CONCENTRATION SOLUTIONS TO SINGULARLY PRESCRIBED GAUSSIAN AND GEODESIC CURVATURES PROBLEM

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Abstract. We consider the following Liouville-type equation with exponential Neumann boundary condition:

\[ \begin{align*}
-\Delta \tilde{u} &= \varepsilon^2 K(x)e^{2\tilde{u}}, \quad x \in D, \\
\frac{\partial \tilde{u}}{\partial n} + 1 &= \varepsilon \kappa(x)e^{\tilde{u}}, \quad x \in \partial D,
\end{align*} \]

where \( D \subset \mathbb{R}^2 \) is the unit disc, \( \varepsilon^2 K(x) \) and \( \varepsilon \kappa(x) \) stand for the prescribed Gaussian curvature and the prescribed geodesic curvature of the boundary, respectively. We prove the existence of concentration solutions if \( \kappa(x) + \sqrt{K(x)} + \kappa(x)^2 \) has a strictly local extremum point, which is a total new result for exponential Neumann boundary problem.

1. Introduction

One of classical problems in conformal geometry is to prescribe the Gaussian curvature on a closed Riemannian surface \((M, \tilde{g})\). Denote by \( g \) the conformal one, and by \( e^{2v} \) the conformal factor. Then \( v \) should comply with the elliptic equation that

\[ -\Delta_{\tilde{g}} v + \tilde{K}(x) = K(x)e^{2v}. \]

Here \( \tilde{K} \) and \( K \) stands for the Gaussian curvature with respect to \( \tilde{g} \) and \( g \) respectively. After the pioneer works \([17, 18]\) by Kazdan and Warner, large amounts of literature are devoted to the solvability of this type of problems, see Chapter 6 in the book \([2]\), where one can also find a comprehensive list of references.

In the case the boundary \( \partial M \neq \emptyset \), other than the Gaussian curvature, it is natural to prescribe also the geodesic curvature on \( \partial M \). Let \( \tilde{\kappa} \), \( \kappa \) be the geodesic curvatures of the boundary with respect to \( \tilde{g} \) and \( g \) respectively. We are then led to the problem

\[ \begin{align*}
-\Delta_{\tilde{g}} v + \tilde{K}(x) &= K(x)e^{2v}, \quad x \in M, \\
\frac{\partial v}{\partial n} + \tilde{\kappa}(x) &= \kappa(x)e^{v}, \quad x \in \partial M.
\end{align*} \]

Brendle \([4]\) found a solution of \((1)\) for constant \( K \) and \( \kappa \) by using the method of flows. When \( K \) and \( \kappa \) are not constants, Cherrier \([8]\) proved the existence of a solution for not big curvatures. Recently, López-Soriano, Malchiodi and Ruiz \([20]\) considered the negative Gaussian curvature case, i.e. \( K < 0 \), and they derived some existence results using a variational approach.

The higher-dimensional analogue of this question is to prescribe scalar curvature of a manifold and mean curvature of the boundary. Moreover, the case of zero scalar curvature has been widely studied, see for instance \([1, 6, 11, 12, 14, 22]\) and the references therein.

When \( M \) is the standard Euclidean disk \( D \subset \mathbb{R}^2 \), this problem may be recognised as a kind of Nirenberg problem to surfaces with boundary. The case \( \kappa = 0 \) has been treated by Chang-Yang \([7]\), which proved the existence of a solution to \((2)\) under the assumption that \( K \) is positive somewhere. Moreover, the case \( K = 0 \) was first treated in Chang-Liu \([5]\) under a suitable conditions on critical points of the geodesic curvature. Also in \([21]\) a solution is found under some symmetric condition. Additionally, a blow up analysis has been given in \([13]\).

For the nonconstant curvatures case, the first work \([1]\) address the existence by setting the problem in a variational framework also under some symmetric assumptions. Recently, Jevnikar and etc \([16]\) gave a complete blow up analysis of the problem \((3)\), where \( K \) and \( \kappa \) are not constant. Precisely, they considered the problem

\[ \begin{align*}
-\Delta u_n &= 2K_n e^{u_n}, \quad \text{in } D, \\
\frac{\partial u_n}{\partial n} + 2 &= 2\kappa_n e^{\frac{u_n}{2}}, \quad \text{on } \partial D,
\end{align*} \]

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where $K_n \to K$ in $C^2(\overline{D})$, $\kappa_n \to \kappa$ in $C^2(\partial D)$ and $K, \kappa$ are not all zeros. If $u_n$ blows up with bounded mass, i.e.
\[
\int_D e^{u_n} + \int_{\partial D} e^{-u_n} \le C < +\infty,
\]
then the blow-up must occur at a critical point of $\varphi(\xi) = H(\xi) + \sqrt{H(\xi)^2 + K(\xi)}$, where $H$ is the harmonic extension of $\kappa$. When finishing our writing the other day, we find that Battaglia-Medina-Pistoia \[3\] considered the inverse problem to \[16\]. They constructed a family of solutions with convergent

\[\text{Theorem 1.1.} \]

Our main result are stated as follows.

Note that the universal constant 1 is the original geodesic curvature of the boundary $\partial D$. Under some non-degeneracy conditions. Note that in their settings, $K$ is not identical to 0.

In this paper, we consider the following perturbation problem
\[
\begin{cases}
-\Delta \tilde{u} = \varepsilon^2 K(x)e^{2\tilde{u}}, & x \in D, \\
\frac{\partial \tilde{u}}{\partial n} + 1 = \varepsilon \kappa(x)e^{\tilde{u}}, & x \in \partial D, 
\end{cases}
\]  
(2)

where $D \subset \mathbb{R}^2$ is the unit disc (centered at $(0,1)$) for convenience, $K(x)$ and $\kappa(x)$ are all positive smooth functions on $\overline{D}$ and $\partial D$ respectively. Here $\varepsilon > 0$ is a small parameter. Note that $\varepsilon^2 K(x)$ and $\varepsilon \kappa(x)$ stand for the prescribed Gaussian curvature and the prescribed geodesic curvature of the boundary.

We mention that the problem \[3\] is the perturbation case of zero curvatures, which do not be included in \[3\] and \[16\]. Note that the universal constant 1 is the original geodesic curvature of the boundary $\partial D$. Our main result are stated as follows.

\[\text{Theorem 1.1.} \]

Assume that the function $\kappa(x) + \sqrt{K(x) + \kappa(x)^2}$ defined on $\partial D$ admits a local extremum point. Then there is an $\varepsilon_0 > 0$ such that for any small $0 < \varepsilon < \varepsilon_0$, the problem \[3\] has a family of single boundary bubbling solutions $u_{\varepsilon}$. Moreover,
\[
\int_D \varepsilon^2 K(x)e^{2\tilde{u}} + \int_{\partial D} \varepsilon \kappa(x)e^{\tilde{u}} = 2\pi.
\]

\[\text{Remark 1.2.} \]

As long as $\kappa(x) + \sqrt{K(x) + \kappa(x)^2}$ is not a constant on $\partial D$, the problem \[3\] admits single boundary bubbling solutions.

The proof of Theorem 1.1 is based on the Lypunov-Schmidt reduction as performed in \[10\]. We first make use of the classification result on $\mathbb{R}^2_+$ \[19\] to build up the approximation solution. For the non-degeneracy of standard bubble, the disk $D$ is transformed to a sphere cap by stereographic projection, and then the non-degeneracy follows from the result in \[10\]. For the solvability of corresponding linearized problem, there seems a gap in getting (53) from (75) in \[10\]. In this paper we give a new proof instead.

This paper is organized as follows. In Section 2, an ansatz of the solution is given. Section 3 is devoted to the invertibility of linearized operator. In Section 4, the nonlinear problem is solved. Variational reduction is then showed in Section 5. In Section 6, the theorem 1.1 is proved. Some computation is listed in the appendix section 7.

2. ANSatz

In this section, we will introduce the approximation solution and give the ansatz of the solution. Let $u(y) = \tilde{u}(\varepsilon y) + 2 \ln \varepsilon$ and $D_\varepsilon = D/\varepsilon$. Then the equation \[3\] is equivalent to
\[
\begin{cases}
-\Delta u = K(\varepsilon y)e^{2u}, & y \in D_\varepsilon, \\
\frac{\partial u}{\partial n} + \varepsilon = \kappa(\varepsilon y)e^u, & y \in \partial D_\varepsilon.
\end{cases}
\]  
(3)

For $\varepsilon \to 0$, it is easy to understand that the limit problem near a boundary point $\xi = \varepsilon \xi'$ is formally like
\[
\begin{cases}
-\Delta u = K(\xi)e^{2u}, & y \in \mathbb{R}^2_+, \\
\frac{\partial u}{\partial n} = \kappa(\xi)e^u, & y \in \partial \mathbb{R}^2_+.
\end{cases}
\]

To introduce the approximation solution, we recall that the unique solution \((\tilde{u}_0)\) to the following half plane problem
\[
\begin{cases}
-\Delta u = ae^{2u}, & x \in \mathbb{R}^2_+, \\
\frac{\partial u}{\partial n} = be^u, & x \in \partial \mathbb{R}^2_+, \quad (a, b > 0 \text{ are two constants})
\end{cases}
\]
is
\[
\tilde{u}_0(x) = \ln \frac{2\lambda}{\sqrt{a} \left( \lambda^2 + (x_1 - s)^2 + (x_2 + \frac{a}{\sqrt{a}}) \lambda^2 \right)}, \quad (s \text{ and } \lambda \text{ are two arbitrary parameters}).
\]
From the above classification result, the first approximation we select is that
\[ U_0(x) = \ln \frac{2\lambda}{\sqrt{K(\xi)}} \left( \lambda^2 \varepsilon^2 + \left| x - \xi + \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right|^2 \right). \]

Here \( \xi \in \partial D \) is undetermined and \( n(\xi) \) is the unit out normal at the point \( \xi \), while \( \lambda \) may be any universal constant. Obviously \( U_0 \) satisfies the problem
\[
\begin{align*}
-\Delta U_0 &= \varepsilon^2 K(\xi) e^{2U_0}, & x &\in \mathbb{R}_+^2, \\
\frac{\partial U_0}{\partial n} &= \varepsilon \kappa(\xi) e^{U_0}, & x &\in \partial \mathbb{R}_+^2.
\end{align*}
\]

It is found that the function \( U_0 \) approximate the equation \( \frac{\partial^2}{\partial x^2} \) near \( \xi \), but far from the boundary condition. As usual, we will modify \( U_0 \) furthermore. On the boundary, one may find that
\[
\begin{align*}
\varepsilon \kappa(\xi) e^{U_0} \frac{\partial U_0}{\partial n} &= \frac{2\kappa(\xi) \lambda \varepsilon}{\sqrt{K(\xi)}} \left( \lambda^2 \varepsilon^2 + \left| x - \xi + \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right|^2 \right) + 2 \left( x - \xi - \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right) \cdot n(x) \\
&= \frac{2(\xi - \xi) \cdot n(x)}{|x - \xi|^2} + \frac{2(\xi - \xi) \cdot n(x)}{\lambda^2 \varepsilon^2 + \left| x - \xi - \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right|^2} + 2 \left( x - \xi - \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right) \cdot n(x) \\
\lambda^2 \varepsilon^2 + \left| x - \xi - \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right|^2 &+ \frac{2\kappa(\xi) \lambda \varepsilon (1 - n(\xi) \cdot n(x))}{\sqrt{K(\xi)}} \left( \lambda^2 \varepsilon^2 + \left| x - \xi - \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \varepsilon n(\xi) \right|^2 \right) \\
&= \frac{2(\xi - \xi) \cdot n(x)}{|x - \xi|^2} + I_1(x) + I_2(x).
\end{align*}
\]

