Lower semicontinuity of global attractors for a class of evolution equations type neural fields in a bounded domain

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Abstract

In this work we consider the nonlocal evolution equation
\[ \frac{\partial u(w,t)}{\partial t} = -u(w,t) + \int_{S^1} J(wz^{-1})f(u(z,t))dz + h, \quad h > 0 \]

which arises in models of neuronal activity, in $L^2(S^1)$, where $S^1$ denotes the unit sphere. We obtain stronger results on existence of global attractors and Lyapunov functional than the already existing in the literature. Furthermore, we prove the result, not yet known in the literature, of lower semicontinuity of global attractors with respect to connectivity function $J$.

Keywords: Neural fields; Lyapunov functional; Lower semicontinuity of attractors

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1 Introduction

We consider initially the nonlocal evolution equation proposed by Wilson and Cowan in [29], which is used to model neuronal activity, that is,

\[
\frac{\partial v(x,t)}{\partial t} = -v(x,t) + \int_{\mathbb{R}} \tilde{J}(x-y)f(v(y,t))dy + h, \quad h > 0.
\]  

(1.1)

In (1.1), \(v(x,t)\) is a real function on \(\mathbb{R} \times \mathbb{R}_+\), \(\tilde{J} \in C^1(\mathbb{R})\) is a non negative even function supported in the interval \([-1,1]\), \(f\) is a non negative nondecreasing function and \(h\) is a positive constant.

In this model, \(v(x,t)\) denotes the mean membrane potential of a patch of tissue located at position \(x\) at time \(t \geq 0\). The connection function \(\tilde{J}\) determines the coupling between the elements at position \(x\) with the element at position \(y\). The non negative nondecreasing function \(f(v)\) gives the neural firing rate, or averages rate at which spikes are generated, corresponding to an activity level \(v\). The parameter \(h\) denotes a constant external stimulus applied uniformly to the entire neural field. We say that the neurons at point \(x\) is active if \(S(x,t) > 0\), where \(S(x,t) = f(v(x,t))\) is the firing rate of a neuron at position \(x\) at time \(t\).

Proceeding as in [26], it is easy to see that the Cauchy problem for (1.1) is well posed in the space of continuous bounded functions, \(C_b(\mathbb{R})\), and that the subspace \(P_{2\tau}\) of 2\(\tau\)-periodic functions is invariant. Thus, defining \(\varphi : \mathbb{R} \to S^1\) by

\[
\varphi(x) = \exp\left(i\frac{\pi}{\tau}x\right)
\]

and, for a 2\(\tau\) periodic function, \(v\), defining \(u : S^1 \to \mathbb{R}\) by \(u(\varphi(x)) = v(x)\), and, in particular, writing

\[
J(\varphi(x)) = \tilde{J}^\tau(x),
\]

where \(\tilde{J}^\tau\) denotes the 2\(\tau\) periodic extension of the restriction of \(J\) to interval \([-\tau,\tau]\), for some
$\tau > 1$, we obtain that: a function $v(x, t)$ is a $2\tau$ periodic solution of (1.1) if and only if $u(w, t) = v(\varphi^{-1}(w), t)$ is a solution of the equation (1.2) below:

$$\frac{\partial u(w, t)}{\partial t} = -u(w, t) + J \ast (f \circ u)(w, t) + h, \quad h > 0,$$

(1.2)

where the $\ast$ above denotes convolution product in $S^1$, that is,

$$(J \ast m)(w) = \int_{S^1} J(wz^{-1})m(z)dz,$$

with $dz = \frac{r}{\pi}d\theta$, where $d\theta$ denotes integration with respect to arc length.

In the literature, there are already several works dedicated to the analysis of this model (see, for example, [1], [6], [9], [10], [13], [14], [15], [16], [22], [24], [25], [26] and [28]). Most of these works have concerned with the existence and stability of characteristic solutions, such as localized excitation (see, for example, [1], [13] and [22]) or traveling front (see, for example, [6], [9] and [10]). Also there are already some works on the global dynamics of this model, (see, for example, [15], [24], [25], [26] and [28]). However, the proof of the lower semicontinuity of global attractors is not yet known, and this proof cannot be given by conventional methods, since we cannot assume that equilibria are all hyperbolic, leaving this property far more attractive from the point of view of mathematical difficulty.

For the sake of clarity and future reference, it is convenient to start with the hypotheses below used in [26] and [28].

(H1) The function $f \in C^1(\mathbb{R})$, $f'$ locally Lipschitz and

$$0 < f'(r) < k_1, \quad \forall \, r \in \mathbb{R},$$

(1.3)

for some positive constant $k_1$.

(H2) $f$ is a nondecreasing function taking values between 0 and $S_{max} > 0$ and satisfies, for
\[ 0 \leq s \leq S_{\text{max}}, \]
\[ \left| \int_0^s f^{-1}(r) \, dr \right| < L < \infty. \]

From (H1) follows that
\[ |f(x) - f(y)| \leq k_1 |x - y|, \quad \forall x, y \in \mathbb{R}, \quad (1.4) \]

and, in particular, there exists constant \( k_2 \geq 0 \) such that
\[ |f(x)| \leq k_1 |x| + k_2. \quad (1.5) \]

In [26] and [28], to obtain results on global attractors and Lyapunov functional, besides the hypotheses (H1) and (H2) above, it is assumed the hypothesis \( k_1 \| J \|_{L^1} < 1 \). Under this assumption, the map \( \Psi : L^2(S^1) \to L^2(S^1) \) given by
\[ \Psi(u) := J * (f \circ u) + h \]
is a contraction. Hence equation \( (P)_J \) bellow has an unique equilibrium \( \bar{u} \), which can leave the attractor to the trivial case of only one point.

In this paper, we organize the results as follows. In Section 2, we conclude that the hypothesis \( k_1 \| J \|_{L^1} < 1 \) is not required to obtain the results from [26] and [28] on global attractors and Lyapunov functional. Therefore, we obtain (see Theorem 2.2 and Proposition 2.3) stronger results in this direction. In Section 3, using the same techniques of [21], we prove the property of lower semicontinuity of the attractors. To the extent of our knowledge, with the exception of [21], the proofs of this property available in the literature assume that \( \text{the equilibrium points are all hyperbolic} \) and therefore isolated (see for example [2], [4], [17] and [18]). However, this property cannot hold true in our case, due to the symmetries present in the equation. In fact, it is a consequence of these symmetries that the nonconstant equilibria arise in families and, therefore, cannot be hyperbolic. This increases the difficulty and the interest
of the problem, since we cannot use results of the type Implicit Function Theorem to prove the continuity of equilibria. To overcome this difficulty we have to replace the hypothesis of hyperbolicity by *normal hyperbolicity* of curves of equilibria. We then used results of [3] on the permanence of *normally hyperbolic invariant manifolds* and use one result of [27] of continuity properties of the local unstable manifolds of the curves of equilibria. Finally, in Section 4, we illustrate our results with a concrete example, which satisfies all hypotheses (H1)-(H4). This does not occur in [21] because there is no proven that the example satisfies the property that imply in normal hyperbolicity.

### 2 Some remarks on global attractor and Lyapunov functional

As proved in [28], under the hypothesis (H1), the map

\[ F(u, J) = -u + J \ast (f(u)) + h \]  

(2.1)

is continuously Frechet differentiable in \( L^2(S^1) \) and, therefore, the equation

\[ \frac{\partial u}{\partial t} = F(u, J) = -u + J \ast (f(u)) + h \]  

(\( P \),\( J \))

generates a \( C^1 \) flow in \( L^2(S^1) \) given, by the variation of constant formula, by

\[ u(w, t) = e^{-t} u(w, 0) + \int_0^t e^{-(t-s)} [J \ast (f \circ u)(w, s) + h] ds. \]

From now on we denote this flow by \( T_J(t) \) to make explicit dependence on the parameter \( J \).

