A COUPLING APPROACH TO RANDOM CIRCLE MAPS EXPANDING ON THE AVERAGE

MIKKO STENLUND AND HENRI SULKU

Abstract. We study random circle maps that are expanding on the average. Uniform bounds on neither expansion nor distortion are required. We construct a coupling scheme, which leads to exponential convergence of measures (memory loss) and exponential mixing. Leveraging from the structure of the associated correlation estimates, we prove an almost sure invariance principle for vector-valued observables. The motivation for our paper is to explore these methods in a nonuniform random setting.

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1. Introduction

In this paper we study random compositions of the form $T_{ω_n} \circ \cdots \circ T_{ω_1}$, where each $T_{ω_i}$ is a $C^2$ circle mapping with no critical points ($T'_{ω_i} \neq 0$ everywhere), drawn independently of the others from a set $\{T_{ω_1} : ω_1 \in Ω\}$ according to a probability distribution $dη(ω_1)$. We do not place uniform bounds on expansion or distortion that would hold from one map to the next. On the contrary, individual maps are allowed to contract locally and distort their images strongly (without a bound). To compensate for such individual freedom, we impose probabilistic conditions on the occurrence of these “bad” maps in the sequence. In particular, we require the maps to be expanding on the average, i.e., $\int \inf |T'_{ω_1}| dη(ω_1) > 1$, with integrable distortion. The precise, somewhat stronger, assumptions are laid out in the next section. We prove statistical properties including the existence of an absolutely continuous invariant measure, exponential memory loss and mixing, as well as an almost sure invariance principle for vector-valued observables. Prior studies entailing similar models include [5,6,11,12,16,17,19]. After finishing the present manuscript, the authors have also learned of the very recent works [1,9] on the subject, as well as the related [28].

The motivation for the paper is twofold. First, we wish to explore the suitability of the coupling method in the above context of nonuniform random maps. Diverting from the papers mentioned, the primary instrument in our analysis is indeed coupling. The coupling method is a soft tool for establishing statistical properties pertaining to the issues of memory loss and correlation decay. In the field of dynamical systems it has been implemented in various works such as [4,7,18,23,25,30] and many others. A transparent introduction to coupling for dynamical systems (in the most elementary setup) can be found in [27]. As to the second motivation, a question that arises naturally is whether other limit laws hold true; we wish to investigate the possibility of proving such laws for the present class of nonuniform random dynamical systems via correlation estimates. It was shown in [8,20,22] that a central limit theorem for Sinai billiards follows from correlation bounds involving suitable classes of observables. In [24] a similar approach was taken to prove an almost sure invariance principle (ASIP) for both random and non-random billiard systems. Here we show that, for our system, an ASIP follows from the

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established correlation estimates with little added work. Yet, the last point is subtle: it depends on the particular form of the correlation estimates, obtained for particular classes of observables. Let us be fully clear that the (averaged) theorems on the Markov chain corresponding to the random maps at issue can certainly be obtained, for example, via spectral methods. Here we present a different approach, which we hope to be of use to other authors beyond the present setup.

Structure of the paper. The paper is organized as follows. In Section 2 we introduce the model precisely and record some mathematical preliminaries necessary for understanding the results and the proofs in the rest of the paper. In Section 3 we present our main results. In the following Sections 4–7 we prove these results in the same order as they appear in Section 3.

2. PRELIMINARIES

Let \( S \) denote the circle obtained by identifying the endpoints of the unit interval \([0, 1]\). The Lebesgue measure on \( S^1 \) is denoted by \( m \).

Given \( \alpha \in (0, 1) \), we denote by \( C^\alpha \) the set of functions \( S \to \mathbb{R} \) (or \( S \to \mathbb{C} \)) that are Hölder continuous with exponent \( \alpha \). The corresponding Hölder constant is denoted by \( |f|_\alpha \). We also introduce the norm

\[
\|f\|_\alpha = |f|_\alpha + \|f\|_\infty.
\]

To define the compositions \( T_{\omega_i} \circ \cdots \circ T_{\omega_1} \) of the Introduction properly, let \((\Omega, \mathcal{F}, \eta)\) be a probability space and, for each \( \omega_1 \in \Omega \), let the map \( T_{\omega_1} : S \to S \) be \( C^2 \) without critical points (with additional assumptions to follow shortly). Then consider compositions of such maps drawn from the product space. We assume the map \( \Omega \times S \to S : (\omega_1, x) \mapsto T_{\omega_1} x \) to be measurable, and define the quantities

\[
\lambda_{\omega_i} = \inf |T'_{\omega_i}| \quad \text{and} \quad \Delta_{\omega_i} = \left\| \frac{T''_{\omega_i}}{(T'_{\omega_i})^2} \right\|_{\infty}.
\]

Notice that \( \lambda_{\omega_i} \) measures the dilation and \( \Delta_{\omega_i} \) the distortion of the map \( T_{\omega_i} \).

Expectations with respect to the “selection distribution” \( \eta \) will often be denoted by angular brackets \( \langle \cdot \rangle \). That is, for any measurable function \( h : \Omega \to \mathbb{R} : h(\omega_1) = h_{\omega_1} \), we write

\[
\langle h \rangle = \int h_{\omega_1} \, d\eta(\omega_1).
\]

Standing assumption. Throughout the paper, we assume that the moment conditions

\[
\langle \lambda^{-2} \rangle < 1 \quad \text{and} \quad \langle \Delta^2 \rangle < \infty \tag{1}
\]

be satisfied.

In particular, \( \langle \lambda \rangle > 1 \), meaning that the composed maps are expanding on the average. An individual map, on the other hand, could have regions of strong contraction, \( T''_{\omega_i} \approx 0 \), as well as those of strong distortion, \( |T''_{\omega_i}| \gg (T'_{\omega_i})^2 \).

The sequence \( (X_n)_{n \geq 0} \) with

\[
X_n(\omega, x) = X_n(\omega_1, \ldots, \omega_n, x) = T_{\omega_n} \circ \cdots \circ T_{\omega_1} (x),
\]

where \( \omega = (\omega_n)_{n \geq 1} \in \Omega^\mathbb{N} \) and \( x \in S \), forms a homogeneous Markov chain with state space \( S \). (We set \( X_0(\omega, x) = x \).) The Markov operator \( Q \) corresponding to \( (X_n)_{n \geq 0} \) has the expression

\[
Qf(x) = \int_{\Omega} f(T_{\omega_1} x) \, d\eta(\omega_1)
\]
for any bounded measurable function \( f \). Let us also define the operator \( \mathcal{P} \) as

\[
\mathcal{P}g(x) = \int \mathcal{L}_{\omega_1} g(x) \, d\eta(\omega_1), \quad g \in L^1(\mathbf{m}),
\]

where \( \mathcal{L}_{\omega_1} : L^1(\mathbf{m}) \to L^1(\mathbf{m}) \) stands for the transfer operator of the map \( T_{\omega_1} \) associated to the Lebesgue measure \( \mathbf{m} \), that is,

\[
\mathcal{L}_{\omega_1} g(x) = \sum_{y \in T_{\omega_1}^{-1}(x)} \frac{g(y)}{|T'_{\omega_1}(y)|}.
\]

As is straightforward to check, it is the dual of \( Q \) in the sense that

\[
\int g \cdot Q f \, d\mathbf{m} = \int \mathcal{P} g \cdot f \, d\mathbf{m}.
\]

A probability distribution \( \mu \) is stationary for the Markov chain \( (X_n)_{n \geq 0} \) if

\[
\int Q f \, d\mu = \int f \, d\mu
\]

for all bounded measurable \( f \). If \( \mu \) is absolutely continuous with density \( \phi \) (with respect to \( \mathbf{m} \)), the stationarity condition reduces to

\[
\mathcal{P} \phi = \phi.
\]

For brevity, we will write \( E \) for the expectation with respect to the product measure \( \mathbb{P} = \eta^\infty \). That is, if \( h : \Omega^n \to \mathbb{R} \) is measurable, then

\[
E[h] = \int h(\omega_1, \ldots, \omega_n) \, d\eta^n(\omega_1, \ldots, \omega_n).
\]

We denote by \( P^\mu \) the measure induced on the path space of the Markov chain \( (X_n)_{n \geq 0} \) with initial measure \( \mu \). The corresponding expectation we denote \( E^\mu \). That is,

\[
E^\mu[h(X_0, \ldots, X_n)] = \int h(X_0, \ldots, X_n) \eta^n(\omega_1, \ldots, \omega_n) \, d\mu(x),
\]

for any bounded measurable \( h : S^{n+1} \to \mathbb{R} \) and any \( n \geq 0 \).

Finally, \( \sigma \) will denote the usual left shift on \( \Omega^\infty \). That is,

\[
(\sigma \omega)_n = \omega_{n+1}, \quad n \geq 1.
\]

3. Results

The next theorem is our first result.

Theorem 1. The Markov chain \( (X_n)_{n \geq 0} \) admits an absolutely continuous stationary probability distribution \( \mu \) whose density \( \phi \) is Lipschitz continuous and bounded away from zero.

Remark 2. In Corollary 12 we establish a “quantitative” lower bound on \( \phi \) depending only on the “system constants” appearing in (1).

From here on, \( \mu \) and \( \phi \) will always refer to the objects above. Once the existence of \( \phi \) has been established, it is interesting to study convergence of initial densities toward it. To that end, we first work with individual sequences \( \omega \).
Theorem 3. There exists such a constant $\theta \in (0, 1)$ that the following holds. Let $\alpha \in (0, 1)$. For almost every $\omega$, there exists $C(\omega) > 0$ such that
\[
\| \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} (\psi^1 - \psi^2) \|_{L^1(m)} \leq C(\omega) \max(\|\psi^1\|_{\alpha}, \|\psi^2\|_{\alpha}) \theta^\alpha n
\] (2)
for all $n \geq 0$ and all probability densities $\psi^1, \psi^2 \in C^\alpha$. Moreover, given a probability distribution $d\nu = \psi \, dm$ with $\psi \in C^\alpha$,
\[
\left| \int f \cdot g \circ T_{\omega_n} \circ \ldots \circ T_{\omega_1} \, d\nu - \int f \, d\nu \int g \circ T_{\omega_n} \circ \ldots \circ T_{\omega_1} \, d\nu \right| \leq C(\omega) \|\psi\|_\alpha \|f\|_\alpha \|g\|_\infty \theta^\alpha n
\] (3)
for all $n \geq 0$, and all complex-valued functions $f \in C^\alpha$ and $g \in L^\infty$.

