Lower Bounds of Concurrence for Multipartite States

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Abstract

We study the entanglement of multipartite quantum states. Some lower bounds of the multipartite concurrence are reviewed. We further present more effective lower bounds for detecting and qualifying entanglement, by establishing functional relations between the concurrence and the generalized partial transpositions of the multipartite systems.

PACS numbers: 03.67.Mn, 03.65.Ud
I. 1. INTRODUCTION

Entanglement is a distinctive feature of quantum mechanics, and an indispensable ingredient in various kinds of quantum information processing applications such as quantum computation [1], quantum teleportation [2], dense coding [3], quantum cryptographic schemes [4], entanglement swapping [5] and remote states preparation (RSP) [6]. These effects based on quantum entanglement have been demonstrated in many pioneering experiments.

An important theoretical challenge in the theory of quantum entanglement is to give a proper description and quantification of quantum entanglement for given quantum states. For bipartite quantum systems, entanglement of formation (EOF) [7] and concurrence [8, 9] are two well defined quantitative measures of quantum entanglement. For two-qubit systems it has been proved that EOF is a monotonically increasing function of the concurrence and an elegant formula for the concurrence was derived analytically by Wootters [10]. However with the increasing dimensions of the subsystems the computation of EOF and concurrence become formidable difficult. A few explicit analytic formulae for EOF and concurrence have been found only for some special symmetric states [11–15].

The first analytic lower bound of concurrence that can be tightened by numerical optimization over some parameters was derived in [16]. In [17, 18] analytic lower bounds on EOF and concurrence for any dimensional mixed bipartite quantum states have been presented by using the positive partial transposition (PPT) and realignment separability criteria. These bounds are exact for some special classes of states and can be used to detect many bound entangled states. In [19] another lower bound on EOF for bipartite states has been presented from a new separability criterion [20]. A lower bound of concurrence based on local uncertainty relations (LURs) criterion is derived in [21]. This bound is further optimized in [22]. In [23, 24] the authors presented lower bounds of concurrence for bipartite systems in terms of a different approach. It has been shown that this lower bound has a close relationship with the distillability of bipartite quantum states. In Ref. [25] an explicit analytical lower bound of concurrence is obtained by using positive maps, which is better than the ones in Refs. [18, 19] in detecting some quantum entanglement. These bounds give rise to a good quantitative estimation of concurrence. They are supplementary in detecting quantum entanglement for bipartite systems.

When referring to multipartite systems, we focus on multipartite concurrence, since the
EOF is only defined for bipartite systems. With the increasing of the number of quantum systems, quantifying multipartite entanglement has become a much difficult task and only few results are obtained. In this paper, we first give a brief review of the lower bounds for multipartite concurrence in section 2. We present some new lower bounds of multipartite concurrence in sections 3-5. These new bounds give rise to better estimations of multipartite concurrence and are more effective in detecting multipartite entanglement. Conclusions and remarks are given in section 6.

II. 2. LOWER BOUNDS OF MULTIPARTITE CONCURRENCE

We first recall the definition and some lower bounds of the multipartite concurrence. Let \( \mathcal{H}_i, i = 1, \ldots, N \), be Hilbert spaces with \( d_i \) dimensions. The concurrence of an \( N \)-partite state \( |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \) is defined by [26]

\[
C_N(|\psi\rangle\langle\psi|) = 2^{1-N} \sqrt{(2^N - 2) - \sum_\alpha \text{Tr}[\rho_\alpha^2]},
\]

where \( \alpha \) labels all different reduced density matrices.

Up to constant factor (1) can be also expressed in another way. Set \( d_i = d, i = 1, 2, \ldots, N \). The \( N \)-partite pure state \( |\psi\rangle \) is generally of the form,

\[
|\psi\rangle = \sum_{i_1, i_2, \ldots, i_N=1}^d a_{i_1, i_2, \ldots, i_N} |i_1, i_2, \ldots, i_N\rangle, \quad a_{i_1, i_2, \ldots, i_N} \in \mathbb{C},
\]

with \( \sum_{i_1, i_2, \ldots, i_N=1}^d a_{i_1, i_2, \ldots, i_N} a_{i_1, i_2, \ldots, i_N}^* = 1 \).

