HIGHER RANK GRAPH $C^*$-ALGEBRAS

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Abstract. Building on recent work of Robertson and Steger, we associate a $C^*$–algebra to a combinatorial object which may be thought of as a higher rank graph. This $C^*$–algebra is shown to be isomorphic to that of the associated path groupoid. Sufficient conditions on the higher rank graph are found for the associated $C^*$–algebra to be simple, purely infinite and AF. Results concerning the structure of crossed products by certain natural actions of discrete groups are obtained; a technique for constructing rank 2 graphs from “commuting” rank 1 graphs is given.

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In this paper we shall introduce the notion of a higher rank graph and associate a $C^*$–algebra to it in such a way as to generalise the construction of the $C^*$–algebra of a directed graph as studied in $\text{CK}$, $\text{KPRR}$, $\text{KPR}$ (amongst others). Graph $C^*$–algebras include up to strong Morita equivalence Cuntz–Krieger algebras and AF algebras. The motivation for the form of our generalisation comes from the recent work of Robertson and Steger $\text{RS1}$, $\text{RS2}$, $\text{RS3}$. In $\text{RS1}$ the authors study crossed product $C^*$–algebras arising from certain group actions on $A_2$-buildings and show that they are generated by two families of partial isometries which satisfy certain relations amongst which are Cuntz–Krieger type relations $\text{RS1}$, Equations (2), (5) as well as more intriguing commutation relations $\text{RS1}$, Equation (7). In $\text{RS2}$ they give a more general framework for studying such algebras involving certain families of commuting $0 - 1$ matrices. In particular the associated $C^*$–algebras are simple, purely infinite and generated by a family of Cuntz–Krieger algebras associated to these matrices. It is this framework which we seek to cast in graphical terms to include a wider class of examples (including graph $C^*$–algebras).

What follows is a brief outline of the paper. In the first section we introduce the notion of a higher rank graph as a purely combinatorial object: a small category $\Lambda$ gifted with a degree map $d : \Lambda \rightarrow \mathbb{N}^k$ (called shape in $\text{RS2}$) playing the role of the length function. No detailed knowledge of category theory is required to read this paper. The associated $C^*$–algebra $C^*(\Lambda)$ is defined as the universal $C^*$–algebra generated by a family of partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ satisfying relations similar to those of $\text{KPR}$ (our standing assumption is that our higher rank graphs satisfy conditions analogous to a directed graph being row–finite and having no sinks). We then describe some basic examples and indicate the relationship between our formalism and that of $\text{RS2}$.

In the second section we introduce the path groupoid $\mathcal{G}_\Lambda$ associated to a higher rank graph $\Lambda$ (cf. $\text{RS3}$, $\text{KPRR}$). Once the infinite path space $\Lambda^\infty$ is formed (and a few elementary facts are obtained) the construction...
is fairly routine. It follows from the gauge-invariant uniqueness theorem (Theorem 3.4) that $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$.

By the universal property $C^*(\Lambda)$ carries a canonical action of $T^k$ defined by

$$\alpha_t(s_\lambda) = t^d(\lambda)s_\lambda$$

called the gauge action. In the third section we prove the gauge-invariant uniqueness theorem, which is the key result for analysing $C^*(\Lambda)$ (cf. BPRS, aHR, see also CK, RS2, where similar techniques are used to prove simplicity). It gives conditions under which a homomorphism with domain $C^*(\Lambda)$ is faithful: roughly speaking, if the homomorphism is equivariant for the gauge action and nonzero on the generators then it is faithful.

This theorem has a number of interesting consequences, amongst which are the isomorphism mentioned above and the fact that the higher rank Cuntz–Krieger algebras of RS2 are isomorphic to $C^*$–algebras associated to suitably chosen higher rank graphs.

In the fourth section we characterise, in terms of an aperiodicity condition on $\Lambda$, the circumstances under which the groupoid $\mathcal{G}_\Lambda$ is essentially free. This aperiodicity condition allows us to prove a second uniqueness theorem analogous to the original theorem of CK. In §8 and §9 we obtain conditions under which $C^*(\Lambda)$ is simple and purely infinite respectively which are similar to those in [KPR] but with the aperiodicity condition replacing condition (L).

In the next section we show that, given a functor $c: \Lambda \to G$ where $G$ is a discrete group, then as in [KP] one may construct a skew product $G \times_c \Lambda$ which is also a higher rank graph. If $G$ is abelian then there is a natural action $\alpha_c^\natural: \hat{G} \to \text{Aut} C^*(\Lambda)$ such that

$$\alpha_c^\natural(s_\lambda) = \langle \chi, c(\lambda) \rangle s_\lambda;$$

moreover $C^*(\Lambda) \rtimes_{\alpha_c^\natural} \hat{G} \cong C^*(G \times_c \Lambda)$. Comparing (1) and (2) we see that the gauge action $\alpha$ is of the form $\alpha^\natural$ and as a consequence we may show that the crossed product of $C^*(\Lambda)$ by the gauge action is isomorphic to $C^*(\mathbb{Z}^k \times_d \Lambda)$; this $C^*$–algebra is then shown to be AF. By Takai duality $C^*(\Lambda)$ is strongly Morita equivalent to a crossed product of this AF algebra by the dual action of $\mathbb{Z}^k$. Hence $C^*(\Lambda)$ belongs to the bootstrap class $\mathcal{N}$ of $C^*$–algebras for which the UCT applies (see [RS2]) and is consequently nuclear. If a discrete group $G$ acts freely on a $k$–graph $\Lambda$, then the quotient object $\Lambda/G$ inherits the structure of a $k$–graph; moreover (as a generalisation of [BS, Theorem 2.2.2]) there is a functor $c: \Lambda/G \to G$ such that $\Lambda \cong G \times_c (\Lambda/G)$ in an equivariant way. This fact allows us to prove that

$$C^*(\Lambda) \rtimes G \cong C^*(\Lambda/G) \otimes K \left( \mathcal{F}(G) \right)$$

where the action of $G$ on $C^*(\Lambda)$ is induced from that on $\Lambda$. Finally in §6 a technique for constructing a 2-graph from “commuting” 1-graphs $A, B$ with the same vertex set is given. The construction depends on the choice of a certain bijection between pairs of composable edges: $\theta: (a, b) \mapsto (b', a')$ where $a, a' \in A^1$ and $b, b' \in B^1$; the resulting 2-graph is denoted $A *_{\theta} B$. It is not hard to show that every 2-graph is of this form.

Throughout this paper we let $\mathbb{N} = \{0, 1, \ldots\}$ denote the monoid of natural numbers under addition. For $k \geq 1$ regard $\mathbb{N}^k$ as an abelian monoid under addition with identity 0 (it will sometimes be useful to regard $\mathbb{N}^k$ as a small category with one object) and canonical generators $e_i$ for $i = 1, \ldots, k$; we shall also regard $\mathbb{N}^k$ as the positive cone of $\mathbb{Z}^k$ under the usual coordinatewise partial order: thus $m \leq n$ if and only if $m_i \leq n_i$ for all $i$ where $m = (m_1, \ldots, m_k)$, and $n = (n_1, \ldots, n_k)$ (this makes $\mathbb{N}^k$ a lattice).

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1. Higher rank graph $C^*$–algebras

In this section we first introduce what we shall call a higher rank graph as a purely combinatorial object (we do not know whether this concept has been studied before). Our definition of a higher rank graph is modelled on the path category of a directed graph (see §II.7 and Example 1.3). Thus a higher rank graph will be defined to be a small category gifted with a degree map (called shape in RS2) satisfying a certain factorization property. We then introduce the associated $C^*$–algebra whose definition is modelled on that of the $C^*$–algebra of a graph as well as the definition of RS2.

**Definitions 1.1.** A $k$–graph (rank $k$ graph or higher rank graph) $(\Lambda, d)$ consists of a countable small category $\Lambda$ (with range and source maps $r$ and $s$ respectively) together with a functor $d: \Lambda \to \mathbb{N}^k$ satisfying the factorisation property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements
μ, ν ∈ Λ such that λ = μν and d(μ) = m, d(ν) = n. For n ∈ Nk we write \( \Lambda^n := d^{-1}(n) \). A morphism between k-graphs \((Λ_1, d_1)\) and \((Λ_2, d_2)\) is a functor \( f : Λ_1 \to Λ_2 \) compatible with the degree maps.

**Remarks 1.2.** The factorisation property of \([1,3]\) allows us to identify \( \text{Obj}(Λ) \), the objects of Λ with \( Λ^0 \). Suppose \( λα = μα \) in Λ then by the the factorisation property \( λ = μ \); left cancellation follows similarly. We shall write the objects of Λ as \( u, v, w, \ldots \) and the morphisms as greek letters \( λ, μ, ν, \ldots \). We shall frequently refer to Λ as a k-graph without mentioning d explicitly.

It might be interesting to replace \( N^k \) in Definition \([1,1]\) above by a monoid or perhaps the positive cone of an ordered abelian group.

Recall that \( λ, μ \in Λ \) are composable if and only if \( r(μ) = s(λ) \), and then \( λμ \in Λ \); on the other hand two finite paths \( λ, μ \) in a directed graph may be composed to give the path \( λμ \) provided that \( r(λ) = s(μ) \); so in \([1,3]\) below we will need to switch the range and source maps.

**Example 1.3.** Given a 1-graph Λ, define \( E^0 = Λ^0 \) and \( E^1 = Λ^1 \). If we define \( s_E(λ) = r(λ) \) and \( r_E(λ) = s(λ) \) then the quadruple \((E^0, E^1, r_E, s_E)\) is a directed graph in the sense of \([KPR, KF]\). On the other hand, given a directed graph \( E = (E^0, E^1, r_E, s_E) \), then \( E^* = \bigcup_{n \geq 0} E^n \), the collection of finite paths, may be viewed as small category with range and source maps given by \( s(λ) = r_E(λ) \) and \( s(λ) = s_E(λ) \). If we let \( d : E^* \to N \) be the length function (i.e. \( d(λ) = n \) if \( λ \in E^n \)) then \((E^*, d)\) is a 1-graph.