Set \( \mathcal{D}(\xi) = \frac{\kappa(\xi)}{\sqrt{K(\xi)}} \) in what follows for convenience. It is checked that
\[
|I_1(x)| = \left| \frac{2\varepsilon (\xi - \xi) \cdot n(x)}{|x - \xi|^2} \frac{2\mathcal{D}(\xi)(x - \xi) \cdot n(\xi) - (1 + \mathcal{D}(\xi)^2)\lambda \varepsilon}{\left( \lambda^2 \varepsilon^2 + |x - \xi - \mathcal{D}(\xi) \varepsilon n(\xi)|^2 \right)^\frac{1}{2}} \right| \leq C \lambda \varepsilon + C \lambda^2 \varepsilon^2 \frac{1}{\lambda^2 \varepsilon^2 + |x - \xi - \mathcal{D}(\xi) \varepsilon n(\xi)|^2},
\]

since \( \frac{2(\xi - \xi) \cdot n(x)}{|x - \xi|^2} \equiv 1 \) for \( x \in \partial D \). So it is easily to get, by assuming \( \xi = (0,0) \) from the symmetry and using \( x - \xi = \lambda \varepsilon z \) , that
\[
\left| \int_{\partial D} I_1 \right| \leq C \lambda \varepsilon \int_{\partial D \varepsilon^2} \frac{dz}{1 + |z - \mathcal{D}(0) n(0)|^2} + C \lambda \varepsilon \leq C \lambda \varepsilon.
\]

Also it holds that
\[
I_2(x) = O(\lambda \varepsilon), \quad \left| \int_{\partial D} I_2 \right| \leq C \lambda \varepsilon.
\]

Next, we define a constant \( d = O(1) \) by
\[
d \int_{\partial D} \frac{\lambda^2 \varepsilon^2 dx}{\lambda^2 \varepsilon^2 + |x - \xi|^2} = \int_{\partial D} (I_1 + I_2).
\]
Thus it is easy to see, owing to \( \int_{\partial D} \left( \frac{2x - \xi}{|x - \xi|^2} - 1 \right) \, dx = 0 \), that
\[
\int_{\partial D} \left( \varepsilon \kappa(\xi) e^{U_0} - \frac{\partial U_0}{\partial n} - 1 - \frac{\lambda^2 \varepsilon^2 \varepsilon}{\lambda^2 \varepsilon^2 + |x - \xi|^2} \right) \, dx = 0.
\]
So one may define a function \( H_0(x) \) satisfying
\[
\begin{align*}
-\Delta H_0 &= 0, \\
\frac{\partial H_0}{\partial n} &= \varepsilon \kappa(\xi) e^{U_0} - \frac{\partial U_0}{\partial n} - 1 - \frac{\lambda^2 \varepsilon^2 \varepsilon}{\lambda^2 \varepsilon^2 + |x - \xi|^2}, \quad x \in \partial D.
\end{align*}
\]
Of course \( H_0 \) is unique up to a constant. Obviously \( (\mathbb{B}) \) implies that
\[
\frac{\partial H_0}{\partial n} = I_1 + I_2 - \frac{\lambda^2 \varepsilon^2 \varepsilon}{\lambda^2 \varepsilon^2 + |x - \xi|^2}, \quad x \in \partial D.
\]
And it holds that, for any \( p > 1 \),
\[
\begin{align*}
\|I_1\|_{L^p(\partial D)} &\leq C \lambda \varepsilon + C \lambda^2 \varepsilon^2 \left( \int_{\partial D} \frac{dx}{\lambda^2 \varepsilon^2 + |x - \xi|^2} \right)^{\frac{1}{p}} = O(\varepsilon^{\frac{1}{p}}), \\
\|I_2\|_{L^p(\partial D)} &= O(\varepsilon^{\frac{1}{p}}), \quad \left\| \frac{\lambda^2 \varepsilon^2 \varepsilon}{\lambda^2 \varepsilon^2 + |x - \xi|^2} \right\|_{L^p(\partial D)} = O(\varepsilon^{\frac{1}{p}}).
\end{align*}
\]
Thus by \( L^p \) theory we have, for \( 0 < s < \frac{1}{p} \), that
\[
\|\nabla H_0\|_{W^{s,p}(B)} = O(\varepsilon^{\frac{1}{p}}).
\]
In what follows, we always choose \( H_0 \) such that \( \int_D H_0 = 0 \), which means that
\[
H_0(x) = O(\varepsilon^s), \quad (\alpha = 1/p \in (0, 1) \text{ is arbitrary})
\]
We then choose the second approximation \( U(x) = U_0(x) + H_0(x) \). It is easy to see that
\[
\begin{align*}
-\Delta U &= \varepsilon^2 K(\varepsilon) e^{2U_0}, \\
\frac{\partial U}{\partial n} + 1 + \frac{d \lambda^2 \varepsilon^2}{\lambda^2 \varepsilon^2 + |x - \xi|^2} &= \varepsilon \kappa(\xi) e^{U_0}, \quad x \in \partial D.
\end{align*}
\]
We will seek a solution to \( (\mathbb{B}) \) of the form \( V(x) = \phi(y) \) where \( V(y) = U(\varepsilon y) + 2 \ln \varepsilon \). Thus the problem can be stated as to find a solution \( \phi \) of
\[
\begin{align*}
-\Delta \phi - 2K(\varepsilon y) e^{2V} \phi &= R_1(\varepsilon y) + K(\varepsilon y) e^{2V} (\varepsilon^2 \phi - 1 - 2 \phi), \quad \text{in} \ D_{\varepsilon}, \\
\frac{\partial \phi}{\partial n} - \kappa(\varepsilon y) e^{V} \phi &= R_2(\varepsilon y) + \kappa(\varepsilon y) e^{V} (\varepsilon^2 \phi - 1 - \phi), \quad \text{on} \ \partial D_{\varepsilon},
\end{align*}
\]
where the error terms
\[
R_1(\varepsilon y) = \Delta V + K(\varepsilon y) e^{2V}, \quad R_2(\varepsilon y) = -\frac{\partial V}{\partial n} - \varepsilon + \kappa(\varepsilon y) e^{V}.
\]

**Lemma 2.1.** Let \( \delta > 0 \) be small and fixed. In \( B_{\frac{\delta}{\varepsilon}}(\xi') \cap D_{\varepsilon} \), we have
\[
R_1(\varepsilon y) = O(\varepsilon^4), \quad R_2(\varepsilon y) = O(\varepsilon^2).
\]

**Proof.** Direct computation shows that, for \( |y - \xi'| \leq \frac{\delta}{\varepsilon} \),
\[
R_1(\varepsilon y) = \Delta V + K(\varepsilon y) e^{2V} = K(\varepsilon y) e^{2(U_0(\varepsilon y) + H_0(\varepsilon y) + 2 \ln \varepsilon)} - K(\xi) e^{2U_0(\varepsilon y) + 2 \ln \varepsilon}
\]
\[
\begin{align*}
&= \frac{4 \lambda^2}{K(\xi)(\lambda^2 + |y - \xi' - \mathcal{D}(\xi) \lambda n(\xi)|^2)} (K(\varepsilon y) e^{2H_0(\varepsilon y)} - K(\xi)) \\
&= \frac{4 \lambda^2}{(\lambda^2 + |y - \xi' - \mathcal{D}(\xi) \lambda n(\xi)|^2)} \left( \frac{K(\varepsilon y)}{K(\xi)} e^{O(\varepsilon^s)} - 1 \right).
\end{align*}
\]
important role thereafter. For any positive numbers $z$ where $\mathcal{D}(\xi)2\lambda = \frac{\mathcal{D}(\xi)2\lambda}{(\lambda^2 + |y - \xi'|^2)} [O(\varepsilon|y - \xi'|) + O(\varepsilon^\alpha)],$

and

\[ R_2(y) = -\frac{\partial V}{\partial n} - \varepsilon + \kappa(\varepsilon y) e^V = \kappa(\varepsilon y) e^V - \kappa(\xi) e^{V_0(\varepsilon y)} + 2 \ln \varepsilon + \varepsilon \frac{d\lambda^2}{\lambda^2 + |y - \xi'|^2} \]

Using Proposition 3.2 in [15], we get the desired result. \hfill \square

In the last of this section, we build the non-degeneracy of the standard bubble $\tilde{U}_0$, which plays an important role thereinafter. For any positive numbers $a$, $b$, $\lambda$, let

\[ z_0(x) = \frac{1}{2\lambda} - \frac{\lambda + \frac{b}{\sqrt{a}}(x_2 + \frac{b}{\sqrt{a}} \lambda)}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2}, \quad z_1(x) = \frac{x_1}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2}, \quad x = (x_1, x_2) \in \mathbb{R}^2_+. \] (6)

Then the following non-degeneracy holds.

**Lemma 2.2.** Any bounded solutions of

\[
\begin{cases}
\Delta \phi + \frac{8\lambda}{(\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2)} \phi = 0, & \text{in } \mathbb{R}^2_+,
\end{cases}
\]

\[
\begin{cases}
\frac{\partial \phi}{\partial n} = \frac{2\lambda}{\sqrt{a}} \left( \frac{x_1}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2} \right)^2 \phi = 0, & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
\]

is a linear combination of $z_0$ and $z_1$.

**Proof.** Let $\Pi$ be the stereographic projection from the unit sphere in $\mathbb{R}^3$ centered at $(0, -\frac{b}{\sqrt{a}} \lambda, 0)$ onto $\mathbb{R}^2$. More specifically, let $\eta = (\eta_1, \eta_2, \eta_3)$ be the coordinates of $\mathbb{R}^3$ taking $(0, -\frac{b}{\sqrt{a}} \lambda, 0)$ as its origin and $x = (x_1, x_2)$ be the coordinates of $\mathbb{R}^2$, we have

\[ \eta_1 = \frac{2x_1}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2}, \quad \eta_2 = \frac{2(x_2 + \frac{b}{\sqrt{a}} \lambda)}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2}, \quad \eta_3 = \frac{x_3^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2 - \lambda^2}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2}. \]

Let $\Sigma = \Pi^{-1}(\mathbb{R}^2_+)$. It is a spherical cap on $\mathbb{S}^2$.

