New Extension of Unified Family of Apostol-Type of Polynomials and Numbers

B. S. El-Desouky, R. S. Gomaa

Department of Mathematics, Faculty of Science, Mansoura University, 35516 Mansoura, Egypt

Abstract

The purpose of this paper is to introduce and investigate a new unification of unified family of Apostol-type polynomials and numbers based on results given in [24] and [25]. Also, we derive some properties for these polynomials and obtain some relationships between the Jacobi polynomials, Laguerre polynomials, Hermite polynomials, Stirling numbers and some other types of generalized polynomials.

Key words: Generalized Euler, Bernoulli and Genocchi polynomials; Stirling numbers; generalized Stirling numbers; Laguerre polynomials; Hermite polynomials; Jacobi polynomials.

AMS Subject Classification: 05A10, 11B68, 11B73, 11B83, 11M06, 11M35, 33E20

1. Introduction

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Euler polynomials are defined by (see [23]):

$$\left( \frac{t}{e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < 2\pi; 1^\alpha := 1) \quad (1.1)$$

and

$$\left( \frac{t}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!}, \quad (|t| < \pi; 1^\alpha := 1), \quad (1.2)$$

where $\mathbb{C}$ denote set of complex numbers.

Recently, Luo and Srivastava [14] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ and the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ as follows:

*Corresponding author
Email address: b_desouky@yahoo.com (B. S. Desouky) (R. S. Gomaa)
Definition 1.1. (Luo and Srivastava [14]) The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by the generating function

$$
\left( \frac{t}{\lambda e^t - 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}
$$

($|t| < 2\pi$ when $\lambda = 1$; $|t| < |\log \lambda|$, when $\lambda \neq 1$; $1^\alpha := 1$). \hspace{1cm} (1.3)

Definition 1.2. (Luo [15]) The generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x; \lambda)$ of order $\alpha \in \mathbb{C}$ are defined by the generating function

$$
\left( \frac{t}{\lambda e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda) \frac{t^n}{n!}
$$

($|t| < \pi$ when $\lambda = 1$; $|t| < |\log(-\lambda)|$, when $\lambda \neq 1$; $1^\alpha := 1$). \hspace{1cm} (1.4)

Natalini and Bernardini [17] defined the new generalization of Bernoulli polynomials in the following form.

Definition 1.3. The generalized Bernoulli polynomials $B_n^{[m-1]}(x)$, $m \in \mathbb{N}$, are defined, in a suitable neighbourhood of $t = 0$ by means of generating function

$$
\frac{t^m e^{xt}}{e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x) \frac{t^n}{n!}.
$$

(1.5)

Recently, Tremblay et al. [26] investigated a new class of generalized Apostol-Bernoulli polynomials. These are defined as follows.

Definition 1.4. The generalized Apostol-Bernoulli polynomials $B_n^{[m-1,\alpha]}(x; \lambda)$ of order $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, are defined, in a suitable neighbourhood of $t = 0$ by means of generating function

$$
\left( \frac{t^m}{\lambda e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x; \lambda) \frac{t^n}{n!}.
$$

(1.6)

Also, Srivastava et al. [24] introduced a new interesting class of Apostol-Bernoulli polynomials that are closely related to the new class that we present in this paper. They investigated the following form.

Definition 1.5. Let $a, b, c \in \mathbb{R}^+(a \neq b)$ and $n \in \mathbb{N}_0$. Then the generalized Bernoulli polynomials $B_n^{(\alpha)}(x; \lambda; a, b, c)$ of order $\alpha \in \mathbb{C}$ are defined by the following generating function:

$$
\left( \frac{t}{\lambda b^t - a^t} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}
$$

($|t \log \left( \frac{a}{b} \right) | < |\log \lambda|$; $1^\alpha := 1$). \hspace{1cm} (1.7)
In this sequel to the work by Sirvastava et al. [25] introduced and investigated a similar generalization of the family of Euler polynomials defined as follows.

**Definition 1.6.** Let \( a, b, c \in \mathbb{R}^+(a \neq b) \) and \( n \in \mathbb{N}_0 \). Then the generalized Euler polynomials \( E_n^{(\alpha)}(x; \lambda; a, b, c) \) of order \( \alpha \in \mathbb{C} \) are defined by the following generating function:

\[
\left( \frac{t}{\lambda b^t + a^t} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x; \lambda; a, b, c) \frac{t^n}{n!}
\]

\[
\left( t \log \left( \frac{a}{b} \right) \right) < | \log(-\lambda) |; \quad 1^\alpha := 1 .
\] (1.8)

It is easy to see that setting \( a = 1 \) and \( b = c = e \) in (1.8) would lead to Apostol-Euler polynomials defined by (1.4). The case where \( \alpha = 1 \) has been studied by Luo et al. [12].

