SIMPLE RESTRICTED MODULES FOR NEVEU-SCHWARZ ALGEBRA

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Abstract. In this paper, we give a construction of simple modules generalizing and including both highest weight and Whittaker modules for the Neveu-Schwarz algebra, in the spirit of the work of Mazorchuk and Zhao on simple Virasoro modules. We establish a 1-1 correspondence between simple restricted Neveu-Schwarz modules and simple modules of a family of finite dimensional solvable Lie superalgebras associated to the Neveu-Schwarz algebra. Moreover, for two of these superalgebras all simple modules are classified.

1. Introduction

It is well known that the Virasoro algebra plays significant roles in diverse areas of mathematics and physics. An important class of modules over the Virasoro algebra is the class of weight modules have been the main focus (cf. [17] and references therein). All simple weight modules with finite dimensional weight spaces were classified by Mathieu in [27]. Whittaker modules are non-weight modules defined over finite dimensional simple Lie algebras, first appeared in [3, 21]. These modules have been studied subsequently in a variety of different settings (cf. [4, 5, 6, 11, 25]). Whittaker modules for the Virasoro algebra were studied in [13, 14, 32, 36]. Recently, Mazorchuk and Zhao [30] proposed a uniform construction of simple Virasoro modules generalizing and including both highest weight and various versions of Whittaker modules. They also characterize simple Virasoro modules that are locally finite over a positive part. Motivated by [30], new simple modules for the Virasoro algebra and its extensions have been studied [8, 9, 15, 26, 28, 37].

Superconformal algebras have a long history in mathematical physics. The simplest examples, after the Virasoro algebra itself (corresponding
to $N = 0$) are the $N = 1$ superconformal algebras: the Neveu-Schwarz algebra and the Ramond algebra. These infinite-dimensional Lie superalgebras are also called the super-Virasoro algebras as they can be regarded as natural super generalizations of the Virasoro algebra. Weight modules for the super-Virasoro algebras have been extensively investigated (cf. [12, 15, 16, 19, 31, 34]). The authors [24] introduced Whittaker type modules over the super-Virasoro algebras and obtain necessary and sufficient conditions for irreducibility of these modules. The aforementioned results demonstrate that the Whittaker modules satisfy some properties that their non-super analogues do. However, there are several differences and some features that are new in the super case [4, 33]. This leads to an additional challenge for generalizing Lie algebra results in the Lie superalgebra setting.

It is known that both simple highest weight modules and simple Whittaker modules for the Neveu-Schwarz algebra are restricted Neveu-Schwarz modules. In view of this, naturally one would want to find a unified characterization for these simple modules. This is part of our motivation for this paper. Whittaker modules have been studied in the framework of vertex operator algebra theory in [1, 2, 18, 35]. Irreducibility of certain weak modules for cyclic orbifold vertex algebras have been established. It is well known (cf. [20, 22]) that there is an isomorphism between the category of restricted Neveu-Schwarz modules and the category of weak modules for the Neveu-Schwarz vertex operator superalgebras.

In the present paper, we focus on classification problem for simple restricted Neveu-Schwarz modules. We give a construction of simple modules generalizing and including both highest weight and Whittaker modules for the Neveu-Schwarz algebra, in the spirit of the work of Mazorchuk and Zhao on simple Virasoro modules. We establish a 1-1 correspondence between simple restricted Neveu-Schwarz modules and simple modules of a family of finite dimensional solvable Lie superalgebras associated to the Neveu-Schwarz algebra. Furthermore we a classify all simple modules for the first and the second members in this family. The classification problem for simple modules of the other non-trivial members is still open as far as we know. Note that any restricted Ramond module is a weak $\sigma$-twisted module for the Neveu-Schwarz vertex operator superalgebras, where $\sigma$ is the canonical automorphism (cf.
Classification problem for simple restricted Ramond modules can be studied similarly.

This paper is organized as follows: In Section 2, we recall some notations, and collect the known facts about the Neveu-Schwarz algebra. In Section 3, we construct simple Neveu-Schwarz modules for generalizing and including both highest weight and Whittaker modules. In Section 4, we provide a characterization of simple restricted modules for Neveu-Schwarz algebra, which reduces the problem of classification of simple restricted Neveu-Schwarz modules to classification of simple modules over a family of finite dimensional solvable Lie superalgebras. In Section 5, we discuss classification of simple modules over a family of finite dimensional solvable Lie superalgebras. We recover the Whittaker modules for the Neveu-Schwarz algebra and produce new simple Neveu-Schwarz modules.

Throughout the paper, we shall use \( \mathbb{C}, \mathbb{N}, \mathbb{Z}_+ \) and \( \mathbb{Z} \) to denote the sets of complex numbers, non-negative integers, positive integers and integers respectively.

2. Preliminaries

In this section, we recall some definitions and results for later use.

