Graph-based Local Elimination Algorithms in Discrete Optimization*

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1 Introduction

The use of discrete optimization (DO) models and algorithms makes it possible to solve many practical problems in scheduling theory, network optimization, routing in communication networks, facility location, optimization in enterprise resource planning, and logistics (in particular, in supply chain management [36]). The field of artificial intelligence includes aspects like theorem proving, SAT in propositional logic (see [23], [50]), robotics problems, inference calculation in Bayesian networks [66], scheduling, and others.

Many real-life DO problems contain a huge number of variables and/or constraints that make the models intractable for currently available DO solvers. \( NP \)-hardness refers to the worst-case complexity of problems. Recognizing problem instances that are better (and easier for solving) than these "worst cases" is a rewarding task given that better algorithms can be used for these easy cases.

Complexity theory has proved that universality and effectiveness are contradictory requirements to algorithm complexity. But the complexity of some class of problems decreases if the class may be divided into subsets and the special structure of these subsets can be used in the algorithm design.

To meet the challenge of solving large scale DO problems (DOPs) in reasonable time, there is an urgent need to develop new decomposition approaches [22], [82], [75]. Large-scale DOPs are characterized not only by huge size but also by special or sparse structure. The block form of many DO problems is usually caused by the weak connectedness of subsystems of real systems. One of the first examples of large sparse linear programming (LP) problems which Dantzig started to study was a class of staircase LP prob-

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lems for dynamic planning [27], [29], [28]. Further examples of staircase linear programs (see Fourer [12]) for multiperiod planning, scheduling, and assignment, and for multistage structural design, are included in a set of staircase test problems collected by Ho & Loute [57]. Staircase linear programs have also been derived in connection with linearly constrained optimal control and stochastic programming [103]. Problems of optimal hotel apartments assignment, linear dynamic programming, labor resources allocation, control on hierarchic structures (usually having tree-like structure), multistage integer stochastic programming, network problems may be considered as examples of DO problems which have staircase structure (see [89], [90]). The well known SAT problem stems from classical investigations by logicians of propositional satisfiability and has over 90 years of history. It is possible to represent a SAT problem as a sparse DO problem [58]. Some applied facility location problems can be formulated as set covering problems, set packing problems, node packing problems [73]. Another class of sparse DO problems is a production lotsizing problem [23]. The frequency assignment problem (FAP) [65] in mobile telephone systems communication is a hard problem as it is closely related to the graph coloring problem. One of the well known decomposition approaches to solving DOPs is Lagrangean decomposition that consists of isolating sets of constraints to obtain separate and easy to solve DO problems. Lagrangean decomposition removes the complicating constraints from the constraint set and inserts them into the objective function. Most Lagrangean decomposition methods deal with special row structures. Block angular structures with complicating variables and with complicating variables and constraints can be decomposed using Benders decomposition [13] and cross decomposition [99]. The Dantzig-Wolfe decomposition principle of LP has its equivalent in integer programming [98]. This approach uses the reformulation that gives rise to an integer master problem, whose typically large number of variables is dealt with implicitly by using an integer programming column generation procedure, also known as branch-and-price algorithm [9] that allows solving large-scale DOPs in recent years. Nemhauser ([74], p. 9) mentioned, however, that

\[ \text{... the overall idea of using branch and bound with linear programming relaxation has not changed.} \]

Usually, DOPs from applications have a special structure, and the matrices of constraints for large-scale problems have a lot of zero elements (sparse matrices). Among decomposition approaches appropriate for solving such problems we mention poorly known local decomposition algorithms using the special block matrix structure of constraints and half-forgotten nonserial dynamic programming algorithms (NSDP) (Berstele & Brioschi [14], [15], [16], Dechter [31], [32], [33], [34], Hooker [58]) which can exploit sparsity in the dependency graph of a DOP and allow to compute a solution in stages such that each of them uses results from previous stages. Recently, there has been growing interest in graph-based approaches to decomposition [19]; one of them is tree decomposition (TD). Courcelle
and Arnborg et al. [6] showed that several NP-hard problems posed in monadic second-order logic can be solved in polynomial time using dynamic programming techniques on input graphs with bounded treewidth. Thus graph-based decomposition approaches have gained importance. Graph-based structural decomposition techniques, e.g., nonserial dynamic programming (NSDP) (Bertele, Brioschi [16], Esogbue & Marks [37], Hooker [58], Martelli & Montanari [68], Mitten & Nemhauser [71], Neumaier & Shcherbina [76], Rosenthal [86], Shcherbina [91]), Wilde & Beightler [101] and its modifications (bucket elimination [32], Seidel's invasion method [87]), tree decomposition combined with dynamic programming [35], [21] and its variants [77], hypertree [47] and hinge decomposition [60], [49] are promising decomposition approaches that allow exploiting the structure of discrete problems in constraint satisfaction (CS) [43] and DO.

It is important that aforementioned methods use just the local information (i.e., information about elements of given element’s neighborhood) in a process of solving discrete problems. It is possible to propose a class of local elimination algorithms as a general framework that allows to calculate some global information about a solution of the entire problem using local computations [62], [63], [95]. Note that a main feature in aforementioned problems is the locality of information, a definition of elements’ neighborhoods and studying them.

The use of local information (see [104], [105], [39], [94], [97]) is very important in studying complex discrete systems and in the development of decomposition methods for solving large sparse discrete problems; these problems simultaneously belong to the fields of discrete optimization [73], [10], [78], [79], [88], artificial intelligence [32], [48], [72], [81], and databases [10]. In linear algebra, multifrontal techniques for solving sparse systems of linear equations were developed (see [85]); these methods are also of the decomposition nature. In [104], local algorithms for computing information are introduced. A local algorithm \( A \) examines the elements in the order specified by an ordering algorithm \( A_\pi \), calculates the function \( \phi \) whose value at each step determines the form of the information marks, and labels the element using local information about the elements in its neighborhood. The function \( \phi \) that induces the algorithm depends on two variables: the first ranges over the set of all elements and the second ranges over the set of neighborhoods. Local decomposition algorithms (see [89], [90]) in DO problems have a specific feature. Namely, rather than calculating predicates, they use Bellman’s optimality principle [12] to find optimal solutions of the subproblems corresponding to blocks of the DO problem. A step of the local algorithm \( A \) changes the neighborhood and replaces the index \( p \) by \( p+1 \) (however, one can increment the index by an arbitrary number replacing \( S_p \) by \( S_{p+\rho} \); at each step of the algorithm, for every fixed set of variables of the boundary ring, the values of the variables of the corresponding neighborhood are stored, which is an important difference of the local algorithm \( A \) from \( A \): information about variables in the solutions
of the subproblems is stored rather than information about the predicates. Zhuravlev proposed to call it **indicator information**.

Tree and branch decomposition algorithms have been shown to be effective for DO problems like the traveling salesman problem [24], frequency assignment [65] etc. (see a survey paper [55]). A paper [4] surveys algorithms that use tree decompositions. Most of works based on tree decomposition approach only present theoretical results [61], see the recent surveys [55], [92]. Thus these methods are not yet recognized tools of operations research practitioners.

Some implementations of NSDP are known [16], [38], however, generally, it remains some "obscure" tool for operations research modelers. Usually, tree decomposition approaches and NSDP are considered in the literature separately, without reference to the close relation between these methods. We try to indicate a close relation between these methods.

A need to solve large-scale discrete problems with special structure using graph-based structural decomposition methods provides the main motivation for this chapter. Here we try to answer a number of questions about tree decomposition and NSDP in solving DO problems. What are they? How and where can they be applied? What consists a connection between different structural decomposition methods, such as tree decomposition and nonserial dynamic programming?

