Critical Exponents without $\beta$-Function

Hagen Kleinert*
Institut für Theoretische Physik,
Freie Universität Berlin, Arnimallee 14, 14195 Berlin, Germany

We point out that the recently developed strong-coupling theory enables us to calculate the three main critical exponents $\nu$, $\eta$, $\omega$, from the knowledge of only the two renormalization constants $Z_\phi$ of wave function and $Z_m$ of mass. The renormalization constant of the coupling strength is superfluous, and with it also the $\beta$-function, the crucial quantity of the renormalization group approach to critical phenomena.

1. Some time ago we have shown [1,3] that there exists a simple way of extracting the strong-coupling properties of a $\phi^4$-theory from perturbation expansions. In particular, we were able to find the power behavior of the renormalization constants in the limit of large couplings, and from this all critical exponents of the system. By using the known expansion coefficients of the renormalization constants in three dimensions up to six loops we derived extremely accurate values for the critical exponents.

This is possible because of the fact that the $\phi^4$-theory displays experimentally observed scaling behavior. This implies that the perturbation expansions of the critical exponents, to be denoted collectively by $f(g_B)$, in powers of some dimensionless bare coupling $g_B$ have all a strong-coupling behavior [3] of the form

$$ f(g_B) = f^* - \frac{\text{const}}{g_B^\epsilon} + \ldots. \quad (1) $$

The number $f^*$ is the critical exponent, and the power $\epsilon$ is the exponent observable in the approach to scaling. The parameter $\epsilon = 4 - D$ denotes as usual the deviation of the space dimension from the naive scale-invariant dimension 4.

Apart from $\omega$, there are two independent critical exponents, for instance $\nu$ which rules the divergence of the coherence length $\xi \propto |T - T_c|^{-\nu}$ as the temperature $T$ approaches the critical temperature $T_c$, and the exponent $\gamma$ which does the same thing for the magnetic susceptibility. The purpose of this paper is to point out that the strong-coupling theory developed in [1,3] allows us to calculate all three critical exponents of the perturbation expansions of only the two renormalization constants $Z_\phi$ of wave function and $Z_m$ of mass. There is no need to go through the hardest calculation for the renormalization constant of the coupling strength, and we do not have to know the famous $\beta$-function of the renormalization group approach to critical phenomena, in which the exponent $\omega$ is found from the derivative of the $\beta$-function at its zero.

2. Let us briefly recall the relevant formulas. Given the first $N + 1$ expansion terms of the critical exponents, $f_N(g_B) = \sum_{n=0}^{N} a_n g_B^n$, we assume that the strong-coupling behavior [4, 5] continues systematically as an inverse powe series in $g_B^{-\omega/\epsilon}$, $f_M(g_B) = \sum_{i=0}^{M} b_m(g_B^{-2/q})^m$, with some finite convergence radius $g_0$ (see examples [3]). Then the $N$th approximation to the value $f^*$ is obtained from the formula

$$ f_N^* = \text{opt} \left[ \sum_{j=0}^{N} a_j g_B^j \sum_{k=0}^{N-j} \left( -\left( \frac{qj}{2k} \right) \right) \right] (-1)^k, \quad (2) $$

where the expression in brackets has to be optimized in the variational parameter $g_B$. The optimum is the smoothest of the real extrema. If there are none, the turning points serve the same purpose.

The derivation of this expression is simple: We replace $g_B$ in $f_N(g_B)$ trivially by $g_B \equiv g_B/\kappa^q$ with $\kappa = 1$. Then we rewrite, again trivially, $\kappa^{-q}$ as $(K^2 + \kappa^2 - K^2)^{-q/2}$ with an arbitrary parameter $K$. Each term is now expanded in powers of $r = (K^2 - K^2)/K^2$ assuming $r$ to be of the order $g_B$. We take the limit $g_B \to 0$ at a fixed ratio $g_B \equiv g_B/K^q$, so that $K \to \infty$ like $g_B^{1/2}$ and $r \to -1$, yielding [4]. Since the final result to all orders cannot depend on the arbitrary parameter $K$, we expect the best result to any finite order to be optimal at an extremal value of $K$, i.e., $g_B$.