2.1. Let \( V = V_0 \oplus V_1 \) be any \( \mathbb{Z}_2 \)-graded vector space. Then any element \( u \in V_0 \) (resp. \( u \in V_1 \)) is said to be even (resp. odd). We define \( |u| = 0 \) if \( u \) is even and \( |u| = 1 \) if \( u \) is odd. Elements in \( V_0 \) or \( V_1 \) are called homogeneous. Whenever \( |u| \) is written, it is understood that \( u \) is homogeneous.

Let \( L = L_0 \oplus L_1 \) be a Lie superalgebra, an \( L \)-module is a \( \mathbb{Z}_2 \)-graded vector space \( V = V_0 \oplus V_1 \) together with a bilinear map, \( L \times V \rightarrow V \), denoted \( (x, v) \mapsto xv \) such that

\[
x(yv) - (-1)^{|x||y|} y(xv) = [x, y]v
\]

for all \( x, y \in L, v \in V \), and \( L_i V_j \subseteq V_{i+j} \) for all \( i, j \in \mathbb{Z}_2 \). It is clear that there is a parity change functor \( \Pi \) on the category of \( L \)-modules, which interchanges the \( \mathbb{Z}_2 \)-grading of a module. We use \( U(L) \) to denote the universal enveloping algebra.
Definition 2.1. Let $L$ be a Lie superalgebra and $V$ be an $L$-module and $x \in L$. If for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $x^n v = 0$, then we call that the action of $x$ on $V$ is locally nilpotent. Similarly, the action of $L$ on $V$ is locally nilpotent if for any $v \in V$ there exists $n \in \mathbb{Z}_+$ such that $L^n x = 0$.

Definition 2.2. Let $L$ be a Lie superalgebra and $V$ be an $L$-module and $x \in L$. If for any $v \in V$ we have $\dim(\sum_{n \in \mathbb{Z}_+} C x^n v) < +\infty$, then we call that the action of $x$ on $V$ is locally finite. Similarly, the action of $L$ on $V$ is locally finite if for any $v \in V$ we have $\dim(\sum_{n \in \mathbb{Z}_+} L^n v) < +\infty$.

Remark 2.3. The action of $x$ on $V$ is locally nilpotent implies that the action of $x$ on $V$ is locally finite. If $L$ is a finitely generated Lie superalgebra, then the action of $L$ on $V$ is locally nilpotent implies that the action of $L$ on $V$ is locally finite.

2.2. Neveu-Schwarz algebra.

Definition 2.4. The Neveu-Schwarz algebra is the Lie superalgebra $g = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \bigoplus_{r \in \mathbb{Z}} \mathbb{C} G_r \oplus \mathbb{C} c$ which satisfies the following commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m-n} \frac{m^3 - m}{12} c,$$

$$[L_m, G_r] = \left(\frac{m}{2} - r\right) G_{m+r},$$

$$[G_r, G_s] = 2L_{r+s} + \frac{1}{3} \delta_{r+s,0} \left(r^2 - \frac{1}{4}\right) c,$$

$$[g, c] = 0,$$

for all $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$, where

$|L_n| = \bar{0}$, $|G_r| = \bar{1}$, $|c| = \bar{0}$.

By definition, we have the following decompositions:

$g = g_{\bar{0}} \oplus g_{\bar{1}},$

where

$$g_{\bar{0}} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_n \oplus \mathbb{C} c,$$

$$g_{\bar{1}} = \bigoplus_{r \in \mathbb{Z} + \frac{1}{2}} \mathbb{C} G_r.$$

It is clear that the even part $g_{\bar{0}}$ is isomorphic to the well-known Virasoro algebra Vir. The Neveu-Schwarz algebra $g$ has a $\frac{1}{2}\mathbb{Z}$-grading by the
eigenvalues of the adjoint action of $L_0$. Then $g$ possesses the following triangular decomposition:

$$g = g_+ \oplus g_0 \oplus g_-,$$

where

$$g_\pm = \bigoplus_{n \in \mathbb{Z}_+} \mathbb{C}L_{\pm n} \oplus \bigoplus_{r \in \mathbb{N} + \frac{1}{2}} \mathbb{C}G_{\pm r}, \quad g_0 = \mathbb{C}L_0 \oplus \mathbb{C}c.$$

**Definition 2.5.** If $W$ is a $g$-module on which $c$ acts as a complex scalar $\ell$, we say that $W$ is of central charge $\ell$.

**Definition 2.6.** A $g$-module $W$ is called restricted in the sense that for every $w \in W$,

$$L_i w = G_{i-\frac{1}{2}} w = 0$$

for $i$ sufficiently large.

Let

$$b = \bigoplus_{i \geq 0} \mathbb{C}L_i \oplus \bigoplus_{i \geq 1} \mathbb{C}G_{i-\frac{1}{2}}.$$

It is clear that $b$ is a subalgebra of $g$.

