Photon added nonlinear coherent states for a one mode field in a Kerr medium

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We construct Deformed Photon Added Nonlinear Coherent States (DPANCSs) by application of the deformed creation operator upon the Nonlinear Coherent States obtained as eigenstates of the deformed annihilation operator and by application of a deformed displacement operator upon the vacuum state. We evaluate some statistical properties like the Mandel parameter, Husimi and Wigner functions for these states and analyze their differences, we give closed analytical expressions for them. We found a profound difference in the statistical properties of the DPANCSs obtained from the two above mentioned generalizations.

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I. INTRODUCTION

The term coherent states was introduced by Glauber\textsuperscript{1} in the context of quantum optics to characterize those states of the electromagnetic field that factorize the field coherence function to all orders. The field coherent states can be constructed by any one of the following definitions: i) as eigenstates of the annihilation operator $\hat{a}\langle\alpha| = \alpha\langle\alpha|$; ii) as those states obtained by the application of the displacement operator upon the vacuum state $|\alpha\rangle = D(\alpha)|0\rangle$ with $D(\alpha) = \exp(\alpha \hat{a}\dagger - \alpha^\ast \hat{a})$ and iii) as the quantum states with a minimum uncertainty relationship $(\Delta p)^2(\Delta q)^2 = (1/2)^2$ where the coordinate and momentum operators are defined as $\hat{q} = (1/\sqrt{2})(\hat{a}\dagger + \hat{a})$, $\hat{p} = (i/\sqrt{2})(\hat{a}\dagger - \hat{a})$ with the condition $\Delta p = \Delta q = 1/\sqrt{2}$\textsuperscript{2,3}.

Glauber constructed the field coherent states by using the harmonic oscillator algebra and concluded that the same states are obtained from any one of the three mathematical definitions.

Photon added coherent states or PACS were introduced by Agarwal and Tara\textsuperscript{4} as states that interpolate between the field coherent states $|\alpha\rangle$ (the most classical-like quantum states) and the number states $|n\rangle$ (purely quantum states). The unnormalized PACS are defined as $|\alpha^{(m)}\rangle = \hat{a}^{\dagger m}|\alpha\rangle$ where the operator $\hat{a}\dagger$ is the creation operator of a boson (the usual creation operator in the harmonic oscillator algebra) and $m$ is a positive integer that specifies the number of photons added to the coherent state $|\alpha\rangle$. These states show nonclassical features like squeezing, sub Poissonian statistics and negativities in their Wigner function due to the photons added to the coherent state. From the experimental point of view, these states were generated by Zavatta et. al.\textsuperscript{5} using parametric down conversion in a nonlinear crystal. They reported the generation of a single-photon-added coherent state $|\alpha,1\rangle$ and its characterization by quantum tomography and reconstructed the corresponding Wigner function. An alternative form to generate a single photon added coherent state was given in\textsuperscript{6} where, by passing a fast exited two level atom through a very high quality factor cavity which contains a coherent field, and, after the atom exits, measuring it in the ground state the system is found in a superposition of displaced number states. It has been demonstrated that single photon addition and subtraction when applied to classical states of light produce highly nonclassical and non-Gaussian states\textsuperscript{7–11}. More recently, Zavatta et. al. gave an experimental demonstration of the bosonic commutation relation via superpositions of quantum operations on thermal light fields\textsuperscript{12,13}. Single photon-added thermal states are such that their quasiprobability function $P(\alpha)$ does not show singularities and it may be accessible from experimental data allowing for the experimental reconstruction of a non classicality quasiprobability for a single photon added thermal state\textsuperscript{14}.

