Semiclassical analysis for a Schrödinger operator with a U(2) artificial gauge: the periodic case

A. Morame\textsuperscript{1} and F. Truc\textsuperscript{2}

January 27, 2014

\textsuperscript{1} Université de Nantes, Faculté des Sciences, Dpt. Mathématiques, UMR 6629 du CNRS, B.P. 99208, 44322 Nantes Cedex 3, (FRANCE), E.Mail: Abderemane.Morame@univ-nantes.fr
\textsuperscript{2} Université de Grenoble I, Institut Fourier, UMR 5582 CNRS-UJF, B.P. 74, 38402 St Martin d’Hères Cedex, (France), E.Mail: Francoise.Truc@ujf-grenoble.fr

Abstract

We consider a Schrödinger operator with a Hermitian 2x2 matrix-valued potential which is lattice periodic and can be diagonalized smoothly on the whole $\mathbb{R}^n$. In the case of potential taking its minimum only on the lattice, we prove that the well-known semiclassical asymptotic of first band spectrum for a scalar potential remains valid for our model.

Keywords : semiclassical asymptotic, spectrum, eigenvalues, Schrödinger, periodic potential, BKW method, width of the first band, magnetic field.

AMS MSC 2000 : 35J10, 35P15, 47A10, 81Q10, 81Q20.

Contents

1 Introduction 2

2 Preliminary: the artificial gauge model 3

3 Proof of Theorem 2.1 5

4 Asymptotic of the first band 6

5 B.K.W. method for the Dirichlet ground state 11
1 Introduction

Schrödinger operators with periodic matrix-valued potentials appear in many models in physics. Such models have been used recently to describe the motion of an atom in optical fields (Co, Co-Da, Da-ai), see also Ca-Yu. The aim of this paper is to investigate their spectral properties using semiclassical analysis. We focus on the first spectral band and assume that the potential has a non degenerate minimum. The Schrödinger operators with a non-Abelian gauge potential are Hamiltonian operators on $L^2(\mathbb{R}^n; \mathbb{C}^m)$ of the following form:

$$H^h = h^2 \sum_{k=1}^{n} (D_{x_k} I - A_k)^2 + V + hQ + h^2 R = P^h(x, hD).$$

(1.1)

The classical symbol of $P^h(x, hD)$, $P^h(x, \xi)$, for $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, is given by

$$P^h(x, \xi) = \sum_{k=1}^{n} \left\{ (\xi_k I - h A_k(x))^2 + i h^2 \partial_{x_k} A_k(x) \right\} + V(x) + hQ(x) + h^2 R(x),$$

(1.2)

$I$ is the identity $m \times m$ matrix, $V$, $Q$, $R$ and the $A_k$ are hermitian $m \times m$ matrix with smooth coefficients and $\Gamma$ periodic:

\[
\{ \begin{array}{l}
A_k = (a_{k,ij}(x))_{1 \leq i,j \leq m}, \\
V = (v_{ij}(x))_{1 \leq i,j \leq m}, \\
Q = (q_{ij}(x))_{1 \leq i,j \leq m}, \\
R = (r_{ij}(x))_{1 \leq i,j \leq m}; \\
ak_{k,ij}, v_{ij}, q_{ij}, r_{ij} \in C^\infty(\mathbb{R}^n; \mathbb{C}), \\
a_{k,ji} = a_{k,ij}, v_{ji} = v_{ij}, q_{ji} = q_{ij}, r_{ji} = r_{ij}, \\
ak_{k,ij}(x - \gamma) = a_{k,ij}(x), v_{ij}(x - \gamma) = v_{ij}(x), \\
q_{ij}(x - \gamma) = q_{ij}(x) \text{ and } r_{ij}(x - \gamma) = r_{ij}(x) \forall \gamma \in \Gamma; \\
\end{array} \right.
\]

(1.3)

$\Gamma$ is a lattice of $\mathbb{R}^n$, $\Gamma = \{ \sum_{k=1}^{n} m_k \beta_k; m_k \in \mathbb{Z} \}$,

$\beta_1, \beta_2, \ldots, \beta_n \in \mathbb{R}^n$ form a basis, $\text{det}(\beta_1, \beta_2, \ldots, \beta_n) \neq 0$.

We use the notation $D = (D_{x_1}, \ldots, D_{x_n})$ where $D_{x_k} = -i \partial_{x_k}$, $k = 1 \ldots n$,

so $D^2 = -\Delta$ is the Laplacian operator on $L^2(\mathbb{R}^n)$.

The dual basis $\{ \beta_1^*, \ldots, \beta_n^* \}$ of the reciprocal lattice $\Gamma^*$, is the basis of $\mathbb{R}^n$ defined by the relations

$$\beta_j^* \beta_k = 2\pi \delta_{jk} : \Gamma^* = \{ \sum_{k=1}^{n} m_k \beta_k^*; m_k \in \mathbb{Z} \}.$$

The fundamental cell, the Wigner-Seitz cell,

$$\mathcal{W}^n = \{ \sum_{k=1}^{n} x_k \beta_k; x_k \in [ -\frac{1}{2}, \frac{1}{2} ) \}.$$
will be identified with the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n / \Gamma$ and the dual cell, the Brillouin zone, is defined by

$$ \mathbb{B}^n = \left\{ \sum_{k=1}^{n} \theta_k \beta_k^*; \theta_k \in \left[ \frac{1}{2}, \frac{1}{2}\right] \right\} . $$

We will identify $L^2(\mathbb{T}^n; \mathbb{C}^m)$ with $\Gamma$ periodic functions of $L^2_{loc}(\mathbb{R}^n; \mathbb{C}^m)$ provided with the norm of $L^2(W^k(\mathbb{T}^n; \mathbb{C}^m))$, with $k \in \mathbb{N}$, may be identified with $\Gamma$ periodic functions of $W^k_{loc}(\mathbb{R}^n; \mathbb{C}^m)$ provided with the norm of $W^k(\mathbb{W}^n; \mathbb{C}^m)$.

By Floquet theory, (see [EA] or [Re-Si]), we have

$$ H^h = \int_{\mathbb{B}^n} H^{h, \theta} d\theta , $$

with $H^{h, \theta}$ the partial differential operator $P_h(x, h(D - \theta))$ on $L^2(\mathbb{T}^n; \mathbb{C}^m)$.

The ellipticity of $P_h(x, h(D - \theta))$ implies that the spectrum of $H^{h, \theta}$ is discrete

$$ \text{sp}(H^{h, \theta}) = \{ \lambda_j^{h, \theta}; j \in \mathbb{N}^* \}, \quad \lambda_1^{h, \theta} \leq \lambda_2^{h, \theta} \leq \ldots \leq \lambda_j^{h, \theta} \leq \lambda_{j+1}^{h, \theta} \leq \ldots \quad (1.4) $$

each $\lambda_j^{h, \theta}$ is an eigenvalue of finite multiplicity and each eigenvalue is repeated according to its multiplicity.

