Semi-Inclusive Distributions in Statistical Models

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Abstract

The semi-inclusive properties of the system of neutral and charged particles with net charge equal to zero are considered in the grand canonical, canonical and micro-canonical ensembles as well as in micro-canonical ensemble with scaling volume fluctuations. Distributions of neutral particle multiplicity and charged particle momentum are calculated as a function of the number of charged particles. Different statistical ensembles lead to qualitatively different dependencies. They are being compared with the corresponding experimental data on multi-hadron production in $p + p$ interactions at high energies.

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In relativistic high-energy collisions many properties of produced particles follow simple rules of statistical mechanics. The single particle momentum spectrum approximately has the Boltzmann form, \( dN/d^{3}p \sim \exp[-(p^2+m^2)^{1/2}/T] \), in the local rest frame of produced matter [1]. The mean particle multiplicity of heavy particles \((m \gg T)\) is also governed by the Boltzmann factor, \( \langle N \rangle \sim \exp(-m/T) \). Here \( T, p, \) and \( m \) are the temperature parameter, the particle momentum, and its mass, respectively. The temperature parameter extracted from the data on \( p+p \) interactions is in the range of 160-190 MeV [2]. Thus, almost all particles are produced at low transverse momenta, \( p_T \), and with low masses \((p_T, m \leq 2 \text{ GeV})\).

However, the standard statistical approach fails to reproduce the KNO scaling [3] of multiplicity distributions observed in the data on \( p+p, p+\bar{p}, \) and \( e^+e^- \) collisions [4, 5, 6]. The other problems are a power law behavior of the single particle transverse momentum spectrum at large transverse momenta, \( dN/d^{3}p \sim p_T^{-K_p} \), and a power law dependence of a mean multiplicity of heavy particles, \( \langle N \rangle \sim m^{-K_m} \), where \( K_m \cong K_p - 3 \) [7]. In our previous paper [8] an extension of the standard statistical approach was suggested to a region of large transverse momenta and/or large particle masses \((p_T, m \geq 3 \text{ GeV})\) by taking into account volume fluctuations. The proposed model, the statistical ensemble with scaling volume fluctuations (MCE/sVF), allows to solve the above mentioned problems of the statistical approach.

Hadron production in high-energy collisions is characterized by two types of quantities: inclusive and semi-inclusive ones. Statistical models are usually used to describe inclusive quantities like mean multiplicity or mean transverse momentum. They are calculated by the summation over all microstates of the system with corresponding statistical weights. In the present paper we study selected properties of semi-inclusive quantities within statistical models. In this case the statistical summation is restricted by the additional condition, e.g. by a requirement that charged hadron multiplicity is fixed. The grand canonical, canonical and micro-canonical ensembles as well as MCE/sVF will be used. In particular, the mean multiplicity of neutral particles and average transverse momentum of charged particles are considered at a fixed charged particle multiplicity. The obtained model predictions are compared to the trends observed in the experimental data.

For the sake of simplicity, the system of non-interacting massless Boltzmann particles – neutral, positively charged and negatively charged – with the total net charge equal to zero \( Q = N_+ - N_- = 0 \) is considered. The degeneracy factors are assumed to be \( g_0 = g_+ = g_- = 1 \) and the temperature parameter is set to \( T = 160 \text{ MeV} \) for quantitative calculations.
The paper is organized as follows. The joint multiplicity distributions of neutral and negatively charged particles, correlations, semi-inclusive averages, and the effects of quantum statistic are calculated in Section II. Semi-inclusive momentum spectra are obtained and discussed in Section III. A comparison with available data is presented in Section IV. Summary presented in Section V closes the paper.

II. MULTIPlicitY DISTRIBUTIONS

A. Grand Canonical Ensemble

The grand canonical ensemble (GCE) is defined by the system volume $V$, temperature $T$, and charge chemical potential $\mu_Q$. The chemical potential $\mu_Q$ regulates an average value of the conserved charge $Q$. For the system with zero net charge considered here $\mu_Q$ is equal to zero. The mean particle multiplicities and the average energy in the GCE are:

$$\langle N_0 \rangle_{gce} = \langle N_+ \rangle_{gce} = \langle N_- \rangle_{gce} \equiv \overline{N} = V T^3/\pi^2,$$

$$\langle E \rangle_{gce} = 3T \langle N_0 \rangle_{gce} + 3T \langle N_- \rangle_{gce} + 3T \langle N_+ \rangle_{gce} \equiv \overline{E} = 9T \overline{N}.$$  

In the GCE the neutral and charged multiplicities $N_0$ and $N_+, N_-$ are uncorrelated and obey the Poisson distribution. Thus, the joint distribution of neutral $N_0$ and negatively charged particles $N_-$ is given by the product of two Poisson distributions which can be approximated by the product of two Gauss distributions at $\overline{N} \gg 1$:

$$P_{gce}(N_0, N_-) = \frac{\overline{N}^{N_0}}{N_0!} \exp(-\overline{N}) \times \frac{\overline{N}^{N_-}}{N_-!} \exp(-\overline{N})$$

$$\cong (2\pi \overline{N})^{-1/2} \exp \left[ -\frac{(N_0 - \overline{N})^2}{2\overline{N}} \right] \times (2\pi \overline{N})^{-1/2} \exp \left[ -\frac{(N_- - \overline{N})^2}{2\overline{N}} \right].$$

B. Canonical Ensemble

The canonical ensemble (CE) is described by the variables $V, T, Q$. The GCE expressions for average quantities remain valid in the CE at $\overline{N} \gg 1$. From the assumption $Q = 0$ follows that $N_+ = N_-$. Consequently, the distribution of $N_+$ and $N_-$ in the CE is narrower than in the GCE. The CE distribution of neutral particles remains the same as in the GCE as it is not constrained by charge conservation law. The joint distribution of neutral and negatively
charged particles is given by (4):

\[
P_{\text{ce}}(N_0, N_-) = \frac{N_0^{N_0} N_-^{2N_-}}{N_0! (N_-)!} \exp \left( -\frac{1}{2} \frac{N^2 - (N_0 - N_-)^2}{N} \right) \times \frac{I_0(2\sqrt{N})}{(N_-)!} \times \frac{1}{4\sqrt{\pi} N^{3/2}} \exp \left[ -\frac{(N_0 - N)^2}{2N} \right] \times \frac{1}{4\sqrt{\pi} N^{3/2}} \exp \left[ -\frac{(N_0 - N)^2}{2N} \right],
\]

where \( I_0 \) is the modified Bessel function.

