The relation between $F(R)$ gravity and Einstein–conformally invariant Maxwell source

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In this paper, we consider the special case of $F(R)$ gravity, in which $F(R) = R^N$ and obtain its topological black hole solutions in higher dimensions. We show that, the same as higher dimensional charged black hole, these solutions may be interpreted as black hole solutions with two event horizons, extreme black hole and naked singularity provided the parameters of the solutions are chosen suitably. But, the presented black hole is different from the standard higher-dimensional Reissner-Nordström solutions. Next, we present the conformally invariant Maxwell field coupled to Einstein gravity and discuss about its black hole solutions. Comparing these two class of solutions, shows that there is a correspondence between the Einstein-conformally invariant Maxwell solutions and the solutions of $F(R)$ gravity without matter field in arbitrary dimensions.

I. INTRODUCTION

Various proposals of diverse characters have been suggested by the physicists during the past decades for going beyond, or modifying, Einstein general relativity and often for few viable reasons. The most important motivation coming from high-energy physics for adding higher order invariants to the gravitational action, as well as a general motivation coming from cosmology and astrophysics [1] for seeking generalizations of Einstein gravity. The so-called modified gravities constructed by adding correction terms to the usual Einstein-Hilbert action (e.g: [2, 3, 4]), is only one endeavor among others to go beyond Einstein general relativity and have opened a new window to study the origin of the current accelerated expansion of the Universe [5, 6, 7]. For example in Lovelock gravity, there have been some attempts for understanding the role of the higher curvature terms from various points of view [4]. As another example, a special case of considering the effect of higher curvature corrections, we will deal with in this paper the so-called $F(R)$ gravity [2] (and for a review, see, [8]), whose action is an arbitrary function of curvature scalar $R$. When $F(R) = R$, the Einstein’s general relativity is recovered. Some of the main reasons to consider $F(R)$ gravity

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are as follows: First of all, there is simplicity: \( F(R) \) actions are sufficiently general to encapsulate some of the basic characteristics of higher-order gravity, but at the same time they are simple enough to be easy to handle. Second, there are serious reasons to believe that \( F(R) \) theories are unique among higher-order gravity theories, in the sense that they seem to be the only ones which can avoid the long known and fatal Ostrogradski instability \[9\]. Third, \( F(R) \) theories have no ghosts \[10\], and the stability condition \( F''(R) \geq 0 \) of \[11\] essentially amounts to guarantee that the scalaron is not a ghost.

In \( F(R) \) gravity, Einstein equations possess extra terms induced from geometry which, when moved to the right hand side, may be interpreted as a matter source represented by the energy-momentum tensor \( T^\text{Curv}_{\mu\nu} \), see equation \[3, 4\]. In a similar fashion, the Space-Time-Matter (STM) theory, discussed below, results in Einstein equations in \( d \)-dimension with some extra matter terms showed by the energy-momentum tensor \( T^\text{matt}_{\mu\nu} \). It therefore seems plausible to make a correspondence between the matter terms in STM theory and geometrical terms in \( T^\text{Curv}_{\mu\nu} \) resulting in \( F(R) \) gravity.

From the other point of view, straightforward generalization of the electromagnetic field to higher dimensions one essential property of it is lost, namely, conformal invariance. In Ref. \[12\], it has been shown that the conformal excitations of the extra-dimensional space components have the form of massive scalar fields living in the external (our) spacetime. Maxwell theory can be studied in a gauge which is invariant under conformal rescalings of the metric, and firstly, has been proposed by Eastwood and Singer \[13\]. Also, there exists a conformally invariant extension of the Maxwell action in higher dimensions (Generalized Maxwell Field, GMF), if one uses the lagrangian of the \( U(1) \) gauge field in the form \[14, 15, 16, 17, 18\]

\[
I_{GMF} = \kappa \int d^{n+1}x \sqrt{-g} (F_{\mu\nu}F^{\mu\nu})^s, \tag{1}
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the Maxwell tensor and \( \kappa \) is an arbitrary constant. It is straightforward to show that for \( s = (n+1)/4 \), the action \[1\] is invariant under conformal transformation \((g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \text{ and } A_\mu \rightarrow A_\mu)\) and for \( n = 3 \), the action \[1\] reduces to the Maxwell action as it should be. The idea is to take advantage of the conformal symmetry to construct the analogues of the four-dimensional Reissner-Nordström black hole solutions in higher dimensions.

