Witnessed Entanglement

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We present a new measure of entanglement for mixed states. It can be approximately computable for every state and can be used to quantify all different types of multipartite entanglement. We show that it satisfies some of the usual properties of a good entanglement quantifier and derive relations between it and other entanglement measures.

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I. INTRODUCTION

The quantification of entanglement in a multipartite quantum system is one of the most important and challenging topics of quantum information theory. The first attempt of measuring entanglement was based on the violation of the so-called Bell’s inequalities [1]. It was thought that the amount of non-classical correlations in a bipartite quantum system was intrinsically related to the level of violation of these family of inequalities. Nonetheless, the generalization of such inequalities to multipartite states is not possible in general. Furthermore, it was already shown that, even for the bipartite case, the CHSH (Clauser-Horne-Shimony-Holt) form of Bell’s inequalities does not constitute a good measure of quantum correlations in the sense that there are states which do not violate it, but can be purified by local operations and classical communication (LOCC) to a state which does [2]. Therefore, different measures of entanglement were introduced based on the concept of asymptotic distillability [3].

For pure states, the von Neumann entropy [4] of the reduced density matrix $T_{RB}(|\psi\rangle_{AB}|\psi\rangle_B)$ is exactly the number of Bell states which can be asymptotically distilled by LOCC from $|\psi\rangle_{AB}$. It satisfies all desired properties (normalization, convexity, local unitary operations invariance, etc...) of an entanglement measure [6]. Several attempts were made in order to generalize this measure (the Entropy Entanglement - $E_H$) for mixed states. The most successful are the entanglement cost ($E_C$), the entanglement of formation ($E_F$), the relative entropy of entanglement ($E_{RE}$) and the distillable entanglement $E_D$ [6]. They all coincide with the entropy of entanglement in the pure states set, showing their relationship with asymptotic distillability properties, and satisfies most of the desirable properties. However, none of these have an operational expression, i.e., they cannot be calculated for arbitrary states. Furthermore, they are defined only for bipartite states, not expressing the interesting properties of multipartite systems [21].

Other measures of entanglement for mixed states, not related with $E_E$, have also been proposed. The robustness of entanglement ($R(\rho|S)$) quantifies the minimal amount of mixing with separable states needed to destroy the original entanglement presented in $\rho$ [8]. It follows easily from its definition that it can be used to quantify multipartite entanglement. However, it is calculable only for a few specialized bipartite density operators. The negativity ($N(\rho)$) is the only example of a computable quantifier [9]. Nonetheless, since it is based on the negativity of the eigenvalues of the partial transpose matrix ($\rho^T_A$) of the system, it does not quantify the entanglement of positive-partial-transpose (PPT) states. Moreover, it is defined only for bipartite correlations.

In this paper we introduce a new computable measure for entanglement, which quantifies all different kinds of multipartite entanglement. Its definition is based on the concept of entanglement witness. According to [10], an $n$-partite density operator $\rho_{1\ldots n} \in B(H_1 \otimes \ldots \otimes H_n)$ (the space of bounded operators acting on the Hilbert space $H_1 \otimes \ldots \otimes H_n$) is non-separable iff there exists a self-adjoint operator $W \in B(H_1 \otimes \ldots \otimes H_n)$ which detects its entanglement, i.e., such that $Tr(W\rho_{1\ldots n}) < 0$ and $Tr(W\sigma_{1\ldots n}) \geq 0$ for all $\sigma_{1\ldots n}$ separable. This condition follows from the fact that the set of separable states is convex and closed in $B(H_1 \otimes \ldots \otimes H_n)$. Therefore, as a conclusion of the Hahn-Banach theorem, for all entangled states there is a linear functional which separates it from the set. It will be necessary to consider only normalized entanglement witnesses such that $Tr(W) = 1$. 

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II. DEFINITION

Definition 1. A Hermitian operator $W_{\rho_{1...n}} \in \mathcal{B}(H_1 \otimes \ldots \otimes H_n)$ is an optimal EW (OEW) for the density operator $\rho_{1...n}$ if

$$Tr(W_{\rho_{1...n}}\rho_{1...n}) \leq Tr(W\rho_{1...n})$$

(1)

for every EW $W$.

Although the above definition is different from the one introduced in [11], the optimal EWs of both criteria are equal.

