Thermodynamics of an Ising-like XXZ chain in a longitudinal magnetic field in the framework of the quantum transfer matrix approach

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Abstract. Taking the Ising chain as a reference model we have derived a perturbative expression for the free energy density of the Heisenberg–Ising chain with strong easy-axis anisotropy. All calculations are performed on the ground of the quantum transfer matrix approach. The obtained result agrees with the direct high-temperature expansion. It also agrees with the low-temperature cluster expansion in the special subregime when quantum fluctuations are weak against thermodynamical ones.

Keywords: quantum integrability (Bethe ansatz), spin chains, ladders and planes

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1. Introduction

One of the basic models of one-dimensional quantum magnetism is the XXZ spin chain in a longitudinal magnetic field $h$ [1]. It corresponds to the Hamiltonian

$$\hat{H} = \sum_{n=1}^{N} \left[ \frac{J}{2} (S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) + J_z \left( S_n^z S_{n+1}^z - \frac{1}{4} I \right) - h S_n^z \right], \quad h \geq 0, \quad (1)$$

where $S_n^z$ and $S_n^\pm = S_n^x \pm i S_n^y$ are the spin-1/2 operators. At $J = 0$ the model (1) reduces to the Ising chain solvable by an elementary machinery [2].

Being purely classical ($[S_n^z, S_m^z] = 0$) the Ising model is rather poor (for example the Ising magnons are dispersionless). Nevertheless it was suggested for a number of magnetic compounds [3–5]. However since the condition $J = 0$ is not associated with any symmetry it is natural to suppose that a more adequate model for these compounds is the Ising-like chain described by Hamiltonian (1) supplemented by the condition

$$\Delta \equiv \frac{J_z}{J} \gg 1. \quad (2)$$

The model (1) and (2) was also suggested for some real compounds [6, 7]. Being quantum it has a more rich physical behavior and at the same time should allow a perturbative treatment around the Ising model.

In the last two decades a new machinery for evaluation of thermodynamics of quantum spin chains was suggested on the base of the quantum transfer matrix (QTM) approach (see reviews [8–10] and references therein). This approach has two main stages. Within the former one a system of integral equations on specially introduced auxiliary functions is derived. Within the latter one an integral representations for the free energy density

$$f(T, h) = -\frac{1}{\beta N \ln N} \ln Z_N(T, h), \quad \beta \equiv \frac{1}{k_B T}, \quad (3)$$

(as usual $Z_N(T, h)$ is the partition function) and correlation functions are obtained in this framework.
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In the Ising $J = 0$ case the corresponding free energy density has the form

$$f^{\text{Is}}(T, h) = -\frac{h}{2} - \frac{1}{\beta} \ln \left( 1 + e^{-\beta h} + \sqrt{(1 - e^{-\beta h})^2 + 4 e^{\beta(J_z - h)}} \right).$$  \hspace{1cm} (4)

Formula (4) readily follows from the representation [2]

$$f^{\text{Is}}(T, h) = -\frac{1}{\beta} \ln \Lambda_-(\beta, h),$$  \hspace{1cm} (5)

where

$$\Lambda_{\pm}(\beta, h) = \cosh \frac{\beta h}{2} \pm \sqrt{\sinh^2 \frac{\beta h}{2} + e^{\beta J_z}},$$  \hspace{1cm} (6)

are eigenvalues of the special transfer matrix

$$T = \begin{pmatrix} e^{\beta h/2} & e^{\beta J_z/2} \\ e^{\beta J_z/2} & e^{-\beta h/2} \end{pmatrix},$$  \hspace{1cm} (7)

related to the Hamiltonian (1) at $J = 0$. As it has been mentioned in [11] formula (4) also may be obtained within the approach of [8–10].

In the present paper we extend the result of [11] studying the model (1) and (2) at the vicinity of the Ising point in the first two perturbation orders. The paper is organized as follows. In section 2 following [11] we introduce the basic auxiliary functions and corresponding integral equations. We use however rather different notations which seem us to be more convenient for our treatment. We also show how to account singularities in the kernels of the integral equations. In section 3 we extract the auxiliary functions related to the Ising model [11] and reduce the integral equations to the form convenient for perturbative expansion. In section 4 we calculate the first two terms of the perturbation expansion and evaluate the corresponding correction to the free energy density. In section 5 treating the high-temperature regime

$$h, |J_z|, |J| \ll k_B T \iff \beta \cdot \max(h, |J_z|, |J|) \ll 1,$$  \hspace{1cm} (8)

we compare the first order terms of the high-temperature expansion related to the obtained formula for $f(T, h)$ with the one which directly follows from the definition (3). Being convinced that both approaches give the same result we analytically confirm correctness of the approach [8–10] at high temperatures [12]. In section 6 we study the low-temperature regime in the phase related to the ferromagnetically polarized ground state

$$|\emptyset\rangle = \cdots \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes |\uparrow\rangle \otimes \ldots, \quad S^x|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle,$$  \hspace{1cm} (9)
Thermodynamics of an Ising-like XXZ chain in a longitudinal magnetic field in the framework of the quantum transfer matrix approach and non-zero magnon gap energy [13]

\[ E_{\text{gap}} = h - J_z - |J| > 0. \]  

(10)

Comparing the calculated perturbative result with the one obtained previously by the low-temperature cluster expansion [13] we demonstrate that these two approaches totally disagree with each other in the extreme low-temperature subregime

\[ k_B T \ll \frac{E_{\text{width}}}{2}, \]  

(11)

where

\[ E_{\text{width}} = 2|J|, \]  

(12)

is the magnon band width. At the same time they give a similar result in the scaled low-temperature subregime

\[ \frac{E_{\text{width}}}{2} \ll k_B T \ll E_{\text{gap}} + \frac{E_{\text{width}}}{2}. \]  

(13)

Under a suggestion that low-temperature quantum fluctuations arise from transitions within the magnon band and hence are governed by \( E_{\text{width}} \) we conclude that the scaled low-temperature regime corresponds to weakness of quantum fluctuations against thermodynamical ones.

Finally in appendix A we give a concise but consistent introduction into the QTM machinery specially adapted to the content of the paper.

2. Foundations of the \( b_+ - b_- \)-formalism

Within the QTM approach the free energy density may be represented in the following form [11]

\[ f(T, h) = \frac{J_z}{2} \tanh \frac{\eta}{2} \sum_{m=-\infty}^{\infty} \frac{e^{-\eta|m|}}{\cosh(\eta m)} - \frac{1}{2\pi \beta} \int_{-\pi/2}^{\pi/2} dx \, d(x) \ln(\mathcal{B}_+(x)\mathcal{B}_-(x)). \]  

(14)

Here

\[ \mathcal{B}_\pm(x) = 1 + b_\pm(x), \]  

(15)

while the functions \( b_+(x) \) and \( b_-(x) \) (which in fact are \( b(x) \) and \( \bar{b}(x) \) in notations of [11]) satisfy the following system of equations

\[ \ln b_\pm(x) + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dy \, (\kappa_\pm(x-y) \ln \mathcal{B}_\pm(y) - \kappa(x-y) \ln \mathcal{B}_\pm(y)) = \mp \frac{\beta h}{2} - J_z \beta c(x), \]  

(16)
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$$
\lim_{T \to \infty} b_\pm(x, T, h) = 1.
$$

(17)

Here

$$
d(x) = \sum_{m=-\infty}^{\infty} \frac{e^{2imx}}{\cosh(\eta m)}, \quad c(x) = \frac{d(x) \tanh \eta}{2}, \quad \kappa(x) = \sum_{m=-\infty}^{\infty} \frac{e^{2imx}}{1+e^{2|\eta|m}},
$$

(18)

$$
\kappa_\pm(x) = \kappa(x \pm i \eta \pm i \epsilon) = \sum_{m=-\infty}^{\infty} \frac{e^{\mp 2(\eta-\epsilon)m}}{1+e^{2|\eta|m}} e^{2imx},
$$

and it is assumed that

$$
cosh \ \eta = \frac{J_z}{J}, \ \eta > 0.
$$

(19)

Functions $\kappa_\pm(x)$ have singular parts. In order to extract them we shall use the following extended representations

$$
\kappa_+(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{e^{2ix-\eta m}}{1+e^{2|\eta|m}} + \sum_{m=-\infty}^{-1} \left(1 - \frac{1}{1+e^{2|\eta|m}}\right) e^{2ix+\epsilon m},
$$

$$
\kappa_-(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \left(1 - \frac{1}{1+e^{2|\eta|m}}\right) e^{2ix+\epsilon m} + \sum_{m=-\infty}^{-1} \frac{e^{2ix+\eta m}}{1+e^{2|\eta|m}}.
$$

(20)

Hence for a function

$$
\varphi(x) = \varphi^{(0)}(x) + \varphi^{(+)}(x) + \varphi^{(-)}(x),
$$

(21)

where $\varphi^{(0)}(x)$ is a number and

$$
\varphi^{(+)}(x) = \sum_{m=1}^{\infty} \varphi^{(m)} e^{2imx}, \quad \varphi^{(-)}(x) = \sum_{m=1}^{\infty} \varphi^{(-m)} e^{-2imx},
$$

(22)

one has

$$
\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \kappa_\pm(x-y) \varphi(y) dy = \varphi^{(0)}(x) \frac{1}{2} + \varphi^{(+)}(x) + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \tilde{\kappa}_\pm(x-y) (\varphi^{(+)}(y) + \varphi^{(+)}(y)) dy,
$$