The aim of this paper is to provide a review of structural decomposition methods and to give a unified framework in the form of **local elimination algorithms** [94]. We propose here the general approach which consists of viewing a decomposition of some DO problem as being represented by a DAG whose nodes represent subproblems that only contain local information. The nodes are connected by arcs that represent the dependency of the local information in the subproblems. A subproblem that is higher in the hierarchy may use the information (or knowledge) obtained in the dependent subproblems.

This paper is organized as follows: In section 2 we introduce local elimination algorithms for solving discrete problems. In Section 3 we survey necessary terminology and notions for discrete optimization problems and their graph representations. In Section 4 we consider local variable elimination schemes for solving DO problems with constraints and discuss a classification of dynamic programming (DP) computational procedure. Elimination Game is introduced. Application of the bucket elimination algorithm from CS to solving DO problems is done. Then, in Section 5 we consider a local block elimination scheme and related notions. As a promising abstraction approach of solving DOPs we define clustering that merges several variables into a single meta-variable. This allows us to create a quotient (condensed) graph and apply a local block elimination algorithm. In Section 6 a tree decomposition scheme is introduced. Connection of the local elimination algorithmic schemes with tree decomposition and a way of transforming the DAG of computational local elimination procedure to tree decomposition are discussed.
2 Local elimination algorithms for solving discrete problems

The structure of discrete optimization problems is determined either by the original elements (e.g., variables) with a system of neighborhoods specified for them and with the order of searching through those elements using a local elimination algorithm or by various derived structures (e.g., block or tree-block structures). Both original and derived structures can be specified by the so called structural graph. The structural graph can be the interaction graph of the original elements (for example, between the variables of the problem) or the quotient (condensed) graph. The quotient graph can be obtained by merging a set of original elements (for example, a subgraph) into a condensed element. The original subset (subgraph) that formed the condensed element is called the detailed graph of this element.

A local elimination algorithm (LEA) eliminates local elements of the problem's structure defined by the structural graph by computing and storing local information about these elements in the form of new dependencies added to the problem. Thus, the local elimination procedure consists of two parts:

A. The forward part eliminates elements, computes and stores local solutions, and finally computes the value of the objective function;
B. The backward part finds the global solution of the whole problem using the tables of local solutions; the global solution gives the optimal value of the objective function found while performing the forward part of the procedure.

The LEA analyzes a neighborhood $ Nb(x) $ of the current element $ x $ in the structural graph of the problem, applies an elimination operator (which depends on the particular problem) to that element, calculates the function $ h(Nb(x)) $ that contains local information about $ x $, and finds the local solution $ x^*(Nb(x)) $. Next, the element $ x $ is eliminated, and a clique is created from the elements of $ Nb(x) $. The elimination of elements and the creation of cliques changes the structural graph and the neighborhoods of elements. The backward part of the local elimination algorithm reconstructs the solution of the whole problem based on the local solutions $ x^*(Nb(x)) $.

The algorithmic scheme of the LEA is a DAG in which the vertices correspond to the local subproblems and the edges reflect the informational dependence of the subproblems on each other.

3 Discrete optimization problems and their graph representations

3.1 Notions and definitions

Consider a sparse DOP in the following form
\[ F(x_1, x_2, \ldots, x_n) = \sum_{k \in K} f_k(X^k) \rightarrow \max \] (1)

subject to the constraints

\[ g_i(X_{S_i}) \leq R_i, \quad i \in M = \{1, 2, \ldots, m\}, \] (2)

\[ x_j \in D_j, \quad j \in N = \{1, \ldots, n\}, \] (3)

where \( X = \{x_1, \ldots, x_n\} \) is a set of discrete variables, \( X^k \subseteq \{x_1, x_2, \ldots, x_n\}, k \in K = \{1, 2, \ldots, t\}, t - \) number of components in the objective function, \( S_i \subseteq \{1, 2, \ldots, n\}, R_i \in \{\leq, =, \geq\}, i \in M; D_j \) is a finite set of admissible values of variable \( x_j, \ j \in N \). Functions \( f_k(X^k), \ k \in K \) are called components of the objective function and can be defined in tabular form. We use here the notation: if \( S = \{j_1, \ldots, j_q\} \) then \( X_S = \{x_{j_1}, \ldots, x_{j_q}\} \).

In order to avoid complex notation, without loss of generality, we consider further a DOP with linear constraints and binary variables:

\[ \max_X f(X) = \max_X \sum_{k \in K} f_k(X^k), \] (4)

subject to

\[ A_{iS_i}X_{S_i} \leq b_i, \quad i \in M = \{1, 2, \ldots, m\}, \] (5)

\[ x_j = 0, 1, \quad j \in N = \{1, \ldots, n\}. \] (6)

We shall consider further a linear objective function (7):

\[ f(x_1, \ldots, x_n) = f(X) = C_N X_N = \sum_{j=1}^{n} c_j x_j \rightarrow \max \] (7)

**Definition 1.** Variables \( x \in X \) and \( y \in X \) interact in DOP with constraints (we denote \( x \sim y \)) if they both appear either in the same component of the objective function, or in the same constraint (in other words, if variables are both either in a set \( X^k \), or in a set \( X_{S_i} \)).

Introduce a graph representation of the DOP. Description of the DOP structure may be done with various detailization. The structural graph of the DOP defines which variables are in which constraints. Structure of a DOP can be defined either by interaction graph of initial elements (variables in the DOP) or by various derived structures, e.g., block structures, block-tree structures defined by so called quotient (condensed or compressed [7], [8], [54]) graph.

Concrete choice of a structural graph of the DOP defines different local elimination schemes: nonserial dynamic programming, block decomposition, tree decomposition etc.

If the DOP is divided into blocks corresponding to subsets of variables (meta-variables) or to subsets of constraints (meta-constraints), then block
structure can be described by a structural quotient (condensed) graph, whose meta-nodes correspond to subsets of the variables of blocks and meta-edges correspond to adjacent blocks (see below, in section 5.1).

An interaction graph \[16\] (dependency graph by HOOKER \[58\]) represents a structure of the DOP in a natural way.

**Definition 2.** \[16\]. Interaction graph of the DOP is an undirected graph \(G = (X, E)\), such that

1. Vertices \(X\) of \(G\) correspond to variables of the DOP;  
2. Two vertices of \(G\) are adjacent iff corresponding variables interact.

Further, we shall use the notion of vertices that correspond one-to-one to variables.

**Definition 3.** Set of variables interacting with a variable \(x \in X\) is denoted by \(N_b(x)\) and called the neighborhood of the variable \(x\). For corresponding vertices a neighborhood of a vertex \(x\) is a set of vertices of interaction graph that are linked by edges with \(x\). Denote the latter neighborhood as \(N_{bG}(x)\).

Introduce the following notions:

1. Neighborhood of a set \(S \subseteq X\), \(N_{bG}(S) = \bigcup_{x \in S} N_{bG}(x) \setminus S\).
2. Closed neighborhood of a set \(S \subseteq X\), \(N_{bG}[S] = N_{bG}(S) \cup S\).

### 4 Local variable elimination algorithms in discrete optimization

#### 4.1 Nonserial dynamic programming and classification of DP formulations

NSDP exploits only local computations to solve global discrete optimization problems and is, therefore, a particular instance of local elimination algorithm. It appeared in 1961 with Aris [3] (see [11], [14], [15], [71]) but is poorly known to the optimization community. This approach is used in Artificial Intelligence under the names “Variable Elimination” or “Bucket Elimination” \[32\]. NSDP being a natural and general decomposition approach to sparse problems solving, considers a set of constraints and an objective function as recursively computable function \[58\]. This allows to compute a solution in stages such that each of them uses results from previous stages. This requires a reduced effort to find the solution. Thus, the DP algorithm can be applied to find the optimum of the entire problem by using the connected optimizations of the smaller DO subproblems with the aid of existing optimization solvers.

