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Ja Kyung Koo, Dong Hwa Shin, and Dong Sung Yoon*

On a problem of Hasse and Ramachandra

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Abstract: Let \( K \) be an imaginary quadratic field, and let \( f \) be a nontrivial integral ideal of \( K \). Hasse and Ramachandra asked whether the ray class field of \( K \) modulo \( f \) can be generated by a single value of the Weber function. We completely resolve this question when \( f = (N) \) for any positive integer \( N \) excluding 2, 3, 4 and 6.

Keywords: class field theory, complex multiplication, Weber function

MSC: Primary 11R37; Secondary 11G15, 11G16

1 Introduction

Let \( K \) be an imaginary quadratic field with ring of integers \( \mathcal{O}_K \), and let \( E \) be an elliptic curve with complex multiplication by \( \mathcal{O}_K \). When \( E \) is given by the affine model

\[
y^2 = 4x^3 - g_2x - g_3 \quad \text{with} \quad g_2 = g_2(\mathcal{O}_K) \quad \text{and} \quad g_3 = g_3(\mathcal{O}_K),
\]

the Weber function \( h : \mathbb{C}/\mathcal{O}_K \to \mathbb{P}^1(\mathbb{C}) \) is defined by

\[
h(z) = \begin{cases} 
\frac{g_2^3}{\Delta} \varphi(z)^2 & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\
\frac{g_3}{\Delta} \varphi(z)^3 & \text{if } K = \mathbb{Q}(\sqrt{-3}), \\
\frac{g_2 g_3}{\Delta} \varphi(z) & \text{otherwise,}
\end{cases}
\]

(1)

where \( \Delta = g_3^2 - 27g_2^3 \) and \( \varphi(z) = \varphi(z; \mathcal{O}_K) \). This map gives rise to an isomorphism of \( E/\text{Aut}(E) \) onto \( \mathbb{P}^1(\mathbb{C}) \) ([8, Theorem 7 in Chapter 1]).

Let \( f \) be a proper nontrivial ideal of \( \mathcal{O}_K \). We denote by \( H \) the Hilbert class field of \( K \), and by \( K_f \) the ray class field of \( K \) modulo \( f \). As a consequence of the main theorem of the theory of complex multiplication, Hasse proved in [4] that

\[
H = K(f) \quad \text{with} \quad j = 1728 \frac{g_2^3}{\Delta} \quad \text{and} \quad K_f = H(h(z_0)) \quad \text{for some} \quad z_0 \in f^{-1}.
\]

(2)

See also [8, Chapter 10]. In his letter to Hecke, Hasse further asked whether \( K_f \) can be generated by a single value of \( h \) without the \( j \)-invariant ([3, p. 91]), and Ramachandra also mentioned this problem later in [10]. It was Sugawara who first gave a partial answer to this question ([12] and [13]), however, it still remains an open question.

In this paper, through careful understanding about the characters on class groups and the second Kronecker limit formula, we shall eventually resolve Hasse-Ramachandra's problem for \( f = (N) \) with any positive integer \( N \) excluding 2, 3, 4 and 6 (Theorem 5.1).

Ja Kyung Koo: Department of Mathematical Sciences, KAIST, Daejeon 34141, Republic of Korea, E-mail: jkoo@math.kaist.ac.kr
Dong Hwa Shin: Department of Mathematics, Hankuk University of Foreign Studies, Yongin-si, Gyeonggi-do 17035, Republic of Korea, E-mail: dhshin@hufs.ac.kr
*Corresponding Author: Dong Sung Yoon: Department of Mathematics Education, Pusan National University, Busan 46241, Republic of Korea, E-mail: dsyoon@pusan.ac.kr

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2 The second Kronecker limit formula

For \( v = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in (\mathbb{Q} \setminus \mathbb{Z})^2 \), we define the (first) Fricke function \( f_v(\tau) \) on the upper half-plane \( \mathbb{H} \) by

\[
f_v(\tau) = \frac{g_2(\tau)g_3(\tau)}{\Delta(\tau)} \varphi(r_1 \tau + r_2),
\]

where \( g_1(\tau) = g_1([r, 1]), g_2(\tau) = g_2([r, 1]), \Delta(\tau) = \Delta([r, 1]) \) and \( \varphi(z) = \varphi(z; [r, 1]) \). This function depends only on \( \pm v \mod \mathbb{Z}^2 \), and is holomorphic on \( \mathbb{H} \) ([8, Chapters 3 and 6]). Furthermore, we define the Siegel function \( g_v(\tau) \) on \( \mathbb{H} \) by the following infinite product

\[
g_v(\tau) = -e^{\pi i r_2 (r_1 - 1)} q^{(r_2^2 - r_1 + 1)/2} (1 - q^e) (1 - q^{n_1} \tau e^{2n_2}) \prod_{n=1}^{\infty} (1 - q^{n_1} \tau e^{2n_2}),
\]

where \( q = e^{2\pi i}. \) If \( N \) is a positive integer so that \( Nv \in \mathbb{Z}^2 \), then \( g_v(\tau)^{12N} \) depends only on \( \pm v \mod \mathbb{Z}^2 \), and has neither zeros nor poles on \( \mathbb{H} \) ([6, §2.1]).

