Innovation and Strategic Network Formation

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Abstract

We study a model of innovation with a large number of firms that create new technologies by combining several discrete ideas. These ideas can be acquired by private investment or via social learning. Firms face a choice between secrecy, which protects existing intellectual property, and openness, which facilitates social learning. These decisions determine interaction rates between firms, and these interaction rates enter our model as link probabilities in a resulting learning network. Higher interaction rates impose both positive and negative externalities on other firms, as there is more learning but also more competition. We show that the equilibrium learning network is at a critical threshold between sparse and dense networks. A corollary is that at equilibrium, the positive externality from interaction dominates: the innovation rate and even average firm profits would be dramatically higher if the network were denser. So there are large returns to increasing interaction rates above the critical threshold—but equilibrium remains critical even after natural interventions. One policy solution is to introduce informational intermediaries, such as public innovators who do not have incentives to be secretive. These intermediaries can facilitate a high-innovation equilibrium by transmitting ideas from one private firm to another.

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1 Introduction

A growing body of empirical research suggests that interactions between inventors are an important part of innovation\(^1\). New technologies are often produced by combining individual insights with learning from peers\(^2\). This confers benefits on firms and inventors engaged in such learning. When highly-connected clusters of firms emerge in a location, as in the technology industry in Silicon Valley, inventors in these areas are much more productive. But frequent collaboration and learning are not assured even when inventors in a given industry co-locate (Saxenian 1996).

Rather, interaction patterns depend on firms’ endogenous decisions about how much to collaborate with each other. More collaborative firms are better positioned to learn from other firms and inventors. But there are also downsides to openness, as secrecy allows firms to prevent potential competition by protecting intellectual property. This presents a choice for firms and inventors between openness and secrecy. This paper uses network-theory techniques to study firms’ decisions about how much to interact and their consequences for information flows, the rate of innovation, and related policy decisions.

To do so, we use a framework inspired by recombinant growth (Weitzman 1998) to explicitly model the creation of new technologies. Technologies are modeled as finite sets of distinct ideas. Ideas can be acquired in two ways: (1) via private investment and (2) via social learning. Firms generate profits by combining ideas to produce new technologies, but the profits from a technology are erased by competition if another firm also knows the component ideas in that technology. There are a large number of firms, and each chooses how much to invest in R&D as well as how open to be. Their choices of levels of openness determine interaction rates between firms, and the probability that one firm learns from another is equal to the interaction rate between the two firms. If a given firm is more open, that firm is more likely to learn from others but also more exposed to others learning its ideas.

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\(^1\)The benefits to interactions between inventors and movement of inventors have been quantified empirically by Akcigit, Caicedo, Miguelez, Stantcheva, and Sterzl (2018), Kerr (2008), Samila and Sorenson (2011), among others.

\(^2\)See Bessen and Nuvolari (2016) for historical examples and Chesbrough (2003) for examples in the technology industry.
Our first contribution, which is methodological, is to develop a theory of endogenous formation of random networks in the context of our economic application. Learning opportunities are random events, and their realizations determine a learning network. We therefore consider link formation decisions with uncertainty, while the leading approach in the literature on network-formation games focuses on deterministic models (Jackson and Wolinsky 1996 and Bala and Goyal 2000). Since we take actions to be continuous choices that translate to interaction rates, optimal behavior satisfies first-order conditions rather than a high-dimensional system of combinatorial inequalities.

A key feature of our model is that ideas can spread several steps through this network: when one firm learns from another, the information transferred can include ideas learned from a third firm. We refer to this as indirect learning. By contrast, existing work on strategic formation of random networks largely focuses on direct connections (Currarini, Jackson, and Pin 2009). Under indirect learning, firms’ incentives depend on the global structure of the network. Firms would like to learn many ideas, since then the firm can combine these ideas to produce a large number of new technologies, and much of this learning can be indirect.

This analysis leads to our second contribution, which is to characterize equilibrium and quantify the associated externalities. Learning outcomes depend dramatically on whether the learning network is sparsely connected or densely connected. If firms’ interaction rates are below a critical threshold, the learning network consists of many small clusters of firms who learn few ideas. Above the threshold, the learning network has a giant component asymptotically: a large group of firms learning a large number of ideas and thus producing many new technologies. We analyze an individual firm’s decision problem in each of these two domains, i.e. when other firms form a sparse or dense network.

In our baseline model, we show that the equilibrium outcome is at the critical threshold between sparse and dense networks. Firms deviate to interact more if the network is likely to be sparse and deviate to interact less if the network will be dense. Intuitively, in sparse networks, firms seek to fill ‘structural holes’ by combining ideas learned via different interactions (Burt 1992). As others interact more, these structural holes disappear, and indeed

\[\text{Golub and Livne (2010)}\] go a step further, allowing payoffs to depend on distance one and two connections. An important feature of our model is that firms’ decisions depend on the global network structure rather than only local connections.
firms tend to learn the same ideas repeatedly from different interactions. So the incentives to be more open are weaker relative to the incentives to be secretive.

When equilibrium is at the critical threshold, there are unboundedly large profitability and welfare gains (as the number of firms grows large) from increasing interaction rates above equilibrium levels. To rephrase in terms of the underlying economic forces, the benefits from more learning outweigh the lost profits from additional competition, even if only producers’ interests are considered. A consequence is that increasing interaction rates is a first-order concern in designing policy. By contrast, policies targeting decisions about private investment rather than interaction, such as subsidies to R&D, have minimal effect at equilibrium—but can be valuable if paired with interventions to increase openness.

We discuss one policy change that can induce more productive interaction patterns, which is to introduce public innovators who do not have incentives to be secretive. For example, governments could fund academic researchers who are especially willing to interact with other researchers, including in industry. The key is that public innovators can serve as informational intermediaries, transmitting ideas between private firms. Thus high-innovation clusters form around the public innovators.

We next show that the prediction of critical equilibrium does not depend on several stylized assumptions in the baseline model. One such assumption is that learning probabilities that are symmetric across pairs of firms. We show that equilibrium remains critical even when firms have different propensities to learn from others. A second assumption in the baseline model is that profits are additive across technologies. Equilibrium remains critical if there are increasing or slightly decreasing returns to producing many technologies, and indeed the key property is that payoffs are convex in the number of ideas known to a firm. The structure of competition is important, however, and the baseline results described above assume zero profits in competitive markets. If profits under competition are instead positive, perhaps because of collusion between firms, then incentives toward secrecy will be weaker and so equilibria will be above critical threshold.

Our final results ask how formal intellectual property rights change the incentives to interact. Consider the consequences of granting patents to a positive fraction of ideas, e.g., allowing hardware but not software ideas to be patented. Patents mitigate firms’ incentives
to be secretive, but can also discourage exchange of ideas. Firms with patents are more open but are also less desirable partners in interactions (at least when ideas are only transmitted directly). The resulting adverse selection in interaction can deter firms from collaborating with others. We show that patent rights can therefore prevent any productive interactions at equilibrium. If indirect learning is important, firms with patents will be informational intermediaries, like the public innovators above. In this case there are benefits to allowing patents, but the optimal policy is often to only allow patents for a very small fraction of ideas.

At a technical level, this paper develops tools for analyzing decisions in network settings with complementarities between indirect connections. Classical results in graph theory characterize the component structure of the learning network, and thus the number of ideas firms will learn, asymptotically (Karp, 1990 and Luczak, 1990). But we will find that firms’ incentives also depend on how these ideas are learned, e.g. via many interactions or a few interactions, so additional new techniques are needed. We derive an expression relating behavior and the extent to which technologies combine ideas from distinct interactions. Establishing these findings requires a careful analysis of the graph branching process governing the number of ideas learned from each interaction.

1.1 Related Literature

Existing models of innovation incorporating interactions between firms generally model these interactions as either mechanical spillovers or learning via imitation. A common approach is to choose a convenient functional form for spillovers, usually motivated by tractability within a macroeconomic (e.g., Kortum, 1997) or network-theory (König, Battiston, Napoletano, and Schweitzer, 2012) framework. By microfounding these spillovers, which arise endogenously within the innovative process, we can study how spillovers vary across policy environments.

In a second approach, which relies on a quality-ladders framework, interactions give firms a chance to catch up as innovation proceeds vertically (e.g., Akcigit, Caicedo, Miguelez, Stantcheva, and Sterzi, 2018 and König, Lorenz, and Zilibotti, 2016). We instead explicitly model horizontal innovation combinations of distinct ideas, which serve as building blocks. Related models appear in Weitzman (1998) and Acemoglu and Azar (2019), which focus on
the evolution of the total amount of innovation over time and do not involve learning or informational spillovers between firms. We introduce tradeoffs between secrecy and learning and find that the resulting incentives push toward an equilibrium at a critical threshold.4

At a more theoretical level, we develop a theory of strategic network formation with probabilistic links. A large literature since Jackson and Wolinsky (1996) and Bala and Goyal (2000) considers endogenous network formation assuming that agents can choose their links exactly.5 Because equilibrium is then characterized by a large system of inequalities, these models illustrate key externalities in special cases but remain largely or entirely intractable in many others. By incorporating uncertainty, we obtain a smooth model of link formation that can be solved via basic optimization techniques combined with analyses of random graphs.6

Under this random-network approach to network formation, incentives to form links depend on the ‘phase transitions’ between sparse and dense networks. Related interactions between phase transitions and optimal behavior have been recently explored in the context of diffusion processes by Campbell (2013), Sadler (2019), and Akbarpour, Malladi, and Saberi (2018), who let adoption and seeding decisions depend on component structure in an underlying network. We instead study equilibria of a game in which agents endogenously make decisions about how much to interact with others, and find there is a subtle interplay between strategic incentives and the global network structure.

4 By assuming a continuum of firms, macroeconomic models of imitation often implicitly restrict to subcritical interaction patterns.

5 The pairwise stability solution concept from Jackson and Wolinsky (1996) and variants have been applied to network formation in many settings, including innovation (König, Battiston, Napoletano, and Schweitzer, 2011, König, Battiston, Napoletano, and Schweitzer, 2012).

6 An alternate approach to smooth network formation is to consider weighted networks, so that each link has an intensity (Baumann, 2017 and Griffith, 2019). By analyzing random networks, we can study continuous link-formation decisions without requiring network weights.
2 Baseline Model

2.1 Basic Setup

There are \( n > 2 \) firms \( 1, \ldots, n \). Each firm \( i \) can potentially discover a distinct idea, also denoted by \( i \). We let \( I \subset \{1, \ldots, n\} \) be the set of ideas that are discovered.

The firm chooses a probability \( p_i \in [0, 1) \) of discovering this idea and pays investment cost \( c(p_i) \). We will assume that \( c \) is continuously differentiable, increasing, and convex with \( c(0) = 0 \) and \( \lim_{p \to 1^-} c(p) = \infty \). The realizations of discoveries are independent.

A technology \( t = \{i_1, \ldots, i_k\} \) consists of \( k \) ideas \( i_1, \ldots, i_k \in I \), where \( k > 1 \) represents the complexity of technologies. Each idea \( i \in t \) must be discovered by the corresponding firm to be included in a technology. There are therefore \( \binom{n}{k} \) potential technologies, and a firm \( i \) can produce more than one technology.

Each firm \( i \) chooses a level of openness \( q_i \in [0, 1] \). Given firms \( i \) and \( j \)'s choices \( q_i \) and \( q_j \), the interaction rate between \( i \) and \( j \) is

\[ \iota(q_i, q_j) = q_i q_j. \]

The timing of the model is simultaneous: firms choose actions \( p_i \) and \( q_i \) and then all learning occurs. We denote actions by \((p, q)\). When actions are symmetric, we will refer to \( p_i \) by \( p \) and \( q_i \) by \( q \).

Given actions \( p \) and \( q \), we denote the set of ideas that firm \( i \) learns from others by \( I_i(p, q) \subset I \). This is a random set depending on realizations of links and discoveries. We now describe how learning occurs.

With probability \( \iota(q_i, q_j) \), firm \( i \) learns directly from firm \( j \). In this case, firm \( i \) learns idea \( j \) if \( j \in I \). If firm \( i \) learns directly from firm \( j \), then with probability \( \delta \in [0, 1] \), firm \( i \) also learns indirectly through firm \( j \). In this case, firm \( i \) also learns all ideas in \( I_j(p, q) \).

All realizations of direct and indirect learning are independent, and in particular, firm \( i \) can learn from firm \( j \) without \( j \) learning from \( i \).

We will call the case \( \delta = 0 \) direct learning and the case \( \delta > 0 \) indirect learning.

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7 The model will extended in Appendix C to allow firms to potentially discover multiple ideas.
8 In the baseline model, the parameter \( k \) is the same for all firms.
When \( \delta > 0 \) we define a directed network, which we call the indirect-learning network, with nodes 1, \ldots, n and a link from node \( j \) to node \( i \) if firm \( i \) learns indirectly through firm \( j \).

A firm \( i \) receives payoff 1 from each technology \textbf{proprietary technology} \( t \). A technology \( t \) is a proprietary for firm \( i \) if (1) \( i \in t \) and (2) \( i \) is the unique firm such that \( j \in \{i\} \cup I_i(p, q) \) for all \( j \in t \). In words, the technology contains firm \( i \)'s idea and firm \( i \) is the unique firm that knows all ideas in the technology.

If \( t \) is not a proprietary technology for firm \( i \), then firm \( i \) receives payoff 0 from the technology \( t \). In Section 5, we will consider more general payoff structures in which (1) payoffs are not additive across technologies and (2) firms instead receive payoff \( f(m) \) if \( m > 0 \) other firms know all ideas contained in a technology.

### 2.2 Example

To illustrate the mechanics of the model, we describe a simple example with \( n = 4 \) firms and complexity \( k = 3 \). Suppose that realizations are such that (1) ideas are discovered by firms in \( I = \{1, 3, 4\} \) and (2) firm 1 learns indirectly through firm 2 and directly from firm 3, firm 3 learns indirectly through firm 1, and firm 3 learns directly from firm 4.

The network and ideas are shown in Figure 1. Black circles correspond to firms with ideas \( i \in I \), i.e. firms that discover ideas, while white circles correspond to firms with ideas \( i \notin I \), i.e. firms that do not discover ideas. Solid arrows denote indirect learning links, while dashed arrows indicate only direct learning occurred.

Since \( k = 3 \), the unique technology \( t \) consisting of ideas in \( I \) is \( t = \{1, 3, 4\} \). We have

\[
I_1(p, q) = \{3\}, I_2(p, q) = \emptyset, I_3(p, q) = \{1, 4\}, I_4(p, q) = \emptyset.
\]

Because firm 3 is the unique firm such that \( t \subset I_i(p, q) \cup \{i\} \) and we have \( 3 \in t \), firm 3 produces the technology \( t \) and receives monopoly profit of one for that technology. There are no profits from any other technologies.

Suppose instead that firm 1 also learns indirectly through firm 3, as shown in Figure 2.
Figure 1: Network with four firms and $k = 3$. Black circles are firms that discover ideas while white circles do not discover ideas. Dashed lines indicate direct learning and solid lines indicate indirect learning. The only technology produced is $t = \{1, 3, 4\}$ and firm 3 receives monopoly profit.

Figure 2: Network with four firms and $k = 3$. Black circles are firms that discover ideas while white circles do not discover ideas. Dashed lines indicate direct learning and solid lines indicate indirect learning. The only technology produced is $t = \{1, 3, 4\}$, and there are no profits because firms 1 and 3 both produce $t$. 
Then we have

\[ I_1(p, q) = \{3, 4\}, I_2(p, q) = \emptyset, I_3(p, q) = \{1, 4\}, I_4(p, q) = \emptyset. \]

The only potential technology remains \( t = \{1, 3, 4\} \). We now have \( t \subset I_i(p, q) \cup \{i\} \) for both firm 1 and firm 3, so both receive the competitive profit of zero for that technology. There are also no profits from other technologies.

### 2.3 Interpretation and Discussion

Before expressing expected payoffs of firms and defining equilibrium, we discuss interpretation and assumptions in the model.

**Actions:** Firm actions are choices \((p_i, q_i)\). The first component \(p_i\) corresponds to a level of investment in R&D. A small probability of a discovery is cheap, while probabilities close to one are very expensive.

The second component \(q_i\) corresponds to a level of openness or secrecy in interactions with other firms. The action \(q_i\) can include decisions such as whether to locate near other firms and how much to send employees to conferences. An important feature of the model is that increasing \(q_i\) increases the probability that firm \(i\) learns from other firms but also increases the probability that other firms learn from \(i\). The baseline model assumes that learning probabilities are symmetric: firm \(i\) learns from firm \(j\) with the same probability that firm \(j\) learns from firm \(i\). In Section 4, we allow firms to have heterogeneous propensities to learn across firms and find this symmetric structure does not drive results.

The downside to interaction for firm \(i\) is exposure to other firms learning from \(i\) rather than fixed link-formation costs. Because the costs of links are an endogenous feature of the model, our equilibrium characterization does not depend on functional forms of costs, as it would with exogenous link costs separate from the innovation process.

In Appendix D, we describe a related model in which firms instead face a tradeoff between learning and private investment. In this extension, the probability that firm \(i\) learns from firm \(j\) depends only on firm \(i\)’s action, and the downside to interaction comes from a budget

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9Stein (2008) gives a microfoundation for bilateral communication in innovation.
constraint on the interaction rate and private investment. The techniques we use extend easily, and we find equilibrium depends on the rate at which firms can substitute between interaction and private investment.

**Formal and Informal Interactions:** The model is meant to primarily describe informal interactions between employees or firms, rather than more formal arrangements such as licensing agreements or joint R&D ventures. As such, our results are most applicable to industries where formal property rights are imperfectly enforced. Because information transmitted via informal interactions can often spread several steps, an analysis considering global network structure is particularly relevant.

In Appendix C, we compare the payoffs to firms with different numbers of private ideas. This analysis can be equivalently interpreted as measuring the value of formal contracting arrangements allowing multiple firms to share ideas frictionlessly. We find that as the number of firms grows large, the benefits to such an arrangement are small compared to a firm’s profits.

**Interaction Rate:** The multiplicative interaction rate

\[ \iota(q_i, q_j) = q_i q_j \]

has the feature that firm \( i \)'s probability of learning from another firm and that firm’s probability of learning from \( i \) are both proportional to \( q_i \). Thus, this is the (unique up to rescaling) interaction rate that arises from a random matching process in which all agents choose a search intensity and the probability of learning in both directions is proportional to that intensity. See Cabrales, Calvó-Armengol, and Zenou (2011) for a microfoundation for a closely related deterministic model.

Two key properties of the interaction rate are:

1. \( \iota(q, q') = \iota(q', q) \) for all \( q \) and \( q' \)
2. \( \iota(q, 0) = 0 \) for all \( q \).

Property (1) says that the interaction rate is symmetric, as discussed above. Property (2) says that firms can choose not to interact with others. Much of the analysis, including our

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\[ \text{Storper and Venables (2004)} \] discuss the importance of one form of informal interactions.
existence and characterization results for symmetric equilibria, generalizes to any strictly increasing and continuously differentiable interaction rate $\iota : [0, 1] \times [0, 1] \to \infty$ satisfying these properties.

**Learning Network:** A useful assumption is that if firm $i$ learns indirectly through firm $j$, then firm $i$ learns all ideas known to $j$. This ensures that there is a well-defined learning network, and this network is a central object in our analysis. If indirect learning were not perfectly correlated across ideas, there would be a separate learning network for each idea.

**Firm Profits:** The positive payoffs from producing technologies correspond to monopoly payoffs, which we normalize to $1$.\(^{11}\) If multiple firms know all ideas contained in $t$, then there is a competitive market and firms receive zero profits. This baseline payoff structure, which we generalize in Section 5.2, corresponds to Bertrand competition.

Our setup requires that monopolist firms must have privately developed one of the ideas in a technology to produce that technology, but competitors need not. To start a new market, some expertise and/or confidence in the quality of the relevant idea is needed. Once a market exists, however, entrants do not require this expertise, perhaps because relevant details can be obtained from the competitor’s technology.

### 2.4 Payoffs

Given actions $(p, q)$, we define the proprietary technologies $PT_i(p, q)$ for $i$ to be the set of technologies $t$ such that $i \in t$, firm $i$ learns all other ideas $j \in t$, and no other firm knows all ideas in $t$. Note that this set is a random object depending on link realizations. Then the expected payoff to firm $i$ is

$$U_i(p, q) = \mathbb{E}[|PT_i(p, q)|] - c(p_i).$$

To further illustrate payoffs, we write the cardinality of $PT_i(p, q)$ explicitly when $\delta = 1$. Recall that $I(p, q)$ is the set of ideas learned by firm $i$ given actions $(p, q)$. Like $PT_i(p, q)$, this is also a random object.

