Quantum quench dynamics

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Quench dynamics is an active area of study encompassing condensed matter physics and quantum information, with applications to cold-atomic gases and pump-probe spectroscopy of materials. Recent theoretical progress in studying quantum quenches is reviewed. Quenches in interacting one dimensional systems as well as systems in higher spatial dimensions are covered. The appearance of non-trivial steady states following a quench in exactly solvable models is discussed, and the stability of these states to perturbations is described. Proper conserving approximations needed to capture the onset of thermalization at long times are outlined. The appearance of universal scaling for quenches near critical points, and the role of the renormalization group in capturing the transient regime, are reviewed. Finally the effect of quenches near critical points on the dynamics of entanglement entropy and entanglement statistics is discussed. The extraction of critical exponents from the entanglement statistics is outlined.
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I. INTRODUCTION

The quantum quench studies how a many particle system prepared initially in the ground state of a Hamiltonian \( H_i \), evolves unitarily in time following the sudden change of the parameters to a final Hamiltonian \( H_f \)\[1, 2\]. Dynamics generated by a quantum quench has become an active topic of research because it poses many fundamental questions that can also be studied by current generation experiments.

Among experimental systems, the most striking are cold-atomic gases trapped in tunable optical lattices which can realize ideal Hamiltonians \[3, 4\]. In addition, the parameters of the Hamiltonian, such as interaction strength between particles, and lattice parameters, can be tuned rapidly in time. Being well isolated from the surroundings, it is a good approximation to assume the absence of an external thermalizing bath. In fact these many particle systems are usually far from equilibrium, with long thermalization times. This is particularly true for cold atoms in one dimensional (1\(d\)) optical lattices where thermalization times can be unobservably long \[5–8\].

Some questions that naturally arise in the field of quantum quenches, and that have the scope of being tested experimentally are, what are the mechanisms and time-scales for thermalization of closed quantum systems \[9–11]\? What is the difference in the dynamics of generic interacting systems and special Hamiltonians known to be integrable \[12–15]\? Since degrees of freedom are not counted the same way in classical and quantum systems, how does the notion of integrability generalize to quantum systems \[16]\? Are there any quantum generalizations to the Kolmogorov-Arnold-Moser (KAM) theorem \[17, 18]\? How does the phenomena of many body localization \[19–21]\, often described by the emergence of quasi-local integrals of motion \[22]\, depend on spatial dimensions, range of interactions and disorder?

Besides the formal notion of integrability, from the condensed matter perspective, natural questions that also arise is if the initial Hamiltonian or final Hamiltonian or both, can support collective order such as ferromagnetism or superconductivity or for that matter topological order, how does this order respond to or develop under a quench? Is there some universality in the dynamics when the quench is in the vicinity of a critical point? If so can one generalize powerful theoretical methods such as the renormalization group (RG) to study quantum quenches?
In attempting to understand how thermalization occurs in closed quantum systems, a useful quantity to study is the reduced density matrix. This is obtained from the full density matrix by tracing or integrating out some subset of the full Hilbert space. While the full density matrix can never show thermalization under unitary time evolution, the reduced density matrix can look effectively thermal with the remaining (integrated out) system acting like an effective reservoir for it. Understanding the dynamics of the reduced density matrix in the context of quantum quenches has in turn lead to a synergy between two different fields, that of condensed matter and quantum information [23–28]. Questions that interest both communities are how the entanglement entropy evolves in time [29–31]? What features of the entanglement entropy and entanglement statistics can differentiate between ergodic and non-ergodic systems [32, 33]? The study of how entanglement scales with size of the sub-system, and with time after the quench, has in turn lead to the development of novel numerical and variational methods for studying quench dynamics [34–36].

The discussion above shows that the field of quantum quenches is very broad, touching upon many concepts. It is difficult to do justice to all these topics in their entirety. This review will discuss some of these topics, with emphasis based on the author’s own research contribution in the field.

The outline of the paper is as follows. To orient the readers, in Section II, quantum quenches in some exactly solvable models of free fermions and free bosons in 1d will be presented. The key features of the resulting nonequilibrium steady states will be discussed. The advantage of studying quenches in 1d is that even effectively free theories capture non-trivial correlations [37]. This can in turn help build intuition about dynamics of correlated states in higher spatial dimensions.

In Section III various perturbations around the free limit will be considered. The perturbations will be non-linear in a way as to break the underlying integrability. The emphasis will be on developing methods that are general to non-integrable systems and also applicable to higher spatial dimensions. In Section III A it will be shown that even in the presence of integrability breaking non-linear terms, it is helpful to think of the dynamics in terms of an intermediate time prethermal (or collisionless) regime [38–41]. It will be shown that for quenches to a critical point, universality emerges in the prethermal dynamics, and an RG approach can be used to study its properties.

Finally the long time collisional or inelastic scattering dominated regime will be discussed
in Section III B. Here it will be outlined how a two-particle irreducible formalism can be used to obtain quantum kinetic equations that preserve conservation laws. Thermalization times obtained from the quantum kinetic equation for a 1d system with interactions and weak disorder will be discussed.

