NOTE ON THE DEFINITION OF NEUTROSOPHIC LOGIC

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Abstract. Smarandache introduced a new logic called "neutrosophic logic". Its definition contains many misuses of nonstandard analysis, and its description is entirely hand-waving. In this note, we describe a rigorous definition of neutrosophic logic and correct all the errors in the original definition. We point out some problems concerning neutrosophic logic. Furthermore, we formulate neutrosophic logic with no use of nonstandard analysis.

1. Introduction

Smarandache [5] introduced a new logic called "neutrosophic logic". In this logic, each proposition takes a value of the form \((T, I, F)\), where \(T, I, F\) are subsets of the nonstandard unit interval \([-0, 1^+]\) and represent all possible values of Truthness, Indeterminacy and Falsity of the proposition, respectively. Unfortunately, its definition contains many misuses of nonstandard analysis. Furthermore, the description is entirely hand-waving.

In section 2, we describe a rigorous definition of neutrosophic logic. All the errors involved in the original definition are corrected. We point out some problems concerning neutrosophic logic, e.g., the paradox where complex propositions may have strange truth values. In section 3, we give an alternative definition of neutrosophic logic without any use of nonstandard analysis. Note that we mainly focus on mathematical correctness, but not on efficacy to mathematics, philosophy, engineering, or any other areas.

2. Correction of the definition

2.1. Confusion of notation. Smarandache used the symbols \(-a\) and \(b^+\) as particular hyperreal numbers.

Let \(\varepsilon > 0\) be a such infinitesimal number. [...] Let’s consider the nonstandard finite numbers \(1^+ = 1 + \varepsilon\), where "1" is its standard part and "\(\varepsilon\)" its non-standard part, and \(-0 = 0 - \varepsilon\), where "0" is its standard part and "\(\varepsilon\)" its non-standard part. ([5] p. 141; [6] p. 9)

Smarandache also used the symbols "\(-a\)" and "\(b^+\)" as particular sets of hyperreal numbers.

Actually, by "\(-a\)" one signifies a monad, i.e., a set of hyper-real numbers in non-standard analysis:

\[
(-a) = \{ a - x \in \mathbb{R}^* \mid x \text{ is infinitesimal} \},
\]

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and similarly \(b^+\) is a hyper monad:
\[
(b^+) = \{ b + x \in \mathbb{R}^* \mid x \text{ is infinitesimal} \}.
\]
([5] p. 141; [6] p. 9)

This confusion of notation can be found thereafter. In [6] pp.10–11, \(-a\) and \(b^+\) were used as hypermonads. On the other hand, in [6] p.13, 1\(^+\) was used as a particular hyperreal number. Note that the definitions of (one-sided) monads have minor errors. The correct definitions are the following:
\[
(-a) = \{ a - x \in \mathbb{R}^* \mid x \text{ is positive infinitesimal} \},
\]
\[
(b^+) = \{ b + x \in \mathbb{R}^* \mid x \text{ is positive infinitesimal} \}.
\]

2.2. Ambiguity of the definition of the nonstandard unit interval. Smarandache did not give any explicit definition of the notation \([-0, 1^+]\) in [5] (or the notation \([-0, 1^+]\) in [6]). He only said:

Then, we call \([-0, 1^+]\) a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. ([5] p. 141; [6] p. 9)

Here -0 and 1\(^+\) are particular real numbers defined in the previous paragraph: 
\(-0 = 0 - \varepsilon\) and \(1^+ = 1 + \varepsilon\), where \(\varepsilon\) is a fixed non-negative infinitesimal. (Note that the phrase “infinitely small but less than 0” is intended to mean “infinitely close to but less than 0”.) Similarly, the phrase “infinitely small but greater than 1” is intended to mean “infinitely close to but greater than 1".) There are two possible definitions of the nonstandard unit interval:

1. \([-0, 1^+] = \{ x \in \mathbb{R}^* \mid -0 < x < 1^+ \}\) following a usual (but rare) notation of open interval;
2. \([-0, 1^+] = \{ x \in \mathbb{R}^* \mid 0 < x \leq 1 \}\).

In the first definition, it is false that nonstandard numbers infinitely close to but less than 0 or infinitely close to but greater than 1, belong to the nonstandard unit interval. \(0 - 2\varepsilon\) is infinitely close to 0 but not in \([-0, 1^+]\). Similarly for \(1 + 2\varepsilon\). Thus the first definition is not compatible with the above-quoted sentence. The second definition is better than the first one. If we adopt the first definition, the resulting logic depends on the choice of the positive infinitesimal \(\varepsilon\). It is not beautiful. In our corrected definition, we shall adopt the second definition.

