Para, pseudo, and orthosupersymmetric quantum mechanics and their bosonization

C. Quesne

PNTPM, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

Abstract

We consider the problem of bosonizing supersymmetric quantum mechanics (SSQM) and some of its variants, i.e., of realizing them in terms of only boson-like operators without fermion-like ones. In the SSQM case, this is realized in terms of the generators of the Calogero-Vasiliev algebra (also termed deformed Heisenberg algebra with reflection). In that of the SSQM variants, this is done by considering generalizations of the latter algebra, namely the $C_{\lambda}$-extended oscillator algebras, where $C_{\lambda}$ is the cyclic group of order $\lambda$.

1 Introduction

Supersymmetry has established an elegant symmetry between bosons and fermions and is one of the cornerstones of modern theoretical physics. Its application to quantum mechanics has provided a powerful method of generating solvable quantum mechanical models. On the other hand, exotic quantum statistics have received considerable attention due to their possible relevance to the fractional quantum Hall effect and anyon superconductivity.

By combining both concepts within the framework of quantum mechanics, one gets variants of SSQM: paraSSQM [1, 2, 3], pseudoSSQM [4, 5], and orthoSSQM [6, 7]. They can be realized in terms of bosons and parafermions [7], pseudofermions [4, 5], or orthofermions [8], respectively.

*E-mail: cquesne@ulb.ac.be
By using the Calogero-Vasiliev algebra [9], Plyushchay showed [10] that SSQM can be described in terms of only boson-like operators without fermion-like ones (see also [11]).

In the present communication, we shall consider generalizations of the Calogero-Vasiliev algebra, namely the $C_\lambda$-extended oscillator algebras (where $C_\lambda = Z_\lambda$ is the cyclic group of order $\lambda$) [12, 13, 14]. We shall show that they have some interesting applications to variants of SSQM [12, 14], as they provide a bosonization of the latter analogous to that obtained by Plyushchay for SSQM.

2 Generalized deformed and $\mathcal{G}$-extended oscillator algebras

The generalized deformed oscillator algebras (GDOAs) (see e.g. Refs. [15, 16] and references quoted therein) arose from successive generalizations of the Arik-Coon [17] and Biedenharn-Macfarlane [18, 19] $q$-oscillators. Such algebras, denoted by $\mathcal{A}_q(G(N))$, are generated by the unit, creation, annihilation, and number operators $I$, $a^\dagger$, $a$, $N$, satisfying the Hermiticity conditions $(a^\dagger)\dagger = a$, $N^\dagger = N$, and the commutation relations

$$\left[ N, a^\dagger \right] = a^\dagger, \quad \left[ N, a \right] = -a, \quad \left[ a, a^\dagger \right]_q = a a^\dagger - q a^\dagger a = G(N),$$

where $q$ is some real number and $G(N)$ is some Hermitian, analytic function.

On the other hand, $\mathcal{G}$-extended oscillator algebras, where $\mathcal{G}$ is some finite group, appeared in connection with $n$-particle integrable models. For the Calogero model [20], for instance, $\mathcal{G}$ is the symmetric group $S_n$ [21, 22].

For two particles, the $S_2$-extended oscillator algebra $\mathcal{A}_\kappa^{(2)}$, where $S_2 = \{ I, K \mid K^2 = I \}$, is generated by the operators $I$, $a^\dagger$, $a$, $N$, $K$, subject to the Hermiticity conditions $(a^\dagger)\dagger = a$, $N^\dagger = N$, $K^\dagger = K^{-1}$, and the relations

$$\left[ N, a^\dagger \right] = a^\dagger, \quad \left[ N, K \right] = 0, \quad K^2 = I,$$
$$\left[ a, a^\dagger \right] = I + \kappa K \quad (\kappa \in \mathbb{R}), \quad a^\dagger K = -K a^\dagger,$$

(2)

together with their Hermitian conjugates.
When the $S_2$ generator $K$ is realized in terms of the Klein operator $(-1)^N$, $A^{(2)}_\kappa$ becomes a GDOA characterized by $q = 1$ and $G(N) = I + \kappa(-1)^N$, and known as the Calogero-Vasiliev oscillator algebra [9].

