The distribution of second degrees in the Bollobás–Riordan random graph model

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1 Introduction

In this paper we consider some properties of random graphs. The standard random graph model $G(n,m)$ was introduced by Erdős and Rényi in [7]. In this model we randomly choose one graph from all graphs with $n$ vertices and $m$ edges. The similar model $G(n,p)$ was suggested by Gilbert in [9]. Here $n$ vertices are joined independently with probability $0 < p < 1$. Many papers deal with the classical models. Main results can be found in [3], [8], [11].

Recently there has been interest in modeling complex real-world networks. Real structures differ from standard random graphs. One of the main characteristics of random graphs is their degree sequence. In many real-world structures the degree sequence has a power law distribution. Standard random graph models do not have this property. So Barabási and Albert suggested a new model in [2]. Then Bollobás and Riordan gave more precise definition of this model. Many models of real-world networks and main results can be found in [4].

This paper deals with the Bollobás–Riordan model. Now let us describe this model. Let $n$ be a number of vertices in our graph and $m$ be a fixed parameter. We begin with the case $m = 1$. We inductively construct a random graph $G_{n}^{1}$. Start with $G_{1}^{1}$ the graph with one vertex and one loop. Similarly we can start with $G_{1}^{0}$ the graph with no vertices. Assume that we already constructed the graph $G_{t-1}^{1}$. At the next step we add one vertex $t$ and one edge between vertices $t$ and $i$, where $i$ is chosen randomly with

$$P(i = s) = \begin{cases} \frac{d_{G_{t-1}^{1}}(s)}{(2t-1)} & \text{if } 1 \leq s \leq t - 1, \\ \frac{1}{(2t-1)} & \text{if } s = t. \end{cases}$$

Here $d_{G_{t}^{1}}(s)$ is the degree of the vertex $s$ in $G_{t}^{1}$. By $d(s)$ denote the degree of $s$ in the graph $G_{n}^{1}$. In other words, the probability that a new vertex will be connected to the vertex $i$ is proportional to the current degree of $i$. Therefore this process is said to be preferential attachment. To obtain $G_{m}^{n}$ with $m > 1$ we construct $G_{1}^{mn}$. Then we identify the vertices $1, \ldots, m$ to form the first vertex; we identify the vertices $m + 1, \ldots, 2m$ to form the second vertex; and so on. After this procedure, edges from $G_{1}^{n}$ connect “big” vertices in $G_{m}^{n}$. Let $\mathfrak{G}_{m}^{n}$ be the probability space of constructed graphs.

Many papers deal with the Bollobás–Riordan model. The diameter of this random graph was considered in [6]. In [5] Bollobás and Riordan proved that the degree sequence has a power law distribution.

**Theorem 1** If $m \geq 1$ is fixed, then there exists a function $\varphi(n) = o(n)$ such that for any $m \leq d \leq n^{1/15}$ we have

$$\lim_{n \to \infty} P\left( \left| \#_{m}^{n}(d) - \frac{2nm(m+1)}{d(d+1)(d+2)} \right| > \frac{\varphi(n)}{d(d+1)(d+2)} \right) = 0.$$ 

Here $\#_{m}^{n}(d)$ is the number of vertices in $G_{m}^{n}$ with degree equal to $d$. 

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Recently Grechnikov substantially improved Theorem 1 (see [10]).

In this paper we consider second degrees of vertices in $G_m^n$. We estimate the expectation of the number of vertices with second degree equal to $d$. Also we prove a concentration result. This paper is organized as follows. In section 2 we give main definitions and results. In section 3 we prove all theorems.

## 2 Definitions and results

In this paper we study the random graph $G_1^n$. When we write $ij \in G_1^n$ we mean that $G_1^n$ has the edge $ij$; when we write $t \in G_1^n$ we simply mean that $t$ is a vertex of $G_1^n$. Given a vertex $t \in G_1^n$ we say that the second degree of the vertex $t$ is

$$d_2(t) = \# \{ ij : i \neq t, j \neq t, it \in G_1^n, ij \in G_1^n \}.\]$$

In other words, the second degree of $t$ is the number of edges adjacent to the neighbors of $t$ except for the edges adjacent to the vertex $t$.

Let $M_1^n(d)$ be the expectation of the number of vertices with degree $d$ in $G_1^n$:

$$M_1^n(d) = M(\# \{ t \in G_1^n : d_{G_1^n}(t) = d \}).$$

By $X_n(d)$ denote the number of vertices with second degree $d$ in $G_1^n$. By definition, put $M_2^n(d) = M X_n(d)$.

The aim of this paper is to prove the following results.

### Theorem 2
For any $k > 1$ we have

$$M_2^n(k) = \frac{4n}{k^2} \left( 1 + O \left( \frac{\ln^2 k}{k} \right) + O \left( \frac{k^2}{n} \right) \right).$$

### Theorem 3
For any $\varepsilon > 0$ there exists a function $\varphi(n) = o(n)$ such that

$$\lim_{n \to \infty} P \left( \left| X_n(k) - M_2^n(k) \right| \geq \frac{\varphi(n)}{k^2} \right) = 0$$

for any $1 \leq k \leq n^{1/6-\varepsilon}$.

This is a concentration result which means that the distribution of second degrees does also obey (asymptotically) a power law.

To prove Theorem 2, we need the following definition. Let $N_n(l, k)$ be the number of vertices in $G_1^n$ with degree $l$, with second degree $k$, and without loops:

$$N_n(l, k) = \# \{ t \in G_1^n : d(t) = l, d_2(t) = k, tt \notin G_1^n \}.\]$$

We shall prove the following theorem.