\(^{11}\)We also assume that all technologies give the same monopoly profits and that these profits are deterministic. It would be equivalent to take monopoly profits to be randomly drawn from any distribution with finite mean, as long as firms have no information about the realizations a priori.
When $\delta = 1$, the expected payoffs to firm $i$ are:

$$U_i(p, q) = p_i \cdot E \left[ \left( \left| I_i(p, q) \right| \right) \right] \cdot \prod_{j \neq i} (1 - \epsilon(q_i, q_j)) - c(p_i).$$

A technology $t$ that $i$ profits from consists of $i$’s private idea, which is developed with probability $p_i$, and a choice of $(k - 1)$ other ideas known to $i$. The firm $j$ faces competition if and only if some firm learns all of $i$’s ideas, and the probability that this does not occur is $\prod_{j \neq i} (1 - \epsilon(q_i, q_j))$. Finally, the private investment cost is $c(p_i)$.

In general, a firm can face competition for a technology $t$ in two ways. First, a firm $j$ can learn all of firm $i$’s ideas via indirect learning. Second, a firm $j$ can learn $i$’s private idea directly and then the other ideas in the technology $t$ from links with firms other than $i$. The probability of the second possibility is more difficult to express in closed form, and in general depends on the technology $t$. We will show that when if there is not too much interaction, then most competition comes via the first channel.

Payoffs in this model depend on the number of ideas known to a firm in a particular combinatorial manner. We will consider allow payoffs to be a more general function of the number of ideas learned in Section 5.1.

### 2.5 Solution Concept

We now define our solution concept:

**Definition 1.** An equilibrium $(p^*, q^*)$ is a pure-strategy Nash equilibrium. An equilibrium $(p^*, q^*)$ is an **investment equilibrium** if $p_i^* > 0$ for all $i$.

Because all choices $p_i$ and $q_i$ are probabilities of discoveries or interactions, we restrict to pure strategies.

If $p_i = 0$ for all $i$, then any $q$ will give an equilibrium: if no other firms are investing, there is no reason to invest and so payoffs are zero. It is easy to see these trivial equilibria always exist, and we will focus on investment equilibria.

For some of our results, it will also be useful to make the stronger assumption that private investment is non-vanishing asymptotically. We consider a sequence of equilibria as
the number of firms $n \to \infty$.

**Definition 2.** A sequence of equilibria has **non-vanishing investment** if

$$\liminf_{n} \min_{i} p_{i}^{*} > 0.$$ 

Depending on $c(\cdot)$, there may be equilibria at which all firms choose very low levels of private investment because others are investing very little. The definition excludes these partial coordination failures as well.

### 3 Equilibrium

In this section, we characterize equilibrium in our baseline model. Two assumptions that facilitate this analysis are that firms are homogeneous (which we will relax in several ways, including Section 4) and that profits are equal to the number of proprietary technologies (which we relax in Section 5).

We first briefly describe investment equilibria under direct learning ($\delta = 0$). The remainder of this section will characterize investment equilibria under indirect learning ($\delta > 0$).

#### 3.1 Direct Learning

We summarize results with $\delta = 0$ here, and give a full analysis in Appendix B. In this case, ideas can spread at most one step.

There exists a symmetric investment equilibrium for $n$ large, and at any sequence of symmetric investment equilibria the interaction rate is

$$\iota(q^{*}, q^{*}) \approx \left( \frac{k - 1}{n} \right)^{\frac{1}{k}}.$$ 

Since the interaction rate is of order $n^{-\frac{1}{k}}$, the probability that a generic firm knows all the ideas in a given technology is of order $\frac{1}{n}$. It follows that the probability that there exists competition on a given technology is constant.
We will see that interaction rates are higher than in the indirect-learning case. With only direct learning much more interaction is needed to generate a substantial risk of competition, so the interaction rate must be higher for potential competition to meaningfully deter openness.

A key feature of the direct learning environment is that interaction between firms \( j \) and \( j' \) imposes only a negative externality on a third firm \( i \) by increasing potential competition. Therefore, decreasing openness would increase average profits. Formally, at the symmetric equilibrium \( (p^*, q^*) \) we have:

\[
\lim_{n \to \infty} \frac{\partial U_i(p^*, q)}{\partial q}(q^*) < 0.
\]

Once indirect learning is introduced, interaction between firms \( j \) and \( j' \) also imposes a positive externality by facilitating indirect learning by firm \( i \). We will compare the magnitudes of these positive and negative externalities.

### 3.2 Indirect Learning

Our main focus is the indirect learning case \((\delta > 0)\) in which ideas can spread multiple steps. Asymptotically, the structure of equilibrium will depend on the global structure of the indirect-learning network. To better understand this dependence, we let the number of firms \( n \to \infty \) and begin with outcomes under a sequence of symmetric actions.

We say that an event occurs a.a.s. (asymptotically almost surely) if the probability of this event converges to 1 as \( n \to \infty \). To simplify notation, we often omit the index \( n \) (e.g., from the actions \((p_i, q_i)\)).

Recall that \( \overline{q} = \max_i q_i \) and \( \underline{q} = \min_i q_i \) are the minimum and maximum of the level of openness \( q_i \) across players.

**Definition 3.** A sequence of symmetric actions with openness \( q \) is:

- **Subcritical** if \( \limsup_n \iota(q, q) \delta n < 1 \)
- **Critical** if \( \lim_n \iota(q, q) \delta n = 1 \)
- **Supercritical** if \( \liminf_n \iota(q, q) \delta n > 1 \)
The expected number of firms with links to $i$ in the indirect-learning network is

$$\nu(q, q)\delta(n - 1),$$

so the three cases distinguish networks where each firm learns indirectly less than once, approximately once, and more than once in expectation. In the subcritical case, it follows that the expected number of firms that learn a given idea is a finite constant. In the supercritical case, there is a positive probability that a given idea is learned by a large number of firms (i.e., a number growing linearly in $n$).

This intuition is formalized by results from the theory of random directed graphs [Karp 1990 and Luczak 1990]. Adapting their results to this setting, we have the following result.

A component of a directed network is a strongly-connected component, i.e. a maximal set of nodes such that there is a path from any node in the set to any other.

**Lemma 1** (Theorem 1 of Luczak 1990). (i) If the indirect-learning network is subcritical, then a.a.s. every component has size $O(\log n)$.

(ii) If the indirect-learning network is supercritical, then a.a.s. there is a unique component of size at least $\tilde{\alpha}n$ for a constant $\tilde{\alpha} \in (0, 1)$ depending on $\lim n \nu(q, q)\delta n$, and all other components have size $O(\log n)$.

These asymptotic results each imply that large finite graphs have the relevant component structure with high probability. It follows from the lemma that in a subcritical sequence of equilibria, all firms learn at most $O(\log n)$ ideas a.a.s. In a supercritical sequence of equilibria, there is a positive fraction of firms learning a constant fraction of all ideas a.a.s. At a critical equilibrium, the number of ideas learned lies between the subcritical and supercritical cases.

To generalize the notion of criticality to arbitrary strategies, consider the matrix $(\nu(q_i, q_j)\delta)_{ij}$. The entry $(i, j)$ is equal to the probability that firm $i$ learns indirectly from firm $j$. Let $\lambda$ be the largest eigenvalue of this matrix.

**Definition 4.** A sequence of symmetric actions with openness $q$ is:

- **Subcritical** if $\limsup_n \lambda < 1$
- **Critical** if $\lim_n \lambda = 1$
• **Supercritical** if \( \liminf_n \lambda > 1 \)

We will see that, as in Lemma 1, the critical threshold corresponds to the emergence of a giant component. To show this, we will combine the results of Bloznelis, Götze, and Jaworski (2012) with analysis of multi-type branching processes.

Our existence result establishes that there are equilibria with non-zero investment and communication. Our characterization result shows that asymptotically, equilibrium is on the threshold between sparse and dense networks:

**Theorem 1.** For \( n \) sufficiently large, there exists a symmetric investment equilibrium. Any sequence of investment equilibria is critical.

At a sequence of symmetric investment equilibrium, the theorem implies that

\[
\iota(q^*, q^*) \to \frac{1}{\delta n},
\]

and in particular symmetric investment equilibria are asymptotically unique.

We are able to drop the assumption of symmetric strategies, which is standard in settings involving random networks (e.g., Currarini, Jackson, and Pin, 2009; Sadler, 2019; and Golub and Livne, 2010), and show any equilibrium is at the critical threshold. Asymmetric equilibria could feature firms with \( \iota(q^*_i, q^*_i) \) above and below \( \frac{1}{\delta n} \).

While the proof of Theorem 1 relies on existing mathematical results on large random graphs, there are several obstacles to applying these results. First, there are complementarities between ideas, so payoffs do not simply depend on the number of ideas learned. Second, link probabilities are endogenous, which in particular means that lower-order terms in link probabilities and vanishing-probability events can matter asymptotically. We now discuss the key ideas in the proof.

**Proof Intuition.** We describe the basic idea of the proof in the case \( \delta = 1 \), and the general argument is similar. We also begin by discussing symmetric strategies.

We will use the first-order condition for \( q_i \) and the assumption of symmetry to characterize equilibrium behavior. When \( \delta = 1 \), whether a firm \( i \) faces competition depends only on whether another firm \( j \) has learned from \( i \). Since profits are zero when a firm \( j \) has learned
from $i$, we can take firm $i$’s first-order condition in $q_i$ conditioning on the event that no firm has yet learned from $i$. We can also condition on firm $i$ discovering its private idea since profits are zero otherwise.

The first-order condition says that at any best response, the cost to firm $i$ of having a firm $j$ learn from $i$ and is equal to the benefit from learning from an additional firm $j$. Applying this to a firm $i$ at equilibrium, we obtain:

$$\mathbb{E} \left[ \left| I_i(p^*, q^*) \right| \right] = \frac{1}{q^*(n-1)} \cdot \mathbb{E} \left[ \frac{\partial \left( \left| I_i(p^*, q_i, q^*_j) \right| \right)}{\partial q_i} \left( q^* \right) \right]$$

(1)

Recall that $I_i(p^*, q^*)$ is the set of ideas learned by firm $i$, which is a random variable. The left-hand side is the cost of an outgoing link, which erases the monopoly payoffs from any technologies produced by $i$. The right-hand side is the marginal benefit from increasing the probability of learning from each other firm by $\frac{1}{n-1}$, which is approximately the marginal benefit from an additional outgoing link.

A key feature of equation (1) is that the left-hand side and right-hand side both depend on the distribution of the number of ideas learned from a given link. We will exploit this symmetry between costs and benefits to solve for $q^*n$. We are able to do so because of the endogenous downside to outgoing links, which depends on the number of ideas that firm $i$ learns.

We use the first-order condition in equation (1) to obtain an expression for $q^*n$ in terms of the number of incoming links used to learn the ideas in an average proprietary technology. Consider a technology $t$ such that $i$ produces $t$ and gets monopoly profits. This technology is a combination of ideas learned from different links. For example, if $k = 4$, an example technology could consist of $i$’s private idea, two ideas learned indirectly from firm $j$, and one idea learned directly from firm $j''$. In this example, the technology would combine ideas from three different links.

More generally behavior will depend on the number of links utilized in learning the ideas in a technology $t$. We refer to this number of links as $\tau(t)$, so that $\tau(t) = 3$ in the example in the previous paragraph. The key tool, which we state in the subcritical region, is:

**Lemma 2.** Along any sequence of symmetric investment equilibria with $\lim \sup \delta_i(q^*, q^*)n < 17$...
\[ \delta \mu(q^*, q^*) n \sim E_{t \in PT_i(p^*, q^*)}[\tau(t)] \]

for all \( i \).

Lemma \(^2\) says that the expected number of other firms from whom \( i \) learns is equal to the expected value of \( \tau(t) \) for a random proprietary technology \( t \). We give a brief intuition for the lemma. If \( \tau(t) \) is higher, then there are stronger complementarities between links, because produced technologies combine ideas from more links. In this case, if a firm has a few existing links, an additional link will be more valuable than an existing link due to these complementarities. Since additional links are relatively more valuable, firms are willing to interact more.

Since \( \tau(t) \) is always at least one, Lemma \(^2\) implies that

\[ \lim_n \mu(q^*, q^*) n \geq 1, \]

so there cannot be a subcritical equilibrium.

In the supercritical region, almost all proprietary technologies \( t \in PT_i(p^*, q^*) \) are created by combining a private idea with \( (k - 1) \) ideas learned from observing the giant component. In particular, payoffs are determined up to lower order terms by whether firm \( i \) has a link that provides a connection to the giant component. Given such a link, additional links add little value. Thus there are not substantial complementarities between links, and indeed are potential redundancies.

But because firms have more to lose from an outgoing link in the supercritical region, complementarities between links are needed to sustain high interaction rates. Since these complementarities are not present, there is not a supercritical equilibrium either. We check this intuition formally by straightforward algebra.

Extending results to asymmetric equilibria presents several additional technical obstacles. One is that existing mathematical results, e.g., [Bloznelis, Götzte, and Jaworski (2012)], prove

\(^{12}\)This would not be the case if firms could produce technologies of any complexity. Then payoffs grow at an exponential rather than polynomial rate in the number of ideas learned, so additional ideas can be very valuable, as in the growth model of [Acemoglu and Azar (2019)].

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the component structure has certain properties asymptotically almost surely. But this does not remove the possibility that vanishing-probability events distort incentives in an unknown direction. To rule this out, we show that an arbitrary subcritical sequence of equilibria,

\[ \lim_{n \to \infty} \mathbb{P}[|I_i(p, q)| = y] \]

decays exponentially in \( y \). The proof bounds \(|I_i(p, q)|\) above with the number of nodes in a multi-type Poisson branching process and then analyzes this branching process.

We prove the existence result and characterization of symmetric equilibria for any strictly increasing and continuously differentiable interaction rate \( \iota : [0, 1] \times [0, 1] \to [0, 1] \) satisfying:

1. \( \iota(q, q') = \iota(q', q) \) for all \( q \) and \( q' \)
2. \( \iota(q, 0) = 0 \) for all \( q \).

We rely on the multiplicative functional form \( \iota(q_i, q_j) = q_i q_j \) to extend the characterization from symmetric equilibria to arbitrary equilibria.

Theorem 1 makes a sharp prediction about equilibrium. This depends on the specification of payoffs, which are linear in the number of monopoly technologies produced by a firm. We consider how our equilibrium characterization extends to more general payoffs in Section 5.

### 3.3 Welfare and Policy Implications

We next discuss welfare consequences of Theorem 1. At any critical sequence of equilibria, the number of ideas \(|I_i(p^*, q^*)|\) learned by each firm is \( o(n) \) asymptotically almost surely. Since each firm can produce at most \( (|I_i(p^*, q^*)|) \) proprietary technologies,

\[ U_i(p^*, q^*) = o(n^{k-1}). \]

Suppose instead that all firms choose \((p, q)\) where \( p > 0 \) and \( \lim_n \iota(q, q)\delta n \in (1, \infty) \). Then if \( \alpha \in (0, 1) \) is fraction of ideas that are learned by all firms in the giant component
asymptotically almost surely,

$$U_i(p, q) = (p)^k \alpha \left( \binom{\alpha n}{k-1} (1 - \delta \cdot \iota(q, q) - (1 - \delta) \cdot \iota(q, q) \cdot \alpha)^{n-1} - c(p) + o(n^{k-1}) \right).$$

(2)

Asymptotically almost surely, a firm learns all ideas learned by the giant component with probability $\alpha$. In this case, the firm learns $\alpha n + o(n)$ ideas and therefore can produce approximately $\binom{\alpha n}{k-1}$ potential technologies. As we show in the proof of Theorem 1, the probability of facing competition on a given technology is approximately

$$(1 - \delta \cdot \iota(q, q) - (1 - \delta) \cdot \iota(q, q) \cdot \alpha)^{n-1}.$$ 

Since $\iota(q, q)\delta n$ converges, this probability converges to a positive constant. So the growth rate of $U_i(p, q)$ is of order $n^{k-1}$.

Thus, average payoffs are much higher in the supercritical region than the critical region when the number of firms is large. Theorem 1 showed that nevertheless individual incentives to be secretive lead to a critical equilibrium.

Interactions impose two externalities on a firm $i$ that is not directly involved: (1) there is a positive externality as these interactions can lead to more indirect learning by $i$, and (2) there is a negative externality, because these interactions can lead to potential competition with $i$. In the subcritical and critical range, the positive externality dominates. This is because most competition comes from firms that learn indirectly through $i$ and thus the negative externality has little impact.

We could also consider social welfare by including the surplus obtained by consumers from monopoly products and competitive products. It follows easily that the socially optimal outcome will be in the supercritical range, like the outcome maximizing average firm profits.\footnote{More formally, suppose that consumer surplus is $W_c$ from each product with a competitive market and $W_m$ from each product with a monopoly, where $W_c \geq W_m \geq 0$. We can then define social welfare to be the sum of total producer payoffs and consumer surplus. The expected number of competitive products and the expected number of total products produced are both increasing in $q$ when firms choose symmetric strategies. Therefore, the findings in the section that increasing $q$ will increase average profits extend to social welfare.}

We can relate these findings to Saxenian’s (1996) study of the Route 128 and Silicon Valley technology industries, which found that Silicon Valley had much more open firms
and grew faster. In the terminology of our model, Route 128’s secrecy corresponds to equilib-rium behavior. But institutional features of Silicon Valley (including non-enforcement of non-compete clauses and common ownership of firms by venture capital firms) may have constrained firms’ actions to prevent high levels of secrecy (subcritical or critical choices of \( q_i \)).

Theorem 1 and equation \( \text{(2)} \) have several policy implications, which we state informally and then as formal results:

- At or near equilibrium outcomes, there are large gains to policies (e.g. non-enforcement of non-compete clauses, establishing innovation clusters) that encourage or require more interaction between firms and thus shift outcomes to the supercritical region.
- Policies to increase private investment (e.g. subsidies for R&D) will not shift outcomes to the supercritical region, and thus have much smaller benefits at equilibrium.
- But once outcomes are in the supercritical region, policies to increase private investment will have large gains.

We can formalize the first bullet point:

**Corollary 1.** For any sequence of equilibria \((p^*, q^*)\) with non-vanishing investment and any \( \epsilon > 0 \),

\[
\lim_{n \to \infty} U_i(p^*, (1 + \epsilon)q^*) U_i(p^*, q^*) = \infty.
\]

The first corollary says that at equilibrium increasing all \( q_i \) by any multiplicative factor has a very large effect on payoffs asymptotically. The proof shows that such an increase will change payoffs from \( o(n^{k-1}) \) to a polynomial of order \( n^{k-1} \).

We also give the second and third bullet points in a corollary:

**Corollary 2.** For any sequence of equilibria \((p^*, q^*)\) with non-vanishing investment and any \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \frac{\partial U_i(p^* + x1, (1 + \epsilon)q^*)}{\partial x}(0) \frac{\partial U_i(p^* + x1, q^*)}{\partial x}(0) = \infty.
\]
| Best Response $q_i$ | Subcritical | Critical | Supercritical |
|---------------------|-------------|----------|---------------|
| Average Payoffs     | High        | Intermediate | Low           |
| Increasing $q$      | Large Benefit | Large Benefit | Ambiguous     |
| Increasing $p$      | Small Benefit | Intermediate | Large Benefit |

Table 1: Best responses and policy implications when firms choose symmetric strategies $(p, q)$ in the subcritical, critical, and supercritical regions.

If all firms are choosing private investment level $p^*$, increasing private investment slightly has a much larger effect in the supercritical region than at equilibrium. The proof shows that the effect of increasing $p$ is $o(n^{k-1})$ at equilibrium but polynomial of order $n^{k-1}$ in this supercritical region.

### 3.4 Public Innovators

Corollary 1 showed there are large gains to increasing interaction rates above equilibrium levels. A natural question is whether these gains be realized via policy interventions other than directly restricting firms’ strategy spaces.

We now show that introducing **public innovators** who are not concerned with secrecy leads to learning and innovation at the same rate as in the supercritical region. These public innovators correspond to academics or government researchers with incentives or motivations other than profiting from producing and selling technologies.

A public innovator $i$ pays investment cost $c(p_i)$ and receives a payoff of one for each technology $t$ such that: (1) $i \in t$ and (2) $j \in \{i\} \cup I_i(p, q)$ for all $j \in t$. We will rely on the fact that for public innovators there is no downside to interactions, but not on the exact incentive structure.

All firms have the same incentives as in the baseline model, and public innovators and firms interact as in the baseline model. We now call an equilibrium symmetric if all public innovators choose the same action and the same holds for all private firms.

**Proposition 1.** Suppose a non-vanishing share of agents are public innovators. Then there exists a sequence of symmetric equilibria with non-vanishing investment, and at any sequence
of equilibria with non-vanishing investment

$$\liminf_n \frac{U_i(p^*, q^*)}{(n-1)(k-1)} > 0$$

for all firms $i$.

The proposition says that at equilibrium, firms’ payoffs are at least a constant fraction of the maximum achievable profits $(n-1)/(k-1)$. Thus payoffs are of order $n^{k-1}$, as in the supercritical region without public innovators. This holds for any positive share of public innovators, and indeed could be extended to a slowly vanishing share of public innovators.

