The topological realization of a simplicial presheaf
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Introduction

While preparing for EUROPROJ’s Trento school on stacks (September 1996) it became apparent that an obvious thing that one would like to say about a stack—to take its topological realization—was not altogether obvious to define or to handle. The same question can be posed more generally for a presheaf of spaces or (as it is common to say) a simplicial presheaf.

We give a definition of the topological realization of a presheaf of spaces on a site $X$, with respect to a covariant functor $F : X \to \text{Top}$. It is a topological space defined in a fairly obvious way. In our usual case $X$ will be a site of schemes over $\text{Spec}(\mathbb{C})$ and $F$ is the functor which to a scheme $X$ associates its underlying usual topological space $X^{\text{top}}$. If $G$ is representable by an object $X$ then the realization of $G$ is (homotopic to) $X^{\text{top}}$.

One recovers the topological realization of a simplicial presheaf by first realizing over each object into a presheaf of spaces, and then applying our definition. One recovers the topological realization of a stack (of groupoids) by first strictifying into a presheaf of groupoids then taking the realization of the nerve of the groupoid over each object to get a presheaf of spaces.

Once the theory of $n$-stacks is well off the ground, the same remark will hold for the topological realization of an $n$-stack (whereas for now one must replace the notion of $n$-stack by $n$-truncated presheaf of spaces, and from this point of view one can directly take the topological realization as defined below).

The main theorem is the invariance of topological realization under Illusie weak equivalence. Recall that if $G$ is a presheaf of spaces on the site $X$ then we obtain a presheaf of sets $\pi_0^{\text{pre}}(G)$ on $X$ and for any $g \in G(X)$ a presheaf of groups $\pi_i^{\text{pre}}(G|_{X/X}, g)$ on $X/X$. Then $\pi_0(G)$ (resp. $\pi_i(G|_{X/X}, g)$) is defined to be the sheafification of $\pi_0^{\text{pre}}(G)$ (resp. the sheafification of $\pi_i^{\text{pre}}(G|_{X/X}, g)$ on $X/X$). A morphism $\psi : G \to G'$ of presheaves of spaces is an Illusie weak equivalence if it induces an isomorphism on the homotopy sheaves

$$\pi_0(G) \xrightarrow{\cong} \pi_0(G')$$

and for any $X \in X$ and $g \in G(X)$ an isomorphism

$$\pi_i^{\text{pre}}(G|_{X/X}, g) \xrightarrow{\cong} \pi_i^{\text{pre}}(G'|_{X/X}, \psi(g)).$$
Heuristically, the correct homotopy type of presheaves of spaces on $\mathcal{X}$ is the homotopy type up to Illusie weak equivalence (this is how the Grothendieck topology is taken into account). More concretely in the case of stacks for example, one often has a presentation of a stack as the quotient of a scheme by a relation. Taking the resulting simplicial scheme and then realizing over each object to get a presheaf of spaces gives something which is Illusie equivalent to the presheaf of spaces associated to the stack. Thus we would like to define the realization in terms of a presheaf of spaces but we would like to know that it is an invariant of the homotopy type, i.e. we would like to know that the morphism of realizations induced by an Illusie weak equivalence is a weak homotopy equivalence of spaces—this is the result of Theorem 3.1.

We give several examples of calculations of topological realizations. These generally make use of Theorem 3.1. The calculation that I originally wanted to look at for the Trento school is that of the realization of a one-dimensional Deligne-Mumford stack.

At the end we sketch how to extend this theory to the case of functors with values in an $n$-topos (although we don’t yet know very clearly what an $n$-topos is). This section resulted from email conversations with K. Behrend and C. Telemann, who were both talking about topological realizations but also pullbacks of stacks from one site to another.

Caution: At the time of writing this first version, I don’t know if the main theorem has already appeared somewhere else. In the case of presheaves of sets over a topological space, I think that it is already known as the statement that the “espace étalé” of a presheaf of sets is the same as the “espace étalé” of the associated sheaf. I imagine that our main theorem has already been used at least in the case of stacks, because it is such an obvious statement that it is easy to use it without thinking to prove it (in particular the reader will probably find that the calculations given in most of our examples are nothing new). So if anybody knows of a reference where the theorem (or some version of it) has already been proved, please let me know (by email: carlos@picard.ups-tlse.fr)!

Let $\text{Top}$ denote the category of topological spaces. Let $\Delta^n$ denote the standard $n$-simplex. In general $*$ denotes the one-point space, or a constant functor to $\text{Top}$ whose values are the one-point space.

1. The definition of topological realization

Suppose $\mathcal{X}$ is a category, and suppose $F : \mathcal{X} \to \text{Top}$ is a covariant functor and $G : \mathcal{X} \to \text{Top}$ a contravariant functor. Then we define the realization of $G$ with respect to
F denoted by \( \mathcal{R}_X(F, G) \) as the quotient

\[
\mathcal{R}_X(F, G) = \coprod_{\phi : X_0 \to X_n} F(X_0) \times G(X_n) \times \Delta^n / \sim
\]

by the equivalence relation \( \sim \) to be explained below. The notation \( \phi : X_0 \to X_n \) means a composable sequence of \( n \)-morphisms

\[
X_0 \xrightarrow{\phi_1} X_1 \ldots X_{n-1} \xrightarrow{\phi_n} X_n
\]
in \( X \).

The equivalence relation \( \sim \) (the obvious one in our situation) is defined by saying that if \( t \in \Delta^m \) is a point and if \( \eta : \Delta^m \to \Delta^n \) is a face, then \((a, b, \eta(t)) \in F(X_0) \times G(X_n) \times \Delta^n \) (indexed by \( \phi \)) is equivalent to \((a', b', t) \in F(X_i) \times G(X_j) \times \Delta^m \) (indexed by \( \phi' \)), where \( i \) and \( j \) are the numbers of the first and last vertices of the face \( \eta \), \( \phi' : X_i \to X_j \) is the composable sequence obtained from \( \phi \) by applying the face \( \eta \), and \( a' \) is the image of \( a \) under \( F(X_0) \to F(X_i) \) and \( b' \) is the image of \( b \) under \( G(X_n) \to G(X_j) \).

The space \( \mathcal{R}_X(F, G) \) is topologized as the quotient of the disjoint union of the spaces \( F(X_0) \times G(X_n) \times \Delta^n \). Note that if we order the addition of “cells” according to \( n \), then a “cell” of the form \( F(X_0) \times G(X_n) \times \Delta^n \) is added onto the previous part via an attaching map from \( F(X_0) \times G(X_n) \times (\partial \Delta^n) \) to the previous part; the attaching map is defined by the equivalence relation \( \sim \).

A morphism \( F \to F' \) (resp. \( G \to G' \)) is said to be an object-by-object equivalence if it induces a weak homotopy equivalence \( F(X) \cong F'(X) \) (resp. \( G(X) \cong G'(X) \)) for each \( X \in \mathcal{X} \). If \( F \to F' \) and \( G \to G' \) are object-by-object equivalences then they induce a weak homotopy equivalence \( \mathcal{R}_X(F, G) \to \mathcal{R}_X(F', G') \).

If \( \pi : \mathcal{X} \to \mathcal{Y} \) is a functor and if \( F, G : \mathcal{Y} \to Top \) are as above, then we obtain a natural transformation

\[
\mathcal{R}_X(\pi^*(F), \pi^*(G)) \to \mathcal{R}_Y(F, G).
\]

This is compatible with compositions \( \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \) in the obvious way.

**Lemma 1.1** Suppose \( \mathcal{X} \) is a category with final object \( U \). Suppose \( F : \mathcal{X} \to Top \) is a functor. Then the natural morphism

\[
F(U) \to \mathcal{R}_X(F, *)
\]

is a weak homotopy equivalence.
Proof: In $\mathcal{R}_X(F, \ast)$ we homotope a cell of the form $F(X_0) \times \Delta^n$ corresponding to a composable $n$-tuple $\phi : X_0 \to X_n$ to $F(U)$ through the cell $F(X_0) \times \Delta^{n+1}$ corresponding to $(\phi, \beta)$ where $\beta : X_n \to U$ is the canonical morphism. Note that the last vertex of the cell $F(X_0) \times \Delta^{n+1}$ gets attached to $F(U)$. This homotopy is a retraction from $\mathcal{R}_X(F, \ast)$ to $F(U)$. \qed

Relative realizations

If $\pi : \mathcal{X} \to \mathcal{Y}$ is a functor (when nothing is specified that means covariant!) and if $F, G : \mathcal{X} \to Top$ are as above, then we can define the standard relative realization which is a covariant functor

$$\mathcal{R}_{\mathcal{X}/\mathcal{Y}}(F, G) : \mathcal{Y} \to Top.$$  

It is defined by

$$\mathcal{R}_{\mathcal{X}/\mathcal{Y}}(F, G)(Y) := \mathcal{R}_{\mathcal{X}/\mathcal{Y}}(F|_{\mathcal{X}/\mathcal{Y}}, G|_{\mathcal{X}/\mathcal{Y}}).$$

Here $\mathcal{X}/\mathcal{Y}$ is the category of pairs $(X, f)$ where $X \in \mathcal{X}$ and $f : \pi(X) \to Y$ is a morphism in $\mathcal{Y}$. Given a morphism $a : Y \to Y'$ we get a functor $\alpha : \mathcal{X}/\mathcal{Y} \to \mathcal{X}/\mathcal{Y}'$. Furthermore

$$F|_{\mathcal{X}/\mathcal{Y}} = \alpha^*(F|_{\mathcal{X}/\mathcal{Y}'}) , \quad G|_{\mathcal{X}/\mathcal{Y}} = \alpha^*(G|_{\mathcal{X}/\mathcal{Y}'}).$$

This and the remark of the previous paragraph gives a morphism

$$\mathcal{R}_{\mathcal{X}/\mathcal{Y}}(F|_{\mathcal{X}/\mathcal{Y}}, G|_{\mathcal{X}/\mathcal{Y}}) \to \mathcal{R}_{\mathcal{X}/\mathcal{Y}'}(F|_{\mathcal{X}/\mathcal{Y}'}, G|_{\mathcal{X}/\mathcal{Y}'}),$$

which is the morphism of functoriality for $\mathcal{R}_{\mathcal{X}/\mathcal{Y}}(F, G)$.