We shall associate a \( C^*\)-algebra to a k-graph in such a way that for \( k = 1 \) the associated \( C^*\)-algebra is the same as that of the directed graph. We shall consider other examples later.

**Definitions 1.4.** The k-graph \( Λ \) is **row finite** if for each \( m \in N^k \) and \( v ∈ Λ^0 \) the set \( Λ^m(v) := \{ λ ∈ Λ^m : r(λ) = v \} \) is finite. Similarly Λ has **no sources** if \( Λ^m(v) \neq ∅ \) for all \( v ∈ Λ^0 \) and \( m ∈ N^k \).

Clearly if \( E \) is a directed graph then \( E \) is row finite (resp. has no sinks) if and only if \( E^* \) is row finite (resp. has no sources). Throughout this paper we will assume (unless otherwise stated) that any k-graph \( Λ \) is row finite and has no sources, that is

\[
0 < \#Λ^n(v) < ∞ \text{ for every } v ∈ Λ^0 \text{ and } n ∈ N^k.
\]

The Cuntz–Krieger relations \([CK, p.253]\) and the relations given in \([KPR, §1]\) may be interpreted as providing a representation of a certain directed graph by partial isometries and orthogonal projections. This view motivates the definition of \( C^*(Λ) \).

**Definitions 1.5.** Let \( Λ \) be a row finite k-graph with no sources. Then \( C^*(Λ) \) is defined to be the universal \( C^*\)-algebra generated by a family \( \{s_λ : λ ∈ Λ\} \) of partial isometries satisfying:

(i) \( \{s_v : v ∈ Λ^0\} \) is a family of mutually orthogonal projections,
(ii) \( s_{λμ} = s_λ s_μ \) for all \( λ, μ ∈ Λ \) such that \( s(λ) = r(μ) \),
(iii) \( s_λ^* s_λ = s_{s(λ)} \) for all \( λ ∈ Λ \),
(iv) for all \( v ∈ Λ^0 \) and \( n ∈ N^k \) we have \( s_v = \sum_{λ ∈ Λ^n(v)} s_λ s_λ^* \).

For \( λ ∈ Λ \), define \( p_λ = s_λ s_λ^* \) (note that \( p_v = s_v \) for all \( v ∈ Λ^0 \)). A family of partial isometries satisfying (i)–(iv) above is called a \( *\)-**representation** of \( Λ \).

**Remarks 1.6.**
(i) If \( \{t_λ : λ ∈ Λ\} \) is a \( *\)-representation of Λ then the map \( s_λ ↦ t_λ \) defines a \( *\)-homomorphism from \( C^*(Λ) \) to \( C^*(\{t_λ : λ ∈ Λ\}) \).
(ii) If \( E^* \) is the 1-graph associated to the directed graph \( E \) (see \([1,3]\), then by restricting a \( *\)-representation to \( E^0 \) and \( E^1 \) one obtains a Cuntz–Krieger family for \( E \) in the sense of \([KPR, §1]\)). Conversely every Cuntz–Krieger family for \( E \) extends uniquely to a \( *\)-representation of \( E^* \).
(iii) In fact we only need the relation (iv) above to be satisfied for \( n = e_i \in N^k \) for \( i = 1, \ldots, k \), the relations for all \( n \) will then follow (cf. \([RS2, Lemma 3.2]\)). Note that the definition of \( C^*(Λ) \) given in \([1,7]\) may be extended to the case where there are sources by only requiring that relation (iv) hold for \( n = e_i \) and then only if \( Λ^{e_i}(v) \neq ∅ \) (cf. \([KPR, Equation (2)]\)).
(iv) For \( λ, μ ∈ Λ \) if \( s(λ) \neq s(μ) \) then \( s_λ s_μ^* = 0 \). The converse follows from \([2,11]\).
(v) Increasing finite sums of \( p_u \)'s form an approximate identity for \( C^*(\Lambda) \) (if \( \Lambda^0 \) is finite then \( \sum_{u \in \Lambda^0} p_u \) is the unit for \( C^*(\Lambda) \)). It follows from relations (i) and (iv) above that for any \( n \in \mathbb{N}^k \), \( \{ p_x : d(\lambda) = n \} \) forms a collection of orthogonal projections (cf. [RS2 3.3]); likewise increasing finite sums of these form an approximate identity for \( C^*(\Lambda) \) (see [23]).

(vi) The above definition is not stated most efficiently. Any family of operators \( \{ s_\lambda : \lambda \in \Lambda \} \) satisfying the above conditions must consist of partial isometries. The first two axioms could also be replaced by:

\[
s_{\lambda \mu} = \begin{cases} s_{\lambda \mu} & \text{if } s(\lambda) = r(\mu) \\ 0 & \text{otherwise.} \end{cases}
\]

**Examples 1.7.**

(i) If \( E \) is a directed graph, then by (i) and (ii) we have \( C^*(E^* \cong C^*(E) \) (see [1,3]).

(ii) For \( k \geq 1 \) let \( \Omega = \Omega_k \) be the small category with objects \( \text{Obj}(\Omega) = \mathbb{N}^k \), and morphisms \( \Omega = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n \} \); the range and source maps are given by \( r(m, n) = m \), \( s(m, n) = n \). Let \( d : \Omega \to \mathbb{N}^k \) be defined by \( d(m, n) = n - m \). It is then straightforward to show that \( \Omega_k \) is a \( k \)-graph and \( C^*(\Omega_k) \cong \mathcal{K}(\ell^2(\mathbb{N}^k)) \).

(iii) Let \( T = T_k \) be the semigroup \( \mathbb{N}^k \) viewed as a small category, then if \( d : T \to \mathbb{N}^k \) is the identity map \( (T, d) \) is a \( k \)-graph. It is not hard to show that \( C^*(T) \cong C(T^k) \), where \( s_{\lambda \nu} \) for \( 1 \leq i \leq k \) are the canonical unitary generators.

(iv) Let \( \{ M_1, \ldots, M_k \} \) be square \( \{0,1\} \) matrices satisfying conditions (H0)–(H3) of [RS2] and let \( A \) be the associated \( C^* \)-algebra. For \( m \in \mathbb{N}^k \) let \( W_m \) be the collection of undecorated words in the finite alphabet \( \mathcal{A} \) of shape \( m \) as defined in [RS2] then let

\[
W = \bigcup_{m \in \mathbb{N}^k} W_m.
\]

Together with range and source maps \( r(\lambda) = o(\lambda), s(\lambda) = t(\lambda) \) and product defined in [RS2, Definition 0.1] \( W \) is a small category. If we define \( d : W \to \mathbb{N}^k \) by \( d(\lambda) = \sigma(\lambda) \), then one checks that \( d \) satisfies the factorisation property, and then from the second part of (H2) we see that \((W, d)\) is an irreducible \( k \)-graph in the sense that for all \( u, v \in W \) there is \( \lambda \in W \) such that \( s(\lambda) = u \) and \( r(\lambda) = v \).

We claim that the map \( s_\lambda \mapsto s_{\lambda, s(\lambda)} \) for \( \lambda \in W \) extends to a \(*\)-homomorphism \( C^*(W) \to \mathcal{A} \) for which \( s_{\lambda \mu} \mapsto s_{\lambda, \mu} \) (since these generate \( \mathcal{A} \) this will show that the map is onto). It suffices to verify that \( \{ s_{\lambda, s(\lambda)} : \lambda \in W \} \) constitutes a \(*\)-representation of \( W \). Conditions (i) and (iii) are easy to check, (iv) follows from [RS2 0.1c,3.2] with \( u = v \in W^0 \). We check condition (ii): if \( s(\lambda) = r(\mu) \) apply [RS2 3.2]

\[
s_{\lambda, s(\lambda)} s_{\mu, s(\mu)} = \sum_{W^d(\rho)(s(\lambda))} s_{\lambda \rho, \mu} s_{\rho, s(\mu)} = s_{\lambda \mu, s(\lambda \mu)}
\]

where the sum simplifies using [RS2 3.1, 3.3]. We shall show below that \( C^*(W) \cong \mathcal{A} \).

We may combine higher rank graphs using the following fact, whose proof is straightforward.

**Proposition 1.8.** Let \( \Lambda_1, d_1 \) and \( \Lambda_2, d_2 \) be rank \( k_1 \) and \( k_2 \) graphs respectively, then \( (\Lambda_1 \times \Lambda_2, d_1 \times d_2) \) is a rank \( k_1 + k_2 \) graph where \( \Lambda_1 \times \Lambda_2 \) is the product category and \( d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \to \mathbb{N}^{k_1 + k_2} \) is given by \( d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2} \) for \( \lambda_1 \in \Lambda_1 \) and \( \lambda_2 \in \Lambda_2 \).

An example of this construction is discussed in [RS2 Remark 3.11]. It is clear that \( \Omega_{k+\ell} \cong \Omega_k \times \Omega_\ell \) for \( k, \ell > 0 \).

**Definition 1.9.** Let \( f : \mathbb{N}^\ell \to \mathbb{N}^k \) be a monoid morphism, then if \( (\Lambda, d) \) is a \( k \)-graph we may form the \( \ell \)-graph \( f^*(\Lambda) \) as follows: (the objects of \( f^*(\Lambda) \) may be identified with those of \( \Lambda \) and) \( f^*(\Lambda) = \{ (\lambda, n) : d(\lambda) = f(n) \} \) with \( d(\lambda, n) = n \), \( s(\lambda, n) = s(\lambda) \) and \( r(\lambda, n) = r(\lambda) \).

**Examples 1.10.**

(i) Let \( \Lambda \) be a \( k \)-graph and put \( \ell = 1 \), then if we define the morphism \( f_i(n) = ne_i \) for \( 1 \leq i \leq k \), we obtain the coordinate graphs \( \Lambda_i := f_i^*(\Lambda) \) of \( \Lambda \) (these are \( 1 \)-graphs).

(ii) Suppose \( E \) is a directed graph and define \( f : \mathbb{N}^2 \to \mathbb{N} \) by \( (m_1, m_2) \mapsto m_1 + m_2 \); then the two coordinate graphs of \( f^*(E^*) \) are isomorphic to \( E^* \). We will show below that \( C^*(f^*(E^*)) \cong C^*(E^*) \cong C(T) \).