Following this, in Section IV results will be presented in higher spatial dimensions $d > 2$ where a quench to a critical point will be considered. Universality in the dynamics will be highlighted and the success of RG in capturing key features of the scaling will be discussed. In Section IV A, the time evolution of the entanglement entropy and entanglement statistics for the critical quench in $d > 2$ will be discussed, and how universal physics may be extracted from the entanglement statistics will be described. Finally in Section V we present our outlook.

II. QUENCHES IN 1d EFFECTIVELY FREE THEORIES: NONEQUILIBRIUM STEADY STATES

In this section we will consider quenches in free theories to highlight the new phenomena that can arise due to a quench. We will consider a spatially inhomogeneous quench, a homogeneous quench, and finally a quench in a system with interactions and disorder.

A. Inhomogeneous quench

Let us first study a spatially inhomogeneous quench in a simple model of non-interacting fermions in 1d. We will consider the spin 1/2 Heisenberg $XX$ chain, which, after a Jordan Wigner transformation, maps to free spinless fermions with nearest neighbor hopping. We will show that even this simple model can exhibit some remarkably interesting nonequilibrium states generated by a quench.

Imagine initially applying a magnetic field along the $\hat{z}$ direction, that varies linearly along the chain, changing sign at the center. Thus the initial Hamiltonian is

$$H_i = -J \sum_j \left[ S_j^x S_{j+1}^x + S_j^y S_{j+1}^y - \frac{j Fa}{J} S_j^z \right],$$

where $j$ is the position index, $a$ is the lattice spacing, and the magnetic field varies linearly as $j Fa$. The ground state of $H_i$, which will act as the initial state for the quench, is a
domain wall centered at \( j = 0 \), and connects regions with maximum positive and negative polarization along \( +\hat{z} \) on opposite ends of the chain. The quench will be the sudden switch off of the magnetic field so that the time evolution is simply due to the homogeneous \( XX \) chain.

After the quench, the domain wall is no longer a stable state, and acquires a time evolution. In the language of fermions, the initial state has a density imbalance due to a spatially varying chemical potential. The quench involves the switching off of the chemical potential so that the density imbalance is no longer stable, and as time evolves, there is flow of particles from one side of the chain to the other. In the language of spins, there is a flow of spin current from one side of the chain to the other. This current causes the domain wall to broaden ballistically with a velocity set by the lattice parameters [42–44].

We take the length \( L \) of the chain to be the largest scale in the problem. Effectively this involves studying the dynamics before the width of the domain wall becomes comparable to the system size. We find that the transverse spin correlation function at a time \( t \) after the quench, defined as,

\[
C_{xx}(j, j + n, t) = \langle \Psi_i | e^{iH_f t} S_j^x S_{j+n}^x e^{-iH_f t} | \Psi_i \rangle,
\]

reaches a nonequilibrium steady state of the form [44]

\[
C_{xx}(j, j + n, \frac{2vt}{na} > 1) \rightarrow C_{xx}^{eq}(n) \cos \left( \frac{2\pi n}{\lambda} \right).
\]

\( j, j + n \) correspond to points in the central region of the chain where the magnetization has dropped to zero. \( C_{xx}^{eq} \) is the ground state correlation of the uniform \( XX \) chain, and falls off as a power-law in position, \( C_{xx}^{eq} \sim \frac{1}{\sqrt{n}} \) [45]. The only knowledge the system has of the initial state is through the spatially oscillating prefactor of wavelength \( \lambda \). It’s physical origin is the spin current flowing through the region, and corresponds to spins twisting in a spiral pattern in the \( XY \) plane, \( S_j^+ = S_j^x + iS_j^y \sim S_0^+ e^{2\pi ij/\lambda} \). The strength of the spin current is proportional to the twist rate of the in-plane spins, and depends on the polarization \( \pm m_0 \) at the two extremes of the domain wall as, \( \lambda = \frac{2m_0}{a} \). In the example discussed above, the domain wall polarization at the extremes are maximal \( m_0 = 1/2 \), so that the wavelength of the oscillation is \( \lambda = 4a \).

Not just domain walls, but other kinds of initial density inhomogeneities relax via the creation of such transient current carrying states. For hard core bosons released from a trap, for example, they imply appearance of quasi-condensates at finite momentum [46, 47].
There are two interesting questions one can address here. One is that in 1d, powerful methods such as bosonization work well in capturing the low energy properties. How well do these methods capture the quench dynamics? Secondly, how do non-trivial interactions such as $J_z \sum_i S_i^z S_{i+1}^z$ affect the results above [48–54]? In particular, how does the domain wall expansion velocity change? How is the steady state spin current affected? The last question is related to transport in interacting models, where bounds on the optical conductivity, such as Mazur’s inequalities exist [55]. How are these bounds, established for systems in thermal equilibrium, affected by a quench?

Here we will only address what features of the nonequilibrium state above is captured by bosonization. Within a bosonization picture, one may encode the effect of forward scattering interactions from non-zero $J_z$ into the Luttinger parameter $K$, while the back-scattering interaction are captured by a non-linear interaction in the form of a cosine field. In the phase where back-scattering is RG irrelevant, the homogeneous $XXZ$ chain is [37]

$$H_{LL} = \frac{u}{2\pi} \int dx \left[ K (\partial_x \theta)^2 + \frac{1}{K} (\partial_x \phi)^2 \right].$$

The domain wall density is encoded in the bosonic variable $\partial_x \phi$, while the current density is in the variable, $\partial_x \theta$, where $\phi, \frac{1}{\pi} \partial_x \theta$ are canonically conjugate. We adopt a convention where Luttinger parameter $K = 1$ is $J_z = 0$ or non-interacting fermions, while $K < 1$ are repulsive fermions and $K > 1$ attractive fermions.

