Inertial Navigation Method of Spacecraft Based on Geodesic Deviation Equation

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Abstract. A new method of spacecraft inertial navigation is proposed in paper. The method uses the earth gravitational field model and the on-board accelerometer to measure data, so the real-time deviation of the spacecraft relative to the geodetic orbit is calculated. Based on the post-Newtonian approximate gravity theory of general relativity, the high precision geodesic deviation equation around the earth is derived, and the equation of the spacecraft relative to the geodesic orbit is transformed into orbital element space. The position of the spacecraft is determined by calculating the deviation between the spacecraft and the nearest geodetic orbit. Because the reference points are selected different in the adjacent following inertia system during navigation, the accumulation error of the traditional inertial navigation is avoided, and the navigation precision can be effectively improved.

1. Introduction

The military satellite, space station and other important spacecraft are the important guarantee for implementing the space strategic deterrence and the space-day integration military force, which is very high in the measurement accuracy of the navigation system. There are three kinds of space navigation methods for spacecrafts at home and abroad, inertial navigation, satellite navigation and pulsar navigation. Three methods have the advantages and disadvantages. At present, the navigation accuracy is improved by the combination mode navigation or the improvement of the hardware equipment, and there is no new breakthrough in the theoretical algorithm. As far as inertial navigation is concerned, the main problem existing in the existing technology is the acceleration needs to be integrated twice, which leads to the accumulation error with time and affects the long-term positioning accuracy, and the spacecraft navigation model theory involves a wide range of gravitational field distribution and space-time bending effect, in which case the relativity effect can’t be ignored. Based on the post-Newton (PN) approximate gravity theory of general relativity, the follow-up inertial reference system is established around the spacecraft, and the relative state equation of the spacecraft is constructed in the orbital element space by using the time asynchronous measurement method. The reference points in the inertial reference system are adjusted continuously in the navigation process, and the error of time accumulation is avoided, so as to achieve higher navigation accuracy.

2. Inertial Reference System in Gravitational Field

The research is based on the ground-centred reference system (GCRS). Let \( \{ x', x^2, x^3 \} = \{ x, y, z \} \) is the usual spatial coordinate system of GCRS, is the time coordinates of geocentric (TCG), so the 4-dimensional space-time coordinates are \( (ct, r) = (x', \mu = 0,1,2,3) \).

According to the post-Newton (PN) approximate gravity theory of general relativity, if the reference
frame of motion in the gravitational field around the earth is an inertial system, the following two equations must be satisfied:

\[
\frac{D^2 x^\mu}{d\tau^2} = 0, \quad \frac{D S^\mu}{d\tau} = 0, 
\]

in which \( x^\mu \), \( S^\mu \), \( \tau \) are the 4-dimensional space-time coordinate of the inertial observer, the 4-dimensional spin and inherent time, and \( D \) is an absolute differential symbol.

The first formula of equation (1) is geodesic equation. If the gravitational potential produced by the spherical earth, the gravitational potential of the multistage moment of the earth and the external potential produced by the sun and the moon (mainly the tidal potential) is considered, the vector potential of the gravitational field is ignored, then the only geodetic line is determined.

\[
\gamma(t) = \gamma(r, v_0, t),
\]

The second formula of equation (1) is a vector translation condition, after neglecting the relativity effect, the formula degenerates into a classical non-rotational condition \[1,2\].

\[
\frac{dS}{d\tau} = 0.
\]

It can be seen that an inertial reference system is formed, when the observer \( \gamma(t) \), which measures the movement of the earth in the gravitational field around the earth, carrying a spatial frame which is relatively fixed with (or parallel to) the GCRS spatial coordinate axis. This paper uses a measurement of asynchronous time, so that the reference system is always in the neighborhood of the tested spacecraft. So, for the spacecraft, it is the inertial system.

