RELATIVE COMPLETIONS AND THE COHOMOLOGY OF LINEAR GROUPS OVER LOCAL RINGS

KEVIN P. KNUDSON

For a discrete group $G$ there are two well-known completions. The first is the Malcev (or unipotent) completion. This is a prounipotent group $U$, defined over $\mathbb{Q}$, together with a homomorphism $\psi: G \to U$ that is universal among maps from $G$ into prounipotent $\mathbb{Q}$-groups. To construct $U$, it suffices to consider the case where $G$ is nilpotent; the general case is handled by taking the inverse limit of the Malcev completions of the $G/\Gamma_r G$, where $\Gamma^r G$ denotes the lower central series of $G$. If $G$ is abelian, then $U = G \otimes \mathbb{Q}$. We review this construction in Section 2.

The second completion of $G$ is the $p$-completion. For a prime $p$, we set $G^{\wedge p} = \varprojlim G/\Gamma^r_p G$, where $\Gamma^r_p G$ is the $p$-lower central series of $G$. If $G$ is a finitely generated abelian group, then $G^{\wedge p} = \mathbb{Z}_p \otimes G$, where $\mathbb{Z}_p$ is the ring of $p$-adic integers. The group $G^{\wedge p}$ is a pro-$p$-group and each $G/\Gamma^r_p G$ is nilpotent provided $H_1(G, \mathbb{F}_p)$ is finite dimensional.

Both of these completions are instances of a general construction. Let $k$ be a field. The unipotent $k$-completion of a group $G$ is a prounipotent $k$-group $U_k$ together with a homomorphism $G \to U_k$. The group $U_k$ is required to satisfy the obvious universal mapping property. The Malcev completion is the case $k = \mathbb{Q}$ and the $p$-completion is the case $k = \mathbb{F}_p$. This construction for other fields $k$ is probably well-known to the experts, but it does not seem to be in the literature.

One reason to study such completions is that they may be used to gain cohomological information about the groups $G$ and $U$. Indeed, the restriction map $H^2_{\text{cts}}(U, k) \to H^2(G, k)$ is injective (the definition of $H^2_{\text{cts}}$ will be recalled below). This allows one to obtain either a lower bound for $\dim_k H^2(G, k)$ or an upper bound for $\dim_k H^2_{\text{cts}}(U, k)$.

Unfortunately, the group $U_k$ may be trivial (e.g., $G$ perfect, or more generally, if $H_1(G, k) = 0$). To circumvent this, Deligne suggested the notion of relative completion. Suppose that $\rho: G \to S$ is a representation of $G$ in a reductive group $S$ defined over $k$. Assume that the image of $\rho$ is Zariski dense. The completion of $G$ relative to $\rho$ is a proalgebraic

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group $G$ over $k$ which is an extension of $S$ by a prounipotent group $U$ together with a map $G \rightarrow G$. The group $G$ should satisfy the obvious universal mapping property.

The basic theory of relative completion in characteristic zero was worked out by R. Hain [6]. Many of his results remain valid in positive characteristic. We shall review this in Section 4 below. For examples of relative completions in characteristic zero beyond those presented here, the reader is referred to Hain’s study [6] of the completion of the mapping class group $\Gamma_{g,r}$ of a surface $S$ of genus $g$ with $r$ marked points relative to its symplectic representation as the group of automorphisms of $H_1(S,\mathbb{Z})$. Other examples, due to the author [12], are the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$ relative to their obvious representations in $SL_n(\mathbb{Z})$. Recent work by Hain and M. Matsumoto [7] tackles the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ relative to the cyclotomic character $\chi : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \mathbb{Z}_\ell^\times$.

In this paper we study the completions of groups such as $SL_n(\mathcal{O}_{X,x})$, where $\mathcal{O}_{X,x}$ is the local ring of a closed point $x$ on a smooth affine curve $X$. We also study the completion of $SL_n(\mathbb{Z})$ relative to the reduction map $SL_n(\mathbb{Z}) \rightarrow SL_n(\mathbb{F}_p)$. We use these completions to compute the second continuous cohomology of special linear groups over complete local rings. A sample result is the following.

**Theorem 5.5.** If $k$ is a finite field or a number field, then for all $n \geq 3$, $H^2_{\text{cts}}(SL_n(k[[T]]), k) = 0$.

As far as the author knows, this is the first calculation of continuous cohomology with coefficients of the same characteristic as $k$.

This paper is organized as follows. In Section 1, we review Hain’s construction of the completion of $\rho : G \rightarrow S$. In Section 2, we discuss in detail the case where $S$ is the trivial group; this is the unipotent completion mentioned above. In Section 3, we present several examples of unipotent completions. Section 4 deals with the basic theory of relative completion and the computation of examples. Finally, in Section 5 we carry out some cohomology calculations.

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1. A Construction of the Relative Completion

The following construction is due to R. Hain [6]. Let $G$ be a group and suppose that $\rho : G \rightarrow S$ is a Zariski dense representation in a
reductive $k$-group $S$. Consider all commutative diagrams of the form

$$
\begin{array}{ccccccc}
1 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & S & \longrightarrow & 1 \\
& & \uparrow \rho & & \uparrow \tilde{\rho} & & \downarrow \rho & & \\
& & G & & & & & \\
\end{array}
$$

where $E$ is a linear algebraic group over $k$, $U$ is a unipotent subgroup of $E$, and the image of $\tilde{\rho}$ is Zariski dense. It is clear how to define morphisms of such diagrams. Moreover, the collection of such diagrams forms an inverse system [6], Prop. 2.1. The completion $\mathcal{G}$ of $G$ relative to $\rho$ is defined to be the inverse limit

$$
\mathcal{G} = \lim_{\leftarrow} E
$$

over all the above commutative diagrams. The kernel of the map $\mathcal{G} \to S$ will be called the *prounipotent radical* of $\mathcal{G}$.

The group $\mathcal{G}$ satisfies the following universal mapping property [6], Prop. 2.3. Suppose that $\mathcal{E}$ is a proalgebraic group over $k$ and that $\mathcal{E} \to S$ is a homomorphism of proalgebraic groups with prounipotent kernel. If $\phi : G \to \mathcal{E}$ is a map whose composition with $\mathcal{E} \to S$ is $\rho$, then there is a unique map $\Phi : \mathcal{G} \to \mathcal{E}$ in the category of proalgebraic $k$-groups such that the diagram

$$
\begin{array}{cccccc}
G & \longrightarrow & \mathcal{G} & \longrightarrow & S \\
\uparrow \phi & & \downarrow \Phi & & \downarrow \phi \\
\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E} \\
\end{array}
$$

commutes.

2. **Unipotent Completion**

The unipotent completion of a group is the case where $S$ is the trivial group. The following construction of the unipotent completion is due to D. Quillen [13], although he considered only the characteristic zero case.

Let $k$ be a field. For a group $G$, denote by $J_k$ the augmentation ideal of the group ring $kG$; i.e., $J_k$ is the kernel of the map $\varepsilon : kG \to k$ defined by $\varepsilon(g) = 1$. The map $g \mapsto (g - 1) + J_k^2$ induces an isomorphism

$$
H_1(G, k) \xrightarrow{\cong} J_k/J_k^2.
$$

Denote by $kG^\wedge$ the completed algebra

$$
kG^\wedge = \varprojlim kG/J_k^1;
$$
it is a complete Hopf algebra ($kG$ is a Hopf algebra via $\Delta(g) = g \otimes g$).

Denote the completion of $J_k$ by $\hat{J}_k$.

Consider the set of grouplike elements

$$\mathcal{P} = \{x \in kG^\wedge : \Delta(x) = x \otimes x, \varepsilon(x) = 1\}.$$ 

This is a subgroup of the group of units of $kG^\wedge$. Define a filtration of $\mathcal{P}$ by

$$\mathcal{P}^i = \mathcal{P} \cap (1 + \hat{J}_k^i)$$

and set

$$\mathcal{P}_i = \mathcal{P} / \mathcal{P}^{i+1}.$$ 

Observe that $[\mathcal{P}^i, \mathcal{P}^j] \subseteq \mathcal{P}^{i+j}$.

**Lemma 2.1.** If $H_1(G, k)$ is finite dimensional, then $\mathcal{P}_i$ may be given the structure of a linear algebraic group over $k$.