Let \( \alpha \) and \( \alpha' \) (resp. \( \beta \) and \( \beta' \)) be subsets of the subindices of \( a \), associated to the same sub Hilbert spaces but with different summing indices. \( \alpha \) (or \( \alpha' \)) and \( \beta \) (or \( \beta' \)) span the whole space of the given sub-index of \( a \). The generalized concurrence of \( |\psi\rangle \) is then given by [9],

\[
C_d^N(|\psi\rangle) = \sqrt{\frac{d}{2m(d-1)} \sum_p \sum_{\alpha, \alpha', \beta, \beta'} |a_{\alpha \beta} a_{\alpha' \beta'} - a_{\alpha \beta'} a_{\alpha' \beta}|^2},
\]

where \( m = 2^{N-1} - 1 \), \( \sum_p \) stands for the summation over all possible combinations of the indices of \( \alpha \) and \( \beta \).

For a mixed multipartite quantum state, \( \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \), the
corresponding concurrence is given by the convex roof:

\[ C_N(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_N(|\psi_i\rangle). \]  

(4)

In [27] the lower bound of concurrence for tripartite systems has been studied by exploring the connection between the generalized partial transposition criterion and concurrence. Let \( \mathcal{H}_A, \mathcal{H}_B \) and \( \mathcal{H}_C \) be three finite dimensional Hilbert spaces associated with the subsystems \( A, B \) and \( C \), with dimensions \( \dim A = m, \dim B = n \) and \( \dim C = p \). Define that \( T_{rk} \) (resp. \( T_{ck} \)), \( k = A, B, C, AB, BC, AC \) to be the row (resp. column) transpositions with respect to the subsystems \( k \). Consider three classes: 1) \( y_i = \{c_k, r_k\} \), where \( i = 1, 2, 3 \) for \( k = A, B, C \) respectively; 2) \( y_4 = \{c_A, r_{BC}\}, y_5 = \{c_{AB}, r_C\}, y_6 = \{c_{AC}, r_B\} \); 3) \( y_7 = \{c_A, r_B\}, y_8 = \{c_A, r_C\}, y_9 = \{c_B, r_C\} \).

For any \( m \otimes n \otimes p \) \((m \leq n, p)\) tripartite mixed quantum state \( \rho \), the concurrence \( C(\rho) \) defined in (1) satisfies

\[ C_N(\rho) \geq \max\{\sqrt{\frac{1}{m(m-1)}(||\rho^{T_{ya}}|| - 1)}, \sqrt{\frac{1}{n(n-1)}(||\rho^{T_{yb}}|| - 1)}, \sqrt{\frac{1}{r(r-1)}(||\rho^{T_{yc}}|| - 1)}\}. \]

where \( q = \min(n, mp) \) and \( r = \min(p, mn) \), \( y_a = y_1 \) or \( y_4 \), \( y_b = y_2 \) or \( y_6 \), \( y_c = y_3 \) or \( y_5 \).

In [28, 29], the definition of multipartite concurrence defined in (1) is re-expressed as \( C(|\psi\rangle) = \sqrt{\langle\psi| \otimes \langle\psi|A|\psi\rangle \otimes |\psi\rangle} \), with \( A = 4(P_+ - P_+^{(1)} \otimes \cdots \otimes P_+^{(N)}) \). \( P_+ \) (resp. \( P_- \)) is the projector onto the globally symmetric (resp. antisymmetric) space. The authors have obtained that the multipartite concurrence satisfies

\[ [C_N(\rho)]^2 \geq Tr(\rho \otimes \rho V), \]

with \( V = 4(P_+ - P_+^{(1)} \otimes \cdots \otimes P_+^{(N)}) - (1 - 2^{1-N})P_- \).

