ON HARD QUADRATURE PROBLEMS
FOR MARGINAL DISTRIBUTIONS OF SDES
WITH BOUNDED SMOOTH COEFFICIENTS

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Abstract. In recent work of Hairer, Hutzenthaler and Jentzen, see [9], a stochastic
differential equation (SDE) with infinitely often differentiable and bounded coeffi-
cients was constructed such that the Monte Carlo Euler method for approximation of
the expected value of the first component of the solution at the final time converges
but fails to achieve a mean square error of a polynomial rate. In the present paper we
show that this type of bad performance for quadrature of SDEs with infinitely often
differentiable and bounded coefficients is not a shortcoming of the Euler scheme in
particular but can be observed in a worst case sense for every approximation method
that is based on finitely many function values of the coefficients of the SDE. Even
worse we show that for any sequence of Monte Carlo methods based on finitely many
sequential evaluations of the coefficients and all their partial derivatives and for every
arbitrarily slow convergence speed there exists a sequence of SDEs with infinitely of-
ten differentiable and bounded by one coefficients such that the first order derivatives
of all diffusion coefficients are bounded by one as well and the first order derivatives
of all drift coefficients are uniformly dominated by a single real-valued function and
such that the corresponding sequence of mean absolute errors for approximation of
the expected value of the first component of the solution at the final time can not
converge to zero faster than the given speed.

1. Introduction

Let \(d, m \in \mathbb{N}\) and consider a \(d\)-dimensional system of autonomous SDEs
\begin{align*}
  dX^{a,b}(t) &= a(X^{a,b}(t)) \, dt + b(X^{a,b}(t)) \, dW(t), \quad t \in [0, 1], \\
  X^{a,b}(0) &= 0,
\end{align*}
(1)

with an \(m\)-dimensional Brownian motion \(W\) and infinitely often differentiable, bounded
coefficients \(a: \mathbb{R}^d \to \mathbb{R}^d\) and \(b: \mathbb{R}^d \to \mathbb{R}^{d \times m}\). In particular, there exists a unique strong
solution \(X^{a,b} = (X^{a,b}_1, \ldots, X^{a,b}_d)\) of (1), see, e.g., Theorem 3.1.1 in [24], and we have
\(E[|X^{a,b}(1)|^p] < \infty\) for every \(p \geq 1\). Let \(f: \mathbb{R}^d \to \mathbb{R}\) be a measurable function that satisfies
a polynomial growth condition. We study the computational task to approximate
the quantity
\[ S(a, b, f) = E[f(X^{a,b}(1))] \]
by means of an algorithm that uses function values of \( a, b \) and \( f \) and, eventually, their partial derivatives \( D^\alpha a, D^\alpha b, D^\alpha f \) at finitely many points in \( \mathbb{R}^d \).

A classical method of this type is given by the Monte Carlo quadrature rule \( \hat{S}_n^E \) based on \( n \) repetitions of the Euler scheme \( \left( \hat{X}^{a,b}(\ell/n) \right)_{\ell=0,\ldots,n} \) with time step \( 1/n \), i.e.,

\[
\hat{S}_n^E(a, b, f) = \frac{1}{n} \sum_{i=1}^{n} f(Y_{i}^{a,b}),
\]

where \( Y_1^{a,b}, \ldots, Y_n^{a,b} \) are independent and identically distributed as \( \hat{X}^{a,b}(1) \) and the scheme \( \hat{X}^{a,b} \) is recursively defined by \( \hat{X}^{a,b}(0) = 0 \) and

\[
\hat{X}^{a,b}(\frac{\ell}{n}) = \hat{X}^{a,b}(\frac{\ell-1}{n}) + a(\hat{X}^{a,b}(\frac{\ell-1}{n})) \cdot \frac{1}{n} + b(\hat{X}^{a,b}(\frac{\ell-1}{n})) \cdot \left( W(\frac{\ell}{n}) - W(\frac{\ell-1}{n}) \right)
\]

for \( \ell = 1, \ldots, n \). Then it is easy to see that

\[
\mathbb{E}\left[ |S(a, b, f) - \hat{S}_n^E(a, b, f)|^2 \right] \leq c_1 \cdot \|f\|_{\text{Lip}}^2 \cdot \exp\left( c_2 \cdot \max_{a \in \mathbb{N}_0^d, |a|_1 = 1} (\|D^\alpha a\|_{\infty} + \|D^\alpha b\|_{\infty}) \right) \cdot \frac{1}{n},
\]

where \( \|f\|_{\text{Lip}} \) denotes the Lipschitz seminorm of \( f \), \( |a|_1 = \sum_{i=1}^d |\alpha_i| \) and \( c_1 \) and \( c_2 \) are positive reals, which only depend on the dimensions \( d \) and \( m \). Thus, if the first order partial derivatives of the coefficients \( a \) and \( b \) are also bounded and the integrand \( f \) is Lipschitz continuous then the sequence of Monte Carlo Euler approximations \( \hat{S}_n^E(a, b, f) \) achieves a polynomial rate of root mean square error convergence of order \( 1/4 \) in terms of the total number \( 2n(n-1) + n \) of evaluations of the coefficients \( a \) and \( b \) and the integrand \( f \).

On the other hand, Hairer, Hutzenthaler and Jentzen have presented in \([9]\) an equation \( \text{(I)} \) with \( d = 4, m = 1 \) and infinitely often differentiable, bounded coefficients \( a, b \) such that for the integrand \( f(x_1, \ldots, x_d) = x_1 \) and every \( \kappa \in (0, \infty) \),

\[
\lim_{n \to \infty} n^\kappa \cdot \mathbb{E}\left[ |S(a, b, f) - \hat{S}_n^E(a, b, f)|^2 \right] = \infty.
\]

Hence the sequence of Monte Carlo Euler approximations \( \hat{S}_n^E(a, b, f) \) might not achieve a polynomial rate of root mean square error convergence in terms of the number of evaluations of the coefficients \( a \) and \( b \) and the integrand \( f \) if the first order partial derivatives of the coefficients are not bounded as well.

It seems natural to ask, whether the latter result demonstrates only a particular fallacy of the Monte Carlo Euler method and a polynomial rate of convergence could always be achieved for equations \( \text{(I)} \) with infinitely often differentiable and bounded coefficients \( a, b \) if only a more advanced approximation scheme than the Euler scheme would be employed. In fact, there is a variety of strong approximation schemes available in the literature that have been constructed to cope with non-Lipschitz continuous coefficients and have been shown to achieve a polynomial rate of convergence, in terms of the number of time steps, for suitable classes of such equations. See, e.g., \([11, 10, 27]\).
for equations with globally monotone coefficients and see, e.g., [3, 8, 7, 1, 20, 12, 14, 6] for equations with possibly non-monotone coefficients.

However, the following result, Theorem 1, which is a straightforward consequence of Corollary 1 in Section 4.1, shows that for \( d \geq 4 \) the pessimistic alternative is true in a worst case sense with respect to the coefficients \( a \) and \( b \). For every sequence of Monte Carlo methods based on some kind of Itô-Taylor scheme there exists a sequence of equations (1) with infinitely often differentiable and bounded by one coefficients such that the resulting sequence of mean absolute errors for approximating the expected value of the first component of the solution at the final time does not converge to zero with a polynomial rate.

To state this finding in a more formal way let

\[
I = \bigcup_{k=1}^{\infty} \{0, 1\}^k
\]

denote the set of all finite sequences of zeros and ones, put \( W_0(t) = t \) for \( t \in [0, 1] \), and for \( \beta \in I, n \in \mathbb{N} \) and \( \ell \in \{1, \ldots, n\} \) let

\[
J_{n,\ell}^\beta = \int_{\frac{\ell-1}{n}}^{\frac{\ell}{n}} \ldots \int_{\frac{u_1-1}{n}}^{\frac{u_1}{n}} 1 \, dW_{\beta_1}(u_1) \ldots dW_{\beta_\ell}(u_\ell)
\]

denote the corresponding iterated Itô-integral over the time interval \( [\frac{\ell-1}{n}, \frac{\ell}{n}] \).

**Theorem 1.** Let \( d = 4, m = 1 \) and let \( \varphi: (\mathbb{R}^4 \times \mathbb{R}^4)^{\mathbb{N}_0} \times \mathbb{R}^f \to \mathbb{R}^4 \) be a measurable mapping. For all infinitely often differentiable functions \( a: \mathbb{R}^4 \to \mathbb{R}^4 \) and \( b: \mathbb{R}^4 \to \mathbb{R}^4 \) and every \( n \in \mathbb{N} \) define the scheme \( (\hat{X}_{n}^{a,b}(\ell))_{\ell=0,\ldots,n} \) by \( \hat{X}_{n}^{a,b}(0) = 0 \) and

\[
\hat{X}_{n}^{a,b}(\ell) = \hat{X}_{n}^{a,b}(\ell-1) + \varphi((D^a a(\hat{X}_{n}^{a,b}(\ell-1)), D^b b(\hat{X}_{n}^{a,b}(\ell-1))))_{\alpha \in \mathbb{N}_0^f}, (J_{n,\ell}^\beta)_{\beta \in I})
\]

and let \( Y_{n,1}^{a,b}, \ldots, Y_{n,n}^{a,b} \) be independent and identically distributed as \( \hat{X}_{n}^{a,b}(1) \).

Then there exist sequences \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n)_{n \in \mathbb{N}} \) of infinitely often differentiable functions \( a_n: \mathbb{R}^4 \to \mathbb{R}^4 \) and \( b_n: \mathbb{R}^4 \to \mathbb{R}^4 \) with \( \|a_n\|_\infty \leq 1 \) and \( \|b_n\|_\infty \leq 1 \) such that for every \( \kappa \in (0, \infty) \),

\[
\lim_{n \to \infty} n^\kappa \cdot \mathbb{E} \left[ |E[X_{1}^{a_n,b_n}(1)] - \frac{1}{n} \sum_{k=1}^{n} Y_{n,k}^{a_n,b_n}| \right] = \infty.
\]

The latter result neither covers Multilevel Monte Carlo schemes nor the case of a non-uniform discretization of time. Moreover, one might argue that a result like Theorem 1 is not surprising since the order one partial derivatives of the chosen coefficients \( a_n \) and \( b_n \) in the theorem are not required to simultaneously satisfy some kind of growth condition. However, from Corollary 1 in Section 4.1 we even obtain that the coefficients \( a_n \) and \( b_n \) in Theorem 1 can be chosen in such a way that the order one partial derivatives of \( b_n \)
are bounded by one as well and the order one partial derivatives of $a_n$ are dominated by the function $x \mapsto 1 + \exp(|x|^3)$, and that furthermore the statement of the theorem extends to any sequence of Monte Carlo methods based on sequential evaluation of the coefficients $a$ and $b$ and all their partial derivatives $D^a a$, $D^b b$ at finitely many points in $\mathbb{R}^d$. More formally, we have the following theorem as an immediate consequence of Corollary 1.

**Theorem 2.** Let $d = 4$, $m = 1$ and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For every $n \in \mathbb{N}$ let $\psi_{n,1}: \Omega \to \mathbb{R}^d$ as well as

$$\psi_{n,i}: (\mathbb{R}^4 \times \mathbb{R}^4)^{N_0 \times \{1, \ldots, i-1\}} \times \Omega \to \mathbb{R}^d, \quad i = 2, \ldots, n,$$

and

$$\varphi_n: (\mathbb{R}^4 \times \mathbb{R}^4)^{N_0 \times \{1, \ldots, n\}} \times \Omega \to \mathbb{R}$$

be measurable mappings. For all infinitely often differentiable functions $a: \mathbb{R}^d \to \mathbb{R}^d$ and $b: \mathbb{R}^d \to \mathbb{R}^d$ and for every $n \in \mathbb{N}$ define random variables $Z_{n,1}^{a,b}, \ldots, Z_{n,n}^{a,b}: \Omega \to (\mathbb{R}^4 \times \mathbb{R}^4)^{N_0}$ by $Z_{n,i}^{a,b}(\omega) = ((D^a a, D^b b)(\psi_{n,1}(\omega)))_{\alpha \in N_0^i}$ and

$$Z_{n,i}(\omega) = ((D^a a, D^b b)(\psi_{n,i}(Z_{n,i}(\omega), \ldots, Z_{n,i-1}(\omega), \omega)))_{\alpha \in N_0^i}, \quad i = 2, \ldots, n.$$

Then for every $n \in \mathbb{N}$ there exist infinitely often differentiable functions $a_n, b_n: \mathbb{R}^4 \to \mathbb{R}^4$ with $\|a_n\|_{\infty} \leq 1$, $\|b_n\|_{\infty} \leq 1$ and $\|D^a b_n\|_{\infty} \leq 1$, $\|D^a a_n/(1 + \exp(|\cdot|^3))\|_{\infty} \leq 1$ for all $\alpha \in N_0^i$ with $|\alpha|_1 = 1$ such that for every $\kappa \in (0, \infty)$,

$$\lim_{n \to \infty} n^\kappa \cdot \mathbb{E}[|E[X_1^{a_n,b_n}(1)] - \varphi_n(Z_{n,1}^{a_n,b_n}, \ldots, Z_{n,n}^{a_n,b_n}, \cdot)|] = \infty.$$

Perhaps even more surprising we obtain from Corollary 2 in Section 4.1 that for every such sequence of Monte Carlo methods and for every arbitrarily slow convergence speed there exists a strictly increasing and continuous function $u: [0, \infty) \to [0, \infty)$ and a sequence of infinitely often differentiable and bounded by one coefficients $a_n$, $b_n$ such that the order one partial derivatives of $b_n$ are bounded by one as well, the order one partial derivatives of $a_n$ are dominated by the function $1 + u(|\cdot|)$ and the resulting sequence of mean absolute errors for computing the expectation of the first component of the solution at the final time can not converge to zero faster than the given speed of convergence. This finding is formally stated in Theorem 3, which follows from Corollary 2 in Section 4.1.

**Theorem 3.** Let $d = 4$, $m = 1$ and let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. For every $n \in \mathbb{N}$ let $\psi_{n,1}: \Omega \to \mathbb{R}^d$ and let

$$\psi_{n,i}: (\mathbb{R}^4 \times \mathbb{R}^4)^{N_0 \times \{1, \ldots, i-1\}} \times \Omega \to \mathbb{R}^d, \quad i = 2, \ldots, n,$$

as well as

$$\varphi_n: (\mathbb{R}^4 \times \mathbb{R}^4)^{N_0 \times \{1, \ldots, n\}} \times \Omega \to \mathbb{R}$$
be measurable mappings. For all infinitely often differentiable functions $a : \mathbb{R}^4 \to \mathbb{R}^4$ and $b : \mathbb{R}^4 \to \mathbb{R}^4$ and for every $n \in \mathbb{N}$ define random variables $Z_{n,1}^{a,b}, \ldots, Z_{n,n}^{a,b} : \Omega \to (\mathbb{R}^4 \times \mathbb{R}^4)^{N_0}$ by $Z_{n,i}^{a,b} = ((D^\alpha a, D^\alpha b)(\psi_{n,1}(\omega)))_{\alpha \in \mathbb{N}_0^n}$ and

$$Z_{n,i}(\omega) = ((D^\alpha a, D^\alpha b)(\psi_{n,i}(Z_{n,1}(\omega), \ldots, Z_{n,i-1}(\omega), \omega)))_{\alpha \in \mathbb{N}_0^n}, \quad i = 2, \ldots, n.$$ 

Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals with $\lim_{n \to \infty} \varepsilon_n = 0$.

