Cloaking using complementary media for the Helmholtz equation and a three spheres inequality for second order elliptic equations

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Abstract

Cloaking using complementary media was suggested by Lai et al. in [11]. This was proved in [21] in the quasistatic regime. One of the difficulties in the study of this problem is the appearance of the localized resonance, i.e., the fields blow up in some regions and remain bounded in some others as the loss goes to 0. To this end, the author introduced the technique of removing localized singularity and used a standard three spheres inequality. The method used in [21] also works for the Helmholtz equation. However, it requires small size of the cloaked region for large frequency due to the use of the (standard) three spheres inequality. In this paper, we give a proof of cloaking using complementary media in the finite frequency regime without imposing any condition on the cloaked region; hence the cloak works for all frequency. To successfully apply the approach in [21], we establish a new three spheres inequality. A modification of the cloaking setting to obtain illusion optics is also discussed.

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Key words: cloaking, illusion optics, superlensing, three spheres inequality, localized resonance, negative index materials, complementary media.

1 Introduction

Negative index materials (NIMs) were investigated theoretically by Veselago in [36]. The existence of such materials was confirmed by Shelby, Smith, and Schultz in [35]. The study of NIMs has attracted a lot attention in the scientific community thanks to their interesting properties and applications. One of the appealing one is cloaking using complementary media.

Cloaking using NIMs or more precisely cloaking using complementary media was suggested by Lai et al in [11]. Their work was inspired from the notion of complementary media suggested by Pendry and Ramakrishna in [32]. Cloaking using complementary media was established in [21].
in the quasistatic regime using slightly different schemes from \[11\]. Two difficulties in the study of cloaking using complementary media are as follows. Firstly, this problem is unstable since the equations describing the phenomenon have sign changing coefficients, hence the ellipticity is lost. Secondly, the localized resonance, i.e., the field blows up in some regions and remains bounded in some others, might appear. To handle these difficulties, in \[21\] the author introduced the technique of removing localized singularity and used a standard three spheres inequality. The approach in \[21\] also involved the reflecting technique introduced in \[18\]. The method in \[21\] also works for the Helmholtz equation; however since the largest radius in the (standard) three spheres inequality is small as frequency is large (see Section \[2\] for further discussion), the size of the cloaked region is required to be small for large frequency.

In this paper, we present a proof of cloaking using complementary media in the finite frequency regime. Our goal is not to impose any condition on the size of the cloaked region (Theorem \[1\]); hence the cloak works for all frequency. To successfully apply the approach in \[21\], we establish a new three spheres inequality for the second order elliptic equations which holds for arbitrary radius (Theorem \[2\] in Section \[2\]). This inequality is inspired from the unique continuation principle and its proof is in the spirit of Protter in \[34\]. A modification of the cloaking setting to obtain illusion optics is discussed in in Section \[3\] (Theorem \[3\]). This involves the idea of superlensing in \[19\]. Cloaking using complementary media for electromagnetic waves is investigated in \[23\].

In addition to cloaking using complementary media, other application of NIMs are superlensing using complementary media as suggested in \[20\] \[30\] \[33\] (see also \[28\]) and confirmed in \[19\] \[22\], and cloaking via anomalous localized resonance \[15\] (see also \[3\] \[10\] \[20\]). Complementary media were studied in a general setting in \[18\] \[22\] and play an important role in these applications see \[17\] \[19\] \[20\] \[21\] \[22\] \[25\].

Let us describe the problem more precisely. Assume that the cloaked region is the annulus \(B_{\gamma r_2} \setminus B_r\) for some \(r_2 > 0\) and \(1 < \gamma < 2\) in which the medium is characterized by a matrix \(a\) and a function \(\sigma\). The assumption on the cloaked region by all means imposes no restriction since any bounded set is a subset of such a region provided that the radius and the origin are appropriately chosen. The idea suggested by Lai et al. in \[11\] in two dimensions is to construct a complementary media in \(B_{r_2} \setminus B_{r_1}\) for some \(0 < r_1 < r_2\).

In this paper, instead of taking the schemes of Lai et al., we use a scheme from \[21\] which is inspired but different from the ones from \[11\]. Following \[21\], the cloak contains two parts. The first one, in \(B_{r_2} \setminus B_{r_1}\), makes use of complementary media to cancel the effect of the cloaked region and the second one, in \(B_{r_1}\), is to fill the space which “disappears” from the cancellation by the homogeneous media. Concerning the first part, instead of \(B_{\gamma r_2} \setminus B_{r_2}\), we consider \(B_{r_3} \setminus B_{r_2}\) with \(r_3 = 2r_2\) (the constant 2 considered here is just a matter of simple representation) as the cloaked region in which the medium is given by

\[
\hat{a}, \hat{\sigma} = \begin{cases} a, \sigma & \text{in } B_{\gamma r_2} \setminus B_{r_2}, \\ I, & \text{in } B_{r_3} \setminus B_{\gamma r_2}. \end{cases}
\]

The complementary media in \(B_{r_2} \setminus B_{r_1}\) is given by

\[
-F_*^{-1} \hat{a}, -F_*^{-1} \hat{\sigma},
\]
where \( F : \partial B_{r_2} \setminus \bar{B}_{r_1} \to B_{r_3} \setminus \bar{B}_{r_2} \) is the Kelvin transform with respect to \( \partial B_{r_2} \), i.e.,

\[
F(x) = \frac{r_2^2}{|x|^2} x.
\]

(1.1)

Here

\[
T_* \hat{a}(y) = \frac{DT(x) \hat{a}(x) DT^T(x)}{J(x)} \quad \text{and} \quad T_* \hat{\sigma}(y) = \hat{\sigma}(x) \frac{J(x)}{J(x)},
\]

where \( x = T^{-1}(y) \) and \( J(x) = |\det DT(x)| \) for a diffeomorphism \( T \). It follows that

\[
r_1 = \frac{r_2^2}{r_3}.
\]

(1.2)

Concerning the second part, the medium in \( B_{r_1} \) is given by

\[
\left( \frac{r_2^2}{r_3^2} \right)^{d-2} I, \left( \frac{r_2^2}{r_3^2} \right)^d.
\]

(1.3)

The reason for this choice will be explained later.

With the loss, the medium is characterized by \( s_\delta A, s_0 \Sigma \), where

\[
A, \Sigma = \begin{cases} 
\hat{a}, \hat{\sigma} & \text{in } B_{r_3} \setminus B_{r_2}, \\
F_*^{-1} \hat{a}, F_*^{-1} \hat{\sigma} & \text{in } B_{r_2} \setminus B_{r_1}, \\
\left( \frac{r_2^2}{r_3^2} \right)^{d-2} I, \left( \frac{r_2^2}{r_3^2} \right)^d & \text{in } B_{r_1}, \\
I, 1 & \text{otherwise},
\end{cases}
\]

(1.4)

and

\[
s_\delta = \begin{cases} 
-1 + i \delta & \text{in } B_{r_2} \setminus B_{r_1}, \\
1 & \text{otherwise}.
\end{cases}
\]

(1.5)

Physically, the imaginary part of \( s_\delta A \) is the loss of the medium (more precisely the loss of the medium in \( B_{r_2} \setminus B_{r_1} \)). Here and in what follows, we assume that,

\[
\frac{1}{\Lambda} |\xi|^2 \leq a(x) \xi \cdot \xi \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \text{ for a.e. } x \in B_{\gamma r_2} \setminus B_{r_2},
\]

(1.6)

for some \( \Lambda \geq 1 \). In what follows, we assume in addition that

\( \hat{a} \) is Lipschitz in \( B_{r_3} \setminus B_{r_1} \).

(1.7)

We can verify that medium \( s_0 A \) is of reflecting complementary property, a concept introduced in [18, Definition 1], by considering diffeomorphism \( G : \mathbb{R}^d \setminus \bar{B}_{r_3} \to B_{r_3} \setminus \{0\} \) which is the Kelvin transform with respect to \( \partial B_{r_3} \), i.e.,

\[
G(x) = \frac{r_3^2 x}{|x|^2}.
\]

(1.8)

It is important to note that

\[
G_* F_* A = I \text{ in } B_{r_3}
\]

(1.9)
since $G \circ F(x) = (r_3^2/r_2^2)x$. This is the reason for choosing $A$ in (1.3).

