TOPOLOGY OF FATOU COMPONENTS FOR ENDOMORPHISMS OF CP^k: LINKING WITH THE GREEN’S CURRENT

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Abstract. Little is known about the global topology of the Fatou set \( U(f) \) for holomorphic endomorphisms \( f : \mathbb{CP}^k \to \mathbb{CP}^k \), when \( k > 1 \). Classical theory describes \( U(f) \) as the complement in \( \mathbb{CP}^k \) of the support of a dynamically-defined closed positive \((1,1)\) current. Given any closed positive \((1,1)\) current \( S \) on \( \mathbb{CP}^k \), we give a definition of linking number between closed loops in \( \mathbb{CP}^k \setminus \text{supp} \) and the current \( S \). It has the property that if \( \text{lk}(\gamma, S) \neq 0 \), then \( \gamma \) represents a non-trivial homology element in \( H_1(\mathbb{CP}^k \setminus \text{supp} S) \).

As an application, we use these linking numbers to establish that many classes of endomorphisms of \( \mathbb{CP}^2 \) have Fatou components with infinitely generated first homology. For example, we prove that the Fatou set has infinitely generated first homology for any polynomial endomorphism of \( \mathbb{CP}^2 \) for which the restriction to the line at infinity is hyperbolic and has disconnected Julia set. In addition we show that a polynomial skew product of \( \mathbb{CP}^2 \) has Fatou set with infinitely generated first homology if some vertical Julia set is disconnected. We then conclude with a section of concrete examples and questions for further study.

1. Introduction

Our primary interest in this paper is the topology of the Fatou set for holomorphic endomorphisms of \( \mathbb{CP}^k \) (written as \( \mathbb{P}^k \) in the remainder of the paper). We develop a type of linking number that in many cases allows one to conclude that a given loop in the Fatou set is homologically non-trivial. One motivation is to find a generalization of the fundamental dichotomy for polynomial (or rational) maps of the Riemann sphere: the Julia set is either connected, or has infinitely many connected components. Further, this type of result paves the way to an exploration of a potentially rich algebraic structure to the dynamics on the Fatou set.

Given a holomorphic endomorphism \( f : \mathbb{P}^k \to \mathbb{P}^k \), the Fatou set \( U(f) \) is the maximal open set on which the iterates \( \{f^n\} \) form a normal family. The Julia set \( J(f) \) is the complement, \( J(f) = \mathbb{P}^k \setminus U(f) \). The standard theory \([14, 21, 35]\) gives a convenient description of these sets in terms the Green’s current \( T \). Specifically, \( T \) is a dynamically defined closed positive \((1,1)\) current with the property that \( J(f) = \text{supp}(T) \). We provide relevant background about the Green’s current in Section 2. Throughout this paper we assume the degree of \( f \) is at least two (i.e. that the components of a lift of \( f \) to \( \mathbb{C}^{k+1} \), with no common factors, have degree at least two).

\[\text{Date: February 10, 2022.}\]
\[\text{^1Research supported in part by a grant from the University of Wisconsin Milwaukee.}\]
\[\text{^2Research was supported in part by startup funds from the Department of Mathematical Sciences at Indiana University Purdue University Indianapolis.}\]
Motivated by this description of the Fatou set, in Section 3 we define a linking number $\text{lk}(\gamma, S)$ between a closed loop $\gamma \subset \mathbb{P}^k \setminus \text{supp} S$ and a closed positive $(1,1)$ current $S$. In Proposition 3.2 we will show that it depends only on the homology class of $\gamma$, and that it defines a homomorphism\
$$\text{lk}(\cdot, S) : H_1(\mathbb{P}^k \setminus \text{supp} S) \to \mathbb{R}/\mathbb{Z}.$$\
In particular, a non-trivial linking number in $\mathbb{R}/\mathbb{Z}$ proves that the homology class of $\gamma$ is non-trivial. The techniques are based on a somewhat similar theory in [32].

This linking number can also be restricted to loops within any open $\Omega \subset \mathbb{P}^k \setminus \text{supp} S$, giving a homomorphism $\text{lk}(\cdot, S) : H_1(\Omega) \to \mathbb{R}/\mathbb{Z}$. If $\Omega$ is the basin of attraction for an attracting periodic point of a holomorphic endomorphism $f : \mathbb{P}^k \to \mathbb{P}^k$ and $S$ is the Green’s current, we will show in Proposition 3.8 that the image of this homomorphism is contained in $\mathbb{Q}/\mathbb{Z}$. This provides a natural setting to show that, under certain hypotheses, the Fatou set $U(f)$ has infinitely generated first homology:

**Theorem 1.1.** Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism and $\Omega \subset U(f)$ is a union of basins of attraction of attracting periodic points for $f$. If there are $c \in H_1(\Omega)$ with linking number $\text{lk}(c, T) \neq 0$ arbitrarily close to 0 in $\mathbb{Q}/\mathbb{Z}$, then $H_1(\Omega)$ is infinitely generated.

(We prove Theorem 1.1 in Section 3) Note that the hypotheses of Theorem 1.1 are satisfied if there are piecewise smooth loops $\gamma \subset \Omega$ with $\text{lk}(\gamma, T) \neq 0$ arbitrarily close to 0 in $\mathbb{Q}/\mathbb{Z}$. In our applications, we often find a loop $\gamma_0$ with nontrivial linking number, and then take an appropriate sequence of iterated preimages $\gamma_n$ under $f^n$ so that $\text{lk}(\gamma_n, T) \to 0$ in $\mathbb{Q}/\mathbb{Z}$.

In order to apply this theory to specific examples, one needs a detailed knowledge of the geometry of the Green’s Current $T$. In the second half of the paper we consider two situations in which it can be readily applied to provide examples of endomorphism $f$ of $\mathbb{P}^2$ having Fatou set $U(f)$ with infinitely generated homology.

The first situation is for polynomial endomorphisms of $\mathbb{P}^2$, that is, holomorphic maps of $\mathbb{P}^2$ that are obtained as the extension a polynomial map $f(z, w) = (p(z, w), q(z, w))$ on $\mathbb{C}^2$. Such mappings (and their generalizations to $\mathbb{P}^k$) were studied in [3]. Given a polynomial endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$, the line at infinity, denoted by $\Pi$, is totally invariant and superattracting. Therefore the restriction of $T$ to $\Pi$ can be understood using the dynamics of the resulting rational map of $f|_\Pi$ and its Julia set $J_\Pi$. In Section 4 we prove the following theorem.

**Theorem 1.2.** Suppose that $f$ is a polynomial endomorphism of $\mathbb{P}^2$ with restriction $f|_\Pi$ to the line at infinity $\Pi$. If $f|_\Pi$ is hyperbolic and $J_\Pi$ is disconnected, then the Fatou set $U(f)$ has infinitely generated first homology.

This theorem provides for many examples of polynomial endomorphisms $f$ of $\mathbb{P}^2$ with interesting homology of $U(f)$. We present one concrete family in Example 4.3.

We then consider the special family of polynomial endomorphisms known as polynomial skew products. While Theorem 1.2 applies to certain polynomial skew products, we develop additional sufficient criteria for $U(f)$ to have interesting homology.

A polynomial skew product is a polynomial endomorphism having the form $f(z, w) = (p(z), q(z, w))$, where $p$ and $q$ are polynomials. We assume that $\text{deg}(p) = \text{deg}(q) = d$ and $p(z) = z^d + O(z^{d-1})$ and $q(z) = w^d + O(z^{d-1})$, where we have normalized leading
Fatou Components for Endomorphism of \( \mathbb{CP}^k \): linking with the Green’s Current coefficients. Since \( f \) preserves the family of vertical lines \( \{ z \} \times \mathbb{C} \), one can analyze \( f \) via the collection of one variable fiber maps \( q_z(w) = q(z, w) \), for each \( z \in \mathbb{C} \). In particular, one can define fiber-wise filled Julia sets \( K_z \) and Julia sets \( J_z := \partial K_z \) with the property that \( w \in \mathbb{C} \setminus K_z \) if and only if the orbit of \( (z, w) \) escapes vertically to a superattracting fixed point \([0 : 1 : 0]\) at infinity.

For this reason, polynomial skew products provide an accessible generalization of one variable dynamics to two variables and have been previously studied by many authors, including Jonsson in [25] and DeMarco, together with the first author of this paper, in [11].

In Section 5 we provide the basic background on polynomial skew products and prove:

**Theorem 1.3.** Suppose \( f(z, w) = (p(z), q(z, w)) \) is a polynomial skew product. 
- If \( J_{z_0} \) is disconnected for any \( z_0 \in J_p \), then \( W^s([0 : 1 : 0]) \) has infinitely generated first homology.
- Otherwise, \( W^s([0 : 1 : 0]) \) is homeomorphic to an open ball.

The first statement is obtained by using Theorem 1.1 while the second is obtained using Morse Theory.

For any endomorphism there is also the measure of maximal entropy \( \mu = T \land T \). Thus another candidate for the name “Julia set” is \( J_2 := \text{supp}(\mu) \). The Julia set that is defined as the complement of the Fatou set is sometimes denoted by \( J_1 \), to distinguish it from \( J_2 \).

The condition from Theorem 1.3 that for some \( z_0 \in \mathbb{C} \), \( J_{z_0} \) is disconnected might seem somewhat unnatural. A seemingly more natural condition might be that \( J_2 \) is disconnected, since for polynomial skew products it is known (see [25]) that \( J_2 = \bigcup_{z \in J_p, J_z} J_z \). However, in Example 6.1 we present certain polynomial skew products with \( J_2 \) connected, but with the Fatou set having infinitely generated first homology. (These examples are obtained by applying Theorem 1.3 to examples from [25] and [11].) In fact, some of these examples persist over an open set within a one-variable holomorphic family of polynomial skew products. Therefore, for polynomial skew products, connectivity of the fiber Julia sets \( J_z \) is at least as important as the connectivity of \( J_2 \) to understanding the homology of the Fatou set.

