A numerical exploration of Miranda’s dynamical history

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Accepted 2013 July 30. Received 2013 July 12; in original form 2013 February 18

ABSTRACT

The Uranian satellite Miranda presents a high inclination (4° 338) and evidence of resurfacing. For the past 20 years it has been accepted that this inclination is due to the past trapping into the 3:1 resonance with Umbriel. These last years there is a renewal of interest for the Uranian system since the Hubble Space Telescope permitted the detection of an inner system of rings and small embedded satellites, their dynamics being of course ruled by the main satellites. For this reason, we here propose to revisit the long-term dynamics of Miranda, using modern tools like intensive computing facilities and new chaos indicators [Mean Exponential Growth factor of Nearby Orbits (MEGNO) and frequency map analysis]. As in the previous studies, we find the resonance responsible for the inclination of Miranda and the secondary resonances associated, likely to have stopped the rise of Miranda’s inclination at 4:5, identify with the frequency analysis tool the libration arguments of the secondary resonances involved, and show in particular that capture into a 3:1 secondary resonance and subsequent capture into a 2:1 secondary resonance may have disrupted the primary resonance with an inclination of Miranda of 4° 395.

Key words: celestial mechanics – planets and satellites: dynamical evolution and stability – planets and satellites: individual: Miranda.

1 INTRODUCTION

In the 1980s, the Voyager 2 spacecraft gave us better knowledge of Uranus and its satellites (see e.g. Smith et al. 1986). It revealed in particular that Miranda and Ariel have been resurfaced. Moreover, orbital models compared to astrometric observations showed that Miranda had a significant inclination $i_M$ of the order of 4° 338 according to GUST86¹ (Laskar 1986). These facts induced several dynamical studies in the late 1980s and early 1990s (Dermott, Malhotra & Murray 1988; Malhotra 1988; Tittemore & Wisdom 1989; Henrard & Sato 1990; Malhotra 1990; Tittemore & Wisdom 1990) showing that the current inclination of Miranda is probably due to a former 3:1 resonance with Umbriel. Indeed, tidal interactions with Uranus are supposed to push the satellites outwards, meeting orbital resonances. Once the system is trapped into this 3:1 resonance, the inclination of Miranda is pumped and the amplitude of libration of the resonant argument rises as the trajectory meets several secondary resonances. The capture in one of these resonances leads the trajectory to the edge of the primary resonance, involving the exit of the latter. Eventually, the inclination of Miranda ceases to be pumped and the two satellites resume tidal migration independently of each other. This scenario of evolution has been intensively studied by the above authors. With this work, we confirm the ideas developed in the past and introduce modern numerical tools to update the problem.

The reason is that there is a renewal of interest over the last few years for the Uranian system. First, the Hubble Space Telescope allowed the discovery of a whole system of rings and inner satellites (see e.g. Showalter & Lissauer 2006), whose dynamics is of course widely influenced by the main satellites. These inner satellites present interesting dynamical configurations and mysteries, for instance, the poorly understood dynamics of Mab (Kumar, de Pater & Showalter 2011) or the instability of Cupid, Belinda, Cressida and Desdemona on a time-scale of $10^7–10^8$ yr (French & Showalter 2012). In particular, Cressida and Desdemona are close to a 3:1 mean motion resonance with Miranda (Duncan & Lissauer 1996). So, their dynamical history cannot be studied without considering the history of the main satellites. Secondly, a new scenario has recently been proposed by Boué & Laskar (2010) to explain the huge obliquity of Uranus ($\approx$98°). This scenario involves a former giant satellite, whose gravitational torque was strong enough to tilt the planet (Morbidelli et al. 2012). We also mention the work of Deienno et al. (2011) showing that the presence of the current main satellites is consistent with the migration of Uranus as predicted by the Nice model. Finally, Uranus and its satellites are the target of the proposal of the space mission Uranus Pathfinder (Arridge et al. 2012). This mission could reveal interesting surface features and physical properties of the main Uranus satellites, as Cassini did for the icy moons of Saturn. Thus, theoretical studies are required to...

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¹ General Uranus Satellite Theory.
identify the bodies who are the most likely to present evidence of a past and/or a current activity. For all these reasons, we propose to revisit the dynamical history of the main Uranian satellites with powerful numerical means and tools, such as frequency analysis (Laskar 1993) and the Mean Exponential Growth factor of Nearby Orbits (MEGNO) chaos indicator (Cincotta & Simó 2000).

This article is split into the following sections. First, the key features of the system are summarized (cf. Section 2). We present the system and the 3:1 mean motion resonance between Miranda and Umbriel that the system should have encountered in its history. The details of the initial conditions used throughout this study are given in order to group the information needed to redo the numerical analysis. Secondly, based primarily on the work of Tittemore & Wisdom (1988) and Malhotra (1990), we present the full Hamiltonian (cf. Section 3) and its averaged form (cf. Section 4). These two sections remind the description of the system via Hamiltonian formalism. The introduction of canonical variables as Delaunay variables allows us to develop a perturbative theory and to obtain a Hamiltonian with two degrees of freedom (angle-action) by averaging over the fast angles. We also introduce the equations related to the tidal effects in the two models.

We then present the new numerical tools implemented to study the system. The numerical integrations of the full system on a sufficient large time-scale were a major problem in the previous studies. We perform numerical integrations over 1 Myr on the system of Uranus with its five main satellites and confirm previous results. We use numerical tools like the chaos detector MEGNO to represent the global dynamics of the system (cf. Section 5). We also improve the resolution and the details of the maps by studying the variations of the orbital elements involved in the mean motion resonance. These scales are introduced in Section 5 and the results on the system are summarized in Section 6. These new tools allow us to extend previous studies by the introduction of new visualizations of the phase plane of the considered problem. We show that the combination of numerical methods like chaos maps and frequency analysis (cf. Section 6) allows us to detect some particular behaviours of the system. We find two regions surrounding the centre of libration where two secondary resonances are superimposed. Thanks to the frequency analysis, the detailed study of the secondary resonant arguments shows that a 2:1 secondary resonance may disrupt the primary resonance with an inclination of Miranda of 4.395. This is part of our conclusions and perspectives presented in the last section (cf. Section 7).

