HOMOLOGY OF $I$–ADIC TOWERS

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Abstract. Let $R$ be a commutative ring and $I \triangleleft R$ an ideal generated by a regular sequence. Then it is known that the natural sequences

$$0 \to \operatorname{Tor}^R_*(R/I, I^s) \to \operatorname{Tor}^R_*(R/I, I^s/I^{s+1}) \to \operatorname{Tor}^R_{s-1}(R/I, I^{s+1}) \to 0$$

are short exact sequences of graded free $R/I$–modules, for any $s \geq 0$. The aim of this paper is to give a proof which accounts for the structural simplicity of the statement. It relies on a minimum of technicalities and exposes the phenomenon in a transparent way as a consequence of the regularity assumption. The ideas discussed here are used in [10] to obtain a better qualitative understanding of $I$-adic towers in algebraic topology.

Introduction

Let $R$ be a commutative ring with unit and assume that $I \triangleleft R$ is an ideal generated by a finite regular sequence. Consider the $I$-adic filtration of $R$

$$\cdots \subseteq I^{s+1} \subseteq I^s \subseteq \cdots \subseteq I \subseteq R.$$  

Theorem 1. The short exact sequences

$$(0.1) \quad 0 \longrightarrow I^{s+1} \longrightarrow I^s \longrightarrow I^s/I^{s+1} \longrightarrow 0$$

induce short exact sequences of graded free $R/I$–modules

$$(0.2) \quad 0 \longrightarrow \operatorname{Tor}^R_*(R/I, I^s) \longrightarrow \operatorname{Tor}^R_*(R/I, I^s/I^{s+1}) \longrightarrow \operatorname{Tor}^R_{s-1}(R/I, I^{s+1}) \longrightarrow 0.$$  

This result appears to have been known to commutative algebraists for a while, but the author knows of no convenient self-contained account in the algebraic literature. He encountered the statement in Baker’s preprint [1], which was written in view of applications to algebraic topology. A published proof of Theorem 1 based on Baker’s ideas can be found in [9].

In this paper, we investigate the situation from a new point of view. One might say that the moral content of Theorem 1 is that the powers of the ideal $I$, which by birth are firmly rooted in the world of $R$-modules, are in the given situation in fact very neatly organized with respect to each other in the derived category $\mathcal{D}_R$ of $R$. This is made precise in Remark 9. We aim to accommodate this view by giving a proof which exposes the statement in a conceptually simple way as a consequence of the regularity assumption and which uses only a minimum of technicalities. In particular, we completely avoid the use of explicit chain complexes other than Koszul complexes. The motivation for our work comes from algebraic topology, as we briefly indicate now.

The basic objects of study in stable homotopy theory are ring spectra. They represent cohomology theories which are equipped with a multiplicative structure. One of the great achievements of modern stable homotopy theory is the construction of a derived category $\mathcal{D}_E$ of module spectra over $E$ for highly structured ring spectra $E$ (see [4,6] for two different accounts). This category shares many formal
Now the crucial observation is that \( \delta \) of the ideal \( I \) is a derivation with respect to the natural biaction of \( \Lambda \)-algebra on \( I/I^2 \) in a simple manner. This follows from regarding the maps \( R \) and some Proposition 2, which are unable to find in the literature.

Applying to (0.4), this shows that \( \delta^s \) is determined by \( \delta^1 \), as follows. Regularity of the ideal \( I \) implies that \( \text{gr}^1_1(R) = \bigoplus_{s \geq 0} I^s/I^{s+1} \) is isomorphic to the symmetric algebra on \( I/I^2 \). Therefore \( \text{Tor}^R_*(S, \text{gr}^1_1(R)) \) is the free graded commutative algebra over \( \text{Tor}^R_*(S, S) \) generated by \( \text{Tor}^R_*(S, I/I^2) \). So we are left to identify \( \delta^1 \) and to show that the sequence (0.3) that it determines is exact. We do this by identifying it with a well-known relative injective resolution of \( S \) over the coalgebra \( \Lambda_* \), which we call the model complex. It is obtained by rearranging the Koszul complex for the regular sequence \((-x_1 + I^2, \ldots, -x_n + I^2) \) on \( \text{gr}^1_1(R) \). We also explain at this point what the structure on the Koszul complex is which makes (0.3) a sequence of \( \Lambda_* \)-comodules. The statement in Theorem 1 about the freeness of the Tor-groups over \( S \) is an easy consequence of the identification of (0.3) as the model complex.