**Lemma 2.1.** Let \( v, u \in (\mathbb{Q} \setminus \mathbb{Z})^2 \) such that \( u \neq \pm v \mod \mathbb{Z}^2 \). Then we have the relation

\[
(f_u(\tau) - f_v(\tau))^6 = \frac{i(\tau)^2(\tau - 1728)^3}{2^{10}3^{24}} g_u(\tau)^6 g_v(\tau)^6 g_{u-v}(\tau)^6 g_{u+v}(\tau)^6.
\]

**Proof.** See [8, Theorem 2 in Chapter 18] and [6, p. 29 and p. 51].

Let \( K \) be an imaginary quadratic field, let \( \mathfrak{o} \) be a proper nontrivial ideal of \( \mathcal{O}_K \) and let \( N > 1 \) be the smallest positive integer in \( \mathfrak{o} \). We denote by \( \text{Cl}(\mathfrak{o}) \) the ray class group of \( K \) modulo \( \mathfrak{o} \). Then \( \text{Gal}(K_{/K}) \) is isomorphic to \( \text{Cl}(\mathfrak{o}) \) via the Artin map \( \sigma = \sigma_{\mathfrak{o}} : \text{Cl}(\mathfrak{o}) \rightarrow \text{Gal}(K_{/K}) \). Let \( C \in \text{Cl}(\mathfrak{o}) \). Take any integral ideal \( \mathfrak{c} \) in the class \( C \) and express

\[
\mathfrak{c}^{-1} = [\omega_1, \omega_2] \text{ for some } \omega_1, \omega_2 \in \mathbb{C} \text{ such that } \omega = \frac{\omega_1}{\omega_2} \in \mathbb{H},
\]

\[
1 = r_1 \omega_1 + r_2 \omega_2 \text{ for some } r_1, r_2 \in (1/N)\mathbb{Z}.
\]

We define the Fricke invariant \( f_{\mathfrak{c}}(C) \) and the Siegel-Ramachandra invariant \( g_{\mathfrak{c}}(C) \) by

\[
f_{\mathfrak{c}}(C) = f_{[r_1]}(\omega) \quad \text{and} \quad g_{\mathfrak{c}}(C) = g_{[r_1]}(\omega)^{12N},
\]

respectively. These values depend only on the class \( C \), not on the choices of \( c, \omega_1 \) and \( \omega_2 \) ([8, §6.2 and §6.3] and [6, §2.1 and 11.1]).

**Proposition 2.2.** The invariants \( f_{\mathfrak{c}}(C) \) and \( g_{\mathfrak{c}}(C) \) belong to \( K_{/K} \). Furthermore, they satisfy

\[
f_{\mathfrak{c}}(C^{\mathfrak{c}}) = f_{\mathfrak{c}}(CC') \quad \text{and} \quad g_{\mathfrak{c}}(C^{\mathfrak{c}}) = g_{\mathfrak{c}}(CC') \quad \text{for all } C' \in \text{Cl}(\mathfrak{o}).
\]

**Proof.** See [6, Theorem 1.1 in Chapter 11].

Let \( \chi \) be a nonprincipal character of \( \text{Cl}(\mathfrak{o}) \). We define the Stickelberger element \( S(\chi) = S_{\mathfrak{o}}(\chi) \) by

\[
S(\chi) = \sum_{C \in \text{Cl}(\mathfrak{o})} \chi(C) \ln |g_{\mathfrak{o}}(C)|,
\]

and the \( L \)-function \( L_{\mathfrak{o}}(s, \chi) \) by

\[
L_{\mathfrak{o}}(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi([\mathfrak{a}])}{N_{K/\mathbb{Q}}(\mathfrak{a})^s} \quad (s \in \mathbb{C}),
\]

where \( a \) runs over all nontrivial ideals of \( \mathcal{O}_K \) prime to \( \mathfrak{o} \) and \([\mathfrak{a}] \) stands for the class in \( \text{Cl}(\mathfrak{o}) \) containing the ideal \( a \). We shall denote by \( \mathfrak{f}_\chi \) the conductor of the character \( \chi \).
Proposition 2.3. Let $\chi_0$ be the primitive character of $\chi$ on $\text{Cl}(f_K)$. If $f_\chi \neq \mathfrak{O}_K$, then we obtain the relation

