Characteristic Lie Algebra and Classification of Semi-Discrete Models

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Abstract

Characteristic Lie algebras of semi-discrete chains are studied. The attempt to adopt this notion to the classification of Darboux integrable chains has been undertaken.

1 Introduction

Investigation of the class of hyperbolic type differential equations of the form

\[ u_{xy} = f(x, y, u, u_x, u_y) \] (1)

has a very long history. Various approaches have been developed to look for particular and general solutions of these kind equations. In the literature one can find several definitions of integrability of the equation. According to one given by G. Darboux, equation (1) is called integrable if it is reduced to a pair of ordinary (generally nonlinear) differential equations, or more exactly, if its any solution satisfies the equations of the form [1], (see also [2])

\[ F(x, y, u, u_x, u_{xx}, \ldots, D_x^m u) = a(x), \quad G(x, y, u, u_y, u_{yy}, \ldots, D_y^n u) = b(y), \] (2)

for appropriately chosen functional parameters \(a(x)\) and \(b(y)\), where \(D_x\) and \(D_y\) are operators of differentiation with respect to \(x\) and \(y\), \(u_x = D_x u, u_{xx} = D_x u_x\) and so on. Functions \(F\) and \(G\) are called \(x\)- and \(y\)-integrals of the equation respectively.

An effective criterion of Darboux integrability has been proposed by G. Darboux himself. Equation (1) is integrable if and only if the Laplace sequence of the linearized equation terminates at both ends. A rigorous proof of this statement has been found only recently [3], [4].

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An alternative method of investigation and classification of the Darboux integrable equations has been developed by A. B. Shabat based on the notion of characteristic Lie algebra. Let us give a brief explanation of this notion. Begin with the basic property of the integrals. Evidently each $y$-integral satisfies the condition: $D_y F(x, y, u, u_x, u_{xx}, \ldots, D^m_x u) = 0$. Taking the derivative by applying the chain rule one defines a vector field $X_1$ such that

$$X_1 F = \left( \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + f \frac{\partial}{\partial u_x} + D_x(f) \frac{\partial}{\partial u_{xx}} + \ldots \right) F = 0.$$  

(3)

So the vector field $X_1$ solves the equation $X_1 F = 0$. But in general, the coefficients of the vector field depend on the variable $u_y$ while the solution $F$ does not. This puts a severe restriction on $F$, actually $F$ should satisfy one more equation $X_2 F = 0$, where $X_2 = \frac{\partial}{\partial u_y}$. Now the commutator of these two operators will also annulate $F$. Moreover, for any $X$ from the Lie algebra generated by $X_1$ and $X_2$ one gets $XF = 0$. This Lie algebra is called characteristic Lie algebra of the equation (1) in the direction of $y$. Characteristic algebra in the $x$-direction is defined in a similar way. Now by virtue of the famous Jacobi theorem, equation (1) is Darboux integrable if and only if both of its characteristic algebras are of finite dimension. In [5] and [6], the characteristic Lie algebras for the systems of nonlinear hyperbolic equations and their applications are studied.

In this article we will study semi-discrete chains of the following form

$$t_{1x} = f(t, t_1, t_x)$$  

(4)

from the Darboux integrability point of view. Here the unknown $t = t(n, x)$ is a function of two independent variables: one discrete $n$ and one continuous $x$. It is assumed that $\frac{\partial f}{\partial t_x} \neq 0$. Subindex means shift or derivative, for instance, $t_1 = t(n + 1, x)$ and $t_x = \frac{\partial}{\partial x} t(n, x)$. Below we use $D$ to denote the shift operator and $D_x$ to denote the $x$-derivative: $Dh(n, x) = h(n + 1, x)$ and $D_x h(n, x) = \frac{\partial}{\partial x} h(n, x)$. For the iterated shifts use the subindex $D^j h = h_j$.

Introduce now notions of the integrals for the semi-discrete chain (1). The $x$-integral is defined similar to the continuous case. We call a function $F = F(x, n, t, t_1, t_2, \ldots)$ depending on a finite number of shifts $x$-integral of the chain (1), if the following condition is valid $D_x F = 0$. It is natural, in accordance with the continuous case, to call a function $I = I(x, n, t, t_x, t_{xx}, \ldots)$ $n$-integral of the chain (1) if it is in the kernel of the difference operator: $(D - 1) I = 0$. In other words $n$-integral should still unchanged under the
action of the shift operator $DI = I$, (see also [7]). One can write it in an enlarged form

$$I(x, n + 1, t_1, f, f_x, f_{xx}, ...) = I(x, n, t, t_x, t_{xx}, ...). \quad (5)$$

Notice that it is a functional equation, the unknown is taken at two different "points". This circumstance causes the main difficulty in studying discrete chains. Such kind problems appear when one tries to apply the symmetry approach to discrete equations (see [8], [9]). However the concept of the Lie algebra of characteristic vector fields provides an effective tool to investigate chains.

Introduce vector fields in the following way. Concentrate on the main equation (5). Evidently the left hand side of it contains the variable $t_1$ while the right hand side does not. Hence the total derivative of the function $DI$ with respect to $t_1$ should vanish. In other words the $n$-integral is in the kernel of the operator $Y_1 := D^{-1} \frac{\partial}{\partial t_1} D$. Similarly one can check that $I$ is in the kernel of the operator $Y_2 := D^{-2} \frac{\partial}{\partial t_1} D^2$. Really, the right hand side of the equation $D^2 I = I$ which immediately follows from (5) does not depend on $t_1$, therefore the derivative of the function $D^2 I$ with respect to $t_1$ vanishes. Proceeding this way one can easily prove that for any natural $j$ the operator $Y_j = D^{-j} \frac{\partial}{\partial t_1} D^j$ solves the equation $Y_j I = 0$.

So far we shifted the argument $n$ forward, shift it now backward and use the main equation (5) written as $D^{-1} I = I$. Rewrite the original equation (4) in the form

$$t_{-1x} = g(t, t_{-1}, t_x). \quad (6)$$

This can be done because of the condition $\frac{\partial f}{\partial t_x} \neq 0$ assumed above. In the enlarged form the equation $D^{-1} I = I$ looks like

$$I(x, n - 1, t_{-1}, g, g_x, g_{xx}, ...) = I(x, n, t, t_x, t_{xx}, ...). \quad (7)$$

The right of the last equation does not depend on $t_{-1}$ so the total derivative of $D^{-1} I$ with respect to $t_{-1}$ is zero, i.e. the operator $Y_{-1} := D \frac{\partial}{\partial t_{-1}} D^{-1}$ solves the equation $Y_{-1} I = 0$. Moreover, the operators $Y_{-j} := D^j \frac{\partial}{\partial t_{-1}} D^{-j}$ also satisfy similar conditions $Y_{-j} I = 0$.

Summarizing the reasonings above one can conclude that the $n$-integral is annulled by any operator from the Lie algebra $\tilde{L}_n$ generated by the operators [10]

$$..., Y_{-2}, Y_{-1}, Y_0, Y_1, Y_2, ... \quad (8)$$
where \( Y_0 = \frac{\partial}{\partial t_1} \) and \( Y_{-0} = \frac{\partial}{\partial t_{-1}} \). The algebra \( \tilde{L}_n \) consists of the operators from the sequence \( (S) \), all possible commutators, and linear combinations with coefficients depending on the variables \( n \) and \( x \). Evidently equation (4) admits a nontrivial \( n \)-integral only if the dimension of the algebra \( \tilde{L}_n \) is finite. But it is not clear that the finiteness of dimension \( \tilde{L}_n \) enough for existence of \( n \)-integrals. By this reason we introduce another Lie algebra called the characteristic Lie algebra of the equation (4). First we define in addition to the operators \( Y_1, Y_2, ... \) differential operators \( X_j = \frac{\partial}{\partial t_{-j}} \) for \( j = 1, 2, ... \).

The following theorem allows us to define the characteristic Lie algebra.

**Theorem 1.1.** Equation (4) admits a nontrivial \( n \)-integral if and only if the following two conditions hold:
1) Linear envelope of the operators \( \{Y_j\}_1^{\infty} \) is of finite dimension, denote this dimension \( N \);
2) Lie algebra \( L_n \) generated by the operators \( Y_1, Y_2, ..., Y_N, X_1, X_2, ..., X_N \) is of finite dimension. We call \( L_n \) the characteristic Lie algebra of (4).

**Remark 1.2.** It is easy to prove that if dimension of \( \{Y_j\}_1^{\infty} \) is \( N \) then the set \( \{Y_j\}_1^N \) constitute a basis in the linear envelope of \( \{Y_j\}_1^{\infty} \).

## 2 Characteristic Lie Algebra \( L_n \)

Study some properties of the characteristic Lie algebra introduced in the theorem 1.1 above. Begin with the proof of the remark 1.2. It will immediately follow from the following lemma.

