The Biased Sampling Profit Extraction Auction

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Abstract

We give an auction for downward-closed environments that generalizes the random sampling profit extraction auction for digital goods of Fiat et al. (2002). The mechanism divides the agents into a market and a sample using a biased coin and attempts to extract the optimal revenue from the sample from the market. The latter step is done with the downward-closed profit extractor of Ha and Hartline (2012). The auction is a 11-approximation to the envy-free benchmark in downward-closed permutation environments. This is an improvement on the previously best known results of 12.5 for matroid and 30.4 for downward-closed permutation environments that are due to Devanur et al. (2012) and Ha and Hartline (2012), respectively.

Economic mechanisms that are less dependent on the assumptions of the environment are more likely to be relevant (cf. Wilson, 1987). The area of prior-free mechanism design attempts to remove the distributional assumption on agents while, at the same time, guaranteeing a good approximation of the optimal revenue.

The performance of a prior-free mechanism is measured with respect to a benchmark. Recently, Hartline and Yan (2011) proposed the envy-free benchmark, denoted by $\text{EFO}(v)$ where $v$ is the valuation vector of the agents. This benchmark is the maximum revenue attainable given that the allocation and payment vectors are envy-free: no agent prefers another’s outcome to her own. A downward-closed environment is one where given a feasible set of agents, all subsets are feasible. A permutation environment is one where the agent identities are randomly permuted with respect to the feasibility constraint. In downward-closed permutation environments, Hartline and Yan (2011) provide detailed justification for the approximation of the envy-free benchmark.

The main approaches to prior-free auctions for digital goods generalize to downward-closed permutation environments. Hartline and Yan (2011) generalized the random sampling auction; Ha and Hartline (2012) generalized the consensus estimate profit extraction auction; and in the present paper we generalize the random sampling profit extraction auction from Fiat et al. (2002). The random sampling profit extraction auction splits the agents into a market and a sample, estimates the optimal profit from the sample, and then attempts to extract that profit from the market.

Ha and Hartline (2012) give a profit extractor for the envy-free benchmark in downward-closed permutation environments. This profit extractor is parameterized by a target valuation profile $\tilde{v}$ and on actual valuation profile $v$ is able to extract at least the profit of the envy-free benchmark $\text{EFO}(\tilde{v})$ when $v$ pointwise dominates $\tilde{v}$, i.e., $v_i \geq \tilde{v}_i$ for all $i$, denoted $v \geq \tilde{v}$ where both $v$ and $\tilde{v}$ are sorted in non-increasing order.

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1For technical reasons the benchmark considered is $\text{EFO}(v^{(2)})$ where $v^{(2)} = (v_2, v_3, \ldots, v_n)$ is same as $v$ but with the highest value lowered to the second highest value.
Lemma 1 (Ha and Hartline, 2012). For downward-closed permutation environments there is a profit extractor parameterized by $\tilde{v}$ that obtains from $v$ at least the envy-free optimal revenue for $\tilde{v}$ if $v \geq \tilde{v}$ and otherwise rejects all agents.

An unbiased partitioning of agents into a market $M$ and sample $S$ would be very unlikely to satisfy pointwise dominance $v^M \geq v^S$ as necessary for the profit extractor of Lemma 1 on $v^M$ to give revenue at least $\text{EFO}(v^S)$. On the other hand, a simple probability of ruin analysis shows that a biased partitioning satisfies the requisite pointwise dominance property with constant probability.

Lemma 2. For partitioning of $N$ into $S$ (with probability $p < 1/2$) and $M$ (otherwise) satisfies $\Pr[v^M \nless v^S] \leq \frac{1}{2}$ and $\Pr[v^M \nless v^S | 1 \in M] \leq \left(\frac{p}{1-p}\right)^2$.

Proof. Consider the following infinite random walk on a straight line: starting from position 0, with probability $p$, move backward one step; otherwise, move forward one step. The position of this random walk describes precisely the difference between the number of agents in $M$ and $S$, where positive value means $M$ has more agents than $S$. The event $v^M \nless v^S$ happens when there exists a time that $M$ has less agents than $S$. Let $r$ be the probability that the random walk eventually takes one step backward from the initial position, we have $r = p + (1-p)r^2$. The first component is the probability of taking one step backward in the first step, and the second component is the probability of the first step being a forward step, then eventually take two steps backward. Solving this equation for $r \in (0,1)$ gives $r = \frac{p}{1-p}$. When we condition on $1 \in M$, our initial position is 1 not 0 and the probability of ruin is $r^2$. If we stop the random walk after finite number $n$ of steps, it only improves the probability of ruin. \hfill \Box

The random sampling profit extraction auction is formally given below with a few modifications for improved performance.

Definition 1 (BSPE$_p$). The biased sampling profit extraction auction parameterized by $p < 0.5$ works as follow.

1. Randomly assign each of the agents to one of three groups $A$, $B$, and $C$ independently with probabilities $p$, $p$, and $1-2p$, respectively.

2. Assume without loss of generality that of the highest valued agent in $A$ has value at least that of the highest valued agent in $B$. Define the market $M = A \cup C$ and sample $S = B$. (If this highest valued agent in $A$ wins in Step 4 and the second highest valued agent in $A \cup B$ is in $B$ increase her payment to this second highest value.)

