Abstract

The two dominant processes of lepton flavor violation, $\mu \to 3e$ and $\mu \to e$ conversion in atomic nuclei caused by neutral KK gauge boson exchanges at tree level, are studied in the context of five dimensional gauge-Higgs unification scenario. The key point of the flavor violation is a fact that the bulk masses and the brane ones cannot be simultaneously diagonalized. We estimate the branching ratio of each processes and obtain the lower bound of compactification scale around weak scale from the current experimental data. We discuss the reasons why the final result is not so severe although the large mixing in the lepton sector seems to give large lepton flavor violation processes.
1 Introduction

It was a great triumph that a Higgs boson was discovered by ATLAS [1] and CMS [2] collaborations at the CERN LHC, but it is still unclear that the observed Higgs boson is a predicted one in the Standard Model (SM) or the physics beyond the SM. Of the various physics beyond the SM, the gauge-Higgs unification (GHU) [3] is one of the fascinating scenarios as a solution to the hierarchy problem [4]. In GHU, Higgs boson is identified with extra spatial components of the gauge boson in higher dimensional gauge theories. This identification immediately forbids a local mass term for Higgs boson by the higher dimensional gauge symmetry, which makes us expect quantum corrections to Higgs mass to be finite in compactified theories. However, we have to check the expectations explicitly since the theories under consideration are non-renormalizable. In fact, the finiteness of Higgs mass has been checked so far in various types of compactifications [4, 5] and up to 2-loops of perturbations [6]. The finite observables other than Higgs mass and potential have been also known so far, such as the gluon fusion production and diphoton decay of Higgs boson [7], the g-2 and EDM [8, 9], and the deviation of Yukawa coupling from the gauge coupling [10].

The observables in the gauge and Higgs sector of the theory are well controlled by the gauge symmetry and predictable, but a matter sector is troublesome in GHU because of a feature such that Yukawa coupling is originated from the gauge coupling. Therefore, issues of Yukawa hierarchy, flavor mixings and CP violation are very nontrivial in GHU. As for Yukawa hierarchy, the fermion mass is provided by W-boson mass to start with. The light fermion Yukawa couplings are realized by the overlap integral of wave functions localized at different point in extra space. Changing the location of the wave functions is realized by $Z_2$ odd bulk mass parameters in a five dimensional case considered in this paper. A realistic Yukawa couplings except for top quark is obtained by $O(1)$ tuning of the $Z_2$ odd bulk mass parameters. Top yukawa coupling is realized by embedding to the large dimensional representation. The mechanism of generating the flavor mixing in GHU was proposed in [11, 12], and was applied to the FCNC processes in the quark sector [12, 13, 14]. To obtain flavor mixings, both the bulk masses and the brane ones are necessary, where the bulk masses are for Yukawa hierarchy as mentioned before and the brane masses are for removing massless exotic fermions as explained later. In general, these bulk masses and the brane ones cannot be simultaneously diagonalized, which results in the flavor mixing. As for CP violation in the context of GHU, some mechanisms have been proposed, where CP is spontaneously broken by the vacuum expectation value (VEV) of CP odd extra component of higher dimensional gauge boson in odd extra dimensions [9], by compactification with a complex structure in even dimensions [15] and by the phases in the flavor mixing matrices for quark doublets [16].

In this paper, we study the lepton flavor violation in the context of GHU. Similar to the
quark sector, the lepton flavor violation appears in the presence of both the bulk masses and the brane ones and by nonzero Kaluza-Klein (KK) Z boson and photon exchange at tree level. We have a naive guess that the lepton flavor violation is larger than the flavor mixing in the quark sector because of the large mixing in the lepton sector contrary to the small mixing in the quark sector. We therefore expect that the lepton flavor violation processes put more severe constraints for parameters of the theory. As typical examples, we consider two processes in the context of GHU, namely $\mu \rightarrow 3e$ and $\mu \rightarrow e$ conversion in nuclei.

This paper is organized as follows. We introduce our model and explain how the lepton flavor mixings are obtained in the next section. In section 3, we estimate the processes of $\mu \rightarrow 3e$ and $\mu \rightarrow e$ conversion in nuclei and derive the lower bound for the compactification scale from the experimental data. Summary is given in section 4. In appendix A, we provide the model parameters used in the calculation.

2 Model

We consider a five dimensional $SU(3) \times U'(1) \times SU(3)_{\text{color}}$ gauge-Higgs unification model compactified on an orbifold $S^1/Z_2$ with a compactification radius $R$. The electroweak symmetry $SU(2)_L \times U(1)_Y$ is embedded in $SU(3) \times U(1)'$ gauge group and $SU(3)_{\text{color}}$ is a QCD color group. Since the quark sector of the model was discussed in detail by the present authors in [12, 13, 14], we briefly summarize our model.

The leptons and quarks except for the top quark are embedded in $\mathbf{3}$ and $\mathbf{\bar{6}}$ representations of $SU(3)$, and the top quark only needs a higher dimensional representation such as the $\mathbf{10}$ of $SU(3)$ for the following reason [17]. In the gauge-Higgs unification scenario, the yukawa couplings are originated from the higher-dimensional gauge coupling so that the various light fermion masses are realized by the overlap integral of wave functions for quarks and leptons, in which the wave function is localized at different point in the fifth dimension. On the other hand, the top quark mass around twice of the W boson mass is realized by the normalization factor of higher dimensional representation. The matter content in our model is summarized as

$$q(\mathbf{3}, 0, 3)^i = Q_3^i + d^i \quad (i = 1, 2, 3),$$  \hspace{1cm} (2.1)  
$$q(\mathbf{\bar{6}}, 0, 3)^i \supset Q_6^i + u^i \quad (i = 1, 2),$$  \hspace{1cm} (2.2)  
$$q(\mathbf{10}, -2/3, 3) \supset Q_{10} + t,$$  \hspace{1cm} (2.3)  
$$l(\mathbf{3}, -2/3, 1)^i = L_3^i + e^i \quad (i = 1, 2, 3),$$  \hspace{1cm} (2.4)  
$$l(\mathbf{\bar{6}}, -2/3, 1)^i \supset L_6^i + \nu^i \quad (i = 1, 2, 3),$$  \hspace{1cm} (2.5)  

where the numbers in the parenthesis stand for the representations of $SU(3), U'(1)$ and $SU(3)_{\text{color}}$. The $Q$ and $L$ represent the quark and lepton $SU(2)_L$ doublets, respectively.
and the $u, d, t, \nu$ and $e$ represent the $SU(2)_L$ singlets of the quarks and leptons. $\mathbb{C}$ means that the representations except for the fundamental representation have fields irrelevant for the SM fermions. The $U'(1)$ charges of each representations are adjust for each SM fields to have appropriate hypercharges.

