η-Ricci Solitons on 3-dimensional Trans-Sasakian Manifolds

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ABSTRACT

In this paper, we study η-Ricci solitons on 3-dimensional trans-Sasakian manifolds. Firstly we give conditions for the existence of these geometric structures and then observe that they provide examples of η-Einstein manifolds. In the case of φ-Ricci symmetric trans-Sasakian manifolds, the η-Ricci soliton condition turns them to Einstein manifolds. Afterward, we study the implications in this geometric context of the important tensorial conditions \( R \cdot S = 0 \), \( S \cdot R = 0 \), \( W_2 \cdot S = 0 \) and \( S \cdot W_2 = 0 \).

RESUMEN

En este artículo estudiamos solitones η-Ricci en variedades trans-Sasakianas tridimensionales. En primer lugar damos condiciones para la existencia de estas estructuras geométricas y luego observamos que ellas dan ejemplos de variedades η-Einstein. En el caso de variedades trans-Sasakianas φ-Ricci simétricas, la condición de solitón η-Ricci las convierte en variedades Einstein. A continuación estudiamos las implicancias en este contexto geométrico de las importantes condiciones tensoriales \( R \cdot S = 0 \), \( S \cdot R = 0 \), \( W_2 \cdot S = 0 \) y \( S \cdot W_2 = 0 \).

Keywords and Phrases: Trans-Sasakian manifold, η-Ricci solitons.

2010 AMS Mathematics Subject Classification: 53C21, 53C25, 53C44.
1 Introduction

In 1982, the notion of the Ricci flow was introduced by Hamilton [10] to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for Riemannian metric $g(t)$ on a smooth manifold $M$ given by

$$\frac{\partial}{\partial t} g(t) = -2S.$$ 

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics $g(t)$, parameterized by $t$ in a non-degenerate interval $I$, on a smooth manifold $M$ satisfying the Ricci flow equation. If $I$ has an initial point $t_0$, then $(M, g(t_0))$ is called the initial condition of or the initial metric for the Ricci flow (or of the solution) [14].

Ricci solitons and $\eta$-Ricci solitons are natural generalizations of Einstein metrics. A Ricci soliton on a Riemannian manifold $(M, g)$ is defined by

$$S + \frac{1}{2} L_X g = \lambda g$$

where $L_X g$ is the Lie derivative along the vector field $X$, $S$ is the Ricci tensor of the metric and $\lambda$ is a real constant. If $X = \nabla f$ for some function $f$ on $M$, the Ricci soliton becomes gradient Ricci soliton. Ricci solitons appear as self-similar solutions to Hamilton’s Ricci flow and often arise as limits of dilations of singularities in the Ricci flow [11]. A soliton is called shrinking, steady and expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

In 2009, the notion of $\eta$-Ricci soliton was introduced by J.C. Cho and M. Kimura [6]. J.C. Cho and M. Kimura proved that a real hypersurface admitting an $\eta$-Ricci soliton in a non-flat complex space form is a Hopf-hypersurface [6]. An $\eta$-Ricci soliton on a Riemannian manifold $(M, g)$ is defined by the following equation

$$2S + L_\xi g + 2\lambda g + 2\mu \eta \otimes \eta = 0, \quad (1.1)$$

where $L_\xi$ is the Lie derivative operator along the vector field $\xi$, $S$ is the Ricci tensor of the metric and $\lambda$, $\mu$ are real constants. If $\mu = 0$, then $\eta$-Ricci soliton becomes Ricci soliton.

