On the uniqueness for the heat equation on complete Riemannian manifolds

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Abstract
We prove some uniqueness result for solutions to the heat equation on Riemannian manifolds. In particular, we prove the uniqueness of $L^p$ solutions with $0 < p < 1$ and improves the $L^1$ uniqueness result of Li (J Differ Geom 20:447–457, 1984) by weakening the curvature assumption.

Keywords Uniqueness problem · Heat equation on manifolds · Complete noncompact manifolds

1 Introduction

In this article, we consider the uniqueness problem for solutions to the heat equation on complete Riemannian manifolds $(M, g)$:

$$(\partial_t - \Delta) f = 0,$$

where $\Delta$ is the Laplace–Beltrami operator with respect to the metric $g$.

It is well-known that uniqueness may fail in general unless we restrict the solutions on some suitable class of functions. A example is the set of functions bounded from below. In [7], the uniqueness of nonnegative solutions to the heat equation has been established under the quadratic Ricci lower bound assumption

$$\text{Ric}(x) \geq -C(r(x) + 1)^2,$$

where $r(x)$ is the geodesic distance from some fixed point and $C$ is a nonnegative constant.

Another typical of class where uniqueness holds is the set of functions with appropriate growth rate in the spirit of [10]. For solutions with $L^2$ integrals on geodesic balls or
parabolic cylinders growing under certain rate, the uniqueness was proved in [1, 3]. The same result holds if $L^2$ is replaced by $L^p$ with $1 < p \leq 2$, and for a special class of manifolds when $p = 1$ [8]. These results imply uniqueness for solutions with suitable pointwise growth rate, provided that the manifold has some volume growth constraint. A case of particular interest is for bounded solutions, see [2] for a survey.

Our first theorem is an improvement in the results in [1, 3]. Namely, we allow the integral to be weighted by a positive power of the time variable. We will also demonstrate an example in Sect. 3.

**Theorem 1.1** Let $M$ be a complete Riemannian manifold, and let $f(x, t)$ be a nonnegative subsolution to the heat equation on $M \times (0, 1]$ with initial data $f(x, 0) = 0$ in the sense of $L^2_{\text{loc}}(M)$. Suppose for some point $q \in M$, and constant $a > 0$,

$$\int_0^1 r^a \int_{B_r(q)} f^2 \leq e^{L(r)}, \quad \forall r > 0,$$

where $L(r)$ is a positive nondecreasing function satisfying

$$\int_1^{\infty} \frac{r}{L(r)} \, dr = \infty.$$  \hspace{1cm} (1.2)

Then $f \equiv 0$ on $M \times (0, 1]$.

In [5], Li considered the uniqueness for $L^p$ solutions to the heat equation. When $p > 1$, the uniqueness holds without further assumption. However when $p = 1$ the uniqueness may fail on sufficiently negatively curved manifolds, it was proved in [5] that the uniqueness for $L^1$ solutions holds under the assumption (1.1).

As an application of Theorem 1.1, we prove the following theorem which can be applied to improve the $L^1$ uniqueness result for the heat equation in [5]. It also implies uniqueness of $L^p$ solutions with $0 < p < 1$. The curvature assumption (1.3) and (1.4) is slightly more general than (1.1) since functions such as $r \ln r$ are allowed.

**Theorem 1.2** Let $(M^n, g)$ be a complete Riemannian manifold with

$$\text{Ric}(x) \geq -k^2(r_q(x)),$$  \hspace{1cm} (1.3)

where $r_q(x)$ is the distance function to a fixed point $q \in M$, and $k(r)$ is a positive nondecreasing function satisfying

$$\int_1^{\infty} \frac{1}{k(r)} \, dr = \infty.$$  \hspace{1cm} (1.4)

Suppose $f$ is a nonnegative subsolution to the heat equation on $M \times (0, 1]$, with initial data $f(0) = 0$ in sense of $L^2_{\text{loc}}(M)$. If for some $0 < p \leq 1$,

$$\|f\|_{L^p(B_r(q) \times [0, 1])} \leq e^{Crk(r)},$$

for any $r > 0$, for some constant $C$, then $f \equiv 0$ on $M \times (0, 1]$. 

For the proof of Theorem 1.2, we use mean value inequality to get a pointwise bound for the solution, which is non-uniform and blows up as $t \to 0$, and we can verify that the assumptions in Theorem 1.1 are satisfied.

To prove the uniqueness of solutions to the heat equation, we can consider a solution starting with 0 initial data and apply Theorems 1.1 or 1.2 to its absolute value which is a nonnegative subsolution.

