On total $H$-irregularity strength of graphs

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Abstract. Let $G$ be a graph with vertex set $V$ and edge set $E$. Total labeling $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, ..., l\}$ called a total $l$-labeling of a graph $G$. The total $l$-labeling is a total $H$-irregular $l$-labeling of graph $G$ if for $H \subseteq G$, the total $H$-weights $wt_l(H) = \sum_{v \in V(H)} f(v) + \sum_{e \in E(H)} f(e)$ are distinct. The irregularity strength $s(G)$ of a graph $G$ is known as the minimum $k$ for which $G$ has an irregular assignment using labels at most $k$. The total $H$-irregular $a$-labeling from the minimum where the graph $G$ is called the total $H$-irregularity strength of $G$, is denoted by $tHS(G)$. In this paper, we have obtained $tHS$ from linegrid, buttrely, hexagonal and diamond graphs. To obtain the $tHS$, we begin to study the total irregularity strength of graph $G$ with subgraph $H$.

1. Introduction

The graphs that we discussed in this paper are mainly plane graphs which are simple, connected and limited graphs. Let $G$ be a graph that has a set of vertex $V(G)$ and a set of edge $E(G)$. The assignment of integration into vertices or edges, or both is subject to certain conditions called graph labeling [4]. Map that contain vertices and edges until positive integers to the number $k$ are called total labeling [11]. Graph $G$ contains a $H$-cover if each subgraph of $H_j$ is isomorphic, which conditions each vertex $E(G)$ is included in at least one of the subgraphs $H_j, j = 1, 2, ..., s$ [2] For each of the two different edge $l_1$ and $l_2$, it holds $w(l_1) \neq w(l_2)$ the total $a$ labeling is said to be the irregular edge of total $a$-labeling on graph $G$. Total edge $H$-irregularity strength $G$, symbolized by the $tES(G)$ is a minimum where graph $G$ has an irregular total edge labeling [3]. The total $a$-labeling is said to be a total $H$-irregular $a$-labeling of the graph $G$ if for $HG$, the total $H$-weights $W(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e)$, are distinct.

If the domain is the vertex-set or the edge-set, the labelings are called respectively vertex labelings or edge labelings. If the domain is $V(G)E(G)$ then we call the labeling total labeling. The most complete recent survey of graph labelings is [12]. The assignment of positive label integers $1, 2, ..., l$ for both of vertices and edges is called a label $l$ number of irregular so that the weight is calculated on a different vertices [1].

The edge irregular total $k$-labelling of the graph $G$ is the total $k$-labelling for every two different edges $e$ and $f$ of $G$ there is $wt(e)wt(f)$. And vertex irregular total $k$-labelling of $G$ is the total $k$-labelling if for every two distinct vertices $x$ and $y$ of $G$ there is $wt(x)wt(y)$.

The irregular assignments and the irregularity strength of graphs is the definition of the total edge irregularity strength introduced by Chartrand et.al.[5]. The $k$-labeling of the edges...
Theorem 1. [2] Let \( E_1, E_2, \ldots, E_k \) such that the sum of the labels of edges incident with a vertex is different for all the vertices of \( G \) and the smallest \( k \) for which there is the an irregular assignment, an irregular assignments is the irregularity strength, \( s(G) \) [7].

The irregularity strength was introduced in [5] by Chartrand et al. The irregularity strength of regular graphs was considered by Faudree and Lehel in [9]. The total edge irregularity strength of \( G, tes(G) \) is the minimum \( k \) for which the graph \( G \) has an edge irregular total \( k \)-labeling.

The total \( H \)-irregular \( a \)-labeling from the minimum where the graph \( G \) is called the total \( H \)-irregularity strength of \( G \), is denoted by \( tHs(G) \). The minimum \( k \) where \( G \) has an edge labeled \( k \) is an irregular total \( k \) is the definition of the total irregularity, \( tes(G) \). The irregularity strength of the edge \( G \), denoted by \( es(G) \) is a minimum \( K \) where graph \( G \) has an irregular labeled \( k \) [12]. The minimum where graph \( G \) has an irregular total labeling subgraph is the total Irregularity strength of \( G, tHs(G) \).