The main scope of this work is to present the correspondence between \( F(R) \) gravity and Einstein-conformally invariant Maxwell theory. As we show later, these solutions have some interesting properties, specially in the electromagnetic fields, which do not occur in Einstein gravity in the present of ordinary Maxwell field.
The outline of our paper is as follows. In section II we present a short review of field equations of \((n+1)\)-dimensional \(F(R)\) gravity. In section III the field equations of \(F(R) = R^N\) gravity are solved in the absence of matter field and the resulting solutions are interpreted as black hole. The solutions of Einstein-conformally invariant Maxwell source are considered in section IV. Conclusions are drawn in the last section.

II. BASIC FIELD EQUATIONS OF \(F(R)\) GRAVITY:

The action of \(F(R)\) gravity, in the presence of matter field has the form of

\[
\mathcal{I}_G = -\frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left[ F(R) + L_{\text{matt}} \right],
\]

where \(R\) is the scalar curvature and \(F(R)\) is an arbitrary function of \(R\), and \(L_{\text{matt}}\) is the Lagrangian of matter fields. Variation with respect to metric \(g_{\mu\nu}\), leads to the field equations

\[
G_{\mu\nu} = T_{\mu\nu}^{\text{Curv}} + \frac{T_{\mu\nu}^{\text{matt}}}{F'(R)},
\]

where \(G_{\mu\nu}\) is the Einstein tensor and the gravitational stress-energy tensor is

\[
T_{\mu\nu}^{\text{Curv}} = \frac{1}{F'(R)} \left( \frac{1}{2} g_{\mu\nu} (F(R) - RF'(R)) + F'(R) g_{\alpha\beta} (g_{\alpha\mu} g_{\beta\nu} - g_{\mu\nu} g_{\alpha\beta}) \right)
\]

with \(F'(R) \equiv dF(R)/dR\) and \(T_{\mu\nu}^{\text{matt}}\) the standard matter stress-energy tensor derived from the matter Lagrangian \(L_{\text{matt}}\) in the action (2). One can consider geometrical terms in the left hand side of the field equation, and therefore Eq. (3) reduces to

\[
R_{\mu\nu} F'(R) - \nabla_\mu \nabla_\nu F'(R) + \left( \Box F'(R) - \frac{1}{2} F(R) \right) g_{\mu\nu} = T_{\mu\nu}^{\text{matt}}.
\]

The trace of Eq. (5) reduces to

\[
n \Box F'(R) + RF'(R) - \frac{n+1}{2} F(R) = T
\]

Here, we consider the special case of \(F(R)\) gravity, namely, \(R^N\) gravity and therefore the field equation (5) reduces to

\[
\left[ NR_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] R^{N-1} + N \left[ g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right] R^{N-1} = T_{\mu\nu}^{\text{matt}}
\]

It is easy to show that for \(N = 1\), Eq. (7) reduces to familiar Einstein gravity.
III. BLACK HOLE SOLUTIONS OF $R^N$ GRAVITY WITHOUT MATTER FIELD:

Here we want to obtain the $(n+1)$-dimensional static solutions of Eqs. (7) without any matter field ($\mathcal{L}_m = 0$, and then $T_{\mu\nu}^{\text{mat}} = 0$) with $N \in \mathbb{N}$. We assume that the metric has the following form

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2d\Omega_k^2,$$

(8)

where

$$d\Omega_k^2 = \begin{cases} 
  d\theta_1^2 + \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = 1 \\
  d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \sin^2 \theta_1 \sum_{i=2}^{n-1} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 , \\
  \sum_{i=1}^{n-1} d\phi_i^2 & k = 0 
\end{cases}$$