$$
\left( \prod_{p : \text{prime ideals of } \mathfrak{O}_K \text{ such that } p \nmid f_\chi} (1 - \frac{\chi_0((p)))}{f_{\chi}(1, \chi_0)} = -\frac{\pi \chi_0((\gamma \mathfrak{O}_K f_\chi))}{3N(f_\chi)^2 / d_K} \omega(f_\chi) \mathcal{T}_s(\chi_0) \mathcal{S}(\chi_0),
$$

where $\mathfrak{O}_K$ is the different ideal of the extension $K/\mathbb{Q}_p$, $\gamma$ is an element of $K$ so that $\gamma \mathfrak{O}_K f_\chi$ is a nontrivial ideal of $\mathfrak{O}_K$ prime to $f_\chi$, $N(f_\chi)$ is the least positive integer in $f_\chi$, $\omega(f_\chi) = |\{a \in \mathfrak{O}_K | a \equiv 1 \pmod{f_\chi}\}|$ and

$$
T_s(\chi_0) = \sum_{\alpha + f_\chi \in (\mathfrak{O}_K / f_\chi)^*} \omega(\mathfrak{O}_K) e^{2\pi i n \mathfrak{O}_K(a \gamma)}.
$$

Proof. See [11, Theorem 9 in Chapter II] or [6, Theorem 2.1 in Chapter II].

Remark 2.4. Since $\chi_0$ is a nonprincipal character of $\text{Cl}(f_\chi)$ by the assumption $f_\chi \neq \mathfrak{O}_K$, we have $L_{f_\chi}(1, \chi_0) \neq 0$ ([5, Theorem 10.2 in Chapter VI]). Thus, if every prime ideal factor of $f_\chi$ divides $f_\chi$, then we derive by Proposition 2.3 that $S(\chi_0) \neq 0$.

3 Differences of Weber functions

For an imaginary quadratic field $K$, fix an element $\tau_K$ of $\mathbb{H}$ so that $\mathfrak{O}_K = [\tau_K, 1]$. From now on, we assume that $K$ is different from $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, and let $N > 1$. We then have $J(\tau_K) \neq 0$, 1728 ([1, p. 261]) and

$$
h(r_1 \tau_K + r_2) = f_{[r_1] / [r_2]}(\tau_K) \text{ for all } \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in (\mathbb{Q} \setminus \mathbb{Z})^2
$$

by the definitions (1) and (3).

Let $H_N$ be the ring class field of the order of conductor $N$ in $K$. Then we have a tower of fields

$$
K \subseteq H \subseteq H_N \subseteq K(N)
$$

([1, §7]). For an integer $t$ prime to $N$, by $C_t = C_{N, t}$ we mean the class in the ray class group $\text{Cl}(N)$ of $K$ modulo $(N)$ containing the ideal $(t)$. Note that $C_1$ is the identity element of $\text{Cl}(N)$.

Lemma 3.1. If $t$ is an integer prime to $N$, then we get

$$
f_{(N)}(C_t) = f_{[t/N]}(\tau_K) \text{ and } g_{(N)}(C_t) = g_{[t/N]}(\tau_K)^{12N}.
$$

Proof. Since

$$
(N \mathfrak{O}_K)(t \mathfrak{O}_K)^{-1} = (N/t) \mathfrak{O}_K = [N \tau_K/t, N/t] \text{ and } 1 = O(N \tau_K/t) + (t/N)(N/t),
$$

we deduce the lemma by the definition (4). □

For an intermediate field $F$ of the extension $K(N)/K$, we shall denote by $\text{Cl}(K(N)/F)$ the subgroup of $\text{Cl}(N)$ corresponding to $\text{Gal}(K(N)/F)$.

Lemma 3.2. We have

$$
\text{Cl}(K(N)/H_N) = \{ C_t | t \in (\mathbb{Z}/N\mathbb{Z})^* / \{ \pm 1 \} \} \cong (\mathbb{Z}/N\mathbb{Z})^* / \{ \pm 1 \}.
$$

Proof. See [2, Proposition 3.8]. □
Let \( t \) be an integer such that
\[
\gcd(N, t) = 1 \quad \text{and} \quad t \not\equiv \pm 1 \pmod{N}.
\]

Note that such an integer \( t \) always exists except for the four cases \( N = 2, 3, 4, 6 \). Express \((t + 1)/N\) and \((t - 1)/N\) as
\[
\frac{t + 1}{N} = n_+ \quad \text{and} \quad \frac{t - 1}{N} = n_-,
\]
where \( n_+, n_-, n_0, n_2, n_3, n_5 \) are integers such that \( n_+, n_0 > 0 \) and \( \gcd(n_+, n_0) = \gcd(n_-, n_5) = 1 \). Observe that the condition \( t \not\equiv \pm 1 \pmod{N} \) is equivalent to saying that neither \( n_+ \), nor \( n_- \), is equal to \( 1 \).

