The Fermion Determinant, its Modulus and Phase

Hisashi KIKUCHI

Ohu University
Koriyama 963, Japan

(kikuchi@yukawa.kyoto-u.ac.jp)

Abstract

We consider path integration of a fermionic oscillator with a one-parameter family of boundary conditions with respect to the time coordinate. The dependence of the fermion determinant on these boundary conditions is derived in a closed form with the help of the self-adjoint extension of differential operators. The result reveals its crucial dependence on them, contrary to the conventional understanding that this dependence becomes negligible over sufficiently long time evolution. An example in which such dependence plays a significant role is discussed in a model of supersymmetric quantum mechanics.

PACS: 11.30.Fs; 11.30.Pb; 03.65.-w; 02.70.Hm;

Keywords: path-integral; boundary condition; supersymmetric quantum mechanics;

The determinant arising from fermionic path integral over Grassmann variables brings us important and critical information on models in field theory. Anomalies [1, 2, 3, 4] in gauge theories are a typical example of this kind. They reveal that the symmetry of the classical action breaks down when the fermion fields are integrated out to yield the determinant [3]. They critically restrict the model in its fermion content [4]. Thus any question about the determinant deserves careful investigation. The issue of this letter is the dependence of the determinant on boundary conditions with respect to the time coordinate.

It is useful to formulate fermions in field theory by putting the system in a finite spatial box. Owing to this treatment the one-particle states can be labeled with an integer. To each one-particle state, we assign a fermionic oscillator to distinguish the vacant and occupied states. The fermions are then described by a set of the oscillators which interact mutually through their
coupling to background bosonic degrees of freedom. For example, in Euclidean path integral formalism, a chiral two-component fermion field in a gauge theory is defined by the Lagrangian

\[ L = \sum_{a,b} \left[ \bar{\psi}_a \frac{d}{d\tau} \psi_a + \bar{\psi}_a \left[ \nu_{ab}(\tau) + i\omega_{ab}(\tau) \right] \psi_b \right], \tag{1} \]

where \( \bar{\psi}_a \) and \( \psi_a \) are anti-commuting fermionic variables which correspond to the creation and annihilation operators, respectively, of the \( a \)-th one-particle state, \( \nu_{ab} \) is the matrix element of the Hamiltonian \( i[\vec{D} + i\vec{A}(x)]\vec{\sigma} \) between the \( a \)-th and the \( b \)-th one-particle states. Similarly, \( \omega_{ab} \) arises from the temporal component of the gauge field. To fully comprehend fermions in field theory, we will explicitly calculate the determinant of a fermionic oscillator, i.e. a fermion with only one one-particle state, with a variety of boundary conditions.

In the following, we restrict the frequency of the oscillator, \( \nu(\tau) + i\omega(\tau) \), which is the one-component counterpart of the matrices in Eq. (1) to be real. This imitates a calculation in the temporal gauge (\( \omega = 0 \)). Since the frequency is real, two quantum states, vacant and occupied, of the oscillator have real and positive-definite transition amplitudes for imaginary time evolution. The path integral over anti-periodic configurations is well known to yield the sum of the amplitudes of the two states. Thus the determinant is real and positive-definite. The periodic boundary condition also yields the sum of their amplitudes, but in which each one is weighted by \( (-)^F \), where \( F \) denotes the occupation number of the corresponding states [6]. If the amplitude of the occupied state is larger than that of the vacant, the determinant becomes negative. In a previous study [7], we considered the determinant with a one-parameter family of boundary conditions which includes the periodic and anti-periodic conditions. There, our motivation was to clarify the role of a zero-mode in the path integral. We have shown the boundary condition which admits it is not the same as the anti-periodic condition and is generally different from the periodic one. The determinant becomes zero under this condition.

Naively, one might think that the dependence of the determinant on boundary conditions becomes negligibly small as the time interval goes to infinity. The difference in the values of the determinant, however, does not disappear. We extend the calculation to a wider family of
boundary conditions than we did in the previous study [7] by letting the parameter be complex, thus confirming its dependence. We will also provide an example in supersymmetric quantum mechanics that shows a crucial dependence on the boundary conditions when analyzed in path integration.

