k-spectrally monomorphic tournaments

Abderrahim Boussaïri, Imane Souktani, Imane Talbaoui, Mohamed Zouagui

December 13, 2021

Abstract

A tournament is k-spectrally monomorphic if all the $k \times k$ principal submatrices of its adjacency matrix have the same characteristic polynomial. Transitive $n$-tournaments are trivially $k$-spectrally monomorphic. We show that there are no other for $k \in \{3, \ldots, n-3\}$. Furthermore, we prove that for $n \geq 5$, a non-transitive $n$-tournament is $(n-2)$-spectrally monomorphic if and only if it is doubly regular. Finally, we give some results on $(n-1)$-spectrally monomorphic regular tournaments.

Keywords: Tournament; adjacency matrix; characteristic polynomial; spectral monomorphy.

MSC Classification: 05C20; 05C50.

1 Introduction

An $n$-tournament $T$ is a digraph with $n$ vertices in which every pair of vertices is jointed by exactly one arc. If the arc joining two given vertices $u$ and $v$ of $T$ is directed from $u$ to $v$, we say that $u$ dominates $v$. A tournament $T$ is transitive if whenever $u$ dominates $v$ and $v$ dominates $w$, then $u$ dominates $w$. With respect to an ordering $v_1, \ldots, v_n$ of the vertices of $T$, the adjacency matrix of $T$ is the $n \times n$ matrix $A = (a_{ij})$ in which $a_{ij}$ is 1 if $v_i$ dominates $v_j$ and 0 otherwise. The skew-adjacency matrix of $T$ is $S := A - A^\top$, where $A^\top$ is the transpose of $A$. Clearly, $A + A^\top = J_n - I_n$, where $I_n$ and $J_n$ denote respectively the $n \times n$ identity matrix and the all-ones matrix. The characteristic polynomial and the skew-characteristic polynomial of $T$ are the characteristic polynomial of $A$ and $S$ respectively.

Following [1], a square matrix is $k$-spectrally monomorphic if all its $k \times k$ principal submatrices have the same characteristic polynomial. The main result of [1] is the characterization of $k$-spectrally monomorphic Hermitian matrices for $k \in \{3, \ldots, n-3\}$.

A tournament is $k$-spectrally monomorphic if its adjacency matrix is $k$-spectrally monomorphic. Spectral monomorphy is a weakening of monomorphy for tournaments. An $n$-tournament is said to be $k$-monomorphic if all its $k$-subtournaments are isomorphic. For example, transitive tournaments are $k$-monomorphic for every $k$. Conversely, it follows from a combinatorial lemma of Pouzet [14] that for any integer $n \geq 6$ and any
$k \in \{3, \ldots, n-3\}$, $k$-monomorphic $n$-tournaments are transitive. Using similar techniques, we prove that the $k$-monomorphy can be relaxed to $k$-spectral monomorphy.

Jean [7] and later Pouzet [15] proved that a non-transitive $n$-tournament is $(n-2)$-monomorphic if and only if it is arc-symmetric, that is, its automorphism group acts transitively on the set of its arcs. Arc-symmetric tournaments were characterized independently by Kantor [8] and Berggren [2]. They showed that an arc-symmetric tournament is isomorphic to one of the quadratic residue tournaments, defined on a finite field $\mathbb{F}_{p^r}$, $p^r \equiv 3 \pmod{4}$ by the following rule: $x$ dominates $y$ if and only if $x-y$ is a non-zero square in $\mathbb{F}_{p^r}$. In a quadratic residue tournament, all pairs of vertices jointly dominate the same number of vertices. Tournaments with this property are called doubly regular. There are many structural and spectral characterizations of doubly regular tournaments [4, 12, 17, 16]. In this paper, we obtain the following.

**Theorem 1.1.** An $n$-tournament with $n \geq 5$ is $(n-2)$-spectrally monomorphic if and only if it is transitive or doubly regular. Moreover, the common characteristic polynomial of all the $(n-2)$-subtournaments of a doubly regular $n$-tournament is

$$(z^2 + z + \frac{n+1}{4})^{\frac{n-5}{2}} (z^3 - \frac{n-5}{2}z^2 - \frac{n-5}{2}z - \frac{n-3}{4}).$$

The characterization of $(n-1)$-spectrally monomorphic $n$-tournaments seems difficult. It is perhaps of comparable difficulty as the open problem of characterizing $(n-1)$-monomorphic $n$-tournaments [3, Problem 43 p. 252]. Doubly regular $n$-tournaments are $(n-1)$-spectrally monomorphic. Indeed, given a doubly regular $n$-tournament, Theorem 1.1 and Theorem 2.7 of [13] imply that the characteristic polynomial of all its $(n-1)$-subtournaments is $(z^2 + z + \frac{n+1}{4})^{\frac{n-3}{2}} (z^2 + \frac{3-n}{2}z + \frac{3-n}{4})$. As for regular $n$-tournaments, they are not always $(n-1)$-spectrally monomorphic.