Here (2) states that, for typical sequences $\omega$, the $L^1$-distance between the push-forwards of two Hölder continuous densities tends to zero exponentially. The bound in (3) states that, with respect to any probability measure having a Hölder continuous density, the random variables $f$ and $g \circ T_{\omega_n} \circ \ldots \circ T_{\omega_1}$ become asymptotically decorrelated at an exponential rate. The method of proof we use is coupling. Theorem 3 can also be obtained by different means, namely that of thermodynamic formalism and Hilbert projective cones; see [11, 12] and, for a piecewise smooth case, [5].

Once the sequence-wise bounds have been obtained, related results can be established for the Markov chain $(X_n)_{n \geq 0}$:

Theorem 4. There exist a constant $\theta \in (0, 1)$ and, for any $\alpha \in (0, 1)$, a constant $C > 0$ such that
\[
\| P^n \psi - \phi \|_{L^1(m)} \leq C \|\psi\|_\alpha \theta^\alpha n
\] (4)
for all $n \geq 0$ and all probability densities $\psi \in C^\alpha$. Moreover,
\[
\left| \int f \cdot Q^n g \, d\mu - \int f \, d\mu \int g \, d\mu \right| \leq C \|f\|_\alpha \|g\|_\infty \theta^\alpha n
\] (5)
for all $n \geq 0$, and all complex-valued functions $f \in C^\alpha$ and $g \in L^\infty$.

By (4), the push-forwards of Hölder continuous initial densities converge in $L^1$ to the Lipschitz continuous invariant density at an exponential rate, while pair correlations with respect to the stationary distribution decay exponentially by (5). In the present formulation, Theorem 4 does strictly speaking not follow from Theorem 3, because we do not claim that $C(\omega)$ has finite expectation. Rather, we will prove the two results in parallel, as consequences of common intermediate bounds.

By Theorem 4, the measure $\mu$ is ergodic. It is standard that distinct ergodic measures are mutually singular. Since $\mu$ is equivalent to $m$ by Theorem 1, we get the following corollary:

Corollary 5. The measure $\mu$ is the unique absolutely continuous ergodic measure.
Theorem 6. Fix a positive integer $d$. Let $f : S \to \mathbb{R}^d$ be Hölder continuous with $\int f \, d\mu = 0$, and denote briefly
\[ A_n = f \circ X_n. \] (6)

There exists such a symmetric, semi-positive-definite, $d \times d$ matrix $\Sigma^2$ that the following hold:

1. The matrix $\Sigma^2$ is the limit covariance of $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} A_k$. That is,
\[ \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \left( \sum_{k=0}^{n-1} A_k \otimes \sum_{k=0}^{n-1} A_k \right) = \Sigma^2. \]

2. The random variables $\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} A_k$ converge in distribution, as $n \to \infty$, to a centered $\mathbb{R}^d$-valued normal random variable with covariance $\Sigma^2$.

3. Given any $\lambda > \frac{1}{4}$, there exists a probability space together with two $\mathbb{R}^d$-valued processes $(A^*_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ on it, for which the following statements are true:
   (a) $(A_n)_{n \geq 0}$ and $(A^*_n)_{n \geq 0}$ have the same distribution.
   (b) The random variables $B_n$, $n \geq 0$, are independent, centered, and normally distributed with covariance $\Sigma^2$.
   (c) Almost surely, $\left| \sum_{k=0}^{n-1} A_k^* - \sum_{k=0}^{n-1} B_k \right| = o(n^\lambda)$.

Item (2) of the theorem is called an averaged (or annealed) central limit theorem and item (3) a vector-valued almost sure invariance principle with covariance $\Sigma^2$ and error exponent $\lambda$. The “almost surely” in item (c) refers to the probability space on which the processes $(A^*_n)_{n \geq 0}$ and $(B_n)_{n \geq 0}$ are defined. Note that $\sum_{k=0}^{n-1} B_k$ can be interpreted as the location of an $\mathbb{R}^d$-valued Brownian motion at time $n$. The almost sure invariance principle implies several other limit results, which we do not list here; see [3,13,21,26].

A standard computation in item (1) yields the formula
\[ \Sigma^2 = \int f \otimes f \, d\mu + \sum_{m=1}^{\infty} \int (f \otimes Q^m f + Q^m f \otimes f) \, d\mu. \]

The question arises whether this matrix is non-degenerate.

Lemma 7. Consider a nonzero vector $v \in \mathbb{R}^d$. The matrix $\Sigma^2$ is degenerate in the direction $v$, i.e., $v^T \Sigma^2 v = 0$, if and only if there exists a Hölder continuous function $g : S \to \mathbb{R}$ satisfying
\[ v^T f(x) = g(x) - g(T_{\omega_1} x) \]
for all $x$ and almost all $\omega_1$. (Here the superscript $T$ denotes transposition.)

The preceding lemma places a serious obstruction to degeneracy. In particular, up to a negligible set of $\omega$’s,
\[ \sum_{k=0}^{p-1} v^T f(T_{\omega_k} \circ \cdots \circ T_{\omega_1}(x)) = 0 \]
whenever $T_{\omega_p} \circ \cdots \circ T_{\omega_1}(x) = x$ (periodic trajectory). Hence, having a degenerate covariance matrix $\Sigma^2$ amounts to a very exceptional choice of $f$. 


4. PROOF OF THEOREM 1

The strategy of proving Theorem 1 is to find $\phi$ as an accumulation point of $(n^{-1} \sum_{k=0}^{n-1} P^k 1)_{n \geq 1}$ by showing that $\|P^n 1\|_{C^1}$ is uniformly bounded for $n \geq 0$.

Given a sequence $(\omega_i)_{i \geq 1}$, denote

$$S_n = \prod_{i=1}^{n} \lambda_{\omega_i}^{-1} \quad \text{and} \quad R_n = \sum_{i=1}^{n} \Delta_{\omega_i} \prod_{j=i+1}^{n} \lambda_{\omega_j}^{-1}. \tag{7}$$

Lemma 8. For any $C^1$-function $\psi$, the bound

$$|(L_{\omega_n} \cdots L_{\omega_1} \psi)'| \leq S_n \cdot L_{\omega_n} \cdots L_{\omega_1} |\psi'| + R_n \cdot L_{\omega_n} \cdots L_{\omega_1} |\psi|$$

holds.

Proof. The straightforward bound

$$|(L_{\omega_i} \psi)'| \leq \lambda_{\omega_1}^{-1} L_{\omega_1} |\psi'| + \Delta_{\omega_1} L_{\omega_1} |\psi|, \quad \omega_1 \in \Omega,$$

holds for a $C^1$-function $\psi$. Iterating this bound yields the claim. \( \square \)

First of all, Lemma 8 implies

$$\|(L_{\omega_n} \cdots L_{\omega_1} \psi)'\|_{L^1(m)} \leq S_n \|\psi'\|_{L^1(m)} + R_n \|\psi\|_{L^1(m)},$$

because $L_{\omega_i}$ is a contraction in $L^1(m)$. Setting $\psi = 1$ yields $\|(L_{\omega_n} \cdots L_{\omega_1})'\|_{L^1(m)} \leq R_n$. Thus,

$$\|(L_{\omega_n} \cdots L_{\omega_1})'\|_{L^1(m)} \leq 1 + \|(L_{\omega_n} \cdots L_{\omega_1})'\|_{L^1(m)} \leq 1 + R_n,$$ \( \tag{8} \)

because $L_{\omega_n} \cdots L_{\omega_1}$ is a $C^1$ probability density. Another consequence of Lemma 8 is

$$\|(L_{\omega_n} \cdots L_{\omega_1})'\|_{L^\infty} \leq R_n \|L_{\omega_n} \cdots L_{\omega_1}\|_{L^\infty} \leq R_n (1 + R_n).$$

In particular,

$$\|P^n 1\|_{C^1} \leq E[(1 + R_n)^2].$$

Lemma 9. There exists such a constant $C_R > 0$ that

$$\sup_{n \geq 1} E[(1 + R_n)^2] \leq C_R.$$

Proof. By Jensen’s inequality, it is enough to check that $E[R_n^2]$ is uniformly bounded. But

$$R_n^2 = \sum_{i=1}^{n} \sum_{\ell=1}^{n} \Delta_{\omega_i} \Delta_{\omega_\ell} \prod_{j=i+1}^{n} \lambda_{\omega_j}^{-1} \prod_{k=\ell+1}^{n} \lambda_{\omega_k}^{-1}$$

$$= \sum_{i=1}^{n} \Delta_{\omega_i} \prod_{j=i+1}^{n} \lambda_{\omega_j}^{-2} + 2 \sum_{1 \leq i < \ell \leq n} \Delta_{\omega_i} \Delta_{\omega_\ell} \prod_{j=i+1}^{n} \lambda_{\omega_j}^{-1} \prod_{k=\ell+1}^{n} \lambda_{\omega_k}^{-1}$$

$$= \sum_{i=1}^{n} \Delta_{\omega_i} \prod_{j=i+1}^{n} \lambda_{\omega_j}^{-2} + 2 \sum_{1 \leq i < \ell \leq n} \Delta_{\omega_i} \Delta_{\omega_\ell} \lambda_{\omega_\ell}^{-1} \prod_{j=i+1}^{\ell-1} \lambda_{\omega_j}^{-1} \prod_{k=\ell+1}^{n} \lambda_{\omega_k}^{-2}.$$

Therefore,

$$E[R_n^2] = \langle \Delta^2 \rangle \sum_{i=1}^{n} \langle \lambda^{-2} \rangle^{n-i} + 2 \langle \Delta \rangle \langle \lambda^{-1} \Delta \rangle \sum_{1 \leq i < \ell \leq n} \langle \lambda^{-1} \rangle^{\ell-i} \langle \lambda^{-2} \rangle^{n-\ell}.$$  

