Notes on quantum coherence with $l_1$-norm and convex-roof $l_1$-norm

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Abstract
In this work, we evaluate quantum coherence using the $l_1$-norm and convex-roof $l_1$-norm and obtain several new results. First, we provide some new general triangle-like inequalities of quantum coherence, with results better than existing ones. Second, for some special three-dimensional quantum states, a method for calculating the convex-roof $l_1$-norm is presented. Lastly, we offer distinct upper bounds in the $l_1$-norm measure of coherence based on the quantum state itself.

Keywords Quantum coherence · $l_1$-Norm · Triangle-like inequalities · Convex-roof $l_1$-norm

1 Introduction
Quantum coherence stems from the superposition of quantum states; it reflects the ability of quantum states to display quantum coherence effects. It is the most basic property distinguishing quantum physics from classical physics, and it is also an important quantum mechanical property that is widely used in quantum information processing and quantum computing. Consequently, quantum information science and the derived quantum technology offer advantages that the corresponding classical informatics and technology cannot provide, such as the theoretical security of quantum communication [1], ultra-fast quantum computing [2], and quantum precision measurements with accuracies exceeding the classical limit [3]. Additionally, quantum coherence is widely used in quantum biology [4,5], quantum thermodynamics [6,7], and quantum computing [8,9]. Therefore, quantifying the quantum coherence in a resource framework...
is important and meaningful, and the study of quantum coherence plays an important role in furthering the frontiers of physics research.

Quantum coherence and entanglement are two characteristics of the quantum world. Quantum entanglement describes bipartite or multiple systems, whereas quantum coherence is defined for single systems. With the rapid developments in quantum information science, researchers have found that, similar to entanglement, quantum coherence can be treated as a physical resource. However, a core issue in this field is the quantification of coherence. Adopting some of the methods used with entanglement, Baumgratz et al. established a quantitative theory of coherence as a resource along with four necessary conditions that should be satisfied by any appropriate measure of coherence [10].

Most of the measures of quantum coherence are based on distances, defined in terms of the minimum distance between the selected quantum state and the given set of incoherent states \( \tilde{I} \). These incoherent states are quantum states that are diagonalizable under a certain reference basis \( \{|i\rangle\}_{i=1}^{d} \) in the Hilbert space, that is, \( \delta = \sum_{i=1}^{d} \delta_i |i\rangle \langle i| \).

A measure of coherence is defined as follows:

\[
C_D (\rho) = \min_{\delta \in \tilde{I}} D (\rho, \delta),
\]

where \( D (\rho, \delta) \) denotes a certain measure of the distance between \( \rho \) and \( \delta \).

Baumgratz and his coworkers found that the relative entropy of coherence

\[
C_{\text{re}} (\rho) = S(\rho_{\text{diag}}) - S(\rho),
\]

where \( S \) is the von Neumann entropy and \( \rho_{\text{diag}} \) denotes the state obtained from \( \rho \) by deleting all off-diagonal elements and the \( l_1 \)-norm of coherence

\[
C_{l_1} (\rho) = \sum_{i \neq j} |\rho_{ij}|,
\]

are both appropriate measures of coherence.

Thus far, many suitable measures have been reported, such as the robustness of coherence [11], maximum relative entropy of coherence [12], entanglement-based coherence measurement [13], average quantum coherence [14], and set coherence [15]. However, the \( l_1 \)-norm of coherence has continued to be significant for research. Recently, it was discovered that the \( l_1 \)-norm of coherence can be used to describe wave-particle duality [16].

Reference [17] showed that if a rank-2 state \( \rho \) can be expressed as a convex combination of two pure states, i.e.,

\[
\rho = p_1 |\psi_1\rangle \langle \psi_1| + p_2 |\psi_2\rangle \langle \psi_2|,
\]
a triangle-like inequality can be established as follows:

\[
\left| E \left( \sqrt{p_1} |\psi_1\rangle \right) - E \left( \sqrt{p_2} |\psi_2\rangle \right) \right| \leq E(\rho) \\
\leq E \left( \sqrt{p_1} |\psi_1\rangle \right) + E \left( \sqrt{p_2} |\psi_2\rangle \right),
\]

where \( E \) can be considered as either a coherence measure or an entanglement concurrence.

Reference [18] provided a general triangle-like inequality satisfied by the \( l_1 \)-norm measure of coherence for a convex combination of \( n \) arbitrary pure states of a quantum state \( \rho \), thus verifying the conclusions presented in Ref. [17].

Both Refs. [17] and [18] estimated the coherence of a quantum state by using state decomposition. We can also evaluate the coherence of a quantum state based on its properties. As mentioned in Ref. [19], for a \( d \)-dimensional quantum state, its \( l_1 \)-norm is not more than \( d - 1 \).

To date, many scholars have estimated the coherence of a quantum state, providing lower and upper bounds in different forms. However, we find that some of these estimates can be optimized, and we can measure the coherence of a quantum state from different perspectives, thus affording distinct conclusions. In this study, we select the \( l_1 \)-norm and convex-roof \( l_1 \)-norm to quantify coherence. Based on the state decomposition of a quantum state and its properties, we provide some new lower and upper bounds for the \( l_1 \)-norm and convex-roof \( l_1 \)-norm.

