Abstract

The invariant projections of the energy-momentum tensors of Lagrangian densities for tensor fields over differentiable manifolds with contravariant and covariant affine connections and metrics \((\mathcal{E}_n, g)\)-spaces are found by the use of an non-null (non-isotropic) contravariant vector field and its corresponding projective metrics. The notions of rest mass density, momentum density, energy current density and stress tensor are introduced as generalizations of these notions from the relativistic continuum media mechanics. The energy-momentum tensors are represented by means of the introduced notions and the corresponding identities are found. The notion of covariant differential operator along a contravariant tensor field is introduced. On its basis, as a special case, the notion of contravariant metric differential operator is proposed. The properties of the operators are considered. By the use of these operators the notion of covariant divergency of a mixed tensor field is determined. The covariant divergency of tensor fields of second rank of the types 1 and 2 is found. Invariant representations of the covariant divergency of the energy-momentum tensor are obtained by means of the projective metrics of a contravariant non-isotropic (non-null) vector field and the corresponding rest mass density, momentum density, and energy flux density. An invariant representation of the first Noether identity is found as well as relations between the covariant divergencies of the different energy-momentum tensors and their structures determining covariant local conserved quantities.
1 Invariant projections of a mixed tensor field of second rank

In the relativistic continuum media mechanics notions are introduced as generalizations of the same notions of the classical continuum media mechanics. This has been done by means of the projections of the (canonical, symmetric of Belinfante or symmetric of Hilbert) energy-momentum tensors along or orthogonal to a non-isotropic (non-null) contravariant vector field.

There are possibilities for using the projections for finding out the physical interpretations of the determined energy-momentum tensors. In an analogous way as in (pseudo) Riemannian spaces without torsion ($V_n$-spaces), the different relations between the quantities with well known physical interpretations can be considered as well as their application for physical systems described by means of mathematical models over differentiable manifold with affine connections and metrics ($L_n, g$)-spaces.

The energy-momentum tensors are obtained as mixed tensor fields of second rank of type 1 [by the use of the procedure on the grounds of the method of Lagrangians with covariant derivatives (MLCD) [1]]

\[ G = G^\alpha_\beta \cdot e_\beta \otimes e^\alpha = G^i_j \cdot \partial_j \otimes dx^i \]  

in contrast to the mixed tensor fields of second rank of type 2

\[ \overline{G} = \overline{G}^\beta_\alpha \cdot e^\alpha \otimes e_\beta = \overline{G}^j_i \cdot dx^i \otimes \partial_j . \]  

\{e^\alpha\} and \{e_\alpha\} are non-co-ordinate (non-holonomic) covariant and contravariant basic vector fields respectively, \{(\alpha, \beta = 1, ..., n)\}, \{dx^i\} and \{\partial_i\} are co-ordinate (holonomic) covariant and contravariant basic vector fields respectively \(i, j = 1, ..., n\), \(\text{dim}(L_n, g) = n\).

To every covariant basic vector field a contravariant basic vector field can be juxtaposed and vice versa by the use of the contravariant and covariant metric tensor fields

\[ g(e_\gamma) = g_{\gamma\tau} \cdot e^\alpha, \quad g(\partial_j) = g_{\gamma j} \cdot dx^i, \quad \overline{g}(e^\gamma) = g^{\alpha\tau} \cdot e_\alpha, \quad \overline{g}(dx^j) = \overline{g}^{\alpha j} \cdot \partial_\alpha . \]  

On this basis a tensor field of the type 2 can be related to a tensor field of the type 1 by the use of the covariant and contravariant tensor fields \(g = g_{ij} \cdot dx^i, dx^j\) and \(\overline{g} = g^{ij} \cdot \partial_i \partial_j\) \[dx^i, dx^j = (1/2)(dx^i \otimes dx^j + dx^j \otimes dx^i), \partial_i, \partial_j = (1/2)(\partial_i \otimes \partial_j + \partial_j \otimes \partial_i)] \n
\[ \overline{G} = g(G)\overline{g} = \overline{G}^\beta_\alpha \cdot e^\alpha \otimes e_\beta = g_{\alpha\tau} \cdot G^\gamma_\delta \cdot g^{\beta\gamma} \cdot e^\alpha \otimes e_\beta , \]  

\[ \overline{G}^\beta_\alpha = g_{\alpha\tau} \cdot G^\gamma_\delta \cdot g^{\beta\gamma} , \]  

\[ G = \overline{g}(G)g = G^\beta_\alpha \cdot e_\beta \otimes e^\alpha = g^{\beta\gamma} \cdot \overline{G}^\gamma_\delta \cdot g_{\tau\alpha} \cdot e_\beta \otimes e^\alpha . \]
\[
G_{\alpha}^{\beta} = g^{\beta\gamma} \cdot G_{\gamma} \cdot g_{\alpha} \cdot . \quad (7)
\]

The Kronecker tensor field appears as a mixed tensor field of second rank of the type 1

\[
Kr = g_{\beta}^{\alpha} \cdot e_{\alpha} \otimes e^{\beta} = g^{j}_{i} \cdot \partial_{i} \otimes dx^{j}
\]

and can be projected by means of the non-isotropic (non-null) contravariant vector field \(u\) and its projection metrics \(h_{u}\) and \(h^{u}\) \([h_{u} = g - \frac{1}{e} \cdot g(u) \otimes g(u), h^{u} = g^{u} - \frac{1}{e} \cdot u \otimes u, e = g(u, u) \neq 0]\)

\[
Kr = \varepsilon_{Kr} \cdot u \otimes g(u) + u \otimes g^{(K\pi)} + Kr s \otimes g(u) + (K S) g ,
\]

where

\[
\varepsilon_{Kr} = \frac{1}{e} \cdot [g(u)](Kr)u = \frac{1}{e} \cdot u_{\gamma} \cdot u^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot u^{\beta} =
\]

\[
\frac{1}{e} \cdot f^{\beta} \cdot f^{\sigma} \cdot f^{\delta} \cdot g_{\sigma\delta} \cdot u^{\gamma} \cdot u^{\beta} = \frac{1}{e} \cdot u^{\gamma} \cdot u^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot u^{\gamma} = \frac{1}{e} \cdot k,
\]

\[
k = \frac{1}{e} \cdot [g(u)](Kr)u = \frac{1}{e} \cdot \frac{1}{e} \cdot u_{\gamma} \cdot u^{\gamma} , \quad u^{\gamma} = g_{\gamma\gamma} \cdot u^{\gamma} , \quad (8)
\]

\[
K_{\pi}^{\gamma} = \frac{1}{e} \cdot [g(u)](Kr) h^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} , e_{\alpha} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot h_{\gamma} \cdot e_{\gamma} \cdot e_{\alpha} =\]

\[
= \frac{1}{e} \cdot u_{\gamma} \cdot h_{\gamma} \cdot e_{\gamma} = K_{\pi}^{\gamma} \cdot e_{\alpha} ,
\]

\[
K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} , \quad K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} ,
\]

\[
= \frac{1}{e} \cdot u_{\gamma} \cdot h_{\gamma} \cdot e_{\gamma} , \quad K_{\pi}^{\gamma} = \frac{1}{e} \cdot u_{\gamma} \cdot h_{\gamma} , \quad (9)
\]

\[
K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} , \quad K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} , \quad \]

\[
K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} , \quad K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} \cdot u^{\gamma} \cdot h_{\gamma} , \quad (10)
\]

\[
K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} h_{\gamma} , \quad K_{\pi}^{\gamma} = \frac{1}{e} \cdot g_{\gamma\gamma} h_{\gamma} , \quad \]

\[
K_{\pi}^{\gamma} = \frac{1}{e} \cdot h_{\gamma} , \quad e_{\alpha} = k , \quad (11)
\]

\[
K_{t}^{\gamma} = \frac{1}{e} \cdot h^{\gamma} , \quad e_{\alpha} = k , \quad (12)
\]

\[
K_{t}^{\gamma} = \frac{1}{e} \cdot h^{\gamma} , \quad e_{\alpha} = k , \quad (13)
\]

\[
K_{t}^{\gamma} = \frac{1}{e} \cdot h^{\gamma} , \quad e_{\alpha} = k , \quad (14)
\]

\[
K_{t}^{\gamma} = \frac{1}{e} \cdot h^{\gamma} , \quad e_{\alpha} = k , \quad (15)
\]

\[
K_{t}^{\gamma} = \frac{1}{e} \cdot h^{\gamma} , \quad e_{\alpha} = k , \quad (16)
\]

\[
K_{t}^{\gamma} = \frac{1}{e} \cdot h^{\gamma} , \quad e_{\alpha} = k , \quad (17)
\]

The corresponding to the Kronecker tensor field mixed tensor field of the type 2

\[
\overline{Kr} = g(Kr)_{\gamma} = g_{\alpha\gamma} \cdot g_{\beta\gamma} \cdot e_{\alpha} \otimes e_{\beta} = g_{\alpha\beta} \cdot g_{\gamma\gamma} \cdot dx^{i} \otimes \partial_{j}
\]

does not appear in general as a Kronecker tensor field.
Special case: $S = C : f^i = g^i_j$, \((f^n \beta = g^n_\beta)\):

\[
\begin{align*}
K_r &= g^\alpha_\beta \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i \otimes \partial_j, \\
k = 1, \quad \varepsilon_{Kr} &= 1, \quad K_r \pi = 0, \quad K_r S = h_u, \quad (Kr\pi) = \pi, \quad (Kr)\pi = \pi.
\end{align*}
\]

(18)

The representation of the tensor fields of the type 1 by the use of the non-isotropic (non-null) contravariant vector field $u$ and its projective metrics $h_u$ and $h_u$ corresponds in its form to the representation of the viscosity tensor and the energy-momentum tensors in the continuum mechanics in $V_3$- or $V_4$-spaces, where $\varepsilon_G$ is the inner energy density, $G\pi$ is the conductive momentum, $e. G\pi$ is the conductive energy flux density and $G S$ is the stress tensor density. An analogous interpretation can also be accepted for the projections of the energy-momentum tensors found by means of the method of Lagrangians with covariant derivatives (MLCD).

2 Energy-momentum tensors and the rest mass density

The covariant Noether identities (generalized covariant Bianchi identities) can be considered as identities for the components of mixed tensor fields of second rank of the first type. The second covariant Noether identity

\[
\theta - sT \equiv Q,
\]

(19)

can be written in the form

\[
\theta - sT \equiv Q,
\]

where

\[
\begin{align*}
\theta &= \theta^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i, \\
sT &= sT^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i, \\
Q &= Q^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i.
\end{align*}
\]

\[
\theta \equiv \theta^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i, \\
sT \equiv sT^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i, \\
Q \equiv Q^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i.
\]

\[
\theta \equiv \theta^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i, \\
sT \equiv sT^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i, \\
Q \equiv Q^{\alpha}_{\beta} \cdot e^\beta_\alpha \cdot \partial_j \cdot dx^i.
\]

$\theta$ is the generalized canonical energy-momentum tensor (GC-EMT) of the type 1; $sT$ is the symmetric energy-momentum tensor of Belinfante (S-EMT-B) of the type 1; $Q$ is the variational energy-momentum tensor of Euler-Lagrange (V-EMT-EL) of the type 1.

The second covariant Noether identity for the energy-momentum tensors of the type 1 is called second covariant Noether identity of type 1.

By means of the non-isotropic contravariant vector field $u$ and its corresponding projective metric the energy-momentum tensors can be represented in an analogous way as the mixed tensor fields of the type 1.

The structure of the generalized canonical energy-momentum tensor and the symmetric energy-momentum tensor of Belinfante for the metric and non-metric tensor fields has similar elements and they can be written in the form

\[
G = kG - L \cdot K_r, \\
\theta = k\theta - L \cdot K_r, \\
sT = T - L \cdot K_r
\]

(21)
where
\[ k\theta = k\overline{\theta}_\alpha^\beta \cdot e_\beta \otimes e^\alpha = k\overline{\theta}_i^j \cdot \partial_j \otimes dx^i. \] (22)

On the analogy of the notions of the continuum media mechanics, \( kG \) is called viscosity tensor field.

\( G \) and \( kG \) can be written by means of \( u, h^a \) and \( h_u \) in the form
\[ G = \varepsilon_G \cdot u \otimes g(u) + u \otimes g(G\pi) + G_s \otimes g(u) + (G\pi)g, \]
\[ kG = \varepsilon_k \cdot u \otimes g(u) + u \otimes g(k\pi) + k_s \otimes g(u) + (k\pi)g. \] (23)

From the relation (21) the relations between the different projections of \( G \) and \( kG \) follow. If we introduce the abbreviation \( \varepsilon_G = \rho_G \), then
\[ \varepsilon_k = \rho_G + \frac{1}{e} \cdot L \cdot k, \quad k\pi = G\pi + L \cdot Kr \pi, \quad k_s = G_s + L \cdot Kr S, \] (24)
and \( G \) can be written by means of (21) in the form
\[ G = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g(k\pi) + k_s \otimes g(u) + (k\pi)g, \] (25)
where
\[ \rho_G = \frac{1}{e^2} \cdot [g(u)](G)(u) \]
is the rest mass density of the energy-momentum tensor \( G \) of the type 1. This type of representation of a given energy-momentum tensor \( G \) by means of the projective metrics of \( u \) and \( \rho_G \) is called representation of \( G \) by means of the projective metrics of the contravariant non-isotropic (non-null) vector field \( u \) and the rest mass density \( \rho_G \).

There are other possibilities for representation of \( G \) by means of \( u \) and its corresponding projective metrics.

If we introduce the abbreviation
\[ p_G = \rho_G \cdot u + G\pi, \] (26)
where \( p_G \) is the momentum density of the energy-momentum tensor \( G \) of the type 1, then \( G \) can be written in the form
\[ G = u \otimes g(p_G \cdot u + G\pi) + G_s \otimes g(u) + (G\pi)g, \]
\[ G = u \otimes g(p_G) + G_s \otimes g(u) + (G\pi)g. \] (27)

The representation of \( G \) by means of the last relation is called representation of \( G \) by means of the projective metrics of the contravariant non-isotropic contravariant vector field \( u \) and the momentum density \( p_G \).

By the use of the relations
\[ g(G\pi, u) = 0, \quad (G\pi)[g(u)] = 0, \] (28)
valid (because of their constructions) for every energy-momentum tensor \( G \) and the definitions
\[ \varepsilon_G = G(u) = (G)(u) = e \cdot (\rho_G \cdot u + G\pi), \quad g(u, u) = e \neq 0, \] (29)
where $e_G$ is the energy flux density of the energy-momentum tensor $G$ of the type 1, the tensor field $G$ can be written in the form

$$
G = (\rho_G \cdot u + G_s) \otimes g(u) + u \otimes g(G\pi) + (G S) g, \\
G = \frac{1}{e} \cdot e_G \otimes g(u) + u \otimes g(G\pi) + (G S) g.
$$

(30)

The representation of $G$ by means of the last expression is called representation of $G$ by means of the projective metrics of the contravariant non-isotropic vector field $u$ and the energy flux density $e_G$.

