Networks of cosmological histories, crossing of the phantom divide line and potentials with cusps

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We discuss the phenomenon of the smooth dynamical gravity induced crossing of the phantom divide line in a framework of simple cosmological models where it appears to occur rather naturally, provided the potential of the unique scalar field has some kind of cusp. The behavior of cosmological trajectories in the vicinity of the cusp is studied in some detail and a simple mechanical analogy is presented. The phenomenon of certain complementarity between the smoothness of the spacetime geometry and matter equations of motion is elucidated. We introduce a network of cosmological histories and qualitatively describe some of its properties.

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I. INTRODUCTION

The discovery of cosmic acceleration \(^1\) has stimulated a construction of a class of dark energy models \(^2\) describing this effect. This dark energy should possess a negative pressure such that the relation between pressure and energy density \(w\) is less than \(-1/3\). Some observations indicate that the present day value of the parameter \(w < -1\) provides the best fit. The corresponding dark energy has been named phantom dark energy \(^3\).

According to some authors, the analysis of observations, permits to specify the existence of the moment when the universe changes the value of the parameter \(w\) from that the region \(w > -1\) to \(w < -1\) \(^4\). This transition is called the crossing of the phantom divide line”.

It is easy to see, that the standard minimally coupled scalar field cannot give rise to the phantom dark energy, because in this model the absolute value of energy density is always greater than that of pressure, i.e. \(|w| < 1\). A possible way out of this situation is the consideration of the scalar field models with the negative kinetic term. Thus, the important problem arising in connection with the phantom energy is the crossing of the phantom divide line. The general belief is that while this crossing is not admissible in simple minimally coupled models its explanation requires more complicated models such as multifield ones or models with non-minimal coupling between scalar field and gravity (see e.g. \(^5\)).

In our preceding paper \(^1\) we have described the phenomenon of the change of sign of the kinetic term of the scalar field implied by Einstein equations. Now we try to answer the question how general is this phenomenon. In other words, is it some curiosity arising due to a very particular choice of the form of a potential of a scalar field and of initial conditions or it is rather a typical phenomenon. In doing this, we introduce a notion of the network of cosmological histories and qualitatively describe its properties. Admittedly, our research looks purely academic, but we stress that the persisting signals in favor of the dark energy equation of state parameter \(w < -1\) \(^6\) justifies the interest in this topic and, in particular, in our systematic approach. The structure of the paper is as follows: Sec. II is devoted to a brief recapitulation of the technique of reconstruction of scalar (phantom) potentials and the transformation of the unique scalar-phantom field (i.e. the crossing of the phantom divide line); in Sec. III we discuss the role of initial conditions for the phantom divide line crossing phenomenon; Sec. IV is devoted to the analysis of a simple mechanical system with the potential with a cusp, which is strictly motivated by the requirement of the possibility of crossing of the phantom divide line within a framework which allows a unique scalar field; in Sec. V we introduce the notion of network of cosmological histories; the section VI contains concluding remarks In the Appendix A we shall present simple solvable examples of such networks. In the Appendix B we shall describe some potentials as functions of time or scalar field, for which exact solution of the Einstein equations is possible.

II. RECONSTRUCTION OF SCALAR (PHANTOM) POTENTIALS AND DYNAMICAL CROSSING OF THE PHANTOM DIVIDE LINE

Let us remember the main points of our approach. We have considered the minimally coupled scalar field with an exponential potential

\[
V_0(\phi) \sim \exp\left(-\frac{3\sqrt{1+w\phi}}{2}\right).
\]  (1)

Such potentials are widely studied in cosmology \(^12\) and have a particular solution corresponding to a special choice of initial conditions, which implies the power-law behavior of the cosmological radius

\[
a(t) = a_0 t^{\frac{2}{3(1+w)}}\),
\]  (2)
or, in other terms,
\[ h(t) = \frac{2}{3(1 + w)t^3}. \] (3)

where \( h = \dot{a}/a \) is the Hubble variable. Changing the initial conditions (i.e. the value of \( \dot{\phi} \) corresponding to some value of \( \phi \)) one shall have other laws of cosmological evolution, which cannot be presented explicitly, but which have the same qualitative behavior as (2).