Public innovators are valuable primarily as informational intermediaries rather than for their private ideas. Because public innovators do not face costs to interaction, they will choose $q_i = 1$ at equilibrium. Therefore, public innovators can learn many ideas via interactions and transmit these ideas to other public innovators or to private firms (e.g., academics learning ideas from conferences and collaborations and then consulting for private industry). Conversely, the proposition would remain unchanged if all public innovators instead choose $p_i = 0$ and $q_i = 1$.

Empirical research on collaboration between academia and industry supports the value of academic researchers as informational intermediaries between firms. Azoulay, Graff Zivin, and Sampat (2012) study movement of star academics, and find that moves increase patent-to-patent and patent-to-article citations locally. Moreover, Jong and Slavova (2014) find that firms that disclose high-quality R&D through publications with academics are more innovative, suggesting information flows exhibit symmetry properties within interactions.

Proposition 1 assumes that firms cannot differentially interact with public innovators and private firms. In Appendix E we show the same result holds when interactions can be directed toward public innovators or private firms.

4 Asymmetric Learning Probabilities

The baseline model assumes that information flows are symmetric across pairs of firms. In practice, firms may have heterogeneous probabilities of learning from others, even give a
fixed interaction rate.

Suppose instead that firm $i$ directly learns from firm $j$ with probability

$$\beta_i(q_i, q_j),$$

where the propensity to learn $\beta_i \in [\beta, 1)$ for some $\beta \in (0, 1)$. Learning otherwise occurs as in the baseline model, including indirect learning.

It is straightforward to extend Definition 4 to allow heterogeneous secrecy. We now let $\lambda$ be the spectral radius of the matrix $(\beta_i(q_i, q_j)\delta)_{ij}$. As before entry $(i, j)$ is equal to the probability that firm $i$ learns indirectly from firm $j$. Let $\lambda$ be the spectral radius of this matrix.

**Definition 5.** A sequence of symmetric actions with openness $q$ is:

- **Subcritical** if $\limsup_n \lambda < 1$
- **Critical** if $\lim_n \lambda = 1$
- **Supercritical** if $\liminf_n \lambda > 1$

Again, the critical threshold corresponds to the emergence of a giant component.

**Theorem 2.** Suppose firms have propensities to learn $\beta$. There exists an investment equilibrium for $n$ large and any sequence of investment equilibria is critical.

Equilibria remain critical even when the directed link probabilities are asymmetric across pairs. The characterization result extends immediately to the case in which $\beta_i$ are chosen endogenously at a cost $\tilde{c}_i(\beta_i)$, which can vary across firms. In this case, firms can now control the likelihood of learning along two dimensions. First, higher interaction rates allow a firm to learn more from others at the expense of a higher probability of its ideas leaking. Second, firms can pay an exogenous cost to increase the probability of learning from others at a given interaction rate, and some firms may be able to do so more cheaply than others.

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14 This choice can be made simultaneously with or prior to the choice of $q_i$. 

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The proof of Theorem 2 shows that decisions with asymmetric learning probabilities are similar to decisions in the baseline model. Recall that at equilibrium, the first-order condition for the openness \( q_i \) relates the number the value of the ideas already known to firm \( i \) with the value of increasing the interaction rate. Fixing \( q_i \), a higher \( \beta_i \) increases both sides of this first-order condition because firms with higher propensities to learn have already learned more ideas but also will learn more from an additional interaction.

In the subcritical region, these two forces cancel out and a firm’s equilibrium choice of openness \( q_i^* \) is approximately independent of that firm’s cost of secrecy \( \tilde{c}_i(\beta_i) \). While the opposing effects do not entirely cancel in the supercritical region, we show that even firms with high \( \beta_i \) will still choose low \( q_i^* \) because those firms have more to lose from outgoing links.

## 5 Structure of Profits

In Sections 2 and 3, we studied equilibrium when expected payoffs were

\[
U_i(p, q) = \mathbb{E} [\|PT_i(p, q)\|] - c(p_i).
\]

Firms’ utility function had two properties:

1. Payoffs are linear in the number of proprietary technologies, and
2. Payoffs do not depend on technologies for which the firm faces competition.

We now relax each of these assumptions. We find that equilibrium remains critical when the returns to producing more technologies are increasing. Changing the profit structure in competitive markets, however, leads to supercritical or subcritical equilibria.

### 5.1 Concavity of Profits

The baseline model assumed that a firm’s profits are linear in the number of proprietary technologies. In practice, there may be increasing or decreasing returns to producing more
technologies. Suppose that the payoffs to firm $i$ are instead

$$|PT_i(p, q)|^\rho - c(p_i),$$

where $\rho > 0$.

The baseline model is the case $\rho = 1$. When $\rho > 1$, there are increasing returns to controlling more monopolies. When $\rho < 1$, there are decreasing returns to controlling more monopolies. Note that these increasing or decreasing returns to scale are not determined by the innovative process, but rather by production costs or other market conditions.

**Proposition 2.** There exists $\rho \leq 1$ such that for any $\rho \geq \rho$, any sequence of symmetric investment equilibria is critical. When $k > 2$, we have $\rho < 1$.

The proposition shows that the prediction of critical equilibria is not knife-edge with respect to $\rho$. In particular, increasing returns to scale cannot move interactions above the critical threshold. As long as $k \neq 2$, slightly decreasing returns to scale will not move interactions below the critical threshold either.

Consider a firm $i$ that does not face competition. We show that under the conditions of the proposition, the firm’s profits are convex in $|I_i(p, q)|$. As a result, learning additional ideas is more appealing relative to protecting existing ideas, so openness will not decrease below the critical region. Checking concavity is delicate when $\rho < 1$, because in this case firm profits are the composition of the binomial coefficient $\binom{|I_i(p, q)|}{k-1}$, which is convex, and the polynomial, $|PT_i(p, q)|^{\rho}$, which is concave.

We also show that for any $\rho$, openness will not increase enough to push equilibrium into the supercritical region either. At a potential supercritical sequence of investment equilibria, profits are driven by the event that firm $i$ learns from the giant component and produces

$$(p^*)^k \binom{\alpha n}{k-1}$$

proprietary technologies, where $\alpha$ is the share of ideas learned by the giant component.

Firm $i$ chooses $q_i$ to maximize the probability of this event. Asymptotically the optimal $q_i$ is independent of the payoffs from this event since these payoffs are very large, and therefore

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the optimal $q_i$ is independent of $\rho$. Given this, the calculation is the same as in the case $\rho = 1$ (Theorem 1), where there is no supercritical sequence of investment equilibria.

More generally, the proof shows that our criticality result relies on two features of the payoff function. First, payoffs for a firm $i$ that does not face competition are convex in the number of ideas $|I_i(p, q)|$ learned by $i$. Second, payoffs grow at a polynomial rate in the number of ideas learned by $i$.

We can state this formally when $\delta = 1$, so that when firm $i$ learns from $j$ it will learn all ideas known to firm $i$. In this case, we let the profits for firm $i$ be $\phi(|I_j(p, q)|)$ when firm $i$ discovers its private idea ($i \in I$) and no firm learns from $i$, and 0 otherwise. We will assume that $\phi(\cdot)$ is strictly increasing and continuously differentiable.

**Proposition 3.** Suppose $\delta = 1$ and payoffs when no firm learns from $i$ and $i \in I$ are equal to $\phi(|I_i(p, q)|)$, where $\phi(x)$ is convex and

$$\frac{\phi(x_j)}{\phi(x'_j)} \to 1$$

along any sequence of $(x_j, x'_j)$ such that $\frac{x_j}{x'_j} \to 1$. Then any sequence of symmetric equilibria with non-vanishing investment is critical.

In particular, the assumption that

$$\frac{\phi(x_i)}{\phi(x'_i)} \to 1$$

along any sequence of $(x_i, x'_i)$ such that $\frac{x_i}{x'_i} \to 1$ holds if

$$\phi(x) =Cx^d + O(x^d)$$

for any $C > 0$ and $d \geq 1$. If payoffs instead grow at an exponential rate in the number of ideas, then a supercritical equilibrium is possible because an additional idea may be very valuable (see Acemoglu and Azar 2019 for a related effect).
5.2 Profits Under Competition

We found in Theorem 1 that equilibrium lies on the critical threshold. This result is robust to different payoffs structures for monopolist firms. We now show that Theorem 1 does depend on the structure of competition, and show that altering payoffs from competitive markets can lead to supercritical or subcritical outcomes.

To generalize the payoff from technologies, we will now assume that firm $i$ receives payoffs $f(m)$ from a technology $t$ such that $i \in t$, firm $i$ learns all other ideas in $t$, and $m$ other firms learn all ideas in $t$, where $f(\cdot)$ is weakly decreasing. We take the normalization $f(0) = 1$.

A simple case is $f(m) = a < 1$ for all $m > 0$. The analysis in previous sections corresponded to the case $a = 0$.

We can also allow $f(m) < 0$, which could correspond to a fixed cost of production that must be paid before competition is known. We assume that firms make a single decision about whether to produce the technologies that they learn.\footnote{\textsuperscript{15} If firms can condition their production decision on the flow of ideas, the analysis becomes more complicated.}

**Proposition 4.** (i) If $0 < f(1) < 1$ and $f(m) \geq 0$ for all $m$, then there exists a symmetric investment equilibrium for $n$ large and any sequence of symmetric investment equilibria is supercritical.

(ii) If $f(m) < 0$ for all $m > 0$, then any sequence of symmetric investment equilibria is subcritical.

Part (i) says that if the potential downside to enabling competitors is not as large, then firms will be more willing to interact. This pushes the equilibrium from the critical threshold into the supercritical region. Cournot competition, for example, would correspond to $f(m)$ satisfying the conditions of part (i) of the proposition.

Proposition 4(i) introduces an additional force to classic debates on whether firms are more innovative in more competitive markets (see Cohen and Levin 1989 for a survey). While much of this literature considers how competition changes firms’ private incentives to conduct R&D, the proposition considers its effect on interaction and learning between firms.

Part (ii) says that increasing the costs of competition discourages interaction, and pushes
the equilibrium to the subcritical region. There need not be an investment equilibrium if payoffs under competition are sufficiently negative.

The proof of (ii) is more involved, as we must characterize payoffs at a critical sequence of equilibria. The key is the following lemma, which states that at the critical threshold, most of firm $i$'s proprietary technologies only include ideas learned from one other firm.

**Lemma 3.** *For any critical sequence of symmetric actions with $p > 0$,*

$$\lim_{n \to \infty} \mathbb{E}_{t \in \mathcal{PT}_i(p,q)}[\tau(t)] = 1.$$  

We use a pair of coupling arguments to show that most profits come from rare events in which a single link (indirectly) lets a firm learn many ideas. Comparing critical equilibria to subcritical equilibria near the critical threshold, we find that the expected number of ideas learned by a firm grows large. Then by comparing critical equilibria to supercritical equilibria near the threshold, we verify that the probability of learning a large number of ideas is small. To complete the proof of the lemma, we show that few technologies are produced by two or more of these rare events occurring simultaneously.

### 6 Patent Rights

In the baseline model, technologies could only be protected via secrecy. We now consider the possibility that a positive fraction of firms receive patents on their ideas. As motivation for this setup, suppose that firms discover different types of ideas and patent law determines which types are patentable. For example, [Bessen and Hunt (2007)](#) discuss the boundaries of patent law in the software industry and how those boundaries have changed over time.

More precisely, a fraction $b \in (0,1)$, of firms receive a **patent** on their private ideas. In this case, other firms cannot use this private idea, either as monopolists or competitors. Formally, a firm $i$ receives payoff 1 from each technology $t$ such that (1) idea $i \in t$; (2) firm $i$ knows all $j \in t$ and no other $j \in t$ receive patents; and (3) either $i$ receives a patent or $i$ is the unique firm that knows all $j \in t$. Else the firm receives payoff 0 from the technology $t$.

If firms choose their level of openness before knowing if their idea is patentable, then
any sequence of investment equilibria is supercritical. Patents substitute for secrecy, and firms can now interact more because with positive probability there will be no downside to interactions. The result and proof are similar to Proposition 4(i), and we omit the details.

Suppose instead that firms can condition their level of openness on whether their idea is patentable. A firm now chooses levels of openness $q_i(0)$, which is the action without a patent, and $q_i(1)$, which is action with a patent.\footnote{We do not allow firms to discriminate in their interactions based on others' patents.} We will refer to the choices at symmetric equilibria as $q^*(0)$ and $q^*(1)$.

For the first part of the following result ($\delta = 0$), we will also assume that each firm $i$ pays cost $\epsilon > 0$ for each realized link. The purpose of this cost is to break near-indifferences in favor of lower interaction rates. We observe in Appendix B that without patents, a small link cost $\epsilon$ has little effect on the equilibrium.

**Proposition 5.** Suppose a fraction $b \in (0, 1)$ of firms receive patents. If $\delta = 0$, then with $k = 2$ and any link cost $\epsilon > 0$, there does not exist an investment equilibrium for $n$ large. If $\delta > 0$, then $\iota(q^*(0), q^*(0))$ is $o(1/n)$ along sequence of any symmetric investment equilibria.

With only direct learning ($\delta = 0$), the proposition says that positive investment cannot be sustained at equilibrium for $n$ large.\footnote{Formally, for all $n$ sufficiently large, all equilibria have $p^* = 0$. When there is no private investment, firms are indifferent to all choices of interaction rates.} This is because of an adverse-selection effect that discourages social interactions.

Because firms receiving patents have no need for secrecy, firms choose very high interaction rates $q_i(1)$. Thus, most interactions are with firms with patents. On the other hand, firms with patents are undesirable to interact with because their ideas cannot be used by others. Because of this adverse selection in the matching process, firms without patents will have much lower expected profits than in the model without patents. When $k = 2$ and there is an arbitrarily small cost to links, this has the effect of shutting down all interaction and investment.

This contrasts with our results on direct learning with no patent rights (Section 3.1 and Appendix B), where there is an investment equilibrium with substantial interaction. With $k > 2$ and patent rights, the adverse selection effect persists but no longer prevents any
equilibrium investment. In this case, the interaction rate between firms without patents is much lower asymptotically than in the result with no patent rights (Appendix B.1).

The direct learning result also suggests a more general adverse-selection effect in strategic network formation. Suppose that agents with lower link formation costs are also less valuable partners for connections. If agents cannot discriminate in their link formation decisions, the composition of the pool of potential partners will discourage connections.

With indirect learning, firms with patents still do not provide private ideas to others, but can now serve as informational intermediaries. The equilibrium learning network now has the following form: there is a clique of firms with patents, as there is no downside to interaction for such firms and so \( q^*(1) = 1 \). Firms without patents now have some interactions with firms without patents, who can transmit ideas from other firms without patents. Interactions between pairs of firms without patents, however, are rare.

Proposition 5 relates to several strands of literature on patent rights. A theoretical and empirical literature, considers firms’ choices between formal and informal intellectual property protections, particularly patents versus secrecy (e.g., Anton and Yao, 2004, Kultti, Takalo, and Toikka, 2006, and the survey Hall, Helmers, Rogers, and Sena, 2014). We focus not on the choice between formal and intellectual property rights but on the interplay between the two. The proposition finds that in markets with some patent rights, firms must sacrifice more learning to achieve a given level of secrecy.

A second contrast is to theoretical findings on patents and follow-up innovation (e.g., Scotchmer, 1991, Scotchmer and Green, 1990, Bessen and Maskin, 2009). This literature investigates when granting patent rights for an idea decreases follow-up innovations involving that idea. In our random-interactions setting, patent rights can not only decrease follow-up innovations involving patent ideas but also decrease follow-up innovations involving other unprotected ideas.

We can use Theorem 1 and Proposition 5 to ask when patent rights improve welfare and what the optimal fraction \( b \) of patentable ideas would be. In the direct-learning case, the proposition gives conditions under which patents are harmful.

In the indirect-learning case, average firm profits and social welfare are higher with interior patent rights \( b \in (0, 1) \) than no patent rights because firms with patents are valuable...
as intermediaries. Under indirect learning, we can ask what value of $b$ maximizes average payoffs. Any positive $b$ provides the benefits of information intermediaries, and so there is a tradeoff between the higher private profits obtained by firms with patents and the social benefits provided by firms without patents. The optimal value of $b$ can be interior asymptotically for low $k$, but for high $k$ the optimal value of $b$ converges to zero as $n$ grows large.

This is easiest to see when $\delta = 1$. In this case, we can compute that

$$e^{-bq^*(0)n} \approx \frac{1}{2},$$

so the average firm profits are approximately

$$(p^*)^{k-1}\left(\frac{1}{2}(1-b)n\right)(b + \frac{1}{4} \cdot (1 - b)).$$

We graph these profits for $n$ large and different values of $k$ in Figure 3. The value of $b^*$ maximizing average firm profits converges to $\frac{1}{3}$ as $n \to \infty$ when $k = 2$ and converges to $\frac{1}{5}$ as $n \to \infty$ when $k = 3$. For $k \geq 4$, the optimal share of patents $b^* \to 0$ as $n \to \infty$. 

Figure 3: Average profits from monopoly products for $n$ large as a function of the patent share $b$, when $\delta = 1$ and $k = 2, 3, 5, \text{ and } 10.$
7 Conclusion

We have studied strategic network formation in large random graphs in the context of an economic application to innovation and social learning. The model is particularly suited to analysis of informal interactions, e.g. between employees of firms, which cannot be fully governed by formal contracts. We find that in these settings, if there are many firms and ideas can travel multiple steps, the global structure of the learning network has stark consequences for incentives and payoffs. In particular, expected payoffs and welfare are much higher in dense learning networks rather than sparse learning networks.

While we have focused on a network-formation game with a tradeoff between secrecy and learning, we have developed more broadly applicable tools for network settings, particularly with complementarities between connections. In Appendix D we show that our analysis extends easily to a related model where the key tradeoff is between private investment and interaction. Beyond this particular extension, the techniques apply more generally to network formation games in large random graphs. Outside of network formation, the same techniques can also be applied to optimizing diffusion processes, e.g. determining the optimal number of seeds for a new product or technology. This includes settings with complementarities across adopters, such as diffusion of a new social media app.

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A Proofs

We say that \( f(n) \sim g(n) \) if \( f(n)/g(n) \to 1 \) as \( n \to \infty \).

A.1 Proof of Theorem 1

We will first prove Theorem 1. We begin by describing the analysis of symmetric equilibria before extending the techniques to arbitrary equilibria.

We begin by showing that we cannot have \( \lim \sup \delta q n < 1 \) along any subsequence of investment equilibria. Our first lemma shows that at a potential sequence of subcritical investment equilibria, the probability of learning a large number of ideas decays exponentially.

Lemma A1. Along any sequence of investment equilibria with \( \delta i(\bar{q}, \bar{q}) n < 1 \), there exists \( C > 0 \) and \( y \) such that the probability that

\[
P[|I_i(p^*, q^*)| = y] \leq e^{-Cy}
\]

for all \( y \geq y \) and all \( i \) and \( n \).

Proof. Let \( ID_i(q^*) \) be the set of firms \( j \) such that there is a path from \( i \) to \( j \) in the indirect-learning network. Then \( I_i(p^*, q^*) \) is the set of ideas such that idea \( j \) is discovered \( (j \in I) \) and \( j \in ID_i(q^*) \) or some firm in \( ID_i(q^*) \) learns directly from \( j \). The probability each firm in \( ID_i(q^*) \) learns directly from \( j \) is at most \( \iota(\bar{q}, \bar{q}) \), so \( |I_i(p^*, q^*)| \) is first-order stochastically dominated by the sum of \( ID_i(q^*) \) random variables distributed as \( \text{Binom}(n, \iota(\bar{q}, \bar{q})) \).

We claim that \( |ID_i(q^*)| \) is first-order stochastically dominated by the number of nodes in the Poisson branching process with parameter \( \iota(\bar{q}, \bar{q}) \delta \). To prove this, it is sufficient to show that a random variable with distribution \( \text{Poisson}(\iota(\bar{q}, \bar{q}) \delta n) \) first-order stochastically dominates a random variable with distribution \( \text{Binom}(n, \iota(\bar{q}, \bar{q}) \delta) \), as \( ID_i(q^*) \) is the set of nodes in a branching process with the distribution of offspring first-order stochastically dominated by \( \text{Binom}(n, \iota(\bar{q}, \bar{q}) \delta) \). By Theorem 1(f) of [Klenke and Mattner (2010)](#), this holds if

\[
(1 - \delta q^*)_n \leq e^{-\iota(\bar{q}, \bar{q}) \delta n}.
\]
Letting $C' = \iota(q, \bar{q})\delta n$, we observe that $(1 - \frac{C'}{n})^n$ is increasing in $n$ and converges to $e^{-C'}$, so the inequality holds.