Remark 2.1. Smarandache mistakenly believes that the monad can be described as an open intervals of \(\mathbb{R}^*\).

We can consider \((-a)\) equals to the open interval \((a - \varepsilon, a)\), where \(\varepsilon\) is a positive infinitesimal number. Thus:
\[
(-a) = (a - \varepsilon, a)
\]
\[
(b^+) = (b, b + \varepsilon)
\]
\[
(-a^+) = (a - \varepsilon_1, a) \cup (a, a + \varepsilon_2),\]
where \(\varepsilon_1, \varepsilon_2\) are positive infinitesimal numbers. ([6] p. 10)

Obviously it is wrong. Suppose, on the contrary, that \((-a)\) can be expressed in the form \((a - \varepsilon, a)\). Then \(a - \varepsilon\) does not belong to \((-a)\). On the other hand, \(a - \varepsilon\) is less than but infinitely close to \(a\), so \(a - \varepsilon \in (-a)\), a contradiction. This false belief well explains why Smarandache fell into the confusion of notation and why he gave
only an ambiguous definition to the nonstandard unit interval: if the monad could be described like above, two definitions of the unit interval would be equivalent.

2.3. Misuse of nonstandard analysis. Let us continue to read the definition.

Let $T, I, F$ be standard or non-standard real subsets of $]-0,1+[,$
with $\sup T = t_{\sup}, \inf T = t_{\inf},$
$\sup I = i_{\sup}, \inf I = i_{\inf},$
$\sup F = f_{\sup}, \inf F = f_{\inf},$
and $n_{\sup} = t_{\sup} + i_{\sup} + f_{\sup},$
$n_{\inf} = t_{\inf} + i_{\inf} + f_{\inf}.$
The sets $T, I, F$ are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countably or uncountably) infinite; union or intersection of various subsets; etc. ([5] pp. 142–143; [6] p. 12)

Subsets of $\mathbb{R}^\ast,$ even bounded, may have neither infima nor suprema, because the transfer principle ensures the existences of infima and suprema only for internal sets. External sets may lack suprema and/or infima. For instance, the monad $\mu(1/2) = \{ x \in \mathbb{R}^\ast \mid x \approx 1/2 \}$ has neither the infimum nor the supremum. To see this, suppose, on the contrary, that $\mu(1/2)$ has the infimum $L = \inf \mu(1/2).$ Let $\varepsilon$ be any positive infinitesimal. Then there is an $x \in \mu(1/2)$ such that $x \leq L + \varepsilon.$ Since every point infinitely close to $\mu(1/2)$ belongs to $\mu(1/2),$ we have that $x - 2\varepsilon \in \mu(1/2).$ Hence $L \leq x - 2\varepsilon \leq L - \varepsilon,$ a contradiction. Similarly for the supremum.

There are two prescriptions:

1. inserting the sentence “assume that $T, I, F$ are internal” or “assume that $T, I, F$ have infima and suprema”; or
2. giving up the use of infima and suprema in formulating neutrosophic logic.

When we adopt the first prescription, the whole interval $]-0,1+[,$ cannot be a value of any proposition. So we cannot consider “completely ambiguous” propositions, none of whose truthness, indeterminacy and falsity are (even roughly) determined. The second prescription is better than the first one. The first reason is that none of infima and suprema are necessary to formulate neutrosophic logic. The second one is that we want to consider propositions with external values such as $]-0,1+[.$ In our definition, we shall adopt the second prescription.

2.4. Rigorous definition of neutrosophic logic. Now let us correct the definition of neutrosophic logic. We define the nonstandard unit interval as follows:

$]-0,1+[ = \{ x \in \mathbb{R}^\ast \mid 0 \leq x \leq 1 \}.$

Let $\mathcal{V}$ be the power set of $]-0,1+[,$ the collection of all subsets of $]-0,1+[.$ Define binary operators on $\mathcal{V}$ as follows:

$A \otimes B = \{ ab \mid a \in A, b \in B \},$
$A \oplus B = \{ a + b - ab \mid a \in A, b \in B \},$
$A \oslash B = \{ c - a + ab \mid a \in A, b \in B, c \in 1^+ \}.$

We need to verify the following.

