On the resolution of cosmic coincidence problem and phantom crossing with triple interacting fluids

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Abstract

We here investigate a cosmological model in which three fluids interact with each other involving certain coupling parameters and energy exchange rates. The motivation of the problem stems from the puzzling ‘triple coincidence problem’ which naively asks why the cosmic energy densities of matter, radiation and dark energy are almost of the same order of magnitude at the present time. In our model, we determine the conditions under triple interacting fluids will cross the phantom divide.

Keywords: Interacting dark energy; cosmological constant; cosmic coincidence problem; phantom energy; phantom crossing.
1 Introduction

Despite several successes of the standard big bang cosmology based on Friedmann-Robertson-Walker (FRW) model, still a series of problems to be resolved like the horizon problem, flatness problem, dark matter (or missing mass) problem, structure formation, topological defects, matter-antimatter asymmetry and the cosmic coincidence problem etc. Most of these are related with the cosmic past of the observable universe while the cosmic coincidence problem has its origin in the recent time since it naively asks why certain cosmological phenomena are occurring in our presence or in our times. Recent astrophysical observations give convincing evidence of an accelerating universe caused by dark energy characterized by the equation of state (EoS) parameter $\omega \simeq -1$. It is yet unknown why the present energy density of the dark energy is approximately equal to that of dust-like matter. It is termed as the cosmic coincidence problem [1]. Till now several models have been proposed in an attempt to solve this problem such as the ‘tracker field’ [2], oscillating dark energy [3] and the variable constants approach [4], to name a few. It appears that the energy density of the radiation component is also almost equivalent to that of the matter and the dark energy i.e. $\rho_m \sim \rho_r \sim \rho_\Lambda$ or $\Omega_m \sim \Omega_r \sim \Omega_\Lambda$, the so-called ‘cosmic triple coincidence problem’ [5]. The question is: why this happens in the current or in recent times. The history of the parameter $\omega$ suggests that it is no more a constant but possesses a parametric form $\omega(z)$, where $z$ is the redshift parameter. Thus in the past $\omega = 1/3$ corresponds to radiation and then $\omega = 0$ for matter. Later it evolved to quintessence $\omega < -1/3$ to cosmological constant $\omega = -1$. This behavior suggests that in future, $\omega$ will be super-negative i.e. $\omega < -1$, which corresponds to the phantom energy. Thus in totality, we have the transition from $\omega > -1$ to $\omega < -1$ the so-called ‘phantom crossing’ or ‘phantom divide’ scenario, while we are observing $\omega = -1$ at the current time [6]. The coincidence problem in this context is rephrased as ‘why is $\omega = -1$ now?’

In recent years, the usual coincidence problem is addressed by proposing an exotic interaction between dark energy and matter in which energy from $\rho_\Lambda$ is diluted or decayed into the $\rho_m$ [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. It is recently proposed that if these two components interact then some energy might dissipate into a third component $\rho_x$ which is as yet hypothetical [19]. The third component can be known form of matter or an altogether exotic fluid in which case some new physics will be required to explain the interaction. If we assume $\rho_x = \rho_r$ then the interaction between three fluids i.e. matter, radiation and dark energy will be quite interesting. It is well-known that matter and radiation were decoupled at the time of emission of cosmic microwave background (CMB) radiation at a redshift $z \sim 1100$. Thus both matter and radiation are almost non-interacting components but it can be anticipated that these two components do supposedly interact with the dark energy. Thus if dark energy and matter interact, the energy dissipated in the interaction is assumed to transfer to the radiation component and vice
versa for the radiation and the dark energy interaction. This dynamic interaction than
hugely alters the effective equation of state of the three interacting fluids. Our analysis in
this paper suggests that effective EoS of the three fluids will be interlinked to each other
considerably and also constrained by coupling parameters. Moreover, the interaction
naturally leads to the phantom crossing scenario.

We here present a model in which three cosmic fluids (radiation, matter and dark energy)
interact with each other with equal degree of freedom. Hence there are three coupling
parameters involved which can take arbitrary positive or negative values not all zero
simultaneously. The positive and negative values correspond to back and forth nature of
energy exchange between the fluids. We emphasis here that the coupling parameters in
our model are only constrained by the choices of effective EoS of the fluids. Thus in our
model, exact EoS of the fluids like in the non-interacting FRW model is not possible. As
the exact EoS for the dark energy is unknown, its interaction with other fluids creates
ambiguities in the determination of the exact EoS of the other fluids. Hence we also stress
that precise values of the EoS’s can be deduced only empirically from the phenomenology
of the interacting fluids.