2 KEY FEATURES OF THE SYSTEM

The system of Uranus has five main satellites, from closest to farthest with respect to Uranus: Miranda, Ariel, Umbriel, Titania and Oberon. Although these satellites are not currently locked into orbital resonances, there are many clues indicating probable passages through mean motion resonances in the past. We observe the important resurfacing of Miranda and Ariel and some abnormalities in the current orbital elements. In view of these orbital elements of the satellites, in particular the high inclination of Miranda (cf. Table 1), we analyse a mean motion resonance acting on the inclinations. The first one encountered by the system in the past is the 3:1 mean motion resonance between Miranda and Umbriel defined by six possible resonant arguments:

\[
\begin{align*}
\dot{\theta}_1 &= \lambda_M - 3\lambda_U + 2\Omega_M, \\
\dot{\theta}_2 &= \lambda_M - 3\lambda_U + \Omega_M + \Omega_U, \\
\dot{\theta}_3 &= \lambda_M - 3\lambda_U + 2\omega_U, \\
\dot{\theta}_4 &= \lambda_M - 3\lambda_U + 2\pi_U, \\
\dot{\theta}_5 &= \lambda_M - 3\lambda_U + \pi_M + \pi_U, \\
\dot{\theta}_6 &= \lambda_M - 3\lambda_U + 2\pi_M,
\end{align*}
\]

where in the left-hand side \( \dot{\theta}_i \) are the resonant arguments for the primary resonances with \( \lambda \), the mean longitudes, \( \Omega \), the ascending nodes and \( \pi \), the pericentres. The indices \( M \) and \( U \) stand, respectively, for Miranda and Umbriel. The right-hand side is the type of the resonance and corresponds to the first non-zero term associated with the cosine of the angle \( \theta_i \) in the perturbative potential (cf. equation 6 in the following section). Fig. 1 locates the six primary resonances considering the six different periods versus the semimajor axis of Miranda. These periods have been computed analytically. The frequency of the angle at the exact resonance is given by

\[
\dot{\theta} = n_M - 3n_U,
\]

where \( \theta = \lambda_M - 3\lambda_U \) is the exact resonant angle for the 3:1 mean motion resonance between Miranda and Umbriel and \( n_M \) and \( n_U \) are the mean motions of the two satellites. This frequency is corrected by the precession rates of the nodes or pericentres due to the oblateness of Uranus given by the expressions (e.g. Murray & Dermott 1999)

\[
\begin{align*}
\dot{\pi}_i &= n_i \left[ J_2 \left( \frac{R_p}{a_i} \right)^2 \right] + O(e_i^2, I_i^2, J_i), \\
\dot{\Omega}_i &= -n_i \left[ J_2 \left( \frac{R_p}{a_i} \right)^2 \right] + O(e_i^2, I_i^2, J_i),
\end{align*}
\]

where \( J_i \) gives the effect of the oblateness of the planet and \( R_p \) is the radius of Uranus. The periods plotted in Fig. 1 are derived from these expressions. These six resonances are not well separated because the oblateness of Uranus is rather small (cf. Table 3) which results in that the isolated resonance theory is only applicable for

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**Table 1.** Mean orbital elements of the five main satellites (Laskar & Jacobson 1987): \( a \) is the semimajor axis, \( e \) is the eccentricity, \( \omega \) is the pericentre, \( M \) is the mean anomaly, \( I \) is the inclination, \( \Omega \) is the ascending node and \( \pi \) is the mean motion. The variables \( P \) and \( P_\Omega \) stand for the orbital and the node periods, respectively.

| Satellites | \( a \) (km) | \( e \) | \( \omega \) (°) | \( M \) | \( I \) | \( \Omega \) | \( \pi \) (deg/d) | \( P \) (yr) | \( P_\Omega \) (yr) |
|------------|--------------|-------|-----------------|-------|------|-------|--------------|---------|--------|
| Miranda    | 129 900      | 0.0013| 68.312          | 311.330| 4.338| 326.438| 254.6906576 | 2.520   | 17.727 |
| Ariel      | 190 900      | 0.0012| 115.349         | 39.481 | 0.041| 22.394| 142.8356579 | 4.144   | 57.248 |
| Umbriel    | 266 600      | 0.0039| 84.709          | 12.469 | 0.128| 33.485| 86.8688879  | 8.706   | 126.951|
| Titania    | 436 300      | 0.0011| 284.400         | 24.614 | 0.079| 99.771| 41.3514246  | 13.46   | 195.369|
| Oberon     | 583 500      | 0.0014| 104.400         | 283.088 | 0.068| 279.771| 26.7394888  | 1.413   | 195.37 |

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Table 2. Physical parameters and corresponding uncertainties of the five main satellites: $\mathcal{G}M$ is given by Jacobson (2007) and the mean radius of the satellites $R$ by Thomas (1988).

| Satellite | $\mathcal{G}M$ (km$^3$ m$^{-2}$) | $R$ (km) |
|----------|-------------------------------|----------|
| Miranda  | $4.4 \pm 0.4$                | $235.8 \pm 0.7$ |
| Ariel    | $86.4 \pm 5.0$               | $578.9 \pm 0.6$ |
| Umbriel  | $81.5 \pm 5.0$               | $584.7 \pm 2.8$ |
| Titania  | $228.2 \pm 5.0$              | $788.9 \pm 1.8$ |
| Oberon   | $192.4 \pm 7.0$              | $761.4 \pm 2.6$ |

small eccentricities and inclinations (Dermott et al. 1988; Tittemore & Wisdom 1988; Malhotra 1990). When the inclination of Miranda increases, this classical theory breaks down. It has been shown that the exit of the resonance at high inclination is due to a commensurability between the libration frequency of the resonant argument and the circulation frequency of a close primary resonance, that is, in other words, due to the passage in a secondary resonance zone (Malhotra 1990; Tittemore & Wisdom 1990). Thus, assuming that the resonance is isolated does not make sense.

Our main purpose being the study of the high inclination of Miranda, we focus on the primary resonance of type $I_2$. The mixed primary resonance $I_3[I_2]$ implies also the rise of the inclination of Umbriel, which is currently in the same order of inclination as of the three other satellites Ariel, Titania and Oberon. For this reason, we do not focus on this type of resonance.

For the numerical analysis of the system, we use as initial conditions the values given by the JPL. All the parameters used are given in Table 1 for the mean orbital elements of the satellites, in Table 2 for the physical parameters of the satellites and in Table 3 for the physical parameters of Uranus.

The initial conditions in our numerical experiments are the current values, except for the inclination of the satellite Miranda and the ratio of semimajor axes of the two satellites Miranda and Umbriel. Considering tidal evolution, we modify these elements to reproduce the capture in the 3:1 mean motion resonance and to study the impact on the system. Since Miranda is the closest main satellite of Uranus, we also use its periods to determine the integration steps. All these points are detailed in each numerical experiment presented in the following sections.

3 P1: FULL PROBLEM

The full problem, i.e. the $N$-body problem with the gravitational perturbations of each body in the system, the effect of the oblateness of the planet and the tidal effect on the semimajor axes and eccentricities, is described in this section and will be denoted by P1 in subsequent sections. We use a planetocentric reference frame, and consider the perturbations of $N$ satellites seen as point masses, and the spherical harmonics $J_2$ and $J_4$ of the gravity field of Uranus. In this framework, the equations of the problem are

$$\mathbf{\ddot{r}}_i = -\mathcal{G}(M + m_i) \frac{\mathbf{r}_i}{r_i^3} + \sum_{j=1, j \neq i}^{N} \mathcal{G}m_j \left( \frac{r_j - r_i}{r_{ij}^3} - \frac{r_i}{r_i^3} \right) + \mathcal{G}M \nabla_i U_i \, ,$$

with

$$U_i = -\sum_{n=1}^{2} R_{2n}^{R_{2n}} J_2^{2n} P_{2n}(\sin \phi_i) \, ,$$

$\mathcal{G}$ being the gravitational constant, $R_e$ the equatorial radius of the planet, $\phi_i$ the latitude of the satellite $i$ in a frame connected to Uranus and $P_{2n}$ the classical Legendre polynomial. The gravitational potential (6) only considers the known spherical harmonics $J_2$ and $J_4$ for the Uranian system (cf. Table 3).