The operator $K$ may be alternatively considered as the generator of the cyclic group $C_2$ of order two, since the latter is isomorphic to $S_2$. By replacing $C_2$ by the cyclic group of order $\lambda$, $C_\lambda = \{ I, T, T^2, \ldots, T^{\lambda-1} \mid T^\lambda = I \}$, one then gets a new class of $G$-extended oscillator algebras [12, 13, 14], generalizing that describing the two-particle Calogero model.

### 3 $C_\lambda$-extended oscillator algebras

Let us consider the algebras generated by the operators $I$, $a^\dagger$, $a$, $N$, $T$, satisfying the Hermiticity conditions $(a^\dagger)^\dagger = a$, $N^\dagger = N$, $T^\dagger = T^{-1}$, and the relations

$$
\begin{align*}
[N, a^\dagger] &= a^\dagger, & [N, T] &= 0, & T^\lambda &= I, \\
[a, a^\dagger] &= I + \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu, & a^\dagger T &= e^{-i2\pi/\lambda} T a^\dagger,
\end{align*}
$$

(3)

together with their Hermitian conjugates [12]. Here $T$ is the generator of (a unitary representation of) the cyclic group $C_\lambda$ (where $\lambda \in \{ 2, 3, 4, \ldots \}$), and $\kappa_\mu$, $\mu = 1, 2, \ldots, \lambda - 1$, are some complex parameters restricted by the conditions $\kappa_\mu^* = \kappa_{\lambda-\mu}$ (so that there remain altogether $\lambda - 1$ independent real parameters).

$C_\lambda$ has $\lambda$ inequivalent, one-dimensional matrix unitary irreducible representations (unirreps) $\Gamma^\mu$, $\mu = 0, 1, \ldots, \lambda - 1$, which are such that $\Gamma^\mu (T^\nu) = \exp(i2\pi \mu \nu/\lambda)$ for any $\nu = 0, 1, \ldots, \lambda - 1$. The projection operator on the carrier space of $\Gamma^\mu$ may be written as

$$
P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{-i2\pi \mu \nu/\lambda} T^\nu,
$$

(4)

and conversely $T^\nu$, $\nu = 0, 1, \ldots, \lambda - 1$, may be expressed in terms of the $P_\mu$’s as

$$
T^\nu = \sum_{\mu=0}^{\lambda-1} e^{i2\pi \mu \nu/\lambda} P_\mu.
$$

(5)

The algebra defining relations (3) may therefore be rewritten in terms of $I$, $a^\dagger$, $a$, $N$, and $P_\mu = P_\mu^\dagger$, $\mu = 0, 1, \ldots, \lambda - 1$, as

$$
\begin{align*}
[N, a^\dagger] &= a^\dagger, & [N, P_\mu] &= 0, & \sum_{\mu=0}^{\lambda-1} P_\mu &= I,
\end{align*}
$$

(3)
\[
[a, a^\dagger] = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, \quad a^\dagger P_\mu = P_{\mu+1} a^\dagger, \quad P_\mu P_\nu = \delta_{\mu,\nu} P_\mu, \quad (6)
\]

where we use the convention \( P_{\mu'} = P_\mu \) if \( \mu' - \mu = 0 \) mod \( \lambda \) (and similarly for other operators or parameters indexed by \( \mu, \mu' \)). Equation (3) depends upon \( \lambda \) real parameters \( \alpha_\mu = \sum_{\nu=1}^{\lambda-1} \exp(i2\pi \mu \nu / \lambda) \kappa_\nu, \mu = 0, 1, \ldots, \lambda - 1 \), restricted by the condition \( \sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0 \). Hence, we may eliminate one of them, for instance \( \alpha_{\lambda-1} \), and denote \( C_\lambda \)-extended oscillator algebras by \( \mathcal{A}_{\alpha_0, \ldots, \alpha_{\lambda-2}}^{(\lambda)} \).

The cyclic group generator \( T \) and the projection operators \( P_\mu \) can be realized in terms of \( N \) as

\[
T = e^{i2\pi N/\lambda}, \quad P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{i2\pi \nu(N-\mu)/\lambda}, \quad \mu = 0, 1, \ldots, \lambda - 1, \quad (7)
\]

respectively. With such a choice, \( \mathcal{A}_{\alpha_0, \ldots, \alpha_{\lambda-2}}^{(\lambda)} \) becomes a GDOA, \( \mathcal{A}^{(\lambda)}(G(N)) \), characterized by \( q = 1 \) and \( G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \), where \( P_\mu \) is given in Eq. (7).

