Spontaneous Polarization of the $\mathbb{Z}_n$-Baxter Model

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Abstract

We show that correlation functions of the $\mathbb{Z}_n$-Baxter model in the principal regime satisfy a system of difference equations. We obtain the spontaneous polarization of the $\mathbb{Z}_n$-Baxter model as a solution of the simplest difference equation.

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1 Introduction

It is widely recognized that systems of holonomic difference equations commonly appear in integrable quantum field theories and solvable lattice statistical models. Smirnov [1] found that form factors in two dimensional integrable massive models satisfy difference equations. Frenkel and Reshetikhin [2] showed that correlation functions of the intertwining operators of the quantum affine algebra obey difference equations, and that the resulting connection matrices give elliptic solutions [3, 4] to the Yang-Baxter equation (YBE). In [5, 6] it was proposed that correlations of the $XXZ$ model in the anti-ferromagnetic regime can be formulated in terms of the trace of products of the intertwining operators of $U_q(sl_2)$. The higher spin analog of the $XXZ$ model [7] and the higher rank generalization [8, 9] were also studied. The integral formula of correlation functions of the $XXZ$ model is given in [6]. That of the matrix element of product of intertwining operators of $U_q(sl_2)$ for arbitrary level is obtained in [10].

It was remarked in [2] that even if the trigonometric $R$-matrices is replaced by the elliptic ones, integrability of difference equations survives. Jimbo, Miwa and Nakayashiki [11] actually obtained difference equations for correlations of the eight-vertex model in the principal regime. In the present paper we generalize their results to the case of the $Z_n$-Baxter model [12].

The $Z_n$-Baxter model is a vertex model on a two-dimensional square lattice $L$ such that the state variables take on $Z_n$-spin. Each oriented line of $L$ carries a spectral parameter varying from line to line. We assign $Z_n$-valued local states on edges. Set

$$R(z_1 - z_2)_{ik}^{jl} := j \quad \text{ if } i + k = j + l, \mod n,$$

otherwise,

(1.1)

where

$$\theta^{(j)}(z) := \sum_{m \in Z} \exp \left\{ \pi \sqrt{-1} (m + \frac{1}{2} - \frac{j}{n})^2 n \tau + 2 \pi \sqrt{-1} (m + \frac{1}{2} - \frac{j}{n})(z + \frac{1}{2}) \right\},$$
\[
    h(z) := \prod_{j=0}^{n-1} \theta^{(j)}(z) / \prod_{j=1}^{n-1} \theta^{(j)}(0),
\]
and \( w \neq 0 \mod \mathbb{Z} + \mathbb{Z} \tau \), is a constant. Now we assume that \( 0 < t < q < u < 1 \), where \( t := \exp(\pi \sqrt{-1} \tau) \), \( q := \exp(\pi \sqrt{-1} w) \), and \( u := \exp(-\pi \sqrt{-1} z) \). Then the elliptic theta functions are expressed in terms of the product series

\[
    \theta^{(j)}(z) = \sqrt{-1} \omega_{j}/t^{n(1/2-j/n)^2} u^{-1+2j/n}(t^{2n}; t^{2n})_\infty (t^{2j} u^2; t^{2n})_\infty (t^{2(n-j)} u^{-2}; t^{2n})_\infty,
\]

\[
    h(z) = \sqrt{-1} t^{n/4} \frac{(t^{2n}; t^{2n})_\infty^3 u^{-1}(u^2; t^2)_\infty(t^2 u^{-2}; t^2)_\infty},
    \tag{1.2}
\]

where

\[
    (a; q_1, \ldots, q_k)_\infty := \prod_{m_1=0}^{\infty} \cdots \prod_{m_k=0}^{\infty} (1 - a q_1^{m_1} \cdots q_k^{m_k}).
\]

Following Baxter \[14\] we call such domain of parameters the principal regime. Note that (1.1) is weights of the eight-vertex model when \( n = 2 \).

The plan of this paper is as follows. In section 2 we summarize our result on the unitarity and the crossing symmetry of the \( \mathbb{Z}_n \)-Baxter model \[15, 16\]. In section 3 we introduce the normalized \( R \)-matrices by the partition function per site in the thermodynamic limit. In section 4 we show that correlation functions defined by using this normalized \( R \)-matrices satisfy \( q \)-difference equations. In section 5 solving the simplest one we obtain the spontaneous polarization. By setting \( n = 2 \), our result reproduces that of \[11\]. We can also get Koyama’s result \[9\] by taking the limit \( t \to 0 \). In section 6 we give some remarks.