Lemma 1.2 Suppose $\pi : \mathcal{X} \to \mathcal{Y}$ is a functor. Suppose that $F : \mathcal{X} \to Top$ is a covariant functor and $G : \mathcal{Y} \to Top$ a contravariant functor. Then there is a natural weak homotopy equivalence

$$\mathcal{R}_X(F, \pi^*(G)) \cong \mathcal{R}_Y(\mathcal{R}_{\mathcal{X}/\mathcal{Y}}(F, \ast), G).$$

The proof is left to the reader since we shall not use it below. \qed

Actually we don’t use the above standard version; instead, we need two more special versions of the relative realization. These are a covariant version depending only on $F$ and a contravariant version depending only on $G$. 

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Suppose that $\pi : X \to Y$ is a split fibered category, in other words it comes from a functor $\Phi : Y \to \text{Cat}$. The objects of $X$ are pairs of the form $(Y, U)$ with $U \in \Phi(Y)$. Note that any $\Phi(Y)$ may be considered as a subcategory of $X$. Suppose that $F : X \to \text{Top}$ is a covariant functor. Then we define the special covariant relative realization which is a covariant functor denoted

$$\mathcal{R}^{-}_{X/Y}(F, *) : Y \to \text{Top}$$

(note—in spite of the notation—that there is only one variable $F$). It is defined by

$$\mathcal{R}^{-}_{X/Y}(F, *)(Y) := \mathcal{R}_{\Phi(Y)}(F|_{\Phi(Y)}, *_{\Phi(Y)}).$$

The second variable $*_{\Phi(Y)}$ is the constant functor on $\Phi(Y)$ associating to each object the one point space. If $a : Y \to Y'$ is a morphism in $Y$ then we get a functor $\Phi(a) : \Phi(Y) \to \Phi(Y')$. The structure of $F$ as functor on $X$ gives a morphism of functors

$$F|_{\Phi(Y)} \to \Phi(a)^{*}(F|_{\Phi(Y')}).$$

We obtain the first of the following two morphisms:

$$\mathcal{R}_{\Phi(Y)}(F|_{\Phi(Y)}, *_{\Phi(Y)}) \to \mathcal{R}_{\Phi(Y)}(\Phi(a)^{*}(F|_{\Phi(Y')}), *_{\Phi(Y)}) \to \mathcal{R}_{\Phi(Y)}(F|_{\Phi(Y')} ,*_{\Phi(Y')}),$$

the second being the standard morphism of functoriality (noting obviously that $*_{\Phi(Y)} = \Phi(a)^{*}(*_{\Phi(Y')})$). The composition of the above two morphisms gives the morphism of functoriality for $\mathcal{R}^{-}_{X/Y}(F, *)$.

**Lemma 1.3** Suppose in the above situation that $G : Y \to \text{Top}$ is a presheaf of spaces on $Y$. Then we have a natural equivalence

$$\mathcal{R}_{X}(F, \pi^{*}(G)) \cong \mathcal{R}_{Y}(\mathcal{R}^{-}_{X/Y}(F, *, G)).$$

**Proof:** The realization on the right, with a realization in the argument, can be seen as the realization of a bisimplicial space. If we take the diagonal simplicial space this gives the realization on the left. The natural inclusion of the realization of the diagonal into the realization of a bisimplicial space is an equivalence. \[\square\]

We now get to the special contravariant version. Suppose this time that $\pi : X \to Y$ is a split fibered category in the other direction corresponding to a contravariant functor $\Phi : Y \to \text{Cat}$. The objects of $X$ are again pairs of the form $(Y, U)$ with $U \in \Phi(Y)$, and
any $Φ(Y)$ may be considered as a subcategory of $X$. Suppose that $G : X → Top$ is a contravariant functor. Then we define the special contravariant relative realization which is a contravariant functor denoted

$$\mathcal{R}^{←}_{X/Y}(\ast, G) : Y → Top$$

(as before there is only one variable $G$). It is defined by

$$\mathcal{R}^{←}_{X/Y}(\ast, G)(Y) := \mathcal{R}_Φ(\ast(Φ(Y)), G|_{Φ(Y)}).$$

The first variable $\ast(Φ(Y))$ is the constant functor on $Φ(Y)$ associating to each object the one point space. If $a : Y → Y'$ is a morphism in $Y$ then we get a functor $Φ(a) : Φ(Y') → Φ(Y)$. The structure of $G$ as presheaf on $X$ gives a morphism of functors

$$G|_{Φ(Y')} → Φ(a)^*(G|_{Φ(Y)}).$$

We obtain the first of the following two morphisms:

$$\mathcal{R}_Φ(\ast(Φ(Y')), G|_{Φ(Y'))} → \mathcal{R}_Φ(\ast(Φ(Y')), Φ(a)^*(G|_{Φ(Y)}) → \mathcal{R}_Φ(\ast(Φ(Y)), G|_{Φ(Y)}),$$

the second being the standard morphism of functoriality (noting as before that $\ast(Φ(Y)) = Φ(a)^*(\ast(Φ(Y')))$. The composition of the above two morphisms gives the morphism of functoriality for $\mathcal{R}^{←}_{X/Y}(\ast, G)$.

**Lemma 1.4** Suppose in the above situation that $F : Y → Top$ is a covariant functor. Then we have a natural equivalence

$$\mathcal{R}_X(\pi^*(F), G) \cong \mathcal{R}_{Y'}(F, \mathcal{R}^{←}_{X/Y}(\ast, G)).$$

*Proof:* This situation is a actually the same as the previous one after taking the opposite categories, interchanging the roles of $F$ and $G$ in the realization. $\square$

### 2. A descent condition for $F$

In this and the remaining sections we assume that $X$ has a Grothendieck topology, i.e. $X$ is a site. Note however that the definition of $\mathcal{R}_X(F, G)$ does not depend on the Grothendieck topology.
Suppose $F$ is a covariant functor $\mathcal{X} \to \text{Top}$. We say that $F$ satisfies covariant descent if, for any object $X \in \mathcal{X}$ and any sieve $\mathcal{B} \subset \mathcal{X}/X$, the natural morphism

$$\mathcal{R}_\mathcal{B}(F|_\mathcal{B}, *) \to \mathcal{R}_{\mathcal{X}/X}(F|_{\mathcal{X}/X}, *)$$

is a weak homotopy equivalence. Note that the space on the right is equivalent to $F(X)$ by Lemma 1.1. The morphism occurring above comes from our general discussion using the observation that $F|_\mathcal{B}$ is the pullback from $\mathcal{X}/X$ to $\mathcal{B}$ of $F|_{\mathcal{X}/X}$.

The following theorem gives the main example for our purposes.

**Theorem 2.1** Suppose $\mathcal{X}$ is the site of schemes of finite type over $\text{Spec}(\mathbb{C})$ with the fppf topology (or any weaker topology such as the etale or Zariski topology). The functor $F : \mathcal{X} \to \text{Top}$ defined by setting $F(X) = X^{\text{top}}$ (the usual topological space underlying the analytic space associated to $X$) satisfies covariant descent.

**Proof:** Suppose $\mathcal{B}$ is a sieve over $X$. Let $B$ denote the disjoint union of the schemes in $\mathcal{B}$ with a morphism $B \to X$ (considered as a non-noetherian scheme locally of finite type). This morphism is fppf surjective. Let $N(B/X)$ denote the standard simplicial scheme whose components are $B \times_X \ldots \times_X B$. Let $B^{\text{top}}$ denote the associated topological space (mapping to $X^{\text{top}}$ and let $N(B^{\text{top}}/X^{\text{top}})$ again denote the simplicial space whose elements are fiber products. There is no confusion in the notation because the functor $F$ commutes with fiber products, so we can first take $B^{\text{top}}$ then take the nerve, or vice-versa getting the same answer. We have

$$\mathcal{R}_\mathcal{B}(F|_\mathcal{B}, *) \cong |N(B^{\text{top}}/X^{\text{top}})|.$$

It is not too hard to see (by stratifying everything and so on) that $X^{\text{top}}$ admits a triangulation as a simplicial complex which we denote $S$ such that every simplex lifts into $B^{\text{top}}$ (in the notation below we will replace $X^{\text{top}}$ by $S$). Let $\tilde{S}$ denote the simplicial complex which is the disjoint union of the simplices of $S$. We can choose a map $\tilde{S} \to B^{\text{top}}$. On the other hand, the morphism

$$|N(\tilde{S}/S)| \to S$$

is a Serre fibration and a weak equivalence. Consider the bisimplicial space $N_i(B^{\text{top}}/S) \times_S N_j(\tilde{S}/S)$. If we fix the variable $j$ then in the variable $i$ it is the same as the nerve of the map

$$B^{\text{top}} \times_S N_j(\tilde{S}/S) \to N_j(\tilde{S}/S).$$
Since this map admits a section, the realization of its nerve is weakly equivalent to the base. Thus if we realize first in the $i$ direction and then in the $j$ direction we obtain something mapping by a weak equivalence to $S$. On the other hand if we fix $i$ then the realization in the $j$ variable is just

$$N_i(B^{\text{top}}/S) \times_S |N_i(\tilde{S}/S)|,$$

which maps by a weak equivalence to $N_i(B^{\text{top}}/S)|$. Thus if we realize first in the $j$-direction and then in the $i$-direction we obtain something mapping by a weak equivalence to $|N_i(B^{\text{top}}/X^{\text{top}})|$. This proves that the map

$$|N_i(B^{\text{top}}/X^{\text{top}})| \to X^{\text{top}}$$

is a weak equivalence. 