The generalized canonical energy-momentum tensor $\theta$ can be represented, in accordance with the above described procedure, by the use of the projective metrics of $u$ and the rest mass density $\rho_0$

$$
\theta = k \theta - L \cdot Kr, \\
\theta = \theta + L \cdot Kr,
$$

(31)

$$
\theta = (\rho_0 + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g(\theta\pi) + \theta S \otimes g(u) + (\theta S) g,
$$

(32)

where

$$
k \theta = k \bar{\theta} = \frac{\delta \alpha}{e} \cdot e_\beta \otimes e^\alpha, \\
k \bar{\theta} = \frac{\delta \alpha}{e} - K_\alpha \beta - \bar{W}_\alpha \beta \gamma + L \cdot g^\beta_\alpha,
$$

(33)

$$
\rho_0 = \frac{1}{e^2} \cdot [g(u)](\theta)(u), \\
\rho_0 = \frac{1}{e} \cdot [g(u)](K_r)(u),
$$

(34)

$$
\rho_0 = \frac{1}{e^2} \cdot g_{\alpha \beta} \cdot u^\beta \cdot \bar{\theta}^\alpha \cdot u^\gamma = \frac{1}{e} \cdot \bar{\theta}^\gamma \cdot u^\alpha \cdot u^\gamma = \frac{1}{e} \cdot \bar{\theta}^\gamma \cdot u_j \cdot \bar{\theta}^i \cdot u^\gamma = \frac{1}{e} \cdot \bar{\theta}^i \cdot u_j \cdot u^\gamma,
$$

(35)

$$
\epsilon \cdot k \theta = \rho_0 + \frac{1}{e} \cdot L \cdot k,
$$

(36)

$$
\theta \pi = \frac{1}{e} \cdot [g(u)](k \theta) h^u = \theta \pi^\alpha \cdot e_\alpha = \theta \pi^\alpha \cdot \partial_i,
$$

(37)

$$
\theta \pi^\beta = \frac{1}{e} \cdot k \bar{\theta}^\alpha \cdot u_\alpha \cdot h^\gamma, \\
\theta \pi^\alpha = \frac{1}{e} \cdot k \bar{\theta}^j \cdot u_j \cdot h^\gamma
$$

(38)

$$
\theta \pi = \frac{1}{e} \cdot h^u (g)(k \theta)(u) = \theta \pi^\alpha \cdot e_\alpha = \theta \pi^\alpha \cdot \partial_i
$$

(39)

$$
\theta \pi^\beta = \frac{1}{e} \cdot h^\alpha \cdot g^\gamma_\alpha \cdot \bar{\theta}^\gamma \cdot u_i, \\
\theta \pi^\alpha = \frac{1}{e} \cdot h^i \cdot g^\gamma_i \cdot k \bar{\theta}^\gamma \cdot u_i
$$

(40)

$$
\theta \pi = \frac{1}{e} \cdot h^u (g)(k \theta)(u) = \theta \pi^\alpha \cdot e_\alpha \otimes e_\beta = \theta \pi^\alpha \otimes \partial_i \otimes \partial_j,
$$

(41)

$$
\theta \pi^\alpha \otimes g^i_\alpha \cdot k \bar{\theta}^\alpha \cdot k \bar{\theta}^\beta \cdot h_{i k} = \theta \pi^i \cdot \partial_i \otimes \partial_j,
$$

(42)

$$
\theta \pi = \theta \pi^\alpha \cdot e_\alpha \otimes e_\beta = \theta \pi^\alpha \cdot \partial_i \otimes \partial_j,
$$

(43)

$$
\theta \pi^\alpha \otimes g^i_\alpha \cdot k \bar{\theta}^\alpha \cdot g^\gamma_\alpha \cdot \bar{\theta}^\gamma \cdot \theta \pi^j = \theta \pi^i \cdot \partial_i \otimes \partial_j,
$$

(44)

$$
g(\theta) = \rho_0 + \frac{1}{e} \cdot L \cdot k \cdot g(u) \otimes g(u) - L \cdot g(Kr) + g(u) \otimes g(\pi) + g(\theta \pi) \otimes g(u) + g(\theta \pi) \cdot g(u) + g(\theta \pi) g,
$$

(45)
\[ g(\theta) = \theta_{\alpha\beta} \cdot e^\alpha \otimes e^\beta = \theta_{ij} \cdot dx^i \otimes dx^j, \]
\[ \theta_{\alpha\beta} = g_{\alpha\gamma} \cdot \overline{\theta}_{\beta}^\gamma, \quad \theta_{ij} = g_{\overline{\alpha}\overline{\beta}} \cdot \overline{\theta}_{j}^k, \]
\[ g(Kr) = g_{\alpha\gamma} \cdot e^\alpha \otimes e^\beta = g_{ij} \cdot dx^i \otimes dx^j, \quad g_{\overline{\alpha}\overline{\beta}} = g_{\alpha\gamma} \cdot f^\gamma \beta = g_{\overline{\alpha}\overline{\beta}} = g_{\overline\gamma}^\alpha, \]
\[ g_{ij} = g_{\alpha\gamma}, \quad (Kr)_{\overline{\alpha\overline{\beta}}} = K_r(\overline{\gamma}) = g_{\overline{\alpha\overline{\beta}}} \cdot e_\alpha \otimes e_\beta = g_{ij}^\gamma \cdot \partial_i \otimes \partial_j. \]

In a co-ordinate basis \( \theta \), \( (\theta) \overline{\alpha\overline{\beta}} \) and \( g(\theta) \) can be represented in the forms
\[ \overline{\theta}^{i}_{j} = (\rho_{\theta} + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^j - L \cdot g_{ij}^i \cdot u^j + \theta_{ij} \cdot \theta_{kl} \cdot \theta_{ji}, \quad (54) \]
\[ \theta^{ij} = \theta_{k} \cdot \overline{\theta}^{i}_{j} \cdot g_{ij} = (\rho_{\theta} + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^j - L \cdot g_{ij}^i \cdot u^j + \theta_{ij} - \theta_{ij} \cdot \theta_{kl} \cdot \theta_{ji}, \quad (55) \]
\[ \theta_{ij} = g_{\overline{\alpha\overline{\beta}}} \overline{\theta}^{j}_{i} = (\rho_{\theta} + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^j - L \cdot g_{ij}^i \cdot u^j + \theta_{ij} \cdot \theta_{kl} \cdot \theta_{ji}, \quad (56) \]

where
\[ u_i = g_{ij}^i \cdot u^j, \quad \theta_{ij} = g_{\overline{\alpha\overline{\beta}}} \cdot \theta_{kl} \cdot \theta_{ji}, \quad \theta_{ij} = g_{\overline{\alpha\overline{\beta}}} \cdot \theta_{kl} \cdot \theta_{ji}. \]

The symmetric energy-momentum tensor of Belinfante \( sT \) can be represented in an analogous way by the use of the projective metrics \( h^u \) and \( h_u \) and the rest mass density \( \rho_T \) in the form
\[ sT = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g(T_{\overline{\alpha\overline{\beta}}} + T_{\overline{\alpha\overline{\beta}}} \otimes g(u) + (T_{\overline{\gamma}} g), \quad (57) \]

where
\[ sT = sT_{\alpha\beta} \cdot e_\beta \otimes e_\alpha = sT_{ij} \cdot \partial_j \otimes dx^i, \quad (58) \]
\[ sT = sT_{K} - L \cdot Kr, \quad skT = sT + L \cdot Kr = T, \]
\[ sT_{K} = L \cdot k, \]
\[ \rho_T = \frac{1}{e} \cdot [g(u)](sT)(u) = \frac{1}{e} \cdot g_{\overline{\alpha\overline{\beta}}} \cdot u^\beta \cdot sT_{\alpha} \cdot u^\alpha = \frac{1}{e} \cdot sT_{\alpha} \cdot u^\alpha \cdot u^\alpha = \frac{1}{e} \cdot sT_{\alpha} \cdot u^\alpha \cdot u^\alpha, \]
\[ T_{\overline{\alpha\overline{\beta}}} = \frac{1}{e} \cdot [g(u)](T)(u)h^u = \frac{1}{e} \cdot [g(u)](skT)h^u = T_{\overline{\alpha\overline{\beta}}} \cdot e_\alpha = T_{\overline{\alpha\overline{\beta}}} \cdot \partial_i, \]
\[ T_{\overline{\alpha\overline{\beta}}} = \frac{1}{e} \cdot T_{\overline{\beta}} \cdot u^\beta \cdot h_u = \frac{1}{e} \cdot T_{\overline{\beta}} \cdot u^\beta \cdot h_u, \]
\[ T_{\overline{\alpha\overline{\beta}}} = \frac{1}{e} \cdot h_u \cdot g((g)(T)(u) = T_{\overline{\alpha\overline{\beta}}} \cdot e_\alpha = T_{\overline{\alpha\overline{\beta}}} \cdot \partial_i, \]
\[ T_{\overline{\alpha\overline{\beta}}} = h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot u^\beta \cdot h_u = \frac{1}{e} \cdot h^i \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot u^\beta \cdot h_u, \]
\[ T_{\overline{\alpha\overline{\beta}}} = h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u = \frac{1}{e} \cdot h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u, \]
\[ T_{\overline{\alpha\overline{\beta}}} = h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u = \frac{1}{e} \cdot h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u, \]
\[ T_{\overline{\alpha\overline{\beta}}} = h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u = \frac{1}{e} \cdot h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u, \]
\[ T_{\overline{\alpha\overline{\beta}}} = h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u = \frac{1}{e} \cdot h_u \cdot g_{\overline{\alpha\overline{\beta}}} \cdot T_{\overline{\beta}} \cdot h^\alpha \cdot h_u, \]
\[ (sT)g = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes u - L \cdot K r(g) + u \otimes T_\pi + T_\pi \otimes u + T_\pi g = \]
\[ = sT^{\alpha \beta} \cdot e_\alpha \otimes e_\beta = sT^{ij} \cdot \partial_i \otimes \partial_j , \]
\[ (62) \]
\[ sT^{\alpha \beta} = g^{\beta \gamma} \cdot sT_\gamma ^\alpha , \quad sT^{ij} = sT_k^i \cdot g^{jk} , \]
\[ (63) \]
\[ g(sT) = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot g(u) \otimes g(u) - L \cdot g(Kr) + g(u) \otimes g(T_\pi) + \]
\[ + g(T_\pi) \otimes g(u) + g(T_\pi g) , \]
\[ g(sT) = sT^{\alpha \beta} \cdot e^\alpha \otimes e^\beta = sT_{ij} \cdot dx^i \otimes dx^j , \]
\[ sT^{\alpha \beta} = g_{\alpha \gamma} \cdot sT_\gamma ^\beta , \quad sT_{ij} = g_{\alpha \gamma} \cdot sT_j ^\gamma , \]
\[ (65) \]

In a co-ordinate basis \( sT \), \((sT)g\) and \(g(sT)\) can be represented in the forms
\[ sT_i^j = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u^j - L \cdot g_i^j + T_\pi i^j \cdot u^j + u_i \cdot T_\pi^j + g_{\pi k} \cdot T_\pi^j k \cdot g_{ij} , \]
\[ (66) \]
\[ sT^{ij} = sT_k^i \cdot g^{jk} = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u^i \cdot u^j - L \cdot g_i^j + T_\pi i^j + T_\pi^j \cdot u^j + T_\pi^j k \cdot g_{ij} , \]
\[ (67) \]
\[ sT_{ij} = g_{\pi k} \cdot sT_j ^k = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u_i \cdot u_j - L \cdot g_{ij} + u_i \cdot T_\pi j + T_\pi j^i \cdot u_j + g_{\pi k} \cdot T_\pi^j \cdot g_{ij} , \]
\[ (68) \]

where
\[ T_\pi i = g_{\pi k} \cdot T_\pi^j k , \quad T_\pi i = g_{\pi k} \cdot T_\pi^j , \quad T_\pi^j i = g_{\pi k} \cdot T_\pi^j k \cdot g_{ij} . \]

The variational energy-momentum tensor of Euler-Lagrange \( Q \) can be represented in the standard manner by the use of the projective metrics \( h_i^u \), \( h_u \) and the rest mass density \( \rho_Q \) in the form
\[ Q = -\rho_Q \cdot u \otimes g(u) - u \otimes g(Q \pi) - Q_s \otimes g(u) - (Q S) g , \]
\[ (69) \]
where
\[ \rho_Q = -\frac{1}{e^2} \cdot [g(u)](Q)(u) , \]
\[ (70) \]
\[ \rho_Q = -\frac{1}{e^2} \cdot \bar{g}_{\pi \gamma} \cdot u^\beta \cdot \bar{Q}_\gamma ^\alpha \cdot u^\gamma = -\frac{1}{e^2} \cdot \bar{Q}_\gamma ^\alpha \cdot \bar{u}_\pi \cdot u^\gamma = \]
\[ = -\frac{1}{e^2} \cdot \bar{g}_{ij} \cdot u^i \cdot \bar{Q}_k ^j \cdot u^k = -\frac{1}{e^2} \cdot \bar{Q}_k ^j \cdot \bar{u}_i \cdot u^j , \]
\[ Q_i = -\frac{1}{e} \cdot [g(u)](Q)(u) , \]
\[ (71) \]
\[ Q_i = -\frac{1}{e} \cdot [g(u)](Q)(u) = Q_{\pi \alpha} \cdot e_\alpha = Q_{\pi i} \cdot \partial_i , \]
\[ (72) \]
\[ Q_{\pi \alpha} = -\frac{1}{e} \cdot \bar{Q}_\gamma ^\beta \cdot u_\beta \cdot h_{\pi \alpha} , \quad Q_{\pi i} = -\frac{1}{e} \cdot \bar{Q}_j ^k \cdot u_k \cdot h_{\pi i} , \]
\[ (73) \]
\[ Q_s = -\frac{1}{e} \cdot h^u(g)(Q)(u) = Q_s \otimes e_\alpha = Q_{s i} \cdot \partial_i , \]
\[ (74) \]
\[ Q_s = -\frac{1}{e} \cdot h^u(g)(Q)(u) \]
\[ Q_s = -\frac{1}{e} \cdot h^u(g)(Q)(u) \]
\[ Q_S = -h^u(g)(Q) h_u = Q_S \otimes e_\alpha = Q_{S i} \cdot \partial_i \otimes \partial_j , \]
\[ Q_S = -h^u(g)(Q) h_u = Q_S \otimes e_\alpha = Q_{S i} \cdot \partial_i \otimes \partial_j , \]
\[ (76) \]
\[ Q_S = -h^u(g)(Q) h_u = Q_S \otimes e_\alpha = Q_{S i} \cdot \partial_i \otimes \partial_j , \]
\[ (77) \]
\[
(Q\tilde{g}) = -\rho_Q \cdot u \otimes u - u \otimes Q_\pi - Q_s \otimes u - Q_S = \\
= Q^{ij} \cdot \partial_i \otimes \partial_j ,
\]

\[
Q^{\alpha\beta} = g^{\gamma\delta} \cdot \bar{Q}_\gamma^\alpha \cdot g^{\delta\tau}, \quad Q^{ij} = \bar{Q}_k^i \cdot g^{kji} ,
\]

\[
g(Q) = -\rho_Q \cdot g(u) \otimes g(u) - g(u) \otimes g(Q_{\pi}) - g(Q_s) \otimes g(u) - g(Q_S) g ,
\]

In a co-ordinate basis \( Q \), \((Q)\tilde{g}\) and \( g(Q) \) can be represented in the forms

\[
\bar{Q}_{ij} = -\rho_Q \cdot u^i \cdot u^j - Q_{\pi i} \cdot u^j - u_i \cdot Q_{\pi j} - S_i^j \cdot Q_{Sij} ,
\]

\[
Q_{ij} = g_{ik} \cdot \bar{Q}_j^k = -\rho_Q \cdot u_i \cdot u_j - u_i \cdot Q_{\pi j} - S_i^j \cdot u_j - Q_{Sij} S_{kli} \cdot g_{lj} ,
\]

where

\[
Q_{\pi i} = g_{ik} \cdot Q_{\pi k} , \quad Q_{Si} = g_{ij} \cdot Q_{Sij} , \quad S_{ij} = g_{ik} \cdot Q_{Sij} S_{kli} \cdot g_{lj} .
\]

The introduced abbreviations for the different projections of the energy-momentum tensors have their analogous forms in \( V_3 \)- and \( V_4 \)-spaces, where their physical interpretations have been proposed \([2] \quad [3] \quad (S.383-385)\). The stress tensor in \( V_3 \)-spaces has been generalized to the energy-momentum tensor \( sT \) in \( V_4 \)-spaces. The viscosity stress tensor \( s_k T \) appears as the tensor \( T \) in the structure of the symmetric energy-momentum tensor of Belinfante \( sT \).

