ABELIAN EXTENSIONS OF GLOBAL FIELDS
WITH CONSTANT LOCAL DEGREES

HERSHY KISILEVSKY AND JACK SONN
Concordia University, Montreal and Technion, Haifa

ABSTRACT. Given a global field $K$ and a positive integer $n$, there exists an abelian extension $L/K$ (of exponent $n$) such that the local degree of $L/K$ is equal to $n$ at every finite prime of $K$, and is equal to two at the real primes if $n = 2$. As a consequence, the $n$-torsion subgroup of the Brauer group of $K$ is equal to the relative Brauer group of $L/K$.

1. Introduction

Let $K$ be a field, $Br(K)$ its Brauer group. If $L/K$ is a field extension, then the relative Brauer group $Br(L/K)$ is the kernel of the restriction map $res_{L/K} : Br(K) \to Br(L)$. Relative Brauer groups have been studied by Fein and Schacher (see e.g. [4,5,6].) Every subgroup of $Br(K)$ is a relative Brauer group $Br(L/K)$ for some extension $L/K$ [4], and the question arises as to which subgroups of $Br(K)$ are algebraic relative Brauer groups, i.e. of the form $Br(L/K)$ with $L/K$ an algebraic extension. For example if $L/K$ is a finite extension of number fields, then $Br(L/K)$ is infinite [5], so no finite subgroup of $Br(K)$ is an algebraic relative Brauer group. In [2] the question was raised as to whether or not the $n$-torsion subgroup $Br_n(K)$ of the Brauer group $Br(K)$ of a field $K$ is an algebraic relative Brauer group. For example, if $K$ is a ($p$-adic) local field, then $Br(K) \cong \mathbb{Q}/\mathbb{Z}$, so $Br_n(K)$ is an algebraic relative Brauer group for all $n$. A counterexample was given in [2] for $n = 2$ and $K$ a formal power series field over a local field. For global fields $K$, the problem is a purely arithmetic one, because of the fundamental local-global description of the Brauer group of a number field. In particular, for a Galois extension $L/K$ of global fields, if the local degree of $L/K$ at every finite prime is equal to $n$, and is equal to 2 at the real primes for $n$ even, then $Br(L/K) = Br_n(K)$. In [2], it was proved that $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all squarefree $n$. In [8], the arithmetic criterion above was verified for any number field $K$ Galois over $\mathbb{Q}$ and any $n$ prime to the class number of $K$, so in particular, $Br_n(\mathbb{Q})$ is an algebraic relative Brauer group for all $n$. In [9], Popescu proved that for a global function field $K$ of characteristic $p$, the arithmetic criterion holds for $n$ prime to the order of the non-$p$ part of the Picard group of $K$.

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In this paper we settle the question completely, by verifying the arithmetic criterion for all \( n \) and all global fields \( K \). In particular, the \( n \)-torsion subgroup of the Brauer group of \( K \) is an algebraic relative Brauer group for all \( n \) and all global fields \( K \). The proof, an extension of the ideas in [8], reduces to the case \( n \) a prime power \( \ell^r \). We first carry out the proof for number fields \( K \). The proof for the function field case when \( \ell \neq \text{char}(K) \) is essentially the same as the proof in the number field case. The proof for \( \ell = \text{char}(K) \) appears in [9].

2. A Splitting Lemma

Let \( K \) be a number field, \( p \) a finite prime of \( K \), \( I_p \) the group of fractional ideals prime to \( p \), \( P_p \) the group of principal fractional ideals in \( I_p \), \( P_{p,1} \) the group of principal fractional ideals in \( P_p \) generated by elements congruent to \( 1 \mod p \). Then \( Cl_K \cong I_p/P_p \) is the class group of \( K \), \( Cl_{K, p} \cong I_p/P_{p,1} \) is the ray class group with conductor \( p \), and \( \overline{P}_p = P_p/P_{p,1} \) is the principal ray with conductor \( p \). We have a short exact sequence

\[
1 \rightarrow \overline{P}_p \rightarrow Cl_{K,p} \rightarrow Cl_K \rightarrow 1. \quad (*)
\]

Let \( \ell \) be a prime dividing the orders of all three terms of (*) and consider the exact sequence of \( \ell \)-primary components

\[
1 \rightarrow \overline{P}_p^{(\ell)} \rightarrow Cl_{K,p}^{(\ell)} \rightarrow Cl_K^{(\ell)} \rightarrow 1. \quad (*\ell)
\]

We are interested in primes \( p \) for which the sequence \((*\ell)\) splits. Let \( a_1, \ldots, a_s \in I_p \) such that their images \( \overline{a}_i \) in \( Cl_K^{(\ell)} \) form a basis of the finite abelian \( \ell \)-group \( Cl_K^{(\ell)} \). Let \( \ell^{m_i} \) be the order of \( \overline{a}_i \), \( i = 1, \ldots, s \). Then \( a_i^{\ell^{m_i}} = (a_i) \in P_p, i = 1, \ldots, s \). Let \( K_1 = K(\sqrt[\ell^{m_i}]{a_i}, 1 \leq i \leq s)K(\mu_{\ell^{m_i}}) \) with \( m = \max\{m_1, \ldots, m_s\} \).

