Discontinuity of topological entropy for Lozi maps

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Abstract. Recently, Buzzi [Maximal entropy measures for piecewise affine surface homeomorphisms. Ergod. Th. & Dynam. Sys. 29 (2009), 1723–1763] showed in the compact case that the entropy map \( f \to h_{\text{top}}(f) \) is lower semi-continuous for all piecewise affine surface homeomorphisms. We prove that topological entropy for Lozi maps can jump from zero to a value above 0.1203 as one crosses a particular parameter and hence it is not upper semi-continuous in general. Moreover, our results can be extended to a small neighborhood of this parameter showing the jump in the entropy occurs along a line segment in the parameter space.

1. Introduction

There have been some recent developments in the study of piecewise affine surface homeomorphisms. In the compact case, Buzzi proved that under the assumption of positive topological entropy, there are finitely many ergodic measures maximizing the entropy [1]. He also showed that topological entropy is lower semi-continuous for these maps. The following question was asked by Buzzi.

Question 1. Prove or disprove the upper semi-continuity of entropy for piecewise affine homeomorphisms of the plane.

Our goal is to answer Buzzi’s above question in the non-compact case by showing that topological entropy of the Lozi map is not upper semi-continuous at a given parameter. Moreover, our results can be extended to show that there is a line segment in the parameter space along which the topological entropy is not upper semi-continuous.

Let us start with a review of the subject.

Piecewise affine homeomorphisms. Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a homeomorphism, where \( n \in \mathbb{Z}^+ \). An affine subdivision of \( f \) is a finite collection \( \mathcal{U} = \{ U_1, \ldots, U_N \} \) of pairwise disjoint non-empty open subsets of \( \mathbb{R}^n \) such that their union is dense in \( \mathbb{R}^n \) and \( f|_{U_i} = A_i|_{U_i} \).
for each $i = 1, \ldots, N$, where $A_i : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible affine map. A *piecewise affine homeomorphism* is a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ for which there exists an affine subdivision.

**Example 1.** Lozi maps are piecewise affine homeomorphisms of the plane given by

$$L = L_{a,b} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 - a|x| + by \\ x \end{pmatrix}, \quad b \in \mathbb{R}, b \neq 0.$$ 

Note that $\mathcal{U} = \{U_1, U_2\}$, where $U_1 = \{(x, y) \in \mathbb{R} \mid x > 0\}$ and $U_2 = \{(x, y) \in \mathbb{R} \mid x < 0\}$.

Let us first review some of the results about continuity properties of entropy in different dimensions. Throughout this paper, we will denote the topological entropy of a map $f$ by $h(f)$.

In one dimension, one can work with piecewise monotone functions. Let $I$ denote a compact interval of $\mathbb{R}$. A map $T : I \to I$ is called a *piecewise monotone* function if there exists a partition of $I$ into finitely many subintervals on each of which the restriction of $T$ is continuous and strictly monotone. Two piecewise monotone maps $T_1$ and $T_2$ are said to be $\varepsilon$-close, if they have the same number of intervals of monotonicity and the graph of $T_2$ is contained in an $\varepsilon$-neighborhood of the graph of $T_1$ considered as subsets of $\mathbb{R}^2$.

It was proved by Misiurewicz and Szlenk [13] that the entropy map $f \to h(f)$ is lower semi-continuous for piecewise monotone continuous maps. They also gave upper bounds for the jumps up of the entropy. For unimodal maps, entropy is continuous for all maps for which it is positive [12].

There are also some continuity results in higher dimensions. Let $C^r(M^n)$ denote the set of $C^r$ self maps of an $n$-dimensional compact manifold. It is a classical result of Katok [9] that the entropy map is lower semi-continuous for $C^{1+\alpha}$ diffeomorphisms on compact surfaces. Yomdin [18] and Newhouse [14] proved that entropy is upper semi-continuous in $C^\infty(M^n)$ for $n \geq 1$. Combining these two results, one can get the continuity of entropy in $C^\infty(M^2)$. This result does not hold for homeomorphisms on surfaces [16]. Also, Misiurewicz [10] constructed examples showing that entropy is not continuous in $C^\infty(M^n)$ for $n \geq 4$, as well as examples [11] showing that entropy is not upper semi-continuous in $C^r(M^n)$, where $r < \infty$ and $n \geq 2$.

For piecewise affine surface homeomorphisms, the following Katok-like theorem (see [8]) was given by Buzzi [1].

**Theorem 1.1.** Let $f : M \to M$ be a piecewise affine homeomorphism of a compact affine surface. Let $S$ be the singularity locus of $M$, that is, the set of points $x$ which have no neighborhood on which the restriction of $f$ is affine. For any $\varepsilon > 0$, there is a compact invariant set $K \subset M \setminus S$ such that $h(f|K) > h(f) - \varepsilon$. Moreover, $f : K \to K$ is topologically conjugate to a subshift of finite type.

The lower semi-continuity of the entropy in the compact case follows from the above theorem. This result may also hold in the non-compact case but it requires more work. The goal of this paper is to disprove the upper semi-continuity in the non-compact case by showing a jump up of the entropy in Lozi maps. Our results can be summarized as follows.
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Theorem 1.2. (Main theorem) In general, the topological entropy of the Lozi map does not depend continuously on the parameters: there exists some \( \epsilon_+ > 0 \) such that for all \( 0 < \epsilon_1 < \epsilon_+ \) and \( |\epsilon_2| < \epsilon_+ \):

(i) the topological entropies of the Lozi maps with \( (a, b) = (1.4 + \epsilon_2, 0.4 + \epsilon_2) \), \( h(L_{1.4+\epsilon_2,0.4+\epsilon_2}) \), are zero; and

(ii) the topological entropies of the Lozi maps \( h(L_{1.4+\epsilon_1+\epsilon_2,0.4+\epsilon_2}) \) have a lower bound of 0.1203.