C. Micro-Canonical Ensemble

The micro-canonical ensemble (MCE) is described by the variables \( V, E, Q \). The MCE partition function for \( N_0 \) neutral and \( N_+ = N_- \) positively and negatively charged massless particles reads (5):

\[
\Omega_{N_0, N_-}(E, V) = \frac{1}{N_0!} \frac{1}{(N_-)!} \left( \frac{V}{\pi^2} \right)^{N_0 + 2N_-} \frac{E^{3N_0 + 6N_- - 1}}{\Gamma(3N_0 + 6N_-)},
\]

where \( \Gamma \) is the Euler gamma function. The joint probability distribution of \( N_0 \) and \( N_- \) in the MCE is,

\[
P_{\text{mce}}(N_0, N_-) = \frac{\Omega_{N_0, N_-}(E, V)}{\Omega(E, V)},
\]

where \( \Omega(E, V) = \sum_{N_0, N_-} \Omega_{N_0, N_-}(E, V) \).

D. Average Multiplicities, Fluctuations and Correlations

The mean quantities in different statistical ensembles can be expressed as

\[
\langle X \rangle = \sum_{N_0, N_-} X(N_0, N_-) P(N_0, N_-).
\]

For the MCE distribution (6) one obtains:

\[
\langle N_0 \rangle_{\text{mce}} \cong \langle N_- \rangle_{\text{mce}} \cong \frac{1}{3\sqrt{3\pi}} (VE^3)^{1/4}.
\]

If the MCE energy equals the average energy of the GCE and CE, \( E = \overline{E} \), then the average MCE multiplicities (5) become equal to those in the GCE and CE (1). This reflects the equivalence of the GCE, CE, and MCE in the thermodynamic limit. However, the multiplicity distributions are different in these ensembles even in the thermodynamic limit. As shown in the Appendix A the MCE distribution (6) for \( N \gg 1 \) can be approximated as

\[
P_{\text{mce}}(N_0, N_-) \cong \frac{\sqrt{2}}{\pi N} \exp \left[ -\frac{(N_0 - N)^2}{N} - \frac{2(N_0 - N)(N_+ - N)}{N} - \frac{3(N_+ - N)^2}{N} \right].
\]
The distributions $P(N_0, N_-)$ in the GCE (3), CE (4), and MCE (6) can be written in a general form of the bivariate normal distribution,

$$P(N_0, N_-) = \frac{1}{2\pi \omega^0 \cdot \omega^- (1 - \rho^2)} \exp \left[ -\frac{1}{2} \left( \frac{(N_0 - \langle N_0 \rangle)^2}{\omega^0} - 2 \rho \frac{(N_0 - \langle N \rangle)(N_- - \langle N_- \rangle)}{\sqrt{\omega^0 \cdot \omega^-}} + \frac{(N_- - \langle N_- \rangle)^2}{\omega^-} \right) \right],$$

(10)

where $\omega^0$ and $\omega^-$ are the scaled variances defined as:

$$\omega^0 \equiv \frac{\langle N_0^2 \rangle - \langle N_0 \rangle^2}{\langle N_0 \rangle}, \quad \omega^- \equiv \frac{\langle N_-^2 \rangle - \langle N_- \rangle^2}{\langle N_- \rangle},$$

(11)

and $\rho$ is the correlation coefficient:

$$\rho \equiv \rho^{0-} \equiv \frac{\langle N_0 N_- \rangle - \langle N_0 \rangle \langle N_- \rangle}{\sqrt{[\langle N_0^2 \rangle - \langle N_0 \rangle^2] \cdot [\langle N_-^2 \rangle - \langle N_- \rangle^2]}} = \frac{\langle N_0 N_- \rangle - \langle N_0 \rangle \langle N_- \rangle}{\sqrt{\omega^0 \cdot \langle N_0 \rangle \cdot \omega^- \cdot \langle N_- \rangle}}.$$  

(12)

The averaging in Eqs. (11,12) is expressed according to Eq. (7).

The scaled variances in the GCE correspond to the uncorrelated Poisson distributions (3):

$$\omega^{0}_{gce} = \omega^{-}_{gce} = \omega^{+}_{gce} = 1,$$

(13)

and the correlation coefficient (12) is obviously equal to zero. One can similarly introduce the coefficients $\rho^{0+}$ and $\rho^{+-}$. They are also equal to zero in the GCE.

In the CE,

$$\omega^{0}_{ce} = 1, \quad \omega^{-}_{ce} = \omega^{+}_{ce} = \frac{1}{2},$$

(14)

for the distribution (4). The strong correlation, $N_+ = N_-$, in each microscopic state of the CE leads to the largest possible value of the correlation coefficient:

$$\rho^{+-}_{ce} = \frac{\langle N_+ N_- \rangle_{ce} - \langle N_+ \rangle_{ce} \langle N_- \rangle_{ce}}{\sqrt{\omega^{+}_{ce} \cdot \langle N_+ \rangle_{ce} \cdot \omega^{-}_{ce} \cdot \langle N_- \rangle_{ce}}} = 1.$$  

(15)

However, similar to the GCE, there are no correlations between neutral and charged particles, $\rho^{0\pm}_{ce} = 0$.