In order to compare results with the spin chain we modify $H_{LL}$ above by introducing a spatially varying chemical potential or magnetic field, and the quench involves switching it off. Studying domain wall dynamics using bosonization reveals that indeed a steady state emerges as the domain wall broadens. This state has the same property as in Eq. (3), namely the correlator has the same power-law correlations $1/r^{1/(2K)}$ as in the ground state of $H_{LL}$, but modulated in space due to the resulting current carrying state $\partial_x \theta \propto 1/\lambda$. The associated spiraling in-plane component of the spin has a wavelength [44],

$$\lambda = \frac{2K}{m_0}. \quad (5)$$

Thus bosonization does manage to capture the main features, although, as expected it does not capture the detailed microscopics such as precise domain wall velocity (in the continuum there is only one velocity, whereas on the lattice there are a range of velocities), and also the detailed profile of the domain wall [56, 57]. The above dependence of $\lambda$ on the
interaction parameter $\mathcal{K}$ or equivalently the anisotropy $J_z$ obtained from bosonization, was also recovered in numerical studies provided the currents (or equivalently the domain wall polarization) was not too large \cite{58}.

The situation described above cannot be probed at very long times numerically as eventually the domain wall width will reach the system size. Thus interesting questions such as how the magnitude of the current depends on irrelevant operators and inelastic scattering remain unanswered. Carrying out such a study analytically is also an open problem.

B. Homogeneous quench

A well studied nonequilibrium steady state is that arising after an interaction quench from $\mathcal{K}_0 \rightarrow \mathcal{K}$ in the Luttinger liquid \cite{59–61} and lattice models \cite{62–64}. Thus the initial state is the zero temperature ground state of free bosons with interaction parameter $\mathcal{K}_0$, while the time evolution is with respect to Eq. (4). To preserve Galilean invariance, we will also impose that the initial and final velocities change as follows $u_0 \mathcal{K}_0 \rightarrow u \mathcal{K}$.

The initial and final Hamiltonians are diagonalized by two different sets of bosonic operators that are related to each other by a Bogoliubov transformation. This transformation, together with the initial state, completely determine the occupation probabilities of the bosonic modes of the final Hamiltonian. Clearly there is no thermalization in a Gaussian theory, but only relaxation to a steady state, the latter being well described by a Generalized Gibbs Ensemble (GGE) \cite{65} that accounts for the conservation of population of the bosonic modes. For the example being discussed, the steady state continues to be a power-law but with a new exponent that depends on the initial and final Luttinger parameters. In particular,

$$C_{\phi\phi}(x,t) = \langle \psi_i | e^{iH_f t} \cos(\phi(x)) \cos(\phi(0)) e^{-iH_f t} | \psi_i \rangle,$$

$$t \rightarrow \infty, L \rightarrow \infty \frac{1}{x^{2K_{neq}}},$$

where $K_{neq} = \frac{1}{8} \mathcal{K}_0 \left(1 + \frac{\mathcal{K}_0^2}{\mathcal{K}_0^2} \right)$. This power-law decay in position is faster than in the ground state of the final Hamiltonian, the latter following from $K_{neq}$ by setting $\mathcal{K}_0 = \mathcal{K}$. Yet the post quench steady state is not a thermal state as for the latter the decay would be exponential in position. A popular way to describe this is via a mode-dependent temperature. Note that, despite the power-law, the steady state is not described by a new Luttinger parameter.
as the duality between the $\phi$ and $\theta$ correlators, a hallmark of the Luttinger liquid, is lost in this nonequilibrium steady state [66, 67].

We should also mention that quenches for free bosons with a gapless spectrum do not always generate power-laws, even when the initial state is the zero temperature ground state. We highlight this in the next sub-section via an example where we switch on a disorder potential as a quench.

An active area of study which we will not cover is to generalize the GGE to integrable models that cannot be written as free bosons or free fermions. Much progress has been made in this direction where GGEs have been constructed that capture the conservation of non-local operators [68–70].

**C. Disorder quench**

An interesting question concerns the modification of the above results for free and clean systems, by disorder. To address this, we start with an initial Hamiltonian corresponding to Eq. (4), and switch on disorder. In the language of bosonization, the disorder potential can be split into a forward scattering part and a backward scattering part [37]. The latter makes the problem non-linear, and its effect will be discussed later. For now let us consider only the effect of random forward scattering. Thus the final Hamiltonian is

$$H_f = H_{LL} - \frac{1}{\pi} \int dx \eta(x) \partial_x \phi(x),$$  

(8)

where $\eta$ is a Gaussian distributed disorder $\eta(x)\eta^*(x') = D_f \delta(x-x')$. In equilibrium, one may completely remove the effect of forward scattering by a simple change of variables $\phi(x) \rightarrow \phi(x) - \frac{K}{\pi} \int^x dy \eta(y)$. Thus forward scattering does not affect the two point correlations. In contrast for a quench, the initial state being the ground state of the fields before the shift, and thus a highly excited state for the shifted fields, the averages with respect to the initial state now do depend on the forward scattering disorder potential.