(9)

which represents the line element of an $(n-1)$-dimensional hypersurface with constant curvature $(n-1)(n-2)k$ and volume $V_{n-1}$. To find the function $g(r)$, one may use any components of Eq. (7). The solution which satisfies all the components of the gravitational field equations (7), can be written as

$$g(r) = k - \frac{m}{r^{n-2}} + \frac{\lambda(N-1)}{r^{n-1}},$$

(10)

where $m$ and $\lambda$ are integration constants which proportional to the mass and charge parameter respectively. In order to study the general structure of this solution, we first look for the curvature singularities. It is easy to show that the Kretschmann scalar $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ diverges at $r = 0$, it is finite for $r > 0$ and goes to zero as $r \to \infty$. Thus, there is an essential singularity located at $r = 0$. The event horizon(s), if there exists any, is (are) located at the root(s) of $g'' = g(r) = 0$

$$kr_+^{n-1} - mr_+ + \lambda = 0.$$

The temperature may be obtained through the use of the definition of surface gravity. One obtains

$$T = \frac{g'(r_+)}{4\pi} = \frac{(n-2)mr_+ - (n-1)\lambda}{4\pi r_+^{n}} = \frac{(n-2)r_+^{n-1} - \lambda}{4\pi r_+^{n}}.$$  

(11)

Using the fact that the temperature of the extreme black hole is zero, it is easy to show that the condition for having an extreme black hole is that the mass parameter is equal to $m_{\text{ext}}$, where $m_{\text{ext}}$ is given as

$$m_{\text{ext}} = (n-1)\left(\frac{\lambda}{n-2}\right)^{(n-2)/(n-1)}.$$ 

(12)
FIG. 1: \( g(r) \) versus \( r \) with for \( N = 2, \lambda = 2, k = 1, n = 4 \) and \( m = 2 < m_{\text{ext}} \) (continuous line), \( m = m_{\text{ext}} = 3 \) (dashed line) and \( m = 4 > m_{\text{ext}} \) (bold line).

The metric of Eqs. (8), (9) and (10) presents a black hole solution with inner and outer horizons, provided the mass parameter \( m \) is greater than \( m_{\text{ext}} \), an extreme black hole for \( m = m_{\text{ext}} \), and a naked singularity otherwise (see Fig. 1 for more details). This behavior is the same as Reissner-Nordström black holes, but, the metric function \( g(r) \), presented here, differ from the standard higher-dimensional Reissner-Nordström solutions since the electric charge term in the metric function is proportional to \( r^{-(n-1)} \) and in the standard higher dimensional charged black hole solutions is proportional to \( r^{-2(n-2)} \). It is notable that for \( N = 1 \) (\( F(R) = R \): Einstein gravity), the third term in Eq. (10) vanishes and the solutions reduce to Schwarzschild like solutions.

IV. THE SOLUTIONS OF EINSTEIN GRAVITY IN THE PRESENCE OF NONLINEAR MAXWELL SOURCE:

In this section, we consider the \((n+1)\)-dimensional action in which gravity is coupled to nonlinear electrodynamics field with an action

\[
\mathcal{I}_G' = -\frac{1}{16\pi} \int_{\partial M} d^{n+1}x \sqrt{-g} \left[ R - \alpha (F_{\alpha\beta}F^{\alpha\beta})^s \right],
\]

where \( \alpha \) is a coupling constant and the exponent \( s \) represented the nonlinear power of the electromagnetic field. Varying the action (13) with respect to the metric tensor \( g_{\mu\nu} \) and the electromagnetic field \( A_{\mu} \), the equations of gravitational and electromagnetic fields may be obtained
as

\[ G_{\mu \nu} = T_{\mu \nu}^{\text{matt}}, \]  

\[ \partial_\mu \left[ \sqrt{-g} F_{\mu \nu} (F_{\alpha \beta} F^{\alpha \beta})^{s-1} \right] = 0, \]  

where

\[ T_{\mu \nu}^{\text{matt}} = 2\alpha \left[ s F_{\mu \rho} F_{\nu}^\rho (F_{\alpha \beta} F^{\alpha \beta})^{s-1} - \frac{1}{4} g_{\mu \nu} (F_{\alpha \beta} F^{\alpha \beta})^s \right] \]  

