DYNAMICS OF A DIFFUSIVE PREY-PREDATOR SYSTEM WITH STRONG ALLEE EFFECT GROWTH RATE AND A PROTECTION ZONE FOR THE PREY

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(Communicated by Sze-Bi Hsu)

ABSTRACT. In this paper, a diffusive prey-predator model with strong Allee effect growth rate and a protection zone Ω₀ for the prey is investigated. We analyze the global existence, long time behaviors of positive solutions and the local stabilities of semi-trivial solutions. Moreover, the conditions of the occurrence and avoidance of overexploitation phenomenon are obtained. Furthermore, we demonstrate that the existence and stability of non-constant steady state solutions branching from constant semi-trivial solutions by using bifurcation theory. Our results show that the protection zone is effective when Allee threshold is small and the protection zone is large.

1. Introduction. One important interaction between biological species is the prey-predator interaction. Previously, the great mass of prey-predator models are expressed in terms of ordinary differential equations where spatial effects are ignored [9, 13, 17]. As we all know that the spatial diffusion of species can affect the complexity of ecosystems, such as ecological invasion and pattern formation, reaction-diffusion prey-predator models have emerged and been widely researched [14, 15, 19, 21, 25, 29]. Considering the spatial distribution of the populations, a prototypical prey-predator system [8] is of the form

\[
\begin{align*}
    u_t - d_1 \Delta u &= f(u) - mh(u)v, \\
    v_t - d_2 \Delta v &= g(v) + ch(u)v,
\end{align*}
\]

where \( u \) and \( v \) represent the densities of prey and predator respectively. The parameters \( d_1, d_2, m \) and \( c \) are positive constants. Here \( f(u) \) and \( g(v) \) represent the growth of \( u \) and \( v \) respectively when the other species is absent. In most articles of prey-predator models, the prey is assumed to grow at a logistic form, i.e., \( f(u) = u(1 - u/K) \), where \( K \) is the carrying capacity. However, due to the factors of mate limitation, cooperative defense and environmental conditioning, it has been recently recognized that the prey species may have a growth rate with Allee effect [1, 21, 22]. Generally, the strong Allee growth rate is taken in the form

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2010 Mathematics Subject Classification. Primary: 92D25, 92D50; Secondary: 35B32, 35B35, 35B40, 35K57.

Key words and phrases. Ratio-dependent prey-predator model, reaction-diffusion, Allee effect, protection zone, bifurcation.

This work was supported by NSFC Grant 11371113.

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where $0 < b < 1$ represents Allee threshold value. The function $h(u)$ represents the functional response of the predator, which was first examined by Holling [11]. The Holling-type functional response is widely used by many researchers [16, 30, 31]. However, there are growing biological and physiological evidences [2, 3, 23, 25] that a functional response depending on the ratio of prey to predator abundance is a suitable representation in some situations, for example, when predators have to search, share and compete for food. Essentially, the ratio-dependent theory asserts that the per capita predator growth rate should be a function of the ratio of prey to predator abundance. In this case, it is necessary to change $h(u)$ to $h(u/v)$ in (1). Specially, we take $h(u/v)$ in the form of $u/(u + v)$. For the sake of simplicity, we suppose that $g(v) = -\theta v$ where $\theta > 0$ is the death rate of the predator. Based on the above arguments, a diffusive prey-predator model [24] with strong Allee effect in prey can be written as follows:

\[
\begin{align*}
    u_t - d_1 \Delta u &= u(1 - u)(u - b) - muv/(u + v), & x \in \Omega, \ t > 0, \\
    v_t - d_2 \Delta v &= -\theta v + c(uv)/(u + v), & x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial n} &= 0, & x \in \partial \Omega, \ t > 0, \\
    \frac{\partial v}{\partial n} &= 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x) > 0, \ v(x, 0) = v_0(x) > 0, & x \in \bar{\Omega},
\end{align*}
\]

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, $u_0(x), v_0(x) \in C^{2+\alpha}(\bar{\Omega})$ and $\partial u_0(x)/\partial n = \partial v_0(x)/\partial n = 0$ on $\partial \Omega$. The reference [24] shows that for any given $u_0(x)$, both of the prey and predator will be extinct if $v_0(x)$ is large enough. Such a feature is known as overexploitation phenomenon which is a distinctive character of dynamics of the prey-predator system with strong Allee effect in the prey growth. This result makes us think about how to save the endangered species. Then a idea comes out that setting up a protection zone for the prey. The prey species can enter and leave the protection zone freely but not the predator. Consequently, it is natural to have two questions: Does the protection zone protect the two species from the extinction caused by overexploitation? How do the Allee effect and protection zone affect the spatiotemporal dynamics of the species? In this paper, we will try to answer these two questions. Very recently, some efforts have been devoted to investigating the impact of protection zone on the prey-predator models [5, 7, 8]. They all showed the existence of a critical patch size of the protection zone and demonstrated that the ultimate fate of species changed with this critical patch size. The competition models with a protection zone also have been studied [6, 27]. Compared with the prey-predator model, the protection zone had some essentially different effects on the fine dynamics of the competition model.

Considering a protection zone in (2), we rewrite it as the following system

\[
\begin{align*}
    u_t - d_1 \Delta u &= u(1 - u)(u - b) - m(x)uv/(u + v), & x \in \Omega, \ t > 0, \\
    v_t - d_2 \Delta v &= -\theta v + c(x)uv/(u + v), & x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial n} &= 0, & x \in \partial \Omega, \ t > 0, \\
    \frac{\partial v}{\partial n} &= 0, & x \in \partial \Omega, \ t > 0, \\
    u(x, 0) &= u_0(x) > 0, & x \in \Omega, \\
    v(x, 0) &= v_0(x) > 0, & x \in \Omega.
\end{align*}
\]
where $\Omega_\ast := \Omega \setminus \overline{\Omega}_0$ and $\Omega_0$ is a smooth interior subdomain of $\Omega$. The larger region $\Omega$ is the habitat of the prey with the protection zone $\Omega_0$. The predator species is initially driven out from $\Omega_0$ and prohibited from entering $\Omega_0$ again while the prey species can enter and leave $\Omega_0$ freely. A no-flux boundary condition is assumed for both species on $\partial \Omega$, so the prey and predator live in a closed ecosystem. Moreover, the no-flux boundary condition is imposed for the predator on $\partial \Omega_0$. We may think of a barrier along $\partial \Omega_0$ that blocks $v$ but not $u$. The initial data $u_0(x)$ is same as that in (2) while the initial data $v_0(x) \in C^{2+\alpha}(\overline{\Omega}_\ast)$ is positive and satisfies $\partial v_0(x)/\partial n = 0$ on $\partial \Omega_\ast$. The function $m(x)$ is zero for $x \in \Omega_0$, which implies that the prey species enjoys predation-free growth in $\Omega_0$. Note that though $v$ is not defined for $x \in \Omega_0$, the interaction term in the first equation of (3) can still be regarded as properly defined since $m(x) = 0$ in $\Omega_0$. The function $c(x) = h(x) \cdot m(x)$ for $x \in \overline{\Omega}_\ast$, where $0 < h(x) \leq 1$ is the conversion rate. Particularly, throughout this article, we assume that

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
m(x) \in C(\overline{\Omega}_\ast), & \quad m^* = \max_{\overline{\Omega}_\ast} m(x), \quad m_\ast = \min_{\overline{\Omega}_\ast} m(x), \\
c(x) \in C(\overline{\Omega}_\ast), & \quad c^* = \max_{\overline{\Omega}_\ast} c(x), \quad c_\ast = \min_{\overline{\Omega}_\ast} c(x).
\end{array}
\right.
\end{align*}
$$

(4)

The meanings of other parameters are the same as those in (2).