Furthermore, the above results imply the maximum principle. For instance, if $u$ is a subsolution to the heat equation with $u(0) \leq 0$, then one can apply Theorems 1.1 or 1.2 to $(u - 0)_{+}$ to show that $u(t) \leq 0$ provided that the assumptions are met.

## 2 Proof

**Proof of Theorem 1.1** The proof is a modification of the arguments due to Karp–Li [3] and Grigor’yan [1]. Define the function

$$
\xi(x, t) = -\frac{(r(x) - R)^2}{4(T - t)},
$$

where $r(x)$ is the distance function to the fixed point $q$, then it is a direct calculation to check

$$
\partial_t \xi + |\nabla \xi|^2 \leq 0.
$$

For any $R > 0$, let $\psi(r)$ be a nonincreasing cut-off function with $|\psi'| \leq \frac{4}{R}$ and

$$
\psi(r) = 
\begin{cases}
1, & r \leq \frac{3}{2}R; \\
0, & r > 2R.
\end{cases}
$$

Let $\phi(x) = \psi^m(r(x))$ where $m > 0$ is some large number to be chosen later, then

$$
|\nabla \phi|^2 \leq \frac{100m^2}{R^2} \phi^{2-2/m}.
$$

For any $0 < \tau < T \leq 1$, we have

$$
\int_\tau^T \int_{\Omega} \phi^2 e^{\frac{\tau}{2}} f^2 \partial_t \xi + 2 \phi^2 e^{\frac{\tau}{2}} f \partial f
\leq \int_\tau^T \int_{\Omega} \phi^2 e^{\frac{\tau}{2}} f^2 \partial_t \xi + 2 \phi^2 e^{\frac{\tau}{2}} f \Delta f
= \int_\tau^T \int_{\Omega} \phi^2 e^{\frac{\tau}{2}} f^2 \partial_t \xi - 2 \phi^2 e^{\frac{\tau}{2}} f \langle \nabla \xi, \nabla f \rangle - 2 \phi^2 e^{\frac{\tau}{2}} |\nabla f|^2 - 4 \phi e^{\frac{\tau}{2}} f \langle \nabla \phi, \nabla f \rangle
\leq \int_\tau^T \int_{\Omega} \phi^2 e^{\frac{\tau}{2}} f^2 (\partial_t \xi + |\nabla \xi|^2) + 4 |\nabla \phi|^2 e^{\frac{\tau}{2}} f^2
\leq 4 \int_\tau^T \int_{\Omega} |\nabla \phi|^2 e^{\frac{\tau}{2}} f^2.
$$

By the choice of $\phi$, using similar arguments as in [4], we can apply the Hölder inequality and Young’s inequality to show the following.
\[
\int |\nabla \phi|^2 e^\xi f^2 \leq \frac{100m^2}{R^2} \int_{\text{spt}(\nabla \phi)} \phi^{2-2/m} e^\xi f^2 \\
\leq \frac{100m^2}{R^2} \left( \int_{\text{spt}(\nabla \phi)} \phi^2 e^\xi f^2 \right)^{(1-1/m)} \left( \int_{\text{spt}(\nabla \phi)} e^\xi f^2 \right)^{1/m} \\
\leq \frac{1}{4t} \left( \int_{\text{spt}(\nabla \phi)} \phi^2 e^\xi f^2 \right) + \frac{C(m)^{m-1}}{R^{2m}} \left( \int_{\text{spt}(\nabla \phi)} e^\xi f^2 \right),
\]

where \( C(m) = 400^{2m-1}m^{2m} \) and \( \text{spt}(\nabla \phi) \) is the compact support of \( \nabla \phi \). Combines with the previous inequality, we obtain

\[
\frac{1}{T} \int \phi^2 e^\xi f^2(T) - \frac{1}{\tau} \int \phi^2 e^\xi f^2(\tau) = \int_\tau^T \int \phi^2 e^\xi f^2 \partial_\xi + 2\phi^2 e^\xi f \partial f - \int_\tau^T \int \phi^2 e^\xi f^2 \]

\[
\leq \frac{C(m)}{R^{2m}} \int_\tau^T \int_{\text{spt}(\nabla \phi)} e^\xi f^2.
\]

On \( \text{spt}(\nabla \phi) \subset B_q(2R) \backslash B_q(\frac{3}{2}R) \), we have

\[
\xi \leq -\frac{R^2}{16(T - \tau)}.
\]

Therefore, if we choose \( m > a + 2 \), the growth assumption on \( f \) will imply

\[
\int_\tau^T \int_{\text{spt}(\nabla \phi)} e^\xi f^2 \leq T^{m-a-2} e^{-\frac{R^2}{16(T - \tau)}} L(2R).
\]