On the analogy of the physical interpretation of the different projections, the following definitions can be proposed for the quantities in the representations of the different energy-momentum tensors:

**A. Generalized canonical energy-momentum tensor of the type 1. \( \theta \)**

(a) Generalized viscous energy-momentum tensor of the type 1. \( \kappa \theta \)

(b) Rest mass density of the generalized canonical energy-momentum tensor \( \theta \). \( \rho_0 \)

(c) Conductive momentum density of the generalized canonical energy-momentum tensor \( \theta \). \( \rho_\pi \)

(d) Conductive energy flux density of the generalized canonical energy-momentum tensor \( \theta \). \( e_\pi \)

(e) Stress tensor of the generalized canonical energy-momentum tensor \( \theta \). \( s_\pi \)

**B. Symmetric energy-momentum tensor of Belinfante of the type 1. \( sT \)**

(a) Symmetric viscous energy-momentum tensor of the type 1. \( T \)

(b) Rest mass density of the symmetric energy-momentum tensor of Belinfante \( sT \). \( \rho_T \)

(c) Conductive momentum density of the symmetric energy-momentum tensor of Belinfante \( sT \). \( T \pi \)
Conductive energy flux density of the symmetric energy-momentum tensor of Belinfante $\text{e} \cdot T_s$.

Stress tensor of the symmetric energy-momentum tensor of Belinfante $\text{e} \cdot T_s$.

C. Variational (active) energy-momentum tensor of Euler-Lagrange $\rho Q$.

(a) Rest mass density of the variational energy-momentum tensor of Euler-Lagrange $\rho Q$.

(b) Conductive momentum density of the variational energy-momentum tensor of Euler-Lagrange $\pi$.

(c) Conductive energy flux density of the variational energy-momentum tensor of Euler-Lagrange $\text{e} \cdot Q_s$.

(d) Stress tensor of the variational energy-momentum tensor of Euler-Lagrange $Q_S$.

The projections of the energy-momentum tensors have properties which are due to their construction, the orthogonality of the projective metrics $h_u$ and $h^u$ correspondingly to the vector fields $u$ and $g(u)$, $h^u[g(u)] = 0$.

\begin{align}
  g(u, \theta \pi) &= g(\theta \pi, u) = 0, \quad g(u, T \pi) = 0, \quad g(u, Q \pi) = 0, \\
  g(u, \theta s) &= g(\theta s, u) = 0, \quad g(u, T s) = 0, \quad g(u, Q s) = 0, \\
  g(u, \theta S) &= g(\theta S, u) = 0, \quad g(u, T S) = 0, \quad g(u, Q S) = 0.
\end{align}

From the properties of the different projections it follows that the conductive momentum density $\pi$ (or $\pi$) is a contravariant vector field orthogonal to the vector field $u$. The conductive energy flux density $\text{e} \cdot s$ (or $\text{e} \cdot \pi$) is also a contravariant vector field orthogonal to $u$. The stress tensor $S$ (or $S$) is orthogonal to $u$ independently of the side of the projection by means of the vector field $u$.

The second covariant Noether identity $\theta - s T \equiv Q$ can be written by the use of the projections of the energy-momentum tensors in the form

\begin{align}
  (\rho_\theta - \rho_T + \rho_Q) \cdot u \otimes g(u) + u \otimes g(\theta \pi - T \pi + Q \pi) + \\
  + (\theta s - T s + Q s) \otimes g(u) + (\theta S - T S + Q S) g(u) = 0.
\end{align}

After contraction of the last expression consistently with $u$ and $\pi$ and taking into account the properties (84) $\div$ (86) the second covariant Noether identity disintegrates in identities for the different projections of the energy-momentum tensors

\begin{align}
  \rho_\theta &= \rho_T - \rho_Q, \quad \theta \pi = T \pi - Q \pi, \quad \theta s = T s - Q s, \quad \theta S = T S - Q S.
\end{align}

If the covariant Euler-Lagrange equations of the type $\delta_v L/\delta V^A \equiv 0$ are fulfilled for the non-metric tensor fields of a Lagrangian system and $\rho Q = 0$, then the variational energy-momentum of Euler-Lagrange $Q = \text{e} Q + \varphi Q$ is equal to zero. This fact leads to vanishing the invariant projections of $Q$ ($\rho Q = 0$, $\pi = 0$, $\text{e} \cdot s = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0$, $\text{e} \cdot \pi = 0$, $\text{e} \cdot S = 0
\( Q_{\pi} = 0, Q_s = 0, Q_S = 0 \). The equality which follows between \( \theta \) and \( sT \) has as corollaries the identities

\[
\rho_\theta \equiv \rho_T, \quad \delta\pi \equiv T\pi, \quad \delta\pi \equiv T\pi, \quad \delta\pi \equiv T\pi. \quad (89)
\]

From the first identity \((\rho_\theta \equiv \rho_T)\) and the identity \((88)\) for \( \rho \), it follows that the covariant Euler-Lagrange equations of the type \( \delta_v L/\delta V^A = 0 \) for non-metric fields \( V \) and \( gQ = 0 \) appear as sufficient conditions for the unique determination of the notion of rest mass density \( \rho \) for a given Lagrangian system.

**Proposition 1** The necessary and sufficient condition for the equality

\[ \rho_\theta = \rho_T \quad (90) \]

is the condition \( \rho_Q = 0 \).

Proof. It follows immediately from the first identity in \((88)\).

The condition \( \rho_Q \neq 0 \) leads to the violation of the unique determination of the notion of rest mass density and to the appearance of three different notions of rest mass density corresponding to the three different energy-momentum tensors for a Lagrangian system. Therefore, the violation of the covariant Euler-Lagrange equations \( \delta_v L/\delta V^A = 0 \) for the non-metric tensor fields or the existence of metric tensor fields in a Lagrangian density with \( gQ \neq 0 \) induce a new rest mass density (a new rest mass respectively) for which the identity \((88)\) is fulfilled.

The identity \( \rho_\theta \equiv \rho_T - \rho_Q \) can be related to the physical hypotheses about the inertial, passive and active gravitational rest mass densities in models for describing the gravitational interaction. To every energy-momentum tensor a non-null rest mass density corresponds. The existence of the variational energy-momentum tensor of Euler-Lagrange is connected with the existence of the gravitational interaction in a Lagrangian system in Einstein’s theory of gravitation [4] and therefore, with the existence of a non-null active gravitational rest mass density. When a Lagrangian system does not interact gravitationally, the active gravitational rest mass density is equal to zero and the principle of equivalence between the inertial and the passive rest mass density is fulfilled [5] - [8].

From the second covariant Noether identity of the type 1, by means of the relations

\[
\bar{G} = g(G)\bar{g}, \quad G = \bar{g}(G)g,
\]

one can find the second covariant Noether identity of the type 2, for the energy-momentum tensors of the type 2, in the form

\[
\bar{\theta} - sT \equiv \bar{Q}, \quad (91)
\]

where

\[
\bar{\theta} = g(\theta)\bar{g}, \quad sT = g(sT)\bar{g}, \quad \bar{Q} = g(Q)\bar{g}. \quad (92)
\]
The invariant representation of the energy-momentum tensors by means of the projective metrics \( h^u \), \( h_u \) and the rest mass density allows a comparison of these tensors with the well known energy-momentum tensors from the continuum media mechanics (for instance, with the energy-momentum tensor of an ideal liquid in \( V_4 \)-spaces:

\[
_sT_{ij} = \left( \rho + \frac{1}{e} \cdot p \right) \cdot u_i \cdot u^j - p \cdot g^j_i, \quad e = \text{const.} \quad \neq 0, \quad k = 1 \). \tag{93}
\]

It follows from the comparison that the Lagrangian invariant \( L \) can be interpreted as the pressure \( p = L \) characterizing the Lagrangian system. This possibility for another physical interpretation than the usual one (in the mechanics \( L \) is interpreted as the difference between the kinetic and the potential energy) allows a description of Lagrangian systems on the basis of phenomenological investigations determining the dependence of the pressure on other dynamical characteristics of the system. If these relations are given, then by the use of the method of Lagrangians with covariant derivatives (MLCD) the corresponding covariant Euler-Lagrange equations can be found as well as the energy-momentum tensors.

The use of a contravariant non-isotropic (non-null) vector field \( u \) and its corresponding projective metrics \( h^u \) and \( h_u \) is analogous to the application of a non-isotropic (time-like) vector field in the s. c. monad formalism [(3 + 1)-decomposition] in \( V_4 \)-spaces for description of dynamical systems in Einstein’s theory of gravitation (ETG) \[8\] - \[17\], \[3\], \[18\].

The contravariant time-like vector field has been interpreted as a tangential vector field at the world line of an observer determining the frame of reference (the reference frame) in the space-time. By the use of this reference frame a given physical system is observed and described. The characteristics of the vector field determine the properties of the reference frame. Moreover, the vector field is assumed to be an absolute element in the scheme for describing the physical processes, i.e. the vector field is not an element of the model of the physical system. It is introduced as a priory given vector field which does not dependent on the Lagrangian system. In fact, the physical interpretation of the contravariant non-isotropic vector field \( u \) can be related to two different approaches analogous to the method of Lagrange and the method of Euler in describing the motion of liquids in the hydrodynamics \[19\].

In the method of Lagrange, the object with the considered motion appears as a point (particle) of the liquid. The motion of this point is given by means of equations for the vector field \( u \) interpreted as the velocity of the particle. The solutions of these equations give the trajectories of the points in the liquid as basic characteristics of the physical system. In this case, the vector field \( u \) appears as an element of the model of the system. It is connected with the motions of the system’s elements. Therefore, a Lagrangian system (and respectively its Lagrangian density) could contain as an internal characteristic a contravariant vector field \( u \) obeying equations of the type of the Euler-Lagrange equations and describing the evolution of the system.
In the method of Euler, the object with the considered motion appears as the model of the continuum media. Instead of the investigation of the motion of every (fixed by its velocity and position) point (particle), the kinematic characteristics in every immovable point in the space are considered as well as the change of these characteristics after moving on to another space point. The motion is assumed to be described if the vector field $u$ is considered as a given (or known) velocity vector field.

The $(n - 1) + 1$ decomposition (monad formalism) can be related to the method of Euler or to the method of Lagrange:

(a) Method of Euler. The vector field $u$ is interpreted as the velocity vector field of an observer who describes a physical system with respect to his vector field (his velocity). This physical system is characterized by means of a Lagrangian system (Lagrangian density).

The motion of the observer (his velocity vector field) is given independently of the motion of the considered Lagrangian system.

(b) Method of Lagrange. The vector field $u$ is interpreted as the velocity vector field of a continuum media with a co-moving with it observer. The last assumption means that the velocity vector field of the observer is identical with the velocity vector field of the media where he is situated.

The motion of the observer (his velocity vector field) is determined by the characteristics of the (Lagrangian) system. Its velocity vector field is, on the other side, determined by the dynamical characteristics of the system by means of equations of the type of the Euler-Lagrange equations.

3 Energy-momentum tensors and the momentum density

In the first section of this paper the notion of momentum density of an energy-momentum tensor $G$ has been introduced as

$$ \rho_G \cdot u + \pi_G $$

where $\rho_G \cdot u$ is the convective momentum density of the energy-momentum tensor $G$; $\pi_G$ is the conductive momentum density of the energy-momentum tensor $G$. The tensor field $G$ can be represented by means of the projective metrics $h^u$ and $h_u$ in the form

$$ G = u \otimes g(p_G) + G_s \otimes g(u) + (G S) g. $$

At the same time the relations are fulfilled

$$ (G) \pi = u \otimes p_G + G_s \otimes u + G S = G^{\alpha \beta} \cdot e_\alpha \otimes e_\beta = G^{ij} \cdot \partial_i \otimes \partial_j, $$

$$ g(G) = g(u) \otimes g(p_G) + g(G_s) \otimes g(u) + g(G S) g, $$

13
\[ p_G = \frac{1}{e} \cdot [g(u)](G)(\mathcal{G}) . \]  

The Kronecker tensor can be represented in the form

\[ Kr = u \otimes g(p_{Kr}) + Kr_s \otimes g(u) + (Kr S)g , \]  

where

\[ p_{Kr} = \frac{1}{e} \cdot k \cdot u + Kr \pi , \quad L \cdot Kr = u \otimes g(L \cdot p_{Kr}) + L \cdot Kr_s \otimes g(u) + (L \cdot Kr S)g . \]  

The generalized canonical energy-momentum tensor \( \theta \) will have then the form

\[ \theta = u \otimes g(p_{\theta}) + \theta s \otimes g(u) + (\theta S)g , \]

where

\[ p_{\theta} = \rho_{\theta} \cdot u + \theta \pi = \rho_{\theta} \cdot u + \theta \pi - L \cdot Kr \pi . \]  

In a co-ordinate basis \( \theta \), \( (\theta)g \) and \( g(\theta) \) can be written in the forms

\[ \theta_{ij} = \theta^k_i \cdot u^j + \theta^j_i \cdot \theta s^k + g_{ik} \cdot \theta S^{ij} , \]  

\[ \theta_{ij} = g_{ik} \cdot \theta^k_i \cdot \theta j^k = u_i \cdot \theta s_j + g_{ij} \cdot \theta S^i \theta \]  

The symmetric energy-momentum tensor of Belinfante \( sT \) can be represented by means of the projective metrics \( h^u, h_u \) and the momentum density \( p_T \) in the form

\[ sT = u \otimes g(p_T) + T s \otimes g(u) + (T S)g = sT_{\alpha \beta} \cdot e_\alpha \otimes e_\beta = sT_{ij} \cdot \partial_i \otimes \partial_j \],

where

\[ p_T = \rho_T \cdot u + T \pi = \rho_T \cdot u + T \pi - L \cdot Kr \pi , \]  

\[ (sT)g = u \otimes p_T + T s \otimes u + T S = sT^{\alpha \beta} \cdot e_\alpha \otimes e_\beta = sT_{ij} \cdot dx^i \otimes dx^j \],

\[ g(sT) = g(u) \otimes g(p_T) + g(T s) \otimes g(u) + g(T S)g = sT_{\alpha \beta} \cdot e_\alpha \otimes e_\beta = sT_{ij} \cdot dx^i \otimes dx^j \],

\[ g(u, p_T) = \rho_T \cdot e , \quad g(u, T \pi) = 0 , \quad h_u(p_T) = h_u(T \pi) , \]
\[ h^u(h_u) = h^u(g) = \mathcal{F}(h_u), \quad h_u(G\pi) = (G\pi)h_u = g(G\pi), \quad (G\pi)g, \quad (109) \]

\[ h_u(pg) = g(G\pi), \quad G\pi = \mathcal{F}(h_u(pg)) , \quad (110) \]

\[ g(T\pi) = h_u(p_T), \quad T\pi = \mathcal{F}(h_u(p_T)) . \]

In a co-ordinate basis, \( sT, (sT)\mathcal{F} \) and \( g(sT) \) can be written in the forms

\[ sT^i_j = p_T^i \cdot u^j + u_i \cdot T s^j + g(\pi) \cdot T S^{jk}, \quad (111) \]

\[ p_T = p_T^i \cdot \partial_i, \quad p_T^j = g(\pi) \cdot p_T^k, \]

\[ sT^{ij} = sT_k^i \cdot g(\pi) = u^i \cdot p_T^j + T s^i \cdot u^j + T S^{ij}, \quad (112) \]

\[ sT_{ij} = g(\pi) \cdot sT^k_j = u_i \cdot p_T^j + T s_i \cdot u_j + T S_{ij}. \quad (113) \]