With a view to studying non-constant positive solutions of (3), we shall consider the stationary problem of (3), i.e., the following semi-linear elliptic problem:

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
-d_1 \Delta u = u(1-u)(u-b) - \frac{m(x)uv}{u+v}, & \quad x \in \Omega, \\
-d_2 \Delta v = -\theta v + \frac{c(x)uv}{u+v}, & \quad x \in \Omega_\ast, \\
\frac{\partial u}{\partial n} = 0, & \quad x \in \partial \Omega, \\
\frac{\partial v}{\partial n} = 0, & \quad x \in \partial \Omega_\ast.
\end{array}
\right.
\end{align*}
$$

(5)

Recall the following results. For any given $q \in L^\infty(\Omega)$, we define $\lambda_1^D(q,O)$ and $\lambda_1^N(q,O)$ to be the principal eigenvalues of $-\Delta + q$ over the region $O$ with Dirichlet and Neumann boundary conditions respectively. If the region $O$ is omitted in the notation, then we understand that $O = \Omega$. If the function $q$ is omitted, then we understand that $q = 0$. It is well known that the properties of $\lambda_1^D(q,O)$ and $\lambda_1^N(q,O)$ [26, 29]:

1. $\lambda_1^D(q,O) > \lambda_1^N(q,O)$;
2. $\lambda_1^D(q_1,O) > \lambda_1^D(q_2,O)$ if $q_1 \geq q_2$ and $q_1 \neq q_2$, for $B = D, N$;
3. $\lambda_1^D(q,O_1) > \lambda_1^D(q,O_2)$ if $O_1 \subset O_2$.

The rest of the paper is arranged as follows. In Section 2, we present basic dynamic properties of (3), such as the global existence and long time behaviors of positive solutions, and the local stabilities of semi-trivial solutions. In Section 3, the conditions of the occurrence and avoidance of overexploitation phenomenon are obtained. We demonstrate that the overexploitation phenomenon will probably disappear when Allee threshold $b$ is small and the protection zone $\Omega_0$ is large. If initial prey population is large enough, then the protection zone is effective. Unfortunately, we also prove that the prey and predator are destined to extinction when Allee threshold $b$ is large or the protection zone $\Omega_0$ is small. In Section 4,
Moreover, by the strong maximum principle we also have

\[ \text{Theorem 2.1.} \quad \text{If } u_0(x) > 0 \text{ for } x \in \Omega \text{ and } v_0(x) > 0 \text{ for } x \in \Omega_*, \text{ then the problem} (3) \text{ has a unique global solution} (u(x,t), v(x,t)) \text{ such that} u(x,t) > 0 \text{ for} (x,t) \in \Omega_+ \times [0, \infty) \text{ and} v(x,t) > 0 \text{ for} (x,t) \in \Omega_* \times [0, \infty). \]

**Proof.** Define

\[
M(u, v) = u(1 - u)(u - b) - \frac{m(x)uv}{u + v}, \quad N(u, v) = -\theta v + \frac{c(x)uv}{u + v}.
\]

It is easy to see that, in the domain \( \{u \geq 0, v \geq 0 \mid u + v \neq 0\} \), the problem (3) is a mixed quasi-monotone system from \( M(u, v) \leq 0 \) and \( N(u, v) \geq 0 \). Take \( \underline{u}(x,t) = 0 \) and \( \bar{u}(x,t) = u^*(t) \), where \( u^*(t) \) is the unique positive solution of

\[
\begin{cases}
  u' = u(1 - u)(u - b), & t > 0, \\
  u(0) = u^*,
\end{cases}
\]

with \( u^* = \max_{x \in \Omega} u_0(x) > 0 \). Let \( \bar{v}(t) \) and \( \underline{v}(t) \) be the unique positive solution of

\[
\begin{cases}
  v' = -\theta v + \frac{c^* \bar{u}v}{\bar{u} + v}, & t > 0, \\
  v(0) = v^*,
\end{cases}
\]

and

\[
\begin{cases}
  v' = -\theta v, & t > 0, \\
  v(0) = v_*,
\end{cases}
\]

respectively, with \( v^* = \max_{x \in \Omega} v_0(x) > 0 \) and \( v_* = \min_{x \in \Omega} v_0(x) > 0 \). We claim that \( (\bar{u}(x,t), \underline{v}(t)) \) and \( (\underline{u}(x,t), \bar{v}(t)) \) are the coupled ordered upper and lower solutions to (3) respectively. Actually,

\[
\begin{align*}
\bar{u}_t - d_1 \Delta \bar{u} - M(\bar{u}, \bar{v}) & \geq 0 = \underline{u}_t - d_1 \Delta \underline{u} - M(\underline{u}, \underline{v}), \\
\bar{v}_t - d_2 \Delta \bar{v} - N(\bar{u}, \bar{v}) & \geq 0 = \underline{v}_t - d_2 \Delta \underline{v} - N(\underline{u}, \underline{v}),
\end{align*}
\]

and

\[
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \forall x \in \partial \Omega, \quad t \geq 0, \quad \frac{\partial \bar{v}}{\partial n} = \frac{\partial \underline{v}}{\partial n} = 0, \forall x \in \partial \Omega_*, \quad t \geq 0,
\]

and

\[
u^* \geq v_0(0) > 0, \forall x \in \Omega_*, \quad v^* \geq v(0) \geq v_*, \forall x \in \Omega_.*
\]

Therefore, (3) has a unique global solution \( (u(x,t), v(x,t)) \) satisfying

\[
0 = \underline{u}(x,t) \leq u(x,t) \leq u^*(t), \quad \forall x \in \Omega_+, \quad t \geq 0,
\]

\[
0 < \underline{v}(t) \leq v(x,t) \leq \bar{v}(t), \quad \forall x \in \Omega_*, \quad t \geq 0.
\]

Moreover, by the strong maximum principle we also have \( u(x,t) > 0 \) for \( x \in \Omega_+ \) and \( t \geq 0 \).

Before we study the long time behavior of solutions to (3), we first give a useful lemma in [29].
Lemma 2.2. Let $f(s)$ be a positive $C^1$ function for $s \geq 0$, and let $d > 0$, $\beta \geq 0$ be constants. Further, let $T \in [0, \infty)$ and $w \in C^{2,1}(\Omega \times (T, \infty)) \cap C^{1,0}(\overline{\Omega} \times [T, \infty))$ be a positive function.

(i) If $w$ satisfies
\[
\begin{cases}
  w_t - d\Delta w \leq (\geq) w^{1+\beta} f(w)(\alpha - w), & (x,t) \in \Omega \times (T, \infty), \\
  \frac{\partial w}{\partial n} = 0, & (x,t) \in \partial \Omega \times [T, \infty),
\end{cases}
\]
and the constant $\alpha > 0$, then
\[
\limsup_{t \to \infty} \max_{\overline{\Omega}} w(\cdot, t) \leq \alpha \ (\liminf_{t \to \infty} \min_{\overline{\Omega}} w(\cdot, t) \geq \alpha).
\]

(ii) If $w$ satisfies
\[
\begin{cases}
  w_t - d\Delta w \leq w^{1+\beta} f(w)(\alpha - w), & (x,t) \in \Omega \times (T, \infty), \\
  \frac{\partial w}{\partial n} = 0, & (x,t) \in \partial \Omega \times [T, \infty),
\end{cases}
\]
and the constant $\alpha \leq 0$, then
\[
\limsup_{t \to \infty} \max_{\overline{\Omega}} w(\cdot, t) \leq 0.
\]

Theorem 2.3. For any solution $(u(x,t), v(x,t))$ of (3),
\[
\limsup_{t \to \infty} \max_{\overline{\Omega}} u(\cdot, t) \leq 1, \quad \limsup_{t \to \infty} \max_{\overline{\Omega}} v(\cdot, t) \leq \max\{0, (c^* - \theta)/\theta\}. \tag{7}
\]
Thus, for any $\varepsilon > 0$, the rectangle $[0, 1+\varepsilon) \times [0, \max\{0, (c^* - \theta)/\theta\} + \varepsilon)$ is a global attractor of (3) in $\mathbb{R}^2_+$. 