Proposition 2. Assume we are given a singular extension of \( R \)-algebras

\[
0 \rightarrow J \rightarrow A \rightarrow A' \rightarrow 0
\]

and some \( R \)-algebra \( T \). Then the connecting homomorphism

\[
\partial : \text{Tor}^R_*(T, A') \rightarrow \text{Tor}^R_{*-1}(T, J)
\]

is a derivation with respect to the natural biaction of \( \text{Tor}^R_*(T, A') \) on \( \text{Tor}^R_*(T, J) \).
The paper is organized as follows. In Section 1, we recall basic material on Koszul complexes and regular sequences. In Section 2, we discuss the model complex mentioned above. In Section 3, we prove Proposition 2. In Section 4, we identify the sequence (0.3) as the model complex (Proposition 7) and deduce Theorem 1. We finish by explaining how the theorem leads to a characterization of the $I$-adic filtration in the derived category $\mathcal{D}_R$.

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**Conventions.** All rings are assumed to be commutative and to have a unit. All algebras are required to have a unit. We use the convention $M^* = M_{-*}$ for graded objects and write $1$ for all identity maps. An unlabelled $\otimes$ means $\otimes_R$, where $R$ is some ring which is specified in the context.

1. Koszul complexes and regular sequences

Let $(r_1, \ldots, r_n)$ be a sequence of elements of a ring $R$, generating an ideal $I \subseteq R$. Consider the exterior algebra $\Lambda^R(M)$ on an $R$–module $M$. It is naturally graded, with $M$ as the submodule of degree one. We denote the degree of a homogeneous element $e \in \Lambda^R(M)$ by $|e|$ and the product on $\Lambda^R(M)$ by $\wedge$. Recall that $\Lambda^R(M)$ is a graded bialgebra, i.e. it supports a coproduct

\[
\Delta: \Lambda^R(M) \longrightarrow \Lambda^R(M) \otimes \Lambda^R(M)
\]

which is a map of graded algebras. On a homogeneous element $m$ of degree one, it is defined by $\Delta(m) = 1 \otimes m + m \otimes 1$. The coproduct is graded cocommutative, in the sense that $\Delta = \tau \circ \Delta$. Here $\tau$ denotes the twist map, defined by $\tau(e \otimes f) = (-1)^{|e||f|} f \otimes e$ for two homogeneous elements $e$ and $f$.

Now assume that $U$ is a free $R$–module of rank $n$ with fixed basis $e_1, \ldots, e_n$. The $R$–linear map $d_1: U \to R$ which sends $e_j$ to $r_j$ has a unique extension to a graded $R$–derivation

\[
d_*: \Lambda^R(U) \longrightarrow \Lambda^R_{-1}(U).
\]

So the image of a product of two homogeneous elements $e$ and $f$ is given by

\[
d_*(e \wedge f) = d_*(e) \wedge f + (-1)^{|e|} e \wedge d_*(f).
\]

Formally, we may interpret $d_*$ as the expression

\[
d_* = \sum_{j=1}^n r_j \frac{\partial}{\partial e_j}.
\]

It follows easily that $d_*$ is a differential. To verify this, it suffices to show that $(d_* \circ d_*)(g) = 0$ for all homogeneous $g$. We prove this by induction over $|g|$. The case $|g| = 1$ is trivial. For $|g| > 1$, write $g$ as a linear combination of elements of the form $e \wedge f$, with $e$ and $f$ homogeneous and $|e| = 1$. Then the inductive step follows from (1.2).

We do not distinguish between the concepts of differential graded modules and chain complexes. In particular, we regard $\Lambda^R_*(U)$ as a chain complex.

Another formulation of the fact that $d_*$ is a derivation is the following. Let $(\Lambda^R_*(U) \otimes \Lambda^R_*(U), d^2)$ be the tensor product of the chain complex $(\Lambda^R_*(U), d_*)$ with itself. Its components are given by

\[
(\Lambda^R_*(U) \otimes \Lambda^R_*(U))_n = \bigoplus_{p+q=n} \Lambda^R_p(U) \otimes \Lambda^R_q(U).
\]
The value of its differential $d^g_*$ on an element of the form $e \otimes f$, where $e$ and $f$ are homogeneous, is given by