Lemma 2.2. $\mathcal{V}$ is closed under these operations.
Proof. Consider the following standard operations on $\mathbb{R}$:

\[
\begin{align*}
  f(a, b) &= ab, \\
  g(a, b) &= a + b - ab, \\
  h(c, a, b) &= c - a + ab.
\end{align*}
\]

Notice that they are continuous everywhere. The standard unit interval $[0, 1]$ is closed under $f, g$ and $h(1, \cdot, \cdot)$. Here we only prove the case of $h(1, \cdot, \cdot)$. Since $h(1, a, b)$ is monotonically decreasing for $a$, we have that $\min_{(a,b) \in [0,1]^2} h(1, a, b) = \min_{b \in [0,1]} h(1, 1, b) = 0$. Similarly, since $h(1, a, b)$ is monotonically increasing for $b$, we have that $\max_{a \in [0,1]} h(1, a, 1) = 1$. Thus $h(\{1\} \times [0,1] \times [0,1]) \subseteq [0, 1]$.

Now, let $a, b \in ]^{-0,1^+}$. Choose $a', b' \in [0, 1]$ infinitely close to $a, b$, respectively. Of course, $c$ is infinitely close to $1$. By the nonstandard characterisation of continuity (see Theorem 4.2.7 of [1]), $f^*(a, b), g^*(a, b), h^*(c, a, b)$ are infinitely close to $f(a', b'), g(a', b'), h(1, a', b') \in [0, 1]$, respectively. Hence $f^*(a, b), g^*(a, b), h^*(c, a, b) \in ]^{-0,1^+}$.

Let $A, B \in \mathbb{V}$. Then $A \ominus B = f^*(A \times B)$, $A \oslash B = g^*(A \times B)$ and $A \oslash B = h^*(1^+ \times A \times B)$ are contained in $\mathbb{V}$.

According to the original definition ([5] p. 143), neutrosophic logic is the $\mathbb{V}^3$-valued (extensional) logic. Each proposition takes a value of the form $(T, I, F) \in \mathbb{V}^3$, where $T$ represents possible values of truthness, $I$ indeterminacy, and $F$ falsity. The logical connectives $\land, \lor, \to$ are interpreted as follows:

\[
\begin{align*}
  (T_1, I_1, F_1) \land (T_2, I_2, F_2) &= (T_1 \land T_2, I_1 \land I_2, F_1 \land F_2), \\
  (T_1, I_1, F_1) \lor (T_2, I_2, F_2) &= (T_1 \lor T_2, I_1 \lor I_2, F_1 \lor F_2), \\
  (T_1, I_1, F_1) \to (T_2, I_2, F_2) &= (T_1 \to T_2, I_1 \to I_2, F_1 \to F_2).
\end{align*}
\]

Remark 2.3. Smarandache used the following operations instead of $\ominus$ and $\oslash$ ([5] p. 145):

\[
\begin{align*}
  A \oslash B &= (A \oplus B) \ominus (A \ominus B), \\
  A \oslash B &= 1^+ \ominus A \ominus (A \ominus B),
\end{align*}
\]

where $\ominus, \odot, \oplus$ are the elementwise subtraction, multiplication and addition of sets. There are at least two reasons why the original definition is not good. The first is pre-mathematical. When calculating, for example, $A \oslash' B = (A \oplus B) \ominus (A \ominus B)$, the second and the third occurrences of $A$ can take different values, despite that they represent the same proposition. The same applies to $B$ and the calculation of $A \oslash' B$ obviously. The second is mathematical. $\mathbb{V}$ is not closed under those operations:

\[
\begin{align*}
  2 &= 1 + 1 - 0 \cdot 1 \in \{0, 1\} \otimes \{1\}, \\
  2 + \varepsilon &= 1 + \varepsilon - 0 + 1 \cdot 1 \in \{0, 1\} \otimes \{1\},
\end{align*}
\]

where $\varepsilon$ is positive infinitesimal. Because of this, Smarandache was forced to do an ad-hoc workaround:

[... if, after calculations, one obtains number $< 0$ or $> 1$, one replaces them by $-0$ or $1^+$, respectively. ([5] p. 145)]
2.5. **Paradoxical phenomena.** Consider the $\mathbb{V}$-valued logic, where each proposition takes a *truth* value $T \in \mathbb{V}$. Each neutrosophic logical connectives was defined componentwise. In other words, neutrosophic logic is the 3-fold product of the $\mathbb{V}$-valued logic. Hence neutrosophic logic cannot be differentiated from the $\mathbb{V}$-valued logic by equational properties (see Lemma 11.3 of [2] Chapter II).