However, one can consider the potential (1) not as a function of the scalar field \( \phi \), but as a function of the cosmic time parameter \( t \) as was done in [11, 13, 14, 15]. In this case the Einstein (Friedmann) equation is equivalent to the linear second-order differential equation
\[ \ddot{\psi} = 9V(t)\psi, \] (4)

for the volume function
\[ \psi(t) \equiv a^3(t). \] (5)

The potential \( V(t) \) corresponding to the exponential potential (1) and to the choice of the initial conditions providing the evolution (2) is
\[ V_1(t) = \frac{2(1 - w)}{9(1 + w)^2t^2}. \] (6)

Equation (4) with the potential (5) has a general solution
\[ \psi(t) = \psi_1 t^{\alpha_1} + \psi_2 t^{\alpha_2}, \]
\[ \alpha_1 = \frac{2}{1 + w}, \]
\[ \alpha_2 = \frac{w - 1}{1 + w}, \] (7)

where \( \psi_1 \) and \( \psi_2 \) are nonnegative constants. If \( \psi_2 = 0 \), one reproduces the cosmological evolution (2), while if \( \psi_1 = 0 \) the cosmological evolution requires the presence of the phantom scalar field with the negative sign of the kinetic term. The most interesting situation reveals when both the coefficients \( \psi_1 \) and \( \psi_2 \) are nonzero. Simple calculation shows that in this case the Hubble variable has a maximum point where, the kinetic term is forced to change its sign.\(^1\)

Now, fixing the relation between these coefficients one can ask what is the corresponding form of the potential as a function of the scalar field \( \phi \). The explicit form of this function cannot be found, but we have studied [11] its asymptotic behavior at singularities and in the point \( t_0 \), where the kinetic term changes the sign and (de-)phantomization occurs, or, in other words, where the universe crosses the phantom divide line. In the neighborhood of the moment \( t_0 \), which depend on the relation between the coefficients \( \psi_1 \) and \( \psi_2 \) [11], the potential has the form
\[ V(\phi) = \frac{2(1 - w)}{9(1 + w)^2 \left( t_0 + \left( \frac{27}{8w} \phi \right)^{2/3} \right)^2}, \]
\[ H = \frac{\sqrt{8(1 - w)(3 - w)^2} \psi_1 \psi_2 t_0^{w - 1}}{(1 + w)^3 \left( \psi_1 t_0^{w - 1} + \psi_2 t_0^w \right)^2}. \] (8)

The scalar field behaves correspondingly as
\[ \phi \sim (t - t_0)^{3/2}. \] (9)

III. INITIAL CONDITIONS AND PHANTOM DIVIDE LINE CROSSING

In the previous section we have implicitly chosen such initial conditions so that simultaneously the scalar field and its time derivatives vanish at the point \( t_0 \), which in turn implies the crossing of the phantom divide line. However, let us fix the form of the potential as a function of the scalar field \( \phi \) and see how the behavior of the universe will change depending on the initial conditions. Suppose that when the cosmic parameter is close to \( t_0 \) the potential behaves as
\[ V(\phi) = \frac{1}{(A + B\phi^{2/3})^2}. \] (10)

The scalar field can have the following form:
\[ \phi(t) = \phi_0 + \phi_1 (t - t_0)^\alpha, \] (11)

where \( \alpha > 1 \) to provide the vanishing of the time derivative of the scalar field at the point \( t_0 \). The situation described above and corresponding to the (de) - phantomization of the scalar field [11] realizes when \( \phi_0 = 0 \) and \( \alpha = 3/2 \). Let us consider the cases when \( \phi_0 \neq 0 \), analyzing the Klein-Gordon equation
\[ \ddot{\phi} + 3h \dot{\phi} + \frac{dV}{d\phi} = 0. \] (12)

The terms here behave as
\[ \ddot{\phi} = \alpha(\alpha - 1)\phi_1 (t - t_0)^{\alpha - 2}, \] (13)
\[ \dot{\phi} = \alpha \phi_1 (t - t_0)^{\alpha - 1}, \] (14)
\[ \frac{dV}{d\phi} = -\frac{4B}{3\phi^{1/3}(A + B\phi^{2/3})^3}. \] (15)

\(^1\) Notice that in papers [16] the technique of the reconstruction of the potentials as functions of the redshift parameter was developed. The reconstruction approach is (to a large extent) model independent. So, it may show that the phantom boundary is crossed in nature, but do not specify a class of potentials as functions of \( \phi \) in which this can be described in a consistent way
Let us consider the case $\alpha > 2$. In this case the first and the second terms of Eq. (12) vanish at $t \to 0$ while the term (13) is a constant. Thus, we have come to a contradiction and this case should be disregarded.

