Spatially dispersive surface impedance in the
electrodynamics of conductors without dc dissipation

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Abstract. We derive general frequency dependencies of the surface impedance modulus for
conductors without the dc dissipation, i.e. for superconductors or perfect conductors. The
frequency-dependent surface impedance was applied for the solutions corresponding to the
spatially dispersive eigenvalues of the permittivity operator for conductors. We demonstrate
that appropriately taken into account effects of the spatial dispersion can give the general
frequency dependence of the surface impedance for the obtained solutions including that
for superconductor. It is shown that an incorporation of the spatial dispersion leads to
an appearance of the Meissner effect in perfect conductors in the same manner as in
superconductors.

1. Introduction
Recently the frequency-dependent surface impedance was calculated for the spatially dispersive
eigenvalues of the permittivity operator in conductors [1]. It is reasonable to give a
brief description these formulations valid for both conductors and superconductors The
electrodynamics of superconductors is supposed to be in principle reduced to those for conductors
as the temperature approaches the critical temperature \(T_c\) and beyond \(T_c\). However, such
reduction is not rather straightaway. Early the problem resulted in a supplement of the Maxwell
equations by postulating additional London equations for an explanation the Meissner effect
[2]. It discriminates the perfect conductor in non-zero-field cooling behavior as compared to the
superconductor. Here we demonstrate that appropriately taken into account effects of the spatial
dispersion can give general frequency dependencies of the surface impedance for the obtained in
[1] solutions including those both for the normal conductor and for the superconductor or perfect
conductor. It is shown that an incorporation of the spatial dispersion leads to an appearance of
the Meissner effect in perfect conductors in the same manner as in superconductors.

2. Frequency dependencies
To evaluate the general frequency dependence of the eigenvalue modulus of absolute permittivity
operator \(|\tilde{\varepsilon}_a|\) and of the surface impedance modulus \(|\tilde{Z}|\), let us again [1] consider at first only
the spatial effects in permittivity, assuming the problem is stationary \(\omega = 0\) and taking into
account only the spatial field inhomogeneity in the form of a wave number \( k'' \). Preliminary results concerning the frequency dependencies were discussed in [3]. The designations of symbols in the equations below are generally accepted [1, 3–9] and can be found in [1, 3, 7, 8].

2.1. Normal conductivity

The first of Maxwell equations can be written as

\[
\text{rot } \vec{H} = \left( \frac{\partial \vec{D}}{\partial t} \right) + \vec{j},
\]

(1)

The frequency \( \omega'' = k'' v_F \) can be associated with the spatial inhomogeneity \( k'' \) [1], where \( v_F \) is the Fermi velocity, i.e. the velocity of propagation of the external perturbation. Using the ambiguity of the representation of a right side of the previous expression [4–6] and integrating it over the time one can get from the constitutive equation [1] \( \vec{D}(\vec{r}, t) = \vec{\varepsilon}_a \vec{E}(\vec{r}, t) \) an expression [7]

\[
\vec{\varepsilon}_a = \vec{\varepsilon}_p \varepsilon_0 - i \tilde{\sigma} / \omega_k = \vec{\varepsilon}_p \varepsilon_0 - i \tilde{\sigma} / (k'' v_F),
\]

(2)

where \( \tilde{\sigma} \) is the static eigenvalue of conductivity operator, \( \vec{\varepsilon}_p \) is the eigenvalue of relative permittivity operator of a lattice. The solutions found in [1] for conductors and superconductors have the fixed arguments of complex numbers \( \vec{\varepsilon}_a \) and \( \vec{Z} \sim 1/\vec{k} \) determined by conditions of the spatial force resonances. So for a finite frequency \( \omega \) of an external field the spatial inhomogeneity \( k'' \) is determined, according to [1, 7], by the frequency and by the eigenvalue of permittivity operator

\[
k'' \sim \omega(\vec{\varepsilon}_a \mu_0)^{1/2} = \omega/\vec{v}_{ph},
\]

(3)

that may be assigned to a mean field approximation. The conductivity in the extremely anomalous limit is determined by the frequency \( \tilde{\sigma} \sim 1/\omega \) also [8]. The substitution of \( k'' \) and \( \tilde{\sigma} \) into the expression (2) for \( \vec{\varepsilon}_a \) results in the proportionality \( (\vec{\varepsilon}_a)^{3/2} \sim \omega^{-2} \) that corresponds to the frequency dependence of the impedance modulus \( |\vec{Z}| \sim \omega^{2/3} \) for all solutions found for the conductor except that for the superconductor, as it was shown in [1].