Under hypothesis (H1), we proved in our previous work [26] that the Cauchy problem for (\( P \)), in \( L^2(S^1) \), is well posed and, assuming hypothesis (H1) and that \( k_1 \| J \|_{L^1} < 1 \), we proved the existence and upper semicontinuity of the global compact attractor in the sense of [11]. Recently, in [28], assuming the hypotheses (H1), (H2) and that \( k_1 \| J \|_{L^1} < 1 \), we prove
that the flow of \((P)J\) is of class \(C^1\) and that it is gradient, in the sense of [11], with Lyapunov functional \(F: L^2(S^1) \to \mathbb{R}\) given by

\[
\mathcal{F}(u) = \int_{S^1} \left[ -\frac{1}{2} S(w) \int_{S^1} J(wz^{-1})S(z)dz + \int_0^{S(w)} f^{-1}(r)dr - hS(w) \right] dw, \tag{2.2}
\]

where \(S(w) = f(u(w))\).

It follows from Lemma below that we can obtain stronger versions of Theorems 8 of [26] and Proposition 4.6 of [28], eliminating the hypothesis \(k_1\|J\|_{L^1} < 1\) which is stronger used these previous works.

**Lemma 2.1.** Assume that (H1) and (H2) hold. Let \(R = 2\tau\|J\|_\infty S_{max} + h\). Then the ball with center at the origin of \(L^2(S^1)\) and radius \(R\sqrt{2\tau}\) is an absorbing set for the flow generated by \((P)_J\).

**Proof** Let \(u(w, t)\) be the solution of \((P)_J\) with initial condition \(u(w, 0)\), then

\[
u(w, t) = e^{-t}u_0(w) + \int_0^t e^{-(t-s)}[J \ast (f \circ u)(w, s) + h]ds.
\]

Using hypothesis (H2) it follows that

\[
|u(w, t)| \leq e^{-t}|u_0(w)| + \int_0^t e^{-(t-s)}|J \ast (f \circ u)(w, s) + h|ds
\]

\[
\leq e^{-t}|u_0(w)| + \int_0^t e^{-(t-s)}[2\tau\|J\|_\infty S_{max} + h]
\]

\[
\leq e^{-t}|u_0(w)| + 2\tau\|J\|_\infty S_{max} + h
\]

\[
= e^{-t}|u_0(w)| + R.
\]

Hence,

\[
\|u(\cdot, t)\|_{L^2} \leq \|e^{-t}|u_0| + R\|_{L^2}
\]

\[
\leq e^{-t}\|u_0\|_{L^2} + R\sqrt{2\tau}.
\]
Therefore, $u(\cdot, t) \in B(0, R + \varepsilon)$ for $t > \ln \left( \frac{\|u_0\|_2}{\varepsilon} \right)$, and the result is proved.

From Lemma 2.1 the Theorem 8 of \cite{26} can be rewritten as:

**Theorem 2.2.** Suppose that the hypotheses (H1) and (H2) hold. Then there exists a global attractor $A_J$ for the flow $T_J(t)$ in $L^2(S^1)$, which is contained in the ball of radius $(2\tau \|J\|_\infty S_{\text{max}} + h)\sqrt{2\tau}$.

And from Theorem 2.2 the Proposition 4.6 of \cite{28} can be rewritten as:

**Proposition 2.3.** Assume that the hypothesis (H1) and (H2) hold. Then the flow generated by equation $(P)_J$ is gradient, with Lyapunov functional given by (2.2).

### 3 Lower semicontinuity of the attractors

As mentioned in the introduction, an additional difficulty we encounter in the proof of lower semicontinuity is that, due to the symmetries present in our model, the nonconstant equilibria are not isolated. In fact, as we will see shortly, the equivariance property of the map $F$ defined in \cite{21} implies that the nonconstant equilibria appear in curves, (see Lemma 3.3) and, therefore, cannot be hyperbolic preventing the use of tools like the Implicit Function Theorem to obtain their continuity with respect to parameters.

In this section we prove the lower semicontinuity property of attractors, $\{A_J\}$ at $J_0 \in \mathcal{J}$, where

$$\mathcal{J} = \{ J \in C^1(\mathbb{R}), \text{ even non negative, supported in } [-1, 1], \|J\|_{L^1} = 1 \}.$$ 

Let us recall that a family of subsets $\{A_J\}$, is lower semicontinuous at $J_0$ if

$$\text{dist}(A_{J_0}, A_J) \rightarrow 0, \text{ as } J \rightarrow J_0.$$
where

\[
\text{dist}(A_{J_0}, A_J) = \sup_{x \in A_{J_0}} \text{dist}(x, A_J) = \sup_{x \in A_{J_0}} \inf_{y \in A_J} \|x - y\|_{L^2}.
\] (3.1)

In order to obtain the lower semicontinuity we will need the following additional hypotheses:

\textbf{(H3)} For each $J_0 \in J$, the set $E$, of the equilibria of $T_{J_0}(t)$, is such that $E = E_1 \cup E_2$, where

(a) the equilibria in $E_1$ are (constant) hyperbolic equilibria;

(b) the equilibria in $E_2$ are nonconstant and, for each $u_0 \in E_2$, zero is simple eigenvalue of the derivative of $F$, with respect to $u$, $DF_u(u_0, J_0) : L^2(S^1) \to L^2(S^1)$, given by

\[
DF_u(u_0, J_0)v = -v + J_0 \ast (f'(u_0)v).
\]

\textbf{(H4)} The function $f \in C^2(\mathbb{R})$.

We start with some remarks on the spectrum of the linearization around equilibria.

\textbf{Remark 3.1.} A simple computation shows that, if $u_0$ is a nonconstant equilibria of $T_{J_0}(t)$ then zero is always an eigenvalue of the operator

\[
DF_u(u_0, J_0)v = -v + J_0 \ast (f'(u_0)v)
\]

with eigenfunction $u_0'$. Therefore, the hypothesis (H3)-b says that we are in the ‘simplest’ possible situation for the linearization around nonconstant equilibria.

\textbf{Remark 3.2.} Let $u_0 \in E_2$. It is easy to show that $DF_u(u_0, J_0)$ is a self-adjoint operator with respect to the inner product

\[
(u, v) = \int_{S^1} u(w)v(w)d(w).
\]

Since

\[
v \to J_0 \ast (f'(u_0)v)
\]

is a compact operator in $L^2(S^1)$, it follows from (H3) that

\[
\sigma(DF_u(u_0, J_0)) \setminus \{0\}
\]
contains only real eigenvalues of finite multiplicity with $-1$ as the unique possible accumulation point.

Now we enunciate a result on the structure of the sets of nonconstant equilibria. The proof of this result is omitted because it is very similar to the proof of the Lemma 3.1 of [21].

**Lemma 3.3.** Suppose that, for some $J_0 \in \mathcal{J}$, (H1), (H3) and (H4) hold. Given $u \in E_2$ and $\alpha \in S^1$, define $\gamma(\alpha; u) \in L^2(S^1)$ by

$$\gamma(\alpha; u)(w) = u(\alpha w), \quad w \in S^1.$$  

Then $\Gamma = \gamma(S^1; u)$ is a closed, simple $C^2$ curve of equilibria of $T_{J_0}(t)$ which is isolated in the set of equilibria, that is, no point of $\Gamma$ is an accumulation point of $E_{J_0} \setminus \Gamma$.