Now, considering the case $\alpha = 2$, we see that while the term containing the first derivative of the scalar field vanishes the term with the second derivative becomes a constant ($2\phi_1$) to be equated to the term (15) up to the sign. Therefore, we obtain the relation

$$\phi_1 = \frac{2B}{3\phi^{1/3}(A + B\phi^{2/3})^3}. \quad (17)$$

It is easy to see that in this case the time derivative of the Hubble parameter $\dot{h}$ behaves as

$$\dot{h} \sim (t-t^0)^2 \quad (18)$$

in contrast to the behavior $\dot{h} \sim t-t^0$, corresponding to the case $\phi_0 = 0$, considered in [11]. Apparently, the behavior (18) means that the Hubble variable has at the moment $t_0$ an inflection point instead of the extremum point and hence, the kinetic term for the scalar field conserves its sign.

In the case when $1 < \alpha < 2$ the second derivative term diverges while other terms are finite, so this behavior cannot be realized.

Until now we have considered the cosmological trajectories having at some moment the vanishing time derivative $\dot{\phi}$ of the scalar field, while $\phi$ could have zero or nonzero values. However, there is also another class of trajectories characterized by the fact that at some moment $t_j$ the scalar field vanishes, while its first time derivative is different from zero. Such trajectories are characterized by the equation:

$$\phi(t) = v_1(t-t_1) + v_{5/3}(t-t_1)^{5/3}, \quad (19)$$

where the constants $v_1$ and $v_{5/3}$ satisfy the condition

$$v_{5/3} = - \frac{6B}{5A^{4/3}v_1^{1/3}}, \quad (20)$$

which can be easily obtained from Eqs. (12) and (15). Calculating the second time derivative of the Hubble variable, one can see that it behaves as $h(t) \sim (t-t_1)^{-1/3}$ that means that higher curvature invariants including the curvature derivatives diverge and one encounters the so called soft singularity. The problem of removal of such singularities is of interest by itself, but will not be considered here.

Thus, let us summarize the results of the above consideration. We have found three types of the trajectories. For the first family of the trajectories the time derivative of the scalar field vanishes at the moment $t_0$. This family is a one-parameter one and is parameterized by the value of the scalar field $\phi_0$, which should be different from zero at the moment $t_0$. All the nonzero values of $\phi_0$ require $\alpha = 2$, the value of $\phi_1$ is fixed by relation (17), the Hubble variable has an inflection point and the kinetic term conserves its sign.

The second family contains the only trajectory with $\phi_0 = 0$. Now the term (15) becomes singular and behaves as $(t-t_0)^{-\alpha/3}$, the term with the first derivative vanishes, while the second derivative term behaves like $(t-t_0)^{\alpha-2}$. Consistency requires $\alpha = 3/2$ and the value of $\phi_1$ is uniquely fixed. The Hubble variable has at the moment $t_0$ an extremum and crossing of the phantom divide line occurs. The third family is characterized by Eqs. (19), (20) and suffers from a soft cosmological singularity.

It seems that in some sense one can consider the possibility of the crossing of the phantom divide line as was described in [11] as exceptional and the corresponding initial conditions as having measure zero. Nevertheless, we would like to show that considering broader set of potentials and initial conditions, one can come to the conclusion, that conditions providing the crossing are, roughly speaking, commensurable to those excluding it. We postpone the corresponding consideration until the fifth section, while the next section will be devoted to an analysis of the simple mechanical model of the particle moving in the potential with a cusp.

IV. MECHANICAL ANALOG: A PARTICLE MOVING IN A POTENTIAL WITH A CUSP

Let us consider a one-dimensional problem of a classical point particle moving in the potential

$$V(x) = \frac{V_0}{(1+x^{2/3})^2}, \quad (21)$$

where $V_0 > 0$. The equation of motion is

$$\ddot{x} - \frac{4V_0}{3(1+x^{2/3})^3x^{1/3}} = 0. \quad (22)$$

We consider three classes of possible motions characterized by the value of the energy $E$. The first class consists of the motions when $E < V_0$. Apparently, the particle with $x < 0, \dot{x} > 0$ or with $x > 0, \dot{x} < 0$ cannot reach the point $x = 0$ and stops at the points $\mp \left(\sqrt{\frac{V_0}{E}} - 1\right)^{3/2}$ respectively. This class of motions corresponds to the class of cosmological evolutions described in the preceding chapter when the Hubble variable has an inflection point and the universe does not cross the phantom divide line.

The second class includes the trajectories when $E > V_0$. In this case the particle crosses the point $x = 0$ with nonvanishing velocity and this case correspond to
the cosmological evolution given by Eq. (19), but naturally there is nothing singular in it, because in our simple mechanical problem, we do not have nothing similar to the Hubble variable and the spacetime geometry defined by its behavior.