### Theorem 4
In $G_1^n$ we have

$$MN_n(l, k) = n c(l, k) (1 + \theta(n, l, k)),\]$$

where $|\theta(n, l, k)| < (2l + k - 1)^2/n$. The constants $c(l, k)$ are defined as follows:

$$c(l, 0) = c(0, k) = 0,\]$$

$$c(1, k) = \frac{2k^2 + 14k}{(k + 1)(k + 2)(k + 3)(k + 4)},\]$$

$$c(l, k) = c(l - 1) \frac{l + k - 1}{2l + k + 2} + c(l - 1, k) \frac{l - 1}{2l + k + 2}, \quad k > 0, l > 1.$$

We shall use the following lemmas to prove these theorems.
Lemma 1 Let $d \geq 1$ be natural; then
\[ M_n^k(d) = \frac{4n}{d(d+1)(d+2)} \left( 1 + \tilde{\theta}(n, d) \right), \]
where $|\tilde{\theta}(n, d)| < d^2/n$.

Denote by $P_n(l, k)$ the number of vertices in $G_1^n$ with a loop, with degree $l$, and with second degree $k$.

Lemma 2 For any $n$ we have
\[ MP_n(l, k) \leq p(l, k), \]
where
\[
\begin{align*}
p(2, 0) &= 1, \\
p(l, k) &= p(l, k-1) \frac{l+k-3}{2l+k-2} + p(l-1, k) \frac{l-1}{2l+k-2}, \quad l \geq 3, k \geq 0.
\end{align*}
\]
For the other values of $l$ and $k$ we have $p(l, k) = 0$.

The next section is organized as follows. First we prove Theorem 4 and Theorem 2; then we prove the lemmas. Finally we give a proof of Theorem 3.

3 Proofs

3.1 Proof of Theorem 4

From the definition of $G_1^n$ it follows that $N_n(l, 0) = N_n(0, k) = 0$. Indeed, since we have no vertices of degree 0, we see that $N_n(0, k) = 0$. Since vertices with loops are not counted in $N_n(l, k)$, it follows that we have no vertices of second degree 0 and $N_n(l, 0) = 0$. Therefore we have $MN_n(l, 0) = MN_n(0, k) = 0$.

Let us prove that $MN_n(1, k) = n c(1, k) (1 + \theta(n, 1, k))$. The proof is by induction on $k$. For $k = 0$ there is nothing to prove. Now assume that for $j < k$ we have
\[ MN_n(1, j) = n c(1, j) (1 + \theta(n, 1, j)), \]
where
\[
|\theta(n, 1, j)| < (j+1)^2/n, \quad c(1, j) = \frac{2j^2 + 14j}{(j+1)(j+2)(j+3)(j+4)}.
\]

Denote by $N_i(l)$ the number of vertices with degree $l$ in $G_1^n$.

We need some additional notation. Let $X$ be a function on $n$ (the number of vertices), $l$ (the first degree we are interested in), $k$ (the second degree we are interested in); then denote by $\theta_1(X)$, $\theta_2(X)$, $\theta_3(X)$ ... some functions on $n$, $l$, $k$ such that $|\theta_i(X)| < X$.

Obviously, $MN_i(1, k) = 0$. For $i \geq 1$ we have
\[
\begin{align*}
MN_{i+1}(1, k) | N_i(1, k), N_i(1, k-1), N_i(k) &= N_i(1, k) \left( 1 - \frac{k+2}{2i+1} \right) + \frac{kN_i(1, k-1)}{2i+1} + \frac{kN_i(k)}{2i+1}. \quad (1)
\end{align*}
\]

Let us explain this equality. Suppose we have $G_1^n$. We add one vertex and one edge. There are $N_i(1, k)$ vertices with degree 1 and with second degree $k$ in $G_1^n$. The probability that we “spoil” one of these vertices is $(k+2)/(2i+1)$. Also we have $N_i(1, k-1)$ vertices with degree 1 and with second degree $k-1$. The probability that one of these vertices has degree 1 and second degree $k$ in $G_1^{i+1}$ is $k/(2i+1)$. Finally, with probability equal to $kN_i(k)/(2i+1)$ the vertex $i+1$ has necessary degrees in $G_1^{i+1}$.
Using (1), Lemma 1, and inductive assumption we get

\[
MN_{i+1}(1, k) = MN_i(1, k) \frac{2i - k - 1}{2i + 1} + \frac{kMN_i(1, k - 1)}{2i + 1} + \frac{kM_1(k)}{2i + 1} =
\]

\[
= MN_i(1, k) \frac{2i - k - 1}{2i + 1} + \left( \frac{ikc(1, k - 1)}{2i + 1} + \frac{4i}{(2i + 1)(k + 1)(k + 2)} \right) (1 + \theta_1 \left( k^2 / i \right)) .
\]

Let us introduce some notation:

\[
a_i = \frac{2i - k - 1}{2i + 1},
\]

\[
b_i = \frac{2i}{2i + 1} \left( 1 + \theta_1 \left( k^2 / i \right) \right),
\]

\[
m = \frac{c(1, k - 1)k}{2} + \frac{2}{(k + 1)(k + 2)} .
\]

Using this notation, we have

\[
MN_{i+1}(1, k) = MN_i(1, k) a_i + m b_i .
\]

Let us prove the following equality by induction on \( n \):

\[
MN_n(1, k) = \frac{2mn}{k + 4} (1 + \theta(n, 1, k)) .
\]

For \( n = 1 \) we have \( MN_1(1, k) = 0 \). Since we have the condition \( |\theta(1, 1, k)| < (k + 1)^2 \), we can take \( \theta(1, 1, k) = -1 \).