2.2. Superconductors and perfect conductors

To estimate the frequency dependencies of moduli \( |\vec{Z}| \) and \( |\vec{\varepsilon}_a| \) for the solution found for superconductors or perfect conductors having the phase of \( \vec{\varepsilon}_a \) equal to \( \beta_1 = \pi \) and the phase of \( \vec{Z} \) equal to \( \psi_1 = \pi/2 \) (and additionally for the solution with \( \alpha_0 = 0 \) and \( \varphi_0 = 0 \)) [1] one should note that for these solutions the real parts of eigenvalue \( \tilde{\sigma} \) of the conductivity operator in the equation (2) equal zero and the direct current does not dissipate with time. In this case one should take into account an influence of spatial inhomogeneity of the field with the largest possible value of a wave number \( k'' \) or of a frequency \( \omega'' = k'' v_F \), regarding an absence of the dc dissipation, for all possible spatial scales, but not only for those corresponding to the frequency of an incident wave (cf. the Eq. (3)). The velocity of propagation of a perturbation in the quasi stationary environment without the dc dissipation could correspond to the spatially dispersive value \( v \), which should not be necessarily associated with the Fermi velocity \( v_F \), but may be equal to the speed of light \( c \), for example, as in the Langmuir plasma frequency. The highest value of spatial inhomogeneity of the field with the largest possible value of a wave number \( k'' \) or of an associated frequency \( \omega'' = k'' v \) in this case can approach the vertical-asymptote values equal to the \( k''_0 \) or to the frequency proportional to \( \omega''_p \) or \( k''_0 v_F \), corresponding to the Langmuir plasma frequency

\[
\omega_p = k''_0 c = c/\lambda_0,
\]

(4)

where

\[
\lambda_0 = (k''_0)^{-1} = (m/\mu_0 e^2)^{1/2}
\]

(5)
is the London penetration depth. The absence of the dc dissipation should result in the feasibility of "acoustic" conditions \( v > 0 \) for the spatially dispersive plasmon polariton frequency \( \omega''_{kp} = k''v \).

This spatial inhomogeneity will correspond to the highest value of the Abraham force (see Eq.(13) below and Ref. [1]). So it should result in the spatial structure dominating over the others possible structures [1, 7]. A value of this spatial inhomogeneity will not depend on frequency values \( \omega \) of any incident microwaves, while those could be considered as quasi stationary when \( \omega < \omega''_{kp0} \approx \omega_p \). This can be illustrated by an expression of a current density after the end of an external perturbation obtained with the account of the spatial dispersion [8]. If one substitutes the spatial factor in Eq.(3) of Ref.[8] in the form of

\[
j_0(\theta, r) = j_0(\theta)e^{i\kappa r}, \tag{6}
\]

where \( \theta \) is the reduced temperature \( \theta = T/T_c \) [8, 9], it is reexpressed as the current density

\[
j(\theta, r, t) = j_0(\theta)e^{-\omega'_k t}e^{i(\kappa r - \omega'_p t)}, \tag{7}
\]

which remains after switching off an electric field \( E = E_0e^{i(kr+\omega t)} \). In a perfect conductor the direct current does not decay with time \( \omega'_k = 0 \) and the stationary current solution (6) may be reconstructed from (7) by the proper time averaging of plasma oscillations with positive and negative signs of the plasmon polariton frequency \( \omega''_k = \omega''_{kp0} \), by a finite time shift in their starting moments, for example.