It is easy to show that when \( s \) goes to 1, the Eqs. (13)–(15), reduce to the in Einstein-standard Maxwell gravity in higher dimensions. The Maxwell equation (15) with metric (8) can be integrated immediately to give

\[ F_{tr} = \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{-q}{r}, & s = \frac{n}{2} \\ \frac{(n-2s)q}{(2s-1)r^{(n-1)/(2s-1)}}, & \text{otherwise} \end{cases} \]  

where \( q \), an integration constant where the electric charge of the spacetime is related to this constant. Inserting the Maxwell fields (17) and the metric (8) in the field equation (14), one can show that these equations have the following solutions

\[ g(r) = k - \frac{m}{r^{n-2}} + \alpha \times \begin{cases} 0, & s = 0, \frac{1}{2} \\ \frac{(-1)^{3n/2}q^{n/2}r^{n} \ln r}{r^{n-2}}, & s = \frac{n}{2} \\ \frac{(-1)^{s}(2s-1)^2}{(n-1)(2s-n)^2(2s-n+1)(2s-1)} \left[ \frac{2(2s-n)^2q^2}{(2s-1)^2} \right]^s, & \text{Otherwise} \end{cases} \]  

where \( m \) is the integration constant which is related to mass parameter. In the linear case \((s = 1)\), the solutions reduce to the higher dimensional Reissner-Nordström solutions with linear Maxwell source as they should be. Straightforward calculation of Kretschmann scalar shows that there is an curvature singularity located at \( r = 0 \).

Before studying in details the spacetime, we first specify the sign of the coupling constant \( \alpha \) in term of the exponent \( s \) in order to ensure a physical interpretation of our future solutions. In fact, the sign of the coupling constant in the action (13) can be chosen such that the energy density, i.e. the \( T_{00} \) component of the energy-momentum tensor in the orthonormal frame, is positive

\[ T_{00} = (-1)^{s+1} \alpha (2s - 1) (2F_{tr}^2)^s > 0. \]  

This condition selects two branches depending on the range of the nonlinear parameter \( s \),

\[ \text{sgn}(\alpha) = \begin{cases} (-1)^{1-s}, & s > \frac{1}{2} \\ (-1)^{-s}, & s < \frac{1}{2} \end{cases} \]
while the cases \( s = 0, 1/2 \) is excluded because in these cases \( F_{tr} \) (and charge term in Eq. (13)) vanishes.

Now, we want to investigate the special case, such that the electromagnetic field equation be invariant under conformal transformation \((g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu} \text{ and } A_\mu \rightarrow A_\mu)\). The idea is to take advantage of the conformal symmetry to construct the analogues of the four dimensional Reissner-Nordström solutions in higher dimensions. It is easy to show that for Lagrangian in the form \( L(F = F_{\alpha\beta}F^{\alpha\beta}) \) in \((n + 1)\)-dimensions, \( T^\mu_\mu \propto \left[F_{df} - \frac{n+1}{4}L\right] \); so \( T^\mu_\mu = 0 \) implies \( L(F) = \text{Constant} \times F^{(n+1)/4} \). Indeed, in our case \( L(F) \propto F^s \), for \( s = (n+1)/4 \), the Maxwell action (14) enjoys the conformal symmetry in arbitrary dimensions. In this case the metric function \( f(r) \) and electromagnetic field \( F_{tr} \) reduce to

\[
g(r) = k - \frac{m}{r^{n-2}} + \frac{2^{(n-3)/4}q^{(n+1)/2}}{r^{n-1}}, \tag{21}
\]

\[
F_{tr} = \frac{q}{r^2}. \tag{22}
\]

Here, we calculate the electric charge of the Einstein-conformally invariant Maxwell solutions. To determine the electric field, we should consider the projections of the electromagnetic field tensor on special hypersurfaces. The electric charge per unit volume of the black hole can be found by calculating the flux of the electromagnetic field at infinity, obtaining

\[
Q = \frac{2^{(n-3)/4}(n+1)q^{(n-1)/2}}{16\pi}, \tag{23}
\]

which confirm that \( q \) is related to the electrical charge of the spacetime. Comparing Eq. (10) with Eq. (21) (and let \( \lambda = [2^{(n-3)/4}q^{(n+1)/2}]/(N - 1) \)), show that there is a correspondence between the Einstein-conformally invariant Maxwell solutions and the solutions of \( F(R) \) gravity without matter field. One can think about the same effects of \( T^\mu_\nu \) in \( F(R) \) gravity (Eq. (3) without \( T^\mu_\nu \)) and \( T^\mu_\nu \) in Einstein-conformally invariant Maxwell gravity (Eq. (14)) and therefore \( \lambda \) is related to the charge parameter \( q \).

