We compute the leading-order low-energy constants of the $\Delta S = 1$ effective weak Hamiltonian in the quenched approximation of QCD with up, down, strange, and charm quarks degenerate and light. They are extracted by comparing the predictions of finite volume chiral perturbation theory with lattice QCD computations of suitable correlation functions carried out with quark masses ranging from a few MeV up to half of the physical strange mass. We observe a $\Delta I = 1/2$ enhancement in this corner of the parameter space of the theory. Although matching with the experimental result is not observed for the $\Delta I = 1/2$ amplitude, our computation suggests large QCD contributions to the physical $\Delta I = 1/2$ rule in the GIM limit, and represents the first step to quantify the rôle of the charm quark-mass in $K \to \pi \pi$ amplitudes. The use of fermions with an exact chiral symmetry is an essential ingredient in our computation.

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a few MeV, which turns out to be essential for a robust extrapolation to the chiral limit. Our results reveal a clear hierarchy between the low-energy constants, which in turn implies the presence of a $\Delta I = 1/2$ enhancement in this corner of the parameter space of (quenched) QCD.

Since we are looking for an order-of-magnitude effect, and since simulations with dynamical fermions are very expensive, it is appropriate for us to first perform the computation in quenched QCD. The latter is not a systematic approximation of the full theory\(^1\). However, when quenched results can be compared with experimental measurements, discrepancies of $\mathcal{O}(10\%)$ are found in most cases\([21]\). In the past there were several attempts to attack the problem by using quenched lattice QCD\([22, 23, 24, 25, 26, 27]\). In particular, in Refs.\([25, 26]\), a fermion action with an approximate chiral symmetry was used and, despite the fact that the charm was integrated out and therefore an ultraviolet power-divergent subtraction was needed, the authors observed a good statistical signal for the subtracted matrix elements in a range of quark masses of about half the physical strange quark-mass. Several computations of $A_I$ which use models to quantify QCD non-perturbative contributions in these amplitudes can also be found in the literature (see Refs.\([28, 29]\) and references therein).

**THE $\Delta S = 1$ EFFECTIVE HAMILTONIAN**

In the SU(4) degenerate case and with GW fermions, the CP-even $\Delta S = 1$ effective Hamiltonian is\([1, 2, 7]\)

$$H_w = \frac{g_w^2}{4M_W^2} V_{us}^* V_{ud} \left\{ \kappa_1^+ Q_1^+ + \kappa_1^- Q_1^- \right\}, \quad (3)$$

where

$$Q_1^\pm = Z_{13}^\pm \left\{ \bar{s} \gamma_\mu P \cdot \bar{u} (\bar{u} \gamma_\mu P \cdot \bar{d}) \right\} \pm (\bar{s} \gamma_\mu P \cdot \bar{d}) (\bar{u} \gamma_\mu P \cdot \bar{u}) - [u \to c], \quad (4)$$

and any further unexplained notation in the paper can be found in Ref.\([7]\). We are interested in the ratios of correlation functions

$$R^\pm(x_0, y_0) = \frac{C^\pm(x_0, y_0)}{C(x_0)C(y_0)}, \quad (5)$$

where

$$C(x_0) = \sum_x \langle [J_0(x)]_{\alpha\beta} [J_0(0)]_{\beta\alpha} \rangle, \quad (6)$$

$$C_{1}^\pm(x_0, y_0) = \sum_{x, \bar{y}} \langle [J_0(x)]_{\alpha\beta} [Q_1^\pm(0)] [J_0(y)]_{au} \rangle, \quad (7)$$

\[^1\] On the other hand the ambiguity in the definition of the LECs pointed out by the Golterman and Pallante\([21]\) is not present in the GIM limit.
TABLE I: Results for $aM_\rho$ and $R^\pm,\text{bare}$ as obtained from 746 and 197 gauge configurations in the $\epsilon$ and $p$ regimes, respectively.

