With a state-of-the-art numerical method used for solving the integral-differential equation of radiative transfer, we investigate the flux of the Ly$\alpha$ photon $v_0$ emergent from an optically thick halo containing a central light source. Our focus is on the time-dependent effects of the resonant scattering. We first show that the frequency distribution of photons in the halo is quickly approaching a locally thermalized state around the resonant frequency, even when the mean intensity of the radiation is highly time dependent. Since initial conditions are forgotten during the thermalization, some features of the flux, such as the two-peak structure of its profile, are actually independent of the intrinsic width and time behavior of the central source, if the emergent photons are mainly from photons in the thermalized state. In this case, the difference $|\nu_k - \nu_0|$, where $\nu_k$ are the frequencies of the two peaks of the flux, cannot be less than 2 times the Doppler broadening. We then study the radiative transfer in the case where the light emitted from the central source is a flash. We calculate the light curves of the flux from the halo. It shows that the flux is still a flash. The time duration of the flash for the flux, however, is independent of the original time duration of the light source but depends on the optical depth of the halo. Therefore, the spatial transfer of resonant photons is a diffusion process, even though it is not a purely Brownian diffusion. This property enables an optically thick halo to trap and store thermalized photons around $v_0$ for a long time after the cessation of the central source emission. The photons trapped in the halo can yield delayed emission, of which the profile also shows typical two-peak structure as that from locally thermalized photons. Possible applications of these results are addressed.

**Key words:** dark ages, reionization, first stars – early universe – galaxies: halos

1. INTRODUCTION

The transfer of Ly$\alpha$ radiation in an optically thick medium is fundamentally important for the understanding of the physics of halos around Ly$\alpha$ photon sources or clouds nearby these sources. It includes Ly$\alpha$ forest, damped Ly$\alpha$ system, Ly$\alpha$ blob, Ly$\alpha$ emitter, and fluorescent Ly$\alpha$ emission of galaxies and quasars, as well as the optical afterglow of gamma-ray bursts (GRBs). The profiles of the emission and absorption of the Ly$\alpha$ radiation from these sources are powerful tools to constrain the mass density, velocity, temperature, and the fraction of neutral hydrogen of intergalactic medium (IGM) at various redshifts (Miralda-Escude & Rees 1998; Miralda-Escude 1998; Haiman & Cen 2005; Tasitsiomi 2006; Totani et al. 2006; McQuinn et al. 2007).

It is well known that the resonant scattering of Ly$\alpha$ photons with neutral hydrogen atoms has a profound effect on the time, space, and frequency dependences of the transfer of Ly$\alpha$ photons. An analytical solution of the integro-differential equation of the resonant radiative transfer revealed that the resonant scattering leads to a local Boltzmann distribution of photons in a small frequency range around the Ly$\alpha$ frequency $v_0$ (Wouthuysen 1952; Field 1958, 1959). The temperature of the Boltzmann distribution is equal to the kinetic temperature of the neutral hydrogen atoms. The width of the local Boltzmann distribution increases with time. This is the so-called Wouthuysen–Field effect, which is important in 21 cm cosmology (Fang 2009).

Besides Field’s solution, no other time-dependent analytical solutions of the integro-differential equation are available. All other analytical solutions (Harrington 1973; Neufeld 1990; Dijkstra et al. 2006) are based on the time-independent Fokker–Planck equation, which is the diffusion approximation of the radiation transfer. These time-independent solutions are important but can only be used to describe the “limiting asymptotic behavior” of the radiative transfer (Adams 1972). It tells us nothing about the time development of the local Boltzmann distribution. The Monte Carlo numerical method has also been used to solve the radiative transfer of Ly$\alpha$ photons (Lee 1974; Zheng & Miralda-Escude 2002; Ahn et al. 2002; Cantalupo et al. 2005; Verhamme et al. 2006). Yet, the time development of the Wouthuysen–Field effect is also absent in this approach.

The time-independent approximation would be reasonable if the length scale $l$ and time scale $t$ of the problem considered satisfy the condition $l/t \ll c$. In this case, one can take $c \rightarrow \infty$, and then the time derivative term $\partial/\partial t$ of the radiative transfer equation can be dropped. The condition $l/t \ll c$, however, cannot be satisfied if the size of the neutral hydrogen halo is large and the time scale of the Ly$\alpha$ photon source is small. For instance, the time scale $t_{after}$ of the optical afterglow of GRBs is of the order of a few tens of hours (e.g., Tanvir et al. 2009; Salvaterra et al. 2009) or a few days (Vreeswijk et al. 2004), while the column number density of neutral hydrogen is as large as $10^{21}$–$10^{22}$ cm$^{-2}$ and the number volume density is about $10^2$–$10^4$ cm$^{-3}$ (Vreeswijk et al. 2004). Therefore, the size of the neutral hydrogen halos should be much larger than $t_{after}$. It is improper to treat the short-time problems with the solutions of “limiting asymptotic behavior.”

Recently, a state-of-the-art numerical solver for the kinetic equations has been developed. This solver is based on the weighted essentially non-oscillatory (WENO) scheme (Jiang & Shu 1996). It has been developed to solve the Boltzmann equations (Carrillo et al. 2003, 2006) and radiative transfer (Qiu et al. 2006, 2007, 2008). This numerical solver has successfully passed the tests of the conservation of photon number, Field’s solutions, and the Wouthuysen–Field effect, etc. and has been properly used as the solver of the transfer of resonant photons (Roy et al. 2009a, 2009b, 2009c).
We will study, in this paper, the time-dependent behavior of the Lyα radiation transfer in an optically thick medium. We will not try to explain any specific observed Lyα spectrum, instead we will study the physical features of the resonant photon transfer, which cannot be addressed with previous solvers. For instance, we will show that the frequency distribution of the Lyα photons can keep in a locally thermalized state even when the intensity and flux are highly time dependent. This feature is essentially important, as thermalization generally will lead to the initial conditions being forgotten. Our solver is also able to study the transfer of a light flash in an optically thick halo. It shows that the nature of the transfer is a diffusion process, though it is not a purely Brownian diffusion. This property leads to the trap and store of photons thermalized around the Lyα frequency for a long time after the cease of the central source emission.

