A survey on NIST PQ signatures

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July 26, 2021

Shor’s shockingly fast quantum algorithm for solving the period-finding problem is a threat for the most common public-key primitives, as it can be efficiently applied to solve both the Integer Factorisation Problem and the Discrete Logarithm Problem. In other words, as soon as a large-enough quantum computer is born, many once-secure protocols have to be replaced by still-secure alternatives. Instead of relying, for example, on the RSA protocol, the Diffie-Hellman key-exchange or the (Elliptic Curve) Digital Signature Algorithm, many researchers moved their attention to the design and analysis of primitives which are yet to be broken by quantum algorithms.

The urgency of the threat imposed by quantum computers led the U.S. National Institute of Standards and Technology (NIST) to open calls for both Post-Quantum Public-Keys Exchange Algorithms and Post-Quantum Digital Signature Algorithms \cite{32}. This new NIST standardisation process started in 2016, has involved hundreds of researchers, has seen 37 early submissions for a total of 82 proposals, and has recently reached its third round of analyses.

In this brief survey we focus on the round 3 finalists and alternate candidates for Digital Signatures, announced on July 22, 2020:

| Finalists          | Alternate Candidates |
|--------------------|----------------------|
| CRYSTALS-DILITHIUM | SPHINCS\textsuperscript{+} |
| FALCON \textsuperscript{16} | GeMSS \textsuperscript{10} |
| Rainbow \textsuperscript{13} | Picnic \textsuperscript{11} |

These schemes are designed to address distinct security levels, known as Security Level I, III and V. These levels correspond to, respectively, 128, 192 and 256 bits of security against collisions. Among the six schemes above, only Falcon cannot be instantiated to all three security levels. In order to present these primitives we start, in Section 1 with an introduction to their underlying mathematical objects and the related problems, i.e. lattices, polynomial ideals, one-way functions and zero-knowledge proofs. Then, Section 2 describes the six digital signatures and lists their algorithms for key-generation, signing and verification. Finally, in Section 3 we conclude with a comparison between the different schemes.

1 Preliminaries

1.1 Digital Signatures

A digital signature is a public-key protocol that acts as the digital counterpart of a traditional signature. Formally, the properties that a digital signature must achieve are
the following [22].

1) **Authentication**: the receiver of the document must be sure of the identity of the sender.
2) **Integrity**: the signed document should not be altered when transmitted.
3) **Non-repudiation**: the signer of the document cannot deny having signed the document.
4) **Non-reusability**: the signature must be used only once.
5) **Unforgeability**: only the signer of the message should be able to give a valid signature.

A signature scheme is usually composed by three algorithms:

- **Keys Generation Algorithm**: using the global parameters defined at the beginning of the scheme, this algorithm generates a private key and the corresponding public key.
- **Signing Algorithm**: using the private key and the message needed to be signed, this algorithm outputs a signature.
- **Verification Algorithm**: using the public key and the message, the receiver is able to decide whether the signature obtained is valid or not.

A detailed description of digital signature schemes can be found in [41, 29].

1.2 Lattice Theory

**Definition 1.** Let $m$ be a positive integer. A discrete additive subgroup $L$ of $\mathbb{R}^m$ is called a (Euclidean) lattice. An equivalent definition may be given in terms of linear algebra: given a finite set of linearly independent vectors $B = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^m$, a lattice $L(B)$ is the set of the linear combinations with integer coefficients of the set $B$. The set $B$ is called a basis of the lattice $L$ and $n$ is the dimension of the lattice. It is possible to write a $n \times m$ matrix $A$ associated to a lattice, in which the rows of the matrix are the coordinates of the vectors of the basis.

In order to expose the most famous lattice problems, on which lattice cryptography is based, we need to introduce the minimal distance $\lambda_1(L)$, that is $\lambda_1(L) = \min_{v \in L \setminus \{0\}} ||v||$. Analogously it is possible to define the successive minimum $\lambda_i(L)$ for $2 \leq i \leq n$ as

$$\lambda_i(L) = \min_r \{v_1, \ldots, v_i \text{ independent in } L : ||v_j|| \leq r \text{ for } 1 \leq j \leq i\}.$$  

Moreover we need to consider other algebraic structures: given a polynomial $\phi(x) \in \mathbb{Z}[x]$, usually $\phi(x) = x^n - 1$ or $\phi(x) = x^n + 1$, and a prime $q > 2$, we define $R = \mathbb{Z}[x]/(\phi(x))$ and $R_q = R/qR = \mathbb{Z}_q[x]/(\phi(x))$.

Some famous examples of hard lattice problems are the following:

- **(SVP)** Given a basis $B$ of a lattice $L$, find a vector $v \in L$ such that $||v|| = \lambda_1(L)$.
- **(Approx-SVP) $\gamma$** Given a basis $B$ of a lattice $L$ find a vector $v \in L$ such that $||v|| \leq \gamma(n)\lambda_1(L)$, where the constant $\gamma(n)$ depends on the dimension of the lattice $n$.
- **(GapSVP) $\gamma$** Given a basis $B$ of a lattice $L$ and a constant $d$, decide if $\lambda_1(L) \leq d$ or $\lambda_1(L) > \gamma d$.
- **(SIVP) $\gamma$** Given a basis $B$ of a lattice $L$ find linearly independent vectors $v_1, \ldots, v_n \in L$ such that $||v_i|| \leq \gamma \lambda_n(L)$ for $1 \leq i \leq n$.  

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• (SIS\(_\beta\), [2, 30]) Given a matrix \( A \in M_{n \times m}(\mathbb{Z}_q) \), find a vector \( z \in \mathbb{Z}_q^m \) such that \( Az \equiv 0 \mod q \) and \( ||z|| \leq \beta \).

• (RSIS\(_\beta\), [27, 24]) Given a vector \( a \in R_q^n \), find a vector \( z \in R_q^m \) such that \( \langle z, b \rangle = 0 \) and \( ||z|| \leq \beta \).

