k \geq 1$. Then

$$\sum_{k \geq 1} D_k^2 = \infty$$

and clearly $D_k \to 0$ as $k \to \infty$. By Lemma 8.2 we conclude that $e(\mathcal{A}) = \mathbb{R}$. \hfill $\Box$

**Remark 8.7.** In a similar way, one may prove the Bernoulli case on a general finite state space that is partially extending the results of [25, 12] concerning the half-stationary two-state space. Note that their results hold also without assuming the Doeblin condition. Consider the following setting: Let $\mathcal{S} = \{0, 1, \ldots, d - 1\}$ for some $d \in \mathbb{N}$ and $X = \mathcal{S}^\mathbb{Z}$ with a product measure

$$\mu = \prod_{n \in \mathbb{Z}} \mu_n$$

where

$$\mu_n = (p_n(0), p_n(1), \ldots, p_n(d - 1))$$

for $n \in \mathbb{Z}$. Suppose that $\mu$ satisfies the Doeblin condition and that the shift $T : (X, \mu) \to (X, \mu)$ is nonsingular and conservative. Denote

$$\lim_{|n| \to \infty} p_n(s) = q(s) \quad \text{for } s \in \mathcal{S}.$$  

If the shift is not of type II$_1$, then there exists $\alpha \in \mathcal{S}$ such that without loss of generality

$$\sum_{n \geq 1} (p_n(\alpha) - q(\alpha))^2 = \infty.$$ 

Take a subsequence $(j_k : k \geq 1)$ of positive integers with $j_{k+1} - j_k \geq 3$ such that $s := \text{sign}(p_{i_k}(\alpha) - q(\alpha))$ is constant in $k \geq 1$ and $\sum_{k \geq 1} (p_{i_k}(\alpha) - q(\alpha))^2 = \infty$. It is easy to see that for every $k \geq 1$ there exists $\beta_k \in \mathcal{S}$ such that

$$\text{sign}(p_{j_k}(\beta_k) - q(\beta_k)) = -s.$$
Then choose a sequence \((i_k : k \geq 1)\) of negative integers with \(i_k - i_{k+1} \geq 3\) and \(\text{sign}(p_{i_k}(\beta_k) - p_{i_k}(\beta_k)) = s\) is constant for \(k \geq 1\). Then the sequence of admissible pairs \((B_k, B'_k)\) for \(B_k = [\alpha, \alpha, \alpha]\), \(B'_k = [\alpha, \beta_k, \alpha]\), \(k \geq 1\), satisfies that

\[
D_k = \log \left( \frac{P_{i_k}(B'_k)P_{j_k}(B_k)}{P_{i_k}(B_k)P_{j_k}(B'_k)} \right) \asymp (p_{j_k}(\alpha) - p_{i_k}(\alpha)) + (p_{i_k}(\beta_k) - p_{j_k}(\beta_k)).
\]

Then in the same way as in the proof of Theorem 4 we we conclude that the Bernoulli shift is of type \(\text{III}_1\). \(\square\)

**Proof of Theorem 5.**

Let \(T : (X_G, \mu) \to (X_G, \mu)\) be a nonsingular and conservative Golden Mean SFT, where \(\mu\) is defined by the transition matrices

\[
P_n = \begin{pmatrix} p_n & 0 & 1 - p_n \\ 0 & 1 - p'_n & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad n \in \mathbb{Z}.
\]

Assume that \(\lim_{|n| \to \infty} P_n = Q\) for

\[
Q = \begin{pmatrix} q & 0 & 1 - q \\ q' & 0 & 1 - q' \\ 0 & 1 & 0 \end{pmatrix}.
\]

If the shift is not of type \(\text{II}_1\), then without loss of generality we have that

\[
\sum_{n \geq 1} (p_n - q)^2 + \sum_{n \geq 1} (p'_n - q')^2 = \infty.
\]

**Case 1.** Suppose that

\[
\sum_{n \geq 1} (p_n - q)^2 = \infty.
\]

As the shift is nonsingular, by the same reasoning we used in the fullshift there is a subsequence \(I_0 \subset \mathbb{N}\) with \(\text{sign}(p_n - q) = \text{sign}(p_{n+1} - q)\) for all \(n \in I_0\) such that \(\sum_{n \in I_0} (p_n - q)^2 = \infty\). Let

\[
I := \{ n \in I_0 : \text{sign}(p_n - q) = \text{sign}(p_{n+2} - q) \} \subset I_0.
\]