If we have a fine tuning such that \( E = V_0 \), we encounter an exceptional case. Now the trajectory satisfying Eq. (22) in the vicinity of the point \( x = 0 \) can behave as

\[
x = C(t_0 - t)^{3/2}, \quad (23)
\]

where

\[
C = \pm \left( \frac{16V_0}{9} \right)^{3/4} \quad (24)
\]

and \( t \leq t_0 \). It is easy to see that independently of the sign of \( C \) in Eq. (24) the signs of the particle coordinate \( x \) and of its velocity \( \dot{x} \) are opposite and hence, the particle can arrive in finite time to the point of the cusp of the potential \( x = 0 \).

Another solution reads as

\[
x = C(t - t_0)^{3/2}, \quad (25)
\]

where \( t \geq t_0 \). This solution describes the particle going away from the point \( x = 0 \). Thus, we can combine the branches of the solutions (23) and (25) in four different manners and there is no way to choose if the particle arriving to the point \( x = 0 \) should go back or should pass the cusp of the potential (21). It can stop at the top as well. Such a “degenerate” behavior of the particle in this third case is connected with the fact that this trajectory is the separatrix between two one-parameter families described above. At the moment there is not yet any strict analogy between this separatrix and the cosmological evolution describing the phantom divide line. In order to establish a closer analogy and to understand what is the crucial difference between mechanical consideration and general relativistic one, we can try to introduce a friction term into the Newton equation (22)

\[
\ddot{x} + \gamma \dot{x} - \frac{4V_0}{3(1 + x^{2/3})^3x^{1/3}} = 0. \quad (26)
\]

It is easy to check that if the friction coefficient \( \gamma \) is a constant one does not have a qualitative change in respect to the discussion above. Let us assume for \( \gamma \) the dependence

\[
\gamma = 3\sqrt{\frac{\dot{x}^2}{2} + V(x)}. \quad (27)
\]

then

\[
\dot{\gamma} = -\frac{3}{2} \dot{x}^2 \quad (28)
\]

and

\[
\ddot{\gamma} = -3\ddot{x}. \quad (29)
\]

The trajectory arriving to the cusp with vanishing velocity is still described by the solution (23). Consider the particle coming to the cusp from the left \( (C < 0) \). It is easy to see that the value of \( \text{variama} \) at the moment \( t_0 \) tends to zero, while its second derivative \( \ddot{\gamma} \) given by Eq. (29) is

\[
\ddot{\gamma}(t_0) = \frac{9}{8} C^2 > 0. \quad (30)
\]

Thus, it looks like the friction coefficient \( \gamma \) reaches its minimum value at \( t = t_0 \). Let us suppose now that the particle is coming back to the left from the cusp and its motion is described by Eq. (25) with negative \( C \). A simple check shows that in this case

\[
\ddot{\gamma}(t_0) = -\frac{9}{8} C^2 < 0. \quad (31)
\]

Thus, from the point of view of the subsequent evolution this point looks as a maximum for the function \( \gamma(t) \). In fact, it means simply that the second derivative of the friction coefficient has a jump at the point \( t = t_0 \). It is easy to check that if instead of choosing the motion to the left, we shall move forward our particle to the right from the cusp \( (C > 0) \), the sign of \( \text{variama}(t_0) \) remains negative as in Eq. (31) and hence we have the jump of this second derivative again. If one would like to avoid this jump, one should try to change the sign in Eq. (25). To implement it in a self-consistent way one can substitute Eq. (27) by

\[
\gamma = 3\sqrt{-\frac{\dot{x}^2}{2} + V(x)} \quad (32)
\]

and Eq. (26) by

\[
\ddot{x} + \gamma \dot{x} + \frac{4V_0}{3(1 + x^{2/3})^3x^{1/3}} = 0. \quad (33)
\]

In fact, it is exactly that what happens automatically in cosmology, when we change the sign of the kinetic energy term for the scalar field, crossing the phantom divide line. Naturally, in cosmology the role of \( \gamma \) is played by the Hubble variable \( h \). The jump of the second derivative of the friction coefficient \( \gamma \) corresponds to the divergence of the third time derivative of the Hubble variable, which represents some kind of soft cosmological singularity (this singularity is even softer than that considered in the preceding section for the family of cosmologies characterized by Eq. (19) where already the second time derivative of the Hubble variable was divergent).