Now, a standard result shows that there are $y$ nodes in the Poisson branching process with probability

$$e^{-C'y} (C'y)^{y-1} \frac{1}{y!}$$

(Theorem 11.4.2 of Alon and Spencer 2004). Using Stirling’s approximation, we can approximate this probability as

$$\frac{1}{\sqrt{2\pi y}} y^{-\frac{3}{2}} (C')^{-1} (C' e^{1-C'})^y.$$

In particular, this probability decays exponentially because $C' e^{1-C'} < 1$ for positive $C' \neq 1$.

Since $|I_i(p^*, q^*)|$ is first-order stochastically dominated by the sum of $ID_i(q^*)$ random variables distributed as $Binom(n, \iota(q, \bar{q}))$, by the central limit theorem, the probability that

$$|I_i(p^*, q^*)| = y$$

also decays exponentially in $y$.

Our second lemma expresses the first-order condition for $q_i$ at a subcritical sequence of investment equilibria.

**Lemma A2.** Along any sequence of investment equilibria with $\delta \iota(q, \bar{q}) n < 1$,

$$\delta \sum_{j \neq i} \left. \frac{\partial \iota(q_i, q^*_j)}{\partial q_i} \right|_{q^*_i} \cdot \mathbb{E} \left[ \left( |I_i(p^*, q^*_i)| \right) \frac{k}{k-1} \right] \sim \mathbb{E} \left[ \left. \frac{\partial \left( |I_i(p^*, q^*_i)| \right)}{\partial q_i} \right|_{q^*_i} \right]$$

for each $i$.

**Proof.** Suppose other players choose actions $p_{-i}$ and $q_{-i}$.

We claim that due to the assumption that $\delta \iota(q, \bar{q}) n < 1$, competition that is not based on learning of all of firm $i$’s ideas indirectly is lower order. More formally, let $T_i(p, q)$ be the set of technologies $t$ such that $i \in t$ and firm $i$ learns all other ideas $j \in t$.\[18\] The claim

\[18\]Recall that $PT_i(q, q_{-i}) \subset T_i(p, q)$ is the subset of technologies $t$ such that no other firm learns all ideas in $t$. 

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is that if there does not exist a link from firm \(i\) to another firm \(j\) in the indirect-learning network, the conditional probability

\[
\mathbb{E}_{t \in T_i(p^*, q^*)}[1_{t \in PT_i(p^*, q^*)}]
\]

that \(t \in PT_i(p^*, q^*)\) for a technology \(t \in T_i(p^*, q^*)\) chosen at random converges to one. Here, each technology \(t \in T_i(p^*, q^*)\) under a given realization of all random variables is chosen with probability proportional to the probability of that realization.

Suppose \(t \in T_i(p^*, q^*)\) and choose some \(j \in t\) distinct from \(i\). By Lemma A1, the probability that a given firm \(j'\) learns \(y \geq \underline{y}\) ideas decays exponentially in \(y\) at a rate independent of \(n\). By independence, the probability that firm \(j'\) learns ideas \(i\) and \(j\) is at most \(o(\frac{1}{n})\). Therefore, the probability that any firm \(j'\) learns ideas \(i\) and \(j\) is at most \(o(1)\). The technology \(t \in PT_i(p, q)\) if there is no such \(j'\) for any \(j \in t\) distinct from \(i\), so this proves the claim.

Thus, we can express the expected utility of player \(i\) choosing \(p_i \in [0, 1)\) and \(q_i \geq 0\) as

\[
p_i \mathbb{E}
\left[
\frac{|I_i(p^*, q^*)|}{k-1}\prod_{j \neq i}(1 - \delta \cdot \iota(q_i, q_j)) - c(p_i) + o(1),
\right]
\]

We first note that the optimal \(q_i\) does not depend on \(p_i\) or \(p_{-i}\), but instead is chosen to maximize

\[
\mathbb{E}
\left[
\frac{|I_i(p^*, q^*)|}{k-1}\prod_{j \neq i}(1 - \delta \cdot \iota(q_i, q_j)) + o(1).
\right]
\]

The first-order condition gives

\[
\delta \mathbb{E}
\left[
\frac{|I_i(p^*, q^*)|}{k-1}\right]
\sum_{j \neq i} \frac{\partial \iota(q_i, q_j^*)}{\partial q_i}(q_i^*)(1 - \delta \cdot \iota(q_i, q_j))^{-1} \sim \partial \mathbb{E}
\left[
\frac{(1_{I_i(p, q^*)_k})}{k-1}\right](q_i^*).
\]

Finally, we have

\[
\sum_{j \neq i} \frac{\partial \iota(q_i, q_j^*)}{\partial q_i}(q_i^*)(1 - \delta \cdot \iota(q_i, q_j))^{-1} \rightarrow \sum_{j \neq i} \frac{\partial \iota(q_i, q_j^*)}{\partial q_i}(q_i^*)
\]

because \(\limsup \delta \iota(q, q) n < 1\).  \(\square\)
Given $t \in PT_i(p^*, q^*)$, let $\tau(t)$ be the smallest number of (direct or indirect) links such that firm $i$ would still know all technologies $j \in t$ with only $\tau(t)$ of its links.

We next prove Lemma 2, which states that along any sequence of symmetric investment equilibria with $\limsup \delta_t(q^*, q^*)n < 1$,

$$\delta_t(q^*, q^*)n \sim \mathbb{E}_{t \in PT_i(p^*, q^*)}[\tau(t)]$$

for all $i$.

As above, each technology $t \in PT_i(p^*, q^*)$ under a given realization of all random variables is chosen with probability proportional to the probability of that realization.

**Proof of Lemma 2.** We will apply Lemma A2, which gives

$$\delta \cdot \frac{\partial \mu(q_i, q^*)}{\partial q_i}(q^*) \cdot \mathbb{E} \left[ \left( \frac{|I_i(p^*, q^*)|}{k-1} \right) \right] \sim \frac{1}{n} \frac{\partial \mathbb{E} \left[ \left( \frac{|I_i(p^*, q^*)|}{k-1} \right) \right]}{\partial q_i}(q^*)$$

at a symmetric equilibrium.

Let $\Gamma$ be the set of weakly increasing tuples $\gamma = (\gamma_1, \ldots, \gamma_\tau)$ of integers such that $\sum_j \gamma_j = k - 1$. We will write $l(\gamma)$ for the length of the tuple $\gamma$.

Let $X_j$ be i.i.d. random variables with distribution given by the number of ideas that firm $i$ would learn from a firm $j'$ conditional on learning directly from $j'$. That is, $X_j$ is distributed as the sum of a Bernoulli random variable with success probability $p^*$ (corresponding to direct learning) and a random variable distributed as $|I_j(p^*, q^*)|$ with probability $\delta$ and zero otherwise.

Then we claim that

$$\mathbb{E} \left[ \left( \frac{|I_i(p^*, q^*)|}{k-1} \right) \right] = \sum_{\gamma \in \Gamma} \binom{n-1}{l(\gamma)} \mu(q^*, q^*)^{l(\gamma)} \mathbb{E} \left[ \prod_{j=1}^{l(\gamma)} \binom{X_j}{\gamma_j} \right] + o(1). \quad (3)$$

The right-hand side counts the expected number of choices of $k - 1$ ideas learned (directly or indirectly) via different neighbors $j$, allowing for the same idea to be chosen multiple times via distinct neighbors. To show the claim, we must argue that the contribution from choices of $k - 1$ ideas including such repetitions is $o(1)$.

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We first bound the probability that there exists a firm \( j \) such that there are two paths from \( i \) to \( j \) with distinct first edges. Given \( j', j'' \in N_i \), we want to bound the probability there is a \( j \) in the intersection

\[
I_{j'}(p^*, q^*) \cap I_{j''}(p^*, q^*)
\]

above. The expected number of firms with a path to \( j' \) in the indirect learning network is at most \( \frac{1}{1 - \iota(q^*, q^*)n\delta} \), and the same holds for \( j'' \). Therefore, each learns from at most \( \frac{1 + \iota(q^*, q^*)n}{1 - \iota(q^*, q^*)n\delta} \) firms. By independence, the probability of an intersection is thus at most \( \frac{1}{n^2(1 + \iota(q^*, q^*)^2)} \). Because \( \iota(q^*, q^*)\delta n \) is bounded away from one above and \( \delta \) is finite, this implies that the probability there is a firm \( j \) observed via any such \( j' \) and \( j'' \) is bounded above by \( O\left(\frac{1}{n}\right) \).

By Lemma A1, the probability that \( |I_i(p^*, q^*)| = y \) decays exponentially in \( y \). Since \( \binom{|I_i(p^*, q^*)|}{k-1} \) is polynomial in \( |I_i(p^*, q^*)| \), it follows that the contribution to the expectation \( E\left[\binom{|I_i(p^*, q^*)|}{k-1}\right] \) from any \( O\left(\frac{1}{n}\right) \)-probability event is \( o(1) \). This completes the proof of the claim in equation (3).

We can express the right-hand side of Lemma A2 similarly. Recall the right-hand side counts the number of additional sets of \( k-1 \) distinct ideas that would be known to \( i \) if \( i \) added a direct link to an additional random agent. We then have:

\[
\frac{1}{n} E \left[ \frac{\partial I_i(p^*, q^*)}{\partial q_i} \right] = \sum_{\gamma \in \Gamma} \binom{n-2}{l(\gamma) - 1} \iota(q^*, q^*)^{l(\gamma) - 1} E \left[ l(\gamma) \prod_{j=1}^{l(\gamma)} \binom{X_j}{\gamma_j} \right] + o(1).
\]

The same argument shows that the contribution from choices of \( k-1 \) ideas at least one of which is learned via multiple links is \( o(1) \).

Substituting into Lemma A2 we find that

\[
\iota(q^*, q^*)\delta \sim E \left[ \binom{n-2}{l(\gamma) - 1} \right],
\]

where the expectation is taken over all \((k-1)\)-element sets of ideas in \( I_i(p^*, q^*) \), and for each such set, \( l(\gamma) \) is the number of direct links on a path to at least one idea in the set.
Thus,
\[
\lim_{n \to \infty} \nu(q^*, q^*) \delta n = \lim_{n \to \infty} \mathbb{E}[\nu(\gamma)] = \lim_{n \to \infty} \mathbb{E}_{t \in P_T(p^*, q^*)}[\tau(t)].
\]

We now show that we cannot have \(\lim \inf \nu(q^*, q^*) \delta n > 1\) along any subsequence of investment equilibria.

Suppose \(\lim \inf \nu(q^*, q^*) \delta n > 1\) and pass to a convergent subsequence under which \(\lim \nu(q^*, q^*) \delta n\) exists. Theorem 1 of [Karp (1990)] shows that a.a.s. the number of firms that all firms in the giant component learn from is \(\alpha n + o(n)\) for a constant \(\alpha\) increasing in \(\lim q^* n\) and that the number of agents outside the giant component observed by any agent is \(o(n)\).

We have
\[
\mathbb{E} \left[ \left( \left| I_i(p^*, (q_i, q^*_{-i})) \right| \right) \right] = (p^*)^k (1 - (1 - \delta \nu(q_i, q^*))^{\alpha(n-1)+o(n)})((\alpha(n-1))^{k-1} + o(n^{k-1})).
\]

In particular, to solve for firm \(i\)'s choice of \(q_i\) to first order, we need only consider technologies consisting of \(i\)'s private idea and \((k-1)\) ideas learned by the giant component. The probability that such a technology faces competition is
\[
(1 - \delta \cdot \nu(q_i, q^*) - (1 - \delta) \cdot \nu(q_i, q^*) \cdot \alpha)^{n-1} + o(1).
\]

The term \(\delta \cdot \nu(q_i, q^*)\) corresponds to the possibility of a firm \(j\) indirectly learning all of firm \(i\)'s ideas. The term \((1 - \delta) \cdot \nu(q_i, q^*) \cdot \alpha\) corresponds to the possibility of a firm \(j\) directly learning firm \(i\)'s idea (but not indirectly learning from \(i\)) and indirectly learning the ideas learned by the giant component.

Thus, we are looking for \(q_i\) maximizing:
\[
\mathbb{E} \left[ \left( \left| I_i(p^*, (q_i, q^*_{-i})) \right| \right) \right] (1 - \delta \cdot \nu(q_i, q^*) - (1 - \delta) \cdot \nu(q_i, q^*) \cdot \alpha)^{n-1}. \tag{4}
\]

We want to find \(q_i\) maximizing expression \(4\) asymptotically. We claim the derivative of expression \(4\) in \(q_i\) is equal to zero at \(q^*\) only if \(\nu(q^*, q^*) \delta n \leq 1\). A fortiori, we can instead
show this for
\[ \mathbb{E} \left( \frac{\left| I_i(p^*, (q_i, q^*_i)) \right|}{k-1} \right) (1 - \delta \cdot \iota(q_i, q^*))^{n-1}. \]

This is because for \( n \) large, the derivative of this expression in \( q_i \) is positive if the derivative of expression [4] is positive.

The first-order condition for the latter expression at \( q_i = q^* \) implies that
\[
(p^*)^k n (\alpha(n-1))^{k-1} (1-\delta \iota(q^*, q^*))^{n-1} (\alpha(1-\delta \iota(q^*, q^*))(1-\delta \iota(q^*, q^*))^{\alpha(n-1)-1}-(1-(1-\delta \iota(q^*, q^*))))^{\alpha(n-1)}
\]
is \( o(n^{k-1}) \). Solving for \( \iota(q^*, q^*) \) such that this holds, we obtain
\[
\lim_{n} (1 - \delta \iota(q^*, q^*))^{\alpha(n-1)-1} = \frac{1}{1+\delta \alpha}.
\]

Thus,
\[
\lim_{n} e^{-\delta \iota(q^*, q^*)n} = \frac{1}{1+\delta \alpha}. \tag{5}
\]

The left-hand side is the asymptotic probability that a firm does not indirectly learn from a firm which learns all ideas known to all firms in the giant component, and this is \( 1 - \alpha \). So
\[
1 - \alpha = \frac{1}{1+\delta \alpha},
\]
or equivalently
\[
\alpha(\alpha + 1 - \delta) = 0.
\]

Thus, any sequence of solutions to equation [5] must have \( \alpha = 0 \), and therefore cannot be supercritical.

We have now shown Theorem 1 for symmetric investment equilibria. We can now prove that there exists a symmetric investment equilibrium for \( n \) large. We will then complete the proof of the theorem by extending the preceding characterization to arbitrary equilibria.

**Existence**: To check existence of a symmetric investment equilibrium, let \( BR_i(q) \) be the set of best responses \( q_i \) for firm \( i \) when all other firms choose \( q_j = q \) and \( p_j = p > 0 \). Note that the set \( BR_i(q) \) does not depend on the value of \( p \). Because payoffs are continuous in \( q \), the correspondence \( BR_i(q) \) has closed graph.
We have shown that when \( \iota(q, q) < \frac{1}{\delta n} \), any element of \( BR_i(q) \) is equal to the expectation of \( \tau(t) \) over proprietary technologies.\(^{19}\) Because \( \tau(t) \geq 1 \) for all \( t \), when \( \iota(q, q) < \frac{1}{\delta n} \) we have

\[
\iota(BR_i(q), q) \subset \left[ \frac{1}{2\delta n}, \frac{k}{\delta n} \right]
\]

for \( n \) sufficiently large.\(^{20}\)

Suppose that \( \iota(q, q) \geq \frac{1}{2\delta n} \). Then firm \( i \) can achieve a positive payoff by choosing \( q_i \) such that \( \sum_{j \neq i} \iota(q_i, q) = 1 \). On the other hand expected payoffs to firm \( i \) vanish along any sequence of \( q_i \to 0 \). Therefore 0 is not in the closure of \( BR_i(q) \) whenever \( \iota(q, q) \geq \frac{1}{2\delta n} \).

By compactness we can choose \( \epsilon(n) \) such that

\[
\iota(BR_i(q), q) \geq \frac{\epsilon(n)}{\delta n}
\]

when \( \frac{1}{2\delta n} \leq \iota(q, q) \). We therefore have

\[
\iota(BR_i(q), q) \subset \left[ \frac{\epsilon(n)}{\delta n}, 1 \right] \text{ when } \iota(q, q) \in \left[ \frac{\epsilon(n)}{\delta n}, 1 \right]
\]

for \( n \) sufficiently large. So by Kakutani’s fixed-point theorem, for \( n \) sufficiently large there exists a fixed point of \( BR_i(q) \) at which \( \iota(q, q) \in \left[ \frac{\epsilon(n)}{\delta n}, 1 \right] \).

Fix any fixed point \( q^* \) of \( BR_i(q) \). Let the potential proprietary technologies \( PPT_i(q) \) be the set of technologies \( t \) such that firm \( i \) will receive monopoly profits for \( t \) if all ideas in the technology \( t \) are discovered. This is a random object depending on the realizations of interactions but not on the realizations of private investment, and \( PPT_i(q) \cap I = PT_i(p, q) \).

We have shown that \( q^* \) is critical, so \( \mathbf{E}[|PPT_i(q)|] \to \infty \).

A symmetric equilibrium corresponds to \( p^* \) satisfying

\[
p^* = \arg\max_p p_i(p^*)^{k-1} \mathbf{E}[|PPT_i(q)|] - c(p).
\]

\(^{19}\)We stated this result above at equilibrium, but only used that firm \( i \) was choosing a best response.\(^{20}\) Indeed, by the lemma we could take any open interval containing \( \left[ \frac{1}{\delta n}, \frac{k-1}{\delta n} \right] \) instead of \( \left[ \frac{1}{2\delta n}, \frac{k}{\delta n} \right] \).
Taking the first-order condition, a symmetric equilibrium corresponds to \( p^* \) satisfying
\[
c'(p^*) = (p^*)^{k-1}E[|PPT_i(q)|].
\]
Because \( c(\cdot) \) is continuously differentiable and convex with \( c'(0) \geq 0 \) and \( c(p) \to \infty \) as \( p \to 1 \), while \( E[|PPT_i(q)|] \to \infty \), there exists a solution for \( n \) sufficiently large. So there exists a symmetric investment equilibrium for \( n \) sufficiently large.

To complete the proof of Theorem 1, it remains to extend our characterization to arbitrary equilibria. It is sufficient to show that we cannot have a supercritical sequence of investment equilibria or a subcritical sequence of investment equilibria, and we treat each case separately.

**Supercritical Case:** Because the sequence of actions is supercritical, we can assume that the matrix \((\iota(q_i, q_j)\delta)_{ij}\) has spectral radius at most \( \lambda > 1 \) for all \( n \) sufficiently large.

We first claim that there exists \( \alpha > 0 \) such that for all \( n \), there is a component of the learning network containing at least \( \alpha n \) firms a.a.s. It is sufficient to show this after decreasing \( q_i \) for some \( i \), and therefore also \( \lambda \). So we can assume without loss of generality that there are at most \( K \) choices of \( q_i \) for each \( n \). Here the number of distinct actions \( K \) can depend on the initial upper bound \( \Lambda \). We denote the number of firms choosing \( q_i \) by \( n(q_i) \).

By Theorem 1 of Bloznelis, Götze, and Jaworski (2012), the largest component has at least \( \alpha n + o(n) \) nodes a.a.s., where \( \alpha \) is the extinction probability of the multi-type branching process with types corresponding to choices of \( q_i \) and the number of successors of type \( q_{i'} \) of a node of type \( q_i \) distributed as a Poisson random variable with mean \( \delta \iota(q_i, q_{i'})n(q_{i'}) \). By Theorem 2 of Section V.3 of Athreya and Ney (1972), this extinction probability \( \alpha > 0 \) for \( n \) large since \( \lambda > \Lambda > 1 \). This proves the claim, and we now return to studying the original actions \( q \).

Each firm can choose \( q_i \) such that the expected number of learning opportunities is equal to 1. Because the probability other firms learn the ideas in the giant component is increasing in their actions \( q_j \), the probability of learning all ideas learned by the component of \( \alpha n \) firms when such a component exists is at least \( 1/e \). So firm payoffs under this choice of \( q_i \) are of order \( n^{k-1} \). Because the probability of a firm indirectly learning from a firm \( i \) grows exponentially in \( q_i \), it follows that we can choose \( C \) such that \( \overline{q} \) is at most \( \frac{C}{\sqrt{n}} \) for all \( n \).
Because there is a component of the learning network containing at least $\alpha n$ firms with probability at least $\epsilon$, we can also choose $C$ such that $q$ is at most $\frac{1}{\sqrt{n}}$ for all $n$. To see this, note that the payoffs to choosing $q_i = \frac{1}{\sqrt{n}}$ are of order $n^{k-1}$. On the other hand, the payoffs to choosing $q_i = \frac{C}{\sqrt{n}}$ are bounded above by $(1 - e^{-\frac{C}{C}})n^{k-1}$. So for $C$ sufficiently small, the expected payoffs to choosing $q_i = \frac{C}{\sqrt{n}}$ conditional on any realization of all links between firms other than $i$ are less than the expected payoffs to choosing $q_i = \frac{1}{\sqrt{n}}$ at equilibrium.