Here

$$\langle \lambda^{-1} \rangle \leq \langle \lambda^{-2} \rangle^{1/2}, \quad \langle \Delta \rangle \leq \langle \Delta^2 \rangle^{1/2} \quad \text{and} \quad \langle \lambda^{-1} \Delta \rangle \leq \langle \lambda^{-2} \rangle^{1/2} \langle \Delta^2 \rangle^{1/2}$$

by Jensen’s and Hölder’s inequalities. Thus, by (1),

$$\sum_{1 \leq i < \ell \leq n} \langle \lambda^{-1} \rangle^{\ell-i} \langle \lambda^{-2} \rangle^{n-\ell} = \sum_{\ell=2}^{n} \left( \sum_{i=1}^{\ell-1} \langle \lambda^{-1} \rangle^{\ell-i} \right) \langle \lambda^{-2} \rangle^{n-\ell} \leq \frac{1}{1 - \langle \lambda^{-1} \rangle} \frac{1}{1 - \langle \lambda^{-2} \rangle} < \infty.$$

This proves the lemma. \( \square \)
It is standard that the existence of a Lipschitz continuous stationary distribution as an accumulation point of the sequence \((n^{-1} \sum_{k=1}^{n-1} \mathcal{P}^k \mathbf{1})_{n \geq 1}\) follows by a compactness argument (see, e.g., [27]). The distribution is strictly positive. To see this, first observe that \(\phi > 0\) on an arc \(I \subset \mathbb{S}\). Also, there exists a \(\lambda > 1\) such that \(\eta(\omega_1 \in \Omega : \lambda_{i_1} \geq \lambda) > 0\). Thus, we have \(\mathcal{P}^n(\phi | I) > 0\) for some sufficiently large \(n\). Now \(\phi = \mathcal{P}^n(\phi) \geq \mathcal{P}^n(\phi | I) > 0\).

The proof of Theorem 1 is now complete. \(\square\)

5. Proofs of Theorems 3 and 4

5.1. Regularity of push-forward densities. The following distortion estimate is standard. It will be needed for controlling the regularity of the push-forward distributions under the dynamics.

**Lemma 10.** Let \(n \in \mathbb{N}\) be arbitrary. For any \(x, y \in \mathbb{S}\),

\[
e^{-R_n d(x,y)} \leq \frac{(T_{\omega_1} \cdots \circ T_{\omega_i})(((T_{\omega_n} \cdots \circ T_{\omega_1})_i^{-1})x)}{(T_{\omega_n} \cdots \circ T_{\omega_1})(((T_{\omega_n} \cdots \circ T_{\omega_1})_{i-1}y)} \leq e^{R_n d(x,y)},
\]

(9)

Here \(R_n\) is as defined earlier and \((T_{\omega_n} \cdots \circ T_{\omega_1})_i^{-1}\) is the \(i\)th branch of the inverse of \(T_{\omega_n} \cdots \circ T_{\omega_1}\) on a given arc \(J \subset \mathbb{S}\) of length \(|J| \leq \frac{1}{2}\) containing both \(x\) and \(y\).

**Proof.** For brevity, let \(x_{-n+i}\) and \(y_{-n+i}\) denote the preimages of \(x\) and \(y\), respectively, along the same branch of the inverse of \(T_{\omega_n} \cdots \circ T_{\omega_1}\). Note that

\[
(T_{\omega_n} \cdots \circ T_{\omega_1})'(x_{-n}) = \prod_{i=1}^{n} T'_{\omega_i} \circ S_{\omega_i}(x_{-n+i}),
\]

where \(S_{\omega_i}\) stands for an appropriate inverse branch of \(T_{\omega_i}\). Hence,

\[
\log \frac{(T_{\omega_n} \cdots \circ T_{\omega_i})(x_{-n})}{(T_{\omega_n} \cdots \circ T_{\omega_i})(y_{-n})} \leq |\log((T_{\omega_n} \cdots \circ T_{\omega_1})(x_{-n})) - \log((T_{\omega_n} \cdots \circ T_{\omega_1})(y_{-n}))|
\]

\[
\leq \sum_{i=1}^{n} |\log T'_{\omega_i} \circ S_{\omega_i}(x_{-n+i}) - \log T'_{\omega_i} \circ S_{\omega_i}(y_{-n+i})|
\]

\[
\leq \sum_{i=1}^{n} \|\log T'_{\omega_i} \circ S_{\omega_i}\|_{\infty} d(x_{-n+i}, y_{-n+i}) \leq \sum_{i=1}^{n} \Delta_{\omega_i} d(x_{-n+i}, y_{-n+i})
\]

\[
\leq \sum_{i=1}^{n} \Delta_{\omega_i} \prod_{j=i+1}^{n} \lambda_{\omega_j}^{-1} d(x, y) = R_n d(x, y).
\]

A similar estimate is obtained by interchanging \(x\) and \(y\), which proves the claim. \(\square\)

**Proposition 11.** Suppose \(\psi\) is a strictly positive probability density and that \(\log \psi \in C^\alpha\). Then \(\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} \psi\) inherits these properties for every \(n \in \mathbb{N}\) and

\[
|\log \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} \psi|_\alpha \leq S_n^\alpha |\log \psi|_\alpha + R_n,
\]

where \(R_n\) and \(S_n\) have been defined earlier.

**Proof.** Let \(J \subset \mathbb{S}\) be an arc with \(|J| \leq \frac{1}{2}\). Given an initial probability density \(\psi\), we introduce the notation

\[
\psi_{n,i}(x) = \frac{\psi((T_{\omega_n} \cdots \circ T_{\omega_1})_i^{-1}x)}{(T_{\omega_n} \cdots \circ T_{\omega_1})'_i((T_{\omega_n} \cdots \circ T_{\omega_1})_i^{-1}x)}, \quad x \in J,
\]

Then

\[
\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} \psi(x) = \sum_{i=1}^{w} \psi_{n,i}(x),
\]
where \( w \) is the number of inverse branches of \( T_{\omega_1} \circ \cdots \circ T_{\omega_i} \) on \( J \). Next, let \( x, y \in \mathbb{S} \) be arbitrary. Without loss of generality, we may assume both points belong to \( J \). Therefore,

\[
\left| \log \frac{\psi_{n,i}(x)}{\psi_{n,i}(y)} \right| \leq \left| \log \frac{\psi((T_{\omega_1} \circ \cdots \circ T_{\omega_i})^{-1}x)}{\psi((T_{\omega_1} \circ \cdots \circ T_{\omega_i})^{-1}y)} \right| + \left| \log \frac{(T_{\omega_1} \circ \cdots \circ T_{\omega_i})((T_{\omega_1} \circ \cdots \circ T_{\omega_i})^{-1}y)}{(T_{\omega_1} \circ \cdots \circ T_{\omega_i})((T_{\omega_1} \circ \cdots \circ T_{\omega_i})^{-1}x)} \right|
\]

Taking logarithms yields the desired bound.

For brevity, denote \( B_n = (S_n^1 | \log \psi|_\alpha + R_n) d(x, y) \). Then

\[
e^{-B_n} \psi_{n,i}(y) \leq \psi_{n,i}(x) \leq e^{B_n} \psi_{n,i}(y).
\]

Summing over \( i \),

\[
e^{-B_n} \mathcal{L}_{\omega_1} \cdots \mathcal{L}_{\omega_1} \psi(y) \leq \mathcal{L}_{\omega_1} \cdots \mathcal{L}_{\omega_1} \psi(x) \leq e^{B_n} \mathcal{L}_{\omega_1} \cdots \mathcal{L}_{\omega_1} \psi(y).
\]

Taking logarithms yields the desired bound.

Before proceeding, we prove the lower bound alluded to below Theorem 1 on the stationary density \( \phi \) in terms of system constants:

**Corollary 12.** There exists a constant \( c > 0 \), depending only on the moments appearing in (1), for which

\[
\inf \phi \geq c.
\]

**Proof.** Recall that \( \phi \) is an accumulation point of the sequence \((n^{-1} \sum_{k=0}^{n-1} \mathcal{P}^{k} 1)_{n \geq 1}\). From Proposition 11 we get \( \log \mathcal{L}_{\omega_1} \cdots \mathcal{L}_{\omega_1} 1 \geq -R_n \) for all \( n \geq 0 \). Applying Jensen’s inequality,

\[
\mathcal{P}^{n} 1 = \mathbb{E}(\mathcal{L}_{\omega_1} \cdots \mathcal{L}_{\omega_1} 1) \geq \mathbb{E}(e^{-R_n}) \geq e^{-\mathbb{E}(R_n)}.
\]

Lemma 9, together with another application of Jensen’s inequality, finishes the proof.

\[\square\]

5.2. **Coupling argument.** We are now ready to explain the coupling step. In what follows, we assume that \( \alpha > 0 \) has been fixed once and for all.

We introduce the notation

\[ \mathcal{H}_K = \{ \psi : X \to \mathbb{R} \text{ a probability density, } \psi > 0, \ | \log \psi|_\alpha \leq K \} \]

with \( K > 0 \). The following lemma will turn out useful.

**Lemma 13.** Fix any \( K > 0 \) and set

\[
\kappa = \frac{1}{2} \exp(-K) \quad (10)
\]

\[
K' = \exp(4K) \quad (11)
\]

Then

1. \( \psi \geq 2\kappa > 0 \) holds for every \( \psi \in \mathcal{H}_K \).
2. \( \tilde{\psi} := (\psi - \kappa)/(1 - \kappa) \in \mathcal{H}_{K'} \) for all \( \psi \in \mathcal{H}_K \).

**Proof.** We have the elementary bounds (see [27])

\[
\exp(-| \log \psi|_\alpha) \leq \psi(x) \leq \exp(| \log \psi|_\alpha)
\]

and

\[
|\psi|_\alpha \geq | \log \psi|_\alpha \exp(| \log \psi|_\alpha)
\]

for probability densities \( \psi \). Thus, for \( \psi \in \mathcal{H}_K \),

\[
\psi(x) \geq \exp(-| \log \psi|_\alpha) \geq \exp(-K) = 2\kappa.
\]

\[\square\]
Therefore,
\[
\left| \log \left( \frac{\psi(x) - \kappa}{1 - \kappa} \right) - \log \left( \frac{\psi(y) - \kappa}{1 - \kappa} \right) \right| \leq \sup_x \frac{1}{\psi(x) - \kappa} |\psi(x) - \psi(y)| \leq \frac{1}{\kappa} |\psi_\alpha| d(x, y)^\alpha
\]
\[
\leq 2K \exp(2K) d(x, y)^\alpha \leq \exp(4K) d(x, y)^\alpha,
\]
which proves the lemma. \(\square\)

From here on, we will assume that \(K > 0\) is fixed once and for all. (The value of \(K\) will be determined later.) This also fixes \(\kappa\) and \(K'\).