The variational energy-momentum tensor of Euler-Lagrange \( Q \) can be represented by means of the projective metrics \( h^u, h_u \) and the momentum density \( pQ \) in the forms

\[ Q = -u \otimes g(pQ) - Q^s \otimes g(u) - (Q^S)g = \frac{\partial}{\partial x^i} , \quad (114) \]

\[ (Q)\mathcal{F} = -u \otimes pQ - Q^s \otimes u - Q^S = Q^{ij} \cdot \partial_i \otimes \partial_j , \quad (115) \]

\[ g(Q) = -g(u) \otimes g(pQ) - g(Q^s) \otimes g(u) - g(Q^S)g = \frac{\partial}{\partial x^i} , \quad (116) \]

where

\[ h_u(Q\pi) = g(Q\pi), \quad g(u, Q\pi) = 0, \quad h_u(pQ) = g(Q\pi), \]

\[ pQ = pQ \cdot u + Q\pi, \quad Q^s = \mathcal{F}(h_u(pQ)), \quad h_u(Q^s) = g(Q^s), \]

\[ h_u(h^u) = h^u(h_u) = h_u(\mathcal{F}) = g(h^u). \quad (117) \]

In a co-ordinate basis \( Q, (Q)\mathcal{F} \) and \( g(Q) \) will have the forms

\[ \overline{Q}^i_j = -pQ_i \cdot u^j - u_i \cdot Q^s_j - g(\pi) \cdot Q S^{jk}, \quad (118) \]

\[ Q^{ij} = \overline{Q}_k^i \cdot g(\pi) = -u^i \cdot p_Q^j - Q^s_j \cdot u^j - Q S^{ij}, \quad (119) \]

\[ Q_i_j = g(\pi) \overline{Q}_k^j = -u_i \cdot pQ_j - Q^s_i \cdot u_j - Q S_{ij}, \quad (120) \]
\[ p_{Qi} = g_{i}^{k} p_{Q}^{k}, \quad Q s_{i} = g_{i}^{k} Q s_{i}^{k}, \quad Q S_{ij} = g_{i}^{k} Q S^{kl} g_{lj}. \]

On the analogy of the notions in the continuum media mechanics one can introduce the following definitions connected with the notion momentum density of a given energy-momentum tensor:

A. Generalized canonical energy-momentum tensor \( \theta \)

(a) Momentum density of \( \theta \)
\[ p_{\theta} = \frac{1}{e} \cdot [g(u)](\theta)(\overline{\theta}) = \rho_{\theta} \cdot u + \theta \pi. \]

(b) Convective momentum density of \( \theta \)
\[ \rho_{\theta} \cdot u. \]

B. Symmetric energy-momentum tensor of Belinfante \( sT \)

(a) Momentum density of \( sT \)
\[ p_{T} = \frac{1}{e} \cdot [g(u)](sT)(\overline{\theta}) = \rho_{T} \cdot u + T \pi. \]

(b) Convective momentum density of \( sT \)
\[ \rho_{T} \cdot u. \]

C. Variational energy-momentum tensor of Euler-Lagrange \( Q \)

(a) Momentum density of \( Q \)
\[ p_{Q} = -\frac{1}{e} \cdot [g(u)](Q)(\overline{\theta}) = -\rho_{Q} \cdot u - Q \pi. \]

(b) Convective momentum density of \( Q \)
\[ \rho_{Q} \cdot u. \]

From the covariant Noether identities for the rest mass and the conductive momentum densities, it follows the identity for the momentum density of the different energy-momentum tensors
\[ p_{\theta} \equiv p_{T} - p_{Q}. \quad (121) \]

If the Euler-Lagrangian equations \( \delta_{c} L / \delta V_{A} = 0 \) for the non-metric tensor fields are fulfilled and \( g_{Q} = 0 \), then
\[ p_{\theta} = p_{T}. \quad (122) \]

The representations of the energy-momentum tensors of the type 2 is analogous to the representations of the energy-momentum tensors of the type 1 by the use of \( h^{u}, h_{u} \) and the momentum density.

4 Energy-momentum tensors and the energy flux density

In the first section the notion of energy flux density has been introduced for a given energy-momentum tensor \( G \) as
\[ e_{G} = G(u) = e \cdot (\rho_{G} \cdot u + G s). \]
The energy-momentum tensors can be now represented by the use of the projective metrics $h^u$, $h_u$ and the energy flux density in the forms

$$G = \frac{1}{e} \cdot e_G \otimes g(u) + u \otimes g(G\pi) + (G\pi)g =
= \theta_{ij} \cdot \partial_i \otimes dx^j ,$$

$$(G)G = \frac{1}{e} \cdot e_G \otimes u + u \otimes G\pi + G\pi = \theta_{ij} \cdot e_{ij} = \theta_{ij} \cdot \partial_i \otimes \partial_j ,$$

(123)

$$g(G) = \frac{1}{e} \cdot g(e_G) \otimes g(u) + g(u) \otimes g(G\pi) + g(G\pi)g =
= \theta_{ij} \cdot \partial_i \otimes \partial_j \cdot dx^i \otimes dx^j .$$

(124)

The Kronecker tensor $K\rho$, where

$$e_K = K\rho(u) = k \cdot u + e \cdot K\rho = e \cdot \left( \frac{1}{e} \cdot k \cdot u + K\rho \right) .$$

(125)

The structure of $e_G$ allows the introduction of the abbreviations:

(a) Convective energy flux density of the energy-momentum tensor $G$ ................................................................. $\rho G \cdot e \cdot u$.

(b) Conductive energy flux density of the energy-momentum tensor $G$ ................................................................. $e \cdot s$.

The generalized canonical energy-momentum tensor $\theta$ can be represented in the form

$$\theta = \frac{1}{e} \cdot e_G \otimes g(u) + u \otimes g(\theta\pi) + (G\pi)g =
= \theta_{ij} \cdot \partial_i \otimes dx^j ,$$

(126)

where

$$e_G = \theta(u) = e \cdot (\rho G \cdot u + \theta S) = e_G \cdot e_{ij} = e_G \cdot \partial_i .$$

or by means of the forms

$$\theta_G = \frac{1}{e} \cdot e_G \otimes u + u \otimes \theta\pi + \theta S = \theta_{ij} \cdot \partial_i \otimes \partial_j ,$$

(127)

$$\theta G = \frac{1}{e} \cdot g(e_G) \otimes g(u) + g(u) \otimes g(\theta\pi) + g(\theta S)g =
= \theta_{ij} \cdot \partial_i \otimes \partial_j \cdot dx^i \otimes dx^j .$$

(128)

In a co-ordinate basis $\theta$, $(\theta)G$ and $g(\theta)$ will have the forms

$$\theta_{ij} = \frac{1}{e} \cdot \partial_i \cdot e_{ij} + \theta\pi \cdot u_j + g_{\theta j} \cdot \theta S_{ij} ,$$

(129)

$$\theta_{ij} = \theta_{ij} \cdot \theta_{ij} = \frac{1}{e} \cdot e_G \cdot u_j + u_i \cdot \theta\pi_j + \theta S_{ij} ,$$

(130)

$$\theta_{ij} = \theta_{ij} \cdot \theta_{ij} = \frac{1}{e} \cdot e_G \cdot u_j + u_i \cdot \theta\pi_j + \theta S_{ij} ,$$

(131)

$$e_{\theta i} = \theta_{ij} \cdot e_{\theta j} = \theta S_{ij} = \theta S_{ij} \cdot \theta S_{ij} .$$

(132)
The symmetric energy-momentum of Belinfante $sT$ can be represented by the use of $e_T$ in the form

$$sT = \frac{1}{\rho} \cdot e_T \otimes g(u) + u \otimes g(T_\pi) + (T S)g = sT \alpha \beta \cdot e_\alpha \otimes e_\beta = sT_i^j \cdot \partial_j \otimes dx^i,$$

(132)

where

$$e_T = sT(u) = e \cdot \left( \rho_T \cdot u + T s \right) = e_T^\alpha \cdot e_\alpha = e_T^i \cdot \partial_i ,$$

(133)

or by means of the forms

$$\left( sT \right) \overline{\overline{g}} = \frac{1}{\rho} \cdot e_T \otimes u + u \otimes T_\pi + T S = sT \alpha \beta \cdot e_\alpha \otimes e_\beta = sT_i^j \cdot \partial_i \otimes \partial_j ,$$

(134)

In a co-ordinate basis $sT$, $(sT)\overline{\overline{g}}$ and $g(sT)$ will have the forms

$$sT_i^j = \frac{1}{\rho} \cdot u_i \cdot e_T^j + \pi u_i \cdot e_T^j + g_{\overline{\overline{g}}} \cdot sT^k, (136)$$

$$sT_i^j = g_{\overline{\overline{g}}} \cdot sT_i^j = \frac{1}{\rho} \cdot e_T^{ij} \cdot u_i + u_j \cdot \pi^i + T S^{ij} ,$$

(137)

$$sT_i^j = g_{\overline{\overline{g}}} \cdot sT_j^k = \frac{1}{\rho} \cdot e_T^{ij} \cdot u_i + u_j \cdot \pi^i + T S^{ij} ,$$

(138)

$$e_T^i = g_{\overline{\overline{g}}} \cdot e_T^j, \quad T S_{ij} = g_{\overline{\overline{g}}} \cdot T S^{kl} \cdot g_{ij}.$$  

The variational energy-momentum tensor of Euler-Lagrange $Q$ can be represented in an analogous way by the use of the energy flux density $e_Q$ in the forms

$$Q = -\frac{1}{\rho} \cdot e_Q \otimes g(u) - u \otimes g(Q_\pi) - (Q S)g = Q \alpha \beta \cdot e_\alpha \otimes e_\beta = Q_i^j \cdot \partial_j \otimes dx^i ,$$

(139)

where

$$e_Q = - Q(u) = e \cdot \left( \rho_Q \cdot u + Q s \right) = e_Q^\alpha \cdot e_\alpha = e_Q^i \cdot \partial_i ,$$

(140)

or by means of the forms

$$\left( Q \right) \overline{\overline{g}} = -\frac{1}{\rho} \cdot e_Q \otimes u - u \otimes Q_\pi - Q S = Q \alpha \beta \cdot e_\alpha \otimes e_\beta = Q_i^j \cdot \partial_i \otimes \partial_j ,$$

(141)

In a co-ordinate basis $Q$, $(Q)\overline{\overline{g}}$ and $g(Q)$ will have the forms

$$\overline{Q_i}^j = -\frac{1}{\rho} \cdot u_i \cdot e_Q^j - Q_\pi \cdot u^j - g_{\overline{\overline{g}}} \cdot Q S^{ij} ,$$

(142)
\[ Q^{ij} = g^k_i \cdot g^j_k = -\frac{1}{e} \cdot e^i_Q \cdot w^j - w^i \cdot Q \pi^j - Q S^{ij}, \]

\[ Q_{ij} = g_{ik} \cdot Q^j_k = -\frac{1}{e} \cdot e_Q \cdot u_j - u_i \cdot Q \pi_j - Q S_{ij}, \]

\[ e_{Q_i} = g_{ik} \cdot e^k_Q, \quad Q_{S_{ij}} = g_{ik} \cdot Q S^{k_l} \cdot g_{lj}. \]

On the analogy of the determined notions, the following abbreviations can be introduced for a given energy-momentum tensor: 

A. Generalized canonical energy-momentum tensor \( \theta \).  
(a) Energy flux density of \( \theta \) \( \cdots \cdots \cdots e_\theta \).  
(b) Convective energy flux density of \( \theta \) \( \cdots \cdots \cdots \rho \theta \cdot e \cdot u \).

B. Symmetric energy-momentum tensor of Belinfante \( sT \).  
(a) Energy flux density of \( sT \) \( \cdots \cdots \cdots e_T \).  
(b) Convective energy flux density of \( sT \) \( \cdots \cdots \cdots \rho T \cdot e \cdot u \).

C. Variational energy-momentum tensor of Euler-Lagrange \( Q \).  
(a) Energy flux density of \( Q \) \( \cdots \cdots \cdots e_Q \).  
(b) Convective energy flux density of \( Q \) \( \cdots \cdots \cdots \rho Q \cdot e \cdot u \).

By means of the covariant Noether identities for the rest mass density and for the conductive energy flux density the identity for the energy flux density follows in the form

\[ e_\theta \equiv e_T - e_Q. \quad (143) \]

The relation between the momentum density and the energy flux density follows from the structure of their definitions

\[ p_G = \rho G \cdot u + G \pi, \quad e_G = (\rho G \cdot u + G s) \cdot e, \quad G \sim (\theta, sT, Q). \]

From the last expressions the relations

\[ p_G = \frac{1}{e} \cdot e_G + G \pi - G s, \quad (144) \]

\[ e_G = (p_G + G s - G \pi) \cdot e \quad (145) \]

follow.

By the use of the different representations of the energy-momentum tensors the different physical processes can be investigated. The application of an representation will depend on the role of the considered quantity (rest mass density, momentum density or energy flux density) in the dynamical process.

The physical interpretation of the introduced notions has been used for describing Lagrangian systems in \( V_4 \)-spaces, where the contravariant vector field \( u \) has been interpreted as a time-like vector field tangential to the trajectories of the moving in the space-time particles. A \( V_4 \)-space is considered as a model of the space-time.
5 Covariant divergency of a mixed tensor field

The operation of the covariant differentiation along a contravariant vector field can be extended to covariant differentiation along a contravariant tensor field.

The Lie derivative $\mathcal{L}_\xi u$ of a contravariant vector field $u$ along a contravariant vector field $\xi$ can be expressed by the use of the covariant differential operators $\nabla_\xi$ and $\nabla_u$ in the form

$$\mathcal{L}_\xi u = \nabla_\xi u - \nabla_u \xi - T(\xi, u) , \xi, u \in T(M),$$

where $T(\xi, u)$ is the contravariant torsion vector field

$$T(\xi, u) = T_{\alpha\beta}^\gamma \cdot \xi^\alpha \cdot u^\beta \cdot e_\gamma = T_{ij}^k \cdot \xi^i \cdot u^j \cdot \partial_k ,$$

constructed by means of the components $T_{\alpha\beta}^\gamma$ (or $T_{ij}^k$) of the contravariant torsion tensor field $T$.

The Lie derivative $\mathcal{L}_\xi V$ of a contravariant tensor field $V = V^A \cdot e_A = V^A \cdot \partial_A \in \otimes^m(M)$ along a contravariant vector field $\xi$ can be written on the analogy of the relation for $\mathcal{L}_\xi u$ and by the use of the covariant differential operator $\nabla_\xi$ and an operator $\nabla_V$ in the form

$$\mathcal{L}_\xi V = \nabla_\xi V - \nabla_V \xi - T(\xi, V) , \xi \in T(M) , V \in \otimes^m(M),$$

where

$$\nabla_V \xi = -\xi^\alpha /\beta \cdot S_{Ba}^{A\beta} \cdot V^B \cdot e_A = -\xi^i /j \cdot S_{Ci}^{Aji} \cdot V^C \cdot \partial_A ,$$

$$T(\xi, V) = T_{B\gamma}^A \cdot \xi^\gamma \cdot V^B \cdot e_A = T_{Ck}^A \cdot \xi^k \cdot V^C \cdot \partial_A ,$$

$$T_{B\gamma}^A = T_{\beta\gamma}^{A\alpha} \cdot S_{Ba}^{A\beta} , T_{Ck}^A = T_{j\gamma}^{1 A \cdots} \cdot S_{Ci}^{Aji} .$$

$\nabla_V \xi$ appears as a definition of the action of the operator $\nabla_V$ on the vector field $\xi$. Let we now consider more closely this operator and its properties.

