Metastable states of graphene Dirac fermion in potential barrier

S Kuwata¹,³ and D Takehisa²
¹Graduate School of Information Sciences, Hiroshima City University, Asaminami-ku, Hiroshima 731-3194, Japan
²Department of System Engineering, Hiroshima City University, Asaminami-ku, Hiroshima 731-3194, Japan

E-mail: kuwata@hiroshima-cu.ac.jp

Abstract. Weak quantization of the Dirac equation under a potential barrier (with height $V_0$) indicates that while the bound state of Dirac fermion (with energy $E_0 \in \mathbb{R}$ and mass $m$) is allowed for $|E_0| < m$, the Dirac fermion is metastable for $|E_0| > m$. Under the condition $V_0 \gg m$, which is well satisfied for typical graphene, it is found that the tunneling time $\Delta t$ and the lifetime $\tau$ for the metastable state are related as $\Delta t \gtrsim 2 \tau$. As $m$ decreases, it follows that $\Delta t/\tau$ tends to increase and behaves as $\log(1/m)$ up to higher order logarithmic correction.

1. Introduction

Graphene, due to its novel electric and magnetic properties [1], has a significant application in nanotechnology [2]. Ideal graphene can be represented by the 2D massless Dirac equation. As contrasted with the Schrödinger equation, the Dirac equation has a somewhat mysterious property, called the Klein paradox [3] or the Klein tunneling [4] (if the width of a potential barrier is finite), where electrons can propagate the potential barrier without damping even if the potential height is large enough, compared with the incident energy. The Klein paradox can be resolved by interpreting antiparticles as particles with negative energy propagating backward in time [5], or more strictly, by second quantization [6–9].

However, from a practical point of view, it may be convenient to understand the Klein paradox without resorting to second quantization. As such a method, weak quantization [10,11] has been known, where the energy eigenvalues for the Dirac equation are treated as complex numbers. As in the case of confining potential [10,11], electrons in a potential barrier turn out to be metastable, with the density of states given by the exact Breit-Wigner spectrum.

The aim of this article is to show that, based on weak quantization, if the potential height is large enough compared with the effective mass for the corresponding 1D Dirac fermion, then the tunneling time is larger than twice the lifetime for the metastable state. This implies that $P \leq e^{-2} (\approx 0.135)$, where $P$ represents the probability that electrons in the metastable state remain their state during the tunneling time. In Sec. 2, we deal with the 1D Dirac equation under a potential barrier, where the Weyl function is introduced to calculate the density of states. In Sec. 3, we calculate the density of states, from which the lifetime and tunneling time can be derived. Sec. 4 is devoted to conclusion.

2. Methods

We begin with the 1D Dirac equation under a potential barrier. In units of $\hbar = c = 1$, we have
\[ \psi'(x) - \left[ E - V(x) i \sigma_y + m \sigma_x \right] \psi(x) = 0, \]  
\text{(1)}

where \( \sigma_x \) and \( \sigma_y \) represent the Pauli matrices. For non-normal incidence of 2D massless Dirac fermion on the potential barrier, the problem can be reduced to that of the 1D massive Dirac fermion, with the effective mass \( m \) being proportional to the transverse momentum \([12]\). The potential barrier \( V(x) \) (with height \( V_0 \) and width \( 2a \)) is given by

\[
V(x) = \begin{cases} 
V_0 & (|x| < a), \\
0 & (|x| > a), 
\end{cases}
\]

with \( a, V_0 > 0 \). Considering that \( \psi \) depends on the energy eigenvalue \( E \), we can write \( \psi \) as \( \psi_E \).

To obtain the density of states, it may be convenient to introduce two Weyl functions \( W_+, W_- : \mathbb{C} \rightarrow \mathbb{C} \) such that \([10]\)

\[
\text{Im } W_{\pm}(E) = (\text{Im } E) \int_{a \pm} |\psi_E(x)|^2 \, dx,
\]

where \( W_+ = [0, \infty), W_- = (-\infty, 0] \), and \( E \) is chosen as a complex number (note that in Ref. \([10]\), \( E \) was defined as \(-E\)). Then the density of states, \( \rho \), of energy \( E_0 \in \mathbb{R} \) can be given by

\[
\rho(E_0) = \lim_{\nu \rightarrow 0^+} [W_+(E_0 + iv) + W_-(E_0 + iv)].
\text{(2)}
\]