Now, we define
\[
\xi_t = (h(t/N) - h(1/N))^{12N} = \left( f\left[0+\right]_N (r_K) - f\left[0-\right]_N (r_K) \right)^{12N}. \tag{6}
\]

Furthermore, for a character \( \chi \) of \( \text{Cl}(N) \) we denote by
\[
S(\chi, \xi_t) = \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left| \xi_t^{s(C)} \right|.
\]

**Lemma 3.3.** If \( \chi \) is nontrivial on \( \text{Cl}(K(N)/H) \), then we obtain
\[
S(\chi, \xi_t) = (N/N_+) \sum_{B_+ \in \text{Cl}(N) \pmod{\text{Cl}(K_0/K_{0*})}} \chi(B_+) \ln \left| \sigma(C_{n_+}) \right| \sum_{A_+ \in \text{Cl}(K_0/K_{0*})} \chi(A_+)
\]
\[
+ (N/N_-) \sum_{B_- \in \text{Cl}(N) \pmod{\text{Cl}(K_0/K_{0*})}} \chi(B_-) \ln \left| \sigma(C_{n_-}) \right| \sum_{A_- \in \text{Cl}(K_0/K_{0*})} \chi(A_-)
\]
\[
- 2(\chi(C_0) + 1)S(\chi).
\]

**Proof.** We derive that
\[
S(\chi, \xi_t) = \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left( \frac{(j(r_K)^{4N}(j(r_K) - 1728)^{6N})^{348N}}{260N^3} \right) \sigma(C)
\]
\[
+ \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left( g\left[0+\right]_N (r_K^{12N}) \right) \sigma(C) \sum_{A \in \text{Cl}(K_0/K_{0*})} \chi(A)
\]
\[
- \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left( g\left[0-\right]_N (r_K^{24N}) \right) \sigma(C) \sum_{A \in \text{Cl}(K_0/K_{0*})} \chi(A)
\]
\[
= \sum_{B_+ \in \text{Cl}(N) \pmod{\text{Cl}(K_0/H)}} \chi(AB) \ln \left( \frac{(j(r_K)^{4N}(j(r_K) - 1728)^{6N})^{348N}}{260N^3} \right) \sigma(AB)
\]
\[
+ (N/N_+) \sum_{B_+ \in \text{Cl}(N) \pmod{\text{Cl}(K_0/K_{0*})}} \chi(A_+ \cdot B_+) \ln \left| \sigma(C_{n_+}) \right| \sum_{A_+ \in \text{Cl}(K_0/K_{0*})} \chi(A_+ \cdot B_+)
\]
\[
+ (N/N_-) \sum_{B_- \in \text{Cl}(N) \pmod{\text{Cl}(K_0/K_{0*})}} \chi(A_+ \cdot B_-) \ln \left| \sigma(C_{n_-}) \right| \sum_{A_- \in \text{Cl}(K_0/K_{0*})} \chi(A_+ \cdot B_-)
\]
\[
- 2 \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left| \sigma(C_{n_0}) \right| - 2 \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left| \sigma(C_{n_5}) \right| \quad \text{by Lemma 3.1}
\]
\[
= \sum_B \chi(B) \ln \left( \frac{(j(r_K)^{4N}(j(r_K) - 1728)^{6N})^{348N}}{260N^3} \right) \sigma(B) \sum_{A} \chi(A)
\]
\[+(N/N+) \sum_{B_+} \chi(B+) \ln |g(N,)(C_{N,n})^{a(B+)}| \sum_{A_+} \chi(A+)
\]
\[+(N/N-) \sum_{B_-} \chi(B-) \ln |g(N,)(C_{N,n})^{a(B-)}| \sum_{A_-} \chi(A_-)
\]
\[-2\chi(C) \sum_{C} \chi(C) \ln |g(N,)(C|C)| - 2 \sum_{C} \chi(C) \ln |g(N,)(C)| \quad \text{by (2) and Proposition 2.2}
\]
\[= (N/N) \sum_{B_+} \chi(B+) \ln |g(N,)(C_{N,n})^{a(B+)}| \sum_{A_+} \chi(A+)
\]
\[+(N/N-) \sum_{B_-} \chi(B-) \ln |g(N,)(C_{N,n})^{a(B-)}| \sum_{A_-} \chi(A_-)
\]
\[-2(\chi(C) + 1)S(\chi)\]

by the assumption that \( \chi \) is nontrivial on \( \text{Cl}(K_{(N)}/H) \) and the definition (5).