(iii) Suppose \( E \) and \( F \) are directed graphs and define \( f : \mathbb{N} \to \mathbb{N}^2 \) by \( f(m) = (m, m) \) then \( f^*(E^* \times F^*) = (E \times F)^* \) where \( E \times F \) denotes the cartesian product graph (see [KP, Def. 2.1]).
Proposition 2.11. Let \( \Lambda \) be a \( k \)-graph and \( f : \mathbb{N}^k \to \mathbb{N}^k \) a monoid morphism, then there is a *-homomorphism \( \pi_f : C^* (f^* (\Lambda)) \to C^* (\Lambda) \) such that \( s_{(\lambda, n)} \mapsto s_{\lambda} \); moreover if \( f \) is surjective, then \( \pi_f \) is too.

Proof. By \[\text{(i)}\] it suffices to show that this is a *-representation of \( f^* (\Lambda) \). Properties (i)–(iii) are straightforward to verify and property (iv) follows by observing that for fixed \( n \in \mathbb{N}^k \) and \( v \in \Lambda^0 \) the map \( f^* (\Lambda)^n (v) \to \Lambda^{f(n)} (v) \) given by \( (\lambda, n) \mapsto \lambda \) is a bijection. If \( f \) is surjective, then it is clear that every generator \( s_{\lambda} \) of \( C^* (\Lambda) \) is in the range of \( \pi_f \).

Later in \[3.5\] we will also show that \( \pi_f \) is injective if \( f \) is injective.

2. The path groupoid

In this section we construct the path groupoid \( \mathcal{G}_\Lambda \) associated to a higher rank graph \( (\Lambda, d) \) along the lines of \[\text{KPRR}, \S2\]. Because some of the details are not quite the same as those in \[\text{KPRR}, \S2\] we feel it is useful to sketch the construction. First we introduce the following analog of an infinite path in a higher rank graph:

Definitions 2.1. Let \( \Lambda \) be a \( k \)-graph, then

\[ \Lambda^\infty = \{ x : \Omega_k \to \Lambda : x \text{ is a } k \text{-graph morphism} \}, \]

is the infinite path space of \( \Lambda \). For \( v \in \Lambda^0 \) let \( \Lambda^\infty (v) = \{ x \in \Lambda^\infty : x(0) = v \} \). For each \( p \in \mathbb{N}^k \) define \( \sigma^p : \Lambda^\infty \to \Lambda^\infty \) by \( \sigma^p (x(m, n)) = x(m + p, n + p) \) for \( x \in \Lambda^\infty \) and \( (m, n) \in \Omega \). (Note that \( \sigma^{p+q} = \sigma^p \circ \sigma^q \).)

By our standing assumption \[\[\text{(i)}\] one can show that for every \( v \in \Lambda^0 \) we have \( \Lambda^\infty (v) \neq \emptyset \). Our definition of \( \Lambda^\infty \) is related to the definition of \( W_\infty \), the space of infinite words, given in the proof of \[\text{RS2}, \text{Lemma 3.8}\]. If \( E^* \) is the 1-graph associated to the directed graph \( E \) then \( (E^*)^\infty \) may be identified with \( E^\infty \).

Remarks 2.2. By the factorisation property the values of \( x(0, m) \) for \( m \in \mathbb{N}^k \) completely determine \( x \in \Lambda^\infty \).

To see this, suppose that \( x(0, m) \) is given for all \( m \in \mathbb{N}^k \) then for \( (m, n) \in \Omega \), \( x(m, n) \) is the unique element \( \lambda \in \Lambda \) such that \( x(0, n) = x(0, m) \).

More generally, let \( \{ n_j : j \geq 0 \} \) be an increasing cofinal sequence in \( \mathbb{N}^k \) with \( n_0 = 0 \), then \( \lambda \in \Lambda^\infty \) is completely determined by the values of \( x(0, n_j) \) (for example one could take \( n_j = jp \) where \( p = (1, \ldots, 1) \in \mathbb{N}^k \)). Moreover, given a sequence \( \{ \lambda_j : j \geq 1 \} \in \Lambda \) such that \( s(\lambda_j) = r(\lambda_{j+1}) \) and \( d(\lambda_j) = n_j - n_{j-1} \) there is a unique \( x \in \Lambda^\infty \) such that \( x(n_j, n_j) = \lambda_j \). For \( (m, n) \in \Omega \) we define \( x(m, n) \) by the factorisation property as follows: let \( j \) be the smallest index such that \( n \leq n_j \), then \( x(m, n) \) is the unique element of degree \( n - m \) such that \( \lambda_1 \cdots \lambda_j = \mu x(m, n) \nu \) where \( d(\mu) = m \) and \( d(\nu) = n_j - n \). It is straightforward to show that \( x \) has the desired properties.

We now establish a factorisation property for \( \Lambda^\infty \) which is an easy consequence of the above remarks:

Proposition 2.3. Let \( \Lambda \) be a \( k \)-graph. For all \( \lambda \in \Lambda \) and \( x \in \Lambda^\infty \) with \( x(0) = s(\lambda) \), there is a unique \( y \in \Lambda^\infty \) such that \( x = \sigma^{d(\lambda)} y \) and \( \lambda = y(0, d(\lambda)) \); we write \( y = \lambda x \). Note that for every \( x \in \Lambda^\infty \) and \( p \in \mathbb{N}^k \) we have \( x = x(0, p) \sigma^p x \).

Proof. Fix \( \lambda \in \Lambda \) and \( x \in \Lambda^\infty \) with \( x(0) = s(\lambda) \). The sequence \( \{ n_j : j \geq 0 \} \) defined by \( n_0 = 0 \) and \( n_j = (j-1)p + d(\lambda) \) for \( j \geq 1 \) is cofinal. Set \( \lambda_1 = \lambda \) and \( \lambda_j = x((j-2)p, (j-1)p) \) for \( j \geq 2 \) and let \( y \in \Lambda^\infty \) be defined by the method given in \[\[2.2\] \]. Then \( y \) has the desired properties.

Next we construct a basis of compact open sets for the topology on \( \Lambda^\infty \) indexed by \( \Lambda \):

Definitions 2.4. Let \( \Lambda \) be a \( k \)-graph. For \( \lambda \in \Lambda \) define

\[ Z(\lambda) = \{ \lambda x \in \Lambda^\infty : s(\lambda) = x(0) \} = \{ x : x(0, d(\lambda)) = \lambda \}. \]

Remarks 2.5. Note that \( Z(v) = \Lambda^\infty (v) \) for all \( v \in \Lambda^0 \). For fixed \( n \in \mathbb{N}^k \) the sets \( \{ Z(\lambda) : d(\lambda) = n \} \) form a partition of \( \Lambda^\infty \) (see \[\[6.6\] \]) since \( d(\lambda) = n \).

For every \( \lambda \in \Lambda \) we have

\[ Z(\lambda) = \bigcup_{r(\mu) = s(\lambda), \ d(\mu) = n} Z(\lambda \mu). \]

We endow \( \Lambda^\infty \) with the topology generated by the collection \( \{ Z(\lambda) : \lambda \in \Lambda \} \). Note that the map given by \( \lambda x \mapsto x \) induces a homeomorphism between \( Z(\lambda) \) and \( Z(s(\lambda)) \) for all \( \lambda \in \Lambda \). Hence, for every \( p \in \mathbb{N}^k \) the map \( \sigma^p : \Lambda^\infty \to \Lambda^\infty \) is a local homeomorphism.
Lemma 2.6. For each $\lambda \in \Lambda$, $Z(\lambda)$ is compact.

Proof. By $\square$ it suffices to show that $Z(v)$ is compact for all $v \in \Lambda^0$. Fix $v \in \Lambda^0$ and let $\{x_n\}_{n \geq 1}$ be a sequence in $Z(v)$. For every $m$, $x_n(0, m)$ may take only finitely many values (by $\square$). Hence there is a $\lambda \in \Lambda^m$ such that $x_n(0, m) = \lambda$ for infinitely many $n$. We may therefore inductively construct a sequence $\{\lambda_j : j \geq 1\}$ in $\Lambda^p$ such that $s(\lambda_j) = r(\lambda_{j+1})$ and $x_n(0, j\lambda) = \lambda_1 \cdots \lambda_j$ for infinitely many $n$ (recall $p = (1, \ldots, 1) \in \mathbb{N}^k$). Choose a subsequence $\{x_n\}$ such that $x_n(0, j\lambda) = \lambda_1 \cdots \lambda_j$. Since $\{j\lambda\}$ is cofinal, there is a unique $y \in \Lambda^\infty(v)$ such that $y((j-1)p, j\lambda) = \lambda_j$ for $j \geq 1$; then $x_{n_j} \to y$ and hence $Z(v)$ is compact. \qed

Note that $\Lambda^\infty$ is compact if and only if $\Lambda^0$ is finite.

Definition 2.7. If $\Lambda$ is $k$-graph then let

$$G_\Lambda = \{(x, n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty : \sigma^i x = \sigma^m y, n = \ell - m\}. $$

Define range and source maps $r, s : G_\Lambda \to \Lambda^\infty$ by $r(x, n, y) = x$, $s(x, n, y) = y$. For $(x, n, y), (y, \ell, z) \in G_\Lambda$ set $(x, n, y)(y, \ell, z) = (x, n + \ell, z)$, and $(x, n, y)^{-1} = (y, -n, x)$; $G_\Lambda$ is called the path groupoid of $\Lambda$ (cf. $\square$).

One may check that $G_\Lambda$ is a groupoid with $\Lambda^\infty = G^0_\Lambda$ under the identification $x \mapsto (x, 0, x)$. For $\lambda, \mu \in \Lambda$ such that $s(\lambda) = s(\mu)$ define

$$Z(\lambda, \mu) = \{(\lambda z, d(\lambda) - d(\mu), \mu z) : z \in \Lambda^\infty(s(\lambda))\}. $$

We collect certain standard facts about $G_\Lambda$ in the following result:

Proposition 2.8. Let $\Lambda$ be a $k$-graph. The sets $\{Z(\lambda, \mu) : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\}$ form a basis for a locally compact Hausdorff topology on $G_\Lambda$. With this topology $G_\Lambda$ is a second countable, $r$-discrete locally compact groupoid in which each $Z(\lambda, \mu)$ is a compact open bisection. The topology on $\Lambda^\infty$ agrees with the relative topology under the identification of $\Lambda^\infty$ with the subset $G^0_\Lambda$ of $G_\Lambda$.