It is fortunate that, due to the parity conservation for \( \psi(x) \) by \( V(-x) = V(x) \), \( W_+ \) and \( W_- \) are simply related as \([10,11]\)

\[
W_+(E) \ W_-(E) = -1.
\text{(3)}
\]

It should be noticed that while \( W_+ \) and \( W_- \) are analytic for \( \text{Im } E > 0 \) by their definition, they have singular points for \( \text{Im } E \leq 0 \). Suppose that all the singular points are contributed from an (infinite) number of poles of order one. Then \( \rho(E_0) \) can be regarded as a Breit-Wigner spectrum, with the central energy \( \langle E \rangle \) and the half width \( \Gamma \) given by the pole of \( W_{\pm}(E) \) at \( E = \langle E \rangle - i\Gamma/2 \) (see Appendix A).

3. Results

Following the procedure for calculating \( W_+ = W_+(E) \), we obtain (see Appendix B)

\[
W_+ = \frac{\alpha + i \beta}{\gamma + i \delta}, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} -\lambda \cos \theta & \mu \sin \theta \\ \frac{\lambda \sin \theta}{\mu} & \cos \theta \end{pmatrix},
\text{(4)}
\]

where \( \lambda, \mu \) and \( \theta \) are given by

\[
\lambda = \frac{\omega_{\text{ex}}}{E + m}, \quad \mu = \frac{\omega_{\text{in}}}{E - V_0 + m}, \quad \theta = \alpha \omega_{\text{in}},
\]

with \( \omega_{\text{ex}} = \sqrt{E + m \sqrt{E - m}} \) and \( \omega_{\text{in}} = \sqrt{(V_0 - E) + m \sqrt{(V_0 - E) - m}} \). We address the problem for the cases of \( |E_0| < m \) and \( |E_0| > m \) separately.

3.1. \( |E_0| < m \)

In this case, \( W_+(E_0) \) has one or more singular points (poles of order one). This implies that there is a possibility that \( \rho(E_0) \) has a spectrum of delta function type through the well-known formula

\[
\lim_{\nu \rightarrow 0^+} \frac{1}{\pi} \int \frac{1}{x} - i \nu \delta(x) \quad (\text{where } P \text{ represents the principal value)},
\]

indicating the existence of a bound state. The existence of the bound state, in itself, can be understood by the charge conjugate symmetry of equation (1), mentioned in Remark 2 below. Recalling that the charge conjugate state \( \psi_e \) has energy \( (E) \) under the potential \( (V) \), we find that the bound state can be realized for \( -E_0 - m > -V_0 \) (the subtraction by \( m \) comes from the elimination of the rest mass). Thus the bound state can be allowed
for $E_0$ such that
\[ -m < E_0 < \min(m, V_0 - m) \quad (V_0 > 0). \]

Considering further that the peak positions of $\rho(E_0)$ are (almost) uniformly distributed (not accumulated), as in the case for $|E_0| > m$ below, we find that as $m$ decreases, the number of the bound states tends to decrease. Hence in the limit of $m \to 0$, the number of the bound state turns out to be, at most, unity. In this sense, we can neglect the bound state for $m \ll 1$.

- **Remark 1** A more detailed analysis indicates that even for sufficiently small but non-vanishing $m$, there always exists a single bound state for $|E_0| < m$.

### 3.2. $|E_0| > m$

In this case, $W_\pm(E_0)$ has no singular point, so that $\lim_{V \to 0+}$ and $\text{Im}$ in equation (2) can be exchanged with each other. As a consequence, $\rho(E_0)$ can be calculated as
\[
\rho(E_0) = -\left[ \left| \frac{\alpha}{\gamma} \frac{\beta}{\delta} \right| \left( \frac{1}{\gamma^2 + \delta^2} + \frac{1}{\alpha^2 + \beta^2} \right) \right]_{E \to E_0}
\]
\[
= \left[ \frac{\omega_{\text{ex}}(E - V_0 + m)}{\omega_{\text{ex}} - V_0(E - m \cos 2\omega_{\text{in}})} + (m \to -m) \right]_{E \to E_0}
\] 

from which it is found that $\rho(E_0)$ is invariant under $m \to -m$, and also under $(E_0, V_0) \to (-E_0, -V_0)$.

