Waves propagation in two-layer system over a non-rational submerged hump

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Abstract. Multiple layered ocean system is a field that is crucial to understand. It exist with the existing of the difference in sea temperature and salinity that will form a density layer in the ocean. This phenomena is crucial to understand and very significance especially to offshore company such as oil related industry and logistics for avoiding casualties at the workplace. Thus, in this paper, the different density of the ocean is focused by simplified the multilayer ocean into two-layer model with the existing of a submerged hump on the seabed. By using the two-layer shallow water waves equation, this problem is solved analytically. For the verification process, the new solution is compared with the existing solution, and the two model is hardly distinguishable Then, the effect of the density and hump profiles to the wave propagation and wave refraction and diffraction are analyzed. The hump and the density profiles have a very significance effect to the relative wave amplitude and wave refraction. The bigger α resulting in the higher relative amplitude and the denser fluid give the weaker relative wave amplitude because of the stronger restoring force.

1. Introduction
In the ocean, the existing of the difference sea temperature and salinity will form a different layer of density. The density of seawater has a significance role in causing ocean currents and circulating heat because of the fact that the dense water sinks below less dense. This type of wave is called internal wave. In a simple model, internal waves can be assume as a two-layer fluid. The propagation of waves in a two-layer fluid with both a free surface and an interface in a flat bottom was pioneered by Stokes [1]. Since then, this study attracted other researchers such as Keulegan [2], Long [3], and Benjamin [4].

Internal waves also produces distortion on water surface that effects directly towards surface spectra of the sea [5]. The distortion of wave leads to the physical phenomena called the wave refraction and diffraction. The powerful equation to study the refraction and diffraction was introduced by Berkhoff [6] which is the mild-slope wave equation. This equation is widely used by many researchers to study either single-layer or two-layer fluid [7-11]. More recently, Niu and Yu[12] studied the long wave propagating over a submerged hump with variable hump profiles, their study found that the effect of the hump shape give a significance contribution to the wave amplitude corresponding to energy. Zhu and Harun [13], constructed the two-layer mild-slope equation with the rigid lid approximation is used for the upper-layer. Replacing the free surface with a rigid-lid approximation is reasonable in many cases, especially at the regional ocean scale, because “internal-
wave mode" only induces small deformation on the free surface and thus a rigid-lid approximation would exclude the fast mode associated with barotropic free surface waves and greatly simplify the theoretical analysis without loss of a great deal of accuracy. Based on the equation obtained, they then derived the analytical solution for long waves propagating over a circular hump located at the bottom of a two-layer ocean. Then, they also used this equation to study wave refraction and diffraction in two-layer fluid as discussed in [14-16].

Thus, in this paper, the different density of the ocean is focused by simplified the multilayer ocean into two-layer model with the existing of a submerged hump on the seabed, defined in [12] but focusing only to the non-rational value of $\alpha$. By using the two-layer mild-slope equation introduced by [13], this problem is solved analytically. For the verification process, the new solution is compared with the existing solution. Then, the effect of the density and hump profiles to the wave propagation and wave refraction and diffraction are analyzed.

2. Analytic solution
In this section, the formulation for waves propagating in two-layer fluid over a circular hump with rigid-lid approximation in the free surface wave being imposed is presented. Consider, the two-layer long wave propagating over a gradually varying topography with constant depth, $h_1$ and $h_2$ and the densities for upper and lower layer are denoted by $\rho_1$ and $\rho_2$ as shown in Figure 1.

The geometry of the hump which locate at the lower layer is as described in [12] and [15] and is defined by

$$h_2(r) = \begin{cases} h_2, & r \geq b, \\ h_0 + \beta r^\alpha, & r < b. \end{cases}$$

(1)

Here, $h_0$ is the hump height measured form the origin, $b$ is the radius of the hump in the horizontal plane, $\alpha$ and $\beta$ are two independent parameters which control the shape of the hump. Here, $\beta$ must always satisfied the relation $\beta = (h_2 - h_0)/b^\alpha$. The geometry of the hump with different value of $\alpha$ is presented in Figure 2. As can be seen, the bigger $\alpha$ implies a flat crest of the hump.

2.1. Two-layer waves propagating over a submerged hump
In this paper, the wave is considered as a small amplitude and its decay is negligible. Thus, the long wave version of the mild slope equation [13] is then applicable:

$$\nabla \cdot \left( \frac{(\rho_2 - \rho_1) h_1 h_2}{\rho_1 h_2 + \rho_2 h_1} \nabla \eta \right) + \frac{\sigma^2}{g} \eta = 0,$$

(2)

where $\nabla$ is a del operator in the horizontal plane, $\eta$ is the surface elevation, $g$ is the gravitational acceleration and $\sigma$ is the angular velocity given by

![Figure 1. Definition sketch of the problem.](image1)

![Figure 2. Hump with different value of $\alpha$.](image2)
\[ \sigma^2 = \frac{g k^2 (\rho_2 - \rho_1) h_1 h_2}{\rho_1 h_2 + \rho_2 h_1}, \]

with \( k \) as a wave number.