Then there exist $c \in (0, \infty)$ and a strictly increasing, continuous function $u : [0, \infty) \to [0, \infty)$ as well as sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ of infinitely often differentiable functions $a_n, b_n : \mathbb{R}^4 \to \mathbb{R}^4$ with $\|a_n\|_\infty \leq 1$, $\|b_n\|_\infty \leq 1$ and $\|D^\alpha a_n/(1 + u(\cdot))\|_\infty \leq 1$, $\|D^\alpha b_n\|_\infty \leq 1$ for all $\alpha \in \mathbb{N}_0^4$ with $|\alpha|_1 = 1$, such that for every $n \in \mathbb{N}$,

$$\mathbb{E} \left[ \mathbb{E}[X^{a_n,b_n}_1(1)] - \varphi_n(Z_{n,1}^{a_n,b_n}, \ldots, Z_{n,n}^{a_n,b_n}) \right] \geq c \cdot \varepsilon_n.$$

In Theorems [1,3] the integrand $f$ is fixed to be a coordinate projection and lower bounds are provided for the worst case mean absolute error of a Monte Carlo quadrature rule on subclasses of equations [1] with infinitely often differentiable coefficients that are bounded by one. On the other hand, one can fix a specific equation [1] with infinitely often differentiable and bounded coefficients $a$ and $b$ and study the worst case mean absolute error of a Monte Carlo quadrature rule with respect to a class of integrands $f$. In the latter setting a negative result of the type stated in Theorems 2 and 3, which holds for any sequence of Monte Carlo quadrature rules that are based on finitely many evaluations of the integrand $f$, can of course not be true. In fact, consider the direct simulation method $\hat{S}_n^{\text{ds}}$ based on $n$ repetitions of the solution $X^{a,b}_1(1)$ of the fixed equation [1] at the final time, i.e.,

$$\hat{S}_n^{\text{ds}}(a, b, f) = \frac{1}{n} \sum_{i=1}^n f(V_i^{a,b}),$$

where $V_1^{a,b}, \ldots, V_n^{a,b}$ are independent and identically distributed as $X^{a,b}_1(1)$. Clearly, if $f$ is bounded by one then

$$\mathbb{E} \left[ |S(a, b, f) - \hat{S}_n^{\text{ds}}(a, b, f)|^2 \right] \leq \frac{1}{n}.$$ 

However, if only deterministic quadrature rules are considered then we obtain again negative statements in the spirit of Theorems [2,3] even for the seemingly easy problem of computing the expected value $\mathbb{E}[f(W(1))]$ for a one-dimensional Brownian motion $W$ and infinitely often differentiable integrands $f : \mathbb{R} \to \mathbb{R}$ that are bounded by one. For instance, we can show that for any sequence of deterministic quadrature rules that are based on evaluations of the integrand $f$ and all its derivatives at finitely many points in $\mathbb{R}$ and for every arbitrarily slow convergence speed there exists a strictly increasing and continuous function $u : [0, \infty) \to [0, \infty)$ and a sequence of infinitely often differentiable and bounded by one integrands $f_n : \mathbb{R} \to \mathbb{R}$ such that the order one partial
derivatives of $f_n$ are dominated by the function $1 + u(|\cdot|)$ and the resulting sequence of approximation errors for computing the expectation $E[f_n(W(1))]$ can not converge to zero faster than the given speed of convergence. This finding, which is formally stated in the following theorem, is a straightforward consequence of Corollary 5 in Section 4.2.

**Theorem 4.** Assume that $W$ is a one-dimensional Brownian motion. For every $n \in \mathbb{N}$ let $x_{n,1}, \ldots, x_{n,n} \in \mathbb{R}$ and let $\varphi_n: \mathbb{R}^{n_0 \times \{1, \ldots, n\}} \to \mathbb{R}$ be a measurable mapping. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive reals with $\lim_{n \to \infty} \varepsilon_n = 0$. Then there exists $c \in (0, \infty)$, a strictly increasing, continuous function $u: [0, \infty) \to [0, \infty)$ and a sequence of infinitely often differentiable functions $f_n: \mathbb{R} \to \mathbb{R}$ with $\|f_n\|_{\infty} \leq 1$ and $\|f_n^{(1)}/(1 + u(|\cdot|))\|_{\infty} \leq 1$ such that for every $n \in \mathbb{N}$,

$$|E[f_n(W(1))] - \varphi_n((f_n^{(k)}(x_{n,1}), \ldots, f_n^{(k)}(x_{n,n}))_{k \in \mathbb{N}_0})| \geq c \cdot \varepsilon_n.$$

The findings stated in Theorems 1 – 3 are worst case results for randomized quadrature rules with respect to a given class of equations (1). It remains an open question whether these results can be strengthened in the sense that for every sequence of Monte Carlo methods for quadrature of the first component of the solution, which are based on finitely many sequential evaluations of the coefficients and all their partial derivatives, there exists a single equation with infinitely often differentiable and bounded coefficients, which leads to the prescribed slow convergence rate of the corresponding sequence of mean absolute errors. Up to now, a positive answer to this question is only known for the sequence of Euler Monte Carlo schemes, see [9] and [2]. Similarly, it is unclear, whether Theorem 4 can be strengthened in the sense that for every sequence of deterministic quadrature rules for quadrature with respect to the one-dimensional standard normal distribution, which are based on finitely many sequential evaluations of the integrand and all its derivatives, there exists a single infinitely often differentiable and bounded integrand leading to the prescribed slow convergence rate of the corresponding sequence of absolute errors. We conjecture that both questions can be answered to the positive and we will address these issues in future research.

We add that there is a number of results on worst case lower error bounds for quadrature of marginals of SDEs in the case of coefficients $a, b$ that satisfy a uniform global Lipschitz condition and integrands $f$ with first order partial derivatives that satisfy a uniform polynomial growth condition, see [23, 17, 22, 19].

We further add that recently in [15] equations (1) with infinitely often differentiable and bounded coefficients $a, b$ have been constructed that can not be approximated at the final time in the pathwise sense with a polynomial rate by any approximation method based on finitely many evaluations of the driving Brownian motion. In the present paper we use a construction, which is conceptually similar to the one from [15] but specifically tailored to the analysis of the quadrature problem.
We briefly describe the content of the paper. In Section 2 we fix some notation with respect to the regularity of coefficients and integrands. In Section 3 we set up the framework for studying worst case errors of randomized and deterministic algorithms for the approximation of nonlinear functionals on function spaces. In particular, we establish lower error bounds for the corresponding minimal randomized and deterministic errors that generalize classical results of Bakhvalov [2] and Novak [21] for linear integration problems. In Section 4 we use the framework from Section 3 to study quadrature problems for SDEs. Section 4.1 is devoted to lower bounds for worst case errors with respect to the coefficients, while Section 4.2 contains our results on worst case errors with respect to the integrands. The proofs of the main results, Theorems 5 and 6, are carried out in Section 5.

2. Notation

Let \( k, \ell_1, \ell_2 \in \mathbb{N} \). For a vector \( x \in \mathbb{R}^k \) and a matrix \( M \in \mathbb{R}^{\ell_1 \times \ell_2} \) we use \( |x| \) and \( |M| \) to denote the maximum norm of \( x \) and \( M \), respectively. For a function \( h : \mathbb{R}^k \to \mathbb{R}^{\ell_1 \times \ell_2} \) we put \( \|h\|_\infty = \sup_{x \in \mathbb{R}^k} |h(x)| \). By \( C^\infty(\mathbb{R}^k; \mathbb{R}^{\ell_1 \times \ell_2}) \) we denote the set of all functions \( h : \mathbb{R}^k \to \mathbb{R}^{\ell_1 \times \ell_2} \) that are infinitely often differentiable and for \( h \in C^\infty(\mathbb{R}^k; \mathbb{R}^{\ell_1 \times \ell_2}) \) and a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}_0^k \) we use

\[
D^\alpha h = \frac{\partial^{\alpha_1 + \ldots + \alpha_k} h}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}} : \mathbb{R}^k \to \mathbb{R}^{\ell_1 \times \ell_2}
\]

to denote the corresponding partial derivative of \( h \). For every \( \nu \in \mathbb{N}_0 \) we use

\[
C^{\infty, \nu}(\mathbb{R}^k; \mathbb{R}^{\ell_1 \times \ell_2}) = \left\{ h \in C^\infty(\mathbb{R}^k; \mathbb{R}^{\ell_1 \times \ell_2}) : \max_{\alpha \in \mathbb{N}_0^k, \alpha_1 + \ldots + \alpha_k \leq \nu} \|D^\alpha h\|_\infty \leq 1 \right\}
\]

to denote all functions \( h \in C^\infty(\mathbb{R}^k; \mathbb{R}^{\ell_1 \times \ell_2}) \) that are bounded by one and have partial derivatives up to order \( \nu \) that are bounded by one as well.

3. Approximation of nonlinear functionals on function spaces and lower worst case error bounds

Let \( A \) and \( B \) be nonempty sets, let \( \mathcal{G} \subset B^A \) be a nonempty set of functions \( g : A \to B \) and let

\[
S : \mathcal{G} \to \mathbb{R}.
\]

We study the approximation of \( S(g) \) for \( g \in \mathcal{G} \) by means of a deterministic or randomized algorithm that is based on finitely many evaluations of the mapping \( g \) at points in \( A \). Our goal is to provide lower bounds for the worst case mean error of any such algorithm in terms of its worst case average number of function evaluations.

A generalized randomized algorithm for this problem is specified by a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) and a triple \((\psi, \nu, \varphi)\),
where

- \( \psi = (\psi_k)_{k \geq 1} \) is a sequence of mappings
  \[ \psi_k : B^{k-1} \times \Omega \to A, \]
  which are used to sequentially determine random evaluation nodes in \( A \) for a given input \( g \in \mathcal{G} \),
- the mapping
  \[ \nu : \mathcal{G} \times \Omega \to \mathbb{N} \]
determines the random total number of evaluations of a given input \( g \in \mathcal{G} \), and
- \( \varphi = (\varphi_k)_{k \geq 1} \) is a sequence of mappings
  \[ \varphi_k : B^k \times \Omega \to \mathbb{R}, \]
  which are used to obtain for every input \( g \in \mathcal{G} \) a random approximation to \( S(g) \) based on the observed function values of \( g \).

To be more precise, we define for every \( k \in \mathbb{N} \) a mapping

\[ N^\psi_k : \mathcal{G} \times \Omega \to B^k \]

by

\[ N^\psi_k = (y_1, \ldots, y_k), \]

where

\[ y_1(g, \omega) = g(\psi_1(\omega)) \]

and

\[ y_\ell(g, \omega) = g(\psi_\ell(y_1(g, \omega), \ldots, y_{\ell-1}(g, \omega), \omega)), \quad \ell = 2, \ldots, k. \]

For a given \( \omega \in \Omega \) and a given input \( g \in \mathcal{G} \) the algorithm specified by \( (\psi, \nu, \varphi) \) sequentially performs \( \nu(g, \omega) \) evaluations of \( g \) at the points

\[ \psi_1(\omega), \psi_2(y_1(g, \omega), \omega), \ldots, \psi_{\nu(g, \omega)}(y_1(g, \omega), \ldots, y_{\nu(g, \omega)-1}(g, \omega), \omega) \in A \]

and finally applies the mapping \( \varphi_{\nu(g, \omega)}(\cdot, \omega) : B^{\nu(g, \omega)} \to \mathbb{R} \) to the data \( N^\psi_{\nu(g, \omega)}(g, \omega) \) to obtain the real number

\[ \hat{S}^\psi_{\nu, \varphi}(g, \omega) = \varphi_{\nu(g, \omega)}(N^\psi_{\nu(g, \omega)}(g, \omega), \omega) \]

as an approximation to \( S(g) \). The induced mapping

\[ \hat{S}^\psi_{\nu, \varphi} : \mathcal{G} \times \Omega \to \mathbb{R} \]
is called a generalized randomized algorithm if for every \( g \in \mathcal{G} \) the mappings

\[ \hat{S}^\psi_{\nu, \varphi}(g, \cdot) : \Omega \to \mathbb{R} \]
and \( \nu(g, \cdot) : \Omega \to \mathbb{N} \)

are random variables.
We use \( \mathcal{A}^{\text{ran}} \) to denote the class of all randomized algorithms. The error and the cost of \( \hat{S} \in \mathcal{A}^{\text{ran}} \) are defined in the worst case sense by

\[
e(\hat{S}) = \sup_{g \in G} \mathbb{E}|S(g) - \hat{S}(g, \cdot)|
\]

and

\[
\text{cost}(\hat{S}) = \inf_{\psi, \nu, \varphi} \left\{ \sup_{g \in G} \mathbb{E} \nu(g, \cdot): \hat{S} = \hat{S}_{\psi, \nu, \varphi} \right\},
\]

respectively. Thus the definition of the cost of \( \hat{S} \) takes into account that the representation \( \hat{S} = \hat{S}_{\psi, \nu, \varphi} \) is not unique in general.

A generalized randomized algorithm \( \hat{S} \in \mathcal{A}^{\text{ran}} \) is called deterministic if the random variable \( \hat{S}(g, \cdot) \) is constant for all \( g \in G \). In this case we have \( \hat{S} = \hat{S}_{\psi, \nu, \varphi} \) with mappings

\[
\psi_k: B^{k-1} \to A, \ \nu: G \to \mathbb{N}, \ \varphi_k: B^k \to \mathbb{R},
\]

and it is easy to see that

\[
\text{cost}(\hat{S}) = \inf_{\psi, \nu, \varphi} \left\{ \sup_{g \in G} \nu(g): \hat{S} = \hat{S}_{\psi, \nu, \varphi} \right\}.
\]

The class of all generalized deterministic algorithms is denoted by \( \mathcal{A}^{\text{det}} \).

Let \( n \in \mathbb{N} \). The crucial quantities for our analysis are the \( n \)-th minimal errors

\[
e^{\text{det}}_n(G; S) = \inf \{ e(\hat{S}): \hat{S} \in \mathcal{A}^{\text{det}}, \text{cost}(\hat{S}) \leq n \}
\]

and

\[
e^{\text{ran}}_n(G; S) = \inf \{ e(\hat{S}): \hat{S} \in \mathcal{A}^{\text{ran}}, \text{cost}(\hat{S}) \leq n \},
\]

i.e., the smallest possible worst case error that can be achieved by generalized deterministic algorithms based on at most \( n \) function values of \( g \in G \) and the smallest possible worst case mean error that can be achieved by generalized randomized algorithms that use at most \( n \) function values of \( g \in G \) on average, respectively. Clearly, \( e^{\text{det}}_n(G; S) \geq e^{\text{ran}}_n(G; S) \).