In the last few years, many authors have worked on Ricci solitons and their generalizations in different Contact metric manifolds in [1], [7], [8], [9], [12] etc. In 2014, B. Y. Chen and S. Deshmukh have established the characterizations of compact shrinking trivial Ricci solitons in [5]. Also, in [2], A. Bhattacharyya, T. Dutta, and S. Pahan studied the torqued vector field and established some applications of torqued vector field on Ricci soliton and conformal Ricci soliton. A.M. Blaga [3], D. G. Prakasha and B. S. Hadimani [17] observed $\eta$-Ricci solitons on different contact metric manifolds satisfying some certain curvature conditions.
In this paper we study the existence of $\eta$-Ricci soliton on 3-dimensional trans-Sasakian manifold. Next we show that $\eta$-Ricci soliton on 3-dimensional trans-Sasakian manifolds becomes $\eta$-Einstein Manifold under some conditions. Next we prove that $\phi$-Ricci symmetric trans-Sasakian manifold $(M, g)$ manifold satisfying an $\eta$-Ricci soliton becomes an Einstein manifold. Next we give an example of an $\eta$-Ricci soliton on 3-dimensional trans-Sasakian manifold with $\lambda = -2$ and $\mu = 6$. Later we obtain some different types of curvature tensors and their properties under certain conditions.

2 Preliminaries

The product $\tilde{M} = M \times \mathbb{R}$ has a natural almost complex structure $J$ with the product metric $G$ being Hermitian metric. The geometry of the almost Hermitian manifold $(\tilde{M}, J, G)$ gives the geometry of the almost contact metric manifold $(M, \phi, \xi, \eta, g)$. Sixteen different types of structures on $M$ like Sasakian manifold, Kenmotsu manifold etc are given by the almost Hermitian manifold $(\tilde{M}, J, G)$.

The notion of trans-Sasakian manifolds was introduced by Oubina [15] in 1985. Then J. C. Marrero [13] have studied the local structure of trans-Sasakian manifolds. In general a trans-Sasakian manifold $(M, \phi, \xi, \eta, g, \alpha, \beta)$ is called a trans-Sasakian manifold of type $(\alpha, \beta)$. An $n (= 2m + 1)$ dimensional Riemannian manifold $(M, g)$ is called an almost contact manifold if there exists a $(1,1)$ tensor field $\phi$, a vector field $\xi$, and a 1-form $\eta$ on $M$ such that

\begin{align}
\phi^2(X) &= -X + \eta(X)\xi, \quad (2.1) \\
\eta(\xi) &= 1, \eta(\phi X) = 0, \quad (2.2) \\
\phi \xi &= 0, \quad (2.3) \\
\eta(X) &= g(X, \xi), \quad (2.4) \\
g(\phi X, \phi Y) &= g(X, Y) - \eta(Y)\eta(X), \quad (2.5) \\
g(X, \phi Y) + g(Y, \phi X) &= 0, \quad (2.6)
\end{align}

for any vector fields $X, Y$ on $M$. A 3-dimensional almost contact metric manifold $M$ is called a trans-Sasakian manifold if it satisfies the following condition

\begin{align}
(\nabla_X \phi)(Y) &= \alpha(g(X, Y)\xi - \eta(Y)\xi) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \quad (2.7)
\end{align}

for some smooth functions $\alpha, \beta$ on $M$ and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$. For 3-dimensional trans-Sasakian manifold, from (2.7) we have,

\begin{align}
\nabla_X \xi &= -\alpha \phi X + \beta(X - \eta(X)\xi), \quad (2.8)
\end{align}
\[(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).\] (2.9)

In a 3-dimensional trans-Sasakian manifold, we have

\[
R(X, Y)Z = \left[ \frac{r}{2} - 2(\alpha^2 - \beta^2 - \xi \beta) \right] [g(Y, Z)X - g(X, Z)Y] \\
- \left[ \frac{r}{2} - 3(\alpha^2 - \beta^2) + \xi \beta \right] [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \xi \\
+ [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] [\phi \text{grad } \alpha - \text{grad } \beta] \\
- \left[ \frac{r}{2} - 3(\alpha^2 - \beta^2) + \xi \beta \right] [\eta(Y)X - \eta(X)Y] \\
- [Z\beta + (\phi Z)\alpha] [\eta(Y)X - \eta(X)Y] \\
- [X\beta + (\phi X)\alpha] [g(Y, Z)\xi - \eta(Z)Y] \\
- [Y\beta + (\phi Y)\alpha] [g(X, Z)\xi - \eta(Z)X],
\]

\[
S(X, Y) = \left[ \frac{r}{2} - (\alpha^2 - \beta^2 - \xi \beta) \right] g(X, Y) \\
- \left[ \frac{r}{2} - 3(\alpha^2 - \beta^2) + \xi \beta \right] \eta(X) \eta(Y) \\
- [Y\beta + (\phi Y)\alpha] [\eta(Y)X - \eta(X)Y] \\
- [X\beta + (\phi X)\alpha] [\eta(Y)X - \eta(X)Y].
\]