The minimum positive integer \( k \) for which graph \( G \) has an edge-irregular \( k \)-labelling is called the irregularity strength of the graph \( G \), denoted by \( s(G) \). An edge-irregular klabelling of \( G \) is the edge labelling \( : E(G)1, 2, \ldots, k \) if every two distinct \( x \) and \( y \) in \( V(G) \) satisfy \( wt(x) \neq wt(y) \). The total \( l \)-labeling \( (\varphi) \) which has different weights in every two disparate subgraphs is called total \( H \)-irregular labeling, where \( H_1 \) and \( H_2 \) are isomorphic to \( H \) there is \( wt_{\varphi}(H_1) = wt_{\varphi}(H_2) \).

Slamin et al. [11] prove that the strength of the total vertex irregularity of the merge is separate from the sun’s graph Rajasingh et al. [8].

The total \( l \)-labeling \( V(G) \cup E(G) \rightarrow \{1, 2, \ldots, l\} \) to the total irregular \( l \)-label is the labeling of the irregular \( l \)-total edge on the graph \( G \) if each two different edges have different weights \[12\]. Rajasingh et al. [8].

The total \( l \)-labeling \( (\varphi) \) which has different weights in every two disparate subgraphs is called total \( H \)-irregular labeling, where \( H_1 \) and \( H_2 \) are isomorphic to \( H \) there is \( wt_{\varphi}(H_1) \neq wt_{\varphi}(H_2) \). We determine \( H \)-weight as

\[
wt_{\varphi}(H) = \sum_{v \in V(H)} \varphi(v) + \sum_{e \in E(H)} \varphi(e),
\]

for the subgraph \( H \subseteq G \) under the total \( l \)-labeling \( (\varphi) \).

The total Irregularity strength of the graph \( G \) \( (tHs(G, H)) \). To find the total \( H \)-irregularity strength \( (tHs(G, H)) \) We use plane graphs such as grid graph and triangular ladder graph [10]. The smallest value of \( l \) that is owned by graph \( G \) has the total \( l \)-labeling \( H \)-irregular is the total \( H \)-irregularity strength graph \( G \) \( (tHs(G, H)) \). We use theorem 1 as the lower bound of the total \( H \)-irregularity strength.

**Theorem 1.** [2] Let \( G \) be a graph that accepts the \( H \)-covering given by \( t \) isomorphic subgraphs to \( H \). Then

\[
tHs(G, H) \geq \left\lfloor \frac{t - 1}{|V(H)| + |E(H)|} \right\rfloor
\]

In this paper, we study about the total \( H \)-irregularity strength of several graphs, namely the line grid graph, the butterfly graph, the hexagonal graph and the diamond graph. From this paper, we study the \( tHs(G) \) with \( H \)-covering irregularity strength \( (tHs(G, H)) \) of each of these different graphs. Line grid graphs is a denoted by \( GLn \).

**2. Results**

In this segment, we present the results of the total \( H \)-irregularity strength in our graph research field, as follows.
Theorem 2. Let $GLn(3, n), n \geq 2$, be a line grid graph admitting a $C_4$-covering. The total $H$-irregularity strength of $GLn(3, n)$ is $\left\lceil \frac{n+7}{4} \right\rceil$.