• (MSIS\(_\beta\), [26]) Given a matrix \( A \in M_{n \times m}(R_q) \), find a vector \( z \in R_q^m \) such that \( Az \equiv 0 \mod q \) and \( ||z|| \leq \beta \).

For a vector \( s \in \mathbb{Z}_q^n \) and a discrete Gaussian distribution \( \chi \), with width \( \alpha q \) for some \( \alpha < 1 \), the LWE distribution \( A_{s,\chi} \) is sampled by choosing \( a \in \mathbb{Z}_q^n \) uniformly at random, \( e \) drawn with \( \chi \) and outputting the pair \( \langle a, b = \langle a, s \rangle + e \mod q \rangle \).

The MQ (multivariate quadratic) problem consists of finding a solution \( \bar{x} \in \mathbb{F}_q^n \) of the system:
\[
p_1(x_1, \ldots, x_n) = p_2(x_1, \ldots, x_n) = \ldots = p_m(x_1, \ldots, x_n) = 0.
\]

When the system (1.1) is random, i.e. for all \( i = 1, \ldots, m \) the coefficients of \( p_i \) are chosen uniformly at random, MQ has been proven to be an NP-hard problem [33]. Since a quantum algorithm for solving MQ problem does not exist, the multivariate protocols are very used in Post-Quantum cryptography.

The field \( \mathbb{F}_q \) is not algebraically closed, therefore a solution of (1.1) definitely belongs to \( \overline{\mathbb{F}_q}^n \) if the field equations are added to the system. So, given the polynomial ideal \( I = (p_1, \ldots, p_m) \), if the solution of (1.1) is unique, solving the MQ problem is equivalent to find the only point of the variety \( V(I) \cap \overline{\mathbb{F}_q}^n = V(I + (x_1^q - x_1, \ldots, x_n^q - x_n)) \) [24].

In terms of functions, solving the MQ problem is equivalent to invert the function \( \mathcal{P} : (x_1, \ldots, x_n) \mapsto (p_1(x_1, \ldots, x_n), \ldots, p_m(x_1, \ldots, x_n)) \), i.e. given \( d = \mathcal{P}(z) \in \mathbb{F}_q^m \), it is unfeasible to recover \( z \in \mathbb{F}_q^n \) if it is not known the way in which the polynomials \( p_i \) are generated.

In a multivariate public key cryptosystem (MPKC), \( \mathcal{P} \) is obtained using a secret set of \( m \) quadratic polynomials with random coefficients \( \{f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\} \) and composing the quadratic map
\[
\mathcal{F} : (x_1, \ldots, x_n) \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)).
\]
with two affine maps $S$ and $T$. The quadratic map $F$ is easy to invert \cite{12}, but the action of the affine maps makes difficult to invert $P$, because the polynomials $p_i$ induced by $P$ are approximately random \cite{5}.

A system induced from the previous construction is known as bipolar system \cite{12}, and it determines different MPKCs depending on the particular choice of $S, F$ and $T$.

### 1.4 One-Way functions

A one-way function is any function which can be efficiently computed but whose pseudo-inverse is hard to find. More formally:

**Definition 3.** A function $f$ is said to be one-way if it can be computed in polynomial time on any input $x$ and if any polynomial-time probabilistic algorithm used to solve $f(x) = y$ knowing $f$ and $y$ succeeds with negligible probability.

Examples of one-way functions are (cryptographically secure) block ciphers and hash functions. These primitives can therefore be safely used in the design of post-quantum digital signatures, since the only known speed-up for a quantum computer is Grover’s search algorithm \cite{18}, which however is not capable of determining in polynomial time a pre-image of a one-way function with more than negligible probability.

### 1.5 Zero-Knowledge proofs

Zero-knowledge proofs (ZKP) are protocols in which a prover can convince a verifier that a statement is true, without disclosing any information apart that the statement is true. The three classic properties that a ZKP needs are

- completeness: honest verifiers will be convinced by honest provers.
- soundness: no malicious prover can prove (with non-negligible probability) a false statement.
- zero-knowledge: no verifier learns anything other than the fact that the statement is true.

### 2 Signature Schemes

#### 2.1 Rainbow

Rainbow \cite{13} is a generalisation of the Unbalanced Oil and Vinegar (UOV) signature scheme \cite{24}, obtained by considering multiple UOV layers. The security of Rainbow is linked to the NP-hard problem of solving a multivariate polynomial system of quadratic equations over the field $\mathbb{F} = \mathbb{F}_2^s$. The fundamental parameters of Rainbow are $s$, three positive integers $v_1$, $o_1$ and $o_2$, and a hash function $H$ whose digest is $(o_1 + o_2) \cdot 2^s$ bits long.