Then the bulk Lagrangian consists of three parts;

$$L_B = L_q + L_1 + L_G.$$  \hfill (2.6)

Each Lagrangians are given as follows;

$$L_q = \bar{q}_i^j (i \partial_3 - M^i_q(y)) q_j^i + \bar{q}_6^3 (i \partial_y - M^i_q(y)) q_6^i + \bar{q}_6^3 i \partial_y q_6^i + \bar{q}_5^3 i \partial_{15} q_5^3.$$ \hfill (2.7a)

$$L_1 = \bar{l}_i^j (i \partial_3 - M^i_q(y)) l_j^i + \bar{l}_6^3 (i \partial_y - M^i_q(y)) l_6^j,$$ \hfill (2.7b)

$$L_G = - \frac{1}{2} \text{Tr} G_{MN} G^{MN} - \frac{1}{4} B_{MN} B^{MN} - \frac{1}{2} \text{Tr} F_{MN} F^{MN},$$ \hfill (2.7c)

where the index $i, j$ denotes a generation: $i = 1, 2, 3$ and $j = 1, 2$. The covariant devatives and the field strength of gauge bosons are defined by

$$\partial = \Gamma^M (\partial_M - ig A_M - ig s G_M),$$ \hfill (2.8a)

$$\partial' = \Gamma^M (\partial_M - ig A_M - ig' B_M - ig s G_M),$$ \hfill (2.8b)

$$G_{MN} = \partial_M G_N - \partial_N G_M - ig s [G_M, G_N],$$ \hfill (2.8c)

$$B_{MN} = \partial_M B_N - \partial_N B_M,$$ \hfill (2.8d)

$$F_{MN} = \partial_M A_N - \partial_N A_M - ig [A_M, A_N].$$ \hfill (2.8e)

The $G_M, B_M$ and $A_M$ stand for the $SU(3)_{\text{color}}, U'(1)$ and $SU(3)$ gauge fields respectively and the gauge fields $G_M$ and $A_M$ are written in a matrix form, e.g. $A_M = A^\mu_M \lambda^\mu / 2$ in terms of Gell-Mann matrices $\lambda^\mu$. $M, N = 0, 1, 2, 3, 5$ and the five dimensional gamma matrices are given by $\Gamma^M = (\gamma^\mu, i \gamma^5) \ (\mu = 0, 1, 2, 3)$. $g_s, g$ and $g'$ are 5D gauge coupling constants of $SU(3)_{\text{color}}, SU(3)$ and $U'(1)$, respectively. $M^i$ are generation dependent bulk mass parameters of the fermions accompanied by the sign function $\epsilon(y)$ with respect to the fifth dimensional coordinate $y$. Note that the gauge fixing terms and ghost fields are needed to describe our model, however, we omitted them to avoid complexity.

The theory has a periodicity along with the $y$-direction and $Z_2$ parities are assigned on the fields. We adopt the $Z_2$ parities for the gauge fields

$$G_\mu(-y) = G_\mu(y), \quad G_y(-y) = -G_y(y),$$ \hfill (2.9a)

$$B_\mu(-y) = B_\mu(y), \quad B_y(-y) = -B_y(y),$$ \hfill (2.9b)

$$A_\mu(-y) = PA_\mu(y)P^{-1}, \quad A_y(-y) = -PA_y(y)P^{-1},$$ \hfill (2.9c)

where the orbifolding matrix is defined as $P = \text{diag}(-, -, +)$ and operated in the same way at the fixed points $y = 0, \pi R$. For the matter fields, the parity assignments is as follows.

$$q^i(3) = \{ Q^i_{3L}(+, +) + Q^i_{3R}(-, -) \} \oplus \{ d^i_{3L}(-, -) + d^i_{3R}(+, +) \} \quad (i = 1, 2, 3).$$ \hfill (2.9d)
where the sign in the parenthesis represents the eigenvalues of the $Z_2$ parities at $y = 0$ and $\pi R$, respectively. Here we can see that the gauge symmetry $SU(3)$ is explicitly broken into $SU(2) \times U(1)$ and a chiral theory is realized in the zero mode sector by $Z_2$ orbifolding.

The mode functions of gauge fields are easily obtained with respect to the $Z_2$ parities. The $Z_2$ even (odd) components are expanded in terms of $C_n(y) = \frac{1}{\sqrt{\pi R}} \cos \frac{n}{R} y, S_n(y) = \frac{1}{\sqrt{\pi R}} \sin \frac{n}{R} y$. Here we concentrate on the zero mode sector which plays an important role on the flavor mixing. The five dimensional fermion $\chi^i (\phi^i)$ with $Z_2$ even (odd) parity includes the left- (right-) handed chiral fermions as

$$\chi^i \supset \chi^i_L (0) f^i_L (y), \phi^i \supset \phi^i_R (0) f^i_R (y),$$

where

$$f^i_L (y) = \sqrt{\frac{M^i}{1 - e^{-2\pi R M^i} e^{-M^i |y|}}}, f^i_R (y) = \sqrt{\frac{M^i}{e^{2\pi R M^i} - 1} e^{M^i |y|}}.$$

The above mode functions lead to the light fermion mass by strong suppressions for Yukawa couplings.

Since the $U(1)$ gauge boson includes a gauge anomaly in this model, we discuss the mixing between $U(1)$ and $U'(1)$ gauge bosons. The mixing between them has been already discussed in [18], a hypercharge $U(1)_Y$ is identified with some linear combination of $U(1)$ and $U'(1)$ gauge groups. The other orthogonal $U(1)$ is anomalous, and therefore the corresponding gauge boson obtain a large mass around the cutoff scale $\Lambda$. Thus, the original $U(1)$ gauge bosons are separated into an anomalous one $Z'$ and the hypercharge gauge boson $A_Y$ as follows:

$$U(1) \text{ part } \supset \frac{g}{2} (\lambda_3 A^3 + \lambda_8 A^8) + g' Y' B$$

$$= \frac{g}{2} \lambda_3 A^3 + \left( \frac{g}{2} \cos \theta \lambda_8 - g' \sin \theta Y' \right) A_Y + \left( \frac{g}{2} \sin \theta \lambda_8 + g' \cos \theta Y' \right) Z',$$

where $A_Y, Z'$ are defined as follows:

$$\begin{cases} A^8 = \cos \theta A_Y + \sin \theta Z', \\ B = \cos \theta Z' - \sin \theta A_Y. \end{cases}$$

We assign the $U'(1)$ charge $-2/3$ on $3, \bar{6}$ and the $U(1)_Y$ hypercharge is identified with a sum of $U(1)$ and $U'(1)$ charges. In this case, the $U(1)$, $U'(1)$, and $U(1)_Y$ charges are provided as

$$\lambda_8 = \frac{1}{\sqrt{3}} \text{diag} (1, 1, -2), Y' = \text{diag} \left( -\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3} \right), Y = \text{diag} \left( \frac{1}{2}, \frac{1}{2}, -1 \right).$$
from which we find
\[ g_Y Y = \frac{g}{2} \cos \theta \lambda_8 - g' \sin \theta \theta', \] (2.15)
and
\[ \cos \theta = \frac{g'}{\sqrt{3g^2 + g'^2}}, \quad \sin \theta = -\frac{\sqrt{3g}}{\sqrt{3g^2 + g'^2}}, \quad g_Y = \sqrt{3g} \cos \theta. \] (2.16)
These results tell us photon and Z-bosons like
\[ \text{U(1) part} = \left( \frac{g}{2} \cos \theta \lambda_3 - g_Y \sin \theta \lambda_Y \right) \frac{Z}{Z'}, \] (2.17)
where \( Q = \text{diag}(0, -1, -1), e = g_Y \cos \theta_W. \) The Weinberg angle \( \theta_W \) in this case is defined by
\[ \sin^2 \theta_W = \frac{3}{4 + 3g^2/g'^2}, \] (2.18)
which shows that the correct Weinberg angle is obtained by choosing a free parameter \( U'(1) \) gauge coupling \( g' \) appropriately.