### 2 Generalized triangular inequality for quantum coherence based on \( l_1 \) norm

On basis of the state decomposition of a quantum state, Ref. [17] provides the following conclusion:

**Lemma 1** If a quantum state \( \rho \) can be expressed as a convex combination of two states, that is, \( \rho = p_1 \rho_1 + p_2 \rho_2 \), then we have

\[
\left| p_1 C_{l_1}(\rho_1) - p_2 C_{l_1}(\rho_2) \right| \leq C_{l_1}(\rho).
\]

Moreover, Ref. [18] provides a triangle-like inequality in the \( l_1 \)-norm measure of coherence.

**Lemma 2** If the state \( \rho \) can be expressed as a convex combination of \( n (n \geq 2) \) states, that is, \( \rho = \sum_{i=1}^{n} p_i \rho_i \), then \( C_{l_1}(\rho) \) satisfies the following triangle-like inequality

\[
\frac{1}{n} \left| \sum_{k=1}^{n} G_k^{(n-1)} - p_k C_{l_1}(\rho_k) \right| \leq C_{l_1}(\rho) \leq \sum_{k=1}^{n} p_k C_{l_1}(\rho_k),
\]

where \( G_k^{(n-1)} \), \( k = 1, 2, \ldots, n \) are the lower bounds of

\[
\sum_{i\neq k} p_i \left( \sum_{j \neq k} p_j \rho_j / \sum_{j \neq k} p_j \right).
\]
We can further develop the conclusion of Lemma 2. However, we first introduce a useful lemma followed by a new triangle-like inequality regarding the $l_1$-norm considering the state decomposition of a quantum state.

**Lemma 3** Let $a_i \geq 0 (i = 1, 2, \ldots, n)$ and not be complete zeros. If $b \geq a_i, i = 1, 2, \ldots, n$, then we obtain

$$b \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i} \geq \frac{1}{n} \sum_{i=1}^{n} a_i.$$  

**Proof** Based on these assumptions, we have

$$\sum_{i=1}^{n} a_i^2 \leq \sum_{i=1}^{n} a_i b = b \sum_{i=1}^{n} a_i.$$  

Note that $a_1, \ldots, a_n$ are not complete zeros; thus, $\sum_{i=1}^{n} a_i > 0$. Therefore, it holds that

$$b \geq \frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i}.$$  

Moreover, using the inequality

$$(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2),$$

we can directly obtain

$$\frac{\sum_{i=1}^{n} a_i^2}{\sum_{i=1}^{n} a_i} \geq \frac{1}{n} \sum_{i=1}^{n} a_i.$$  

Next, we present the general triangle-like inequality based on the $l_1$-norm measure of coherence.

**Theorem 1** If the state $\rho$ can be expressed as a convex combination of $n (n \geq 2)$ states, that is, $\rho = \sum_{i=1}^{n} p_i \rho_i$, then $C_{l_1}(\rho)$ satisfies

$$\frac{\sum_{i=1}^{n} A_i^2}{\sum_{i=1}^{n} A_i} \leq C_{l_1}(\rho) \leq \frac{1}{n} \sum_{k=1}^{n} \left[ G_k^{(n-1)} + p_k C_{l_1}(\rho_k) \right],$$  

where

$$G_k^{(n-1)} = (1 - p_k)C_{l_1}(\frac{\rho - p_k \rho_k}{1 - p_k}), \quad k = 1, 2, \ldots, n;$$

$$A_i = \left| G_i^{(n-1)} - p_i C_{l_1}(\rho_i) \right|.$$  

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It should be noted that we only consider the case where
\[ G_k^{(n-1)} - p_k C_{l_1}(\rho_k), k = 1, 2, \ldots, n \]
are not all zeros. If
\[ G_k^{(n-1)} - p_k C_{l_1}(\rho_k) = 0, k = 1, 2, \ldots, n, \]
then 0 can act as a lower bound of \( C_{l_1}(\rho) \).

**Proof** First, we analyze the right-hand side of Eq. (1). In the case of \( n = 2 \), it holds that
\[
G_1^{(1)} = p_2 C_{l_1}(\rho_2), \quad G_2^{(1)} = p_1 C_{l_1}(\rho_1).
\]
(2)

Therefore, from Lemma 2, we have
\[
\frac{1}{2} \left[ (G_1^{(1)} + p_1 C_{l_1}(\rho_1)) + (G_2^{(1)} + p_2 C_{l_1}(\rho_2)) \right]
\]
\[
= \frac{1}{2} \left[ 2(p_1 C_{l_1}(\rho_1) + p_2 C_{l_1}(\rho_2)) \right]
\]
\[
= p_1 C_{l_1}(\rho_1) + p_2 C_{l_1}(\rho_2) \geq C_{l_1}(\rho).
\]

Hence, the conclusion is true when \( n = 2 \).