(23)

where

$$
\tilde{\kappa}_+(x) = \sum_{m=1}^{\infty} \frac{e^{2ix-\eta m}}{1+e^{2|\eta|m}} - \sum_{m=-\infty}^{-1} \frac{e^{2imx}}{1+e^{2|\eta|m}}, \quad \tilde{\kappa}_-(x) = -\sum_{m=1}^{\infty} \frac{e^{2ixm}}{1+e^{2|\eta|m}} + \sum_{m=-\infty}^{-1} \frac{e^{2ix+\eta m}}{1+e^{2|\eta|m}}.
$$

(24)
3. The Ising solution and around

At the Ising point
\[ J = 0 \iff \eta = \infty, \]  
(25)
one has from (18) and (24)
\[ \alpha^s(x) = 1, \quad c^s(x) = \frac{1}{2}, \quad \kappa^s(x) = \frac{1}{2}, \quad \kappa^s(x) = 0, \]  
(26)
so the right side of (16) does not depend on \( x \) and a substitution
\[ b_\pm(x) \to b^s_\pm, \quad \mathfrak{B}_\pm(x) \to \mathfrak{B}^s_\pm = 1 + b^s_\pm \]  
(27)
yields the following system of algebraic equations
\[
\ln b^s_\pm + \frac{\ln \mathfrak{B}^s_\pm - \ln \mathfrak{B}^s_\mp}{2} = \mp \frac{\beta h}{2} - \frac{J_z \beta}{2} \iff \left( b^s_\pm \right)^2 + 1 = e^{\beta(-J_z \mp h)},
\]  
(28)
or in an equivalent form
\[
b^s_\pm b^s_\mp = e^{-\beta J_z}, \quad \frac{\left( b^s_\pm \right)^2 + e^{-\beta J_z} b^s_\mp}{b^s_\pm + 1} = e^{\beta(-J_z \mp h)}. 
\]  
(29)
At the same time formula (14) in the Ising case (25) reduces to
\[
f^s(T, h) = -\frac{J_z}{2} - \frac{1}{2\beta} \ln (\mathfrak{B}^s_\pm \mathfrak{B}^s_\mp) .
\]  
(30)
System (17) and (29) has the single solution
\[
b^s_\pm = e^{-\beta(J_z \pm h/2)} \left( \sqrt{\sinh^2 \frac{\beta h}{2} + e^{\beta J_z} \pm \sinh \frac{\beta h}{2}} \right).
\]  
(31)
According to (15) and (29)
\[
\mathfrak{B}^s_\pm \mathfrak{B}^s_\mp = 1 + e^{-\beta J_z} + b^s_\pm + b^s_\mp,
\]  
(32)
and a substitution of (31) into (32) gives
\[
\mathfrak{B}^s_\pm \mathfrak{B}^s_\mp = 1 + e^{-\beta J_z} \left( 2 \cosh \frac{\beta h}{2} \cdot \sqrt{\sinh^2 \frac{\beta h}{2} + e^{\beta J_z} + \cosh^2 \frac{\beta h}{2} + \sinh^2 \frac{\beta h}{2}} \right)
\]  
\[
= e^{-\beta J_z} \left( \cosh \frac{\beta h}{2} + \sqrt{\sinh^2 \frac{\beta h}{2} + e^{\beta J_z}} \right)^2.
\]  
(33)
So (4) really follows from (30) and (33) [11].
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Let us now extract the Ising solution taking

\[ b_\pm(x) = b_\pm^{Is} + \tilde{b}_\pm(x), \quad f(T, h) = f^{Is}(T, h) + \tilde{f}(T, h), \] (34)

as well as

\[ d(x) = d^{Is} + \tilde{d}(x), \quad c(x) = c^{Is} + \tilde{c}(x), \quad \kappa(x) = \kappa^{Is} + \tilde{\kappa}(x), \] (35)

where according to (18), (26) and (35)

\[ \tilde{d}(x) = 2 \sum_{m=1}^{\infty} \cos 2mx \cosh \eta m, \quad \tilde{c}(x) = \tanh \eta \sum_{m=1}^{\infty} \cos 2mx \cosh(\eta m) - \frac{e^{-2\eta}}{1 + e^{-2\eta}}, \]

\[ \tilde{\kappa}(x) = 2 \sum_{m=1}^{\infty} \cos 2mx \frac{1}{1 + e^{2\eta m}}. \] (36)

From (24) and (36) readily follows that

\[ \int_{-\pi/2}^{\pi/2} \tilde{d}(x) dx = \int_{-\pi/2}^{\pi/2} \tilde{\kappa}(x) dx = \int_{-\pi/2}^{\pi/2} \tilde{\kappa}_\pm(x) dx = 0. \] (37)

Now using (34)–(37) one may reduce (16) and (24) to the forms

\[ \ln \left( 1 + \frac{\tilde{b}_\pm(x)}{b_\pm^{Is}} \right) + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} dy \left[ \kappa_\pm(x - y) \ln \left( 1 + \frac{\tilde{b}_\pm(y)}{b_\pm^{Is}} \right) - \kappa(x - y) \ln \left( 1 + \frac{\tilde{b}_\pm(y)}{b_\pm^{Is}} \right) \right] = -J_z \beta \tilde{c}(x), \] (38)

and

\[ \tilde{f}(T, h) = J_z \left( \frac{1 - \tanh \eta}{2} - 2 \tanh \eta \sum_{m=1}^{\infty} \frac{e^{-2\eta m}}{1 + e^{-2\eta m}} \right) \]

\[ - \frac{1}{2\pi \beta} \int_{-\pi/2}^{\pi/2} dx \left[ \ln \left( 1 + \frac{\tilde{b}_\pm(x)}{b_\pm^{Is}} \right) + \ln \left( 1 + \frac{\tilde{b}_\pm(x)}{b_\pm^{Is}} \right) \right], \] (39)

more convenient for the perturbative series expansion.

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4. Series expansion near the Ising point

Suggesting the series expansions

\[ \tilde{b}_\pm(x) = \sum_{j=1}^{\infty} e^{-\eta_j} b^{(j)}_\pm(x), \tag{40} \]

\[ \tilde{f}(T, h) = \sum_{j=1}^{\infty} e^{-\eta_j} f^{(j)}(T, h), \tag{41} \]

and representing each \( b^{(j)}_\pm(x) \) in the separated form (21)

\[ b^{(j)}_\pm(x) = b^{(j; 0)}_\pm(x) + b^{(j; +)}_\pm(x) + b^{(j; -)}_\pm(x), \tag{42} \]

where according to (22)

\[ b^{(j; +)}_\pm(x) = \sum_{m=1}^{\infty} b^{(j; 2m)}_\pm e^{2imx}, \quad b^{(j; -)}_\pm(x) = \sum_{m=1}^{\infty} b^{(j; -2m)}_\pm e^{-2imx}, \tag{43} \]

we shall calculate \( b^{(j)}_\pm(x) \) and \( f^{(j)}(T, h) \) for \( j = 1, 2 \). Since

\[ \frac{1}{\Delta} \equiv \frac{1}{\cosh \eta} = 2e^{-\eta} + o(e^{-2\eta}), \tag{44} \]

formula

\[ \tilde{f}(T, h) = e^{-\eta} f^{(1)}(T, h) + e^{-2\eta} f^{(2)}(T, h) + o(e^{-2\eta}), \tag{45} \]

is equivalent to

\[ \tilde{f}(T, h) = \frac{f^{(1)}(T, h)}{2\Delta} + \frac{f^{(2)}(T, h)}{4\Delta^2} + o\left( \frac{1}{\Delta^2} \right). \tag{46} \]

For evaluation of the first two terms in the right sides of (40) and (41) we also need the expansions

\[ \tilde{c}(x) = 2e^{-\eta} \cos 2x + (2 \cos 4x - 1)e^{-2\eta} + o(e^{-2\eta}), \]

\[ \tilde{\kappa}(x) = 2e^{-2\eta} \cos 2x + o(e^{-2\eta}), \]

\[ \tilde{\kappa}_\pm(x) = -e^{2(\mp ix - \eta)} + o(e^{-2\eta}), \]

\[ d(x) = 1 + 4(e^{-\eta} \cos 2x + e^{-2\eta} \cos 4x) + o(e^{-2\eta}), \]

\[ \frac{1 - \tanh \eta}{2} - 2 \tanh \eta \sum_{m=1}^{\infty} \frac{e^{-2\eta m}}{1 + e^{-2\eta m}} = -e^{-2\eta} + o(e^{-2\eta}), \tag{47} \]

which directly follow from (24) and (36).
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Using (47) one readily gets in the order \( e^{-\eta} \)

\[
f^{(1)}(T, h) = -\frac{1}{2\pi\beta} \int_{-\pi/2}^{\pi/2} \left( \frac{b_+^{(1)}(x)}{B_+^l} + \frac{b_-^{(1)}(x)}{B_-^l} \right) dx = -\frac{1}{2\beta} \left( \frac{b_+^{(1;0)}}{B_+^l} + \frac{b_-^{(1;0)}}{B_-^l} \right),
\]

and

\[
\frac{b_+^{(1)}(x)}{b_+^l} + \frac{b_-^{(1;\pm)}(x)}{B_+^l} + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left[ \frac{b_+^{(1)}(y)}{B_+^l} - \frac{b_-^{(1)}(y)}{B_-^l} \right] dy = -2J_z\beta \cos 2x,
\]

or in an expanded form

\[
\frac{b_+^{(1;0)}}{b_+^l} + \frac{1}{2} \left[ \frac{b_+^{(1;0)}}{B_+^l} - \frac{b_-^{(1;0)}}{B_-^l} \right] = 0,
\]

\[
\frac{b_+^{(1;\pm)}(x)}{b_+^l} = -J_z\beta, \quad \frac{b_-^{(1;\pm)}(x)}{b_-^l} = -J_z\beta,
\]

\[
\frac{b_-^{(1;\pm)}(x)}{b_-^l} = -J_z\beta, \quad \frac{b_-^{(1;\pm)}(x)}{b_-^l} = -J_z\beta.
\]