As $n$ grows large, the probability that $i \in t$ for a uniformly chosen $t \in PT_i(q_i, q_{-i})$ approaches zero. Similarly the probability that $\bar{i} \in t$ for a uniformly chosen $t \in PT_i(q_i, q_{-i})$ approaches zero. We will show that for any sequence of best responses $q_i$ for firm $i$, the expected number of interactions

$$\sum_{j \neq i} \iota(q_i^*, q_j^*)$$

has a unique limit.

To show this, we next claim that there is at most one component of linear size a.a.s. To do so, we will use equation (5) of Bloznelis, Götze, and Jaworski (2012). In the notation of Bloznelis, Götze, and Jaworski (2012), the type space $S$ will be $S = [C, \bar{C}]$, and the kernel $\kappa(s, s') = ss'$. We will identify the type of an agent $i$ with $q_i \sqrt{n}$.

The space of distributions $\Delta(S)$ over types is compact. Fix such a distribution. Relabelling so that $q_i$ are increasing in $i$, we can generate a random network for each $n$ by taking the action $q_i$ of agent $i$ will be $s_i / \sqrt{n}$, where $s_i$ is the $i^{th}$ quantile of the distribution. As $n \to \infty$, by equation (5) of Bloznelis, Götze, and Jaworski (2012), the largest component of the learning network learns $\alpha n + o(n)$ ideas for some $\alpha \in [0, 1]$. It follows from Theorem 1 and the same approximation techniques used in that paper that the second largest component learns $o(n)$ ideas. Because the space of distributions $\Delta(S)$ is compact, this convergence of component sizes is uniform. So passing to a convergent subsequence if necessary, we can assume that there is a unique giant component learning $\alpha n + o(n)$ ideas a.a.s., where $\alpha > 0$.

The payoffs to choosing $q_i$ are then equal to

$$(p^*)^k \binom{\alpha n}{k - 1}$$

times the probability that firm $i$ learns all ideas known to the giant component and no firm
$j$ learns $i$'s idea and all ideas known to the giant component, plus a term of order $o(n^{k-1})$. Formally, if $G_1$ is the set of firms that learn all ideas known to the giant component, the action $q_i$ is chosen to maximize:

$$\left(\frac{\alpha n}{k-1}\right) \left(1 - \prod_{j \in G_1} (1 - \delta \iota(q_i, q_j^*)) \right) \prod_{j \in G_1} (1 - \iota(q_i, q_j^*)) \prod_{j \notin G_1} (1 - \delta \iota(q_i, q_j^*)) + o(n^{k-1}).$$

Taking the first-order condition, we must have:

$$\delta \sum_{j \in G_1} q_j^* \sim \left(1 - \prod_{j \in G_1} (1 - \iota(q_i, q_j^*)) \right) \left(\sum_{j \in G_1} q_j^* + \delta \sum_{j \notin G_1} q_j^* \right).$$

Since the right-hand side is increasing in $q_i$, the solution has a unique limit

$$\lim_{n} \sum_{j \neq i} \iota(q_i^*, q_j^*).$$

This limit does not depend on $i$. Passing to a subsequence if necessary, we can assume the limit

$$\lim_{n \to \infty} \sum_{j} \iota(q_i, q_j^*)$$

exists and is independent of $i$. Moreover, this limit must be greater than $\frac{1}{\delta}$ for equilibrium to be supercritical.

But then the same calculation as in the symmetric case shows that the best response $q_i$ for all firms is at most $\frac{\delta}{\sqrt{n}}$ asymptotically, which gives a contradiction.

**Subcritical Case:** The largest component of the learning network has at most $o(n)$ nodes a.a.s. We will derive an asymmetric version of the characterization in Lemma 2.

By the same argument given in Case 1, we can choose $C$ such that $\overline{q}$ is at most $\frac{C}{\sqrt{n}}$ for all $n$. We now proceed to derive a characterization of equilibrium as in Lemma 2. We will then use this characterization to show the result.

We first claim, as in the proof of Lemma 2, that the contribution to

$$\mathbb{E} \left[ \left( \frac{|I_i(P^*, q^*)|}{k - 1} \right) \right]$$


from ideas that are learned via multiple direct connections is lower order.

The key to the claim is the following lemma:

**Lemma A3.** Consider a subcritical sequence of actions such that \( \iota(q_i, q_i)n \) is bounded above uniformly. Then

\[
\lim_{n \to \infty} P[|I_i(p^*, q^*)| = y] 
\]

decreases at an exponential rate in \( y \).

**Proof.** Because the sequence of actions is subcritical, we can assume that the matrix \((\iota(q_i, q_j)\delta)_{ij}\) has spectral radius at most \( \overline{\lambda} < 1 \) for all \( n \) sufficiently large. Increasing \( q_i \) for some \( i \) and therefore also \( \overline{\lambda} \), we can assume without loss of generality that there are at most \( K \) choices of \( q_i \) for each \( n \). Here the number of distinct actions \( K \) can depend on the initial upper bound \( \overline{\lambda} \). We denote the number of firms choosing \( q_i \) by \( n(q_i) \).

We will bound the number of firms \( j \) with a path from \( j \) to \( i \) in the indirect learning network above by the number of nodes in a multi-type branching process with the number of successors distributed as Poisson random variables. The types will correspond to the (at most \( K \)) choices of \( q_i \).

For each firm \( i \), the number of firms choosing \( q_j \) that firm \( i \) learns from indirectly is a binomial random variable with success probability \( \delta \iota(q_i, q_j) \) and at most \( n(q_j) \) trials. We showed in the proof of Lemma A1 that such a random variable is first-order stochastically dominated by a Poisson random variable with parameter \( \delta \iota(q_i, q_j)n(q_j) \).

Therefore, the number of firms that firms \( j \) with a path from \( j \) to \( i \) in the indirect learning network is first-order stochastically dominated by the number of nodes in the multi-type branching process such that the number of successors of a node of type \( q_i \) of each type \( q_j \) is distributed as a a Poisson random variable with mean \( \delta \iota(q_i, q_j)n(q_j) \). Call this number of nodes \( y' \).

We want to show that \( y' < \infty \) with probability one and the probability that \( y' = y \) decays exponentially in \( y \). The (at most \( K \times K \)) matrix \( \delta \iota(q_i, q_j)n(q_j) \) has spectral radius at most \( \overline{\lambda} \) because \((\iota(q_i, q_j)\delta)_{ij}\) does. Therefore, by Theorem 2 of Section V.3 of Athreya and Ney (1972), the probability that \( y' = \infty \) is zero.

Let \( Z_j \) be the number of nodes in the \( j^{th} \) generation of the branching process. By Theorem
1 of Section V.3 of [Athreya and Ney (1972)], the probability that \( Z_j > 0 \) decays is of order at most \( \lambda_i \). So the probability that \( Z_T > 0 \) decays exponentially in \( T \).

Dropping nodes with zero probability of interaction if necessary, we can assume that all \( q_i > 0 \). By the Perron-Frobenius theorem, there exists an eigenvector of \( (\delta_i(q_i, q_j))_{ij} \) with positive real entries and eigenvalue equal to the spectral radius of this matrix. We call this eigenvector \( \mathbf{v} \).

We claim that the probability that there are more than \( T v_i \) nodes of some type \( q_i \) in one of the generations 1, \ldots, \( T \) decays exponentially in \( T \). There is one node in generation zero. Suppose that there are at most \( T v_i \) nodes of each type \( q_i \) in generation \( j \). Then by our construction of \( \mathbf{v} \), the number of nodes of each type \( q_i \) in generation \( j + 1 \) is Poisson with mean at most \( T v_i \lambda \). A Poisson random variable of mean \( T v_i \lambda \) is the sum of \( T \) Poisson random variables of mean \( v_i \lambda \). So by the central limit theorem, the probability that there are at least \( T v_i \) such nodes decays exponentially in \( T \), independent of \( j \). This implies the claim.

We have completed the proof that \( y' < \infty \) with probability one and the probability that \( y' = y \) decays exponentially in \( y \). Finally, \( |I_j(p^*, q^*)| \) is first-order stochastically dominated by the sum of \( y' \) Bernoulli random variables with \( n \) trials and success probability \( \max_i \epsilon(q_i, q_i) \). The statement of the lemma now follows by the central limit theorem.

This is because the probability that there are links from \( j \) to \( i \) in the indirect learning network for \( l \) firms \( j \) decays exponentially in \( l \). On the other hand, the probability of learning the ideas in a component multiple times converges to zero because each component is \( o(n) \) a.a.s. This gives the claim.

We now let \( X_j \) be i.i.d. random variables with distribution given by the number of ideas that firm \( i \) would learn from a firm \( j \) conditional on learning directly from \( j \). That is, \( X_j \) is distributed as the sum of a Bernoulli random variable with success probability \( p^*_j \) (corresponding to direct learning) and a random variable with the distribution of \( |I_j(p^*, q^*)| \) with probability \( \delta \) and equal to zero otherwise.
\[
\mathbb{E} \left[ \left( |I_i(p^*, q^*)| \right) \right] = \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{|\gamma|} \neq i} l(\gamma) \prod_{r=1}^{l(\gamma)} q^*_j q^*_r \mathbb{E} \left[ \prod_{r=1}^{l(\gamma)} X_{jr} ( \gamma_r ) \right] + o(\mathbb{E} \left[ \left( |I_i(p^*, q^*)| \right) \right]).
\]

The second summation is over choices of \(l(\gamma)\) of distinct firms other than \(i\).

\[
\frac{1}{n} \mathbb{E} \left[ \frac{\partial (l_i(p^*, q^*, q^*))}{\partial q_i} (q^*) \right] = \frac{1}{n} \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{|\gamma|} \neq i} l(\gamma) \prod_{r=1}^{l(\gamma)} q^*_j q^*_r \mathbb{E} \left[ \prod_{r=1}^{l(\gamma)} X_{jr} ( \gamma_r ) \right] + o(\mathbb{E} \left[ \left( |I_i(p^*, q^*)| \right) \right]).
\]

By Lemma \(\text{A2}\) we have

\[
\delta(\sum_{j \neq i} q^*_j) \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{|\gamma|} \neq i} l(\gamma) \prod_{r=1}^{l(\gamma)} q^*_j q^*_r \mathbb{E} \left[ \prod_{r=1}^{l(\gamma)} X_{jr} ( \gamma_r ) \right] \sim \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{|\gamma|} \neq i} l(\gamma) \prod_{r=1}^{l(\gamma)} q^*_j q^*_r \mathbb{E} \left[ \prod_{r=1}^{l(\gamma)} X_{jr} ( \gamma_r ) \right].
\]

Rearranging,

\[
\sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{|\gamma|} \neq i} l(\gamma) \prod_{r=1}^{l(\gamma)} q^*_j q^*_r \mathbb{E} \left[ \prod_{r=1}^{l(\gamma)} X_{jr} ( \gamma_r ) \right] (\delta q^*_i(\sum_{j \neq i} q^*_j) - l(\gamma)) \sim 0.
\]

In particular, we have

\[
\delta q^*_i(\sum_{j \neq i} q^*_j) \sim \mathbb{E}_{t \sim G_i(p^*, q^*)}[\tau(t)]
\]

where the expectation is taken with respect to the appropriate distribution \(G_i(p^*, q^*)\) over technologies.

As in the proof of Theorem \(\text{I}\) this implies that \(\liminf_{n \to \infty} \delta q^*_i(\sum_{j \neq i} q^*_j) \geq 1\) for each \(i\). So the limit inferior of the row sums of \((\delta q^*_i, q^*_j)\) is at least one. Thus the spectral radius of this matrix also satisfies

\[
\limsup_n \lambda \geq 1,
\]

which contradicts our assumption that the sequence of equilibria is subcritical.

We conclude that any sequence of investment equilibria must be critical, which proves Theorem \(\text{I}\).
A.2 Remaining Proofs

Proof of Corollary 1. By Theorem 1, each firm learns \( o(n) \) ideas at the equilibrium \((p^*, q^*)\). Therefore, \( U_i(p^*, q^*) = o(n^{k-1}) \).

The spectral radius \( \lambda \to 1 \) at a sequence of investment equilibria by Theorem 1. Let \( \lambda' \) be the spectral radius under actions \((p^*, (1 + \epsilon)q^*)\) for each \( n \).

Let \( \epsilon > 0 \). Because the first partial derivative of \( \iota(\cdot, \cdot) \) is a strictly positive and continuous function on a compact set, we can bound the first partial derivative of \( \iota(\cdot, \cdot) \) above and away from zero from below. It follows that we can choose \( 0 < \epsilon < \tau \) such that

\[
1 + \epsilon < \lambda' < 1 + \epsilon
\]

for \( n \) sufficiently large.

Therefore, as shown in the proof of Theorem 1 when actions are \((p^*, (1 + \epsilon)q^*)\), there is a giant component of firms learning at least \( \alpha n \) ideas for some \( \alpha > 0 \). The payoffs to firm \( i \) from the event that \( i \) learns all ideas known to the giant component and no other firm learns from \( i \) grows at rate proportional to \( n^{k-1} \). Firm \( i \) can ensure this event has non-vanishing probability, for example by setting \( q_i = \frac{1}{\sqrt{n}} \). So at any sequence of equilibria with non-vanishing investment, expected profits

\[
U_i(p^*, (1 + \epsilon)q^*)
\]

must grow at rate proportional to \( n^{k-1} \). The result follows from comparing the two growth rates.

\( \square \)

Proof of Corollary 2. Recall the potential proprietary technologies \( PPT_i(q) \) are the set of technologies \( t \) such that firm \( i \) will receive monopoly profits for \( t \) if all ideas in the technology \( t \) are discovered. This is a random object depending on the realizations of interactions but not on the realizations of private investment, and \( PPT_i(q) \cap I = PT_i(p, q) \).
Given actions \((p, q)\)

\[ U_i(p, q) = \mathbb{E} \left[ \sum_{t \in PPT_i(q)} \prod_{j \in t} p_j \right] - c(p_i). \]

Because \(\mathbb{E}[|PPT_i(q^*)|]\), the first-order condition in \(p_i\) implies that we must have \(p_i^*\).

Along a sequence of equilibria with non-vanishing investment, we must have \(p^* \to 1\) since \(\mathbb{E}[|PPT_i(q^*)|]\) \(\to\) \(\infty\). Therefore,

\[ \frac{\partial U_i(p^* + x1, q^*)}{\partial x}(0) \sim k\mathbb{E}[|PPT_i(q^*)|] - c'(p_i^*). \]

By the first-order condition for \(p_i\),

\[ \mathbb{E}[|PPT_i(q^*)|] \sim c'(p_i^*). \]

Combining these approximate equalities, it follows that

\[ \frac{\partial U_i(p^* + x1, q^*)}{\partial x}(0) \sim (k - 1)\mathbb{E}[|PPT_i(q^*)|] \]

The same arguments as in the proof of Corollary \(\square\) show that \(\mathbb{E}[|PPT_i(q^*)|]\) is \(o(n^{k-1})\) while \(\mathbb{E}[|PPT_i((1 + \epsilon)q^*)|]\) is a polynomial of order \(n^{k-1}\).

Hence we also have

\[ \mathbb{E}[|PPT_i((1 + \epsilon)q^*)|] > \mathbb{E}[|PPT_i(q^*)|] \sim c'(p^*) \]

for \(n\) large, so

\[ \frac{\partial U_i(p + x1, (1 + \epsilon)q^*)}{\partial x}(0) > (k - 1)\mathbb{E}[|PPT_i((1 + \epsilon)q^*)|] \]

for \(n\) large.

The corollary follows from the growth rates of \(\mathbb{E}[|PPT_i(q^*)|]\) and \(\mathbb{E}[|PPT_i((1 + \epsilon)q^*)|]\). \(\square\)

Proof of Proposition \(\square\) Let \(b(n)\) be the share of public innovators for each \(n\).
We first show that

\[
\liminf_{n} \frac{U_i(p^*, q^*)}{\binom{n-1}{k-1}} > 0
\]

for all \(i\) at any investment equilibrium.

It is weakly dominant and strictly preferred at any investment equilibrium for all public innovators to choose \(q_i = 1\). Therefore, all public innovators are in the same component of the learning network. Private investment \(p_i\) by public innovators is non-vanishing, so asymptotically almost surely all firms in this component learn at least \(\alpha n\) ideas for some \(\alpha > 0\).

Let \(\bar{q}\) and \(\underline{q}\) be the maximum and minimum levels of openness \(q_i^*\) chosen by private firms, respectively. Because the probability that no firm learns indirectly from \(i\) vanishes exponentially in \(\mathcal{U}(q^*_i, 1)n\) while payoffs are \(O(n^{k-1})\), the quantity \(\mathcal{U}(\bar{q}, 1)n\) must be bounded at equilibrium.

Therefore, a.a.s. a given firm \(i\)'s links are all with public innovators. Since learning indirectly from a public innovator implies learning at least \(\alpha n\) ideas, it follows that \(\mathcal{U}(q, 1)n\) does not vanish asymptotically. Therefore, the expected payoff \(U_i(p^*, q^*)\) has order \(n^{k-1}\) for each firm \(i\). This proves the characterization of investment equilibria.

It remains to show there exists a sequence of symmetric equilibria with non-vanishing investment. Recall that we now call an equilibrium symmetric if all public innovators choose the same action and the same holds for all private firms.

Suppose that all public innovators choose \(p_i \geq \frac{1}{2}\) and \(q_i = 1\) and all firms other than \(i\) choose \((p, q)\) with \(p \geq \frac{1}{2}\) and \(\delta qn \leq 1\). If \(q_i\) is the best response for \(i\), then \(\lim_{n} q_i n\) exists and is independent of \((p, q)\). This is because the probability of interactions between \(i\) and other firms vanishes asymptotically, while the best response does not depend on the number of ideas learned by the unique giant component.

Therefore, we can choose \(\epsilon > 0\) such that if \(q \in [\frac{\epsilon}{\delta n}, \frac{1}{\delta n}]\), then for \(n\) large so is any best response \(q_i\) for firm \(i\). We claim that for \(n\) large, given \(p\), there exists \(q\) that is a best response to \((p, q)\). This follows from Kakutani’s fixed point theorem as in the proof of Theorem 1.

We call this choice of openness \(q(p)\).

Given such \((p, q)\), each firm has a non-vanishing probability of learning a linear number
of ideas. Therefore, $E[|I_i(p, q)|] \to \infty$. So any best response $p_i$ for each public innovator and each firm $i$ has $p_i \geq \frac{1}{2}$. By Kakutani’s fixed point theorem, there exist symmetric actions $(p, q(p))$ such that $p_i$ is also a best response for each $i$. Thus there exists a sequence of symmetric equilibria with non-vanishing investment.

**Proof of Theorem 3** We first characterize equilibria by showing we cannot have a supercritical and then subcritical sequence of investment equilibria. We then show there exists an investment equilibrium for $n$ large.

**Supercritical Case:** Suppose there is a supercritical sequence of investment equilibria with choices $\beta^*_i$ of secrecy.

Passing to a subsequence if necessary, we can assume that all firms in the giant component learn $\tilde{\alpha}n + o(n)$ ideas for some $\tilde{\alpha}$ and that the number of firms learning all ideas learned by the giant component is $\alpha n + o(n)$ for some $\alpha$\footnote{Because link probabilities are no longer symmetric within pairs, we do not assume that $\alpha = \bar{\alpha}$.} The argument is the same as in the proof of Theorem 1.

For each $i$, let $\alpha_i$ be the probability that firm $i$ learns all ideas learned by all firms in the giant component. Finally, we let

$$\overline{\beta} = \frac{\sum_j \beta^*_j q^*_j}{\sum_j q^*_j}.$$  

As $n \to \infty$, this converges to the derivative of the number of firms that learn from firm $i$ in $q_i$ divided by $\sum_{j \neq i} q^*_j$.

The first-order condition for firm $i$ then implies:

$$\delta \alpha_i \overline{\beta} \leq (1 - \alpha_i) \beta^*_i \alpha + o(1). \tag{7}$$

Suppose we increase $q_i / \sum_{j \neq i} q^*_j$ infinitessimally. We can condition on the event that no firm has learned indirectly from $i$. The left-hand side is the probability that a firm $i$ has learned indirectly from the giant component times the probability that a firm learns indirectly from $i$ after this increase. The right-hand side is the probability that firm $i$ has not learned indirectly from the giant component times the probability that firm $i$ learns indirectly from the giant component after this increase.
We have $\alpha_i = 1 - e^{-\alpha \beta_i \delta \sum_{j \neq i} \ell(q_i^*, q_j^*)}$. Substituting into equation (7),

$$(1 - e^{-\alpha \beta_i \delta \sum_{j \neq i} \ell(q_i^*, q_j^*)}) \beta^* \leq e^{-\alpha \beta_i \delta \sum_{j \neq i} \ell(q_i^*, q_j^*)} \beta^* \alpha + o(1).$$

Therefore,

$$\delta \sum_{j \neq i} \ell(q_i^*, q_j^*) \leq \frac{\log(1 + \alpha \beta_i \delta / \beta^*)}{\alpha \beta_i^*} + o(1).$$

We claim the limit superior of the right-hand side as $n \to \infty$ is at most one. Since $\beta_i \leq 1$, this will contradict our assumption that equilibrium is supercritical.