Given a \(K'' > 0\) and a sequence \(\omega\), we say that the

**Coupling condition** \(C(K'', \omega, n)\) is satisfied if
\[
S_n K'' + R_n \leq K. \tag{12}
\]

This definition is natural, because Proposition 11 implies that
\[
\mathcal{L}_\omega \cdots \mathcal{L}_\omega \psi \in \mathcal{H}_K \quad \forall \psi \in \mathcal{H}_{K''} \quad \text{if } C(K'', \omega, n) \text{ is satisfied.}
\]
Let \(\psi^1\) and \(\psi^2\) be arbitrary densities in \(\mathcal{H}_{K''}\) and suppose \(C(K'', \omega, n)\) is satisfied for some \(n\). Then, by the above observation and by Lemma 13,
\[
\psi^j_n \equiv \mathcal{L}_\omega \cdots \mathcal{L}_\omega \psi^j = \kappa + (1 - \kappa) \tilde{\psi}^j_n,
\]
where
\[
\tilde{\psi}^j_n = (\psi^j_n - \kappa)/(1 - \kappa) \in \mathcal{H}_{K'}. \tag{13}
\]
Thus,
\[
\|\psi^1_n - \psi^2_n\|_{L^1(m)} \leq (1 - \kappa)\|\tilde{\psi}^1_n - \tilde{\psi}^2_n\|_{L^1(m)}.
\]
In other words, the coupling condition allowed us to “couple” a \(\kappa\) fraction of the \(n\)-step push-forwards of the densities \(\psi^1\) and \(\psi^2\). Obviously, the procedure can be continued inductively, treating \(\tilde{\psi}^j_n\) as the initial densities: since (13) holds, we can couple a \(\kappa\) fraction of the \(\tau\)-step push-forwards \(\mathcal{L}_{\omega_{i+\tau}} \cdots \mathcal{L}_{\omega_{i+1}} \tilde{\psi}^j_n\) and \(\mathcal{L}_{\omega_{i+\tau}} \cdots \mathcal{L}_{\omega_{i+1}} \tilde{\psi}^2_n\) assuming that \(C(K', \sigma'' \omega, \tau)\) holds, and again the “normalized remainder densities” are in \(\mathcal{H}_{K'}\) by (13).

Let us formalize the above procedure. Given \(K'' > 0\) and \(\omega\), define
\[
\tau_0(\omega) = 0 \quad \text{and} \quad \tau_1(\omega) = \inf\{n \geq 0 : C(K'', \omega, n) \text{ is satisfied}\},
\]
and
\[
\tau_k(\omega) = \inf\{n \geq 0 : C(K', \sigma^{\tau_{k-1}}(\omega), n) \text{ is satisfied}\}
\]
for all \(k \geq 1\). We use here the convention that the infimum of the empty set is \(\infty\). Next, set
\[
n_0(\omega) = 0 \quad \text{and} \quad n_k(\omega) = \sum_{j=1}^k \tau_j(\omega),
\]
for \(k \geq 1\). Now \(n_k \in \mathbb{N} \cup \{\infty\}\) is the time at which the \(k\)th coupling will occur and \(\tau_k \in \mathbb{N} \cup \{\infty\}\) is the \(k\)th inter-coupling time. (Both depend on \(K''\) and \(\omega\), but we suppress this from the notation.) In particular, using the above coupling argument in combination with the \(L^1(m)\)-contractivity of each \(\mathcal{L}_\omega\), and the obvious fact that \(\|\psi^1 - \psi^2\|_{L^1(m)} \leq 2\|\psi^1 - \psi^2\|_{L^1(m)} \leq 2(1 - \kappa)^k\) for all \(n \geq n_k\) holds for all pairs \(\psi^1, \psi^2 \in \mathcal{H}_{K''}\). Alternatively, writing
\[
N_n(\omega) = \max\{k \geq 0 : n_k(\omega) \leq n\}
\]
for the number of couplings by time \(n\) for the sequence \(\omega\),
\[
\|\psi^1_n - \psi^2_n\|_{L^1(m)} \leq 2(1 - \kappa)^{N_n(\omega)} \quad \forall n \geq 0.
\]

A COUPLING APPROACH TO RANDOM CIRCLE MAPS EXPANDING ON THE AVERAGE 9
In brief, the $L^1$-distance between the densities converges exponentially to zero as a function of the number of couplings that has occurred. To make use of this, it is necessary to study the statistical properties of $N_n$.

### 5.3. Coupling time analysis.

In this section we analyze the tail behavior of the inter-coupling times $\tau_k$, and subsequently obtain crucial information about the distribution of $N_n$. The task will boil down to studying a pair of random difference equations.

For notational simplicity, let us write

$$A_n = \lambda_{\omega_n}^{-1} \quad \text{and} \quad B_n = \Delta_{\omega_n}. $$

Given $\alpha \in (0, 1)$, Proposition 11 states that

$$|\log L_{\omega_n} \cdots L_{\omega_1} \psi|_{\alpha} \leq S^\alpha_n |\log \psi|_{\alpha} + R_n,$$

where, according to (7),

$$S_n = \prod_{i=1}^{n} A_i \quad \text{and} \quad R_n = \sum_{i=1}^{n} B_i \prod_{j=i+1}^{n} A_j.$$

Starting with $|\log \psi|_{\alpha} = \xi$, we can perform a coupling when $S^\alpha_n \xi + R_n \leq K$; see (12). Here $\xi > 0$ is an arbitrary initial condition and $K > 0$ a large (non-random) constant to be fixed later.

Note that $R_n$ and $\tilde{S}_n = S^\alpha_n \xi$ satisfy the random difference equations

$$R_n = A_n R_{n-1} + B_n, \quad n \geq 1,$$

and

$$\tilde{S}_n = A_n^\alpha \tilde{S}_{n-1}, \quad n \geq 1.$$

Our objective is to control the random time when the coupling condition

$$Z_n = R_n + \tilde{S}_n \leq K$$

is first satisfied, given the initial condition

$$(R_0, \tilde{S}_0) = (0, \xi).$$

Indeed,

$$|\log L_{\omega_n} \cdots L_{\omega_1} \psi|_{\alpha} \leq Z_n$$

for all $n \geq 0$. This objective is complicated by the fact that $\tilde{S}_n$ and $R_n$ are not independent random variables, and because — unlike $(\tilde{S}_n)_{n \geq 0}$ and $(R_n)_{n \geq 0}$ separately — the sequence $(Z_n)_{n \geq 0}$ of their sums does not satisfy a simple recursion relation.

To remedy the above situation, we begin with the observation that

$$Z_n \leq R_n + \max(\tilde{S}_n, 1) \leq R_n + \max(\tilde{S}_n^{1/\alpha}, 1) \leq R_n + \tilde{S}_n^{1/\alpha} + 1.$$

Since $\tilde{S}_n^{1/\alpha} = A_n \tilde{S}_n^{1/\alpha}$, the sums

$$L_n = R_n + \tilde{S}_n^{1/\alpha}$$

satisfy the random difference equation

$$L_n = A_n L_{n-1} + B_n, \quad n \geq 1, \quad (14)$$

with the initial condition

$$L_0 = \xi^{1/\alpha}.$$

This is of interest because (14) is “simple” and because of the dominating property

$$Z_n \leq L_n + 1, \quad n \geq 0.$$

The coupling condition is therefore certainly satisfied if

$$L_n \leq K - 1.$$
Thus, let
\[ T = \inf\{ k \geq 0 : L_k \leq K - 1 \} \]
be the first time the Markov chain \((L_n)_{n \geq 0}\) dips below level \(K - 1\). The utility of \(T\) to our proof lies in the fact that it dominates the inter-coupling times \(\tau_k\) when \(\xi\) is chosen properly.

For the following, note that the assumptions in (1) imply
\[ \langle A_1 \rangle < 1 \quad \text{and} \quad \langle B_1 \rangle < \infty. \]

**Proposition 14.** Fix any
\[ K > \frac{\langle B_1 \rangle}{1 - \langle A_1 \rangle} + 1. \]
Then
\[ q = \langle A_1 \rangle + \frac{\langle B_1 \rangle}{K - 1} < 1. \]
Starting the Markov chain \((L_n)_{n \geq 0}\) at an arbitrary level \(L_0 = \ell > K - 1\),
\[ P^\ell(T > n) \leq \frac{\ell q^n}{K - 1}, \]
for all \(n \geq 0\). (Here \(P^\ell\) is the path measure of \((L_n)_{n \geq 0}\) starting at \(\ell\).)

We point out that the number \(\frac{\langle B_1 \rangle}{1 - \langle A_1 \rangle}\) appearing in the lemma above is the expected value of the stationary limit distribution of the chain \((L_n)_{n \geq 0}\); see [29].

**Proof.** The key idea of the proof is to dominate the chain \((L_n)_{n \geq 0}\) with another chain \((\tilde{L}_n)_{n \geq 0}\) whose value decays below level \(K - 1\) quickly. Note that we can rewrite (14) as
\[ L_m = U_m L_{m-1} \]
where
\[ U_m = A_m + \frac{B_m}{L_{m-1}}. \]
Given \(L_0 = \ell > K - 1\), we have \(T \geq 1\) and \(L_{m-1} > K - 1\) for all \(m \in [1, T]\). Therefore,
\[ U_m < A_m + \frac{B_m}{K - 1} = V_m \]
and
\[ L_m \leq V_m L_{m-1}, \quad 1 \leq m \leq T. \]
Defining a Markov chain \((\tilde{L}_m)_{m \geq 0}\) such that \(\tilde{L}_0 = \ell\) and
\[ \tilde{L}_m = V_m \tilde{L}_{m-1}, \quad m \geq 0, \]
we have
\[ L_m \leq \tilde{L}_m, \quad 0 \leq m \leq T. \]
That is, \(\tilde{L}_m\) dominates \(L_m\) for as long as the chain \((L_m)_{m \geq 0}\) remains above level \(K - 1\). In particular, \(T > n \Rightarrow \min_{0 \leq m \leq n} L_m > K - 1 \Rightarrow \tilde{L}_n > K - 1\), so that
\[ P^\ell(T > n) \leq P^\ell(\tilde{L}_n > K - 1). \]
Since \(\tilde{L}_n = \ell \prod_{m=1}^n V_m\), Markov’s inequality now yields
\[ P^\ell(\tilde{L}_n > K - 1) < \frac{1}{K - 1} E^\ell(\tilde{L}_n) = \frac{\ell}{K - 1} (E(V_1))^n = \frac{\ell q^n}{K - 1}, \]
where
\[ q = \langle A_1 \rangle + \frac{\langle B_1 \rangle}{K - 1} < 1 \]
as we assume that \(K > \frac{\langle B_1 \rangle}{\langle A_1 \rangle} + 1\). \(\square\)
Proposition 15. Let $K > 0$ be as in Proposition 14, and $\alpha \in (0, 1)$ and $K'' > 0$ be given. There exist such constants $t \in (0, 1)$, $\vartheta \in (0, 1)$ independent of $\alpha$, and $D > 0$ that
\[
\mathbb{P}(N_n < \lceil \alpha n \rceil) \leq D \vartheta^n
\]
holds for all $n \geq 0$.