The path integral of the fermionic oscillator is given by

\[
I = \int [d\bar{\psi}d\psi] \exp \left[ - \int_0^T d\tau \bar{\psi} \mathcal{D} \psi \right],
\]

where \( \mathcal{D} = d/d\tau + \nu(\tau) \) and \( \nu(\tau) \) is the time-dependent real frequency induced by background bosonic degrees of freedom. The integration over the anti-commuting variables, \( \psi \) and \( \bar{\psi} \), is implemented by Berezin \[8, 9, 10, 11\], and the result is called the determinant of \( \mathcal{D} \). Since \( \mathcal{D} \) is not a finite-dimensional matrix, its determinant is not simply the product of its eigenvalues. Difficulty arises because \( \mathcal{D} \) is not hermitian: we use the inner product of vectors, say \( \varphi \) and \( \phi \), defined by \( (\varphi, \phi) \equiv \int_0^T d\tau \varphi^* \phi \) in the Hilbert space composed of square-integrable functions of \( \tau \) in the interval \([0, T]\); the adjoint \( \mathcal{D}^\dagger = -d/d\tau + \nu(\tau) \) is then different from \( \mathcal{D} \). We do not know a proper setup in the eigenvalue problem of non-hermitian operators that provides us with such a useful tool for path integral as a complete orthonormal set of eigenvectors. Even the meaning of the eigenvalues of \( \mathcal{D} \) is not very clear: under the boundary conditions we deal with in the following, the linear manifold on which \( \mathcal{D} \) operates is generally different from that which results from its operation.

Fujikawa has proposed a method which is useful in this situation \[12\]. Following him, we solve the eigenvalue problems of \( \mathcal{D}^\dagger \mathcal{D} \) and \( \mathcal{D} \mathcal{D}^\dagger \). Appropriate boundary conditions make these operators self-adjoint and non-negative. Then their normalized eigenvectors \( \varphi^{(n)} \) and \( \phi^{(n)} \) \((n = 1, 2, 3, \ldots)\),

\[
\mathcal{D}^\dagger \mathcal{D} \varphi^{(n)} = \lambda_n \varphi^{(n)} , \quad \mathcal{D} \mathcal{D}^\dagger \phi^{(n)} = \lambda_n \phi^{(n)} ,
\]

have a one-to-one correspondence to each other,

\[
\frac{1}{\sqrt{\lambda_n}} \mathcal{D} \varphi^{(n)} = \phi^{(n)} , \quad \frac{1}{\sqrt{\lambda_n}} \mathcal{D}^\dagger \phi^{(n)} = \varphi^{(n)} .
\]
This relation holds for all pairs of eigenvectors with a positive eigenvalue, and their spectra, the sets of eigenvalues, are identical except a possible difference at zero eigenvalue. Using the expansions by these eigenvectors,

\[ \psi(\tau) = \sum_n a_n \varphi^{(n)}(\tau), \quad \bar{\psi}(\tau) = \sum_n \bar{a}_n \phi^{(n)*}(\tau), \]

we transform the integral measure \([d\bar{\psi}d\psi]\) to \([d\bar{a}da]\) and obtain

\[ I = N \int \prod_n [d\bar{a}_n da_n] e^{-\sum_n \sqrt{\lambda_n} \bar{a}_n a_n} = N \left[ \det(\mathcal{D}^\dagger \mathcal{D}) \right]^{1/2}, \]

where \(N\) is the Jacobian between the two measures, and \([\det(\mathcal{D}^\dagger \mathcal{D})]^{1/2} \equiv \Pi_n \sqrt{\lambda_n}\) is the infinite product of the square-root of the eigenvalues. Although the latter is divergent, it gives a finite result to \(I\) when combined with \(N\).

The boundary conditions we use are parametrized by one complex variable \(\beta\). The conditions for \(\mathcal{D}^\dagger \mathcal{D}\) are given by

\[ \varphi(0) + \beta \varphi(T) = 0, \quad \beta^* \mathcal{D} \varphi(0) + \mathcal{D} \varphi(T) = 0, \]

and those for \(\mathcal{D} \mathcal{D}^\dagger\) are given by

\[ \beta^* \phi(0) + \phi(T) = 0, \quad \mathcal{D}^\dagger \phi(0) + \beta \mathcal{D}^\dagger \phi(T) = 0. \]

