NON-SEPARABLE TREE-LIKE BANACH SPACES AND ROSENTHAL’S \( \ell_1 \)-THEOREM

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Abstract. We introduce and investigate a class of non-separable tree-like Banach spaces. As a consequence, we prove that we can not achieve a satisfactory extension of Rosenthal’s \( \ell_1 \)-theorem to spaces of the type \( \ell_1(\kappa) \), for \( \kappa \) an uncountable cardinal.

1. Introduction

Rosenthal’s \( \ell_1 \)-theorem [8] is one of the most remarkable results in Banach space geometry. It provides a fundamental criterion for the embedding of \( \ell_1 \) into Banach spaces.

Theorem 1.1 (Rosenthal’s \( \ell_1 \)-theorem). Let \( (x_n) \) be a bounded sequence in the Banach space \( X \) and suppose that \( (x_n) \) has no weakly Cauchy subsequence. Then \( (x_n) \) contains a subsequence equivalent to the usual \( \ell_1 \)-basis.

A satisfactory extension of Theorem 1.1 to spaces of the type \( \ell_1(\kappa) \), for \( \kappa \) an uncountable cardinal, would be desirable, since it would provide a useful criterion for the embedding of \( \ell_1(\kappa) \) into Banach spaces. Naturally, therefore, R. G. Haydon [6] posed the following problem: Let \( \kappa \) be an uncountable cardinal. Suppose that \( X \) is a Banach space, \( A \) is a bounded subset of \( X \) whose cardinality is equal to \( \kappa \) and such that \( A \) does not contain any weakly Cauchy sequence. Can we deduce that \( A \) has a subset equivalent to the usual \( \ell_1(\kappa) \)-basis?

Before the question was posed, Haydon [5] had already presented a counterexample for the case where the cardinal \( \kappa \) is equal to \( \omega_1 \). A completely different counterexample for the case of \( \omega_1 \) had also been obtained by J. Hagler [3]. Finally, the complete solution to the aforementioned problem was given by C. Gryllakis [2] who proved that the answer is always negative with only one exception, namely when both \( \kappa \) and \( \text{cf}(\kappa) \) are strong limit cardinals.

In this paper, we first introduce for any infinite cardinal \( \kappa \) a tree-like Banach space \( X_\kappa \). Our construction is motivated by the well-known James Tree space (JT) [7] and Hagler Tree space (HT) [3]. We also study in detail various properties of the space \( X_\kappa \) and we mostly focus on a family of continuous functionals defined on \( X_\kappa \).

As a consequence of our investigation we give a very simple answer to Haydon’s problem.

Closing this introductory section, we recall some definitions for the sake of completeness. A sequence \( (x_n)_{n \in \mathbb{N}} \) in a Banach space \( X \) is weakly Cauchy if the scalar sequence \( (f(x_n))_{n \in \mathbb{N}} \) converges for every \( f \) in \( X^* \). A subset \( A \subset X \) with cardinality...
\( \kappa \) is equivalent to the usual \( \ell_1(\kappa) \)-basis if there are constants \( C_1, C_2 > 0 \) such that 
\[ C_1 \sum_{i=1}^{n} |a_i| \leq \| \sum_{i=1}^{n} a_i x_i \| \leq C_2 \sum_{i=1}^{n} |a_i|, \]
for any \( n \in \mathbb{N} \), any \( x_1, x_2, \ldots, x_n \in A \) and any scalars \( a_1, \ldots, a_n \).

Finally, we should mention that this is not the first time non-separable tree-like Banach spaces have been defined (e.g. see \( \text{II} \) and \( \text{III} \)).

2. THE BASIC CONSTRUCTION

Suppose that \( \kappa \) is an infinite cardinal. Then we set
\[
\Gamma = \{0,1\}^\kappa = \{ a : \{ \xi < \kappa \} \rightarrow \{0,1\} \} = \{(a_\xi)_{\xi < \kappa} | a_\xi = 0 \text{ or } 1 \}
\]
\[
\mathcal{D} = \{0,1\}^{<\kappa} = \bigcup \left\{ \{0,1\}^\eta | \operatorname{Ord}(\eta), \eta < \kappa \right\}
\]
\[
= \left\{ (a_\xi)_{\xi < \eta} | \eta \text{ is an ordinal, } \eta < \kappa, \ a_\xi = 0 \text{ or } 1 \right\}.
\]

The set \( \mathcal{D} \) is called the (standard) tree. The elements \( s \in \mathcal{D} \) are called nodes. The elements of the set \( \Gamma = \{0,1\}^\kappa \) are called branches.

If \( s \) is a node and \( s \in \{0,1\}^\eta \), we say that \( s \) is on the \( \eta \)-th level of \( \mathcal{D} \). We denote the level of \( s \) by \( \text{lev}(s) \). The initial segment partial ordering on \( \mathcal{D} \), denoted by \( \preceq \), is defined as follows: if \( s = (a_\xi)_{\xi < \eta_1} \) and \( s' = (b_\xi)_{\xi < \eta_2} \) belong to \( \mathcal{D} \) then \( s \preceq s' \) if and only if \( \eta_1 \leq \eta_2 \) and \( a_\xi = b_\xi \) for any \( \xi < \eta_1 \). We also write \( s < s' \) if \( s \preceq s' \) and \( s \neq s' \). By \( s \perp s' \) we mean that \( s \) and \( s' \) are incomparable, that is neither \( s \preceq s' \) nor \( s' \preceq s \). If \( s \preceq s' \) we say \( s' \) is a follower of \( s \). Further, the nodes \( s \cup \{0\} \) and \( s \cup \{1\} \) are called the successors of \( s \), that is we reserve the word successor as meaning immediate follower. However, we observe that a node does not need to have an immediate predecessor.

