The Classification of Regular Surfaces Isogenous to a Product of Curves with $\chi(\mathcal{O}_S) = 2$

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1 Introduction

A complex surface $S$ is said to be \textit{isogenous to a product} if $S$ is a quotient

$$S = (C_1 \times C_2)/G,$$

where the $C_i$'s are curves of genus at least two, and $G$ is a finite group acting freely on $C_1 \times C_2$. Due to Catanese [Cat00] there are two possibilities how $G$ can act on the product $C_1 \times C_2$:

- For all $g \in G$ and $(z, w) \in C_1 \times C_2$ we have $g(z, w) = (g(z), g(w))$. In this case the action is called \textit{diagonal}.

- There exists $g \in G$, such that $g(z, w) = (g(w), g(z))$ for all $(z, w) \in C_1 \times C_2$. In this case the curves $C_1$ and $C_2$ are isomorphic.

If the action of $G$ is diagonal, we say that $S$ is of \textit{unmixed type}, else we say that $S$ is of \textit{mixed type}.

Since these surfaces were introduced by Catanese [Cat00] there has been produced a considerable amount of literature. In particular the surfaces isogenous to a product with $\chi(\mathcal{O}_S) = 1$ are completely classified: The Bogomolov-Miyaoka-Yau inequality $K_S^2 \leq 9\chi(\mathcal{O}_S)$ together with Debarre’s inequality $K_S^2 \geq 2p_g(S)$ gives $0 \leq q(S) = p_g(S) \leq 4$. By Beauville [Be82] all minimal surfaces $S$ of general type with $p_g(S) = q(S) = 4$ are a product of two genus two curves. A minimal surface $S$ of general type with $p_g(S) = q(S) = 3$ is either the symmetric square of a genus three curve or $S = (F_2 \times F_3)/\tau$, where $F_g$ is a curve of genus $g$ and $\tau$ is of order two acting on $F_2$ as an elliptic involution and on $F_3$ as a fixed point free involution [CCML98, Pir02, HP02]. The classifications of surfaces isogenous to a product in the remaining cases are: the case $p_g = 0$, $q = 0$, due to Bauer, Catanese, Grunewald [BCG08], $p_g = 1$, $q = 1$, due to Carnovale, Polizzi [CP09] and $p_g = 2$, $q = 2$, due to Penegini [Pe10].
Our aim is to give a classification in the case \( \chi(\mathcal{O}_S) = 2 \) under the assumption that \( S \) regular and of unmixed type. We want to mention that these surfaces have the invariants \( q(S) = 0 \) and \( p_g(S) = 1 \) like \( K3 \) surfaces and there are recent constructions of \( K3 \) surfaces with non-symplectic automorphisms as product quotient surfaces by Garbagnati and Penegini [GP13]. Our main result is the following (see also the table in 5.9):

**Theorem 1.1.** There are exactly 49 families of regular surfaces isogenous to a product of unmixed type with \( \chi(\mathcal{O}_S) = 2 \).

We will now explain how the paper is organized. In section 2 we explain the basics about surfaces isogenous to a product of curves. In section 3 we recall Riemann’s existence theorem and introduce the necessary tools from group theory and combinatorics. These facts are used to show that there is an entirely group theoretic description of surfaces isogenous to a product. In section 4 we recall the description of the moduli space of surfaces isogenous to a product, due to Catanese. In section 5 we give an algorithm, which we use to classify all regular surfaces isogenous to a product of unmixed type with \( \chi(\mathcal{O}_S) = 2 \). The computations are performed with the computer algebra system MAGMA [Mag]. In particular the Database of Small Groups and the Database of Perfect Groups is used. Afterwards we discuss the output of the computation. Finally we show the classification result.

Acknowledgments: The author thanks Ingrid Bauer and Sascha Weigl for several suggestions, useful discussions and very careful reading of the paper.

2 Surfaces Isogenous to a Product

In this section we explain some basic facts about surfaces isogenous to a product. We work over the field of complex numbers \( \mathbb{C} \) and use the standard notations from the theory of complex surfaces, see for example [Be83]. The self-intersection of the canonical class is denoted by \( K_S^2 \), the topological Euler number by \( e(S) = \sum_{i=1}^4 (-1)^i h^i(S, \mathbb{C}) \), \( \kappa(S) \) is the Kodaira dimension. The holomorphic-Euler-Poincaré-characteristic is defined as \( \chi(\mathcal{O}_S) := 1 - q(S) + p_g(S) \), where \( q(S) := h^1(S, \mathcal{O}_S) \) and \( p_g(S) := h^2(S, \mathcal{O}_S) \). The basic objects we consider are the following:

\[\text{http://www.staff.uni-bayreuth.de/~bt300503/}^{1}\]
Definition 2.1. A surface $S$ is said to be isogenous to a product if $S$ is a quotient

$$S = (C_1 \times C_2)/G,$$

where the $C_i$'s are smooth projective curves of genus at least two, and $G \leq \text{Aut}(C_1 \times C_2)$ is a finite group of automorphisms acting freely on $C_1 \times C_2$.

Immediate consequences of this definition are: The surface $S$ is smooth, projective and of general type, i.e. $\kappa(S) = 2$. The canonical class $K_S$ is ample. In particular $S$ is minimal.

The self-intersection of the canonical class $K_S^2$, the topological Euler number and the holomorphic-Euler-Poincaré-characteristic can be expressed in terms of the genera $g(C_i)$:

Proposition 2.2. \cite{Cat00} Theorem 3.4] Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product, then

$$K_S^2 = \frac{8(g(C_1) - 1)(g(C_2) - 1)}{|G|}, \quad e(S) = \frac{1}{2}K_S^2 \quad \text{and} \quad \chi(O_S) = \frac{1}{8}K_S^2.$$

In our case $p_g(S) = 1$ and $q(S) = 0$ we get $K_S^2 = 16$ and $e(S) = 8$, moreover we have the useful relation

$$2|G| = (g(C_1) - 1)(g(C_2) - 1). \quad (1)$$

Remark 2.3. For the rest of the paper we consider the unmixed case, where the action of $G$ on the product $C_1 \times C_2$ is diagonal, i.e.

$$G = G \cap (\text{Aut}(C_1) \times \text{Aut}(C_2)).$$

Since we consider unmixed actions only, we obtain two Galois coverings

$$f_1 : C_1 \rightarrow C_1/G, \quad f_2 : C_2 \rightarrow C_2/G.$$  

By \cite{Cat00} Proposition 3.13] one can assume without loss of generality, that $G$ acts faithfully on $C_1$ and on $C_2$. We want to relate the invariants $p_g(S)$ and $q(S)$ with the genera of the curves. To do this, we need the following theorem.