\[\Box\]

4 Lemmas on characters of class groups

If we set
\[F = K \left(h(1/N)\right) = K \left(f_{1/N}(\tau_K)\right),\]
then we obtain by (2) that
\[\text{Cl}(K_{(N)}/H) \cap \text{Cl}(K_{(N)}/F) = \text{Cl}(K_{(N)}/HF) = \text{Cl}(K_{(N)}/K_{(N)}) = \{C_1\}.\]

In this section, we shall prove the existence of certain characters of class groups under the assumption that
\(F\) is properly contained in \(K_{(N)}\).

**Lemma 4.1.** Assume that
\[\text{gcd}(72, N) \in \{1, 8, 9, 72\}.\]

Then, there is a character \( \chi \) of \( \text{Cl}(N) \) satisfying the following properties:

(A1) It is trivial on \( \text{Cl}(K_{(N)}/H_N) \).

(A2) \( \chi(C) \neq 1 \) for any chosen \( C \in \text{Cl}(K_{(N)}/H) \setminus \text{Cl}(K_{(N)}/H_N) \).

(A3) Every prime ideal factor of \( (N) \) divides the conductor \( (N) \).

**Proof.** See [7, Lemma 3.4 and Remark 4.5].\[\Box\]

**Lemma 4.2.** Suppose that \( F \) is properly contained in \( K_{(N)} \). Then, there is a character \( \rho \) of \( \text{Cl}(N) \) satisfying the following properties:

- It is trivial on \( \text{Cl}(K_{(N)}/H) \), and so \( (N)_{\rho} = \mathcal{O}_K \).
- It is nontrivial on \( \text{Cl}(K_{(N)}/F) \).

Here, \( (N)_{\rho} \) stands for the conductor of the character \( \rho \).

**Proof.** Since \( |\text{Cl}(K_{(N)}/F)| \geq 2 \) and \( \text{Cl}(K_{(N)}/H) \cap \text{Cl}(K_{(N)}/F) = \{C_1\} \) by (7), one can take a class \( C \in \text{Cl}(K_{(N)}/F) \setminus \text{Cl}(K_{(N)}/H) \). Thus, if we let \( \mu : \text{Cl}(N) \to \text{Cl}(N)/\text{Cl}(K_{(N)}/H) \) be the canonical homomorphism, then there is a character \( \psi \) of \( \text{Cl}(N)/\text{Cl}(K_{(N)}/H) \) such that \( \psi(\mu(C)) \neq 1 \).

Now, defining a character \( \rho \) of \( \text{Cl}(N) \) by \( \rho = \psi \circ \mu \), we see that it is trivial on \( \text{Cl}(K_{(N)}/H) \). Since
\[\text{Cl}(N)/\text{Cl}(K_{(N)}/H) \simeq \text{Cl}(H/K) = \text{Cl}(\mathcal{O}_K),\]
we get \( (N)_{\rho} = \mathcal{O}_K \). Moreover, \( \rho(C) = \psi(\mu(C)) \neq 1 \) implies that \( \rho \) is nontrivial on \( \text{Cl}(K_{(N)}/F) \).\[\Box\]
**Proposition 4.3.** Assume that
\[ \gcd(72, N) \in \{1, 8, 9, 72\} \text{ and } F \text{ is properly contained in } K(N). \tag{8} \]

Then, there is a character \( \chi \) of \( \Cl(N) \) and an integer \( t \) which satisfy the following properties:

(A1) \( \chi \) is nontrivial on \( \Cl(K(N)/F) \).
(A2) \( \gcd(N, t) = 1 \) and \( t \not\equiv \pm 1 \text{ (mod } N) \).
(A3) \( S(\overline{\chi}, \xi) \neq 0 \).