Proof. Let $GLn(3, n), n \geq 2$, be a line grid graph with the vertex set $V(L(Gn(3, n))) = \{x_j; 1 \leq i \leq n - 1, 1 \leq j \leq n, i = \text{odd}\} \cup x_j^i; 1 \leq i \leq n, 1 \leq i \leq n, i = \text{even}\}$ and the cardinality is $|V(L(Gn(3, n)))| = 3(n-1) - 2(n)$. The set of edges is $E(L(Gn)) = \{x_j^i x_j^{i+1}; 1 \leq i \leq n, 1 \leq j \leq n \cup (x_j^i x_{j+1}^i; 1 \leq i \leq n, i \text{odd}, 1 \leq j \leq n - 1) \cup (x_j^i x_{j-1}^i; 1 \leq i \leq n, i \text{even}, 2 \leq j \leq n\}$ and the cardinality is $|E(L(Gn(3, n)))| = 8n - n$. The line grid graph $G(3, n)n \geq 2$, contains a $C_4$-covering with exactly $2n - 2$ cycles $C_4$. The lower bound that we get from the theorem 2, $tHS(L(Gn), C_4) \geq \left\lceil \frac{n+7}{4} \right\rceil$. Put $l tHS(L(Gn), C_4) \geq \left\lceil \frac{n+7}{4} \right\rceil$. We define $C_4$-irregular total $l$-labeling $\varphi_4 : V(L(Gn(3, n))) \cup E(L(Gn(3, n))) \rightarrow \{1, 2, \ldots, l\}$ with aim the indicating to that $l$ is the upper bound for total $C_4$-irregular strength $L(Gn)$.

A $C_4$-irregular total $l$-labeling $\varphi_4 : V(L(Gn)) \cup E(L(Gn)) \rightarrow \{1, 2, \ldots, l\}$ is as follows: for $j = 1, 2, \ldots, l$,

\[
\begin{align*}
f(x_1^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 1 ; 8t - 6 \leq i \leq 8t - 4 \\
3t ; 8t - 3 \leq i \leq 8t - 1 \\
3t + 1 ; 8t \leq i \leq 8t + 1 
\end{cases} \\
f(x_2^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 2 ; i = 8t - 6 \\
3t - 1 ; 8t - 5 \leq i \leq 8t - 4 \\
3t ; 8t - 3 \leq i \leq 8t - 1 \\
3t + 1 ; 8t \leq i \leq 8t + 1 
\end{cases} \\
f(x_3^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 1 ; 8t - 6 \leq i \leq 8t - 4 \\
3t ; 8t - 3 \leq i \leq 8t - 2 \\
3t + 1 ; 8t - 1 \leq i \leq 8t + 1 
\end{cases} \\
f(x_4^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 2 ; i = 8t - 7 \\
3t - 1 ; 8t - 6 \leq i \leq 8t - 4 \\
3t ; 8t - 3 \leq i \leq 8t - 2 \\
3t + 1 ; 8t - 1 \leq i \leq 8t + 1 
\end{cases} \\
f(x_5^i) &= \begin{cases} 
1 ; i = 1 \\
13t - 2 ; 8t - 7 \leq i \leq 8t - 6 \\
3t - 1 ; 8t - 5 \leq i \leq 8t - 3 \\
3t ; 8t - 2 \leq i \leq 8t - 1 \\
3t + 1 ; i = 8t + 1 
\end{cases} \\
f(x_1^i x_2^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 2 ; i = 8t - 6 \\
3t - 1 ; 8t - 5 \leq i \leq 8t - 3 \\
3t ; 8t - 2 \leq i \leq 8t - 1 \\
3t + 1 ; i = 8t + 1 
\end{cases} \\
f(x_2^i x_3^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 2 ; i = 8t - 7 \\
3t - 1 ; 8t - 6 \leq i \leq 8t - 4 \\
3t ; 8t - 3 \leq i \leq 8t - 2 \\
3t + 1 ; i = 8t + 1 
\end{cases} \\
f(x_3^i x_4^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 2 ; i = 8t - 7 \\
3t - 1 ; 8t - 5 \leq i \leq 8t - 3 \\
3t ; 8t - 2 \leq i \leq 8t - 1 \\
3t + 1 ; i = 8t + 1 
\end{cases} \\
f(x_4^i x_5^i) &= \begin{cases} 
1 ; i = 1 \\
3t - 2 ; i = 8t - 7 \\
3t - 1 ; 8t - 5 \leq i \leq 8t - 3 \\
3t ; 8t - 2 \leq i \leq 8t - 1 \\
3t + 1 ; i = 8t + 1 
\end{cases}
\end{align*}
\]
For each vertex and edge label under the $\varphi_4$-labeling is almost $l$. We can see and observe that every vertex and edge under $\varphi_4$-labeling are almost $l$. We get the $C_4$-weight of $C_4^t$, $l = 1, 2, \ldots, 2m - 2$, under the total labeling $\varphi_4$, we get

