Kerr Solution Consistent Motion of Spin Particles in the General Relativity

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Abstract

The paper presents equations determining the particle spin evolution in the post-Newtonian approximation in the problem of motion of two mass and spin possessing particles. The equations are derived with the Einstein-Infeld-Hoffmann method from the condition of metric tensor symmetry. The consideration uses the condition of coordinate harmonicity and the metric coincidence nearby the particles with expansions of the Kerr solution written in the harmonic coordinates. For gyroscopes on satellites Gravity Probe B, for example, the equations yield a deviation of the axis of revolution, which is, first, two times as small as that obtained by J.L. Anderson in paper gr-qc/0511093 and, second, the deviation is of opposite sense. From the equations it follows that the total angular momentum of the spin particle system, generally speaking, is not conserved, beginning even with the post-Newtonian approximation. The results are briefly discussed.

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1. Introduction

A system of two particles possessing masses and intrinsic angular momentums is considered. The description of motion of these particles within the general relativity requires two types of dynamic equations: the ones determining the motion of center of masses of particles and spin evolution for each of them.

The equations of the first type can be derived either with Fock method or Einstein-Infeld-Hoffmann (EIH) method. As a rule, no disagreements in regard to these equations arise.

The situation regarding the spin evolution describing equations is absolutely different. There are several versions of these equations for self-consistent motion of two spin particles (e.g., [1]-[4]) and plenty of versions of equations describing the spin dynamics of the trial spin particle (e.g., [5]-[17]). All the versions differ from each other, and it is far from being evident what of these versions is proper.

We have solved the problem of self-consistent motion of two spin particles in the post-Newtonian approximation (PN\textsuperscript{2}) with the EIH method. The distinctive features of our consideration are the following.

1. All the operations provided for by the EIH method are performed using de Donder condition for metric

$$
\left(\sqrt{-g} \ g^{\lambda \sigma}\right)_{,\sigma} = 0,
$$

which coincides with the condition of coordinate harmonicity.

2. In the approximation expressions of the EIH method the unknown coefficients involve expansions of the Kerr solution written in harmonic coordinates in [18], [19]. The Kerr solution expansion coefficients are taken from ref. [20].

3. Dynamic equations for the spins result from the conditions of the metric symmetry in the approximations under consideration.

The dynamic equations for particle spins that have been derived by us and the ones for trial spin particle which follow from them differ from all others in that they are consistent with the starting general relativity postulate of metric tensor symmetry. Ensuring the metric symmetry, we encounter an alternative: either the metric symmetry or the law of conservation of angular

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\textsuperscript{2}By the PN approximation is meant an approximation, where corrections to the Newtonian approximation begin to appear irrespective of the order of smallness of the corrections.
momentum. As a result of the choice made we arrive at nonconservation of
the vector of total angular momentum of the particle system even in the PN
approximation. This issue is interesting by itself, and we discuss it briefly.
Besides, estimations for gyroscopes, the experiments with which were con-
ducted on satellite Gravity Probe B (GP-B), are presented and discussed.

We guess that the reader is familiar with the EIH method, so we restrict
ourselves to minimum elucidations in this regard. Also, cumbersome trea-
tments on EIH technique are not included in this paper. All these treatments
and calculation details will be presented in a separate paper.

2. Notation

This paper solves the equations of general relativity with zero energy-mo-
momentum tensor. Like in refs. [21], [22], write the equations using \( \eta_{\alpha\beta} \), the
metric tensor of Minkowski space.

\[
R_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} (\eta^{\mu\nu} R_{\mu\nu}) = 0.
\]

Here \( R_{\alpha\beta} \) is Ricci tensor specified in the standard manner. In the back-
ground space the Cartesian coordinates will be used, therefore

\[
\eta_{\alpha\beta} = \eta^{\alpha\beta} = \text{diag} [-1, 1, 1, 1].
\]

Two types of quantities are used: \( h_{\alpha\beta}, \gamma_{\alpha\beta} \). These quantities are given
by:

\[
g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad \gamma_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} (\eta^{\mu\nu} h_{\mu\nu}).
\]

The raising and lowering of index in \( h_{\alpha\beta} \) and \( \gamma_{\alpha\beta} \) is made using tensors
\( \eta^{\alpha\beta} \). Thus, by \( h^{\alpha\beta} \) is meant \( h^{\alpha\beta} \equiv \eta^{\alpha\mu} \eta^{\beta\nu} h_{\mu\nu} \).