Now if we require that

\[
(T - \tau) \leq \frac{R^2}{16L(2R)},
\]

then

\[
\frac{1}{T} \int_{B(R)} f^2(T) - \frac{1}{\tau} \int_{B(2R)} f^2(\tau) \leq \frac{C(m)T^{m-a-2}}{R^{2m}}.
\]

To proceed, we take an increasing sequence of \( R_i \), and a decreasing sequence of \( \tau_i \) in the following way. Let \( R_i = 2^i R, \tau_0 = \tau \) and take \( \tau_{i+1} \) such that

\[
\tau_i - \frac{R_i^2}{16L(2R)} \leq \tau_{i+1} < \tau_i.
\]

Then for any \( N \), apply (2.4) inductively we have
\[
\frac{1}{\tau} \int_{B(R)} f^2(\tau) = \frac{1}{\tau_N} \int_{B(R_N)} f^2(\tau_N) + \sum_{i=0}^{N-1} \left( \frac{1}{\tau_i} \int_{B(R_i)} f^2(\tau_i) - \frac{1}{\tau_{i+1}} \int_{B(R_{i+1})} f^2(\tau_{i+1}) \right)
\]
\[
\leq \frac{1}{\tau_N} \int_{B(R_N)} f^2(\tau_N) + \sum_{i=0}^{N-1} C(m)\tau_i^{m-1} \frac{R_i^m}{R_i^m}
\]
\[
\leq \frac{1}{\tau_N} \int_{B(R_N)} f^2(\tau_N) + \frac{2C(m)\tau_i^{m-1} \frac{R_i^m}{R_i^m}}{R_i^m}.
\]

By the assumption on \(L(r)\) (1.2), we must have
\[
\sum_{i=0}^{\infty} \frac{R_i^m}{L(R_i)} = \infty,
\]
hence we can choose the sequence \(\{\tau_i\}\) such that \(\tau_i\) becomes zero in finite steps.

To show that the first term in the last line of (2.5) can be dropped, we claim that for any \(R > 0\), we have
\[
\lim_{t \to 0^+} \frac{1}{t} \int_{B(R)} f^2(t) = 0.
\]

To prove the claim. For any cut-off function \(\phi\), since \(\lim_{t \to 0^+} \int f^2\phi^2(t) = 0\), we have
\[
0 \geq \int_0^t \int \phi^2 f(\partial_t f - \Delta f)
= \frac{1}{2} \int \phi^2 f^2(t) + \int_0^t \int \phi^2 |\nabla f|^2 + 2 \int_0^t \int \phi \langle \nabla \phi, \nabla f \rangle
\]
\[
\geq \frac{1}{2} \int \phi^2 f^2(t) - \int_0^t \int |\nabla \phi|^2 f^2.
\]

Choose a cut-off function \(\phi\) similarly as before such that \(|\nabla \phi| \leq C\phi^{1-1/m}\) for some \(m \geq 2\), then the above inequality yields
\[
\int \phi^2 f^2(t) \leq C \int_0^t \int (\phi^2 f^2)^{\frac{m-1}{m}} f^2
\]
\[
\leq C \int_0^t \left( \int \phi^2 f^2 \right)^{\frac{m-1}{m}} \left( \int_{spt(\phi)} f^2 \right)^{\frac{1}{m}}
\]
\[
\leq C \sup_{s \in (0,t)} \|f(s)\|_{L^2(spt(\phi))} \int_0^t \left( \int \phi^2 f^2 \right)^{\frac{m-1}{m}} t.
\]

Since the RHS is nondecreasing in \(t\), we have
\[
\left( \sup_{s \in (0,t)} \int \phi^2 f^2(s) \right)^{\frac{1}{m}} \leq C \sup_{s \in (0,t)} \|f(s)\|_{L^2(spt(\phi))} t,
\]
and hence $\int \phi^2 f^2(t) = o(t^m)$. Since the cut-off function $\phi$ is chosen for an arbitrary radius, this proves the claim.

By letting $R \to \infty$ in (2.5), we show that $f(\tau) \equiv 0$ for any $\tau \in (0, 1]$. This completes the proof.