Proof. Since $u$ satisfies
\[
\begin{cases}
  u_t - d_1\Delta u \leq u(1-u)(u-b), & x \in \Omega, \ t > 0, \\
  \frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x,0) = u_0(x) > 0, & x \in \Omega,
\end{cases}
\]
and $u^*(t)$ is the unique solution of (6), it is deduced by the comparison principle that $u(x,t) \leq u^*(t)$. Evidently, the first inequality of (7) holds. As a result, for any $\varepsilon > 0$, there exists $T > 0$ such that $u(x,t) \leq 1 + \varepsilon$ for all $x \in \overline{\Omega}$ and $t \geq T$. It follows that $v$ satisfies
\[
\begin{cases}
  v_t - d_2\Delta v \leq -\theta v + \frac{c^*(1+\varepsilon)v}{1+\varepsilon + v} = v \left(\frac{c^* - \theta}(1+\varepsilon) - \theta\frac{v}{1+\varepsilon + v}\right), & x \in \Omega^*, \ t \geq T, \\
  \frac{\partial v}{\partial n} = 0, & x \in \partial \Omega^*, \ t \geq T.
\end{cases}
\]
If $\theta < c^*$, in view of Lemma 2.2 and the arbitrariness of $\varepsilon$, we have
\[
\limsup_{t \to \infty} \max_{\overline{\Omega}} v(\cdot, t) \leq \frac{c^* - \theta}{\theta}.
\]
If $\theta \geq c^*$, we have the differential inequality
\[
v_t - d_2\Delta v \leq -\frac{\theta v^2}{1+\varepsilon + v}.
\]
Similarly, Lemma 2.2 suggests that
\[
\limsup_{t \to \infty} \max_{\overline{\Omega}} v(\cdot, t) \leq 0.
\]
In either case, the second inequality of (7) holds.

\[ \text{Theorem 2.4.} \quad \text{Let the parameters } d_1, d_2, \theta > 0 \text{ and } 0 < b < 1. \text{ Suppose that } m(x) \text{ and } c(x) \text{ satisfy (4). If } 0 < u_0(x) \leq b \text{ for } x \in \Omega \text{ and } v_0(x) > 0 \text{ for } x \in \Omega_*, \text{ then } \lim_{t \to \infty} u(x,t) = 0 \text{ uniformly on } \Omega \text{ and } \lim_{t \to \infty} v(x,t) = 0 \text{ uniformly on } \Omega_*.
\]

\[ \text{Proof.} \text{ The proof is divided into three cases.}
\]

\[ \text{Case 1. } u_0(x) < b \text{ on } x \in \Omega.
\]

Recalling the proof of Theorem 2.3, we have \( u(x,t) \leq u^*(t) \), where \( u^*(t) \) is the unique solution of (7). It follows from \( u_0(x) < b \) that \( \lim_{t \to \infty} u(x,t) = 0 \) uniformly on \( \Omega \). Accordingly, for any given \( \varepsilon > 0 \), there exists \( T > 0 \) such that

\[ u(x,t) < \varepsilon, \quad \forall x \in \Omega, \ t \geq T.
\]

It follows that \( v \) satisfies

\[
\begin{aligned}
&v_t - d_2 \Delta v \leq -\theta v + \frac{c^* \varepsilon v}{\varepsilon + v} \leq -\theta v + c^* \varepsilon, \quad x \in \Omega_*, \ t \geq T, \\
&\frac{\partial v}{\partial n} = 0, \quad x \in \partial \Omega_*, \ t \geq T.
\end{aligned}
\]

Then Lemma 2.2 implies \( \limsup_{t \to \infty} \max_{\Omega_*} v(x,t) \leq \frac{c^*}{\theta} \varepsilon \). The arbitrariness of \( \varepsilon \) gives that \( \lim_{t \to \infty} v(x,t) = 0 \) uniformly on \( x \in \Omega_* \).

\[ \text{Case 2. } u_0(x) \leq b \text{ and } u_0(x) \neq b \text{ on } x \in \Omega.
\]

Similar to Case 1, \( u(x,t) \leq u^*(t) \). Consequently, \( 0 < u(x,t) \leq b \). Let \( w(x,t) = b - u(x,t) \), then \( 0 \leq w(x,t) < b \) satisfies

\[
\begin{aligned}
w_t - d_1 \Delta w &= w(b - w)(1 - b + w) + \frac{m(x)v(b - w)}{b - w + v}, \quad x \in \Omega, \ t > 0, \\
\frac{\partial w}{\partial n} &= 0, \quad x \in \partial \Omega, \ t > 0, \\
w(x,0) &= b - u_0(x) \geq 0 \neq 0, \quad x \in \Omega.
\end{aligned}
\]

It follows from the strong maximum principle that

\[ w(x,t) > 0, \quad \forall x \in \Omega, \ t > 0.
\]

Therefore, \( u(x,t) < b \) for \( x \in \Omega, \ t > 0 \). As demonstrated in Case 1, we have \( \lim_{t \to \infty} u(x,t) = 0 \) uniformly on \( x \in \Omega \) and \( \lim_{t \to \infty} v(x,t) = 0 \) uniformly on \( x \in \Omega_* \).

\[ \text{Case 3. } u_0(x) \equiv b \text{ on } x \in \Omega.
\]

Similar to Case 2, it easy to see that \( u(x,t) \leq b \) for \( x \in \Omega, \ t \geq 0 \). If \( u(x,t) \equiv b \), we obtain a contradiction since \( v(x,t) > 0 \) for \( x \in \Omega_*, \ t > 0 \). Hence there exists \( t_0 > 0 \) such that \( 0 < u(x,t_0) \leq b \) and \( u(x,t_0) \neq b \). As proved in Case 2, we get \( \lim_{t \to \infty} u(x,t) = 0 \) uniformly on \( x \in \Omega \) and \( \lim_{t \to \infty} v(x,t) = 0 \) uniformly on \( x \in \Omega_* \).

\[ \text{Theorem 2.5.} \quad \text{Let the parameters } d_1, d_2, \theta > 0 \text{ and } 0 < b < 1. \text{ Suppose that } m(x) \text{ and } c(x) \text{ satisfy (4). If } \theta \geq c^*, \text{ then } (u(x,t), v(x,t)) \text{ tends to } (u_*(x), 0) \text{ uniformly as } t \to \infty, \text{ where } u_*(x) \text{ is a non-negative solution of}
\]

\[
d_1 \Delta u + u(1 - u)(u - b) = 0, \quad x \in \Omega; \quad \frac{\partial u}{\partial n} = 0, \quad x \in \partial \Omega.
\]
Proof. On account of the second inequality of (7), if \( \theta \geq c^* \), then \( \lim_{t \to \infty} v(x,t) = 0 \) uniformly for \( x \in \Omega \). Now the equation of \( u(x,t) \) is asymptotically autonomous [18], and its limit behavior is determined by the parabolic equation:

\[
\begin{align*}
& v_t = d_1 \Delta v + u(1-u)(u-b), \quad x \in \Omega, \quad t > 0, \\
& \frac{\partial v}{\partial n} = 0, \quad x \in \partial \Omega, \quad t > 0.
\end{align*}
\]

It is well known that (9) is a gradient system and its every orbit converges to a steady state of corresponding elliptic problem for (9) [10]. Then from the theory of asymptotically autonomous dynamical system, the solution \((u(x,t),v(x,t))\) of (3) converges to \((u_a(x),0)\) as \( t \to \infty \).

It is easy to check that the system (3) has semi-trivial solutions \((1,0)\), \((b,0)\) and \((u_s(x),0)\), where \( u_s(x) \) is a non-constant positive solution to (8) and the existence of \( u_s(x) \) has been discussed in Subsection 3.2 of [28]. In the following, we will study the stabilities of the semi-trivial solutions to (3).

**Theorem 2.6.** Let the parameters \( d_1, d_2, \theta > 0 \) and \( 0 < b < 1 \). Suppose that \( m(x) \) and \( c(x) \) satisfy (4). Then

(i) The semi-trivial solution \((b,0)\) is always unstable;

(ii) The semi-trivial solution \((1,0)\) is locally asymptotically stable when \( \theta > \theta_* \) and unstable when \( \theta < \theta_* \), where

\[
\theta_* = -d_2 \lambda^N_1 \left( -\frac{c(x)}{d_2}, \Omega_* \right) > 0;
\]

(iii) The non-constant semi-trivial solution \((u_s(x),0)\) is unstable if the domain \( \Omega \) is convex.

Proof. (i) The linearized eigenvalue problem of (5) at \((b,0)\) can be written as

\[
\begin{align*}
& d_1 \Delta h + (b-b^2)h - m(x)k + \mu h = 0, \quad x \in \Omega, \\
& d_2 \Delta k + (c(x)-\theta)k + \mu k = 0, \quad x \in \Omega_*,
\end{align*}
\]

\[
\begin{align*}
& \frac{\partial h}{\partial n} = 0, \quad x \in \partial \Omega, \\
& \frac{\partial k}{\partial n} = 0, \quad x \in \partial \Omega_*,
\end{align*}
\]

where \( \mu \) is an eigenvalue and \((h,k)\) is the corresponding eigenfunction of \( \mu \). It is obvious that \( \mu = b(b-1) > 0 \) is an eigenvalue with corresponding eigenfunction \((\phi_1,0)\), where \( \phi_1 \) is the positive eigenfunction corresponding to the first eigenvalue of \(-\Delta\) with Neumann boundary condition. Consequently, \((b,0)\) is always unstable.