**Proof.** Since $H_1(G, k)$ is finite dimensional and the graded algebra

$$\text{Gr}^* kG^\wedge = \bigoplus_{i \geq 0} \hat{J}_k^i / \hat{J}_k^{i+1}$$

is generated by

$$\text{Gr}^1 kG^\wedge = \hat{J}_k / \hat{J}_k^2 \cong H_1(G, k),$$

the quotient algebra $kG^\wedge / \hat{J}_k^{i+1}$ is a finite dimensional $k$-algebra. Define a homomorphism $\varphi : \mathcal{P} \to \text{Aut}(kG^\wedge / \hat{J}_k^{i+1})$ by $\varphi(x)(u) = xu$. Since the elements of $\mathcal{P}^{i+1}$ act trivially on $kG^\wedge / \hat{J}_k^{i+1}$, we have an induced embedding $\overline{\varphi} : \mathcal{P}_i \to \text{Aut}(kG^\wedge / \hat{J}_k^{i+1})$. This endows $\mathcal{P}_i$ with the structure of an algebraic group over $k$. Since $\mathcal{P}_i$ consists of those $x$ satisfying the polynomial conditions $\varepsilon(x) = 1$ and $\Delta(x) = x \otimes x$, we see that $\overline{\varphi}(\mathcal{P}_i)$ is a closed subvariety of $\text{Aut}(kG^\wedge / \hat{J}_k^{i+1})$. 

Recall that an algebraic group $U \subset GL(V)$ over $k$ is called **unipotent** if there is a filtration

$$V = V^0 \supset V^1 \supset \cdots \supset V^m \supset 0$$

of $V$ such that each element of $U$ acts trivially on the quotients $V^i / V^{i+1}$. We claim that each $\mathcal{P}_i$ is unipotent. To see this, note that we have a filtration of $kG^\wedge / \hat{J}_k^{i+1}$ by powers of $\hat{J}_k$

$$kG^\wedge / \hat{J}_k^{i+1} \supset \hat{J}_k / \hat{J}_k^{i+1} \supset \cdots \supset \hat{J}_k^i / \hat{J}_k^{i+1} \supset 0.$$ 

If $g \in \mathcal{P}_i$, then $g - 1 \in \hat{J}_k$ and hence $(g - 1)\hat{J}_k^i \subseteq \hat{J}_k^{i+1}$. Thus, if $x \in \hat{J}_k^i$, we have $gx = x + v$ for some $v \in \hat{J}_k^{i+1}$; i.e., $g$ acts trivially on

$$\hat{J}_k^i / \hat{J}_k^{i+1} \cong (\hat{J}_k^i / \hat{J}_k^{i+1}) / (\hat{J}_k^{i+1} / \hat{J}_k^{i+1}).$$
Thus, \( P_l \) is a unipotent group over \( k \). As such, it is isomorphic to a subgroup of some \( U_n(k) \) (§, p. 87), the subgroup of \( GL_n(k) \) consisting of upper triangular matrices with 1’s on the diagonal. In particular, each \( P_l \) is nilpotent.

We shall need the following fact about nilpotent groups.

**Lemma 2.2.** Let \( f : M \to N \) be a homomorphism of nilpotent groups such that the induced map \( f_* : H_1(M, \mathbb{Z}) \to H_1(N, \mathbb{Z}) \) is surjective. Then \( f \) is surjective.

**Proof.** Denote by \( \Gamma^* M \) and \( \Gamma^* N \) the lower central series of \( M \) and \( N \), respectively. Consider the induced map of associated graded algebras

\[
\text{Gr}(f) : \text{Gr}^* M \to \text{Gr}^* N,
\]

where \( \text{Gr}^* G = \bigoplus_{i \geq 1} \Gamma^i G / \Gamma^{i+1} G \). This algebra is a Lie algebra over \( \mathbb{Z} \) with bracket induced by the commutator in \( G \). Note that as an algebra \( \text{Gr}^* G \) is generated by \( \text{Gr}^1 G = G / \Gamma^2 G = H_1(G, \mathbb{Z}) \). Consider the commutative diagram

\[
\begin{array}{ccc}
(\text{Gr}^1 M)^{\otimes l} & \xrightarrow{f^{\otimes l}} & (\text{Gr}^1 N)^{\otimes l} \\
\downarrow & & \downarrow \\
\text{Gr}^l M & \xrightarrow{\text{Gr}^l(f)} & \text{Gr}^l N.
\end{array}
\]

Since \( f^{\otimes l} \) is surjective by hypothesis and the vertical arrows are also surjective, we see that \( \text{Gr}^l(f) \) is surjective.

Now, say that \( \Gamma^{m+1} M = \{1\} \), but \( \Gamma^m M \neq \{1\} \). Then since \( \text{Gr}^l(f) \) is surjective for all \( l \), we must have \( \Gamma^{m+1} N / \Gamma^{m+2} N = 0 \); that is, \( \Gamma^{m+1} N = \Gamma^{m+2} N \). This can occur only if \( \Gamma^{m+1} N = \{1\} \) (since \( N \) is nilpotent). Consider the commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & \Gamma^m M \\
\downarrow \text{Gr}^m(f) & & \downarrow \text{Gr}^m(f) \\
1 & \longrightarrow & \Gamma^m N
\end{array}
\quad
\begin{array}{ccc}
\Gamma^m M & \xrightarrow{\Gamma^m(f)} & \Gamma^m N \\
\downarrow \text{Gr}^m(f) & & \downarrow \text{Gr}^m(f) \\
\Gamma^m M & \xrightarrow{\Gamma^m(f)} & \Gamma^m N
\end{array}
\]

This shows that \( \Gamma^{m-1} M \) surjects onto \( \Gamma^{m-1} N \). A simple induction then shows that \( f : M \to N \) is surjective.

Now, let \( G \) be a group with \( H_1(G, k) \) finite dimensional. We have an inclusion \( G \to kG \). Composing with the completion map \( kG \to kG^\wedge \) gives a map \( G \to kG^\wedge \). The image of this map lies in \( P \) and hence for each \( l \) there is a homomorphism \( G \to P_l \). We claim that \( P = \varprojlim P_l \) is the unipotent \( k \)-completion of \( G \). Before proving this, we establish a few facts about the groups \( P \) and \( P_l \).
Recall the filtration $\mathcal{P}^\bullet$ of $\mathcal{P}$ defined by
$$\mathcal{P}^i = \mathcal{P} \cap (1 + \hat{j}_k^i).$$
This induces a filtration on each $\mathcal{P}_i$:
$$\mathcal{P}_i = \mathcal{P}/\mathcal{P}^{i+1} \supset \mathcal{P}^2/\mathcal{P}^{i+1} \supset \cdots \supset \mathcal{P}^l/\mathcal{P}^{l+1} \supset \{1\}.$$ Denote the $i$th term $\mathcal{P}^i/\mathcal{P}^{i+1}$ by $L_i$.

Let us determine the structure of the graded algebra $\text{Gr}^\bullet \mathcal{P} = \bigoplus_{i \geq 1} \mathcal{P}^i/\mathcal{P}^{i+1}$.

**Lemma 2.3.** There is an isomorphism
$$\text{Gr}^\bullet \mathcal{P} \xrightarrow{\cong} \bigoplus_{i \geq 1} \hat{j}_k^i/\hat{j}_k^{i+1}.$$ 

**Proof.** For each $i$, define a map $p_i : \mathcal{P}^i \to \hat{j}_k^i/\hat{j}_k^{i+1}$ by
$$p_i(x) = (x - 1) + \hat{j}_k^i.$$ 

The map $p_i$ is a homomorphism since
$$p_i(xy) = (xy - 1) + \hat{j}_k^{i+1} = (x - 1) + (y - 1) + (x - 1)(y - 1) + \hat{j}_k^{i+1} = (x - 1) + (y - 1) + \hat{j}_k^{i+1} = p_i(x) + p_i(y).$$ 

The kernel of $p_i$ is precisely the subgroup $\mathcal{P}^{i+1}$. We therefore have an induced injection
$$\overline{p}_i : \mathcal{P}^i/\mathcal{P}^{i+1} \longrightarrow \hat{j}_k^i/\hat{j}_k^{i+1}.$$ 

We claim that the map $\overline{p}_i$ is also surjective. To see this, consider the composite
$$H_1(G, k) \longrightarrow H_1(\mathcal{P}, k) \longrightarrow \mathcal{P}/\mathcal{P}^2 \xrightarrow{\overline{p}_i} \hat{j}_k/\hat{j}_k^2,$$
(the map $H_1(\mathcal{P}, k) \to \mathcal{P}/\mathcal{P}^2$ is the quotient map $(\mathcal{P}/\Gamma^2\mathcal{P}) \otimes k \to \mathcal{P}/\mathcal{P}^2$). This composite is clearly the canonical isomorphism $H_1(G, k) \xrightarrow{\cong} \hat{j}_k/\hat{j}_k^2$. It follows that the map $\overline{p}_i$ is surjective. Now consider the commutative diagram

$$\begin{array}{ccc}
(\mathcal{P}/\mathcal{P}^2)^{\otimes i} & \xrightarrow{\overline{p}_i^{\otimes i}} & (\hat{j}_k/\hat{j}_k^2)^{\otimes i} \\
\downarrow & & \downarrow \psi \\
\mathcal{P}^i/\mathcal{P}^{i+1} & \xrightarrow{\overline{p}_i} & \hat{j}_k^i/\hat{j}_k^{i+1}
\end{array}$$
Since \( \psi \) is surjective and \( \beta_i \) is injective, a diagram chase shows that \( \beta_i \) is surjective.