Let a mixed tensor field $K \in \otimes^k l(M)$ be given in a non-co-ordinate (or co-ordinate) basis

$$K = K^{C}^{D} \cdot e_C \otimes e_D = K^{C_1 \cdots}_{D_\beta} \cdot e_{C_1} \otimes \cdots \otimes e_{C_m} ,$$

$$e_C = e_{C_1} \otimes \cdots \otimes e_{C_m} , \quad e_D = e_{D_1} \otimes \cdots \otimes e_{D_m} .$$

The action of the operator $\nabla_V$ on the mixed tensor field $K$ can be defined on the analogy of the action of $\nabla_V$ on a contravariant vector field $\xi$

$$\nabla_V K = -K^{C_1 \cdots}_{D_\beta} \cdot S_{Ba}^{A\beta} \cdot V^B \cdot e_{C_1} \otimes \cdots \otimes e_{C_m} ,$$

where $\nabla_V$ is the covariant differential operator along a contravariant tensor field $V$

$$\nabla_V : K \Rightarrow \nabla_V K , \quad K \in \otimes^k l(M) , \quad V \in \otimes^m(M) , \quad \nabla_V K \in \otimes^{k-1+m} l(M) .$$

Remark. There is another possibility for a generalization of the action of $\nabla_V$ on a mixed tensor field

$$\nabla_V K = -\sum_{m=1}^{k} K^{\alpha_1 \cdots \alpha_{m-1} \cdots \alpha_1}_{D_\beta} \cdot S_{Ba}^{A\beta} \cdot V^B \cdot e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{m-1}} \otimes e_{\alpha_{m+1}} \otimes \cdots \otimes e_{\alpha_k} \otimes e_D \otimes e_A .$$
The result of the action of this operator on a contravariant vector field \( \xi \) is identical with the action of the above defined operator \( \nabla_V \).

**Remark.** A covariant differential operator along a contravariant tensor field can also be defined through its action on mixed tensor fields in the form

\[
\nabla_V K = K^{C}_{\ D/\beta} \cdot V^{A_{1} \beta} \cdot e_C \otimes e_D \otimes e_{A_1}.
\]

The operator \( \nabla_V \) differs from \( \nabla_V \) in its action on a contravariant vector field and does not appear as a generalization of the already defined operator by its action on a contravariant vector field. It appears as a new differential operator acting on mixed tensor fields.

The **covariant differential operator** \( \nabla_V \) has the properties:

(a) Linear operator

\[
\nabla_V (\alpha \cdot K_1 + \beta \cdot K_2) = \alpha \cdot \nabla_V K_1 + \beta \cdot \nabla_V K_2,
\]

\( \alpha, \beta \in \mathbb{R} \) (or \( \mathbb{C} \)), \( K_1, K_2 \in \otimes^k (M) \).

The proof of this property follows immediately from (146) and the linear property of the covariant differential operator along a basic contravariant vector field.

(b) Differential operator (not obeying the Leibniz rule)

\[
\nabla_V (K \otimes S) = \nabla_{e_\alpha} K \otimes \overline{S}^\beta + K \otimes \nabla_V S,
\]

\( K = K^{A}_{\ B} \cdot e_A \otimes e_B \), \( \nabla_{e_\alpha} K = K^{A}_{\ B/\alpha} \cdot e_A \otimes e_B \),

\( S = \overline{S}^C_{\ D} \cdot e_C \otimes e_D = \overline{S}^{C_1}_{\ D} \cdot e_{C_1} \otimes e_D \),

\( \overline{S}^\beta = -\overline{S}^{C_1}_{\ D} \cdot S_{E\alpha} \cdot \overline{V}^{F\beta} \cdot e_{C_1} \otimes e_D \).

The proof of this property follows from the action of the defined in (146) operator \( \nabla_V \) and the properties of the covariant derivative of the product of the components of the tensor fields \( K \) and \( S \).

If the tensor field \( V \) is given as a contravariant metric tensor field \( g \), then the covariant differential operator \( \nabla_V \) (\( V = g \)) will have additional properties connected with the properties of the contravariant metric tensor field.

**Definition 2** **Contravariant metric differential operator** \( \nabla_g \). **Covariant differential operator** \( \nabla_V \) for \( V = g \).

By means of the relations

\[
-S_{B\alpha} \cdot g^B \cdot e_A = (g^B_{\alpha} \cdot g^{\beta\kappa} + g^{\kappa}_{\alpha} \cdot g^{\beta\sigma}) \cdot e_\sigma \otimes e_\kappa = (g^B_{\alpha} \cdot g^{\beta\kappa} + g^{\kappa}_{\alpha} \cdot g^{\beta\sigma}) \cdot e_\sigma \cdot e_\kappa,
\]

\[
\ e_\sigma \cdot e_\kappa = \frac{1}{2} (e_\sigma \otimes e_\kappa + e_\kappa \otimes e_\sigma),
\]

\[
K^{C_1}_{\ D/\beta} \cdot g^\alpha_{\ D} = K^{C_1}_{\ D/\beta} \cdot g^\alpha_{\ D},
\]

\[
(147)
\]

\[
(148)
\]
the action of the contravariant metric differential operator on a mixed tensor field $K$ can be represented in the form

$$
\nabla_K = (g_{\alpha}^\beta \cdot g^{\beta \gamma} + \nabla^\beta \cdot g^{\beta \gamma}) \cdot e_{C_1} \otimes e_D \otimes e_\sigma \otimes e_\kappa = (g_{\alpha}^\beta \cdot g^{\beta \gamma} + \nabla^\beta \cdot g^{\beta \gamma}) \cdot e_{C_1} \otimes e_D \otimes e_\sigma \otimes e_\kappa .
$$

(149)

The properties of the operator $\nabla_K$ are determined additionally by the properties of the contravariant metric tensor field of second rank:

(a) $\nabla_K : K \Rightarrow \nabla_K K$, $K \in \otimes^k l(M)$, $\nabla_K K \in \otimes^{k+1} l(M)$.

(b) Linear operator

$$\nabla_K (\alpha \cdot K_1 + \beta \cdot K_2) = \alpha \cdot \nabla_K K_1 + \beta \cdot \nabla_K K_2 .
$$

(c) Differential operator (not obeying the Leibniz rule)

$$\nabla_K (K \otimes S) = \nabla_{e_\beta} K \otimes \nabla_S + K \otimes \nabla_{K} S ,$$

$$K \in \otimes^k l(M) , \quad S \in \otimes^m l(M) ,$$

$$K = K^{A \beta} B \cdot e_A \otimes e_B , \quad \nabla_{e_\beta} K = K^{A \beta} B \cdot e_A \otimes e_B ,$$

$$S = S^{C \gamma} D \cdot e_C \otimes e_D , \quad \nabla_{K} S = \nabla^{C \gamma} D \cdot e_C \otimes e_D ,$$

$$\nabla_S = (\nabla^{C \gamma} D \cdot g^{\beta \gamma} + \nabla^{C \gamma} D \cdot g^{\beta \gamma}) \cdot e_{C_1} \otimes e_D \otimes e_S .
$$

(150)

**Remark.** The definition of $\nabla_K$ in (149) differs from the definition in [6], where $\nabla = \nabla_K$, i.e. the contravariant metric differential operator is defined in the last case as a special case of the covariant differential operator $\nabla_V$ for $V = g$.

The notion of covariant divergency of a mixed tensor field has been used in $V_4$-spaces for the determination of conditions for the existence of local conserved quantities and in identities of the type of the first covariant Noether identity.

Usually, the covariant divergency of a contravariant or mixed tensor field has been given in co-ordinate or non-co-ordinate basis in the form

$$\delta K = K^{A \beta} B / \beta \cdot e_A \otimes e_B = K^{C i} D / i \cdot \partial_C \otimes d_{D} ,
$$

(151)

where

$$K^{A \beta} B / \beta = K^{A \beta} B / \gamma \cdot g_{\beta}^{\gamma} , \quad K^{C i} D / i = K^{C i} D / k \cdot g_{\gamma}^{k} .
$$

(152)

For full anti-symmetric covariant tensor fields (differential forms) the covariant divergency (called also codifferential) $\delta$ is defined by means of the Hodge operator $\ast$, its reverse operator $\ast^{-1}$ and the external differential operator $a \tilde{\nabla}$ in the form [20, 21] (pp. 147-149)

$$\delta = \ast^{-1} \circ a \tilde{\nabla} \circ \ast .
$$

(153)

**Remark.** The Hodge operator is constructed by means of the permutation (Levi-Clivita) symbols. It maps a full covariant anti-symmetric tensor of rank $(0, p) \equiv a \otimes p \ M \equiv \Lambda^p (M)$ in a full covariant anti-symmetric tensor of rank $(0, n - p) \equiv a \otimes_{n - p} \ M \equiv \Lambda^n - p (M)$, where dim $M = n$,

$$\ast : a A \rightarrow \ast A , \quad a A \in \Lambda^p (M) , \quad \ast A \in \Lambda^n - p (M) ,$$

(147-149)
with
\[ aA = A_{[i_1...i_p]} dx^{i_1} \wedge ... \wedge dx^{i_p}, \quad * aA = * A_{[j_1...j_{n-p}]} dx^{j_1} \wedge ... \wedge dx^{j_{n-p}}, \]
\[ * A_{[j_1...j_{n-p}]} = \frac{1}{p!} \varepsilon_{i_1...i_p j_1...j_{n-p}} A_{[i_1...i_p]}, \quad A_{[i_1...i_p]} = g^{i_1} g^{i_2} ... g^{i_p} A_{[i_1...i_p]}; \]
\[ *^{-1} = (-1)^{p(n-p)} * . \]

By the use of the contravariant metric differential operator \( \nabla g \), the covariant metric tensor field \( g \) and the contraction operator one can introduce the notion of covariant divergency of a mixed tensor field \( K \) with finite rank.

**Definition 3**  Covariant divergency \( \delta K \) of a mixed tensor field \( K \)

\[ \delta K = \frac{1}{2} [\nabla g K] g = K^{A\beta} B_{/\beta} e_A \otimes e^B = K^{C_i} D_{/i} \partial_{C_i} \otimes dx^D, \]

where
\[ K = K^{A\beta} B_{/\beta} e_A \otimes e^B = K^{C_i} D_{/i} \partial_{C_i} \otimes dx^D, \]
\[ K \in \otimes^k T(M), \quad k \geq 1. \]

\( \delta \) is called operator of the covariant divergency

\[ \delta : K \Rightarrow \delta K, \quad K \in \otimes^k T(M), \quad \delta K \in \otimes^{k-1} T(M), \quad k \geq 1. \]

**Remark.** The symbol \( \delta \) has also been introduced for the variation operator. Both operators are different from each other and can easily be distinguished. Ambiguity would occur only if the symbol \( \delta \) is used out of the context. In such a case, the definition of the symbol \( \delta \) is necessary.

The properties of the operator of the covariant divergency \( \delta \) are determined by the properties of the contravariant metric differential operator, the contraction operator and the metric tensor fields \( g \) and \( \nabla g \)

(a) The operator of the covariant divergency \( \delta \) is a linear operator

\[ \delta (\alpha \cdot K_1 + \beta \cdot K_2) = \alpha \cdot \delta K_1 + \beta \cdot \delta K_2, \]
\[ \alpha, \beta \in R \ (or \ C), \ K_1, K_2 \in \otimes^k T(M). \quad (154) \]

The proof of this property follows immediately from the definition of the covariant divergency.

(b) Action on a tensor product of tensor fields

\[ \delta (K \otimes S) = \nabla_S K + K \otimes \delta S, \quad (155) \]

where
\[ K = K^{A\beta} B_{/\beta} e_A \otimes e^B, \quad S = S^{C\beta} D_{/\beta} e_A \otimes e_C \otimes e^D, \]
\[ \nabla_S K = K^{A\beta} B_{/\beta} S^{C\beta} D_{/\beta} e_A \otimes e^B \otimes e_C \otimes e^D \quad (see \ above \ \nabla_V). \quad (156) \]

The proof of this property follows from the properties of \( \nabla g \) and from the relations

\[ \frac{1}{2} [\nabla e_\beta K \otimes S^{\beta}] g = K^{A\beta} B_{/\beta} S^{C\beta} D_{/\beta} e_A \otimes e^B \otimes e_C \otimes e^D, \quad (157) \]
\[
\frac{1}{2} [K \otimes \nabla_S]g = K \otimes \frac{1}{2} \cdot [\nabla_S]g = K \otimes \delta S .
\]

(c) Action on a contravariant vector field \(u\)
\[
\delta u = \frac{1}{2} \cdot [\nabla_S]g = u^\beta \cdot _\beta = u^i \cdot _i .
\]

(d) Action on the tensor product of two contravariant vector fields \(u\) and \(v\)
\[
\delta (u \otimes v) = \nabla_v u + \delta u \cdot v , \quad \nabla_v u = \nabla_v u .
\]

(e) Action of the product of an invariant function \(L\) and a mixed tensor field \(K\)
\[
\delta(LK) = \nabla_K L + L \cdot \delta K ,
\]
\[
\nabla_K L = L_{/\beta} : K^{A\beta} B \cdot e_A \otimes e_B , \quad \delta K = K^{A\beta} B_{/\beta} \cdot e_A \otimes e_B ,
\]
\[
L_{/\beta} = e_\beta L , \quad L_{ij} = L_{ij} ,
\]
\[
K = K^{A\beta} B \cdot e_A \otimes e_\beta \otimes e_B \in \otimes^k_1(M) .
\]

Special case: Action of the product of an invariant function \(L\) and the contravariant metric tensor \(g\)
\[
\delta (Lg) = (L_{/\beta} \cdot g^{\alpha\beta} + L \cdot g^{\alpha\beta}_{/\beta}) = (L_{ij} \cdot g^{ij} + L \cdot g^{ij}_{/ij}) .
\]

(f) Action on an anti-symmetric tensor product of two contravariant vector fields \(u\) and \(v\)
\[
\delta (u \wedge v) = \frac{1}{2} \cdot (\nabla_u u - \nabla_u v + \delta v \cdot u - \delta u \cdot v ) = \]
\[
= -\frac{1}{2} \cdot [\nabla_u v + T(u,v) + \delta u \cdot v - \delta v \cdot u] .
\]

(g) Action on a full anti-symmetric contravariant tensor field \(A\) of second rank
\[
\delta A = \frac{1}{2} \cdot (A^{\alpha\beta} - A^{\beta\alpha})_{/\beta} \cdot e_\alpha = \frac{1}{2} \cdot (A^{ij} - A^{ji})_{/ij} \cdot \partial_i ,
\]
\[
A = A^{\alpha\beta} \cdot e_\alpha \wedge e_\beta = A^{ij} \cdot \partial_i \wedge \partial_j , \quad A^{\alpha\beta} = - A^{\beta\alpha} .
\]

(h) Action on a tensor product of a contravariant vector field \(u\), multiplied with an invariant function, and a covariant vector field \(g(v)\) with the contravariant vector field \(v\)
\[
\delta (\varepsilon \cdot u \otimes g(v)) = (u\varepsilon) \cdot g(v) + \varepsilon \cdot [\delta u \cdot g(v) + (\nabla_u g)(v) + g(\nabla_u v)] = \]
\[
= [u\varepsilon + \varepsilon \cdot \delta u] \cdot g(v) + \varepsilon \cdot [(\nabla_u g)(v) + g(\nabla_u v)] , \quad \varepsilon \in C^\infty(M) , \quad \varepsilon'(x^k) = \varepsilon(x^k) , \quad u,v \in T(M) .
\]

Special case: \(v \equiv u\)
\[
\delta (\varepsilon \cdot u \otimes g(u)) = [u\varepsilon + \varepsilon \cdot \delta u] \cdot g(u) + \varepsilon \cdot [(\nabla_u g)(u) + g(\nabla_u u)] , \quad \nabla_u u = a .
\]

Special case: \(\varepsilon = 1\)
\[
\delta (u \otimes g(v)) = \delta u \cdot g(v) + (\nabla_u g)(v) + g(\nabla_u v) .
\]
5.1 Covariant divergency of a mixed tensor field of second rank