Proof. Observe that
\[
\mathbb{P}(N_n < \lceil \alpha n \rceil) = \mathbb{P}\left( \sum_{j=1}^{\lceil \alpha n \rceil} \tau_j > n \right),
\]
where the equality holds because the sum appearing on the right side is just the time when the $\lceil \alpha n \rceil$th coupling occurs. The variables $\tau_j, j \geq 1$, are independent and $\tau_j, j \geq 2$, are also identically distributed. Thus, for any $p \in (0, 1)$,
\[
\mathbb{P}\left( \sum_{j=1}^{\lceil \alpha n \rceil} \tau_j > n \right) \leq p^n \mathbb{E}(p^{-\tau_1}) = p^n \mathbb{E}(p^{-\tau_1}) \mathbb{E}(p^{-\tau_2})^{\lceil \alpha n \rceil - 1} \leq p^n \mathbb{E}(p^{-\tau_1}) \mathbb{E}(p^{-\tau_2})^{\lceil \alpha n \rceil}
\]
By Proposition 14, each of the random variables $\tau_j$ has an exponential tail: more precisely, there exists $q \in (0, 1)$ such that
\[
\mathbb{P}(\tau_1 > m) \leq \frac{(K')^{1/\alpha}}{K - 1} q^m \quad \text{and} \quad \mathbb{P}(\tau_2 > m) \leq \frac{(K')^{1/\alpha}}{K - 1} q^m
\]
for $m \geq 0$. Thus, fixing any $p \in (q, 1),
\[
\mathbb{E}(p^{-\tau_1}) = \int_1^\infty \mathbb{P}(p^{-\tau_1} > x) \, dx = \int_1^\infty \mathbb{P}(\tau_1 > \frac{\log x}{\log(p^{-1})}) \, dx < \infty.
\]
In fact, writing
\[
c = \int_1^\infty x^{-\log q/\log p} \, dx,
\]
we have
\[
\mathbb{E}(p^{-\tau_1}) \leq \frac{(K'')^{1/\alpha} c}{K - 1} \quad \text{and} \quad \mathbb{E}(p^{-\tau_2}) \leq \frac{(K')^{1/\alpha} c}{K - 1}.
\]
We then have
\[
\mathbb{P}(N_n < \lceil \alpha n \rceil) \leq \frac{(K'')^{1/\alpha} c}{K - 1} \left( p \left( \frac{(K'')^{1/\alpha} c}{K - 1} \right)^u \right)^n.
\]
Moreover, with the choices
\[
p = q^{2\beta}, \quad \beta = \frac{K - 1}{2K}, \quad u = t\alpha
\]
we have $c = K - 1$ and
\[
\mathbb{P}(N_n < \lceil \alpha n \rceil) \leq (K'')^{1/\alpha} (q^{2\beta} (K'))^n.
\]
Now we choose $t > 0$ so that $(K')^t = q^{-\beta}$, which yields $\vartheta = q^3 < 1$ and $D = (K'')^{1/\alpha}$. \qed

Proposition 15 implies the bound
\[
\mathbb{P}(N_n < \lceil \alpha n \rceil \text{ for some } n \geq m) = \mathbb{P}\left( \bigcup_{n \geq m} \{ N_n < \lceil \alpha n \rceil \} \right) \leq \sum_{n \geq m} \mathbb{P}(N_n < \lceil \alpha n \rceil)
\]
\[
\leq \sum_{n \geq m} D \vartheta^n = D' \vartheta^m
\]
for all $m \geq 0$. Next, define the random time $\tilde{n} = \tilde{n}(K'', \omega)$ by
\[
\tilde{n} = \inf\{ m \geq 0 : N_n \geq \lceil \alpha n \rceil \text{ for all } n \geq m \}.
In words, given a sequence, the number of couplings by time \( n \) is at least \( \lceil t\alpha n \rceil \) for every \( n \geq \hat{n} \). Then
\[
\mathbb{P}(\hat{n} > k) = \mathbb{P}(N_n < \lceil t\alpha n \rceil \text{ for some } n \geq k) \leq D' \vartheta^k.
\]
In particular, the expected value of \( \hat{n} \) is finite.

5.4. **Proofs of the theorems.** In this section we patch together the results of the previous sections. This leads to Theorems 3 and 4.

Given two probability densities \( \psi_1, \psi_2 \in \mathcal{H}_{K''} \), we have
\[
\|\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} (\psi_1 - \psi_2)\|_{L^1(m)} \leq 2(1 - \kappa)^{\lceil t\alpha n \rceil} \leq 2(1 - \kappa)^{t\alpha n - 1}
\]
for all \( n \geq \hat{n} \). Thus, setting
\[
\chi(n) = \chi(n; K'', \omega) = 1_{\{n > \hat{n}\}} + (1 - \kappa)^{t\alpha n - 1},
\]
the bound
\[
\|\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} (\psi_1 - \psi_2)\|_{L^1(m)} \leq 2\chi(n)
\]
holds true for any \( n \geq 0 \). This implies (2) for the restricted class of densities.

Next, we relax the regularity condition. To this end, given an arbitrary probability density \( \psi \in C'^{\alpha} \), define
\[
\psi_h = \frac{\psi + h}{1 + h}, \quad h > 0.
\]
Since \( \psi + h \geq h \), we have
\[
|\log \psi_h(x) - \log \psi_h(y)| = |\log(\psi(x) + h) - \log(\psi(y) + h)| \leq \frac{1}{h} |\psi(x) - \psi(y)| \leq \frac{|\psi|_{\alpha}}{h} d(x, y)^{\alpha}.
\]
Thus, we obtain
\[
\psi_h \in \mathcal{H}_1, \quad h \geq |\psi|_{\alpha}.
\]
Recall \( |\log \phi|_{\alpha} \leq \text{Lip}(\phi) < \infty \). Setting \( h = |\psi|_{\alpha} + \text{Lip}(\phi) \), both \( \psi_h, \phi_h \) \( \in \mathcal{H}_1 \), so that
\[
\|\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} (\psi - \phi)\|_{L^1(m)} \leq 2(1 + h) \chi(n; 1, \omega).
\]
As \( \|\psi\|_{\infty} \geq 1 \), we can estimate \( 1 + h \leq (1 + \text{Lip}(\phi)) \|\psi\|_{\alpha} \). Taking expectations,
\[
\|P^n(\psi - \phi)\|_{L^1(m)} \leq 2(1 + \text{Lip}(\phi)) \|\psi\|_{\alpha} (D' \vartheta^n + (1 - \kappa)^{t\alpha n - 1}).
\]
In other words, we have proved (2) and (4).

Let us continue our analysis of individual sequences \( \omega \). Suppose \( g \in L^\infty \) is complex-valued and \( f \in C'^{\alpha} \) is real-valued with \( \int f \, d\mathbb{m} = 0 \). Define
\[
\tilde{f} = \frac{f + 2|f|_{\alpha}}{2|f|_{\alpha}}.
\]
Since \( \|f\|_{\infty} \leq |f|_{\alpha} \), it is easy to check that \( \tilde{f} \in \mathcal{H}_1 \). Therefore,
\[
\left| \int f \cdot g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\mathbb{m} \right| = \left| \int \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} f \cdot g \, d\mathbb{m} \right| \leq \|g\|_{\infty} \|\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} f\|_{L^1(m)}
\]
\[
= 2|f|_{\alpha} \|g\|_{\infty} \|\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} (\tilde{f} - 1)\|_{L^1(m)} = 4|f|_{\alpha} \|g\|_{\infty} \chi(n; 1, \omega).
\]
In general, \( \int f \, d\mathbb{m} = 0 \) fails, in which case the preceding bound yields
\[
\left| \int f \cdot g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\mathbb{m} - \int f \, d\mathbb{m} \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\mathbb{m} \right| \leq 4|f|_{\alpha} \|g\|_{\infty} \chi(n; 1, \omega).
\]
We can also change the measure in the integrals above. Indeed, let \( \psi \in C'^{\alpha} \) be a probability density and denote \( d\nu = \psi \, d\mathbb{m} \). Then readily
\[
\left| \int f \cdot g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\nu - \int f \, d\nu \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\nu \right| \leq 4|\psi|_{\alpha} |f|_{\alpha} \|g\|_{\infty} \chi(n; 1, \omega),
\]
because \( |f \psi|_\alpha \leq \|\psi\|_\alpha \|f\|_\alpha \). On the other hand,
\[
\left| \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\nu - \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, dm \right|
= \left| \int \psi \cdot g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, dm - \int \psi \, dm \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, dm \right|
\leq 4|\psi|_\alpha \|g\|_\infty \chi(n; 1, \omega).
\]
Collecting the bounds,
\[
\left| \int f \cdot g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\nu - \int f \, d\nu \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\nu \right| \leq 8\|\psi\|_\alpha \|f\|_\alpha \|g\|_\infty \chi(n; 1, \omega).
\]
Hence, we have proved (3) for real-valued \( f \). For complex-valued \( f \), a similar bound follows from the one above with a larger prefactor. This proves Theorem 3.