Define two constants $m = o_1 + o_2$ and $n = v_1 + o_1 + o_2 = m + v_1$ and let $V_1 = \{1, \ldots, v_1\}$, $V_2 = \{1, \ldots, v_1 + o_1\}$, $O_1 = \{v_1 + 1, \ldots, v_1 + o_1\}$ and $O_2 = \{v_1 + o_1 + 1, \ldots, n\}$ be four sets of integers determined by the parameters, and let $S : \mathbb{F}^m \rightarrow \mathbb{F}^m$ and $T : \mathbb{F}^n \rightarrow \mathbb{F}^n$ be two invertible affine maps. For each $k \in O_1 \cup O_2$ define the map $f_k : \mathbb{F}^n \rightarrow \mathbb{F}$ according
to the formula
\[
f_k(x_1, \ldots, x_n) = \sum_{i,j\in V_l, i \leq j} \alpha_{k,i,j} x_i x_j + \sum_{i\in V_l, j\in O_l} \beta_{k,i,j} x_i x_j + \sum_{i\in V_l \cup O_l} \gamma_{k,i,j} x_i + \delta_k,
\]
where \( l \in \{1, 2\} \) is the unique index for which \( k \in O_l \) and \( \alpha_{k,i,j}, \beta_{k,i,j}, \gamma_{k,i,j}, \delta_k \in \mathbb{F} \) are randomly generated parameters. The \( m \) functions in \( n \) variables \( f_{v_1+1}, \ldots, f_n \) are used to define a quadratic map \( \mathcal{F} : \mathbb{F}^n \rightarrow \mathbb{F}^m \), such that
\[
\mathcal{F}(x_1, \ldots, x_n) = (f_{v_1+1}(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)).
\]
Due to the structure of \( \mathcal{F} \), given \( d \in \mathbb{F}^m \) it is possible to find in a reasonable amount time a value \( \bar{z} \in \mathbb{F}^n \) such that \( \mathcal{F}(\bar{z}) = d \), employing an algorithm that fixes the first variables and then applies Gaussian elimination. This property is used to efficiently compute a value \( z \in \mathbb{F}^n \) such that \( \mathcal{P}(z) = \mathcal{S} \circ \mathcal{F} \circ \mathcal{T}(z) = d \). On the other hand, given \( z \) and \( \mathcal{P} \) it is easy to compute \( d = \mathcal{P}(z) \), but from \( d \in \mathbb{F}^m \) it is unfeasible to obtain a value \( z \in \mathbb{F}^n \) for which \( \mathcal{P}(z) = d \) without knowing \( \mathcal{F} \) if \( \mathcal{S}, \mathcal{T}, \) and \( \mathcal{F} \) are random.

Given the parameters \((s, v_1, o_1, o_2, H)\) described above, the protocol works as follows.

### 2.1.1 Key generation

1. Randomly choose \( \mathcal{S}, \mathcal{T} \) and \( \mathcal{F} \) as defined above, choosing the maps’ coefficients uniformly at random in \( \mathbb{F} \).
2. The private key consists of \((\mathcal{S}, \mathcal{F}, \mathcal{T})\).
3. The public key is the composition \( \mathcal{P} = \mathcal{S} \circ \mathcal{F} \circ \mathcal{T} \).

### 2.1.2 Signing

Given a key-pair \((\mathcal{S}, \mathcal{F}, \mathcal{T}), \mathcal{P}\) and a message digest \( d \), compute the signature performing the following steps:

1. Choose uniformly at random a bit string \( r \) with the same length of \( d \).
2. Compute \( h = H(d || r) \) interpreted as a vector of \( \mathbb{F}^m \).
3. Compute \( x = \mathcal{S}^{-1}(h) \).
4. Compute \( y = \mathcal{F}^{-1}(x) \).
5. Compute \( z = \mathcal{T}^{-1}(y) \).
6. The signature is the pair \((z, r)\).

### 2.1.3 Verification

To verify a signature \((z, r)\) on a message digest \( d \) perform the following steps:

1. Compute \( h = H(d || r) \) interpreted as a vector of \( \mathbb{F}^m \).
2. Compute \( h' = \mathcal{P}(z) \) and check if \( h' = h \).
2.2 GeMSS

GeMSS [10] is a multivariate signature scheme, based on a system of polynomial equations over the field \( \mathbb{F}_2 \). The fundamental parameters of GeMSS are the following: \( m \) the number of equations, \( \Delta \) and \( v \) that determine the number of total variables, and a hash function \( H \) whose digest is \( k \) bits long.

Fix \( n = m + \Delta \) and let \( S \in GL_{n+v}(\mathbb{F}_2) \) and \( T \in GL_n(\mathbb{F}_2) \) be two invertible matrices. Define \( F \in \mathbb{F}_2^n[X, v_1, \ldots, v_v] \), a polynomial of degree \( D \), with the following structure:

\[
F(X, v_1, \ldots, v_v) = \sum_{0 \leq j < i < n} A_{i,j} X^{2^j} + \sum_{0 \leq i < n} \beta_i(v_1, \ldots, v_v) X^{2^i} +
\]

where \( A_{i,j} \in \mathbb{F}_2^n \), each \( \beta_i : \mathbb{F}_2^v \rightarrow \mathbb{F}_2^n \) is linear and \( \gamma(v_1, \ldots, v_v) : \mathbb{F}_2^v \rightarrow \mathbb{F}_2^n \) is quadratic.

Let \( (\theta_1, \ldots, \theta_n) \in \mathbb{F}_2^n \) be a basis of \( \mathbb{F}_2^n \) over \( \mathbb{F}_2 \). Given \( E = \sum_{k=1}^n e_k \cdot \theta_k \in \mathbb{F}_2^n \), define the following function:

\[
\Phi : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n \quad E \mapsto \Phi(E) = (e_1, \ldots, e_n).
\]

Starting from \( F \), it is possible to define \( n \) multivariate polynomials \( f_k \in \mathbb{F}_2[x_1, \ldots, x_{n+v}] \), such that \( F(\sum_{k=1}^n \theta_k x_k, v_1, \ldots, v_v) = \sum_{k=1}^n \theta_k f_k \). The public key \( P \) is derived from \( f_1, \ldots, f_n \) and it consists of the first \( m \) components of

\[
(p_1, \ldots, p_n) = (f_1 ((x_1, \ldots, x_{n+v}) \cdot S), \ldots, f_n ((x_1, \ldots, x_{n+v}) \cdot S)) \cdot T,
\]

which is reduced modulo the field equations, that is \( \text{mod}(x_1^2 - x_1, \ldots, x_n^2 - x_n) \). Due to the structure of \( F \), given \( d \in \mathbb{F}_2^m \) and \( r \in \mathbb{F}_2^{n-m} \) randomly chosen, it is possible, with a procedure that fixes the last \( v \) variables and then applies Berlekamp’s algorithm on the resulting univariate polynomial, to find a root of \( F - \Phi^{-1}((d, r) \cdot T^{-1}) \) in a reasonable amount time \( \mathcal{O}(nD) \). This property is used to efficiently compute a value \( z \in \mathbb{F}_2^{n+m} \) such that \( P(z) = d \). On the other hand, given \( z \) and \( P \) it is easy to compute \( d = P(z) \), but from \( d \in \mathbb{F}_2^m \) it is unfeasible to obtain a value \( z \in \mathbb{F}_2^{n+m} \) for which \( P(z) = d \) without knowing \( F \), if \( S, T \) and \( F \) are random.

