Index Theory and Adiabatic Limit in QFT. Part II: A Generalization of Mackey’s Theory of Induced Representations

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Abstract

The paper has a technical character in which we extend Mackey theory of induced representations so as to embrace representations of the covering of the Poincaré group in Krein spaces (with Gupta-Bleuler operator) such as the Lopuszański representation encountered in construction of free fields with non trivial gauge freedom, e.g. electromagnetic field. Proposed extension allows us to pursue explicit spectral-operator construction of space-time in the Fock space of free fields – a crucial point in the solution of the representation problem (viz. convergence problem of the formal perturbation series) in the perturbative algebraic QFT (developed mainly by Dütsch, Fredenhagen and Brunetti) which we have proposed in our previous paper. The proposal of our previous paper is in fact an extension of the general program proposed by Fredenhagen, Dütsch and Brunetti whose main point lies in separating the algebraic aspect – by interpreting the perturbation series as a formal deformation from the analytic aspect tied to the representation problem. The representation problem cited to above is immediately connected to the general problem in QFT of proving the existence of interacting fields with non trivial gauge freedom. Presented generalization of Mackey theory is given an independent character so as the “subgroup theorem” and “the Kronecker product theorem” are proved for general separable locally compact groups and for their Krein-isometric not necessary bounded representations induced by Krein-unitary (bounded) representations.

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1 Introduction

In perturbative algebraic QFT the infra-red-divergence (IR) problem is clearly separated from the ultra-violet-divergence (UV) problem by using a space-time function \( x \mapsto g(x) \) with compact support as coupling “constant” – a microlocal perturbative construction of interacting fields. The UV-problem is essentially solved – the origin of infinite counter terms of the renormalization scheme is well understood by now, i. e. using the counter terms (renormalization) is equivalent to the microlocal perturbative construction of the perturbative series of Epstein-Glaser-type, where no infinite counter terms appear but instead one uses recurrence rules for the construction of the chronological product of fields regarded as operator-valued distributions. The renormalization scheme is now incorporated into the following recurrence rules for the chronological product [158]:

1) causality,
2) symmetry,
3) unitarity,
4) Poincaré covariance,

2 Preliminaries

[158]
5) Ward identities – quantum version of gauge invariance (e. g. in case of QED),

6) preservation of the Steinmann scaling degree,

part of the remaining freedom may be reduced by imposing the natural field equations for the interacting field (which is always possible for the standard gauge fields) and the rest of the remaining freedom is pertinent to the Stückelberg-Petermann renormalization group. All the recurrence rules should be regarded as important physical laws which incorporate the whole content of the standard pragmatic approach including the renormalization scheme. Causality implies locality for perturbatively constructed (using the Epstein-Glaser method [23]) algebras of localized fields $\mathcal{F}(O)$ regarded as “smeared out” operator-valued distributions, where $g$ is constant (equal to the electric charge in case of QED) within the open space-time region $O$ – the only step where the UV-problem shows up and is solved by the use of Epstein-Glaser method. The IR-problem is solved only partially, i. e. nets $O \mapsto \mathcal{F}(O)$ of algebras $\mathcal{F}(O)$ of local (unbounded) operator localized fields have likewise been constructed perturbatively [18], but in the sense of formal power series only.

The most important and still open problems are the following.

(a) To construct physically relevant states, which in fact involves the connected essentially equivalent problem of existence of the adiabatic limit $(g \mapsto \text{constant function over the whole space-time})$ in the sense of [23].

This is the IR-problem.

(b) The convergence of the formal power series. This is the representation problem.

Existence of the adiabatic limit has been proved by Epstein and Glaser but only for massive (gauge-free) fields. Existence of the adiabatic limit for Green functions and for Wightman functions has been proved for QED by Blanchard and Seneor [2]. However in gauge theories with the so called confinement phenomenon (e. g. QCD) divergences are much more difficult to control, and the direct Epstein-Glaser-proof of the existence of the adiabatic limit does not work. However it was conjectured for a long time by all those who have tried it, that the problem (a) cannot be solved within the perturbative framework, and that (a) requires the solution of (b) – compare the discussion placed below in this Section.