By the nonsingularity of the shift, using Corollary 5.3 and the Cauchy–Schwartz inequality we see that

\[
\sum_{n \in I} (p_n - q)^2 \leq \sum_{n \in I} (p_n - p_{n+2})^2 < \infty.
\]
We then get that

\[ \sum_{n \in I} (p_n - q)^2 = \infty \]

on an index set \( I \subset \mathbb{N} \) with the property

\[ \text{sign}(p_n - q) = \text{sign}(p_{n+1} - q) = \text{sign}(p_{n+2} - q), \quad n \in I. \]

Write \( s := \text{sign}(p_n - q) \) for any \( n \in I \). We can find a subsequence \((j_k : k \geq 1) \subset I\) and a sequence \((i_k : k \geq 1)\) of negative integers with the same properties as in the fullshift. Since the adjacency matrix \( G \) of the Golden Mean SFT satisfies \( G^M > 0 \) for \( M = 3 \) and the admissible pairs we will find will have length \( L = 4 \), we further require that \( j_k - j_{k-1} \geq 7 \) and \( i_k - i_{k+1} \geq 7 \) for all \( k \geq 1 \). This can be done using the same considerations as in the fullshift. Consider the admissible pairs

\[ (B_0, B'_0) \text{ for } B_0 = [0, 0, 0, 0], B'_0 = [0, 2, 1, 0] \]

and

\[ (B_1, B'_1) \text{ for } B_1 = [0, 0, 0, 2], B'_1 = [0, 2, 1, 2]. \]

For every \( k \geq 1 \) we have that

\[
D_{0,k} := \log \left( \frac{P_{i_k}(B'_0)P_{j_k}(B_0)}{P_{i_k}(B_0)P_{j_k}(B'_0)} \right) = \log \frac{p_{i_k}}{p_{i_k}} + \log \frac{p_{j_k+1}}{p_{i_k+1}} + \log \frac{1 - p_{i_k}}{1 - p_{j_k}} + \log \frac{p_{j_k+2}}{p_{i_k+2}} + \log \frac{p'_{i_k+2}}{p'_{j_k+2}}
\]

and that

\[
D_{1,k} := \log \left( \frac{P_{i_k}(B'_1)P_{j_k}(B_1)}{P_{i_k}(B_1)P_{j_k}(B'_1)} \right) = \log \frac{p_{i_k}}{p_{i_k}} + \log \frac{p_{j_k+1}}{p_{i_k+1}} + \log \frac{1 - p_{i_k}}{1 - p_{j_k}} + \log \frac{1 - p_{j_k+2}}{1 - p_{i_k+2}} + \log \frac{1 - p'_{i_k+2}}{1 - p'_{j_k+2}}.
\]

Define, for \( k \geq 1 \),

\[
g(k) = \begin{cases} 
0, & \text{sign}(\text{log} \frac{p_{j_k+2}}{p_{i_k+2}}) = s, \\
1, & \text{sign}(\text{log} \frac{1 - p_{i_k+1}}{1 - p_{j_k+1}}) = s.
\end{cases}
\]

Similarly to the fullshift, by the approximation in (3.0.1) we have that

\[
\log \frac{p_{j_k}}{p_{i_k}} \asymp \log \frac{p_{j_k+1}}{p_{i_k+1}} \asymp p_{j_k} - p_{i_k};
\]

we also have that

\[
\text{sign} \left( \log \frac{1 - p_{i_k}}{1 - p_{j_k}} \right) = \text{sign} \left( \log \frac{p_{j_k+1}}{p_{i_k+1}} \right) = s, \quad k \geq 1;
\]
and by the nonsingularity of the shift

\[
\log \frac{1 - p_{ik}}{1 - p_{jk}} + \log \frac{1 - p_{ik}'}{1 - p_{jk}'} \approx (p_{jk} - p_{jk+2}) + (p_{ik+2} - p_{ik})
\]

is a square-summable sequence. Then the very same proof of Claim 8.6 shows that the sequence \( D_k := D_{g(k),k}, k \geq 1 \), satisfies

\[
D_k \xrightarrow{k \to \infty} 0 \quad \text{and} \quad \sum_{k \geq 1} D_k^2 = \infty.
\]

Thus, the admissible configuration \((B_{g(k)}(i_k), B'_{g(k)}(j_k)), k \geq 1\), satisfies the conditions of Lemma 8.2 and we conclude that \( e(\mathcal{R}_G) = \mathbb{R} \).