Finally, it is possible to iterate \( t \) times a part of the signature to increase the security level \( \lambda \), indeed in this way it is possible to apply the hash function \( H \) and at the same time to combine the actions of \( S \) and \( T \) on the variables more than once.

Given the parameters \((m, \Delta, v, D, H, t)\) described above, the protocol works as follows.

2.2.1 Key generation

1. Randomly choose \( S, T \) and \( F \) choosing the coefficients of \( F \) uniformly at random in \( \mathbb{F}_2^n \) and the elements of \( S \) and \( T \) in \( \mathbb{F}_2 \).
2. The private key consists of \((S, T, F)\).
3. Compute \( p = (p_1, \ldots, p_n) \) as defined in (2.2).
4. The public key is \( P = (p_1, \ldots, p_m) \), the first \( m \) components of \( p \).
2.2.2 Signing

Given a key-pair $((S,T,F),P)$ and a message digest $h$, compute the signature performing the following steps:

1. Set $S_0 = 0 \in \mathbb{F}_2^m$.

2. Repeat for $i = 1$ to $t$ the following steps:
   
   (a) Get $D_i$ the first $m$ bits of $h$ and compute $D'_i = D_i \oplus S_{i-1}$.
   
   (b) Randomly choose $(v_1,\ldots,v_v) \in \mathbb{F}_2^v$ and $r \in \mathbb{F}_2^{n-m}$.
   
   (c) Compute $A_i = \phi^{-1}((D'_i,r) \cdot T^{-1})$ as described in (2.1).
   
   (d) Compute a root $Z$ of $F - A_i$.
   
   (e) Compute $(S_i,X_i) = (\phi(Z),v_1,\ldots,v_v) \cdot S^{-1} \in \mathbb{F}_2^m \times \mathbb{F}_2^{n+v-m}$.
   
   (f) Compute $h = H(h)$.

3. The signature is $z = (S_t,X_t,\ldots,X_1)$.

2.2.3 Verification

To verify a signature $z$ on a message digest $h$ perform the following steps:

1. Repeat for $i = 1$ to $t$
   
   (a) Get $D_i$ the first $m$ bits of $h$.
   
   (b) Compute $h = H(h)$.

2. Repeat for $i = t-1$ to $0$
   
   (a) Compute $S_i = P(S_{i+1},X_{i+1}) \oplus D_{i+1}$.

3. Check if $S_0 = 0$.

2.3 CRYSTALS-DILITHIUM

CRYSTALS-DILITHIUM [4] is a lattice-based signature built on the hardness of two problems: MLWE and SelfTargetMSIS problem [23], a variation of the MSIS problem. The first problem is defined over a polynomial ring $R_q = \mathbb{Z}_q[x]/(x^{256} + 1)$, where $q$ is a prime such that $q \equiv 1 \pmod{512}$. This condition on $q$ allows to use the NTT (Number Theoretic Transform, a generalization of the discrete Fourier transform over a finite field) representation. Given $H$ a hash function and a vector $x$, SelfTargetMSIS consists in finding a vector $z' = (z,c)$ with small coefficients such that $H(x||f(z')) = c$ with $w(c) = 60$ (where $w$ denotes the Hamming weight), where $f$ is a linear function.

In order to define an ordering relation in $\mathbb{Z}_q$, we will consider the embedding $\eta : \mathbb{Z}_q \rightarrow \mathbb{Z}$, where $\eta(z) \equiv z \pmod{q}$ and $-\frac{q-1}{2} \leq \eta(z) \leq \frac{q-1}{2}$. For any $z_1, z_2 \in \mathbb{Z}_q$ we say that $z_1 \leq z_2$ if and only if $\eta(z_1) \leq \eta(z_2)$.

Let $w = w_0 + w_1 x + \ldots + w_{255} x^{255}$ be a polynomial in $R_q$, the norm $\|w\|_{\infty} := \max_{i} |w_i|$ is used to check some conditions related to the security and correctness, for this reason it is introduced a parameter $\beta \in \mathbb{Z}$ as a bound for the norm of some quantities. Let $A \in M_{k,l}(R_q)$ be a matrix and set $\bar{w} = Ay$, where $y \in R_q^l$ is a vector such that $\|y\|_{\infty} \leq \gamma_1$. 


(with $\gamma_1 \in \mathbb{Z}$ another parameter), we distinguish between the high-order and low-order parts of $\tilde{w}$ as follows: for each component $w'$ of $\tilde{w}$

$$w' = w'_1 \cdot 2\gamma_2 + w'_0,$$

(2.3)

where $\|w'_0\|_\infty \leq \gamma_2$, where $\gamma_2 \in \mathbb{Z}$ is another parameter. We call $w'_1$ the high-order part, while $w'_0$ is the low-order part of $w'$. We denote with $\text{HB}(\tilde{w})$ (HighBits) the vector comprising all $w'_1$'s, thus is the high-order part of $\tilde{w}$, and with $\text{LB}(\tilde{w})$ (LowBits) the low-order part of $\tilde{w}$.

For storage efficiency, instead of generating and storing the entire matrix $A$, the protocol makes use of a secure PRNG and the NTT: using the NTT, it is possible to identify $a \in R$ and $\tilde{a} \in \mathbb{Z}_q^{256}$, where $\tilde{a}$ is the NTT representation of $a$, while if $A \in R^{k \times l}$ we denote with $\text{NTT}(A)$ the matrix where each coefficient of $A$ is identified with an element of $\mathbb{Z}_q^{256}$. To obtain the matrix $A$, every element $\tilde{a}_{i,j}$ of $\text{NTT}(A)$ is generated from a 256 bit random seed $\rho$.