It is known that the nonlinear coherent states (NLCS) constructed with the formalism of an f-deformed algebra\textsuperscript{15} also show nonclassical properties\textsuperscript{16–18}. The NLCS may be obtained as eigenstates of a deformed annihilation operator $A(\alpha, f) = \alpha(\alpha, f)$, where $A = \hat{a}\dagger f(\hat{a})$ a deformation function of the number operator $\hat{a} = \hat{a}\dagger\hat{a}$ and also by the application of a deformed displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{A}\dagger - \alpha^\ast\hat{A})$ upon the vacuum state i.e. $|\alpha\rangle_D = \hat{D}(\alpha)|0\rangle$ with $f(n)$ a deformation function of the number operator $\hat{a} = \hat{a}\dagger\hat{a}$ and also by the application of a deformed displacement operator $\hat{D}(\alpha) = \exp(\alpha\hat{A}\dagger - \alpha^\ast\hat{A})$ upon the vacuum state i.e. $|\alpha\rangle_D = \hat{D}(\alpha)|0\rangle$. These two options correspond to the generalization of Glauber’s first two definitions for the construction of a field coherent state. Recall that the different generalizations applied to systems with dynamical properties different from those of the harmonic oscillator yield to non-equivalent states\textsuperscript{19–22}. Sivakumar has shown that the PACS may be regarded as NLCS for a particular deformation function\textsuperscript{22} and in a recent publication the deformed photon added nonlinear coherent states (DPANCSs) $|\alpha^{(m)}, f\rangle$ were introduced\textsuperscript{24}. These states also show nonclassical features and for the case $f(n) = 1$ reduce to the usual PACS.
The statistical properties of a given state may be analyzed by means of standard parameters like the Mandel parameter, the second order correlation function (to test bunching or antibunching effects), probability distributions like the Husimi function and quasi probability distributions like the Wigner function. The presence of negativities in the Wigner function is a characteristic of nonclassical states [26,27].

In this work we construct DPANCSs for the case of a Kerr Hamiltonian. We have chosen this particular system because its spectrum contains an infinite number of bound states and its nonlinear coherent states may be obtained exactly either as eigenstates of a deformed annihilation operator or by application of a deformed displacement operator upon the vacuum state [30].

The paper is organized as follows: In section II we briefly present the PACSs and the DPANCSs based in references [4] and [24] respectively and in the subsection [15] we consider the specific case of a Kerr Hamiltonian and construct the corresponding nonlinear coherent states and their deformed photon added nonlinear coherent states. In section III we analyze the statistical properties mentioned above, that is; the Mandel parameter, Husimi Q-function, and Wigner function. Finally, in section IV we give our conclusions.

II. THEORY

Like the Nonlinear Coherent States [15,17] the PACS show nonclassical properties like squeezing and sub-Poissonian statistics. They are defined as [4]:

\[ |\alpha^{(m)}\rangle = N_{\alpha}^{(m)} a^{|m\rangle} |\alpha\rangle = N_{\alpha}^{(m)} \sum_{k=0}^{m} \binom{m}{k} \sqrt{k!} a^{|m-k\rangle} |\alpha, k\rangle \]  

with normalization constant

\[ N_{\alpha}^{(m)} = 1/(\langle |\alpha|^2 a^{|m\rangle} |\alpha\rangle)^{1/2} = 1/(L_m(-|\alpha|^2 m)!)^{1/2} \]

where \(L_m(x)\) is the Laguerre polynomial of order \(m\) and where \(|\alpha, k\rangle = D(\alpha)|k\rangle\) is a displaced number state. We have changed the notation given in Ref. [4] to avoid confusion between the PACS and the displaced number states.

The state \( |\alpha^{(m)}\rangle\) can be written in terms of Fock states as:

\[ |\alpha^{(m)}\rangle = \exp(-|\alpha|^2/2) \langle L_m(-|\alpha|^2 m)! \rangle^{1/2} \sum_{n=0}^{\infty} \frac{|\alpha|^n \sqrt{(n+m)!}}{n!} |n+m\rangle. \]  

If instead of using a coherent state \(|\alpha\rangle\) one uses a nonlinear coherent state, for instance one constructed as eigenstate of a deformed annihilation operator \(A = \tilde{a} f(\tilde{n})\) then the DPANCSs (Deformed Photon Added Non Linear Coherent States) would be defined as [24]:

\[ |\alpha^{(m)}, f\rangle = \frac{A^{1m} |\alpha, f\rangle}{(\langle \alpha, f | A^m A^{1m} |\alpha, f\rangle)^{1/2}} \]  

where the nonlinear coherent state \(|\alpha, f\rangle\) is given by [15]:

\[ |\alpha, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n! f(n)!}} |n\rangle. \]  