**Lemma 2.1.** If for some integer \( N \) the operator \( Y_{N+1} \) is a linear combination of the operators with less indices:

\[
Y_{N+1} = \alpha_1 Y_1 + \alpha_2 Y_2 + ... + \alpha_N Y_N \tag{9}
\]

then for any integer \( j > N \), we have a similar expression

\[
Y_j = \beta_1 Y_1 + \beta_2 Y_2 + ... + \beta_N Y_N. \tag{10}
\]

**Proof.** Due to the property \( Y_{k+1} = D^{-1} Y_k D \), we have from (9)

\[
Y_{N+2} = D^{-1}(\alpha_1)Y_2 + D^{-1}(\alpha_2)Y_3 + ... + D^{-1}(\alpha_N)(\alpha_1 Y_1 + ... + \alpha_N Y_N). \tag{11}
\]

Now by using induction one can easily complete the proof of the lemma.
Lemma 2.2. The following commutativity relations take place:

\[ [Y_0, Y_{-0}] = 0, \quad [Y_0, Y_1] = 0, \quad [Y_{-0}, Y_{-1}] = 0. \]

Proof. The first of the relations is evident. In order to prove two others find the coordinate representation of the operators \( Y_1 \) and \( Y_{-1} \) acting in the class of locally smooth functions of the variables \( x, n, t, t_x, t_{xx}, \ldots \). By direct computations

\[
Y_1 I = D^{-1} \frac{d}{dt_1} DI
\]

\[
= D^{-1} \frac{d}{dt_1} I(t_1, f, f_x, \ldots)
\]

\[
= \left\{ \frac{\partial}{\partial t} + D^{-1} \frac{\partial f}{\partial t_1} \frac{\partial}{\partial t_x} + D^{-1} \frac{\partial f_x}{\partial t_1} \frac{\partial}{\partial t_{xx}} + \ldots \right\} I(t, t_x, t_{xx}, \ldots)
\]

(12)

one gets

\[
Y_1 = \frac{\partial}{\partial t} + D^{-1} \left( \frac{\partial f}{\partial t_1} \right) \frac{\partial}{\partial t_x} + D^{-1} \left( \frac{\partial f_x}{\partial t_1} \right) \frac{\partial}{\partial t_{xx}} + \ldots
\]

(13)

Now notice that all of the functions \( f, f_x, f_{xx}, \ldots \) depend on the variables \( t_1, t, t_x, t_{xx}, \ldots \) and do not depend on \( t_2 \) hence the coefficients of the vector field \( Y_1 \) do not depend on \( t_1 \) and therefore the operators \( Y_1 \) and \( Y_0 \) commute.

In a similar way by using the explicit coordinate representation

\[
Y_{-1} = \frac{\partial}{\partial t} + D \left( \frac{\partial g}{\partial t_{-1}} \right) \frac{\partial}{\partial t_x} + D \left( \frac{\partial g_x}{\partial t_{-1}} \right) \frac{\partial}{\partial t_{xx}} + \ldots
\]

(14)

one can prove that \([Y_{-0}, Y_{-1}] = 0\).

The following statement turned out to be very useful for studying the characteristic Lie algebra \( L_n \).

Lemma 2.3. Suppose that the vector field

\[
Y = \alpha(0) \partial_t + \alpha(1) \partial_{t_x} + \alpha(2) \partial_{t_{xx}} + \ldots,
\]

(15)

where \( \alpha_x(0) = 0 \) solves the equation \([D_x, Y] = 0\), then \( Y = \alpha(0) \partial_t \).
The proof is based on the following formula

\[ [D_x, Y] = (\alpha_x(0) - \alpha(1))\partial_t + (\alpha_x(1) - \alpha(2))\partial_{tx} + \ldots \]  \hfill (16)

So if \( a_x(0) = 0 \), then \( a(1) = 0 \), but if \( a_x(1) = 0 \) then \( a(2) = 0 \) and hence \( a(j) = 0 \) for all \( j > 0 \).

In the formula (12) we have already given an enlarged coordinate form of the operator \( Y_1 \). One can check that the operator \( Y_2 \) is a vector field of the form

\[ Y_2 = D^{-1}(Y_1(f))\partial_t + D^{-1}(Y_1(f_x))\partial_{tx} + D^{-1}(Y_1(f_{xx}))\partial_{txx} + \ldots \]  \hfill (17)

It immediately follows from the equation \( Y_2 = D^{-1}Y_1D \) and the coordinate representation (12). By induction one can prove similar formulas for arbitrary \( j \):

\[ Y_{j+1} = D^{-1}(Y_j(f))\partial_t + D^{-1}(Y_j(f_x))\partial_{tx} + D^{-1}(Y_j(f_{xx}))\partial_{txx} + \ldots \]  \hfill (18)

**Lemma 2.4.** For the operators \( D_x, Y_1, Y_{-1} \) considered on the space of smooth functions of \( t, t_x, t_{xx}, \ldots \) the following commutativity relations take place:

\[ [D_x, Y_1] = pY_1 \quad , \quad [D_x, Y_{-1}] = qY_{-1}, \]

where \( p = -D^{-1}(\partial f / \partial t_1) \) and \( q = -D(\partial g / \partial t_{-1}) \).

**Proof.** Recall that

\[ Y_1 = \frac{\partial}{\partial t} + D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\frac{\partial}{\partial t_x} + D^{-1}\left(\frac{\partial f_x}{\partial t_1}\right)\frac{\partial}{\partial t_{xx}} + \ldots \]  \hfill (19)

We find \([D_x, Y_1]\) by using (16) as

\[ [D_x, Y_1] = -D^{-1}(f_{t_1})\partial_t + D^{-1}(D_x(f_{t_1}) - f_{xt_1})\partial_{tx} + \ldots \]  \hfill (20)

For arbitrary function \( H \), we have

\[ [D_x, \partial_{t_1}]H(t, t_1, t_x, t_{xx}, \ldots) = D_xH_{t_1} - \frac{\partial}{\partial t_1}D_xH \]

\[ = (H_{tt_1}t_x + H_{t_1t_1}t_{1x} + \ldots) - \frac{\partial}{\partial t_1}(H_{tx}t_x + H_{t_1t_1}t_{1x} + \ldots) \]

\[ = -H_{t_1}f_{t_1}. \]  \hfill (21)
By taking \( H = f \) and \( H = f_x \), one gets \([D_x, \partial_t]f = -f_t f_t, [D_x, \partial_t]f_x = -f_{xt} f_t, \) and so on. We insert these equations into (20) to find

\[
[D_x, Y_1] = -D^{-1}\left(\frac{\partial f}{\partial t_1}\right)\left\{ \frac{\partial}{\partial t} + D^{-1}\left(\frac{\partial f}{\partial t_1}\right) \frac{\partial}{\partial t_x} + D^{-1}\left(\frac{\partial f_x}{\partial t_1}\right) \frac{\partial}{\partial t_x} + \ldots \right\} = -D^{-1}\left(\frac{\partial f}{\partial t_1}\right) Y_1. \tag{22}
\]

In a similar way one can prove that \([D_x, Y_{-1}] = -D\left(\frac{\partial g}{\partial t_{-1}}\right) Y_{-1}. \]

Let us prove theorem 1.1. Suppose that there exists a nontrivial \( n \)-integral \( F = F(t, t_x, \ldots, t_{[N]}) \) for the equation \( \Pi \), here \( t_{[j]} = D_j^j t \) for any \( j \geq 0 \). Then all the vector fields from the Lie algebra \( M \) generated by \( \{Y_j, X_k\} \) for \( j = 1, 2, \ldots \) and \( k = 1, \ldots, N_2 \) where \( N_2 \) is chosen arbitrarily satisfying \( N_2 \geq N \) annihilate \( F \). We will show that \( \dim M < \infty \). Consider first the projection of the algebra \( M \) given by the operator \( P_N \):

\[
P_N\left( \sum_{i=-N_2}^{-1} x(i) \partial_{t_i} + \sum_{i=0}^{\infty} x(i) \partial_{t_{[i]}} \right) = \sum_{i=-N_2}^{-1} x(i) \partial_{t_i} + \sum_{i=0}^{N} x(i) \partial_{t_{[i]}}. \tag{23}
\]

Let \( L_n(N) \) be the projection of \( M \). Then evidently the equation \( Z_0 F = 0 \) is satisfied for any \( Z_0 \) in \( L_n(N) \). Evidently, \( \dim L_n(N) < \infty \). Let the set \( \{Z_{01}, Z_{02}, \ldots, Z_{0N_1}\} \) forms a basis in \( L_n(N) \). One can represent any \( Z_0 \) in \( L_n(N) \) as a linear combination

\[
Z_0 = \alpha_1 Z_{01} + \alpha_2 Z_{02} + \ldots + \alpha_{N_1} Z_{0N_1}. \tag{24}
\]

Suppose that the vector fields \( Z, Z_1, \ldots, Z_{N_1} \) in \( M \) are connected with the operators \( Z_0, Z_{01}, \ldots, Z_{0N_1} \) in \( L_n(N) \) by the formulas \( P_N(Z) = Z_0, P_N(Z_1) = Z_{01}, \ldots, P_N(Z_{N_1}) = Z_{0N_1} \). We have to prove that

\[
Z = \alpha_1 Z_1 + \alpha_2 Z_2 + \ldots + \alpha_{N_1} Z_{N_1}. \tag{25}
\]

In the proof, we use the following lemma.

**Lemma 2.5.** Let \( F_1 = D_x F \) and \( F \) is an \( n \)-integral. Then for each \( Z \) in \( M \) we have \( ZF_1 = 0 \).