Theorem 2.4. Let $X$ be a smooth projective variety and $G$ be a finite group acting faithfully on $X$. If $Y = X/G$ is smooth, then

$$H^0(Y, \Omega_X^p) \simeq H^0(X, \Omega_X^p)^G.$$
A proof of this result can be found in [Griff76].

By K"unneth’s formula [GH78, p.103-104]:

\[
H^0(C_1 \times C_2, \Omega^1_{C_1 \times C_2})^G = H^0(C_1, \Omega^1_{C_1})^G \oplus H^0(C_2, \Omega^1_{C_2})^G.
\]

According to the previous theorem \(q(S) = h^0(C_1 \times C_2, \Omega^1_{C_1 \times C_2})^G\). Since \(q(S) = 0\) by assumption, we conclude that \(g(C_i/G) = 0\) for both \(i = 1, 2\). Thus the holomorphic maps \(f_i\) from above are Galois coverings of \(\mathbb{P}^1_\mathbb{C}\).

\section{Group theory, Riemann surfaces and combinatorics}

To give a purely group theoretic description of surfaces isogenous to a product, we introduce the required notation from group theory and combinatorics and recall Riemann’s existence theorem.

\begin{definition}
Let \(T = [m_1, ..., m_r] \in \mathbb{N}^r\) be an \(r\)-tuple. We define

\[
\Theta(T) := -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)
\]

and in case \(\Theta(T) \neq 0\)

\[
\alpha(T) := \frac{4}{\Theta(T)}.
\]

For \(r \geq 3\) we denote by \(\mathcal{N}_r\) the set of all \(r\)-tuples \([m_1, ..., m_r]\) with the following properties:

- \(2 \leq m_1 \leq ... \leq m_r\)
- \(\Theta([m_1, ..., m_r]) > 0\)
- \(\alpha([m_1, ..., m_r]) \in \mathbb{N}\)
- \(m_i \mid \alpha([m_1, ..., m_r])\) for all \(1 \leq i \leq r\).

The union of all \(\mathcal{N}_r\) is defined as \(\mathcal{N} := \bigcup_{r \geq 3} \mathcal{N}_r\). An element in \(\mathcal{N}\) is called a type. A type \(T\) contained in \(\mathcal{N}_r\) is said to be of length \(r\), we write \(l(T) = r\). Moreover for a type \(T = [m_1, ..., m_r]\) we use the notation \(T = [m_1, ..., m_r]_{\alpha(T)}\).
For simplicity we write
\[
[a_1^{k_1}, \ldots, a_r^{k_r}] := [a_1, \ldots, a_1, \ldots, a_r, \ldots, a_r].
\]

In the following lemma we give a classification of all types which satisfy the conditions from definition 3.1 above. This is the starting point of the classification of regular surfaces isogenous to a product with \( \chi(\mathcal{O}_S) = 2 \) (see also 3.3).

**Lemma 3.2.** There are no types of length \( r \) if \( r = 7 \) or \( r \geq 9 \). The set \( \mathcal{N} \) is finite and given by:

\[
\mathcal{N} = \left\{ \begin{array}{c}
[2, 3, 7]_{168}, [2, 3, 8]_{96}, [2, 4, 5]_{80}, [2, 3, 9]_{72}, [2, 3, 10]_{60}, \\
[2, 3, 12]_{48}, [2, 4, 6]_{48}, [3^2, 4]_{48}, [2, 3, 14]_{42}, [2, 5^2]_{40}, \\
[2, 3, 18]_{36}, [2, 4, 8]_{32}, [2, 3, 30]_{30}, [2, 5, 6]_{30}, [3^2, 5]_{30}, \\
[2, 4, 12]_{24}, [2, 6^2]_{24}, [3, 4^2]_{24}, [3^2, 6]_{24}, [3^3, 3]_{24}, \\
[3^2, 7]_{21}, [2, 4, 20]_{20}, [2, 5, 10]_{20}, [2, 6, 9]_{18}, [3^2, 9]_{18}, \\
[2, 8^2]_{16}, [4^3]_{16}, [2^3, 4^2]_{16}, [3, 5^2]_{15}, [3^2, 15]_{15}, \\
[2, 7, 14]_{14}, [2, 12^2]_{12}, [3, 4, 12]_{12}, [3, 6^2]_{12}, [4^2, 6]_{12}, \\
[2^3, 6]_{12}, [2^2, 3^2]_{12}, [5^3]_{10}, [2^3, 10]_{10}, [3, 9^2]_{9}, \\
[4, 8^2]_{8}, [2^2, 4^2]_{8}, [2^5]_{8}, [7^3]_{7}, [2^2, 6^3]_{6}, \\
[2^3, 6^2]_{6}, [3^4]_{6}, [2^4, 3]_{6}, [4^4]_{4}, [2^3, 4^2]_{4}, \\
[2^6]_{4}, [3^3]_{3}, [2^8]_{2}
\end{array} \right\}
\]

**Proof.** We use the fourth property in the case \( i = r \):

From \( m_r \left| -2 + \sum_{i=1}^{r} \left( \frac{4}{m_i} \right) \right| \) it follows \( \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \leq 2 + \frac{4}{m_r} \) (*).

Since \( m_i \geq 2 \) for all \( 1 \leq i \leq r \), we get

\[
r - 2 \leq \sum_{i=1}^{r} \left( \frac{1}{m_i} + \frac{4}{m_r} \right) \leq \frac{r}{2} + 2,
\]

and therefore \( r \leq 8 \). We now investigate two cases: \( r = 3 \) and \( r \geq 4 \).