Here, \(N_f\) is a normalization constant, \(f(n)! = f(n)f(n-1) \cdots f(0)\) and \(f(0)! = 1\). Successive applications of the deformed creation operator upon the nonlinear coherent state \(|\alpha, f\rangle\) yield:

\[ \hat{A}^{1m} |\alpha, f\rangle = N_f \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(n+m)!}}{n! f(n)!^2} f(n+m)! |n+m\rangle. \]  

so that the DPANCSs can be written as:

\[ |\alpha^{(m)}, f\rangle = \frac{N_f \sum_{n=0}^{\infty} \alpha^n \sqrt{(n+m)!}}{n! f(n)!^2} f(n+m)! |n+m\rangle. \]  

It is clear that the states given in \(|\alpha^{(m)}\rangle\) and \(|\alpha^{(m)}, f\rangle\) depend upon the specific form of the deformation function. In some cases, like for instance the Morse potential, the number of bound states supported by the potential is finite and the states obtained from Eqs. [4] and [24] are approximate [25].

A. Anharmonic oscillator Hamiltonian

Let us consider a Hamiltonian of the form [35]

\[ H_D = \hbar \Omega \hat{A} \hat{A}^\dagger \]  

that has been written in terms of deformed creation \(\hat{A}^\dagger\) and annihilation \(\hat{A}\) operators defined as \(\hat{A} = \hat{a} f(\hat{n})\) and \(\hat{A}^\dagger = f(\hat{n}) \hat{a}\) where the operators \(\hat{a}, \hat{a}^\dagger, \hat{n} = \hat{a}^\dagger \hat{a}\) are the usual harmonic oscillator operators and the function \(f(\hat{n})\) is a real function of the number operator. The Hamiltonian given above written in terms of the number operator takes the form

\[ H_D = \hbar \Omega \hat{n} f^2(\hat{n}), \]  

if the function \(f(\hat{n}) = 1\) we recover the harmonic oscillator Hamiltonian otherwise we have a deformed oscillator with deformation function \(f(\hat{n})\).

If we now choose the deformation function as:

\[ f^2(\hat{n}) = 1 + \frac{\chi}{\Omega} \hat{n}, \]  

the Hamiltonian given by Eq. [8] yields

\[ H_D = \hbar \Omega \hat{n} + \hbar \chi \hat{n}^2 = \hbar \omega_0 \left[ (1 - \chi/\omega_0) \hat{n} + (\chi/\omega_0) \hat{n}^2 \right] \]

which is the Hamiltonian for a single mode field of frequency \(\omega_0\) in a Kerr medium with a real anharmonicity parameter \(\chi\) related to the optical properties of the
medium\textsuperscript{36–38}. It has been shown by Yurke and Stoler\textsuperscript{39} that an initial coherent state $|\alpha\rangle$ will evolve under the influence of an anharmonic oscillator Hamiltonian of the form

$$H = \omega \hat{n} + \chi \hat{n}^2$$

(11)

into a coherent superposition of a finite number of coherent states which are distinguishable when $|\alpha|$ is large. Notice that Eq.\textsuperscript{11} reduces to Eq.\textsuperscript{10} for $k = 2$.

Once we have specified the deformation function the NLCS can be constructed explicitly using Eq.\textsuperscript{4}. For the case under consideration we obtain\textsuperscript{32}:

$$|\alpha, f\rangle = N_f(\alpha) \sum_{n=0}^{\infty} \left(\frac{\omega_0 - \chi}{\chi}\right)^{n/2} \frac{\alpha^n}{\sqrt{n! \left(\omega_0/\chi\right)^n}} |n\rangle$$

(12)

where $N_f(\alpha) = \Gamma(\alpha + n) / \Gamma(\alpha)$ is the Pochhammer symbol and the normalization constant is given by $N_f(\alpha) = 1 / \sqrt{\alpha F_3(\omega_0/\chi; (\omega_0 - \chi)/\chi) |\alpha|^2}$. Successive applications of the deformed creation operator upon the nonlinear coherent state $|\alpha, f\rangle$ yield the deformed photon added nonlinear coherent state:

$$|\alpha^{(m)}, f\rangle = \left(N_f^{(m)}(\alpha)\right)^{-1/2} \sum_{n=0}^{\infty} \alpha^n \left(\frac{\omega_0 - \chi}{\chi}\right)^{n/2} \frac{\alpha^n}{\sqrt{n! \left(\omega_0/\chi\right)^n}} |n + m\rangle$$

(13)

with the normalization constant $N_f^{(m)}(\alpha) = 2F_3(\omega_0/\chi + m, m + 1; \omega_0/\chi, \omega_0/\chi, 1) |\alpha|^2 (\omega_0 - \chi)/\chi$.