Lemma 2.1. In order that the sequence \((*\ell)\) split, it is sufficient that \( p \) split completely in \( K_1 \).

Proof. Suppose \( p \) splits in \( K_1 \). Then \( a_i \) is locally an \( \ell^{m_i} \)-th power at \( p \), so there exists \( a_i \in K^* \) such that \( a_i^{\ell^{m_i}} \equiv a_i \mod p \), \( i = 1, \ldots, s \). Let \( \alpha_i = a_i^{-1} \). Then \( \overline{b}_i = \overline{a}_i \) \( \ell^{m_i} \), \( i = 1, \ldots, s \), and \( b_i = a_i^{\ell^{m_i}} = a_i^{\ell^{m_i}} \alpha_i^{-\ell^{m_i}} = a_i^{(\alpha_i^{-\ell^{m_i}} \equiv 1 \mod p)} \), so \( b_i \in P_{p,1} \). Let \( \overline{b}_i \) be the image of \( b_i \) in \( Cl_{K,p}^{(\ell)} \). We have just seen that its order is \( \ell^{m_i} \).

Hence the \( \overline{b}_i \) together generate a subgroup of order at most \( \ell^{\sum m_i} \). Since the \( \overline{b}_i = \overline{a}_i \) generated the \( \ell \)-class group \( Cl_{K, p}^{(\ell)} \), of order \( \ell^{\sum m_i} \), it follows that the \( b_i \) together generate a complement to \( \overline{P}_p^{(\ell)} \) in \( Cl_{K,p}^{(\ell)} \). \( \square \)

Note. In the proof of Lemma 2.1, we could have taken \( a_1, \ldots, a_s \in I_p \) to be prime ideals outside of any given finite set of primes of \( K \), by virtue of the generalized Dirichlet density theorem.
3. $\ell^r$-torsion in the Brauer group

**Theorem 3.1.** Let $K$ be a number field, $\ell$ a prime number, $r$ a positive integer. Then there exists an abelian $\ell$-extension $L/K$ of exponent $\ell^r$ such that the local degree $[L_K^p : K_p]$ is equal to $\ell^r$ for every finite prime $p$ of $K$, and is equal to 2 at every real prime if $\ell = 2$.

**Proof.** Let $H_K$ be the Hilbert class field of $K$, and let $H_{K}(\ell)$ be the $\ell$-primary part of $H_K$. Let $\ell^t$ be the exponent of $Cl(\ell)_K$. Let $S$ be the set of primes of $K$ that split completely in the extension $F$ of $K$ generated by $H_{K}(\ell)$, all $\ell^{r+t}$-th roots of all units (including roots of unity) of $K$, and the field $K_1$ of Lemma 2.1. For $p \in S$, $\mathcal{P}_p \cong (\mathcal{O}_K/p)^*$ modulo the image of the unit group $E_K$ of $K$.

By definition of $S$, every unit of $K$ is an $\ell^{r+t}$-th power in $K^*_p$, so the cyclic group $\mathcal{P}_{p}^{(\ell)}$ has order divisible by $\ell^{r+t}$. Let $R(p,\ell)$ denote the $\ell$-ray class field with conductor $p$. It follows that the subfield of $R(p,\ell)$ fixed by the $\ell^t$-torsion subgroup of $Gal(R(p,\ell)/K)$ is a cyclic extension of $K$ of degree divisible by $\ell^r$. Let $L^p$ be the subfield of this field of degree exactly $\ell^r$ over $K$. $L^p/K$ is totally ramified at $p$ and $p$ is the only prime of $K$ ramifying in $L^p$.

For each $p \in S$ the splitting of the short exact sequence $K_{\ell}$ allows us to fix a splitting map $f_p : Cl_{K,p}^{(\ell)} \rightarrow \mathcal{P}_{p}^{(\ell)}$.

By [3, Theorem 5, p. 105], there exists a cyclic extension $L_0/K$ of degree $\ell^r$ having local degree $\ell^r$ at the primes $l_1, ..., l_g$ dividing $\ell$, and local degree 2 at the real primes if $\ell = 2$.