**Proof.** We divide the proof into three cases in accordance with \( \gcd(72, N) \).

**Case 1.** First, consider the case where \( \gcd(72, N) \in \{8, 72\} \). Let \( C \) be the class in \( \Cl(N) \) containing the ideal \( ((N/2)\tau_K + 1) \). We observe by Lemma 3.2 that
\[ C \in \Gal(K(N)/K(N/2)) \setminus \Gal(K(N)/H_N). \tag{9} \]
Then, by Lemma 4.1 there is a character \( \chi \) of \( \Cl(N) \) satisfying (A1)–(A3). If \( \chi \) is trivial on \( \Cl(K(N)/F) \), then we replace \( \chi \) by \( \rho \), where \( \rho \) is a character of \( \Cl(N) \) given in Lemma 4.2. The new character \( \chi \) is nontrivial on \( \Cl(K(N)/F) \) and preserves the properties (A1)–(A3). Take any integer \( t \) such that \( \gcd(N, t) = 1 \) and \( t \not\equiv \pm 1 \text{ (mod } N) \). Since \( N, t + 1 \) and \( t - 1 \) are all even, we see that \( N_+ \) and \( N_\cdot \) divide \( N/2 \), from which it follows that
\[ \Cl(K(N)/K(N/2)) \subseteq \Cl(K(N)/K(N_+)) \cap \Cl(K(N)/K(N_\cdot)). \tag{10} \]
We then achieve that
\[ S(\overline{\chi}, \xi) = (N/N_+ \sum_{B_+ \in \Cl(N) (\text{mod } \Cl(K(N/K(N_+))))} \overline{\chi}(B_+) \ln |g_{(N_+)}(C_{N_+}, n_+)^{\text{ord}(B_+)}| \sum_{A_+ \in \Cl(K(N)/K(N_+))} \overline{\chi}(A_+) \]
\[ + (N/N_- \sum_{B_- \in \Cl(N) (\text{mod } \Cl(K(N/K(N_-))))} \overline{\chi}(B_-) \ln |g_{(N_-)}(C_{N_-}, n_-)^{\text{ord}(B_-)}| \sum_{A_- \in \Cl(K(N)/K(N_-))} \overline{\chi}(A_-) \]
\[ - 2(\chi(C_t) + 1)S(\overline{\chi}) \] by Lemma 3.3
\[ = -2(\chi(C_t) + 1)S(\overline{\chi}) \text{ since } \chi \text{ is nontrivial on } \Cl(K(N)/K(N_\cdot)) \text{ and } \Cl(K(N)/K(N_+)) \]
by (9), (10) and (A2)
\[ = -4S(\overline{\chi}) \] by (A1) and Lemma 3.2
\[ \neq 0 \] by Proposition 2.3 and Remark 2.4.

**Case 2.** Second, consider the case where \( \gcd(72, N) = 9 \). If we let \( C \) be the class in \( \Cl(N) \) containing the ideal \( ((N/3)\tau_K + 1) \), then we see that
\[ C \in \Gal(K(N)/K(N/3)) \setminus \Gal(K(N)/H_N) \tag{11} \]
by Lemma 3.2. By Lemma 4.1, there exists a character \( \chi \) of \( \Cl(N) \) satisfying (A1)–(A3). In a similar way to the above Case 1, we may assume that \( \chi \) is nontrivial on \( \Cl(K(N)/F) \). Take \( t = 2 \), and then we get
\[ n_+ = 1, N_+ = \frac{N}{3} \text{ and } n_- = 1, N_- = N. \]
So, we derive that
\[ S(\overline{\chi}, \xi) = 3 \sum_{B_+ \in \Cl(N) (\text{mod } \Cl(K(N/K(N_+))))} \overline{\chi}(B_+) \ln |g_{(N/3)}(C_{N/3}, 1)^{\text{ord}(B_+)}| \sum_{A_+ \in \Cl(K(N)/K(N/3))} \overline{\chi}(A_+) \]
\[ + S(\overline{\chi}) - 2(\chi(C_t) + 1)S(\overline{\chi}) \] by Lemma 3.3
\[ = -(2\chi(C_t) + 1)S(\overline{\chi}) \text{ since } \chi \text{ is nontrivial on } \Cl(K(N)/K(N/3)) \text{ by (11) and (A2)} \]
\[ = -3S(\overline{\chi}) \] by (A1) and Lemma 3.2
\[ \neq 0 \] by Proposition 2.3 and Remark 2.4.
Case 3. Lastly, consider the case where \( \gcd(72, N) = 1 \). By Lemma 4.1, there is a character \( \chi \) of \( \text{Cl}(N) \) satisfying (A1)–(A3) for any chosen \( C \in \text{Cl}(K(N)/H) \setminus \text{Cl}(K(N)/H_N) \). In like manner as above, we may assume that \( \chi \) is nontrivial on \( \text{Cl}(K(N)/F) \). Take \( t = 2 \), then it follows that

\[
 n_+ = 3, \quad N_+ = N \quad \text{and} \quad n_- = 1, \quad N_- = N.
\]

