Horizon area–angular momentum–charge–magnetic flux inequalities in the 5D Einstein–Maxwell-dilaton gravity

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Abstract
In this paper, we consider 5D spacetimes satisfying the Einstein–Maxwell–dilaton gravity equations which are $U(1)^2$ axisymmetric but otherwise highly dynamical. We derive inequalities between the area, the angular momenta, the electric charge and the magnetic fluxes for any smooth stably outer marginally trapped surface.

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1. Basic notions and setting the problem

The study of inequalities between the horizon area and the other characteristics of the horizon has attracted a lot of interest recently. Within the general theory of relativity, lower bounds for the area of dynamical horizons in terms of their angular momentum or/and charge were given in [1–8], generalizing the similar inequalities for the stationary black holes [9–11]. These remarkable inequalities are based solely on general assumptions and they hold for any axisymmetric but otherwise highly dynamical horizon in general relativity. For a nice review on the subject, we refer the reader to [12]. The relationship between the proofs of the area–angular momentum–charge inequalities for quasi-local black holes and stationary black holes is discussed in [13–15]. Inequalities between the horizon area, the angular momentum and the charges were also studied in some four-dimensional (4D) alternative gravitational theories [16].

A generalization of the 4D horizon area–angular momentum inequality to $D$-dimensional vacuum Einstein gravity with a $U(1)^{D-3}$ group of spatial isometries was given in [17]. The purpose of this work is to derive some inequalities between the horizon area, horizon angular momentum, horizon charges and horizon magnetic fluxes in the five-dimensional (5D) Einstein–Maxwell-dilaton gravity including as a particular case the 5D Einstein–Maxwell gravity. It should be stressed that the derivation of the mentioned inequalities in the higher dimensional Einstein–Maxwell and Einstein–Maxwell-dilaton gravity is much more difficult and is not so straightforward as in the higher dimensional vacuum gravity even in spacetimes
admitting the $U(1)^{D-3}$ isometry group. The main reason behind this is the lack of the nontrivial group of hidden symmetries for the dimensionally reduced Einstein–Maxwell-dilaton equations in the general case\(^1\) [18]. In contrast, the dimensionally reduced vacuum Einstein equations (in spacetimes with the $U(1)^{D-3}$ isometry group) possess the nontrivial group of hidden symmetries, namely $SL(D-2, \mathbb{R})$ and a matrix sigma model presentation is possible. Some of the difficulties due to the presence of a Maxwell field can be circumvented by following a method similar to that used in the 4D Einstein–Maxwell-dilaton gravity [16] as we have shown below.

Let $(\mathcal{M}, g_{ab}, F_{ab}, \phi)$ be a 5D spacetime satisfying the Einstein–Maxwell-dilaton equations

\begin{align}
G_{ab} &= 2\partial_a\phi\partial_b\phi - \nabla^c\phi\nabla_c g_{ab} - 2V(\phi)g_{ab} + 2e^{-2\phi}\left(F_{ac}F^c_b - \frac{g_{ab}}{4}F_{cd}F^{cd}\right), \\
\nabla_a(e^{-2\phi}F^{ab}) &= 0 = \nabla[bF_{bc}], \\
\nabla_a\nabla^a\phi &= \frac{dV(\phi)}{d\phi} - \frac{\alpha}{2}e^{-2\phi}F_{cd}F^{cd},
\end{align}

where $g_{ab}$ is the spacetime metric, $\nabla_a$ is its Levi-Civita connection, $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ is the Einstein tensor and $F_{ab}$ is the Maxwell field. The dilaton field is denoted by $\phi$, $V(\phi)$ is its potential and $\alpha$ is the dilaton coupling parameter. We assume that the dilaton potential is non-negative, $V(\phi) \geq 0$. The Einstein–Maxwell gravity is recovered by first putting $\alpha = 0$ and $V(\phi) = 0$ and then $\phi = 0$.

As an additional technical assumption, we require the spacetime to admit a $U(1)^2$ group of spatial isometries. The commuting Killing fields are denoted by $\eta_1$ and $\eta_2$, and they are normalized to have a period $2\pi$. We also require the Maxwell and the dilaton fields to be invariant under the flow of the Killing fields, i.e. $\xi_{\eta_1}F = \xi_{\eta_2}\phi = 0$.

Let us further consider a compact closed smooth submanifold $B$ of dimension $\dim B = 3$ invariant under the action of $U(1)^2$. The induced metric on $B$ and its Levi-Civita connection are denoted by $\eta_{ab}$ and $D_a$, respectively. The future directed null normals to $B$ will be denoted by $n$ and $l$ with the normalization condition $g(n, l) = -1$ and with $-l$ pointing outward. In what follows, we require $B$ to be a stably outer-marginally trapped surface which means that $\Theta_n = 0$ and $\xi \Theta_n \leq 0$ with $\Theta_n$ being the expansion of $n$ on $B$.

As a three-dimensional compact manifold with an action of $U(1)^2$, $B$ is topologically either $S^3$, $S^2 \times S^1$ or a lens space $L(p, q)$ with $p$ and $q$ being co-prime integers [22, 23]. Moreover, the factor space $\hat{B} = B/U(1)^2$ can be identified with the closed interval $[-1, +1]$. As was shown in [22, 23], certain linear combinations of the Killing fields $\eta_i$, with integer coefficients, vanish at the ends of the factor space. In other words, there exist integer vectors $a_k \in \mathbb{Z}^3$ such that $a_k^I\eta_l \to 0$ at $x = \pm 1$, where $x$ is the coordinate parameterizing the factor space. Equivalently, the Gram matrix defined by

\[ H_{IJ} = g(\eta_I, \eta_J) \]

is invertible in the interior of the interval $[-1, 1]$ and has a one-dimensional kernel at the interval end points, i.e. $H_{IJ}a_k^I \to 0$ at $x = \pm 1$.