A system of two particles is considered. In the background space we
choose Cartesian coordinates \((ct, x_k)\) in an arbitrary way. Denote the particle
radius-vector coordinates by \( \xi_k; \eta_k \), respectively. Denote the length of the
vector with coordinates \( R_k = \eta_k - \xi_k \) by \( R \). Denote the derivative with respect
to \( ct \) either by point above the quantity (for example, \( \dot{\xi}_k \)) or as subscript “0”
separated with comma (for example, \( \xi_k,0 \equiv \ddot{\xi}_k \)).

The masses of the first and second particles are denoted by \( \tilde{M}, \tilde{m} \), respec-
tively. Quantities
\[ M = \frac{GM}{c^2}; \quad m = \frac{G\dot{m}}{c^2} \]

are equal to halves of Schwarzschild radii of the particles. Here \( G \) is the gravitational constant, \( c \) is light speed. In what follows quantities \( M, m \) will be referred to as masses, although they have the length dimension. Write the axial vectors of the intrinsic angular momentums of the particles in form

\[ \tilde{M}cS_k; \quad \tilde{m}cs_k. \]

The \( S_k; s_k \) have the length dimension; we will call them the reduced vectors of intrinsic angular momentums or simply the spins.

Equations (2) in the \( \lambda^k \)-order are denoted as \([00; \lambda^2], [0k; \lambda^3], [mn; \lambda^4], \) etc. Hereinafter the coordinate conditions are denoted as \([c.c.; 0; \lambda^3], [c.c.; k; \lambda^4], \) etc.

### 3. Smallness parameters

In the EIH method the principal smallness parameter is

\[ \lambda = \frac{v}{c} \]

where \( v \) is the characteristic relative particle velocity. Alongside the parameter \( \lambda \), the system under discussion has four more dimensionless and a priori independent parameters:

\[ u \equiv \frac{M}{r}; \quad v \equiv \frac{S}{r}; \quad m/r; \quad s/r. \]

With large enough values of the radial variable \( r \) these parameters become small. However, the relative relationships among them are arbitrary in principle as they are independent of \( r \) and depend on the particle characteristics. In this paper the consideration will be under the following assumptions:

\[ \frac{M}{r} \sim \lambda^2; \quad \frac{m}{r} \sim \lambda^2, \]

\[ \frac{S}{r} \sim \lambda; \quad \frac{s}{r} \sim \lambda. \]
We will not dwell on elucidation of the physical meaning of these assumptions; these elucidations are presented in many papers and are of standard nature. Note only that in many real situations assumptions (7), (8) are satisfied not in full measure. Nevertheless, the dynamic equations derived under assumptions (7), (8) can be used as a source of information about the motion of spin particles in these cases as well.

4. Arrangement of orders of smallness

Certain assumptions as to the arrangement of the orders of smallness of some terms appearing in functions $\gamma_{00}$, $\gamma_{0k}$, $\gamma_{mn}$ should be made. The lowest-order approximation term construction is determined uniquely by the assortment of parameters (6). The orders of smallness of the other terms are taken so that the EIH procedure be self-consistent. Our consideration will use the following arrangements of the orders of smallness:

$$
\begin{align*}
\gamma_{00} &= \gamma_{00}^0 + \gamma_{00}^4 + \gamma_{00}^5 + \ldots, \\
\gamma_{0k} &= \gamma_{0k}^3 + \gamma_{0k}^5 + \gamma_{0k}^6 + \ldots, \\
\gamma_{mn} &= \gamma_{mn}^4 + \gamma_{mn}^5 + \gamma_{mn}^6 + \ldots
\end{align*}
$$

(9)

Typically, in the EIH method, the expansions of functions $\gamma_{00}$, $\gamma_{0k}$, $\gamma_{mn}$ are composed of terms of the same parity in $\lambda$. In our case the results of [20] are taken into consideration, from which it follows that the expansions of functions $\gamma_{00}, \gamma_{0k}, \gamma_{mn}$ should contain terms of all orders of smallness, beginning with $\gamma_{00}^4, \gamma_{0k}^5, \gamma_{mn}^6$, respectively.