The parameters of CRYSTALS-DILITHIUM are $q, d, k, l, \eta, \Omega, H, G$, where $\gamma_1, \gamma_2, k, l$ and $q$ are defined as above, $d \in \mathbb{Z}_q$, $\eta$ and $\Omega$ are other bounds and $G, H$ are hash functions.

Given the parameters $(q, k, l, \eta, G, H, d, \gamma_1, \gamma_2, \beta, \Omega)$ described above, the protocol works as follows.

### 2.3.1 Key generation

1. Choose uniformly at random two bit strings $\rho$ and $\theta$ of length 256.
2. Choose uniformly at random $(s_1, s_2) \in R^{l} \times R^{k}$ with $|s_i| \leq \eta$.
3. Compute $A \in R^{k \times l}$ from $\rho$ using NTT representation.
4. Compute $t = As_1 + s_2$.
5. Compute $t_0 = t \mod 2^d$ and $t_1 = \frac{t - t_0}{2^d}$.
6. The public key is $P = (\rho, t_1)$.
7. The private key is $S = (\rho, \theta, G(\rho || t_1), s_1, s_2, t_0)$.

### 2.3.2 Signing

Given a key-pair $(S, P)$ and a message $M$ compute the signature performing the following steps:

1. Compute $A$ from $\rho$ as described above.
2. Compute $\mu = G(G(\rho || t_1) || M)$ and $\mu' = G(\theta || \mu)$.
3. Compute uniformly at random $y \in R^l$ with $\|y\|_\infty < \gamma_1$, starting from seed $\rho'$ using NTT representation.
4. Compute $w = Ay$ and $w_1 = \text{HB}(w)$.
5. Compute $c = H(\mu || w_1)$ and $z = y + cs_1$.
6. Compute $r_1 = \text{HB}(w - cs_2)$ and $r_0 = \text{LB}(w - cs_2)$.
7. Check if all the following conditions are satisfied else repeat from step 3:
   (a) $\|z\|_\infty < \gamma_1 - \beta$.
   (b) $\|r_0\|_\infty < \gamma_2 - \beta$.
   (c) $r_1 = w_1$.

8. Compute $h = (h_1, \ldots, h_k) = r_1 \oplus \text{HB}(w - cs_2 + ct_0)$.

9. Compute $\Omega' = \omega(h)$ and check if $\Omega' \leq \Omega$ else repeat from step 3.

10. The signature is $(z, h, c)$.

2.3.3 Verification

To verify a signature $(z, h, c)$ on a message $M$ perform the following steps:

1. Compute $A$ and $\mu$ as described in the signing process.

2. Compute $w'_1 = \text{HB}(w - cs_2)$ knowing $\text{HB}(Az - ct_1 \cdot 2^d) = \text{HB}(w - cs_2 + ct_0)$ and $h$ that allows to remove the error generated by $ct_0$.

3. Check if all the following conditions are satisfied:
   (a) $\|z\|_\infty < \gamma_1 - \beta$.
   (b) $c = H(\mu || w'_1)$.
   (c) Compute $\Omega' = \omega(h)$ and check if $\Omega' \leq \Omega$.

Given the parameter $d \in \mathbb{Z}_q$ and computed $z = y + cs_1$ with $s_1 \in R^d$, it is possible to define $t_1, t_0 \in \mathbb{Z}_q$ such that $t = t_1 \cdot 2^d + t_0$ ($t_1$ is the high order part of $t$) and compute $\text{HB}(\bar{w} - cs_2 + ct_0)$. Indeed:

$$Az - ct_1 \cdot 2^d = Ay + cAs_1 - c(t - t_0) = Ay - cs_2 + ct_0 = \bar{w} - cs_2 + ct_0.$$ 

Starting from $r_1 = \text{HB}(\bar{w} - cs_2 + ct_0)$, it is easy to obtain $\text{HB}(\bar{w} - cs_2)$ knowing $h = r_1 \oplus \text{HB}(\bar{w} - cs_2)$, indeed it is sufficient to check which bits of $h$ have value 1 to find the error bits in $r_1$ and changing their value. The arithmetic modulus $\frac{q-1}{2^k}$ is required to modify $r_1$ depending on the sign of $\text{LB}(\bar{w} - cs_2 + ct_0)$. Besides, the parameter $\Omega$ is the maximum Hamming weight that $h$ can assume and thanks to the condition $\|z\|_\infty < \gamma_1 - \beta$, it is possible to make the correction of error bits successfully, in a safe way. On the other hand, it is infeasible to recover $z$ without knowing $y$ (so $\bar{w}$ cannot be computed) and $s_1$.

2.4 FALCON

FALCON \cite{16} is a particular lattice-based signature, which is based on solving the SIS problem over the NTRU lattices. Given $n = 2^k$, $q \in \mathbb{N}^*$ and defined $R$ using $\phi(x) = x^n + 1$, the problem consists in determining $f, g, G, F \in R$ such that $f$ is invertible modulus $q$ (this condition is equivalent to require that $\text{NTT}(f)$ does not contain 0 as a coefficient) and such that the following equation (NTRU equation) is satisfied:

$$fG - gF = q \mod \phi.$$

(2.4)
If \( h := g \cdot f^{-1} \mod q \), it is possible to verify that the matrices \( P = \begin{bmatrix} 1 & h \\ 0 & q \end{bmatrix} \) and 
\( Q = \begin{bmatrix} f & g \\ F & G \end{bmatrix} \) generate the same lattice:
\[
\Lambda(P) = \{ zP \mid z \in R_q \} = \{ zQ \mid z \in R_q \} = \Lambda(Q),
\]
but, if \( f \) and \( g \) are sufficiently small, then \( h \) should seem random, so, given \( h \), the hardness of this problem consists of finding \( f \) and \( g \). Each coefficient of the polynomials \( f = \sum_{i=0}^{n-1} f_i x^i \)
and \( g = \sum_{i=0}^{n-1} g_i x^i \) is generated from a distribution close to a Gaussian of center 0 and standard deviation \( \sigma \in [\sigma_{\min}, \sigma_{\max}] \) (where \( \sigma, \sigma_{\min}, \sigma_{\max} \) are parameters). The following general property is fundamental to solve the NTRU equation \([2.4]\) in particular if \( f = \sum_{i=0}^{n-1} a_i x^i \in \mathbb{Q}[x] \), \( f \) can be decomposed in a unique way as:
\[
f(x) = f_0(x^2) + x f_1(x^2),
\]
(2.5)
where \( f_0 = \sum_{i=0}^{n/2-1} a_{2i} x^i \) and \( f_1 = \sum_{i=0}^{n/2-1} a_{2i+1} x^i \).

