WELL-POSEDNESS RESULTS FOR FRACTIONAL SEMI-LINEAR
WAVE EQUATIONS

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Dedicated to Prof. Juan J. Nieto on the occasion of his 60th birthday

Abstract. This work is concerned with well-posedness results for nonlocal
semi-linear wave equations involving the fractional Laplacian and fractional
derivative operator taken in the sense of Caputo. Representations for solutions,
existence of classical solutions, and some $L^p$-estimates are derived, by consid-
ering a quasi-stationary elliptic problem that comes from the realisation of the
fractional Laplacian as the Dirichlet-to-Neumann map for a non-uniformly el-
liptic problem posed on a semi-infinite cylinder. We derive some properties
such as existence of global weak solutions of the extended semi-linear integro-
differential equations.

1. Introduction. In this paper we study well-posedness results for nonlocal semi-
linear integro-differential wave equations which involve both the fractional Laplacian
(in space) and the Caputo fractional derivative operator (in time), as follows:

\[
\begin{aligned}
\mathbb{D}_t^\alpha u + (-\Delta)^s u &= f(u), \quad u = u(x,t), \quad (x, t) \in \mathbb{R}^d \times (0, \infty), \\
u(0) &= \varphi, \quad \varphi = \varphi(x) \quad x \in \mathbb{R}^d, \\
u_t(0) &= \psi, \quad \psi = \psi(x) \quad x \in \mathbb{R}^d, 
\end{aligned}
\]

(1)

where $s \in (0, 1)$, $\alpha \in (1, 2]$, the integer $d > 2s$, and $f \in \mathcal{C}^1(\mathbb{R})$. We fix the variables
$t$, $x$ to be in the spaces $t > 0$, $x \in \mathbb{R}^d$, and we consider nonnegative solutions $u$ of (1).

The integro-partial differential equation (1) interpolates between the fractional
heat equation ($\alpha = 1$) and the fractional wave equation ($\alpha = 2$). The problem
(1) has been intensively studied by several authors (see e.g., [28, 29, 42, 3] and

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references therein), in the case where \( f \) is linear. Some recent results on global existence in time in the classical case \((s = 1 \text{ and } \alpha = 2)\) have been obtained in [44].

The interest in these nonlocal models lies in the fact that some nonlocal phenomena cannot be captured by the classical theory of partial differential equations and yet fit well into nonlocal frameworks. Phenomena with nonlocality in the form of memory effects include, for example, anomalous diffusion and wave propagations, and the modelling of neuronal transmission in Purkinje cells, whose malfunctioning is known to be related to the lack of voluntary coordination and the appearance of tremors [16]. The paper [1] covers many of these applications and motivation on the usefulness of nonlocal operators.

In the literature, it is well known that the linear analysis of fractional PDEs might provide the mathematical tools needed to study semi-linear or nonlinear equations. For the semi-linear problem, a theory of mild solutions for the classical semi-linear diffusion and wave equations with a Lipschitz nonlinearity is developed in [23, 25], where the authors provided existence and regularity results for semi-linear wave equations with a polynomial-like nonlinearity for the Laplacian in the whole space domain. The general well-posedness theory for linear non-homogeneous equations associated with (1) has been studied in [22, 28, 29, 45, 32, 4] and references therein. In [30, 32], a theory of integral solutions has been developed for the semi-linear problem (1) when \( s = 1 \). In [13, 32], the authors consider the Laplacian operator and find a “critical” exponent in order to deduce the global existence of integral solutions for small data in low space dimensions. There is also a similar analysis but in a more general setting in [4].

At this point it is important to clarify the scope of the present work and its relevance within the well-posedness context. We shall adopt the strategy presented in [28, 32, 4] to derive existence, uniqueness, behaviour of the fundamental solution of (1) and some associated estimates. These results are obtained by assuming that the nonlinearity of \( f \in C^1(\mathbb{R}) \) can be controlled by the following assumptions.

Firstly, that

\[
  f(0) = 0 \quad \text{and} \quad |f'(\tau)| \leq C(1 + |\tau|^{\sigma}) \quad \forall \tau \in \mathbb{R}, \sigma > \frac{1}{\alpha - 1} > 1, C > 0. \quad (H_1)
\]

Secondly, that there exists \( \epsilon > 0 \) such that, for the two monotone increasing real-valued functions \( g_1, g_2 \geq 0 \) defined by

\[
  g_1(\zeta) = \sup_{0 \leq |\tau| \leq \zeta} |f'(\tau)| + \epsilon \zeta \quad \text{and} \quad g_2(\zeta) = \sup_{0 \leq |\tau| \leq \zeta} |f(\tau)| + \epsilon \zeta, \quad \zeta \geq 0,
\]

we have

\[
  f(0) = 0 \quad \text{and} \quad |f'(\tau)| \leq g_1(|\tau|), \quad |f(u)| \leq g_2(|\tau|), \quad \forall \tau \in \mathbb{R}. \quad (H_2)
\]

Furthermore, we also study global weak solutions and regularity properties for the solution of the problem (1), by using the Caffarelli-Sylvester extension technique.

Indeed, problem (1) being fully nonlocal, applying local PDE techniques to treat this type of semilinear problem should be done delicately. The difficulty in the analysis of (1) to derive further of its properties is due to its nonlocality in both the space and time variables involved [11, 31, 34, 46, 47]. As a consequence, some new techniques as compared to classical analysis might be considered to overcome the nonlocality introduced by this fractional operator, giving rise to some limitations in the fractional derivative order as well as fractional Laplacian. This difficulty has been overcome for the fractional Laplacian in [11] (respectively for the fractional derivative operator in [7, 9]) by localizing the respective operator: for the fractional
Laplacian, this involves a Dirichlet-to-Neumann map for an extension problem to the upper half-space \( \mathbb{R}^{d+1}_+ \), which corresponds to a nonuniformly elliptic PDE. This result was later extended in \([10, 48]\) to bounded domains \( \Omega \subset \mathbb{R}^d \) and to more general operators, thereby obtaining an extension problem posed on \( C := \Omega \times (0, \infty) \). See also \([12, 17, 43, 41]\) and references therein for further applications and reading.

In this work, we consider the extension technique in the sense of Caffarelli and Sylvestre \([11]\) in order to rewrite the problem (1) as the following quasi-stationary elliptic problem with a dynamic boundary condition \([52, 43, 41, 5, 14, 15]\):

\[
\begin{cases}
- \text{div} \left( y^{1-2s} \nabla U \right) = 0 \text{ in } C \times (0, T), \\
U = 0 \text{ on } \partial L \times (0, T), \\
c_s \partial_t^s \U + \partial_{y}^{-2s} U = c_s f(U) \text{ on } (\Omega \times \{0\}) \times (0, T),
\end{cases}
\]

(2)

to be solved for the function \( U(x, y, t) \) on \( \Omega \times (0, \infty) \times (0, T) \) with the initial conditions

\[
U = \varphi \text{ on } \Omega \times \{0\}, \quad t = 0, \quad \partial_t U = \psi \text{ on } \Omega \times \{0\}, \quad t = 0,
\]

(3)

where \( \partial L = \partial \Omega \times [0, \infty) \) corresponds to the lateral boundary of \( C \), \( c_s = 2^{1-2s} \Gamma(1-s)/\Gamma(s) \) is a constant, and the conormal exterior derivative of \( U \) at \( \Omega \times \{0\} \) is

\[
\partial_{y}^{-2s} U = - \lim_{y \to 0^+} y^{1-2s} U_y = c_s (-\Delta)^s u,
\]

(4)

with the limit being understood in the sense of distributions \([11, 12, 48]\). We also denote by \( y \) the extended variable so that (2)–(3) with mixed boundary conditions can be thought as the harmonic extension of \( u \) into \( 2 - 2\alpha \) extra dimensions \([11]\). With the solution \( U \) to the extension problem (2)–(3) at hand, one can find the solution to (1) through setting \([10, 11, 12, 48, 43]\):

\[
u = U|_{y=0},
\]

by applying some local techniques of classical PDEs. We note that the solvability of the problem (1) and its extended version (2)–(3) – in the linear framework, i.e., \( f \) linear – has been well presented and analyzed in \([43]\), where the authors proved existence and uniqueness results and derived regularity estimates both in time and space in weighted Sobolev spaces.

Our main contribution below (Theorem 2.11) establishes global weak solutions and regularity properties of (2)–(3) with respect to the extended variable \( y \in (0, \infty) \). The result states that if \( \varphi \) is in the domain of the fractional operator \((-\Delta)^{s^*}\) for \( s^* := \frac{1}{\alpha} \), \( \psi \in L^2(\Omega) \) and \( f \) satisfies suitable assumptions given by \((H_1)\) and \((H_2)\) below, then (2)–(3) has a unique weak solution on \((0, T^*)\) for some \( T^* > 0 \). In our analysis, the critical value \( s^* \) will play here an essential role for finding classes of unique weak solutions that satisfy a certain energy identity of the solutions.

The motivation for deriving such global weak solutions and regularity results is due to the fact that some studies of a numerical scheme to approximate the solution (1) through the resolution of problem (2) must be concerned with the regularity of its solution. This was already observed in \([43]\) for the well-posedness of the linear fractional wave equation.

This work is organised as follows. In Section 2, we introduce some preliminary results, and the main results of this paper are stated. In Section 3, we establish the existence of non negative solution for problem (1) and recall the known result on the solution representation for (2)–(3) in the linear setting. Section 4 is devoted to the proof of some basic properties of the fundamental solutions related to (1) in the
linear setting, such as providing \( L^p - L^r \) estimates. Finally, in Section 5 we prove the main results of this paper presented in 2.2.

2. Preliminaries and main results.

2.1. Notations and definitions. First of all, let us fix our notation and terminology for the spaces and operators which we shall use throughout this paper.

We will use some standard notations taken from [19]. Let \( X \) be a real Banach space, endowed with norm \( \| \cdot \| \). The symbol \( L^p(0,T;X) \) will denote the Banach space of all measurable functions \( u : [0,T] \rightarrow X \) such that:

\[
\| u \|_{L^p(0,T;X)} = \left\{ \left( \int_0^T \| u(t) \|^p \right)^{1/p} \right\} < \infty
\]

The symbol \( C_b(I;X) \) denotes the space of bounded continuous functions defined on an interval \( I \) and taking values in the Banach space \( X \), equipped with the norm \( \sup_{t \in I} \| \cdot \|_X \).

We will use some standard notations taken from [19]. Let \( \widehat{\phi}, \psi \in \mathcal{S}(\mathbb{R}^d) \) be the Fourier transform of \( \phi, \psi \). For any \( \phi, \psi \in \mathcal{S}(\mathbb{R}^d) \), we have the following inequality for a convolution on the space \( \mathcal{S}'(\mathbb{R}^d) \):

\[
\| \hat{g} \ast \hat{h} \|_{L^p(\mathbb{R}^d)} \leq \| g \|_{L^p(\mathbb{R}^d)} \| h \|_{L^r(\mathbb{R}^d)}, \quad g \in L^p(\mathbb{R}^d), \quad h \in L^r(\mathbb{R}^d).
\]

Let \( \mathcal{S}'(\mathbb{R}^d) \) be the set of all tempered distributions, that is the topological dual of \( \mathcal{S}(\mathbb{R}^d) \) (space of \textbf{Schwartz functions}) as in [21]. For any \( \varphi \in \mathcal{S}(\mathbb{R}^d) \), the function \( \widehat{\varphi}(\xi), \xi \in \mathbb{R}^d \), denotes the Fourier transform of \( \varphi \) and \( \mathcal{F}^{-1}(\varphi) \) its inverse (see A).

**Definition 2.1** (Sobolev space – first definition). For any real number \( s \) and any natural number \( d \), the \( s \)th Sobolev space on \( \mathbb{R}^d \) is defined to be

\[
H^s(\mathbb{R}^d) := \left\{ \phi \in \mathcal{S}'(\mathbb{R}^d) : \widehat{\phi} \in L^2_{\text{loc}}(\mathbb{R}^d), \| \phi \|_{H^s} < \infty \right\},
\]

where the Sobolev norm \( \| \cdot \|_{H^s} \) is defined by

\[
\| \phi \|_{H^s} := \left( \int_{\mathbb{R}^d} |\widehat{\phi}(\xi)|^2 \left( 1 + |\xi|^2 \right)^s \, d\xi \right)^{1/2}.
\]
\textbf{Definition 2.2} (Sobolev space – second definition). For any \( s \in (0, 1) \) and \( d > 2s \), the \( s \)th fractional Sobolev space on \( \mathbb{R}^d \) is defined to be

\[
H^s(\mathbb{R}^d) = \left\{ \phi \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi(x) - \phi(z)|^2}{|x-z|^{d+2s}} \, dx \, dz < \infty \right\},
\]

endowed with the norm

\[
\|\phi\|_s = \left( \int_{\mathbb{R}^d} |\phi(x)|^2 \, dx + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\phi(x) - \phi(z)|^2}{|x-z|^{d+2s}} \, dx \, dz \right)^{1/2}.
\]

\textbf{Remark 1.} The equivalence of Definition 2.1 and Definition 2.2 is stated and proven in [40].