**Corollary 3.4.** Let $M$ a closed connected curve of equilibria in $E_2$ and $u_0 \in M$. Then $M = \Gamma$, where $\Gamma = \gamma(S^1, u_0)$.

**Proof** Suppose that $\Gamma \not\subset M$. Then there exist equilibria in $M \setminus \Gamma$ accumulating at $u_0$ contradicting Lemma 3.3. Therefore $\Gamma \subseteq M$. Since $\Gamma$ is a simple closed curve, it follows that $M = \Gamma$.

In order to prove our main result, we need some preliminary results, which we present in the next three subsections.

### 3.1 Continuity of the equilibria

The upper semicontinuity of the equilibria is a consequence of the upper semicontinuity of global attractors (see Theorem 11 of [26]). The lower semicontinuity of the hyperbolic equilibria is usually obtained via the Implicit Function Theorem. However, this approach fails here since the equilibria may appear in families as we have shown in Lemma 3.3. To overcome this difficulty, we need the concept of normal hyperbolicity, (see [3]).
Recall that, if $T(t) : X \to X$ is a semigroup, a set $M \subset X$ is invariant under $T(t)$ if $T(t)M = M$, for any $t > 0$.

**Definition 3.5.** Suppose that $T(t)$ is a $C^1$ semigroup in a Banach space $X$ and $M \subset X$ is an invariant manifold for $T(t)$. We say that $M$ is normally hyperbolic under $T(t)$ if

(i) for each $m \in M$ there is a decomposition

$$X = X^c_m \oplus X^u_m \oplus X^s_m$$

by closed subspaces with $X^c_m$ being the tangent space to $M$ at $m$.

(ii) for each $m \in M$ and $t \geq 0$, if $m_1 = T(t)(m)$

$$DT(t)(m)|_{X^c_m} : X^c_m \to X^c_{m_1}, \quad \alpha = c, u, s$$

and $DT(t)(m)|_{X^u_m}$ is an isomorphism from $X^u_m$ onto $X^u_{m_1}$.

(iii) there is $t_0 \geq 0$ and $\mu < 1$ such that for all $t \geq t_0$

$$\mu \inf \left\{ \| DT(t)(m)x^u \| : x^u \in X^u_m, \| x^u \| = 1 \right\} > \max \{ 1, \| DT(t)(m)|_{X^c_m} \| \} , \quad (3.2)$$

$$\mu \min \{ 1, \inf \{ \| DT(t)(m)x^c \| : x^c \in X^c_m, \| x^c \| = 1 \} \} > \| DT(t)(m)|_{X^s_m} \|. \quad (3.3)$$

The condition (3.2) suggests that near $m \in M$, $T(t)$ is expansive in the direction of $X^u_m$ and at rate greater than on $M$, while (3.3) suggests that $T(t)$ is contractive in the direction of $X^s_m$, and at a rate greater than that on $M$.

The following result has been proved in [3].

**Theorem 3.6. (Normal Hyperbolicity)** Suppose that $T(t)$ is a $C^1$ semigroup on a Banach space $X$ and $M$ is a $C^2$ compact connected invariant manifold which is normally hyperbolic under $T(t)$, (that is (i) and (ii) hold and there exists $0 \leq t_0 < \infty$ such that (iii) holds for all
\( t \geq t_0 \). Let \( \tilde{T}(t) \) be a \( C^1 \) semigroup on \( X \) and \( t_1 > t_0 \). Consider \( N(\varepsilon) \), the \( \varepsilon \)-neighborhood of \( M \), given by

\[
N(\varepsilon) = \{ m + x^u + x^s, \ x^u \in X^u_m, \ x^s \in X^s_m, \ \|x^u\|, \|x^s\| < \varepsilon \}.
\]

Then, there exists \( \varepsilon^* > 0 \) such that for each \( \varepsilon < \varepsilon^* \), there exists \( \sigma > 0 \) such that if

\[
\sup_{u \in N(\varepsilon)} \left\{ \|\tilde{T}(t_1)u - T(t_1)u\| + \|D\tilde{T}(t_1)(u) - DT(t_1)(u)\| \right\} < \sigma
\]

and

\[
\sup_{u \in N(\varepsilon)} \|\tilde{T}(t)u - T(t)u\| < \sigma, \text{for } 0 \leq t \leq t_1,
\]

there is an unique compact connected invariant manifold of class \( C^1 \), \( \tilde{M} \), in \( N(\varepsilon) \). Furthermore, \( \tilde{M} \) is normally hyperbolic under \( \tilde{T}(t) \) and, for each \( t \geq 0 \), \( \tilde{T}(t) \) is a \( C^1 \)-diffeomorphism from \( \tilde{M} \) to \( \tilde{M} \).

**Proposition 3.7.** Assume that the hypotheses (H1), (H2) and (H3) hold. Then, for each \( J \in \mathcal{J} \), any curve of equilibria of \( T_J(t) \) is a normally hyperbolic manifold under \( T_J(t) \).

**Proof** Here we follow closely a proof of [21]. Let \( M \) be a curve of equilibria of \( T_J(t) \) and \( m \in M \). From (H3) it follows that

\[
Ker(DF_u(m, J)) = \text{span}\{m'\}.
\]

Let \( Y = \mathcal{R}(DF_u(m, J)) \) the range of \( DF_u(m, J) \). Since \( DF_u(m, J) \) is self-adjoint and Fredholm of index zero, it follows from (H3) that

\[
\sigma(DF_u(u_0, J)|_Y) = \sigma_u \cup \sigma_s,
\]

where \( \sigma_u, \sigma_s \) correspond to the positive and negative eigenvalues respectively.
From (H1) and (H2), it follows that $T_J(t)$ is a $C^1$ semigroup. Consider the linear autonomous equation

$$\dot{v} = (DF_u(m, J)|_Y)v.$$  \hfill (3.4)

Then $DT_J(t)v_0$ is the solution of (3.4) with initial condition $v_0$, that is $DT_J(t)(m)v_0 = e^{(DF_u(m, J)|_Y)t}v_0$. In particular $DT_J(t)(m)|_Y \equiv D(T_J(t)|_Y)(m) = e^{(DF_u(m, J)|_Y)t}$.

Let $P_u$ and $P_s$ be the spectral projections corresponding to $\sigma_u$ and $\sigma_s$. The subspaces $X^u_m = P_u Y$, $X^s_m = P_s Y$ are then invariant under $DT_J(t)$ and the following estimates hold (see [8], p. 73, 81).

$$\|DT_J(t)|_Y v\| \leq Ne^{-\nu t}\|v\|, \text{ for } v \in X^s_m \text{ and } t \geq 0, \hfill (3.5)$$

$$\|DT_J(t)|_Y v\| \leq Ne^{\nu t}\|v\|, \text{ for } v \in X^u_m \text{ and } t \leq 0, \hfill (3.6)$$

for some positive constant $\nu$ and some constant $N > 1$.

It is clear that $DT_J(t) \equiv 0$ when restricted to $X^c_m = \text{span}\{m'\}$. Therefore, we have the decomposition

$$L^2(S^1) = X^c_m \oplus X^u_m \oplus X^s_m.$$  

Since $DF_u(m, J)|_Y$ is an isomorphism

$$DF_u(m, J)|_{X^\alpha_m} : X^\alpha_m \to X^\alpha_m, \ \alpha = u, s,$$

is an isomorphism. Consequently, the linear flow

$$DT_J(t)(m)|_{X^\alpha_m} : X^\alpha_m \to X^\alpha_m$$

is also an isomorphism.