$$ wt_{\varphi_4}(C_4^t) = \sum_{v \in V(C_4^t)} \varphi(v) + \sum_{e \in E(C_4^t)} \varphi(e), $$

From the $wt_{\varphi_4}(C_4^t)$ that the amount of label vertexes and edges, we obtain the sequence increases. And it is enough to prove that $wt_{\varphi_4}(C_4^t) < wt_{\varphi_4}(C_4^{l+1})$, $l = 1, 2, \ldots, n$.

The function label of vertex and edge $wt_{\varphi_4}$ point are variable periodic functions $n$. For each positive integer $l$, $l = 1, 2, 3, \ldots$, the $wt_{\varphi_4}$ function is used to label the vertex and edge of the graph $G(Ln)$. For each weight $G(Ln)$:

$$ w_1 = \varphi_4(x_1^4) + \varphi_4(x_1^3) + \varphi_4(x_1^2) + \varphi_4(x_1^1) + \varphi_4(x_1^0) + \varphi_4(x_1^0)(x_1^1) + \varphi_4(x_1^1)(x_1^2), $$

$$ w_2 = \varphi_4(x_0^4) + \varphi_4(x_0^3) + \varphi_4(x_0^2) + \varphi_4(x_0^1) + \varphi_4(x_0^0) + \varphi_4(x_0^0)(x_0^1) + \varphi_4(x_0^1)(x_0^2), $$

$$ w_3 = \varphi_4(x_3^4) + \varphi_4(x_3^3) + \varphi_4(x_3^2) + \varphi_4(x_3^1) + \varphi_4(x_3^0) + \varphi_4(x_3^0)(x_3^1) + \varphi_4(x_3^1)(x_3^2), $$

for each value $s$, weights are obtained

$$ w_1 = 8, 11, 14, \ldots, 3n + 5 $$

$$ w_2 = 10, 13, 16, \ldots, 3n + 7 $$

$$ w_3 = 9, 12, 15, \ldots, 3n + 6 $$

The based on the equation obtained $wt_{\varphi_4}(L(Gn)) = 1 + wt_{\varphi_4}(G(n))$. total weight of $L(Gn)$ is

$wt_{\varphi_4}(L(Gn)) = wt_{\varphi_4}(C_4^t) = \sum_{v \in V(C_4^t)} \varphi(v) + \sum_{e \in E(C_4^t)} \varphi(e) = 8, 9, 10, 11, 12, \ldots, 3n + 7$.

we respect to $wt_{\varphi_3}(C_4^t) < wt_{\varphi_4}(C_4^{l+1})$, $l = 1, 2, \ldots, n$ then $wt_{\varphi_3}(C_4^{n+1}) = 2 + wt_{\varphi_3}(C_4^t)$ we can observed the illustration of total $C_3$-irregularity strength of the Figures 1.

**Theorem 3.** Let $Wd(3, n)$, $n \geq 2$, be a butterfly graph recognizing a $C_3$-covering. The total $H-$irregularity strength of $Wd(3, n)$ is