In other words, we show that the entropy is zero on the line segment \( l = \{(1.4 + \epsilon_2, 0.4 + \epsilon_2) : |\epsilon_2| < \epsilon_+\} \) and it is above 0.1203 for the parameters immediately to the right of that segment.

2. Topological entropy

Topological entropy is a quantitative measurement of how complicated a map is.

Definition 2.1. Let \( f : X \to X \) be a continuous map on a compact metric space \( (X, d) \) with a metric \( d \). Two distinct points \( x, y \in X \), \( x \neq y \), are called \((n, \epsilon)\)-separated for a positive integer \( n \) and \( \epsilon > 0 \) if there is \( m \in \{0, 1, \ldots, n - 1\} \) such that \( d(f^m(x), f^m(y)) > \epsilon \).

A set \( U \subset X \) is called an \((n, \epsilon)\)-separated set if every pair of distinct points \( x, y \in U \), \( x \neq y \), is \((n, \epsilon)\)-separated.

Let \( r(n, \epsilon, f) \) be the maximum cardinality of an \((n, \epsilon)\)-separated set \( U \subset X \). By compactness, this number is always finite. Define

\[
h(\epsilon, f) = \limsup_{n \to \infty} \frac{\log(r(n, \epsilon, f))}{n}.
\]

Then the topological entropy of \( f \), \( h(f) \), is defined as

\[
h(f) = \lim_{\epsilon \to 0, \epsilon > 0} h(\epsilon, f).
\]

Remark. Note that the Lozi map is defined on \( \mathbb{R}^2 \), which is not compact. To be able to investigate the topological entropy of the Lozi map, we take one-point compactification of \( \mathbb{R}^2 \) and extend the map continuously to this set. For more details about this continuous extension, see [7].

3. Lower bound techniques

There are some computer-assisted techniques to give rigorous lower bounds for the topological entropy of maps, like Hénon [4] and Ikeda [5]. They were first introduced by Zygliczyński [19] and developed in [2, 3]. There are also more recent methods by Newhouse et al [15] which give better lower bounds for the Hénon map.

Let us review the following ideas which were used in [2].

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a continuous map and \( N_1, N_2, \ldots, N_p \) be \( p \) pairwise disjoint quadrilaterals. Note that we can parametrize each \( N_i \) with the unit square \( I^2 = [0, 1] \times [0, 1] \) by choosing a homeomorphism \( h_i : I^2 \to N_i \). We call the edges \( h_i([0] \times [0, 1]) \) ‘vertical’ and the edges \( h_i([0, 1] \times [0]) \) and \( h_i([0, 1] \times \{1\}) \) ‘horizontal’. We define a covering relation between two quadrilaterals in the following way (see Figure 1).
Definition 3.1. We say $N_i$ $f$-covers $N_j$ and write $N_i \Rightarrow N_j$ if:

(i) $f|_{N_i}$ is one-to-one;
(ii) for each $\rho \in [0, 1]$, there are exactly two numbers $t_1^\rho, t_2^\rho \in (0, 1)$ such that $f(h_i([t_1^\rho] \times \{\rho\}))$ lies in one of the vertical edges of $N_j$ and $f(h_i([t_2^\rho] \times \{\rho\}))$ lies in the other vertical edge of $N_j$, and for all $t_1^\rho < t < t_2^\rho$, $f(h_i([t] \times \{\rho\})) \in N_j$; and

(iii) for $0 \leq t < t_1^\rho$ and $t_2^\rho < t \leq 1$, $f(h_i([t] \times \{\rho\})) \cap N_j$ is empty.

If one can show the existence of these quadrilaterals and associated covering relations, they can be used to give rigorous lower bounds for the topological entropy of $f$.

Theorem 3.2. [2] Let $N_1, N_2, \ldots, N_p$ be pairwise disjoint quadrilaterals and $f : \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. Let $A = (a_{ij})$ be a square matrix, where $1 \leq i, j \leq p$ and

$$a_{ij} = \begin{cases} 1 & \text{if } N_i \Rightarrow N_j, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f$ contains a Cantor set on which it is topologically conjugate to the subshift of finite type with transition matrix $A$. In particular, $h(f) \geq \log(\lambda_1)$, where $\lambda_1$ is the largest magnitude eigenvalue ($\lambda_1 \geq |\lambda_j|$ for all eigenvalues of $A$).

Note that there is no easy way to detect these systems of quadrilaterals. They are usually found by trial and error; for example, see [2, 15].

4. Discontinuity of entropy for Lozi maps

Buzzi’s results [1] about lower semi-continuity of the entropy of piecewise affine homeomorphisms on compact surfaces cannot be applied directly to Lozi maps which are defined on the plane. These results should also hold in the non-compact case, but more work is required. On the other hand, nothing much is known about upper semi-continuity. For Lozi maps, there are some monotonicity results (see [6, 17]) around $b = 0$. It is also known that $h(L_{a,b})$ depends continuously on the parameters $(a, b)$ at all points $(a, 0)$, where $a > 1$: first note that $h(L_{a,0}) = \min(\log a, \log 2)$ for $a > 1$, as in the tent map. By the monotonicity results in [6], $h(L_{a-N|b|,0}) \leq h(L_{a,b}) \leq h(L_{a+N|b|,0})$ for some $N > 0$ and $|b|$ small. So, continuity follows.

We first prove that the entropy jumps from zero to a positive value if the parameters are slightly changed from $(a, b) = (1.4, 0.4)$ to $(a, b) = (1.4 + \epsilon_1, 0.4)$, where $\epsilon_1$ is positive and small.
THEOREM 4.1. There exists some $\epsilon_* > 0$ such that for all $0 < \epsilon_1 < \epsilon_*$.