We present two types of lower bounds for the minimal errors \( e^{\text{det}}_n(G; S) \) and \( e^{\text{ran}}_n(G; S) \), which generalize classical results of Bakhvalov and Novak for the case of \( S \) being a linear functional on a space \( G \) of real-valued functions \( g: A \to \mathbb{R} \), see \cite{2, 21}.

**Proposition 1.** Let \( \varepsilon > 0 \), \( m \in \mathbb{N} \), \( b^* \in B \) and assume that there exist \( 2m \) functions

\[
g_{1,+}, g_{1,-}, \ldots, g_{m,+}, g_{m,-}: A \to B
\]

with the following properties.

(i) The sets

\[
\{g_{1,+} \neq b^*\} \cup \{g_{1,-} \neq b^*\}, \ldots, \{g_{m,+} \neq b^*\} \cup \{g_{m,-} \neq b^*\}
\]

are pairwise disjoint,
We have \( g_1^+, g_1^-, \ldots, g_m^+, g_m^- \in G \),

(iii) We have \( S(g_i^+) - S(g_i^-) \geq \varepsilon \) for \( i = 1, \ldots, m \).

Then, for every \( n \in \mathbb{N} \),

\[
e_n^{\text{ran}}(G; S) \geq \frac{m - 16n}{8m} \varepsilon.
\]

**Proposition 2.** Let \( B \) be a linear space. Let \( \varepsilon > 0 \), \( m \in \mathbb{N} \), \( b^* \in B \) and assume that there exist \( 2m \) functions \( g_1^+, g_1^-, \ldots, g_m^+, g_m^- : A \to B \)

with the following properties.

(i) The sets \( \{g_1^+ \neq b^*\} \cup \{g_1^- \neq b^*\}, \ldots, \{g_m^+ \neq b^*\} \cup \{g_m^- \neq b^*\} \)

are pairwise disjoint,

(ii) We have \( g_1^+, g_1^-, \ldots, g_m^+, g_m^- \in G \) and for all \( \delta_1, \ldots, \delta_m \in \{+,-\} \) we have

\[
\sum_{i=1}^{m} g_{i,\delta_i} \in G \quad \text{and} \quad S\left(\sum_{i=1}^{m} g_{i,\delta_i}\right) = \sum_{i=1}^{m} S(g_{i,\delta_i}),
\]

(iii) We have \( S(g_{i,+}) - S(g_{i,-}) \geq \varepsilon \) for \( i = 1, \ldots, m \).

Then, for every \( n \in \mathbb{N} \),

\[
e_n^{\text{det}}(G; S) \geq \frac{m - n}{2} \varepsilon
\]

and for every \( n \leq m/4 \),

\[
e_n^{\text{ran}}(G; S) \geq \sqrt{\frac{m - 4n}{128}} \varepsilon.
\]

For the proof of the lower bounds for the \( n \)-th minimal randomized errors in Propositions 1 and 2 we employ a classical averaging principle of Bakhvalov, see [2]. Consider a probability measure \( \mu \) on the power set \( P(G) \) of \( G \) with finite support. For a deterministic algorithm \( \widehat{S} \in A^{\text{det}} \) we define the average error and the average cost of \( \widehat{S} \) with respect to \( \mu \) by

\[
e(\widehat{S}, \mu) = \int_{G} |S(g) - \widehat{S}(g)| \mu(dg)
\]

and

\[
\text{cost}(\widehat{S}, \mu) = \inf \left\{ \int_{G} \nu(g) \mu(dg) : \widehat{S} = \widehat{S}_{\psi,\nu,\varphi}, (\psi, \nu, \varphi) \text{ satisfies } [3] \right\}.
\]
The smallest possible average case error with respect to \( \mu \) that can be achieved by any generalized deterministic algorithm based on at most \( n \) function evaluations on average with respect to \( \mu \) is then given by
\[
e_{n}^{\text{det}}(\mu) = \inf \{ e(\bar{S}, \mu) : \bar{S} \in \mathcal{A}^{\text{det}}, \text{cost}(\bar{S}, \mu) \leq n \}.
\]

**Lemma 1.** For every probability measure \( \mu \) on \( \mathcal{P}(G) \) and every \( n \in \mathbb{N} \) we have
\[
e_{n}^{\text{ran}}(G; S) \geq \frac{1}{2} e_{2n}^{\text{det}}(\mu).
\]

For convenience of the reader we provide a proof of Lemma 1.

**Proof of Lemma 1.** Let \( \hat{S} \in \mathcal{A}^{\text{ran}} \) with \( \text{cost}(\hat{S}) \leq n \). Let \( \rho > 0 \) and choose \((\psi, \nu, \varphi)\) such that \( \hat{S} = \hat{S}_{\psi, \nu, \varphi} \) and \( \sup_{g \in G} \mathbb{E}[\nu(g, \cdot)] \leq n + \rho \). Put
\[
\Omega_1 = \{ \omega \in \Omega : \int_{G} \nu(g, \omega) \, d\mu(g) \leq 2n \}.
\]
Then \( 2n \mathbb{P}(\Omega_1^c) \leq \sup_{g \in G} \mathbb{E}[\nu(g, \cdot)] \leq n + \rho \), and therefore, \( \mathbb{P}(\Omega_1) \geq 1/2 - \rho/(2n) \). For every \( \omega \in \Omega_1 \) we have \( \hat{S}(\cdot, \omega) \in \mathcal{A}^{\text{det}} \) and \( \text{cost}(\hat{S}(\cdot, \omega), \mu) \leq 2n \), which implies
\[
\int_{G} |S(g) - \hat{S}(g, \omega)| \mu(dg) \geq e_{2n}^{\text{det}}(\mu).
\]
Hence
\[
e(\hat{S}) \geq \int_{G} \mathbb{E}[|S(g) - \hat{S}(g, \cdot)|] \mu(dg)
\]
\[
\geq \int_{\Omega_1} \int_{G} |S(g) - \hat{S}(g, \omega)| \mu(dg) \mathbb{P}(d\omega) \geq (1/2 - \rho/(2n)) \cdot e_{2n}^{\text{det}}(\mu)
\]
Letting \( \rho \) tend to zero completes the proof.

**Proof of Proposition 1.** Let \( \mu \) denote the uniform distribution on
\[
\tilde{G} = \{ g_{1,+}, g_{1,-}, \ldots, g_{m,+}, g_{m,-} \}.
\]
We show that
\[
e_{n}^{\text{det}}(\mu) \geq \frac{m - 8n}{4m} \varepsilon,
\]
which jointly with Lemma 1 yields the lower bound in Proposition 1.

In order to prove 1, let \( \hat{S} \in \mathcal{A}^{\text{det}} \) with \( \text{cost}(\hat{S}, \mu) \leq n \). Let \( \rho > 0 \) and choose \((\psi, \nu, \varphi)\) satisfying 3 such that \( \hat{S} = \hat{S}_{\psi, \nu, \varphi} \) and \( \int_{G} \nu(g) \, d\mu(g) \leq n + \rho \). Put
\[
\tilde{G}_1 = \{ g \in \tilde{G} : \nu(g) \leq 4n \}
\]
and let
\[
I = \{ i \in \{ 1, \ldots, m \} : g_i,+, g_i,- \in \tilde{G}_1 \}.
Then $4n \mu(\tilde{G}_1) \leq n + \rho$, and therefore $\mu(\tilde{G}_1) \geq 3/4 - \rho/(4n)$. Since $|\tilde{G}_1| = \mu(\tilde{G}_1) \cdot 2m$, we conclude that

$$|I| \geq (1/2 - \rho/(2n)) \cdot m.$$ 

Let

$$K = \{\psi_1, \psi_2(b^*), \ldots, \psi_{4n}(b^*, \ldots, b^*)\}$$

denote the set of the first $4n$ nodes in $A$ that are produced by the sequence $(\psi_k)_{k \in \mathbb{N}}$ for evaluating the constant function $x \mapsto b^*$ on $A$, and put

$$J = \{i \in I: K \cap \{\{g_i, + \neq b^*\} \cup \{g_i, - \neq b^*\}\} = \emptyset\}.$$ 

Clearly, $\hat{S}_{\psi, \nu, \varphi}(g_{i_1, \delta_1}) = \hat{S}_{\psi, \nu, \varphi}(g_{i_2, \delta_2})$ for all $i_1, i_2 \in J$ and all $\delta_1, \delta_2 \in \{-, +\}$, and, observing property (i), we conclude $|J| \geq (1/2 - \rho/(2n))m - 4n$. Thus, by property (iii),

$$\int_g |S(g) - \hat{S}(g)| \mu(dg) \geq \frac{1}{2m} \sum_{i \in J} (|S(g_{i, +}) - \hat{S}(g_{i, +})| + |S(g_{i, -}) - \hat{S}(g_{i, -})|)$$

$$\geq \frac{1}{2m} \sum_{i \in J} |S(g_{i, +}) - S(g_{i, -})|$$

$$\geq \frac{|J|}{2m} \varepsilon \geq (1/4 - \rho/(4n) - 2n/m) \varepsilon.$$ 

Letting $\rho$ tend to zero yields

$$e(\hat{S}, \mu) \geq (1/4 - 2n/m) \varepsilon,$$

which completes the proof. \qed

**Proof of Proposition 2.** We first prove the lower bound for the $n$-th minimal error of deterministic methods. Let $\hat{S} \in A_{\text{det}}$ with cost($\hat{S}$) $\leq n$. Choose $(\psi, \nu, \varphi)$ satisfying (3) such that $\hat{S} = \hat{S}_{\psi, \nu, \varphi}$ and sup$_{g \in \mathcal{G}} \nu(g) \leq n$. Consider the function

$$g = \sum_{i=1}^m g_{i, +} \in \mathcal{G}$$

and let $K$ denote the set of at most $n$ nodes in $G$ that are used by $\hat{S}_{\psi, \nu, \varphi}$ for evaluating $g$. Put

$$J = \{i \in \{1, \ldots, m\}: K \cap \{\{g_i, + \neq b^*\} \cup \{g_i, - \neq b^*\}\} = \emptyset\}$$

and let

$$h = \sum_{i \notin J} g_{i, +} + \sum_{i \in J} g_{i, -}.$$
Clearly, $\hat{S}_{\psi,\nu,\varphi}(g) = \hat{S}_{\psi,\nu,\varphi}(h)$, and, observing property (i), $|J| \geq m - n$. Thus, by properties (ii) and (iii),

\[
e(\hat{S}) \geq \frac{1}{2} |S(g) - \hat{S}(g)| + |S(h) - \hat{S}(h)| \geq \frac{1}{2}|S(g) - S(h)| \\
\geq \frac{1}{2} \sum_{i \in J} (S(g_{i,+}) - S(g_{i,-})) \geq \frac{|J|}{2} \varepsilon \geq (m - n) \cdot \varepsilon / 2,
\]

which completes the proof of the lower bound for the minimal deterministic error.

We turn to the proof of the lower bound for the minimal randomized error. Let $\mu$ denote the uniform distribution on $\tilde{G} = \{ \sum_{i=1}^{m} g_{i,\delta_{i}} : \delta_{1}, \ldots, \delta_{m} \in \{+, -\} \}$. We show that

\[ (5) \quad e_{n}^{\text{det}}(\mu) \geq \sqrt{\frac{m - 2n}{32}} \varepsilon, \]

which jointly with Lemma 1 yields the desired lower bound in Proposition 2.

In order to prove (5), let $\hat{S} \in A_{\text{det}}$ with cost($\hat{S}, \mu$) $\leq n$. Let $\rho > 0$ and choose $(\psi, \nu, \varphi)$ satisfying (3) such that $\hat{S} = \hat{S}_{\psi,\nu,\varphi}$ and $\int_{G} \nu(f) \mu(df) \leq n + \rho$. Put $\tilde{G}_{1} = \{ g \in \tilde{G} : \nu(g) \leq 2n \}$.

Then $2n \mu(\tilde{G}_{1}) \leq n + \rho$ and therefore $\mu(\tilde{G}_{1}) \geq 1/2 - \rho/(2n)$. Hence

\[ (6) \quad |\tilde{G}_{1}| = \mu(\tilde{G}_{1}) \cdot 2^{m} \geq (1 - \rho/n) \cdot 2^{m-1}. \]

Consider the function $N_{2n}^{\psi} : G \to B^{2n}$ and put

$\mathcal{Y} = \{ N_{2n}^{\psi}(g) : g \in \tilde{G}_{1} \}$

as well as

$\tilde{G}_{1,y} = \{ g \in \tilde{G}_{1} : N_{2n}^{\psi}(g) = y \}$

for all $y \in \mathcal{Y}$. Note that

$\tilde{G}_{1} = \bigcup_{y \in \mathcal{Y}} \tilde{G}_{1,y}.$

Fix $y \in \mathcal{Y}$, let $K_{y}$ denote the set of the nodes in $A$ that are used by $\hat{S}_{\psi,\nu,\varphi}$ for evaluating each of the functions $g \in \tilde{G}_{1,y}$ and put

$J_{y} = \{ i \in \{1, \ldots, m\} : K_{y} \cap (\{ g_{i,+} \neq b^{*} \} \cup \{ g_{i,-} \neq b^{*} \}) = \emptyset \}.$

For $g = \sum_{i=1}^{m} g_{i,\delta_{i}} \in \tilde{G}_{1,y}$ let $g_{1} = \sum_{i \in J_{y}} g_{i,\delta_{i}}$ and put

$\tilde{G}_{1,y}(g_{1}) = \{ g_{1} + \sum_{i \in J_{y}} g_{i,\delta_{i}} : (\delta_{i})_{i \in J_{y}} \in \{-, +\}^{|J_{y}|} \}.$
Let $\tilde{G}^y = \{ g_1 : g \in \tilde{G}_{1,y} \}$. Clearly, $\tilde{G}_{1,y}(g_1) \cap \tilde{G}_{1,y}(h_1) = \emptyset$ for all $g_1, h_1 \in \tilde{G}^y$ with $g_1 \neq h_1$ and

$$\tilde{G}_{1,y} = \bigcup_{g_1 \in \tilde{G}^y} \tilde{G}_{1,y}(g_1).$$

We show that for every $h \in \tilde{G}_{1,y}$,

$$\sum_{g \in \tilde{G}_{1,y}(h_1)} |S(g) - \hat{S}(g)| \geq 2^{\frac{1}{2} |J_y|} (m - 2n)^{1/2} \varepsilon. \quad (7)$$

Using (6) and (7) we may then conclude that

$$\int_G |S(g) - \hat{S}(g)| \mu(dg) \geq 2^{-m} \sum_{g \in \tilde{G}_{1,y}} |S(g) - \hat{S}(g)| = 2^{-m} \sum_{y \in \mathcal{Y}} \sum_{h_1 \in \tilde{G}^y} \sum_{g \in \tilde{G}_{1,y}(h_1)} |S(g) - \hat{S}(g)|$$

$$\geq 2^{-m} \sum_{y \in \mathcal{Y}} \sum_{h_1 \in \tilde{G}^y} 2^{\frac{1}{2} |J_y|} (m - 2n)^{1/2} \varepsilon$$

$$= 2^{-m-3/2} |\tilde{G}_{1,y}| (m - 2n)^{1/2} \varepsilon$$

$$\geq 2^{-5/2} (1 - \rho/n) (m - 2n)^{1/2} \varepsilon.$$

Letting $\rho$ tend to zero yields

$$e(\hat{S}, \mu) \geq 2^{-5/2} (m - 2n)^{1/2} \varepsilon,$$

which in turn implies (5).

