Grid functions remapping method for completely conservative Lagrangian operator-difference scheme for astrophysical MHD problems

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Abstract. We represent conservative procedure of grid functions remapping for completely conservative Lagrangian operator-difference scheme on a triangular grid of variable structure. The grid functions remapping procedure based on the conditional minimization of specially constructed functionals is developed. The described procedure is applied to the simulation of MHD astrophysical problems.

1. Introduction

The numerical simulations is one of the important tools for study astrophysical problems. A number of astrophysical problems can be simulated by numerical solution of the magnetohydrodynamical (MHD) equations. There are many different numerical approaches for the simulation of MHD equations. In this paper we consider the completely conservative Lagrangian operator-difference scheme on a triangular grid of variable structure suggested in \cite{1}, \cite{2}.

Lagrangian numerical schemes are free of the artificial transfer of angular momentum, and free boundary conditions at the outer boundary of the cloud can be satisfied exactly. In the case when in the flow there are nonuniform contraction or expansion, vortexes or shear flows the Lagrangian cells become flattened soon after the beginning of the simulations, thus significantly reducing the accuracy of calculations and preventing continuing the simulations. In such situation the grid remapping allows to overcome this drawback of the Lagrangian schemes.

The grid remapping procedure can be divided by two subsequent steps: correction of the grid structure (for example removing the triangles with very sharp angles, local refining or rarefying the grid); interpolation of grid functions on a new grid structure.

2. Completely conservative operator-difference method on a triangular grid of variable structure

The operator-difference scheme used in our simulation is a first-order implicit conservative Lagrangian scheme on an irregular triangular grid. For further details of the scheme, see
We briefly describe the idea of the method for 2D cylindrical coordinates case. Define in the computational domain $\Omega$ a triangular grid $\omega = \omega_x + \omega_\triangle$, which consists of a set of knots $\omega_x$ and a set of triangular cells $\omega_\triangle$; define in particular the set of boundary knots $\omega_\gamma$. Introduce the following numeration for the grid: the cells of the grid are numbered by the index $i$ and are designated $\triangle_i$, the knots of the grid are numbered by the index $j$ and are designated $\pi_j = \{r_j, z_j\}$. Let the points adjacent to the knot $\pi_j$ have the indices $k = 1, 2, \ldots, K_j$, increasing counter-clockwise around the point $\pi_j$. Here $K_j$ is the total number of points adjacent to the point $\pi_j$. The points adjacent to $\pi_j$ and the cells touching this point are called the vicinity of the point $\pi_j$. The points which are apices of the cell $\triangle_i$ are numbered by the index $l = 1, 2, 3, \ldots$ increasing counter-clockwise along the boundary of the triangle. The boundary knots are also numbered counter-clockwise by the index $q$. Introduce linear spaces $B_{x,1} (B_{\gamma,1})$ of the vector functions defined in the knots (boundary knots) $\pi_j \in \omega_x (\pi_q \in \omega_\gamma)$ and linear spaces $B_{\triangle,0}^0 (B_{\gamma,0})$ of scalar functions defined in the cells (boundary knots) $\triangle_i \in \omega_\triangle (\pi_q \in \omega_\gamma)$. Introduce also the linear space $B_{\triangle,0}$ of scalar functions defined in the cells and boundary knots. The following relation is valid for the introduced spaces:

$$B_{\triangle,0} = B_{\triangle,0}^0 \oplus B_{\gamma,0}. \tag{1}$$

Define scalar multiplications in the spaces described above:

$$(u,v)_x = \sum_{x_j \in \omega_x} V_j^x u_j v_j, \quad u,v \in B_{x,1}, \tag{2}$$

$$(u,v)_\triangle = \sum_{\triangle_i \in \omega_\triangle} V_i^\triangle u_i v_i, \quad u,v \in B_{\triangle,1}^0, \tag{3}$$

where

$$V_i^\triangle = \tilde{r} s_i, \quad \tilde{r} = \frac{1}{3} \sum_{l=1}^{3} r_l; \quad V_j^x = \frac{1}{3} \sum_{k=1}^{K_j} V_k^\triangle, \tag{4}$$

and analogously for the spaces $B_{\gamma,0}, B_{\gamma,1}$. The values $V_i^\triangle, V_j^x$ are the ”volumes” of the cell $\triangle_i$ and of the knot $\pi_j$ respectively.