When \(\alpha\) and \(\beta\) are constants the above equations reduce to,

\[
R(\xi, X)\xi = (\alpha^2 - \beta^2)(\eta(X)\xi - X),
\] (2.10)

\[
S(X, \xi) = 2(\alpha^2 - \beta^2)\eta(X),
\] (2.11)

\[
R(\xi, X)Y = (\alpha^2 - \beta^2)(g(X, Y)\xi - \eta(Y)X).
\] (2.12)

\[
R(X, Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y).
\] (2.13)

**Definition 2.1.** A trans-Sasakian manifold \(M^3\) is said to be \(\eta\)-Einstein manifold if its Ricci tensor \(S\) is of the form

\[S(X, Y) = a g(X, Y) + b \eta(X) \eta(Y),\]

where \(a, b\) are smooth functions.
3 $\eta$-Ricci solitons on trans-Sasakian manifolds

To study the existence conditions of $\eta$-Ricci solitons on 3-dimensional trans-Sasakian manifolds, we prove the following theorem.

**Theorem 3.1:** Let $(M, g, \phi, \eta, \xi, \alpha, \beta)$ be a 3-dimensional trans-Sasakian manifold with $\alpha$, $\beta$ constants ($\beta \neq 0$). If the symmetric $(0, 2)$ tensor field $h$ satisfying the condition $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu \eta(X)\eta(Y)$ is parallel with respect to the Levi-Civita connection associated to $g$. Then $(g, \xi, \mu)$ becomes an $\eta$-Ricci soliton.

**Proof:** We consider a symmetric $(0,2)$-tensor field $h$ which is parallel with respect to the Levi-Civita connection ($\nabla h = 0$). Then it follows that

$$h(R(X,Y)Z, W) + h(R(X,Y)Z, W) = 0,$$

(3.1)

for an arbitrary vector field $W$, $X$, $Y$, $Z$ on $M$. Put $X = Z = W = \xi$, we get

$$h(R(X,Y)\xi, \xi) = 0,$$

(3.2)

for any $X$, $Y \in \chi(M)$ By using the equation (2.13)

$$h(Y, \xi) = g(Y, \xi)h(\xi, \xi),$$

(3.3)

for any $Y \in \chi(M)$. Differentiating the equation (3.3) covariantly with respect to the vector field $X \in \chi(M)$ we have

$$h(\nabla_X Y, \xi) + h(Y, \nabla_X \xi) = g(\nabla_X Y, \xi)h(\xi, \xi) + g(Y, \nabla_X \xi)h(\xi, \xi),$$

(3.4)

Using the equation (2.8) we have

$$\beta h(X, Y) - \alpha h(\phi X, Y) = -\alpha g(\phi X, Y)h(\xi, \xi) + \beta h(\xi, \xi)g(X, Y).$$

(3.5)

Interchanging $X$ by $Y$ we have

$$\beta h(X, Y) - \alpha h(X, \phi Y) = -\alpha g(X, \phi Y)h(\xi, \xi) + \beta h(\xi, \xi)g(X, Y).$$

(3.6)

Then adding the above two equations we get

$$\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \beta h(\xi, \xi)g(X, Y).$$

(3.7)

We see that $\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)]$ is a symmetric tensor of type $(0, 2)$. Since $\mathcal{L}_\xi g(X, Y)$, $S(X, Y)$, $\eta(X) = g(X, \xi)$ and $\eta(Y) = g(Y, \xi)$ are symmetric tensors of type $(0, 2)$ and $\lambda$, $\mu$ are real constants, the sum $\mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu \eta(X)\eta(Y)$ is a symmetric tensor of type $(0, 2)$. 

Therefore, we can take the sum as an another symmetric tensor field of type (0, 2). Hence for we can assume that \( \beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu \eta(X)\eta(Y) \).