Let $\Omega$ be a smooth open subset of $\mathbb{R}^d$ ($d = 2, 3$) such that $B_{r_3} \subset \subset \Omega$. Given $f \in L^2(\Omega)$, let $u_\delta, u \in H^1_0(\Omega)$ be respectively the unique solution to

$$\text{div}(s_\delta A \nabla u_\delta) + s_0 k^2 \Sigma u_\delta = f \quad \text{in} \quad \Omega,$$  \hfill (1.10)

and

$$\Delta u + k^2 u = f \quad \text{in} \quad \Omega. \hfill (1.11)$$

As in [18], we assume that equation (1.11) with $f = 0$, has only zero solution in $H^1_0(\Omega)$.

$$\hfill (1.12)$$

Our result on cloaking using complementary media is:

**Theorem 1.** Let $d = 2, 3$, $f \in L^2(\Omega)$ with supp $f \subset \Omega \setminus B_{r_3}$ and let $u$ and $u_\delta$ in $H^1_0(\Omega)$ be the unique solution to (1.10) and (1.11) resp. There exists $\gamma_0 > 1$, depending only on $\Lambda$ and the Lipschitz constant of $\hat{a}$ such that if $1 < \gamma < \gamma_0$ then

$$u_\delta \to u \text{ weakly in } H^1(\Omega \setminus B_{r_3}) \text{ as } \delta \to 0. \hfill (1.13)$$

For an observer outside $B_{r_3}$, the medium in $B_{r_3}$ looks like the homogeneous one by (1.13) (and also (1.11)): one has cloaking.

**Remark 1.** The case $k = 0$ was established in [21].

The proof of Theorem 1 is given in Section 3. It is based on the removing localized singularity technique introduced in [21] and uses a new three sphere inequality (Theorem 2) discussed in the next section. The discussion on illusion optics is given in Section 4.

## 2 Three spheres inequalities

Let $v$ be an holomorphic function defined in $B_{r_3}$, Hadamard in [8] proved the following famous three spheres inequality:

$$\|v\|_{L^\infty(\partial B_{r_2})} \leq \|v\|_{L^\infty(\partial B_{r_1})}^{\alpha} \|v\|_{L^\infty(\partial B_{r_3})}^{1-\alpha}, \hfill (2.1)$$

for all $0 < R_1 < R_2 < R_3$ where

$$\alpha = \log \left(\frac{R_3}{R_2}\right) / \log \left(\frac{R_3}{R_1}\right).$$

A three spheres inequality for general elliptic equations was proved by Landis [13] using Carleman type estimates. Landis proved [13, Theorem 2.1] that if $v$ is a solution to

$$\text{div}(M \nabla v) + \vec{b} \cdot \nabla v + cv = 0 \text{ in } B_{r_3}, \hfill (2.2)$$

\footnote{In fact, [13] Theorem 2.1] deals with the non-divergent form; however since $M$ is assumed $C^2$, the two forms are equivalent.}
where $M$ is elliptic, symmetric, and of class $C^2$, $b,c \in C^1$, and $c \leq 0$, then there is a constant $C > 0$ such that
\[
\|v\|_{L^\infty(\partial B_{R_2})} \leq C\|v\|_{L^\infty(\partial B_{R_1})}^{\alpha}\|v\|_{L^\infty(\partial B_{R_3})}^{1-\alpha},
\]  
(2.3)

for some $\alpha \in (0,1)$ depending only on $R_2/R_1, R_2/R_3$, the ellipticity constant of $M$, and the regularity constants of $M, b, c$. The assumption $c \leq 0$ is crucial and this is discussed in the next paragraph. Another proof was obtained by Agmon [1] in which he used the logarithmic convexity. Garofalo and Lin in [6] established similar results where the $L^\infty$-norm is replaced by the $L^2$-norm, and $M$ is of class $C^1$, $\overline{b}$ and $c$ are in $L^\infty$:
\[
\|v\|_{L^2(\partial B_{R_2})} \leq C\|v\|_{L^2(\partial B_{R_1})}^{\alpha}\|v\|_{L^2(\partial B_{R_3})}^{1-\alpha},
\]  
(2.4)

using the frequency function.

A typical example of (2.2) when $c > 0$ is the Helmholtz equation:
\[
\Delta v + k^2v = 0 \text{ in } B_{R_3}.
\]  
(2.5)

Given $k > 0$, neither (2.4) nor (2.3) holds for all $R_1 < R_2 < R_3$. Indeed, first consider the case $d = 2$. It is clear that for $n \in \mathbb{Z} \setminus \{0\}$, the function $J_n(kr)e^{in\theta}$ is a solution to (2.3) in $\mathbb{R}^2 \setminus \{0\}$, where $J_n$ is the Bessel function of order $n$. By taking $R_1, R_2$, and $R_3$ such that $J_n(kR_1) = 0 \neq J_n(kR_2)$, one reaches the fact that neither (2.4) nor (2.3) is valid. The same conclusion holds in the higher dimensional case by similar arguments. In the case $c > 0$, (2.4) holds under the smallness of $R_3$ (see e.g., [2, Theorem 4.1]); this condition is equivalent to the smallness of $c$ for a fixed $R_3$ by a scaling argument.

In this paper, we establish a new type of three spheres inequalities without imposing the smallness condition on $R_3$. This inequality will play an important role in the proof of Theorem 1. Define
\[
\|v\|_{H(\partial B_r)} = \|v\|_{H^{1/2}(\partial B_r)} + \|M\nabla v \cdot \nu\|_{H^{-1/2}(\partial B_r)}.
\]  
(2.6)

Here and in what follows, $\nu$ denotes the outward normal vector on a sphere.

Our result on three spheres inequalities is:

**Theorem 2.** Let $d \geq 2$, $c_1, c_2 > 0$, $0 < R_3 < R_1 < R_2 < R_3 < R^*$, and let $M$ be a Lipschitz uniformly elliptic symmetric matrix-valued function defined in $B_{R^*}$. Assume $v \in H^1(B_{R_3} \setminus \overline{B}_{R_1})$ satisfies
\[
|\text{div}(M\nabla v)| \leq c_1|\nabla v| + c_2|v|, \quad \text{in } B_{R_3} \setminus \overline{B}_{R_1}.
\]  
(2.7)

Then, for any $\lambda_0 > 1$ with $R_2 \in (\lambda_0 R_1, R_3/\lambda_0)$, there exist a constant $C$ and $q \geq 1$, depending on the elliptic and the Lipschitz constant of $M$, $C$ also depends on $c_1, c_2, R_3, R^*, d$, and $\lambda_0$ such that
\[
\|v\|_{H(\partial B_{R_3})} \leq C\|v\|_{H(\partial B_{R_1})}^\alpha\|v\|_{H(\partial B_{R_3})}^{1-\alpha},
\]  
(2.8)

where
\[
\alpha := \frac{R_2^{-q} - R_3^{-q}}{R_1^{-q} - R_3^{-q}}.
\]  
(2.9)
In Theorem 2, one does not impose any smallness condition on \( R_1, R_2, R_3 \) and the exponent \( \alpha \) is independent of \( c_1 \) and \( c_2 \). The proof of Theorem 2 is inspired from the approach of Protter in [34]. Nevertheless, different test functions are used. The ones in [34] are too concentrated at 0 and not suitable for our purpose. The connection between three spheres inequalities and the unique continuation principle, and the application of three spheres inequalities for the stability of Cauchy problems can be found in [2].

The proof of Theorem 2 is presented in the next two subsections.

2.1 Preliminaries

This section contains several lemmas used in the proof of Theorem 2. These lemmas are in the spirit of [34]. Nevertheless, the test functions used here are different from there. Let \( 0 < R_1 < R_3 < +\infty \). In this section, we assume that \( M \) is a Lipschitz symmetric matrix-valued function defined in \( \overline{B}_{R_3 \setminus B_{R_1}} \) and satisfies

\[
\frac{1}{\Lambda} |\xi|^2 \leq M(x)\xi \cdot \xi \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^d,
\]

for a.e. \( x \in \overline{B}_{R_3 \setminus B_{R_1}} \), for some \( \Lambda \geq 1 \). Set

\[
L := \|M\|_{L^\infty} + R_3 \|\nabla M\|_{L^\infty}.
\]

The first lemma is:

**Lemma 1.** Let \( d \geq 2 \) and \( z \in H^2(B_{R_3 \setminus \overline{B}_{R_1}}) \). We have

\[
\int_{B_{R_3 \setminus \overline{B}_{R_1}}} (x \cdot M \nabla z) \text{ div}(M \nabla z) \geq - \int_{B_{R_3 \setminus \overline{B}_{R_1}}} C L^2 |\nabla z|^2 - \int_{\partial(B_{R_3 \setminus \overline{B}_{R_1}}} C L^2 r |\nabla z|^2,
\]

for some positive constant \( C \) depending only on \( d \).

