Topology at zero and finite $T$ in $SU(2)$ Yang-Mills theory

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Abstract

We determine the topological susceptibility $\chi$ at $T = 0$ and its behaviour at finite $T$ across the deconfining transition in pure $SU(2)$ gauge theory. We use an improved topological charge density operator. $\chi$ goes to zero above $T_c$, but more slowly than in $SU(3)$ gauge theory.
1 Introduction

The anomalous breaking of the flavour singlet axial symmetry in QCD is driven by instanton effects [1]. This breaking brings about a large mass for the pseudoscalar singlet [2, 3]

\[ m_{\eta'}^2 = \frac{2N_f}{f^2_\pi} \chi - m_{\eta'}^2 + 2 m_K^2. \]  

(1)

\( \chi \) is the topological susceptibility of the pure gauge theory

\[ \chi \equiv \int d^4x \langle 0 | T(Q(x)Q(0)) | 0 \rangle_{\text{quenched}}, \]  

(2)

with

\[ Q(x) = \frac{g^2}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x). \]  

(3)

The prediction of eq. (1) is \( \chi \approx (180 \text{ MeV})^4 \).

The behaviour of \( \chi \) at high temperature \( T \) has also physical relevance (see for instance [4]). Debye screening in the quark–gluon plasma is expected to produce a suppression of the topological susceptibility above the deconfining temperature \( T_c \) [5].

In [6] we determined, by a numerical simulation on the lattice, \( \chi \) at zero and finite temperature in the pure SU(3) gauge theory. At \( T = 0 \) we obtained the value \( (\chi)^{1/4} = 175(5) \text{ MeV} \), which is consistent with previous determinations [7, 8] and with the prediction of eq. (1). We also showed that \( \chi \) keeps approximately constant below \( T_c \) and has a sharp drop at the transition point \( T = T_c \). At \( T/T_c \approx 1.4 \) the susceptibility \( \chi \) reduces to a few per cent of its value before the transition.

In this paper we determine \( \chi \) for SU(2) pure gauge theory both at \( T = 0 \) and at the deconfining transition with the same technique used in ref. [6].

In Section 2 we review the method. Our results are presented in Section 3. In section 4 we give some concluding remarks.

2 The method

The topological charge was measured on the lattice with the operators [9, 10]

\[ Q^{(i)}_L(x) = -\frac{1}{2\pi^2} \sum_{\mu\nu\rho\sigma=\pm1}^4 \tilde{\epsilon}_{\mu\nu\rho\sigma} \text{Tr} \left( \Pi^{(i)}_{\mu\nu}(x) \Pi^{(i)}_{\rho\sigma}(x) \right). \]  

(4)
\( \tilde{\epsilon}_{\mu\nu\rho\sigma} \) is the standard Levi-Civita tensor for positive directions while for negative ones the relation \( \tilde{\epsilon}_{\mu\nu\rho\sigma} = -\tilde{\epsilon}_{-\mu\nu\rho\sigma} \) holds. \( \Pi^{(i)}_{\mu\nu} \) is the plaquette in the \( \mu - \nu \) plane constructed with \( i \)-times smeared links \( U_{\mu}^{(i)}(x) \). These smeared links are defined recursively starting from the ordinary link \( U_{\mu}(x) \) as

\[
U_{\mu}^{(0)}(x) = U_{\mu}(x),
\]

\[
U_{\mu}^{(i)}(x) = (1 - c)U_{\mu}^{(i-1)}(x) + \frac{c}{6} \sum_{\alpha = \pm 1}^{\pm 4} U_{\alpha}^{(i-1)}(x)U_{\mu}^{(i-1)}(x + \hat{\alpha})U_{\alpha}^{(i-1)}(x + \hat{\mu})^\dagger,
\]

\[
U_{\mu}^{(i)}(x) = \frac{\overline{U}_{\mu}^{(i)}(x)}{\left(\frac{1}{2} \text{Tr} \overline{U}_{\mu}^{(i)}(x) \overline{U}_{\mu}^{(i)}(x)\right)^{1/2}}.
\] (5)

The parameter \( c \) can be tuned in order to optimize the improvement of the operator. We choose \( c = 0.85 \) and measure the topological charge for \( i = 0, 1, 2 \).

The topological susceptibility from the \( i \)-smeared operator on the lattice is calculated by

\[
\chi_{L}^{(i)} = \langle \sum x Q_{L}^{(i)}(x)Q_{L}^{(i)}(0) \rangle. \] (6)

\( \chi_{L}^{(i)} \) mixes with the continuum susceptibility \( \chi \) and with all renormalization group invariant operators of dimension \( \leq 4 \), i.e. the trace of the energy-momentum tensor and the unity operator [11]. The mixing to continuum \( \chi \) is described by the square of the (finite) multiplicative renormalization of the topological charge operator \( Q_{L}^{(i)} \) to the continuum operator \( Q \) [12]:

\[
Q_{L}^{(i)} = Z^{(i)}(\beta)Qa^4 + O(a^6). \] (7)

Therefore the following relation holds

\[
\chi_{L}^{(i)} = Z^{(i)}(\beta)a(\beta)^4 \chi + M^{(i)}(\beta) + O(a^6), \] (8)

where \( M^{(i)}(\beta) \) is the mixing to the trace of the energy-momentum tensor and to the unity operator. As usual \( \beta \equiv 2N_c/g^2 \) where \( N_c \) is the number of colours and \( g \) the gauge coupling.