- If \( r = 3 \), we claim that \( m_2 \geq 3 \). Suppose \( m_2 = 2 \), then also \( m_1 = 2 \) and

\[
0 < \Theta(m_1, m_2, m_3) = 1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} = -\frac{1}{m_3},
\]

a contradiction. The inequality (*) in the case \( r = 3 \) reads

\[
\frac{5}{m_3} \geq 1 - \frac{1}{m_1} - \frac{1}{m_2} \geq 1 - \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.
\]

From this we conclude \( m_3 \leq 30 \).
In the second case $r \geq 4$, we use the formula (*) again:

$$r - \sum_{i=1}^{r} \frac{1}{m_i} \leq 2 + \frac{4}{m_r}$$

and get

$$\frac{5}{m_r} \geq r - 2 - \sum_{i=1}^{r-1} \frac{1}{m_i} \geq \frac{r - 3}{2}.$$  

Because of this, we always have $10 \geq \frac{10}{r-3} \geq m_r$ in that case.

Now, it suffices to check only a finite number of types. This can be easily done with a computer. \hfill \Box

**Definition 3.3.** Let $G$ be a finite group, $2 \leq m_1 \leq \ldots \leq m_r$ integers. A spherical system of generators of $G$ of type $[m_1, \ldots, m_r]$ is an $r$-tuple $A = (g_1, \ldots, g_r)$ of elements of $G$, such that:

- $G = \langle g_1, \ldots, g_r \rangle$, $g_1 \cdot \ldots \cdot g_r = 1_G$.

- There exists a permutation $\tau \in \mathfrak{S}_r$, such that $\text{ord}(g_i) = m_{\tau(i)}$.

The stabilizer set of $A$ is defined as

$$\Sigma(A) := \bigcup_{h \in G} \bigcup_{j \in \mathbb{Z}} \bigcup_{i=1}^{r} \{h g_j^i h^{-1}\}.$$  

A pair $(A_1, A_2)$ of spherical systems of generators of $G$ is called disjoint, if and only if

$$\Sigma(A_1) \cap \Sigma(A_2) = \{1_G\}.$$  

The geometry behind this definition is known as **Riemann’s existence theorem**. A detailed explanation can be found in [Mir, chapter III, sections 3 and 4]. We will use the following version of this theorem:

**Theorem 3.4 (Riemann’s existence theorem).** A finite group $G$ acts as a group of automorphisms on a compact Riemann surface $C$ of genus $g(C) \geq 2$, such that $C/G \simeq \mathbb{P}^1$, if and only if there exists a spherical system of generators $A$ of $G$ of type $T = [m_1, \ldots, m_r]$, such that the following Riemann-Hurwitz formula holds:

$$2g(C) - 2 = |G| \Theta(T).$$
By Riemann’s existence theorem we have a group theoretical description of surfaces isogenous to a product: Given \( S = (C_1 \times C_2)/G \), isogenous to a product, we can attach a disjoint pair of spherical systems of generators

\[
(A_1(S), A_2(S)) \text{ of type } (T_1(S), T_2(S)).
\]

Geometrically, disjoint means that \( G \) acts without fixed points on \( C_1 \times C_2 \). Conversely, the data above determine a surface isogenous to a product.

Next, we want to show that the types \((T_1(S), T_2(S))\), attached to a regular surface \( S \) isogenous to a product with \( p_g(S) = 1 \) of unmixed type, satisfy the conditions of \([3.1]\). The proof of this fact is similar to the proof given in [BCG08]. For convenience of the reader we will present the proof.

**Theorem 3.5.** Let \( S \) be a surface isogenous to a product of curves of unmixed type with \( p_g(S) = 1 \) and \( q(S) = 0 \). Let \( T_1(S) = \left[ m_1, ..., m_r \right] \) and \( T_2(S) = \left[ n_1, ..., n_s \right] \) be the corresponding types, then

- \( \Theta(T_i(S)) > 0 \) for \( i = 1, 2 \).
- \( \alpha(T_i(S)) \in \mathbb{N} \) for \( i = 1, 2 \).
- \( m_i | \alpha(T_1(S)) \) for all \( 1 \leq i \leq r \) and \( n_i | \alpha(T_2(S)) \) for all \( 1 \leq i \leq s \).

**Proof.** We consider the holomorphic maps \( f_i : C_i \to C_i/G \) and apply the Riemann-Hurwitz formula

\[
2g(C_i) - 2 = |G| \Theta(T_i(S)), \quad i = 1, 2.
\]

This already shows the first claim, since \( g(C_1) \geq 2 \) and \( g(C_2) \geq 2 \). From (2) and \( 2|G| = (g(C_1) - 1)(g(C_2) - 1) \) \([1]\), we deduce

\[
\alpha(T_1(S)) = \frac{4}{\Theta(T_1(S))} = g(C_2) - 1 \quad \text{and} \quad \alpha(T_2(S)) = \frac{4}{\Theta(T_2(S))} = g(C_1) - 1
\]

so the second claim follows. It remains to prove the third claim. Let \( A_1(S) = \langle g_1, ..., g_r \rangle \) be a corresponding ordered spherical system of generators of \( G \) of type \( T_1(S) \). The cyclic group \( \langle g_i \rangle \) of order \( m_i \) acts on \( C_1 \) with at least one fixed point, but the action on the product \( C_1 \times C_2 \) is free. Therefore \( \langle g_i \rangle \) acts on \( C_2 \) freely. The map \( C_2 \to C \) of degree \( m_i \), where \( C := C_2/\langle g_i \rangle \) is unramified. In this case we have

\[
2g(C) - 2 = \frac{2\deg(C_2) - 2}{m_i} > 0, \quad \text{due to Riemann-Hurwitz. Hence}
\]

\[
g(C_2) - 1 = \alpha(T_1(S)) = m_i (g(C) - 1),
\]

for all \( i = 1, ..., r \). With the same argument we can show that \( n_i | \alpha(T_2(S)) \) for all \( i = 1, ..., s \). \( \square \)
4 Moduli Spaces

In this section we want to describe the moduli space of surfaces isogenous to a product. We follow the papers [BCG08, S31,S8-9] and [Pe11, Appendix]. Due to the work of Gieseker [Gie] there exists a quasi-projective moduli space of minimal smooth projective surfaces of general type with fixed invariants $K_S^2$ and $\chi(O_S)$, which is denoted by $M_{(\chi(O_S),K_S^2)}$. For a fixed finite group $G$ and a fixed pair of types $(T_1, T_2)$ we denote the subset of $M_{(2,16)}$ of isomorphism classes of surfaces isogenous to a product, which admit a disjoint pair of spherical systems of generators $(A_1, A_2)$ of type $(T_1, T_2)$, by $M_{(G,T_1,T_2)}$.