2 The model of triple interacting fluids

We start by assuming the background to be spatially flat, homogeneous and isotropic
FRW spacetime

\[ ds^2 = -dt^2 + a^2(t) \left[ dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \]

(1)

where \( a(t) \) is the scale factor. We consider three fluids having the equations of state
(EoS) \( p_i = \omega_i \rho_i, \ i = 1, 2, 3 \) where \( p_i \) and \( \rho_i \) are the corresponding pressures and the energy
densities of the fluids, respectively. Also \( \omega_i \) are the dimensionless EoS parameters. For
the sake of simplicity, we assume the three fluids to be perfect fluid-like since in general,
interaction between fluids might lead to local inhomogeneities, which are ignored in the
present paper. The equations governing the interaction of three fluids are expressed as

\[ \dot{\rho}_1 + 3H(1 + \omega_1)\rho_1 = Q_3 - Q_2, \]

(2)

\[ \dot{\rho}_2 + 3H(1 + \omega_2)\rho_2 = Q_1 - Q_3, \]

(3)

\[ \dot{\rho}_3 + 3H(1 + \omega_3)\rho_3 = Q_2 - Q_1. \]

(4)

Here \( Q_i \) are the energy exchange (or dissipative) terms to be put ad hoc in the above
equations. The explicit form of \( Q_i \) should be determined only from the phenomenological
and empirical results. However, from dimensional considerations, the quantity \( Q_i \) should
have dimensions of density into the time inverse. Choosing the later one to be Hubble
parameter, we notice that \( Q_i \) can be of the following forms: \( Q_i \approx H \rho_i \). This approximation
can be saturated to equality by inserting a dimensionless parameter (say $\lambda_i$). Thus we can write \[ Q_i = \lambda_i H \rho_{io} a^{-3(1+\omega_i)}, \quad i = 1, 2, 3. \] (5)

In the last expression, $\lambda_i$ are the coupling constants which can take positive or negative values to yield two-sided energy exchange rather than one-sided. Also Eqs. (2) to (5) show that this a coupled system of three differential equations which needs to be solved. Further $\rho_{io}$ are the constant energy densities at some reference time $t = t_o$. Also $H \equiv \dot{a}/a$ is Hubble parameter which determines the rate of expansion of the universe. Sum of Eqs. (2) to (4) yield the combined energy conservation
\[
\sum_{i=1}^{3} [\dot{\rho}_i + 3H(1 + \omega_i)\rho_i] = 0, \tag{6}
\]

Also the energy conservation for the individual component (for the case of non-interacting fluids) yields
\[
\rho_i' = \rho_{io} a^{-3(1+\omega_i)}, \quad i = 1, 2, 3. \tag{7}
\]

Here $\rho_{io}$ are integration constants. Combining Eqs. (2), (3) and (4), we arrive at the density evolution of the interacting fluids as
\[
\begin{align*}
\rho_1 &= C_1 a^{-3(1+\omega_1)} + \frac{\lambda_2 \rho_{2o}}{3} \frac{a^{-3(1+\omega_2)}}{\omega_2 - \omega_1} + \frac{\lambda_3 \rho_{3o}}{3} \frac{a^{-3(1+\omega_3)}}{\omega_1 - \omega_3}, \tag{8} \\
\rho_2 &= C_2 a^{-3(1+\omega_2)} + \frac{\lambda_3 \rho_{3o}}{3} \frac{a^{-3(1+\omega_3)}}{\omega_3 - \omega_2} + \frac{\lambda_1 \rho_{1o}}{3} \frac{a^{-3(1+\omega_1)}}{\omega_2 - \omega_1}, \tag{9} \\
\rho_3 &= C_3 a^{-3(1+\omega_3)} + \frac{\lambda_1 \rho_{1o}}{3} \frac{a^{-3(1+\omega_1)}}{\omega_1 - \omega_3} + \frac{\lambda_2 \rho_{2o}}{3} \frac{a^{-3(1+\omega_2)}}{\omega_3 - \omega_2}. \tag{10}
\end{align*}
\]

Here $C_i$’s are constants of integration. Now addition of Eqs. (8), (9) and (10) results in
\[
\rho = \rho_1' X_1 + \rho_2' X_2 + \rho_3' X_3, \tag{11}
\]

where
\[
\begin{align*}
X_1 &= 1 + \frac{\lambda_1}{3} \frac{\omega_2 - \omega_3}{(\omega_1 - \omega_3)(\omega_2 - \omega_1)}, \tag{12} \\
X_2 &= 1 + \frac{\lambda_2}{3} \frac{\omega_3 - \omega_1}{(\omega_2 - \omega_1)(\omega_3 - \omega_2)}, \tag{13} \\
X_3 &= 1 + \frac{\lambda_3}{3} \frac{\omega_1 - \omega_2}{(\omega_1 - \omega_3)(\omega_3 - \omega_2)}. \tag{14}
\end{align*}
\]