**Proof.** An integration by parts gives

\[
\int_{B_{R_3 \setminus \overline{B}_{R_1}}} (x \cdot M \nabla z) \text{ div}(M \nabla z) = - \int_{B_{R_3 \setminus \overline{B}_{R_1}}} \nabla (x \cdot M \nabla z) \cdot M \nabla z + \int_{\partial(B_{R_3 \setminus \overline{B}_{R_1}}} (x \cdot M \nabla z) M \nabla z \cdot \nu.
\]

Using the symmetry of \( M \), we have \(^2\)

\[
\frac{\partial}{\partial x_i}(x \cdot M \nabla z) = \frac{\partial}{\partial x_i} \left( M_{kj} x_j \frac{\partial z}{\partial x_k} \right) = M_{kj} x_j \frac{\partial^2 z}{\partial x_i \partial x_k} + M_{ki} \frac{\partial z}{\partial x_k} + x_j \frac{\partial M_{kj}}{\partial x_i} \frac{\partial z}{\partial x_k},
\]

and

\[
- \int_{B_{R_3 \setminus \overline{B}_{R_1}}} 2 x_j M_{kj} \frac{\partial^2 z}{\partial x_i \partial x_k} M_{il} \frac{\partial z}{\partial x_l} = - \int_{B_{R_3 \setminus \overline{B}_{R_1}}} x_j M_{kj} M_{il} \frac{\partial}{\partial x_k} \left( \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_l} \right) \frac{\partial z}{\partial x_l} = \int_{B_{R_3 \setminus \overline{B}_{R_1}}} \frac{\partial (x_j M_{kj} M_{il})}{\partial x_k} \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_l} - \int_{\partial(B_{R_3 \setminus \overline{B}_{R_1}}} x_j M_{kj} M_{il} \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_l} \nu. \quad (2.13)
\]

\(^2\)In what follows, the repeated summation is used.
We derive from (2.12) and (2.13) that
\[
- \int_{B_{R_3} \setminus \overline{B}_{R_1}} \nabla (x \cdot M \nabla z) \cdot M \nabla z \geq - \int_{B_{R_3} \setminus \overline{B}_{R_1}} CL^2 |\nabla z|^2 - \int_{\partial (B_{R_3} \setminus \overline{B}_{R_1})} CL^2 r |\nabla z|^2. \tag{2.14}
\]
The conclusion now follows from (2.11) and (2.14).

The second lemma is

**Lemma 2.** Let \( d \geq 2, \beta \in \mathbb{R}, \) and \( z \in H^2(B_{R_3} \setminus \overline{B}_{R_1}). \) There exists \( p_{\Lambda, L} \geq 1 \) such that if \( p \geq p_{\Lambda, L} \) and \( |\beta| R_3^{-p} \geq 2 \) then
\[
\int_{B_{R_3} \setminus \overline{B}_{R_1}} e^{\beta r^{-p}} (M x \cdot \nabla |z|^2) \text{div}(M \nabla e^{-\beta r^{-p}}) + \int_{\partial (B_{R_3} \setminus \overline{B}_{R_1})} CL^2 p^2 \beta^2 r^{-2p-1} |z|^2 \\
\geq \int_{B_{R_3} \setminus \overline{B}_{R_1}} \frac{1}{2} \Lambda^{-2} p^3 \beta^2 r^{-2p-2} |z|^2 - \int_{B_{R_3} \setminus \overline{B}_{R_1}} CL^2 |\nabla z|^2.
\]
for some positive constant \( C \) depending only on \( d. \)

**Proof.** A computation yields
\[
\text{div}(M \nabla e^{-\beta r^{-p}}) = p \beta e^{-\beta r^{-p}} [p \beta r^{-2p-4} - (p + 2) r^{-p-4}] x \cdot M x + p \beta r^{-p-2} e^{-\beta r^{-p}} \text{div}(M x).
\]
An integration by parts gives
\[
\int_{B_{R_3} \setminus \overline{B}_{R_1}} e^{\beta r^{-p}} (M x \cdot \nabla |z|^2) \text{div}(M \nabla e^{-\beta r^{-p}}) = P + Q. \tag{2.15}
\]
Here
\[
P = P_1 + P_2 + P_3
\]
with
\[
\begin{align*}
P_1 &= - \int_{B_{R_3} \setminus \overline{B}_{R_1}} p^2 \beta^2 |z|^2 \text{div} [r^{-2p-4}(x \cdot M x) M x], \\
P_2 &= \int_{B_{R_3} \setminus \overline{B}_{R_1}} p \beta (p + 2) |z|^2 \text{div} [r^{-p-4}(x \cdot M x) M x], \\
P_3 &= \int_{B_{R_3} \setminus \overline{B}_{R_1}} 2p \beta r^{-p-2} \text{div}(M x) z \nabla z \cdot M x,
\end{align*}
\]
and
\[
Q = \int_{\partial (B_{R_3} \setminus \overline{B}_{R_1})} p \beta |z|^2 \left( [p \beta r^{-2p-4} - (p + 2) r^{-p-4}] x \cdot M x \right) M x \cdot \nu.
\]
We next estimate \( P \) and \( Q. \) A computation yields
\[
- \text{div} [r^{-2p-4}(x \cdot M x) M x] = (2p + 4)(x \cdot M x)^2 r^{-2p-6} - r^{-2p-4} \text{div} [(x \cdot M x) M x].
\]
This implies
\[
P_1 \geq \int_{B_{R_3} \setminus \overline{B}_{R_1}} p^2 \beta^2 r^{-2p-2} |z|^2 [(2p + 4) \Lambda^{-2} - CL^2]. \tag{2.16}
\]
Similarly,
\[ P_2 \geq - \int_{B_{R_3} \setminus \overline{B}_R} (p + 2)p|\beta| r^{-p-2} |z|^2 [(p + 4)\Lambda^{-2} + CL^2]. \]  
(2.17)

A combination of (2.16) and (2.17) yields
\[ P_1 + P_2 \geq \int_{B_{R_3} \setminus \overline{B}_R} \Lambda^{-2} p^3 \beta^2 r^{-2p-2} |z|^2. \]  
(2.18)

Here we used the fact that \( p \geq p_{\Lambda,L} \) and \( |\beta| R_3^{-p} \geq 2 \). On the other hand, using Cauchy’s inequality, we have

\[ |P_3| \leq \int_{B_{R_3} \setminus \overline{B}_R} p^2 \beta^2 r^{-2p-2} L^2 |z|^2 + \int_{B_{R_3} \setminus \overline{B}_R} CL^2 |\nabla z|^2. \]

It follows from (2.18) that
\[ P \geq \int_{B_{R_3} \setminus \overline{B}_R} \frac{1}{2} p^2 \beta^2 \Lambda^{-2} r^{-2p-2} |z|^2 - \int_{B_{R_3} \setminus \overline{B}_R} CL^2 |\nabla z|^2, \]  
(2.19)

provided that \( p \geq 2\Lambda^2 L^2 \). Since
\[ |Q| \leq \int_{\partial(B_{R_3} \setminus \overline{B}_R)} 2\Lambda^2 p^2 \beta^2 r^{-2p-1} |z|^2. \]

the conclusion follows. \( \square \)

Using Lemmas 1 and 2, we can prove the following result.