A subset \( T \) of \( \mathcal{D} \) is called a subtree if it is order isomorphic to \( \{0,1\}^{<\lambda} \) for some cardinal \( \lambda \leq \kappa \). In this paper, we only use countable subtrees of \( \mathcal{D} \), that is subtrees order isomorphic to \( \{0,1\}^{<\aleph_0} \). In the case \( T \) is countable, we enumerate its elements as \( T = \{t_1, t_2, t_3, \ldots \} \) where \( t_1 \) is the minimum element of \( T \) and for each \( m \in \mathbb{N} \), \( t_{2m}, t_{2m+1} \) are the successors (on the tree \( T \)) of \( t_m \).

A linearly ordered subset \( \mathcal{I} \) of \( \mathcal{D} \) is called a segment if for every \( s < t < s' \), \( t \) is contained in \( \mathcal{I} \) provided that \( s, s' \) belong to \( \mathcal{I} \). Consider now a non-empty segment \( \mathcal{I} \). Let \( \eta_1 \) be the least ordinal such that there exists a node \( s \in \mathcal{D} \) with \( \text{lev}(s) = \eta_1 \) and \( s \in \mathcal{I} \). Suppose further that there are an ordinal \( \eta \) and a node \( s' \) on the \( \eta \)-th level so that \( s \preceq s' \) for every \( s \in \mathcal{I} \). Let \( \eta_2 \) be the least ordinal satisfying this property. Then we say that \( \mathcal{I} \) is an \( \eta_1 \)\(-\eta_2 \) segment. A segment is called initial if \( \eta_1 = 0 \), that is \( \emptyset \in \mathcal{I} \).

We next define admissible families of segments in the sense of Hagler \( \text{III} \). Suppose that \( \{I_j\}_{j=1}^{r} \) is a finite family of segments. This family is called admissible if the following conditions are satisfied:

1. there exist ordinals \( \eta_1 < \eta_2 \) such that \( I_j \) is an \( \eta_1 \)-\( \eta_2 \) segment for each \( j = 1, \ldots, r \);
2. \( I_i \cap I_j = \emptyset \) provided that \( i \neq j \).

Consider now the vector space \( c_{00}(\mathcal{D}) \) of finitely supported functions \( x : \mathcal{D} \rightarrow \mathbb{R} \).

For any segment \( \mathcal{I} \) of \( \mathcal{D} \), we set \( \mathcal{I}^* : c_{00}(\mathcal{D}) \rightarrow \mathbb{R} \) with \( \mathcal{I}^*(x) = \sum_{s \in \mathcal{I}} x(s) \). Then,
for any $x \in c_00(D)$, we define the norm
\[ \|x\| = \sup \left[ \sum_{j=1}^{r} |T_j^r(x)|^2 \right]^{1/2} \]
where the supremum is taken over all finite, admissible families $\{T_j\}_{j=1}^r$ of segments.

The space $X_\kappa$ is the completion of the normed space $(c_00(D), \|\cdot\|)$ we have just defined.

For every node $s \in D$, we define $e_s : D \to \mathbb{R}$ with $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$ otherwise. Clearly, $\|e_s\| = 1$ for any $s \in D$.

We come now to the final definition. Suppose that $\{s_i \mid i \in I\}$ is a family of nodes of the tree $D$. This family is called \textit{strongly incomparable} (see [3]) if the following hold:

1. $s_i \perp s_j$ provided that $i \neq j$;
2. if $\{S_1, \ldots, S_r\}$ is any admissible family of segments, then at most two nodes of the $s_i$’s, $i \in I$, are contained in $S_1 \cup \ldots \cup S_r$.

There is a standard way for constructing strongly incomparable families of nodes. Suppose that $(s_\xi)_{\xi < \eta}$ is a set of nodes, where $\eta < \kappa$, such that $s_0 < s_1 < \ldots$. For any ordinal $\xi < \eta$, let $t_\xi$ be the successor of $s_\xi$ with $t_\xi \perp s_{\xi+1}$. Then, the family $\{t_\xi \mid \xi < \eta\}$ is strongly incomparable.

Concerning strongly incomparable sets of nodes, we quote the following proposition whose proof is straightforward.

**Proposition 2.1.** Suppose that $\{s_i \mid i \in I\}$ is a strongly incomparable set of nodes on the tree $D$. Then the family $\{e_{s_i} \mid i \in I\}$ is equivalent to the usual basis of $c_0(I)$. More precisely, for any $n \in \mathbb{N}$, any $i_1, \ldots, i_n \in I$ and any scalars $a_1, \ldots, a_n$, we have
\[ \max_{1 \leq k \leq n} |a_k| \leq \left\| \sum_{k=1}^{n} a_k e_{s_{i_k}} \right\| \leq \sqrt{2} \max_{1 \leq k \leq n} |a_k|. \]

3. The main results

Suppose that $B = (a_\xi)_{\xi < \kappa} \in \Gamma$ is any branch. Then $B$ can be naturally identified with a maximal segment of $D$, namely $B = \{s_0 < s_1 < \ldots < s_\eta < \ldots\}$ where $s_0 = \emptyset$ and $s_\eta = (a_\xi)_{\xi < \eta}$ for any ordinal $\eta < \kappa$. In Section 2 we defined the linear functional $B^* : c_00(D) \to \mathbb{R}$ by setting $B^*(x) = \sum_{s \in B} x(s)$. Clearly, $\|B^*\| = 1$. This functional can be extended to a bounded functional on $X_\kappa$, having the same norm and which is denoted again by $B^*$. Let also $\Gamma^*$ denote the set which contains the functionals $B^*$ defined above. Then $\Gamma^*$ is a bounded subset of $X_\kappa^*$ whose cardinality is equal to $2^\kappa$.

This section is devoted to the study of the family $\Gamma^*$. Towards this direction, we first prove the following.

**Theorem 3.1.** Suppose that $(B_n)_{n \in \mathbb{N}}$ is a sequence of branches such that $B_n \neq B_m$ for $n \neq m$. Then $(B_n^*)_{n \in \mathbb{N}}$ contains a subsequence equivalent to the usual $l_1$-basis.