In Section 2, we introduce the new extension of unified family of Apostol-type polynomials and numbers that are defined in [7]. Also, we determine relation between some results given in [23, 24, 10, 11, 26] and our results and introduce some new identities for polynomials defined in [7]. In Section 3, we give some basic properties of the new unification of Apostol-type polynomials and numbers. Finally in Section 4, we introduce some relationships between the new unification of Apostol-type polynomials and other known polynomials.

2. Unification of multiparameter Apostol-type polynomials and numbers

**Definition 2.1.** Let \( a, b, c \in \mathbb{R}^+(a \neq b) \), \( n \in \mathbb{N}_0 \) and \( m \in \mathbb{N} \). Then the new unification of Apostol-type polynomials \( M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) \) are defined, in a suitable neighbourhood of \( t = 0 \) by means of generating function

\[
F_{\alpha_r}^{[m-1,r]} = \frac{t^{km} 2r^{m(1-k)} e^{xt}}{\prod_{i=0}^{r-1} (\alpha_i b^t - a^t \sum_{t=0}^{m-1} t^i / i!)} = \sum_{n=0}^{\infty} M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) \frac{t^n}{n!}
\]

\[
\left( t \log \left( \frac{b}{a} \right) \right) < 2\pi \text{ when } m = 1 \text{ and } \alpha_i = 1; \quad \left( t \log \left( \frac{b}{a} \right) \right) < | \log(\alpha_i) |
\]

when \( m = 1 \) and \( \alpha_i \neq 1; \forall i = 0, 1, ..., r - 1 \),

\[ (2.1) \]

where \( k \in \mathbb{N}_0; r \in \mathbb{C}; \alpha_r = (\alpha_0, \alpha_1, ..., \alpha_{r-1}) \) is a sequence of complex numbers.
The generating function in (2.1) gives many types of polynomials as special cases, for example, see the following table:

| Setting | Equation | Description |
|---------|----------|-------------|
| 1       | $k = 1, \alpha_i = \lambda, i = 0, 1, ..., r - 1, \text{ hence if } m = 1$ | $M_{n}^{(m,r)}(x;\lambda) = \mathfrak{B}_{n}^{(m,r)}(x;\lambda)$ (generalized Bernoulli polynomials of order $r$, see [25]) |
| 2       | $k = 0, \alpha_i = -\lambda, i = 0, 1, ..., r - 1, \text{ hence if } m = 1$ | $M_{n}^{(m,r)}(x;\lambda) = (-1)^r E_{n}^{(m,r)}(x;\lambda)$ (generalized Euler polynomials of order $r$, see [25]) |
| 3       | $\alpha_i = \beta, i = 0, 1, ..., r - 1, c = b, \text{ hence if } m = 1$ | $M_{n}^{(m,r)}(x;k;a,b;\beta) = y_{n}^{(r)}(x;k;a,b)$ (unification of Apostol-type polynomials of order $r$, see [21]) |
| 4       | $k = 1, t = t \ln a, x = \frac{t}{\ln a}, \alpha_i = \lambda, i = 0, 1, ..., r - 1$, $a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(\frac{t}{\ln a};1,1,c,m;r;\lambda) = (\ln a)^m B_{n}^{(m-1,r)}(x;c,a;\lambda)$ (generalized Bernoulli polynomials of order $r$, see [11]) |
| 5       | $k = 0, t = t \ln a, x = \frac{t}{\ln a}, \alpha_i = -\lambda, i = 0, 1, ..., r - 1$, $a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(\frac{t}{\ln a};0,1,c,m;r;\lambda) = (-1)^r (\ln a)^m E_{n}^{(m-1,r)}(x;c,a;\lambda)$ (generalized Euler polynomials of order $r$, see [11]) |
| 6       | $k = 1, \alpha_i = 1, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } r = 1$ | $M_{n}^{(m-1,r)}(1;1,e,1,e,1) = B_{n}^{(m-1)}(x)$ (generalized Bernoulli polynomials, see [17]) |
| 7       | $k = 0, \alpha_i = -1, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } r = 1$ | $M_{n}^{(m-1,r)}(0;1,1,e,e,1) = -E_{n}^{(m-1)}(x)$ (generalized Euler polynomials, see [17]) |
| 8       | $k = 1, \alpha_i = 1, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(1;1,e,e,1) = B_{n}^{(m-1,r)}(x)$ (generalized Bernoulli polynomials of order $r$, see [11]) |
| 9       | $k = 0, \alpha_i = -1, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(0;1,1,e,e,1) = (-1)^r E_{n}^{(m-1,r)}(x)$ (generalized Euler polynomials of order $r$, see [11]) |
| 10      | $k = 1, \alpha_i = -1, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(1;1,e,e,1) = (-1)^r (\frac{1}{2})^{r+m} C_{n}^{(m-1,r)}(x)$ (generalized Genocchi polynomials of order $r$, see [11]) |
| 11      | $k = 1, \alpha_i = \lambda, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(1;1,e,e,1) = B_{n}^{(m-1,r)}(x;\lambda)$ (generalized Apostol-Bernoulli polynomials of order $r$, see [11]) |
| 12      | $k = 0, \alpha_i = -\lambda, i = 0, 1, ..., r - 1, a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(0;1,1,e,e,1) = (-1)^r E_{n}^{(m-1,r)}(x;\lambda)$ (generalized Apostol-Euler polynomials of order $r$, see [11]) |
| 13      | $m = 1, a = 1, b = e, c = e, \text{ hence if } m = 1$ | $M_{n}^{(m-1,r)}(k;1,e,e,1) = M_{n}^{(m-1,r)}(x;k,e,e)$ (a new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, see [11]) |
Remark 2.1. If we set $x = 0$ in (2.1), then we obtain the new unification of multiparameter Apostol-type numbers, as