$$d^g_*(e \otimes f) = d_*(e) \otimes f + (-1)^{|e|} e \otimes d_*(f).$$

The reader may check that we can write

$$(1.3) \quad d^g_* = d_* + 1 + \tau \circ (d_* \otimes 1) \circ \tau.$$ 

To say that $d_*$ is a derivation is equivalent to state that the product

$$\wedge: \Lambda^R(U) \otimes \Lambda^R(U) \longrightarrow \Lambda^R(U)$$

is a map of chain complexes. The coproduct $\Delta$, on the other hand, is not compatible with the differential in such a way. Instead, we have

$$(1.4) \quad d^g_* \circ \Delta = 2(\Delta \circ d_*).$$

This is a consequence of the formula

$$(1.5) \quad \Delta \circ d_* = (d_* \otimes 1) \circ \Delta,$$

which we prove in an instant. Namely, cocommutativity of $\Lambda^R(U)$ and (1.5) imply

$$\tau \circ (d_* \otimes 1) \circ \tau \circ \Delta = \tau \circ (d_* \otimes 1) \circ \Delta = \tau \circ \Delta \circ d_* = \Delta \circ d_*.$$

Applying (1.5) again, (1.4) follows from (1.3).

So let us prove (1.5). We abbreviate $\Lambda^R_*(U)$ by $\Lambda_*$ in the following. Regard $\Lambda_* \otimes \Lambda_*$ as a $\Lambda_*$–bimodule by restricting scalars along $\Delta$. So the left (right) action of an element $x \in \Lambda_*$ is given by left (right) multiplication by $\Delta(x)$. We claim that both sides of (1.5) are derivations $\Lambda_* \rightarrow (\Lambda_* \otimes \Lambda_*)_{*,-1}$ with respect to this bimodule structure. For the left hand side, note that as an algebra map, $\Delta$ is a map of $\Lambda_*$–bimodules. As $d_*$ is a derivation, this implies by naturality that $\Delta \circ d_*$ is a derivation. For the right hand side, we use naturality in the first argument. Namely, it can be checked that $d_* \otimes 1$ is a derivation with respect to the canonical $\Lambda_* \otimes \Lambda_*$–bimodule structure on $\Lambda_* \otimes \Lambda_*$. Also, $\Delta$ is an algebra map. Hence $(d_* \otimes 1) \circ \Delta$ is a derivation with respect to the $\Lambda_*$–bimodule structure obtained by restricting scalars along $\Delta$. Now it is easily checked that both sides of (1.5) coincide on $e_j$ and hence on all elements of degree one. As these generate $\Lambda_*$, the statement follows.

The complex $(\Lambda^R(U), d_*)$ is called the Koszul complex associated to the sequence $(r_1, \ldots, r_n)$; we denote it by $K(r_1, \ldots, r_n)$. The projection $\varepsilon: \Lambda^0_0(U) \rightarrow R / I$ defines an augmentation of $K(r_1, \ldots, r_n)$ over $R/I$.

**Proposition 3** ([7, Th. 16.5]). For a regular sequence $(r_1, \ldots, r_n)$, the augmented differential graded algebra $K(r_1, \ldots, r_n)$ defines an $R$–free resolution of $R/I$.

As applying $R/I \otimes -$ to $K(r_1, \ldots, r_n)$ kills all the differentials, this implies

**Corollary 4.** There is an isomorphism of graded algebras

$$\text{Tor}^R_*(R/I, R/I) \cong R/I \otimes \Lambda^R_*(U).$$

Let $\text{gr}_I^*(R)$ be the graded algebra associated to the $I$–adic filtration of $R$. Its components are the $R/I$–modules $\text{gr}_I^*(R) = I^s / I^{s+1}$, where by convention $I^0 = R$. Let $\{r_j\}$ denote the residue class of $r_j$ in $I/I^2$. For a ring $S$ and an $S$–module $M$, let $\text{Sym}^S_*(M)$ be the symmetric algebra on $M$. If $M$ is free of rank $n$, $\text{Sym}^S_*(M)$ is isomorphic to a polynomial ring over $S$ in $n$ variables. Just as the exterior algebra on $M$, $\text{Sym}^S_*(M)$ admits a graded bialgebra structure. On homogeneous elements $m$ of degree one, the coproduct is given by $\Delta(m) = 1 \otimes m + m \otimes 1$.

**Proposition 5** ([7, Th. 16.2]). For a regular sequence $(r_1, \ldots, r_n)$, $I/I^2$ is freely generated as an $R/I$–module by $\{r_1, \ldots, r_n\}$, and there is an isomorphism of graded algebras

$$\text{gr}_I^*(R) \cong \text{Sym}^R_*(I/I^2).$$
Assume now that the ground ring \( R \) is graded and that \( M \) is a graded \( R \)-module. Then \( \Lambda^R(M) \) and \( \text{Sym}^R(M) \) are bigraded \( R \)-bialgebras. If \( r_1, \ldots, r_n \) are homogeneous elements, the ideal \( I = (r_1, \ldots, r_n) \) is graded and \( \text{gr}_f^1(R) \) is a bigraded \( R/I \)-algebra.