Proof. One may check that the sets $Z(\lambda, \mu)$ form a basis for a topology on $G_\Lambda$. To see that multiplication is continuous, suppose that $(x, n, y)(y, \ell, z) = (x, n + \ell, z) \in Z(\gamma, \delta)$. Since $(x, n, y), (y, \ell, z)$ are composable in $G_\Lambda$ there are $\kappa, \nu \in \Lambda$ and $t \in \Lambda^\infty$ such that $x = \gamma t\kappa$, $y = \nu t\delta$ and $z = \delta t\kappa$. Hence $(x, k, y) \in Z(\gamma \kappa, \nu)$ and $(y, \ell, z) \in Z(\nu, \delta \kappa)$ and the product maps $G^0_\Lambda \cap (Z(\gamma \kappa, \nu) \times Z(\nu, \delta \kappa))$ into $Z(\gamma, \delta)$. The remaining parts of the proof are similar to those given in [KPRR, Proposition 2.6]. \qed

Note that $Z(\lambda, \mu) \cong Z(s(\lambda))$, via the map $(\lambda z, d(\lambda) - d(\mu), \mu z) \mapsto z$. Again we note that in the case $k = 1$ we have $\Lambda = E^*$ for some directed graph $E$ and the groupoid $G_{E^*} \cong GE$, the graph groupoid of $E$ which is described in detail in [KPRR, §2].

Proposition 2.9. Let $\Lambda$ be a $k$-graph and let $f : \mathbb{N}^l \to \mathbb{N}^k$ be a morphism. The map $x \mapsto f^*(x)$ given by $f^*(x)(m, n) = (x(f(m), f(n)), n - m)$ defines a continuous surjective map $f^* : \Lambda^\infty \to f^*(\Lambda)^\infty$. Moreover, if the image of $f$ is cofinal (equivalently $f(p)$ is strictly positive in the sense that all of its coordinates are nonzero) then $f^*$ is a homeomorphism.

Proof. Given $x \in f^*(\Lambda)^\infty$ choose a sequence $\{m_i\}$ such that $n_j = \sum_{i=1}^j m_i$ is cofinal in $\mathbb{N}^l$. Set $n_0 = 0$ and let $\lambda_j \in \Lambda^f(n_j)$ be defined by the condition that $x(n_{j-1}, n_j) = (\lambda_j, n_j)$. We must show that there is an $x' \in \Lambda^\infty$ such that $x'(f(n_{j-1}), f(n_j)) = \lambda_j$. It suffices to show that the the intersection $\cap_j Z(\lambda_1 \cdots \lambda_j) \neq \emptyset$. But this follows by the finite intersection property. One checks that $x = f^*(x')$. Furthermore the inverse image of $Z(\lambda, n)$ is $Z(\lambda)$ and hence $f^*$ is continuous.

Now suppose that the image of $f$ is cofinal, then the procedure defined above gives a continuous inverse for $f^*$. Given $x \in f^*(\Lambda)^\infty$, then since $f(n_j)$ is cofinal, the intersection $\cap_j Z(\lambda_1 \cdots \lambda_j)$ contains a single point $x'$. Note that $x'$ depends on $x$ continuously. \qed

For higher rank graphs of the form $f^*(\Lambda)$ with $f$ surjective (see $\square$), the associated groupoid $G_{f^*(\Lambda)}$ decomposes as a direct product as follows:

Proposition 2.10. Let $\Lambda$ be a $k$-graph and let $f : \mathbb{N}^l \to \mathbb{N}^k$ be a surjective morphism. Then

$$G_{f^*(\Lambda)} \cong G_\Lambda \times \mathbb{Z}^{l-k}. $$
Proof. Since \( f \) is surjective, the map \( f^* : \Lambda^\infty \to f^*(\Lambda)^\infty \) is a homeomorphism (see 2.4). The map \( f \) extends to a surjective morphism \( f : \mathbb{Z}^k \to \mathbb{Z}^k \). Let \( j : \mathbb{Z}^k \to \mathbb{Z}' \) be a section for \( f \) and let \( i : \mathbb{Z}'^\ell - k \to \mathbb{Z}' \) be an identification of \( \mathbb{Z}'^\ell - k \) with \( \ker f \). Then we get a groupoid isomorphism by the map

\[
((x, n, y), m) \mapsto (f^* x, i(m) + j(n), f^* y),
\]

where \(((x, n, y), m) \in \mathcal{G}_\Lambda \times \mathbb{Z}'^\ell - k\).

Finally, as in \[RS2\] Lemma 3.8 we demonstrate that there is a nontrivial \( * \)-representation of \( (\Lambda, d) \).

**Proposition 2.11.** Let \( \Lambda \) be a \( k \)-graph then there exists a representation \( \{ S_\lambda : \lambda \in \Lambda \} \) of \( \Lambda \) on a Hilbert space with all partial isometries \( S_\lambda \) nonzero.

**Proof.** Let \( \mathcal{H} = \ell^2(\Lambda^\infty) \), then for \( \lambda \in \Lambda \) define \( S_\lambda \in \mathcal{B}(\mathcal{H}) \) by

\[
S_\lambda e_y = \begin{cases} e_{\lambda y} & \text{if } s(\lambda) = y(0), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( \{ e_y : y \in \Lambda^\infty \} \) is the canonical basis for \( \mathcal{H} \). Notice that \( S_\lambda \) is nonzero since \( \Lambda^\infty(s(\lambda)) \neq \emptyset \); one then checks that the family \( \{ S_\lambda : \lambda \in \Lambda \} \) satisfies conditions 2.3(i)–(iv).

## 3. The gauge invariant uniqueness theorem

By the universal property of \( C^*(\Lambda) \) there is a canonical action of the \( k \)-torus \( T^k \), called the **gauge action**: \( \alpha : T^k \to \text{Aut} C^*(\Lambda) \) defined for \( t = (t_1, \ldots, t_k) \in T^k \) and \( s_\lambda \in C^*(\Lambda) \) by

\[
\alpha_t(s_\lambda) = t^{d(\lambda)} s_\lambda
\]

where \( t^m = t_1^{m_1} \cdots t_k^{m_k} \) for \( m = (m_1, \ldots, m_k) \in \mathbb{N}^k \). It is straightforward to show that \( \alpha \) is strongly continuous. As in \[CK\] Lemma 2.2 and \[RS2\] Lemma 3.6 we shall need the following:

**Lemma 3.1.** Let \( \Lambda \) be a \( k \)-graph. Then for \( \lambda, \mu, \alpha, \beta \in \Lambda \) and \( q \in \mathbb{N}^k \) with \( d(\lambda), d(\mu) \leq q \) we have

\[
S_*^\alpha S_*^\beta = \sum_{\lambda \alpha = \mu \beta, d(\lambda \alpha) = q} s_\alpha s_*^\beta.
\]

Hence every nonzero word in \( s_\alpha, S_*^\mu \) may be written as a finite sum of partial isometries of the form \( s_\alpha S_*^\beta \) where \( s(\alpha) = s(\beta) \); their linear span then forms a dense \(*\)-subalgebra of \( C^*(\Lambda) \).

**Proof.** Applying [2.3](iv) to \( s(\lambda) \) with \( n = q - d(\lambda) \), to \( s(\mu) \) with \( n = q - d(\mu) \) and using \[2.3\] (ii) we get

\[
S_*^\alpha S_*^\beta = P_{s(\lambda)} S_*^\alpha S_*^\beta P_{s(\mu)} = \left( \sum_{\lambda \alpha = \beta \gamma} s_\alpha S_*^\alpha \left( \sum_{\lambda \alpha = \beta \gamma} s_\gamma S_*^\gamma \right) \right) S_*^\beta = \left( \sum_{\lambda \alpha = \beta \gamma} s_\gamma S_*^\gamma \right) \left( \sum_{\lambda \alpha = \beta \gamma} s_\gamma S_*^\gamma \right).
\]

By [2.3](iv) if \( d(\lambda \alpha) = d(\mu \beta) \) but \( \lambda \alpha \neq \mu \beta \), then the range projections \( p_{\lambda \alpha}, p_{\mu \beta} \) are orthogonal and hence one has \( S_{\lambda \alpha} S_{\mu \beta} = 0 \). If \( \lambda \alpha = \mu \beta \) then \( S_{\lambda \alpha} S_{\mu \beta} = p_v \) where \( v = s(\alpha) \) and so \( s_\alpha S_{\lambda \alpha} s_{\mu \beta} S_*^\beta = s_\alpha p_v S_*^\beta = s_\alpha S_*^\beta \); formula (7) then follows from formula (6). The rest of the proof is now routine.

Following \[RS2\] §4: for \( m \in \mathbb{N}^k \) let \( \mathcal{F}_m \) denote the \( C^* \)-subalgebra of \( C^*(\Lambda) \) generated by the elements \( s_{\lambda \mu} S_*^\nu \) for \( \lambda, \mu, \nu \in \Lambda^m \) where \( s(\lambda) = s(\mu) \), and for \( v \in \Lambda^0 \) denote \( \mathcal{F}_m(v) \) the \( C^* \)-subalgebra generated by \( s_{\lambda \mu} S_*^\nu \) where \( s(\lambda) = v \).