By introducing Jacobian coordinates, we write the usual Hamiltonian to the first order on satellite masses (Tittemore & Wisdom 1988):

$$\mathcal{H} = \sum_{i=1}^{N} \frac{\mathcal{G}Mm_i}{2a_i} \left[ 1 + \sum_{n=1}^{2} J_{2n} \left( \frac{R_e}{a_i} \right)^{2n} P_{2n}(\sin \phi_i) \right] - \mathcal{R} \, ,$$

where $a_i$ is the semimajor axis of the satellite $i$ and $\mathcal{R}$ the disturbing function expressed as (e.g. Champenois 1998)

$$\mathcal{R} = -\frac{1}{a_j} \mathcal{G}m_j \left( \frac{a_j}{r_{ij}} - a_i \frac{r_i}{r_j} \right)$$

for the external perturbation by a satellite $j$ on a satellite $i$. This complete Hamiltonian (7) considers the mutual gravitational interactions between the $N$ satellites of the system as well as the oblateness of Uranus.

As a dissipation effect, we add the tidal effect on the eccentricities and semimajor axes of the satellite $i$ via Kaula formulations (see...
\[ \frac{d\theta_i}{dt} = 3 \frac{k_i^2 n_i m_i R_i^5}{Q_i a_i^7 M} \left( 1 + \frac{51}{4} \epsilon_i^2 \right) - \frac{21 k_i n_i M R_i^5}{Q_i a_i^7 m_i} \epsilon_i^2, \]  
\tag{9} \]

\[ \frac{de_i}{dt} = \frac{57}{8} \frac{k_i^2 n_i m_i}{Q_i a_i^5} \left( \frac{R_i}{a_i} \right)^5 e_i^2 - \frac{21 k_i n_i M R_i^5}{Q_i a_i^7 m_i} \left( \frac{R_i}{a_i} \right)^5 e_i, \]
\tag{10} \]

where \( R_p \) is the mean radius of Uranus (\( R_p \neq R_0 \)), and \( R_i \) and \( n_i \) represent, respectively, the mean radius and the mean motion of the satellite \( i \). We observe that these formulations depend on the Love number \( k_i \) and on the dissipation function \( Q \). These are secular equations assuming that the satellites are in synchronous rotation, as expected from their tidal despinning (Gladman et al. 1996). Moreover, due to our poor knowledge of the relevant values, the dissipation functions are assumed to be constant with respect to the tidal frequencies. Practically, the tidal effect is added on the orbital elements \( a_i \) and \( e_i \), which are thereafter converted into the Cartesian coordinates before the integration of equations (5).

### 4 P2: AVERAGED PROBLEM

In this section, we introduce an averaged version of the problem P1 that will henceforth be denoted by P2. To reduce the computation time and to facilitate an analytical perturbative theory of the problem, following Tittemore & Wisdom (1988), we perform an analytical averaging process on the short period angles \( \lambda_i \). As at the lowest order there is no coupling between the resonances in eccentricity and in inclination, the averaged model considers a circular-inclined approximation for the inclination resonance (Tittemore & Wisdom 1989) which allows us to write the Hamiltonian (7) like

\[ H = H_{kep} + H_{db} + H_{es} + H_{osc}, \]  
\tag{11} \]

splitting into the Keplerian, the effects of the oblateness of the planet parts, the resonant Hamiltonian and the secular one. We finally obtain a Hamiltonian with two degrees of freedom (\( J_M, J_U, \theta_M, \theta_U \)), these variables being canonically conjugate (Malhotra 1990):

\[ H = v_1 J_M + v_2 J_U - \beta (J_M + J_U)^2 \]
\[ + 2\epsilon_4 (J_M J_U)^{1/2} \cos \left( \frac{\theta_M - \theta_U}{2} \right) + 2\epsilon_5 J_M \cos \theta_M \]
\[ + 2\epsilon_6 (J_M J_U)^{1/2} \cos \left( \frac{\theta_M + \theta_U}{2} \right) + 2\epsilon_7 J_U \cos \theta_U, \]  
\tag{12} \]

where

\[ \theta_M = \theta_1, \]  
\tag{13} \]

\[ \theta_U = \theta_5, \]  
\tag{14} \]

\[ J_M = \frac{1}{2} m_M [G M a_M]^{1/2} I_M^2, \]  
\tag{15} \]

\[ J_U = \frac{1}{2} m_U [G M a_U]^{1/2} I_U^2, \]  
\tag{16} \]

where \( \theta_i \) are the resonant angles for the 3:1 resonance between Miranda and Umbriel described in Section 2 and \( J_i \) the associated variables given to the lowest order in \( m_i \) and \( J_i \). Following Malhotra (1988), we define the parameters depending on the problem:

\[ v_1 = v_0 + \Delta v_1, \quad v_2 = v_0 + \Delta v_2, \]  
\tag{17} \]

with

\[ v_0 = \frac{1}{2} \left[ 3n_U \left[ 1 + 3 J_2 \left( \frac{R_U}{a_U} \right)^2 + \frac{m_M}{M} \left( 1 + \alpha \frac{d}{d\alpha} \right) b_1^{(0)}(\alpha) \right] \right. \]
\[ - n_M \left[ 1 + 3 J_2 \left( \frac{R_M}{a_M} \right)^2 - \frac{m_U}{M} \alpha \frac{d}{d\alpha} b_1^{(0)}(\alpha) \right] \right], \]  
\tag{18} \]

and

\[ \Delta v_1 = \frac{3}{2} J_2 \left( \frac{R_U}{a_U} \right)^2 + \frac{m_U}{4 M} \alpha \beta_1^{(1)}(\alpha) \right] n_M, \]  
\tag{19} \]

\[ \Delta v_2 = \frac{3}{2} J_2 \left( \frac{R_U}{a_U} \right)^2 + \frac{m_M}{4 M} \alpha \beta_1^{(1)}(\alpha) \right] n_U, \]  
\tag{20} \]

\[ \beta = \frac{3}{8} \left( 1 + 9 \frac{m_M/m_U}{\alpha} \right) \frac{1}{m_M M}. \]  
\tag{21} \]

The expression \( v_0 = 0 \) corresponds to the exact 3:1 commensurability between the mean motions of Miranda and Umbriel, and \( \Delta v_i \) are the corrections of the secular precession rates of the nodes \( \Omega_i \) on the resonant combination of the mean motions of the satellites.