**Case 2.** Suppose that \( \sum_{n \geq 1} (p_n - q)^2 < \infty \). Then by Theorem 5.2 we can assume that \( p_n = q \) for all \( n \geq 1 \) without changing the equivalence class of the measure. Note that in this case we have that \( \sum_{n \geq 1} (p'_n - q)^2 = \infty \). Choose sequences \((i_k : k \geq 1)\) and \((j_k : k \geq 1)\) in the same way we chose in the first case. Consider the admissible pair

\[
(B, B') \quad \text{for} \quad B = [1, 0, 0, 2], \quad B' = [1, 2, 1, 2]
\]

for which

\[
D_k := \log \frac{P_{ik}(B')P_{jk}(B)}{P_{ik}(B)P_{jk}(B')}
\]

\[
= \log \frac{1 - p_{ik}'}{1 - p_{jk}'} + \log \frac{p_{ik}}{p_{jk}} + \log \frac{1 - p_{ik}'}{1 - p_{jk}'} + \log \frac{q}{p_{ik+1}} + \log \frac{1 - q}{1 - p_{ik+2}}.
\]

We can choose the sequence \((i_k : k \geq 1)\) such that \( p_{ik} \xrightarrow{k \to \infty} q \) fast enough so that

\[
\sum_{k \geq 1} \left( \log \frac{q}{p_{ik+1}} \right)^2 + \sum_{k \geq 1} \left( \log \frac{1 - q}{1 - p_{ik+2}} \right)^2 < \infty,
\]

using the nonsingularity of the shift as in Corollary 5.3. Then since

\[
\text{sign}(p_{ik}' - p_{jk}') = \text{sign}(p_{ik+1}' - p_{jk+1}') = \text{sign}(p_{ik+2}' - p_{jk+2}')
\]

is constant for \( k \geq 1 \) it follows that \( \sum_{k \geq 1} D_k^2 = \infty \). It is clear that \( D_k \xrightarrow{k \to \infty} 0 \) so that \((B(i_k), B'(j_k)), k \geq 1,\) is an admissible configuration that satisfies the conditions of Lemma 8.2 so that \( e(\mathcal{R}_G) = \mathbb{R} \). \( \square \)
9 Examples

We introduce a construction of a class of Markov SFT’s for which the shift is nonsingular and conservative, providing a class of examples to our Theorems 4 and 5. This method of construction is due to Kosloff, first introduced in [26] and then was used in [24] to construct conservative Anosov diffeomorphisms of the 2-dimensional torus without a Lebesgue a.c.i.m. In the original construction further effort has been made to show that the shift is of type III$_1$. Using our results, to conclude that the shift is of type III$_1$ we only need to show that the shift is nonsingular and conservative and that the measure is not equivalent to a homogeneous Markov measure.

Throughout the construction we fix some $0 < q < 1$. To define a Markov measure on $X = \{0, 1\}^\mathbb{Z}$ or on the Golden Mean SFT $X_G \subset \{0, 1, 2\}^\mathbb{Z}$, we take an input of the form $\{p_k, M_k, N_k : k \geq 1\}$, where $1 > p_k \rightarrow 1/2$ and $M_k$ and $N_k$ are positive integers satisfying

$$1 = M_0 < N_1 < M_1 < \cdots < M_{k-1} < N_k < M_k < \cdots .$$

For such input denote the stochastic matrices

$$Q_k = \begin{pmatrix} p_k & 1 - p_k \\ q & 1 - q \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 1/2 \\ q & 1 - q \end{pmatrix}$$

for the fullshift $X$, and the stochastic matrices

$$Q_k = \begin{pmatrix} p_k & 0 & 1 - p_k \\ q & 0 & 1 - q \\ 0 & 1 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1/2 & 0 & 1/2 \\ q & 0 & 1 - q \\ 0 & 1 & 0 \end{pmatrix}$$

for the Golden Mean SFT $X_G$. Then let $\mu_{\{p_k, M_k, N_k : k \geq 1\}}$ be the Markov measure with transition matrices $(P_n : n \in \mathbb{Z})$ defined by

$$P_n = \begin{cases} Q_k, & n \in [M_{k-1}, N_k) \\ Q, & \text{otherwise} \end{cases}, \text{ for } n \in \mathbb{Z},$$

with an appropriate definition of the transition matrices $Q_k$ depending on whether we consider $X$ or $X_G$. The coordinate distributions $(\pi_n : n \in \mathbb{Z})$ can be chosen arbitrarily as long as they satisfy the consistency condition $\pi_n P_n = \pi_{n+1}$ for all $n \in \mathbb{Z}$. This defines a Markov measure with the Doeblin condition on $X$ or on $X_G$ that fits the convergent scenario we discussed in Theorems 4 and 5.
Remark 9.1. Note that the particular choice of $q$ that has been made in [26] was designed only to compute essential values, which we do not need, hence we consider an arbitrary $q$. Further note that in the Golden Mean SFT it is obvious that the Markov measure $\mu_{\{p_k,M_k,N_k:k \geq 1\}}$ is not equivalent to a non-homogeneous product measure. But also in the fullshift, as

$$P_n \xrightarrow{n \to \infty} \begin{pmatrix} 1/2 & 1/2 \\ q & 1-q \end{pmatrix},$$

by choosing $q \neq 1/2$ and using Corollary 5.4 we ensure that $\mu_{\{p_k,M_k,N_k:k \geq 1\}}$ cannot be equivalent to any product measure. Thus, this construction provides a class of examples for Markov fullshift and Markov SFT with Markov measures that are not equivalent to product measures and are of type III$_1$.