Representing homogeneous linear system (50) as

\[
(2 + b_\pm^l)t_\pm + b_\pm^l t_\mp = 0, \quad t_\pm = \frac{b_\pm^l}{1 + b_\pm^l},
\]

one readily concludes that its nontrivial solvability implies condition \( b_+^l + b_-^l = -2 \) which cannot be fulfilled because both \( b_\pm^l \) in (31) are positive. Hence (50) yields

\[
b_\pm^{(1;0)} = 0,
\]

and according to (48) and (53)

\[
f^{(1)}(T, h) = 0.
\]

The remaining system (51) gives

\[
b_+^{(1;2)} = -J_z\beta b_+^l, \quad b_-^{(1;2)} = -J_z\beta b_-^l, \quad b_+^{(1;\pm)} = -J_z\beta b_\pm^l, \quad b_-^{(1;\pm)} = -J_z\beta b_\pm^l.
\]
Turning to \( f^{(2)}(T, h) \) and accounting (47) one readily gets from (39)

\[
f^{(2)}(T, h) = -J_z + \frac{1}{2\pi\beta} \int_{-\pi/2}^{\pi/2} \left[ -4 \left( \frac{b_+^{(1)}(x)}{\mathcal{B}_+^{(1s)}} + \frac{b_-^{(1)}(x)}{\mathcal{B}_-^{(1s)}} \right) \cos 2x + \frac{1}{2} \left( \frac{b_+^{(1)}(x)}{\mathcal{B}_+^{(1s)}} \right)^2 - \frac{b_-^{(1)}(x)}{\mathcal{B}_-^{(1s)}} + \frac{1}{2} \left( \frac{b_+^{(1)}(x)}{\mathcal{B}_+^{(1s)}} \right)^2 - \frac{b_-^{(2)}(x)}{\mathcal{B}_-^{(1s)}} \right] dx = -J_z + \frac{1}{2\beta} \left[ -2 \left( \frac{b_+^{(1;2)}(x)}{\mathcal{B}_+^{(1s)}} + \frac{b_-^{(1;2)}(x)}{\mathcal{B}_-^{(1s)}} + \frac{b_+^{(1;2)}(x)}{\mathcal{B}_+^{(1s)}} + \frac{b_-^{(1;2)}(x)}{\mathcal{B}_-^{(1s)}} \right) \right. \\
\left. - \frac{b_+^{(2;0)}(x)}{\mathcal{B}_+^{(2s)}} - \frac{b_-^{(2;0)}(x)}{\mathcal{B}_-^{(2s)}} + \frac{b_+^{(1;2)}(x) b_-^{(1;2)}(x)}{(\mathcal{B}_+^{(1s)})^2} + \frac{b_-^{(1;2)}(x) b_-^{(1;2)}(x)}{(\mathcal{B}_-^{(1s)})^2} \right], \tag{56}
\]

So for an evaluation of \( f^{(2)}(T, h) \) we additionally need only \( b_+^{(2;0)}(x) \). The latter may be extracted from the equation

\[
\frac{b_+^{(2;0)}(x)}{b_+^{(2s)}} - \frac{b_+^{(1;2)}(x) b_-^{(1;2)}(x)}{(b_+^{(1s)})^2} + \frac{1}{2} \left[ \frac{b_+^{(2;0)}(y)}{b_+^{(1s)}} - \frac{b_+^{(1;2)}(y) b_-^{(1;2)}(y)}{(b_+^{(1s)})^2} - \frac{b_-^{(2;0)}(y) b_-^{(1;2)}(y)}{(b_-^{(1s)})^2} + \frac{b_-^{(1;2)}(y) b_-^{(1;2)}(y)}{(b_-^{(1s)})^2} \right] dy = J_z \beta (1 - 2 \cos 4x), \tag{57}
\]

which follows from (38) in the order \( e^{-2x} \). According to (57)

\[
\frac{b_+^{(2;0)}(x)}{b_+^{(2s)}} - \frac{b_+^{(1;2)}(x) b_-^{(1;2)}(x)}{(b_+^{(1s)})^2} + \frac{1}{2} \left[ \frac{b_+^{(2;0)}(x)}{b_+^{(1s)}} - \frac{b_+^{(1;2)}(x) b_-^{(1;2)}(x)}{(b_+^{(1s)})^2} - \frac{b_-^{(2;0)}(x) b_-^{(1;2)}(x)}{(b_-^{(1s)})^2} + \frac{b_-^{(1;2)}(x) b_-^{(1;2)}(x)}{(b_-^{(1s)})^2} \right] = J_z \beta. \tag{58}
\]

Now taking

\[
b_+^{(2;0)} \equiv \mathcal{B}_+^{(1s)} x_+, \quad A_+ \equiv 2J_z \beta + J_z^2 \beta^2 \left( \frac{2}{(b_+^{(1s)})^2} + \frac{(b_+^{(1s)})^2 b_-^{(1s)}}{(b_+^{(1s)})^2} - \frac{(b_-^{(1s)})^2 b_-^{(1s)}}{(b_-^{(1s)})^2} \right), \tag{59}
\]

and substituting \( b_+^{(1;2)} \) and \( b_-^{(1;2)} \) from (55) one reduces (58) to the form

\[
\left( \frac{2}{b_+^{(1s)}} + 1 \right) x_+ + x_+ = A_+, \tag{60}
\]
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\[ x_\pm = \frac{b_+^{(ls)} b_-^{(ls)}}{\mathcal{B}_+^{(ls)} + \mathcal{B}_-^{(ls)}} \left( \frac{A_\pm + A_- - A_+}{2} \right). \]  

(62)

So according to (59) and (62)

\[ \frac{b_+^{(2;0)}}{\mathcal{B}_+^{(ls)}} + \frac{b_-^{(2;0)}}{\mathcal{B}_-^{(ls)}} = x_+ + x_- = \frac{A_+ b_+^{ls} + A_- b_-^{ls}}{\mathcal{B}_+^{(ls)} + \mathcal{B}_-^{(ls)}}, \]  

(63)

and a direct substitution of (60) into (63) gives

\[ A_+ b_+^{ls} + A_- b_-^{ls} = 2J_z \beta (U - 2) + J_z^2 \beta^2 \left[ 2 \left( \frac{U(U-1)}{\mathcal{V}} - 2 \right) + \frac{(1-\mathcal{V})(4U^2 - 4\mathcal{V})}{\mathcal{V}^2} \right], \]  

(64)

where

\[ U = \mathcal{B}_+^{(ls)} + \mathcal{B}_-^{(ls)}, \quad \mathcal{V} = \mathcal{B}_+^{(ls)} \mathcal{B}_-^{(ls)}. \]  

(65)

Now from (63) and (64) it follows that

\[ \frac{b_+^{(2;0)}}{\mathcal{B}_+^{(ls)}} + \frac{b_-^{(2;0)}}{\mathcal{B}_-^{(ls)}} = 2J_z \beta \left( 1 - \frac{2}{U} \right) + J_z^2 \beta^2 \left( \frac{U-2}{\mathcal{V}} - \frac{4}{U\mathcal{V}} + \frac{U}{\mathcal{V}^2} \right). \]  

(66)

At the same time according to (55)

\[ - \frac{b_+^{(1;2)}}{\mathcal{B}_+^{(ls)}} - \frac{b_-^{(1;2)}}{\mathcal{B}_-^{(ls)}} - \frac{b_+^{(1;-2)}}{\mathcal{B}_+^{(ls)}} - \frac{b_-^{(1;-2)}}{\mathcal{B}_-^{(ls)}} = 2J_z \beta \frac{b_+^{ls} + b_-^{ls} + b_+^{ls} b_-^{ls}}{\mathcal{B}_+^{(ls)} + \mathcal{B}_-^{(ls)}} \]

\[ = 2J_z \beta \left( 1 - \frac{1}{\mathcal{V}} \right) \frac{b_+^{(1;2)}(x) b_-^{(1;-2)}(x)}{(\mathcal{B}_+^{(ls)})^2} + \frac{b_+^{(1;2)}(x) b_-^{(1;-2)}(x)}{(\mathcal{B}_-^{(ls)})^2} \]

\[ = J_z^2 \beta^2 \left( \frac{(b_+^{ls})^2}{\mathcal{B}_+^{(ls)}} + \frac{(b_-^{ls})^2}{\mathcal{B}_-^{(ls)}} \right) = J_z^2 \beta^2 \left( \frac{U-4}{\mathcal{V}} + \frac{U}{\mathcal{V}^2} \right), \]  

(67)

and a substitution of (66) and (67) into (56) results in

\[ f^{(2)}(T, h) = 2J_z \left( \frac{1}{U} - \frac{1}{\mathcal{V}} \right) + J_z^2 \beta \left( \frac{2}{U\mathcal{V}} - \frac{1}{\mathcal{V}^2} \right). \]  

(68)

According to (15), (31), (32) and (65)

\[ U = 2 + \mathcal{V}, \quad \mathcal{V} = 1 + e^{-\beta J_z} + \mathcal{V}, \]  

(69)
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$$\Xi = 2 e^{-\beta J_z} \left( \cosh \beta h \sqrt{\frac{\sinh^2 \beta h}{2} + e^{\beta J_z} + \sinh^2 \frac{\beta h}{2}} \right).$$  \hspace{1cm} (70)$$

With the use of (69) one may rewrite (68) in the form

$$f^{(2)}(T, h) = 2J_z \hat{f}_1 + J_z^2 \hat{f}_2,$$  \hspace{1cm} (71)

where

$$\hat{f}_1 = \frac{e^{-\beta J_z} - 1}{(2 + \Xi)(1 + e^{-\beta J_z} + \Xi)}, \quad \hat{f}_2 = -\frac{\beta \Xi}{(2 + \Xi)(1 + e^{-\beta J_z} + \Xi)}.$$  \hspace{1cm} (72)

Formulas (70)–(72) express the main result of the paper.