For a $m$ deformed photon added nonlinear coherent state $|\alpha^{(m)}, f\rangle$, the probability of finding the $k^{th}$ excited state in the distribution is given by $P_{k,m}(\alpha) = |\langle k | \alpha^{(m)}, f\rangle|^2$, we obtain:

$$P_{k,m}(\alpha) = \left|\frac{\alpha^{(m)}(\alpha)}{\sqrt{n! \left(\omega_0/\chi\right)^n}}\right|^2 \frac{k! (\omega_0/\chi)^{k-m}}{m! (\omega_0/\chi)^m [(k-m)!]^2}$$

(14)

Notice that due to the factorial in the denominator states with $k < m$ are not allowed as was the case for the $m$ photon added coherent state\textsuperscript{34}.

If we construct a deformed displacement operator by the replacement of the usual operators $\hat{a}, \hat{a}^\dagger$ by their deformed counterparts $\hat{A}, \hat{A}^\dagger$, we face the problem that the commutator between them is, in general, not a scalar and it is not possible to write the exponential of a sum as a product of exponentials. However, for the case we are considering here the commutation relations are:

$$[\hat{A}, \hat{A}^\dagger] = 1 + \frac{\chi}{\omega_0 - \chi} + \frac{2\chi}{\omega_0 - \chi} \hat{n}, \quad [\hat{A}, \hat{n}] = \hat{A},$$

$$[\hat{A}^\dagger, \hat{n}] = -\hat{A}^\dagger$$

(15)

and the set of operators $\{\hat{A}, \hat{A}^\dagger, \hat{n}, 1\}$ forms a closed Lie algebra enabling us to write the deformed displacement operator in a product form\textsuperscript{30}. If we now apply the deformed displacement operator to the vacuum state $|0\rangle$ we obtain the nonlinear coherent states:

$$|\zeta(\alpha)\rangle = (1 - |\zeta(\alpha)|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} |n\rangle$$

(16)

where $\zeta(\alpha) = e^{i\phi} \tanh (|\alpha| / \sqrt{\omega_0 - \chi})$ and $\alpha = |\alpha| e^{i\phi}$. We can now apply $m$ times the deformed creation operator upon this nonlinear coherent state and obtain the $m$ photon added nonlinear coherent state:

$$|\zeta(\alpha), m\rangle = \left(N_{\zeta(\alpha)}^m\right)^{-1/2} \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} |n + m\rangle$$

(17)

with the normalization constant $N_{\zeta(\alpha)}^m = 2F_3(\omega_0/\chi + m, m + 1; \omega_0/\chi, \omega_0/\chi, 1; |\alpha|^2)$. For the case of a $m$-photon added nonlinear coherent state constructed from the states $|\zeta(\alpha)\rangle$, the probability of finding the $k^{th}$ excited state in the distribution is given by:

$$P_{k,m}(\zeta(\alpha)) = \frac{1}{N_{\zeta(\alpha)}^m} \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} |n + m\rangle$$

(18)

In figure\textsuperscript{4} we show the distribution probabilities with $m = 1$ for the states $|\alpha^{(m)}\rangle$ (yellow), $|\alpha^{(m)}, f\rangle$ (blue) and $|\zeta(\alpha), m\rangle$ (green) for $\alpha = 3, \chi/\omega_0 = 0.1$. It can be seen that the distribution for the states $|\zeta(\alpha), m\rangle$ is broader than that of the $|\alpha^{(m)}, f\rangle$ with that for the states $|\alpha^{(m)}\rangle$ in between. A more detailed analysis of this conduct will be presented later in this work when we evaluate several statistical properties like the Mandel parameter, the Husimi Q-function and the Wigner function.
III. STATISTICAL PROPERTIES

We now analyze some statistical properties of the states we constructed in the previous section. Several criteria have been developed to analyze non-classical states \[21\]. Consider for instance the Mandel parameter \[22\] defined as:

\[
Q = \frac{\langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle}, \tag{19}
\]