Assume $\phi$ is a bounded solution of problem (6), we define a function $\Phi(\eta)$ on $\Sigma$ by

\[ \phi(x) = \Phi(\eta) \frac{2\lambda}{\lambda^2 + x_1^2 + (x_2 + \frac{b}{\sqrt{a}} \lambda)^2}, \]

then

\[
\begin{cases}
\frac{\Delta \Phi + 2\Phi}{\partial n} = 0, \\
\frac{\partial \Phi}{\partial \nu} = \frac{b}{\sqrt{a}} \Phi = 0.
\end{cases}
\]

Using Proposition 3.2 in [15], we get the desired result. \hfill \square

### 3. The Linearized Operator

In the section, the invertibility of the linearized operator is studied. The main result is the solvability of the following linear problem. For given $f$ and $h$, find $(\phi, c_1)$ such that

\[
\begin{cases}
-\Delta \phi - 2K(\varepsilon y) e^{2V} \phi = f + c_1 \chi_\varepsilon Z_{1, \varepsilon}, & \text{in } D_\varepsilon, \\
\frac{\partial \phi}{\partial n} - \kappa(\varepsilon y) e^{V} \phi = h, & \text{on } \partial D_\varepsilon \\
\int_{D_\varepsilon} \chi_\varepsilon Z_{1, \varepsilon} \phi = 0.
\end{cases}
\]

where $f \in L^\infty(D_\varepsilon)$, $h \in L^\infty(\partial D_\varepsilon)$ and $Z_{0, \varepsilon}$, $Z_{1, \varepsilon}$, $\chi_\varepsilon$ are defined as follows. From now on we denote $z_0$, $z_1$ with $a = K(\xi)$, $b = \kappa(\xi)$ in [15]. Around the point $\xi' = \xi/\varepsilon \in \partial D_\varepsilon$, we consider a smooth change of variables

\[ F_\varepsilon(y) = \frac{1}{\varepsilon} F(\varepsilon y), \]
where $F': B_p(\xi) \to M$ is a diffeomorphism and $M$ is an open neighborhood of the origin such that $F(B \cap B_p(\xi)) = \mathbb{R}_+^n \cap M$, $F(\partial B \cap B_p(\xi)) = \partial \mathbb{R}_+^n \cap M$. Also we can choose $F$ preserving area. Define $Z_0, z_0(y) = z_0(F(y))$, $Z_1, z_1(y) = z_1(F(y))$.

Next, we select a large but fixed number $R_0$ and non-negative smooth function $\chi : \mathbb{R} \to \mathbb{R}$ such that $\chi(r) = 1$ for $r \leq R_0$ and $\chi(r) = 0$ for $r \geq R_0 + 1, 0 \leq \chi \leq 1$. Then set $\chi_\varepsilon(y) = \chi(F(y))$.

For simplicity, let us denote $W_1(y) = 2K(\varepsilon y)e^{2V}$ and $W_2(y) = \kappa(\varepsilon y)e^V$, then

$$W_1(y) = \frac{2K(\varepsilon y)}{K(\xi)} \left(1 + O(\varepsilon|y - \xi'|) + O(\varepsilon^\alpha)\right),$$

$$W_2(y) = \frac{\kappa(\varepsilon y)}{\sqrt{K(\xi)}} \left(1 + O(\varepsilon|y - \xi'|) + O(\varepsilon^\alpha)\right),$$

for any $\alpha \in (0, 1)$.

For $f \in L^\infty(D_\varepsilon)$ and $h \in L^\infty(\partial D_\varepsilon)$ we define two norms

$$\|f\|_{*, D_\varepsilon} = \sup_{y \in D_\varepsilon} |f(y)| \cdot (1 + |y - \xi'|^2 \ln^3(1 + |y - \xi'|)),$$

$$\|h\|_{*, \partial D_\varepsilon} = \sup_{y \in \partial D_\varepsilon} |h(y)| \cdot (1 + |y - \xi'| \ln^3(1 + |y - \xi'|)).$$

**Lemma 3.1.** Choose $R_1$ large enough such that for any $\varepsilon > 0$ small enough, there exists a smooth and positive function

$$\psi : D_\varepsilon \setminus B_{R_1}(\xi') \to \mathbb{R}$$

so that

$$-\Delta \psi - W_1 \psi \geq \frac{1}{1 + |y - \xi'|^2 \ln^3(1 + |y - \xi'|)} \quad \text{in} \quad D_\varepsilon \setminus B_{R_1}(\xi'),$$

$$\frac{\partial \psi}{\partial n} - W_2 \psi \geq \frac{1}{1 + |y - \xi'| \ln^3(1 + |y - \xi'|)} \quad \text{on} \quad \partial D_\varepsilon \setminus B_{R_1}(\xi'),$$

$$\psi > 0, \quad \text{in} \quad D_\varepsilon \setminus B_{R_1}(\xi'),$$

$$\psi \geq 1, \quad \text{on} \quad D_\varepsilon \cap \partial B_{R_1}(\xi').$$

The positive constants $C$ is independent of $\varepsilon, R_1$ and $\psi$ is bounded uniformly

$$0 < \psi \leq C \quad \text{in} \quad D_\varepsilon \setminus B_{R_1}(\xi').$$

**Proof.** Without loss of generality we may assume that $\xi$ is the original point. We take

$$\psi_1 = \frac{-y_2}{r \ln^3 r}, \quad y = (y_1, y_2), \quad r = |y - \mathcal{D}(\xi)\lambda n(\xi)|,$$

then

$$\nabla \psi_1 = \frac{(\ln^3 r + 3 \ln^2 r)y_2}{r^2 \ln^6 r} \left(\frac{y_1}{r}, \frac{y_2 + \lambda \mathcal{D}(\xi)}{r}\right) - \left(0, \frac{1}{r \ln^3 r}\right).$$

Direct computation gives that

$$-\Delta \psi_1 - W_1 \psi_1 = O \left(\frac{1}{r^2 \ln^3 r}\right),$$

$$\frac{\partial \psi_1}{\partial n} = \nabla \psi_1 \cdot (\varepsilon y_1, \varepsilon y_2 - 1) = \frac{\varepsilon y_2 y_1^2 + (\varepsilon y_2 - 1)y_2^2}{r^3 \ln^4 r} + \frac{1 - \varepsilon y_2}{r \ln^3 r} + O \left(\frac{1}{r \ln^3 r}\right),$$

for $R_1$ large and on the boundary

$$W_2 \psi_1 = O \left(\frac{1}{r^2 \ln^3 r}\right).$$

Let

$$\psi_2 = 1 - \frac{1}{\ln r},$$

then

$$\nabla \psi_2 = \frac{1}{r \ln^3 r} \left(\frac{y_1}{r}, \frac{y_2 + \lambda \mathcal{D}(\xi)}{r}\right)$$

from which we can obtain

$$-\Delta \psi_2 - W_1 \psi_2 = \frac{2}{r^2 \ln^3 r} + O \left(\frac{1}{r}\right).$$
and
\[ \frac{\partial \psi_2}{\partial n} - W_2 \psi_2 = \frac{y_2}{r^2 \ln^2 r} + O\left(\frac{1}{r^2}\right). \]
Now we can define \( \psi_0 = \psi_1 + C \psi_2 \) then
\[ -\Delta \psi_0 - W_1 \psi = \frac{2C}{r^2 \ln^3 r} + O\left(\frac{1}{r^2 \ln^3 r}\right) \geq \frac{1}{r^2 \ln^3 r}, \]
where we choose \( C \) large but independent of \( \varepsilon, R_1 \). Note that now \( C \) is fixed.
\[ \frac{\partial \psi_0}{\partial n} - W_2 \psi_0 = \frac{\varepsilon y_2^2 + (\varepsilon y_2 - 1)y_2^2}{r^2 \ln^3 r} + \frac{1 - \varepsilon y_2}{r \ln^3 r} + \frac{y_2^2}{r^2 \ln^3 r} + O\left(\frac{1}{r \ln^3 r}\right). \]
If \( \varepsilon y_2 \geq \frac{1}{8} \), using \( (y_1, y_2) \in \partial B_{\frac{1}{2}} \) to get \( \frac{y_2}{r} \geq \frac{1}{8r} \) for \( \varepsilon \) small. From \( 0 \leq y_2 \leq \frac{2}{r}, |y_1| \leq r, \)
\[ \left| \frac{\varepsilon y_2^2 + (\varepsilon y_2 - 1)y_2^2}{r^2 \ln^3 r} + \frac{1 - \varepsilon y_2}{r \ln^3 r} \right| \leq \frac{4}{r \ln^3 r}. \]
Hence
\[ \frac{\partial \psi_0}{\partial n} - W_2 \psi_0 \geq \frac{C}{20r \ln^2 r} - \frac{4}{r \ln^3 r} + O\left(\frac{1}{r \ln^3 r}\right) \geq \frac{1}{r \ln^3 r}. \]
If \( 0 \leq y_2 \leq \frac{1}{8} \), then \( \frac{y_2^2}{r^2 y_1 y_2} \leq \frac{y_2^2}{r} \leq \frac{1}{16} \) for \( (y_1, y_2) \in \partial B_{\frac{1}{2}} \). Obviously \( \frac{\varepsilon y_2 y_2^2}{r^3 \ln^3 r} + \frac{C y_2^2}{r^2 \ln^2 r} \geq 0, \)
\[ \frac{\partial \psi_0}{\partial n} - W_2 \psi_0 \geq \frac{1 - \varepsilon y_2}{r \ln^3 r} \left( 1 - \frac{y_2^2}{r^2} \right) + O\left(\frac{1}{r \ln^3 r}\right) \geq \frac{7}{16} + \frac{15}{16r \ln^3 r} + O\left(\frac{1}{r \ln^3 r}\right) \geq \frac{1}{2r \ln^3 r}. \]
where \( R_1 \) is large. Finally, letting \( \psi = 2 \psi_0 \), we get the lemma. \( \square \)

**Lemma 3.2.** There is an \( \varepsilon_0 > 0 \), for any \( \varepsilon \in (0, \varepsilon_0) \) and any solution \( \phi \) of
\[
\begin{aligned}
-\Delta \phi - W_1 \phi &= f, \quad \text{in } D_{\varepsilon}, \\
\frac{\partial \phi}{\partial n} - W_2 \phi &= h, \quad \text{on } \partial D_{\varepsilon}, \\
\int_{D_{\varepsilon}} \chi_{\varepsilon} Z_{1,\varepsilon} \phi &= 0, \quad i = 0, 1.
\end{aligned}
\]
(8)
It holds that
\[ \|\phi\|_{L^\infty(D_{\varepsilon})} \leq C(\|f\|_{*, D_{\varepsilon}} + \|h\|_{*, \partial D_{\varepsilon}}). \]
where \( C \) is independent of \( \varepsilon \).

Proof. Choose \( R_0 = 2R_1, R_1 \) being the constant of Lemma 3.1. Then according to the properties of the barrier \( \psi \) and Lemma 2.2, we can finish the proof very similar to that of Lemma 4.2 in [11]. Here we omit the details. \( \square \)

Next we will establish an a priori estimate for solutions to the above equation under the orthogonality condition with respect to \( Z_{1,\varepsilon} \) only.

**Lemma 3.3.** For \( \varepsilon \) sufficiently small, if \( \phi \) solves
\[
\begin{aligned}
-\Delta \phi - W_1 \phi &= f, \quad \text{in } D_{\varepsilon}, \\
\frac{\partial \phi}{\partial n} - W_2 \phi &= h, \quad \text{on } \partial D_{\varepsilon}, \\
\int_{D_{\varepsilon}} \chi_{\varepsilon} Z_{1,\varepsilon} \phi &= 0,
\end{aligned}
\]
(9)
then
\[ \|\phi\|_{L^\infty(D_{\varepsilon})} \leq C(\|f\|_{*, D_{\varepsilon}} + \|h\|_{*, \partial D_{\varepsilon}}). \]
(10)
where \( C \) is independent of \( \varepsilon \).