Given a $b$-module $V$ and $\ell \in \mathbb{C}$, consider the corresponding induced module

$$\text{Ind}(V) := U(g) \otimes_{U(b)} V$$

and denote

$$\text{Ind}_\ell(V) = \text{Ind}(V)/(c - \ell)\text{Ind}(V).$$

Denote by $\mathbb{M}$ the set of all infinite vectors of the form $i := (\ldots, i_2, i_1)$ with entries in $\mathbb{N}$, satisfying the condition that the number of nonzero entries is finite, and $\mathbb{M}_1 := \{i \in \mathbb{M} \mid i_k = 0, 1, \forall k \in \mathbb{Z}_+\}$.

Let $0$ denote the element $(\ldots, 0, 0) \in \mathbb{M}$ and for $i \in \mathbb{Z}_+$ let $\epsilon_i$ denote the element $(\ldots, 0, 1, 0, \ldots, 0) \in \mathbb{M}$, where $1$ is in the $i$'th position from right. For any $i \in \mathbb{M}$, we write

$$w(i) = \sum_{k \in \mathbb{Z}_+} k \cdot i_k,$$

which is a nonnegative integer. For any nonzero $i \in \mathbb{M}$, let $p$ be the smallest integer such that $i_p \neq 0$ and define $i' = i - \epsilon_p$. 
Definition 2.7. Denote by $\prec$ the reverse lexicographical total order on $\mathbb{M}$, defined as follows: for any $i, k \in \mathbb{M}$, set

$$i \prec j \iff \text{there exists } r \in \mathbb{Z}^+ \text{ such that } i_r < j_r \text{ and } i_s = j_s, \forall 1 \leq s < r.$$

Now we can induce a principal total order on $\mathbb{M} \times \mathbb{M}_1$, still denoted by $\prec$: $(i, k) \prec (j, m)$ if and only if one of the following conditions is satisfied:

1. $w(i) + w(k) < w(j) + w(m)$;
2. $w(i) + w(k) = w(j) + w(m)$ and $(k, w(k)) \prec (m, w(m))$;
3. $k = m$ and $(i, w(i)) \prec (j, w(j)), \forall i, j \in \mathbb{M}, k, m \in \mathbb{M}_1$.

Let $V$ be a simple $\mathfrak{b}$-module. For $k \in \mathbb{M}_1, i \in \mathbb{M}$, we denote

$$G^kL^i = \ldots G^{k_2}_{-2+\frac{1}{2}} G^{k_1}_{-1+\frac{1}{2}} \ldots L^k_{-2} L^1_{-1} \in U(\mathfrak{g}_-).$$

According to the PBW Theorem, every element of $\text{Ind}_\ell(V)$ can be uniquely written in the following form

$$\sum_{k \in \mathbb{M}_1, i \in \mathbb{M}} G^kL^i_{v_{k,i}},$$

where all $v_{k,i} \in V$ and only finitely many of them are nonzero. For any $v \in \text{Ind}_\ell(V)$ as in (2.1), we denote by $\text{supp}(v)$ the set of all $(i, k) \in \mathbb{M} \times \mathbb{M}_1$ such that $v_{k,i} \neq 0$. For a nonzero $v \in \text{Ind}_\ell(V)$, we write $\text{deg}(v)$ the maximal (with respect to the principal total order on $\mathbb{M} \times \mathbb{M}_1$) element in $\text{supp}(v)$, called the degree of $v$. Note that here and later we make the convention that $\text{deg}(v)$ only for $v \neq 0$.

### 3. Construction of simple restricted $\mathfrak{g}$-modules

In this section, we give a construction of simple restricted $\mathfrak{g}$-modules.

**Theorem 3.1.** Let $V$ be a simple $\mathfrak{b}$-module and assume that there exists $t \in \mathbb{Z}_+$ satisfying the following two conditions:

(a) the action of $L_t$ on $V$ is injective;
(b) $L_i V = 0$ for all $i > t$.

Then

(i) $G_{j-\frac{1}{2}} V = 0$ for all $j > t$.
(ii) For $\ell \in \mathbb{C}$, the induced module $\text{Ind}_\ell(V)$ is a simple $\mathfrak{g}$-module.
Proof. (i) Assume $L_t V = 0$ for all $i > t$. For $j \geq t$, by $G^2_{j+\frac{1}{2}} V = L_{2j+1} V = 0$, we have $W = G_{j+\frac{1}{2}} V$ is a proper subspace of $V$. For $r \in \mathbb{Z}_+$, we have

$$G_{r-\frac{1}{2}} W = G_{r-\frac{1}{2}} G_{j+\frac{1}{2}} V = L_{r+j} V - G_{j+\frac{1}{2}} G_{r-\frac{1}{2}} V \subset G_{j+\frac{1}{2}} V = W,$$

$$2L_{t} W = [G_{r-\frac{1}{2}}] G_{j+\frac{1}{2}} V = G_{r-\frac{1}{2}} G_{j+\frac{1}{2}} V + G_{r-\frac{1}{2}} G_{j+\frac{1}{2}} V \subset W.$$

It follows that $W$ is a proper submodule of $V$. Then $W = G_{j+\frac{1}{2}} V = 0$ for $j \geq t$ since $V$ is simple.

In order to prove (ii), we need the following claim.