$$ iHs(Wd, C_3) = \left\lceil \frac{n + 5}{6} \right\rceil $$

**Proof.** Let $Wd(3, n)$, $n \geq 2$, be a butterfly graph with the vertex set $V(Wd, c3) = \{x_i; 1 \leq i \leq n + 1\} \cup \{y_i; 1 \leq i \leq n\} \cup \{z_i; 1 \leq i \leq n + 1\}$ and the cardinality is $|V(Wd, c3)| = 3n + 2$. The set of edges is $E(Wd, C_3) = \{x_iy_i; 1 \leq i \leq n\} \cup \{y_ix_{i+1}; 1 \leq i \leq n\} \cup \{x_iz_i; 1 \leq i \leq n + 1\} \cup \{z_iy_i; 1 \leq i \leq n\} \cup \{y_{i+1}z_i; 1 \leq i \leq n\}$ and the cardinality is $|E(Wd, C_3)| = 5n + 1$. 
The butterfly graph $(W_d, C_3)$, contains a $C_3$-covering with exactly $2n - 2$ cycles $C_3$. The lower bound that we get from the theorem 3, $tHs(W_d, C_3) \geq \lceil \frac{n+5}{6} \rceil$. Put $\ell tHs(W_d, C_3) \geq \lceil \frac{n+5}{6} \rceil$.

We specify a $C_3$-irregular total $l$-labeling $\varphi_3 : V(W_d, C_3) \cup E(W_d, C_3) \rightarrow \{1, 2, \ldots, l\}$ is prove that $\alpha$ as an upper bound for the total $W_d$-irregularity strength of $W_d$.

A $C_3$-irregular total $l$-labeling $\varphi_3 : V(L(Gn)) \cup E(L(Gn)) \rightarrow \{1, 2, \ldots, l\}$ is as follows:

$$
\begin{align*}
  f(x_i) &= \left\lfloor \frac{i+2}{3} \right\rfloor, \\
  f(y_i) &= \left\lfloor \frac{i+2}{3} \right\rfloor, \\
  f(z_i) &= \left\lfloor \frac{i+1}{3} \right\rfloor, \\
  f(x_iy_i) &= \left\lfloor \frac{i+1}{3} \right\rfloor, \\
  f(x_iz_i) &= \left\lfloor \frac{i}{3} \right\rfloor, \\
  f(y_iz_i+1) &= \left\lfloor \frac{i}{3} \right\rfloor
\end{align*}
$$

We get the upper bound from the function of $C_3$-irregular total $W_d(3, n)$-labelling. We get to present to the upper bound of the graph in the theorem 3, $tHs((W_d)_{n+2}, C_3) \leq \left\lfloor \frac{i+2}{3} \right\rfloor$.

Based on the labeling above, we can show the all weights are different by the following equation:

$$
\begin{align*}
  wt\varphi_n(W_d^{i+2}) - wt\varphi_n(W_d^i) &= \varphi_3(x_{i+1}) + \varphi_3(y_{i+1}) + \varphi_3(z_{i+1}) + \varphi_3(x_iz_{i+1}) + \varphi_3(x_{i}y_{i+1}) + \\
  &\quad \varphi_3(x_{i}y_{i+1}) - \varphi_3(x_{i}) - \varphi_3(y_{i}) - \varphi_3(z_{i}) - \varphi_3(x_{i}z_{i}) - \varphi_3(x_{i}y_{i}) - \\
  &\quad \varphi_3(z_{i}y_{i}) \\
  &= 2
\end{align*}
$$

for every $w$ odd

$$
\begin{align*}
  wt\varphi_n(W_d^{i+3}) - wt\varphi_n(W_d^{i+1}) &= \varphi_3(x_{i+2}) + \varphi_3(y_{i}) + \varphi_3(z_{i+2}) + \varphi_3(x_{i+1}y_{i+1}) + \varphi_3(z_{i+1}y_{i+1}) + \\
  &\quad \varphi_3(x_{i+1}z_{i+2}) - \varphi_3(x_{i+1}) - \varphi_3(y_{i}) - \varphi_3(z_{i+1}) - \varphi_3(x_{i+1}y_{i}) - \\
  &\quad \varphi_3(x_{i+1}y_{i+1}) - \varphi_3(x_{i+1}y_{i+1}) -
\end{align*}
$$
Figure 2. Illustration of butterfly graph (Wd)

\[ \varphi_3(z_{i+1}y_i) - \varphi_3(x_iz_{i+1}) = 2 \]