## 2 Unitarity and Crossing Symmetry

Let \( V = \mathbb{C}^n \) and \( \{v_i\}_{i \in \mathbb{Z}_n} \) be the standard orthonormal basis of \( V \). Then the \( R \)-matrix \( R(z) \) whose \((ik, jl)\)-element is \( R(z)_{ik}^{jl} \) gives a linear map \( V \otimes V \to V \otimes V \)

\[
    R(z)(v_j \otimes v_l) = \sum_{i,k \in \mathbb{Z}_n} (v_i \otimes v_k) R(z)_{ik}^{jl}.
\]

The \( \mathbb{Z}_n \)-Baxter model has the \( \mathbb{Z}_n \)-symmetry

\[
    \begin{align*}
        (i) & \quad R(z)_{ik}^{jl} = 0, \quad \text{unless } i + k = j + l, \text{ mod } n, \\
        (ii) & \quad R(z)_{i+j+p,l+p}^{i+k+p} = R(z)_{ij}^{kl}, \quad \text{for every } i, j, k, l \text{ and } p \in \mathbb{Z}_n.
    \end{align*}
    \tag{2.1}
\]

In terms of two linear map in \( V \)

\[
    gv_i = \omega^i v_i, \quad h v_i = v_{i-1}, \tag{2.2}
\]
depends only upon the difference of rapidities $\omega\ z$, where, $P$ is the permutation as a linear map $R$, Owing to (2.3) the $K$ as a linear map $R$ of tensor product of some V’s. For $R$ is the permutation $V_{z_1} \otimes V_{z_2} \rightarrow V_{z_2} \otimes V_{z_1}$. In this notation, YBE (2.4) reads as follows:

\[(I \otimes \tilde{R}^{V_{z_1},V_{z_2}}(\tilde{R}^{V_{z_1},V_{z_3}} \otimes I))(I \otimes \tilde{R}^{V_{z_2},V_{z_3}}) = (\tilde{R}^{V_{z_2},V_{z_3}} \otimes I)(I \otimes \tilde{R}^{V_{z_1},V_{z_3}})(\tilde{R}^{V_{z_1},V_{z_2}} \otimes I),\]  

(2.5)
as a linear map $V_{z_1} \otimes V_{z_2} \otimes V_{z_3} \rightarrow V_{z_3} \otimes V_{z_2} \otimes V_{z_1}$.

Next we extend this formulation for arbitrary spaces $K$ and $L$ of tensor product of some $V$’s. For $K = V_{z_1} \otimes \cdots \otimes V_{z_k}$ and $L = V_{z_1}^{i_1} \otimes \cdots \otimes V_{z_1}^{i_l}$, it is very natural to define a linear map $\tilde{R}^{K,L} : K \otimes L \rightarrow L \otimes K$ as follows [17, 13]:

\[\tilde{R}^{K,V_{z_1}} := \tilde{R}^{V_{z_1},V_{z_1}} \cdots \tilde{R}^{V_{z_1},V_{z_1}},\]
\[\tilde{R}^{K,L} := \tilde{R}^{V_{z_1},V_{z_1}} \cdots \tilde{R}^{V_{z_1},V_{z_1}}\]

YBE holds for $\tilde{R}^{K,L}$ by virtue of YBE for $\tilde{R}^{V,V}$ (2.3):

\[(I \otimes \tilde{R}^{K,L})(\tilde{R}^{K,M} \otimes I)(I \otimes \tilde{R}^{L,M}) = (\tilde{R}^{L,M} \otimes I)(I \otimes \tilde{R}^{K,M})(\tilde{R}^{K,L} \otimes I),\]  

(2.6)
as a linear map $K \otimes L \otimes M \rightarrow M \otimes L \otimes K$.