**Theorem 1.2.** For every integer of the form $n = 3^k \cdot 7$, there exists a non $(n-1)$-spectrally monomorphic regular $n$-tournament.

### 2 Properties of $k$-spectrally monomorphic matrices

Let $M$ be an $n \times n$ matrix. The characteristic polynomial of $M$ is $P_M(z) := \det(zI_n - M)$. For a subset $\alpha$ of $[n] := \{1, \ldots, n\}$, we denote by $M[\alpha]$ the principal submatrix of $M$ whose rows and columns are indexed by the elements of $\alpha$. For $i \in [n]$, we denote $M[[n] \setminus \{i\}]$ simply by $M_i$. Let $P_M(z) = z^n + a_1z^{n-1} + a_2z^{n-2} + a_3z^{n-3} + \cdots + a_n$. It is well-known (see for example [10, p. 494]), that

$$a_k = (-1)^k \sum_{\alpha \in \binom{[n]}{k}} \det M[\alpha]$$

for every $k$ in $[n]$.

**Remark 2.1.** If $M$ is the adjacency matrix of a tournament $T$, then $a_1 = a_2 = 0$ and $-a_3$ is the number of 3-cycles of $T$.  

The derivative of the characteristic polynomial of $M$ is equal to the sum of the characteristic polynomials of its $(n-1) \times (n-1)$ principal submatrices [13, Theorem 1]. Applying this fact iteratively, we get

$$P_M^{(n-k)}(z) = (n-k)! \sum_{\alpha \in \binom{[n]}{k}} P_{M[\alpha]}(z), \quad (2)$$

where $P_M^{(i)}(z)$ denotes the $i$-th derivative of $P_M(z)$. As a consequence, we have

**Corollary 2.2.** Suppose that $M$ is $k$-spectrally monomorphic and let $Q$ be the common characteristic polynomial of its $k \times k$ principal submatrices. Then

$$P_M^{(n-k)}(z) = \frac{n!}{k!} Q(z).$$

In [1], Attas et al. showed that spectrally monomorphic matrices enjoy a hereditary property.

**Proposition 2.3.** $k$-spectrally monomorphic $n \times n$ complex matrices are $l$-spectrally monomorphic for each $l \in \{1, \ldots, \min(k, n-k)\}$.

A slight modification of the proof of this proposition yields the following result, which plays a key role in our study of $k$-spectrally monomorphic tournaments.

**Proposition 2.4.** Let $M$ be an $n \times n$ complex matrix, $k \in \{3, \ldots, n-1\}$ and $p \leq k$. If $M$ is $k$-spectrally monomorphic, then for every subset $\beta \subseteq [n]$ of size at most $n-k$, the number $\sum_{\alpha \supseteq \beta, |\alpha| = p} \det(M[\alpha])$ depends only on the cardinality of $\beta$.

## 3 $k$-spectrally monomorphic tournaments for $k \in \{3, \ldots, n-2\}$

Obviously, all tournaments are 2-spectrally monomorphic, so we will only consider $k$-spectrally monomorphic tournaments for $k \geq 3$.

**Proposition 3.1.** $k$-spectrally monomorphic $n$-tournaments are transitive for every $n \geq 6$ and $k \in \{3, \ldots, n-3\}$.

**Proof.** Let $k \in \{3, \ldots, n-3\}$ and let $T$ be a $k$-spectrally monomorphic $n$-tournament. As $n \geq 6$, by Proposition 2.3, $T$ is 3-spectrally monomorphic. The two tournaments with three vertices, namely the 3-cycle and the transitive tournament, have different characteristic polynomials. Moreover, every tournament with more than three vertices contains a transitive 3-tournament. Hence $T$ is necessarily transitive because, by hypothesis, all its 3-subtournaments have the same characteristic polynomial. \qed

Let $T$ be an $n$-tournament with vertex set $V$ and let $x \in V$. The out-neighborhood of the vertex $x$ is $N_T^+(x) := \{y \in V : x \text{ dominates } y\}$ and its in-neighborhood is $N_T^-(x) := \{y \in V : y \text{ dominates } x\}$. The out-degree (resp. the in-degree) of $x$ is $\delta_T^+(x) := |N_T^+(x)|$ (resp. $\delta_T^-(x) := |N_T^-(x)|$). The tournament $T$ is regular if there exists $k \geq 1$ such that
the out-degree of all its vertices is \( k \). In this case, \( n \) is odd and \( k = \frac{n-1}{2} \). The tournament \( T \) is near-regular if \( n \) is even and the out-degree of every vertex is \( \frac{n-2}{2} \) or \( \frac{n}{2} \).