In particular, we can choose \( \nu = \mu \). Since \( f \) and \( g \) are bounded, we can therefore estimate
\[
\left| \int f \cdot Q^n g \, d\mu - \int f \, d\mu \int g \, d\mu \right| = \left| \int f \cdot Q^n g \, d\mu - \int f \, d\mu \int Q^n g \, d\mu \right|
= \left| \int f \cdot E[g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1}] \, d\mu - \int f \, d\mu \int E[g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1}] \, d\mu \right|
\leq E\left[ \int f \cdot g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\mu - \int f \, d\mu \int g \circ T_{\omega_n} \circ \cdots \circ T_{\omega_1} \, d\mu \right]
\leq C\|\phi\|_\alpha \|f\|_\alpha \|g\|_\infty E[\chi(n; 1, \omega)] \leq C\|\phi\|_\alpha \|f\|_\alpha \|g\|_\infty (D^h \vartheta^m + (1 - \kappa)^{m-1}).
\]
This proves (5) and Theorem 4. \( \square \)

6. Proof of Theorem 6

Our proof of Theorem 6 is based on providing exponential bounds, uniform in \( n \), on multiple correlation functions of the form
\[
E^\mu[FG_n] - E^\mu[F]E^\mu[G_n],
\] (15)
where, for certain Hölder continuous functions \( g_i \in C^\alpha, i \geq 0, \)
\[
F = g_0 \circ X_0 \cdots g_m \circ X_m
\]
\[
G_n = g_{m+1} \circ \ldots \circ X_{m+k+n} \cdots g_{m+k} \circ X_{m+k+n}.
\] (16)
The main ingredient for obtaining such bounds will be the pair correlation bound in (5) of Theorem 4. Here, beside the uniform exponential rate, the crucial bit of information is that the function \( g \) appearing in (5) is only required to be in \( L^\infty \) and that the bound depends on \( g \) only through its \( L^\infty \) norm.

Fix \( \alpha \in (0, 1), H > 0 \) and \( \varepsilon > 0 \). Let \( f_k, k \geq 0, \) be real-valued functions such that
\[
\sup_{k \geq 0} |f_k|_\alpha \leq H.
\] (17)
Let \( t_k, k \geq 0, \) be real numbers satisfying
\[
\sup_{k \geq 0} |t_k| \leq \varepsilon
\] (18)
and define the functions
\[
g_k = e^{it_k f_k}, \quad k \in \mathbb{N}.
\]
These are the functions we use in (16). Notice immediately that
\[
|g_k| = 1 \quad \text{and} \quad |g_k|_\alpha \leq \varepsilon H.
\] (19)

For what follows, we define the operator \( \hat{\mathcal{P}} \) by setting
\[
\hat{\mathcal{P}} g = \phi^{-1} \mathcal{P}(\phi g).
\]
This will be convenient for manipulating integrals with resect to the invariant measure $\mu$, as
\[
\int \hat{P} g \cdot f \, d\mu = \int P(\phi g) \cdot f \, dm = \int g \cdot Q f \, d\mu.
\]
We also introduce the operators $\hat{P}_g$ and $Q_g$ which act according to
\[
\hat{P}_g(h) = \hat{P}(gh) \quad \text{and} \quad Q_g(h) = Q(gh).
\]

**Lemma 16.** Defining
\[
\tilde{G}_n(\omega, x) = g_{m+1} \circ X_{1+n}(\sigma^m \omega, x) \cdots g_{m+k} \circ X_{k+n}(\sigma^m \omega, x),
\]
we have
\[
E^\mu[FG_n] = E^\mu \left[ g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \cdot \tilde{G}_n \right].
\]
(20)

Above, the operator product $\hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0}$ acts on the constant function $1$. Moreover,
\[
E^\mu[G_n] = E^\mu \left[ \tilde{G}_n \right] \quad \text{and} \quad E^\mu[F] = E^\mu \left[ g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \right].
\]
(21)

**Proof.** We can write
\[
X_{m+i}(\omega, x) = X_i(\sigma^m \omega, X_m(\omega, x)).
\]
Accordingly, $\tilde{G}_n(\omega, X_m(\omega, x)) = G_n(\omega, x)$. Because the Markov chain $(X_n)_{n \geq 0}$ is stationary, $X_m$ has distribution $\mu$. Now, since $\tilde{G}_n(\omega, \cdot)$ only depends on $\omega_i, i > m$, the first identity in (21) follows. Next, integrating with respect to the variables $\omega_i, 1 \leq i \leq m$, in ascending order of the index $i$, we get
\[
E^\mu[FG_n] = E^\mu \left[ g_0 Q_{g_1} Q_{g_2} \cdots Q_{g_m} \tilde{G}_n \right].
\]
Using duality repeatedly, starting with the first $Q$ from the left, then the second, and so on, we arrive ultimately at (20). The second identity in (21) is proved in a similar fashion. $\square$

As a consequence, the difference in (15) equals
\[
E^\mu \left[ g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \cdot \tilde{G}_n \right] - E^\mu \left[ g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \right] E^\mu \left[ \tilde{G}_n \right].
\]

Note also that $g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1$ does not depend on $\omega$ at all. Thus,
\[
E^\mu \left[ g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \right] = \int g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \, d\mu.
\]

We can also integrate out the $\omega$-dependence of $\tilde{G}_n$:
\[
\int \tilde{G}_n \, d\eta^{k+n}(\omega_{m+1}, \cdots, \omega_{m+k+n}) = Q^n Q_{g_{m+1}} Q_{g_{m+2}} \cdots Q_{g_{m+k}} 1.
\]

Here the $\omega_i$-integrals were done in descending order of the index $i$. The resulting expression only depends on $x$. This leaves us with
\[
E^\mu[FG_n] - E^\mu[F] E^\mu[G_n] + E^\mu[G_n] - E^\mu[FG_n] = \int g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \cdot Q^n Q_{g_{m+1}} Q_{g_{m+2}} \cdots Q_{g_{m+k}} 1 \, d\mu
\]
\[
- \int g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1 \, d\mu \cdot \int Q^n Q_{g_{m+1}} Q_{g_{m+2}} \cdots Q_{g_{m+k}} 1 \, d\mu.
\]
(22)

In order to take advantage of (5) directly, we will need to bound $Q_{g_{m+1}} Q_{g_{m+2}} \cdots Q_{g_{m+k}} 1$ in the supremum norm and $g_m \hat{P}_{g_{m-1}} \cdots \hat{P}_{g_1} \hat{P}_{g_0} 1$ in the Hölder norm. Bounds in the supremum norm are immediate, because $Q$ and $P$ are increasing operators and because $\|g_i\|_\infty = 1$. Indeed,
\[
||Q_{g_i} h|| \leq ||Q_{g_i} h|| \leq ||h||_\infty Q1 = ||h||_\infty
\]
and
\[
||\hat{P}_{g_i} h|| \leq \phi^{-1} P(\phi g_i h) \leq ||h||_\infty \phi^{-1} P(\phi) = ||h||_\infty
\]
for any \( h : \mathbb{S} \to \mathbb{C} \), so that
\[
\|Q_{g_{m+1}}Q_{g_{m+2}} \cdots Q_{g_{m+k}} 1\|_{\infty} \leq 1
\]  
(23) 
and
\[
\|g_{m}\hat{P}_{g_{m-1}} \cdots \hat{P}_{g_{i}} 1\|_{\infty} \leq \|\hat{P}_{g_{m-1}} \cdots \hat{P}_{g_{i}} 1\|_{\infty} \leq 1.
\]  
(24) 
We now proceed to bounding the Hölder constant \( |g_{m}\hat{P}_{g_{m-1}} \cdots \hat{P}_{g_{i}} 1|_{\alpha} \), which is more subtle.

Note that
\[
\hat{P}_{g_{m-1}} \cdots \hat{P}_{g_{i}} 1 = \phi^{-1}P_{g_{m-1}} \cdots P_{g_{i}} \phi = \phi^{-1} \int L_{\omega_{n},g_{m-1}} \cdots L_{\omega_{1},g_{i}} \phi \, d\eta(\omega_{1}, \ldots, \omega_{n}), \tag{25}
\]
where
\[
L_{\omega_{1},g} = L_{\omega_{1}}(gh).
\]
The following identity will be convenient, because the right side involves a composition of the “usual” transfer operators \( L_{\omega_{1}} \):

**Lemma 17.** For any \( h \),
\[
L_{\omega_{n},g_{m-1}} \cdots L_{\omega_{1},g_{i}} h = L_{\omega_{n}} \cdots L_{\omega_{1}} (e^{V_{n}}h),
\]
where \( V_{n} = V_{n}(\omega) = \sum_{k=0}^{n-1} it_{k} f_{k} \circ T_{\omega_{k}} \circ \cdots \circ T_{\omega_{1}} \).

**Proof.** This holds for \( n = 1 \). Assume that it holds for \( n = k \). Then, for \( n = k + 1 \) and any \( u \),
\[
\int u \cdot L_{\omega_{n},g_{m-1}} \cdots L_{\omega_{1},g_{i}} h \, dm \overset{\text{induction}}{=} \int u \cdot L_{\omega_{n},g_{m-1}} \cdots L_{\omega_{1}} (e^{V_{n}}h) \, dm
\]
\[
= \int u \circ T_{\omega_{n}} e^{it_{n-1} f_{n-1}} L_{\omega_{n-1}} \cdots L_{\omega_{1}} (e^{V_{n-1}}h) \, dm
\]
\[
= \int u \circ T_{\omega_{n}} \circ \cdots \circ T_{\omega_{1}} \exp(it_{n-1} f_{n-1} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_{1}}) \cdot e^{V_{n-1}}h \, dm
\]
\[
= \int u \circ T_{\omega_{n}} \circ \cdots \circ T_{\omega_{1}} \cdot e^{V_{n}}h \, dm = \int u \cdot L_{\omega_{n}} \cdots L_{\omega_{1}} (e^{V_{n}}h) \, dm.
\]
Thus, the induction principle proves the lemma. \( \square \)

**Lemma 18.** For any complex-valued \( h \in C^{\alpha} \),
\[
|L_{\omega_{n},g_{m-1}} \cdots L_{\omega_{1},g_{i}} h|_{\alpha} \leq (1 + R_{n}) \left( \sum_{j=1}^{n} \frac{\|h\|_{\infty} \Delta_{\omega_{j}}}{} + \sum_{k=0}^{n-1} \frac{\varepsilon H \|h\|_{\infty}}{\lambda_{\omega_{k+1}}^{\alpha}} + \frac{|h|_{\alpha}}{\lambda_{\omega_{1}}^{\alpha}} \right).
\]