For a Poissonian distribution the Mandel parameter is \( Q = 1 \) corresponding to a field coherent state \( |\alpha\rangle \), for \( Q > 1 \) we have super Poissonian statistics (classical states) and for \( Q < 1 \) sub Poissonian statistics (non classical states). The Mandel parameter as a function of \( \alpha \) using the m-photon added coherent states \( |\alpha^{(m)}\rangle \) as well as m-photon added non linear coherent states \( |\alpha^{(m)}, f\rangle \) and \( |\zeta(\alpha), m\rangle \) is shown in figure 2 for the case where \( \chi/\omega_0 = 0.15 \). As mentioned above, the Mandel parameter for a field coherent state \( |\alpha\rangle \) is a constant equal to one. For a photon added coherent state with \( m \) different from zero the Mandel parameter is zero when \( \alpha \) is zero and increases with \( \alpha \) tending asymptotically to one, in the figure we present the case when \( m = 1 \).

![Mandel parameter Q vs. alpha](image)

**FIG. 2.** (Color online) Mandel parameter \( Q \) with \( m = 0, 1 \) as a function of \( \alpha \) for a m-photon added coherent state \( |\alpha^{(m)}\rangle \) (red, purple), and deformed m-photon added non linear coherent states \( |\alpha^{(m)}, f\rangle \) (dark blue, blue) and \( |\zeta(\alpha), m\rangle \) (dark green, green) with the parameter \( \chi/\omega_0 = 0.15 \).

The Mandel parameter using the nonlinear coherent states \( |\alpha^{(m)}, f\rangle \) and \( |\zeta(\alpha), m\rangle \) is also shown in figure 2. For the case with \( m = 0 \) (no photon added) and for small values of \( \alpha \) both calculations converge to the harmonic result, that is, they start at one. As the \( \alpha \) value increases their behavior separates from the harmonic one. For the nonlinear states \( |\alpha^{(0)}, f\rangle \) the Mandel parameter takes values smaller than one indicating a non classical behavior in contrast with the nonlinear states \( |\zeta(\alpha), 0\rangle \) whose Mandel parameter is larger than one indicating a super Poissonian statistics. For \( m \neq 0 \) (in the figure we present the case \( m=1 \)), their Mandel parameter starts at zero in accordance with the PACS \( |\alpha^{(m)}\rangle \), however as the \( \alpha \) value increases, the Mandel parameter for the states \( |\alpha^{(1)}, f\rangle \) approaches 0.6 asymptotically in contrast with the PACS whose asymptotic value is that corresponding to a coherent state. That means that independently of the value of the parameter \( \alpha \) these deformed photon added nonlinear coherent states remain nonclassical. The Mandel parameter for the non linear states \( |\zeta(\alpha), 1\rangle \) starts at zero as the other two cases but attains values larger than one indicating a classical behavior for large enough values of \( \alpha \).

It is worth mentioning that for the Hamiltonian given by Eq. 10 both nonlinear coherent states \( |\alpha, f\rangle \) and \( |\zeta(\alpha)\rangle \) are exact because the number of bound states is infinite.

The Husimi-Q function is always positive, and in some cases is convenient to use for simple representations of quantum states. It is defined as the Fourier transform of the anti-normal-order characteristic function

\[
Q^{[\rho]}(\alpha) = \frac{1}{\pi} \int d^{2}\lambda e^{(\alpha \lambda^* - \alpha^* \lambda)} C^{[\rho]}_{m,n}(\lambda),
\]

which is simply related to the expectation value of the density operator \( \rho \) in state \( |\alpha\rangle \) by:

\[
Q^{[\rho]}(\alpha) = \frac{1}{\pi} \text{Tr} [\rho |\alpha\rangle \langle \alpha|] = \frac{1}{\pi} |\langle \alpha|\rho|\alpha\rangle| \tag{20}
\]

which can also be written as:

\[
Q^{[\rho]}(\alpha) = \frac{1}{\pi} \text{Tr} [0|D(-\alpha)\rho D(\alpha)|0] = \frac{1}{\pi} \text{Tr} [(00)\rho D(-\alpha)\rho D(\alpha)]. \tag{21}
\]

The \( Q \) function is thus the average of the projector onto the vacuum state, in the field displaced in phase space by \(-\alpha\). Eq. (21) is the basis of the \( Q \)-function reconstruction method described in Refs. 26, 27.