Theorem 4.1. [BCG08, Remark 5.1]

• The subset $M_{(G,T_1,T_2)} \subset M_{(2,16)}$ consists of a finite number of connected components of the same dimension, which are irreducible in the Zariski topology.

• The dimension $d(G,T_1,T_2)$ of any component in $M_{(G,T_1,T_2)}$ is

$$d(G,T_1,T_2) = l(T_1) - 3 + l(T_2) - 3.$$

The problem to determine the number $n$ of the connected components of $M_{(G,T_1,T_2)}$ can be translated in a group theoretical problem. We recall the following definition:

Definition 4.2. Let $r \in \mathbb{N}$ be a positive integer. We define the Artin-Braid group $B_r$ as

$$B_r := \left\langle \sigma_1, ..., \sigma_{r-1} \left| \begin{array}{c}
\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \\
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } i = 1, ..., r - 2
\end{array} \right. \right\rangle.$$

Let $G$ be a finite group and $T$ be a type of length $l(T) = r$. We denote the set of spherical systems of generators of $G$ of type $T$ by $B(G, T)$. The Artin-Braid group $B_r$ acts on $B(G, T)$ as follows:

$$\sigma_i(A) := (g_1, ..., g_{i-1}, g_i \cdot g_{i+1} \cdot g_i^{-1}, g_{i+2}, ..., g_r),$$

for all $A = (g_1, ..., g_r) \in B(G, T)$ and $1 \leq i \leq r - 1$. This determines a well-defined action, which is called the Hurwitz action. There is also a natural action of $Aut(G)$ on $B(G, T)$:

$$\varphi(A) := (\varphi(g_1), ..., \varphi(g_r)),$$
for all $\varphi \in Aut(G)$. Let $(\gamma_1, \gamma_2, \varphi) \in B_r \times B_s \times Aut(G)$ be a triple and $(A_1, A_2) \in B(G, T_1) \times B(G, T_2)$, we define

$$(\gamma_1, \gamma_2, \varphi) \cdot (A_1, A_2) := (\varphi(\gamma_1(A_1)), \varphi(\gamma_2(A_2))).$$

It is easy to verify that this defines an action of $B_r \times B_s \times Aut(G)$ on $B(G, T_1) \times B(G, T_2)$. We denote this action by $\Sigma$, and its restriction to the first factor $B(G, T_1)$ by $\Sigma_1$ and to the second factor $B(G, T_2)$ by $\Sigma_2$.  \[2\]

**Theorem 4.3.** [BCG08, Proposition 5.2] Let $S$ and $S'$ be surfaces isogenous to a product of unmixed type with $q(S) = q(S') = 0$. The surfaces $S$ and $S'$ are in the same irreducible component if and only if

- $G(S) \simeq G(S')$,
- $(T_1(S), T_2(S)) = (T_1(S'), T_2(S'))$,
- $(A_1(S), A_2(S))$ and $(A_1(S'), A_2(S'))$ are in the same $\Sigma$-orbit, or $(A_1(S), A_2(S))$ and $(A_2(S'), A_1(S'))$ are in the same $\Sigma$-orbit.

In theory we now have a method to compute the number of connected components of $M_{(G, T_1, T_2)}$: first we compute a representative $(A_1, A_2)$ for each orbit of the action $\Sigma$. Then we determine the pairs, where the intersection $\Sigma(A_1) \cap \Sigma(A_2)$ is trivial. If $T_1(S) = T_2(S)$ and there is more than one pair, we have to consider the $\mathbb{Z}_2$ action corresponding to the exchange of the curves. \[3\] The number of the remaining pairs is the number of connected components of $M_{(G, T_1, T_2)}$. However, the set $B(G, T_1) \times B(G, T_2)$ can be very large. Even with a computer it is not possible, or at least very time consuming, to perform this calculation. To improve the speed of the calculation we use an idea of Penegini and Rollenske, which is based on the following lemma:

**Lemma 4.4.** [Pe11, Appendix 6.1] Let $(A_1, A_2), (B_1, B_2) \in B(G, T_1) \times B(G, T_2)$.

- If $A_i$ and $B_i$ are in the same orbit of the Hurwitz action, then also the pairs $(A_1, A_2)$ and $(B_1, B_2)$ are in the same $\Sigma$-orbit.
- If $A_1$ and $B_1$ are in different $\Sigma_1$-orbits, then also $(A_1, A_2)$ and $(B_1, B_2)$ are in different $\Sigma$-orbits.

\[2\]We want to stress that the action of $Aut(G)$ and the Hurwitz action commute.

\[3\]This happens in only one of our examples. Thus we consider only the $\Sigma$ action in our program and treat the exceptional example separately. \[5.8\]
Now we have an effective algorithm:

- Compute a set \( R_1 \) of representatives of the action \( \Sigma_1 \) on \( B(G, T_1) \).
- Compute a set \( R_2 \) of representatives of the Hurwitz action on \( B(G, T_2) \).
- Determine the set of tuples \( (A_1, A_2) \in R_1 \times R_2 \) which satisfy:
  \[
  \Sigma(A_1) \cap \Sigma(A_2) = \{1_G\}.
  \]

This set is denoted by \( R \).

We achieve the following: Every orbit of \( \Sigma \) has at least one representative in \( R \) by \[4.4\]. Hence we have an upper bound for the number \( n \) of \( \Sigma \)-orbits. We also have a lower bound for \( n \). The pairs \( (A_1, A_2) \) and \( (B_1, B_2) \) are in different orbits of \( \Sigma \), if

1. \( A_1 \neq B_1 \) or
2. \( A_2 \) and \( B_2 \) are in different orbits of \( \Sigma_2 \) (cf. \[4.4\]).