Finally, the estimates (3.2) and (3.3) follow from estimates (3.5) and (3.6) above. \hfill $\Box$
Remark 3.8. For $u, v \in L^2(S^1)$, from (3.7) follows that

$$\|f'(u)v\| \leq k_1\|v\|_{L^2}. \quad (3.7)$$

Proposition 3.9. Suppose that the hypotheses (H1)-(H2) hold. Let $DT_J(t)(u)$ be the linear flow generated by the equation

$$\frac{\partial v}{\partial t} = -v + J_0 \ast (f'(u)v).$$

Then, for a fixed $J_0 \in J$, we have

$$\|T_J(t)u - T_{J_0}(t)u\|_{L^2(S^1)} + \|DT_J(t)(u) - DT_{J_0}(t)(u)\|_{L(L^2(S^1), L^2(S^1))} \to 0 \text{ as } \|J - J_0\|_{L^1} \to 0,$$

uniformly for $u$ in bounded sets of $L^2(S^1)$ and $t \in [0, b], b < \infty$.

**Proof** From Lemma 10 of [26] it follows that

$$\|T_J(t)u - T_{J_0}(t)u\|_{L^2(S^1)} \to 0 \text{ as } \|J - J_0\|_{L^1} \to 0,$$

for $u$ in bounded sets of $L^2(S^1)$ and $t \in [0, b]$.

By the variation of constants formula, we have

$$DT_J(t)(u)v = e^{-t}v + \int_0^t e^{-(t-s)}J \ast (f'(u)v)ds.$$

Thus, using Young’s inequality, we obtain

$$\|DT_J(t)(u)v - DT_{J_0}(t)(u)v\|_{L^2} \leq \int_0^t e^{-(t-s)}\|J - J_0\|_{L^1} \|f'(u)v\|_{L^2}ds \leq \int_0^t e^{-(t-s)}\|J - J_0\|_{L^1} k_1 \|v\|_{L^2}ds.$$

Using (3.7), it follows that

$$\|DT_J(t)(u)v - DT_{J_0}(t)(u)v\|_{L^2} \leq k_1 \|J - J_0\|_{L^1} \|v\|_{L^2}.$$
Therefore
\[
\|DT_J(t)(u) - DT_{J_0}(t)(u)\|_{L^2(S^1)} = \sup_{\|v\| = 1} \|DT_J(t)v - DT_{J_0}(t)v\|_{L^2(S^1)} 
\]
\[
\leq \sup_{\|v\| = 1} k_1 \|J - J_0\|_1 \|v\|_2 
\]
\[
= C(J),
\]
with \(C(J) \to 0\), as \(\|J - J_0\|_1 \to 0\). This completes the proof. 

The proof of the theorem below follows closely the proof of Theorem 3.4 of [21].

Theorem 3.10. Suppose that the hypotheses (H1)-(H4) hold. Then the set \(E_J\) of the equilibria of \(T_J(t)\) is lower semi-continuous with respect to \(J\) at \(J_0\).

Proof. The continuity of the constant equilibria follows from the Implicit Function Theorem and the hypothesis of hyperbolicity.

Suppose now that \(m\) is a nonconstant equilibrium and let \(\Gamma = \gamma(\alpha; m)\) be the isolated curve of equilibria containing \(m\) given by Lemma 3.3. We want to show that, for every \(\varepsilon > 0\), there exists \(\delta > 0\) so that, if \(J \in \mathcal{J}\), there exists \(\Gamma_J \in \mathcal{E}_J\) such that \(\Gamma \subset \Gamma^\varepsilon_J\), where \(\Gamma^\varepsilon_J\) is the \(\varepsilon\)-neighborhood of \(\Gamma_J\).

From Lemma 3.3 and Propositions 3.7 and 3.9, the assumptions of the Normal Hyperbolicity Theorem are met. Thus, given \(\varepsilon > 0\), there is \(\delta > 0\) such that, if \(\|J - J_0\|_1 < \delta\) there is an unique \(C^1\) compact connected invariant manifold \(\Gamma_J\) normally hyperbolic under \(T_J(t)\), such that \(\Gamma_J\) is \(\varepsilon\)-close and \(C^1\)-diffeomorphic to \(\Gamma\).

Since \(T_J(t)\) is gradient and \(\Gamma_J\) is compact, there exists at least one equilibrium \(m_J \in \Gamma_J\). In fact, the \(\omega\) limit of any \(u \in \Gamma_J\) is nonempty and belongs to \(\Gamma_J\) by invariance. From Lemma 3.8.2 of [11], it must contain an equilibrium. Since \(\Gamma_J\) is \(\varepsilon\)-close to \(\Gamma\), there exists \(m \in \Gamma\) such that \(\|m - m_J\|_{L^2(S^1)} < \varepsilon\).
Let $\tilde{\Gamma}_J$ be the curve of equilibria given by $\tilde{\Gamma}_J \equiv \{ \gamma(\alpha; m_J), \alpha \in S^1 \}$ which is a normally hyperbolic invariant manifold under $T_J(t)$ by Proposition 3.7. Then, for each $\alpha \in S^1$, we have

$$\| \gamma(\alpha; m_J) - \gamma(\alpha; m) \|_{L^2}^2 = \int_{S^1} |\gamma(\alpha; m_J)(w) - \gamma(\alpha; m)(w)|^2 dw$$

$$= \int_{S^1} |m_J(\alpha w) - m(\alpha w)|^2 dw$$

$$= \| m_J - m \|_{L^2}.$$

Thus

$$\| \gamma(\alpha; m_J) - \gamma(\alpha; m) \|_{L^2} = \| m_J - m \|_{L^2}$$

$$< \varepsilon.$$

and $\Gamma$ is $\varepsilon$-close to $\tilde{\Gamma}_J$. Since there are only a finite number of curves of equilibria the result follows immediately.

3.2 Existence and continuity of the local unstable manifolds

Let us return to equation $(P)_J$. Recall that the unstable set $W^u_J = W^u_J(u_J)$ of an equilibrium $u_J$ is the set of initial conditions $\varphi$ of $(P)_J$, such that $T_J(t) \varphi$ is defined for all $t \leq 0$ and $T_J(t) \varphi \to u_J$ as $t \to -\infty$. For a given neighborhood $V$ of $u_J$, the set $W^u_J \cap V$ is called a local unstable set of $u_J$.

Using results of [27] we now show that the local unstable sets are actually Lipschitz manifolds in a sufficiently small neighborhood and vary continuously with $J$. More precisely, we have

$$\| \gamma(\alpha; m_J) - \gamma(\alpha; m) \|_{L^2} = \| m_J - m \|_{L^2}$$

$$< \varepsilon.$$
Lemma 3.11. If $u_0$ is a fixed equilibrium of $(P)_J$ for $J = J_0$, then there is a $\delta > 0$ such that, if $\|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} < \delta$ and

$$U_\delta^J := \{ u \in W^u_J(u_J) : \|u - u_J\|_{L^2} < \delta \}$$

then $U_\delta^J$ is a Lipschitz manifold and

$$\text{dist}(U_\delta^J, U^J_{J_0}) + \text{dist}(U^J_{J_0}, U_\delta^J) \to 0 \quad \text{as} \quad \|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} \to 0,$$

with dist defined as in (3.1).