**Proof.** It is easy to check that \( F_1 \) is also an \( n \)-integral, really \( DF_1 = DD_x F = D_x DF = D_x F = F_1 \). It was shown above that any \( Z \) in \( M \) annihilates \( n \)-integrals.
Apply the operator \( Z - \alpha_1 Z_1 - \alpha_2 Z_2 - ... - \alpha_N Z_N \) to the function \( F_1 = F_1(t, t_x, t_{xx}, ..., t_{[N+1]}) \),

\[
(Z - \alpha_1 Z_1 - \alpha_2 Z_2 - ... - \alpha_N Z_N) F_1 = 0. \tag{26}
\]

We can write (26) as

\[
(Z_0 - \alpha_1 Z_{01} - \alpha_2 Z_{02} - ... - \alpha_N Z_{0N}) F_1 + (X(N + 1) - \alpha_1 X_1(N + 1) - \alpha_2 X_2(N + 1) - ... - \alpha_N X_N(N + 1)) \frac{\partial}{\partial t_{[N+1]}} F_1 = 0,
\tag{27}
\]

where \( X(N + 1), X_1(N + 1), ..., X_N(N + 1) \) are the coefficients before \( \frac{\partial}{\partial t_{[N+1]}} \) of the vector fields \( Z, Z_1, Z_2, ..., Z_N \). The first summand in (27) vanishes by (24). In the second one the factor \( \frac{\partial}{\partial t_{[N+1]}} F_1 = \frac{\partial}{\partial t_{[N]}} F \) is not zero. So we have

\[
X(N + 1) = \alpha_1 X_1(N + 1) + \alpha_2 X_2(N + 1) + ... + \alpha_N X_N(N + 1). \tag{28}
\]

Equation (28) shows that

\[
P_{N+1}(Z) = \alpha_1 P_{N+1}(Z_1) + \alpha_2 P_{N+1}(Z_2) + ... + \alpha_N P_{N+1}(Z_N). \tag{29}
\]

So by applying mathematical induction, one can prove the formula (25). Thus the Lie algebra \( M \) is of finite dimension. Now construct the characteristic algebra \( L_n \) by using \( M \). Since \( \dim M < \infty \), the linear envelope of the vector fields \( \{Y_j\}_1^\infty \) is of finite dimension. Choose a basis in this linear space consisting of \( Y_1, Y_2, ..., Y_K \) for \( K \leq N \leq N_2 \). Then the algebra generated by \( Y_1, Y_2, ..., Y_K, X_1, X_2, ..., X_K \) is of finite dimension, because it is a subalgebra of \( M \). This algebra is just characteristic Lie algebra of the equation (I).

Suppose that conditions (1) and (2) of the theorem are satisfied. So there exists a finite dimensional characteristic Lie algebra \( L_n \) for the equation (I). Show that in this case equation (I) admits a nontrivial \( n \)-integral. Let \( N_1 \) be the dimension of \( L_n \) and \( N \) is the dimension of the linear envelope of the vector fields \( \{Y_j\}_j^\infty \). Take the projection \( L_n(N_2) \) of the Lie algebra \( L_n \) defined by the operator \( P_{N_2} \) defined by the formula (23) with \( N_2 \) instead of \( N \).

Evidently, \( L_n(N_2) \) consists of the finite sums \( Z_0 = \sum_{i=-N}^{-1} x(i) \partial_{t_i} + \sum_{i=0}^{N_2} x(i) \partial_{t_i[1]} \) where \( N = N_1 - N_2 \). Let \( Z_01, ..., Z_{0N_1} \) form a basis in \( L_n(N_2) \). Then we have
\( N_1 = N + N_2 \) equations \( Z_0 G = 0, j = 1, \ldots, N_1 \), for a function \( G \) depending on \( N + N_2 + 1 = N_1 + 1 \) independent variables. Then due to the well-known Jacobi theorem, there exists a function \( G = G(t_{-N}, t_{-N+1}, \ldots, t_{-1}, t, t_x, t_{xx}, \ldots, t_{[N_2]}) \), which satisfies the equation \( ZG = 0 \) for any \( Z \) in \( L_n \). But really it does not depend on \( t_{-N}, \ldots, t_{-1} \) because \( X_1 G = 0, X_2 G = 0, \ldots, X_N G = 0 \). Thus the function \( G \) is \( G = G(t, t_x, t_{xx}, \ldots, t_{[N_2]}) \). Such a function is not unique but any other solution of these equations, depending on the same set of the variables, can be represented as \( h(G) \) for some function \( h \).

Note one more property of the algebra \( L_n \). Let \( \pi \) be a map which assigns to each \( Z \) in \( L_n \) its conjugation \( D^{-1} Z D \). Evidently, the map \( \pi \) acts from the algebra \( L_n \) into its central extension \( L_n \oplus \{ X_{N+1} \} \), because for the generators of \( L_n \) we have \( D^{-1} Y_j D = Y_{j+1} \) and \( D^{-1} X_j D = X_{j+1} \). Evidently, \( [X_{N+1}, Y_j] = 0 \) and \( [X_{N+1}, X_j] = 0 \) for any integer \( j \leq N \). Moreover \( X_{N+1} F = 0 \) for the function \( G = G(t, t_x, \ldots, t_{[N_2]}) \) mentioned above. This fact implies that \( ZG_1 = 0 \) for \( G_1 = DG \) and for any vector field \( Z \) in \( L_n \). Really, for any \( Z \) in \( L_n \) one has a representation of the form \( D^{-1} ZD = \bar{Z} + \lambda X_{N+1} \) where \( \bar{Z} \) in \( L_n \) and \( \lambda \) is a function. So

\[
ZG_1 = ZDG = D(D^{-1} Z DG) = D(\bar{Z} + \lambda X_{N+1})G = 0. \tag{30}
\]

Therefore \( G_1 = h(G) \) or \( DG = h(G) \). In other words function \( G = G(n) \) satisfies an ordinary difference equation of the first order. Its general solution can be written as \( G = H(n, c) \) where \( H \) is a function of two variables and \( c \) is an arbitrary constant. By solving the equation \( G = H(n, c) \) with respect to \( c \) one gets \( c = F(G, n) \). The function \( F = F(G, n) \) found is just \( n \)-integral searched. Actually, \( DF(G, n) = Dc = c = F(G, n) \). So \( DF = F \). This completes the proof of the theorem 1.1.

## 3 Restricted classification

Different approaches are known to classify integrable nonlinear differential (pseudo-differential) equations. One of the most popular and powerful is based on higher symmetries. For the first time the theoretical aspects of this method have been formulated in the famous paper by N. Kh. Ibragimov and A. B. Shabat [11]. Several classes of nonlinear models were tested by this method in [12], [13]. The symmetry approach allowed R. I. Yamilov to find all integrable chains of the Volterra type [14]: \( u_t(n) = f(u(n-1), u(n), u(n+1)) \). The consistency approach to the classification of integrable discrete equations
has been studied by Adler, Bobenko and Suris in [15]. Classification based on the notion of the recursion operator is studied in [16]-[19].

In this paper we undertake an attempt to adopt the notion of characteristic Lie algebra to the problem of classification of Darboux integrable discrete equations of the form (14). The classification problem consists of describing of all chains admitting finite dimensional characteristic Lie algebras in both directions. Actually the problem of studying the algebra generated by the operators (3) seems to be quite difficult. That is why we will start with a very simple case.

Formulation of the problem. Study the problem of finding all of the equations (4) for which the Lie algebra generated by the operators $Y_1$ and $Y_{-1}$ is two dimensional.

Let us denote $Y_{1,-1} = [Y_1, Y_{-1}]$. We require that the following relation takes place $Y_{1,-1} = \lambda Y_1 + \mu Y_{-1}$. It follows from the explicit formulas (13) and (14) that the vector field $Y_{1,-1}$ does not contain a summand with the term $\frac{\partial}{\partial t}$ hence $\mu = -\lambda$. The commutators of the basic vector fields with the operator of the total derivative admit simple expressions (see lemma 2.4 above). Evaluate the commutator $[D_x, Y_{1,-1}]

\begin{align*}
[D_x, Y_{1,-1}] & = [Y_1, [D_x, Y_{1,-1}]] - [Y_{-1}, [D_x, Y_1]] \\
& = [Y_1, qY_{-1}] - [Y_{-1}, pY_1] \\
& = Y_1(q)Y_{-1} + qY_{1,-1} - Y_{-1}(p)Y_1 + pY_{1,-1} \\
& = (p + q)Y_{1,-1} + Y_1(q)Y_{-1} - Y_{-1}(p)Y_1.
\end{align*}

Remind that due to the reasoning above a coefficient $\lambda = \lambda(n, x)$ should exist such that

\begin{equation}
Y_{1,-1} = \lambda(Y_1 - Y_{-1}).
\end{equation}

The problem is in finding of $f$ in the equation $t_{1x} = f(t, t_1, t_x)$ for which the constraint (31) is valid.