Given \( f \) and \( g \), it is easy to obtain \((F,G)\) solution of \([2.4]\), indeed there is a recursive procedure that uses the previous property and allows to solve a NTRU equation in the ring \( \mathbb{Z} = \mathbb{Z}[x]/(x + 1) \) and then transforms this solution \((F,G) \in \mathbb{Z} \times \mathbb{Z}\) into two polynomials of \( \mathbb{Z}[x]/(\phi) \). Thanks to the FFT, it is possible to define the matrix 
\[
\bar{B} = \begin{bmatrix} \text{FFT}(g) & \text{FFT}(-f) \\ \text{FFT}(G) & \text{FFT}(-F) \end{bmatrix}.
\]
Moreover we also need to consider the LDL decomposition of \( \mathcal{G} = \bar{B} \cdot \bar{B}^T = LDL^T \), where \( L = \begin{bmatrix} 1 & 0 \\ L & 1 \end{bmatrix} \) and \( D = \begin{bmatrix} D_{11} & 0 \\ 0 & D_{22} \end{bmatrix} \).

Starting from \( G \in M_{2,2}(\mathbb{Q}[x]/(\phi)) \), it is possible to construct the so-called FALCON tree \( T \): the root of \( T \) is \( L \) and its two child-nodes \( G_0, G_1 \in M_{2,2}(\mathbb{Q}[x]/(x^{n/2} + 1)) \) are obtained considering the decomposition of \( D_{11} \) and \( D_{22} \) as described in \([2.5]\). Iterating this procedure on \( G_0 \) and \( G_1 \), it is possible to obtain the whole tree \( T \), where each leaf \( l \in \mathbb{Q} \) is normalized, i.e \( l' = \frac{a}{q} \).

FALCON uses a particular hash function \( H \), which transforms a string modulus \( q \) in a polynomial \( c \in \mathbb{Z}_q[x]/(\phi) \).

In addition to the standard deviations described above, the parameters of FALCON are \( k, q \) and two other constants \( \beta \in \mathbb{Q}_+ \) and \( b_l \in \mathbb{N}^* \) that will be described later.

Given a solution of \( \bar{B}t = c \), there exists a recursive procedure (Fast Fourier sampling), which applies a randomized rounding on the coefficients of \( t \in \mathbb{Q}[x]/(\phi) \) to obtain a polynomial \( z \in R \), using the information stored in \( T \).

Let \( a, b \in \mathbb{Q}[x]/(\phi) \), it is possible define the following inner product and its associated norm:
\[
\langle a, b \rangle = \frac{1}{n^2} \sum_{i \in \mathbb{C} : \phi(i) = 0} a(i) \cdot \overline{b(i)}.
\]

Let \( \beta \in \mathbb{Q}_+ \), then it is possible to compute \( s = (t - z)\bar{B} \) with \( ||s||^2 \leq ||\beta^2|| \) and using the inverse of FFT it is easy to compute \( s_1, s_2 \in R \), which satisfy:
\[
s_1 + s_2 h = c \mod q.
\]
(2.6)
On the other hand, given \((s_1, s_2)\) it is unfeasible to recover \(s\) without knowing \(\bar{B}\) and \(T\).

Finally, FALCON uses a compression algorithm, which transforms \(s_2\) in a byte string \((8 \cdot b_l - 328)\) long.

Given the parameters \((k, q, \sigma_{\text{min}}, \sigma_{\text{max}}, \sigma, \beta, b_l)\) described above, the protocol works as follows.

### 2.4.1 Key generation

1. Compute \(f = \sum_{i=0}^{n-1} f_i x_i\) and \(g = \sum_{i=0}^{n-1} g_i x_i\), generating \(f_i\) and \(g_i\) from Gaussian distribution \(D_{0, \sigma}\).
2. Check that \(f\) is invertible modulus \(q\), else restart from step 1.
3. Find \((F, G)\) solution of the NTRU equation (2.4).
4. Compute \(\bar{B}\) as described above.
5. Compute \(G = B \cdot \bar{B}^T\), obtain the FALCON tree \(T\) using LDL decomposition and normalize its leaves.
6. Compute \(h = g f^{-1} \mod q\).
7. The private key is \((\bar{B}, T)\).
8. The public key is \(h\).

### 2.4.2 Signing

Given a key-pair \((h, (\bar{B}, T))\) and a message \(m\), compute the signature performing the following steps:

1. Choose uniformly at random a bit string \(r\) 320 long.
2. Compute \(c = H(r || m, q, n)\) and solve \(\bar{B} t = c\).
3. Compute \(z\) randomized rounding of \(t\) as described above.
4. Compute \(s = (t - z) \bar{B}\).
5. Check that \(\|s\|^2 \leq \|\beta^2\|\) else repeat from step 3.
6. Compute \((s_1, s_2)\) satisfying (2.6).
7. By compressing \(s_2\), compute a string \(s'\) of \((8 \cdot b_l - 328)\) bytes.
8. The signature is \((r, s')\).