V. CONCLUSIONS

In the presented paper, we considered the special case of \( F(R) \) gravity, so-called \( R^N \) gravity and obtained its topological black hole solutions in higher dimensions. We found that, such as charged black hole solutions, these solutions may be interpreted as black hole solutions with two event horizons, extreme black hole and naked singularity provide that the mass parameter \( m \) is greater than an extremal value \( m_{\text{ext}} \), \( m = m_{\text{ext}} \) and \( m < m_{\text{ext}} \) respectively. But the presented solutions
differ from the standard higher-dimensional Reissner-Nordström solutions since the electric charge term in obtained metric function is proportional to $r^{-(n-1)}$ and in the standard higher dimensional charged black hole solutions is proportional to $r^{-2(n-2)}$.

Next, we presented topological black hole solutions in Einstein gravity with nonlinear electromagnetic field. Then, we restricted ourself to the special case, namely, the conformally invariant Maxwell field coupled to Einstein gravity and discuss about the correspondence between the Einstein-conformally invariant Maxwell solutions and the solutions of $F(R)$ gravity without matter field in arbitrary dimensions. It is easy to show that presented solutions reduce to Reissner-Nordström black hole in four dimension.

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[1] S. Capozziello, Int. J. Mod. Phys. D 11 (2002) 483; S. Capozziello, V. F. Cardone, S. Carloni and A. Troisi, Int. Mod. Phys. D 12 (2003) 1969; S. Capozziello, S. Nojiri, S.D. Odintsov and A. Troisi, Phys. Lett. B 639 (2006) 135; S. Nojiri and S. D. Odintsov, J. Phys. Conf. Ser. 66 (2007) 012005; S. Nojiri and S. D. Odintsov, J. Phys. A 40 (2007) 6725; F. Briscese, E. Elizalde, S. Nojiri and S. D. Odintsov, Phys. Lett. B 646 (2007) 105; S. Nojiri, S.D. Odintsov and H. Stefancic, Phys. Rev. D 74, (2006) 086009; M. E. Soussa and R. P. Woodard, Gen. Rel. Grav. 36 (2004) 855; R. Dick, Gen. Rel. Grav. 36 (2004) 217; A. E. Dominguez and D. E. Barraco, Phys. Rev. D 70, (2004) 043505; V. Faraoni, Phys. Rev. D 75 (2007) 067302; D. A. Easson, Int. J. Mod. Phys. A 19 (2004) 5343; G. J. Olmo, Phys. Rev. Lett. 95 (2005) 261102; G. Allemandi, M. Francaviglia, M. L. Ruggiero and A. Tartaglia, Gen. Rel. Grav. 37 (2005) 1891; S. Capozziello and A. Troisi, Phys. Rev. D 72 (2005) 044022; T. Clifton and J.D. Barrow, Phys. Rev. D 72 (2005) 103005; T. P. Sotiriou, Gen. Rel. Grav. 38 (2006) 1407; S. Capozziello, A. Stabile and A. Troisi, Mod. Phys. Lett. A 21 (2006) 2291; A. Dolgov and D. N. Pelliccia, Nucl. Phys. B 734 (2006) 208; S. Nojiri and S. D. Odintsov, Int. J. Geom. Meth. Mod. Phys. 4 (2007) 115.

[2] M. Akbar and Rong-Gen Cai, Phys. Lett. B635 (2006) 7; M. Akbar and Rong-Gen Cai, Phys. Lett. B648 (2007) 243; J. C. C. de Souza and V. Faraoni, Class. Quant. Grav. 24 (2007) 3637; K. Atazadeh and H.R. Sepangi, Int. J. Mod. Phys. D 16 (2007) 687; K. Atazadeh, M. Farhoudi and H. R. Sepangi, Phys. Lett. B 660 (2008) 275; S. Nojiri and S.D. Odintsov, Phys. Rev. D 78 (2008) 046006; G. Cognola,
E. Elizalde, S. Nojiri, S.D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D77 (2008) 046009; K. Bamba and S.D. Odintsov, JCAP 0804 (2008) 024.

[3] S. Nojiri and S.D. Odintsov, Phys. Rev. D68 (2003) 123512; S. M. Carroll, V. Duvvuri, M. Trodden and M. S. Turner, Phys. Rev. D70 (2004) 043528.