Here we have assumed $C_i = \rho_{io}$ and $\rho \equiv \rho_1 + \rho_2 + \rho_3$ is the total energy density of the interacting fluids. Note that the case of non-interacting fluids is obtained by choosing
\( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) which yield \( X_1 = X_2 = X_3 = 1 \) i.e. \( \rho = \rho_1' + \rho_2' + \rho_3' \). Using the first FRW equation

\[
H^2 = \frac{8\pi G}{3} \rho. \tag{15}
\]

Differentiating Eq. (7) w.r.t \( t \), we obtain

\[
\dot{\rho}_i' = -3(1 + \omega_i)H\rho_{io}a^{-3(1+\omega_i)}, \quad i = 1, 2, 3. \tag{16}
\]

Differentiating Eq. (15) w.r.t \( t \) and then using Eqs. (11) and (16), we find

\[
\dot{H} = -H\sqrt{24\pi G} \sum_{i=1}^{3} [\rho_i'(1 + \omega_i)X_i] \left( \sum_{i=1}^{3} \rho_i'X_i \right)^{-1/2}. \tag{17}
\]

### 2.1 Role of parameter \( \dot{H} \)

The parameter \( \dot{H} \) is significant as its possible signature governs the dynamics of the universe [31]. The slowing down in the expansion takes place when

\[
\omega_i > -1, \quad \lambda_i > \frac{3(\omega_1 - \omega_2)(\omega_2 - \omega_1)}{\omega_3 - \omega_2}, \tag{18}
\]

\[
\omega_2 > -1, \quad \lambda_2 > \frac{3(\omega_2 - \omega_1)(\omega_3 - \omega_2)}{\omega_1 - \omega_3}, \tag{19}
\]

\[
\omega_3 > -1, \quad \lambda_3 > \frac{3(\omega_1 - \omega_3)(\omega_3 - \omega_2)}{\omega_2 - \omega_1}. \tag{20}
\]

Thus \( \dot{H} \) will be negative when all the \( \omega_i > -1 \). This result corresponds to the quintessence dominated universe.

Note that the vanishing \( \dot{H} \) in Eq. (17) will yield a de Sitter universe or a cosmological constant dominated universe i.e.

\[
\dot{H} = 0 \implies \omega_1 = \omega_2 = \omega_3 = -1. \tag{21}
\]

Moreover \( \dot{H} > 0 \) corresponds to an accelerating universe. This situation arises in our model when

\[
(1 + \omega_1)X_1 < 0, \tag{22}
\]

\[
(1 + \omega_2)X_2 < 0, \tag{23}
\]

\[
(1 + \omega_3)X_3 < 0. \tag{24}
\]

The above Eqs. (22), (23) and (24) yield respectively

\[
\omega_1 < -1, \quad \lambda_1 > \frac{3(\omega_1 - \omega_3)(\omega_2 - \omega_1)}{\omega_3 - \omega_2}. \tag{25}
\]
\[ \omega_2 < -1, \quad \lambda_2 > \frac{3(\omega_2 - \omega_1)(\omega_3 - \omega_2)}{\omega_1 - \omega_3}, \quad (26) \]
\[ \omega_3 < -1, \quad \lambda_3 > \frac{3(\omega_1 - \omega_2)(\omega_3 - \omega_2)}{\omega_2 - \omega_1}. \quad (27) \]

Thus \( \dot{H} \) will be positive when all the \( \omega_i < -1 \). This result corresponds to the phantom energy dominated universe. Note that the coupling parameters \( \lambda_i \) have to be positive both for the decelerating and the accelerating universe. This result turns out to be consistent with [22] that coupling parameters cannot be negative to avoid violation of second law of thermodynamics. Moreover the same investigation shows that small positive values for the coupling parameters are motivated from the empirical results. Therefore the universe evolves from the earlier quintessence to cosmological constant and then later to the phantom energy dominated universe. Also the conditions (18)-(20), and the analysis of this section determines the super-acceleration and not the acceleration.