Proposition 9.2. Let $\mu = \mu_{\{p_k,M_k,N_k:k \geq 1\}}$ be as above, either for the fullshift or for the Golden Mean SFT. Then we have the following:

1. The shift is nonsingular with respect to $\mu$ if, and only if,

   $$(9.2.1) \quad \sum_{k \geq 1} (p_k - 1/2)^2 < \infty.$$  

2. The only homogeneous Markov measure that can be equivalent to $\mu$ is the Markov measure $\nu$ defined by the transition matrix $Q$, and $\mu$ is not equivalent to $\nu$ if, and only if,

   $$(9.2.2) \quad \sum_{k \geq 1} (N_k - M_{k-1})(p_k - 1/2)^2 = \infty.$$  

Proof. For the nonsingularity we see that

$$\sum_{n \geq 1} \sum_{s,t \in S} (\sqrt{P_n(s,t)} - \sqrt{P_{n-1}(s,t)})^2 = \sum_{k \geq 1} \sum_{s,t \in S} (\sqrt{Q_k(s,t)} - \sqrt{Q(s,t)})^2$$

and that

$$(9.2.3) \quad (\sqrt{Q_k(s,t)} - \sqrt{Q(s,t)})^2 \preceq (p_k - 1/2)^2$$

for all $s, t \in S$, so the criteria follows from Corollary 5.3. For the equivalence of $\mu$ to $\nu$, we see that

$$\sum_{n \geq 1} \sum_{s,t \in S} (\sqrt{P_n(s,t)} - \sqrt{Q(s,t)})^2$$

$$= \sum_{k \geq 1} \sum_{s,t \in S} (N_k - M_{k-1})(\sqrt{Q_k(s,t)} - \sqrt{Q(s,t)})^2.$$  

Then by the approximation (9.2.3) the criteria follows from Corollary 5.4. 

$\Box$
The following proposition was proved by Kosloff as part of [26, Theorem 13] for the Golden Mean SFT. It is based on a calculation that can be carried out for both the fullshift and the Golden Mean SFT.

**Proposition 9.3** (Kosloff). Let \( \mu = \mu\{p_k, M_k, N_k : k \geq 1\} \) be as above, either for the fullshift or for the Golden Mean SFT. Let us denote \( \lambda(p) = p_1 - p \) for \( p \in (0, 1) \). Then the shift is conservative with respect to \( \mu \) if

\[
(9.3.1) \quad 1 < \lambda(p_k)^{M_k-1} \leq e^{r_k} \quad \text{where } \sum_{k \geq 1} r_k < \infty,
\]

and

\[
(9.3.2) \quad \sum_{k \geq 1} (M_k - 2N_k)\lambda(p_1)^{-N_k} = \infty.
\]

Note that with condition (9.3.1) we have \( (p_k - 1/2)^2 \asymp (1 - \lambda(p_k))^2 = o(r_k^2) \) as well as \( \sum_{k \geq 1} r_k < \infty \), so this implies condition (9.2.1) for the nonsingularity. The conditions of Propositions 9.2 and 9.3 can be fulfilled simultaneously for various choices of \( \{p_k, M_k, N_k : k \geq 1\} \) and it may be constructed inductively as follows. Fix some sequence \( (r_k : k \geq 1) \) of positive numbers with \( \sum_{k \geq 1} r_k < \infty \). Set an arbitrary \( 1/2 < p_1 < 1 \) and arbitrary positive integers \( M_0 < N_1 < M_1 \). Assume that \( \{p_j, M_j, N_j : 1 \leq j \leq k\} \) has been defined for some \( k \in \mathbb{N} \). Define \( p_{k+1}, N_{k+1} \) and \( M_{k+1} \) as follows:

1. choose \( p_{k+1} \) such that \( 1 < \lambda(p_{k+1})^{M_k} \leq e^{r_k} \);
2. choose \( N_{k+1} > M_k \) such that \( (N_{k+1} - M_k)(1 - \lambda(p_{k+1}))^2 \geq 1 \); and,
3. choose \( M_{k+1} > N_{k+1} \) such that \( (M_{k+1} - 2N_{k+1})\lambda(p_1)^{-N_{k+1}} \geq 1 \).