The terms \( b_1^{(i)}(\alpha) \) are the Laplace coefficients, with \( \alpha = \Omega_M/\Omega_U \). Their expressions have been numerically computed from the integral given by, e.g. Murray & Dermott (1999):

\[ b_1^{(i)}(\alpha) = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos j \psi d\psi}{(1 - 2\alpha \cos \psi + \alpha^2)^{3/2}}. \]  
\tag{22} \]

We also define the expressions of \( \epsilon_i \), related to the inclination resonance \( I_M^2 \) (Malhotra 1990):

\[ \epsilon_1 = -n_M \frac{m_U}{M} \alpha f_1(\alpha), \]  
\tag{23} \]

\[ \epsilon_2 = -n_M \left[ \frac{m_M m_U}{M M} \right]^{1/2} \alpha^{3/4} f_2(\alpha), \]  
\tag{24} \]

\[ \epsilon_3 = -n_M \frac{m_M}{M} \alpha^{3/2} f_3(\alpha), \]  
\tag{25} \]

| Physical parameters and corresponding uncertainties of Uranus: | \( \frac{GM}{(\text{km}^3 \text{s}^{-2})} \) | \( R_e \) (km) | \( R_p \) (km) | \( J_2 \times 10^6 \) | \( J_4 \times 10^6 \) |
|---|---|---|---|---|---|
| Uranus | $5793964 \pm 6$ | $26200$ | $25362$ | $3341.29 \pm 0.72$ | $-30.44 \pm 1.02$ |

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Table 3. Physical parameters and corresponding uncertainties of Uranus: \( GM \) is given by Jacobson (2007). The parameters \( R_e \) and \( R_p \) stand for the equatorial (Jacobson 2007) and the mean radius (Seidelman et al. 2007) of the planet, respectively, and \( J_2 \) and \( J_4 \) for the spherical harmonics associated with the oblateness of the planet (Jacobson 2007).
\[ \epsilon_4 = -n_M \left( \frac{m_M m_u}{M M_u} \right)^{1/2} \alpha^{5/4} f_4(\alpha) \]  

(26)

with the following expressions for \( f_i(\alpha) \):

\[ f_1(\alpha) = \frac{1}{8} \alpha b_{0i}^{(2)}(\alpha) = 0.0828, \]
\[ f_2(\alpha) = -\frac{1}{4} \alpha b_{0i}^{(2)}(\alpha) = -0.1656, \]
\[ f_3(\alpha) = \frac{1}{8} \alpha b_{0i}^{(2)}(\alpha) = 0.0828, \]
\[ f_4(\alpha) = -\frac{1}{4} \alpha b_{0i}^{(2)}(\alpha) = 0.2845, \]

where we neglect the indirect perturbation of order \( m_M/M \). The Laplace coefficients are evaluated at the exact 3:1 mean motion resonance (\( \alpha = 3^{-2/3} \)).

To follow a trajectory through the 3:1 resonance in inclination with the tidal effects, we also consider Kaula’s formulations (10) where we assume a circular approximation. We obtain a tidal perturbation on the semimajor axis of both satellites, Miranda and Umbriel, expressed as

\[ \frac{d\alpha}{dr} = 3 \frac{k_i^2 n_m R_i^5}{Q_i \alpha^4 M}. \]

(27)

where the dissipation factor of the satellite \( (k_i/Q_i) \) is neglected in the case of the averaged problem. Because of the circular approximation, we have a tidal perturbation only on the semimajor axes of the satellites.

## 5 NUMERICAL METHODS

A major obstacle in previous studies was the numerical integrations of the full equations of P1 on a long enough time-scale. The rate of the tidal evolution of the system being rather slow, the CPU time needed is quite high.

Using new computing tools,\(^3\) we have integrated numerically the equations of motion of the \( N \)-body problem in Cartesian coordinates (cf. equation 5) with the Adams–Bashforth–Moulton 10th-order predictor–corrector integrator (Hairer, Norsett & Wanner 1993), and the outputs have been successfully compared to those of the well-known software SWIFT (Levison & Duncan 1994) on the Uranian system.

In this section, we introduce several methods used to obtain our numerical results. We first present the chaos indicator called MEGNO (Cincotta & Simó 2000). We perform a set of numerical integrations on the problems P1 and P2 to represent the phase plane of the problem with a chaos map.

Secondly, we study the variations of the orbital elements of the satellite Miranda. According to the type of variations, different details of the eye of the 3:1 resonance between Miranda and Umbriel, which are not visible with the chaos indicator, appear. More information follows in Section 5.2.

\(^3\)The computations were performed on an HPC cluster which powers the Interuniversity Scientific Computing Facility centre (ISCF: http://www.ptci.unamur.be) located at the University of Namur (Belgium).

### 5.1 Chaos indicator

To study the stability and the structure of the phase space of our system, we implement the chaos indicator called MEGNO elaborated by Cincotta & Simó (2000) and used by different authors (i.e. Goździewski et al. 2001; Cincotta, Giordano & Simó 2003; Valk et al. 2009; Delsate 2011). The indicator MEGNO is defined by

\[ Y(t) = \frac{2}{t} \int_0^t \delta(t') \frac{d}{dt'} Y(t') \, dt', \]

(28)

where \( \delta = ||X|| \) \( X \) being the tangent vector \((\delta p, \delta q)\) of the vector angle-action \((p, q)\) and \( \delta \) the time derivative of \( \delta \). Given equation (28), the running time average of the MEGNO, \( \overline{Y}(t) \), is defined by the following integral:

\[ \overline{Y}(t) = \frac{1}{t} \int_0^t Y(t') \, dt'. \]

(29)

where, as explained by Cincotta et al. (2003), \( \overline{Y}(t) \) gives quickly the behaviour of the system, i.e.

\[ \lim_{t \to \infty} \overline{Y}(t) = 0: \text{stable and periodic trajectory}, \]
\[ \lim_{t \to \infty} \frac{\overline{Y}(t)}{t} > 0: \text{chaotic trajectory}. \]

Goździewski et al. (2001) propose an efficient way to compute the MEGNO, by adding two differential equations to the system:

\[ \frac{dy}{dt} = \frac{\delta \cdot \delta}{\delta \cdot \delta}, \]

(30)
\[ \frac{dw}{dt} = \frac{y(t)}{t}, \]

(31)

and by computing, at each step of the integration, the MEGNO and its averaged form by

\[ Y(t) = \frac{2}{t} \frac{y(t)}{t}, \]

(32)
\[ \overline{Y}(t) = \frac{w(t)}{t}. \]

(33)

In a first step, integrating equations (30) and (31), we compute the indicator using the definitions (32) and (33) on the problem P1 with Uranus and its five main satellites perturbing each other. In a second step, again on the problem P1, we compute the indicator by considering a three-body problem consisting of Uranus, Miranda and Umbriel, the latter two satellites being involved in the mean motion resonance studied. In both cases, the initial tangent vector is chosen randomly. We notice that the main points of the global dynamics are similar in the two simulations and conclude that considering a three-body problem is sufficient to represent the evolution of the system during the passage in the 3:1 resonance.

To obtain the chaos map (cf. Fig. 2) in the plane semimajor axis \( d_M \) versus resonant argument \( \theta_M \), we perform \( 10^5 \) numerical integrations over 1500 yr, each one associated with an initial condition of the phase plane. This time-span corresponds approximately to 100 regression periods of the node of Miranda \( \Delta_{25} \), which is a characteristic period of the system. We fix the integration step at around 1/80th of the smallest period of the five main satellites (cf.
the change. We base our new colour scales, named orbital element variations, on these previous facts and define the following colour scales for the map:

(i) The variation of the semimajor axis of Miranda, $\delta_a$ (km),
(ii) The variation of the inclination of Miranda, $\delta_i$ (°),
(iii) the variation of the eccentricity of Miranda, $\delta_e$.