- In most cases it is possible to determine the number \( n \) of orbits of \( \Sigma \) using this method. If this is not possible, then we exchange the pairs \( T_1 \) and \( T_2 \) and compute the set \( R \) again. If it is still not possible to determine \( n \) we have to identify the pairs of the smaller set using the action \( \Sigma \).

## 5 The algorithm and the classification result

In this section we explain our algorithm, which allows us to classify all regular surfaces isogenous to a product of curves of unmixed type with \( \chi(O(S)) = 2 \). For the implementation of the algorithm the computer algebra system MAGMA [Mag] is used. The program is based on the program in the appendix of [BCGP]. After the explanation of the algorithm, we discuss the output of the computations. Finally we give our classification result.

Let \( S = (C_1 \times C_2)/G \) be a regular surface isogenous to a product of unmixed type with \( \chi(O_S) = 2 \). From \( 2|G| = (g(C_1) - 1)(g(C_2) - 1) \), \( \alpha(T_1(S)) = g(C_2) - 1 \) and \( \alpha(T_2(S)) = g(C_1) - 1 \) for the attached types \( T_i(S) \), it follows

\[
|G| = \frac{1}{2} \alpha(T_1) \alpha(T_2).
\]
According to the list in lemma 3.2, $\alpha(T_i) \leq 168$ and therefore $|G| \leq 14112$. The group order is also bounded in terms of the genera $g(C_i)$, in fact $|\text{Aut}(C_i)| \leq 84(g(C_i) - 1)$ due to Hurwitz’ famous theorem. For small $g(C_i)$ there are better bounds. In Breuer’s book [Br00, p.91] there is a table which gives the maximum order of $|\text{Aut}(C_i)|$ in case $2 \leq g(C_i) \leq 48$.

**Definition 5.1.** Let $A = [m_1, ..., m_r]$ be an $r$-tuple of integers $m_i \geq 2$. The polygonal group $T(m_1, ..., m_r)$ is defined as

$$T(m_1, ..., m_r) = \langle t_1, ..., t_r \mid t_1 \cdot ... \cdot t_r = t_1^{m_1} = ... = t_r^{m_r} = 1 \rangle.$$ 

A group $G$ admitting a spherical system of generators of type $[m_1, ..., m_r]$ is a quotient of $T(m_1, ..., m_r)$. The following lemma will be used in the sequel. The proof of it is elementary and will be omitted.

**Lemma 5.2.** Let $G$ be a group and $H$ a quotient of $G$, then:

- $H^{ab}$ is a quotient of $G^{ab}$.
- The commutator subgroup $[H, H]$ is a quotient of $[G, G]$.
- If $G$ is a quotient of $T(2, 3, 7)$, then $G$ is perfect.

We can now describe the algorithm briefly. We perform the following steps:

**Step 1:** The program computes the set $N$ of types given in lemma 3.2. For every integer $m \leq 14112$ we compute the set of all triples of the form $(m, T_1, T_2)$ up to permutation of $T_1$ and $T_2$, where $T_1, T_2 \in N$ and $m = \frac{1}{2} \alpha(T_1) \alpha(T_2)$.

**Step 2:** For every triple $(m, T_1, T_2)$ the script computes $g(C_2) = \alpha(T_1(S)) + 1$ and $g(C_1) = \alpha(T_2(S)) + 1$. If $2 \leq g(C_i) \leq 48$ for at least one $i$, we check if $m$ is less or equal to the maximum group order of the automorphism group $\text{Aut}(C_i)$ allowed by Breuer’s table.

**Step 3:** For every triple $(m, T_1, T_2)$ passing this test, the script searches the list of groups of order $m$ for a group admitting a spherical system of generators of type $T_1$ and one of type $T_2$.

**Step 4:** For each triple $(m, T_1, T_2)$ and each group $G$ of order $m$, admitting a spherical system of generators of type $T_1$ and of type $T_2$, the script computes the number of orbits of the action $\mathcal{F}$ on $\mathcal{B}(G, T_1) \times \mathcal{B}(G, T_2)$, using the method explained after lemma 4.4.
In Step 3 we face two computational difficulties:

- In MAGMA’s Database of Small Groups all groups of order $|G| \leq 2000$ are contained, except the groups of order $|G| = 1024$ (which can not occur). In the case $2001 \leq |G| \leq 14112$, there is no MAGMA Database containing all groups of these orders. Only the perfect groups with $|G| \leq 50000$ are contained in MAGMA’s Database of Perfect Groups.

- Groups of order

$$|G| \in \{1920, 1152, 768, 512, 384, 256\}$$

can occur. Despite the fact, that we have access to all groups of these orders, it is not efficient to search through all of them for spherical systems of generators, because the number of these groups is too high.\footnote{Indeed there are 12,059,590 groups $G$, such that $|G| \in \{1920, 1152, 768, 512, 384, 256\}$.}

Due to these difficulties we split the program into two main routines namely Mainloop1 and Mainloop2.

- The function Mainloop1 treats the cases where

$$|G| \geq 2001 \text{ or } |G| \in \{1920, 1152, 768, 512, 384, 256\}.$$

i) The case $m = |G| \geq 2001$. For all triples $(m, T_1, T_2)$, we search the Database of Perfect Groups for a perfect group of order $m$, admitting a spherical system of generators of type $T_1$ and $T_2$. If neither $T_1 = [2, 3, 7]$ nor $T_2 = [2, 3, 7]$ we can not yet decide if there is a non-perfect group of order $m$, admitting a spherical system of generators of type $T_i$ (cf. 5.2). The script then saves the triple in the file exceptional.txt. We have to investigate these cases with theoretical arguments (see subsection 5.1), and we will show these do not occur.