Commute each side of the equation (31) with the operator $D_x$

\begin{align*}
[D_x, Y_{1,-1}] & = [D_x, \lambda Y_1] - [D_x, \lambda Y_{-1}] \\
& = (p + q)\lambda(Y_1 - Y_{-1}) + Y_1(q)Y_{-1} - Y_{-1}(p)Y_1 \\
& = D_x(\lambda)Y_1 + \lambda pY_1 - D_x(\lambda)Y_{-1} - \lambda qY_{-1}.
\end{align*}

Compare now two different expressions for the commutator. This gives rise to conditions

\begin{align*}
q\lambda - Y_{-1}(p) = D_x(\lambda) & , & p\lambda - Y_1(q) = D_x(\lambda),
\end{align*}
which form an over-determined system of equations for unknown $\lambda$, it should satisfy simultaneously two equations. By solving them with respect to $\lambda$ and $D_x(\lambda)$ we obtain equations

$$
\lambda = \frac{Y_{-1}(p) - Y_1(q)}{q - p}, \quad D_x(\lambda) = \frac{qY_1(q) - pY_{-1}(p)}{p - q}
$$

(32)

which immediately yield

$$
D_x\left(\frac{Y_{-1}(p) - Y_1(q)}{q - p}\right) = \frac{pY_{-1}(p) - qY_1(q)}{q - p}.
$$

(33)

First of all note that this equation contains both $f$ and its inverse $g$. Exclude $g$. Recall that $t_{1x} = f(t, t_1, t_x)$ and $t_x = f(t_{-1}, t, t_{1x})$ where $t_{-1x} = g(t, t_{-1}, t_x)$. Differentiating the identity $t_x = f(t_{-1}, t, g(t, t_{-1}, t_x))$ with respect to $t_{-1}$ one gets

$$
D^{-1}\left(\frac{\partial f}{\partial t}(t, t_1, t_x)\right) + D^{-1}\left(\frac{\partial f}{\partial t_x}(t, t_1, t_x)\right) \frac{\partial g}{\partial t_{-1}} = 0
$$

(34)

which implies that

$$
g_{t_{-1}} = -D^{-1}\left(\frac{f_t}{f_{tx}}\right),
$$

(35)

so $D(g_{t_{-1}}) = -\frac{f_t}{f_{tx}}$. Let us write the equation (33) explicitly. Evaluate first $Y_1(q)$ and $Y_{-1}(p)$, where $p = -D^{-1}(f_{t_1})$ and $q = \frac{f_t}{f_{tx}}$.

$$
Y_1(q) = \left\{ \partial_t + D^{-1}(f_{t_1})\partial_{t_x} + D^{-1}(f_{x_1})\partial_{t_{xx}} + \ldots \right\} \frac{f_t}{f_{tx}}
$$

$$
= \left(\frac{f_t}{f_{tx}}\right) t + D^{-1}(f_{t_1})\left(\frac{f_t}{f_{tx}}\right) t_x.
$$

(36)

$$
Y_{-1}(p) = -\left\{ \partial_t - \frac{f_t}{f_{tx}}\partial_{t_x} - D\left(\frac{\partial g_x}{\partial t_{-1}}\right)\partial_{t_{xx}} - \ldots \right\} D^{-1}(f_{t_1})
$$

$$
= -\left(D^{-1}(f_{t_1})\right) t + \frac{f_t}{f_{tx}}\left(D^{-1}(f_{t_1})\right) t_x.
$$

(37)
By inserting these equations into (33) one gets a long expression

\[
D_x \left\{ \frac{-(D^{-1}(f_{t_1}))_t + \frac{f_t}{f_{t_2}}(D^{-1}(f_{t_1}))_t}{\frac{f_t}{f_{t_2}} + D^{-1}(f_{t_1})} - \left( \frac{f_t}{f_{t_2}} + D^{-1}(f_{t_1}) \left( \frac{f_t}{f_{t_2}} \right)_t \right) \right\}
\]

\[
= \frac{D^{-1}(f_{t_1}) \left( (D^{-1}(f_{t_1}))_t - \frac{f_t}{f_{t_2}}(D^{-1}(f_{t_1}))_t \right) - \frac{f_t}{f_{t_2}} \left( \left( \frac{f_t}{f_{t_2}} \right)_t + D^{-1}(f_{t_1}) \left( \frac{f_t}{f_{t_2}} \right)_t \right)}{\frac{f_t}{f_{t_2}} + D^{-1}(f_{t_1})}
\]

(38)

Equation (38) is rather difficult to study and we put one more restriction on \( f \). Suppose that \( f = a(t) + b(t) + c(t_x) \). Find the variables \( p, q, Y_1(q), Y_{-1}(p) \) in terms of \( a, b, c \)

\[
p = -D^{-1}(f_{t_1}) = -b'(t),
\]

\[
q = -D(g_{t_{-1}}) = \frac{f_t}{f_{t_2}} = \frac{a'(t)}{c'(t_x)}.
\]

\[
Y_1(q) = (\partial_t + b'(t)\partial_{t_x}) \frac{a'(t)}{c'(t_x)} = \frac{a''(t)}{c'(t_x)} - \frac{b'(t)a'(t)c''(t_x)}{(c'(t_x))^2},
\]

\[
Y_{-1}(p) = (\partial_t - \frac{a'(t)}{c'(t_x)}\partial_{t_x}) (-b'(t)) = -b''(t).
\]

Substitute these expressions into (33)

\[
D_x G(t, t_x) = \frac{b'(t)b''(t) - \frac{a'(t)}{c'(t_x)} \left[ \frac{a''(t)}{c'(t_x)} - \frac{b'(t)a'(t)c''(t_x)}{(c'(t_x))^2} \right]}{\frac{a'(t)}{c'(t_x)} + b'(t)}
\]

(39)

where,

\[
G(t, t_x) = \left[ \frac{-b''(t) - \frac{a''(t)}{c'(t_x)} + \frac{b'(t)a'(t)c''(t_x)}{(c'(t_x))^2}}{\frac{a'(t)}{c'(t_x)} + b'(t)} \right].
\]

(40)

Evidently the left hand side of the equation (39) is of the form \( \frac{\partial G}{\partial t} t_x + \frac{\partial G}{\partial t_x} t_{xx} \) and it contains the variable \( t_{xx} \) while the right hand side does not contain it. This gives an additional constraint \( \frac{\partial G}{\partial t_x} = 0 \).
Investigation of the equation \((39)\) is tediously long. Thus we will only give the answers.

**Theorem 3.1.** If the equation \((4)\) with particular choice of \(f(t,t_1,t_x) = a(t) + b(t_1) + c(t_x)\) has the operators \(Y_1\) and \(Y_{-1}\) such that the Lie algebra generated by these two operators is two-dimensional then \(f(t,t_1,t_x)\) is of one of the forms:

1) \(f(t,t_1,t_x) = \gamma(t_x + \beta \sinh(\alpha t + \lambda)) + \beta \cosh t_1 + \eta,\)
2) \(f(t,t_1,t_x) = \gamma(t_x + \beta e^{\alpha t}) + \beta e^{\alpha t_1} + \eta,\)
3) \(f(t,t_1,t_x) = c(t_x) + \gamma t_1 + \beta,\)
4) \(f(t,t_1,t_x) = \gamma \ln |t_x| + \frac{1}{\gamma} \ln(e^t - e) + \beta,\)
5) \(f(t,t_1,t_x) = \gamma t_x^2 + \beta t_x + \alpha t + \eta,\)

where \(c(t_x)\) is an arbitrary function and \(\alpha, \beta, \gamma, \lambda, \eta\) are arbitrary constants.

Additionally, if in the cases 1), 2), 3), the corresponding characteristic Lie algebras are also two-dimensional then the equations are of the forms:

\[i)\] \(t_{1x} = t_x + \beta \sinh(t) + \beta \cosh(t_1),\)
\[ii)\] \(t_{1x} = t_x + e^t + e^{t_1},\)
\[iii)\] \(t_{1x} = t_x,\)
and they have the \(n\)-integrals respectively:

\[a)\] \(I = \frac{\beta^2}{2} \cosh^2 t - \beta t_x \cosh t + \frac{t^2}{2} + \beta t_x \sinh t - t_{xx} + \frac{\beta^2}{2} n,\)
\[b)\] \(I = \frac{e^{2t}}{2} + \frac{t^2}{2} - t_{xx},\)
\[c)\] \(I = t_{xx}.\)

Note that \(ii)\) has also \(x\)-integral as

\[F = e^{t_1-t} + e^{2t_1-t_2-t} + e^{t_1-t_2}.\]  \((41)\)

Thus, \(t_{1x} = t_x + e^t + e^{t_1}\) is a discrete analog of the Liouville equation. The proof of the theorem 3.1 is given in the next section.
4 The Proof of the Theorem 3.1

Here we consider the cases which satisfy the condition (39) and prove the theorem 3.1.

Case 1. $a' = 0$:

The condition (39) turns out to be

$$D_x \left( \frac{-b''}{b'} \right) = b''.$$  \hfill (42)

This gives us $b' = \gamma$ so $b(t_1) = \gamma t_1 + \beta$. There is no condition on $c(t_x)$. Hence $f$ becomes

$$f(t, t_1, t_x) = c(t_x) + \gamma t_1 + \beta,$$  \hfill (43)

where $\beta$ and $\gamma$ are arbitrary constants and $c(t_x)$ is an arbitrary function of $t_x$.

Case 2. $b' = 0$:

The condition (39) turns out to be

$$D_x \left( \frac{a''}{a'} \right) = \left( \frac{a'''a' - (a'')^2}{(a')^2} \right) t_x$$

$$= \frac{a''}{a'}.$$

Since only $c$ depends on $t_x$, we have $c' = \frac{\gamma}{t_x}$ which implies

$$c(t_x) = \gamma \ln |t_x| + \beta.$$  \hfill (44)

The remaining equation gives $a(t)$ as

$$a(t) = -\ln(-1 + e^{\lambda t + \lambda \sigma + \xi}).$$  \hfill (45)

Thus, in a simplified form $f$ is

$$f(t, t_1, t_x) = \gamma \ln |t_x| + \frac{1}{\gamma} \ln(e^t - e) + \beta,$$  \hfill (46)

where $\beta$ and $\gamma$ are arbitrary constants.
Case 3. $c''' = 0$:

Note that we should have
\[ \frac{\partial G}{\partial t_x}(t,t_x) = 0 \] (47)
in the condition (39) since there is no $t_x$ explicitly in the right hand side. If we expand (47) we obtain
\[ c'''c'[b'(a')^2+c'(b')^2a']-(c'')^2[b'(a')^2+2c'(b')^2a']+(c')^2c''[a''b'-a'b''] = 0. \] (48)

Under the case $c''' = 0$ we have $c'' = A$ and naturally $c' = At_x + B$ where $A$, $B$ are constants, we have
\[ A^2t_x^2[a''b' - a'b''] + 2At_x[B(a''b' - a'b'') - A(b')^2a'] + [-Ab'(a')^2 - 2AB(b')^2a' + B^2a''b' - B^2a'b''] = 0. \]
The coefficient of $t_x^2$ should vanish, so $a''b' - a'b'' = 0$. We should also have
\[ [B(a''b' - a'b'') - A(b')^2a'] = 0 \] (49)
from the coefficient of $t_x$. Since $a''b' - a'b'' = 0$, the equation becomes
\[ A(b')^2a' = 0. \] (50)
To provide this equation to be satisfied we have three choices.