Considering the variables water depth \( r < b \) and because our system is axi-symmetric with respect to the \( z \)-axis, it is convenient to adopt a cylindrical coordinates system \( (r, \theta, z) \). Equation (2) can be constructed via separation of variables as

\[ \eta = \sum_{n=0}^{\infty} R_n(r) \cos(n\theta), \]

with \( R_n(r) \) satisfying

\[

t^2 (w_1 r^{2\alpha} + w_2 r^\alpha + w_3) R_n''(r) + r (w_4 r^{2\alpha} + w_5 r^\alpha + w_6) R_n'(r)
+ \left( (w_5 r^{2\alpha} + w_6 r^\alpha + w_7) \mu r^2 - n^2 (w_1 r^{2\alpha} + w_2 r^\alpha + w_3) \right) R_n(r) = 0. \tag{3}
\]

where

\[ w_1 = (\rho_1 - \rho_2) \beta^2 \rho_1 h_1, \]
\[ w_2 = (\rho_1 - \rho_2) (2h_0 \rho_1 + h_1 \rho_2) \beta h_1, \]
\[ w_3 = h_0 h_1 (\rho_1 - \rho_2) (h_0 \rho_1 + h_1 \rho_2), \]
\[ w_4 = h_1 (\rho_1 - \rho_2) (a h_1 \rho_2 + 2h_0 \rho_1 + h_1 \rho_2) \beta, \]
\[ w_5 = -\beta^2 \rho_1^2, \]
\[ w_6 = 2 \beta \rho_1 (h_0 \rho_1 + h_1 \rho_2), \]
\[ w_7 = -(h_0 \rho_1 + h_1 \rho_2)^2, \]
\[ \mu = \frac{\sigma^2}{\rho_1 h_1}. \]

Eq. (3) is in the form of the ordinary differential equation with regular singular point. Thus, Eq. (3) can be solved by Frobenius series method:

\[ R_n(r) = \sum_{m=0}^{\infty} a_m r^{m+c}, \tag{4} \]

where \( c \) is the root from the indicial equation and \( a_m \) are the recurrence relation to be determined. Substituting Eq. (4) into (3), results in

\[
\sum_{m=0}^{\infty} a_m r^{m+c} \left\{ ((m+c)^2 - n^2) w_1 r^{2\alpha} + \left( ((m+c)(m+c-1) - n^2) w_2 + (m+c) w_4 \right) r^\alpha 
+ \left( (m+c)^2 - n^2 \right) w_5 r^{2\alpha+2} + w_6 r^{\alpha+2} + w_7 r^2 \mu \right\} = 0. \tag{5}
\]

Reindexing and collecting the same power of \( r \), we then find the roots, \( c \), as:

\[ c = \pm n. \]

Because the difference on these two roots is an integer, the solution of \( R_n(r) \) can be written as:

\[ R_{1n}(r) = \sum_{m=0}^{\infty} a_m r^{m+n}, \tag{6} \]

and

\[ R_{2n}(r) = R_{1n} \ln r + \sum_{m=0}^{\infty} b_m r^{m-n}, \tag{7} \]

where \( b_m \) are another constant to be determined. Noting that \( R_{2n} \) becomes singular at \( r \to 0 \), it has to be discarded with the imposition of the condition that water surface elevation must be finite at the origin. Thus, by substituting Eq. (6) into (5), we find the recurrence relation as:

\[
a_m = -\frac{1}{m(m+2n) a_2} \left\{ w_2 a_{m-(2\alpha+2)} + w_6 a_{m-(\alpha+2)} + ((m+n-2\alpha)^2 - n^2) w_1 a_{m-2\alpha} + w_2 a_{m-2} + \left( ((m+n-\alpha)(m+n-1-\alpha) - n^2) w_2 + (m+n-\alpha) w_4 \right) a_{m-\alpha} \right\}, \tag{8}
\]

where \( \alpha \geq 1 \) and \( m = 0,1,2, \ldots \).
2.2. Two-layer wave refraction around a submerged hump

For the general solution in the constant water depth, \( r \geq b \), the solution is given by:

\[ \eta_1 = \eta_I + \eta_S, \]  

(9)

where \( \eta_I \) and \( \eta_S \) correspond to the incident wave and the scattered wave given by:

\[ \eta_I(r, \theta) = A_I e^{i k_0 x} = A_I \sum_{n=0}^{\infty} i^n \varepsilon_n J_n(k_0 r) \cos(n \theta), \]