ii) The case $|G| \in \{1920, 1152, 768, 512, 384, 256\}$. Each group $G$, admitting a spherical system of generators of type $T_1 = [n_1, ..., n_r]$ and of type $T_2 = [m_1, ..., m_s]$ is a quotient of

$$\mathbb{T}(T_1) := \mathbb{T}(n_1, ..., n_r) \text{ and } \mathbb{T}(T_2) := \mathbb{T}(m_1, ..., m_s).$$

According to lemma 5.2 the group $G^{ab}$ is a quotient of

$$\mathbb{T}(n_1, ..., n_r)^{ab} \text{ and } \mathbb{T}(m_1, ..., m_s)^{ab}.$$
• The function \textit{Mainloop2} treats the case where
\[ |G| \leq 2000 \quad \text{and} \quad |G| \not\in \{1920, 1152, 768, 512, 384, 256\}. \]
The output is written in the file \textit{loop2.txt}. Due to the high use of memory, if one of the types is \([2^8]\), we split the computation into two parts:

i) With the command \textit{Mainloop2}(n_1, n_2, 1) the script classifies, in the sense of above, all surfaces where \( n_1 \leq |G| \leq n_2 \) and \( T_1 \neq [2^8] \).

ii) With the command \textit{Mainloop2}(n_1, n_2, 0) the script classifies all surfaces where \( n_1 \leq |G| \leq n_2 \) and \( T_1 = [2^8] \).

Also in this main routine, if \( T_1 = [2^8] \) we have to treat some cases separately. Our workstation has not enough memory to compute the set \( B(G, [2^8]) \), if \( |G| \in \{168, 96, 48\} \).

The occurring triples \((n, T_1, T_2)\), where \( n \) is one of the group orders above are marked by the script as exceptional. We treat these cases in subsection 5.2.

Before we discuss the exceptional cases of the output, we explain some notation from group theory, that will be used in this and in the next section.

• We use the following MAGMA notation: \( \langle a, b \rangle \) denotes the group of order \( a \) having number \( b \) in the database of Small Groups [Mag].

• The group \( U(4, 2) \leq GL(4, \mathbb{F}_2) \) is defined to be the subgroup of upper triangle matrices \( A = (a_{ij}) \) with \( a_{ii} = 1 \), for all \( 1 \leq i \leq 4 \).

• The group \( G(128, 36) := \langle 128, 36 \rangle \) is given in a polycyclic presentation:
\[
\langle 128, 36 \rangle = \left\{ g_1, \ldots, g_7 \left| \begin{array}{l}
g_1^2 = g_4, \quad g_2^2 = g_5, \quad g_2^{g_1} = g_2 g_3 \\
g_3^{g_1} = g_3 g_6, \quad g_3^{g_2} = g_3 g_7, \quad g_4^{g_2} = g_4 g_6 \\
g_5^{g_1} = g_5 g_7
\end{array} \right. \right\}
\]

Here \( g_i^{g_j} \) means \( g_j^{-1} g_i g_j \). The squares of the generators \( g_1, \ldots, g_7 \), which are not mentioned in the presentation are equal to 1. If \( g_i^{g_i} = g_i \), this relation is omitted in the presentation.

5.1 Exceptional cases for \textit{Mainloop1}:

The following cases are skipped by the program, because the group order is greater than 2000, and saved in the file \textit{exceptional.txt}:
Our aim is to show, that the cases above cannot occur.

**Proposition 5.3.** There are no finite groups $G$ admitting a disjoint pair of spherical systems of generators $(A_1, A_2)$ of type $(T_1, T_2)$, where $|G|$, $T_1$ and $T_2$ are in the table above.

**Proof.** Our MAGMA code has already excluded the cases where $G$ is perfect. We only treat the case $|G| = 4608$, $T_1 = [2, 3, 8]$ and $T_2 = [2, 3, 8]$. The other cases can be excluded using similar methods. The ideas we use are from [BCG08]. The group $G$ is a quotient of $\mathcal{T}(2, 3, 8)$ and there is a surjective homomorphism $\phi^{ab}: \mathcal{T}(2, 3, 8)_{ab} \rightarrow G_{ab}$. Since $G$ is not perfect, $G^{ab} \neq \{1_G\}$. Similarly, the commutator subgroup $G' = [G, G]$ is also a quotient of $\mathcal{T}(2, 3, 8)' = [\mathcal{T}(2, 3, 8), \mathcal{T}(2, 3, 8)]$. We have

$$\mathcal{T}(2, 3, 8)^{ab} \simeq \mathbb{Z}_2, \text{ and } \mathcal{T}(2, 3, 8)' \simeq \mathcal{T}(3, 3, 4).$$

This implies $G^{ab} \simeq \mathbb{Z}_2$, thus $|G'| = 2304$ and the group $G'$ is a quotient of $\mathcal{T}(3, 3, 4)$.

$$\mathcal{T}(3, 3, 4)^{ab} \simeq \mathbb{Z}_3, \text{ and } \mathcal{T}(3, 3, 4)' \simeq \mathcal{T}(4, 4, 4).$$

Since $|G'| = 2^8 \cdot 3^2$ the group $G'$ is solvable, due to Burnside’s theorem [B04]. We have $G'' \simeq \mathbb{Z}_3$, thus: $|G''| = 768$ and $G''$ is a quotient of $\mathcal{T}(4, 4, 4)$. This is impossible according to the following computational fact, which can be verified with MAGMA.

**Lemma 5.4.** [BCG08, Lemma 4.11] There are 1090235 groups of order 768. None of them is a quotient of $\mathcal{T}(4, 4, 4)$. 

\[14\]
5.2 Exceptional cases for Mainloop2:

Here \( T_1 = [2^8] \) and \( T_2 \in \mathcal{N} \). We have \( |G| = \frac{1}{2} \alpha(T_1) \alpha(T_2) = \alpha(T_2) \leq 168 \). Since our workstation has not enough memory to compute the set \( B(G, [2^8]) \) if \( |G| \in \{168, 96, 48\} \), these cases are marked as exceptional. We receive the following output:

| \( T_2 \) | \( G \) | No. of \( \mathcal{T} \)-orbits |
| --- | --- | --- |
| \([2, 3, 7]\) | \((168, 42)\) | 0 | exceptional case |
| \([2, 3, 8]\) | \((96, 64)\) | 0 | exceptional case |
| \([2, 4, 6]\) | \((48, 48)\) | 2 | exceptional case |
| \([3, 4^2]\) | \((24, 12)\) | 1 | |
| \([2^4, 4]\) | \((16, 11)\) | 2 | |
| \([2^5]\) | \((8, 5)\) | 1 | |

Next, we will investigate the exceptional cases.