Now we summarize the unitarity and the crossing symmetry of the $\mathbb{Z}_n$-Baxter model.
(1) The unitarity The unitarity or the first inversion relation \([13, 16]\) is given by
\[
\hat{R}^{V_{21},V_{22}} \hat{R}^{V_{22},V_{21}} = \frac{\hbar(z_1 - z_2 + w)\hbar(-z_1 + z_2 + w)}{\hbar^2(w)} I \otimes I.
\] (2.7)

(2) The crossing symmetry Let \(V^*\) be the dual space of \(V\) and \(\{v_i^*\}_{i \in \mathbb{Z}_n}\) be the dual basis of \(\{v_i\}_{i \in \mathbb{Z}_n}\). Then we have the isomorphism \(C : V^* \to \Lambda^{n-1}(V)\)
\[
Cv_i^* = \sum_{i_1, \ldots, i_{n-1}} \frac{\epsilon_{i_1 \cdots i_{n-1}}}{\sqrt{(n-1)!}} v_{i_1} \otimes \cdots \otimes v_{i_{n-1}},
\] (2.8)
where \(\epsilon_{i_1 \cdots i_{n-1}}\) is the \(n\)-th order completely antisymmetric tensor.

The \(R\)-matrices corresponding to the collision between a particle and an antiparticle is given as follows:
\[
\hat{R}^{V_{21},V_{22}} = (C \otimes I)^{-1} \hat{R}^{V_{21},V_{22}+(n-1)w} \otimes \cdots \otimes V_{22+w} (I \otimes C),
\]
\[
\hat{R}^{V_{21},V_{22}} = (I \otimes C)^{-1} \hat{R}^{V_{21}+(n-1)w} \otimes \cdots \otimes V_{21+w} \otimes V_{22} (C \otimes I).
\] (2.9)

The matrix elements of these are
\[
\hat{R}^{V_{21},V_{22}}(v_j \otimes v_i^*) = \sum_{i',k} (v_k^* \otimes v_k)(\hat{R}^{V_{21},V_{22}})^{i'k}_{i,j},
\]
\[
\hat{R}^{V_{21},V_{22}}(v_j^* \otimes v_i) = \sum_{i,k} (v_i \otimes v_k)(\hat{R}^{V_{21},V_{22}})^{ik*}_{j,i},
\]
which meet the crossing symmetry \([13, 10]\)
\[
(\hat{R}^{V_{21},V_{22}})^{i'k}_{j,l} = (\hat{R}^{V_{22},V_{21}})^{kl}_{ij} \prod_{p=2}^{n-1} \frac{\hbar(-z_1 + z_2 + pw)}{\hbar(w)},
\]
\[
(\hat{R}^{V_{21},V_{22}})^{kl*}_{i,j} = (\hat{R}^{V_{22},V_{21}+nw})^{ij*}_{kl} \prod_{p=1}^{n-2} \frac{\hbar(-z_1 + z_2 - pw)}{\hbar(w)}.
\] (2.10)

From (2.7) and (2.10), we have the following second inversion relation \([13, 10]\)
\[
\sum_{j,l} \hat{R}^{il}_{kl}(z) \hat{R}^{k'l}_{j'i}(z - nw) = \frac{\hbar(-z)\hbar(z + nw)}{\hbar^2(w)} \delta_{i'i} \delta_{k'k}.
\] (2.11)

The \(R\)-matrix corresponding to antiparticle-antiparticle scattering is also defined by
\[
\hat{R}^{V_{21},V_{22}} = (C \otimes C)^{-1} \hat{R}^{V_{21}+(n-1)w} \otimes \cdots \otimes V_{21+w},V_{22+(n-1)w} \otimes \cdots \otimes V_{22+w} (C \otimes C).
\]
3 Partition Function and S-Matrices

Let $\kappa(z)$ be the partition function per site in the thermodynamic limit. With the help of two inversion relations (2.7), (2.11) and Baxter’s corner transfer matrix method [14], we can get the functional equations for $\kappa(z)$:

$$
\kappa(z)\kappa(-z) = \frac{h(z + w)h(-z + w)}{h^2(w)},
\kappa(z)\kappa(-z - nw) = \frac{h(-z)h(z + nw)}{h^2(w)}.
$$

(3.1)

Hereafter $\kappa(z)$ is often denoted by $\kappa(u)$ through the relation $u = \exp(-\pi \sqrt{-1}z)$.