3. Geodetic Deviation Equation

In the following discussion, Einstein’s summation convention is used, that is, the upper and lower indicators repeating means the summation of the index, and the range of values of the index is stipulated as follows:

\[
\begin{align*}
\mu, \nu, \alpha, \beta &= 0, 1, 2, 3, \\
i, j, k, l &= 1, 2, 3, \\
m, n, p, q &= 1, \ldots, 6.
\end{align*}
\]

Let the spacecraft \( S' \) in GCRS and the adjacent geodetic reference points \( S \) of 4-dimensional space-time coordinates are \( (x'^\mu) \) and \( (x^\mu) \), the 4-dimensional displacement of the spacecraft with respect to the reference point is \( \delta x^\mu = x'^\mu - x^\mu \).

\[
\delta t = t' - t, \quad \delta r = r'(t') - r(t).
\]

Accordingly, \( \delta \dot{t} \) and \( \delta v \) are respectively the relative time change rate and relative velocity.

\[
\delta \dot{t} = \frac{d(\delta t)}{d\tau}, \quad \delta \dot{v} = \frac{d(\delta r)}{d\tau} = v'(t') - v(t).
\]

According to the theory of relativity\[3\], the 4-dimensional deviation equation of spacecraft relative to geodesic is got:

\[
\frac{D^2 (\delta x^\mu)}{d\tau^2} + R^\mu_{\nu\rho\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} \delta x^\rho = F^\mu.
\]

in which \( R^\mu_{\nu\rho\sigma} \) is the Riemann curvature tensor of the gravitational field, \( dx^\mu/d\tau \) is the four-dimensional
velocity of the reference point, $F^r$ is a 4-dimensional non-gravity applied to the unit mass of a spacecraft. According to the differential relation between solid and coordinate time,

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt}\right)^2 \left(\frac{d^2}{dt^2} \frac{d}{dt}\right)$$

(8)

using equation (1) and (7) with the upper formula, the three-dimensional relative equation of motion is obtained under the post-Newton approximation as follows [4]:

$$\frac{d^2(\delta r)}{dt^2} = W \cdot \delta r + V \cdot \delta v + 2g\delta i + f.$$  

(9)

And the component of second-order tensor $W$ and $V$ are expressed in the formula:

$$
\begin{align*}
W_i'(r) &= \left(1 + \frac{v_i^2}{c^2}\right) w_k(r) - \frac{2v_i'v_i}{c^2} w_k(r), \\
V_i'(r) &= -\frac{2}{c^2} \left[ \delta^c_i (g \times v)' + 2\delta^c_i (g \cdot v) \right]
\end{align*}
$$

(10)

in which $w_k = \partial_i \hat{\partial}_j w$ is the component of the gravitational gradient tensor of the reference points in GCRS, $g = g_0 + g_x$ is the gravitational field strength of the reference points, $f$ is a three-dimensional Newton non-gravity applied to the unit mass of spacecraft, including solar light pressure, air resistance and driving force, etc. $\epsilon_{ij}$ and $\delta_{ij}$ respectively are three-dimensional Levi-Civita and Kronecker symbols.

Under the condition of $\delta x \ll \delta x'$, the equation (9) and (10) are strictly. When applied to the inertial navigation of spacecraft around the earth, there are something to discuss. The first term on the right of the equation (9) is the main part, which is essentially the difference between the gravitational intensity of the spacecraft and the reference point, when the space displacement is not very small, it should be represented by the following integral:

$$
\int_{x-r}^{x+r} W(x) \cdot dx = \left[ W(r) + \frac{1}{2} (\delta r \cdot \nabla W) \right] \cdot \delta r + R \left( \frac{\mu}{r^3} |\delta r|^3 \right).
$$

(11)

in which $R(x)$ is the remainder of $x$ in the order of magnitude. So actually $|\delta r|$ and its higher order term is omitted of the first item on the right of the equation (9). If $|R| < 10^{-10} \text{m/s}^2$ is required, then the relative displacement should be limited to the $|\delta r| < 10^{-1} \text{m}$. In order to relax the restrictions, and using $W(r)$ replaced the average of the reference point and spacecraft's gravity gradient:

$$\frac{1}{2} \left[ W(r) + W(r') \right] = W(r) + \frac{1}{2} (\delta r \cdot \nabla W) + \cdots.$$  