We have a similar analytic version. Let $\mathcal{X}^{\text{an}}$ denote the site whose underlying category is that of complex analytic spaces, and whose topology is given by saying that a family is surjective if it admits sections locally on the base. Let $F : \mathcal{X}^{\text{an}} \to \text{Top}$ be the functor associating to an analytic space, its underlying topological space.

**Theorem 2.2** With the above notations, $F : \mathcal{X}^{\text{an}} \to \text{Top}$ satisfies covariant descent.

**Proof:** Use the same proof as above but with $S$ just as $X^{\text{top}}$ and $\tilde{S}$ as an open covering of $S$ admitting a lifting to $B^{\text{top}}$. The main step, that $|N_i(\tilde{S}/S)| \to S$ is a Serre fibration and weak equivalence, still holds. 

**Remark:** If we just want Theorem 2.1 for the etale or Zariski topologies then we can use the “open” version as in 2.2, this avoids the use of a triangulation.

**3. Statement of the main theorem**

Suppose as above that $\mathcal{X}$ is a site. Recall that a morphism $G \to G'$ of presheaves of topological spaces (i.e. contravariant functors $\mathcal{X} \to \text{Top}$) is called an Illusie weak equivalence if it induces isomorphisms of the sheaves associated to the homotopy presheaves $\pi_0(G)$ or $\pi_i(G,g)$ (this is explained further in the introduction).
Theorem 3.1 Suppose $\mathcal{X}$ is a site and $F : \mathcal{X} \to \text{Top}$ is a covariant functor which satisfies covariant descent. Then any Illusie weak equivalence $G \to G'$ between contravariant functors induces a weak homotopy equivalence of realizations

$$\mathcal{R}_X(F, G) \xrightarrow{\simeq} \mathcal{R}_X(F, G').$$

Even though the theorem can be stated without referring to the closed model category structure or homotopy sheafification, these notions are essential in our proof. Actually I think that there also exists a proof which doesn’t use these ideas but which proceeds by Postnikov induction. However, one needs to start at degree zero with the same result for presheaves of sets, and as it seems that the argument needed for presheaves of sets is essentially the same as our argument below, we have preferred to go straight through with this argument in general. The following section treats the generalities we will need, the subsequent section gives the main lemma.

4. Homotopy-sheafification

Suppose $G$ is a presheaf of spaces on a category $\mathcal{C}$ (i.e. a contravariant functor to $\text{Top}$). A section of $G$ over $\mathcal{C}$ is a function $g$ which to each composable sequence $\phi : X_0 \to X_n$ associates $g(\phi) : \Delta^n \to G(X_0)$ such that if $\eta : \Delta^m \to \Delta^n$ is a face, and if $\phi' : X_i \to X_j$ is the composable sequence obtained by applying the face to $\phi$, then

$$g(\phi) \circ \eta = g(\phi').$$

The space of sections can be topologized using the compact-open topology on the space of maps $\Delta^n \to G(X_0)$, taking the Tychonoff topology on the product over all $\phi$ of these spaces of maps, and then considering the space of sections as a subspace of the product. Let $\Gamma(\mathcal{C}, G)$ denote this space of sections.

If $\pi : \mathcal{C} \to \mathcal{D}$ is a functor and if $G$ is a presheaf of spaces over $\mathcal{D}$ then there is an induced map $\Gamma(\mathcal{D}, G) \to \Gamma(\mathcal{C}, \pi^*(G))$.

If $\mathcal{C}$ has a final object $U$ then by Lemma 1.1 the map $\Gamma(\mathcal{C}, G) \to G(U)$ is a weak homotopy equivalence.

Now we get back to our site $\mathcal{X}$. For any object $X \in \mathcal{X}$ we have the weak equivalence $\Gamma(\mathcal{X}/X, G|_{\mathcal{X}/X}) \simeq G(X)$. If $\mathcal{B} \subset \mathcal{X}/X$ is a sieve then we get a natural map

$$\Gamma(\mathcal{X}/X, G|_{\mathcal{X}/X}) \to \Gamma(\mathcal{B}, G|_{\mathcal{B}}).$$
We say that $G$ is a homotopy-sheaf if for any object $X \in \mathcal{X}$ and any sieve $\mathcal{B} \subset \mathcal{X}/X$ this restriction map is a weak homotopy equivalence.

Recall from [8] that for any presheaf of spaces $G$ (which can be considered as a simplicial presheaf by taking the singular simplicial set over each object, for example) there is an essentially canonical morphism $G \to G'$ to a fibrant presheaf of spaces. While not getting into the precise definition of Jardine’s fibrant condition here, we note that the essential point is that a fibrant simplicial presheaf corresponds to a presheaf of spaces which is a homotopy sheaf. To prove this we can interpret $\Gamma(\mathcal{B}, G|_{\mathcal{B}})$ as a morphism space $Hom(\ast'_{\mathcal{B}}, G)$ where $\ast'_{\mathcal{B}}$ is a presheaf equivalent to the presheaf of sets $\ast_{\mathcal{B}}$ represented by $\mathcal{B}$, i.e. it gives a contractible space for each object of $\mathcal{B}$ and empty otherwise. The presheaf of spaces $\ast'_{\mathcal{B}}$ is Illusie weak-equivalent to $\ast_{\mathcal{X}/X}$ so (by the closed model structure [8]) the morphism

$$Hom(\ast'_{\mathcal{B}}, G) \to Hom(\ast_{\mathcal{X}/X}, G)$$

is a weak homotopy equivalence of spaces.

**Proposition 4.1** The replacement of $G$ by a homotopy-sheaf $G'$ in the $n$-truncated case (resp. fibrant object in the non-truncated case) together with an Illusie weak equivalence $G \to G'$ is unique up to object-by-object equivalence. More precisely if $G'$ and $G''$ are two such replacements with Illusie weak equivalences $G \to G'$ and $G \to G''$ then there is a diagram

$$
\begin{array}{ccc}
G & \to & G'' \\
\downarrow & & \downarrow \\
G' & \to & G^{(3)}
\end{array}
$$

where the bottom horizontal arrow and the right vertical arrow are object-by-object equivalences. Furthermore this diagram is unique up to homotopy.

**Proof:** We can assume that $G'$ and $G''$ are fibrant in the sense of Jardine, for in the $n$-truncated case if $H$ is a homotopy-sheaf then the replacement by a fibrant object $H \to H'$ is an object-by-object weak equivalence.

(To see this note that it suffices to show that $H \times_{H'} Z \to Z$ is an object-by-object equivalence for any $Z \in \mathcal{X}$ and $Z \to H'$; but this fiber product preserves the homotopy-sheaf condition so we may effectively assume that $H'$ is represented by $Z$; then arguing inductively on $n$ we show that the top homotopy-group presheaf is trivial so $H$ is $n-1$-truncated, eventually we get down to $H$ 0-truncated and then an Illusie weak equivalence between sheaves of sets is an isomorphism.)
Let $F$ be the pushout of $G \to G'$ and $G \to G''$, then let $F \to G^{(3)}$ be a replacement by a fibrant object. The morphism $G' \to G^{(3)}$ is an Illusie weak equivalence between fibrant objects. Therefore this morphism is invertible up to homotopy, in particular it is an object-by-object equivalence. The same goes for $G'' \to G^{(3)}$. □

If $G'$ is a homotopy sheaf and $G \to G'$ is an Illusie weak equivalence then we say that $G'$ is the homotopy-sheafification of $G$.

Suppose now that $G$ is $n$-truncated (i.e. the homotopy groups of the $G(X)$ vanish in degrees strictly bigger than $n$). Then we propose an explicit method for obtaining the homotopy sheafification by applying $n + 2$ times a certain process denoted $\gamma$. In the case of presheaves of sets ($n = 0$) this is essentially the same thing as the standard construction of the sheafification of a presheaf of sets by doing the obvious operation 2 times. We will define $\gamma(G)$ below with a morphism $G \to \gamma(G)$ (an Illusie weak equivalence). It was shown in [10] that if $G$ is $n$-truncated then $\gamma^{n+2}G$ is a homotopy sheaf. We give a sketch of proof below also, since [10] is not published.

Note that Jardine’s proof of the closed model structure [8] gives, of course, a method for obtaining $G \to G'$ with $G'$ fibrant (and thus obtaining the homotopy-sheafification). His method, while explicit, is not very useful for proving things about the notion of Illusie weak equivalence because it consists of taking the pushout over all diagrams $U \xleftarrow{a} V \to G$ with $a$ an Illusie weak equivalence (and then iterating this an infinite number of times using transfinite induction).

Using our operation $\gamma$ we have the following outline of the proof of Theorem 3.1. The main lemma is to prove, using the explicit definition of $\gamma$, that the morphism $G \to \gamma(G)$ induces a weak equivalence of realizations

$$\mathcal{R}_X(F, G) \to \mathcal{R}_X(F, \gamma(G)).$$

The same is of course true for the iteration $G \to \gamma^{n+2}(G)$. Now suppose $G \to G'$ is an Illusie weak equivalence between $n$-truncated presheaves of spaces. Then we obtain a diagram

$$
\begin{array}{ccc}
G & \to & \gamma^{n+2}(G) \\
\downarrow & & \downarrow \\
G' & \to & \gamma^{n+2}(G'')
\end{array}
$$

giving a corresponding diagram on the level of realizations. The horizontal arrows induce weak equivalences of the realizations. On the other hand, all arrows are Illusie weak
equivalences and the spaces on the right are homotopy-sheaves. Thus by the uniqueness result, the right vertical arrow is an object-by-object equivalence, in particular it induces a weak equivalence of realizations. We conclude that the vertical arrow on the left induces a weak equivalence of realizations, which is Theorem 3.1.

To complete the proof of Theorem 3.1 we will just need a little argument to go from the $n$-truncated case to the general case (since the hypothesis of $n$-truncation is not present in the statement of 3.1). We give this below.