**Lemma 3.** Let \( d \geq 2 \), \( \beta \in \mathbb{R} \), and \( v \in H^2(B_{R_3} \setminus \overline{B}_R) \). There exists a positive constant \( p_{\Lambda,L} \geq 1 \) such that if \( p \geq p_{\Lambda,L} \) and \( |\beta| R_3^{-p} \geq 2 \) then

\[ \int_{B_{R_3} \setminus \overline{B}_R} \frac{p^{p+2} e^{2\beta r^{-p}}}{2p|\beta|} \left[ \text{div}(M \nabla v) \right]^2 + \int_{B_{R_3} \setminus \overline{B}_R} CL^2 e^{2\beta r^{-p}} |\nabla v|^2 \]
\[ + \int_{\partial(B_{R_3} \setminus \overline{B}_R)} CL^2 \beta^2 r^{-2p-1} e^{2\beta r^{-p}} |v|^2 + \int_{\partial(B_{R_3} \setminus \overline{B}_R)} CL^2 e^{2\beta r^{-p}} |\nabla v|^2 \]
\[ \geq \int_{B_{R_3} \setminus \overline{B}_R} \frac{1}{2} \Lambda^{-2} p^3 \beta^2 r^{-2p-2} e^{2\beta r^{-p}} |v|^2; \]

for some positive constant \( C \) depending only on \( d \).

**Proof.** By considering the real part and the imaginary part of \( v \) separately, one might assume that \( v \) is real. Set
\[ z = e^{\beta r^{-p}} v \quad \text{equivalently} \quad v = e^{-\beta r^{-p}} z. \]

Since \( \text{div}(M \nabla (gh)) = 2\nabla h \cdot M \nabla g + h \text{div}(M \nabla g) + g \text{div}(M \nabla h) \) (\( M \) is symmetric), it follows that
\[ \text{div}(M \nabla v) = 2\beta pr^{-p-2} e^{-\beta r^{-p}} x \cdot M \nabla z + e^{-\beta r^{-p}} \text{div}(M \nabla z) + z \text{div}(M \nabla e^{-\beta r^{-p}}). \]
Using the inequality \((a + b + c)^2 \geq 2a(b + c)\), we obtain
\[
\frac{1}{2} [\text{div}(M \nabla v)]^2 \geq 2|β|pr^{-p-2}e^{-br^{-p}} (x \cdot M \nabla z) \left( e^{-br^{-p}} \text{div}(M \nabla z) + z \text{div}(M \nabla e^{-br^{-p}}) \right).
\]
This implies
\[
\int_{B_R \setminus B_{R_1}} \frac{r^{p+2}e^{2br^{-p}}}{2|β|} [\text{div}(M \nabla v)]^2 \geq \int_{B_R \setminus B_{R_1}} 2(x \cdot M \nabla z) \text{div}(M \nabla z) + \int_{B_R \setminus B_{R_1}} e^{br^{-p}} (M x \cdot \nabla |z|^2) \text{div}(M \nabla e^{-br^{-p}}).
\]
Applying Lemmas 1 and 2 we have
\[
\int_{B_R \setminus B_{R_1}} \frac{r^{p+2}e^{2br^{-p}}}{2|β|} [\text{div}(M \nabla v)]^2 \geq \int_{B_R \setminus B_{R_1}} \left( \Lambda^{-2}p^3β^2r^{-2p-2}|z|^2 - CL^2|∇z|^2 \right)
- \int_{∂(B_R \setminus B_{R_1})} \left( CL^2p^2β^2r^{-2p-1}|z|^2 + CL^2r|∇z|^2 \right). \tag{2.20}
\]
Since \(z = e^{br^{-p}}v\),
\[
|∇z|^2 \leq 2e^{2br^{-p}}(|∇v|^2 + p^2β^2r^{-2p-2}|v|^2). \tag{2.21}
\]
A combination of (2.20) and (2.21) yields, since \(p \geq p_{Λ,L}^+\),
\[
\int_{B_R \setminus B_{R_1}} \frac{r^{p+2}e^{2br^{-p}}}{2|β|} [\text{div}(M \nabla v)]^2 \geq \int_{B_R \setminus B_{R_1}} e^{br^{-p}} \left( \frac{1}{2} \Lambda^{-2}p^3β^2r^{-2p-2}|v|^2 - CL^2|∇v|^2 \right)
- \int_{∂(B_R \setminus B_{R_1})} e^{2br^{-p}} (CL^2p^2β^2r^{-2p-1}|v|^2 + CL^2r|∇v|^2).
\]
The conclusion follows. \(□\)

We also have

**Lemma 4.** Let \(d \geq 2\), \(β ∈ \mathbb{R}\), and \(v ∈ H^2(B_R \setminus B_{R_1})\). There exists a positive constant \(p_{Λ,L}^+ \geq 1\) such that if \(p \geq p_{Λ,L}^+\) and \(|β|R_1^{-p} \geq 2\) then
\[
\int_{B_R \setminus B_{R_1}} e^{br^{-p}}v \text{div}(M \nabla v) + \int_{B_R \setminus B_{R_1}} e^{br^{-p}}|∇v|^2 \\
\leq \int_{B_R \setminus B_{R_1}} Cβ^2p^2r^{-2p-2}e^{2br^{-p}}|v|^2 + \int_{∂(B_R \setminus B_{R_1})} Ce^{2br^{-p}}(r|∇v|^2 + r^{-1}|v|^2),
\]
for some positive constant \(C\) depending only on \(d\), \(Λ\), and \(L\).
Proof. We have
\[- \int_{B_R \setminus B_R} e^{2\beta r - p} \text{div}(M \nabla v) = \int_{B_R \setminus B_R} M \nabla v \cdot \nabla (e^{2\beta r - p} v) - \int_{\partial (B_R \setminus B_R)} e^{2\beta r - p} v M \nabla v \cdot \nu. \] (2.22)
On the other hand,
\[\int_{B_R \setminus B_R} M \nabla v \cdot \nabla (e^{2\beta r - p} v) = \int_{B_R \setminus B_R} e^{2\beta r - p} M \nabla v \cdot \nabla v - 2 \beta \text{pr}^{-2p - 2} e^{2\beta r - p} v M \nabla v \cdot x \] (2.23)
and
\[\int_{\partial (B_R \setminus B_R)} e^{2\beta r - p} v M \nabla v \cdot \nu \leq \int_{\partial (B_R \setminus B_R)} e^{2\beta r - p} (r|\nabla v|^2 + L^2 r^{-1}|v|^2). \] (2.24)
Since
\[2 \beta \text{pr}^{-2p - 2} v M \nabla v \cdot x \leq \frac{1}{2} \Lambda^{-1}|\nabla v|^2 + 8 \beta^2 p^2 L^2 \Lambda r^{-2p - 2}|v|^2,\]
we derive from (2.22), (2.23), and (2.24) that
\[\int_{B_R \setminus B_R} e^{2\beta r - p} v \text{div}(M \nabla v) + \int_{B_R \setminus B_R} \frac{1}{2} \Lambda^{-1} e^{2\beta r - p} |\nabla v|^2 \leq \int_{B_R \setminus B_R} C \beta^2 p^2 r^{-2p - 2} e^{2\beta r - p} |v|^2 + \int_{\partial (B_R \setminus B_R)} C e^{2\beta r - p} (r|\nabla v|^2 + r^{-1}|v|^2). \]
The conclusion follows. \qed

Lemma 5. Let \(d \geq 2, \beta \in \mathbb{R}, \) and \(v \in H^2(B_R \setminus B_R).\) There exists a positive constant \(p_{\Lambda, L} \geq 1\) such that if \(p \geq p_{\Lambda, L}\) and \(|\beta| \geq 2 R_3^{-p}\) then
\[\int_{B_R \setminus B_R} e^{2\beta r - p} |\beta| p^3 \beta^2 r^{-2p - 2} e^{2\beta r - p} |v|^2 + |\nabla v|^2 \leq C \int_{B_R \setminus B_R} p r^{p+2} e^{2\beta r - p} |\text{div}(M \nabla v)|^2 \]
\[+ C \int_{\partial (B_R \setminus B_R)} |\beta| p e^{2\beta r - p} (r|\nabla v|^2 + p^2 \beta^2 r^{-2p - 1}|v|^2). \] (2.25)
for some positive constant \(C\) depending only on \(d, \Lambda, \) and \(L.\)

Proof. Note that
\[|v \text{div}(M \nabla v)| \leq p|\beta|^2 |v|^2 r^{-2p - 2} + \frac{4}{p|\beta|} |\text{div}(M \nabla v)|^2 r^{p+2}.\]
The conclusion now follows from Lemmas \(\text{3}\) and \(\text{4}\) The details are left to the reader. \qed
2.2 Proof of Theorem 2

Let

\[ 1 < \lambda < \lambda_0, \]

(which will be defined later) and set

\[ D = B_{\lambda R_3} \setminus \overline{B}_{R_3/\lambda}. \]