This paper is organized as follows. Section 2 presents the theoretical background of the Lyα photon transfer in an optically thick medium. Section 3 gives the solution of Lyα photons emergent from optically thick spherical halos with a steady source located at the center of the halo. Section 4 explains the solutions when the central source is a flash. A discussion and conclusion is given in Section 5. The details of the numerical implementation are given in the Appendix.

2. THEORY OF LYα RADIATIVE TRANSFERS IN OPTICALLY THICK HALOS

2.1. Optically Thick Halos

The property of the halo around individual luminous object depends on the luminosity, the spectrum of UV photon emission, and the time evolution of the center object. For luminous objects at high redshift, the halos generally consist of three spheres (Cen 2006; Liu et al. 2007). The innermost region is the highly ionized Strömgren sphere, or the H II region, which is optically thin of Lyα photons. The second region, which is just outside the H II region, is optically thick of Lyα photons. The temperature of the baryon gas is about 10^4 K, which is due to the heating of the UV photons. The third region is outside the heated region. It is unheated and therefore the temperature of the baryon gas can be as low as 10^2 K.

In this context, we will study the transfer of Lyα photons in a radius R spherical halo of neutral hydrogen with temperature T in the range of 10^2–10^4 K. Assuming that the uniformly distributed H I gas has number density n_{HI}, the optical depth over a light path dl is dτ(ν) = σ(ν)n_{HI}dl, where σ(ν) is the cross section of the resonant scattering of Lyα photons by hydrogen, given as

\[ σ(x) = σ_0φ(x,a), \]

where x is the dimensionless frequency defined by \( x ≡ (ν - ν_0)/Δν_ν \), ν₀ being the resonant frequency, \( Δν_ν = ν₀(ν_T/ε) \) is the Doppler broadening and \( ν_T = √{2k_B T/m} \). Therefore, x measures the frequency deviation \( Δν = |ν - ν_0| \) in units of \( Δν_ν = 1.06 × 10^{11}/(T/10^4)^{1/2} \) Hz. \( σ_0 = πe^4 f/m_e c Δν_ν = 1.10 × 10^{-2} cm^2 \) is the cross section of the resonant scattering at the frequency ν₀ = 2.46 × 10^{15} s⁻¹. The function \( φ(x,a) \) in Equation (1) is the normalized profile given by the Voigt function as (Hummer 1965)

\[ φ(x,a) = \frac{a}{π^{3/2}} \int_{−∞}^{∞} dy \frac{e^{−y^2}}{(x−y)^2 + a^2}. \]

The parameter a in Equation (2) is the ratio of the natural to the Doppler broadening. For the Lyα line, \( a = 4.7 × 10^{-4}(T/10^4)^{−1/2} \). The optical depth of the halo with column number density of neutral hydrogen \( N_{HI} = n_{HI}R \) is

\[ τ(x) = N_{HI} σ(x) = τ_0 φ(x,a). \]

Since \( φ(0,a) = 1/√π \) when \( a ≪ 1 \), the line-center optical depth is then \( τ(0) = τ_0/√π \) and

\[ τ_0 = 1.04 × 10^{7} \left( \frac{T}{10^4} \right)^{−1/2} \left( \frac{N_{HI}}{10^{20} cm^{-2}} \right). \]

2.2. Radiative Transfer Equation in Spherical Halo

The radiative transfer of Lyα photons in spherical halo is described by the equation of the specific intensity \( I(ν, r, x, μ) \) as

\[ \frac{∂I}{∂η} + μ \frac{∂I}{∂r} + (1−μ^2) \frac{∂I}{∂μ} - γ ∂I/∂x = −φ(x,a)I + \int R(x,x';a)I(ν, r', x', μ')dμ'dx' + S, \]

where we use dimensionless time \( η \) defined as \( η = cν_{HI}/σ_0f \) and dimensionless coordinate \( r \) defined as \( r = r_0ν_{HI}/σ_0f_r \), with \( r_0 \) being the physical radial coordinate. That is, \( η \) and \( r \) are, respectively, in the units of mean free flight time and mean free path of photon \( ν_0 \). For a signal propagated in the radial direction with the speed of light, we have \( r = η + const \). In Equation (5), \( μ = cos θ \) is the direction relative to the radial vector \( r \).

The redistribution function \( R(x,x';a) \) gives the probability of a photon absorbed at the frequency \( x' \) and re-emitted at the frequency \( x \). It depends on the details of the scattering (Henyey 1941; Hummer 1962, 1969). If we consider coherent scattering without recoil, the redistribution function with the Voigt profile (Equation (2)) is

\[ R(x,x';a) = \frac{1}{π^{1/2}} \int_{−π/2}^{π/2} e^{−a^2} \times \left[ \tan^{-1} \left( \frac{x_{min} + a}{a} \right) − \tan^{-1} \left( \frac{x_{max} - a}{a} \right) \right] du, \]

where \( x_{min} = min(x,x') \) and \( x_{max} = max(x,x') \). In the case of \( a = 0 \), i.e., considering only the Doppler broadening, the redistribution function is

\[ R(x,x') = \frac{1}{2} \text{erfc}[max(|x|, |x'|)]. \]

The redistribution function of Equation (7) is normalized as \( \int_{−∞}^{∞} R(x,x')dx' = φ(x,0) = π^{−1/2}e^{−x^2} \). With this normalization, the total number of photons is conserved in the evolution described by Equation (5). That is, the destruction processes of Lyα photons, such as the two-photon process (Spitzer & Greenstein 1951; Osterbrock 1962), are ignored in Equation (5). In Equations (6) and (7), we also do not consider the recoil of atoms. It is equal to assuming that the mass of atom is very large. The effect of recoil actually is under control (Roy et al. 2009c). We will address it in next section.