- **Remark 2** The invariances of $\rho(E_0)$ under $m \to -m$ and $(E_0, V_0) \to (-E_0, -V_0)$ come from the mass inversion symmetry and charge conjugate symmetry of equation (1), respectively; $\psi_m = \sigma_y \psi$ and $\psi_c = \sigma_x \psi^*$ satisfy equation (1) with $m \to -m$ and $(E_0, V_0) \to (-E_0, -V_0)$, respectively.

Furthermore, $\rho$ has the following limit:
\[
\lim_{E_0 \to m^+} \rho(E_0) = 0 \quad \left[ m(1 \pm \cos 2\omega_{\text{in}}^{(0)}) \neq 0 \right],
\] 

where $\omega_{\text{in}}^{(0)} = \omega_{\text{in}}|_{E \to m}$. As an example of $\rho(E_0)$, see figure 1, in which the spectrum is well represented by the Breit-Wigner distribution. Notice that $\rho(E_0)$ satisfies equation (6) and that the peak positions are (almost) uniformly distributed, with the energy difference between neighboring peaks being around $\frac{\pi^2}{\gamma} \approx 1.5708 \times 10^{-2}$ (see the second column in table 1).

![Figure 1. Density of states, $\rho(E_0)$, for $(a, m, V_0) = (100, 1, 100)$](image)

Recalling that the central energy $\langle E \rangle$ and the half width $\Gamma$ are given by the pole (of order one) of $W_+(E)$ and $W_-(E)$ at $E = \langle E \rangle - i\Gamma/2$, we obtain, together with equation (3), the following relations:
\[
\begin{align*}
\begin{cases}
\frac{E - m}{\omega_{\text{ex}}} \sin \theta + i \frac{E - V_0 - m}{\omega_{\text{in}}} \cos \theta = 0, \\
\frac{E - m}{\omega_{\text{ex}}} \cos \theta + i \frac{E - V_0 - m}{\omega_{\text{in}}} \sin \theta = 0,
\end{cases}
\end{align*}
\]

\(E = \langle E \rangle - i \Gamma / 2\),

(7)

corresponding to the zero point of \(\gamma + i \delta\) and that of \(\alpha + i \beta\), respectively. For the values of \(\langle E \rangle\) and \(\Gamma\) in figure 1, see table 1, where use has been made of equation (7) numerically.

Table 1. Values of \(\langle E \rangle - m\) and \(\Gamma\) for the \(n\)-th metastable state for the same values of \((a, m, V_0)\) as in figure 1. For later convenience, \(\kappa = 2a \Gamma / v_k\), where \(v_k = \sqrt{\langle E \rangle^2 - m^2}\).

| \(n\) | \(\langle E \rangle - m\) | \(\Gamma\) | \(\kappa\) |
|------|-----------------|------|------|
| 1    | \(3.34021 \times 10^{-3}\) | \(8.10331 \times 10^{-4}\) | \(1.98782\) |
| 2    | \(1.90475 \times 10^{-2}\) | \(1.92952 \times 10^{-3}\) | \(2.00531\) |
| 3    | \(3.47548 \times 10^{-2}\) | \(2.60249 \times 10^{-3}\) | \(2.02532\) |
| 4    | \(5.04621 \times 10^{-2}\) | \(3.13153 \times 10^{-3}\) | \(2.04531\) |

Before obtaining a general solution to equation (7), we consider a special solution such that \(\Gamma = 0\). In this case, it follows that

\[(\Gamma = 0) \Rightarrow \begin{cases}
\langle E \rangle - m = 0 = \cos a \omega_{\text{in}}^{(0)}, \\
\langle E \rangle - m = 0 = \sin a \omega_{\text{in}}^{(0)},
\end{cases}
\]

satisfying \(1 + \cos 2a \omega_{\text{in}}^{(0)} = 0\) and \(1 - \cos 2a \omega_{\text{in}}^{(0)} = 0\), respectively. Recalling the condition in the bracket in equation (6), we find that \(\lim_{E_0 \rightarrow m} \rho(E_0)\) is not vanishing in this case.

\bullet \quad \textbf{Remark 3} The relation of equation (8) is known as a super-critical condition [13], in which the Klein tunneling can be realized.

However, the relation of \(\langle E \rangle = m\) implies that the incident momentum on the potential barrier is vanishing, so that such a state is difficult to observe. In this sense, we neglect the case of \(\langle E \rangle = m\).