(ii) The linearized eigenvalue problem of (5) at \((1,0)\) is

\[
\begin{align*}
& d_1 \Delta h + (b-1)h - m(x)k + \mu h = 0, \quad x \in \Omega, \\
& d_2 \Delta k + (c(x)-\theta)k + \mu k = 0, \quad x \in \Omega_*,
\end{align*}
\]

\[
\begin{align*}
& \frac{\partial h}{\partial n} = 0, \quad x \in \partial \Omega, \\
& \frac{\partial k}{\partial n} = 0, \quad x \in \partial \Omega_*,
\end{align*}
\]

(10)
where $\mu$ is an eigenvalue and $(h, k)$ is the corresponding eigenfunction of $\mu$. If $k \neq 0$, it is derived from the second equation of (10) and the assumption $\theta > \theta_*$ that

$$\mu \geq \theta + d_2 \lambda_1^N \left( -\frac{c(x)}{d_2}, \Omega_* \right) > 0.$$ 

If $k \equiv 0$, then it follows that $h \neq 0$. According to the first equation of (10), we have $\mu \geq 1 - b > 0$. In conclusion, $\mu > 0$ and $(1, 0)$ is linearly stable when $\theta > \theta_*$ while $(1, 0)$ is unstable when $\theta < \theta_*$. 

(iii) According to the well-known results Theorem 2 in [4], the non-constant solution $u_*(x)$ of (8) is unstable if $\Omega$ is convex. Hence, $(u_*(x), 0)$ is also an unstable solution of (5) when $\Omega$ is convex. □

3. Overexploitation. In this section, we will investigate the occurrence and avoidance of the overexploitation phenomenon in (3). For this purpose, we first give two useful lemmas in [5].

**Lemma 3.1.** Suppose that $d_1 > 0$, $0 < b < 1$ and $\Omega_0$ is a bounded domain with a smooth boundary in $\mathbb{R}^N$ ($N \geq 1$). For the Dirichlet boundary problem

$$\begin{cases}
  d_1 \Delta u + u(1-u)(u-b) = 0, & x \in \Omega_0, \\
  u = 0, & x \in \partial \Omega_0,
\end{cases} 
(11)$$

the following statements hold true.

(i) If $1/2 \leq b < 1$, then for any $d_1 > 0$, the only non-negative solution of (11) is $u = 0$.

(ii) If $0 < b < 1/2$ and $d_1 > (1-b)^2[4\lambda_1^D(\Omega_0)]^{-1}$, the only non-negative solution of (11) is $u = 0$.

(iii) If $0 < b < 1/2$, then there exists a constant $D_0 \in (0, (1-b)^2[4\lambda_1^D(\Omega_0)]^{-1}]$ such that for $0 < d_1 < D_0$, the system (11) has at least two positive solutions. Moreover, the system (11) has a maximal solution $\bar{U}(x)$ such that for any solution $u(x)$ of (11), $\bar{U}(x) > u(x)$ for $x \in \Omega_0$.

**Lemma 3.2.** Suppose that $\beta$, $A$ are positive constants, $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 1$), and $\Omega_1$ is a smooth interior subdomain of $\Omega$. Then for any $\varepsilon > 0$, there exist $\hat{\beta} > 0$ and $K > 0$ such that when $\beta > \hat{\beta}$, the unique positive solution $u(x)$ of the modified Helmholtz’s equation

$$\begin{cases}
  \Delta u - \beta^2 u = 0, & x \in \Omega \setminus \overline{\Omega_1}, \\
  \frac{\partial u}{\partial n} = 0, & x \in \partial \Omega, \\
  u = A, & x \in \partial \Omega_1,
\end{cases}$$

satisfies

$$0 < u(x) < \varepsilon, \text{ for } x \in \Omega \setminus \overline{\Omega_1} \text{ and } d(x, \partial \Omega_1) \geq K\beta^{-1},$$

$$0 < u(x) < (A + \varepsilon)e^{-\beta d(x, \partial \Omega_1)}, \text{ for } x \in \Omega \setminus \overline{\Omega_1} \text{ and } d(x, \partial \Omega_1) < K\beta^{-1},$$

where $d(x, \partial \Omega_1)$ is the distance from $x$ to $\partial \Omega_1$.

**Theorem 3.3.** Assume that $m(x)$, $c(x)$ satisfy (4) and the parameters $d_2$, $\theta > 0$ and $m_*> (1-b)^2/4$. Suppose that $b$ and $d_1$ satisfy one of the following conditions:

(i) $1/2 \leq b < 1$ and $d_1 > 0$; (ii) $0 < b < 1/2$ and $d_1 > (1-b)^2[4\lambda_1^D(\Omega_0)]^{-1}$. Then for a given initial prey population $u_0(x) > 0$, there exists a constant $v_0^*$ which depends
on \( b, \theta, d_1, m_* \) and \( u_0(x) \) such that when the initial predator population \( v_0(x) \geq v_0^* \), the corresponding solution \((u(x,t), v(x,t))\) of (3) satisfies \( \lim_{t \to \infty} u(x,t) = 0 \) uniformly on \( \Omega \) and \( \lim_{t \to \infty} v(x,t) = 0 \) uniformly on \( \Omega_* \).

Proof. In order to get above result, it suffices to prove that there exists \( T^* > 0 \) such that \( u(x,T^*) < b \) for \( x \in \Omega \) in view of Theorem 2.4. In the following, we will find two time intervals \( I_1 \) and \( I_2 \) satisfying \( I_1 \supset I_2 \). We will prove that

\[
   u(x,t) < \frac{b}{2}, \ (x,t) \in \Omega_* \times I_1 \text{ and } u(x,t) < \frac{b}{2^2}, \ (x,t) \in \Omega_0 \times I_2.
\]

The proof is divided into five steps.

Step 1. We claim that for any \( \varepsilon > 0 \), there exists \( T_1 > 0 \), for any \( T_2 > 0 \), \( u_0(x) > 0 \) and \( v_0(x) \geq v_0^* \), satisfying

\[
   u(x,t) \leq 1 + \varepsilon, \ (x,t) \in \Omega \times [T_1, \infty), \quad (12)
\]

\[
   v(x,t) \geq v_0^* e^{-\theta(T_1 + T_2)}, \ (x,t) \in \Omega_* \times [0, T_1 + T_2]. \quad (13)
\]

On account of Theorem 2.3, for any \( \varepsilon > 0 \), there exists \( T_1 > 0 \) such that \( u(x,t) \leq 1 + \varepsilon \) for any \( x \in \Omega \) and \( t \geq T_1 \). Let \( v_1(t) \) be the solution to

\[
   \begin{cases}
   v' = -\theta v, & 0 < t \leq T_1 + T_2, \\
   v(0) = v_0^*, 
\end{cases}
\]

where \( v_0^* \) will be chosen later. From the assumption \( v_0 \geq v_0^* \) and the comparison principle, we have

\[
   v(x,t) \geq v_1(t) = v_0^* e^{-\theta T_1} \geq v_0^* e^{-\theta(T_1 + T_2)}, \ (x,t) \in \Omega_* \times [0, T_1 + T_2].
\]

Step 2. We will prove that for the fixed \( \varepsilon > 0 \) in Step 1, there exists \( T_3 > 0 \) such that

\[
   u(x,t) \leq \max\{2\varepsilon, (1 + 2\varepsilon)e^{-\beta d(x,\partial \Omega_0)} + \varepsilon\}, \ (x,t) \in \Omega_* \times [T_1 + T_3, T_1 + T_2],(14)
\]

where \( \beta \) satisfies

\[
   d_1 \beta^2 = \frac{m_* v_0^* e^{-\theta(T_1 + T_2)}}{1 + \varepsilon + v_0^* e^{-\theta(T_1 + T_2)}} - \frac{(1 - b)^2}{4}.
\]