$$M_n^{[m-1,r]}(0; k, a, b, c; \overline{\alpha}_r) = M_n^{[m-1,r]}(k, a, b, c; \overline{\alpha}_r).$$

(2.2)

Remark 2.2. From No.13 in Table1 and [7, Table1], we can obtain the polynomials and the numbers given in [4, 7, 9, 12, 13].

3. Some basic properties for the polynomial $M_n^{[m-1,r]}(x; k, a, b, c; \overline{\alpha}_r)$

Theorem 3.1. Let $a, b, c \in \mathbb{R}^+(a \neq b)$ and $x \in R$. Then

$$M_n^{[m-1,r]}(x + y; k, a, b, c; \overline{\alpha}_r) = \sum_{\ell=0}^{n} \binom{n}{\ell} x^{n-\ell} (\ln c)^{n-\ell} M_{\ell}^{[m-1,r]}(y; k, a, b, c; \overline{\alpha}_r).$$

(3.1)

$$M_n^{[m-1,r]}(x + r; k, a, b, c; \overline{\alpha}_r) = M_n^{[m-1,r]}(x; k, a, b, c; \overline{\alpha}_r).$$

(3.2)

Proof. For the first equation, from (2.1)

$$\sum_{n=0}^{\infty} M_n^{[m-1,r]}(x + y; k, a, b, c; \overline{\alpha}_r) \frac{t^n}{n!} = \frac{t^{rm} \gamma^{(1-k)}(x)}{\prod_{i=0}^{r-1} \left( \alpha_i b^i - a^i \sum_{t=0}^{m-1} \frac{\ell!}{\ell!} \right)} c^{xt}$$

$$= \sum_{j=0}^{\infty} \left( ty \ln c \right)^j \sum_{l=0}^{\infty} M_{l}^{[m-1,r]}(y; k, a, b, c; \overline{\alpha}_r) \frac{t^j}{j!},$$

using Cauchy product rule, we can easily obtain (3.1).

For the second equation (3.2), from (2.1)

$$\sum_{n=0}^{\infty} M_n^{[m-1,r]}(x + r; k, a, b, c; \overline{\alpha}_r) \frac{t^n}{n!} = \frac{t^{rm} \gamma^{(1-k)}(x)}{\prod_{i=0}^{r-1} \left( \alpha_i b^i - \left( \frac{a}{c} \right)^i \sum_{t=0}^{m-1} \frac{\ell!}{\ell!} \right)} c^{xt}$$

$$= \sum_{n=0}^{\infty} M_n^{[m-1,r]}(x; k, a, b, c; \overline{\alpha}_r) \frac{t^n}{n!},$$

Equating coefficient of $\frac{t^n}{n!}$ on both sides, yields (3.2).