From the definition of the covariant divergency \( \delta K \) of a mixed tensor field \( K \), the explicit form of the covariant divergency of tensor fields of second rank of the type 1. or 2. follows as

\[
\delta G = \frac{1}{2} [\nabla \varphi] g = G^{\alpha \beta} \cdot e^{\alpha} = G^{i j} \cdot dx^i ,
\]

\[
\delta G = \frac{1}{2} [\nabla \varphi] g = G^{\alpha \beta} \cdot e^{\alpha} = G^{i j} \cdot dx^i .
\]

By the use of the relations (155)÷(164), (166)÷(168), and the expression [see (155)÷(158)]

\[
\nabla_v (g(u)) = \nabla_v (g(u)) = (\nabla_v g)(u) + g(\nabla_v u) ,
\]

\[
\delta (g(u) \otimes v) = \delta v \cdot g(u) + (\nabla_v g)(u) + g(\nabla_v u) ,
\]

\[
\delta ((^G S)g) = (g_{\alpha \beta} \cdot G S_{\beta \gamma})_{/ \beta} \cdot e^{\alpha} ,
\]

the covariant divergency of the representation of \( G \) by means the rest mass density \( \rho_G (\varepsilon_G = \rho_G) \)

\[
G = \rho_G \cdot u \otimes g(u) + u \otimes g(^G \pi) + ^G S \otimes g(u) + (^G S)g,
\]

can be found in the form \( (\nabla u u = a) \)

\[
\delta G = \rho_G \cdot g(a) + (u \rho_G + \rho_G \cdot \delta u + \delta ^G S) \cdot g(u) + \delta u \cdot g(^G \pi) + g(\nabla_u G^\pi) + g(\nabla_u G^\pi)
\]

\[
+ \nabla_u a + \rho_G \cdot (\nabla u g)(u) + (\nabla_u g)(^G \pi) + (\nabla_a g)(u) + \delta ((^G S)g) .
\]

\[
\overline{g}(\delta G) \text{ will have the form}
\]

\[
\overline{g}(\delta G) = \rho_G \cdot a + (u \rho_G + \rho_G \cdot \delta u + \delta ^G S) \cdot u + \delta u \cdot G^\pi + \nabla_u G^\pi + \nabla_a u + \rho_G \cdot (\nabla u g)(u) + (\nabla_u g)(^G \pi) + (\nabla_a g)(u) + \delta ((^G S)g) .
\]

In a co-ordinate basis \( \delta G \) and \( \overline{g}(\delta G) \) will have the forms

\[
G_{i j} = \rho_G \cdot a_{i j} + (\rho_G \cdot u_j + \rho_G \cdot u_j + \delta ^G S) \cdot u_{i j} + u_{i j} \cdot G^\pi + \nabla_{a_{i j}} + g_t (\nabla_{t_{i j}} + u_k + \delta ^G S) + g_t \cdot \rho_G \cdot u_{i j} + u_{i j} \cdot G^\pi + \nabla_{a_{i j}}
\]

\[
+ g_t \cdot g_t (G^\pi + ^G S) + g_t \cdot u_{i j} \cdot G^\pi + \nabla_{a_{i j}} + g_t (\nabla_{t_{i j}} + u_k + \delta ^G S) + g_t \cdot \rho_G \cdot u_{i j} + u_{i j} \cdot G^\pi + \nabla_{a_{i j}} + g_t \cdot g_t (G^\pi + ^G S) + g_t \cdot u_{i j} \cdot G^\pi + \nabla_{a_{i j}}
\]

\[
\delta G = \rho_G \cdot a_{i j} + (\rho_G \cdot u_j + \rho_G \cdot u_j + \delta ^G S) \cdot u_{i j} + u_{i j} . G^\pi + \nabla_{a_{i j}} + g_t (\nabla_{t_{i j}} + u_k + \delta ^G S) + g_t \cdot \rho_G \cdot u_{i j} + u_{i j} . G^\pi + \nabla_{a_{i j}} + g_t \cdot g_t (G^\pi + ^G S) + g_t \cdot u_{i j} \cdot G^\pi + \nabla_{a_{i j}}
\]

\[
\delta G = \rho_G \cdot a_{i j} + (\rho_G \cdot u_j + \rho_G \cdot u_j + \delta ^G S) \cdot u_{i j} + u_{i j} . G^\pi + \nabla_{a_{i j}} + g_t (\nabla_{t_{i j}} + u_k + \delta ^G S) + g_t \cdot \rho_G \cdot u_{i j} + u_{i j} . G^\pi + \nabla_{a_{i j}} + g_t \cdot g_t (G^\pi + ^G S) + g_t \cdot u_{i j} \cdot G^\pi + \nabla_{a_{i j}}
\]
The relation between $\delta G$ and $\delta \overline{G}$ follows from the relation between $G$ and $\overline{G}$

$$\overline{G} = g(G)\overline{g} ; \delta \overline{G} = \delta (g(G)\overline{g}) = \frac{1}{2} [\nabla \pi (g(G)\overline{g})] . \quad (178)$$

The covariant divergency of the Kronecker tensor field can be found in an analogous way as the covariant divergency of a tensor field of second rank of the type 1, since $Kr = g_{\beta}^{\alpha} \cdot e_{\alpha} \otimes e^{\beta} = g_{j}^{i} \cdot \partial_{i} \otimes dx^{j}$

$$\delta Kr = \frac{1}{2} [\nabla \pi Kr] g = g_{\alpha}^{\beta} \cdot e^{\alpha} = g_{i,j}^{\alpha} \cdot dx^{i} . \quad (179)$$

If we use the representation of $Kr$

$$Kr = \frac{1}{e} \cdot k \cdot u \otimes g(u) + u \otimes g(Kr_{\pi}) + Kr_{s} \otimes g(u) + (Kr_{S})g ,$$

then the covariant divergency $\delta Kr$ can be written in the form

$$\delta Kr = \frac{1}{e} \cdot k \cdot g(a) + [u(\frac{1}{e} \cdot k) + \frac{1}{e} \cdot k \cdot \delta u + \delta Kr_{s}] \cdot g(u) +$$

$$+ \delta u \cdot (Kr_{\pi}) + g(\nabla_{u} Kr_{\pi}) + g(\nabla_{Kr_{s}}u) +$$

$$+ \frac{1}{e} \cdot k \cdot (\nabla_{u} g)(u) + (\nabla_{u} g)(Kr_{\pi}) + (\nabla_{Kr_{s}}g)(u) + \delta ((Kr_{S})g) , \quad (180)$$

or in the form

$$\overline{g}(\delta Kr) = \frac{1}{e} \cdot k \cdot a + [u(\frac{1}{e} \cdot k) + \frac{1}{e} \cdot k \cdot \delta u + \delta Kr_{s}] \cdot u +$$

$$+ \delta u \cdot Kr_{\pi} + \nabla_{u} Kr_{\pi} + \nabla_{Kr_{s}}u +$$

$$+ \frac{1}{e} \cdot k \cdot \overline{g}(\nabla_{u} g)(u) + \overline{g}(\nabla_{u} g)(Kr_{\pi}) + \overline{g}(\nabla_{Kr_{s}}g)(u) + \overline{g}(\delta ((Kr_{S})g)) . \quad (181)$$

In a co-ordinate basis $\delta Kr$ and $\overline{g}(\delta Kr)$ will have the forms

$$g^{i,j} = \frac{1}{e} \cdot k \cdot a_{i} + [(\frac{1}{e} \cdot k)_{j} \cdot u^{i} + \frac{1}{e} \cdot k \cdot u^{i} \cdot j + Kr_{s}^{i,j}] \cdot u_{i} +$$

$$+ u^{i} \cdot j, Kr_{\pi} + g_{ij}^{\alpha} \cdot (Kr_{\pi}^{i,j} \cdot k \cdot u_{k} + u_{k} \cdot j, Kr_{s}^{i}) +$$

$$+ g_{ij,k} \cdot (\frac{1}{e} \cdot k \cdot u^{i} \cdot j + Kr_{\pi}^{i,j} \cdot u_{k} + u_{k} \cdot j, Kr_{s}^{i}) + (g_{\pi}, Kr_{s}^{i,j})_{j} , \quad (182)$$

$$g^{ik}, g_{k}^{i,j} = \frac{1}{e} \cdot k \cdot a^{i} + [(\frac{1}{e} \cdot k)_{i} \cdot u^{j} + \frac{1}{e} \cdot k \cdot u^{j} \cdot i + Kr_{s}^{j,i} \cdot u^{i} +$$

$$+ u^{j} \cdot i, Kr_{\pi}^{i,j} + Kr_{\pi}^{i,j} \cdot u_{k} + u_{k} \cdot j, Kr_{s}^{i} +$$

$$+ g^{ik} \cdot g_{i,k} \cdot (\frac{1}{e} \cdot k \cdot u^{i} \cdot j + Kr_{\pi}^{i,j} \cdot u_{k} + u_{k} \cdot j, Kr_{s}^{i}) + g^{ik} \cdot (g_{ik}, Kr_{s}^{i,j})_{j} . \quad (183)$$

6 Covariant divergency of the energy-momentum tensors and the rest mass density

The covariant divergency of the energy-momentum tensors can be represented by the use of the projective metrics $h^{u}$, $h_{u}$ of the contravariant vector field $u$ and the rest mass density for the corresponding energy-momentum tensor. In this case the representation of the energy-momentum tensor is in the form

$$G = (\rho_{G} + \frac{1}{e} \cdot L \cdot k) \cdot u \otimes g(u) - L \cdot Kr + u \otimes g^{(s)} + k \cdot s \otimes g(u) + (k \cdot S)g ,$$

26
\[
\delta G = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot g(a) + \\
+ [u(\rho_G + \frac{1}{e} \cdot L \cdot k) + (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta G^\alpha \cdot g(u) - \\
- KrL - L \cdot \delta Kr + u \cdot g(G^\pi) + g(\nabla_u G^\pi) + g(\nabla_\pi g(u)] + \\
+ (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_\pi g)(G^\pi) + (\nabla_\pi g)(u) + \\
+ \delta(G^\pi)g) ,
\]

By the use of the relations

\[
u(\rho_G + \frac{1}{e} \cdot L \cdot k) = u\rho_G + L \cdot u(\frac{1}{e} \cdot k) + \frac{1}{e} \cdot k \cdot (uL) = \\
\]

\[
= [\rho_G/\alpha + L \cdot (\frac{1}{e} \cdot k)/\alpha + \frac{1}{e} \cdot L \cdot k \cdot L/\alpha \cdot u^\alpha = \\
= [\rho_G, L \cdot (\frac{1}{e} \cdot k), J + \frac{1}{e} \cdot L \cdot K, J] \cdot u^J ,
\]

\[
K_r L = L/\alpha \cdot e^\alpha = L_i \cdot dx^i = \nabla_{Kr} L ,
\]

\[
\delta(L \cdot Kr) = KrL + L \cdot \delta Kr ,
\]

\[
\delta(G^\pi)g = (g_{\alpha\beta} \cdot G^{\beta\gamma}) \cdot g \cdot e^\gamma [\text{see (172)}] ,
\]

\[\delta G\] and \[\bar{\pi}(\delta G)\] can be found in the forms

\[
\delta G = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot g(a) + \\
+ [u(\rho_G + \frac{1}{e} \cdot L \cdot k) + (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta G^\alpha \cdot g(u) - \\
- KrL - L \cdot \delta Kr + u \cdot g(G^\pi) + g(\nabla_u G^\pi) + g(\nabla_\pi g(u)] + \\
+ (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot (\nabla_u g)(u) + (\nabla_\pi g)(G^\pi) + (\nabla_\pi g)(u) + \\
+ \delta(G^\pi)g) ,
\]

In a co-ordinate basis \[\delta G\] and \[\bar{\pi}(\delta G)\] will have the forms

\[
G_{ij} \cdot \delta G = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot a_i + \\
+ [(\rho_G + \frac{1}{e} \cdot L \cdot k), j \cdot u^j + (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u^j, j + G^{ij} \cdot a_j - \\
- L_{ij} - g_{ij}^{l} \cdot u^l + G_{ij} \cdot a_j + g_{ij, k} \cdot (G_{ij} \cdot u^k + u^j \cdot G_{ik}) + \\
+ g_{ij, k, l} \cdot [(\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u^j, k + G_{ij} \cdot u^k + u^j \cdot G_{ik}] + \\
+ (g_{ik} \cdot G^{ij})]_j ,
\]

\[
\bar{\pi}(\delta G) = (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot a_i + \\
+ [(\rho_G + \frac{1}{e} \cdot L \cdot k), j \cdot u^j + (\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u^j, j + G^{ij} \cdot a_j - \\
- L_{ij} \cdot g^{ij} + L \cdot g^{ij} \cdot k_{ij} + u^j, l + G^{ij} \cdot a_j + \\
+ g^{ij} \cdot g_{ij, k} \cdot [(\rho_G + \frac{1}{e} \cdot L \cdot k) \cdot u^j, k + G_{ij} \cdot u^k + u^j \cdot G_{ik}] + \\
+ g^{ij} \cdot (g_{ik} \cdot G^{ij})]_j .
\]
On the grounds of the representations of $\delta G$ and $\mathcal{G}(\delta G)$ the representation of the different energy-momentum tensors $\theta$, $\mathcal{T}$ and $Q$ can be found.

The covariant divergency $\delta \theta$ of the generalized canonical energy-momentum tensor (GC-EMT) $\theta$ can be written in the form

$$
\delta \theta = (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot g(u) + 
\begin{align*}
+ [u(\rho_\theta + \frac{1}{c} \cdot L \cdot k) + (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta^\mathcal{G} g] \cdot g(u) - \\
- K_r L - L \cdot \delta K_r + \delta u \cdot g(Tg) + g(\nabla u Tg) + g(\nabla u g(u)) + \\
+ (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla u)g(u) + (\nabla u g(Tg) + (\nabla u g)(u) + \\
+ \delta (Tg)]
\end{align*}
$$

or in the form

$$
\mathcal{G}(\delta \theta) = (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot a + 
\begin{align*}
+ [u(\rho_\theta + \frac{1}{c} \cdot L \cdot k) + (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta^\mathcal{G} g] \cdot u - \\
- \mathcal{G}(K_r L) - L \cdot \mathcal{G}(\delta K_r) + \delta u \cdot u + \nabla u \cdot u + \nabla u u + \\
+ (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot \nabla u g(u) + \mathcal{G}(\nabla u g(Tg) + \mathcal{G}(\nabla u g))(u) + \\
+ \mathcal{G}(\delta (Tg))
\end{align*}
$$

In a co-ordinate basis $\delta \theta$ and $\mathcal{G}(\delta \theta)$ will have the forms

$$
\mathcal{G}_{i,j} \cdot \delta \theta = (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot a_{i,j} + 
\begin{align*}
+ [(\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot u_i + (\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot u_{i,j} + \delta^\mathcal{G} g] \cdot u_i - \\
- L_i \cdot g_{i,j} + u_{i,j} \cdot \nabla g + \mathcal{G}(\nabla g) + \mathcal{G}(\nabla G) + \\
+ g_{i,j} \cdot [(\rho_\theta + \frac{1}{c} \cdot L \cdot k) \cdot u_i + u_{i,j} + \delta^\mathcal{G} g] + \\
+ (\mathcal{G}_i \cdot \delta^\mathcal{G} g)_{i,j}
\end{align*}
$$

