Diffeomorphism Groups and Nonlinear Quantum Mechanics

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Abstract. This talk is dedicated to my friend and collaborator, Prof. Dr. Heinz-Dietrich Doebner, on the occasion of his 80th birthday. I shall review some highlights of the approach we have taken in deriving and interpreting an interesting class of nonlinear time-evolution equations for quantum-mechanical wave functions, with few equations; more detail may be found in the references. Then I shall comment on the corresponding hydrodynamical description.

1. Diffeomorphism groups and quantum mechanics

The set $G = \text{Diff}_c(\mathbb{R}^3)$ of compactly-supported $C^\infty$ diffeomorphisms of physical space $\mathbb{R}^3$ is a group under composition, and a topological group in the topology of uniform convergence in all derivatives on compact sets. By the support of a diffeomorphism $\phi$, one means the smallest closed set outside of which $\phi$ acts trivially, i.e. $\phi(\mathbf{x}) = \mathbf{x}$. There is then a sense in which $G$ may be regarded as a fundamental local symmetry group of physical space for quantum mechanics [1], with closed subgroups of $G$ naturally associated with compact sets according to the support of the diffeomorphisms in them.

As an infinite-dimensional group, $G$ is generated by a certain local current algebra that was introduced by Dashen and Sharp in 1968, and which in turn may be obtained from canonical fields [2, 3]. Distinct continuous unitary representations of $G$ then describe and predict a wide variety of distinct quantum-mechanical systems, so that the classification and interpretation of these representations becomes of physical importance in quantum theory [4, 5]. In particular, one obtains descriptions of infinite as well as finite quantum systems, and one describes particle statistics and topological effects in a mathematically natural way. This approach (but replacing $\mathbb{R}^3$ by $\mathbb{R}^2$) led also to an independent prediction in 1980-81 of the possibility of intermediate or “anyonic” statistics in two-space, by Menikoff, Sharp, and me, confirming the idea conjectured by Leinaas and Myrheim in 1977 [6, 7, 8].

Diffeomorphism groups also arise naturally through a method of quantization on configuration space manifolds developed in the early 1980s by Doebner and Tolar and their collaborators [9, 10]. Here the diffeomorphism group that is represented is the group of compactly-supported $C^\infty$ diffeomorphisms of the $N$-particle configuration space. For $N = 1$, where the configuration space consists of the family of 1-point subsets of the physical space, $G$ coincides with this group (of course); while for $N > 1$, $G$ can be regarded naturally as a subgroup.

Henceforth, let us limit our discussion to the case of a single quantum particle. To construct a class of representations of $G$, one considers its natural semidirect product with the additive
topological group of real-valued, compactly-supported $C^\infty$ functions on $\mathbb{R}^3$. The corresponding infinite-dimensional Lie algebra is then a certain (equal-time) algebra of quantum-mechanical mass density (or probability density) operators $\rho(x)$ and momentum density (or probability flux density) operators $J(x)$. The most elementary representations of this algebra lead immediately to ordinary quantum mechanics; but a certain class of unitarily inequivalent representations proved difficult to interpret. In these representations, the current density operator $J(x)$ is modified from its usual representation $J_0(x)$ by the addition of a diffusion current:

$$J(x) = J_0(x) - D \nabla \rho(x), \quad (1)$$

where $D$ is a real number having the dimensions of a diffusion coefficient. Combining this equation with the (standard) equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J(x) = 0, \quad (2)$$

and inserting the usual formulas for the density and the current in terms of the wave function $\psi(x)$, Doebner and the I were led directly to a certain family of nonlinear modifications of the Schrödinger equation [11, 12, 13, 14, 15, 16]. These may provide descriptions of possible irreversible, dissipative, or diffusive quantum-mechanical systems, discussed further in Sec. 4 below. It is interesting to consider them either as phenomenological descriptions approximating more complicated but linear quantum theories, or as possible fundamental processes involving actual deviations from linearity (e.g., at Planck scales).

2. Motivations for nonlinearity in quantum mechanics

Since we have arrived at, and shall discuss, a particular family of nonlinear derivative Schrödinger equations, which includes linearizable Schrödinger equations as special cases, it is worthwhile to step back and inquire about the plausibility of fundamental quantum theories that incorporate such nonlinearities in the time-evolution, and consider our reasons for investigating them. Two important general motivations may be put forth for such investigation.