In the principal regime using (1.2) the following expression solves (3.1) [13]

$$
\kappa(u) = u^{-(n-2)/n} (u^2; t^2)_\infty^2 (t^2 u^{-2}; t^2)_\infty^2 \bar{k}(u),
$$

(3.2)

where

$$
\bar{k}(u) = \frac{(q^2 u^2; t^2, q^{2n})_\infty^2 (q^2 n u^{-2}; t^2, q^{2n})_\infty^2 (t^2 q^{-2} u^2; t^2, q^{2n})_\infty^2 (t^2 q^{2n} u^{-2}; t^2, q^{2n})_\infty^2}{(q^{2+2n} u^{-2}; t^2, q^{2n})_\infty^2 (u^2; t^2, q^{2n})_\infty^2 (t^2 q^{-2+2n} u^{-2}; t^2, q^{2n})_\infty^2 (t^2 u^2; t^2, q^{2n})_\infty^2}.
$$

For later convenience, we define $\tilde{S}^{K,L} = PS^{K,L}$ by

$$
\tilde{S}^{V_{z_1},V_{z_2}} = \kappa(z_1 - z_2)^{-1} \tilde{R}^{V_{z_1},V_{z_2}},
\tilde{S}^{V_{z_1},V_{z_2}^*} = \prod_{p=1}^{n-1} \kappa(z_1 - z_2 - pw)^{-1} \tilde{R}^{V_{z_1},V_{z_2}^*},
\tilde{S}^{V_{z_1}^*,V_{z_2}} = \prod_{p=1}^{n-1} \kappa(z_1 - z_2 + pw)^{-1} \tilde{R}^{V_{z_1}^*,V_{z_2}},
\tilde{S}^{V_{z_1}^*,V_{z_2}^*} = \prod_{p,q=1}^{n-1} \kappa(z_1 - z_2 + (p - q)w)^{-1} \tilde{R}^{V_{z_1}^*,V_{z_2}^*}.
$$

(3.3)

Then the unitarity and the crossing symmetry for $\tilde{S}^{K,L}$ hold:

$$
\tilde{S}^{K,L} \tilde{S}^{L,K} = \text{id.},
(\tilde{S}^{V_{z_1},V_{z_2}})_{ij,k}^{*k} = (\tilde{S}^{V_{z_2},V_{z_1}})_{ij}^{kl},
(\tilde{S}^{V_{z_1}^*,V_{z_2}})_{i,j}^{kl} = (\tilde{S}^{V_{z_2}^*,V_{z_1}+nw})_{ij}^{ik}.
$$

(3.4)

The first one is obvious. The last two follow from

$$
\kappa(-z)^{-1} \prod_{j=1}^{n-1} \kappa(z - jw) = \prod_{j=2}^{n-1} \frac{h(-z + jw)}{h(w)},
$$

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which can be checked by explicit calculation. The relations (3.4) are visualized as

\[ j^* \quad \Rightarrow \quad \begin{array}{c} i^* \\ j \quad \Rightarrow \quad z_2^* \\ l^* \end{array} \quad = \quad \begin{array}{c} i \\ z_1 \end{array} \quad \begin{array}{c} k \\ l \end{array} \]

where \( \tilde{z} = z + nw \). Note that \( z (z^*) \) in the above figure represents the space \( V_z (V_z^*) \), while \( i (i^*) \) denotes \( v_i (v_i^*) \).

**Remark** Speaking in terms of Young tableau, \( V = \square \), the fundamental representation of \( sl_n \), while \( V^* \) corresponds to \( n-1 \) vertical \( \square \)'s, the space of antisymmetric tensors in \( V^* \otimes V^{n-1} \). Thus both \( V \) and \( V^* \) are indispensable in our formulation. For \( n = 2 \) \[\text{from (2.8) and (2.9), one can do without } V^* \text{ by identifying } v_i^* = (-1)^i v_{1-i}, V_z^* = V_{2z+w}.\]

### 4 Difference Equations for Correlation Functions

In the principal regime the Boltzmann weights of the types \( \hat{S}(z)^{ii+1}_{ii+1} \) dominates the others. Thus in the low temperature limit \( t, q \to 0 \), only the configuration such that the spin variables take the same value in the direction from NE to SW and increase by one in the direction from NW to SE, is possible. We call it a configuration of the ground states. There are \( n \) ones labeled by \( m \in \mathbb{Z}_n \). In what follows, we fix one of them (say, \( m \)) and define all the correlation functions in terms of the low-temperature series expansion (i.e. the formal power series of \( t \) and \( q \)). Then the lowest order of them comes from the \( m \)-th ground state configuration. Furthermore, any finite order contribution is derived from the configurations which differ from
that of the $m$-th ground state by altering a finite number of spins. It is equivalent to taking
the GNS representation obtained from the $m$-th ground state ($m$-th GNS representation) as
the Hilbert space. It is expected that the correlation function defined in such a way is an
analytic function which has a finite convergence radious if there exists the phase transition
at a finite temperature.

