Deformations of Hom-Alternative and Hom-Malcev algebras

Mohamed Elhamdadi *
University of South Florida

Abdenacer Makhlouf †
Université de Haute Alsace

Abstract

The aim of this paper is to extend Gerstenhaber formal deformations of algebras to the case of Hom-Alternative and Hom-Malcev algebras. We construct deformation cohomology groups in low dimensions. Using a composition construction, we give a procedure to provide deformations of alternative algebras (resp. Malcev algebras) into Hom-alternative algebras (resp. Hom-Malcev algebras). Then it is used to supply examples for which we compute some cohomology invariants.

Keywords. Hom-alternative algebra, Hom-Malcev algebra, formal deformation, cohomology.

AMS Classification 17A30, 17D10, 17D15, 16S80

Introduction

The Hom-Lie algebras were introduced in [23, 24] to study the deformations of Witt algebra, which is the complex Lie algebra of derivations of the Laurent polynomials in one variable, and the deformation of the Virasoro algebra, a one-dimensional central extension of Witt algebra. Since then Hom-structures in different settings (algebras, coalgebras, Hopf algebras, Leibniz algebras, n-ary algebras etc) were investigated by many authors (see for example [1, 3, 4, 5, 6, 13, 23, 24, 26, 29, 30, 42, 43]). The Hom-alternative algebras were introduced by the second author in [27], while Hom-Malcev algebras were introduced by D. Yau in [44], where connections between Hom-alternative algebras and Hom-Malcev algebras are given.

A deformation of a mathematical object (for example, analytic, geometric or algebraic structures) is a family of similar structures which depend on some parameter. One usually asks does there exist a parameter family of similar structures, such that for an initial value of the parameter one gets the same structure one started with.

In the fifties, Kodaira and Spencer developed a more or less systematic theory of deformations of complex structures of higher dimensional manifolds. The concept of deformations of complex structures, or of a family of complex structures depending differentiably on a parameter, can be defined in terms of structure tensors determining the complex structures. Soon in the sixties, Gerstenhaber generalized this to the context of algebraic and homological setting [17, 18, 19, 20]. He gave a unified treatment of the subjects of deformations of algebras and the cohomology modules on which the analysis of deformations depends. See also [8] [9] [12] [25] [32] [33] [35] for deformation theory.

The deformations of Hom-algebras were initiated by the second author and S. Silvestrov in [31], where the deformations of Hom-associative and Hom-Lie algebras were investigated. Further developments on the

*Email: emohamed@math.usf.edu
†Email: Abdenacer.Makhlouf@uha.fr
cohomology of Hom-associative and Hom-Lie algebras were given in [2], where cochain complexes were provided. Independently the cochain complex in the case of Hom-Lie algebras was given in [38].

In this paper we extend the theory of formal deformation à la M. Gerstenhaber to the context of Hom-alternative and Hom-Malcev algebras. This can be seen as an extension of the work done by the authors in [11]. Using the general procedure of opposite multiplication, one turns left Hom-alternative algebras into right Hom-alternative algebras and vice-versa. Thus we will restrict ourselves to the case of left Hom-alternative algebras since any statement with left Hom-alternative algebra has its corresponding statement for right Hom-alternative algebra.

The paper is organized as follows. In Section 1, we review the basic definitions and properties of Hom-alternative algebras, Hom-Malcev algebras and give examples. In Section 2, we establish the formal deformation theory of Hom-alternative algebras and also give some elements of a cohomology of these algebras. In Section 3, we provide a way to obtain formal deformations of alternative algebras into Hom-alternative algebras using a composition process. Section 4 is dedicated to supplying examples and computations of derivations and cocycles of 4-dimensional Hom-alternative algebras. We conclude the paper by Section 5 in which we study, in a similar way, formal deformations of Malcev and Hom-Malcev algebras. Some computations in this paper were done using a software system of computer algebra.

1 Preliminaries

We start by recalling the notions of Hom-alternative and Hom-Malcev algebras. Throughout this paper $\mathbb{K}$ is an algebraically closed field of characteristic zero and $A$ is a vector space over $\mathbb{K}$. We mean by a Hom-algebra a triple $(A, \mu, \alpha)$ consisting of a vector space $A$, a bilinear map $\mu$ and a linear map $\alpha$. In all the examples involving multiplication, the unspecified products are either given by skewsymmetry, when the algebra is skewsymmetric, or equal to zero.

First we recall the definition and some properties of Hom-alternative algebras.

Definition 1.1 ([27]) A left Hom-alternative algebra (resp. right Hom-alternative algebra) is a triple $(A, \mu, \alpha)$ consisting of a $\mathbb{K}$-vector space $A$, a multiplication $\mu : A \otimes A \to A$ and a linear map $\alpha : A \to A$ satisfying the left Hom-alternative identity, that is for any $x, y$ in $A$,

$$\mu(\alpha(x), \mu(x, y)) = \mu(\mu(x, x), \alpha(y)), \quad (1)$$

respectively, right Hom-alternative identity, that is

$$\mu(\alpha(x), \mu(y, y)) = \mu(\mu(x, y), \alpha(y)) \quad (2)$$

A Hom-alternative algebra is one which is both left and right Hom-alternative algebra.

Definition 1.2 Let $(A, \mu, \alpha)$ and $(A', \mu', \alpha')$ be two Hom-alternative algebras. A linear map $f : A \to A'$ is said to be a morphism of Hom-alternative algebras if the following holds

$$\mu' \circ (f \otimes f) = f \circ \mu \quad \text{and} \quad f \circ \alpha = \alpha' \circ f.$$ 

Remark 1.3 ([28]) Notice that Hom-associative algebras are Hom-alternative algebras. Since a Hom-associative algebra is a Hom-algebra $(A, \mu, \alpha)$ satisfying $\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$, for any $x, y, z$ in $A$. 
The following is an example of Hom-associative algebra which of course is an alternative algebra.

**Example 1.4** Let $A = \mathbb{K}^3$ be the 3-dimensional vector space over $\mathbb{K}$ with a basis $\{e_1, e_2, e_3\}$. The following multiplication $\mu$ and linear map $\alpha$ on $A$ define Hom-associative algebras.

\[
\begin{align*}
\mu(e_1, e_1) &= a e_1, & \mu(e_2, e_2) &= b e_2, \\
\mu(e_1, e_2) &= \mu(e_2, e_1) = a e_2, & \mu(e_2, e_3) &= b e_3, \\
\mu(e_1, e_3) &= \mu(e_3, e_1) = b e_3,
\end{align*}
\]

and

\[
\alpha(e_1) = a e_1, \quad \alpha(e_2) = a e_2, \quad \alpha(e_3) = b e_3,
\]

where $a, b$ are parameters in $\mathbb{K}$.

When $a \neq b$ and $b \neq 0$, the equality $\mu(\mu(e_1, e_1), e_3)) - \mu(e_1, \mu(e_1, e_3)) = (a - b)be_3$ makes this algebra neither associative nor left alternative.

Now, we give another example of left Hom-alternative algebra. More general examples will be supplied further in the paper.

**Example 1.5** Let $A = \mathbb{K}^4$ be the 4-dimensional vector space over $\mathbb{K}$ with a basis $\{e_0, e_1, e_2, e_3\}$. The following multiplication $\mu$ and linear map $\alpha$ on $A$ define a left Hom-alternative algebra.

\[
\begin{align*}
\mu(e_0, e_0) &= e_0 + e_2, & \mu(e_2, e_0) &= 2 e_2, & \mu(e_3, e_0) &= e_2, \\
\alpha(e_0) &= e_0 + e_2, & \alpha(e_1) &= 0, & \alpha(e_2) &= 2 e_2, & \alpha(e_3) &= e_2,
\end{align*}
\]

This Hom-alternative algebra is not alternative since

\[
\mu(\mu(e_0 + e_3, e_0 + e_3), e_0) - \mu(e_0 + e_3, \mu(e_0 + e_3, e_0)) = -e_2 \neq 0.
\]

**Remark 1.6** The Hom-associator of a Hom-algebra $(A, \mu, \alpha)$ is a trilinear map denoted by $\text{as}_\alpha$, and defined for any $x, y, z \in A$ by

\[
\text{as}_\alpha(x, y, z) = \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)).
\]

In terms of Hom-associator, the identities (1) and (2) may be written: $\text{as}_\alpha(x, x, y) = 0$ and $\text{as}_\alpha(y, x, x) = 0$. They are also equivalent, by linearization, to

\[
\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(y), \mu(x, z)) - \mu(\mu(y, x), \alpha(z)) = 0.
\]

respectively,

\[
\mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) + \mu(\alpha(x), \mu(z, y)) - \mu(\mu(x, z), \alpha(y)) = 0.
\]

**Remark 1.7** The multiplication could be considered as a linear map $\mu : A \otimes A \to A$, then the condition (4) and (5) can be written

\[
\mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) \circ (\text{id} \otimes \sigma_1) = 0,
\]

respectively

\[
\mu \circ (\alpha \otimes \mu - \mu \otimes \alpha) \circ (\text{id} \otimes \sigma_2) = 0.
\]

where $\text{id}$ stands for the identity map and $\sigma_1$ and $\sigma_2$ stand for trilinear maps defined for any $x, y, z \in A$ by $\sigma_1(x \otimes y \otimes z) = y \otimes x \otimes z$ and $\sigma_2(x \otimes y \otimes z) = x \otimes z \otimes y$.