Thus, one seems to confront the problem of choosing between two alternatives: 1) to encounter a weak singularity in the spacetime geometry; 2) to change the sign of the kinetic term for matter field. In this paper we pursue the second alternative insofar as we privilege the smoothness of spacetime geometry and consider equations of motion for matter as less fundamental than the Einstein equations (see also more detailed discussion in [11]).
V. POTENTIAL AS A COSMIC TIME FUNCTION AND THE NETWORK OF COSMOLOGICAL HISTORIES

Indeed, all the previous considerations were based on the treatment of the potential as a fixed function of the scalar field and changing initial conditions for this scalar field. As was already emphasized, one can use an alternative treatment of the potential \( V \) as a function of time. In the example considered in Figure 1, it was shown that for \( V(t) \) instead of the initial conditions which exclude the crossing of the phantom divide line are exceptional.

Let us try to imagine a set of possible cosmologies as a two-dimensional surface (see the picture below) where the vertical lines represent the potentials as functions of \( t \) while the horizontal lines represent the potentials as functions of \( \phi \).

![Network of cosmological histories](image)

Network of cosmological histories

The crossing of the phantom divide line.

Now, we undertake a similar analysis showing that for a horizontal line (potential as a function of \( t \)) almost all the initial conditions imply the crossing effect. Let us consider a potential \( V_2(t) \) corresponding to the horizontal line intersecting the vertical line \( V_2(\phi) \) at the point \( C \). We shall look for a cosmological evolution where the Hubble variable has an extremum at the moment \( t_0 \). A regular potential can be represented around this moment as

\[
V(t) = \frac{1}{9}(U_0 + U_1(t - t_0)).
\]

Then Eq. (4) acquires the form

\[
\frac{d^2\psi}{dt^2} = (U_0 + U_1 \tau)\psi,
\]

where \( \tau \equiv t - t_0 \). We shall look for two independent solutions in the vicinity of \( \tau = 0 \); in the form

\[
\psi(\tau) = \sum_{n=0}^{\infty} a_n \tau^n.
\]

The recurrent relation between the coefficients \( a_n \) is

\[
(n + 2)(n + 1)a_{n+2} = U_0 a_n + U_1 a_{n-1}.
\]

We define the first independent solution by fixing of the first two coefficients as \( a_0 = 1, a_1 = 1 \) while for the second solution we choose \( a_0 = 1, a_1 = 0 \) (one can easily check that these solutions are independent, calculating the first terms of their Wronskian). The leading terms of the general solution of Eq. (35) has the following form:

\[
\psi(\tau) = \psi_1 \left(1 + \tau + \frac{U_0 \tau^2}{2} + \frac{(U_0 + U_1) \tau^3}{6}\right)
+ \psi_2 \left(1 + \frac{U_0 \tau^2}{2} + \frac{U_1 \tau^3}{6}\right),
\]

where \( \psi_1 \) and \( \psi_2 \) are nonnegative. We shall need also the expressions for the first and second time derivatives of the solution (35):

\[
\dot{\psi} = \psi_1 \left(1 + V_0 \tau + \frac{(V_0 + V_1) \tau^2}{2}\right) + \psi_2 \left(V_0 \tau + \frac{V_1 \tau^2}{2}\right),
\]

\[
\ddot{\psi} = \psi_1 (U_0 + (U_0 + U_1) \tau) + \psi_2 (U_0 + U_1 \tau).
\]

The time derivative of the Hubble variable has the form

\[
\dot{h} = \frac{3}{\psi^2} \dot{\psi} \ddot{\psi} - \dot{\psi}^2.
\]

It is enough to study the numerator of expression (41):

\[
\dot{\psi} \ddot{\psi} - \dot{\psi}^2 = (\psi_1 + \psi_2)^2 U_0 - \psi_1^2
+ (\psi_1 + \psi_2) U_1 \tau.
\]
To have at the moment $\tau = 0$ the extremum of the function $h(\tau)$ its time derivative should behave as $\dot{h} \sim \tau$, that means that Eq. (42) implies that

$$ (\psi_1 + \psi_2)U_0 = \psi_1^2, \quad (43) $$

and

$$ U_1 \neq 0. \quad (44) $$

The condition (43) determines the relation between the coefficients $\psi_1$ and $\psi_2$ in terms of the coefficient $U_0$. Thus, for a potential $V(t)$ fixing the moment of time $t_0$ we fix the constants $U_0$ and $U_1$. If the condition (44) is satisfied, choosing the constants $\psi_1$ and $\psi_2$ satisfying condition (43), one obtains a cosmological evolution undergoing the crossing of the phantom divide line.

VI. CONCLUDING REMARKS

Thus, we have seen that considering the potential as a function of the scalar field having a cusp we should choose exceptional initial conditions to describe a smooth cosmological evolution undergoing the phantom divide line crossing. We have seen also that there is some kind of complementarity between the smoothness of the space-time geometry and that of the structure of matter action.