**Proof.** Consider the set $\mathcal{A}$ consisting of all ordinals $\eta < \kappa$ which satisfy the following: there are nodes $\varphi \neq t$ with $\text{lev}(\varphi) = \text{lev}(t) = \eta$ and there are positive integers $m_1 \neq m_2$ such that $\varphi \in B_{m_1}$, $t \in B_{m_2}$. Clearly $\mathcal{A}$ is a non-empty set, therefore we can consider its least element, say $\eta$. Then $\eta$ can not be a limit ordinal. Indeed, let $\varphi = (a_\xi)_{\xi < \eta}$ and $t = (b_\xi)_{\xi < \eta}$ be as above. Since $\varphi \neq t$, there exists $\eta_1 < \eta$
with \( a_{\eta_1} \neq b_{\eta_1} \). We set \( \tilde{\varphi} = (a_{\xi})_{\xi < \eta_1 + 1} \) and \( \tilde{t} = (b_{\xi})_{\xi < \eta_1 + 1} \). Now we observe that \( \tilde{\varphi} \neq \tilde{t} \), these nodes are placed on the same level and \( \tilde{\varphi} \leq \varphi, \tilde{t} \leq t \). Hence, \( \tilde{\varphi}, \tilde{t} \in B_{m_1} \), \( t \in B_{m_2} \). By the minimality of \( \eta \), we conclude that \( \eta = \eta_1 + 1 \).

Furthermore, the minimality of \( \eta \) also implies that there exists a node \( s_1 \) on the level \( \eta_1 \), so that \( s_1 \in B_m \), for every \( m \in \mathbb{N} \), and the nodes \( \varphi, t \) on the level \( \eta = \eta_1 + 1 \) are precisely the successors of \( s_1 \). Now, we set \( \varphi_1 = \varphi \) and \( t_1 = t \). We may assume that there are infinitely many terms of the sequence \( (B_m)_{m \in \mathbb{N}} \) which pass through the node \( \varphi_1 \). Then we choose a branch \( B_{t_1} \) passing through the node \( t_1 \) (clearly such a branch does exist). \( B_{t_1} \) is just the first term of the desired subsequence.

We next set \( N_1 = \{ m \in \mathbb{N} \mid m > l_1 \} \) and \( \varphi_1 \in B_m \). Then \( N_1 \) is an infinite subset of \( \mathbb{N} \). Repeating the previous argument to the branches \( (B_m)_{m \in N_1} \), we find an ordinal \( \eta_2 > \eta_1 + 1 \) and a node \( s_2 \) on the \( \eta_2 \)-th level with successors \( \varphi_2 \) and \( t_2 \), such that

- all branches \( B_m, m \in N_1 \), pass through the node \( s_2 \);
- infinitely many branches of the sequence \( (B_m)_{m \in N_1} \) pass through \( \varphi_2 \) and the set \( \{ m \in N_1 \mid t_2 \in B_m \} \) is non-empty.

We also choose a branch \( B_{t_2} \) so that \( t_2 \in B_{t_2} \).

Continue in the obvious manner. We inductively construct a sequence \( s_1 < s_2 < \ldots \) of nodes of \( D \), with the successors of \( s_i \) denoted by \( \varphi_i \) and \( t_i \), and a sequence \( l_1 < l_2 < \ldots \) of positive integers such that the following hold:

1. \( s_1 < \varphi_1 \leq s_2 < \varphi_2 \leq s_3 \ldots \);
2. \( s_i \in B_{t_j} \) for any \( j \geq i \), however the branches \( B_{t_j}, j > i \), pass through the node \( \varphi_i \) while the branch \( B_{t_i} \) passes through the node \( t_i \).

We prove now that the sequence \( (B_{t_m}^\ast)_{m \in \mathbb{N}} \) is equivalent to the usual \( \ell_1 \)-basis. Let \( M \in \mathbb{N} \) and \( a_1, \ldots, a_M \in \mathbb{R} \) be given. We set \( x = \sum_{i=1}^{M} \operatorname{sgn}(a_i) e_{t_i} \). Condition (1) of the above construction implies that the sequence \( (t_i) \) is strongly incomparable. Hence by Proposition 3.1 we have \( \|x\| = \sqrt{2} \). Furthermore, condition (2) implies that \( t_i \in B_{t_j} \cup \{ t_j \mid j \neq i \} \), thus \( B_{t_j}(e_{t_i}) = \delta_{ij} \). Therefore:

\[
\left\| \sum_{i=1}^{M} a_i B_{t_i}^\ast \right\| \geq \frac{1}{\|x\|} \left\| \sum_{i=1}^{M} a_i B_{t_i}^\ast(x) \right\| = \frac{1}{\sqrt{2}} \left\| \sum_{i=1}^{M} a_i \operatorname{sgn}(a_i) \right\| = \frac{1}{\sqrt{2}} \sum_{i=1}^{M} |a_i|.
\]

Clearly, we have \( \| \sum_{i=1}^{M} a_i B_{t_i}^\ast \| \leq \sum_{i=1}^{M} |a_i| \) and the proof is complete.

\[\square\]

**Corollary 3.1.** The set \( \Gamma^\ast \) contains no weakly Cauchy sequence.

We pass now to the second result concerning the set of functionals \( \{ B^\ast \mid B \in \Gamma \} \).

**Theorem 3.2.** There exists no subset of \( \Gamma^\ast \) which is equivalent to the usual \( \ell_1(\kappa^+) \)-basis.

For the proof of the above theorem we need to establish some lemmas. Before proceeding, let us introduce some notation. First of all, if \( A \) is any set, then \( |A| \) denotes the cardinality of \( A \). Suppose now that \( \Delta \subseteq \Gamma \) is a set of branches. For any node \( s \in D \), we denote \( \Delta_s \), the set of all branches \( B \in \Delta \) passing through \( s \), that is \( \Delta_s = \{ B \in \Delta \mid s \in B \} \). We also set \( \Delta_s^\ast = \Delta \setminus \Delta_s = \{ B \in \Delta \mid s \notin B \} \).