In Section 6.2 we provide an example of a family of polynomial skew products \( f_a \) depending on a single complex parameter \( a \) with the following property: if \( a \) is in the Mandelbrot set \( \mathcal{M} \), then the Fatou set \( U(f_a) \) is homeomorphic to the union of three open balls, while if \( a \) is outside of \( \mathcal{M} \) then \( H_1(U(f_a)) \) is infinitely generated.

Since neither of the sufficient conditions from Theorems 1.2 and 1.3 extend naturally to general endomorphisms of \( \mathbb{P}^k \), it remains a mystery what is an appropriate condition for endomorphism to have non-simply connected Fatou set. We conclude Section 6 and this paper, with a discussion of a few potential further applications of the techniques of this paper to holomorphic endomorphisms of \( \mathbb{P}^k \).

**Acknowledgments.** We thank John H. Hubbard for bringing us and some central ideas together at the start of this project. We have benefited greatly from discussions with many people, including Eric Bedford, Greg Buzzard, Laura DeMarco, Mattias Jonsson, Sarah Koch, Lex Oversteegen, Rodrigo Perez, Han Peters, Enrique Pujals and Nessim Sibony. The second author thanks Mikhail Lyubich and Ilia Binder for their mathematical guidance and financial support while he was a postdoctoral fellow.

We thank the anonymous referee for many helpful comments, particularly those encouraging us to prove stronger statements in Theorems 1.3 and 6.3.
2. The Green’s current $T$

We provide a brief reminder of the properties of the Green’s current that will be needed later in this paper. We refer the reader who would like to see more details to [14, 21, 36]. While the following construction works more generally for generic (algebraically stable) rational maps having points of indeterminacy, we restrict our attention to globally holomorphic maps of $\mathbb{P}^k$.

Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$ is holomorphic and that the Jacobian of $f$ does not identically vanish on $\mathbb{P}^k$. Then $f$ lifts to a polynomial map $F : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ each of whose coordinates is a homogeneous polynomial of degree $d$ and so that the coordinates do not have a common factor. It is a theorem that

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log ||F^n(z)||$$

converges to a plurisubharmonic function $G : \mathbb{C}^{k+1} \to [-\infty, \infty)$ called the Green’s function associated to $f$. Since $f$ is globally well-defined on $\mathbb{P}^k$ we have that $F^{-1}(0) = \emptyset$. It has been established that $G$ is Holder continuous and locally bounded on $\mathbb{C}^{k+1} \setminus \{0\}$.

If $\pi : \mathbb{C}^{k+1} \setminus \{0\} \to \mathbb{P}^k$ is the canonical projection, there is a unique positive closed $(1,1)$ current $T$ on $\mathbb{P}^k$ satisfying $\pi^*T = \frac{1}{2\pi}dd^c G$. (This normalization is not uniform—many authors do not divide by $2\pi$.) More explicitly, consider any open set $V \subset \mathbb{P}^k$ that is “small enough” so that a holomorphic section $\sigma : V \to \mathbb{C}^{k+1}$ of $\pi$ exists. Then, on $V$ we have that $T$ is given by $T = \frac{1}{2\pi}dd^c(G \circ \sigma)$. Choosing appropriate open sets covering $\mathbb{P}^k$ and sections of $\pi$ on each of them, the result extends to all of $\mathbb{P}^k$ producing a single closed positive $(1,1)$ current on $\mathbb{P}^k$ independent of the choice of open sets and sections used. See [34, Appendix A.4]. By construction, the Green’s current satisfies the invariance $f^*T = d \cdot T$. (See Section 3.3 for the definition of the pull-back $f^*T$.)

Recall that the Fatou set $U(f)$ is the maximal open set in $\mathbb{P}^k$ where the family of iterates $\{f^n\}$ form a normal family and that the Julia set of $f$ is given by $J(f) = \mathbb{P}^k \setminus U(f)$. A major motivation for studying the Green’s current is the following.

**Theorem 2.1.** Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a holomorphic endomorphism and let $T$ be the Green’s current corresponding to $f$. Then, $J(f) = \text{supp } T$.

See, for example, [14, Proposition 4.5] or [36, Theorem 2.2].

**Remark 2.2.** If $f$ is a polynomial endomorphism, another form of Green’s function, given by

$$G_{\text{affine}}(z) = \lim \frac{1}{d^n} \log ||f^n(z)||$$

is often considered in the literature. (Here $\log_+ = \max\{\log, 0\}$.) The result is again a PSH function $G : \mathbb{C}^k \to [0, \infty)$.

We can relate $G_{\text{affine}}$ to $G$ in the following way. Consider the open set $V = \mathbb{C}^k \subset \mathbb{P}^k$. Using the section $\sigma(z_1, \cdots, z_k) = (z_1, \cdots, z_k, 1)$, we find $G_{\text{affine}}(z_1, \cdots, z_k) = G \circ \sigma(z_1, \cdots, z_k)$ because $||F^k \circ \sigma||$ only differs from $||f^k||$ by a bounded amount for each iterate $k$.

Therefore, if $f$ is a polynomial endomorphism of $\mathbb{P}^k$, one can compute $T$ on $\mathbb{C}^k$ using the formula $T = \frac{1}{2\pi}dd^c G_{\text{affine}}$.

\[^{1}\text{We will often use the abbreviation PSH in place of plurisubharmonic and we use the convention that PSH functions cannot be identically equal to } -\infty.\]
Remark 2.3. Note that formulae (1) and (2) are independent of the norm \( \| \cdot \| \) that is used since any two norms are equivalent up to a multiplicative constant.

Remark 2.4. When \( k = 1 \), the resulting Green’s current is precisely the measure of maximal entropy \( \mu_f \) whose support is the Julia set \( J(f) \subset \mathbb{P}^1 \). If \( f \) is a polynomial, then \( \mu_f \) also coincides with the harmonic measure on \( K(f) \), taken with respect to the point at infinity.

3. Linking with a closed positive \((1,1)\) current in \( \mathbb{P}^k \).

Suppose that \( S \) is an (appropriately normalized) closed positive \((1,1)\) current on \( \mathbb{P}^k \) and \( \gamma \subset \mathbb{P}^k \setminus \text{supp}(S) \) is a piecewise smooth closed loop. We will define a linking number \( \text{lk}(\gamma, S) \in \mathbb{R}/\mathbb{Z} \), depending only on the homology class \( [\gamma] \in H_1(\mathbb{P}^k \setminus \text{supp}(S)) \).

3.1. Classical linking numbers in \( \mathbb{S}^3 \). Classically one considers the linking number of two oriented loops \( c \) and \( d \) in \( \mathbb{S}^3 \). The linking number \( \text{lk}(c, d) \in \mathbb{Z} \) is found by taking any oriented surface \( \Gamma \) with oriented boundary \( c \) and defining \( \text{lk}(c, d) \) to be the signed intersection number of \( \Gamma \) with \( d \) as in Figure 1. For this and many equivalent definitions of linking number in \( \mathbb{S}^3 \) see [33, pp. 132-133], [8, pp. 229-239], and [31, Problems 13 and 14].

![Figure 1](image_url)  
Here \( \text{lk}(c, d) = +2 \).

To see that this linking number is well-defined notice that assigning \( \text{lk}(c, d) = [\Gamma] \cdot [d] \), where \( \cdot \) indicates the intersection product on \( H_* (\mathbb{S}^3, c) \), coincides with the classical definition. (For background on the intersection product on homology, see [3] pages 366-372.) If \( \Gamma' \) is any other 2-chain with \( \partial \Gamma' = c \) then \( \partial (\Gamma - \Gamma') = [c] - [c] = 0 \) and \( (\Gamma - \Gamma') \) represents a homology class in \( H_2(\mathbb{S}^3) \). Since \( H_2(\mathbb{S}^3) = 0 \), \( [\Gamma - \Gamma'] = 0 \) forcing \( [\Gamma - \Gamma'] \cdot [d] = 0 \). Therefore: \( [\Gamma] \cdot [d] = [\Gamma'] \cdot [d] \), so that \( \text{lk}(c, d) \) is well defined.

3.2. Generalization. Given any closed positive \((1,1)\) current \( S \) on \( \mathbb{P}^k \) and any piecewise smooth two chain \( \sigma \) in \( \mathbb{P}^k \) with \( \partial \sigma \) disjoint from \( \text{supp} S \), we can define

\[
\langle \sigma, S \rangle = \int_\sigma \eta_S
\]

where \( \eta_S \) is a smooth approximation of \( S \) within it’s cohomology class in \( \mathbb{P}^k - \partial \sigma \), see [16] pages 382-385. The resulting number \( \langle \sigma, S \rangle \) will depend only on the cohomology class of \( S \) and the homology class of \( \sigma \) within \( H_2(\mathbb{P}^k, \partial \sigma) \). (Note that if \( S \) is already a smooth form, one need not require that \( \partial \sigma \) be disjoint from \( \text{supp} S \).)

Notice that \( H_2(\mathbb{P}^k) \) is generated by the class of any complex projective line \( L \subset \mathbb{P}^k \). Since \( S \) is non-trivial, \( \langle L, S \rangle \neq 0 \), so that after an appropriate rescaling we can assume that
\( (L, S) = 1 \). In the remainder of the section we assume this normalization. (It is satisfied by the Green’s Current from Section 2.)

What made the linking numbers in \( S^3 \) well-defined, independent of the choice of \( \Gamma \), is that \( H_2(S^3) = 0 \). One cannot make the immediately analogous definition that \( lk(\gamma, S) = \langle \Gamma, S \rangle \) in \( \mathbb{P}^k \), since \( H_2(\mathbb{P}^k) \neq 0 \) implies that \( \langle \Gamma, S \rangle \) can depend on the choice of \( \Gamma \). For example, given \( \Gamma \) with \( \partial \Gamma = \gamma \) then \( \partial \Gamma' = \gamma \) for \( \Gamma' = \Gamma + L \), however \( \langle \Gamma', S \rangle - \langle \Gamma, S \rangle = \langle L, S \rangle = 1 \neq 0 \).