### 2.2 Behavior of deceleration parameter

The deceleration parameter \( q \approx \dot{H} + H^2 = -8\pi G \sum_{i=1}^{3} \rho_i \left( \frac{2}{3} + \omega_i \right) X_i \). The parameter \( q \) is significant as its possible signature governs the dynamics of the universe. For instance \( q < 0 \) represents the acceleration phase of the expanding universe. This slowing down in the expansion takes place when

\[ \omega_1 > \frac{2}{3}, \quad \lambda_1 > \frac{3(\omega_1 - \omega_3)(\omega_2 - \omega_1)}{\omega_3 - \omega_2}, \quad (28) \]
\[ \omega_2 > \frac{2}{3}, \quad \lambda_2 > \frac{3(\omega_2 - \omega_1)(\omega_3 - \omega_2)}{\omega_1 - \omega_3}, \quad (29) \]
\[ \omega_3 > \frac{2}{3}, \quad \lambda_3 > \frac{3(\omega_1 - \omega_3)(\omega_3 - \omega_2)}{\omega_2 - \omega_1}. \quad (30) \]

Thus \( q \) will be negative when all the \( \omega_i > -\frac{2}{3} \). This result corresponds to the quintessence dominated universe.

Note that the vanishing \( q \) in Eq. (17) will yield the following state equations

\[ q = 0 \Rightarrow \omega_1 = \omega_2 = \omega_3 = -\frac{2}{3}. \quad (31) \]

Moreover \( q > 0 \) corresponds to an decelerating universe. This situation arises in our model when

\[ \left( \frac{2}{3} + \omega_1 \right) X_1 < 0, \quad (32) \]
The above Eqs. (32), (33) and (34) yield respectively

\[ \omega_1 < -\frac{2}{3}, \quad \lambda_1 > \frac{3(\omega_1 - \omega_3)(\omega_2 - \omega_1)}{\omega_3 - \omega_2}, \]

\[ \omega_2 < -\frac{2}{3}, \quad \lambda_2 > \frac{3(\omega_2 - \omega_1)(\omega_3 - \omega_2)}{\omega_1 - \omega_3}, \]

\[ \omega_3 < -\frac{2}{3}, \quad \lambda_3 > \frac{3(\omega_1 - \omega_3)(\omega_3 - \omega_2)}{\omega_2 - \omega_1}. \]

Thus \( q \) will be negative when all the \( \omega_i < -\frac{2}{3} \). The strength of the interaction is determined by the coupling parameters \( \lambda_i \). These coupling constants are expressed in terms of the equation of state parameters \( \omega_i \). We here stress that the exact nature of the interaction is largely unknown i.e. the mediating particles of the interaction are not yet identified. Any interacting dark energy model should, in principle, be motivated from the particle physics or the corresponding phenomenology, however there are as yet no sound theoretical models which could identify the particle interactions. There are some arguments that the phantom-like dark energy can decay into at least one ordinary particle and some other phantom-like particles [24].

3 Stability analysis

To perform stability analysis, we write Eqs. (2-5) as

\[ \dot{\rho}_1 + 3H(1 + \rho_1)\rho_1 = 3H\lambda_1(\rho_3 - \rho_2), \]

\[ \dot{\rho}_2 + 3H(1 + \rho_2)\rho_2 = 3H\lambda_2(\rho_1 - \rho_3), \]

\[ \dot{\rho}_3 + 3H(1 + \rho_3)\rho_3 = 3H\lambda_3(\rho_2 - \rho_1). \]

Here we have used the constraint equations on the coupling parameters by: \( \lambda_3 - \lambda_2 = \lambda_1, \lambda_1 - \lambda_3 = \lambda_2 \) and \( \lambda_2 - \lambda_1 = \lambda_3 \). Since the above system is autonomous i.e. there is no explicit time dependent term in the equations, we can analyze the system by first finding its critical points and later on, checking the stability of the system about those points. A critical point (also called fixed point) is the one that satisfies the dynamical system (like Eqs. 2-5) when equated to zero. Similarly, a critical point becomes an attractor if the solution of the system converges to that point for large values of the time parameter. Such kind of analysis is generally performed to check whether the dynamical system possesses any solution which is stable against small perturbations [25, 26, 27, 28, 29, 30].
Further, defining the dimensionless density parameters as

\[ u_1 = \frac{\rho_1}{\rho_{cr}} = \Omega_1, \]  
\[ u_2 = \frac{\rho_2}{\rho_{cr}} = \Omega_2, \]  
\[ u_3 = \frac{\rho_3}{\rho_{cr}} = \Omega_3. \]  

(41)  
(42)  
(43)