Assuming that the conclusion is true when \( n = m \), let us analyze the case of \( n = m + 1 \). For \( \forall k \in \{1, 2, \ldots, m+1\} \),
\[
\sum_{i \neq k} \frac{p_i}{1-p_k} \rho_i
\]
remains a density matrix. According to Lemma 2, we have
\[
C_{l_1}(\rho) = C_{l_1}((1 - p_k)(\sum_{i \neq k} \frac{p_i}{1-p_k} \rho_i) + p_k \rho_k)
\]
\[
\leq (1 - p_k)C_{l_1}(\sum_{i \neq k} \frac{p_i}{1-p_k} \rho_i) + p_k C_{l_1}(\rho_k)
\]
\[
= G_k^{(m)} + p_k C_{l_1}(\rho_k).
\]

For any arbitrary \( k \), it holds that
\[
(m + 1)C_{l_1}(\rho) = \sum_{k=1}^{m+1} C_{l_1}(\rho) \leq \sum_{k=1}^{m+1} (G_k^{(m)} + p_k C_{l_1}(\rho_k)).
\]
In other words,

\[ C_{l_1}(\rho) \leq \frac{1}{m+1} \sum_{k=1}^{m+1} (G_k^{(m)} + p_k C_{l_1}(\rho_k)). \]

Hence, the inequality also holds when \( n = m + 1 \). Thus, the right side of Eq. (1) is true.

Next, we consider the left side of Eq. (1).

First, we analyze the case of \( n = 2 \). From Eq. (2), we have

\[
\frac{|p_2 C_{l_1}(\rho_2) - p_1 C_{l_1}(\rho_1)|^2 + |p_1 C_{l_1}(\rho_1) - p_2 C_{l_1}(\rho_2)|^2}{2|p_1 C_{l_1}(\rho_1) - p_2 C_{l_1}(\rho_2)|} = |p_1 C_{l_1}(\rho_1) - p_2 C_{l_1}(\rho_2)| \leq C_{l_1}(\rho).
\]

Thus, the conclusion holds when \( n = 2 \).

Assuming that the conclusion holds when \( n = m \), we now consider the case of \( n = m + 1 \).

For \( \forall k \in \{1, 2, \ldots, m+1\} \), according to Lemma 1, we have

\[
C_{l_1}(\rho) = C_{l_1}((1 - p_k)(\sum_{i \neq k}^{n} \frac{p_i}{1-p_k} \rho_i) + p_k \rho_k)
\geq |(1 - p_k)C_{l_1}(\sum_{i \neq k}^{n} \frac{p_i}{1-p_k} \rho_i) - p_k C_{l_1}(\rho_k)|
= |G_k^{(m)} - p_k C_{l_1}(\rho_k)|.
\]

Based on Lemma 3, the conclusion is true for the case of \( n = m + 1 \).

Combined with the above analysis, Eq. (1) is true for any arbitrary natural number \( n \).

Furthermore, we can prove that our estimate of coherence is more accurate than that provided by Lemma 2.

**Theorem 2** If \( \rho \) can be expressed as a convex combination of \( n \) states, that is, \( \rho = \sum_{i=1}^{n} p_i \rho_i \), then

\[
\frac{1}{n} \sum_{k=1}^{n} |G_k^{(n-1)} - p_k C_{l_1}(\rho_k)| \leq \frac{\sum_{i=1}^{n} A_i^2}{\sum_{i=1}^{n} A_i}, \tag{3}
\]

\[
\frac{1}{n} \sum_{k=1}^{n} (G_k^{(n-1)} + p_k C_{l_1}(\rho_k)) \leq \sum_{k=1}^{n} p_k C_{l_1}(\rho_k), \tag{4}
\]

where \( A_i = |G_i^{(n-1)} - p_i C_{l_1}(\rho_i)|, i = 1, 2, \ldots, n \).
Proof Assume that $\rho$ can be expressed as a convex combination of $n$ states, i.e., $\rho = \sum_{i=1}^{n} p_i \rho_i$. Based on Lemma 3, Eq. (3) is true.

We also have

$$
\frac{1}{n} \sum_{k=1}^{n} (G_k^{(n-1)} + p_k C_{l_1}(\rho_k)) \\
= \frac{1}{n} \sum_{k=1}^{n} G_k^{(n-1)} + \frac{1}{n} \sum_{k=1}^{n} p_k C_{l_1}(\rho_k) \\
= \frac{1}{n} \sum_{k=1}^{n} (1 - p_k) C_{l_1} \left( \sum_{i \neq k} \frac{p_i}{1-p_k} \rho_i \right) + \frac{1}{n} \sum_{k=1}^{n} p_k C_{l_1}(\rho_k) \\
\leq \frac{1}{n} \sum_{k=1}^{n} (1 - p_k) \sum_{i \neq k} \frac{p_i}{1-p_k} C_{l_1}(\rho_i) + \frac{1}{n} \sum_{k=1}^{n} p_k C_{l_1}(\rho_k) \\
= \frac{1}{n} \sum_{k=1}^{n} \sum_{i \neq k} p_i C_{l_1}(\rho_i) + \frac{1}{n} \sum_{k=1}^{n} p_k C_{l_1}(\rho_k) \\
= \frac{(n-1)}{n} \sum_{k=1}^{n} p_k C_{l_1}(\rho_k) + \frac{1}{n} \sum_{k=1}^{n} p_k C_{l_1}(\rho_k) \\
= \sum_{k=1}^{n} p_k C_{l_1}(\rho_k).
$$