Recall some “techniques” used in the EIH method.

- In the EIH method, in the differentiation of any function in $x^0$ its order of smallness increases by one as against the one that the function had before the differentiation. In the differentiation with respect to spatial coordinates $x^k$ the function order of smallness is preserved.
- All the expressions must be symmetric about the replacement of the first particle by the second and the second by the first, that is about simultaneous replacements:

$$
\begin{align*}
M \rightarrow m; & \quad m \rightarrow M; \quad S \rightarrow s; \quad s \rightarrow S; \\
\xi_k \rightarrow \eta_k; & \quad \eta_k \rightarrow \xi_k; \quad R_k \rightarrow -R_k.
\end{align*}
$$

(10)
• With vanishing parameters $m, s$ the expressions derived with the EIH method should be the same as the associated Kerr solution expansions written for the particle with parameters $M, S$.

• If inequalities (7), (8) hold for the first particle and the parameters of the second particle satisfy inequalities

$$\frac{m}{r} \sim \lambda^4; \quad \frac{s}{r} \sim \lambda^3;$$

then in the PN approximation the second particle can be considered as a trial particle. In this case the presence of the second particle in the system impacts in no way on the dynamic equations for the first particle; as for the second particle, it moves in the field of gravity generated by the first particle like in the external field. In so doing the second particle is a trial spin particle.

Input equations (2) are written in each order of approximation in the form of one equation $00$, three equations $0^k$, and six equations $mn$. The consistency of arrangement of orders of smallness (9) with equations (2) shows up in the fact that written-out equation chain $[00; \lambda^{2k}], [0^k; \lambda^{2k+1}], [mn; \lambda^{2k+2}]$ allows us to sequentially determine $\gamma^{00}_{2k}, \gamma^{0k}_{2k+1}, \gamma^{mn}_{2k+2}$. It is therewith assumed that coordinate conditions consistent with this procedure are used.

5. Explicit form of the Kerr solution expansion

Let us present the explicit form of the principal terms of the expansions of $\gamma_{\alpha\beta}$ in smallness parameters $u = m/r, \ v = a/r$, where $a \equiv \sqrt{(s_k s_k)}$. The expansions are borrowed from ref. [20]. In the expressions for $\gamma_{00}, \gamma_{0k}, \gamma_{mn}$ we restrict ourselves to writing-out only those terms that contain multipliers given by $u, u^2, vu, vu^2, v^2u, v^3u$.

In the Kerr solution, the tensor of intrinsic angular momentum reduced to unit mass, $s_{mn}$, has as few as one nonzero component $s_{12}$, whose value coincides with $a$. Thus,

$$s_{12} = a; \quad s_{23} = s_{31} = 0; \quad s_1 = s_2 = 0; \quad s_3 = a. \quad (12)$$

Axial momentum vector $s_k$ is related to tensor $s_{mn}$ by the standard relation
\[ s_{mn} = \varepsilon_{mnc}s_c; \quad s_c = \frac{1}{2}\varepsilon_{cab}s_{ab}; \quad s_3 = a. \] (13)

From (12), (13) it follows that the following relations take place:

\[ 1 = \frac{(s_c s_c)}{a^2}. \] (14)

\[ \cos^2 \theta = \frac{(s_a x_a)(s_b x_b)}{a^2 r^2}. \] (15)

The expression for \( \gamma_{00} \) in Cartesian coordinates is given by:

\[
\gamma_{00} = 4u + u^2 - 6\frac{s_a s_b}{a^2}\left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right) \cdot v^2 \cdot u + ... \\
= 4\frac{m}{r} + \frac{m^2}{r^2} - 6\frac{m}{r} s_a s_b \left(\frac{x_a x_b}{r^2} - \frac{1}{3}\delta_{ab}\right) + ... . \] (16)

The expressions for \( \gamma_{01}, \gamma_{02}, \gamma_{03} \) in Cartesian coordinates are given by:

\[
\gamma_{0k} = \frac{2m(s_k x_a)}{r^3} - 2\frac{m^2(s_k x_a)}{r^4} - 5\frac{m(s_k x_a)(s_k x_a)}{r^3} + \frac{m(s_k x_a)}{r^4}(s_a s_a) + ... . \] (17)

The expressions for \( \gamma_{11}, \gamma_{22}, \gamma_{33}, \gamma_{12}, \gamma_{13}, \gamma_{23} \) in Cartesian coordinates are given by:

\[
\gamma_{mn} = \frac{m^2 x_m x_n}{r^4} - 2\frac{m^2}{r^2} \delta_{mn} + \frac{m^2}{r^5} \left[(s_m x_c)x_n + (s_n x_c)x_m\right] + ... . \] (18)

With vanishing \( s_{mn} \) expressions (16) - (18) transfer to Schwarzschild solution expansions.