We also recall the spectral definition for the fractional Sobolev space [8, 43]. Let \( \lambda, \phi \in \mathbb{R} \times H^1_0(\Omega) \setminus \{0\} \) be respectively the eigenvalue and eigenfunction associated to the eigenvalue problem

\[
(-\Delta)\phi = \lambda\phi \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega. \tag{9}
\]

The above eigenvalue problem (9) has a countable collection of solutions \( \{\lambda_k, \phi_k\}_{k \in \mathbb{N}} \) such that \( \{\phi_k\}_{k=1}^\infty \) is an orthonormal basis of \( L^2(\Omega) \). We define, for \( s \geq 0 \), the space

\[
\mathbb{H}^s(\Omega) = \left\{ v = \sum_{k=1}^\infty v_k\phi_k : \|v\|_{\mathbb{H}^s(\Omega)}^2 := \sum_{k=1}^\infty \lambda_k^s |v_k|^2 < \infty \right\}, \tag{10}
\]

and denote by \( \mathbb{H}^{-s}(\Omega) \) the dual space of \( \mathbb{H}^s(\Omega) \). Throughout this paper, the duality pairing between the aforementioned spaces will be denoted by \( \langle \cdot, \cdot \rangle \).

Furthermore, notice that, if \( s \in (0, \frac{1}{2}) \), we have \( \mathbb{H}^s(\Omega) = H^s(\Omega) = H^s_0(\Omega) \), while, for \( s \in (\frac{1}{2}, 1) \), \( \mathbb{H}^s(\Omega) \) can be characterized by [35, 38, 43]

\[
\mathbb{H}^s(\Omega) = \left\{ v \in H^s(\Omega) : v = 0 \text{ on } \partial\Omega \right\}.
\]

If \( s = \frac{1}{2} \), we have that \( \mathbb{H}^{\frac{1}{2}}(\Omega) \) is the so-called Lions–Magenes space \( H^{\frac{1}{2}}_{00}(\Omega) \) [35, 49, 43].

Next we recall some properties of the weighted Sobolev spaces, borrowed from [41, 43, 17].

\textbf{Definition 2.3} (see [41, 43, 17]). Given \( s \in (0, 1) \), a bounded open set \( \Omega \subset \mathbb{R}^d \), and the associated semi-infinite cylinder \( C := \Omega \times (0, \infty) \), we denote

\[
L^2(\gamma^{1-2s}; C) := \left\{ U : C \to \mathbb{R}, \int_C y^{1-2s} |U|^2 \, dt \, dx < +\infty \right\},
\]

endowed with the norm

\[
\|U\|_{L^2(\gamma^{1-2s}; C)} := \left( \int_C y^{1-2s} |U|^2 \, dt \, dx \right)^{1/2}.
\]

We also denote

\[
H^1(\gamma^{1-2s}; C) := \left\{ U \in L^2(\gamma^{1-2s}; C) : \nabla U \in L^2(\gamma^{1-2s}; C) \right\},
\]

with the induced norm

\[
\|U\|_{H^1(\gamma^{1-2s}; C)} := \left( \int_C y^{1-2s} (|U|^2 + |\nabla U|^2) \, dt \, dx \right)^{1/2}.
\]
Both extensions, the one by Caffarelli and Silvestre [11] and the ones in [10, 12, 48] for \( \Omega \) bounded and general elliptic operators, require us to deal with a local but non-uniformly elliptic problem. We define the weighted Sobolev space

\[
\tilde{H}^1(y^\alpha, C) = \left\{ v \in H^1(y^{1-2s}, C) : v = 0 \text{ on } \partial \mathcal{C} \right\}.
\]

Since \( 1-2s \in (-1, 1) \), \( |y|^{1-2s} \) belongs to the Muckenhoupt class \( A_2 \) [39, 50], and thus \( H^1(y^{1-2s}, C) \) is a Hilbert space and \( C^\infty(\Omega) \cap H^1(y^{1-2s}, C) \) is dense in \( H^1(|y|^{1-2s}, C) \) (cf. [50, Proposition 2.1.2]). It has been shown that the following weighted Poincaré inequality holds:

\[
\|u\|_{L^2(y^\alpha, C)} \leq \|\nabla v\|_{L^2(y^\alpha, C)} \quad \forall v \in \tilde{H}^1(y^\alpha, C).
\]

Next we recall that for \( v \in H^1(y^{1-2s}, C) \), \( \text{tr}_\Omega v \) denotes its trace onto \( \Omega \times \{0\} \), and from [42, Prop. 2.5],

\[
\text{tr}_\Omega \tilde{H}^1(y^{1-2s}, C) = \mathbb{H}^s(\Omega), \quad \|\text{tr}_\Omega v\|_{\mathbb{H}_s(\Omega)} \leq \|v\|_{H^1(y^{1-2s}, C)}.
\]

We recall the definition of the fractional Laplacian operator of order \( s \in (0, 1) \) via the Fourier approach as follows.

**Definition 2.4.** For any \( u \in \mathcal{S}(\mathbb{R}^d) \) and for \( s \in (0, 1) \), the fractional Laplacian operator \( (-\Delta)^s : \mathcal{S} \to L^2(\mathbb{R}^d) \) is defined by

\[
(-\Delta)^s u(x) = \mathcal{F}^{-1}(|\xi|^{2s}\hat{u}(\xi)),
\]

or equivalently in term of its singular integral representation

\[
(-\Delta)^s u(x) = c(d, s)\text{P.V.} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{|x-y|^{d+2s}} \, dy,
\]

where P.V. stands for the Cauchy principal value and \( c \) is a constant depending only on \( d \) and \( s \).

In the above definition, one could notice that the fractional Laplacian operator \( (-\Delta)^s \) can be viewed as a pseudo-differential operator of symbol \( |\xi|^{2s} \). The equivalence relation between these two formulations is standard and it can be found in many papers (see, for instance, [40, 20, 51] or [49, Chapter 16]).

Furthermore, in the spectral approach, the fractional Laplacian \( (-\Delta)^s \) is defined by:

\[
(-\Delta)^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega), \quad (-\Delta)^s v := \sum_{k=1}^{\infty} \lambda_k^s v_k \phi_k, \quad s \in (0, 1),
\]

where \( (\lambda_k, \phi_k) \in \mathbb{R} \times H^1_0(\Omega) \setminus \{0\} \) are solutions to the eigenvalue problem (9) discussed above, and we used the notation \( v_k = (v, \phi_k) \) for the scalar product of \( v \) and \( \phi_k \).

Finally we recall the definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative respectively.

**Definition 2.5** (Riemann-Liouville fractional integral). The Riemann-Liouville fractional integral \( J_0^\alpha \) of order \( \alpha \geq 0 \) is defined for \( \alpha = 0 \) as \( J_0^0 := I \), where \( I \) denotes the identity operator, and for \( \alpha > 0 \) as

\[
J_0^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) \, d\tau \quad \text{for all } t \in (0, T).
\]
Definition 2.6 (Caputo fractional derivative). The Caputo fractional derivative $D_t^\alpha$ of order $\alpha$ is defined by

$$D_t^\alpha u(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-1-\alpha} u^{(n)}(\tau) \, d\tau, \quad \text{for all } t \in (0, T),$$

where $n - 1 \leq \alpha < n$, $n \in \mathbb{N}$, and $u^{(n)}$ denotes the derivative of order $n$ of $u$.

When $\alpha = 1$ and $\alpha = 2$, we get from (17) that $D_t^1 = \partial_t$ and $D_t^2 = \partial_t^2$ respectively. Also, it follows directly from (17) and (16) that

$$D_t^\alpha u(t) = J_n^{\alpha} \left( u^{(n)} \right)(t).$$

Definition 2.7. Let $1 < \alpha \leq 2$ and $0 < s \leq 1$. Suppose $\phi, \psi \in C(\mathbb{R}^d)$ and $f \in C([0, \infty) \times \mathbb{R}^d)$ linear. Then a function $u \in C([0, \infty) \times \mathbb{R}^d)$ is a classical solution of the Cauchy problem

$$
\begin{cases}
D_t^\alpha u(t, x) + (\Delta)^s u(t, x) = f, & \text{in } (0, \infty) \times \mathbb{R}^d, \\
u(0, x) = \varphi(x) & \text{and } u'(0, x) = \psi(x), & \text{in } \mathbb{R}^d,
\end{cases}
$$

if the following conditions are all valid:

(i) $F^{-1}(|\xi|^{2s}\hat{u}(\xi))$ defines a continuous function of $x$ for each $t > 0$;

(ii) for every $x \in \mathbb{R}^d$, the fractional integral $J^{1-\alpha}u$, as defined in (16), is continuously differentiable with respect to $t > 0$;

(iii) the function $u(t, x)$ satisfies the integro-partial differential equation (19) for every $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and the initial condition (19) for every $x \in \mathbb{R}^d$.

2.2. Main results. Our first theorem states that a classical solution of (19) has an integral representation. Since we defined the fractional Laplacian via the Fourier transform, we need to guarantee that the inverse Fourier transform $F^{-1}(|\xi|^{2s}\hat{u}(\xi))$ determines a continuous function in $x$. By the Riemann-Lebesgue lemma, this is true if $|\cdot|^{2s}\hat{u}(t, \cdot) \in L^1(\mathbb{R}^d)$. Furthermore, we also need that $u(\cdot, x)$ is a continuous function up to 0 for all $x \in \mathbb{R}^d$. For these purposes, in the case that the forcing term $f$ is a linear function, following [28] we impose the condition

$$|\hat{f}(t, \xi)| \leq C|\hat{g}_2(\xi)|,$$

where the function $\hat{g}_2$ satisfies

$$(1 + |\cdot|^{2s})\hat{g}_2(\cdot) \in L^1(\mathbb{R}^d),$$

and $C > 0$ is a constant which is uniform in time. Hence the representation theorem reads as follows (and it is proved in section 3 below).

Theorem 2.8. Let $\varphi, \psi \in L^1(\mathbb{R}^d)$ be such that $\hat{\varphi}, \hat{\psi} \in L^1(\mathbb{R}^d)$, and let $f$ be a function satisfying $f(t, \cdot) \in L^1(\mathbb{R}^d)$ for all $t \geq 0$ and the condition (20) with $g$ satisfying (21). Define

$$Z(t, x) = (\pi)^{-d/2} \left|x\right|^{-d} H_{2,3}^{\alpha} \left[ 2^{-2s} |x|^{2s} t^{-\alpha} \right] \left[ (1, 1), (2, 0) \right] \left[ (d/2, s), (1, 1), (1, s) \right],$$

$$W(t, x) = (\pi)^{-d/2} \left|x\right|^{-d} H_{2,3}^{\alpha} \left[ 2^{-2s} |x|^{2s} t^{-\alpha} \right] \left[ (1, 1), (2, 0) \right] \left[ (d/2, s), (1, 1), (1, s) \right],$$

$$Y(t, x) = \pi^{-d/2} \left|x\right|^{-d} t^{\alpha-1} H_{2,3}^{\alpha} \left[ 2^{-2s} t^{-\alpha} |x|^{2s} \right] \left[ (1, 1), (\alpha, 0) \right] \left[ (d/2, s), (1, 1), (1, s) \right].$$
where $H_{2,3}^{2,1}$ is the Fox $H$-function defined in $C$. Then the function

$$u(t, x) = \int_{\mathbb{R}^d} Z(t, x - y) \varphi(y) \, dy + \int_{\mathbb{R}^d} W(t, x - y) \psi(y) \, dy$$

$$+ \int_0^t \int_{\mathbb{R}^d} Y(t - \tau, x - y) f(\tau, y) \, dy \, ds$$

is a classical solution to the problem (19).

The following existence-uniqueness result is our first main result.

**Theorem 2.9.** Let $1 < \alpha < 2$, $0 < s < 1$ and $\sigma > 0$. Let also $1 \leq p_0 = \alpha \sigma d/2s \leq p$. Assume that $f(u)$ satisfies condition (H$_3$). Then, for $\varepsilon > 0$ and for any small initial data $(\varphi, \psi)$ in the space $A$ defined in (7) which satisfy $\|(\varphi, \psi)\|_A \leq \varepsilon$, there exists a maximal time $T^* > 0$ and a function $u$ which satisfies the following conditions.

(i) $u \in C((0, T^*); L^p(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d))$ satisfies the integro-differential equation (1), and this solution is uniquely determined in the class

$$\left\{ u \in C((0, T^*); L^p(\mathbb{R}^d)) : \sup_{t \in (0, T^*)} t^{1/q} \|u(t)\|_{L^r} < \infty \right\},$$

for any $r \geq p$ and $q$ satisfying (5).