Using Eqs. (41-43), we can rewrite Eqs (38-40) as

\[ \frac{du_1}{dx} = 3 u_1 \{(1 + \omega_2) u_2 + (1 + \omega_3) u_3\} + 3 \lambda_1 (u_3 - u_2), \]  
\[ \frac{du_2}{dx} = 3 u_2 \{(1 + \omega_1) u_1 + (1 + \omega_3) u_3\} + 3 \lambda_2 (u_1 - u_3), \]  
\[ \frac{du_3}{dx} = 3 u_3 \{(1 + \omega_1) u_1 + (1 + \omega_2) u_2\} + 3 \lambda_3 (u_2 - u_1). \]  

(44)  
(45)  
(46)

Above the time derivative has been replaced with the differentiation with respect to a new parameter \( x = \ln a \). This parameter corresponds to the number of e-foldings which is convenient to use for the dynamics of dark energy. By equating above three equations to zero, we obtain the critical points (see below). To find whether the system approaches to any critical point, we check the stability about these points, by producing small perturbations \( \delta u_1, \delta u_2 \) and \( \delta u_3 \) around the critical point \((u_{1c}, u_{2c}, u_{3c})\) i.e.

\[ u_1 = u_{1c} + \delta u_1, \quad u_2 = u_{2c} + \delta u_2, \quad u_3 = u_{3c} + \delta u_3. \]  

(47)

Now in order to linearize our dynamical system, we take the \( \delta \) variation of the above equations (44-46) and evaluate them at the critical points \( \{(u_{1c,j}, u_{2c,j}, u_{3c,j})\} \), where \( j \) is an indexing parameter:

\[ \frac{d\delta u_1}{dx} = 3 \delta u_1 \{(1 + \omega_2) u_{2c,j} + (1 + \omega_3) u_{3c,j}\} + 3 \delta u_2 \{u_{1c,j} (1 + \omega_2) - \lambda_1\} \]  
\[ + 3 \delta u_3 \{u_{1c,j} (1 + \omega_3) + \lambda_1\}, \]  
\[ \frac{d\delta u_2}{dx} = 3 \delta u_2 \{(1 + \omega_1) u_{1c,j} + (1 + \omega_3) u_{3c,j}\} + 3 \delta u_3 \{u_{2c,j} (1 + \omega_3) - \lambda_2\} \]  
\[ + 3 \delta u_1 \{u_{2c,j} (1 + \omega_1) + \lambda_2\}, \]  
\[ \frac{d\delta u_3}{dx} = 3 \delta u_3 \{(1 + \omega_2) u_{2c,j} + (1 + \omega_1) u_{1c,j}\} + 3 \delta u_1 \{u_{3c,j} (1 + \omega_1) - \lambda_3\} \]  
\[ + 3 \delta u_2 \{u_{3c,j} (1 + \omega_2) + \lambda_3\}. \]  

(48)  
(49)  
(50)

We then next construct a matrix consisting of the coefficients of the perturbation parameters and compute the eigenvalues of that matrix. The stability of the critical points depends on the nature of the corresponding eigenvalues and it leads to three specific cases:
if the real parts of all the eigenvalues are negative, then the following critical point is a stable node; if the real parts of the eigenvalues are all positive, then that critical point is an unstable node, in all other cases, the critical points will be saddle points. In our analysis, we shall be interested in only those critical points that yield stable nodes.

In order to obtain some physical interpretation of our results, we assign dust (or matter), radiation and dark energy with subscripts 1, 2 and 3 respectively. Thus the corresponding EoS parameters take values $\omega_1 = 0$, $\omega_2 = 1/3$ and $\omega_3 = -1/3$ or $-1$ depending whether it is quintessence or cosmological constant. From here onwards, we divide our dynamical system (44-46) into two parts: one for $\omega_3 = -1$ and other for $\omega_3 = -1/3$.

It is not possible to obtain the critical points of the full system (44-46). The only way out is by choosing at least one parameter $\lambda_i$ to be zero. In the next sections, we shall use only two coupling parameters while taking the third one to be zero. For instance, taking $\lambda_1 = 0$ implies that radiation and dark energy are mutually non-interacting while there are interactions between dark energy and matter and between matter and radiation. Similar interpretations can be made when choosing either $\lambda_2$ or $\lambda_3$ to be zero.

### 3.1 Critical points for $\lambda_1 = 0$

Note that the following critical points are obtained by setting $\lambda_1 = 0$.