The state of affairs is in fact the end product of a long process of learning how to incorporate the renormalization prescriptions within a consistent mathematical framework, which started with the investigations of Stückelberg and Bogoliubov [5]. Especially contributions to the solution of (a) for gauge-type fields within the perturbative algebraic QFT are of special interest for us. Dütsch and Fredenhagen [18] constructed the interacting QED fields as deformation (in the Bordemann-Waldmann sense [6]) of the free theory. They constructed the observable algebra and the physical Hilbert space out of the local fields and the Krein space (Hilbert space with the Gupta-Bleuler operator)
for the free theory and then they show that the structures, i.e. the observable algebra and the physical Hilbert space acted on by the algebra, is stable under deformation, i.e. is preserved by the interacting fields in the sense that all the linear-algebraic structures over $\mathbb{C}$ are replaced by the corresponding structures over the power series division ring, where the operators are regarded as formal power series operators acting in the formal Krein space over the power series division ring. The whole point of the approach of Dütsch and Fredenhagen [18] lies in the replacement of the fundamental field $\mathbb{C}$ of complex numbers by the power series division ring, and in separating the “algebraic level” by using the algebraic structures over the formal power series-type fields instead of $\mathbb{C}$ from the analytic representation problem (b) similarly as in the Weyl-type deformation. This approach enables us to separate the local (UV-problem) from the “global” adiabatic problem (a). The same phenomenon we encounter in the Weyl-type deformation of symplectic manifold, where the introduction of formal power series ring enables us to construct deformations of symplectic manifolds with non-trivial topology using a (formal) but essentially local construction.

All the above recurrence rules of the microlocal perturbation series are incorporated within the properties of the deformation proposed by Dütsch and Fredenhagen. In particular an implication of the existence of the adiabatic limit (as is the case e.g. for the massive gauge fields) may equivalently be formulated within the deformation formalism of Dütsch and Fredenhagen in the following way: the deformation formal power series of the elements of the observable algebra has the coefficients in this algebra. The same stability holds for the $\ast$-algebra equal to the kernel of the BRST transformation as well as for the ideal being the image of the BRST transformation, compare [18]. This is an extraordinary and non trivial stability property very important in the solution of the problems (a) and (b). In order to explain it, let us suppose that we have an algebra $\mathcal{A}$ (in fact we need a more specific conditions\footnote{It should be an algebra of operators commuting with the Gupta-Bleuler operator, commutative and involutive, with the involution represented by the Krein adjoint equal to the ordinary adjoint as the the operators of the algebra commute with the Gupta-Bleuler operator, so let us suppose that it is a $C^\ast$-algebra or pre-$C^\ast$-algebra.\footnote{In fact we have to know how to apply the deformation to all ingredients of the spectral triple defining the manifold structure.}} of bounded operators acting in the Fock (Krein) space of free fields which moreover fulfils the conditions:

1. $\mathcal{A}$ has the Gelfand spectrum $\text{Spec } \mathcal{A}$ with a smooth finite dimensional manifold structure – for example let the manifold structure be encoded within the Connes-type spectral operator format.

2. Suppose moreover that we know how to apply the perturbation series to the elements\footnote{In fact we have to know how to apply the deformation to all ingredients of the spectral triple defining the manifold structure.} of the algebra $\mathcal{A}$, which gives us the deformation of Dütsch and Fredenhagen of the elements of the algebra.

3. Suppose that the algebra $\mathcal{A}$ is canonically related to the algebra of observables, so that the stability property remarked to above holds true for $\mathcal{A}$: the deformation formal power series associated to any element of that
algebra has coefficients in this algebra whenever this holds true for the
algebra of observables, i.e. whenever the adiabatic limit does exists.

(4) Suppose finally that the deformation of \( A \) induced by the perturbation
series has the following locality property\(^3\): if \( a \in A \) with \( a \) represented by
a smooth function with compact support contained within a compact set
\( K \subset \text{Spec} \ A \) and if the formal power series defining the deformation \( \tilde{a} \) of
\( a \) has the coefficients represented by the functions \( a_0, a_1, a_2 \ldots \), then all
\( a_0, a_1, a_2 \ldots \) have compact supports contained in \( K \).

Or alternatively if (3) is not fulfilled but instead we have

(3') \( A \) is the center of the involutive algebra \( \mathcal{C} \) generated by the coefficients of
deformations of elements of \( A \), and let \( \mathcal{A} \cong S(M) \cong \mathcal{C} \) as linear Fréchet
spaces respecting localization of \( \mathcal{C} \) with respect to \( \mathcal{A} \).