Therefore, the identities (4) and (5) are equivalent respectively to

\[
\text{as}_\alpha + \text{as}_\alpha \circ \sigma_1 = 0 \quad \text{and} \quad \text{as}_\alpha + \text{as}_\alpha \circ \sigma_2 = 0.
\]
Hence, for any $x, y, z \in A$, we have
\[ a_\alpha(x, y, z) = -a_\alpha(y, x, z) \quad \text{and} \quad a_\alpha(x, y, z) = -a_\alpha(x, z, y). \]

The identity (9) leads to the following characterization of Hom-alternative algebras

**Proposition 1.8** A triple $(A, \mu, \alpha)$ defines a Hom-alternative algebra if and only if the Hom-associator $\alpha(x, y, z)$ is an alternating function of its arguments, that is
\[ a_\alpha(x, y, z) = -a_\alpha(y, x, z) = -a_\alpha(x, z, y) = -a_\alpha(z, y, x). \]

The proof can be found in [27]. Further properties of Hom-alternative algebras could be found in [27] and [44]. Now, we give the definition of Hom-Malcev algebras and their connection to Hom-alternative algebras.

**Definition 1.9 ([44])** A Hom-Malcev algebra is a triple $(A, [\ , \ ], \alpha)$ where $[\ , \ ] : A \times A \to A$ is a skewsymmetric bilinear map and $\alpha : A \to A$ a linear map satisfying for all $x, y, z \in A$
\[ J_\alpha(\alpha(x), \alpha(y), [x, z]) = [J_\alpha(x, y, z), \alpha^2(x)] \quad \text{(Hom-Malcev identity)} \]
where $J_\alpha$ is the Hom-Jacobiator which is a trilinear map defined by $J_\alpha(x, y, z) = \bigcirc_{x,y,z} [x, y], \alpha(z), ]$ with $\bigcirc_{x,y,z}$ denoting the summation over the cyclic permutation on $x, y, z$.

Likewise when $\alpha$ is the identity the Hom-Malcev identity reduces to classical Malcev identity which is equivalent, using skewsymmetry to
\[ [x, y], [x, z] = [[[x, y], z], x] + [[[y, z], x], x] + [[[z, x], x], y] \]

**Remark 1.10** Recall that a Hom-Lie algebra is a Hom-algebra $(A, [\ , \ ], \alpha)$ where the bracket is skewsymmetric and satisfies, for $x, y, z \in A$, $J_\alpha(x, y, z) = 0$. Then any Hom-Lie algebra is a Hom-Malcev algebra.

**Example 1.11 ([44])** Let $A = \mathbb{K}^4$ be the 4-dimensional vector space over $\mathbb{K}$ with a basis $\{e_0, e_1, e_2, e_3\}$. The following bracket $[-, -]$ and linear map $\alpha$ on $A$ define a Hom-Malcev algebra.
\[
\begin{align*}
[e_0, e_1] & = -(b_1 e_1 + b_2 e_2 + a_2 b_1 e_3), & [e_0, e_2] & = -c e_2, & [e_0, e_3] & = b_1 c e_3, \\
[e_1, e_0] & = b_1 e_1 + b_2 e_2 + a_2 b_1 e_3, & [e_1, e_2] & = 2 b_1 c e_3, & [e_1, e_0] & = c e_2, \\
[e_2, e_1] & = -2 b_1 c e_3, & [e_3, e_0] & = -b_1 c e_3.
\end{align*}
\]

and
\[
\begin{align*}
\alpha_1(e_0) & = e_0 + a_2 e_2 + a_3 e_3, & \alpha_1(e_1) & = b_1 e_1 + b_2 e_2 + a_2 b_1 e_3, \\
\alpha_1(e_2) & = c e_2, & \alpha_1(e_3) & = b_1 c e_3.
\end{align*}
\]

We show in the following the connection between Hom-alternative algebras and Hom-Malcev algebras given in [44]. The Hom-alternative algebras are related to Hom-Malcev algebras as Hom-associative algebras to Hom-Lie algebras (see [28]).

**Theorem 1.12 ([44])** Let $(A, \mu, \alpha)$ be a Hom-alternative algebra. Then $(A, [\ , \ ], \alpha)$, where the bracket is defined for all $x, y \in A$ by
\[ [x, y] = \mu(x, y) - \mu(y, x) \]
is a Hom-Malcev algebra.

We refer to [44] for more properties on Hom-Malcev algebras and Hom-Malcev-admissible algebras.
2 Formal Deformations of Hom-Alternative algebras

We develop, in this section, a deformation theory for Hom-Alternative algebras by analogy with Gerstenhaber deformations ([17, 13, 19, 20]). Heuristically, a formal deformation of an algebra \( A \) is 1-parameter family of multiplication (of the same sort) obtained by perturbing the multiplication of \( A \). Let \(( A, \mu_0, \alpha_0)\) be a left Hom-alternative algebra. Let \( \mathbb{K}[[t]] \) be the power series ring in one variable \( t \) and coefficients in \( \mathbb{K} \) and let \( A[[t]] \) be the set of formal power series whose coefficients are elements of \( A \) (note that \( A[[t]] \) is obtained by extending the coefficients domain of \( A \) from \( \mathbb{K} \) to \( \mathbb{K}[[t]] \)). Then \( A[[t]] \) is a \( \mathbb{K}[[t]] \)-module. When \( A \) is finite-dimensional, we have \( A[[t]] = A \otimes_{\mathbb{K}} \mathbb{K}[[t]] \). One notes that \( A \) is a submodule of \( A[[t]] \). Given a \( \mathbb{K} \)-bilinear map \( f : A \times A \rightarrow A \), it admits naturally an extension to a \( \mathbb{K}[[t]] \)-bilinear map \( f : A[[t]] \otimes A[[t]] \rightarrow A[[t]] \), that is, if \( x = \sum_{i \geq 0} a_i t^i \) and \( y = \sum_{j \geq 0} b_j t^j \) then
\[
f(x \otimes y) = \sum_{i \geq 0, j \geq 0} t^{i+j} f(a_i \otimes b_j).
\]

**Definition 2.1** Let \(( A, \mu_0, \alpha_0)\) be a left Hom-alternative algebra. A **formal left Hom-alternative deformation** of \( A \) is given by the \( \mathbb{K}[[t]] \)-bilinear map \( \mu_t : A[[t]] \otimes A[[t]] \rightarrow A[[t]] \) and the \( \mathbb{K}[[t]] \)-linear map \( \alpha_t : A[[t]] \rightarrow A[[t]] \), such that \( \mu_t = \sum_{i \geq 0} \mu_i t^i \), where each \( \mu_i \) is a \( \mathbb{K} \)-bilinear map \( \mu_i : A \otimes A \rightarrow A \) (extended to be \( \mathbb{K}[[t]] \)-bilinear), and \( \alpha_t = \sum_{i \geq 0} \alpha_i t^i \), where each \( \alpha_i \) is a \( \mathbb{K} \)-linear map \( \mu_i : A \rightarrow A \) (extended to be \( \mathbb{K}[[t]] \)-linear) such that for \( x, y, z \in A \), the following formal left Hom-alternativity identity holds
\[
\mu_t(\alpha_t(x) \otimes \mu_t(y \otimes z)) - \mu_t(\mu_t(x \otimes y) \otimes \alpha_t(z)) + \mu_t(\alpha_t(y) \otimes \mu_t(x \otimes z)) - \mu_t(\mu_t(y \otimes x) \otimes \alpha_t(z)) = 0. \tag{11}
\]

The identity (11) is called the deformation equation of the Hom-alternative algebras. Notice that here both the multiplication and the linear map are deformed.