(i) The topological entropy of the Lozi map with $(a, b) = (1.4, 0.4)$, $h(L_{1.4,0.4})$, is zero.

(ii) The topological entropies of the Lozi maps $h(L_{1.4+\epsilon_1,0.4})$ have a lower bound of 0.1203.

Proof of Theorem 4.1(i). Let us denote $L_{1.4,0.4} = L$. We will prove that $h(L^4) = 0$. By direct calculation of $L^4$, one can solve the equation $L^4(x, y) = (x, y)$ for $(x, y) \in \mathbb{R}^2$ to see that $L^4$ has the following fixed points (see the Appendix A):

(i) fixed points of $L$: $p_1 = (1/2, 1/2)$ and $p_2 = (-5/4, -5/4)$;

(ii) the closed line segment $\ell_1$ which connects $(-20/29, 35/29)$ to $(0, 15/29) = L^2(-20/29, 35/29)$; and

(iii) the closed line segment which connects $(15/29, -20/29)$ to $(35/29, 0) = L^2(15/29, -20/29)$, i.e. $L(\ell_1)$.

Note that $p_1$ is a saddle fixed point and $v^s_1 = (\lambda^s_1, 1)$, where $\lambda^s_1 = (-7 + \sqrt{89})/10$, is a stable direction at $p_1$, and $W^s_+(p_1) = \{p_1 + v^s_1t \mid t > 0\}$ is invariant under $L$ and therefore $L^4$. Similarly, $p_2$ is a saddle point and $v^u_2 = (-\lambda^u_2, -1)$, where $\lambda^u_2 = (7 + \sqrt{89})/10$ is an unstable direction at $p_2$ and $W^u_+(p_2) = \{p_2 + v^u_2t \mid t > 0\}$ is invariant under $L^4$.

Let us call the left and the right connected components of the unstable manifold at $p_1$ $W_L(p_1)$ and $W_R(p_1)$, respectively (see Figure 2). We want to show that $W_L(p_1)$ is attracted by $\ell_1$ and $W_R(p_1)$ is attracted by $L(\ell_1)$. But let us first explain how to conclude the proof of Theorem 4.1(i) from that claim. Let $U = \mathbb{R}^2 \setminus M$, where $M = W^s_+(p_1) \cup \{p_1\} \cup W^u_+(p_2) \cup \{p_2\} \cup W_L(p_1) \cup \ell_1 \cup W_R(p_1) \cup L(\ell_1)$. Note that $U$ is invariant by construction and it is simply connected since the complement of $U$ in the extended plane, i.e. $M \cup \{\infty\}$, is connected. Also, note that $M \cup \{\infty\}$ is compact because of the claim that $W_L(p_1)$ is attracted by $\ell_1$ and $W_R(p_1)$ is attracted by $L(\ell_1)$. This implies $U$ is homeomorphic to the open unit disk (by Riemann mapping theorem) which is homeomorphic to $\mathbb{R}^2$. Since $L^4$ has no fixed points in $U$ and it is orientation preserving, Brouwer’s translation theorem implies that $L^4$ has no non-wandering points in $U$. This shows the non-wandering set of $L^4$ only consists of the fixed points of $L^4$. So, $h(L^4) = 4h(L) = 0$.

$W_L(p_1)$ is attracted to $\ell_1$. Now, let $Z$ be the intersection of the half line $m = \{p_1 + v^s_1t \mid t > 0\}$ and the x-axis, where $v^u_1 = (-\lambda^u_1, -1)$ and $\lambda^u_1 = (-7 - \sqrt{89})/10$ (see Figure 3). Note that

$$W_L(p_1) = \bigcup_{n=0}^{\infty} L^{4n}(-p_1 - v^u_1t | 0.1 > t > 0),$$

i.e. forward iterations of a small piece in the unstable direction. Let the portion of $W_L(p_1)$ which connects $L(Z)$ and $L^2(Z)$ be called $W$. It is not hard to see that $W_L(p_1) = \bigcup_{n=-\infty}^{\infty} L^{4n}(W)$. We want to show that every $x \in W$ (so every $x \in W_L(p_1)$) is attracted to $\ell_1$.

Remark. Note that all points in $\ell_1$ have a neutral direction (along $\ell_1$) and a contracting direction with slope $-5/2$. This gives an immediate basin of attraction up to the interaction with the singularity lines of $L^4$. The basin (trapping region) intersects and therefore
FIGURE 2. Several components of the invariant manifolds of the fixed points $p_1$ and $p_2$ given together with the two line segments (darker) of period-4 points: $\ell_1$, which connects $(-20/29, 35/29)$ to $(0, 15/29)$, and $L(\ell_1)$, which connects $(15/29, -20/29)$ to $(35/29, 0)$. $U$ is the complement of all the points shown in the picture.

captures a large part of $W_\ell(p_1)$, but not all, since that set extends to the left and right. Below, we show that these left and right parts are also eventually attracted to the trapping region.