**Proposition 5.5.** The group \( (168, 42) \) has no disjoint pair of spherical systems of generators of type \( ([2^8], [2, 3, 7]) \).

**Proof.** A MAGMA computation shows, that this group has only one conjugacy class of elements of order 2. Hence, for every pair \( (A_1, A_2) \) of generators of type \( ([2^8], [2, 3, 7]) \) the intersection \( \Sigma(A_1) \cap \Sigma(A_2) \) is nontrivial. \( \square \)

**Proposition 5.6.** The group \( (96, 64) \) has no disjoint pair of spherical systems of generators of type \( ([2^8], [2, 3, 8]) \).

**Proof.** A MAGMA calculation shows, that this group has two conjugacy classes of elements of order 2. We denote them by \( K_1 \) and \( K_2 \). We have \( |K_2| = 3 \) and \( \langle K_2 \rangle \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \). Since \( |\langle K_2 \rangle| = 4 \), there is no spherical system of generators \( A_1 = (h_1, ..., h_8) \) of \( G \) of type \( [2^8] \), with \( h_i \in K_2 \) for all \( 1 \leq i \leq 8 \). Every spherical system of generators of \( G \) contains elements of \( K_1 \). A further MAGMA calculation shows, that there is no spherical system of generators \( A_2 = (g_1, g_2, g_3) \) of \( G \) of type \( [2, 3, 8] \), with \( g_1 \in K_2 \). \( \square \)

**Proposition 5.7.** The group \( (48, 48) \) admits disjoint pairs of spherical systems of generators of type \( ([2^8], [2, 4, 6]) \). The number of \( \mathcal{T} \)-orbits is two.

**Proof.** The group \( G \) has 19 elements of order 2, they are contained in 5 conjugacy classes. We denote them by \( K_1, ..., K_5 \) and the set of elements of order two by \( M \).
class | rep | length
---|---|---
$K_1$ | $G.2$ | 1
$K_2$ | $G.4$ | 3
$K_3$ | $G.2 \ast G.4$ | 3
$K_4$ | $G.1$ | 6
$K_5$ | $G.1 \ast G.2$ | 6

A MAGMA calculation shows, that $g_1 \in K_4 \cup K_5$ for each spherical system of generators $(g_1, g_2, g_3) \in \mathcal{B}(G, [2, 4, 6])$. Since we are interested in disjoint pairs of spherical systems of generators, it is not necessary to compute the whole set $\mathcal{B}(G, [2^8])$. All elements $(h_1, ..., h_8) \in \mathcal{B}(G, [2^8])$, which contain some $h_i \in K_4$, as well as some $h_j \in K_5$ are irrelevant. For each $1 \leq l \leq 5$ we define subsets

$$\mathcal{B}_l := \{ [h_1, ..., h_8] \in \mathcal{B}(G, [2^8]) \mid h_i \in M \setminus K_l \}$$

of $\mathcal{B}(G, [2^8])$ and compute them in the cases $l = 4$ and $l = 5$. We have $|\mathcal{B}_4| = |\mathcal{B}_5| = 9,213,120$. Since the Hurwitz action acts via conjugation and permutation of elements, we can restrict it to $\mathcal{B}_4$ and to $\mathcal{B}_5$. Next we compute for each orbit of the restricted actions a representative. The sets of representatives are denoted by $\mathcal{R}_4$ and $\mathcal{R}_5$. We have $|\mathcal{R}_4| = |\mathcal{R}_5| = 10$. All elements in $\mathcal{B}(G, [2, 4, 6])$ are contained in two orbits of the Hurwitz action. We denote by $A_1$ and $A_2$ two representatives of these orbits. There are two disjoint pairs in $\{A_1, A_2\} \times \mathcal{R}_4$, and two disjoint pairs in $\{A_1, A_2\} \times \mathcal{R}_5$. According to [1,4] we have at least one representative for each $\mathcal{T}$-orbit. Using the function ”Orbi” we can identify the pairs above, which are $\mathcal{T}$-equivalent. We find two equivalence classes. To verify the above computations a source code can be found at [http://www.staff.uni-bayreuth.de/~bt300503/script.txt](http://www.staff.uni-bayreuth.de/~bt300503/script.txt).

**Remark 5.8.**

From the output files we can see that there is exactly one occurrence with $T_1(S) = T_2(S)$ and $n \geq 2$. In this case $G = G(128, 36)$. The types are $T_i = [4, 4, 4]$ and $n = 2$. We denote by $(A_1, A_2)$ and $(B_1, B_2)$ the representatives for the $\mathcal{T}$-orbits from the output file loop2.txt. It remains to check if $(A_2, A_1)$ and $(B_1, B_2)$ are in the same $\mathcal{T}$-orbit. A MAGMA computation shows that this is not the case. The source code for this computation is available at [http://www.staff.uni-bayreuth.de/~bt300503/script.txt](http://www.staff.uni-bayreuth.de/~bt300503/script.txt).