Corollary 3.1. If $y = 0$ in (3.1), we have

$$M_n^{[m-1,r]}(x; k, a, b, c; \overline{\alpha}_r) = \sum_{\ell=0}^{n} \binom{n}{\ell} x^{n-\ell} (\ln c)^{n-\ell} M_{\ell}^{[m-1,r]}(k, a, b, c; \overline{\alpha}_r).$$

(3.3)

$$= \sum_{\ell=0}^{n} \binom{n}{n-\ell} x^{\ell} (\ln c)^{\ell} M_{n-\ell}^{[m-1,r]}(k, a, b, c; \overline{\alpha}_r).$$

(3.4)
Theorem 3.2. The following identity holds true, when \( m = 1 \) and \( \alpha_i \neq 0 \) in (2.1)
\[
\forall i = 0, 1, \ldots, r - 1 \quad \frac{(-1)^{r(1-k)+n}}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=0}^{n} \binom{n}{m} \left( r \ln \left( \frac{ab}{c} \right) \right)^{n-m} M_{m}^{[0, r]} \left( x; k; a, b, c; \frac{1}{\alpha_r} \right). \quad (3.5)
\]

Proof. From (2.1)
\[
\sum_{n=0}^{\infty} M_{n}^{[0, r]}(r - x; k; a, b, c; \alpha_r) \frac{t^n}{n!} = \frac{t^{r-k}x^{r(1-k)}(r-x)^{r}}{\prod_{i=0}^{r-1} \alpha_i} \left( ab^r - a^r \right)
\]
\[
= \frac{(-1)^{r(1-k)} (r-t)^{r-k}x^{r(1-k)}(r-x)^{r}}{\prod_{i=0}^{r-1} \alpha_i} \left( \frac{1}{\alpha_i} - a^r \right)
\]
\[
= \frac{(-1)^{r(1-k)} (ba^r)}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=0}^{\infty} M_{m}^{[0, r]} \left( x; k; a, b, c; \frac{1}{\alpha_r} \right) \frac{(-t)^m}{m!}
\]
\[
= \frac{(-1)^{r(1-k)} (ba^r)}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=0}^{\infty} \frac{\left( r \ln \left( \frac{ab}{c} \right) \right)^{t}}{\ell!} \sum_{m=0}^{\infty} M_{m}^{[0, r]} \left( x; k; a, b, c; \frac{1}{\alpha_r} \right) \frac{(-t)^m}{m!}
\]
Hence, we can easily obtain (3.5).

Remark 3.1. If we put \( \alpha_i = \beta, i = 0, 1, \ldots, r - 1 \), \( c = b \) and \( r = v \) in (3.5), then it gives [24, Eq. (34)],
\[
M_{n}^{[0, v]}(v - x; k; a, b, b; \beta) = \frac{(-1)^{v(1-k)+n}}{(\beta)^{v}} \sum_{m=0}^{n} \binom{n}{m} \left( v \ln a \right)^{n-m} M_{m}^{[0, v]} \left( x; k; a, b, b; \beta^{-1} \right),
\]
where \( M_{m}^{[0, v]} \left( x; k; a, b, b; \beta^{-1} \right) \) is the unification of the Apostol-type polynomials.

Theorem 3.3. The unification of Apostol-type numbers satisfy
\[
M_{n}^{[m-1, r]}(k; a, b, c; \alpha_r) = \sum_{l=0}^{n} \binom{n}{l} M_{l}^{[m-1, l]}(k; a, b, c; \alpha_r) M_{n-l}^{[m-1, r-l]}(k; a, b, c; \alpha_r). \quad (3.6)
\]

Proof. When \( x = 0 \) in (2.1), we have
\[
\sum_{n=0}^{\infty} M_{n}^{[m-1, r]}(k; a, b, c; \alpha_r) \frac{t^n}{n!} = \frac{t^{r-k}x^{r(1-k)}(r-x)^{r}}{\prod_{i=0}^{r-1} \alpha_i} \left( ab^r - a^r \right)
\]
\[
= \frac{(-1)^{r(1-k)} (r-t)^{r-k}x^{r(1-k)}(r-x)^{r}}{\prod_{i=0}^{r-1} \alpha_i} \left( \frac{1}{\alpha_i} - a^r \right)
\]
\[
= \frac{(-1)^{r(1-k)} (ba^r)}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=0}^{\infty} M_{m}^{[0, r]} \left( x; k; a, b, c; \frac{1}{\alpha_r} \right) \frac{(-t)^m}{m!}
\]
Using Cauchy product rule, we obtain (3.6).
Theorem 3.4. The following relationship holds true

\[
\sum_{k_1+k_2+\ldots+k_\ell=n} \prod_{i=1}^\ell \frac{M_{k_i}^{[m-1,r_i]}(x_i; k; a, b, c; \overline{\alpha}_r)}{k_1! k_2! \ldots k_\ell!} = \frac{1}{n!} M_n^{[m-1,r]}([x]; k; a, b, c; \overline{\alpha}_r),
\]