**Lemma 2.4.** The homomorphism \( G \to \mathcal{P}_l \) induces an isomorphism \( H_1(G, k) \cong H_1(\mathcal{P}_l, k) \).

**Proof.** Recall the filtration \( L^* \) of \( \mathcal{P}_l \) defined by \( L^i = \mathcal{P}_l / \mathcal{P}_l^{i+1} \). By Lemma 2.3, the graded quotients satisfy \( L^i / L^{i+1} \cong \hat{\mathcal{J}}_k^i / \hat{\mathcal{J}}_k^{i+1} \). Hence, the algebra \( \text{Gr}^i \mathcal{P}_l \) is generated by \( \text{Gr}^1 \mathcal{P}_l \cong \mathcal{P} / \mathcal{P}^2 \).

We have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}_l / \mathcal{P}_l^2 & \to & \mathcal{P} / \mathcal{P}^2 \\
\downarrow & & \downarrow \\
\Gamma^i \mathcal{P}_l / \Gamma^{i+1} \mathcal{P}_l & \overset{\varphi}{\to} & \mathcal{L}^i / \mathcal{L}^{i+1}
\end{array}
\]

The top horizontal arrow is surjective since \( \Gamma^2 \mathcal{P}_l \subseteq \mathcal{L}_2 \), and the vertical arrows are surjective since the algebras \( \text{Gr}^i \mathcal{P}_l \) and \( \text{Gr}^1 \mathcal{P}_l \) are generated by \( \text{Gr}^1 \). It follows that the map \( \varphi \) is surjective.

We have an exact sequence

\[
\Gamma^i \mathcal{P}_l / \Gamma^{i+1} \mathcal{P}_l \overset{\varphi}{\to} \mathcal{L}^i / \mathcal{L}^{i+1} \to \mathcal{L}^i / \mathcal{L}^{i+1} \Gamma^i \mathcal{P}_l \to 0.
\]

Since \( \varphi \) is surjective, we see that \( \mathcal{L}^i = \mathcal{L}^{i+1} \Gamma^i \mathcal{P}_l \). This implies that \( \mathcal{L}^i = \Gamma^i \mathcal{P}_l \) for all \( m \geq 1 \). But since \( \mathcal{L}^{i+1} = \{1\} \), we see that \( \mathcal{L}^i = \Gamma^i \mathcal{P}_l \) for all \( i \). In particular, \( H_1(\mathcal{P}_l, k) \cong \hat{\mathcal{J}}_k^i / \hat{\mathcal{J}}_k^{i+1} \). Thus we have the commutative diagram

\[
\begin{array}{ccc}
H_1(G, k) & \to & H_1(\mathcal{P}_l, k) \\
\downarrow & & \downarrow \\
\hat{\mathcal{J}}_k^i / \hat{\mathcal{J}}_k^{i+1} & \cong & \hat{\mathcal{J}}_k^i / \hat{\mathcal{J}}_k^{i+1}
\end{array}
\]

which shows that the map \( H_1(G, k) \to H_1(\mathcal{P}_l, k) \) is an isomorphism.

**Lemma 2.5.** If \( H_1(G, k) \) is finite dimensional, then the image of the map \( G \to \mathcal{P}_l \) is Zariski dense.

**Proof.** Denote by \( \mathcal{Z}_l \) the Zariski closure of the image of \( G \) in \( \mathcal{P}_l \). Since the composite

\[
H_1(G, k) \to H_1(\mathcal{Z}_l, k) \to H_1(\mathcal{P}_l, k)
\]

is an isomorphism (Lemma 2.4), the map \( H_1(\mathcal{Z}_l, k) \to H_1(\mathcal{P}_l, k) \) is surjective. Note that if \( U \) is a unipotent group over \( k \), then \( H_1(U, \mathbb{Z}) = U / [U, U] \) is an abelian \( k \)-group and hence \( H_1(U, \mathbb{Z}) \cong H_1(U, k) \). Lemma 2.2 implies that the map \( \mathcal{Z}_l \to \mathcal{P}_l \) is surjective; that is, \( \mathcal{Z}_l = \mathcal{P}_l \).
Proposition 2.6. If $H_1(G, k)$ is finite dimensional, then the map $G \to \mathcal{P}$ is the unipotent $k$-completion.

Proof. Note that $\mathcal{P} = \varprojlim \mathcal{P}_l$ is a prounipotent group over $k$ and that the image of $G$ in $\mathcal{P}$ is Zariski dense. Denote by $\mathcal{U}$ the unipotent $k$-completion of $G$. By the universal mapping property, there is a unique map $\varphi : \mathcal{U} \to \mathcal{P}$ making the diagram

\[
\begin{array}{ccc}
G & \longrightarrow & \mathcal{U} \\
\downarrow & & \downarrow \\
\mathcal{P} & \longleftarrow & \varphi
\end{array}
\]

commute. We show that $\varphi$ is an isomorphism.

Since the map $G \to \mathcal{P}$ has Zariski dense image, so does the map $\mathcal{U} \to \mathcal{P}$. It follows that $\varphi$ is surjective since $\varphi(\mathcal{U})$ is a closed subgroup of $\mathcal{P}$ [2], p. 47.

To see that $\varphi$ is injective, suppose that $\mathcal{U}$ is a unipotent group over $k$. Then $\mathcal{U}$ is a subgroup of some $U_n \subset GL_n(k)$ for some $n$ (here, $U_n$ is the subgroup of upper triangular unipotent matrices). A representation $p : G \to U$ induces a ring homomorphism $\tilde{p} : kG \to gl_n(k)$ defined by $\tilde{p}(\sum \alpha g) = \sum \alpha g p(g)$. The image of $J_k$ under $\tilde{p}$ lies in the subalgebra $n$ of nilpotent upper triangular matrices. Thus, the kernel of $\tilde{p}$ contains $J_k^n$ and hence $\tilde{p}$ induces a map

\[
\mathcal{P} : kG/J_k^n \longrightarrow gl_n(k).
\]

Since $\tilde{p}(J_k) \subseteq n$, we see that $\mathcal{P}_{n-1} \subseteq 1 + J_k/J_k^n$ is contained in $U_n$. If the image of $G$ is Zariski dense in $U$, then $\mathcal{P}_{n-1} \subseteq U$. Thus, the diagram

\[
\begin{array}{ccc}
G & \longrightarrow & \mathcal{P}_{n-1} \\
\downarrow & & \downarrow \\
\mathcal{P} & \longleftarrow & \mathcal{U}
\end{array}
\]

commutes. Now, $\mathcal{U} = \varprojlim U_\alpha$, with each $U_\alpha$ unipotent over $k$. Applying the above construction to the compositions $G \to \mathcal{U} \to U_\alpha$, we see that $\varphi$ is injective as follows. If $\varphi(u) = 1$, then $\varphi(u)$ maps to 1 in each $\mathcal{P}_{n-1}$. But the diagram

\[
\begin{array}{ccc}
\mathcal{U} & \longrightarrow & U_\alpha \\
\varphi & & \downarrow \\
\mathcal{P} & \longrightarrow & \mathcal{P}_{n-1}
\end{array}
\]

\[
\begin{array}{ccc}
u & \longrightarrow & u_\alpha \\
\downarrow & & \downarrow \\
\varphi(u) = 1 & \longrightarrow & 1
\end{array}
\]

shows that $u_\alpha = 1$ for each $\alpha$; i.e., $u = 1$. \qed
Corollary 2.7. If \( k \subset F \) is a field extension, then the map \( \mathcal{U}_F \to \mathcal{U}_k(F) \) is an isomorphism.

Proof. The group \( \mathcal{U}_F \) is the set of grouplike elements in \( FG^\wedge \). On the other hand, the group \( \mathcal{U}_k(F) \) is the set of grouplike elements of \( kG^\wedge \otimes F = FG^\wedge \).

Proposition 2.8. The map \( G \to \mathcal{P} \) induces an isomorphism
\[ H_1(G, k) \to H_1(\mathcal{P}, k). \]

Proof. Let \( A \) be a \( k \)-vector space with basis \( \{ e_i \}_{i=1}^n \). Then we have a bijection
\[ \text{Hom}_{\text{groups}}(G, A) = \text{Hom}_{\text{k-vect. sp.}}(H_1(G, k), A). \]
On the other hand, if \( \sum \alpha_i e_i \) is a vector in \( A \), the map
\[ \sum \alpha_i e_i \mapsto \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \]
identifies \( A \) as a unipotent group over \( k \). The universal mapping property of unipotent completion then provides a bijection
\[ \text{Hom}_{\text{groups}}(G, A) = \text{Hom}_{\text{k-vect. sp.}}(H_1(\mathcal{P}, k), A); \]
that is, \( H_1(G, k) \) and \( H_1(\mathcal{P}, k) \) both represent maps of \( G \) into \( A \). It follows that the natural map \( H_1(G, k) \to H_1(\mathcal{P}, k) \) is an isomorphism.