Define linear operators $\nabla_x^0$, $\nabla_\triangle$ and $\Phi_\gamma$ in the following form:

$$\left(\nabla_\triangle \cdot \mathbf{v}\right)_i = \frac{1}{3V_i} \sum_{l=1}^{3} N_{i,l+1} (\mathbf{v}_i R_{l,l+1} + \mathbf{v}_{l+1} R_{l+1,l}), \quad \mathbf{v} \in B_{x,1}, \triangle_i \in \omega_\triangle. \tag{5}$$

$$\nabla_x^0 p_j = \frac{1}{3V_j} \sum_{k=1}^{K_j} N_{k}^\prime p_k, \quad p \in B_{\triangle,0}^0, \pi_j \in \omega_x. \tag{6}$$

$$\Phi_\gamma p_q = \frac{1}{3V_q} N_{q}^\prime p_q, \quad p \in B_{\gamma,0}, \pi_q \in \omega_\gamma, \tag{7}$$

where $R_{j1,j2} = r_{j1} + r_{j2}/2$,

$$N_{j1,j2} = \begin{pmatrix} z_{j2} - z_{j1} \\ r_{j2} - r_{j1} \end{pmatrix} \text{ is a normal vector to the segment } [x_{j1}, x_{j2}], \tag{9}$$

$$N_{q}^\prime = -(N_{q,k} R_{j,k} + N_{j,k+1} R_{j,k+1}), \quad N_{q} = (N_{q,q+1} R_{q,q+1} + N_{q-1,q} R_{q,q-1}). \tag{10}$$
Introduce also the grid operator $\nabla_\times$ in the following form:
\[
\forall \tilde{p} = p + p_\gamma \in B_{\Delta,0}, \quad p \in B^0_{\Delta,0}, \quad p_\gamma \in B_{\gamma,0}:
\]
\[
\nabla_\times \tilde{p} = \begin{cases} 
\nabla_\times^0 p + \Phi_\gamma p_\gamma, & \exists_j \in \omega_x, \\
\nabla_\times^0 p, & \exists_j \in \omega_x \setminus \omega_\gamma.
\end{cases}
\]
\[
\text{(11)}
\]
The operator $\nabla_\Delta :$ is a grid analog for the differential operator $\text{div}$, $\nabla_\times$ is a grid analog for grad. The following relation is valid for the introduced operators:
\[
(\nabla_\times p, v) + (\nabla_\Delta \cdot v, p)_{\Delta} = (\Phi_\gamma p_\gamma, u_\gamma),
\]
\[
\text{where } p \in B^0_{\Delta,0}, \quad v \in B_{\times,1}, \quad p_\gamma \in B_{\gamma,0}, \quad u_\gamma \in B_{\gamma,1}. \text{ Introduce the operator } S_\times:
\]
\[
(S_\times f)_j = \frac{1}{3} \sum_{k=1}^{K_j} f_k, \quad \forall f \in B_{\Delta,0}(B_{\Delta,1}).
\]
\[
\text{(14)}
\]

The grid linear operators $\nabla_\times^0, \nabla_\Delta, \Phi_\gamma,$ are defined analogously to (2), (3), (3'). Assume that $x, u, \Phi$ are defined in knots and $p, \rho$ — in cells. Define the time grid $\omega_t = \{n \Delta t, n = 0, 1, \ldots\};$ for the arbitrary grid function defined on $\omega_t,$ introduce the following specifications:
\[
f(t_n) = f^n = f, \quad f^{n+1} = \hat{f}, \quad f_t = \frac{\hat{f} - f}{\Delta t}. \quad f^\sigma = \sigma \hat{f} + (1 - \sigma) f.
\]

For simplicity we consider set of gas dynamical equations with self-gravitation:
\[
\begin{align*}
\frac{dx}{dt} &= u, \quad \frac{d\rho}{dt} + \rho \text{div} u = 0, \\
\rho \frac{du}{dt} &= -\text{grad}p - \rho \text{grad} \Phi, \\
\rho \frac{d\varepsilon}{dt} + p \text{div} u &= 0, \quad \eta = \frac{1}{\rho} \frac{T^R}{p}, \quad \varepsilon = \frac{T^R}{\gamma - 1},
\end{align*}
\]
\[
\Delta \Phi = 4\pi G \rho.
\]
\[
\text{(15)}
\]

where $\frac{d}{dt}$ is total time $t$ derivative, $\times = (r, z),$ $u = (u^r, u^z, u^r)$ is velocity vector, $p$ is pressure, $\varepsilon$ is internal energy, $\Phi$ is gravitational potential, $\rho$ is density, $T$ is temperature, $G$ is gravitational constant, $R$ is universal gas constant, $\gamma$ is adiabatic index. Axial and cylindrical symmetry are assumed.