The scaled variances in the MCE are:

$$\omega^{0}_{mce} = \frac{3}{4}, \quad \omega^{-}_{mce} = \omega^{+}_{mce} = \frac{1}{4}.$$  

(16)

They reflect the suppression of fluctuations of neutral particles in the MCE comparing to the GCE and CE, and stronger suppression of fluctuations of charged particles comparing to the CE. The correlation coefficient, $\rho^{+-}_{mce} = 1$, is the same as in the CE (15). The exact energy
conservation in the MCE leads to a rather strong anti-correlation between neutral and charged particles:

\[ \rho_{mce}^{0-} = \rho_{mce}^{0+} = -\frac{1}{\sqrt{3}} \approx -0.577. \]  (17)

In the large volume limit the multiplicity distribution of negatively charged particles in the GCE, CE, and MCE can be approximated by normal distribution [11]:

\[ P(N_{-}) = \sum_{N_{0}} P(N_{-}, N_{0}) \approx (2\pi\omega^{-N})^{-1/2} \exp \left[ -\frac{(N_{-} - N)^2}{2\omega^{-N}} \right], \]  (18)

with \( \omega_{gce}^{-} = 1 \), \( \omega_{ce}^{-} = 1/2 \), and \( \omega_{mce}^{-} = 1/4 \) in the GCE, CE and MCE ensembles, respectively.

The \( N_{-} \) distribution in these ensembles is presented in Fig. 1 left. As the considered system has zero charge, the distributions \( P(N_{+}) \) are equal to \( P(N_{-}) \) ones in all statistical ensembles.

FIG. 1: (Color online) Examples of the multiplicity distributions (left) and the inclusive momentum spectra (right) of negatively charged particles obtained within the GCE, CE, MCE and MCE/sVF. The distributions are calculated assuming \( \overline{N} = 10 \) and \( T = 160 \) MeV (see text for details).

The neutral particle multiplicity distribution reads:

\[ P(N_{0}) = \sum_{N_{-}} P(N_{-}, N_{0}) \approx (2\pi\omega^{0-N})^{-1/2} \exp \left[ -\frac{(N_{0} - \overline{N})^2}{2\omega^{0-N}} \right], \]  (19)

with \( \omega_{gce}^{0} = \omega_{ce}^{0} = 1 \) and \( \omega_{mce}^{0} = 3/4 \).
E. MCE with scaling Volume Fluctuations

The MCE with scaling volume fluctuations (MCE/sVF) \[8\] is described by the variables \(E\), \(Q\) and \(V\) as well as the distribution function defining the scaling volume fluctuations\(^1\). All quantities calculated within the MCE/sVF will be denoted by the subscript \(\alpha\). For a description of the volume fluctuations it is convenient to introduce an auxiliary variable \(y\) as:

\[ y \equiv (V/V_0)^{1/4}, \tag{20} \]

and describe the scaling volume fluctuations by the scaling function \(\psi_\alpha(y)\) (see Ref. [8] for details). Experimental data on the multiplicity distribution of charged hadrons in \(p + p\) interactions suggest a simple analytical form of the \(\psi_\alpha(y)\) function \([8, 14, 15]\):

\[ \psi_\alpha(y) = \frac{k^k}{\Gamma(k)} y^{k-1} \exp(-k y), \tag{21} \]

with \(k = 4\) and \(\Gamma(k)\) being the Euler gamma function.

The joint \(N_0\) and \(N_-\) distribution in the MCE/sVF equals to:

\[ P_\alpha(N_0, N_-) = \int_0^\infty dy \ P_{mce}(N_0, N_-) \psi_\alpha(y), \tag{22} \]

where \(P_{mce}(N_0, N_-)\) is given by Eq. (6). The analytical approximations for \(P_\alpha(N_0, N_-)\) are discussed in Appendix \[B\]. The inclusive mean multiplicities in the MCE/sVF are:

\[ \langle N_- \rangle_\alpha = \sum_{N_-,N_0} N_- P_\alpha(N_0, N_-) \cong \overline{N} , \quad \langle N_0 \rangle_\alpha = \sum_{N_-,N_0} N_0 P_\alpha(N_0, N_-) \cong \overline{N} , \tag{23} \]

and, thus, they coincide with those in the GCE, CE, and MCE at \(\overline{N} \gg 1\). The inclusive multiplicity distributions in the MCE/sVF are:

\[ P_\alpha(N_-) = \sum_{N_0} P_\alpha(N_0, N_-) \cong \frac{1}{\overline{N}} \psi_\alpha\left(\frac{N_-}{\overline{N}}\right) , \tag{24} \]

\[ P_\alpha(N_0) = \sum_{N_-} P_\alpha(N_0, N_-) \cong \frac{1}{\overline{N}} \psi_\alpha\left(\frac{N_0}{\overline{N}}\right). \tag{25} \]

The \(P_\alpha(N_-)\) distribution is shown in Fig. [11]. It is significantly broader than the corresponding distributions for the GCE, CE, and MCE. The scaled variance for negatively charged and neutral particles is:

\[ \omega_-^\alpha \cong \frac{1}{k} \overline{N} + \omega_-^{mce} , \quad \omega_0^\alpha \cong \frac{1}{k} \overline{N} + \omega_0^{mce} . \tag{26} \]

\(^1\) Statistical ensembles with fluctuating extensive quantities are discussed in recent papers [12, 13].
Thus, in the MCE/sVF, because of the scaling volume fluctuations, the scaled variance increases in proportion to the mean multiplicity, while the scaled variance is approximately independent of mean multiplicity, $\omega \approx const$, in the GCE, CE, and MCE.

For illustration of the previously discussed properties the joint $N_0$ and $N_-$ distributions calculated within the GCE (3), CE (4), MCE (6), and MCE/sVF (22) are shown in Fig. 2. The multiplicities of neutral and negatively charged particles are uncorrelated in the GCE and CE (see Fig. 2 top panels). They are anti-correlated and correlated in the MCE and MCE/sVF,
respectively. A positive correlation between $N_0$ and $N_-$ in the MCE/sVF is caused by the scaling volume fluctuations. Note that finite size effects are seen in Fig. 2. For example, the Poisson distribution (3) is significantly asymmetric for large deviations from $\overline{N}$.

![Graphs of N_0 vs N_- for different ensembles](image)

FIG. 3: (Color online) Examples of the dependence of the neutral particle mean multiplicity on the multiplicity of negatively charged particles calculated within the GCE (top left), CE (top right), MCE (bottom left) and MCE/sVF (bottom right). The distributions are calculated assuming $\overline{N} = 10$ (see text for details).