Now put \( t = k + 1 \). This is needed for the sequel. Assume that

\[
MN_i(1, k) = \frac{2mi}{t + 3} (1 + \theta(i, 1, t - 1)) .
\]

Then

\[
MN_{i+1}(1, k) = MN_i(1, k) a_i + m b_i =
\]

\[
= \frac{2mi(2i - t)}{(2i + 1)(t + 3)} (1 + \theta_2(2i^2 / i)) + \frac{2mi}{2i + 1} (1 + \theta_1 ((t - 1)^2 / i)) =
\]

\[
= \frac{2m}{t + 3} \left( i + 1 - \frac{1}{2i + 1} + \theta_3 \left( \frac{(2i - t)t^2}{2i + 1} \right) + \theta_4 \left( \frac{(t - 1)^2(t + 3)}{2i + 1} \right) \right) .
\]

If \( t \geq 1 \) and \( 2i - t \geq 0 \), then

\[
\frac{1}{2i + 1} + \frac{t^2|2i - t|}{2i + 1} + \frac{(t - 1)^2(t + 3)}{2i + 1} < t^2 .
\]

Therefore,

\[
MN_{i+1}(1, k) = \frac{2m(i + 1)}{t + 3} \left( 1 + \theta_5 \left( i^2 / (i + 1) \right) \right) .
\]

In this case, we can put \( \theta(i + 1, 1, k) = \theta_5 \left( i^2 / (i + 1) \right) \).

If \( t \geq 1 \) and \( 2i - t \leq -2 \), then we do not have enough edges in \( G_i \) and \( MN_{i+1}(l, k) = 0 \). In this case, we can put \( \theta(i + 1, 1, k) = -1 \).

We consider the case \( 2i - t = -1 \) later.

We get

\[
MN_n(1, k) = \frac{2mn}{k + 4} (1 + \theta(n, 1, k)) .
\]

Note that

\[
\frac{2m}{k + 4} = \frac{4}{(k + 1)(k + 2)(k + 4)} + \frac{2c(1, k - 1)k}{2k + 4} =
\]
This completes the proof for $M_{N_n}(1,k)$.

Consider the case $l,k > 1$. Assume that for all $i < l,j < k$ we have $M_{N_n}(i,j) = nc(i,j)(1 + \theta(n,i,j))$. Put $t = 2l+k-1$. Obviously, $M_{N_1}(l,k) = 0$. For $i \geq 1$ we have

$$M_{N_{i+1}}(l,k) = M_{N_i}(l,k) \left( 1 - \frac{2l+k}{2i+1} \right) + \frac{(l-1) M_{N_i}(l-1,k)}{2i+1} + \frac{(l+k-1) M_{N_i}(l,k-1)}{2i+1} =$$

$$= M_{N_i}(l,k) \frac{2i-t}{2i+1} + \left( \frac{(l-1)c(l-1,k)}{2i+1} + \frac{(l+k-1)c(l,k-1)}{2i+1} \right) (1 + \theta_1 \left( (t-1)^2/i \right)).$$

Introduce some notation:

$$a_i = \frac{2i-t}{2i+1},$$

$$b_i = \frac{2i}{2i+1} (1 + \theta_1 \left( (t-1)^2/i \right)), $$

$$m = \frac{(l-1)c(l-1,k)}{2} + \frac{(l+k-1)c(l,k-1)}{2}.$$  

We have

$$M_{N_{i+1}}(l,k) = M_{N_i}(l,k) a_i + m b_i.$$  

It remains to prove the following statement by induction on $n$:

$$M_{N_n}(l,k) = \frac{2mn}{t+3} \left( 1 + \theta_5 \left( t^2/n \right) \right) = \frac{2mn}{t+3} \left( 1 + \theta(n,l,k) \right).$$

The proof is the same as in the case of $l = 1$. In this case we have

$$\frac{2m}{t+3} = \frac{(l-1)c(l-1,k)}{2l+k+2} + \frac{(l+k-1)c(l,k-1)}{2l+k+2} = c(l,k).$$

Now we need to consider only the case $2i-t = -1$. We need to show that $M_{N_{i+1}}(l,k) = (i+1)c(l,k)(1 + \theta(i+1,l,k))$. We have $2(i+1) = 2l+k$. In our graph $G^{i+1}_l$ we have $i+1$ edges. Therefore the sum of all degrees is equal to $2l+k$. Suppose we have at least one vertex with degree $l$ and second degree $k$. We do not count vertices with a loop in $N_{i+1}(l,k)$. Consequently $l$ edges go out from this vertex. And there are $k/2$ edges between the neighbors of our vertex. And we have no other edges. Hence our vertex is joined to all other vertices in $G^{i+1}_l$. So $l = i$. Thus $k = 2$. It follows that we consider the vertex 2. And there is one edge from the vertex 2 to the vertex 1; also edges from the vertices 3, . . . , $i+1$ go to the vertex 2. So, there is only one graph with $N_{i+1}(l,k) \neq 0$. This graph has only one vertex with degree $l$ and second degree $k$. Therefore the probability of this graph is equal to $M_{N_{i+1}}(l,2)$. We have $M_{N_{i+1}}(l,2) = \frac{(2l-1)!}{(2l+1)}!!$.

Recall that $l = i$ and $k = 2$. Now we must only prove that

$$M_{N_{i+1}}(l,2) = (i+1)c(l,2)(1 + \theta(l+1,l,2)).$$

Let us prove the inequality

$$c(l,2) \geq \frac{24(2l-1)!}{5(2l+4))!!}.$$ 

It follows from the definition of $c(l,k)$ that

$$c(1,2) = \frac{1}{10},$$

$$c(l,2) \geq c(l-1,2) \frac{l-1}{2l+4}, \ l \geq 2.$$  

Obviously, $\theta(l+1,l,2) \geq -1$. Let us obtain the following upper bound:

$$\theta(l+1,l,2) + 1 = \frac{M_{N_{i+1}}(l,2)}{(l+1)c(l,k)} \leq \frac{2(2l-1)!5(2l+4))!!}{(2l+1)!!(l+1)24(2l-1)!} = \frac{5(2l+4))!!}{12(2l+1)!!(l+1)} \leq \frac{(2l+1)^2}{(l+1)}.$$ 

This completes the proof.
3.2 Proof of Theorem 2

From Theorem 4 we have the constants \(c(l, k)\). Imagine that we have a table with \(c(l, k)\), where \(l\) is the number of a row and \(k\) is the number of a column. The sum of all numbers in the table is equal to 1. The sum of numbers in \(l\)-th row is equal to \(\frac{4}{n(l+1)(l+2)}\). It can easily be checked using the definition of \(c(l, k)\). But we need to calculate \(M_n^2(k)\), so we are interested in the sum of all numbers in \(k\)-th column. More precisely,

\[
M_n^2(k) = \sum_{l=1}^{\infty} M N_n(l, k) + \sum_{l=1}^{\infty} M P_n(l, k).
\]