**Trapping region.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a map. A neighborhood $U$ of an $f$-invariant set $A \subset \mathbb{R}^n$ is called a **trapping region** for $A$ if for all $m > 0$, $f^m(U) \subset U$ and $\bigcap_{m>0} f^m(U) = A$. Below, we introduce a trapping region $R$ around $\ell_1$ such that any
point \( x \in R \) is attracted to a point in \( \ell_1 \) under forward iterations of \( L \). Let

\[
R_1 = (-20/29, 35/29 + 0.2),
R_2 = (-20/29 + 0.1, 35/29 - 0.25),
R_3 = (0, 15/29 - 0.25),
R_4 = (-0.2, 15/29 + 0.5).
\]

Let us call the left and right end points of \( \ell_1 \) \( F_1 \) and \( F_2 \), respectively. Note that \( F_1 = (-20/29, 35/29) \) and \( F_2 = L^2(F_1) = (0, 15/29) \). Let \( R \) be the hexagon with vertices \( R_1, F_1, F_2, R_3 \) and \( R_4 \). The sides \( F_1R_2 \) and \( F_2R_4 \) are parallel to each other with slope \(-5/2\) and they are stable directions at \( F_1 \) and \( F_2 \), respectively. Since \( R_1 \) is in the stable manifold of a point in \( \ell_1 \), it is attracted to \( \ell_1 \) under iterations of \( L^4 \). Similarly, \( R_4 \) is attracted to \( F_2 \) since it is in the stable manifold of \( F_2 \). So, the quadrilateral with vertices \( R_1, F_1, F_2 \) and \( R_4 \) is mapped to thinner and thinner quadrilaterals for which one of the sides is always \( \ell_1 = F_1F_2 \). Similarly, the quadrilateral with vertices \( F_1, R_2, R_3 \) and \( F_2 \) is mapped towards \( \ell_1 \) (see Figure 5). So, \( R \) is a trapping region.

Figure 4. This figure shows a portion of the left unstable manifold of the fixed point \( p_1 \). Note that all the points on the line segment connecting \( F_1 \) to \( F_2 \) are period-4 points of \( L \).

We want to show that all the points in \( W \) are eventually mapped into \( R \) under forward iterations of \( L^4 \). Let us start with the part of \( W \) which connects \( L(Z) \) and \( L^3(Z) \). The image of this line segment (under \( L^4 \)) is the portion of \( W_{\ell}(p_1) \) which connects \( L^5(Z) \) and \( L^3(Z) \) (see Figure 4). Let us call this portion \( \overline{W} \). \( L^3(Z) \) and \( L^3(Z) \) are both in \( R \) but there is a part of \( \overline{W} \) which is still outside of \( R \), which we denote by \( \overline{W} \backslash R \), i.e. \( \overline{W} \) is the closure of \( \overline{W} \backslash R \). Note that \( \ell_c : y = 1 - 1.4(1 + 1.4x + 0.4y) + 0.4x \) is a critical line for \( L^4 \) around \( F_1 \), i.e. images of lines which transversally intersect \( \ell_c \) are broken lines.
Let $\ell_c = \mathcal{L}^4(\ell_c)$. Also, let $W \cap R_1 F_1 = W_{R_1F_1}$, $W \cap R_2 F_1 = W_{R_2F_1}$, $W \cap \ell_c = W_{\ell_c}$ and the intersection point of $W$ and $\ell_c$ which stays below $\ell_c$ be $W_{\ell_c}$.

$\overline{W}$ consists of two parts: the line segment which connects $W_{R_1F_1}$ and $W_{\ell_c}$, and the line segment which connects $W_{\ell_c}$ and $W_{R_2F_1}$ (see Figure 6). Note that $\mathcal{L}^4(\ell_c)$ is a broken line that stays in $R$ since $\ell_c$ intersects $\ell_c$, which is a critical line for $\mathcal{L}^4$. So, all points on the line segment connecting $W_{R_1F_1}$ and $W_{\ell_c}$ are also mapped into $R$.

On the other hand, $W_{\ell_c}$ is mapped to a point on $\ell_c$. So, the line segment connecting $W_{\ell_c}$ and $W_{\ell_c}$ is also completely mapped into $R$ under $\mathcal{L}^8$.

The only part left is the portion that connects $W_{\ell_c}$ and $W_{R_2F_1}$. But note that $W_{R_2F_1}$ is on the stable direction so forward iterations move towards $F_1$. $W_{\ell_c}$ is mapped between $W_{\ell_c}$ and $F_1$. So, one can repeat the same argument for this line segment connecting $\mathcal{L}^4(W_{R_2F_1})$. 

Figure 5. Trapping region $R$ (gray), and images $\mathcal{L}^4(R)$ (darker) and $\mathcal{L}^8(R)$ (darkest).
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Figure 6. The set $\overline{W}$ (thickest solid broken line) and the part of the images $L^4(\overline{W})$ (thinner) and $L^8(\overline{W})$ (thinnest) which stay outside of $R$. Note that everything above $\ell_c$ is mapped into $R$ under $L^4$.

and $L^4(W_{\ell_c})$. So, by induction, the portion that connects $W_{\ell_c}$ and $W_{R_2F_1}$ is also mapped into $R$ eventually. This completes the proof that $\overline{W}$ is mapped into $R$.

The above analysis explains that forward images of $\overline{W}$ consist of some parts which are mapped into $R$ and some parts which stay outside of $R$. However, the parts outside of $R$ are eventually attracted by $R$ (see Figure 6).

Now, for the other portion of $W$ (connecting $L^3(Z)$ and $L^3(Z)$), similar arguments can be applied; this time, the critical line $\ell_c$ is the $y$-axis and the parts outside of $R$ are either mapped into $R$ or attracted by $F_2$.

Finally, note that $W_\ell(p_1)$ is attracted to $\ell_1$ implies that $W_\ell(p_1) = L(W_\ell(p_1))$ is attracted to $L(\ell_1)$.

Proof of Theorem 4.1(ii). We want to show that for any $\epsilon_1$ positive and small, Theorem 3.2 applies with an appropriate subshift of finite type yielding the lower bound for the map $L_{(1.4+\epsilon_1,0.4)}$.

Fix an $\epsilon_1 > 0$ and denote $L_{\epsilon_1} = L_{(1.4+\epsilon_1,0.4)}$. Note that the line segment connecting $F_1 = (-20/29, 35/29)$ and $F_2 = (0, 15/29)$ consists of period-4 points of $L_{(1.4,0.4)}$.