The study of the variations of the orbital elements allows the visualization of structures of the phase space invisible with the chaos indicator MEGNO: these structures are not chaotic and/or are too barely perceptible because the variations are very small. We also have the association of the structures with the orbital element involved. To illustrate this, we consider the full problem P1 and represent the phase plane of the 3:1 resonance between Miranda and Umbriel (cf. Fig. 3) obtained by the same integration process as shown in Fig. 2, but with the inclination (cf. Fig. 3a) and the eccentricity (cf. Fig. 3b) variations for the colour scale. What we observe is as follows: we have the eye of the resonance with the inclination variation for the colour scale, whereas it is almost erased in the eccentricity one. Moreover, with the eccentricity variation for the colour scale, we have another structure that does not appear in the inclination variation map: this is another primary resonance involving the pericentre of Umbriel $\sigma_U$, the primary resonance $\theta_4$ (cf. Fig. 1). The semimajor axis variation map brings together the structures of the inclination and the eccentricity variation maps, giving a global visualization of the phase space plane (cf. Fig. 7).

![Figure 2. Chaos map in the plane semimajor axis $a_M$ versus resonant argument $\theta_M$ resulting from the numerical integrations of the three-body problem between Uranus, Miranda and Umbriel with the Adams–Bashforth–Moulton integrator over 1500 yr. The integration step is set to 1/80 d. The initial conditions are the current ones (cf. Tables 1–3) except for the mean anomaly $M_M$, the semimajor axis $a_M$ and the inclination $I_M$ of the satellite Miranda. The first two variables are set, respectively, in the range $[0°–360°]$ and $[127 845–127 895 \text{ km}]$. The initial inclination is $4°/8$.](https://academic.oup.com/mnras/article-abstract/435/2/1776/1046411)

Figure 2. Chaos map in the plane semimajor axis $a_M$ versus resonant argument $\theta_M$ resulting from the numerical integrations of the three-body problem between Uranus, Miranda and Umbriel with the Adams–Bashforth–Moulton integrator over 1500 yr. The integration step is set to 1/80 d. The initial conditions are the current ones (cf. Tables 1–3) except for the mean anomaly $M_M$, the semimajor axis $a_M$ and the inclination $I_M$ of the satellite Miranda. The first two variables are set, respectively, in the range $[0°–360°]$ and $[127 845–127 895 \text{ km}]$. The initial inclination is $4°/8$. 

Table 1. For each numerical simulation, we compute the solutions of the equations of motion of the three-body problem P1, without adding the tidal perturbation, and the averaged MEGNO value (33) to have the third dimension for the colour scale. The tidal effect is not considered because instead of following one trajectory in time, we select a region of initial conditions which covers the entire dynamics ($a_M \in [127 845–127 895 \text{ km}]$ and $M_M \in [0°–360°]$). We do not need the effect of tides to push the pair of satellites in the resonance. This method allows us to have a global visualization of the eye of the resonance.

In Fig. 2, we obtain the phase plane given by the averaged MEGNO value. The indicator identifies the chaotic and the stable structures of the phase plane (respectively, dark and light colours): it detects the external separatrix, but also a smaller one defining the boundary of secondary resonance zones. Indeed, we note first the wide separatrix between the libration and the circulation zones. The stable zone in the large separatrix (top right-hand side in the figure) is the following primary resonance in inclination considering the nodes of Miranda $\Omega_M$ and Umbriel $\Omega_U$, represented by the resonant argument $\theta_2$ (cf. Fig. 1). The other structures appearing in the circulation zones are the 3:1 primary resonances in eccentricities $\theta_i$, with $i = 4, 5, 6$. The visualization of all these structures in a small range of semimajor axis confirms the well-known small separation between the resonances of the system due to the small oblateness of the planet. Secondly, we distinguish another separatrix in the libration zone, suggesting the presence of secondary resonance zones (middle of the figure) that will be examined more closely in the following sections.

5.2 Orbital element variations

The second method in our numerical analysis considers the variation in orbital elements of the satellite Miranda. When the bodies pass through a resonance, they cross the separatrix between the circulation and libration zones, inducing changes in the nature of the orbit: we observe modifications in eccentricities or inclinations of the satellites involved in the resonance. In contrast, in the centre of the resonance, the variations are negligible as the orbital elements are constant. Indeed, the closer is the separatrix, the stronger is
6 RESULTS AND COMMENTS

We apply the numerical methods presented in Section 5 to the three-body problem between Uranus, Miranda and Umbriel to study the 3:1 mean motion commensurability between Miranda and Umbriel. We consider both the models P1 and P2, the latter allowing us to speed up the CPU time while maintaining the global evolution of the system.

We have two different types of results. The first one groups the trajectories that evolve in the time with a tidal dissipation on a large time-scale, typically 1 Myr. The usual problem in this case is the dissipation function $Q_2$ for the planet which is value is not known, but recent studies suggest smaller values for planet dissipation factor than ever in the case of Jupiter and Saturn. Following Lainey et al. (2009), the value of $k_3/Q = (1.102 \pm 0.203) \times 10^{-5}$ for Jupiter and according to Lainey et al. (2012), the value of $k_3/Q = (2.3 \pm 0.7) \times 10^{-4}$ for Saturn. To our knowledge, there are no similar recent studies for the approximation of the dissipation factor of Uranus. We can, however, mention the works of Gavrilov & Zharkov (1977) and Goldreich & Soter (1966) which give an upper bound with $k_3/Q = 2.08 \times 10^{-5}$ considering the migration of Miranda from the synchronous orbit to its current position over a period of 4.5 billion years. The lower bound is given by Tittermore & Wisdom (1989) with $k_3/Q = 2.67 \times 10^{-6}$ assuming the passage through the 3:1 resonance. We implement a set of tests with different values for $k_3/Q$ and we note similar behaviours on different time-scales. We choose the value of $5.2 \times 10^{-5}$ for the planet because it produces the rise of inclination of Miranda significantly faster than the current one, allowing us to study the overall process in a moderate CPU time. The strengthening of the parameters $k_3/Q$ allows us to speed up the integrations, provided that the trajectories remain adiabatic. Malhotra (1991) provides good comments on the adiabatic evolution depending on the characteristic dynamical time-scale $\tau$. We write the variable $\tau$ due to the effects of the tidal dissipation (Malhotra 1991):

$$\tau = \left( \frac{\omega}{\dot{\omega}} \right)^{-1} \approx 10^{12} \text{yr}, \quad (34)$$

where $\omega$ is the exact resonance frequency and $\dot{\omega}$ its derivative. The variable $\tau$ characterizes the evolution of the resonant frequency. We observe that the periods of the resonant arguments are smaller than the characteristic time-scale (cf. Fig. 1) and conclude that the evolution remains adiabatic with our choice of $k_3/Q$.

Secondly, we represent the global evolution via maps depending on a colour-scale MEGNO to represent the separatrix or depending on orbital element variations, principally the inclination one, to study the resonance in more detail. In this case, as a reminder, we represent the phase plane semimajor axis versus resonant argument determined by a set of numerical integrations over 1 500 yr without taking into account the tidal effect. The detailed phase plane shows precisely two regions surrounding the libration centre which are identified as secondary resonance zones with the frequency analysis tool.