Let \( x \) and \( y \) be distinct vertices of \( T \). The out-degree (resp. the in-degree) of \( (x, y) \) is \( \delta^+_T(x, y) := |N^+_T(x) \cap N^-_T(y)| \) (resp. \( \delta^-_T(x, y) := |N^-_T(x) \cap N^+_T(y)| \)). With these notations, the tournament \( T \) is doubly regular if the out-degree of all pairs of vertices is the same.

Recall the following fundamental properties of doubly regular tournaments (for the proof, see [12]).

**Proposition 3.2.** Let \( T \) be a doubly regular \( n \)-tournament. Then \( T \) is regular and there exists \( t \geq 0 \) such that \( n = 4t + 3 \). Moreover, if \( x \) and \( y \) are two distinct vertices of \( T \) such that \( x \) dominates \( y \) then

\[
|N^+_T(x) \cap N^-_T(y)| = |N^-_T(x) \cap N^+_T(y)| = |N^+_T(x) \cap N^-_T(y)| = t \\
\text{and } |N^-_T(x) \cap N^+_T(y)| = t + 1.
\]

Notice that \( |N^-_T(x) \cap N^+_T(y)| \) is the number of 3-cycles containing \( x \) and \( y \). A tournament in which the number of 3-cycles containing any pair of vertices is a positive constant, say \( k \), is called homogeneous. Kotzig [9] proved that such a tournament has \( 4k - 1 \) vertices. Reid and Brown [4] proved that homogeneous tournaments are doubly regular. Consequently

**Proposition 3.3.** Let \( T \) be a tournament. The following statements are equivalent

i) Every pair of vertices is contained in the same number of 3-cycles.

ii) \( T \) is transitive or doubly regular.

Throughout this paper, the zero and the all-ones \( p \times q \) matrices are denoted by \( O_{p,q} \) and \( J_{p,q} \). The matrices \( O_{p,p} \) and \( J_{p,p} \) are simply written as \( O_p \) and \( J_p \). The zero and the all-ones column vectors of size \( p \) are denoted respectively by \( 0_p \) and \( 1_p \).

Let \( A \) be the adjacency matrix of an \( n \)-tournament \( T \) with respect to an ordering \( x_1, \ldots, x_n \) of its vertices. For any \( i, j \in \{1, \ldots, n\} \), the \((i, j)\)-entry of \( AA^\top \) is

\[
\begin{cases} 
\delta^+_T(x_i) & \text{if } i = j, \\
\delta^-_T(x_i, x_j) & \text{if } i \neq j.
\end{cases}
\] (3)

If \( T \) is doubly regular with \( n = 4t + 3 \) vertices then \( AA^\top = tJ_n + (t + 1)I_n \). This equality was used in [5] Proposition 3.1] to prove that

\[
P_A(z) = \left(z - \frac{n-1}{2}\right)\left(z^2 + z + \frac{n+1}{4}\right)^{\frac{n-1}{2}}.
\] (4)

**Proof of Theorem [14]** Let \( T \) be an \((n-2)\)-spectrally monomorphic \( n \)-tournament with vertex set \( \{v_1, \ldots, v_n\} \) and adjacency matrix \( A \). Let \( v_i \) and \( v_j \) be two distinct vertices of \( T \). From Proposition 2.4 the sum \( \sum_{\ell \in [n] \setminus \{i,j\}} \det(A[i, j, l]) \), which is exactly the number of 3-cycles containing \( v_i \) and \( v_j \), does not depend on \( v_i \) and \( v_j \). Then, by Proposition 3.3 \( T \) is doubly regular or transitive.