In fact, the integer vectors $a_k$ determine the topology of $B$. By a global $SL(2, \mathbb{Z})$ redefinition of the Killing fields [22, 23], we may present $a_k$ in the form $a_k = (1, 0)$ and $a_+ = (p, q)$ with $p$ and $q$ being the co-prime integers. The topology of $B$ is then $S^3$ when $(p = \pm 1, q = 0)$, $S^2 \times S^1$ when $(p = 0, q = \pm 1)$ and that of a lens space $L(p, q)$ in the other cases.

\(^1\) Fortunately, there are sectors in the Einstein–Maxwell gravity which are completely integrable [19–21].
Proceeding further, we consider a small neighborhood \( O \) of \( B \). When the neighborhood is sufficiently small it can be foliated by the two-parametric copies \( B(u, r) \) of \( B = B(0, 0) \) and parameterized by the so-called null Gauss coordinates defined by a well-known procedure \([18]\). In the null Gauss coordinates the metric in \( O \) can be written in the form

\[
g = -2 \, du (dr - r^2 \, \Upsilon \, du - \beta_a \, dy^a) + \gamma_{ab} \, dy^a \, dy^b ,
\]

where \( n = \frac{\alpha}{\eta^2} \), \( l = \frac{\beta}{\eta^2} \), and the function \( \Upsilon \) and the metric \( \gamma \) are invariantly defined on each \( B(u, r) \). Using these coordinates one can show that on \( B \) it holds

\[
R_\gamma - D^a \beta_a - \frac{1}{2} \beta^a \beta_a - 2 \tilde{G}_{ab} n^a p^b = -2 \rho \Theta_n \geq 0,
\]

where it has been taken into account that \( \Theta_n = 0 \) on \( B \). Here \( R_\gamma \) and \( D^a \) are the Ricci scalar curvature and Levi-Civita connection, respectively, with respect to the metric \( \gamma_{ab} \) on \( B \). Taking into account that the dilaton potential is non-negative, this inequality can be rewritten in the form

\[
R_\gamma - D^a \beta_a - \frac{1}{2} \beta^a \beta_a - 2 \tilde{G}_{ab} n^a p^b = 2V(\phi) - 2 \rho \Theta_n \geq 0,
\]

where \( \tilde{G}_{ab} = G_{ab} + 2V(\phi) g_{ab} \).

Making use of (7), for every axisymmetric function \( f \) (i.e. every function \( f \) invariant under the isometry group), we have

\[
0 \leq \int_B \left( -D^a \beta_a - \frac{1}{2} \beta^a \beta_a + R_\gamma - 2 \tilde{G}_{ab} n^a p^b \right) f^2 \, dS = \int_B \left( 2 f \beta^a D_a f - \frac{1}{2} \beta^a \beta_a f^2 + R_\gamma f^2 - 2 \tilde{G}_{ab} n^a p^b f^2 \right) \, dS.
\]

where \( dS \) is the surface element on \( B \). Now, we consider the unit tangent vector \( N^a \) on \( B \) which is orthogonal to \( \eta_1 \). With its help and taking into account that \( (\gamma^{ab} - N^a N^b) D_b f = 0 \), we find

\[
\beta_a \beta^a = (\gamma^{ab} - N^a N^b) \beta_a \beta^a + (N^a \beta_a)^2 \quad \text{and} \quad 2 f \beta^a D_a f = 2 (f N^a \beta_a) (N^b D_b f) \]

which gives

\[
0 \leq \int_B \left[ 2 (f N^a \beta_a) (N^b D_b f) - \frac{1}{2} (N^a \beta_a)^2 f^2 - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_a \beta_b f^2 + R_\gamma f^2 - 2 \tilde{G}_{ab} n^a p^b f^2 \right] \, dS.
\]

Finally, taking into account that

\[
2 (f N^a \beta_a) (N^b D_b f) - \frac{1}{2} (N^a \beta_a)^2 f^2 \leq 2 (N^b D_b f)^2
\]

and that \( (N^b D_b f)^2 = (\gamma^{ab} - N^a N^b) D_a f D_b f + N^a N^b D_a f D_b f = \gamma^{ab} D_a f D_b f = D_a f D^b f \), we obtain the important inequality

\[
0 \leq \int_B \left[ 2 D_a f D^a f - \frac{1}{2} \gamma^{ab} \beta_a \beta_b f^2 + R_\gamma f^2 - 2 \tilde{G}_{ab} n^a p^b f^2 \right] \, dS.
\]

In order to extract the constructive information from this inequality, we should perform a dimensional reduction and express the inequality as an inequality on the factor space \( \vec{B} = B / U^2(1) = [-1, 1] \). The dimensional reduction can be performed along the lines of \([18]\). So we shall give here only some basic steps and results without going into detail. As the first step, it is very convenient to present the Killing fields \( \eta_j \) in the adapted coordinates, i.e. \( \eta_j = \frac{\partial}{\partial \phi^j} \), where the coordinates \( \phi^j \) are \( 2\pi \)-periodic. Then the induced metric \( \gamma_{ab} \) on \( B \) takes the form

\[
\gamma = \frac{dx^2}{\tilde{C}^2 r^2} + H_{ij} \, d\phi^i \, d\phi^j ,
\]

\[\text{\textsc{Class. Quantum Grav. 30 (2013) 115010}}\]
where \( C > 0 \) is a constant and \( h = \det(H_{ij}) \). The absence of conical singularities requires the following condition to be satisfied:

\[
\lim_{x \to \pm 1} C^2 \frac{h}{1 - x^2} \frac{H_{ij}d_x a_i d_x a_j}{1 - x^2} = 1.
\]