It remains to prove the estimate (7). For $\delta \in \{+, -\}$ we define

$$\bar{\delta} = \begin{cases} +, & \text{if } \delta = -, \\ -, & \text{if } \delta = +, \end{cases}$$

and for $g = h_1 + \sum_{i \in J_y} g_{i, \delta_i} \in \tilde{G}_{1,y}(h_1)$ we put

$$\bar{g} = h_1 + \sum_{i \in J_y} g_{i, \delta_i}.$$
Clearly, \( \hat{S}_{\psi,\nu,\varphi}(g) = \hat{S}_{\psi,\nu,\varphi}(\bar{g}) \) for all \( g \in \bar{G}_{1,y}(h_1) \). Thus, by property (ii), the Khintchine inequality, see [28], and property (iii),

\[
\sum_{g \in \bar{G}_{1,y}(h_1)} |S(g) - \hat{S}(g)| = \frac{1}{2} \sum_{g \in \bar{G}_{1,y}(h_1)} (|S(g) - \hat{S}(g)| + |S(\bar{g}) - \hat{S}(\bar{g})|) \\
\geq \frac{1}{2} \sum_{g \in \bar{G}_{1,y}(h_1)} |S(g) - S(\bar{g})| \\
= \frac{1}{2} \sum_{\delta \in \{-1,1\}^{|J_y|}} \left| \sum_{i \in J_y} (S(g_i,\delta_i) - S(g_i,\delta_i)) \right| \\
= \frac{1}{2} \sum_{\delta \in \{-1,1\}^{|J_y|}} \left| \sum_{i \in J_y} \delta_i (S(g_{i,+}) - S(g_{i,-})) \right| \\
\geq \frac{2^{2|J_y| - 1/2}}{2} \left( \sum_{i \in J_y} (S(g_{i,+}) - S(g_{i,-}))^2 \right)^{1/2} \\
\geq 2^{2|J_y| - 3/2} |J_y|^{1/2} \varepsilon.
\]

Using \( |J_y| \geq m - 2n \) completes the proof of (7) and hereby finishes the proof of Proposition [2]. \( \square \)

4. LOWER WORST CASE ERROR BOUNDS FOR QUADRATURE OF SDEs

We consider a class of equations (11) specified by a class

\[
\mathcal{E} \subset C^{\infty,0}(\mathbb{R}^d,\mathbb{R}^d) \times C^{\infty,1}(\mathbb{R}^d,\mathbb{R}^{d \times m})
\]

of coefficients \((a,b)\) and a class

\[
\mathcal{F} \subset C^{\infty}(\mathbb{R}^d;\mathbb{R})
\]

of integrands \( f \) satisfying a polynomial growth condition, and we study the problem of approximating

\[
\mathbb{E}[f(X^{a,b}(1))]
\]

for all \((a,b) \in \mathcal{E}\) and all \( f \in \mathcal{F}\) by means of a randomized or a deterministic algorithm that may use function values of \( a, b, f \) and all partial derivatives \( D^a a, D^b b, D^a f \) at finitely many points in \( \mathbb{R}^d \). Our goal is to establish a lower bound for the smallest possible worst case mean error over the classes \( \mathcal{E} \) and \( \mathcal{F} \) that can be achieved by any such algorithm if, on average, at most \( n \) evaluation nodes in \( \mathbb{R}^d \) may be used.

We formalize this problem in terms of the framework specified in Section [3] as follows. Put

\[
A = \mathbb{R}^d, \quad B = (\mathbb{R}^d \times \mathbb{R}^{d \times m} \times \mathbb{R})^{[a_{id},b_{id}]},
\]
let \( G(\mathcal{E}, \mathcal{F}) \subset B^4 \) be given by
\[
G(\mathcal{E}, \mathcal{F}) = \{(D^\alpha a, D^\alpha b, D^\alpha f)_{\alpha \in \mathbb{N}^d_0}: (a, b) \in \mathcal{E}, f \in \mathcal{F}\}
\]
and define \( S^{\text{nde}}: G(\mathcal{E}, \mathcal{F}) \to \mathbb{R} \) by
\[
S^{\text{nde}}((D^\alpha a, D^\alpha b, D^\alpha f)_{\alpha \in \mathbb{N}^d_0}) = E[f(X^{a,b}(1))].
\]
The corresponding classes of deterministic and randomized algorithms for approximating \( E[f(X^{a,b}(1))] \) based on finitely many sequential evaluations of \( (D^\alpha a, D^\alpha b, D^\alpha f)_{\alpha \in \mathbb{N}^d_0} \) are given by \( A^{\text{det}} \) and \( A^{\text{ran}} \), respectively, and the resulting minimal errors are denoted by \( e^{\text{det}}_n(G(\mathcal{E} \times \mathcal{F}); S^{\text{nde}}) \) for deterministic methods and \( e^{\text{ran}}_n(G(\mathcal{E} \times \mathcal{F}); S^{\text{nde}}) \) for randomized methods. Note that \( A^{\text{ran}} \) contains in particular any Monte Carlo method or multilevel Monte Carlo method that is based on a strong or weak Itô-Taylor scheme of arbitrary order.

4.1. Worst case analysis with respect to coefficients. In this section we take \( d = 4, m = 1 \) and we consider the following two types of classes of equations \( \mathcal{E} \). For a function \( u: [0, \infty) \to [0, \infty) \) we put
\[
\mathcal{E}_u = \left\{(a, b) \in C^{\infty,0}(\mathbb{R}^4; \mathbb{R}^4) \times C^{\infty,1}(\mathbb{R}^4; \mathbb{R}^4): \max_{\alpha \in \mathbb{N}^4_0, |\alpha| = 1} \sup_{x \in \mathbb{R}^4} \frac{|D^\alpha a(x)|}{1 + u(|x|)} \leq 1 \right\}.
\]
Thus, \( u \) is used to impose a growth condition on the first order partial derivatives of a drift coefficient \( a \). Furthermore, we consider the class of equations
\[
\mathcal{E}_{\text{lin}} = \left\{(a, b) \in C^{\infty,0}(\mathbb{R}^4; \mathbb{R}^4) \times C^{\infty,1}(\mathbb{R}^4; \mathbb{R}^4): \max_{\alpha \in \mathbb{N}^4_0, |\alpha| = 1} \sup_{x \in \mathbb{R}^4} \frac{|D^\alpha a(x)|}{1 + |x|} < \infty \right\},
\]
where the drift coefficient \( a \) is required to satisfy a linear growth condition. Clearly,
\[
\mathcal{E}_{\text{lin}} = \bigcup_{k=1}^{\infty} \mathcal{E}_{u_k},
\]
where \( u_k(x) = k \cdot (1 + x) \) for \( x \in [0, \infty) \).

The class of integrands is given by
\[
\mathcal{F} = \{\pi_1\},
\]
where
\[
\pi_1: \mathbb{R}^4 \to \mathbb{R}, \quad (x_1, \ldots, x_4) \mapsto x_1,
\]
is the projection on the first coordinate. We thus study the computation of \( E[X^{a,b}_1(1)] \) for all \((a, b) \in \mathcal{E}_u\) or all \((a, b) \in \mathcal{E}_{\text{lin}}\).

The following result provides lower bounds for the minimal errors \( e^{\text{ran}}_n(G(\mathcal{E}_u \times \{\pi_1\}); S^{\text{nde}}) \) in case that the function \( u \) is continuous, strictly increasing and satisfies the condition
\[
\lim_{x \to \infty} u(x)/x > 0.
\]
See Section 5.1 for a proof.
Theorem 5. There exist \( c_1, c_2 \in (0, \infty) \) and \( c_3 \in [1, \infty) \) such that for all continuous and strictly increasing functions \( u: [0, \infty) \to [0, \infty) \), for all \( \delta \in (0, \infty) \), for all \( x_\delta \in (0, \infty) \) with \( \inf_{x \geq x_\delta} u(x)/x \geq \delta \) and for all \( n \in \mathbb{N} \),

\[
e_n^\text{ran} \left( G(E_u \times \{ \pi_1 \}); S^{sde} \right) \geq c_1 \cdot \exp \left(-c_2 \cdot \left( u^{-1}(c_3 \cdot \delta^{-1} \cdot \max(1, u^2(x_\delta + 1)) \cdot n^4) \right)^2 \right).
\]

Due to Lemma 5 in [15] we have

\[
\forall q, c \in (0, \infty): \lim_{n \to \infty} n^q \cdot \exp \left(-c \cdot (u^{-1}(c \cdot n^4))^2 \right) = \infty
\]

if and only if

\[
(8) \quad \forall q \in (0, \infty): \lim_{x \to \infty} u(x) \cdot \exp(-q \cdot x^2) = \infty.
\]

Clearly, (8) implies \( \lim_{x \to \infty} u(x)/x > 0 \). As an immediate consequence of Theorem 5 we thus get a non-polynomial decay of the minimal errors \( e_n^\text{ran}(G(E_u \times \{ \pi_1 \}); S^{sde}) \) if \( u \) satisfies the exponential growth condition (8).

Corollary 1. Assume that \( u: [0, \infty) \to [0, \infty) \) is continuous, strictly increasing and satisfies (8). Then for all \( q > 0 \),

\[
\lim_{n \to \infty} n^q \cdot e_n^\text{ran}(G(E_u \times \{ \pi_1 \}); S^{sde}) = \infty.
\]

The following result shows that the minimal errors \( e_n^\text{ran}(G(E_u \times \{ \pi_1 \}); S^{sde}) \) may decay arbitrary slow.

Corollary 2. For every sequence \( (\epsilon_n)_{n \in \mathbb{N}} \subset (0, \infty) \) with \( \lim_{n \to \infty} \epsilon_n = 0 \) there exists \( c \in (0, \infty) \) and a strictly increasing and continuous function \( u: [0, \infty) \to [0, \infty) \) such that for all \( n \in \mathbb{N} \) we have

\[
e_n^\text{ran}(G(E_u \times \{ \pi_1 \}); S^{sde}) \geq c \cdot \epsilon_n.
\]

Proof. Without loss of generality we may assume that the sequence \( (\epsilon_n)_{n \in \mathbb{N}} \) is strictly decreasing.

Choose \( c_1, c_2 \in (0, \infty) \) and \( c_3 \in [1, \infty) \) according to Theorem 5. Choose \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) we have

\[
(9) \quad \epsilon_n \leq \exp(-4c_2)
\]

and put

\[
b_n = \left( \frac{1}{c_2} \cdot \ln \frac{1}{\epsilon_n} \right)^{1/2}
\]

for \( n \geq n_0 \). Note that \( (b_n)_{n \geq n_0} \) is strictly increasing and satisfies \( \lim_{n \to \infty} b_n = \infty \).

Next, define \( u: [0, \infty) \to [0, \infty) \) recursively by

\[
u(x) = \begin{cases} x, & \text{if } x \leq b_{n_0}, \\ u(b_n) + (x - b_n) \cdot \max(1, \frac{c_3^4(n+1)^4 - u(b_n)}{b_{n+1} - b_n}), & \text{if } x \in (b_n, b_{n+1}] \text{ and } n \geq n_0.\end{cases}
\]
Then $u$ is continuous, strictly increasing and satisfies
\[ \inf_{x>0} u(x)/x \geq 1. \]
Moreover, for all $n \geq n_0 + 1$ we have
\[ u(b_n) \geq 4c_3 \cdot n^4. \]
Note that $b_{n_0} \geq 2$ due to (9). Hence $u(2) = 2$. Applying Theorem 5 with $\delta = 1$ and $x_\delta = 1$ and observing (10) as well as the fact that $u^{-1}$ is increasing we thus obtain for $n \geq n_0 + 1$ that
\[
\epsilon_n^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_u \times \{\pi_1\}); S^{\text{sde}} \right) \geq c_1 \cdot \exp(-c_2 \cdot (u^{-1}(c_3 \cdot 4 \cdot n^4))^2) \\
\geq c_1 \cdot \exp(-c_2 \cdot b_n^2) = c_1 \cdot \epsilon_n.
\]
Finally, for all $n \in \{1, 2, \ldots, n_0\}$,
\[
\epsilon_n^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_u \times \{\pi_1\}); S^{\text{sde}} \right) \geq \epsilon_{n_0+1}^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_u \times \{\pi_1\}); S^{\text{sde}} \right) \\
\geq c_1 \cdot \epsilon_{n_0+1} \geq \frac{c_1 \epsilon_{n_0+1}}{\epsilon_1} \cdot \epsilon_n,
\]
which completes the proof. \qed

As a further consequence of Theorem 5 it turns out that the class of equations $\mathcal{E}_\text{lin}$ is too large to obtain convergence of the corresponding minimal errors to zero at all.