(12)

So, the order of magnitude of the relative displacement can be relaxed to $|\delta r| < 10^1 \text{m}$. In addition, in the expression of $W_i'$, the factor of $c^2$, is the relativity effect, the order of magnitude is only the gradient of $10^{-4}$. Considering that the accuracy of gravitational gradient measurement is about $10^{10} / \text{s}^2$, these factors can be ignored. So, using the average value of gravity gradient as $W_i'(r)$ in the equation (9).

$$\hat{W}_i(r, r') = \frac{1}{2} \left[ w_{ik}(r) + w_{ik}(r') \right].$$

(13)

This not only does not change the linear relationship between relative acceleration and relative displacement, but also ensures higher position accuracy in the case of large deviation.
The second term on the right of the equation (9) is the Relativism effect, which is of small order of magnitude. The gravitational field on the geodesic can be approximate represented by the spherical gravity field $g_0$:

\[
\frac{2}{c^2} g \times v \approx -\frac{R}{r^3} h.
\]  

(14)

in which $h=r \times v$ is the angular momentum of geodetic orbit, $R_g=2\mu/c^2=8.87$mm is the radius of gravity. For spacecraft navigation, the second factor $(r \cdot v \approx 0)$ can also be omitted, so there is

\[
V'_i(r) = \frac{R}{r^3} \varepsilon \omega h^i.
\]  

(15)

The third term on the right of the equation (9) is due to the relative time change rate $\delta \dot{t}$ between the spacecraft and the reference point, called time change term. The classical relative equation of motion is discussed at the same time $(\delta t=0)$, so there is no time change term for the relative displacement of adjacent spacecraft. But in fact, the time variation term in the relative motion equation is very useful in navigation calculation. In order to reduce the calculation error, we always want to select the smaller distance between reference point and the space point, but the reference point which close to the space point is not necessarily at the same time. Taking into account the time change term, we can compare the relative displacement between the point to be measured and the reference point at different times, and avoid the error caused by the inappropriate selection of the reference point. For example, the periods of orbiting orbit and geodesic orbit are set to be $T'$ and $T$ ($T' \neq T$), if the reference point is selected to synchronize with the spacecraft time, then the relative displacement value of the space will increase gradually. If when spacecraft is at time $t'$, the time of the reference point on the geodetic orbit is $t = t' T / T'$. It is possible that the relative displacement does not exceed the limit condition, in which case the time change rate is $\delta \dot{t} = (T' - T) / T'$. Synthesize the above discussion and define the 6-dimensional state vector

\[
X = \begin{bmatrix} x^i & x^j & x^k & v^i & v^j & v^k \end{bmatrix}^T,
\]  

(16)

then the equation (9) can be expressed in a 6-dimensional form.

\[
\delta \dot{X} = U \cdot \delta X + 2G \delta \dot{t} + F.
\]  

(17)

\[
U(r, r') = \begin{bmatrix} 0_{3 \times 3} & I_{3 \times 3} \\ \tilde{W}(r, r') & V(r) \end{bmatrix},
\]

\[
G(r) = \begin{bmatrix} 0_{3 \times 1} \\ g(r) \end{bmatrix},
\]

\[
F(r') = \begin{bmatrix} 0_{3 \times 1} \\ f(r') \end{bmatrix}.
\]  

(18)

in which the dot sign in the formula indicates the derivation of the TCG, $\tilde{W}(r, r')$ and $V(r)$ is respectively determined by the equation (13) and (15), $\tilde{W}(r')$ and $f(r')$ is quantities related to the state of the spacecraft[4], and is measured by airborne inertial devices of spacecraft.