(13)

The area \( \mathcal{A} \) of \( B \) can be easily found from (12) and the result is

\[
\mathcal{A} = 8\pi^2 C^{-1}.
\]

(14)

Therefore, condition (13) can be rewritten in the form

\[
\mathcal{A} \times \frac{1}{8\pi^2} = \lim_{x \to \pm 1} \left( \frac{h}{1 - x^2} \frac{H_{ij}d_x a_i d_x a_j}{1 - x^2} \right)^{1/4}
\]

(15)

Since the factor space \( O/U(1)^2 \) is simply connected, we can introduce the electromagnetic potentials \( \Phi_I \) and \( \Psi \) invariant under the isometry group and defined by

\[
d\Phi_I = i_{\eta_i} F, \quad d\Psi = e^{-2\omega} i_{\eta_1} i_{\eta_2} \star F.
\]

(16)

The Maxwell 2-form can then be written in the form

\[
F = H^{ij} \eta_i \wedge d\Psi + h^{-1} e^{2\omega} \star (d\Psi \wedge \eta_1 \wedge \eta_2).
\]

(17)

Using the field equations, one can show that there exist potentials \( \chi_I \) invariant under the isometry group such that the twist \( \omega_I = \star (\eta_1 \wedge \eta_2 \wedge d\eta_I) \) satisfies

\[
\omega_I = d\chi_I + 2\Phi_I d\Psi - 2\Psi d\Phi_I.
\]

(18)

By the direct computation of the twist using the metric (5), one finds that on \( B \) it holds

\[
\beta_i = i_{\eta_i} \beta = C_1 \pi \omega_I \text{ or in the explicit form}
\]

\[
\beta_i = \partial_i \chi_I + 2\Phi_I \partial_i \Psi - 2\Psi \partial_i \Phi_I.
\]

(19)

Also, one can show that on \( B \) we have

\[
\tilde{G}_{ab} e^{\varphi} h^a b = D_a \varphi D^b \varphi + 2e^{-2\omega} H^{ij} D_i \Phi_J D^b \Phi_J + h^{-1} e^{2\omega} D_a \Psi D^b \Psi.
\]

(20)

Using the explicit form (12) of the metric induced on \( B \), by the direct computation, we find

\[
R_y = C^2 h \left[ \frac{-\partial_j h}{h} + \frac{1}{4} h^{-2} (\partial_i h)^2 - \frac{1}{4} \text{Tr} \left( H^{-1} \partial_i h \right)^2 \right].
\]

(21)

Finally, choosing

\[
f = \left( \frac{1 - x^2}{h} \right)^{1/2},
\]

(22)

substituting (19), (20) and (21) into (11) and taking into account that \( dS = C^{-1} dx \prod_i dx_i \), we obtain

\[
\int_{-1}^{1} \left( 1 - x^2 \right) \left[ \frac{1}{8} \text{Tr} (H^{-1} \partial_i H)^2 + \frac{1}{8} h^{-2} (\partial_i h)^2 \right.
\]

\[
+ \frac{1}{4} h^{-1} H^{ij} \left( \partial_i \chi_J + 2\Phi_j \partial_i \Psi - 2\Psi \partial_i \Phi_J \right) (\partial_j \chi_I + 2\Phi_i \partial_j \Psi - 2\Psi \partial_j \Phi_I)
\]

\[
+ e^{-2\omega} H^{ij} \partial_i \Phi_I \partial_j \Phi_J + e^{2\omega} h^{-1} (\partial_i \Psi)^2 + (\partial_i \varphi)^2 \left] - \frac{1}{1 - x^2} \right) dx \leq 0.
\]

(23)

Now, we can introduce the strictly positive definite metric \( G_{AB} \) given by

\[
G_{AB} dx^A dx^B = \frac{1}{8} \text{Tr} (H^{-1} dH)^2 + \frac{1}{8} h^{-2} (dh)^2
\]

\[
+ \frac{1}{4} h^{-1} H^{ij} (d\chi_I + 2\Phi_i d\Psi - 2\Psi d\Phi_I) (d\chi_J + 2\Phi_j d\Psi - 2\Psi d\Phi_J)
\]

\[
+ e^{-2\omega} H^{ij} d\Phi_I d\Phi_J + e^{2\omega} h^{-1} (d\Psi)^2 + (d\varphi)^2
\]

(24)
on the nine-dimensional manifold $\mathcal{N} = \{(H_{IJ}(I \leq J), \chi_I, \Phi_I, \Psi, \phi) \in \mathbb{R}^9; h > 0\}$. In terms of this metric, the inequality (23) takes the form
\[
I_\sigma[X^A] = \int_{-1}^{1} \left(1 - x^2\right) G^{AB} \frac{dx^A}{dx} \frac{dx^B}{dx} - \frac{1}{1 - x^2} \right] dx \leq 0.
\]

In order to transform this inequality into an inequality for the area, we use condition (15) which combined with (25) gives
\[
A \geq 8\pi^2 x(\sigma^1),
\]
where
\[
I[X^A] = I_\sigma[X^A] + \frac{1}{4} \chi \ln \left[ \frac{h}{1 - x^2} H_{IJ}(\sigma(\chi)) d\sigma(\chi) \right]_{x=1}^{x=-1}
\]
with $d\sigma(\chi)$ defined by $d\sigma(\chi) = \frac{1}{2}(1 + \chi) d\alpha + \frac{1}{2}(1 - \chi) d\alpha$. We should note that there is an ambiguity in defining the functional $I[X^A]$. For example, we can define it by
\[
I[X^A] = aI_\sigma[X^A] + \frac{1}{4} \chi \ln \left[ \frac{h}{1 - x^2} H_{IJ}(\sigma(\chi)) d\sigma(\chi) \right]_{x=1}^{x=-1},
\]
where $a$ is an arbitrary positive number. This ambiguity, however, does not affect the final results, since $I[X^A] = 0$ as we show below.