Thus, Eq. (4) holds. \qed

3 Generalized triangular inequality for quantum coherence based on convex-roof $l_1$-norm

The convex-roof $l_1$-norm is another method for measuring coherence [20]. The convex-roof $l_1$-norm of a mixed state is defined as

$$
\widetilde{C}_{l_1}(\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i C_{l_1}(|\psi_i\rangle).
$$

However, we are unfamiliar with the convex-roof $l_1$-norm owing to the difficulty in its calculation. To date, we can only calculate the convex-roof $l_1$-norm of a two-dimensional state [20,21] and some special high-dimensional states [22]. In this study, we prove that, for certain three-dimensional quantum states, $C_{l_1}(\rho) = \widetilde{C}_{l_1}(\rho)$ holds. We also provide a distinct triangle-like inequality in the convex-roof $l_1$-norm of coherence. To this end, we begin with the two lemmas mentioned in Refs. [20] and [17].
Lemma 4  For any quantum state $\rho$, we have

$$C_{l_1}(\rho) \leq \tilde{C}_{l_1}(\rho).$$

Lemma 5  If $\rho$ is a pure state, then

$$C_{l_1}(\rho) = \tilde{C}_{l_1}(\rho).$$

According to Lemma 4, we obtain the following theorem.

Theorem 3  Let $\rho$ be a three-dimensional quantum state with $\text{rank}(\rho) = 3$. If the matrix of $\rho$ under reference basis $\{|i\rangle\}_{i=1}^d$ has the following form

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 \\ \rho_{12} & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix},$$

then

$$C_{l_1}(\rho) = \tilde{C}_{l_1}(\rho).$$

Proof  Select $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$, and $p_1, p_2, p_3$ as follows:

$$|\psi_1\rangle = \frac{1}{\sqrt{p_1}} \left( \sqrt{\rho_{11}} |1\rangle + \frac{|\rho_{12}|}{\sqrt{\rho_{11}}} e^{-i\arg \rho_{12}} |2\rangle \right),$$

$$|\psi_2\rangle = |2\rangle, \quad |\psi_3\rangle = |3\rangle,$$

$$p_1 = \rho_{11} + \frac{|\rho_{12}|^2}{\rho_{11}}, \quad p_2 = \rho_{22} - \frac{|\rho_{12}|^2}{\rho_{11}}, \quad p_3 = \rho_{33}. $$

Thus,

$$\rho = p_1 |\psi_1\rangle \langle \psi_1 | + p_2 |\psi_2\rangle \langle \psi_2 | + p_3 |\psi_3\rangle \langle \psi_3 |,$$

and

$$p_1 C_{l_1} (|\psi_1\rangle) + p_2 C_{l_1} (|\psi_2\rangle) + p_3 C_{l_1} (|\psi_3\rangle) = C_{l_1} (\rho).$$

According to the definition of the convex-roof $l_1$-norm, we have

$$C_{l_1}(\rho) \geq \tilde{C}_{l_1}(\rho).$$

Thus, it holds that

$$C_{l_1}(\rho) = \tilde{C}_{l_1}(\rho).$$

$\square$
Reference [18] provides the following triangle-like inequality regarding the convex-roof $l_1$-norm by using pure state decomposition.

**Lemma 6** If $\rho$ can be expressed as a convex combination of $n$ linearly independent pure states, that is, $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle \psi_i|$, then the convex-roof $l_1$-norm satisfies the following triangle-like inequality

$$\frac{1}{n} \sum_{k=1}^{n} \left| \widetilde{C}_{i_k}^{(n-1)} - p_k \widetilde{C}_{i_1} (|\psi_k\rangle) \right| \leq \sum_{i} p_i \widetilde{C}_{i_1} (|\psi_i\rangle),$$

(5)

where $\widetilde{C}_{i_1} (|\psi_i\rangle) = C_{i_1} (|\psi_i\rangle)$ is the convex-roof $l_1$-norm of the pure state $|\psi_i\rangle$ ($1 \leq i \leq n$) and $\widetilde{G}_k^{(n-1)}$ ($1 \leq k \leq n$) is the lower bound of $\sum_{i \neq k} p_i \widetilde{C}_{i_1} \left( \frac{\sum_{j \neq k} p_j \rho_j}{\sum_{i \neq k} p_i} \right)$.

More importantly, while the correctness of the left side of Eq. (5) can be discussed, the right side of Eq. (5) can be improved.

Next, we present a new general triangle-like inequality for the convex-roof $l_1$-norm measure of coherence.