First we estimate \(\sum_{l=1}^{\infty} c(l, k)\). Recall that

\[
c(l, 0) = 0, \quad c(1, k) = \frac{2k^2 + 14k}{(k + 1)(k + 2)(k + 3)(k + 4)},
\]

\[
c(l, k) = c(l, k - 1) \frac{l + k - 1}{2l + k + 2} + c(l - 1, k) \frac{l - 1}{2l + k + 2}, \quad k > 0, l > 1.
\]

Note that there exists a function \(C(k) \geq 0\) such that for all \(l \geq k \geq 0\) and \(l \geq 1\) the inequality

\[
c(l, k) \leq C(k) 2^{-l} \frac{(l - 1)!}{(l - k)!}
\]

holds. Indeed, the case of \(k = 0\) is obvious with \(C(k) = 0\). In the case of \(k \geq 1\) we define \(C(k)\) so that \(C(k) \geq C(k - 1)\) and (2) holds for \(l = k\). We have

\[
(2l + k + 2) \frac{c(l, k)}{C(k)} \leq \frac{C(k - 1)}{C(k)} (l + k - 1) 2^{-l} \frac{(l - 1)!}{(l - k + 1)!} + 2^{-l+1} \frac{(l - 1)!}{(l - k)!} \leq 2^{-l} \frac{(l - 1)!}{(l - k)!} \left( \frac{l + k - 1}{l - k + 1} + 2(l - k) \right) \leq (2l + k + 2) 2^{-l} \frac{(l - 1)!}{(l - k)!}.
\]

This proves (2).

In particular, the series \(\sum_{l=1}^{\infty} l N c(l, k)\) converges for all \(N\) and \(k\).

Let us make some transformations:

\[
(2l + k + 2) c(l, k) = (l + k - 1) c(l, k - 1) + (l - 1) c(l - 1, k),
\]

\[
\sum_{l=2}^{\infty} (2l + k + 2) c(l, k) = \sum_{l=2}^{\infty} (l + k - 1) c(l, k - 1) + \sum_{l=1}^{\infty} l c(l, k),
\]

\[
\sum_{l=2}^{\infty} (l + k + 2) c(l, k) = \sum_{l=2}^{\infty} (l + k - 1) c(l, k - 1) + c(1, k).
\]

Put \(x_k = \sum_{l=2}^{\infty} c(l, k)\). Then \(x_0 = 0\) and for \(k \geq 1\) we have

\[
(k + 2) x_k = (k - 1) x_{k-1} + c(1, k) + \sum_{l=2}^{\infty} l (c(l, k - 1) - c(l, k)),
\]

\[
(k + 2)(k + 1) k x_k = (k - 1)(k + 1) k x_{k-1} + (k + 1) k c(1, k) + \sum_{l=2}^{\infty} l (k(k + 1) c(l, k - 1) - k(k + 1) c(l, k)),
\]

\[
(k + 2)(k + 1) k x_k = \sum_{s=1}^{k} (s(s + 1)(s + 2) x_s - (s - 1) s(s + 1) x_{s-1}) =
\]
\[ \sum_{s=1}^{k} s(s+1)c(1, s) + \sum_{l=2}^{\infty} l \left( \sum_{s=1}^{k} (s(s+1)c(l, s-1) - s(s+1)c(l, s)) \right) = \]
\[ = \sum_{s=1}^{k} s(s+1)c(1, s) + \sum_{l=2}^{\infty} l \left( \sum_{s=1}^{k} ((s+1)(s+2) - s(s+1))c(l, s) - (k+1)(k+2)c(l, k) \right) = \]
\[ = \sum_{s=1}^{k} s(s+1)c(1, s) + \sum_{l=2}^{\infty} l \left( \sum_{s=1}^{k} 2(s+1)c(l, s) - (k+1)(k+2)c(l, k) \right), \]
\[ x_k = \frac{1}{k(k+1)(k+2)} \sum_{s=1}^{k} s(s+1)c(1, s) + \frac{2}{k(k+1)(k+2)} \sum_{l=2}^{\infty} l \left( \sum_{s=1}^{k} (s+1)c(l, s) \right) - \frac{1}{k} \sum_{l=2}^{\infty} lc(l, k). \quad (3) \]

Put \( y_k = \sum_{l=2}^{\infty} lc(l, k) \). Then
\[ x_k = \frac{1}{k(k+1)(k+2)} \sum_{s=1}^{k} s(s+1)c(1, s) + \frac{2}{k(k+1)(k+2)} \sum_{s=1}^{k} (s+1)y_s - \frac{1}{k} y_k. \]

Make some transformations:
\[ (2l + k + 2)lc(l, k) = (l + k - 1)lc(l, k - 1) + l(l-1)c(l - 1, k), \]
\[ \sum_{l=2}^{\infty} (2l + k + 2)lc(l, k) = \sum_{l=2}^{\infty} (l + k - 1)lc(l, k - 1) + \sum_{l=1}^{\infty} l(l+1)c(l, k), \]
\[ \sum_{l=2}^{\infty} (l + k + 1)lc(l, k) = \sum_{l=2}^{\infty} (l + k - 1)lc(l, k - 1) + 2c(1, k), \]
\[ ky_k + \sum_{l=2}^{\infty} (l + 1)lc(l, k) = (k - 2)y_{k-1} + \sum_{l=2}^{\infty} l(l+1)c(l, k - 1) + 2c(1, k), \]
\[ k(k-1)y_k = \sum_{s=1}^{k} (s(s-1)y_s - (s-1)(s-2)y_{s-1}) = \]
\[ = \sum_{s=1}^{k} \left( (s-1) \sum_{l=2}^{\infty} l(l+1)c(l, s-1) - (s-1) \sum_{l=2}^{\infty} (l+1)lc(l, s) + 2(s-1)c(1, s) \right) = \]
\[ = 2 \sum_{s=1}^{k} (s-1)c(1, s) + \sum_{l=2}^{\infty} l(l+1) \sum_{s=1}^{k} c(l, s) - k \sum_{l=2}^{\infty} l(l+1)c(l, k). \]