There is a simple modification: Given any \( \Gamma \) and \( \Gamma' \) both having boundary \( \gamma \), \( [\Gamma' - \Gamma] \in H_2(\mathbb{P}^k) \) so that \( [\Gamma' - \Gamma] \sim k \cdot [L] \) for some \( k \in \mathbb{Z} \). Since \( S \) is normalized, this gives that \( \langle \Gamma', S \rangle = \langle \Gamma, S \rangle \) (mod 1).

**Definition 3.1.** Let \( S \) be a normalized closed positive \((1,1)\) current on \( \mathbb{P}^k \) and let \( \gamma \) be a piecewise smooth closed curve in \( \mathbb{P}^k \setminus \supp(S) \). We define the linking number \( lk(\gamma, S) \) by
\[
lk(\gamma, S) := \langle \Gamma, S \rangle \pmod{1}
\]
where \( \Gamma \) is any piecewise smooth two chain with \( \partial \Gamma = \gamma \).

Unlike linking numbers between closed loops in \( S^3 \), it is often the case that that \( \langle \Gamma, S \rangle \notin \mathbb{Z} \), resulting in non-zero linking numbers (mod 1). See Subsection 3.4 for an explicit example.

**Proposition 3.2.** If \( \gamma_1 \) and \( \gamma_2 \) are homologous in \( H_1(\mathbb{P}^k \setminus \supp(S)) \), then \( lk(\gamma_1, S) = lk(\gamma_2, S) \).

**Proof.** Let \( \Gamma \) be any piecewise smooth two chain contained in \( \mathbb{P}^k \setminus \supp S \) with \( \partial \Gamma = \gamma_1 - \gamma_2 \). Then, since \( \mathbb{P}^k \setminus \supp S \) is open and \( \Gamma \) is compact subset, \( \Gamma \) is bounded away from the support of \( S \). Consequently for any smooth approximation \( \eta_S \) of \( S \) supported in a sufficiently small neighborhood of \( S \), we have \( lk(\gamma_1, S) - lk(\gamma_2, S) = \int_{\Gamma} \eta_T = 0 \). \( \square \)

**Corollary 3.3.** If \( \gamma \in \mathbb{P}^k \setminus \supp S \) with \( lk(\gamma, S) \neq 0 \), then \( \gamma \) is a homologically non-trivial loop in \( \mathbb{P}^k \setminus \supp S \).

Since \( lk(\gamma, S) \) depends only on the homology class of \( \gamma \) and the pairing \( \langle \cdot, S \rangle \) is linear in the space of chains \( \sigma \) (having \( \partial \sigma \) disjoint from \( \supp S \)), the linking number descends to a homomorphism:
\[
lk(\cdot, S) : H_1(\mathbb{P}^k \setminus \supp S) \to \mathbb{R}/\mathbb{Z}.
\]
Similarly \( \lk(\cdot, S) : H_1(\Omega) \to \mathbb{R}/\mathbb{Z} \) for any open \( \Omega \subset \mathbb{P}^k \setminus \supp S \).

**Remark 3.4.** (Topological versus Geometric linking numbers.) The classical linking number, and also Definition 3.1 depend only on the homology class of the loop \( \gamma \) (in the complement of some other loop of the support of some current, respectively.)

A linking number depending on the geometry of \( \gamma \) is given by
\[
\hat{lk}(\gamma, T) := \langle \Gamma, S - \Omega \rangle \in \mathbb{R},
\]
where \( \partial \Gamma = \gamma \) and \( \Omega \) is (normalization of) the Kähler form defining the Fubini-Study metric on \( \mathbb{P}^k \). Given any \( \Gamma \) and \( \Gamma' \) both having boundary \( \gamma \) we have that \( \langle \Gamma - \Gamma', T - \Omega \rangle = 0 \), since \( S \) and \( \Omega \) are cohomologous. (In the language of \( \text{[32]} \), p. 132, we say that \( T - \Omega \) is in the “linking kernel of \( \mathbb{P}^k \).”)

Because \( \supp \Omega = \mathbb{P}^k \), the statement of Proposition 3.2 does not apply. Rather, \( \hat{lk}(\gamma, S) \) depends on the geometry of \( \gamma \subset \mathbb{P}^k \setminus \supp S \). In fact, similar linking numbers were used in \( \text{[17]} \) \( \text{[18]} \) to determine if a given real-analytic \( \gamma \) has the appropriate geometry to be the boundary of a positive holomorphic 1-chain (with bounded mass).
Remark 3.5. (Other manifolds.) Suppose that $M$ is some other compact complex manifold with $H_2(M)$ of rank $k$, generated by $\sigma_1, \ldots, \sigma_k$. If $\langle \sigma_1, S \rangle, \ldots, \langle \sigma_k, S \rangle$ are rationally related, then $S$ can be appropriately rescaled so that Definition 3.1 provides a well-defined linking number between any piecewise smooth closed curve $\gamma \in M \setminus \text{supp} S$ and $S$. If $H_2(M)$ has rank $k > 1$, this provides a rather restrictive cohomological condition on $S$. (It is similar to the restriction of being in the “linking kernel” described in [32].)

3.3. Invariance and restriction properties of $\langle \cdot, \cdot \rangle$. Suppose that $\Omega, \Lambda$ are open subsets of $\mathbb{C}^j$ and $\mathbb{C}^k$, and $f : \Omega \to \Lambda$ is a (possibly ramified) analytic mapping. Let $S$ be a closed positive $(1,1)$ current given on $\Omega$ by $S = dd^c u$ for some PSH function $u$. If $f(\Omega)$ is not contained in the polar locus of $u$, then the pull-back of $S$ under $f$ is defined by pulling back the potential: $f^*(S) := dd^c(u \circ f)$. Since $u \circ f$ is not identically equal to $-\infty$, it is also a PSH function, and $f^*(S)$ is a well-defined closed positive $(1,1)$ current.

Suppose that $M$ and $N$ are complex manifolds and that $S$ is a closed positive $(1,1)$ current on $N$. If $f : M \to N$ is a holomorphic map with $f(M)$ not entirely contained in the polar locus of $S$, then the pull-back $f^*S$ can be defined by taking local charts and local potentials for $S$. See [34, Appendix A.7] and [21, p. 330-331] for further details.

Proposition 3.6. Suppose that $S$ is a closed positive $(1,1)$ current on $N$ and $f : M \to N$, with $f(M)$ not contained in the polar locus of $S$. If $\sigma$ is a piecewise smooth two chain in $M$ with $\partial \sigma$ disjoint from $\text{supp} f^*S$, then $\langle f_*\sigma, S \rangle = \langle \sigma, f^*S \rangle$.

Proof. Since $f(M)$ is not contained in the polar locus of $S$, $f^*S$ is well-defined. Since $\partial \sigma$ is disjoint from $\text{supp} f^*S$, $\partial f(\sigma)$ is disjoint from $\text{supp} S$. Let $\eta_S$ be a smooth approximation of $S$ in the same cohomology class as $S$ and having support disjoint from $\partial f(\sigma)$. Then, $\langle f_*\sigma, S \rangle = \int_{f_*\sigma} \eta_S = \int_\sigma f^* \eta_S = \langle \sigma, f^*S \rangle$, since $f^* \eta_S$ is a smooth approximation of $f^*S$. □

In the case that $M$ is an analytic submanifold of $N$ not entirely contained in the polar locus of $S$, the restriction of $S$ to $M$ is defined by $S|_M := \iota^* S$, where $\iota : M \to N$ is the inclusion. When computing linking numbers, we will often choose $\Gamma$ within some one-dimensional curve $M$ in $N$, with $M$ not contained in the polar locus of $S$. In that case $S|_M$ is a positive measure on $M$ and we can use the following:

Corollary 3.7. Let $S$ be a positive closed $(1,1)$ current on $N$ and $M$ be an analytic curve in $N$ that is not entirely contained in the polar locus of $S$. If $\Gamma$ is a piecewise smooth two chain in $M$ with $\iota(\partial \Gamma)$ disjoint from $\text{supp} S$, then

\begin{equation}
\langle \iota(\Gamma), S \rangle = \int_\Gamma S|_M.
\end{equation}

Proof. Proposition 3.6 gives $\langle \iota(\Gamma), S \rangle = \langle \iota_* \Gamma, S \rangle = \langle \Gamma, \iota^* S \rangle = \langle \Gamma, S|_M \rangle$. Any positive $(1,1)$ current on $M$ is a positive measure. Thus, $\int_\Gamma S|_M$ is defined, and coincides with the result obtained by first choosing a smooth approximation to $S|_M$. Thus $\langle \Gamma, S|_M \rangle = \int_\Gamma S|_M$. □

In the remainder of the paper, we will not typically distinguish between $\Gamma$ and $\iota(\Gamma)$.

3.4. Linking with the Green’s Current. We conclude the section with some observations specific to the Green’s current $T$, including the proof of Theorem [11], as well as an example illustrating the definitions given above. It is worth noting that the Green’s current has empty polar locus, since $G$ is locally bounded on $\mathbb{C}^{k+1} \setminus \{0\}$, so that the hypotheses of Proposition 3.6 and Corollary 3.7 are easy to check.
Proposition 3.8. Suppose that \( f : \mathbb{P}^k \to \mathbb{P}^k \), \( W^s(\zeta) \subset U(f) \) is the basin of attraction of some attracting periodic cycle \( \zeta \), and \( T \) is the Green’s Current of \( f \). Then
\[
\text{lk}(\cdot, T) : H_1(W^s(\zeta)) \to \mathbb{Z}[1/d]/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}.
\]

Proof. Suppose that \( \zeta \) is of period \( N \). Then, the basin of attraction \( W^s(\zeta) \) contains a union of small open balls \( B_0, \ldots, B_{N-1} \) centered at each point \( \zeta, \ldots, f^{N-1}(\zeta) \) of the orbit \( \zeta \). Since \( H_1(W^s(\zeta)) \) is generated by the classes of piecewise smooth loops, it is sufficient to consider a single such loop \( \gamma \). Since \( \gamma \) is a compact subset of \( W^s(\zeta) \), there is some \( n \) so that \( f^n(\gamma) \) is contained in \( \bigcup B_i \), giving that \( f^n(\gamma) \) has trivial homology class in \( H_1(W^s(\zeta)) \). In particular, \( \text{lk}(f^n(\gamma), T) = 0 \pmod{1} \), so that for any \( \Gamma \) with \( \partial \Gamma = \gamma \) we have \( \langle f^n(\Gamma), T \rangle = k \) for some integer \( k \).