2. A Certain Exact Sequence

Let \( V \) and \( W \) be free modules of rank \( n \) over a given ring \( S \), with bases \( e_1, \ldots, e_n \) and \( f_1, \ldots, f_n \) respectively. The sequence \( (f_1, \ldots, f_n) \) is regular on the graded \( S \)-algebra \( R = \text{Sym}_S(W) \). It generates an ideal \( I \) such that \( R/I \cong S \). The Koszul complex

\[
K(f_1, \ldots, f_n) = \Lambda^R(R \otimes_S V)
\]

provides by Proposition 3 an \( R \)-free resolution of \( S \). By the remark at the end of Section 1, \( K(f_1, \ldots, f_n) \) is bigraded. Clearly, \( K(f_1, \ldots, f_n) \) is obtained as a bigraded \( S \)-bialgebra from \( \Lambda_* = \Lambda^S(V) \) by extending scalars, i.e.

\[
K(f_1, \ldots, f_n) \cong R \otimes S \Lambda_*
\]

In particular, the coproduct \( \Delta \) on \( K(f_1, \ldots, f_n) \) corresponds to \( 1 \otimes \Delta' \), where \( \Delta' \) is the coproduct on \( \Lambda_* \). We denote the differential of \( R \otimes S \Lambda_* \) corresponding under (2.2) to the differential \( d_* \) of \( K(f_1, \ldots, f_n) \) by \( d_* \) as well.

Schematically, the complex (2.1) can be depicted as

\[
\begin{array}{cccccc}
\cdots & R^0 \otimes_S \Lambda_1 & \cdots \\
R^0 \otimes_S \Lambda_2 & R^1 \otimes_S \Lambda_2 & \cdots \\
R^0 \otimes_S \Lambda_1 & R^1 \otimes_S \Lambda_1 & R^2 \otimes_S \Lambda_1 & \cdots \\
R^0 \otimes_S \Lambda_0 & R^1 \otimes_S \Lambda_0 & R^2 \otimes_S \Lambda_0 & R^3 \otimes_S \Lambda_0 & \cdots
\end{array}
\]

Note that \( R^0 = \Lambda_0 = S \). Clearly, the complex (2.1) is a resolution of \( S \) precisely because each diagonal sequence of length at least two is exact. Grouping the diagram by columns instead of rows and prefixing the inclusion \( \varepsilon: S \to \Lambda_* \), we obtain an exact sequence of \( S \)-modules

\[
0 \to S \xrightarrow{\varepsilon} \Lambda_* \xrightarrow{\partial^0} R^1 \otimes_S \Lambda_{*-1} \xrightarrow{\partial^1} R^2 \otimes_S \Lambda_{*-2} \xrightarrow{\partial^2} \cdots
\]

We call (2.4) the model complex under \( S \) of rank \( n \) and write its differential as

\[
\sum_{j=1}^{n} f_i \otimes \frac{\partial}{\partial e_i}
\]

Here \( f_i \) means multiplication by \( f_i \).

We have more structure on (2.4). Recall that an extended right \( \Lambda_* \)-comodule is one of the form \( M \otimes_S \Lambda_* \), where \( M \) is an \( S \)-module and where the coaction is given by \( 1 \otimes \Delta \). More generally, a relative injective \( \Lambda_* \)-comodule is a retract of an extended one. Now endow \( S \) with the trivial coaction and the other terms of (2.4) with the extended coaction. Then we claim that the coaugmentation \( \varepsilon \) and the differentials \( \partial^j \) are \( \Lambda_* \)-colinear. This is clear for \( \varepsilon \). For the \( \partial^j \), it is a reformulation
of equation (1.5) for the coproduct $\Delta$ on $K_s = K(f_1, \ldots, f_n)$. To see this, consider

$$
\begin{array}{c}
K_s \\
\Delta \downarrow \downarrow \\
R \otimes_S \Lambda_s \\
\downarrow d_s \downarrow \\
R \otimes_S \Lambda_{s-1} \\
\Delta \downarrow \downarrow \\
K_{s-1} \otimes_R K_s \\
\downarrow 1 \otimes \Delta' \downarrow \\
R \otimes_S \Lambda_s \otimes_S \Lambda_s \\
\downarrow d_s \otimes 1 \downarrow \\
R \otimes_S \Lambda_{s-1} \otimes_S \Lambda_s .
\end{array}
$$

By the remark after (2.2), the top, bottom and lateral faces of the cube commute. The back face commutes by equation (1.5). This forces the front face to commute, which implies that the $\partial^j$ are maps of $\Lambda_s$-comodules. The following proposition summarizes our observations.