Proof. Let \( \phi \) be the solution of (8). Let \( R > R_0 + 1 \) be large and fixed. Set
\[ h_1(x) = \frac{\ln \frac{2}{\varepsilon} - \ln |x|}{\ln \frac{2}{\varepsilon} - \ln R} \quad \text{and} \quad h_2(x) = \frac{\ln \frac{2}{\varepsilon} - \left(\frac{2}{\varepsilon}\right)^\sigma}{\ln \frac{2}{\varepsilon} - \left(\frac{2}{\varepsilon}\right)^\sigma} \quad \text{for} \quad R \leq |x| \leq \frac{1}{\varepsilon}, \]
and
\[ \hat{h}_1(y) = h_1(F_{\varepsilon}(y)), \quad \hat{h}_2(y) = h_2(F_{\varepsilon}(y)), \]
(10)
where $\sigma \in (0, 1)$ is small and to be determined later. Obviously

$$\Delta h_1 \equiv 0, \quad \Delta h_2(x) = \frac{\sigma^2}{|x|^\sigma + (\frac{|x|}{\gamma})^\sigma}.$$ 

Let $\eta_0, \eta_1, \eta_2$ be radial non-negative smooth cut-off functions on $\mathbb{R}^2$ so that

$$\eta_1(|x|) \equiv 1 \text{ in } B_R(0), \quad \eta_1(|x|) \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R+1}(0), \quad \eta_0(|x|) = \eta_1(|x| + 2)$$

and

$$\eta_2 \equiv 1 \text{ in } B_{\frac{R}{2}}(0), \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{\frac{R}{2}}(0), \quad |\nabla \eta_2| \leq \frac{C \varepsilon}{\delta}, \quad |\nabla^2 \eta_2| \leq \frac{C \varepsilon^2}{\delta^2} \text{ in } B_{\frac{R}{4}}(0) \setminus B_{\frac{R}{4}}(0),$$

where $C$ is independent of $\sigma, \delta, \varepsilon$.

Again we write

$$\dot{\eta}_0(y) = \eta_0(F_\varepsilon(y)), \quad \dot{\eta}_1(y) = \eta_1(F_\varepsilon(y)), \quad \dot{\eta}_2(y) = \eta_2(F_\varepsilon(y)).$$

and define

$$\tilde{Z}_{0,\varepsilon} = \tilde{\eta}_0 Z_{0,\varepsilon} + (1 - \tilde{\eta}_0)\tilde{\eta}_1 \hat{h}_1 Z_{0,\varepsilon} + (1 - \tilde{\eta}_0)(1 - \tilde{\eta}_1)\tilde{\eta}_2 \hat{h}_2 Z_{0,\varepsilon}.$$ 

Given $\phi$, the solution to (1), let

$$\tilde{\phi} = \phi + d_0 \tilde{Z}_{0,\varepsilon}, \quad \text{where } d_0 = -\frac{\int_{D_{\varepsilon}} Z_{0,\varepsilon} \chi_{\varepsilon} \phi}{\int_{D_{\varepsilon}} Z_{0,\varepsilon} \chi_{\varepsilon}}.$$ 

Then estimate (11) is a direct consequence of the following claim.

**Claim.** Choose $R$ large and fixed, $\sigma$ small enough and then fixed, there exists $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$,

$$|d_0| \leq C(\|f\|_{*,D_{\varepsilon}} + \|h\|_{*,\partial D_{\varepsilon}}).$$ 

To prove the claim, observe, with the notation $L = -\Delta - W_1$, that

$$L(\tilde{\phi}) = f + d_0 L(\tilde{Z}_{0,\varepsilon}) \quad \text{in } D_{\varepsilon}$$

and

$$\frac{\partial \tilde{\phi}}{\partial n} - W_2 \tilde{\phi} = h + d_0 \left( \frac{\partial}{\partial n} - W_2 \right) \tilde{Z}_{0,\varepsilon} \quad \text{on } \partial D_{\varepsilon}.$$ 

Then by Lemma 3.2, we have

$$\|\tilde{\phi}\|_{L^\infty(D_{\varepsilon})} \leq C|d_0| \left( \left\| L(\tilde{Z}_{0,\varepsilon}) \right\|_{*,D_{\varepsilon}} + \left\| \left( \frac{\partial}{\partial n} - W_2 \right) \tilde{Z}_{0,\varepsilon} \right\|_{*,\partial D_{\varepsilon}} \right) + C(\|f\|_{*,D_{\varepsilon}} + \|h\|_{*,\partial D_{\varepsilon}}). \quad (13)$$

Multiplying the equation (12) by $\tilde{Z}_{0,\varepsilon}$ and integrating by parts we find that

$$d_0 \left[ \int_{D_{\varepsilon}} L(\tilde{Z}_{0,\varepsilon}) \tilde{Z}_{0,\varepsilon} + \int_{\partial D_{\varepsilon}} \tilde{Z}_{0,\varepsilon} \left( \frac{\partial}{\partial n} - W_2 \right) \tilde{Z}_{0,\varepsilon} \right]$$

$$= -\int_{D_{\varepsilon}} f \tilde{Z}_{0,\varepsilon} + \int_{\partial D_{\varepsilon}} \tilde{\phi} \left( \frac{\partial}{\partial n} - W_2 \right) \tilde{Z}_{0,\varepsilon} - \int_{\partial D_{\varepsilon}} h \tilde{Z}_{0,\varepsilon} - \int_{D_{\varepsilon}} \tilde{\phi} L(\tilde{Z}_{0,\varepsilon})$$

$$\leq C(\|f\|_{*,D_{\varepsilon}} + C\|\tilde{\phi}\|_{L^\infty} \cdot \left( \left\| \frac{\partial \tilde{Z}_{0,\varepsilon}}{\partial n} - W_2 \tilde{Z}_{0,\varepsilon} \right\|_{*,\partial D_{\varepsilon}} + \|L(\tilde{Z}_{0,\varepsilon})\|_{*,D_{\varepsilon}} \right) + C\|h\|_{*,\partial D_{\varepsilon}}$$

$$\leq C \left( \|f\|_{*,D_{\varepsilon}} + \|h\|_{*,\partial D_{\varepsilon}} \right) \left( \left\| \frac{\partial \tilde{Z}_{0,\varepsilon}}{\partial n} - W_2 \tilde{Z}_{0,\varepsilon} \right\|_{*,\partial D_{\varepsilon}} + \|L(\tilde{Z}_{0,\varepsilon})\|_{*,D_{\varepsilon}} \right)$$

$$+ C|d_0| \left( \left\| \frac{\partial \tilde{Z}_{0,\varepsilon}}{\partial n} - W_2 \tilde{Z}_{0,\varepsilon} \right\|_{*,\partial D_{\varepsilon}}^2 + \|L(\tilde{Z}_{0,\varepsilon})\|_{*,D_{\varepsilon}}^2 \right).$$

due to (13) and the constant $C$ doesn’t depend on $\varepsilon, \sigma$.

For $R$ large and fixed and for any $\sigma \in (0, 1)$ and small $\varepsilon$, we may easily achieve the claim (11) by building the ensuing estimates.

$$\int_{D_{\varepsilon}} L(\tilde{Z}_{0,\varepsilon}) \tilde{Z}_{0,\varepsilon} + \int_{\partial D_{\varepsilon}} \tilde{Z}_{0,\varepsilon} \left( \frac{\partial}{\partial n} - W_2 \right) \tilde{Z}_{0,\varepsilon} \geq \epsilon_0 \sigma - C \left( \frac{1}{\ln \varepsilon} + \varepsilon^\sigma \right), \quad (14)$$

$$\|L(\tilde{Z}_{0,\varepsilon})\|_{*,D_{\varepsilon}} \leq C \left( \sigma + \frac{1}{\ln \varepsilon} \right), \quad (15)$$
where the positive constants $c_0, C$ are independent of $\sigma, \varepsilon$.

**Proof of (14).** Denote $x = F_\varepsilon(y)$ hereinafter and recall that this map preserves area. It is divided that
\[
\int_{D_\varepsilon} L(\tilde{Z}_{0,\varepsilon}) \tilde{Z}_{0,\varepsilon} = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 := \sum_{i=1}^6 \int_{B_i} L(\tilde{Z}_{0,\varepsilon}) \tilde{Z}_{0,\varepsilon},
\]
where
\[
B_1 = (F_\varepsilon)^{-1}(\{r < R - 2\} \cap \mathbb{R}^2_+), \quad B_2 = (F_\varepsilon)^{-1}(\{R - 2 < r < R - 1\} \cap \mathbb{R}^2_+),
\]
\[
B_3 = (F_\varepsilon)^{-1}(\{R - 1 < r < R\} \cap \mathbb{R}^2_+), \quad B_4 = (F_\varepsilon)^{-1}(\{R < r < R + 1\} \cap \mathbb{R}^2_+),
\]
and
\[
B_5 = (F_\varepsilon)^{-1}(\left\{ R + 1 < r < \frac{\delta}{4\varepsilon} \right\} \cap \mathbb{R}^2_+), \quad B_6 = (F_\varepsilon)^{-1}(\left\{ \frac{\delta}{4\varepsilon} < r < \frac{\delta}{3\varepsilon} \right\} \cap \mathbb{R}^2_+)
\]
with $r = |x|$.

Then we estimate term by term. In $x$, the operator $L$ has the form that
\[
\tilde{L} = -\Delta + O(\varepsilon|x|) \nabla^2 + O(\varepsilon) \nabla - W_1((F_\varepsilon)^{-1} x) = -\Delta + O(\varepsilon|x|) \nabla^2 + O(\varepsilon) \nabla - W_1(x + O(\varepsilon|x|))
\]
\[
= -\Delta - \frac{8\lambda}{\lambda^2 + x_1^2 + (x_2^2 + \frac{1}{\sqrt{\lambda}})^2} + O(\varepsilon|x|) \nabla^2 + O(\varepsilon) \nabla + O\left( \frac{\varepsilon^\alpha + \varepsilon|x|}{1 + |x|^4} \right)
\]
with $a = K(\xi), b = \kappa(\xi)$.

**Estimate of $I_1$.** In $B_1$, $\tilde{Z}_{0,\varepsilon} = Z_{0,\varepsilon}$. So we get that
\[
I_1 = \int_{\{0 \leq r \leq 2\} \cap \mathbb{R}^2_+} \tilde{L}(z_0) z_0 = \int_{\{0 \leq r \leq 2\} \cap \mathbb{R}^2_+} \left( -\Delta z_0 + O(\varepsilon|x|) \nabla^2 z_0 + O(\varepsilon) \nabla z_0 - W_1(\zeta + x + O(\varepsilon|x|)) z_0 \right)_0 = O(\varepsilon^\alpha).
\]

**Estimate of $I_2$.** In $B_2$, $\tilde{Z}_{0,\varepsilon} = Z_{0,\varepsilon} + (1 - \eta_0)(\hat{h}_1 - 1)Z_{0,\varepsilon}$. Let $\tilde{z}_0 = z_0 + (1 - \eta_0)(h_1 - 1)z_0$. Then it holds that
\[
I_2 = \int_{\{R-2 \leq r \leq R-1\} \cap \mathbb{R}^2_+} \tilde{L}(\tilde{z}_0) \tilde{z}_0 = \int_{\{R-2 \leq r \leq R-1\} \cap \mathbb{R}^2_+} \left\{ -z_0 \Delta[(1 - \eta_0)(h_1 - 1)] - 2\nabla z_0 \cdot \nabla[(1 - \eta_0)(h_1 - 1)] + O(\varepsilon|x|) \nabla^2 z_0 + O(\varepsilon) \nabla z_0 \right\}_0 = O(\varepsilon^\alpha).
\]