Conversely, let \( T \) be a doubly regular tournament with \( n = 4t + 3 \) vertices and let \( x \) and \( y \) be two distinct vertices of \( T \) such that \( x \) dominates \( y \). Denote by \( R \) the tournament
obtained from $T$ by removing $x$ and $y$. The vertex set of $R$ is partitioned into the following four subsets: $N_T^+(x) \cap N_T^+(y)$, $N_T^-(x) \cap N_T^+(y)$, $N_T^+(x) \cap N_T^-(y)$ and $N_T^-(x) \cap N_T^-(y)$. Let $B$ be the adjacency matrix of $R$. Using (3) and Proposition 3.2 we get

$$BB^\top - (t + 1)I_{4t+1} = \begin{pmatrix} tJ_t & tJ_{t,t+1} & tJ_t & tJ_t \\ tJ_{t+1,t} & (t-1)J_{t+1} & tJ_{t+1,t} & (t-1)J_{t+1,t} \\ tJ_t & tJ_{t,t+1} & (t-1)J_t & (t-1)J_t \\ tJ_t & (t-1)J_{t+1,t} & tJ_t & (t-2)J_t \end{pmatrix}.$$  

As in the proof of Proposition 3.1 of [5], we have

$$P_B^2(z) = \det (zI_{4t+1} - B) \det (zI_{4t+1} - B^\top)$$

$$= \det (z^2I_{4t+1} - z (B + B^\top) + BB^\top)$$

$$= \det (z^2I_{4t+1} - z (J_{4t+1} - I_{4t+1}) + BB^\top).$$

Then $P_B^2(z)$, and a fortiori $P_B(z)$, do not depend on the choice of $x$ and $y$. Hence $T$ is $(n-2)$-spectrally monomorphic.

Let $Q$ be the common characteristic polynomial of the $(n-2)$-subtournaments of $T$. Using (4) and Corollary 2.2 we get

$$Q(z) = \frac{1}{n(n-1)} P_A^{(2)}(z)$$

$$= \left( z^2 + z + \frac{n+1}{4} \right)^{\frac{n-5}{2}} \left( z^3 - \frac{n-5}{2} z^2 - \frac{n-5}{2} z - \frac{n-3}{4} \right).$$

\[\square\]

4 A family of regular $n$-tournaments which are not $(n-1)$-spectrally monomorphic

As mentioned in the introduction, regular $n$-tournaments are not always $(n-1)$-spectrally monomorphic. The smallest counter-example has 7 vertices, its adjacency matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

To obtain an infinite family of counter-examples, we use the following construction. Let $T_1$, $T_2$ and $T_3$ be three regular $n$-tournaments with disjoint vertex sets $V_1 = \{v_1, \ldots, v_n\}$, $V_2 = \{v_{n+1}, \ldots, v_{2n}\}$ and $V_3 = \{v_{2n+1}, \ldots, v_{3n}\}$ respectively. Consider the $3n$-tournament $T$ with vertex set $V = V_1 \cup V_2 \cup V_3$, obtained from $T_1$, $T_2$ and $T_3$ by adding arcs from $V_1$ to $V_2$, $V_2$ to $V_3$ and $V_3$ to $V_1$.

**Proposition 4.1.** The $3n$-tournament $T$ is regular. Moreover, if at least one of the three tournaments $T_1$, $T_2$ and $T_3$ is not $(n-1)$-spectrally monomorphic then $T$ is not $(3n - 1)$-spectrally monomorphic.
The proof of this proposition requires the following lemma.

**Lemma 4.2.** Let $T$ be a regular $n$-tournament with adjacency matrix $A$. Then, for every $i \in [n]$, we have

$$P_A(z) = \frac{1}{2} (z - \frac{n-1}{2}) [(n + 2z + 1) P_{A_1}(z) + (n - 2z - 1) P_{A_1}(-z - 1)].$$

**Proof.** Without loss of generality, we assume that $i = 1$. The matrix $A$ is partitioned as follows

$$A = \begin{pmatrix} 0 & w^\top \\ 1_{n-1} - w & A_1 \end{pmatrix},$$

where $w$ is an $(n - 1)$-column vector with coordinates in $\{0, 1\}$. Since $T$ is regular, $n = 2k + 1$ for some positive integer $k$, moreover, $A_1 1_n = k 1_n$ and $1_n^\top A = k 1_n^\top$. Then

$$w^\top 1_{n-1} = 1_{n-1}^\top w = k; \quad (5)$$

$$A_1 1_{n-1} = (k - 1) 1_{n-1} + w; \quad (6)$$

$$1_{n-1}^\top A_1 = k 1_{n-1}^\top - w^\top. \quad (7)$$

We have

$$P_A(z) = \det (zI_n - A) = \begin{vmatrix} z & -w^\top \\ w - 1_{n-1} & zI_{n-1} - A_1 \end{vmatrix}.$$ 