It is important to stress the close relation between Bell’s inequalities and entanglement witnesses. Since an EW is a Hermitian operator, it is associated with a possible experiment whose expectation value in all separable states is positive. Therefore, one may consider witness operators as generalized Bell’s inequalities, where the violation of the latter is replaced by a negative expectation value. Of course Bell’s inequalities have an important role in the refutation of hidden local variables theories for quantum mechanics, not shared by EWs. Nonetheless, in what concerns the detection of entanglement, entanglement witnesses can be considered just generalized Bell’s inequalities, with the locality condition of the later relaxed. Thus, the optimal entanglement witness of a state $\rho_{1...n}$ can be identified with the experiment which most detects its entanglement, i.e., which has the lowest possible expectation value. We will show that the result of this optimal experiment might be used to quantify entanglement.

Definition 2. The witnessed entanglement $E_W(\rho_{1...n})$ of a multipartite state $\rho_{1...n} \in \mathcal{B}(H_1 \otimes \ldots \otimes H_n)$ is given by

$$E_W(\rho_{1...n}) = \max \left\{ 0, -Tr(W_{\rho_{1...n}}\rho_{1...n}) \right\}$$

(2)

The lack of an operational procedure to calculate entanglement measures in general is ultimately related to the complexity of distinguishing entangled from separable mixed states, which was shown to be NP-HARD [12]. Since an operational measure would also be a necessary and sufficient test for separability, we should not expect to find one. Nevertheless, a procedure to calculate OEW for all states with arbitrary probability and precision was recently introduced [13]. Although the optimization of entanglement witnesses is also NP-HARD [14], the optimization of approximate EW, operators which are positive for almost all separable states, can be realized very efficiently with a chosen probability. As the amount of violation (the percentage of negative expectation values over the separable states set) can also be chosen, it is possible to reach the exact OEW with any desired precision. Thus, the witnessed entanglement (WE) introduced earlier can be approximately calculated for all mixed states.

III. BASIC PROPERTIES

Besides having an interesting physical interpretation and an efficient numerical method of computation, the witnessed entanglement fulfills most of the usual requirements of a good entanglement measure [15]. These properties can be summarized as:

(i) $E(\sigma) = 0$ if, and only if, $\sigma$ is separable.

(ii) Local unitary operations leave $E(\sigma)$ invariant, i.e,

$$E(\sigma) = E(U_1^\dagger \otimes \ldots \otimes U_n^\dagger \rho U_1 \otimes \ldots \otimes U_n).$$

(3)

(iii) (Monotonicity under LOCC) Entanglement cannot increase under local operations and classical communication (LOCC) protocols.

$$E(\Lambda(\sigma)) \leq E(\sigma),$$

(4)

where $\Lambda$ is a superoperator implementable by LOCC.

(iv) (Continuity) For every $\epsilon \geq 0$ and density operators $\rho$ and $\sigma$, there is a real number $C \geq 0$ such as

$$||\rho - \sigma||_L \leq \epsilon \Rightarrow |E(\rho) - E(\sigma)| \leq C(L)\epsilon,$$

(5)

where $|| ||_L$ is any norm defined for finite dimensional systems. Note that $C$ depends of the chosen norm.

(v) (Convexity) Mixing of states does not increase entanglement.

$$E(\lambda \rho + (1-\lambda)\sigma) \leq \lambda E(\rho) + (1-\lambda)E(\sigma),$$

(6)

for all $\rho$, $\sigma$ and $0 \leq \lambda \leq 1$. 
Proposition 1. The Witnessed Entanglement satisfies properties (i), (ii), (iv) and (v).

Proof. (i): From the definition of entanglement witnesses we have that \( \text{Tr}(W_\sigma \sigma) \geq 0 \) for all separable states \( \sigma \). Therefore,

\[
E_W(\sigma) = \max \{ 0, -\text{Tr}(W_\rho \rho \rho \rho \ldots) \} = 0.
\]

(7)

Conversely, as there exists an EW for every entangled state, \( E_W \) is positive for any non-separable density operator.