2. Minimizer existence lemma

In order to put a lower bound on the area, we should find the minimum of the function $I[X^A]$ with appropriate boundary conditions if the minimum exists. Below we show that in certain cases the minimum exists. The natural class of functions for the minimizing problem is given by $\sigma = -\ln \left( \frac{1 + x}{1 - x} \right) \in C^\infty[-1, 1], \ln \left( \frac{h + a(\phi)}{h - a(\phi)} \right) \in C^\infty[-1, 1], (\chi_I, \Phi_I, \Psi, \phi) \in C^\infty[-1, 1]$ with boundary conditions $\sigma(\pm 1) = \sigma^\pm, (\chi_I(\pm 1), \Phi_I(\pm 1), \Psi(\pm 1)) = (\chi_I^\pm, \Phi_I^\pm, \Psi^\pm, \phi^\pm)$. Since the electromagnetic potentials and the twist potential are defined up to a constant, without loss of generality, we can choose
\[
\chi^+_I = -\chi^-_I, \quad \Phi^+_I = -\Phi^-_I, \quad \Psi^+_I = -\Psi^-_I.
\]

Lemma 1. For the dilaton coupling parameter satisfying $0 \leq \gamma^2 \leq \frac{8}{3}$, there exists a unique smooth minimizer of the functional $I[X^A]$ with the prescribed boundary conditions.

Proof. Let us consider the truncated functional
\[
I_\sigma[X^A][\sigma, \phi] = \int_{-1}^{1} \left(1 - x^2\right) G_{AB} \frac{dx^A}{dx} \frac{dx^B}{dx} - \frac{1}{1 - x^2} \right] dx,
\]
with $-1 < \sigma < \phi < 1$. By introducing a new variable $t = \frac{1}{4} \ln \left( \frac{1 + x}{1 - x} \right)$, the truncated functional takes the form
\[
I_\sigma[X^A][\sigma, \phi] = \int_{-1}^{1} G_{AB} \frac{dx^A}{dt} \frac{dx^B}{dt} - 1\right] dt,
\]
which is just a modified version of the geodesic functional in the Riemannian space $(\mathcal{N}, G_{AB})$. Consequently, the critical points of the functional are geodesics in $\mathcal{N}$. It was shown in [18] that for $0 \leq \alpha^2 \leq \frac{8}{3}$ the Riemannian space $(\mathcal{N}, G_{AB})$ is simply connected, geodesically complete and with negative sectional curvature. Therefore, for the fixed points $X^I(t_I)$ and $X^A(t_2)$ there exist a unique minimizing geodesic connecting these points. Therefore, the global minimizer of $I_\sigma[X^A][\sigma, \phi]_I$ exists and is unique for $0 \leq \alpha^2 \leq \frac{8}{3}$. Since $(\mathcal{N}, G_{AB})$ is geodesically complete,
the global minimizer of $I_{\epsilon}[X^A]\|t_1; t_1\rangle$ can be extended to a global minimizer of $I_{\epsilon}[X^A]\|t_1; t_1\rangle$. Indeed, let us take $x_1(\epsilon) = -1 + \epsilon$ and $x_2(\epsilon) = 1 - \epsilon$ (i.e., $t_1(\epsilon) = -t_2(\epsilon) = \frac{1}{2} \ln \left(\frac{1}{\epsilon^2}\right)$) with $\epsilon$ being a small positive number and consider the truncated functional

$$I_{\epsilon}^*[X^A] = \int_{x_1(\epsilon)}^{x_2(\epsilon)} \left[ (1 - x^2)G_{AB} \frac{dX^A}{dx} \frac{dX^B}{dx} - \frac{1}{1 - x^2} \right] dx,$$

with the boundary conditions $X^A(x_1(\epsilon))$ and $X^A(x_2(\epsilon))$. Consider now the unique minimizing geodesic $\Gamma_\epsilon$ between the points $X^A(x_1(\epsilon))$ and $X^A(x_2(\epsilon))$. Then, we have

$$I_{\epsilon}^*[X^A] \geq I_{\epsilon}^*[X^A]|_{\Gamma_\epsilon},$$

where the right-hand side of the above inequality is evaluated on the geodesic $\Gamma_\epsilon$. Taking into account that $\lambda_\epsilon^2 = G_{AB} \frac{dx^A}{dr} \frac{dx^B}{dr}$ is constant on the geodesic $\Gamma_\epsilon$, we obtain

$$I_{\epsilon}^*[X^A]|_{\Gamma_\epsilon} = \int_{x_1(\epsilon)}^{x_2(\epsilon)} \left[ G_{AB} \frac{dX^A}{dr} \frac{dX^B}{dr} - 1 \right] dr = (\lambda_\epsilon^2 - 1) (t_2(\epsilon) - t_1(\epsilon)). \tag{34}$$