#### 3.1.1 Quintessence $\omega_3 = -1/3$

The dynamical system yields the following critical points $\{(u_{1c_j}, u_{2c_j}, u_{3c_j}), j = 1, 2, 3, 4\}$.

\[
\begin{align*}
\text{u}_{1c_1} &= 0, \\
\text{u}_{2c_1} &= \frac{3}{2}\lambda_2, \\
\text{u}_{3c_1} &= -\frac{3}{4}\lambda_3, \\
\text{u}_{1c_2} &= \frac{1}{48\lambda_2 - 24\lambda_3} \left( -8\lambda_2^2 + \lambda_3 \left( -35\lambda_3 + \sqrt{16\lambda_2^2 + 568\lambda_2\lambda_3 + \lambda_3^2} \right) \\
&\quad -2\lambda_2 \left( -35\lambda_3 + \sqrt{16\lambda_2^2 + 568\lambda_2\lambda_3 + \lambda_3^2} \right) \right), \\
\text{u}_{2c_2} &= \frac{1}{32} \left( 4\lambda_2 - \lambda_3 + \sqrt{16\lambda_2^2 + 568\lambda_2\lambda_3 + \lambda_3^2} \right), \\
\text{u}_{3c_2} &= \frac{1}{16} \left( -4\lambda_2 + \lambda_3 - \sqrt{16\lambda_2^2 + 568\lambda_2\lambda_3 + \lambda_3^2} \right), \\
\text{u}_{1c_3} &= \frac{1}{48\lambda_2 - 24\lambda_3} \left( -8\lambda_2^2 + \lambda_3 \left( -35\lambda_3 + \sqrt{16\lambda_2^2 + 568\lambda_2\lambda_3 + \lambda_3^2} \right) \\
&\quad -2\lambda_2 \left( -35\lambda_3 + \sqrt{16\lambda_2^2 + 568\lambda_2\lambda_3 + \lambda_3^2} \right) \right).
\end{align*}
\]
Now we substitute above critical points in Eqs. (38-40) and obtain eigenvalues. Since analytical expressions for the eigenvalues are not possible, so we shall resort to estimate the numerical values of eigenvalues for specific choices of coupling parameters. Fixing $\lambda_2 = 0.5$ and $\lambda_3 = 0.7$ (both positive), $\lambda_2 = 0.5$ and $\lambda_3 = -0.7$ (one positive) and $\lambda_2 = -0.5$ and $\lambda_3 = -0.7$ (both negative) no stable node is obtained from the above critical points. However, for $\lambda_2 = -0.5$ and $\lambda_3 = 0.7$, the first critical point $\{(u_{1c_1}, u_{2c_1}, u_{3c_1})\}$ becomes a stable node.

\section{Cosmological constant $\omega_3 = -1$}

\begin{align*}
  u_{1c_1} &= \lambda_3, \\
  u_{2c_1} &= 0, \\
  u_{3c_1} &= \lambda_3, \\
  u_{4c_1} &= 0, \\
  u_{4c_2} &= 0, \\
  u_{4c_3} &= 0.
\end{align*}

For all the choices of $\lambda_2$ and $\lambda_3$ as adopted in the previous subsection, no critical point arises as a stable node.

\section{Critical points for $\lambda_2 = 0$}

Setting $\lambda_2 = 0$, we obtain the following critical points:

\subsection{Quintessence $\omega_3 = -1/3$}

\begin{align*}
  u_{1c_1} &= -\frac{1}{2} \lambda_1.
\end{align*}
\begin{align}
    u_{e1} &= 0, \quad \ (70) \\
    u_{e1} &= \lambda_3, \quad \ (71) \\
    u_{e2} &= \frac{1}{48} \left( -9\lambda_1 - 2\lambda_3 + \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right), \quad \ (72) \\
    u_{e2} &= \frac{1}{64(3\lambda_1 - 2\lambda_3)} \left( 27\lambda_1^2 - 2\lambda_3 \left( -2\lambda_3 + \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right) \\
    &\quad \quad \quad -3\lambda_1 \left( 72\lambda_3 + \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right) \right), \quad \ (73) \\
    u_{e3} &= \frac{1}{32} \left( 9\lambda_1 + 2\lambda_3 - \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right), \quad \ (74) \\
    u_{e1} &= \frac{1}{48} \left( -9\lambda_1 - 2\lambda_3 - \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right), \quad \ (75) \\
    u_{e2} &= \frac{1}{64(3\lambda_1 - 2\lambda_3)} \left( 27\lambda_1^2 + 3\lambda_1 \left( -72\lambda_3 + \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right) \\
    &\quad \quad \quad +2\lambda_3 \left( 2\lambda_3 + \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right) \right), \quad \ (76) \\
    u_{e3} &= \frac{1}{32} \left( 9\lambda_1 + 2\lambda_3 + \sqrt{81\lambda_1^2 + 1476\lambda_1\lambda_3 + 4\lambda_3^2} \right), \quad \ (77) \\
    u_{e1} &= 0, \quad \ (78) \\
    u_{e2} &= 0, \quad \ (79) \\
    u_{e3} &= 0. \quad \ (80)
\end{align}

For \(\lambda_1 = 0.5\) and \(\lambda_3 = -0.7\), the stable node arises at second critical point \(\{(u_{e1}, u_{e2}, u_{e3})\}\). For other values of \(\lambda_1\) and \(\lambda_3\), no other stable node arises.