**Claim.** For any $v \in \text{Ind}_c(V) \setminus V$, let $\text{deg}(v) = (i, k)$, $\hat{k} = \min\{s : k_s \neq 0\}$ if $k \neq 0$ and $i = \min\{s : i_s \neq 0\}$ if $i \neq 0$. Then

1. If $k \neq 0$, then $\hat{k} > 0$ and $\text{deg}(G_{k+t-\frac{1}{2}} v) = (i, \hat{k})$;
2. If $k = 0, i \neq 0$, then $\hat{i} > 0$ and $\text{deg}(L_{i+t} v) = (i', 0)$.

To prove this, we assume that

$$v = \sum_{m \in M_1, j \in M} G^m L^j v_{m,j},$$

(3.1)

where all $v_{m,j} \in V$ and only finitely many of them are nonzero.

1. It suffices to consider those $v_{j,m}$ with

$$G^m L^j v_{j,m} \neq 0.$$

Note that $G_{k+t-\frac{1}{2}} v_{j,m} = 0$ for any $(j, m) \in \text{supp}(v)$. One can easily check that

$$G_{k+t-\frac{1}{2}} G^m L^j v_{j,m} = [G_{k+t-\frac{1}{2}}, G^m] L^j v_{j,m} + G^m [G_{k+t-\frac{1}{2}}, L^j] v_{j,m}.$$

Clearly $L_t v_{j,m} \neq 0$ by (a).

If

$$w(j) + w(m) < w(i) + w(k),$$

then

$$\text{deg} G_{k+t-\frac{1}{2}} G^m L^j v_{j,m} < (i, k').$$

Now we suppose that $w(i) + w(k) = w(j) + w(m)$ and $m < k$ and denote

$$\text{deg}(G_{k+t-\frac{1}{2}} G^m L^j v_{j,m}) = (j_1, m_1) \in M \times M_1.$$

Let $\hat{m} := \min\{s : m_s \neq 0\} > 0$. If $\hat{m} > \hat{k}$, it is easy to see that $w(m') < w(m) - \hat{k} = w(k')$. If $\hat{m} = \hat{k}$, we can similarly deduce
\((j_1, m_1) = (i, m')\). Since \(m' < k'\), we have \(\deg(G_{k+t-\frac{1}{2}}G^mL^jv_{j,m}) = (j_1, m_1) \prec (i, k')\).

If \(m = k\), it is easy to see that
\[
\deg([G_{k+t}, G^m]L^jv_{j,m}) = (j, k') \prec (i, k'),
\]
\[
\deg(G^m[G_{k+t}, L^j]v_{j,m}) = (j_1, m_1) \prec (i, k'),
\]
where the equality holds if and only if \(j = i\).

Combining all the arguments above we conclude that \(\deg(G_{k+t-\frac{1}{2}}v) = (i, k')\), as desired.

(2) We consider \(v_{j,0}\) with
\[
L_{i+t}L^jv_{j,0} \neq 0.
\]
Since \(L_{i+t}v_{j,0} = 0\) for any \((j, 0) \in \text{supp}(v)\), then we have
\[
L_{i+t}L^jv_{j,0} = [L_{i+t}, L^j]v_{j,0}.
\]

If \(j = i\), it is easy to get that
\[
\deg(L_{i+t}L^jv_{j,0}) = (j', 0) = (i', 0).
\]
Now suppose \((j, w(j)) \prec (i, w(i))\), then we write
\[
\deg(L_{i+t}L^jv_{j,0}) = (j_1, 0). \quad (3.2)
\]
If \(w(j) < w(i)\), then \(w(j_1) \leq w(j) - \hat{i} < w(i) - \hat{i} = w(i')\), which shows that \((j_1, 0) \prec (i, 0)\).

Then we suppose \(w(j) = w(i)\) and \(j \subset i\). Let \(\hat{j} := \min\{s : j_s \neq 0\} > 0\). If \(\hat{j} > \hat{i}\), we obtain \(w(j') < w(i) - \hat{i} = w(i')\). If \(\hat{j} = \hat{i}\), we can similarly check that \((j_1, 0) = (j', 0)\). By \(j' \prec i'\), we have \(\deg(L_{j+t}L^jv_{j,0}) = (j_1, 0) \prec (i', 0)\).

Consequently, we conclude that \(\deg(L_{j+t}v) = (i', 0)\). This proves the claim.

Using the claim repeatedly, from any nonzero element \(v \in \text{Ind}_\ell(V)\) we can reach a nonzero element in \(U(g)v \cap V \neq 0\), which implies that the simplicity of \(\text{Ind}_\ell(V)\).

\[\square\]

**Remark 3.2.** In Theorem 3.1 note that the actions of \(L_i, G_{k+t-\frac{1}{2}}\) on \(\text{Ind}_\ell(V)\) for all \(i > t\) are locally nilpotent. It follows that \(\text{Ind}_\ell(V)\) is a simple restricted \(g\)-module of central charge \(\ell\).
4. Characterization of simple restricted g-modules

In this section, we give a characterization of simple restricted g-modules of central charge $\ell$.