Note that these conditions keep the one-to-one relation between \(\varphi\) and \(\phi\) in Eq. (4). They define two linear sub-manifolds in the Hilbert space. These linear manifolds are the domains of \(\mathcal{D}^\dagger \mathcal{D}\) and \(\mathcal{D} \mathcal{D}^\dagger\), respectively\(^1\). We can readily verify equations \((\mathcal{D}^\dagger \mathcal{D} \varphi_1, \varphi_2) = (\mathcal{D} \varphi_1, \mathcal{D} \varphi_2) = (\varphi_1, \mathcal{D}^\dagger \mathcal{D} \varphi_2)\) for \(\varphi_1\) and \(\varphi_2\) which are arbitrarily chosen in the domain of \(\mathcal{D}^\dagger \mathcal{D}\). Thus \(\mathcal{D}^\dagger \mathcal{D}\) is at least symmetric and non-negative; and so is \(\mathcal{D} \mathcal{D}^\dagger\). They are in fact self-adjoint so as to have a complete orthonormal basis of eigenvectors, and the expansions in Eq. (5) cover all possible configurations. The proof is given based on the mathematical theory of the self-adjoint extension of symmetric operators \([13]\); it will be detailed in a separate publication \([14]\).
A different value of \( \beta \) provides a different domain for \( \mathcal{D}^\dagger \mathcal{D} \). Thus its determinant depends on \( \beta \) as well as the background \( \nu(\tau) \), \( I = I_\beta[\nu] \).

We calculate the modulus of \( I \) first and its phase next. However, prior to that, we view how the determinant has phase, since it is helpful to have a perspective on our calculations. Let us start at Eq. (4). We always take the positive root of \( \sqrt{\lambda_n} \) in this equation. The relative phase between \( \varphi \) and \( \phi \) is then fixed. The consequent determinant \([\det(\mathcal{D}^\dagger \mathcal{D})]^{1/2}\) in Eq. (6) becomes a product of positive quantities, and it does not induce phase. Hence only the Jacobian \( \mathcal{N} \) contains the phase of \( I \).

Now assume temporarily that \( \beta \) is real. We can then choose real functions for the eigenvectors \( \varphi^{(n)}(\tau) \) and \( \phi^{(n)}(\tau) \). These functions can be regarded as the “transformation matrix” between \( \psi(\tau)(\bar{\psi}(\tau)) \) and \( a_n(a_n) \), according to Eq. (5). The Jacobian \( \mathcal{N} \) is related to their “determinant” and is real for this case. At \( \beta = 1 \) the determinant is positive-definite, and thus the phase in \( \mathcal{N} \) is zero independently of \( \nu(\tau) \).

For a complex value of \( \beta \), the eigenvectors cannot be real and \( \mathcal{N} \) becomes complex. To obtain its phase, we compare two different measures, one is related to the coefficients in expansion (3) where the eigenvectors are solved for the complex \( \beta \), and the other is obtained from those for \( \beta = 1 \) but with the same frequency \( \nu(\tau) \). The value of \( \mathcal{N} \) differs depending on to which measure the original one \([d\bar{\psi}d\psi]\) is transformed. Let \( \varphi^{(n)\nu} \) and \( \phi^{(n)\nu} \) be normalized eigenvectors and \([d\bar{a}d\bar{b}]\) be their related measure for the value of \( \beta \) of interest; similarly \( \varphi^{(n)}, \phi^{(n)} \) and \([d\bar{a}d\bar{a}]\) are obtained for \( \beta = 1 \). These measures are related to each other by \([d\bar{a}d\bar{a}] = J[d\bar{b}d\bar{b}]\), where

\[
J \equiv [\det(\varphi^{(n)}, \varphi^{(m)\nu})]^{-1} \times [\det(\phi^{(n)*}, \phi^{(m)*\nu})]^{-1}.
\] (9)

The expressions \((\varphi^{(n)}, \varphi^{(m)\nu})\) and \((\phi^{(n)*}, \phi^{(m)*\nu})\) in Eq. (4) are the inner product of the \( n \)-th and the \( m \)-th eigenvectors; each of them is the element of the transformation matrix between the two different orthonormal complete bases. The power \((-1)\) in Eq. (4) comes from the fact that the measure is for the integration of Grassmann variables \([8]\). Since the bases are complete and orthonormal, the transformation matrices between them are unitary, and \( J \) is a phase factor.
$N$ changes by $J$ when the different bases are used, and thus the phase in $N$ is that of $J$, which is also the phase of $I$.