(ii) For any $r > p$, the solution $u$ for integro-differential equation (1) satisfies

$$t^{\frac{1}{q}} \|u(t)\|_{L^r} \to 0, \quad \text{as} \quad t \to 0.$$

(25)

(iii) If $T^* < \infty$, then for any $p \leq r \leq \infty$, we have

$$\lim_{t \to T^*} \|u(t)\|_{L^r} = \infty,$$

and furthermore

$$\|u\|_{L^r} \geq C (T^* - t)^{\alpha + (1 + \frac{\alpha}{2s}) \frac{p}{q}}.$$

**Definition 2.10** (Global weak solution for the extended problem). Let $s^* = \frac{1}{\alpha} \in (\frac{1}{2}, 1)$ and let $\alpha \in (1, \alpha_\star)$, for $\alpha_\star = 4s/d$ and $2s < d < 4s$. A function $U$ is said to be a weak solution of problem (2)–(3) on $(0, T)$, for some $T > 0$, if for initial values $\text{tr}_\Omega U(\cdot, 0, 0) = \varphi$, $\partial_t \text{tr}_\Omega U(\cdot, 0, 0) = \psi$ and for almost every $t \in (0, T)$,

$$\begin{cases}
U \in C([0, T]; \mathbb{H}^{s_\star}(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \\
\text{tr}_\Omega \mathcal{D}^s U \in C([0, T]; \mathbb{H}^{-s_\star}(\Omega)),
\end{cases}$$

(27)

and

$$\lim_{t \to 0^+} \|\text{tr}_\Omega U(\cdot, 0, 0) - \varphi\|_{\mathbb{H}^{s^*_\star}} = 0, \quad \lim_{t \to 0^+} \|\partial_t \text{tr}_\Omega U(\cdot, 0, 0) - \psi\|_{\mathbb{H}^{-s^*_\star}} = 0,$$

(28)

for some

$$\frac{1}{\alpha} > s^*_\star > 0 \quad \text{and} \quad \frac{\alpha - 1}{\alpha} \geq s^*_2 > 0.$$

$$(\text{tr}_\Omega \mathcal{D}_s^\alpha U, \text{tr}_\Omega \phi) + E(U, \phi) = (f(U), \text{tr}_\Omega \phi) \quad \forall \phi \in \mathring{H}^1(y^{1-2s}, C),$$

(29)

where $(\cdot, \cdot)$ denotes the the duality pairing between $\mathbb{H}^{-s_\star}(\Omega)$ and $\mathbb{H}^{s_\star}(\Omega)$ and we define the bilinear form $E : \mathring{H}^1(y^{1-2s}, C) \times \mathring{H}^1(y^{1-2s}, C) \to \mathbb{R}$

$$E(U, \phi) := \frac{1}{c_s} \int_{\mathbb{C}} y^{1-2s} (\nabla U \cdot \nabla \phi) \, dx.$$
Theorem 2.11. Let the solution is given by global weak solution. If the above properties hold for any $T > 0$, then we say that $U$ is an extended global weak solution.

The following well-posedness result is our second main result.

**Theorem 2.11.** Let $\alpha \in (1, \alpha_*)$, for $\alpha_* = 4s/d$ and $2s < d < 4s$. Let $s^* \in (1/2, 1)$, $\varphi \in H^{s^*}(\Omega)$ and $\psi \in L^2(\Omega)$, and suppose that $f$ satisfies (H2). Then there is a time $T^* > 0$ (depending only on $\varphi$ and $\psi$) such that the extended problem (2)--(3) has a unique weak solution on $(0, T^*)$ in the sense of Definition 2.10. Furthermore, the solution is given by

$$U(x, y, t) = \sum_{k=1}^{\infty} \varphi_k E_{\alpha,1}(-\lambda_k^s t^\alpha)\phi_k(x)\vartheta_k(y) + \sum_{k=1}^{\infty} \psi_k t E_{\alpha,2}(-\lambda_k^s t^\alpha)\phi_k(x)\vartheta_k(y) + \sum_{k=1}^{\infty} \left(\int_0^t f_{k,\mathcal{U}}(y, \tau)(t - \tau)^{\alpha - 1} E_{\alpha,\alpha}(-\lambda_k^s (t - \tau)^\alpha) \, d\tau\right) \phi_k(x)\vartheta_k(y), \quad (31)$$

where the eigenpairs $(\lambda_k, \phi_k)$ are defined as above (9) and the function $\vartheta_k$ is defined by (44) below. We adopt the notation $\varphi_k = (\varphi, \phi_k)_{L^2(\Omega)}$ and $\psi_k = (\psi, \phi_k)_{L^2(\Omega)}$, the scalar products of $\varphi$ and $\psi$ respectively with $\phi_k(x)$, and similarly $f_{k,\mathcal{U}}(y, t) = f\left(\mathcal{U}(\cdot, y, t), \phi_k\right)_{L^2(\Omega)}$.

3. Representation formula for the integro-differential equation. In this section, we establish the existence of a non-negative solution to the problem (1) in the linear setting given in Theorem 2.8, which justifies calling fundamental solutions for the equation (1) and also for the semilinear problem in the extended form (2)--(3) with mixed boundary condition. In connection with these results, we note the existence of some previous papers on fundamental solutions for fractional partial differential equations, such as [6], in which some similar techniques were used.

3.1. Solution representation formula for problem (1). In this case, we consider the function $f$ to be linear and adopt the method of Laplace-Fourier transforms to provide the solution representation of (1). The proof follows the strategy of [28, 18].

**Proof of Theorem 2.8.** We divide the proof into three steps as there are three requirements in the definition of the classical solution 2.7.

**Step I.** Let $1 < \alpha \leq 2$ and suppose $u$ satisfies (1). By taking the Fourier transform of (1) we get

$$\left\{\begin{array}{l}
\mathcal{D}_t^\alpha \widehat{u}(t, \xi) + |\xi|^{2s} \widehat{u}(t, \xi) = \widehat{f}(t, \xi), \quad \widehat{u}(0) = \widehat{\varphi}, \\
\widehat{\vartheta}(0) = \widehat{\psi},
\end{array}\right. \quad (t, \xi) \in (0, \infty) \times \mathbb{R}^d,$$

where $\xi \in \mathbb{R}^d$ is the dual variable of $x$. We rewrite the differential equation (32) as:

$$\widehat{u}(t, \xi) = |\xi|^{2s} (J^\alpha_t - 1) * \widehat{u}(t, \xi) + t\widehat{\psi}(\xi) + \int (J^\alpha_t - 1) * \widehat{f}(t, \xi), \quad t > 0, \quad (33)$$

Suppose also that $(J^\alpha_t + f)(t)$ is exponentially bounded. By taking the Laplace transform on both sides of (33) and assuming that $\{\lambda^\alpha : \Re(\lambda) > |\xi|^{1/2s}\} \subset \text{dom}((-\Delta)^\alpha)$, we have

$$\widehat{u}(\lambda, \xi) = \lambda^\alpha - (\lambda^\alpha + |\xi|^{2s})^{-1} \varphi(\xi) + \lambda^{\alpha - 2}(\lambda^\alpha + |\xi|^{2s})^{-1} \psi(\xi) + (\lambda^\alpha + |\xi|^{2s})^{-1} \widehat{f}(\lambda, \xi), \quad (34)$$
where $\lambda$ is the dual variable of $t$ and $\Re(\lambda) > |\xi|^{1/2s}$, and where $\tilde{v}$ denotes the Laplace transform of $v$.

To invert (34), it is convenient to first invert the Laplace transform. From the properties of the Mittag-Leffler transform given in (77)–(78), we deduce that

$$\int_0^\infty e^{-\lambda t} E_{\alpha,1} \left( -|\xi|^2 s t^\alpha \right) dt = \lambda^{\alpha-1} \left( \lambda^\alpha + |\xi|^2 s \right)^{-1}, \quad \Re(\lambda) > |\xi|^{1/2s};$$

$$\int_0^\infty e^{-\lambda t} t E_{\alpha,2} \left( -|\xi|^2 s t^\alpha \right) dt = \lambda^{\alpha-2} \left( \lambda^\alpha + |\xi|^2 s \right)^{-1}, \quad \Re(\lambda) > |\xi|^{1/2s} ;$$

$$\int_0^\infty e^{-\lambda t} t^{\alpha-1} E_{\alpha,\alpha} \left( -|\xi|^2 s t^\alpha \right) dt = \left( \lambda^\alpha + |\xi|^2 s \right)^{-1}, \quad \Re(\lambda) > |\xi|^{1/2s} .$$

Therefore, to obtain the fundamental solution, it is needed to find integrable functions $Z(x,t), W(x,t), Y(x,t) \in L^1(\mathbb{R}^d)$ satisfying respectively

$$\hat{Z}(\xi,t) = E_{\alpha,1} \left( -|\xi|^2 s t^\alpha \right),$$

$$\hat{W}(\xi,t) = t E_{\alpha,2} \left( -|\xi|^2 s t^\alpha \right),$$

$$\hat{Y}(\xi,t) = t^{\alpha-1} E_{\alpha,\alpha} \left( -|\xi|^2 s t^\alpha \right).$$

Notice that to deal with the asymptotic behaviour of (38), it is convenient to express the Mittag-Leffler function $E_{\alpha}(z)$ in terms of the Fox $H$-function. Using equations (75)–(76) from B, we can express each term in (35)–(37) as:

$$\hat{Z}(t,\xi) := E_{\alpha,1} \left( -|\xi|^2 s t^\alpha \right) = H^{1,1}_{1,2} \left[ |\xi|^2 s t^\alpha \right]_{(0,1), (1,0), (0,\alpha)}^{(0,1), (0,1), (\alpha,0)},$$

$$\hat{W}(t,\xi) := t E_{\alpha,2} \left( -|\xi|^2 s t^\alpha \right) = t E_{\alpha,2} \left( -|\xi|^2 s t^\alpha \right) = t H^{1,1}_{1,2} \left[ |\xi|^2 s t^\alpha \right]_{(0,1), (1,0), (-1,\alpha)}^{(0,1), (0,1), (1-\alpha,0)},$$

$$\hat{Y}(t,\xi) := t^{\alpha-1} E_{\alpha,\alpha} \left( -|\xi|^2 s t^\alpha \right) = t^{\alpha-1} H^{1,1}_{1,2} \left[ |\xi|^2 s t^\alpha \right]_{(0,1), (0,1), (\alpha,0)}^{(0,1), (1-\alpha,0), \alpha}.$$

Replacing each term above in (34), we get

$$\hat{u}(t,\xi) = \hat{Z} \ast \hat{\varphi}(t,\xi) + \hat{W} \ast \hat{\psi}(t,\xi) + \hat{Y} \ast \hat{f}(t,\xi).$$

**Step II.** This step is dedicated to the inversion of the Fourier transform for $\hat{u}(t,\xi)$.

Notice that the function $\hat{u}(t,\xi)$ is a radial function of $\xi$. So to be able to get the inverse of $\hat{u}(t,\xi)$, we need first of all to prove that $\mathcal{F}^{-1}(|\xi|^2 s \hat{u}(t,\xi))$ is a continuous function with respect to $x$ for each $t > 0$. This strategy has been applied in [28]. Indeed, from the asymptotic behaviour of the Mittag-Leffler function given by (79), we obtain

$$|\hat{Z}(t,\xi)| \leq \frac{C}{1 + |\xi|^2 s t^\alpha}, \quad t > 0.$$
So \(|\xi|^{2s} \hat{W}(t, \xi)|^2\) is bounded for all \(t > 0\) and by using the assumption that \(\hat{\psi} \in L^1\) we obtain
\[
|\xi|^{2s} \mathcal{F}_{x \to \xi} (W \ast \psi)(t, \cdot) = |\xi|^{2s} \hat{W}(t, \xi) \hat{\psi}(\xi) \in L^1(\mathbb{R}^d). \tag{40}
\]

Next we discuss the estimation of the function \(Y\). From the asymptotic estimate [28]
\[
\hat{Y}(t, \xi) \sim \frac{t^{\alpha-1}}{1 + |\xi|^{4s+2\alpha}}, \tag{41}
\]
and the assumption (20), we obtain
\[|\hat{f}(\tau, \xi)| \leq |g_2(\xi)|\]
with \(|\xi|^{2s} g_2(\xi) \in L^1(\mathbb{R}^d)\). Combining the above estimate with (41), we obtain
\[
|\xi|^{2s} \mathcal{F}(Y \ast f)(t, \xi) = |\xi|^{2s} \int_0^t \hat{Y}(t - \tau, \cdot) \hat{f}(\tau, \cdot) d\tau
\]
\[
\leq |\xi|^{2s} |g_2(\xi)| \int_0^t \frac{(t - \tau)^{\alpha-1}}{1 + (t - \tau)^{2s+2\alpha}} d\tau,
\]
which establishes that \(|\cdot|^{2s} \mathcal{F}(Y \ast f)(t, \cdot) \in L^1\). This together with (39) and (40) gives, again by the Riemann-Lebesgue lemma, that \(\mathcal{F}^{-1}(|\xi|^{2s} \hat{u}(t, \xi))\) is a continuous function, as required. From this continuity property, for \(\varphi, \psi \in S(\mathbb{R}^d)\) and for \(f \in L^1((0, \infty); \mathcal{S}(\mathbb{R}^d))\), we can write (38) in the form
\[
u(t, x) = \int_{\mathbb{R}^d} Z(t, x - \xi) \varphi(\xi) d\xi + \int_{\mathbb{R}^d} W(t, x - \xi) \psi(\xi) d\xi d\tau
\]
\[+ \int_0^t \int_{\mathbb{R}^d} Y(t, x - \xi) f(\tau, \xi) d\xi d\tau, \tag{42}\]
where
\[
Z(t, x) = \mathcal{F}^{-1} \left( E_{\alpha, 1}(-|\xi|^{2s} t^\alpha) \right),
\]
\[
W(t, x) = \mathcal{F}^{-1} \left( t E_{\alpha, 2}(-|\xi|^{2s} t^\alpha) \right),
\]
\[
Y(t, x) = \mathcal{F}^{-1} t^{\alpha-1} \left( E_{\alpha, \alpha}(-|\xi|^{2s} t^\alpha) \right).
\]

We proceed to calculate each of these inverse Laplace transforms in turn.