In Equation (5), the term with the parameter \( γ \) is due to the expansion of the universe. If \( n_f \) is equal to the mean of the number density of cosmic hydrogen, we have \( γ = τ_{GP} \), where \( τ_{GP} \) is the Gunn–Peterson optical depth. Since Gunn–Peterson optical depth is of the order of 10^6 at high redshift (e.g., Roy et al. 2009c), we will take the parameter \( γ = 10^{-5}–10^{-6} \).
2.3. Eddington Approximation

When the optical depth is large, we can take the Eddington approximation as

\[ I(\eta, r, x, \mu) \simeq J(\eta, r, x) + 3\mu F(\eta, r, x), \]  

where \( J(\eta, r, x) \) is the angularly averaged specific intensity and \( F(\eta, r, x) \) is the flux. Defining \( j = r^2 J \) and \( f = r^2 F \), Equation (5) yields the equations of \( j \) and \( f \) as

\[ \frac{\partial j}{\partial \eta} + \frac{2f}{\partial r} = -\phi(x; a)j + \int R(x, x'; a)jdx' + \gamma \frac{\partial j}{\partial x} + r^2 S, \]  

\frac{\partial f}{\partial \eta} + \frac{1}{3} \frac{\partial j}{\partial r} - \frac{2}{3} \frac{j}{r} = -\phi(x; a)f.  

(9)

(10)

The mean intensity \( j(\eta, r, x) \) describes the \( x \) photons trapped in the halo by the resonant scattering, while the flux \( f(\eta, r, x) \) describes the photons in transit.

The source term \( S \) of the Equations (9) and (10) can be described by a boundary condition of \( j \) and \( f \) at \( r = r_0 \). We can take \( r_0 = 0 \), as \( r_0 \) is not important if the optical depth of the halo is large. Thus, we have

\[ j(\eta, 0, x) = 0, \quad f(\eta, 0, x) = S_0 \phi_s(x), \]  

where \( S_0 \) and \( \phi_s(x) \) are, respectively, the intensity and normalized frequency profile of the sources. Since Equations (9)–(11) are linear, the intensity \( S_0 \) can be taken as any constant. That is, the solution \( f(x) \) of \( S_0 = S \) is equal to \( S f_1(x) \), where \( f_1(x) \) is the solution of \( S = 1 \). On the other hand, Equations (9) and (10) are not linear with respect to the function \( \phi_s(x) \), i.e., the solution \( f(x) \) of \( \phi_s(x) \) is not equal to \( \phi_s(x)f_1(x) \), where \( f_1 \) is the solution of \( \phi_s(x) = 1 \).

In the range outside the halo, \( r > R \), no photons propagate in the direction \( \mu < 0 \). Therefore, the boundary condition at \( r = R \) given by \( \int_0^{-1} \mu J(\eta, R, x, \mu)d\mu = 0 \) is then (Unno 1955)

\[ j(\eta, R, x) = 2f(\eta, R, x). \]  

(12)

There is no photon in the field before \( t = 0 \). Therefore, the initial condition is

\[ j(0, r, x) = f(0, r, x) = 0. \]  

(13)

We solve Equations (9) and (10) with the numerical method developed recently (Roy et al. 2009a, 2009b, 2009c). Some details of this method are given in the Appendix. We first solve the problems when the sources \( S \) is steady.

3. SOLUTIONS OF STEADY SOURCES

3.1. Time Scale of Escape

First we consider steady sources. That is, the parameter \( S_0 \) in Equation (11) is time independent after the switch-on of the sources at \( \eta = 0 \) (Equation (13)). A typical solution of the flux \( f(\eta, r, x) \) given by Equations (9) and (10) is shown in Figure 1, for which the source is taken to be \( S_0 = 1 \) and \( \phi_s(x) = (1/\sqrt{\pi})e^{-x^2} \), i.e., the emission line width is equal to the Doppler broadening. The parameters \( a \) and \( \gamma \) are taken to be \( 10^{-3} \) and \( 10^{-5} \), respectively. The effect of \( \gamma = 10^{-5} \) actually is ignorable in these solutions. The left panel is the solutions \( f(\eta, r, x) \) at radial \( r = 10^2 \) and time \( \eta = 500, 1000, 2000, \) and \( 3000 \) with the boundary condition (Equations (11) and (12)). The solutions approach a stable state at time \( \eta \geq 2000 \).

The right panel of Figure 1 is the total flux \( F_t(\eta) \equiv \int f(\eta, r, x)dx \). It again shows that total flux approaches a stable state with \( F_t = 1 \) at time \( \eta \geq 2000 \). If the Ly\( \alpha \) photon transfer is due only to spatial diffusion, the time scale of a spatial transfer with an optical depth \( \tau_0 = 10^2 \) should be as large as \( \eta \sim \tau_0^2 = 10^4 \). However, Figure 1 shows that photons have already escaped from the \( \tau_0 = 10^2 \) hole within the time \( \eta \sim 2000 \), which is much less than the time scale of a purely Brownian diffusion. Therefore, the transfer of photons should not be a process of purely Brownian diffusion. This point is very well known in early studies on the escape of resonant photons from opaque clouds (Osterbrock 1962; Harrington 1973; Avery & House 1968). The time scale of escape shown in Figure 1 is also consistent with the estimation given by Monte Carlo simulations (Adams, 1975; Bonilha et al. 1979). However, these works are mainly based on the time scale of the escape of photons with frequency, at which the photon can take a “single longest excursion” (Adams 1972). Figure 1 shows that the escape time scale \( \eta = 2000 \) is available not only for photons.
that can take a “single longest excursion” but also for all photons with frequency around $v_0$ or $x = 0$.