Note that each determinant factor in Eq. (9) is in fact not well-defined separately. The eigenvectors have arbitrariness in their phase, but each factor is not invariant under the change in the phase. It is the one-to-one correspondence between $\varphi$ and $\phi$, Eq. (4), that correlates the phase of $\varphi$ to that of $\phi$ and makes $J$ invariant despite this arbitrariness.

The modulus of the determinant is readily obtained by slightly extending the calculation in the previous study [7] to the case where the parameter $\beta$ can be complex. To obtain the modulus, we define a two-by-two matrix

$$M_\beta(z) \equiv \begin{pmatrix} u_1(z; 0) + \beta u_1(z; T) & u_2(z; 0) + \beta u_2(z; T) \\ \beta^* D u_1(z; 0) + D u_1(z; T) & \beta^* D u_2(z; 0) + D u_2(z; T) \end{pmatrix},$$

(10)

where $u_1$ and $u_2$ are two linearly independent solutions of a $z$-parametrized differential equation

$$D^\dagger D u_i(z; \tau) = zu_i(z; \tau),$$

(11)
solved with initial conditions $u_1(z; 0) = 1, \dot{u}_1(z; 0) \equiv du_1(z; 0)/d\tau = 0, u_2(z; 0) = 0$, and $\dot{u}_2(z; 0) = 1$. The complex parameter $z$ in Eq. (11) becomes an eigenvalue of $D^\dagger D$ if and only if some linear combination of $u_1$ and $u_2$ meets the boundary conditions in Eq. (7). In other words, there exists a two-component non-zero vector $\gamma_i$ that satisfies $[M_\beta(z)]_{ij}\gamma_j = 0$. This is why we choose the particular form for $M_\beta$ in Eq. (10). The condition that the $z$ is an eigenvalue of $D^\dagger D$ is equivalent to $\det M_\beta(z) = 0$.

From this behavior of $M_\beta$, we can prove the identity

$$\frac{\det(D^\dagger D - z)}{\det(D^\dagger D - z)} \equiv \prod_{n=1}^{\infty} \frac{\lambda_n - z}{\lambda_n - z} = \frac{\det M_\beta(z)}{\det M_\beta(z)}$$

(12)

where the tildes used here distinguish two different operators, matrices and eigenvalues which are defined with different frequencies, say $\nu(\tau)$ and $\tilde{\nu}(\tau)$, but with the same value of the parameter $\beta$. Two fractional expressions, far left and far right in (12), are meromorphic functions of $z$. Owing to the behavior of $M_\beta$, they have poles and zeros at same values of $z$, i.e. zeros.
at eigenvalues of $\mathcal{D}^\dagger \mathcal{D}$ and poles at those of $\tilde{\mathcal{D}}^\dagger \tilde{\mathcal{D}}$. Both functions converge to 1 asymptotically as $|z|$ goes to infinity except along the real positive axis \(^7\). The ratio of the two meromorphic functions is thus an analytic function of $z$ that goes to 1 as $z$ goes to infinity in any direction, and is necessarily a constant \(^{-1}\). This concludes the proof of Eq. (12).

The identity \(^1\) means that the ratio $|N[\det(\mathcal{D}^\dagger \mathcal{D})]^{1/2}|/|\det M_\beta(0)|^{1/2}$ does not depend on the $\nu(\tau)$ which is used to define $\mathcal{D}$ and $M_\beta$. We can thereby obtain the functional dependence of the modulus on $\nu(\tau)$ by calculating the determinant of $M_\beta$. Putting the solutions at $z = 0$,

$$u_1(0; \tau) = \exp \left[ -\int_0^\tau d\tau' \nu(\tau') \right] \times \left\{ 1 + \nu(0) \int_0^\tau d\tau' \exp \left[ 2 \int_0^{\tau'} d\tau'' \nu(\tau'') \right] \right\},$$

$$u_2(0; \tau) = \exp \left[ -\int_0^\tau d\tau' \nu(\tau') \right] \int_0^\tau d\tau' \exp \left[ 2 \int_0^{\tau'} d\tau'' \nu(\tau'') \right],$$