On the basis of the introduced operators, the following implicit difference scheme for (15) can be written [4]:
\[
\rho_i V_i = \rho_i^{0} V_i^{0} = m_i, \quad \forall \Delta_i \in \omega_\Delta,
\]
\[
\times_{jt} = u_{j}^{(0,5)}, \quad \forall \exists_j \in \omega_x,
\]
\[
\rho_j u_{jt} = - (\nabla_\times g)_j - q_{\rho_j} (S_\times (\nabla_\Delta \Phi))_j, \quad \forall \exists_j \in \omega_x,
\]
\[
\rho_i \varepsilon_{it} = - g_i (\nabla_\Delta \cdot u^{(0,5)})_i, \quad \forall \Delta_i \in \omega_\Delta,
\]
\[
g = p^{(a)} + \omega, \quad \omega = - \frac{\nu}{\eta_i}, \quad \forall \Delta_i \in \omega_\Delta,
\]
\[
\eta_i = 1/\rho_i = T_i/p_i, \quad \varepsilon_i = T_i/(\gamma - 1), \quad \forall \Delta_i \in \omega_\Delta,
\]
\[
(\nabla_\times \cdot \nabla_\Delta \Phi)_j = \rho_j,
\]
\[
\rho_j = (S_\times \rho)_j = \frac{1}{3Y_j} \sum_{k=1}^{K_j} \rho_k V^\Delta_k, \quad \forall \exists_j \in \omega_x.
\]
Here $\nu$ is a coefficient of artificial viscosity; its value was locally (for knot $j$ and its vicinity) adjusted by the formula $\nu = 2/\sqrt{\lambda_{\text{max}}}$ where $\lambda_{\text{max}}$ is the local (for knot $j$ and its vicinity) maximal eigenvalue for the acoustical analogue of the operator of the difference scheme. This formula was based on the method described in [5]. In that work, the coefficient $\nu$ was locally adjusted to satisfy the condition of quickest damping of the highest frequency eigen-oscillations of the acoustical discrete model. That adjustment of $\nu$ allows us to take into account spatial properties of the operators of the difference scheme and variability of the Lagrangian grid. The application of that methodology for the definition of $\nu$ allows spreading the shock front over 2 or 3 grid intervals. The total time derivatives for the velocity vector in the difference form are:

$$u_{jt} = \{u_{jt}^r - \frac{u_{j}^\varphi(0.5)}{r_{j}(0.5)}u_{j}^\varphi(0.5); u_{jt}^\varphi + \frac{u_{j}^\varphi(0.5)}{r_{j}(0.5)}u_{j}^r(0.5); u_{jt}^z\}.$$

Substituting the boundary values known from the boundary conditions into (4), we get a set of non-linear operator equations for the grid functions $x, u \in B_{\times, 1}, \rho, T \in B_{\Delta, 0}$ at every time step, which is solved by a specially adjusted variant of the Newton method. The gravitational potential $\Phi$ was calculated explicitly at the beginning of every time step.

3. Remapping method for the Lagrangian triangular grid of variable structure

3.1. Triangular grid structure reconstruction procedure

One of the important problems for application of difference schemes in Lagrangian variables is grid distortion which prevents to continue the simulations and could lead also to the loss of accuracy of the results. The grid distortions of such kind appear when Lagrangian numerical schemes are applied to the simulations which contains vortexes, slip surfaces, nonuniform contraction or expansion, etc. In situations of this kind, the difference grid becomes significantly nonuniform, i.e. contains triangular cells with a very small angle between neighboring sides, or triangular cells of substantially different sizes adjoin to each other. Such grid distortions prevent calculations due to loss of accuracy at the advanced stage of the solution.

In order to overcome this problem in the papers [2, 3] and some other papers of these authors the grid remapping procedure for a triangular grid was developed. Here the remapping procedure was modified what allows to carry out simulations in the presence of large differences in flow parameters [6, 7].