Then we compute
\[
\beta h(\xi, \xi) g(X, Y) = \mathcal{L}_\xi g(X, Y) + 2\lambda g(X, Y) + 2\mu \eta(X)\eta(Y).
\]

As \( h \) is parallel so, \( h(\xi, \xi) \) is constant. Hence, we can write \( h(\xi, \xi) = -\frac{\lambda}{2} \) where \( \beta \) is constant and \( \beta \neq 0 \).

So, from the equation (3.7) we have
\[
\beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = -2\lambda g(X, Y),
\]
for any \( X, Y \in \chi(M) \). Therefore \( \mathcal{L}_\xi g(X, Y) + 2S(X, Y) + 2\mu \eta(X)\eta(Y) = -2\lambda g(X, Y) \) and so \( (g, \xi, \mu) \) becomes an \( \eta \)-Ricci soliton.

**Corollary 3.2:** Let \( (M, g, \phi, \eta, \xi, \alpha, \beta) \) be a 3-dimensional trans-Sasakian manifold with \( \alpha, \beta \) constants \( (\beta \neq 0) \). If the symmetric (0, 2) tensor field \( h \) admitting the condition \( \beta h(X, Y) - \frac{\alpha}{2}[h(\phi X, Y) + h(X, \phi Y)] = \mathcal{L}_\xi g(X, Y) + 2S(X, Y) \) is parallel with respect to the Levi-Civita connection associated to \( g \) with \( \lambda = 2n \). Then \( (g, \xi) \) becomes a Ricci soliton.

Next theorem shows the necessary condition for the existence of \( \eta \)-Ricci soliton on 3-dimensional trans-Sasakian manifolds.

**Theorem 3.3:** If 3-dimensional trans-Sasakian manifold satisfies an \( \eta \)-Ricci soliton then the manifold becomes \( \eta \)-Einstein manifold with \( \alpha \) and \( \beta \) constants.

**Proof:** From the equation (1.1) we get
\[
2S(X, Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X, Y) - 2\mu \eta(X)\eta(Y).
\]

By using the equation (2.8) we get
\[
S(X, Y) = -(\beta + \lambda)g(X, Y) + (\beta - \mu)\eta(X)\eta(Y)
\]
and
\[
S(X, \xi) = -(\lambda + \mu)\eta(X).
\]

Also from (2.11) we have
\[
\lambda + \mu = 2(\beta^2 - \alpha^2).
\]

The Ricci operator \( Q \) is defined by \( g(QX, Y) = S(X, Y) \). Then we get
\[
QX = (\mu - \beta + 2(\alpha^2 - \beta^2))X + (\beta - \mu)\eta(X)\xi.
\]
Then we can easily see that the manifold is an \( \eta \)-Einstein manifold.

We know a manifold is \( \phi \)-Ricci symmetric if \( \phi^2 \circ \nabla Q = 0 \). Now we prove the next theorem.

**Theorem 3.4:** If a \( \phi \)-Ricci symmetric trans-Sasakian manifold \((M, g)\) satisfies an \( \eta \)-Ricci soliton then \( \mu = \beta, \lambda = 2(\beta^2 - \alpha^2) - \beta \) and \((M, g)\) is an Einstein manifold.

**Proof:** From the equation (3.13) we have

\[
(\nabla_X Y) = \nabla_X Y - Q(\nabla_X Y)
\]

\[
= -\alpha(\beta - \mu)\eta(Y)\phi X + \beta(\beta - \mu)\eta(Y)X - (\beta - \mu)\eta(Y)\eta(X)e
\]

\[
+(\beta - \mu)[-\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y)]e.
\]

Now applying \( \phi^2 \) both sides we have \( \mu = \beta, \lambda = 2(\beta^2 - \alpha^2) - \beta \) and \((M, g)\) is an Einstein manifold.

We construct an example of \( \eta \)-Ricci soliton on 3-dimensional trans-Sasakian manifolds in the next section.