2.1 Lepton flavor mixing

In the gauge-Higgs unification, it is not trivial to generate the flavor mixing since Yukawa coupling is originated from the gauge coupling which is flavor universal. In the paper [12], it was proposed that the flavor mixing in the context of gauge-Higgs unification can be generated in a situation that both of the bulk and brane mass terms are present, because these masses cannot be simultaneously diagonalized in a flavor space. The mechanism was applied to the quark sector in [12, 13, 14]. In this subsection, we apply the mechanism to the lepton sector. As was seen before, we have two lepton doublets \( L^{(0)}_L \) and \( L^{(0)}_R \) per a generation, we must make one of the linear combination of them massive by introducing the brane-localized mass terms and the brane-localized fermions with charges conjugate to leptons \( \bar{l}_R \) as was done in the quark sector. Here is such mass terms localized at \( y = 0 \).

\[ \mathcal{L}_{BM} = \int_{-\pi_R}^{\pi_R} dy \sqrt{2\pi R} \delta(y) \tilde{L}_R^i(x) [\eta_{ij} L^{(0)}_{3L}(x, y) + \lambda_{ij} L^{(0)}_{6L}(x, y)] \]
\[ \supset \sqrt{2\pi R} \tilde{L}_R^i(x) [\eta_{ij} f^i_L(0), \lambda_{ij} f^j_L(0)] \left( \begin{array}{c} L^{(0)}_{3L} \\ L^{(0)}_{6L} \end{array} \right) \]
\[ = \sqrt{2\pi R} \tilde{L}_R^i [m_{diag}, 0] \left( \begin{array}{c} L^H \\ L^{SM} \end{array} \right), \] (2.19)
where \( f^i_L(0) \) is a zero mode function of the \( i \)-th generational lepton doublet evaluated at \( y = 0 \). It shows that the \( L^H \) become massive and decouple from the low-energy effective theory. The other lepton doublet \( L^{SM} \) which corresponds to the standard model leptons is given by
\[ \left( \begin{array}{c} L^{(0)}_{3L} \\ L^{(0)}_{6L} \end{array} \right) = U \left( \begin{array}{c} L^H \\ L^{SM} \end{array} \right) = \left( \begin{array}{cc} U_1 & U_3 \\ U_2 & U_4 \end{array} \right) \left( \begin{array}{c} L^H \\ L^{SM} \end{array} \right) = \left( \begin{array}{c} U_1 L^H + U_3 L^{SM} \\ U_2 L^H + U_4 L^{SM} \end{array} \right) \rightarrow \left( \begin{array}{c} U_3 L^{SM} \\ U_4 L^{SM} \end{array} \right), \] (2.20)
where $U$ stands for $6 \times 6$ unitarity matrix. Then the lepton Yukawa couplings is obtained from their gauge interactions as

$$\mathcal{L}_{\text{Lepton Yukawa}} \supset -\frac{g}{2} \bar{e}_i H L_3 + \frac{g}{\sqrt{2}} \bar{\nu}_R H^T (i\sigma_2)^* L_i + (\text{h.c.})$$

$$\supset -\frac{g}{2} \frac{h}{\sqrt{2} \pi R} \bar{e}_i H (0) U^{ij}_R I^{(0)}_{RL} \nu^{i (0)}_L + \frac{g}{\sqrt{2} \pi R} \bar{\nu}_R H^T (i\sigma_2)^* L_i + (\text{h.c.}),$$

(2.21)

where $H$ is a SM Higgs doublet and $h$ is its neutral component. $I^{(0)}_{RL}$ is an overlap integral of lepton zero mode functions;

$$I^{(0)}_{RL} = \int_{-\pi R}^{\pi R} dy f^i_L(y) f^j_R(y).$$

(2.22)

The mass eigenstates

$$\begin{align*}
\tilde{e}^i_R & = V^{\prime ij}_e \tilde{e}^j_R, \\
\bar{e}_i^L & = V^{\prime ij}_e \bar{e}^j_L,
\end{align*}$$

$$\begin{align*}
\tilde{\nu}^i_R & = V^{\prime ij}_\nu \tilde{\nu}^j_R, \\
\bar{\nu}_i^L & = V^{\prime ij}_\nu \bar{\nu}^j_L,
\end{align*}$$

(2.23)

are obtained by the ordinary bi-unitary transformations as

$$\begin{align*}
&\begin{cases}
\frac{1}{2} V^{(0)}_{eRL} U^T_3 V^T_{eL} = \frac{M_e}{M_W}, \\
\frac{1}{\sqrt{2}} V^{(0)}_{\nu RL} U^T_4 V^T_{\nu L} = \frac{M_\nu}{M_W},
\end{cases}
\end{align*}$$

(2.24)

where the eigenvalues are $M_e = \text{diag}(m_e, m_\mu, m_\tau)$, $M_\nu = \text{diag}(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})$. The model parameters are fitted by the above lepton masses and Maki-Nakagawa-Sakata (MNS) mixing matrix $V_{\text{MNS}}$ obtained by $V_{\text{MNS}} = V^T_{eL} V_{\nu L}$. The results of numerical calculation are listed in the appendix A.

### 2.2 Lepton flavor violation

In the previous subsection, we discussed how the lepton flavor mixing occurs. These flavor mixings cause the FCNC vertices with the KK mode gauge bosons since the gauge couplings are flavor dependent. From now, we focus on the neutral current sector and derive the FCNC interactions which are necessary to calculate $\mu \to 3e$ and $\mu \to e$ conversion processes. Extracting the photon and $Z$-boson interactions, we obtain the following expression.

$$\mathcal{L}_1 \supset \bar{\nu}_6 [\frac{g}{2} \lambda^a A^a_M + g' Y^{a'} A^{a'}_M] \Gamma^M \nu^6_3 + \text{Tr} \bar{\nu}_6 \left[ -g \lambda^a A^a_M + g' Y^{a'} A^{a'}_M \right] \Gamma^M \nu^6_6$$

$$\supset e [-\bar{e}_3 \gamma^\mu e_3 - \bar{e}_6 \gamma^\mu e_6 - \bar{e} \gamma^\mu e] \gamma_\mu$$

$$+ \frac{1}{2} g \cos \theta_W \left[ (-1 + \tan^2 \theta_W) e_3 \gamma^\mu e_3 + 2 \tan^2 \theta_W \bar{e} \gamma^\mu e \right] Z^\mu$$

$$+ g \cos \theta_W \left[ \frac{1}{2} (-1 + \tan^2 \theta_W) \bar{e}_6 \gamma^\mu e_6 \right] Z^\mu,$$