By adding the sum of the last $(n - 1)$ columns of the matrix $zI_n - A$ to its first column, we get

$$P_A(z) = \begin{vmatrix} z - w^\top 1_{n-1} & -w^\top \\ w - 1_{n-1} + (zI_{n-1} - A_1) 1_{n-1} & zI_{n-1} - A_1 \end{vmatrix}.$$ 

Adding the sum of the last $(n - 1)$ rows of the above matrix to its first row, we get

$$P_A(z) = \begin{vmatrix} z - k + 1_{n-1}^\top (z - k) 1_{n-1} - w^\top + 1_{n-1}^\top (zI_{n-1} - A_1) \\ (z - k) 1_{n-1} \end{vmatrix}.$$ 

Now, subtracting the first row from each of the last $n - 1$ rows, we obtain

$$P_A(z) = n \begin{vmatrix} (z - k) \frac{1}{n} (z - k) 1_{n-1}^\top & zI_{n-1} - A_1 \\ (z - k) 1_{n-1} \end{vmatrix}.$$ 

We conclude by the first assertion of lemma below.
A family of regular $n$-tournaments which are not $(n - 1)$-spectrally monomorphic

**Lemma 4.3.** [6] Let $T$ be an $n$-tournament with adjacency matrix $A$. For any scalar $\lambda$, we have

i) The characteristic polynomial of $A + \lambda J_n$ is

$$P_{A + \lambda J_n}(z) = (\lambda + 1)P_A(z) - (-1)^n \lambda P_A(-z - 1).$$

ii) If $T$ is regular, then

$$(z - \frac{n-1}{2}) P_{A + \lambda J_n}(z) = (z - n \lambda - \frac{n-1}{2}) P_A(z).$$

**Remark 4.4.** Lemma 4.2 shows that the characteristic polynomial of a regular $n$-tournament is determined by the characteristic polynomial of any of its $(n - 1)$-subtournaments. The technique we used in the proof is borrowed from that of [13, Theorem 1.1].

Now, we are able to prove Proposition 4.1.

**Proof of Proposition 4.1.** Obviously, the tournament $T$ is regular. Let $A$, $B$ and $C$ be the adjacency matrices of $T_1$, $T_2$ and $T_3$ respectively. The adjacency matrix of $T$ is

$$M = \begin{pmatrix} A & J_n & O_n \\ O_n & B & J_n \\ J_n & O_n & C \end{pmatrix}.$$  

Without loss of generality, suppose that $T_3$ is not $(n - 1)$-spectrally monomorphic and let $i \in [n]$. The adjacency matrix of $T - v_{i+2n}$ is

$$M_{i+2n} = \begin{pmatrix} A & J_n & O_{n,n-1} \\ O_n & B & J_{n,n-1} \\ J_{n-1,n} & O_{n-1,n} & C_i \end{pmatrix},$$

where $C_i$ is the adjacency matrix of $T_3 - v_{i+2n}$. The characteristic polynomial of $M_{i+2n}$ is

$$P_{M_{i+2n}}(z) = \begin{vmatrix} zI_n - A & -J_n & O_{n,n-1} \\ O_n & zI_n - B & -J_{n,n-1} \\ -J_{n-1,n} & O_{n-1,n} & zI_{n-1} - C_i \end{vmatrix}.$$  

Multiplying the last $n - 1$ rows by $(z - \frac{n-1}{2})$, we get

$$P_{M_{i+2n}}(z) = \frac{1}{(z - \frac{n-1}{2})^{n-1}} \begin{vmatrix} zI_n - A & -J_n & O_{n,n-1} \\ O_n & zI_n - B & -J_{n,n-1} \\ -J_{n-1,n} & O_{n-1,n} & (z - \frac{n-1}{2})(zI_{n-1} - C_i) \end{vmatrix}.$$  

As $T_1$ is regular, the sum of the first $n$ rows is

$$(z - \frac{n-1}{2}, \ldots, z - \frac{n-1}{2}, -n, \ldots, -n, 0, \ldots, 0).$$
A family of regular \( n \)-tournaments which are not \((n - 1)\)-spectrally monomorphic

Adding this sum to each of the last \( n - 1 \) rows, we obtain

\[
P_{M_i+2n}(z) = \frac{1}{\left(\frac{n-1}{2}\right)^{n-1}} \begin{vmatrix} zI_n - A & -J_n & O_{n,n-1} \\ O_n & zI_n - B & -J_{n,n-1} \\ O_{n-1,n} & -nJ_{n-1,n} & \left( z - \frac{n-1}{2} \right) \left( zI_{n-1} - C_i \right) \end{vmatrix} = \frac{p_A(z)}{\left(\frac{n-1}{2}\right)^{n-1}} \begin{vmatrix} zI_n - B & -J_{n,n-1} \\ -nJ_{n-1,n} & \left( z - \frac{n-1}{2} \right) \left( zI_{n-1} - C_i \right) \end{vmatrix}.\]