6.1 Type of results 1: the tracking of a trajectory

Let us consider the tracking of a trajectory in its capture in the 3:1 resonance between Miranda and Umbriel. The equations of motion derive from the model P1 (cf. Section 3). At the beginning of the simulation, the initial orbital elements are set to the current ones (cf. Table 1) except for the semimajor axes of Miranda and Umbriel and the inclination of Miranda which are fixed at lower values. We add the tidal effect on the semimajor axes and eccentricities of Miranda and Umbriel following Kaula’s formulations (10). The ratios $k_3/Q$ are fixed to $5.2 \times 10^{-5}$ for Uranus and $10^{-4}$ for the satellites. We also assume a dissipation factor $Q$ smaller for Miranda than Umbriel as Miranda shows an important resurfacing (Smith et al. 1986) suggesting an important past thermal history, but this kind of aspect will be studied in a following paper.

The result of the numerical integration gives us the evolution of the resonant argument $\theta_M$ during the capture in the resonance. In Fig. 4, the tidal evolution of the resonant argument is plotted versus the semimajor axis ratio $a_M/a_U$ versus the rise of inclination of Miranda $I_M$. We observe first a circulation of the angle $\theta_M$. The capture in the resonance $I_M^2$ occurs when the commensurability 1/3 between the semimajor axes is approaching, corresponding to a ratio

$$\left( \frac{a_M}{a_U} \right)^{3/2} = (0.4807)^{3/2} = 0.3333.$$

Then, we enter in a libration zone and the inclination starts to increase. The trajectory is later captured in a secondary resonance zone which implies the exit of the primary resonance zone. We note a passage followed by a capture in a secondary resonance at, respectively, $I_M = 2.8$ and $I_M = 4.1$ which are the same critical points as those in Malhotra (1990).

6.2 Type of results 2 : global visualizations via phase planes

6.2.1 Averaged problem P2 versus full problem P1

Let us consider the averaged system consisting of a three-body problem involving Uranus, Miranda and Umbriel in a circular-inclined approximation described by the model P2 (cf. Section 4). To compare the quality of the averaged problem P2 versus the full problem P1, we implement the same process as explained in Section 5.2 on the two problems. The integration step for the problem P2 is 17/300 yr, i.e. 1/300th of the smallest nodal period. In our case, it is a period of 17 yr related to the node of Miranda (cf. Table 1). Fig. 5 compares two eyes of the resonance obtained by numerical integrations of the equations of motion of the complete model P1 for the first one (a) and the equations of motion of the averaged model P2 for the second one (b) with the inclination variation in degrees for the colour scale.

We observe that as a whole the two eyes are similar: we distinguish the libration and the circulation zones, and the stable zone for the mixed resonance $\theta_2$ also appears in the two models. We observe also the two zones of secondary resonances, confirming the possibility of their studies with the averaged problem P2. Obviously the details and the precision of the eye in the complete model P1 are blurred in the averaged problem P2, but the global dynamics is well represented with the approximation. We note that the colour scale is identical. It is due to the fact that the dynamics of the inclination are ruled by the resonant (or quasi-resonant out of the separatix) argument, present in the two systems (P1 and P2). To check our assumption, we use the NAFF (Numerical Analysis of the Fundamental Frequencies) algorithm based on Laskar’s original idea [see, for instance, Laskar (1993) for the method, and Laskar (2005) for the convergence proofs]. It aims at identifying the coefficients $a_k$...
Figure 4. Evolution of the resonant argument θ_M versus the semimajor axis ratio a_M/a_U versus the inclination of Miranda I_M. The initial conditions are the current values (cf. Tables 1–3) except for the semimajor axis of Umbriel, a_U = 265 200 km and the semimajor axis of Miranda, a_M = 127 400 km, which evolves with the tidal effect on the semimajor axes and on the eccentricities. The initial inclination of Miranda is 0°001. The integration step is 1/60 d. The ratio k_2/Q is 5.2 × 10^{-3} for Uranus and 10^{-4} for the satellites. We note the capture in the primary resonance when the commensurability 1/3 is approached and the consequent rise of inclination of Miranda. We also observe the disruption of the primary resonance following the capture of a secondary resonance.

and ω_k of a complex signal f(t) obtained numerically over a finite time-span [−T, T] and verifying

\[ f(t) \approx \sum_{k=1}^{n} a_k \exp(i\omega_k t), \tag{35} \]

where ω_k are real frequencies and a_k complex coefficients. Using this tool, we check that the variations of the inclination have the same period as this resonant angle in the two problems P1 and P2.

A last thing in the comparison of the two problems is the following: in Figs 5(a) and (b), we notice a shift of the eye of the resonance in the two problems. Indeed the two ranges of semimajor axis for Miranda are not the same in the two problems. This shift is due to the definitions of the variables J_M (15) and J_U (16) which imply that the semimajor axis of Miranda in the averaged problem differs by a quantity of order (\sin I_M/2)^2 from the complete one. This can be important as the inclination increases. Based on the second chapter in Malhotra (1988), we show that

\[ \sqrt{a_M} = \sqrt{a_M (1 - 3 s_M^2) + \sqrt{a_M} a^{-1} s_U^2} \]

\[ (1 - s_M^2 + 3 s_M^2 s_U^2), \tag{36} \]

where a_M is the semimajor axis of Miranda in the averaged problem and s_j = \sin I_j/2, j being M for Miranda and U for Umbriel. Fig. 6 shows the shift of the eye of the resonance with increase of the inclination of Miranda I_M, which is significant when the inclination is high.

6.2.2 Global evolution of the system

The representations in three dimensions of the phase plane (semimajor axis versus resonant argument with a colour scale) is not sufficient to represent the entire dynamics as it evolves with the rise of inclination of Miranda. We know that the chaoticity in the system increases with the inclination of Miranda (Tittemore & Wisdom 1988). As we want to present the whole dynamics, we propose to follow the evolution of the system with a successive set of maps in the semimajor axis variations for the colour scale resulting from numerical integrations of the problem P1.

Fig. 7 shows a set of six phase planes of semimajor axis a_M versus resonant argument θ_1 with the variation in semimajor axis in colour scale. The initial inclination of Miranda is set to 1°, 2°, 2°, 8°, 4°, 1°, 4°, 338 and 4°, respectively. When the inclination increases, the separatrix broadens to become a layer of chaotic motion, in particular when two resonances meet. This increase of chaos is due to the overlap of two close separatrices, a consequence of the closeness of the resonances. Fig. 9 in Moons & Henrard (1994) shows ‘the landscape’ of the problem with the location of the separatrix of the primary resonance and the centres of the secondary resonances in a particular plane. It also shows the location of the chaotic layers around the separatrix. We see that these layers become larger as the inclination of Miranda is high and we observe exactly the same feature in our maps.

The different zones of the phase planes evolve too and, in particular, the centre of libration presents different structures moving with time: we distinguish some zones of secondary resonances which appear and move with the rise of inclination. Some of them have
already been detected by Tittemore & Wisdom (1990) and extensively studied by Malhotra (1990) and Moons & Henrard (1994). As we were intrigued by the different zones in the libration centre and their evolution with the rise of inclination of Miranda, we tried to make a zoom-in of this centre. Fig. 8 shows an enlargement of the centre of the eye when the initial inclination of Miranda is set to 4.338. We clearly see three zones: the centre of libration and two other zones surrounding the centre.