(ii): We will prove by absurd that \( E_W \) is invariant under local unitary operations. Suppose that \( E_W(\sigma) > E_W(\rho) \), where

\[
\sigma = U_1^\dagger \otimes \ldots \otimes U_n^\dagger \rho U_1 \otimes \ldots \otimes U_n.
\]

Then,

\[
E_W(\sigma) = \{ -\text{Tr}(W_\sigma \sigma), 0 \} = \max \{ -\text{Tr}(W_\rho U_1^\dagger \otimes \ldots \otimes U_n^\dagger \rho U_1 \otimes \ldots \otimes U_n), 0 \}
\]

(9)

where \( W = U_1 \otimes \ldots \otimes U_n W_\sigma U_1^\dagger \otimes \ldots \otimes U_n^\dagger \) is an entanglement witness. But equation (9) contradicts the fact that \( W_\rho \) is the OEW for \( \rho \). If we suppose that \( E_W(\sigma) < E_W(\rho) \), the same argument can be applied to \( \rho = U_1 \otimes \ldots \otimes U_n \sigma U_1^\dagger \otimes \ldots \otimes U_n^\dagger \).

Therefore, \( E_W(\rho) = E_W(\sigma) \) must hold.

(iv): Let \( W_\rho \) and \( W_\sigma \) be optimal entanglement witnesses for \( \rho \) and \( \sigma \), respectively. If either \( \rho \) or \( \sigma \) is separable, the result follows trivially. We then assume that

\[
\text{Tr}(W_\rho \rho) < 0, \quad \text{Tr}(W_\sigma \sigma) < 0. \quad (a)
\]

(10)

As the intersection of the Ews set with the set of Hermitian matrices with unity trace is compact, we have that for some norm \( || ||_{L'} \),

\[
\max_{W \in \mathcal{M}} ||W||_{L'} = D, \quad (b)
\]

(11)

where \( D \geq 0 \) is a real number. Every norm of finite dimensional operators are equivalent, i.e., for every finite dimensional operator \( A \) and any two chosen norms \( || ||_{L'} \) and \( || ||_L \), there always exists real numbers \( n \) and \( m \) such that

\[
m ||A||_L \leq ||A||_{L'} \leq n ||A||_L \quad (c).
\]

(12)

Thus, one sees that equation (14) is valid for every norm.

We can assume without loss of generality that \( E(\rho) \geq E(\sigma) \) (d). Hence,

\[
|E(\sigma) - E(\rho)| = |\text{Tr}(W_\rho \rho) - \text{Tr}(W_\sigma \sigma)| = |\text{Tr}[W_\rho (\rho - \sigma)] + \text{Tr}(W_\sigma \sigma) - \text{Tr}(W_\sigma \sigma)|.
\]

(13)

Since \( W_\sigma \) is a OEW for \( \sigma \), \( \text{Tr}(W_\rho \rho) - \text{Tr}(W_\sigma \sigma) \geq 0 \). From (d) we also have \( |\text{Tr}[W_\rho (\rho - \sigma)]| \geq |\text{Tr}[(W_\rho - W_\sigma) \sigma]|. \)

Therefore,

\[
|E(\sigma) - E(\rho)| = |\text{Tr}[W_\rho (\rho - \sigma)] + \text{Tr}(W_\rho \sigma) - \text{Tr}(W_\sigma \sigma)| \leq |\text{Tr}[W_\rho (\rho - \sigma)]| \leq ||W_\rho||_{HS}||\rho - \sigma||_{HS},
\]

(14)

where we have used the Cauchy-Schwartz inequality for the Hilbert-Schmidt inner product. From Eq. (17) and conditions (b) and (c) we then have that for every norm \( || ||_L \),

\[
|E(\sigma) - E(\rho)| \leq ||W_\rho||_{HS}||\rho - \sigma||_{HS} \leq a ||W_\rho||_L ||\rho - \sigma||_L \leq C \epsilon,
\]

(15)

for some real numbers \( a, C \geq 0 \).

(v): The convexity follows straightforwardly from definition 2 and the concavity and convexity of the max and min functions, respectively.