**Proposition 6.** The model complex (2.4) is a complex of right $\Lambda_s$-comodules. It provides a relative injective resolution of the trivial $\Lambda_s$-comodule $N$.

3. **Tor of a Singular Extension of Algebras**

Let $T$ be an algebra over a ring $R$. The functor $\text{Tor}_s^R(T, -)$ maps $R$-modules to graded $T$-modules and $R$-algebras to graded $T$-algebras. If $R$ is a graded ring and $T$ a graded $R$-algebra, $\text{Tor}_s^R(T, -)$ takes graded $R$-modules to bigraded $T$-modules and graded $R$-algebras to bigraded $T$-algebras.

Let $p: A \to A'$ be a surjection of $R$-algebras. Denote the kernel of $p$ by $J$ and the inclusion $J \to A$ by $i$. Assume that the extension

$$
0 \to J \xrightarrow{i} A \xrightarrow{p} A' \to 0
$$

(3.1)

is singular, i.e. that the multiplication on $A$ restricted to $J$ is trivial. An equivalent condition is that the $A$–bimodule structure on $J$ lifts to $A'$. This action induces a bimodule structure of $\text{Tor}_s^R(T, A')$ on $\text{Tor}_s^R(T, J)$; the action maps are given by

$$
\begin{align*}
\text{Tor}_s^R(T, A') \otimes_T \text{Tor}_s^R(T, J) &\to \text{Tor}_s^R(T, A' \otimes J) \\
\text{Tor}_s^R(T, J) \otimes_T \text{Tor}_s^R(T, A') &\to \text{Tor}_s^R(T, J \otimes A')
\end{align*}
$$

where the first maps are Küneth maps and the second maps are induced by the left and the right $A'$-actions on $J$.

We need some preparation for the proof of Proposition 2. First, recall the “Fundamental Lemma” from homological algebra. Let $M$ and $N$ be $R$-modules, let $P_\ast \to M$ be a complex over $M$ with projective components and let $Y_\ast \to N$ be a resolution of $N$. Then a given map $f: M \to N$ can be lifted to a chain map $F: P_\ast \to Y_\ast$ covering $f$, and this lift is unique up to homotopy [3, Prop. V.1.1].

Now according to the Horseshoe Lemma [3, Prop. V.2.2], it is possible to choose projective resolutions $P_\ast \to J$, $Q_\ast \to A$ and $Q'_\ast \to A'$ of $J$, $A$ and $A'$ respectively which fit into a short exact sequence of chain complexes

$$
\begin{array}{c}
0 \to P_\ast \xrightarrow{i} Q_\ast \xrightarrow{\pi} Q'_\ast \to 0 \\
\downarrow \downarrow \downarrow \\
0 \to J \xrightarrow{i} A \xrightarrow{p} A' \to 0.
\end{array}
$$

A tensor product $P \otimes Q$ of two projective modules $P$ and $Q$ is projective, because $\text{Hom}_R(P \otimes Q, -) = \text{Hom}_R(P, \text{Hom}_R(Q, -))$ is exact. So the Fundamental Lemma
We may view their direct sum as an endomorphism $\delta$ to the $(3.3)$

commute in the homotopy category of chain complexes; similarly for the analogous diagrams involving the right actions.

Proof of Proposition 2. Let $\alpha = \{x'\} \in \text{Tor}^R(T, A')$ be the residue class of some $x' \in T \otimes Q'_p$. Let $x$ be a lift of $x'$ to $T \otimes \tilde{Q}_p$ and $v$ a lift of $d(x)$ to $T \otimes p_{p-1}$, where $d$ is the differential of $T \otimes Q_*$. For $\beta \in \text{Tor}^R(T, A')$, choose elements $y', y$ and $w$ in a similar way. By definition, $\partial(\alpha)$ and $\partial(\beta)$ are represented by $v$ and $w$ respectively. Because the diagram (3.2) above commutes, $\partial(\alpha \cdot \beta)$ is represented by a lift of $d(\mu(x \otimes y))$ to $T \otimes p_{p+q-1}$. As $Q_*$ is a differential graded algebra, we have