**Computation of \(\mathcal{F}^{-1}(|\xi|^{2s} \hat{Z}(t, \xi))(x)\).**

Notice that the function \(Z\) is a radial function of \(\xi\), and for radial functions we have from (74) in A that
\[
\mathcal{F}^{-1}(|\xi|^{2s} \hat{Z}(t, \xi)) = (2\pi)^{-\frac{d}{2}} |x|^\frac{2-d}{2} \int_0^\infty \xi^\frac{d}{2} J_{d-2}(|x|\xi) E_{\alpha, 1}(-|\xi|^{2s} t^\alpha) (\xi, t) d\xi,
\]
where \(J_{(d-2)/2}\) is the modified Bessel function (see [53]). So
\[
Z(t, x) = (2\pi)^{-\frac{d}{2}} |x|^\frac{2-d}{2} \int_0^\infty \xi^\frac{d}{2} J_{d-2}(|x|\xi) H_{1, 1}^{(1), (0, 1), (0, \alpha)} \left( |\xi|^{2s} t^\alpha \right) d\xi
\]
\[
= (2\pi)^{-\frac{d}{2}} |x|^{1-d} \int_0^\infty (|x|\xi)^\frac{d}{2} J_{d-2}(|x|\xi) H_{1, 1}^{(1), (0, 1), (0, \alpha)} \left( |\xi|^{2s} t^\alpha \right) d\xi.
\]

We now use the properties (iv) and (v) for the Fox \(H\)-function given in [28, Lemma 2.14] or in [33, Section 3] to get
\[ Z(x, t) = (\pi)^{-d/2} |x|^{-d} H_{3,2}^{1,2} \left[ 2^{2s} |x|^{-2s} t^\alpha \right] \begin{pmatrix} (1-d/2, s), (0, 1), (0, s) \\ (0, 1) \end{pmatrix} \]
\[ = (\pi)^{-d/2} |x|^{-d} H_{3,2}^{0,2} \left[ 2^{2s} |x|^{-2s} t^\alpha \right] \begin{pmatrix} (1-d/2, s), (0, 0) \\ (0, 0) \end{pmatrix} \]
\[ = (\pi)^{-d/2} |x|^{-d} H_{3,2}^{2,0} \left[ -2^{-2s} |x|^{2s} t^{-\alpha} \right] \begin{pmatrix} (1, \alpha) \\ (d/2, s) \end{pmatrix} \]
\[ = (\pi)^{-d/2} |x|^{-d} H_{3,2}^{1,1} \left[ -2^{-2s} t^{-\alpha} |x|^{2s} \right] \begin{pmatrix} (1, 1) \\ (d/2, s) \end{pmatrix} \]

**Computation of \( \mathcal{F}^{-1}(|\xi|^{2s} \hat{W}(t, \xi))(x) \).**

We proceed as we did in the case of \( Z \). The function \( W \) being a radial function of \( \xi \), we can compute the inverse Fourier transform as

\[ \mathcal{F}^{-1}(|\xi|^{2s} \hat{W}(t, \xi)) = (2\pi)^{-d/2} |x|^{-d} \int_0^\infty t^{d/2} J_{d/2-2} (|\xi|x) E_{\alpha, 2} (-|\xi|^{2s} t^\alpha)(\xi, t) d\xi, \]

so that

\[ W(t, x) = (2\pi)^{-d/2} |x|^{-d} \int_0^\infty (|\xi|x)^{d/2} J_{d/2-2} (|\xi|x) H_{1,2}^{1,1} \left[ |\xi|^{2s} t^\alpha \right] \begin{pmatrix} (0, 1) \\ (0, 1), (-1, \alpha) \end{pmatrix} d\xi \]
\[ = (2\pi)^{-d/2} |x|^{-d} \int_0^\infty (|\xi|x)^{d/2} J_{d/2-2} (|\xi|x) H_{1,2}^{1,1} \left[ |\xi|^{2s} t^\alpha \right] \begin{pmatrix} (0, 1) \\ (0, 1), (-1, \alpha) \end{pmatrix} d\xi. \]

We now use the property (iii) for the Fox \( H \)-function given in [28, Lemma 8.10] to obtain

\[ W(t, x) = (\pi)^{-d/2} |x|^{-d} H_{3,2}^{1,2} \left[ 2^{2s} t^{-\alpha} |x|^{-2s} \right] \begin{pmatrix} (1-d/2, s), (0, 1), (0, s) \\ (0, 1), (-1, \alpha) \end{pmatrix} \]
\[ = (\pi)^{-d/2} |x|^{-d} H_{3,2}^{1,1} \left[ 2^{-2s} |x|^{2s} t^{-\alpha} \right] \begin{pmatrix} (1, 1), (2, \alpha) \\ (d/2, s), (1, 1), (1, s) \end{pmatrix}. \]

**Computation of \( \mathcal{F}^{-1}(|\xi|^{2s} \hat{Y}(t, \xi))(x) \):**

Notice that \( \hat{Y} \) is a radial function of \( \xi \), and we calculate from the properties (v) and (vi) for Fox-\( H \) functions [28, Lemma 8.10] that:

\[ \mathcal{F}^{-1} \left( |\xi|^{2s} \hat{Y}(t, \xi) \right) \]
\[ = (2\pi)^{-d/2} |x|^{-d} \int_0^\infty |\xi|^{d/2+2s} J_{d/2-2} (|\xi|x) H_{1,2}^{1,1} \left[ |\xi|^{2s} t^\alpha \right] \begin{pmatrix} (0, 1) \\ (0, 1), (1, \alpha) \end{pmatrix} d\xi \]
\[ = \frac{2^{2s}}{\pi^{d/2}} |x|^{-d-2s} t^{-\alpha-1} H_{3,2}^{1,2} \left[ 2^{2s} |x|^{-2s} t^\alpha \right] \begin{pmatrix} (1-d/2-2s, s), (0, 1), (-s, \alpha) \\ (0, 1), (-1, \alpha) \end{pmatrix} \]
\[ = \frac{\pi^{-d/2}}{d/2} |x|^{-d-2s} t^{-\alpha-1} H_{3,2}^{1,1} \left[ 2^{2s} |x|^{-2s} t^\alpha \right] \begin{pmatrix} (1-d/2, s), (1, 1), (0, \alpha) \\ (0, 1), (0, \alpha) \end{pmatrix} \]
\[ = \pi^{-d/2} |x|^{-d-2s} t^{-\alpha-1} H_{3,2}^{1,1} \left[ 2^{2s} t^{-\alpha} |x|^{2s} \right] \begin{pmatrix} (1, 1) \\ (d/2, s) \end{pmatrix} \]

In order to finish the proof, we have to show that the function \( u \) is a jointly continuous function in \([0, \infty) \times \mathbb{R}^d \). This has been done in [28, Theorem 2.26 (Step III and Step IV)]. Indeed, following their argument, the continuity at \( t = 0 \) is established. Furthermore, for \( t > 0 \), the continuity in both variables follows from the conditions given for \( \phi, \psi \) and \( f \), which guarantees that \( \phi, \psi \) and \( f \) are continuous and uniformly bounded. Finally, the asymptotic behaviour of \( Z, W \) and \( Y \) together
with the Lebesgue dominated convergence theorem imply the continuity. Hence the desired result and the end of the proof. □

3.2. Solution representation formula for the extended semilinear problem. We briefly explain how to derive the solution representation for the extended problem (2)–(3). The strategy is based on the eigenpairs defined in Section 2 and the proof of the solution representation of the linear case, which was already obtained in [43].

Proceed assuming by separation of variables $U(x, y, t) = q(t)v(x, y)$. The equation (2) with respect to the time variable becomes

$$
\mathbb{D}_t^s q(t) + \lambda^s q(t) = f(q(t)), \quad t > 0, \quad U(x, 0, 0) := q(0) = \varphi, \quad q'(0) = \psi,
$$

and the solution has the form

$$
q(t) = q(0)E_{\alpha, 1}(-\lambda^s t^\alpha) + q'(0)tE_{\alpha, 2}(-\lambda^s t^\alpha)
+ \int_0^t (t - \tau)^{\alpha - 1}E_{\alpha, \alpha}(-\lambda^s(t - \tau)^\alpha)f_k(U(0, \tau)d\tau.
$$

The function $v(x, y)$ corresponds to the known eigenvalue-eigenfunction problem [12]. Applying $v(x, y)$ in (2), the only solution are for $\phi_k \partial_k(y)$ and $\lambda^s = \lambda^s_k > 0$ for some $k$, where $\phi_k$ are eigenfunctions of the fractional Laplacian and $\lambda^s_k$ are the corresponding eigenvalues. Then the general solution will be written as an infinite linear combination of all the possibilities:

$$
U(x, y, t) = \sum_{k=1}^{\infty} a_k E_{\alpha, 1}(-\lambda^s_k t^\alpha)\partial_k(y) + \sum_{k=1}^{\infty} b_k tE_{\alpha, 2}(-\lambda^s_k t^\alpha)\phi_k(x)\partial_k(y)
+ \sum_{k=1}^{\infty} \left( \int_0^t f_k(U(y, t)(t - \tau)^{\alpha - 1}E_{\alpha, \alpha}(-\lambda^s_k(t - \tau)^\alpha)d\tau \right) \phi_k(x)\partial_k(y). \quad (43)
$$

The functions $\partial_k(y)$ are solutions of the following system:

$$
\begin{cases}
\frac{d^2}{dy^2} \partial_k(y) + \frac{1 - 2s}{y} \frac{d}{dy} \partial_k(y) = \lambda^s_k \partial_k(y), & y \in (0, \infty), \\
\partial_k(0) = 1, \quad \lim_{y \to \infty} \partial_k(y) = 0. \quad (44)
\end{cases}
$$

If $s \in (0, 1) \setminus \{\frac{1}{2}\}$, then [12, Proposition 2.1], [43]

$$
\partial_k(y) = c_s \left( \sqrt{\lambda^s_k y} \right)^s K_s \left( \sqrt{\lambda^s_k y} \right), \quad (45)
$$

and if $s = \frac{1}{2}$, we thus have $\partial_k(y) = \exp(-\sqrt{\lambda^s_k y})$, where $c_s = 2^{1-s}/\Gamma(s)$ and $K_s$ denotes the modified Bessel function of the second kind. For a comprehensive treatment of the Bessel function $K_s$, we refer the reader to [2, Chapter 9.6].

The coefficients $a_k, b_k$ in (43) are found thanks to the initial value conditions for $t = 0, y = 0, U(x, 0, 0) = u(x, 0) = \varphi(x)$ and $U'(x, 0, 0) = u'(x, 0) = \psi(x)$. They are simply the linear combination with $a_k$ coefficients and the corresponding eigenfunctions (resp. $b_k$ coefficients and the corresponding eigenfunctions), which will be normalized to the unitary norm in $L^2(\Omega)$. So if $\varphi, \psi \in L^2(\Omega)$, we can express them as linear combinations of the eigenfunctions:

$$
\varphi(x) = \sum_{k=1}^{\infty} \varphi_k \phi_k(x), \quad \psi(x) = \sum_{k=1}^{\infty} \psi_k \phi_k(x), \quad ||\phi_k||_{L^2(\Omega)} = 1,
$$

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and adopt the notation \( \varphi_k = (\varphi, \phi_k)_{L^2(\Omega)}, \psi_k = (\psi, \phi_k)_{L^2(\Omega)} \) and 
\( f_k, \mathcal{U}(y, t) = f(\mathcal{U}(\cdot, t), \phi_k)_{L^2(\Omega)} \).