Let \( u_1(x,t) \) be the solution to

\[
   \begin{cases}
   u_1_t = d_1 \Delta w - d_1 \beta^2 w, & x \in \Omega_*, \ t > T_1, \\
   \frac{\partial w}{\partial n} = 0, & x \in \partial \Omega, \ t > T_1, \\
   w(x,0) = 0, & x \in \partial \Omega_0, \ t > T_1, \\
   w(x,T_1) = u(x,T_1), & x \in \Omega_*. \quad (15)
\end{cases}
\]

Thanks to (12), (13) and \( (1 - u)(u - b) \leq (1 - b)^2/4 \), it is easy to check that

\[
   (1 - u)(u - b) - \frac{m(x)u}{u + v} \leq -d_1 \beta^2, \ (x,t) \in \Omega_* \times [T_1, T_1 + T_2].
\]

Then it follows from the comparison principle that

\[
   u(x,t) \leq u_1(x,t), \ (x,t) \in \Omega_* \times [T_1, T_1 + T_2].
\]

It well known that (15) is gradient system and its all positive solutions tend to solutions of corresponding elliptic problem. With help of Lemma 3.2, the corresponding elliptic problem of (15) has a unique positive solution which denoted with
$u_2(x)$. Thus $u_2(x)$ is globally asymptotically stable. Then there exists $T_3 > 0$ such that $u_1(x, t) \leq u_2(x) + \varepsilon$ for $x \in \Omega_\delta$ and $T_1 + T_3 \leq t \leq T_1 + T_2$. Reusing Lemma 3.2, it follows that for any $x \in \Omega_\delta$ and $T_1 + T_3 \leq t \leq T_1 + T_2$,

$$u(x, t) \leq u_1(x, t) \leq u_2(x) + \varepsilon \leq \max\{2\varepsilon, (1 + 2\varepsilon)e^{-\beta d(x, \partial \Omega_\delta)} + \varepsilon\}.$$  

**Step 3.** Now our task is to prove that there exists $T_4 > 0$, $\delta > 0$ and $\beta > 0$, such that

$$u(x, t) < \frac{b}{2}, \quad (x, t) \in \bar{\Omega}_\delta \times [T_1 + T_3 + T_4, T_1 + T_2],$$

where $\Omega_\delta = \{x \in \Omega : d(x, \Omega_0) < \delta\}$.  

For $\delta > 0$, it is easy to see that $\Omega_\delta \supset \Omega_0$ and $\lim_{\delta \to 0} \Omega_\delta = \Omega_0$. Let $u_3(x, t)$ be the solution to

$$\begin{cases}
    w_t = d_1 \Delta w + \rho(w(1 - w)(w - b)), & x \in \Omega_\delta, \ t > T_1 + T_3, \\
    w(x, t) = \rho, & x \in \partial \Omega_\delta, \ t > T_1 + T_3, \\
    w(x, T_1 + T_3) = u(x, T_1 + T_3), & x \in \Omega_\delta,
\end{cases}  \tag{16}$$

where

$$\rho = \rho(\delta, \beta, \varepsilon) = \max\{2\varepsilon, (1 + 2\varepsilon)e^{-\beta \delta} + \varepsilon\} > 0.$$  

From (14) and the comparison principle, we have

$$u(x, t) \leq u_3(x, t), \quad (x, t) \in \bar{\Omega}_\delta \times [T_1 + T_3, T_1 + T_2].$$

By the assumptions and Lemma 3.1, the only non-negative solution of (11) is $u \equiv 0$. Therefore, for sufficient small $\delta > 0$, the system (11) with the domain $\Omega_\delta$ has no positive solution. We choose such a small $\delta_0 > 0$, then $\rho(\delta_0, \beta, \varepsilon) \to 0$ as $\varepsilon \to 0$ and $\beta \to \infty$. From the implicit function theorem, for sufficiently small $\rho > 0$, the corresponding elliptic problem of (16) has the unique solution $u_4(\rho, x)$ near $u \equiv 0$. Obviously, $0 < u_4(\rho, x) < \rho$. We choose large enough $\beta_0 > 0$ and $0 < \varepsilon_0 \ll b/4$ such that $\rho(\delta_0, \beta_0, \varepsilon_0) \ll b/4$ is sufficiently small and $u_4(x) = u_4(\rho(\delta_0, \beta_0, \varepsilon_0), x)$ is the only steady state solution of (16). Because the system (16) is a gradient system, $u_3(x, t)$ must converge to the unique positive steady state solution $u_4(x)$ as $t \to \infty$. Therefore, there exists $T_4 > 0$ such that

$$u(x, t) \leq u_3(x, t) \leq u_4(x) + \varepsilon_0 < b/2, \quad x \in \bar{\Omega}_{\delta_0} \times [T_1 + T_3 + T_4, T_1 + T_2].  \tag{17}$$

**Step 4.** We will prove that if $\beta > \max\{\hat{\beta}, \beta_0, \beta_1\}$, then

$$u(x, t) < \frac{b}{2}, \quad (x, t) \in \bar{\Omega} \times [T_1 + T_3 + T_4, T_1 + T_2],  \tag{18}$$

where $\hat{\beta}$ is defined in Lemma 3.2; $\beta_0$ is chosen as in Step 3; $\beta_1 = \delta_0^{-1}[\ln(4 + 8\varepsilon_0) - \ln b]$.

Let $\delta_0, \beta_0, \varepsilon_0, T_1, T_3$ and $T_4$ be chosen as above and $T_2 = T_3 + T_4$. Owing to (14), we obtain that if $\beta > \max\{\hat{\beta}, \beta_0, \beta_1\}$, $x \in \bar{\Omega} \setminus \Omega_{\delta_0}$ and $T_1 + T_3 \leq t \leq T_1 + T_2$, then

$$u(x, t) \leq \max\{2\varepsilon_0, (1 + 2\varepsilon_0)e^{-\beta d(x, \partial \Omega_\delta)} + \varepsilon_0\} < \max\{2\varepsilon_0, (1 + 2\varepsilon_0)e^{-\beta \delta_0} + \varepsilon_0\} < b/2.$$  

Then (18) can be deduced from (17) and (19).
Step 5. Take \( \beta_* \) satisfying \( \beta_* > \max\{\hat{\beta}, \beta_0, \beta_1\} \) and \( \beta_* \neq d_1^{-1}[m_* - (1-b)^2/4] \). Let \( v_0^\ast \) satisfy
\[
d_1\beta_*^2 = \frac{m_*v_0^\ast e^{-\theta(T_1+T_2)}}{1 + \varepsilon_0 + v_0^\ast e^{-\theta(T_1+T_2)}} - \frac{(1-b)^2}{4},
\]
where \( T_1, T_2 \) and \( \varepsilon_0 \) are chosen as above. It is clearly that \( v_0^\ast \) depends on \( b, \theta, d_1, m_* \) and \( u_0(x) \). Based on above arguments, taking \( T^* = T_1 + T_2 \), we have
\[
u(x, T^*) < b/2, \quad x \in \Omega.
\]
Applying Theorem 2.4, it follows that \( \lim_{t \to \infty} u(x, t) = 0 \) uniformly on \( \Omega \), \( \lim_{t \to \infty} v(x, t) = 0 \) uniformly on \( \Omega_* \).

**Theorem 3.4.** Let the parameters \( d_2, \theta > 0, 0 < b < 1/2 \) and the subdomain \( \Omega_0 \) be fixed, and \( D_0 \) be defined in Lemma 3.1. Define
\[
\hat{u}(x) = \begin{cases} 
\tilde{U}(x), & x \in \Omega_0, \\
0, & x \in \Omega_*,
\end{cases}
\]
where \( \tilde{U}(x) \) is the maximal positive solution of (11). Assumed that \( m(x), c(x) \) satisfy (4) and \( (u(x, t), v(x, t)) \) is the positive solution of (3). If \( 0 < d_1 < D_0 \) and \( u_0(x) \geq \hat{u}(x) \), then for any initial predator population \( v_0(x) > 0 \), we have \( u(x, t) \geq \hat{u}(x) \) for all \( x \in \Omega \) and \( t > 0 \).