**Corollary 3.** There exists $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$ we have
\[ \epsilon_n^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_\text{lin} \times \{\pi_1\}); S^{\text{sde}} \right) \geq c. \]

**Proof.** Choose $c_1, c_2 \in (0, \infty)$ and $c_3 \in [1, \infty)$ according to Theorem 5. Let $n \in \mathbb{N}$. Define $u_n : [0, \infty) \to [0, \infty)$ by
\[
u_n(x) = \begin{cases} \frac{c_3 \cdot x,}{2c_3 + (4c_3^2 \cdot n^4 - 2c_3) \cdot (x - 2)}, & \text{if } x \in [0, 2], \\
\end{cases}
\]
Clearly, $u_n$ is continuous and strictly increasing, and for all $x \in [0, \infty)$ we have
\[ u_n(x) \leq 2c_3 + 4c_3^2 \cdot n^4 \cdot x. \]
Hence $\mathcal{E}_u \subset \mathcal{E}_\text{lin}$, and therefore
\[
\epsilon_n^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_\text{lin} \times \{\pi_1\}); S^{\text{sde}} \right) \geq \epsilon_n^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_u \times \{\pi_1\}); S^{\text{sde}} \right).
\]
Clearly, $u_n(x) \geq c_3 \cdot x$ for all $x \in [0, \infty)$. Moreover, we have $u_n(2) = 2c_3$ and $u_n(3) = 4c_3^2 \cdot n^4$. Applying Theorem 5 with $\delta = c_3$ and $x_\delta = 1$ we obtain
\[
\epsilon_n^{\text{ran}} \left( \mathcal{G}(\mathcal{E}_u \times \{\pi_1\}); S^{\text{sde}} \right) \geq c_1 \cdot \exp(-c_2 \cdot (u^{-1}(1, u_n^2(2) \cdot n^4))^2) \\
= c_1 \cdot \exp(-c_2 \cdot (u_n^{-1}(4c_3^2 \cdot n^4))^2) = c_1 \cdot \exp(-9c_2),
\]
which completes the proof. \qed
Remark 1. Negative results of the type stated in Corollaries can not hold in the case of ordinary differential equations, i.e., the presence of a stochastic part is essential to have the described slow convergence phenomena. In fact, let \( d \in \mathbb{N} \), let \( a \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d) \) and consider the ordinary differential equation
\[
 dX^a(t) = a(X^a(t)) \, dt, \quad t \in [0, 1],
\]

\[
 X^a(0) = 0.
\]

Let \( n \in \mathbb{N} \) and let \( (\hat{X}_n^a(\ell/n))_{\ell=0,...,n} \) denote the corresponding Euler scheme with time step \( 1/n \), i.e., \( \hat{X}_n^a(0) = 0 \) and for \( \ell = 1, \ldots, n \),
\[
 \hat{X}_n^a(\ell/n) = \hat{X}_n^a((\ell-1)/n) + a(\hat{X}_n^a((\ell-1)/n)) \cdot \frac{1}{n}.
\]

Since \( \|a\|_\infty \leq 1 \) we have \( |X^a(t)| \leq 1 \) for all \( t \in [0, 1] \) and \( |\hat{X}_n^a(\ell/n)| \leq 1 \) for all \( \ell = 0, \ldots, n \). Using the latter two facts it is straightforward to see that for all Lipschitz continuous functions \( f: \mathbb{R}^d \to \mathbb{R} \) we have
\[
 |f(X^a(1)) - f(\hat{X}_n^a(1))| \leq c_1 \cdot \|f\|_{\text{Lip}} \cdot \exp\left(c_2 \cdot \max_{\alpha \in \mathbb{N}^d,|\alpha|=1} |D^\alpha a(x)| \right) \cdot \frac{1}{n},
\]
where \( c_1 \) and \( c_2 \) are positive reals, which only depend on the dimension \( d \).

4.2. Worst case analysis with respect to integrands. In this section we take \( d = m = 1 \) and we study the quadrature problem for the trivial equation
\[
 dX(t) = dW(t), \quad t \in [0, 1],
\]

\[
 X(0) = 0.
\]

The class of equations is thus given by the singleton
\[
 \mathcal{E} = \{(0, 1)\}.
\]

For a function \( u: [0, \infty) \to [0, \infty) \) we consider the class of integrands given by
\[
 \mathcal{F}_u = \{ f \in C^\infty_0(\mathbb{R}; \mathbb{R}) : \|f'/\left(1 + u(|\cdot|)\right)\|_\infty \leq 1 \}.
\]

We thus study the computation of \( E[f(W(1))] \) for all functions \( f: \mathbb{R} \to \mathbb{R} \) that are infinitely often differentiable, bounded by one and have a derivative \( f' \), which satisfies the growth condition specified by \( u \).

The following result provides lower bounds for the minimal errors \( e_n^\text{det}(\mathcal{G}(\{(0, 1)\} \times \mathcal{F}_u); S^{sde}) \) of deterministic algorithms in case that the function \( u \) is continuous, strictly increasing and unbounded. See Section 5.2 for a proof.

**Theorem 6.** There exists \( c \in (0, \infty) \) such that for all continuous and strictly increasing functions \( u: [0, \infty) \to [0, \infty) \) with \( \lim_{x \to \infty} u(x) = \infty \) and every \( n \in \mathbb{N} \) we have
\[
 e_n^\text{det}(\mathcal{G}(\{(0, 1)\} \times \mathcal{F}_u); S^{sde}) \geq c \cdot \exp\left(-u^{-1}(\max(n, u(0))^2)\right).
\]
By Lemma 5 in [15],

\[ \lim_{n \to \infty} n^q \cdot \exp\left( -(u^{-1}(n))^2 \right) = \infty \]

for all \( q > 0 \) if and only if (8) holds for all \( q > 0 \). Thus, Theorem 3 implies that, analogously to Corollary 1, we can not have a polynomial rate of convergence of the minimal errors \( e_n^{\text{det}}(\{(0,1)\} \times F_u; S^{\text{sde}}) \) if \( u \) satisfies the exponential growth condition (8).

**Corollary 4.** Assume that \( u: [0, \infty) \to [0, \infty) \) is continuous, strictly increasing and satisfies (8). Then for all \( q > 0 \),

\[ \lim_{n \to \infty} n^q \cdot e_n^{\text{det}}(\{(0,1)\} \times F_u; S^{\text{sde}}) = \infty. \]

Similar to Corollary 2 we may also have an arbitrary slow decay of the minimal errors \( e_n^{\text{det}}(\mathcal{G}(\{(0,1)\} \times F_u); S^{\text{sde}}) \).

**Corollary 5.** For every sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \subset (0, \infty) \) with \( \lim_{n \to \infty} \varepsilon_n = 0 \) there exists \( c \in (0, \infty) \) and a strictly increasing and continuous function \( u: [0, \infty) \to [0, \infty) \) such that for all \( n \in \mathbb{N} \) we have

\[ e_n^{\text{det}}(\{(0,1)\} \times F_u; S^{\text{sde}}) \geq c \cdot \varepsilon_n. \]

**Proof.** Without loss of generality we may assume that the sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \) is strictly decreasing.

Choose \( c \in (0, \infty) \) according to Theorem 3 and put

\[ b_n = \left( \ln \frac{1}{\varepsilon_n} \right)^{1/2} \]

for \( n \in \mathbb{N} \). Note that \( (b_n)_{n \geq n_0} \) is strictly increasing and satisfies \( \lim_{n \to \infty} b_n = \infty \).

Next, define \( u: [0, \infty) \to [0, \infty) \) recursively by

\[ u(x) = \begin{cases} 
1 + x, & \text{if } x \leq b_1, \\
u(b_n) + (x - b_n) \cdot \max(1, \frac{n+1-u(b_n)}{b_{n+1}-b_n}), & \text{if } x \in (b_n, b_{n+1}] \text{ and } n \geq 1.
\end{cases} \]

Then \( u \) is continuous, strictly increasing and for all \( n \geq 1 \) we have

\[ u(b_n) \geq n. \tag{11} \]

Note that \( u(0) = 1 \). Applying Theorem 5 and observing (11) as well as the fact that \( u^{-1} \) is increasing we thus obtain for \( n \geq 1 \) that

\[ e_n^{\text{det}}(\mathcal{G}(\{(0,1)\} \times F_u); S^{\text{sde}}) \geq c \cdot \exp(-(u^{-1}(n))^2) \geq c \cdot \exp(-b_n^2) = c \cdot \varepsilon_n, \]

which finishes the proof. \( \square \)

5. Proofs

We prove Theorem 5 in Section 5.1 and Theorem 6 in Section 5.2.
5.1. **Proof of Theorem [5]**. Throughout the following let \( u : [0, \infty) \to [0, \infty) \) be strictly increasing, continuous and satisfy the condition \( \liminf_{x \to \infty} u(x)/x > 0 \).

In order to construct unfavourable equations (12) in the class \( \mathcal{E}_u \) we employ a particular construction of functions in \( C^{\infty,0}(\mathbb{R}; \mathbb{R}) \) and \( C^{\infty,1}(\mathbb{R}; \mathbb{R}) \), which is established in the following two lemmas.

**Lemma 2.** Let \( \tau, \tau_1, \tau_2, v, v_1, v_2 \in \mathbb{R} \) with \( v \in (0, \infty), \tau_1 < \tau_2, v_1 < v_2 \), and consider the functions

\[
\eta_{\tau,v} : \mathbb{R} \to (0, v], \quad x \mapsto \begin{cases} v \cdot \left(1 - \exp \left( \frac{1}{x-\tau} \right) \right), & \text{if } x < \tau, \\ v, & \text{if } x \geq \tau \end{cases}
\]

and

\[
\theta_{\tau_1,\tau_2,v_1,v_2} : \mathbb{R} \to [v_1, v_2], \quad x \mapsto \begin{cases} v_1, & \text{if } x \leq \tau_1, \\ v_1 + \frac{v_2 - v_1}{1 + \exp \left( \frac{\tau_2 - \tau_1}{\tau_2 - x} \right)}, & \text{if } x \in (\tau_1, \tau_2), \\ v_2, & \text{if } x \geq \tau_2. \end{cases}
\]

Then

(i) \( \eta_{\tau,v}, \theta_{\tau_1,\tau_2,v_1,v_2} \in C^\infty(\mathbb{R}; (0, v]), \theta_{\tau_1,\tau_2,v_1,v_2} \in C^\infty(\mathbb{R}; [v_1, v_2]) \),

(ii) \( \forall k \in \mathbb{N}, [\tau, \infty) \subset \{ \eta_{\tau,v}^{(k)} = 0 \}, \mathbb{R} \setminus (\tau_1, \tau_2) \subset \{ \theta_{\tau_1,\tau_2,v_1,v_2}^{(k)} = 0 \} \),

(iii) \( \eta_{\tau,v} \) is strictly increasing on \( (-\infty, \tau] \), \( \theta_{\tau_1,\tau_2,v_1,v_2} \) is strictly increasing on \( [\tau_1, \tau_2] \),

(iv) \( \| \eta_{\tau,v} \|_{\infty} \leq 4 \exp(-2) \cdot v, \| \theta_{\tau_1,\tau_2,v_1,v_2} \|_{\infty} \leq 4 \cdot \frac{v_2 - v_1}{\tau_2 - \tau_1} \).

**Proof.** Property (i) is well-known and Properties (ii) and (iii) are obvious. In order to prove Property (iv) we first note that

\[
(12) \quad \max_{y > 0} (y^2 \cdot \exp(-y)) = 4 \cdot \exp(-2), \quad \min_{y \in (0,1)} \left( \exp\left(\frac{1}{y}\right) + \exp\left(-\frac{1}{1-y}\right) \right) = 2 \cdot \exp(-2).
\]

For \( x \in (-\infty, \tau) \) we have

\[
\eta'_{\tau,v}(x) = \frac{v}{(x-\tau)^2} \cdot \exp\left(\frac{1}{x-\tau}\right),
\]

which jointly with the first equality in (12) yields the first inequality in (iv). Next, define

\[
\theta : (0, 1) \to \mathbb{R}, \quad x \mapsto \frac{1}{1 + \exp\left(\frac{1}{y} - \frac{1}{1-y}\right)}.
\]

Then for all \( x \in (\tau_1, \tau_2) \) we have

\[
\theta_{\tau_1,\tau_2,v_1,v_2}(x) = v_1 + (v_2 - v_1) \cdot \theta\left(\frac{x-\tau_1}{\tau_2 - \tau_1}\right),
\]

and therefore

\[
(13) \quad \theta'_{\tau_1,\tau_2,v_1,v_2}(x) = \frac{v_2 - v_1}{\tau_2 - \tau_1} \cdot \theta'\left(\frac{x-\tau_1}{\tau_2 - \tau_1}\right).
\]
For all \( y \in (0, 1) \) we have
\[
\theta'(y) = \left( \frac{1}{y^2} + \frac{1}{(1-y)^2} \right) \cdot \frac{1}{(1 + \exp(\frac{1}{y} - \frac{1}{1-y}))^2} \cdot \exp\left( \frac{1}{y} - \frac{1}{1-y} \right)
\]
\[
\leq \frac{1}{y^2} \cdot \frac{1}{1 + \exp(\frac{1}{y} - \frac{1}{1-y})} + \frac{1}{(1-y)^2} \cdot \frac{\exp(\frac{1}{y} - \frac{1}{1-y})}{1 + \exp(\frac{1}{y} - \frac{1}{1-y})}
\]
\[
= \left( \frac{1}{y^2} \cdot \exp\left( \frac{1}{y} \right) + \frac{1}{(1-y)^2} \exp\left( \frac{1}{1-y} \right) \right) \cdot \frac{1}{\exp\left( \frac{1}{y} \right) + \exp\left( \frac{1}{1-y} \right)},
\]
which jointly with (12) and (13) yields the second inequality in (iv). \( \square \)

**Lemma 3.** Consider the functions
\[
\rho_1 = 1/8 - \theta_{0,1/2,0,1/8}, \quad \rho_2 = \theta_{1/2,1,0,1/8}
\]
as well as the function
\[
\rho_3: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x \cdot \exp(-x^2),
\]
and put
\[
c_{\rho_1} = \int_{0}^{\frac{1}{2}} (\rho_1(x))^2 \, dx, \quad c_{\rho_2} = \int_{\frac{1}{2}}^{1} \rho_2(x) \, dx.
\]
Then
(i) \( \rho_1, \rho_2, \rho_3 \in C^{\infty,1}(\mathbb{R}; \mathbb{R}) \),
(ii) \( \{\rho_1 = 0\} = [1/2, \infty), \quad \{\rho_2 = 0\} = (-\infty, 1/2], \)
(iii) \( \forall x \in [0, \infty): 0 \leq \rho_3(x) = -\rho_3(-x), \)
(iv) \( \forall x \in [1/2, 1]: \rho_3(x) \geq \rho_3(1)/2 > 0, \)
(v) \( c_{\rho_1} \in [1/1024, 1/128], \quad c_{\rho_2} \in [1/64, \infty). \)

**Proof.** Property (i) is an immediate consequence of the definition of \( \rho_1 \) and \( \rho_2 \), see Lemma 2 and Property (iii) is obvious.

By Lemma 2(i) we have \( \rho_1, \rho_2 \in C^{\infty,0}(\mathbb{R}; \mathbb{R}) \). By Lemma 2(iii) we have \( \|\rho_i\|_\infty \leq 4 \cdot (1/8)/(1/2) = 1 \) for \( i = 1, 2 \). Hence \( \rho_1, \rho_2 \in C^{\infty,1}(\mathbb{R}; \mathbb{R}) \). Clearly, \( \rho_3 \in C^{\infty}(\mathbb{R}; \mathbb{R}) \). Furthermore, \( \rho_3'(x) = (1-2x^2) \exp(-x^2) \) for all \( x \in \mathbb{R} \), which yields \( \|\rho_3\|_\infty = \rho_3(1/\sqrt{2}) \leq 1 \).

For every \( x \in [-1, 1] \) we have \( |1-2x^2| \leq 1 \leq \exp(x^2) \), while for every \( x \in \mathbb{R} \) with \( |x| > 1 \) we get
\[
|1-2x^2| = 2x^2 - 1 = 1 + \int_{1}^{x^2} 2 \, dy \leq 1 + \int_{1}^{x^2} \exp(y) \, dy = 1 + \exp(x^2) - \exp(1) \leq \exp(x^2).
\]
Hence \( \|\rho_3\|_\infty \leq 1 \) and we conclude that \( \rho_3 \in C^{\infty,1}(\mathbb{R}; \mathbb{R}) \). Thus Property (i) is proven.