*The definition of $\gamma$*

Suppose $\mathcal{X}$ is a site and $G$ is a presheaf of spaces on $\mathcal{X}$. Let $\pi : \mathcal{E} \to \mathcal{X}$ be the fibered category associated to the contravariant functor $X \mapsto Sv(X)$ where $Sv(X)$ denotes the category (directed set, really) of sieves $\mathcal{B} \subset \mathcal{X}/X$. The objects of $\mathcal{E}$ are pairs $(X, \mathcal{B})$ where $X \in \mathcal{X}$ and $\mathcal{B} \subset \mathcal{X}/X$ is a sieve.

We define the contravariant functor $\tilde{\gamma}(G) : \mathcal{E} \to \text{Top}$ by

$$\tilde{\gamma}(G)(X, \mathcal{B}) := \Gamma(\mathcal{B}, G|_{\mathcal{B}}).$$

We are in the situation of the contravariant relative realization discussed at the start; we put

$$\gamma'(G) := \mathcal{R}_{\mathcal{E}/\mathcal{X}}^{\mathcal{X}}(\ast, \gamma'(G)).$$

On the other hand note that there is a canonical section

$$\xi : \mathcal{X} \hookrightarrow \mathcal{E}$$

defined by setting $\xi(X) := (X, \mathcal{X}/X)$. Put

$$\sigma(G) := \xi^*(\tilde{\gamma}(G)).$$

Thus

$$\sigma(G)(X) := \Gamma(\mathcal{X}/X, G).$$

There is a natural weak equivalence

$$\sigma(G) \xrightarrow{\sim} G.$$

On the other hand note trivially that

$$\sigma(G) = \mathcal{R}_{\xi(\mathcal{X})/\mathcal{X}}^{\mathcal{X}}(\ast, \tilde{\gamma}(G)|_{\xi(\mathcal{X})}).$$
and from this and a functoriality of the realization in terms of the base category we obtain a natural morphism
\[ \sigma(G) \to \gamma'(G). \]

Finally let \( \gamma(G) \) denote the push-out (object-by-object) of the diagram
\[
\begin{array}{c}
\sigma(G) \quad \to \quad \gamma'(G) \\
\downarrow \\
G
\end{array}
\]

We obtain a morphism \( \gamma(G) \) which is object-by-object weak equivalent to the morphism \( \sigma(G) \to \gamma'(G) \).

Remark: We have
\[ \gamma'(G)(X) = \lim_{\to, \mathcal{B} \subset X/X} \Gamma(\mathcal{B}, G|_{\mathcal{B}}). \]

In particular, \( \gamma'(G) \) (or, up to object-by-object weak equivalence, \( \gamma(G) \)) is the same as the presheaf of spaces constructed in [10].

Remark: The operation \( \gamma \) is functorial in \( G \). This is completely clear from the definition since at each step we apply a functor.

Remark: If \( G \) is \( n \)-truncated then so is \( \gamma(G) \).

The homotopy-sheafification \( \gamma^{n+2} \)

**Theorem 4.2** Suppose \( G \) is an \( n \)-truncated presheaf of spaces. Then the morphism obtained by iterating the operation \( \gamma \)
\[ G \to \gamma^{n+2}(G) \]
is the homotopy-sheafification of \( G \). In Jardine’s terms this means that the morphism is object-by-object weak equivalent to the replacement of \( G \) by a fibrant object.

**Proof:** For now we refer to [10] for the proof. \( \square \)

5. The main lemma

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We treat the situation we will encounter in the main lemma, in a general context first. Suppose \( C \) is a category with a projector, that is a functor \( P : C \to C \) such that \( P^2 = I \) (where \( I \) denotes the identity functor of \( C \)). Let \( \mathcal{D} = P(C) \subset C \) denote the image subcategory. Suppose we have a natural morphism of functors \( \psi : P \to I \) such that \( \psi \mid D \) is the identity of \( P \mid D = I_D \). Suppose that \( F : \mathcal{D} \to \text{Top} \) is a covariant functor and \( G : C \to \text{Top} \) is a contravariant functor (presheaf of spaces). Then \( \psi \) induces a morphism of presheaves of spaces \( G \to P^*(G) \). On the other hand, \( P \) induces a morphism of realizations \( \mathcal{R}_C(P^*(F), P^*(G \mid D)) \to \mathcal{R}_D(F, G \mid D) \). Composing these we obtain a morphism

\[
a : \mathcal{R}_C(P^*(F), G) \to \mathcal{R}_D(F, G \mid D).
\]

On the other hand, note that \( P^*(F) \mid D = F \). The inclusion \( D \subset C \) induces a morphism

\[
b : \mathcal{R}_D(F, G \mid D) \to \mathcal{R}_C(P^*(F), G).
\]

**Lemma 5.1** With the above notations, \( ab \) is equal to the identity and \( ba \) is homotopic to the identity. In particular \( a \) and \( b \) are homotopy equivalences of spaces.

**Proof:** The hard part is the case of \( ba \). Note that

\[
\mathcal{R}_C(P^*(F), G) = \coprod_{\phi : X_0 \to X_n \in C} F(P(X_0)) \times G(X_n) \times \Delta^n / \sim.
\]

Given a point \((\phi, f, g, t)\) here, the image by \( a \) is the point \((P(\phi), f, \psi^*_{X_n}(g), t)\) in the component \( F(P(X_0)) \times G(P(X_n)) \times \Delta^n \) of

\[
\mathcal{R}_D(F, G \mid D) = \coprod_{\rho : Y_0 \to Y_n \in \mathcal{D}} F(Y_0) \times G(Y_n) \times \Delta^n / \sim.
\]

Since \( b \) is just induced by the inclusion, the image of this point by \( b \) can be written in the same way as \((P(\phi), f, \psi^*_{X_n}(g), t)\), a point in the component \( F(P(X_0)) \times G(P(X_n)) \times \Delta^n \) of

\[
\mathcal{R}_C(P^*(F), G) = \coprod_{\rho : Y_0 \to Y_n \in C} F(Y_0) \times G(Y_n) \times \Delta^n / \sim.
\]

To conclude, we can write

\[
ba(\phi, f, g, t) = (P(\phi), f, \psi^*_{X_n}(g), t).
\]
We would like to define a homotopy from \( ba \) to the identity. To do this note that the realization of the diagram

\[
P(X_0) \to P(X_1) \to \ldots \to P(X_n)
\]

\[
\downarrow \psi_{X_0} \quad \downarrow \psi_{X_1} \quad \downarrow \psi_{X_n}
\]

\[
X_0 \to X_1 \to \ldots \to X_n
\]

is naturally isomorphic to \([0, 1] \times \Delta^n\) (this is an example of the division of the product of simplices which can be accomplished via the nerves of categories). From this diagram we obtain a map

\[
H_{\phi} : [0, 1] \times \Delta^n \times F(P(X_0)) \times G(X_n) \to \mathcal{R}(P^*(F), G),
\]

such that

\[
H_{\phi}(0, t, f, g) = (P(\phi), f, \psi_{X_n}^*(g), t)
\]

and

\[
H_{\phi}(1, t, f, g) = (\phi, f, g, t).
\]

We leave it to the reader to check that these maps fit together with the glueing relations \(\sim\) to give a map

\[
H : [0, 1] \times \mathcal{R}(P^*(F), G) \to \mathcal{R}(P^*(F), G)
\]

such that \(H(0) = ba\) and \(H(1)\) is the identity.

It is easy to see that \(ab\) is the identity, using the above formula for \(a\) and the fact that \(\psi|_D\) is the identity on \(P|_D = I_D\). \(\square\)

We can now state and prove the main lemma.

**Lemma 5.2** Suppose \(G\) is a presheaf of spaces, and suppose \(F\) is a covariant functor \(\mathcal{X} \to Top\) satisfying covariant descent. Then the morphism \(G \to \gamma(G)\) induces a weak equivalence of realizations \(\mathcal{R}_X(F, G) \cong \mathcal{R}_X(F, \gamma(G))\).

**Proof:** Let \(\mathcal{C}\) be the category of triples \((X, \mathcal{B}, Y)\) where \(X \in \mathcal{X}, \mathcal{B} \subset \mathcal{X}/X\) is a sieve, and \(Y \in \mathcal{B}\) is an object (together with morphism to \(X\)). We have an inclusion of categories \(i : \mathcal{X} \rightarrow \mathcal{C}\) defined by \(i(X) = (X, \mathcal{X}/X, X)\). Let \(\mathcal{D}\) be the image of \(i\) (it is equal to \(\mathcal{X}\)). Let \(P : \mathcal{C} \to \mathcal{C}\) be the morphism defined by

\[
P(X, \mathcal{B}, Y) := i(Y).
\]
The morphism $Y \to X$ (which comes with the data of $Y$) gives a morphism $i(Y) \to (X, B, Y)$. This is natural, so it provides our natural transformation $\psi : P \to I$. Define a presheaf of spaces $G'$ on $C$ by

$$G'(X, B, Y) := \Gamma(B, G|_B).$$

Define the presheaf of spaces $F'$ on $C$ by

$$F'(X, B, Y) := F(Y).$$

Note that $F' = P^*(F')$ already. Apply Lemma 5.1 to $F'$ and $G'$ on $C$ with $P$ and $\psi$ as above. We obtain that the inclusion induces a homotopy equivalence (denoted $b$ in Lemma 5.1)

$$\mathcal{R}_X(i^*(F'), i^*(G')) = \mathcal{R}_D(F', G'|_D) \to \mathcal{R}_C(P^*(F'), G') \xrightarrow{\cong} \mathcal{R}_D(F', G'|_D).$$

