Page curve and symmetries

Pak Hang Chris Lau, a Toshifumi Noumi, a Yuhei Takii a and Kotaro Tamaoka b

a Department of Physics, Kobe University, Rokko, Kobe 657-8501, Japan
b Department of Physics, College of Humanities and Sciences, Nihon University, Sakura-josui, Tokyo 156-8550, Japan
E-mail: phcl2@panda.kobe-u.ac.jp, tnoumi@phys.sci.kobe-u.ac.jp, yuhei.takii@stu.kobe-u.ac.jp, tamaoka.kotaro@nihon-u.ac.jp

Abstract: Motivated by the quantum process of black hole evaporation and its implications for symmetries, we consider a qubit system with a random dynamics as a toy model of black hole. We compute its symmetry-resolved entropies and discuss its implications. We first consider the case where charges are conserved and compute the symmetry-resolved entropies. We derive a symmetry-resolved analogue of the Page curve. We then consider the case where symmetry is explicitly broken and charges are no longer conserved. It serves as a toy model for global symmetry breaking in black hole evaporation. Despite the simple framework, the symmetry-resolved entropies capture various interesting features during the analogous process of black hole evaporation in our qubit model.

Keywords: Black Holes, Random Systems, Gauge Symmetry, Global Symmetries

ArXiv ePrint: 2206.09633
1 Introduction

Thought experiments on black hole evaporation offer various lessons on symmetries in quantum gravity. The most famous one is on the absence of global symmetries, which states that symmetries in quantum gravity have to be either gauged or explicitly broken [1–3]. It is then natural to ask how large gauge interactions have to be and also how much symmetries have to be broken. Such quantitative questions are crucial when exploring phenomenological implications of quantum gravity constraints. A lot of research activities are ongoing toward this direction, especially along the line of the conjectured swampland criteria [4, 5]. See, e.g., [6–8] for review articles.

More recently, significant progress has been made on the long-standing problem in quantum gravity, whether or not black hole evaporation follows the unitary time evolution [9, 10], in the light of entanglement entropy and gravitational path integral [11–14]. Notably, this progress also made implications for global symmetries in quantum gravity [15–19]. It is then of great interests if one can use these developments to quantify required gauge interactions or symmetry breaking. We would like to set up a framework for such a quantitative problem.

According to the progress mentioned above, the time evolution of the entanglement entropy in the evaporation process follows the so-called unitary Page curve. The original Page curve [20, 21] was introduced as entanglement entropy with random unitary time evolution, assuming that the dynamics of black holes are highly chaotic. Such random time
evolution is also used in the Hayden-Preskill protocol [22], which gives insight into information recovery in the evaporation process. Several recent studies [23, 24] have discussed information recovery in the Hayden-Preskill protocol when there are conserved quantities.

In this paper, inspired by these developments, we discuss a quantity called symmetry-resolved (entanglement) entropy [25–27] (see also [28–34, 34, 35–81] for further development) in similar random systems as a toy model for studying the relationship between symmetries in quantum gravity and the dynamics of black holes. See [19] for a seminal work in this direction in the context of two dimensional gravity. The symmetry-resolved entropy is a finer decomposition of the entanglement entropy into superselection sectors of conserved charges. We will find that it is useful to introduce an analogue of the Page curve in the presence of symmetries. We also find that this perspective is useful to quantify effects of symmetry breaking on entanglement entropy.

The organization of this paper is as follows. In section 2, we review the entanglement measures relevant in this paper, especially the symmetry-resolved entropy. In section 3, we introduce our toy model of black hole evaporation with conserved charges and then derive a symmetry-resolved analog of the Page curve. In section 4, we introduce explicit symmetry breaking and study its effects on the Page curve. In section 5, we conclude by discussing the implications of our results for the evaporation process with charged Hawking radiation.

2 Symmetry-resolved entropy

This section reviews the necessary entanglement measures used in this paper. Let us consider a system in a pure state represented by a density matrix \( \rho = |\psi \rangle \langle \psi | \). We split the total system into a subregion \( A \) and its complement \( \bar{A} \). A basic entanglement measure between the two regions is the entanglement entropy. It is defined to be the von Neumann entropy of the reduced density matrix,

\[
S_A = -\text{Tr} \rho_A \log \rho_A ,
\]

(2.1)

where \( \rho_A = \text{Tr}_{\bar{A}} \rho \) is the reduced density matrix of the subregion \( A \) constructed by tracing the total density matrix over the complement region \( \bar{A} \). Entanglement entropy has the property that \( S_A = S_{\bar{A}} \) for a system in a pure state. When the two regions are maximally entangled, the entanglement entropy attains its maximal value of \( S_A = \min(\log d_A, \log d_{\bar{A}}) \), where \( d_A \) and \( d_{\bar{A}} \) are the dimensions of the Hilbert space of the region \( A \) and \( \bar{A} \), respectively.

A powerful tool to compute the entanglement entropy is the replica trick. We first compute a generalization of the entanglement entropy called the Rényi entropy,

\[
S_n \equiv \frac{1}{1-n} \log \text{Tr}[\rho_A^n] .
\]

(2.2)