The requirement of an independence of the spatial inhomogeneity \( k'' \) on the frequency \( \omega \) should result, according to the equation

\[
k'' = \omega(\varepsilon_a|\mu_0|^{1/2} = \omega/|\varepsilon_{ph}|, \tag{8}
\]

in the inverse ratio \( \sqrt{\varepsilon_a} \sim 1/\omega \) and in the dependence of the surface impedance (its modulus) linear on a frequency \( \omega \)

\[
|Z| = \frac{\mu_0}{|\varepsilon_a|} = \frac{\omega|\mu_0|}{k''}. \tag{9}
\]

The frequency dependence of the eigenvalue \( \varepsilon_a \) in the right side of the Eq. (8) is determined by the requirement of the independent on \( \omega \) wavenumber \( k'' \) in the left side, inversely to their correlations in the Eq. (3), where the wavenumber \( k'' \) is determined by the eigenvalue \( \varepsilon_a \) according to the mean-field approximation. The further temperature issue of how the electrodynamics of superconductors can be reduced to those for conductors as the temperature approaches and beyond the critical temperature may be clearly stated using, e. g., the generalized two-fluid principles [8,9] or rigorous modeling [9,10].

3. The spatial dispersion and Meissner effect in superconductors and perfect conductors

In the surface impedance approximation [4–7,9,10] let it will be the coordinate \( z \) with the basis vector \( k \), which are directed into the conductor together with the unitary normal \( n \) to the boundary surface \( z = 0 \). A monochromatic transverse field at the normal incidence to the plane surface of a conductor will be considered. An electric field strength vector is directed tangentially along the \( x \)-axis \( (\mathbf{E}(z) = \mathbf{E}_z(z)i) \), and a magnetic induction vector is directed along the \( y \)-axis \( (\mathbf{B}(z) = \mathbf{B}_y(z)j) \). Here \( i \) and \( j \) are the basis vectors. The fields decrease into the depth of a conductor as \( \mathbf{B}_y(z) = \mathbf{B}_y(0)e^{-z/\delta} \) and \( \mathbf{E}_z(z) = \mathbf{E}_z(0)e^{-z/\delta} \), where \( \delta \) is the complex penetration depth, which defines the surface impedance by the relation \( Z = i\omega|\mu_0\delta| [4–7,9,10] \), and \( \mu_0 \) is the
The expression (13) shows that Abraham force at all frequencies on the boundary surface [7, 9]
\[ \vec{Z} = \frac{\vec{E}_x(0)}{H_y(0)} = \frac{\vec{B}_y(0)}{D_x(0)}, \]
the zero limit of the surface impedance \( \vec{Z}|_{\omega \to 0} = 0 \) at the zero frequency, deduced from the Eq. (9), means that the stationary tangential component of an electric field (or of a magnetic induction) should be equal zero \( \vec{E}_x(0)|_{\omega \to 0} = 0 \) (or \( \vec{B}_y(0)|_{\omega \to 0} = 0 \), but with a possible finite value of \( H_y(0)|_{\omega \to 0} \neq 0 \) (or \( D_x(0)|_{\omega \to 0} \neq 0 \)).