### 2.4.3 Verification

Given a public key \(h\), to verify a signature \((r, s')\) on a message digest \(c\) perform the following steps:

1. By decompressing \(s'\), compute \(s_2\).
2. Compute \(s_1 = c - s_2 h \mod q\).
3. Check if \(\|(s_1, s_2)\|^2 \leq \|\beta^2\|\).
2.5 SPHINCS+

SPHINCS+ is based on hash functions and it is nothing more than an opportune union of three signature schemes: WOTS+, XMSS and FORS. SPHINCS+ works with two main tree structures: a Hypertree and a FORS tree. The Hypertree consists of \(d\) Merkle trees of height \(h'\). On each of these trees is applied an XMSS signature scheme. XMSS, in turn, consists of a one-time signature WOTS+ applied on the root of the previous layer plus the authentication path of the randomly chosen leaf. On the other hand, a FORS tree is made up of \(k\) parallel trees of height \(a\) and, contrary to Hypertrees, this kind of trees is used only on signature generation and verification, but not for key generation.

SPHINCS+ uses the FORS scheme to generate a hash value that relates the message to \(k\) FORS roots. After that, a Hypertree signature is applied to the hash returned by the FORS signature to generate a SPHINCS+ signature.

The security of this scheme derives from the security of the hash function involved. In particular SPHINCS+ uses the so called tweakable hash functions, which allow us to approach the details of how exactly the nodes are computed.

The choice of the hash function strongly influences the security of the signature, in fact the length \(n\) of every hash value of this protocol is fundamental to determinate the security level, the authors have chosen SHAKE256 as the hash function family.

The parameters \(k\) and \(a\) determine the performance and security of FORS, so it is necessary to balance the value of these two parameters to avoid getting too large or too slow signatures. Instead, the height of the Hypertree \(h'd\) determines the number of XMSS instances, so this value has a direct impact on security: a taller Hypertree gives more security. Remark that the number of layers \(d\) is a pure performance trade-off parameter and does not influence security.

Finally, the Winternitz parameter \(w\) is a trade-off parameter (greater \(w\) means shorter signatures but slower signing), which determines the number and length of the hash chains per WOTS+ instance.

The privacy of SPHINCS+ is guaranteed by the pseudorandom generation of WOTS+ and FORS secret keys (this operation randomizes the choice of FORS and WOTS+ leaves used to sign).

Given the parameters \((n, h', d, k, a, w)\) described above, the protocol works as follows.

2.5.1 Key generation

The description of key generation assumes the existence of a function secRand which on input \(n\) returns \(n\) bytes of cryptographically strong randomness.

1. Compute SK.seed=secRand\((n)\), which is used to generate all the WOTS+ and FORS private key elements.
2. Compute SK.prf=secRand\((n)\), which is used to generate a randomization value for the randomized message hash.
3. Compute PK.seed=secRand\((n)\), which is the public seed.
4. Compute PK.root, which is the hypertree root, i.e. the XMSS root of the tree on the top level.
5. The private key is: SK=(SK.seed, SK.prf, PK.seed, PK.root).
6. The public key is: PK=(PK.seed, PK.root).
2.5.2 Signing

Given the private key SK and a message M, compute the signature performing the following steps:

1. Compute R, an \( n \)-bytes string pseudorandomly generated starting from SK.prf and M.
2. Compute the digest of M.
3. Compute SIGFORS, which is a FORS signature applied to the first \( ka \) bits of the digest.
4. Starting from SIGFORS, derive PKFORS i.e. the public key associated to the FORS signature.
5. Compute HTSIG, which is an hypertree signature applied to PKFORS.
6. The SPHINCS\(^\dagger\) signature is: \( \text{SIG} = (R, \text{SIGFORS}, \text{HTSIG}) \).

2.5.3 Verification

To verify a signature SIG on a message M perform the following steps:

1. Get R, which corresponds to the first \( n \) bytes of SIG.
2. Get SIGFORS, the following \( k(a + 1) \cdot n \) bytes of SIG.
3. Compute the digest of M.
4. Starting from SIGFORS and the first \( ka \) bits of the digest, derive PKFORS.
5. Starting from PKFORS, check the hypertree verification.

2.6 Picnic

Picnic \[11\] is a signature scheme whose security is based on the one-wayness of a block cipher and the pseudo-random properties of an extensible hash function.

In particular the construction relies upon the fact that a digital signature is essentially a non-interactive zero knowledge proof of knowledge of the preimage of a one-way function output, where the challenge inside the proof is tied to the message that is being signed. In other words, the signer creates a transcript that demonstrates the knowledge of the private key whose image through the one-way function is the public key, without revealing any information about the private key itself. Moreover this transcript is indissolubly bound to the message.

Starting from this general idea, Picnic instantiates a signature scheme using classical general-purpose primitives: a block cipher, a secure multi-party computation protocol (MPC), and an extensible cryptographic hash function (also known as extensible output function or XOF). The zero-knowledge proof (ZKP) is derived from the hash and the MPC protocol, exploiting the security properties of the latter. The prover computes the one-way function using its multi-party decomposition, controlling every party. The security of the MPC protocol allows the disclosure of the complete view of some (in this case all but one) parties without revealing anything about the secret input, so a ZKP may be constructed committing to every view and randomly selecting which ones to reveal (the challenge). The commitment (built from the hash) binds the prover to the views (i.e. they cannot be
changed after the commitment) without revealing them yet (the commitment is "hiding"). Soundness can be achieved repeating this process for a few iterations, so that the verifier can be convinced that the prover could not have successfully produced the views without actually knowing the MPC input, except with negligible probability.

The protocol just described is interactive, but there are fairly simple techniques that allow to transform it into a non-interactive one, i.e. a transcript produced by the prover that by itself can convince a verifier. These techniques use a deterministic pseudo-random generator (the hash) to derive the challenges from the public values, i.e. the public key, the commitments and the message. Assuming the (quantum) random oracle model \[6, 43\] (i.e. the hash is modeled through an oracle that outputs random values on new inputs, but does not change answer when a query is repeated), we maintain soundness even without interaction, and the message is tightly fastened to the transcript, so that it is infeasible to adapt this signature for another message without knowing the private key.