**Subcase 3.i. $A = 0$:**

In this case $c'' = 0$ and so $c' = B$ where $B$ is constant. The condition (39) becomes
\[ D_x \left[ \frac{a'' + Bb''}{a' + Bb'} \right] = \frac{a'a'' - Bbb''}{a' + Bb'}. \] (51)
Hence
\[ \frac{a'' + Bb''}{a' + Bb'} = \alpha, \] (52)
where $\alpha$ is constant. Thus we have two unknowns and two equations which are
\[ a' = \sigma e^{\alpha t} - Bb', \] (53)
and

\[(a')^2 = B^2(b')^2 + \rho,\]  \hspace{1cm} (54)

where \(\sigma\) and \(\rho\) are constants. We take square of the first equation and subtract from the second one. We get

\[\sigma^2 e^{2\alpha t} - 2\sigma Bb'e^{\alpha t} - \rho = 0\]  \hspace{1cm} (55)

whose solution is

\[b = \frac{\sqrt{\rho}}{B\alpha} \cosh(\alpha t + \log \frac{\sigma}{\sqrt{\rho}}) + \delta.\]  \hspace{1cm} (56)

In a similar way we find \(a\) as

\[a = \frac{\sqrt{\rho}}{B\alpha} \sinh(\alpha t + \log \frac{\sigma}{\sqrt{\rho}}) + \tau.\]  \hspace{1cm} (57)

Here \(\rho \neq 0\) and \(\delta, \tau\) are constants. Since \(c' = B\), we find \(c(t_x) = Bt_x + \zeta\).

Finally, after appropriate transformation we obtain

\[f(t, t_1, t_x) = \gamma(t_x + \beta \sinh(\alpha t + \lambda)) + \beta \cosh t_1 + \eta,\]  \hspace{1cm} (58)

where \(\alpha, \beta, \eta, \gamma\) and \(\lambda\) are constants.

If \(\rho = 0\), then the equation (55) gives us

\[b = \frac{\sigma}{2B\alpha} e^{\alpha t}.\]  \hspace{1cm} (59)

By using this equation we obtain \(a\) as

\[a = \frac{\gamma}{2\alpha} e^{\alpha t}.\]  \hspace{1cm} (60)

We have also \(c(t_x) = Bt_x + \xi\). Thus simplified form of \(f\) is

\[f(t, t_1, t_x) = \gamma(t_x + \beta e^{\alpha t}) + \beta e^{\alpha t_1} + \eta,\]  \hspace{1cm} (61)

where \(\alpha, \beta, \gamma\) and \(\eta\) are arbitrary constants.
Subcase 3.ii. $a' = 0$:

In the Case 1. we have analyzed this case. Here,

$$f(t, t_1, t_x) = a(t) + b(t_1) + c(t_x) = \gamma t_x^2 + \beta t_x + \alpha t_1 + \eta,$$

where $\alpha$, $\beta$, $\gamma$ and $\eta$ are arbitrary constants.

Subcase 3.iii. $b' = 0$:

Here $c(t_x) = \gamma t_x^2 + \beta t_x + \lambda$, but for $b' = 0$ we have the following equation

$$D_x\left(\frac{a''}{a'}\right) = \left(\frac{a''a' - (a'')^2}{(a')^2}\right)t_x = \frac{a''}{c'},$$

which cannot be satisfied by this $c(t_x)$. Hence $a'' = 0$. So we obtain,

$$f(t, t_1, t_x) = a(t) + b(t_1) + c(t_x) = \gamma t_x^2 + \beta t_x + \alpha t + \eta,$$

where $\alpha$, $\beta$, $\gamma$ and $\eta$ are arbitrary constants.

The functions given in these cases may not give us $n$-integral $I$ even though they satisfy the condition \([39]\). Let us see an example to this fact. This example belongs to Case 1.

Example.

$$t_{1x} = f(t, t_1, t_x) = t_1 + t_x. \quad (62)$$

By using the definitions of the operators $Y_j$, we obtain

$$Y_1 = \partial_t + \partial_{t_1} + \partial_{t_{xx}} + \partial_{t_{xxx}} + ...$$

$$Y_2 = \partial_{t_x} + 2\partial_{t_{xx}} + 3\partial_{t_{xxx}} + ...$$

$$Y_3 = \partial_{t_x} + 3\partial_{t_{xx}} + 6\partial_{t_{xxx}} + ...$$
and so on. As we see that the set \( Y_1, Y_2, Y_3, ..., Y_n \) is linearly independent which means that the algebra is infinite dimensional. By the theorem 1.1, \( n \)-integral does not exist.

Now we will focus on three problems which yield from the above considerations. We will study on \( n \)-integrals of these problems.

**Problem 1.** The equation for the function \( f \) is

\[
t_{1x} = f(t, t_1, t_x) = \lambda t_x + \lambda \beta \sinh t + \beta \cosh t_1 + \gamma.
\]  

(63)

Hence,

\[
t_{-1x} = g(t, t_x, t_{-1}) = \frac{1}{\lambda}(t_x - \lambda \beta \sinh t_{-1} - \beta \cosh t - \gamma).
\]

(64)

To obtain the operators \( Y_1 \), we find the equations written below:

\[
\begin{align*}
t_{1x} &= \beta \sinh t_1, \\
f_x &= \lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1, \\
f_{xt_1} &= \beta \cosh t_1 (\lambda t_x + \lambda \beta \sinh t + \beta \cosh t_1 + \gamma) + \beta^2 \sinh^2 t_1, \\
f_{xx} &= \lambda t_{xxx} + \lambda \beta (t_x)^2 \sinh t + \lambda \beta t_{xx} \cosh t + \beta (t_{1x})^2 \cosh t_1 \\
&\quad + \beta \sinh t_1 (\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1), \\
f_{xx1} &= 3\beta^2 (t_{1x}) \sinh t_1 \cosh t_1 + \beta (t_{1x})^2 \sinh t_1 + \beta^3 \sinh^3 t_1 \\
&\quad + \beta \cosh t_1 (\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1),
\end{align*}
\]
\[ f_{xxx} = \lambda t_{xxx} + \lambda \beta t_{xxx} \cosh t + 3\lambda \beta t_x t_{xx} \sinh t + \lambda \beta t_x^3 \cosh t \]
\[ + \ 3\beta \cosh t_1(t_1 x)[\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1] + \beta(t_1 x)^3 \sinh t_1 \]
\[ + \ \beta \sinh t_1 \{\lambda t_{xxx} + \lambda \beta t_{xx} \cosh t + \lambda \beta t_x^2 \sinh t + \beta(t_1 x) \cosh t \}
\[ + \ [\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1] \beta \sinh t_1 \}, \]
\[ f_{xxx1} = 3\beta[\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1][\beta \sinh t_1 \cosh t_1 + t_{1x} \sinh t_1] \]
\[ + \ 3\beta t_{1x} \cosh t_1[\beta^2 \sinh^2 t_1 + \beta t_{1x} \cosh t_1] + 3\beta^2(t_{1x})^2 \sinh^2 t_1 \]
\[ + \ \beta \cosh t_1[\lambda t_{xxx} + \lambda \beta t_{xx} \cosh t + \lambda \beta t_x^2 \sinh t + \beta t_{1x}^2 \cosh t \]
\[ + \ \beta \sinh t_1(\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1)] + \beta t_{1x}^3 \cosh t_1 \]
\[ + \ \beta \sinh t_1[\beta \cosh t_1(\lambda t_{xx} + \lambda \beta t_x \cosh t + \beta t_{1x} \sinh t_1) + \beta^3 \sinh^3 t_1 \]
\[ + \ 3\beta^2 t_{1x} \cosh t_1 \sinh t_1 + \beta t_{1x}^2 \sinh t_1], \]
\[ : \]
Thus,
\[ D^{-1}(f_{t_1}) = \beta \sinh t, \]
\[ D^{-1}(f_{xt_1}) = \beta t_x \cosh t + \beta^2 \sinh^2 t, \]
\[ D^{-1}(f_{xxt_1}) = 3\beta^2 t_x \sinh t \cosh t + \beta t_x^2 \sinh t + \beta t_{xx} \cosh t + \beta^3 \sinh^3 t, \]
\[ D^{-1}(f_{xxx}) = 4\beta^2 t_x^2 \sinh^2 t + 6\beta^3 t_x \cosh t \sinh^2 t + 4\beta^2 t_{xx} \sinh t \cosh t \]
\[ + 3\beta^2 t_x^2 \cosh^2 t + \beta t_x^3 \cosh t + 3\beta t_x t_{xx} \sinh t \]
\[ + \beta t_{xxx} \cosh t + \beta^4 \sinh^4 t, \]