Then the entanglement entropy can be recovered by analytic continuation and taking the limit \( S_A = \lim_{n \to 1} S_n \). The Rényi entropy is non-increasing in \( n \), i.e., \( S_{n_1} \leq S_{n_2} \), \( \forall \ n_1 > n_2 \). In this paper we utilize this property of the Rényi entropy and study the second Rényi entropy \( (S_2) \) instead of the entanglement entropy. It provides the tightest lower bound to the entanglement entropy and allows us to compute the Page curves.
Another focus of this paper is the existence of symmetry in the system which leads to conserved charges. The Hilbert space in such a system naturally splits into superselection sectors, labelled by the charge. We will explore the consequence of symmetry to the entanglement properties of the system. By assuming that the dynamics of the system is random, we can model the reduced density matrix using a probability distribution and decompose it into the form,

\[
\rho_A = \sum_q p(q)\rho_A(q) + \Delta\rho, \quad \text{Tr}_q\rho_A(q) = 1, \quad (2.3)
\]

where \(\rho_A(q)\) is the reduced density matrix in each charge sector and \(p(q)\) is the probability of finding the reduced density matrix with charge \(q\). \(\text{Tr}_q\) is the trace over the charge \(q\) subspace of \(A\). We also include scenarios of explicit symmetry breaking which leads to violation of charge conservation. The term \(\Delta\rho\) encodes such violation effects and allows transition between different charge sectors. We will first consider the case with \(\Delta\rho = 0\) in section 3 and then study the case with \(\Delta\rho \neq 0\) in section 4.

The existence of conserved charges allows one to define a finer entanglement measure for each charge sector. Let us set \(\Delta\rho = 0\) and plug in the form of (2.3) into the definition of entanglement entropy,

\[
S_A = -\text{Tr}\rho_A \log \rho_A \\
= -\sum_q p(q)\text{Tr}_q\rho_A(q) \log \rho_A(q) - \sum_q p(q) \log p(q) \\
= \sum_q p(q)S_A(q) + S_{c.f.}, \quad (2.4)
\]

where the first term contains the symmetry-resolved entanglement entropy defined as the von Neumann entropy for the reduced density matrix in the charge \(q\) sector: \(S_A(q) \equiv -\text{Tr}_q\rho_A(q) \log \rho_A(q)\). The second term \(S_{c.f.} \equiv -\sum_q p(q) \log p(q)\) is the charge fluctuation entropy [82]. Since each term in \(S_A\) is semi-positive definite, there is an obvious relation \(S_A \geq S_{c.f.}\). A symmetry-resolved Rényi entropy can similarly be defined and the symmetry-resolved entanglement entropy can be recovered by taking the limit \(n \to 1:\)

\[
S_n(q) = \frac{1}{1 - n} \log \text{Tr}[\rho_A(q)^n], \quad S_A(q) = \lim_{n \to 1} S_n(q). \quad (2.5)
\]

3 Symmetry-resolved Page curves

We first compute the symmetry resolved entropy in a random system with a conserved charge. Although our results are applicable to general quantum systems with finite dimensional Hilbert space, we will illustrate the calculation using a simple qubit model.

3.1 Setup

We consider an \(N\) qubits model with each qubit representing either a +1 or −1 charged particle. The allowed charge in this toy model is bounded by the number of qubits as \(Q = -N, -N + 2, \ldots, N\). To model the time evolution of a Page curve, we consider a
time dependent bipartition of the system. At the initial time $t = 0$, the region $A$ is empty and the whole system is in the complement region $\bar{A}$. At each time step, we perform a random time-evolution and then one qubit of the system is selected to become part of $A$. We interpret this setup as a toy model of black hole evaporation by regarding $A$ and $\bar{A}$ as Hawking radiation and the black hole, respectively.

More explicitly, at an intermediate time $t < N$, the state $|\psi\rangle$ is split into a subsystem $A$ with $t$ qubits and a complement subsystem $\bar{A}$ with $N-t$ qubits. For a state with charge $Q$ and a bipartition of the system into charge $q$ and charge $\bar{q}$ subregions, we employ the following notation:

$$
|\psi\rangle = \sum_{q+\bar{q}=Q} \sum_{a,\bar{a}} \frac{e^{i\delta(q_a,\bar{q}_a)}}{\sqrt{d(Q)}} |q_a\rangle_A \otimes |\bar{q}_\bar{a}\rangle_{\bar{A}},
$$

(3.1)

where $|q_a\rangle_A$ denotes a charge $q$ state labeled by $a$ in the region $A$ and similarly for $|\bar{q}_\bar{a}\rangle_{\bar{A}}$. Note that the charge $q$ of the subregion $A$ takes the value $q = -t, -t+2, \ldots, t$ and similarly for $\bar{q}$. The labels $a$ and $\bar{a}$ account for the degeneracy of the states in each charge sector. $\delta(q_a,\bar{q}_\bar{a})$ is a random phase whose average will be taken in the computation of entropy afterwards. $d(Q)$ is the dimension of the charge $Q$ superselection sector. More explicitly,

$$
d(Q) = \sum_{q+\bar{q}=Q} d_A(q)d_{\bar{A}}(\bar{q}),
$$

(3.2)

where $d_A(q)$ is the number of charge $q$ states in $A$ and similarly for $d_A(\bar{q})$. In our present qubit setup, we have

$$
d_A(q) = \left( \frac{t}{t+q} \right), \quad d_{\bar{A}}(\bar{q}) = \left( \frac{N-t}{N-t+\bar{q}} \right), \quad d(Q) = \left( \frac{N}{N+Q} \right).
$$

(3.3)

The advantage of the notation (3.1) is that it can be easily modified to accommodate a more general situation, e.g., the scenario with violation of charge conservation studied in section 4.

### 3.2 Calculation of Page curve

We consider an initial state with a total charge $Q$. The state after a random time-evolution populates equally over all the charge $Q$ states and takes the form of (3.1). The reduced density matrix of the subregion $A$ reads

$$
\rho_A = \text{Tr}_{\bar{A}}|\psi\rangle\langle\psi| = \sum_q p(q)\rho_A(q)
$$

(3.4)

with $p(q)$ and $\rho_A(q)$ given by

$$
p(q) = \frac{d_A(q)d_{\bar{A}}(Q-q)}{d(Q)},
$$

(3.5)

$$
\rho_A(q) = \frac{1}{d_A(q)} 1_q + \sum_{a \neq b} \frac{e^{i(\delta(q_a,\bar{q}_a)-\delta(q_b,\bar{q}_\bar{a}))}}{d_A(q)d_{\bar{A}}(Q-q)} |q_a\rangle\langle q_b|.
$$

(3.6)

Here $1_q$ is the identity matrix in the charge $q$ subspace of $A$. 