In [22, 28, 29], it is shown that the multipartite concurrence defined in (1) satisfies

\[ C_N(\rho) \geq \sqrt{(4 - 2^{3-N})Tr\{\rho^2\} - 2^{2-N} \sum_\alpha Tr\{\rho^2_\alpha\}}. \]  

(5)

We derived an effective lower bound for multipartite quantum systems in [30]. First for tripartite case,
Theorem 1 For an arbitrary $d \times d \times d$ mixed state $\rho$ in $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, the concurrence $C(\rho)$ defined in (3) satisfies
\[
\tau_3(\rho) \equiv \frac{d}{6(d-1)} \sum_{\alpha} \sum_{\beta} \left[ (C_{\alpha\beta}^{123}(\rho))^2 + (C_{\alpha\beta}^{132}(\rho))^2 + (C_{\alpha\beta}^{231}(\rho))^2 \right] \leq C^2(\rho),
\]
where $\tau_3(\rho)$ is a lower bound of $C(\rho)$,
\[
C_{\alpha\beta}^{123}(\rho) = \max\{0, \lambda(1)_{\alpha\beta}^{123} - \lambda(2)_{\alpha\beta}^{123} - \lambda(3)_{\alpha\beta}^{123} - \lambda(4)_{\alpha\beta}^{123}\},
\]
$\lambda(1)_{\alpha\beta}^{123}, \lambda(2)_{\alpha\beta}^{123}, \lambda(3)_{\alpha\beta}^{123}, \lambda(4)_{\alpha\beta}^{123}$ are the square roots of the four nonzero eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho^{123}_{\alpha\beta}$ with $\rho^{123}_{\alpha\beta} = S^{123}_{\alpha\beta} \rho^* S^{123}_{\alpha\beta}$. $C_{\alpha\beta}^{132}(\rho)$ and $C_{\alpha\beta}^{231}(\rho)$ are defined in a similar way to $C_{\alpha\beta}^{123}(\rho)$.

Theorem 1 can be directly generalized to arbitrary multipartite case.

Theorem 2 For an arbitrary $N$-partite state $\rho \in \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$, the concurrence defined in (3) satisfies:
\[
\tau_N(\rho) \equiv \frac{d}{2m(d-1)} \sum_p \sum_{\alpha\beta} (C^{p\alpha\beta}(\rho))^2 \leq C^2(\rho),
\]
where $\tau_N(\rho)$ is the lower bound of $C(\rho)$, $\sum_p$ stands for the summation over all possible combinations of the indices of $\alpha, \beta$, $C^{p\alpha\beta}(\rho) = \max\{0, \lambda(1)^p_{\alpha\beta} - \lambda(2)^p_{\alpha\beta} - \lambda(3)^p_{\alpha\beta} - \lambda(4)^p_{\alpha\beta}\}$, $\lambda(i)^p_{\alpha\beta}, i = 1, 2, 3, 4$, are the square roots of the four nonzero eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho^{p\alpha\beta}$ where $\rho^{p\alpha\beta} = S^{p\alpha\beta} \rho^* S^{p\alpha\beta}$.

In [31] we further obtained lower bound of multipartite concurrence by bipartite partitions of the whole quantum systems. For a pure $N$-partite quantum state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, $\dim \mathcal{H}_i = d_i, \ i = 1, \ldots, N$, the concurrence of bipartite decomposition between subsystems $12 \cdots M$ and $M+1 \cdots N$ is defined by
\[
C_2(|\psi\rangle\langle\psi|) = \sqrt{2(1 - \text{Tr}\{\rho_{12\cdots M}^2\})},
\]
where $\rho_{12\cdots M} = \text{Tr}_{M+1\cdots N}\{|\psi\rangle\langle\psi|\}$ is the reduced density matrix of $\rho = |\psi\rangle\langle\psi|$ by tracing over the subsystems $M+1 \cdots N$.

For a mixed multipartite quantum state, $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$, the corresponding concurrence of (9) is then given by the convex roof:
\[
C_2(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_2(|\psi_i\rangle\langle\psi_i|),
\]
which will be called the bipartite concurrence.

The relation between the concurrences in (4) and the bipartite concurrence in (10) can be directly given by the following theorem.

**Theorem 3** For a multipartite quantum state \( \rho \in \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}_N \) with \( N \geq 3 \), the following inequality holds,

\[
C_N(\rho) \geq \max \left\{ 2^{\frac{3-N}{2}} C_2(\rho) \right\},
\]

where the maximum is taken over all kinds of bipartite concurrence.

In terms of the lower bounds of bipartite concurrence derived from PPT, realignment of the density matrix, local uncertainty relation and the covariance matrix separability criterion in [18, 21, 22], and (11), we get the following theorem.