The first focuses on the likeliness of an actual nonlinearity. There are many physical contexts in which wave equations occur – e.g., sound waves, water waves, electromagnetic waves, vibrating strings and membranes – yet in all of these, the description by means of a linear wave equation is approximate, and nonlinear modifications provide greater precision. Thus it is natural to consider linearity in quantum mechanics – in particular, the superposition principle – as an approximation, perhaps a very good approximation, but not necessarily a perfect one. Of course, quantum mechanics itself is an approximate theory (as are the classical descriptions of sound, light, and vibrating objects). Then it is even more plausible that one might improve on the approximation by introducing nonlinearity in the time-evolution.

The second general motivation focuses on the unlikeliness of an actual nonlinearity. If we take the evidence for strict quantum-mechanical linearity to be convincing, so that (unlike other wave theories) quantum mechanical time-evolution is, in fact, strictly linear, it is natural to ask why. That is, can we identify some physical principle or law whereby nature has “selected” linear quantum mechanics from a wider family of possible nonlinear theories? But to make sense of this question, one must know how to express quantum mechanics in a wider, nonlinear framework. This leads to a set of interesting questions pertaining to the foundations of quantum mechanics; questions that are often bypassed by taking linearity as an axiom rather than considering it to be a possible consequence of other properties within a wider framework.

My perspective is not that of a believer or disbeliever. I think we learn much in science from inquiring into the consequences of relaxing standard assumptions. Here we have, in addition, a mathematical reason to pursue this direction. Let us then explore some aspects of nonlinear quantum mechanics.
3. Assumptions of linearity in quantum mechanics

The assumption of linearity does not enter quantum mechanics just once. In fact, there is a set of interrelated but logically partially independent ways in which it enters. When we consider weakening or eliminating one of these assumptions, it is too strong a constraint to expect all of the others to remain as is. Indeed, the work that Doebner and I have done together goes considerably beyond the formulation of a natural family of nonlinear time-evolution equations. We have, in addition, explored certain concrete ways in which other assumptions of linearity in quantum mechanics should be relaxed.

The following are some assumptions of linearity usually made in quantum mechanics:

0. Pure states form a linear space (over the complex numbers \( \mathbb{C} \)), on which a sesquilinear form defines an inner product (leading to the Hilbert space \( \mathcal{H} \)).
1. The time-evolution in \( \mathcal{H} \) is linear, leading to the superposition principle.
2. Quantum-mechanical observables are described by self-adjoint (hence linear) operators. The spectrum of such an operator corresponds to the set of possible measurement outcomes.
3. Gauge transformations are implemented by linear (in fact, unitary) operators.
4. Projection postulate: At the time a measurement is performed, a pure state (which then serves as the initial condition for the subsequent time-evolution) is obtained by linear, orthogonal projection of the state vector onto the subspace associated (via the spectral theorem) with the observable’s measured value or range of values.
5. Mixed states are described by density matrices, obtained as sums or integrals of linear one-dimensional projection operators onto pure states.
6. N-particle quantum mechanics is obtained by linear extension of the operators describing all the 1-particle observables to the tensor product of the 1-particle Hilbert spaces. More generally, composite systems are described by linearly extending the operators describing the component observables to the tensor product of the component state-spaces.
7. Multiparticle wave functions describing identical particles (bosons or fermions) are obtained by linear symmetrization or, respectively, antisymmetrization of sets of 1-particle wave functions.

Next let us write just a few equations of nonlinear quantum mechanics, allowing us to highlight how certain of these assumptions are to be modified.

4. Nonlinear quantum mechanics, measurement, and gauge transformations

The equation for the wave function \( \psi \) obtained from (1)-(2) is
\[
\frac{i\hbar}{\hbar} \frac{\partial}{\partial t} \psi = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right\} \psi + \frac{i}{2} \hbar \frac{\nabla^2 \rho}{\rho} \psi ,
\]
where \( \rho = \overline{\psi} \psi \). Note that the new, nonlinear term is a purely imaginary homogenous functional of \( \psi \). It is only the imaginary part of such new, nonlinear terms, that is fixed by Eq. (1). As the equation is nonlinear, there remains no fundamental reason to exclude real, homogeneous nonlinear functionals of \( \psi \) from entering the time-evolution.