(10)

\[ \eta_S(r, \theta) = \sum_{n=0}^{\infty} B_n H_{1,n}(k_0 r) \cos(n \theta), \]

(11)

in which \( A_I \) is the incident wave amplitude, \( i = \sqrt{-1} \), \( k_0 \) is the wave number in the constant water depth region, \( J_n \) is the Bessel function of the first kind, \( B_n \) is a set of complex constant to be determined, \( H_{1,n} \) is the Hankel function of the first kind and \( \varepsilon_n \) is the Jacobi symbol defined by:

\[ \varepsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n \geq 1. \end{cases} \]

(12)

For the general solution in a variable water depth, \( r < b \), the water surface elevation is given by:

\[ \eta_2(r, \theta) = \sum_{n=0}^{\infty} C_n R_{1,n}(k_0 r) \cos(n \theta), \]

(13)

where \( C_n \) is a set of complex constant to be determined. To make sure that these solution is continuous, the solution in these two sub region must be matched on the \( r = b \), require

\[ \frac{\partial \eta_1}{\partial r} = \frac{\partial \eta_2}{\partial r}, \]

(14)

Therefore, substituting Eqs. (14) and (15) into Eqs. (9)-(13), the complex constants \( B_n \) and \( C_n \) can be determined as:

\[ B_n = -\frac{A_I i^n \varepsilon_n \left[ R_{1n}(b) J_n'(k_0 b) - R_{1n}'(b) J_n(k_0 b) \right]}{R_{1n}(b) k_0 H_{1n}'(k_0 b) - R_{1n}'(b) H_{1n}(k_0 b)}, \]

(16)

\[ C_n = \frac{\pi b [R_{1n}(b) k_0 H_{1n}'(k_0 b) - R_{1n}'(b) H_{1n}(k_0 b)]}{2 A_I i^{n+1} \varepsilon_n}. \]

(17)

in which the prime denote the derivatives. Substitute Eqs. (16) and (17) into Eqs. (9)-(13), the water surface elevation for entire domain can be calculated.

3. Results and discussions

In this section, the validation of the newly derived solution is presented. Then the effect of the density and hump profile to the wave refraction are analysed.

![Figure 3. Comparison with single-layer fluid when \( \rho_1 = 0 \) along x-axis.](image-url)
3.1. Comparison with the existing solution
Single-layer fluid is the special case for the two-layer fluids when the density for the upper layer, \( \rho_1 = 0 \). Thus, as part of the verification process, we compare our model with those of [8] and [12] when \( n = 2 \). Figure 3 shows the comparison between the single layer model and two-layer model with \( \rho_1 = 0, \rho_2 = 5, h_1 = h_2 = 4.8, h_0 = 3.2, b = 0.5 \) along x-axis. As expected, the two models agree well.

3.2. Effect of the hump profiles and the density ratios
Next, the hump profiles and the density ratios of the layers are examined. First, the contour plots for the different hump profiles \( \alpha = 1, 2, 4 \) and 10 with \( \rho_1 = 1, \rho_2 = 5, h_1 = h_2 = 4, h_0 = 2.8, \) and \( b = 0.5 \) are presented in Figure 4. It can be seen that as the \( \alpha \) values getting bigger, the refraction effect over the hump becomes obvious because of wave distortion from the wider flat crest of the hump. The relative wave amplitude also is the highest for the biggest \( \alpha \). A better comparison on wave amplitude with different value of \( \alpha \) with \( \rho_1/\rho_2 = 1/5 \) and \( \rho_1/\rho_2 = 3/5 \) is presented in Figures (5) and (6) along x-axis. From both figures, it can obviously see that the biggest \( \alpha \) gives the highest relative wave amplitude. These phenomena is due to the flat crest of the hump profiles for the bigger \( \alpha \), making the propagating wave “feel” the hump more compare to smaller \( \alpha \). Besides, when the fluid is denser, the restoring force is weaker, thus the relative wave amplitude in a less dense fluid has a higher amplitude when compared to the denser fluid, as shown in Figs. (5) and (6).

![Contour plot of the wave amplitude at different values of \( \alpha \).](image)

4. Conclusion
A new analytical solution for two-layer fluid model for a different shape of hump is derived. The solution is validate with the existing single layer model from [8] and [12], and the two model is hardly distinguishable. The hump and the density profiles have a very significance effect to the relative wave amplitude and wave refraction. The bigger \( \alpha \) resulting in the higher relative amplitude and the denser fluid give the weaker relative wave amplitude because of the stronger restoring force.
Figure 5. Relative wave amplitude at different values of $\alpha$ with $\rho_1/\rho_2 = 1/5$.

Figure 6. Relative wave amplitude at different values of $\alpha$ with $\rho_1/\rho_2 = 3/5$.

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