---

JHEP10(2022)015
Then, the symmetry-resolved second Rényi entropy is
\[
e^{-S_2(q)} = \text{Tr} \left[ \rho_A(q)^2 \right] = \frac{1}{d_A(q)} + \frac{1}{d_A(\bar{q})} - \frac{1}{d_A(q)d_A(\bar{q})},
\]
where we have taken an average over the random phases.\(^1\) In this language, the ordinary second Rényi entropy reads
\[
e^{-S_2} = \sum_q p(q)^2 e^{-S_2(q)}.
\]
Note that the expressions (3.7)–(3.8) in terms of the dimension of each Hilbert space are applicable generally, independent of fine details of the Hilbert space. In figure 1 and figure 2, we plot the symmetry-resolved second Rényi entropy (3.7) and the second Rényi entropy (3.8) in our concrete qubit model with (3.3). In particular, the former gives a symmetry-resolved analogue of the Page curve.

### 3.3 Thermodynamic limit

It is easy to derive an analytic expression of the symmetry-resolved second Rényi entropy (3.7) in the thermodynamic limit defined by \(N \to \infty\) with \(t/N\), \(q/N\), and \(Q/N\) fixed. For this, we introduce a symmetry-resolved analogue of the Page time \(t^*(q)\) for each \(q\) such that \(d_A(t^*(q)) = d_A(N - t^*(q))\). Then, for \(t < t^*(q)\), we find
\[
S_2(q) \simeq \log d_A(q) \\
\simeq \frac{(t + q)}{2} \log \frac{2t}{t + q} + \frac{(t - q)}{2} \log \frac{2t}{t - q}.
\]
\(^1\)Such dephasing is an analog of the average over unitary operators with the Haar measure. To be precise, details of the sub-leading term (the third term with a minus sign) under the thermodynamic limit depend on the choice of ensemble. Refer to recent work [83–85] that mentioned these sub-leading terms in the context of AdS/CFT correspondence.
Similarly, for $t > t_\ast(q)$, we have

$$S_2(q) \simeq \log d_{\tilde{A}}(Q - \bar{q}) \simeq \frac{(N - t + Q - q)}{2} \log \frac{2(N - t)}{N - t + Q - q} + \frac{(N - t - Q + q)}{2} \log \frac{2(N - t)}{N - t - Q + q}.$$  

(3.10)

4 Explicit symmetry breaking

Next we generalize the analysis to systems with explicit symmetry breaking. As we have mentioned, violation of charge conservation induces a charge-block-off-diagonal component $\Delta \rho$ of the reduced density matrix, which leads to a new contribution to the second Rényi entropy:

$$e^{-S_2} = \text{Tr} \rho_{\tilde{A}}^2 = \sum_q p(q)^2 e^{-S_2(q)} + \text{Tr} \left( \Delta \rho^2 \right),$$  

(4.1)

where the second term quantifies entanglement between different charge sectors. Our main focus here is to clarify when and how each contribution becomes dominant.

4.1 Setup

Let us generalize our previous setup (3.1) as follows:

$$|\psi\rangle = \sum_{q, \bar{q}} \sqrt{w(q, \bar{q})} \sum_{\alpha, \bar{\alpha}} e^{i\delta(q_\alpha, \bar{q}_{\bar{\alpha}})} |q_{\alpha} \rangle_A \otimes |\bar{q}_{\bar{\alpha}} \rangle_{\bar{A}},$$  

(4.2)

where the weight function $w(q, \bar{q}) \geq 0$ gives a probability of finding a state in the charge $(q, \bar{q})$ sector and quantifies violation of charge conservation. It is normalized as

$$\sum_{q, \bar{q}} w(q, \bar{q}) = 1.$$  

(4.3)
A convenient parametrization of $w(q, \bar{q})$ for our calculations in this paper is

$$w(q, \bar{q}) = \frac{d_A(q)d_A(\bar{q})}{\mathcal{N}} f(q, \bar{q}) ,$$  \hspace{0.5cm} (4.4)

$$\mathcal{N} = \sum_{q, \bar{q}} d_A(q)d_A(\bar{q}) f(q, \bar{q}) ,$$  \hspace{0.5cm} (4.5)

where $\mathcal{N}$ is the normalization constant and $f(q, \bar{q})$ is the distribution function controlling the degree of symmetry violation. For example, the weight function $w(q, \bar{q})$ for the charge conserving case (3.1) is localized in the subspace $q + \bar{q} = Q$ and the corresponding distribution function is

$$f(q, \bar{q}) = \delta_{q+\bar{q}, Q} , \quad \mathcal{N} = \sum_{q, \bar{q}} d_A(q)d_A(\bar{q})(Q - q) \equiv d(Q) .$$  \hspace{0.5cm} (4.6)

In contrast, if the symmetry is completely broken, every state appears randomly independent of $q + \bar{q}$, so that

$$f(q, \bar{q}) = 1 , \quad \mathcal{N} = \sum_{q, \bar{q}} d_A(q)d_A(\bar{q}) \equiv d$$ \hspace{0.5cm} (4.7)

where $d$ is the dimension of the total Hilbert space, i.e.,

$$d = \sum_{Q} d(Q) .$$  \hspace{0.5cm} (4.8)

In our qubit model $d = 2^N$. Also note that the charge distribution $p(q)$ is the binomial distribution and therefore it becomes Gaussian in the large $N$ limit. We will introduce a family of weight functions interpolating the two profiles (4.6) and (4.7), but we keep $w(q, \bar{q})$ general to derive a general formula of entropy in the meantime.