For \( k \geq 2 \) we have
\[ y_k = \frac{2}{k(k-1)} \sum_{s=1}^{k} (s-1)c(1, s) + \frac{1}{k(k-1)} \sum_{l=2}^{\infty} l(l+1) \sum_{s=1}^{k} c(l, s) - \frac{1}{k-1} \sum_{l=2}^{\infty} l(l+1)c(l, k). \]

Let \( z_k = \sum_{l=2}^{\infty} l(l+1)c(l, k) \). Then for \( k \geq 2 \)
\[ y_k = \frac{2}{k(k-1)} \sum_{s=1}^{k} (s-1)c(1, s) + \frac{1}{k(k-1)} \sum_{s=1}^{k} z_s - \frac{1}{k-1} z_k. \]

Make similar transformations
\[ (2l + k + 2)l(l+1)c(l, k) = (l + k - 1)l(l+1)c(l, k - 1) + (l+1)l(l-1)c(l - 1, k), \]
Since \( c(1,s) = O \left( \frac{1}{s^2} \right) \), we have

\[
0 \leq z_k \leq \frac{1}{k} \sum_{l=2}^{\infty} (l + k) l(l + 1) c(l, k) = O \left( \frac{1}{k} \sum_{s=1}^{k} \frac{1}{s^2} \right) = O \left( \frac{1}{k} \right),
\]

\[
\sum_{s=1}^{k} (s - 1) c(1, s) = O \left( \sum_{s=1}^{k} \frac{1}{s} \right) = O(\ln k),
\]

\[
y_k = O \left( \frac{\ln k}{k^2} \right),
\]

\[
\sum_{s=1}^{k} (s + 1) y_s = O \left( \sum_{s=1}^{k} \frac{\ln s}{s} \right) = O(\ln^2 k),
\]

\[
x_k = \frac{1}{k(k + 1)(k + 2)} \sum_{s=1}^{k} s(s + 1) c(1, s) + O \left( \frac{\ln^2 k}{k^3} \right).
\]

Finally, \( c(1, s) = \frac{2}{s(s+1)} + O \left( \frac{1}{s^3} \right), \) so \( \sum_{s=1}^{k} s(s + 1) c(1, s) = 2k + O(\ln k) \) and

\[
x_k = \frac{2}{(k + 1)(k + 2)} + O \left( \frac{\ln^2 k}{k^3} \right) = \frac{2}{k^2} + O \left( \frac{\ln^2 k}{k^3} \right),
\]

\[
\sum_{l=1}^{\infty} c(l, k) = c(1, k) + x_k = \frac{4}{k^2} + O \left( \frac{\ln^2 k}{k^3} \right).
\]

Now we can estimate \( M_n^2(k) \):

\[
M_n^2(k) = \sum_{l=1}^{\infty} c(l, k) n(1 + \theta(n, l, k)) + \sum_{l=1}^{\infty} MP_n(l, k).
\]

The first sum:

\[
\sum_{l=1}^{\infty} c(l, k) n = \frac{4n}{k^2} + O \left( \frac{n \ln^2 k}{k^3} \right).
\]

The second sum:

\[
\sum_{l=1}^{\infty} c(l, k) n |\theta(n, l, k)| \leq \sum_{l=1}^{\infty} c(l, k) (2l + k)^2 = \sum_{l=1}^{\infty} 4l^2 c(l, k) + \sum_{l=1}^{\infty} 4lkc(l, k) + \sum_{l=1}^{\infty} k^2 c(l, k) =
\]

\[
= 4c(1, k) + \sum_{l=2}^{\infty} 4l(l + 1)c(l, k) - \sum_{l=2}^{\infty} 4l c(l, k) + 4kc(1, k) + \sum_{l=2}^{\infty} 4lkc(l, k) + \sum_{l=1}^{\infty} k^2 c(l, k) =
\]
We are interested in $M$ they only looked at $\tilde{\theta}$. In [5] Bollobás and Riordan computed the expectation of the number of vertices with degree $d$. But they only looked at $d \leq n^{1/15}$ and proved that

$$\text{MM}_n^1(d) \sim \frac{4n}{d(d+1)(d+2)}.$$ 

We are interested in $\text{MM}_n^1(d)$ for any $d$. In addition, we want to estimate $|\tilde{\theta}(n, d)|$. Therefore we compute $M_i^1(d)$ in this paper.

The proof is by induction on $d$. First we need to consider 2 cases: $d = 1$ and $d = 2$.

Consider the case $d = 1$. Obviously, $M_0^1(1) = 0$. Since $M_i^1(1) = \frac{2i}{3} \left( 1 + \tilde{\theta}(i, 1) \right)$, then

$$M_{i+1}^1(1) = M_i^1(1) \left( 1 - \frac{1}{2i+1} \right) + \frac{2i}{2i+1} = \frac{2i}{3} \left( 1 + \tilde{\theta}(i, 1) \right) \frac{2i}{2i+1} + \frac{2i}{2i+1} =$$

$$= \frac{2}{3} \left( i + 1 - \frac{1}{2i+1} + \frac{2i^2}{2i+1} \tilde{\theta}(i, 1) \right) = \frac{2(i+1)}{3} \left( 1 - \frac{1}{(2i+1)(i+1)} + \frac{2i^2}{(2i+1)(i+1)} \tilde{\theta}(i, 1) \right).$$

Put $\tilde{\theta}(i+1, 1) = \frac{2i}{(2i+1)(i+1)} \tilde{\theta}(i, 1) - \frac{1}{(2i+1)(i+1)}$. Note that

$$|\tilde{\theta}(i+1, 1)| \leq \frac{2i}{(2i+1)(i+1)} + \frac{1}{(2i+1)(i+1)} \leq 1/(i+1).$$

This completes the proof for $d = 1$.