Our next step is to evaluate $\lambda_\epsilon^2$ and this can be done by evaluating $G_{AB} \frac{dx^A}{dr} \frac{dx^B}{dr}$ at the boundary points which are in a small neighborhood of the poles $x = \pm 1$. For this purpose, we first write $\lambda_\epsilon^2$ in the form

$$\lambda_\epsilon^2 = \frac{(1 - x^2)^2}{8} \text{Tr} \left( H^{-1} \frac{dH}{dx} \right)^2 + \frac{(1 - x^2)^2}{8} h^{-2} \left( \frac{dh}{dx} \right)^2$$

$$+ \frac{(1 - x^2)^2}{4} h^{-1} H^{ij} \left( \frac{d\chi_i}{dx} + 2\Phi_i \frac{d\psi}{dx} - 2\psi \frac{d\Phi_i}{dx} \right) \left( \frac{d\chi_j}{dx} + 2\Phi_j \frac{d\psi}{dx} - 2\psi \frac{d\Phi_j}{dx} \right)$$

$$+ (1 - x^2)^2 e^{-2\psi} h^{-1} \left( \frac{d\chi_i}{dx} + 2\Phi_i \frac{d\psi}{dx} - 2\psi \frac{d\Phi_i}{dx} \right) \left( \frac{d\chi_j}{dx} + 2\Phi_j \frac{d\psi}{dx} - 2\psi \frac{d\Phi_j}{dx} \right)^2. \tag{35}$$

Within the class of functions we consider, we have

$$\frac{(1 - x^2)^2}{8} h^{-2} \left( \frac{dh}{dx} \right)^2 = \frac{1}{2} + O(\epsilon) \tag{36}$$

in a small neighborhood of the poles.

In order to estimate the term associated with $H$, we take into account that $H^{-1} \frac{dH}{dx}$ satisfies its own characteristic equation, namely $\text{Tr} \left( H^{-1} \frac{dH}{dx} \right)^2 = h^{-2} \left( \frac{dh}{dx} \right)^2 - 2h^{-1} \det \frac{dH}{dr}$. Hence, we find

$$\frac{(1 - x^2)^2}{8} \text{Tr} \left( H^{-1} \frac{dH}{dx} \right)^2 = \frac{1}{2} + O(\epsilon). \tag{37}$$

Proceeding further, we note that $\partial / \partial \chi_i$ are the Killing fields for the metric $G_{AB}$ and consequently we have the following constants of motion on the geodesics $\Gamma_\epsilon$:

$$\frac{1}{2} h^{-1} H^{ij} \left( \frac{d\chi_i}{dr} + 2\Phi_i \frac{d\psi}{dr} - 2\psi \frac{d\Phi_i}{dr} \right) = \frac{1}{2} h^{-1} H^{ij} \left( \frac{d\chi_j}{dx} + 2\Phi_j \frac{d\psi}{dx} - 2\psi \frac{d\Phi_j}{dx} \right) = c_i. \tag{38}$$

Hence, we obtain

$$\frac{(1 - x^2)^2}{4} h^{-1} H^{ij} \left( \frac{d\chi_i}{dx} + 2\Phi_i \frac{d\psi}{dx} - 2\psi \frac{d\Phi_i}{dx} \right) \left( \frac{d\chi_j}{dx} + 2\Phi_j \frac{d\psi}{dx} - 2\psi \frac{d\Phi_j}{dx} \right)$$

$$= h H^{ij} c_i c_j = O(\epsilon). \tag{39}$$
For the remaining terms, it is easy to see that they behave as
\[
(1 - x^2)^2 e^{-2\omega \Psi H} H \frac{d\Phi_i}{dx} \frac{d\Phi_j}{dx} = O(\epsilon),
\]
(40)
\[
(1 - x^2)^2 e^{2\omega \Psi H^{-1}} \left( \frac{d\Psi}{dx} \right)^2 = O(\epsilon),
\]
(41)
\[
(1 - x^2)^2 \left( \frac{d\phi}{dx} \right)^2 = O(\epsilon^2).
\]
(42)

Summarizing the results so far, we conclude that the behavior of \( \lambda^2 \) for small \( \epsilon \) is
\[
\lambda^2 = 1 + O(\epsilon).
\]
(43)

Therefore, we have
\[
\lim_{\epsilon \to 0} I_\epsilon^* [X^A] = 0.
\]
(44)
which, in view of (33), gives
\[
I_\epsilon [X^A] = \lim_{\epsilon \to 0} I_\epsilon^* [X^A] \geq 0.
\]
(45)

Therefore, there exists a unique global minimizer of the functional \( I_\epsilon [X^A] \). Since the functionals \( I[X^A] \) and \( I_\epsilon [X^A] \) differ in boundary terms, the global minimizer of \( I_\epsilon [X^A] \) is also a global minimizer of \( I[X^A] \). This completes the proof.

It should be noted that from (25) and (45) immediately follows that \( I_\epsilon^* [X^A] = 0 \).

The extremal stationary near horizon geometry is in fact defined by the same variational problem with the same boundary conditions and by the same class of functions. Therefore, as a direct consequence of the proven lemma, we obtain the following.

**Corollary.** For every dilaton coupling parameter \( \alpha \) in the range \( 0 \leq \alpha^2 \leq \frac{8}{3} \), the area \( A \) of \( B \) satisfies the inequality
\[
A \geq A_{ENHG},
\]
(46)
where \( A_{ENHG} \) is the area associated with the extremal stationary near horizon geometry of the Einstein–Maxwell-dilaton gravity with \( V(\phi) = 0 \), for the corresponding \( \alpha \). The equality is saturated only for the area associated with the extremal stationary near horizon geometry with \( V(\phi) = 0 \).