Let us consider the probability

$$P^{(m)} \left( \frac{z_1, \cdots, z_N}{z'_1, \cdots, z'_N} \right)^{i_1, \cdots, i_N} := \sum_{\text{config}} \text{config}$$

in the inhomogeneous $\mathbb{Z}_n$-Baxter model. Here $m$ specify the boundary condition such that the
spin on the reference edge equals to $m$ in the ground state configuration. The reference edge
is the next left to the one with the spectral parameter $z_1$. The symbol of the sum denotes the
statistical sum over the $m$-th GNS representation. If we assign the weight $S$ to each vertex,
the statistical sum gives just the probability in the thermodynamic limit.

The YBE, the unitarity and the crossing symmetry permit the following manipulations
\( (m) \hspace{1cm} i \hspace{1cm} \cdots \hspace{1cm} \Rightarrow \hspace{1cm} (m + 1) \hspace{1cm} i^* \hspace{1cm} \cdots \)

\( z \Rightarrow z^* \)

\( \cdots \)

\( (m) \hspace{1cm} i^* \hspace{1cm} \cdots \hspace{1cm} \Rightarrow \hspace{1cm} (m) \hspace{1cm} i \hspace{1cm} \cdots \)

\( \bar{z} \Rightarrow \bar{\bar{z}} \)

\( \cdots \)

\( (m) \hspace{1cm} i^* \hspace{1cm} \cdots \hspace{1cm} \Rightarrow \hspace{1cm} (m - 1) \hspace{1cm} i \hspace{1cm} \cdots \)

\( z^* \Rightarrow \bar{z} \)

where \( \bar{z} = z + nw \).
Thus we conclude that the probability $P^{(m)} \left( z_1, \ldots, z_N \right)_{i_1, \ldots, i_N}$ is given by the following correlation on the dislocated lattice

$$F^{(m)}(z_1', \ldots, z_N', z_N^*, \ldots, z_1^*)_{i_1', \ldots, i_N', i_N^*, \ldots, i_1^*} = \sum_{\text{config}} \sum_{\text{config}}$$

If we set $z_\nu = z_\nu'$, $i_\nu = i_\nu'$ ($1 \leq \nu \leq N$) in the above expression, we can get the probability such that the spins on $N$ successive vertical spins located in the same row take the values $i_1, \ldots, i_N$.

We also define the correlations

$$F^{(m)}(\zeta_1', \ldots, \zeta_{2N})_{\iota_1, \ldots, \iota_{2N}} = \sum_{\text{config}} \sum_{\text{config}}$$

where the set of $\zeta_j$'s is a permutation of $z_1', \ldots, z_N', z_N^*, \ldots, z_1^*$, and the set of $\iota_j$'s is that of $i_1', \ldots, i_N', i_1^*, \ldots, i_N^*$. Consider the $V_{\zeta_1} \otimes \cdots \otimes V_{\zeta_{2N}}$-valued correlators

$$F^{(m)}(\zeta_1, \ldots, \zeta_{2N}) = \sum_{\iota_1, \ldots, \iota_{2N}} v_{\iota_1} \otimes \cdots \otimes v_{\iota_{2N}} F^{(m)}(\zeta_1, \ldots, \zeta_{2N})_{\iota_1, \ldots, \iota_{2N}}$$

(4.1)

where $V_{z^*} = V_{z}$ and $v_i = v_i^*$. 