### 5. The high-temperature regime

Rigorously speaking validity of the QTM approach was proved only for high temperatures [12]. That is why a comparison between the direct high-temperature expansion and the one which follows from (70)–(72) may be considered only as a good check of the calculations.

At high temperatures one has

$$f(T, h) = -\lim_{N \to \infty} \frac{1}{\beta N} \ln \text{tr} \left( I - \beta \hat{H} + \frac{(\beta \hat{H})^2}{2} \right) + o(\beta).$$  \hspace{1cm} (73)

Since

$$\text{tr} I = 2^N, \quad \text{tr} \hat{H} = 2^N \frac{N}{4} \text{tr} H,$$

$$\text{tr} \hat{H}^2 = 2^N \frac{N}{4} \left( \text{tr} H^2 + \text{tr} H_{12} H_{23} + \frac{N - 3}{4} (\text{tr} H)^2 \right),$$  \hspace{1cm} (74)

where the $4 \times 4$ matrix (here $I_n$ is $n \times n$ identity matrix and $I = I_2^\otimes$)

$$H = \frac{J}{2} \left( S^+ \otimes S^- + S^- \otimes S^+ \right) + J_z \left( S^z \otimes S^z - \frac{1}{4} I_4 \right) - \frac{h}{2} (S^z \otimes I_2 + I_2 \otimes S^z),$$  \hspace{1cm} (75)

is the Hamiltonian density related to (1) and

$$H_{12} = H \otimes I_2, \quad H_{23} = I_2 \otimes H.$$  \hspace{1cm} (76)
Substituting (74) into (73) and expanding the logarithm one readily gets
\[
f(T, h) = -\ln \frac{2}{\beta} - \lim_{N \to \infty} \frac{1}{\beta N} \ln \left[ 1 - \frac{\beta N}{4} \text{tr} H + \frac{\beta^2 N}{8} \right]
\times \left( \text{tr}(H^2 + H_{12}H_{23}) + \frac{N - 3}{4} (\text{tr} H)^2 \right) + o(\beta) = -\frac{\ln 2}{\beta} + \frac{\text{tr} H}{4}
\]
\[- \beta \left( \frac{\text{tr} H^2 + \text{tr} H_{12}H_{23}}{8} - \frac{3}{32} (\text{tr} H)^2 \right) + o(\beta),
\]
\[
(77)
\]
or
\[
f(T, h) = -\frac{2}{\beta} + \frac{\text{tr} H}{4} + \frac{\beta}{32} \left( 3(\text{tr} H)^2 - 4[\text{tr} H^2 + \text{tr}(H_{12}H_{23})] \right) + o(\beta).
\]
\[
(78)
\]
At the same time from (75) it follows that
\[
\text{tr} H = -J_z, \quad \text{tr} H^2 = \frac{J_z^2 + J_z^2 + h^2}{2}, \quad \text{tr}(H_{12}H_{23}) = \frac{J_z^2 + h^2}{2}.
\]
\[
(79)
\]
So a substitution of (79) into (77) yields
\[
\tilde{f}(T, h) = -\frac{\beta J_z^2}{16\Delta^2} + o(\beta),
\]
\[
(80)
\]
and according to (46)
\[
f^{(2)}(T, h) = -\frac{\beta J_z^2}{4}.
\]
\[
(81)
\]
This formula may be also readily obtained from (71) and (72) and the high temperature expansion
\[
\Xi = 2 - J_z \beta + o(\beta),
\]
\[
(82)
\]
which directly follows from exact formula (70).

6. The low-temperature polarized regime

Before evaluating the low-temperature expansion for the free energy density of the Ising-like chain in the polarized phase (9) and (10) we shall study the pure Ising chain. A low-temperature expansion of the free energy density (4) under the condition (10) (with $J = 0$) results in the formula
\[
f^{(a)}(T, h) = e_0(h) + f_{\text{magn}}^{(a)}(T, h) + f_{\text{bound}}^{(a)}(T, h) + f_{\text{scatter}}^{(a)}(T, h) + o(e^{2\beta(J_z - h)}),
\]
\[
(83)
\]
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$$e_0(h) = \frac{h}{2},$$  \hspace{1cm} (84)

is the ground state energy density in the polarized regime and

$$f_{\text{magn}}^\text{Is}(T, h) = -\frac{e^{-\beta E_{\text{magn}}^\text{Is}}}{\beta}, \quad f_{\text{bound}}^\text{Is}(T, h) = -\frac{e^{-\beta E_{\text{bound}}^\text{Is}}}{\beta},$$

$$f_{\text{scatt}}^\text{Is}(T, h) = \frac{3e^{-\beta E_{\text{scatt}}^\text{Is}}}{2\beta}. \hspace{1cm} (85)$$

The parameters

$$E_{\text{magn}}^\text{Is} = h - J_z, \quad E_{\text{bound}}^\text{Is} = 2h - J_z, \quad E_{\text{scatt}}^\text{Is} = 2(2h - J_z), \hspace{1cm} (86)$$

are the energies of a one magnon state (a single excited spin $\cdots \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \cdots$), a two-magnon bound state (two neighboring excited spins) and a two-magnon scattering states (two isolated excited spins). Physical meaning of the expansion (83) is quite clear and expresses a subdivision of the energy spectrum on independent subsectors.

More specific is hierarchy of the terms in (83). Of course the condition (10) yields $E_{\text{magn}}^\text{Is} < E_{\text{bound}}^\text{Is}, E_{\text{scatt}}^\text{Is}$ so that $f_{\text{magn}}^\text{Is}(T, h)$ is the leading term of the expansion ($e_0(h)$ is a constant term)

$$f_{\text{lead}}^\text{Is}(T, h) = f_{\text{magn}}^\text{Is}(T, h). \hspace{1cm} (87)$$

The subleading terms are however different for ferromagnets and (magnetically polarized) antiferromagnets. Namely as it follows from (85) and (86)

$$f_{\text{sublead}}^\text{Is}(T, h) = f_{\text{bound}}^\text{Is}(T, h), \quad J_z < 0,$$

$$f_{\text{sublead}}^\text{Is}(T, h) = f_{\text{scatt}}^\text{Is}(T, h), \quad h > J_z > 0. \hspace{1cm} (88)$$

Let us now again turn back to the Ising-like chain. First of all we note that reproducing the cluster expansion result of [13] one has to account that the Hamiltonian (17) of [13] turns into (1) only after changing signs of the couplings $J_z \rightarrow -J_z, J \rightarrow -J$ and renormalization of the Zeeman term $S_z n - 1/2 I \rightarrow S_z n$. Under this procedure the low-temperature cluster expansion formula for the free energy density of the XXZ spin chain in the polarized phase takes the form (83) however with

$$f_{\text{magn}}(T, h) = -\frac{e^\beta(J_z-h)}{2\pi \beta} \int_0^\pi e^{-\beta J \cos k} dk = -\frac{e^\beta(J_z-h)}{2\pi \beta} \int_{-\pi}^{\pi} e^{\beta J \cos k} dk, \hspace{1cm} (90)$$

$$f_{\text{bound}}(T, h) = -\frac{e^\beta(J_z-2h)}{2\pi \beta} \int_{-\pi}^{2\pi} dk \Theta(J_z^2 - J^2 \cos^2 k/2) e^{-\beta J^2/J_z \cos^2 (k/2)}, \hspace{1cm} (91)$$

$$f_{\text{scatt}}(T, h) = \frac{e^{2\beta(J_z-h)}}{4\pi \beta} \int_0^{2\pi} dk \left( e^{-2\beta J \cos k} + \frac{J_z}{\pi} \int_{-\pi}^{\pi} dk \frac{e^{2\beta J \cos k/2 \cos k}}{J_z - J e^{-i\omega} \cos k/2} \right). \hspace{1cm} (92)$$

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(Here (91) is an improved version of (77) in [13].) Accounting that in the Ising-like case \( \Theta(J^2_z - J^2 \cos^2 k/2) = 1 \) one readily reduces (91) to the form

\[
f_{\text{bound}}(T, h) = -\frac{e^{\beta(J_z - 2h)}}{2\pi \beta} \int_0^{2\pi} dk \, e^{-\beta J^2_z \cos^2(k/2)}.
\]

(93)

As in the Ising case the magnon contribution \( f_{\text{magn}}(T, h) \) gives the leading low-temperature asymptotics to the free energy density while \( f_{\text{bound}}(T, h) \) and \( f_{\text{scatt}}(T, h) \) give a subleading one correspondingly in ferromagnetic and polarized antiferromagnetic cases.

The expansion (83) as well as the corresponding one based on (90)–(93) are efficient only in the low-temperature regime governed by the inequality

\[
\beta(h - J_z) \gg 1,
\]

(94)
or according to (10) and (12)

\[
k_B T \ll E_{\text{gap}} + E_{\text{width}}.
\]

(95)

In the extreme low-temperature subregime when (94) or (95) are replaced by a more strict inequality

\[
\beta |J| \gg 1 \iff k_B T \ll \frac{E_{\text{width}}}{2},
\]

(96)
the saddle-point integration reduces (90) to an approximative asymptotic expression

\[
f_{\text{magn}}(T, h) \approx f_{\text{magn}}^{\text{extr}}(T, h) = -\frac{e^{\beta(J_z + |J| - h)}}{2\pi \beta} \int_{-\infty}^{\infty} e^{-\beta |J| k^2/2} dk = -\frac{e^{-\beta E_{\text{gap}}}}{\sqrt{2\pi \beta^3 |J|}}.
\]

(97)

Since the right side of (97) is singular at \( J = 0 \) it can not be represented in the power expansion form (41), which corresponds to the perturbation theory around the point \( J = 0 \) based on the QTM approach.