**Lemma 3.3.** Let \( \Delta \subseteq \Gamma \) be a set of branches with \( |\Delta| = \kappa^+ \). Then there exists a node \( s \in D \) such that \( |\Delta_{s \cup \{0\}}| = |\Delta_{s \cup \{1\}}| = \kappa^+ \).
Proof. Assume that the assertion is not true. Then for every node $s \in D$ there is a successor $s \cup \{\epsilon\}$ of $s$, where $\epsilon = 0$ or 1, such that $|\Delta_{s \cup \{\epsilon\}}| < \kappa^+$. With this assumption and using transfinite induction we construct a branch $B = \{s_\eta\}_\eta < \kappa = \{s_0 < s_1 < \ldots\}$ with the property that $|\Delta_{s_\eta}| = \kappa^+$ for any $\eta < \kappa$.

We start with $s_0 = \emptyset$. Clearly, $|\Delta_\emptyset| = |\Delta| = \kappa^+$. Suppose now that $\eta$ is an ordinal, $\eta < \kappa$, and we have defined the nodes $\{s_\xi\}_{\xi < \eta}$ with $lev(s_\xi) = \xi$ and $|\Delta_{s_\xi}| = \kappa^+$ for any $\xi < \eta$.

If $\eta = \eta_0 + 1$, then by the inductive hypothesis we have $|\Delta_{s_{\eta_0}}| = \kappa^+$. Clearly, $\Delta_{s_{\eta_0}} = \Delta_{s_{\eta_0} \cup \{0\}} \cup \Delta_{s_{\eta_0} \cup \{1\}}$. Therefore, there exists a successor $s_{\eta_0} \cup \{\epsilon\}$ (where $\epsilon = 0$ or 1) of $s_{\eta_0}$ such that $|\Delta_{s_{\eta_0} \cup \{\epsilon\}}| = \kappa^+$. Let $s_\eta = s_{\eta_0} \cup \{\epsilon\}$.

If $\eta$ is a limit ordinal, we set $s_\eta = \cup_{\xi < \eta} s_\xi$. Then $s_\eta$ is a node on the $\eta$-th level of $D$. It remains to show that $|\Delta_{s_\eta}| = \kappa^+$. Since, $\Delta = \Delta_{s_\eta} \cup \Delta_{s_\eta}^c$, it suffices to prove that $|\Delta_{s_\eta}^c| \leq \kappa$.

Let us consider a branch $B$ belonging to $\Delta_{s_\eta}^c$, that is $s_\eta \notin B$. We also denote $S$ the initial segment $\{s_\xi\}_{\xi \leq \eta}$. We consider now the set $A$ containing all ordinals $\xi \leq \eta$ such that at the $\xi$-th level of $D$, the segments $B$ and $S$ do not pass through the same node. The set $A$ is non-empty as $\eta \in A$. Therefore $A$ has a minimum element, say $\xi_0$. The minimality of $\xi_0$ implies that $\xi_0$ can not be a limit ordinal. Hence $\xi_0 = \xi + 1$. Further, it follows by the minimality of $\xi_0$ that at the level $\xi$, we have $s_\xi \in B$ and $s_\xi \in S$, while at the level $\xi + 1$, $s_{\xi + 1} \in S$ and $s_{\xi + 1} \notin B$. Consequently,

$$\Delta_{s_{\xi}}^c = \cup_{\xi < \eta} \{B \in \Delta \mid s_\xi \in B \text{ and } s_{\xi + 1} \notin B\} = \cup_{\xi < \eta} (\Delta_{s_\xi} \cap \Delta_{s_{\xi + 1}}^c).$$

Observe that $s_{\xi + 1}$ is a successor of $s_\xi$, $|\Delta_{s_\xi}| = |\Delta_{s_{\xi + 1}}| = \kappa^+$ and $\Delta_{s_\xi} \cap \Delta_{s_{\xi + 1}}^c$ consists of all branches $B \in \Delta$ which pass through the other successor of $s_\xi$. By our assumption in the beginning of the proof, we have $|\Delta_{s_\xi} \cap \Delta_{s_{\xi + 1}}^c| \leq \kappa$. Therefore $|\Delta_{s_\xi}^c| \leq \sum_{\xi < \eta} \kappa = \kappa$.

Therefore a branch $B = \{s_\eta\}_\eta < \kappa$ has been constructed with the property $|\Delta_{s_\eta}| = \kappa^+$ for any $\eta < \kappa$. To complete the proof of the lemma, we only need to repeat our last argument. Consider a branch $B \in \Delta$ with $B \neq B$. Let $\xi_0$ be the minimum ordinal such that at the $\xi_0$-th level the branches $B, B$ do not pass through the same node. The minimality of $\xi_0$ implies that $\xi_0 = \xi + 1$, $s_\xi \in B$ and $s_{\xi + 1} \notin B$. Therefore

$$\Delta \subseteq \{B\} \cup \left(\cup_{\xi < \kappa} (\Delta_{s_\xi} \cap \Delta_{s_{\xi + 1}}^c)\right).$$

Since $|\Delta_{s_\xi} \cap \Delta_{s_{\xi + 1}}| \leq \kappa$, it follows that $|\Delta| \leq \kappa$ and we have reached a contradiction. \hfill \Box

**Lemma 3.4.** Let $\Delta \subset \Gamma$ be a set of branches with $|\Delta| = \kappa^+$. Then there exists a countable subtree $T$ of $D$, $T = \{t_1, t_2, t_3, \ldots\}$, such that the following hold:

1. $|\Delta_m| = \kappa^+$ for any node $t_m \in T$;
2. for any node $t_m \in T$ there exists a node $s_m \in D$, so that $t_m \leq s_m$ and $t_{2m}, t_{2m + 1}$ are the successors of $s_m$ (that is, when we look at the tree $D$, then the successors of $t_m$ still remain the successors of some node $s_m \in D$).

