Negative dimensional group extrapolation and dualities in $N = 1$ supersymmetric gauge theories

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Abstract

We point out that the similarities in $N = 1$ supersymmetric $SO$, $SP$ gauge theories can be explained by using the trick of extrapolating the groups to the negative dimensions. One of the advantages of this trick is that anomaly matching is automatically satisfied.
1 Introduction

In a couple of years, the understanding of non-perturbative properties of supersymmetric gauge theories has been rapidly enlarged. Especially, duality proposed by Seiberg [1] gives a clue to analyze the strong interaction dynamics. Many authors [2] have suggested dualities which have various gauge groups ( $SU, SO, SP$ group and their product group ), various matter contents (at most 2-index tensor ).

In this paper we point out that the similarities of dualities in $N = 1$ supersymmetric $SO, SP$ gauge theories can be explained by using the trick of negative dimensional groups. In constructing the duality in $N = 1$ supersymmetric gauge theories one of the key idea is anomaly matching. Anomaly matching conditions heavily depend on the representations and the charges of matter fields in the model. Therefore we have to find the matter content of the model by trial and error. However if we know one $SO(SP)$ duality we can easily obtain another $SP(SO)$ duality which automatically satisfies anomaly matching conditions by using the trick of negative dimensional groups.

The idea of negative dimensional groups is not new and goes back to Penrose [4] who has constructed the $SU(2)$ ( = $SP(2)$ ) representations in terms of $SO(-2)$. Since then many relations have been observed among the expressions for the $SU(N)$, $SO(N)$ and $SP(N)$ group invariants under the substitution $N \rightarrow -N$ [5]. On the other hand, Parisi and Sourlas [6] have observed that a Grassmann space of dimension $N$ can be interpreted as an ordinary space of dimension $-N$. In supersymmetric theories the $N \rightarrow -N$ relations are, in a sense, built in and we are going to utilize this property in this paper.

2 Similarities in $SO$ and $SP$ duality

In order to show the advantages of our trick of negative dimensional groups we focus here on dualities with $SO$ and $SP$ gauge groups [4]. It is well known that these models strongly resemble each other in appearance. As an example, we shall take the model proposed by Intriligator [7]. In Ref.[4], duality in supersymmetric $SO$ and $SP$ gauge theories are discussed. The electric theory of $SO$ dual model is a $N = 1$ supersymmetric $SO(2N_c)$ gauge theory with $2N_f$ fields $Q_i$ in the fundamental representation and a

\footnote{As will be explained later, $SU(N)$ group is self-dual under $N \rightarrow -N$. Therefore we don’t discuss $SU$ group here.}

\footnote{In order to see the relation with the $SP$ groups we are restricting our discussion to the even dimensional $SO(2N_c)$ groups leaving aside the $SO(2N_c + 1)$ groups}
symmetric traceless tensor $X$. The anomaly free global symmetries are $SU(2N_f) \times U(1)_R$ with the fields transforming as

$$Q^i \begin{pmatrix} 2N_f, 1 - \frac{2(N_c - k)}{(k + 1)N_f} \end{pmatrix},$$

$$X \begin{pmatrix} 1, \frac{2}{k + 1} \end{pmatrix}. \quad (1)$$

The superpotential is

$$W = g_k Tr X^{k+1}. \quad (2)$$

The magnetic theory is $N = 1$ supersymmetric $SO(2N_c)$ gauge theory, where $\tilde{N}_c \equiv k(N_f + 2) - N_c$, with $2N_f$ fields $q^i$ in the fundamental representation, a symmetric traceless tensor $Y$ and singlets $M_j (j = 1, \cdots, k)$. The anomaly free global symmetries are $SU(2N_f) \times U(1)_R$ with the fields transforming as

$$q^i \begin{pmatrix} 2N_f, 1 - \frac{2(\tilde{N}_c - k)}{(k + 1)N_f} \end{pmatrix},$$

$$Y \begin{pmatrix} 1, \frac{2}{k + 1} \end{pmatrix},$$

$$M_j \begin{pmatrix} N_f(2N_f + 1), \frac{2(j + k)}{(k + 1)} - \frac{4(N_c - k)}{(k + 1)N_f} \end{pmatrix}. \quad (3)$$

The superpotential is

$$W = Tr Y^{k+1} + \sum_{j=1}^{k} M_j q^i Y^{k-j} q. \quad (4)$$

On the other hand, the electric theory of $SP$ dual model is a $N = 1$ supersymmetric $SP(2N_c)$ gauge theory with $2N_f$ fields $Q^i$ in the fundamental representation and an antisymmetric traceless tensor $X$. The global symmetries are $SU(2N_f) \times U(1)_R$ with fields transforming as

$$Q^i \begin{pmatrix} 2N_f, 1 - \frac{2(N_c + k)}{(k + 1)N_f} \end{pmatrix},$$

$$X \begin{pmatrix} 1, \frac{2}{k + 1} \end{pmatrix}. \quad (5)$$

$^3$In this paper, we denote the symplectic group as $SP(2N_c)$ whose fundamental representation is $2N_c$ dimensional.
The superpotential is

\[ W = g_k \text{Tr} X^{k+1}. \]  