**Proof.** Let \( w(x, t) \) be the solution of the Dirichlet boundary value problem
\[
\begin{cases}
w_t = d_1 \Delta w + w(1-w)(w-b), & x \in \Omega_0, \ t > 0, \\
w(x, t) = 0, & x \in \partial \Omega_0, \ t > 0, \\
w(x, 0) = u_0(x) \geq 0, & x \in \Omega_0.
\end{cases}
\] (20)
Obviously, \( w(x, t) \) exists globally for all \( x \in \Omega_0 \) and \( t > 0 \). It is derived from \( u_0(x) \geq \tilde{U}(x) \) and the comparison principle that \( w(x, t) \geq \tilde{U}(x) \) for all \( x \in \Omega_0 \) and \( t > 0 \). On the other hand, \( u(x, t) \) satisfies the first equation in (20) on \( \Omega_0 \), \( u(x, t) > 0 \) for \( x \in \partial \Omega_0 \), and \( u(x, 0) = u_0(x) \). Thus \( u(x, t) \) is a upper solution of (20). Then \( u(x, t) \geq w(x, t) \geq \tilde{U}(x) \) for all \( x \in \Omega_0 \) and \( t > 0 \). In conclusion, \( u(x, t) \geq \hat{u}(x) \) for all \( x \in \Omega \) and \( t > 0 \). \[]

4. Bifurcation from semi-trivial solutions \((0, 1)\) and \((0, b)\). In this section, we will investigate the bifurcation from semi-trivial solutions \((1, 0)\) and \((b, 0)\) using bifurcation theory. We fix \( d_1, d_2 > 0 \) and \( 0 < b < 1 \), and take \( \theta \) as the main bifurcation parameter. For \( p > 1 \), we define
\[
X_1 = \{ u \in W^{2,p} (\Omega) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \}, \quad Y_1 = L^p(\Omega),
\]
and
\[
X_2 = \{ v \in W^{2,p} (\Omega_*) : \frac{\partial v}{\partial n} = 0 \text{ on } \partial \Omega_* \}, \quad Y_2 = L^p(\Omega_*).
\]
For a given operator \( A \), we denote the kernel and range of \( A \) with by \( \mathcal{N}(A) \) and \( \mathcal{R}(A) \) (sometimes, we simply write them as \( \mathcal{N} \mathcal{A} \) and \( \mathcal{R} \mathcal{A} \), respectively.

Before we study the bifurcation of (5), we first give a useful lemma.
Theorem 4.2. Let the parameters $\lambda_1(q)$ be the first eigenvalue of the operator $-\Delta + q$ in $\Omega$ with the homogeneous Neumann boundary condition, and $\phi_1$ be the corresponding eigenfunction of $\lambda_1(q)$. Furthermore, if $f \in L^2(\Omega)$ satisfying $\int_{\Omega} f \phi_1 dx = 0$ and $\lambda_1(q) = 0$, then the Neumann boundary value problem
\[
\begin{cases}
-\Delta u + q(x)u = f, & x \in \Omega, \\
\frac{\partial u}{\partial n} = 0, & x \in \partial \Omega
\end{cases}
\]
adopts a solution.

Proof. Let $\{\lambda_i(q)\}_{i=1}^{\infty}$ and $\{\phi_i\}_{i=1}^{\infty}$ be the eigenvalues and corresponding eigenfunctions of $-\Delta + q$ in $\Omega$ with homogeneous Neumann boundary condition. We may assume that $\|\phi_i\|_2 = 1$ for $i \geq 1$. Decompose $f$ as $f = \sum_{i=1}^{\infty} c_i \phi_i$. It follows from $\int_{\Omega} f \phi_i dx = 0$ that $c_1 = 0$. Due to the fact that the remainder of proof is similar to Theorem 2.7.2 in [26], we omit it.

Theorem 4.2. Let the parameters $d_1$, $d_2 > 0$ and $0 < b < 1$ be fixed. Assume that $m(x)$ and $c(x)$ satisfy (4) and $\theta_*$ is defined in Theorem 2.6.

(i) The point $(\theta_*, 1, 0)$ is a bifurcation point of the coexistence state of (5). Moreover, when $0 < s \ll 1$, the bifurcating coexistence state $(\theta(s), u(s), v(s))$ from $(\theta_*, 1, 0)$ takes the form
\[
\begin{cases}
\theta(s) = \theta_* + \theta_1 s + O(s^2), \\
u(s) = s \Phi_* + O(s^2), \\
v(s) = s \Psi_* + O(s^2),
\end{cases}
\]
where $\Phi_*$ is the positive eigenfunction corresponding to $\theta_*$ with $\int_{\Omega} \Phi_*^2 dx = 1$ and
\[
\Phi_* = (-d_1 \Delta + 1 - b)^{-1}[-m(x)\Phi_*] < 0.
\]
By substituting $(\theta(s), u(s), v(s))$ into the second equation of (5), we obtain $\theta_1 = -\int_{\Omega} c(x) \Psi_*^2 dx$. Furthermore, $\theta = \theta_*$ is the unique bifurcation value of the bifurcating positive solution from $(\theta_*, 1, 0)$.

(ii) The point $(\theta_*, b, 0)$ is a bifurcation point of the coexistence state of (5). Moreover, when $0 < s \ll 1$, the bifurcating coexistence state $(\tilde{\theta}(s), \tilde{u}(s), \tilde{v}(s))$ from $(\theta_*, b, 0)$ takes the form
\[
\begin{cases}
\tilde{\theta}(s) = \theta_* + \tilde{\theta}_1 s + O(s^2), \\
\tilde{u}(s) = b + s \Phi_* + O(s^2), \\
\tilde{v}(s) = s \Psi_* + O(s^2),
\end{cases}
\]
where $\Phi_*$ is the positive eigenfunction corresponding to $\theta_*$ with $\int_{\Omega} \Phi_*^2 dx = 1$ and
\[
\Phi_* = [-d_1 \Delta - b(1 - b)]^{-1}[-m(x)\Phi_*].
\]
By substituting $(\tilde{\theta}(s), \tilde{u}(s), \tilde{v}(s))$ into the second equation of (5), we obtain $\tilde{\theta}_1 = -\int_{\Omega} c(x) \Psi_*^2 dx$. Furthermore, $\theta = \theta_*$ is the unique bifurcation value of the bifurcating positive solution from $(\theta_*, b, 0)$.

Proof. Because the proof of (ii) is similar to (i), we will only prove (i). Define a mapping $F: \mathbb{R} \times X_1 \times X_2 \to Y_1 \times Y_2$ by
\[ F(\theta, u, v) = \left( \frac{d_1 \Delta u + u(1-u)(u-b) - \frac{m(x)uv}{u+v}}{d_2 \Delta v - \theta v + \frac{c(x)uv}{u+v}} \right). \]

It is easy to see that \( F_\theta(\theta_*, 1, 0) = 0 \). For any \((\xi, \eta) \in X_1 \times X_2\), a simple calculation yields
\[ F_{(u,v)}(\theta, u, v)(\xi, \eta) = \left( \frac{d_1 \Delta \xi + [-3u^2 + 2(b+1)u - b] \xi - \frac{m(x)v^2}{(u+v)^2} \xi - \frac{m(x)u^2}{(u+v)^2} \eta}{d_2 \Delta \eta - \theta \eta + \frac{c(x)v^2}{(u+v)^2} \xi + \frac{c(x)u^2}{(u+v)^2} \eta} \right). \]

Consequently,
\[ F_{(u,v)}(\theta_*, 1, 0)(\xi, \eta) = \left( \frac{d_1 \Delta \xi + (b-1)\xi - m(x)\eta}{d_2 \Delta \eta - \theta \eta + c(x)\eta} \right). \]

**Step 1.** We shall prove
\[ \dim(NF_{(u,v)}(\theta_*, 1, 0)) = 1, \quad NF_{(u,v)}(\theta_*, 1, 0) = \text{span}\{(\Phi_*, \Psi_*)\}. \]

In fact, if there exists \((0,0) \neq (\xi_1, \eta_1) \in X_1 \times X_2\) such that \( F_{(u,v)}(\theta_*, 1, 0)(\xi_1, \eta_1) = (0,0) \), then
\[ \begin{cases}
  d_1 \Delta \xi_1 + (b-1)\xi_1 - m(x)\eta_1 = 0, & x \in \Omega, \\
  d_2 \Delta \eta_1 - \theta \eta_1 + c(x)\eta_1 = 0, & x \in \Omega_*, \\
  \frac{\partial \xi_1}{\partial n} = 0, & x \in \partial \Omega, \\
  \frac{\partial \eta_1}{\partial n} = 0, & x \in \partial \Omega_*. 
\end{cases} \tag{21} \]

Since the operator \( d_1 \Delta + (b-1) \) is invertible, we have \( \eta_1 \neq 0 \). From \( \theta_* = -d_2 \lambda^N \left( -\frac{c(x)}{d_2}, \Omega_* \right) \), we have \( \eta_1 \in \text{span}\{\Psi_*\} \), i.e., \( \eta_1 = l\Psi_* \) for some constant \( l \neq 0 \). Hence, \( \xi_1 = l\Phi_* \in \text{span}\{\Phi_*\} \). Therefore, \((\xi_1, \eta_1) \in \text{span}\{(\Phi_*, \Psi_*)\} \).