### 4.2 General formula

First, the reduced density matrix reads

$$\rho_A = \text{Tr}_{\bar{A}} |\psi\rangle\langle \psi| = \sum_q p(q)\rho_A(q) + \Delta \rho$$  \hspace{0.5cm} (4.9)

with $p(q)$, $\rho_A(q)$ and $\Delta \rho$ given by

$$p(q) = \sum_q w(q, \bar{q}) ,$$  \hspace{0.5cm} (4.10)

$$\rho_A(q) = \frac{1}{d_A(q)} + \frac{1}{p(q)} \sum_{\bar{q}, \bar{\bar{q}}} \sum_{a \neq b} \frac{w(q, \bar{q})e^{i(\delta(q_a, \bar{q}_a) - \delta(q_b, \bar{q}_b))}}{d_A(q)d_A(\bar{q})} |q_a\rangle\langle q_b| ,$$  \hspace{0.5cm} (4.11)

$$\Delta \rho = \sum_{q \neq q'} \sum_{a,b} \sum_{\bar{q}, \bar{\bar{q}}} \frac{\sqrt{w(q, \bar{q})w(q', \bar{q})}}{d_A(q)d_A(q')d_A(\bar{q})^2} e^{i(\delta(q_a, \bar{q}_a) - \delta(q'_a, \bar{q}'_a))} |q_a\rangle\langle q'_b| .$$  \hspace{0.5cm} (4.12)
Then, the two contributions to the second Rényi entropy (4.1) are

\[
\sum_q p(q)^2 e^{-S_2(q)} = \sum_{q,q'} \frac{w(q,q')w(q',\bar{q})}{d_A(q)} + \sum_{q\bar{q}} \frac{w(q,\bar{q})^2}{d_A(q)} \left[ 1 - \frac{1}{d_A(q)} \right], \tag{4.13}
\]

\[
\text{Tr}(\Delta \rho)^2 = \sum_{q,q',\bar{q}} \frac{w(q,q')w(q',\bar{q})}{d_A(q)} - \sum_{q,\bar{q}} \frac{w(q,\bar{q})^2}{d_A(q)}, \tag{4.14}
\]

where we took an average over the random phases. Note that the total contribution is

\[
e^{-S_2} = \sum_{q,q',\bar{q}} \frac{w(q,q')w(q',\bar{q})}{d_A(q)} + \sum_{q,q',\bar{q}} \frac{w(q,q')w(q',\bar{q})}{d_A(q)} - \sum_{q,\bar{q}} \frac{w(q,\bar{q})^2}{d_A(q)d_{\bar{A}}(q)}. \tag{4.15}
\]

### 4.3 Illustrative examples

For illustration, let us consider several examples for the weight function \(w(q,\bar{q})\). We begin by the extreme case (4.7), where the symmetry is completely broken and time-evolution happens randomly without depending on \(q + \bar{q}\). Then, the two contributions (4.13)–(4.14) read

\[
\sum_q p(q)^2 e^{-S_2(q)} = \frac{1}{d_A} \left[ 1 + \sum_q \frac{d_A(q)^2}{d_A} - \frac{1}{d_A} \right], \tag{4.16}
\]

\[
\text{Tr}(\Delta \rho)^2 = \frac{1}{d_{\bar{A}}} \left[ 1 - \sum_q \frac{d_A(q)^2}{d_{\bar{A}}} \right], \tag{4.17}
\]

where we introduced

\[
d_A = \sum_q d_A(q), \quad d_{\bar{A}} = \sum_q d_{\bar{A}}(\bar{q}). \tag{4.18}
\]

We also have

\[
e^{-S_2} = \frac{1}{d_A} + \frac{1}{d_{\bar{A}}} - \frac{1}{d_A d_{\bar{A}}}. \tag{4.19}
\]

Note that (4.16)–(4.17) and (4.19) do not depend on the initial value \(Q\) since the symmetry is completely broken. In figure 3, we show the \(t\)-dependence of the contribution of (4.17) to the entropy (4.19) with a curved line. There we find that the charge-off-diagonal contribution \(\text{Tr}(\Delta \rho)^2\) becomes dominant after the Page time. The reason is actually simple: before the Page time, diagonal components of the reduced density matrix \(\rho_A\) control the entropy, so that \(\Delta \rho\) does not play a role. In contrast, after the Page time, off-diagonal components become important. In models without charge conservation, the number of charge-off-diagonal components is much larger than that of off-diagonal components in the charge-diagonal sector, hence the charge-off-diagonal contribution \(\text{Tr}(\Delta \rho)^2\) becomes dominant. See also figure 4 for comparison of \(S_2\) in the present model and the symmetry preserving case studied in the previous section.
We expect that other symmetry breaking scenarios will lie in between the two extreme cases with symmetry being preserved or completely broken. We will study a particular case where the $f(q, \bar{q})$ in the weight function is chosen to be a Gaussian distribution,

$$f(q, \bar{q}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(q+\bar{q}-Q)^2}{2\sigma^2}},$$

where the parameter $\sigma$ is the standard deviation. By taking the limit of $\sigma \to 0$, the distribution function reduces to the symmetry preserving case in (4.6) and $\Delta \rho = 0$ as discussed in section 3. On the other hand, when $\sigma \to \infty$, symmetry is completely broken and the distribution function reduces to (4.7).

Now, to consider the contribution of (4.14) to the second Rényi entropy for different values of $\sigma$, we plot their $t$-dependence in figure 3 with dotted lines. These figures tell us that the contribution from the charge-off-diagonal component becomes significant after the Page time. As expected, the Gaussian profile (4.20) interpolates the two extreme cases.