**3. Horizon area–angular momenta–charge–magnetic flux inequality for the critical dilaton coupling parameter**

For the critical coupling \( \alpha^2 = \frac{8}{3} \), the Riemannian space \((N, G_{AB})\) is an \( SL(4, \mathbb{R})/O(4) \) symmetric space [16] and, therefore, there exists a matrix \( M \) such that the metric \( G_{AB} \) can be written in the form
\[
G_{AB}dX^A dX^B = \frac{1}{4} \text{Tr}(M^{-1}dM)^2,
\]
(47)
where \( M \) is a positive definite and \( M \in SL(4, \mathbb{R}) \). Finding the explicit form of the matrix \( M \) is a tedious task and here we present only the final result. The matrix \( M \) is given by
\[
M = \begin{pmatrix}
E_{2 \times 2} & 0 \\
S^T & E_{2 \times 2}
\end{pmatrix}
\begin{pmatrix}
N & 0 \\
0 & Y
\end{pmatrix}
\begin{pmatrix}
E_{2 \times 2} & S \\
0 & E_{2 \times 2}
\end{pmatrix} = \begin{pmatrix}
N & NS \\
S^T N & S^T NS + Y
\end{pmatrix},
\]
(48)
where \( E_{2 \times 2} \) is the unit \( 2 \times 2 \) matrix and \( S, N \) and \( Y \) are \( 2 \times 2 \) matrices which have the following explicit form:

\[
S = \begin{pmatrix} 2\Phi_1 & 2\Phi_2 \\ x_1 + 2\Phi_1\psi & x_2 + 2\Phi_2\psi \end{pmatrix},
\]

\[
N = e^{\sqrt{\frac{3}{2}}h^{-1}} \begin{pmatrix} e^{-4\sqrt{\frac{3}{2}}h + 4\psi^2} & -2\psi \\ 2\psi & 1 \end{pmatrix},
\]

\[
Y = e^{\sqrt{\frac{3}{2}}h}.
\]

In terms of the matrix \( M \), the Euler–Lagrange equations are

\[
\frac{d}{dx} \left[ (1 - x^2)M^{-1} \frac{dM}{dx} \right] = 0.
\]

Hence, we obtain

\[
(1 - x^2)M^{-1} \frac{dM}{dx} = 2A,
\]

where \( A \) is a constant matrix with \( \text{Tr}A = 0 \), since \( \det M = 1 \). Integrating further, we find

\[
M = M_0 \exp \left( \ln \frac{1 + x}{1 - x} A \right),
\]

with \( M_0 \) being a constant matrix with the same properties as \( M \) and satisfying \( A^T M_0 = M_0 A \). As a positive definite matrix, \( M_0 \) can be written in the form \( M_0 = B B^T \) for some constant matrix \( B \) with \( |\det B| = 1 \) and this presentation is up to an orthogonal matrix \( O \), i.e it is invariant under the transformation \( B \rightarrow BO \). This freedom can be used to diagonalize the symmetric matrix \( B^T AB^{-1} \). So we can take \( B^T AB^{-1} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \) and we obtain

\[
M = B \begin{pmatrix} \frac{1 + x}{1 - x}^{\lambda_1} & 0 & 0 & 0 \\ 0 & \frac{1 + x}{1 - x}^{\lambda_2} & 0 & 0 \\ 0 & 0 & \frac{1 + x}{1 - x}^{\lambda_3} & 0 \\ 0 & 0 & 0 & \frac{1 + x}{1 - x}^{\lambda_4} \end{pmatrix} B^T.
\]

The eigenvalues \( \lambda_i \) can be found by comparing the singular behavior of the left-hand side and the right-hand side of (55) at \( x \rightarrow \pm 1 \). Taking into account that only the matrix \( N \) in \( M \) is singular at \( x \rightarrow \pm 1 \), we find that \( \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = \lambda_4 = 0 \). Even more, if we write the matrix \( B \) in the block form

\[
B = \begin{pmatrix} B_1 & R \\ L & B_2 \end{pmatrix},
\]

where \( B_1, B_2, R \) and \( L \) are \( 2 \times 2 \) matrices, from the singular behavior at \( x \rightarrow \pm 1 \), we find

\[
B_1 E_{\pm} B_1^T = \frac{1}{4} N_{\pm},
\]

\[
B_1 E_{\pm} L^T = \frac{1}{4} N_{\pm} S_{\pm},
\]

\[
L E_{\pm} L^T = \frac{1}{4} S_{\pm}^T N_{\pm} S_{\pm}.
\]
Here, the matrices \( E_\pm \), \( N_\pm \) and \( S_\pm \) are defined as follows:

\[
E_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]

\[
N_\pm = \lim_{x \to \pm 1} (1 - x^2)N = e^{\sqrt{\gamma}v_+ + \sigma_+} \begin{pmatrix} 4\Psi^\pm & -2\Psi^\pm \\ -2\Psi^\pm & 2\Phi^\pm \end{pmatrix},
\]

\[
S_\pm = \lim_{x \to \pm 1} S = \begin{pmatrix} 2\Phi^\pm \\ \chi_1^\pm + 2\Phi^\mp \Psi^\pm \end{pmatrix} \begin{pmatrix} 2\Phi^\pm \\ \chi_2^\pm + 2\Phi^\mp \Psi^\pm \end{pmatrix}.
\] (60)

In order to explore the regular part of \( M \) at \( x \to 1 \), we consider the matrix \((1-x)M\).