For every \( x \in [1/2, 1] \) we have \( x \cdot \exp(-x^2) \geq \frac{1}{2} \cdot \exp(-1) = \frac{1}{2} \cdot \rho_3(1) \), which proves Property (iv) in Lemma 2.
Since \(0 \leq \rho_1 \leq 1/8\) we get
\[
\int_0^{1/2} (\rho_1(x))^2 \, dx \leq \frac{1}{128}.
\]
Since \(\rho_1\) is decreasing on \([0,1/2]\) we have
\[
\int_0^{1/2} (\rho_1(x))^2 \, dx \geq \int_0^{1/4} (\rho_1(x))^2 \, dx \geq \frac{1}{4} \cdot (\rho_1(1/4))^2 = \frac{1}{4} \cdot \left(\frac{1}{16}\right)^2 = \frac{1}{1024}.
\]
Furthermore, since \(\rho_2\) is increasing on \([1/2,1]\) we get
\[
\int_{1/2}^1 \rho_2(x) \, dx \geq \int_{3/4}^1 \rho_2(x) \, dx \geq \frac{1}{4} \cdot \rho_2(3/4) = \frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64},
\]
which yields Property (v) and completes the proof of the lemma.

Using the functions \(\rho_1, \rho_2, \rho_3\) from Lemma 4 we construct a subclass of unfavourable equations \(E_u^* \subset E_u\). Put
\[
\mathcal{H} = \{ h \in C^{\infty,1}(\mathbb{R}; \mathbb{R}): \{ h \neq 0 \} \subset [0,1/2]\},
\]
\[
\mathcal{V}_u = \{ v \in C^{\infty}(\mathbb{R}; \mathbb{R}): |v|, |v'| \leq 1 + u(| \cdot |) \}.
\]
We use a single diffusion coefficient \(b\) given by
\[
b: \mathbb{R}^4 \to \mathbb{R}^4, \quad (x_1, \ldots, x_4) \mapsto (0, \rho_1(x_4), 0, 0),
\]
and for all \(h \in \mathcal{H}\) and \(v \in \mathcal{V}_u\) we define a drift coefficient \(a^{h,v}\) by
\[
a^{h,v}: \mathbb{R}^4 \to \mathbb{R}^4, \quad (x_1, \ldots, x_4) \mapsto (\rho_2(x_4) \cdot \rho_3 \left(\frac{x_3}{1+x_3} \cdot v(x_2)\right), 0, h(x_4), 1).
\]
Put
\[
E_u^* = \{(a^{h,v}, b): h \in \mathcal{H}, v \in \mathcal{V}_u\}.
\]

**Lemma 4.** We have \(E_u^* \subset E_u\) and, consequently, for every \(n \in \mathbb{N}\),
\[
e_n^{ran}(\mathcal{G}(E_u \times \{ \pi_1 \}); S^{sde}) \geq e_n^{ran}(\mathcal{G}(E_u^* \times \{ \pi_1 \}); S^{sde}).
\]

**Proof.** From \(\rho_1 \in C^{\infty,1}(\mathbb{R}; \mathbb{R})\) we immediately get \(b \in C^{\infty}(\mathbb{R}^4; \mathbb{R}^4)\) as well as
\[
\max_{\alpha \in \mathbb{N}_0^4: \sum \alpha_i \leq 1} \| D^\alpha b \|_\infty = \max(\| \rho_1 \|_\infty, \| \rho_1' \|_\infty) \leq 1.
\]
From \(\rho_2, \rho_3, h, v \in C^{\infty}(\mathbb{R}; \mathbb{R})\) it is clear that \(a^{h,v} \in C^{\infty}(\mathbb{R}^4; \mathbb{R}^4)\). Furthermore, we have
\[
\| a^{h,v} \|_\infty \leq \max(\| \rho_2 \|_\infty, \| \rho_3 \|_\infty, \| h \|_\infty, 1) = 1 \quad \text{since} \quad \rho_2, \rho_3, h \in C^{\infty,1}(\mathbb{R}; \mathbb{R}).
\]
For every
Let \( x = (x_1, \ldots, x_4) \in \mathbb{R}^4 \) we have

\[
D^\alpha a^{h,v}(x) = \begin{cases} 
0, & \text{if } \alpha = (1, 0, 0, 0), \\
(\rho_2(x_4) \cdot \frac{x_4}{1 + x_4^2} \cdot v'(x_2) \cdot \rho_3(\frac{x_4}{1 + x_4^2} \cdot v(x_2)), 0, 0, 0), & \text{if } \alpha = (0, 1, 0, 0), \\
(\rho_2(x_4) \cdot \frac{1 - x_4^2}{(1 + x_4^2)^2} \cdot v(x_2) \cdot \rho_3(\frac{x_4}{1 + x_4^2} \cdot v(x_2)), 0, 0, 0), & \text{if } \alpha = (0, 0, 1, 0), \\
(\rho'_2(x_4) \cdot \rho_3(\frac{x_4}{1 + x_4^2} \cdot v(x_2)), 0, h'(x_4), 0), & \text{if } \alpha = (0, 0, 0, 1),
\end{cases}
\]

and therefore

\[
|D^\alpha a^{h,v}(x)| \leq \begin{cases} 
0, & \text{if } \alpha = (1, 0, 0, 0), \\
|v'(x_2)|, & \text{if } \alpha = (0, 1, 0, 0), \\
|v(x_2)|, & \text{if } \alpha = (0, 0, 1, 0), \\
1, & \text{if } \alpha = (0, 0, 0, 1), \\
\leq 1 + u(|x_2|) \leq 1 + u(|x|)
\end{cases}
\]

since \( \rho_2, \rho_3, h \in C^\infty_1(\mathbb{R}; \mathbb{R}) \), \( v \in \mathcal{V}_u \) and \( u \) is increasing.

Hence \((a^{h,v}, b) \in \mathcal{E}_u\), which finishes the proof. \( \square \)

Next, we determine the values of \( G^{\text{de}} \) on \( \mathcal{G}(\mathcal{E}^*_u, \{\pi_1\}) \).

**Lemma 5.** For every \( h \in \mathcal{H} \) and every \( v \in \mathcal{V}_u \) the solution \( X^{a^{h,v},b} \) of (1) with the drift coefficient \( a = a^{h,v} \) given by (15) and the diffusion coefficient \( b \) given by (14) satisfies

\[
E[X^{a^{h,v},b}(1)] = \frac{c_{\rho_2}}{\sqrt{2\pi c_{\rho_1}}} \int_\mathbb{R} \rho_3(\frac{\int_0^{1/2} h(t) \, dt}{1 + (\int_0^{1/2} h(t) \, dt)^2} \cdot v(x)) \cdot \exp(-\frac{x^2}{2c_{\rho_1}}) \, dx.
\]

**Proof.** Let \( h \in \mathcal{H} \) and \( v \in \mathcal{V}_u \), and write \( X \) in place of \( X^{a^{h,v},b} \). By definition of \( a^{h,v} \) and \( b \) we have

\[
X_1(t) = 1_{[1/2,1]}(t) \cdot \rho_3(\frac{X_3(1/2)}{1 + (X_3(1/2))^2} \cdot v(X_2(1/2))) \cdot \int_1^t \rho_2(s) \, ds,
\]

\[
X_2(t) = \int_0^{\min(t,1/2)} \rho_1(s) \, dW(s),
\]

\[
X_3(t) = \int_0^{\min(t,1/2)} h(s) \, ds,
\]

\[
X_4(t) = t
\]

for all \( t \in [0, 1] \), and therefore

\[
X_1(1) = c_{\rho_2} \cdot \rho_3(\frac{\int_0^{1/2} h(t) \, dt}{1 + (\int_0^{1/2} h(t) \, dt)^2} \cdot v(X_2(1/2))).
\]

Furthermore,

\[
X_2(1/2) = \int_0^{1/2} \rho_1(s) \, dW(s) \sim N(0, c_{\rho_1}),
\]
there exists
\[ (17) \]
Below we show
\[ \forall S_{\text{sde}} \in \mathcal{A}_{\text{sde}} \exists \hat{S}_{\text{sde}} \in \mathcal{A}_{\text{int}} : \text{cost}(\hat{S}_{\text{int}}) \leq \text{cost}(\hat{S}_{\text{sde}}) \] and \( \hat{S}_{\text{sde}} = \hat{S}_{\text{sde}} \circ T \).

Clearly, \((16)\) and \((17)\) jointly imply that for every \( \hat{S}_{\text{sde}} \in \mathcal{A}_{\text{sde}} \) with \( \text{cost}(\hat{S}_{\text{sde}}) \leq n \), there exists \( \hat{S}_{\text{sde}} \in \mathcal{A}_{\text{int}} \) such that \( \text{cost}(\hat{S}_{\text{int}}) \leq n \) and
\[ e(\hat{S}_{\text{sde}}) = \sup_{g \in \mathcal{G}(\mathcal{E}_u \times \{\pi_1\})} \mathbb{E}|S_{\text{sde}}(g) - \hat{S}_{\text{sde}}(g)| \]
\[ \geq \sup_{g \in \mathcal{G}((a^{h,v,b}; k \in \mathcal{H}) \times \{\pi_1\})} \mathbb{E}|S_{\text{sde}}(g) - \hat{S}_{\text{sde}}(g)| \]
\[ = \sup_{g \in \mathcal{G}(\mathcal{H})} \mathbb{E}|S_{\text{sde}} \circ T(g) - \hat{S}_{\text{sde}} \circ T(g)| = e(\hat{S}_{\text{int}}). \]

Hence \( e_n^{\text{ran}}(\mathcal{G}(\mathcal{E}_u \times \{\pi_1\}); S_{\text{sde}}) \geq e_n^{\text{ran}}(\mathcal{G}(\mathcal{H}); \hat{S}_{\text{sde}}) \).

It remains to prove \((17)\). To this end we first note that there exists a mapping \( \widetilde{T} : \mathbb{R}^4 \times \mathbb{R}^n_0 \to (\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R})^n_0 \) such that \( Tg(x) = \widetilde{T}(x, g(x_4)) \) for all \( g \in \mathcal{G}(\mathcal{H}) \) and all

\[ E(X_1(1)) = \frac{c_{\rho_2}}{2\pi c_{\rho_1}} \cdot \int_{\mathbb{R}} \rho_3 \left( \frac{\rho_2^{1/2} h(t) dt}{1 + (\rho_2^{1/2} h(t) dt)^2} \cdot v(x) \right) \cdot \exp(-\frac{x^2}{2c_{\rho_1}^2}) \, dx \]
as claimed. \[ \square \]
where cost(\hat{S}^{\text{ran}}_{\psi,\nu,\varphi}) ≤ cost(\hat{S}^{\text{int}}_{\psi,\nu,\varphi}) ≤ cost(\hat{S}^{\text{ran}}_{\psi,\nu,\varphi}) and thus finishes the proof of (17).

In view of Lemma 7 it suffices to establish appropriate lower bounds for the minimal errors \epsilon_{\text{ran}}^m(\mathcal{G}(\mathcal{H}); S^\text{int}_v). To this end we first construct unfavourable functions \( h \in \mathcal{H} \) in Lemma 7 and then employ Proposition 1 in Lemma 8.

Lemma 7. For every \( m \in \mathbb{N} \) there exist \( h_1, \ldots, h_m \in \mathcal{H} \) with the following properties.

(i) The sets \( \{h_1 \neq 0\}, \ldots, \{h_m \neq 0\} \) are pairwise disjoint.

(ii) For all \( i \in \{1, \ldots, m\} \) we have

\[
\int_0^\frac{1}{2} h_i(t) \, dt = \frac{1}{(12m)^2}.
\]
Proof. Let
\[ h : \mathbb{R} \to [0, 1/(6m)], \quad x \mapsto \begin{cases} \theta \frac{1}{6m}, \frac{1}{6m} \cdot (x), & \text{if } x \leq \frac{2}{6m}, \\ \frac{1}{6m} - \theta \frac{3}{6m}, \frac{1}{6m} \cdot (x), & \text{if } x > \frac{2}{6m}, \end{cases} \]
and put
\[ c_0 = \int_0^{\frac{3}{6m}} h(x) \, dx. \]
For \( i = 1, \ldots, m \) we define
\[ h_i : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \frac{1}{c_0 \cdot (12m)^2} \cdot h \left( x - \frac{i-1}{2m} \right). \]

Let \( i \in \{1, \ldots, m\} \). From Lemma \( \ref{lem:smoothness}(i) \) we get \( h \in C^\infty(\mathbb{R}; \mathbb{R}) \) and therefore we have \( h_i \in C^\infty(\mathbb{R}; \mathbb{R}) \). By the definition of \( h \) we have \( h(x) = 1/(6m) \) for all \( x \in [1/(6m), 2/(6m)] \), which implies
\[ c_0 \geq \int_0^{\frac{2}{6m}} h(x) \, dx = \frac{1}{(6m)^2}. \]
It follows that
\[ \|h_i\|_\infty = \frac{1}{c_0 \cdot (12m)^2} \cdot \|h\|_\infty \leq \frac{1}{4 \cdot 6m} \leq 1 \]
and, using Lemma \( \ref{lem:smoothness}(iv) \),
\[ \|h'_i\|_\infty = \frac{1}{c_0 \cdot (12m)^2} \cdot \|h'\|_\infty \leq \frac{\|h'\|_\infty}{4} = 1. \]

Hence \( h_i \in C^{\infty,1}(\mathbb{R}; \mathbb{R}) \).

By the definition of \( h \) we have \( \{h \neq 0\} = (0, 3/(6m)) \), which implies
\[ \{h_i \neq 0\} = \left( \frac{i-1}{2m}, \frac{i}{2m} \right) \subseteq [0, 1/2]. \]
Thus, \( h_i \in \mathcal{H} \) and statement (i) of the lemma holds.

Finally,
\[ \int_0^{\frac{1}{2}} h_i(x) \, dx = \frac{1}{c_0 \cdot (12m)^2} \cdot \int_0^{\frac{3}{6m}} h(x) \, dx = \frac{1}{(12m)^2}, \]
which proves statement (ii) of the lemma and finishes the proof. \( \square \)

Lemma 8. For every \( v \in \mathcal{V}_u \) and every \( n \in \mathbb{N} \) we have
\[ e_n^v(G(\mathcal{H}); S_v^{\text{int}}) \geq \frac{c_{\rho_2}}{68 \sqrt{2\pi c_{\rho_1}}} \cdot \int_{\mathbb{R}} \rho_3 \left( \frac{4(102n)^2}{1+16(102n)^4} \cdot v(x) \right) \cdot \exp \left( -\frac{x^2}{2c_{\rho_1}} \right) \, dx. \]
Proof. Let $v \in \mathcal{V}_n$ and $n \in \mathbb{N}$ and choose $h_1, \ldots, h_{17n} \in \mathcal{H}$ according to Lemma 7 with $m = 17n$. By Lemma 7(ii) and the fact that $\rho_3(t) = -\rho_3(-t)$ for all $t \geq 0$ we obtain

$$S^i_{v} = \int_{[1, \infty)} \rho_3 t \cdot \exp(\frac{-x^2}{2\overline{v}(t)}) \, dx$$

for $i = 1, \ldots, 17n$. It remains to apply Proposition 11 with $A = \mathbb{R}$, $B = \mathbb{R}^n$, $G = G(\mathcal{H})$, $m = 17n$, $g_i = \rho_3 h_i^{(k)}$, $g_i = \rho_3(-h_i^{(k)})$ and $b^* = 0 \in \mathbb{R}^n$.