The necessity of applying the procedure for the grid remapping is connected with the fact, that when the nodes of a triangular Lagrangian grid are moving, according to the laws of magnetic hydrodynamics, there are various distortions mesh. For example, when vortex motions of the environment occur, it is possible to get "overlap" the grid. Distortion of the grid is also possible with a nonuniform compression or expansion of matter or in shear flows. The remapping of the grid involves performing an analysis of the quality of the grid, its correction and adaptation (reorganization of the grid). Necessity correction of the quality of the grid is determined by the possible appearance of "bad" cells, i.e. triangles very different from equilateral. Dynamic adaptation of the grid allows to thicken grid in those parts of the calculated area where it is necessary to increase spatial resolution, and rarely the grid where the flow is smooth, which allows us to significantly reduce the dimension used grid and, therefore, reduce the costs of computer time.

The procedure for the remapping the grid, in turn, consists of two stages. The first is a local change in the structure of the grid. The second is to determine the values of the grid functions in the areas of the changed structure. The local change in the structure of the computational grid is reduced to three local operations:

(i) the diagonal of a quadrangle formed by two triangles is replaced by another diagonal (Fig.1);
Figure 1. Elementary remapping - quadrangle formed by two triangles diagonal replacement on another diagonal.

Figure 2. Deleting a node is the union of two neighboring grid nodes.

Figure 3. Adding a node - adding a new node to the middle connections between neighboring nodes.
(ii) combining two adjacent grid nodes (Fig.2);
(iii) adding a new node to the middle of the link between neighboring nodes (Fig.3).

The structure of the grid is corrected using the first two operations. Mesh adaptation occurs through operations 2 and 3.

3.2. Grid functions remapping method for completely conservative Lagrangian operator-difference scheme for astrophysical MHD problems.

At the second stage of the of the grid remapping, the values of the grid functions in the nodes and cells involved in changing the structure mesh are calculated. The use of simple reinterpolation in recalculating the grid functions leads to violation of conservation laws and essential errors in calculations in regions with large gradients of the solution (for example, on shock waves). It is important to minimize the numerical error introduced by the recounting procedure. For this, not only the error was minimized, introduced into the values of the grid functions, but also in the gradients of these functions functions. In addition, it is necessary to fulfill the conservation laws (mass, momentum, energy) in the vicinity of the local restructuring. The method of recalculating grid functions is based on minimizing functionals containing the values of the functions to be recalculated, their gradients and grid analogs of conservation laws [2]. Let us describe the technique of recalculating grid functions, based on minimization such functionals [2].

Suppose that in the computational domain we have a mesh \( \omega \) consisting of the set of triangular cells \( \omega_{\triangle} \) and the set of nodes \( \omega_{\times} \) - vertices of triangles. In the centers of grid cells, the following grid functions are given: \( \rho \) - density, \( p \) is the pressure, \( T \) is the temperature, \( \mathbf{H} = (H_r, H_\varphi, H_z) \) - magnetic field strength, \( \varepsilon \) - internal energy, \( m, V \) - mass and volume of the grid cell, respectively. At the grid nodes, \( \mathbf{u} = (u_r, u_\varphi, u_z) \) is the velocity vector, \( \mathbf{x} = (r, z) \) is the coordinate.

We consider a simply-connected part of the \( \Gamma \) grid bounded by a simply connected closed broken line consisting of sides of cells. Consider the situation where the reorganization of the grid is done internally \( \Gamma_{\text{gamma}} \). When adding a new node, the value of the node functions in the new nodes are determined by linear interpolation. Values of the cell functions on the new grid are found from the minimization conditions errors in gradients and conservation laws. Let’s \( I = \{\triangle_1, \ldots, \triangle_n\} \) - is a subset from \( \Gamma \), and \( J = \{x_1, \ldots, x_k\} \) - is a set of nodes from \( \Gamma \). After remapping procedure we have \( \tilde{I} = \{\tilde{\triangle}_1, \tilde{\triangle}_2, \ldots, \tilde{\triangle}_m\} \), \( \tilde{J} = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_l\} \) - is a set of the cells and nodes from \( \Gamma \). By the symbol \( \tilde{\cdot} \) we will denote operators and grid functions defined on a new grid structure. New values of the density \( \tilde{\rho}_\triangle \) for \( \triangle_i \in \tilde{I} \) are defined by the minimization of the following functional:

\[
\min_{\tilde{\rho}} \left\{ \sum_{x_j \in J \cap \tilde{J}} \frac{(\nabla_j \rho - \tilde{\nabla}_j \tilde{\rho})^2}{(\nabla_j \rho)^2 + \alpha_1 \sum_{x_j \in J \cap \tilde{J}} (\nabla_j \rho)^2 + \alpha_n} + \alpha_{\rho 1} \left[ \sum_{\triangle_i \in I} \rho_i V_i - \sum_{\triangle_i \in \tilde{I}} \tilde{\rho}_i \tilde{V}_i \right]^2 \right. \\
+ \alpha_{\rho 2} \sum_{\triangle_i \in \tilde{I}} (\tilde{\rho}_i - \rho_i)^2 \right\} \tag{16}
\]