(When $m = 1$ and $V = Q = R = A_k = 0$, $(\frac{1}{\sqrt{|\mathbb{T}^n|}} e^{i\omega \cdot x})_{\omega \in \Gamma^*}$ is the Hilbert basis of $L^2(\mathbb{T}^n)$ which is composed of eigenfunctions of $h^2(D - \theta)^2$.

The Floquet theory guarantees that

$$ \text{sp}(H^h) = \bigcup_{\theta \in \mathbb{R}^n} \text{sp}(H^{h, \theta}) = \bigcup_{j=1}^{\infty} b_j^h , \quad (1.5) $$

where $b_j^h$ denotes the $j$-th band $b_j^h = \{ \lambda_j^{h, \theta}, \theta \in \mathbb{B}^n \}$.

In the sequel $h_0$ will be a non negative small constant, $h$ will be in $[0, h_0]$, and any non negative constant which doesn’t depend on $h$ will invariably be denoted by $C$.

2 Preliminary: the artificial gauge model

We will be interested in the model of artificial gauge considered in [Co], [Co-Da] and [Da-al]

$$ m = 2, \quad V = vI + W, \quad A_k = Q = R = 0, \quad \forall k, $$

$$ W = w.\sigma, \quad \text{with} \quad w = (w_1, w_2, w_3), \quad v \quad \text{and} \quad w_j \quad \text{are in} \quad C^\infty(\mathbb{R}^n; \mathbb{R}), \quad (2.1) $$

we denote $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, where the $\sigma_j$ are the Pauli matrices.
Let us remark that 

\[ V = vI + W, \quad W = w.\sigma, \quad W^2 = |w|^2I. \tag{2.2} \]

In the sequel we will assume that

\[ |w(x)| > 0 \quad \text{and} \quad v(x) - |w(x)| \]

has a unique non degenerate minimum on \( \mathbb{T}^n \). \tag{2.3}

Due to the invariance of the Laplacian by translation and by the action of \( O(n) \), we can assume, up to a composition by a translation of the potentials, that

\[ v(\gamma) - |w(\gamma)| < v(x) - |w(x)|, \quad \forall x \in \mathbb{R}^n \setminus \Gamma \quad \text{and} \quad \forall \gamma \in \Gamma, \]

\[ v(x) - |w(x)| = E_0 + \sum_{k=1}^{\frac{n}{2}} \tau_k^2 x_k^2 + \mathcal{O}(|x|^3), \quad \text{as} \quad |x| \to 0, \tag{2.4} \]

\( (\tau_k > 0, \ \forall k) \).

Due to the special form of the matrix-valued potential \( V = vI + w.\sigma \), there exists \( U \in \mathbb{U}(2) \), ( a unitary \( 2 \times 2 \) matrix), such that

\[ U^* V U = \tilde{V} = \begin{bmatrix} v - |w| & 0 \\ 0 & v + |w| \end{bmatrix}. \tag{2.5} \]

As \( |w| \) never vanishes, \( U = U(x) \) can be chosen smooth and \( \Gamma \) periodic: \( U \in C^\infty(\mathbb{T}^n; \mathbb{U}(2)) \). More precisely

\[ U = (\alpha, \beta, \rho).\sigma, \quad \text{with} \quad (\alpha, \beta, \rho) \in C^\infty(\mathbb{T}^n; \mathbb{R}^3 \setminus \{0\}). \tag{2.6} \]

Firstly let us expand the formula of the operator

\[ \tilde{H}^h = U^* H^h U = h^2 D^2 I + U^* V U - 2ih^2 \sum_{k=1}^{n} \left( (U^* \partial_{x_k} U)D_{x_k} - h^2 U^* \partial_{x_k}^2 U \right) \]

which can be rewritten as

\[ \tilde{H}^h = U^* H^h U = h^2 \sum_{k=1}^{n} (D_{x_k} I - A_k)^2 + \tilde{V} + h^2 R, \tag{2.7} \]

where

\[ A_k = iU^* \partial_{x_k} U = [(\partial_{x_k} \alpha, \partial_{x_k} \beta, \partial_{x_k} \rho) \wedge (\alpha, \beta, \rho)].\sigma, \tag{2.8} \]

and

\[ R = \sum_{k=1}^{n} \left\{ (U^* \partial_{x_k} U)^2 + (\partial_{x_k} U^*). (\partial_{x_k} U) \right\}. \tag{2.9} \]

So we can assume that \( H^h \) is of the form \( \square \) with \( m = 2, \quad Q = 0, \quad A_k \) and \( R \) given by \( 2.8 \) and \( 2.9 \), with \( U \) defined by \( 2.6 \), and \( V = \tilde{V} \) a diagonal matrix given by \( 2.5 \).
Theorem 2.1 Under the above assumptions, the first bands \( b_j^h \), \( j = 1, 2, \ldots \), of \( H^h \) are concentrated around the value \( h \mu_j + E_0 \), \( j = 1, 2, \ldots \), in the sense that, there exist \( N_0 > 1 \) and \( h_0 > 0 \) such that

\[
distance(h \mu_j + E_0, b_j^h) \leq Ch^2, \quad \forall j < N_0 \text{ and } \forall h, 0 < h < h_0,
\]

where \( \mu_j = \sum_{k=1}^n (2j_k + 1) \tau_k \), \( j_k \in \mathbb{N} \), the \((\mu_i)_{i \in \mathbb{N}}^\star\) is the increasing sequence of the eigenvalues of the harmonic oscillator \(-\Delta + \sum_{k=1}^n \tau_k^2 x_k^2\).

3 Proof of Theorem 2.1

Proof. According to the above discussion, we can assume that

\[
H^h = P^h(x, hD), \quad \text{with } P^h(x, hD) = \begin{pmatrix} P_{11}^h(x, hD) & P_{12}^h(x, hD) \\ P_{21}^h(x, hD) & P_{22}^h(x, hD) \end{pmatrix}, \tag{3.1}
\]

with

\[
\begin{align*}
P_{11}^h(x, hD) &= h^2(D - a_{11}(x))^2 + v(x) - |w(x)| + h^2 r_{11}(x) \\
P_{12}^h(x, hD) &= -h^2 a_{12}(x)(D + a_{11}(x)) - h^2 a_{12}(x)(D - a_{11}(x)) \\
P_{21}^h(x, hD) &= -h^2 a_{21}(x)(D - a_{11}(x)) - h^2 a_{21}(x)(D + a_{11}(x)) \\
P_{22}^h(x, hD) &= +ih^2 \text{div}(a_{12}(x)) + h^2 r_{21}(x)
\end{align*} \tag{3.2}
\]

\((D = (D_{x_1}, D_{x_2}, \ldots, D_{x_n}) \text{ and } a_{i,j}(x) = (a_{1,ij}(x), a_{2,ij}(x), \ldots, a_{n,ij}(x))\).

(We used that \( a_{2,11} = -a_{11} \).

Let us denote by \( H_{11}^{h,\theta} \) and \( H_{22}^{h,\theta} \) the operators associated with \( P_{11}^h(x, h(D - \theta)) \) and \( P_{22}^h(x, h(D - \theta)) \) on \( L^2(\mathbb{T}^n; \mathbb{C}) \).