The following lemma provides a technical tool for the construction of unfavorable functions $v_n \in \mathcal{V}_n$ carried out in Lemma 10.

**Lemma 9.** Let $\delta > 0$ and let $x_\delta \in (0, \infty)$ satisfy $\inf_{x \geq x_\delta} u(x)/x \geq \delta$. Then the function

$$(18) \quad \zeta_u: [1, \infty) \to [0, \infty), \quad x \mapsto \left(\int_0^{x-1} u(y) \, dy\right)^{1/2}$$

satisfies

(i) $\zeta_u$ is strictly increasing, differentiable on $(1, \infty)$ and $\zeta_u([1, \infty)) = [0, \infty)$,

(ii) $\forall x \in [1, \infty)$: $\zeta_u(x) \leq \frac{1}{\sqrt{\delta}} u(\max(x - 1, x_\delta))$,

(iii) $\forall x \in [x_\delta + 1, \infty)$: $\zeta_u'(x) \geq \sqrt{\delta}/2$,

(iv) $\forall y \in [\zeta_u(2), \infty)$: $\zeta_u^{-1}(y) \leq 4u^{-1}(y^2)$.

Proof. Since $u$ is continuous and positive, the mapping $\zeta_u$ is strictly increasing on $[1, \infty)$ and continuously differentiable on $(1, \infty)$ with

$$\zeta_u'(x) = \frac{u(x-1)}{2} \cdot \left(\int_0^{x-1} u(y) \, dy\right)^{-1/2}$$

for every $x \in (1, \infty)$. Clearly, $\zeta_u(1) = 0$. Moreover, for $x \in [x_\delta + 2, \infty)$ we have $\zeta_u'(x) \geq \int_{x-2}^{x-1} u(y) \, dy \geq u(x-2) \geq \delta \cdot (x-2)$, which implies $\lim_{x \to \infty} \zeta_u(x) = \infty$ and completes the proof of (i).

For $x \in [1, \infty)$ we have

$$\zeta_u^2(x) = \int_0^{\max(x-1,x_\delta)} u(y) \, dy \leq \max(x-1,x_\delta) \cdot u(\max(x-1,x_\delta))$$

$$\leq \delta^{-1} \cdot u^2(\max(x-1,x_\delta)),$$

which implies (ii).

Next, let $x \in [x_\delta + 1, \infty)$. Then $x - 1 \geq x_\delta$, and therefore $u(x-1) \geq \delta \cdot (x-1)$. It follows

$$\zeta_u'(x) \geq \frac{u(x-1)}{2} \cdot ((x-1) \cdot u(x-1))^{-1/2} = \frac{1}{2} \left(\frac{u(x-1)}{x-1}\right)^{1/2} \geq \frac{\sqrt{\delta}}{2}.$$
which proves (iii).

Finally, let $y \in [\zeta_u(2), \infty)$. Then $\zeta_u^{-1}(y) \geq 2$, which implies $\zeta_u^{-1}(y) - 2 \geq \zeta_u^{-1}(y)/4$. It follows

$$y^2 = (\zeta_u(\zeta_u^{-1}(y)))^2 \geq \int_{\zeta_u^{-1}(y) - 2}^{\zeta_u^{-1}(y) - 1} u(x) \, dx \geq u(\zeta_u^{-1}(y) - 2) \geq u(\zeta_u^{-1}(y)/4),$$

which shows (iv) and finishes the proof of the lemma.

\[ \square \]

**Lemma 10.** Let $\delta > 0$, let $x_\delta \in (0, \infty)$ satisfy $\inf_{x \geq x_\delta} u(x)/x \geq \delta$, let

$$\kappa_\delta = \frac{8}{\sqrt{\delta}} \cdot \max(u(x_\delta + 1), 1)$$

and let $\zeta_u: [1, \infty) \to [0, \infty)$ be the function given by (18). Let $n \in \mathbb{N}$ and let

$$\alpha_n = \zeta_u^{-1}\left(\frac{\kappa_\delta \cdot \frac{1+16 \cdot (102n)^4}{4 \cdot (102n)^2}}{}\right).$$

Then there exists a function $v_n \in C^\infty(\mathbb{R}; (0, \infty))$ such that

(i) $v_n$ is strictly increasing,

(ii) $v_n, v'_n \leq 1 + u(|\cdot|),$

(iii) $\int_{0}^{\alpha_n} \rho_\delta \left(\frac{4 \cdot (102n)^2}{1+16 \cdot (102n)^2} \cdot v_n(x)\right) \, dx \geq \frac{\rho_\delta(1)}{4}.$

**Proof.** Note that $\alpha_n$ is well-defined due to Lemma 9(i). We put

$$\beta_n = \frac{1 + 16 \cdot (102n)^4}{4 \cdot (102n)^2}.$$

We first show that

(19) \hspace{1cm} \alpha_n \geq x_\delta + 2

and

(20) \hspace{1cm} \zeta_u(\alpha_n) - \zeta_u(\alpha_n - 1) \geq \frac{\sqrt{\delta}}{2}.

By Lemma 9(ii) and the fact that $\kappa_\delta \geq u(x_\delta + 1)/\sqrt{\delta}$ we have

$$\zeta_u(x_\delta + 2) \leq \frac{1}{\sqrt{\delta}} u(x_\delta + 1) \leq \kappa_\delta \leq \kappa_\delta \cdot \beta_n,$$

which yields (19) since $\zeta_u$ is increasing. From (19) and Lemma 9(iii) we obtain that $\inf_{x \in [\alpha_n - 1, \alpha_n]} \zeta'_u(x) \geq \sqrt{\delta}/2$, which implies (20) by the mean value theorem.

Note that $\alpha_n > 2$ due to (19) and put

$$k_n = \max\{k \in \mathbb{N}: \alpha_n - k > 1\}.$$
Clearly,

\[(21) \quad \alpha_n - k_n \in (1, 2].\]

Observing \[20\] and the fact that \(\zeta_u\) is increasing we may define a function \(v_n : \mathbb{R} \to \mathbb{R}\) by

\[
v_n(x) = \frac{1}{\kappa_\delta} \times \begin{cases} 
\eta_{\alpha_n - k_n, \zeta_u(\alpha_n - k_n)}(x), & \text{if } x \in (-\infty, \alpha_n - k_n), \\
\theta_{\alpha_n + k, \alpha_n + k + 1, \zeta_u(\alpha_n + k), \zeta_u(\alpha_n + k + 1)}(x), & \text{if } x \in [\alpha_n + k, \alpha_n + k + 1) \quad \text{and } k \in \{-k_n, \ldots, -2\} \cup \mathbb{N}_0, \\
\theta_{\alpha_n - 1, \alpha_n - 1/2, \zeta_u(\alpha_n - 1), \zeta_u(\alpha_n) - \sqrt{\delta}/4}(x), & \text{if } x \in [\alpha_n - 1, \alpha_n - 1/2), \\
\theta_{\alpha_n - 1/2, \alpha_n \zeta_u(\alpha_n) - \sqrt{\delta}/4, \zeta_u(\alpha_n)}(x), & \text{if } x \in [\alpha_n - 1/2, \alpha_n). 
\end{cases}
\]

By Lemma \[21\](i),(ii) we immediately get \(v_n \in C^\infty(\mathbb{R}; (0, \infty))\). Furthermore, Property (i) is a straightforward consequence of Lemma \[21\](iii) together with Lemma \[9\](i).

We turn to the proof of Property (ii). First assume that \(x \in (-\infty, \alpha_n - k_n)\). Using Lemma \[21\](iii),(iv) as well as \[21\], the fact that \(\zeta_u\) is increasing and Lemma \[9\](ii) we get

\[
v_n(x) + v_n'(x) = \frac{1}{\kappa_\delta} \cdot (\eta_{\alpha_n - k_n, \zeta_u(\alpha_n - k_n)}(x) + \eta_{\alpha_n - k_n, \zeta_u(\alpha_n - k_n)}'(x)) \\
\leq \frac{1 + 4 \exp(-2)}{\kappa_\delta} \cdot \zeta_u(\alpha_n - k_n) \leq \frac{2}{\kappa_\delta} \cdot \zeta_u(2) \\
\leq \frac{2}{\kappa_\delta} \cdot \zeta_u(x_\delta + 2) \leq \frac{2}{\kappa_\delta} \cdot \frac{u(x_\delta + 1)}{\sqrt{\delta}} \leq \frac{1}{4} \leq 1.
\]

Next, assume that \(x \in [\alpha_n + k, \alpha_n + k + 1)\) with \(k \in \{-k_n, \ldots, -2\} \cup \mathbb{N}_0\). Using Lemma \[21\](iii),(iv) and the fact that \(\zeta_u\) is non-negative, we obtain

\[
v_n(x) + v_n'(x) = \frac{\theta_{\alpha_n + k, \alpha_n + k + 1, \zeta_u(\alpha_n + k), \zeta_u(\alpha_n + k + 1)}(x) + \theta_{\alpha_n + k, \alpha_n + k + 1, \zeta_u(\alpha_n + k), \zeta_u(\alpha_n + k + 1)}'(x)}{\kappa_\delta} \\
\leq \frac{\zeta_u(\alpha_n + k + 1) + 4(\zeta_u(\alpha_n + k + 1) - \zeta_u(\alpha_n + k))}{\kappa_\delta} \leq \frac{5\zeta_u(\alpha_n + k + 1)}{\kappa_\delta}.
\]

Note that \(\alpha_n + k + 1 \geq \alpha_n - k_n + 1 \geq 2\). Using Lemma \[9\](ii) and the fact that \(u\) is increasing we may therefore conclude that

\[
\frac{5 \cdot \zeta_u(\alpha_n + k + 1)}{\kappa_\delta} \leq \frac{5 \cdot u(\max(\alpha_n + k, x_\delta))}{8 \cdot \max(u(x_\delta + 1), 1)} \leq \max(1, u(\alpha_n + k)) \leq 1 + u(x).
\]
Next, assume that \( x \in [\alpha_n - 1, \alpha_n - 1/2] \). Using Lemma 2(iii),(iv), the fact that \( \zeta_u \geq 0 \), (19), Lemma 9(ii) and the fact that \( u \) is increasing we get

\[
\max(v_n(x), v'_n(x)) = \max\left(\theta_{\alpha_n - 1, \alpha_n - 1/2, \zeta_u(\alpha_n)}(x), \theta'_{\alpha_n - 1, \alpha_n - 1/2, \zeta_u(\alpha_n)}(x)\right)
\leq \frac{\kappa}{\delta} \max(\zeta_u(\alpha_n) - \sqrt{\delta}/4, \zeta_u(\alpha_n) - \sqrt{\delta}/4 - \zeta_u(\alpha_n - 1)))
\leq \frac{8\zeta_u(\alpha_n)}{\kappa \delta} \leq u(\alpha_n - 1) \leq u(x).
\]

Finally, assume that \( x \in [\alpha_n - 1/2, \alpha_n] \). Using Lemma 2(iii),(iv) as well as (20), Lemma 9(ii), (19) and the fact that \( u \) is increasing we get

\[
\max(v_n(x), v'_n(x)) = \max\left(\theta_{\alpha_n - 1/2, \alpha_n, \zeta_u(\alpha_n)}(x), \theta'_{\alpha_n - 1/2, \alpha_n, \zeta_u(\alpha_n)}(x)\right)
\leq \frac{\kappa}{\delta} \max(\zeta_u(\alpha_n), 2\sqrt{\delta}) \leq \frac{4\zeta_u(\alpha_n)}{\kappa \delta} \leq \frac{1}{2} \cdot u(\alpha_n - 1) \leq u(x).
\]

It remains to prove Property (iii). Since \( \rho_3 \geq 0 \) on \([0, \infty)\), see Lemma 3(iii), we have

\[
\int_0^{\alpha_n} \rho_3\left(\frac{v_n(x)}{\beta_n}\right) dx \geq \int_{\alpha_n - 1/2}^{\alpha_n} \rho_3\left(\frac{v_n(x)}{\beta_n}\right) dx.
\]

Let \( x \in [\alpha_n - 1/2, \alpha_n] \). Then

\[
\frac{v_n(x)}{\beta_n} \leq \frac{\zeta_u(\alpha_n)}{\beta_n \cdot \kappa \delta} = 1.
\]

Furthermore, by Lemma 2(iii),

\[
\frac{v_n(x)}{\beta_n} \geq \frac{\zeta_u(\alpha_n)}{\beta_n \cdot \kappa \delta} - \frac{\sqrt{\delta}/4}{\beta_n \cdot \kappa \delta} = 1 - \frac{\sqrt{\delta}}{4\beta_n \cdot \kappa \delta} \geq 1 - \frac{\delta}{32 \cdot u(x_\delta + 1)}.
\]

Since \( u(x_\delta + 1) \geq \delta \cdot (x_\delta + 1) \geq \delta \) we conclude that \( v_n(x)/\beta_n \in [1/2, 1] \). Therefore, by Lemma 3(iv),

\[
\rho_3\left(\frac{v_n(x)}{\beta_n}\right) \geq \frac{\rho_3(1)}{2}.
\]

Hence

\[
\int_{\alpha_n - 1/2}^{\alpha_n} \rho_3\left(\frac{v_n(x)}{\beta_n}\right) dx \geq \frac{\rho_3(1)}{4},
\]

which completes the proof of the lemma.
Proposition 3. Let \( \delta \in (0, \infty) \) and let \( x_\delta \in (0, \infty) \) satisfy \( \inf_{x \geq x_\delta} u(x)/x \geq \delta \) and let \( \kappa_\delta = \frac{8}{\sqrt{\delta}} \cdot \max(u(x_\delta + 1), 1) \)

Then for every \( n \in \mathbb{N} \),

\[
e_{n}^{\text{ran}}(G(E_u \times \{\pi_1\}); S^{\text{sde}}) \geq \frac{1}{17 \cdot 2^7 \cdot e \cdot \sqrt{\pi}} \cdot \exp(-2^{13} \cdot (u^{-1}(17^2 \cdot 56^4 \cdot \kappa_\delta^2 \cdot n^4))^2).
\]

Proof. Let \( \zeta_u : [1, \infty) \to [0, \infty) \) be the function given by (18), let \( n \in \mathbb{N} \) and choose a function \( v_n : \mathbb{R} \to (0, \infty) \) according to Lemma 10. Note that \( v_n \in V_u \) due to Lemma 10(ii). Hence, by Lemmas 4, 6, 8, 10 and Lemma 3

\[
e_{n}^{\text{ran}}(G(E_u \times \{\pi_1\}); S^{\text{sde}}) \geq e_{n}^{\text{ran}}(G(H); S^{\text{int}}_{v_n})
\]

\[
\geq \frac{c_{p_2}}{68 \sqrt{2\pi c_{p_1}}} \cdot \int_{\mathbb{R}} \rho_3 \left( \frac{4(102n)^2}{1+16(102n)^2} \cdot v_n(x) \right) \cdot \exp\left(-\frac{x^2}{2c_{p_1}}\right) dx
\]

\[
\geq \frac{c_{p_2} \cdot \rho_3(1)}{272 \sqrt{2\pi c_{p_1}}} \cdot \exp\left(-\frac{\alpha_n^2}{2c_{p_1}}\right) \geq \frac{1}{17 \cdot 2^7 \cdot e \cdot \sqrt{\pi}} \cdot \exp(-2^9 \cdot \alpha_n^2).
\]

Put

\[
\beta_n = \frac{1 + 16 \cdot (102n)^4}{4 \cdot (102n)^2}.\]

By (19) we have \( \alpha_n > 2 \). Hence \( \kappa_\delta \cdot \beta_n \geq \zeta_u(2) \) and we may apply Lemma 9(iv) to obtain

\[
\alpha_n = \zeta_u^{-1}(\kappa_\delta \cdot \beta_n) \leq 4u^{-1}(\kappa_\delta^2 \cdot \beta_n^2).
\]

Use \( \beta_n \leq 17/4 \cdot (102n)^2 \) and the fact that \( u^{-1} \) is increasing to complete the proof. \( \square \)

Clearly, Proposition 3 implies Theorem 5.