At the same time if

\[
\frac{|J|}{h - J_z} = \frac{E_{\text{width}}}{2E_{\text{gap}} + E_{\text{width}}} \ll 1,
\]

(98)
then there is additionally the scaled low-temperature subregime governed by the supplemental condition

\[
\beta |J| \ll 1 \iff k_B T \gg \frac{E_{\text{width}}}{2}.
\]

(99)
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In this case the first two terms of the power expansion for the exponent in (90) yield

\[ f_{\text{magn}}(T, h) \approx f_{\text{magn}}^{\text{scal}}(T, h) = -\frac{e^{\beta(J_z-h)}}{\beta} \left( 1 + \frac{\beta^2 J^2}{4} \right) = -\frac{e^{\beta(J_z-h)}}{\beta} \left( 1 + \frac{\beta^2 J^2}{4\Delta^2} \right). \]  (100)

In a similar manner (93) gives

\[ f_{\text{bound}}(T, h) \approx f_{\text{bound}}^{\text{scal}}(T, h) = -\frac{e^{\beta(J_z-2h)}}{\beta} \left( 1 - \frac{J_z\beta}{2\Delta^2} \right). \]  (101)

Derivation of an analogous expansion for \( f_{\text{scatt}}(T, h) \) is a bit more cumbersome. Namely

\[
\frac{J_z}{2\pi} \int_{-\pi}^{\pi} \frac{d\kappa}{J_z - e^{-i\kappa} \cos k/2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\kappa}{e^{-i\kappa} \cos k/2} \left( 1 + \frac{e^{-i\kappa}}{\Delta} \cos \frac{k}{2} + \frac{e^{-2i\kappa}}{\Delta^2} \cos^2 \frac{k}{2} + o \left( \frac{1}{\Delta^2} \right) \right) \\
\times \left( 1 - 2\beta J \cos \frac{k}{2} \cos \kappa + 2\beta^2 J^2 \cos^2 \frac{k}{2} \cos^2 \kappa + o(\beta^2 J^2) \right)
\]

\[ = 1 + \frac{\beta^2 J^2}{\Delta^2} \cos^2 \frac{k}{2} + o(\beta^2 J^2) + o \left( \frac{1}{\Delta^2} \right). \]  (102)

So according to (92) and (102)

\[ f_{\text{scatt}}(T, h) \approx f_{\text{scatt}}^{\text{scal}}(T, h) = \frac{e^{2\beta(J_z-h)}}{\beta} \left( \frac{3}{2} + \frac{2\beta^2 J^2 - \beta J_z}{2\Delta^2} \right). \]  (103)

Let us now reproduce the asymptotic formulas (100), (101) and (103) by the QTM approach. Representing (70) in the form

\[ \Xi = e^{-\beta J_z} (e^{\beta h} - 1) \left[ 1 + \frac{1 + e^{-\beta h}}{2} \left( \sqrt{1 + \frac{e^{\beta J_z}}{\sinh^2(\beta h/2)} - 1} \right) \right], \]  (104)

one readily gets at \( \beta \to \infty \)

\[ \Xi = e^{\beta(h-J_z)} - e^{-\beta J_z} + 1 + o(\min(1, e^{-\beta J_z})). \]  (105)

So according to (72) and (105)

\[ \tilde{f}_2 = \beta \left( -e^{\beta(J_z-h)} + 4 e^{2\beta(J_z-h)} \right) + o(e^{2\beta(J_z-h)}). \]  (106)

The corresponding expression for \( \tilde{f}_1 \) cardinally depends on the sign of \( J_z \). Namely

\[ \tilde{f}_1 = \begin{cases} 
  e^{\beta(J_z-2h)} + o(e^{\beta(J_z-2h)}), & J_z < 0 \\
  -e^{2\beta(J_z-h)} + o(e^{2\beta(J_z-h)}), & J_z > 0.
\end{cases} \]  (107)
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Figure 1. Plots of \( g_{\text{extr}}(\zeta) \) and \( g_{\text{scal}}(\zeta) \).

Now a substitution of (106) and (107) into (71) reproduces (100), (101) and (103).

In order to visualize the difference between the low-temperature and scaled low-temperature regimes we may introduce according to (90), (97) and (100) the following two universal functions

\[
g_{\text{extr}}(\zeta) \equiv \frac{f_{\text{extr}}^{magn}(T, h)}{f_{\text{magn}}^{magn}(T, h)} = \frac{\sqrt{2\pi}e^{\zeta}}{\sqrt{\zeta} \int_{-\pi}^{\pi} e^{\zeta} \cos k \, dk},
\]

\[
g_{\text{scal}}(\zeta) \equiv \frac{f_{\text{scal}}^{magn}(T, h)}{f_{\text{magn}}^{magn}(T, h)} = \frac{\pi(4 + \zeta^2)}{2 \int_{-\pi}^{\pi} e^{\zeta} \cos k \, dk},
\]

(108)

where \( \zeta = \beta |J|. \) According to (96), (99) and (108) the extreme low-temperature regime corresponds to \( \zeta \gg 1 \) and \( g_{\text{extr}}(\zeta) \approx 1 \), while the scaled low-temperature regime corresponds to \( \zeta \ll 1 \) and \( g_{\text{scal}}(\zeta) \approx 1 \). Namely, as it readily follows from (108)

\[
\lim_{\zeta \to \infty} g_{\text{extr}}(\zeta) = \lim_{\zeta \to 0} g_{\text{scal}}(\zeta) = 1.
\]

(109)

The plots of \( g_{\text{extr}}(\zeta) \) and \( g_{\text{scal}}(\zeta) \) are presented in figure 1.

From figure 1 we see that \( f_{\text{magn}}^{\text{extr}}(T, h) \) gives a better approximation to \( f_{\text{magn}}^{magn}(T, h) \) than \( f_{\text{magn}}^{\text{scal}}(T, h) \) only for \( \zeta > \zeta_c \approx 1.83 \).

In order to give a more detailed characterization of these approximations we suggest two relative errors functions

\[
\varepsilon_a(\zeta) \equiv \left| \frac{f_{a}^{magn}(T, h) - f_{\text{magn}}^{\text{magn}}(T, h)}{f_{\text{magn}}^{magn}(T, h)} \right| = |g_a(\zeta) - 1|, \quad a = \text{extr, scal}.
\]

(110)

Plots of these functions are presented in figures 2 and 3.
7. Summary and discussion

In the present paper extending the result of [11] belonging to the Ising chain we studied within the QTM approach a highly anisotropic Heisenberg–Ising chain and have suggested the perturbative formula (71) for the free energy density. At high temperatures the result agrees with the direct high-temperature expansion. At low temperatures an agreement with the cluster expansion was studied only for the polarized case (9) and (10) and has been established only in the special scaled regime (13) which may be realized only under the condition (98) and for which quantum fluctuations are small against
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the thermodynamical ones. We suggest that the obtained result produce an analytical confirmation for a correctness of the QTM approach in the mentioned above two temperature regimes. Hence it is naturally to suppose that for ferromagnetically polarized chains (under condition (10)) the QTM approach is also correct in the whole range above the scaled low-temperature regime. Of course it will be instructive to compare the obtained QTM result with alternative analytical calculations for an antiferromagnetic chain in a low field when the condition (10) fails. However in this case the corresponding cluster expansion is not yet developed.

Recently it was suggested an interesting new approach to dynamics of Ising-like $XXZ$ chains in zero magnetic field [14, 15]. Its main idea is to subdivide a highly anisotropic Hamiltonian (1) at $h = 0$ into a sum

$$\hat{H} = J(\hat{H}_F + \hat{F} + \hat{F}^\dagger + \Delta \hat{H}_I),$$

where

$$\hat{H}_I = \sum_{n=1}^{N} \left( S_n^z S_{n+1}^z - \frac{1}{4} I \right),$$

$$\hat{F} = \frac{1}{2} \sum_{n=1}^{N} \left[ (S_{n+1}^z S_n^z + S_n^z S_{n+1}^z) \left( \frac{1}{4} - S^z_{n-1} S^z_{n+2} \right) + \frac{1}{2} (S_n^z S_{n+1}^z - S_n^z S_{n+1}^z) \left( S_{n+2}^z - S_{n-1}^z \right) \right],$$

$$\hat{H}_F = \sum_{n=1}^{N} (S_n^z S_{n+1}^z + S_n^z S_{n+1}^z) \left( \frac{1}{4} + S^z_{n-1} S^z_{n+2} \right),$$

and it may be readily proved that

$$[\hat{H}_I, \hat{H}_F] = 0, \quad [\hat{H}_I, \hat{F}] = F.$$

The term $\hat{F} + \hat{F}^\dagger$ may be excluded from (111) by a unitary transformation $\hat{H} \to U \hat{H} U^{-1}$ similar to the one that was used for the Hubbard model [16]. Despite the operator $U$ was not presented explicitly it was shown that under the conditions

$$U = e^{S}, \quad S = \sum_{m=1}^{\infty} \frac{S_m}{\Delta^m}, \quad S_1 = F - F^\dagger,$$

one has

$$U \hat{H} U^{-1} = J(\hat{H}_F + \Delta \hat{H}_I + O(\Delta^{-1})).$$

As it was suggested in [15] the ‘folded Hamiltonian’ $\hat{H}_F$ is responsible for pre-relaxation behavior of the model (for example it describes a time evolution after a quantum quench in the asymptotic limit $1 \ll Jt/\hbar \ll \Delta \ll N \to \infty$). After a rather cumbersome non-local transformation of spin operators Hamiltonian in the right side of (115) may be transformed into a more tractable form with a spectrum rater similar to the spectrum of XX chain in a magnetic field readily solvable within different machineries (see [17] and references therein).