2.1 Deformations equation

Now we investigate the deformation equation. We give conditions on \( \mu_i \) in order for the deformation \( \mu_t \) to be Hom-alternative. Expanding the left hand side of the equation (11) and collecting the coefficients of \( t^k \) yields an infinite system of equations given, for any nonnegative integer \( k \), by
\[
\sum_{i=0}^{k} \sum_{j=0}^{k-i} \mu_t(\alpha_j(x) \otimes \mu_{k-i}(y \otimes z)) - \mu_t(\mu_{k-i}(x \otimes y) \otimes \alpha_j(z)) + \mu_t(\alpha_j(y) \otimes \mu_{k-i}(x \otimes z)) - \mu_t(\mu_{k-i}(y \otimes x) \otimes \alpha_j(z)) = 0. \tag{12}
\]

The first equation corresponding to \( k = 0 \), is the left Hom-alternativity identity of \(( A, \mu_0, \alpha_0)\). The second equation corresponding to \( k = 1 \) can be written
\[
\begin{align*}
[\mu_0 \circ (\mu_1 \otimes \alpha_0 - \alpha_0 \otimes \mu_1) + \mu_1 \circ (\mu_0 \otimes \alpha_0 - \alpha_0 \otimes \mu_0)] \\
+ [\mu_0 \circ (\mu_0 \otimes \alpha_1 - \alpha_1 \otimes \mu_0)] \circ (\text{id} \otimes \sigma_1) = 0,
\end{align*} \tag{13}
\]
where \( \sigma_1 \) is defined on \( A \otimes A \) by \( \sigma_1(x \otimes y \otimes z) = y \otimes x \otimes z \).
2.2 Elements of Cohomology

We provide some elements of a cohomology theory motivated by formal deformation theory in the case when $α_0$ is not deformed. Then the deformation equation becomes

\[
\sum_{i=0}^{k} μ_i(α(x) ⊗ μ_{k-i}(y) ⊗ z)) - μ_i(μ_{k-i}(x ⊗ y) ⊗ α(z)) + μ_0(α(y) ⊗ μ_{k-i}(x ⊗ z)) - μ_i(μ_{k-i}(y ⊗ x) ⊗ α(z)) = 0. \tag{15}
\]

Therefore, the equation \(13\) is reduced to

\[
[μ_0 ∘ (μ_1 ⊗ α_0 - α_0 ⊗ μ_1) + μ_1 ∘ (μ_0 ⊗ α_0 - α_0 ⊗ μ_0)] ∘ (id ⊗ 3 + σ_1) = 0,
\]

which suggests that $μ_1$ should be a 2-cocycle for a certain left Hom-alternative algebra cohomology. In the sequel we define first and second coboundary operators fitting with deformation theory. Let $(A, μ, α)$ be a Hom-alternative algebra and let $C^1(A, A)$ be the set of linear maps $f : A → A$ which commute with $α$, and $C^2(A, A)$ be the set of bilinear maps on $A$. We define the first differential $δ^1 f ∈ C^2(A, A)$ by

\[
δ^1 f = μ ∘ (f ⊗ id) + μ ∘ (id ⊗ f) - f ∘ μ. \tag{17}
\]

The map $f$ is said to be a 1-cocycle if $δ^1 f = 0$. We remark that the first differential of a left Hom-alternative algebra is similar to the first differential map of Hochschild cohomology of associative algebras. The 1-cocycles are derivations. Let $φ ∈ C^2(A, A)$, we define the second differential $δ^2 φ ∈ C^3(A, A)$ by

\[
δ^2 φ = [μ ∘ (φ ⊗ α - α ⊗ φ) + φ ∘ (μ ⊗ α - α ⊗ μ)] ∘ (id ⊗ 3 + σ_1). \tag{18}
\]

where $σ_1$ is defined on $A ⊗ 3$ by $σ_1(x ⊗ y ⊗ z) = y ⊗ x ⊗ z$.

**Proposition 2.2** The composite $δ^2 ∘ δ^1$ is zero.

**Proof.** Let $x, y, z ∈ A$ and $f ∈ C^1(A, A)$,

\[
δ^1 f(x ⊗ y) = μ(f(x) ⊗ y) + μ(x ⊗ f(y)) - f(μ(x ⊗ y)).
\]

In order to simplify the notation, the multiplication is denoted by concatenation of terms and the tensor product is removed on the right hand side. Then

\[
δ^2(δ^1 f)(x ⊗ y ⊗ z) = α(x)f(yz) + f(α(x))(yz) - f[α(x)(yz)] + α(x)[yf(z) + f(y) z - f(yz)] +
+ α(y)f(xz) + f(α(y))(xz) - f[α(y)(xz)] + α(y)[xf(z) + f(x) z - f(xz)] +
- [(xy)f(α(z)) + f(xy)α(z) - f((xy)α(z))] - [xf(y) + f(x) y - f(xy)α(z)] +
- [(yx)f(α(z)) + f(yx)α(z) - f((yx)α(z))] - [yf(x) + f(y) x - f(yx)α(z)],
= 0,
\]

since $f$ and $α$ commute and the multiplication $μ$ is Hom-alternative. □

The group of the images of $δ^1$, denoted $B^2(A, A)$, corresponds to the 2-coboundaries and the kernel of $δ^2$, denoted $Z^2(A, A)$, gives the 2-cocycles. The 2-cohomology group is defined as the quotient $H^2(A, A) = Z^2(A, A)/B^2(A, A)$. 

Remark 2.3 To define higher order differentials one needs to consider left Hom-alternative multiplicative $p$-cochain, which are a linear maps $f : A^p \to A$, satisfying
\[
\alpha \circ f(x_0, ..., x_{n-1}) = f(\alpha(x_0), \alpha(x_1), ..., \alpha(x_{n-1})) \text{ for all } x_0, x_1, ..., x_{n-1} \in A.
\]

In [2], it is shown that Proposition 2.4 Equation (21) is equivalent to a complex for Hom-associative algebras. It turns out that the operad of alternative algebras is not Koszul [10]. We conjecture that it is also the case for Hom-alternative algebras. Obviously one may set $\delta^p = 0$ for $p > 3$ but we expect that there is a nontrivial minimal model.

The deformation equation (15) may be written using coboundary operators as
\[
\delta^2 \mu_k (x \otimes y \otimes z) = - \sum_{i=1}^{k-1} \mu_i (\alpha(x) \otimes \mu_{k-i}(y \otimes z)) - \mu_i (\mu_{k-i}(x \otimes y) \otimes \alpha(z)) + \\
\mu_i (\alpha(y) \otimes \mu_{k-i}(x \otimes z)) - \mu_i (\mu_{k-i}(y \otimes x) \otimes \alpha(z)),
\]
where $k$ is any nonnegative integer. Hence we have

Proposition 2.4 Let $(A, \mu_0, \alpha_0)$ be a Hom-alternative algebra and $(A, \mu_t, \alpha_t)$ be a deformation such that $\mu_t = \sum_{i \geq 0} \mu_i t^i$. Then $\mu_1$ is a 2-cocycle, that is $\delta^2 \mu_1 = 0$.

2.3 Equivalent and trivial deformations

In this section, we characterize equivalent as well as trivial deformations of left Hom-alternative algebras.

Definition 2.5 Let $(A, \mu_0, \alpha_0)$ be a left Hom-alternative algebra and let $(A_t, \mu_t, \alpha_t)$ and $(A'_t, \mu'_t, \alpha'_t)$ be two left Hom-alternative deformations of $A$, where $\mu_t = \sum_{i \geq 0} t^i \mu_i$, $\mu'_t = \sum_{i \geq 0} t^i \mu'_i$, with $\mu_0 = \mu'_0$, and $\alpha_t = \sum_{i \geq 0} t^i \alpha_i$, $\alpha'_t = \sum_{i \geq 0} t^i \alpha'_i$, with $\alpha_0 = \alpha'_0$.

We say that the two deformations are equivalent if there exists a formal isomorphism $\rho_t : A[[t]] \to A[[t]]$, i.e. a $\mathbb{K}[[t]]$-linear map that may be written in the form $\rho_t = \sum_{i \geq 0} t^i \rho_i = id + t \rho_1 + t^2 \rho_2 + \ldots$, where $\rho_i \in \text{End}_\mathbb{K}(A)$ and $\rho_0 = id$ are such that the following relations hold
\[
\rho_t \circ \mu_t = \mu'_t \circ (\rho_t \otimes \rho_t) \quad \text{and} \quad \alpha'_t \circ \rho_t = \rho_t \circ \alpha_t.
\]

A deformation $A_t$ of $A_0$ is said to be trivial if and only if $A_t$ is equivalent to $A_0$ (viewed as a left Hom-alternative algebra on $A[[t]]$).