Thus we set \( \mathbf{j} = (1/2i)\overline{\psi} \nabla \psi - (\nabla \overline{\psi}) \psi \), where the probability flux \( \mathbf{j} = (\hbar/m) \mathbf{j} \), and define the homogeneous nonlinear functionals
\[
R_1 = \frac{\nabla \cdot \mathbf{j}}{\rho}, \quad R_2 = \frac{\nabla^2 \rho}{\rho}, \quad R_3 = \frac{\mathbf{j} \cdot \mathbf{j}}{\rho^2}, \quad R_4 = \frac{\mathbf{j} \cdot \nabla \rho}{\rho^2}, \quad R_5 = \frac{(\nabla \rho)^2}{\rho^2} .
\]
A more general nonlinear Schrödinger equation of the desired type, natural from several points of view, is then
\[
\frac{i\hbar}{\hbar} \frac{\partial}{\partial t} \psi = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(x) \right\} \psi + \frac{i}{2} \hbar D \sum_{j=1}^5 R_j [\psi] \psi ,
\]

where the real parameters $D_j'(j = 1, \ldots, 5)$ are a further set of diffusion coefficients.

At this stage, one might think that the nonlinear terms in Eq. (5) represent (possibly small) perturbations of linear quantum mechanics describing distinct physical effects, and that the effect magnitudes are governed by the diffusion coefficients. However, we shall see shortly that a different interpretation is needed.

Finally, we expand the Laplacian of $\psi$ in the basis of functionals provided by the $R_j$, 

$$\frac{\nabla^2 \psi}{\psi} = iR_1 + \frac{1}{2}R_2 - R_3 - \frac{1}{4}R_5;$$

we introduce general external scalar and vector potentials into Eq. (5); and we allow slightly more general real functionals proportional to the modulus and the phase of $\psi$. There results the very general nonlinear Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = \left[ \sum_{j=1}^{2} \nu_j R_j + \frac{\nabla \cdot (A\rho)}{\rho} \right] \psi + \sum_{j=1}^{5} \mu_j R_j + U + \frac{\nabla \cdot (A_1 \rho)}{\rho} + \frac{A_2 \cdot \vec{A}}{\rho} + \alpha_1 \ln \rho + \alpha_2 S, \quad (7)$$

where $\nu_j (j = 1, 2)$, $\mu_j (j = 1, \ldots, 5)$, and $\alpha_j (j = 1, 2)$ are real coefficients; $A = A(x, t)$, $A_1 = A_1(x, t)$, and $A_2 = A_2(x, t)$ are three independent external real vector potentials; $U = U(x, t)$ is an external real scalar potential; and $S$ is the phase of the wave function.

The usual, linear Schrödinger equation describing a particle of mass $m$ and charge $q$ moving in an external electromagnetic potential $(\Phi, A)$, with an additional external nonelectric scalar potential $V$, corresponds to the values $\nu_1 = -\hbar/2m$, $\nu_2 = 0$, $\mu_1 = 0$, $\mu_2 = -\hbar/4m$, $\mu_3 = \hbar/2m$, $\mu_4 = 0$, $\mu_5 = \hbar/8m$, $\alpha_1 = \alpha_2 = 0$; $A = (q/2mc)A$; $A_1 = 0$; $A_2 = -(q/mc)A$; and $U = (1/\hbar)[V + q\Phi] + (q^2/2mc)(A \cdot A)$. The coefficient $D$ in Eq. (5) corresponds to $\nu_2 = D/2$.

But more general values, including time-dependent values, of all the coefficients are possible. This family of nonlinear Schrödinger equations includes as special cases, or intersects with, not only the linear Schrödinger equation but numerous specific but often ad hoc proposals that have been made over the years for nonlinear modifications of quantum mechanics – proposals by Kibble, Guerra and Pusterla, Sabatier and Auberson, Bialynicki-Birula and Mycielski, Kostin, Haag and Bannier, and Schuch; for citations and further discussion, see Ref. [13].

Note that there is no longer an easy way to write the Fourier transform of Eq. (5) or (7) so as to obtain a quantum time-evolution in momentum space. Indeed, positional space has been privileged here – fundamentally privileged, as a consequence of our having started with the group of its diffeomorphisms. This immediately points toward a way to modify the usual quantum-mechanical theory of measurement so that it can apply in such a nonlinear context.

