Some Boundary Value Problems for the Hyperbolic: Hyperbolic type Equation with Two Line of Degeneration in Special Domain

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Abstract
In the present paper we study unique solvability of the analogues of problem Bitsadze for the degenerating hyperbolic-hyperbolic type equation. Uniqueness and existence theorem for solution of this problems are proven with principle extremum and by the method of integral equations.

Keywords: Boundary value problem; Existence and uniqueness of solution; Degenerating equation; Hyperbolic-hyperbolic type; A principle an extremum; Method of integral equations

Introduction
Last years the increasing attention of mathematicians is involved with problems of correctness of the boundary value problems (BVP) for the degenerating equations of the mixed parabolic-hyperbolic, elliptic-hyperbolic and hyperbolic-hyperbolic types. It is closely connected with appendices of such problems to the decision of problems of mechanics, gas dynamics, biology and in other material sciences. The first basic researches under the theory of the degenerating equations of the mixed and mixed-compound type are Tricomi’s [1], Gellerstedt’s et al. [2], Bitsadze’s [3] and Salahitdinov’s [1,4,5] works. The degenerating and singularity equations possess that nature, that for them the correctness of some classical problems not always takes place. This fact rather for the degenerating equations of elliptic type, in the first has been noticed of MKeldych [6], and concerning the degenerating equations of hyperbolic type of Gellerstedt. In this cases Bitsadze has suggested to study modify problems Cauchy for the degenerating equation of hyperbolic type because the problems Cauchy for such equations it is put incorrectly. Since Bitsadze’s [2] works, in the theory partial differential equations there was a new direction, in which the analogue of problem Tricomi for the first time is formulated and investigated in double connected domain for the modeling equations of the third order of elliptic-hyperbolic type [7], and uniqueness of solution of the problem for the degenerating equations of hyperbolic-hyperbolic type in double-connected domain was proved by Islomov et al. [4].

The Statement of Problems
In the present work the analogues of problem A.V. Bitsadze [2] is formulated and investigated for the hyperbolic-hyperbolic type equation with two degenerating lines of the following kind:

\[(−y)^r u_n − x^r u_\tau = 0, n = const > 0\]  \hspace{1cm} (1)

in the special domain $\Omega$, bounded at $y<0$ with characteristics

\[A_j C_j : \left(−1\right)^{j+1} x^{\alpha_j} + \left(−y\right)^{\beta_j} = q_j, j = 1, 2\]

\[B_j C_j : \left(−1\right)^{j+1} x^{\alpha_j} + \left(−y\right)^{\beta_j} = 1; (j = 1, 2), \left(0 < q < 1\right)\]

of the equation (1), and at $y = 0$ with segments $A_j B_j$ where $A_j \left(−1\right)^{j+1} q_j, 0\right)$ $B_j \left(−1\right)^{j+1} 0, j = 1, 2$.

We introduce the following notations:
Through $\Omega_1$ and $\Omega_2$ we will designate characteristic triangles $A_j B_j E_j$ and $C_j F_j C_j$ ($j = 1, 2$), accordingly, and through $\Omega_3$ we will designate characteristic quadrangles $A_j E_j F_j C_j$ ($j = 1, 2$). In the section 2 we have formulated and proved unique solvability of a problems I(I*) and II (II*) in the domain of , which, consist of four characteristic triangles and from two quadrangles. The result, which is obtained in this section shows that when we will investigate problems I(I*) and II(II*), in each sub domains, we find the solution of equation (1)in an explicit form In the section 3 we studying uniqueness and existence of solution of a problem III (III’), Uniqueness of solution of problem III (III’’) are proven with principle extremum. Existence of the solution of problem III (III’’) we have proved, by method integral equations. The main result of this section shows that when we will studying existence of the solution of problem III(III’’), we have singularity integral equation, which regularities by the method of Karleman’s-Vekua [8], to the integral equation of Fredgolm of the second kind. Unique solvability of the problems I(I’’) and II(II’’)

Problem I: Find a function $u(x, y)$ in the domain $\Omega$ with following properties:

1) $u(x, y) \in C \left(\Omega_1 \cap \Omega_2^2 \right)$;

2) $u(x, y)$ satisfies the equation (1) in domains $\Omega_1$, $\Omega_2$, $\Omega_3$, $\Omega_4$ ($j = 1, 2$);

3) $u(x, y)$ satisfies the following conditions

$u(x, 0) = \tau_j (x), x \in \overline{A_j B_j}$ \hspace{1cm} (2)

$u(x, y)|_{\partial \Omega_3^j} = \varphi_j (x), x \in [0, 1]$ \hspace{1cm} (3)

$u(x, y)|_{\partial \Omega_4^j} = \psi_j (x), x \in [-q, 0]$ \hspace{1cm} (4)