**Estimate of $I_3$.** In $B_3$, $\tilde{Z}_{0,\varepsilon} = \hat{h}_1 Z_{0,\varepsilon}$. We arrive at
\[
I_3 = \int_{\{R-1 \leq r \leq R\} \cap \mathbb{R}^2_+} \tilde{L}(h_1 z_0) h_1 z_0 = \int_{\{R-1 \leq r \leq R\} \cap \mathbb{R}^2_+} \left( -2\nabla z_0 \cdot \nabla h_1 + O(\varepsilon|x|) \nabla^2(h_1 z_0) + O(\varepsilon) \nabla(h_1 z_0) \right)_0 h_1 z_0 + O(\varepsilon^\alpha)
\]
\[
= O\left( \frac{\varepsilon^\alpha + \varepsilon|x|}{1 + |x|^2} \right).
\]

**Estimate of $I_4$.** Now $\tilde{Z}_{0,\varepsilon} = \eta_1 \hat{h}_2 \tilde{Z}_{0,\varepsilon}$ and set again $\tilde{z}_0(x) = h_1 z_0(x) + (1 - \eta_1)(h_2 - 1)h_1 z_0(x)$. Then
\[
I_4 = \int_{\{R \leq r \leq R+1\} \cap \mathbb{R}^2_+} \tilde{L}(\tilde{z}_0) \tilde{z}_0
\]
\[
\int_{(r<r<R+1) \cap \mathbb{R}^2_+} \{ -2\nabla h_1 \cdot \nabla z_0 - \Delta [(1 - \eta_1)(h_2 - 1)h_1]z_0 - 2\nabla [(1 - \eta_1)(h_2 - 1)h_1] \cdot \nabla z_0 \} z_0 + O(\varepsilon^6)
\]

\[
= \int_{(r<r<R+1) \cap \mathbb{R}^2_+} -\Delta [(1 - \eta_1)(h_2 - 1)h_1]z_0 + O(\frac{\sigma}{R^2} + \frac{1}{\ln \varepsilon} + \varepsilon^6).
\]

The functions \(\eta_1, h_2\) are both radial symmetric and decreasing with \(h_2^2(r) = \frac{-\sigma}{|r|^\sigma - |r'|^{\sigma}}\), \(\eta_1'(R) = \eta_1(R + 1) = \eta'(R + 1) = 0, \eta_1(R) = 1\), \(\frac{\partial \eta}{\partial \sigma}\big|_{x_2 = 0} = \frac{2m}{\partial \sigma}ig|_{x_2 = 0} = 0\). This leads to

\[
\int_{(r<r<R+1) \cap \mathbb{R}^2_+} -\Delta [(1 - \eta_1)(h_2 - 1)h_1]z_0 \left[ h_1z_0(x) + (1 - \eta_1)(h_2 - 1)h_1z_0(x) \right]
\]

\[
= - \int_{\mathcal{O}(r<r<R+1) \cap \mathbb{R}^2_+} \frac{\partial (h_1h_2 - 1)h_1}{\partial \nu} h_1h_2z_0^2 + O \left( \varepsilon^6 + \varepsilon^6 R^2 + \frac{1}{\ln \varepsilon} \right)
\]

\[
\]
From the above six estimates, we conclude that
\[
\int_{D_ε} L(\tilde{Z}_{0,ε}) \tilde{Z}_{0,ε} \geq γ₀σ + O \left( ε^α + \frac{1}{\ln \frac{1}{ε}} + \frac{σ^2}{R} + \frac{σ}{R^2} \right).
\] (18)

Next we estimate \( \int_{∂D_ε} \tilde{Z}_{0,ε} \left( \frac{∂}{∂n} - W_2 \right) \tilde{Z}_{0,ε} \). It is checked, writing \( x = (x₁, x₂) \), that
\[
\int_{∂D_ε} \tilde{Z}_{0,ε} \left( \frac{∂}{∂n} - W_2 \right) \tilde{Z}_{0,ε} = \int_{∂R^2} \tilde{Z}_0 \left[ B(\tilde{z}_0) - W_2(\xi' + x₁ + O(ε|x₁|)\tilde{z}_0) \right] j(x)
\]
where \( \tilde{z}_0 = \tilde{Z}_{0,ε}(R⁻¹(ε)) \) and \( j(x) \) is a positive function arising from the change of variables bounded uniformly in \( ε \). \( B \) is a differential operator of order one on \( ∂R^2_+ \). Rotating \( ∂D_ε \) so that \( \nabla F_ε(ξ') = I \) we get the following expansion
\[
B = -\frac{∂}{∂x₂} + O(ε|x₁|)\nabla.
\]
Recall that
\[
W_2(\xi' + x₁ + O(ε|x₁|)) = \frac{2λ}{\sqrt{K(ξ)} \lambda^2 + |y - ξ'| - D(ξ)λn(ξ)} \left[ 1 + O(ε|y - ξ'|) + O(ε^α) \right] = \frac{2bλ}{\sqrt{a} (\lambda^2 + x₁^2 + \frac{b^2x₂^2}{a})} \left[ 1 + O(ε|x₁| + ε^α) \right].
\]
Thus
\[
\int_{∂R^2_+ \cap \{ |x₁| ≤ R - 2 \}} \tilde{Z}_0 \left[ B(z₀) - W_2(\xi' + x₁ + O(ε|x₁|)z₀) \right] j(x) = O(ε^α + ε ln R).
\]
Note that \( \tilde{z}_0 = z₀ + (1 - η₀)(h₁ - 1)z₀ \) for \( R - 2 ≤ |x| ≤ R - 1 \), using the fact that \( η₀, h₁ \) has zero normal derivative on \( R^2_+ \), we have
\[
B(\tilde{z}_0) = -\frac{∂z₀}{∂x₂} + (1 - η₀)(h₁ - 1)\frac{∂z₀}{∂x₂} + O(ε|x₁|)\nabla \tilde{z}_0 \quad \text{with} \quad x₂ = 0.
\]
It is easy to see that
\[
\int_{∂R^2_+ \cap \{ R - 2 ≤ |x₁| ≤ R - 1 \}} \tilde{Z}_0 \left[ B(z₀) - W_2(\xi' + x₁ + O(ε|x₁|)z₀) \right] j(x) = O \left( ε^α + \frac{1}{\ln \frac{1}{ε}} \right).
\]
Similarly,
\[
\int_{∂R^2_+ \cap \{ R - 1 ≤ |x₁| ≤ R \}} h₁z₀ \left[ B(h₁z₀) - W_2(\xi' + x₁ + O(ε|x₁|)h₁z₀) \right] j(x) = O(ε^α)
\]
and
\[
\int_{∂R^2_+ \cap \{ R ≤ |x₁| ≤ R + 1 \}} \tilde{Z}_0 \left[ B(z₀) - W_2(\xi' + x₁ + O(ε|x₁|)z₀) \right] j(x) = O(ε^α)
\]
with \( \tilde{z}_0 = η₁h₁z₀ + (1 - η₁)h₁h₂z₀ \). Also we obtain that
\[
\int_{∂R^2_+ \cap \{ R + 1 ≤ |x₁| ≤ R + 2 \}} h₁h₂z₀ \left[ B(h₁h₂z₀) - W_2(\xi' + x₁ + O(ε|x₁|)h₁h₂z₀) \right] j(x)
\]
\[
= O \left( ε^α + ε ln \frac{1}{ε} + \frac{1}{\ln \frac{1}{ε}} + ε^σ R^σ \right),
\]
and
\[
\int_{∂R^2_+ \cap \{ R + 2 ≤ |x₁| ≤ R + 3 \}} η₂h₁h₂z₀ \left[ B(η₂h₁h₂z₀) - W_2(\xi' + x₁ + O(ε|x₁|)η₂h₁h₂z₀) \right] j(x)
\]
\[
= O \left( ε^α + R^2σ ln^{-2} \frac{1}{ε} \right).
\]
Finally combing the above estimate and (18), we have that
\[
\int_{D_ε} L(\tilde{Z}_{0,ε}) \tilde{Z}_{0,ε} + \int_{∂D_ε} \tilde{Z}_{0,ε} \left( \frac{∂}{∂n} - W_2 \right) \tilde{Z}_{0,ε} \geq γ₀σ + O \left( \frac{σ^2}{R} + \frac{σ}{R^2} + \frac{1}{\ln \frac{1}{ε}} + ε^σ R^σ \right).
\]
Now take \( R \) large enough, we know (14) holds.

**Proof of (15).** For \( y ∈ B₁ \), \( \tilde{Z}_{0,ε} = Z_{0,ε} \). Obviously we have
and hence
\[ \|L(\tilde{Z}_{0,\varepsilon})\|_{**,B_1} \leq C\varepsilon^\alpha. \]

For \( y \in B_2 \), then \( \tilde{z}_0 = z_0 + (1 - \eta_0)(h_1 - 1)z_0 \). So
\[
L(\tilde{Z}_{0,\varepsilon}) = L(z_0) = -\Delta z_0 + O(\varepsilon) \nabla^2 z_0 - \nabla \left( (\varepsilon' + x + O(\varepsilon|x|)) z_0 \right) = O(\varepsilon) \nabla^2 z_0 + O(\varepsilon) \nabla z_0 + O(\varepsilon^\alpha)z_0 = O(\varepsilon|x| + \varepsilon^\alpha),
\]
which leads to
\[ \|L(\tilde{Z}_{0,\varepsilon})\|_{**,B_2} \leq \frac{C}{\ln \frac{1}{\varepsilon}}. \]

For \( y \in B_3 \), it holds that \( \tilde{z}_0 = h_1z_0 \). Then
\[
L(\tilde{Z}_{0,\varepsilon}) = \tilde{L}(z_0) = -h_1\Delta z_0 - 2\nabla h_1 \cdot \nabla z_0 + O(\varepsilon) \nabla^2 z_0 + O(\varepsilon) \nabla z_0 - W_1 ( (\varepsilon' + x + O(\varepsilon|x|)) \tilde{z}_0 = O \left( \varepsilon^\alpha + \frac{1}{\ln \frac{1}{\varepsilon}} + \sigma \right),
\]
from which we can get that
\[ \|L(\tilde{Z}_{0,\varepsilon})\|_{**,B_3} \leq \frac{C}{\ln \frac{1}{\varepsilon}} + C\sigma. \]

For \( y \in B_4 \), we see that \( \tilde{z}_0 = h_1h_2z_0 \). Thus it is checked that
\[
O(\varepsilon|x|) \nabla^2 (h_1h_2z_0) = O(\varepsilon|x|) \left[ \nabla^2 h_1h_2z_0 + h_1 \nabla^2 h_2z_0 + h_1h_2 \nabla^2 z_0 + 2 \nabla h_1 \nabla h_2z_0 + 2 \nabla h_1h_2 \nabla z_0 + 2h_1h_2 \nabla z_0 \right]
= O \left( \frac{\varepsilon}{|x|^{1+\sigma}} \right),
\]
\[
O(\varepsilon) \nabla (h_1h_2z_0) = O \left( \frac{\varepsilon}{\ln \frac{1}{\varepsilon}|x|} \left[ \frac{1}{|x|} - \frac{\varepsilon}{\delta} \right] \sigma + \frac{\ln \frac{1}{\varepsilon} - \ln |x|}{\ln \frac{1}{\varepsilon}} \frac{\varepsilon\sigma}{|x|^{1+\sigma}} + \frac{\varepsilon}{|x|^{2+\sigma}} \right) = O \left( \frac{\varepsilon}{|x|^{1+\sigma}} \right),
\]
and
\[
-\Delta (h_1h_2z_0) - W_1 ( (\varepsilon' + x + O(\varepsilon|x|)) h_1h_2z_0 = O \left( \frac{\sigma}{\ln \frac{1}{\varepsilon}|x|^{2+\sigma}} + \frac{\sigma^2}{|x|^{2+\sigma}} + \frac{1}{\ln \frac{1}{\varepsilon}|x|^3} + \frac{\sigma}{|x|^{4+\sigma}} + \frac{\varepsilon|x| + \varepsilon^\alpha}{1 + |x|^4} \right).
\]
It is easy to get that
\[ \|L(\tilde{Z}_{0,\varepsilon})\|_{**,B_5} \leq C\sigma + C\varepsilon^\sigma \ln^3 \frac{1}{\varepsilon} + C\varepsilon^\alpha + \frac{C}{\ln \frac{1}{\varepsilon}}. \]