The right panel of Figure 1 also plays the role of testing our algorithm. Because $\int \phi(x, a) j \, dx = \int R(x, x'; a) j \, dx \, dx'$, Equation (9) yields

$$F_t(\eta, r) = F_t(\eta, 0) = S_0$$

when the solution approached the stable state. Equation (14) is the conservation of photon number. Since numerical errors which have accumulated over a long time evolution could be huge, Equation (14) is useful to check the reliability of the code.

The right panel of Figure 1 shows perfectly $F_t(\eta, r = 10^2) = F_t(\eta, 0) = S_0 = 1$ at stable state.

### 3.2. Time Scale of Local Thermalization

Figure 2 shows a solution of the mean intensity $j(\eta, r, x)$ of Equations (9) and (10) at $r = 10^2$ and time $\eta = 200, 300$, and $500$. The source is the same as Figure 1, namely $S_0 = 1$ and $\phi_0(x) = (1/\sqrt{\pi})e^{-x^2}$. Other parameters are also the same as Figure 1.

A remarkable feature of the solutions is to show a flat plateau around $x = 0$. As has been shown by Roy et al. (2009c), the flat plateau actually is the Boltzmann statistical equilibrium distribution around $x = 0$ when the atomic mass is infinite. If the mass is finite, i.e., considering the recoil in the redistribution functions (6) or (7), the flat plateau will become $e^{-2bt}$, where $b = h v_0/m v_T c$. This is the local Boltzmann distribution required by the Wouthuysen–Field effect (Roy et al. 2009b).

Without resonant scattering, the pure absorption will lead to the flux of $v_0$ photons at $r = 10^2$ to be attenuated by a factor $e^{-r/20}$, and so on. Therefore, Figure 2 shows that a major effect of resonant scattering is to restore $v_0$ photons in optically thick halos. According to the redistribution function $\mathcal{R}(x, x')$ (Equation (5)), the probability of transferring a $x'$ photon to $a [x] < [x']$ electron is larger than that from $x'$ to $[x] > [x']$. Therefore, the net effect of resonant scattering is to bring photons with frequency $x' \neq 0$ to the central range $x \sim 0$ of frequency space. Photons of $x \sim 0$ are then effectively restored. Meanwhile, the restored photons are thermalized. We see from Figure 2 that in the time range from $\eta = 200$ to $500$, the mean intensity $j$ at $x = 0$ quickly increases by a factor larger than 10. At the same time, the flat plateau or the local thermalization around $x = 0$ is perfectly held. Therefore, the time scales $t_{\text{ther}}$ of $v_0$ photon restoring and thermalization should be less than $\eta = 200$, which is much smaller than the time scale $t_{\text{escape}}$ of escaping from a halo of $r = 10^2$.

The result of $t_{\text{ther}} \ll t_{\text{escape}}$ in optically thick halos is very important. One can conclude that the photons of the flux $f$ around $x = 0$ emergent from optically thick halos actually come from the thermalized photons, regardless of the initial distribution of the photons. The initial conditions of the photon source have already been forgotten during the thermalization. Therefore, one can expect that the profile of flux $f$ has to be independent of the sources. We demonstrate this point with Figure 3.

Figure 3 presents the flux $f$ given by Equations (9) and (10) with the same parameters as the solution of Figure 1, and the source profiles are $\phi_0(x) = (1/\sqrt{\pi})e^{-x^2}$ (left panel) and $\phi_0(x) = (1/\sqrt{\pi/2})e^{-2x^2}$ (right panel). That is, the line widths are, respectively, 1 and $1/\sqrt{2}$. We see that the left and right panels of Figure 3 are almost identical within $|x| \leq 3$. Therefore, the initial distribution of photon frequency is already forgotten, and the photons of the left and right cases at $\eta > 200$ actually are from the same thermalized sources. The profiles of the flux are also held if the line width is broader than 1. We will show this point when the source has a continuant spectrum (Section 3.4).
are not completely thermalized yet. With the dimensionless variables, \( ar \) is equal to \( \alpha \tau \). A line \( |x_{\pm}| = (ar)^{1/3} \) is also shown in Figure 4. It shows that our numerical result of \( |x_{\pm}| \) is significantly different from the \((a\tau_0)^{1/3}\) law if \( ar < 30 \). That is, in the range \( ar < 30 \), photons of the flux \( f \) are dominated by the locally thermalized photons. The frequency \( x_{\pm} \) actually is dependent on the width of the flat plateau or the locally thermalized distribution of \( j \). Therefore, \( x_{\pm} \) is always larger than two. This feature has also been addressed in Bonilha et al. (1979).

The curve of Figure 4 is approximately available for \( a = 10^{-3} \) and \( 10^{-4} \). Thus, \( |x_{\pm}| \) is larger than \((ar)^{1/3}\) if \( r < 3 \times 10^5 \) and \( 3 \times 10^6 \) for \( a = 10^{-3} \) and \( 10^{-4} \), respectively.

Figure 4 shows a slow increase of \( |x_{\pm}| \) with \( r \) in the range \( ar \lesssim 30 \), and then it approaches \((ar)^{1/3}\) for larger \( ar \). When \( r \) is large, more photons of the flux are attributed to the resonant scattering by the Lorentzian wing of the Voigt function. \( |x_{\pm}| \) is then approaching \((ar)^{1/3}\). It should be emphasized once again that all these results are independent of the intrinsic width of Ly\( \alpha \) emission from the central source.

3.4. Absorption Spectrum

If the radiation from the sources has a continuous spectrum, the effect of neutral hydrogen halos is to produce an absorption line at \( \nu = \nu_0 \). The profile of the absorption line can also be found by solving Equations (9) and (10), but replacing the boundary Equation (11) by

\[
j(\eta, 0, x) = 0, \quad f(\eta, 0, x) = S_0. \tag{15}\]

That is, we assume that the original spectrum is flat in the frequency space.