In a semiclassical picture, the quench creates excited quasiparticles that when propagating through the system pick up random phases associated with the disorder potential. Before averaging over disorder, the correlators decay with the same power-law as in the ground state of the clean system $H_{LL}$, but in addition carry random phases. For example, the $C_{\phi\phi}$
correlator has the following form before disorder averaging [71],

\[ C_{\phi\phi}(x, t) = C_{\phi\phi}^{eq}(x) e^{-i \frac{K}{2} \sum_{\epsilon=\pm} \left[ \int_{x+\epsilon ut}^{x+ut} dy \eta(y) - \int_{x}^{x+ut} dy \eta(y) \right]} \]  

(9)

\( C_{\phi\phi}^{eq} \) is the ground state correlator of \( H_{LL} \). The accompanying phase factors show that the operator at position \( x \) and a time \( t \) after the quench acquires contributions from random phases accumulated from the region \([x, x + ut]\) for left moving quasiparticles, and the region \([x, x - ut]\) for right moving quasiparticles.

On disorder-averaging this result, the random phases lead to an exponential decay of the correlations. There is an interesting light-cone behavior where for \( nut < r \) the disorder averaged correlators decay exponentially in time, while for \( nut > r \), a crossover to a steady state with exponential decay in position occurs. Note \( n = 1 \) or \( 2 \) depending on the correlator being studied. In particular \( n = 1 \) for \( C_{\phi\phi} \), while \( n = 2 \) for the correlator for the conjugate operators, \( C_{\theta\theta} \) [71].

Thus, even though there is no inelastic scattering and hence no generation of a temperature, the correlators decay exponentially in position at long times. This is an example of elastic dephasing. In subsequent sections we will discuss the effect of non-linearity arising from the back-scattering potential. We will discuss its effect on a quench when it is irrelevant and marginally irrelevant.

### III. QUENCHES IN 1d INTERACTING FIELD THEORIES

In this section we discuss the effect of non-linear perturbations on the free fermion and free boson steady states discussed above. We will consider the effect of the non-linearities for the cases where they are RG irrelevant or marginal, as this will allow us to carry out controlled perturbative schemes. Various non-linearities are of interest. We will primarily consider the role of a back-scattering interaction. These could arise due to a local impurity potential, an underlying lattice, and even a disorder potential.

#### A. Universal scaling in prethermal regime

Consider an initial state which is the ground state of the Luttinger liquid with an interaction parameter \( K \) as described by Eq. (4). The quench corresponds to the sudden switch
on of a non-linear term corresponding to back-scattering from a commensurate lattice. Thus the final Hamiltonian is,

\[ H_f = H_{\text{LL}} + V; \quad V = -g \int dx \cos(2\phi). \]  

The prequench Hamiltonian is described by free bosons, while the post quench Hamiltonian is the celebrated sine-Gordon model. Note that many different lattice models can lead to the same continuum field theory. For example, the post quench Hamiltonian could represent both a spin-chain, as well as the Bose Hubbard model. This may lead to confusion as one model is integrable, and the other is not. In the phase where the cosine term is irrelevant, the differences in the two models show up at the level of how the lattice or UV physics is introduced.

The model above has a quantum critical point. For the spin chain the critical point separates the gapless XX phase from the gapped Ising phase, while for the Bose Hubbard model, it separates the superfluid phase from the Mott insulating phase. The transition between the two phases is of the Berezenskii-Kosterlitz-Thouless (BKT) kind. For \( g \to 0 \), the critical point is located at \( K = 2 \), with \( K < 2 \) being the gapped phase where the cosine potential is RG relevant. In contrast for \( K > 2 \), the cosine potential is RG irrelevant and the spectrum stays gapless [37]. At the critical point, the system in the zero temperature ground state shows UV independent universal physics. A cautionary note here is that the universality is not in all quantities, but some special set of correlators directly related to the “order-parameter” [72, 73].

We study quenches in the gapless phase where the cosine term is irrelevant and at the critical point where it becomes marginally irrelevant, perturbatively in the cosine potential. We show that when the parameters of \( H_f \) are tuned to be at the critical point, an intermediate time prethermal phase exists which shows UV independent universal physics.

The two point correlation functions which show the onset of universal behavior in equilibrium are defined in Eq. (6). For the spin chain it represents the staggered anti-ferromagnetic \((-1)^n S_i^z S_{i+n}^z\) correlation function, while for the Bose Hubbard model it measures the superconducting phase fluctuations. We study how \( C_{\phi \phi} \) evolves in time perturbatively in the cosine potential and find that when \( K = 2 \), the leading non-zero correction grows logarithmically as follows (below we set velocity \( u = 1 \), position and time are in units of a lattice
scale, and we give results only for macroscopic position and times \( r, t \gg 1 \) \[74\],

\[
C_{\phi\phi}(r, t) = C^0_{\phi\phi}(r) \left[ 1 + \pi g \left( \ln r^2 + \ln t^2 - \ln \sqrt{64t^2 + (4t^2 - r^2)^2} \right) \right]. \quad (11)
\]

In the absence of the cosine potential \((g = 0)\), this correlator decays as \( C^0_{\phi\phi}(r) \sim \frac{1}{r} \).