Now, let $N_1$ be the quadrilateral given by the four vertices

\[
A = (0, 15/29 - \epsilon_1), \\
B = (\epsilon_1, 15/29 + (7/2)\epsilon_1), \\
C = ((5/2)\epsilon_1, 15/29 + (5/2)\epsilon_1), \\
D = ((3/2)\epsilon_1, 15/29 - 2\epsilon_1).
\]
Also, let $N_2$ be the quadrilateral whose vertices are

\[
E = (-3\epsilon_1, 15/29 + (7/2)\epsilon_1),
F = (-2\epsilon_1, 15/29 + (5/6)\epsilon_1),
G = (0, 15/29 - (1/2)\epsilon_1),
H = (-\epsilon_1, 15/29 + (13/6)\epsilon_1).
\]

For $N_1$, let the sides $AB$ and $CD$ be ‘vertical’ and the other two sides be ‘horizontal’. Similarly, for $N_2$, let $EF$ and $GH$ be ‘vertical’ and the other two sides be ‘horizontal’. Note that the images of $N_1$ and $N_2$ under $\mathcal{L}_\epsilon^4$ are also quadrilaterals since $N_1$ and $N_2$ are chosen away from the singularity locus of $\mathcal{L}_\epsilon^4$. Moreover, vertical edges are contracted since they are close to the stable directions around $(0, 15/29)$ and $(-20/29, 35/29)$.

By direct calculation, it can be shown that the images of the vertices under the map $\mathcal{L}_\epsilon^4$ ares given by (see Figure 7):

\[
\mathcal{L}_\epsilon^4(A) = \left(\frac{30476}{18125} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} - \frac{6363}{3625} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(1.68\epsilon_1, \frac{15}{29} - 1.75\epsilon_1\right).
\]

\[
\mathcal{L}_\epsilon^4(B) = \left(\frac{6188}{3625} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} - \frac{1319}{725} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(1.70\epsilon_1, \frac{15}{29} - 1.81\epsilon_1\right).
\]

\[
\mathcal{L}_\epsilon^4(C) = \left(-\frac{4769}{1450} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} + \frac{847}{290} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(-3.28\epsilon_1, \frac{15}{29} + 2.92\epsilon_1\right).
\]

\[
\mathcal{L}_\epsilon^4(D) = \left(-\frac{120153}{3625} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} + \frac{21639}{7250} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(-3.31\epsilon_1, \frac{15}{29} + 2.98\epsilon_1\right).
\]

\[
\mathcal{L}_\epsilon^4(E) = \left(-\frac{9283}{18125} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} + \frac{1554}{3625} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(-0.51\epsilon_1, \frac{15}{29} + 0.42\epsilon_1\right).
\]

\[
\mathcal{L}_\epsilon^4(F) = \left(-\frac{23209}{54375} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} + \frac{3792}{10875} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(-0.42\epsilon_1, \frac{15}{29} + 0.34\epsilon_1\right).
\]

\[
\mathcal{L}_\epsilon^4(G) = \left(\frac{36363}{18125} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} - \frac{7494}{3625} \epsilon_1 + O(\epsilon_1^2)\right)
\approx \left(2.00\epsilon_1, \frac{15}{29} - 2.06\epsilon_1\right).
\]
Figure 7. This figure shows the quadrangles $N_1$ and $N_2$ and their images (thinner boxes). Notice the covering relations: $N_1 \Rightarrow N_1$, $N_1 \Rightarrow N_2$ and $N_2 \Rightarrow N_1$.

\[
\mathcal{L}_{\epsilon_1}^4(H) = \left( \frac{113584}{54375} \epsilon_1 + O(\epsilon_1^2), \frac{15}{29} - \frac{22917}{10875} \epsilon_1 + O(\epsilon_1^2) \right)
\approx \left( 2.08 \epsilon_1, \frac{15}{29} - 2.10 \epsilon_1 \right).
\]

It is not hard to see that we have the following covering relations: $N_1 \Rightarrow N_1$, $N_1 \Rightarrow N_2$ and $N_2 \Rightarrow N_1$. So, the transition matrix is given by

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix},
\]

where the largest magnitude eigenvalue is $(\sqrt{5} + 1)/2$. Since we are using $\mathcal{L}_{\epsilon_1}^4$ during the process, $h(\mathcal{L}_1) = (1/4)h(\mathcal{L}_{\epsilon_1}^4) \geq (1/4) \log((\sqrt{5} + 1)/2) > 0.1203$ by Theorem 3.2.

Remark. We would like to point out that the jump up in the entropy explained above is somewhat similar to the following one-dimensional case: let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = -2|x|$. All the initial points except the fixed point at $x = 0$ go to infinity under further iterations of $T$ so the entropy of $T$ is zero. Note that the graph of $T(x)$ stays below the diagonal line $y = x$. On the other hand, the perturbed map $T_\delta(x) = -2|x| + \delta$, where $\delta > 0$ has entropy $\log 2$ (similar to the standard tent map), and the graph of $T_\delta(x)$ crosses the diagonal line. One can see a similar kind of behavior at the images of $N_1$ and $N_2$ under the maps $\mathcal{L}^4$ and $\mathcal{L}_{\epsilon_1}^4$ (see Figure 8). Images of $N_1$ and $N_2$ under $\mathcal{L}^4$ stay on the left of the critical line $x = 0$ and the entropy is zero. On the other hand, under $\mathcal{L}_{\epsilon_1}^4$, these images cross...
the critical line and the entropy jumps up. We would like to thank Newhouse for pointing out this similarity between the one-dimensional and two-dimensional cases.

Now, we can extend our results from \((a, b) = (1.4, 0.4)\) to \((a, b) = (1.4 + \epsilon_2, 0.4 + \epsilon_2)\), where \(|\epsilon_2|\) is small.
Proof of Theorem 1.2. Let $\mathcal{L}$ denote $\mathcal{L}((4+\varepsilon_2, 0.4+\varepsilon_2))$.