We discuss in the following the equivalence of two deformations. The two identities in (20) may be written as
\[
\rho_t (\mu_t(x \otimes y)) = \mu'_t (\rho_t(x) \otimes \rho_t(y)), \quad \forall x, y \in A.
\]
and
\[
\rho_t (\alpha_t(x)) = \alpha'_t (\rho_t(x)), \quad \forall x \in A.
\]

Equation (21) is equivalent to
\[
\sum_{i,j \geq 0} \rho_t (\mu_j (x \otimes y)) t^{i+j} = \sum_{i,j,k \geq 0} \mu'_t (\rho_j(x) \otimes \rho_k(y)) t^{i+j+k}.
\]
By identification of the coefficients, one obtains that the constant coefficients are identical, i.e.
\[ \mu_0 = \mu'_0 \quad \text{because} \quad \rho_0 = id. \]

For the coefficients of \( t \)
and since \( \varphi_0 = id \), it follows that
\[ \begin{align*}
\mu_1(x, y) + \rho_1(\mu_0(x \otimes y)) &= \mu'_1(x \otimes y) + \mu_0(\rho_1(x) \otimes y) + \mu_0(x \otimes \rho_1(y)).
\end{align*} \] (24)

Consequently,
\[ \begin{align*}
\mu'_1(x \otimes y) &= \mu_1(x \otimes y) + \rho_1(\mu_0(x \otimes y)) - \mu_0(\rho_1(x) \otimes y) - \mu_0(x \otimes \rho_1(y)).
\end{align*} \] (25)

The homomorphism condition of equation (22) is equivalent to
\[ \begin{align*}
\sum_{i,j \geq 0} \rho_i(\alpha_j(x)) t^{i+j} &= \sum_{i,j \geq 0} \alpha'_i(\rho_j(x)) t^{i+j},
\end{align*} \]
which gives \( \alpha_0 = \alpha'_0 \) modulo \( t \), and
\[ \begin{align*}
\rho_1 \circ \alpha_0 + \rho_0 \circ \alpha_1 &= \alpha'_0 \circ \rho_1 + \alpha'_1 \circ \rho_0 \quad \text{modulo} \ t^2.
\end{align*} \] (26)

The second order conditions of the equivalence between two deformations of a left Hom-alternative algebra is given by (25) which may be written as
\[ \begin{align*}
\mu'_1(x \otimes y) &= \mu_1(x \otimes y) - \delta^1 \rho_1(x \otimes y).
\end{align*} \] (27)

In general, if the deformations \( \mu_t \) and \( \mu'_t \) of \( \mu_0 \) are equivalent then \( \mu'_1 = \mu_1 + \delta^1 f_1 \). Therefore, we have the following

**Proposition 2.6** Let \((A, \mu_0, \alpha_0)\) be a Hom-alternative algebra and \((A, \mu_t, \alpha_0)\) be a deformation such that \( \mu_t = \sum_{i \geq 0} \mu_i t^i \). The integrability of \( \mu_1 \) depends only on its cohomology class.

When we deform only the multiplication, the elements of \( H^2(A, A) \) give the infinitesimal deformations (\( \mu_t = \mu_0 + t \mu_1 \)). We have also the following

**Proposition 2.7** Let \((A, \mu_0, \alpha_0)\) be a left Hom-alternative algebra. There is, over \( \mathbb{K}[t]/t^2 \), a one-to-one correspondence between the elements of \( H^2(A, A) \) and the infinitesimal deformation of \( A \) defined by
\[ \begin{align*}
\mu_t(x \otimes y) &= \mu_0(x \otimes y) + t \mu_1(x \otimes y), \quad \forall x, y \in A.
\end{align*} \] (28)

**Proof 1** The deformation equation is equivalent to \( \delta^2 \mu_1 = 0 \).

### 3 Deformation by Composition

In this section, we construct deformations of left alternative algebras using the composition Theorem given in [27], which provides a method of obtaining left Hom-alternative algebras starting from a left alternative algebra and an algebra endomorphism. Also, we use the notion of \( n \)th derived Hom-algebras introduced in [44] to construct other deformation of left alternative algebras viewed as Hom-algebras. We start by recalling the composition theorems of left alternative algebras.
Theorem 3.1 ([27]) Let \((A, \mu)\) be a left alternative algebra and \(\alpha : A \to A\) be an algebra endomorphism. Set \(\mu_{\alpha} = \alpha \circ \mu\), then \((A, \mu_{\alpha}, \alpha)\) is a left Hom-alternative algebra.

Moreover, suppose that \((A', \mu')\) is another left alternative algebra and \(\alpha' : A' \to A'\) is an algebra endomorphism. If \(f : A \to A'\) is an algebra morphism that satisfies \(f \circ \alpha = \alpha' \circ f\) then

\[ f : (A, \mu_{\alpha}, \alpha) \longrightarrow (A', \mu'_{\alpha'}, \alpha') \]

is a morphism of left Hom-alternative algebras.

Remark 3.2 Theorem (3.1) gives a procedure of constructing Hom-alternative algebras using ordinary alternative algebras and their algebra endomorphisms. More generally, given a left Hom-alternative algebra \((A, \mu, \alpha)\), one may ask whether this Hom-alternative algebra is induced by an ordinary alternative algebra \((A, \tilde{\mu})\), that is \(\alpha\) is an algebra endomorphism with respect to \(\tilde{\mu}\) and \(\mu = \alpha \circ \tilde{\mu}\). This question was addressed and discussed for Hom-associative algebras in [14, 15, 16]. Now, if \(\alpha\) is an algebra endomorphism with respect to \(\tilde{\mu}\) then \(\alpha\) is also an algebra endomorphism with respect to \(\mu\). Indeed,

\[ \mu(\alpha(x), \alpha(y)) = \alpha \circ \tilde{\mu}(\alpha(x), \alpha(y)) = \alpha \circ \alpha \circ \tilde{\mu}(x, y) = \alpha \circ \mu(x, y). \]

If \(\alpha\) is bijective then \(\alpha^{-1}\) is also an algebra automorphism. Therefore one may use an untwist operation on the Hom-alternative algebra in order to recover the alternative algebra \((\tilde{\mu} = \alpha^{-1} \circ \mu)\).

The previous procedure was generalized in [44] to \(n\)th derived Hom-algebras. We split the definition given in [44] into two types of \(n\)th derived Hom-algebras.

Definition 3.3 ([44]) Let \((A, \mu, \alpha)\) be a multiplicative Hom-algebra and \(n \geq 0\). The \(n\)th derived Hom-algebra of type 1 of \(A\) is defined by

\[ A^n = (A, \mu^{(n)} = \alpha^n \circ \mu, \alpha^{n+1}), \quad (29) \]

and the \(n\)th derived Hom-algebra of type 2 of \(A\) is defined by

\[ A^n = (A, \mu^{(n)} = \alpha^{2n-1} \circ \mu, \alpha^{2n}), \quad (30) \]

Note that in both cases \(A^0 = A\), \(A^1 = (A, \mu^{(1)} = \alpha \circ \mu, \alpha^2)\) and \(A^{n+1} = (A^n)^1\).

Observe that for \(n \geq 1\) and \(x, y, z \in A\) we have

\[ \mu^{(n)}(\mu^{(n)}(x, y), \alpha^{n+1}(z)) = \alpha^n \circ \mu(\alpha^n \circ \mu(x, y), \alpha^{n+1}(z)) = \alpha^{2n} \circ \mu(\mu(x, y), \alpha(z)). \]

Therefore, following [44], one obtains the following result.

Theorem 3.4 Let \((A, \mu, \alpha)\) be a multiplicative left Hom-alternative algebra (resp. Hom-alternative algebra). Then the \(n\)th derived Hom-algebra of type 1 is also a left Hom-alternative algebra (resp. Hom-alternative algebra). The same holds for multiplicative Hom-associative algebras.

Now, we use Theorem 3.1 and Theorem 3.4 to obtain a procedure of deforming left Hom-alternative algebras by composition.
Proposition 3.5 Let \((A, \mu_0)\) be a left alternative algebra and \(\alpha_t\) be an algebra endomorphism of the form 
\[ \alpha_t = \text{id} + \sum_{i \geq 1} t^i \alpha_i, \]
where \(\alpha_i\) are linear maps on \(A\), \(t\) is a parameter in \(\mathbb{K}\) and \(p\) is an integer. Set \(\mu_t = \alpha_t \circ \mu\).
Then \((A, \mu_t, \alpha_t)\) is a left Hom-alternative algebra which is a deformation of the alternative algebra viewed as a Hom-alternative algebra \((A, \mu_0, \text{id})\).

Moreover, the \(n\)th derived Hom-algebra of type \(1\)
\[ A^n_t = \left( A, \mu^{(n)}_t = \alpha_t^n \circ \mu_t, \alpha_t^{n+1} \right), \tag{31} \]
is a deformation of \((A, \mu, \text{id})\).