The covariant divergency $\delta T$ of the symmetric energy-momentum tensor of Belinfante (S-EMT-B) $\mathcal{T}$, represented in the form

$$
\mathcal{T} = (\rho_T + \frac{1}{c} \cdot L \cdot k) \cdot u \cdot g(u) - L \cdot K_r u + u \cdot g(Tg) + Tg \otimes g(u) + (Tg) g
$$

can be found in the form

$$
\delta \mathcal{T} = (\rho_T + \frac{1}{c} \cdot L \cdot k) . g(u) + 
\begin{align*}
+ [u(\rho_T + \frac{1}{c} \cdot L \cdot k) + (\rho_T + \frac{1}{c} \cdot L \cdot k) \cdot \delta u + \delta^T g] \cdot g(u) - \\
- K_r L - L \cdot \delta K_r + \delta u \cdot g(Tg) + g(\nabla u Tg) + g(\nabla u g(u)) + \\
+ (\rho_T + \frac{1}{c} \cdot L \cdot k) \cdot (\nabla u)g(u) + (\nabla u g)(Tg) + (\nabla u g)(u) + \\
+ \delta (Tg)]
\end{align*}
$$
or in the form

\[
\mathcal{F}(\delta_s T) = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot a + \\
+ [u(\rho_T + \frac{1}{e} \cdot L \cdot k) + (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot \delta u + \delta^T \pi] \cdot u - \\
- \mathcal{F}(KrL) - L \cdot \mathcal{F}(\delta^rKr) + \delta u \cdot T \pi + \nabla_u T \pi + \nabla_T \pi u + \\
+ (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot \mathcal{F}(\nabla_u g)(u) + \mathcal{F}(\nabla_u g)(\delta^T \pi) + \mathcal{F}(\nabla_T \pi g)(u) + \\
+ \mathcal{F}(\delta((T S) g)) .
\]

(200)

In a co-ordinate basis \( \delta_s T \) and \( \mathcal{F}(\delta_s T) \) will have the forms

\[
\mathcal{F}^{\pi} \cdot T_{i}^{j} = (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot a_i + \\
[\rho_T + \frac{1}{e} \cdot L \cdot k] \cdot u^j + (\rho_T + \frac{1}{e} \cdot L \cdot k) \cdot u^{j} \cdot u^{j} + T \pi_{i,j} \cdot u_i - \\
-u_{i} \cdot L \cdot g_{i,j} + u^{j} \cdot T \pi_{i,j} + g_{i,j} \cdot (T \pi_{i,k} \cdot u^k + u^{j} \cdot T \pi_{i,k}) + \\
+ g_{i,j,k} \cdot ([\rho_T + \frac{1}{e} \cdot L \cdot k] \cdot u^j \cdot u^k + T \pi_{i,j} \cdot u^k + u^{j} \cdot T \pi_{i,k}) + \\
+ (g^{T} \cdot T \pi S_{jk})_{i,j} .
\]

(201)

The covariant divergancy \( \delta Q \) of the variational energy-momentum tensor of Euler-Lagrange (V-EMT-EL) \( Q \), represented in the form

\[
Q = - \rho Q \cdot u \otimes g(u) - u \otimes g(\pi) - Q_s \otimes g(u) - (Q S) g ,
\]

follows in the form

\[
\delta Q = - \rho Q \cdot g(u) - (u \rho_Q \cdot g(\pi) + \rho_Q \cdot \delta u + \delta^Q s) \cdot g(u) - \\
\delta u \cdot g(\pi) - g(\nabla_u \pi) - g(\nabla_T \pi u) - \\
- \rho_Q \cdot (\nabla_u g)(u) - (\nabla_u g)(\pi) - (\nabla_T g)(u) - \delta((Q S) g) ,
\]

(203)

or in the form

\[
\mathcal{F}(\delta Q) = - \rho Q \cdot a - (u \rho_Q + \rho_Q \cdot \delta u + \delta^Q s) \cdot u - \\
- \delta u \cdot Q \pi - \nabla_u Q \pi - \nabla_T \pi u - \\
- \rho_Q \cdot \mathcal{F}(\nabla_u g)(u) - \mathcal{F}(\nabla_u g)(\pi) - \mathcal{F}(\nabla_T g)(u) - \mathcal{F}(\delta((Q S) g)) .
\]

(204)

In a co-ordinate basis \( \delta Q \) and \( \mathcal{F}(\delta Q) \) will have the forms

\[
\mathcal{F}^{\pi} \cdot Q_{i}^{j} = - \rho Q \cdot a_i - (\rho_Q \cdot u^j + \rho_Q \cdot u^{j} \cdot u^{j} + Q \pi_{i,j}) \cdot u_i - \\
- u_{i}^{j} \cdot Q \pi_{i,j} - g_{i,j} \cdot (Q \pi_{i,k} \cdot u^k + u^{j} \cdot Q \pi_{i,k}) - \\
+ g_{i,j,k} \cdot (\rho_Q \cdot u^j \cdot u^k + Q \pi_{i,j} \cdot u^k + u^{j} \cdot Q \pi_{i,k} - (g^{T} \cdot Q S_{jk})_{i,j} ,
\]

(205)

\[
\mathcal{F}^{T} \cdot Q_{k}^{j} = - \rho Q \cdot a^j - (\rho_Q \cdot u^j + \rho_Q \cdot u^{j} \cdot u^{j} + Q \pi_{j} \cdot Q \pi_{j}) \cdot u^j - \\
- u_{j}^{j} \cdot Q \pi_{j} - Q \pi_{j} \cdot u^j - u^{j} \cdot Q \pi_{j} - \\
- (Q \pi_{j} \cdot Q \pi_{j}) \cdot u^j - Q \pi_{j} - (g^{T} \cdot Q S_{jk})_{i,j} ,
\]

(206)

29
7 Covariant divergency of the energy-momentum tensors and the momentum density

In the previous chapter the energy-momentum tensors are represented by the use of the projective metrics $h^u$, $h_u$ and the momentum density $p$. For this type of representation of a tensor field $G$ of the type 1., which has the form

$$G = u \otimes g(p_G) + \ G s \otimes g(u) + (G S)g,$$

the covariant divergency $\delta G$ can be calculated in the form

$$\begin{align*}
\delta G &= g(\nabla_u p_G) + \delta u \cdot g(p_G) + \delta G s \cdot g(u) + g(\nabla_s u) + (\nabla_u g)(p_G) + \\
&\quad + (\nabla_s g)(u) + \delta ((G S)g),
\end{align*}
$$

(207)

or in the form

$$\begin{align*}
\overline{\nabla}(\delta G) &= \nabla_u p_G + \delta u \cdot p_G + \delta G s \cdot u + \nabla_s u + \overline{\nabla}(\nabla_u g)(p_G) + \\
&\quad + \overline{\nabla}(\nabla_s g)(u) + \overline{\nabla}(G S)g).
\end{align*}
$$

(208)

In a co-ordinate basis $G_{ij} ; = \delta_G (\delta_G G_{ij}) \cdot \delta u \cdot g(p_G) + \delta G s \cdot g(u) + g(\nabla_s u) + (\nabla_u g)(p_G) + \\
+ \delta G s \cdot (\nabla_s g)(u) + \delta ((G S)g),
$$

(209)

$$
\begin{align*}
G_{ij} ; & = \frac{\partial}{\partial x^i} \cdot (p_G^{ij} \cdot u^k + u^j \cdot G^{ik}) + \\
&\quad + g_{ij} \cdot (p_G^{ij} \cdot u^k + u^j \cdot G^{ik}) + \frac{\partial}{\partial x^j} \cdot (g_{ij} \cdot G^{ij} k),
\end{align*}
$$

(210)

On the other side, the covariant divergency of $(G)\overline{\nabla}$ can be found. $(G)\overline{\nabla}$ is represented in the form

$$(G)\overline{\nabla} = u \otimes p_G + \ G s \otimes u + \ G S = G_{\alpha \beta} \cdot e_\alpha \otimes e_\beta = G^{ij} \cdot \partial_i \otimes \partial_j.$$

(211)

$$\delta((G)\overline{\nabla}) \text{ will have the form}
$$

$$\begin{align*}
\delta((G)\overline{\nabla}) &= \nabla_u G s + \nabla_{p_G} u + \delta p_G \cdot u + \delta u \cdot G s + \delta G S,
\end{align*}
$$

(212)

$$
\begin{align*}
G_{\alpha \beta \gamma} / \beta &= G s_{\alpha \beta} \cdot u^\gamma + u^\gamma / \beta \cdot G s_{\alpha \beta} + p_G / \beta \cdot u^\alpha \cdot u^\gamma / \beta \cdot G s^\gamma + G S_{\alpha \beta / \gamma} / \beta, \\
G^{ij} ; & = G s_{ij} \cdot u^i + u^i \cdot G s_{ij} + p_G_{ij} \cdot u^i + u^i \cdot G s_{ij} + G S_{ij} ,
\end{align*}
$$

(213)

(214)

$$
\begin{align*}
G_{ij} = G s_{ij} \cdot u^i + u^i \cdot G s_{ij} + p_G_{ij} \cdot u^i + u^i \cdot G s_{ij} + G S_{ij} ,
\end{align*}
$$

(215)

The covariant divergency $\delta (G)\overline{\nabla}$ of the generalized canonical energy-momentum tensor $\theta$ follows from (207) and (208) in the form

$$\begin{align*}
\delta \theta &= g(\nabla_u p_\theta) + \delta u \cdot g(p_\theta) + \delta s \cdot g(u) + g(\nabla_s u) + (\nabla_u g)(p_\theta) + \\
&\quad + (\nabla_s g)(u) + \delta ((G S)g),
\end{align*}
$$

(215)
or in the form
\[
\mathfrak{g}(\delta \theta) = \nabla_u p_\theta + \delta u \cdot p_\theta + \delta^s s \cdot u + \nabla_s u + \mathfrak{g}(\nabla_u g)(p_\theta) + \nabla(\nabla_s g)(u) + \mathfrak{g}(\delta((^s Term))) .
\] (216)

In a co-ordinate basis \( \delta \theta \) and \( \mathfrak{g}(\delta \theta) \) will have the forms
\[
\mathfrak{g}_i ^j T_j = g_{ijkl} \cdot (p_{\alpha k} \cdot u^k + u^k \cdot p_{\alpha k} + \theta s^k \cdot u^j + u^j \cdot \theta s^k) + g_{ijkl} \cdot (p_{\alpha k} \cdot u^k + u^k \cdot \theta s^k) + (g_{ijkl} \cdot \theta S^j k)_{ij} ,
\] (217)
\[
g^k T_j \cdot \mathfrak{g}_k ^j T_j = p_{\alpha k} \cdot u^j + u^j \cdot p_{\alpha k} + \theta s^j \cdot u^i + u^i \cdot \theta s^j + g^k \cdot (g_{ijkl} \cdot \theta S^j k)_{ij} .
\] (218)

The covariant divergency \( \delta_s T \) of the symmetric energy-momentum tensor of Belinfante \( s \) can be written in the form
\[
\delta_s T = g(\nabla_u p_T) + \delta u \cdot g(p_T) + \delta^T S \cdot g(u) + g(\nabla_x u) + (\nabla_u g)(p_T) + (\nabla_x g)(u) + \mathfrak{g}(\delta((^T S))) ,
\] (219)
or in the form
\[
\mathfrak{g}(\delta_s T) = \nabla_u p_T + \delta u \cdot p_T + \delta^T S \cdot u + \nabla_x u + \mathfrak{g}(\nabla_u g)(p_T) + \mathfrak{g}(\nabla_x g)(u) + \mathfrak{g}(\delta((^T S))) .
\] (220)

In a co-ordinate basis \( \delta_s T \) and \( \mathfrak{g}(\delta_s T) \) will have the forms
\[
s_{T} i ^j T_j = g_{ijkl} \cdot (p_{\alpha k} \cdot u^k + u^k \cdot p_{\alpha k} + T s^k \cdot u^j + u^j \cdot T s^k) + g_{ijkl} \cdot (p_{\alpha k} \cdot u^k + u^k \cdot T s^k) + (g_{ijkl} \cdot \theta S^j k)_{ij} ,
\] (221)
\[
g^k T_j \cdot s_{T} k ^j T_j = p_{\alpha k} \cdot u^j + u^j \cdot p_{\alpha k} + T s^j \cdot u^i + u^i \cdot T s^j + g^k \cdot (g_{ijkl} \cdot \theta S^j k)_{ij} .
\] (222)

The covariant divergency \( \delta Q \) of the variational energy-momentum tensor of Euler-Lagrange \( Q \) can be found in the forms
\[
\delta Q = - g(\nabla_u p_Q) - \delta u \cdot g(p_Q) - \delta^Q S \cdot g(u) - g(\nabla_s u) - (\nabla_u g)(p_Q) - (\nabla_s g)(u) - \delta((^Q S)) ,
\] (223)
\[
\mathfrak{g}(\delta Q) = - \nabla_u p_Q - \delta u \cdot p_Q - \delta^Q S \cdot u - \nabla_s u - \mathfrak{g}(\nabla_u g)(p_Q) - \mathfrak{g}(\nabla_s g)(u) - \mathfrak{g}(\delta((^Q S))) .
\] (224)

In a co-ordinate basis \( \delta Q \) and \( \mathfrak{g}(\delta Q) \) will have the forms
\[
\mathfrak{g}_i ^j Q_j = - g_{ijkl} \cdot (p_{\alpha k} \cdot u^k + u^k \cdot p_{\alpha k} + Q s^k \cdot u^j + u^j \cdot Q s^k) - g_{ijkl} \cdot (p_{\alpha k} \cdot u^k + u^k \cdot Q s^k) + (g_{ijkl} \cdot \theta S^j k)_{ij} ,
\] (225)
\[
g^k T_j \cdot \mathfrak{g}_k ^j Q_j = - p_{\alpha k} \cdot u^j - u^j \cdot p_{\alpha k} - Q s^j \cdot u^i - u^i \cdot Q s^j - g^k \cdot (g_{ijkl} \cdot \theta S^j k)_{ij} .
\] (226)
8 Covariant divergency of the energy-momentum tensors and the energy flux density

In the previous chapter the notion of energy flux density $e_G$ has been introduced. By means of it a mixed tensor field $G$ of second rank and of type 1. can be represented in the form

$$ G = \frac{1}{e} \cdot e_G \otimes g(u) + u \otimes g^{(G)} + (G^S)g. $$

The covariant divergency $\delta G$ of a given through $e_G$ tensor field $G$ can be found by the use of the relation

$$ \delta(\frac{1}{e} \cdot e_G \otimes g(u)) = [e_G(\frac{1}{e}) + \frac{1}{e} \cdot \delta e_G] \cdot g(u) + \frac{1}{e} \cdot [(\nabla e_G)g(u) + g(\nabla e_G)u] \quad (227) $$

and the property (155) of $\delta$ in the form

$$ \delta G = g(\nabla u \cdot g^{(G)}u) + \frac{1}{e} \cdot g(\nabla e_G)u + [e_G(\frac{1}{e}) + \frac{1}{e} \cdot \delta e_G] \cdot g(u) + \delta u \cdot g^{(G)} + + \frac{1}{e} \cdot (\nabla e_G)g)(u) + (\nabla u)g(\nabla ^{(G)}u) + \delta((G^S)g), \quad (228) $$

or in the form

$$ \overline{g}(\delta G) = \nabla u e_G + \frac{1}{e} \cdot \nabla e_G u + [u \cdot e_G(\frac{1}{e}) + \frac{1}{e} \cdot \delta e_G] \cdot u + \delta u \cdot G + + \frac{1}{e} \cdot \overline{g}(\nabla e_G)g(u) + \overline{g}(\nabla u)g(\nabla ^{(G)}u) + \overline{g}(\delta((G^S)g)) \quad (229) $$

On the other side, the covariant divergency of $(G)\overline{g}$ can be found in the form

$$ \delta((G)\overline{g}) = \frac{1}{e} \cdot \nabla u e_G + \nabla e_G u + [u \cdot e_G(\frac{1}{e}) + \frac{1}{e} \cdot \delta u] \cdot e_G + \delta G \cdot u + \delta((G^S) \overline{g}). \quad (230) $$

In a co-ordinate basis $\delta G$ and $\overline{g}(\delta G)$ will have the forms

$$ G_i^{\cdot j} = \frac{1}{e} \cdot G^i_{\cdot j} = g^{(G \cdot i}_{\cdot j} \cdot u^k + \frac{1}{e} \cdot u^i_{\cdot j} \cdot e_G^k + \frac{1}{e} \cdot \delta e_G^k \cdot u^i_{\cdot j} + u^k_{\cdot j} \cdot G^i_{\cdot j} + + g_i^{\cdot j}(G \cdot S_{\cdot j}, \delta u^i_{\cdot j} \cdot e_G^k + G^i_{\cdot j} \cdot u^k_{\cdot j} + \delta u \cdot G, \delta((G^S) \overline{g}))) \quad (231) $$

and

$$ g^{\cdot i}_{\cdot j} = g^{(G \cdot i}_{\cdot j} \cdot u^i + \frac{1}{e} \cdot u^i_{\cdot j} \cdot e_G^l + \frac{1}{e} \cdot (G_{\cdot i} \cdot \nabla u^i_{\cdot j} + \frac{1}{e} \cdot e_G^k \cdot u^i_{\cdot j} + u^i_{\cdot j} \cdot G^i_{\cdot j} + + g^{\cdot i}_{\cdot j}(S_{\cdot j}) \cdot u^k_{\cdot j} + (g^i_{\cdot j} \cdot G \cdot S_{\cdot j})_{\cdot j}, \quad (232) $$

The different possibilities for a representation of the covariant divergency of the energy-momentum tensors are related to the different possibilities for a representation of the first covariant Noether identity for a given Lagrangian system with its corresponding energy-momentum tensors.