**Theorem 4** For any \( N \)-partite quantum state \( \rho \), we have:

\[
C_N(\rho) \geq 2^{\frac{3-N}{2}} \max \{B1, B2, B3\},
\]

where

\[
B1 = \max_i \sqrt{\frac{2}{M_i(M_i - 1)}} \left[ \max(||T_A(\rho^i)||, ||R(\rho^i)||) - 1 \right],
\]

\[
B2 = \max_i \frac{2||C(\rho^i)|| - (1 - \text{Tr}\{((\rho_A^i)^2)\}) - (1 - \text{Tr}\{((\rho_B^i)^2)\})}{\sqrt{2M_i(M_i - 1)}},
\]

\[
B3 = \max_i \sqrt{\frac{8}{M_i^3 N_i^2 (M_i - 1)}} \left( ||T(\rho^i)|| - \sqrt{M_i N_i (M_i - 1)(N_i - 1)} \right),
\]

\( \rho^i \) are all possible bipartite decompositions of \( \rho \), \( M_i = \min \{d_{s_1}d_{s_2} \cdots d_{s_m}, d_{s_{m+1}}d_{s_{m+2}} \cdots d_{s_N}\} \), \( N_i = \max \{d_{s_1}d_{s_2} \cdots d_{s_m}, d_{s_{m+1}}d_{s_{m+2}} \cdots d_{s_N}\} \).

### III. 3. IMPROVED LOWER BOUNDS OF THE MULTIPARTITE CONCURRENCE

In this section, we will derive a new bound for multipartite quantum systems by using the following lemma.
Lemma 5 For a bipartite density matrix $\rho \in H_A \otimes H_B$, one has

\begin{align}
1 - Tr\{\rho^2_{AB}\} & \geq (1 - Tr\{\rho^2_A\}) - (1 - Tr\{\rho^2_B\}), \\
1 - Tr\{\rho^2_{AB}\} & \geq (1 - Tr\{\rho^2_B\}) - (1 - Tr\{\rho^2_A\}),
\end{align}

where $\rho_{A|B} = Tr_A\{\rho_B\}$, $\rho_B = Tr_B\{\rho_A\}$.

Theorem 6 For a multipartite quantum state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ with $N \geq 3$, the following inequality holds:

$$C_N(\rho) \geq \max_{\{M=1,2,\ldots,N-1\}} \left\{ \left( 2^{\frac{1-N}{2}} \sqrt{2^{N-M} + 2^M - 2} \right) C_2(\rho_M) \right\},$$

where the maximum takes over all kinds of bipartite concurrences.

Proof. For a pure multipartite state $|\varphi\rangle \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$, one has $Tr\{\rho^2_{12\ldots M}\} = Tr\{\rho^2_{M+1\ldots N}\}$ for all $M = 1, 2, \ldots, N-1$.

From (13) and (14), we obtain

\begin{align}
1 - Tr\rho^2_{12\ldots Mi_1\ldots i_p} & \geq (1 - Tr\rho^2_{12\ldots M}) - (1 - Tr\rho^2_{i_1\ldots i_p}), \\
1 - Tr\rho^2_{j_1\ldots j_q M+1\ldots N} & \geq (1 - Tr\rho^2_{M+1\ldots N}) - (1 - Tr\rho^2_{j_1\ldots j_q}),
\end{align}

and

$$C^2_N(|\varphi\rangle\langle\varphi|) = 2^{2-N} \left[ (2^N - 2) - \sum_{\alpha} Tr\rho^2_{\alpha} \right] = 2^{2-N} \left( \sum_{k=1}^{2^{N-2}} (1 - Tr\rho^2_k) \right)$$

\begin{align}
& \geq 2^{2-N} \left\{ (2^{N-M} - 1)(1 - Tr\rho^2_{12\ldots M}) + (2^M - 1)(1 - Tr\rho^2_{M+1\ldots N}) \right\} \\
& = 2^{2-N} \left\{ (2^{N-M} + 2^M - 2)(1 - Tr\rho^2_{12\ldots M}) \right\} \\
& = 2^{2-N} \left\{ (2^{N-M} + 2^M - 2) \frac{C_2(|\varphi\rangle M\langle\varphi|)}{2} \right\},
\end{align}

i.e. $C_N(|\varphi\rangle\langle\varphi|) \geq \max_{\{M=1,2,\ldots,N-1\}} \left( 2^{\frac{1-N}{2}} \sqrt{2^{N-M} + 2^M - 2} \right) C_2(|\varphi\rangle M\langle\varphi|)$. 