(2.25)
where $e_3$ and $e_6$ stand for the electron which are included in the $L_3$ and $L_6$. Integrate out the fifth extra dimension, four dimensional gauge interactions are obtained as follows:

$$L_{4D}^{GI} = \int_{-\pi R}^{\pi R} dy \ L_1$$

$$\dot{e} - e \sum_n \left\{ \bar{\tilde{e}} \gamma^\mu \left[ LV_{L}^\dagger (U_{I L}^3 U_3 + U_{I L}^4 U_4) V_{eL} + RV_{eR}^\dagger I_n (-1)^n V_{eR} \right] \tilde{e}_{\gamma}^{(n)} + \frac{1}{2} g \cos \theta_W \bar{\tilde{e}} \gamma^\mu \left[ L (-1 + \tan^2 \theta_W) V_{L}^\dagger (U_{I L}^3 U_3 + U_{I L}^4 U_4) V_{eL} \right.$$

$$+ R 2 \tan^2 \theta_W V_{eR}^\dagger I_n (-1)^n V_{eR} \left. \right] \tilde{e}_{\gamma}^{(n)} + R 2 \tan^2 \theta_W V_{eR}^\dagger I_n (-1)^n V_{eR} \right] \tilde{e}_{\gamma}^{(n)} \right\},$$

where the chiral projection operators are used $L = (1 + \gamma_5)/2$ and $R = (1 - \gamma_5)/2$. The $I_n^L$ are the integration of the profiles of KK mode gauge boson and zero mode leptons and their explicit form will be given in the next section. The obtained vertices indicate that the FCNC process appears by the neutral KK $Z$ and photon exchange at tree level as in the case of KK gluon exchange in the quark sector [12, 13, 14].

### 3 Lepton flavor violation processes

We are interested in several lepton flavor violation processes such as $\mu \rightarrow e\gamma$, $\mu \rightarrow 3e$ and $\mu \rightarrow e$ conversion in atomic nuclei which are focused in the ILC experiments. In this paper, we concentrate on the $\mu \rightarrow 3e$ and $\mu \rightarrow e$ conversion for the following reason. The $\mu \rightarrow e\gamma$ process appears at one loop contributions in contrast with the others arise from tree-level contributions, therefore we expect that the $\mu \rightarrow e\gamma$ process is suppressed compared with the other two processes.

#### 3.1 $\mu \rightarrow 3e$ process

Now we are ready to calculate a lepton flavor violation (LFV) process: $\mu \rightarrow 3e$ which is one of the main issue of the ILC experiment. The recent experiment [19] puts a upper bound for $\mu \rightarrow 3e$

$$\text{Br}(\mu^+ \rightarrow e^+e^-e^+) = \frac{\Gamma(\mu^+ \rightarrow e^+e^-e^+)}{\Gamma_{\text{total}}} < 1.0 \times 10^{-12}. \quad (3.1)$$

Since the ordinary muon decay process $\mu \rightarrow \nu_\mu e\bar{\nu}_e$ is dominated within the total decay width of muon $\Gamma_{\text{total}}$, it can be replaced with $\Gamma(\mu \rightarrow \nu_\mu e\bar{\nu}_e)$ in a good approximation.

To calculate the decay width $\Gamma(\mu \rightarrow e^-e^+e^-)$, we first consider the general formulae of $\mu \rightarrow 3e$ process. The generalized vertex functions in our model are

$$G^{(n)} = \gamma^\mu (A_n^{GL} L + A_n^{GR} R), \quad (3.2)$$
The photon and Z-boson vertex functions are found as follows.

\[ G^{\omega} = \gamma^\mu (B_n^{GL} L + B_n^{GR} R), \quad (3.3) \]

\[ G^{\omega} = \eta_{\mu\nu} \frac{q^2 - (M_n^G)^2}{q^2}. \quad (3.4) \]

We adopt the abbreviation \( G \) in the superscript of \( A, B \) as an intermediate neutral gauge boson: \( G = \gamma, Z \). The photon and Z-boson vertex functions are found as follows.

\[ A_n^L = -e \left[ V_{eL}^\dagger (U_3 J_n^L U_3 + U_4 J_n^L U_4) V_{eL} \right]_{\mu e}, \quad (3.5) \]
\[ A_n^R = -e [V_{eR}^\dagger J_n^L (-1)^n V_{eR}]_{\mu e}, \quad (3.6) \]
\[ B_n^L = -e \left[ V_{eL}^\dagger (U_3 J_n^L U_3 + U_4 J_n^L U_4) V_{eL} \right]_{ee}, \quad (3.7) \]
\[ B_n^R = -e [V_{eR}^\dagger J_n^L (-1)^n V_{eR}]_{ee}, \quad (3.8) \]
\[ A_n^{ZL} = \frac{1}{2} g \cos \theta_W (-1 + \tan^2 \theta_W) \left[ V_{eL}^\dagger (U_3 J_n^L U_3 + U_4 J_n^L U_4) V_{eL} \right]_{\mu e}, \quad (3.9) \]
\[ A_n^{ZR} = g \cos \theta_W \tan^2 \theta_W [V_{eR}^\dagger J_n^L (-1)^n V_{eR}]_{\mu e}, \quad (3.10) \]
\[ B_n^{ZL} = \frac{1}{2} g \cos \theta_W (-1 + \tan^2 \theta_W) \left[ V_{eL}^\dagger (U_3 J_n^L U_3 + U_4 J_n^L U_4) V_{eL} \right]_{ee}, \quad (3.11) \]
\[ B_n^{ZR} = g \cos \theta_W \tan^2 \theta_W [V_{eR}^\dagger J_n^L (-1)^n V_{eR}]_{ee}. \quad (3.12) \]

The overlap integrals of mode functions between the zero mode leptons and the KK photons (KK Z-bosons) are

\[ I_n^L = \int_{-\pi R}^{\pi R} dy (f_L^i)^2 C_n = J_n^L, \quad I_n^R = \int_{-\pi R}^{\pi R} dy (f_R^i)^2 C_n = (-1)^n J_n^R, (i = e, \mu, \tau) \quad (3.13) \]

where \( C_n \) and \( f_L^i \) are the mode functions of gauge bosons and zero mode fermions which are defined by the section 2 and

\[ J_n^i = \frac{1}{\sqrt{\pi R}} \frac{(2RM^i)^2}{1 - e^{-2\pi RM^i}} \frac{1 - (-1)^n e^{-2\pi RM^i}}{n^2 + (2M^i R)^2}. \quad (3.14) \]

The numerical values used in the above calculations are found in appendix A.