We pursue the direction taken by Mielnik [17], where positional measurements are taken to be fundamental. The absolute square of the wave function in (positional) configuration space continues as in linear quantum mechanics to predict the probability distribution for the outcome of a positional measurement. But any other measurement (including measurement of momentum, angular momentum, etc.) is then to be described as derived from a succession of positional measurements, taken at different times, with the possible intervening imposition of external fields. The derivation from positional measurements must, of course, take account of the time-evolution theory.

For this purpose, we adopt a preferred representation – the positional configuration space – over which the wave function is defined. For a single particle, $|\psi(x, t)|^2$ describes the probability density for finding the particle at $x$ at time $t$, with $\int_{\mathbb{R}^3} |\psi|^2 dx = 1$. We further assume a (positional) “projection postulate”: when an idealized, nondestructive measurement localizes the particle described by $\psi$ at time $t_0$ to a region $B$, the initial state for the subsequent time-evolution is given by

$$\frac{\chi_B(x)\psi(x, t_0)}{\int_B |\psi(x, t_0)|^2 dx}, \quad (8)$$
where \( \chi_B(x) \) is the indicator function with respect to the region \( B \). Physically, this means that such a localization of the particle’s position in a region of space does not change the (local) current density in that region.

With the measurement theory thus adapted, we next consider gauge transformations. As in linear quantum mechanics, these are transformations of the wave function that are local in space and time, but leave the outcomes of all measurements invariant. As in the linear case, the time-evolution equation can also change. However, in the framework of nonlinear quantum mechanics, there is no longer any reason to restrict ourselves to linear gauge transformations. We thus arrive at a larger gauge group, one that includes the local \( U(1) \) gauge transformations of electromagnetism, but whose elements can act nonlinearly on wave functions.

Writing \( \psi = R \exp iS \), where the modulus \( R(x,t) \) and the phase \( S(x,t) \) are real, a nonlinear gauge transformation is specified by smooth, real-valued functions \( (\Lambda, \gamma, \theta) \), with \( \Lambda \neq 0 \); where \( \Lambda = \Lambda(t), \gamma = \gamma(t), \) and \( \theta = \theta(x,t) \). Under such a transformation \( \psi \to \psi' = R' \exp iS' \), with

\[
R' = R, \quad S' = \Lambda S + \gamma \ln R + \theta.
\] (9)

These obey the semidirect product group law

\[
(\Lambda_1, \gamma_1, \theta_1)(\Lambda_2, \gamma_2, \theta_2) = (\Lambda_1 \Lambda_2, \gamma_1 + \Lambda_1 \gamma_2, \theta_1 + \Lambda_1 \theta_2).
\] (10)

The subgroup defined by \( \gamma(t) \equiv 0 \) and \( \Lambda(t) \equiv 1 \), is the usual group of linear gauge transformations in quantum mechanics.

Then if \( \psi \) obeys a nonlinear Schrödinger equation belonging to the class described by Eq. (7) above, with some choice of coefficients \( \nu_j, \mu_j, \alpha_j \) and external fields \( U, A, A_1, A_2 \), then \( \psi' \) obeys a nonlinear Schrödinger equation in the same class, but with transformed coefficients and external fields. The two theories are physically equivalent! In particular, this means that there is family of nonlinear quantum theories linearizable via nonlinear gauge transformations, and therefore physically equivalent to ordinary quantum mechanics.

The probability density \( \rho(x,t) \) is manifestly gauge-invariant under (9). One also introduce a gauge-invariant current, \( J^g = -2\nu_1 \hat{j} - 2\nu_2 \nabla \rho - 2\rho A \). Now the physical content of a nonlinear quantum theory must be described not by the coefficients and fields in Eq. (7), but by gauge-invariant combinations of coefficients, and by gauge-invariant fields. Explicit formulas for the action of nonlinear gauge transformations on coefficients and fields, and for gauge-invariant quantities, may be found (for example) in [14], so I shall not reproduce them here. The notion of physical equivalence classes (under nonlinear gauge transformation) of the nonlinear quantum time-evolutions in Eq. (7) then provides a further unification of the many historically-proposed nonlinear quantum theories, together with some newer ones.