6.2.3 The secondary resonance zones

Dermott et al. (1988) and Tittemore & Wisdom (1989) show that the role of secondary resonances in the 3:1 resonance is crucial: a capture into a secondary resonance can explain the escape of the primary with a high inclination for Miranda due to the chaotic layers present near the separatrix. Tittemore & Wisdom (1989) use a circular inclined model and follow one trajectory captured in the \( I^\*=2 \) primary resonance. At different critical points of the evolution, analysing surfaces of section, they show that the capture in a commensurability 1:2 between the libration frequency of \( \theta_1 \) and the circulation frequency of \( \theta_2 \) can lead to an exit of the primary resonance with a high inclination \( (\theta_1^M = 4.6) \). Malhotra (1990) identifies some secondary resonances of the problem with a simple perturbed pendulum model and shows that the capture into a 3:1 secondary resonance between the libration frequency \( \theta_1 \) and the circulation frequency of \( \theta_2 \) leads to the exit of the primary resonance at an inclination close to 4.6°.

If we go back to Fig. 8 (top panel), it clearly presents two zones surrounding the centre of libration of the primary resonance \( \theta_1 \), which we can suspect to be secondary resonance zones. To identify these zones, we implement the method described by Laskar (1990). This method has the advantage to identify definitely the frequencies of the complete problem P1 and allows us to plot the libration frequencies of the secondary resonances of the problem like in Noyelles & Vienne (2007). Let us note that the frequencies in the problem P1 are more numerous than in the averaged one P2. We therefore select an initial condition in one of the zones [symbol 'X' in Fig. 8 (top panel)] and use the frequency analysis tool to study the frequencies and their combinations of the resulting trajectory. To sum up the method, the main frequency of the oscillation of the primary resonance \( \theta_1 \) is extracted from the trajectory chosen in Fig. 8, and combined with other frequencies of the problem to reconstruct the arguments of distinct secondary resonances. The libration argument \( \theta_1 \) is written as

\[
\theta_1 = \theta_0 + A_1 \cos \Phi_1 + \text{other terms of smaller amplitudes,}
\]

where \( \theta_0 = \pi \) is the value of \( \theta_1 \) at the libration equilibrium, \( A_1 \) is the largest amplitude in the quasi-periodic expansion and

\[
\Phi_1 = \omega_1 t + \Phi_{10},
\]

where \( \Phi_{10} \) is the initial phase. The main frequency of the oscillation is \( \omega_1 \). The other two angles \( \theta_2 \) and \( \theta_3 \), considered as circulating, are simply approximated by the largest term (in amplitude) of their quasi-periodic expansion:

\[
\theta_j = A_j \cos \Phi_j + \text{other terms of smaller amplitudes,}
\]

where \( A_j \) are the amplitudes,

\[
\Phi_j = \omega_j t + \Phi_{j0},
\]
Miranda’s dynamical history

Figure 7. Phase planes of the resonance from the three-body problem Uranus–Miranda–Umbriel. The integrator, the integration step, the model and the initial conditions are the same as Fig. 2. The initial inclinations of Miranda $I_M$ are $1^\circ$, $2^\circ$, $3^\circ$, $4^\circ$, $5^\circ$ and $6^\circ$, respectively, in panels (a), (b), (c), (d), (e) and (f). The third dimension considers the variations in the semimajor axis of Miranda $a_M$ (km). As the inclination of Miranda increases, the separatrix widens, approaching the next primary resonances involving chaos by overlap. The zones of secondary resonances seem to evolve with the rise of the inclination.

where $\Phi_\beta$ are the initial phases and $\omega_j$ the frequencies of the selected terms for $j = 2, 3$. A secondary resonance is a linear combination of $\omega_1$ with $\omega_2$ or $\omega_3$ close to 0.

Fig. 8 (bottom panels) shows the result of two particular combinations of $\omega_1$, first with $\omega_2$ (left-hand panel) and secondly with $\omega_3$ (right-hand panel). We note that these two combinations are in libration at the same time, indicating a secondary resonance between the libration argument of $\theta_1$ and the circulation arguments of the primary resonances $\theta_2$ and $\theta_3$, respectively. The first one is of type 2:1 between $\omega_1$ and $\omega_2$ and has already been identified (Malhotra 1990). The second one is a secondary resonance of type 4:1, between $\omega_1$ and $\omega_3$, reported by Tittemore & Wisdom (1989). We insist on the definitive identification of these secondary resonances, thanks to the frequency analysis tool as we plot the libration arguments.

In the problem of the 3:1 resonance between Miranda and Umbriel, it is obvious that different scenarios can occur, but the most interesting one is the trajectory which gives an exit of the primary resonance at an inclination close to 4.5° for Miranda. Fig. 9 shows this particular trajectory associated with our frequency analysis results. The top panel (a) shows the evolution of the inclination of Miranda $I_M$ versus the time $t$ during the capture in the $I_{23}$ resonance: the angle $\theta_1$ is in a libration regime during the rise in inclination (cf. Fig. 9b). The inclination of Miranda value is 4.395 at the exit of the resonance. The two bottom panels show the frequency analysis results. During the capture, two combinations are successively in libration. Like in Malhotra (1990), the libration amplitude of $\theta_1$ decreases until the capture in a 3:1 secondary resonance between the libration frequency $\omega_1$ and the circulation frequency $\omega_2$ (Fig. 9b and d). At time $t \approx 420$ kyr, this secondary resonance is disrupted and the trajectory enters a chaotic zone. In Tittemore & Wisdom (1989), when the inclination of Miranda increases, they observe an enlargement of the chaotic region and the tangle of the secondary resonance islands in the separatrix at high inclination for Miranda. In Fig. 7, we also observe this enlargement and we can distinguish the secondary resonance zones in the separatrix in the last panel (f). This explains the chaotic phase in this step of evolution.

Eventually, the trajectory is captured in the 2:1 secondary resonance until the exit of the primary resonance (cf. Fig. 9b and c). If we refer to fig. 7 in Malhotra (1990), this different scenario remains consistent with the explanation obtained by the averaged model presented by the author. We invite the reader to relate to fig. 7 (p. 461) for the following explanation: at an inclination of Miranda close to 2.8°, the amplitude of libration of $\theta_1$ is small. The trajectory follows the 3:1 curve until an inclination of Miranda close to 3.4° is reached i.e. until the entrance in the chaotic zone (hatched area in the figure) where we have the disruption of the 3:1 secondary resonance. At this point, our scenario is different from Malhotra (1990). Next, our trajectory oscillates between the two secondary resonances and is finally captured in the 2:1 resonance between the libration frequency $\omega_1$ and the circulation frequency $\omega_2$ until the exit of the primary resonance $\theta_1$ at a high inclination of Miranda.
Figure 8. Zoom-in of the centre of libration from the three-body problem between Uranus, Miranda and Umbriel (top panel). The integrator, the integration step, the model and the initial conditions are the same as Fig. 2 except for the semimajor axis of Miranda $a_M \in [127,860–127,880 \text{ km}]$. The initial inclination of Miranda $I_M$ is 4°338. The symbol X in the map represents the initial condition for the trajectory analysed by frequency analysis (bottom panels). Two different combinations of frequencies seem to librate. We have the first one with the circulation frequency of $\theta_2$ (a) and the second one with the circulation argument $\theta_3$ (b).