**Proof 2** The first assertion follows from Theorem 3.1. In particular for an infinitesimal deformation of the identity \(\alpha_t = \text{id} + t\alpha_1\), we have \(\mu_t = \mu + t\alpha_1 \circ \mu\).

The proof of the left Hom-alternativity of the \(n\)th derived Hom-algebra \((A, \mu_t, \alpha_t)\) follows from the fact that, for \(n \geq 1\) and \(x, y, z \in A\), we have
\[
\mu_t^{(n)}(\mu_t^{(n)}(x, y), \alpha_t^{n+1}(z)) = \alpha_t^n \circ \mu_t(\alpha_t^n \circ \mu_t(x, y), \alpha_t^{n+1}(z)) = \alpha_t^{2n} \circ \mu_t(\mu_t(x, y), \alpha_t(z)).
\]

In case \(n = 1\) and \(\alpha_t = \text{id} + t\alpha_1\) the multiplication is of the form
\[
\mu_t^{(1)} = (\text{id} + t\alpha_1) \circ (\text{id} + t\alpha_1) \circ \mu = \mu + 2t\alpha_1 \circ \mu + t^2\alpha_1^2 \circ \mu
\]
and the twist is \(\alpha_t^2 = (\text{id} + t\alpha_1)^2 = \text{id} + 2t\alpha_1 + t^2\alpha_1^2\). Therefore we get another deformation of the alternative algebra viewed as a Hom-alternative algebra \((A, \mu_0, \text{id})\). The proof in the general case is similar.

**Remark 3.6** Proposition 3.5 is valid for Hom-associative algebras, G-Hom-associative algebras and Hom-Lie algebras.

**Remark 3.7** More generally, if \((A, \mu, \alpha)\) is a multiplicative left Hom-alternative algebra where \(\alpha\) may be written of the form \(\alpha = \text{id} + t\alpha_1\), then the \(n\)th derived Hom-algebra of type \(1\)
\[ A^n_t = \left( A, \mu^{(n)} = \alpha^n \circ \mu, \alpha^{n+1} \right), \tag{32} \]
gives a one parameter formal deformation of \((A, \mu, \alpha)\). But for any \(\alpha\) one obtains just new left Hom-alternative algebras.

## 4 Examples of Deformations and Computations

In this section we provide examples of deformations of left alternative algebras. Using Proposition 3.5 we construct left Hom-alternative formal deformations of the 4-dimensional left alternative algebras which are not associative (see [21]). These algebras are viewed as left Hom-alternative algebras with identity map as a twist. To this end, for each algebra, we provide all the algebra endomorphisms which are infinitesimal deformations of the identity, that is of the form \(\alpha_t = \text{id} + t\alpha_1\), where \(\alpha_1\) is a linear map. Therefore, left
Hom-alternative algebras are obtained from left alternative algebras and correspond to left Hom-alternative formal deformations of these left alternative algebras.

There are exactly two alternative but not associative algebras of dimension 4 over any field \([21]\). With respect to a basis \(\{e_0, e_1, e_2, e_3\}\), one algebra is given by the following multiplication

\[
\mu_{4,1}(e_0, e_0) = e_0, \quad \mu_{4,1}(e_0, e_1) = e_1, \quad \mu_{4,1}(e_2, e_0) = e_2,
\]

\[
\mu_{4,1}(e_2, e_3) = e_1, \quad \mu_{4,1}(e_3, e_0) = e_3, \quad \mu_{4,1}(e_3, e_2) = -e_1.
\]

The other algebra is given by

\[
\mu_{4,2}(e_0, e_0) = e_0, \quad \mu_{4,2}(e_0, e_2) = e_2, \quad \mu_{4,2}(e_0, e_3) = e_3,
\]

\[
\mu_{4,2}(e_1, e_0) = e_1, \quad \mu_{4,2}(e_2, e_3) = e_1, \quad \mu_{4,2}(e_3, e_2) = -e_1.
\]

These two alternative algebras are anti-isomorphic, that is the first one is isomorphic to the opposite of the second one.

In the following we characterize the homomorphisms which induce left Hom-alternative structures on the 4-dimensional left alternative algebras, given by (33) and (34) which are not associative and also on the Octonions algebra (see \([7]\) or \([27]\) for the multiplication table). Straightforward calculations give

**Proposition 4.1** The only left Hom-alternative algebra structure on 4-dimensional left alternative algebras defined by the multiplications (33) or (34) is given by the identity homomorphism. The same holds for the octonions algebra.

### 4.1 Examples of Deformations by Composition

We provide for the 4-dimensional left alternative algebras defined by the multiplications (33) or (34) the algebra endomorphisms which may be viewed as infinitesimal deformations of the identity. Then according to Theorem (3.1), the linear maps yield Hom-alternative infinitesimal deformations of the left alternative algebras defined by \(\mu_{4,1}\) and \(\mu_{4,2}\).

**Proposition 4.2** The infinitesimal deformations \(\alpha\) of the identity which are algebra endomorphisms of the alternative algebra \(\mu_{4,1}\), defined above by the equation (33), are given with respect to the same basis by

\[
\alpha(e_0) = e_0 + t(a_1 e_1 + a_2 e_2 + a_3 e_3),
\]

\[
\alpha(e_1) = e_1 + t(-a_4 a_5 + a_6 + a_7 + a_6 a_7) e_1,
\]

\[
\alpha(e_2) = e_2 + t[(a_3 - a_2 a_5 + a_3 a_6)e_1 + a_6 e_2 + a_5 e_3],
\]

\[
\alpha(e_3) = e_3 + t[(-a_2 + a_3 a_4 - a_2 a_7)e_1 + a_4 e_2 + a_7 e_3].
\]

where \(a_1, \cdots, a_7 \in \mathbb{K}\) are free parameters.

Hence, the linear map \(\alpha\) and the following multiplication \(\mu_{4,1}\) defined by

\[
\mu_{4,1}(e_0, e_0) = e_0 + t(a_1 e_1 + a_2 e_2 + a_3 e_3),
\]

\[
\mu_{4,1}(e_0, e_1) = e_1 + t(-a_4 a_5 + a_6 + a_7 + a_6 a_7) e_1,
\]

\[
\mu_{4,1}(e_2, e_0) = e_2 + t[(a_3 - a_2 a_5 + a_3 a_6)e_1 + a_6 e_2 + a_5 e_3],
\]

\[
\mu_{4,1}(e_2, e_3) = e_1 + t(-a_4 a_5 + a_6 + a_7 + a_6 a_7) e_1,
\]

\[
\mu_{4,1}(e_3, e_0) = e_3 + t[(-a_2 + a_3 a_4 - a_2 a_7)e_1 + a_4 e_2 + a_7 e_3],
\]

\[
\mu_{4,1}(e_3, e_2) = e_1 - t(-a_4 a_5 + a_6 + a_7 + a_6 a_7) e_1.
\]
where $\text{b}$

Proposition 4.5

and as Hom-alternative algebras are given with respect to the same basis by

Remark 4.4

In the following we compute the derivations and the 2-cocycles of some 4-dimensional alternative algebras.

4.2 Derivations and cocycles

In the following we compute the derivations and the 2-cocycles of some 4-dimensional alternative algebras.

Recall that a derivation of a left Hom-alternative algebra $(A, \mu, \alpha)$ is given by a linear map $f : A \to A$ satisfying

$$\mu(f(x) \otimes y) + \mu(x \otimes f(y)) - f(\mu(x \otimes y)) = 0.$$

Proposition 4.5

The derivations of the 4-dimensional alternative algebras defined by (33) and (34) viewed as Hom-alternative algebras are given with respect to the same basis by

$$f(e_0) = b_1 e_1 + b_2 e_2 + b_3 e_3,$$
$$f(e_1) = (b_4 + b_5) e_1,$$
$$f(e_2) = b_3 e_1 + b_4 e_2 + b_6 e_3,$$
$$f(e_3) = -b_2 e_1 + b_7 e_2 + b_5 e_3,$$

where $b_1, \cdots, b_7 \in \mathbb{K}$ are free parameters.
Recall that a 2-cocycle of a left Hom-alternative algebra \((A, \mu, \alpha)\) is given by a linear map \(\varphi : A \times A \to A\) satisfying
\[
\delta^2 \varphi(x, y, z) = \mu \circ (\varphi(x, y), \alpha(z)) - \mu(\alpha(x), \varphi(y, z)) + \varphi \circ (\mu(x, y), \alpha(z)) - \varphi(\alpha(x), \mu(y, z)) + \mu \circ (\varphi(y, x), \alpha(z)) - \mu(\alpha(y), \varphi(x, z)) + \varphi \circ (\mu(y, x), \alpha(z)) - \varphi(\alpha(y), \mu(x, z))
\]
(35)
\[
\varphi(x, y, z) = 0.
\]