It seems that equation (7) is still somewhat more difficult than to solve in an analytical way. To simplify it, consider the condition such that

\[V_0 \gg m,\]

(9)

which holds for typical graphene, where \(V_0 \sim 1\) eV. In this case, \(m\) in \((V_0 - E \pm m)\) may be neglected, so that \(\omega_{\text{in}}\) can be approximated as

\[\omega_{\text{in}} = \omega_{\text{in}}^{(r)} + i \omega_{\text{in}}^{(i)} \approx (V_0 - \langle E \rangle + i \Gamma / 2) \quad \left(\omega_{\text{in}}^{(r)}, \omega_{\text{in}}^{(i)} \in \mathbb{R}\right).
\]

As a consequence, equation (7) is reduced to

\[
\begin{align*}
\langle E \rangle - m \sin \theta - i \omega_{\text{ex}} \cos \theta & \approx 0, \\
\langle E \rangle - m \cos \theta - i \omega_{\text{ex}} \sin \theta & \approx 0,
\end{align*}
\]

\(E = \langle E \rangle - i \Gamma / 2\).

(10)

Denoting the real and imaginary parts of \(\omega_{\text{ex}}\) as \(\omega_{\text{ex}} = e + i f\) \((e, f \in \mathbb{R})\), for the time being, we can rewrite equation (10) using a matrix form as

\[
\begin{pmatrix}
\sin \omega_{\text{in}}^{(r)} \\
\cos \omega_{\text{in}}^{(r)} \\
\cos \omega_{\text{in}}^{(r)} \\
\sin \omega_{\text{in}}^{(r)}
\end{pmatrix}
\begin{pmatrix}
M_1 \\
M_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix},
\]

where \(M_1\) and \(M_2\) are related as
\[
\begin{align*}
\begin{cases}
\frac{M_1 + M_2}{2} = \left( E_{\text{NR}} \cosh a\hat{r} - e \sinh a\hat{r} & - f \cosh a\hat{r} - f \sinh a\hat{r} \right), \\
\frac{M_1 - M_2}{2} = \left( f \sinh a\hat{r} + e \cosh a\hat{r} & + E_{\text{NR}} \sinh a\hat{r} - e \cosh a\hat{r} \right).
\end{cases}
\end{align*}
\]

with \( E_{\text{NR}} := \langle E \rangle - m \) and \( \hat{r} := \gamma / 2 \), so that their determinants amount to

\[
\begin{align*}
\det M_1 &= (f\hat{r} - eE_{\text{NR}}) \cosh 2a\hat{r} + \frac{1}{2}(e^2 + f^2 + E_{\text{NR}}^2 + \hat{r}^2) \sinh 2a\hat{r}, \\
\det M_2 &= -\det M_1.
\end{align*}
\]

Hence from \( \det M_1 = 0 \) or \( \det M_2 = 0 \) by \( (\sin \omega_{\text{in}}^{(r)} \cos \omega_{\text{in}}^{(r)}) \neq (0 \quad 0) \), we obtain the following relations:

\[
\tanh 2a\hat{r} = \frac{2(eE_{\text{NR}} - f\hat{r})}{e^2 + f^2 + E_{\text{NR}}^2 + \hat{r}^2} = \frac{\sqrt{\Phi_m(\hat{r}, E_{\text{NR}})}}{\sqrt{2(E_{\text{NR}} + m)}},
\]

where \( \Phi_m(\hat{r}, E_{\text{NR}}) = -\hat{r}^2 + 2mE_{\text{NR}} + E_{\text{NR}}^2 + \left( f^2 + E_{\text{NR}}^2 \right) \left[ f^2 + (2m + E_{\text{NR}})^2 \right] \). In the second line in equation (11), we have replaced \( e \) and \( f \) using \( \hat{r} \) and \( E_{\text{NR}} \) through the relation of \( e + if = \sqrt{E - m} / \sqrt{E + m} \), with \( E = (E_{\text{NR}} + m) - i\hat{r} \). It may be convenient to derive the necessary condition for equation (11) as

\[
(E_{\text{NR}} + m)^2 X^4 + (\hat{r}^2 - 2mE_{\text{NR}} - E_{\text{NR}}^2) X^2 - \hat{r}^2 = 0 \quad (X = \tanh 2a\hat{r}),
\]

which has been obtained from squaring both sides of equation (11), shifting the terms \( -\hat{r}^2 + 2mE_{\text{NR}} + E_{\text{NR}}^2 \), and finally squaring both sides of the resultant relation.