This result shows once again the importance of the separatrix and of the extended chaotic zone associated with it: the future of a trajectory captured in the 3:1 mean motion resonance between Miranda and Umbriel is not fully determined as it depends on a chaotic passage during the evolution. The final value of the inclination of Miranda depends on the capture into a set or another of secondary resonances.

7 CONCLUSIONS AND PERSPECTIVES

In this work, we focus on the 3:1 mean motion resonance between Miranda and Umbriel and try to explain the high inclination of Miranda by studying the unaveraged and averaged equations of motion. This problem was studied by numerous authors 20 years ago, but the update of some results with new numerical tools gives new views of it. We retrieve the main results and improve the understanding of the problem with new powerful numerical methods.

The chaos detector MEGNO has never been applied in the case of the main satellites of Uranus. We show that the combination of the chaos detector and the orbital element maps brings a new visualization of the phase planes of the problem. The use of maps and frequency analysis on particular trajectories allows the detection of unusual zones in these phase planes where a set of secondary resonances are combined.

In particular, we show that a trajectory captured in the $I_M^2$ resonance can be disrupted at an inclination for Miranda of 4°395 by the 2:1 secondary resonance between the libration frequency $\omega_1$ and the circulation frequency $\omega_2$. The use of the frequency analysis tool allows the detection and the definitive determination of the type of secondary resonance inside the primary resonance. We also observe that there is a chaotic step in the evolution inside the resonance as the secondary resonance islands are located inside the separatrix when the inclination of Miranda is high. The future of the trajectory and the final value of the inclination of Miranda depend on the capture into one or another secondary resonance.

The dynamical aspects of the Uranian system are numerous and full of interest. In the future, based on an idea of Moons & Henrard (1994) we will present an analytical approach based on the angle-action formalism to model the problem and its set of secondary resonances. We will also study a combination of this dynamical model with an intern evolution of the satellites. This combination is interesting by many points, but the main one according to us is the understanding of a dynamical abnormality (high inclination of Miranda) and a geological anomaly (differentiation of Miranda) through a resonance phenomenon.

ACKNOWLEDGEMENTS

The work of EV is supported by an FNRS PhD Fellowship. The work of BN is supported by an FNRS Postdoctoral Research Fellowship. This research used resources of the Interuniversity Scientific Computing Facility located at the University of Namur,
Figure 9. An example of evolution of the inclination of Miranda $i_M$ with the complete problem P1. The top panel (a) shows the evolution of the inclination of Miranda $i_M$ versus the time $t$ during the capture in the $I_2^M$ resonance. Panel (b) is the evolution of the angle $\theta_1$ versus the time $t$. Panels (c) and (d) show the frequency analysis results for two combinations of frequency during the capture in the $I_2^M$ resonance. In panel (b), we observe the decrease of the libration amplitude of $\theta_1$ until the capture in a 3:1 secondary resonance between the libration frequency $\omega_1$ and the circulation frequency $\omega_2$ (panel d). At time $t = 420\, \text{kyr}$, this secondary resonance is disrupted. Eventually, the trajectory is captured in the 2:1 secondary resonance until the exit of the primary resonance at an inclination of Miranda of 4° 395.

Belgium, which is supported by the FRS-FNRS under convention no. 2.4617.07. The authors would like to thank Nicolas Delsate for his valuable advice and proofreading, and Françoise Rémus for an interesting discussion on the dissipation in Uranus.

REFERENCES

Arridge C. S. et al., 2012, Exp. Astron., 33, 753
Boué G., Laskar J., 2010, ApJ, 721, L44
Champenois S., 1998, PhD thesis, Paris Observatory
Cincotta P. M., Simò C., 2000, A&A, 147, 205
Cincotta P. M., Giordano C. M., Simò C., 2003, Phys. D, 182, 151
Deienno R., Yokoyama T., Nogueira E. C., Callegari N., Jr, Santos M. T., 2011, A&A, 536, A57
Delsate N., 2011, PhD thesis, Facultés Universitaires Notre-Dame de la Paix Namur
Dermott S. F., Malhotra R., Murray C. D., 1988, Icarus, 76, 295
Duncan M. J., Lissauer J. J., 1996, Icarus, 125, 1
French R. S., Showalter M. R., 2012, Icarus, 220, 911
Gavrilov S. V., Zharkov V. N., 1977, Icarus, 32, 443
Gladman B., Quinn D. D., Nicholson P., Rand R., 1996, Icarus, 122, 166
Goldreich P., Soter S., 1966, Icarus, 5, 375
Goździewski K., Bois E., Maciejewski A. J., Kisevela-Eggleton L., 2001, A&A, 378, 569
Hairer E., Nørsett S. P., Wanner G., 1993, Solving Ordinary Differential Equations I, Springer, Berlin
Henrard J., Sato M., 1990, Celest. Mech. Dyn. Astron., 47, 391
Jacobson R. A., 2007, BAAS, 39, 453
Kumar K., de Pater I., Showalter M. R., 2011, Orbital Dynamics of Mab, EPSC Abstracts, 6, EPSC-DPS2011-669-1
Lainey V., Arlot J. E., Karatekin O., Van Hoolst T., 2009, Nat, 459, 957
Laskar J., 1986, A&A, 166, 349
Laskar J., 1990, Icarus, 88, 266
Laskar J., 1993, Celest. Mech. Dyn. Astron., 56, 191
Laskar J., 2005, in Benest et al., eds, Frequency Map Analysis and Quasiperiodic Decompositions, in Hamiltonian Systems and Fourier Analysis: New Prospects for Gravitational Dynamics. Cambridge Sci. Publ., Cambridge, p. 99
Laskar J., Jacobson R. A., 1987, A&A, 188, 212
Levison H. F., Duncan M. J., 1994, Icarus, 108, 18
Malhotra R., 1988, PhD thesis, Cornell University
Malhotra R., Dermott S. F., 1990, Icarus, 85, 444
Malhotra R., 1991, Icarus, 94, 399
Moons M., Henrard J., 1994, Celest. Mech. Dyn. Astron., 59, 129
Morbidelli A., Tsiganis K., Batygin K., Crida A., Gomes R., 2012, Icarus, 219, 737
Murray C. D., Dermott S. F., 1999, Solar System Dynamics. Cambridge Univ. Press, Cambridge
Noyelles B., Vienne A., 2007, Icarus, 190, 594
Seidelmann P. K. et al., 2007, Celest. Mech. Dyn. Astron., 98, 155
Showalter M. R., Lissauer J. J., 2006, Sci, 311, 973
Smith B. A. et al., 1986, Sci, 233, 43
Thomas, 1988, Icarus, 73, 427
Tittemore W. C., Wisdom J., 1988, Icarus, 74, 172
Tittemore W. C., Wisdom J., 1989, Icarus, 78, 63
Tittemore W. C., Wisdom J., 1990, Icarus, 85, 394
Valk S., Delsate N., Lemaitre A., Carletti T., 2009, Adv. Space Res., 43, p. 1509
Yoder C. F., Peale S. J., 1981, Icarus, 47, 1

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