Then, if \( c_0 = \min |w(x)| \) and \( c_1 = \max \| R(x) \| \),

\[
\text{sp}(H_{11}^{h,\theta}) \subset [E_0 - h^2 c_1, +\infty[ \quad \text{and} \quad \text{sp}(H_{22}^{h,\theta}) \subset [E_0 - h^2 c_1 + 2c_0, +\infty[.
\]

To prove the theorem it is then enough to prove the proposition below.

Proposition 3.1 Let us consider a constant \( c \), \( 0 < c < c_0 \). Then there exists \( C_0 > 0 \) such that, for any \( E^h \in ] - \infty, E_0 + 2c[ \), we have

\[
\begin{align*}
E^h \in \text{sp}(H_{11}^{h,\theta}) &\Rightarrow \text{distance}(E^h, \text{sp}(H_{11}^{h,\theta})) \leq C_0 h^2 \\
E^h \in \text{sp}(H_{11}^{h,\theta}) &\Rightarrow \text{distance}(E^h, \text{sp}(H_{11}^{h,\theta})) \leq C_0 h^2.
\end{align*} \tag{3.3}
\]
Proof. For such $E^h$, $(H_{22}^{h,\theta} - E^h)^{-1}$ exists and, thanks to semiclassical pseudodifferential calculus of $[\text{Ro}]$ (see also $[\text{Di-Sj}]$), for $h_0 > 0$ small, if $0 < h < h_0$ then

$$
\|(H_{22}^{h,\theta} - E^h)^{-1}\|_{L^2(\mathbb{T}^n)} + \|h(D - \theta)(H_{22}^{h,\theta} - E^h)^{-1}\|_{L^2(\mathbb{T}^n)} + \|h(D - \theta)(H_{22}^{h,\theta} - E^h)^{-1}h(D - \theta)\|_{L^2(\mathbb{T}^n)} \leq C,
$$

and then

$$
\|P_{12}^h(x, h(D - \theta))(h_{22}^{h,\theta} - E^h)^{-1}P_{21}^h(x, h(D - \theta))u_1^h\|_{L^2(\mathbb{T}^n)} \leq h^2 C.
$$

So if $E^h \in \text{sp}(H^{h,\theta})$, then $u^h = (u_1^h, u_2^h) \neq (0, 0)$ is an eigenfunction of $H^{h,\theta}$ associated with $E^h$ if

$$
\begin{align*}
H_{11}^{h,\theta}u_1^h + P_{12}^h(x, h(D - \theta))u_2^h &= E^h u_1^h, \\
u_2^h &= -(H_{22}^{h,\theta} - E^h I)^{-1}P_{21}^h(x, h(D - \theta))u_1^h.
\end{align*}
$$

(3.4)

In fact $E^h \in \mathbb{C}$, $E_0 + c\mathbb{C}$ will be an eigenvalue of $H^{h,\theta}$ iff there exists $u_1^h$ in the Sobolev space $W^2(\mathbb{T}^n; \mathbb{C})$, $\|u_1^h\|_{L^2(\mathbb{T}^n)} \neq 0$, such that

$$
H_{11}^{h,\theta}u_1^h - P_{12}^h(x, h(D - \theta))(H_{22}^{h,\theta} - E^h I)^{-1}P_{21}^h(x, h(D - \theta))u_1^h = E^h u_1^h,
$$

then we get the first part of Proposition 3.1.

If $E^h$ is an eigenvalue of $H_{11}^{h,\theta}$ satisfying the assumption of Proposition 3.1, then $u_1^h$ an associated eigenfunction, then with

$$
u^h = (u_1^h, -(H_{22}^{h,\theta} - E^h)^{-1}P_{21}^h(x, h(D - \theta))u_1^h),$$

one has

$$
\|(H^{h,\theta} - E^h I)u^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)} = \|P_{12}^h(x, h(D - \theta))(H_{22}^{h,\theta} - E^h)^{-1}P_{21}^h(x, h(D - \theta))u_1^h\|_{L^2(\mathbb{T}^n; \mathbb{C})}
$$

$$
\leq h^2 C\|u^h\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)},
$$

we get the second part of Proposition 3.1.

Theorem 2.1 follows from Proposition 3.1 and \cite{Si-1, Si-2, He-Sj-1} and \cite{He-Sj-2} results, (see also \cite{He}), which guarantee that the sequence of eigenvalues of $H_{11}^{h,\theta}$, $(\lambda_j(H_{11}^{h,\theta}))_{j \in \mathbb{N}}$, satisfies $\forall N_0 > 1$, $\exists h_0 > 0$, $C_0 > 0$ s.t. $0 < h < h_0$ and $\forall j \leq N_0$, $|\lambda_j(H_{11}^{h,\theta}) - (h\mu_j^h + E_0)| \leq C_0 h^2 \square$

4 Asymptotic of the first band

For any real Lipschitz $\Gamma$ periodic function $\phi$, and for any $u \in W^2(\mathbb{T}^n; \mathbb{C}^2)$, we have the identity

$$
\begin{align*}
\text{Re} \left( \langle P^h(x, h(D - \theta))u | e^{2\phi/h} u \rangle_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \right) &= \sum_{k=1}^n h^2 \|(D_{x_k} - \theta_k)I - A_k)e^{\phi/h} u\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)}^2, \\
+ \langle (\tilde{V} - |\nabla \phi|^2 I + h^2 R)u | e^{2\phi/h} u \rangle_{L^2(\mathbb{T}^n; \mathbb{C}^2)}.
\end{align*}
$$

(4.1)
This identity enables us to apply the method used in \[\text{He-Sj-1}\], (see also \[\text{He}\] and \[\text{On}\]). We define the Agmon distance on \(\mathbb{R}^n\)

\[
\begin{aligned}
d(y, x) &= \inf_{\gamma} \int_0^1 \sqrt{v(\gamma(t)) - |w(\gamma(t))| - E_0} |\dot{\gamma}(t)| dt,
\end{aligned}
\]

(4.2)

the inf is taken among paths such that \(\gamma(0) = y\) and \(\gamma(1) = x\).

For common properties of the Agmon distance, one can see for example \[\text{Hi-Si}\].

We will use that, for any fixed \(y \in \mathbb{R}^n\), the function \(d(y, x)\) is a Lipschitz function on \(\mathbb{R}^n\) and \(|\nabla_x d(y, x)|^2 \leq v(x) - |w(x)| - E_0\) almost everywhere on \(\mathbb{R}^n\).

Using that the zeros of \(v(x) - w(x) - E_0\) are the elements of \(\Gamma\) and are non degenerate, we get that the real function \(d_0(x) = d(0, x)\) satisfies, (see \[\text{He-Sj-1}\]), \(|\nabla d_0(x)|^2 = v(x) - |w(x)| - E_0\) in a neighbourhood of 0.