The above shows light-cone dynamics where, when \( r \gg 2t \), the correlator decays in position primarily as in the initial state. Since our initial state was gapless, the correlator at these short times has a power-law in position decay. Under the effect of the cosine, this result is corrected by logarithms that grow in time as \( C_{\phi\phi}(r \gg 2t) \sim \frac{1}{r} \left[ 1 + 2\pi g \ln t \right] \). In the opposite limit of long times \( r \ll 2t \), the logarithmic correction is cut off by the distance so that \( C_{\phi\phi}(r \ll 2t) \sim \frac{1}{r} \left[ 1 + 2\pi g \ln r \right] \). However right on the light-cone, \( r = 2t \), the logarithmic corrections are qualitatively different. Quite generically one expects that on the light cone the correlators will be enhanced due to contributions from ballistically propagating, and initially entangled quasiparticles reaching the the positions \( 0, r \) simultaneously. Our result indicates that for a quench to the critical point this enhancement appears as a further enhancement of the logarithmic correction \( C_{\phi\phi}(r = 2t) \sim \frac{1}{r} \left[ 1 + 2\pi g \left( \frac{3}{2} \right) \ln t \right] \), note the change in the prefactor of the logarithm from \( 2\pi g \rightarrow 2\pi g(3/2) \).

Being perturbative, we do not expect these results to be valid at long times. Nevertheless, the UV independence of the perturbative treatment indicates that there is an intermediate time regime where an RG analysis may be performed to capture the quench dynamics.

As in equilibrium, we employ a Wilson Fisher approach to gradually integrate out short wavelength degrees of freedom, and study how parameters renormalize. For the case under discussion where a periodic potential was quenched, the couplings \( g, K \) obey the same RG equations as in equilibrium, which are of the BKT type. The difference now is that the RG flows have to be cut-off by \( \text{min}(2t, r) \), as is clear from the perturbative results for the correlation function presented above. In addition, the correlations themselves can be shown to obey a Callan-Szymanzik equation that relates correlators at long distances and/or times after the quench with the short distance and short time correlators of a weak coupling problem, where for the latter a perturbative treatment can be used.

Employing such a procedure, we arrive at the following results [74],

\[
C_{\phi\phi}(r, t) \sim \begin{cases} 
\sqrt{\ln t} & \text{if } 2t < r \\
\ln t & \text{if } 2t = r \\
\sqrt{\ln r} & \text{if } 2t > r 
\end{cases} 
\quad (12)
\]
The last long time result is the same as in the ground state of $H_f$ [72, 73]. At this level of perturbation theory, our calculation does not account for the long time limit where the energy injected by the quench will lead to inelastic scattering. The above results show that a prethermal regime exists where although full scale inelastic scattering is absent, yet dephasing effects from interactions can strongly modify the correlators from a naive free fermion or free boson model.

B. Thermalization via multi-particle scattering

Eventually for a generic nonintegrable system thermalization sets in. When carrying out the RG, the indication of the long time inelastic regime appears as the generation of additional terms in the Keldysh action [61, 75, 76]. For bosons, if $\phi_q$ is the quantum field, and $\phi_c$, the classical fields [77, 78], the new terms generated by the RG are of the kind $\phi_q \partial_t \phi_c$ and $\phi_q^2$. The former corresponds to dissipation, while the latter to an effective temperature.

Thus although the full system evolving unitarily will never show thermalization, the RG shows that the integrated out short wavelength modes can act as an effective reservoir for the long wavelength modes. When the system is out of equilibrium, for example due to a quench, this reservoir is where the long wavelength modes dissipate their energy. It is important to note that dissipative terms are generated even when the interactions are RG irrelevant indicating that irrelevant operators are responsible for thermalization to set in, and therefore cannot be neglected at long times.

In order to study the full dynamics of this thermal phase, we develop a quantum kinetic equation approach. Naïve kinetic equations do not work in 1$d$ as two particle scattering cannot relax a distribution function. In addition for our Hamiltonian, the non-linear term is not $\phi^4$ but is more complex, being proportional to $\cos \phi$. Moreover, when $\cos \phi$ is RG irrelevant, the $\phi$ field has strong fluctuations, and therefore one may not Taylor expand $\cos \phi$.

We adopt a two particle irreducible (2PI) formalism that allows one to treat generic interactions such as the cosine term, and also allows us to make conserving approximations. The principle is the following. One writes a Keldysh field theory or action $e^{i\Gamma}$ as a functional $\Gamma(G)$ of two particle averages, the Green’s function $G$. The solution for $G$ is the saddle point of this field theory $\delta \Gamma/\delta G = 0$. It can be shown that the saddle point condition is equivalent
to the Dyson equation in Keldysh space [78–81]. Next, one makes suitable approximations for the 2PI action, that translates to approximations for the self-energy, Dyson equations, and consequently the Green’s functions.