We end this section by comparing $S_2$ for various values of $\sigma$ with a fixed $Q$, which is a complementary calculation to figure 2 and the result is presented in figure 4. We find that $S_2$ for a finite $\sigma$ scans a region surrounded by the symmetry completely broken case $\sigma \to \infty$ (the upper bound) and the charge conserving case $\sigma = 0$ (the lower bound). Note that we chose a nonzero value of $Q$ simply to lower the peak value of the second Rényi entropy and make the $\sigma$-dependence clearer, but the qualitative behavior does not depend on the value of $Q$.

### 5 Discussion

To conclude, we discuss possible implications of the results of our qubit model for black hole evaporation and symmetries. First, we consider the model with charge conservation in section 3. Also, we set the total charge $Q = 0$ and assume the number of qubits $N \gg 1$, having neutral black holes in mind. At the earlier time $t$ satisfying $1 \ll t \ll N$, the probability (3.5) of finding a charge $q$ state is well approximated as

$$p(q) \simeq \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{q^2}{2\sigma^2}\right), \quad \sigma = \sqrt{t},$$

Figure 3. The contribution of $\text{Tr}(\Delta \rho)^2$ to $e^{-S_2}$ for different values of $\sigma$ and $N = 400, Q = 0$. 

We expect that other symmetry breaking scenarios will lie in between the two extreme cases with symmetry being preserved or completely broken. We will study a particular case where the $f(q, \bar{q})$ in the weight function is chosen to be a Gaussian distribution,
Figure 4. Our results of second Rényi entropy $S_2$ for various values of $\sigma$ with $Q = 20$. The three finite $\sigma$ cases are bounded in between by the two extreme cases ($\sigma \to 0$ and $\sigma \to \infty$). $N = 50$ is used.

which reproduces the Gaussian charge distribution of free Hawking particles. Note that our model did not include neutral particles such as gravitons and photons, but the generalization is straightforward and the result in the qubit model remains qualitatively the same.

Here it is instructive to explore the consequence if we naively extrapolate the Gaussian charge distribution (5.1) to late time. This is analogous to extrapolating semiclassical analysis of Hawking radiation to late time of black hole evaporation. Assuming charge conservation, hence $\Delta \rho = 0$, we find that

$$e^{-S_2} = \sum_q p(q)^2 e^{-S_2(q)} \leq \sum_q p(q)^2 \sim t^{-1/2}.$$  \hspace{1cm} (5.2)

Here we have neglected a $t$-independent constant that gives a subleading correction to the entropy, at the most right-hand-side. This implies that the second Rényi entropy monotonically increases in $t$:

$$S_2 \geq \frac{1}{2} \log t,$$  \hspace{1cm} (5.3)

which implies unitarity violation since $S_2$ is bounded by the system size as

$$S_2 \leq \min (\log d_A, \log d_{\bar{A}}) \sim \min (t, N - t).$$  \hspace{1cm} (5.4)

Note that unitarity violation in this argument happens much later than the Page time $t \sim N/2$, because the second inequality in (5.2) is a loose bound (recall that we typically have $S_2(q) \gg 1$ for $t, N \gg 1$) and accordingly the monotonic growth (5.3) is logarithmic.

This observation implies that either (A) the Gaussian charge distribution has to be modified or (B) the assumption of charge conservation has to be relaxed, in order to recover unitarity. See a recent interesting paper [19] for earlier discussion on this point in the context of two dimensional gravity, where the charge distribution of matters (charged radiation) was essentially given by (5.1) in the presence of exact global symmetry and possible ways out were discussed. Actually, our simple qubit models provide illustrative examples for the above two scenarios. First, the model in section 3 provides an example for the case (A), where the charge distribution (3.5) is no more Gaussian once the finite $N$
effects are taken into account. In this case, the off-diagonal contribution in $\rho_A(q)$ becomes dominant after the Page time and it plays a role to decrease the second Rényi entropy, which is reminiscent of the island formula. It is also worth mentioning that gauge interactions may modify the charge distribution (3.5) into a non-Gaussian form. For example, in two dimensional gravity, the gauge force becomes stronger at IR and large charge emission is disfavored [19]. It would be of great interests to see if a similar effect appears in higher dimensions.

On the other hand, our qubit model with the distribution function (4.7) studied in section 4 provides an illustrative example for the case (B). In particular, in the limit $\sigma \to \infty$ of the model (4.20), the symmetry is completely broken and the time-evolution is random in the charge space. As a consequence, the charge distribution $p(q)$ is Gaussian (in the macroscopic limit $N \gg 1$). However, the new contribution, $\text{Tr}(\Delta \rho^2)$ in (4.1), originating from the charge-off-diagonal part $\Delta \rho$ of the reduced density matrix is responsible for decreasing the entanglement entropy to make it consistent with unitarity. It would be interesting to evaluate $\text{Tr}(\Delta \rho^2)$ in concrete gravity systems, where global symmetry should be explicitly broken at least through non-perturbative effects, to see more direct relation between our observation and the standard lore about absence of global symmetries in quantum gravity.

**Note added.** After we completed the project, ref. [86] appeared on arXiv, which has some overlap with our section 3. In ref. [86], they studied the symmetry-resolved entanglement entropy and introduced a symmetry-resolved analogue of the Page curve in qubit models similar to our model in section 3 with charge conservation. In particular, our (16) and their (33) are consistent with each other.

**Acknowledgments**

We would like to thank the organizers and participants of the Kagoshima Workshop on Quantum Aspects of Gravitation, where this project was initiated. P.H.C.L. is supported in part by JSPS KAKENHI Grant No. 20H01902. T.N. is supported in part by JSPS KAKENHI Grant No. 20H01902 and No. 22H01220, and MEXT KAKENHI Grant No. 21H00075, No. 21H05184 and No. 21H05462. K.T. is supported in part by JSPS KAKENHI Grant No. 21K13920 and MEXT KAKENHI Grant No. 22H05265.