6. Construction of the Einstein equation solution corresponding to two Kerr particles

6.1. An approach to using the EIH method in the problem of two Kerr particles

The approach differs basically in nothing from the one used for two particles having nonzero mass, but possessing no intrinsic angular momentums (Schwarzschild particles). The problem specificity consists in the following:
First, in all orders of smallness functions $\gamma_{00}$, $\gamma_{0k}$, $\gamma_{mn}$ split into two parts,

\[
\begin{align*}
\gamma_{00} &= \hat{\gamma}_{00} + \bar{\gamma}_{00} \\
\gamma_{0k} &= \hat{\gamma}_{0k} + \bar{\gamma}_{0k} \\
\gamma_{mn} &= \hat{\gamma}_{mn} + \bar{\gamma}_{mn}
\end{align*}
\]  \hspace{1cm} (19)

The first parts (with circumflex accent) are solutions to the associated problem for two Schwarzschild particles. The second parts (with bar) are additions that owe their origin to the particle intrinsic momentums. Splitting (19) can be performed always without loss of generality.

Second, each of the parts of function (19) is expanded in smallness parameters so that they coincide with the exact Kerr solution expansions in the two smallness parameters used in the EIH procedure. (Recall that in the problem of two Schwarzschild particles one smallness parameter was used.) The following arrangement of orders of smallness will be used according to exact solution expansions (9):

\[
\begin{align*}
\gamma_{00} &= \left[ \hat{\gamma}_{00} + \frac{\hat{\gamma}}{4} + \frac{\hat{\gamma}}{6} + \ldots \right] + \left[ \bar{\gamma}_{00} + \frac{\bar{\gamma}}{4} + \frac{\bar{\gamma}}{6} + \ldots \right] \\
\gamma_{0k} &= \left[ \hat{\gamma}_{0k} + \frac{\hat{\gamma}}{5} + \frac{\hat{\gamma}}{6} + \ldots \right] + \left[ \bar{\gamma}_{0k} + \frac{\bar{\gamma}}{5} + \frac{\bar{\gamma}}{6} + \ldots \right] \\
\gamma_{mn} &= \left[ \hat{\gamma}_{mn} + \frac{\hat{\gamma}}{4} + \frac{\hat{\gamma}}{5} + \frac{\hat{\gamma}}{6} + \ldots \right] + \left[ \bar{\gamma}_{mn} + \frac{\bar{\gamma}}{4} + \frac{\bar{\gamma}}{5} + \frac{\bar{\gamma}}{6} + \ldots \right]
\end{align*}
\]  \hspace{1cm} (20)

In the expansion of $\gamma_{mn}$ for the exact solution the odd order appears in the $\lambda^5$ approximation.

Third, the exact solutions are written with the same coordinate conditions as in the EIH procedure. In our consideration these will be the harmonic coordinate conditions.

Fourth, functions (20) are substituted both into the dynamic equations and coordinate conditions and the dynamic equations and coordinate conditions, which the second parts of the functions (with circumflex accent) have to satisfy, are determined.

Fifth, the resultant equations and conditions are solved. In so doing the conditions of solvability of the dynamic equations and coordinate conditions are satisfied as well.
6.2. Determination of $\gamma_{0k}$

The EIH procedures can be applied essentially in their standard form until $\gamma_{0k}$ has to be determined. We will not write out the associated expressions as they are cumbersome. These expressions along with the details of their derivation will be presented in a preprint to be published.