Let us define \( \beta \sinh t = \psi \). Hence we can write the operator \( Y_1 \) as
\[
Y_1 = \partial_t + \psi \partial_x + (\psi^2 + \psi_x) \partial_{tx} + [(\psi^2 + \psi_x)x + \psi(\psi^2 + \psi_x)] \partial_{txx}
\]
\[ + \{[(\psi^2 + \psi_x)x + \psi(\psi^2 + \psi_x)]_x + \psi[(\psi^2 + \psi_x)x + \psi(\psi^2 + \psi_x)] \} \partial_{txxx}
\]
\[ + \ldots. \] (65)

Now we will obtain the operator \( Y_{-1} \). From (64) we have,
\[
g_{t_{-1}} = -\beta \cosh t_{-1},
\]
\[
g_x = [t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t]/\lambda,
\]
\[
g_{xt_{-1}} = \beta^2 \cosh^2 t_{-1} - \beta t_{-1x} \sinh t_{-1},
\]
\[
g_{xx} = [t_{xxx} - \beta \cosh t_{-1}(t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t)]
\]
\[ - \beta t_{xx} \sinh t - \beta t_x^2 \cosh t - \lambda \beta (t_{-1x})^2 \sinh t_{-1}] / \lambda,
\]
\[
g_{xxt_{-1}} = [3\lambda \beta^2 (t_{-1x}) \sinh t_{-1} \cosh t_{-1} - \lambda \beta (t_{-1x})^2 \cosh t_{-1} - \lambda \beta^3 \cosh^3 t_{-1}
\]
\[ - \beta \sinh t_{-1}(t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t)] / \lambda,
\]

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\[ g_{xxx} = \left\{ t_{xxxx} + -3\beta t_{-1x} \sinh t_{-1} (t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t) \right. \]
\[ \left. - \lambda \beta (t_{-1x})^3 \cosh t_{-1} - \beta t_{xxx} \sinh t - 3\beta t_x t_{xx} \cosh t - \beta t_x^3 \sinh t \right. \]
\[ \left. - \beta \cosh t_{-1} [t_{xxx} - \beta \cosh t_{-1} (t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t) \right. \]
\[ \left. - \lambda \beta (t_{-1x})^2 \sinh t_{-1} - \beta t_{xx} \sinh t - \beta t_x^2 \cosh t] \right\} / \lambda, \]
\[ g_{xxxt_{-1}} = \left\{ (t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t) (4\beta^2 \cosh t_{-1} \sinh t_{-1} - 3\beta t_{-1x} \cosh t_{-1}) \right. \]
\[ \left. - 3\beta t_{-1x} \sinh t_{-1} (2\lambda \beta^2 \cosh^2 t_{-1} - \lambda \beta t_{-1x} \sinh t_{-1}) \right. \]
\[ \left. - \beta \sinh t_{-1} [t_{xxx} - \beta \cosh t_{-1} (t_{xx} - \lambda \beta t_{-1x} \cosh t_{-1} - \beta t_x \sinh t) \right. \]
\[ \left. - \lambda \beta (t_{-1x})^2 \sinh t_{-1} - \lambda \beta (t_{-1x})^3 \sinh t_{-1} - \beta t_{xx} \sinh t - \beta t_x^2 \cosh t] \right. \]
\[ \left. + \lambda \beta^4 \cosh^4 t_{-1} + 4\lambda \beta^2 t_{-1x}^2 \cosh^2 t_{-1} \right\} / \lambda, \]
\[ \vdots \]

From these equations we get
\[ D(g_{t_{-1}}) = -\beta \cosh t, \]
\[ D(g_{xt_{-1}}) = \beta^2 \cosh^2 t - \beta t_x \sinh t, \]
\[ D(g_{xxt_{-1}}) = 3\beta^2 t_x \sinh t \cosh t - \beta t_x^2 \cosh t - \beta t_{xx} \sinh t - \beta^3 \cosh^3 t, \]
\[ D(g_{xxx\tau}) = 4\beta^2 t_{xx} \sinh t \cosh t - 3\beta t_{x xx} \cosh t - 6\beta^3 t_x \cosh^2 t \sinh t \]
\[ + 4\beta^2 t_x^2 \cosh^2 t + 3\beta^2 t_x^2 \sinh^2 t - \beta t_x^3 \sinh t \]
\[ - \beta t_{xxx} \sinh t + \beta^4 \cosh^4 t, \]
\[ : \]

If we define \(-\beta \cosh t = \varphi\) we write the operator \(Y_{-1}\) as
\[ Y_{-1} = \partial_t + \varphi \partial_{t_x} + (\varphi^2 + \varphi_x) \partial_{t_{xx}} + [\varphi^2 + \varphi_x \cosh t + \varphi(\varphi^2 + \varphi_x)] \partial_{t_{xxx}} \]
\[ + \{[\varphi^2 + \varphi_x \cosh t + \varphi(\varphi^2 + \varphi_x)]x + \varphi[\varphi^2 + \varphi_x \cosh t + \varphi(\varphi^2 + \varphi_x)] \partial_{t_{xxxx}} \]
\[ + \ldots \] (66)

Now let us only take the first three terms of \(Y_1\) and \(Y_{-1}\) and find \(n\)-integral \(I\) such that both equations \(Y_1 I = 0\) and \(Y_{-1} I = 0\) are satisfied. The first three terms of \(Y_1\) are
\[ Y_1 = \partial_t + \beta \sinh t \partial_{t_x} + (\beta \cosh t_x + \beta^2 \sinh^2 t) \partial_{t_{xx}}. \] (67)

We will use the method of characteristic,
\[ \frac{dt}{I} = \frac{dt_x}{\beta \sinh t} = \frac{dt_{xx}}{(\beta \cosh t_x + \beta^2 \sinh^2 t)} = \frac{dI}{0}. \] (68)

By the first and second terms we have
\[ \beta \cosh t - t_x = c_1. \] (69)

By the first and third terms we have
\[ \beta t_x \sinh t - t_{xx} = c_2. \] (70)

Hence from \(Y_1 I = 0\) we get the solution \(I = F(c_1, c_2)\) where \(F\) is an arbitrary function. Now by using the transformation \(\tilde{t} = t, c_1 = \beta \cosh t - t_x, c_2 = \ldots \)
$\beta t_x \sinh t - t_{xx}$, we will write $Y_-$ in terms of $\bar{t}$, $c_1$ and $c_2$. Note that

$$
\partial_t = \frac{\partial \bar{t}}{\partial t} \frac{\partial}{\partial \bar{t}} + \frac{\partial c_1}{\partial t} \frac{\partial}{\partial c_1} + \frac{\partial c_2}{\partial t} \frac{\partial}{\partial c_2} = \partial_t + \beta \sinh \bar{t} \partial_{c_1} + \beta t_x \cosh \bar{t} \partial_{c_2},
$$

$$
\partial_{t_x} = \frac{\partial \bar{t}}{\partial t_x} \frac{\partial}{\partial \bar{t}} + \frac{\partial c_1}{\partial t_x} \frac{\partial}{\partial c_1} + \frac{\partial c_2}{\partial t_x} \frac{\partial}{\partial c_2} = -\partial c_1 + \beta \sinh \bar{t} \partial_{c_2},
$$

$$
\partial_{t_{xx}} = \frac{\partial \bar{t}}{\partial t_{xx}} \frac{\partial}{\partial \bar{t}} + \frac{\partial c_1}{\partial t_{xx}} \frac{\partial}{\partial c_1} + \frac{\partial c_2}{\partial t_{xx}} \frac{\partial}{\partial c_2} = -\partial c_2.
$$

Under these transformations $Y_-$ becomes

$$
Y_- = \partial_t + \beta (\sinh \bar{t} + \cosh \bar{t}) \partial_{c_1} - \beta c_1 (\sinh \bar{t} + \cosh \bar{t}) \partial_{c_2}. \quad (71)
$$

Again by using the method of characteristic,

$$
\frac{d\bar{t}}{dI} = \frac{dc_1}{\beta (\sinh \bar{t} + \cosh \bar{t})} = \frac{dc_2}{-\beta c_1 (\sinh \bar{t} + \cosh \bar{t})} = 0 \quad (72)
$$

we obtain

$$
\beta (\sinh \bar{t} + \cosh \bar{t}) - c_1 = \tilde{c}_1 \quad (73)
$$

and

$$
c_1^2 + c_2 = \tilde{c}_2. \quad (74)
$$

Thus from $Y_- I = 0$ we obtain the solution

$$
I = G(\beta (\sinh \bar{t} + \cosh \bar{t}) - c_1, \frac{c_1^2}{2} + c_2). \quad (75)
$$

But we search for the common solution of $Y_1 I = 0$ and $Y_- I = 0$. Since the solution of $Y_1 I = 0$ only depends on $c_1$ and $c_2$ we take the solution $I = H(u)$ where $u = \frac{c_1^2}{2} + c_2$. Note that we may simply take

$$
I = \frac{c_1^2}{2} + c_2 = \frac{\beta^2}{2} \cosh^2 t - \beta t_x \cosh t + \frac{t_x^2}{2} + \beta t_x \sinh t - t_{xx}. \quad (76)
$$

**Remark 4.1.** We should also check that $DI = I$ to be sure that $I$ is n-integral.

$$
DI = \frac{\lambda^2}{2} t_x^2 + \frac{\lambda^2 \beta^2}{2} \cosh^2 t - \frac{\lambda^2 \beta^2}{2} + \frac{\gamma^2}{2} + \lambda^2 \beta t_x \sinh t + \lambda \gamma t_x + \lambda \gamma \sinh t - \lambda t_{xx} - \lambda \beta t_x \cosh t.
$$
To make $DI$ similar to $I$, we have to choose $\lambda = 1$ and $\gamma = 0$. In this case,

$$DI = I - \frac{\beta^2}{2}.$$  

(77)

So $I$ is not $n$-integral, but it helps to find $n$-integral. Really,

$$F = \frac{\beta^2}{2} \cosh^2 t - \beta t_x \cosh t + \frac{t_x^2}{2} + \beta t_x \sinh t - t_{xx} + \frac{\beta^2}{2} n,$$

(78)

solves the equation $DF = F$.