The distributions in Fig. 2 are rather different. Thus, it is obvious that the dependence on $N_-$ of the semi-inclusive mean multiplicity of neutral particles defined as:

$$\langle N_0 \rangle^* \equiv \frac{\sum_{N_0} N_0 \, P(N_0, N_-)}{\sum_{N_0} P(N_0, N_-)} ,$$

(27)

is different in various ensembles. Namely, it is independent of $N_-$ in the GCE and CE:

$$\langle N_0 \rangle^*_{gce} = \langle N_0 \rangle^*_{ce} = \overline{N} .$$

(28)
In the MCE, $\langle N_0 \rangle_{mce}$ monotonically decreases with increasing $N_-$ and equals approximately (see Appendix C):

$$\langle N_0 \rangle_{mce}^* \cong \frac{4}{3} \left( \frac{N_-}{3N} \right)^3 . \quad (29)$$

Finally, the positive correlation between $N_0$ and $N_-$ in the MCE/sVF leads to an approximately linear increase$^2$ $\langle N_0 \rangle_\alpha^*$ with increasing $N_-:

$$\langle N_0 \rangle_\alpha^* \cong N_- . \quad (30)$$

The semi-inclusive mean multiplicities of neutral particles calculated within the GCE, CE (28), MCE (29), and MCE/sVF (30) are shown as functions of $N_-$ in Fig. 3.

### F. Quantum Statistics

In this Subsection we illustrate the effects of quantum statistics in the GCE, CE, and MCE using the micro-correlator method of Ref. [16]. The mean multiplicities in the CE or MCE are approximately the same as in the GCE. They are given by the sum of mean occupation numbers with momentum $p$ [17]:

$$\langle N^a \rangle_{ce} \cong \langle N^a \rangle_{mce} \cong \langle N^a \rangle_{gce} \equiv \sum_p \langle n_{ap} \rangle = \sum_p \frac{1}{\exp (\epsilon_p / T) - \gamma} . \quad (31)$$

where $a$ is +, −, or 0 and denotes positive, negative or neutral particles, $\epsilon_p = p$ is one particle energy for massless particles, $\gamma = +1$ for Bose statistics, $\gamma = -1$ for Fermi statistics, and $\gamma = 0$ corresponds to the Boltzmann approximation used throughout the present paper. We study a neutral system, thus, chemical potentials are zero in Eq. (31), and the average occupation numbers $\langle n_{ap} \rangle \equiv \langle n_p \rangle$ are therefore the same for neutral and charged particles. The summation over discrete levels can be substituted by the integration in the thermodynamic limit:

$$\sum_p \ldots \cong \frac{V}{2\pi^2} \int_0^\infty p^2 dp \ldots . \quad (32)$$

The fluctuations and correlations in the GCE, CE and MCE are very different, nevertheless they can be expressed in terms of the fluctuations of the occupation numbers of a single momentum level in the GCE,

$$\langle (\Delta n_{ap}^2)_{gce} \rangle \equiv v_{ap}^2 = \langle n_p \rangle (1 + \gamma \langle n_p \rangle) . \quad (33)$$

$^2$ A linear increase of $\langle N_0 \rangle_{mce}^*$ with $N_-$ is due to the assumption of massless particles. For non-zero value of mass $m$ the relation (30) is changed at large $N_-$. The maximum value of $N_-$ is $N_-^{max} = E/2m$. Eq. (30) remains approximately valid for $N_- \ll N_-^{max}$, but $\langle N_0 \rangle_\alpha^*$ approaches zero at $N_- \rightarrow N_-^{max}$.
This is a main advantage of the micro-correlator method. It allows to calculate the fluctuations and correlations using the following micro-correlators [18]:

\[
\langle \Delta n^a_p \Delta n^b_k \rangle_{gce} = v^a_p v^b_k \delta_{pk} \delta_{ab},
\]
\[
\langle \Delta n^a_p \Delta n^b_k \rangle_{ce} = v^a_p v^b_k \delta_{pk} \delta_{ab} - q^a q^b \sum_{p,a} v^a_p v^2_{p,a} q^{a2},
\]
\[
\langle \Delta n^a_p \Delta n^b_k \rangle_{mce} = v^a_p v^b_k \delta_{pk} \delta_{ab} - \frac{v^a_{2b} v^b_{2k}}{|A|} \left[ q^a q^b \sum_{p,a} \epsilon^2_p + \epsilon_p \epsilon_k \sum_{p,a} v^a_p v^{a2} q^{a2} \right],
\]

where \( \delta_{pk} \) and \( \delta_{ab} \) are the Kronecker delta symbols, \( q^a \), \( q^b \) are particle charges, \( \pm 1 \) or 0, and

\[
|A| \equiv \left( \sum_{p,a} v^a_p \epsilon^2_p \right) \cdot \left( \sum_{p,a} v^a_p q^{a2} \right)
\]

is the correlation determinant.

The variance and correlations in the GCE, CE and MCE are calculated as the sums (integrals) over momentum of the corresponding micro-correlators [34-36]:

\[
\langle (\Delta N^2_a) \rangle = \sum_{p,k} \langle \Delta n^a_p \Delta n^a_k \rangle, \quad \langle \Delta n_a^a \Delta n_b^b \rangle = \sum_{p,k} \langle \Delta n^a_p \Delta n^b_k \rangle.
\]