For any GDOA \( \mathcal{A}_q(G(N)) \), one may define a so-called structure function \( F(N) \), which is the solution of the difference equation \( F(N + 1) - qF(N) = G(N) \), such that \( F(0) = 0 \) \([14]\). For \( \mathcal{A}^{(\lambda)}(G(N)) \), we find

\[
F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu, \quad \beta_0 \equiv 0, \quad \beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu \quad (\mu = 1, 2, \ldots, \lambda - 1). \quad (8)
\]

At this point, it is worth noting that for \( \lambda = 2 \), we obtain \( T = K, P_0 = (I+K)/2, P_1 = (I-K)/2, \) and \( \kappa_1 = \kappa_1^* = \alpha_0 = -\alpha_1 = \kappa \), so that \( \mathcal{A}_{\alpha_0}^{(2)} \) coincides with the \( S_2 \)-extended oscillator algebra \( \mathcal{A}_K^{(2)} \) and \( \mathcal{A}^{(2)}(G(N)) \) with the Calogero-Vasiliev algebra.

In Ref. [14], it was shown that \( \mathcal{A}^{(\lambda)}(G(N)) \) (and more generally \( \mathcal{A}_{\alpha_0, \ldots, \alpha_{\lambda-2}}^{(\lambda)} \)) has only two different types of unirreps: infinite-dimensional bounded from below unirreps and finite-dimensional ones. Among the former, there is the so-called bosonic Fock space representation, wherein \( a^\dagger a = F(N) \) and \( aa^\dagger = F(N + 1) \). Its carrier space \( \mathcal{F} \) is spanned by the eigenvectors \( |n\rangle \) of the number operator \( N \), corresponding to the eigenvalues \( n = 0, 1, 2, \ldots \), where \( |0\rangle \) is a vacuum state, i.e., \( a|0\rangle = N|0\rangle = 0 \) and \( P_\mu|0\rangle = \delta_{\mu,0}|0\rangle \). The eigenvectors can be written as

\[
|n\rangle = \mathcal{N}_n^{-1/2} (a^\dagger)^n |0\rangle, \quad n = 0, 1, 2, \ldots, \quad (9)
\]

\[
\sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu = \delta_{\mu,\nu} P_\mu, \quad (6)
\]
where $N_n = \prod_{i=1}^{n} F(i)$. The creation and annihilation operators act upon $|n\rangle$ in the usual way, i.e.,

\[ a^\dagger |n\rangle = \sqrt{F(n+1)} |n+1\rangle, \quad a|n\rangle = \sqrt{F(n)} |n-1\rangle, \quad (10) \]

while $P_\mu$ projects on the $\mu$th component $F_\mu = \{|k\lambda + \mu\rangle \mid k = 0, 1, 2, \ldots\}$ of the $Z_\lambda$-graded Fock space $F = \sum_{\mu=0}^{\lambda-1} F_\mu$. It is obvious that such a bosonic Fock space representation exists if and only if $F(\mu) > 0$ for $\mu = 1, 2, \ldots, \lambda - 1$. This gives the following restrictions on the algebra parameters $\alpha_\mu$:

\[ \sum_{\nu=0}^{\mu-1} \alpha_\nu > -\mu, \quad \mu = 1, 2, \ldots, \lambda - 1. \quad (11) \]

In the bosonic Fock space representation, one may consider the bosonic oscillator Hamiltonian, defined as usual by

\[ H_0 \equiv \frac{1}{2} \{a, a^\dagger\}. \quad (12) \]

It can be rewritten as

\[ H_0 = a^\dagger a + \frac{1}{2} \left( I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \right) = N + \frac{1}{2} I + \sum_{\mu=0}^{\lambda-1} \gamma_\mu P_\mu, \quad (13) \]

where $\gamma_0 \equiv \frac{1}{2} \alpha_0$ and $\gamma_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu + \frac{1}{2} \alpha_\mu$ for $\mu = 1, 2, \ldots, \lambda - 1$.