$$d(\mu(x \otimes y)) = \mu(d(x) \otimes y) + (-1)^p \mu(x \otimes d(y)).$$

From commutativity of (3.2) and (3.3), it follows that

$$\{\mu(x \otimes d(y))\} = \{\mu(x \otimes w)\} = \{\gamma_1(x' \otimes w)\} = \alpha \cdot \partial(\beta)$$

and similarly that $\{\mu(d(x) \otimes y)\} = \partial(\alpha) \cdot \beta$. Altogether we have proved

$$\partial(\alpha \cdot \beta) = \partial(\alpha) \cdot \beta + (-1)^p \alpha \cdot \partial(\beta),$$

which was the claim. \hfill \Box

4. The $I$–adic tower

Let $(r_1, \ldots, r_n)$ be a regular sequence of a ring $R$, generating an ideal $I$. Put $S = R/I$. Consider the short exact sequences of $R$–modules

$$0 \to I^{s+1}/I^{s+2} \to I^s/I^{s+2} \to I^s/I^{s+1} \to 0$$

for $s \geq 0$, where the maps are the canonical injections and projections. Associated to the $F^s$ are connecting homomorphisms

$$\delta^s : \text{Tor}^R_I(S, I^s/I^{s+1}) \to \text{Tor}^R_{i-1}(S, I^{s+1}/I^{s+2}).$$

We may view their direct sum as an endomorphism $\delta^s$ of the bigraded $S$–module

$$\text{Tor}^R_I(S, \text{gr}^s_I(R)) \cong \bigoplus_{s \geq 0} \text{Tor}^R_I(S, I^s/I^{s+1})$$
of bidegree \((-1, 1)\). Let \(p\) be the projection \(R \to S\) and \(\eta\) the composition

\[
\eta: S \cong \text{Tor}_*^R(S, R) \xrightarrow{\partial_*} \text{Tor}_*^R(S, S).
\]

Let \(W\) be a free \(S\)-module of rank \(n\) with fixed basis \(f_1, \ldots, f_n\). Proposition 5 implies that we obtain an isomorphism \(I/I^2 \cong W\) of \(S\)-modules by mapping the residue class \(\{r_j\}\) to \(-f_j\). This induces an isomorphism \(\text{Sym}_S^*(I/I^2) \cong \text{Sym}_S^*(W)\) of algebras. Precomposing it with the algebra isomorphism \(\psi\) from Corollary 4, we obtain an isomorphism of bigraded algebras

\[
\varphi: \text{gr}_1^*(R) \cong \text{Sym}_S^*(I/I^2).
\]

The reason for the minus sign will become clear soon. Combining \(\varphi\) with the isomorphism from Corollary 4, we obtain an isomorphism of bigraded algebras

\[
\psi_*: \text{Tor}_*^R(S, \text{gr}_1^*(R)) \xrightarrow{\cong} \Lambda_*^{\text{Sym}_S^*(W)}(\text{Sym}_S^*(W) \otimes U),
\]

where \(U\) is a free \(R\)-module of rank \(n\). We fix a basis \(e_1, \ldots, e_n\) of \(U\) and write \(e_j\) for the image of \(e_j\) in \(V = S \otimes U\) as well. Recall that we have defined a differential on the right side of (4.3) in (2.1).

**Proposition 7.** The endomorphism \(\delta^*\) of \(\text{Tor}_*^R(S, \text{gr}_1^*(R))\) corresponds under the isomorphism \(\psi_*\) from (4.3) to the differential \(d_*\) of the Koszul complex. In other words, the sequence of graded \(S\)-modules

\[
0 \to S \xrightarrow{\eta} \text{Tor}_1^R(S, S) \xrightarrow{\delta_0} \text{Tor}_2^R(S, I/I^2) \xrightarrow{\delta_1} \cdots
\]

is mapped under \(\psi_*\) to the model complex (2.4) under \(S\). In particular, (4.4) is exact.