In all, the solution is
\[
\mathcal{U}(x, y, t) := \sum_{k \geq 1} u_k(t) \phi_k(x) \vartheta_k(y) 
= \sum_{k=1}^{\infty} \varphi_k E_{\alpha,1}(-\lambda_k^s t^\alpha) \phi_k(x) \vartheta_k(y) + \sum_{k=1}^{\infty} \psi_k t E_{\alpha,2}(-\lambda_k^s t^\alpha) \phi_k(x) \vartheta_k(y) 
+ \sum_{k=1}^{\infty} \left( \int_0^t f_k, \mathcal{U}(y, t)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^s (t-\tau)^\alpha) \, d\tau \right) \phi_k(x) \vartheta_k(y). \tag{46}
\]

4. Some \( L^p - L^r \) estimates of the fundamental solution for the problem (1).

4.1. Asymptotic behaviour of the fundamental solutions \( Z, W, \) and \( Y \). Before moving into providing the exact behaviour of the fundamental solutions \( Z, W, \) and \( Y \), we first consider their asymptotic behaviour as \( t \to 0 \), recalling the following three theorems stated and proved in [32].

**Theorem 4.1 ([32])**. Let \( 1 \leq \alpha \leq 2, 0 < s \leq 1 \). Denote \( R := |x|^{2s}t^{-\alpha} \). Then the function \( Z \) satisfies the following estimates.

(i) If \( R \geq 1 \), then
\[ Z(t, x) \leq t^\alpha |x|^{-d-2s} \]

(ii) If \( R \leq 1 \), then
\[ Z(t, x) \leq \begin{cases} 
  t^{-\alpha/d} & \text{if } d < 2s \\
  t^{-\alpha} \left( 1 + \ln \left( |x|^{2s}t^{-\alpha} \right) \right) & \text{if } 2s = d \\
  |x|^{-d+2s}t^{-\alpha} & \text{if } d > 2s.
\end{cases} \]

Furthermore,

(iii) If \( 1 < \alpha < 2 \), then for \( R \geq 1 \)
\[ Z(t, x) \leq |x|^{-d}. \]

(iv) If \( \alpha \in \mathbb{N} \), then for \( R \leq 1 \)
\[ Z(t, x) \leq \begin{cases} 
  t^{-\alpha/d} & \text{if } d < 4s \\
  t^{-2\alpha} \left( 1 + \log \left( |x|^{2s}t^{-\alpha} \right) \right) & \text{if } d = 4s \\
  t^{-2\alpha} |x|^{-d+4} & \text{if } d > 4s.
\end{cases} \]

(v) If \( s \in \mathbb{N} \), then there exists a constant \( c = c(\alpha, s) > 0 \) such that for \( R \geq 1 \),
\[ Z(t, x) \leq |x|^{-d} \exp \left\{ -c(\alpha, s) |x|^{2s} \frac{t^{-\alpha}}{|x|^{2s}} \right\}. \]

**Theorem 4.2 ([32])**. Let \( 1 \leq \alpha \leq 2, 0 < s \leq 1 \). Denote \( R := |x|^{2s}t^{-\alpha} \). Then the function \( W \) satisfies the following estimates.

(i) If \( R \geq 1 \), then
\[ W(t, x) \leq t|x|^{-d} \]

(ii) If \( R \leq 1 \), then
\[ W(t, x) \leq \frac{t^{\alpha/d}}{|x|^{d}} \begin{cases} 
  1 & \text{if } d < 2s \\
  \ln \left( |x|^{2s}t^{-\alpha} \right) & \text{if } 2s = d \\
  |x|^{d-2s}t^{-\alpha} & \text{if } d > 2s.
\end{cases} \]
(ii) If $R \leq 1$, then
\[
W(t, x) \leq \begin{cases} 
1 - \frac{d}{2} & \text{if } d < 2s \\
1 - \alpha \left( 1 + | \log \left( |x|^{2s} t^{-\alpha} \right) \right) & \text{if } d = 2s \\
1 - \alpha |x|^{-d+2s} & \text{if } d > 2s.
\end{cases}
\]

Furthermore,
(iii) If $\alpha - 1 \in \mathbb{N}$, then for $R \geq 1$
\[
W(t, x) \leq t^{\alpha+1} |x|^{-d-2s}
\]
(iv) If $\alpha - 1 \in \mathbb{N}$, then for $R \leq 1$
\[
W(t, x) \leq \begin{cases} 
1 - \frac{d}{2} & \text{if } d < 4s \\
1 - 2\alpha \left( 1 + | \log \left( |x|^{2s} t^{-\alpha} \right) \right) & \text{if } d = 4s \\
1 - 2\alpha |x|^{-d+4s} & \text{if } d > 4s.
\end{cases}
\]

(iv) If $s \in \mathbb{N}$, then there exists a constant $c = c(\alpha, s) > 0$ such that for $R \geq 1$,
\[
W(t, x) \leq t|x|^{-d} \exp \left\{-c(t^{-\alpha}|x|^{2s})^{\frac{1}{d-\alpha}} \right\}.
\]

**Theorem 4.3 ([32]).** Let $\alpha \in (0, 2)$, $s \in (0, 1)$. Denote $R := |x|^{2s} t^{-\alpha}$. Then the function $Y$ satisfies the following estimates.

(i) If $R \geq 1$, then
\[
Y(t, x) \leq t^{\alpha-1} |x|^{-d-2s}
\]

(ii) If $R \leq 1$, then
\[
Y(t, x) \leq \begin{cases} 
1 - \frac{d}{2} & \text{if } d < 4s \\
1 - \alpha \left( 1 + | \log \left( |x|^{2s} t^{-\alpha} \right) \right) & \text{if } d = 4s \\
|x|^{-d+2s t^{-\alpha}} & \text{if } d > 4s.
\end{cases}
\]

We note that these estimates have been studied by [28] in the case $\alpha \in (0, 1)$ and for a diffusion problem, but here, since we are working in the range $\alpha \in (0, 2)$, the asymptotic behaviour is different. Hence the motivation of taking into consideration the estimates derived in [32].

4.2. **$L^p - L^r$ estimates of the fundamental solutions $Z$, $W$ and $Y$.** When proving the decay estimates, we will need the following asymptotic estimates for the fundamental solutions $Z$, $W$, and $Y$. We begin by studying the function $Z$.

**Lemma 4.4 ($L^p$ estimate of $Z$).** Let $d \geq 1$, $1 < \alpha \leq 2$, $0 < s \leq 1$, and $1 \leq p < \Lambda_1(s, d)$ where
\[
\Lambda_1 = \Lambda_1(s, d) := \begin{cases} 
\frac{d}{d-2s} & \text{if } d > 2s, \\
\infty & \text{otherwise}.
\end{cases}
\]

Then $Z(t, \cdot) \in L^p(\mathbb{R}^d)$ for any $t > 0$ and
\[
\|Z(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{d}{2} \left( 1 - \frac{1}{d} \right)}, \quad t > 0.
\]
Moreover, if $1 = d \leq 2s$, then (48) holds for all $p \in [1, \infty]$. Finally, for $d > 2s$ and $1 < \alpha < 2$, we obtain
\[
\|Z(t, \cdot)\|_{L^p} \leq t^{-\alpha}, \quad t > 0.
\]
Proof. We follow the strategy of the proof of [28, Lemma 5.1]. We begin by decomposing the $L^p$-integral of $Z$ as
\[ \|Z(t,\cdot)\|_{L^p}^p = \int_{\{R \geq 1\}} |Z(t,x)|^p \, dx + \int_{\{R \leq 1\}} |Z(t,x)|^p \, dx. \]

In view of Lemma 4.1, we have for all dimensions $d$ and for all values $1 \leq p < \infty$ that
\[ \int_{\{R \geq 1\}} |Z(t,x)|^p \, dx \leq \int_{\{R \geq 1\}} t^\alpha |x|^{-dp-2sp} \, dx \leq \int_{t^{\frac{p}{2}}}^\infty t^\alpha p^{-dp-2sp} r^{d-1} \, dr \leq t^{-\frac{dp}{2}}(p-1), \]

thus
\[ \left( \int_{\{R \geq 1\}} |Z(t,x)|^p \, dx \right)^{\frac{1}{p}} \leq t^{-\frac{dp}{2}}(1-\frac{1}{p}) \quad \text{for all } 1 < p < \infty \text{ and } t > 0. \] (49)

Next, we estimate the integral where $R \leq 1$. In the case $\alpha = 1$ or $2s > d$ and $0 < \alpha < 1$, we have that for all $1 < p < \infty$
\[ \int_{\{R \leq 1\}} |Z(t,x)|^p \, dx \leq \int_{\{R \leq 1\}} t^{-\frac{dp}{2}} \, dx \leq \int_{0}^{t^{\frac{p}{2}}} t^{-\frac{dp}{2}} r^{d-1} \, dr \leq t^{-\frac{dp}{2}} \frac{d}{2}. \]

So that,
\[ \left( \int_{\{R \leq 1\}} |Z(t,x)|^p \, dx \right)^{\frac{1}{p}} \leq t^{-\frac{dp}{2}}(1-\frac{1}{p}) \quad \text{for all } 1 < p < \infty \text{ and } t > 0. \]

If $2s = d$ and $1 < \alpha < 2$ we estimate
\[ \int_{\{R \leq 1\}} |Z(t,x)|^p \, dx \leq \int_{\{R \leq 1\}} t^{-\alpha p}(|\log(|x|^{2s}t^{-\alpha})| + 1)^p \, dx \]
\[ \leq \int_{0}^{t^{\frac{p}{2}}} t^{-\alpha p} \left(|\log(r^{2s}t^{-\alpha})| + 1\right)^p r^{d-1} \, dr \]
\[ \leq \int_{0}^{1} t^{-\alpha p + d/2s} \left(|\log(z^{2s})| + 1\right)^p z^{d-1} \, dz \]
\[ \leq t^{-\alpha p + d/2s} = t^{-\frac{dp}{2}}(p-1), \]

for all $1 \leq p < \infty$.

Finally, if $0 < 2s < d$ and $1 < \alpha < 2$ we have
\[ \int_{\{R \leq 1\}} |Z(t,x)|^p \, dx \leq \int_{\{R \leq 1\}} t^{-\alpha p} |x|^{-dp+2sp} \, dx \leq \int_{0}^{t^{\frac{p}{2}}} t^{-\alpha p(-d+2s)p} r^{d-1} \, dr \]
\[ \leq t^{-\alpha p + d/2s(d-2s)p} \leq t^{-\frac{dp}{2}}(p-1), \]

whenever the last integral is finite, that is, whenever
\[ p < \frac{d}{d-2s} = \Lambda_1(s,d). \]

Hence combining the previous estimates we have that
\[ \left( \int_{\{R \leq 1\}} |Z(t,x)|^p \, dx \right)^{\frac{1}{p}} \leq t^{-\frac{dp}{2}}(1-\frac{1}{p}) \quad \text{for all } 1 \leq p < \Lambda(s,d) \text{ and } t > 0. \] (50)
Furthermore from Theorem 4.1 we have \( Z(t, \cdot) \in L^\infty(\mathbb{R}) \) for all \( t > 0 \), provided \( \alpha = 2 \) or \( 2s < d \), and moreover, we have the estimate
\[
\|Z(t, x)\|_{L^\infty} \leq t^{-\frac{d}{2s}},
\]
which proves the second statement.

For the weak-\( L^p \)-estimate we set \( p = \frac{d}{d-2s} \). We need to estimate
\[
\|Z(t, \cdot)\|_{L^p, \infty} = \sup \left\{ \zeta \, d_{Z(t,x)}(\zeta)^{\frac{1}{p}} : \zeta > 0 \right\},
\]
where
\[
d_{Z(t,x)}(\zeta) = |\{x \in \mathbb{R}^d : Z(t,x) > \zeta\}|
\]
denotes the distribution function of \( Z(t,x) \). The proof of this part follows line by line the one in [28, Lemma 5.1]. This finishes the proof.

\[\square\]

Lemma 4.5 \( (L^p \) estimate of \( W \)). Let \( d \geq 1 \), \( 1 < \alpha \leq 2 \), \( 0 < s \leq 1 \), and \( 1 \leq p < \Lambda_2(s,d) \) where
\[
\Lambda_2 = \Lambda_2(s,d) := \begin{cases} \frac{d}{2s} & \text{if } d > 2s, \\ \infty & \text{otherwise}. \end{cases}
\]

Then \( W(t, \cdot) \in L^p(\mathbb{R}^d) \) for any \( t > 0 \) and
\[
\|W(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq t^{1 - \frac{sd}{2p}(1 - \frac{1}{p})}, \quad t > 0.
\]

Moreover, if \( 1 = d \leq 2s \), then (52) holds for all \( p \in [1, \infty] \). Finally, for \( d > 2s \) and \( 1 < \alpha \leq 2 \), we obtain
\[
\|W(t, \cdot)\|_{L^\infty, \infty} \leq t^{-\alpha}, \quad t > 0.
\]

Proof. The proof is very similar to that of Lemma 4.4 for the function \( Z \). We omit the details here.

Next we state the following decay result.

Proposition 1. Let \( d \geq 1 \), \( 1 < \alpha \leq 2 \) and \( 0 < s \leq 1 \). Assume that \( u \) is the mild solution of equation (1) with \( f \equiv 0 \) and \( \varphi, \psi \in L^p(\mathbb{R}^d) \), where \( 1 \leq p \leq \infty \).