In the rest of this section, we deal with the three following regimes: \( 2a\hat{r} \ll 1, 2a\hat{r} \gg 1 \), and \( 2a\hat{r} = \mathcal{O}(1) \), to claim that \( 2a\hat{r} \geq v_g \), where \( v_g \) represents the group velocity, mentioned in table 1. Notice that the relation \( 2a\hat{r} \geq v_g \) can be rewritten as \( \Delta t \geq 2\tau \), where \( \Delta t := 2a / v_g \) and \( \tau := \gamma^{-1} \) represent the tunneling time and the lifetime of the metastable state, respectively.

3.2.1. Case for \( 2a\hat{r} \ll 1 \). In this limit, \( X \) can be expanded as

\[
\tanh 2a\hat{r} = 2a\hat{r} + O \left( (2a\hat{r})^3 \right) \quad (2a\hat{r} \to 0).
\]

Neglecting the higher-order terms, we obtain from equations (12) and (13)

\[
2a\hat{r} = \frac{1 + (2a)^2 (E_{\text{NR}} - m^2)}{1 + (2a)^2} \left( \frac{E_{\text{NR}}^2}{\text{NR}} + m^2 \right) \quad (2a\hat{r} \ll 1).
\]

Here we have eliminated the case of \( 2a\hat{r} = 0 \), because \( 2a\hat{r} = 0 \) leads to \( E_{\text{NR}} = 0 \) by equation (11), super-critical solution. Then the ratio \( \kappa := \frac{2a\gamma}{v_g} \) can be rewritten as a function of \( \langle E \rangle \) as

\[
\kappa = \frac{2a\gamma}{v_g} \quad = 2\sqrt{g(\langle E \rangle)}, \quad g(\langle E \rangle) := \left( 1 + (2a)^2 (\langle E \rangle^2 - m^2) \right) \left( \frac{\langle E \rangle^2}{\langle E \rangle^2 - m^2} \right).
\]

Considering that \( g(\langle E \rangle) \) is a monotonically decreasing function for \( \langle E \rangle > m \), and that \( g(\langle E \rangle) \to 1 \) for \( \langle E \rangle \to \infty \), we eventually find that

\[
\kappa > 2.
\]

- **Remark 4** By equation (14), the relation \( 2a\hat{r} \ll 1 \) is realized for \( m \gg a^{-1}, E_{\text{NR}} \). Thus, the
limit of \( m \to 0 \) is incompatible with \( 2a \ell^0 \ll 1 \). Actually, in the limit of \( m \to 0 \), the right-hand side of equation (14) amounts to unity, so that the expansion equation (13) is not a good approximation.

3.2.2. Case for \( 2a \ell^0 \gg 1 \). In this limit, \( X^2 \) can be parameterized as

\[
X^2 = 1 - \varepsilon \quad (0 < \varepsilon \ll 1).
\]

Substituting equation (15) into equation (12) and neglecting \( O(\varepsilon^2) \), we obtain

\[
\varepsilon = \frac{m^2}{\ell^2 + E_{NR}^0}.
\]

Hence, the lower bound for \( \kappa \) can be written as

\[
\kappa \geq 2a \ell
\]

\[
= 2a \tanh^{-1} \sqrt{1 - \varepsilon}
\]

\[
= 4 \log(4/\varepsilon) + O(\varepsilon) \quad (0 < \varepsilon \ll 1)
\]

where the first inequality comes from \( v_g \leq 1 \). Hence we obtain \( \kappa \gg 1 \) for \( 2a \ell^0 \gg 1 \).

3.2.3. Case for \( 2a \ell^0 = O(1) \). By \( v_g \leq 1 \), it follows from \( \kappa \geq 2a \ell \) that

\[
\kappa \gg 2.
\]

In either case, \( \kappa \) satisfies \( \kappa \geq 2 \). Considering that \( \tanh 2a \ell^0 \) is a monotonically increasing function of \( 2a \ell^0 \), we claim from equation (11) that \( \kappa \gg 2 \) should hold for all values of \( 2a \ell^0 \).