**Lemma 3.2.** For \( m \in \mathbb{N}^k \), \( v \in \Lambda^0 \) there exist isomorphisms

\[
\mathcal{F}_m(v) \cong K\left( \ell^2(\{ \lambda \in \Lambda^m : s(\lambda) = v \}) \right)
\]

and \( \mathcal{F}_m \cong \bigoplus_{v \in \Lambda^0} \mathcal{F}_m(v) \). Moreover, the \( C^* \)-algebras \( \mathcal{F}_m, m \in \mathbb{N}^k \), form a directed system under inclusion, and \( \mathcal{F}_\Lambda = \cup \mathcal{F}_m \) is an AF \( C^* \)-algebra.
Proof. Fix $v \in \Lambda^0$ and let $\lambda, \mu, \alpha, \beta \in \Lambda^m$ be such that $s(\lambda) = s(\mu)$ and $s(\alpha) = s(\beta)$, then by \ref{iv} we have
\begin{equation}
(s_{\lambda}s_{\mu}^*) (s_{\alpha}s_{\beta}^*) = \delta_{\mu,\alpha}s_{\lambda}s_{\beta}^*,
\end{equation}
so that the map which sends $s_{\lambda}s_{\mu}^* \in F_m(v)$ to the matrix unit $e_{\lambda,\mu}^v \in K(\ell^2(\{\lambda \in \Lambda^m : s(\lambda) = v\}))$ for all $\lambda, \mu \in \Lambda^m$ with $s(\lambda) = s(\mu) = v$ extends to an isomorphism. The second isomorphism also follows from \ref{iv} (since $s(\mu) \neq s(\alpha)$ implies $\mu \neq \alpha$). We claim that $F_m$ is contained in $F_n$ whenever $m \leq n$. To see this we apply \ref{iv} to give
\begin{equation}
s_{\lambda}s_{\mu}^* = s_{\lambda}s_{\mu}^* = \sum_{\Lambda^t(s(\lambda))} s_{\lambda}s_{\gamma}s_{\gamma}^* = \sum_{\Lambda^t(s(\lambda))} s_{\lambda}s_{\gamma}s_{\gamma}^*
\end{equation}
where $\ell = n - m$. Hence the $C^*$–algebras $F_m, m \in \mathbb{N}^k$, form a directed system as required. 

Note that $F_\Lambda$ may also be expressed as the closure of $\cup_{j=1}^{\infty} F_{jp}$ where $p = (1, \ldots, 1) \in \mathbb{N}^k$.

Clearly for $t \in T^k$ the gauge automorphism $\alpha_t$ defined in \ref{v} fixes those elements $s_{\lambda}s_{\mu}^* \in C^*(\Lambda)$ with $d(\lambda) = d(\mu)$ (since $\alpha_t(s_{\lambda}s_{\mu}^*) = t(d(\lambda) - d(\mu))s_{\lambda}s_{\mu}^*$) and hence $F_\Lambda$ is contained in the fixed point algebra $C^*(\Lambda)^{\alpha}$. Consider the linear map on $C^*(\Lambda)$ defined by
\[ \Phi(x) = \int_{T^k} \alpha_t(x) dt \]
where $dt$ denotes normalised Haar measure on $T^k$ and note that $\Phi(x) \in C^*(\Lambda)^{\alpha}$ for all $x \in C^*(\Lambda)$. As the proof of the following result is now standard, we omit it (see \cite[Proposition 2.11]{cK}, \cite[Lemma 3.3]{RS2}, \cite[Lemma 2.2]{BPRS}).

**Lemma 3.3.** Let $\Phi, F_\Lambda$ be as described above.

(i) The map $\Phi$ is a faithful conditional expectation from $C^*(\Lambda)$ onto $C^*(\Lambda)^{\alpha}$.

(ii) $F_\Lambda = C^*(\Lambda)^{\alpha}$.

Hence the fixed point algebra $C^*(\Lambda)^{\alpha}$ is an AF algebra. This fact is key to the proof of the gauge–invariant uniqueness theorem for $C^*(\Lambda)$ (see \cite[Theorem 2.1]{BPRS}, \cite[Theorem 2.3]{HR}, see also \cite{cK, RS2} where a similar technique is used in the proof of simplicity).

**Theorem 3.4.** Let $B$ be a $C^*$–algebra, $\pi : C^*(\Lambda) \to B$ be a homomorphism and let $\beta : T^k \to Aut(B)$ be an action such that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in T^k$. Then $\pi$ is faithful if and only if $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.

Proof. If $\pi(p_v) = 0$ for some $v \in \Lambda^0$ then clearly $\pi$ is not faithful. Conversely, suppose that $\pi$ is equivariant and that $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$; we first show that $\pi$ is faithful on $C^*(\Lambda)^{\alpha} = \bigcup_{j \geq 0} F_{jp}$. For any ideal $I$ in $C^*(\Lambda)^{\alpha}$, we have $I = \bigcup_{j \geq 0} (I \cap F_{jp})$ (see \cite[Lemma 3.1]{cK}, \cite[Lemma 1.3]{ALNR}). Thus it is enough to prove that $\pi$ is faithful on each $F_n$. But by \ref{iv} it suffices to show that it is faithful on $F_n(v)$, for all $v \in \Lambda^0$. Fix $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda^n$ with $s(\lambda) = s(\mu) = v$; we need only show that $\pi(s_{\lambda}s_{\mu}^*) \neq 0$. Since $\pi(p_v) \neq 0$ we have
\[ 0 \neq \pi(p_v^2) = \pi(s_{\lambda}s_{\lambda}s_{\mu}s_{\mu}^*) = \pi(s_{\lambda}^2)\pi(s_{\lambda}s_{\mu}^*)\pi(s_{\mu}). \]

Hence $\pi(s_{\lambda}s_{\mu}^*) \neq 0$ and $\pi$ is faithful on $C^*(\Lambda)^{\alpha}$. Let $a \in C^*(\Lambda)$ be a nonzero positive element; then since $\Phi$ is faithful $\Phi(a) \neq 0$ and as $\pi$ is faithful on $C^*(\Lambda)^{\alpha}$ we have
\[ 0 \neq \pi(\Phi(a)) = \pi\left( \int_{T^k} \alpha_t(a) dt \right) = \int_{T^k} \beta_t(\pi(a)) dt; \]

hence, $\pi(a) \neq 0$ and $\pi$ is faithful on $C^*(\Lambda)$ as required.

**Corollary 3.5.**

(i) Let $(\Lambda, d)$ be a $k$–graph and let $G_\Lambda$ be its associated groupoid, then there is an isomorphism $C^*(\Lambda) \cong C^*(G_\Lambda)$ such that $s_\lambda \mapsto 1_{Z(\lambda, s(\lambda))}$ for $\lambda \in \Lambda$. Moreover the canonical map $C^*(G_\Lambda) \to C^*(G_\Lambda)$ is an isomorphism.

(ii) Let $\{M_1, \ldots, M_k\}$ be a collection of matrices satisfying (H0)–(H3) of \cite{RS2} and $W$ the $k$–graph defined in \ref{iv}, then $C^*(W) \cong A$, via the map $s_\lambda \mapsto s_\lambda s(\lambda)$ for $\lambda \in W$. 

\[ \square \]
(iii) If $\Lambda$ is a $k$-graph and $f : \mathbb{N}^k \to \mathbb{N}^k$ is injective then the $*$-homomorphism $\pi_f : C^*(f^*(\Lambda)) \to C^*(\Lambda)$ (see 1.11) is injective. In particular the $C^*$-algebras of the coordinate graphs $\Lambda_i$ for $1 \leq i \leq k$ form a generating family of subalgebras of $C^*(\Lambda)$. Moreover, if $f$ is surjective then $C^*(f^*(\Lambda)) \cong C^*(\Lambda) \otimes C(\mathbb{T}^{2-k})$.

(iv) Let $(\Lambda_i, d_i)$ be $k_i$-graphs for $i = 1, 2$, then $C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ via the map $s_{(\lambda_1, \lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$ for $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.

Proof. For (i) we note that $s_{\alpha} \mapsto 1_{Z(\alpha s(\Lambda))}$ for $\lambda \in \Lambda$ is a $*$-representation of $\Lambda$; hence there is a $*$-homomorphism $\pi : C^*(\Lambda) \to C^*(\mathcal{G}_\Lambda)$ such that $\pi(s_{\lambda}) = 1_{Z(\alpha s(\Lambda))}$ for $\lambda \in \Lambda$ (see 1.6(i)). Let $\beta$ denote the $\mathbb{T}^k$-action on $C^*(\mathcal{G}_\Lambda)$ induced by the $\mathbb{Z}^k$-valued $1$-cocycle defined on $\mathcal{G}_\Lambda$ by $(x, k, y) \mapsto \beta$ (see 1.11.5.1); one checks that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in \mathbb{T}^k$. Clearly, if $v \in \Lambda^0$ we have $1_{Z(v, v)} \neq 0$; since $\Lambda^\infty(v) = \emptyset$ and $\pi$ is injective. Surjectivity follows from the fact that $\pi(s_{\lambda}s_{\mu}^*) = 1_{Z(\lambda, \mu)}$ together with the observation that $C^*(\mathcal{G}_\Lambda) = \text{span}(1_{Z(\lambda, \mu)})$. The same argument shows that $C^*_v(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$ and so $C^*_v(\mathcal{G}_\Lambda) \cong C^*(\mathcal{G}_\Lambda)$.

For (ii) we note that there is a surjective $*$-homomorphism $\pi : C^*(\mathcal{G}_\Lambda) \to \mathcal{A}$ such that $\pi(s_{\lambda}) = s_{\lambda s(\Lambda)}$ for $\lambda \in \Lambda$ (see 1.7(iv)) which is clearly equivariant for the respective $\mathbb{T}^k$-actions. Moreover by [RS2, Lemma 2.9] we have $s_{\alpha v} \neq 0$ for all $v \in \mathcal{A}_0 = \mathcal{A}$ and so the result follows.