Proof. As already mentioned in the previous section, assuming the hypothesis (H1), the map $F : L^2(S^1) \times J \to L^2(S^1)$,

$$F(u, J) = -u + J * (f(u)) + h$$

defined by the right-hand side of $(P)_J$ is continuously Frechet differentiable. Let $u_J$ be an equilibrium of $(P)_J$. Writing $u = u_J + v$, it follows that $u$ is a solution of $(P)_J$ if and only if $v$ satisfies

$$\frac{\partial v}{\partial t} = L(J)v + r(u_J, v, J), \quad (3.8)$$

where $L(J)v = \frac{\partial}{\partial u} F(u_J, J) = -v + J * (f'(u_J)v)$ and $r(u_J, v, J) = F(u_J + v, J) - F(u_J, J) - L(J)v$. We rewrite equation (3.8) in the form

$$\frac{\partial v}{\partial t} = L(J_0)v + g(v, J), \quad (3.9)$$

where $g(v, J) = [L(J) - L(J_0)]v + r(u_J, v, J)$ is the “non linear part” of (3.9). Note that now the “linear part” of (3.9) does not depend on the parameter $J$, as required by Theorems 2.5 and 3.1 from [27].
Note that
\[
\| [L(J) - L(J_0)]v \|_{L^2} \leq \| (J - J_0) \ast (f'(u_J)v) \|_{L^2} + \| J_0 \ast [f'(u_J) - f'(u_{J_0})]v \|_{L^2}.
\]

But, using Holder inequality, we have
\[
| J_0 \ast [f'(u_J) - f'(u_{J_0})](w)v(w) | \leq \int_{S^1} J_0(wz^{-1}) |f'(u_J(z)) - f'(u_{J_0}(z))| |v(z)| dz
\]
\[
\leq \| J_0 \|_\infty \int_{S^1} |f'(u_J(z)) - f'(u_{J_0}(z))| |v(z)| dz
\]
\[
\leq \| J_0 \|_\infty \| f'(u_J) - f'(u_{J_0}) \|_{L^2} \| v \|_{L^2}.
\]

Thus, remembering that we are assuming the notation of our previous work ([26] and [28]), where the measure of $S^1$ is $2\tau$, we obtain
\[
\| J_0 \ast [f'(u_J) - f'(u_{J_0})]v \|_{L^2} \leq \sqrt{2\tau} \| J_0 \|_\infty \| f'(u_J) - f'(u_{J_0}) \|_{L^2} \| v \|_{L^2}.
\]

Hence, using Young inequality and hypothesis (H1), we have
\[
\| [L(J) - L(J_0)]v \|_{L^2} \leq k_1 \| J - J_0 \|_{L^1} \| v \|_{L^2} + \sqrt{2\tau} \| J_0 \|_\infty \| f'(u_J) - f'(u_{J_0}) \|_{L^2} \| v \|_{L^2}.
\]

But, keeping $u_{J_0} \in L^2(S^1)$, from Theorem 3.10 follows that $\| u_J - u_{J_0} \|_{L^2} \to 0$, as $\| J - J_0 \|_{L^1} \to 0$. It follows that $u_J(w) \to u_{J_0}(w)$ almost everywhere in $S^1$. From (H1) follows that, there exists $M > 0$ such that
\[
| f'(u_J(w)) - f'(u_{J_0}(w)) | \leq M | u_J(w) - u_{J_0}(w) |, \quad \text{almost everywhere.}
\]

Then
\[
\| f' \circ u_J - f' \circ u_{J_0} \|_{L^2}^2 = \int_{S^1} | f'(u_J(w)) - f'(u_{J_0}(w)) |^2 dw
\]
\[
\leq \int_{S^1} M^2 | u_J(w) - u_{J_0}(w) |^2 dw
\]
\[
= M^2 \| u_J - u_{J_0} \|_{L^2}^2. \tag{3.10}
\]
Therefore, using (3.10), we obtain

\[ \| [L(J) - L(J_0)]v \|_{L^2} \leq k_1 \| J - J_0 \|_{L^1} \| v \|_{L^2} + \sqrt{2} \tau \| J_0 \|_\infty M \| u_J - u_{J_0} \|_{L^2} \| v \|_{L^2}. \] (3.11)

Now, note that,

\[ r(u_J, v, J) - r(u_{J_0}, v, J_0) = F(u_J + v, J) - F(u_J, J) - L(J)v \]

\[ - F(u_{J_0} + v, J_0) + F(u_{J_0}, J_0) + L(J_0)v \]

\[ = J * f(u_J + v) - J * f(u_J) + J_0 * f(u_{J_0}) - J_0 * f(u_{J_0} + v) \]

\[ - [L(J) - L(J_0)]v \]

\[ = J * [f(u_J + v) - f(u_J)] + J_0 * [f(u_{J_0}) - f(u_{J_0} + v)] \]

\[ - [L(J) - L(J_0)]v. \]

But

\[ J * [f(u_J + v) - f(u_J)] = J * [f'(\bar{v})v] \]

and

\[ J_0 * [f(u_{J_0} + v) - f(u_{J_0})] = J_0 * [f'(\bar{\bar{v}})v], \]

for some \( \bar{v} \) in the segment defined by \( J * f(u_J) \) and \( J * f(u_J + v) \) and for some \( \bar{\bar{v}} \) in the segment defined by \( J_0 * f(u_{J_0}) \) and \( J_0 * f(u_{J_0} + v) \). Then

\[ J * [f(u_J + v) - f(u_J)] + J_0 * [f(u_{J_0}) - f(u_{J_0} + v)] = J * [f'(\bar{v})v] - J_0 * [f'(\bar{\bar{v}})v] \]

\[ = J * [f'(\bar{v})v] - J_0 * [f'(\bar{\bar{v}})v] \]

\[ + J_0 * [f'(\bar{\bar{v}})v] - J_0 * [f'(\bar{\bar{v}})v] \]

\[ = (J - J_0) * f'(\bar{\bar{v}})v \]

\[ + J_0 * [f'(\bar{\bar{v}}) - f'(\bar{\bar{v}})]v. \]
Thus
\[ r(u_J, v, J) - r(u_{J_0}, v, J_0) = (J - J_0) * f'(\tilde{v})v + J_0 * [f'(\tilde{v}) - f'(\tilde{v})]v + [L(J_0) - L(J)]v. \]

Hence
\[ \| r(u_J, v, J) - r(u_{J_0}, v, J_0) \|_{L^2} \leq \| J - J_0 \|_{L^1} \| f'(\tilde{v})v \|_{L^2} + \| J_0 * [f'(\tilde{v}) - f'(\tilde{v})]v \|_{L^2} + \| [L(J) - L(J_0)]v \|_{L^2}. \]

But, from hypothesis (H4), there exists \( M > 0 \) such that
\[ |f'(\tilde{v})(z) - f'(\tilde{\tilde{v}})(z)| \leq M|\tilde{v}(z) - \tilde{\tilde{v}}(z)|, \quad \forall z \in S^1, \]

thus
\[ |J_0 * [f'(\tilde{v})(w) - f'(\tilde{\tilde{v}})(w)]v(w)| \leq \int_{S^1} |J_0(wz^{-1})| |f'(\tilde{v})(z) - f'(\tilde{\tilde{v}})(z)| |v(z)|dz \leq \int_{S^1} \| J_0 \|_{\infty} M |\tilde{v}(z) - \tilde{\tilde{v}}(z)| |v(z)|dz \leq \| J_0 \|_{\infty} M \| \tilde{v} - \tilde{\tilde{v}} \|_{L^2} \| v \|_{L^2}. \]

Then, remembering again that we are assuming the notation of our previous work ([20] and [28]), where the measure of \( S^1 \) is \( 2\tau \), we obtain
\[ \| J_0 * [f'(\tilde{v}) - f'(\tilde{\tilde{v}})]v \|_{L^2} \leq \| J \|_{\infty} M \sqrt{2\tau} \| \tilde{v} - \tilde{\tilde{v}} \|_{L^2} \| v \|_{L^2}, \quad (3.12) \]