The eigenvectors of $H_0$ are the states $|n\rangle = |k\lambda + \mu\rangle$, defined in Eq. (9), and their eigenvalues are given by

\[ E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \frac{1}{2}, \quad k = 0, 1, 2, \ldots, \quad \mu = 0, 1, \ldots, \lambda - 1. \quad (14) \]

In each $F_\mu$ subspace of the $Z_\lambda$-graded Fock space $F$, the spectrum of $H_0$ is therefore harmonic, but the $\lambda$ infinite sets of equally spaced energy levels, corresponding to $\mu = 0, 1, \ldots, \lambda - 1$, may be shifted with respect to each other by some amounts depending upon the algebra parameters $\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}$, through their linear combinations $\gamma_\mu, \mu = 0, 1, \ldots, \lambda - 1$.

For the Calogero-Vasiliev oscillator, i.e., for $\lambda = 2$, the relation $\gamma_0 = \gamma_1 = \kappa/2$ implies that the spectrum is very simple and coincides with that of a shifted harmonic oscillator. For $\lambda \geq 3$, however, it has a much richer structure. According to the parameter values, it may be nondegenerate, or may exhibit some $(\nu+1)$-fold...
degeneracies above some energy eigenvalue, where \( \nu \) may take any value in the set \( \{1, 2, \ldots, \lambda - 1\} \). In Ref. [13], the complete classification of nondegenerate, twofold and threefold degenerate spectra was obtained for \( \lambda = 3 \) in terms of \( \alpha_0 \) and \( \alpha_1 \).

In the remaining part of this communication, we will show that the bosonic Fock space representation of \( \mathcal{A}(\lambda)(G(N)) \) and the corresponding bosonic oscillator Hamiltonian \( H_0 \) have some useful applications to variants of SSQM.

4 Application to parasupersymmetric quantum mechanics of order \( p \)

In SSQM with two supercharges, the supersymmetric Hamiltonian \( \mathcal{H} \) and the supercharges \( Q^\dagger, Q = (Q^\dagger)^\dagger \), satisfy the sqm(2) superalgebra, defined by the relations

\[
Q^2 = 0, \quad [\mathcal{H}, Q] = 0, \quad \{Q, Q^\dagger\} = \mathcal{H},
\]

(15)
together with their Hermitian conjugates. Such a superalgebra is most often realized in terms of mutually commuting boson and fermion operators.

Plyushchay [10], however, showed that it can alternatively be realized in terms of only boson-like operators, namely the generators of the Calogero-Vasiliev algebra \( \mathcal{A}^{(2)}(G(N)) \) (see also Ref. [11]). The SSQM bosonization can be performed in two different ways, by choosing either \( Q = a^\dagger P_1 \) (so that \( \mathcal{H} = H_0 - \frac{1}{2}(K + \kappa) \)) or \( Q = a^\dagger P_0 \) (so that \( \mathcal{H} = H_0 + \frac{1}{2}(K + \kappa) \)). The first choice corresponds to unbroken SSQM (all the excited states are twofold degenerate while the ground state is nondegenerate and at vanishing energy), and the second choice describes broken SSQM (all the states are twofold degenerate and at positive energy).

SSQM was generalized to parasupersymmetric quantum mechanics (PSSQM) of order two by Rubakov and Spiridonov [1], and later on to PSSQM of arbitrary order \( p \) by Khare [2]. In the latter case, Eq. (15) is replaced by

\[
Q^{p+1} = 0 \quad \text{(with } Q^p \neq 0),
\]

\[
[H, Q] = 0,
\]

\[
Q^pQ^\dagger + Q^{p-1}Q^\dagger Q + \cdots + QQ^\dagger Q + Q^\dagger Q^p = 2pQ^{p-1}H,
\]

(16)
and is retrieved in the case where \( p = 1 \). The parasupercharges \( Q, Q^\dagger \), and the parasupersymmetric Hamiltonian \( \mathcal{H} \) are usually realized in terms of mutually commuting boson and parafermion operators.

A property of PSSQM of order \( p \) is that the spectrum of \( \mathcal{H} \) is \((p+1)\)-fold degenerate above the \((p-1)\)th energy level. This fact and Plyuschay’s results for \( p = 1 \) hint at a possibility of representing \( \mathcal{H} \) as a linear combination of the bosonic oscillator Hamiltonian \( H_0 \) associated with \( A^{(p+1)}(G(N)) \) and some projection operators.