3. Examples of Unipotent Completions

3.1. \( G = \mathbb{Z} \). Let \( k \) be a field. The group algebra \( kG \) is the ring of Laurent polynomials \( k[t, t^{-1}] \). The augmentation ideal is principal and is generated by \( t - 1 \). The completion \( kG^\wedge \) is the power series ring \( k[[T]] \) with augmentation \( T \mapsto 0 \); the ideal \( J_k \) is the principal ideal \( (T) \). The canonical map \( kG \to kG^\wedge \) sends \( t \) to \( 1 + T \). If \( k \) has characteristic zero, then \( \mathcal{P} = \{(1 + T)^\alpha : \alpha \in k\} \), where \( (1 + T)^\alpha = \exp(\alpha \log(1 + T)) \).

Also, \( \mathcal{P} \cap (1 + J_k^l) = \{1\} \) for \( l \geq 2 \) and hence \( \mathcal{P} = \mathcal{P}_l \) for all \( l \). Thus, \( \mathcal{P} \cong k \cong k \otimes_{\mathbb{Z}} G \), as one would expect.

Let us realize each \( \mathcal{P}_l \) as a group of matrices over \( k \). The group \( \mathcal{P}_l \) is a subgroup of \( \text{Aut}(kG^\wedge / J_k^{l+1}) \) acting via left multiplication. In this case \( \mathcal{P} \cong \mathcal{P}_l \) for each \( l \). The element \( (1 + T)^\alpha \in \mathcal{P} \) is the power series
\[ 1 + \alpha T + \left(\frac{\alpha^2}{2} - \frac{\alpha}{2}\right)T^2 + \left(\frac{\alpha^3}{3} - \frac{\alpha^2}{2} + \frac{\alpha^3}{6}\right)T^3 + \cdots. \]
With respect to the basis $1, T, T^2, \ldots, T^l$ of $k[[T]]/(T^{l+1})$, this element acts via the lower triangular matrix

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
\alpha & 1 & 0 & \cdots & 0 \\
c_2(\alpha) & \alpha & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
c_{l-1}(\alpha) & \alpha & 1 & \cdots & 0 \\
c_l(\alpha) & c_{l-1}(\alpha) & \cdots & c_2(\alpha) & \alpha & 1
\end{pmatrix}
$$

where $c_i(\alpha)$ is the coefficient of $T^i$ in the power series $(1 + T)^\alpha$. Thus, we have realized $\mathcal{P} \cong k$ as a closed subgroup of $\text{Aut}(kG^\wedge/j_k^{l+1})$.

If $k = \mathbb{F}_p$, then the situation is much different. Since $(1 + T)^p = 1 + T$ as $1 + T$ has finite order in each $\mathcal{P}_l = \mathcal{P}/\mathcal{P} \cap (1 + j_k^{l+1})$. In fact, $1 + T$ has order $p^d$ in $\mathcal{P}_{p^d}, \mathcal{P}_{p^{d+1}}, \ldots, \mathcal{P}_{p^{d+1-1}}$. Note also that the maps $\mathbb{Z} \to \mathcal{P}_l$ are surjective since there are no proper Zariski dense subgroups of an $\mathbb{F}_p$-group (the Zariski topology is discrete). Each of the homomorphisms

$$
\mathcal{P}_{p^d} \leftarrow \mathcal{P}_{p^{d+1}} \leftarrow \cdots \leftarrow \mathcal{P}_{p^{d+1-1}}
$$

is the identity $\mathbb{Z}/p^d \to \mathbb{Z}/p^d$. It follows that $\mathcal{P} = \varprojlim \mathcal{P}_l \cong \mathbb{Z}_p$. Thus, $\mathcal{P} \cong \mathbb{Z}_p \otimes \mathbb{Z} G$.

Note that this must be correct. If one hoped that the answer were $\mathbb{F}_p \cong \mathbb{F}_p \otimes \mathbb{Z} G$, then the universal mapping property would imply the existence of a nontrivial homomorphism $\mathbb{F}_p \to \mathbb{Z}_p$.

Let us make the case $p = 2$ explicit. The group $\mathcal{P}_l$ is a subgroup of $\text{Aut}(kG^\wedge/j_k^{l+1})$, acting as left multiplication by elements of $\mathcal{P}$. We shall write down the coordinate rings of the first few $\mathcal{P}_l$. Note that with respect to the basis $1, T, T^2, \ldots, T^l$ of $k[[T]]/(T^{l+1})$, the element $1 + T$ acts via the lower triangular $(l + 1) \times (l + 1)$ matrix

$$
\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 1 & 1
\end{pmatrix}
$$

For $l = 1$, this matrix has order 2. The coordinate ring of $\mathcal{P}_1$ is

$$
\mathcal{O}(\mathcal{P}_1) = \mathbb{F}_2[T_{11}, T_{12}, T_{21}, T_{22}]/(T_{12}, T_{11} - 1, T_{22} - 1).
$$
For $l = 2, 3$, the above matrix has order 4. The coordinate ring of $\mathcal{P}_2$ is

$$\mathcal{O}(\mathcal{P}_2) = \mathbb{F}_2[T_{ij}, 1 \leq i, j \leq 3]/(T_{ij}, i < j; T_{ii} - 1; T_{21} - T_{32}),$$

and that of $\mathcal{P}_3$ is

$$\mathcal{O}(\mathcal{P}_3) = \mathbb{F}_2[T_{ij}, 1 \leq i, j \leq 4]/I,$$

where $I$ is the ideal generated by $T_{ij}, i < j; T_{ii} - 1; T_{21} - T_{32}; T_{32} - T_{43}; T_{31} - T_{42}; T_{21} - T_{41}$. Note that each of these is a two-dimensional $\mathbb{F}_2$-algebra. For $l = 4$, the matrix representing $1 + T$ has order 8. The coordinate ring of $\mathcal{P}_4$ is

$$\mathcal{O}(\mathcal{P}_4) = \mathbb{F}_2[T_{ij}, 1 \leq i, j \leq 5]/I$$

where $I$ is the ideal generated by $T_{ij}, i < j; T_{ii} - 1; T_{21} - T_{32}; T_{32} - T_{43}; T_{43} - T_{54}; T_{31} - T_{42}; T_{42} - T_{53}; T_{41} - T_{52}$. It is a three-dimensional $\mathbb{F}_2$-algebra. To obtain the $\mathbb{F}_2$-unipotent completion of $\mathbb{Z}$, we take the inverse limit of the groups $\text{Hom}_{\mathbb{F}_2}(\mathcal{O}(\mathcal{P}_l), \mathbb{F}_2)$.

3.2. $G = \mathbb{Z}/n$. Here the group algebra is $kG = k[t]/(t^n - 1)$; the augmentation ideal is $J_k = (t - 1)$. If $k$ has characteristic zero, then since

$$(t - 1)^2 = t^2 - 2t + 1 = t^n + t^2 - 2t = t(t^{n-2} + \cdots + t + 2)(t - 1),$$

and since both $t$ and $t^{n-2} + \cdots + t + 2$ are units in $kG$, we have $J_k^2 = J_k$. Thus, the completion $kG^\wedge$ is the algebra $\lim \mathcal{O}(\mathcal{P}_l) / J_k^l = k$. The only grouplike element is 1, and hence the unipotent $k$-completion is trivial.

If $k = \mathbb{F}_p$, then we must consider separately two cases. First assume that $p$ is prime to $n$. In this case, the above factorization shows again that $kG^\wedge = k$ and that $\mathcal{P} = 1$. If $p$ divides $n$, then we may assume that $n = p^d$ for some $d \geq 1$ (note that the completion of $G \times H$ is the product of the completions of $G$ and $H$). In this case we have $J_k^p = 0$ since $(1 - t)^p = (1 - t^{np}) = (1 - t^n) = 0$. Thus, $kG^\wedge \cong kG$ and $\mathcal{P}$ is the collection of the $t^i, 0 \leq i \leq p^d$. This group is $\mathbb{Z}/p^d$.