We need to evaluate the determinant

\[
\begin{vmatrix} zI_n - B & -J_{n,n-1} \\ -nJ_{n-1,n} & \left( z - \frac{n-1}{2} \right) \left( zI_{n-1} - C_i \right) \end{vmatrix} = n^{n-1} \begin{vmatrix} zI_n - B & -J_{n,n-1} \\ -J_{n-1,n} & \left( z - \frac{n-1}{2} \right) \left( zI_{n-1} - C_i \right) \end{vmatrix}.\]

Repeating the previous process, we get

\[
\begin{vmatrix} zI_n - B & -J_{n,n-1} \\ -J_{n-1,n} & \left( z - \frac{n-1}{2} \right) \left( zI_{n-1} - C_i \right) \end{vmatrix} = \frac{1}{\left(\frac{n-1}{2}\right)^{n-1}} \begin{vmatrix} zI_n - B & -J_{n,n-1} \\ -nJ_{n-1,n} & \left( z - \frac{n-1}{2} \right)^2 \left( zI_{n-1} - C_i \right) - nJ_{n-1,n} \end{vmatrix} P_B(z) \det \left( \frac{1}{n} \left( z - \frac{n-1}{2} \right)^2 \left( zI_{n-1} - C_i \right) - nJ_{n-1,n} \right).
\]

Hence

\[
P_{M_i+2n}(z) = p_A(z)P_B(z) \det \left( zI_{n-1} - C_i - \frac{n^2}{\left( z - \frac{n-1}{2} \right)^2} J_{n-1} \right).
\]

Using the first assertion of Lemma 4.3, we have

\[
P_{M_{i+2n}}(z) = p_A(z)P_B(z) \left( \left( \frac{n^2}{\left( z - \frac{n-1}{2} \right)^2} \right) + 1 \right) P_{C_i}(z) - \left( \frac{n^2}{\left( z - \frac{n-1}{2} \right)^2} \right) P_{C_i}(-z - 1) \right).\]

Applying Lemma 4.2 on the tournament \( T_3 \), we can write \( P_{M_{i+2n}}(z) \) as follows

\[
P_{M_{i+2n}}(z) = F(z)P_{C_i}(z) + G(z), \tag{8}
\]

where \( F \) and \( G \) are rational fractions that depend only on \( n, p_A(z), p_B(z), \) and \( p_{C_i}(z) \).

As \( T_3 \) is not \((n - 1)\)-spectrally monomorphic, there exists \( j \in [n] \setminus \{i\} \) such that \( P_{C_j}(z) \neq P_{C_i}(z) \). By (8), we get \( P_{M_{i+2n}}(z) \neq P_{M_{j+2n}}(z) \). Hence, \( T \) is not \((3n - 1)\)-spectrally monomorphic.

\( \square \)
5 Skew-spectral monomorphy

We can consider another kind of spectral monomorphy as follows. A tournament $T$ with adjacency matrix $A$ is $k$-skew-spectrally monomorphic if its skew-adjacency matrix $S = A - A^\top$ is $k$-spectrally monomorphic. The $k$-skew-spectral monomorphy is a weakening of $k$-spectral monomorphy. Indeed, the following equality [6, (5.1)] shows that the characteristic polynomial of $A$ determines that of $S$.

\[
P_S(z) = 2^{n-1} \left[ P_A \left( \frac{z + 1}{2} \right) + (-1)^{n-1} P_A \left( \frac{z - 1}{2} \right) \right]
\]  

(9)

Obviously, every tournament is 3-skew-spectrally monomorphic. Moreover, a tournament $T$ is $k$-skew-spectrally monomorphic if and only if the Hermitian matrix $iS$ is $k$-spectrally monomorphic. Then, the results of [1] provide a characterization of $k$-skew-spectrally monomorphic $n$-tournaments for $k \in \{4, \ldots, n-3\}$. In order to state this characterization, we need to define the switching of tournaments. The switch of a tournament $T$, with respect to a subset $X$ of $V$, is the tournament obtained from $T$ by reversing all the arcs between $X$ and $V \setminus X$. Two tournaments with the same vertex set are switching equivalent if and only if their skew-adjacency matrices are $\{\pm 1\}$-diagonally similar [11]. It follows that the switching operation preserves $k$-skew-spectral monomorphy. In particular, every switch of a transitive tournament is $k$-skew-spectrally monomorphic. Conversely, using [1, Corollary 3.5], we obtain

Proposition 5.1. Let $n$ and $k$ be integers such that $n \geq 8$ and $4 \leq k \leq n-4$. An $n$-tournament is $k$-skew-spectrally monomorphic if and only if it is switching equivalent to a transitive tournament.