For (iii) note that the injection $f : \mathbb{N}^k \to \mathbb{N}^k$ extends naturally to a homomorphism $f : \mathbb{Z}^k \to \mathbb{Z}^k$ which in turn induces a map $\hat{f} : \mathbb{T}^k \to \mathbb{T}^k$ characterised by $\hat{f}(t)^p = f(t)(\mathbb{T}^k)$ for $p \in \mathbb{N}^k$. Let $B$ be the fixed point algebra of the gauge action of $\mathbb{T}^k$ on $C^*(\Lambda)$ restricted to the kernel of $\hat{f}$. The gauge action restricted to $B$ descends to an action of $\mathbb{T}^k = \mathbb{T}^k/\ker f\mathbb{N}^k$ on $B$ which we denote $\pi$. Observe that for $t \in \mathbb{T}^k$ and $(\lambda, n) \in f^*(\Lambda)$ we have

$$
\alpha_t(\pi_f(s_{\lambda n})) = \pi_f(t)(\mathbb{N}^k) = \hat{f}(t)^n s_{\lambda};
$$

hence $\text{Im} \pi_f \subseteq B$ (if $t \in \text{Ker} \hat{f}$ then $\hat{f}(t)^n = 1$). By the same formula we see that $\pi_f \circ \alpha = \sigma \circ \pi_f$ and the result now follows by 3.4. The last assertion follows from part (i) together with the fact that $\mathcal{G}_{\hat{f}^*}(\Lambda) \cong \mathcal{G}_\Lambda \times \mathbb{Z}^{2-k}$ (see 2.10).

For (iv) define a map $\pi : C^*(\Lambda_1 \times \Lambda_2) \to C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ given by $s_{(\lambda_1, \lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$; this is surjective as these elements generate $C^*(\Lambda_1) \otimes C^*(\Lambda_2)$. We note that $C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ carries a $\mathbb{T}^{k_1+k_2}$-action $\beta$ defined for $(t_1, t_2) \in \mathbb{T}^{k_1+k_2}$ and $(\lambda_0, \lambda_1) \in \Lambda_1 \times \Lambda_2$ by $\beta(t_1, t_2)(s_{\lambda_0} \otimes s_{\lambda_1}) = \alpha_{t_1} s_{\lambda_1} \otimes \alpha_{t_2} s_{\lambda_2}$. Injectivity then follows by 3.4, since $\pi$ is equivariant and for $(v, w) \in (\Lambda_1 \times \Lambda_2)^{0}$ we have $p_v \otimes p_w \neq 0$.

Henceforth we shall tacitly identify $C^*(\Lambda)$ with $C^*(\mathcal{G}_\Lambda)$.

**Remark 3.6.** Let $\Lambda$ be a $k$-graph and suppose that $f : \mathbb{N}^k \to \mathbb{N}^k$ is an injective morphism for which $H$, the image of $f$, is cofinal. Then $\pi_f$ induces an isomorphism of $C^*(f^*(\Lambda))$ with its range, the fixed point algebra of the restriction of the gauge action to $H^\perp$.

4. Aperiodicity and its consequences

The aperiodicity condition we study in this section is an analog of condition (L) used in [KPR]. We first define what it means for an infinite path to be periodic or aperiodic.

**Definitions 4.1.** For $x \in \Lambda^\infty$ and $p \in \mathbb{Z}^k$ we say that $p$ is a period of $x$ if for every $(m, n) \in \Omega$ with $m + p \geq 0$ we have $x(m + p, n + p) = x(m, n)$. We say that $x$ is periodic if it has a nonzero period. We say that $x$ is eventually periodic if $\sigma^n x$ is periodic for some $n \in \mathbb{N}^k$, otherwise $x$ is said to be aperiodic.

**Remarks 4.2.** For $x \in \Lambda^\infty$ and $p \in \mathbb{Z}^k$, $p$ is a period of $x$ if and only if $\sigma^m x = \sigma^n x$ for all $m, n \in \mathbb{N}^k$ such that $p = m - n$. Similarly $x$ is eventually periodic, with eventual period $p \neq 0$ if and only if $\sigma^m x = \sigma^n x$ for some $m, n \in \mathbb{N}^k$ such that $p = m - n$.

**Definition 4.3.** The $k$-graph $\Lambda$ is said to satisfy the aperiodicity condition (A) if for every $v \in \Lambda^0$ there is an aperiodic path $x \in \Lambda^\infty(v)$.

**Remark 4.4.** Let $E$ be a directed graph which is row finite and has no sinks then the associated $1$-graph $E^*$ satisfies the aperiodicity condition if and only if every loop in $E$ has an exit (i.e. satisfies condition (L) of [KPR]). However, if we consider the $2$-graph $f^*(E^*)$ where $f : \mathbb{N}^2 \to \mathbb{N}$ is given by $f(m_1, m_2) = m_1 + m_2$ then $p = (1, -1)$ is a period for every point in $f^*(E^*)^\infty$ (even if $E$ has no loops).

---

1This can be also deduced from the amenability of $\mathcal{G}_\Lambda$ (see 3.3).
Proposition 4.5. The groupoid $G_\Lambda$ is essentially free (i.e. the points with trivial isotropy are dense in $G_\Lambda^0$) if and only if $\Lambda$ satisfies the aperiodicity condition.

Proof. Observe that if $x \in \Lambda^\infty$ is aperiodic then $\sigma^m x = \sigma^n x$ implies that $m = n$ and hence $x \in \Lambda^\infty = G_\Lambda^0$ has trivial isotropy, and conversely. Hence $G_\Lambda$ is essentially free if and only if aperiodic points are dense in $\Lambda^\infty$. If aperiodic points are dense in $\Lambda^\infty$ then $\Lambda$ clearly satisfies the aperiodicity condition, for $Z(v) = \Lambda^\infty(v)$ must then contain aperiodic points for every $v \in \Lambda^0$. Conversely, suppose that $\Lambda$ satisfies the aperiodicity condition, then for every $\lambda \in \Lambda$ there is $x \in \Lambda^\infty(s(\lambda))$ which is aperiodic. Hence the aperiodic points are dense in $\Lambda^\infty$.

The isotropy group of an element $x \in \Lambda^\infty$ is equal to the subgroup of its eventual periods (including 0).

Theorem 4.6. Let $\pi : C^*(\Lambda) \to B$ be a $*$-homomorphism and suppose that $\Lambda$ satisfies the aperiodicity condition. Then $\pi$ is faithful if and only if $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.

Proof. If $\pi(p_v) = 0$ for some $v \in \Lambda^0$ then clearly $\pi$ is not faithful. Conversely, suppose $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$; then by Proposition 4.9(i) we have $C^*(\Lambda) = C^*_v(\mathcal{G}_\Lambda)$ and hence from [KPR, Corollary 3.6] it suffices to show that $\pi$ is faithful on $C_0(\mathcal{G}_\Lambda^0)$. If the kernel of the restriction of $\pi$ to $C_0(\mathcal{G}_\Lambda^0)$ is nonzero, it must contain the characteristic function $1_{Z(\lambda)}$ for some $\lambda \in \Lambda$. It follows that $\pi(s_\lambda^* s_\lambda) = 0$ and hence $\pi(s_\lambda) = 0$; in which case $\pi(p_{s(\lambda)}) = \pi(s_\lambda^* s_\lambda) = 0$, a contradiction.

Definition 4.7. We say that $\Lambda$ is cofinal if for every $x \in \Lambda^\infty$ and $v \in \Lambda^0$ there is $\lambda \in \Lambda$ and $n \in \mathbb{N}^k$ such that $s(\lambda) = x(n)$ and $r(\lambda) = v$.

Proposition 4.8. Suppose $\Lambda$ satisfies the aperiodicity condition, then $C^*(\Lambda)$ is simple if and only if $\Lambda$ is cofinal.

Proof. By Proposition 4.3(ii) $C^*(\Lambda) = C^*_v(\mathcal{G}_\Lambda)$; since $\mathcal{G}_\Lambda$ is essentially free, $C^*(\Lambda)$ is simple if and only if $\mathcal{G}_\Lambda$ is minimal. Suppose that $\Lambda$ is cofinal and fix $x \in \Lambda^\infty$ and $\lambda \in \Lambda$; then by cofinality there is a $\mu \in \Lambda$ and $n \in \mathbb{N}^k$ so that $s(\mu) = x(n)$ and $r(\mu) = s(\lambda)$. Then $y = \lambda \mu \sigma^n x \in Z(\lambda)$ and $y$ is in the same orbit as $x$; hence all orbits are dense and $\mathcal{G}_\Lambda$ is minimal.

Conversely, suppose that $\mathcal{G}_\Lambda$ is minimal and that $x \in \Lambda^\infty$ and $v \in \Lambda^0$ then there is $y \in Z(v)$ such that $x, y$ are in the same orbit. Hence there exist $m, n \in \mathbb{N}^k$ such that $\sigma^n x = \sigma^m y$; then it is easy to check that $\lambda = y(0, m)$ and $n$ have the desired properties.

Notice that second hypothesis used in the following corollary is the analog of the condition that every vertex connects to a loop and it is equivalent to requiring that for every $v \in \Lambda^0$, there is an eventually periodic $x \in \Lambda^\infty(v)$ with positive eventual period (i.e. the eventual period lies in $\mathbb{N}^k \setminus \{0\}$). The proof follows the same lines as [KPR, Theorem 3.9]:

Proposition 4.9. Let $\Lambda$ satisfy the aperiodicity condition. Suppose that for every $v \in \Lambda^0$ there are $\lambda, \mu \in \Lambda$ with $d(\mu) \neq 0$ such that $r(\lambda) = v$ and $s(\lambda) = r(\mu) = s(\mu)$ then $C^*(\Lambda)$ is purely infinite in the sense that every hereditary subalgebra contains an infinite projection.

Proof. Arguing as in [KPR, Lemma 3.8] one shows that $\mathcal{G}_\Lambda$ is locally contracting. The aperiodicity condition guarantees that $\mathcal{G}_\Lambda$ is essentially free, hence by [A-D, Proposition 2.4] (see also [LS]) we have $C^*(\Lambda) = C^*_v(\mathcal{G}_\Lambda)$ is purely infinite.