Let \( u_1 \in H^1(D \setminus \partial B_{R_1}) \) and \( u_3 \in H^1(D \setminus \partial B_{R_3}) \) be respectively the unique solution to

\[
\begin{aligned}
\text{div}(M \nabla u_1) &= 0 & & \text{in } D \setminus \partial B_{R_1}, \\
[u_1] &= v; & [M \nabla u_1 \cdot \nu] &= M \nabla v \cdot \nu & & \text{on } \partial B_{R_1}, \\
u_1 &= 0 & & \text{on } \partial D,
\end{aligned}
\]

and

\[
\begin{aligned}
\text{div}(M \nabla u_3) &= 0 & & \text{in } D \setminus B_{R_3}, \\
[u_3] &= v; & [M \nabla u_3 \cdot \nu] &= M \nabla v \cdot \nu & & \text{on } \partial B_{R_3}, \\
u_3 &= 0 & & \text{on } \partial D.
\end{aligned}
\]

Here and in what follows, \( [\cdot] \) denotes the jump across a sphere and \( \nu \) denotes the unit outward normal vector on a sphere. It follows that

\[
\begin{aligned}
\|u_1\|_{H^1(D \setminus \partial B_{R_1})} &\leq C\|v\|_{H^2(\partial B_{R_1})}, & & (2.26) \\
\|u_1\|_{H^{3/2}(\partial B_{R_1/\gamma})} &\leq C\|v\|_{H^2(\partial B_{R_1})},
\end{aligned}
\]

and

\[
\begin{aligned}
\|u_3\|_{H^1(D \setminus \partial B_{R_3})} &\leq C\|v\|_{H^2(\partial B_{R_3})}, & & (2.27) \\
\|u_3\|_{H^{3/2}(\partial B_{R_3/\gamma})} &\leq C\|v\|_{H^2(\partial B_{R_3})}.
\end{aligned}
\]

Here and in what follows in this proof, \( C \) denotes a positive constant depending only on the elliptic and the Lipschitz constant of \( M, c_1, c_2, \lambda_0, R_s, R_s, \) and \( d. \) Set

\[ d_1 = (\lambda - 1)R_1 \text{ and } d_3 = (\lambda - 1)R_3/\lambda. \]

Let \( \varphi_1, \varphi_3 \in C_c^2(\mathbb{R}^d) \) be such that

\[
\varphi_1 = \begin{cases} 
1 & \text{in } B_{R_1 + d_1/3} \setminus B_{R_1}, \\
0 & \text{in } \mathbb{R}^d \setminus (B_{R_1 + d_1/2} \setminus B_{R_1/\lambda}),
\end{cases} \quad \text{and} \quad
\varphi_3 = \begin{cases} 
1 & \text{in } B_{R_3} \setminus B_{R_3 - d_3/3}, \\
0 & \text{in } \mathbb{R}^d \setminus (B_{\lambda R_3} \setminus B_{R_3 - d_3/2}).
\end{cases}
\]

Define

\[
V = \begin{cases} 
v - \varphi_1 u_1 - \varphi_3 u_3 & \text{in } B_{R_3} \setminus \overline{B}_{R_1}, \\
-\varphi_1 u_1 - \varphi_3 u_3 & \text{in } D \setminus (B_{R_3} \setminus \overline{B}_{R_1}).
\end{cases} \quad (2.28)
\]

Applying Lemma 5 we obtain, for \( |\beta| > 2(\gamma R_3)^p, \)

\[
C \int_D e^{2\beta r - p} |\beta|^2 |V|^2 + |\nabla V|^2 \leq \int_D e^{2\beta r - p} |\text{div}(M \nabla V)|^2 + \int_{\partial D} |\beta|e^{2\beta r - p} (|\nabla V|^2 + \beta^2 |V|^2). \quad (2.29)
\]

The proof is now quite standard and divided into two cases.
Define $\alpha$. This implies $C$ holds for any large $\alpha \in (0, 1)$ and $\beta > 0$ as follows:

$$\alpha' = \frac{R_2^{-p} - \hat{R}_3^{-p}}{\hat{R}_1^{-p} - \hat{R}_3^{-p}} \quad \text{and} \quad \beta (R_2^{-p} - \hat{R}_1^{-p}) = (1 - \alpha') \ln \left( \frac{\|v\|_{H(\partial B_{R_3})}}{\|v\|_{H(\partial B_{R_1})}} \right)^{1/2}$$

Note that $0 < \alpha' < 1$ since $R_2 < R_3/\gamma$. We assume that $\|v\|_{H(\partial B_{R_3})} > C \|v\|_{H(\partial B_{R_1})}$ for some large $C$ such that $\beta \geq \max \{2R_3^{-p}, 2, \beta_0\}$ since if $\|v\|_{H(\partial B_{R_3})} < C \|v\|_{H(\partial B_{R_1})}$, the conclusion holds for any $\alpha \in (0, 1)$ by taking $\beta = \max \{2R_3^{-p}, 2, \beta_0\}$ in (2.31). It follows from (2.31) and the choice of $\alpha'$ and $\beta$ that

$$\|v\|_{H(\partial B_{R_2})} \leq C \beta \|v\|_{H(\partial B_{R_3})}^{\alpha'} \|v\|_{H(\partial B_{R_1})}^{1-\alpha'} \quad (2.32)$$

Define $\alpha := \frac{R_2^{-2p} - R_3^{-2p}}{\hat{R}_1^{-2p} - \hat{R}_3^{-2p}}$.

It is clear that $\alpha < \frac{R_2^{-p} - \hat{R}_3^{-p}}{\hat{R}_1^{-p} - \hat{R}_3^{-p}}$. Hence, by choosing $\lambda$ close to 1,

$$\alpha < \alpha' \quad \forall R_2 \in (\gamma_0 R_1, R_3/\gamma_0). \quad (2.34)$$

A combination of (2.22) and (2.31) implies

$$\|v\|_{H(\partial B_{R_2})} \leq C \|v\|_{H(\partial B_{R_3})}^{\alpha} \|v\|_{H(\partial B_{R_1})}^{1-\alpha} \quad (2.35)$$

Case 2: $\|v\|_{H(\partial B_{R_1})} \geq \|v\|_{H(\partial B_{R_3})}$. The proof is similar to the previous case by considering $\beta < -2(\gamma R_3)^{-p}$. The details are left to the reader. The proof is complete.

3 Cloaking using complementary media. Proof of Theorem 1

This section containing three subsections is devoted to the proof of Theorem 1. In the first subsection, we present two useful lemmas. The proof of Theorem 1 is given in the second subsection.

\[\text{Here we assume that } \|v\|_{H(\partial B_{R_1})} \neq 0 \text{ since otherwise } v = 0. \text{ This fact is a consequence of the unique continuation principle and can be obtained from (2.31) by letting } \beta \to \infty.\]
3.1 Preliminaries

In this section, we present two lemmas which will be used in the proof of Theorems 1 and 3. The first lemma is on a change of variables and follows from [18, Lemma 1].

**Lemma 6.** Let \( d \geq 2, k > 0 \) with \( 0 < R_1 < R_2 < R_3 \) and \( R_3 = R_2^2 / R_1 \). Let \( a \in [L^\infty(B_{R_3} \setminus \overline{B}_{R_2})]^{d \times d} \) be a matrix valued function, \( \sigma \in L^\infty(B_{R_3} \setminus \overline{B}_{R_2}) \) be a complex function, and \( K : B_{R_2} \setminus \overline{B}_{R_1} \rightarrow B_{R_3} \setminus \overline{B}_{R_2} \) be the Kelvin transform with respect to \( \partial B_{R_2} \), i.e.,

\[
K(x) = R_2^2 x / |x|^2.
\]

For \( v \in H^1(B_{R_3} \setminus \overline{B}_{R_2}) \), define \( w = v \circ K^{-1} \). Then

\[
\text{div}(a \nabla v) + k^2 \sigma v = 0 \text{ in } B_{R_3} \setminus \overline{B}_{R_2}
\]

if and only if

\[
\text{div}(K_a \nabla w) + k^2 K_a \sigma w = 0 \text{ in } B_{R_2} \setminus \overline{B}_{R_1}.
\]

Moreover,

\[
w = v \quad \text{and} \quad K_a \nabla w \cdot \nu = -a \nabla v \cdot \nu \text{ on } \partial B_{R_2}.
\]

The second lemma is a stability estimate for solutions of (1.10).