In addition to the switch of transitive tournaments, there is another class of $k$-skew-spectrally monomorphic tournaments for $k = n-3$, which arises from skew-conference matrices. Recall that a skew-conference matrix $C$ is an $n \times n$ skew-symmetric matrix with 1 and $-1$ off the diagonal, such that $C^\top C = (n-1)I_n$. Let $T$ be an $n$-tournament whose skew-adjacency matrix is a skew-conference matrix. Proposition 4.1 of [1] implies that $T$ is $k$-skew-spectrally monomorphic for $k \in \{n-3,n-2,n-1\}$. The following proposition provides a complete characterization of $(n-3)$-skew-spectrally monomorphic $n$-tournaments. It is a consequence of [1, Theorem 4.2].

Proposition 5.2. A tournament with $n \geq 7$ vertices is $(n-3)$-skew-spectrally monomorphic if and only if it is switching equivalent to a transitive tournament or its skew-adjacency matrix is a skew-conference matrix.

For $n \in \{3,4,5\}$, all $n$-tournaments are $(n-2)$-skew-spectrally monomorphic. For $n \geq 6$, we have three classes of $(n-2)$-skew-spectrally monomorphic $n$-tournaments:

1. The switches of transitive tournaments.
2. The switches of doubly regular tournaments.
3. Tournaments whose skew-adjacency matrix is a skew-conference matrix.

Problem 5.3. Are there other examples of $(n-2)$-skew-spectrally monomorphic $n$-tournaments?
As we have seen above, $k$-spectrally monomorphic $n$-tournaments are $k$-skew-spectrally monomorphic. For every $n \geq 4$, we give an example of an $(n - 1)$-skew-spectrally monomorphic $n$-tournament that is not $(n - 1)$-spectrally monomorphic.

Consider the transitive tournament $T_n$ with vertex set $V = \{v_1, \ldots, v_n\}$ such that $v_i$ dominates $v_j$ if $i < j$. Denote by $R_n$ the tournament obtained from $T_n$ by reversing the arc $(v_1, v_n)$.

**Proposition 5.4.** The tournament $R_n$ is $(n - 1)$-skew-spectrally monomorphic but not $(n - 1)$-spectrally monomorphic.

**Proof.** Let $A$ be the adjacency matrix of $R_n$ and let $P_A(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be its characteristic polynomial. By (1), we have

$$a_k = (-1)^k \sum_{\alpha \in \binom{[n]}{k}} \det(A[\alpha]).$$

Let $k \in \{3, \ldots, n\}$ and let $\alpha \in \binom{[n]}{k}$. Consider the set $Z = \{v_i : i \in \alpha\}$. The adjacency matrix of the subtournament $R_n[Z]$ is $A[\alpha]$. If $v_1 \notin Z$ or $v_n \notin Z$ then $R_n[Z]$ is transitive and hence $\det(A[\alpha]) = 0$. If $\{v_1, v_n\} \subseteq Z$ then $R_n[Z]$ is isomorphic to $R_k$. By expanding along the first column, the determinant of the adjacency matrix of $R_k$ is $(-1)^{k+1}$. Then

$$\sum_{\alpha \in \binom{[n]}{k}} \det(A[\alpha]) = (-1)^{k+1} \binom{n-2}{k-2}.$$

Consequently,

$$P_A(z) = z^n - \sum_{k=3}^{n} \binom{n-2}{k-2} z^{n-k}.$$

By deleting the vertex $v_1$ or $v_n$ from the tournament $R_n$, we obtain a transitive tournament with characteristic polynomial $z^{n-1}$. If we delete any vertex from the set $\{v_2, \ldots, v_{n-1}\}$, we obtain a tournament isomorphic to $R_{n-1}$ with characteristic polynomial $z^{n-1} - \sum_{k=3}^{n-1} \binom{n-1}{k-2} z^{n-1-k}$. This proves that $R_n$ is not $(n - 1)$-spectrally monomorphic.

The switch of the tournament $R_n$ with respect to $\{v_1\}$ is transitive. Hence $R_n$ is $(n - 1)$-skew-spectrally monomorphic.