In this section we only present the explicit expression for $\gamma_{0k}$. The function $\gamma_{0k}$ is determined from equation $[0k; \lambda^5]$ and coordinate condition $[c.c.; 0k; \lambda^5]$. The calculation of surface integrals from the coordinate condition leads to the following solvability conditions:

$$M = \frac{1}{2} M \left( \dot{\xi}_i \dot{\xi}_j \right) - \frac{1}{2} \frac{m M}{R} ; \quad m = \frac{1}{2} m \left( \dot{\eta}_i \dot{\eta}_j \right) - \frac{1}{2} \frac{m M}{R} .$$

As a result we arrive at:

$$\frac{\hat{\gamma}_0}{k} = \frac{\gamma_{0k}}{k} + 8 \frac{m M \dot{\eta}_i}{R r_1} - 4 \frac{m M \dot{\xi}_i}{R r_1} - 4 \frac{M (\dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{r_1} + 2 \frac{M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{r_1^2}$$

$$- \frac{M (X_i \dot{\xi}_i)}{r_1^2} + 2 \frac{M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{r_1^2} - 2 \frac{m M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{R^3 r_1}$$

$$- 6 \frac{m M (R_i \dot{\eta}_i)}{R^3 r_1} + 8 \frac{m M (X_i \dot{\eta}_i \dot{\eta}_k \dot{\eta}_k)}{R^3 r_1} + \frac{8 m M (X_i \dot{\eta}_i \dot{\eta}_k \dot{\eta}_k)}{R^3 r_1} .$$

$$\frac{\gamma_{0k}}{k} = \frac{2 m \left( S_{ka} X_a \right)}{r_1^2} - 5 \frac{M(\dot{S}_{ka} X_a) \langle S_{ka} X_a \rangle (S_{ka} X_a)}{r_1^2} + \frac{M(\dot{S}_{ka} X_a)}{r_1^2} \langle S_a S_a \rangle$$

$$+ 6 \frac{M(S_{ka} X_a) \langle X_i \dot{\xi}_i \rangle S_{ka}}{R^3 r_1} - 2 \frac{M(S_{ka} X_a) \langle X_i \dot{\xi}_i \rangle S_{ka}}{r_1^2} + 2 \frac{M(S_{ka} X_a) \langle X_i \dot{\xi}_i \rangle S_{ka}}{r_1^2}$$

$$- 2 \frac{M(\dot{S}_{ka} X_a)}{R^3 r_1} - 3 \frac{M(\dot{S}_{ka} X_a)}{R^3 r_1} - 3 \frac{M(\dot{S}_{ka} X_a)}{R^3 r_1}$$

$$+ 4 \frac{m M (S_{ka} R_i)}{R^3 r_1} + 4 \frac{m M (S_{ka} R_i)}{R^3 r_1} + 4 \frac{m M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{R^3 r_1}$$

$$+ 4 \frac{m M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{R^3 r_1} + 12 \frac{m M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k) R_i}{R^3 r_1} + 6 \frac{m M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k) R_i}{R^3 r_1}$$

$$+ 6 \frac{M(X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k) R_i}{R^3 r_1} + 4 \frac{m M (S_{ka} X_a) \langle S_{ka} X_a \rangle}{R^3 r_1} - 6 \frac{m M (X_i \dot{\xi}_i \dot{\xi}_k \dot{\xi}_k)}{R^3 r_1} .$$

7. A method for derivation of the dynamic equations in the PN approximation

The above expression for $\gamma_{0k}$ is needed to derive the dynamic equations determining the motion of centers of masses of particles as well as equations
determining time derivatives of spins, i.e. $\dot{S}_{3\,mn}$ and $\dot{s}_{3\,mn}$.

The expression for $\gamma_{0k}$ involves not $\dot{S}_{3\,mn}$, $\dot{s}_{3\,mn}$, but $S_{3\,mn}$, $s_{3\,mn}$. The equations for $\dot{S}_{3\,mn}$ and $\dot{s}_{3\,mn}$ appear after $\gamma_{0k}$ has been substituted into the coordinate condition relating $\gamma_{0k,0}$ and $\gamma_{k,s,s}$, and this condition is used to derive $\gamma_{mn}$. But we need not determine the complete expression for $\gamma_{mn}$, it will suffice to only determine the anti-symmetric part of $\gamma_{mn}$ and require that it vanish. It is this condition that will give the dynamic equations for the spin evolution.