We summarize the properties of the Agmon distance we will need:

\[
\begin{aligned}
i) &\quad \exists R_0 > 0 \ s.t. \ d_0(x) \in C^\infty(B_0(R_0)) \\
ii) &\quad |\nabla d_0(x)|^2 = v(x) - |w(x)| - E_0, \ \forall x \in B_0(R_0) \\
iii) &\quad |\nabla d_0(x)|^2 \leq v(x) - |w(x)| - E_0 \\
iv) &\quad |\nabla d_\Gamma(x)|^2 \leq v(x) - |w(x)| - E_0
\end{aligned}
\]

(4.3)

where \(d_0(x) = d(0, x)\), \(B_0(r) = \{x \in \mathbb{R}^n; \ d_0(x) < r\}\) and \(d_\Gamma(x) = d(\Gamma, x) = \inf_{\omega \in \Gamma} d(\omega, x)\).

The least Agmon distance in \(\Gamma\) is

\[
S_0 = \inf_{1 \leq k \leq n} d_0(\beta_k) = \inf_{\rho \neq \omega, (\omega, \rho) \in \Gamma^2} d(\omega, \rho).
\]

(4.4)

The Agmon distance on \(\mathbb{T}^n\), \(d_\Gamma^n(\cdot, \cdot)\), is defined by its \(\Gamma\)-periodic extension on \((\mathbb{R}^n)^2\)

\[
d_\Gamma^n(y, x) = \min_{\omega \in \Gamma} d(y, x + \omega).
\]

Then

\[
\frac{S_0}{2} = \sup_r \{ r > 0 \ s.t. \ {x \in \mathbb{T}^n; \ d_\Gamma^n(x_0, x) < r}\} \text{ is simply connected},
\]

(4.5)

where \(x_0\) is the single point in \(\mathbb{T}^n\) such that \(v(x_0) - |w(x_0)| = E_0\). The \(\Gamma\)-periodic function on \(\mathbb{R}^n\), \(d_\Gamma(x)\) is the one corresponding to the extension of \(d_\Gamma^n(x_0, x)\).

If \(\lambda^{h, \theta}\) is an eigenvalue of \(H^{h, \theta}\) and if \(u^{h, \theta}\) is an associated eigenfunction, then by \(\text{(3.1)}\) one gets as in the scalar case considered in \[\text{He-Sj-1}\] and \[\text{He-Sj-2}\],

\[
\begin{aligned}
\sum_{k=1}^n h^2 ||(D_{x_k} - \theta_k)I - A_k)e^{\phi/h}u^{h, \theta}||_{L^2(\mathbb{T}^n; \mathbb{C}^2)}^2
\end{aligned}
\]

\[
+ \left< [\bar{V} - |\nabla \phi|^2I + h^2R - \lambda^{h, \theta}I]^+_{h, \theta}u^{h, \theta} | e^{2\phi/h}u^{h, \theta} \right>_{L^2(\mathbb{T}^n; \mathbb{C}^2)} > 0
\]

\[
\leq \left< [\bar{V} - |\nabla \phi|^2I + h^2R - \lambda^{h, \theta}I]^-_{h, \theta}u^{h, \theta} | e^{2\phi/h}u^{h, \theta} \right>_{L^2(\mathbb{T}^n; \mathbb{C}^2)};
\]

(4.6)
so, when \( \phi(x) = d^{T_n}(x_0, x) \), necessarily \( \lambda^{h, \theta} - E_0 + O(h^2) > 0 \),
and if \( h/C < \lambda^{h, \theta} - E_0 < hC \), then \( u^{h, \theta} \) is localized in energy near \( x_0 \), for any
\( \eta \in ]0,1[ \); \( \exists C_\eta > 0 \) such that

\[
\sum_{k=1}^{n} h^2 \|((D_{x_k} - \theta_k)I - A_k)e^{\eta d^{T_n}(x_0, x)/h}u^{h, \theta}\|_{L^2(T^n, \mathbb{C}^2)} + (1 - \eta) < (v - |w| - E_0)u^{h, \theta} \| e^{2\eta d^{T_n}(x_0, x)/h}u^{h, \theta} > L^2(T^n, \mathbb{C}^2) \\
\leq hC_\eta \int_{\{x \in T^n; d^{T_n}(x_0, x) < \sqrt{hC}\}} |u^{h, \theta}(x)|^2 dx.
\]

(4.7)

Let \( \Omega \subset T^n \) an open and simply connected set with smooth boundary satisfying, for some \( \eta \), \( 0 < \eta < S_0/2 \),
\[
\{x \in T^n; d^{T_n}(x_0, x) < \frac{S_0 - \eta}{2}\} \subset \Omega \subset \{x \in T^n; d^{T_n}(x_0, x) < S_0/2\} \quad (4.8)
\]
Let \( H^h_\Omega \) be the selfadjoint operator on \( L^2(\Omega, \mathbb{C}^2) \) associated with \( P^h(x, hD) \) with
Dirichlet boundary condition. We denote by \( (\lambda_j(H^h_\Omega))_{j \in \mathbb{N}} \) the increasing sequence of eigenvalues of \( H^h_\Omega \). Using the method of [He-Sj-1], we get easily the following results.

**Theorem 4.1** For any \( \eta \), \( 0 < \eta < S_0/2 \), there exist \( h_0 > 0 \) and \( N_0 > 1 \) such that, if \( 0 < h < h_0 \) and \( j \leq N_0 \), then
\[
\forall \theta \in \mathbb{B}^n, \quad 0 < \lambda_j(H^h_\Omega) - \lambda_j^{h, \theta} \leq C e^{-(S_0 - \eta)/(2h)} ;
\]
so the length of the band \( b_{j}^h \) satisfies \( |b_{j}^h| \leq C e^{-(S_0 - \eta)/(2h)} \).

For the first band, we have the following improvement
\[
|b_{1}^h| \leq C e^{-(S_0 - \eta)/h}.
\]
(4.10)

**Sketch of the proof.**

As \( \Omega \) is simply connected and the one form \( \theta dx \) is closed, there exists a smooth
real function \( \psi_\theta(x) \) on \( \overline{\Omega} \) such that \( e^{-i\psi_\theta(x)} P^h(x, hD)e^{i\psi_\theta(x)} = P^h(x, h(D - \theta)) \) : the
Dirichlet operators on \( \Omega \), \( H^h_\Omega \) and \( H^{h, \theta}_\Omega \) associated to \( P^h(x, hD) \) and \( P^h(x, h(D - \theta)) \) are gauge equivalent, so they have the same spectrum.

Therefore the min-max principle says that
\[
0 < \lambda_j(H^{h, \theta}_\Omega) - \lambda_j^{h, \theta} = \lambda_j(H^h_\Omega) - \lambda_j^{h, \theta}.
\]
But the exponential decay of the eigenfunction \( \varphi_j^{h, \theta}(x) \) associated with \( \lambda_j^{h, \theta} \), given by (4.7) implies that
\[
\|(P^h(x, h(D - \theta)) - \lambda_j^{h, \theta}) \varphi_j^{h, \theta}(x)\|_{L^2(\Omega, \mathbb{C}^2)} \leq C e^{-(S_0 - \eta + \epsilon)/(2h)} ;
\]
for some \( \epsilon > 0 \), and for a smooth cut-off function \( \chi \) supported in \( \Omega \) and \( \chi(x) = 1 \) if \( d^{T_n}(x_0, x) \leq (S_0 - \eta + \epsilon)/2 \).
So distance($\lambda^h, \theta, sp(H^h_\Omega)) \leq C e^{-(S_0 - \eta + \epsilon)/(2h)}$.