**Proof.** Let $t_1 = \emptyset$. By Lemma 3.3 there exists a node $s_1 \in D$, with $t_2 \leq s_1$ such that $|\Delta_{s_1 \cup \{0\}}| = |\Delta_{s_1 \cup \{1\}}| = \kappa^+$. We set $t_2 = s_1 \cup \{0\}$ and $t_3 = s_1 \cup \{1\}$. Then
\(t_2, t_3\) are the successors of \(t_1\) in \(T\) and they are the successors of \(s_1\) when we look at the tree \(D\).

Applying Lemma 3.3 to the family \(\Delta_{s_1 \cup \{0\}} = \Delta_{t_2}\) we find a node \(s_2 \in D\), with \(t_2 \leq s_2\), such that \(|\Delta_{s_2 \cup \{0\}}| = |\Delta_{s_2 \cup \{1\}}| = \kappa^+\). Then the successors of \(t_2\) in \(T\) are the nodes \(t_4 = s_2 \cup \{0\}\) and \(t_5 = s_2 \cup \{1\}\). We continue in the obvious manner. \(\square\)

**Proof of Theorem 3.2.** Assume that \(\Delta \subseteq \Gamma\) is a set of branches with \(|\Delta| = \kappa^+\) and \(\Delta^* = \{B^* \mid B \in \Delta\}\) is equivalent to the usual \(\ell_1(\kappa^+)-\)basis. Then there exists a constant \(\delta > 0\) such that for any \(n \in \mathbb{N}\), any \(B_1, \ldots , B_n \in \Delta\) and any scalars \(a_1, \ldots , a_n\),

\[
\delta \sum_{i=1}^n |a_i| \leq \left\| \sum_{i=1}^n a_i B_i^* \right\| \leq \sum_{i=1}^n |a_i|.
\]

Let \(T\) be the countable subtree of \(D\) given by Lemma 3.4 and let \(n \in \mathbb{N}\) be any positive integer. Then we choose branches \(B_1, \ldots , B_n\) and \(B_{n+1}, \ldots , B_{2n}\) belonging to \(\Delta\) as follows. We work at the \(n\)-th level of \(T\) which consists of the nodes \(t_{2n}, t_{2n+1}, t_{2n+2}, \ldots , t_{2n+1-1}\). If we consider the pair \(t_{2n}, t_{2n+1}\), the construction of the tree \(T\) implies that these nodes are the successors of some node of the tree \(D\). Therefore they belong to the same level of \(D\), say the level \(\xi_1\). Similarly the nodes \(t_{2n+2}, t_{2n+3}\) are placed on the same level of \(D\), say \(\xi_2\), and so on. Finally, let \(\xi_{2^n-1} = lev(t_{2^n+1}) = lev(t_{2^n+1-1})\). We may assume, without loss of generality, that \(\xi_1 = \max\{\xi_k \mid 1 \leq k \leq 2^{n-1}\}\). Then we choose branches \(B_1\) and \(B_{n+1}\) of the family \(\Delta\) such that \(B_1\) passes through \(t_{2n}\) and \(B_{n+1}\) passes through \(t_{2n+1}\) (such branches exist by Lemma 3.4). If \(\psi_1\) denotes the immediate predecessor (on the tree \(D\)) of the nodes \(t_{2n}, t_{2n+1}\), then the branches \(B_1, B_{n+1}\) coincide up to the level of \(\psi_1\) and they separate each other at the next level.

The nodes \(t_{2^{n}}, t_{2^{n}+1}\) are followers of the node \(t_2\) in the tree \(T\). We now forget the followers of \(t_2\) and we repeat the previous procedure to the nodes belonging to the \(n\)-th level of \(T\) which are followers of \(t_3\). That is, we detect the pair, say \(t_{2^n+2k}, t_{2^n+2k+1}\), which is placed on the greatest level of \(D\) (if this is not unique, we simply choose one). Then we choose branches \(B_2, B_{n+2}\) belonging to \(\Delta\) such that \(B_2\) passes through the left-hand node of the pair, i.e. the node \(t_{2^n+2k}\), and \(B_{n+2}\) passes through the right-hand node \(t_{2^n+2k+1}\). Let \(\psi_2\) denote the immediate predecessor of \(t_{2^n+2k}, t_{2^n+2k+1}\) on the tree \(D\). Then \(lev(\psi_1) \geq lev(\psi_2)\). The branches \(B_2, B_{n+2}\) coincide up to the level of \(\psi_2\). We also notice that the branches \(B_1, B_2\) separate each other before the level of \(t_2, t_3\) and this happens for the branches \(B_{n+1}, B_{n+2}\). The nodes \(t_{2^n+2k}, t_{2^n+2k+1}\) are followers either of \(t_6\) or \(t_7\). If \(t_6\) is a predecessor of \(t_{2^n+2k}, t_{2^n+2k+1}\), then we forget the followers of \(t_6\) and we continue with the nodes belonging to the \(n\)-th level of \(T\) which are followers of \(t_7\).

After \(n-1\) iterated applications of the previous argument, we find branches \(B_1, \ldots , B_{n-1}\) and \(B_{n+1}, \ldots , B_{2n-1}\) of the family \(\Delta\) and nodes \(\psi_1, \ldots , \psi_{n-1}\) of \(D\). At this stage only one pair of nodes on the \(n\)-th level of \(T\) has been left. Let \(\psi_n\) be the immediate predecessor on \(D\) of these nodes. We choose \(B_n, B_{2n}\) \(\in \Delta\) such that \(B_n\) passes through the left-hand node and \(B_{2n}\) passes through the right-hand node.

Now we observe that the branches \(B_1, \ldots , B_n\) are pairwise disjoint below the level of \(\psi_n\) and this is also true for the branches \(B_{n+1}, \ldots , B_{2n}\). Therefore, if \(\eta_1 = lev(\psi_n)\) and \(\eta_2 = lev(\psi_1)\), then the following hold.