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP³ supports the goals of the International Year of Basic Sciences for Sustainable Development.

**References**

[1] T. Banks and L.J. Dixon, *Constraints on String Vacua with Space-Time Supersymmetry*, *Nucl. Phys. B* **307** (1988) 93 [SPIRE].

[2] T. Banks and N. Seiberg, *Symmetries and Strings in Field Theory and Gravity*, *Phys. Rev. D* **83** (2011) 084019 [arXiv:1011.5120] [SPIRE].
[3] D. Harlow and H. Ooguri, *Symmetries in quantum field theory and quantum gravity*, *Commun. Math. Phys.* **383** (2021) 1669 [arXiv:1810.05338] [inSPIRE].

[4] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, *The String landscape, black holes and gravity as the weakest force*, *JHEP* **06** (2007) 060 [hep-th/0601001] [inSPIRE].

[5] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys. B* **766** (2007) 21 [hep-th/0605264] [inSPIRE].

[6] E. Palti, *The Swampland: Introduction and Review*, *Fortsch. Phys.* **67** (2019) 1900037 [arXiv:1903.06239] [inSPIRE].

[7] M. van Beest, J. Calderón-Infante, D. Mirfendereski and I. Valenzuela, *Lectures on the Swampland Program in String Compactifications*, [arXiv:2102.01111] [inSPIRE].

[8] D. Harlow, B. Heidenreich, M. Reece and T. Rudelius, *The Weak Gravity Conjecture: A Review*, [arXiv:2201.08380] [inSPIRE].

[9] S.W. Hawking, *Particle Creation by Black Holes*, *Commun. Math. Phys.* **43** (1975) 199 [Erratum ibid. **46** (1976) 206] [inSPIRE].

[10] S.W. Hawking, *Breakdown of Predictability in Gravitational Collapse*, *Phys. Rev. D* **14** (1976) 2460 [inSPIRE].

[11] G. Penington, *Entanglement Wedge Reconstruction and the Information Paradox*, *JHEP* **09** (2020) 002 [arXiv:1905.08255] [inSPIRE].

[12] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *JHEP* **12** (2019) 063 [arXiv:1905.08762] [inSPIRE].

[13] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *Replica Wormholes and the Entropy of Hawking Radiation*, *JHEP* **05** (2020) 013 [arXiv:1911.12333] [inSPIRE].

[14] G. Penington, S.H. Shenker, D. Stanford and Z. Yang, *Replica wormholes and the black hole interior*, *JHEP* **03** (2022) 205 [arXiv:1911.11977] [inSPIRE].

[15] D. Harlow and E. Shaghoulian, *Global symmetry, Euclidean gravity, and the black hole information problem*, *JHEP* **04** (2021) 175 [arXiv:2010.10539] [inSPIRE].

[16] Y. Chen and H.W. Lin, *Signatures of global symmetry violation in relative entropies and replica wormholes*, *JHEP* **03** (2021) 040 [arXiv:2101.06005] [inSPIRE].

[17] P.-S. Hsin, L.V. Iliesiu and Z. Yang, *A violation of global symmetries from replica wormholes and the fate of black hole remnants*, *Class. Quant. Grav.* **38** (2021) 194004 [arXiv:2011.09444] [inSPIRE].

[18] A. Belin, J. De Boer, P. Nayak and J. Sonner, *Charged eigenstate thermalization, Euclidean wormholes and global symmetries in quantum gravity*, *SciPost Phys.* **12** (2022) 059 [arXiv:2012.07875] [inSPIRE].

[19] A. Milekhin and A. Tajdini, *Charge fluctuation entropy of Hawking radiation: a replica-free way to find large entropy*, [arXiv:2109.03841] [inSPIRE].

[20] D.N. Page, *Information in black hole radiation*, *Phys. Rev. Lett.* **71** (1993) 3743 [hep-th/9306083] [inSPIRE].

[21] D.N. Page, *Time Dependence of Hawking Radiation Entropy*, *JCAP* **09** (2013) 028 [arXiv:1301.4998] [inSPIRE].
[22] P. Hayden and J. Preskill, *Black holes as mirrors: Quantum information in random subsystems*, JHEP 09 (2007) 120 [arXiv:0708.4025] [inSPIRE].

[23] Y. Nakata, E. Wakakuwa and M. Koashi, *Black holes as clouded mirrors: the Hayden-Preskill protocol with symmetry*, arXiv:2007.00895 [inSPIRE].

[24] H. Tajima and K. Saito, *Universal limitation of quantum information recovery: symmetry versus coherence*, arXiv:2103.01876 [inSPIRE].

[25] N. Laflorencie and S. Rachel, *Spin-resolved entanglement spectroscopy of critical spin chains and Luttinger liquids*, J. Stat. Mech. 1411 (2014) P11013.

[26] M. Goldstein and E. Sela, *Symmetry-resolved entanglement in many-body systems*, Phys. Rev. Lett. 120 (2018) 200602 [arXiv:1711.09418] [inSPIRE].

[27] J.C. Xavier, F.C. Alcaraz and G. Sierra, *Equipartition of the entanglement entropy*, Phys. Rev. B 98 (2018) 041106 [arXiv:1804.06357] [inSPIRE].

[28] R. Bonsignori, P. Ruggiero and P. Calabrese, *Symmetry resolved entanglement in free fermionic systems*, J. Phys. A 52 (2019) 475302 [arXiv:1907.02084] [inSPIRE].