We have $\beta^* \leq 1$ since $\beta_j^* \leq 1$ for all $j$, so it is sufficient to show that

$$\frac{\log(1 + \alpha \beta_i \delta)}{\alpha \beta_i^*} < 1.$$  

This is a special case of the general elementary property $x > \log(1 + x)$ for $x > 0$.

Therefore, we have

$$\delta \beta_i^* \sum_{j \neq i} \ell(q_i^*, q_j^*) < 1$$

for $n$ sufficiently large. Since this holds for all $i$, the spectral radius of the matrix $(\delta \beta_i^* \ell(q_i^*, q_j^*))_{i,j}$ of indirect learning probabilities is less than one for all $n$ sufficiently large. This contradicts our assumption that the sequence of equilibria is supercritical.

**Subcritical Case:** Suppose there is a subcritical sequence of investment equilibria. Introducing choices of secrecy, Lemma A2 states that

$$\mathbb{E} \left[ \left| I_i(p^*, q^*) \right| \right] = \frac{1}{n} \mathbb{E} \left[ \frac{\partial (I_i(p^*, (q_i^*, q_j^*))}{\partial q_i} (q^*) \right] + o(\mathbb{E} \left[ \left| I_i(p^*, q^*) \right| \right]).$$

We now let $X_j$ be i.i.d. random variables with distribution given by the number of ideas that firm $i$ would learn from a firm $j$ conditional on learning directly from $j$. That is, $X_j$ is distributed as the sum of a Bernoulli random variable with success probability $p_j^*$ (corresponding to direct learning) and a random variable with the distribution of $|I_j(p^*, q^*)|$ with probability $\delta$ and equal to zero otherwise.
By the same arguments as in the proof of Theorem 1, we have:

$$E \left[ \left( |I_i(p^*, q^*)| \right) \right] = \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{(\gamma)} \neq i} \prod_{r=1}^{l(\gamma)} \beta_i q_j^* \prod_{r' \neq r} \left( X_{jr} \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right).$$

and also

$$\frac{1}{n} \left[ \frac{\partial \left( \frac{1}{k-1} \prod_{r=1}^{l(\gamma)} \beta_i q_j^* \prod_{r' \neq r} \left( X_{jr} \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right) \right)}{\partial q_i} \right] = \frac{1}{n} \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{(\gamma)} \neq i} \left( \sum_{r=1}^{l(\gamma)} \beta_i q_j^* \prod_{r' \neq r} \left( X_{jr} \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right) \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right).$$

Therefore,

$$\delta \left( \sum_{j \neq i} q_j^* \beta_j \right) \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{(\gamma)} \neq i} \prod_{r=1}^{l(\gamma)} q_j^* \prod_{r' \neq r} \left( X_{jr} \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right) \sim \sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{(\gamma)} \neq i} \left( \sum_{r=1}^{l(\gamma)} q_j^* \prod_{r' \neq r} q_j^* \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right) \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right).$$

Note that the $\beta_i$ terms cancel. Rearranging,

$$\sum_{\gamma \in \Gamma} \sum_{j_1, \ldots, j_{(\gamma)} \neq i} \prod_{r=1}^{l(\gamma)} q_j^* \prod_{r' \neq r} \left( X_{jr} \right) \prod_{r=1}^{l(\gamma)} \left( \frac{1}{E} \left[ \left( \left( |I_i(p^*, q^*)| \right) \right) \right] \right) \left( \delta q_i^* \sum_{j \neq i} q_j^* \beta_j \right) - l(\gamma) \sim 0.$$

In particular, we have

$$\delta q_i^* \sum_{j \neq i} q_j^* \beta_j \sim E_{t \sim G_i(p^*, q^*)}[\tau(t)]$$

where the expectation is taken with respect to the appropriate distribution $G_i(p^*, q^*)$ over technologies. Because $\tau(t) \geq 1$ for all $t$, the limit superior of the expected number of firms that learn from $i$ is at least one. This contradicts our assumption that the sequence of equilibria is subcritical.

**Existence:** We will show there exists an investment equilibrium for $n$ large. To do so, we first fix $p$ with

$$\liminf_n \min_i p_i > 0$$

for all $i$. Given such a $p$, we consider the set of best responses $BR(q_{-i})$ for firm $i$ when other firms choose actions $(p_{-i}, q_{-i})$. Note that unlike in the proof of Theorem 1 since the equilibrium is no longer symmetric, the best response $q_i$ can depend on others’ levels of
private investment.

First suppose that a sequence of opponents’ actions \((p_{-i}, q_{-i})\) is subcritical. Our analysis above showed that for \(n\) large, the best response \(BR(p_{-i}, q_{-i}, \beta_{-i})\) has

\[q_i(\sum_{j \neq i} q_j \beta_j) \in \left[\frac{1}{2\delta n}, \frac{k}{\delta n}\right].\]

Next, suppose that along a sequence of opponents’ actions \((p_{-i}, q_{-i}, \beta_{-i})\), the matrix of link probabilities for firms other than \(i\) has spectral radius \(\lambda > \frac{1}{2}\). Then firm \(i\) can achieve a positive payoff by choosing \(q_i\) and \(\beta_i\) such that \(\beta_i \sum_{j \neq i} \iota(q, q) = 1\). On the other hand expected payoffs to firm \(i\) vanish or are negative along any sequence of best-responses such that \(\beta_i q_i \to 0\). Therefore 0 is not in the closure of \(BR_i(q)\) whenever \(\iota(q, q) \geq \frac{1}{2\delta n}\).

Because 0 is not in the closure of \(BR_i(q_{-i})\) for any \(q_{-i} > 0\), by compactness we can choose \(\epsilon(n)\) such that

\[BR_i(q_{-i}) \geq \frac{\epsilon(n)}{\sqrt{n}}\]

when \(\lambda > \frac{1}{2}\). We therefore have

\[BR_i(q_{-i})(\sum_{j \neq i} q_j \beta_j) \in \left[\frac{\epsilon(n)}{\delta n}, 1\right] \text{ when } q_j(\sum_{j' \neq j} q_{j'} \beta_{j'}) \in \left[\frac{\epsilon(n)}{\delta n}, 1\right]\]

for \(n\) sufficiently large. So by Kakutani’s fixed-point theorem, for \(n\) sufficiently large there exists a fixed point of \(q \mapsto (BR_i(q_{-i}))\). We will call this fixed point \(q(p)\) to indicate the dependence on \(p\). It remains to show that there exists \(p\) such that \(p_i\) is a best response under actions \((p, q(p))\) for all \(i\).

Suppose that \(p_i \geq \frac{1}{2}\) for all \(i\) and all \(n\). Our analysis of the subcritical and supercritical regions above extends immediately to this fixed point, as we did not rely on \(p\) being chosen optimally. Therefore, the sequence of outcomes \(q(p)\) must be critical. In particular, expected payoffs at \((p, q(p))\) converge to \(\infty\) for all such sequences of \(p\). \[22\] We extend our definition of criticality to the restriction of the random network to agents other than \(i\).
The best response $p_i$ maximizes

$$p_i \mathbb{E}_{t \in \text{PPT}_i}(q) \left[ \prod_{j \in t \setminus \{i\}} p_j \right] - c(p_i)$$

and therefore satisfies

$$c'(p_i) = \mathbb{E}_{t \in \text{PPT}_i}(q) \left[ \prod_{j \in t \setminus \{i\}} p_j \right].$$

(8)

Because $c(p_i)$ is strictly increasing and strictly convex with $c'(0) \geq 0$ and $c(p) \to \infty$ as $p \to 1$, there exists a solution.

Since $p_j \geq \frac{1}{2}$ for all $j$, for $n$ large the optimal $p_i \geq \frac{1}{2}$ as well since the expected number of potential proprietary technologies converges to infinity by equation (8). So by Kakutani’s fixed point theorem, for $n$ large there exists $p \in [\frac{1}{2}, 1]^n$ such that $p_i$ is optimal under actions $(p, q)$. This is a symmetric investment equilibrium.

Proof of Proposition. We first show there is no sequence of supercritical symmetric investment equilibria for any $\rho > 0$. To do so, we consider firm $i$’s choice of $q_i$ in the supercritical region. As in the proof of Theorem

$$\mathbb{E} \left[ \left| I_t(p^*_i, q_i, q^{*}_{-i}) \right| \right] = (p^*)^k (1 - \delta \iota(q_i, q^*) \alpha^{(n-1)+o(n)}) ((\alpha(n-1))^{k-1} + o(n^{k-1})).$$

In particular, to solve for firm $i$’s choice of $q_i$ to first order, we need only consider technologies consisting of $i$’s private idea and $(k-1)$ ideas learned by the giant component. The probability that such a technology faces competition is

$$(1 - \delta \cdot \iota(q_i, q^*) - (1 - \delta) \cdot \iota(q_i, q^*) \cdot \alpha)^{n-1} + o(1).$$

The term $\delta \cdot \iota(q_i, q^*)$ corresponds to the possibility of a firm $j$ indirectly learning all of firm $i$’s ideas. The term $(1 - \delta) \cdot \iota(q_i, q^*) \cdot \alpha$ corresponds to the possibility of a firm $j$ directly learning firm $i$’s idea (but not indirectly learning from $i$) and indirectly learning the ideas learned by the giant component.
Thus, we are looking for $q_i$ maximizing:

$$
\mathbb{E} \left[ \left( |I_i(p^*, (q_i, q^*_{-i}))| \right)^{\rho} \right] (1 - \delta \cdot \nu(q_i, q^*) - (1 - \delta) \cdot \nu(q_i, q^*) \cdot \alpha)^{n-1}.
$$

This expression is equal to:

$$
((p^*)^k (1 - (1 - \delta \nu(q_i, q^*))^{(n-1) + o(n)}) ((\alpha(n - 1))^{k-1} + o(n^{k-1})))^{\rho} (1 - \delta \cdot \nu(q_i, q^*) - (1 - \delta) \cdot \nu(q_i, q^*) \cdot \alpha)^{n-1}.
$$

Therefore, asymptotically the optimal $q_i$ will be a maximizer of:

$$
(p^*)^{\rho(k-1)+1} (1 - \delta \nu(q_i, q^*))^{(n-1) + o(n)} ((\alpha(n - 1))^{k-1} + o(n^{k-1}))^{\rho} (1 - \delta \cdot \nu(q_i, q^*) - (1 - \delta) \cdot \nu(q_i, q^*) \cdot \alpha)^{n-1}.
$$

The terms containing $q_i$ do not depend on $\rho$ to first order. Therefore, the optimization problem is the same as in Theorem 1 and the same argument shows there is no supercritical sequence of symmetric investment equilibria.

It remains to define $\rho$ suitably and show there is no subcritical sequence of symmetric investment equilibria when for $\rho \geq \rho$. We will choose $\rho$ to satisfy the conditions in the following lemma:

**Lemma A4.** If $k = 2$ and $\rho \geq 1$, then $\binom{y}{k-1}^\rho$ is convex in $y > 0$. If $k > 2$, there exists $\rho < 1$ such that $\binom{y}{k-1}^\rho$ is convex in $y > 0$ for all $\rho \geq \rho$.

**Proof.** We can assume $y \geq k - 1$. We want to determine the sign of:

$$
\frac{d^2}{dy^2} \left( \binom{y}{k-1}^\rho \right) = \frac{d^2}{dy^2} \left( \left( \frac{\prod_{j=0}^{k-2} (y - j)}{(k-1)!} \right)^\rho \right).
$$

This has the same sign as

$$
\frac{d}{dy} \left( \rho \left( \prod_{j=0}^{k-2} (y - j) \right)^{\rho-1} \sum_{i=0}^{k-2} \prod_{j \neq i} (y - j) \right).
$$
This derivative is equal to
\[
\rho (\rho - 1) \left( \prod_{j=0}^{k-2} (y - j) \right)^{\rho-2} \left( \sum_{i=0}^{k-2} \prod_{j \neq i} (y - j) \right)^2 + \rho \left( \prod_{j=0}^{k-2} (y - j) \right)^{\rho-1} \sum_{i=0}^{k-2} \sum_{i' \neq i, i'} \prod_{j \neq i, i'} (y - j). \quad (9)
\]

If \( \rho \geq 1 \), both the first and second term are non-negative for \( y \geq k - 1 \), so expression \((9)\) is non-negative as well.

Suppose \( k > 2 \). Expression \((9)\) has the same sign as
\[
(\rho - 1) \left( \sum_{i=0}^{k-2} \prod_{j \neq i} (y - j) \right)^2 + \left( \prod_{j=0}^{k-2} (y - j) \right)^{k-2} \sum_{i=0}^{k-2} \sum_{i' \neq i, i'} \prod_{j \neq i, i'} (y - j). \quad (10)
\]

The first term may be negative if \( \rho < 1 \), while the second term is positive for \( y \geq k - 1 \). Both are polynomials of degree \( 2k - 2 \) in \( y \). Therefore, we can choose \( y \) and \( \rho < 1 \) sufficiently close to 1 such that expression \((10)\) is positive for \( \rho > \rho \) and \( y > y \).

We want the expression to be positive for \( k - 1 \leq y \leq y \). There are finitely many values, and for each expression \((10)\) is positive when \( \rho \) is sufficiently close to one or at least one. Therefore, increasing \( \rho \) if needed, we find that expression \((10)\) is positive for \( \rho > \rho \) and \( y \geq k - 1 \). This proves the lemma.

Let \( \rho \geq \rho \), where \( \rho = 1 \) when \( k = 2 \) and \( \rho \) is chosen as in Lemma \((A4)\) for \( k > 2 \).

Suppose there exists a sequence of symmetric investment equilibria with \( \limsup_n \delta \cdot \iota(q^*, q^*)n < 1 \). Passing to a subsequence if necessary, we can assume that \( \delta \cdot \iota(q^*, q^*)n \) converges.

We claim that for \( n \) sufficiently large
\[
\frac{\delta}{\partial q_i} \left( \left| I_i(p^*,q^*) \right| \right) \cdot \mathbb{E} \left[ \left( \left| I_i(p^*,q^*) \right| \right)^\rho \right] < \frac{1}{n - 1} \mathbb{E} \left[ \frac{\partial (|I_i(p^*,q^*)|)}{\partial q_i} \right] \left( q^* \right) \quad (11)
\]
for all \( i \). Both sides of the inequality converge because \( \delta \cdot \iota(q^*, q^*)n \) converges to a limit less than one.

Let \( X_1 \) be a random variable equal to \( |I_i(p,q)| \) with probability \( \delta \) and 0 with probability
1 − δ. Then the left-hand side of equation (11) is equal to

\[ \mathbb{E} \left[ \left( \frac{X_1}{k-1} \right)^{\rho} \right] \]

asymptotically.

Let \( X_2 \) be the random variable with distribution equal to the change in \(|(I_i(p,q))|\) if firm \( i \) learned from an additional firm \( j \) chosen uniformly at random. Then the right-hand side of equation (11) is equal to

\[ \mathbb{E} \left[ \left( \frac{|I_i(q,q)| + X_2}{k-1} \right)^{\rho} - \left( \frac{|I_i(q,q)|}{k-1} \right)^{\rho} \right] \]

asymptotically.

In this case, with probability \( 1 - \delta \), the firm \( i \) only learns directly from firm \( j \). With probability \( \delta \), firm \( i \) learns indirectly through firm \( j \), and then learns

\[ |I_j(p,q)| - |I_i(p,q) \cap I_j(p,q)| \]

additional ideas.

The expected cardinality

\[ |I_i(p,q) \cap I_j(p,q)| \]

is \( o(1) \), by the same independence argument given in the proof of Lemma A2. Therefore, we can ignore the intersection term in computing the limit of the right-hand side of equation (11).

Let \( \tilde{X}_2 \) be the random variable with distribution equal to the number of ideas firm \( i \) learns from firm \( j \), including any ideas firm \( i \) already knows, i.e. \( X_2 \) without this intersection term.

Then \( \tilde{X}_2 \) first-order stochastically dominates \( X_1 \), and is one higher with non-vanishing probability. By Lemma A4, this implies

\[ \mathbb{E} \left[ \left( \frac{X_1}{k-1} \right)^{\rho} \right] < \mathbb{E} \left[ \left( \frac{|I_i(q,q)| + \tilde{X}_2}{k-1} \right)^{\rho} - \left( \frac{|I_i(q,q)|}{k-1} \right)^{\rho} \right] \]

for \( n \) large. It follows that the same inequality holds with \( X_2 \) replacing \( \tilde{X}_2 \), which proves the
claim.

So along any sequence of symmetric investment equilibria with \( \limsup \delta_i(q^*, q^*) < 1 \), for \( n \) sufficiently large

\[
\delta_i(q^*, q^*) n > \mathbb{E}_{t \in p_{T_i(p^*, q^*)}}[\tau(t)]
\]

for all \( i \). The proof is the same as the proof of Lemma 2, with the approximate equality from Lemma A2 replaced by the inequality from equation (11).

In particular, \( \delta_i(q^*, q^*) n > 1 \) for \( n \) large, which contradicts the assumption of subcriticality. So any sequence of symmetric investment equilibria is critical.

Proof of Proposition 3. The proof follows the same basic outline as the proof of Proposition 2, with the function

\[
\left| \frac{I_i(p, q)}{k} \right| - 1
\]

replaced by \( \phi(\left| I_i(p, q) \right|) \).

We first show there is no sequence of supercritical symmetric equilibria with \( \liminf_n p^*/n > 0 \). To do so, we consider firm \( i \)'s choice of \( q_i \) in the supercritical region. As in the proof of Theorem 1, the payoffs to firm \( i \) are:

\[
\mathbb{E}[U_i(p^*, (q_i, q_{-i}))] = (1-(1-\iota(q_i, q^*))^{\alpha(n-1)+o(n)})(1-\iota(q_i, q^*))^{n-1}p^*(\alpha(n-1) + o(n^{k-1})) - c(p^*)
\]

when the giant component has size \( \alpha n + o(n) \).

We will bound

\[
\phi(p^*(\alpha(n-1) + y)),
\]

where \( y \) is \( o(n) \). This expression is less than or equal to

\[
\phi(p^*(\alpha(n-1))) + p^*y\phi'(p^*(\alpha(n-1) + y)).
\]

By the assumption of non-vanishing investment, we have \( \lim_n p^* > 0 \). By our assumption that \( \frac{\phi(x_j)}{\phi(x_j')} \to 1 \) when \( x_j/x_j' \to 1 \), we can conclude

\[
\phi(p^*(\alpha(n-1) + y)) = \phi(p^*(\alpha(n-1))) + o(\phi(p^*(\alpha(n-1)))).
\]
Therefore, \( q_i \) is chosen to maximize:

\[
(1 - (1 - \iota(q_i, q^*))^{\alpha(n-1) + o(n)})(1 - \iota(q_i, q^*))^{n-1} + o(1).
\]

The maximization is the same as in Theorem 1 with \( \delta = 1 \), and the same calculation shows there is no supercritical sequence of symmetric investment equilibria.

The proof that there is no the subcritical sequence of symmetric equilibria with \( \lim \inf_n p^*/n > 0 \) is the same as in Proposition 2, with \( (|I_i(p, q|)^\rho \) replaced by \( \phi(|I_i(p, q|) \). We no longer need to prove Lemma A4 as we assume that \( \phi(\cdot) \) is convex. \( \square \)

**Proof of Proposition 4.** Proof of (i): We will use Lemma 3, which we now prove, to show we cannot have a critical sequence of symmetric investment equilibria.

**Proof of Lemma 3.** We can assume without loss of generality that \( p \) is bounded away from zero, because \( \mathbb{E}_{t \in PT_i(p, q)}[\tau(t)] \) does not depend on the value of \( p \) as long as \( p \) is non-negative.

Let \( \epsilon > 0 \). The probability that firm \( i \) learns from \( d \) firms decays exponentially in \( d \). Because the number of possible proprietary technologies is bounded above by \( \binom{n}{k-1} \), we can choose \( \overline{d} \) such that the contribution to \( \mathbb{E}_{t \in PT_i(p, q)}[\tau(t)] \) from the event that firm \( i \) learns from more than \( \overline{d} \) other firms is at most \( \epsilon \) for \( n \) large.

Since \( \epsilon \) is arbitrary, we can restrict our analysis to the event that firm \( i \) learns from at most \( \overline{d} \) other firms.