For coherent states \( |z\rangle \) it is given by:

\[
Q^{[\alpha]}(z) = \frac{1}{\pi} \langle z|\alpha\rangle \langle \alpha|z\rangle = \frac{1}{\pi} \exp (-|z - \alpha|^2). \tag{22}
\]

with \( |z\rangle \) and \( |\alpha\rangle \) usual harmonic oscillator coherent states. For the case of a coherent state \( |z\rangle \) and an \( m \) photon added coherent state \( |\alpha^{(m)}\rangle \) the Husimi function is defined by \[4\] :

\[
Q^{[\alpha^{(m)}]}(z) = \frac{1}{\pi} \langle z|\alpha^{(m)}\rangle \langle \alpha^{(m)}|z\rangle
= \frac{1}{\pi} \frac{|z|^{2m}}{m!L_m(-|\alpha|^2)} \exp (-|z - \alpha|^2) \tag{23}
\]

We now consider the deformed \( m \) photon added nonlinear coherent state \( |\alpha^{(m)}, f\rangle \) given by Eq. 13. The Husimi function is in this case:

\[
Q^{[\alpha^{(m)}, f]}(z) = \frac{1}{\pi} \langle z|\alpha^{(m)}, f\rangle \langle \alpha^{(m)}, f|z\rangle \tag{24}
\]
and taking the products we get:

$$Q^{|\alpha^{(m)},f\rangle\langle\alpha^{(m)},f\rangle|}(z) = \frac{1}{\pi} \frac{e^{-|z|^2}}{N_{\alpha}^m \cdot m!} \times \sum_{n=0}^{\infty} \frac{(z^* \alpha m)^n}{n!} \left( \frac{\omega_0 - \chi}{\chi} \right)^{n/2} \frac{\sqrt{(\omega_0/\chi + m)_n}}{(\omega_0/\chi)_n} (25)$$

with the normalization constant $N_{\alpha}^m = 2^m F_2(\omega_0/\chi + m, m + 1; (\omega_0/\chi, 1; (\omega_0 - \chi)/\chi|^2)$.

In figure 3[4] we show the Husimi function $Q^{|\alpha^{(1)},f\rangle\langle\alpha^{(1)},f\rangle|}(z)$ (1 photon added) for $\alpha = 1.1, z = x + iy, \chi/\omega_0 = 0.15$.

For the case of deformed photon added nonlinear coherent states obtained by application of the deformed displacement operator upon the vacuum state $|\zeta(\alpha), m\rangle$ (see Eq. 17) we obtain:

$$Q^{|\zeta(\alpha), m\rangle\langle\zeta(\alpha), m\rangle|}(z) = \frac{1}{\pi} \frac{e^{-|z|^2}}{N_{\zeta(\alpha)}^m \cdot m!} \times \sum_{n=0}^{\infty} \frac{(z^* \zeta(\alpha))^n}{n!} \left( \frac{\omega_0/\chi + m}_n \right) (26)$$

with normalization constant $N_{\zeta(\alpha)}^m = 2^m F_2(\omega_0/\chi + m, m + 1; (\omega_0/\chi, 1; (\omega_0 - \chi)/\chi|^2)^2$.

In figure 4[5] we show its Husimi function with the same set of parameters used in figure 3. Notice that both distributions are qualitatively similar even though the nonlinear coherent states $|\alpha^{(m)}, f\rangle$ and $|\zeta(\alpha), m\rangle$ differ significantly in their Mandel parameter. We also calculated the Husimi function for no-photon added nonlinear coherent states and the differences between them are too small to be noticed in a graph for this value of the parameter $\alpha$.