On the hand, considering the potential as a function of time we encounter just the following situation: initial conditions implying the crossing of the phantom divide line are typical. That means that roughly speaking in the set of all possible cosmological trajectories those crossing and non-crossing the phantom divide line have measures of the same order. We are not able to construct a rigorous description of this set of cosmological histories, but the treatment undertaken above permits to make some qualitative remarks. First of all let us notice that a network of vertical lines representing the potential as a function of the scalar field and horizontal lines, representing potentials as functions of the cosmic time, is not closed. Namely, let us leave a vertical line at the point $A$ and make some walk along the horizontal line to the point $B$. At the point $B$ we shall have a cosmology with the crossing. Then let us make some walk along the vertical line from the point $B$ to some point $C$. The cosmology corresponding to the point $C$ is free from crossing. Now, traveling along the horizontal line from point $C$ we shall always have trajectories with crossing and, hence, we shall not have an opportunity to return to the initial vertical line, which should have cosmologies without crossing. (The empty circle $D$ in our picture signifies the absence of intersection between the respective vertical and horizontal lines).

There is another interesting question. It is well known that for the exponential potential (11) all the initial conditions imply the cosmologies free from the crossing phenomenon. The explanation of this fact is a very simple one: the effect of smooth crossing (11) can have place only for potentials which at some point have a divergent derivative with respect to the scalar field, while the derivative of the exponential potential is always regular.

In conclusion we would like to say that the above investigation shows that the smooth crossing of the phantom divide line in simple cosmological models could be considered as a rather natural phenomenon more than an exotic anomaly. The network of cosmological histories constructed for the illustration of this fact seems to be interesting object on its own. The points of this network represent functions $h(t)$ on which two transformations generated by means of scalar field potentials, treated as functions of $\phi$ or of $t$. Graphically they were represented as displacements along vertical and horizontal lines. The network discussed in this paper was constructed beginning from the simple cosmological evolution (3). Notice that our initial point of network (4) is characterized by some value of the equation of state parameter $w$ and is nothing but a power-law evolution (2). The points lying on a horizontal line intersecting this initial power-law point (the point $A$ on our picture), are described explicitly (11) and have more complicated structure. As far as other points of the network are concerned we have studied above some of their local properties, but we are not able to describe their global topology. It seems unlikely that two different power-law points belong to the same network. If this statement is confirmed one has got a family of networks characterized by different values of the initial constant equation of state parameter $w$. In the Appendix A we give an explicit illustration of the simplest network containing an exponential expansion, while in the Appendix B we give some other examples of potentials for which it is possible to find exact solutions.

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Appendix A. Examples of simple networks of cosmological histories

We present here a particularly simple and in some way degenerate example of the network of cosmic histories which is discrete and contains only three points. Let us consider de Sitter universe with

$$ h(t) = H_0. \quad (45) $$

The corresponding potential as is well-known and as can be easily obtained from the equation (see e.g. (11) and references therein)

$$ V = \frac{\dot{h}}{3} + h^2, \quad (46) $$

References
has the form
\[ V(\phi) = H_0^2, \]
while the initial condition is
\[ \dot{\phi} = 0. \]

This initial condition can be defined at any moment of time and the evolution shall conserve it, because the Klein-Gordon equation for the constant potential has the form
\[ \ddot{\phi} + 3h \dot{\phi} = 0. \]

Notice that in this case the Klein-Gordon equation has the same form for usual scalar and phantom. Now, treating the potential as a constant function of time we can easily resolve for this case Eq. (48):
\[ \psi = \psi_1 \exp(3H_0t) + \psi_2 \exp(-3H_0t). \]

The Hubble function correspondingly is
\[ h(t) = H_0 \frac{\psi_1 \exp(3H_0t) - \psi_2 \exp(-3H_0t)}{\psi_1 \exp(3H_0t) + \psi_2 \exp(-3H_0t)}. \]

while its time derivative is
\[ \dot{h}(t) = \frac{12H_0^2\psi_1\psi_2}{(\psi_1 \exp(3H_0t) + \psi_2 \exp(-3H_0t))^2}. \]