By using these expressions, we can write down the amplitude \( \mathcal{M} \) of \( \mu \to 3e \) process as

\[ \mathcal{M} = \sum_{G} \sum_{n=1}^{\infty} \sum_{p_1} \bar{e}(p_1) \gamma^\mu (A_n^{GL} L + A_n^{GR} R) \mu(k) \bar{e}(p_2) \gamma^\mu (B_n^{GL} L + B_n^{GR} R) e(p_2) \frac{1}{(k - p_1)^2 - (M_n^G)^2}. \quad (3.15) \]

\footnote{Note that we must take into account the anomalous \( U(1) \) gauge boson \( Z' \) in \( G \), however, we can neglect the contributions for the following reason. The anomaly in this model appears at the fixed points, so that it yields the brane localized mass term of the anomalous gauge boson [20]. Since the profile of the anomalous gauge boson behaves as \( \sin |y| \), the couplings to the zero mode leptons are expected to be small due to the locality of the zero mode leptons.}
In this calculation, we treat the spin of the initial state ($\mu$) and the final state ($e$) as “spin average” and “spin sum”, respectively. After the straightforward lengthy algebra, we obtain the partial decay width of $\Gamma(\mu \to 3e)$ as

$$\Gamma(\mu \to 3e) = \frac{m_\mu}{128\pi^3} \left[ -2m_e^3m_\mu S_1 - \frac{1}{48}m_e m_\mu^3 S_2 + \frac{1}{3}m_e^2 m_\mu^2 S_3 + \frac{2}{15}m_\mu^4 (S_4 + S_5) \right] \quad (3.16)$$

where the mode summations are defined as

$$S_1 = \sum_{G,G',n,n'} \frac{1}{(M_G^d)^2(M_{G'}^d)^2} (A_n^{GR} A_{n'}^{G'Ls} + A_n^{GL} A_{n'}^{G'R*})(B_n^{GR} B_{n'}^{G'Ls} + B_n^{GL} B_{n'}^{G'R*}), \quad (3.17)$$

$$S_2 = \sum_{G,G',n,n'} \frac{1}{(M_G^d)^2(M_{G'}^d)^2} (A_n^{GR} A_{n'}^{G'Ls} + A_n^{GL} A_{n'}^{G'R*})(B_n^{G'R*} B_{n'}^{GL} + B_n^{GR} B_{n'}^{G'Ls}), \quad (3.18)$$

$$S_3 = \sum_{G,G',n,n'} \frac{1}{(M_G^d)^2(M_{G'}^d)^2} (A_n^{GL} A_{n'}^{G'Ls} + A_n^{GR} A_{n'}^{G'R*})(B_n^{G'R*} B_{n'}^{GL} + B_n^{GR} B_{n'}^{G'Ls}), \quad (3.19)$$

$$S_4 = \sum_{G,G',n,n'} \frac{1}{(M_G^d)^2(M_{G'}^d)^2} (A_n^{GL} A_{n'}^{G'Ls} B_n^{GR} B_{n'}^{G'R*} + A_n^{GR} A_{n'}^{G'R*} B_n^{GL} B_{n'}^{G'Ls}), \quad (3.20)$$

$$S_5 = \sum_{G,G',n,n'} \frac{1}{(M_G^d)^2(M_{G'}^d)^2} (A_n^{GL} A_{n'}^{G'Ls} B_n^{G'R*} B_{n'}^{GL} + A_n^{GR} A_{n'}^{G'R*} B_n^{GR} B_{n'}^{G'Ls}). \quad (3.21)$$

These mode summations are very complicated and we calculated them numerically as

$$S_1 = 2.38 \times 10^{-16} g^4 R^4 \sim 4.45 \times 10^{-15} e^4 R^4, \quad (3.22)$$

$$S_2 = -3.04 \times 10^{-17} g^4 R^4 \sim -5.69 \times 10^{-16} e^4 R^4, \quad (3.23)$$

$$S_3 = -9.68 \times 10^{-15} g^4 R^4 \sim -1.81 \times 10^{-13} e^4 R^4, \quad (3.24)$$

$$S_4 = 3.38 \times 10^{-16} g^4 R^4 \sim 6.33 \times 10^{-15} e^4 R^4, \quad (3.25)$$

$$S_5 = 6.92 \times 10^{-14} g^4 R^4 \sim 1.29 \times 10^{-12} e^4 R^4. \quad (3.26)$$

We note that the $SU(2)$ gauge coupling $g$ is replaced by the $U(1)_{em}$ gauge coupling $e$ through the relation $e = g \sin \theta_W$. We finally obtain the partial decay width of $\mu \to 3e$ in the gauge-Higgs unification

$$\Gamma(\mu \to 3e) \sim 4.45 \times 10^{-13} (RM_W)^4, \quad (3.27)$$

which tells us the lower bound for the compactification scale $R^{-1} \geq 0.81 M_W \sim 65 \text{GeV}$.

### 3.2 $\mu \to e$ conversion

In this subsection, we turn to another process of lepton flavor violation at tree level in gauge-Higgs unification, e.g. $\mu \to e$ conversion in nuclei. The four dimensional effective Lagrangian describing $\mu \to e$ conversion process is given by the following 4-Fermi interactions [21, 22]

$$\mathcal{L}_{\mu\to e} \supset \frac{G_F}{\sqrt{2}} \left[ \bar{e} \gamma^\mu (v - a \gamma_5) \mu \sum_{q=u,d} \bar{q} (v^q - a^q \gamma_5) q \right], \quad (3.28)$$
where the lepton current part is the same form used in the calculations of \( \mu \to 3e \) with factors
\[
v = \frac{1}{2}(A_{n}^{GL} + A_{n}^{GR}), \quad a = \frac{1}{2}(A_{n}^{GL} - A_{n}^{GR}) \quad (G = \gamma_\mu, Z_\mu).
\] (3.29)

The four dimensional effective neutral current of the quark sector in (3.28) can be found from the photon and the \( Z \)-boson currents of the model in [14].

\[
\mathcal{L}_{\text{NC}}^{4D} = \bar{u}_N^\gamma \left( L_N^\gamma U_n^L + R_N^\gamma R_n \right) u_\mu^{(n)} + \bar{d}_N^\gamma \left( L_N^\gamma L_n^d + R_N^\gamma R_n \right) d_\mu^{(n)}
\]
\[
+ \bar{u}_N^\gamma \left( L_N^\gamma U_n^L + R_N^\gamma R_n \right) u_\mu^{(n)} + \bar{d}_N^\gamma \left( L_N^\gamma L_n^d + R_N^\gamma R_n \right) d_\mu^{(n)},
\] (3.30)

where

\[
B_{n}^{L\mu} = \frac{2}{3} e \left[ V_{nL}^L (U_{q3}^{U} I_{q3}^{U} U_{q3}^{Q} + U_{q4}^{U} I_{q4}^{U} U_{q4}) V_{nL} \right]_{uu},
\]
\[
B_{n}^{R\mu} = \frac{2}{3} e \left[ V_{nR}^L (U_{q3}^{U} I_{q3}^{U} U_{q3}^{Q} + U_{q4}^{U} I_{q4}^{U} U_{q4}) V_{nR} \right]_{uu},
\]
\[
B_{n}^{I\mu} = \frac{1}{3} e \left[ V_{n}^{L} (U_{q3}^{U} I_{q3}^{U} U_{q3}^{Q} + U_{q4}^{U} I_{q4}^{U} U_{q4}) V_{n} \right]_{dd},
\]
\[
B_{n}^{I\mu} = \frac{1}{3} e \left[ V_{n}^{R} (U_{q3}^{U} I_{q3}^{U} U_{q3}^{Q} + U_{q4}^{U} I_{q4}^{U} U_{q4}) V_{n} \right]_{dd}.
\]

The \( U_{q3} \) and \( U_{q4} \) are the mixing matrices representing how much the \( Q_{SM} \) is included in the \( Q_{3}, Q_{6} \) and \( Q_{15} \), which correspond to the (2.20) in the lepton sector. \( V_{nL(R)} \) and \( V_{nL(R)} \) are the unitarity matrices diagonalizing up-type and down-type yukawa couplings, respectively. The parameters in the quark sectors are fitted similarly to the leptons and they are summarized in the appendix A. Then the factors in the eq. (3.28) can be read off from eq. (3.30) as
\[
v^{q} = \frac{1}{2} \left( B_{n}^{GLq} + B_{n}^{GRq} \right), \quad a^{q} = \frac{1}{2} \left( B_{n}^{GLq} - B_{n}^{GRq} \right).
\] (3.39)