Let $q_1, ..., q_e$ be the primes of $K$ not dividing $\ell$, which ramify in $L_0$, ordered in such a way that $L_0/K$ has local degree $\ell^r$ at $q_1, ..., q_d$, $0 \leq d \leq e$, and $L_0/K$ has local degree less than $\ell^r$ at $q_{d+1}, ..., q_e$.

Using the Note following the proof of Lemma 2.1, we may assume without loss of generality that the ideals $a_1, ..., a_s$ in Lemma 2.1, representing the elements of the class group, are prime ideals, distinct from $l_1, ..., l_g$ and from $q_1, ..., q_e$.

Suppose $d < e$. Let $q = q_{d+1}$, and let $\ell^a, \ell^b$ be the ramification index and inertia degree respectively of $q$ in $L_0/K$.

We seek a prime $p \in S$ such that

1. $p$ splits completely in $L_0$,
2. $l_1, ..., l_g$ and $q_1, ..., q_d$ split completely in $L^p$, and
3. the local degree of $L_0L^p/K$ at $q$ is $\ell^r$.

(1) is a Chebotarev splitting condition compatible with the Chebotarev condition $p \in S$. (2) is equivalent to a Chebotarev splitting condition on $p$, by a “reciprocity argument” (cf.
Lemma 2.2. Let \( a \) be any prime of \( K \), and let \( p \) be any prime in \( S \) different from \( a \). Let \( L' \) be a subfield of \( L^p \) containing \( K \), of degree \( \ell^s \) over \( K \). Let \( \ell^m \) be the order of the image of \( a \) in \( Cl_{K,p}^{(\ell)} \), and let \( a^{\ell m} = (\alpha) \in P_p \). Then \( a \) splits completely in \( L' \) if and only if \( p \) splits completely in \( K(\mu_{\ell^m+s}, \sqrt[\ell^m+s]{\alpha}) \).

Proof. Let \( \widehat{\beta} = f_p(\widehat{a}) \in \overline{P}_p^{(\ell)} \), where \( f_p : Cl_{K,p}^{(\ell)} \rightarrow \overline{P}_p^{(\ell)} \) is the splitting map above, and \( \widehat{a} \) denotes the image of \( a \) in \( Cl_{K,p}^{(\ell)} \). Then

\[
\widehat{\beta}^{\ell^m} = f_p(\widehat{a})^{\ell^m} = f_p(\widehat{a}^{\ell^m}) = f_p((\widehat{\alpha})) = (\widehat{\alpha}).
\]

If \( p \in S \), the cyclic group \( \overline{P}_p^{(\ell)} \) has order divisible by \( \ell^{r+t} \) hence by \( \ell^{m+s} \) (because \( m \leq t \) and \( s \leq r \)). Now \( a \) splits completely in \( L' \) if and only if the Frobenius of \( a \) in \( \text{Gal}(L'/K) \) is trivial, which holds if and only if \( \widehat{\beta} = f_p(\widehat{a}) \) is an \( \ell^s \)th power in \( \overline{P}_p^{(\ell)} \), which holds if and only if \( \widehat{\alpha} = \widehat{\beta}^{\ell^m} \) is an \( \ell^{m+s} \)th power in \( \overline{P}_p^{(\ell)} \) (since \( \overline{P}_p^{(\ell)} \) has order divisible by \( \ell^{m+s} \)), which holds if and only if \( \alpha \) is an \( \ell^{m+s} \)th power mod \( p \), which holds if and only if the polynomial \( x^{\ell^{m+s}} - \alpha \) has a root mod \( p \), which holds if and only if \( x^{\ell^{m+s}} - \alpha \) factors into linear factors mod \( p \) (p splits completely in \( K(\mu_{\ell^{m+s}}) \)), which holds if and only if \( p \) splits completely in \( K(\mu_{\ell^m+s}, \sqrt[\ell^m+s]{\alpha}) \). \( \square \)