**Theorem 4** Let $\rho$ be a mixed state that can be expressed as a convex combination of $n$ ($n \geq 2$) linearly independent pure states: $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$, i.e., $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle \psi_i|$. If we know some off-diagonal element $\rho_{st}$ of $\rho$, then

$$\frac{|\rho_{st}|^2}{B} \leq \widetilde{C}_{i_1} (\rho) \leq \frac{1}{n} \sum_{k=1}^{n} \left[ \widetilde{G}_k^{(n-1)} + p_k \widetilde{C}_{i_1} (|\psi_k\rangle) \right],$$

(6)

where

$$B = n \left( \sum_{k=1}^{n} p_k C_{i_1} (|\psi_k\rangle) + 1 \right),$$

$$\widetilde{G}_k^{(n-1)} = (1 - p_k) \widetilde{C}_{i_1} \left( \sum_{l \neq k} \frac{p_l}{1 - p_k} |\psi_l\rangle \right), k = 1, \ldots, n.$$

**Proof** We use mathematical induction to prove this theorem. First, we consider the case of $n = 2$. Assume that $\rho = p_1 \rho_1 + p_2 \rho_2$, where $\rho_1$ and $\rho_2$ are two linearly independent pure states. According to the definition of the convex-roof $l_1$-norm, we have

$$\widetilde{C}_{i_1} (\rho) \leq p_1 \widetilde{C}_{i_1} (\rho_1) + p_2 \widetilde{C}_{i_1} (\rho_2).$$

According to the definition, we can directly obtain

$$\widetilde{G}_1^{(1)} = p_2 \widetilde{C}_{i_1} (\rho_2), \widetilde{G}_2^{(1)} = p_1 \widetilde{C}_{i_1} (\rho_1).$$
Furthermore, we have
\[
\tilde{C}_1(\rho) \leq p_1 \tilde{C}_1(\rho_1) + p_2 \tilde{C}_1(\rho_2)
= \frac{1}{2} \left[ (\tilde{G}_1^{(1)} + p_1 \tilde{C}_1(\rho_1)) + (\tilde{G}_2^{(1)} + p_2 \tilde{C}_1(\rho_2)) \right].
\]

Therefore, the conclusion is true when \( n = 2 \).

Assuming that the conclusion holds in the case of \( n = m \), we now analyze the case of \( n = m + 1 \).

For \( \forall k \in \{1, 2, \ldots, m+1\} \), we have
\[
\tilde{C}_1(\rho) \leq (1 - p_k) \tilde{C}_1(\sum_{i \neq k} \frac{p_i}{1-p_k} \rho_i) + p_k \tilde{C}_1(\rho_k)
= \tilde{G}_k^{(n-1)} + p_k \tilde{C}_1(\rho_k). \tag{7}
\]

Summing over all the \( k \) in Eq. (7), we have
\[
\tilde{C}_1(\rho) \leq \frac{1}{m+1} \sum_{k=1}^{m+1} (\tilde{G}_k^{(m)} + p_k \tilde{C}_1(\rho_k)).
\]

In other words, the conclusion is true in the case of \( n = m + 1 \). Thus, the right side of Eq. (6) holds.

Next, we analyze the left side of Eq. (6). Assume that \( \{q_k, |\phi_k\rangle\} \) is the optimal decomposition of \( \rho \), i.e.,
\[
\tilde{C}_1(\rho) = \sum_{k=1}^{n} q_k C_1(\langle \phi_k |).
\]

Let \( |\phi_k\rangle = \sum_{j=1}^{d} \phi_{kj} e^{i\theta_{kj}} |j\rangle, k = 1, 2, \ldots, n \), where \( \phi_{kj} \) are nonnegative real numbers and \( \theta_{kj} \) are real numbers. Thus,
\[
|\phi_k\rangle \langle \phi_k | = \left( \sum_{j=1}^{d} \phi_{kj} e^{i\theta_{kj}} |j\rangle \right) \left( \sum_{t=1}^{d} \phi_{kt} e^{-i\theta_{kt}} \langle t | \right),
= \left( \sum_{j=1}^{d} \sum_{t=1}^{d} \phi_{kj} \phi_{kt} e^{i(\theta_{kj} - \theta_{kt})} |j\rangle \langle t | \right).
\]

Therefore,
\[
C_1(|\phi_k\rangle) = \sum_{i \neq j} \phi_{ki} \phi_{kj}. \tag{8}
\]
Then, we have
\[
\tilde{C}_l^1(\rho) = \sum_{k=1}^{n} q_k C_l^1(|\phi_k\rangle) = \sum_{k=1}^{n} q_k \sum_{i \neq j} \phi_{ki} \phi_{kj}
\]
\[
\geq \sum_{k=1}^{n} q_k \sum_{i \neq j} \phi_{ki} \phi_{kj} \geq \sum_{k=1}^{n} q_k (1 + \sum_{i \neq j} \phi_{ki} \phi_{kj})
\]
\[
\geq \sum_{k=1}^{n} q_k^2 \sum_{i \neq j} \phi_{ki} \phi_{kj} \geq \sum_{k=1}^{n} \sum_{i \neq j} \left(\sqrt{q_k \phi_{ki}} \phi_{kj}\right)^2
\]
\[
\geq \sum_{i \neq j} \sum_{k=1}^{n} \left(\sqrt{q_k \phi_{ki}} \phi_{kj}\right)^2 \geq \sum_{i \neq j} \left|\rho_{ij}\right|^2 / B \geq \frac{\left|\rho_{st}\right|^2}{B}.
\]

\[\square\]

We can also prove that the upper bound provided by Theorem 4 is better than that provided by Lemma 6.