The \( \mu \to e \) conversion rate in the light nuclei is precisely discussed in the paper [21, 22]
\[
B_{\text{conv}} = \frac{2p_{e}E_{e}F_{p}^{2}m_{n}^{3}e^{3}Z_{2}\alpha^{3}}{\pi^{3}Z_{2}\Gamma_{\text{cap}}^{\text{eff}}}
\left[ (v_{N}^{N} - a_{Z}^{N})Q_{N}^{N} + (v_{N}^{Z} - a_{Z}^{Z})Q_{N}^{Z} \right]^{2} + \left[ (v_{N}^{N} - a_{Z}^{N})Q_{N}^{N} + (v_{N}^{Z} + a_{Z}^{Z})Q_{N}^{Z} \right]^{2},
\]
\[
B_{\text{conv}} = \frac{2p_{e}E_{e}F_{p}^{2}m_{n}^{3}e^{3}Z_{2}\alpha^{3}Z_{2}^{2}}{\pi^{3}Z_{2}\Gamma_{\text{cap}}^{\text{eff}}}
\left[ (v_{N}^{N} - a_{Z}^{N})Q_{N}^{N} + (v_{N}^{Z} - a_{Z}^{Z})Q_{N}^{Z} \right]^{2} + \left[ (v_{N}^{N} - a_{Z}^{N})Q_{N}^{N} + (v_{N}^{Z} + a_{Z}^{Z})Q_{N}^{Z} \right]^{2},
\] (3.40)

where \( Q_{N}^{N} = [v_{N}^{N}(2Z + N) + v_{N}^{d}(2N + Z)]|_{G=Z} \) and \( Q_{N}^{Z} = [v_{N}^{N}(2Z + N) + v_{N}^{d}(2N + Z)]|_{G=Z} \). 

\( Z \) and \( N \) represent the atomic number and neutron number of the target nuclei. The
most sensitive experimental result is the case for the $^{48}_{22}$Ti. The parameters in the above expressions are

$$E_e \sim p_e \sim m_{\mu}, \ F_p \sim 0.55, \ Z_{\text{eff}} \sim 17.61, \ \Gamma_{\text{capl}} \sim 2.6 \times 10^6 [s^{-1}]. \quad (3.41)$$

Putting them into eq. (3.40) and summing up the KK modes of internal gauge bosons, we obtain the following results

$$B_{\text{conv}} = \begin{cases} 2.89 \times 10^{-4} R^4 & \text{(for the case } R_u = 1_{3\times3}) \ , \\
1.46 \times 10^{-4} R^4 & \text{(for the case } R_d = 1_{3\times3}) \ , \end{cases} \quad (3.42)$$

which lead to the lower bounds of compactification scale as

$$R^{-1} \geq \begin{cases} 147.5 \text{GeV} & \text{(for the case } R_u = 1_{3\times3}) \ , \\
69.89 \text{GeV} & \text{(for the case } R_d = 1_{3\times3}) \ , \end{cases} \quad (3.43)$$

from the experimental data $\text{Br}(\mu \to e)_{\text{Ti}} < 6.1 \times 10^{-13}$ [23].

4 Summary

In this paper, we have discussed the lepton flavor violation within the gauge-Higgs unification model. Yukawa couplings are essentially universal since they are gauge coupling in this scenario. The flavor violation is achieved by the interplay between the fermion bulk mass terms localizing the leptons and quarks at fixed points and the brane localized mass terms removing the extra massless fermions. We have no flavor violation on the neutral gauge interaction in the zero mode sector due to the universality of the gauge coupling, but the tree level FCNC vertex appears in the KK mode gauge boson sector since the gauge interactions in the KK mode sector are found to be flavor dependent.

These tree level FCNC interactions may give rise to the large lepton flavor violation processes such as the $\mu \to 3e$ decay processes and the $\mu \to e$ conversion in atomic nuclei, which is one of the main purpose of the future ILC experiment. Though these process takes place at tree level, they are rather small against our expectation and we obtain the lower bound of compactification scale as $1/R \geq \mathcal{O}(M_W)$.

The reason why the lepton violations are suppressed is that the lepton flavor symmetry is almost conserved in this model. It is the general feature of this model that the differences between the eigenstates of bulk masses and brane masses of fermions is the only source of the flavor violation in contrast to the other models such as the supersymmetric model, in which the large flavor violation other than Yukawa coupling is in general present in soft SUSY breaking parameters. If the neutrino masses are degenerate, the lepton flavor violation completely disappears. Taking into account this fact, the final results roughly receive the suppression factor as $\Delta m_{\mu}^2 R^2$ reflecting the flavor structure,
where $\Delta m^2_\nu$ denotes the differences of neutrino mass squared. For example, the $\mu \to 3e$ processes which was argued in the main text are naively estimated as

$$\text{Br}(\mu \to 3e) \sim (m_W R)^4 \leq 10^{-12}. \quad (4.1)$$

Without such kind of suppressions, we find more stringent bound $1/R \sim 10^3 M_W$ than our result. However, if the factor $\Delta m^2_\nu R^2$ is taken into account, we obtain the branching ratio as

$$\text{Br}(\mu \to 3e) \sim \Delta m^2_\nu R^2 (m_W R)^4 \leq 10^{-12} \quad (4.2)$$

which gives the lower bound $1/R \sim M_W$ with the observed neutrino mass $\Delta m_\nu \leq \mathcal{O}(10\text{MeV})$. This is the physical reason that the lepton flavor violation considered in this paper is unexpectedly suppressed although it happens even at tree level. From this observation, it is very important to study a loop-induced process $\mu \to e\gamma$ in the gauge-Higgs unification since it is expected to provide a stronger bound on model parameters. This issue will be left for our future work.

**Acknowledgments**

This work was supported in part by the Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, No. 24540283 (N.M.).

**A Model parameters**

In this appendix A, the details of parameters used in the calculations are summarized.

**A.1 lepton sector**

Yukawa coupling and MNS matrix in the lepton sector are given by

$$\hat{Y}_e = V_{eR}^\dagger \hat{J}^{(00)}_{RL} U_3 V_{eL}, \quad \hat{Y}_\nu = V_{\nu R}^\dagger \sqrt{2} \hat{J}^{(00)}_{RL} U_4 V_{\nu L}, \quad (A.1)$$

$$V_{\text{MNS}} = V_{eL}^\dagger V_{\nu L} = \begin{pmatrix} c_{12} c_{13} & s_{12} c_{13} & s_{13} \\ -s_{12} c_{23} - c_{12} s_{23} s_{13} & c_{12} c_{23} - s_{12} s_{23} s_{13} & s_{23} c_{13} \\ s_{12} s_{23} - c_{12} c_{23} s_{13} & -c_{12} s_{23} - s_{12} c_{23} s_{13} & c_{23} c_{13} \end{pmatrix}, \quad (A.2)$$