The equations of translational motion of particles are derived as before through integration of equation $[mn; \lambda^0]$. A new point is that, first, a different decomposition of the terms of this equation to the curl combination and the part denoted by $\sum_{i=1}^{18} \alpha_i$ (a different amount of $\alpha_i$ and a different explicit form of them) takes place. Second, in the integration of $\sum_{i=1}^{18} \alpha_i$ it should be kept in mind that the contributions are made not only by Schwarzschild parts $\bar{\gamma}_{\mu
u}$, but also by Kerr parts $\bar{\gamma}_{\mu
u}$. The strategy of derivation of the equations of translational motion of particles consists in determination of $\dot{\gamma}_{\mu
u}$, $\bar{\gamma}_{\mu
u}$, which make a contribution to the integral of $[mn; \lambda^0]$, and in calculation of the integral. Note that the contribution of $\gamma_{mn}$ to the integral is zero, so for this purpose $\gamma_{mn}$ need not be determined in the explicit form.

The resultant dynamic equations determining the motion of center of masses of particles coincide with those presented in many papers, e.g., [1]. Therefore, we do not write them out. Note only that the laws of conservation of energy and particle momentum vector follow from these equations.

8. The dynamic equations for spins

The resultant equations determining the spin dynamics that have been derived by the above method are given by:

$$
\dot{S}_{3\,c} = 9\frac{m}{R^3} \left( R_l \dot{\xi}_l \right) S_c + 2\frac{m}{R^3} \left( R_l \dot{\eta}_l \right) S_c - 2\frac{m}{R^3} \left( S_l \dot{\xi}_l \right) R_c. \tag{24}
$$

$$
\dot{s}_{3\,k} = -9\frac{M}{R^3} \left( R_l \dot{\eta}_l \right) s_k - 2\frac{M}{R^3} \left( R_l \dot{\xi}_l \right) s_k + 2\frac{M}{R^3} \left( s_l \dot{\eta}_l \right) R_k. \tag{25}
$$
9. Time variation in total momentum

The substitution of the resultant dynamic equations for spins into the equations for time derivative of total momentum of the particle system leads to the following expression for $\dot{M}_c$:

$$
\dot{M}_c = +\frac{mM}{R_c^3} \left\{ -2 \left( S_i \dot{\xi}_l \right) R_c + 9 \left( R_l \dot{\xi}_l \right) S_c + 2 \left( R_l \dot{\eta}_l \right) S_c \right\} \\
-\frac{mM}{R_c^3} \left\{ -2 \left( s_l \dot{\eta}_l \right) R_c + 9 \left( R_l \dot{\eta}_l \right) s_c + 2 \left( R_l \dot{\eta}_l \right) s_c \right\} \\
+\varepsilon_{cmn} \left\{ -6 \frac{mM(s_m R_l)(R_l \dot{R}_l)}{R_c^5} R_n - 4 \frac{mM(s_m \dot{R}_l)}{R_c^3} R_n + 3 \frac{mM(s_m \ddot{R}_l)}{R_c^3} R_n \right\} \\
+\varepsilon_{cmn} \left\{ -6 \frac{mM(S_m R_l)(R_l \dot{R}_l)}{R_c^5} R_n - 3 \frac{mM(S_m \dot{R}_l)}{R_c^3} R_n + 4 \frac{mM(S_m \ddot{R}_l)}{R_c^3} R_n \right\} \\
-3 \frac{mM}{R_c^3} \left( (S_l + s_l) R_l \right) \left( S_{cl} + s_{cl} \right) R_l
$$

(26)

The analysis of the structure of the right-hand side of the equation for $\dot{M}_c$ suggests that no re-determination of the total angular momentum of the particle system can get the law of conservation of momentum to be obeyed in the PN approximation.

For a spin particle system located in plane-asymptotic background space it may be possible to construct a conserved quantity similar to the total angular momentum. As it follows from our consideration, however, the quantity will not reduce definitely to the sum of orbital moments of the particles and their intrinsic angular momentums.

The obtained result of nonconservation of the total angular momentum of the particle system in the PN approximation appears unusual, however, this result seems not to contradict the first principles of general relativity. Most probably, similar effects will appear with respect to energy and momentum as well at some approximation step.