Despite the approach [14, 15] reveals some new dynamical features of the $XXZ$ chain at zero magnetic field a study of static ones was not carried out. Nevertheless a
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A correspondence between the QTM and the cluster expansion results at extremely low temperatures in the present paper was not established. Probably this problem may be attacked by some alternative non-perturbative approaches. In fact this subject has a special interest because the QTM approach by itself is based on some postulates which were proved only for finite-size systems or at high temperatures [9, 12]. We also suppose that the suggested approach may be extended on evaluation of correlation functions.

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Appendix A. Foundations of the QTM approach

Here without deepening on details and following [9, 18] we shall briefly outline proofs of (14) and (16) from the first principles [19].

- We start from the $R$-matrix

\[
\tilde{R}(\lambda) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & c(\lambda) & b(\lambda) & 0 \\
0 & b(\lambda) & c(\lambda) & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad b(\lambda) = \frac{\sinh(\lambda)}{\sinh(\lambda + \eta)}, \quad c(\lambda) = \frac{\sinh \eta}{\sinh(\lambda + \eta)},
\]

(A.1)

which satisfies the Braid group Yang–Baxter equation

\[
\tilde{R}_{12}(\lambda - \mu)\tilde{R}_{23}(\lambda)\tilde{R}_{12}(\mu) = \tilde{R}_{23}(\mu)\tilde{R}_{12}(\lambda)\tilde{R}_{23}(\lambda - \mu).
\]

(A.2)

The latter also may be represented in two equivalent forms

\[
\tilde{R}_{12}(\lambda - \mu)L_{13}(\lambda)L_{23}(\mu) = L_{13}(\mu)L_{23}(\lambda)\tilde{R}_{12}(\lambda - \mu),
\]

(A.3)

\[
\tilde{R}_{12}(\lambda - \mu)L_{31}^{t_1}(-\lambda)L_{32}^{t_1}(-\mu) = L_{31}^{t_1}(-\mu)L_{32}^{t_1}(-\lambda)\tilde{R}_{12}(\lambda - \mu),
\]

(A.4)

where the index $t_1$ denotes transposition with respect to the first space (for example $(A \otimes B)^{t_1} = A^t \otimes B$),

\[
L(\lambda) = P\tilde{R}(\lambda),
\]

(A.5)

and $P$ is the permutation operator in $\mathbb{C}^2 \otimes \mathbb{C}^2$ ($P\xi \otimes \zeta = \zeta \otimes \xi$ for $\xi, \zeta \in \mathbb{C}^2$).

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Additionally one has

\[
\hat{R}(\lambda) = I + \frac{2\lambda}{J_z \tanh \eta} H^{(0)} + o(\lambda), \tag{A.6}
\]

where

\[
H^{(0)} = \frac{J}{2} \left( S^+ \otimes S^- + S^- \otimes S^+ \right) + J_z \left( S^z \otimes S^z - \frac{1}{4} I \right), \quad J_z = J \cosh \eta, \tag{A.7}
\]

is the Hamiltonian density related to (1) with reduced Zeeman term (equivalently at \( h = 0 \)). So according to (A.5) and (A.6) the two transfer matrices

\[
t(\lambda) = \text{tr}_a T_a(\lambda), \quad \tilde{t}(\lambda) = \text{tr}_a \tilde{T}_a(\lambda), \tag{A.8}
\]

where

\[
T_a(\lambda) = L_{a,N}(\lambda) \ldots L_{a,1}(\lambda), \quad \tilde{T}_a(\lambda) = L_{1,a}(\lambda) \ldots L_{N,a}(\lambda), \tag{A.9}
\]

have the following expansions at the vicinity of \( \lambda = 0 \)

\[
t(\lambda) = U \left( I + \frac{2\lambda}{J_z \tanh \eta} \hat{H}^{(0)} \right) + o(\lambda), \quad \tilde{t}(\lambda) = U^{-1} \left( I + \frac{2\lambda}{J_z \tanh \eta} \hat{H}^{(0)} \right) + o(\lambda), \tag{A.10}
\]

where \( \hat{H}^{(0)} = \sum_{n=1}^{N} H^{(0)}_{n,n+1} \) and \( U \) is the shift operator

\[
(U \xi)_{i_1,i_2,\ldots,i_N} = \sum_{j_1,j_2,\ldots,j_N} \delta_{i_1,j_N} \delta_{i_2,j_{N-1}} \delta_{i_3,j_{N-2}} \ldots \delta_{i_N,j_1} \xi_{j_1,j_2,\ldots,j_N} = \xi_{i_2,i_3,\ldots,i_1}. \tag{A.11}
\]

Hence according to the translation invariance of the model \( [\hat{H}, U] = 0 \) one has

\[
\tilde{t}(\lambda - \nu)t(-\nu - \lambda) = I - \frac{4\nu}{J_z \tanh \eta} \hat{H}^{(0)} + o(1). \tag{A.12}
\]

Introducing now the Trotter number \( M \to \infty \) and taking

\[
\nu = \frac{\beta J_z \tanh \eta}{4M}, \tag{A.13}
\]

one readily has according to the Trotter–Suzuki formula [20]

\[
\lim_{M \to \infty} \left( \tilde{t}(\lambda - \nu)t(-\nu - \lambda) \right)^M \bigg|_{\lambda=0} = \lim_{M \to \infty} \left( I - \frac{\beta \hat{H}^{(0)}}{M} \right)^M = e^{-\beta \hat{H}^{(0)}}. \tag{A.14}
\]
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At the same time for total Hamiltonian (1) one has

\[ e^{-\beta \hat{H}} = e^{-\beta \hat{H}^0} \prod_{n=1}^{N} e^{\beta h S^z_n}. \]  

(A.15)

Since operator \( e^{-\beta \hat{H}} \) acts on the tensor product of \( N \) spaces \( \mathbb{C}^2 \) attached to sites of the chain it will convenient to represent this formula in an expanded form

\[ (e^{-\beta \hat{H}})_{1\ldots N} = \prod_{n=1}^{N} \lim_{M \to \infty} \text{Tr}_{1\ldots 2M} F_{1\ldots N}^{1\ldots 2M}(\lambda) \bigg|_{\lambda=0}, \]  

(A.16)

where

\[ F_{1\ldots N}^{1\ldots 2M}(\lambda) \equiv \left( \hat{T}_{2M}(\lambda - \nu)T_{2M-1}(\nu - \lambda) \ldots \hat{T}_2(\nu - \lambda)T_1(-\nu - \lambda) \right)_{1\ldots N}. \]  

(A.17)

- The trace of \( F_{1\ldots N}^{1\ldots 2M} \) in (A.16) (not \( F_{1\ldots N}^{1\ldots 2M} \) by itself!) may be transformed into an equivalent form

\[ \text{Tr}_{1\ldots 2M} F_{1\ldots N}^{1\ldots 2M}(\lambda) = \text{Tr}_{1\ldots 2M} \left( T_{1}^{\text{QTM}(0)}(\lambda) \ldots T_{N}^{\text{QTM}(0)}(\lambda) \right)_{1\ldots 2M}, \]  

(A.18)

where the quantum monodromy matrix \( T_{N}^{\text{QTM}(0)}(\lambda) \) is defined as follows

\[ \left( T_{j}^{\text{QTM}(0)}(\lambda) \right)_{1\ldots 2M} \equiv L_{j\,2M}(\lambda - \nu)L_{2M-1\, j}(\nu - \lambda) \ldots L_{j\,2}(\lambda - \nu)L_{1\, j}(\nu - \lambda). \]  

(A.19)

It is instructive to check (A.18) in the simplest case \( N = 2M = 2 \). Numerating for convenience the auxiliary spaces by indices \( \tilde{1} \) and \( \tilde{2} \) one has from (A.9) and (A.17)

\[ F_{12}^{\tilde{1}\tilde{2}}(\lambda) = L_{12}(\lambda - \nu)L_{22}(\lambda - \nu)L_{12}(\nu - \lambda)L_{11}(\nu - \lambda). \]  

(A.20)

Using invariance of trace under transposition and commutativity between \( L_{11}^\dagger(\nu - \lambda) \) and \( L_{22}(\lambda - \nu) \) one reduces the trace of (A.20) to

\[ \text{Tr}_{12} F_{12}^{\tilde{1}\tilde{2}}(\lambda) = \text{Tr}_{12} \left( L_{12}(\lambda - \nu)L_{22}(\nu - \lambda)L_{12}(\nu - \lambda) \right). \]  

(A.21)

Looking now at (A.19) we see that the right side of (A.21) is just \( \text{Tr}_{12}(T_{1}^{\text{QTM}(0)}(\lambda)T_{2}^{\text{QTM}(0)}(\lambda))_{\tilde{1}\tilde{2}} \) in total agreement with (A.18).