In the characteristic zero case, then, we see that the completion of $\mathbb{Z}/n$ is $\mathbb{Z}/n \otimes_\mathbb{Z} k$ and in the characteristic $p$ case it is $\mathbb{Z}/n \otimes_\mathbb{Z} \mathbb{Z}_p$. So, if $G$ is a finitely generated abelian group, the $\mathbb{Q}$-completion of $G$ is $G \otimes_\mathbb{Z} \mathbb{Q}$ and the $\mathbb{F}_p$-completion is $G \otimes_\mathbb{Z} \mathbb{Z}_p$. 
3.3. $p$-completion v. $\mathbb{F}_p$-completion. Suppose that $H_1(G, \mathbb{F}_p)$ is finite dimensional. Then the $\mathbb{F}_p$-completion of $G$ is the $p$-completion. Indeed, the $p$-completion of $G$ is the inverse limit
\[ G^{\wedge p} = \lim_{\leftarrow} G/\Gamma^*_p G, \]
where $\Gamma^*_p G$ is the $p$-lower central series of $G$. Each group $G/\Gamma^*_p G$ is a finite $p$-group if $H_1(G, \mathbb{F}_p)$ is finite dimensional. On the other hand, each group $P_l$ is a unipotent group over $\mathbb{F}_p$. Such groups are nilpotent and since $P_l$ is a closed subgroup of some $U_n(\mathbb{F}_p)$, it is finite. Thus, we have a homomorphism
\[ G^{\wedge p} \longrightarrow P. \]
But since $G^{\wedge p}$ is a pronipotent group over $\mathbb{F}_p$, we have a unique map
\[ P \longrightarrow G^{\wedge p}, \]
and it is clear that these maps are inverse isomorphisms. Alternatively, both $G^{\wedge p}$ and $P$ satisfy the same universal mapping property and are therefore isomorphic via a unique isomorphism.

3.4. Coordinate rings of curves. Let $X$ be a smooth affine curve over $k$ and let $x$ be a closed point on $X$. Denote by $O_{X,x}$ the local ring at $x$; it has maximal ideal $m_x$. We have a split extension
\[ 1 \longrightarrow K(O_{X,x}) \longrightarrow SL_n(O_{X,x}) \xrightarrow{\text{mod } m_x} SL_n(k) \longrightarrow 1. \]
The group $K(O_{X,x})$ is filtered by powers of the maximal ideal $m_x$; denote this filtration by $K^*(O_{X,x})$. By a theorem of W. Klingenberg [10], $K^*(O_{X,x})$ is the lower central series of $K(O_{X,x})$ provided $\text{char}(k) \neq 2$ or $k \neq \mathbb{F}_3$. The graded quotients satisfy
\[ K^l/K^{l+1} \cong \mathfrak{s}l_n(k); \]
in particular, $H_1(K(O_{X,x}), \mathbb{Z}) \cong \mathfrak{s}l_n(k)$. This is a special case of the following.

**Proposition 3.1.** Let $G$ be a group such that the graded quotients $\Gamma^l G/\Gamma^{l+1} G$ are finite dimensional $k$-vector spaces. Let $\hat{G} = \lim_{\leftarrow} G/\Gamma^l G$. Then the map $G \rightarrow \hat{G}$ is the unipotent $k$-completion.

**Proof.** This is essentially the same as Proposition 2.6. Each $G/\Gamma^l G$ is a unipotent $k$-group by hypothesis and since $G$ surjects onto each $G/\Gamma^l G$, the image of $G$ in $\hat{G}$ is Zariski dense. Since $H_1(G, k)$ is finite dimensional, the unipotent $k$-completion of $G$ is the group $P \subset kG^{\wedge}$. The universal mapping property provides a unique map $\Phi : P \rightarrow \hat{G}$ of pronipotent $k$-groups; it is surjective since $\Phi(P)$ is a closed Zariski dense subgroup of $\hat{G}$. The injectivity of $\Phi$ follows since the composition $G \rightarrow P \rightarrow P_l$ must factor through some $G/\Gamma^l G$. \qed
Now, for the group $K(\mathcal{O}_{X,x})$, $\text{char}(k) \neq 2, k \neq \mathbb{F}_3$, the lower central series satisfies the hypotheses of the proposition. Thus, the unipotent $k$-completion is the group

$$K(\hat{\mathcal{O}}_{X,x}) = \ker\{SL_n(\hat{\mathcal{O}}_{X,x}) \to SL_n(k)\},$$

where $\hat{\mathcal{O}}_{X,x}$ is the $m_x$-adic completion of $\mathcal{O}_{X,x}$.

3.5. $G = SL_n(\mathbb{Z})$. We have the exact sequence

$$1 \to \Gamma(n,p) \to SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_p) \to 1.$$ 

If $n \geq 3$, the filtration of $\Gamma(n,p)$ by powers of $p$ is the lower central series $H$. Since the graded quotients $\Gamma(n,p^l)/\Gamma(n,p^{l+1})$ are isomorphic to $\mathfrak{s}l_n(\mathbb{F}_p)$, Proposition 3.1 implies that the $\mathbb{F}_p$-completion of $\Gamma(n,p)$ is the group

$$\hat{\Gamma} = \varprojlim \Gamma(n,p)/\Gamma(n,p^l).$$

This group fits into the exact sequence

$$1 \to \hat{\Gamma} \to SL_n(\mathbb{Z}_p) \to SL_n(\mathbb{F}_p) \to 1.$$ 

Note that the $\mathbb{F}_p$-completion of $SL_n(\mathbb{Z})$ is trivial. Also, if $\ell \neq p$ the $\mathbb{F}_\ell$-completion of $\Gamma(n,p)$ is trivial since $H_1(\Gamma(n,p), \mathbb{F}_\ell) = 0$.

If $k$ is any field of characteristic $p$, then the unipotent $k$-completion of $\Gamma(n,p)$ is obtained by taking the $k$-form of $\hat{\Gamma}$. This is constructed as follows. The first group in the inverse system is $U_2 = \mathfrak{s}l_n(k)$. The next is a nonsplit extension

$$0 \to \mathfrak{s}l_n(k) \to U_3 \to \mathfrak{s}l_n(k) \to 0.$$ 

The group $U_3$ is the unipotent group defined by the diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathfrak{s}l_n(\mathbb{F}_p) & \to & \Gamma(n,p)/\Gamma(n,p^2) & \to & \mathfrak{s}l_n(\mathbb{F}_p) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{s}l_n(k) & \to & U_3 & \to & \mathfrak{s}l_n(k) & \to & 0.
\end{array}$$

In general, we have a diagram of central extensions

$$\begin{array}{ccccccccc}
0 & \to & \mathfrak{s}l_n(\mathbb{F}_p) & \to & \Gamma(n,p)/\Gamma(n,p^r) & \to & \Gamma(n,p)/\Gamma(n,p^{r-1}) & \to & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathfrak{s}l_n(k) & \to & U_r & \to & U_{r-1} & \to & 1
\end{array}$$

and the $k$-completion of $\Gamma(n,p)$ is $U_k = \varprojlim U_r$. 
4. Relative Completions

Let $S$ be a reductive algebraic group over $k$ and let $\rho : \Gamma \to S$ be a representation with Zariski dense image. Denote by $\mathcal{G}$ the completion of $\Gamma$ with respect to $\rho$; it is an extension

$$1 \to U \to \mathcal{G} \to S \to 1.$$ 

In this section we construct some examples of relative completions. We first prove a few basic facts.

First, let $A$ be a rational representation of $S$ and suppose that we have an extension in the category of algebraic groups over $k$:

$$0 \to A \to G \to S \to 1.$$ 

Then this extension splits [2], p. 158. This extends to unipotent kernels; a proof of the following may be found in [3], Prop. 4.3.

**Proposition 4.1.** If $S$ is a reductive algebraic group over $k$ and if

$$1 \to U \to G \to S \to 1$$

is an extension in the category of algebraic groups over $k$, where $U$ is unipotent, then the extension splits.

From this, one easily deduces that the relative completion

$$1 \to U \to \mathcal{G} \to S \to 1$$

is a split extension.

Denote the kernel of $\rho$ by $T$ and the unipotent $k$-completion of $T$ by $\mathcal{T}$. Then the map $T \to U$ induces a unique map $\Phi : \mathcal{T} \to U$. Denote by $K$ the kernel of $\Phi$ and by $L$ the image of $\rho$.

**Proposition 4.2.** Suppose that $H_1(T, k)$ is finite dimensional. If the action of $L$ on $H_1(T, k)$ extends to a rational representation of $S$ (e.g., $L = S$), then $K$ is central in $\mathcal{T}$.