(4') All coefficients \( a_0, a_1, \ldots \) of deformation \( \tilde{a} = a_0 + e a_1 + e^2 a_2 + \ldots \) of \( a \in A \)
depend (linearly) and locally on \( a \), i.e. if \( b = a + e \in \mathcal{A} \) where the support
of \( e \) regarded as a function does not intersect the support of \( a \), then the
localization of the coefficients \( a_0 = a, a_1, \ldots \) at \( \mathcal{O} \subset K \subset \text{Spec} \ A \) is equal
to the localization of the coefficients \( b_0 = b, b_1, \ldots \) at \( \mathcal{O} \).

In this situation the Fedosov necessary and sufficient condition\(^{25} \) (in a
slightly generalized form with the Fedosov formal index for families) for the
existence of the so called asymptotic faithful involutive representation of the
defformation of \( A \) in a Hilbert space over \( \mathbb{C} \) is immediately applicable to the
Dütsch and Fredenhagen deformation of \( A \).

This means that the formal power series in the deformation of the elements
of \( \mathcal{A} \) are convergent iff the Fedosov-type conditions are fulfilled for \( \mathcal{A} \).

Then the coefficients \( a_1, a_2 \ldots \) of the deformation \( \tilde{a} \) of \( a \in \mathcal{A} \) may be represented
by differential operators \( D_1, D_2, \ldots \) acting on \( a \) regarded as a function
and the the product \( a \star b = \tilde{a} b \) in the first case is represented by

\[
a \cdot b + e(a \cdot D_1 b + D_1 a \cdot b) + e^2(a \cdot D_2 b + D_1 a \cdot D_1 b + D_2 a \cdot b) + \ldots,
\]

where dot stands for the point wise multiplication and in the second situation

\[
a \star b = \tilde{a} b = ab + eB_1(a, b) + e^2B_2(a, b) + \ldots
\]

where \( B_k \) are bidifferential operators acting on \( a, b \), which are non symmetric in
general: \( B_k(a, b) \neq B_k(b, a) \). We thus obtain a deformation of a kind to which
the the standard formal deformation of the algebra of smooth (or Schwartz)
functions on Poisson manifolds belong. In the first case we obtain a “symmetric”
defformation with symmetric \( B_1(a, b) = B_1(a, b) = a \cdot D_1 b + D_1 a \cdot b, \)
\( B_2(b, a) = B_2(a, b), \ldots \), and the property that the (norm) convergence of the

\(^3\) This makes sense because by assumption the algebra \( \mathcal{A} \) may be represented (up to possible
infinite uniform multiplicity) by multiplication operators by smooth functions on a smooth
manifold on the Hilbert space of square integrable functions on the manifold.
series in deformation \( \hat{a} = a + a_1 + \ldots = a + D_1 a + \ldots \) regarded as operators gives a representation in the Fock space of the deformed algebra \( \hat{A} \) regarded as an abstract formal algebra with the product \( \ast \). This is not the case in the second situation with (3') and (4') instead of (3) and (4), where the formal algebra plays a more profound role and the representation and conditions necessary and sufficient for existence of an ordinary Hilbert space representation of this formal algebra should be constructed like in the algebraic deformation theory (compare Fedosov index). In order to explain it let us assume for a while (just for simplicity) that in addition \( C \) is spatially isomorphic to \( A \otimes B \), with the corresponding structure \( \mathcal{H} \cong \mathcal{H}_A \otimes \mathcal{H}_B \) of the Fock space \( \mathcal{H} \), with the symmetric part

\[
\hat{a} = e^1 a_1 \otimes 1 + e^2 a_2 \otimes 1 + \ldots
\]

in the deformation of \( a \in A \):

\[
\hat{a} = a + e^1 a_1 \otimes b_1 + e^2 a_2 \otimes b_2 + \ldots
\]

separated of from the deformation

\[
\hat{a} = a + e^1 1 \otimes b_1 + e^2 1 \otimes b_2 + \ldots = a + e^1 B_1(a, b) + \ldots.
\]