A combination of (A.16) and (A.18) gives

\[ \left( e^{-\beta \hat{H}} \right)_{1\ldots N} = \lim_{M \to \infty} \text{Tr}_{1\ldots 2M} \left( T_{1}^{\text{QTM}(0)}(\lambda) \ldots T_{N}^{\text{QTM}(0)}(\lambda) \right) \bigg|_{\lambda=0}, \]  

(A.22)

where

\[ T_{N}^{\text{QTM}(0)}(\lambda) = e^{\beta h S^z} T_{N}^{\text{QTM}(0)}(\lambda). \]  

(A.23)

Both \( T_{N}^{\text{QTM}(0)}(\lambda) \) in (A.22) and \( T_{N}^{\text{QTM}(0)}(\lambda) \) in (A.18) are treated as \( 2 \times 2 \) matrices whose elements act in the auxiliary space.
The main advantage of (A.22) follows from the fact that the quantum monodromy matrices satisfy the Sklyanin permutation relations

\[ \hat{R}_{12}(\lambda - \mu)T_{1}^{QTM}(\lambda)T_{2}^{QTM}(\mu) = T_{2}^{QTM}(\mu)T_{1}^{QTM}(\lambda)\hat{R}_{12}(\lambda - \mu). \]  

(A.24)

Proof of (A.24) for \( T^{QTM(0)}(\lambda) \) directly follows from (A.3), (A.4) and (A.19) while its generalization on \( T^{QTM}(\lambda) \) is a consequence of commutativity between \( e^{\beta h(S_{1}^{z} + S_{2}^{z})} \) and \( \hat{R}_{12}(\lambda) \). Taking in (A.23) trace with respect to the physical space one readily gets a 2\( N \otimes 2^{N} \) QTM

\[ t^{QTM}(\lambda) = \text{tr} T^{QTM}(\lambda). \]  

(A.25)

According to (A.22) and (A.25)

\[ Z(\beta, N) = \lim_{M \to \infty} \text{Tr}_{1,\ldots,2M} (t^{QTM(0)})^{N}. \]  

(A.26)

Hence in the thermodynamical limit

\[ f(\beta) = -\frac{1}{\beta} \lim_{N \to \infty} \frac{\ln Z(\beta, N)}{N} = -\frac{\ln \Lambda_{\text{max}}(0)}{\beta}, \]  

(A.27)

where \( \Lambda_{\text{max}}(\lambda) \) is the maximal eigenvalue of \( t^{QTM}(\lambda) \).

- In the next stage under an utilization of the usual algebraic Bethe ansatz formalism as well as some conjectures based on finite-M calculations it is postulated that the eigenvalue \( \Lambda_{\text{max}}(\lambda) \) corresponds to \( N = 2M \) and an eigenvector

\[ |\lambda_{1}, \ldots, \lambda_{M}\rangle = B(\lambda_{1}) \ldots B(\lambda_{M})|\uparrow\rangle_{N} |\downarrow\rangle_{N-1} \ldots |\uparrow\rangle_{2} |\downarrow\rangle_{1}, \]  

(A.28)

where \( B(\lambda) \) is an entry of

\[ T^{QTM}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}. \]  

(A.29)

The set of numbers \( \lambda_{1}, \ldots, \lambda_{M} \) satisfy a system of Bethe equations

\[ \Phi(\lambda_{j}) = 0, \quad j = 1, \ldots, M, \]  

(A.30)

where

\[ \Phi(\lambda) \equiv e^{\beta h/2} \phi \left( \lambda + \frac{\eta}{2} \right) \prod_{k=1}^{M} \sinh(\lambda - \lambda_{k} - \eta) + e^{-\beta h/2} \phi \left( \lambda - \frac{\eta}{2} \right) \prod_{k=1}^{M} \sinh(\lambda - \lambda_{k} + \eta), \]  

(A.31)

and

\[ \phi(\lambda) \equiv \left[ \sinh \left( \lambda - \nu + \frac{\eta}{2} \right) \sinh \left( \lambda + \nu - \frac{\eta}{2} \right) \right]^{M}. \]  

(A.32)
The corresponding expression for eigenvalue of $t^{\text{QTM}}(\lambda)$ is

$$
\Lambda(\lambda; \lambda_1, \ldots, \lambda_M) = \frac{\Phi(\lambda)}{[\sinh(\lambda - \nu + \eta) \sinh(\lambda + \nu - \eta)]^M \prod_{j=1}^M \sinh(\lambda - \lambda_j)}.
$$

(A.33)

According to (A.31) and (A.32) $e^{3M\lambda} \Phi(\lambda)$ is a degree $3M$ polynomial of $e^{2\lambda}$ and at the same time

$$
\lim_{\lambda \to \infty} e^{-3M\lambda} \Phi(\lambda) = e^{\beta h/2} + e^{-\beta h/2}.
$$

(A.34)

Hence

$$
\Phi(\lambda) = (e^{\beta h/2} + e^{-\beta h/2}) \prod_{j=1}^M \sinh(\lambda - \lambda_j) \prod_{l=1}^{2M} \sinh(\lambda - w_l),
$$

(A.35)

where $\lambda_j (j = 1, \ldots, M)$ are the Bethe roots while the adjoint $2M$ parameters $w_l$ are the so called hole-type roots. From (A.33) and (A.35) it also follows that

$$
\Lambda(\lambda; \lambda_1, \ldots, \lambda_M) = \frac{(e^{\beta h/2} + e^{-\beta h/2}) \prod_{l=1}^{2M} \sinh(\lambda - w_l)}{[\sinh(\lambda - \nu + \eta) \sinh(\lambda + \nu - \eta)]^M}.
$$

(A.36)

Introducing now two $i\pi$-periodic functions

$$
a(\lambda) \equiv e^{-\beta h} \frac{\phi(\lambda - \eta/2)}{\phi(\lambda + \eta/2)} \prod_{j=1}^M \frac{\sinh(\lambda - \lambda_j + \eta)}{\sinh(\lambda - \lambda_j - \eta)}, \quad \mathfrak{A}(\lambda) \equiv 1 + a(\lambda),
$$

(A.37)

one readily see that $\Phi(\lambda) = 0 \Leftrightarrow a(\lambda) = -1 \Leftrightarrow \mathfrak{A}(\lambda) = 0$. So according to (A.35) the function $\mathfrak{A}(\lambda)$ has zeroes at the Bethe roots $\lambda_j (j = 1, \ldots, M)$ and adjoint (hole-type) roots $w_l (l = 1, \ldots, 2M)$. It also has simple poles at $\lambda_j + \eta$ and two $M$-fold poles at $\nu - \eta$ and $-\nu$. At the same time according to (A.37) $\lim_{\lambda \to \infty} \mathfrak{A}(\lambda) = 1 + e^{-\beta h}$. Hence $\mathfrak{A}(\lambda)$ may be represented in the following factorized form

$$
\mathfrak{A}(\lambda) = \frac{(1 + e^{-\beta h}) \prod_{j=1}^M \sinh(\lambda - \lambda_j) \prod_{l=1}^{2M} \sinh(\lambda - w_l)}{[\sinh(\lambda + \nu) \sinh(\lambda - \nu + \eta)]^M \prod_{j=1}^M \sinh(\lambda - \lambda_j - \eta)}.
$$

(A.38)

It was postulated in [9] that all the Bethe roots $\lambda_j (j = 1, \ldots, M)$ are different, purely imaginary and accumulate to zero at $\beta \to 0$. For all adjoint (hole-type) one has $|\text{Re } w_l| \geq \eta/2$ ($l = 1, \ldots, 2M$). In the $\beta \to 0$ limit they accumulate to $\pm \eta$.

According to (A.37) and (A.32) function $a(\lambda)$ is meromorphic and in the fundamental strip $D$ defined by equalities $-\pi/2 < \text{Im } \lambda \leq \pi/2$ has $M$ simple poles at $\lambda = \lambda_j + \eta$ and two $M$-fold poles at $\lambda = -\nu$ and $\lambda = \nu - \eta$. At the same time according to (A.37) and (A.38) function $\mathfrak{A}(\lambda)$ also is meromorphic in $D$ and has here $3M$ zeroes at $\lambda_j$ and $w_l$. Hence the following integral in the complex plane

$$
I(\lambda) \equiv \frac{1}{2\pi i} \oint_C \ln \mathfrak{A}(z) \frac{\sinh(\lambda - z)}{d\lambda} dz = -\frac{1}{2\pi i} \oint_C \ln \mathfrak{A}(z) \frac{\sinh(\lambda - z)}{dz} dz,
$$

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where

\[
\oint_C \equiv \int_{\eta/2-i\pi/2}^{\eta/2+i\pi/2} + \int_{-\eta/2+i\pi/2}^{-\eta/2-i\pi/2} + \int_{\eta/2-i\pi/2}^{\eta/2+i\pi/2} + \int_{-\eta/2+i\pi/2}^{-\eta/2-i\pi/2},
\]

is well defined because along the contour \( C \) function \( \ln A(z) \) has zero winding number\(^1\). Using integration by parts and accounting zeroes and a pole of the function \( A(z) \) one readily gets

\[
I(\lambda) = \frac{1}{2\pi i} \oint_C \ln \sinh(\lambda - z) d(\ln A(z)) = \sum_{j=1}^{M} \ln \sinh(\lambda - \lambda_j) - M \ln \sinh(\lambda + \nu).
\]

Hence

\[
I(\lambda + \eta) - I(\lambda - \eta) = \ln \left( \frac{\sinh(\lambda + \nu - \eta)}{\sinh(\lambda + \nu + \eta)} \right)^M + \ln \left[ \prod_{j=1}^{M} \left( \frac{\sinh(\lambda - \lambda_j + \eta)}{\sinh(\lambda - \lambda_j - \eta)} \right) \right],
\]

and a combination of (A.32), (A.37) and (A.42) gives

\[
\ln a(\lambda) = -\beta h + \ln \left( \frac{\sinh(\lambda - \nu) \sinh(\lambda + \nu + \eta)}{\sinh(\lambda + \nu) \sinh(\lambda - \nu + \eta)} \right)^M + I(\lambda + \eta) - I(\lambda - \eta).
\]