This achieves the proof of (4.9).

Let us denote $E^h, \theta$, (respectively $E^h, \Omega$), the first eigenvalue $\lambda^h, \theta$, (respectively $\lambda^1(H^h_\Omega)$), and $\varphi^h, \theta(x)$, (respectively $\varphi^h, \Omega(x)$), the associated normalized eigenfunctions. Let $\chi$ be a cut-off function satisfying the same properties as before. Then

$$P^h(x, h(D - \theta))(e^{-i\varphi(x)}\chi(x)\varphi^h, \Omega(x)) = \lambda^1(H^h_\Omega)e^{-i\varphi(x)}\chi(x)\varphi^h, \Omega(x) + e^{-i\varphi(x)}r^h_0(x)$$

with, thanks to the same identity as (4.7) for Dirichlet problem on $\Omega$,

$$\|r^h_0\|_{L^2(\mathbb{T}^n; \mathbb{C}^2)} \leq C e^{-(S_0 - \eta)/(2h)}.$$

The same argument used in [He-Sj-1], (see also [He]), gives this estimate

$$|E^h, \theta - E^h, \Omega - \tau h| \leq C e^{-(S_0 - \eta)/(2h)}.$$

As $\tau h = < r^h_0 | \chi \varphi^h, \Omega >_{L^2(\mathbb{T}^n; \mathbb{C}^2)}$ does not depend on $\theta$, so

$$\forall \theta \in \mathbb{B}^n, \quad |E^h, \theta - E^h, \Omega - \tau h| \leq C e^{-(S_0 - \eta)/(2h)}.$$

This estimate ends the proof of (4.10) □

As for the tunnel effect in [He-Sj-1] and [Si-2], we have an accuracy estimate for the first band, like the scalar case in [Si-3] and in [On] (see also [He]).

**Theorem 4.2** There exists $h_0 > 0$ such that, if $0 < h < h_0$ then

$$|b^h| \leq C e^{-S_0/h}.$$

**Sketch of the proof.** Instead of comparing $H^h, \theta$ with an operator defined in a subset of $\mathbb{T}^n$, we have to work on the universal cover $\mathbb{R}^n$ of $\mathbb{T}^n$.

We take $\Omega \subset \mathbb{R}^n$ an open and simply connected set with smooth boundary satisfying, for some $\eta_0$, $0 < \eta_0 < \eta_1 < S_0/2$,

$$B_0((S_0 + \eta_0)/2) \subset \Omega \subset B_0((S_0 + \eta_1)/2). \quad (4.11)$$

So $\Omega$ contains the Wigner set $\mathbb{W}^n$, more precisely

$$\mathbb{W}^n \subset \Omega \subset 2\mathbb{W}^n \quad \text{and} \quad \Omega \cap \Gamma = \{0\}.$$

We let also denote $H^h_\Omega$ the Dirichlet operator on $L^2(\Omega; \mathbb{C}^2)$ associated with $P^h(x, hD)$, and $E^h_\Omega$ its first eigenvalue. The associated eigenfunction is also denoted by $\varphi^h, \Omega(x)$.

In the same way as to get (4.7), we have

$$\sum_{k=1}^{n} h^2 \| (D_{x_k} - A_k) e^{i\varphi^h, \Omega} \|_{L^2(\Omega; \mathbb{C}^2)}^2 \leq hC \int_{B_0(\sqrt{\eta}C)} |\varphi^h, \Omega(x)|^2 dx, \quad (4.12)$$
then the Poincaré estimate gives
\[
\int_{\Omega} e^{2d_0(x)/h} |\varphi^{h,\Omega}(x)|^2 \, dx \leq h^{-1} C \int_{B_0(\sqrt{h}C)} |\varphi^{h,\Omega}(x)|^2 \, dx.
\] (4.13)

Let \( \chi \) a smooth cut-off function satisfying
\[
\chi(x) = 1 \text{ if } d_0(x) \leq (S_0 + \eta_0)/2 \quad \text{and} \quad \chi(x) = 0 \text{ if } x \notin \Omega.
\]
Then the function
\[
\varphi^h(x) = \sum_{\omega \in \Gamma} e^{i\theta(x-\omega)} \chi(x-\omega) \varphi^{h,\Omega}(x-\omega)
\]
is \( \Gamma \)-periodic and satisfies
\[
\left\{ \begin{aligned}
(P^h(x, h(D - \theta)) - E^h_\Omega)\varphi^h(x) &= r^{h,\theta} \\
\|r^{h,\theta}\|_{L^2(W^n;C^2)} &\leq Ce^{-(S_0 + \eta_0)/(2h)}\|\varphi^h(x)\|_{L^2(W^n;C^2)}
\end{aligned} \right. 
\] (4.14)
and
\[
<r^{h,\theta} | \varphi^h(x) >_{L^2(W^n;C^2)} =
\sum_{\omega, \rho \in \Gamma_0} e^{i\theta(\rho - \omega)} \int_{W^n} ([P^h(x, hD); \chi] \varphi^{h,\Omega})(x - \omega). \overline{(\chi \varphi^{h,\Omega})(x - \rho)} \, dx
\]
with \( \Gamma_0 = \{0, \pm \beta_1, \ldots, \pm \beta_n\} \) and
\[
[P^h(x, hD); \chi] = -2h^2i \sum_{k=1}^n \partial_{x_k} \chi (D_{x_k} I - A_k) - h^2 \Delta \chi I.
\]
So
\[
\left| \frac{1}{\|\varphi^h(x)\|_{L^2(W^n;C^2)}^2} < r^{h,\theta} | \varphi^h(x) >_{L^2(W^n;C^2)} \right| \leq Ce^{-S_0/h}.
\] (4.15)
The proof comes easily from (4.12) and (4.13) as in [Ou] or in [He].

Using the same argument of [He-Sj-1] as in the proof of (4.10), we get that
\[
|E^{h,\theta} - E^h_\Omega - \tau^{h,\theta}| \leq Ce^{-(S_0 + \eta_0)/h},
\] (4.16)
with \( \tau^{h,\theta} = \frac{1}{\|\varphi^h(x)\|_{L^2(W^n;C^2)}^2} < r^{h,\theta} | \varphi^h(x) >_{L^2(W^n;C^2)} \).