Let me remark that if \( \Lambda \equiv 1 \), Eq. (9) specifies a well-defined action of the nonlinear gauge transformation on wave functions; but where \( \Lambda \neq 1 \), this is not the case and further discussion is needed. However, the action of nonlinear gauge-transformations on the coefficients and fields of Eq. (7) is well-defined in all cases.

A question that remains open is the following. Given a set of \( N \) possibly distinct single-particle nonlinear time-evolutions, what extensions to \( N \)-particle nonlinear time-evolutions (if any) respect the "no signal" property of linear quantum mechanics? In a sense this is the fundamental question of nonlinear quantum mechanics, as it pertains to the relationship between linearity (or, more precisely, linearizability under gauge transformation) and local causality.

5. Hydrodynamical variables

Let me close with a brief discussion of hydrodynamical variables in the context of nonlinear quantum mechanics, based on ideas that go back to Madelung’s 1927 formulation [18].
In the usual, most elementary formulation, one introduces a velocity field \( \mathbf{v} = \mathbf{j}/\rho = (\hbar/m) \nabla S \). In regions where \( S \) and \( \nabla S \) are smooth and nonsingular, one then has \( \nabla \times \mathbf{v} = 0 \), and \( \oint_{\Gamma} \mathbf{v} \cdot d\ell = 0 \) around closed paths \( \Gamma \). But this is not so in general, as \( S \) is undefined at zeroes of the wave function \( \psi \). Rather, single-valuedness of \( \psi \) (whereby \( S \) is well-defined up to \( 2\pi n \)) motivates the “quantization” condition \( \oint_{\Gamma} (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = (2\pi n \hbar/m) \), where \( \sigma \) is an oriented, bounded surface.

Then the Schrödinger equation, expressed in terms of the hydrodynamical variables \( \rho \) and \( \mathbf{v} \), becomes the coupled system

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})
\]

\[
\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left[ \frac{1}{2} \mathbf{v} \cdot \mathbf{v} + \frac{\hbar^2}{2m^2} \left( -\frac{\nabla^2 \rho}{\rho} + \frac{1}{4} \frac{|\nabla \rho|^2}{\rho^2} \right) + \frac{1}{m} V \right].
\]  

(11)

The first equation is the equation of continuity. The terms with the coefficient \( \hbar^2/2m^2 \) in the second equation are called the “quantum potential”; as \( \hbar/m \to 0 \) this quantum potential disappears (classical limit), leaving the Euler equation for an idealized fluid.

More general hydrodynamical equations result from the general nonlinear time-evolution described by Eq. (7). In fact, defining the gauge-invariant velocity field \( \mathbf{v}^{gi} = \mathbf{J}^{gi}/\rho \), one obtains

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}^{gi})
\]

\[
\frac{\partial \mathbf{v}^{gi}}{\partial t} = \nabla \left[ \frac{1}{2} \mathbf{v}^{gi} \cdot \mathbf{v}^{gi} + 2\tau_2 \frac{\nabla^2 \rho}{\rho} + 2\tau_3 \frac{|\nabla \rho|^2}{\rho^2} - \frac{1}{m} V \right] + \nabla \left[ 2\tau_1 \mathbf{v}^{gi} \cdot \mathbf{v}^{gi} + (2\tau_1[1 + \tau_3] - \tau_4) \mathbf{v}^{gi} \cdot \frac{\nabla \rho}{\rho} + 2 \mathbf{v}^{gi} \cdot \frac{\rho \mathbf{A}^{gi}_1}{\rho} - 2\tau_3 \mathbf{v}^{gi} \cdot \mathbf{A}^{gi}_2 + 2\beta_1 \ln \rho \right]
\]

\[+ \frac{q}{m} \mathbf{E} - \beta_2 \mathbf{v}^{gi},
\]

together with \( \nabla \times \mathbf{v}^{gi} = (q/m) \mathbf{B} \), and a quantization condition for \( \mathbf{v}^{gi} \); where \( \mathbf{E} \) and \( \mathbf{B} \) are gauge-invariant external fields exerting electric and magnetic forces respectively on the particle; the \( \tau_j, j = 1, \ldots, 5 \), are independent, gauge-invariant parameters derived from the coefficients \( \nu_j \) and \( \mu_j \); the \( \beta_j, j = 1, 2 \), are independent, gauge-invariant parameters derived from the coefficients \( \alpha_j \); and \( \mathbf{A}^{gi}_1 \) and \( \mathbf{A}^{gi}_2 \) are independent, gauge-invariant external vector fields coupling nonlinearly to the wave function.