10. Numerical estimations for the gyroscopes on satellite GPB

For the numerical estimations we will use the input data for gyroscopes that are presented in ref. [3]. According to that paper, the gyroscopes were spheres of radius $r_g = 1.9cm$ and mass $m = 75g$. The initial angular velocity was $\omega = 27 000 \text{ rad/s}$. The satellite was launched onto a nearly ideal polar
orbit of radius $R = 7027 \text{ km}$. The orbital velocity of the gyroscopes is $v = 7.5 \cdot 10^5 \text{ cm/s}$. Ratio $v/c$ is $v/c = 2.5 \cdot 10^{-5}$. The ratio of half the Schwarzschild radius of the Earth to the gyroscope orbit radius is

$$\frac{M}{R} = 6.5 \cdot 10^{-10}. \tag{27}$$

The gyroscope axes were in the satellite orbit plane.

The equation used in ref. [3] to estimate the gyroscope axis of revolution deviation is derived in that paper itself using Landau-Lifshitz energy-momentum pseudo-tensor [23]. It is given by:

$$\dot{s}_k = \frac{1}{5} \frac{M}{R^3} \left\{ - \left( s_l \dot{R}_l \right) R_k - 19 \left( s_l \dot{R}_l \right) \dot{R}_k + 16 s_k \left( R_l \dot{R}_l \right) \right\}. \tag{28}$$

Superpose the satellite orbit plane to plane $(x, y)$ and write the change in radius-vector in form

$$R_1 = R \cos (\Omega t); \quad R_2 = R \sin (\Omega t); \quad \Omega = \frac{2\pi}{T}. \tag{29}$$

From these conditions it follows that

$$\dot{s}_1 = -R \Omega \sin (\Omega t); \quad \dot{s}_2 = R \Omega \cos (\Omega t). \tag{30}$$

At the initial time the gyroscope axis of revolution orientation coincides with axis $x$,

$$s_{1(0)} = s_0; \quad s_{21(0)} = 0. \tag{31}$$

From Anderson’s equation it follows that the contribution to the time derivatives from the spin is made by the two first terms in the right-hand side of (28). Upon averaging of these terms over the orbit for the time of one revolution we obtain that

$$\frac{\dot{s}_1}{s_0} T = 0; \quad \frac{\dot{s}_2}{s_0} T = \frac{M}{R}. \tag{32}$$

Thus, in one revolution around the Earth the gyroscope axes gets displaced by angle $\Delta \theta$ of

$$\Delta \theta = 2 \frac{M}{R} = 1.27 \cdot 10^{-9}. \tag{33}$$
In its functioning the satellite GPB had to perform about 5000 revolutions, hence, the gyroscope deviation angles of the order of $6 \cdot 10^{-6}$ radian had to be measured in the experiment.

From equations (24), (25) for spins that were derived with the method of this paper it follows that, first, the deviation must be 2 times as small and, second, the gyroscope axes deviate to the side opposite to the one following from equations (28).

11. Conclusions

As far as we know, the procedure of expansion of the Kerr solution written in the harmonic coordinates had been performed by nobody before our paper [20], hence, had not been used for normalization of expressions derived with the EIH method. This entitles us to suggest that the particle spin motion results presented elsewhere should be validated through comparison with the Kerr solution expansions in Section 5.

Another test for correctness of the equations for spins that are presented in the literature is the absence of any anti-symmetric part in the expression for $\gamma_{mn}$, which can appear from coordinate condition \[c.c.; \lambda^6\].

Now as for the amount of the gyroscope axis of revolution deviation on satellite GP-B. From the consideration presented in Section 10 it follows that our equations (24), (25) give $\Delta \theta = -M/R$ for the deviation (per revolution around the Earth), while equations (28) give $\Delta \theta = +2M/R$. We guess that the general relativity predicts the effect that agrees with our equations (24), (25). It is extremely interesting to learn the results of the forthcoming gyroscope experiment on satellite Gravity Probe B!

From the obtained results it follows that the case in point should be not only the effects of the deviation from motion along the geodesic, precession and axis of revolution orientation variation, but also the change in lengths of intrinsic momentums, i.e. values of angular velocity of rotation. Gyroscope rotation acceleration/deceleration on different segments of the world line could be noted in the laboratory without orbiting as well. To do this sufficient statistics should be compiled regarding the amount of revolutions made by the gyroscope about its axis for one and the same time on segments, where the spin projection onto the radius-vector is positive and negative. The comparison of the revolution amount on these two segments can serve a test for the general relativity validation.
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