Taking into account (57), we find that at \( x \to 1 \), we have

\[
(1-x)N = \frac{1}{2}N_+ + (1-x)RR^T + \frac{1}{8} (1-x)^2 N_-
\]

\[
(1-x)NS = \frac{1}{2}N_+ S_+ + (1-x)RB^T + \frac{1}{8} (1-x)^2 N_- S_-,
\]

\[
(1-x)S^TNS + (1-x)Y = \frac{1}{2}S^T N_+ S_+ + (1-x)B^T B^T + \frac{1}{8} (1-x)^2 S^T N_- S_-.
\] (61)

Using these relations after long but straightforward calculations, we obtain

\[
\lim_{x \to 1} \frac{\hbar}{1 - x^2} H_{\pm} = \frac{e^{-\sqrt{\gamma}v_+ - \sigma_+}}{16} \left[ s^T_+(a_+) - s^T_-(a_-) \right] N_+ [ s_+(a_+) - s_-(a_-) ],
\]

where

\[
s_{\pm}(a_+) = S_{\pm} a_+ = \begin{pmatrix} 2\Phi^\pm \\ \chi_1^\pm + 2\Phi^\mp \Psi^\pm \end{pmatrix} \begin{pmatrix} 2\Phi^\pm \\ \chi_2^\pm + 2\Phi^\mp \Psi^\pm \end{pmatrix} = \begin{pmatrix} 2\Phi^\pm a^T_+ \\ \chi_1^\pm a^+_1 + 2\Phi^\mp d^T_+ \Psi^\pm \end{pmatrix}.
\] (62)

By similar considerations one can show that

\[
\lim_{x \to 1} \frac{\hbar}{1 - x^2} H_{\pm} = \frac{e^{-\sqrt{\gamma}v_+ - \sigma_+}}{16} \left[ s^T_-(a_-) - s^T_+(a_+) \right] N_- [ s_+(a_-) - s_-(a_+) ],
\]

where

\[
s_{\pm}(a_-) = S_{\pm} a_- = \begin{pmatrix} 2\Phi^\pm \\ \chi_1^\pm + 2\Phi^\mp \Psi^\pm \end{pmatrix} \begin{pmatrix} 2\Phi^\pm \\ \chi_2^\pm + 2\Phi^\mp \Psi^\pm \end{pmatrix} = \begin{pmatrix} 2\Phi^\pm a^T_- \\ \chi_1^\pm a^-_1 + 2\Phi^\mp d^T_- \Psi^\pm \end{pmatrix}.
\] (63)

The above results combined with (26) and (27), when \( L\text{=}X^4 \) is zero is taken into account, give the following inequality:

\[
\sum \geq 8\pi^2 (Z_+ Z_-)^{1/4},
\]

where

\[
Z_+ = \frac{1}{16} [ s^T_+(a_+) - s^T_+(a_-) ] \Sigma_+ [ s_+(a_+) - s_-(a_-) ],
\]

\[
Z_- = \frac{1}{16} [ s^T_-(a_-) - s^T_-(a_+) ] \Sigma_+ [ s_+(a_-) - s_-(a_+) ],
\]

and

\[
\Sigma_\pm = e^{-\sqrt{\gamma}v_+ - \sigma_+} N_\pm = \begin{pmatrix} 4\Psi^\pm & -2\Psi^\pm \\ -2\Psi^\pm & 2\Phi^\pm \end{pmatrix}.
\] (64)

In order to express the inequality in the more compact form, we should relate the potential values at \( x = \pm 1 \) with the angular momenta, charges and magnetic fluxes. The full angular momenta \( J \) associated with \( B \) is given by

\[
J_i = \frac{\pi}{4} \int_B \epsilon_{ij} \left( b^T \right)_j \Psi = \frac{\pi}{2} \int_B ( \Phi_i \Psi = \Psi \Phi_i )
\]

\[
= \frac{\pi}{4} \int_B \omega \eta - \frac{\pi}{2} \int_B ( \Phi_i \Psi = \Psi \Phi_i ) = \frac{\pi}{4} \int_B \omega d\Omega,
\] (74)
where the first integral is the contribution of the gravitational field, while the second one reflects the contribution of the electromagnetic field. The direct calculation gives the following expressions for $J_I$, namely:

$$J_I = \frac{\pi}{4} (\chi^+ - \chi^-) = \frac{\pi}{2} \chi^+. \quad (75)$$

The electric charge is given by

$$Q = \frac{1}{2\pi^2} \int_B e^{-2\alpha \Phi} \star F = 2 (\Psi^+ - \Psi^-) = 4\Psi^+. \quad (76)$$

In this way, we obtain

$$A \geq 8\pi \sqrt{|J_\pm + \frac{1}{8} Q \mathcal{F}_\pm||J_\mp - \frac{1}{8} Q \mathcal{F}_\mp|}, \quad (77)$$

where

$$J_\pm = J_I a_\pm, \quad \mathcal{F}_\pm = 2\pi (\Phi^+_I - \Phi^-_I) a_\pm. \quad (78)$$

The quantities $\mathcal{F}_\pm$ can be interpreted as the magnetic fluxes through appropriately defined two-surfaces $D_\pm$. We define $D_\pm$ in the following way. First, we uplift the factor space interval $\hat{B} = [-1, 1]$ to a curve in the spacetime manifold $M$, and then we act with the isometries generated by the Killing field $a_\pm$. It is not difficult to see that the so-constructed two-dimensional surfaces $D_\pm$ have $S^2$-topology for $a_+ = \pm a_-$ and disc topology in the other cases. The magnetic fluxes through $D_\pm$ are given by

$$\mathcal{F}_\pm = \int_{D_\pm} F = 2\pi \int_B i_\pm d_i F = 2\pi d^I_\pm \int_B i_i F = 2\pi d^I_\pm \int_B^1 d\Phi_I = 2\pi a^I_\pm \int_{\hat{B}}^1 d\Phi_I = 2\pi a^I_\pm (\Phi^+_I - \Phi^-_I), \quad (79)$$

and obviously coincide with the previously defined quantities $\mathcal{F}_\pm$. In the case when the topology of $D_\pm$ is the spherical one, the magnetic fluxes are in fact (up to sign) the magnetic (dipole) charge associated with $\mathcal{B}$.