5. Skew products and group actions

Let $G$ be a discrete group, $\Lambda$ a $k$-graph and $c : \Lambda \to G$ a functor. We introduce an analog of the skew product graph considered in [KPR, §2] (see also [CT]); the resulting object, which we denote $G \times_c \Lambda$, is also a $k$-graph. As in [KP] if $G$ is abelian the associated $C^*$-algebra is isomorphic to a crossed product of $C^*(\Lambda)$ by the natural action of $G$ induced by $c$ (more generally it is a crossed product by a coaction — see [Ma, KQR]). As a corollary we show that the crossed product of $C^*(\Lambda)$ by the gauge action, $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k$, is isomorphic to $C^*(\mathbb{Z}^k \times \Lambda)$, the $C^*$-algebra of the skew-product $k$-graph arising from the degree map. It will then follow that $C^*(\Lambda) \rtimes_\alpha \mathbb{T}^k$ is AF and that $\mathcal{G}_\Lambda$ is amenable.
Definition 5.1. Let $G$ be a discrete group, $(\Lambda, d)$ a $k$-graph. Given $c : \Lambda \to G$ a functor then define the skew product $G \times_c \Lambda$ as follows: the objects are identified with $G \times \Lambda^0$ and the morphisms are identified with $G \times \Lambda$ with the following structure maps

$$s(g, \lambda) = (gc(\lambda), s(\lambda)) \quad \text{and} \quad r(g, \lambda) = (g, r(\lambda)).$$

If $s(\lambda) = r(\mu)$ then $(g, \lambda)$ and $(gc(\lambda), \mu)$ are composable in $G \times_c \Lambda$ and

$$(g, \lambda)(gc(\lambda), \mu) = (g, \lambda\mu).$$

The degree map is given by $d(g, \lambda) = d(\lambda)$.

One must check that $G \times_c \Lambda$ is a $k$-graph. If $k = 1$ then any function $c : E^1 \to G$ extends to a unique functor $c : E^* \to G$ (as in [KP, §2]). The skew product graph $E(c)$ of $E^*$ is related to our skew product in a simple way: $G \times_c E^* = E(c)^*$. A key example of this construction arises by regarding the degree map $d$ as a functor with values in $\mathbb{Z}^k$.

The functor $c$ induces a cocycle $\tilde{c} : \mathcal{G}_\Lambda \to G$ as follows: given $(x, \ell - m, y) \in \mathcal{G}_\Lambda$ so that $\sigma^\ell x = \sigma^m y$ then set

$$\tilde{c}(x, \ell - m, y) = c(x(0, \ell))c(y(0, m))^{-1}.$$

As in [KP] one checks that this is well-defined and that $\tilde{c}$ is a (continuous) cocycle; regarding the degree map $d$ as a functor with values in $\mathbb{Z}^k$, we have $d(x, n, y) = n$ for $(x, n, y) \in \mathcal{G}_\Lambda$. In the following we show that the skew product groupoid obtained from $\tilde{c}$ (as defined in [KP]) is the same as the path groupoid of the skew product (cf. [KP, Theorem 2.4]):

Theorem 5.2. Let $G$ be a discrete group, $\Lambda$ a $k$-graph and $c : \Lambda \to G$ a functor. Then $\mathcal{G}_{G \times_c \Lambda} \cong \mathcal{G}_\Lambda(\tilde{c})$ where $\tilde{c} : \mathcal{G}_\Lambda \to G$ is defined as above.

Proof. We first identify $G \times \Lambda^\infty$ with $(G \times_c \Lambda)^\infty$ as follows: for $(g, x) \in G \times \Lambda^\infty$ define $(g, x) : \Omega \to G \times_c \Lambda$ by

$$(g, x)(m, n) = (gc(x(0, m)), x(m, n));$$

it is straightforward to check that this defines a degree-preserving functor and thus an element of $(G \times_c \Lambda)^\infty$.

Under this identification $\sigma^n(g, x) = (gc(x(0, n)), \sigma^n x)$ for all $n \in \mathbb{N}^k$, $(g, x) \in (G \times_c \Lambda)^\infty$. As in the proof of [KP, Theorem 2.4] define a map $\phi : \mathcal{G}_\Lambda(\tilde{c}) \to \mathcal{G}_{G \times_c \Lambda}$ as follows: for $x, y \in \Lambda^\infty$ with $\sigma^x = \sigma^y$ set

$$\phi([x, \ell - m, y]) = [x', \ell - m, y']$$

where $x' = (g, x)$ and $y' = (g\tilde{c}(x, \ell - m, y), y)$. Note that

$$\sigma^m y' = \sigma^m(g\tilde{c}(x, \ell - m, y), y) = \sigma^m(gc(x(0, \ell))c(y(0, m))^{-1}, y) = (gc(x(0, \ell)), \sigma^m y) = (gc(x(0, \ell)), \sigma^x x') = \sigma'(g, x) = \sigma'(x', \ell - m, y'),$$

and hence $(x', \ell - m, y') \in \mathcal{G}_{G \times_c \Lambda}$. The rest of the proof proceeds as in [KP, Theorem 2.4] mutatis mutandis.

Corollary 5.3. Let $G$ be a discrete abelian group, $\Lambda$ a $k$-graph and $c : \Lambda \to G$ a functor. There is an action $\alpha^c : \widehat{G} \to \text{Aut}(C^*(\Lambda))$ such that for $\chi \in \widehat{G}$ and $\lambda \in \Lambda$

$$\alpha^c(\chi)(s_\lambda) = (\chi, c(\lambda))s_\lambda.$$ 

Moreover $C^*(\Lambda) \rtimes_{\alpha^c} \widehat{G} \cong C^*(G \times_c \Lambda)$. In particular the gauge action is of the form, $\alpha = \alpha^d$, and so $C^*(\Lambda) \rtimes_{\alpha} \mathbb{T}^k \cong C^*(\mathbb{Z}^k \times_d \Lambda)$.

Proof. Since $C^*(\Lambda)$ is defined to be the universal $C^*$-algebra generated by the $s_\lambda$’s subject to the relations [F], and $\alpha^c$ preserves these relations it is clear that it defines an action of $\widehat{G}$ on $C^*(\Lambda)$. The rest of the proof follows in the same manner as that of [KP, Corollary 2.5] (see [KP, II.5.7]).

In order to show that $C^*(\Lambda) \rtimes_{\alpha} \mathbb{T}^k$ is AF, we need the following lemma:

Lemma 5.4. Let $\Lambda$ be a $k$-graph and suppose there is a map $b : \Lambda^0 \to \mathbb{Z}^k$ such that $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$ for all $\lambda \in \Lambda$, then $C^*(\Lambda)$ is AF.

Proof. For every $n \in \mathbb{Z}^k$ let $A_n$ be the closed linear span of elements of the form $s_\lambda s_\mu^*$ with $b(s(\lambda)) = n$. Fix $\lambda, \mu \in \Lambda$ with $b(s(\lambda)) = b(s(\mu)) = n$ we claim that $s_\lambda^* s_\mu = 0$ if $\lambda \neq \mu$. If $s_\lambda^* s_\mu \neq 0$ then by [F] there are $\alpha, \beta \in \Lambda$ with $s(\lambda) = r(\alpha)$ and $s(\mu) = r(\beta)$ such that $\lambda \alpha = \mu \beta$; but then we have

$$d(\alpha) + n = d(\alpha) + b(s(\lambda)) = b(s(\alpha)) = b(s(\mu)) = d(\beta) + b(s(\mu)) = d(\beta) + n.$$
Thus \(d(\alpha) = d(\beta)\) and hence by the factorisation property \(\alpha = \beta\). Consequently \(\lambda = \mu\) by cancellation and the claim is established. It follows that for each \(v\) with \(b(v) = n\) the elements \(s_\lambda s_\mu^*\) with \(s(\lambda) = s(\mu) = v\) form a system of matrix units and two systems associated to distinct \(v\)'s are orthogonal (see \(3.2\)). Hence we have

\[
A_n \cong \bigoplus_{b(v) = n} \mathcal{K} (\ell^2(s^{-1}(v))).
\]

By an argument similar to that in the proof of Lemma \(3.2\), if \(n \leq m\) then \(A_n \subseteq A_m\) (see equation \([3]\)); our conclusion now follows.

Note that \(A_n\) in the above proof is the \(C^*\)-algebra of a subgroupoid of \(G_\Lambda\) which is isomorphic to the disjoint union

\[
\bigsqcup_{b(v) = n} R_v \times \Lambda^\infty(v)
\]

where \(R_v\) is the transitive principal groupoid on \(s^{-1}(v)\). Since \(G_\Lambda\) is the increasing union of these elementary groupoids, it is an AF-groupoid and hence amenable (see \([R, III.1.1]\)). The existence of such a function \(b : \Lambda^0 \to Z^k\) is not necessary for \(C^*(\Lambda)\) to be AF since there are 1–graphs with no loops which do not have this property (see \([KPR, Theorem 2.4]\)).

**Theorem 5.5.** Let \(\Lambda\) be a k-graph, then \(C^*(\Lambda) \rtimes_\alpha T^k\) is AF and the groupoid \(G_\Lambda\) is amenable. Moreover, \(C^*(\Lambda)\) falls in the bootstrap class \(\mathcal{N}\) of \([RSc]\) and is therefore nuclear. Hence, if \(C^*(\Lambda)\) is simple and purely infinite (see \(\S 5\)), then it may be classified by its \(K\)-theory.

**Proof.** Observe that the map \(b : (Z^k \times_d \Lambda)^0 \to Z^k\) given by \(b(n, v) = n\) satisfies

\[
b(s(n, \lambda)) - b(r(n, \lambda)) = b(n + d(\lambda), \lambda) - b(n, r(\lambda)) = n + d(\lambda) - n = d(n, \lambda),
\]

The first part of the result then follows from \(5.2\) and \(5.3\). To show that \(G_\Lambda\) is amenable we first observe that \(G_\Lambda(d) \cong G_{Z^k \times_d \Lambda}\) is amenable. Since \(Z^k\) is amenable, we may apply \([R, Proposition II.3.8]\) to deduce that \(G_\Lambda\) is amenable. Since \(C^*(\Lambda)\) is strongly Morita equivalent to the crossed product of an AF algebra by a \(Z^k\)-action, it falls in the bootstrap class \(\mathcal{N}\) of \([RSc]\). The final assertion follows from the Kirchberg-Phillips classification theorem (see \([K, P]\)).