Recall that \( f^*T = dT \), where \( d \) is the algebraic degree of \( f \). Proposition 3.6 gives that \( k = \langle f^n(\Gamma), T \rangle = \langle \Gamma, (f^*)^nT \rangle = d^n \langle \Gamma, T \rangle \). In particular, \( \text{lk}(\gamma, T) \equiv k/d^n \pmod{1} \). \( \square \)

Using Proposition 3.8, Theorem 1.1 presents a general strategy for showing that \( H_1(U(f)) \) is infinitely generated.

Proof of Theorem 1.1: Since \( \Omega \) is a union of basins of attraction for attracting periodic points of \( f \), Proposition 3.8 gives that \( \text{lk}(\cdot, T) : H_1(\Omega) \to \mathbb{Q}/\mathbb{Z} \). There are homology classes \( c \in H_1(\Omega) \) with \( \text{lk}(c, T) \neq 0 \) arbitrarily close to zero, so, since \( \text{lk}(\cdot, T) \) is a homomorphism, the image of \( \text{lk}(\cdot, T) : H_1(\Omega) \to \mathbb{Q}/\mathbb{Z} \) is dense in \( \mathbb{Q}/\mathbb{Z} \). Because any dense subgroup of \( \mathbb{Q}/\mathbb{Z} \) is infinitely generated, the image of \( \text{lk}(\cdot, T) \) is infinitely generated, hence \( H_1(\Omega) \) is, as well. \( \square \)

Example 3.9. Consider the polynomial skew product \( (z, w) \mapsto (z^2, w^2 + 0.3z) \), for which the Fatou set consists of the union of basins of attraction for three super-attracting fixed points: \([0 : 1 : 0], [0 : 0 : 1], \) and \([1 : 0 : 0] \). In Figure 2 we show a computer generated image of the intersection of \( W^s([0 : 1 : 0]) \) (lighter grey) and \( W^s([0 : 0 : 1]) \) (dark grey) with the vertical line \( z = z_0 = 0.99999 \). In terms of the fiber-wise Julia sets that were mentioned in the introduction, \( K_{z_0} \) is precisely the closure of the dark grey region and \( J_{z_0} \) is its boundary.

We will see in Proposition 5.1 that \( T_{|z=z_0} \) is precisely the harmonic measure on \( K_{z_0} \). Using this knowledge, and supposing that the computer image is accurate, we illustrate how the above definitions can be used to show that the smooth loop \( \gamma \) shown in the figure represents a non-trivial homology class in \( H_1(W^s([0 : 1 : 0])) \).

Suppose that we use the two chain \( \Gamma_1 \) that is depicted in the figure to compute \( \text{lk}(\gamma, T) \). The harmonic measure on \( K_{z_0} \) is supported in \( J_{z_0} \) and equally distributed between the four symmetric pieces with total measure of \( K_{z_0} \) is 1. Therefore (using Corollary 3.7), we see that \( \text{lk}(\gamma, T) = \int_{\Gamma_1} T_{|z=z_0} = \frac{1}{4} \pmod{1} \), because \( \Gamma_1 \) covers exactly 1 these 4 pieces of \( K_{z_0} \).

If instead we use \( \Gamma_2 \), the disc “outside of \( \gamma \)” within the projective line \( z = z_0 \) with the orientation chosen so that \( \partial \Gamma_2 = c \) as depicted, then \( \text{lk}(\gamma, T) = \int_{\Gamma_2} T_{|z=z_0} = -\frac{3}{4} \pmod{1} \), because \( \Gamma_2 \) covers 3 of the 4 symmetric pieces of \( K_{z_0} \), but with the opposite orientation than that of \( \Gamma_1 \). However, \( -\frac{3}{4} \pmod{1} = \frac{1}{4} \pmod{1} \), so we see that the computed linking number does come out the same.

Since \( \text{lk}(\gamma, T) \neq 0 \pmod{1} \), Corollary 3.3 gives that it is impossible to have any 2-chain \( \Lambda \) within \( W^s([0 : 1 : 0]) \) (even outside of the vertical line \( z = z_0 \)) so that \( \partial \Lambda = c \). Thus \( \langle \gamma \rangle \neq 0 \in H_1(W^s([0 : 1 : 0])) \).
4. **Application to Polynomial Endomorphisms of \( \mathbb{P}^2 \)**

Having developed the linking numbers in Section 3, Theorem 1.2 will be a consequence of the following well-known result:

**Theorem 4.1.** [1, Thm. 5.7.1] Let \( g : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map. Then, if \( J(g) \) is disconnected, it contains uncountably many components, and each point of \( J(g) \) is an accumulation point of infinitely many distinct components of \( J(g) \).

Let us begin by studying the Fatou set of one-dimensional maps:

**Proposition 4.2.** If \( g : \mathbb{P}^1 \to \mathbb{P}^1 \) is a hyperbolic rational map with disconnected Julia set \( J(g) \), then the Fatou set \( U(g) \) has infinitely generated first homology.

**Remark 4.3.** When reading the proof of Proposition 4.2 it is helpful to keep in mind two examples. The first is the polynomial \( r(z) = z^3 - 0.48z + (0.706260 + 0.502896i) \) for which one of the critical points escapes to infinity, while the other is in the basin of attraction for a cycle of period 3. The result is a filled Julia set with infinitely many non-trivial connected components, each of which is homeomorphic to the Douady’s rabbit. (See [30].)

The second example are maps of the form \( f(z) = z^n + \lambda/z^h \), which were considered in [28]. For suitable \( n, h \), and \( \lambda \) the Julia set is a Cantor set of nested simple closed curves.

**Proof of Proposition 4.2.** Since \( g \) is hyperbolic, \( U(g) \) consists of the basins of attraction of finitely many attracting periodic points. Therefore, according to Theorem 1.1 it is
sufficient to find elements of $H_1(U(g))$ having non-zero linking numbers with $T = \mu_g$ that are arbitrarily close to 0.

Theorem 4.1 will allow us to find a sequence of piecewise smooth two chains $\Gamma_1, \Gamma_2, \ldots$ so that $0 < \langle \Gamma_{n-1}, \mu_G \rangle < \langle \Gamma_n, \mu_G \rangle < 1$ and $\partial \Gamma_n \subset U(g)$, as follows.

For each $n$, $\Gamma_n$ will be a union of disjoint positively-oriented closed discs in $\mathbb{P}^1$, each counted with weight one. Since $J(g)$ is disconnected, we can find a piecewise smooth oriented loop $\gamma_1 \subset U(g)$ that separates $J(g)$. Let $\Gamma_1$ be the positively-oriented disc in $\mathbb{P}^1$ having $\gamma_1$ as its oriented boundary. Since $\mu_g$ is normalized and $\gamma_1$ separates $J(g) = \text{supp}(\mu_g)$, we have $0 < \langle \Gamma_1, \mu_g \rangle < 1$. Now suppose that $\Gamma_1, \ldots, \Gamma_{n-1}$ have been chosen. Since $\langle \Gamma_{n-1}, \mu_g \rangle < 1$, we have $J(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1}) \neq \emptyset$. Then, according to Theorem 4.1 there is more than one component of $J(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1})$, so we can choose an oriented loop $\gamma_n \subset U(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1})$ so that at least one component of $J(g) \cap (\mathbb{P}^1 \setminus \Gamma_{n-1})$ is on each side of $\gamma_n$. Then, we let $\Gamma_n$ be the union of oriented discs in $\mathbb{P}^1$ consisting of the points inside of $\gamma_n$ and any discs from $\Gamma_{n-1}$ that are not inside of $\gamma_n$.

Considering the homology class $[\partial \Gamma_n - \partial \Gamma_{n-1}] \in H_1(U(g))$ we have that
$$lk([\partial \Gamma_n - \partial \Gamma_{n-1}],\mu_g) = \langle \Gamma_n, \mu_G \rangle - \langle \Gamma_{n-1}, \mu_G \rangle \pmod{1}$$
is non-zero for each $n$. However, since
$$\sum_n \langle \Gamma_n, \mu_G \rangle - \langle \Gamma_{n-1}, \mu_G \rangle$$
is bounded by 1, we have that $lk([\partial \Gamma_n - \partial \Gamma_{n-1}],\mu_g) \to 0$ in $\mathbb{Q}/\mathbb{Z}$. Theorem 4.1 then gives that $H_1(U(g))$ is infinitely generated. \square

Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a polynomial endomorphism given in projective coordinates by
\begin{equation}
(4) \quad f([Z : W : T]) = [P(Z, W, T) : Q(Z, W, T) : T^d].
\end{equation}
Since $f : \mathbb{P}^2 \to \mathbb{P}^2$ is assumed globally holomorphic, $P(Z, W, T), Q(Z, W, T)$, and $T^d$ have no common zeros other than $(0, 0, 0)$.

The (projective) line at infinity $\Pi := \{T = 0\}$ is uniformly super-attracting and the restriction $f_\Pi$ is given in homogeneous coordinates by
\begin{equation}
(5) \quad f_\Pi : ([Z : W]) \to [P_0(Z, W) : Q_0(Z, W)].
\end{equation}
where $P_0 := P(Z, W, 0)$ and $Q_0 := Q(Z, W, 0)$.

Let $U(f)$ be the Fatou set for $f$ and $U(f_\Pi)$ the Fatou set for $f_\Pi$. The former is an open set in $\mathbb{P}^2$, while the latter is an open set in the line at infinity $\Pi$.