**Proof.** We need to identify the maps \(\partial^*\) from (2.4) with the connecting homomorphism \(\delta^*\) from (4.2) under the isomorphism \(\psi_*\). Recall that the direct sum over the \(\partial^*\) is the differential \(d_*\) of the Koszul complex

\[
\Lambda_*^{\text{Sym}_S^*(W)}(\text{Sym}_S^*(W) \otimes V).
\]

We have constructed \(d_*\) as the unique graded derivation which maps the elements \(e_j\) to \(f_j\). So it suffices to show that the direct sum over the \(\delta^*\) has the corresponding properties. To see this, consider the singular extension of algebras

\[
0 \to \bigoplus_{s \geq 0} I^{s+1}/I^{s+2} \to \bigoplus_{s \geq 0} I^s/I^{s+2} \to \bigoplus_{s \geq 0} I^s/I^{s+1} \to 0
\]

over \(\bigoplus_{s \geq 0} I^s/I^{s+1} = \text{gr}_1^*(R)\). Its connecting homomorphism

\[
\delta^*: \text{Tor}_*^R(S, \bigoplus_{s \geq 0} I^s/I^{s+1}) \to \text{Tor}_{*+1}^R(S, \bigoplus_{s \geq 0} I^{s+1}/I^{s+2})
\]

is the direct sum of the \(\delta^*\). By Proposition 2, \(\delta^*\) is a derivation. So we are left to show that \(\delta^0\) maps the element \(e_j \in \text{Tor}_0^R(S, S)\) to \(-\{r_j\} \in I/I^2 \cong \text{Tor}_0^S(S, I/I^2)\), which corresponds to \(f_j\) under \(\psi_*\). This is the content of the lemma below. \(\square\)

The following lemma shows why we had to define \(\psi_*\) in the way we did.

**Lemma 8.** The connecting homomorphism

\[
\delta^0: \text{Tor}_*^R(S, S) \to \text{Tor}_{*+1}^R(S, I/I^2)
\]

maps the element \(e_j\) to the residue class \(-\{r_j\}\) of \(-r_i\) in \(I/I^2\).
Proof. We have free $R$–resolutions $\Lambda^R(U) \to S$ and $I/I^2 \otimes \Lambda^R(U) \to I/I^2$. By the Horseshoe Lemma, we may construct a differential $d'_s$ and an augmentation $\varepsilon'$ on $(I/I^2 \oplus R) \otimes \Lambda^R(U)$ over $R/I^2$ such that the diagram

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I/I^2 \otimes \Lambda^R(U) & \to & (I/I^2 \oplus R) \otimes \Lambda^R(U) & \to & \Lambda^R(U) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & I/I^2 & \to & R/I^2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0 \\
\end{array}
\]

is a short exact sequence of acyclic complexes. We can set $\varepsilon'((\{r_j\} \cup \{s_j\}) \cup \{\{r_j\} \cap \{s_j\}) \to R \oplus I/I^2$.

This shows that $\delta^0(e_j) = -\{r_j\}$. \hfill \square

The projections $p_{s+1}: I^{s+1} \to I^{s+1}/I^{s+2}$ and $I^s \to I^s/I^{s+2}$ induce a morphism $\mathcal{E}^s: \begin{array}{ccccccccc} 0 & & I^{s+1} & & I^s & & I^s/I^{s+1} & & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & I^{s+1}/I^{s+2} & & I^s/I^{s+2} & & I^s/I^{s+1} & & 0 \\
\end{array}$ of short exact sequences. It exhibits $\mathcal{F}^s$ as the pushout $(p_{s+1})_* (\mathcal{E}^s)$ of $\mathcal{E}^s$. So by naturality, the connecting homomorphism $\delta^s$ of $\mathcal{F}^s = (p_{s+1})_* (\mathcal{E}^s)$ factors into

$\text{Tor}^R(S, I^s/I^{s+1}) \xrightarrow{\varepsilon^s} \text{Tor}^R_{s-1}(S, I^{s+1}) (p_{s+1})_* \xrightarrow{(p_{s+1})_*} \text{Tor}^R_{s-1}(S, I^{s+1}/I^{s+2})$,  

where $\varepsilon^s$ is the connecting homomorphism associated to $\mathcal{E}^s$.