**Problem 2.** The equation for the function $f$ is

$$t_{1x} = f(t, t_1, t_x) = \lambda(t_x + e^t) + e^{t_1} + c.$$  

(79)

Hence

$$t_{-1x} = g(t, t_x, t_{-1}) = \frac{1}{\lambda}(t_x - e^t + \lambda e^{t-1} - c).$$  

(80)

To obtain the operator $Y_1$ we find the equations written below:

$$f_{t_1} = e^{t_1},$$

$$f_x = \lambda t_{xx} + \lambda t_x e^t + t_1 e^{t_1},$$

$$f_{xt_1} = t_1 e^{t_1} + e^{2t_1},$$

$$f_{xx} = \lambda t_{xxx} + \lambda(t_x)^2 e^t + \lambda t_{xx} e^t + (\lambda t_{xx} + \lambda t_x e^t + t_1 e^{t_1}) e^{t_1} + (t_1 x)^2 e^{t_1},$$

$$f_{xxt_1} = e^{t_1} (\lambda t_{xx} + \lambda t_x e^t + t_1 e^{t_1}) + 3t_1 e^{2t_1} + e^{3t_1} + (t_1 x)^2 e^{t_1}.$$
\[
f_{xxx} = \lambda t_{xxx} + 2\lambda t_x t_{xx} e^t + \lambda t_x^2 e^t + \lambda t_x t_{xx} e^t \\
+ 3t_{1x}(\lambda t_{xx} + \lambda t_x e^t + t_{1x} e^{t_1}) + (t_{1x}^3)e^{t_1} \\
+ e^{t_1}\{\lambda t_{xxx} + \lambda t_{xx} e^t + \lambda t_x^2 e^t + [\lambda t_{xx} + \lambda t_x e^t + t_{1x} e^{t_1}] e^{t_1} + (t_{1x}^2)e^{t_1}\};
\]

\[
f_{xxx1} = (\lambda t_{xx} + \lambda t_x e^t + t_{1x} e^{t_1})(3t_{1x} e^{t_1} + 3e^{2t_1}) + e^{t_1} t_{1x}(6t_{1x} e^{t_1} + 3e^{2t_1}) \\
+ e^{t_1}\{\lambda t_{xxx} + \lambda t_{xx} e^t + \lambda t_x^2 e^t + [\lambda t_{xx} + \lambda t_x e^t + t_{1x} e^{t_1}] e^{t_1} + (t_{1x}^2)e^{t_1}\} \\
+ e^{t_1}\{[\lambda t_{xx} + \lambda t_x e^t + t_{1x} e^{t_1}] e^{t_1} + 3e^{2t_1} t_{1x} + e^{3t_1} + (t_{1x})^2 e^{t_1}\} + (t_{1x})^3 e^{t_1},
\]

Hence

\[
D^{-1}(f_{t_1}) = e^t,
\]

\[
D^{-1}(f_{xt_1}) = t_x e^t + e^{2t},
\]

\[
D^{-1}(f_{xxt_1}) = t_{xx} e^t + 3t_x e^{2t} + e^{3t} + t_x^2 e^t,
\]

\[
D^{-1}(f_{xxxt_1}) = 3t_x t_{xx} e^t + 7t_x^2 e^{2t} + 4t_{xx} e^{2t} + 6t_x e^{3t} + t_x^3 e^t + t_{xxx} e^t + e^{4t},
\]

\[
\vdots
\]

Now we are ready to write the operator \( Y_1 \):

\[
Y_1 = \partial_t + e^t \partial_{t_x} + (e^t t_x + e^{2t}) \partial_{t_{xx}} + (t_{xx} e^t + 3t_x e^{2t} + e^{3t} + t_x^2 e^t) \partial_{t_{xxx}} \\
+ (3t_x t_{xx} e^t + 7t_x^2 e^{2t} + 4t_{xx} e^{2t} + 6t_x e^{3t} + t_x^3 e^t + t_{xxx} e^t + e^{4t}) \partial_{t_{xxxx}} \\
+ \ldots
\]

If we define \( t_x + e^t = \psi \), \( Y_1 \) becomes

\[
(81)
\]
\[ Y_1 = \partial_t + e^t \partial_{t_x} + e^t \psi \partial_{t_{xx}} + e^t (\psi^2 + \psi x) \partial_{t_{xxx}} + e^t [(\psi^2 + \psi x)_x + \psi (\psi^2 + \psi x)] \partial_{t_{xxxx}} + \ldots \] (82)

Now we will obtain the operator \( Y_{-1} \). From (80) we have,

\[
\begin{align*}
  g_{t-1} &= -e^{t-1}, \\
  g_x &= \left[ t_{xx} - t_x e^t - \lambda t_{-1x} e^{t-1} \right]/\lambda, \\
  g_{xt-1} &= -t_{-1x} e^{t-1} + e^{2t-1}, \\
  g_{xx} &= \left[ t_{xxx} - t_x^2 e^t - t_{xx} e^t - (t_{xx} - t_x e^t - t_{-1x} e^{t-1}) e^{t-1} - \lambda (t_{-1x})^2 e^{t-1} \right]/\lambda, \\
  g_{xxx} &= \left[ t_{xxxx} - 3t_x t_{xx} e^t - t_{x}^3 e^t + 3t_{-1x} e^{t-1} (t_{xx} - t_x e^t - t_{-1x} e^{t-1}) - \lambda (t_{-1x})^3 e^{t-1} \\
  &\quad - e^{t-1} \left\{ t_{xxx} - t_{xx} e^t - t_{x}^2 e^t - (t_{xx} - t_x e^t - \lambda t_{-1x} e^{t-1}) e^{t-1} - \lambda (t_{-1x})^2 e^{t-1} \right\} \right]/\lambda, \\
  g_{xxxx} &= \left[ \lambda e^{t-1} t_{-1x} (7t_{-1x} e^{t-1} - 6e^{2t-1}) \\
  &\quad + (t_{xx} - t_x e^t - \lambda t_{-1x} e^{t-1}) (-3t_{-1x} e^{t-1} + 4e^{2t-1}) \\
  &\quad - e^{t-1} \left\{ t_{xxx} - t_{xx} e^t - t_{x}^2 e^t - (t_{xx} - t_x e^t - \lambda t_{-1x} e^{t-1}) e^{t-1} - \lambda (t_{-1x})^2 e^{t-1} \right\} \\
  &\quad + e^{2t-1} (t_{xx} - t_x e^t - \lambda t_{-1x} e^{t-1}) e^{t1} + \lambda e^{4t-1} - \lambda (t_{-1x})^3 e^{t-1} \right]/\lambda \\
  &\vdots
\end{align*}
\]
From these equations we get

\[ D(g_{t-1}) = -e^t, \]
\[ D(g_{xt-1}) = e^{2t} - t_x e^t, \]
\[ D(g_{xtt-1}) = -e^t t_{xx} + 3t_x e^{2t} - e^{3t} - (t_x)^2 e^t, \]
\[ D(g_{xxtt-1}) = -3t_x t_{xx} e^t + 7t_x^2 e^{2t} + 4t_{xx} e^{2t} - 6t_x e^{3t} - t_x^3 e^t - t_{xxxx} e^t + e^{4t}, \]
\[ \vdots \]

Thus the operator \( Y_{-1} \) is

\[ Y_{-1} = \partial_t - e^t \partial_{t_x} - (e^t t_x - e^{2t}) \partial_{t_{xx}} + (-e^t t_{xx} + 3t_x e^{2t} - e^{3t} - (t_x)^2 e^t) \partial_{t_{xxxx}} + \ldots \]

If we define \( t_x - e^t = \varphi \), \( Y_{-1} \) becomes

\[ Y_{-1} = \partial_t - e^t \partial_{t_x} - e^t \varphi \partial_{t_{xx}} - e^t (\varphi^2 + \varphi_x) \partial_{t_{xxxx}} - e^t [(\varphi^2 + \varphi_x) x + \varphi (\varphi^2 + \varphi_x)] \partial_{t_{xxxxx}} + \ldots \]

(84)

Now we will only take the first three terms of \( Y_1 \) and \( Y_{-1} \) and find the integral \( I \) such that both \( Y_1 I = 0 \) and \( Y_{-1} I = 0 \) are satisfied. The first three terms of \( Y_1 \) are

\[ Y_1 = \partial_t + e^t \partial_{t_x} + (e^{2t} + t_x e^t) \partial_{t_{xx}}. \]

(85)

The method of characteristic gives us

\[ \frac{dt}{1} = \frac{dt_x}{e^t} = \frac{dt_{xx}}{e^{2t} + t_x e^t} = \frac{dI}{0}. \]

(86)

By the first and second terms we have

\[ e^t - t_x = c_1. \]