### 3.2.2 Cosmological constant \(\omega_3 = -1\)

\begin{align}
    u_{e1} &= 0, \quad \ (81) \\
    u_{e2} &= -\frac{3}{4}\lambda_3, \quad \ (82) \\
    u_{e3} &= -\frac{3}{4}\lambda_3, \quad \ (83) \\
    u_{e1} &= 0, \quad \ (84) \\
    u_{e2} &= 0, \quad \ (85) \\
    u_{e3} &= 0. \quad \ (86)
\end{align}

Here for \(\lambda_1 = 0.5\) and \(\lambda_3 = -0.7\), the stable node arises at second critical point \(\{(u_{e1}, u_{e2}, u_{e3})\}\).
3.3 Critical points for $\lambda_3 = 0$

Now we perform similar analysis for $\lambda_3 = 0$, we get the critical points

3.3.1 Quintessence $\omega_3 = -1/3$

\[
\begin{align*}
    u_{1c_1} &= \frac{3}{4} \lambda_1, \\
    u_{2c_1} &= -\lambda_2, \\
    u_{3c_1} &= 0, \\
    u_{1c_2} &= \frac{1}{48} \left( -9\lambda_1 - 8\lambda_2 + \sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} \right), \\
    u_{2c_2} &= \frac{1}{64} \left( 9\lambda_1 + 8\lambda_2 - \sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} \right), \\
    u_{3c_2} &= \frac{1}{96\lambda_1 - 128\lambda_2} \left( 27\lambda_1^2 + 396\lambda_1\lambda_2 + 32\lambda_2^2 - 3\lambda_1\sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} - 4\lambda_2\sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} \right), \\
    u_{1c_3} &= \frac{1}{48} \left( -9\lambda_1 - 8\lambda_2 - \sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} \right), \\
    u_{2c_3} &= \frac{1}{64} \left( 9\lambda_1 + 8\lambda_2 + \sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} \right), \\
    u_{3c_3} &= \frac{1}{96\lambda_1 - 128\lambda_2} \left( 27\lambda_1^2 + 396\lambda_1\lambda_2 + 32\lambda_2^2 + 3\lambda_1\sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} + 4\lambda_2\sqrt{81\lambda_1^2 + 4176\lambda_1\lambda_2 + 64\lambda_2^2} \right), \\
    u_{1c_4} &= 0, \\
    u_{2c_4} &= 0, \\
    u_{3c_4} &= 0.
\end{align*}
\]

Among the above critical points, the only stable node is produced for the first one for values $\lambda_1 = -0.5$ and $\lambda_2 = 0.7$.

3.3.2 Cosmological constant $\omega_3 = -1$

\[
\begin{align*}
    u_{1c_1} &= \frac{3}{4} \lambda_1, \\
    u_{2c_1} &= -\lambda_2, \\
    u_{3c_1} &= 0, \\
    u_{1c_4} &= \frac{3}{4} \lambda_1, \\
    u_{2c_4} &= -\lambda_2, \\
    u_{3c_4} &= 0.
\end{align*}
\]
\[
\begin{align*}
  u_{1e_2} &= \frac{7\lambda_1 \lambda_2}{3\lambda_1 + 4\lambda_2}, \\
  u_{2e_2} &= -\frac{21\lambda_1 \lambda_2}{4(3\lambda_1 + 4\lambda_2)}, \\
  u_{3e_2} &= \frac{7\lambda_1 \lambda_2 (-9\lambda_1 + 16\lambda_2)}{4(3\lambda_1 + 4\lambda_2)^2}, \\
  u_{1e_3} &= 0, \\
  u_{2e_3} &= 0, \\
  u_{3e_3} &= 0.
\end{align*}
\] (102) (103) (104) (105) (106) (107)

Among the above critical points, the only stable node is produced for the first one for values \(\lambda_1 = -0.5\) and \(\lambda_2 = 0.7\).

In figures 1-6, we provide the schematic representation of triple interacting fluids, taking two at a time, with various choices of the coupling parameters and initial conditions. The initial conditions are chosen arbitrarily but these must satisfy the constraint \(u_1(0) + u_2(0) + u_3(0) = 1\) (the Friedmann equation). In figures 1 and 4, we assume first and second interacting fluids whereas in figures 2 and 5 and 3 and 6, we assume second and third and first and third interacting fluids respectively. The three solutions \(u_1, u_2\) and \(u_3\) always satisfy the constraint \(u_1(0) + u_2(0) + u_3(0) = 1\).