One obtains for Bosons:

\[
\omega^\pm_{gce} = \frac{\sum_p v^2_p}{\sum_p \langle n_p \rangle} \approx \frac{\int_0^\infty p^2 dp \, e^{p/T} (e^{p/T} - 1)^{-1}}{\int_0^\infty p^2 dp \, (e^{p/T} - 1)^{-1}} = \frac{\pi^2}{6 \zeta(3)} \approx 1.368,
\]
\[
\omega^\pm_{ce} = \frac{\sum_p v^2_p}{\sum_p \langle n_p \rangle} - \frac{\left( \sum_p v^2_p \right)^2}{\sum_p \langle n_p \rangle \sum_{p,a} v^2_p q^{a2}} = \frac{\sum_p v^2_p}{2 \sum_p \langle n_p \rangle} = \frac{1}{2} \omega^\pm_{gce} \approx 0.684,
\]
\[
\omega^\pm_{mce} = \frac{\sum_p v^2_p}{\sum_p \langle n_p \rangle} - \frac{\left( \sum_p v^2_p \right)^2}{\sum_p \langle n_p \rangle \sum_{p,a} v^2_p q^{a2}} - \frac{\left( \sum_p v^2_p \epsilon_p \right)^2}{\sum_p \langle n_p \rangle \sum_{p,a} v^2_p \epsilon^2_p}
\]
\[
= \frac{\pi^2}{12 \zeta(3)} - \frac{45 \zeta(3)}{2 \pi^4} \approx 0.407,
\]

where \( \zeta(3) \approx 1.202 \) is the zeta Riemann function. In calculating [39-41] we use \( \langle n^a_p \rangle \equiv \langle n_p \rangle \) and \( v^+_{p2} = v^-_{p2} = v^{02} \equiv v^0_p \) for a neutral system. This gives: \( \sum_{p,a} v^2_p q^{a2} = 2 \sum_p v^2_p \) and \( \sum_{p,a} v^2_p \epsilon^2_p = 3 \sum_p v^2_p \epsilon^2_p \). Similarly one can get the results for Fermions:

\[
\omega^\pm_{gce}^{\text{Fermi}} = \frac{\pi^2}{9 \zeta(3)} \approx 0.912,
\]
\[
\omega^\pm_{ce}^{\text{Fermi}} = \frac{1}{2} \omega^\pm_{gce}^{\text{Fermi}} \approx 0.456,
\]
\[
\omega^\pm_{mce}^{\text{Fermi}} = \frac{\pi^2}{18 \zeta(3)} - \frac{135 \zeta(3)}{7 \pi^4} \approx 0.218.
\]
The scaled variances for neutral particles are:

\[ \omega_{0 \text{Bose}}^{\text{gce}} = \omega_{0 \text{Bose}}^{\text{ce}} = \omega_{0 \text{Bose}}^{\text{mce}} \approx 1.368, \quad \omega_{0 \text{Fermi}}^{\text{gce}} = \omega_{0 \text{Fermi}}^{\text{ce}} = \omega_{0 \text{Fermi}}^{\text{mce}} \approx 0.912, \quad (45) \]

\[ \omega_{0 \text{Bose}}^{\text{mce}} = \frac{\pi^2}{6 \zeta(3)} - \frac{45 \zeta(3)}{2 \pi^4} \approx 1.091, \quad \omega_{0 \text{Fermi}}^{\text{mce}} = \frac{\pi^2}{9 \zeta(3)} - \frac{135 \zeta(3)}{7 \pi^4} \approx 0.674. \quad (46) \]

The correlation coefficients can be also calculated using micro-correlator method:

\[ \rho^{ab} = \frac{\langle \Delta N_a \Delta N_b \rangle}{\sqrt{\langle N_a \rangle \langle N_b \rangle} \sqrt{\omega_a \omega_b}} = \frac{1}{\sqrt{\omega_a \omega_b}} \frac{\sum_{p,k} \langle \Delta n_a^p \Delta n_b^k \rangle}{\sum_p \langle n_p \rangle \sum_p v_p^2 \epsilon_p^2}. \quad (47) \]

There are no correlations between positively and negatively charged particles in the GCE, \( \rho_{gce}^{+ -} = 0 \), and there is the absolute correlation in the CE and MCE, \( \rho_{ce}^{+ -} = \rho_{mce}^{+ -} = 1 \). These values are the same for any type of statistics. The correlation between charged and neutral particles, \( \rho_{0 -}^{0 -} \), is zero for the GCE and CE, but it has a negative value for the MCE. The correlation coefficient \( \rho_{mce}^{0 -} \) reads:

\[ \rho_{mce}^{0 -} = -\frac{1}{\sqrt{\omega_0 \omega_-}} \frac{\left( \sum_p v_p^2 \epsilon_p^2 \right)^2}{3 \sum_p \langle n_p \rangle \sum_p v_p^2 \epsilon_p^2}. \quad (48) \]

The Eq. (48) gives for Bosons and Fermions:

\[ \rho_{mce}^{0 - \text{Bose}} = -\sqrt{2} \left[ \left( \frac{\pi^6}{135 \zeta(3)^2} - 2 \right) \left( \frac{\pi^6}{135 \zeta(3)^2} - 1 \right) \right]^{-1/2} \approx -0.417, \quad (49) \]

\[ \rho_{mce}^{0 - \text{Fermi}} = -\sqrt{2} \left[ \left( \frac{7 \pi^6}{1215 \zeta(3)^2} - 2 \right) \left( \frac{7 \pi^6}{1215 \zeta(3)^2} - 1 \right) \right]^{-1/2} \approx -0.621. \quad (50) \]

The scaled variances and correlation coefficients for the Boltzmann approximation can be obtained from Eqs. (39-41,48) replacing \( \gamma = 0 \) in Eq. (41):

\[ \omega_{0 \text{Boltz}}^{\text{gce}} = 1, \quad \omega_{0 \text{Boltz}}^{\text{ce}} = \frac{1}{2}, \quad \omega_{0 \text{Boltz}}^{\text{mce}} = \frac{1}{4}, \quad (51) \]

\[ \omega_{0 \text{Boltz}}^{\text{gce}} = \omega_{0 \text{Boltz}}^{\text{ce}} = 1, \quad \omega_{0 \text{Boltz}}^{\text{mce}} = \frac{3}{4}, \quad \rho_{mce}^{0 - \text{Boltz}} = -\frac{1}{\sqrt{3}} \approx -0.577. \quad (52) \]

They, of course, coincide with our previous results \( [13,14,16,17] \) for the Boltzmann statistics.