The following corollary then follows immediately from our Theorems 4 and 5.

**Corollary 9.4.** If \( \mu = \mu\{p_k, M_k, N_k : k \geq 1\} \) satisfies conditions (9.2.2) as well as conditions (9.3.1) and (9.3.2), then \( \mu \) is a Markov measure which is not equivalent to any product measure, and the shift is nonsingular, conservative and of type III_1 with respect to \( \mu \).

The construction of a conservative divergent Markov measure is more subtle. Yet, the class of conservative divergent Markov measures is non-empty as there is the construction of such Bernoulli shift due to Kosloff [25].

**Appendix A. Mixing Properties of Markov Measures**

Let \((X_A, \mu)\) be a topologically-mixing MSFT that satisfies the Doeblin condition \((D)\). The sequence of transition matrices of \( \mu \) will be denoted by \((P_n : n \in \mathbb{Z})\)
and the coordinate distributions by \((\pi_n : n \in \mathbb{Z})\). The integer \(M\) will stand for the first positive integer for which \(A^M > 0\) and the constant \(\delta\) is the positive constant of the Doeblin condition.

**Lemma A.1.** The marginals of \(\mu\) satisfy

\[
\delta^M \leq \pi_n(s) \leq 1 - \delta^M, \quad n \in \mathbb{Z}, s \in S.
\]

Also for every \(N \geq 1\) sufficiently large, specifically \(N \geq M\), we have that

\[
\delta^M \leq \mathbb{P}^{(n,n+N)}(s,t) \leq 1 - \delta^M, \quad n \in \mathbb{Z}, s, t \in S.
\]

**Proof.** We start by bounding the transition matrices. It is an immediate observation that

\[
\mathbb{P}^{(n,n+M-1)}(s,t) \geq \delta^M, \quad n \in \mathbb{Z}, s, t \in S.
\]

Thus, for any \(N \geq M\),

\[
\mathbb{P}^{(n,n+N)}(s,t) = \sum_{u \in S} \mathbb{P}^{(n,n+N-M)}(s,u)\mathbb{P}^{(n+M-1,n+N)}(u,t) 
\geq \sum_{u \in S} \mathbb{P}^{(n,n+N-M)}(s,u)\delta^M = \delta^M, \quad n \in \mathbb{Z}, s, t \in S.
\]

Now we easily deduce the same bound for the coordinate distributions:

\[
\pi_n(s) = \sum_{t \in S} \pi_{n-M}(t)\mathbb{P}^{(n-M,n-1)}(t,s) 
\geq \sum_{t \in S} \pi_{n-M}(t)\delta^M = \delta^M, \quad n \in \mathbb{Z}, s \in S.
\]

Those lower bounds yield the upper bounds so the proof is complete. \(\square\)

**Lemma A.2.** There exists a constant \(C(\delta, M) \in (0, 1)\) depending only on \(M\) and \(\delta\), such that for every pair of Borel sets \(E \in \sigma(\ldots, X_{n-1}, X_n)\) and \(F \in \sigma(X_m, X_{m+1}, \ldots)\), if \(m - n \geq M\) then

\[
C(\delta, M)\mu(E)\mu(F) \leq \mu(E \cap F) \leq C(\delta, M)^{-1}\mu(E)\mu(F).
\]

**Proof.** Observe that for \(E \in \sigma(\ldots, X_{n-1}, X_n)\) and \(m\) with \(m - n \geq M\),

\[
\mu(E \cap \{X_m = s\}) = \sum_{t \in S} \mu(E \cap \{X_{n+1} = t\})\mu(\{X_m = s\} \mid E \cap \{X_{n+1} = t\}) 
= \sum_{t \in S} \mu(E \cap \{X_{n+1} = t\})\mu(\{X_m = s\} \mid \{X_{n+1} = t\}) 
= \sum_{t \in S} \mu(E \cap \{X_{n+1} = t\})\mathbb{P}^{(n+1,m-1)}(t,s) 
\geq \delta^M \sum_{t \in S} \mu(E \cap \{X_{n+1} = t\}) = \delta^M \mu(E), \quad s \in S,
\]
where we used the Markov property and the lower bound of Lemma A.1. If instead we use the upper bound of Lemma A.1 we get that

\[
\mu(E \cap \{X_m = s\}) \leq (1 - \delta^M) \mu(E), \quad s \in S.
\]