Let us consider behaviour of the Abraham force in a perfect conductor (or in a superconductor) in the limit of zero frequency \( \omega \to 0 \). From the expression for the spatial density of the momentum flux [1, 7], which is the corresponding component of the Maxwellian field-stress tensor,
\[ \vec{\Pi} = [\vec{D} \times \vec{B}] = (\mu_0^{1/2} \varepsilon_a^{3/2}) \vec{E}_x^2(0) \mathbf{k}, \]
where \( \mathbf{k} \) is the basis vector of the z-axis, the Abraham force can be derived [4, 6] neglecting the frequency dispersion:
\[
\left(1 - \frac{1}{\varepsilon} \right) \frac{\partial \vec{\Pi}}{\partial t} = \left(1 - \frac{1}{\varepsilon} \right) \frac{\partial}{\partial t} [\vec{D} \times \vec{B}] = \\
= \left(1 - \frac{1}{\varepsilon} \right) \frac{\partial}{\partial t} \left[ \vec{D}_x(0) e^{i\omega t} \mathbf{i} \times \vec{B}_y(0) e^{i\omega t} \mathbf{j} \right] = \\
= 2i\omega \left(1 - \frac{1}{\varepsilon} \right) e^{i2\omega t} \left[ \vec{D}_x(0) \mathbf{i} \times \vec{B}_y(0) \mathbf{j} \right],
\]
where \( \varepsilon = \varepsilon_a / \varepsilon_0 \) is the eigenvalue of relative permittivity operator. The surface impedance (10) defines the relation between electric and magnetic inductions \( \vec{D}_x(0) \) and \( \vec{B}_y(0) \) and may be substituted in the expression (12) for the Abraham force. Then using the Eq. (9) for the modulus of surface impedance and taking account of its phase \( \psi_1 = \pi/2 \) we obtain
\[
2i\omega \left(1 - \frac{1}{\varepsilon} \right) e^{i2\omega t} \left[ \frac{\vec{B}_y(0)}{Z} \mathbf{i} \times \vec{B}_y(0) \mathbf{j} \right] = \\
= 2i\omega \left(1 - \frac{1}{\varepsilon} \right) e^{i2\omega t} \left[ \frac{-ik''_A}{\omega \mu_0} \vec{B}_y(0) \mathbf{i} \times \vec{B}_y(0) \mathbf{j} \right] = \\
= 2 \left(1 - \frac{1}{\varepsilon} \right) e^{i2\omega t} \left[ \frac{k''_A}{\mu_0} \vec{B}_y(0) \mathbf{i} \times \vec{B}_y(0) \mathbf{j} \right] = \\
= 2 \left(1 - \frac{1}{\varepsilon} \right) e^{i2\omega t} \left[ \frac{\text{rot} \vec{H}(0)}{\mu_0} \times \vec{H}(0) \right].
\]
The expression (13) shows that Abraham force at all frequencies \( \omega \ll \omega_p \), as well as in the stationary limit \( \omega \to 0 \), achieves the highest magnitude when the spatial inhomogeneity \( k''_A \) gets the largest, frequency-independent value (see Sec. 2.2) corresponding to the London (or plasma) penetration depth \( \lambda_0 = (k''_A)^{-1} = (m/\mu_0 e^2)^{1/2} \). Such the spatial configuration of the fields related to the reconstruction of an electronic system, which corresponds to the highest magnitude of the Abraham force, is dominant due to the force correlations and the consequential decrease of the free energy [1, 7]. And respectively, the absolutely unstable spatial configuration, according to Eq. (13), corresponds to the geometry of the spatially homogeneous field and to the zero value of the spatial inhomogeneity \( k''_A = 0 \) when the Abraham force is zero with no advantage in the free energy. So the conclusions obtained here in the surface impedance approximation, which is
quite adequate to the general description of Meissner effect [9], show that the spatial dispersion results in an appearance of the Meissner effect in perfect conductors in the same manner as in superconductors contrary to the preceding considerations [2], which do not incorporate the spatial dispersion effects. A driving force of its appearance is the Abraham force.

It should be noted that this conclusion may be derived more formally, without considerations of any frequency dependencies. Abraham force can be reexpressed as

\[
\left( 1 - \frac{1}{\varepsilon} \right) \frac{\partial \Pi}{\partial t} = \left( 1 - \frac{1}{\varepsilon} \right) \frac{\partial}{\partial t} \left[ \mathbf{D} \times \mathbf{B} \right] =
\]

\[
= \left( 1 - \frac{1}{\varepsilon} \right) \left[ \frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B} \right] + \left( 1 - \frac{1}{\varepsilon} \right) \left[ \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t} \right]
\]

For monochromatic fields the Maxwell equations in a homogeneous medium with the spatial dispersion may be written [1, 4–7] as

\[
\text{rot} \mathbf{B} = \mu_0 \text{rot} \mathbf{H} = \mu_0 (\partial \mathbf{D}/\partial t); \quad \text{rot} \mathbf{E} = - (\partial \mathbf{B}/\partial t);
\]

\[
\text{div} \mathbf{D} = \text{div} \mathbf{E} = 0; \quad \text{div} \mathbf{B} = 0.
\]