We then have the following

**Proposition 4.6** The 2-cocycles of the 4-dimensional alternative algebras defined in (34) viewed as Hom-alternative algebras are given with respect to the same basis by

\[
\begin{align*}
\varphi(e_0, e_0) &= \lambda_1 e_0, & \varphi(e_0, e_1) &= \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3, \\
\varphi(e_0, e_2) &= \lambda_4 e_0 - \lambda_5 e_1, & \varphi(e_0, e_3) &= \lambda_6 e_0 - \lambda_7 e_1, \\
\varphi(e_1, e_0) &= \lambda_8 e_0 - \lambda_2 e_2 - \lambda_3 e_3, & \varphi(e_1, e_1) &= \lambda_8 e_1, \\
\varphi(e_1, e_2) &= \lambda_3 e_1, & \varphi(e_1, e_3) &= -\lambda_2 e_1, \\
\varphi(e_2, e_0) &= \lambda_9 e_1 + \lambda_1 e_2, & \varphi(e_2, e_1) &= (\lambda_4 - \lambda_3) e_1 + \lambda_8 e_2, \\
\varphi(e_2, e_2) &= \lambda_4 e_2, & \varphi(e_2, e_3) &= -\lambda_8 e_0 - \lambda_{10} e_1 + (\lambda_2 + \lambda_6) e_2 + \lambda_3 e_3, \\
\varphi(e_3, e_0) &= \lambda_7 e_1 + \lambda_1 e_3, & \varphi(e_3, e_1) &= (\lambda_2 + \lambda_6) e_1 + \lambda_8 e_3, \\
\varphi(e_3, e_2) &= \lambda_8 e_0 + \lambda_{10} e_1 - \lambda_2 e_2 + (\lambda_4 - \lambda_3) e_3, & \varphi(e_3, e_3) &= \lambda_6 e_3.
\end{align*}
\]

where \(\lambda_i \in \mathbb{K}\) with \(1 \leq i \leq 10\) are free parameters.

The cohomology classes of these 2-cocycles are trivial. Hence the Hom-alternative algebra is rigid in the class of alternative algebras.

For the 4-dimensional alternative algebras defined in (34) we obtain

**Proposition 4.7** The 2-cocycles of the 4-dimensional alternative algebras defined in (34) viewed as Hom-alternative algebras are given with respect to the same basis by

\[
\begin{align*}
\varphi(e_0, e_0) &= \lambda_1 e_0, & \varphi(e_0, e_1) &= \lambda_2 e_0 + \lambda_3 e_2 + \lambda_4 e_3, \\
\varphi(e_0, e_2) &= -\lambda_5 e_1 - \lambda_1 e_2, & \varphi(e_0, e_3) &= \lambda_6 e_1 - \lambda_1 e_3, \\
\varphi(e_1, e_0) &= \lambda_1 e_1 - \lambda_3 e_2 - \lambda_4 e_3, & \varphi(e_1, e_1) &= \lambda_2 e_1, \\
\varphi(e_1, e_2) &= (\lambda_7 - \lambda_8) e_1 + \lambda_2 e_2, & \varphi(e_1, e_3) &= (\lambda_9 + \lambda_{10}) e_1 + \lambda_2 e_3, \\
\varphi(e_2, e_0) &= \lambda_7 e_0 + \lambda_5 e_1, & \varphi(e_2, e_1) &= \lambda_8 e_1, \\
\varphi(e_2, e_2) &= \lambda_7 e_2, & \varphi(e_2, e_3) &= -\lambda_2 e_0 + \lambda_{11} e_1 - \lambda_3 e_2 + (\lambda_7 - \lambda_8) e_3, \\
\varphi(e_3, e_0) &= \lambda_9 e_0 - \lambda_6 e_1, & \varphi(e_3, e_1) &= -\lambda_{10} e_1, \\
\varphi(e_3, e_2) &= \lambda_2 e_0 - \lambda_{11} e_1 + (\lambda_9 + \lambda_{10}) e_2 + \lambda_8 e_3, & \varphi(e_3, e_3) &= \lambda_9 e_3.
\end{align*}
\]

where \(\lambda_i \in \mathbb{K}\) with \(1 \leq i \leq 11\) are free parameters.

The cohomology classes of these 2-cocycles are trivial. Hence the Hom-alternative algebra is rigid in the class of alternative algebras.
We compute now the derivation, the 2-cocycles and the 2-cohomology group of the 4-dimensional Hom-
alternative algebras, which are not alternative algebras, given by

\[
\begin{align*}
\mu(e_0, e_0) &= e_0 + ta_1 e_1, & \mu(e_0, e_1) &= e_1 + t(-a_2a_3 + a_4) e_1, \\
\mu(e_2, e_0) &= e_2 + ta_4 e_2 + ta_3 e_3, & \mu(e_2, e_3) &= e_1 + t(-a_2a_3 + a_4) e_1, \\
\mu(e_3, e_0) &= e_3 + ta_2 e_2, & \mu(e_3, e_2) &= -e_1 - t(-a_2a_3 + a_4) e_1.
\end{align*}
\] (36)

\[
\begin{align*}
\alpha(e_0) &= e_0 + ta_1 e_1, & \alpha(e_1) &= e_1 + t(-a_2a_3 + a_4) e_1, \\
\alpha(e_2) &= e_2 + ta_4 e_2 + ta_3 e_3, & \alpha(e_3) &= e_3 + ta_2 e_2.
\end{align*}
\] (37)

where \(a_1, a_2, a_3, a_4 \in \mathbb{K}\) are free parameters. These Hom-
alternative algebras are particular cases of the Hom-alternative algebras constructed in Proposition 4.2.

**Proposition 4.8** The derivations \(f\) of the 4-dimensional Hom-alternative algebras defined in (36) and (37) are given with respect to the same basis by

\[
\begin{align*}
f(e_0) &= b_1 e_1, \\
f(e_1) &= -b_1 \frac{a_2a_3 - a_4}{a_1} e_1, \\
f(e_2) &= -b_1 \frac{a_2a_3 - b_1a_4 + b_2a_1}{a_1} e_2 - \frac{a_3(b_1a_2a_3 - b_1a_4 + 2b_2a_1)}{a_1a_4} e_3, \\
f(e_3) &= -\frac{a_2(b_1a_2a_3 - b_1a_4 + 2b_2a_1)}{a_1a_4} e_2 + b_2 e_3.
\end{align*}
\]

where \(b_1, b_2 \in \mathbb{K}\) are free parameters.

and for the 2-cocycles we obtain

**Proposition 4.9** The only 2-cocycle of the 4-dimensional Hom-alternative algebras defined in (36) and (37) corresponds to the multiplication (36). Moreover there exist a linear \(g\) such that \(\mu = \delta^1 g\) with \(g\) defined by

\[
\begin{align*}
g(e_0) &= e_0, \\
g(e_1) &= \nu_1 e_0 + (\nu_2 + \nu_3 - 1)e_1 + \nu_4 e_2 - \nu_5 e_3, \\
g(e_2) &= \nu_5 e_0 + \nu_2 e_2 + \nu_6 e_3, \\
g(e_3) &= \nu_4 e_0 + \nu_7 e_2 + \nu_8 e_3.
\end{align*}
\]

where \(\nu_i \in \mathbb{K}\) with \(1 \leq i \leq 8\) are free parameters.

Hence the cohomology class of \(\mu\) is zero.

5  **Deformations of Hom-Malcev algebras**

In this section we study the formal deformation of Hom-Malcev algebras. Under the same assumptions as in Section 2 we define the 1-parameter formal deformation of a Malcev algebra as
Definition 5.1 Let $A$ be a $K$-vector space and $(A, [\ , \ ], 0, \alpha_0)$ be a Hom-Malcev algebra. A Hom-Malcev 1-parameter formal deformation is given by the $K[[t]]$-bilinear map $[\ , \ ]_t: A[[t]] \times A[[t]] \to A[[t]]$ and $K[[t]]$-linear map $\alpha_t: A[[t]] \to A[[t]]$ of the form

$$[\ , \ ]_t = \sum_{i\geq 0} [\ , \ ]_i t^i$$

and $\alpha_t = \sum_{i\geq 0} \alpha_i t^i$ (38)

where each $[\ , \ ]_i: A \times A \to A$ is a $K$-bilinear map (extended to be $K[[t]]$-bilinear) and each $\alpha_i: A \to A$ is a $K$-linear map (extended to be $K[[t]]$-linear), satisfying for $x, y, z \in A$ the following identity:

$$J_{\alpha_t}(\alpha_t(x), \alpha_t(y), [x, z]) = [J_{\alpha_t}(x, y, z), \alpha^2_t(x)]_t$$

(39)

where $J_{\alpha_t}$ is the Hom-Jacobiator defined for $x, y, z \in A$ by $J_{\alpha_t}(x, y, z) = \circ_{x,y,z} [[x, y]_t, \alpha_t(z)]_t$.