1. All segments \(B_i \cap \{s \mid lev(s) \geq \eta_2 + 1\}, i = 1, 2, \ldots , 2n\), are pairwise disjoint.
(2) The segments \( B_i \cap \{ s \mid \eta_1 + 1 \leq \text{lev}(s) \leq \eta_2 \} \) for \( i = 1, 2, \ldots, n \) are pairwise disjoint. Hence they are admissible \((\eta_1 + 1)-(\eta_2 + 1)\) segments. Similarly, 
\( B_i \cap \{ s \mid \eta_1 + 1 \leq \text{lev}(s) \leq \eta_2 \}, i = n + 1, \ldots, 2n, \) form an admissible family.

(3) \( B_i \cap \{ s \mid \text{lev}(s) \leq \eta_1 \} = B_{n+i} \cap \{ s \mid \text{lev}(s) \leq \eta_1 \} \) for any \( i = 1, 2, \ldots, n \).

Let us also denote \( S_i = B_i \cap \{ s \mid \text{lev}(s) \leq \eta_1 \} \).

After the choice of \((B_i)_{i=1}^{2n}\) has been completed, our next purpose is to estimate the norm of the functional \( \sum_{i=1}^{2n} a_i B_i^* \) for any scalars \( a_1, \ldots, a_{2n} \), and to contradict the assumption that \( \Delta^* \) is equivalent to the usual \( \ell_1(\kappa^+) \)-basis. For this reason, we consider a finitely supported vector \( x = \sum_{s \in B} \lambda_s e_s \in X_\kappa \) with \( \|x\| \leq 1 \). We can write \( x = x_1 + x_2 + x_3 \), where \( x_1 = \sum_{\text{lev}(s) \leq \eta_1} \lambda_s e_s \), \( x_2 = \sum_{\eta_1 + 1 \leq \text{lev}(s) \leq \eta_2} \lambda_s e_s \) and \( x_3 = \sum_{\eta_2 + 1 \leq \text{lev}(s)} \lambda_s e_s \). Clearly, \( \|x_j\| \leq \|x\| = 1 \) for any \( j = 1, 2, 3 \). Then

\[
\left| \sum_{i=1}^{2n} a_i B_i^*(x) \right| \leq \left| \sum_{i=1}^{2n} a_i B_i^*(x_1) \right| + \left| \sum_{i=1}^{2n} a_i B_i^*(x_2) \right| + \left| \sum_{i=1}^{2n} a_i B_i^*(x_3) \right|
\]

Now we have,

\[
\left| \sum_{i=1}^{2n} a_i B_i^*(x_3) \right| \leq \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{2n} |B_i^*(x_3)|^2 \right)^{1/2} \leq \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2}
\]

\[
\left| \sum_{i=1}^{2n} a_i B_i^*(x_2) \right| \leq \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} |B_i^*(x_2)|^2 \right)^{1/2} \leq \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2}
\]

\[
\left| \sum_{i=1}^{2n} a_i B_i^*(x_1) \right| = \left| \sum_{i=1}^{n} (a_i B_i^*(x_1) + a_{n+i} B_{n+i}^*(x_1)) \right|
\]

\[
= \sum_{i=1}^{n} \left| a_i + a_{n+i} \right| S_i^*(x_1) \leq \sum_{i=1}^{n} |a_i + a_{n+i}| S_i^*(x_1) \leq \sum_{i=1}^{n} |a_i + a_{n+i}|.
\]

Summarizing the above, for any finitely supported \( x \in X_\kappa \) with \( \|x\| \leq 1 \) we have

\[
\left| \sum_{i=1}^{2n} a_i B_i^*(x) \right| \leq (\sqrt{2} + 1) \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2} + \sum_{i=1}^{n} |a_i + a_{n+i}|.
\]

Therefore, \( \| \sum_{i=1}^{2n} a_i B_i^* \| \leq (\sqrt{2} + 1) \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2} + \sum_{i=1}^{n} |a_i + a_{n+i}| \). On the other hand, \( \Delta^* \) is equivalent to the usual \( \ell_1(\kappa^+) \)-basis. It follows that

\[
\delta \sum_{i=1}^{2n} |a_i| \leq (\sqrt{2} + 1) \left( \sum_{i=1}^{2n} a_i^2 \right)^{1/2} + \sum_{i=1}^{n} |a_i + a_{n+i}|.
\]

If we choose \( a_1 = \ldots = a_n = 1 \) and \( a_{n+1} = \ldots = a_{2n} = -1 \), then we obtain \( \delta \leq \frac{2}{\sqrt[2n]{2}} \) for any \( n \in \mathbb{N} \) and we reach a contradiction. \( \square \)
4. The non-separable version of Rosenthal’s $\ell_1$-theorem

In this section, we show that we can not achieve a satisfactory extension of Rosenthal’s $\ell_1$-theorem to spaces of the type $\ell_1(\kappa)$, for $\kappa$ an uncountable cardinal. As it was mentioned in the introduction, this extension is possible in only one case, namely when both $\kappa$ and $\text{cf}(\kappa)$ are strong limit cardinals. For the proof of this result we refer to [2] and we shall discuss the other cases.

Suppose first that $\kappa$ is not a strong limit cardinal. This means that there exists a cardinal $\lambda < \kappa$ with $\kappa \leq 2^\lambda$. We now consider the space $X_\lambda$ and the corresponding family of functionals $\Gamma^* \subset X^*_\lambda$. Then, $\Gamma^*$ is a bounded subset of $X^*_\lambda$ whose cardinality is equal to $2^\lambda \geq \kappa$. Further, by Corollary 3.1, the set $\Gamma^*$ contains no weakly Cauchy sequence and, by Theorem 3.2 no subset of $\Gamma^*$ is equivalent to the usual $\ell_1(\kappa)$-basis.

We next consider the case where $\kappa$ is strong limit but $\text{cf}(\kappa)$ is not a strong limit cardinal. This case is not so simple as the previous one, however it is essentially based on the arguments developed in Section 3.