where

$$U_4 = R_\nu \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{13} \end{bmatrix}, \quad U_3 = R_e \begin{bmatrix} \sqrt{1 - a_{11}^2} & 0 & 0 \\ 0 & \sqrt{1 - a_{12}^2} & 0 \\ 0 & 0 & \sqrt{1 - a_{13}^2} \end{bmatrix}, \quad (A.3)$$

$$R_\nu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta'_{12} & \sin \theta'_{12} \\ 0 & -\sin \theta'_{12} & \cos \theta'_{12} \end{bmatrix}, \quad R_e = \begin{bmatrix} \cos \theta'_{13} & 0 & \sin \theta'_{13} \\ 0 & 1 & 0 \\ -\sin \theta'_{13} & 0 & \cos \theta'_{13} \end{bmatrix} \begin{bmatrix} \cos \theta'_{11} & -\sin \theta'_{11} & 0 \\ \sin \theta'_{11} & \cos \theta'_{11} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (A.4)$$
\[
R_c = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{12} & \sin \theta_{12} \\
0 & -\sin \theta_{12} & \cos \theta_{12}
\end{bmatrix}
\begin{bmatrix}
\cos \theta_{13} & 0 & \sin \theta_{13} \\
0 & 1 & 0 \\
-\sin \theta_{13} & 0 & \cos \theta_{13}
\end{bmatrix}
\begin{bmatrix}
\cos \theta_{11} & -\sin \theta_{11} & 0 \\
\sin \theta_{11} & \cos \theta_{11} & 0 \\
0 & 0 & 1
\end{bmatrix},
\]
(A.5)

\[
I_{RL}^{(00)} = \text{diag}[b_1^l, b_2^l, b_3^l], \quad \left(b_i^l = \frac{\pi R M_i^l}{\sinh \pi R M_i^l}\right)
\]
(A.6)

\[
s_{ij} \equiv \sin \phi_{ij}, \quad c_{ij} \equiv \cos \phi_{ij}.
\]
(A.7)

\(V_{\nu L(R)}\) are the unitary matrices diagonalizing the matrices \(\hat{Y}_{\nu}^{\dagger} \hat{Y}_{\nu}(\hat{Y}_{\nu}^{\dagger})\) and \(V_{\nu L(R)}\) are the unitary matrices diagonalizing the matrices \(\hat{Y}_{\nu}^{\dagger} \hat{Y}_{\nu}(\hat{Y}_{\nu}^{\dagger})\). In the MNS matrix, CP phase is neglected since CP violation is not an issue in this paper.

The input parameters for physical observables we should fit are \([24]\)

\[
\begin{align*}
\hat{m}_{\nu_e} &= (2.00 \times 10^{-9})/m_W, \quad \hat{m}_{\nu_\mu} = (1.90 \times 10^{-4})/m_W, \quad \hat{m}_{\nu_\tau} = 0.0182/m_W, \quad \text{(A.8)} \\
\hat{m}_e &= (5.11 \times 10^{-4})/m_W, \quad \hat{m}_{\mu} = 0.106/m_W, \quad \hat{m}_\tau = 1.78/m_W, \quad \text{(A.9)} \\
m_{\nu e} &\sim 2\text{eV}, \quad m_{\nu_\mu} \sim 190\text{keV}, \quad m_{\nu_\tau} \sim 18.2\text{MeV}, \quad \text{(A.10)} \\
\sin^2 \phi_{12} &= 0.306, \sin^2 \phi_{23} = 0.42, \sin^2 \phi_{13} = 0.021. \quad \text{(A.11)}
\end{align*}
\]

We found a set of numerical solutions reproducing the above physical observables in the special case \(R_c = I_{3\times3}\).

\[
\begin{align*}
\sin \theta_{11} &= 5.53 \times 10^{-1}, \sin \theta_{12} = 6.48 \times 10^{-1}, \sin \theta_{13} = 1.50 \times 10^{-1}, \\
\theta'_{11} &= \theta'_{12} = \theta'_{13} = 0, \\
a_{11} &= 2.77 \times 10^{-6}, \quad a_{12} = 1.27 \times 10^{-3}, \quad a_{13} = 7.24 \times 10^{-3}, \quad \text{(A.12)} \\
b_1^l &= 6.35 \times 10^{-6}, \quad b_2^l = 1.31 \times 10^{-3}, \quad b_3^l = 2.21 \times 10^{-2}. \quad \text{(A.13)}
\end{align*}
\]

Then, Yukawa coupling and their mixing matrix in terms of these numerical solutions are listed below.

\[
Y_\nu = \sqrt{2} I_{RL}^{(00)} U_4 = \begin{pmatrix}
2.48 \times 10^{-11} & 0 & 0 \\
0 & 2.36 \times 10^{-6} & 0 \\
0 & 0 & 2.26 \times 10^{-4}
\end{pmatrix},
\]
(A.16)

\[
V_{\nu L} = V_{\nu R} = I_{3\times3},
\]

\[
Y_e = I_{RL}^{(00)} U_3 = \begin{pmatrix}
5.23 \times 10^{-6} & -3.47 \times 10^{-6} & 9.52 \times 10^{-7} \\
4.47 \times 10^{-4} & 9.03 \times 10^{-4} & 8.41 \times 10^{-4} \\
-1.00 \times 10^{-2} & -1.05 \times 10^{-2} & 1.66 \times 10^{-2}
\end{pmatrix},
\]

\[
V_{e L} = \begin{bmatrix}
0.824 & 0.340 & -0.454 \\
-0.547 & 0.688 & -0.477 \\
0.150 & 0.641 & 0.753
\end{bmatrix},
\]
(A.17)

\[
V_{e R} = \begin{bmatrix}
1.00 & -2.14 \times 10^{-8} & -1.83 \times 10^{-9} \\
2.14 \times 10^{-8} & 1.00 & -1.48 \times 10^{-6} \\
1.83 \times 10^{-9} & 1.48 \times 10^{-6} & 1.00
\end{bmatrix},
\]
(A.18)

\[
U_3 = \begin{pmatrix}
0.824 & -0.547 & 0.150 \\
0.340 & 0.688 & 0.641 \\
-0.454 & -0.477 & 0.753
\end{pmatrix}, \quad U_4 = \begin{pmatrix}
2.77 \times 10^{-6} & 0 & 0 \\
0 & 1.27 \times 10^{-3} & 0 \\
0 & 0 & 7.24 \times 10^{-3}
\end{pmatrix}.
\]
(A.19)
A.2 quark sector

As for the quark sector, numerical solutions were studied in detail in [14]. Therefore, only the results are listed. Yukawa couplings and their mixing matrices are

\[
\begin{align*}
\hat{Y}_d &= \text{diag}(\hat{m}_d, \ldots) = V_{dR}^\dagger I_{RL}^{(00)} U_{q3} V_{dL}, \\
\hat{Y}_u &= \text{diag}(\hat{m}_u, \ldots) = V_{uR}^\dagger W I_{RL}^{(00)} U_{q1} V_{uL},
\end{align*}
\]

where \( W = \text{diag}(\sqrt{2}, \sqrt{2}, 2) \) which is originated form the normalization factor of \( \mathbf{1S} \).