Since the Husimi function is always positive it is best to compute the Wigner function for the purpose of getting information related with the non classicality of the state [28]. To that end we consider the work of Refs. [40, 42] where the Wigner function is written as an infinite series in terms of the displaced number-state expectation values

$$W(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \hat{\rho} | \alpha, k \rangle (27)$$

and where the states $|\alpha, k\rangle$ are defined as [6, 43, 45]:

$$|\alpha, k\rangle = \hat{D}(\alpha) |k\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \left( \frac{|k|!}{n!} \right)^{1/2} \alpha^k \mathcal{L}_n^{k-n} (|\alpha|^2) |n\rangle,$$

where $\mathcal{L}_n^{k-n}(x)$ is the associated Laguerre polynomial. The displaced number state is a generalization of the usual field coherent state, they have been studied theoretically and experimentally. It is worth to mention that the displaced Fock states of the electromagnetic field were synthesized by overlapping the single-photon Fock state $|1\rangle$ with coherent states on a high-reflection beam splitter and completely characterized by means of quantum homodyne tomography [46] and more recently displaced number states of a vibrational mode of a single trapped ion were created by means of a combination of optical and electrical manipulation of the ion [47].

The density matrix for a coherent state is $\hat{\rho} = |\beta\rangle \langle \beta|$, so that the Wigner function can be written as [40]:

$$W(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \hat{\rho} | \beta, \alpha, k \rangle = \frac{2}{\pi} \exp(-2|\beta - \alpha|^2) (28)$$

For the construction of the Wigner function correspond-
ing to the nonlinear coherent state $|\beta, f\rangle$ we get:

$$W_f(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \beta, f \rangle \langle \beta, f | \alpha, k \rangle$$

(29)

where $|\beta, f\rangle$ is the nonlinear coherent state obtained either as eigenstate of the deformed annihilation operator (Eq. 12) or by displacement of the vacuum state by the generalized displacement operator (Eq. 10).

Since the states $|\alpha^{(m)}, f\rangle$ and $|\zeta(\alpha), m\rangle$ show different statistical behavior, we will consider them separately. First we take the density matrix corresponding to a nonlinear photon added coherent state obtained as an eigenstate of the annihilation operator. The Wigner function is:

$$W_{f,A}^m(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \beta^{(m)}, f \rangle \langle \beta^{(m)}, f | \alpha, k \rangle$$

(30)

where the state $|\beta^{(m)}, f\rangle$ is that given by Eq. 13. As a result we obtain:

$$W_{f,A}^m(\alpha) = \frac{2e^{-|\alpha|^2}}{\pi N_m^f(\beta)} \sum_{k=0}^{\infty} (-1)^k |\alpha|^{2(m-k)} \sum_{l=0}^{k} \frac{(-|\alpha|^2)^l}{l!} \times \sum_{n=0}^{\infty} \frac{(\alpha^* \beta)^n n!}{n!} \frac{\omega_0 - \chi}{\chi} \frac{n^2}{k+1} \left(\frac{n+m}{k-l}\right) \times \frac{\sqrt{\frac{(\omega_0/\chi+m)_n}{\omega_0/\chi}}}{\omega_0/\chi} \beta^2.$$  

(31)

The Wigner function for m-photon added nonlinear coherent states $|\zeta(\beta), m\rangle$ is:

$$W_{f,D}^m(\alpha) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \langle \alpha, k | \zeta(\beta), m \rangle \langle \zeta(\beta), m | \alpha, k \rangle$$

(32)

with the states $|\zeta(\beta), m\rangle$ given by Eq. 17. As a result we get:

$$W_{f,D}^m(\alpha) = \frac{2e^{-|\alpha|^2}}{\pi N_m^{\zeta(\beta)}(\alpha)} \sum_{k=0}^{\infty} (-1)^k |\alpha|^{2(m-k)} \sum_{l=0}^{k} \frac{(-|\alpha|^2)^l}{l!} \times \sum_{n=0}^{\infty} \frac{(\alpha^* \zeta(\beta))^n n!}{n!} \frac{\omega_0 - \chi}{\chi} \frac{(n+m)(k-l)}{k-l} \times \frac{\sqrt{(\omega_0/\chi + m)_n}}{\omega_0/\chi} |\alpha|^2.$$  

(33)

When we evaluated the Mandel parameter, we found that the states $|\zeta(\alpha)\rangle$ have a super-Poissonian statistics while $|\alpha, f\rangle$ a sub-Poissonian one and both coincide with the usual coherent states when $\alpha$ is small enough. This means that nonclassical behavior is expected for $|\alpha, f\rangle$ and not for $|\zeta(\alpha)\rangle$. For the photon added nonlinear coherent states $|\zeta(\alpha), m\rangle$ and $|\alpha^{(m)}, f\rangle$ the Mandel parameter is smaller than one for small values of the parameter $\alpha$. We should expect a nonclassical behavior for both nonlinear coherent states in this region of $\alpha$.