**Theorem 5** If a mixed state \(\rho\) can be expressed as a convex combination of \(n\) pure states, \(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\), that is,
\[
\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle \psi_i|,
\]
then it holds that
\[
\frac{1}{n} \sum_{k=1}^{n} \left[\tilde{G}_k^{(n-1)} + p_k \tilde{C}_l^1(|\psi_k\rangle)\right] \leq \sum_{k=1}^{n} p_k \tilde{C}_l^1(|\psi_k\rangle).
\]

**Proof** Assume that a mixed state \(\rho\) can be expressed as a convex combination of \(n\) pure states \(|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\), that is,
\[
\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle \psi_i|.\]
Thus, we have
\[
\frac{1}{n} \sum_{k=1}^{n} \left[ \tilde{G}_k^{(n-1)} + p_k \tilde{C}_{l_1}(|\psi_k\rangle) \right] = \frac{1}{n} \sum_{k=1}^{n} \tilde{G}_k^{(n-1)} + \frac{1}{n} \sum_{k=1}^{n} p_k \tilde{C}_{l_1}(|\psi_k\rangle) \\
= \frac{1}{n} \sum_{k=1}^{n} (1 - p_k) \tilde{C}_{l_1} \left( \sum_{i \neq k} \frac{p_i}{1 - p_k} |\psi_i\rangle \right) + \frac{1}{n} \sum_{k=1}^{n} p_k \tilde{C}_{l_1}(|\psi_k\rangle) \\
\leq \frac{1}{n} \sum_{k=1}^{n} (1 - p_k) \sum_{i \neq k} \frac{p_i}{1 - p_k} \tilde{C}_{l_1}(|\psi_i\rangle) + \frac{1}{n} \sum_{k=1}^{n} p_k \tilde{C}_{l_1}(|\psi_k\rangle) \\
= \frac{(n - 1)}{n} \sum_{k=1}^{n} p_k \tilde{C}_{l_1}(|\psi_k\rangle) + \frac{1}{n} \sum_{k=1}^{n} p_k \tilde{C}_{l_1}(|\psi_k\rangle) \\
= \sum_{k=1}^{n} p_k \tilde{C}_{l_1}(|\psi_k\rangle). \tag*{□}
\]

4 Different upper bounds of coherence

In this section, we detail the evaluation of the coherence of a quantum state using its properties. Reference [19], using the dimension of a quantum state, recorded the following conclusion:

**Lemma 7** For any d-dimensional quantum state \( \rho \), it holds that \( C_{l_1}(\rho) \leq d - 1 \). Moreover, \( C_{l_1}(\rho) \) reaches the upper bound when \( \rho \) is in the maximally mixed state:

\[
|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle.
\]

However, in some cases, the evaluation provided by Lemma 7 is highly inaccurate. For example,

\[
\rho = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_d\}.
\]

With additional conditions, Lemma 7 can be improved. To this end, we introduce the following lemma [23].

**Lemma 8** Consider that \( A = [a_{ij}] \) is positive semi-definite and that for some \( k \in \{1, \ldots, n\} \), it holds that \( a_{kk} = 0 \). Then, for each \( i \in \{1, \ldots, n\} \), we have \( a_{ik} = a_{ki} = 0 \).
Theorem 6 Let $\rho$ be a $d$-dimensional quantum state. If there exists $r$ null elements in the main diagonal of $\rho$, then

$$C_{l_1}(\rho) \leq d - r - 1. \quad (9)$$

Proof Assume that $\rho$ is a $d$-dimensional quantum state satisfying

$$\rho_{k_i,k_i} \neq 0, i = 1, 2, \ldots, d - r (k_1 < k_2 < \cdots < k_{d-r}).$$

Let the matrix of $\rho$ under reference basis $\{|i\rangle\}_{i=1}^d$ be

$$\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \cdots & \rho_{1d} \\
\rho_{21} & \rho_{22} & \cdots & \rho_{2d} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{d1} & \rho_{d2} & \cdots & \rho_{dd}
\end{pmatrix}.
$$

Define

$$\rho' = \begin{pmatrix}
\rho_{k_1,k_1} & * & \cdots & * \\
* & \rho_{k_2,k_2} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & \rho_{k_{d-r},k_{d-r}}
\end{pmatrix},$$

where $\rho'$ is obtained via the permutation of each element of $\rho$ whose row index or column index is $k_i, i = 1, 2, \ldots, d-r$ in the same order. Furthermore, all the principal minors of $\rho'$ are the principal minors of $\rho$. Therefore, all the principal minors of $\rho'$ are nonnegative, given that $\rho$ is positive semi-definite.