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The correlators (4.1) have the $S$-matrix symmetry:

$$F^{(m)}(\cdots, \zeta_{j+1}, \zeta_j, \cdots) = S_{j,j+1}^{V_{j+1}V_j} F^{(m)}(\cdots, \zeta_j, \zeta_{j+1}, \cdots).$$

Furthermore, it satisfies

$$F^{(m\pm 1)}(\zeta_2, \cdots, \zeta_{2N}, \tilde{\zeta}) = P_{2N-1,2N} \cdots P_{1,2} F^{(m)}(\zeta_1, \zeta_2, \cdots, \zeta_{2N}).$$

In the left hand side we should take + from the double signs if $\zeta_1 = \zeta'_\nu, (1 \leq \exists \nu \leq N)$, - if $\zeta_1 = z^*_\mu, (1 \leq \exists \mu \leq N)$. The symbol $\tilde{\zeta}$ means $z + nw$ if $\zeta = z$, $(z + nw)^*$ if $\zeta = z^*$. By setting $n = 2$ eqs. (4.2) and (4.3) give those of the eight-vertex model [11]. In [1] these are two of axioms that the form factors of integrable massive models should obey. As for the derivation of them, see [1] [11].

From (4.2) and (4.3) we obtain the difference equation

$$F^{(m\pm 1)}(\cdots, \tilde{\zeta}_j, \cdots) = (S_{j,j+1}^{V_{j+1}V_j})^{-1} \cdots (S_{2N-1,2N}^{V_{2N}V_{2N-1}})^{-1} \times \times S_{1,j}^{V_{1}V_j} \cdots S_{j-1,j}^{V_{j-1}V_j} F^{(m)}(\cdots, \zeta_j, \cdots),$$

where the interpretation of the left hand side is the same as that of (4.3).

## 5 Spontaneous Polarization

In this section let us concentrate to $F^{(m)}(z', z^*)$. Since it depends only upon $u = \exp(-\pi \sqrt{-1}(z - z'))$, we denote $F^{(m)}(z', z^*)$ by $F^{(m)}(u)$. Set

$$G^{(m)}(u) := \sum_{l=0}^{n-1} \omega^{ml} F^{(l)}(u)_{00^*}. \tag{5.1}$$

Then we have

$$\frac{G^{(m)}(uq^{-n})}{G^{(m)}(u)} = \sum_{j=0}^{n-1} \omega^{m(j-1)} (S_{j,j}^{V_{j}V_{j}})^{00^*}_{00^*}, \tag{5.2}$$

where we use the $Z_n$-symmetry $F^{(m)}(u)_{jj^*} = F^{(m+p)}(u)_{j+p(j+p)^*}$. Let us solve (5.2) under the condition $\lim_{u,q \to 0} G^{(m)}(1) = \omega^{-m}$, which follows from that $F^{(m)}(z', z^*)_{jj^*}|_{z = z'} \to \delta_j^{m+1}$ in the low temperature limit.

Using the crossing symmetry for $S$-matrix, we get

$$(S_{j,j}^{V_{j}V_{j}})^{00^*}_{00^*} = (S_{j,j}^{V_{j}V_{j}})^{00}_{0j} = \frac{u^{(n-2)/n} (t^{2n}, t^{2n})_\infty (q^2, t^2)_\infty (t^2 q^{-2}, t^2)_\infty}{\tilde{\kappa}(u) (t^2, t^2)_\infty (u^2, t^{2n})_\infty (t^{2n} u^{-2}, t^{2n})_\infty} (t^{2} q^{-2}, t^2)_\infty (t^2 q^{-2}, t^2)_\infty \theta^{(j)}(z - z' + w).$$
It follows from the transformation properties for the theta functions,
\[
\sum_{j=0}^{n-1} \omega^{m(j-1)} \theta^{(j)}(z - z' + w) = n \omega^{-m} h((z - m)/n + w) \prod_{l \neq m} h((-z + l)/n) \\
= u^{-(n-2)/n} \prod_{l \neq m} h(l/n) \times (\omega^{-m} q^2 u^{-2/n}; t^2)_{\infty} (t^2 \omega^{-m} u^{-2/n}; t^2)_{\infty} \times \\
\times (\omega^{-m} q^2 u^{-2/n}; t^2)_{\infty} (t^2 \omega^{-m} u^{-2/n}; t^2)_{\infty},
\]
where we use (1.2) at the second equality. Therefore (5.2) reduces to
\[
\frac{G^m(uq^{-n})}{G^m(u)} = \frac{1}{\tilde{\kappa}(u)} \frac{(\omega^{-m} q^2 u^{-2/n}; t^2)_{\infty} (t^2 \omega^{-m} u^{-2/n}; t^2)_{\infty}}{(\omega^{-m} u^{2/n}; t^2)_{\infty} (t^2 \omega^{-m} u^{-2/n}; t^2)_{\infty}}. \tag{5.3}
\]
Note that the relation
\[
\varphi(uq^{-n}) = \frac{1}{\tilde{\kappa}(u)} \varphi(u),
\]
where
\[
\varphi(u) := g(uq^{n/2}) g(u^{-1} q^{n/2}), \quad g(u) := \frac{(q^{3n} u^{-2}; t^2, q^{2n} q^{2n})_{\infty} (t^2 q^{3n} u^{-2}; t^2, q^{2n} q^{2n})_{\infty}}{(q^{2n+1} u^2; t^2, q^{2n} q^{2n})_{\infty} (t^2 q^{2n+1} u^2; t^2, q^{2n} q^{2n})_{\infty}}.
\]
Then we obtain
\[
G^{(m)}(u) = \omega^{-m} \varphi(u) \frac{(q^2; q^2)_{\infty} (t^2 \omega^m u^{-2/n}; t^2)_{\infty} (t^2 \omega^{-m} u^{-2/n}; t^2)_{\infty}}{\varphi(1) (t^2; t^2)_{\infty} (q^2 \omega^m u^{2/n}; q^2)_{\infty} (q^2 \omega^{-m} u^{-2/n}; q^2)_{\infty}}. \tag{5.4}
\]
The uniqueness of this solution is ensured under the assumption of the analyticity.