A solution of the time evolution of \( j \) and \( f \) at \( r = 10^2 \) with the source Equation (15) is shown in Figure 5. The optical depths at the frequency \( |x| > 3 \) are small, and therefore the Eddington approximation would no longer be proper. However, those photons do not strongly involve the resonant scattering, and hence they do not significantly affect the solution around \( x = 0 \). Therefore, the solution is still useful to study the profiles of \( j \) and \( f \) around \( x = 0 \). The profile of \( j \) typically is an absorption line with width given by the position of the two-peak structure. As expected, the profile in the range \( |x| < 4 \) is the same as the left panel of Figure 1. It shows again that the two-peak structure is independent of the frequency spectrum of the central source.

The flux \( f \) of Figure 5 has a flat wing in both sides of \( x > 3 \) and \( x < -3 \), because the effect of resonant scattering is negligible.

3.3. Two Peaks in the Flux Profile

The profiles of either \( j \) or \( f \) shown in Figures 1–3 have two-peak structure. The flux \( f \) is dominated by photons with frequency \( x_{\pm} \approx (2-3) \). The two-peak structure has been recognized in all the time-independent solutions of the Fokker–Planck approximation (Harrington 1973; Neufeld 1990; Dijkstra et al. 2006) and Monte Carlo simulations (Lee 1974; Zheng & Miralda-Escude 2002; Ahn et al. 2002; Cantalupo et al. 2005; Verhamme et al. 2006). A point we would like to emphasize is that this structure is independent of the profile of the source \( S \). It is because the initial properties of the central sources have been forgotten during the local thermalization.

Since the shape of the locally thermalized distribution is time independent, the frequency of the two peaks, \( |x_{\pm}| \), at a given \( r \) is also time independent. When the photons of the flux \( f \) mainly come from the locally thermalized photons, the frequency of the two peaks, \( |x_{\pm}| \), should not be described by the relation \( |x_{\pm}| = (ar)^{1/3} \), because this relation is from a solution of the time-independent Fokker–Planck equation, which does not describe the thermalization (Neufeld 1990).

Figure 4 presents the peak frequency \( |x_{\pm}| \) as a function of \( ar \) for solutions given by Equations (9) and (10) with \( a = 10^{-2} \). We consider only \( r \gtrsim 10^2 \), as in the case \( r \lesssim 10^2 \) photons do not undergo a large enough number of scattering and therefore

![Figure 4](image-url) Positions of the peaks \( |x_{\pm}| \) as a function of \((ar)^{1/3}\) for the solutions of Equations (9) and (10) with \( a = 10^{-2} \). Other parameters are the same as in Figure 1. The dashed line is for \( x_{\pm} = \pm (ar)^{1/3} \).

![Figure 5](image-url) Solutions of \( j(\eta, r, x) \) (left) and \( f(\eta, r, x) \) (right) of Equations (9) and (10) at \( r = 10^2 \) and \( \eta = 200, 300, \) and 500. The source is given by Equation (16). The parameters are \( a = 10^{-3} \) and \( \gamma = 10^{-5} \).
for photons with frequency $|\nu| > 4$, and they can freely transfer from $r = 0$ to $10^2$. On the other hand, the mean intensity $j$, which is the isotropic component of the intensity $J$ (Equation (8)), does not have wings at $|\nu| > 4$. That is, resonant scattering cannot store photons with $|\nu| > 4$ in the halo. Nevertheless, the mean intensity $j$ still has a flat plateau around $x = 0$. It means that in the period from $\eta = 200$ to $\eta = 500$, the halo trapped and stored more and more photons of $|\nu| < 4$. These photons are in the locally thermalized state.

3.5. The Estimation of the H\textsc{i} Column Density

As an application of the absorption spectrum in Figure 5, we study the profile of the red damping wing of the optically thick IGM at high redshifts. Since the variable $x$ is independent of redshift, the profile of a red damping wing is directly given by the flux $f(\eta, r, x)$ at the wing of low frequency $x \leq 0$.

If the hydrogen clouds are located far from the Ly\textsc{a} sources, the red damping wing can be modeled as the absorption of optically thick halos. The red damping wing of damped Ly\textsc{a} system (DLA) model is then given by $f(x) = e^{-\tau(x)}$ and $x < 0$, where $\tau(x)$ is from the Voigt function Equation (2) as

$$\tau(x) = \tau_0 \phi(x, a) = \tau_0 \frac{a}{\pi^{3/2}} \int_{-\infty}^{\infty} dy \frac{e^{-y^2}}{(y-x)^2 + a^2}. \quad (16)$$

The column number density of neutral hydrogen atoms, $N_{H\textsc{i}}$, generally is estimated with fitting profile Equation (16) with observation, and then, $N_{H\textsc{i}}$ can be found from $\tau_0$ by Equation (4). If the light source is located in a hydrogen cloud, the column number density given by the fitting of Equation (16) should be underestimated, because resonant scattering helps the transfer of resonant photons.

An example Figure 6 gives (a) the red damping wings of Equation (16) with $\tau_0 = 2 \times 10^2$ and (b) a solution of Section 3.4 at $r = 10^4$ and $\eta = 9 \times 10^4$. The profiles of panels (a) and (b) are very different from each other. The former is quickly dropping when $|x|$ is less than about 3, while the later at $|x| < 3$ is softly dependent on $x$. The curve of panel (b) approaches a much higher bottom at $x = 0$.

More importantly, Figure 6 shows that the curve (a) with $\tau_0 = 2 \times 10^2$ is more or less close to the curve (b). That is, the DLA model at $\tau_0 = 2 \times 10^2$ may give a similar data fitting as the solution with resonant scattering. Therefore, for optical depth $\tau_0 = 10^3$, the DLA model of Equation (16) will underestimate the column number density of neutral hydrogen by about two orders.