It is worth noting that NSDP is implicit in Hammer and Rudeanu’s “basic method” for pseudoboolean optimization \[52\]. Crama, Hansen, and Jaudard \[26\] discovered that the basic method can exploit the structure of a
DOP with the usage of so-called co-occurrence graph (interaction graph). It was found that the complexity of the algorithm depends on induced width of this graph, which is defined for a given ordering of the variables. Consideration of the variables in the right order may result in a smaller induced width and faster solution [59].

In [16] mostly DO problems without constraints were considered. Here, we consider an application of NSDP variable elimination algorithm to solving DO problems with constraints.

One of the most useful graph-based interpretations is a representation of computational DP procedure as a direct acyclic graph (DAG) whose vertices are associated with subproblems and whose edges express information interdependence between subproblems.

Every DP algorithm has an underlying DAG structure that usually is implicit [30]: the dependencies between subproblems in a DP formulation can be represented by a DAG. Each node in the DAG represents a subproblem. A directed edge from node A to node B indicates that the solution to the subproblem represented by node A is used to compute the solution to the subproblem represented by node B (Fig. 1). The DAG is explicit only when we have a graph optimization problem (say, a shortest path problem). Having nodes $u_1, \ldots, u_k$ point to $v$ means “subproblem $v$ can only be solved once the solutions to $u_1, \ldots, u_k$ are known” (Fig. 2). Thus, the DP formulation can be described by the DAG of the computational procedure of a DP algorithm (underlying DAG [30]). Li & Wah [100] proposed to classify various DP computational procedures or DP formulations on the basis of the dependencies between subproblems from the underlying DAG.

The nodes of the DAG can be organized into levels such that subproblems at a particular level depend only on subproblems at previous levels. In this case, the DP procedure (formulation) can be categorized as follows. If subproblems at all levels depend only on the results of subproblems at the immediately preceding levels, the procedure (formulation) is called a serial DP procedure (formulation), otherwise, it is called a nonserial DP procedure (formulation).

Example 1. The simplest optimization problem is the serial unconstrained discrete optimization problem [16].
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Problem

Fig. 2. Underlying DAG of subproblems.

\[
\max_X f(X) = \max_X \sum_{i \in K} f_i(X^i),
\]

where \( X = \{x_1, \ldots, x_n\} \) is a set of discrete variables.

\[ K = \{1, 2, \ldots, n-1\}; \quad X^i = \{x_i, x_{i+1}\}. \]

In fig. 3 it is shown an interaction graph of the serial DO problem.

Fig. 3. Interaction graph for the serial formulation of unconstrained DOP.

4.2 Discrete optimization problem with constraints

Consider the DOP (7), (5), (6) and suppose without loss of generality that variables are eliminated in the order \( x_1, \ldots, x_n \). Using the local variable elimination scheme eliminate the first variable \( x_1 \). \( x_1 \) is in a set of constraints with the indices in \( U_1 \):

\[ U_1 = \{i \mid x_1 \in S_i\} \]
Together with \(x_1\), constraints in \(U_1\) contain variables from \(Nb(x_1)\).

The following subproblem \(P_1\) corresponds to the variable \(x_1\) of the DOP:

\[
h_{x_1}(Nb(x_1)) = \max_{x_1} \{c_1x_1 | A_iS_iX_{S_i} \leq b_i, \ i \in U_1, \ x_j = 0,1, \ x_j \in Nb[x_1]\}
\]

Then the initial DOP can be transformed in the following way:

\[
\max_{x_1,\ldots,x_n} \left\{ \sum \{C_NX_N | A_iS_iX_{S_i} \leq b_i, \ i \in M, \ x_j = 0,1, \ j \in N \} \right\} = \\
\max_{x_2,\ldots,x_n} \left\{ \{C_N−\{1\}X_N−\{1\}\} + h_{x_1}(Nb(x_1)) | A_iS_iX_{S_i} \leq b_i, \ i \in M-U_1, \ x_j = 0,1, \ j = 2,\ldots,n \right\}
\]

The last problem has \(n-1\) variables; from the initial DOP were excluded constraints with the indices in \(U_1\) and from the objective function the term \(c_1x_1\); there appeared a new objective function term \(h_{x_1}(Nb(x_1))\). Due to this fact the interaction graph associated with the new problem is changed: a vertex \(x_1\) is eliminated and its neighbors have become connected (due to the appearance a new term \(h_{x_1}(Nb(x_1))\) in the objective). It can be noted that a graph induced by vertices of \(Nb(x_1)\) is complete, i.e. is a clique. Denote the new interaction graph \(G^1\) and find all neighborhoods of variables in \(G^1\). NSDP eliminates the remaining variables one by one in an analogous manner. We have to store tables with optimal solutions at each stage of this process. At the stage \(n\) of the described process we eliminate a variable \(x_n\) and find an optimal value of the objective function. Then a backward step of the local elimination procedure is performed using the tables with solutions.

### 4.3 Elimination game, combinatorial elimination process, and underlying DAG of the LAE computational procedure

Consider a sparse discrete optimization problem (1) — (3) whose structure is described by an undirected interaction graph \(G = (X,E)\). Solve this problem with a local elimination algorithm (LEA). LEA uses an ordering \(\alpha\) of \(X\) \([84]\):

Given a graph \(G = (X,E)\) an ordering \(\alpha\) of \(X\) is a bijection \(\alpha : X \leftrightarrow \{1,2,\ldots,n\}\) where \(n = |X|\).

\(G_\alpha\) and \(X_\alpha\) are correspondingly an ordered graph and an ordered vertex set.

Sometimes the ordering will be denoted as \(x_1,\ldots,x_n\), i.e. \(\alpha(x_i) = i\) and \(i\) will be considered as an index of the vertex \(x_i\).

In \(G_\alpha\), a **monotone neighborhood** \(\overline{Nb_\alpha}(x_i)\) \([18,84]\) of \(x_i \in X\) is a set of vertices **monotonely adjacent** to a vertex \(x_i\), i.e.

\[
\overline{Nb_\alpha}(x_i) = \{x_j \in Nb_\alpha(x_i)|j > i\}.
\]

The graph \(G_\alpha\) \([85]\) obtained from \(G = (X,E)\) by

(i) adding edges so that all vertices in \(Nb_G(x)\) are pairwise adjacent, and

(ii) deleting \(x\) and its incident edges
is the \textit{x–elimination} graph of $G$. This process is called the \textbf{elimination} of the vertex $x$.

Given an ordering $x_1, x_2, \ldots, x_n$, the LEA proceeds in the following way: it subsequently eliminates $x_1, x_2, \ldots, x_n$ in the current graph and computes an associated local information about vertices from $h_{x_i}(Nb(x_i))$ \cite{94}. This can be described by the \textbf{combinatorial elimination process} \cite{53}:

$$G_0 = G, G_1, \ldots, G_{j-1}, G_j, \ldots, G_n$$

where $G_j$ is the $x_j$–elimination graph of $G_{j-1}$ and $G_n = \emptyset$.