Now we conclude:

\[
\mu(E \cap F) = \sum_{s \in S} \mu(E \cap \{X_m-1 = s\}) \mu(F \mid E \cap \{X_m-1 = s\})
\]

\[
= \sum_{s \in S} \mu(E \cap \{X_m-1 = s\}) \mu(F \mid \{X_m-1 = s\})
\]

\[
\geq \delta^M \mu(E) \sum_{s \in S} \mu(F \mid \{X_m-1 = s\})
\]

\[
= \delta^M \mu(E) \sum_{s \in S} \frac{\mu(F \cap \{X_m-1 = s\})}{\pi_{m-1}(s)}
\]

\[
\geq \frac{\delta^M}{1 - \delta^M} \mu(E) \sum_{s \in S} \mu(F \cap \{X_m-1 = s\})
\]

\[
= \frac{\delta^M}{1 - \delta^M} \mu(E) \mu(F),
\]

where we used the Markov property, the above observations and the lower bound of Lemma A.1. A similar use of the upper bound of Lemma A.1 shows that

\[
\mu(E \cap F) \leq \frac{1 - \delta^M}{\delta^M} \mu(E) \mu(F).
\]

Then the Lemma holds for the constant \(C(\delta, M) = \delta^M/(1 - \delta^M) > 0.\) □

**Appendix B. A criteria for equivalence of Markov measures**

Let \(S\) be a finite state space and let \(X_A\) be a topologically-mixing SFT in \(S^\mathbb{Z}\). Let \(\nu\) and \(\mu\) be a pair of Markov measures on \(X_A\) defined by \((\pi_n, P_n : n \in \mathbb{Z})\) and \((\lambda_n, Q_n : n \in \mathbb{Z})\), respectively. Recall that \((\pi_n, \hat{P}_n : n \in \mathbb{Z})\) is the sequence of the reversed sequence of transitions of \(\nu\),

\[
\hat{P}_n(s, t) = \nu(X_n-1 = t \mid X_n = s) = \frac{\pi_{n-1}(t)}{\pi_n(s)} P_{n-1}(t, s), \quad s, t \in S, n \in \mathbb{Z}.
\]

Then \(\pi_n \hat{P}_n = \pi_{n-1}\) for \(n \in \mathbb{Z}\). Clearly, a Markov measure that is specified by \((P_n : n \in \mathbb{Z})\) with the usual convention of the dependence direction is also specified by \((\hat{P}_n : n \in \mathbb{Z})\) for the reversed dependence direction.

For the sake of completeness we repeat the formulation of Theorem 5.2.
Theorem B.1. Let \( \nu \) and \( \mu \) be Markov measures on a topologically-mixing SFT \( X_A \) defined by \((P_n : n \in \mathbb{Z})\) and \((Q_n : n \in \mathbb{Z})\), respectively. Suppose that both satisfy the Doeblin condition \((D)\). Then \( \nu \ll \mu \) if, and only if,

\[
\sum_{n \geq 1} \sum_{s,t,u,v \in S} d_n^2[\nu, \mu](s, u, v, t) < \infty,
\]

where for \( n \geq 1 \) and \( s, t, u, v \in S \) we denote the numbers

\[
d_n^2[\nu, \mu](s, u, v, t) := \left( \sqrt{P_n(u, s)} - \sqrt{Q_n(u, s)} \right)^2.
\]

In order to prove Theorem 5.2 we use a significant generalization of the Kakutani dichotomy, established by Kabanov–Lipcer–Shiryaev (see [36] and references therein). Their theorem is formulated in the following setting. Let \( X \) be a standard Borel space and fix \((A_n : n \geq 1)\) a filtration of \( X \). Let \( \nu \) and \( \mu \) be probability measures on \( X \). For every \( n \geq 1 \) let \( \nu_n \) and \( \mu_n \) be the restriction of \( \nu \) and \( \mu \) to \( A_n \), respectively, and suppose that \( \nu_n \ll \nu_n \) for all \( n \geq 1 \). Let

\[
m_n(x) := \frac{d\nu_n}{d\mu_n}(x) \quad \text{for } n \geq 1.
\]

Then \((m_n : n \geq 1)\) is a martingale with respect to the natural filtration and it satisfies

\[
\int_X m_n(x) d\mu(x) = 1
\]

for every \( n \geq 1 \), so by the martingale convergence theorem

\[
m_\infty(x) := \lim_{n \to \infty} m_n(x) \text{ exists for } \mu\text{-a.e. } x \in X.
\]

In fact, this limit exists also for \( \nu\text{-a.e. } x \in X \). See also [37, Chapter 6] for a comprehensive representation.