The covariant divergency $\delta \theta$ of the generalized canonical energy-momentum tensor $\theta$ can be written by the use of the energy flux density $e_\theta$ in the form

$$ \delta \theta = g(\nabla \theta) + \frac{1}{e} \cdot g(\nabla e_G)u + [e_\theta(\frac{1}{e}) + \frac{1}{e} \cdot \delta e_\theta] \cdot g(u) + \delta u \cdot g^{(\theta)} + + \frac{1}{e} \cdot (\nabla e_G)g(u) + (\nabla u)g(\nabla ^{(\theta)}u) + \delta((\theta^S)g), \quad (233) $$

32
or in the form

$$\mathbf{\nabla}(\delta \theta) = \nabla_u \delta \pi + \frac{1}{\epsilon} \cdot \nabla_{e \cdot u} + \left[ \epsilon \delta \phi \left( \frac{1}{\epsilon} \right) + \frac{1}{\epsilon} \cdot \delta \epsilon \phi \right] \cdot u + \delta u \cdot \theta \pi +$$

$$\frac{1}{\epsilon} \cdot \mathbf{\nabla}(\nabla_{e \cdot g}(u) + \mathbf{\nabla}(\nabla_{u} g)^{\delta \pi} + \mathbf{\nabla}(\delta(S) g)) .$$

(234)

In a co-ordinate basis $\delta \pi$ and $\mathbf{\nabla}(\delta \theta)$ will have the forms

$$\mathbf{\nabla}_i \iota \cdot j \delta = g_{ij} \cdot \left\{ \delta \theta \pi \cdot j \cdot k \cdot u^k + \frac{1}{\epsilon} \cdot u^i \cdot k \cdot c_\theta \cdot \delta \epsilon \phi \left( \frac{1}{\epsilon} \right) + \left[ \frac{1}{\epsilon} \right] \cdot c_\phi \cdot \delta \epsilon \phi \right\}$$

$$+ g_{ij} \cdot \left[ \frac{1}{\epsilon} \cdot u^i \cdot k \cdot \theta \pi \cdot j \cdot k \cdot u^k \right] + (g_{i \pi} \cdot \delta S) \cdot ) \cdot j \cdot .$$

(235)

$$g^{ik} \cdot g_{k \cdot i} \cdot j \cdot \pi = \frac{\theta \pi \cdot i \cdot j \cdot k \cdot u^k + \frac{1}{\epsilon} \cdot u^i \cdot j \cdot k \cdot c_\theta \cdot \delta \epsilon \phi \left( \frac{1}{\epsilon} \right) + \left[ \frac{1}{\epsilon} \right] \cdot c_\phi \cdot \delta \epsilon \phi \right\}$$

$$+ g_{ij} \cdot \left[ \frac{1}{\epsilon} \cdot u^i \cdot k \cdot \theta \pi \cdot i \cdot j \cdot k \cdot u^k \right] + g^{ij} \cdot \left( g_{i \pi} \cdot \delta S \right) \cdot ) \cdot j \cdot .$$

(236)

The covariant divergency $\delta \pi T$ of the symmetric energy-momentum tensor of Belinfante $\pi T$ is found by the use of $e \pi T$ in the form

$$\delta \pi T = g(\nabla_u \cdot T \pi) + \frac{1}{\epsilon} \cdot g(\nabla_{e \cdot T \pi} u) + \left[ e_T \left( \frac{1}{\epsilon} \right) + \frac{1}{\epsilon} \cdot \delta e_T \right] \cdot g(u) + \delta u \cdot g(T \pi) =$$

$$+ \frac{1}{\epsilon} \cdot g(\nabla_{u \cdot g}(u) + \nabla_{u} g)^{\delta T \pi} + \mathbf{\nabla}(\delta(S) g)) .$$

(237)

or in the form

$$\mathbf{\nabla}(\delta \pi T) = \nabla_u \cdot T \pi + \frac{1}{\epsilon} \cdot \nabla_{e \cdot T \pi} u + \left[ e_T \left( \frac{1}{\epsilon} \right) + \frac{1}{\epsilon} \cdot \delta e_T \right] \cdot u + \delta u \cdot T \pi +$$

$$+ \frac{1}{\epsilon} \cdot \mathbf{\nabla}(\nabla_{e \cdot g}(u) + \nabla_{u} g)^{\delta (T \pi)} + \mathbf{\nabla}(\delta(S) g)) .$$

(238)

In a co-ordinate basis $\pi T$ and $\mathbf{\nabla}(\delta \pi T)$ will have the forms

$$s_{T1} \iota \cdot j = g_{ij} \cdot \left( \cdot T \pi \cdot j \cdot k \cdot u^k + \frac{1}{\epsilon} \cdot u^i \cdot k \cdot e_\phi \cdot \delta \epsilon \phi \right\}$$

$$+ g_{ij} \cdot \left[ \frac{1}{\epsilon} \cdot u^i \cdot k \cdot T \pi \cdot j \cdot k \cdot u^k \right] + (g_{i \pi} \cdot T \cdot S \cdot j) \cdot ) \cdot j \cdot .$$

(239)

$$g^{ik} \cdot s_{T k \cdot i} \cdot j = \cdot T \pi \cdot j \cdot i \cdot j \cdot k \cdot u^k + \frac{1}{\epsilon} \cdot u^i \cdot j \cdot k \cdot e_\phi \cdot \delta \epsilon \phi \right\}$$

$$+ g_{ij} \cdot \left[ \frac{1}{\epsilon} \cdot u^i \cdot j \cdot k \cdot T \pi \cdot j \cdot k \cdot u^k \right] + g^{ij} \cdot \left( g_{i \pi} \cdot T \cdot S \cdot j \right) \cdot ) \cdot j \cdot .$$

(240)

The covariant divergency $\delta Q$ of the variational energy-momentum tensor of Euler-Lagrange $Q$ can be found by means of $e Q$ in the forms

$$\delta Q = - \left( g(\nabla_u \cdot Q \pi) - \frac{1}{\epsilon} \cdot g(\nabla_{e \cdot Q \pi} u) - \left[ e_Q \left( \frac{1}{\epsilon} \right) + \frac{1}{\epsilon} \cdot \delta e_Q \right] \cdot g(u) - \delta u \cdot g(Q \pi) -$$

$$- \frac{1}{\epsilon} \cdot \mathbf{\nabla}(\nabla_{e \cdot Q \pi} g) \cdot ) \cdot ) \cdot ) .$$

(241)

$$\mathbf{\nabla}(\delta Q) = - \left( \nabla_u \cdot Q \pi - \frac{1}{\epsilon} \cdot \nabla_{e \cdot Q \pi} u - \left[ e_Q \left( \frac{1}{\epsilon} \right) + \frac{1}{\epsilon} \cdot \delta e_Q \right] \cdot u - \delta u \cdot Q \pi -$$

$$- \frac{1}{\epsilon} \cdot \mathbf{\nabla}(\nabla_{e \cdot Q \pi} g) \cdot ) \cdot ) \cdot ) .$$

(242)

In a co-ordinate basis $\delta Q$ and $\mathbf{\nabla}(\delta Q)$ will have the forms

$$Q_{\iota \cdot j} = - g_{ij} \cdot \left( \cdot Q \pi \cdot j \cdot k \cdot u^k + \frac{1}{\epsilon} \cdot u^i \cdot k \cdot e_\phi \cdot \delta \epsilon \phi \right\}$$

$$+ g_{ij} \cdot \left[ \frac{1}{\epsilon} \cdot u^i \cdot k \cdot Q \pi \cdot j \cdot k \cdot u^k \right] + (g_{i \pi} \cdot Q \cdot S \cdot j) \cdot ) \cdot j \cdot .$$

(243)
\[ g^{jk} \cdot g_{ik} = -Q\pi^i \cdot u^j - \frac{1}{e_j} \cdot u^i \cdot e_{\|j} - \left[ \left( \frac{1}{e_j} \right) \cdot e_{\|j} + \frac{1}{e_j} \cdot e_{\|j} \right] \cdot u^i \cdot u^j \cdot Q\pi^i - g^{jk} \cdot g_{ij} \cdot \left[ \frac{1}{e_j} \cdot u^j \cdot e_{\|j} + Q\pi^j \cdot u^k \right] - g^{jk} \cdot (g_{ik} \cdot Q\pi^j)_{\|j}. \]  

(244)

By means of the covariant divergency of the energy-momentum tensors the covariant Noether identities can be represented in index-free forms.

9 Covariant Noether’s identities and relations between their structures

The covariant Noether identities

\[ T_\alpha + \bar{T}_{\alpha} = 0, \quad T_i + \bar{T}_i = 0, \]
\[ \bar{T}_\alpha - sT_\alpha = 0, \quad \bar{T}_i - sT_i = 0, \]

for the mixed tensor fields of second rank of the type 1. \(\theta, sT\) and \(Q\) can be written in index-free form by the use of the covariant divergency as

\[ F + \delta \theta = 0, \quad \theta - sT = Q, \]
\[ \bar{\theta}(F) + \bar{\theta}(\delta \theta) = 0, \quad (\theta - sT)\bar{\theta} = (Q)\bar{\theta}. \]  

(245)

where

\[ F = vF + gF, \]
\[ vF = vF + gW, \quad gF = gFW + gW, \quad aF = vF + gW, \]
\[ W = vF + W, \quad F = vF + \frac{\delta gL}{\delta g_{\beta\gamma}} \cdot g_{\beta\gamma} \cdot e^\alpha, \quad aW = vF + \frac{\delta gL}{\delta g_{\beta\gamma}} \cdot g_{\beta\gamma} \cdot e^\alpha, \]
\[ gW = gW + gW, \quad gW = gW + gW, \quad gW = gW + gW. \]  

(246) (247) (248)

From the second Noether identity \((\theta - sT = Q)\) the relation between the covariant divergencies of the energy-momentum tensors \(\theta, sT\) and \(Q\) follows

\[ \delta \theta \equiv \delta sT + \delta Q, \delta sT \equiv \delta \theta - \delta Q. \]

Definition 4 Local covariant conserved quantity \(G\) of the type of an energy-momentum tensor of the type 1. Mixed tensor field \(G\) of the type 1. with vanishing covariant divergence, i.e. \(\delta G = 0, G_{\alpha}^{\beta} = 0, G_{ij} = 0, G_{ij} = 0.\)

If a given energy-momentum tensor has to fulfil conditions for a local covariant conserved quantity, then relations follow from the covariant Noether identities (CNIs) between the covariant divergencies of the other energy-momentum tensors and the covariant vector field \(F\)

| No. | Condition for \(\delta G\) | Condition for \(F\) | Corollaries from CNIs |
|-----|--------------------------|---------------------|----------------------|
| 1.  | \(\delta \theta = 0\)    | \(F = 0\)           | \(\delta sT = - \delta Q\) |
| 2.  | \(\delta sT = 0\)        | \(F \neq 0\)        | \(\delta \theta = \delta Q = - F\) |
|     |                          | \(F = 0\)           | \(\delta \theta = \delta Q\) |
| 3.  | \(\delta Q = 0\)        | \(F \neq 0\)        | \(\delta \theta = \delta sT = - F\) |
|     |                          | \(F = 0\)           | \(\delta \theta = \delta sT = 0\) |

34
Special case:

\[
\frac{\delta_v L}{\delta V^A_B} = 0, \quad \frac{\delta_g L}{\delta g_{\alpha\beta}} = 0.
\]

\[
\begin{align*}
vaF &= 0, \quad gaF &= 0, \quad Q = 0, \quad vF = vW, \quad gF = gW, \\
aF &= vaF + gaF = 0.
\end{align*}
\]

For \( W = 0 \):

\[
\frac{\delta \theta}{\delta s\, T} = 0, \quad \frac{\delta \theta}{\delta s\, T} = -W.
\]

The finding out the covariant Noether identities for a given Lagrangian density \( L = \sqrt{-\det g} L \) along with the energy-momentum tensors \( \theta, s\, T \) and \( Q \) allow the construction of a rough scheme of the structures of a Lagrangian theory over a differentiable manifold with contravariant and covariant affine connections and a metric:

\[
\begin{array}{ccc}
\leftarrow & \downarrow & \rightarrow \\
L & \delta L/\delta K^A_B & \\
\theta & \rightarrow & Q \\
\theta\cdot s\, T = Q & \leftarrow & \downarrow \\
F & \leftarrow & \downarrow \\
F + \delta \theta = 0 & \leftarrow & \\
\end{array}
\]

\textbf{Fig. 1. Scheme of the main structure of a Lagrangian theory}

The symmetric energy-momentum tensor of Hilbert \( g_{sh} T \) appears as a construction related to the functional variation of the metric field variables \( g_{\alpha\beta} \) [as a part of the variables \( K^A_B \sim (V^A_B, \ g_C) \)] and interpreted as a symmetric energy-momentum tensor of a Lagrangian system. This tensor does not exist as a relevant element of the scheme for obtaining Lagrangian structures by the method of Lagrangians with covariant derivatives (MLCD). It takes in the scheme a separate place and has different than the usual for the other elements relations.
The field theories involve relations between the different structures of the Lagrangian systems. For the most part of Lagrangian systems equations of the type of the Euler-Lagrange equations have been imposed and the symmetric energy-momentum tensor of Hilbert has been used. In Einstein’s theory of gravitation (ETG) the existing relations among the different structure’s elements are very peculiar. They require additional considerations.

In the Fig. 3, L is the Lagrangian invariant of the material distribution. \( L_g \) is the Lagrangian invariant of the gravitational field.

There are other possible relations between the structures of two Lagrangian densities than the relations between \( L_g \) and \( L_m \) in ETG. For instance, the relations between the variational energy-momentum tensors \( gQ \) and \( mQ \) of both Lagrangian densities and between the vector fields \( gF \) and \( mF \).
Fig. 4. Possible relations between two Lagrangian systems

On the grounds of relations of this type a model for describing the gravitational interaction in \( V_4 \)-spaces is considered different from ETG [22].

The covariant Noether identities can be used for a generalization of notions of the continuum media mechanics related to the notions of force density, density of the moment of the force (density of the inertial momentum) and angular momentum density for Lagrangian systems described by models over differentiable manifolds with affine connections and metrics.

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