The process of interaction graph transformation corresponding to the LEA scheme is known as \textbf{Elimination Game} which was first introduced by Parter \cite{80} as a graph analogy of Gaussian elimination. The input of the elimination game is a graph $G$ and an ordering $\alpha$ of $G$ (i.e. $\alpha(x) = i$ if $x$ is $i$-th vertex in the ordering $\alpha$). Elimination Game according to \cite{53} consists in the following. At each step $i$, the neighborhood of vertex $x_i$ is turned into a clique, and $x_i$ is deleted from the graph. This is referred to as eliminating vertex $x_i$.

We obtain a graph $G^{(i)}$. The filled graph $G^{+}_\alpha = (X, E^{+}_\alpha)$ is obtained by adding to $G$ all the edges added by the algorithm. The resulting filled graph $G^{+}_\alpha$ is a triangulation of $G$ (Fulkerson & Gross \cite{44}), i.e., a chordal graph.

Let us introduce the notion for the \textbf{elimination tree} (etree) \cite{67}. Given a graph $G = (X, E)$ and an ordering $\alpha$, the elimination tree is a directed tree $T_\alpha$ that has the same vertices $X$ as $G$ and its edges are determined by a parent relation defined as follows: the parent $x$ is the first vertex (according to the ordering $\alpha$) of the monotone neighborhood $Nb^+_\alpha(x)$ of $x$ in the filled graph $G^{+}_\alpha$.

Using the parent relation introduced above we can define a directed filled graph $\overrightarrow{G}^{+}_\alpha$.

The underlying DAG of a local variable elimination scheme can be constructed using Elimination Game. At step $i$, we represent the computation of the function $h_{x_i}(Nb_{G_{i-1}}(x_i))$ as a node of the DAG (corresponding to the vertex $x_i$). Then, this node containing variables $(x_i, Nb^{(i-1)}_{G_{i-1}}(x_i))$ is linked with a first $x_j$ (accordingly to the ordering $\alpha$) which is in $Nb_{G^{(i-1)}_{i-1}}(x_i)$.

It is easy to see that the elimination tree is the DAG of the computational procedure of the LEA.

\textbf{Example 2.} Consider a DOP (P) with binary variables:

\begin{align*}
2x_1 &+ 3x_2 + x_3 + 5x_4 + 4x_5 + 6x_6 + x_7 & \rightarrow \text{max} \\
3x_1 &+ 4x_2 + x_3 & \leq 6, \quad (C_1) \\
2x_2 &+ 3x_3 + 3x_4 & \leq 5, \quad (C_2) \\
2x_2 &+ 3x_5 & \leq 4, \quad (C_3) \\
2x_3 &+ 3x_6 + 2x_7 & \leq 5, \quad (C_4) \\
x_j &\geq 0, \; j = 1, \ldots, 7.
\end{align*}
The interaction graph is shown in Fig. 4 (a). Elimination Game results and graphs $G_s^{(i)}$ are in Fig. 5. Associated underlying DAG of NSDP procedure for the variable ordering \{x_5, x_2, x_1, x_3, x_6, x_7\} is shown in Fig. 4 (b).

![Interaction Graph](image)

**Fig. 4.** Elimination tree of the DOP (a) Computing the information while eliminating variables in the LEA computational procedure (b) (example 2).

### 4.4 Bucket elimination

Bucket elimination (BE) is proposed in [32] as a version of NSDP for solving CSPs. Now, we consider a modification of the BE algorithm for solving DOPs. The BE algorithm works as follows: Assume we are given an order $x_1, \ldots, x_n$ of the variables of the DOP. BE starts by creating $n$ ”buckets”, one for each variable $x_j$. BE algorithm uses as input ordered set of variables and a set of constraints. To each variable $x_j$ is corresponded a bucket $\Sigma^{(x_j)}$, i.e., a set of constraints and components of objective function built as follows: In the bucket $\Sigma^{(x_j)}$ of variable $x_j$ we put all constraints that contain $x_j$ but do not contain any variable having a higher index. We now iterate on $j$ from $n$ to 1, eliminating one bucket at a time. Algorithm finds new components of the objective applying so called ”elimination operator” (in our case the latter consists on solving associated DO subproblems) to all constraints and components of the objective function of the bucket under consideration. New components of the objective function reflecting an impact of variable $x_j$ on the rest part of the DO problem, are located in corresponding lower buckets. Consider an application of BE to solving the DOP with constraints from Example 2. We use an elimination ordering $\alpha : \{x_5, x_2, (x_1, x_4), x_3, (x_6, x_7)\}$. Variables $(x_1, x_4)$ shall be eliminated in block since they are indistinguishable. Build buckets (subsets of constraints) beginning from last (due order $\alpha$) block $(x_6, x_7)$. A bucket $\Sigma^{(x_6,x_7)}$ includes all constraints of the DOP containing the variables $x_6$, $x_7$, i.e., the bucket $\Sigma^{(x_6,x_7)}$ consists of constraint $C_4$: $\Sigma^{(x_6,x_7)} = \{C_4\}$. Similarly: $\Sigma^{(x_5)} = \{C_1, C_2\}$, $\Sigma^{(x_1,x_4)} = \emptyset$, $\Sigma^{(x_2)} = \{C_3\}$, $\Sigma^{(x_3)} = \emptyset$. 
We solve a DO subproblem associated with the bucket $\Sigma^{(x_6,x_7)}$:
For each binary assignment $x_3$, we compute values $x_6, x_7$ such that

$$h_{x_6,x_7}(x_3) = \max_{x_6,x_7} \{ 6x_6 + x_7 \mid 2x_3 + 3x_6 + 2x_7 \leq 5, x_j \in \{0,1\} \}.$$
The function \( h_{x_6,x_7}(x_3) \) is placed in the bucket \( \Sigma(x_3) \). Consider the DO subproblem associated with this bucket
\[
\begin{align*}
h_{x_3}(x_1, x_2, x_4) &= \max_{x_3} [x_3 + h_{x_6,x_7}(x_3)] \\
3x_1 + 4x_2 + x_3 &\leq 6, \\
2x_2 + 3x_3 + 3x_4 &\leq 5,
\end{align*}
\]
x_3 = 0, 1, j = 1, 2, 3, 4.

We place the function \( h_{x_3}(x_1, x_2, x_4) \) in the bucket \( \Sigma(x_1,x_4) \) and solve the problem
\[
\begin{align*}
h_{x_1,x_4}(x_2) &= \max_{x_1,x_4} \{2x_1 + 5x_4 + h_{x_3}(x_1, x_2, x_4) \mid x_j \in \{0,1\}\}.
\end{align*}
\]

Build the corresponding table 3.

Function \( h_{x_1,x_4}(x_2) \) is placed in the bucket \( \Sigma(x_2) \). A new DO subproblem left to be solved
\[
\begin{align*}
h_{x_2}(x_5) &= \max_{x_2} \{3x_2 + h_{x_1,x_4}(x_2) \mid 2x_2 + 3x_5 \leq 4, x_j \in \{0,1\}\}
\end{align*}
\]

Place \( h_{x_2}(x_5) \) in the last bucket \( \Sigma(x_5) \). The new subproblem is:
\[
\begin{align*}
h_{x_5} &= \max_{x_5} \{4x_5 + h_{x_2}(x_5) \mid x_j \in \{0,1\}\}.
\end{align*}
\]
its solution is \( h_5 = 18, \ x_5^* = 1 \) and the maximal objective value is 18.
To find the optimal values of the variables, it is necessary to do backward step of the BE procedure: from the last table 4 using \( x_5 = 1 \) we have \( x^*_2 = 0 \). Considering the table 3 we have for \( x_2 = 0 \): \( x^*_1 = 1 \), \( x^*_4 = 1 \). From the table 2: \( x_1 = 1 \), \( x_2 = 0 \), \( x_4 = 1 \) \( x^*_3 = 0 \). Table 1: \( x_3 = 0 \Rightarrow x^*_6 = 1 \), \( x^*_7 = 1 \).