(13)

into the expression of $M_\beta(0)$ in Eq. (10) and taking its determinant, we obtain

$$|I| = \mathcal{N}' \left[ (\mathcal{M}_0 + \beta \mathcal{M}_1)(\mathcal{M}_0 + \beta^* \mathcal{M}_1) \right]^{1/2},$$

(14)

where $\mathcal{M}_0 = \exp \left[ -(1/2) \int_0^\tau d\tau \nu(\tau) \right]$, $\mathcal{M}_1 = \exp \left[ -(1/2) \int_0^\tau d\tau' \nu(\tau') \right]$ and $\mathcal{N}'$ is a positive constant that may depend only on $\beta$.

Eq. (14) implies $I$ vanishes for $\beta = -\mathcal{M}_0^2$ (note $\mathcal{M}_1 = \mathcal{M}_0^{-1}$). This is because the boundary conditions with this value of $\beta$ admit a zero mode in the domain. Its effect on the determinant calculation has been clarified in \(^7\).

We next proceed to the calculation of the phase, $\ln J$. Our plan is to integrate infinitesimal variations induced in the phase while the parameter $\beta$ moves continuously from 1 to some value of interest. $\nu(\tau)$ is kept fixed in this process. We use the formula $\delta \ln \det M = \text{Tr} M^{-1} \delta M$ that holds for an infinitesimal variation of any matrix $M$. Applying this formula to the variation $\delta \ln J$ with $J$ in Eq. (9), and using the fact that a set of eigenvectors is complete, we obtain

$$\delta \ln J = \sum_n \left[ (\phi_n^{(n)}, \phi_n^{(n)'}) - (\phi_n^{(n)}, \phi_n^{(n)')} \right] = \sum_n \left[ (\phi_n^{(n)}, \phi_n^{(n)'}) - (\phi_n^{(n)}, \phi_n^{(n)')} \right],$$

(15)

\(^2\)The same argument is used in Ref. \(^11\) to calculate a different determinant.
where \( \varphi^{(n)} \) and \( \phi^{(n)} \) are eigenvectors at \( \beta \) and primed ones are those at \( \beta + \delta \beta \). We put the one-to-one correspondence \( \varphi^{(n)} = D \varphi^{(n)}/\sqrt{\lambda_n} \) into (13). After integrating by parts and using the similar one-to-one relation for \( \phi^{(n)'} \) and the boundary condition (7), we obtain

\[
\delta \ln \mathcal{J} = \sum_n \frac{1}{\lambda_n} \left[ \frac{\delta \beta^*}{\beta^*} \varphi^{(n)\ast}(T) D \varphi^{(n)}(T) + \frac{1}{2} \delta \lambda_n \right],
\]

(16)

where \( \delta \lambda_n \) is the variation of the \( n \)-th eigenvalue under the infinitesimal change of \( \beta \). \( \delta \lambda_n \) is calculated by the relation

\[
\delta \lambda_n (\varphi^{(n)}, \phi^{(n)'}) = (\varphi^{(n)}, D^\dagger D \phi^{(n)\prime}) - (D^\dagger D \varphi^{(n)}, \varphi^{(n)\prime})
\]

(17)

We then obtain

\[
\delta \ln \mathcal{J} = \sum_n \frac{1}{2 \lambda_n} \left[ \frac{\delta \beta^*}{\beta^*} \varphi^{(n)\ast}(T) D \varphi^{(n)}(T) - \frac{\delta \beta}{\beta} D \varphi^{(n)*}(T) \phi^{(n)}(T) \right].
\]

(18)

This expression manifestly shows that \( \delta \ln \mathcal{J} \) is purely imaginary as it should be.