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Where \( T(x), \varphi(x), \Psi(x) \) - given functions, and 
\[ \tau_j(x) \in C(A_i,A_j) \cap C^1(A_i,A_j) \]
\[ r_1(x) = \varphi(1), r_2(-q) = \Psi(-q) \]  
(5)
\[ \varphi(x) \in C[0,1] \cap C^1(0,1), \psi(x) \in C[-q,0] \cap C^1(-q,0) \]  
(5)

**Problem 1:** Find a function \( u(x,y) \) in domain \( \Omega \), satisfies to all conditions problem I(1), except (2) (1) and (2) which are replaced with conditions:
\[ u(x,0) = v(x), x \in A_j \]  
(6)
where \( v(x) \) - given functions, and
\[ v_j(x) \in C(A_i,A_j), (j=1,2) \]  
(7)

**Problem II (II!):** Find a function \( u(x,y) \) in domain \( \Omega \), satisfies to all conditions problem I(1), except (3) (1) and (4) which are replaced with conditions:
\[ u[B,E] = \mu(x), x \in \partial \Omega \]  
(8)

And
\[ u[A,C] = \psi(x), x \in \partial \Omega \]  
(9)
where \( \mu(x), \psi(x) \)- given functions, and:
\[ u(x) \in C(\partial \Omega), \partial \Omega \cap C(\partial \Omega), \cap C(\partial \Omega) \]  
(10)

**Theorem 1:** If conditions (5) (5) and (7)) are satisfied that the solution of problem I(1) is unique solvability.

Proof: Is known, that the solution of problem Cauchy in domain \( \Omega \), for the equation (1) satisfying to conditions (2) (1) and (6) looks like
\[ u(x,y) = \int_{t-q}^{t} \left[ r(z (t-q)) (t-q) - \right] dt \]  
(11)

From here, by virtue condition (3), considering (51) (71)) it is easily possible to define unknown function \( v(x) (T_i(x)) \), hence, owing to uniqueness of the solution of problem Cauchy, the solution of the problem I(1) in domain \( \Omega \) is uniquely defined.

Further, designating through \( h(x) \) a trace of the solution of problem Cauchy-Goursat 1 (Cauchy-Goursat 2) from domain \( \Omega \), on the characteristic \( A_i \) and \( E \) considering the condition (3)) taking into account (5), by the method of Riemann, we restore the solution of the problem I(1) in domain of the \( \Omega \), as the solution of problem Cauchy and this solution is given by the formula:
\[ u(x,y) = 2 \int_{t-q}^{t} \left[ r(z (t-q)) (t-q) - \right] dt \]  
(11)

- \[ \int_{t-q}^{t} \left[ h \left( \left( \frac{q^{2n+2}}{2} + \right) \left( \frac{t+1-q}{2} \right)^{2} \right) \right] dt \]  
(13)

Where
\[ h(x) = \frac{1}{1-x} \]  
(14)

**Problem III (III!):** Find a function \( u(x,y) \) in domain \( \Omega \) satisfies to all conditions problem I(1), except (3) which are replaced with conditions:
\[ u[B,E] = \varphi^*(x), x \in \left[ \frac{1-q}{2}, 1 \right] \]  
(13)
\[ u[F,C] = \psi^*(x), x \in \left[ \frac{q-1}{2}, 0 \right] \]  
(14)

where \( \varphi^*(x), \psi^*(x) \) - given functions, and
\[ \varphi^*(x) \in C \left[ \frac{1-q}{2}, 1 \right], \cap C \left[ \frac{q-1}{2}, 0 \right], \ \cap C \left[ \frac{q-1}{2}, 0 \right] \]  
(15)

**Theorem 3:** If conditions (5) (51) (71) are satisfied and (15) that solution of a problem III(1) exists and is unique.

Proof: Is known, that the solution of problem Cauchy in domain \( \Omega \), for the equation (1) satisfying to conditions (0), (1) and (6) looks like:
\[ u(x,y) = \int_{t-q}^{t} \left[ r(z (t-q)) (t-q) - \right] dt \]  
(11)

Further, designating through \( h(x) \) a trace of the solution of problem Cauchy-Goursat 1 (Cauchy-Goursat 2) from domain \( \Omega \), on the characteristic \( A_i \) and \( E \) considering the condition (3)) taking into account (5), by the method of Riemann, we restore the solution of the problem I(1) in domain of the \( \Omega \), as the solution of problem Cauchy and this solution is given by the formula:
\[ u(x,y) = 2 \int_{t-q}^{t} \left[ r(z (t-q)) (t-q) - \right] dt \]  
(11)

Where
\[ \varphi^*(x) \in C \left[ \frac{1-q}{2}, 1 \right], \cap C \left[ \frac{q-1}{2}, 0 \right], \ \cap C \left[ \frac{q-1}{2}, 0 \right] \]  
(15)
From here, owing to condition $u(x, y)|_{y^*} = h_2(y^*)$ and (14) taking into account properties of integro-differential operators of fractional order [8,9] accordingly we will receive:

$$v(y) = \frac{y}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1} D_1, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)$$  \hspace{1cm} (18)$$

And

$$v(y) = \frac{y}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1} D_1, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)$$  \hspace{1cm} (19)$$

where $h_2(y)$ - a trace of solution of the Goursat in domain $\Omega_{12}$, satisfying condition (13) and

$$u(x, y)|_{y^*} = h_2(y^*)$$

and $y^* = \frac{y}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1}$, $r_1(y) = r\left(y^{2\beta - 1}\right)$.