In addition,

$$tr(\rho') = tr(\rho) = 1.$$

Hence, $\rho'$ is a density matrix, the dimension of which is $d - r$.

According to Lemmas 7 and 8, we can directly obtain

$$C_{l_1}(\rho) = C_{l_1}(\rho') \leq d - r - 1.$$

$\square$

Furthermore, we provide another two distinct upper bounds of the $l_1$-norm.

Theorem 7 Assume that $\rho$ is a $d$-dimensional quantum state. For convenience, the matrix of $\rho$ under the reference basis $\{|i\rangle\}_{i=1}^d$ is still denoted by $\rho$. All integers $k(2 \leq k \leq d)$ are assembled such that $\rho_{k,1}, \rho_{k+1,2}, \ldots, \rho_{d,d-k+1}$ are not complete zeros in a set $S_\rho$. Defining the number of elements in $S_\rho$ as $t$, we have

$$C_{l_1}(\rho) \leq 2t.$$
**Proof** Assume that $\rho$ is a $d$-dimensional quantum state. Let

$$S_\rho = \{r_1, r_2, \ldots, r_t\}.$$ 

All the principal minors of $\rho$ are nonnegative because $\rho$ is positive semi-definite, that is,

$$\rho \begin{pmatrix} i & j \\ i & j \end{pmatrix} = \begin{vmatrix} \rho_{ii} & \rho_{ij} \\ \rho_{ji} & \rho_{jj} \end{vmatrix} = \rho_{ii}\rho_{jj} - |\rho_{ij}|^2 \geq 0.$$ 

Furthermore, we have

$$|\rho_{ij}| \leq \sqrt{\rho_{ii}\rho_{jj}} \leq \frac{1}{2}(\rho_{ii} + \rho_{jj}).$$

It should be noted that $\rho$ is Hermite; hence, it follows that

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| = 2 \sum_{i=1}^{t} \sum_{j=1}^{d-r_i+1} |\rho_{r_i+j-1,j}| \leq 2 \sum_{i=1}^{t} \sum_{j=1}^{d-r_i+1} \frac{(\rho_{r_i+j-1,r_i+j-1} + \rho_{jj})}{2} \leq 2 \sum_{j=1}^{d} \frac{\rho_{jj}}{2} + 2 \sum_{i=1}^{t} \sum_{j=1}^{d-r_i+1} \frac{\rho_{r_i+j-1,r_i+j-1}}{2} = t + \sum_{i=1}^{t} \sum_{j=1}^{d-r_i+1} \rho_{r_i+j-1,r_i+j-1}.$$ 

We declare that, among the following $t(d+1) - (r_1 + r_2 + \ldots + r_t)$ numbers

$$\rho_{r_1,r_1}, \rho_{r_1+1,r_1+1}, \ldots, \rho_{d,d}, \ldots, \rho_{r_t,r_t}, \rho_{r_t+1,r_t+1}, \ldots, \rho_{d,d},$$

the same one appears $t$ times at most; thus,

$$C_{l_1}(\rho) \leq t + \sum_{i=1}^{t} \sum_{j=1}^{d-r_i+1} \rho_{r_i+j-1,r_i+j-1} \leq t + t(\rho_{11} + \cdots + \rho_{dd}) = 2t.$$ 

$\square$
Theorem 8 Let \( \rho \) be a \( d \)-dimensional (\( d \geq 2 \)) quantum state. If there exists \( \rho_{ij} \) such that \( |\rho_{ij}| = 0 \), then we have

\[
C_{l_1}(\rho) \leq \sqrt{[d(d - 1) - 1]} \mu \leq d - 1, \tag{10}
\]

where \( \mu = \sum_{k=1}^{d} \lambda_k^2 - \sum_{k=1}^{d} \rho_{kk}^2 \) and \( \lambda_k (k = 1, 2, \ldots, d) \) are the eigenvalues of \( \rho \).

Proof Let \( \rho \) be a \( d \)-dimensional quantum state. According to the spectral decomposition theorem, there exists a unitary matrix \( U \) such that

\[
\rho = U \begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_d,
\end{pmatrix}
U^+.
\]

Then, \( \rho^2 \) can be expressed as

\[
\rho^2 = U \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_d \\
\end{pmatrix}
U^+ U \begin{pmatrix}
\lambda_1 \\
\vdots \\
\lambda_d \\
\end{pmatrix}
U^+
\]

\[
= U \begin{pmatrix}
\lambda_1^2 \\
\vdots \\
\lambda_d^2 \\
\end{pmatrix}
U^+.
\]

Following the expression of \( \rho^2 \) established above, we obtain

\[
tr(\rho^2) = \sum_{i=1}^{d} \sum_{j=1}^{d} |\rho_{ij}|^2 = \sum_{k=1}^{d} \lambda_k^2.
\]

Furthermore, we have

\[
\sum_{i \neq j} |\rho_{ij}|^2 = \sum_{k=1}^{d} \lambda_k^2 - \sum_{k=1}^{d} \rho_{kk}^2.
\]

Now, we introduce the modified Cauchy inequality [24]. Assume that \( V \) is a product space; thus, for any \( x, y, z \in V \), where \( \|z\| = 1 \), it holds that

\[
|(x, y)|^2 \leq \|x\|^2 \|y\|^2 - G(x, y, z),
\]

where

\[
G(x, y, z) = (\|x\| (y, z) - \|y\| (x, z))^2.
\]
The equality holds only if $x$, $y$, $z$ are linearly dependent.