In Ref. [14] (see also Ref. [12]), it was proved that PSSQM of order \( p \) can indeed be bosonized in terms of the generators of \( A^{(p+1)}(G(N)) \) for any allowed (i.e., satisfying Eq. (11)) values of the algebra parameters \( \alpha_0, \alpha_1, \ldots, \alpha_{p-1} \). For such a purpose, ansätze of the type

\[
Q = \sum_{\nu=0}^{p} \sigma_{\nu} a^\dagger P_{\nu}, \quad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^{p} r_{\nu} P_{\nu},
\]

were chosen. Here \( \sigma_{\nu} \) and \( r_{\nu} \) are some complex and real constants, respectively, to be determined in such a way that Eq. (16) is fulfilled. It was found that there are \( p + 1 \) families of solutions, which may be distinguished by an index \( \mu \in \{0, 1, \ldots, p\} \) and from which one may choose the following representative solutions

\[
Q_\mu = \sqrt{2} \sum_{\nu=1}^{p} a^\dagger P_{\mu+\nu},
\]

\[
\mathcal{H}_\mu = N + \frac{1}{2} (2\gamma_{\mu+2} + r_{\mu+2} - 2p + 3) I + \sum_{\nu=1}^{p} (p+1-\nu) P_{\mu+\nu},
\]

where

\[
r_{\mu+2} = \frac{1}{p} \left[ (p-2) \alpha_{\mu+2} + 2 \sum_{\nu=3}^{p} (p-\nu+1) \alpha_{\mu+\nu} + p(p-2) \right].
\]

The eigenvectors of \( \mathcal{H}_\mu \) are the states (9) and the corresponding eigenvalues are easily found. All the energy levels are equally spaced. For \( \mu = 0 \), PSSQM is unbroken, otherwise it is broken with a \((\mu+1)\)-fold degenerate ground state. All the excited states are \((p+1)\)-fold degenerate. For \( \mu = 0, 1, \ldots, p-2 \), the ground state energy may be positive, null, or negative depending on the parameters, whereas for \( \mu = p-1 \) or \( p \), it is always positive.

Khare [2] showed that in PSSQM of order \( p \), \( \mathcal{H} \) has in fact \( 2p \) (and not only two) conserved parasupercharges, as well as \( p \) bosonic constants. In other words,
there exist $p$ independent operators $Q_r$, $r = 1, 2, \ldots, p$, satisfying with $\mathcal{H}$ the set of equations (16), and $p$ other independent operators $I_t$, $t = 2, 3, \ldots, p+1$, commuting with $\mathcal{H}$, as well as among themselves. In Ref. [14], a realization of all such operators was obtained in terms of the $\mathcal{A}^{(p+1)}(G(N))$ generators.

As a final point, let us note that there exists an alternative approach to PSSQM of order $p$, which was proposed by Beckers and Debergh [3], and wherein the multilinear relation in Eq. (16) is replaced by the cubic equation

$$\left[Q, \left[Q^\dagger, Q\right]\right] = 2Q\mathcal{H}. \quad (20)$$

In Ref. [12], it was proved that for $p = 2$, this PSSQM algebra can only be realized by those $\mathcal{A}^{(3)}(G(N))$ algebras that simultaneously bosonize Rubakov-Spiridonov-Khare PSSQM algebra.

5 Application to pseudosupersymmetric quantum mechanics

Pseudosupersymmetric quantum mechanics (pseudoSSQM) was introduced by Beckers, Debergh, and Nikitin [4, 5] in a study of relativistic vector mesons interacting with an external constant magnetic field. In the nonrelativistic limit, their theory leads to a pseudosupersymmetric oscillator Hamiltonian, which can be realized in terms of mutually commuting boson and pseudofermion operators, where the latter are intermediate between standard fermion and $p = 2$ parafermion operators.

It is then possible to formulate a pseudoSSQM [4, 5], characterized by a pseudosupersymmetric Hamiltonian $\mathcal{H}$ and pseudosupercrace operators $Q, Q^\dagger$, satisfying the relations

$$Q^2 = 0, \quad [\mathcal{H}, Q] = 0, \quad QQ^\dagger Q = 4c^2Q\mathcal{H}, \quad (21)$$

and their Hermitian conjugates, where $c$ is some real constant. The first two relations in Eq. (21) are the same as those occurring in SSQM, whereas the third one is similar to the multilinear relation valid in PSSQM of order two. Actually, for $c = 1$ or $1/2$, it is compatible with Eq. (16) or (20), respectively.