In case the Poisson structure \( \{a, b\} \) equal to the limit of

\[
\frac{a \ast b - b \ast a}{e}
\]

as \( e \to 0 \), is regular, i.e. locally

\[
\{a, b\} = t^{ij} \partial_i a \partial_j b
\]

with the tensor \( t^{ij} \) having a constant rank all over the manifold \( M = \text{Spec } M \), i.e. when \( M \) possesses a smooth foliation into symplectic leaves, then the Fedosov necessity and sufficiency conditions for existence of the asymptotic representation are almost immediately applicable (with the only exception that instead of just the formal index of Fedosov we have the formal index for families corresponding to the smooth foliation of the manifold \( M \) into symplectic leaves). Namely one constructs the representation locally by using the symbol calculus for pseudo differential operators and then the "global" representation is defined by interlacing the local representations. The local representations define a Čech 2-cocycle on the manifold \( \text{Spec } A = M \) whose integrality as a cohomology class is the necessary and sufficient condition for the existence of the ("global") colated representation of the deformed algebra. This can be formulated in terms of the formal \( K \)-theory of the formal deformation of \( A \) and formal Fedosov integrality for the index pairing for the deformed algebra. We expect the same kind of Fedosov necessity and sufficiency conditions for the algebra of operators fulfilling (1), (2), (3') and (4') even if the splitting property \( C = A \otimes B \) with \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) does not hold. The crucial property is that \( \text{Spec } A \) possess a natural finite dimensional manifold structure, with the perturbation defining its deformation preserving locality.
The whole point lies in proving the existence of $\mathcal{A}$ or in constructing the algebra $\mathcal{A}$. Moreover the algebra $\mathcal{A}$ (before deformation) should be sufficiently rich in order to distinguish all relevant states, e.g. it should annihilate a "small" subspace of the Fock space. The best we can expect of $\mathcal{A}$ is that it annihilates only the vacuum state, but it would be still interesting if the annihilated subspace consisted say of the vacuum and one-particle states only. This is important if we want to achieve any chance for recovering the asymptotic states in order to preserve any contact with the existing interpretation in terms of particle states along the lines of the algebraic QFT (compare the Haag-Ruelle scattering theory within the algebraic QFT). In our previous paper [74] we have noticed that the algebra of space-time coordinates should be constructible (with the possible uniform infinite multiplicity) within the operator-algebra-Connes-type format in the Fock-Krein space of local free fields as a natural generalization of the so called DHR theory of global gauge groups ([35], Chap. IV), and as an answer to a problem set up by Staruszkiewicz [70]. We have noticed there that it should fulfil the requirements posed on $\mathcal{A}$ with the conditions (3') and (4') instead of (3) and (4). Strictly speaking we do not know if $\mathcal{A}$ lies in the center of $\mathcal{C}$, but we expect that linear Fréchet isomorphism $\mathcal{A} \to \mathcal{C}$ of condition (3') preserves localization, where we say an $A \in \mathcal{C}$ is a local operator localized at $\mathcal{O}$ whenever it is a function of local fields over the open region $\mathcal{O}$. In the case of the free theory the space-time structure is indeed constructible naturally in that way using the free fields and the representation of the double cover of the Poincaré group acting in the Fock-Krein space of free fields over the flat Minkowski space-time. This construction is natural and suggested by the interrelation between one-particle states and the spectrum of the translation generators. Recall that the translation generators commute with the Gupta-Bleuler operator by the very properties of free gauge fields, say the free theory underlying the QED. We construct the algebra $\mathcal{A}$ of operators and the accompanying spectral triple (the Dirac operator, the fundamental symmetry operator and the gradation, with a uniform infinite multiplicity of the whole spectral triple) in the Fock-Krein space using the method of harmonic analysis on symmetric spaces which are manifolds acted on by the regular representations (in the sense of Mackey-Loomis) of classical semi-simple Lie groups. Let us explain the construction in more detail.