The MPC protocols are much more sensitive to the number of AND operations on two secret bits than to XOR operations, since the masking of AND gates requires extra information to keep consistency. This, in turn, causes the MPC views (and thus the signature) to grow in size, therefore Picnic selected as block cipher LowMC \[^1\], an algorithm designed to minimize such operations for a given security level. LowMC employs a classic substitution-permutation structure with \(n\)-bit blocks (where \(n\) essentially defines the security level of the whole signature) and \(r\) rounds in which \(s\) parallel 3-bit S-boxes are applied (note that they do not necessarily cover the entire block), followed by a linear permutation (defined by a different matrix for each round), and a round-key addition (the round-keys are derived multiplying the key by \(r + 1\) different matrices: one for the initial key-whitening and again one per round).

Picnic’s NIST submission defines various parameters sets that, besides optimizing LowMC parameters for the three security levels, employ different MPC protocols and techniques to obtain a non-interactive zero-knowledge proof (NIZKP). More specifically, picnic-LX-FS (where \(X \in \{1, 3, 5\}\) is the security level defined by NIST) uses the proof system ZKB++ (an optimized version of ZKBoo \[^17\], a ZKP for boolean circuits based on an MPC called “circuit decomposition”) that simulates \(T\) parallel MPC executions between 3 parties, and uses the Fiat-Shamir transform \[^15\] to obtain a NIZKP. The picnic-LX-full variant changes the LowMC parameters: uses a full S-box layer that allows to reduce the number of rounds. The parameter sets picnic-LX-UR use again ZKB++ but with the Unruh transform \[^12, 13, 14\], which expands the signature size but is provable secure in the stronger quantum random oracle model (unlike the FS transform in general). Finally the sets picnic3-LX bring along various optimizations: like picnic-LX-full they use a full S-box layer and the Fiat-Shamir transform, but they use a different ZKP and employ various optimizations to reduce signature size. The ZKP used in picnic3-LX is the KKW protocol \[^21\], which simulates \(T\) parallel MPC executions between \(N\) parties (\(N = 16\) in the chosen parameters sets). Each execution is divided into an offline preprocessing phase and an online phase where the shares are broadcast and the output reconstructed. In KKW the challenger chooses \(u\) executions for which the online phase will be revealed for all but one party, whereas for the other executions only the preprocessing phases will be revealed (for all parties).

Note that the MPC executions assume that each party consumes some random bits read from an input tape. These tapes are deterministically generated from seeds through the XOF, and in turn those seeds are generated from a master seed, which is generated alongside a salt (used as extra input in every other derivation to prevent multi-target attacks such as in \[^13\]) from the secret key, the message, the public key, and the length.
parameter $S$ (and optionally an extra random input to randomize signatures), always through the XOF. The picnic3-LX parameters sets employ a tree structure to derive the seeds in order to reduce the amount of information needed to be included in the signature to reveal the MPC executions. Moreover they use Merkle trees to compute the commitments, so the signatures can be compressed further.

All parameter sets use as XOF an instance of the SHAKE family \cite{shake} (specifically SHAKE128 for security level L1 and SHAKE256 for L3 and L5) employing domain separation techniques to differentiate the uses as different random oracles.

Given the parameters $(S, n, s, r, T, u)$ described above, the protocol works as follows.

2.6.1 Key generation

1. Choose a random $n$-bit string $p$, and a random $n$-bit string $k$.

2. Using LowMC with parameters $(n, s, r)$, compute the encryption of $p$ with $k$, denoted $C = E(k, p)$.

3. The private key is $k$.

4. The public key is $(C, p)$.

2.6.2 Signing

Given a key-pair $((C, p), k)$ and a message $M$, compute the signature performing the following steps:

1. Derive the master seed and the salt from $k, M, (C, p), S$ (and possibly a random input of size $2S$), then derive the individual seeds.

2. Simulate $T$ executions of the MPC protocols, producing for each party their view and output, starting from their seed.

3. Compute the commitments to every seed and corresponding view.

4. Compute the NIZKP challenge $e$ from the MPC outputs, the commitments, the salt, the public key and the message.

5. Compute the NIZKP response by selecting for each MPC execution the appropriate outputs and decommitments to reveal, according to $e$.

6. Assemble the signature $\sigma$ by including: $e$, the salt, the NIZKP response, and the commitments not derivable from the response.

2.6.3 Verification

Given a public key $(C, p)$, to verify a signature $\sigma$ on a message $M$ perform the following steps:

1. Deserialize $\sigma$ extracting the NIZKP challenge $e$, the salt, the NIZKP response, and the commitments.

2. Parse the NIZKP response to obtain, for each of the $T$ MPC executions, the outputs and the decommitments prescribed by $e$. 
3. Use the seeds included in the decommitments to derive (with the salt) the tapes of the revealed parties, then use these values and the rest of the response to simulate the MPC executions that compute the LowMC encryption of $p$ with output $C$, computing the views for which the commitments are not included in the signature.

4. Complete the commitments deriving the missing values from the results of the previous step, then derive the challenge $e'$ as in signing.

5. The signature is valid if every parsing/deserialization succeeds, the MPC computations are correct, and $e' = e$.

### 3 Comparison

![Signature-Public Key Size Comparison](image_url)

In Figure 1, we summarize the dimensions in bytes of the public keys and the corresponding dimensions of the signatures of all the schemes presented in this survey, as well as those of the two classical schemes ECDSA and RSA. It is interesting to notice that the multivariate schemes have small signatures, but the size of their public keys is the largest among all the schemes. On the other hand, SPHINCS+ and Picnic have small public keys, but large signatures, while the algorithms based on lattices have intermediate values in terms of both public keys and signatures. Finally it is worth to point out that, among all the schemes depicted, the best compromise in terms of dimension is still obtained by the non-quantum scheme of ECDSA.
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