We now interpret these elements in terms of $F$ and $G$. The left hand side first. Note that $i^*(F') = F$ whereas $i^*(G') = \sigma(G)$ in the notations used in the definition of $\gamma$. We obtain that the inclusion $i$ induces a weak homotopy equivalence

$$\mathcal{R}_X(F, \sigma(G)) \xrightarrow{\cong} \mathcal{R}_C(P^*(F'), G').$$

Next, define an intermediate category $E$ to be the category of pairs $(X, B)$, with a functor $q : E \to X$ defined by $q(X, B) = X$. This is the same as the category used to define $\gamma'$ and $\gamma$. The presheaf $G'$ is pulled back from the one we denoted $\tilde{\gamma}(G)$ on $E$. We have

$$\mathcal{R}_C(P^*(F'), G') = \mathcal{R}_E(\mathcal{R}_{\mathcal{C}/E}(P^*(F'), *), \tilde{\gamma}(G)).$$

There is a natural morphism

$$\mu : \mathcal{R}_{\mathcal{C}/E}(P^*(F'), *) \to q^*(F).$$

The condition that $F$ satisfies covariant descent is exactly the condition that $\mu$ is an object-by-object weak equivalence. We obtain a natural weak equivalence

$$c : \mathcal{R}_C(P^*(F'), G') \xrightarrow{\cong} \mathcal{R}_E(q^*(F), \tilde{\gamma}(G)).$$

The map $i$ composed with the projection to $E$ is the map $\xi : X \to E$ defined by $\xi(X) = (X, X/X)$. Note that $\xi^*(q^*(F)) = F$ whereas $\xi^*(\tilde{\gamma}(G)) = \sigma(G)$. Also $\xi^*(P^*(F)) = F$ and the composition

$$F \to \xi^*\mathcal{R}_{\mathcal{C}/E}(P^*(F'), *) \to F$$

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is the identity, so the composition of our above morphism \( c \) with the inclusion \( b \) gives the map

\[
\mathcal{R}_X(F, \sigma(G)) \to \mathcal{R}_E(q^*(F), \tilde{\gamma}(G))
\]

induced by \( \xi \). Again this map is a weak equivalence (the fact that \( b \) is a weak equivalence is Lemma 5.1 while the fact that \( c \) is a weak equivalence is the covariant descent condition for \( F \)).

Finally we have a natural weak equivalence

\[
\mathcal{R}_E(q^*(F), \tilde{\gamma}(G)) \cong \mathcal{R}_X(F, \mathcal{R}_{E/X}^\ast(\ast, \tilde{\gamma}(G))).
\]

By definition

\[
\mathcal{R}_{E/X}^\ast(\ast, \tilde{\gamma}(G)) = \gamma'(G),
\]

and the morphism \( \sigma(G) \to \mathcal{R}_{E/X}^\ast(\ast, \tilde{\gamma}(G)) \) induced by \( \xi \) is the morphism constructed above. We conclude by composing this with the previous morphism to obtain the weak equivalence

\[
\mathcal{R}_X(F, \sigma(G)) \to \mathcal{R}_X(F, \gamma'(G)).
\]

Finally note that the pushout diagram for \( \gamma(G) \) gives (by functoriality of the realization) a diagram

\[
\begin{array}{ccc}
\mathcal{R}_X(F, \sigma(G)) & \to & \mathcal{R}_X(F, \gamma'(G)) \\
\downarrow & & \downarrow \\
\mathcal{R}_X(F, G) & \to & \mathcal{R}_X(F, \gamma(G)).
\end{array}
\]

The vertical arrows are obtained from object-by-object weak equivalences in the pushout diagram of \( \gamma(G) \), so they are weak equivalences of spaces here. Thus the bottom arrow is a weak equivalence. This completes the proof of the main lemma. \( \Box \)

**Proof of Theorem 3.1**

We have treated above the case where \( G \) is \( n \)-truncated. For any \( G \) we have the morphism \( G \to \tau_{\leq n}^\text{pre} G \) truncating the homotopy groups below \( n \), object-by-object. If \( G \to G' \) is an Illusie weak equivalence then for any \( n \), \( \tau_{\leq n}^\text{pre} G \to \tau_{\leq n}^\text{pre} G' \) is an Illusie weak equivalence of \( n \)-truncated presheaves of spaces. By the main theorem for the truncated case,

\[
\mathcal{R}_X(F, \tau_{\leq n}^\text{pre} G) \to \mathcal{R}_X(F, \tau_{\leq n}^\text{pre} G').
\]
is a weak equivalence. Note now that the morphism $G \to \tau_{\leq n}^{\text{pre}} G$ may be obtained by adding on balls of dimension $\geq n + 2$. Hence the morphism 

$$\mathcal{R}_X(F, G) \to \mathcal{R}_X(F, \tau_{\leq n}^{\text{pre}} G)$$

is obtained by adding on pieces which are products of balls of dimension $\geq n + 2$ with something else. Thus for $i \leq n$ the above morphism induces isomorphisms 

$$\pi_i(\mathcal{R}_X(F, G), y) \xrightarrow{\cong} \pi_i(\mathcal{R}_X(F, \tau_{\leq n}^{\text{pre}} G), y).$$

Combining with the result of the theorem for the truncated case we get that 

$$\pi_i(\mathcal{R}_X(F, G), y) \xrightarrow{\cong} \pi_i(\mathcal{R}_X(F, G'), y')$$

for any $i \leq n$. Since this is true now for any $n$, we get that $\mathcal{R}_X(F, G) \to \mathcal{R}_X(F, G')$ is a weak equivalence, which is the conclusion of Theorem 3.1. \hfill \Box

6. Applications

**Lemma 6.1** Suppose $F : \mathcal{X} \to \text{Top}$ is a covariant functor and suppose $G : \mathcal{X} \to \text{Sets} \subset \text{Top}$ is a presheaf of sets represented (as a presheaf) by a disjoint union of objects of $\mathcal{X}$, 

$$G = \coprod_{i \in I} X_i.$$  

Then 

$$\mathcal{R}_X(F, G) \cong \coprod_{i \in I} F(X_i).$$

**Proof:** It is immediate that 

$$\mathcal{R}_X(F, G) \cong \coprod_{i \in I} \mathcal{R}_X(F, X_i).$$

On the other hand, by Lemma [1] we have that $\mathcal{R}_X(F, X_i) \cong X_i^{\text{top}}$. \hfill \Box

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Lemma 6.2 Suppose $F : \mathcal{X} \to \text{Top}$ is a covariant functor and suppose $A$ is a simplicial presheaf of sets on $\mathcal{X}$. Let $\mathcal{R}_\mathcal{X}(F, A)$ be the simplicial space whose components are the realizations of the component presheaves of $A$. Let $G$ be the realization of $A$ (i.e. $G(X)$ is the realization of the simplicial set $A(X)$ for each object $X$). Then

$$\mathcal{R}_\mathcal{X}(F, G) = \mathcal{R} (\mathcal{R}_\mathcal{X}(F, A)).$$

Proof: Both sides are the realizations of the same bisimplicial set. \qed

The preceding two lemmas allow us to identify $\mathcal{R}_\mathcal{X}(F, G)$ with the naive construction one would make: we can express $G$ (up to object-by-object equivalence) as the realization of a simplicial presheaf of sets, and we can even suppose that the component presheaves are disjoint unions of objects of $\mathcal{X}$; then we can take the realization of each component as simply the disjoint union of the $F(X_i)$ with $X_i$ making up the component; and finally we can take the realization of the resulting simplicial topological space. The two previous lemmas show that this process actually gives $\mathcal{R}_\mathcal{X}(F, G)$ up to weak homotopy equivalence. Our main result now shows (under the hypothesis that $F$ satisfies covariant descent) that this construction is unchanged if we replace $G$ by an Illusie-equivalent $G'$. This doesn’t seem to be easy to see directly from the point of view of the construction given in this paragraph.

In view of the discussion above, we propose an alternative shorter notation. If the site $\mathcal{X}$ in question is understood from context, then we denote by $F(G)$ the realization $\mathcal{R}_\mathcal{X}(F, G)$. This is compatible with the original notation for $F$ by Lemma 6.1.

We can apply this discussion to 1-stacks. An algebraic stack $\mathcal{G}$ is typically given by a presentation, starting with a surjective morphism $X \to \mathcal{G}$ from a scheme of finite type $X$ (this being a surjection of stacks but not pre-stacks!). The definition of algebraic stack is such that $R := X \times_{\mathcal{G}} X$ is represented by a scheme of finite type. We obtain a simplicial scheme

$$Z_n := X \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} X,$$

with $Z_0 = X$ and $Z_1 = R$. In the case of stacks over $\text{Spec}(\mathbb{C})$ we would like to define the realization $|\mathcal{G}|$ to be the realization of the simplicial topological space $\{Z_n^{\text{top}}\}$. The realization of the simplicial presheaf $|Z|$ is a presheaf of spaces which is Illusie weak equivalent to the 1-truncated presheaf of spaces $G$ corresponding to the stack $\mathcal{G}$ (in fact, the presheaf of spaces corresponding to $\mathcal{G}$ is the fibrant object associated to $|Z|$). Our main theorem coupled with Lemmas 6.1 and 6.2 shows that the realization of the simplicial
space \( \{Z^n_{\text{top}}\} \) is weak homotopy equivalent to the realization \( R_X(F,G) \) as defined above (with \( F \) as in Theorem 2.1). In particular it is independent of the choice of surjection \( X \to G \).

7. Algebraic to analytic

Let \( \mathcal{X} \) denote the category of schemes of finite type over \( \text{Spec}(\mathbb{C}) \) and let \( \mathcal{X}^{\text{an}} \) denote the category of complex analytic spaces. Suppose \( G \) is a presheaf of spaces on \( \mathcal{X} \). We will define a presheaf of spaces \( G^{\text{an}} \) on \( \mathcal{X}^{\text{an}} \), extending the construction of the complex analytic space associated to a scheme of finite type.