The finding of a minimum of this functional reduces to solving the system algebraic equations. If this system of equations is linear, then its solved directly. In the case where the system of the
equations obtained from the conditions of the extremum of the functional is nonlinear, to find its solution, the Newton method is used. Similarly the values of the pressure, the components of the magnetic field in the new cells are calculated. This approach allows the recalculation of cell functions at reorganization of a grid qualitatively enough.

However, the application of functionals containing gradients of the values recalculated functions and grid analogs of conservation laws, is not sufficiently effective in recalculating the grid functions, for the simulation of some astrophysical problems. For example, for the simulation of iron core collapse and neutron star formation, for the simulations of magnetorotational supernova explosions.

The large gradients of the grid functions that arise when solving such problems lead to the occurrence of large errors in the solution during the reorganization of the grid. In addition, when recalculating the cell functions according to the method described above, a significant accumulation of errors in the conservation laws occurs. In the problems of the collapse and magnetorotational explosion of a supernova, the exact implementation of conservation laws is important, since in the calculations it is necessary to do a lot of time steps, which can lead to accumulation of errors. The specificity of this type of problem is that even a weak violation of the conservation laws in numerical calculation can lead to a qualitative distortion of the results. For the calculation of astrophysical problems such as the problem of the collapse of cold protostellar clouds, the problem of the collapse of the iron core and the formation of a neutron star, and also for the simulation of magnetorotational supernovae, the technique for reconstructing the grid and recalculating the grid functions was modified. Astrophysical problems have the following features: a large difference in parameters, the presence of a free boundary, the need for exact conservation laws to be satisfied when recalculating the grid functions.

For the simulation of astrophysical problems, a method of conditional minimization of functionals was proposed and implemented, which ensures the exact implementation of conservation laws. To calculate the density in new cells, it was suggested to look for the conditional minimum of the following functional [6]:

\[
\min_{\tilde{\rho}_i} \sum_I \frac{\tilde{\rho}_i - \rho_{ci}}{\rho_{ci}/\tilde{V}_i} \left( \sum_I (\tilde{H}_{ri} - H_{rci}) \right)^2, \tag{17}
\]

under the condition of mass conservation of vicinity of the remapping region:

\[
\sum_I \tilde{\rho}_i \tilde{V}_i = \sum_I \rho_i V_i = M_{\text{vicinity}}. \tag{18}
\]

Here \(\rho_{ci}\) - is a linear interpolation of the density to a new grid cells. Applying the Lagrange multiplier method to minimize the conditional functional, we obtain:

\[
\tilde{\rho}_i = \rho_{ci} \frac{M_{\text{vicinity}}}{\sum_I \rho_{ci} \tilde{V}_i}. \tag{19}
\]

It is easy to show that the density recalculation by the formula (19) allows us to fulfill the law of conservation of mass in the remapping region.

For the calculations of new values of the poloidal magnetic field components \(H_r, H_z\) we use the following conditional minimization of the functionals [6]:

\[
\min_{\tilde{H}_{ri}} \sum_I \left( (\tilde{H}_{ri} - H_{rei})^2 \right) \tilde{V}_i, \tag{20}
\]
\[
\min_{\tilde{H}_{zi}} \sum_{i} \left\{ (\tilde{H}_{zi} - H_{zci})^2 \right\} \tilde{V}_i, \tag{21}
\]

under the conditions of conservation of every component of the magnetic energy in the remapping region:

\[
\sum_{i} \tilde{H}_{ri}^2 \tilde{V}_i = \sum_{i} H_{ri}^2 V_i = E_{mag, r \text{ vicinity}},
\]

\[
\sum_{i} \tilde{H}_{zi}^2 \tilde{V}_i = \sum_{i} H_{zi}^2 V_i = E_{mag, z \text{ vicinity}}.
\]

Here \(H_{rci}, H_{zci}\) - is a linear interpolation of poloidal magnetic field components to a new grid cells.

Conditional minimum of these functionals is found by the Lagrange multiplier method.