| $am$ | $aM_\rho$ | $R^+,\text{bare}$ | $R^-,\text{bare}$ | $(R^+\cdot R^-)\text{bare}$ |
|------|-------------|-------------------|-------------------|-----------------------------|
| $0.002$ | -           | $0.600(43)$       | $2.42(13)$        | $1.45(15)$                  |
| $0.003$ | -           | $0.603(41)$       | $2.40(12)$        | $1.44(14)$                  |
| $0.020$ | $0.1960(28)$| $0.654(40)$       | $2.20(12)$        | $1.44(12)$                  |
| $0.030$ | $0.2302(25)$| $0.691(33)$       | $1.93(9)$         | $1.33(9)$                   |
| $0.040$ | $0.2598(24)$| $0.723(31)$       | $1.75(8)$         | $1.26(8)$                   |
| $0.060$ | $0.3110(24)$| $0.772(30)$       | $1.51(7)$         | $1.17(8)$                   |

periodic boundary conditions by standard Monte Carlo techniques. The topological charge and the quark propagators are computed following Ref. [38]. The statistical variance of the estimates of correlation functions has been reduced by implementing a generalization of the low-mode averaging technique proposed in [39], which turns out to be essential to get a signal for the lighter quark masses. The lattice has a bare coupling constant $\beta \equiv 6/g_0^2 = 5.8485$, which corresponds to a lattice spacing $a \sim 0.12$ fm, and a volume of $V a^{-4} = 16^3 \times 32$. The list of simulated bare quark masses, together with the corresponding results for pion masses and unrenormalized ratios $R^\pm,\text{bare} = Z^2 R^\pm / Z_{11}^\pm$, are reported in Table I. Further technical details will be provided in a forthcoming publication.

The values in Table I show that $R^+,\text{bare}$ exhibit a pronounced mass dependence, which is more marked in $R^-,\text{bare}$. We have explored several fit strategies, attempting to minimize the systematic uncertainties due to neglected higher orders in ChPT. The structure of Eqs. (10) and (11) indeed suggests that it is possible to cancel large NLO ChPT corrections by constructing suitable combinations of $R^\pm,\text{bare}$. We observe that the product $g_1^+ g_1^-$ is very robust with respect to the details of the fit strategy. The simplest way to extract this quantity is from a fit to the combination $(R^+ R^-)\text{bare}$, where NLO ChPT corrections cancel in the limit $m \rightarrow 0$. We obtain

\begin{equation}
(g_1^+ g_1^-)\text{bare} = 1.47(12) .
\end{equation}

To extract $g_1^+,\text{bare}$ and $g_1^-,\text{bare}$ separately we then fit $R^+,\text{bare}$ to NLO ChPT, taking the value of $F$ from a fit to the two-point functions as in Ref. [39] and the bare $\Sigma$ from Ref. [10]. Putting the result together with Eq. (13) we get

\begin{equation}
g_1^+,\text{bare} = 0.63(4)(8) ,
g_1^-,\text{bare} = 2.33(11)(30) ,
\end{equation}

where the first error is statistical and the second is an estimate of the systematic uncertainty from the spread of the central values obtained from fits to different quantities and/or mass intervals. The physical LECs are given by

\begin{equation}
g_1^+ = k_1^+ \left[ \frac{R_{11}^+,\text{RGI}}{R_{11}^+,\text{bare}} \right]_{\text{ref}} g_1^+,\text{bare} ,
\end{equation}

where $k_1^\pm$ are the renormalization group-invariant (RGI) Wilson coefficients [1, 2, 3, 4, 7]. The RGI quantities

\begin{equation}
R_{11}^+,\text{RGI} = R_{11}^+,\text{bare} \quad \left[ r_0^2 M_R^2 = r_0^2 M_R^2 \right]
\end{equation}

at the pseudoscalar $r_0^2 M_R^2 = 1.5736$ are taken from Refs. [41, 42, 43, 44], and $r_0$ is a low-energy reference scale widely used in quenched QCD computations [45]. This procedure, analogous to the one proposed for the scalar density in Ref. [46], provides values of the LECs that are non-perturbatively renormalized, as explained in detail in Ref. [44].