3. Pad the valuation vectors of $M$ and $S$ with 0’s so that they are equal in length. Let the padded vectors be $v^M$ and $v^S$ respectively.

4. Run the profit extractor parameterized by $v^S$ on the market $M$.

5. If all agents are rejected by the profit extractor and it is feasible to serve agent 1 (the highest valued agent over all), serve her and charge her $v_2$.

Lemma 3 (Incentive Compatibility). For all probabilities $p$, BSPE$_p$ is incentive compatible.
Proof. Fix the partitioning of $A$, $B$, and $C$. No agent in $M$ can change the definition of sets $M$ and $S$ without losing (thus obtaining zero utility). No agent in $S$ can change the definition of sets $M$ and $S$ without obtaining a payment of at least her value (from the parenthetical in Step 2 thus obtaining non-positive utility). Therefore no agent wants to manipulate the definition of $M$ and $S$. For given $M$ and $S$ this mechanism is the profit extraction mechanism which is incentive compatible for fixed $M$ and $S$. Only the highest valued agent would want to win in Step 5; furthermore, she cannot cause dominance to fail without lowering her bid (and forfeiting her status as the highest bidder).

Lemma 4. The envy-free benchmark $\text{EFO}(v)$ for a random sample $S$ of $N$ with each element selected independently with probability $p$ satisfies $E[\text{EFO}(v^S)] \geq p \text{EFO}(v)$.

Proof. Consider the envy-free optimal outcome for $v$. Clearly if we restrict attention only to agents in $S$ there is still no envy. Therefore, $\text{EFO}(v^S) \geq \text{EFO}_S(v)$ where $\text{EFO}_S(v)$ is short-hand notation for the contribution from agents in $S$ to the envy-free optimal revenue on $v$. Of course, $E[\text{EFO}_S(v)] = p \text{EFO}(v)$.

Lemma 5. For any downward-closed permutation environment and any probability $p < 0.5$,

1. $\text{BSPE}_p$ approximates $\text{EFO}(v_{-1})$ to within a factor of $p - \left(\frac{p}{1-p}\right)^2$ where $v_{-1} = (v_2, v_3, \ldots, v_n)$.

2. $\text{BSPE}_p$ approximates $\text{EFO}(v_2)$ to within a factor of $p + (1-p)p^3$ when there are $n \geq 5$ agents.

Proof. To show part 1 of the lemma, we will focus on the revenue obtainable via the profit extraction step. Lemma 4 says that we would obtain at least $\text{EFO}(v^S)$ when $v^M \geq v^S$. Thus the expected revenue is at least:

$$E[\text{BSPE}_p(v)] \geq E[\text{EFO}(v^S) \mid v^M \geq v^S] \cdot \Pr[v^M \geq v^S]$$

$$= E[\text{EFO}(v^S)] - E[\text{EFO}(v^S) \mid v^M \not\geq v^S] \cdot \Pr[v^M \not\geq v^S]$$

$$\geq p \text{EFO}(v_{-1}) - \text{EFO}(v_{-1}) \cdot \Pr[v^M \not\geq v^S]$$

$$= [p - \left(\frac{p}{1-p}\right)^2] \cdot \text{EFO}(v_{-1}).$$

The second inequality warrants some explanation: the first term follows from applying Lemma 4 to $v_{-1}$, the second term follows from monotonicity of EFO.

To show part 2 of the lemma, we analyze the event that player 2 is in the sample and the event that the market pointwise dominates the sample. With probability $p$ agent 2 is in the sample and $\text{EFO}(v^S) \geq \text{EFO}(v_2)$. If the market pointwise dominates the sample then the mechanism obtains this revenue; otherwise, the revenue from agent 1 via Step 5 is at least $\text{EFO}(v_2)$. With probability $(1-p)$ agent 2 is in the market and the probability of $v^M \not\geq v^S$ (implying that the profit extractor fails) is at least $p^3$ by stepping backward three steps in a row (possible when there are more than 5 agents); in this case again the revenue from agent 1 via Step 5 is at least $\text{EFO}(v_2)$. To conclude, the revenue of the mechanism it at least:

$$E[\text{BSPE}_p(v)] \geq [p + (1-p)p^3] \cdot \text{EFO}(v_2).$$

\[2\]The first part of this lemma is non-trivial only for $p < 0.38.$
**Theorem 1.** For any downward-closed permutation environment with probability \( p < 0.5 \) and \( n \geq 5 \) agents, \( \text{BSPE}_p \) approximates \( \text{EFO}(v^{(2)}) \) within a factor of \( \frac{1}{r_1 r_2} \) where \( r_1 = p - \left( \frac{p}{1-p} \right)^2 \) and \( r_2 = p + (1-p)p^3 \). This factor is minimized at 11 when \( p = 0.26 \).

**Proof.** \( \text{EFO}(v_2) + \text{EFO}(v_{-1}) \geq \text{EFO}(v^{(2)}) \) due to subadditivity of EFO function as shown by Hartline and Yan (2011). Combining with the above lemma, we have the desired ratio. \( \Box \)

**References**

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\(^3\)For \( n \leq 4 \) agents the 1-unit Vickrey auction is a 4-approximation to \( \text{EFO}(v^{(2)}) \).