For regular tournaments, we have

**Proposition 5.5.** A regular $n$-tournament is $(n - 1)$-spectrally monomorphic if and only if it is $(n - 1)$-skew-spectrally monomorphic.

We will prove the following stronger result.

**Proposition 5.6.** Let $T$ be a regular $n$-tournament. For $i, j \in \{1, \ldots, n\}$, we have

$$P_{A_j}(z) - P_{A_i}(z) = \frac{2^n z^{2n-1}}{2z-n+1} \left( P_{S_j}(2z + 1) - P_{S_i}(2z + 1) \right),$$

where $A$ and $S$ are respectively the adjacency and the skew-adjacency matrices of $T$. 

Proof. From (9), we get
\[ P_{S_i}(2z + 1) = 2^{n-2}[P_{A_i}(z) + P_{A_i}(-z - 1)]; \]
\[ P_{S_j}(2z + 1) = 2^{n-2}[P_{A_j}(z) + P_{A_j}(-z - 1)]. \]

So,
\[ P_{S_j}(2z + 1) - P_{S_i}(2z + 1) = 2^{n-2}[P_{A_j}(z) - P_{A_i}(z) + P_{A_j}(-z - 1) - P_{A_i}(-z - 1)]. \]

By Lemma 4.2, we have
\[ (n + 2z + 1) (P_{A_j}(z) - P_{A_i}(z)) + (n - 2z - 1) (P_{A_j}(-z - 1) - P_{A_i}(-z - 1)) = 0 \]

Combining the last two equalities, we get the desired result. \( \square \)

References

[1] K. Attas, A. Boussaïri, and I. Souktani. Characterization of \( k \)-spectrally monomorphic hermitian matrices, 2021.

[2] J.L. Berggren. An algebraic characterization of finite symmetric tournaments. Bulletin of the Australian Mathematical Society, 6(1):53–59, 1972.

[3] J.A. Bondy and U.S.R. Murty. Graph theory with applications. North-Holland, New York, 1976.

[4] E. Brown and K.B. Reid. Doubly regular tournaments are equivalent to skew hadamard matrices. Journal of Combinatorial Theory, Series A, 12(3):332–338, 1972.

[5] D. De Caen, D.A. Gregory, S.J. Kirkland, N.J. Pullman, and J.S. Maybee. Algebraic multiplicity of the eigenvalues of a tournament matrix. Linear algebra and its applications, 169:179–193, 1992.

[6] D.A. Gregory, S.J. Kirkland, and B.L. Shader. Pick’s inequality and tournaments. Linear algebra and its applications, 186:15–36, 1993.

[7] M. Jean. Line-symmetric tournaments. Recent Progress in Combinatorics, Academic Press, New York, pages 265–271, 1969.

[8] W.M. Kantor. Automorphism groups of designs. Mathematische Zeitschrift, 109(3):246–252, 1969.

[9] A. Kotzig. Sur les tournois avec des 3-cycles régulièrement placés. Matematický časopis, 19(2):126–134, 1969.

[10] C.D. Meyer. Matrix analysis and applied linear algebra, volume 71. Siam, 2000.

[11] G.E. Moorhouse. Two-graphs and skew two-graphs in finite geometries. Linear Algebra and its Applications, 226:529–551, 1995.
[12] V. Müller and J. Pelant. On strongly homogeneous tournaments. *Czechoslovak Mathematical Journal*, 24(3):378–391, 1974.

[13] H. Nozaki and S. Suda. A characterization of skew hadamard matrices and doubly regular tournaments. *Linear Algebra and its Applications*, 437(3):1050–1056, 2012.

[14] M. Pouzet. Application d’une propriété combinatoire des parties d’un ensemble aux groupes et aux relations. *Mathematische Zeitschrift*, 150(2):117–134, 1976.

[15] M. Pouzet. Sur certains tournois reconstructibles application a leurs groupes d’automorphismes. *Discrete Mathematics*, 24(2):225–229, 1978.

[16] P. Rowlinson. On 4-cycles and 5-cycles in regular tournaments. *Bulletin of the London Mathematical Society*, 18(2):135–139, 1986.

[17] N.Z. Salvi. Some properties of regular tournament matrices (italian. english summary). In *Proceedings of the Conference on Combinatorial and Incidence Geometry: Principles and Applications (La Mendola, 1982)*, *Rend. Sem. Mat. Brescia*, volume 7, pages 635–643.

[18] A.J. Schwenk. Spectral reconstruction problems. *Annals of the New York Academy of Sciences*, 328(1):183–189, 1979.