Thus, using (3.7), (3.11) and (3.12), and the fact that \( \| \tilde{v} - \tilde{\tilde{v}} \|_{L^2} \to 0 \), as \( \| J - J_0 \|_{L^1} \to 0 \), follows that
\[ \| r(u_J, v, J) - r(u_{J_0}, v, J_0) \|_{L^2} \leq k_1 \| J - J_0 \|_{L^1} \| v \|_{L^2} + \| J \|_{\infty} M \sqrt{2\tau} \| \tilde{v} - \tilde{\tilde{v}} \|_{L^2} \| v \|_{L^2} \]
\[ + k_1 \| J - J_0 \|_{L^1} \| v \|_{L^2} + \sqrt{2\tau} \| J_0 \|_{\infty} M \| u_J - u_{J_0} \|_{L^2} \| v \|_{L^2} \]
\[ = C_1(J) \| v \|_{L^2}, \quad (3.13) \]
with $C_1(J) \to 0$, as $\|J - J_0\|_{L^1} \to 0$.

Now, since

$$g(v, J) - g(v, J_0) = [L(J) - L(J_0)]v + r(u_J, v, J) - r(u_{J_0}, v, J_0),$$

using (3.11) and (3.13), we obtain

$$\|g(v, J) - g(v, J_0)\| \leq \|L(J) - L(J_0)\|_{L^2} + \|r(u_J, v, J) - r(u_{J_0}, v, J_0)\|_{L^2} \leq C_2(J),$$

(3.14)

where $C_2(J) \to 0$ as $\|J - J_0\|_{L^2} \to 0$.

In a similar way, we obtain for any $v_1, v_2$ with $\|v_1\|_{L^2(S^1)}$ and $\|v_2\|_{L^2(S^1)}$ smaller than $\rho$

$$\|g(v_1, J) - g(v_2, J)\|_{L^2} \leq \nu(\rho)\|v_1 - v_2\|_{L^2},$$

(3.15)

where $\nu(\rho) \to 0$, as $\rho \to 0$.

Therefore, the conditions of Theorems 2.5 and 3.1 from [27] are satisfied and we obtain the existence of locally invariant sets for (3.9) near the origin, given as graphs of Lipschitz functions which depend continuously on the parameter $J$ near $J_0$. Using uniqueness of solutions, we can easily prove that these sets coincide with the local unstable manifolds of (3.9).

Now, noting that the translation

$$u \to (u - u_J)$$

sends an equilibrium $u_J$ of $(P)_J$ into the origin (which is an equilibrium of (3.9)), the results claimed follow immediately.

Using the compactness of the set of equilibria, one can obtain an ‘uniform version’ of Lemma 3.11 that will be needed later.

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Lemma 3.12. Let $J = J_0$ be fixed. Then, there is a $\delta > 0$ such that, for any equilibrium $u_0$ of $(P)_{J_0}$, if $\|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} < \delta$ and

$$U^\delta_J := \{ u \in U_J(u_J) : \|u - u_J\|_{L^2(S^1)} < \delta \}$$

then $U^\delta_J$ is a Lipschitz manifold and

$$\sup_{u_0 \in E_{J_0}} \text{dist}(U^\delta_J, U^\delta_{J_0}) + \text{dist}(U^\delta_{J_0}, U^\delta_J) \to 0 \quad \text{as} \quad \|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} \to 0,$$

with $\text{dist}$ defined as in (3.1)

Proof. From Lemma 3.11, we know that, for any $u_0 \in E_{J_0}$, there is a $\delta = \delta(u_0)$ such that $U^\delta_J$ is a Lipschitz manifold, if $\|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} < 2\delta$. Thus, in particular, $U^\delta_J$ is a Lipschitz manifold, if $\|J - J_0\|_{L^1} + \|\tilde{u}_0 - u_J\|_{L^2} < \delta$, for any $\tilde{u}_0 \in E_{J_0}$ with $\|\tilde{u}_0 - u_0\|_{L^2} < \delta$. Taking a finite subcovering of the covering of $E_{J_0}$ by balls $B(u_0, \delta(u_0))$, with $u_0$ varying in $E_{J_0}$, the first part of the result follows with $\delta$ chosen as the minimum of those $\delta(u_0)$.

Now, if $\varepsilon > 0$ and $u_0 \in E_{J_0}$, there exists, by Lemma 3.11, $\delta = \delta(u_0)$ such that, if $\|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} < 2\delta$, then

$$\text{dist}(U^\delta_J, U^\delta_{J_0}) + \text{dist}(U^\delta_{J_0}, U^\delta_J) < \varepsilon/2.$$ 

If $\tilde{u}_0 \in E_{J_0}$ is such that $\|\tilde{u}_0 - u_0\|_{L^2} < \delta$ and $\|J - J_0\|_{L^1} + \|\tilde{u}_0 - u_J\|_{L^2} < \delta$ then, since $\|J - J_0\|_{L^1} + \|u_0 - u_J\|_{L^2} < 2\delta$

$$\text{dist}(U^\delta_J(u_J), U^\delta_{J_0}(\tilde{u}_0)) + \text{dist}(U^\delta_{J_0}(\tilde{u}_0), U^\delta_J(u_J))$$

$$< \text{dist}(U^\delta_J(u_J), U^\delta_{J_0}(u_0)) + \text{dist}(U^\delta_{J_0}(u_0), U^\delta_J(u_J)) + \text{dist}(U^\delta_J(\tilde{u}_0), U^\delta_J(u_J))$$

$$+ \text{dist}(U^\delta_J(u_0), U^\delta_J(\tilde{u}_0)) < \varepsilon.$$

By the same procedure above of taking a finite subcovering of the covering of $E_{J_0}$ by balls $B(u_0, \delta(u_0))$, and $\delta$ the minimum of those $\delta(u_0)$, we conclude that

$$\text{dist}(U^\delta_J(u_J), U^\delta_{J_0}(\tilde{u}_0)) + \text{dist}(U^\delta_{J_0}(\tilde{u}_0), U^\delta_J(u_J)) < \varepsilon.$$
if $\|J - J_0\|_{L^1} + \|\tilde{u}_0 - u_J\|_{L^2} < \delta$, for any $\tilde{u}_0 \in E_{J_0}$. This proves the result claimed. \hfill \Box

### 3.3 Characterization of the attractor

As a consequence of its gradient structure (see Remark 4.7 of [28]), the attractor of the flow generated by $(P)_J$ is given by the union of the unstable set of the set of equilibria. We prove below a more precise characterization.

As is well known in the literature, an equation of the form

$$\dot{x} + Bx = g(x),$$

where $B$ is a bounded linear operator on a Banach space $X$ and $g : X \to X$ is a $C^2$ function, may be rewritten in the form

$$\dot{x} + Ax = f(x),$$

(3.16) where $A = B - g'(x_0)$ and $f(x) = g(x_0) + r(x)$, with $r$ differentiable and $r(0) = 0$.

The following result has been proven in [12].

**Theorem 3.13.** Suppose the spectrum $\sigma(A)$ contains 0 as a simple eigenvalue, while the remainder of the spectrum has real part outside some neighborhood of zero. Let $\gamma$ be a curve of equilibria of the flow generated by (3.10), of class $C^2$. Then there exists a neighborhood $U$ of $\gamma$ such that, for any $x_0 \in U$ whose positive orbit is precompact and whose $\omega$-limit set $\omega(x_0)$ belongs to $\gamma$, there exists a unique point $y(x_0) \in \gamma$ with $\omega(x_0) = y(x_0)$. Similarly, for any $x_0 \in U$ with bounded negative orbit and $\alpha$-limit set $\alpha(x_0)$ in $\gamma$, there exists a unique point $y(x_0) \in \gamma$ such that $\alpha(x_0) = y(x_0)$.