(i) We have
\[
\|u(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq \|\varphi\|_{L^p(\mathbb{R}^d)} + t\|\psi\|_{L^p(\mathbb{R}^d)}, \quad t > 0.
\]

(ii) If \( p = \infty \), then
\[
\|u(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)} + t\|\psi\|_{L^\infty(\mathbb{R}^d)}, \quad t > 0;
\]

(iii) If \( 1 \leq p < \infty \) and \( d > 4sp \), then for every \( r \in \left[p, \frac{pd}{d-4sp}\right) \), we have
\[
\|u(t, \cdot)\|_{L^r(\mathbb{R}^d)} \leq t^{-\frac{sd}{2r}(1 - \frac{1}{r})} \left(\|\varphi\|_{L^p(\mathbb{R}^d)} + t\|\psi\|_{L^p(\mathbb{R}^d)}\right), \quad t > 0.
\]

If in addition \( 1 < \alpha < 2 < d \), then
\[
\|u(t, \cdot)\|_{L^{\frac{pd}{d-4sp}}(\mathbb{R}^d)} \leq t^{-\alpha}, \quad t > 0,
\]
and specifically for \( q = 1 \),
\[
\|u(t, \cdot)\|_{L^{\frac{pd}{d-4sp}}(\mathbb{R}^d)} \leq t^{-\alpha}, \quad t > 0.
\]

(iv) If \( 1 \leq p < \infty \) and \( d = 2sp \), then the estimate (53) holds for every \( r \in [p, \infty) \).

(v) If \( d < 2sp \) or \( \alpha = 1 \), then the estimate (53) holds for every \( r \in [p, \infty] \).
Proof. The proof of these estimates are based on an application of the integral form of the Minkowski inequality, the Young inequality for convolutions, and the results of Lemmas 4.4 and 4.5. Indeed, by setting $f \equiv 0$, equation (42) becomes

$$u(t, x) = \int_{\mathbb{R}^d} Z(t, x - y) \varphi(y) dy + \int_{\mathbb{R}^d} W(t, x - y) \psi(y) dy dr.$$  \hspace{1cm} (54)

Next, for $p, q$ and $r$ defined via (5), we use the Cauchy-Schwarz inequality, Young’s inequality for convolutions, and Minkowski inequalities respectively to obtain

$$\|u(t, \cdot)\|_{L^r(\mathbb{R}^d)} = \|Z(t, \cdot) \ast \varphi(\cdot) + W(t, \cdot) \ast \psi(\cdot)\|_{L^r} \leq \|Z(t, \cdot)\|_{L^p} \|\varphi\|_{L^p} + \|W(t, \cdot)\|_{L^q} \|\psi\|_{L^q}.$$  \hspace{1cm} (55)

From Lemma 4.4 and Lemma 4.5 we have respectively

$$\|Z(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{qd}{d} - \frac{1}{2}}$$  \hspace{1cm} (56)

and for $1 = q < \Lambda_2$, where

$$\Lambda_2 = \begin{cases} \frac{d}{d - 2q} & \text{if } d > 4s, \\ \infty & \text{otherwise.} \end{cases}$$

Next, for $d > 4sq$, for $r \in [q, \frac{qd}{d-4sq}] \supset [q, \frac{qd}{d-2sq}]$ we get from (5) the corresponding $q \in [1, \frac{d}{d-4sq})$. Furthermore, if $d \leq 4sq$, the values of $q \in [1, \Lambda_2)$ give the corresponding choices of $r$ in $[q, \infty)$ and thus in both cases we may use (56) for the following estimate.

Hence we obtain

$$\|W(t, \cdot)\|_{L^p(\mathbb{R}^d)} \|\psi\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{qd}{d} - \frac{1}{2}} \|\psi\|_{L^p(\mathbb{R}^d)}.$$  \hspace{1cm} (57)

Putting (57) into (55) gives

$$\|u(t, \cdot)\|_{L^r(\mathbb{R}^d)} \leq t^{-\frac{qd}{d} - \frac{1}{2}} \|\varphi\|_{L^p(\mathbb{R}^d)} + t^{1 - \frac{qd}{d} - \frac{1}{2}} \|\psi\|_{L^p(\mathbb{R}^d)}.$$  \hspace{1cm} (58)

Now to get our different claims (i)–(v), we only need to consider different cases corresponding to the different choices of the parameters $r, p$ and $q$.

(i) If $q = 1, r = p$, then (58) becomes

$$\|u\|_{L^p(\mathbb{R}^d)} \leq \|\varphi\|_{L^p(\mathbb{R}^d)} + t \|\psi\|_{L^p(\mathbb{R}^d)}.$$  \hspace{1cm} (59)

(ii) Choose $q = 1, r = \infty, p = \infty$ in (5) to get:

$$\|u\|_{L^\infty(\mathbb{R}^d)} \leq \|\varphi\|_{L^\infty(\mathbb{R}^d)} + t \|\psi\|_{L^\infty(\mathbb{R}^d)}.$$  \hspace{1cm} (60)

(iii) Taking $r \in [p, \frac{pd}{d - 2sp}]$, we have $1 < q < \frac{d}{d - 2s}$ so that:

$$\|u\|_{L^r(\mathbb{R}^d)} \leq t^{-\frac{qd}{d} - \frac{1}{2}} \left( \|\varphi\|_{L^p(\mathbb{R}^d)} + t \|\psi\|_{L^p(\mathbb{R}^d)} \right).$$  \hspace{1cm} (61)

(iv) Let $d > 2sp$ and $r = \frac{pd}{d - 2sp}$; we have from (5) that $q = \frac{d}{d - 2s}$. Making use of the second parts of Lemma 4.4 and Lemma 4.5 respectively, for $d > 2s$ and $1 < \alpha < 2$, we obtain

$$\|Z(t, \cdot)\|_{L^{\frac{d}{d - 2s}}(\mathbb{R}^d)} \leq t^{-\alpha}, \quad t > 0,$$

and for $1 = d \leq 2s$,

$$\|W(t, \cdot)\|_{L^{\frac{d}{d - 2s}}(\mathbb{R}^d)} \leq t^{-1 - \alpha}, \quad t > 0,$$
which together with Young’s inequality for weak $L^p$-spaces gives [28] 
\[
\|u(t, \cdot)\|_{L^{\frac{d}{d - 2s}}(\mathbb{R}^d)} \leq \|Z(t, \cdot)\|_{L^{\frac{d}{d - 2s}}(\mathbb{R}^d)} \|\varphi\|_{L^p(\mathbb{R}^d)} + \|W(t, \cdot)\|_{L^{\frac{d}{d - 2s}}(\mathbb{R}^d)} \|\psi\|_{L^p(\mathbb{R}^d)} 
\leq t^{-\alpha} + t^{-1 - \alpha} \leq t^{-\alpha},
\]
as required.

(v) Observe that by inserting $d = 2sp$ and $p \in [1, \frac{d}{d - 2s})$ in (5) we get $r \in [p, \infty)$. Similarly, inserting $q \in (d/2s, \infty]$ and $p \in [1, \frac{d}{d - 2s})$ in (5) gives $r \in [q, \infty]$. This yields the first claim of (v). Finally if $\alpha = 2$, by Lemma 4.4 and for any $1 \leq p \leq \infty$ we use Young’s inequality as in (55) to obtain the claim for any $r \in [p, \infty]$.

Next we provide the following time-space estimates or weak-$L^p$ spaces estimates for the free part of the solution of (42) (i.e. by setting $f \equiv 0$). This type of estimates are also connected with the so-called Marcinkiewicz interpolation theorem.

The Marcinkiewicz interpolation theorem (see D) was proved even for pointwise quasi-additive linear or sublinear operators $T$, i.e. for $T$ such that there exists a constant $\delta \geq 1$ such that for any measurable functions $f, g$ we have the following inequality [36]:
\[
|T(f + g)(t)| \leq \delta(|Tf(t)| + |Tg(t)|),
\]
(59)

**Proposition 2.** Let $(p, q, r)$ be any admissible triplet as defined in (5). Assume that $d > 2sp$, $f \equiv 0$ and $\varphi$, $\psi \in L^p(\mathbb{R}^d)$. Then we have
\[
u \in C_b(0, T; L^p) \cap L^q(0, T; L^r),
\]
(60)
and
\[
\|u\|_{L^p(0, T; L^q(\mathbb{R}^d))} \leq \|\varphi\|_{L^p(\mathbb{R}^d)} + t\|\psi\|_{L^p(\mathbb{R}^d)}.
\]
(61)

**Proof.** The first statement in the proposition follows directly from the continuity of $u$ discussed in the first step of the proof of Theorem 2.8, by using claim (i) in Proposition 1 and the continuity of $u$ with respect to $t$. We then consider only the case $p > r$. We recall the form of $u$ as a solution to (1), in the case $f \equiv 0$, as
\[
u = Z \ast \varphi + W \ast \psi.
\]
For $t \in [0, \infty)$, we define nonlinear operators $T_1(t), T_2(t)$ from the space of measurable functions $L^p(\mathbb{R}^d)$ to itself as follows:
\[
T_1(t)\varphi \equiv \|Z(\cdot, t)\ast \varphi\|_p, \quad T_2(t)\psi \equiv \|W(\cdot, t)\ast \psi\|_p.
\]
Clearly, the sublinear operators $T_1(t)$ and $T_2(t)$ respectively satisfy the inequality (59). From the claim (iii) of Proposition 1, we have that
\[
T_1(t)\varphi \leq t^{-\frac{a d}{2s} (\frac{1}{p} - \frac{1}{r})} \|\varphi\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{1}{p^*}},
\]
\[
T_2(t)\psi \leq t^{1 - \frac{a d}{2s} (\frac{1}{p} - \frac{1}{r})} \|\psi\|_{L^p(\mathbb{R}^d)} \leq t^{-\frac{1}{q^{**}}},
\]
for $t > 0$, where
\[
\frac{1}{q^1} = \frac{ad}{2s} \left(1 \frac{1}{p} - \frac{1}{r}\right), \quad \frac{1}{q^2} = -1 + \frac{ad}{2s} \left(1 \frac{1}{p} - \frac{1}{r}\right).
\]
So from (81) in D, we obtain:
\[
m_1 \{t : |T_1(t)\varphi| > \lambda\} \leq m_1 \left\{t : \|C t^{-\frac{1}{p^*}} \varphi\|_{L^p(\mathbb{R}^d)} > \lambda\right\}
\]
Lemma 4.6. Let
\[ m_1 \left\{ t : t < \left( \frac{C \| \varphi \|_{L^p(\mathbb{R}^d)}}{\lambda} \right)^{q_1^*} \right\} \]
\[ \leq \left( \frac{C \| \varphi \|_{L^p(\mathbb{R}^d)}}{\lambda} \right)^{q_1^*}; \]
\[ m_2 \left\{ t : |T_2(t)\psi| > \lambda \right\} \leq m_2 \left\{ t : \| C t^{-\frac{d}{2}} \psi \|_{L^p(\mathbb{R}^d)} > \lambda \right\} \]
\[ = m_2 \left\{ t : \| C \| \psi \|_{L^p(\mathbb{R}^d)} \|^{q_2^*} \right\} \]
\[ \leq \left( \frac{C \| \psi \|_{L^p(\mathbb{R}^d)}}{\lambda} \right)^{q_2^*}, \]
where \( m_i(v) \) for \( i = 1, 2 \) is a Lebesgue measure of \( v \) in \([0, \infty)\). Therefore, \( T_1(t) \) and \( T_2(t) \) are weak type \((p, q_1^*)\) and \((p, q_2^*)\) operators respectively. Furthermore, from the claim (i) of Proposition 1
\[ \sup_t T_1(t) \varphi \leq \| \varphi \|_{L^p(\mathbb{R}^d)}, \quad \text{and} \quad \sup_t T_2(t) \varphi \leq t \| \psi \|_{L^p(\mathbb{R}^d)}, \]
this means that \( T(t) := T_1(t) + T_2(t) \) is also a weak type \((p, \infty)\) operator.

Next we consider any admissible triplet \((r, q_1^*, p)\) (resp. \((r, q_2^*, p)\)); we also take \( \epsilon > 0 \) sufficiently small and consider a second admissible triplet \((r, q_1^*, p - \epsilon)\) (resp. \((r, q_2^*, p - \epsilon)\)) such that for \( d > 4sp \), we have
\[ \frac{1}{q_1^* - \epsilon} - \frac{1}{p_2^*} \]
Since \( q_1^* - \epsilon < q_1^* \) (resp. \( q_2^* - \epsilon < q_2^* \)) and \( 1 < p - \epsilon < p < r \), one has that \( T(t) \) is a weak type \((r, \infty)\) and \((p - \epsilon, q_1^*)\) operator (resp. \((r, \infty)\) and \((p - \epsilon, q_2^*)\) operator). Hence the Marcinkiewicz interpolation theorem implies that \( T(t) \) is a strong \((p, q^*)\) type operator, that is,
\[ \| u \|_{L^q(0, T; L^r(\mathbb{R}^d))} \leq \| Z \ast \varphi \|_{L^{q_1^*}(0, T; L^r(\mathbb{R}^d))} + \| W \ast \psi \|_{L^{q_2^*}(0, T; L^r(\mathbb{R}^d))} \]
\[ \leq \| \varphi \|_{L^p(\mathbb{R}^d)} + t \| \psi \|_{L^p(\mathbb{R}^d)}. \]
Hence the desired results.