[29] A. Belin, L.-Y. Hung, A. Maloney, S. Matsuura, R.C. Myers and T. Sierens, *Holographic Charged Rényi Entropies*, JHEP 12 (2013) 059 [arXiv:1310.4180] [inSPIRE].

[30] P. Caputa, M. Nozaki and T. Numasawa, *Charged Entanglement Entropy of Local Operators*, Phys. Rev. D 93 (2016) 105032 [arXiv:1512.08132] [inSPIRE].

[31] J.S. Dowker, *Conformal weights of charged Rényi entropy twist operators for free scalar fields in arbitrary dimensions*, J. Phys. A 49 (2016) 145401 [arXiv:1512.01135] [inSPIRE].

[32] J.S. Dowker, *Charged Rényi entropies for free scalar fields*, J. Phys. A 50 (2017) 165401 [arXiv:1512.01135] [inSPIRE].

[33] E. Cornfeld, M. Goldstein and E. Sela, *Imbalance entanglement: Symmetry decomposition of negativity*, Phys. Rev. A 98 (2018) 032302 [arXiv:1804.00632] [inSPIRE].

[34] H. Barghathi, C.M. Herdman and A. Del Maestro, *Rényi Generalization of the Accessible Entanglement Entropy*, Phys. Rev. Lett. 121 (2018) 150501.

[35] H. Barghathi, E. Casiano-Diaz and A. Del Maestro, *Operationally accessible entanglement of one-dimensional spinless fermions*, Phys. Rev. A 100 (2019) 022324 [arXiv:1905.03312] [inSPIRE].

[36] N. Feldman and M. Goldstein, *Dynamics of Charge-Resolved Entanglement after a Local Quench*, Phys. Rev. B 100 (2019) 235146 [arXiv:1905.10749] [inSPIRE].

[37] E. Cornfeld, L.A. Landau, K. Shtengel and E. Sela, *Entanglement spectroscopy of non-Abelian anyons: Reading off quantum dimensions of individual anyons*, Phys. Rev. B 99 (2019) 115429 [arXiv:1810.01853] [inSPIRE].

[38] S. Fraenkel and M. Goldstein, *Symmetry resolved entanglement: Exact results in 1D and beyond*, J. Stat. Mech. 2003 (2020) 033106 [arXiv:1910.08459] [inSPIRE].

[39] P. Calabrese, M. Collura, G. Di Giulio and S. Murciano, *Full counting statistics in the gapped XXZ spin chain*, Europhys. Lett. 129 (2020) 60007 [arXiv:2002.04367] [inSPIRE].

[40] K. Monkman and J. Sirker, *Operational entanglement of symmetry-protected topological edge states*, Phys. Rev. Res. 2 (2020) 043191.
[41] D. Azses and E. Sela, *Symmetry-resolved entanglement in symmetry-protected topological phases*, Phys. Rev. B 102 (2020) 235157 [arXiv:2008.09332] [insPIRE].

[42] D. Azses, R. Haenel, Y. Naveh, R. Raussendorf, E. Sela and E.G. Dalla Torre, *Identification of Symmetry-Protected Topological States on Noisy Quantum Computers*, Phys. Rev. Lett. 125 (2020) 120502 [arXiv:2002.04620] [insPIRE].

[43] H. Barghathi, J. Yu and A. Del Maestro, *Theory of noninteracting fermions and bosons in the canonical ensemble*, Phys. Rev. Res. 2 (2020) 043206.

[44] M.T. Tan and S. Ryu, *Particle number fluctuations, Rényi entropy, and symmetry-resolved entanglement entropy in a two-dimensional Fermi gas from multidimensional bosonization*, Phys. Rev. B 101 (2020) 235169 [arXiv:1911.01451] [insPIRE].

[45] S. Murciano, G. Di Giulio and P. Calabrese, *Symmetry resolved entanglement in gapped integrable systems: a corner transfer matrix approach*, SciPost Phys. 8 (2020) 046 [arXiv:1911.09588] [insPIRE].

[46] X. Turkeshi, P. Ruggiero, V. Alba and P. Calabrese, *Entanglement equipartition in critical random spin chains*, Phys. Rev. B 102 (2020) 014455 [arXiv:2005.03331] [insPIRE].

[47] M. Kiefer-Emmanouilidis, R. Unanyan, J. Sirker and M. Fleischhauer, *Bounds on the entanglement entropy by the number entropy in non-interacting fermionic systems*, SciPost Phys. 8 (2020) 083.

[48] S. Murciano, P. Ruggiero and P. Calabrese, *Symmetry resolved entanglement in two-dimensional systems via dimensional reduction*, J. Stat. Mech. 202008 (2020) 083102 [arXiv:2003.11453] [insPIRE].

[49] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer and J. Sirker, *Evidence for Unbounded Growth of the Number Entropy in Many-Body Localized Phases*, Phys. Rev. Lett. 124 (2020) 243601 [arXiv:2003.04849] [insPIRE].

[50] L. Capizzi, P. Ruggiero and P. Calabrese, *Symmetry resolved entanglement entropy of excited states in a CFT*, J. Stat. Mech. 202007 (2020) 073101 [arXiv:2003.04670] [insPIRE].

[51] S. Murciano, G. Di Giulio and P. Calabrese, *Entanglement and symmetry resolution in two dimensional free quantum field theories*, JHEP 08 (2020) 073 [arXiv:2006.09069] [insPIRE].

[52] D.X. Horváth and P. Calabrese, *Symmetry resolved entanglement in integrable field theories via form factor bootstrap*, JHEP 11 (2020) 131 [arXiv:2008.08553] [insPIRE].

[53] R. Bonsignori and P. Calabrese, *Boundary effects on symmetry resolved entanglement*, J. Phys. A 54 (2021) 015005 [arXiv:2009.08508] [insPIRE].