[4] D. Kastor and R. B. Mann, JHEP 0604 (2006) 048; M. H. Dehghani and S. H. Hendi, Phys. Rev. D 73 (2006) 084021; R. G. Cai and N. Ohta, Phys. Rev. D74 (2006) 064001; M. H. Dehghani and S. H. Hendi, Int. J. Mod. Phys. D 16 (2007) 1829; M. H. Dehghani, N. Alinejad and S. H. Hendi, Phys. Rev. D 77 (2008) 104025; S. H. Hendi and M. H. Dehghani, Phys. Lett. B 666 (2008) 116; M. H. Dehghani, N. Bostani and S. H. Hendi, Phys. Rev. D 78 (2008) 064031; R. G. Cai, L. M. Cao, Y. P. Hu and S. P. Kim, Phys. Rev. D78 (2008) 124012; S. H. Mazharimousavi and M. Halilsoy, Phys. Lett. B665 (2008) 125; Q. Exirifard and M. M. Sheikh-Jabbari, Phys. Lett. B661 (2008) 158; G. A. S. Dias, S. Gao and J. P. S. Lemos, Phys. Rev. D75 (2007) 024030; M. Farhoudi, Gen. Rel. Grav.41 (2009) 117.

[5] I. Zlatev, L. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82 (1999) 896; P. J. Steinhardt, L. Wang, I. Zlatev, Phys. Rev. D 59 (1999) 123504; M. S. Turner, Int. J. Mod. Phys. A 17S1 (2002) 180; V. Sahni, Class. Quant. Grav. 19 (2002) 3435.

[6] R. R. Caldwell, M. Kamionkowski, N. N. Weinberg, Phys. Rev. Lett. 91 (2003) 071301; R. R. Caldwell, Phys. Lett. B 545 (2002) 23; P. Singh, M. Sami, N. Dadhich, Phys. Rev. D 68 (2003) 023522; J. G. Hao, X. Z. Li, Phys. Rev. D 67 (2003) 107303.

[7] C. Armendariz-Picon, T. Damour and V. Mukhanov, Phys. Lett. B 458 (1999) 209; M. Malquarti, E. J. Copeland, A. R. Liddle and M. Trodden, Phys. Rev. D 67 (2003) 123503; T. Chiba, Phys. Rev. D 66 (2002) 023514.

[8] T. P. Sotiriou and V. Faraoni, to appear in Rev. Mod. Phys., [arXiv:0805.1726]; S. Nojiri and S.D. Odintsov, [arXiv:0807.0685]; V. Faraoni, [arXiv:0810.2602]; N. Straumann, [arXiv:0809.5148].

[9] R. P. Woodard, Lect. Notes Phys. 720 (2007) 403.

[10] I. L. Buchbinder, S. D. Odintsov and I. L. Shapiro, Effective Actions in Quantum Gravity (IOP Publishing, Bristol (1992)); M. Ferraris, M. Francoaviglia and G. Magnano, Class. Quant. Grav. 5 (1988) L95; K. S. Stelle, Gen. Rel. Grav. 9 (1978) 353; K. S. Stelle, Phys. Rev. D16 (1977) 953; A. Strominger, Phys. Rev. D30 (1984) 2257; R. Utiyama and B. S. DeWitt, J. Math. Phys. 3 1962 608; G. A. Vilkovisky, Class. Quant. Grav. 9 (1992) 895.

[11] A. D. Dolgov and M. Kawasaki, Phys. Lett. B573 (2003) 1; V. Faraoni, Phys. Rev. D74 (2006) 104017; S. Nojiri and S.D. Odintsov, Phys. Lett. B652 (2007) 343; S. Nojiri and S.D. Odintsov, Phys. Lett. B657 (2007) 238.

[12] U. Gunther, A. Starobinsky and A. Zhuk, Phys. Rev. D69 (2004) 044003.

[13] M. Eastwood and M. Singer, Phys. Lett. A 107 (1985) 73.

[14] M. Hassaine and C. Martinez, Phys. Rev. D 75 (2007) 027502.

[15] M. Hassaine and C. Martinez, Class. Quant. Grav. 25 (2008) 195023.

[16] H. Maeda, M. Hassaine and C. Martinez, Phys. Rev. D 79 (2009) 044012.
[17] S. H. Hendi and H. R. Rastegar-Sedehi, Gen. Rel. Grav., 41 (2009) 1355.

[18] S. H. Hendi, Phys. Lett. B 677 (2009) 123.