This causes some counterintuitive phenomena. Let $A$ be a (true) proposition with value $(\{1\}, \{0\}, \{0\})$ and let $B$ be a (false) proposition with value $(\{0\}, \{0\}, \{1\})$. Usually we expect that the falsity of the conjunction $A \land B$ is $\{1\}$. However, its actual falsity is $\{0\}$. We expect that the indeterminacy of the negation $\neg A$ is $\{0\}$. However, its actual indeterminacy is $1^+$ (see [5] p. 145 for the definition of the negation). These phenomena propose to modify and improve the definition of neutrosophic logic.

3. **Neutrosophic logic without nonstandard analysis**

3.1. **Nonarchimedean fields.** Let $K$ be an ordered field. It is well-known that the ordered semiring of natural numbers $\mathbb{N}$ can be canonically embedded into $K$ by sending $n \mapsto 1_K + \cdots + 1_K$. This embedding can be uniquely extended to the ordered ring of integers $\mathbb{Z}$ and to the ordered field of rational numbers $\mathbb{Q}$. Thus we may assume without loss of generality that $\mathbb{Q} \subseteq K$. An element $x \in K$ is said to be *infinitesimal* (relative to $\mathbb{Q}$) if $|x| \leq q$ for any positive $q \in \mathbb{Q}$. For instance, the unit of the addition $0_K$ is trivially infinitesimal. The ordered field $K$ is called *nonarchimedean* if it has nonzero infinitesimals.

**Example 3.1.** Every ordered subfield of the real field $\mathbb{R}$ is archimedean. Conversely, every archimedean ordered field can be (uniquely) embedded into $\mathbb{R}$ (see Theorem 10.21 of [1]).

**Example 3.2.** The hyperreal field $\mathbb{R}^*$ is a nonarchimedean ordered field. Generally, every proper extension $K$ of $\mathbb{R}$ is nonarchimedean: Let $x \in K \setminus \mathbb{R}$. There are two cases. Case I: $x$ is *infinite* (i.e. its absolute value $|x|$ is an upper bound of $\mathbb{R}$). Then its reciprocal $1/x$ is nonzero infinitesimal. Case II: $x$ is finite. Then the set $\{y \in \mathbb{R} | y < x\}$ is nonempty and bounded in $\mathbb{R}$. So it has the supremum $x^\ast$. The difference $x - x^\ast$ is nonzero infinitesimal. Hence $K$ is nonarchimedean.

**Example 3.3** (cf. [3] pp. 15–16). Let $K$ be an ordered field. Let $K(X)$ be the field of rational functions over $K$. Define an ordering on $K(X)$ by giving a positive cone:

$$0 \leq \frac{f(X)}{g(X)} \iff 0 \leq \frac{\text{the leading coefficients of } f(X)}{\text{the leading coefficients of } g(X)}.$$

Then $K(X)$ forms an ordered field having nonzero infinitesimals (relative to not only $\mathbb{Q}$ but also $K$) such as $1/X$ and $1/X^2$. Hence $K(X)$ is nonarchimedean.

3.2. **Alternative definition of neutrosophic logic.** Comparing with other nonarchimedean fields, one of the essential features of the hyperreal field $\mathbb{R}^*$ is the transfer principle, which states that $\mathbb{R}^*$ has the same first order properties as $\mathbb{R}$. On the other hand, neutrosophic logic does not depend on transfer, so the use of nonstandard analysis is not essential for this logic, and can be eliminated from its definition.

Fix a nonarchimedean ordered field $K$. Let $x, y \in K$. $x$ and $y$ are said to be *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal. We say that $x$ is *roughly*
smaller than $y$ (and write $x \lessdot y$) if $x < y$ or $x \approx y$. For $a, b \in \mathbb{K}$ the set $]-a, b+[_{\mathbb{K}}$ is defined as follows:

$$]-a, b+[_{\mathbb{K}} = \{ x \in \mathbb{K} \mid a \lessdot x \lessdot b \}.$$  

Let $\mathcal{V}_{\mathbb{K}}$ be the power set of $]-0, 1+[_{\mathbb{K}}$. A new neutrosophic logic can be defined as the $\mathcal{V}_{\mathbb{K}}^3$-valued logic. The rest of the definition is completely the same as the case $\mathbb{K} = \mathbb{R}^*$.  

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