7
Assuming that $\rho = \sum_i p_i |\varphi_i\rangle \langle \varphi_i|$ attains the minimal decomposition of the multipartite concurrence, one has

$$C_N(\rho) = \sum_i p_i C_N(|\varphi_i\rangle \langle \varphi_i|)$$

$$\geq 2^{1-N} \sqrt{2^{N-M} + 2M - 2} \sum_i p_i C_2(|\varphi_i\rangle_M \langle \varphi_i|)$$

$$\geq 2^{1-N} \sqrt{2^{N-M} + 2M - 2} \min_{\{p_i, |\varphi_i|\}} \sum_i p_i C_2(|\varphi_i\rangle_M \langle \varphi_i|)$$

$$= \left(2^{1-N} \sqrt{2^{N-M} + 2M - 2}\right) C_2(\rho_M).$$

Therefore we have

$$C_N(\rho) \geq \max_{\{M=1,2,...,N-1\}} \left\{ \left(2^{1-N} \sqrt{2^{N-M} + 2M - 2}\right) C_2(\rho_M) \right\}.$$  

IV. 4. FUNCTIONAL RELATIONS BETWEEN CONCURRENCE AND THE GENERALIZED PARTIAL TRANSPOSITIONS

Let us consider an $N$-qubit state, the generalized $W$ state,

$$|\varphi\rangle = a_1|10...0\rangle + a_2|01...0\rangle + ... + a_N|00...1\rangle.$$  

(18)

\textbf{Theorem 7} For any $N$-qubit mixed state with decomposition on the generalized $W$ states, $\rho = \sum_i p_i |\varphi_i\rangle \langle \varphi_i|$, such that $|\varphi_i\rangle$ can be written in the form (18) for all $i$, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq 2^{1-N} \max \left\{ |\rho^{T_{\Gamma_1^1}}| - 1, \max_M \left\{ \sqrt{2^{N-M} + 2M - 2} - 1\right\} \right\},$$

(19)

where $\Gamma_1^1$, $\Gamma_2^1$ denote two subsets of the indices $\{1,2,...,N\}$, $\Gamma_1^1 \cap \Gamma_2^1 = \emptyset$, $\Gamma_1^1 \cup \Gamma_2^1 = \{1,2,...,N\}$, $\alpha = 1, ..., d$, $M = \{1,2,...,N-1\}$ is the number of elements of $\Gamma_1^1$.

\textbf{Proof.} An $N$-qubit $W$ state can be viewed as $d$ different bipartite systems. From the results for bipartite systems [18], these $d$ bipartite separations give rise to, respectively

$$1 - Tr\{\rho_{\Gamma_1^1}^2\} \geq \frac{1}{2} (|R_{\Gamma_1^1,\Gamma_2^2}(\rho)| - 1)^2, \alpha = 1, ..., d.$$
Hence

\[
C(\langle \varphi | \varphi \rangle) = 2^{1-N} \sqrt{d - \sum_{\alpha=1}^d Tr\{\rho_{\Gamma_1}^2\}}
\]

\[
= 2^{1-N} \sqrt{\frac{2d - \sum_{\alpha=1}^d Tr\{\rho_{\Gamma_1}^2\} - \sum_{\alpha=1}^d Tr\{\rho_{\Gamma_2}^2\}}{2}}
\]

\[
\geq 2^{1-N} \max_M \sqrt{\frac{2^{N-M} + 2^M - 2}{4}} (1 - Tr\{\rho_{\Gamma_1}^2\})
\]

\[
\geq 2^{1-N} \max_M \sqrt{\frac{2^{N-M} + 2^M - 2}{4}} (|R_{\Gamma_1,\Gamma_2}(\rho)| - 1).
\]

