A NOTE ON REGULARIZED BERNOULLI DISTRIBUTIONS AND $p$-ADIC DIRICHLET EXPANSIONS

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Abstract. We consider Bernoulli distributions and their regularizations, which are measures on the $p$-adic integers $\mathbb{Z}_p$. It is well known that their Mellin transform can be used to define $p$-adic $L$-functions. We show that for $p > 2$ one of the regularized Bernoulli distributions is particularly simple and equal to a measure on $\mathbb{Z}_p$ that takes the values $\pm \frac{1}{2}$ on clopen balls. We apply this to $p$-adic $L$-functions for Dirichlet characters of $p$-power conductor and obtain Dirichlet series expansions similar to the complex case. Such expansions were studied by D. Delbourgo, and this contribution provides an approach via $p$-adic measures.

1. Introduction

$p$-adic $L$-functions are $p$-adic analogues of complex $L$-functions. They have a long history and the primary constructions going back to by Kubota-Leopoldt [5] and Iwasawa [3] are via the interpolation of special values of $L$-functions. $p$-adic $L$-functions can be defined via the unbounded Haar distribution on $\mathbb{Z}_p^*$, which yields a Volkenborn integral. However, it is also possible to use $p$-adic measures.

Let $p > 2$ be a prime number and $k \geq 0$ an integer. For $x \in \mathbb{Z}_p$, we denote by $\{x\}_p$ the unique representative of $x \mod p^n$ between 0 and $p^n - 1$.

The Bernoulli distributions $E_k$ on $\mathbb{Z}_p$ are defined by

$$E_k(a + p^n \mathbb{Z}_p) = p^{n(k-1)}B_k\left(\frac{\{a\}_p}{p^n}\right),$$

where $B_k(x)$ is the $k$-th Bernoulli polynomial and $B_k = B_k(0)$ are the Bernoulli numbers (see [4]). For $k = 0$, $B_0(x) = 1$ and $E_0$ is the Haar distribution. For $k = 1$, one has $B_1(x) = x - \frac{1}{2}$. Choose $c \in \mathbb{Z}$ with $c \neq 1$ and $c \notin p\mathbb{Z}$. Then the regularization of $E_k$ is defined by

$$E_{k,c}(a + p^n \mathbb{Z}_p) = E_k(a + p^n \mathbb{Z}_p) - c^k E_k\left(\frac{\{a\}_c}{p^n} + p^n \mathbb{Z}_p\right).$$

One shows that the regularized Bernoulli distributions $E_{k,c}$ are measures. Now let $\chi : \mathbb{Z} \to \overline{\mathbb{Q}} \subset \mathbb{C}_p$ be a Dirichlet character of conductor $f_\chi = p^m$, $m \geq 0$. We denote by $\omega$ the Teichmüller character modulo $p$. For $a \in \mathbb{Z}_p^*$, we write $\langle a \rangle = \frac{a}{\omega(a)} \in 1 + p\mathbb{Z}_p$.

It is well known that the $p$-adic $L$-function $L_p(s, \chi)$ is a Mellin-transform of $E_{1,c}$ (see [6] Theorem 12.2):

$$L_p(s, \chi) = \frac{-1}{1 - \chi(c)\langle c\rangle^{-s+1}} \int_{\mathbb{Z}_p^*} \chi\omega^{-1}(a)\langle a \rangle^{-s}dE_{1,c}.$$
$L_p(s, \chi)$ interpolates the complex values at $s = 0, -1, -2, \ldots$ up to a factor, i.e., for integers $k \geq 1$ (see [6] Theorem 5.11):

$$L_p(1 - k, \chi) = -\left(1 - \chi \omega^{-k}(p)p^{k-1}\right)\frac{B_{k, \chi \omega^{-k}}}{k}$$

2. Choosing a $p$-adic measure

The regularization of the Bernoulli distributions depends on a parameter $c$ and there seems to be no preferred choice. However, we will show below that $E_{1, c}$ is particularly simple for $c = 2$.

**Definition 2.1.** Let $p \neq 2$ be a prime. Then

$$\mu(a + p^n\mathbb{Z}_p) = (-1)^{(a)_p^n}$$

defines a normalized measure on $\mathbb{Z}_p$. We call $\mu$ the alternating measure, since the value on all clopen balls is $\pm 1$.

It is very easy to verify that $\mu$ is in fact a measure. Note that we excluded $p = 2$.

The Theorem below shows that $\mu$ is (up to the factor $\frac{1}{2}$) equal to the regularized Bernoulli distribution $E_{1, 2}$.