**Proof of Theorem 1.2** By \cite{9}, the curvature assumption (1.3) implies that there is a Sobolev inequality in the following form:

$$
\left( \int \phi^{2n} \right)^{\frac{n-2}{n}} \leq \frac{R^2 e^{C(n)Rk(R)}}{Vol(B_q(R))^2} \int (|\nabla \phi|^2 + R^{-2} \phi^2),
$$

for any smooth function $\phi$ compactly supported in the geodesic ball $B_q(R)$. With the Sobolev inequality, we can apply Nash–Moser iteration to prove a mean value inequality for $f$ (see Chapter 19 of \cite{6}), for any $t \in (0, 1]$, where $\alpha, \beta, \gamma$ are positive constants depending on $n$ and $p$. Without loss of generality, we can assume $R > 2$ and hence

$$
Vol(B_q(R/2)) \geq v_0 := Vol(B_q(1)).
$$

Now the assumption of the theorem implies

$$
|f(x, t)| \leq \frac{e^{Cr(x)|2(x)|}}{t^a}, \quad t \in (0, 1],
$$

for some constants $C$ depending on $n, p, v_0$, and the constant $a$ only depends on $n$ and $p$. By (1.1) and volume comparison theorem, we have the volume growth estimate

$$
Vol(B_q(R)) \leq C(n)e^{C(n)Rk(R)},
$$

for any $R > 0$, see for example \cite{2}. Hence, we have

$$
\int_0^1 t^a \int_{B(R)} f^2 \leq e^{L(R)},
$$

with

$$
L(R) = CRk(2R),
$$

for some constant $C$. By (1.4), the function $L(R)$ satisfies (1.2). The result now follows from Theorem 1.1.

□
3 Example

In this section, we describe the construction of a solution to the heat equation, which belongs to the uniqueness class of Theorem 1.1, but not in that of [1, 3] or [8]. Intuitively, we want to construct a solution which has a sequence of ‘spikes’ with fast growing heights, while supported on decaying domains so that we have some integral control of the solution locally.

Take \( M = \mathbb{R}^n \) with \( n \geq 3 \), and we will make several assumptions for simplicity; however, the same method can be used to construct more complicated examples.

To start with, let \( \tilde{u}_0 \) be a continuous function on \( \mathbb{R}^n \) with growth rate slower than \( e^{C|\cdot|} \).

For simplicity, we take \( \tilde{u}_0 \geq 0 \) and \( \tilde{u}_0 \in L^1(\mathbb{R}^n) \).

We will construct a “spiked” initial function \( u_0 \) by modifying \( \tilde{u}_0 \): for each positive integer \( i = 1, 2, 3, \ldots \), choose a geodesic ball

\[
B(p_i, r_i) \subset B(0, i + 1) \setminus B(0, i),
\]

where the radii is chosen to be

\[
r_i = \left( \frac{1}{\omega_n i^2 e^{3i}} \right)^{\frac{1}{n}}.
\]

Here \( \omega_n \) is the volume of the unit ball in \( \mathbb{R}^n \). Denote

\[
\bar{r}_i = \frac{r_i}{2^{1/n}}.
\]

We now modify \( \tilde{u}_0 \) in each \( B(p_i, r_i) \) to obtain the desired initial data \( u_0 \). Define

\[
\begin{aligned}
  u_0 &= \begin{cases} 
  e^{3i}, & \text{on } B(p_i, \bar{r}_i), \\
  \text{continuous and } e^{3i}, & \text{on } B(p_i, r_i) \setminus B(p_i, \bar{r}_i), \\
  \tilde{u}_0, & \text{otherwise}.
  \end{cases}
\end{aligned}
\]

The new function \( u_0 \) is a continuous function which is \( L^1 \) on the modified region \( \bigcup_{i=1}^{\infty} B(p_i, r_i) \).

Solve the Cauchy problem of the heat equation with initial function \( u_0 \) by convoluting with the heat kernel:

\[
u(x, t) = \int \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy.
\]

For each \( x \in B(p_i, \bar{r}_i) \) and \( t > 0 \),

\[
u(x, t) \geq \frac{1}{(4\pi t)^{n/2}} \int_{B(p_i, \bar{r}_i)} e^{-\frac{|x-y|^2}{4t}} u_0(y)dy \\
\geq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{a^2}{4t}} e^{3i} \omega_n \bar{r}_i^n \\
= \frac{1}{2i^2 (4\pi t)^{n/2}} e^{-\frac{j^2}{4t}}.
\]

Hence
Thus $u$ violates the assumption in either [3] or [1] when $n \geq 3$. For $L^p$ integrals with $p > 1$ one can compute similarly.

On the other hand, since we assumed $u_0$ to be $L^1$, we have

$$|u(x,t)| \leq \frac{\|u_0\|_{L^1(\mathbb{R}^n)}}{(4\pi t)^{n/2}},$$

hence it satisfies the assumption of Theorem 1.1.

To construct examples which are not in $L^1$, we can start with $\tilde{u} \equiv 1$ instead of a $L^1$ function; and to construct examples not bounded from either side, we can add a sequence of “negative spikes” to $u_0$ sufficiently far away from the positive one.

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