It follows from the Lemma that the second condition is a Chebotarev splitting condition on \( p \), hence compatible with the previous conditions. The third condition, that the local degree of \( L_0L^p/K \) at \( q \) is \( \ell^r \), will hold if \( q \) is unramified in \( L^p/K \) with degree of inertia exactly \( \ell^{r-a} \). This condition is equivalent, by Lemma 2.2, to the condition that \( p \) splits completely in \( K(\mu_{\ell^m+s}, \sqrt[\ell^m+s]{\alpha}) \) but not in \( K(\mu_{\ell^m+s+1}, \sqrt[\ell^m+s+1]{\alpha}) \), where \( q \) plays the role of \( a \) in Lemma 2.2. \( m \) is the order of \( q \) in the class group. Note that \( a < r \) here, so \( m + a + 1 \leq r + t \). In order that this last Chebotarev condition be compatible with the preceding ones, it is necessary and sufficient that \( \sqrt[\ell^{m+a+1}]{\alpha} \) not lie in the composite of all the fields in which \( p \) has been required to split. This follows from ramification considerations, since the ramification index of \( q \) in the composite of all the fields in which \( p \) has been required to split, is exactly \( \ell^a \), whereas the ramification index of \( q \) in \( K(\mu_{\ell^m+s+1}, \sqrt[\ell^m+s+1]{\alpha}) \) is \( \ell^{a+1} \). We have therefore proved the existence of a prime (in fact infinitely many) \( p = p_1 \) satisfying the three conditions.

Similarly, there exists a prime \( p_2 \in S \), such that

(4) \( p_2 \) splits completely in \( L_0 \) and \( L_1 := L^{p_1} \),

(5) \( l_1, ..., l_g, q_1, ..., q_d, q_{d+1}, p_1 \) split completely in \( L_2 := L^{p_2} \), and

(6) the local degree of \( L_0L^{p_2}/K \) at \( q_{d+2} \) is \( \ell^r \).
Proceeding in this manner, we get \( p_1, \ldots, p_{e-d} \), such that

(7) \( p_i \) splits completely in \( L_j \) for \( i \neq j \),

(8) \( l_i, \ldots, l_g, q_1, \ldots, q_e \) split completely in \( L_1, \ldots, L_{e-d} \), and

(9) the local degree of the composite \( L_0L_1 \cdots L_{e-d} \) is equal to \( \ell^r \) at each of the primes \( l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, \ldots, p_{e-d} \).

Let us now extend \( l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, \ldots, p_{e-d} \) to an enumeration

\[ l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, \ldots, p_{e-d}, a_1, a_2, \ldots, a_s, a_{s+1}, \ldots \]

of all the finite primes of \( K \), where \( a_1, a_2, \ldots, a_s \) are the primes of Lemma 2.1.

We now seek a prime \( p = p_{e-d+1} \in S \) such that

(10) \( p \) splits completely in \( L_0, L_1, \ldots, L_{e-d} \),

(11) \( l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, \ldots, p_{e-d}, a_1, a_2, \ldots, a_s, a_{s+1}, \ldots \) split completely in \( L^p \), and

(12) \( a = a_1 \) is inert in \( L^p \).

(10) and (11) are equivalent to saying that \( p \) splits completely in a Galois extension (a composite of \( L_i \)'s and fields used in proving compatibility of previous Chebotarev conditions) in which \( a \) is unramified. (12) is equivalent to the condition that \( a \) does not split completely in the subfield of \( L^p \) of degree \( \ell \) over \( K \). By Lemma 2.2 applied to this \( a \), this condition is equivalent to the (Chebotarev) condition that \( p \) does not split completely in \( K(\mu_{\ell^{m+1}}, \sqrt[\ell^{m+1}]{\alpha}) \), where \( m \) corresponds to this \( a \). Since \( a \) ramifies in \( K(\mu_{\ell^{m+1}}, \sqrt[\ell^{m+1}]{\alpha}) \), this last Chebotarev condition is compatible with the others.

We remove \( p = p_{e-d+1} \) from the enumeration, denote \( L^p \) by \( L_{e-d+1} \), and move to the next ideal \( a_2 \). A similar argument yields a prime \( p = p_{e-d+2} \) and a corresponding \( L_{e-d+2} \), such that

(13) \( p \) splits completely in \( L_0, L_1, \ldots, L_{e-d+1} \),

(14) \( l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, \ldots, p_{e-d+1} \) split completely in \( L^p \), and

(15) \( a = a_2 \) is inert in \( L^p = L_{e-d+2} \).

Continuing in this way, a sequence \( \{p_i\} \) of primes of \( K \) is generated, with a corresponding sequence of fields \( \{L_i\} \).

Form the composite \( L := L_0L_1L_2 \cdots \). The local degree of \( L/K \) at every finite prime is \( \ell^r \). In fact, the local degree of \( L_0L_1 \cdots L_{e-d} \) is equal to \( \ell^r \) at each of the primes \( l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, \ldots, p_{e-d} \), as we showed earlier; these primes split completely in \( L_j \) for \( j > e - d \); at every \( p_i \), the local extension is cyclic and totally ramified of degree \( \ell^r \); and at every \( a \notin \{l_1, \ldots, l_g, q_1, \ldots, q_e, p_1, p_2, \ldots \} \), the local extension is cyclic and unramified of degree \( \ell^r \). The local degree at the real primes is 2 since this is true already for \( L_0 \). \( \Box \)
4. The function field case.