For \( y \in B_6 \), \( \tilde{z}_0 = \eta h_1h_2z_0 \), \( h_1 = O \left( \frac{1}{\ln \frac{1}{\varepsilon}} \right), h_2 = O(\varepsilon^\alpha) \). Hence
\[
L(\tilde{Z}_{0,\varepsilon}) = \tilde{L}(z_0) = -\Delta z_0 + O(\varepsilon) \nabla^2 z_0 + O(\varepsilon) \nabla z_0 - W_1 ( (\varepsilon' + x + O(\varepsilon|x|)) \tilde{z}_0 = O \left( \frac{\varepsilon^2+\varepsilon}{\ln \frac{1}{\varepsilon}} + \frac{\varepsilon|x| + \varepsilon^\alpha}{1 + |x|^4} \right),
\]
and
\[ \|L(\tilde{Z}_{0,\varepsilon})\|_{**,B_6} \leq C\varepsilon^\sigma \ln^2 \frac{1}{\varepsilon} + C\varepsilon^4. \]

From the above estimates, (15) is finally concluded.
Proof of (16). It is known, in the proof of (14), that \( \frac{\partial Z_0}{\partial n} - W_2 Z_0, e \) is transformed to \( B(\tilde{z}_0) - W_2(\xi' + x_1 + O(\xi|x_1|)) \tilde{z}_0 \). We still estimate it in \( B_1 \) respectively. For \( y \in B_1, \tilde{z}_0 = z_0 \).

\[
B(z_0) - W_2(\xi' + x_1 + O(\xi|x_1|))z_0 = O(\varepsilon^\alpha) .
\]

For \( y \in B_2, \tilde{z}_0 = z_0 + (1 - \eta_0)(h_1 - 1)z_0 \).

\[
B(\tilde{z}_0) - W_2(\xi' + x_1 + O(\xi|x_1|))\tilde{z}_0 = O\left( \varepsilon^\alpha + \frac{1}{\ln \frac{1}{\varepsilon}} \right) .
\]

For \( y \in B_3, \tilde{z}_0 = h_1 z_0 \).

\[
B(h_1 z_0) - W_2(\xi' + x_1 + O(\xi|x_1|))h_1 z_0 = O(\varepsilon^\alpha) .
\]

For \( y \in B_4, \tilde{z}_0 = h_1 z_0 + (1 - \eta_1)(h_2 - 1)h_1 z_0 \).

\[
B(\tilde{z}_0) - W_2(\xi' + x_1 + O(\xi|x_1|))\tilde{z}_0 = O(\varepsilon^\alpha) .
\]

For \( y \in B_5, \tilde{z}_0 = h_1 h_2 z_0 \).

\[
B(h_1 h_2 z_0) - W_2(\xi' + x_1 + O(\xi|x_1|))h_1 h_2 z_0 = O\left( \varepsilon^\alpha + \frac{1}{\ln \frac{1}{\varepsilon}} \right) .
\]

And for \( y \in B_6, \tilde{z}_0 = \eta_2 h_1 h_2 z_0, h_1 = O\left( \frac{1}{\ln \frac{1}{\varepsilon}} \right), h_2 = O(\varepsilon^\alpha), \)

\[
B(\eta_2 h_1 h_2 z_0) - W_2(\xi' + x_1 + O(\xi|x_1|))\eta_2 h_1 h_2 z_0 = O\left( \varepsilon^\alpha + \frac{1}{\ln \frac{1}{\varepsilon}} \right) .
\]

Combining all these estimates and the definition of \( || \cdot ||_{\ast, \partial D_\varepsilon} \), we get the desired (16). \( \square \)

**Proposition 3.4.** There exist an \( \varepsilon_0 > 0 \) and a positive number \( C \) such that for any \( \xi' \in \partial D_\varepsilon \) and \( f \in L^\infty(\partial D_\varepsilon), h \in L^\infty(\partial D_\varepsilon), \) there is a unique solution \( \phi \in L^\infty(\partial D_\varepsilon), c_1 \in \mathbb{R} \) to

\[
\begin{cases}
-\Delta \phi - 2K(ey) e^{2\nu} \phi = f + c_1 \chi_\varepsilon Z_{1,\varepsilon}, & \text{in } D_\varepsilon, \\
\frac{\partial \phi}{\partial n} - \kappa(ey)e^{\nu} \phi = h, & \text{on } \partial D_\varepsilon , \\
\int_{D_\varepsilon} \chi_\varepsilon Z_{1,\varepsilon} \phi = 0.
\end{cases}
\]

Moreover,

\[
||\phi||_{L^\infty(D_\varepsilon)} \leq C(||f||_{\ast,D_\varepsilon} + C||h||_{\ast,\partial D_\varepsilon}).
\]

**Proof.** By Lemma 3.3, any solution to (16) satisfies

\[
||\phi||_{L^\infty(D_\varepsilon)} \leq C(||f||_{\ast,D_\varepsilon} + ||h||_{\ast,\partial D_\varepsilon});
\]

Therefore, for the estimate, it is enough to prove

\[
|c_1| \leq C(||f||_{\ast,D_\varepsilon} + ||h||_{\ast,\partial D_\varepsilon}).
\]

In the following, let \( \tilde{\eta}_2 \) be defined as before. Multiplying the first equation of (19) by \( \tilde{\eta}_2 Z_{1,\varepsilon} \) and integrating by parts we can get that

\[
c_1 \int_{D_\varepsilon} \chi_\varepsilon Z_{1,\varepsilon}^2 = -\int_{D_\varepsilon} f \tilde{\eta}_2 Z_{1,\varepsilon} + \int_{D_\varepsilon} [-\Delta(\tilde{\eta}_2 Z_{1,\varepsilon}) - W_1 \tilde{\eta}_2 Z_{1,\varepsilon}] \phi \\
- \int_{\partial D_\varepsilon} h \tilde{\eta}_2 Z_{1,\varepsilon} + \int_{\partial D_\varepsilon} \frac{\partial \tilde{\eta}_2}{\partial n} Z_{1,\varepsilon} + \int_{\partial D_\varepsilon} \tilde{\eta}_2 \left( \frac{\partial Z_{1,\varepsilon}}{\partial n} - W_2 Z_{1,\varepsilon} \right) .
\]

From \( |\nabla \tilde{\eta}_2| \leq C \varepsilon, |\nabla^2 \tilde{\eta}_2| \leq C \varepsilon^2, Z_{1,\varepsilon} = O(\frac{1}{1+|y| - \varepsilon}) \), direct computation leads to

\[
|\nabla \tilde{\eta}_2| - \int_{D_\varepsilon} f \tilde{\eta}_2 Z_{1,\varepsilon} - \int_{\partial D_\varepsilon} h \tilde{\eta}_2 Z_{1,\varepsilon}| \leq C(||f||_{\ast,D_\varepsilon} + ||h||_{\ast,\partial D_\varepsilon}),
\]

\[
\int_{D_\varepsilon} [-\Delta(\tilde{\eta}_2 Z_{1,\varepsilon}) - W_1 \tilde{\eta}_2 Z_{1,\varepsilon}] \phi = O\left( \varepsilon^\alpha + \varepsilon \ln \frac{1}{\varepsilon} \right) ||\phi||_{L^\infty(D_\varepsilon)};
\]

and

\[
\int_{\partial D_\varepsilon} \frac{\partial \tilde{\eta}_2}{\partial n} Z_{1,\varepsilon} + \int_{\partial D_\varepsilon} \tilde{\eta}_2 \left( \frac{\partial Z_{1,\varepsilon}}{\partial n} - W_2 Z_{1,\varepsilon} \right) = O\left( \varepsilon^\alpha + \varepsilon \ln \frac{1}{\varepsilon} \right) ||\phi||_{L^\infty(D_\varepsilon)}.
\]

Then it is easy to obtain

\[
|c_1| \leq C(||f||_{\ast,D_\varepsilon} + ||h||_{\ast,\partial D_\varepsilon}) + C\left( \varepsilon^\alpha + \varepsilon \ln \frac{1}{\varepsilon} \right) ||\phi||_{L^\infty(D_\varepsilon)},
\]
from which we can get
\[ \| \phi \|_{L^\infty(D_\varepsilon)} \leq C(\| f \|_{* \cdot D_\varepsilon} + \| h \|_{* \cdot \partial D_\varepsilon}). \] (20)

Now consider the Hilbert space
\[ H = \left\{ \phi \in H^1(D_\varepsilon) : \int_{D_\varepsilon} \chi_\varepsilon Z_{1,\varepsilon} \phi = 0 \right\} \]
with the norm \( \| \phi \|_H^2 = \int_{D_\varepsilon} |\nabla \phi|^2 + |\phi|^2 \).
Equation (15) is equivalent to find \( \phi \in H \) such that
\[ \int_{D_\varepsilon} (\nabla \phi \cdot \nabla \psi + \phi \psi) - \int_{D_\varepsilon} (\psi_1 + W_1 \phi \psi) - \int_{\partial D_\varepsilon} W_2 \phi \psi = \int_{D_\varepsilon} f \psi + \int_{\partial D_\varepsilon} h \psi, \quad \forall \psi \in H. \]
By Fredholm’s alternative it is enough to show the uniqueness of solutions to the problem (19), which is
guaranteed by Lemma 3.3.

Remark 3.5. The result of Proposition 3.4 implies that the unique solution \( \phi = T(f, h) \) of (22) defines a continuous linear map. For later purposes, the differentiability of the operator \( T \) with respect to \( \xi' \) is necessary. The proof is almost the same as that in (10). Here we omit the details.