To sum up the terms in Eq. (18), we use the resolvent \( R_z \equiv (D^\dagger D - z)^{-1} \), which has the expression

\[
R_z(\tau, \sigma) = \sum_n \frac{1}{\lambda_n - z} \varphi^{(n)}(\tau) \varphi^{(n)\ast}(\sigma)
\]

(19)

as an integral kernel. We notice readily that contour integration of \( DR_z/z \) over \( z \) along a contour that goes around all eigenvalues of \( D^\dagger D \) clockwise in the complex \( z \)-plain gives the relevant part of the sum. The contour we use here is one that goes below the real positive axis towards the origin from infinity until it passes all the eigenvalues and then goes back moving above the real axis. Without changing the value of the integral, we can add to it another contour that almost makes a circle at infinity but does not cross the real axis, because the value of the integral along the latter contour is zero. We now have the integral along the closed contour \( C \) (see Figure 1), in which only the pole at the origin contributes to the integral. We obtain

\[
\delta \ln \mathcal{J} = \frac{1}{2} \left[ \left. \frac{\delta \beta^*}{\beta^*} DR_0(\tau, \sigma) \right|_{\tau=\sigma=T} - \text{(c.c.)} \right]
\]

(20)

8
as the result of the sum.

We have assumed that $\beta$ always stays off the value at which one eigenvalue becomes zero and $R_0$ becomes singular; the phase of $I$ is obviously meaningless when $I$ is zero.

In order to calculate the resolvent $R_0$ in Eq. (20), we have to know some mathematical details about the self-adjoint extension of differential operators. Following a textbook [13], we briefly describe a specific extension of the operator $D^\dagger D$ to the extent where our calculation of $R_0$ appears convincing; the self-contained description will be given in [14]. The operator that is extended to be self-adjoint is $D^\dagger D$ defined, however, in a domain $D_0$ under more restrictive boundary conditions than Eq. (7). These conditions are $\varphi(0) = \varphi(T) = \dot{\varphi}(0) = \dot{\varphi}(T) = 0$. $D^\dagger D$ defined in $D_0$ is symmetric but is not self-adjoint: the domain of its adjoint is not restricted by a boundary condition and is obviously larger than $D_0$. We have to extend $D_0$ by adding two appropriate vectors in order to make $D^\dagger D$ self-adjoint, that is, the operator itself and its adjoint have the same domain. These two vectors have the form

$$w_i(z) = U_{ij} u_j(z^*) - u_i(z) \quad (i, j = 1, 2),$$

where $u_i$ is the solution to Eq. (11) with an arbitrary but non-real parameter $z$, and $u_i(z^*) = u_i^*(z)$ is its complex conjugate. The two-by-two matrix $U$ in Eq. (21) has to be “unitary” in the sense that the mapping defined by $u_i \rightarrow U_{ij}u_j^*$ from the two-dimensional vector space composed of $u_i$s to that of $u_i^*$s is unitary. $D^\dagger D$ becomes self-adjoint in this extended domain $D \equiv D_0 \oplus \{w_1, w_2\}$ [13].

The unitarity of $U$ solely does not determine $U$, but boundary conditions do. The requirement that any vector in $D$ should meet the boundary conditions in Eq. (7) solves $U$ as

$$[U_\beta(z)]_{ij} = [M_\beta(z^*)]^{-1}_{jk} [M_\beta(z)]_{ki}$$

for the parameters $\beta$ and $z$. The mapping induced by this particular $U_\beta(z)$ is shown to be unitary [14]. The self-adjoint operator $D^\dagger D$ that we have used is in fact this extended operator.
Having described the structure of the domain $D$, we can now resume the calculation of the resolvent. What we need to know in Eq. (20) is its value only at the end of the time interval. Among all the vectors in $D$, only the two $w_i$s in Eq. (21) can contribute to $R_z$ at the end. We examine the operation of $(D^\dagger D - z)$ on $w_i$ and find

$$(D^\dagger D - z)w_i(z) = (z^* - z)[U_\beta(z)]_{ij}u_j(z^*),$$  \hfill (23)$$

which enables us to obtain the necessary information on $R_z$. Since zero is not an eigenvalue, we can take the limit $z \to 0$ in Eq. (23) to obtain $R_0 u_i = \tilde{w}_i$, where

$$\tilde{w}_i \equiv \lim_{z \to 0} \frac{w_i(z)}{z^* - z} = \lim_{z \to 0} \frac{1}{z^* - z} \{[U_\beta(z)]_{ij}u_j(z^*) - u_i(z)\},$$  \hfill (24)$$

and we have used $U_\beta(0) = 1$ (see Eq. (22)). As an integral kernel, $R_0$ is written as

$$R_0(\tau, \sigma) = \sum_{i=1,2} \tilde{w}_i(\tau)\tilde{u}_i(\sigma) + ..., \hfill (25)$$