For any \( Z \in \mathcal{X}^{\text{an}} \) put

\[
G^{\text{an}, \text{pre}}(Z) := \lim_{\rightarrow, B \subset \Gamma(Z, \mathcal{O})} G(\text{Spec}(B)).
\]

The limit is a homotopy limit (which can be thought of as a realization over the indexing category, for example). The limit is taken over all subalgebras \( B \subset \Gamma(Z, \mathcal{O}) \) of finite type over \( \mathbb{C} \). Let \( G^{\text{an}} \) be the homotopy sheaf associated to \( G^{\text{an}, \text{pre}} \).

Suppose \( G \) is a sheaf of sets represented by a disjoint union of schemes of finite type

\[
G = \coprod_{i \in I} X_i.
\]

Then

\[
G^{\text{an}, \text{pre}} \cong \coprod_{i \in I} X_i^{\text{an}},
\]

that is \( G^{\text{an}} \) is (homotopic to) the sheaf of sets represented by the disjoint union of the associated analytic spaces.

If \( \{A^i\}_{i \in I} \) is a directed system of simplicial presheaves of spaces, then

\[
\lim_{\rightarrow, i \in I} |A^i| \cong |(\lim_{i \in I} A^i)|,
\]

where \( |A| \) denotes the object-by-object realization of a simplicial presheaf of spaces into a presheaf of spaces. If \( B \) is a simplicial presheaf of spaces and if \( B' \) denotes the homotopy sheafification at each stage, then

\[
|B| \to |B'|
\]

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is an Illusie weak equivalence (proof: it is the realization of a morphism of simplicial presheaves of spaces which is an Illusie weak equivalence at each stage; by the cohomological interpretation of Illusie weak equivalence [4] we obtain the stated result) and in particular induces an equivalence of homotopy sheafifications. Putting these together we find that if \( G \) is a simplicial presheaf of spaces on \( \mathcal{X} \) then \((|G|)_\text{an}\) is the homotopy sheafification of \(|G|\). We can apply this in the case where \( A \) is a simplicial presheaf of sets whose components \( G_n \) are represented by disjoint unions of schemes of finite type, say

\[
A_n = \prod_{i \in I_n} X_{n,i}.
\]

Let \( G = |A| \) be the realization into a presheaf of spaces. We find that \( G_\text{an} \) is the homotopy sheafification of \( A_\text{an} \) with

\[
A_\text{an}_n = \prod_{i \in I_n} X_\text{an}_{n,i}.
\]

We can apply the preceding discussion to obtain a comparison theorem between the realizations of \( G \) and \( G_\text{an} \).

**Theorem 7.1** Suppose \( F : \mathcal{X} \to \text{Top} \) and \( F : \mathcal{X}_\text{an} \to \text{Top} \) are the covariant functors given in Theorems 2.1 and 2.2 (no confusion results from keeping the same notation for the two). Then for any presheaf of spaces \( G \) on \( \mathcal{X} \) we have a weak homotopy equivalence

\[
\mathcal{R}_\mathcal{X}(F, G) \cong \mathcal{R}_{\mathcal{X}_\text{an}}(F, G_\text{an}).
\]

**Proof:** Any presheaf of spaces \( G \) is Illusie weak equivalent (or in fact object-by-object weak equivalent) to the realization of a presheaf of sets \( A \) of the form discussed above (the components being disjoint unions of schemes). On the other hand, we have

\[
\mathcal{R}_\mathcal{X}(F, |A|) \cong |\mathcal{R}_\mathcal{X}(F, A)|,
\]

and similarly

\[
\mathcal{R}_{\mathcal{X}_\text{an}}(F, |A_\text{an}|) \cong |\mathcal{R}_\mathcal{X}(F, A_\text{an})|.
\]

Finally, if \( A_n = \prod_{i \in I_n} X_{i,n} \) then

\[
\mathcal{R}_\mathcal{X}(F, A_n) = \prod_{i \in I_n} X_{i,n}^{\text{top}} = \mathcal{R}_{\mathcal{X}_\text{an}}(F, A_\text{an}_n).
\]

We obtain that

\[
\mathcal{R}_\mathcal{X}(F, |A|) \cong \mathcal{R}_{\mathcal{X}_\text{an}}(F, |A_\text{an}|).
\]
If $G = |A|$ then by the above discussion $G^{an}$ is Illusie weak-equivalent to $|A^{an}|$ so by Theorem 3.1 we get

$$\mathcal{R}_X(F, G) \cong \mathcal{R}_{X^{an}}(F, G^{an})$$

as claimed. \[
\]

8. Examples

Quotient stacks

Consider the case of the site $\mathcal{X}$ of schemes over $\mathbb{C}$. Suppose $X$ is a scheme and $H$ is an algebraic group acting on $X$. Let $G := X/H$ be the quotient pre-stack, in other words $G(Y)$ is the groupoid corresponding to the action of $H(Y)$ on $X(Y)$. Let $G'$ be the quotient stack, which is the stack associated to the pre-stack $G$, i.e. the homotopy-sheafification. The morphism $G \to G'$ is an Illusie weak equivalence. We are interested in calculating

$$(X/H)^{\text{top}} := \mathcal{R}_X(F, G')$$

where $F$ is the functor $Y \mapsto Y^{\text{top}}$. By Theorem 3.1 it suffices to calculate $\mathcal{R}_X(F, G')$. Note that $G'$ is the object-by-object realization of the simplicial presheaf

$$A_n := X \times H \times \ldots \times H.$$ 

We get that $(X/H)^{\text{top}}$ is the realization of the simplicial space

$$A_n^{\text{top}} = X^{\text{top}} \times H^{\text{top}} \times \ldots \times H^{\text{top}}.$$ 

Let $EH^{\text{top}}$ denote a contractible space on which $H^{\text{top}}$ acts topologically freely, so that the quotient is $BH^{\text{top}}$. Then put

$$S_n := A_n \times EH^{\text{top}}$$

with simplicial structure modified to reflect the diagonal action of $H^{\text{top}}$ on $X^{\text{top}} \times EH^{\text{top}}$ (this action is topologically free). We have a morphism of simplicial spaces $S_n \to A_n^{\text{top}}$ which is a weak equivalence at each stage; thus it gives a weak equivalence of realizations. But the realization of $S$ is just the quotient $(X^{\text{top}} \times EH^{\text{top}})/H^{\text{top}}$. The conclusion is that there is a weak homotopy equivalence

$$(X/H)^{\text{top}} \cong (X^{\text{top}} \times EH^{\text{top}})/H^{\text{top}}.$$ 

Not very surprising, after all...
Gerbs

Suppose \( Z \) is a connected scheme and \( Y \to Z \) is a morphism of algebraic stacks. Recall that \( Y \) is a gerb or gerb over \( Z \) if there is an etale covering \( U \to Z \) and a smooth group scheme \( H_U \) over \( U \) such that \( Y \times_Z U \) is equivalent to the stack associated to the pre-stack \( K(H_U/U, 1) \) (this notation means the stack which to a point \( X \to U \) associates the groupoid with one object and automorphism group \( H_U(X) \)). Note that in this case \( H_U \) descends to a section \( \mathcal{H} \) of the sheaf of sets of isomorphism classes of group schemes over \( Z \) (which we can think of as being a group scheme over \( Z \) but which is only defined up to isomorphism—and this etale locally).

We say that \( Y \) is a split gerb if there is a group scheme \( H \) over \( Z \) and if \( Y \) is the stack associated to the pre-stack \( K(H/Z, 1) \) over \( Z \).

**Lemma 8.1** If \( Z \) is a connected scheme of finite type and \( Y \to Z \) is a split gerb with group scheme \( H \) which is smooth and a fibration over \( Z \), then \( Y^{top} \to Z^{top} \) is homotopic to a fibration with fiber \( B(H^{top}_{z}) \) where \( H_z \) is the fiber of \( H \) over a basepoint \( z \in Z \) and \( B(H^{top}_{z}) \) denotes the classifying space of the topological group \( H_z \).

**Proof:** Express \( Y \) as the realization of the simplicial set

\[
A_n = H \times_Z \ldots \times_Z H.
\]

Then \( Y^{top} \) is equivalent to the realization \( |A_n^{top}| \) where here, since the components of \( A_n \) are schemes, we can just take the usual topological realization at each stage. This realization is a fibration (since \( H \) is a a fibration over \( Z \)) and the fiber is just the standard expression for \( B(H^{top}_{z}) \).

\( \square \)

**Caution:** If we take \( Z \) equal to the affine line and let \( H' \) be the constant group scheme with fiber \( GL(n) \) for example, then let \( H \) be obtained by blowing up the identity element over the origin in the affine line and throwing out the part at infinity, we obtain a smooth group scheme over the affine line whose generic point is \( GL(n) \) and whose special point is \( Lie(GL(n)) \) considered as an abelian group. In this case the topological type of the fiber changes so our realization will no longer be a fibration over \( Z^{top} \). This is the reason for the condition that \( H \to Z \) should be a fibration, in the hypothesis of the lemma.

Go back to the case of a general gerb \( Y \to Z \) and let \( U \to Z \) be the etale covering over which it splits, given by the definition. Assume as before that the group scheme type
\( \mathcal{H} \) over \( Z \) is smooth and a fibration (i.e. the group scheme \( H_U \) is smooth and a fibration over \( U \)).

We obtain a simplicial stack

\[
A_n := Y \times_Z U \times_Z \ldots \times_Z U
\]

(or equivalently a simplicial presheaf of spaces). The original stack \( Y \) is Illusie-equivalent to the object-by-object realization \( |A| \). By 6.2 we can calculate the global realization of \( |A| \) by taking the global realization at each stage and then realizing the resulting simplicial space,

\[
|A|^{\text{top}} = |A^{\text{top}}|.
\]

By the main theorem 3.1 we get

\[
Y^{\text{top}} \cong |A^{\text{top}}|.
\]

On the other hand let \( C_n := U \times_Z \ldots \times_Z U \). It is a simplicial scheme realizing \( Z \), so again applying Theorem 3.1 and the previous discussion we have

\[
|C^{\text{top}}| \cong Z^{\text{top}}.
\]

Via these equivalences, the morphism \( Y^{\text{top}} \to Z^{\text{top}} \) is homotopic to the morphism

\[
|A^{\text{top}}| \to |C^{\text{top}}|
\]

induced by the morphism of simplicial stacks \( A \to C \).