Therefore, we obtain

\[
 S(\chi, \xi) = \chi(C_\ell)S(\chi) + S(\chi) - 2(\chi(C_\ell) + 1)S(\chi)
\]

by Lemma 3.3

\[
 = -2S(\chi) \quad \text{by (A1) and Lemma 3.2}
\]

\[
 \neq 0 \quad \text{by Proposition 2.3 and Remark 2.4.}
\]

This proves the lemma.

\[\square\]

**Lemma 4.4.** Assume that

\[
 \gcd(72, N) \in \{2, 3, 4, 6, 12, 18, 24, 36\} \quad \text{and} \quad N \neq 2, 3, 4, 6. \tag{12}
\]

Then, there exists an integer \( t \) satisfying the following properties:

(C1) \( \gcd(N, t) = 1 \) and \( t \equiv \pm 1 \pmod{N} \).

(C2) There are prime factors \( p_+, p_- \) of \( N \) (not necessarily distinct) such that \( \gcd(p_+, N_+) = 1 \) (Note that \( N_+ \) depends on the choice of \( t \)).

**Proof.** Let \( \ell \) be an integer such that \( \ell > 1 \) and \( \gcd(6, \, \ell) = 1 \). One can take \( t \) as listed in Table 1.

**Table 1:** An integer \( t \) satisfying (C1) and (C2)

| \( N \) | \( t \) | \( N_+ \) | \( N_- \) | \( p_+ \) | \( p_- \) |
|-------|-------|-------|-------|-------|-------|
| 12    | 5     | 2     | 3     | 3     | 2     |
| 18    | 5     | 3     | 9     | 2     | 2     |
| 24    | 7     | 3     | 4     | 2     | 3     |
| 36    | 17    | 2     | 9     | 3     | 2     |
| \( 2\ell \) | \( \ell + 2 \) | \( \ell \) | \( \ell \) | 2     | 2     |
| \( 4\ell \) | \( 2\ell + 1 \) | \( \ell \) | 2     | 2 \quad \text{a prime factor of } \ell

\[
2^a 3^b \ell \text{ with } a \geq 0, \ b \geq 1 \quad \begin{cases} x \equiv 1 \pmod{2^a \ell}, \\ x \equiv -1 \pmod{3^b} \end{cases}
\]

a divisor of \( 2^a \ell \) \ a divisor of \( 3^b \) \ a prime factor of \( \ell \)

Let \( (N) = \prod_p p^{n_p} \) be the prime ideal factorization of \( N \). Then we get

\[
 [K(N) : H] = \frac{\omega(N)}{2} \prod_{p \mid (N)} (N_{K/Q(p)} - 1) N_{K/Q(p)}^{n_p-1},
\]

where \( \omega(N) \) is the number of roots of unity in \( K \) which are congruent to 1 modulo \( (N) \) ([9, Theorem 1 in Chapter VI]). One can then readily deduce that

\[
 K_{(N)} = K_{(M)} \text{ for a proper divisor } M \text{ of } N \iff 2 \mid N \text{ and } 2 \text{ splits in } K.
\]

In this case, we have

\[
 K_{(N)} = K_{(N/2)}. \tag{13}
\]
Furthermore, it is well known that

\[
[H_N : H] = N \prod_{p \mid N} \left(1 - \left(\frac{d_K}{p}\right) \frac{1}{p}\right),
\]

(14)

where \((d_K/p)\) is the Legendre symbol for an odd prime \(p\), and \((d_K/2)\) is the Kronecker symbol ([1, Theorem 7.24]).

**Lemma 4.5.** Assume that if \(2 \parallel N\), then \(2\) does not split in \(K\). Let \(p\) be a prime factor of \(N\) with \(p^e \parallel N\). Then, there is a nontrivial character \(\chi_p\) of \(\text{Cl}(N)\) satisfying the following properties:

- It is trivial on \(\text{Cl}(K_{N}/H_{p^{e}})\), and so \((N)_{\chi_p}\) divides \((p^e)\).
- \((N)_{\chi_p}\) is divisible by every prime ideal factor of \((p)\).

**Proof.** Note that the assumption implies \([H_{p^{e}} : H] \geq 2\) by (14). Therefore, the lemma is an immediate consequence of [7, Lemma 3.3].

**Proposition 4.6.** Assume that

\(N\) satisfies (12) and \(F\) is properly contained in \(K_{(N)}\).

Under this assumption instead of (8), Proposition 4.3 also holds.