The case $d = 2$ is somewhat different. Obviously, $M_0^1(2) = 0$. Suppose $M_i^1(2) = \frac{i}{6} \left( 1 + \tilde{\theta}(i, 2) \right)$. Then

$$M_{i+1}^1(2) = M_i^1(2) \left( 1 - \frac{2}{2i+1} \right) + M_i^1(1) \frac{1}{2i+1} + \frac{1}{2i+1} =$$
3.4 Proof of Lemma 2

Obviously, \( MP_n(0, k) = MP_n(1, k) = 0 \). For all \( k > 0 \) we have \( MP_n(2, k) = 0 \). For \( k = 0 \) we have

\[
MP_n(2, 0) = \sum_{i=1}^{n} \frac{1}{2i - 1} \prod_{j=i+1}^{n} \frac{2j - 3}{2j - 1} = \sum_{i=1}^{n} \frac{1}{2n - 1} = \frac{n}{2n - 1} \leq 1.
\]

The rest of the proof is by induction. Consider \( l \geq 3, k \geq 0 \). Suppose that for \( i < l \) and \( j < k \) we have \( MP_n(i, j) \leq p(i, j) \).
Trivially, \( P_1(l, k) = 0 \). It is easily shown that \( MP_{i+1}(l, k) = 0 \) if \( 2i + 4 < 2l + k \).

If \( 2i + 4 = 2l + k \) and \( P_{i+1}(l, k) \neq 0 \), then \( l = i + 2 \) and \( k = 0 \). And we have only one graph with \( P_{i+1}(l, k) \neq 0 \). Arguing as in the end of Section 3.1, we see that the probability of this graph is \( \frac{(l - 1)!}{(2l - 1)!!} \).

From the recurrent relation we have \( p \geq \frac{1}{2^{l-2}} \). In our case we get

\[
MP_{i+1}(l, k) = \frac{(l - 1)!}{(2l - 1)!!} < \frac{1}{2^{l-2}} = p(l, 0).
\]

If \( 2i + 3 \geq 2l + k \), then

\[
MP_{i+1}(l, k) = MP_i(l, k) \left( 1 - \frac{2l + k - 2}{2i + 1} \right) + MP_i(l, k - 1) \frac{l + k - 3}{2i + 1} + MP_i(l - 1, k) \frac{l - 1}{2i + 1}.
\]

Using the recurrent relation for \( p(l, k) \) and induction on \( i \) it is easy to prove that \( MP_n(l, k) \leq p(l, k) \). This concludes the proof of Lemma 2.

### 3.5 Proof of Theorem 3

This proof is similar to the proof given in [5]. But our case is more complicated. We need the Azuma–Hoeffding inequality (see [1]):

**Lemma 3** Let \( (X_i)_{i=0}^n \) be a martingale such that \( |X_i - X_{i-1}| \leq c \) for any \( 1 \leq i \leq n \). Then

\[
P(|X_n - X_0| \geq x) \leq 2e^{-\frac{x^2}{2cn}}
\]

for any \( x > 0 \).

Suppose we are given an \( \varepsilon > 0 \). Fix \( n \geq 3 \) and \( k: 1 \leq k \leq n^{1/6-\varepsilon} \). Consider the random variables \( X^i(k) = M(X_n(k)|G^i_1), i = 0, \ldots, n \). Let us explain the notation \( M(X_n(k)|G^i_1) \). We construct the graph \( G^n_1 \in \mathcal{G}_1^n \) by induction. For any \( t \leq n \) there exists a unique \( G^i_t \in \mathcal{G}_t^i \) such that \( G_t^n \) is obtained from \( G^i_t \). So \( M(X_n(k)|G^i_1) \) is the expectation of the number of vertices with second degree \( k \) in \( G_t^n \) if at the step \( t \) we have the graph \( G^i_t \).

Note that \( X^0(k) = MX_n(k) \) and \( X^n(k) = X_n(k) \). From the definition of \( G^n_1 \) it follows that \( X^i(k) \) is a martingale.

We will prove below that for any \( i = 1, \ldots, n \)

\[
|X^i(k) - X^{i-1}(k)| \leq 10k \ln n.
\]

Theorem 3 follows from this statement immediately. Put \( c = 10k \ln n \). Then from Azuma–Hoeffding inequality it follows that

\[
P\left(|X_n(k) - MX_n(k)| \geq k \sqrt{\ln n} \right) \leq 2\exp \left\{ -\frac{n k^2 \ln^2 n}{200 n k^2 \ln^2 n} \right\} = o(1).
\]

If \( k \leq n^{1/6-\varepsilon} \), then the value of \( n/k^2 \) is considerably greater than \( k \ln^2 n \sqrt{n} \). This means that we have

\[
(k \sqrt{n \ln^2 n}) / (n/k^2) = o(1).
\]

It remains to estimate the quantity \( |X^i(k) - X^{i-1}(k)| \). The proof is by a direct calculation. Fix \( 1 \leq i \leq n \) and some graph \( G^{i-1}_1 \). Note that

\[
|\mathbb{E} \left(X_n(k)|G^i_1\right) - \mathbb{E} \left(X_n(k)|G^{i-1}_1\right)| \leq \max_{\tilde{G}^i_1 \in \mathcal{G}^{i-1}_1} \left\{ \mathbb{E} \left(X_n(k)|\tilde{G}^i_1\right) \right\} - \min_{\tilde{G}^i_1 \in \mathcal{G}^{i-1}_1} \left\{ \mathbb{E} \left(X_n(k)|\tilde{G}^i_1\right) \right\}.
\]