**Step 2.** We will show that \( \text{Codim}(\mathcal{R}F_{(u,v)}(\theta_*, 1, 0)) = 1 \).

In fact, if \((\xi^*, \eta^*) \in \mathcal{R}F_{(u,v)}(\theta_*, 1, 0) \), then there exists \((\xi_2, \eta_2) \in X_1 \times X_2\) such that
\[ \begin{cases}
  d_1 \Delta \xi_2 + (b-1)\xi_2 - m(x)\eta_2 = \xi^*, & x \in \Omega, \\
  d_2 \Delta \eta_2 - \theta \eta_2 + c(x)\eta_2 = \eta^*, & x \in \Omega_*, \\
  \frac{\partial \xi_2}{\partial n} = 0, & x \in \partial \Omega, \\
  \frac{\partial \eta_2}{\partial n} = 0, & x \in \partial \Omega_. 
\end{cases} \tag{22} \]

Multiplying \( \eta_2 \) and \( \eta_1 \) to the second equation of (21) and (22), respectively, and integrating them over \( \Omega \) and then subtracting the results, we have
\[ \int_{\Omega_*} \eta_1 \eta^* dx = 0. \]

It follows from \( \eta_1 = l\Psi_* \) that \( \int_{\Omega_*} \Psi_* \eta^* dx = 0 \). Consequently, \((\xi^*, \eta^*) \) is orthogonal to \((0, \Psi_*) \).
Conversely, if \((\xi^*, \eta^*)\) is orthogonal to \((0, \Psi_*)\), then the second equation of (22) has a solution \(\eta\) on account of Lemma 4.1. Then the first equation of (21) admits a solution \(\xi\) since the operator \(d_1\Delta + b - 1\) is invertible. Thus \((\xi^*, \eta^*) \in R \mathcal{F}_{(u,v)}(\theta_1, 1, 0)\), and hence \(\text{Codim}(R \mathcal{F}_{(u,v)}(\theta_1, 1, 0)) = 1\).

**Step 3.** Since \(R \mathcal{F}_{(u,v)}(\theta_1, 1, 0)\) is orthogonal to \((0, \Psi_*)\), and
\[
\mathcal{F}_{\theta,(u,v)}(\theta_1, 1, 0)(\Psi_*) = (0, -\Psi_*),
\]
we have
\[
\mathcal{F}_{\theta,(u,v)}(\theta_1, 1, 0)(\Phi_*, \Psi_*) \notin R \mathcal{F}_{(u,v)}(\theta_1, 1, 0).
\]

Finally, applying the bifurcation theorem (Theorem 3.2.2 in [20]), we arrive at the desired conclusions.

In the sequel part, we will prove the uniqueness of bifurcation value \(\theta_*\). Assume that \(\theta_0\) is another bifurcation value from the semi-trivial solution \((1,0)\), then (5) exists positive sequence \(\{(\theta_n, u_n, v_n)\}^\infty_{n=1}\) such that
\[
\lim_{n \to \infty} (\theta_n, u_n, v_n) = (\theta_0, 1, 0) \in \mathbb{R} \times X_1 \times X_2.
\]
Putting \((\theta_n, u_n, v_n)\) into the second equation of (5) and dividing it by \(\|v_n\|_p\), it follows that for any \(n \geq 1\),
\[
-d_2\Delta \left(\frac{v_n}{\|v_n\|_p}\right) = -\theta_n \frac{v_n}{\|v_n\|_p} + \frac{c(x)u_n}{u_n + v_n} \cdot \frac{v_n}{\|v_n\|_p},
\]
which is equivalent to
\[
d_2 \left(\frac{v_n}{\|v_n\|_p}\right) = (\theta_0 - \theta_n)(-\Delta)^{-1} \left(\frac{v_n}{\|v_n\|_p}\right) + (-\Delta)^{-1} \left(\frac{c(x)u_n}{u_n + v_n} - \theta_0\right) \frac{v_n}{\|v_n\|_p}.
\]
Since \((-\Delta)^{-1}\) is compact operator, there exists a subsequence of \(v_n/\|v_n\|_p\), denoted by itself, and a function \(\varphi \in W^{2,p}(\Omega_*)\), such that
\[
\lim_{n \to \infty} \frac{v_n}{\|v_n\|_p} = \varphi > 0.
\]
Take the limit on both sides of (23), we get
\[
d_2 \varphi = (-\Delta)^{-1}(c(x) - \theta_0)\varphi.
\]
Then we have
\[
\begin{cases}
  d_2\Delta \varphi - \theta_0 \varphi + c(x)\varphi = 0, & x \in \Omega_*, \\
d_2 \frac{\partial \varphi}{\partial n} = 0, & x \in \partial \Omega_*. 
\end{cases}
\tag{24}
\]
Multiplying \(\varphi\) and \(\eta_1\) to the second equation of (21) and the first equation (24), respectively, and integrating them over \(\Omega_*\), and then subtracting the results, we have
\[
(\theta_* - \theta_0) \int_{\Omega_*} \varphi \eta_1 dx = 0.
\]
Hence, \(\theta_0 = \theta_*\). The proof is completed.

**Theorem 4.3.** Assume that the conditions in Theorem 4.2 hold. If \(0 < s \ll 1\), then the local bifurcation coexistence state \((u(s), v(s))\) bifurcating from \((\theta_*, 1, 0)\) is non-degenerate. Moreover, \((u(s), v(s))\) is linearly stable.
Proof. For ease of notation, we denote \( \theta(s) = \theta, \) \((u(s), v(s)) = (u, v)\). The linearized problem at \((u, v)\) can be written as

\[
\mathcal{L}(s)(\xi, \eta) = \mu(s)(\xi, \eta),
\]

where

\[
\mathcal{L}(s) = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
\]

\[
A_{11} = -d_1 \Delta + 3u^2 - 2(b + 1)u + b + \frac{m(x)v^2}{(u + v)^2}, \quad A_{12} = \frac{m(x)u^2}{(u + v)^2},
\]

\[
A_{21} = -c(x)v^2, \quad A_{22} = -d_2 \Delta + \theta - \frac{c(x)u^2}{(u + v)^2}.
\]

It easy to see that, as \( s \to 0 \),

\[
\mathcal{L}(s) \to \mathcal{L}_0 := \begin{pmatrix}
-d_1 \Delta + 1 - b & m(x) \\
0 & -d_2 \Delta + \theta - c(x)
\end{pmatrix}.
\]

We claim that \( 0 \) is the smallest eigenvalue of \( \mathcal{L}_0 \) with the corresponding eigenfunction \((\Phi_*, \Psi_*)\), where \( \Phi_* \) and \( \Psi_* \) are defined in Theorem 4.2. In fact, we suppose that \( \tilde{\mu} < 0 \) is the principal eigenvalue of \( \mathcal{L}_0 \) with the corresponding eigenfunction \((\tilde{\xi}, \tilde{\eta})\). If \( \tilde{\eta} \equiv 0 \) then we have \( \tilde{\mu} = 0 \) according to \( \theta_* = -d_2 \lambda_*^2 \left( -\frac{c(x)}{d_2}, \Omega_* \right) \). This is a contradiction, and so \( \tilde{\eta} \equiv 0 \). If \( \tilde{\xi} \not\equiv 0 \), we get a contradiction since \( \tilde{\mu} = 1 - b > 0 \). On the other hand, \( 0 \) is the first eigenvalue of \(-d_2 \Delta + \theta_* - c(x)\) with the corresponding eigenfunction \(\Psi_*\). Therefore, \( 0 \) is the smallest eigenvalue of \( \mathcal{L}_0 \) with the corresponding eigenfunction \((\Phi_*, \Psi_*)\), and all the other eigenvalues of \( \mathcal{L}_0 \) are positive.