**Lemma 7.** Let \( 0 < \delta < 1 \), \( f \in L^2(\Omega) \), and let \( A \in [L^\infty(\Omega)]^{d \times d} \) and \( \Sigma \in L^\infty(\Omega, \mathbb{C}) \) be such that \( A \) is Lipschitz and uniformly elliptic and \( \Im(\Sigma) \geq 0 \). Assume that \( u_\delta \in H^1_0(\Omega) \) is the unique solution to (1.10). Then

\[
\|u_\delta\|_{H^1(\Omega)}^2 \leq C \left( \delta^{-1} \|f\|_{L^2(\Omega)} \|u_\delta\|_{L^2(\text{supp} f)} + \|f\|_{L^2(\Omega)}^2 \right),
\]

for some positive constant \( C \) independent of \( \delta \) and \( f \).

Lemma 7 is a variant of [18, Lemma 1]. The case \( k = 0 \) and its variant in the case \( k > 0 \) were considered in [21] and [19] respectively. The proof is similar to the one of [18, Lemma 1]. For the convenience of the reader, we present the proof.

**Proof.** The existence and uniqueness of \( u_\delta \) are given in [18]. We only establish (3.1) by contradiction. Assume that (3.1) is not true. Then there exist \( \delta_n \rightarrow 0 \) and \( (f_n) \in L^2(\Omega) \) such that

\[
\|u_n\|_{H^1(\Omega)} = 1 \quad \text{and} \quad \frac{1}{\delta_n} \|f_n\|_{L^2(\Omega)} \|u_n\|_{L^2(\text{supp} f_n)} + \|f_n\|_{L^2(\Omega)}^2 \rightarrow 0,
\]

as \( n \rightarrow \infty \), where \( u_n \in H^1_0(\Omega) \) is the unique solution to

\[
\text{div}(s_{\delta_n} A \nabla u_n) + k^2 s_0 \Sigma u_n = f_n \text{ in } \Omega.
\]

Without loss of generality, one may assume that \( u_n \rightharpoonup u \) weakly in \( H^1(\Omega) \) and strongly in \( L^2(\Omega) \); moreover, \( u \in H^1_0(\Omega) \) and \( u \) satisfies

\[
\text{div}(s_0 A \nabla u) + k^2 s_0 \Sigma u = 0 \text{ in } \Omega.
\]
Multiplying equation (3.3) by $\bar{u}_n$ (the conjugate of $u_n$) and integrating on $\Omega$, we have

$$
\int_{\Omega} s_{\delta_n} A \nabla u_n \cdot \nabla \bar{u}_n \, dx - \int_{\Omega} k^2 s_0 \Sigma |u_n|^2 \, dx = - \int_{\Omega} f_n \bar{u}_n \, dx.
$$

Considering the imaginary part and using the fact that

$$
\frac{1}{\delta_n} \left| \int_{\Omega} f_n \bar{u}_n \right| \leq \frac{1}{\delta} \| f_n \|_{L^2(\Omega)} \| u_n \|_{L^2(\text{supp} f_n)} \to 0 \text{ as } n \to \infty \text{ by (3.2)},
$$

we obtain, by (1.6),

$$
\| \nabla u_n \|_{L^2(B_r_2 \setminus B_{r_1})} \to 0 \text{ as } n \to \infty.
$$

(3.5)

Since $\text{div}(A \nabla u_n) + k^2 \Sigma u_n = f_n$ in $B_r_2 \setminus B_{r_1}$ and $f_n \to 0$ in $L^2(\Omega)$, it follows from that $u_n \to 0$ in the distributional sense. This in turn implies

$$
\| u_n \|_{L^2(B_{r_2} \setminus B_{r_1})} \to 0 \text{ as } n \to \infty.
$$

(3.6)

A combination of (3.5) and (3.6) yields

$$
\| u_n \|_{H^1(B_{r_2} \setminus B_{r_1})} \to 0 \text{ as } n \to \infty.
$$

(3.7)

Hence

$$
u = 0 \text{ in } B_{r_2} \setminus B_{r_1},
$$

and

$$
\| u_n \|_{H^{1/2}(\partial B_{r_2})} + \| u_n \|_{H^{1/2}(\partial B_{r_1})} + \| A \nabla u_n \cdot \nu \|_{H^{-1/2}(\partial B_{r_2})} + \| A \nabla u_n \cdot \nu \|_{H^{-1/2}(\partial B_{r_1})} \to 0 \text{ as } n \to \infty.
$$

(3.8)

Since $u = 0$ in $B_{r_2} \setminus B_{r_1}$ and $u$ satisfies (3.4), it follows from the unique continuation principle that $u = 0$ in $\Omega$. Hence, since $u_n \to u$ in $L^2(\Omega)$,

$$
u_n \to 0 \text{ in } L^2(\Omega) \text{ as } n \to \infty.
$$

(3.9)

Multiplying (3.3) by $\bar{u}_n$ and integrating on $\Omega \setminus B_r$, we have

$$
\int_{\Omega \setminus B_r} A \nabla u_n \cdot \nabla \bar{u}_n \, dx - \int_{\Omega \setminus B_r} k^2 s_0 \Sigma |u_n|^2 \, dx = - \int_{\Omega} f_n \bar{u}_n \, dx + \int_{\partial B_r} A \nabla u_n \cdot \nu \bar{u}_n.
$$

Using (3.8) and (3.9), we obtain

$$
\| \nabla u_n \|_{L^2(\Omega \setminus B_r)} \to 0 \text{ as } n \to \infty.
$$

(3.10)

Similarly,

$$
\| \nabla u_n \|_{L^2(B_{r_1})} \to 0 \text{ as } n \to \infty.
$$

(3.11)

A combination of (3.7), (3.9), (3.10), and (3.11), we obtain

$$
\| u_n \|_{H^1(\Omega)} \to 0 \text{ as } n \to \infty.
$$

which contradicts (3.2). The proof is complete.
3.2 Proof of Theorem 1

We use the approach in [21] with some modifications from [19] so that the same proof also
give the result on illusion optics (Theorem 3 in Section 4). However, instead of applying
the standard three sphere inequality as in [21], we use Theorem 2.

We have, by Lemma 7,
\[ \|u_\delta\|_{H^1(\Omega)}^2 \leq C \left( \delta^{-1} \|f\|_{L^2(\Omega)} \|u_\delta\|_{L^2(\Omega \setminus B_{r^3})} + \|f\|_{L^2(\Omega)} \right). \] (3.12)

As in [21], let \(u_{1,\delta}\) be the reflection of \(u_\delta\) through \(\partial B_{r_2}\) by \(F\), i.e.,
\[ u_{1,\delta} = u_\delta \circ F^{-1} \text{ in } \mathbb{R}^d \setminus \bar{B}_{r_2} \] (3.13)
and let \(u_{2,\delta}\) be the reflection of \(u_{1,\delta}\) through \(\partial B_{r_2}\) by \(G\), i.e.,
\[ u_{2,\delta} = u_{1,\delta} \circ G^{-1} \text{ in } B_{r_3}. \] (3.14)

By Lemma 6,
\[ \text{div}(A \nabla u_{1,\delta}) + \frac{1}{1 - i\delta} k^2 \Sigma u_{1,\delta} = 0 \text{ in } B_{r_3} \setminus B_{r_2}, \] (3.15)
\[ \Delta u_{2,\delta} + k^2 u_{2,\delta} = 0 \text{ in } B_{r_3}. \] (3.16)

Applying Lemma 6 again and using the fact that \(F^* A = A\) in \(B_{r_3} \setminus B_{r_2}\), we have
\[ u_{1,\delta} = u_\delta \big|_+ \text{ on } \partial B_{r_2} \quad \text{and} \quad (1 - i\delta) A \nabla u_{1,\delta} \cdot \nu = A \nabla u_\delta \cdot \nu \big|_+ \text{ on } \partial B_{r_2}. \] (3.17)

Let \(V_{1,\delta} \in H^1(B_{r_3} \setminus B_{r_2})\) be the unique solution to
\[
\begin{cases}
\text{div}(A \nabla V_{1,\delta}) + k^2 \Sigma V_{1,\delta} = -\frac{i\delta}{1 - i\delta} k^2 \Sigma u_{1,\delta} \text{ in } B_{r_3} \setminus B_{r_2}, \\
A \nabla V_{1,\delta} \cdot \nu - ik V_{1,\delta} = 0 \quad \text{on } \partial B_{r_2}, \\
V_{1,\delta} = 0 \quad \text{on } \partial B_{r_3}.
\end{cases}
\] (3.18)