Conservation laws follow naturally in the 2PI approach from Noether’s theorem. For example deviations that correspond to globally conserved quantities do not change the action. On the other hand, local variations of these conserved quantities change the 2PI action. By requiring that these local changes occur around a saddle point, and therefore vanish to leading order, one may derive hydrodynamic equations. Thus one may obtain consistent definitions for the stress momentum tensor if energy and momentum are conserved, and the diffusion equation, if total number is conserved. Any approximations made to the 2PI action translate to consistent approximations to the Green’s functions, the hydrodynamic equations, and conserved densities [82].

We carried out this procedure for the Luttinger liquid with a cosine potential. We studied the case when the cosine was due to an underlying regular lattice [83] and also when it was due to a disorder potential [71]. We briefly discuss the results for the case of disorder. The effect of the forward scattering disorder has already been discussed in the previous section. The backward scattering disorder can be written as,

\[ V_{\text{dis}} = \int dx \left[ \xi^* e^{2i\phi} + \xi e^{-2i\phi} \right], \]

where we assume the disorder to be Gaussian distributed \( \xi(x) \xi^*(x') = D \delta(x - x') \). The ground state of \( H = H_{\text{LL}} + V_{\text{dis}} \) is well studied, showing a BKT transition from the superfluid to the Bose-glass phase at Luttinger parameter close to \( K = 3/2 \) [37]. We study quench dynamics when the disorder is suddenly switched on. The forward scattering disorder causes dephasing, while backward scattering causes inelastic scattering.

Under conditions that this cosine potential is RG irrelevant, we constructed the 2PI action to second order in the disorder potential. Due to the interaction vertex being of the form \( e^{2i\phi} \), a key feature of the kinetic equation is that, even the second order correction allows for multi-particle scattering between bosons as the exponential keeps track of all powers of the \( \phi \) field.

In solving the kinetic equation, we can make further approximations such as keeping only the leading order Wigner expansion, making the kinetic equation local in time. Some very general results can then be obtained for thermalization times, and how it is affected as one
approaches the superfluid-Bose glass quantum critical point. We discuss some key results below.

In general the relaxation of the initial out of equilibrium population is not a pure exponential. Nevertheless by matching the long time behavior of the kinetic equation with an exponential relaxation rate, we found the thermalization time $t_{th}$ to be

$$t_{th}^{-1} \sim D_b \left[ T_{eff} \right]^{K-1},$$

where $D_b$ is the strength of the backscattering potential, while $T_{eff}$ is a measure of density of excited quasiparticles generated by the quench. In particular, the bosonic density at long wavelengths $\hbar/p$, is $n_p = T_{eff} / (u|p|)$ where $T_{eff} = \mathcal{O} (D_b, D_f)$. The dependence on $D_f, b$ reflects the fact that the sudden switch on of the disorder creates a non-zero density of bosonic excitations.

The above result for $t_{th}$ shows that on decreasing $K$ from a large value $K \gg 1$ towards the critical point located at $K = 3/2$, the relaxation rate increases. This indicates that thermalization "critically speeds up" as a quantum critical point is approached from the superfluid side. This is a very general result that occurs when the system is out of equilibrium, and when the leading non-linearity responsible for thermalization is also the one driving the phase transition. This result for "critical speeding up" is also borne out in experiments [84].

The thermalization dynamics captured by a kinetic equation, as described above, effectively captures local equilibration. Reaching global equilibrium is however more difficult as conserved quantities have to be transported over long distances. Setting up a hydrodynamic approach to study this long time limit is an important open question. The hydrodynamic regime is particularly intriguing in $1d$ as simple RG arguments [85] indicate that non-linearities of the Kardar-Parisi-Zhang (KPZ) type [86] are relevant and will modify the power-law associated with the hydrodynamic long time tails from their naive Gaussian predictions [87]. Studies performed on the $1d$ Bose-Hubbard model indicate that such a regime may be identified numerically [85] where the long time averages of various observable were found to fall off in time as $t^{-1/2}$ (the Gaussian prediction) with subleading corrections proportional to $t^{-3/4}$ (whose origin is consistent with KPZ scaling). Further analytic and numerical studies exploring this physics is an important direction of research.
IV. QUENCHES IN SPATIAL DIMENSION $d > 2$

The methods developed above for 1d systems are rather general, and can be used to study quenches in higher spatial dimensions. In this section we discuss quenches in a bosonic $\phi^4$ theory in general dimension $d$. Time dependent mean-field treatments of various models have indeed shown the existence of dynamical phase transitions where by tuning the parameters of the quench, the system at long times can go from exhibiting disordered to ordered behavior, with the transition between the two characterized by the vanishing of an effective mass [88–92]. Our goal is to uncover universal physics in the quench dynamics when the quench is tuned right at the critical point [91, 93–96]. In 1d there is no finite temperature critical point for a continuously broken symmetry. This limits the possibilities of seeing universal physics in quenches because a quench injects an effective temperature into the system, which cuts-off all power-law correlations. One may see some power-laws persist at intermediate times as discussed in Section III A, but in 1d since the critical points are often of the BKT kind, interactions only give weak logarithmic corrections. We should emphasize that by universal we mean physics that is independent of microscopic details of the system, such as interaction strength. By this definition it excludes power-laws such as those discussed in Section II B, as these power-law decays depend on the microscopic details via the Luttinger parameter.