**Proposition 3.14.** Assume the hypotheses $(H1)$-$\!(H4)$ hold. Let $E_J$ be the set of the equilibria
of $T_J(t)$. For $u \in E_J$, let $W^u_J(u)$ be the unstable set of $u$. Then

$$A_J = \bigcup_{u \in E_J} W^u_J(u).$$

**Proof** From Remark 4.7 of [28], follows that

$$A_J = W^u_J(E_J).$$

There exists only a finite number, \{${u_1, \ldots, u_k}$\} of constant equilibria since they are all hyperbolic. For each nonconstant equilibrium $u \in E_J$, there is a curve $M_u \subset E_J \subset A_J$. From Lemma 3.3 these curves $M_u$ are all isolated and, since $A_J$ is compact, it follows that there exists only a finite number of them; $M_1, \ldots, M_n$. Thus

$$A_J = \left( \bigcup_{i=1}^n W_J^u(M_i) \right) \bigcup \left( \bigcup_{j=1}^k W_J^u(u_j) \right).$$

From Theorem 3.13 follows that

$$W^u_J(M_i) = \bigcup_{v \in M_i} W^u_J(v), \ i = 1, \ldots, n.$$ 

Therefore

$$A_J = \bigcup_{v \in E_J} W^u_J(v),$$

which concludes the proof. \(\square\)

### 3.4 Proof of the lower semicontinuity

Using the results obtained in the previous subsections, the proof of the lower semicontinuity can now be adapted from Lemma 3.8 and Theorem 3.9 of [21], as shown below.
Lemma 3.15. Assume the same hypotheses of Proposition 3.14. Then, given $\varepsilon > 0$, there exists $T > 0$ such that, for all $u \in \mathcal{A}_J \setminus E^\varepsilon_J$

$$T_J (-t) u \in E^\varepsilon_J,$$

for some $t \in [0, T]$, where $E^\varepsilon_J$ is the $\varepsilon$-neighborhood of $E_J$. Furthermore, when $\varepsilon$ is sufficiently small,

$$T_J (-t) u \in U_J (u_0),$$

for some $u_0 \in E_J$, where $U_J (u_0)$ is the local unstable manifold of $u_0 \in E_J$.

Proof. Let $\varepsilon > 0$ be given and $u \in \mathcal{A}_J \setminus E^\varepsilon_J$. From Proposition 3.14 it follows that

$$u \in W^u_{J_0} (\bar{u}) \setminus E^\varepsilon_J,$$

for some $\bar{u} \in E_J$. Thus, there exists $t_u = t_u (\varepsilon) < \infty$ such that $T_J (-t_u) u \in E^\varepsilon_J$. By continuity of the operator $T_J (-t_u)$, there exists $\eta_u > 0$ such that $T_J (-t_u) B (u, \eta_u) \subset E^\varepsilon_J$, where $B (u, \eta_u)$ is the ball of center $u$ and radius $\eta_u$. By compactness, there are $u_1, \ldots, u_n \in A_0 \setminus E^\varepsilon_J$ such that

$$\mathcal{A}_J \setminus E^\varepsilon_J \subset \bigcup_{i=1}^n B (u_i, \eta_u),$$

with $T_J (-t_{u_i}) B (u_i, \eta_u) \subset E^\varepsilon_J$, for $i = 1, \ldots, n$. Let $T = \max \{ t_{u_1}, \ldots, t_{u_n} \}$. Then, for any $u \in \mathcal{A}_J \setminus E^\varepsilon_J$, $T_J (-t) u \in E^\varepsilon_J$, for some $t \in [0, T]$. Since $u \in W^u_{J_0} (\bar{v}) \setminus E^\varepsilon_J$, for some $\bar{v} \in E_J$ and $T_J (-t) u \in E^\varepsilon_J$, to conclude that $T_J (-t) u \in U_J (\bar{u})$, when $\varepsilon$ is sufficiently small, it is enough to show that there exists $\delta > 0$ such that $W^u_{J_0} (v) \cap B (v, \delta) \subset U_J (v)$, for all $v \in E_J$.

Therefore, the conclusion follows immediately from Lemma 3.11.

Theorem 3.16. Assume the hypotheses (H1)-(H4). Then the family of attractors $\{ \mathcal{A}_J \}$ is lower semicontinuous with respect to the parameter $J$ at $J_0 \in \mathcal{J}$. 

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Proof. Let \( \varepsilon > 0 \) be given. From Lemma 3.15, there is \( T > 0 \) such that, for all \( u \in A_{J_0} \setminus E_{J_0}^c \), there exists \( t_u \in [0, T] \) such that

\[
\bar{u} := T_{J_0}(-t_u)u \in U_{J_0}(u_0),
\]

for some \( u_0 \in E_{J_0} \). Since \( T_{J_0}(t) \) is a continuous family of bounded operators, there exists \( \eta > 0 \) such that, for all \( t \in [0, T] \)

\[
\|z - w\|_{L^2} < \eta \Rightarrow \|T_{J_0}(t)z - T_{J_0}(t)w\|_{L^2} < \frac{\varepsilon}{2}.
\]

By the uniform continuity of the equilibria and local unstable manifolds with respect to the parameter \( J \) asserted by Theorem 3.10 and Lemma 3.12, there exists \( \delta^* > 0 \) (independent of \( u \)) such that \( \|J - J_0\|_{L^1} < \delta^* \) implies the existence of \( u_J \in E_J \) and some \( \bar{u}_J \in U_J(u_J) \) with

\[
\|\bar{u}_J - \bar{u}\|_{L^2} < \eta,
\]

where \( U_J(u_J) \) denotes the local unstable manifold of the equilibrium \( u_J \) of \( T_J(t) \). Thus, when \( \|J - J_0\|_{L^1} < \delta^* \) we obtain, from (3.18) and (3.19)

\[
\|T_{J_0}(t)\bar{u}_J - T_{J_0}(t)\bar{u}\|_{L^2} < \frac{\varepsilon}{2} \quad \text{for any} \quad t \in [0, T].
\]

On the other hand, from continuity of the flow with respect to parameter \( J \), (see Lemma 10 of [26]), there exists \( \bar{\delta} > 0 \) such that \( \|J - J_0\|_{L^1} < \bar{\delta} \) implies

\[
\|T_J(t)(u) - T_{J_0}(t)(u)\|_{L^2} < \frac{\varepsilon}{2},
\]

for any \( u \in B(0, 2\tau\|J\|_{L^\infty} S_{max} + h)) \) and \( t \in [0, T] \). In particular, (3.21) holds for \( u = \bar{u}_J \) and \( t = t_u \).