At the same time according to the initial definition (A.39) and an identity

\[
\coth w - \coth(w + v) = \frac{\sinh v}{\sinh w \sinh(w + v)},
\]

applied at \( w = \lambda - z - \eta \) and \( v = 2\eta \) one has

\[
I(\lambda + \eta) - I(\lambda - \eta) = -\oint_C \frac{dz}{2\pi i \sinh(\lambda - z + \eta) \sinh(\lambda - z - \eta)} \ln A(z).
\]

Now a combination of (A.43) and (A.45) yields

\[
\ln a(\lambda) = -\beta h + \ln \left( \frac{\sinh(\lambda - \nu) \sinh(\lambda + \nu + \eta)}{\sinh(\lambda + \nu) \sinh(\lambda - \nu + \eta)} \right)^M - \oint_C \frac{dz}{2\pi i \sinh(\lambda - z + \eta) \sinh(\lambda - z - \eta)} \ln A(z).
\]

\(^1\)In fact the contour presented in figure 1 of [18] is slightly tightened, however this does not affect the final result.
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In the Trotter ($M \to \infty$) limit according to (A.13) one has for an arbitrary $\alpha$
\[
\lim_{M \to \infty} \ln \left( \frac{\sinh(\alpha \pm \nu)}{\sinh(\alpha \mp \nu)} \right)^M = \lim_{M \to \infty} M \ln \left( 1 \pm \frac{\nu}{\tanh \alpha} \right) = \pm J_z \beta \tanh \eta. \tag{A.47}
\]

Using (A.47) and an identity (A.44) applied at $w = \lambda$ and $v = \eta$ one readily reduce (A.46) in the limit $M \to \infty$ to
\[
\ln a(\lambda) = -\beta h - \frac{J_z \beta \sinh \eta \tanh \eta}{2 \sinh \lambda \sinh (\lambda + \eta)} - \int_C \frac{dz}{2\pi i \sinh (\lambda + z + \eta) \sinh (\lambda - z - \eta)} \sinh (2\eta) \ln \mathcal{A}(z) . \tag{A.48}
\]

Additionally to $a(\lambda)$ and $\mathcal{A}(\lambda)$ one may introduce the dual quantities
\[
\bar{a}(\lambda) = 1/a(\lambda), \quad \bar{\mathcal{A}}(\lambda) = 1 + \bar{a}(\lambda). \tag{A.49}
\]

According to (A.37) the corresponding equation for $\bar{a}(\lambda)$ may be obtained from (A.48) under substitutions $h \to -h, \eta \to -\eta$. Namely it takes the form
\[
\ln \bar{a}(\lambda) = \beta h - \frac{J_z \beta \sinh \eta \tanh \eta}{2 \sinh \lambda \sinh (\lambda + \eta)} + \int_C \frac{dz}{2\pi i \sinh (\lambda - z + \eta) \sinh (\lambda - z - \eta)} \sinh (2\eta) \ln \bar{A}(z) . \tag{A.50}
\]

Substituting into (A.50) $\bar{a}(\lambda) = 1/a(\lambda)$ one readily gets an equation on $a(\lambda)$
\[
\ln a(\lambda) = -\beta h + \frac{J_z \beta \sinh \eta \tanh \eta}{2 \sinh \lambda \sinh (\lambda - \eta)} - \int_C \frac{dz}{2\pi i \sinh (\lambda - z + \eta) \sinh (\lambda - z - \eta)} \sinh (2\eta) \ln \bar{A}(z) . \tag{A.51}
\]

Let
\[
b_+(x) \equiv a(\eta/2 + ix), \quad b_-(x) \equiv \frac{1}{a(-\eta/2 + ix)}, \quad \mathcal{B}_+(x) \equiv 1 + b_+(x). \tag{A.52}
\]

According to (A.49) and (A.52)
\[
\bar{A}(\eta/2 + ix) = \frac{\mathcal{A}(\eta/2 + ix)}{a(\eta/2 + ix)} = \frac{\mathcal{B}_+(x)}{b_+(x)}, \quad \bar{A}(-\eta/2 + ix) = \mathcal{B}_-(x). \tag{A.53}
\]

Taking in (A.51) $\lambda = \eta/2 + ix$ and postulating $\eta > 0$ one reduce it to the form
\[
\ln b_+(x) = -\beta h - \frac{J_z \beta \tanh \eta}{2} \tilde{k}_{\pm}(x)
+ \int_{-\pi/2}^{\pi/2} \frac{dy}{2\pi} (\tilde{k}_{\eta}(x-y)(\ln \mathcal{B}_+(y) - \ln b_+(y)) - \tilde{k}_{\eta}(x-i\eta-y) \ln \mathcal{B}_-(y)) , \tag{A.54}
\]

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where
\[ \hat{\kappa}_\eta(x) \equiv -\frac{\sinh 2\eta}{\sinh(i\eta + x) \sinh(i\eta - x)} = -\frac{4 \sinh 2\eta e^{2ix}}{(e^{2ix} - e^{2\eta})(e^{2ix} - e^{-2\eta})}. \] (A.55)

Let us now extract \( b \) from the right side of (A.54) by Fourier transformation
\[ g(x) \rightarrow g_n = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{-2inx} g(x) dx. \] (A.56)

According to (A.55)
\[ \hat{\kappa}_\eta(x) \rightarrow \frac{1}{\pi} \int \frac{2i \sinh 2\eta z^{-n} dz}{(z - e^{2\eta})(z - e^{-2\eta})} = \frac{1}{\pi} \int \frac{2i \sinh 2\eta w^n dw}{(w - e^{2\eta})(w - e^{-2\eta})} = 2 e^{-2\eta |n|}. \] (A.57)

Hence under (A.56) formula (A.54) turns into
\[ (\ln b_+)_n = -\beta h \delta n_0 - J_z \beta \tanh \eta e^{-|n|} + e^{-2\eta |n|}(\ln \mathcal{B}_+ - \ln b_+ - e^{-2\eta n} \ln \mathcal{B}_-)_n, \] (A.58)

or equivalently
\[ (\ln b_+)_n = -\frac{\beta h}{2} \delta n_0 - \frac{\beta J_z}{2 \cosh \eta n} + \frac{1}{1 + e^{2\eta |n|}} (\ln \mathcal{B}_+ - \ln \mathcal{B}_-) e^{-2\eta |n|} \ln \mathcal{B}_-)_n. \] (A.59)

Inverting now (A.56) by
\[ g_n \rightarrow \sum_n e^{2inx} g_n, \] (A.60)

and making the following regularization
\[ \frac{e^{-2\eta n}}{1 + e^{2\eta |n|}} \rightarrow \sum_{n=-\infty}^{\infty} \frac{e^{2(i\eta - \eta + \epsilon)n}}{1 + e^{2\eta |n|}} = \kappa_-(x), \] (A.61)

one readily gets (16) for \( b_+(x) \). Equation (16) for \( b_-(x) \) may be obtained in the same manner.

- According to (A.36) and (A.38)
\[ \mathfrak{A}(\lambda) = e^{-\beta h/2} \Lambda(\lambda; \lambda_1, \ldots, \lambda_M) \left( \frac{\sinh(\lambda + \nu - \eta)}{\sinh(\lambda + \nu)} \right)^M \prod_{j=1}^{M} \frac{\sinh(\lambda - \lambda_j)}{\sinh(\lambda - \lambda_j - \eta)}. \] (A.62)
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Under the mentioned above substitution $h \to -h$, $\eta \to -\eta$ (and according to (A.13) $\nu \to -\nu$) one gets from (A.62)

$$\tilde{A}(\lambda) = e^{\beta h/2} \Lambda(\lambda; \lambda_1, \ldots, \lambda_M) \left( \frac{\sinh(\lambda - \nu + \eta)}{\sinh(\lambda - \nu)} \right)^M \prod_{j=1}^{M} \frac{\sinh(\lambda - \lambda_j)}{\sinh(\lambda - \lambda_j + \eta)}. \quad (A.63)$$

At the same time according to (A.52) and (A.53)

$$\mathcal{B}_+(x) = \tilde{A}(ix + \eta/2), \quad \mathcal{B}_-(x) = \tilde{A}(ix - \eta/2). \quad (A.64)$$

A Substitution (A.62) and (A.63) into (A.64) yields

$$\mathcal{B}_+(x)\mathcal{B}_-(x) = \left( \frac{\sinh(ix - \nu + \eta/2)}{\sinh(ix + \nu - \eta/2)} \right)^M \Lambda(ix + \eta/2)\Lambda(ix - \eta/2). \quad (A.65)$$

Using (A.44) and (A.47) one readily gets from (A.65) in the limit $M \to \infty$.

$$\ln[\mathcal{B}_+(x)\mathcal{B}_-(x)] = -\frac{\beta J_z}{2} \tanh \frac{\eta}{2} x + \ln \Lambda(ix + \eta/2) + \ln \Lambda(ix - \eta/2). \quad (A.66)$$

Applying the Fourier transformation (A.56) to (A.66) one readily gets

$$(\ln \Lambda)_n = \frac{\beta J_z}{2} \tanh \frac{\eta}{2} e^{-\eta |n|} + [\ln(\mathcal{B}_+\mathcal{B}_-)]_n. \quad (A.67)$$

The inverse Fourier transformation (A.60) of (A.67) yields

$$\ln \Lambda(ix) = \frac{\beta J_z}{2} \tanh \frac{\eta}{2} \sum_{n=-\infty}^{\infty} \frac{e^{-\eta |n| + 2i\pi n x}}{\cosh \eta m} + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dy d(x - y) \ln(\mathcal{B}(y)\mathcal{B}(y)). \quad (A.68)$$

Now a substitution of (A.68) at $x = 0$ into (A.27) results in (14).

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