4. The Relative Motion Equation of Root Element Space

We transform the relative motion equation into the Root element space, because both practice and theory show that it is simpler and more effective to use the root element variable to discuss the relative
motion of the spacecraft [5]. The variable \( \sigma \) is defined as a 6-dimensional root number vector, and the set of all root numbers constitute the root element space.

\[
\sigma = [a \ e \ i \ \Omega \ \omega \ M].
\]  

(19)

For spacecraft navigation, geodetic orbit can be expressed as instantaneous ellipse,

\[
r(t) = a(1 - e^2) \left[ \begin{array}{c}
\cos u \cos \Omega - \sin u \sin \Omega \cos i \\
\cos u \sin \Omega + \sin u \cos \Omega \cos i \\
\sin u \cos i
\end{array} \right] (t),
\]  

(20)

where \( u = f + \omega \), all root numbers refer to the value of the current time.

\[
\sigma(t) = \sigma(t_0) + \int_{t_0}^{t} \dot{\sigma} dt,
\]  

(21)

The rate of change is determined by the Gaussian perturbation equation, the perturbation force \( g_x \) is the gravity other than the spherical gravity. According to the existing gravitational field model, the geodetic orbit is accurate.

The transformation relationship of root element space and state space is

\[
\delta X = \frac{\partial X}{\partial \sigma} \cdot \delta \sigma = E \cdot \delta \sigma,
\]  

(22)

in which \( \delta \sigma = \sigma(t') - \sigma(t) \), the variable \( E \) is a \( 6 \times 6 \) transformation matrix \( E^m_n = \partial X^m / \partial \sigma^n \), and its inverse matrix is recorded as \( \hat{E} = E^{-1} \). From the above formula, it can be seen that \( \delta \dot{X} \) is the transformation of the change rate of the state.

\[
\delta \dot{X} = E \cdot \delta \dot{\sigma} + \hat{E} \cdot \delta \sigma,
\]  

(23)

The equations of motion of the root element space are obtained by replacing the 21 and equation (22) with the equation (17).

\[
\delta \dot{\sigma} = \hat{E} \cdot (U \cdot E - \dot{E}) \cdot \delta \sigma + 2 \left( \dot{E} \cdot \dot{G} \right) \delta t + \dot{E} \cdot F,
\]  

(24)

A explanation: according to the Riemann contact in the 6-dimensional state space, the connection of the root number space can be obtained.

\[
\Gamma_p^{nm}(\sigma) = \hat{E}_q^p \frac{\partial E^q_m}{\partial \sigma^n},
\]  

(25)

in which \( \delta X \) is small, the difference between the gravity gradient of the spacecraft and the reference point is ignored, that is the formula \( \vec{W}(r, r') \approx \vec{W}(r) \), then the equation (17) of the state space can be obtained.

\[
U = \hat{E} \cdot \delta X = \frac{\partial}{\partial X} \left( \delta \cdot E \right),
\]  

(26)

Under this condition, the coefficient of the first term to the right of the equation (24) can also be written as following:

\[
\hat{E} \cdot (U \cdot E - \dot{E}) = \nabla \delta - \delta \cdot \Gamma(\sigma) = \frac{\partial \delta}{\partial \sigma},
\]  

(27)
in which the variable $\nabla$ is the absolute derivative symbol of the root element space, $\dot{\sigma}$ is the rate of change of the number on the reference track. It can be seen from here that the reason why we do not adopt $\frac{\partial \hat{\sigma}}{\partial \bar{\sigma}}$ as a coefficient [6], Because the variable $\delta X$ is not smaller, with $U(r, r')$ the inclusion of actual observations $W(r')$ as a coefficient, it is beneficial to improve the measurement accuracy and reduce the accumulation error with time.