Choose \( \delta = \min\{\delta^*, \bar{\delta}\} \) and let \( v_J := T_J(t_u)\bar{u}_J \). Note that \( v_J \in A_J \), since \( \bar{u}_J \in U_J(u_J) \).
Thus, using (3.20) and (3.21) we obtain, when $\|J - J_0\|_{L^1} < \delta$

$$\|v_J - u\|_{L^2} = \|T_J(t_u)\tilde{u}_J - T_{J_0}(t_u)\tilde{u}\|_{L^2} \leq \|T_J(t_u)\tilde{u}_J - T_{J_0}(t_u)\tilde{u}_J\|_{L^2} + \|T_{J_0}(t_u)\tilde{u}_J - T_{J_0}(t_u)\tilde{u}\|_{L^2} < \varepsilon.$$  

When $u \in E^\varepsilon_{J_0} \subset A_{J_0}$ this conclusion follows straightforwardly from the continuity of equilibria. Thus the lower semicontinuity of attractors follows. \(\square\)

4 A concrete example

In this section we illustrate the results of the previous sections to the particular case of (1.1) where $f(x) = (1 + e^{-x})^{-1}$ and

$$\tilde{J}(x) = \begin{cases} 
  e^{\frac{1}{1-x^2}}, & \text{if } |x| < 1, \\
  0, & \text{if } |x| \geq 1.
\end{cases}$$

The function $f$ has been motivated by similar functions in [7], [15] and [29] and the function $\tilde{J}$ has been adapted from a test function in [5].

In this case, we can rewrite equation (1.1) as

$$\frac{\partial v(x,t)}{\partial t} = -v(x,t) + \int_{-1}^{1} e^{\frac{-1}{1-(x-y)^2}}(1 + e^{-v(y)})^{-1}dy + h.$$  \hspace{1cm} (4.1)

As mentioned in the introduction, defining $\varphi : \mathbb{R} \rightarrow S^1$ by $\varphi(x) = e^{x^2}$ and, for $v \in \mathbb{R}$, $u : S^1 \rightarrow \mathbb{R}$ by $u(\varphi(x)) = v(x)$ and writing $J(\varphi(x)) = \tilde{J}(x)$, where $\tilde{J}$ denotes the $2\tau$ periodic extension of the restriction of $J$ to interval $[-\tau, \tau]$, $\tau > 1$, the equation (4.1) is equivalent to equation

$$\frac{\partial u(w,t)}{\partial t} = -u(w,t) + \int_{S^1} J(wz^{-1})(1 + e^{-u(z)})^{-1}dz + h.$$  \hspace{1cm} (4.2)

with now $dz = \frac{\tau}{\pi}d\theta$, where $d\theta$ denotes integration with respect to arc length.
4.1 Check hypotheses

The function $f$ satisfies the hypotheses (H1) and (H2) and (H4), with $k_1 = S_{\text{max}} = 1$, $L = \ln 2$ and $k_2 = \frac{1}{2}$ in [1,5] and the function $J$ satisfies the hypothesis (H3)-b assumed in the Section 3.

In fact, note that $f'(x) = (1 + e^{-x})^{-2}e^{-x} > 0$. Then, since $1 < (1 + e^{-x})^2 \leq 4$, $\forall x \in \mathbb{R}$, follows that

$$
\frac{1}{4} \leq (1 + e^{-x})^{-2} < 1.
$$

Thus

$$
|f(x) - f(y)| < |x - y|.
$$

In particular, since $f(0) = \frac{1}{2}$, we have

$$
|f(x)| < |x| + \frac{1}{2}, \forall x \in \mathbb{R}.
$$

Furthermore, since $f''(x) = 2(1 + e^{-x})^{-3}e^{-2x} - (1 + e^{-x})^{-2}e^{-x}$, we have $|f''(x)| < 3$, $\forall x \in \mathbb{R}$, it implies that $f'$ is locally Lipschitz. Hence (H1) and (H4) are satisfied.

To verify (H2), we begin by noting that $0 < |(1 + e^{-x})^{-1}| < 1$ and $f^{-1}(x) = -\ln(\frac{1-x}{x})$.

Thus by a direct computation we obtain that, for $0 \leq s \leq 1$,

$$
\left| \int_0^s -\ln\left(\frac{1-x}{x}\right)dx \right| \leq \ln 2.
$$

Finally, to verify (H3), fix an equilibrium solution $u_0$ of [1,2], then from Remark 3.1

$$
u'_0 = J * ((f'(u_0)u'_0)),
$$

that is, zero is eigenvalue of $DF_u(u_0)$ with eigenfunction $u'_0$. Now, from Remark 3.2 $DF_u(u_0)$ is self-adjoint operator. Then, to prove that zero is simple eigenvalue, it is enough to show that if $v \in \text{Ker}(DF_u(u_0))$ then, $v = \lambda u_0$ for some $\lambda \in \mathbb{R}$.
For this, let \( v \in L^2(S^1) \) be such that \( DF_u(u_0)(v) = 0 \). Then

\[
v = J \ast ((f' \circ u)v).
\]

Hence, using Holder inequality, for any \( \lambda \in \mathbb{R} \), we have

\[
|v(w) - \lambda u_0'(w)| = |J \ast [f'(u_0)v - \lambda f'(u_0)u_0'](w)| \\
\leq |J \ast [f'(u_0)v - f'(u_0)\lambda u_0']|(w) \\
\leq \sqrt{2\tau} \|J\|_{\infty} \|f'(u_0)v - f'(u_0)\lambda u_0'\|_{L^2}.
\]

But

\[
\|f'(u_0)v - f'(u_0)\lambda u_0'\|_{L^2} = \|f'(u_0)[v - \lambda u_0']\|_{L^2} \\
< k_1 \|v - \lambda u_0'\|_{L^2} \\
= \|v - \lambda u_0'\|_{L^2}.
\]

Then

\[
|v(w) - \lambda u_0'(w)| \leq \sqrt{2\tau} \|J\|_{\infty} \|v - \lambda u_0'\|_{L^2}.
\]

Now, since \( 0 \leq \tilde{J}(x) \leq e^{-1} \), follows that \( \|J\|_{\infty} \leq \frac{1}{e} \). Thus

\[
\|v - \lambda u_0'\|_{L^2} \leq \frac{2\tau}{e} \|v - \lambda u_0'\|_{L^2}.
\]

It implies

\[
(1 - \frac{2\tau}{e}) \|v - \lambda u_0'\|_{L^2} \leq 0.
\]

Thus, choosing \( \tau \) such that \( \frac{2\tau}{e} < 1 \), follows that \( v = \lambda u_0' \) in \( L^2(S^1) \). Hence, zero is simple eigenvalue of \( DF_u(u_0) \).

Therefore all results of Sections 2 and 3 are valid for the flow generated by equation (4.2).
4.2 Concluding remarks

Remark 4.1. In (4.1), this choice for $\tilde{J}$ implies that we are in the case of lateral-inhibition type fields (short-range excitation and long-range inhibition), (see for example, [7], [13], and [22]). Similar connection functions (type "Mexican hat") as $\tilde{J}(x) = e^{-a|x|}$, $a > 0$, $\tilde{J}(x) = 2\sqrt{b}\pi e^{-bx^2}$, $b > 0$ or $\tilde{J}(x) = e^{-a|x|} - e^{-b|x|}$, $0 < a < b$, has been used often in previous work, (see, for example, [9], [10], [19], [20] and [22]). Hoping to make the model more realistically the connectivity existing in the prefrontal cortex, in [16] is considered the synaptic connection function $\tilde{J}(x) = e^{-b|x|}(b\sin|x| + \cos x)$, which changes sign infinitely often.

Remark 4.2. Note that, the equivalence between the equations (4.1) and (4.2), given in the formulation above, implies that the lateral-inhibition type connectivity function (short-range excitation and long-range inhibition) in (4.1), when restrict to space of $2\pi$-periodic functions, results in a recurrent-excitation type connectivity function in (4.2). Therefore, thus as in [16], we hope have a connectivity function $J$ that represents more realistically the connectivity existing in brain activities, since it is known that electrical discharges from brain cells result in a recurrent seizure disorder such as migraine and epilepsy (see, for example, [23]).

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