For the calculation of the poloidal components of the magnetic field \(H_r, H_z\) we get following formulae:

\[
\tilde{H}_{ri} = H_{rci} \frac{E_{mag, r \text{ vicinity}}}{\sum_{i} H_{ri}^2 \tilde{V}_i},
\]

\[
\tilde{H}_{zi} = H_{zci} \frac{E_{mag, z \text{ vicinity}}}{\sum_{i} H_{zi}^2 \tilde{V}_i}.
\]

It is easy to show that the formulas given above calculate new values of the poloidal components of the magnetic field, preserving the values of each of the components of poloidal magnetic energy in the remapping region.

For the calculations of the toroidal component of the magnetic field \(H_\phi\) on a new grid the following conditional minimum of the functional is calculated:

\[
\min_{\tilde{H}_{\phi i}} \sum_{i} \left\{ (\tilde{H}_{\phi i} - H_{\phi ci})^2 \right\} \tilde{S}_i, \tag{22}
\]

under the condition of the conservation of the toroidal magnetic flux \(H_\phi S\) in the remapping region:

\[
\sum_{i} \tilde{H}_{\phi i} \tilde{S}_i = \sum_{i} H_{\phi i} S_i = F_{mag, \phi \text{ vicinity}}.
\]

Finally for the calculation if the toroidal components of the magnetic field \(\tilde{H}_{\phi i}\) we get the following formula:

\[
\tilde{H}_{\phi i} = H_{\phi ci} \frac{F_{mag, \phi \text{ vicinity}}}{\sum_{i} H_{\phi ci} \tilde{S}_i}.
\]

Using this formula one can calculate the values of the toroidal component of the magnetic field in a new cells preserving the toroidal magnetic flux.

However, if among the averaged values of the toroidal component of the magnetic field \(H_{\phi ci}\) there are values of different signs, then the conditional minimum of the functional (22) can lead
to strong distortion of the solution. This is due to the fact that the minimum of such a functional can be achieved in the case when not the physically large positive and negative values of the toroidal magnetic flux $\tilde{H}_\varphi S_i$ in the different cells of the remapping region, compensating each other.

In such a situation, in order to find the values of the toroidal component of the magnetic field in the cells of the remapping region, the conditional minimum of the following functional is sought (by analogy with the poloidal components of the magnetic field):

$$\min_{H_{\varphi i}} \sum_I \{ (\tilde{H}_{\varphi i} - H_{\varphi ci})^2 \} \tilde{V}_i,$$

provided that the toroidal magnetic energy is preserved in the remapping region:

$$\sum_I \tilde{H}^2_{\varphi i} \tilde{V}_i = \sum_I H^2_{\varphi i} V_i = E_{mag, \varphi \text{ vicinity}},$$

In this case, to calculate the toroidal components of the magnetic field $\tilde{H}_{\varphi i}$ we get the following formulae:

$$\tilde{H}_{\varphi i} = H_{\varphi ci} \sqrt{\frac{E_{mag, \varphi \text{ vicinity}}}{\sum_I H^2_{\varphi ci} V_i}}.$$

The application of the above formula allows us to calculate the toroidal component of the magnetic field in the cells of a remapping region with the conservation of the toroidal component of the magnetic energy.

To find the new values of the temperature $\tilde{T}_i$ (the pressure $\tilde{p}_i$) in the remapping region, we seek the conditional minimum of the following functional:

$$\min_{T_{\varphi i}} \sum_I (\tilde{T}_{\varphi i} - T_{\varphi ci})^2 \tilde{V}_i,$$

under the condition of preserving of the internal energy in the remapping region

$$\sum_I \tilde{\varepsilon}_i \tilde{V}_i = \sum_I \varepsilon V_i = E_{int, \text{ vicinity}},$$

Applying the method of Lagrange multipliers for the conditional minimization of this functional, we obtain a system of nonlinear functional equations:

$$\begin{cases}
2(\tilde{T}_i - T_i) \tilde{V}_i + \lambda \left( \frac{\partial \varepsilon}{\partial T_i} \right) \tilde{V}_i = 0, \\
\sum_I \xi_i \tilde{V}_i = E_{int, \text{ vicinity}}.
\end{cases}$$

Here $\lambda$ is the Lagrange multiplier. We solve this system of equations by Newton method.

4. Astrophysical applications

The described numerical method based on a completely conservative Lagrangian operator-difference scheme on a triangular grid of variable structure was successfully applied to a simulation of the collapse of cold rapidly rotating protostellar cloud [8], collapse of a magnetized rotating cloud [9]. Application of the described here conservative grid functions remapping procedure allowed us to simulate magnetorotational core-collapsed supernova explosion [6, 7].
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