We respect to wt_{\varphi_3}(C_n^l) < wt_{\varphi_3}(C_n^{l+1}), l = 1, 2, \ldots, n then wt_{\varphi_3}(C_n^{l+1}) = 2 + wt_{\varphi_3}(C_n^l)

The all H-weights are distinct. This matter concludes that \( tHs(Wd_n) = \left\lceil \frac{n + 5}{6} \right\rceil \). The example of total C3-irregularity of butterfly graph labeling, we can see on Figure 2, and we get

\[ tHs(Wd, C_3) = 2 \]

**Theorem 4.** Let \( Hn(3, n), n \geq 2 \), be a Hexagonal graph recognizing a \( C_6 \)-covering. The total \( H \)-irregularity strength of \( Hn(n) \) is

\[ tHs(Hn, C_6) = \left\lceil \frac{n + 11}{12} \right\rceil \]

**Proof.** Let \( Hn(6, n), n \geq 2 \), be a cycle graph with the vertex set \( V(C_6) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n + 1\} \cup \{z_i; 1 \leq i \leq n + 1\} \cup \{k_i; 1 \leq i \leq n\} \) and the cardinality is \( |V(C_6)| = 4n + 2 \). The set of edges is \( E(C_6) = \{x_iy_i; 1 \leq i \leq n\} \cup \{x_iz_{i+1}; 1 \leq i \leq n\} \cup \{y_iz_i; 1 \leq i \leq n + 1\} \cup \{z_ik_i; 1 \leq i \leq n\} \cup \{z_{i+1}k_i; 1 \leq i \leq n\} \) and the cardinality is \( |E(Hn, C_6)| = 5n + 1 \).

The cycle graph \( (Hn, C_6) \), contains a \( C_6 \)-covering with exactly \( 2n - 2 \) cycles \( C_6 \). The lower bound that we get from the theorem 4, \( tHs(Hn) \geq \left\lceil \frac{n + 11}{12} \right\rceil \). Put \( l tHs(Hn) \geq \left\lceil \frac{n + 11}{12} \right\rceil \). We specify a \( C_6 \)-irregular total l-labeling \( \varphi_6 : V(Hn, C_6) \cup E(C_6) \to \{1, 2, \ldots, l\} \) is prove that \( \alpha \) as an upper bound for the total \( H \)-irregularity strength of \( Hn \).

A \( C_6 \)-irregular total l-labeling \( \varphi_6 : V(C6) \cup E(C6) \to \{1, 2, \ldots, l\} \) is as follows:

\[
\begin{align*}
    f(x_i) &= \left\lceil \frac{i + 10}{12} \right\rceil \\
    f(y_i) &= \left\lceil \frac{i + 7}{12} \right\rceil \\
    f(z_i) &= \left\lceil \frac{i + 8}{12} \right\rceil \\
    f(k_i) &= \left\lceil \frac{i + 6}{12} \right\rceil \\
    f(x_iy_i) &= \left\lceil \frac{i + 5}{12} \right\rceil \\
    f(y_iz_{i+1}) &= \left\lceil \frac{i + 4}{12} \right\rceil \\
    f(x_iz_i) &= \left\lfloor \frac{i}{12} \right\rfloor \\
    f(z_ik_i) &= \left\lceil \frac{i + 3}{12} \right\rceil \\
    f(k_iz_{i+1}) &= \left\lceil \frac{i + 2}{12} \right\rceil
\end{align*}
\]
We get the upper bound from the function of $C_6$–irregular total $H_n$-labelling. We get to present to the upper bound of the graph in the theorem 3, $tHs((H_n), C_6) \leq \left\lceil \frac{i+10}{12} \right\rceil$