### 4 Example of \( \eta \)-Ricci solitons on 3-dimensional trans-Sasakian manifolds

We consider the three dimensional manifold \( M = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\} \) where \((x, y, z)\) are the standard coordinates in \( \mathbb{R}^3 \). The vector fields

\[
e_1 = e^{2x} \frac{\partial}{\partial x}, e_2 = e^{2z} \frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z}
\]

are linearly independent at each point of \( M \). Let \( g \) be the Riemannian metric defined by

\[
g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}
\]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M^3) \). Let \( \phi \) be the \((1,1)\) tensor field defined by \( \phi(e_1) = e_2, \phi(e_2) = -e_1, \phi(e_3) = 0 \). Then using the linearity property of \( \phi \) and \( g \) we have

\[
\eta(e_2) = 1, \phi^2(Z) = -Z + \eta(Z)e_2, \ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),
\]

for any \( Z, W \in \chi(M^3) \). Thus for \( e_2 = \xi, (\phi, \xi, \eta, g) \) defines an almost contact metric structure on \( M \). Now, after some calculation we have,
\[ [e_1, e_3] = -2e_1, [e_2, e_3] = -2e_2, [e_1, e_2] = 0. \]

The Riemannian connection \( \nabla \) of the metric is given by the Koszul’s formula which is

\[
2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g([Y, Z]) - g([X, Z]) + g(Z, [X, Y]).
\]

By Koszul’s formula we get,

\[
\nabla e_1 e_1 = 2e_3, \quad \nabla e_2 e_1 = 0, \quad \nabla e_1 e_2 = 0, \quad \nabla e_2 e_2 = 2e_3, \\
\nabla e_3 e_2 = 0, \quad \nabla e_2 e_3 = -2e_1, \quad \nabla e_1 e_3 = -2e_2, \quad \nabla e_3 e_3 = 0.
\]

From the above it can be easily shown that \( M^3(\phi, \xi, \eta, g) \) is a trans-Sasakian manifold of type \((0, -2)\).

Here

\[
R(e_1, e_2)e_2 = -4e_1, \quad R(e_3, e_2)e_2 = 4e_2, \quad R(e_1, e_3)e_3 = -4e_1, \quad R(e_2, e_3)e_3 = -4e_2,
\]

\[
R(e_3, e_1)e_1 = -4e_2, \quad R(e_2, e_1)e_1 = 4e_3.
\]

So, we have

\[
S(e_1, e_1) = 0, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = -8. \tag{4.1}
\]

From the equation (1.1) we get \( \lambda = -2 \) and \( \mu = 6 \). Therefore, \((g, \xi, \lambda, \mu)\) is an \( \eta \)-Ricci soliton on \( M^3(\phi, \xi, \eta, g) \).

In the next sections we consider \( \eta \)-Ricci Solitons on 3-dimensional trans-Sasakian manifolds satisfying some curvature conditions.

5 \( \eta \)-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying \( R(\xi, X) \cdot S = 0 \)

First we suppose that 3-dimensional trans-Sasakian manifolds with \( \eta \)-Ricci solitons satisfy the condition

\[
R(\xi, X) \cdot S = 0.
\]

Then we have

\[
S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0.
\]
for any \( X, Y, Z \in \chi(M) \).

Using the equations (2.12), (3.10), (3.11) we get

\[
(\beta - \mu)g(X, Y)\eta(Z) + (\beta - \mu)g(X, Z)\eta(Y) - 2(\beta - \mu)\eta(X)\eta(Y)\eta(Z) = 0.
\]

Put \( Z = \xi \) we have

\[
(\beta - \mu)g(X, Y) - (\beta - \mu)\eta(X)\eta(Y) = 0.
\]

Setting \( X = \phi X \) and \( Y = \phi Y \) in the above equation we get

\[
(\beta - \mu)g(\phi X, \phi Y) = 0.
\]

Again using the equation (3.12) we have

\[
\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta.
\]

Also we can easily see that \( M \) is an Einstein manifold. So we have the following theorem.

**Theorem 5.1:** If a 3-dimensional trans-Sasakian manifold \((M, g, \phi, \eta, \xi, \alpha, \beta)\) with \( \alpha, \beta \) constants admitting an \( \eta \)-Ricci soliton satisfies the condition \( R(\xi, X) \cdot S = 0 \) then \( \mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta \) and \( M \) is an Einstein manifold.