The solution is \((1, 0, 0, 1, 1, 1, 1)\), optimal objective value is 18.

5 Block elimination scheme

5.1 Partitions, clustering, and quotient graphs

The local elimination procedure can be applied to elimination of not only separate variables but also to sets of variables and can use the so-called "elimination of variables in blocks" ([15], [90]), which allows to eliminate several variables in block. Local decomposition algorithm [90] actually implements the local block elimination algorithm. If the DOP is divided into blocks corresponding to subsets of variables (meta-variables), then block structure can be described with the aid of a structural condensed graph whose meta-nodes correspond to subsets of the variables or blocks and meta-edges correspond to adjacent blocks.

Applying the method of merging variables into meta-variables allows to obtain condensed or meta-DOPs which have a simpler structure. If the resulting meta-DOP has a nice structure (e.g., a tree structure) then it can be solved efficiently.

The structural graph of the meta-DOP is obtained by collapsing merged nodes into a single meta-node and connecting the meta-node with all nodes that were adjacent with some of the merged nodes. Such a graph usually is called a quotient graph.

An ordered partition of a set \( X \) is a decomposition of \( X \) into ordered sequence of pairwise disjoint nonempty subsets whose union is all of \( X \). Partitioning is a fundamental operation on graphs. One variant of it is to partition the vertex set \( X \) to three sets \( X = U \cup S \cup W \), such that \( U \) and \( W \) are balanced, meaning that neither of them is too small, and \( S \) is small. Removing \( S \) along with all edges incident on it separates the graph into two connected components. \( S \) is called a separator. In general, graph partitioning is \( NP \)-hard. Since graph partitioning is difficult in general, there is a need for approximation algorithms. A popular algorithm in this respect is MeTiS [70], which has a good implementation available in the public domain.

Taking advantage of indistinguishable variables (two variables are indistinguishable if they have the same closed neighborhood [1], [7], [54], [8]) it is possible to compute a quotient (condensed) graph which is formed by merging all vertices with the same neighborhoods into a single meta-node. Let \( x \) be a block of a graph \( G \) [5], i.e., a maximal set of indistinguishable with \( v \) vertices. Clearly, the blocks of \( G \) partition \( X \) since indistinguishability is an equivalence relation defined on the original vertices.
An equivalence relation on a set induces a partition on it, and also any partition induces an equivalence relation. Given a graph \( \Gamma = (X, E) \), let \( X \) be a partition on the vertex set \( X \):

\[
X = \{x_1, x_2, \ldots, x_m\}.
\]

That is, \( \cup_{i=1}^{m} x_i = X \) and \( x_i \cap x_k = \emptyset \) for \( i \neq k \). We define the quotient graph of \( G \) with respect to the partition \( X \) to be the graph

\[ G/X = (X, E), \]

where \((x_i, x_k) \in E\) if and only if \( \text{Nb}_G(x_i) \cap x_k \neq \emptyset \).

The quotient graph \( G(X, E) \) is an equivalent representation of the interaction graph \( G(X, E) \), where \( X \) is a set of blocks (or indistinguishable sets of vertices), and \( E \subseteq X \times X \) be the edges defined on \( X \). A local block elimination scheme is one in which the vertices of each block are eliminated contiguously \[5\]. As an application of a clustering technique we consider below a block local elimination procedure \[16\] where the elimination of the block (i.e., a subset of variables) can be seen as the merging of its variables into a meta-variable.

The merges done define a so called synthesis tree \[102\] on the variables.

**Definition 4.** A synthesis tree of an initial DOP \( P \) is a tree whose leaves correspond to the variables of the initial DOP \( P \), and where each intermediate node is a meta-variable corresponding to the combination of its children nodes.

Using the synthesis tree it is possible to ”decode” meta-variables and find the solution of the initial DOP.

Consider an ordered partition \( X \) of the set \( X \) of the variables into blocks:

\[
X = (x_1, \ldots, x_p), \quad p \leq n,
\]

where \( x_l = X_{K_l} \) (\( K_l \) is a set of indices corresponding to \( x_l \), \( l = 1, \ldots, p \)). For this ordered partition \( X \), the DOP P; \( \{7\}, \{3\}, \{9\} \) can be solved by the LEA using quotient interaction graph \( G \).

**A. Forward part**

Consider first the block \( x_1 \). Then

\[
\max_X \{ C_N X_N | A_{iS_i}, X_{S_i} \leq b_i, \quad i \in M, \quad x_j = 0,1, \quad j \in N \} =
\max_{X_{K_2}, \ldots, X_{K_p}} \{ C_{N-K_1} X_{N-K_1} + h_1 (\text{Nb}(X_{K_1})) | A_{iS_i}, X_{S_i} \leq b_i, \quad i \in M - U_1, \quad x_j = 0,1, \quad j \in N - K_1 \}
\]

where \( U_1 = \{i : S_i \cap K_1 \neq \emptyset\} \) and

\[
h_1 (\text{Nb}(X_{K_1})) = \max_{X_{K_1}} \{ C_{K_1} X_{K_1} | A_{iS_i}, X_{S_i} \leq b_i, \quad i \in U_1, \quad x_j = 0,1, \quad x_j \in \text{Nb}(x_1) \}. 
\]
The first step of the local block elimination procedure consists of solving, using complete enumeration of $X_{K_1}$, the following optimization problem

\[
h_1(Nb(X_{K_1})) = \max_{X_{K_1}} \{ C_{K_1} X_{K_1} | A_{S_1} X_{S_1} \leq b, \ i \in U_1, x_j = 0, 1, x_j \in Nb(x_1) \},
\]

(8)

and storing the optimal local solutions $X_{K_1}$ as a function of the neighborhood of $X_{K_1}$, i.e., $X_{K_1}^* (Nb(X_{K_1}))$.

The maximization of $f(X)$ over all feasible assignments $Nb(X_{K_1})$, is called the elimination of the block (or meta-variable) $X_{K_1}$. The optimization problem left after the elimination of $X_{K_1}$ is:

\[
\max_{X^{-X_{K_1}}} \{ C_{N-K_1} X_{N-K_1} + h_1(Nb(X_{K_1})) | A_{S_1} X_{S_1} \leq b, \ i \in M - U_1, \\
x_j = 0, 1, j \in N - K_1 \}.
\]

Note that it has the same form as the original problem, and the tabular function $h_1(Nb(X_{K_1}))$ may be considered as a new component of the modified objective function. Subsequently, the same procedure may be applied to the elimination of the blocks – meta-variables $x_2 = X_{K_2}, \ldots, x_p = X_{K_p}$, in turn. At each step $j$ the new component $h_{x_j}$ and optimal local solutions $X_{K_j}^*$ are stored as functions of $Nb(X_{K_j} | X_{K_1}, \ldots, X_{K_{j-1}})$, i.e., the set of variables interacting with at least one variable of $X_{K_j}$ in the current problem, obtained from the original problem by the elimination of $X_{K_1}, \ldots, X_{K_{j-1}}$. Since the set $Nb(X_{K_p} | X_{K_1}, \ldots, X_{K_{p-1}})$ is empty, the elimination of $X_{K_p}$ yields the optimal value of objective $f(X)$.