4. Conclusion
We have calculated, based on weak quantization, the lifetime \( \tau \) for the metastable 1D Dirac fermion under a potential barrier (with its height \( V_0 \)), to reveal that \( \tau \) and the tunneling time \( \Delta t \) are related as \( \Delta t \geq 2\tau \) in the limit of \( V_0 \gg m \). The relation of \( \Delta t \geq 2\tau \) indicates that \( P := e^{-\Delta t/\tau} \leq e^{-2} \approx 0.135 \), where \( P \) represents the probability that the electrons in the metastable state remain their state during the tunneling time. As \( m \) decreases, \( P \) tends to be smaller and behaves as \( P \sim (m/\log m)^2 \) by \( e^{-\kappa} \sim e^{-\log(4/\varepsilon)} \sim \varepsilon \sim (m/\ell)^2 \sim (m/\log m)^2 \). Thus in the regime of \( m \to 0 \), which corresponds to the normal incidence in graphene, it can be stated that almost all metastable electrons are decayed during the tunneling time. To obtain the decay mode, however, field theoretical consideration is necessary.

Acknowledgments
The authors are indebted to H Fujisaka for invaluable discussions.

Appendices

- A Breit-Wigner spectrum
Consider the density of states of a Breit-Wigner type as \( \hat{\rho}(E_0 + iv) := \left| \hat{\psi}(E_0 + iv) \right|^2 \), where \( \hat{\psi}(E_0 + iv) = \sum_n \frac{c_n}{E_0 - (E_n + iv + i\gamma_n/2)} \), with \( v > 0, \gamma_n \geq 0 \), and \( E_0, E_n \in \mathbb{R}; c_n \in \mathbb{C} \). In this case, \( \hat{\rho} \) can be rewritten as

\[
\hat{\rho}(E_0 + iv) = \text{Im} \left[ \hat{\mathcal{W}}(E_0 + iv) \right],
\]

where \( \hat{\mathcal{W}}(E_0 + iv) := \sum_n \gamma_n \frac{1}{(E_0 - (E_n + iv + i\gamma_n/2))^2} \), with \( \gamma_n = 2i\epsilon_n \sum_m c_m^* c_n \) and \( E_{n+1} - E_n - il\gamma_n/2 \). Notice that \( \hat{\mathcal{W}}(E) \) (for \( E \in \mathbb{C} \)) is analytic for \( \text{Im} E > 0 \), and that it has a pole (of order one) at \( E = (E_n) - i\gamma_n/2 \).

- B Calculation of \( W_+(E) \)
Let \( \psi^{(i)}(x) \)'s \( (i = 1, 2) \) be two solutions to equation (1) for \( x \geq 0 \) such that \( \psi^{(1)}(0) = (1)^0 \) and
\( \psi^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). The function \( W_+(E) \) is given by the boundary condition such that

\[
\Psi(x) := \psi^{(1)}(x) + W_+(E) \psi^{(2)}(x) \quad (x \geq 0)
\]
satisfies \( \lim_{x \to 0^+} \Psi(x) = 0 \) (for \( \text{Im} \ E > 0 \)). Taking account of the boundary condition of \( \psi^{(i)}(x) \) \( (i = 1, 2) \) at \( x = a \), we obtain equation (4).

**References**

[1] Katsnelson M I 2012 *Graphene: Carbon in Two Dimensions* (Cambridge: Cambridge University Press)

[2] Castro Neto A H, Guinea F, Peres N M R, Novoselov K S and Geim A K 2009 *Rev. Mod. Phys.* **81** 109-62

[3] Klein O 1929 *Z. Phys.* **53** 157-65

[4] Calogeracos A and Dombey N 1999 *Int. J. Mod. Phys. A* **14** 631-43

[5] Holstein B R 1998 *Am. J. Phys.* **66** 507-12

[6] Hund F 1941 *Z. Phys.* **117** 1-17

[7] Schwinger J 1951 *Phys. Rev.* **82** 664-79

[8] Krekora P, Su Q and Grobe R 2004 *Phys. Rev. Lett.* **92** 040406

[9] Gavrilov S P and Gitman D M 2016 *Phys. Rev. D* **93** 045002

[10] Titchmarsh E C 1961 *Proc. London Math. Soc.* **S3-11** 159-68

[11] Giachetti R 2008 *Phys. Rev. Lett.* **101** 190401

[12] Stander N, Huard B and Goldhaber-Gordon D 2009 *Phys. Rev. Lett.* **102** 026807

[13] Dombey N, Kennedy P and Calogeracos A 2000 *Phys. Rev. Lett.* **85** 1787-90