Theorem B.2 (Kabanov–Lipcer–Shiryaev). In the above setting, let

\[
M_n(x) := m_n(x)m_{n-1}^{-1}(x), \quad \text{where } M_n(x) := 0 \text{ if } m_{n-1}(x) = 0.
\]

Then

\[
\nu \ll \mu \iff \sum_{n \geq 1} \left( 1 - \mathbb{E}_\mu(\sqrt{M_n(x)} \mid A_{n-1}) \right) < \infty \text{ for } \nu\text{-a.e. } x \in X.
\]

Consider a topologically-mixing SFT \( X_A \) and let \((A_n : n \geq 1)\) be the natural filtration defined by \( A_n = \sigma(X_k : |k| \leq n) \). Let \( \nu \) and \( \mu \) be Markov measures on \( X_A \) defined by \((P_n : n \in \mathbb{Z})\) and \((Q_n : n \in \mathbb{Z})\), respectively. Suppose that both \( \nu \) and \( \mu \) satisfy the Doeblin condition. The following result was established in [31, 30] and we deduce it from Theorem B.2 by a straightforward calculation.
**Corollary B.3.** In the above setting we have

\[ v \ll \mu \iff \sum_{n \geq 1} \sum_{s,t \in S} d_n^2[v, \mu](s, x_{-(n-1)}, x_{n-1}, t) < \infty \]

\[(B.3.1)\]

for \( v \)-a.e. \( x \in X_A \),

where for \( n \geq 1 \) and \( s, t \in S \) we denote the random variables

\[ d_n^2[v, \mu](s, x_{-(n-1)}, x_{n-1}, t) := \left( \sqrt{\hat{P}_{-(n-1)}(u, x_{-(n-1)}) \hat{P}_{n}(x_{n-1}, t)} - \sqrt{\hat{Q}_{-(n-1)}(x_{-(n-1)}, s) \hat{Q}_{n}(x_{n-1}, t)} \right)^2. \]

**Proof.** It is clear that every such \( v \) and \( \mu \) satisfy \( v_n \ll \mu_n \) for all \( n \geq 1 \). Let us calculate \( E_{\mu} [\sqrt{\mathcal{M}_n} \mid A_{n-1}] \). The Radon–Nikodym derivatives are given by

\[ m_n(x) := \frac{dv_n}{d\mu_n}(x) = \frac{\pi_{-(n-1)}(x_{n-1})}{\lambda_{-(n-1)}(x_{n-1})} \prod_{i=-n}^{-1} P_{i}(x_i, x_{i+1}) \frac{P_{n-1}(x_{n-1}, x_n)}{Q_{n-1}(x_{n-1}, x_n)}, \]

so one can see that

\[ M_n(x) = \frac{\hat{P}_{-(n-1)}(x_{-(n-1)}, x_{n-1}) P_{n-1}(x_{n-1}, x_n)}{\hat{Q}_{-(n-1)}(x_{-(n-1)}, x_{n-1}) Q_{n-1}(x_{n-1}, x_n)}. \]

It follows that

\[ E_{\mu} (\sqrt{M_n(x)} \mid A_{n-1}) = \sum_{s,t \in S} \sqrt{ \frac{\hat{P}_{-(n-1)}(s, x_{-(n-1)}) P_{n-1}(x_{n-1}, t)}{\hat{Q}_{-(n-1)}(s, x_{-(n-1)}) Q_{n-1}(x_{n-1}, t)} } \frac{\hat{Q}_{-(n-1)}(s, x_{-(n-1)}) Q_{n-1}(x_{n-1}, t)}{\hat{Q}_{-(n-1)}(s, x_{-(n-1)}) Q_{n-1}(x_{n-1}, t)} \]

\[ = 1 - \frac{1}{2} \sum_{s,t \in S} \left( \sqrt{\hat{P}_{-(n-1)}(s, x_{-(n-1)}) P_{n-1}(x_{n-1}, t)} - \sqrt{\hat{Q}_{-(n-1)}(s, x_{-(n-1)}) Q_{n-1}(x_{n-1}, t)} \right)^2. \]

This shows that

\[ 1 - E_{\mu} (\sqrt{M_n(x)} \mid A_n) = \frac{1}{2} \sum_{s,t \in S} d_n^2[v, \mu](s, x_{-(n-1)}, x_{n-1}, t), \quad n \geq 1, \]

and from Theorem B.2 the proof is complete.
It follows from Corollary B.3 that if \( \sum_{n \geq 1} \sum_{s,t,u,v \in S} d_n^2[v, \mu](s, u, v, t) < \infty \) then \( \sum_{n \geq 1} \sum_{s,t,u,v \in S} d_n^2[v, \mu](s, x_{-(n-1)}, x_{n-1}, t) < \infty \) for every \( x \in X_4 \) so that \( v \ll \mu \). Thus, to prove Theorem 5.2 we need to show that the right-hand side of condition (B.3.1) does not hold if \( \sum_{n \geq 1} \sum_{s,t,u,v \in S} d_n^2[v, \mu](s, u, v, t) = \infty \) for every \( x \in X_4 \) so that \( \nu \ll \mu \).