In Ref. [14], it was proved that pseudoSSQM can be bosonized in two different ways in terms of the generators of $\mathcal{A}^{(3)}(G(N))$ for any allowed values of the
parameters $\alpha_0, \alpha_1$. This time, the ansätze
\[
Q = \sum_{\nu=0}^{2} \left( \xi_\nu a + \eta_\nu a^\dagger \right) P_\nu, \quad H = H_0 + \frac{1}{2} \sum_{\nu=0}^{2} r_\nu P_\nu,
\]
were chosen, and the complex constants $\xi_\nu, \eta_\nu$, and the real ones $r_\nu$ were determined in such a way that Eq. (21) is fulfilled.

The first type of bosonization corresponds to three families of two-parameter solutions, labelled by an index $\mu \in \{0, 1, 2\}$,
\[
Q_\mu(\eta_{\mu+2}, \varphi) = \left( \eta_{\mu+2} a^\dagger + e^{i\varphi} \sqrt{4c^2 - \eta_{\mu+2}^2} a \right) P_{\mu+2},
\]
\[
H_\mu(\eta_{\mu+2}) = N + \frac{1}{2} (2\gamma_{\mu+2} + r_{\mu+2} - 1) I + 2P_{\mu+1} + P_{\mu+2},
\]
where $0 < \eta_{\mu+2} < 2|c|$, $0 \leq \varphi < 2\pi$, and
\[
r_{\mu+2} = \frac{1}{2c^2} \left( 1 + \alpha_{\mu+2} \right) \left( |\eta_{\mu+2}|^2 - 2c^2 \right).
\]
Choosing for instance $\eta_{\mu+2} = \sqrt{2}|c|$, and $\varphi = 0$, hence $r_{\mu+2} = 0$ (producing an overall shift of the spectrum), leads to
\[
Q_\mu = c\sqrt{2} \left( a^\dagger + a \right) P_{\mu+2},
\]
\[
H_\mu = N + \frac{1}{2} (2\gamma_{\mu+2} - 1) I + 2P_{\mu+1} + P_{\mu+2}.
\]
A comparison between Eq. (23) or (25) and Eq. (18) shows that the pseudosupersymmetric and $p = 2$ parasupersymmetric Hamiltonians coincide, but that the corresponding charges are of course different. The conclusions relative to the spectrum and the ground state energy are therefore the same as in Sec. 4.

The second type of bosonization corresponds to three families of one-parameter solutions, again labelled by an index $\mu \in \{0, 1, 2\}$,
\[
Q_\mu = 2|c| a P_{\mu+2},
\]
\[
H_\mu(r_\mu) = N + \frac{1}{2} (2\gamma_{\mu+2} - \alpha_{\mu+2}) I + \frac{1}{2} (1 - \alpha_{\mu+1} + \alpha_{\mu+2} + r_\mu) P_\mu + P_{\mu+1},
\]
where $r_\mu \in \mathbb{R}$ changes the Hamiltonian spectrum in a significant way. The levels are indeed equally spaced if and only if $r_\mu = (\alpha_{\mu+1} - \alpha_{\mu+2} + 3) \text{ mod } 6$. If $r_\mu$ is small enough, the ground state is nondegenerate, and its energy is negative for $\mu = 1,$
or may have any sign for $\mu = 0$ or $2$. On the contrary, if $r_\mu$ is large enough, the ground state remains nondegenerate with a vanishing energy in the former case, while it becomes twofold degenerate with a positive energy in the latter. For some intermediate $r_\mu$ value, one gets a two or threefold degenerate ground state with a vanishing or positive energy, respectively.

6 Application to orthosupersymmetric quantum mechanics of order two

Mishra and Rajasekaran \[8\] introduced order-$p$ orthofermion operators by replacing the Pauli exclusion principle by a more stringent one: an orbital state shall not contain more than one particle, whatever be the spin direction. The wave function is thus antisymmetric in spatial indices alone with the order of the spin indices frozen.