The quench we study is one where initially the system is in the ground state of a bosonic system with a large mass. Thus the initial state has short range correlations that decay within a few lattice sites. The quench corresponds to reducing the mass from this large value to a very small value such that the final Hamiltonian is tuned close to criticality. The interactions are always present, but if the initial mass is sufficiently large, the effect of interactions on the initial state is negligible. Thus we may model the initial state as the ground state of a free boson Hamiltonian $H_i$ with mass $\Omega_0^2$, while the time evolution is due to an interacting Hamiltonian $H_f$ with mass $r \ll \Omega_0^2$ and interaction strength $u$,

$$H_i = \int_x \left[ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \vec{\phi})^2 + \frac{\Omega_0^2}{2} \vec{\phi}^2 \right],$$

$$H_f = H_i(\Omega_0^2 \rightarrow r) + \frac{u}{4!N} \int_x \left( \vec{\phi}^2 \right)^2,$$

above the fields are vectors $\vec{\phi}(\vec{x}) = (\phi_1, \ldots, \phi_N)$ with $N$ components and the model has an $O(N)$ symmetry. For $N = 2$ this model captures the transition from a Mott insulator.
to a superfluid phase, while for \( N = 3 \) it describes a transition from a paramagnet to an antiferromagnet.

A quench injects energy into the system which translates into an effective temperature. This is seen even at the level of free bosons \((u = 0)\), where the density of bosonic excitations at momentum \( k \) are

\[
 n_k = \frac{T_{\text{eff}}}{\omega_k} \quad \text{with} \quad \omega_k = \sqrt{k^2 + r} \quad \text{and} \quad T_{\text{eff}} = \Omega_0/4.
\]

This temperature causes the system to approach a finite temperature classical (rather than the zero temperature quantum) critical point. Yet the time evolution towards such a state is itself unitary, so that the full quantum dynamics needs to be accounted for.

On studying the time evolution perturbatively in the interactions, logarithmic corrections are found at the classical upper critical dimension \( d = 4 \). An RG allows us to resum these logarithms and obtain scaling forms valid for general \( \epsilon = 4 - d \), provided \( d > 2 \) \((d = 2 \) being the lower critical dimension of the theory) \([93, 95]\). The RG results can be benchmarked against exact solutions obtained from solving the Hartree-Fock equations in the \( N = \infty \) limit \([94]\).

A critical quench is characterized by an effective mass \( r_{\text{eff}}(t) = r + \frac{u}{6} \left( \phi(t) \right)^2 \) which approaches zero at long times. For the model under discussion, and at the upper critical dimension, this is found to vanish as \( r_{\text{eff}}(t) \to a/t^2 \). Due to the vanishing gap, there is no scale in the problem other than the time after the quench. In addition the lack of inelastic scattering implies the system retains memory of the initial quench. Thus the solutions show aging which is characterized by the emergence of universal scaling behavior.

The results for the correlation \((G_K(k, t, t') = -i \langle [\phi_k(t), \phi_{-k}(t')] \rangle)\) and the response function \((G_R(k, t, t') = -i \langle [\phi_k(t), \phi_{-k}(t')] \rangle)\) at two times \( t \) and \( t' \) after the quench, as a function of the wavevector \( k \) are found to be,

\[
 G_K(k, t, t') = \frac{1}{k^{2 - 2\theta_N}} G_K(kt, kt'), 
\]

\[
 G_R(k, t, t') = \frac{1}{k} \left( \frac{t'}{t} \right)^{\theta_N} G_R(kt, kt'),
\]

with \( \theta_N = (N + 2)\epsilon/(4(N + 8)) \), and the scaling functions obeying, \( G_K(x, y) \sim (xy)^{1 - \theta_N} \) for \( x, y \ll 1 \) and \( G_K(x, y) \sim 1 \) for \( x, y \gg 1 \), while \( G_R(x, y) \sim x \) for \( y \ll x \ll 1 \) \([95]\).

The exponent \( \theta_N \) appearing above is referred to as the initial slip exponent, and was first identified in quenches in classical systems coupled to a thermalizing reservoir \([97-99]\). The physical meaning of this exponent is clarified as follows. Imagine modifying the quench by imposing a tiny magnetic field \( \vec{\phi} \cdot \vec{h} \) in \( H_i \). At \( t > 0 \), the quench involves time-evolving the
system according to $H_f$ given above. The difference now is that the initial state has a small magnetization $M_0 = \langle \tilde{\phi} \rangle$ which acquires a time evolution.

Generic quenches into the gapped phase will cause the magnetization to decay exponentially in time (note that our model does not conserve $\langle \tilde{\phi} \rangle$). For a critical quench on the other hand, the susceptibility is enhanced. Thus the initial response to $M_0$ is the growth of magnetization with time $M(t) = M_0 t^{\theta_N}$. Thus $\theta_N$ is the scaling dimension of the initial magnetic field. Analogies can also be made with equilibrium phase transitions where spatial translation invariance is broken by a boundary, with fields on the boundary acquiring a different scaling dimension than the bulk fields [100]. The $\theta_N$ exponent can also be associated with renormalization of fields that now sit on the temporal boundary.