**Corollary 5.2:** A 3-dimensional trans-Sasakian manifold with \( \alpha, \beta \) constants satisfies the condition \( R(\xi, X) \cdot S = 0 \), there is no Ricci soliton with the potential vector field \( \xi \).

6  \quad \eta\text{-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying } S(\xi, X) \cdot R = 0

We consider 3-dimensional trans-Sasakian manifolds with \( \eta \)-Ricci solitons satisfying the condition

\[
S(\xi, X) \cdot R = 0.
\]
So we have

\[ S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W - S(\xi, Y)R(X, Z)W \\
+ S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W + S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0. \]

Taking inner product with \( \xi \) then the above equation becomes

\[ S(X, R(Y, Z)W) \xi = - S(\xi, R(Y, Z)W)X + S(X, Y)\eta(R(\xi, Z)W) - S(\xi, Y)\eta(R(X, Z)W) \\
+ S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0. \]  

(6.1)

Put \( W = \xi \) and using the equations (2.10), (2.12), (3.10), (3.11) we get

\[ (\beta + \lambda)g(X, R(Y, Z)\xi) + (\lambda + \mu)\eta(R(Y, Z)X) = 0. \]  

(6.2)

Also we have

\[ \eta(R(Y, Z)X) = -g(X, R(Y, Z)\xi). \]

So from the equation (6.2) we get

\[ (\beta + 2\lambda + \mu)g(X, R(Y, Z)\xi) = 0. \]

Again using the equation (3.12) we have

\[ \mu = \beta + 4(\beta^2 - \alpha^2), \; \lambda = -2(\beta^2 - \alpha^2) + \beta]. \]

So we have the following theorem.

**Theorem 6.1:** If a 3-dimensional trans-Sasakian manifold \( (M, g, \phi, \eta, \xi, \alpha, \beta) \) with \( \alpha, \beta \) constants admitting an \( \eta \)-Ricci soliton satisfies the condition \( S(\xi, X) \cdot R = 0 \) then \( \mu = \beta + 4(\beta^2 - \alpha^2), \; \lambda = -2(\beta^2 - \alpha^2) + \beta]. \).
Corollary 6.2: A 3-dimensional trans-Sasakian manifold with $\alpha$, $\beta$ constants satisfies the condition $S(\xi, X) \cdot R = 0$, there is no Ricci soliton with the potential vector field $\xi$.

7 $\eta$-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying $W_2(\xi, X) \cdot S = 0$

Definition 7.1. Let $M$ be 3-dimensional trans-Sasakian manifold with respect to semi-Symmetric metric connection. The $W_2$-curvature tensor of $M$ is defined by [16]

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{2}(g(X, Z)QY - g(Y, Z)QX).$$ (7.1)

We assume 3-dimensional trans-Sasakian manifolds with $\eta$-Ricci solitons satisfying the condition $W_2(\xi, X) \cdot S = 0$.

Then we have

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0$$

for any $X, Y, Z \in \chi(M)$.

Using the equations (2.12), (3.10), (3.11), (7.1) we get

$$\begin{align*}
\left[-\frac{(\beta + \lambda)}{2}(\lambda + \mu) + \frac{(\beta + \lambda)^2}{2} + (\beta - \mu)(\alpha^2 - \beta^2) + (\lambda + \mu)\frac{(\beta - \mu)}{2}\right]g(X, Y)\eta(Z) \\
+\left[\frac{(\beta + \lambda)^2}{2} - \frac{(\beta + \lambda)}{2}(\lambda + \mu) + (\beta - \mu)(\alpha^2 - \beta^2) + (\lambda + \mu)\frac{(\beta - \mu)}{2}\right]g(X, Z)\eta(Y) \\
+\left[-(\beta + \lambda)(\beta - \mu) - 2(\beta - \mu)(\alpha^2 - \beta^2) - (\beta - \mu)(\lambda + \mu)\eta(X)\eta(Y)\eta(Z) = 0.\right]
\end{align*}$$