**Lemma 4.4.** If $f_\Pi$ is hyperbolic then $U(f_\Pi) \subset U(f)$.

*Proof.* Since $f_\Pi$ is hyperbolic, $U(f_\Pi)$ is in the union of the basins of attraction $W^s_\Pi(\zeta_i)$ of a finite number of periodic attracting points $\zeta_1, \ldots, \zeta_k$. The line at infinity $\Pi$ is transversally superattracting, so each $\zeta_i$ is superattracting in the transverse direction to $\Pi$ and (at least) geometrically attracting within $\Pi$. Let $W^s(\zeta_i) \subset \mathbb{P}^2$ be the basin of attraction for $\zeta_i$ under $f$. Then, $W^s_\Pi(\zeta_i) \subset W^s(\zeta_i)$, giving $U(f_\Pi) \subset U(f)$. \square

Let $T$ be the Green’s current for $f$ and let $\mu_\Pi$ be the measure of maximal entropy for the restriction $f_\Pi$.

**Lemma 4.5.** The restriction $T|_\Pi$ coincides with $\mu_\Pi$. 

Proof. Consider the lift $F_{\Pi} : \mathbb{C}^2 \to \mathbb{C}^2$ of the rational map $f_{\Pi} : \mathbb{P}^1 \to \mathbb{P}^1$. As observed in Remark 2.4,
\[ G_{\Pi}(Z, W) = \lim_{n \to \infty} \frac{1}{2\pi} \log ||F_{\Pi}^n(Z, W)|| \]
is the potential for $\mu_{\Pi}$, meaning that $\pi^*\mu_{\Pi} = \frac{1}{2\pi} dd^c G_{\Pi}$.

The restriction $T_{\Pi}$ is obtained by restricting of the potential $G$ to $\pi^{-1}(\Pi) = \{(Z, W, 0) \in \mathbb{C}^3\}$. Specifically, it is defined by $\pi^*(T_{\Pi}) = \frac{1}{2\pi} dd^c(G(Z, W, 0))$. Therefore, it suffices to show that $G(Z, W, 0) = G_{\Pi}(Z, W)$. However, this follows directly from the fact that $F(Z, W, 0) = F_{\Pi}(Z, W)$. (Here $F$ is the lift of $f$ to $\mathbb{C}^3$, as given by 4 when considered in non-projective coordinates $[Z, W, T]$.) \qed

Proof of Theorem 1.2. As in the proof of Proposition 4.2, we can find a sequence of 1-cycles $c_n$ in $U(f_{\Pi})$ having linking numbers with $\mu_{\Pi}$ arbitrarily close to 0 in $\mathbb{Q}/\mathbb{Z}$. Since $f_{\Pi}$ is hyperbolic, Lemma 1.4 gives that each $c_n$ is in the union of basins of attraction for finitely many attracting periodic points of $f$. In particular, $lk(c_n, T)$ is well-defined for each $n$. Lemma 1.5 gives that $T_{\Pi} = \mu_{\Pi}$, so that $lk(c_n, T)$ (considering $c_n$ in $\mathbb{P}^2$) coincides with $lk(c_n, \mu_{\Pi})$ (considering $c_n$ in the projective line $\Pi$). Therefore, $lk(c_n, T) \neq 0$ and $lk(c_n, T) \to 0$ in $\mathbb{Q}/\mathbb{Z}$. Theorem 1.1 gives that the union of these basins has infinitely generated first homology, and hence $U(f)$ does as well. \qed

Example 4.6. We embed the polynomial dynamics of $r(z)$ from Remark 4.3 as the dynamics on the line at infinity $\Pi$ for a polynomial endomorphism of $\mathbb{P}^2$. Let $R(Z, W) = Z^3 - 0.48ZW^2 + (0.706260 + 0.502896i)W^3$ be the homogeneous form of $r$, and let $P(Z, W, T)$ and $Q(Z, W, T)$ be any homogeneous polynomials of degree 2. Then
\[ f([Z : W : T]) = [R(Z, W) + T \cdot P(Z, W, T) : W^3 + T \cdot Q(Z, W, T) : T^3] \]
is a polynomial endomorphism with $f_{\Pi} = r$. In this case, Theorem 1.2 gives that the basin of attraction of $[1 : 0 : 0]$ for $f$ has infinitely generated first homology.

Remark 4.7. Suppose that $f : \mathbb{P}^k \to \mathbb{P}^k$ is a holomorphic endomorphism having an invariant projective line $\Pi$. Lemma 1.5 can be extended to give that $T_{\Pi} = \mu_{\Pi}$, where $\mu_{\Pi}$ is the measure of maximal entropy for the one-dimensional map $f_{\Pi}$. If $\Pi$ is at least geometrically attracting transversally, $f_{\Pi}$ is hyperbolic, and $J(f_{\Pi})$ is disconnected, then essentially the same proof as that of Theorem 1.2 gives that the Fatou set $U(f)$ has infinitely generated first homology.

Using this observation, one can inductively create sequences of polynomial endomorphisms $f_k : \mathbb{P}^k \to \mathbb{P}^k$, for every $k$, each having Fatou set with infinitely generated first homology. One begins with a hyperbolic polynomial endomorphism $f_1$ of the Riemann sphere $\mathbb{P}^1$ having disconnected Julia set. Then, for each $k$, one can let $f_k : \mathbb{P}^k \to \mathbb{P}^k$ be any polynomial endomorphism whose dynamics on the hypersurface $\mathbb{P}^{k-1}$ at infinity is given by $f_{k-1}$. (When $k = 2$, the construction of $f_2 : \mathbb{P}^2 \to \mathbb{P}^2$ is similar to that from Example 4.3.) The resulting maps each have a totally-invariant projective line $\Pi$ that is transversally superattracting with $f_{k,\Pi} = f_1$ hyperbolic with disconnected Julia set. Thus, the Fatou set $U(f_k)$ has infinitely generated first homology.
5. Application to Polynomial skew products

A polynomial skew product is a polynomial endomorphism of the form

$$f(z, w) = (p(z), q(z, w))$$

with $p$ and $q$ polynomials of degree $d$ where $p(z) = z^d + O(z^{d-1})$ and $q(z) = w^d + O_z(w^{d-1})$.

(See Jonsson [25].) Theorem 1.2 can be applied to many polynomial skew products $f$ to show that that $H_1(U(f))$ is infinitely generated; for example, $f(z, w) = (z^2, w^2 + 10z^2)$, which has $J_\Pi$ a Cantor set. Next we will find alternative sufficient conditions under which a polynomial skew product has Fatou set with infinitely generated first homology, proving Theorem 1.3. This theorem will apply to many maps to which Theorem 1.2 does not apply; for example, $f(z, w) = (z^2, w^2 - 3z)$, for which $J_\Pi$ is equal to the unit circle.

5.1. Preliminary background on polynomial skew products. The Green’s current for any polynomial endomorphism can be computed in the affine coordinates on $\mathbb{C}^2$ as $T := \frac{1}{2\pi i} dd^c G_{\text{affine}}$, where $G_{\text{affine}}$ is the (affine) Green’s function defined in Remark 2.2.

The “base map” $p(z)$ has a Julia set $J_p \subset \mathbb{C}$ and, similarly, a Green’s function $G_p(z) := \lim_{n \to \infty} \frac{1}{n} \log ||p^n(z)||$. Furthermore, one can define a fiber-wise Green’s function\footnote{For the purist: the Green’s functions $G_p$ and $G_z$ should also have the subscript “affine”, but it is dropped here for ease of notation. See Section 2 for the distinction.} by:

$$G_z(w) := G_{\text{affine}}(z, w) - G_p(z).$$

For each fixed $z$, $G_z(w)$ is a subharmonic function of $w$ and one defines the fiber-wise Julia sets by $K_z := \{G_z(w) = 0\}$ and $J_z := \partial K_z$.

The extension of $f$ to $\mathbb{P}^2$ is given by

$$f([Z : W : T]) = [P(Z, T) : Q(Z, W, T) : T^d],$$

where $P(Z, T)$ and $Q(Z, W, T)$ are the homogeneous versions of $p$ and $q$. The point $[0 : 1 : 0]$ that is “vertically at infinity” with respect to the affine coordinates $(z, w)$ is a totally-invariant super-attracting fixed point and $(z, w) \in W^s([0 : 1 : 0])$ if and only if $w \in \mathbb{C} \setminus K_z$.

Suppose that $(z, w) \in W^s([0 : 1 : 0])$ and $(z_n, w_n) := f^n(z, w)$. Then,

$$G_{\text{affine}}(z, w) = \lim_{n \to \infty} \frac{1}{n} \log ||f^n(z, w)||_\infty = \lim_{n \to \infty} \frac{1}{n} \log |w_n|$$

and

$$G_z(w) = G_{\text{affine}}(z, w) - G_p(z) = \lim_{n \to \infty} \frac{1}{n} \log |w_n| - \lim_{n \to \infty} \frac{1}{n} \log |z_n|.$$
Proof. Notice that
\[
T|_{z=z_0} = \frac{1}{2\pi} d\overline{d} G_{\text{affine}}|_{z=z_0} = \frac{1}{2\pi} d\overline{d} G_{\text{affine}}(z_0, w) = \frac{1}{2\pi} d\overline{d} \left( G_{\text{affine}}(z_0, w) - G_p(z_0) \right) = \frac{1}{2\pi} d\overline{d} G_{z_0}(w).
\]
According to [25, Thm 2.1], \( G_{z_0} \) is the Green’s function for \( K_z \) with pole at infinity. We have thus obtained that \( \mu_{z_0} \) is exactly the harmonic measure \( \mu_{z_0} \) on \( K_{z_0} \).

5.2. Topology of the basin of attraction \( W^s([0 : 1 : 0]) \).

Proposition 5.2. If \( \zeta \) is a totally-invariant (super)-attracting fixed point for a holomorphic \( f : \mathbb{CP}^k \to \mathbb{CP}^k \), then \( W^s(\zeta) \) is path-connected.