By Fredholm’s theory,
\[ \|V_{1,\delta}\|_{H^1(B_{r_3} \setminus B_{r_2})} \leq C \delta \|u_\delta\|_{H^1(\Omega)}. \] (3.19)

Define \(U_{1,\delta}\) in \(B_{r_3} \setminus B_{r_2}\) as follows
\[ U_{1,\delta} = u_\delta - u_{1,\delta} - V_{1,\delta}. \] (3.20)

Then \(U_{1,\delta} \in H^1(B_{r_3} \setminus B_{r_2})\) and \(U_{1,\delta}\) satisfies
\[ \text{div}(A \nabla U_{1,\delta}) + k^2 \Sigma U_{1,\delta} = 0 \text{ in } B_{r_3} \setminus B_{r_2}, \]
\[ \|U_{1,\delta}\|_{H^{1/2}(\partial B_{r_2})} + \|A \nabla U_{1,\delta} \cdot \nu\|_{H^{1/2}(\partial B_{r_2})} \leq C \delta \|u_\delta\|_{H^1(\Omega)}, \]
and
\[ \|U_{1,\delta}\|_{H^{1/2}(\partial B_{r_3})} + \|A \nabla U_{1,\delta} \cdot \nu\|_{H^{1/2}(\partial B_{r_3})} \leq C \|u_\delta\|_{H^1(\Omega)}. \]
Applying Theorem 2, we have

\[ \|U_{1,\delta}\|_{H^{1/2}(\partial B_{r_2})} + \|A\nabla U_{1,\delta} \cdot \nu\|_{H^{1/2}(\partial B_{r_2})} \leq C\delta^\alpha \|u_\delta\|_{H^1(\Omega)} \quad (3.21) \]

where \( \alpha \) is given in (2.9) with \( R_1 = r_2, R_2 = \gamma r_2, R_3 = r_3 \). By first taking \( q \) large enough and then choosing \( \gamma_0 \) close to 1 enough, from (2.9), we can assume that

\[ \alpha > \frac{1}{2}. \quad (3.22) \]

Here is the place where the condition \( \gamma < \gamma_0 \) is required. A combination of (3.19) and (3.21) yields

\[ \|u_\delta - u_{1,\delta}\|_{H^{1/2}(\partial B_{\gamma r_2})} + \|A\nabla (u_\delta - u_{1,\delta}) \cdot \nu\|_{H^{-1/2}(\partial B_{\gamma r_2})} \leq C\delta^\alpha \|u_\delta\|_{H^1(\Omega)}. \quad (3.23) \]

In what follows, we assume that \( k = 1 \) for notational ease. Define \( U_{2,\delta} \) in \( B_{r_3} \setminus B_{\gamma r_2} \) as follows

\[ U_{2,\delta} = u_{1,\delta} - u_{2,\delta} + V_{1,\delta}. \]

Then

\[ \Delta U_{2,\delta} + U_{2,\delta} = 0 \quad \text{in } B_{r_3} \setminus B_{\gamma r_2}, \quad (3.24) \]

and

\[ U_{2,\delta} = 0 \quad \text{and} \quad \partial_r U_{2,\delta} = -\frac{i\delta}{1 - i\delta} \partial_r u_{1,\delta} + \partial_r V_{1,\delta} \quad \text{on } \partial B_{r_3}. \quad (3.25) \]

**Case 1: \( d = 2 \).** As in [19], define

\[ \hat{J}_n(r) = 2^n n! J_n(r) \quad \text{and} \quad \hat{Y}_n(r) = \frac{\pi i}{2^{n(n - 1)}} Y_n(r), \]

where \( J_n \) and \( Y_n \) are the Bessel and Neumann functions of order \( n \). It follows from [5] (3.57) and (3.58) that

\[ \hat{J}_n(t) = t^n[1 + o(1)] \quad (3.26) \]

and

\[ \hat{Y}_n(t) = t^{-n}[1 + o(1)], \quad (3.27) \]

as \( n \to +\infty \). From [324] one can represent \( U_{2,\delta} \) as follows

\[ U_{2,\delta} = a_0\hat{J}_0(|x|) + b_0\hat{Y}_0(|x|) + \sum_{n=1}^{\infty} \sum_{\pm} [a_{n,\pm}\hat{J}_n(|x|) + b_{n,\pm}\hat{Y}_n(|x|)] e^{\pm in\theta} \quad \text{in } B_{r_3} \setminus B_{\gamma r_2}, \quad (3.28) \]

for \( a_0, b_0, a_{n,\pm}, b_{n,\pm} \in \mathbb{C} \quad (n \geq 1) \). Assume that

\[ \partial_r U_{2,\delta} = c_0 + \sum_{n \geq 1} \sum_{\pm} c_{n,\pm} e^{\pm in\theta} \quad \text{on } \partial B_{r_3}. \]
Then, by (3.18), (3.19), and (3.25),
\[
|c_0|^2 + \sum_{n \geq 1} \sum_{\pm} n^{-1} |c_{n, \pm}|^2 \sim \| \partial_r U_{2, \delta} \|_{H^{-1/2}(\partial B_{r_3})}^2 \leq C \delta^2 \| u \|_{H^1(\Omega)}^2.
\] (3.29)

Using (3.25) again, we have
\[
\begin{cases}
    a_{n, \pm} \hat{J}_n(r_3) + b_{n, \pm} \hat{Y}_n(r_3) = 0 \\
    a_{n, \pm} \hat{J}'_n(r_3) + b_{n, \pm} \hat{Y}'_n(r_3) = c_{n, \pm}
\end{cases}
\]
for \( n \geq 0 \).

Here we denote \( a_{0, \pm} = a_0/2; b_{0, \pm} = b_0/2; \) and \( c_{0, \pm} = c_0/2 \). It follows that
\[
\begin{cases}
    a_{n, \pm} = c_{n, \pm} AC_n, \\
    b_{n, \pm} = c_{n, \pm} BC_n,
\end{cases}
\]
for \( n \geq 0 \). \( (3.30) \)

where
\[
AC_n = -\frac{\hat{Y}_n}{\hat{J}_n Y''_n - \hat{J}'_n Y'_n} \quad \text{and} \quad BC_n = -\frac{\hat{J}_n}{\hat{Y}_n Y''_n - \hat{Y}'_n \hat{J}_n}.
\]

Using (3.26) and (3.27), we derive that
\[
AC_n = -\frac{1}{2n} r_3^{1-n} \left(1 + o(1)\right) \quad \text{and} \quad BC_n = \frac{1}{2n} r_3^{1+n} \left(1 + o(1)\right).
\]

We now make use the removing of localized singularity technique introduced in [19, 21]. Set
\[
\hat{u}_\delta(x) = \sum_{n=1}^{\infty} \sum_{\pm} b_{n, \pm} \hat{Y}_n(|x|) e^{\pm in\theta} \quad \text{in} \ B_{r_3} \setminus B_{r_2},
\]

We claim that, for \( \gamma r_2 \leq r \leq r_3 \),
\[
\| U_{2, \delta} - \hat{u}_\delta \|_{H^{1/2}(\partial B_r)} + \| \partial_r U_{2, \delta} - \partial_r \hat{u}_\delta \|_{H^{-1/2}(\partial B_r)} \leq C \delta^2 \| u_\delta \|_{H^1(\Omega)}.
\] (3.31)

Indeed, for \( \gamma r_2 \leq r \leq r_3 \),
\[
\| U_{2, \delta} - \hat{u}_\delta \|_{H^{1/2}(\partial B_r)}^2 = \| \sum_{n \geq 0} \sum_{\pm} a_{n, \pm} \hat{J}_n(|x|) e^{in\theta} \|_{H^{1/2}(\partial B_r)}^2 \sim \sum_{n \geq 0} \sum_{\pm} (n+1) |a_{n, \pm}|^2 |\hat{J}_n(|x|)|^2 \\
\quad \sim \sum_{n \geq 0} \sum_{\pm} (n+1) |c_{n, \pm} AC_n|^2 |\hat{J}_n(|x|)|^2 \leq C \sum_{n \geq 0} \sum_{\pm} (n+1)^{-1} |c_{n, \pm}|^2 (r/r_3)^{2n}.
\]