Put $Z = \xi$ in the above equation we get

$$\begin{align*}
\left[-\frac{(\beta + \lambda)}{2}(\lambda + \mu) + \frac{(\beta + \lambda)^2}{2} + (\beta - \mu)(\alpha^2 - \beta^2) + (\lambda + \mu)\frac{(\beta - \mu)}{2}\right]g(X, Y) \\
+\left[\frac{(\beta + \lambda)^2}{2} - \frac{(\beta + \lambda)}{2}(\lambda + \mu) + (\beta - \mu)(\alpha^2 - \beta^2) + (\lambda + \mu)\frac{(\beta - \mu)}{2}\right]g(X, Z)\eta(Y) \\
+\left[-(\beta + \lambda)(\beta - \mu) - 2(\beta - \mu)(\alpha^2 - \beta^2) - (\beta - \mu)(\lambda + \mu)\eta(X)\eta(Y)\eta(Z) = 0.\right]
\end{align*}$$
\[-(\beta + \lambda)(\beta - \mu) - 2(\beta - \mu)(\alpha^2 - \beta^2) - (\beta - \mu)(\lambda + \mu)]\eta(X)\eta(Y) = 0.\]

Setting \(X = \phi X\) and \(Y = \phi Y\) in the above equation we get
\[(\beta - \mu)((\beta + 2\lambda + \mu + 2(\alpha^2 - \beta^2))g(\phi X, \phi Y) = 0.\]

Again using the equation (3.12) we have
\[
\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta
\]
or
\[
\mu = 2(\beta^2 - \alpha^2) + \beta, \quad \lambda = -\beta.
\]

So we have the following theorem.

**Theorem 7.1:** If a 3-dimensional trans-Sasakian manifold \((M, g, \phi, \eta, \xi, \alpha, \beta)\) with \(\alpha, \beta\) constants admitting an \(\eta\)-Ricci soliton satisfies the condition \(W_2(\xi, X) \cdot S = 0\) then \(\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta\) or \(\mu = 2(\beta^2 - \alpha^2) + \beta, \quad \lambda = -\beta\).

**Corollary 7.2:** A 3-dimensional trans-Sasakian manifold with \(\alpha, \beta\) constants satisfies the condition \(W_2(\xi, X) \cdot S = 0\), there is no Ricci soliton with the potential vector field \(\xi\).

### 8 \(\eta\)-Ricci solitons on 3-dimensional trans-Sasakian manifolds satisfying \(S(\xi, X) \cdot W_2 = 0\)

Suppose that 3-dimensional trans-Sasakian manifolds with \(\eta\)-Ricci solitons satisfy the condition
\[S(\xi, X) \cdot W_2 = 0.\]

So we have
\[
S(X, W_2(Y, Z)V)\xi - S(\xi, W_2(Y, Z)V)X + S(X, Y)W_2(\xi, Z)V - S(\xi, Y)W_2(X, Z)V
+ S(X, Z)W_2(Y, \xi)V - S(\xi, Z)W_2(Y, X)V + S(X, V)W_2(Y, Z)\xi - S(\xi, V)W_2(Y, Z)X = 0.
\]
Taking inner product with $\xi$, then the above equation becomes

\[
S(X, W_2(Y, Z) V) - S(\xi, W_2(Y, Z) V) \eta(X) + S(X, Y) \eta(W_2(\xi, Z) V) \\
- S(\xi, Y) \eta(W_2(X, Z) V) + S(X, Z) \eta(W_2(Y, \xi) V) - S(\xi, Z) \eta(W_2(Y, X) V) \\
+ S(X, V) \eta(W_2(Y, Z) \xi) - S(\xi, V) \eta(W_2(Y, Z) X) = 0. \tag{8.1}
\]

Put $V = \xi$ and using the equations (2.10), (2.12), (3.10), (3.11), (7.1) we get

\[- (\beta + \lambda) g(X, W_2(Y, Z) \xi) + (\lambda + \mu) \eta(W_2(Y, Z) X) = 0. \tag{8.2}\]

Using the equations (3.10), (3.11), (7.1) then the equation (8.2) becomes

\[
[(\beta + \lambda)^2 + (\lambda + \mu)^2 + 2(\alpha^2 - \beta^2)(\beta + 2\lambda + \mu)] g(X, R(Y, Z) \xi) = 0.
\]

Using the equation (3.12) we have

\[
\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta
\]

or

\[
\mu = 2(\beta^2 - \alpha^2) + \beta, \quad \lambda = -\beta.
\]

So we have the following theorem.