A nearly identical statement is proven for \( \mathbb{CP}^2 \) in Theorem 1.5.9 from [22]. We refer the reader to their proof since it is nearly identical for \( \mathbb{CP}^k \). In particular, for any skew product \( W^s([0 : 1 : 0]) \) is path connected.

Although \( G_z(w) \) is subharmonic in \( w \) for any fixed \( z \), it does not form a PSH function of both \( z \) and \( w \). Consider the points \( (z, w) \in W^s([0 : 1 : 0]) \) for which \( z \in J_p \). At these points \( G_{\text{affine}} \) is pluriharmonic, i.e. \( d\overline{d} G_{\text{affine}} = 0 \), but \( G_p(z) \) is not pluriharmonic, i.e. \( d\overline{d} G_p(z) > 0 \). Therefore, at these points \( d\overline{d} G_z(w) < 0 \), so \( G_z(w) \) is not PSH.

Lemma 5.3. The function \( -G_z(w) \) is PSH at all points \( (z, w) \in W^s([0 : 1 : 0]) \cap \mathbb{C}^2 \) and it extends to a PSH function on all of \( W^s([0 : 1 : 0]) \). The resulting function is pluriharmonic on \( W^s([0 : 1 : 0]) \) except at points for which \( Z/T \in J_p \).

Proof. Since \( -G_z(w) = G_p(z) - G_{\text{affine}}(z, w) \), with \( G_{\text{affine}}(z, w) \) pluriharmonic in \( W^s([0 : 1 : 0]) \) and \( G_p(z) \) PSH everywhere, the result is PSH in \( W^s([0 : 1 : 0]) \cap \mathbb{C}^2 \).

Jonsson proves in [25, Lemma 6.3] that \( G_z(w) \) extends as a PSH function in a suitable neighborhood of \( \Pi \setminus \{[0 : 1 : 0]\} \) and his proof immediately gives that the result is pluriharmonic in a (possibly smaller) neighborhood within \( W^s([0 : 1 : 0]) \) of \( \Pi \setminus \{[0 : 1 : 0]\} \). Therefore, \( -G_z(w) \) is also pluriharmonic in the same neighborhood.

Thus, \( -G_z(w) \) extends to a PSH on \( W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\} \) and, assigning \(-\infty \) to \([0 : 1 : 0]\), gives the desired extension to all of \( W^s([0 : 1 : 0]) \). The result will be pluriharmonic except at \([0 : 1 : 0]\) and at the points in \( W^s([0 : 1 : 0]) \cap \mathbb{C}^2 \) where \( d\overline{d}(-G_z(w)) > 0 \), that is the points where \( Z/T \in J_p \).

Proof of Theorem 1.3: We first suppose that \( J_{z_0} \) is disconnected for some \( z_0 \in J_p \). Let \( z_1, z_2, \ldots \) be any sequence of iterated preimages of \( z_0 \) so that \( p^n(z_n) = z_0 \).

Consider the vertical line \( \{z_0\} \times \mathbb{C} \). Since \( J_{z_0} \) is disconnected, so is \( K_{z_0} \), and we can choose two disjoint positively-oriented piecewise smooth loops \( \eta_1, \eta_2 \subset \{z_0\} \times (\mathbb{C} \setminus K_{z_0}) \) each enclosing a proper subset of \( K_{z_0} \).

Perturbing \( \eta_1, \eta_2 \) within \( \{z_0\} \times (\mathbb{C} \setminus K_{z_0}) \), if necessary, we can suppose that none of the \( d-1 \) critical values of \( f|_{\{z_1\} \times \mathbb{C}} : \{z_1\} \times \mathbb{C} \to \{z_0\} \times \mathbb{C} \) (counted with multiplicity) are on \( \eta_1 \) or \( \eta_2 \). Since the regions enclosed by \( \eta_1 \) and \( \eta_2 \) are disjoint, at least one of them contains at most \( d-2 \) of these critical values. Let \( \gamma_0 \) be this curve.

Since \( \gamma_0 \subset \{z_0\} \times (\mathbb{C} \setminus K_{z_0}) \), \( \gamma_0 \subset W^s([0 : 1 : 0]) \). Because \( \gamma_0 \) is compact, it is bounded away from \( \text{supp}(T) \), and the linking number \( \text{lk}(\gamma_0, T) \) is a well defined invariant of the
homology class \([\gamma]\) within \(H_1(W^s([0:1:0]))\). We let \(\Gamma_0\) be the closed disc in \(\{(z_0) \times \mathbb{C}\) having \(\gamma_0\) as its oriented boundary. Since \(\Gamma_0\) contains some proper subset of \(K_{z_0}\) (and hence of \(J_{z_0}\)) with \(\text{supp}(\mu_{z_0}) = J_{z_0}\) we have that

\[
0 < \langle \Gamma_0, T \rangle = \int_{\Gamma_0} \mu_{z_0} < 1.
\]

Therefore, \(lk(\gamma_0, T) = \langle \Gamma_0, T \rangle \pmod{1} \neq 0 \pmod{1}\), giving that \([\gamma_0]\) is non-trivial.

Consider the preimages \(D_1, \ldots, D_j\) of \(\Gamma_0\) under \(f|_{\{(z_1) \times \mathbb{C}\}} : \{z_1\} \times \mathbb{C} \to \{z_0\} \times \mathbb{C}\). Since at most \(d - 2\) critical values of the degree \(d\) ramified cover \(f|_{\{(z_1) \times \cup D_i\}}\) are contained in \(\Gamma_0\), it is a consequence of the Riemann-Hurwitz Theorem that the Euler characteristic of \(\cup D_i\) is greater than or equal to \(2\). Because each \(D_i\) is a domain in \(\mathbb{C}\), at least two components \(D_1\) and \(D_2\) are discs.

The total degree of \(f|_{\{(z_1) \times \mathbb{C}\}} : \cup D_i \to \Gamma_0\) is \(d\), so \(f|_{\{(z_1) \times \mathbb{C}\}} : D_i \to \Gamma_0\) a ramified covering of degree \(k_i \leq d - 1\) for each \(i\). Proposition 3.6 and the basic invariance \(f^*T = d \cdot T\) for the Green’s current give that

\[
\langle D_i, T \rangle = \frac{1}{d} \langle D_1, f^*T \rangle = \frac{1}{d} \langle f_*D_i, T \rangle = \frac{1}{d} \langle k_i, T \rangle \leq \frac{d - 1}{d} \langle \Gamma_0, T \rangle
\]

for each \(i\).

As before, we can perturb the boundaries of \(D_1\) and \(D_2\) within \(\{z_1\} \times (\mathbb{C} \setminus K_{z_1})\) so that none of the critical values of \(f|_{\{(z_2) \times \mathbb{C}\}}\) lie on their boundaries and so that \(D_1\) and \(D_2\) remain disjoint. (It will not affect the pairings given by (9)). At least one of the discs \(D_1, D_2\) contains at most \(d - 2\) critical values of \(f|_{\{(z_2) \times \mathbb{C}\}}\). We let \(\Gamma_1\) be that disc and \(\gamma_1 = \partial \Gamma_1\). Then

\[
0 < \langle \Gamma_1, T \rangle \leq \frac{d - 1}{d} \langle \Gamma_0, T \rangle \leq \frac{d - 1}{d}.
\]

Continuing in the same way, we can find a sequence of discs \(\Gamma_0, \Gamma_1, \ldots\) so that

- \(\Gamma_n \subset \{z_n\} \times \mathbb{C}\),
- \(\gamma_n = \partial \Gamma_n \subset W^s([0:1:0])\),
- \(\Gamma_n\) contains at most \(d - 2\) critical values of \(f|_{\{z_{n+1}\} \times \mathbb{C}}\) (counted with multiplicity), and
- \(\langle \Gamma_n, T \rangle \leq \frac{d - 1}{d} \langle \Gamma_{n-1}, T \rangle\).

Consequently,

\[
0 < \langle \Gamma_n, T \rangle \leq \left(\frac{d - 1}{d}\right)^n,
\]

giving that \(lk(\gamma_n, T) \to 0\) in \(\mathbb{Q}/\mathbb{Z}\). Therefore, Theorem 11 gives that \(H_1(W^s([0:1:0]))\) is infinitely generated.

We will now show that if \(J_z\) is connected for every \(z \in J_p\), then \(W^s([0:1:0])\) is homeomorphic to an open ball. Consider the local coordinates \(z' = Z/W, t' = T/W, \) chosen so that \((z', t') = (0, 0)\) corresponds to \([0:1:0]\). In these coordinates

\[
f(z', t') = \left(\frac{P(z', t')}{Q(z', 1, t')} \cdot \frac{t'^d}{Q(z', 1, t')}\right),
\]
where $P$ and $Q$ are the homogeneous versions of $p$ and $q$ appearing in Equation (4). The assumption that $q(z) = w^d + O_z(w^{d-1})$ and $p(z) = z^d + O(z^{d-1})$ imply that we have the expansion

$$f(z', t') = (P(z', t'), t''d) + g(t', z'),$$

with $(P(z', t'), t''d)$ non-degenerate of degree $d$ and $g(t', z')$ consisting of terms of degree $d+1$ and larger.

Therefore, we can construct a potential function\footnote{The potential function $h$ is sometimes also be called the Green’s function of the point $(0, 0)$.} for the superattracting point $(0, 0)$:

$$h(z', t') := \lim_{n \to \infty} \frac{1}{d^n} \log \| f^n(z', t') \|_\infty.$$\label{eq:potential}\index{potential function}

The result is a continuous pluri-subharmonic function with logarithmic singularity at $(z', t') = (0, 0)$ having the property that $(z', t') \in W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\}$ if and only if $h(z', t') < 0$. In particular,

$$h : W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\} \to (-\infty, 0)$$

is proper.