Let \( K \) be a global function field of characteristic \( p \). Let \( \ell \) be a prime different from \( p \). Theorem 3.1 holds also for \( K \), with essentially the same proof, except simpler, since there are no primes dividing \( \ell \). We indicate how this works.

By Chebotarev’s density theorem for function fields (see e.g. [10]), there exists a prime \( p \) which is inert in the constant degree \( \ell \) extension of \( K \). This property is equivalent to \( p \) having degree prime to \( \ell \) (see e.g. [9]). Fix one such prime \( p_\infty \). Let \( \mathcal{O} \) be the ring of all elements of \( K \) which are integral at all primes \( p \neq p_\infty \) of \( K \). \( \mathcal{O} \) is a Dedekind domain.

Call \( p_\infty \) the infinite prime of \( K \) and all the others finite primes of \( K \). There is an exact sequence

\[
1 \rightarrow \text{Pic}^0(K) \rightarrow \text{Pic}(\mathcal{O}) \rightarrow \mathbb{Z}/d_\infty \mathbb{Z} \rightarrow 0
\]

with \( \text{Pic}^0(K) \) the group of divisor classes of degree zero of \( K \), \( \text{Pic}(\mathcal{O}) \) the ideal class group of \( \mathcal{O} \), and the third arrow is the degree mod \( d_\infty \) map, where \( d_\infty \) denotes the degree of \( p_\infty \). As \( d_\infty \) and \( \ell \) are relatively prime, the \( \ell \)-primary components of \( \text{Pic}^0(K) \) and \( \text{Pic}(\mathcal{O}) \) are isomorphic; we denote them by \( \text{Cl}_{\ell}^0(K) \). If \( p \) is a finite prime of \( K \), let \( R^{(p)} \) be the maximal abelian extension of \( K \) with conductor dividing \( p \), in which \( p_\infty \) splits completely, i.e. the ray class field mod \( p \). \( \text{Gal}(R^{(p)}/K) \) is canonically isomorphic, via the reciprocity map, to \( \text{Cl}_{K,p} \), the ray class group mod \( p \) (See [7, p. 204].

The proofs of Lemma 2.1, Lemma 2.2 and Theorem 3.1 proceed as in the number field case, except that there are no primes \( l_1, ..., l_g, q_1, ..., q_e \) to deal with; the field \( L_0 \) is replaced with the constant degree \( \ell^r \) extension of \( K \), and \( e = 0 \). \( L \) is the composite of the fields \( L_0, L_1, L_2, ... \). We therefore have

**Theorem 4.1.** Let \( K \) be a global function field of characteristic \( p \), \( \ell \) a prime different from \( p \), \( r \) a positive integer. Then there exists an abelian \( \ell \)-extension \( L/K \) of exponent \( \ell^r \) such that the local degree \([LK_p : K_p] \) is equal to \( \ell^r \) for every prime \( p \) of \( K \).

5. The \( n \)-torsion subgroup of the Brauer group of \( K \)

**Theorem 5.1.** Given a global field \( K \) and a positive integer \( n \), there exists an abelian extension \( L/K \) (of exponent \( n \)) such that the \( n \)-torsion subgroup of the Brauer group of \( K \) is equal to the relative Brauer group of \( L/K \).

**Proof.** Consider the case \( n = \ell^r \), \( \ell \) prime. By Theorems 3.1 and 4.1, if \( \text{char}(K) \neq \ell \), there exists an abelian \( \ell \)-extension \( L/K \) whose local degree at every nonarchimedean prime is equal to \( \ell^r \), and is equal to 2 at the real primes if \( K \) is a number field and \( \ell = 2 \). If \( \text{char}(K) = \ell \), the same result holds by [9]. It follows from the fundamental theorem of class field theory on the Brauer group of a number field that \( L \) splits every algebra class of order dividing \( \ell^r \), and conversely, any algebra class split by \( L \) has order dividing \( \ell^r \). For general \( n \), the theorem follows from a straightforward reduction to the prime power case (see [2]).
Remark. Adrian Wadsworth has pointed out to us that in the case \( \text{char}(K) = \ell \), 
\( Br_{\ell^r}(K) \) is the relative Brauer group of a purely inseparable extension, namely 
\( L = \ell^r\sqrt{K} \), by results of Albert [1, p. 109].

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