We have

$$J_{\alpha_t}(x, y, z) = \circ_{x,y,z} [[x, y]_t, \alpha_t(z)]_t$$

$$= \sum_{i,j,k\geq 0} \circ_{x,y,z} [[x, y]_i, \alpha_k(z)]_j t^{i+j+k}.$$

We introduce the following notation $J^{i,j}_{\alpha_t}$ which is a trilinear map defined by

$$J^{i,j}_{\alpha_t}(x, y, z) = \circ_{x,y,z} [[x, y]_i, \alpha(z)]_j,$$

where $\alpha$ is a linear map and $[\ , \ ]_i$ and $[\ , \ ]_j$ are bilinear maps.

Therefore the left hand side of the identity (39) gives

$$J_{\alpha_t}(\alpha_t(x), \alpha_t(y), [x, z]) = \sum_{i,j,k,p,q,r\geq 0} J^{i,j}_{\alpha_t}(\alpha_p(x), \alpha_q(y), [x, z]_k) t^{i+j+k+p+q+r}.$$ (41)

While the right hand side of (39) gives

$$[J_{\alpha_t}(x, y, z), \alpha^2_t(x)]_t = \sum_{i,j,k,p,q,r\geq 0} ([\circ_{x,y,z} [[x, y]_i, \alpha_r(z)]_j, \alpha_p \circ \alpha_q(x)]_k) t^{i+j+k+p+q+r}$$ (42)

Equating (41) and (42) yields Hom-Malcev identity of the original Hom-Malcev algebra, for the degree 0 terms. The terms of degree 1 lead to a 12 terms identity which reduces, when the twist map $\alpha_0$ is not deformed to the following identity

$$J^{0,0}_{\alpha_0}(\alpha_0(x), \alpha_0(y), [x, z]) + J^{0,1}_{\alpha_0}(\alpha_0(x), \alpha_0(y), [x, z]]_0 + J^{0,0}_{\alpha_0}(\alpha_0(x), \alpha_0(y), [x, z]_1)$$

$$- [J^{0,0}_{\alpha_0}(x, y, z), \alpha^2_0(x)]_0 - [J^{0,1}_{\alpha_0}(x, y, z), \alpha^2_0(x)]_0 - [J^{0,0}_{\alpha_0}(x, y, z), \alpha^2_0(x)]_1 = 0$$ (43)

This identity and the study of trivial and equivalent deformation suggests to introduce the following 1-coboundary and 2-coboundary operators for a Hom-Malcev algebra $(A, [\ , \ ], \alpha)$. Let $C^0_s(A, A)$ be the set of skewsymmetric $\alpha$-multiplicative $n$-linear maps on $A$. We define the first differential $\delta^1 f \in C^0_s(A, A)$ by

$$\delta^1 f = [\ , \ ] \circ (f \otimes \text{id}) + [\ , \ ] \circ (\text{id} \otimes f) - f \circ [\ , \ ].$$ (44)

The second differential $\delta^2 \phi \in C^2_s(A, A)$ where $\phi \in C^2_s(A, A)$ and denoted by $\phi = [\ , \ ]_1$, is defined by

$$\delta^2 \phi(x, y, z) = J^{1,0}_{\alpha_0}(\alpha_0(x), \alpha_0(y), [x, z]) + J^{0,1}_{\alpha_0}(\alpha_0(x), \alpha_0(y), [x, z]_0) + J^{0,0}_{\alpha_0}(\alpha_0(x), \alpha_0(y), [x, z]_1)$$

$$- [J^{0,0}_{\alpha_0}(x, y, z), \alpha^2_0(x)]_0 - [J^{0,1}_{\alpha_0}(x, y, z), \alpha^2_0(x)]_0 - [J^{0,0}_{\alpha_0}(x, y, z), \alpha^2_0(x)]_1.$$ (45)
5.1 Deformation by composition

In the sequel we give a procedure of deforming Malcev algebras into Hom-Malcev algebras using the following two Theorems.

**Theorem 5.2** ([44]) Let \((A, [, ])\) be a Malcev algebra and \(\alpha\) be an algebra endomorphism on \(A\). Then the Hom-algebra \((A, \alpha \circ [, ], \alpha)\) induced by \(\alpha\) is a Hom-Malcev algebra.

**Theorem 5.3** ([44]) Let \((A, [, ], \alpha)\) be a Hom-Malcev algebra. Then the \(n\)th derived Hom-algebra of type 2

\[ (A, [, ])^{(n)} = \alpha^{2^n - 1} \circ [, ], \alpha^{2^n} \]

is also a Hom-Malcev algebra.

We provide a procedure to deform Malcev algebras by composition.

**Proposition 5.4** Let \((A, [, ])\) be a Malcev algebra and \(\alpha_t\) be an algebra endomorphism of the form \(\alpha_t = \text{id} + \sum_{i=1}^{p} t^i \alpha_i\), where \(\alpha_i\) are linear maps on \(A\), \(t\) is a parameter in \(\mathbb{K}\) and \(p\) is an integer.

Let \([,] = \alpha_t \circ [, ]\), then \((A, [, ], \alpha_t)\) is a Malcev algebra which is a deformation of the Malcev algebra viewed as a Hom-Malcev algebra \((A, [, ], \text{id})\).

Moreover, the \(n\)th derived Hom-algebra of type 2

\[ A^n_t = \left( A, [, ]^{(n)}_t = \alpha_t^{2^n - 1} \circ [, ], \alpha_t^{2^n} \right) \]

is a deformation of \((A, [, ], \text{id})\).

**Proof 3** The first assertion follows from Theorem 5.2. In particular for an infinitesimal deformation of the identity \(\alpha_t = \text{id} + t\alpha_1\), we have \([, ]_t = [, ] + t\alpha_1 \circ [, ]\).

The proof of the Hom-Malcev identity of the \(n\)th derived Hom-algebra \((A, [, ], \alpha_t)\) follows from the Theorem 5.3. In case \(n = 1\) and \(\alpha_t = \text{id} + t\alpha_1\) the bracket is

\[ [, ]_t^{(1)} = (\text{id} + t\alpha_1) \circ (\text{id} + t\alpha_1) \circ [, ] = [, ] + 2t\alpha_1 \circ [, ] + t^2 \alpha_1^2 \circ [, ] \]

and the twist map is \(\alpha_t^2 = (\text{id} + t\alpha_1)^2 = \text{id} + 2t\alpha_1 + t^2 \alpha_1^2\). Therefore we get another deformation of the Malcev algebra viewed as a Hom-Malcev algebra \((A, [, ], \text{id})\). The proof in the general case is similar.

**Remark 5.5** More generally, if \((A, [, ], \alpha)\) is a multiplicative Hom-Malcev algebra where \(\alpha\) may be written of the form \(\alpha = \text{id} + t\alpha_1\), then the \(n\)th derived Hom-algebra of type 1

\[ A^n_t = \left( A, \mu^{(n)} = \alpha^n \circ [, ], \alpha^{n+1} \right) \]

gives a one parameter formal deformation of \((A, [, ], \alpha)\). But for any \(\alpha\) one obtains just new Hom-Malcev algebras.

**Example 5.6** The example 1.11 is obtained in [44] by computing the algebra endomorphism of the Malcev algebra defined with respect to \(\{e_0, e_1, e_2, e_3\}\) by

\[ [e_0, e_1] = -e_1, \quad [e_0, e_2] = -e_2, \quad [e_0, e_3] = e_3, \quad [e_1, e_2] = 2e_3. \]
By specializing the parameters, we consider the following class of the algebra endomorphism which are deformation of the identity map

\[ \alpha(e_0) = e_0 + t(e_2 + e_3), \quad \alpha(e_1) = e_1 + t(e_1 + e_2 + e_3) + t^2 e_3, \]
\[ \alpha(e_2) = e_2 + t e_2, \quad \alpha(e_3) = e_3 + 2t e_3 + t^2 e_3. \]

By Theorem 5.4 we obtain the Hom-Malcev deformation of the Malcev algebra (48) given by

\[ [e_0, e_1]_t = -e_1 - e_1 + e_2 + e_3 - t^2 e_3, \quad [e_0, e_2]_t = -e_2 - t e_2, \]
\[ [e_0, e_3]_t = e_3 + 2t e_3 + t^2 e_3, \quad [e_1, e_2]_t = 2e_3 + 4t e_3 + 2t^2 e_3. \]

and the linear map defined in (49).