B. Backward part.

This part of the procedure consists of the consecutive choice of $X_{K_p}^*$, $X_{K_{p-1}}^*, \ldots, X_{K_1}^*$, i.e., the optimal local solutions from the stored tables $X_{K_1}^* (Nb(X_{K_1})), X_{K_2}^* (Nb(X_{K_2} | X_{K_1})), \ldots, X_{K_p}^* | X_{K_{p-1}} \ldots, X_{K_1}$.

Block elimination game and underlying DAG

It is possible to extend EG to the case of the block elimination. The input of extended EG is an initial interaction graph $G$ and a partition $X = \{x_1, \ldots, x_p\}$ of vertices of $G$. At each step $\nu$ $(1 \leq \nu \leq p)$ of EG, the neighborhood $Nb(x_\nu)$ of $x_\nu$ is turned into a clique, and $x_\nu$ is deleted from the graph $G$. The filled graph $G^+_X = (X, E^+)$ is obtained by adding to $G$ all the edges added by the algorithm. The resulting filled graph $G^+_X$ is a triangulation of $G$, i.e., a chordal graph $G^+$. Underlying DAG of the local block elimination procedure contains nodes corresponding to computing of functions $h_{x_\nu} (Nb_{G^+_X}(x_\nu))$ and is a generalized elimination tree.

Example 3. Local block elimination for unconstrained DOP.

Consider an unconstrained DOP

$$\max_X [f_1(x_1, x_2, x_3) + f_2(x_2, x_3, x_4) + f_3(x_2, x_5) + f_4(x_3, x_6, x_7)]$$
where
\[ X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \]

and functions \( f_1, f_2, f_3, f_4 \) are given in the following tables.

| Table 5. \( f_1 \) | Table 6. \( f_2 \) |
|-------------------|-------------------|
| \( x_1 \) \( x_2 \) \( x_3 \) | \( f_1 \) | \( x_1 \) \( x_2 \) \( x_3 \) | \( f_2 \) |
| 0 0 0 2 | 0 0 0 3 |
| 0 0 1 3 | 0 0 1 1 |
| 0 1 0 4 | 0 1 0 5 |
| 0 1 1 0 | 0 1 1 2 |
| 1 0 0 5 | 1 0 0 4 |
| 1 0 1 2 | 1 0 1 1 |
| 1 1 0 4 | 1 1 0 3 |
| 1 1 1 1 | 1 1 1 0 |

| Table 7. \( f_3 \) | Table 8. \( f_4 \) |
|-------------------|-------------------|
| \( x_2 \) \( x_3 \) | \( f_3 \) | \( x_3 \) \( x_6 \) \( x_7 \) | \( f_4 \) |
| 0 0 6 | 0 0 0 5 |
| 0 1 2 | 0 0 1 2 |
| 1 0 4 | 0 1 0 3 |
| 1 1 5 | 0 1 1 4 |
| 1 0 0 2 | 1 0 1 1 |
| 1 1 0 3 | 1 1 1 6 |

Consider an ordered partition of the variables of the set into blocks:

- \( x_1 = \{x_5\} \)
- \( x_2 = \{x_1, x_2, x_4\} \)
- \( x_3 = \{x_6, x_7\} \)
- \( x_4 = \{x_3\} \)

Interaction graph for this problem is the same as in Fig. 4 (a).

For the ordered partition \( X = \{x_1, x_2, x_3, x_4\} \), this unconstrained DO problem may be solved by the LEA. Initial interaction graph with partition presented by dashed lines is shown in Fig. 6 (a), quotient interaction graph is in Fig. 6 (b), and the DAG of the block local elimination computational procedure is shown in Fig. 7.

**A. Forward part**

Consider first the block \( x_1 = \{x_5\} \). Then \( \text{Nb}(x_1) = \{x_2\} \). Solve using complete enumeration the following optimization problem

\[ h_{x_1}(\text{Nb}(x_1)) = \max_{x_5} f_3(x_2, x_5), \]
and store the optimal local solutions $x_1$ as a function of a neighborhood, i.e., $x_1^*(Nb(x_1))$.

Eliminate the block $x_1$ and consider the block $x_2 = \{x_1, x_2, x_4\}$. $Nb(x_2) = \{x_3\}$. Now the problem to be solved is

$$h_{x_2}(x_3) = \max_{x_1, x_2, x_4} \{h_{x_1}(x_2) + f_1(x_1, x_2, x_3) + f_2(x_2, x_3, x_4)\}.$$ 

Build the corresponding table 10.

**Table 9. Calculation of $h_{x_1}(x_2)$**

| $x_2$ | $h_{x_1}(x_2)$ | $x_2^*$ |
|-------|---------------|---------|
| 0     | 6             | 0       |
| 1     | 5             | 1       |

**Table 10. Calculation of $h_{x_2}(x_3)$**

| $x_3$ | $h_{x_2}(x_3)$ | $x_3^*$ |
|-------|---------------|---------|
| 0     | 14            | 1 0 0   |
| 1     | 14            | 0 0 0   |
Eliminate the block \( x_2 \) and consider the block \( x_3 = \{x_6, x_7\} \). The neighbor of \( x_3 \) is \( x_3 \): \( Nb(x_3) = \{x_3\} \). Solve the DOP containing \( x_3 \):

\[
h_{x_3}(x_3) = \max_{x_6, x_7} \{f_4(x_3, x_6, x_7), x_j \in \{0, 1\}\}
\]

and build the table 11.

| \( x_3 \) | \( h_{x_3}(x_3) \) | \( x_6^* \) | \( x_7^* \) |
|-----|-----------------|-----|-----|
| 0   | 5  | 0  | 0  |
| 1   | 6  | 0  | 1  |

Eliminate the block \( x_3 \) and consider the block \( x_4 = \{x_3\} \). \( Nb(x_4) = \emptyset \). Solve the DOP:

\[
h_{x_4} = \max_{x_3} \{h_{x_2}(x_3) + h_{x_3}(x_3), x_j \in \{0, 1\}\} = 20,
\]

where \( x_4^* = 1 \).

**B. Backward part.**

Consecutively find \( x_3^*, x_2^*, x_1^* \), i.e., the optimal local solutions from the stored tables 11, 10, 9:

- \( x_3^*_3 = 1 \Rightarrow x_6^* = 1, x_7^* = 1 \) (table 11);
- \( x_3^*_3 = 1 \Rightarrow x_4^*_1 = 0, x_2^*_2 = 0, x_4^*_4 = 0 \) (table 10);
- \( x_3^*_3 = 0 \Rightarrow x_3^*_4 = 0 \) (table 9).

We found the optimal solution to be \( (0, 0, 1, 0, 0, 1, 1) \), the maximum objective value is 20.

**Example 4. Local block elimination for constrained DOP**

Consider the DOP from example 2 and an ordered partition of the variables of the set into blocks:

\[
x_1 = \{x_5\}, \ x_2 = \{x_1, x_2, x_4\}, \ x_3 = \{x_6, x_7\}, \ x_4 = \{x_3\}.
\]

For the ordered partition \{\( x_1, x_2, x_3, x_4 \)\}, this constrained optimization problem may be solved by the LEA.