The additive renormalization \( M^{(i)}(\beta) \) can be determined by thermalizing the short range fluctuations starting from a zero field configuration, without changing the (zero) topological content of it [13, 14, 15, 16, 6]. We
start with the flat configuration (all links \( U_\mu(x) = 1 \)) and create a sample of configurations by applying some heat–bath steps. At each step we measure the topological susceptibility. The content of instantons is checked on intermediate steps, by cooling a copy of the configuration. Configurations where instantons or antiinstantons have been produced are eliminated from the sample. A plateau is reached after \( \sim 10 - 20 \) steps that keeps constant along \( \sim 100 \) updating steps. The value for \( M^{(i)}(\beta) \) is the average of this plateau on the ensemble. The number of discarded configurations depends on the value of \( \beta \). At \( \beta = 2.5 \) the rate of discarded configurations per heating step is \( \sim 1\% \); at \( \beta = 2.6 \) it drops to \( \sim 0.5\% \) and is well below \( 0.1\% \) at \( \beta = 2.7 \).

By subtracting \( M^{(i)}(\beta) \) we naturally impose the condition that the physical topological susceptibility be zero on topologically trivial configurations, thus matching the continuum renormalization prescription for \( \chi \).

The multiplicative renormalization \( Z^{(i)}(\beta) \) can be determined in a similar way. We thermalize short range fluctuations starting from a 1 instanton configuration. At each updating step we measure the topological charge. Again checks are performed to eliminate configurations where the topological content of the starting configuration has been changed during the updating procedure. \( Q_L^{(i)} \) is then measured: it stabilizes on a plateau where \( Q_L^{(i)} = Z^{(i)}(\beta)Q \), and the value of \( Z^{(i)}(\beta) \) is extracted from the average on these plateaux.

The statistical accuracy in the determination of the physical value of \( \chi \) strongly depends on \( M^{(i)}(\beta) \) and \( Z^{(i)}(\beta) \), which in their turn depend on the lattice regularization \( Q_L^{(i)} \) used for the topological charge. The improvement of the operator results in a better accuracy [10, 11].

### 3 Results

The zero temperature determination of \( \chi \) has been done on a \( 16^4 \) lattice at three different values of \( \beta \) with the Wilson action. Statistical errors have been estimated by using a standard blocking procedure. To fix the scale of length we refer to ref. [17]. We put

\[
a(\beta) = \frac{1}{\Lambda_L} \lambda(\beta) f(\beta)
\]

(9)
where \( f(\beta) \) is the usual 2–loop scaling function and \( \lambda(\beta) \) is a corrective factor tabulated in Table 1 of ref. [17], which tends to one at large \( \beta \), where asymptotic scaling sets up.

The results for \( \chi/\Lambda_L^4 \) at zero temperature for 0,1,2–smearings are shown in Table I and Figure 1. Scaling is observed in each of them. The three determinations (0,1,2-smearings) agree within errors. The smearing process has indeed improved the result by drastically reducing the error bars.

To convert to physical units we need a determination of \( \Lambda_L \). We get it by using the result of ref. [17] \( T_c/\Lambda_L = 21.45(14) \), and that of ref. [18] \( T_c/\sqrt{\sigma} = 0.62(2) \) (\( \sigma \) is the string tension). The result is \( \Lambda_L = 14.15(42) \) MeV, where, as usual, we have assumed \( \sqrt{\sigma} = 440 \) MeV. Combining the 2–smeared results at the three \( \beta \) values we obtain \( (\chi)^{1/4} = (198\pm2\pm6) \) MeV, where the first error comes from our determination and the second from the uncertainty on \( \Lambda_L \).

We have made a comparison with existing determinations of \( \chi \) at \( T = 0 \). We agree with ref. [19] where \( (\chi)^{1/4} = 200(15) \) MeV is obtained by use of improved cooling. In ref. [20] 230(30) MeV is quoted for the same quantity, which is computed by a different (improved) action and with an improved geometric algorithm. The result of refs. [11, 21], when converted in MeV by use of the same scale as in this paper, is \( (178 \pm 1 \pm 5) \) MeV. However there the renormalization constants were computed by perturbation theory because the method of ref. [13] did not exist yet. In ref. [22] \( \chi \approx 130 \) MeV is obtained which is lower; we are not able to trace back what part of the difference is due to the different method of computing \( (\chi)^{1/4} \) (cooling) and what part comes from the determination of the scale \( a \). As for ref. [23] the comparison will be done systematically in a forthcoming paper [24].