(3.7)

where \(|r| = r_1+r_2+\ldots+r_\ell\) and \(|x| = x_1+x_2+\ldots+x_\ell\) and \(\overline{\alpha}_r = (\alpha_{\sum_{j=1}^{i-1} r_j}, \alpha_{\sum_{j=1}^{i-1} r_j+1}, \ldots, \alpha_{\sum_{j=1}^i r_j-1})\), \(i = \{1, 2, \ldots, \ell\}\).

Proof. Starting with (2.1), we get

\[
\begin{align*}
\sum_{n=0}^\infty \left( \sum_{k_1=0}^\infty M_{k_1}^{[m-1,r_1]}([x]; k; a, b, c; \overline{\alpha}_r) \right) \frac{t^n}{n!} &= \sum_{n=0}^\infty \left( \prod_{i=0}^{\ell} \frac{t^{r_i} k m^{r_i} m(1-k)m_i}{\prod_{i=0}^{r_i-1} \left( \alpha_i a^t - a t \sum_{i=0}^{m-1} \frac{\overline{\alpha}_r}{r_i} \right)} \right) \frac{t^n}{n!} \\
&= \sum_{k_1=0}^\infty M_{k_1}^{[m-1,r_1]}([x]; k; a, b, c; \overline{\alpha}_r) \prod_{k_1=0}^\infty \left( \sum_{m=0}^{\infty} M_{k_2}^{[m-1,r_2]}([x]; k; a, b, c; \overline{\alpha}_r) \frac{t_{k_2}}{k_2!} \right) \\
&= \sum_{k_1=0}^\infty M_{k_1}^{[m-1,r_1]}([x]; k; a, b, c; \overline{\alpha}_r) \prod_{k_1=0}^\infty \left( \sum_{m=0}^{\infty} M_{k_2}^{[m-1,r_2]}([x]; k; a, b, c; \overline{\alpha}_r) \frac{t_{k_2}}{k_2!} \right)
\end{align*}
\]

Using Cauchy product rule on the right hand side of the last equation and equating coefficients of \(t^n\) on both sides, yields (3.7).

Using No.13 in Table 1, we obtain Nörlund’s results, see [19] and Carlitz’s generalizations, see [2] by our approach in Theorem 3.5 and Theorem 3.6 as follows

Theorem 3.5. For \((\overline{\alpha}_r)^n = (\alpha_0^n, \alpha_1^n, \ldots, \alpha_{\ell-1}^n)\), we have

\[
\prod_{i=1}^{\ell} \sum_{s_i=0}^{r_{i-1} \alpha_i a^t} M_{k-i}^{[0,r_i]} \left( x + \sum_{i=1}^{r_i} \frac{s_i}{n}; k; 1, e, c; \overline{\alpha}_r \right)^n = n^{r-k-\ell} M_{k}^{[0,r]} \left( nx+k; 1, e, c; \overline{\alpha}_r \right).
\]

(3.8)

\[
\prod_{i=1}^{\ell} \sum_{s_i=0}^{r_{i-1} \alpha_i a^t} M_{k-i}^{[0,r_i]} \left( x + \sum_{i=1}^{r_i} \frac{s_i}{n}; k; 1, e, c; \overline{\alpha}_r \right)^n = n^{r(k-1)-\ell} \frac{\ell!}{\ell!} M_{k}^{[0,r]} \left( nx+k; 1, e, c; \overline{\alpha}_r \right).
\]

(3.9)

Proof. For the first equation and starting with (2.1), we get

\[
\sum_{\ell=0}^n \left( \sum_{i=1}^r \alpha_i a^t \right)^\ell \prod_{i=1}^{\ell} \sum_{s_i=0}^{r_{i-1} \alpha_i a^t} M_{k-i}^{[0,r_i]} \left( x + \sum_{i=1}^{r_i} \frac{s_i}{n}; k; 1, e, c; \overline{\alpha}_r \right)^n = \frac{(nt)^{r-\ell} \alpha^{r(1-k)} e^{nt}}{\prod_{i=0}^{r-1} \alpha_i c^t - 1} \prod_{i=1}^{\ell} \sum_{s_i=0}^{r_{i-1} \alpha_i a^t} M_{k-i}^{[0,r_i]} \left( nx+k; 1, e, c; \overline{\alpha}_r \right)^n
\]

\[
= \frac{(nt)^r 2^{r(1-k)} e^{(nx)t}}{\prod_{i=0}^{r-1} \alpha_i c^t - 1} = n^{r-k} M_{k}^{[0,r]} \left( nx+k; 1, e, c; \overline{\alpha}_r \right) \frac{\ell!}{\ell!}
\]
Equating coefficients of $t^\ell$ on both sides, yields (3.3).