Note that there is a sheaf of groups \( H_{C_n} \) on \( C_n \), the pullback of \( H_U \); and we have that \( A_n \) is the stack associated to the pre-stack \( K(H_{C_n}/C_n, 1) \). In particular there is a morphism (Illusie equivalence) \( K(H_{C_n}/C_n, 1) \to A_n \) over \( C_n \). By Theorem 3.1 this induces a weak homotopy equivalence

\[
K(H_{C_n}/C_n, 1)^{\text{top}} \to A_n^{\text{top}}
\]

compatible with the morphism to \( C_n^{\text{top}} \). From our treatment of the case of a split gerb, the map \( A_n^{\text{top}} \to C_n^{\text{top}} \) is a fibration with fiber \( B(H_z^{\text{top}}) \). Note that we can choose the basepoint \( z \in Z \) to lift to any connected component of \( U \), and the fibers of \( H_{C_n} \) over lifts of \( z \) are all isomorphic—so we get a fibration with the same fiber over all connected components. Note also that the morphisms in the simplicial space \( C_n^{\text{top}} \) induce isomorphisms of the fibers. This in turn implies that the realization \( |A^{\text{top}}| \) is a fibration over \( |C^{\text{top}}| \) with fiber \( B(H_z^{\text{top}}) \). We have proved the following statement.
Proposition 8.2 Suppose $Z$ is a connected scheme. If $Y \to Z$ is a gerb with isomorphism-type of group schemes $\mathcal{H}$ smooth and a fibration over $Z$, let $H_z$ be a representative for the fiber of a group scheme of type $\mathcal{H}$ over a basepoint $z \in Z$. Then $Y^\text{top} \to Z^\text{top}$ is a fibration with fiber $B(H_z^\text{top})$.

Look now at the case of Deligne-Mumford gerbs (i.e. gerbs which are Deligne-Mumford stacks). Note that a gerb $Y \to Z$ is Deligne-Mumford if and only if the group-scheme type $\mathcal{H}$ is that of a finite group $H$. In this case $H^\text{top} = H$ and we get that $Y^\text{top} \to Z^\text{top}$ is a fibration with fiber $B(H)$. We have the following converse saying that to give the gerb is the same as to give the fibration. Some notation: $\text{Aut}(BH)$ denotes the $H$-space (or actually $A_\infty$-space) of self-homotopy equivalences of $BH$, and $B(\text{Aut}(BH))$ denotes its delooping. Recall that

$$\pi_0(\text{Aut}(BH)) = \text{Out}(H)$$

is the group of outer automorphisms of $H$ and

$$\pi_1(\text{Aut}(BH)) = Z(H)$$

is the center of $H$. Thus $B(\text{Aut}(BH))$ is a 2-truncated connected space with $\pi_1 = \text{Out}(H)$ and $\pi_2 = Z(H)$ (it is associated to the standard example of a crossed-module $H \to \text{Aut}(H)$).

Proposition 8.3 Suppose $Z$ is a connected scheme and $H$ is a finite group. Then the 2-category of Deligne-Mumford gerbs $Y \to Z$ of type $H$ is equivalent to the 2-category of fibrations over $Z^{\text{top}}$ with fiber $H$ or equivalently the Poincaré 2-category of the space $\text{Hom}(Z^{\text{top}}, B(\text{Aut}(BH)))$.

Proof: This is a comparison theorem between etale cohomology and usual cohomology with coefficients in the 2-category $B(\text{Aut}(BH))$. Using Postnikov devissage on this last 2-category it suffices to prove the comparison theorem for the cohomology with coefficients in $\pi_1(B(\text{Aut}(BH)))$ and $\pi_2(B(\text{Aut}(BH)))$. But these are finite groups so the comparison theorem between etale and usual cohomology holds. \qed

Deligne-Mumford stacks

25
We show how to use the comparison theorem \[7.1\] and Theorem \[3.1\] applied to \(X^{an}\) to give a cut-and-paste description of the topological realization of a Deligne-Mumford stack.

Suppose \(Y\) is a Deligne-Mumford stack. We have the following result:

**Proposition 8.4** There exists a coarse moduli space \(Z\) for \(Y\), in other words a scheme with a morphism \(Y \to Z\) which is universal for morphisms from \(Y\) to schemes. Furthermore, locally in the etale topology of \(Z\) we can express \(Y\) as a quotient stack by a finite group action and \(Z\) as the corresponding quotient scheme.

Mochizuki alludes to this result in \[9\] and gives \[3\] as reference.

**Assumption:** We assume that \(Z\) can be stratified as a disjoint union of connected smooth locally closed subschemes

\[Z = \bigcup Z_\beta\]

such that the stabilizer subgroups of the group action are locally constant over \(Z_\beta\) and \(Z\) (as stratified) is equisingular along the \(Z_\beta\).

This assumption is probably always true, but we don’t try to prove it here since it is not our purpose to get into a long discussion of stratifications.

Let \(Y_\beta\) denote the inverse image of \(Z_\beta\) in \(Y\), and let \(H_\beta\) denote the isomorphism class of stabilizer group over \(Z_\beta\). Note that \(Y_\beta \to Z_\beta\) is a gerb with group \(H_\beta\).

Let \(Z_{\dim \leq n}\) denote the union of strata of dimension \(\leq n\). It is a closed subset, and

\[Z_{\dim = n} := Z_{\dim \leq n} - Z_{\dim \leq n-1}\]

is a disjoint union of the strata of dimension \(n\). Use the same notations in \(Y\).

The topological realization \(Y_{\dim = n}^{\top}\) is a fibration over \(Z_{\dim = n}^{\top}\) whose fiber over a connected component \(Z_{\beta}^{\top}\) is \(K(H_\beta, 1)\). The gerb \(Y_{\dim = n}\) is determined by \(Z_{\dim = n}\) and this fibration.

Let \(T_{\leq n}\) be a good (as usual in the theory of singular spaces and stratifications) tubular neighborhood of \(Z_{\dim \leq n}^{an}\) in \(Z^{an}\). Let \(V_{\leq n}\) be the inverse image of \(T_n\) in \(Y^{an}\).

Let \(T_{= n}\) be the complement of \(T_{\leq n-1}\) in \(T_{\leq n}\) (and we assume that this is nicely arranged so as to be a tubular neighborhood of a subset which is essentially \(Z_{\dim = n}\)). Again let \(V_{= n}\) be the inverse image in the stack.

**Inductive claim:** The retraction of \(T_{\leq n}^{\top}\) to \(Z_{\dim \leq n}^{\top}\) lifts homotopically to a retraction from \(V_{\leq n}^{\top}\) to \(Y_{\dim \leq n}^{\top}\). This retraction preserves \(V_{= n}^{\top}\) over \(T_{= n}^{\top}\).
In particular $V^\top_{\leq n}$ is homotopy equivalent to $Y^\top_{\dim=n}$, which as we have said above is a fibration over $Z^\top_{\dim=n}$ with fibers $K(H_\beta, 1)$.

Applying the main theorem to the presentation of the stack $V_{\leq n}$ corresponding to the covering by $V_{\leq n-1}$ and $V_n$ we find that $V_{\leq n}$ is obtained by glueing $V^\top_{\leq n-1}$ (which we understand) to $V^\top_{\leq n}$ (which we suppose we understand by induction) along a boundary which is homotopic to the inverse image in $Y$ of the boundary of $T_{\leq n-1} \cap Z_{\dim \leq n}$. This boundary piece is again a gerb.

By looking at this closely one can arrange to have the inductive claim for $n$ once it is known for $n-1$.

Finally when we get to $n = \dim(Z)$ we are done and we have constructed $Y^\top_{\dim=n}$ by a sequence of “cut and paste” operations. To recapitulate this sequence of operations, we get $Y^\top_{\leq n}$ from $Y^\top_{\leq n-1}$ by adding on a space which is a fibration over $Z_{\dim=n}$ (the union of open strata of dimension $n$) with fibers of the form $K(H_\beta, 1)$ for various finite groups $H_\beta$.

Remark: The whole of the above discussion goes through equally well for any algebraic stack $Y$, if we assume the existence of a morphism $Y \to Z$, and assuming the existence of a nice stratification as in the assumption stated above (in particular this implies that $Y \to Z$ gives an isomorphism of $Spec(\mathbb{C})$-valued points). Note that this assumption is not automatic, since for example it doesn’t hold for the moduli stacks of vector bundles and the like (for in those cases the coarse moduli space is no longer an isomorphism on points).

Problem: Give a “cut and paste” description of the topological realization of the moduli stacks of vector bundles (as opposed to the quotient space description arising from the expression of the moduli stack as a quotient of the Hilbert scheme).

3. The case of curves

Finally we treat very explicitly what happens in the case of curves. Suppose $Y$ is a smooth Deligne-Mumford stack of dimension 1. Let $\pi : Y \to Z$ be the coarse moduli space. Since $Y$ is normal, the coarse moduli space is also normal so $Z$ is a smooth curve. Let $P_i \in Z$ be the points of ramification of $\pi$ and let $n_i$ be the ramification indices. Define the intermediate stack $Y \to W \to Z$ to be the orbicurve with ramification $n_i$ at $P_i$ over $Z$. It is given by local charts which ramify at $P_i$ with degree $n_i$ and are etale elsewhere (but don’t meet the $P_j$) $j \neq i$). The morphism $Y \to W$ is a gerb with group $H$ where $\pi^{-1}(x) = K(H, 1)$ for a general point $x \in Z$.