4. The Nonlinear Problem

In this section, we will solve the following nonlinear problem that
\[
\begin{cases}
-\Delta \phi - 2K(\varepsilon) e^{2\phi} \phi = R_1(y) + N_1(\phi) + c_1 \chi_{\varepsilon} Z_{1,\varepsilon}, & \text{in } D_\varepsilon, \\
\frac{\partial \phi}{\partial n} - \kappa(\varepsilon) e^{\phi} \phi = R_2(y) + N_2(\phi), & \text{on } \partial D_\varepsilon, \\
\int_{D_\varepsilon} \chi_{\varepsilon} Z_{1,\varepsilon} \phi = 0, & \text{in } \partial D_\varepsilon,
\end{cases}
\] (21)
where
\[ N_1(\phi) = K(\varepsilon) e^{2\phi}(e^{2\phi} - 1 - 2\phi), \quad N_2(\phi) = \kappa(\varepsilon) e^{\phi}(e^{\phi} - 1 - \phi). \]

Recall in Lemma 2.1 that
\[ R_1(y) = \frac{4\lambda^2}{(\lambda^2 + |y - \xi'| - D(\xi)\lambda n(\xi))^2} \left[ O(\varepsilon|y - \xi'|) + O(\varepsilon^2) \right], \]
\[ R_2(y) = \frac{D(\xi)2\lambda}{(\lambda^2 + |y - \xi'| - D(\xi)\lambda n(\xi))^2} \left[ O(\varepsilon|y - \xi'|) + O(\varepsilon^2) \right] + \varepsilon \frac{d\lambda^2}{\lambda^2 + |y - \xi'|^2}. \]
in \( |y - \xi'| \leq \frac{\varepsilon}{2} \), from which we may obtain that
\[ \| R_1(y) \|_{* \cdot D_\varepsilon} \leq C \varepsilon^2, \quad \| R_2(y) \|_{* \cdot \partial D_\varepsilon} \leq C \varepsilon^2, \quad \text{for any } \alpha \in (0, 1). \] (22)

Lemma 4.1. There exists \( \varepsilon_0 > 0 \) and \( C > 0 \), such that for \( 0 < \varepsilon < \varepsilon_0 \) and \( \xi' \in \partial D_\varepsilon \), the problem (24) admits a unique solution \((\phi, c_1)\) such that
\[ \| \phi \|_{L^\infty(D_\varepsilon)} + |c_1| \leq C \varepsilon^2, \quad \text{for any } \alpha \in (0, 1). \] (23)
Furthermore, the function \( \xi' \rightarrow \phi(\xi') \in C(\overline{D}_\varepsilon) \) is \( C^1 \) and
\[ \| D\xi \phi \|_{L^\infty(D_\varepsilon)} \leq C \varepsilon^2. \] (24)

Proof. In terms of the operator \( T \) defined in the previous section, problem (21) becomes
\[ \phi = T(R_1 + N_1(\phi), R_2 + N_2(\phi)) \equiv A(\phi). \] (25)
For a given number \( \gamma > 0 \), let us consider the region
\[ \mathcal{F}_\gamma = \{ \phi \in H : \| \phi \|_{L^\infty(D_\varepsilon)} \leq \gamma \varepsilon^2 \}. \]
From Proposition 3.4 we get
\[
\| A(\phi) \|_{L^\infty(D_\varepsilon)} \leq C \left( \| R_1 + N_1(\phi) \|_{* \cdot D_\varepsilon} + \| R_2 + N_2(\phi) \|_{* \cdot \partial D_\varepsilon} \right) \\
\leq C \left( \| R_1 \|_{* \cdot D_\varepsilon} + \| R_2 \|_{* \cdot \partial D_\varepsilon} + \| N_1(\phi) \|_{* \cdot D_\varepsilon} + \| N_2(\phi) \|_{* \cdot \partial D_\varepsilon} \right) \\
\leq C \varepsilon^2 + C \frac{1}{1 + |y - \xi'|^2 + \frac{1}{1 + |y - \xi'|^2}} \| \xi' \|_{* \cdot \partial D_\varepsilon} \\
\leq C \varepsilon^2 + C \varepsilon^2. \]
which leads to $A(\phi) \in F_\gamma$ for any $\phi \in F_\gamma$ where $\gamma$ is large but fixed. Also, for any $\phi_1, \phi_2 \in F_\gamma$,
\[ \|N_1(\phi_1) - N_1(\phi_2)\|_{\ast, D_\varepsilon} \leq C\gamma \varepsilon \|\phi_1 - \phi_2\|_{L^\infty(D_\varepsilon)} \]
and
\[ \|N_2(\phi_1) - N_2(\phi_2)\|_{\ast, \partial D_\varepsilon} \leq C\gamma \varepsilon \|\phi_1 - \phi_2\|_{L^\infty(D_\varepsilon)} \]
where $C$ is independent of $\gamma, \varepsilon$. Thus the operator $A$ is a contraction mapping on $F_\gamma$ and therefore a unique fixed point of $A$ exists in this region.

The discussion of the differentiability of $\phi$ with respect to $\xi'$ can be got by the similar proof in [10]. □

5. Variational Reduction

After Lemma [12], we already achieved that
\[
\begin{aligned}
& \left\{-\Delta(V + \phi) = K(\varepsilon y)e^{2V + 2\phi}, \quad \text{in } D_\varepsilon, \\
& \quad \frac{\partial(V + \phi)}{\partial n} + \varepsilon = \kappa(\varepsilon y)e^{V + \phi} + c_1 \chi_\varepsilon Z_1, \quad \text{on } \partial D_\varepsilon, \\
& \quad \int_{D_\varepsilon} \chi_\varepsilon Z_1 \phi = 0. \quad (26)
\end{aligned}
\]

To solve our problem, it is sufficient to let $c_1 = 0$ in [21]. In this section, we will see $c_1 = 0$ is equivalent to solving a finite dimensional problem.

Define the energy functional
\[ \bar{E}(U) = \frac{1}{2} \int_D |\nabla U|^2 - \frac{\varepsilon^2}{2} \int_D K e^{2U} + \int_{\partial D} U - \varepsilon \int_{\partial D} \kappa(x) e^U. \]

Also in terms of $V$,
\[ E(V) = \frac{1}{2} \int_{D_\varepsilon} |\nabla V|^2 - \frac{\varepsilon}{2} \int_{D_\varepsilon} K(\varepsilon y) e^{2V} + \int_{D_\varepsilon} \varepsilon V - \int_{\partial D_\varepsilon} \kappa(\varepsilon y) e^V. \]

Notice that $E(V) = \bar{E}(U) + 2 \ln \varepsilon \int_{\partial D_\varepsilon} dx = \bar{E}(U) + 4 \pi \ln \varepsilon$.

**Proposition 5.1.** If $\xi' \in \partial D_\varepsilon$ is a tangentially critical point of $F(\xi') = E(V(\xi') + \phi(\xi'))$ on $\partial D_\varepsilon$. Then $c_1 = 0$.

**Proof.** Note that $\partial_\tau V(\xi') = Z_1, o(1)$ in $L^\infty(D_\varepsilon)$, where $\tau$ is the unit tangent vector at $\xi'$. It is easy to see, since $\|\partial_\tau \phi\|_{L^\infty} = o(1)$, that
\[ 0 = \partial_\tau F(\xi') = E'(V + \phi)(\partial_\tau V + \partial_\tau \phi) = \int_{D_\varepsilon} c_1 \chi_\varepsilon Z_1(\partial_\tau V + \partial_\tau \phi) = \int_{D_\varepsilon} c_1 \chi_\varepsilon Z_1^2 + o(1), \]
from which the proposition is concluded. □

Next we will compute $E(V + \phi)$, on account of Proposition 5.1.

**Proposition 5.2.** It holds that
\[ E(V + \phi) = E(V) + O(\varepsilon^{2\alpha}). \]

**Proof.** On account of (21), we have, for some $t \in (0, 1)$, that
\[
\begin{aligned}
E(V + \phi) &= E(V) + E'(V)\phi + \frac{1}{2} E''(V + t\phi)\phi^2 \\
&= E(V) - \int_{D_\varepsilon} R_1 \phi - \int_{\partial D_\varepsilon} R_2 \phi \\
&\quad + \frac{1}{2} \int_{D_\varepsilon} R_1 \phi + \frac{1}{2} \int_{D_\varepsilon} N_1(\phi) \phi + \int_{D_\varepsilon} K(\varepsilon y) e^{2V} (1 - \varepsilon^{2\phi}) \phi^2 \\
&\quad + \frac{1}{2} \int_{\partial D_\varepsilon} R_2 \phi + \frac{1}{2} \int_{\partial D_\varepsilon} N_2(\phi) \phi + \frac{1}{2} \int_{\partial D_\varepsilon} \kappa(\varepsilon y) e^V (1 - \varepsilon^t \phi) \phi^2.
\end{aligned}
\]

From (22), it is easily checked that
\[
\begin{aligned}
\int_{D_\varepsilon} R_1 \phi &\leq C\|R_1\|_{\ast, D_\varepsilon}\|\phi\|_{L^\infty} = O(\varepsilon^{2\alpha}), \\
\int_{D_\varepsilon} N_1(\phi) \phi &\leq O(\|\phi\|_{L^\infty}^3) = O(\varepsilon^{3\alpha}), \\
\int_{\partial D_\varepsilon} R_2 \phi &\leq O(\varepsilon^{2\alpha}), \\
\int_{\partial D_\varepsilon} N_2(\phi) \phi &\leq O(\varepsilon^{3\alpha}).
\end{aligned}
\]
Theorem 6.1. Let us calculate terms by terms. From the computation in the appendix, we get that
\[
\begin{align*}
&\int_{D_\varepsilon} K(\varepsilon y)e^{2V}(1-\varepsilon^{2\phi})\phi^2 \leq C \int_{\mathbb{R}^2} \frac{1}{(\lambda^2 + |y - \xi|^2)^2} |\phi|^3 = O(\varepsilon^{3\alpha}), \\
&\int_{\partial D_\varepsilon} \kappa(\varepsilon y)e^{V}(1-\varepsilon^{4\phi})\phi^2 = O(\varepsilon^{3\alpha}).
\end{align*}
\]
The proof is complete. \(\square\)

6. Energy expansion and the proof of Theorem

In this section, we will first compute the energy functional and then prove the main theorem.

**Theorem 6.1.** It holds that
\[
E(V) = 2\pi \ln \varepsilon - 2\pi + 2\pi \ln 2 - 2\pi \ln \left(\kappa(\xi) + \sqrt{K(\xi) + \kappa^2(\xi)}\right) + O(\varepsilon^\alpha).
\]

**Proof.** From the equation \(\Box \) of \( U \), we may have that
\[
E(V) = \frac{1}{2} \int_{D_\varepsilon} (-\Delta V) V - \frac{1}{2} \int_{D_\varepsilon} K(\varepsilon y)e^{2V} + \frac{1}{2} \int_{\partial D_\varepsilon} \frac{\partial V}{\partial n} V + \int_{\partial D_\varepsilon} e^V - \int_{\partial D_\varepsilon} \kappa(\varepsilon y)e^V
\]
\[
= \frac{1}{2} \int_{D_\varepsilon} K(\xi)e^{2(U_0(\varepsilon y) + 2\ln \varepsilon)} V - \frac{1}{2} \int_{D_\varepsilon} K(\varepsilon y)e^{2V} + \frac{\varepsilon}{2} \int_{\partial D_\varepsilon} V - \frac{\varepsilon d}{2} \int_{\partial D_\varepsilon} \frac{\lambda^2}{\lambda^2 + |y - \xi|^2} V
\]
\[
+ \frac{1}{2} \int_{\partial D_\varepsilon} \kappa(\xi)e^{U_0(\varepsilon y) + 2\ln \varepsilon} V - \int_{\partial D_\varepsilon} \kappa(\varepsilon y)e^V.
\]
Let us calculate terms by terms. From the computation in the appendix, we get that
\[
\int_{D_\varepsilon} K(\xi)e^{2(U_0(\varepsilon y) + 2\ln \varepsilon)} V
\]
\[
= \int_{D_\varepsilon} \frac{4\lambda^2}{(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)^2} \left( \ln \frac{2\lambda}{\sqrt{K(\xi)(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)} + H_0(\xi y) \right)
\]
\[
= \int_{D_\varepsilon} \frac{4\lambda^2}{(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)^2} \left( \ln \frac{2\lambda}{\sqrt{K(\xi)} + \ln \frac{1}{(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)} + O(\varepsilon^\alpha) \right)
\]
\[
= -2\pi \left[ 2\sinh^{-1} \mathcal{D}(\xi) + 2\ln \lambda - \frac{\mathcal{D}(\xi)}{\sqrt{1 + \mathcal{D}(\xi)^2}} \left( 1 + 2\ln 2\lambda + \ln(1 + \mathcal{D}(\xi)^2) \right) \right]
\]
\[
+ 2\pi \left( 1 - \frac{\mathcal{D}(\xi)}{\sqrt{1 + \mathcal{D}(\xi)^2}} \right) \ln \frac{2\lambda}{\sqrt{K(\xi)}} + O(\varepsilon^\alpha)
\]
\[
= -2\pi \left[ 2\sinh^{-1} \mathcal{D}(\xi) - \mathcal{D}(\xi) \frac{\mathcal{D}(\xi)}{\sqrt{1 + \mathcal{D}(\xi)^2}} \left( 2\ln 2 + \ln(1 + \mathcal{D}(\xi)^2) \right) \right]
\]
\[
+ 2\pi \left( 1 - \frac{\mathcal{D}(\xi)}{\sqrt{1 + \mathcal{D}(\xi)^2}} \right) \left( \ln \frac{2\lambda}{\sqrt{K(\xi)}} - 1 + 2\ln \lambda + O(\varepsilon^\alpha) \right).
\]