Put \( \hat{G}^i_1 = \arg\max_{\tilde{G}^i_1} \mathbb{E} \left(X_n(k)|\tilde{G}^i_1\right), \tilde{G}^i_1 = \arg\min_{\tilde{G}^i_1} \mathbb{E} \left(X_n(k)|\tilde{G}^i_1\right) \). We need to estimate the difference \( \mathbb{E} \left(X_n(k)|\hat{G}^i_1\right) - \mathbb{E} \left(X_n(k)|\tilde{G}^i_1\right) \).
Using the notation $N_t(l, k)$ and $P_t(l, k)$ from Section 2, we get

\[
M(X_n(k)|\hat{G}_t^i) = \sum_{l=1}^{\infty} M(N_t(l, k)|\hat{G}_t^i) + \sum_{l=1}^{\infty} M(P_t(l, k)|\hat{G}_t^i),
\]

\[
M(X_n(k)|\bar{G}_t^i) = \sum_{l=1}^{\infty} M(N_t(l, k)|\bar{G}_t^i) + \sum_{l=1}^{\infty} M(P_t(l, k)|\bar{G}_t^i).
\]

For $i \leq t \leq n$ put

\[
\delta_t(l, k) = M(N_t(l, k)|\hat{G}_t^i) - M(N_t(l, k)|\bar{G}_t^i), \quad \delta'_t(l, k) = \delta_t(l, k)I(\delta_t(l, k) > 0),
\]

\[
\epsilon_t(l, k) = M(P_t(l, k)|\hat{G}_t^i) - M(P_t(l, k)|\bar{G}_t^i), \quad \epsilon'_t(l, k) = \epsilon_t(l, k)I(\epsilon_t(l, k) > 0),
\]

\[
\delta_t(k) = M(N_t(k)|\hat{G}_t^i) - M(N_t(k)|\bar{G}_t^i), \quad \delta'_t(k) = \delta_t(k)I(\delta_t(k) > 0).
\]

Note that

\[
M(X_n(k)|\hat{G}_t^i) - M(X_n(k)|\bar{G}_t^i) = \sum_{l=1}^{\infty} \delta_n(l, k) + \sum_{l=1}^{\infty} \epsilon_n(l, k) \leq \sum_{l=1}^{\infty} \delta'_n(l, k) + \sum_{l=1}^{\infty} \epsilon'_n(l, k) = \sum_{l=1}^{\infty} \sum_{j=0}^{k} (\delta'_n(l, j) + \epsilon'_n(l, j)).
\]

Let us estimate this double sum.

First suppose that $n = i$. Fix $G_1^{i-1}$. Graphs $\hat{G}_1^i$ and $\bar{G}_1^i$ are obtained from the graph $G_1^{i-1}$. We add the vertex $i$ and one edge $iq$ or $i\bar{q}$, respectively. New edge changes only the degree of $q$ or $\bar{q}$ and the second degree of neighbors of $q$ or $\bar{q}$, respectively. Consider $\bar{G}_1^i$. Fix $l$ and $j \leq k$. We are interested in measuring the growth of the number of vertices with degree $l$ and second degree $j$ at the step $i$. First $i$ can become a vertex of second degree $j$ with $j \leq k$. Secondly the vertex $q$ can become a vertex of second degree $j$ with $j \leq k$. Thirdly the second degree of neighbors of $q$ increases. If $q$ has at least $k + 1$ neighbors in $G_1^{i-1}$, then after the step $i$ these vertices have second degree bigger than $k$ and we do not count them. If $q$ has at most $k$ neighbors in $G_1^{i-1}$, then at most $k$ vertices change their second degrees at the step $i$. Arguing as above, we consider $\bar{G}_1^i$. We are interested in measuring the decrease of the values $N_{i-1}(l, j)$ and $P_{i-1}(l, j)$. First $q$ has new degree after the step $i$. Secondly some neighbors of $q$ can have second degree $j \leq k$ in $G_1^{i-1}$ (so the number of the neighbors of $q$ in $G_1^{i-1}$ is not bigger than $k + 1$). Let us sum all the just-mentioned numbers. We have

\[
\sum_{l=1}^{\infty} \sum_{j=0}^{k} (\delta'_n(l, j) + \epsilon'_n(l, j)) \leq 1 + 1 + k + 1 + (k + 1) = 2k + 4.
\]

The case $n = i$ is complete. Now consider $t$: $i \leq t \leq n - 1$. Note that

\[
M(N_{t+1}(1)|G_1^i) = M(N_t(1)|G_1^i) \left(1 - \frac{1}{2t + 1}\right) + \frac{2t}{2t + 1},
\]

\[
M(N_{t+1}(2)|G_1^i) = M(N_t(2)|G_1^i) \left(1 - \frac{2}{2t + 1}\right) + M(N_t(1)|G_1^i) \frac{1}{2t + 1} + \frac{1}{2t + 1},
\]

\[
M(N_{t+1}(j)|G_1^i) = M(N_t(j)|G_1^i) \left(1 - \frac{j}{2t + 1}\right) + M(N_t(j - 1)|G_1^i) \frac{j - 1}{2t + 1}, \quad j \geq 3,
\]

\[
M(N_{t+1}(1, j)|G_1^i) = M(N_t(1, j)|G_1^i) \left(1 - \frac{j + 2}{2t + 1}\right) + j M(N_t(1, j - 1)|G_1^i) \frac{j - 1}{2t + 1} + j M(N_t(j)|G_1^i) \frac{j - 2}{2t + 1},
\]

\[
M(N_{t+1}(l, j)|G_1^i) = M(N_t(l, j)|G_1^i) \left(1 - \frac{2l + j}{2t + 1}\right) + (l - 1) M(N_t(l - 1, j)|G_1^i) \frac{l - 1}{2t + 1} + (l - 2) M(N_t(l - 2, j)|G_1^i) \frac{l - 2}{2t + 1}.
\]
We obtained the same equalities in proofs of Theorem 4, Lemma 1, and Lemma 2. Replace $G_1^i$ by $\hat{G}_1^i$ or $\tilde{G}_1^i$ in these equalities. Substracting the equalities with $\hat{G}_1^i$ from the equalities with $\tilde{G}_1^i$ and using the inequality $(a + b)I(a + b > 0) \leq aI(a > 0) + bI(b > 0)$, we get