Although \( E_W \) is not a monotone under general LOCC maps, we can state the following:

Proposition 2. For every unital LOCC map \( *, \)

\[
E_W(\Lambda(\rho)) \leq E_W(\rho)
\]

(16)
In order to do that, consider the index set $P$ which is an $m$-partite density operator. Then, \( \Lambda(\rho) \) is an entanglement witness if:
\[
\rho = \rho_{AB} \quad \text{and} \quad \rho_{AB} \text{ is the projector in the subspace of the minimum eigenvalue of } \rho^{TA}. \]

Proof. It suffices to show that \( E_W \) is not increasing under separable operations. Let \( \Lambda \) be a unital trace-preserving separable quantum superoperator. Then,
\[
Tr(W_{\rho} \Lambda(\rho)) = Tr(\Lambda^d(W_{\rho})\rho),
\]
where \( \Lambda^d \) is the dual map of \( \Lambda \), which is unital, since \( \Lambda \) is trace preserving, and separable, and \( W_{\rho} \) is the optimal EW for \( \rho = \Lambda(\rho) \). Indeed, \( Tr(\Lambda^d(W_{\rho})\sigma) = Tr(W\Lambda(\sigma)) \geq \) for all \( \sigma \in S \), since \( W \) is an EW and \( \Lambda(\sigma) \in S \). Now we have to prove that \( Tr(\Lambda^d(W_{\rho})) = 1 \). Using that the channel is unital, i.e. \( T(I) = I \),
\[
Tr(\Lambda^d(W_{\rho})) = Tr(W_{\rho}) = 1. \tag{18}
\]

IV. MULTIPARTITE ENTANGLEMENT

A \( n \)-partite density operator \( \rho_{1...n} \) is a \( m \)-separable state if it is possible to find a decomposition for it such that in each pure state term at most \( m \) parties are entangled among each other, but not with any member of the other group of \( n-m \) parties. Since the subspace of \( m \)-separable density operators is convex and closed, it is also possible to apply the Hahn-Banach theorem to it and establish the concept of entanglement witness to \( (m+1) \)-partite entanglement. In order to do that, consider the index set \( P = \{1,2,...,n\} \). Let \( S_p \) be a subset of \( P \) which has at most \( m \) elements. Then \( W \) is an \( (m+1) \)-partite entanglement witness if:
\[
\forall \ P_{1}^{m}, ..., P_{v}^{m} \text{ such that } \bigcup_{k=1}^{v} P_{k}^{m} = P \text{ and } P_{k}^{m} \cap P_{l}^{m} = \{\}
\]

Equation (19) assures that the operator \( W \) is positive for all \( m \)-separable states. Thus, it is also possible to determine optimal \( (m+1) \)-partite entanglement witnesses for every \( (m+1) \)-partite entangled state and establish an order in this set with \( E_W \).

V. CONNECTIONS WITH OTHER ENTANGLEMENT MEASURES

We have considered so far OEW of the most general form. We will show that it is possible to construct another measure of entanglement for non-positive-partial-transpose-states (NPPTES), reducing the search space of EW. Witnesses operators can be classified as decomposable (d-EW) or not decomposable (nd-EW)\[11\]. In the case of bipartite entanglement, a Hermitian operator \( W \) is a \( d-EW \) if it can be written in the form of
\[
W = pP + (1-p)Q^{TA}, \quad 0 \leq p \leq 1 \tag{20}
\]
where \( P \) and \( Q \) are positive operators. It is clear from this definition that these EWs cannot detect PPTES. A \( nd-EW \) is a Hermitian operator which cannot be written as (20). It was already shown that \( W \) is a \( nd-EW \) if, and only if, it detects PPTES [12]. If we restrict our search of optimal EW to \( d-EW \), we will still have a measure of entanglement, the decomposable witnessed entanglement \( (E_{d-W}) \), which satisfies the same properties of \( E_W \), despite the fact that it will be null for both separable and PPTES. The advantage of this new measure is that it can be calculated in a straightforward manner.

**Proposition 3.** \( E_{d-W}(\rho_{AB}) = \log(d^D/d) \ min(\lambda_{\min}(\rho_{AB}^{TA}),0) \), where \( \lambda_{\min}(A) \) stands for the minimum eigenvalue of \( A \).