We continue by studying the above type of results for the inhomogeneous equation. First we need the \( L^p \)-decay estimates for the fundamental solution \( Y \).

**Lemma 4.6.** Let \( d \geq 1, 1 < \alpha \leq 2 \) and \( 0 < s < 1 \). Then \( Y(t, \cdot) \in L^p(\mathbb{R}^d) \) and
\[ \| Y(t, \cdot) \|_{L^p(\mathbb{R}^d)} \leq t^{\alpha - 1 - \frac{d}{2p}(1 - \frac{1}{p})}, \quad t > 0, \] (62)
for every \( 1 \leq p < \Lambda_3 \), where
\[ \Lambda_3 = \Lambda_3(s, d) = \begin{cases} \frac{d}{s - 4s}, & \text{if } d > 4s, \\ \infty, & \text{otherwise.} \end{cases} \]

At the borderline \( p = \Lambda_3 \), we also have for \( d > 4s \) that \( Y(t, \cdot) \) belongs to \( L^{\frac{d}{s} - \alpha}; \infty(\mathbb{R}^d) \) and
\[ \| Y(t, \cdot) \|_{L^{\frac{d}{s} - \alpha}; \infty} \leq t^{1 - \alpha}, \quad t > 0. \]
Finally, if $\alpha = 2$ or $d < 4s$, estimate (62) holds for all $p \in [1, \infty]$.

Proof. The proof in Lemma 4.4 works verbatim, using the above equivalence whenever required. Similar results could also be found in [28].

Next we have the following estimate for the inhomogeneous solution of the integro-differential equation (1), for which the idea of the proof follows the one in [26, Proposition 3.4].

Proposition 3. Let $d \geq 1$, $1 < \alpha \leq 2$ and $0 < s \leq 1$. Assume that $u$ is the mild solution of equation (1) with $\varphi \equiv 0$ and $\psi \equiv 0$.

(i) For any admissible triplet $(p,q,r)$ and $\sigma$ satisfying

$$\max (d(1 + \sigma), \alpha \sigma d) < 2r, \quad \sigma d < 2sr,$$

and for any $f \in L^{\frac{q}{1+\sigma}} \left(0, T; L^{\frac{r}{1+\sigma}} \right)$, we have $u \in L^q \left(0, T; L^r \right)$ and

$$\|u\|_{L^q(0,T;L^r)} \leq CT^{\alpha \left(1 - \frac{\sigma d}{2sr} \right)} \|f\|_{L^{\frac{q}{1+\sigma}}(0,T;L^{\frac{r}{1+\sigma}})},$$

where $C$ is a positive constant independent of $T$ and $f$.

(ii) For any admissible triplet $(p,q_1,r_1)$ and $\sigma$ satisfying

$$\left(1 - \frac{\sigma d}{2rs} \right) > \frac{\sigma}{aq} \quad \frac{1}{r_2} < \frac{2s}{\sigma d}, \quad 2rs (1 + \sigma) > \alpha \sigma d,$$

and for any $f \in L^{\frac{q_1}{aq}} \left(0, T; L^{\frac{r_1}{aq}} \right)$, we have $u \in L^{q_1} \left(0, T; L^{r_1} \right)$ and

$$\|u\|_{L^{q_1}(0,T;L^{r_1})} \leq CT^{\alpha \left(1 - \frac{\sigma d}{2rs} \right)} \|f\|_{L^{\frac{q_1}{aq}}(0,T;L^{\frac{r_1}{aq}})} \||\tilde{f}|\|_{L^{q_1}(0,T;L^{r_1})},$$

where $C$ is a positive constant independent of $T$ and $f$. 

Proof. The strategy of the proof is similar to [26, Proposition 3.4].

5. Proof of the main results. In this section we provide the full proofs of Theorem 2.9 and Theorem 2.11.

5.1. Proof of Theorem 2.9. The underlying fact here is the result in [26, Theorem 1.3], which was also used to solve a similar but more specific problem in the linear setting in [28]. First of all, we note that, since $f(u)$ is a well defined function by assumption $(H_1)$, so too is the right-hand side of the equation (42). Then by Theorem 2.8, the solution of the integro-differential equation given by the representation (42) (for a semilinear function $f$ defined under the hypothesis $(H_1)$) is precisely equivalent to the integral equation

$$u(t, x) = Z \ast \varphi + W \ast \psi + \mathcal{Y} u,$$

where the operator $\mathcal{Y}$ is defined by

$$\mathcal{Y} u := \int_0^t \int_{\mathbb{R}^d} Y(t - \tau, x - y) f(u(\tau, y)) \, d\tau \, dy.$$

In order for Theorem 2.9 to be applicable, the following Lemma establishing some important nonlinear estimates in the spaces $X_{r,q}(0,T)$ (previously defined in Section 2) will be required.
Lemma 5.1. Let \((p, q, r)\) be any admissible triple satisfying \((5)\), and \(f\) satisfy \((H_1)\). For any \(u, v \in X_{r,q}(0,T)\), we have \(Y_u, Y_v \in X_{r,q}(0,T)\), and
\[
\|Y_u\|_{X_{r,q}(0,T)} \leq C \left[ T^\alpha \|u\|_{X_{r,q}(0,T)} + T^{\alpha(1 - \frac{d}{2p})} \|u\|_{X_{r,q}(0,T)}^{1+\sigma} \right], \tag{64}
\]
\[
\|Y_u - Y_v\|_{X_{r,q}(0,T)} \leq C \left[ T^\alpha + T^{\alpha(1 - \frac{d}{2p})} \left( \|u\|_{X_{r,q}(0,T)}^{\sigma} + \|v\|_{X_{r,q}(0,T)}^{\sigma} \right) \right] \times \|u - v\|_{X_{r,q}(0,T)}. \tag{65}
\]

Proof. The proof of this Lemma works verbatim from the proof of \([26, \text{Lemma } 4.1]\), using the above equivalence whenever is required.

Proof of Theorem 2.9. Consider an admissible triplet \((p, q, r)\) as defined by \((5)\), with the assumption that \(r \geq p\). We can rewrite the integro-differential equation \((63)\) as
\[
u(x,t) = \Phi(u(x,t)),
\]
where the functional operator \(\Phi\) is defined by
\[
\Phi(u)(t) := Z \star \varphi + W \star \psi + Yu. \tag{66}
\]
In order for the theorem to hold true, the proof of the following claim will be required.

Claim. There is some \(T \in (0, 1)\) such that, for any \(u, v \in X_{r,q}(0,T)\), we have
\[
\|\Phi(u) - \Phi(v)\|_{X_{r,q}(0,T)} \leq \delta \|u - v\|_{X_{r,q}(0,T)},
\]
where \(\delta \in (0, 1)\) is constant.

Once the claim is proved we obtain:
\[
\|\Phi(u)\|_{X_{r,q}(0,T)} \leq \|\Phi(0)\|_{X_{r,q}(0,T)} + \|\Phi(u) - \Phi(0)\|_{X_{r,q}(0,T)} \leq \|\varphi\|_{L^p} + \|\psi\|_{L^p} + \delta \|u\|_{X_{r,q}(0,T)},
\]
which shows that \(\Phi(u)\) is in \(X_{r,q}(0,T)\).

Proof of Claim. For \(T \in (0, 1)\) and for any \(u, v \in X_{r,q}(0,T)\), we have from \((65)\)
\[
\|\Phi(u) - \Phi(v)\|_{X_{r,q}(0,T)} \leq \|Y_u - Y_v\|_{X_{r,q}(0,T)} \leq C \left[ T^\alpha + T^\alpha 1_{-\frac{d}{2p}} \left( \|u\|_{X_{r,q}(0,T)}^{\sigma} + \|v\|_{X_{r,q}(0,T)}^{\sigma} \right) \right] \|u - v\|_{X_{r,q}(0,T)} \leq C \left[ T^\alpha + 2T^\alpha + 1_{-\frac{d}{2p}} \left( \|\varphi\|_{L^p} + \|\psi\|_{L^p} \right)^\sigma \right] \|u - v\|_{X_{r,q}(0,T)} \tag{67}
\]
But from \((5)\), for \(q = 1\) we have \(p = r\). Using \((6)\) we have that
\[
T \|\psi\|_{L^p} = \sup_{0 < t \leq T} t \|\psi\|_{L^p}.
\]
Next, for \(\varepsilon > 0\) and for any initial data \((\varphi, \psi)\) in the space \(A\) defined by \((7)\), the above inequality in \((67)\) becomes
\[
\|\Phi(u) - \Phi(v)\|_{X_{r,q}(0,T)} \leq C \left[ T^\alpha + 2T^\alpha + 1_{-\frac{d}{2p}} \left( \|\varphi\|_{L^p} + T \|\psi\|_{L^p} \right)^\sigma \right] \times \|u - v\|_{X_{r,q}(0,T)} \leq 3CT^\alpha + 1_{-\frac{d}{2p}} \left( \|\varphi, \psi\|_{A}^\sigma \right) \|u - v\|_{X_{r,q}(0,T)} \leq \delta \|u - v\|_{X_{r,q}(0,T)},
\]
if
\[ T \leq \left( \tilde{C} \|(\varphi, \psi)\|_{A} \right)^{-\nu}, \tag{68} \]
where \( \nu = \left( \frac{2}{q} + \alpha \left( 1 + \frac{d q}{2 q} \right) \right)^{-1} \) and \( \delta \in (0, 1) \). This ends the proof of the claim.

Using the claim we have just proved, by the contraction mapping theorem, there exists a unique solution of \((63)\) on \((0, T)\) such that \( u \in X_{r,q}(0, T) \). Notice that this argument can be extended for any \( r \geq p \), and for \( r = p \), we have that \( u \in C([0, T]; L^p) \).

Furthermore as in [26, Theorem 1.3], one should notice that the above estimates hold true for any \( r \geq p \). For instance, when \( r = p \) we have \( u \in C([0, T]; L^p) \) and \( \|u(T)\|_{L^p} \leq C\|(\varphi, \psi)\|_{A} \) for \( T \) defined as in \((68)\). Since \( T \) can be determined only by \( \|\varphi\|_{L^P} \) and \( \|\psi\|_{L^P} \), independently of \( r \), one can extend this solution \( u \) on the maximal interval \([0, T^*)\), with \( T^* \) independent of \( r \). Fix \( t_0 \in (0, T^*) \), with \( T^* < \infty \), then \( \|u(t_0)\|_{L^r} < \infty \). By an extension procedure of the solution \( u \) on \([t_0, t_1]\) such that \( \tilde{C}(t_1 - t_0)^{\alpha} + \tilde{C}(t_1 - t_0)^{\alpha + (1 + \frac{d q}{2 q})} \leq \delta \) and \( t_1 - t_0 \leq 1 \). So we get the blow-up rate estimate in \((26)\) by interpreting the fact that \( \tilde{C}(t_1 - t_0)^{\alpha + (1 + \frac{d q}{2 q})} \leq \|u(t_0)\|_{L^r}. \)

Next we prove the claim \((25)\). Proceeding as in the proof of the claim above, we fix \( \varphi_1, \varphi_2 \in L^p \). For a given \( \varphi_2, \psi_2 \in L^p \cap L^r \) such that \( \|\varphi_1 - \varphi_2\|_p \leq \epsilon \) and \( \|\psi_1 - \psi_2\|_p \leq \epsilon \), and for \( u \) and \( v \) two solutions of
\[
\begin{align*}
  u &= Z \ast \varphi_1 + W \ast \psi_1 + \mathcal{Y} u, \\
  v &= Z \ast \varphi_2 + W \ast \psi_2 + \mathcal{Y} v
\end{align*}
\]
respectively, we have
\[
\begin{align*}
  \|u(t) - v(t)\|_{X_{r,q}(0,T)} &= \sup_{0 < t \leq T} t^{\frac{q}{r}} \|u(t) - v(t)\|_{L^r} \leq \sup_{0 < t \leq T} t^{\frac{q}{r}} \|Z \ast (\varphi_1 - \varphi_2)\|_{L^r} \\
 &+ \sup_{0 < t \leq T} t^{\frac{q}{r}} \|W \ast (\psi_1 - \psi_2)\|_{L^r} + \sup_{0 < t \leq T} t^{\frac{q}{r}} \|\mathcal{Y} u - \mathcal{Y} v\|_{L^r} \\
 &\leq CT^{\frac{q}{r}} \|\varphi_1 - \varphi_2\|_{L^r} + CT^{\frac{q}{r} + 1} \|\psi_1 - \psi_2\|_{L^r} \\
 &+ \sup_{0 < t \leq T} t^{\frac{q}{r}} \|W \ast (\psi_1 - \psi_2)\|_{L^r} + \sup_{0 < t \leq T} t^{\frac{q}{r}} \|\mathcal{Y} u - \mathcal{Y} v\|_{L^r} \\
 &\leq C \|\varphi_1 - \varphi_2\|_{L^p} + \sup_{0 < t \leq T} t^{\frac{q}{r}} \|\mathcal{Y} u - \mathcal{Y} v\|_{L^r} \\
 &\leq C \|\varphi_1 - \varphi_2\|_{L^p} + \sup_{0 < t \leq T} t^{\frac{q}{r}} \|\mathcal{Y} u - \mathcal{Y} v\|_{L^r} \\
 &\leq C \|\varphi_1 - \varphi_2\|_{L^p} + \sup_{0 < t \leq T} t^{\frac{q}{r}} \|\mathcal{Y} u - \mathcal{Y} v\|_{L^r}.
\end{align*}
\]
For small positive \( T \) as defined in \((68)\), we have that \(\sup_{0 < t \leq T} t^{\frac{q}{r}} \|u(t) - v(t)\|_{L^r} \leq C_2 \epsilon\).