\[
\delta_{t+1}(j) \leq \delta_t(j) \left(1 - \frac{j}{2t+1}\right) + \delta_t(j - 1) \frac{j - 1}{2t+1},
\]

\[
\delta_{t+1}(1, j) \leq \delta_t(1, j) \left(1 - \frac{j}{2t+1}\right) + \frac{j \delta_t(j)}{2t+1},
\]

\[
\delta_{t+1}(l, j) \leq \delta_t(l, j) \left(1 - \frac{2l + j - 2}{2t+1}\right) + \frac{(l - 1) \delta_t(l - 1, j)}{2t+1} + \frac{(l + j - 1) \delta_t(l, j - 1)}{2t+1}, \quad l \geq 2,
\]

\[
\epsilon'_{t+1}(2, 0) \leq \epsilon'_t(2, 0) \left(1 - \frac{2}{2t+1}\right),
\]

\[
\epsilon'_{t+1}(l, j) \leq \epsilon'_t(l, j) \left(1 - \frac{2l + j - 2}{2t+1}\right) + \frac{(l - 1) \epsilon'_t(l - 1, j)}{2t+1} + \frac{(l + j - 3) \epsilon'_t(l, j - 1)}{2t+1}, \quad l \geq 3.
\]

Now we can estimate the sum

\[
\sum_{l=1}^{\infty} \sum_{j=0}^{k} \delta'_{t+1}(l, j) + \sum_{l=2}^{\infty} \sum_{j=0}^{k} \epsilon'_{t+1}(l, j) \leq
\]

\[
\leq \sum_{j=1}^{k} \left(\delta'_t(1, j) \left(1 - \frac{j}{2t+1}\right) + \frac{j \delta'_t(j)}{2t+1} + \epsilon'_t(2, 0) \left(1 - \frac{2}{2t+1}\right) + \right.
\]

\[
\sum_{l=2}^{\infty} \sum_{j=1}^{k} \left(\delta'_t(l, j) \left(1 - \frac{2l + j - 2}{2t+1}\right) + \frac{(l - 1) \delta'_t(l - 1, j)}{2t+1} + \frac{(l + j - 1) \delta'_t(l, j - 1)}{2t+1} \right) +
\]

\[
\sum_{l=3}^{\infty} \sum_{j=0}^{k} \left(\epsilon'_t(l, j) \left(1 - \frac{2l + j - 2}{2t+1}\right) + \frac{(l - 1) \epsilon'_t(l - 1, j)}{2t+1} + \frac{(l + j - 3) \epsilon'_t(l, j - 1)}{2t+1} \right) =
\]

\[
= \sum_{j=1}^{k} \delta'_t(1, j) - \sum_{j=1}^{k} \frac{(j + 1) \delta'_t(1, j)}{2t+1} + \sum_{j=1}^{k} \frac{j \delta'_t(j)}{2t+1} + \sum_{l=2}^{\infty} \sum_{j=1}^{k} \frac{(l + j) \delta'_t(l, j)}{2t+1} +
\]

\[
- \sum_{l=2}^{\infty} \sum_{j=1}^{k} \frac{(2l + j) \delta'_t(l, j)}{2t+1} + \sum_{l=2}^{\infty} \sum_{j=1}^{k} \frac{l \delta'_t(l, j)}{2t+1} + \sum_{l=2}^{\infty} \sum_{j=1}^{k} \frac{(l + j) \delta'_t(l, j)}{2t+1} +
\]

\[
+ \epsilon'_t(2, 0) - \frac{2 \epsilon'_t(2, 0)}{2t+1} + \sum_{l=3}^{\infty} \sum_{j=0}^{k} \epsilon'_t(l, j) - \sum_{l=3}^{\infty} \sum_{j=0}^{k} \frac{(2l + j - 2) \epsilon'_t(l, j)}{2t+1} +
\]

\[
+ \sum_{l=3}^{\infty} \sum_{j=0}^{k} \frac{l \epsilon'_t(l, j)}{2t+1} + \sum_{l=3}^{\infty} \sum_{j=0}^{k} \frac{l \epsilon'_t(l, j)}{2t+1}.
\]
It remains to estimate the sum $\sum_{j=1}^{k} \frac{j\delta'_t(j)}{2t+1}$. Note that for any $t \geq i$ we have $\sum_{j=0}^{k} \delta'_t(j) \leq 3$. It is obvious for $t = i$ (when we add a new vertex $i$, we change only the degree of $\hat{q}$ or $\bar{q}$). If $t + 1 > i$, then

$$\sum_{j=1}^{k} \delta'_{t+1}(j) \leq \sum_{j=1}^{k} \left( \delta'_t(j) \left( 1 - \frac{j}{2t+1} \right) + \delta'_t(j-1) \frac{j-1}{2t+1} \right) = \sum_{j=1}^{k} \delta'_t(j) - \delta'_t(k) \frac{k}{2t+1}.$$ 

In other words, $\sum_{j=1}^{k} \delta'_t(j)$ is not increasing when $t$ is growing.

So we get

$$\sum_{l=1}^{\infty} \sum_{j=0}^{k} \left( \delta'_{l+1}(l, j) + \epsilon'_{l+1}(l, j) \right) \leq \sum_{l=1}^{\infty} \sum_{j=0}^{k} \left( \delta'_t(l, j) + \epsilon'_t(l, j) \right) + \frac{3k}{2t+1}.$$ 

Thus we have

$$|M(X_n(k)|G^i_1) - M(X_n(k)|G^i_1)| \leq \sum_{l=1}^{\infty} \sum_{j=0}^{k} \left( \delta'_t(l, j) + \epsilon'_t(l, j) \right) + \sum_{t=1}^{n-1} \frac{3k}{2t+1} \leq 2k + \frac{3k}{2t+1} \leq 2k + 5 + \frac{3}{2}k \ln n \leq 10k \ln n.$$ 

This concludes the proof of Theorem 3.

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