By the perturbation theory of linear operators [12], we know that, when \( s \) are sufficiently small, \( \mathcal{L}(s) \) has a unique eigenvalue \( \mu(s) \) satisfying \( \lim_{s \to 0} \mu(s) = 0 \) and all the other eigenvalues of \( \mathcal{L}(s) \) have positive real parts and are apart from \( 0 \). In order to simplify the notation, we denote \( \mathcal{L}(s) = \mathcal{L} \) and \( \mu(s) = \mu \) below.

Now we determine the sign of \( \text{Re} \mu \) as \( s > 0 \) is sufficiently small. Let \( \xi, \eta \) be the corresponding eigenfunction to \( \mu \) such that \( (\xi, \eta) \to (\Phi_*, \Psi_*) \). Multiplying the second equation of \( \mathcal{L}(\xi, \eta) = \mu(\xi, \eta) \) by \( \nu \) and integrating over \( \Omega_* \), we get

\[
-d_2 \int_{\Omega_*} v \Delta \eta dx + \theta \int_{\Omega_*} v \eta dx - \int_{\Omega_*} \frac{c(x)u^2v}{(u + v)^2} \eta dx - \int_{\Omega_*} \frac{c(x)v^3}{(u + v)^2} \xi dx = \mu \int_{\Omega_*} v \eta dx.
\]

Multiplying the second equation of (5) by \( \eta \) integrating over \( \Omega_* \), we have

\[
-d_2 \int_{\Omega_*} v \Delta \eta dx = -d_2 \int_{\Omega_*} \xi \eta dx = -\theta \int_{\Omega_*} \eta v dx + \int_{\Omega_*} \frac{c(x)uv}{u + v} \eta dx.
\]

This fact combines with (25) to yield

\[
\mu \int_{\Omega_*} v \eta dx = \int_{\Omega_*} \frac{c(x)uv}{u + v} \eta dx - \int_{\Omega_*} \frac{c(x)u^2v}{(u + v)^2} \eta dx - \int_{\Omega_*} \frac{c(x)v^3}{(u + v)^2} \xi dx.
\]

Noting that \( (u, v) = (1 + s \Phi_* + O(s^2), s \Psi_* + O(s^2)) \) and \( \xi \to \Phi_*, \eta \to \Psi_* \), dividing (26) by \( s^2 \) and letting \( s \to 0^+ \), it is deduced that \( \lim_{s \to 0^+} \frac{\mu}{s^2} = 0 \). Further calculation, we divide (26) by \( s^3 \) and let \( s \to 0^+ \), then it follows that

\[
\lim_{s \to 0^+} \frac{\mu}{s^2} = \frac{-\int_{\Omega_*} c(x) \Phi_* \Psi_*^3 dx}{\int_{\Omega_*} \Psi_*^2 dx} > 0.
\]
This implies that \( \text{Re} \mu \neq 0 \) for \( s \) sufficiently small. Since all the other eigenvalues of \( L \) have positive real parts and are apart from 0, the local bifurcation coexistence state bifurcating from \((\theta^*, 1, 0)\) is non-degenerate. In addition, the limits (27) suggests that the bifurcation coexistence state \((u(s), v(s))\) is linearly stable.

Acknowledgments. We would like to thank reviewers and editors for their valuable comments and recommendations.

REFERENCES

[1] W. C. Allee, *Principles of Animal Ecology*, Saunders, RI, 1949.
[2] H. R. Akcakaya, R. Arditi and L. R. Ginzburg, Ratio-dependent prediction: An abstraction that works, *Ecology*, 76 (1995), 995–1004.
[3] R. Arditi and H. Saiah, Empirical evidence of the role of heterogeneity in ratio-dependent consumption, *Ecology*, 73 (1992), 1544–1551.
[4] R. G. Casten and C. G. Holland, Instability results for reaction diffusion equations with Neumann boundary conditions, *J. Diff. Equat.*, 27 (1978), 266–273.
[5] R. H. Cui, J. P. Shi and B. Y. Wu, Strong Allee effect in a diffusive predator-prey system with a protection zone, *J. Diff. Equat.*, 256 (2014), 108–129.
[6] Y. H. Du and X. Liang, A diffusive competition model with a protection zone, *J. Diff. Equat.*, 244 (2008), 61–86.
[7] Y. H. Du, R. Peng and M. X. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, *J. Diff. Equat.*, 246 (2009), 3932–3956.
[8] Y. H. Du and J. P. Shi, A diffusive predator-prey model with a protection zone, *J. Diff. Equat.*, 229 (2006), 63–91.
[9] S. B. Hsu, On global stability of a predator-prey system, *Math. Biosci.*, 39 (1978), 1–10.
[10] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.
[11] C. S. Holling, Some characteristics of simple types of predation and parasitism, *Can. Ent.*, 91 (1959), 385–398.
[12] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin-New York, 1966.
[13] P. E. Kloeden and C. Pötzsche, Dynamics of modified predator-prey models, *Int. J. Bifurc. Chaos*, 20 (2010), 2657–2669.
[14] Y. Lou, Some reaction diffusion models in spatial ecology, *Sci. Sin. Math.*, 45 (2015), 1619–1634.
[15] Y. Lou and B. Wang, Local dynamics of a diffusive predator-prey model in spatially heterogeneous environment, *J. Fixed Point Theory Appl.*, 19 (2017), 755–772.
[16] Y. Li and M. X. Wang, Stationary pattern of a diffusive prey-predator model with trophic intersections of three levels, *Nonlinear Anal. RWA*, 14 (2013), 1806–1816.
[17] R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton Univ. Press, Princeton, 1973.
[18] K. Mischaikow, H. Smith and H. R. Thieme, Asymptotically autonomous semiflows: Chain recurrence and Lyapunov functions, *Trans. Amer. Math. Soc.*, 347 (1995), 1669–1685.
[19] N. Min and M. X. Wang, Qualitative analysis for a diffusive predator-prey model with a transmissible disease in the prey population, *Comput. Math. Appl.*, 72 (2016), 1670–1689.
[20] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Amer. Math. Soc., Providence, RI, 2001.
[21] W. J. Ni and M. X. Wang, Dynamics and patterns of a diffusive Leslie-Gower prey-predator model with strong Allee effect in prey, *J. Diff. Equat.*, 261 (2016), 4244–4272.
[22] W. J. Ni and M. X. Wang, Dynamicl properties of a Leslie-Gower prey-predator model with strong Allee effect in prey, *Discrete Contin. Dyn. Syst. Ser. B*, 22 (2017), 3409–3420.
[23] P. Y. H. Pang and M. X. Wang, Qualitative analysis of a ratio-dependent predator-prey system with diffusion, *Proc. Roy. Soc. Edinburgh Sect. A*, 133 (2003), 919–942.
[24] F. Rao and Y. Kang, The complex dynamics of a diffusive prey-predator model with an Allee effect in prey, *Ecol. Complex.*, 28 (2016), 123–144.
[25] M. X. Wang, Stationary patterns for a prey-predator model with prey-dependent and ratio-dependent functional responses and diffusion, *Phys. D*, 196 (2004), 172–192.
[26] M. X. Wang, *Nonlinear Elliptic Partial Differential Equations* (in Chinese), Science Press, Beijing, 2010.
[27] Y. X. Wang and W. T. Li, Effect of cross-diffusion on the stationary problem of a diffusive competition model with a protection zone, *Nonlinear Anal. RWA*, 14 (2013), 224–245.

[28] J. F. Wang, J. P. Shi and J. J. Wei, Dynamics and pattern formation in a diffusive predator-prey systems with strong Allee effect in prey, *J. Diff. Equat.*, 251 (2011), 1276–1304.

[29] Q. X. Ye, Z. Y. Li, M. X. Wang and Y. P. Wu, *The Introduction of Reaction-Diffusion Equations* (in Chinese), Science Press, Beijing, 2011.

[30] F. Q. Yi, J. J. Wei and J. P. Shi, Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator-prey system, *J. Diff. Equat.*, 246 (2009), 1944–1977.

[31] J. Zhou and C. L. Mu, Coexistence of a diffusive predator-prey model with Holling type-II functional response and density dependent mortality, *J. Math. Anal. Appl.*, 385 (2012), 913–927.

Received for publication August 2017.

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