**Proof.** If \( \rho_{AB} \) is separable or PPT, then \( Tr(d-W_{\rho_{AB}}\rho_{AB}) \) will be non-negative and \( E_{d-W} \) will be null. If \( \rho_{AB} \) is NPPT, since \( Tr(XY^{TA}) = Tr(X^{TA}Y) \), it is easy to see that the optimal d-EW will be \( P^{TA} \), where \( P \) is the projector in the subspace of the minimum eigenvalue of \( \rho_{AB}^{TA} \). \qed
The negativity (N) is another example of an entanglement measure which is both computable and null for PPTES. It is well known that, for systems of two qubits, N is equal to twice the absolute value of the negative eigenvalue of the partial transpose of the state. Thus, in this dimension, it coincides with \( E_{d-W} \). The decomposable witnessed entropy of entanglement is particularly interesting for 2 x 2 and 2 x 3 systems. Since there are no PPT entangled states in these cases, the optimal \( d-EW \) is the exact OEW and \( E_W \) and \( E_{d-W} \) are equal.

**Proposition 4.** \( E_W(\rho) = E_{d-W}(\rho) \) and \( E_W(\rho) = E_{d-W}(\rho) = N(\rho) \) for all 2 x 3 and 2 x 2 states, respectively.

We have seen that, up to a dimension normalization factor, \( E_W(\rho) \) can be identified with the result of an experiment which has the lowest possible expectation value in \( \rho \), while it has non-negative expectation value in all separable states. \( E_W(\rho) \) is also related to the amount of mixing with the maximally random state \((I/D)\) needed to destroy the original entanglement presented in \( \rho \). This quantity, introduced in [9], is called random robustness \((R(\rho|I/D))\) and is given by the minimum \( s \) for which \( \rho(s) = \frac{1}{1+s}(\rho + sI/D) \) is separable.

**Proposition 5.** \( E_W(\rho) = R(\rho|I/D)/D \), for every multipartite density operator \( \rho \).

*Proof.* Since \( s' = R(\rho|I/D) \) is the minimum value for which \( \rho(s) \) is separable, \( Tr(W_\rho \rho(s')) = \frac{1}{1+s'}(Tr(W_\rho \rho)+s'/D) = 0 \). Therefore, \( R(\rho|I/D) = -DTr(W_\rho \rho) \).

It was proved in [9] that, for bipartite pure states, \( R(\rho|I/D) \) is given by the product of the two biggest Schmidt coefficients with the dimension of the composite Hilbert space.

**Proposition 6.** Given a pure bipartite state \(|\Psi\rangle\) with decreasing ordered Schmidt coefficients \( a_i \), \( E_W(|\Psi\rangle\langle\Psi|) = a_1a_2 \).

Since \( R(\rho|I/D) \) is an upper bound for the robustness \( R(\rho|S) \), it is possible to use the results presented in [9] and establish the following order between different measures:
\[
E_W(\rho) \geq R(\rho|S) \geq N(\rho) \geq E_D(\rho).
\]  

(21)

**VI. EXAMPLE**

As an example, consider the following state of three qubits: \( \rho_{ABC} = (1 - p)|\psi_W\rangle\langle\psi_W| + p|\psi_GHZ\rangle\langle\psi_GHZ| \), where \( |\psi_GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \) and \( |\psi_W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle) \). We have calculated \( E_W \) for genuine tripartite entanglement, bipartite entanglement of the possible partitions (AB-C, A-BC and AC-B), and bipartite entanglement of the reduced density matrices (A-B, A-C and B-C). The results are plotted in Fig. 1. Since the state is symmetric with respect to exchange of the particles, the amount of bipartite entanglement in the three partitions and in the three reduced density matrices are the same. The value of \( E_{d-W} \), which was also calculated for the three partitions, and \( E_W \) are the same, showing that it is not possible to reach a bound entangled state mixing the identity with this family of states. Note in addition that the the witnessed entanglement gives the same order than the relative entropy of entanglement [18] and the geometric measure of entanglement [19] for the W and GHZ states.

**VII. CONCLUSION**

In summary, we have introduced a new measure of entanglement which is operational and can be used to quantify all different types of multipartite entanglement. It has two interesting physical interpretations and is connected to several other different measures.

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FIG. 1: $E_{W}$ as a function of $p$ for the tripartite entanglement (1), for the bipartite entanglement in any of the three partitions (2), for the bipartite entanglement in any of the three reduced density matrices (4), and $E_{d-W}$ as a function of $p$ for the bipartite entanglement in any of the three partitions (3).

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