For $t \in \mathbb{N}$, let

$$m^{(t)} = \bigoplus_{m > t} \mathbb{C}L_m \oplus \bigoplus_{m > t} \mathbb{C}G_{m - \frac{1}{2}},$$

Note that $m^{(0)} = g_+$. 

Proposition 4.1. Let $S$ be a simple $g$-module. Then the following conditions are equivalent:

1. There exists $t \in \mathbb{Z}_+$ such that the actions of $L_i, G_{i - \frac{1}{2}}$ for all $i \geq t$ on $S$ are locally finite.
2. There exists $t \in \mathbb{Z}_+$ such that the actions of $G_{i - \frac{1}{2}}, L_i$ for all $i \geq t$ on $S$ are locally nilpotent.
3. There exist $t \in \mathbb{Z}_+$ such that $S$ is a locally finite $m^{(t)}$-module.
4. There exist $t \in \mathbb{Z}_+$ such that $S$ is a locally nilpotent $m^{(t)}$-module.
5. $S$ is a highest weight module, or there exists $\ell \in \mathbb{C}, t \in \mathbb{Z}_+$ and a simple $b$-module $V$ such that both conditions $(a)$ and $(b)$ in Theorem 3.1 are satisfied and $S \cong \text{Ind}_\ell(V)$.

Proof. First we prove $(1) \Rightarrow (5)$. Suppose that $S$ is a simple $g$-module and there exists $t \in \mathbb{Z}_+$ such that the actions of $G_{i - \frac{1}{2}}, L_i, i \geq t$ are locally finite.

Choose a simple Vir-submodule $S'$ of $S$. Clearly $L_i, i > t$ are locally finite on $S'$. By Proposition 4 in [30], there exist $t' \in \mathbb{Z}_+$ and a simple Vir$_+-$module $W$ such that $S' = \text{Ind}(W)$ as Virasoro module and $L_nW = 0$ for all $n > t'$, where Vir$_+ := \bigoplus_{m > 0} \mathbb{C}L_m$.

Then we can choose a nonzero $w \in W$ such that $L_nw = 0$ for all $n > t'$.

Take any $j \in \mathbb{Z}$ with $j > t'$ and we denote

$$V_G = \sum_{m \in \mathbb{N}} \mathbb{C}L_m^{[m]}G_{j - \frac{1}{2}}w = U(\mathbb{C}L_{t'})G_{j - \frac{1}{2}}w,$$

which are all finite-dimensional. By Definition 2.4 it is clear that $G_{j+(m+1)t'-\frac{1}{2}}w \in V_G$ if $G_{j+mt'-\frac{1}{2}}w \in V_G$. Therefore, by induction on $m$, we obtain $G_{j+mt'-\frac{1}{2}}w \in V_G$ for all $m \in \mathbb{N}$. Then, it follows from
the facts that \( \sum_{m \in \mathbb{N}} \mathbb{C} G_{j + m't' - \frac{1}{2}} w \) are finite-dimensional for any \( j > t' \). Hence,

\[
\sum_{i \in \mathbb{Z}_+} \mathbb{C} G_{t' + i - \frac{1}{2}} w = \sum_{j = t' + 1}^{2t} \left( \sum_{m \in \mathbb{Z}_+} \mathbb{C} G_{j + m't' - \frac{1}{2}} w \right)
\]

is finite-dimensional. In fact, we can take \( p \in \mathbb{Z}_+ \) such that

\[
\sum_{i \in \mathbb{Z}_+} \mathbb{C} G_{t' + i - \frac{1}{2}} w = p \sum_{i = 0}^{p} \mathbb{C} G_{t' + i + \frac{1}{2}} w.
\] (4.1)

Now we write

\[
V' := \sum_{\tilde{i}_0, \ldots, \tilde{i}_p \in \{0, 1\}} \mathbb{C} G_{\tilde{i}_0 t' + \frac{1}{2}} \cdots G_{\tilde{i}_k t' + \frac{1}{2}} w, \text{ which is finite-dimensional by (1). Moreover } \text{ } V' \text{ is a finite-dimensional } m(t')-\text{module}.
\]

It follows that we can choose a minimal \( n \in \mathbb{N} \) such that

\[
(G_m - \frac{1}{2} + a_1 G_{m+1} - \frac{1}{2} + \cdots + a_n G_{m+n} - \frac{1}{2}) V' = 0 \quad (4.2)
\]

for some \( m > t' \) and \( a_i \in \mathbb{C} \). Applying \( L_{2m-1} \) to (4.2), one has

\[
(a_1 [L_{2m-1}, G_m - \frac{1}{2}] + \cdots + a_n [L_{2m-1}, G_{m+n} - \frac{1}{2}]) V' = 0,
\]

which implies \( n = 0 \), that is,

\[
G_m - \frac{1}{2} V' = 0. \quad (4.3)
\]