For the second equation and starting with (2.1), we get

$$
\sum_{\ell=0}^{\infty} \frac{(mt)^\ell}{\ell!} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_{\ell \ell}^{[0,r]} \left( x + \frac{\sum_{i=1}^{r} s_i}{n}; k, 1, e, (\overline{\omega})^n \right) = \frac{n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1}}{\prod_{i=0}^{r-1} \alpha_i e^{x^{n+r-1}}} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^{s_i})^{s_i}
$$

$$
= n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1} \prod_{i=0}^{r-1} \alpha_i e^{s_i}
$$

then, we have

$$
\sum_{\ell=0}^{\infty} \frac{(mt)^\ell}{\ell!} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_{\ell + \ell}^{[0,r]} \left( x + \frac{\sum_{i=1}^{r} s_i}{n}; k, 1, e, (\overline{\omega})^n \right) = n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1} \prod_{i=0}^{r-1} (\alpha_i e^{s_i})^{s_i}
$$

Equating coefficients of $t^\ell$ on both sides, yields (3.3). \hfill \Box

**Theorem 3.6.** For $(\overline{\omega})^n = (\alpha_0^n, \alpha_1^n, \ldots, \alpha_{r-1}^n)$ and $(\overline{\omega})^m = (\alpha_0^m, \alpha_1^m, \ldots, \alpha_{r-1}^m)$ we have

$$
n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1} \prod_{i=0}^{r-1} (\alpha_i e^{s_i})^{s_i}
$$

$$
\sum_{\ell=0}^{\infty} \frac{(mt)^\ell}{\ell!} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_{\ell \ell}^{[0,r]} \left( x + \frac{\sum_{i=1}^{r} s_i}{n}; k, 1, e, (\overline{\omega})^n \right) = \frac{n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1}}{\prod_{i=0}^{r-1} \alpha_i e^{x^{n+r-1}}} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^{s_i})^{s_i}
$$

$$
= \frac{n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1} \prod_{i=0}^{r-1} \alpha_i e^{s_i}}{\prod_{i=0}^{r-1} \alpha_i e^{x^{n+r-1}}} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^{s_i})^{s_i}
$$

**Proof.** For the first equation and starting with (2.1), we get

$$
\sum_{\ell=0}^{\infty} \frac{(mt)^\ell}{\ell!} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1})^{s_i} M_{\ell \ell}^{[0,r]} \left( x + \frac{\sum_{i=1}^{r} s_i}{n}; k, 1, e, (\overline{\omega})^n \right) = \frac{n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1}}{\prod_{i=0}^{r-1} \alpha_i e^{x^{n+r-1}}} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^{s_i})^{s_i}
$$

$$
= \frac{n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1} \prod_{i=0}^{r-1} \alpha_i e^{s_i}}{\prod_{i=0}^{r-1} \alpha_i e^{x^{n+r-1}}} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^{s_i})^{s_i}
$$

$$
= \frac{n^{rk} r^{2r} \alpha^{r(2-k)} e^{r(r-1)k} x^{n+r-1} \prod_{i=0}^{r-1} \alpha_i e^{s_i}}{\prod_{i=0}^{r-1} \alpha_i e^{x^{n+r-1}}} \prod_{i=1}^{r-1} \sum_{s_i=0}^{n-1} (\alpha_{i-1} e^{s_i})^{s_i}
$$

\[(3.10)\]
\begin{align*}
= m^{-k} n^{-k} \sum_{\ell=0}^{\infty} \left( \prod_{i=1}^{\ell} (\alpha_{i-1})^{n_{p_i}} m^{\ell} \right) \left( \frac{x}{m} + \frac{\sum_{i=1}^{r} p_i \cdot n}{m} ; k; 1, e, e (\alpha_r)^m \right) t^\ell.
\end{align*}

Equating coefficients of \( t^\ell \) on both sides, yields \((3.10)\).

Also, It is not difficult to prove \((3.11)\).