If we define \( E^{(m)}(u) = \sum_{j=0}^{n-1} \omega^j F^{(m)}(u)_{jj} \), then \( E^{(m)}(u) = \omega^m G^{(-1)}(u) \). Since it is the expectation value of \( g \) defined in (2.2), the polarization of this model is given as follows:
\[
\langle g \rangle^{(m)} = E^{(m)}(u)|_{u=1} = \omega^{m+1} \frac{(q^2; q^2)_{\infty} (t^2 \omega^1; t^2)_{\infty} (t^2 \omega^{-1}; t^2)_{\infty}}{(t^2; t^2)_{\infty} (q^2 \omega^1; q^2)_{\infty} (q^2 \omega^{-1}; q^2)_{\infty}}. \tag{5.5}
\]
It reproduces the polarization of the eight-vertex model conjectured by Baxter and Kelland \[18\] and confirmed in \[11\] when \( n = 2 \). Taking the limit \( t \to 0 \), it gives the result of \[3\] in which the one point function of the \( \mathfrak{sl}_n \)-analog of the XXZ model is calculated. We can also obtain the expectation value of \( g^l \)
\[
\langle g^l \rangle^{(m)} = \omega^{(m+1)} \frac{(q^2; q^2)_{\infty} (t^2 \omega^l; t^2)_{\infty} (t^2 \omega^{-l}; t^2)_{\infty}}{(t^2; t^2)_{\infty} (q^2 \omega^l; q^2)_{\infty} (q^2 \omega^{-l}; q^2)_{\infty}}. \tag{5.6}
\]
6 Concluding Remarks

In this article we study the symmetry of the correlation function of the inhomogeneous $\mathbb{Z}_n$-Baxter model on the dislocated lattice, which coincide the probability of the appropriate spin configurations if one tune the value of spectral parameters and spin variables. Such correlations can be characterized as solutions of the system of difference equations. By solving the simplest one, we obtain the spontaneous polarization. This result includes various ones already obtained [18, 11, 9] as special cases.

The next problem is to calculate the $N$-point function of the $\mathbb{Z}_n$-Baxter model. In the trigonometric case the free field representation of the $q$-deformed vertex operator of the quantum affine algebra and the resultant integral formulae of $N$-point functions were obtained [6, 10]. Since the elliptic generalization of the quantum affine algebra has not yet been found, it is difficult to apply the strategy in [6, 10] to the present case. Thus it is promising to derive and solve the recursion relation of the correlations.

The critical behavior of this model is also interesting. The six-vertex model with $q^m = 1$, which is related to minimal models of conformal field theories, is the critical limit of the eight-vertex model [14]. Refs. [5, 6, 7] treat the six-vertex model and its higher spin analog in the antiferromagnetic regime ($-1 < q < 0$) because they are based on the corner transfer matrix method [14]. It is valuable to discuss the above trigonometric models with $|q| = 1$ as the critical limit of the $\mathbb{Z}_n$-Baxter model.

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