where $\tilde{u}_i$s are the linear combinations of $u_i$s that satisfy $(\tilde{u}_i, u_j) = \delta_{ij}$, and we have excluded terms that do not contribute to the final result. Both $\tilde{w}_i$ and $\tilde{u}_i$ can be explicitly obtained with the solutions $u_i$s at $z = 0$. Plugging these results into (20) and after a long but straightforward calculation, we obtain

$$\delta \ln J = \frac{1}{2} \left[ \frac{\mathcal{M}_1 \delta \beta}{\mathcal{M}_0 + \beta \mathcal{M}_1} - \frac{\mathcal{M}_1 \delta \beta^*}{\mathcal{M}_0 + \beta^* \mathcal{M}_1} \right].$$  \hfill (26)$$

By integrating this equation under the condition $J = 1$ at $\beta = 1$, we finally obtain

$$J = \left[ \frac{\mathcal{M}_0 + \beta \mathcal{M}_1}{\mathcal{M}_0 + \beta^* \mathcal{M}_1} \right]^{1/2}. \hfill (27)$$

The dependence of the phase factor in Eq. (27) on the parameter $\beta$ shows the advantage of the present calculation where we have allowed $\beta$ to be complex. Let $\beta$ move along the real axis, for example, from 1 to $-1$ and assume the value $-\mathcal{M}_0^2$ is between them. At this point the determinant becomes zero, and there will occur an ambiguity in its sign after $\beta$ passing the point if we know only its modulus. This ambiguity is solved since we can use a path that
circumvents the point along an circle with infinitesimal radius \( \epsilon \) as \( \beta = -\mathcal{M}_0^2 + \epsilon e^{i\theta} \) where \( \theta \) moves from 0 to \( \pi \). The determinant becomes negative after \( \beta \) passes the value.

Combining the modulus (14) and the phase factor (27), we obtain

\[
I_\beta[\nu] = \mathcal{N}'(\mathcal{M}_0 + \beta \mathcal{M}_1) .
\]

(28)

\( \mathcal{M}_0 \) and \( \mathcal{M}_1 \) are in fact the transition amplitudes for the vacant and the occupied states, respectively. The result in Eq. (28) is not only consistent with the known case of the anti-periodic (\( \beta = 1 \)) and the periodic (\( \beta = -1 \)) boundary conditions, but also applies to any complex value of \( \beta \). \( \mathcal{N}' \) is an overall normalization and will not affect the following discussion.

Although we have considered only a single fermionic oscillator, our result indicates that the similar dependence is expected to appear in fermion determinants in field theory, perhaps in a more complicated manner. At least for a simple case in which there are a finite number of fermionic oscillators that share a common value of \( \beta \) for their boundary conditions, we can safely conclude, without seeing anomalies, that the determinant is the trace of the time evolution operator in which the amplitude of each state is weighted by \( \beta^F \), according to its fermion number \( F \). It is hard to imagine that the difference of the determinant caused by such a non-trivial weighting disappears even in the limit \( T \to \infty \). Thus we have to be careful which boundary condition we use in the calculations.

In order to see how the boundary conditions affect the result, let us examine an illuminating example — the supersymmetric double-well quantum mechanics [15]. With \( q \) and \( p \), the coordinate and momentum of bosonic degrees of freedom, \( \psi^\dagger \) and \( \psi \), the creation and annihilation operators for fermion, we write the Hamiltonian of this model as

\[
H = \frac{1}{2} \{Q, Q^\dagger\} = \frac{1}{2}(p^2 + W(q)^2) + \frac{1}{2} \frac{dW(q)}{dq}[\psi^\dagger, \psi],
\]

(29)

where \( Q \equiv (p + iW(q))\psi \), \( Q^\dagger \equiv (p - iW(q))\psi^\dagger \) and \( W \) is chosen to be \( W(q) = q(1 - gq) \) with a coupling constant \( g \). What will happen if one analyzes the model by path integral without paying attention to the boundary conditions? One may try to integrate the fermion first in a
given bosonic background and then to integrate the bosonic part weighting the bosonic measure with the fermion determinant, which is given by Eq. (2) with \( \nu(\tau) = 1 - 2gq(\tau) \). Under the periodic boundary condition, the determinant changes the sign when the background varies between \( q(\tau) \) and \( 1/g - q(\tau) \), for which the bosonic potential \( W(q)^2 \) takes the same value. The contribution of the two configurations cancels each other out, and the path integral becomes zero. One may conclude that the model is ill-defined by interpreting this result as the statistical trace being zero.