\section*{4. Some relations between the polynomials \( M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) \) and other polynomials and numbers}

In this section, we give some relationships between the polynomials \( M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) \) and Laguerre polynomials, Jacobi polynomials, Hermite polynomials, generalized Stirling numbers of second kind, Stirling numbers and Bleimann-Butzer-hahn basic.

\begin{theorem}
For \( \alpha_r = (\alpha_0, \alpha_1, ..., \alpha_r) \in \mathbb{C} \), \( (x; \alpha_r) \ell = (x - \alpha_0)(x - \alpha_1)...(x - \alpha_{\ell-1}) \) and \( n, j \in \mathbb{N}_0 \), we have relationship

\begin{equation}
M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) = \sum_{j=0}^{n} (x; \alpha)_j \sum_{\ell=j}^{n} \binom{n}{n - \ell} (\ln c)^\ell S(\ell, j; \alpha) M_{n-\ell}^{[m-1,r]}(k; a, b, c; \alpha_r)
\end{equation}

\end{theorem}

\begin{proof}
Using \((3.4)\) and from definition of generalized Stirling numbers of second kind, see \([4]\), we easily obtain \((4.1)\).
\end{proof}

\begin{theorem}
For \( \alpha_r = (\alpha_0, \alpha_1, ..., \alpha_r) \in \mathbb{C} \), \( (x; \alpha) \ell = (x-1)...(x-\ell+1) \) and \( n, j \in \mathbb{N}_0 \), we have relationship

\begin{equation}
M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) = \sum_{j=0}^{n} (x; \alpha)_j \sum_{\ell=j}^{n} \binom{n}{n - \ell} (\ln c)^\ell S(\ell, j; \alpha) M_{n-\ell}^{[m-1,r]}(k; a, b, c; \alpha_r)
\end{equation}

\end{theorem}

\begin{proof}
Using \((3.4)\) and from definition of Stirling numbers of second kind, see \([8]\), we easily obtain \((4.2)\).
\end{proof}

\begin{theorem}
The relationship

\begin{equation}
M_{n}^{[m-1,r]}(x; k; a, b, c; \alpha_r) = \sum_{j=0}^{n} \sum_{\ell=0}^{n} (-1)^{j!} \binom{n}{n - \ell} (\ln c)^\ell (\ell + \alpha)\beta^\ell J_j(\alpha) M_{n-\ell}^{[m-1,r]}(k; a, b, c; \alpha_r)
\end{equation}

holds between the new unification of multiparameter Apostol-type polynomials and generalized Laguerre polynomials, see \([24]\, No.(3) \textbf{Table1}\).

\(\square\)
Proof. From (3.4) and substitute
\[ x^\ell = \ell! \sum_{j=0}^{\ell} (-1)^j \binom{\ell + \alpha}{\ell - j} L_j^\alpha(x), \]
then we get (4.3).

\[ \text{Theorem 4.4. For } (\alpha + \beta + j + 1)_{\ell+1} = (\alpha + \beta + j + 1)(\alpha + \beta + j + 2) \ldots (\alpha + \beta + j + \ell + 1). \]

The relationship
\[ M_n^{[m-1,r]}(x; k; a, b, c; \sigma_r) = \sum_{j=0}^{\ell} \sum_{\ell=\ell}^{n} (-1)^{n-j} \binom{n}{n-\ell} (\ln c)^{\ell} \binom{\ell + \alpha}{\ell - j} \frac{\beta + 2j + 1}{(\alpha + \beta + j + 1)_{\ell+1}} P_j^{(\alpha, \beta)}(1 - 2x)M_n^{[m-1,r]}(k; a, b, c; \sigma_r) \] (4.4)
holds between the new unification of Apostol-type polynomials and Jacobi polynomials, see [22, p.49, Eq. (35)].

Proof. From (3.4) and substitute
\[ x^\ell = \ell! \sum_{j=0}^{\ell} (-1)^j \binom{\ell + \alpha}{\ell - j} \frac{\alpha + \beta + 2j + 1}{(\alpha + \beta + j + 1)_{\ell+1}} P_j^{(\alpha, \beta)}(1 - 2x), \]
then we get (4.4).

\[ \text{Theorem 4.5. The relationship} \]
\[ M_n^{[m-1,r]}(x; k; a, b, c; \sigma_r) = \sum_{j=0}^{\ell} \sum_{\ell=\ell}^{n} 2^{-\ell} \binom{n}{n-\ell} \binom{\ell}{2j} \frac{2^{j!}}{j!} (\ln c)^{\ell} H_{\ell-2j}(x)M_n^{[m-1,r]}(k; a, b, c; \sigma_r) \] (4.5)
holds between the new unification of Apostol-type polynomials and Hermite polynomials, see [26, No.(1) Table1].