(6)

The magnetic theory is \( N = 1 \) supersymmetric \( SP(2\tilde{N}_c) \) gauge theory, where \( \tilde{N}_c \equiv k(N_f - 2) - N_c \), with \( 2N_f \) fields \( q^i \) in the fundamental representation, an antisymmetric traceless tensor \( Y \) and singlets \( M_j (j = 1, \cdots, k) \). The global symmetries are \( SU(2N_f) \times U(1)_R \) with the fields transforming as

- \( q^i \) \( \left( \frac{2N_f}{k+1}, 1 - \frac{2(\tilde{N}_c + k)}{(k+1)N_f} \right) \),
- \( Y \) \( \left( 1, \frac{2}{k+1} \right) \),
- \( M_j \) \( \left( N_f(2N_f - 1), \frac{2(j + k)}{(k+1)N_f} - \frac{4(N_c + k)}{(k+1)N_f} \right) \).

(7)

The superpotential is

\[ W = \text{Tr} Y^{k+1} + \sum_{j=1}^{k} M_j q Y^{k-j} q. \]  

(8)

It is easy to recognize that the representations and the charges of fields are quite similar. Furthermore, it can be seen that we can obtain \( SP(SO) \) duality from the \( SO(SP) \) duality by changing the signs of \( N_c, N_f \) into \( -N_c, -N_f \) and exchanging a symmetric (an antisymmetric) tensor for an antisymmetric (a symmetric) tensor. This feature is not specific to these models and is applicable to \( SO, SP \) dual models discovered so far [2].

3 Negative dimensional group

Group theoretically, these can be anticipated by considering the negative dimensional groups first proposed by Penrose [4]. This is a technique to calculate the algebraic invariants. Using this technique, we can find the peculiar relations for dimensions of the irreducible representations of the classical groups \( SU(N), SO(N), SP(N) \) [5]. If \( \lambda_s \) is a Young tableau with \( s \) boxes and if the dimensions of the corresponding irreducible representations of \( SU(N), SO(N) \) and \( SP(N) \) are denoted by \( D(\lambda_s; N) \), \( D[\lambda_s; N] \) and \( D(\lambda_s; N) \), respectively, it was noticed by King [6] that
Here \( \tilde{\lambda} \) stands for the "transposed" (rows and columns interchanged) Young tableau. Moreover, it is useful to give the relations among the generalized Casimirs of the classical groups in totally symmetric and totally antisymmetric representations.

\[
C_p^{SU(N)}(1,1,\ldots,1) = (-1)^{p-1}C_p^{SU(-N)}(r,0,\ldots,0), \quad (11)
\]
\[
C_p^{SO(2N)}(1,1,\ldots,1) = (-1)^{p-1}C_p^{SP(-2N)}(r,0,\ldots,0), \quad (12)
\]
\[
C_p^{SP(2N)}(1,1,\ldots,1) = (-1)^{p-1}C_p^{SO(-2N)}(r,0,\ldots,0), \quad (13)
\]

where \( C_p(1,1,\ldots,1) \) and \( C_p(r,0,\ldots,0) \) mean the p-th order generalized Casimir in totally antisymmetric and totally symmetric rank-r tensor representations respectively.

These relations Eqs.(9)-(13) are necessary to take anomaly matching conditions into account.

We can express these results symbolically as follows [5],

\[
SU(-N) \cong \overline{SU(N)},
\]
\[
SO(-N) \cong \overline{SP(N)},
\]
\[
SP(-N) \cong \overline{SO(N)}, \quad (14)
\]

where the overbar means symmetrization and antisymmetrization are interchanged.

Since the supersymmetric theories are "invariant" under this interchange we can use this technique as a useful method of obtaining a dual model from another dual model through the extrapolation to the negative dimensional groups. The procedures are:

- Change the sign of the group dimension, \( N_c \leftrightarrow -N_c, N_f \leftrightarrow -N_f. \)

- Interchange the symmetrization and antisymmetrization of the representations.

We can actually convince ourselves that with these procedures Eq.(7) follows from Eq.(3). This negative dimensional group technique is very powerful since the anomaly matching is automatically satisfied.
4 Summary and Discussion

In this paper we have pointed out that the similarities of dualities in $N = 1$ supersymmetric $SO$, $SP$ gauge theories can be explained by using the trick of negative dimensional groups. If we know the duality in supersymmetric $SO(SP)$ gauge theory we can easily obtain another duality in supersymmetric $SP(SO)$ gauge theory which automatically satisfies the anomaly matching conditions by extrapolating the groups to the negative dimensions in the model. By this trick we can also know the representations and the charges of the fields. On the other hand, when there is no known duality this trick is powerless.

Explicit application of this trick in finding new dualities is left for future studies. It is interesting to see whether this trick is applicable to other groups. We have described the substitution $N \to N$ just as a useful trick for studying the duality structures of supersymmetric theories leaving aside the direct significance it might have in such theories. Thus it can be interesting to study the symmetries under $N \to N$ directly in the supersymmetric theories where duality is realized explicitly. We hope to report elsewhere these together with the related problems.

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