**Proof.** Consider two points \( x, y \in \mathbb{S} \) and an arc \( J \) containing both \( x \) and \( y \) with \( |J| \leq \frac{\pi}{2} \). Denote by \( S_{n,i} : J \to S_{n,i} J, 1 \leq i \leq w \), the branches of the inverse of \( T_{n} \equiv T_{\omega_{n}} \circ \cdots \circ T_{\omega_{1}} \). Note
\[
\frac{d}{dz} \frac{1}{T'_{n}(S_{n,i} z)} = - \frac{T''_{n}(S_{n,i} z) \cdot S_{n,i}'(z)}{\left( T'_{n}(S_{n,i} z) \right)^{2}} = - \frac{T''_{n}(S_{n,i} z)}{\left( T'_{n}(S_{n,i} z) \right)^{3}}.
\]
Observe also that \( \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1} \) is the transfer operator associated to \( T_n \). Without loss of generality, we assume \( T_n' > 0 \). Then, recalling Lemma 17 and that \( |e^{V_n}| = 1 \),
\[
|\mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1}(e^{V_n}h)(x) - \mathcal{L}_{\omega_n} \cdots \mathcal{L}_{\omega_1}(e^{V_n}h)(y)| = \left| \sum_{i=1}^{w} \left( \frac{e^{V_n}h(S_{n,i}x)}{T'_n(S_{n,i}x)} - \frac{1}{T_n(S_{n,i}y)} \right) (e^{V_n}h)(S_{n,i}x) \right|
\]
\[
\leq \left| \sum_{i=1}^{w} \left( \frac{1}{T'_n(S_{n,i}x)} - \frac{1}{T_n(S_{n,i}y)} \right) (e^{V_n}h)(S_{n,i}x) \right| + \left| \sum_{i=1}^{w} \frac{1}{T_n(S_{n,i}y)} \right| \left| (e^{V_n}h)(S_{n,i}x) - (e^{V_n}h)(S_{n,i}y) \right|
\]
\[
\leq \|e^{V_n}h\|_{\infty} \sum_{i=1}^{w} \left| \frac{1}{T'_n(S_{n,i}x)} - \frac{1}{T_n(S_{n,i}y)} \right| + \sum_{i=1}^{w} \frac{1}{T_n(S_{n,i}y)} \left| (e^{V_n}h)(S_{n,i}x) - (e^{V_n}h)(S_{n,i}y) \right|
\]
\[
\leq \|h\|_{\infty} \sum_{i=1}^{w} \int \frac{T''_n(S_{n,i}z)}{(T'_n(S_{n,i}z))^2} \, d(z, x, y) + \sum_{i=1}^{w} \frac{|e^{V_n}h \circ S_{n,i}|_\alpha d(x, y)^\alpha}{T_n(S_{n,i}y)}
\]
\[
\leq \left( \|h\|_{\infty} \left| \frac{T''_n}{(T'_n)^2} \right|_{\infty} \right) + \sup_{1 \leq i \leq w} |e^{V_n}h \circ S_{n,i}|_\alpha \|L_{\omega_n} \cdots L_{\omega_1}1\|_{\infty}.
\]

Now, we can first estimate \( \sup_{1 \leq i \leq w} |e^{V_n}h \circ S_{n,i}|_\alpha \):
\[
|e^{V_n}h \circ S_{n,i}|_\alpha \leq \|h\|_{\infty} |e^{V_n}S_{n,i}|_\alpha + |h \circ S_{n,i}|_\alpha \leq \|h\|_{\infty} \sum_{k=0}^{n-1} |t_k| e^{V_n} \circ T_k \circ S_{n,i}|_\alpha + |h|_\alpha \lambda_{T_n}^{-\alpha}
\]
\[
\leq \varepsilon H \|h\|_{\infty} \sum_{k=0}^{n-1} \lambda_{\omega_n}^{-\alpha} \cdots \lambda_{\omega_{k+1}}^{-\alpha} + |h|_\alpha \lambda_{\omega_n}^{-\alpha} \cdots \lambda_{\omega_1}^{-\alpha}.
\]

Finally,
\[
\left| \frac{T''_n}{(T'_n)^2} \right|_{\infty} = \left| \frac{1}{T_n} (\log T'_n) \right|_{\infty} = \left| \frac{1}{T_n} \sum_{j=1}^{n} \left( \log T'_{\omega_j} \circ T_{j-1} \right) \right|_{\infty}
\]
\[
\leq \sum_{j=1}^{n} \left| \frac{1}{(T_{\omega_n} \circ \cdots \circ T_{\omega_{j+1}}') \circ T_{j} \cdot T'_{j-1}} \right| \left| T''_{\omega_j} \circ T_{j-1} \cdot T'_{j-1} \right|\right|_{\infty}
\]
\[
= \sum_{j=1}^{n} \left| \frac{1}{(T_{\omega_n} \circ \cdots \circ T_{\omega_{j+1}}') \circ T_{j}} \right| \left| T''_{\omega_j} \cdot T_{j-1} \right|_{\infty} \leq \sum_{j=1}^{n} \lambda_{\omega_n} \cdots \lambda_{\omega_{j+1}} \Delta_{\omega_j}.
\]

Collecting the bounds and recalling (8) finishes the proof. 

\[ \square \]

**Proposition 19.** Given \( \alpha \in (0, 1) \), \( H > 0 \) and \( \varepsilon > 0 \), there exists such a constant \( C > 0 \) that
\[
\sup_{n \geq 0} |g_{n} \hat{P}_{g_{n-1}} \cdots \hat{P}_{g_{0}} 1|_\alpha \leq C
\]
holds for all \( n \geq 0 \), for all choices of \( (f_{k})_{k \geq 0} \) and \( (t_{k})_{k \geq 0} \) satisfying (17) and (18).

**Proof.** First,
\[
|g_{n} \hat{P}_{g_{n-1}} \cdots \hat{P}_{g_{0}} 1|_\alpha \leq |g_{n}|_\alpha \|\hat{P}_{g_{n-1}} \cdots \hat{P}_{g_{0}} 1\|_{\infty} + |\hat{P}_{g_{n-1}} \cdots \hat{P}_{g_{0}} 1|_\alpha
\]
\[
\leq \varepsilon H + |\hat{P}_{g_{n-1}} \cdots \hat{P}_{g_{0}} 1|_\alpha
\]
where (19) and (24) were used. Next, by (25) and Lemma 17,
\[
|\mathcal{P}_{g_{n-1}} \cdots \mathcal{P}_{g_0} 1|_a \leq \left| \phi^{-1} \int \mathcal{L}_{\omega_n,g_{n-1}} \cdots \mathcal{L}_{\omega_1,g_0} \phi \, d\eta^n(\omega_1, \ldots, \omega_n) \right|_a \\
\leq |\phi^{-1}|_a |\phi|_{\mathcal{P}_{g_{n-1}} \cdots \mathcal{P}_{g_0} 1_\infty} + |\phi^{-1}|_\infty \int |\mathcal{L}_{\omega_n,g_{n-1}} \cdots \mathcal{L}_{\omega_1,g_0} \phi|_a \, d\eta^n(\omega_1, \ldots, \omega_n) \\
\leq |\phi|_a |\phi^{-1}|_\infty^2 |\phi|_\infty + |\phi^{-1}|_\infty \int |\mathcal{L}_{\omega_n,g_{n-1}} \cdots \mathcal{L}_{\omega_1,g_0} \phi|_a \, d\eta^n(\omega_1, \ldots, \omega_n).
\]

With the aid of Lemma 18, the bound in (8) and Lemma 9, we can bound the integral in the last line above using Hölder’s inequality and independence. Namely,
\[
\int |\mathcal{L}_{\omega_n,g_{n-1}} \cdots \mathcal{L}_{\omega_1,g_0} \phi|_a \, d\eta^n(\omega_1, \ldots, \omega_n) \\
\leq \mathbb{E} \left[ (1 + R_n) \left( \sum_{j=1}^n \frac{\|\phi\|_\infty \Delta_{\omega_j}}{\lambda_{\omega_n} \cdots \lambda_{\omega_{j+1}}} + \sum_{k=0}^{n-1} \varepsilon H |\phi|_\infty \sum_{j=1}^n \left( \frac{\Delta_{\omega_j}^2}{\lambda_{\omega_n}^2 \cdots \lambda_{\omega_{j+1}}^2} \right) \right) \right]^{1/2} \\
\leq (\mathbb{E} [(1 + R_n)^2])^{1/2} \left\{ |\phi|_\infty \sum_{j=1}^n \left( \mathbb{E} \left[ \frac{\Delta_{\omega_j}^2}{\lambda_{\omega_n}^2 \cdots \lambda_{\omega_{j+1}}^2} \right] \right)^{1/2} + |\phi|_a \left( \mathbb{E} \left[ \frac{1}{\lambda_{\omega_n}^2 \cdots \lambda_{\omega_{j+1}}^2} \right] \right)^{1/2} \right\} \\
\leq C_R^{1/2} \left\{ |\phi|_\infty \sum_{j=1}^n \lambda^{-2} \sum_{j=1}^{n-1} \lambda^{-2} (n-j)/2 + \varepsilon H |\phi|_\infty \sum_{k=0}^{n-1} \lambda^{-2} (n-k)/2 + |\phi|_a \lambda^{-2} n/2 \right\} \\
\leq C_R^{1/2} |\phi|_a \left( \sum_{j=1}^n \lambda^{-2} (n-j)/2 + \varepsilon H \sum_{k=0}^{n-1} \lambda^{-2} (n-k)/2 + \lambda^{-2} n/2 \right). 
\]

By Jensen’s inequality, \( \lambda^{-2} \leq \lambda^{-2} \). Since also \( \lambda^{-2} \leq \lambda^{-2} \) < 1 holds, we can further bound the expression in the last line by
\[
C_R^{1/2}, |\phi|_a \left( \frac{( \sum_{j=1}^n \lambda^{-2} (n-j)/2 + \varepsilon H \sum_{k=0}^{n-1} \lambda^{-2} (n-k)/2 + \lambda^{-2} n/2 }{1 - \lambda^{-2} n/2 } \right). 
\]

Finally, note that |\phi|_a is bounded by the Lipschitz constant of \( \phi \) for all \( \alpha \in (0,1) \). \( \square \)

We are finally in position to state the multiple correlation bound that is needed to prove Theorem 6:

**Theorem 20.** There exist such a constant \( \theta \in (0,1) \) and, given \( \alpha \in (0,1) \), \( H > 0 \) and \( \varepsilon > 0 \), a constant \( C > 0 \) that
\[
|E^n [F G_n] - E^n [F] E^n [G_n]| \leq C \theta^{nm}
\]
holds for all \( n \geq 0 \), for all choices of \( (f_k)_{k \geq 0} \) and \( (t_k)_{k \geq 0} \) satisfying (17) and (18).