(i) The entropy is zero for $\mathcal{L}$. For $|\varepsilon_2|$ small and fixed, we still have two line segments of period-4 points: the line segment connecting

$$F_2^{e_2} = \frac{1 - (0.4 + \varepsilon_2)^2}{(1.4 + \varepsilon_2)(1 + (0.4 + \varepsilon_2)^2)}$$

and the image of this line segment under $\mathcal{L}$. So, we can still find a similar trapping region using the vertical lines and the stable directions at $F_1^{e_2}$ and $F_2^{e_2}$. The rest of the proof is the same as in the case of $(a, b) = (1.4, 0.4)$.

(ii) The lower bound for $(a, b) = (1.4 + \varepsilon_1 + \varepsilon_2, 0.4 + \varepsilon_2)$. Let $\mathcal{L}_{e_1} = \mathcal{L}((1.4 + \varepsilon_1 + \varepsilon_2, 0.4 + \varepsilon_2))$. We need to find two boxes, as in the case of $(a, b) = (1.4, 0.4)$, which give us the covering relations. We slightly modify the points we used before: for $\varepsilon_1$ positive and small, let $N_1$ be the quadrilateral given by the four vertices

\[
\begin{align*}
\tilde{A} &= (0, F_2^{e_2} - \varepsilon_1), \\
\tilde{B} &= (\varepsilon_1, F_2^{e_2} + (7/2)\varepsilon_1), \\
\tilde{C} &= ((5/2)\varepsilon_1, F_2^{e_2} + (5/2)\varepsilon_1), \\
\tilde{D} &= ((3/2)\varepsilon_1, F_2^{e_2} - 2\varepsilon_1).
\end{align*}
\]

Also, let $N_2$ be the quadrilateral whose vertices are

\[
\begin{align*}
\tilde{E} &= (-3\varepsilon_1, F_2^{e_2} + (7/2)\varepsilon_1), \\
\tilde{F} &= (-2\varepsilon_1, F_2^{e_2} + (5/6)\varepsilon_1), \\
\tilde{G} &= (0, F_2^{e_2} - (1/2)\varepsilon_1), \\
\tilde{H} &= (-\varepsilon_1, F_2^{e_2} + (13/6)\varepsilon_1).
\end{align*}
\]

In other words, $15/29$ is replaced with $F_2^{e_2}$. We want to show that we still have the same covering relations and the same lower bound.

Although one can explicitly write down the images of these points under $\mathcal{L}_{e_1}$, for simplicity we only want to point out the differences between this case and the case $(a, b) = (1.4, 0.4)$. For example, $\mathcal{L}_{e_1}(\tilde{A})$ consists of terms including $\varepsilon_1$ and some others not including $\varepsilon_1$. Observe that if $\varepsilon_1$ equals zero, then $F_2^{e_2}$ is a period-4 point, so the terms in $\mathcal{L}_{e_1}(\tilde{A})$ not including $\varepsilon_1$ add up to $F_2^{e_2}$ (this is because when $\varepsilon_1 = 0$, $\tilde{A}$ becomes $(0, F_2^{e_2})$) and so $\mathcal{L}_{e_1}((1.4 + \varepsilon_2, 0.4 + \varepsilon_2))(\tilde{A}) = F_2^{e_2})$. Note that $15/29$ in the proof of $(a, b) = (1.4, 0.4)$ case is now replaced with $F_2^{e_2}$.

On the other hand, the terms in $\mathcal{L}_{e_1}(\tilde{A})$ including $\varepsilon_1$ can be made arbitrarily close to the terms including $\varepsilon_1$ in the $(a, b) = (1.4, 0.4)$ case (i.e. to the terms $(30476/18125)\varepsilon_1$ in the $x$-coordinate and $-(6363/3625)\varepsilon_1$ in the $y$-coordinate of $\mathcal{L}_{e_1}(A)$) by choosing $|\varepsilon_2|$ small. Note that the size of $|\varepsilon_2|$ does not depend on $\varepsilon_1$ but rather depends on the coefficient of $\varepsilon_1$, i.e. $(30476/18125)$ and $-(6363/3625)$.

The same argument can be applied to all other points, so our new boxes also satisfy the previous covering relations, giving the same lower bound (0.1203) for the entropy. \hfill \Box

Remark. The reason why the entropy is zero on the line segment $l = \{(1.4 + \varepsilon_2, 0.4 + \varepsilon_2) : |\varepsilon_2| < \varepsilon_f \}$ and it is above 0.1203 for the parameters to the right of that segment is the fact
that we have a line segment of period-4 points when the parameters are chosen from \( l \). These period-4 points create a trapping region causing the zero entropy. On the other hand, period-4 points suddenly disappear to the right of \( l \), causing enough expansion and allowing us to find the necessary subshift which gives the positive entropy (see Figure 8).

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A. **Appendix**

Here, we explain some of the details in the proof of Theorem 4.1(i). We show that \( L_{1.4,0.4}^4 = L^4 \) has the following fixed points:

(i) fixed points of \( L_{1.4,0.4} = L \): \( p_1 = (1/2, 1/2) \) and \( p_2 = (-5/4, -5/4) \);
(ii) the closed line segment \( \ell_1 \) which connects \((-20/29, 35/29)\) to \((0, 15/29) = L^2(-20/29, 35/29)\);
(iii) the closed line segment which connects \((15/29, -20/29)\) to \((35/29, 0) = L^2(15/29, -20/29)\), i.e. \( L(\ell_1) \).