It has been found experimentally that the single photon added coherent state has a Wigner function that can be negative for small values of the parameter $\alpha$ and the state approaches a coherent state conduct for large enough values of it.

We evaluated the expressions given in Eqs. 31 and 33 and the results are shown in figures 5 and 6.

![FIG. 5. (Color online) Wigner function $W_{f,A}^{\beta -1}(\alpha)$ with a density matrix corresponding to a nonlinear coherent state $|\beta^{(1)}, f\rangle$ obtained as eigenstate of the annihilation operator with $\beta = 1.1, \chi/\omega_0 = 0.15$.](image)

![FIG. 6. (Color online) Wigner function $W_{f,D}^{\beta -1}(\alpha)$ with a density matrix corresponding to a nonlinear coherent state $|\zeta(\beta), 1\rangle$ obtained by displacement of the vacuum state with $\beta = 1.1, \chi/\omega_0 = 0.15$.](image)

We can see from the figures that in both cases the Wigner function presents negativities and in figure these negativities are more pronounced than those in figure. This result is in agreement with what we found when we calculated the Mandel parameter, since for this value of the parameter $\alpha$ the Mandel parameter is in both cases smaller than one, though in the case of the $|\zeta(\alpha), 1\rangle$ it is close to one whereas for the states $|\alpha^{(1)}, f\rangle$ it is about 0.5.
In figure 7 we show the Wigner function for a photon added nonlinear coherent state $|\beta^{(m)}, f \rangle$ for the case with $m = 4$ and it is seen that it takes negative values and the distribution is not symmetrical with respect to the axes.

![Wigner function](image)

**FIG. 7.** (Color online) Wigner function $W_f (\beta)$ of a density matrix corresponding to a photon added nonlinear coherent state $|\beta^{(m)}, f \rangle$ obtained as eigenstate of the deformed annihilation operator with $\beta = 0.5, \chi/\omega_0 = 0.15$ and $m = 4$.

### IV. CONCLUSIONS

In this work we have analyzed the statistical differences between nonlinear coherent states for a Kerr like medium when one makes use of two different generalizations for their construction. On the one hand we have generalized the definition as eigenstates of a deformed annihilation operator (AOCs), on the other, as those states obtained by the application of a generalized displacement operator upon the vacuum state (DOCS). Since the Hamiltonian fulfills a finite Lie algebra and the number of eigenvalues is infinite, both generalizations are exact and thus allow by the measurement of their statistical properties to discern which generalization is most appropriate, at least for this particular Hamiltonian. It would be difficult to obtain this information from the reconstruction of the Husimi or the Wigner functions since these are very similar for the AOCs and the DOCS.

When computing the Mandel parameter we found that the AOCs present a sub-Poissonian statistics corresponding to non classical states while the DOCS present a super-Poissonian statistics corresponding to a thermal state. We constructed also deformed $m$-photon added nonlinear coherent states by application of the deformed creation operator upon these non linear coherent states. We found that the deformed $m$-photon added nonlinear coherent states obtained by application of the deformed annihilation operator upon the $|\alpha, f \rangle$ present a more pronounced non classicality than the $m$-photon added coherent states introduced by Agarwal and Tara and our results are in agreement with those of Ref. [24]. However, when we construct the deformed $m$-photon added nonlinear coherent states by application of the deformed annihilation operator upon $|\zeta(\alpha)\rangle$ we obtain states whose conduct becomes classical for large enough values of the parameter $\alpha$. We stress the fact that for a deformation function $f(\tilde{n})$ corresponding to a Kerr like medium the non linear coherent states $|\alpha, f \rangle$ and $|\zeta(\alpha)\rangle$ present a completely different statistical behavior, in the first case the states are non classical for all values of the parameter $\alpha$, in the second they behave as classical states. When one constructs the deformed photon added nonlinear coherent states with $|\alpha, f \rangle$ they are more non classical than the PACS, while those constructed with the $|\zeta(\alpha)\rangle$ are non-classical only for small $\alpha$ and become classical for large enough values of it.

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