If we let $(z'_n, t'_n) = f^n(z', t')$, then Equation (10) simplifies to

$$h(z', t') = \begin{cases} \lim_{n \to \infty} \frac{1}{d^n} \log |t'_n| & \text{if } z'/t' \in K_p, \\
\lim_{n \to \infty} \frac{1}{d^n} \log |z'_n| & \text{if } z'/t' \notin K_p.\end{cases}$$\label{eq:potential2}

since $z'_{n+1}/t'_{n+1} = p(z'_n/t'_n)$. Equation (8) gives that in the original affine coordinates $(z, w)$ we have

$$h(z, w) = \begin{cases} \lim_{n \to \infty} \frac{1}{d^n} \log |t'_n| = - \lim_{n \to \infty} \frac{1}{d^n} \log |w_n| = -G_z(w) & \text{if } z \in K_p, \\
\lim_{n \to \infty} \frac{1}{d^n} \log |z'_n| = \log |z| - \lim_{n \to \infty} \frac{1}{d^n} \log |w_n| = -G_z(w) & \text{if } z \notin K_p,\end{cases}$$

which is harmonic on the intersection of any vertical line $\{z\} \times \mathbb{C}$ with $W^s([0 : 1 : 0])$ and pluriharmonic except when $z \in J_p$; see Lemma 5.3. A similar calculation shows that $h$ coincides with the extension of $-G_z(w)$ described in Lemma 5.3 and that the restriction of $h$ to $\Pi$ is $-G_{\Pi}$. (Here, $G_{\Pi}$ is the Green’s function for the action $f_{\Pi}$ of $f$ on the line at infinity.)

Therefore, $h(z', t')$ is pluriharmonic on $W^s([0 : 1 : 0]) \setminus \{(z', w') : z'/w' \in J_p\}$ and the restriction of $h(z', t')$ to any line through $(0, 0)$ is harmonic on $W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\}$, with a logarithmic singularity at $(0, 0)$.

Since $J_{z_0}$ is connected for every $z_0 \in J_p$, Proposition 6.3 from [25] gives that $J_z$ is connected for every $z \in \mathbb{C}$ and also $J_{\Pi}$ is connected, or, equivalently, that $G_z(w)$ (for any $z$) and $G_{\Pi}$ have no (escaping) critical points. Therefore, the restriction of $h$ to any complex line through $(0, 0)$ has no critical points in $W^s([0 : 1 : 0])$.

The sublevel set $W_a := h^{-1}([-\infty, a])$ is open for any $a \in (-\infty, 0)$ since $h : W^s([0 : 1 : 0]) \setminus \{[0 : 1 : 0]\} \to (-\infty, 0)$ is continuous with $h(z', t') \to -\infty$ if and only if $(z', t') \to (0, 0)$. 

particular, \( W \) open ball. (See [7, Theorem 11.3.6.1].) Negative, the intersection of \( W \) with \( C \) make a the normalization of \( -\nabla \) to \( G \) singularity at \([0 : 1 : 0]\). Equation 2.2 from [25] implies that

\[
\eta(z', t') = \log|t'| + G_p \left( \frac{z'}{t'} \right) + \eta(z', t') \text{ if } t' \neq 0, \text{ and}
\]

\[
h(z', t') = \log|z'| + G^\#_p \left( \frac{t'}{z'} \right) + \eta(z', t') \text{ if } z' \neq 0,
\]

with \( \eta(z', t') \) becoming arbitrarily small for \((z', t')\) sufficiently small and \( G^\#_p(x) \) obtained by extending \( G_p(1/x) - \log(1/x) \) continuously through \( x = 0 \). Therefore, for \( a \) sufficiently negative, the intersection of \( W_a \) with any complex line through \((0, 0)\) will be convex. In particular, \( W_a \) is an star-convex open subset in \( \mathbb{C}^2 \), implying that it is homeomorphic to an open ball. (See [7] Theorem 11.3.6.1.)

We define a new function \( \tilde{h} \) which agrees with \( h \) except in the interior of \( W_a \), where we make a \( C^\infty \) modification (assigning values less than \( a \)) in order to remove the logarithmic singularity at \([0 : 1 : 0]\).

We will use \( \tilde{h} \) as Morse function to show that \( W_b := h^{-1}([-\infty, b]) \) is diffeomorphic to \( W_a \) for any \( b \in (a, 0) \). The classical technique from Theorem 3.1 of [29] would use the normalization of \( -\nabla \tilde{h} \) to generate a flow whose time \((b - a)\) map gives the desired diffeomorphism. This will not work in our situation, since \( \tilde{h} \) is not differentiable at points for which \( z'/w' \in J_p \). However, essentially the same proof works if we replace \( -\nabla \tilde{h} \) with any \( C^1 \) vector field \( V \) on \( W^s([0 : 1 : 0]) \) having no singularities in \( \tilde{h}^{-1}([a, b]) \) and along which \( \tilde{h} \) is decreasing. Note that, as in [29], we need that \( \tilde{h}^{-1}([a, b]) \) is compact, which follows from \( h \) being proper.

Let \( V \) be the the vector field parallel to each line through \((z', t') = (0, 0)\), obtained within each line as minus the gradient of the restriction of \( \tilde{h} \) to that line. The restriction of \( \tilde{h} \) to each complex line through \((0, 0)\) has no critical points in \( \tilde{h}^{-1}([a, b]) \), so it is decreasing along \( V \). Since \( h \) is pluriharmonic for points with \( z'/t' \notin J_p \), it follows immediately that \( V \) is smooth there. To see that \( V \) is smooth in a neighborhood of points where \( z'/t' \in J_p \), notice that

\[
\nabla_v G_v(w) = \nabla_v G(z, w) - G_p(z) = \nabla_v G(z, w),
\]

with \( G(z, w) \) pluriharmonic on \( W^s([0 : 1 : 0]) \cap \mathbb{C}^2 \).

Therefore, for any \( b \in (a, 0) \), \( W_b \) is homeomorphic to \( W_a \) and thus to an open ball. One can then make a relatively standard construction, using these homeomorphisms for \( b \) increasing to \( 0 \), in order to show that \( W^s([0 : 1 : 0]) = \bigcup_{b<0} W_b \) is homeomorphic to an open ball. \( \square \)

6. Further applications

In this final section we discuss a few examples of maps to which we have applied the results of this paper, and then a few types of maps which we feel would be fruitful to study further with techniques similar to those of this paper.

6.1. Relationship between connectivity of \( J_2 \) and the topology of the Fatou set for polynomial skew products. For polynomial skew products, \( J_2 = \text{supp} (\mu) = \text{supp} (T \wedge T) = \bigcup_{z \in J_2} \bar{J}_z \), which by [25] is also the closure of the set of repelling periodic points. Here we examine to what extent connectivity of \( J_2 \) affects the homology of the Fatou set \( U \).
The following example shows that there are many polynomial skew products $f$ with $J_2$ connected for which $H_1(U(f))$ is non-trivial (in fact infinitely generated.)

**Example 6.1.** Consider $f(z, w) = (z^2 - 2, w^2 + 2(2 - z))$ which has $J_2$ connected and has $J_z$ disconnected over $z = -2 \in J_p$, as shown in [25] Example 9.7. Theorem 1.3 immediately applies, giving that $H_1(U(f))$ is infinitely generated.

In fact, examples of this phenomenon can appear “stably” within a one parameter family. Let $p_n(z) = z^2 + c_n$ be the unique quadratic polynomial with periodic critical point of least period $n$ and $c_n$ real. Then, [11] Theorem 6.1] yields that for $n$ sufficiently large,

$$f_n(z, w) = (p_n(z), w^2 + 2(2 - z))$$

is Axiom A with $J_z$ disconnected for most $z \in J_{p_n}$ and with $J_2$ connected. Suppose that $f_n$ is embedded within any holomorphic one-parameter family $f_{n, \lambda}$ of polynomial skew products. Then, Theorems 4.1 and 4.2 from [11] (see also, [24, Thm C]) give that all maps $f_{n, \lambda}$ within the same hyperbolic component as $f_n$ also have $J_2$ connected, but $J_z$ disconnected over most $z$ in $J_{p_n, \lambda}$. (Here, $p_n$ is the first component of $f_{n, \lambda}$.) An immediate application of Theorem 1.3 yields that $H_1(U(f_{n, \lambda}))$ is infinitely generated for all $f_{n, \lambda}$ within this hyperbolic component.

Next we consider the possibility of $J_2$ being disconnected, but $f$ not satisfying the hypotheses of our Theorem 1.3

**Question 6.2.** Is there a polynomial skew product $f$ with $J_2$ disconnected, but all $J_z$’s connected for all $z \in \mathbb{C}$, such that $H_1(U(f))$ is trivial? More generally, is there any endomorphism of $\mathbb{P}^2$ with $J_2$ disconnected, but with all Fatou components having trivial homology?

By [25] Proposition 6.6], in order for $f$ to satisfy the hypotheses of this question, $J_p$ would have to be disconnected. However, a simple product like $(z, w) \mapsto (z^2 - 100, w^2)$ does not suffice; note for this map, the basin of attraction of $[1 : 0 : 0]$, hence the Fatou set, has nontrivial homology. Not many examples of non-product polynomial skew products are understood, and the current list of understood examples contains no maps which satisfy the hypotheses of this question.

### 6.2. A quadratic family of polynomial skew products.

We now consider the family of examples $f_a(z, w) = (z^2, w^2 + az)$, which are skew products over $p(z) = z^2$.

The geometry and dynamics in $J_p \times \mathbb{C}$ were explored in [11]. For example, there it is established that:

1. [11] Theorem 5.1]: $f_a$ is Axiom A if and only if $g_a(w) := w^2 + a$ is hyperbolic; and
2. [11] Lemma 5.5]: $J_2$ can be described geometrically in the following manner: $J_{\alpha, t}$ is a rotation of angle $t/2$ of $J(z=1)$. That is, start with $J(g_a)$ in the fiber $J_{z=1}$, then as the base point $z = e^{it}$ moves around the unit circle $J_p = S^1$, the corresponding $J_z$’s are rotations of $J(g_a)$ of angle $t/2$, hence the $J_z$’s complete a half turn as $z$ moves once around the base circle.