**Proof.** [6] The group $\Gamma$ acts on the completion $kT^\wedge$ by conjugation. This action preserves the filtration by powers of $\hat{J}$, so $\Gamma$ acts on the associated graded algebra

$$\text{Gr}^J kT^\wedge = \bigoplus_{m \geq 0} \hat{J}^m / \hat{J}^{m+1}.$$ 

If $H_1(T, k)$ is finite dimensional, then each algebra $kT^\wedge / \hat{J}^l$ is finite dimensional. Thus, each of the groups $\text{Aut}(kT / \hat{J}^l)$ is an algebraic group. Since $\text{Gr}^J kT^\wedge$ is generated by $\hat{J} / \hat{J}^2 = H_1(T, k)$, it follows
that Aut\( (kT) \wedge \), the group of augmentation preserving algebra automorphisms of \( kT \wedge \), is a proalgebraic group which is an extension of a subgroup of \( \text{Aut} H_1(T, k) \) by a prounipotent group \( \mathcal{V} \):

\[
1 \longrightarrow \mathcal{V} \longrightarrow \text{Aut}(kT) \wedge \longrightarrow \text{Aut} H_1(T, k).
\]

(1)

If the action of \( \Gamma \) on \( H_1(T, k) \) factors through a rational representation \( S \rightarrow \text{Aut} H_1(T, k) \), then we can form a proalgebraic group extension

\[
1 \longrightarrow \mathcal{V} \longrightarrow E \longrightarrow S \longrightarrow 1
\]
of \( S \) by the prounipotent group \( \mathcal{V} \) by pulling back the extension (1) along \( S \rightarrow \text{Aut} H_1(T, k) \). Since the map \( \Gamma \rightarrow \text{Aut} H_1(T, k) \) factors through \( S \), we can lift the map \( \Gamma \rightarrow \text{Aut}(kT) \wedge \) to a map \( \Gamma \rightarrow E \) whose composition with \( E \rightarrow S \) is \( \rho : \Gamma \rightarrow S \). By the universal mapping property of relative completion, this induces a homomorphism \( \mathcal{G} \rightarrow E \).

Since the composite

\[
\mathcal{T} \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \longrightarrow E \longrightarrow \text{Aut}(kT) \wedge
\]
is the action of \( \mathcal{T} \subset kT \wedge \) on \( kT \wedge \) by inner automorphisms, we see that the kernel of this map is the center of \( \mathcal{T} \). It follows that the kernel of \( \Phi \) is central in \( \mathcal{T} \).

In [6], Hain states the result (due to Deligne) that if \( H^1(L, A) = 0 \) for all rational representations \( A \) of \( S \), then \( \Phi : \mathcal{T} \rightarrow \mathcal{U} \) is surjective. This works well in characteristic zero thanks to the vanishing results for arithmetic groups due to Ragunathan [16]. In positive characteristic, however, this criterion is not very useful. We present here a simplified version which suits our purposes.

**Proposition 4.3.** Assume that \( H_1(T, k) \) is finite dimensional. If \( \rho : \Gamma \rightarrow S \) is surjective, then \( \Phi : \mathcal{T} \rightarrow \mathcal{U} \) is surjective.

**Proof.** We first show that the composition

\[
H_1(T, k) \longrightarrow H_1(\mathcal{T}, k) \xrightarrow{\Phi_*} H_1(\mathcal{U}, k)
\]
is surjective. Let \( A \) be the cokernel of this map; it is a rational representation of \( S \). Pushing out the extension

\[
1 \longrightarrow \mathcal{U} \longrightarrow \mathcal{G} \longrightarrow S \longrightarrow 1
\]
along the map \( \mathcal{U} \rightarrow H_1(\mathcal{U}, k) \rightarrow A \), we obtain an extension of algebraic groups

\[
0 \rightarrow A \longrightarrow G \longrightarrow S \longrightarrow 1.
\]
This splits; say $s : S \to G$ is a splitting. Since the image of $\Gamma$ in $G$ is Zariski dense, the image of $\Gamma$ in $G$ is Zariski dense. But since the diagram
\[
\begin{array}{ccc}
\Gamma & \to & G \\
\rho & \downarrow & \downarrow \\
S & \to & G
\end{array}
\]
commutes, we see that the image of $s$ is Zariski dense in $G$. This forces $A$ to be trivial.

We have shown that the map $\Phi_* : H_1(T, k) \to H_1(U, k)$ is surjective. In particular, both of these vector spaces are finite dimensional ($H_1(T, k)$ is finite dimensional by Proposition 2.8). Also, both $T$ and $U$ are their own unipotent $k$-completions. Thus we have isomorphisms $T \cong P(kT^\wedge)$ and $U \cong P(kU^\wedge)$, where $P(kH^\wedge)$ is the set of grouplike elements of $kH^\wedge$. Now, $H_1(T, k) \cong \hat{J}_T/\hat{j}_T$ and $H_1(U, k) \cong \hat{J}_U/\hat{j}_U$, where $\hat{J}_H$ is the augmentation ideal in $kH^\wedge$. Since $\Phi_*$ is surjective, we see that the map $Gr^*\hat{\Phi} : Gr^*kT^\wedge \longrightarrow Gr^*kU^\wedge$ is surjective. But then the induced map $\hat{\Phi} : kT^\wedge \to kU^\wedge$ is surjective ([13], p. 266). This implies that $\Phi : T \to U$ is surjective since $\hat{\Phi}$ maps grouplike elements to grouplike elements.

We are now able to compute some examples of relative completions.

4.1. Let $X$ be a smooth affine curve over $k$ and let $O_{X,x}$ be the local ring at a closed point $x$ with maximal ideal $m_x$. In the previous section we showed that the unipotent $k$-completion of $K(O_{X,x}) = \{A \in SL_n(O_{X,x}) : A \equiv I \mod m_x\}$ is the group $K(\hat{O}_{X,x})$, provided that $\text{char}(k) \neq 2$ or $k \neq \mathbb{F}_3$. Let $\rho : SL_n(O_{X,x}) \to SL_n(k)$ be reduction modulo $m_x$ and consider the completion relative to $\rho$

\[
1 \longrightarrow U \longrightarrow G \longrightarrow SL_n(k) \longrightarrow 1.
\]

There is a surjective map $\Phi : K(\hat{O}_{X,x}) \to U$ with central kernel. An easy calculation shows that the center of $K(\hat{O}_{X,x})$ is trivial. Since the above extension splits, we see that the group $G$ is the group $SL_n(\hat{O}_{X,x})$.

4.2. Consider the curve $X = \mathbb{A}^1_k$. The coordinate ring is the polynomial algebra $k[t]$. If $k$ is a finite field or a number field, then the filtration of

\[
K(k[t]) = \ker\{SL_n(k[t]) \xrightarrow{\rho} SL_n(k)\}
\]

by powers of $t$ is the lower central series, provided that $n \geq 3$ ([13]). Thus, the unipotent $k$-completion of $K(k[t])$ is the group $K(k[[T]])$. 


Since \( \rho : SL_n(k[t]) \to SL_n(k) \) is surjective, and since \( K(k[[T]]) \) has trivial center, we see that the relative completion is \( SL_n(k[[T]]) \).

4.3. Let \( k \) be any field and consider the group \( SL_n(k[t]/t^l) \). Reduction modulo \( t \) gives a representation \( \rho : SL_n(k[t]/t^l) \to SL_n(k) \). The kernel \( K \) is filtered by powers of \( t \); this filtration is finite in length and is the lower central series. The graded quotients are all isomorphic to \( sl_n(k) \).

It follows that the unipotent \( k \)-completion of \( K \) is \( \varprojlim K/K_l \cong K \) (this holds for all \( n \)). Thus, the relative completion is simply the group \( SL_n(k[t]/t^l) \).

4.4. Consider the extension

\[ 1 \to \Gamma(n, p) \to SL_n(Z) \xrightarrow{\rho} SL_n(F_p) \to 1 \]

for \( n \geq 3 \). The \( F_p \)-completion of \( \Gamma(n, p) \) is the group \( \hat{\Gamma} \):

\[ 1 \to \hat{\Gamma} \to SL_n(Z_p) \to SL_n(F_p) \to 1. \]

Denote the completion of \( \rho : SL_n(Z) \to SL_n(F_p) \) by \( G \) and let \( U \) be its prounipotent radical. The map \( \Phi : \hat{\Gamma} \to U \) is surjective since \( \rho \) is and its kernel is central in \( \hat{\Gamma} \). Clearly, though, the center of \( \hat{\Gamma} \) is trivial and hence \( \Phi \) is an isomorphism. Thus, \( G \) is the semidirect product

\[ G \cong \hat{\Gamma} \rtimes SL_n(F_p). \]

(Recall that the relative completion is a split extension.) This group is not isomorphic to \( SL_n(Z_p) \) since the projection map \( SL_n(Z_p) \to SL_n(F_p) \) does not split. Note also that \( SL_n(Z_p) \) is not proalgebraic over \( F_p \) since the rings \( Z/p^d \) are not \( F_p \)-algebras.

Thus, the group \( G \) is a bit mysterious in the sense that we do not have an explicit description of it in terms of matrices, nor do we have a formula for the homomorphism \( SL_n(Z) \to G \).