By action of \( L_i \) on (4.3)

\[
G_{m+i} - \frac{1}{2} V' = 0, \quad \forall i > t'. \quad (4.4)
\]

For any \( \tilde{k} \in \mathbb{N} \), we consider the vector space

\[
N_{\tilde{k}} = \{ v \in S \mid G_{-\frac{1}{2}} v = L_{\tilde{k}} v = 0 \quad \text{for all } \tilde{k} > \tilde{k} \}.
\]

Clearly, \( N_{\tilde{k}} \neq 0 \) for sufficiently large \( \tilde{k} \in \mathbb{N} \). Thus we can find a smallest nonnegative integer, saying \( s \), with \( V := N_s \neq 0 \). Using \( k > s \) and \( p \geq 1 \), it follows from \( k + p - \frac{1}{2} > s \) and \( k + p - 1 > s \) that we can easily check that

\[
L_k (G_{p-\frac{1}{2}} v) = (p - \frac{k + 1}{2}) G_{k+p-\frac{1}{2}} v = 0
\]

and

\[
G_{-\frac{1}{2}} (G_{p-\frac{1}{2}} v) = 2L_{k+p-1} v = 0,
\]

respectively. Clearly, \( G_{p-\frac{1}{2}} v \in V \) for all \( p \geq 1 \). Similarly, we can also obtain \( L_i v \in V \) for all \( i \in \mathbb{N} \). Therefore, \( V \) is a \( \mathfrak{b} \)-module.

If \( s = 0 \), then by Theorem 1(c) in [29], \( S \) is a highest weight module.
If \( s \geq 1 \), by the definition of \( V \), we can obtain that the action of \( L_s \) on \( V \) is injective by Theorem 3.1. Since \( S \) is simple and generated by \( V \), then there exists a canonical surjective map 
\[ \pi : \text{Ind}(V) \to S, \quad \pi(1 \otimes v) = v, \quad \forall v \in V. \]

Next we only need to show that \( \pi \) is also injective, that is to say, \( \pi \) as the canonical map is bijective. Let \( K = \ker(\pi) \). Obviously, \( K \cap V = 0 \). If \( K \neq 0 \), we can choose a nonzero vector \( v \in K \setminus V \) such that \( \text{deg}(v) = (i, k) \) is minimal possible. Note that \( K \) is a \( \mathfrak{g} \)-submodule of \( \text{Ind}_\ell(V) \). By the claim in proof of Theorem 3.1 we can create a new vector \( u \in K \) with \( \text{deg}(u) \prec (i, k) \), which is a contradiction. This forces \( K = 0 \), that is, \( S \cong \text{Ind}_\ell(V) \). According to the property of induced modules, we see that \( V \) is simple as a \( \mathfrak{b} \)-module.

Moreover, (5) \( \Rightarrow \) (3) \( \Rightarrow \) (1), (5) \( \Rightarrow \) (4) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (1) are clear. This completes the proof. \( \square \)

**Lemma 4.2.** Let \( V \) be a simple restricted \( \mathfrak{g} \)-module. Then there exists \( t \in \mathbb{Z}_+ \) such that the actions of \( L_i, G_{i-\frac{1}{2}} \) for all \( i \geq t \) on \( V \) are locally nilpotent.

**Proof.** Let \( 0 \neq v \in V \), there exists \( s \in \mathbb{Z}_+ \) such that \( L_i v = G_{i-\frac{1}{2}} v = 0 \) for all \( i \geq s \). For \( V = U(\mathfrak{g})v \), every element \( w \) of \( V \) can be uniquely written in the following form
\[ w = \sum_{k \in M, i \in \mathbb{M}} G^k L^i v. \]

Then, for \( i \geq s \), there exists \( N \) sufficiently large such that
\[ L^N_i w = G^N_{i-\frac{1}{2}} w = 0. \]
\( \square \)

From Theorem 4.1 and Lemma 4.2 we are in a position to state our main result.

**Theorem 4.3.** Let \( \ell \in \mathbb{C} \), every simple restricted \( \mathfrak{g} \)-module of central charge \( \ell \) is isomorphic to a simple highest weight module, or a simple module of the form \( \text{Ind}_\ell(V) \), where \( V \) is a simple \( \mathfrak{b}^{(t)} \)-module for some \( t \in \mathbb{Z}_+ \), where \( \mathfrak{b}^{(t)} = \mathfrak{b}/\mathfrak{m}^{(t)} \) is the quotient algebra of \( \mathfrak{b} \) by \( \mathfrak{m}^{(t)} \).
5. Simple $b^{(t)}$-modules and examples

5.1. Classifications. For $t = 0$, the algebra $b^{(0)}$ is commutative and its simple modules are one dimensional. Next we shall classify all simple $b^{(t)}$-module for $t = 1, 2$.

For $t = 1$, $b^{(1)} = b^{(1)}_0 \oplus b^{(1)}_1$ is a 3-dimensional solvable Lie superalgebra with $b^{(1)}_0 = CL_0 \oplus CL_1$ and $b^{(1)}_1 = CG_{1/2}$. All simple $b^{(1)}_0$-modules are constructed by R. Block in [7].