FIG. 1: Mass dependence of $R^\pm,\text{bare}$ and $(R^+ R^-)\text{bare}$.

PHYSICS DISCUSSION

By using the non-perturbative renormalization factors in Ref. [44]

\begin{equation}
\left[ \frac{R_{+},\text{RGI}}{R_{+,\text{bare}}} \right]_{\text{ref}} = 1.15(12) ,
\left[ \frac{R_{-},\text{RGI}}{R_{-,\text{bare}}} \right]_{\text{ref}} = 0.56(6) ,
\end{equation}

and the perturbative values $k_1^+ = 0.708$ and $k_1^- = 1.978$ (see Ref. [2]), we obtain our final results

\begin{equation}
g_1^+ = g_1^- = 2.6(5) ,
g_1^+ g_1^- = 1.2(2) .
\end{equation}

A solid estimate of discretization effects would require simulations at several lattice spacings, which is beyond the scope of this exploratory study. However, computations of $R^\pm$ at different lattice spacings and for masses close to $m_s/2$ [46, 47] indicate that discretization effects may be smaller than the errors quoted above. It is also interesting to note that quenched computations of various physical quantities carried out with Neuberger fermions
show small discretization effects at the lattice spacing of our simulations [48, 49].

The values of $g_8^{\pm}$ in Eq. (18) are the main results of this paper. They reveal a clear hierarchy between the low-energy constants, $g_8^{+} \gg g_8^{-}$, which implies the presence of a $\Delta I = 1/2$ enhancement in the GIM-limit of (quenched) QCD. The strong mass dependence of $R_{s, \text{bare}}$ in Fig. 1 indicates that an extrapolation of data around or above the physical kaon mass to the chiral limit is probably subject to large systematic uncertainties.

When the charm mass $m_c$ is sufficiently heavier than the three light-quark masses, the chiral effective theory has a three-flavour SU(3) symmetry and the LO $\Delta S = 1$ effective Hamiltonian has two unknown LECs, $g_{27}$ and $g_8$. In our strategy these LECs are considered functions of the charm mass, and our normalizations are such that

$$g_{27}(0) = g_8^{+}, \quad g_8(0) = g_8^{+} + \frac{g_8^{+}}{5}. \quad (19)$$

The values of $g_{27}^{\exp}(m_c)$ and $g_8(m_c)$ can be estimated at the physical value of the charm mass $m_c$ by matching the LO ChPT expressions with the experimental results for $|A_0|$ and $|A_2|$. The result is

$$|g_{27}^{\exp}(m_c)| \sim 0.50, \quad |g_8^{\exp}(m_c)| \sim 10.5. \quad (20)$$

These estimates are, of course, affected by systematic errors due to higher-order ChPT contributions [51]. Keeping this in mind, the value of $g_{27}^{\exp}(m_c)$ is in good agreement with our result. Since $g_{27}$ is expected to have a mild dependence on the charm-quark mass (only via the fermion determinant in the effective gluonic action), and barring accidental cancellations among quenching effects and higher-order ChPT corrections, this agreement points to the fact that higher-order ChPT corrections in $|A_2|$ may be relatively small. Our value for $g_8(0)$ differs by roughly a factor of 4 from $g_8^{\exp}(m_c)$ given in Eq. (20). Apart from possible large quenching artefacts, our result suggests that the charm mass dependence and/or higher-order effects in ChPT are large for $|A_0|$. Indeed in this case penguin contractions, which are absent in the GIM limit, can be responsible for a large charm-mass dependence in $g_8$, a dependence that can be studied in the next step of our strategy [3, 50].

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