4.5. Let \( \Gamma \) be the absolute Galois group of the finite field \( F_p \). The group \( \Gamma \) is isomorphic to \( \hat{Z} \), the profinite completion of \( Z \). Let \( \ell \) be a prime different from \( p \) and consider the action of \( \Gamma \) on the \( \ell \)th roots of unity in \( \overline{F}_p \). This defines a homomorphism \( \rho : \Gamma \to \mathbb{F}_\ell^\times \) as follows. Let \( \zeta_\ell \) be a primitive \( \ell \)th root of unity and let \( \sigma \) be an element of \( \Gamma \). Write

\[ \sigma(\zeta_\ell) = \zeta_\ell^{\rho(\sigma)}, \quad \rho(\sigma) \in \{1, 2, \ldots, \ell - 1\}. \]

(Note that \( \rho(\sigma) \neq 0 \) since \( \sigma(1) = 1 \).) This defines the map \( \rho \). Since \( \rho \) maps the Frobenius automorphism to a generator of \( \mathbb{F}_\ell^\times \), we see that \( \rho \) is surjective. The kernel of \( \rho \) consists of those automorphisms that fix \( \mathbb{F}_p(\zeta_\ell) \); that is, \( \ker \rho = \text{Gal}(\overline{F}_p/F_p(\zeta_\ell)) \). This group is also isomorphic to
\( \hat{\mathbb{Z}} \) and the inclusion \( \ker \rho \to \Gamma \) is simply multiplication by \( \ell - 1 \). Note that \( \hat{\mathbb{Z}} \cong \prod_q \mathbb{Z}_q \). Thus, we have an extension

\[
0 \longrightarrow \hat{\mathbb{Z}} \xrightarrow{\ell - 1} \hat{\mathbb{Z}} \xrightarrow{\rho} \mathbb{F}_\ell^\times \longrightarrow 1.
\]

The \( \mathbb{F}_\ell \)-completion of \( \hat{\mathbb{Z}} \) is clearly \( \mathbb{Z}_\ell \) and since \( \rho \) is surjective we have a surjection \( \Phi : \mathbb{Z}_\ell \to \mathcal{U} \), where \( \mathcal{U} \) is the prounipotent radical of the completion of \( \Gamma \) relative to \( \rho \). The kernel of \( \Phi \) is central in \( \mathbb{Z}_\ell \), but that is not a useful piece of information in this case. We claim that \( \Phi \) is injective so that the relative completion is the semidirect product

\[
0 \longrightarrow \mathbb{Z}_\ell \longrightarrow G \xrightarrow{\Phi} \mathbb{F}_\ell^\times \longrightarrow 1,
\]

where \( \mathbb{F}_\ell^\times \) acts via multiplication on \( \mathbb{Z}_\ell \).

Recall that in the general situation we denote by \( L \) the image of \( \rho : \Gamma \to S \). Assume that \( H^1(L, A) \) vanishes for all rational representations \( A \) of \( S \) and that \( H^2(L, A) \) vanishes for all nontrivial \( A \). Consider the extension

\[
(2) \quad 0 \longrightarrow \mathcal{K} \longrightarrow G \longrightarrow L \longrightarrow 1
\]

where \( \mathcal{K} \) is the kernel of \( \Phi : \mathcal{T} \to \mathcal{U} \). Since \( H_1(L, k) = 0 \), there is a universal central extension with kernel a \( k \)-vector space

\[
0 \longrightarrow H_2(L, k) \longrightarrow \mathcal{L} \longrightarrow L \longrightarrow 1.
\]

Its cocycle is the identity map in

\[
\text{Hom}(H_2(L, k), H_2(L, k)) \cong H^2(L, H_2(L, k)).
\]

The extension \( (2) \) is classified by a linear map \( \psi : H_2(L, k) \to \mathcal{K} \). Under the above stated conditions on \( H^1(L, A) \) and \( H^2(L, A) \), the map \( \psi \) is surjective \([\text{4}], \text{Prop. 4.13}\).

In the example under consideration, we have \( L = S = \mathbb{F}_\ell^\times \). If \( A \) is a rational \( S \)-module, then \( H^1(S, A) = 0 = H^2(S, A) \) since \( |S| = \ell - 1 \) is invertible in \( A \) (\( A \) is an \( \mathbb{F}_\ell \)-vector space). Since \( H_2(\mathbb{F}_\ell^\times, \mathbb{F}_\ell) = 0 \), we see that \( \mathcal{K} = 0 \).

### 5. Cohomology

Suppose that \( \pi \) is a projective limit of groups, \( \pi = \varprojlim \pi_\alpha \), and let \( k \) be a field. We define the continuous cohomology of \( \pi \) to be

\[
H^i_{\text{cts}}(\pi, k) = \varprojlim H^i(\pi_\alpha, k).
\]

For example if \( \pi \) is the Galois group of a field extension \( L/F \), then \( H^i_{\text{cts}}(\pi, k) \) is simply the usual Galois cohomology. Note that there is a natural map \( H^i_{\text{cts}}(\pi, k) \to H^i(\pi, k) \). It is obvious that \( H^0_{\text{cts}}(\pi, k) = H^0(\pi, k) \).
Lemma 5.1. Let $G$ be a group and let $U$ be its unipotent $k$-completion. Then the map
\[ H^2_{cts}(U, k) \longrightarrow H^2(G, k) \]
is injective. If $H_1(G, k)$ is finite dimensional, then the map
\[ H^1_{cts}(U, k) \longrightarrow H^1(G, k) \]
is an isomorphism.

Proof. Let $\alpha \in H^2_{cts}(U, k)$. This class corresponds to a central extension
\[ 0 \longrightarrow k \longrightarrow U_\alpha \longrightarrow U \longrightarrow 1 \]
in the category of proalgebraic $k$-groups. Observe that $U_\alpha$ is pronipotent. Suppose that $\alpha$ restricts to 0 in $H^2(G, k)$. Then we have a commutative diagram of extensions
\[ \begin{array}{ccc}
0 & \longrightarrow & k \\
\downarrow & & \downarrow \\
0 & \longrightarrow & G_\alpha \longrightarrow G \longrightarrow 1 \\
\end{array} \]
Since $\alpha$ maps to 0 in $H^2(G, k)$, the bottom extension splits. We then have the composite map $G \longrightarrow G_\alpha \longrightarrow U_\alpha$. By the universal mapping property of $U$, we get a unique map $U \longrightarrow U_\alpha$ making the diagram commute; i.e., the top extension splits. Thus, $\alpha = 0$.

If $H_1(G, k)$ is finite dimensional, then the unipotent $k$-completion of $G$ is the group $\mathcal{P} \subset kG^\wedge$. By Lemma 2.4, the map $H_1(G, k) \rightarrow H_1(\mathcal{P}_l, k)$ is an isomorphism for each $l$. It follows that the map
\[ H^1_{cts}(\mathcal{P}, k) = \lim_{\longrightarrow} H^1(\mathcal{P}_l, k) \longrightarrow H^1(G, k) \]
is an isomorphism.

Remark. The above argument does not work to show that $H^2(U, k)$ injects into $H^2(G, k)$. An element $\alpha$ corresponds to an extension of $U$ by $k$, but this may not be an extension of proalgebraic groups. A class in $H^2_{cts}(U, k)$ actually corresponds to a compatible sequence of extensions of algebraic groups; this allows us to use the universal mapping property of $U$.

Corollary 5.2. The map $H^1_{cts}(U, k) \rightarrow H^1(U, k)$ is an isomorphism.
Proof. We have the commutative diagram
\[
\begin{array}{ccc}
H^1_{cts}(U, k) & \longrightarrow & H^1(U, k) \\
\downarrow \cong & & \downarrow \\
H^1(G, k).
\end{array}
\]
The vertical arrow is an isomorphism by Proposition 2.8.

Corollary 5.3. The map $H^2_{cts}(U, k) \to H^2(U, k)$ is injective.

Corollary 5.4. Let $\rho : \Gamma \to S$ be a split surjective representation and let $\mathcal{G}$ be the completion relative to $\rho$. Assume that $H^1(T, k)$ is finite dimensional ($T = \ker \rho$) and that $\Phi : \mathcal{T} \to \mathcal{U}$ is an isomorphism. Then the restriction map
\[
H^2_{cts}(\mathcal{G}, k) \longrightarrow H^2(\Gamma, k)
\]
is injective.