Khare, Mishra, and Rajasekaran \[6\] then developed orthosupersymmetric quantum mechanics (OSSQM) of arbitrary order $p$ by combining boson operators with orthofermion ones, for which the spatial indices are ignored. OSSQM is formulated in terms of an orthosupersymmetric Hamiltonian $H$, and $2p$ orthosupercharge operators $Q_r, Q_r^\dagger, r = 1, 2, \ldots, p$, satisfying the relations

$$Q_r Q_s = 0, \quad [H, Q_r] = 0, \quad Q_r Q_s + \delta_{r,s} \sum_{t=1}^{p} Q_t^\dagger Q_t = 2\delta_{r,s} H,$$

and their Hermitian conjugates, where $r$ and $s$ run over $1, 2, \ldots, p$.

In Ref. \[14\], it was proved that OSSQM of order two can be bosonized in terms of the generators of some well-chosen $A^{(3)}(G(N))$ algebras. As ansätze, the expressions

$$Q_1 = \sum_{\nu=0}^{2} \left( \xi_{\nu} a + \eta_{\nu} a^\dagger \right) P_{\nu}, \quad Q_2 = \sum_{\nu=0}^{2} \left( \zeta_{\nu} a + \rho_{\nu} a^\dagger \right) P_{\nu},$$

$$H = H_0 + \frac{1}{2} \sum_{\nu=0}^{2} r_\nu P_{\nu},$$

were used, and the complex constants $\xi_{\nu}, \eta_{\nu}, \zeta_{\nu}, \rho_{\nu}$, and the real ones $r_\nu$ were determined in such a way that Eq. \[27\] is fulfilled. There exist two families of two-parameter solutions, labelled by $\mu \in \{0, 1\}$,

$$Q_{1,\mu}(\xi_{\mu+2}, \varphi) = \xi_{\mu+2} a_{\mu+2} + e^{i\varphi} \sqrt{2 - \xi_{\mu+2}^2} a^\dagger P_{\mu},$$
\[ Q_{2,\mu}(\xi_{\mu+2}, \varphi) = -e^{-i\varphi} \sqrt{2 - \xi_{\mu+2}^2} a P_{\mu+2} + \xi_{\mu+2} a^\dagger P_\mu, \]
\[ H_\mu = N + \frac{1}{2}(2\gamma_{\mu+1} - 1)I + 2P_\mu + P_{\mu+1}, \] (29)

where \(0 < \xi_{\mu+2} \leq \sqrt{2}\) and \(0 \leq \varphi < 2\pi\), provided the algebra parameter \(\alpha_{\mu+1}\) is taken as \(\alpha_{\mu+1} = -1\). As a matter of fact, the absence of a third family of solutions corresponding to \(\mu = 2\) comes from the incompatibility of this condition (i.e., \(\alpha_0 = -1\)) with conditions (11).

The orthosupersymmetric Hamiltonian \(H\) in Eq. (29) is independent of the parameters \(\xi_{\mu+2}, \varphi\). All the levels of its spectrum are equally spaced. For \(\mu = 0\), OSSQM is broken: the levels are threefold degenerate, and the ground state energy is positive. On the contrary, for \(\mu = 1\), OSSQM is unbroken: only the excited states are threefold degenerate, while the nondegenerate ground state has a vanishing energy. Such results agree with the general conclusions of Ref. [6].

For \(p\) values greater than two, the OSSQM algebra (27) becomes rather complicated because the number of equations to be fulfilled increases considerably. A glance at the 18 independent conditions for \(p = 3\) led to the conclusion that the \(A^{(4)}(G(N))\) algebra is not rich enough to contain operators satisfying Eq. (27). Contrary to what happens for PSSQM, for OSSQM the \(p = 2\) case is therefore not representative of the general one.

7 Conclusion

In this communication, we showed that the \(S_2\)-extended oscillator algebra, which was introduced in connection with the two-particle Calogero model, can be extended to the whole class of \(C_\lambda\)-extended oscillator algebras \(A^{(\lambda)}_{\alpha_0, \alpha_1 \ldots, \alpha_{\lambda-2}}\), where \(\lambda \in \{2, 3, \ldots\}\), and \(\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}\) are some real parameters. In the same way, the GDOA realization of the former, known as the Calogero-Vasiliev algebra, is generalized to a class of GDOAs \(A^{(\lambda)}(G(N))\), where \(\lambda \in \{2, 3, \ldots\}\), for which one can define a bosonic oscillator Hamiltonian \(H_0\), acting in the bosonic Fock space representation.