Let \( \rho = \sum_i p_i |\varphi_i\rangle\langle \varphi_i |\) attain the minimal decomposition of the multipartite concurrence. Note that \(|R(\rho)| \leq \sum_i p_i |R(\langle \varphi_i | \varphi_i \rangle)| \) [18]. One has

\[
C(\rho) = \sum_i p_i C(\langle \varphi_i | \varphi_i \rangle)
\]

\[
\geq 2^{1-N} \max_M \sqrt{\frac{2^{N-M} + 2^M - 2}{4}} \sum_i p_i (|R_{\Gamma_1,\Gamma_2}(\langle \varphi_i | \varphi_i \rangle)| - 1)
\]

\[
\geq 2^{1-N} \max_M \sqrt{\frac{2^{N-M} + 2^M - 2}{4}} (|R_{\Gamma_1,\Gamma_2}(\rho)| - 1),
\]

From which one gets (19).

\[\Box\]

V. 5. ENTANGLEMENT DETECTING AND ESTIMATION OF CONCURRENCE

In this section, we use the above several lower bounds of multipartite concurrence to detect quantum entanglement. We will show by examples that these bounds provide a better estimation of the multipartite concurrence.

(1) Lower bound and separability

An N-partite quantum state \( \rho \) is fully separable if and only if there exist \( p_i \) with \( p_i \geq 0, \sum_i p_i = 1 \) and pure states \( \rho_i^j = |\psi_i^j\rangle\langle \psi_i^j | \) such that \( \rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2 \otimes \cdots \otimes \rho_i^N \). It is easily verified that for a fully separable multipartite state \( \rho \), \( \tau_N(\rho) \) defined in [8] is zero. Thus \( \tau_N(\rho) > 0 \) indicates that there must be some kinds of entanglement inside the quantum state, which shows that the lower bound \( \tau_N(\rho) \) can be used to recognize entanglement.

As an example we consider a tripartite quantum state [32], \( \rho = \frac{1}{8}I_8 + p|W\rangle\langle W| \), where \( I_8 \) is the \( 8 \times 8 \) identity matrix, and \( |W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \) is the tripartite W-state.
By using the generalized correlation matrix criterion presented in [33], the entanglement of \( \rho \) is detected for 0.3068 < \( p \) ≤ 1. From our theorem, we have that the lower bound \( \tau_3(\rho) > 0 \) for 0.2727 < \( p \) ≤ 1. Therefore our bound detects entanglement better in this case. If we replace W with GHZ state in \( \rho \), the criterion in [33] detects the entanglement of \( \rho \) for 0.35355 < \( p \) ≤ 1, while \( \tau_3(\rho) \) detects, again better, the entanglement for 0.2 < \( p \) ≤ 1.

(2) Estimation of multipartite concurrence

The lower bounds together with some upper bounds can be used to estimate the value of the concurrence. In [22, 28, 29], it is shown that the upper and lower bound of multipartite concurrence satisfy

\[
\sqrt{(4 - 2^{3-N}) \text{Tr}\{\rho^2\} - 2^{2-N} \sum_{\alpha} \text{Tr}\{\rho_\alpha^2\}} \leq C_N(\rho) \leq \sqrt{2^{2-N}[(2^N - 2) - \sum_{\alpha} \text{Tr}\{\rho_\alpha^2\}].
\]

(20)

In fact we can obtain a more effective upper bound for multi-partite concurrence. Let \( \rho = \sum_i \lambda_i |\psi_i\rangle\langle \psi_i| \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N \), where \(|\psi_i\rangle\)'s are the orthogonal pure states and \( \sum_i \lambda_i = 1 \). We have

\[
C_N(\rho) = \min_{\{p_i, |\varphi_i\rangle\}} \sum_i p_i C_N(|\varphi_i\rangle\langle \varphi_i|) \leq \sum_i \lambda_i C_N(|\psi_i\rangle\langle \psi_i|).
\]

(21)

We now show that our upper and lower bounds can be better than that in (5) by detailed examples.