The strong-coupling approach to the limiting value

$$ r = -1 + \kappa^2 / K^2 = -1 + O(g_B^{-2/q}) $$

implies the leading correction to $f_N^*$ to be of the order of $g_B^{-2/q}$. Application of the theory to a function with the strong-coupling behavior (3) requires therefore setting the parameter $q$ equal to $2\epsilon/\omega$ in formula (2).

For $N = 2$ and 3, the strong-coupling limits (3) are very simple. Defining $\rho \equiv 1 + q/2 = 1 + \epsilon / \omega$, one has for $N = 2$:

$$ f_2^* = \text{opt} \left[ a_0 + a_1 \rho g_B + a_2 g_B^2 \right] = a_0 - \frac{1}{4} a_2^\rho, \quad (3) $$

and for $N = 3$:

$$ f_3^* = \text{opt} \left[ a_0 + \frac{1}{3} a_1 \rho (\rho + 1) g_B + a_2 (2\rho - 1) g_B^2 + a_3 g_B^3 \right] $$

$$ = a_0 - \frac{1}{3} \frac{a_1}{a_3} + \frac{2}{27} \frac{a_2^2}{a_3^2} (1 - r), \quad (4) $$

where $r \equiv \sqrt{1 - 3 a_1 a_3 / a_2^2}$ and $a_1 \equiv \frac{1}{3} a_1 \rho (\rho + 1)$ and $a_2 \equiv a_2 (2\rho - 1)$. The positive square root must be taken.
to connect $g^*_B$ smoothly to $g^*_B$ for small $B$. If the square
root is imaginary, the optimum is given by the unique
turning point, leading once more to (3), but with $r = 0$.

Before we can apply formula (3), we must find the expo-
tent $\omega$ describing the approach to scaling. We do this
by studying the strong-coupling limit of the logarithmic
derivative $s(g_B) = g_B f'(g_B)/f(g_B)$ of any critical ex-
ponent $f(g_B)$, again via formula (3). Since $f(g_B)$ ap-
proaches a constant $f^*$ like (3), its logarithmic derivative
$s(g_B)$ must have the same type of behavior with $s^* = 0$.
This equation determines $\omega$. For an expansion of the crit-
ical exponents up to order $g_B^2$, it is easy to find explicit
results. Let us denote the generic expansion of the expo-

\begin{equation}
(\omega - 1) = 1 + \frac{2 + 2 - (n + 8 \epsilon)}{2 (8 + n + 2 n^2) g^*_B}, \tag{13}
\end{equation}

The critical exponent $\nu$ is obtained from the strong-
coupling limit of the function $\nu(g_B) = 1/[2 - \gamma(g_B)]$, whereas
the exponent $\gamma$ is the same limit of the function $\gamma(g_B) = \nu(g_B)/(2 - \eta(g_B))$. These functions have the expansions up to order $g_B^2$:

\begin{align*}
\nu(g_B) &= 1 + \frac{n + 2 + 2 - (n + 8 \epsilon)}{12 (8 + n + 2 n^2) g^*_B} \\
\nu(g_B) &= 1 + \frac{2 + 2 - (n + 8 \epsilon) + 2 (95 + 9 n + n^2) - 2 \epsilon (90 + 2 n^2)}{72 (8 + n + 2 n^2) g^*_B}, \tag{14}
\end{align*}

There is no need to give more expansion coefficients, which can all be downloaded from the internet (URL: ).

4. We begin by calculating the critical exponent $\omega$ from the requirement that the expansions $\nu(g_B)$ has a constant strong-coupling limit. Then also the subtracted function $F(g_B) = \nu(g_B) - \nu(0)$ has a constant limit. From (3) and (4), we find the expansion coefficients of the logarithmic derivative

\begin{equation}
\frac{\rho}{\nu} = \frac{4 (8 + n + 2 n^2) + \epsilon^2 (181 + 24 n + n^2)}{2 (8 + n + \epsilon (n - 3))}. \tag{15}
\end{equation}

The resulting exponents $\omega = \epsilon/\rho - 1$ are plotted against $\epsilon$ in Fig. 4, together with the plots derived in Ref. 3 from the expansion of the renormalized coupling constant $g(g_B)$ in powers of $g_B$. The first two terms of the $\epsilon$-expansion of $\omega$ are, in all expressions

\begin{equation}
\omega = \epsilon - \frac{3 n + 1 + 4 (n + 8) \epsilon^2}{(n + 8) \epsilon^2} + \ldots. \tag{18}
\end{equation}