**Proof.** Let

\[
\chi = \prod_{p \mid N} \chi_p,
\]

where \(\chi_p\) is a character of \(\text{Cl}(N)\) given in Lemma 4.5 for each prime factor \(p\) of \(N\). If \(\chi\) is trivial on \(\text{Cl}(K_{(N)}/F)\), then we replace \(\chi\) by \(\chi_{\rho}\) where \(\rho\) is a character of \(\text{Cl}(N)\) given in Lemma 4.2. Then, \(\chi\) satisfies the following properties:

(i) It is trivial on \(\text{Cl}(K_{(N)}/H_{N})\).
(ii) It is nontrivial on \(\text{Cl}(K_{(N)}/F)\).
(iii) \((N)_{\chi}\) is divisible by every prime ideal factor of \((N)\).

Now, take an integer \(t\) satisfying (C1) and (C2) in Lemma 4.4. We then derive that

\[
S(\chi, \xi) = (N/N_{+}) \sum_{B_{+} \in \text{Cl}(N) (\text{mod } \text{Cl}(K_{(N)}/K_{(N)})]} \chi(B_{+}) \ln \left|g_{(N)}(C_{N_{+}, n_{+}})^{\sigma(B_{+})}\right| \sum_{A_{+} \in \text{Cl}(K_{(N)}/K_{(N)})} \chi(A_{+})
\]

\[
+(N/N_{-}) \sum_{B_{-} \in \text{Cl}(N) (\text{mod } \text{Cl}(K_{(N)}/K_{(N)})]} \chi(B_{-}) \ln \left|g_{(N)}(C_{N_{-}, n_{-}})^{\sigma(B_{-})}\right| \sum_{A_{-} \in \text{Cl}(K_{(N)}/K_{(N)})} \chi(A_{-})
\]

\[-2(\chi(C_{t}) + 1)S(\chi) \quad \text{by Lemma 3.3}
\]

\[= -4S(\chi) \quad \text{because } \chi \text{ is nontrivial on } \text{Cl}(K_{(N)}/K_{(N)}) \text{ and } \text{Cl}(K_{(N)}/K_{(N)})
\]

\[\# 0 \quad \text{by Proposition 2.3, Remark 2.4 and (iii).}
\]

\[
5 \quad \text{Main theorem}
\]

Now, we are ready to prove our main theorem. Note by (2) that the problem of Hasse and Ramachandra is trivial if the class number of \(K\) is one.
Theorem 5.1. Let $K$ be an imaginary quadratic field other than $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, and let $N > 1$ be an integer such that $N \neq 2, 3, 4, 6$. Then we have

$$K(N) = \begin{cases} K(h(1/N)) & \text{if } 2 \mid N \text{ or } 2 \text{ does not split in } K, \\ K(h(2/N)) & \text{otherwise.} \end{cases}$$

Proof. First, consider the case where $2 \mid N$ or 2 does not split in $K$. Suppose on the contrary that $F = K(h(1/N))$ is properly contained in $K(N)$. Then, by Propositions 4.3 and 4.6, there exist a character $\chi$ of $\text{Cl}(N)$ and an integer $t$ such that

(B1) $\chi$ is nontrivial on $\text{Cl}(K(N)/F)$,

(B2) $\gcd(N, t) = 1$ and $t \not\equiv \pm 1 \pmod{N}$,

(B3) $S(\chi, \xi_t) \neq 0$.

On the other hand, since $F$ is a Galois extension of $K$, it contains the Galois conjugate $h(1/N)^{\sigma(C)}$ of $h(1/N)$. We then see by Proposition 2.2 and Lemma 3.1 that

$$h(1/N)^{\sigma(C)} = f\left[\frac{0}{1/N}\right](\tau_K)^{\sigma(C)} = f\left[\frac{0}{N}\right](C_t) = f\left[\frac{0}{t/N}\right](\tau_K) = h(t/N).$$

Thus $F$ contains the element $\xi_t = (h(t/N) - h(1/N))^{12N}$. Now, we derive that

$$S(\chi, \xi_t) = \sum_{C \in \text{Cl}(N)} \chi(C) \ln \left|\xi_t^{\sigma(C)}\right|$$

$$= \sum_{B \in \text{Cl}(N) \mod \text{Cl}(K_{3N}/F)} \sum_{A \in \text{Cl}(K_{3N}/F)} \chi(B) \ln \left|\xi_t^{\sigma(B)}\right| \sum_{A \in \text{Cl}(K_{3N}/F)} \chi(A) \quad \text{because } \xi_t \in F$$

$$= 0 \quad \text{by (B1)},$$

which contradicts (B3). Hence, we have $K(N) = K(h(1/N))$ as desired.

Second, consider the case where $2 \mid N$ and 2 splits in $K$. Then we have

$$K(N) = K(N/2) \quad \text{as mentioned in (13)}$$

$$= K(h(2/N)) \quad \text{by the first case of the theorem.}$$

This completes the proof. $\square$

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