5. The State for Spacecraft Navigation

Let discrete $t'_s$ and $t'_r (s = 0, 1, 2, \cdots)$ are time series of the spacecraft and the reference orbit, the former is the true coordinate time (TCG) experienced by the spacecraft, and the latter is the selected reference point coordinate time in geodesic orbit. If we take the spacecraft observation interval $\Delta \tau$ as a constant, the corresponding reference orbital interval is generally variable:

$$
\begin{align*}
\Delta \tau &= t'_s - t'_{s-1} = \text{const}, \\
\Delta t_s &= t_s - t_{s-1} = \left(1 - \delta t_s\right) \Delta \tau.
\end{align*}
$$

Let $\delta t'_s$ as the relative time change rate in the formula.

$$
\delta t'_s = \frac{\delta t_s - \delta t_{s-1}}{\Delta \tau},
$$

So the below formula is the relationship between the number of changes in the spacecraft relative to the reference orbit and the observation time.

$$
\delta \sigma_s = \delta \sigma_{s-1} + \left(\delta \hat{\sigma}_s + \dot{\sigma} \delta t'_s\right) \Delta \tau,
$$

The equation of state of spacecraft is obtained by replacing the equation (24) with the upper formula.

$$
\delta \sigma_s = A \cdot \delta \sigma_{s-1} + B \delta t'_s + C_s,
$$

After formula derivation, specific expressions of each matrix can be obtained. The equations of state are about six root elements, of which the variable $\delta t'_s$ is not a variable, it can be set according to the actual navigation project. In order to ensure the space distance between the point to be measured on the real track and the reference point of the geodesic track is small, We request that $t'_s$ of the spacecraft and $t'_r$ of reference orbit basically has the same angle (not necessarily the same), thus

$$
\delta t'_s = 1 - \frac{\dot{M}'_s}{\dot{M}_{s-1}}.
$$

In the actual navigation process, after the state of $t'_{s-1}$ is determined, the time change rate is obtained from the above formula, At the same time, the current time is obtained corresponding to reference point time.

$$
t_s = t_{s-1} + \left(1 - \delta t'_s\right) \Delta \tau.
$$
This determines the state of the reference point at the current time $\sigma_s$ and its field amount $W(r_j)$ and $g(r_j)$, plus $W(r_j')$ and $f(r_j')$ the measurements of airborne inertial devices. And, the current state of the spacecraft is determined.

$$\sigma_s' = \sigma_s + \delta\sigma_s.$$  \hfill (35)

6. Example Analysis and Conclusion

To verify the feasibility of the method, there is a simple case for preliminary simulation. Only considering the spherical uniform gravitational field, and the measured gravitational gradient of spacecraft is equal to the gravitational gradient of the reference point. Let spacecraft inherent time intervals is 1 second $\Delta \tau = 1s$, meanwhile set non-gravitational interaction per unit mass as $(f_o, f_r, f_z) = 0.01, 10^3, 10^4)$. At the initial time the difference of the half-long axis between the reference point and the spacecraft point is 2cm, $a = 4.237 \times 10^7 m, e = 0.001, i = 0.001\pi$, actually the values can be taken according to the situation. In this case, the spacecraft state can be calculated at any time, the results of the comparison between the true value and the estimated value of the position deviation is shown in Table 1. As a result, the error of the estimated orbit and the real orbit can reach cm order. In addition, when the position deviation is larger, the reference track can be changed. Or according to the non-gravitational force on the actual spacecraft, multiple reference tracks is designed in advance, covering the area that the spacecraft may experience, so the specific design parameters depending on the positioning accuracy of the space craft.

| Table 1. Comparison between estimated and real value. |
|-----------------|-----------------|-----------------|
| Time [s]       | Font Real value [m] | Estimated value [m] |
| 0              | 0                | 0 |
| 300            | 5.79             | 5.77          |
| 600            | 25.42            | 25.45         |
| 1200           | 99.85            | 99.79         |

In this paper, based on the inertial navigation method of geodesic deviation equation, an inertial system is constructed, which is used as the reference for the measurement value of inertial elements on spacecraft. This reference orbit can be kept around the spacecraft and can be switched to another design orbit when the error is larger, similar to the ideal satellite tracking spacecraft. Therefore, the integration of long time or long distance is not necessary in the positioning calculation, which avoids the accumulation error, and can achieve high navigation accuracy.

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