Let us summarize the results of this section in the following.

**Theorem 1.** Let $B$ be a smooth stably outer marginally trapped surface in a spacetime satisfying 5D Einstein–Maxwell-dilaton equations with a dilaton coupling parameter $\alpha^2 = 8^3$ and having an isometry group $U(1)^2$. If the dilaton potential is non-negative, then the area of $B$ satisfies the inequality

$$A \geq 8\pi \sqrt{|J_\pm + \frac{1}{8} Q \mathcal{F}_\pm||J_\mp - \frac{1}{8} Q \mathcal{F}_\mp|}, \quad (80)$$

with $J_\pm = J_I a_\pm$, where $J_I$, $Q$ and $\mathcal{F}_\pm$ are the angular momenta, the electric charge and the magnetic flux associated with $B$, respectively. The equality is saturated only for the extremal stationary near horizon geometry of the $\alpha^2 = 8^3$ Einstein–Maxwell-dilaton gravity with $V(\Phi) = 0$.

4. Horizon area–angular momenta–charge–magnetic flux inequality for the dilaton coupling parameter $0 \leq \alpha^2 \leq \frac{8}{3}$

Finding a sharp lower bound for the horizon area and for the arbitrary dilaton coupling parameter is very difficult, since the geodesic equations for arbitrary $\alpha$ cannot be integrated explicitly. Nevertheless, an important estimate can be found for a dilaton coupling parameter in the range $0 \leq \alpha^2 \leq \frac{8}{3}$. The inequality is given by the following.
Theorem 2. Let $\mathcal{B}$ be a smooth stably outer marginally trapped surface in a spacetime satisfying 5D Einstein–Maxwell–dilaton equations with a dilaton coupling parameter $0 \leq \alpha^2 \leq \frac{8}{3}$ and having the isometry group $U(1)^2$. If the dilaton potential is non-negative, then the area of $\mathcal{B}$ satisfies the inequality

$$A \geq 8\pi \sqrt{|J_+ + \frac{1}{8} Q_3^+| |J_- - \frac{1}{8} Q_3^-|},$$

(81)

with $J_\pm = J J_\pm$, where $J_T$, $Q$ and $\mathcal{F}_\pm$ are the angular momenta, the electric charge and the magnetic flux associated with $\mathcal{B}$, respectively. The equality is saturated for the extremal stationary near horizon geometry of the $\alpha^2 = \frac{8}{3}$ Einstein–Maxwell–dilaton gravity with $V(\varphi) = 0$.

Proof. The proof is a direct generalization of the one in four dimensions [16]. Let us first consider the case $0 < \alpha^2 \leq \frac{8}{3}$ and define the metric

$$\tilde{G}_{AB} \, dX^A \, dX^B = \frac{1}{8} \text{Tr} \left( H^{-1} \, dH \right)^2 + \frac{1}{8} h^{-2} (dh)^2$$

$$+ \frac{1}{4} h^{-1} H^{IJ} (dX_I + 2 \Phi_I \, d\Psi - 2 \Psi \, d\Phi_I) (dX_J + 2 \Phi_J \, d\Psi - 2 \Psi \, d\Phi_J)$$

$$+ e^{-2\alpha \varphi} H^{IJ} \, d\Phi_I \, d\Phi_J + e^{2\alpha \varphi} h^{-1} (d\Psi)^2 + \frac{3\alpha^2}{8} (d\varphi)^2,$$

(82)

and the associated functional

$$I[X^A] = \int_{-1}^{1} \left[ (1 - x^2) \tilde{G}_{AB} \, dX^A \, dX^B \frac{dx}{dx} - \frac{1}{1 - x^2} \right] dx + \frac{1}{4} x \ln \left[ \frac{h}{1 - x^2} \frac{H_{IJ} (x) a^I (x)}{1 - x^2} \right] |_{x = -1},$$

(83)

It is not difficult to see that $I[X^A] \geq \tilde{I}[X^A]$ which gives

$$A \geq 8\pi^2 \tilde{I}^{[X^A]}.$$

(84)

However, redefining the dilaton field $\varphi = \sqrt{\frac{8}{3}} \alpha \varphi$, we see that the functional $\tilde{I}[X^A]$ reduces to the functional $I[X^A]$ for the critical coupling $\alpha^2 = \frac{8}{3}$. Therefore, we can conclude that

$$A \geq 8\pi \sqrt{|J_+ + \frac{1}{8} Q_3^+| |J_- - \frac{1}{8} Q_3^-|}$$

(85)

for every $\alpha$ in the range $0 < \alpha^2 \leq \frac{8}{3}$. The continuity argument shows that the inequality also holds for the Einstein–Maxwell case $\alpha = 0$.

5. Discussion

In this paper, we derived the inequalities between the area, the angular momenta, the electric charge and the magnetic fluxes for any smooth stably outer marginally trapped surface in the 5D Einstein–Maxwell–dilaton gravity with a dilaton coupling parameter in the range $0 \leq \alpha^2 \leq \frac{8}{3}$. In proving the inequalities, we assumed that the dilaton potential is non-negative and the spacetime is $U(1)^2$ axisymmetric but otherwise highly dynamical. It is worth mentioning that all of our results still hold even in the presence of matter with an axially symmetric energy–momentum tensor satisfying the dominant energy condition.

Since the considerations in this paper are entirely quasi-local, our results can be applied to stationary axisymmetric black holes in asymptotically flat and Kaluza–Klein spacetimes, as well as in spacetimes with de Sitter asymptotic.

The approach of this paper can be easily extended to the case of 5D Einstein–Maxwell–Chern–Simons gravity with with Chern–Simons coefficient $\lambda_{CS}$.
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