For \(\lambda \geq 3\), the spectrum of \(H_0\) has a very rich structure in terms of the algebra parameters \(\alpha_0, \alpha_1, \ldots, \alpha_{\lambda-2}\). This can be exploited to provide a bosonization of PSSQM of order \(p = \lambda - 1\), and, for \(\lambda = 3\), a bosonization of pseudoSSQM and OSSQM of order two.
References

[1] V.A. Rubakov and V.P. Spiridonov, Parasupersymmetric Quantum Mechanics, Mod. Phys. Lett. A3 (1988) 1337.

[2] A. Khare, Parasupersymmetry in Quantum Mechanics, J. Math. Phys. 34 (1993) 1277.

[3] J. Beckers and N. Debergh, Parastatistics and Supersymmetry in Quantum Mechanics, Nucl. Phys. B340 (1990) 767.

[4] J. Beckers, N. Debergh and A.G. Nikitin, On Parasupersymmetries and Relativistic Descriptions for Spin one Particles: II. The Interacting Context with (Electro)Magnetic Fields, Fortschr. Phys. 43 (1995) 81.

[5] J. Beckers and N. Debergh, From Relativistic Vector Mesons in Constant Magnetic Fields to Nonrelativistic (Pseudo)Supersymmetries, Int. J. Mod. Phys. A10 (1995) 2783.

[6] A. Khare, A.K. Mishra, and G. Rajasekaran, Orthosupersymmetric Quantum Mechanics, Int. J. Mod. Phys. A8 (1993) 1245.

[7] Y. Ohnuki and S. Kamefuchi, Quantum Field Theory and Parastatistics, Springer-Verlag, Berlin, 1982.

[8] A.K. Mishra and G. Rajasekaran, Algebra for Fermions with a New Exclusion Principle, Pramana - J. Phys. 36 (1991) 537.

[9] M.A. Vasiliev, Higher Spin Algebras and Quantization on the Sphere and Hyperboloid, Int. J. Mod. Phys. A6 (1991) 1115.

[10] M.S. Plyushchay, Deformed Heisenberg Algebra, Fractional Spin Fields, and Supersymmetry without Fermions, Ann. Phys. (N.Y.) 245 (1996) 339.

[11] J. Beckers, N. Debergh, and A.G. Nikitin, Reducibility of Supersymmetric Quantum Mechanics, Int. J. Theor. Phys. 36 (1997) 1991.

[12] C. Quesne and N. Vansteenkiste, Cλ-Extended Harmonic Oscillator and (Para)Supersymmetric Quantum Mechanics, Phys. Lett. A240 (1998) 21.
[13] C. Quesne and N. Vansteenkiste, *Algebraic Realization of Supersymmetric Quantum Mechanics for Cyclic Shape Invariant Potentials*, Helv. Phys. Acta **72** (1999) 71.

[14] C. Quesne and N. Vansteenkiste, *C_\lambda-Extended Oscillator Algebras and Some of Their Deformations and Applications to Quantum Mechanics*, Int. J. Theor. Phys. **39** (2000) 1175.

[15] C. Quesne and N. Vansteenkiste, *Generalized q-Oscillators and Their Hopf Structures*, J. Phys. **A28** (1995) 7019.

[16] C. Quesne and N. Vansteenkiste, *Representation Theory of Deformed Oscillator Algebras*, Helv. Phys. Acta **69** (1996) 141.

[17] M. Arik and D.D. Coon, *Hilbert Spaces of Analytic Functions and Generalized Coherent States*, J. Math. Phys. **17** (1976) 524.

[18] L.C. Biedenharn, *The Quantum Group SU_q(2) and a q-Analogue of the Boson Operators*, J. Phys. **A22** (1989) L873.

[19] A.J. Macfarlane, *On q-Analogues of the Quantum Harmonic Oscillator and the Quantum Group SU(2)_q*, J. Phys. **A22** (1989) 4581.

[20] F. Calogero, *Solution of the One-Dimensional N-Body Problems with Quadratic and/or Inversely Quadratic Pair Potentials*, J. Math. Phys. **12** (1971) 419.

[21] A.P. Polychronakos, *Exchange Operator Formalism for Integrable Systems of Particles*, Phys. Rev. Lett. **69** (1992) 703.

[22] L. Brink, T.H. Hansson and M.A. Vasiliev, *Explicit Solution to the N-Body Calogero Problem*, Phys. Lett. **B286** (1992) 109.