By using once again Theorem 5.4 we obtain the following Hom-Malcev deformations defined by

\[ [e_0, e_1] = -e_1 - 2t(e_1 + e_2 + e_3) - t^2(e_1 + 2e_2 + 5e_3) - 4t^3 e_3 - t^4 e_3, \]
\[ [e_0, e_2] = -e_2 - 2t e_2 - t^2 e_2, \]
\[ [e_0, e_3] = e_3 + 4t e_3 + 6t^2 e_3 + 4t^3 e_3 + t^4 e_3, \]
\[ [e_1, e_2] = 2e_3 + 8t e_3 + 12t^2 e_3 + 8t^3 e_3 + 2t^4 e_3. \]

\[ \alpha^2(e_0) = e_0 + t(2e_2 + 2e_3) + t^2(e_2 + 2e_3) + t^3 e_3, \]
\[ \alpha^2(e_1) = e_1 + 2t(e_1 + e_2 + e_3) + t^2(e_1 + 2e_2 + 5e_3) + 4t^3 e_3 + t^4 e_3, \]
\[ \alpha^2(e_2) = e_2 + 2t e_2 + t^2 e_2, \]
\[ \alpha^2(e_3) = e_3 + 4t e_3 + 6t^2 e_3 + 4t^3 e_3 + t^4 e_3. \]

Acknowledgment The second author would like to thank the Mathematics Department at the University of South Florida for its hospitality while this paper was being finished.

References

[1] F. Ammar and A. Makhlouf, Hom-Lie algebras and Hom-Lie admissible superalgebras, Journal of Algebra (to appear), arXiv:0906.1668v1.
[2] F. Ammar, Z. Ejbehi and A. Makhlouf, Cohomology and Deformations of Hom-algebras, arXiv:1005.0456.
[3] F. Ammar and A. Makhlouf and S. Silvestrov, Ternary q-Virasoro-Witt Hom-Nambu-Lie algebras, J. Phys. A: Math. Theor. 43 (2010) 265204 (13pp).
[4] J. Arnlind, A. Makhlouf and S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras, Journal of Mathematical Physics, 51, 043515 (2010); doi:10.1063/1.3359004 (2010).
[5] H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n-ary Nambu algebras and beyond, Journal of Mathematical Physics 50, 1 (2009).
[6] S. Caenepeel, I. Goyvaerts, Monoidal Hom-Hopf algebras, arXiv:0907.0187v1 [math.RA] (2009).
[7] J.C. Baez, The octonions, Bull. of the Amer. Math. Soc., 39 2, (2001), 145–2005.
[8] J.S. Carter, A. Crans, M. Elhamdadi, and M. Saito, *Cohomology of the adjoint of Hopf Algebras*, J. Gen. Lie Theory Appl. 2, no 1 (2008), 19–34.

[9] J.S. Carter, A. Crans, M. Elhamdadi, and M. Saito, *Cohomology of categorical self-distributivity*, J. Homotopy Relat. Struct. 3 (2008), no. 1, 13–63.

[10] A. Dzhumadil’daev, P. Zusmanovich, *The alternative operad is not Koszul*, arXiv:0906.1272v1.

[11] M. Elhamdadi, and A. Makhlouf, *Cohomology and Formal deformations of Alternative algebras*, arXiv:0907.1548

[12] A. Fialowski, *Deformation of Lie algebras*, Math USSR Sbornik, vol. 55, (1986), 467–473.

[13] Y. Fregier and A. Gohr *Lie Type Hom-algebras*, arXiv:0903.3393v2 [Math.RA] (2009).

[14] Y. Fregier and A. Gohr, *On unitality conditions for Hom-associative algebras*, Arxi:0904.4874v2 [Math.RA] (2009).

[15] Y. Fregier, A. Gohr, S. Silvestrov, *Unital algebras of Hom-associative type and surjective or injective twistings*, J. Gen. Lie Theory Appl. 3, no. 4, (2009), 285–295.

[16] A. Gohr, *On Hom-algebras with surjective twisting*, arXiv:0906.3270v3 [Math.RA] (2009).

[17] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. (2) 79, (1964), 59-103.

[18] M. Gerstenhaber, *On the deformation of rings and algebras. II*, Ann. of Math. (2) 84 (1966) 1–19.

[19] M. Gerstenhaber, *On the deformation of rings and algebras III*, Ann. of Math. (2) 88 (1968) 1–34.

[20] M. Gerstenhaber, *On the deformation of rings and algebras IV*, Ann. of Math. (2) 99 (1974) 257–276.

[21] E. Goodaire, *Alternative rings of small order and the hunt for Moufang circle loops*, Nonassociative algebra and its applications (São Paulo, 1998), 137–146, Lecture Notes in Pure and Appl. Math., 211, Dekker, New York, 2000.

[22] M. Goze and E. Remm, *A class of nonassociative algebras including flexible and alternative algebras, operads and deformations*, arXiv:0910.0700v1 (2009).

[23] J. T. Hartwig, D. Larsson and S.D. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra 295 (2006), no. 2, 314–361.

[24] D. Larsson and S.D. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra 288 (2005), no. 2, 321–344.

[25] O. A. Laudal, *Formal moduli of algebraic structures*, Lecture Notes in Mathematics, 754, Springer, Berlin, 1979.

[26] A. Makhlouf, *Paradigm of Nonassociative Hom-algebras and Hom-superalgebras*, Proceedings of Jordan Structures in Algebra and Analysis Meeting, Eds: J. Carmona Tapia, A. Morales Campoy, A. M. Peralta Pereira, M. I. Ramirez Ivarez, Publishing house: Circulo Rojo (2010), 145–177.

[27] A. Makhlouf, *Hom-alternative algebras and Hom-Jordan algebras*, arXiv:0909.0326

[28] A. Makhlouf, S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. 2 (2) (2008), 51–64.

[29] A. Makhlouf, S. Silvestrov, *Hom-Lie admissible Hom-coalgebras and Hom-Hopf algebras*, In “Generalized Lie Theory in Mathematics, Physics and Beyond”, Eds V. Abramov, E. Paal, S. Silvestrov and A. Stolin, *Springer* (2008).

[30] A. Makhlouf, S. D. Silvestrov, *Hom-algebras and Hom-coalgebras*, Journal of Algebra and its Applications (to appear), Preprints in Math. Sci., Lund Univ., Center for Math. Sci., 2008. arXiv:0811.0400v2. (2008).
[31] A. Makhlouf, S. Silvestrov, *Notes on Formal Deformations of Hom-Associative and Hom-Lie Algebras*, Forum Mathematicum (to appear), arXiv:0712.3130.

[32] A. Makhlouf, *A comparison of deformations and geometric study of varieties of associative algebras*, International Journal of Mathematics and Mathematical Science Vol. 2007, Article ID 18915,(2007).

[33] A. Makhlouf, *Degeneration, rigidity and irreducible components of Hopf algebras*, Algebra Colloquium, 12, no 2 (2005), 241–254.

[34] A. I. Maltsev, *Analytical loops*, Matem. Sbornik, 36 (1955), 569-576 (in Russian).

[35] M. Markl and J.D. Stasheff, *Deformation theory via deviations*, J. Algebra 170, (1994), 122–155.

[36] K. McCrimmon, *Alternative algebras*, http://www.mathstat.uottawa.ca/~neher/Papers/alternative/

[37] E. Paal, *Note on operadic non-associative deformations*, Journal of Nonlinear Mathematical Physics Volume 13, Supplement (2006), 8792

[38] Y. Sheng, *Representations of Hom-Lie algebras*, ArXiv:1005:0140, (2010).

[39] I.P. Shestakov, *Moufang Loops and alternative algebras*, Proc. of Amer. Math. Soc., 132, no 2, (2003), 313–316.

[40] I.P. Shestakov, *Speciality and Deformations of Algebras*. Preprint

[41] K. Yamaguti, *On the theory of Malcev algebras* Kumamoto J. Sci. Ser. A 6 (1963) 9–45.

[42] D. Yau, *Hom-algebras and homology*, J. Lie Theory 19 (2009), no. 2, 409–421.

[43] D. Yau, *Enveloping algebras of Hom-Lie algebras*, J. Gen. Lie Theory Appl. 2 (2008), no. 2, 95–108.

[44] D. Yau., *Hom-Malcev, Hom-alternative, and Hom-Jordan algebras*, arXiv 1002.3944 (2010).

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF SOUTH FLORIDA,
4202 E FOWLER AVE.,
TAMPA, FL 33620, USA

and

LABORATOIRE DE MATHÉMATIQUES,
INFORMATIQUE ET APPLICATIONS,
UNIVERSITÉ DE HAUTE ALSACE,
FRANCE.