As usual the constants \( \psi_1 \) and \( \psi_2 \) are nonnegative. Correspondingly we have three opportunities. If \( \psi_2 = 0 \) this is our initial point describing infinitely expanding flat de Sitter universe. If \( \psi_1 = 0 \), one has an infinitely contracting flat de Sitter universe. If both the coefficients are positive, the Hubble variable is growing and this situation could be realized only in the universe where the scalar field has a negative kinetic term (phantom). It is easy to understand that a change of the relation \( \psi_2/\psi_1 \) is simply equivalent to the shift of the cosmic time variable \( t \). Thus, we have only one cosmic history described by phantom field with the constant potential. In this history the universe begins its evolution at \( t = -\infty \) with a quasi-de Sitter contraction. It arrives to a minimum possible value of the cosmological radius at some moment \( t_{\text{min}} \). This moment can be easily found from Eq. (51) however, its value does not have a particular meaning just like the relation \( \psi_2/\psi_1 \). After the moment \( t_{\text{min}} \) begins an expansion which at \( t \to \infty \) becomes quasi-de Sitter. Making a convenient shift of the time variable we can write down Eq. (51) in a specially simple form:
\[ h(t) = H_0 \tanh 3H_0t. \]

Thus, instead of a horizontal line we obtain two discrete points.

Now, let us treat the constant potential as a function of \( \phi \) changing the initial condition for the time derivative of this field. It is easy to show that for \( \dot{\phi} \neq 0 \) there are two cosmollogical histories given by the following expression for the Hubble variable:
\[ h(t) = \pm H_0 \coth 3H_0t. \]

These solutions describe a universe which begins its evolution from cosmological singularity at \( t = 0 \) and then is infinitely expanding arriving to quasi-de Sitter phase at \( t \to \infty \) and a universe which begins its evolution at \( t = -\infty \) and then contracts arriving to the Big Crunch type cosmological singularity at \( t = 0 \). It is important to notice that the choice of the initial value of the time derivative of the scalar field is not important, because in the process of evolution this value runs between 0 and \( \pm \infty \). The sign of this time derivative is not important either because the potential of this field is constant.

Thus, we have presented a schematic example of an exactly solvable cosmological network, which contains only five discrete points. Two of these points represent an infinitely expanding and infinitely contracting de Sitter universes, two of them describe universe evolving from the cosmological singularity to an infinite expansion or vice versa, driven by a scalar field with constant potential. The last point represents a universe which passes from contraction to expansion and is driven by a phantom field. All the network is characterized by an absolute value of the constant \( H_0 \). Indeed, beginning from \(-H_0 \) we arrive to the same network, which was constructed from \( H_0 \).

As a last remark we can give two examples of even more trivial networks of cosmological histories. First of them contains only one point. If one has a negative constant potential \( V = -H_0^2 \), than the only cosmological history present in the network is that given by \( h(t) = -H_0 \tan 3H_0t \) describing the evolution from the Big Bang at \( t = -\pi/6H_0 \) to the Big Crunch at \( t = \pi/6H_0 \) passing through the point of maximal expansion at \( t = 0 \). The second network consists of three points. Considering massless scalar field with vanishing potential, one can see that there are three opportunities: if the time derivative of the scalar field is nonzero, then one has either a universe which begins its evolution from the Big Bang singularity and expands infinitely or a universe which contracts finishing in the Big Crunch singularity. If the time derivative of the scalar field is zero, one has a static Minkowski universe.

**Appendix B. Some solvable potentials**

The concept of the network of cosmological histories introduced in this paper is based on the simultaneous use of the notion of potential as function of time and as function of the scalar field. Unfortunately, it is much more difficult to find exactly solvable examples for potentials as functions of scalar fields than for those, treated as functions of time. Thus, we, first give another example of the potential as function of \( \phi \) (see, for additional details [17]).
Let us consider the cosmological evolution given by the Hubble function
\[ h(t) = H_0 \coth \frac{3H_0(1+w)t}{2}, \]  
(55)
while the cosmological radius behaves as
\[ a(t) = a_0 \left( \sinh \frac{3H_0(1+w)t}{2} \right)^{2/3(1+w)}. \]  
(56)
This evolution occurs in the universe filled with two perfect fluids: one of them is a cosmological constant and the other is a barotropic fluid with the equation of state parameter \( w \). The scalar field potential realizing this evolution has the form \( \psi \):
\[ V(\phi) = H_0^2 \left( 1 + \frac{1 - w}{2} \sinh^2 \frac{3\sqrt{1+w}\phi}{2} \right). \]  
(57)
The potential as a function of time is
\[ V(t) = H_0^2 + \frac{(1-w)H_0^2}{2 \sinh^2 \frac{3H_0(1+w)t}{2}}. \]  
(58)
The general solution of Eq. (11) with potential \( V(t) \) can be written as usual as
\[ \psi(t) = \psi_1 \left( \sinh \frac{3H_0(1+w)t}{2} \right)^{2/(1+w)} + \psi_2 \left( \sinh \frac{3H_0(1+w)t}{2} \right)^{2/(1+w)} \times \int dt' \left( \sinh \frac{3H_0(1+w)t'}{2} \right)^{-4/(1+w)}. \]  
(59)
The explicit form of the solution \( \psi(t) \) is very simple for some special values of the parameter \( w \). We shall consider the case \( w = 0 \). Now, the solution is
\[ \psi(t) = \psi_1 \sinh^2 \frac{3H_0t}{2} + \psi_2 \left( \coth \frac{3H_0t}{2} - \sinh 3H_0t \right). \]  
(60)
If \( \psi_2 = 0 \), we come back to our starting point: a cosmological evolution \( \psi_1 \) for \( w = 0 \).