Since $\text{cf}(\kappa)$ is not strong limit, there exists a cardinal $\lambda < \text{cf}(\kappa)$ with $\text{cf}(\kappa) \leq 2^\lambda$. By the definition of $\text{cf}(\kappa)$, there are cardinals $\{\kappa_i \mid i < \text{cf}(\kappa)\}$ such that $\kappa_i < \kappa$, for any ordinal $i < \text{cf}(\kappa)$, and $\kappa = \sum_{i<\text{cf}(\kappa)} \kappa_i$. We next consider the space $X_\kappa$ and we choose a family of branches $A \subset \Gamma$ as follows. We focus on the level $\lambda$ of the tree $D$. This level consists of the nodes $\{0,1\}^\lambda = \{(a_\xi)_{\xi<\lambda} \mid a_\xi = 0 \text{ or } 1\}$. Therefore, there are $2^\lambda$ nodes on the level $\lambda$. Since $\text{cf}(\kappa) \leq 2^\lambda$, we can choose nodes $\{t_i \mid i < \text{cf}(\kappa)\}$ on the level $\lambda$ with $t_i \neq t_j$ provided that $i \neq j$. Now we observe that for any $i < \text{cf}(\kappa)$, the set of all branches passing through the node $t_i$ has cardinality $2^\kappa$. Hence, for any $i < \text{cf}(\kappa)$, we can choose a family of branches $A_i \subset \Gamma$ such that $|A_i| = \kappa_i$ and each branch belonging to $A_i$ passes through the node $t_i$. Finally, let $A = \cup_{i<\text{cf}(\kappa)} A_i$ and let $A^*$ be the family of the corresponding functionals, that is $A^* = \{B^* \mid B \in A\}$.

Clearly, the choice of the family $A$ implies that $|A^*| = |A| = \sum_{i<\text{cf}(\kappa)} \kappa_i = \kappa$. Furthermore, by Corollary 6.1, $A^*$ contains no weakly Cauchy sequence. So, it remains to show that no subset of $A^*$ is equivalent to the usual $\ell_1(\kappa)$-basis.

The proof follows the lines of the proof of Theorem 3.2. We describe briefly the corresponding of Lemma 6.3.

Lemma 4.1. Let $\Delta$ be a subset of $A$ with $|\Delta| = \kappa$. Then there exists a node $s \in D$ such that $\text{lev}(s) < \lambda$ and $|\Delta_{\text{lev}(s)}| = |\Delta_s| = \kappa$. (Recall that $\Delta_s = \{B \in \Delta \mid s \in B\}$.)

Proof. Assuming that the assertion is not true, we construct an initial segment $S = \{s_\eta \mid \eta<\lambda\} = \{s_0 < s_1 < \ldots\}$ such that $|\Delta_{s_\eta}| = \kappa$ for any $\eta < \lambda$. We start with $s_0 = \emptyset$. If $\eta = \eta_0 + 1$, then $s_\eta$ is one of the followers of $s_{\eta_0}$. If $\eta$ is a limit ordinal, then we set $s_\eta = \cup_{\xi<\eta} s_\xi$. Clearly, $s_\eta$ is a node on the $\eta$-th level of $D$. We next show that $\Delta^c_{s_\eta} = \cup_{\xi<\eta} (\Delta^c_{s_\xi} \cap \Delta^c_{s_{\xi+1}})$. Therefore, $|\Delta^c_{s_\eta}| = \sum_{\xi<\eta} |\Delta^c_{s_\xi} \cap \Delta^c_{s_{\xi+1}}| < \kappa$, since $|\Delta_{s_\xi} \cap \Delta^c_{s_{\xi+1}}| < \kappa$ and $\eta < \lambda < \text{cf}(\kappa)$. Hence $|\Delta^c_{s_\eta}| = \kappa$ and this completes the construction of $S$.

Finally, we set $s_1 = \cup_{\xi<\lambda} s_\xi$. Then $s_1$ belongs to the level $\lambda$ and as previously we show $|\Delta_{s_1}| = \kappa$. However, the choice of $A$ indicates that $|\Delta_s| < \kappa$ for any node $s$ on the level $\lambda$ and we have reached a contradiction. \qed
Using Lemma 4.1, we construct a countable subtree $T = \{t_1, t_2, t_3, \ldots\}$ of $D$ such that:

1. $|\Delta_{t_m}| = \kappa$ for any $m = 1, 2, \ldots$ (therefore, $lev(t_m) < \lambda$);
2. the successors $t_{2m}, t_{2m+1}$ of the node $t_m$ are the successors of some node $s_m \in D$.

Finally, we repeat the proof of Theorem 3.2 to show that no subset $\Delta^*$ of $A^*$ is equivalent to the usual $\ell_1(\kappa)$-basis.

5. THE STRUCTURE OF THE SUBSPACES OF $X_\kappa$

The structure of the subspaces of the James Tree space ($JT$) and the Hagler Tree space ($HT$) has been studied extensively, since it has provided answers to several questions about Banach spaces. By analogy, the structure of the subspaces of $X_\kappa$ seems quite interesting. This section is devoted to some remarks concerning this issue.

First of all, $X_\kappa$ contains a lot of subspaces isomorphic to $c_0(\kappa)$. Indeed, let $B = \{s_\eta\}^{\eta < \kappa}$ be any branch and, for any $\eta < \kappa$, let $t_\eta$ be the successor of $s_\eta$ with $t_\eta \neq s_{\eta+1}$. Then $\{t_\eta \mid \eta < \kappa\}$ is a strongly incomparable family of nodes. By Proposition 2.1 it follows that the subspace $span\{e_\eta \mid \eta < \kappa\}$ is isomorphic to $c_0(\kappa)$. Furthermore, it is easy to verify that for any ordinal $\eta < \kappa$ the subspace $span\{e_s \mid s \in \{0,1\}^\eta\}$ is isometrically isomorphic to the space $\ell_2(2^n)$. The main properties of the spaces $JT$ and $HT$ suggest now the following problem about the subspaces of $X_\kappa$.