Although the exponent appearing in the growth of the initial magnetization is the same in the quantum and classical problem, there are important differences between the two systems. One is that for the quantum problem, quasiparticles travel ballistically, whereas for the open classical system, quasiparticles travel diffusively. Thus the quantum dynamics has a clear light-cone, and the precise temporal form of the response and correlation functions are not the same.

The difference between the finite-$N$ and $N = \infty$ limit is that at the longest times, the former thermalizes, while the latter is trapped in a nonequilibrium steady state which corresponds to the prethermal state for the former. The larger is $N$, the more stable is the prethermal regime for the finite-$N$ model, and our perturbative RG scheme accurately captures this prethermal regime. Just like for the 1d problem, our RG also generates dissipative terms in the effective action, with the strength of this self-generated dissipation providing an estimate for the duration of the prethermal regime.

Eq. (15a) shows that the prethermal steady state has a bosonic occupation number which scales not as $1/k^2$ as for the critical free theory, but sightly slower in momentum $1/k^{2-2\theta_N}$. This translates to a real space decay which is faster in position for the critical interacting bosons in comparison to critical free bosons.

It is also useful to note that classically the total number of degrees of freedom in the system is $2N_d$, where $N_d$ of them are the “positions” $\phi_i$, and the other $N_d$ correspond to the “momenta” $\pi_j$. However, the quench dynamics has only $N_d$ conserved quantities corresponding to fixing the commutation relations between $\phi_i, \pi_j$. Thus strictly speaking the dynamics of the model is not integrable [91].
A. Entanglement dynamics

An important question concerns the growth of entanglement entropy after a quench. There are very few analytic and exact results on this topic in $d > 1$ as the technical know-how developed for two point correlation functions cannot be easily generalized to the reduced density matrix. In an interacting system, Wick’s theorem does not hold, therefore to determine the density matrix one needs not only the two point correlation function but all higher moments, making the task of calculating the entanglement statistics quite daunting.

Yet, the $N = \infty$ limit of field theories are effectively free as Hartree-Fock becomes exact. Moreover, the results are not totally trivially related to a non-interacting theory as can be seen from the discussion of the two point correlation above for the $O(N)$ model. Note that the non-trivial scaling exponent $\theta_N$ is entirely due to interactions, and would vanish for non-interacting bosons.

Inspired by this observation, we exploited the quasi-free nature of the $\phi^4$ theory in the $N = \infty$ limit to construct the reduced density matrix after the quench [101]. We considered a real space entanglement cut in the form of a hypercube of length $L$, constructed the two point correlations in that region, and from it constructed the reduced density matrix, the last step following from the fact that Wicks’ theorem holds [102] when $N = \infty$.

In studying the critical quench described in the previous section, we found that the entanglement entropy (EE) grows linearly in time at short times, saturating to a volume law at long times, where the latter is due to the finite density of quasiparticles generated by the quench. This time evolution of the EE could be captured very well by a semiclassical picture of ballistically propagating quasiparticles. Within this picture one assume particles are emitted uniformly in all directions, but only the quasiparticles emitted in opposite directions are initially entangled EPR pairs. The EE at any given time was equated to the number of EPR pairs, where one member is inside the hypercube and the other is outside. Two fitting parameters were used, one was the contribution to the EE by an EPR pair, and the second was the EE at lattice time-scales. This semiclassical picture was found to fit the actual time evolution of the EE very well indicating that the EE is not sensitive to the subtle critical physics associated with aging [101].

In contrast the entanglement spectrum (ES) which are the eigenvalues of the reduced density matrix show scaling which is sensitive to the initial slip exponent $\theta_N$. If $\lambda$ are the
eigenvalues of the correlation matrix in a hypercube of length \( L \), then the eigenvalues of the reduced density matrix \( \omega \) are related to \( \lambda \) as \( 2\lambda = \coth(\omega/2) \). We found the following scaling form for all the \( \lambda \) that constitute the "low-energy" part of the entanglement spectrum \[101\]

\[
\lambda(L, t) = L^{-2\theta_N+1}W(t/L),
\]

(16)

where \( W \) is a scaling function. Note that the scaling was obeyed by many eigenvalues corresponding to a substantial fraction of the ES. When studying two point correlations, there are only two independent correlators that depend on \( \theta_N \). Thus for models where critical exponents are not apriori known, the ES could be a good way to extract them as the same scaling form should be obeyed by a macroscopic number of quantities.

V. OUTLOOK

In the past it was thought that the main feature of nonequilibrium quantum systems was a high effective temperature, and consequently such systems will show no quantum correlated states, with any correlations being fundamentally classical. The recent hectic activity in the field shows this to be not always the case. There are many body localized phases for example where quench dynamics can lead to steady states which are still coherent due to the suppression of inelastic scattering. Even for clean systems, a long lived prethermal window can appear, which is characterized by a low effective temperature, with nonequilibrium power-laws and scaling.

In the light of recent experiments in pump probe spectroscopy \[103\], an important open question is whether quenches can stabilize correlated states, albeit in an intermediate time regime, different from those in thermal equilibrium. For example, due to the existence of several competing superconducting orders in lattices, by simply tuning the quench amplitude one may access superconducting orders of different symmetries \[104\]. Another important question of immediate interest is to study the long time limit after the quench where hydrodynamic modes are important. One expects that the long thermalization times associated with conserved modes can lead to interesting time-dependence of observables.
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