Proof. Note that $\mathcal{G}$ is the inverse limit of the groups $\mathcal{G}_\alpha$ with each $\mathcal{G}_\alpha$ an extension
\[
1 \longrightarrow U_\alpha \longrightarrow \mathcal{G}_\alpha \longrightarrow S \longrightarrow 1.
\]
For each $\alpha$, we have a Hochschild–Serre spectral sequence
\[
E_2^{i,j}(\mathcal{G}_\alpha) = H^i(S, H^j(U_\alpha, k)) \Longrightarrow H^{i+j}(\mathcal{G}_\alpha, k).
\]
Taking the direct limit of these sequences, we get a spectral sequence $E(\mathcal{G})$ satisfying
\[
E_2^{i,j}(\mathcal{G}) = H^i(S, \lim H^j(U_\alpha, k)) \Longrightarrow \lim H^{i+j}(\mathcal{G}_\alpha, k);
\]
that is,
\[
E_2^{i,j}(\mathcal{G}) = H^i(S, H^j_{cts}(U, k)) \Longrightarrow H^{i+j}_{cts}(\mathcal{G}, k).
\]
Since $\Phi : \mathcal{T} \to \mathcal{U}$ is an isomorphism we have $H^1_{cts}(U, k) \cong H^1(T, k)$ (Lemma 5.1). Since the extension
\[
1 \longrightarrow T \longrightarrow \Gamma \longrightarrow S \longrightarrow 1
\]
splits, we have
\[
H^i(S, H^j_{cts}(U, k)) \cong H^i(S, H^j(T, k))
\]
for $j = 0, 1$; that is, we have an isomorphism
\[
E_2^{i,j}(\mathcal{G}) \cong E_2^{i,j}(\Gamma), \quad j = 0, 1,
\]
where $E(\Gamma)$ is the Hochschild–Serre spectral sequence for the above extension. Moreover, the differential $d^2 : E_2^{i,1} \to E_2^{i+2,0}$ vanishes for both spectral sequences since the extensions are split. By Lemma 5.1.
there is an inclusion of $S$-modules $H^2_{\text{cts}}(U, k) \rightarrow H^2(T, k)$ and hence $E^{0,2}_2(G)$ injects into $E^{0,2}_2(\Gamma)$. These facts together imply the following:

1. $E^{2,0}_\infty(G) = E^{2,0}_\infty(\Gamma)$;
2. $E^{1,1}_\infty(G) = E^{1,1}_\infty(\Gamma)$;
3. $E^{0,2}_\infty(G) \hookrightarrow E^{0,2}_\infty(\Gamma)$.

It follows that $H^2_{\text{cts}}(G, k)$ injects into $H^2(\Gamma, k)$. 

5.1. Let $X$ be a smooth affine curve over $k$ and let $O_{X,x}$ be the local ring at $x \in X$. In the previous section, we showed that the completion of $SL_n(O_{X,x}) \rightarrow SL_n(k)$ is the extension

$$1 \rightarrow K(\hat{O}_{X,x}) \rightarrow SL_n(\hat{O}_{X,x}) \rightarrow SL_n(k) \rightarrow 1,$$

provided that $\text{char}(k) \neq 2$ or $k \neq \mathbb{F}_3$ (recall that we need these restrictions on $k$ to guarantee that $K(\hat{O}_{X,x})$ is the unipotent $k$-completion of $K(O_{X,x})$). By Corollary 5.4, we see that the map

$$H^2_{\text{cts}}(SL_n(\hat{O}_{X,x}), k) \rightarrow H^2(SL_n(O_{X,x}), k)$$

is injective for all $n$.

5.2. Let $k$ be a finite field or a number field and consider the completion of $SL_n(k[t]) \rightarrow SL_n(k)$ for $n \geq 3$. This is the split extension

$$1 \rightarrow K(k[[T]]) \rightarrow SL_n(k[[T]]) \rightarrow SL_n(k) \rightarrow 1.$$

**Theorem 5.5.** Let $k$ be a finite field or a number field. Then for all $n \geq 3$, $H^2_{\text{cts}}(SL_n(k[[T]]), k) = 0$.

**Proof.** We have an injection

$$H^2_{\text{cts}}(SL_n(k[[T]]), k) \rightarrow H^2(SL_n(k[t]), k).$$

If $k$ is a number field then by [11], the group $H^2(SL_n(k[t]), k)$ coincides with $H^2(SL_n(k), k)$. We then have the chain of equalities

$$H^2(SL_n(k[t]), k) = H^2(SL_n(k), k) = H_2(SL_n(k), k) = H_2(SL_n(k), \mathbb{Z}) \otimes k = K_2(k) \otimes k.$$

Since $K_2(k)$ is a torsion group when $k$ is a number field, we see that $H^2(SL_n(k[t]), k) = 0$. 

If \( k \) is a finite field, then we use the following chain of equalities

\[
H^2(SL_n(k[t]), k) = H_2(SL_n(k[t]), k) \\
= H_2(SL_n(k[t]), \mathbb{Z} \otimes k) \\
= K_2(k[t]) \otimes k \\
= K_2(k) \otimes k,
\]

where the last equality is the fundamental theorem of algebraic K-theory. Since \( K_2(k) = 0 \) for finite fields, we are done.

**Remark.** It is easy to see that if \( E \) is a field with \( \text{char}(E) \neq \text{char}(k) \), then \( H^\bullet_{\text{cts}}(SL_n(k[[T]]), E) = H^\bullet(SL_n(k), E) \). Also, Theorem 5.3 could be deduced for \( n \geq 5 \) using homological stability for \( SL_n \) together with existing calculations of the groups \( K_2(k[t]/t^l) \). The above proof avoids this (and works for \( n \geq 3 \)).

In [5], Hain calls a pronilpotent group \( \mathcal{P} \) pseudonilpotent if

\[
H^\bullet_{\text{cts}}(\mathcal{P}, k) \xrightarrow{\cong} H^\bullet(\mathcal{P}, k)
\]

(his definition is actually more general). Let \( k \) be a number field and consider the group \( K(k[[T]]) \). We know that \( H^2_{\text{cts}}(K(k[[T]]), k) \) injects into \( H^2(K(k[[T]]), k) \). Suppose that this map is an isomorphism. Then we would have an isomorphism

\[
H^2(SL_n(k[[T]]), k) \xrightarrow{\cong} H^2(SL_n(k[t]), k) \xrightarrow{\cong} H^2(SL_n(k), k).
\]

Dualizing and taking the direct limit over \( n \), we would have an isomorphism

\[
K_2(k[[T]]) \otimes k \xrightarrow{\cong} K_2(k) \otimes k.
\]

Gabber’s rigidity theorem [4] asserts that \( K_2(k[[T]], \mathbb{Z}/n) \cong K_2(k, \mathbb{Z}/n) \). We would therefore have an isomorphism \( K_2(k[[T]]) \cong K_2(k) \). This is not the case, however, since the kernel of the map \( K_2(k[[T]]) \to K_2(k) \) is the relative group \( K_2(k[[T]], (T)) \) and this group is nontrivial [1] (indeed, it is uniquely divisible). It follows that the group \( K(k[[T]]) \) is not pseudonilpotent.

5.3. Consider the commutative diagram of extensions

\[
\begin{array}{c}
1 \longrightarrow \Gamma(n, p) \longrightarrow SL_n(\mathbb{Z}) \longrightarrow SL_n(\overline{\mathbb{F}}_p) \longrightarrow 1 \\
\downarrow \quad \downarrow \quad \downarrow \\
1 \longrightarrow \hat{\Gamma} \longrightarrow SL_n(\mathbb{Z}_p) \longrightarrow SL_n(\mathbb{F}_p) \longrightarrow 1.
\end{array}
\]
We showed (3.5) that $\hat{\Gamma}$ is the $F_p$-completion of $\Gamma(n,p)$ if $n \geq 3$. While Corollary 5.4 does not apply in this case, we can still use the argument in the proof to deduce that the map

$$H^2_{cts}(SL_n(\mathbb{Z}_p), F_p) \rightarrow H^2(SL_n(\mathbb{Z}), F_p)$$

is injective for $n \geq 3$. Note that $H^2(SL_n(\mathbb{Z}), F_p) \cong H_2(SL_n(\mathbb{Z}), F_p)$. The group $H_2(SL_n(\mathbb{Z}), \mathbb{Z})$ is 2-torsion for all $n \geq 3$; it is equal to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ for $n = 3, 4$ [8], and to $\mathbb{Z}/2$ for $n \geq 5$ [14]. It follows that if $p \geq 3$, then $H_2(SL_n(\mathbb{Z}), F_p) = 0$.

**Theorem 5.6.** Let $n \geq 3$. If $p \geq 3$, then $H^2_{cts}(SL_n(\mathbb{Z}_p), F_p) = 0$. When $p = 2$ we have

$$\dim_{\mathbb{F}_2} H^2_{cts}(SL_n(\mathbb{Z}_2), \mathbb{F}_2) \leq \begin{cases} 2 & n = 3, 4 \\ 1 & n \geq 5 \end{cases}$$

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Department of Mathematics, 1150 F/AB, Wayne State University, Detroit, Michigan 48202

*E-mail address*: knudson@math.wayne.edu