Proof. From (3.4) and substitute
\[ x^\ell = 2^{-\ell} \sum_{j=0}^{\ell} \binom{\ell}{2j} \frac{2^{j!}}{j!} H_{\ell-2j}(x), \]
then we get (4.5).

\[ \text{Theorem 4.6. When } m = 1, a = 1, b = e \text{ and } c = e \text{ in } (2.4) \text{ and for } \sigma_r = (\alpha_0, \alpha_1, ..., \alpha_{r-1}), \] 
\[ \alpha_i \neq 0, \ i = 0, 1, ..., r - 1 \text{ and } \beta_m = (\beta_0, \beta_1, ..., \beta_{m-1}), \] 
\[ \beta_i \neq 0, \ i = 0, 1, ..., m - 1, \] 
we have the following relationship
\[ M_n^{(r)}(x; k; \sigma_r) = \frac{n!}{\prod_{i=0}^{r-1} \alpha_i} \sum_{m=r}^{\infty} \frac{2^{(1-k)(r-m)} \prod_{j=0}^{m-1} \beta_j}{(n+k(m-1))!} C(m, r; \alpha_r, \beta_r, \beta_m) M_n^{(m)}(m+k(m-r))(x; k; \beta_m), \] (4.6)
between the new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, and \( C(m, r; \alpha_r, \beta_r, \beta_m) \) (the generalized Lah numbers), see [3].
Proof. From [7, Eq. 2.1],
\[
\sum_{n=0}^{\infty} M_{n}^{(r)}(x; k; \overline{a_r}) \frac{t^n}{n!} = \frac{t^{\alpha} \gamma(1-k) e^{xt}}{\prod_{i=1}^{\alpha} (\alpha_i e^{t} - 1)} \frac{1}{(e^{t}; e^{x} \overline{a_r})_{\infty}}
\]
\[
= \frac{t^{\alpha} \gamma(1-k) e^{xt}}{\prod_{i=1}^{\alpha} (\alpha_i e^{t} - 1)} \left( \sum_{m=0}^{\infty} C(m, r; \alpha_r; e^{x} \overline{a_r}) \frac{(e^{t}; e^{x} \overline{a_r})_{m}}{(e^{t} e^{x} \overline{a_r})_{m}} \right)
\]
\[
= \sum_{m=0}^{\infty} \prod_{i=1}^{\alpha} \frac{\beta_i^{m}}{e^{t} e^{x} \overline{a_r}} \left( \sum_{m=0}^{\infty} \frac{n! \gamma(1-k) e^{xt} m!}{(n-k(m-1))! \prod_{i=1}^{\alpha} \alpha_i} \right) \frac{1}{n!}.
\]
Equating the coefficients of \(t^n\) on both sides, yields (4.6).

Using No.13 in Table 1, see [7] and the definition of the unified Bernstein and Bleimann-Butzer-Hahn basis (see [18]),
\[
\left( \frac{2^{1-k} x^k}{(1 + ax)^{k}} \right)^m \frac{1}{mk!} e^{t(\frac{1+b}{1+ax})} = \sum_{n=0}^{\infty} P_n^{(a,b)}(x; k, m) \frac{t^n}{n!}, \quad (4.7)
\]
where \(k, m \in \mathbb{Z}^+, a, b \in \mathbb{R}, t \in \mathbb{C}\), we obtain the following theorem

**Theorem 4.7.** For \(\alpha_i \neq 0, i = 0, 1, ..., r - 1\), we have relationship
\[
P_n^{(a,b)}(x; k, m) = \frac{r-1}{r!} \prod_{i=0}^{r-1} \frac{\alpha_i}{r} \left( \frac{x}{1 + ax} \right)^{r-k} \sum_{j=0}^{r} s_{r,j} \left( \frac{1}{\alpha_r} \right) \sum_{\ell=0}^{\infty} j^{n-\ell} \left( \frac{n}{\ell} \right) M_{\ell}^{(r)} \left( \frac{1 + bx}{1 + ax}; k; \overline{a_r} \right), \quad (4.8)
\]
between the unified Bernstein and Bleimann-Butzer-Hahn basis, the new unified family of generalized Apostol-Bernoulli, Euler and Genocchi polynomials, see [7] and generalized Stirling numbers of first kind, see [4].

**Proof.** From (2.1) and (4.7) and with some elementary calculation, we easily obtain (4.8).

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