**Problem.** Is it true that there exists no subspace of $X_\kappa$ isomorphic to $\ell_1(\kappa)$?

Concerning the above problem, we prove a partial result. Assume that $B = \{s_\eta\}^{\eta < \kappa}$ is any branch of the tree $D$. Then we show that the subspace generated by this branch, that is the subspace $span\{e_s\}^{\eta < \kappa}$, does not contain any copy of $\ell_1(\kappa)$.

For our convenience, we first define a Banach space isometrically isomorphic to the subspace generated by any branch. Let $\kappa$ be an infinite cardinal. We consider the vector space $c_000(\{\eta \mid \eta < \kappa\})$ consisting of all finitely supported functions $x : \{\eta \mid \eta < \kappa\} \to \mathbb{R}$. For any $x \in c_000(\{\eta \mid \eta < \kappa\})$, we set

$$
\|x\| = \sup\{|S^*\{x\}|\}
$$

where the supremum is taken over all segments $S \subseteq \{\eta \mid \eta < \kappa\}$. If $E_\kappa$ denotes the completion of the normed space we have just defined, then $E_\kappa$ is isometrically isomorphic to the subspace of $X_\kappa$ generated by any branch.

As usual, for any ordinal $\eta < \kappa$, we consider the vector $e_\eta \in E_\kappa$ with $e_\eta(\xi) = 1$ if $\xi = \eta$ and $e_\eta(\xi) = 0$ otherwise. We now define some projections on the space $E_\kappa$. Let $\eta$ be any ordinal, $\eta < \kappa$. We define $P_\eta : span\{e_\xi\}^{\xi < \kappa} \to span\{e_\xi\}^{\xi < \eta}$ as follows: if $x = \sum_{\xi < \kappa} x(\xi)e_\xi$ is finitely supported, then $P_\eta(x) = \sum_{\xi < \eta} x(\xi)e_\xi$. Clearly, $P_\eta$ is a linear projection with $\|P_\eta\| = 1$. We can also extend $P_\eta$ continuously and we obtain a projection $P_\eta : E_\kappa \to E_\kappa$ onto $span\{e_\xi\}^{\xi < \eta}$ with $\|P_\eta\| = 1$. We next prove the following.

**Proposition 5.1.** The space $E_\kappa$ does not contain any isomorphic copy of $\ell_1(\kappa)$.

**Proof.** Suppose, on the contrary, that $\ell_1(\kappa)$ embeds isomorphically into $E_\kappa$. Then we find a subset $\{x_\xi \mid \xi < \kappa\}$ of $E_\kappa$ which is equivalent to the usual $\ell_1(\kappa)$-basis.
Without loss of generality, we may assume that \( x_\xi \) is finitely supported and \( \|x_\xi\| = 1 \) for any \( \xi < \kappa \).

We inductively construct a sequence \((y_m)_{m=0}^\infty\) belonging to \( \text{span}\{e_\xi\}_{\xi<\kappa}\) with the following properties:

1. \( \|y_m\| = 1 \) for each \( m \);
2. if \( A_m \subset \{\xi < \kappa\} \) is the support of \( y_m \) then \( \max A_m < \min A_{m+1} \) for any \( m \);
3. \((y_m)_{m=0}^\infty\) is a block sequence of \((x_\xi)_{\xi<\kappa}\), that is there are ordinals \( \eta_0 < \eta_1 < \ldots \) so that \( y_m \in \text{span}\{x_\xi \mid \eta_m < \xi < \eta_{m+1}\}\).

We start with \( y_0 = x_0, \eta_0 = 0 \) and \( \eta_1 = 1 \). Let \( \xi_1 = \max A_0 + 1 \). We claim that there exists \( y \in \text{span}\{x_\xi\}_{\xi \geq 1}, \ y \neq 0, \) such that \( P_{\xi_1}(y) = 0 \). Indeed, if we assume that \( P_{\xi_1}(y) \neq 0 \) for all \( y \in \text{span}\{x_\xi\}_{\xi \geq 1}, \ y \neq 0, \) then the linear operator \( P_{\xi_1} : \text{span}\{x_\xi\}_{\xi \geq 1} \to \text{span}\{e_\xi\}_{\xi<\xi_1} \) is one-to-one. Since \( \{x_\xi\}_{\xi \geq 1} \) are linearly independent, it follows that the (algebraic) dimension of the vector space \( \text{span}\{e_\xi\}_{\xi<\xi_1} \) is equal to \( \kappa \), which is a contradiction. Therefore, there is \( y \in \text{span}\{x_\xi\}_{\xi \geq 1} \) such that \( y \neq 0 \) and \( P_{\xi_1}(y) = 0 \). We set \( y_1 = y/\|y\| \). Since \( P_{\xi_1}(y) = 0 \), we have \( \max A_0 < \min A_1 \). Moreover, we can choose an ordinal \( \eta_2 > \eta_1 \) such that \( y \in \text{span}\{x_\xi \mid \eta_1 < \xi < \eta_2\} \). Applying repeatedly the previous argument, we construct the desired sequence \((y_m)_{m=0}^\infty\).

Since \((x_\xi)_{\xi<\kappa}\) is equivalent to the usual \( \ell_1(\kappa)\)-basis, it is easy to verify that the sequence \((y_m)_{m=0}^\infty\) is equivalent to the usual \( \ell_1\)-basis. Furthermore, the sequence \((y_m)_{m=0}^\infty\) belongs to \( \text{span}\{e_\xi \mid \xi \in \cup_{m=0}^\infty A_m\} \). The latter space is isometrically isomorphic to \( E_{80} \), which in turn is isomorphic to \( c_0 \) (see [3]). That is, in a space isomorphic to \( c_0 \) we find a copy of \( \ell_1 \), which is a contradiction.

\[ \square \]

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