**A. Forward part**

Consider first the block \( x_1 = \{x_5\} \). Then \( Nb(x_1) = \{x_2\} \). Solve the following problem containing \( x_5 \) in the objective and the constraints:

\[
h_{x_1}(Nb(x_1)) = \max_{x_5} \{4x_5 \mid 2x_2 + 3x_5 \leq 4, x_j \in \{0, 1\}\}
\]

and store the optimal local solutions \( x_1 \) as a function of a neighborhood, i.e., \( x_1^*(Nb(x_1)) \). Eliminate the block \( x_1 \), and consider the block \( x_2 = \{x_1, x_2, x_4\} \). \( Nb(x_2) = \{x_3\} \). Now the problem to be solved is
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\( h_{x_2}(x_3) = \max_{x_1, x_2, x_4} \{ h_{x_1}(x_2) + 2x_1 + 3x_2 + 5x_4 \} \)

subject to

\[
\begin{align*}
3x_1 + 4x_2 + x_3 & \leq 6, \\
2x_2 + 3x_3 + 3x_4 & \leq 5, \\
x_j & = 0, 1, \ j = 1, 2, 3, 4.
\end{align*}
\]

Build the corresponding table 13.

| \( x_2 \) | \( h_{x_1}(x_2) \) | \( x_1^* \) |
|---|---|---|
| 0 | 4 | 1 |
| 1 | 0 | 0 |

| \( x_3 \) | \( h_{x_2}(x_3) \) | \( x_1^* \) | \( x_2^* \) | \( x_4^* \) |
|---|---|---|---|---|
| 0 | 11 | 1 | 0 | 1 |
| 1 | 6 | 1 | 0 | 0 |

Eliminate the block \( x_2 \) and consider the block \( x_3 = \{x_6, x_7\} \). The neighbor of \( x_3 \) is \( x_3 \): \( Nb(x_3) = \{x_3\} \). Solve the DOP containing \( x_3 \):

\[
h_{x_3}(x_3) = \max_{x_6, x_7} \{ h_{x_2} + x_3 + 6x_6 + x_7 \mid 2x_3 + 3x_6 + 2x_7 \leq 5, x_j \in \{0, 1\} \}
\]

and build the table 14.

| \( x_6 \) | \( h_{x_3}(x_3) \) | \( x_1^* \) | \( x_2^* \) |
|---|---|---|---|
| 0 | 18 | 1 | 1 |
| 1 | 12 | 1 | 0 |

Eliminate the block \( x_3 \) and consider the block \( x_4 = \{x_3\} \). \( Nb(x_4) = \emptyset \). Solve the DOP:

\[
h_{x_4} = \max_{x_3} \{ h_{x_3}(x_3), x_j \in \{0, 1\} \} = 18,
\]

where \( x_3^* = 0 \).

**B. Backward part.**

Consecutively find \( x_4^*, x_2^*, x_1^* \), i.e., the optimal local solutions from the stored tables 14, 13, 12. \( x_3^* = 0 \Rightarrow x_6^* = 1, x_7^* = 1 \) (table 14); \( x_3^* = 0 \Rightarrow x_1^* = 1, x_2^* = 0, x_4^* = 1 \) (table 13); \( x_2^* = 0 \Rightarrow x_5^* = 1 \) (table 12). We found the optimal global solution to be \((1, 0, 0, 1, 1, 1, 1)\), the maximum objective value is 18.

### 6 Tree structural decompositions in discrete optimization

Tree structural decomposition methods use partitioning of constraints and use as a meta-tree a structural graph. Dynamic programming algorithm starts at the leaves of the meta-tree and proceeds from the smaller to the larger subproblems (corresponding to the subtrees) that is to say, bottom-up in the rooted tree.
6.1 Tree decomposition and methods of its computing

Aforementioned facts and an observation that many optimization problems which are hard to solve on general graphs are easy on trees make detection of tree structures in a graph a very promising solution. It can be done with such powerful tool of the algorithmic graph theory as a tree decomposition and the treewidth as a measure for the "tree-likeness" of the graph [83]. It is worth noting that in [56] is discussed a number of other useful parameters like branch-width, rank-width (clique-width) or hypertree-width.

Definition 5. Let \( G = (X, E) \) be a graph. A tree decomposition of \( G \) is a pair \( (T; \mathcal{Y}) \) with \( T = (I; F) \) a tree and \( \mathcal{Y} = \{ y_i \mid I \in I \} \) a family of subsets of \( X \), one for each node of \( T \), such that

- (i) \( \bigcup_{i \in I} y_i = X \),
- (ii) for every edge \( (x, y) \in X \) there is an \( i \in I \) with \( x \in y_i, y \in y_i \),
- (iii) (intersection property) for all \( i, j, l \in I \), if \( i < j < l \), then \( y_i \cap y_l \subseteq y_j \).

Note that tree decomposition uses partition of constraints, i.e., it can be considered as a dual structural decomposition method. The best known complexity bounds are given by the "treewidth" \( tw \) (Robertson, Seymour [83]) of an interaction graph associated with a DOP. This parameter is related to some topological properties of the interaction graph. Tree decomposition and the treewidth (Robertson, Seymour [83]) play a very important role in algorithms, for many \( NP \)-complete problems on graphs that are otherwise intractable become polynomial time solvable when these graphs have a tree decomposition with restricted maximal size of cliques (or have a bounded treewidth [6, 20, 21]). It leads to a time complexity in \( O(n \cdot 2^{tw+1}) \). Tree decomposition methods aim to merge variables such that the meta-graph is a tree of meta-vertices.

The procedure to solve a DO problem with bounded treewidth involves two steps: (1) computation of a good tree decomposition, and (2) application of a dynamic programming algorithm that solves instances of bounded treewidth in polynomial time.

Thus, a tree decomposition algorithm can be applied to solving DOPs using the following steps:

(i) generate the interaction graph for a DOP \( (P) \);
(ii) using an ordering for Elimination Game add edges in the interaction graph to produce a (chordal) filled graph;
(iii) build the elimination tree and information flows (see Fig 4(b));
(iv) identify the maximum cliques, apply an absorption and build subproblems;
(v) produce a tree decomposition;
(vi) solve the DO subproblems for each meta-node and combine the results using LEA.

As finding an optimal tree decomposition is \( NP \)-hard, approximate optimal tree decompositions using triangulation of a given graph are often exploited.
Let us list existing methods of computing tree decomposition using a survey of them in [61]. **Optimal triangulations** algorithms have an exponential time complexity. Unfortunately, their implementations do not have much interest from a practical viewpoint. For example, the algorithm described in [41] has time complexity $O(n^4 \cdot (1.9601^n))$ [61]. A paper [46] has shown that the algorithm proposed in [96] cannot solve small graphs (50 vertices and 100 edges). The approach of [46] using a branch and bound algorithm, seems promising for computing optimal triangulations. **Approximation algorithms** approximate the optimum by a constant factor. Although their complexity is often polynomial in the treewidth [2], this approach seems unusable due to a big hidden constant. **Minimal triangulation** computes a set $C'$ such that, for every subset $C'' \subset C'$, the graph $G' = (X, C \cup C')$ is not triangulated. The algorithms LEX-M [84] and LB [17] have a polynomial time complexity of $O(ne)$ with $e'$ the number of edges in the triangulated graph. **Heuristic triangulation** methods build a perfect elimination order by adding some edges to the initial graph. They can be easily implemented and often do this work in polynomial time without providing any minimality warranty. In practice, these heuristics compute triangulations reasonably close to the optimum [64].

Experimental comparative study of four triangulation algorithms, LEX-M, LB, min-fill and MCS was done in [61]. Min-fill orders the vertices from 1 to $n$ by choosing the vertex which leads to add a minimum number of edges when completing the subgraph induced by its unnumbered neighbors. Paper [61] claims that LB and min-fill obtain the best results.