**Lemma B.4.** Let \( (a_n : n \geq 1) \) be a sequence of non-negative numbers such that \( \sum_{n \geq 1} a_n = \infty \). Let \( (\Omega, \mathbb{P}) \) be a probability space and let \( (A_n : n \geq 1) \) be a sequence of events in \( \Omega \). Then

\[
\mathbb{P}\left( \sum_{n \geq 1} a_n 1_{A_n} = \infty \right) \geq \liminf_{n \to \infty} \mathbb{P}(A_n).
\]

**Proof.** Denote \( p := \liminf_{n \to \infty} \mathbb{P}(A_n) \). Excluding trivialities assume \( p > 0 \) and let \( 0 < \epsilon < p \) be arbitrary. Fix \( N \geq 1 \) such that \( \mathbb{P}(A_n) \geq \epsilon \) for every \( n \geq N \). For every \( C > 0 \) let

\[
F_C := \left\{ \sum_{n \geq N} a_n 1_{A_n} \leq C \right\},
\]

and suppose toward a contradiction that \( \mathbb{P}(F_C) > 1 - \epsilon \) for some \( C > 0 \). Then

\[
\mathbb{P}(F_C) - \mathbb{P}(A_n^c) > \mathbb{P}(F_C) - (1 - \epsilon) > 0, \quad n \geq N.
\]

We then have

\[
\begin{align*}
\infty &= \sum_{n \geq N} a_n (\mathbb{P}(F_C) - (1 - \epsilon)) \\
&\leq \sum_{n \geq N} a_n (\mathbb{P}(F_C) - \mathbb{P}(A_n^c)) \\
&\leq \sum_{n \geq N} a_n \mathbb{P}(F_C \cap A_n) = \sum_{n \geq N} a_n \mathbb{E}[1_{F_C} 1_{A_n}] \\
&= \mathbb{E}\left[1_{F_C} \sum_{n \geq N} a_n 1_{A_n}\right] \\
&\leq C \mathbb{P}(F_C),
\end{align*}
\]

where the second inequality is general: \( \mathbb{P}(E \cap F) \geq \mathbb{P}(E) - \mathbb{P}(F^c) \), the next equality is by monotone convergence and the last one is by the definition of \( F_C \). This is a contradiction so that \( \mathbb{P}(F_C) \leq 1 - \epsilon \) for all \( C \geq 0 \). It follows that

\[
\mathbb{P}\left( \sum_{n \geq 1} a_n 1_{A_n} < \infty \right) = \mathbb{P}\left( \sum_{n \geq N} a_n 1_{A_n} < \infty \right) \leq \limsup_{C \to \infty} \mathbb{P}(F_C) \leq 1 - \epsilon.
\]

Since \( 0 < \epsilon < p \) is arbitrary the proof is complete. \( \square \)
**Proof of Theorem 5.2.** One of the implications is immediate from Corollary B.3 as we already mentioned. For the other implication suppose that

\[ \sum_{n \geq 1} \sum_{s, t, u, v \in S} d_n^2[v, \mu](s, u, v, t) = \infty. \]

Then \( \sum_{n \geq 1} \sum_{s, t \in S} d_n^2[v, \mu](s, u_0, v_0, t) = \infty \) for some \( u_0, v_0 \in S \). Consider the sets

\[ A_n := \{ X_{-(n-1)} = u_0, X_{n+1} = v_0 \}, \quad n \geq 1. \]

By the topologically-mixing, the Doeblin condition and Proposition 5.1,

\[ \liminf_{n \to \infty} \nu(A_n) \geq C(\delta, M) \liminf_{n \to \infty} \pi_{-(n-1)}(u) \pi_{n+1}(v) \geq C(\delta, M) \delta^2 > 0. \]

Then by the Lemma B.4 we conclude that

\[ \sum_{n \geq 1} \sum_{s, t \in S} d_n^2[v, \mu](s, x_{-(n-1)}, x_{n-1}, t) \]

\[ = \sum_{n \geq 1} \left( \sum_{s, t, u, v \in S} d_n^2[v, \mu](s, u, v, t) \right) 1_{A_n}(x) = \infty \]

on a \( \nu \)-positive measure set (of measure at least \( C(\delta, M) \delta^2 \)). Then by condition (B.3.1) \( \nu \) is not absolutely continuous with respect to \( \mu \). \( \square \)

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