Also it holds that
\[
\int_{D_\varepsilon} K(\varepsilon y)e^{2V} = \int_{D_\varepsilon} \frac{4\lambda^2 K(\varepsilon y)e^{2U_0(\varepsilon y)}}{K(\xi)(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)^2}
\]
\[
= \int_{D_\varepsilon \cap B_\varepsilon(\xi')} \left( \frac{4\lambda^2}{(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)^2} + O(\varepsilon^\alpha) \right)
\]
\[
= \frac{4\lambda^2}{(\lambda^2 + |y - \xi' - \mathcal{D}(\xi)\lambda n(\xi)|^2)^2} + O(\varepsilon^\alpha)
\]
\[
= 2\pi \left( 1 - \frac{\mathcal{D}(\xi)}{\sqrt{1 + \mathcal{D}(\xi)^2}} \right) + O(\varepsilon^\alpha).
\]
So we obtain that
\[
\frac{1}{2} \int_{D_\varepsilon} K(\xi)e^{2(U_0(\varepsilon y) + 2\ln \varepsilon)} V - \frac{1}{2} \int_{D_\varepsilon} K(\varepsilon y)e^{2V}
\]
From (27)–(30), we achieve that

\[ E(V) = 2\pi \ln \varepsilon - 2\pi + 2\pi \left( \ln \frac{2\lambda}{\sqrt{K(\xi)}} - \sinh^{-1} \mathcal{D}(\xi) - \ln \lambda \right) + O(\varepsilon^\alpha) \]

\[ = 2\pi \ln \varepsilon - 2\pi + 2\pi \ln \varepsilon - 2\pi \ln 2 - 2\pi \ln \sqrt{K(\xi)} - 2\pi \sinh^{-1} \mathcal{D}(\xi) + O(\varepsilon^\alpha) \]

\[ = 2\pi \ln \varepsilon - 2\pi + 2\pi \ln \varepsilon - 2\pi \ln \left( \kappa(\xi) + \sqrt{K(\xi) + \kappa(\xi)^2} \right) + O(\varepsilon^\alpha). \]

The proof is complete. \( \square \)

**Proof of Theorem 4.** Since \( \xi_\ast \) is a local extremum point of \( \kappa(\xi) + \sqrt{K(\xi) + \kappa(\xi)^2} \) on the boundary, there is an open neighborhood \( \Gamma \subset \partial D \) of \( \xi_\ast \) such that

\[ \kappa(\xi) + \sqrt{K(\xi) + \kappa(\xi)^2} < (\text{or } >) \kappa(\xi_\ast) + \sqrt{K(\xi_\ast) + \kappa(\xi_\ast)^2}, \quad \forall \xi \in \partial \Gamma. \]
Recall again that
\[ E(V + \phi)[\xi] = 2\pi \ln \varepsilon - 2\pi + 2\pi \ln 2 - 2\pi \ln \left(\kappa(\xi) + \sqrt{K(\xi) + \kappa(\xi)^2}\right) + O(\varepsilon^\kappa). \]
So we have, from the continuity, that
\[ E(V + \phi)[\xi] < (or >) E(V + \phi)[\xi_*], \quad \forall \xi \in \partial \Gamma. \]
Therefore \( \xi_* \in \partial D \) must be a tangentially critical point of \( E(V + \phi)[\xi] \). The existence result is then from Proposition 5.1. The remaining identity is just due to the Gauss-Bonnet theorem. \( \square \)

7. Appendix

Some useful computation is given in this section.

**Lemma 7.1.** It holds that, for \( \lambda > 0 \),
\[ \int_{D_{\lambda^*}} \frac{4\lambda^2}{(\lambda^2 + |y - \xi'| - D(\xi) \lambda n(\xi))^2} dy = 2\pi \left(1 - \frac{D(\xi)}{\sqrt{1 + D(\xi)^2}}\right) + O(\varepsilon). \]

**Proof.** Without loss of generality, \( \xi \) is assumed to be the origin owing to the symmetry. For a small and fixed \( \delta > 0 \), we have that
\[
\int_{D_{\lambda^*}} \frac{4\lambda^2}{(\lambda^2 + |y - \xi'| - D(\xi) \lambda n(\xi))^2} dy = \int_{D_{\lambda^*}} \frac{4}{(1 + |z - D(0)n(0)|^2)} dz + O(\varepsilon^2)
\]
\[
= \int_{B_{\lambda^*} \cap B_{\lambda^*}(0)} \frac{4}{(1 + |z - D(0)n(0)|^2)} dz - \int_{D_{\lambda^*} \cap B_{\lambda^*}(0)} \frac{4}{(1 + |z - D(0)n(0)|^2)} dz + O(\varepsilon^2)
\]
\[
= \int_{B_{\lambda^*}^+} \frac{4}{(1 + |z - D(0)n(0)|^2)} dz + O(\varepsilon) = 2\pi \left(1 - \frac{D(0)}{\sqrt{1 + D(0)^2}}\right) + O(\varepsilon),
\]
where the last but one equality is due to
\[
\int_{D_{\lambda^*} \cap B_{\lambda^*}(0)} \frac{4}{(1 + |z - D(0)n(0)|^2)} dz \leq \frac{4}{\lambda^2} \int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{\lambda z' - z^2}} \int_0^{\frac{\pi}{2}} \frac{1}{1 + z_1^2 + z_2^2} dz_2
\]
\[
\leq \frac{4}{\lambda^2} \int_0^{\frac{\pi}{4}} \left(\frac{1}{\lambda z' - z^2}\right) dz_1 = O(\lambda\varepsilon). \]

Similarly, the following calculation holds too.

**Lemma 7.2.** We have
\[
\int_{D_{\lambda^*}} \frac{4\lambda^2}{(\lambda^2 + |y - \xi'| - D(\xi) \lambda n(\xi))^2} \ln \left(\frac{1}{\lambda^2 + |y - \xi'| - D(\xi) \lambda n(\xi)^2}\right) dy
\]
\[
= -2\pi \left[2\sinh^{-1} D(\xi) + 1 + 2\ln \lambda - \frac{D(\xi)}{\sqrt{1 + D(\xi)^2}}\left(1 + 2\ln 2\lambda + \ln(1 + D(\xi)^2)\right)\right] + O(\varepsilon).
\]

**Proof.** It holds that
\[
\int_{D_{\lambda^*}} \frac{4\lambda^2}{(\lambda^2 + |y - D(0)n(0)|^2)^2} \ln \left(\frac{1}{\lambda^2 + |y - D(0)n(0)|^2}\right) dy
\]
\[
= \int_{D_{\lambda^*}} \frac{4}{(1 + |z - D(0)n(0)|^2)^2} \left[\ln \left(\frac{1}{1 + |z - D(0)n(0)|^2}\right) - 2\ln \lambda\right] dz
\]
\[
= \int_{R_+^2} \frac{4}{(1 + |z - D(0)n(0)|^2)^2} \ln \left(\frac{1}{1 + |z - D(0)n(0)|^2}\right) dz - 2\ln \lambda \int_{R_+^2} \frac{4}{(1 + |z - D(0)n(0)|^2)^2} dz
\]
\[
+ O(\varepsilon)
\]
\[
= -2\pi \left[2\sinh^{-1} D(0) + 1 - \frac{D(0)}{\sqrt{1 + D(0)^2}}\left(1 + 2\ln 4 + \ln(1 + D(0)^2)\right)\right].
\]
Lemma 7.3. We have
\[-4\pi \ln \lambda \left( 1 - \frac{D(0)}{\sqrt{1 + D(0)^2}} \right) + O(\varepsilon) = -2\pi \left[ 2\sinh^{-1} D(0) + 1 + \ln \lambda - \frac{D(0)}{\sqrt{1 + D(0)^2}} \left( 1 + 2\ln 2\lambda + \ln(1 + D(0)^2) \right) \right] + O(\varepsilon). \]

The next are about the boundary terms.

Lemma 7.4. It holds that
\[\int_{\partial D_\varepsilon} \frac{2\lambda D(\xi)}{\lambda^2 + |y - \xi' - D(\xi)\lambda n(\xi)|^2} \frac{1}{\sqrt{1 + D(\xi)^2}} dy = -2\pi \int_{\partial D_\varepsilon} \frac{2\lambda D(\xi)}{\lambda^2 + |y - \xi' - D(\xi)\lambda n(\xi)|^2} dz + O(\varepsilon). \]

Proof. Direct computation shows that
\[\int_{\partial D_\varepsilon} \frac{2\lambda D(\xi)}{\lambda^2 + |y - \xi' - D(\xi)\lambda n(\xi)|^2} \frac{1}{\sqrt{1 + D(\xi)^2}} dy = -2\pi \int_{\partial D_\varepsilon} \frac{2\lambda D(\xi)}{\lambda^2 + |y - \xi' - D(\xi)\lambda n(\xi)|^2} \frac{1}{\sqrt{1 + D(\xi)^2}} dz + O(\varepsilon) = -2\pi \frac{D(\xi)}{\sqrt{1 + D(\xi)^2}} \left[ \ln 4 + \ln(1 + D(\xi)^2) + 2\ln \lambda \right] + O(\varepsilon^\alpha). \]

Lemma 7.5. It holds that
\[\varepsilon \int_{\partial D_\varepsilon} \ln(\lambda^2 + |y - \xi' - D(\xi)\lambda n(\xi)|^2) = -4\pi \ln \varepsilon + O(\varepsilon). \]

Proof. We may check that
\[\varepsilon \int_{\partial D_\varepsilon} \ln(\lambda^2 + |y - D(0)\lambda n(0)|^2) dy = \lambda \varepsilon \int_{\partial D_\varepsilon} \left[ \ln \lambda^2 + \ln(1 + |z - D(0)\lambda n(0)|^2) \right] dz\]
\[ 4\pi \ln \lambda + \lambda \varepsilon \int_{\partial D} \ln \left( 1 + |z - \mathcal{D}(0)|^2 \right) \, dz \]
\[ = 4\pi \ln \lambda + 2\pi \left( -2 \ln(\lambda \varepsilon) + O(\varepsilon) \right) = -4\pi \ln \varepsilon + O(\varepsilon). \]

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