One can conclude that Bose statistics makes the fluctuations always bigger and Fermi statistics – smaller: \( \omega_{\text{Fermi}} < \omega_{\text{Boltz}} < \omega_{\text{Bose}} \), in all statistical ensembles (GCE, CE, MCE) and for all types of particles (positive, negative, neutral). The strongest effect for the neutral system in equilibrium \( ^3 \) is for the scaled variance of charged Bosons in the MCE: \( \omega_{mce}^{0 - \text{Bose}} / \omega_{mce}^{0 - \text{Boltz}} \approx 1.6. \)

\(^3\) The effects of quantum statistics for fluctuations can be much stronger at non-zero chemical potential. The scaled variance of Bosons may rise up to infinity near the point of Bose-condensation \( [19] \).
The only correlation coefficient which fills an influence of quantum statistics is $\rho^{0\pm}$ in the MCE: $\rho^{0\pm}_{\text{Fermi}} < \rho^{0\pm}_{\text{Boltz}} < \rho^{0\pm}_{\text{Bose}} < 0$. However, the quantum statistics does not change a sign of this correlation. Thus, the main features of the GCE, CE, and MCE fluctuations and correlations found in the Boltzmann approximation – constant values of $\omega^{\pm}$ and $\omega^0$ in the thermodynamic limit, strong correlations, $\rho^{+\pm}_{\text{ce}} = \rho^{+\pm}_{\text{mce}} = 1$, due to the exact charge conservation, and anti-correlation between neutral and charged particles, $\rho^{0\pm}_{\text{mce}} < 0$, due to the exact energy conservation – remain the same for Bose and Fermi statistics. The quantum statistics can not simulate the MCE/sVF effects: an increase of the scaled variances in proportion to the mean multiplicities (26) and a strong positive correlation, $\rho^{0\pm}_a \cong 1$, between neutral and charged particles (30). These new effects take place due to the scaling volume fluctuations in the MCE/sVF.

III. SEMI-INCLUSIVE MOMENTUM SPECTRA

In this section single particle momentum spectra of negatively charged particles are considered. The inclusive spectra are denoted as $F(p)$ and the semi-inclusive at fixed $N_-$ as $F^*(p)$. In both cases the spectra are normalized to unity: $\int_0^{\infty} p^2 dp F(p) = 1$ and $\int_0^{\infty} p^2 dp F^*(p) = 1$.

A. GCE and CE

The inclusive and semi-inclusive momentum spectra in the GCE and CE are equal and read

$$F_{\text{gce}}(p) = F^*_{\text{gce}}(p) = F_{\text{ce}}(p) = F^*_{\text{ce}}(p) = \frac{1}{2T^3} \exp \left( - \frac{p}{T} \right).$$

(53)

This follows from the fact that a single particle momentum spectrum and the particle multiplicity are uncorrelated in these ensembles.

\footnote{This is true for the used here Boltzmann statistics. The form of momentum spectrum in the CE becomes different from that in the GCE for quantum gases in finite volumes. For the isospin conservation this was demonstrated in Ref. [20].}
B. Micro-Canonical Ensemble

The inclusive single particle momentum spectrum of negatively charged particles in the MCE reads:

\[
F_{mce}(p) = \frac{1}{N} \frac{1}{2E^3} \sum_{N_0=0}^{\infty} \sum_{N_-=1}^{\infty} \frac{N_- (3N_0 + 6N_- - 1)!}{(3N_0 + 6N_- - 4)!} \left( 1 - \frac{p}{E} \right)^{3N_0+6N_- - 4} P_{mce}(N_0, N_-), \tag{54}
\]

see also Ref. \[8\]. The spectrum (54) approximately has the Boltzmann form (53) at momenta \( p \) significantly smaller than the total system energy \( E \). However, large deviations from (53)
are observed close to the threshold, \( p = E \), where the MCE spectrum approaches zero. The inclusive spectra \( F(p) \) in the GCE, CE \((53)\), and MCE \((54)\) are shown in Fig. 1 right.

The semi-inclusive momentum spectrum at a fixed number of negatively charged particles is given by:

\[
F^\ast_{mce}(p) = \frac{C}{2E^3} \sum_{N_0=0}^\infty \frac{(3N_0 + 6N_− - 1)!}{(3N_0 + 6N_− - 4)!} \left( 1 - \frac{p}{E} \right)^{3N_0 + 6N_− - 4} P_{mce}(N_0, N_-), \quad (55)
\]

where \( N_- \geq 1 \) and \( C = [\sum_{N_0} P_{mce}(N_0, N_-)]^{-1} \) is the normalization factor. Examples of the \( F^\ast_{mce}(p) \) spectrum for three values of \( N_- \) are shown in Fig. 4 bottom, left. The semi-inclusive spectra in the MCE \((55)\) have the Boltzmann form for \( p \ll E \),

\[
F^\ast_{mce}(p) \sim \frac{1}{2T^\ast_{mce}} \exp \left( -\frac{p}{T^\ast_{mce}} \right), \quad (56)
\]

and the inverse slope parameter \( T^\ast_{mce} \) depends on \( N_- \). This dependence is presented in Fig. 5 for the MCE and other ensembles studied here. In the GCE and CE the \( T^\ast \) is independent of \( N_- \) and equal to the inverse slope parameter of the inclusive spectrum, \( T^\ast_{gcce} = T^\ast_{ce} = T = 160 \text{ MeV} \). In the MCE the inverse slope parameter decreases with increasing \( N_- \) and it crosses the line \( T = 160 \text{ MeV} \) at \( N_- = N \). Thus, the inclusive momentum spectrum \( F_{mce}(p) \) \((54)\) coincides with the semi-inclusive one, \( F^\ast_{mce}(p) \) \((55)\), at the crossing point.