Theorem 4.2 follows from (4.15) and (4.16). \( \square \)
5 B.K.W. method for the Dirichlet ground state

Let $\Omega$ be an open set satisfying (4.8), more precisely $\Omega \subset \mathbb{R}^n$ an open, bounded and simply connected set with smooth boundary satisfying, for some $\eta_1$ and $\eta_2$, $0 < \eta_1 < \eta_2 < S_0/2$,

$$\{ x \in \mathbb{R}^n; \ d_0(x) < \frac{S_0 - \eta_2}{2} \} \subset \Omega \subset \{ x \in \mathbb{R}^n; \ d_0(x) < \frac{S_0 - \eta_1}{2} \} \quad (5.1)$$

Theorem 5.1 The first eigenvalue $E^{h,\Omega} = \lambda_1(H^h_{\Omega})$ of the Dirichlet operator $H^h_{\Omega}$ admits an asymptotic expansion of the form

$$E^{h,\Omega} \simeq \sum_{j=0}^{\infty} h^j e_j ,$$

and if $S_0 - \eta_1$ is small enough, the associated eigenfunction $\varphi^{h,\Omega}$ has also an asymptotic expansion of the form

$$\varphi^{h,\Omega} = e^{-\varphi/h}(f^+_h, f^-_h) , \quad f^\pm_h \simeq \sum_{j=0}^{\infty} h^j f^\pm_j , \quad (f^-_0 = 0).$$

As usual

$$e_0 = E_0 , \quad e_1 = \tau_1 , \quad e_2 = r_{11}(0) + \sum_{k=1}^{n} |a_{k,11}(0)|^2 , \quad (5.2)$$

and $\phi$ is the real function satisfying the eikonal equation

$$|\nabla \phi(x)|^2 = v(x) - |w(x)| - E_0 , \quad (5.3)$$

equal to $d(x)$ in a neighbourhood of $0$.

($r_{11}$ and the $a_{k,11}$ are defined by (1.1) and (1.3). $E_0$ and $\tau_1$ are defined by (2.2) and (2.4)).

Proof. By (2.8) the magnetic field associated with each magnetic potential $a_{.,ij}$ is zero:

$$d \left( \sum_{k=1}^{n} a_{k,ij}(x) dx_k \right) = 0 \quad \forall i, j \in \{1, 2\} . \quad (5.4)$$

As $\Omega$ is simply connected, there exists a smooth real function on $\Omega$, $\psi(x)$ such that $\nabla \psi(x) = a_{.,11}$. Let $J_{\psi}$ be the unitary operator on $L^2(\Omega; \mathbb{C}^2)$,

$$J_{\psi} = \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \quad (5.5)$$
As the spectrum of $H^h_{\Omega}$ and $(J_\psi)^*H^h_{\Omega}J_\psi$ are the same, considering $(J_\psi)^*H^h_{\Omega}J_\psi$ instead of $H^h_{\Omega}$, we can assume that

\[
\begin{align*}
    a_{.11} &= a_{.22} = 0, \\
    a_{.22} &= -2\beta \nabla \alpha + 2\alpha \nabla \beta, \\
    a_{.21} &= \frac{\alpha}{\beta} = \rho \nabla \beta - \beta \nabla \rho + i\alpha \nabla \rho - i\rho \nabla \alpha.
\end{align*}
\] (5.6)

Let us write

\[
P^h(x, hD)(e^{-\phi/h}f_h) = W_0(x)f_h + hW_1(x, D)f_h + h^2W_2(x, D)f_h \] (5.7)

with

\[
W_0(x) = V(x) - |\nabla \phi(x)|^2 I
\]

\[
W_1(x, D) = \Delta \phi I + 2i \sum_{k=1}^{n} \partial_{x_k} \phi(D_{x_k} I - A_k)
\]

\[
W_2(x, D) = \sum_{k=1}^{n} (D_{x_k} I - A_k)^2 + R(x).
\]

We look for an eigenvalue $E^h \simeq \sum_{j=0}^{\infty} h^j e_j$

and an associated eigenfunction $f_h \simeq \sum_{j=0}^{\infty} h^j f_j$, so

\[
e^{\phi/h}(P^h(x, hD) - E^h I)(e^{-\phi/h}f_h) \simeq \sum_{j=0}^{\infty} h^j \kappa_j
\]

with

\[
\begin{align*}
    \kappa_0(x) &= (W_0(x) - e_0 I)f_0(x) \\
    \kappa_1(x) &= (W_1(x, D) - e_1 I)f_0(x) + (W_0(x) - e_0 I)f_1(x) \\
    \kappa_2(x) &= (W_2(x, D) - e_2 I)f_0(x) + (W_1(x, D) - e_1 I)f_1(x) + (W_0(x) - e_0 I)f_2(x) \\
    \kappa_j(x) &= -e_j f_0(x) - \sum_{\ell=1}^{j-3} e_{j-\ell} f_{\ell}(x) + (W_2(x, D) - e_2 I)f_{j-2}(x) \\
    &\quad + (W_1(x, D) - e_1 I)f_{j-1}(x) + (W_0(x) - e_0 I)f_j(x), \quad (j > 2).
\end{align*}
\]

We recall that $f_j = (f_j^+, f_j^-)$ and we want that $\kappa_j(x) = 0, \forall j$.

1) Term of order 0

As $\kappa_0(x) = 0 \iff \begin{cases} (-|\nabla \phi(x)|^2 + v(x) - |w(x)| - e_0) f_0^+(x) = 0 \\
                               (-|\nabla \phi(x)|^2 + v(x) + |w(x)| - e_0) f_0^-(x) = 0, \end{cases}$

choosing $\phi$ satisfying (5.3), then

\[
e_0 = E_0 \text{ and } -|\nabla \phi(x)|^2 + v(x) + |w(x)| - e_0 = 2|w(x)| > 0 \implies \text{ that}
\]

\[
f_0^-(x) = 0, \quad e_0 = E_0 \quad \text{and} \quad W_0(x) - e_0 I = \begin{pmatrix} 0 & 0 \\ 0 & 2|w(x)| \end{pmatrix}. \] (5.8)
2) Term of order 1.
The components of \( \kappa_1(x) = (\kappa_1^+(x), \kappa_1^-(x)) \) become
\[
\begin{align*}
\kappa_1^+(x) &= (\Delta \phi(x) - e_1)f_0^+(x) + 2\nabla \phi(x) \cdot \nabla f_0^+(x) \\
\kappa_1^-(x) &= 2|w(x)|f_1^-(x) - 2i(a_{.21}(x) \cdot \nabla \phi(x))f_0^+(x),
\end{align*}
\]
As \(|\nabla \phi(x)|\) has a simple zero at \(x_0 = 0\), the equation \(\kappa_1^+(x) = 0\) can be solved only when \(e_1 = \Delta \phi(0)\). In this case there exists a unique function \(f_0^+(x)\) such that \(f_0^+(0) = 1\) and \(\kappa_1^+(x) = 0\). We can conclude that the study of the term of order 1 leads to
\[
\begin{align*}
e_1 &= \Delta \phi(0), \\
2\nabla \phi(x) \cdot \nabla f_0^+(x) &= [e_1 - \Delta \phi(x)]f_0^+(x), \\
f_1^-(x) &= \frac{1}{|w(x)|} (\nabla \phi(x), a_{.21}(x)) f_0^+(x). \\
\end{align*}
\]
(5.9)