Now we can give our main theorem, which implies in particular theorem [1,1]
Theorem 5.9. Let $S = (C_1 \times C_2)/G$ be a regular surface isogenous to a product of unmixed type with $\chi(O_S) = 2$. Then $g(C_1)$, $g(C_2)$, the group $G$ and the corresponding types $T_1(S)$, $T_2(S)$ are:

| $g(C_1)$ | $g(C_2)$ | $G$ | $Id$ | $T_1(S)$ | $T_2(S)$ | $n$ |
|----------|----------|-----|------|----------|----------|-----|
| 17       | 43       | $PSL(2, \mathbb{F}_7) \times \mathbb{Z}_2$ | $\langle 336, 209 \rangle$ | $[2, 3, 14]$ | $[4^3]$ | 2 |
| 49       | 9        | $(\mathbb{Z}_2)^3 \rtimes \mathbb{S}_4$ | $\langle 192, 955 \rangle$ | $[2^2, 4^2]$ | $[2, 4, 6]$ | 2 |
| 49       | 8        | $PSL(2, \mathbb{F}_7)$ | $\langle 168, 42 \rangle$ | $[7^3]$ | $[3^2, 4]$ | 2 |
| 17       | 22       | $PSL(2, \mathbb{F}_7)$ | $\langle 168, 42 \rangle$ | $[3^2, 7]$ | $[4^2]$ | 2 |
| 5        | 81       | $(\mathbb{Z}_2)^4 \rtimes \mathbb{D}_5$ | $\langle 160, 234 \rangle$ | $[2, 4, 5]$ | $[4^2]$ | 5 |
| 17       | 17       | $G(128, 36)$ | $\langle 128, 36 \rangle$ | $[4^3]$ | $[4^2]$ | 2 |
| 9        | 31       | $\mathbb{S}_5$ | $\langle 120, 34 \rangle$ | $[2, 5, 6]$ | $[2^2, 4^2]$ | 1 |
| 5        | 49       | $(\mathbb{Z}_2)^4 \rtimes \mathbb{D}_3$ | $\langle 96, 195 \rangle$ | $[2, 4, 5]$ | $[4^2]$ | 1 |
| 25       | 9        | $(\mathbb{Z}_2)^4 \rtimes \mathbb{D}_3$ | $\langle 96, 227 \rangle$ | $[2^2]$ | $[3^2, 4^2]$ | 1 |
| 9        | 17       | $(\mathbb{Z}_2)^3 \rtimes \mathbb{D}_4$ | $\langle 64, 73 \rangle$ | $[2^2, 4]$ | $[2^2, 4^2]$ | 1 |
| 9        | 17       | $U(4, 2)$ | $\langle 64, 138 \rangle$ | $[2^2, 4]$ | $[2^2, 4^2]$ | 1 |
| 13       | 11       | $\mathbb{A}_5$ | $\langle 60, 5 \rangle$ | $[5^3]$ | $[2^2, 3^2]$ | 1 |
| 41       | 4        | $\mathbb{A}_5$ | $\langle 60, 3 \rangle$ | $[3^5]$ | $[2, 5^2]$ | 2 |
| 9        | 16       | $\mathbb{A}_5$ | $\langle 60, 5 \rangle$ | $[3^5, 2]$ | $[2^2]$ | 2 |
| 5        | 31       | $\mathbb{A}_5$ | $\langle 60, 5 \rangle$ | $[3^2, 5^2]$ | $[2^6]$ | 1 |
| 5        | 25       | $\mathbb{S}_4 \times \mathbb{Z}_2$ | $\langle 48, 48 \rangle$ | $[2^3, 3]$ | $[4^2]$ | 1 |
| 9        | 13       | $\mathbb{S}_4 \times \mathbb{Z}_2$ | $\langle 48, 48 \rangle$ | $[2^2, 6]$ | $[2^2, 4^2]$ | 1 |
| 13       | 9        | $\mathbb{S}_4 \times \mathbb{Z}_2$ | $\langle 48, 48 \rangle$ | $[2^3]$ | $[4^2, 6]$ | 1 |
| 3        | 49       | $\mathbb{S}_4 \times \mathbb{Z}_2$ | $\langle 48, 48 \rangle$ | $[2, 4, 6]$ | $[2^5]$ | 2 |
| 9        | 9        | $(\mathbb{Z}_2)^3 \rtimes \mathbb{Z}_4$ | $\langle 32, 22 \rangle$ | $[2^2, 4^2]$ | $[2^2, 4^2]$ | 1 |
| 9        | 9        | $\mathbb{D}_3 \rtimes (\mathbb{Z}_2)^2$ | $\langle 32, 46 \rangle$ | $[2^5]$ | $[2^5]$ | 1 |
| 17       | 5        | $(\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_2$ | $\langle 32, 27 \rangle$ | $[2^3, 4^2]$ | $[2^2, 4]$ | 3 |
| 9        | 9        | $(\mathbb{Z}_2)^4 \rtimes \mathbb{Z}_2$ | $\langle 32, 27 \rangle$ | $[2^2, 4^2]$ | $[2^5]$ | 1 |
| 5        | 13       | $\mathbb{S}_4$ | $\langle 24, 12 \rangle$ | $[2^2, 3^2]$ | $[4^2]$ | 1 |
| 3        | 25       | $\mathbb{S}_4$ | $\langle 24, 12 \rangle$ | $[3, 4^2]$ | $[2^6]$ | 1 |
| 5        | 9        | $\mathbb{D}_3 \rtimes \mathbb{Z}_2$ | $\langle 16, 11 \rangle$ | $[2^5]$ | $[2^3, 4^2]$ | 1 |
| 9        | 5        | $(\mathbb{Z}_2)^2 \rtimes \mathbb{Z}_4$ | $\langle 16, 3 \rangle$ | $[2^3, 4^2]$ | $[2^2, 4^2]$ | 2 |
| 9        | 5        | $(\mathbb{Z}_2)^4$ | $\langle 16, 14 \rangle$ | $[2^6]$ | $[2^5]$ | 2 |
| 3        | 17       | $\mathbb{D}_3 \rtimes \mathbb{Z}_2$ | $\langle 16, 11 \rangle$ | $[2^4, 4]$ | $[2^5]$ | 2 |
| 7        | 4        | $(\mathbb{Z}_3)^2$ | $\langle 9, 2 \rangle$ | $[3^5]$ | $[3^4]$ | 1 |
| 5        | 5        | $(\mathbb{Z}_2)^3$ | $\langle 8, 5 \rangle$ | $[2^6]$ | $[2^6]$ | 1 |
| 3        | 9        | $(\mathbb{Z}_2)^3$ | $\langle 8, 5 \rangle$ | $[2^6]$ | $[2^6]$ | 1 |

Each row in the table corresponds to a union of connected components of the Gieseker moduli space of surfaces of general type with $K_S^2 = 16$ and $\chi = 2$. The number $n$ of these components is given in the last column.
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