**Proposition 5.1.** Any simple $b^{(1)}$-module $V$ is isomorphic to $U \oplus G_{1/2}U$ or an one-dimensional module, where $U$ is a simple $b^{(1)}_0$-module, up to parity-change.

**Proof.** Let $V$ be a simple $b^{(1)}$-module. Choose an simple $b^{(1)}_0$-submodule $U$ of $V$, then $V = U + G_{1/2}U$ and $V_0 = U$.

**Case 1.** $V$ is $G_{1/2}$-torsion free.

In this case, $V_1 = G_{1/2}U$ and $U$ is $L_1$-torsion free since $G_{1/2}^2 = L_1$ in $U(b^{(1)})$. Moreover $V_0 = G_{1/2}V_1$, so $V$ is simple $b^{(1)}$-module if and only if $U$ is simple $a^{(1)}$-module.

**Case 2.** $V$ is $G_{1/2}$-torsion. In this case, $U$ is $L_1$-torsion. It follows that there exists $w \in U$ such that $L_1w = cw$ for some $c \in \mathbb{C}$. If $c = 0$, then $U = \mathbb{C}w$. Replaced $w$ by $G_{1/2}w$ if $G_{1/2}w \neq 0$, we have $V = \mathbb{C}G_{1/2}w$. If $c \neq 0$, then $U = \mathbb{C}[L_0]w$ and $V = U + G_{1/2}U$ and $G_{1/2}(G_{1/2}U) = U$. 

\[\square\]

For $t = 2$, $b^{(2)} = b^{(2)}_0 \oplus b^{(2)}_1$ is a 5-dimensional solvable Lie superalgebra with $b^{(2)}_0 = CL_0 \oplus CL_1 \oplus CL_2$ and $b^{(2)}_1 = CG_{1/2} \oplus CG_{3/2}$. All simple $b^{(2)}_0$-modules are constructed by Mazorchuk and Zhao in [30].

**Proposition 5.2.** Any simple module over $b^{(2)}$ is isomorphic to $U \oplus G_{3/2}U$, where $U$ is a simple $b^{(2)}_0$-module, up to parity-change.

**Proof.** Let $V = V_0 \oplus V_1$ be a simple $b^{(2)}$-module. For any $w \in V_0$, replaced $w$ by $G_{3/2}w$ if $G_{3/2}w \neq 0$, we have $G_{3/2}w = 0$. In this case $V_0 = U(b^{(2)}_0)w$ and $G_{3/2}V_0 = 0$, then $V_1 = G_{3/2}V_0$. \[\square\]

5.2. Examples.
5.2.1. *Highest weight modules.* For $h, \ell \in \mathbb{C}$, let $Cv$ be one-dimensional $g_0$-module defined by $L_0v = hv, cv = \ell v$. Let $g_+$ act trivially on $v$, making $v$ a $(g_0 \oplus g_+)$-module. The Verma module for Neveu-Schwarz algebra (cf. [20]) can be defined by

$$M(h, \ell) = U(g) \otimes_{U(g_0 \oplus g_+)} Cv,$$

The module $M(h, \ell)$ has the unique simple quotient $L(h, \ell)$, the unique (up to isomorphism) simple highest weight module with highest weight $(h, \ell)$. These simple modules correspond to $t = 0$ case in Theorem 4.3.

5.2.2. *Whittaker modules.* Let

$$p(t) = \bigoplus_{m > t} \mathbb{C}L_m \oplus \bigoplus_{m > t} \mathbb{C}G_{m + \frac{1}{2}}.$$

and $\psi$ a Lie superalgebra homomorphism $\psi : p \to \mathbb{C}$. It follows that $\psi(L_i) = 0$ for $i \geq 3$ and $\psi(G_{j - \frac{1}{2}}) = 0$ for all $j \geq 2$. For $\ell \in \mathbb{C}$, let $cw$ be one dimensional $(p \oplus \mathbb{C}c)$-module with $xw = \psi(x)w$ for all $x \in p$ and $cw = \ell w$, then the Whittaker module for Neveu-Schwarz algebra is defined by

$$W(\psi, \ell) = U(g) \otimes_{U(p \oplus \mathbb{C}c)} Cw.$$

By [24], the Whittaker module $W(\psi, \ell)$ is simple if $\psi$ is non-trivial, i.e., $\psi(L_1) \neq 0$ or $\psi(L_2) \neq 0$.

Let $\psi : p \to \mathbb{C}$ be a nontrivial Lie superalgebra homomorphism and $A_\psi = Cw \oplus Cu$ with be a two-dimensional vector space with

$$xw = \psi(x)w, \; G_{\frac{3}{2}}w = u, \; \forall x \in p.$$

Then $A_\psi$ is a simple $g_+$-module. Consider induced module

$$V_\psi = U(b) \otimes_{U(p)} A_\psi \cong \mathbb{C}[L_0]A_\psi.$$

It is straightforward to check that $V_\psi$ is a simple $b$-module. Hence, by Theorem 3.1, we obtain the corresponding simple induced $g$-module $\text{Ind}_c(V_\psi)$. These are exactly the Whittaker modules $W(\psi, \ell)$.