The quantum hydrodynamical Eqs. (12) are gauge-invariant under the full group of nonlinear gauge transformations described above. Particularly interesting features include the fact that the quantum potential term is now governed by two coefficients instead of one; the presence of the term \( 2\tau_1 \mathbf{v}^{gi} \cdot \mathbf{v}^{gi} \), which takes us from Euler to Navier-Stokes hydrodynamics; the possibility of forces exerted by the two new external vector fields; and the explicit frictional term \( -\beta_2 \mathbf{v}^{gi} \).

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References

[1] Goldin, G A 2004 Lectures on diffeomorphism groups in quantum physics. In Contemporary Problems in Mathematical Physics: Proceedings of the Third Int’l. Conference, Cotonou, Benin ed J Govaerts et al (Singapore: World Scientific) pp 3-93
[2] R. Dashen, R and Sharp, D H 1968 Currents as coordinates for hadrons. Phys. Rev. 165 1857–66
[3] Goldin, G A 1971 Non-relativistic current algebras as unitary representations of groups. J. Math. Phys. 12 462–87
[4] Goldin, G A and D. H. Sharp, D H 1970 Lie algebras of local currents and their representations. In Group Representations in Mathematics and Physics: Battelle-Seattle 1969 Rencontres ed V Bargmann. Lecture Notes in Physics 6 (Berlin: Springer) (pp. 300–10)
[5] Goldin, G A, Grodnik, J, Powers, R T and Sharp, D H 1974 Nonrelativistic current algebra in the ‘N/V’ limit. J. Math. Phys. 15 88–100
[6] Leinaas, J M and Myrheim, J 1977 On the theory of identical particles. Nuovo Cimento 37B 1–23
[7] Goldin, G A, Menikoff, R and Sharp, D H 1980 Particle statistics from induced representations of a local current group. J. Math. Phys. 21 650–64
[8] Goldin, G A, Menikoff, R and Sharp, D H 1981 Representations of a local current algebra in non-simply connected space and the Aharonov-Bohm effect. J. Math. Phys. 22 1664–68
[9] Angermann, B, Doebner, H-D and Tolar, J 1983 Quantum kinematics on smooth manifolds. In Nonlinear Partial Differential Operators and Quantization Procedures ed S I Andersson and H-D Doebner. Lecture Notes in Mathematics 1037 (Berlin: Springer) pp 171–208
[10] Doebner, H-D, Stovicek, P and Tolar, J 2001 Quantization of kinematics on configuration manifolds. Rev. Math. Phys. 13 799–845
[11] Doebner, H-D and Goldin, G A 1992 On a general nonlinear Schrödinger equation admitting diffusion currents. Phys. Lett. A 162 397–401
[12] Doebner H-D and Goldin G A 1994 Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations. J. Phys. A: Math. Gen. 27 1771–80
[13] Doebner, H-D and Goldin G A 1996 Introducing nonlinear gauge transformations in a family of nonlinear Schrödinger equations. Phys. Rev. A 54 3764–71
[14] Doebner, H-D, Goldin, G A and Nattermann, P 1999 Gauge transformations in quantum mechanics and the unification of nonlinear Schrödinger equations. J. Math. Phys. 40 49–63
[15] Doebner, H-D and Goldin, G A 2005 On nonlinearity in quantum mechanics and the stationary states of hydrogen and antihydrogen. In Group Theoretical Methods in Physics, Inst. of Physics Conf. Series No. 185 ed G S Pogosyan et al pp. 243–8
[16] Goldin, G A 2008 Nonlinear quantum mechanics: results and open questions. Physics of Atomic Nuclei (Yadernaia Fizika) 71 910–7
[17] Mielnik, B 1974 Generalized quantum mechanics. Commun. Math. Phys. 37 221–56
[18] Madelung, E 1927 Quantentheorie in hydrodynamischer Form. Z. Phys. 40 322–6