**Proof.** This follows immediately from (5) of Lemma 4, once we collect (22), (23), (24) and Proposition 19. \( \square \)

The following theorem is a special case of the main result in [10].

**Theorem 21** (Gouëzel [10]). Let \( (A_n)_{n \geq 0} \) be a stationary sequence of \( \mathbb{R}^d \)-valued random variables which is centered and bounded. Given integers \( n > 0 \), \( m > 0 \), \( 0 \leq b_1 < b_2 < \cdots < b_{n+m+1} \), \( k \geq 0 \), and vectors \( t_1, \ldots, t_{n+m} \in \mathbb{R}^d \), define
\[
X_{n,m}^{(k)} = \sum_{j=n}^{m} t_j \cdot \sum_{l=b_j+k}^{b_{j+1}-1+k} A_l.
\]
Assume that there exist such constants $\varepsilon > 0$, $C > 0$, and $c > 0$ that
\[
E\left(e^{iX_{1,n}^{(0)}+iX_{n+1,n+m}^{(k)}}\right) - E\left(e^{iX_{1,n}^{(0)}}\right)E\left(e^{iX_{n+1,n+m}^{(k)}}\right) \leq Ce^{-ck}\left(1 + \max_{1 \leq j \leq n+m} |b_{j+1} - b_j| \right)^{C(n+m)}
\]
holds for all choices of the numbers $n$, $m$, $b_j$, $k > 0$, and of the vectors $t_j$ satisfying $|t_j| \leq \varepsilon$. Then items (1)–(3) of Theorem 6 (with $E$ in place of $E^\mu$) are true for the process $(A_n)_{n \geq 0}$.

In other words, it is now enough to prove that (26) holds in our case, with $A_n$ as defined in (6). This is immediate, as
\[
t_j \cdot A f(\omega, x) = |t_j| \left( \frac{t_j}{|t_j|} \cdot f \circ T_{\omega_1} \circ \cdots \circ T_{\omega_n} (x) \right),
\]
where the maps $\frac{t_j}{|t_j|} \cdot f : S \rightarrow \mathbb{R}$, $j \geq 0$, are uniformly Hölder continuous:
\[
\left| \frac{t_j}{|t_j|} \cdot f(x) - \frac{t_j}{|t_j|} \cdot f(y) \right| \leq |f(x) - f(y)| \leq |f|_\alpha d(x, y)\alpha.
\]
Therefore, the difference on the left side of (26) is of the general form (15) with $F$ and $G_n$ as in (16). Theorem 20 thus yields the bound $C\theta^m$ on the right side of (26).

The proof of Theorem 6 is complete.

7. Proof of Lemma 7

Let $\Phi$ denote the skew product map $\Phi(\omega, x) = (\sigma \omega, T_{\omega_1} x)$ and $\pi$ the projection $\pi(\omega, x) = x$. Abusing notation, we write $P^\mu = \mathbb{P} \times \mu$ and $E^\mu$ for the corresponding expectation in this section; this measure is invariant for $\Phi$. Setting $A_n = f \circ \pi \circ \Phi^n$ (cf. (6)), observe that
\[
\Sigma^2 = E^\mu (A_0 \otimes A_0) + \sum_{m=1}^{\infty} E^\mu (A_0 \otimes A_m + A_m \otimes A_0).
\]
Denote
\[
S_n = \sum_{m=0}^{n-1} A_m.
\]

Lemma 22. There exists a constant $C \geq 0$ such that
\[
\sup_{n \geq 0} \left\| E^\mu (S_n \otimes S_n) - n \Sigma^2 \right\| \leq C.
\]

Proof. Using invariance,
\[
E^\mu (S_n \otimes S_n) = n E^\mu (A_0 \otimes A_0) + \sum_{m=1}^{n-1} (n - m) E^\mu (A_0 \otimes A_m + A_m \otimes A_0).
\]
Recalling (27), we have
\[
E^\mu (S_n \otimes S_n) - n \Sigma^2 = \sum_{m=1}^{\infty} a_n(m) E^\mu (A_0 \otimes A_m + A_m \otimes A_0),
\]
where $a_n(m) = -m$ for $1 \leq m < n$ and $a_n(m) = -n$ for $m \geq n$. Because $|a_n(m)| \leq m$,
\[
\left\| E^\mu (S_n \otimes S_n) - n \Sigma^2 \right\| \leq \sum_{m=1}^{\infty} m \left\| E^\mu (A_0 \otimes A_m + A_m \otimes A_0) \right\|.
\]
But
\[
E^\mu (A_0 \otimes A_m + A_m \otimes A_0) = \int (f \otimes Q^m f + Q^m f \otimes f) \, d\mu
\]
is exponentially small in $m$ according to Theorem 4, so the sum above converges.
Recall the $\mu$-average of $f$ vanishes. Given a vector $v \in \mathbb{R}^d$, we define $f_v = v^T f$,

\[ X_k = f_v \circ \pi \circ \Phi^k \quad \text{and} \quad S_n = \sum_{k=0}^{n-1} X_k. \]

Since $v^T (A_m \otimes A_n) v = v^T (A_n \otimes A_m) v = X_n X_m$ for all $n, m \geq 0$, (28) gives

\[ |E^{\mu}(S_n^2) - n v^T \Sigma^2 v| = |v^T (E^{\mu}(S_n \otimes S_n) - n \Sigma^2) v| \leq C|v|^2 \]

uniformly for $n \geq 1$.

From here, the proof is similar to [2]. Suppose $\Sigma^2$ is degenerate. In other words, there exists a vector $v \in \mathbb{R}^d$ such that $v^T \Sigma^2 v = 0$. By the bound above, the random variables $S_n$ are uniformly bounded in $L^2(P^\mu)$. By the Banach–Alaoglu theorem, there exists a sequence $(n_k)_{k \geq 1}$ and $S \in L^2(P^\mu)$ such that

\[ \lim_{k \to \infty} E^{\mu}(hS_{n_k}) = E^{\mu}(hS) \]

for all $h \in L^2(P^\mu)$. In particular, if $h$ is independent of the sequence $\omega$ and $g = \mu(S)$,

\[ \lim_{k \to \infty} \sum_{j=0}^{n_k-1} \mu(h \mathcal{Q}^j f_v) = \mu(hg). \]

A similar identity is obtained with $\mathcal{P} h$ in place of $h$. Therefore,

\[ \mu(h(f_v - g + \mathcal{Q} g)) = \mu(h f_v) - \lim_{k \to \infty} \sum_{j=0}^{n_k-1} \mu(h \mathcal{Q}^j f_v) + \lim_{k \to \infty} \sum_{j=1}^{n_k} \mu(h \mathcal{Q}^j f_v) = \lim_{k \to \infty} \mu(h \mathcal{Q}^{n_k} f_v). \]

By Theorem 4, the last limit vanishes. In other words, there exists $g \in L^2(\mu)$ such that

\[ f_v = g - \mathcal{Q} g. \]

Claim. In fact,

\[ f_v(x) = g(x) - g(T_{\omega_1} x) \]

almost surely.

Accepting the Claim for now, pick $\omega_1$ so that the above identity holds for almost every $x$. Standard Livschitz (Livšic) rigidity theory then shows that $g$ is Hölder continuous [14, 15]. In particular, (31) holds for all $x$. This proves the lemma in one direction.

To prove the lemma in the other direction, suppose (31) holds almost surely for some nonzero vector $v$ and some Hölder continuous $g$. Then $S_n = g - g \circ \pi \circ \Phi^n$ holds $P^\mu$-almost-everywhere, and

\[ E^{\mu}(S_n^2) = \|g - g \circ \Phi^n\|^2_{L^2(P^\mu)} \leq 4 \|g\|^2_{L^2(\mu)}. \]

Combining the bound with (29) we get $v^T \Sigma^2 v \leq n^{-1} (C|v|^2 + 4 \|g\|^2_{L^2(\mu)})$ for all $n \geq 1$, which is only possible if $\Sigma^2$ is degenerate in the direction of $v$.

It thus remains to prove the earlier Claim. To that end, we define

\[ G_k = g \circ \pi \circ \Phi^k \]

and

\[ M_n = \sum_{k=0}^{n-1} (X_k - G_k + G_{k+1}) = S_n - G_0 + G_n. \]
Since \((S_n)_{n \geq 1}\) is uniformly bounded in \(L^2(P^\mu)\), so is \((M_n)_{n \geq 1}\). Since (30) holds, the latter sequence is also a martingale adapted to the filtration \((\mathcal{F}_n)_{n \geq 1}\) where \(\mathcal{F}_n\) is the sigma-algebra generated by the random variables \(x, \omega_1, \ldots, \omega_n\). Therefore,

\[
E^\mu(M_n^2) = \sum_{k=0}^{n-1} E^\mu((X_k - G_k + G_{k+1})^2) = nE^\mu((X_0 - G_0 + G_1)^2).
\]

Combining these two facts, it follows that \(X_0 - G_0 + G_1\) vanishes almost surely. The claim is proved.

This finishes the proof of Lemma 7. \(\square\)

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(Mikko Stenlund) Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland.

E-mail address: mikko.stenlund@helsinki.fi

URL: [http://www.math.helsinki.fi/mathphys/mikko.html](http://www.math.helsinki.fi/mathphys/mikko.html)

(Henri Sulku) Department of Mathematics and Statistics, P.O. Box 68, FIN-00014 University of Helsinki, Finland.

E-mail address: henri.sulku@helsinki.fi