We know, however, that the model is well-defined, and further it is an instructive example of the dynamical supersymmetry breaking [13]. The path integral over the periodic configurations is not the statistical trace, but the regularized Witten index \( \text{Tr}(-F e^{-TH}) \) [16, 3]. The fact that it is zero merely implies the possibility that two perturbative zero-energy states, one localized at \( q = 0 \) with \( F = 0 \) \( (F \equiv \psi^\dagger \psi) \) and the other localized at \( q = 1/g \) with \( F = 1 \), may be lifted in pairs by non-perturbative effect of the interaction, and the supersymmetry may be broken. It indeed happens in this model by the valley-instanton effect [17].

This example shows that in any model which exhibits dynamical supersymmetry breaking, its fermion determinant needs to have non-trivial dependence on the boundary conditions. The path integral has to vanish in the case of the periodic fermion fields so that the Witten index is zero, while it should provide a non-vanishing value for anti-periodic fermion fields in order that the partition function is not zero. Thus the careful study of the dependence of fermion determinants on the boundary conditions is important in the search for a model in which supersymmetry breaks dynamically.

**Acknowledgments**

The author thanks H. Aoyama for the discussion on the subject and his comments on the manuscript, K. Funakubo and M. Sato for the discussions with them. He also thanks M. Itoh for the critical reading of the manuscript and his useful suggestions about English usage. D. Jones
and T. Fukube gave the author helpful comments on the English in the manuscript, which he appreciates as well.

References

[1] S. Adler, *Phys. Rev.* **177** (1969) 2426; J. Bell and R. Jackiw, *Nuovo Cim.* **60A** (1969) 47.

[2] S. Adler and W. Bardeen, *Phys. Rev.* **182** (1969) 1517; W. Bardeen, *Phys. Rev.* **184** (1969) 1848.

[3] E. Witten, *Phys. Lett.* **117B** (1982) 324.

[4] L. Alvarez-Gaume and E. Witten, *Nucl. Phys.* **B234** (1983) 269, and references cited therein.

[5] K. Fujikawa, *Phys. Rev. Lett.* **42** (1979) 1195; *Phys. Rev.* **D21** (1980) 2848; *Phys. Rev.* **D22** (1980) 1499(E).

[6] S. Cecotti and L. Girardello, *Phys. Lett.* **110B** (1982) 39.

[7] H. Kikuchi, “Fermion Determinant Calculus”, hep-th/9903247.

[8] F. Berezin, “The Method of Second Quantization,” (Academic Press, New York, 1966).

[9] L. D. Faddeev, in “Methods in Field Theory” ed. by R. Balian and J. Zinn-Justin, (North-Holland, 1975).

[10] Y. Onuki and T. Kashiwa, *Prog. Theor. Phys.* **60** (1978) 548.

[11] S. Coleman, in “The whys of subnuclear physics,” Erice 1977, ed. by A. Zichichi, (Plenum Press, 1979).

[12] K. Fujikawa, *Phys. Rev.* **D29** (1983) 285.
[13] N. I. Akhiezer and I. M. Glazman, “Theory of Linear operators in Hilbert space,” (Dover, New York, 1993). (A republication of two separate Volumes by Frederick Ungar Co., New York in 1961 and 1963.)

[14] H. Kikuchi, in preparation.

[15] E. Witten, *Nucl. Phys.* **B188** (1981) 513.

[16] E. Witten, *Nucl. Phys.* **B202** (1982) 253.

[17] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato, and S. Wada, *Phys. Lett.* **B424** (1998) 93; *Nucl. Phys.* **B553** (1999) 644.
Figure caption: Figure 1. The schematic drawing of the contour $C$ in the complex $z$-plane of $\mathcal{D}R_z/z$. $C$ is a closed contour made of a contour that goes around all eigenvalues of $\mathcal{D}^\dagger \mathcal{D}$ clockwise and another that goes around at infinity counterclockwise, as explained in the text. The crosses indicate positions of the poles of $\mathcal{D}R_z/z$. 
This figure "figure.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/0210003v2