We need to solve \( L^4(x, y) = (x, y) \) for \((x, y) \in \mathbb{R}^2\). Note that this calculation is not trivial since \( L^4 \) has \( 2^4 = 16 \) affine domains to check. We summarize these computations below. Let

\[
\begin{align*}
C &= 1 - 1.4|x| + 0.4y, \\
B &= 1 - 1.4|C| + 0.4x, \\
A &= 1 - 1.4|B| + 0.4(C).
\end{align*}
\]

Note that we need to solve

\[
L^4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - 1.4|A| + 0.4(B) \\ A \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.
\]

**Domains 1 and 2:** \( B \geq 0, C \geq 0, x \geq 0 \). First, let us use the equality of the \( y \)-coordinate of \( L^4 \) to \( y \):

\[
A = y \implies 1 - 1.4(1 - 1.4(1 - 1.4x + 0.4y) + 0.4x) + 0.4(1 - 1.4x + 0.4y) = y \\
\implies 0.056y = 1.96 - 3.864x \implies y = -69x + 35. \quad (A.1)
\]

Now, let us use the equality of the \( x \)-coordinate of \( L^4 \) to \( x \).

**Domain 1.** Assuming also \( A = y \geq 0 \),

\[
1 - 1.4y + 0.4(1 - 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies -1.624y + 0.84 = 0.056x.
\]

Now, solving this equation together with equation (A.1), one gets \( x = y = 0.5 \). This is the right fixed point of \( L_{1.4,0.4} \).

**Domain 2.** Assuming \( A = y < 0 \),

\[
1 + 1.4y + 0.4(1 - 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \\
\implies 1.176y + 0.84 = 0.056x.
\]
Now, solving this equation together with equation (A.1), one gets $x = 15/29$, $y = -20/29$. This is the left end point of $\mathcal{L}(\ell_1)$.

**Domains 3 and 4:** $B \geq 0$, $C \geq 0$, $x < 0$. From the equality of the $y$-coordinate of $\mathcal{L}^4$ to $y$,

$$\mathcal{A} = y \implies 1 - 1.4(1 - 1.4(1 + 1.4x + 0.4y) + 0.4x) + 0.4(1 + 1.4x + 0.4y) = y$$

$$\implies 0.056y = 1.96 + 2.744x \implies y = 49x + 35. \quad (A.2)$$

Now, let us use the equality of the $x$-coordinate of $\mathcal{L}^4$ to $x$.

**Domain 3.** Assuming also $\mathcal{A} = y \geq 0$,

$$1 - 1.4y + 0.4(1 - 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies -1.624y + 0.84 = 1.624x. \quad (A.3)$$

Now, solving this equation together with equation (A.2), one gets $x = -20/29$, $y = 35/29$. This is the left end point of $\ell_1$.

**Domain 4.** Assuming $\mathcal{A} = y < 0$,

$$1 + 1.4y + 0.4(1 - 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies 1.176y + 0.84 = 1.624x. \quad (A.4)$$

Now, solving this equation together with equation (A.2), one gets $x = -3/4$, $y = -7/4$, but note that at this point $C < 0$, so this point is not in Domain 4 and there are no fixed points.

**Domains 5 and 6:** $B \geq 0$, $C < 0$, $x \geq 0$. From the equality of the $y$-coordinate of $\mathcal{L}^4$ to $y$,

$$\mathcal{A} = y \implies 1 - 1.4(1 + 1.4(1 - 1.4x + 0.4y) + 0.4x) + 0.4(1 - 1.4x + 0.4y) = y$$

$$\implies 1.624y = -1.96 + 1.624x \implies y = x - 35/29. \quad (A.3)$$

Now, let us use the equality of the $x$-coordinate of $\mathcal{L}^4$ to $x$.

**Domain 5.** Assuming also $\mathcal{A} = y \geq 0$,

$$1 - 1.4y + 0.4(1 + 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies -1.176y + 1.96 = 1.624x. \quad (A.3)$$

Now, solving this equation together with equation (A.3), one gets $x = 490/261 \approx 1.8773$, $y = 175/261 \approx 0.6704$, but note that at this point $B < 0$, so this point is not in Domain 5 and there are no fixed points.

**Domain 6.** Assuming $\mathcal{A} = y < 0$,

$$1 + 1.4y + 0.4(1 + 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies 1.624y + 1.96 = 1.624x. \quad (A.4)$$

Now, solving this equation together with equation (A.3), one gets $x = x$. So, the part of the line segment $y = x - 35/29$ that stays in Domain 6 is a line segment of fixed points of $\mathcal{L}^4$. Note that this line segment is $\mathcal{L}(\ell_1)$.

**Domains 7 and 8:** $B \geq 0$, $C < 0$, $x < 0$. From the equality of the $y$-coordinate of $\mathcal{L}^4$ to $y$,

$$\mathcal{A} = y \implies 1 - 1.4(1 + 1.4(1 + 1.4x + 0.4y) + 0.4x) + 0.4(1 + 1.4x + 0.4y) = y$$

$$\implies 0.84y = -1.96 - 2.744x \implies y = (49/15)x - 7/3. \quad (A.4)$$

Now, let us use the equality of the $x$-coordinate of $\mathcal{L}^4$ to $x$. 

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*Discontinuity of topological entropy for Lozi maps*
**Domain 7.** Assuming also \( A = y \geq 0, \)
\[
1 - 1.4y + 0.4(1 + 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies -1.176y + 1.96 = 0.056x.
\]
Now, solving this equation together with equation (A.4), one gets \( x = 35/29 \approx 1.2068, \)
\( y = 140/87 \approx 1.6091, \) but note that at this point \( x \geq 0, \) so this point is not in Domain 7
and there are no fixed points.