5.2.3. *High order Whittaker modules.* For $t \in \mathbb{N}$, let

$$p^{(t)} = \bigoplus_{m > t} \mathbb{C}L_m \oplus \bigoplus_{m > t} \mathbb{C}G_{m + \frac{1}{2}}.$$

It is clear that $p^{(0)} = p$. All finite dimensional simple modules over $p^{(0)}$ have been classified in [24]. Now we shall classify all finite-dimensional simple modules over $p^{(t)}$ for $t \in \mathbb{Z}_+$. 
We have the following lemma which can be proved in a way similar to Proposition 3.3 in \[24\], we have

**Lemma 5.3.** Let $V$ be a finite dimensional simple $\mathfrak{p}^{(t)}$-module for $t \in \mathbb{Z}_+$. Then there exists $k \in \mathbb{Z}_+$ such that $\mathfrak{p}^{(k)}V = 0$.

**Proposition 5.4.** Let $S$ be a simple finite dimensional $\mathfrak{p}^{(t)}$-module for $t \in \mathbb{Z}_+$. Then

(i) $\dim S = 1$;

(ii) $[\mathfrak{p}^{(t)}, \mathfrak{p}^{(t)}]S = 0$.

**Proof.** By Proposition 5.3, we have $\mathfrak{p}^{(i)}S = 0$ for some $i \geq t$. Hence $S$ is a simple finite dimensional module over the nilpotent Lie superalgebra $\mathfrak{p}^{(t)}/\mathfrak{p}^{(i)}$. Moreover $[\mathfrak{p}^{(t)}, \mathfrak{p}^{(t)}] \subseteq [\mathfrak{p}^{(0)}, \mathfrak{p}^{(0)}]$, then (i) follows from Lemma 1.37 in \[10\]). As $\dim S = 1$, we also have that the Lie superalgebra $\text{End}_\mathbb{C}(S)$ is commutative, which implies (ii). This completes the proof. \[\Box\]

Let $\psi_k$ be a Lie superalgebra homomorphism $\psi_k : \mathfrak{p}^{(k)} \to \mathbb{C}$ for some $k \in \mathbb{Z}_+$. Then $\psi_k(L_i) = 0$ for $i \geq 2k + 3$ and $\psi_k(G_{j+\frac{1}{2}}) = 0$ for all $j \geq k + 1$. Let $\mathbb{C}w$ be one-dimensional $\mathfrak{p}^{(k)}$-module with $xw = \psi_k(x)w$ for all $x \in \mathfrak{p}^{(k)}$ and $cw = \ell w$ for some $\ell \in \mathbb{C}$. The higher order Whittaker module $W(\psi_k, \ell)$ is given by

$$W(\psi_k, \ell) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^{(k)})} \mathbb{C}w.$$ 

**Proposition 5.5.** For $k \in \mathbb{Z}_+$, the higher order Whittaker module $W(\psi_k, \ell)$ is simple if and only if $\psi_k(L_{2k+1}) \neq 0$ or $\psi_k(L_{2k+2}) \neq 0$.

**Proof.** For $k \in \mathbb{Z}_+$, let $\psi_k : \mathfrak{p}^{(k)} \to \mathbb{C}$ be a Lie superalgebra homomorphism and $A_{\psi_k} = \mathbb{C}w \oplus \mathbb{C}u$ with be a two-dimensional vector space with

$$xw = \psi_k(x)w, \ G_{k+\frac{1}{2}}w = u, \ \forall x \in \mathfrak{p}^{(k)}.$$ 

Then $A_{\psi_k}$ is a simple $\mathfrak{m}^{(k)}$-module if and only if $\psi_k(L_{2k+1}) \neq 0$ or $\psi_k(L_{2k+2}) \neq 0$. Moreover, if $\psi_k(L_{2k+1}) = \psi_k(L_{2k+2}) = 0$, then $A_{\psi_k}$ has a trivial submodule $\mathbb{C}G_{k+\frac{1}{2}}w$.

Let

$$V_{\psi_k} = U(\mathfrak{b}) \otimes_{U(\mathfrak{m}^{(k)})} A_{\psi_k}.$$
It is straightforward to check that $V_{\psi_k}$ is a simple $\mathfrak{b}$-module if and only if $\psi_k(L_{2k+1}) \neq 0$ or $\psi_k(L_{2k+2}) \neq 0$. From Theorem 3.1 we obtain the corresponding induced $\mathfrak{g}$-module $\text{Ind}_\ell(V_{\psi_k})$ is simple if and only if $\psi_k(L_{2k+1}) \neq 0$ or $\psi_k(L_{2k+2}) \neq 0$. These modules are exactly the higher order Whittaker modules $W(\psi_k, \ell)$. □

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