We claim that as \( n \to \infty \), we have

\[
\mathbb{E}[|I_i(p, q)|] \to \infty.
\]

Let \( \lambda > 0 \) and \( q(\lambda) \) is defined by \( \iota(q(\lambda), q(\lambda)) = \frac{1-\lambda}{n} \). For any \( q < q' \), the random variable \( |I_i(p, q)| \) is first-order stochastically dominated by \( |I_i(p, q')| \). So it is sufficient to show that

\[
\lim_{\lambda \to 0} \lim_{n \to \infty} \mathbb{E}[|I_i(p, q)|] \to \infty.
\]

We can bound \( |I_i(p, q)| \) below by the expected number of firms \( j \) with a path from \( j \) to \( i \) in the indirect learning network. By Theorem 11.6.1 of \cite{Alon and Spencer}, the limit
of this quantity as $n \to \infty$ is equal to the number of nodes in a Poisson branching process with parameter $1 - \lambda$. As $\lambda \to 0$, this number of nodes converges to infinity. This proves the claim.

The proof of Lemma 3 will also use the following lemma, which states that learning a large number of ideas at a critical sequence of equilibria is rare for $n$ large:

**Lemma A5.** Let $\omega(n) \to \infty$. Then

$$\Pr[|I_i(p,q)| > \omega(n)] \to 0$$

as $n \to \infty$.

**Proof.** Let $\epsilon > 0$. We want to prove that

$$\Pr[|I_i(p,q)| > \omega(n)] < \epsilon$$

for $n$ large.

Let $q(\lambda)$ be the solution to $\iota(q(\lambda), q(\lambda)) = \frac{1+\lambda}{\delta n}$. Once again, for any $q < q'$, the random variable $|I_i(p,q)|$ is first-order stochastically dominated by $|I_i(p,q')|$. So it is sufficient to show there exists $\lambda > 0$ such that

$$\Pr[|I_i(p,q(\lambda))| > \omega(n)] < \epsilon$$

for $n$ large.

We showed in the proof of Lemma 2 that the number of descendants of $i$ in the indirect learning network is first-order stochastically dominated by the Poisson branching process with parameter $1 + \lambda$. The probability that this Poisson branching process includes infinitely many nodes converges to 0 as $\lambda \to 0$ (by equation 11.8 of Alon and Spencer, 2004), so we can choose $\lambda$ such that this probability is at most $\epsilon/2$.

Given $\lambda$, we can also choose $\overline{y}$ such that the probability that the Poisson branching process has $y$ nodes for any $\overline{y} \leq y < \infty$ is at most $\epsilon/4$. Because $|I_i(p,q)|$ is first-order stochastically dominated by the sum of $y$ Bernoulli random variables with success probability $p$ and $y$
binomial random variables distributed as $\text{Binom}(\frac{p(1+\lambda)}{\delta n}, n - 1)$, we can choose $\overline{y}$ such that the probability that

$$\overline{y} \leq |I_i(p, q)| < \infty$$

is at most $\epsilon/2$.

Then

$$\mathbb{P}[|I_i(p, q)| > \overline{y}] < \epsilon$$

for $n$ large, which implies

$$\mathbb{P}[|I_i(p, q)| > \omega(n)] < \epsilon$$

for $n$ large since $\omega(n) \to \infty$.

Choose $\omega(n) \to \infty$ such that

$$\frac{\omega(n)}{\mathbb{E}[|I_i(p, q)|]} \to 0.$$

We have assumed that $i$ learns from at most $\overline{d}$ other firms. We can order these firms from 1 to $\overline{d}$. For each of these firms $j$, let the additional ideas $AI_j$ be the set of ideas that firm $i$ learns from firm $j$ and has not learned from any previous firm $1, \ldots, j - 1$ in our ordering.

We claim that as $n \to \infty$, a vanishing share of proprietary technologies include ideas that firm $i$ learns only from firms $j$ with $AI_j \leq \omega(n)$. The number of such ideas is bounded above by $\omega(n)\overline{d}$. So the number of proprietary technologies including at least one such idea is bounded above by

$$\mathbb{E}\left[\left(\frac{|I_i(p, q)| + \overline{d}\omega(n)}{k - 2}\right) \cdot (\overline{d}\omega(n))\right] = \overline{d}\omega(n)\mathbb{E}\left[\left(\frac{|I_i(p, q)| + \overline{d}\omega(n)}{k - 2}\right)\right], \quad (12)$$

while the total number of proprietary technologies is on the same order as

$$\mathbb{E}\left[\left(\frac{|I_i(p, q)|}{k - 1}\right)\right] \geq \mathbb{E}\left[\left(\frac{|I_i(p, q)|}{k - 2}\right)\right] \cdot \frac{\mathbb{E}[|I_i(p, q)|]}{k - 1}. \quad (13)$$

Since

$$\frac{\omega(n)}{\mathbb{E}[|I_i(p, q)|]} \to 0,$$

the quotient of expression (12) divided by expression (13) vanishes as $n \to \infty$. 

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Let $\epsilon > 0$. For $n$ sufficiently large, Lemma A5 implies that the probability that $AI_j > \omega(n)$ is at most $\epsilon$. We will show that the contribution to $\mathbb{E}_{t \in PT_i(p, q)}[\tau(t)]$ from the event that $AI_j > \omega(n)$ for more than one $j$ can be taken to be small.

We condition on the event that $AI_j > \omega(n)$ for at least one $j$. Then the probability that $AI_j > \omega(n)$ for at least one other $j$ is bounded above by $d\epsilon$, while the expected number of proprietary technologies increases by a multiplicative factor of less than $d^{k-1}$ in this case. Since $\epsilon$ can be taken to be arbitrarily small, it follows that the contribution to $\mathbb{E}_{t \in PT_i(p^*, q^*)}[\tau(t)]$ from the event that $AI_j > \omega(n)$ for more than one $j$ vanishes as $n \to \infty$.

The remaining technologies in $PT_i(p, q)$ consist of firm $i$’s private idea and $k - 1$ ideas learned from a single firm $j$. We thus have $\tau(t) = 1$ for each of the remaining technologies $t \in PT_i(p, q)$. This shows that $\mathbb{E}_{t \in PT_i(p, q)}[\tau(t)] \to 1$, which proves the lemma.

We claim that if $\lim \inf_n p > 0$ and

$$\lim \sup_n \iota(q, q)n\delta \leq 1,$$

then given symmetric actions $(p, q)$,

$$\frac{\partial U_i(p, q)}{\partial q_i}(q) > 0$$

for $n$ sufficiently large. In words, given symmetric actions in the subcritical or critical region, increasing $q_i$ would increase payoffs. We can assume without loss of generality that $p$ is bounded away from zero, because the sign of this derivative is independent of $p$.

Let $D_i(q)$ be the set of firms $j$ such that there is a path from $i$ to $j$ in the indirect-learning network. We claim that when $\lim \sup_n \iota(q, q)\delta n \leq 1$, a.a.s. a random technology $t \in T_i(p, q)$ is known by $|D_i(q)|$ other firms.

We can write $\iota(q, q)\delta n = 1 + \lambda$, where $\lim \sup_n \lambda \leq 0$. By the ‘No Middle Ground’ claim from p. 210-211 of Alon and Spencer (2004), the probability that a given node in an undirected random network with link probability $\frac{1 + \lambda}{n}$ is contained in a component of cardinality at least $\lambda n$ at most $n^{-2k-1}$ for $n$ large. A standard correspondence states that the size of the component containing a given node in an undirected random graph first-

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order stochastically dominates the number of nodes reachable by a path from that node in a directed random graph with the same link probability (see for example Luczak 1990). So the probability that a given idea is learned indirectly by more than $\lambda n$ firms is at most $n^{-2k-1}$ for $n$ large. Thus, we can choose a constant such that the probability that any idea is learned indirectly by more than $\lambda n$ firms is at most $n^{-2k}$ for $n$ large. Since there are $\binom{n}{k}$ potential technologies, it is without loss of generality to restrict to the event that no idea is learned indirectly by more than $\lambda n$ ideas.

Now, choose $t \in T_i(p, q)$ and let $j \in T_i(p, q)$. The number of firms that learn idea $j$ from each firm that indirectly learns $j$ is bounded above by a Poisson random variable with parameter $1/\delta$ (by the same argument as in the proof of Lemma [A2]). So we can assume that at most $C\lambda n$ firms learn the idea for some constant $C > 0$. For a firm $j' \notin D_i(q)$ to know all ideas in $t$, that firm must learn $j$ and directly learn $i$. Each of the $C\lambda n$ firms that learn $j$ will directly learn $i$ with probability at most $1/\delta n$, so the probability that any of these firms directly learns $i$ vanishes asymptotically. This proves the claim.

Thus, the payoff to firm $i$ is

$$U_i(p, q) = \mathbb{E}[|D_i(q)||T_i(p, q)|] - c(p_i) + o(\mathbb{E}[|PT_i(p, q)|]).$$

Thus,

$$\frac{\partial U_i(p, q)}{\partial q_i}(q) = \mathbb{E}\left[\frac{\partial |D_i(q)|}{\partial q_i}(q)|T_i(p, q)|\right] + \mathbb{E}\left[\frac{\partial |T_i(p, q)|}{\partial q_i}(q)f(|D_i(q)|)\right] + o(\mathbb{E}[|PT_i(p, q)|]).$$

We claim this is equal to

$$\mathbb{E}\left[\frac{\partial |D_i(q)|}{\partial q_i}(q)\right]\mathbb{E}[|T_i(p, q)|]\mathbb{E}\left[\frac{\partial |T_i(p, q)|}{\partial q_i}(q)\right]\mathbb{E}[f(|D_i(q)|)] + o(\mathbb{E}[|PT_i(p, q)|]). \quad (14)$$

The relevant random variables are independent conditional on the event that

$$I_i(p, q) \cap D_i(q) = \emptyset$$

and this intersection remains empty after adding an additional incoming or outgoing link.
Because 
\[ \limsup_n \iota(q, q)n\delta \leq 1, \]
this occurs asymptotically almost surely. We must show the contributions to \( T_i(p, q) \) from the vanishing probability event that this intersection is non-empty vanish as \( n \to \infty \). This follows from the bounds on the probability of this event in the ‘No Middle Ground’ claim from p. 210-211 of [Alon and Spencer (2004)], and the argument is the same as above.

Because \( f \) is non-negative with \( f(0) = 1 \) and \( f(1) > 0 \), we can choose \( \epsilon > 0 \) such that
\[ -\frac{1}{qn} \cdot \mathbb{E} \left[ \partial f(|D_i(q)|) \right] (q) < \mathbb{E} [f(|D_i(q)|)] - \epsilon \]
for all \( n \). Here, the left-hand side is equal to the decrease in \( f(|D_i(q)|) \) when an additional firm learns from firm \( i \), which is at most \( f(|D_i(q)|) \) and will be smaller with non-vanishing probability.

At a subcritical sequence of actions, we have
\[ \limsup_n \iota(q, q)n\delta \leq \liminf_n \mathbb{E}_{t \in PT_i(p, q)} [\tau(t)]. \]
By the same argument used to prove Lemma 2, this implies that
\[ \mathbb{E} [|T_i(p, q)|] < \frac{1}{qn} \cdot \mathbb{E} \left[ \partial |T_i(p, q)| \right] (q) \]
for all \( n \) sufficiently large. At a critical sequence of actions, Lemma 3 shows that
\[ \lim_{n \to \infty} \mathbb{E}_{t \in PT_i(p, q)} [\tau(t)] = 1. \]
At a critical sequence of actions, it follows from the definition of \( \tau(t) \) that
\[ \mathbb{E} [|T_i(p, q)|] \sim \frac{1}{qn} \cdot \mathbb{E} \left[ \partial |T_i(p, q)| \right] (q). \]
In either case, substituting into expression (14) we obtain:

\[
\frac{\partial U_i(p, q)}{\partial q_i}(q) > 0
\]

in the subcritical or critical region for \( n \) sufficiently large. This proves the claim, so any sequence of symmetric investment equilibria is supercritical.

It remains to show there exists a symmetric investment equilibrium. We have shown that for \( n \) sufficiently large, when all other firms choose \( q_j = q \) with \( q \in [\frac{1}{2n}, 1] \), the best response for firm \( i \) is also in \( [\frac{1}{2n}, 1] \). As in the proof of Theorem [1] existence follows by Kakutani’s fixed point theorem.

Proof of (ii): Suppose there exists a critical or supercritical sequence of symmetric investment equilibria. We claim that

\[
\frac{\partial U_i(p^*, q^*)}{\partial q_i}(q^*) < 0,
\]

which will give a contradiction. We can again assume without loss of generality that \( p \) is bounded away from zero, because the sign of this derivative is independent of \( p \).

First suppose there exists a critical sequence of investment equilibria. Lemma [3] shows that

\[
\lim_{n \to \infty} \mathbb{E}_{t \in PT_i(p^*, q^*)}[\tau(t)] = 1.
\]

As a consequence,

\[
\mathbb{E} \left[ \left( \frac{\left| I_i(p^*, (q_i, q_{-i}^*)) \right|}{k - 1} \right) \right] \sim \frac{1}{qn} \cdot \frac{\partial \mathbb{E} \left[ \left( \frac{\left| I_i(p^*, (q_i, q_{-i}^*)) \right|}{k - 1} \right) \right]}{\partial q_i}(q^*).
\]

Therefore,

\[
\frac{\partial \mathbb{E} \left[ \left| PT_i(p^*, (q_i, q_{-i}^*)) \right| \right]}{\partial q_i}(q^*) \sim 0.
\]

On the other hand,

\[
\frac{\partial \mathbb{E} \left[ \left| T_i(p^*, (q_i, q_{-i}^*)) \right| \right]}{\partial q_i}(q^*) > 0
\]

for \( n \) large, and increasing \( q_i \) weakly increases the number of firms \( f(m) \) who know each
technology. Since firm \( i \) receives negative profits from each \( t \in T_i(\mathbf{p}^*, (\mathbf{q}_i, \mathbf{q}^*_{-i})) \) that is not proprietary, firm \( i \)'s profits are decreasing in \( q_i \) at \( q_i = q^* \).

We next suppose there exists a supercritical sequence of equilibria. The expected profits from choosing \( q_i \) are:

\[
(p^*)^k \left( \frac{\alpha n}{k - 1} \right) (1 - \delta_t(q_i, q^*))^{\alpha n} f(\mathbb{E}_{t \in T_i(\mathbf{p}^*, (\mathbf{q}_i, \mathbf{q}^*_{-i}))}[m]) + o(n^{k-1}),
\]

where \( \alpha \) is the share of ideas learned by all firms in the giant component and \( m \) is the number of other firms who know all ideas in a technology \( t \).

Conditional on the event that \( \mathbb{E}_{t \in T_i(\mathbf{p}^*, (\mathbf{q}_i, \mathbf{q}^*_{-i}))}[m] > 0 \), increasing \( q_i \) will weakly decrease expected payoffs because increasing \( \mathbb{E}_{t \in T_i(\mathbf{p}^*, (\mathbf{q}_i, \mathbf{q}^*_{-i}))}[m] \) and increasing \( |I_i(\mathbf{p}^*, (q_i, q^*_{-i}))| \) both weakly decrease payoffs. Moreover, this increase is strict, because the probability of learning the ideas in the giant component is strictly higher under higher \( q_i \).

Suppose \( \mathbb{E}_{t \in T_i(\mathbf{p}^*, (\mathbf{q}_i, \mathbf{q}^*_{-i}))}[m] = 0 \). We showed in the proof of Theorem 1 that when \( f(m) = 0 \) for all \( m > 0 \), the increase in expected payoffs from an additional incoming link is less than the decrease in expected payoffs from an additional outgoing link. Since decreasing \( f(m) \) for \( m > 0 \) does not change the effect of an additional incoming link but decreases the expected payoffs from an additional outgoing link, it follows that increasing \( q_i \) will weakly decrease expected payoffs conditional on \( \mathbb{E}_{t \in T_i(\mathbf{p}^*, (\mathbf{q}_i, \mathbf{q}^*_{-i}))}[m] = 0 \).

Therefore, increasing \( q_i \) will weakly decrease expected payoffs unconditionally, which gives our contradiction. We have checked the critical and subcritical cases, so this completes the proof of Proposition 4.

\[ \square \]

Proof of Proposition 5 (i): Suppose there exists a sequence of investment equilibrium with \( n \to \infty \).

We first consider the first-order condition for \( q_i(1) \). The expected payoff to firm \( i \) with a patent when \( k = 2 \) and link costs are \( \epsilon \) is:

\[
p_i p^* q^*(0) q_i(1)(1 - b)(n - 1) - \epsilon(q_i(1)(q^*(0)(1 - b) + q^*(1)b)(n - 1) - c(p_i).
\]

The first term now does not depend on whether other firms learn the ideas involved in the
technologies that firm \( i \) can produce. The second term is the expected link cost.

The expected payoff is linear in \( q_i(1) \), so the coefficient of \( q_i(1) \) must be non-negative at any investment equilibrium. Therefore:

\[
(p^*)^2 q^i(0)(1 - b) \geq \epsilon (q^i(0)(1 - b) + q^i(1)b).
\]

If equality holds, then firms with patents are indifferent to all choices of interaction rates. But then firms without patents would not choose positive interaction rates, which they must at any investment equilibrium. So the inequality is strict.

Because payoffs are strictly increasing in \( q_i(1) \) on \([0, 1]\), we have \( q^i(1) = 1 \). Thus the inequality

\[
(p^*)^2 q^i(0)(1 - b) > \epsilon (q^i(0)(1 - b) + q^i(1)b)
\]

implies that \( \liminf_n q^i(0) \) is positive. But if all interaction rates are bounded below by constants, the probability that a firm \( j \) without a patent receives monopoly profits from a given technology \( t \) decays exponentially. Since firm \( j \)'s link costs are linear in \( n \), firm \( j \)'s expected payoff is negative. This cannot occur at equilibrium, so we have a contradiction.

(ii): It is weakly dominant for all firms to choose \( q^i(1) = 1 \), and is strictly optimal at any investment equilibrium. So for any \( \delta > 0 \), almost surely all firms with patents are in the same component of the indirect-learning network. All firms in this component learn \( \alpha n \) ideas for some \( \alpha > 0 \).

Suppose that \( q^i(0)n \rightarrow \infty \) and consider firm \( i \) without a patent. The probability that no firm with a patent learns indirectly from firm \( i \) decays exponentially in \( q^i(0)n \), and so profits must be \( o(n^{k-1}) \). But firm \( i \) could receive higher profits by deviating to choose \( q_i(0) = \frac{1}{n^m} \), as this would give profits of order \( n^{k-1} \). So it must be the case that \( q^i(0) \) is \( O(1/n) \), and thus

\[
\iota(q^i(0), q^i(0)) = (q^i(0))^2 = O(1/n^2).
\]
B Direct Learning

We now analyze the baseline model from Section 2 in the case $\delta = 0$. Then firms can only learn directly from other firms, and not indirectly.

**Proposition B1.** When $\delta = 0$, there exists a symmetric investment equilibrium for $n$ large, and at any sequence of symmetric equilibria

$$\lim_n \iota(q^*, q^*) n^{\frac{1}{k}} = (k - 1)^{\frac{1}{k}}.$$

Interaction rates are much higher than in the indirect-learning case, because without indirect learning much more interaction is needed for competition to be a substantial force. The expected number of ideas learned by each firm is now $O\left(\frac{n}{k-1}\right)$, which is still asymptotically lower than in the supercritical case with indirect learning (where the expected number of ideas learned is linear in $n$).

**Proof.** For $n$ large, the expected number of potential technologies that firm $i$ produces and which include firm $i$’s private idea:

$$|T_i((p_i, p_{-i}^*), (q_i, q_{-i}^*))| \sim \frac{1}{(k-1)!} (\iota(q_i, q^*)(n-1))^{k-1}$$

The probability that no other firm produces any such technology is:

$$(1 - \iota(q_i, q^*) \iota(q^*, q^*)^{k-1})^{n-1} + o(1).$$

So $q_i$ is chosen to maximize the number of potential proprietary technologies for $i$:

$$|PT_i((p_i, p_{-i}^*), (q_i, q_{-i}^*))| \sim \frac{1}{(k-1)!} (\iota(q_i, q^*)(n-1))^{k-1} (1 - \iota(q_i, q^*) \iota(q^*, q^*)^{k-1})^{n-1}. \quad (15)$$

The first-order condition is

$$(k-1) \left( \frac{\partial \iota(q^*, q^*)}{\partial q} (q^*) \right) (1 - \iota(q_i, q^*) \iota(q^*, q^*)^{k-1}) \sim (n-1) \iota(q_i, q^*) \left( \frac{\partial \iota(q^*, q^*)}{\partial q} (q_i) \right) \iota(q^*, q^*)^{k-1}.$$
payoffs would vanish asymptotically, but firms could achieve non-vanishing profits by choosing any interaction rate proportional to $\frac{1}{n}$. Thus, the first-order condition implies

$$\lim_{n \to \infty} (n - 1) \iota(q_i, q^*) \iota(q^*, q^*)^{k-1} = k - 1.$$  

At equilibrium, this implies

$$\iota(q^*, q^*) \sim \left( \frac{k - 1}{n - 1} \right)^{\frac{1}{k}}$$

as desired.