4. SOLUTIONS OF FLASH SOURCES

4.1. Frequency Dependence of Ly\textsc{a} Transfer

If the light source is significantly time dependent, like the optical afterglow of GRBs, we can treat the source as a flash described by boundary conditions as

$$j(\eta, 0, x) = 0, \quad f(\eta, 0, x) = \begin{cases} S_0 \phi(x), & 0 < \eta < \eta_0 \\ 0, & \eta > \eta_0. \end{cases} \quad (17)$$

It describes a flash within a time range $0 < \eta < \eta_0$ or the time duration is $\Delta \eta = \eta_0$. We consider the case of $r \geq \eta_0$. That is, the size of the halo is larger than the spatial range lasted by the flash, as photon spatial transfer in an optically thick medium cannot be faster than the speed of light. The initial condition is still the same (Equation (13)).

Figure 7 presents two time-dependent solutions of the mean intensity $j$ and flux $f$; one is for a flash of Equation (17) with $\eta_0 = 100$ at $r = 10^4$; the other is for a flash with $\eta_0 = 500$ at $r = 10^3$. Other parameters are $S_0 = 1$, $a = 10^{-2}$, and $\phi(x) = (1/\sqrt{\pi}) e^{-x^2}$. We still see the typical flat plateau of $j$ at all times, even when the original time duration of the flash is as short as $\Delta \eta = 100$.

The time dependence of $j$ has a rising phase and a decaying phase. The time evolution of $j$ is held in both rising and decaying phases. We see from Figure 7 that the rising and decaying phases are frequency dependent. Photons at the two peaks reach their maximum at about $\eta = 200$ (top right panel) and $\eta = 4000$ (bottom right panel), while the flat plateau reaches their maximum at about $\eta = 500$ (top right panel) and $\eta = 6000$ (bottom right panel). That is, the halo holds a locally thermalized photons for a much longer time than the original time durations $\Delta \eta = 100$ (top panel) and $\Delta \eta = 500$ (bottom panel).

The time evolution of $f$ also consists of a rising phase and a decaying phase. Therefore, the flux emergent from the halo is also a flash. However, the time duration is very different from the original one. For the top panel, we see that the profile of $f$ is almost time independent from $\eta = 300$ to 500. That is, the time duration $500 - 300 = 200$ is much larger than the original one $\Delta \eta = 100$. For the bottom panel, the time independent flux is from $\eta = 3000$ to 5000, and therefore the time duration of the flash is about 2000, which is also much larger than the original time duration $\eta = 500$.

4.2. The Light Curves of a Flash

To further study the feature of time dependence of flash, we plot Figure 8, which gives the light curve of the flux $f$ at $x = 0$ (top panels) and the total flux (bottom panels). The top panels show the light curve of rising and decaying phases of the $v_0$ photon flux $f$. With these light curves, one can find the maximum of the flux $f$ at time $\eta_{\text{max}}$; the rising phase is then $\eta < \eta_{\text{max}}$ and decaying phase is $\eta > \eta_{\text{max}}$. For each curve, one can also define a time duration $\Delta \eta = \eta_2 - \eta_1$, where $\eta_1 < \eta_{\text{max}}$ and $\eta_2 > \eta_{\text{max}}$, and both are given by the condition $f(\eta_1, r, 0) = 0.9 f(\eta_{\text{max}}, r, 0)$. The time duration is then $\Delta \eta = \eta_2 - \eta_1$.

The two top panels of Figure 8 are for a flash source with original time duration $\Delta \eta = 50$. Figure 8 shows that the time duration of the flash, $\Delta \eta$, is $r$-dependent. At $r = 0$, i.e., at the source, $\Delta \eta = 50$. At $r = 10^2$ (top left of Figure 8), $\Delta \eta$ is about...
200, while at $r = 10^3$, we have $\Delta \eta \simeq 2000$. Therefore, the time duration $\Delta \eta$ increases with $r$. This result shows that the time duration seems to be dependent mainly on the size $r$ (or optical depth $r$) of the halo, regardless the initial time duration $\Delta \eta = \eta_0$.

The bottom two panels of Figure 8 are the light curves of the total flux $F_t(\eta, r) = \int f(\eta, r, x) dx$. The peak times $\eta_{\text{max}}$ of total flux generally are less than that of $\nu_0$ photon flux, as the $\nu_0$ photon needs longer restoration time. The time durations given by the total flux are also a little less than that of $\nu_0$ photon flux, but it still shows that the time duration is mainly dependent on the size $r$ of the halo.

Figure 9 presents the light curves of flash sources with time duration $\Delta \eta = \eta_0 = 1, 5, \text{and } 20$. The halo size is $r = 10^2$. For the case of $\eta_0 = 1$, we have $\eta_0 \ll r$. Therefore, the source can be considered as a pulse. The top panels of Figure 9 are for flux of $\nu_0$ photons, while the bottom panels are the corresponding total flux. Although the three flash sources have very different time durations at $r = 0$, their light curves of $f$ at $x = 0$ are very similar. The maximum values of $f$ for $\eta_0 = 1, 5, \text{and } 20$ can even be described by relations as $f_{20} \simeq 4f_5 \simeq 20f_1$. The coefficients 4 and 20 are from the ratio of the total numbers of photons of the three flashes. The three light curves of $F_t$ at the bottom panels of Figure 9 are also similar to each other, although they are not as good as the top three curves $f$. This is because the light curves are frequency dependent. Nevertheless, we still see that the three maximum values of $F_t$ also roughly satisfy the relation $F_{20} \simeq 4F_5 \simeq 20F_1$.