[54] B. Estienne, Y. Ikhlef and A. Morin-Duchesne, *Finite-size corrections in critical symmetry-resolved entanglement*, SciPost Phys. 10 (2021) 054 [arXiv:2010.10515] [insPIRE].

[55] S. Murciano, R. Bonsignori and P. Calabrese, *Symmetry decomposition of negativity of massless free fermions*, SciPost Phys. 10 (2021) 111 [arXiv:2102.10054] [insPIRE].

[56] H.-H. Chen, *Symmetry decomposition of relative entropies in conformal field theory*, JHEP 07 (2021) 084 [arXiv:2104.03102] [insPIRE].

[57] S. Zhao, C. Northe and R. Meyer, *Symmetry-resolved entanglement in AdS$_3$/CFT$_2$ coupled to U(1) Chern-Simons theory*, JHEP 07 (2021) 030 [arXiv:2012.11274] [insPIRE].
[58] K. Weisenberger, S. Zhao, C. Northe and R. Meyer, Symmetry-resolved entanglement for excited states and two entangling intervals in $\text{AdS}_3/\text{CFT}_2$, JHEP 12 (2021) 104 [arXiv:2108.09210] [nSPIRE].

[59] L. Capizzi and P. Calabrese, Symmetry resolved relative entropies and distances in conformal field theory, JHEP 10 (2021) 195 [arXiv:2105.08596] [nSPIRE].

[60] L.Y. Hung and G. Wong, Entanglement branes and factorization in conformal field theory, Phys. Rev. D 104 (2021) 026012 [arXiv:1912.11201] [nSPIRE].

[61] P. Calabrese, J. Dubail and S. Murciano, Symmetry-resolved entanglement entropy in Wess-Zumino-Witten models, JHEP 10 (2021) 067 [arXiv:2106.15946] [nSPIRE].

[62] D.X. Horváth, L. Capizzi and P. Calabrese, U(1) symmetry resolved entanglement in free 1 + 1 dimensional field theories via form factor bootstrap, JHEP 05 (2021) 197 [arXiv:2103.03197] [nSPIRE].

[63] D. Azses, E.G. Dalla Torre and E. Sela, Observing Floquet topological order by symmetry resolution, Phys. Rev. B 104 (2021) L220301 [arXiv:2109.01151] [nSPIRE].

[64] M. Kiefer-Emmanouilidis, R. Unanyan, M. Fleischhauer and J. Sirker, Slow delocalization of particles in many-body localized phases, Phys. Rev. B 103 (2021) 024203 [nSPIRE].

[65] G. Parez, R. Bonsignori and P. Calabrese, Exact quench dynamics of symmetry resolved entanglement in a free fermion chain, J. Stat. Mech. 2009 (2021) 093102 [arXiv:2106.13115] [nSPIRE].

[66] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[67] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[68] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[69] G. Parez, R. Bonsignori and P. Calabrese, Exact quench dynamics of symmetry resolved entanglement in a free fermion chain, J. Stat. Mech. 2009 (2021) 093102 [arXiv:2106.13115] [nSPIRE].

[70] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[71] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[72] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[73] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].

[74] G. Parez, R. Bonsignori and P. Calabrese, Quasiparticle dynamics of symmetry-resolved entanglement after a quench: Examples of conformal field theories and free fermions, Phys. Rev. B 103 (2021) L041104 [arXiv:2010.09794] [nSPIRE].
[75] M. Ghasemi, *Universal Thermal Corrections to Symmetry-Resolved Entanglement Entropy and Full Counting Statistics*, arXiv:2203.06708 [inSPIRE].

[76] S. Scopa and D.X. Horváth, *Exact hydrodynamic description of symmetry-resolved Rényi entropies after a quantum quench*, J. Stat. Mech. 2208 (2022) 083104 [arXiv:2205.02924] [inSPIRE].

[77] G. Parez, R. Bonsignori and P. Calabrese, *Dynamics of charge-imbalance-resolved entanglement negativity after a quench in a free-fermion model*, J. Stat. Mech. 2205 (2022) 053103 [arXiv:2202.05309] [inSPIRE].

[78] H.-H. Chen, *Dynamics of charge imbalance resolved negativity after a global quench in free scalar field theory*, JHEP 08 (2022) 146 [arXiv:2205.09532] [inSPIRE].

[79] F. Ares, P. Calabrese, G. Di Giulio and S. Murciano, *Multi-charged moments of two intervals in conformal field theory*, JHEP 09 (2022) 051 [arXiv:2206.01534] [inSPIRE].

[80] S. Fraenkel and M. Goldstein, *Extensive Long-Range Entanglement in a Nonequilibrium Steady State*, arXiv:2205.12991 [inSPIRE].

[81] N. Iizuka, A. Miyata and T. Ugajin, *A comment on a fine-grained description of evaporating black holes with baby universes*, JHEP 09 (2022) 158 [arXiv:2111.07107] [inSPIRE].

[82] A. Lukin et al., *Probing entanglement in a many-body-localized system*, Science 364 (2019) 256.

[83] B. Freivogel, D. Nikolakopoulou and A.F. Rotundo, *Wormholes from Averaging over States*, arXiv:2105.12771 [inSPIRE].

[84] D. Stanford, Z. Yang and S. Yao, *Subleading Weingartens*, JHEP 02 (2022) 200 [arXiv:2107.10252] [inSPIRE].

[85] K. Goto, Y. Kusuki, K. Tamaoka and T. Ugajin, *Product of random states and spatial (half-)wormholes*, JHEP 10 (2021) 205 [arXiv:2108.08308] [inSPIRE].

[86] S. Murciano, P. Calabrese and L. Piroli, *Symmetry-resolved Page curves*, Phys. Rev. D 106 (2022) 046015 [arXiv:2206.05083] [inSPIRE].