Due to an absence of magnetic charges and an equality \( \partial \mathbf{B}/\partial t = 0 \) in the second of Eqs (15) in the stationary approximation \( \omega = 0 \) the second term in the sum (14) equals zero. Regarding the ambiguity of the representation of the first Maxwell equation (cf. the first of Eqs (15) and the Eq. (1) \( \text{rot} \mathbf{H} = (\partial \mathbf{D}/\partial t) + \mathbf{j} \)), one can replace the partial time derivative in the first term of the sum (14), which may be assigned formally to the "stationary displacement current" or simply to the current density \( \mathbf{j} \), by the \( \text{rot} \mathbf{H} \) from the first of Eqs (15), that results in

\[
\left( 1 - \frac{1}{\varepsilon} \right) \left[ \text{rot} \mathbf{H} \times \mathbf{B} \right] = \left( 1 - \frac{1}{\varepsilon} \right) \left[ \frac{k''}{\mu_0} \mathbf{B}_0 \mathbf{j} \times \mathbf{B}_0 \right].
\]

This expression shows that Abraham force achieves the highest magnitude when the spatial inhomogeneity \( k'' \) gets the largest value. It corresponds to a half of the London penetration depth \( \lambda_0/2 = (2k''_0)^{-1} = (m/\mu_0 \alpha e^2)^{1/2} \) taking into account a factor of 2 that differentiates the Eqs (13) and (17) in the stationary approximation \( \omega = 0 \) (cf. an introduction of the effective constitutive parameters in [3]).

4. London equations and Abraham force

The partial derivative of magnetic induction with respect to time in the second of Maxwell equations (15) may be represented as

\[
(\partial \mathbf{B}/\partial t) = - \text{rot} \mathbf{E} = - \text{rot} (\mathbf{j}/\sigma) = - (i \omega \mu_0/(k''_0)^2) \text{rot} \mathbf{j},
\]

where the first London equation [2, 8, 9] \( d\mathbf{j}(z, t)/dt = d(\mathbf{j}(z)e^{i\omega t})/dt = i \omega \mathbf{j}(z, t) = (1/\mu_0 \lambda_0^2)\mathbf{E} = ((k''_0)^2/\mu_0)\mathbf{E} \) has been used to derive the conductivity \( \sigma \) from Ohm’s law \( \mathbf{j} = \sigma \mathbf{E} \) = \( (1/\omega \mu_0 \lambda_0^2)\mathbf{E} = ((k''_0)^2/\omega \mu_0)\mathbf{E} \). Substituting the Eq. (18) and second London equation [2, 9] \( \text{rot} \mathbf{j} = - (1/\lambda_0^2) \mathbf{H} = - (k''_0)^2 \mathbf{H} \) into the last term of Abraham force (14), one gets the reinforced expression instead of the Eq. (17)

\[
\left( 1 - \frac{1}{\varepsilon} \right) \left[ \frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B} \right] + \left( 1 - \frac{1}{\varepsilon} \right) \left[ \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t} \right] =
\]
Here the modulus (Eq. (9)) and argument $\psi$ of the complex surface impedance $\tilde{Z}$ have been used similarly to the Eq. (13). Comparisons of the Eqs (13), (17) and (19) in the stationary approximation reveal the double ratios $k''_j = 2k''_s = 2k''_0$ in the force expressions. These equalities demonstrate that finding the proper, empirically varified solution of the electrodynamic problem may require an introduction of the effective constitutive parameters (cf. [3]) or postulating the additional phenomenological equations (cf. Sec. 4) depending on the kind of representation of the original problem operator.

5. Conclusions
We derived the general frequency dependence of the frequency-dependent surface impedance for the solutions corresponding to the spatially dispersive eigenvalues of the permittivity operator for conductors for all solutions including that for superconductors. It is shown that an incorporation of the spatial dispersion leads to an appearance of the Meissner effect in perfect conductors in the same manner as in superconductors. This expanded conception is promising for applications in novel nanoelectronic devices exploiting the coherence, nonlocality of the superconducting-like state and for search of approaches to the problem of room temperature superconductivity.

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