### 6.2 Computing tree decompositions for NSDP schemes

Given a triangulated (or chordal) graph, the set of its maximal cliques corresponds to the family of subsets associated with a tree decomposition (so called **clique tree** [18]). When we exploit tree decomposition, we only consider approximations of optimal triangulations by clique trees. Hence, the time complexity is then $O(n \cdot 2^{w^+ + 1})$ ($w + 1 \leq w^+ + 1 \leq n$). The space complexity is $O(n \cdot s \cdot 2^s)$ with $s$ the size of the largest minimal separator [61]. Usually, tree decomposition is considered in the literature separately from NSDP issues. But there is a close connection between these two structural decomposition approaches. Moreover, it is easy to see that a tree decomposition can be obtained from the DAG of the computational NSDP procedure (this fact was noted in [63]).

Consider example 2 and build a tree decomposition associated with the corresponding NSDP procedure. Associated underlying DAG of NSDP procedure for the variable ordering $\{x_5, x_2, x_1, x_4, x_3, x_6, x_7\}$ is shown in Fig. 4 (b). As was mentioned above, this underlying DAG of local variable elimination (the elimination tree) is constructed using Elimination Game. A node $i$ of the DAG is containing variables $(\alpha_i, Nb_{G_{i-1}}^{(i-1)}(x_i))$ is linked with the first $x_j$ (accordingly to the ordering $\alpha$) which is in $Nb_{G_{i-1}}^{(i-1)}(x_i)$. Nodes and edges of
desired tree decomposition correspond one-by-one to nodes and edges of the underlying DAG. Each node of the tree decomposition is indeed a meta-node containing a subset of vertices of the interaction graph $G$. This subset induces a subgraph in $G$ that was condensed to generate the meta-node. Restore these subgraphs for each meta-node of the tree decomposition.

**Proposition 1.** Graph structure obtained by this construction from the underlying DAG of the NSDP procedure is a tree decomposition.

Proof is in [63].

In our example 2, we observe that the first (accordingly to ordering $\alpha$) meta-node corresponds to the variable $x_5$ and contains variables (vertices) $x_2, x_5$ (i.e., $x_5 \cup \text{Nb}(x_5)$). Subgraph induced by these vertices can be constructed using the interaction graph $G$ (Fig. 4a). This subgraph is shown in Fig. 8 (a) — the meta-node $y_1$. Next meta-node $y_2$ of the tree decomposition corresponds to the variable $x_2$ and contains variables $x_1, x_2, x_3, x_4$. The corresponding induced subgraph (clique) is shown inside the meta-node $y_2$ in Fig. 9 (a). Continuing in analogous way we obtain the tree decomposition as shown in Fig. 8 (a).

It is easy to see that some cliques in this tree decomposition are not maximal and can be absorbed by other cliques. In the case, when one clique contains another clique, the second clique can be absorbed into the first one. Thus, the clique corresponding to the meta-node $y_2$ is absorbed by clique $y_3$ (we denote a result of absorption as a clique $y_{2,3}$). The clique $y_5$ is absorbed by clique $y_4$ forming a clique $y_{4,5}$. After absorptions done we obtain a clique tree (Fig. 8 (b)) containing only maximal cliques. These maximal cliques correspond to constraints of the DOP. In Fig. 8 (b) maximal cliques and links between them are shown. Local decomposition algorithm [90] that uses a dynamic programming paradigm can be applied to this clique tree. Other possible way of finding the clique tree is using maximal spanning tree in the dual graph.

### 6.3 Applying the local decomposition algorithm to solving DO problem

To describe how tree decompositions are used to solve problems with the local decomposition algorithm, let us assume we find a tree decomposition of a graph $G$. Since this tree decomposition is represented as a rooted tree $T$, the ancestor/descendant relation is well-defined. We can associate to each meta-node $y$ the subgraph of $G$ made up by the vertices in $y$ and all its descendants, and all the edges between those vertices. Starting at the leaves of the tree $T$, one computes information typically stored in a table, in a bottom-up manner for each bag until we reach the root. This information is sufficient to solve the subproblem for the corresponding subgraph. To compute the table for a node of the tree decomposition, we only need the information stored in the tables of the children (i.e. direct descendants) of this node. The DO problem for the
entire graph can then be solved with the information stored in the table of the root of $T$. Consider example 2 and exploit the tree decomposition (clique tree) shown in Fig. 8 (b). Let us solve the subproblem corresponding to the block $y_1$. Since this block is adjacent to the block $y_{2,3}$, we have to solve a DOP with variables $y_1 - y_{2,3}$ for all possible assignments $y_1 \cap y_{2,3}$. Thus, since $y_1 - y_{2,3} = \{x_5\}$ and $y_1 \cap y_{2,3} = \{x_2\}$, then induced subproblem has the form:

$$h_{y_1}(x_2) = \max_{x_5} \{4x_5\}$$

subject to

$$2x_2 + 3x_5 \leq 4, \quad x_j = 0, 1, j \in \{2, 5\}$$

Solution of the problem can be written in a tabular form (see table 15).

| $x_2$ | $h_{y_1}(x_2)$ |
|-------|----------------|
| 0     | 4              |
| 1     | 1              |

Table 15. Calculation of $h_{y_1}(x_2)$

| $x_3$ | $h_{y_{2,3}}(x_3)$ |
|-------|--------------------|
| 0     | 11                 |
| 1     | 6                  |

Table 16. Calculation of $h_{y_{2,3}}(x_3)$

Since $y_{2,3} - y_4 = \{x_1, x_2, x_3, x_4\} - \{x_3, x_6, x_7\} = \{x_1, x_2, x_4\}$ and $y_{2,3} \cap y_4 = \{x_3\}$, next subproblem corresponding to the leaf (or meta-node) $y_{2,3}$ of the clique tree is
Forming a maximal clique by absorption

Fig. 8. Tree decomposition for the NSDP procedure (example 2) before (a) and after absorption (b).
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$$h_{y_2,3}(x_3) = \max_{x_1,x_2,x_4} \{h_{y_1} + 2x_1 + 3x_2 + 5x_4\}$$

subject to

$$3x_1 + 4x_2 + x_3 \leq 6, \quad 2x_2 + 3x_3 + 3x_4 \leq 5, \quad x_j = 0, 1, \ j \in \{1, 2, 3, 4\}$$

Solution of this subproblem is in table 16. The last problem corresponding to the block $y_{4,5}$ left to be solved is:

$$h_{y_{4,5}} = \max_{x_3,x_6,x_7} \{h_{y_{2,3}}(x_3) + x_3 + 6x_6 + x_7\}$$

s.t.

$$2x_3 + 3x_6 + 2x_7 \leq 5, \quad x_j = 0, 1, \ j \in \{3, 6, 7\}$$

Table 17. Calculation of $h_{y_{4,5}}$

| $h_{y_{4,5}}$ | $x_3^*$ | $x_6^*$ | $x_7^*$ |
|--------------|---------|---------|---------|
| 18           | 0       | 1       | 1       |

The maximal objective value is 18. To find the optimal values of the variables, it is necessary to do a backward step of the dynamic programming procedure: from table 17 we have $x_3^* = 0, \ x_6^* = 1, \ x_7^* = 1$. From the table 16 using the information $x_3^* = 0$ we find $x_1^* = 1, \ x_2^* = 0, \ x_4^* = 1$. Considering table 15 we have for $x_3^* = 0$: $x_5^* = 1$. The solution is $(1, 0, 0, 1, 1, 1, 1)$; the maximal objective value is 18.

7 Conclusion

This paper reviews the main graph-based local elimination algorithms for solving DO problems. The main aim of this paper is to unify and clarify the notation and algorithms of various structural DO decomposition approaches. We hope that this will allow us to apply the aforementioned decomposition techniques to develop competitive algorithms which will be able to solve practical real-life discrete optimization problems.

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