4 Conclusion and discussion

In this paper, we have attempted to resolve the cosmic triple coincidence problem which naively asks why the energy densities of the three major ingredients of the cosmic composition namely matter, radiation and dark energy are of same order at current time. Rephrasing, why \(\omega\) has evolved to \(-1\) in recent times. We here point out that the EoS \(p = \omega \rho\) used in this paper is not, in general, a true EoS for any cosmic fluid. Rather it is a phenomenological relationship suitable for the configuration. The actual EoS may not be that simple and may have dependencies on various other cosmological parameters like redshift, time, Hubble parameter and its derivatives etc [8]. However to a first order approximation, it may be taken for such analysis. Further, the standard non-interacting FRW model cannot resolve the coincidence problem since it predicts a hierarchial system in which radiation density decreases faster compared to matter, while density of dark energy remains either constant (if it is cosmological constant) or increases (if it is phantom energy). Therefore it contradicts with the observations where all the three components have equivalent densities. Model of interacting dark energy, which is a modification of the FRW model, has enormous potential to explain this cosmological conundrum. It naturally predicts that if the cosmic fluids interact with each other, it leads to a scenario
compatible with the observations [23].

We have found that in the interacting fluid model, the three fluids can achieve super-negative equation of state. If matter and radiation are among the components then they will induce negative pressure along with the dark energy to produce accelerated expansion. This result has earlier been shown for the interacting Chaplygin gas model in [7].
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Figure 1: The phase diagram of the triple interacting fluid model with the choice of couplings $[\lambda_1 = 0.2, \lambda_2 = 0.04, \lambda_3 = 0.06], x = -1,..,1$ and EoS parameters $[\omega_1 = 0, \omega_2 = 1/3, \omega_3 = -1]$. Chosen scene is between $u_1$ and $u_2$. The curves correspond to the initial conditions $u_1(0) = 0.2, u_2(0) = 0.6, u_3(0) = 0.2$ (brown); $u_1(0) = 0.4, u_2(0) = 0.2, u_3(0) = 0.4$ (red); $u_1(0) = 0.3, u_2(0) = 0.4, u_3(0) = 0.3$ (green).
Figure 2: The phase diagram of the triple interacting fluid model with the choice of couplings $[\lambda_1 = 0.2, \lambda_2 = 0.04, \lambda_3 = 0.06]$, $x = -1, ..., 1$ and EoS parameters $[\omega_1 = 0, \omega_2 = 1/3, \omega_3 = -1]$. Chosen scene is between $u_2$ and $u_3$. The curves correspond to the initial conditions as taken in Fig.1.
Figure 3: The phase diagram of the triple interacting fluid model with the choice of couplings $[\lambda_1 = 0.2, \lambda_2 = 0.4, \lambda_3 = 0.6], x = -1, \ldots, 1$ and EoS parameters $[\omega_1 = 0, \omega_2 = 1/3, \omega_3 = -1]$. Chosen scene is between $u_1$ and $u_3$. The curves correspond to the initial conditions as taken in Fig.1.
Figure 4: The phase diagram of the triple interacting fluid model with the choice of couplings $[\lambda_1 = 0, \lambda_2 = 0.4, \lambda_3 = -0.4]$, $x = -1, ..., 1$ and EoS parameters $[\omega_1 = 0, \omega_2 = 1/3, \omega_3 = -1/3]$. Chosen scene is between $u_1$ and $u_2$. The curves correspond to the initial conditions as taken in Fig.1.
Figure 5: The phase diagram of the triple interacting fluid model with the choice of couplings $[\lambda_1 = 0, \lambda_2 = 0.4, \lambda_3 = -0.4]$, $x = -1, \ldots, 1$ and EoS parameters $[\omega_1 = 0, \omega_2 = 1/3, \omega_3 = -1/3]$. Chosen scene is between $u_3$ and $u_2$. The curves correspond to the initial conditions as taken in Fig.1.
Figure 6: The phase diagram of the triple interacting fluid model with the choice of couplings $[\lambda_1 = 0, \lambda_2 = 0.4, \lambda_3 = -0.4]$, $x = -1, .., 1$ and EoS parameters $[\omega_1 = 0, \omega_2 = 1/3, \omega_3 = -1/3]$. Chosen scene is between $u_1$ and $u_3$. The curves correspond to the initial conditions as taken in Fig.1.
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