The case when both the coefficients \( \psi_1 \) and \( \psi_2 \) are different from zero is rather cumbersome, while its qualitative behavior can be understood studying the case \( \psi_1 = 0, \psi_2 \neq 0 \).

Let us rewrite the solution \( \psi(t) \) for this case in the form
\[ \psi(t) = \psi_2 \frac{\cosh \frac{3H_0t}{2}}{\sinh \frac{3H_0t}{2}} (2 - \cosh 3H_0t). \]  
(61)
Let us consider first the case \( \psi_2 > 0 \). Then, the volume function \( \psi(t) \) will be nonnegative in the intervals:
\[ 0 \leq t \leq \frac{1}{3H_0} arccosh 2, \]  
(62)
and
\[ -\infty \leq t \leq -\frac{1}{3H_0} arccosh 2. \]  
(63)
The Hubble variable can be expressed as
\[ h(t) = -\frac{3H_0(1 - y + y^2) \text{sign}(t)}{\sqrt{y^2 - 1(2 - y)}}, \]  
(64)
and its time derivative reads
\[ \dot{h}(t) = -\frac{9H_0^2(y^3 + 3y^2 - 6y + 1)}{(y^2 - 1)(2 - y)^2}, \]  
(65)
where
\[ y = \cosh 3H_0t. \]  
(66)
One can see by simple algebraic treatment that the function \( h(t) \) can have the only zero at some moment \( t_0 \) in the interval \( [0,2] \), i.e., when \( 0 < y < 2 \). Such a point describes a transition from a phantom-type contraction to a non-phantom contraction ending in a Big Crunch. The evolution in the interval \( [0,2] \) describes instead a non-phantom contraction beginning at \( t = -\infty \) in a de Sitter stage and ending at \( t = -\frac{1}{3H_0} arccosh 2 \) in the Big Crunch. Analogously, one can show that in the case \( \psi_2 < 0 \), one can have either non-phantom expansion beginning from the Big Bang and ending in an infinite de Sitter universe or the transition from the non-phantom expansion to phantom expansion culminating in a Big Rip singularity at \( t = \frac{1}{3H_0} arccosh 2 \). The inclusion of the term with \( \psi_1 \neq 0 \) can change some features of the cosmological evolution. However, the qualitative and numerical analysis of the corresponding differential equations shows that the number of the moments of time when the universe crosses the phantom divide line does not change.

For example, for the case when both the coefficients \( \psi_1 \) and \( \psi_2 \) are positive and \( \psi_1 = 3\psi_2 \) the evolution is presented in Fig. 1. Here, the universe begins its evolu-

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where
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(66)

FIG. 1: \( h(t) \) dependence at \( \psi_1/\psi_2 = 3 \)

\[ -\infty \leq t \leq -\frac{1}{3H_0} arccosh 2. \]  
(63)

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As we have already mentioned above, the number of the scalar field potentials treated as functions of the scalar field, for which it is possible to find an exact solution is very limited. Instead, one can find some potentials treated as functions of the cosmic time parameter, for which one can find complete exact solutions. It is connected with the linearity of the corresponding equations, which is nothing but a Schrödinger equation for a system with zero energy. Thus, we can borrow some solvable examples of such potentials from non-relativistic quantum mechanics (see e.g. [18, 19, 20]). We shall give here a couple of examples of such potentials. For the potential

\[ V(t) = \frac{A^2}{t^4} \]  

the solution of Eq. (1) looks like

\[ \psi = \psi_1 \exp \left( \frac{A}{t} \right) + \psi_2 t \exp \left( -\frac{A}{t} \right). \]  

For the potential

\[ V(t) = \frac{B^2 + 1}{(1 + t^2)^2} \]  

the solution reads

\[ \psi = \psi_1 \sqrt{1 + t^2} \exp(\text{Barcertant}) + \psi_2 \sqrt{1 + t^2} \exp(-\text{Barcertant}). \]  

The investigation of cosmologies based on the solutions (15) and (16) might represent some interest but it is outside the scope of the present paper.

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