\[
I_{RL}^{(00)} = \text{diag}[b_1^0, b_2^0, b_3^0] \left(b_i^0 = \frac{\pi R M_i^2}{\sinh \pi R M_i^2}\right), \tag{A.21}
\]

\[
U_{q4} = R_u \begin{bmatrix}
a_{q1} & 0 & 0 \\
0 & a_{q2} & 0 \\
0 & 0 & a_{q3}
\end{bmatrix},
\quad U_{q3} = R_d \begin{bmatrix}
\sqrt{1 - a_{q1}^2} & 0 & 0 \\
0 & \sqrt{1 - a_{q2}^2} & 0 \\
0 & 0 & \sqrt{1 - a_{q3}^2}
\end{bmatrix}, \tag{A.22}
\]

where \( R_u \) and \( R_d \) are arbitrary \( 3 \times 3 \) rotation matrices parametrized as

\[
R_u = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta'_{q2} & \sin \theta'_{q2} \\
0 & -\sin \theta'_{q2} & \cos \theta'_{q2}
\end{bmatrix}
\begin{bmatrix}
\cos \theta'_{q3} & 0 & \sin \theta'_{q3} \\
0 & 1 & 0 \\
-\sin \theta'_{q3} & 0 & \cos \theta'_{q3}
\end{bmatrix}
\begin{bmatrix}
\cos \theta'_{q1} & -\sin \theta'_{q1} & 0 \\
\sin \theta'_{q1} & \cos \theta'_{q1} & 0 \\
0 & 0 & 1
\end{bmatrix}, \tag{A.23a}
\]

\[
R_d = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \theta_{q2} & \sin \theta_{q2} \\
0 & -\sin \theta_{q2} & \cos \theta_{q2}
\end{bmatrix}
\begin{bmatrix}
\cos \theta_{q3} & 0 & \sin \theta_{q3} \\
0 & 1 & 0 \\
-\sin \theta_{q3} & 0 & \cos \theta_{q3}
\end{bmatrix}
\begin{bmatrix}
\cos \theta_{q1} & -\sin \theta_{q1} & 0 \\
\sin \theta_{q1} & \cos \theta_{q1} & 0 \\
0 & 0 & 1
\end{bmatrix}. \tag{A.23b}
\]

The two set of numerical solutions were found in [14]. One is a set of solutions with the case \( R_u = \mathbf{1}_{3 \times 3} \) where the up-type quark mixings vanish.

\[
\begin{align*}
a_{q1}^2 &\approx 0.1023, & (b_1^0)^2 &\approx 4.355 \times 10^{-9}, & \sin \theta_{q1} &\approx -2.587 \times 10^{-2} \\
a_{q2}^2 &\approx 0.9887, & (b_2^0)^2 &\approx 1.302 \times 10^{-4}, & \sin \theta_{q2} &\approx 2.224 \times 10^{-2} \\
a_{q3}^2 &\approx 0.9966
\end{align*}
\tag{A.24}
\]

Another is a set of solutions the case \( R_d = \mathbf{1}_{3 \times 3} \) where the down-type quark mixings vanish.

\[
\begin{align*}
a_{q1}^2 &\approx 0.0650, & (b_1^0)^2 &\approx 3.973 \times 10^{-9}, & \sin \theta'_{q1} &\approx 0.6704 \\
a_{q2}^2 &\approx 0.9931, & (b_2^0)^2 &\approx 2.235 \times 10^{-4}, & \sin \theta'_{q2} &\approx -3.936 \times 10^{-2} \\
a_{q3}^2 &\approx 0.9966
\end{align*}
\tag{A.25}
\]

References

[1] G. Aad et al. [ATLAS Collaboration], Phys. Lett. B 716 (2012) 1.
[2] S. Chatrchyan et al. [CMS Collaboration], Phys. Lett. B 716 (2012) 30.

[3] N. S. Manton, Nucl. Phys. B 158, 141 (1979); D. B. Fairlie, Phys. Lett. B 82, 97 (1979); D. B. Fairlie, J. Phys. G 5, L55 (1979); Y. Hosotani, Phys. Lett. B 126, 309 (1983); Y. Hosotani, Phys. Lett. B 129, 193 (1983); Y. Hosotani, Annals Phys. 190, 233 (1989).

[4] H. Hatanaka, T. Inami and C. S. Lim, Mod. Phys. Lett. A 13, 2601 (1998).

[5] I. Antoniadis, K. Benakli and M. Quiros, New J. Phys. 3, 20 (2001); G. von Gersdorff, N. Irges and M. Quiros, Nucl. Phys. B 635, 127 (2002); R. Contino, Y. Nomura and A. Pomarol, Nucl. Phys. B 671, 148 (2003); C. S. Lim, N. Maru and K. Hasegawa, J. Phys. Soc. Jap. 77, 074101 (2008); C. S. Lim, N. Maru and T. Miura, arXiv:1402.6761 [hep-ph].

[6] N. Maru and T. Yamashita, Nucl. Phys. B 754, 127 (2006); Y. Hosotani, N. Maru, K. Takenaga and T. Yamashita, Prog. Theor. Phys. 118, 1053 (2007).

[7] N. Maru and N. Okada, Phys. Rev. D 77, 055010 (2008); Phys. Rev. D 87, 095019 (2013); arXiv:1310.3348 [hep-ph]; N. Maru, Mod. Phys. Lett. A 23, 2737 (2008).

[8] Y. Adachi, C. S. Lim and N. Maru, Phys. Rev. D 76, 075009 (2007); Phys. Rev. D 79, 075018 (2009); Nucl. Phys. B 839, 52 (2010).

[9] Y. Adachi, C. S. Lim and N. Maru, Phys. Rev. D 80, 055025 (2009).

[10] C. S. Lim and N. Maru, arXiv:0904.0304 [hep-ph].

[11] G. Burdman and Y. Nomura, Nucl. Phys. B 656, 3 (2003).

[12] Y. Adachi, N. Kurahashi, C. S. Lim and N. Maru, JHEP 1011 (2010) 150.

[13] Y. Adachi, N. Kurahashi, C. S. Lim and N. Maru, JHEP 1201 (2012) 047.

[14] Y. Adachi, N. Kurahashi, N. Maru and K. Tanabe, Phys. Rev. D 85 (2012) 096001.

[15] C. S. Lim, N. Maru and K. Nishiwaki, Phys. Rev. D 81, 076006 (2010).

[16] Y. Adachi, N. Kurahashi, N. Maru and K. Tanabe, arXiv:1201.2290 [hep-ph].

[17] G. Cacciapaglia, C. Csaki and S. C. Park, JHEP 0603, 099 (2006).

[18] C. A. Scrucca, M. Serone and L. Silvestrini, Nucl. Phys. B 669, 128 (2003).

[19] U. Bellgardt et al. [SINDRUM Collaboration], Nucl. Phys. B 299, 1 (1988).

[20] G. Panico, M. Serone, and A. Wulzer, Nucl. Phys. B 739, 186 (2006).
[21] C. Csaki, Y. Grossman, P. Tanedo and Y. Tsai, Phys. Rev. D 83, 073002 (2011).

[22] W. -F. Chang and J. N. Ng, Phys. Rev. D 71, 053003 (2005).

[23] P. Wintz et al. [SINDRUM II Collaboration],

[24] J. Beringer et al. [Particle Data Group Collaboration], Phys. Rev. D 86, 010001 (2012).