Example 1: Consider the \( 2 \times 2 \times 2 \) Dür-Cirac-Tarrach states defined by [34]:

\[
\rho = \sum_{\sigma = \pm} \lambda_0^\sigma |\psi_0^\sigma\rangle\langle \psi_0^\sigma| + \sum_{j=1}^3 \lambda_j (|\psi_j^+\rangle\langle \psi_j^+| + |\psi_j^-\rangle\langle \psi_j^-|),
\]

where the orthonormal Greenberger-Horne-Zeilinger (GHZ)-basis \(|\psi_j^\pm\rangle \equiv \frac{1}{\sqrt{2}} (|j\rangle_1_2|0\rangle_3 \pm |(3-j)\rangle_1_2|1\rangle_3), |j\rangle_1_2 \equiv |j_1\rangle_1 |j_2\rangle_2 \) with \( j = j_1 j_2 \) in binary notation. From theorem 2 we have that the lower bound of \( \rho \) is \( \frac{1}{3} \). If we mix the state with white noise, \( \rho(x) = (1-x)I_8 + x\rho \), by direct computation we have, as shown in FIG. 11, the lower bound obtained in (5) is always zero, while the lower bound in (12) is larger than zero for 0.425 ≤ \( x \) ≤ 1, which shows that \( \rho(x) \) is detected to be entangled at this situation. And the upper bound (dot line) in (5) is much larger than the upper bound we have obtained in (21) (solid line).

Actually, our new lower bound in (15) is different from the lower bound in (11), which can be seen from the following example.
FIG. 1: Our lower and upper bounds of $C_3(\rho)$ from (12) and (21) (solid line) and the upper bound obtained in (20) (dot line) while the lower bound in (20) is always zero.

**Example 2:** Consider the generalized GHZ state: $|\varphi\rangle = \cos \theta |00...0\rangle + \sin \theta |11...1\rangle$. It is easy to obtain that $Tr\rho^2 = 1 - 2\sin^2 \theta \cos^2 \theta$ for all $i_1 \neq i_2 \neq ... \neq i_m \in 1, 2, ..., N$. Hence we have by definition $C(|\varphi\rangle) = 2^{1 - \frac{N}{2}} \sqrt{(2^N - 2)(2\sin^2 \theta \cos^2 \theta)}$. By our new lower bound in (15), we get

$$C_N(\rho) \geq \max_{\{M=1,2,\ldots,N-1\}} \left\{ \left(2^{\frac{1-N}{2}} \sqrt{2^{N-M} + 2^{M-2}}\right) C_2(\rho_M) \right\}$$

$$= \max_{\{M=1,2,\ldots,N-1\}} \left\{ \left(2^{\frac{1-N}{2}} \sqrt{2^{N-M} + 2^{M-2}}\right) \sqrt{4 \sin^2 \theta \cos^2 \theta} \right\}$$

For example, $N = 4$, we get $C_N(|\varphi\rangle\langle\varphi|) = \sqrt{7} \sin^2 \theta \cos^2 \theta$. From our bound we have $C_N(|\varphi\rangle\langle\varphi|) \geq \sqrt{4 \sin^2 \theta \cos^2 \theta} > \sqrt{2} \sin^2 \theta \cos^2 \theta$, where $\sqrt{2} \sin^2 \theta \cos^2 \theta$ is the bound from [31].

**VI. 6. REMARKS AND CONCLUSIONS**

By establishing functional relations between the concurrence and the generalized partial transpositions of the multipartite systems, we have presented some effective lower bounds for detecting and qualifying entanglement for multipartite systems. These bounds can be also served as separability criteria. They detect entanglement of some states better than some separability criteria.
Generally, to derive a lower bound of multipartite concurrence, we calculate the multipartite concurrence for pure states first. Then by using the convex property of the quantities in the calculation one can directly find a tight lower bound. Mintert et al. in [35] have derived a precise lower bound for bipartite concurrence, which detects mixed entangled states with a positive partial transpose. It would be interesting and challenging to use this approach to derive a lower bound for multipartite concurrence.

Acknowledgments This work is supported by the NSFC 10875081 and PHR201007107.