Let us also calculate $\omega$ from the auxiliary function $h(g_B)$ of Eq. (6) by solving Eq. (5) for $\omega$, which reads more explicitly:

\begin{equation}
\frac{\omega}{\epsilon} - 1 = - \frac{\rho}{\nu - 1} = - \frac{1}{2} \frac{A_3^2 \rho^2}{3 A_3 - 2 A_2^2}. \tag{19}
\end{equation}

yielding $\rho = 1 + \epsilon/\omega$:

\begin{equation}
\rho = 1 + \frac{\epsilon}{\omega} = \frac{1}{2} + \sqrt{\frac{6 A_3}{A_2^2} - \frac{15}{4}}. \tag{20}
\end{equation}

From $F(g_B) = \nu(g_B) - \nu(0)$ we obtain

\begin{equation}
\rho = 1 + \frac{\epsilon}{\omega} = \frac{1}{2} + \sqrt{\frac{6 A_3}{A_2^2} - \frac{15}{4}}. \tag{21}
\end{equation}

\begin{equation}
\frac{\rho}{\nu} = \frac{4 (8 + n + 2 n^2) + \epsilon^2 (181 + 24 n + n^2)}{2 (8 + n + \epsilon (n - 3))}. \tag{15}
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\end{equation}
while $F(g_B) = \gamma(g_B) - \gamma(0)$ yields

$$\rho = 1/2 + \sqrt{\frac{3}{2}} \times \sqrt{\frac{4(8 + n)^2 + \epsilon(352 + 32n - 4n^2) + \epsilon^2(232 + 36n + n^2)}{2(8 + n) + \epsilon(n - 3)}}. \tag{22}$$

The resulting $\omega = \epsilon/(\rho - 1)$ are also plotted in Fig. 1. 

The $\epsilon$-expansions are again the same as in \cite{1}. 

5. With $\omega$ being determined, the critical exponents $\nu$ and $\gamma$ are calculated as before in Ref. \cite{3a} by inserting their power series coefficients into the strong-coupling equation \cite{3}. Depending on the different expressions for the resummed $\omega(\epsilon)$ we obtain from the limiting values $f^*$ various resummed function $\nu(\epsilon)$ and $\gamma(\epsilon)$. Their common $\epsilon$-expansions up to $\epsilon^2$ are

$$\nu = \frac{1}{2} + \frac{1 + n + 2}{4n + 8} \epsilon + \frac{(n + 2)(n + 3)(n + 20)}{8(n + 8)^3} \epsilon^2, \tag{23}$$

$$\gamma = 1 + \frac{1 + n + 2}{2n + 8} \epsilon + \frac{1}{4} \frac{(n + 2)(n^2 + 22n + 52)}{(n + 8)^3} \epsilon^2. \tag{24}$$

Thus we have shown that the $\epsilon$-expansions of all critical exponents including $\omega$ can be obtained from only two renormalization constants $\phi^2/\phi_B^2$ and $m^2/m_B^2$. Thus there is no need to calculate the renormalization constant of the coupling strength.

6. In three dimensions, the two renormalization constants $\phi^2/\phi_B^2$ and $m^2/m_B^2$ have been calculated up to seven loops \cite{1}, and the principal critical exponents have been derived via conventional resummation methods in \cite{0}, \cite{10}, and via variational perturbation theory by the present author in \cite{11}. The latter works, $\omega$ was derived from the expansion of the renormalized coupling constant in powers of $g_B$, which is only known up to six loops. In the spirit of the present discussion, we would like to recalculate $\omega$ from one of the two expansions of the two renormalization constants known up to seven loops. As an example we take the expansion for $\eta_\pi = \eta - \eta_m$. The reason for this choice is that the theorectical large-order behavior can be fitted extremely well to the known seven expansion coefficients, so that higher-order coefficients can be predicted quite reliably. This was done in \cite{12}, and the extrapolated expansion can be found in Table V of that paper. From this we may determine $\omega$ via the condition that $s(g_B) = d\log \eta(g_B)/d\log g_B$ vanishes in the strong-coupling limit, i.e., we the optimum of Eq. \cite{12} for $s^*$ should be zero. We do this only for the universality class of the superfluid transition of helium, proceeding as follows: For a various $\omega$-values we calculate for increasing orders the strong-coupling values $s_N^*$ and extrapolate them to infinite $N$ by a procedure explained in Ref. \cite{3}. From the results we find the $\omega$-value at which $s_N^*$ is zero to be $\omega = 0.790$. The extrapolation of $s_N^*$ for this is shown in Fig. \cite{12}.