At finite temperature we used a \( 32^3 \times 8 \) lattice: at this size the deconfining transition is located at \( \beta_c = 2.5115(40) \) [18]. The temperature \( T \) as a function of \( \beta \) is given by

\[
T = \frac{1}{N_\tau a(\beta)}.
\]  

The results for 1,2–smearings are shown in Table II and figure 2: the data for the 0–smeared operator have very large errors above the deconfining temperature and are not shown in the figure.

At \( T < T_c \) our data are consistent with the value at zero temperature, while a drop is observed above the deconfining transition. This behaviour was also observed in the \( SU(3) \) gauge theory [4]; however in the \( SU(3) \) case
the drop is quite steeper than in the present case. This qualitative difference between the two gauge theories can be well appreciated in Figure 3, where \( \chi/\chi(T=0) \) is plotted versus \( T/T_c \) for both \( SU(2) \) and \( SU(3) \).

4 Concluding remarks

We have determined the topological susceptibility of \( SU(2) \) pure gauge theory at zero temperature and its behaviour through \( T_c \). The improvement of the topological charge density operator \([10]\) has made this determination possible. If, to fix the scale, the \( SU(2) \) string tension is assumed to be the physical one, \( \chi(T=0) \) results slightly larger than for \( SU(3) \). \( \chi \) is approximately constant below \( T_c \), and drops to zero above the transition, however more slowly than for \( SU(3) \).

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Figure captions

Figure 1. $\chi$ at $T = 0$. The straight line is the result of the linear fit of the 2-smeared data. The improvement from $Q_L^{(0)}$ to $Q_L^{(2)}$ is clearly visible. $\Lambda_L = 14.15$ MeV was used to fix the scale by eq. (9).

Figure 2. $\chi/\Lambda_L^4$ versus $T/T_c$ across the deconfining phase transition for 1 and 2 smearings. The horizontal band is the determination at $T = 0$ of Figure 1.

Figure 3. The ratio $\chi/\chi(T=0)$ as a function of $T/T_c$ for $SU(2)$ and $SU(3)$ for the 2-smeared data.

Table captions

Table I. $\chi/\Lambda_L^4$ from the 0,1 and 2-smeared operators at $T = 0$. $\chi^{(i)}$ means the continuum susceptibility obtained from the $i$-smeared operator.

Table II. $T/T_c$, $\chi^{(1)}/\Lambda_L^4$, $\chi^{(2)}/\Lambda_L^4$ as a function of $\beta$. Same notation as in Table I.
Table I

| $\beta$ | $10^{-4} \times \chi^{(0)}/\Lambda_L^4$ | $10^{-4} \times \chi^{(1)}/\Lambda_L^4$ | $10^{-4} \times \chi^{(2)}/\Lambda_L^4$ |
|---------|----------------------------------|----------------------------------|----------------------------------|
| 2.44    | 4.7(2.1)                         | 3.73(30)                         | 3.76(26)                         |
| 2.5115  | 5.6(2.0)                         | 3.96(27)                         | 3.85(19)                         |
| 2.57    | 3.7(2.1)                         | 3.97(39)                         | 3.84(27)                         |

Table II

| $\beta$ | $T/T_c$ | $10^{-4} \times \chi^{(1)}/\Lambda_L^4$ | $10^{-4} \times \chi^{(2)}/\Lambda_L^4$ |
|---------|---------|----------------------------------|----------------------------------|
| 2.40    | 0.695   | 3.88(46)                         | 3.45(40)                         |
| 2.42    | 0.743   | 3.42(34)                         | 3.26(27)                         |
| 2.44    | 0.793   | 3.71(29)                         | 3.66(24)                         |
| 2.48    | 0.904   | 3.45(28)                         | 3.53(24)                         |
| 2.5115  | 1.000   | 3.60(31)                         | 3.54(25)                         |
| 2.54    | 1.095   | 2.99(26)                         | 2.41(18)                         |
| 2.57    | 1.203   | 2.40(16)                         | 2.28(12)                         |
| 2.60    | 1.320   | 1.92(17)                         | 1.81(11)                         |
| 2.65    | 1.538   | 1.46(22)                         | 1.42(17)                         |
| 2.70    | 1.786   | 1.06(17)                         | 0.83(8)                          |
Fig. 2

The graph illustrates the behavior of $\frac{\chi}{\Lambda_L^4}$ as a function of $T/T_c$. The data points are represented for different smearings:

- **1 smear** indicated by circles.
- **2 smear** indicated by squares.

The solid line represents $\frac{\chi}{\Lambda_L^4}$ evaluated at $T=0$. The horizontal shaded region indicates the theoretical expectation for the order parameter at zero temperature.