Now, we attempt to determine an upper bound of $C_{l_1}(\rho)$ using this useful tool. Let $V = \mathbb{R}^n (n \geq 2)$ and the standard inner product be taken as the fixed product. Subsequently, $x$, $y$, $z$ is selected in the following manner:

$$x = (x_1, x_2, \ldots, x_n), \quad y = \left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right),$$

$$z = (0, 0, \ldots, 1, 0, \ldots, 0).$$

where the $i$th element of the $n$-tuple $z$ is 1. Owing to the definition of the standard inner product, we have

$$(x, y) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i, \quad \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}, \quad \|y\| = 1,$$

$$(y, z) = \frac{1}{\sqrt{n}}, \quad (x, z) = x_i.$$

Using the modified Cauchy inequality, we have

$$\frac{1}{n} \left(\sum_{i=1}^{n} x_i\right)^2 \leq \sum_{i=1}^{n} x_i^2 \cdot \left(\sum_{i=1}^{n} x_i^2 \frac{1}{\sqrt{n}} - x_i\right)^2.$$

For the convenience of narration, we introduce the following notations:

$$\sum_{i=1}^{n} x_i := u, \quad \sum_{i=1}^{n} x_i^2 := \lambda.$$

Then,

$$\frac{1}{n} u^2 \leq \lambda - \left(\sqrt{\lambda} \frac{1}{\sqrt{n}} - x_i\right)^2.$$

In other words,

$$\frac{1}{n} u^2 - \lambda + \frac{1}{n} \lambda \leq \frac{2}{\sqrt{n}} \sqrt{\lambda x_i} - x_i^2. \quad (11)$$

Note that Eq. (11) holds for each $i$; hence,

$$\frac{1}{n} u^2 - \lambda + \frac{1}{n} \lambda \leq \min \left\{ \sqrt[4]{\frac{4\lambda}{n}} x_i - x_i^2 \bigg| i = 1, \ldots, n \right\}.$$
If \( j \in \{1, 2, \ldots, n\} \), then \( x_j = 0 \). Therefore,

\[
\frac{1}{n} u^2 - \lambda + \frac{1}{n} \lambda \leq 0.
\]

In other words,

\[
u \leq \sqrt{(n-1)\lambda}.
\]

In summary, it holds that

\[
C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}| \leq \sqrt{[d(d - 1) - 1]} \mu,
\]

where \( \mu = \sum_{k=1}^{d} \lambda_k^2 - \sum_{k,=1}^{d} \rho_{kk}^2 \).

Next, we focus on

\[
\sqrt{[d(d - 1) - 1]} \mu \leq d - 1.
\]

According to the properties of density matrix, we have

\[
\sum_{k=1}^{d} \lambda_k^2 \leq \left( \sum_{k=1}^{d} \lambda_k \right)^2 = 1,
\]

\[
\sum_{k=1}^{d} \rho_{kk}^2 \geq \frac{1}{d} \left( \sum_{k=1}^{d} \rho_{kk} \right)^2 = \frac{1}{d}.
\]

Thus, \( \mu \) can be expressed as

\[
\mu \leq 1 - \frac{1}{d} = \frac{d - 1}{d}.
\]

From Eq. (12), we can directly obtain

\[
\sqrt{[d(d - 1) - 1]} \mu \leq \sqrt{[d(d - 1) - 1]} \frac{d - 1}{d}
\]

\[
= \sqrt{(d - 1)^2 - \frac{d - 1}{d}} \leq d - 1.
\]
5 Conclusion and discussion

We studied the properties of quantum coherence based on the $l_1$-norm and convex-roof $l_1$-norm, thus realizing a better generalized triangular inequality. Furthermore, we proved that, for a certain type of three-dimensional quantum state $\rho$, it holds that $C_{l_1}(\rho) = \tilde{C}_{l_1}(\rho)$. In addition, we offer a few upper bounds for the $l_1$-norm in different forms according to the properties of a quantum state.

For future research, we propose the following conjecture: Let $\rho$ be a three-dimensional quantum state with rank($\rho$) $\neq 2$; then,

$$C_{l_1}(\rho) = \tilde{C}_{l_1}(\rho).$$

The difficulty lies in calculating the convex-roof $l_1$-norm of more general quantum states.

We presented certain previously unreported facts, which we believe will be helpful in the quantitative estimation of quantum coherence and other quantum resources.

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