Based on the labeling above, we can show the all weights are different by the following equation:

\[
wt\varphi_n(H_n^{j+1}) - wt\varphi_n(H_n^j) = \varphi_6(x_i + 1) + \varphi_6(y_i + 1) + \varphi_6(x_i + 2) + f(x_iy_i + 1) + \varphi_6(y_i, x_i + 1) + \varphi_6(z_i + 1) - \varphi_6(x_i) - \varphi_6(y_i) - \varphi_6(x_i + 1) - \varphi_6(z_i) - \varphi_6(k_i) - \varphi_6(z_i + 1) - \varphi_6(x_iy_i) - \varphi_6(y_i, x_i + 1) = 1
\]

We respect to $wt\varphi_n(C_n^l) < wt\varphi_n(C_n^{l+1})$, $l = 1, 2, \ldots, n$ then $wt\varphi_n(C_n^{l+1}) = 2 + wt\varphi_n(C_n^l)$. The all $H$-weights are distinct. This matter concludes that $tHs((H_n), H_n) = \left\lceil \frac{n+11}{12} \right\rceil$. The example of total $H_n$-irregularity of diamond ladder graph labeling, we can see on Figure 3, and we get $tHs(Hn, C_6) = 2$.

**Theorem 5.** Let $Dn(3, n)$, $n \geq 2$, be a diamond graph recognizing a $C_4$-covering. The total $H$–irregularity of $Dn(n)$ is

\[
tHs(Dn, C_4) = \left\lceil \frac{n + 8}{9} \right\rceil
\]

**Proof.** let $Dn(4, n)$, $n \geq 2$, be a diamond graph with the vertex set $V(Dn, C_4) = \{x_i; 1 \leq i \leq n\} \cup \{y_i; 1 \leq i \leq n + 1\} \cup \{z_i; 1 \leq i \leq n + 1\}$ and the cardinality is $|V(Dn, C_4)| = 3n + 1$. The set of edges is $E(Dn, C_4) = \{x_iy_i; 1 \leq i \leq n\} \cup \{x_iz_i; 1 \leq i \leq n\} \cup \{y_iz_i; 1 \leq i \leq n\} \cup \{y_iz_i; 1 \leq i \leq n\} \cup \{y_iy_{i+1}; 1 \leq i \leq n\}$ and the cardinality is $|E(Dn, C_4)| = 5n$. The diamond graph $(Dn, C_4)$, contains a $C_4$-covering with exactly $2n - 2$ cycles $C_4$. The lower bound that we get from the theorem 5, $tHs(Dn, C_4) \geq \left\lceil \frac{n+8}{9} \right\rceil$. Put $l \ tHs(C_4) \geq \left\lceil \frac{n+8}{9} \right\rceil$. We specify a $C_4$-irregular total l-labeling $\varphi_4 : V(Dn, C_4) \cup E(Dn, C_4) \rightarrow \{1, 2, \ldots, l\}$ is prove that $\alpha$ as an upper bound for the total $Dn$–irregularity strength of $Dn$.

A $C_4$-irregular total l-labeling $\varphi_4 : V(C_4) \cup E(C_4) \rightarrow \{1, 2, \ldots, l\}$ is as follows:

\[
f(x_i) = \left\lceil \frac{i+6}{9} \right\rceil \quad f(y_i) = \left\lceil \frac{i+7}{9} \right\rceil
\]
Figure 4. Illustration of Total $C_4$-Irregularity strength of diamond graph ($D_n$)

\[
f(z_i) = \left\lceil \frac{i+5}{9} \right\rceil \quad f(x_iy_i) = \left\lceil \frac{i+4}{9} \right\rceil \\
f(x_iy_{i+1}) = \left\lceil \frac{i+3}{9} \right\rceil \quad f(y_iz_1) = \left\lceil \frac{i+2}{9} \right\rceil \\
f(y_i+1z_i) = \left\lceil \frac{i+1}{9} \right\rceil \quad f(y_iy_{i+1}) = \left\lceil \frac{i+2}{9} \right\rceil
\]

We get the upper bound from the function of $C_4$-irregular total $D_n$-labelling. We get to present to the upper bound of the graph in the theorem 3, $tHs((D_n),C_4) \leq \left\lceil \frac{i+6}{9} \right\rceil$