3) Term of order 2.
The components of \(\kappa_2(x) = (\kappa_2^+(x), \kappa_2^-(x))\) become
\[
\begin{align*}
\kappa_2^+(x) &= -\Delta f_0^+(x) + (|a_{.21}(x)|^2 + r_{11}(x) - e_2)f_0^+(x) \\
&+ (\Delta \phi(x) - e_1)f_1^+(x) + 2\nabla \phi(x) \cdot \nabla f_1^+(x) \\
&- 2i(a_{.12}(x) \cdot \nabla \phi(x))a_1^-(x) \\
\kappa_2^-(x) &= 2|w(x)|f_2^-(x) - 2i(a_{.21}(x) \cdot \nabla \phi(x))f_1^+(x) \\
&+ \Delta f_1^-(x) - 2\nabla \phi(x) \cdot \nabla f_2^-(x) + 2i(a_{.21}(x) \cdot \nabla \phi(x))f_0^+(x) \\
&+ i(div(a_{.21}(x))f_0^+(x) + r_{21}(x)f_0^+(x),
\end{align*}
\]
(5.10)
The unknown function \(f_1^+(x)\) must give \(\kappa_2^+(x) = 0\) in (5.10). This equation, with the initial condition \(f_1^+(0) = 0\), can be solved only if
\[
e_2 = -\Delta f_0^+(0) + |a_{.21}(0)|^2 + r_{11}(0).
\]
(We used that \(f_0^+(0) = 1\)). So \(\kappa_2 = 0\) implies
\[
\begin{align*}
e_2 &= -\Delta f_0^+(0) + |a_{.21}(0)|^2 + r_{11}(0), \\
2\nabla \phi(x) \cdot \nabla f_1^+(x) + (\Delta \phi(x) - e_1)f_1^+(x) &= \Delta f_0^+(x) \\
+ (e_2 - |A_{.21}(x)|^2|a_{.21}(x)|^2 - r_{11}(x))f_0^+(x) + 2i(a_{.12}(x) \cdot \nabla \phi(x))f_1^-(x) \\
f_2^-(x) &= \frac{1}{2|w(x)|} [2i(a_{.21}(x) \cdot \nabla \phi(x))f_1^+(x) - \Delta f_1^-(x) + 2\nabla \phi(x) \cdot \nabla f_2^-(x) \\
&- 2i(a_{.21}(x) \cdot \nabla \phi(x))f_0^+(x) - i(div(a_{.21}(x))f_0^+(x) - r_{21}(x)f_0^+(x)].
\end{align*}
\]

4) Term of order \(j > 2\).
We assume that \(e_\ell\) for \(\ell = 0, 1, \ldots, j-1\), the functions \(f_\ell^\pm(x)\) for \(\ell = 0, 1, \ldots, j-2\), and the one \(f_{j-1}^-\) are well-known, \(f_\ell^+(0) = 0\) when \(0 < \ell < j - 1\).
The equation \(\kappa_j^- = 0\) becomes
\[
\begin{align*}
2\nabla \phi(x) \cdot \nabla f_{j-1}^- + (\Delta \phi(x) - e_1)f_{j-1}^- \\
&= \Delta f_{j-2}^- + (e_2 - |A_{.21}(x)|^2|a_{.21}(x)|^2 - r_{11}(x))f_{j-2}^-(x) - r_{12}(x)f_{j-2}^- \\
&+ 2i(a_{.12}(x) \cdot \nabla \phi(x))f_{j-1}^- + \sum_{\ell=0}^{j-3} e_{j-\ell} f_\ell^+(x).
\end{align*}
\]
(5.11)
This equation has a unique solution \( f_{j-1}^+(x) \) such that \( f_{j-1}^+(0) = 0 \) iff
\[
e_j = -\Delta f_{j-2}^+(0).
\] (5.12)

The equation \( \kappa_j^- = 0 \) gives
\[
f_j^-(x) = \frac{1}{2w(x)}[2i(a_{.,21}(x),\nabla \phi(x))f_{j-1}^+(x)
-\Delta f_{j-1}^-(x) + 2\nabla \phi(x) \nabla f_{j-1}^-(x)\Delta f_{j-2}^-(x) - (|a_{.,21}(x)|^2 - e_2)f_{j-2}^-(x)
-2ia_{.,21}(x) \nabla f_{j-2}^-(x) - i(\text{div}(a_{.,21}(x))f_{j-2}^-(x) - \tau_21(x)f_{j-2}^-(x)).
\]

5) End of the proof.

Let \( \chi(x) \) be a cut-off function equal to 1 in a neighbourhood of 0 and supported in \( \Omega \). Then taking \( \chi(x)f_j(x) \) instead of \( f_j(x) \), we get a function \( \varphi^{h,\Omega} \) satisfying Dirichlet boundary condition and Theorem 5.1 \( \square \)

6 Sharp asymptotic for the width of the first band

Returning to the proof of Theorem 4.2, we have to study carefully the \( \tau^{h,\theta} \) defined in (4.16), using the method of [He-Sj-1] performed in [He] and [Ou].

Using (4.14)–(4.16), we have
\[
\tau^{h,\theta} = \sum_{\omega \in \Gamma_0^+} \left( e^{-i\theta\omega}(\rho_\omega^+ + \rho_\omega^-) + e^{i\theta\omega}(\rho_\omega^+ + \rho_\omega^-) \right),
\] (6.1)

with \( \Gamma_0^+ = \{\beta_1, \ldots, \beta_n\} \),
\[
\rho_\omega^+ = \frac{1}{\|\varphi^h\|^2_{L^2(W^n;C^2)}} \int_{W^n} ([P^h(x, hD); \chi] \varphi^{h,\Omega})(x - \omega) \overline{(\chi \varphi^{h,\Omega})(x)}dx
\]
and
\[
\rho_\omega^- = \frac{1}{\|\varphi^h\|^2_{L^2(W^n;C^2)}} \int_{W^n} ([P^h(x, hD); \chi] \varphi^{h,\Omega})(x) \overline{(\chi \varphi^{h,\Omega})(x + \omega)}dx.
\]

We get from the formula of \( [P^h(x, hD); \chi] \) and from the estimate (4.13) that
\[
\rho_\omega^+ = -\frac{h^2}{\|\varphi^h\|^2_{L^2(W^n;C^2)}} \int_{W^n} (\nabla \chi(x - \omega) \nabla \varphi^{h,\Omega})(x - \omega) \overline{(\chi \varphi^{h,\Omega})(x)}dx + O(h^e - S_0/h)
\] (6.2)
and
\[
\rho_\omega^- = -\frac{h^2}{\|\varphi^h\|^2_{L^2(W^n;C^2)}} \int_{W^n} (\nabla \chi(x) \nabla \varphi^{h,\Omega})(x) \overline{(\chi \varphi^{h,\Omega})(x + \omega)}dx + O(h^e - S_0/h).
\]