7. The reader may wonder why we can so easily discard the renormalized coupling strength $g(g_B)$. The answer is simple: Instead of $g(g_B)$, we can just as well parametrize the coupling strength by the parameter $g_\gamma(g_B) \equiv [\nu(g_B) - \nu(0)]/\nu'(0)$ or $g_\nu(g_B) \equiv [\gamma(g_B) - \gamma(0)]/\gamma'(0)$ which both start out like

$$g_\nu,\gamma = g_B - \frac{1}{3} \left[ \frac{n + 8}{\epsilon} + (2 - n/2) \right] g_B^2 + O(g_B^3),$$

but continue differently. The renormalized coupling constant $g(g_B)$ has the same first terms except for the last parentheses. For either of these expansions we can define $\beta$-type of functions

$$\beta_{\nu,\gamma}(g_B) = -\epsilon g_{\nu,\gamma}(g_B) \frac{d\log g_{\nu,\gamma}(g_B)}{d\log g_B}, \tag{25}$$

and express these in terms of $g_{\nu,\gamma}$. Up to the order $g_B^2$, this yields

$$\beta_{\nu}(g_B) = -\epsilon g_{\nu} + \frac{8 + n}{3} + \frac{3 - n}{6} g_{\nu},$$

$$\beta_{\gamma}(g_B) = -\epsilon g_{\gamma} + \frac{8 + n}{3} + \frac{2 - n}{6} g_{\gamma}, \tag{26}$$

whose zeros determine the critical exponents $\nu, \gamma$ to satisfy

$$g_{\nu,\gamma} = \frac{3\epsilon}{8 + n} + \ldots , \tag{27}$$

in agreement with the first two terms in \cite{23} and \cite{24}. The slopes of these $\beta$-like functions are universally equal to $\omega$.

8. Summarizing we see that no $\beta$-function is needed to calculate the principal critical exponent of $\phi^4$-theories, $\nu, \gamma$, and $\omega$, which can all be obtained from the two renormalization constants of wave function and mass, and thus from the perturbation expansion of the self-energy $\Sigma(p^2)$ in powers of the bare coupling constant. The two desired expansions are extracted from $\Sigma(0)$ and $\Sigma'(0)$. This observation should be useful for possible future automated computer calculations of the critical exponents.

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[1] H. Kleinert, Phys. Rev. D 57, 2264 (1998) (E-Print \textit{aps/1997/num25001}); Addendum ibid. D 58, 1077 (1998) (cond-mat/9803266);

[2] H. Kleinert, \textit{Seven Loop Critical Exponents from Strong-Coupling $\phi^4$-Theory in Three Dimensions}, FU-Berlin preprint 1998 (hep-th/9812197).
In a next approximation, one will have to admit in the
strong-coupling expansion (1) daughter powers $1/g_\nu^\alpha$ with $\alpha \neq \omega$. These are neglected here, this being equiva-
 lent to the neglect of confluent singularities at the
infrared-stable fixed point in the renormalization group
approach discussed by B.G. Nickel, Phyica A 177, 189
(1991); A. Pelissetto and E. Vicari (University of Pisa
preprint IFUP-TH 52/97).

FIG. 1. Plots of the various solutions for $\omega(\epsilon)$ obtained
from the perturbation expansions of $\nu(g_\nu)$, $\gamma(g_\nu)$. The
dotted curve is the universal $\epsilon$-expansion up to $\epsilon^2$ of Eq. (1). The
dashed curves are ordered with increasing dash lengths, from
[2], [3]. The dashed-dotted curves were calcu-
lated in Ref. [1] from the perturbation expansion of $g(g_\nu)$,
only as in [1], [17], and once as in [2], [21]. The dot shows
the accurate value derived from a seven-loop expansion
in three dimensions.