Based on the labeling above, we can show the all weights are different by the following equation:

\[
wt\varphi_4(D_n^{i+1}) - wt\varphi_4(D_n^i) = \varphi_4(x_i+1) + \varphi_4(y_i+1) + \varphi_4(y_i+2) + \varphi_4(z_i+1) + \varphi_4(x_iy_i+1) + \\
\varphi_4(x_iy_{i+1}+1) - \varphi_4(x_i) - \varphi_4(y_i) - \varphi_4(y_i+1) - \varphi_4(z_i) - \varphi_4(x_iy_i) - \\
\varphi_4(x_iy_{i+1}) - \varphi_4(y_i+1) - \varphi_4(y_iy_{i+1}) = 1
\]

We respect to $wt\varphi_4(C_n^i) < wt\varphi_4(C_n^{i+1})$, $i = 1, 2, \ldots, n$ then $wt\varphi_4(C_n^{i+1}) = 2 + wt\varphi_4(C_n^i)$. The all $H$-weights are distinct. This matter concludes that $tHs(D_n) = \left\lceil \frac{n+8}{9} \right\rceil$. The example of total $C_4$-irregularity of diamond graph labeling, we can see on Figure 4, and we get $tHs(D_n,C_4) = 2$.

3. Conclusion
In this paper, We have given the result of total $H$-irregularity strength of linegrid graphs, butterfly graphs, hexagonal graphs and diamond graphs. We recognize $H$-covering on all graphs in this discussion that $H$ is a cycle and fan graph.

Open Problem 1. Find the total $H$–irregularity strength of the graphs with $H \neq C$.

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References
[1] Ahmad A and Baça M 2013 On vertex irregular total labelings Ars Combinatoria 112 pp 129-139
[2] Ashraf F, Bača M, Lászlóková M and Semaničová-Feňovčíková A 2017 On H-irregularity strength of graphs Discussiones Mathematicae: Graph Theory 37 pp 1067-1078

[3] Agustin I H, Dafik, Marsidi, and Albirri E R 2017 On the total H-irregularity strength of graphs: a new notion Journal of Physics: Conf. Series 855 pp 1-9

[4] Gallian J A 2017 A dynamic survey of graph labeling The Electronic Journal of Combinatorics 20 pp 1-432

[5] G. Chartrand, M.S. Jacobson, J. Lehel, O.R. Oellermann, S. Ruiz, F. Saba, Irregular networks, Congr. 64 (1988) 187192.

[6] M. Baca, S. Jendro l, M. Miller and J. Ryan, On irregular total labellings, Discrete Math. 307(2007), 13781388.

[7] M.K. Siddiqui , A. Ahmad, M.F. Nadeem, Y. Bashir, Total edge irregularity strength of the disjoint union of sun graphs, International J. of Math and Soft Compu., 3, No.1 (2013), 21–27

[8] Rajasingh I, Rajan B and Annamma V 2012 On total vertex irregularity strength of triangle related graphs Annals of Pure and Applied Mathematics 1(2) pp 108-116

[9] R.J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs. Colloq. Math. Soc. Janos Bolyai, 52, Combinatorics, Eger. North Holland, Amsterdam, 1987, 247256.

[10] Seoul M A, El-Zekey M and El-Gazar E F 2017 Mean, odd sequential and triangular sum graphs Circulation in Computer Science 2(4) pp 40-52

[11] Slamin, Dafik, and Winnona W 2011 Total vertex irregularity strength of the disjoint union of sun graphs The Electronic Journal of Combinatorics 2012 pp 1-9

[12] Tarawneh I, Hasni R and Ahmad A 2016 On the edge irregularity strength of corona product of cycle with isolated vertices AKCE International Journal of Graphs and Combinatorics 13 pp 213-217

[13] Winarsih T and Indriati D 2018 Total edge irregularity strength of (n,t)-kite graph Journal of Physics: Conference Series 1008 pp 1-5