(87)
By the second and third terms we have

\[ t_x e^t - t_{xx} = c_2. \]  

(88)

Hence from \( Y_1 I = 0 \) we get the solution \( I = F(c_1, c_2) \) where \( F \) is an arbitrary function. Now by using the transformation \( \tilde{t} = t, \ c_1 = e^t - t_x, \ c_2 = t_x e^t - t_{xx} \), we will write \( Y_{-1} \) in terms of \( \tilde{t}, \ c_1 \) and \( c_2 \). Note that

\[
\partial_t = \frac{\partial_t \partial}{\partial_t \partial_{\tilde{t}}} + \frac{\partial_{c_1} \partial}{\partial_t \partial_{c_1}} + \frac{\partial_{c_2} \partial}{\partial_t \partial_{c_2}} = \partial_{\tilde{t}} + e^t \partial_{c_1} + t_x e^t \partial_{c_2},
\]

\[
\partial_{t_x} = \frac{\partial_{t_x} \partial}{\partial_{t_x} \partial_{\tilde{t}}} + \frac{\partial_{c_1} \partial}{\partial_{t_x} \partial_{c_1}} + \frac{\partial_{c_2} \partial}{\partial_{t_x} \partial_{c_2}} = -\partial_{c_1} + e^t \partial_{c_2},
\]

\[
\partial_{t_{xx}} = \frac{\partial_{t_{xx}} \partial}{\partial_{t_{xx}} \partial_{\tilde{t}}} + \frac{\partial_{c_1} \partial}{\partial_{t_{xx}} \partial_{c_1}} + \frac{\partial_{c_2} \partial}{\partial_{t_{xx}} \partial_{c_2}} = -\partial_{c_2}.
\]

Under these transformations \( Y_{-1} \) becomes

\[ Y_{-1} = \partial_{\tilde{t}} + 2e^t \partial_{c_1} - 2c_1 e^t \partial_{c_2}. \]  

(89)

Again by using the method of characteristic,

\[ \frac{d\tilde{t}}{1} = \frac{dc_1}{2e^t} = \frac{dc_2}{-2c_1 e^t} = \frac{dI}{0} \]  

(90)

we obtain

\[ 2e^t - c_1 = \bar{c}_1 \]  

(91)

and

\[ \frac{c_1^2}{2} + c_2 = \bar{c}_2. \]  

(92)

Hence from \( Y_{-1} I = 0 \) we obtain the solution

\[ I = G(2e^t - c_1, \frac{c_1^2}{2} + c_2). \]  

(93)

But we search for the common solution of \( Y_1 I = 0 \) and \( Y_{-1} I = 0 \). Since the solution of \( Y_1 I = 0 \) only depends on \( c_1 \) and \( c_2 \) we take the solution \( I = H(u) \) where \( u = \frac{c_1^2}{2} + c_2 \). Note that we may simply take

\[ I = \frac{c_1^2}{2} + c_2 = \frac{e^{2t}}{2} + \frac{t_x^2}{2} - t_{xx}. \]  

(94)
Remark 4.2. We should also check that $DI = I$.

$$DI = \frac{\lambda^2}{2} t_x^2 + \frac{\lambda^2}{2} e^{2t} + \frac{c^2}{2} + \lambda^2 t_x e^t + \lambda c t_x + \lambda c e^t - \lambda t_{xx} - \lambda t_x e^t.$$ 

If we choose $\lambda = 1$ and $c = 0$, we see that $I$ becomes

$$I = \frac{e^{2t}}{2} + \frac{t_x^2}{2} - t_{xx}$$  \hspace{1cm} (95)

and it satisfies $DI = I$ i.e. $I$ is integral.

Now we will find the $x$-integral of (79) with the above conditions. Hence we try to find $F$ which satisfies,

$$D_x F(t_2, t_1, t) = F_{t_2} t_{2x} + F_{t_1} t_{1x} + F_t t_x = 0.$$  \hspace{1cm} (96)

Here $t_{1x} = t_x + e^t + e^{t_1}$ and $t_{2x} = t_{1x} + e^{t_1} + e^{t_2} = t_x + e^t + 2e^{t_1} + e^{t_2}$. We insert these into (96), we get

$$F_{t_2} (t_x + e^t + 2e^{t_1} + e^{t_2}) + F_{t_1} (t_x + e^t + e^{t_1}) + F_t t_x = 0.$$  \hspace{1cm} (97)

To satisfy this equation the coefficient of $t_x$ and the other terms should vanish separately. Thus we have

$$F_{t_2} + F_{t_1} + F_t = 0,$$

$$F_{t_2} (e^t + 2e^{t_1} + e^{t_2}) + F_{t_1} (e^t + e^{t_1}) = 0.$$

So we get the operators,

$$X_1 = \partial_{t_2} + \partial_{t_1} + \partial_t,$$

$$X_2 = (e^t + 2e^{t_1} + e^{t_2}) \partial_{t_2} + (e^t + e^{t_1}) \partial_{t_1} = 0.$$

The commutator of these operators is $[X_1, X_2] = X_2$.

Now for simplicity, we may consider the operator $\tilde{X}_2 = e^{-t} X_2$. Explicitly,

$$\tilde{X}_2 = (1 + e^{t_1-t}) \partial_{t_1} + (1 + 2e^{t_1-t} + e^{t_2-t}) \partial_{t_2}.$$  \hspace{1cm} (98)

We will make change of variables,

$$\tau = t,$$

$$\tau_1 = t_1 - t,$$

$$\tau_2 = t_2 - t.$$
The operator $\tilde{X}_2$ becomes
\[
\tilde{X}_2 = (1 + e^{\tau_1})\partial_{\tau_1} + (1 + 2e^{\tau_1} + e^{\tau_2})\partial_{\tau_2}.
\] (99)

Now we use the method of characteristics;
\[
\frac{d\tau_1}{1 + e^{\tau_1}} = \frac{d\tau_2}{1 + 2e^{\tau_1} + e^{\tau_2}}.
\] (100)

Equivalently we have
\[
\frac{-de^{-\tau_1}}{1 + e^{\tau_1}} = \frac{-de^{-\tau_2}}{1 + 2e^{\tau_1} + e^{\tau_2}}.
\] (101)

Let $e^{-\tau_1} = u$ and $e^{-\tau_2} = v$. Hence we get
\[
\frac{du}{1 + u} = \frac{dv}{v(1 + \frac{2}{u}) + 1}
\] (102)

which has the solution
\[
v = \frac{u^2}{u + 1} \left[ \tilde{c} - \frac{1}{u} \right].
\] (103)

If we put $e^{-\tau_1} = u$ and $e^{-\tau_2} = v$ in this solution we find $\tilde{c}$ as
\[
\tilde{c} = e^{\tau_1} \left[ 1 + \frac{(e^{\tau_1} + 1)}{e^{\tau_2}} \right].
\] (104)

Hence $F$ is
\[
F = e^{t_1-t} + e^{2t_1-t_2-t} + e^{t_1-t_2}.
\] (105)

Thus, $t_{1x} = t_x + e^t + e^{t_1}$ is a discrete analog of the Liouville equation.

**Problem 3.** The equation for the function $f$ is
\[
t_{1x} = f(t, t_1, t_x) = r(t_x) + \beta t_1.
\] (106)

Hence
\[
t_{-1x} = g(t, t_x, t_{-1}) = r^{-1}(t_x - \beta t).
\] (107)
To obtain the operator $Y_1$ we find the equations written below

$$f_{t_1} = \beta,$$

$$f_x = [r(t_x)]_x + \beta(r(t_x) + \beta t_1),$$

$$f_{xt_1} = \beta^2,$$

$$f_{xx} = [r(t_x)]_{xx} + \beta[r(t_x)]_x + \beta^2(r(t_x) + \beta t_1),$$

$$\vdots$$

Clearly the operator $Y_1$ is

$$Y_1 = \partial_t + \beta \partial_{t_x} + \beta^2 \partial_{t_{xx}} + \beta^3 \partial_{t_{xxx}} + \beta^4 \partial_{t_{xxxx}} + \ldots.$$  \hspace{1cm} (108)

Since the function $g = r^{-1}(t_x - \beta t)$ does not depend on $t_{-1}$, the operator $Y_{-1}$ is

$$Y_{-1} = \partial_t.$$ \hspace{1cm} (109)

**Remark 4.3.** Let us take the first three terms of $Y_1$. Clearly, the operators $Y_1$ and $Y_{-1}$ commutes. From $Y_{-1}I = \partial_t I = 0$ we see that $I$ is independent of $t$. From $Y_1I = 0$ we obtain $I$ as

$$I = f(t_{xx} - \beta t_x).$$ \hspace{1cm} (110)

Simply, we may take $I = t_{xx} - \beta t_x$. Now we check if $DI = I$.

$$DI = t_{xx}r_{t_x}(t_x).$$ \hspace{1cm} (111)

If $\beta = 0$ and $r_{t_x}(t_x) = 1$ the equation $DI = I$ is satisfied. In this case the equation (106) takes the form $t_{1x} = t_x$ and $I = t_{xx}$ is its $n$-integral.

## Acknowledgments

The authors thank Prof. M. Gürses for fruitful discussions. One of the authors (AP) thanks the Scientific and Technological Research Council of Turkey (TUBİTAK) and the other (IH) thanks (TUBİTAK), the Integrated PhD. Program (BDP) and grants RFBR # 05-01-00775 and RFBR # 06-01-92051-CE-a for partial financial support.
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