We now consider free actions of groups on k-graphs (cf. \([KP, \S 3]\)). Let \(\Lambda\) be a k-graph and \(G\) a countable group, then \(G\) acts on \(\Lambda\) if there is a group homomorphism \(G \to \text{Aut} \Lambda\) (automorphisms are compatible with all structure maps, including the degree): write \((g, \lambda) \mapsto g\lambda\). The action of \(G\) on \(\Lambda\) is said to be **free** if it is free on \(\Lambda^0\). By the universality of \(C^*(\Lambda)\) an action of \(G\) on \(\Lambda\) induces an action \(\beta\) on \(C^*(\Lambda)\) such that \(\beta_g s_\lambda = s_{g\lambda}\).

Given a free action of a group \(G\) on a k-graph \(\Lambda\) one forms the quotient \(\Lambda/G\) by the equivalence relation \(\lambda \sim \mu\) if \(\lambda = g\mu\) for some \(g \in G\). One checks that all structure maps are compatible with \(\sim\) and so \(\Lambda/G\) is also a k-graph.

**Remark 5.6.** Let \(G\) be a countable group and \(c : \Lambda \to G\) a functor, then \(G\) acts freely on \(G \times_c \Lambda\) by \((g, h, \lambda) = (gh, \lambda)\); furthermore \((G \times_c \Lambda)/G \cong \Lambda\).

Suppose now that \(G\) acts freely on \(\Lambda\) with quotient \(\Lambda/G\); we claim that \(\Lambda\) is isomorphic, in an equivariant way, to a skew product of \(\Lambda/G\) for some suitably chosen \(c\) (see \([GT, Theorem 2.2.2]\)). Let \(q\) denote the quotient map. For every \(v \in (\Lambda/G)^0\) choose \(v' \in \Lambda^0\) with \((q(v')) = v\) and for every \(\lambda \in \Lambda/G\) let \(\lambda'\) denote the unique element in \(\Lambda\) such that \(q(\lambda') = \lambda\) and \(r(\lambda') = r(\lambda')\). Now let \(c : \Lambda/G \to G\) be defined by the formula

\[
s(\lambda') = c(\lambda)s(\lambda').
\]

We claim that \(c(\lambda\mu) = c(\lambda)c(\mu)\) for all \(\lambda, \mu \in \Lambda\) with \(s(\lambda) = r(\mu)\). Note that

\[
r(c(\lambda)\mu') = c(\lambda)r(\mu') = c(\lambda)s(\lambda') = c(\lambda)s(\lambda') = s(\lambda');
\]

hence, we have \((\lambda\mu)' = \lambda'(c(\lambda)\mu')\) (since the image of both sides agree under \(q\) and \(r\)). Thus

\[
c(\lambda\mu)s(\mu') = c(\lambda\mu)s(\lambda') = s((\lambda\mu)') = c(\lambda)s(\mu') = c(\lambda)c(\mu)s(\mu')
\]

which establishes the desired identity (since \(G\) acts freely on \(\Lambda\)). The map \((g, \lambda) \mapsto g\lambda'\) defines an equivariant isomorphism between \(G \times_c (\Lambda/G)\) and \(\Lambda\) as required.

The following is a generalization of \([KPR, 3.9, 3.10]\) and is proved similarly.
Theorem 5.7. Let $\Lambda$ be a $k$-graph and suppose that the countable group $G$ acts freely on $\Lambda$, then
\[ C^*(\Lambda) \rtimes_G G \cong C^*(\Lambda/G) \otimes K(\ell^2(G)). \]

Equivalently, if $c : \Lambda' \to G$ is a functor, then
\[ C^*(G \times_c \Lambda') \rtimes_G G \cong C^*(\Lambda') \otimes K(\ell^2(G)) \]
where $\beta$, the action of $G$ on $C^*(G \times_c \Lambda')$, is induced by the natural action on $G \times_c \Lambda'$. If $G$ is abelian this action is dual to $\alpha^*$ under the identification of $\mathbb{B}_G$.

Proof. The first statement follows from the second with $\Lambda' = \Lambda/G$; indeed, by Proposition 3.7 there is a functor $c : \Lambda/G \to G$ such that $\Lambda \cong G \times_c (\Lambda/G)$ in an equivariant way. The second statement follows from applying Proposition 3.7 to the natural $G$-action on $G_{G\times,\Lambda'} \cong G_N(c)$. The final statement follows from the identifications
\[ C^*(\Lambda) \rtimes_{\alpha^*} \tilde{G} \cong C^*(G \times_c \Lambda) \cong C^*(\tilde{G}_\Lambda(c)) \]
and [R, II.2.7].

6. 2-graphs

Given a $k$-graph $\Lambda$ one obtains for each $n \in \mathbb{N}^k$ a matrix
\[ M^\Lambda_n(u, v) = \#\{\lambda \in \Lambda^n : r(\lambda) = u, s(\lambda) = v\}. \]

By our standing assumption the entries are all finite and there are no zero rows. Note that for any $m, n \in \mathbb{N}^k$ we have $M^\Lambda_{m+n} = M^\Lambda_m M^\Lambda_n$ (by the factorization property); consequently, the matrices $M^\Lambda_m$ and $M^\Lambda_n$ commute for all $m, n \in \mathbb{N}^k$. If $W$ is the $k$-graph associated to the commuting matrices $\{M_1, \ldots, M_k\}$ satisfying conditions (H0)–(H3) of [RS2] which was considered in Example 1.7(iv), then one checks that $M^W_m = M^I_m$. Further, if $\Lambda = E^n$ is a 1-graph derived from the directed graph $E$, then $M^\Lambda_n$ is the vertex matrix of $E$.

Now suppose that $A$ and $B$ are 1-graphs with $A^0 = B^0 = V$ such the associated vertex matrices commute. Set $A_0 = \{((\alpha, \beta) \in A \times B : s(\alpha) = r(\beta)\}$ and $B_0 = \{((\beta, \beta) \in B \times A : s(\beta) = r(\alpha)\}$; since the associated vertex matrices commute there is a bijection $\theta : (\alpha, \beta) \mapsto (\beta, \alpha')$ from $A_0 \times B_0$ to $B_0 \times A_0$ such that $s(\alpha) = r(\beta')$ and $s(\beta) = s(\alpha')$. We construct a 2-graph $\Lambda$ from $A, B$ and $\theta$. This construction is very much in the spirit of [RS2]; roughly speaking an element in $\Lambda$ of degree $(m, n)$ is given by an element in $\Lambda$ of degree $(m, n)$ for $(i, j) \in V(m, n)$.

An element in $\Lambda^{(m-n)}$ is given by $v(i, j) \in V$ for $(i, j) \in V(m, n)$, $\alpha(i, j) \in A^1$ for $(i, j) \in V(m-1, n)$ and $\beta(i, j) \in B^1$ for $(i, j) \in V(m, n-1)$ (set $W(m, n) = \emptyset$ if $m$ or $n$ is negative) satisfying the following compatibility conditions wherever they make sense:

i) $r(\alpha(i, j)) = v(i, j)$ and $r(\beta(i, j)) = v(i, j)$
ii) $s(\alpha(i, j)) = v(i+1, j)$ and $s(\beta(i, j)) = v(i, j+1)$
iii) $\theta(\alpha(i, j), \beta(i, j)) = (\beta(i, j), \alpha(i+1, j))$;

for brevity and with a slight abuse of notation we regard this element as a triple $(v, \alpha, \beta)$ (note that $\alpha$ disappears if $m = 0$ and $\beta$ disappears if $n = 0$ and $v$ is determined by $\alpha$ and/or $\beta$ if $mn \neq 0$). Set
\[ \Lambda = \bigcup_{(m, n)} \Lambda^{(m-n)} \]
and define $s(v, \alpha, \beta) = v(m, n)$ and $r(v, \alpha, \beta) = v(0, 0)$.

Note that if $\lambda \in A^m$ and $\mu \in B^n$ with $m, n > 0$ such that $s(\lambda) = r(\mu)$ there is a unique element $(v, \alpha, \beta) \in \Lambda^{(m-n)}$ such that $\lambda = (0, 0)\alpha(1, 0) \cdots n(0, m-1, 0)$ and $\mu = (m, 0)\beta(m, 1) \cdots (n-1, 0)$; denote this element $\lambda_{MN}$. Further if $\lambda \in A^m$ and $\mu \in B^n$ with $m, n > 0$ such that $s(\lambda) = s(\mu)$ there is a unique element $(v, \alpha, \beta) \in \Lambda^{(m-n)}$ such that $\lambda = (0, 0)\alpha(1, 0) \cdots n(0, m-1, 0)$ and $\mu = (0, 0)\beta(0, 1) \cdots (0, n-1)$; denote this element $\mu_{MN}$. Using these two facts is not difficult to verify that given elements $(v, \alpha, \beta) \in \Lambda^{(m-n)}$ and $(v', \alpha', \beta') \in \Lambda^{(m'-n'\cdot-n')}$.
Finally, we write $A = A \ast \theta B$. It is straightforward to verify that up to isomorphism any 2-graph may be obtained from its constituent 1-graphs in this way.

If $A = B$, then we may take $\theta = \iota$ the identity map. In that case one has $A \ast \iota A \cong f^* (A)$ where $f : \mathbb{N}^2 \to \mathbb{N}$ is given by $f (m, n) = m + n$. Hence, by Corollary 3.3(i) we have $C^* (A \ast \iota A) \cong C^* (A) \otimes C (T)$.

To further emphasise the dependence of the product $A \ast \theta B$ on the bijection $\theta : A^1 \ast B^1 \to B^1 \ast A^1$ consider the following example:

**Example 6.1.** Let $A = B$ be the 1-graph derived from the directed graph which consists of one vertex and two edges, say $A^1 = \{ e, f \}$ (note $C^* (A) \cong O_2$). Then $A^1 \ast A^1 = \{( e, e), (e, f), (f, e), (f, f) \}$, and we define the bijection $\theta$ to be the flip. It is easy to show that $A \ast \theta A \cong A \times A$; hence,

$$C^* (A \ast \theta A) \cong O_2 \otimes O_2 \cong O_2,$$

where the first isomorphism follows from Corollary 3.3(iv) and the second from the Kirchberg-Phillips classification theorem (see [KPR]). But

$$C^* (A \ast \iota A) \cong O_2 \otimes C (T),$$

hence, $A \ast \theta A \not\cong A \ast \iota A$.

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