[1] See, for example, D.P. Di Vincenzo, Science 270, 255(1995).
[2] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W.K. Wootters, Phys. Rev. Lett. 70, 1895 (1993);
    S.Albeverio, S.M.Fei and Y.L.Yang, Phys. Rev. A 66, 012301(2002).
[3] C.H.Bennett and S.J.Wiesner, Phys. Rev. Lett. 69, 2881(1992).
[4] A. Ekert, Phys. Rev. Lett. 67, 661(1991);
    D. Deutsch et al, Phys. Rev. A 56, 1163(1997).
[5] M. Žukowski, A. Zeilinger, M.A. Horne and A.K. Ekert, Phys. Rev. Lett. 71, 4287(1993);
    S. Bose, V. Vedral and P.L. Knight, Phys. Rev. A 57, 822(1998); 60, 194(1999);
    B.S. Shi, Y.K. Jiang, G.C. Guo, Phys. Rev. A 62, 054301(2000).
[6] C.H. Bennett, D.P. DiVincenzo, P.W. Shor, J.A. Smolin, B.M. Terhal and W.K. Woottter, Phys. Rev. Lett. 87, 077902(2001);
    D. W. Leung and P. W. Shor, Phys. Rev. Lett. 90, 127905(2003).
[7] C. H. Bennett, D. P. DiVincenzo and J. A. Smolin, et al. Phys. Rev. A 54, 3824(1996).
    M. B. Plenio and S. Virmani, Quant. Inf. Comp. 7, 1(2007).
[8] A. Uhlmann Phys. Rev. A 62 032307(2000);
    P. Rungta, V. Bužek, and C. M. Caves, et al. Phys. Rev. A 64, 042315(2001).
[9] S. Albeverio and S. M. Fei, J. Opt. B: Quantum Semiclass Opt, 3, 223-227(2001).
[10] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[11] Terhal B M, Vollbrecht K G H, Phys Rev Lett, 85, 2625-2628(2000).
[12] S.M. Fei, J. Jost, X.Q. Li-Jost and G.F. Wang, Phys. Lett. A 310, 333-338(2003).
[13] S.M. Fei and X.Q. Li-Jost, Rep. Math. Phys. 53, 195-210(2004).
[14] S.M. Fei, Z.X. Wang and H. Zhao, Phys. Lett. A 329, 414-419(2004).
[15] P. Rungta and C.M. Caves, Phys Rev A 67, 012307(2003).
[16] F. Mintert, M. Kus, A. Buchleitner, Phys. Rev. Lett. 92, 167902(2004).
[17] K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. Lett. 95, 210501(2005).
[18] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. 95, 040504(2005).
[19] H.P. Breuer, J. Phys. A 39, 11847(2006).
[20] H.P. Breuer, Phys. Rev. Lett. 97, 080501(2006).
[21] J. I. de Vicente, Phys. Rev. A 75, 052320(2007).
[22] C. J. Zhang, Y. S. Zhang, and S. Zhang, et al. Phys. Rev. A 76, 012334(2007).
[23] E. Gerjuoy, Phys. Rev. A 67, 052308(2003).
[24] Y. C. Ou, H. Fan and S. M. Fei, Phys. Rev. A 78, 012311(2008).
[25] X.S. Li, X.H. Gao, and S.M. Fei, Phys. Rev. A 83, 034303 (2011).
[26] L. Aolita and F. Mintert, Phys. Rev. Lett. 97, 050501(2006);
   A. R. R. Carvalho, F. Mintert, and A. Buchleitner, Phys. Rev. Lett. 93, 230501(2004).
[27] X. H. Gao, S. M. Fei and K. Wu, Phys. Rev. A 74, 050303(R) (2006).
[28] F. Mintert and A. Buchleitner, Phys. Rev. Lett. 98, 140505 (2007).
[29] L. Aolita, A. Buchleitner, and F. Mintert, Phys. Rev. A 78, 022308(2008).
[30] M. Li, S.M. Fei and Z.X. Wang, J. Phys. A, Math. Theor.42, 1453032009.
[31] M. Li, S.M. Fei and Z.X. Wang, Rep. Math. Phys.65, 22010.
[32] A. Acín, D. Bruss, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 87, 040401(2001).
[33] A. S. M. Hassan and P. S. Joag, Quant. Inf. Comp. 8, 0773-0790(2008).
[34] W. Dür, J. I. Cirac, and R. Tarrach, Phys. Rev. Lett. 83, 3562 (1999).
[35] F. Mintert, M. Kús, and A. Buchleitner, Phys. Rev. Lett. 92, 167902 (2004).