**Domain 8.** Assuming \( A = y < 0, \)
\[
1 + 1.4y + 0.4(1 + 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies 1.624y + 1.96 = 0.056x.
\]
Now, solving this equation together with equation (A.4), one gets \( x \approx 0.3485, \) \( y \approx \)
\(-1.1948, \) but note that at this point \( x \geq 0, \) so this point is not in Domain 8 and there
are no fixed points.

**Domains 9 and 10:** \( B < 0, C \geq 0, x \geq 0. \) From the equality of the y-coordinate of \( L^4 \) to \( y, \)
\[
A = y \implies 1 + 1.4(1 - 1.4(1 - 1.4x + 0.4y) + 0.4x) + 0.4(1 - 1.4x + 0.4y) = y
\]
\[
\implies 1.624y = 0.84 + 2.744x \implies y = (49/29)x + 15/29. \tag{A.5}
\]
Now, let us use the equality of the x-coordinate of \( L^4 \) to \( x. \)

**Domain 9.** Assuming also \( A = y \geq 0, \)
\[
1 - 1.4y + 0.4(1 - 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies -1.624y + 0.84 = 0.056x.
\]
Now, solving this equation together with equation (A.5), one gets \( x = 0, \) \( y = 15/29. \) This
is the right end point of \( \ell_1. \)

**Domain 10.** Assuming \( A = y < 0, \)
\[
1 + 1.4y + 0.4(1 - 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies 1.176y + 0.84 = 0.056x.
\]
Now, solving this equation together with equation (A.5), one gets \( x = y = -3/4, \) but note
that at this point \( x < 0, \) so this point is not in Domain 10 and there are no fixed points.

**Domains 11 and 12:** \( B < 0, C \geq 0, x < 0. \) From the equality of the y-coordinate of \( L^4 \)
to \( y, \)
\[
A = y \implies 1 + 1.4(1 - 1.4(1 + 1.4x + 0.4y) + 0.4x) + 0.4(1 + 1.4x + 0.4y) = y
\]
\[
\implies 1.624y = 0.84 - 1.624x \implies y = -x + 15/29. \tag{A.6}
\]
Now, let us use the equality of the x-coordinate of \( L^4 \) to \( x. \)

**Domain 11.** Assuming also \( A = y \geq 0, \)
\[
1 - 1.4y + 0.4(1 - 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies -1.624y + 0.84 = 1.624x.
\]
Now, solving this equation together with equation (A.6), one gets \( x = x. \) So, the part of
the line segment \( y = -x - 15/29 \) that stays in Domain 11 is a line segment of fixed points
of \( L^4. \) Note that this line segment is \( L(\ell_1). \)
Domain 12. Assuming \( \mathcal{A} = y < 0 \),

\[
1 + 1.4y + 0.4(1 - 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies 1.176y + 0.84 = 1.624x.
\]

Now, solving this equation together with equation (A.6), one gets \( x \approx 3.8813, \ y \approx -3.3641 \), but note that at this point \( x \geq 0 \), so this point is not in Domain 12 and there are no fixed points.

Domains 13 and 14: \( B < 0, \ C < 0, \ x \geq 0 \). From the equality of the \( y \)-coordinate of \( \mathcal{L}^4 \) to \( y \),

\[
\mathcal{A} = y \implies 1 + 1.4(1 + 1.4(1 - 1.4x + 0.4y) + 0.4x) + 0.4(1 - 1.4x + 0.4y) = y \implies 0.056y = 4.76 - 2.744x \implies y = -49x + 85.
\]

(A.7)

Now, let us use the equality of the \( x \)-coordinate of \( \mathcal{L}^4 \) to \( x \).

Domain 13. Assuming also \( \mathcal{A} = y \geq 0 \),

\[
1 - 1.4y + 0.4(1 + 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies -1.176y + 1.96 = 1.624x.
\]

Now, solving this equation together with equation (A.7), one gets \( x = 1.75, \ y = -0.75 \), but note that at this point \( y = \mathcal{A} < 0 \), so this point is not in Domain 13 and there are no fixed points.

Domain 14. Assuming \( \mathcal{A} = y < 0 \),

\[
1 + 1.4y + 0.4(1 + 1.4(1 - 1.4x + 0.4y) + 0.4x) = x \implies 1.624y + 1.96 = 1.624x.
\]

Now, solving this equation together with equation (A.7), one gets \( x \approx 1.7241, \ y \approx 0.5172 \), but note that at this point \( y = \mathcal{A} \geq 0 \), so this point is not in Domain 14 and there are no fixed points.

Domains 15 and 16: \( B < 0, \ C < 0, \ x < 0 \). From the equality of the \( y \)-coordinate of \( \mathcal{L}^4 \) to \( y \),

\[
\mathcal{A} = y \implies 1 + 1.4(1 + 1.4(1 + 1.4x + 0.4y) + 0.4x) + 0.4(1 + 1.4x + 0.4y) = y \implies 0.056y = 4.76 + 3.864x \implies y = 69x + 85.
\]

(A.8)

Now, let us use the equality of the \( x \)-coordinate of \( \mathcal{L}^4 \) to \( x \).

Domain 15. Assuming also \( \mathcal{A} = y \geq 0 \),

\[
1 - 1.4y + 0.4(1 + 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies -1.176y + 1.96 = 0.056x.
\]

Now, solving this equation together with equation (A.8), one gets \( x = -35/29, \ y = 50/29 \), but note that at this point \( C \geq 0 \), so this point is not in Domain 15 and there are no fixed points.

Domain 16. Assuming \( \mathcal{A} = y < 0 \),

\[
1 + 1.4y + 0.4(1 + 1.4(1 + 1.4x + 0.4y) + 0.4x) = x \implies 1.624y + 1.96 = 0.056x.
\]

Now, solving this equation together with equation (A.8), one gets \( x = y = -5/4 \). This is the left fixed point of \( \mathcal{L}_{1.4,0.4} \).
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