5.2. Proof of Theorem 6. Throughout the following let \( u : [0, \infty) \to [0, \infty) \) be strictly increasing, continuous and satisfy \( \lim_{x \to \infty} u(x) = \infty \).

Put

\[
H_u = \{h \in C^\infty(\mathbb{R}; \mathbb{R}) : |h'| \leq 1 + u\}.
\]

In the setting of Section 3 we take

\[
A = \mathbb{R}, \quad B = \mathbb{R}^{N_0}, \quad G(H_u) = \{(h^{(k)})_{k \in \mathbb{N}_0} : h \in H_u\},
\]

and we define \( S^{\text{int}} : G(H_u) \to \mathbb{R} \) by

\[
S^{\text{int}}((h^{(k)})_{k \in \mathbb{N}_0}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sin(h(x)) \cdot \exp(-\frac{x^2}{2}) dx.
\]
Lemma 11. For every $n \in \mathbb{N}$ we have
\[ e_n^{\text{det}}(\mathcal{G}((0, 1) \times \mathcal{F}_u); S^{\text{sde}}) \geq e_n^{\text{det}}(\mathcal{G}(H_u); S^{\text{int}}). \]

Proof. We use $A^{\text{det}}_{\text{sde}}$ and $A^{\text{det}}_{\text{int}}$ to denote the classes of deterministic algorithms for the approximation of $S^{\text{sde}}: \mathcal{G}((0, 1) \times \mathcal{F}_u) \rightarrow \mathbb{R}$ and $S^{\text{int}}$, respectively. Note that $\sin \circ h \in \mathcal{F}_u$ for every $h \in H_u$. We can therefore define a mapping $T: \mathcal{G}(H_u) \rightarrow \mathcal{G}((0, 1) \times \mathcal{F}_u)$ by
\[ T((h(k))_{k \in \mathbb{N}_0}) = (0, 1_{\{0\}}(k), (\sin \circ h)^{(k)})_{k \in \mathbb{N}_0}. \]
Clearly,
\[ S^{\text{int}} = S^{\text{sde}} \circ T. \]

Similar to the proof of Lemma 6 it thus remains to show that
\[ \forall \hat{S}^{\text{sde}} \in A^{\text{det}}_{\text{sde}} \exists \hat{S}^{\text{int}} \in A^{\text{det}}_{\text{int}} : \text{cost}(\hat{S}^{\text{int}}) \leq \text{cost}(\hat{S}^{\text{sde}}) \text{ and } \hat{S}^{\text{int}} = \hat{S}^{\text{sde}} \circ T. \]

In order to prove (22) we first note that for every $k \in \mathbb{N}$ there exists a mapping $\rho_k: \mathbb{R}^{k+1} \rightarrow \mathbb{R}$ such that
\[ (\sin \circ h)^{(k)} = \rho_k \circ (h, h^{(1)}, \ldots, h^{(k)}) \]
for every $k$-times differentiable function $h: \mathbb{R} \rightarrow \mathbb{R}$, and we define
\[ \rho: \mathbb{R}^{\mathbb{N}_0} \rightarrow (\mathbb{R} \times \mathbb{R} \times \mathbb{R})^{\mathbb{N}_0}, \quad (x_k)_{k \in \mathbb{N}_0} \mapsto (0, 1_{\{0\}}(k), \rho_k(x_0, \ldots, x_k))_{k \in \mathbb{N}_0}. \]
Next, let $\hat{S}^{\text{sde}} \in A^{\text{det}}_{\text{sde}}$ be given by $\hat{S}^{\text{sde}} = \hat{S}^{\text{sde}}_{\psi, \nu, \varphi}$ with mappings
\[ \psi_k: ((\mathbb{R} \times \mathbb{R} \times \mathbb{R})^{\mathbb{N}_0})^{k-1} \rightarrow \mathbb{R}, \quad k \in \mathbb{N}, \]
\[ \nu: \mathcal{G}((0, 1) \times \mathcal{F}_u) \rightarrow \mathbb{N}, \]
\[ \varphi_k: ((\mathbb{R} \times \mathbb{R} \times \mathbb{R})^{\mathbb{N}_0})^{k} \rightarrow \mathbb{R}, \quad k \in \mathbb{N}, \]
see Section 3. We define mappings
\[ \tilde{\psi}_k: (\mathbb{R}^{\mathbb{N}_0})^{k-1} \rightarrow \mathbb{R}, \quad k \in \mathbb{N}, \]
\[ \tilde{\nu}: \mathcal{G}(H_u) \rightarrow \mathbb{N}, \]
\[ \tilde{\varphi}_k: (\mathbb{R}^{\mathbb{N}_0})^{k} \rightarrow \mathbb{R}, \quad k \in \mathbb{N}, \]
by taking $\tilde{\psi}_1 = \psi_1$ and
\[ \tilde{\psi}_k(\tilde{y}_1, \ldots, \tilde{y}_{k-1}) = \psi_k(\rho(\tilde{y}_1), \ldots, \rho(\tilde{y}_{k-1})), \quad k \geq 2, \]
\[ \tilde{\nu}(g) = \nu(Tg), \]
\[ \tilde{\varphi}_k(\tilde{y}_1, \ldots, \tilde{y}_k) = \varphi(\rho(\tilde{y}_1), \ldots, \rho(\tilde{y}_k)), \quad k \in \mathbb{N}. \]
Put \( \tilde{\psi} = (\tilde{\psi}_k)_{k \in \mathbb{N}} \) and \( \tilde{\varphi} = (\tilde{\varphi}_k)_{k \in \mathbb{N}} \). Then \( \tilde{S}^{\text{int}}_{\tilde{\psi}, \tilde{\varphi}, \tilde{\varphi}} \in \mathcal{A}^{\text{det}}_{\text{int}} \) and by the definition of the mappings \( \tilde{\psi}_k, \tilde{\nu} \) and \( \tilde{\varphi}_k \) we have \( \tilde{S}^{\text{int}}_{\tilde{\psi}, \tilde{\nu}, \tilde{\varphi}}(g) = \tilde{S}^{\text{sde}}_{\psi, \nu, \varphi}(Tg) \) for all \( g \in \mathcal{G}(\mathcal{H}_u) \). Moreover,

\[
\sup_{g \in \mathcal{G}(\mathcal{H}_u)} \tilde{\nu}(g, \cdot) = \sup_{g \in \mathcal{G}(\{(0,1)\} \times \mathcal{F}_u)} \nu(g, \cdot),
\]

which shows \( \text{cost}(\tilde{S}^{\text{int}}_{\tilde{\psi}, \tilde{\nu}, \tilde{\varphi}}) \leq \text{cost}(\tilde{S}^{\text{sde}}_{\psi, \nu, \varphi}) \) and thus finishes the proof of (22).

\[\square\]

**Lemma 12.** Let \( n \in \mathbb{N} \) and put \( z_n = \max(n, u(0)) \). Then there exist functions \( h_{1,+}, h_{1,-}, \ldots, h_{2n,+}, h_{2n,-} \in \mathcal{H}_u \)

with the following properties.

(i) For every \( i = 1, \ldots, 2n \) we have

\[
\{h_{i,+} \neq 0\} \cup \{h_{i,-} \neq 0\} = (u^{-1}(z_n) + i\frac{1}{n}, u^{-1}(z_n) + \frac{i}{n}).
\]

(ii) For all \( \delta_1, \ldots, \delta_{2n} \in \{-, +\} \) we have \( \sum_{i=1}^{2n} h_{i,\delta_i} \in \mathcal{H}_u \) and

\[
\int_{\mathbb{R}} \sin \left( \sum_{i=1}^{2n} h_{i,\delta_i}(x) \right) dx = \sum_{i=1}^{2n} \int_{\mathbb{R}} \sin(h_{i,\delta_i}(x)) dx.
\]

(iii) For every \( i = 1, \ldots, 2n \) we have

\[
\int_{\mathbb{R}} \sin(h_{i,+}(x)) \cdot \exp(-\frac{x^2}{2}) dx \geq \frac{\sin(\frac{1}{12}) \cdot \exp(-4)}{3} \cdot \frac{\exp(-u^{-1}(z_n)^2)}{n}.
\]

**Proof.** Let \( n \in \mathbb{N} \) and define a function \( h : \mathbb{R} \to \mathbb{R} \) by

\[
h(x) = \begin{cases} 
\theta \frac{1}{3n} \frac{1}{12}(x), & \text{if } x \leq \frac{2}{3n}, \\
\frac{1}{12} - \theta \frac{2}{3n} \frac{1}{12}(x), & \text{if } x > \frac{2}{3n}.
\end{cases}
\]

By Lemma 2 we have \( h \in C^\infty(\mathbb{R}; \mathbb{R}) \) with

\[
\{h \neq 0\} = (0, 1/n),
\]

and

\[
0 \leq h \leq \frac{1}{12}
\]

and

\[
\|h'\|_\infty \leq n.
\]

For all \( i \in \{1, \ldots, 2n\} \) and \( \delta \in \{-, +\} \) we define

\[
h_{i,\delta} : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \delta h(x - u^{-1}(z_n) - i\frac{1}{n}).
\]
Clearly, (23) implies Property (i). Moreover, \( h_{i,\delta} \in C^\infty(\mathbb{R}; \mathbb{R}) \), and by (25) and the fact that \( u \) is increasing we get for every \( x \in (u^{-1}(z_n) + \frac{i-1}{n}, u^{-1}(z_n) + \frac{i}{n}) \) that
\[
|h'_{i,\delta}(x)| \leq n \leq z_n = u(u^{-1}(z_n)) \leq u(x),
\]
which proves that \( h_{i,\delta} \in H_u \).

Let \( \delta_1, \ldots, \delta_n \in \{-, + \} \). Then \( \sum_{i=1}^{2n} h_{i,\delta_i} \in C^\infty(\mathbb{R}; \mathbb{R}) \), and using Property (i) as well as (26) we have \( |\sum_{i=1}^{2n} h'_{i,\delta_i}(x)| \leq u(x) \) for every \( x \in \mathbb{R} \). Thus \( \sum_{i=1}^{2n} h_{i,\delta_i} \in H_u \).

Furthermore, Property (i) implies that
\[
\sin \circ \sum_{i=1}^{2n} h_{i,\delta_i} = \sum_{i=1}^{2n} \sin h_{i,\delta_i},
\]
which finishes the proof of Property (ii).

Finally, using Property (i) and (24) we get
\[
\int_{\mathbb{R}} \sin(h_{i,+}(x)) \cdot \exp(-\frac{x^2}{2}) \, dx = \int_{u^{-1}(z_n) + i/n}^{u^{-1}(z_n) + i/n} \sin(h_{i,+}(x)) \cdot \exp(-\frac{x^2}{2}) \, dx
\]
\[
\geq \exp\left(-\frac{(u^{-1}(z_n) + i/n)^2}{2}\right) \cdot \int_{u^{-1}(z_n) + (i-1)/n}^{u^{-1}(z_n) + i/n} \sin(h_{i,+}(x)) \, dx
\]
\[
= \exp\left(-\frac{(u^{-1}(z_n) + i/n)^2}{2}\right) \cdot \int_{0}^{\frac{1}{n}} \sin(h(x)) \, dx
\]
\[
\geq \exp\left(-\frac{(u^{-1}(z_n) + i/n)^2}{2}\right) \cdot \int_{0}^{\frac{2}{3n}} \sin(h(x)) \, dx
\]
\[
= \exp\left(-\frac{(u^{-1}(z_n) + i/n)^2}{2}\right) \cdot \sin\left(\frac{1}{12}\right) \cdot \frac{1}{3n}
\]
\[
\geq \exp(-u^{-1}(z_n)^2 - 4) \cdot \sin\left(\frac{1}{12}\right) \cdot \frac{1}{3n}
\]
for all \( i = 1, \ldots, 2n \), which shows Property (iii) and completes the proof of the lemma.

\[\square\]

**Proposition 4.** For every \( n \in \mathbb{N} \) we have
\[
e^\text{det}_n(G(H_u); \Sigma^{\text{int}}) \geq \sqrt{2} \cdot \sin\left(\frac{1}{12}\right) \cdot \exp(-4) \cdot \frac{1}{6\sqrt{\pi}} \cdot \exp(-u^{-1}(\max(n, u(0)))^2).
\]

**Proof.** Let \( n \in \mathbb{N} \). We may apply Proposition 2 with \( m = 2n, b^* = 0 \in \mathbb{R}^{n_0} \) and \( g_{i,\delta} = (h_{i,\delta}^{(k)})_{k \in \mathbb{N}_0} \) for all \( i \in \{1, \ldots, 2n\} \) and \( \delta \in \{+, -\} \) to obtain the desired lower bound. Indeed, Lemma 12(i),(ii) imply that the conditions (i),(ii) in Proposition 2 are
satisfied. Furthermore, by Lemma 12(iii) we obtain

\[
S^{\mathrm{int}}(g_{i,+}) - S^{\mathrm{int}}(g_{i,-}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left( \sin(h_{i,+}(x)) - \sin(h_{i,-}(x)) \right) \exp\left(-\frac{x^2}{2}\right) dx
\]

\[
= \frac{\sqrt{2}}{\sqrt{\pi}} \int_{\mathbb{R}} \sin(h_{i,+}(x)) \exp\left(-\frac{x^2}{2}\right) dx
\]

\[
\geq \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \sin\left(\frac{1}{12}\right) \cdot \exp(-4) \cdot \frac{\exp\left(-\frac{(u-1(\max(n,u(0))))^2}{3}\right)}{n}
\]

for all \( i \in \{1, \ldots, 2n\} \), which completes the proof. \( \square \)

Clearly, Lemma 11 and Proposition 4 jointly imply Theorem 6.

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