Applying the cut-and-paste construction above, one can see that $W^\top_{\leq n}$ is obtained by glueing the lens spaces $K(\mathbb{Z}/n_i, \mathbb{Z}, 1)$ into the space $(W - \{P_i\})^\top_{\leq n}$ at the punctures $P_i$, with the loop around the puncture going to the standard generator for the fundamental
group of the lens space. Then from the discussion of gerbs, $Y^{\text{top}} \to W^{\text{top}}$ is a fibration with fiber $K(H, 1)$. Lemma 8.3 says that the classifying element $\eta$ for this fibration in the appropriate classifying space $\text{Hom}(W^{\text{top}}, B\text{Aut}(K(H, 1)))$ gives exactly the data of the gerb $Y \to W$. In particular, the data of the stack $Y$ can be given by the curve $Z$, the points $P_i$ and ramification indices $n_i$ (which results up to now in a space $W^{\text{top}}$), the group $H$ (up to isomorphism) and the map $\eta : W^{\text{top}} \to B\text{Aut}(K(H, 1))$. The map $\eta$ yields first a map $\eta_1 : \pi_1(W^{\text{top}}, x) \to \text{Out}(H)$ describing the twisting of the group $H$ as one moves around in $W$, and second, an element $\eta_2$ of $H^2(W^{\text{top}}, Z(H))$ where the coefficient system (the center of $H$) is twisted by $\eta_1$.

9. The generalisation to $n$-topoi

This section is a direct result of conversations (email) with K. Behrend (in the course of preparations for the Trento stack school) and C. Teleman, who is thinking about this type of thing in connection with $G$-bundles on a curve. At the current writing the arguments below are only sketches— I haven’t checked the details.

For this section we make a major change of notation: now $\Delta$ will denote the simplicial category whose objects are ordered sets of the form $[n] = \{0, \ldots, n\}$ and whose morphisms are increasing morphisms of ordered sets. Thus $\Delta^n$ now denotes the cartesian product of the category $\Delta$ with itself $n$ times.

Recall that an $n$-category is a functor $\alpha : \Delta^n \to \text{Sets}$ which satisfies certain conditions \[\text{[11]}\]. The set of objects of $\alpha$ is $\alpha_0, \ldots, 0$. Let $\alpha_m(x_0, \ldots, x_m)$ denote the $n-1$-category of composable $m$-tuples with objects $X_0, \ldots, x_m$.

We assume known a definition of the $n$-category of functors $\text{Hom}(\alpha, \beta)$ between two $n$-categories, such that the composition

$$\text{Hom}(\alpha, \beta) \times \text{Hom}(\beta, \gamma) \to \text{Hom}(\alpha, \gamma)$$

is strictly associative. This seems to exist from some preliminary thoughts on the subject, although noone has yet checked the details.

If $\alpha$ is an $n$-category we can define a new $n$-category $\alpha*$ as the $n$-category with one final object attached. The objects of $\alpha*$ are those of $\alpha$ plus one more object denoted $e$, and

$$\alpha*_{n}(x_0, \ldots, x_j, e, \ldots, e) := \alpha_j(x_0, \ldots, x_j),$$
and $\alpha_n^* \langle x_0, \ldots, x_n \rangle = \emptyset$ if $x_i = e$ and $x_j \neq e$ for $j > i$—this gives the definition of the $n$-category $\alpha^*$.

Similarly we can define the $n$-category $\ast \alpha$ by attaching one initial object.

These satisfy homotopy-universal properties which we leave to be elucidated later. Heuristically these properties will say that if $f \alpha \to \beta$ is a functor of $n$-categories and if $u \in \text{Ob}(\beta)$ is an object with a natural transformation $f \to cu$ (where $cu : \alpha \to \beta$ is the constant functor with values $u$) then there is an essentially unique functor $\alpha^* \to \beta$ sending $e$ to $u$ (or to an object equivalent to $u$ provided with the equivalence). The same goes with arrows reversed for $\ast \alpha$.

In view of these universal properties it is reasonable to say that $\beta$ admits direct limits indexed by $\alpha$ if the functor of $n$-categories (restriction)

$$\text{Hom}(\alpha^*, \beta) \to \text{Hom}(\alpha, \beta)$$

is an equivalence. Similarly we say that $\beta$ admits inverse limits indexed by $\alpha$ if the functor

$$\text{Hom}(\ast \alpha, \beta) \to \text{Hom}(\alpha, \beta)$$

is an equivalence.

Needless to say, the limits we are talking about here correspond to what is sometimes called “holim” in the usual topological situation. The category $\textbf{Top}$ admits a natural structure of $n$-category for any $n$ (denoted $\text{Top}_n$); the objects are the topological spaces and the morphism $n-1$-category from $X$ to $Y$ is defined to be the Poincaré $n-1$ category of the space $\text{Hom}_{\text{top}}(X, Y)$. Note that this structure of $n$-category is what might usually be called the $n$-category of $n-1$-truncated spaces. It is equivalent to the $n$-category of $n-1$-groupoids. The usual holim (direct and inverse) are limits in the above sense, indexed over 1-categories, in the $n$-category $\text{Top}_n$.

We adopt the working definition\footnote{Of course this circumlocution means that we haven’t yet checked the details!} that an $n$-topos is an $n$-category which admits direct and inverse limits indexed by $n$-categories.

The $n$-category $\text{Top}_n$ is the basic example of an $n$-topos, extending the topos of sets (which is the case $n = 1$). More generally the $n$-category of $n-1$-stacks of groupoids over a site $\mathcal{X}$ is an $n$-topos which we call the $n$-topos associated to $\mathcal{X}$.

If $\tau$ is an $n$-topos the existence of limits automatically gives a functor $\text{Top}_n \to \tau$. In the case of the $n$-topos of $n-1$-stacks over $\mathcal{X}$ this is the functor taking an $n-1$-groupoid to the associated constant stack.
Realizations with values in an $n$-topos

We now recast our discussion of topological realization in terms of functors with values in an $n$-topos $\tau$. Suppose $\mathcal{X}$ is a category and $F : \mathcal{X} \to \tau$ is a covariant functor. Suppose that $G$ is an $m$-stack. Then we obtain $\mathcal{R}_\mathcal{X}(F, G) \in \tau$, functorial (in the homotopic sense) in the two variables $F$ and $G$. To define this, we take essentially the same definition as before, but noting that the explicit space that is constructed there is really just a direct holim. More precisely let $Fl(\mathcal{X})$ denote the category of morphisms in $\mathcal{X}$, we have

$$Fl(\mathcal{X}) \to \mathcal{X} \times \mathcal{X}^o$$

but on the other hand $(F, G)$ (and the canonical $Top_m \to \tau$) gives a functor $\mathcal{X} \times \mathcal{X}^o \to \tau$. Composing we obtain the functor $Fl(\mathcal{X}) \to \tau$ and the realization is its direct limit in $\tau$. The morphisms which we could construct explicitly before are now only canonically defined up to canonically defined homotopy up to etc (again the details still have to be worked out carefully).

Suppose $\mathcal{X}$ is a site. As before we say that $F$ satisfies covariant descent if for any $X \in \mathcal{X}$ and sieve $\mathcal{B} \subset \mathcal{X}/X$ the morphism $\mathcal{R}_\mathcal{B}(F|_{\mathcal{B}}, *_{\mathcal{B}}) \to F(X)$ is an equivalence in $\tau$. The same proof as above goes through to give the following.

**Theorem 9.1** If $\tau$ is an $n$-topos, $\mathcal{X}$ a site, and $F : \mathcal{X} \to \tau$ a covariant functor satisfying covariant descent and $G, G' : \mathcal{X} \to Top_m$ two presheaves of $m-1$-truncated spaces with a morphism $G \to G'$ then the induced morphism $\mathcal{R}_\mathcal{X}(F, G) \to \mathcal{R}_\mathcal{X}(F, G')$ is an equivalence in $\tau$. Consequently the functor $F$ extends to a functor from the topos of $m-1$-stacks on $\mathcal{X}$ to $\tau$ commuting with (homotopy!) direct limits.

$\blacksquare$

In the case $\tau = Top_n$ we recover the previous Theorem 3.1.

The following corollary extends to the $n$-topos of $n-1$-stacks the usual definition of pullback of sheaves for a morphism of sites.

**Corollary 9.2** Suppose $\mathcal{X}$ and $\mathcal{Y}$ are sites and $F : \mathcal{X} \to \mathcal{Y}$ is a functor (which is sometimes called a morphism of sites from $\mathcal{Y}$ to $\mathcal{X}$). Then $F$ extends uniquely to a functor from the $n$-topos associated to $\mathcal{X}$ to the $n$-topos associated to $\mathcal{Y}$, compatible with homotopy direct limits.
Question: Where do inverse limits come into things?

Remark: In the case where $\tau$ is the $n$-topos of $n-1$-truncated presheaves of spaces over a site $\mathcal{Y}$ (such as for Corollary 9.2) one can again do everything by hand as in the first part of the paper (the details are pretty much identical) so in this case one doesn’t need to worry about getting all of the foundational material about $n$-categories straight.

C. Telemann has a nice interpretation of the realization obtained from the functor $F : X \mapsto X^{\text{top}}$. He points out that on the analytic site $\mathcal{X}^{\text{an}}$ the topological realization can be defined “internally” as in the etale homotopy theory of Artin-Mazur, Friedlander. In our notations I think that this means that the constant functor $F_0(X) := \ast$ has a canonical replacement by a functor $F$ which satisfies covariant descent (one just enforces covariant descent in a way analogous to the operation $\gamma$ for enforcing the homotopy-sheaf condition). This functor is equivalent to the standard topological realization functor. Then Telemann points out that by pulling back a stack from the algebraic to the analytic site (with pullback defined as in Corollary 9.2) and applying this internal topological realization functor we obtain our topological realization.

Remark: The operation $G \mapsto G^{\text{an}}$ that we have defined slightly differently should be the same as the pullback via the morphism of sites

$$(\text{analytic}) \rightarrow (\text{algebraic})$$

(which corresponds to the standard functor $\mathcal{X} \rightarrow \mathcal{X}^{\text{an}}$).

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