Interactions between firms $j$ and $j'$ now only impose a negative externality on a third firm $i$. The negative externality appears because these interactions can facilitate competition. There is no longer a benefit to firm $i$, because learning between firms $j$ and $j'$ cannot facilitate indirect learning by firm $i$.

A consequence is that increasing all firms’ openness would decrease average profits:

**Corollary B1.** When $\delta = 0$, at any symmetric investment equilibrium $(p^*, q^*)$,

$$\lim_{n \to \infty} \frac{\partial U_i(p^*, q)}{\partial q}(q^*) < 0.$$  

**Proof.** Firm $i$’s optimization problem over $q_i$ given symmetric strategies $(p^*_{-i}, q_{-i})$ by opponents is equivalent to choosing an interaction rate $\iota(q_i, q_{-i})$ with all other firms, given their interaction rates $\iota(q_{-i}, q_{-i})$ with each other.

By the envelope theorem, the derivative of $U_i(p^*, q)$ as we vary $\iota(q_i, q_{-i})$ fixing $\iota(q_{-i}, q_{-i})$ is zero at $q = q^*$. We can see from equation (15) that the derivative of $U_i(p^*, q)$ as we vary $\iota(q_{-i}, q_{-i})$ fixing $\iota(q_i, q_{-i})$ is negative. Therefore, decreasing $q$ symmetrically at equilibrium reduces the payoffs $U_i(p^*, q)$.

The corollary shows that decreasing interaction rates will increase average profits. If firms respond to the new interaction rates by adjusting private investment, the effect on the innovation rate will be more ambiguous: there will be an increase in R&D but a given discovery will be less likely to spread.
We can observe from the proof that adding a constant link cost $\epsilon > 0$ would not change the result of Proposition B1. This contrasts with Proposition 5(i), where there is no investment equilibrium for any positive $\epsilon$. We next discuss direct learning with patent rights when $k > 2$.

### B.1 Patents

Proposition 5 considers granting patent rights to some types of ideas when $\delta = 0$ and $k = 2$. We found that adverse selection interactions prevents the emergence of an investment equilibrium.

We now extend the analysis to $k > 2$. There is now an investment equilibrium, but the same adverse selection effect implies that firms with patents choose much lower levels of openness than without patents.

Suppose that a fraction $b \in (0, 1)$ of firms receive patents, as in Proposition 5. We maintain the assumption that $\delta = 0$.

**Proposition B2.** Suppose a fraction $b \in (0, 1)$ of firms receive patents and $\delta = 0$. For $k > 2$, there exists a symmetric investment equilibrium for $n$ large, and at any sequence of symmetric equilibria

$$\lim_{n} q^*(0)n^{1/k} = \left(\frac{(1 - b)(k - 1)}{b}\right)^{1/k}.$$

**Proof.** It is weakly dominant for patent rights choose $q^*(1) = 1$, and this action is the best response at equilibrium. We claim that at any symmetric investment equilibrium,

$$q^*(0)n \to \infty.$$

The probability that all ideas in a given technology $t$ including $i$ are known to another firm is

$$(1 - q^*(0)^k)bn + o(1).$$

This converges to zero whenever $q^*(0)n^{1/k} \to 0$, so if $q^*(0)n \to 0$ then firm $i$ could profitably deviate to increase $q_i$.

Therefore, by the law of large numbers, for a firm $i$ without patents choosing $p_i$ and $q_i(0)$
against equilibrium actions

\[ I_i((p_i, p^*_i), (q_i, q^*_i)) | \sim p^* q_i q^*(0)(1 - b)n. \]

So the expected number of proprietary technologies for a firm without patents choosing \( p_i \) and \( q_i(0) \) is:

\[ \mathbb{E}[|PT_i((p_i, p^*_i), (q_i, q^*_i))|] \sim p_i(p^*)^{k-1}(1 - q_i(0)q^*(0)^{k-1})b n q_i q^*(0)(1 - b)n. \]

Note that we use our explicit formula \( \iota(q_i, q_j) = q_i q_j \) for the interaction rate here.

We can approximate the binomial coefficient with its highest-order term, so taking the first-order condition and cancelling terms gives:

\[ bn q^*(0)^{k-1}q_i q^*(0)(1 - b)n \sim (k - 1)(1 - b)n q^*(0)(1 - q_i(0)q^*(0)^{k-1}). \]

Taking \( q_i = q^*(0) \) and solving,

\[ q^*(0) \sim \left( \frac{1 - b}{bn} \cdot \frac{k - 1}{n} \right)^{1/k} \]

as claimed above.

As a result, the interaction rate between firms without patents is

\[ \iota(q^*(0), q^*(0)) \sim \left( \frac{1 - b}{b} \cdot \frac{k - 1}{n} \right)^{2/k}, \]

which is lower order than the interaction rate in Proposition B1. The payoffs to firms without patents are therefore of lower order than in Proposition B1, where no patent rights are granted.

The interaction rate between a firm with a patent and a firm without a patent is

\[ \iota(q^*(0), q^*(1)) \sim \left( \frac{1 - b}{b} \cdot \frac{k - 1}{n} \right)^{1/k}, \]

which is the same order as the interaction rate in Proposition B1. The payoffs to firms with
patents are therefore of the same order as in Proposition $\text{B1}$ where no patent rights are granted.

\[ \square \]

C Firm Size

The baseline model assumed that each firm can discover a single idea. In this section, we consider firms that can instead discover $1 < \sigma < k$ private ideas.

A firm with size $\sigma > 1$ can frictionlessly share ideas internally without fear of competition. Equivalently, we can interpret a firm of size $\sigma > 1$ as the entity created by a licensing agreement between $\sigma$ small firms.

More formally, we continue to let the set of firms be $\{1, \ldots, n\}$ but now allow multiple ideas for each firm. Each of the $\sigma$ ideas corresponding to firm $i$ is discovered independently with probability $p_i$. A firm learning $i$ directly from $j$ will learn all private ideas discovered by firm $j$. The analysis from Sections 3 and 5, including Theorem 1 and Propositions 2 and 4, extends easily to any firm size $\sigma < k$.

We will now compare the payoffs of firms of two different sizes $\sigma$ and $\sigma'$. Suppose there are fixed positive shares of firms of each size. We can think of the exercise as measuring the value of increasing firm size.

**Proposition C3.** Assume $f(m) \geq 0$ for all $m$. If firm $i$ can discover $\sigma$ ideas and firm $i'$ can discover $\sigma'$ ideas, then at any sequence of investment equilibria:

$$
\lim_{n \to \infty} \frac{U_i(p^*, q^*)}{U_{i'}(p^*, q^*)} = \frac{\sigma}{\sigma'}.
$$

The proposition says that when payoffs are such that equilibrium is critical or super-critical, then small and large firms obtain the same payoffs per idea asymptotically. An implication is that merging two separate firms would increase their profits by very little for $n$ large.

We can give intuition in the case $\sigma = 1$ and $\sigma' = 2$. Two separate firms of size one can each potentially produce technologies by combining their private idea with $n-1$ ideas learned.

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\[ \text{23} \] The assumption that $\sigma < k$ rules out investment equilibria with no interaction: all firms choose $q_i = 0$ but private investment $p_i > 0$.  

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from others. A firm of size two can also potentially produce technologies by combining either of its private ideas with \( n - 1 \) ideas learned from others. In addition, the firm of size two can produce technologies by combining both of its private ideas with \( n - 2 \) ideas learned from others. These additional technologies generate any excess profits for the larger firm over the two smaller firms. Because \( \binom{y}{k-1} \) is much larger than \( \binom{y}{k-2} \) for \( y \) large, the additional technologies have a small impact on profits in large markets.

When equilibrium is subcritical (e.g. \( f(m) < 0 \) for all \( m > 0 \)), we have

\[
\lim_{n \to \infty} \frac{U_i(p^*, q^*)}{U_i'(p^*, q^*)} > \frac{\sigma}{\sigma'}.
\]

In this case, because firm profits are bounded asymptotically, technologies using multiple private ideas will generate a non-vanishing share of a firm’s profits.

An important assumption in Proposition C3 is that \( \sigma \) and \( \sigma' \) does not depend on \( n \), so that firms are still small relative to the overall market. A firm that can discover a non-vanishing fraction of all ideas can obtain much higher payoffs per idea than small firms, as such a firm will obtain payoffs of order \( n^{k-1} \) even without interacting with other firms.

**Proof of Proposition C3** Suppose that \( f(m) \geq 0 \) for all \( m > 0 \), as in Sections 2 and 3. We can then show that equilibrium is critical or supercritical by a modification of the argument used to prove Theorem 1 and Proposition 4, which we now describe.

A version of Lemma A2 still applies at any subcritical equilibrium. The statement and proof must be modified, as in the proof of the subcritical asymmetric case of Theorem 1 to accommodate heterogeneity in firms. Because \( \sigma < k \), we must have \( \tau(t) \geq 1 \) for all \( t \). Because \( \delta q^* n \) is equal to the expectation of \( \tau(t) \) with respect to a suitable distribution over technologies \( t \), we cannot have a subcritical equilibrium.

Therefore, we have

\[
\lim_{n \to \infty} U_i(p^*, q^*) = \infty
\]

for firms \( i \) of either size.

So for any integer \( y > 0 \) and any \( \epsilon > 0 \), we have

\[
\mathbb{E} \left[ |PT_i(p^*, q^*)|1_{|i(p^*, q^*)| > y} \right] \geq (1 - \epsilon)\mathbb{E} \left[ |PT_i(p^*, q^*)| \right]
\]
for $n$ sufficiently large, where $1$ is the indicator function. That is, almost all of the profits of firm $i$ are generated in the event that firm $i$ learns at least $y$ ideas.

Because

$$\lim_{y \to \infty} \frac{y}{(y-1)!} = 0$$

for all $l > 1$, in expectation at least a share $1 - \epsilon$ of technologies in $PT_i(p^*, q^*)$ include only one private idea developed by firm $i$ (of either size).

Suppose $\sigma < \sigma'$ and fix firms $i$ of size $\sigma$ and $i'$ of size $\sigma'$. The preceding facts imply that by choosing $(p_i, q_i) = (p_{i'}, q_{i'})$, for any $\epsilon > 0$ the firm $i$ can guarantee

$$\mathbb{E}[|PT_i((p_i, p'^* - i), (q_i, q'^* - i))|] \geq (1 - \epsilon)^3 \mathbb{E}[|PT_{i'}(p^*, q^*)|]$$

for $n$ sufficiently large. Here the first two factors of $(1 - \epsilon)$ correspond to the share of proprietary technologies including only one private idea developed by firm $i'$, while we introduce the third because at least a share $1 - \epsilon$ of proprietary technologies for firm $i'$ do not include idea $i$. This implies the result.

This section introduced heterogeneity in firm size. Our results can similarly accommodate heterogeneity in other parameters, such as the complexity $k$ of products produced by a firm or the private investment cost $c(\cdot)$.

### D Investment/Interaction Tradeoffs

Our main model focuses on a tradeoff between learning and secrecy. The same techniques also let us characterize a related model in which firms instead face tradeoffs between learning and investment, and must decide how to allocate resources between these two tasks. In particular, we now model the probability that firm $i$ learns from firm $j$ as depending only on firm $i$’s action rather than depending symmetrically on firm $i$ and firm $j$’s actions.

A firm $i$ continues to choose actions $p_i$ and $q_i$, now subject to the budget constraint that

$$p_i + \lambda q_i n = 1$$
for some $\lambda > 0$. The constant $\lambda$ determines the cost of an additional expected interaction in terms of probability of discovering a private idea.

Firm $i$ learns directly from each firm $j$ with probability $q_i$. As in the main model, in this case firm $j$ learns indirectly through firm $j$ with probability $\delta \in [0,1]$. All realizations of links and private ideas remain independent.

We maintain the baseline payoff structure from Section 2. There is no longer an explicit cost $c(p_i)$ to private investment. So

$$U_i(p, q) = \mathbb{E}[|PT_i(p, q)|],$$

where $PT_i(p, q)$ is the set of technologies for which firm $i$ receives monopoly profits.

The equilibrium characterization depends on the rate at which the firm can substitute between interaction and private investment:

**Theorem D1.** Suppose $\delta > 0$. Any sequence of symmetric equilibria with positive payoffs is:

(i) Subcritical if $\lambda < \frac{\delta}{2}$;

(ii) Critical if $\lambda = \frac{\delta}{2}$; and

(iii) Supercritical if $\lambda > \frac{\delta}{2}$.

As the opportunity cost of interaction decreases, the equilibrium level $q^*$ increases. The key intuition is that, as in the baseline model, the marginal downside to additional interaction is proportional to the current profits. In the baseline model that downside comes from interaction potentially facilitating competition, while now the downside comes from a lower probability of discovering a private idea.

Exploiting the similar structure, we can derive a variant of Lemma 2:

$$\lambda q^* n \sim p^* \mathbb{E}_{t \in PT_i(p^*, q^*)} [\tau(t)].$$

At the critical threshold, we have

$$\delta q^* n \to 1$$

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24 In particular, this learning rate no longer depends on $q_j$. 

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and can show as in the proof of Proposition 4(ii) that \( \mathbb{E}_{t \in PT_i(p^*, q^*)}[\tau(t)] \to 1 \). The theorem follows from these facts and the budget constraint.

One could also show as in the proof of Theorem 1 that there exists an equilibrium with positive payoffs for \( n \) large. We omit the proof here.

**Proof.** We will show an analogue of Lemma A2 in this context.

We claim that along any sequence of symmetric equilibria with positive payoffs and \( \delta \bar{q}n < 1 \) for all \( n \),

\[
\lambda \mathbb{E} \left[ \left( |I_i(p^*, q^*)| \right) \right] \sim \ p^* \mathbb{E} \left[ \frac{\partial (|I_i(p^*, q^*)|)}{\partial q_i} (q^*_i) \right] \quad (16)
\]

for each \( i \).

We can argue as in the proof of Lemma A2 that since \( \delta \bar{q}n < 1 \), competition from indirect learning is lower order. Therefore, we can condition on the event that no firm \( j \) has learned indirectly from firm \( i \), which is independent of \( |I_i(p^*, q^*)| \).

The left-hand side of equation (16) is \( \lambda \) times the benefit from a marginal increase in private investment \( p_i \). The right-hand side of equation (16) is the benefit from a marginal increase in \( q_i \). By the budget constraint

\[
p_i + \lambda q_i n = 1,
\]

these are equal at any interior equilibrium. Payoffs are zero at equilibria with \( p_i = 0 \) or \( p_i = 1 \) for any \( i \).

It follows as in the proof of Lemma 2 that along any sequence of symmetric equilibria with positive payoffs and \( \delta \bar{q}n < 1 \) for all \( n \),

\[
\lambda q^* n \sim p^* \mathbb{E}_{t \in PT_i(p^*, q^*)}[\tau(t)].
\]

At any symmetric subcritical equilibrium we must have

\[
\mathbb{E}_{t \in PT_i(p^*, q^*)}[\tau(t)] > 1.
\]
So we must have

$$\lambda q^* n < p^*$$

for $n$ large. Substituting in the budget constraint, we conclude that $\lambda < \frac{\delta}{2}$.

Because

$$\lambda q n < p^* \mathbb{E}_{t \in PT_i}(p^*, q)[\tau(t)]$$

for $q$ sufficiently small and $n$ large, by continuity there exists a subcritical equilibrium when $\lambda < \frac{\delta}{2}$.

Next, suppose $\delta q^* n > 1$. Then if $\alpha$ is the number of ideas learned by firms in the giant component, the payoff to choosing $q_i$ is approximately proportional to:

$$(1 - \lambda q_i n)(1 - e^{-\delta q^* \alpha n})(1 - \lambda q^* n)^{k-1}\left(\frac{\alpha n}{k-1}\right).$$

Indeed, this is the payoff if no firm learns $i$’s idea and all other ideas learned by the giant component, and the probability this occurs is independent of the choice of $q_i$.

Taking the first-order condition, we have

$$\lambda (1 - e^{-\delta q^* \alpha n}) \sim \delta \alpha (1 - \lambda q^* n) e^{-\delta q^* \alpha n}$$

or equivalently

$$\frac{\lambda}{\delta} \sim \alpha (1 - \lambda q^* n) \cdot \frac{e^{-\delta q^* \alpha n}}{1 - e^{-\delta q^* \alpha n}}.$$  

Because $\alpha \sim 1 - e^{-\delta q^* \alpha n}$, this implies

$$\frac{\lambda}{\delta} \sim (1 - \frac{\lambda}{\delta} q^* n)(1 - \alpha).$$

To have a solution with $\alpha > 0$, and therefore to have a sequence of symmetric supercritical equilibria, requires $\lambda > \frac{\delta}{2}$.

Since the derivative of the payoffs in $q_i$ are negative for a symmetric $q$ with $q$ large and $n$ large, by continuity there exists a symmetric supercritical equilibrium when $\lambda > \frac{\delta}{2}$.

Combining the two arguments above, any sequence of symmetric equilibria with positive payoffs is critical.

\[\square\]
E Public Innovators and Directed Interaction

We now show that the result of Proposition 1 continues to apply if firms can direct their interactions toward private firms or public innovators.

As in Section 3.4, public innovator $i$ pays investment cost $c(p_i)$ and receives a payoff of one for each technology $t$ such that: (1) $i \in t$ and (2) $j \in \{i\} \cup I_i(p, q)$ for all $j \in t$. We will rely on the fact that for public innovators there is no downside to interactions, but not on the exact incentive structure.

All firms have the same incentives as in the baseline model. Public innovators and firms can now choose two interaction rates $q_{i0}$ and $q_{i1}$, where $q_{i0}$ is the interaction rate with public innovators and $q_{i1}$ is the interaction rate with private firms.

We show payoffs again grow at the same rate as in the supercritical region, up to a constant factor:

**Proposition E4.** Suppose a non-vanishing share of agents are public innovators. Then there exists a sequence of symmetric equilibria with non-vanishing investment, and at any sequence of equilibria with non-vanishing investment

$$\liminf_n \frac{U_i(p^*, q^*)}{\binom{n-1}{k-1}} > 0$$

for all firms $i$.

**Proof.** Let $b(n)$ be the share of public innovators for each $n$.

We first show that

$$\liminf_n \frac{U_i(p^*, q^*)}{\binom{n-1}{k-1}} > 0$$

for all $i$ at any investment equilibrium.

It is weakly dominant and strictly preferred at any investment equilibrium for public innovators to choose $q_{i0}^* = q_{i1}^* = 1$. Therefore, all public innovators are in the same component of the learning network. Private investment $p_i$ by public innovators is non-vanishing, so asymptotically almost surely all firms in this component learn at least $\alpha n$ ideas for some $\alpha > 0$. 82
Each private firm can obtain expected payoffs $O(n^{k-1})$ by choosing $q_{i0} = 1$ and $q_{i1} = 0$. This is because then the probability that firm $i$ learns indirectly from a public innovator and no firm $j$ learns from $i$ is non-vanishing, and the payoffs from this event are $O(n^{k-1})$. This shows the desired bound on $\frac{U_i(p^*, q^*)}{\binom{n}{k-1}}$.

It remains to show there exists a sequence of symmetric equilibria with non-vanishing investment. Suppose that all public innovators choose $p_0 \geq \frac{1}{2}$ and $q_{i0} = q_{i1} = 1$ and all firms other than $i$ choose $(p_1, q_0, q_1)$ with $p_1 \geq \frac{1}{2}$, $\delta q_0 n \leq 1$ and $q_1 = 0$. If $q_{i0}$ is the best response for $i$, then $\lim_n q_{i0} n$ exists and is independent of $p_0$ and $(p_1, q_0, q_1)$. This is because the probability of interactions between $i$ and other firms vanishes asymptotically, while the best response does not depend on the number of ideas learned by the unique giant component.

Therefore, we can choose $\epsilon > 0$ such that if $q_0 \in \left[\frac{\epsilon}{\delta n}, \frac{1}{\delta n}\right]$, then so is firm $i$’s best response $q_{i0}$. Because all other private firms choose $q_{j1} = 0$, firm $i$ is indifferent to all choices of $q_{i1}$ and in particular $q_{i1} = 0$ is a best response. We claim that for $n$ large, given $p$, there exists $q$ such that each each $q_i$ is a best response to $(p_0, p_1, q_0, q_1)$. This follows from Kakutani’s fixed point theorem as in the proof of Theorem 1. We call this choices of openness $q_0(p_0, p_1)$.

Given such $(p_0, p_1, q_0(p_0, p_1))$, each firm has a non-vanishing probability of learning a linear number of ideas. Therefore, $\mathbb{E}[|I_i(p, q)|] \to \infty$. So any best response $p_i$ for each public innovator and each firm $i$ has $p_i \geq \frac{1}{2}$. By Kakutani’s fixed point theorem, there exists $(p_0, p_1, q_0(p_0, p_1))$ such that $p_0 \geq \frac{1}{2}$ and $p_1 \geq \frac{1}{2}$ are also best responses. Thus there exists a sequence of equilibria with non-vanishing investment.

\[\square\]