At the proof of the theorem 3 takes place

Lemma: The solution $u(x, y)$ of the problem III (III*) at $\tau = \Phi (t)$ and $\beta = \Psi (t)$, accordingly we will receive $\Phi (t)$, hence in the domain the positive maximum and negative minimum reaches only in points $C_1$ and $C_2$.

Proof: By virtue (20) and considering solutions of problem Cauchy-Goursat (in the domain $\Omega_1$) and Goursat (in the domain $\Omega_2$) for the equation (1) we will receive, that $u(x, y)|_{y^*}=0$ on the characteristic $C_1C$. From here, owing to a principle of extremum for the hyperbolic equations [4,9,10] function $u(x, y)$ reaches the positive maximum and the negative minimum in the domain $\Omega_{12}$ only on the piece $C_1C_2$.

Hence, owing to continuity of solution $u(x, y)$ have received the contradiction, i.e., the function $u(x, y)$ does not reach the positive maximum (the negative minimum) in the interval $C_1C_2$. Hence, function $u(x, y)$ can reach the positive maximum (the negative minimum) only on the points $C_1$ and $C_2$. The lemma is proved. As, $u(x, y)|_{y^*}=0$ on the characteristic $C_1C$, we have that, $u(x, y)|_{y^*}=0$ on the points $C_1$ and $C_2$. From here, owing to continuity of solution $u(x, y)$ in the domain of the problem III (III*) with zeroes dates, has only trivial solution, i.e. uniqueness of the solution of problem III (III*) is roved. Existence of the solution of the problem III (III*) is proved, by method integral equations.

From functional relation

$$r_1(y) = \frac{\pi}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1} D_{12}, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)$$  \hspace{1cm} (21)$$

And

$$r_1(y) = \frac{\pi}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1} D_{12}, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)$$  \hspace{1cm} (22)$$

excluding $r_1(y)$ and considering properties of the integro-differential operators, we will receive singular integral equation.

$$v(y) + \cos \pi(1-2\beta) - \frac{\sin \pi(1-2\beta)}{\pi} \int_{\Delta y} \left( 1-y^* \right)^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y) = \Phi(t)$$  \hspace{1cm} (23)$$

where

$$\Phi(t) = \frac{\pi}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1} D_{12}, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)$$  \hspace{1cm} (24)$$

Entering designations, $a(y^*)=1 + \cos \pi(1-2\beta)$, $b(y^*)=\sin \pi(1-2\beta)$ and

$$K(t, y^*) = \frac{1}{\gamma \beta (2 \beta - 1)} y^{2\beta - 1} D_{12}, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)$$  \hspace{1cm} (25)$$

we will copy the equation (23) in the form of integral Cauchy [11,12]:

$$a(y^*)v(y^*) - b(y^*) = \Phi(y^*)$$  \hspace{1cm} (26)$$

We will estimate the function $\Phi(y^*)$, for this considering properties integro-differential operators, we have from (24)

$$\Phi(y^*) = \frac{d}{\gamma \beta (2 \beta - 1) D_{12}, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)}$$

From here, having executed replacements $s = (1-q^{2\beta} z + q^{2\beta-2})$ in first inner integral and $s = (1-t) z + t$ in second inner integral, we will receive:

$$\Phi(y^*) = \frac{\beta A_{y^*}^{2\beta - 1}}{\gamma \beta (2 \beta - 1) D_{12}, h_2(y^*) (y^* - q)^{2\beta - 1} - q^{2\beta - 1} y^{2\beta - 1} D_{12}, v(y)}$$

Hence, owing to properties $B_{\alpha, \beta}(t, z)$ and taking into account a continuity of functions $h_2(t, z)$, $h_2(t, z)$ and $\psi(t, z)$, $\psi(t, z)$ we will receive the estimate for function $\Phi(y^*)$:

$$\Phi(y^*) \leq const \cdot (1-y^*)^{2\beta - 1}$$

As, $\Phi(y^*) = \Phi(y^*)$ the integral equation (23) is singular integral equation of the normal type, and by virtue (27) and $|K(t, y^*)| \leq const \cdot t^{1-2\beta}$ we have that, index of integral equation (27) is equal to zero. Hence, by virtue of the theory singular integral equations and by the method regularities of Karleman-Vekua [11], the integral equation (26) will be reduced to the integral equation of Fredholm of the second kind with weak singularity. Thus, by virtue uniqueness of solution of the problem III (III*) the function $v(y^*) \in C(1-q; \Omega_{12})$ and $\psi(t, z) = const \cdot (1-y^*)^{2\beta - 1}$ after it is found $v_3(y)$, from (21) and (22) we will found $v_3(y)$
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