C. MCE with scaling Volume Fluctuations

The inclusive single particle momentum spectrum in the MCE/sVF equals to:

\[
F_\alpha(p) = \frac{1}{N} \frac{1}{2E^3} \sum_{N_0=0}^\infty \sum_{N_-=1}^\infty \frac{N_- (3N_0 + 6N_- - 1)!}{(3N_0 + 6N_- - 4)!} \left( 1 - \frac{p}{E} \right)^{3N_0 + 6N_- - 4} P_{mce}(N_0, N_-), \quad (57)
\]

The structure of Eq. \((57)\) is the same as the structure of the corresponding Eq. \((54)\) for the MCE. The only difference is in the form of the multiplicity distribution, namely \( P_{\alpha}(N_0, N_-) \) is used in Eq. \((57)\) instead of \( P_{mce}(N_0, N_-) \) in Eq. \((54)\). The inclusive spectrum \( F_\alpha(p) \) is shown in Fig. 1 right. It can be well approximated by the power law dependence:

\[
F_\alpha(p) \cong \frac{k^4}{2\Gamma(k)} T^{k+1} (p + Tk)^{-k-4} \cong 11.27 \text{ GeV}^5 (p + 4T)^{-8}, \quad (58)
\]

where \( k = 4 \) is used in the last expression (see Ref. [8]).

The semi-inclusive momentum spectrum in the MCE/sVF reads:

\[
F^\ast_\alpha(p) = \frac{C}{2E^3} \sum_{N_0=0}^\infty \frac{(3N_0 + 6N_- - 1)!}{(3N_0 + 6N_- - 4)!} \left( 1 - \frac{p}{E} \right)^{3N_0 + 6N_- - 4} P_{\alpha}(N_0, N_-), \quad (59)
\]
FIG. 5: (Color online) The dependence of the inverse slope parameter of the momentum spectra on the multiplicity of negatively charged particles $N_-$ calculated within the GCE (top left), CE (top right), MCE (bottom left) and MCE/sVF (bottom right). The distributions are calculated assuming $N = 10$ and $T = 160$ MeV (see text for details).

where $C = \left[ \sum_{N_0} P_\alpha (N_0, N_-) \right]^{-1}$. The spectrum $F_\alpha^*(p)$ is plotted in Fig. 4 for several values of $N_-$. Similar to the MCE spectrum (56) the MCE/sVF one can be approximated as:

$$F_\alpha^*(p) \simeq \frac{1}{2T_\alpha^3} \exp \left(-\frac{p}{T_\alpha^*} \right),$$

with the inverse slope parameter $T_\alpha^*$. The dependence of $T_\alpha^*$ on $N_-$ is shown in Fig. 5 bottom right. The MCE/sVF temperature $T_\alpha^*$ decreases with increasing $N_-$. For $N_- = N$ the inverse slope parameter $T^*$ is the same in the MCE and MCE/sVF and equals to the parameter $T$ in the GCE and CE. The analytical approximations of the dependence of $T^*$ on $N_-$ in the MCE and MCE/sVF are presented in Appendix C.
IV. COMPARISON WITH DATA

A quantitative comparison of the discussed statistical models with the experimental data requires a significant additional effort, which is far beyond the scope of this paper. In particular one should introduce proper degrees of freedom and all related conservation laws as well as a longitudinal collective motion of matter. Nevertheless, a qualitative comparison seems to be useful already, and consequently it is presented in this section.

An excellent review of the experimental data on semi-inclusive properties of $p+p$ interactions at high energies can be found in Ref. [21]. The volume scaling function as well as the value of the temperature parameter used in this work in quantitative calculations were selected in order to approximately reproduce the results on the inclusive distributions in $p+p$ interactions. Consequently, a comparison between these data and the model results is justified. Clearly, as the influence of global conservation laws is crucial for the considered statistical approaches the data referring to the semi-inclusive properties measured in the full phase-space are of primary importance. Several features of these data are well established [21]. Two of them are relevant for the comparison with the models discussed here, namely:

(A) the mean multiplicity of produced $\pi^0$ mesons increases with increasing multiplicity of negatively charged particles and

(B) the average transverse momentum of negatively charged particles decreases with increasing multiplicity of these particles.

The property (B) needs several comments. First, it is well experimentally established [21] for charged hadron multiplicity and mean transverse momentum measured in full phase-space in $p+p$ interactions at 6.6-400 GeV/c. Clearly, the full phase-space results are relevant when effects related to the global conservation laws are of interest. Second, the mean transverse momentum increases with increasing multiplicity when mid-rapidity values are considered [22]. An interpretation of this dependence is, however, beyond the scope of this paper.

For the purpose of the comparison between considered models and data the mean transverse momentum was calculated as:

$$
\langle p_T \rangle^* = 2 \int_0^\infty dp_T p_T^2 \int_{-\infty}^\infty dy p_T F^*(p)
$$

$$
= \frac{1}{T^3} \int_0^\infty dp_T p_T^2 \int_{-\infty}^\infty dy p_T \exp \left( - \frac{p_T \cosh y}{T^*} \right) = \frac{3\pi}{4} T^* \approx 2.36 T^*. \quad (61)
$$

The mean multiplicity of neutral particles calculated within the models was identified with the mean $\pi^0$ multiplicity.
FIG. 6: (Color online) The mean multiplicity of neutral particles (left) and the transverse momentum of negatively charged particles (right) as a function of the multiplicity of negatively charged particles. The experimental data on $p + p$ interactions at 205 GeV/c \cite{23}, left and \cite{24}, right are indicated by closed circles. The predictions of the GCE, CE, MCE and MCE/sVF are shown by the lines. The calculations are performed assuming $\overline{N} = 3$ and $T = 160$ MeV (see text for details).

The model predictions are summarized in Fig. 6. In the GCE and CE $\langle N_0 \rangle^*$ and $\langle p_T \rangle^*$ are independent of $N_-$. The MCE reproduces property (B), but leads to decrease of mean multiplicity $\langle N_0 \rangle^*$ with increasing $N_-$. Both features (A) and (B) are qualitatively reproduced in the MCE/sVF.

V. SUMMARY

Semi-inclusive distributions for the system of neutral and charged massless particles with net charge equal to zero are considered in the grand canonical, canonical, micro-canonical ensembles, and in the micro-canonical ensemble with scaling volume fluctuations (MCE/sVF). The MCE/sVF has been included in the present study as it is the only statistical ensemble which reproduces the KNO scaling of multiplicity distributions and the power law behavior of the inclusive transverse momentum spectra measured in $p + p$ interactions. The mean multiplicity of neutral particles and momentum spectra of charged particles are calculated at fixed charged particle multiplicity $N_-$. Different statistical ensembles lead to qualitatively different results for these semi-inclusive quantities even in the large volume limit. In other words, the semi-inclusive quantities can be different in different statistical ensembles despite of the ensemble
thermodynamical equivalence.