Furthermore, following the same strategy as above, we find that
\[
\begin{align*}
  \sup_{0 < t \leq T} t^{\frac{q}{r}} \|u(t)\|_{L^r} \leq CT^{\frac{q}{r}} \|\varphi_2\|_{L^r} + CT^{\frac{q}{r} + 1} \|\psi_2\|_{L^r} + \sup_{0 < t \leq T} t^{\frac{q}{r}} \|\mathcal{Y} v\|_{L^r} \rightarrow 0, \text{ as } T \rightarrow 0.
\end{align*}
\]
The combination of these two estimates gives us the claim \((25)\).
5.2. **Proof of Theorem 2.11.** Having obtained conditions that guarantee the existence and uniqueness of solutions for problems (1) and (2)–(3), we now provide the proof of Theorem 2.11 stated in Section 2.2.

**Proof of Theorem 2.11.** The extended global weak solution $U$ to problem (2)–(3) follows from, on the basis of (46) and (44), the arguments elaborated in the proof of [4, Theorem 3.2 and 4.4]. These arguments, together with [43, Theorem 3.12] and [12, Proposition 2.1], show that for a given function $u \in L^\infty(0, T; \mathbb{H}^{s^*})$, there exists $U \in L^\infty(0, T; \hat{H}^1(y^{1-2s}, \mathcal{C}))$ such that

$$u = \text{tr}_\Omega U.$$  

As a consequence of the above expression, $\text{tr}_\Omega$ is such that

$$L^\infty(0, T; \mathbb{H}^{s^*}) \subseteq \text{tr}_\Omega \left( L^\infty(0, T; \hat{H}^1(y^{1-2s}, \mathcal{C})) \right).$$

We fix $0 < T^* \leq T$, and consider the space

$$\mathcal{X} := \left\{ \text{tr}_\Omega U \in C([0, T^*]; \mathbb{H}^{s^*}(\Omega)) \cap C^1([0, T^*]; L^2(\Omega)) : \text{tr}_\Omega U(\cdot, 0, 0) = \varphi, \right.$$  

$$\left. \text{tr}_\Omega \partial_t \text{tr}_\Omega U(\cdot, 0, 0) = \psi, \text{ and } \left\| \text{tr}_\Omega U(\cdot, y, t) \right\|_{\mathbb{H}^{s^*}(\Omega)} + \left\| \text{tr}_\Omega \partial_t \text{tr}_\Omega U(\cdot, y, t) \right\|_{L^2(\Omega)} \leq C \quad \forall \ t \in [0, T^*) \right\},$$

for some constant $C > 0$. Define the mapping $\Pi$ on $\mathcal{X}$ using the relation (69) by

$$\Pi(U) := \Theta(u),$$

where

$$\Theta(u)(t) = \sum_{k=1}^\infty \varphi_k E_{\alpha,1}(-\lambda_k^* t^\alpha) \phi_k + \sum_{k=1}^\infty \psi_k t E_{\alpha,2}(-\lambda_k^* t^\alpha) \phi_k$$  

$$+ \sum_{k=1}^\infty \left( \int_0^t f_k(u(\tau))(t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_k^* (t - \tau)^\alpha) \, d\tau \right) \phi_k.$$  

Now

$$\left\| \text{tr}_\Omega U \right\|_{\mathcal{X}} := \sup_{t \in [0, T^*)} \left( \left\| U(\cdot, y, t) \right\|_{\mathbb{H}^{s^*}(\Omega)} + \left\| \text{tr}_\Omega \partial_t U(\cdot, y, t) \right\|_{L^2(\Omega)} \right)$$

defines a norm on $\mathcal{X}$, so the space $\mathcal{X}$ is a closed subspace of the Banach space $C([0, T^*]; \mathbb{H}^{s^*}(\Omega)) \cap C^1([0, T^*]; L^2(\Omega))$. So the proof of the existence of a locally defined solution of (2)–(3) and the control of the remaining term tr$_\Omega \nabla^y \Pi(U)$ follows by a fixed point argument for the map $\Pi(U)$ on $\mathcal{X}$, using (H$_2$), from [4, Theorem 3.2] and (69) respectively.

Hence we conclude by saying that the proof of this Theorem works verbatim from the proof of [4, Theorem 4.4], using the equivalence (69) whenever required. 

**Appendix A. Fourier transform of a function in $\mathbb{R}^d$.** The Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ for a function $v \in L^1(\mathbb{R}^d)$ are defined by

$$\hat{v}(\xi) = \mathcal{F}[v](\xi) = \int_{\mathbb{R}^d} v(x) e^{-i\xi \cdot x} \, dx, \quad v(x) = \mathcal{F}^{-1}[\hat{v}](x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \hat{v}(\xi) e^{i\xi \cdot x} \, dx.$$  

(72)
In particular, we need a few facts about the transforms of radially symmetric functions [27]. Using the explicit expression for the integration of $e^{i\omega \cdot x}$ over the unit sphere $S^{d-1}$ in $\mathbb{R}^d$, namely
\[
\int_{S^{d-1}} e^{i\omega \cdot x} d\omega = (2\pi)^{\frac{d}{2}} |x|^{1-\frac{d}{2}} J_{\frac{d}{2}-1}(|x|),
\]
the Fourier transform of a radial function $v(|x|)$ becomes
\[
\mathcal{F}[v](\xi) = (2\pi)^{\frac{d}{2}} |\xi|^{1-\frac{d}{2}} \int_0^\infty r^{\frac{d}{2}} J_{\frac{d}{2}-1}(r|\xi|) v(r) dr.
\] (73)

In the same spirit, the inverse Fourier transform of a radial function $\hat{v}(|\xi|)$ becomes
\[
\mathcal{F}^{-1}[\hat{v}](x) = (2\pi)^{-\frac{d}{2}} |x|^{1-\frac{d}{2}} \int_0^\infty \eta^{\frac{d}{2}} J_{\frac{d}{2}-1}(\eta|x|) \hat{v}(\eta) d\eta.
\] (74)

**Appendix B. Mittag-Leffler functions.** We recall some facts about Mittag-Leffler functions borrowed from [31, 24].

The Mittag-Leffler functions of one and two parameters are defined respectively by:
\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + 1)};
\]
\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(an + \beta)}.
\]

These functions can be related to the Fox $H$-functions of $C$ in the following way:
\[
E_\alpha(z) = H_{1,2}^{1,1} \left[ -z \begin{array}{c} (0,1) \\ (0,1) \end{array} (0,1) , (0,\alpha) \right];
\] (75)
\[
E_{\alpha,\beta}(z) = H_{1,2}^{1,1} \left[ -z \begin{array}{c} (0,1) \\ (0,1) \end{array} (0,1) , (1-\beta,\alpha) \right].
\] (76)

The following formulae for Laplace transforms of Mittag-Leffler functions somewhat justify their importance in the field of fractional calculus:
\[
\mathcal{L} [E_\alpha (\omega t^\alpha)] = \frac{s^{\alpha-1}}{s^\alpha - \omega};
\] (77)
\[
\mathcal{L} [t^{\beta-1} E_{\alpha,\beta} (\omega t^\alpha)] = \frac{s^{\alpha-\beta}}{s^\alpha - \omega}.
\] (78)

The Mittag-Leffler function has asymptotic growth in all directions in the complex plane except for the negative real axis, along which it can be controlled by the following estimates:
\[
E_\alpha(-t^\alpha) \sim \begin{cases} 
\exp \left( \frac{-t^\alpha}{\Gamma(1+\alpha)} \right) & \text{as } t \to 0, \\
\left( \frac{t^{\alpha}}{\Gamma(1-\alpha)} \right) & \text{as } t \to \infty.
\end{cases}
\] (79)
Appendix C. Fox H-functions. We recall some notions on Fox H-functions borrowed from [32, 31].

We start by fixing

\[ d \in \mathbb{N}, \quad \alpha \in (0, 2), \quad s \in (0, \infty). \]

For \( \gamma \in [0, \infty), \sigma \in \mathbb{R} \) and \( z \in \mathbb{C} \) define

\[
H_{\sigma, \gamma}(z) := \frac{\Gamma\left(\frac{d}{2} + \gamma + 2sz\right)\Gamma(1 + z)\Gamma(-z)}{\Gamma(-\gamma - 2sz)\Gamma(1 - \sigma + \alpha z)}.
\]

The H-function \( H_{m,n}^{p,q}(r) \) is defined by

\[
H_{m,n}^{p,q}[r] \left( \begin{array}{c} (a_1, \alpha_1) \ldots (a_p, \alpha_p) \\ (b_1, \beta_1) \ldots (b_q, \beta_q) \end{array} \right] = \frac{1}{2\pi i} \int_L \prod_{k=1}^{m} \Gamma(b_k + \beta_k z) \prod_{k=1}^{n} \Gamma(1 - a_k - \alpha_k z) \prod_{i=m+1}^{q} \Gamma(1 - b_i - \beta_i z) r^{-z} \, dz,
\]

where \( L \) is the Mellin–Barnes contour defined below and \( 0 \leq n \leq p, 0 \leq m \leq q \). In particular, we are interested in the following special case. For \( r \in (0, \infty) \), we define

\[
H_{\sigma, \gamma}(r) := H_{2,3}^{2,1}[r] \left( \begin{array}{c} (1, 1) \\ (\frac{d}{2} + \gamma, 2s) \\ (1, 1) \\ (1 + \gamma, 2s) \end{array} \right] = \frac{1}{2\pi i} \int_L H_{\sigma, \gamma}(z) r^{-z} \, dz.
\] (80)

Here

\[ L = L_{B^r} = \{ z \in \mathbb{C} : \Re[z] = \ell_0 \} \]

and \( \ell_0 \) is chosen to satisfy:

\[
\begin{align*}
\max (-1, -\frac{d}{2s}) < \ell_0 &< 0 & \text{if } \gamma \notin \{0, s\}, \\
\max (-1, -\frac{d}{2s}) < \ell_0 &< 1 & \text{if } \gamma = 0, \\
\max (-2, -1 - \frac{d}{2s}) < \ell_0 &< 0 & \text{if } \gamma = s.
\end{align*}
\]

Appendix D. The Marcinkiewicz interpolation theorem. The Kolmogorov result (1925) and the Riesz-Thorin convexity theorem (1926) showing that the conjugate-function operator is of weak type \((1,1)\) and of strong type \((p,p)\) for \( 1 < p < \infty \) were known by Marcinkiewicz. One of his famous results is known as the Marcinkiewicz interpolation theorem, which reads as follows.

The Marcinkiewicz interpolation theorem [36, 37]. If a linear or sublinear mapping \( T \) is of weak type \((1,1)\) and strong type \((\infty, \infty)\), that is, satisfies the estimates

\[
\lambda m\{ t \in I : |Tf(t)| > \lambda \} \leq A \int_I |f(t)| \, dt \quad \text{for any } \lambda > 0, \quad (81)
\]

\[
\sup_{t \in I} |Tf(t)| \leq B \sup_{t \in I} |f(t)| \quad \text{for any } t \in I, \quad (82)
\]

then the operator \( T \) is of strong type \((p,p)\) for \( 1 \leq p \leq \infty \), i.e. we have the following estimate:

\[
\int_I |Tf(t)|^p \, dt \leq C_{A,B} \int_I |f(t)|^p \, dt, \quad C_{A,B}^{1/p} \leq \frac{2}{(p-1)^{1/p}} A^{1/p} B^{1-1/p}. \quad (83)
\]
Conclusions. In this paper, we have undertaken the problem of obtaining the well-posedness of (1). For the quasi-stationary elliptic problem (2) we derived some properties such as the existence of global weak solutions of the extended semi-linear integro-differential equations, and some $L^p$-estimates.

As for further research in the same direction, we have started to analyse whether one can also obtain a boundary operator for larger values of the parameters involved, namely $s$ and $\alpha$. Our work on these problems will be saved for later publications.

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