Either Figure 8 or Figure 9 reveals that the time scales of the propagation of a flash in halos are mainly dependent on the size $r$, regardless the original time duration. This feature indicates that the spatial transfer of a flash essentially is a diffusion process. As mentioned in Section 3, the spatial transfer of resonant photons cannot be described as a purely Brownian diffusion process, by which the time duration $\Delta \eta$ should increase with $r^2$ or $r^2$. On the other hand, if the spatial transfer of a photon can be, on average, described by a constant speed, the time duration $\Delta \eta$ of a flash should be, on average, equal to the original time duration or, at least, dependent on the original time duration. However, Figures 8 and 9 show that the time duration $\Delta \eta$ is roughly proportional to $r$ and independent of the initial time duration. Therefore, the spatial transfer of resonant photons essentially should still be a diffusion process.

4.3. Delayed Emission Line of Stored Photons

The effect of trapping and storing photons in an optically thick halo can be more clearly revealed with a flash source. Let us consider a flash source with a continuant spectrum, which is described by Equation (17), but we take $S_0 \phi_\eta(x) = 1$. The time evolutions of $j$ and $f$ for $\eta_0 = 50$ at $r = 10^2$ are presented in Figure 10, in which we take the time to be $\eta = 200, 300, \text{and } 500$. 

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**Figure 7.** Profiles of $j(\eta, r, x)$ and $f(\eta, r, x)$ with time-dependent source (Equation (17)). Top panel: $j(\eta, r, x)$ (left) and $f(\eta, r, x)$ (right) of $\eta_0 = 100$ at $r = 10^2$. Bottom panel: $\eta_0 = 500$ at $r = 10^3$. The parameters $a = 10^{-2}$ and $\gamma = 10^{-5}$. 

**Figure 8.** The light curves of $\nu_0$ photon flux for $\eta_0 = 1, 5, \text{and } 20$ at $r = 10^2$. 

**Figure 9.** The light curves of total flux $F_t(\eta, r)$ for $\eta_0 = 1, 5, \text{and } 20$ at $r = 10^2$. 

**Figure 10.** The light curves of $j(\eta, r, x)$ and $f(\eta, r, x)$ for $\eta_0 = 50$ at $r = 10^2$. 

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Figure 8. Top panel: the light curves of $f(\eta,r,x)$ at $x = 0$ at $r = 10^2$ (left) and $r = 10^3$ (right) for flash source with $\eta_0 = 50$. Bottom panel: the light curves of the total flux $F_t(\eta,r) = \int f(\eta,r,x) dx$ at $r = 10^2$ (left) and $r = 10^3$ (right).

Figure 9. Top panel: the time dependence of $f(\eta,r,x)$ at $x = 0$ and at $r = 10^2$ of flash source with $\eta_0 = 1$ (left), 5 (middle), and 20 (right). Bottom panel: the light curves of the total flux $F_t(\eta,r)$ of the corresponding top panel.
The top panels show the evolution of the mean intensity $j$ in the halo. In an early time $\eta = 200$, there are photons in the central range $|x| < 4$ as well as in wings $|x| > 4$. At later time $\eta = 300$, wing photons disappear, because all wing photons from the flash source have already escaped from the $r$ halo. At $\eta = 500$, the mean intensity $j$ of $|x| < 4$ is still about the same as $j$ at $\eta = 300$. The flat plateau of $j$ is shown at all times. Therefore, the locally thermalized photons are stored in the halo at least from time $\eta = 200$ to $500$, which is much longer than the $\eta_0 = 50$.

The evolution of the flux $f$ given by the bottom panels of Figure 10 is more interesting. At the time $\eta = 200$, the flux shows a typical absorption line at $\nu_0$. However, at $\eta = 300$, the flux $f$ becomes a typical emission line with two-peak profile. At time $\eta = 500$, $f$ is still a two-peak emission line. The flux of the emission at $\eta = 500$ is as strong as that at $\eta = 300$. Its light curve is similar to that of Figures 8 and 9. Note that Figures 8 and 9 are for a source of emission line, while Figure 10 is for a continuant spectrum. The similarity of the light curves of Figure 10 with Figures 8 and 9 is again due to the local thermalization and the diffusion in the physical space, both of which lead to the initial frequency spectrum and the time dependence of the photon sources being forgotten.

The emission at $\eta > 300$ is a delayed emission, as the flash of source has already ceased. The photons of the delayed emission are provided by the $|x| < 4$ photons stored in the halo $r < 10^2$. The time duration of the delayed emission is about the same as the time scale of the decaying phase of Figures 8 and 9. Therefore, it is also proportional to $r$, regardless the original time duration. Thus, at large $r$, a flash with a continuant spectrum and very short original time duration $\eta_0$ can produce a two-peak emission with time duration proportional to $r$.

5. DISCUSSIONS AND CONCLUSIONS

The resonant scattering made the transfers of resonant photons in physical space and frequency space to be coupled from each other. It leads to the time scale $\eta$ of the spatial transfer of the resonant photons in the halo with optical depth $\tau \gg 1$ being much faster than a purely Brownian diffusion process requiring $\eta \propto \tau^2$. However, essentially it is still a diffusion process, which can be approximately described by $\eta \propto x^{1/H}$, with the index $H$ less than but very close to 1. It is possible, if we consider the single longest excursion playing the role of a long-range dependence, that this diffusion process has a positive correlation or is a fractal Brownian diffusion (e.g., Beran 1994).

The number of photons basically is conserved. Thus, an optically thick halo is a store of photons with frequency $\sim \nu_0$. The time scale of the store is the same as the time scale of the above-mentioned diffusion, i.e., approximately proportional to the optical depth $\tau$. Moreover, the stored photons are always in the state of local Boltzmann distribution, even when the mean intensity is highly time dependent. The initial conditions are forgotten in the process of approaching the locally thermal equilibrium. The local Boltzmann distribution is independent of the frequency spectrum and time dependence of the source.