Due to the structure of $J_2$, the difference between $f_a$ and the product $h_a(z, w) = (z^2, w^2 + a)$ is one “twist” in $J_2$. In [11] it is shown that $f_a$ and $h_a$ are in the same hyperbolic component if and only if $a$ is in the main cardioid of the Mandelbrot set, $\mathcal{M}$. 
Motion of the open disc parameterized by $z$ basins of attraction is path connected. Since each of these fixed points is totally invariant, Proposition 5.2 gives that each of their basins of attraction for three superattracting fixed points $[1 : 0 : 0], [0 : 1 : 0]$, and $[0 : 0 : 1]$, each of which is path-connected.

Theorem 6.3. The Fatou set of $f_a$ is the union of the basins of attraction of three super-attracting fixed points: $[0 : 0 : 1], [0 : 1 : 0]$, and $[1 : 0 : 0]$, each of which is path-connected. Moreover:

- If $a \notin M$, then $W^s([0 : 1 : 0])$ has infinitely generated first homology.
- If $a \in M$, then each of the three basins of attraction $W^s([0 : 1 : 0]), W^s([0 : 0 : 1])$ and $W^s([1 : 0 : 0])$ is homeomorphic to an open ball.

Proof. For any $a$, the fiberwise Julia set $J_0$ is the unit circle $|z| = 1$. Proposition 4.2 from [32] can be modified to show that there is a local super-stable manifold $W^s_{\text{loc}}(J_0)$ that is obtained as the image of a holomorphic motion of $J_0$ that is parameterized over $D = \{|z| = 1\}$. Proposition 5.2 gives that $W^s_{\text{loc}}(J_0)$ is the zero locus of a pluri-harmonic (hence real-analytic) function. Pulling back $W^s_{\text{loc}}(J_0)$ under iterates of $f_a$, we obtain a global separatrix $W^s(J_0)$ over the entire unit disc $D = \{|z| = 1\}$. Note that $W^s(J_0)$ may not be a manifold, since ramification may occur at points where it intersects the critical locus of $f_a$. For $|z| < 1$, $J_z$ is the intersection of $W^s(J_0)$ with $\{z\} \times \mathbb{C}$ and that $K_z$ is the intersection of $W^s([0 : 0 : 1]) \cup W^s(J_0)$ with $\{z\} \times \mathbb{C}$. Thus, any point $(z, w)$ with $|z| < 1$ is in $W^s([0 : 0 : 1]) \cup W^s(J_0) \cup W^s([0 : 1 : 0])$.

Under the symmetry $S$, each of the above statements about the super-stable manifold of $J_0$ corresponds immediately to a statement about the unit circle $J_0 = \{Z/W = 1\}$ in the line at infinity $\Pi = \{T = 0\}$. Moreover, any point in $\mathbb{P}^2$ with $|T| < |Z|$ is in $W^s([1 : 0 : 0]) \cup W^s(J_0) \cup W^s([0 : 1 : 0])$. Therefore, the Fatou set of $f_a$ is the union of basins of attraction for three superattracting fixed points $[1 : 0 : 0], [0 : 1 : 0]$, and $[0 : 0 : 1]$. Since each of these fixed points is totally invariant, Proposition 5.2 gives that each of their basins of attraction is path connected.

The vertical Julia $J_1$ set over the fixed fiber $z = 1$ is precisely the Julia set of $w \mapsto w^2 + a$, which is connected if and only if $a \in \mathcal{M}$. In particular, if $a \notin \mathcal{M}$, it follows from Theorem 1.3 that $W^s([0 : 1 : 0])$ has infinitely generated first homology.

If $a \in \mathcal{M}$, then, for each $z \in J_p$, $J_z$ is a rotation of the connected set $J_1$ and Theorem 1.3 gives that $W^s([0 : 1 : 0])$ is homeomorphic to an open ball. We will now use Slodkowski’s Theorem on holomorphic motions [35] (see also [23, Section 5.2]) to show that $W^s([0 : 0 : 1])$ and $W^s([1 : 0 : 0]) = S(W^s([0 : 0 : 1]))$ are homeomorphic to the open bidisc.

We will extend (in the parameter $z$) the holomorphic motion whose image is $W^s_{\text{loc}}(J_0)$ to a holomorphic motion of $J_0$ parameterized by $z \in \mathbb{D}$, having the entire separatrix $W^s(J_0)$ as its image. Then, by Slodkowski’s Theorem, this holomorphic motion extends (in the fiber $w$) from $J_0$ to a holomorphic motion of the entire Riemann sphere $\mathbb{P}^1$ that is also parameterized by $z \in \mathbb{D}$. Consequently, $W^s([0 : 0 : 1])$ will be the image of a holomorphic motion of the open disc $\{z = 0, |w| < 1\}$, parameterized by $z \in \mathbb{D}$.
Since $a \in \mathcal{M}_i$, it also follows from [25, Proposition 6.4] that for each $z \in \mathbb{C}$ the fiber-wise critical points

$$C_z := \{ w \in \mathbb{C} : q'_z(w) = 0 \}$$

are in $K_z$. We now check that they are disjoint from $W^s(J_0)$.

The union of these fiber-wise critical points is just the horizontal line $w = 0$ that stays on one side of $W^s(J_0)$, possibly touching at many points. Note, however, that they are disjoint at $z = 0$. Consider the point $z_0$ (with $|z_0| < 1$) of smallest modulus where $w = 0$ and $W^s(J_0)$ touch. Then, there is a neighborhood of $U$ of $z_0$ in $\mathbb{C}^2$ in which $W^s(J_0)$ is given by the zero set of a PSH function $\Psi$. Changing the sign of $\Psi$ (if necessary) we can assume that $\Psi \leq 0$ for points in $K_z \cap U$. The restriction $\psi(z) = \Psi|_{w=0}$ is a non-positive harmonic function in a neighborhood of $z_0$ having $\psi(z_0) = 0$, but $\psi(z) < 0$ for $z$ with $|z| < |z_0|$. This violates the maximum principle. Therefore, the fiber-wise critical points $C_z$ are disjoint from $W^s(J_0)$ for every $z$.

Suppose that $D \subset W^s(J_0)$ is the graph of a holomorphic function $\nu(z)$ defined on $\{ |z| < r \}$, for some $0 < r < 1$. Then, since $W^s(J_0)$ is disjoint from the horizontal critical locus $w = 0$, the Implicit Function Theorem gives that $f_a^{-1}(D)$ is the union of two discs through the pre-images of $\nu(0)$, each given as the graph of a holomorphic function over $\{ |z| < \sqrt{r} \}$.

Let $(0,w) \in J_0$ with preimages $(w_1,0)$ and $(w_2,0)$. Since $f_a(W^s_{\text{loc}}(w_{1,2})) \subset W^s_{\text{loc}}(w)$, the two discs from $f_a^{-1}(W^s_{\text{loc}}(w))$ form extensions of $W^s_{\text{loc}}(w_1)$ and $W^s_{\text{loc}}(w_2)$, as graphs of holomorphic functions of $|z| < \sqrt{r}$.

Therefore, by taking the preimages under $f_a$, the family of local stable discs can be extended, each as the graph of a holomorphic function over $|z| < \sqrt{r}$. Applied iteratively, we can extend them as the graphs of holomorphic functions over discs $|z| < r$ for any $r < 1$. In the limit we obtain global stable curves $W^s(w_0)$ through every $w_0 \in J_0$, each of which is the graph if a holomorphic function of $z \in \mathbb{D}$. Since the global stable curves of distinct points in $J_0$ are disjoint, their union gives $W^s(J_0)$ as the image of a holomorphic motion of $J_0$ parameterized by $z \in \mathbb{D}$. \hfill $\square$

6.3. Postcritically Finite Holomorphic Endomorphisms. Until presenting the conjecture of the previous subsection, this paper has been about endomorphisms with complicated Fatou topology. The opposite extreme is that the Fatou topology may also be trivial in many cases. We suspect one simple case in which Fatou topology is trivial is when the map is postcritically finite (PCF).

**Question 6.4.** Does the Fatou set of a postcritically finite holomorphic endomorphism of $\mathbb{P}^2$ always have trivial homology?

A starting point for investigation into this question could be to attempt to establish it for the postcritically finite examples constructed by Sarah Koch [26, 27]. Heuristic evidence supports that the homology is trivial for Koch’s maps. Her construction provides a class of PCF endomorphisms, containing an infinite number of maps, including the previously studied examples of [13] and [10].

6.4. Other holomorphic endomorphisms of $\mathbb{P}^k$. As we have demonstrated in Sections 4 and 5, given some information about the geometry of the support of $T$, we can apply the techniques of Sections 3 to study the Fatou set of a holomorphic endomorphism of $\mathbb{P}^2$. We would like to be able to apply this theorem to other holomorphic endomorphisms of $\mathbb{P}^k$. 
However, specific examples of holomorphic endomorphisms that are amenable to analytic study are notoriously difficult to generate.

One family of endomorphisms which seem a potentially vast area of study are the Hénon-like endomorphisms. Introduced by Hubbard and Papadopol in [21], and studied a bit further by Fornæss and Sibony in [15], these are holomorphic endomorphisms arising from a certain perturbation of the Hénon diffeomorphisms. The Hénon diffeomorphisms have been deeply studied (e.g., by Bedford Lyubich, and Smillie, [2, 4, 5, 6], Hubbard and Oberste-Vorth [19, 20, and Fornæss and Sibony [12]). A natural question which is thus far quite wide open is: how does the dynamics of a Hénon diffeomorphism relate to the dynamics of the perturbed Hénon endomorphism? Computer evidence suggests the dynamics of Hénon-like endomorphisms is rich and varied.

Specifically concerning the topology of the Fatou set, the main result of [5] is that connectivity of the Julia set is determined by connectivity of a slice Julia set in a certain unstable manifold. We ask whether this result would have implications for the related Hénon endomorphism, which would allow us to use linking numbers to establish some analog of Theorem 1.3 for Hénon endomorphisms.

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