Proof of Theorem 1. Consider the diagram of graded $S$–modules

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\eta} & \text{Tor}^R_s(S, S) & \xrightarrow{\delta^0} & \text{Tor}^R_{s-1}(S, I/I^2) & \xrightarrow{\delta^1} & \text{Tor}^R_{s-2}(S, I^2/I^3) & \xrightarrow{\delta^2} & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \text{Tor}^R_{s-1}(S, I) & & \text{Tor}^R_{s-2}(S, I^2) & & \text{Tor}^R_{s-3}(S, I^3) & & \cdots \\
\end{array}
\]

We know from Proposition 7 that the row is exact. So the claim is that the diagram is isomorphic to

\[
\begin{array}{ccccccccc}
0 & \xrightarrow{\eta} & \text{Tor}^R_s(S, S) & \xrightarrow{\delta^0} & \text{Tor}^R_{s-1}(S, I/I^2) & \xrightarrow{\delta^1} & \text{Tor}^R_{s-2}(S, I^2/I^3) & \xrightarrow{\delta^2} & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
& & \text{coker} \eta \cong \ker \delta^1 & & \text{coker} \delta^0 \cong \ker \delta^2 & & \text{coker} \delta^1 \cong \ker \delta^2 & & \cdots \\
\end{array}
\]

We prove the statement by induction as follows. We know that $\eta$ is injective, hence $\text{Tor}^R_{s-1}(S, I) \cong \text{coker} \eta$. By exactness, we have $\text{coker} \eta \cong \ker \delta^1$. Hence the morphism $(p_1)_*$ is injective. It follows that $\text{Tor}^R_{s-2}(S, I^2)$ is isomorphic to $\text{coker}(p_1)_* = \text{coker}(\delta^0)$, and so on.

For the second claim, we have to show that the submodule $\ker \delta^s \cong \text{Tor}^R_s(S, I^s)$ of the free $S$–module $\text{Tor}^R_s(S, I^s/I^{s+1})$ is free. By Proposition 7, this is equivalent to the kernel of $\partial^s$ from the model complex (2.4) over $S$ being $S$–free. If $S \cong \mathbb{Z}$, this
is automatic. For general $S$, note that the model complex over $S$ can be obtained by applying $S \otimes \mathbb{Z}$ to the model complex over $\mathbb{Z}$. Therefore,

$$\ker(\partial^s): \text{Sym}^S_\mathbb{Z}(W) \otimes_S \Lambda^S_*(U) \to \text{Sym}^S_{\mathbb{Z}+1}(W) \otimes_S \Lambda^S_{*+1}(U)$$

is isomorphic to

$$S \otimes \mathbb{Z} \ker(\partial^s): \text{Sym}^S_\mathbb{Z}(W') \otimes \mathbb{Z} \Lambda^S_*(V') \to \text{Sym}^S_{\mathbb{Z}+1}(W') \otimes \mathbb{Z} \Lambda^S_{*+1}(V')),$$

where $V'$ and $W'$ are free $\mathbb{Z}$-modules of rank $n$. Therefore $\ker(\partial^s)$ is $S$-free.  

\begin{remark}
We can derive from the theorem a characterization of the tower

$$\cdots \overset{I^3}{\rightarrow} \overset{I^2}{\rightarrow} \overset{I}{\rightarrow} R \rightarrow S \rightarrow \Sigma I/I^2 \rightarrow \Sigma^2 I^2/I^3 \rightarrow \cdots,$$

in the derived category $\mathcal{D}_R$ of the ring $R$ (an arrow with a circle indicates a map of degree $-1$). We do not go into detail here as the situation is entirely analogous to the one in topology considered in [10]. What we have proved implies, in the language of injective classes (see for instance [2] or [10]), that the tower is an Adams resolution of $R$ with respect to the injective class associated to $S = R/I$. As such, it is uniquely characterized up to isomorphism by the sequence

$$0 \rightarrow R \rightarrow S \rightarrow \Sigma I/I^2 \rightarrow \Sigma^2 I^2/I^3 \rightarrow \cdots,$$

derived from (4.5) in an obvious way. As $I^s/I^{s+1}$ is a direct sum of suspended copies of $S$, we need only $S$ and its endomorphisms in $\mathcal{D}_R$ to describe the sequence.

We also obtain a characterization of the completion $R_\mathbb{Z}$ of $R$ with respect to $I$ in $\mathcal{D}_R$. In our context, it appears as the homotopy limit $\text{holim}_s R/I^s$. We find that $\text{holim}_s R/I^s$ is the completion $R_\mathbb{Z}$ of $R$ with respect to $S$ in $\mathcal{D}_R$, in the sense of Dwyer and Greenlees [5]. Namely, Prop. 6.14 in this paper states that completion with respect to $S$ is the same as Bousfield localization for the homology theory $S_\mathbb{Z}(-) = H_*(S \otimes^L -)$, where $\otimes^L$ is the derived tensor product. Now all the $R/I^s$ are $S_\mathbb{Z}$-local, and Theorem 1 implies that $R \to \text{holim}_s R/I^s$ is an $S_\mathbb{Z}$-equivalence.

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