It follows from (3.29) that
\[
\| U_{2, \delta} - \hat{u}_\delta \|_{H^{1/2}(\partial B_r)} \leq C \delta \| u_\delta \|_{H^1(\Omega)},
\]
for \( \gamma r_2 \leq r \leq r_3 \). Similarly,
\[
\| \partial_r U_{2, \delta} - \partial_r \hat{u}_\delta \|_{H^{-1/2}(\partial B_r)} \leq C \delta \| u_\delta \|_{H^1(\Omega)},
\]
for \( \gamma r_2 \leq r \leq r_3 \). Similarly,
for \(\gamma r_2 \leq r \leq r_3\). As a consequence of (3.19) and (3.31), we obtain for \(\gamma r_2 \leq r \leq r_3\),

\[
\|u_{1,\delta} - u_{2,\delta} - \hat{u}_\delta\|_{H^{1/2}(\partial B_r)} + \|\partial_{\nu}u_{1,\delta} - \partial_{\nu}u_{2,\delta} - \partial_{\nu}\hat{u}_\delta\|_{H^{-1/2}(\partial B_r)} \leq C\delta\|u_\delta\|_{H^1(\Omega)}. \tag{3.32}
\]

Define

\[
U_\delta = \begin{cases} 
    u_\delta & \text{in } \Omega \setminus B_{r_3}, \\
    u_\delta - \hat{u}_\delta & \text{if } x \in B_{r_3} \setminus B_{r_2}, \\
    u_{2,\delta} & \text{if } x \in B_{r_2}.
\end{cases}
\]

We have

\[
\text{div}(A\nabla U_\delta) + k^2 U_\delta = f \text{ in } \Omega \setminus (\partial B_{r_3} \cup \partial B_{r_2}).
\]

On the other hand, from (3.23) and (3.32), we obtain

\[
\|U_\delta\|_{H^{1/2}(\partial B_{r_2})} + \|[A\nabla U_\delta \cdot \nu]\|_{H^{-1/2}(\partial B_{r_2})} \leq C\delta\|u_\delta\|_{H^1(\Omega)}
\]

and

\[
\|U_\delta\|_{H^{1/2}(\partial B_{r_3})} + \|[A\nabla U_\delta \cdot \nu]\|_{H^{-1/2}(\partial B_{r_3})} \leq C\delta\|u_\delta\|_{H^1(\Omega)}.
\]

Using (3.12), we derive that

\[
\|U_\delta\|_{H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_2}))} \leq C\delta\left(\delta^{-1/2}\|U_\delta\|_{H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_2}))} + \|f\|_{L^2(\Omega)}\right) + C\|f\|_{L^2(\Omega)}.
\]

Since \(\alpha > 1/2\), it follows that \(U_\delta\) is bounded in \(H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_2}))\). Without loss of generality, one may assume that \(U_\delta \rightharpoonup U\) weakly in \(H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_2}))\) as \(\delta \to 0\); moreover, \(U \in H^1(\Omega)\) and

\[
\Delta U + k^2 U = f \text{ in } \Omega \text{ and } U = 0 \text{ on } \partial \Omega.
\]

Hence \(U = u\). Since the limit is unique, we have the convergence for the family \((U_\delta)\) as \(\delta \to 0\).

**Case 2: \(d = 3\).** Define

\[
\hat{j}_n(t) = 1 \cdot 3 \cdots (2n + 1) j_n(t) \quad \text{and} \quad \hat{y}_n = -\frac{y_n(t)}{1 \cdot 3 \cdots (2n - 1)},
\]

where \(j_n\) and \(y_n\) are the spherical Bessel and Neumann functions of order \(n\). Then, as \(n\) large enough, (see, e.g., [5] (2.37) and (2.38))

\[
\hat{j}_n(kr) = r^n \left(1 + O(1/n)\right) \quad \text{and} \quad \hat{y}_n(kr) = r^{-n-1} \left(1 + O(1/n)\right). \tag{3.33}
\]

Thus one can represent \(U_{2,\delta}\) of the form

\[
U_{2,\delta} = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[a_m^n \hat{j}_n(|x|) + b_m^n \hat{y}_n(|x|)\right] Y_m^n(\hat{x}) \quad \text{in } B_{r_3} \setminus B_{r_0}, \tag{3.34}
\]

for \(a_m^n, b_m^n \in \mathbb{C}\) and \(\hat{x} = x/|x|\). The proof now follows similarly as in the case \(d = 2\). The details are left to the reader.
Remark 2. In the proof, we use essentially the fact \((A, \Sigma) = (I, 1)\) in \(B_{r_3} \setminus B_{r_2}\) to use separation of variables in this region. In fact, this condition is not necessary by using the technique of separation of variables for a general structure in [20].

Remark 3. The construction of the cloak given by (1.4) is not restricted to the Kelvin transforms \(F\) (and \(G\)). In fact, one can extend this construction to a general class of reflections considered in [18].

Remark 4. The condition \((F^*A, F^*\Sigma) = (A, \Sigma)\) in \(B_{r_3} \setminus B_{r_2}\) is necessary to ensure that cloaking can be achieved and the localized resonance might take place see [24] (see also [4] for related results).

Remark 5. Cloaking can also be achieved via schemes generated by changes of variables [7, 14, 31]. Resonance might also appear in this context but for specific frequencies see [9, 16]. It is shown in [16] that in the resonance case cloaking might not be achieved and the field inside the cloaked region can depend on the field outside. Cloaking can also be achieved in the time regime via change of variables [26, 27].

4 Illusion optics using complementary media

We next discuss briefly how to obtain illusion optics in the spirit of Lai et al. in [12]. The scheme used here is a combination of the ones used for cloaking and superlensing in [21, 19] and is slightly different from [12]. More precisely, set

\[ m = \frac{r_3^2}{r_2^2}. \]

Let \(a_c \in [L^\infty(B_{r_2/m})]^{d \times d}\) be elliptic and \(\sigma_c \in L^\infty(B_{r_2/r_3}^2, \mathbb{C})\) with \(\Im(\sigma_c) \geq 0\). Define

\[ A_1, \Sigma_1 = \begin{cases} A, \Sigma & \text{in } \Omega \setminus B_{r_2/m}, \\ a_c, \sigma_c & \text{in } B_{r_2/m}, \end{cases} \quad (4.1) \]

and

\[ \hat{A}_1, \hat{\Sigma}_1 = \begin{cases} I, 1 & \text{in } \Omega \setminus B_{r_2}, \\ (r_3/r_2)^2 - d a_c(x/m), (r_3/r_2)^{-d} \sigma_c(x/m) & \text{in } B_{r_2}. \end{cases} \quad (4.2) \]

Recall that \((A, \Sigma)\) is defined in [14]. We assume that the following equation has only zero solution in \(H^1_0(\Omega)\): \[
\text{div}(A_1 \nabla v) + k^2 \Sigma_1 v = 0 \text{ in } \Omega. \quad (4.3)
\]

We obtain the following result on illusion optics:

**Theorem 3.** Let \(d = 2, 3\), \(f \in L^2(\Omega)\) with \(\text{supp } f \subset \Omega \setminus B_{r_3}\) and let \(u\) and \(u_\delta\) in \(H^1_0(\Omega)\) be respectively the unique solution of

\[
\text{div}(s_\delta A_1 \nabla u_\delta) + k^2 s_0 \Sigma_1 u_\delta = f \quad \text{in } \Omega,
\]

and

\[
\text{div}(\hat{A}_1 \nabla u) + k^2 \hat{\Sigma}_1 u = f \quad \text{in } \Omega.
\]
There exists $\gamma_0 > 1$, depending only on $\Lambda$ and the Lipschitz constant of $\hat{a}$ such that if $1 < \gamma < \gamma_0$ then

$$u_\delta \to u \text{ weakly in } H^1(\Omega \setminus B_{r_3}) \text{ as } \delta \to 0.$$  \hspace{1cm} (4.4)

For an observer outside $B_{r_3}$, the medium in $B_{r_3}$ looks like $(\hat{A}_1, \hat{\Sigma}_1)$: one has illusion optics.

**Proof.** The proof is similar to the one of Theorem 1. Note that in the proof of Theorem 1 we do not use the information of the medium inside $B_{r_2/m}$. The details are left to the reader. \qed

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