**Theorem 8.1:** If a 3-dimensional trans-Sasakian manifold $(M, g, \phi, \eta, \xi, \alpha, \beta)$ with $\alpha, \beta$ constants admitting an $\eta$-Ricci soliton satisfies the condition $S(\xi, X) \cdot W_2 = 0$ then $\mu = \beta, \quad \lambda = 2(\beta^2 - \alpha^2) - \beta$ or $\mu = 2(\beta^2 - \alpha^2) + \beta, \quad \lambda = -\beta$.

**Corollary 8.2:** A 3-dimensional trans-Sasakian manifold with $\alpha, \beta$ constants satisfies the condition $S(\xi, X) \cdot W_2 = 0$, there is no Ricci soliton with the potential vector field $\xi$.

**Acknowledgement:** The author wish to express her sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.
References

[1] C. S. Bagewadi, G. Ingalahalli, S. R. Ashoka, A study on Ricci solitons in Kenmotsu Manifolds, ISRN Geometry, (2013), Article ID 412593, 6 pages.

[2] A. Bhattacharyya, T. Dutta, and S. Pahan, Ricci Soliton, Conformal Ricci Soliton And Torqued Vector Fields, Bulletin of the Transilvania University of Brasov Series III: Mathematics, Informatics, Physics,, Vol 10(59), No. 1 (2017), 39-52.

[3] A. M. Blaga, Eta-Ricci solitons on p-Kenmotsu manifolds, Balkan Journal of Geometry and Its Applications, Vol.20, No.1, 2015, pp. 1-13.

[4] C. Călin, M. Crasmareanu, Eta-Ricci solitons on Hopf hypersurfaces in complex forms, Revue Roumaine de Math. Pures et app., 57 (1), (2012), 53-63.

[5] B. Y. Chen, S. Deshmukh, Geometry of compact shrinking Ricci solitons, Balkan Journal of Geometry and Its Applications, Vol.19, No.1, 2014, pp. 13-21

[6] J.C. Cho, M. Kimura Ricci solitons and real hypersurfaces in a complex space form, Tohoku Math. J. 61 (2), (2009), 205-2012.

[7] O. Chodosh, F. T.-H Fong, Rational symmetry of conical Kähler-Ricci solitons, Math. Ann., 364(2016), 777-792.

[8] A. Futaki, H. Ono, G. Wang, Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds, J. Diff. Geom. 83 (3), (2009), 585-636.

[9] S. Golab, On semi-symmetric and quarter-symmetric linear connection, Tensor. N. S., 29(1975), 249-254.

[10] R. S. Hamilton, The formation of singularities in the Ricci flow, Surveys in Differential Geometry (Cambridge, MA, 1993), 2, 7-136, International Press, Combridge, MA, 1995.

[11] R. S. Hamilton, The Ricci flow on surfaces, Mathematical and general relativity, Contemp. math, 71(1988), 237-261.

[12] G. Ingalahalli, C. S. Bagewadi, Ricci solitons on α-Sasakian Manifolds, ISRN Geometry, (2012), Article ID 421384, 13 pages.

[13] J. C. Marrero, The local structure of trans-Sasakian manifolds, Ann. Mat. Pura. Appl., (4), 162(1992), 77-86.

[14] J. Morgan, G. Tian, Ricci Flow and the Poincaré Conjecture, American Mathematical Society Clay Mathematics Institute, (2007).
[15] J. A. Oubina, *New classes of almost contact metric structures*, pub. Math. Debrecen, 20 (1), (2015), 1-13.

[16] G. P. Pokhariyal, R. S. Mishra, *The curvature tensors and their relativistic significance*, Yokohama Math. J., 18(1970), 105-106.

[17] D. G. Prakasha, B. S. Hadimani, *η-Ricci solitons on para-Sasakian manifolds*, Journal of Geometry, (2016), DOI: 10.1007/s00022-016-0345-z, pp 1-10.