**Theorem 2.2.** Let $p > 2$ and $\mu$ the above alternating measure. Then

$$\frac{1}{2}\mu = E_{1, 2}.$$

**Proof.** Let $n \geq 1$. First, we consider even representatives of $a + p^n\mathbb{Z}_p$. Let $a = 2b$ where $b \in \{0, 1, \ldots, \frac{p^n-1}{2}\}$. Then

$$E_{1, 2}(a + p^n\mathbb{Z}_p) = E_1(2b + p^n\mathbb{Z}_p) - 2E_1(b + p^n\mathbb{Z}_p) = \frac{2b}{p^n} - \frac{1}{2} - 2\left(\frac{b}{p^n} - \frac{1}{2}\right) = \frac{1}{2}.$$

Now we look at odd representatives. Let $a = 2b + 1$ where $b \in \{0, 1, \ldots, \frac{p^n-3}{2}\}$. Then we have

$$E_{1, 2}(a+p^n\mathbb{Z}_p) = E_1(2b+1+p^n\mathbb{Z}_p) - 2E_1(b+\frac{1}{2}+p^n\mathbb{Z}_p) = \frac{2b+1}{p^n} - \frac{1}{2} - \frac{2}{2}\left(\frac{b+\frac{1}{2}}{p^n} - \frac{1}{2}\right).$$

Note that $\frac{1}{2}$ has the following $p$-adic representation:

$$\frac{1}{2} = \frac{p+1}{2} + \frac{p-1}{2} p + \frac{p-1}{2} p^2 + \ldots$$

Hence $\left\{\frac{1}{2}\right\}_{p^n} = \frac{p^n+1}{2}$. Since $b = \{b\}_{p^n} \leq \frac{p^n-3}{2}$, we have $\left\{\frac{1}{2}\right\}_{p^n} + \{b\}_{p^n} \leq p^n - 1$. So we can add the representatives and obtain

$$\left\{b + \frac{1}{2}\right\}_{p^n} = b + \frac{p^n+1}{2}.$$ 

This implies our assertion:

$$E_{1, 2}(a + p^n\mathbb{Z}_p) = \frac{2b + 1}{p^n} - \frac{1}{2} - \frac{2b + p^n + 1}{p^n} + 1 = -\frac{1}{2}.$$ 

$\square$
3. Dirichlet series expansion

We can use Theorem 2.2 to obtain a $p$-adic Dirichlet series expansion of $L_p(s, \chi)$.

**Corollary 3.1.** Let $p > 2$ be a prime and $\chi$ a Dirichlet character of $p$-power conductor. Then:

$$L_p(s, \chi) = \frac{-1}{1 - \chi(2)(2)^{-s+1}} \cdot \lim_{n \to \infty} \sum_{a=1 \atop p \nmid a}^n (-1)^a \frac{\chi(a)}{2} \omega^{-1}(a)\langle a \rangle^{-s}$$

$L_p(s, \chi)$ is analytic in $s \in \mathbb{Z}_p$, except a simple pole at $s = 1$ for $\chi = \omega$. For $\chi = \omega^i$ and $i = 0, 1, \ldots, p-2$, we obtain the $p-1$ branches of the $p$-adic zeta function:

$$\zeta_{p,i}(s) = L_p(s, \omega^{1-i}) = \frac{-1}{1 - \omega(2)^{-1-i}(2)^{-s+1}} \cdot \lim_{n \to \infty} \sum_{a=1 \atop p \nmid a}^n (-1)^a \frac{\omega(a)}{2} \omega^{i-1}(a)\langle a \rangle^{-s}$$

**Remark 3.2.** The integral representation of $p$-adic $L$-functions using measures and Iwasawa’s construction using Stickelberger elements (see [3]) suggest that a Dirichlet series expansion is possible. However, the exact coefficients of $\langle a \rangle^{-s}$ are not obvious, even for the $p$-adic zeta function. We see that the complex and the $p$-adic expansions are surprisingly similar. In the $p$-adic case, we have to look at the above subsequence of the series since $|\langle a \rangle^{-s}|_p = 1$.

Dirichlet series expansions were studied by D. Delbourgo in [1], [2]. He considers Dirichlet characters $\chi$ with $\gcd(p, 2f_\chi \phi(f_\chi)) = 1$ and their Teichmüller twists. We obtain similar results, but consider the case $f_\chi = p^m$ and use other methods ($p$-adic measures). For the $p-1$ branches of the $p$-adic zeta function, we obtain the same expansions. The values $\mp \frac{1}{2}$ of the scaled alternating measure $E_{1,2} = \frac{1}{2} \mu$ can be found as coefficients $a_1(\chi)$ and $a_2(\chi)$ in the first row of Table 1 in [2].

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