The obtained model predictions are compared with the experimental data on $p + p$ inelastic interactions at high energies. The MCE/sVF follows the trends observed in the data. This demonstrates the role of volume fluctuations in the system with exact energy and charge conservation. However, the detailed comparison with the experimental data is far beyond the scope of this paper. The conclusive comparison with the experimental results would require inclusion in the statistical model calculations several neglected effects. In particular, the hadron masses and quantum numbers, isospin symmetry, quantum statistics, and resonance decays should be taken into account.

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**APPENDIX A**

The bivariate normal approximation $P_{mce}(N_0, N_-)$ can be derived as follows. Eq. (5) can be rewritten as:

$$\Omega_{N_0,N_-}(E,V) = \frac{1}{E} \exp\left[f(N_0, N_-)\right],$$  \hspace{1cm} (A1)

where

$$f(N_0, N_-) = (N_0 + 2N_-) \ln[A] - \ln[N_0!] - 2 \ln[N_-!] - \ln[(3N_0 + 6N_- - 1)!],$$  \hspace{1cm} (A2)

and $A = VE^3/\pi^2$. Using the Stirling formula, $\ln(N!) \approx (N + 1/2) \ln(N) - N + \ln(2\pi)/2$ at $N \gg 1$, the r.h.s. of Eq. (A2) can be expanded with respect to $N_0$ and $N_-$ near the maximum of $f$. Then, the mean multiplicities can be calculated from the condition $\partial f/\partial N_0 = \partial f/\partial N_- = 0$. Second derivatives of $f$ with respect of $N_0$ and $N_-$ at the point of maximum are:

$$\frac{\partial^2 f}{\partial N_0^2} \approx -\frac{2}{N}, \hspace{1cm} \frac{\partial^2 f}{\partial N_-^2} \approx -\frac{6}{N}, \hspace{1cm} \frac{\partial^2 f}{\partial N_0 \partial N_-} \approx -\frac{2}{N}.$$  \hspace{1cm} (A3)

and Eq. (9) follows.
APPENDIX B

Using the approximation (9) the integration over $y$ can be done analytically. Then the joint $N_0$ and $N_\alpha$ distribution in the MCE/sVF reads:

$$P_\alpha(N_0, N_-) \equiv \int_0^\infty dy \, p_{mce}(N_0, N_-) \, \psi_\alpha(y) \simeq \frac{k^k}{\Gamma(k)} \frac{2\sqrt{2}}{\pi N} \left( \frac{N_0^2 + 2N_0 N_- + 3N_-^2}{6N^2 + Nk} \right)^{(k-1)/2}$$

$$\times \exp(4N_0 + 8N_-) \, K_{1-k} \left[ 2 \sqrt{\left( 6 + \frac{k}{N} \right)} \, \left( N_0^2 + 2N_0 N_- + 3N_-^2 \right) \right],$$

where $K_{1-k}$ is the Bessel function of the second kind. Eq. (B1) can be simplified using the asymptotic expansion:

$$K_{1-k}(x) = \sqrt{\frac{\pi}{2x}} \, e^{-x} \left( 1 + \frac{4k^2 - 8k + 3}{8} \frac{1}{x} + O(x^{-2}) \right), \quad x \gg 1.$$  \hfill (B2)

Consequently,

$$P_\alpha(N_0, N_-) \simeq \frac{k^k}{\Gamma(k)} \frac{1}{\sqrt{\pi N}} \left( \frac{N_0^2 + 2N_0 N_- + 3N_-^2}{6N^2 + Nk} \right)^{(k-1)/2}$$

$$\times \frac{1}{\sqrt{N_0 + 2N_-}} \exp \left[ -k \frac{N_0 + 2N_-}{3N} - \frac{(N_0 - N_-)^2}{3N_-} \right].$$  \hfill (B3)

APPENDIX C

In the MCE with a fixed multiplicity $N_-$ the system temperature $T_{mce}^*$ can be found as follows. The mean multiplicity of neutral particles equals to $\langle N_0^* \rangle_{mce} = V T_{mce}^3 / \pi^2$, and their average energy is $\langle E_0^* \rangle_{mce} = 3T_{mce}^* \langle N_0^* \rangle_{mce}$. Thus, the total energy reads,

$$\frac{3VT_{mce}^4}{\pi^2} + 6N_- T_{mce}^* \approx E. \quad \text{(C1)}$$

The first term in the l.h.s. of Eq. (C1) corresponds to the average energy of neutral particles and the second term to that of charged particles. For the multiplicity of negatively charged particles close to the mean multiplicity one can solve approximately Eq. (C1) with respect to temperature. Denoting $\delta N_- = N_- - \overline{N}$ and $\delta T = T - T_{mce}^*$ the solution reads:

$$\delta T \simeq - T \frac{\delta N_-}{2\overline{N} + \overline{N}} \quad \text{or} \quad \delta T \simeq - T \frac{4}{3} \frac{N_-}{\overline{N}} \quad \text{or} \quad T_{mce}^* \simeq T \left( \frac{4}{3} - \frac{N_-}{\overline{N}} \right). \quad \text{(C2)}$$
Consequently, one gets:

\[
\langle N_0^* \rangle_{mce} = \frac{1}{\pi^2} V T^x_m c e \approx \sqrt{N} \left( \frac{4}{3} - \frac{N_-}{3N} \right)^3 .
\]  

(C4)

In the MCE/sVF at fixed \(N_-\) one finds,

\[
3T^*_\alpha \langle N_0^* \rangle_{\alpha} + 6 N_- T^*_m c e = E .
\]  

(C5)

Using Eq. (30) this gives,

\[
T^*_\alpha \approx T \frac{N}{N_-} .
\]  

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