14
Theorem 6.1 Under the assumption of Theorem 4.2, if for any \( \omega \in \{ \pm \beta_1, \ldots, \pm \beta_n \} \) such that the Agmon distance in \( \mathbb{R}^n \) between 0 and \( \omega \) is the least one, \( d(0, \omega) = S_0 \), there exists one or a finite number of minimal geodesics joining 0 and \( \omega \), then there exists \( \eta_0 > 0 \) and \( h_0 > 0 \) such that
\[
 b^h_1 = \eta_0 h^{1/2} e^{-S_0/h} \left( 1 + O(h^{1/2}) \right), \quad \forall h \in [0, h_0].
\]

Sketch of the proof. Following the proof of splitting in [He-Sj-1] and [He], in (6.2) we can change \( \mathbb{W}^n \) into \( \mathbb{W}^n \cap \mathcal{O} \), where \( \mathcal{O} \) is any neighbourhood of the minimal geodesics between 0 and the \( \pm \beta_k \), such that \( d(x) = d(0, x) \in C^\infty(\mathcal{O}) \). In this case the B.K.W. method is valid in \( \mathbb{W}^n \cap \mathcal{O} \). If \( \varphi_{B.K.W.}^{h} \) is the B.K.W. approximation of \( \varphi^{h,\Omega} \) in \( \mathbb{W}^n \cap \mathcal{O} \), then, thanks to (11), for any \( p_0 > 0 \) there exists \( C_{p_0} \) such that
\[
 h^n \sum_{k=1}^n \| (D_{x_k}I - A_k) e^{k(d(x))} \chi_0 (\varphi^{h,\Omega} - \varphi_{B.K.W.}^{h}) \|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)}
 + \| e^{d(x)} \chi_0 (\varphi^{h,\Omega} - \varphi_{B.K.W.}^{h}) \|^2_{L^2(\mathbb{W}^n;\mathbb{C}^2)} \leq h^{p_0} C_{p_0},
\]
where \( \chi_0 \) is a cut-off function supported in \( \mathbb{W}^n \cap \mathcal{O} \) and equal to 1 in a neighborhood of the minimal geodesics between 0 and the \( \pm \beta_k \). We have assumed that \( \| \varphi^{h,\Omega} \|_{L^2(\mathbb{W}^n;\mathbb{C}^2)} = 1 \) and then \( \| \chi_0 \varphi_{B.K.W.}^{h} \|_{L^2(\mathbb{W}^n;\mathbb{C}^2)} - 1 = O(h^p) \) for any \( p > 0 \).

As (6.2) remains valid if we change \( \varphi^{h,\Omega} \) into \( \chi_0 \varphi^{h,\Omega} \), the estimate (6.3) allows also to change \( \varphi^{h,\Omega} \) into \( \chi_0 \varphi_{B.K.W.}^{h} \). As a consequence, Theorem 6.1 follows easily, if in \( \mathbb{W}^n \cap \mathcal{O} \), \( \chi(x) = \chi_1(d(x)) \) for a decreasing function \( \chi_1 \) on \([0, +\infty[ \) with compact support, equal to 1 in a neighborhood of 0. In this case (6.2) becomes
\[
 \rho^\pm_\omega = \frac{h^{(2-n)/2}}{(2\pi)^{1/2} \prod_{k=1}^n \tau_{k}^{1/2}} \times
\]
\[
 \int_{\mathbb{W}^n \cap \mathcal{O}} \chi_1(d(x-\omega)) \chi_1(d(x)) |\nabla d(x-\omega)|^2 f_0^+(x-\omega) f_0^+(x) e^{-\left(d(x-\omega)+d(x)/h\right)} \, dx + O(he^{-S_0/h})
\]
and
\[
 \rho^-_\omega = \frac{h^{(2-n)/2}}{(2\pi)^{1/2} \prod_{k=1}^n \tau_{k}^{1/2}} \times
\]
\[
 \int_{\mathbb{W}^n \cap \mathcal{O}} \chi_1(d(x)) \chi_1(d(x+\omega)) |\nabla d(x)|^2 f_0^+(x) f_0^+(x+\omega) e^{-\left(d(x)+d(x+\omega)/h\right)} \, dx + O(he^{-S_0/h}).
\]

We remind that for any \( y \) in a minimal geodesic joining 0 to \( \pm \beta_k \), if \( y \neq 0 \) and \( y \neq \pm \), then the function \( d(x) + d(x \mp \beta_k) \), when it is restricted to any hypersurface orthogonal to the geodesic through \( y \), has a non degenerate minimum \( S_0 \) at \( y \).}

**Acknowledgement**

The authors are grateful to Bernard Helffer for many discussions.
References

[Ag] S. Agmon : Lectures in exponential decay of solutions of second-order elliptic equations. Princeton University Press, Math. Notes 29.

[Ca-Yu] M. Cardona, P. Y. Yur : Fundamentals of Semiconductors Physics and Materials Properties. Springer, Berlin 2010.

[Co] N. R. Cooper : Optical Flux Lattices for Ultracold Atomic Gases. Phys. Rev. Lett. 106, (2011), 175301.

[Co-Da] N. R. Cooper, J. Dalibard : Optical flux lattices for two-photon dressed states. Europhysics Letters, 95 (6), (2011), 66004.

[Da-al] J. Dalibard, F. Gerbier, G. Juzeliunas, P. Ohberg : Colloquium: Artificial gauge potentials for neutral atoms. Rev. Mod. Phys. 83 (4), (2011), p. 1523-1543.

[Di-Sj] M. Dimassi, J. Sjöstrand : Spectral asymptotics in the semi-classical limit. Cambridge University Press, 1999.

[Ea] M. S. P. Eastham : The spectral theory of periodic differential equations. Scottish Academic, London 1974.

[He] B. Helffer : Semi-classical analysis for the Schrödinger operator and applications. Lecture Notes in Math. 1336, Springer, 1988.

[He-Sj-1] B. Helffer, J. Sjöstrand : Multiple wells in the semi-classical limit. Comm. in P.D.E., 8 (4), (1984), p.337-408.

[He-Sj-2] B. Helffer, J. Sjöstrand : Effet tunnel pour l’équation de Schrödinger avec champ magnétique. Ann. Scuola Norm. Sup. Pisa, 14 (4), (1987), p. 625-657.

[Hi-Si] P. D. Hislop, I. M. Sigal : Introduction to spectral theory. With applications to Schrödinger operators. Applied Mathematical Sciences, 113. Springer-Verlag, 1996.

[Ou] A. Outassourt : Comportement semi-classique pour l’opérateur de Schrödinger à potentiel périodique. J. of Functional Analysis 72, (1987), p. 65-93.

[Re-Si] M. Reed, B. Simon : Methods of modern mathematical physics, Vol 4. Academic Press, New York 1972.

[Ro] D. Robert : Autour de l’approximation semi-classique. Birkhauser, 1986.
[Si-1] B. Simon : Semi-classical analysis of low lying eigenvalues, I. Ann. Inst. H. Poincaré, 38, (1983), p. 295-307.

[Si-2] B. Simon : Semiclassical analysis of low lying eigenvalues, II. Annals of Math. 120, (1984), p. 89-118.

[Si-3] B. Simon : Semiclassical analysis of low lying eigenvalues, III. Ann. Physics, 158 (2), (1984), p. 295-307.