All these features show that the $\nu \simeq \nu_0$ photons play the central role of radiation transfer with resonant scattering. The major difference between our solutions and some analytical solutions (Harrington 1973; Neufeld 1990; Dijkstra et al. 2006) is also around $x = 0$. Since the analytical solutions are based on the assumption $\phi(x) = a/\sqrt{\pi}x^2$, and the Gaussian core $e^{-x^2}$ is ignored, it generally leads to $J(x = 0) = 0$. With these approximations, the solutions cannot show the effects of restoration and thermalization of photons around $x = 0$.  

Figure 10. Profiles of $j(\eta, r, x)$ (top) and $f(\eta, r, x)$ (bottom) at $r = 10^2$ when a flash source (Equation (17)) with $\eta_0 = 50$ and $S_0\phi_s(x) = 1$. The time is $\eta = 200$ (left), 300 (middle), and 500 (right).
These basic properties found in our numerical solutions yield the following features of Lyα photons emergent from an optically thick halo.

1. At a given $r$, the profile of the two peaks of the flux is time independent and also independent of the initial profile of the photons. Therefore, it is impossible to estimate the line profile of the source from the profile of the flux emergent from an optically thick halo.

2. The frequencies $|x_\pm|$ of the two peaks of the flux are not less than about 2. This would be useful to estimate the kinetic temperature of the neutral hydrogen atoms.

3. The resonant scattering makes the flux of the red damping wing very different from that of the DLA model. The flux is obtained by the fifth-order WENO approximation in an upwind fashion, because the wind direction is fixed. Denote $h_j = f(\eta^n, r_i, x_j)$, $j = -2, -1, \ldots, N + 3$ (A2) with fixed $x$ and $\eta$. The numerical flux from the WENO procedure is obtained by

$$\hat{h}_{j+1/2}^{(1)} = \omega_1 \hat{h}_{j+1/2}^{(1)} + \omega_2 \hat{h}_{j+1/2}^{(2)} + \omega_3 \hat{h}_{j+1/2}^{(3)}$$

where $\hat{h}^{(m)}$ are the three third-order fluxes on three different stencils given by

$$\hat{h}_{j+1/2}^{(1)} = -\frac{1}{6} h_{j-1} + \frac{5}{6} h_j + \frac{1}{3} h_{j+1},$$
$$\hat{h}_{j+1/2}^{(2)} = \frac{1}{3} h_j + \frac{5}{6} h_{j+1} - \frac{1}{6} h_{j+2},$$
$$\hat{h}_{j+1/2}^{(3)} = \frac{11}{6} h_{j+1} - \frac{7}{6} h_{j+2} + \frac{1}{3} h_{j+3},$$

and the nonlinear weights $\omega_m$ are given by

$$\omega_m = \frac{\hat{\alpha}_m}{\lambda^n}, \quad \hat{\alpha}_l = \frac{\gamma_l}{(\epsilon + \beta_l)^2},$$

where $\epsilon$ is a parameter to avoid the denominator to become zero and is taken as $\epsilon = 10^{-3}$. The linear weights $\gamma_l$ are given by

$$\gamma_1 = \frac{3}{10}, \quad \gamma_2 = \frac{3}{5}, \quad \gamma_3 = \frac{1}{10},$$

and the smoothness indicators $\beta_l$ are given by

$$\beta_1 = \frac{13}{12} (h_{j-1} - 2 h_j + h_{j+1})^2 + \frac{1}{4} (h_{j-1} - 4 h_j + h_{j+1})^2,$$
$$\beta_2 = \frac{13}{12} (h_j - 2 h_{j+1} + h_{j+2})^2 + \frac{1}{4} (h_j - h_{j+2})^2,$$
$$\beta_3 = \frac{13}{12} (h_{j+1} - 2 h_{j+2} + h_{j+3})^2 + \frac{1}{4} (3 h_{j+1} - 4 h_{j+2} + h_{j+3})^2.$$

To approximate the $r$-derivatives in the system of Equations (9) and (10), we need to perform the WENO procedure based on a characteristic decomposition. We write the left-hand side of Equations (9) and (10) as

$$\mathbf{u}_r + A \mathbf{u}_r,$$

where $\mathbf{u} = (j, f)^T$ and

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{3} & 0 \end{pmatrix}$$

is a constant matrix. To perform the characteristic decomposition, we first compute the eigenvalues, the right eigenvectors, and the left eigenvectors of $A$ and denote them by $\Lambda$, $R$, and $R^{-1}$. We then project $\mathbf{u}$ to the local characteristic fields $\mathbf{v}$ with $\mathbf{v} = R^{-1}\mathbf{u}$. Now $\mathbf{u}_r + A \mathbf{u}_r$ of the original system is decoupled as two independent equations as $\mathbf{v}_r + \Lambda \mathbf{v}_r$. We approximate the derivative $\mathbf{v}_r$ component by component, each with the correct upwind direction, with the WENO reconstruction procedure similar to the procedure described above for $\frac{\partial}{\partial x}$. In the end, we transform $\mathbf{v}_r$ back to the physical space by $\mathbf{u}_r = R \mathbf{v}_r$. We refer the readers to Cockburn et al. 1998 for more implementation details.
A.2. Adaptive Mesh Procedure for Non-uniform Grid

A fifth-order conservative finite difference WENO scheme can only be applied to a uniform grid or a smoothly varying grid. A smooth transformation,

$$\xi = \xi(r),$$  \hspace{1cm} (A7)

gives us a uniform grid in a new variable $\xi$. In this case, $\xi$ is sufficiently smooth, i.e., it has as many derivatives as the order of accuracy of the scheme. Therefore, the left-hand side of Equations (9) and (10) can be written as

$$u_t + Au,$$  \hspace{1cm} (A8)

where $u = (j, f)^T$ and

$$A = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$$

is transformed to

$$u_t + \xi A u,$$  \hspace{1cm} (A9)

and the WENO derivative approximation is now applied to $u_\xi$.

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