Before we explain the spectral construction of space-time suggested in [74], several remarks on the Connes’ spectral format giving a structure of a pseudo-Riemann manifold to the Gelfand spectrum of a commutative pre-$C^*$-algebra $\mathcal{A}$ of operators acting in a Hilbert space $\mathcal{H}$ are in order. It has been analysed (in the compact case) by Strohmaier [72], where he recalled a result of H. Baum [1] that: 1) the Hilbert space of square integrable sections of the Clifford module naturally associated to the pseudo-Riemann structure on a (not necessary compact) orientable and time orientable pseudo-Riemann manifold admits a fundamental symmetry $\mathcal{J}$ (induced by a space like reflection) which induces in

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4We are talking here of the construction of the undeformed $\mathcal{A}$ and the the whole spectral triple structure encoding the space-time manifold structure in the Fock space of the free theory (up to an infinite uniform multiplicity).
the space of sections of the module the structure of the Krein space; 2) the natural Dirac operator $D$ associated to the module is not self-adjoint with respect to any natural Hilbert space associated to the pseudo-Riemann manifold, but it is self-adjoint in the Krein sense whenever the ordinary Riemann metric associated to the space like reflection is complete (which is automatic for compact manifolds). The main contribution of [72] lies in recognition that for the important class of fundamental symmetries $\mathcal{J}$ the operator $(\mathcal{J}D)^2 + (\mathcal{J}^\dagger)^2$ is an ordinary elliptic operator of Laplace-type with respect to a Riemann metric on the pseudo-Riemann manifold, so that a “Wick-type-rotation-procedure” using the operator $\mathcal{J}$ allows us to construct a class of ordinary riemannian spectral triples naturally associated to the pseudo-Riemann structure with respect to which the manifold is complete. Because the Krein space may be represented as an ordinary Hilbert space $\mathcal{H}$ with an operator $\mathcal{J}$ which is unitary and self-adjoint, in particular it fulfils $\mathcal{J}^2 = I$, the results of Strohmaier lie within the general scheme of introducing additional smooth structures on the manifold with the help of Connes-type-operator format proposed by Fröhlich, Grandjean and Recknagel [27]. Summing up, we have a tuple $(\mathcal{A}, D, \mathcal{H})$ acting in the Hilbert space $\mathcal{H}$ which together with a fundamental symmetry $\mathcal{J}$ composes a Krein space $(\mathcal{H}, \mathcal{J})$, the elements of the involutive algebra $\mathcal{A}$ commute with the (admissible) fundamental symmetry $\mathcal{J}$, the involution in $\mathcal{A}$ is represented by the Krein-adjoint $a^\dagger = \mathcal{J} a^{\ast} \mathcal{J}$ equal to the ordinary adjoint $a^\ast$, as $\mathcal{A}$ commutes with $\mathcal{J}$, $D$ is Krein self-adjoint: $D = \mathcal{J} D^\ast \mathcal{J}$, the operators $[D, a], a \in \mathcal{A}$ have bounded extensions; and there exists a selfadjoint operator $D_3$ whose square is equal to the positive self-adjoint operator $1/2((\mathcal{J}D)^2 + (\mathcal{J}^\dagger)^2)$ such that $(\mathcal{A}, D_3, \mathcal{H})$ composes an ordinary spectral triple fulfilling the first five conditions of Connes [10] characterizing the manifold structure spectrally. Moreover we assume the fundamental symmetry $\mathcal{J}$ to be regular, i.e. lying within the domain of any power of the derivation $\delta(\cdot) = [D_3, \cdot]$, (which by Lemma 13.2 of [10]) is equivalent to the condition that $\mathcal{J}$ lies within the domain of any power of the derivation

$$\delta_1(\cdot) = [D_3^2, \cdot](1 + D_3^2)^{-1/2},$$

with $D_3^2 = 1/2((\mathcal{J}D)^2 + (\mathcal{J}^\dagger)^2)$. Assumption of Connes [10] that $(\mathcal{A}, D_3, \mathcal{H})$ is the spectral triple which respects the first five conditions of [10], §2 in the slightly strengthened form (see the assumptions of the Reconstruction Theorem 1.1 of [10]) is

5In our previous paper we have designated the Hilbert space $\mathcal{H}$ accompanying the Krein space $(\mathcal{H}, \mathcal{J})$ and corresponding to $\mathcal{J}$ by $\delta_3$, and the Krein space $(\mathcal{H}, \mathcal{J})$ just by $\delta$. We hope this changing of notation, justified in the next section, will not cause any misunderstandings.

6With the regularity condition 3 and the orientability condition 4 fulfilled in the slightly stronger form (see [10], §2).

7Recall that the $\mathcal{J}$-modulus $|D_3|$ of $D$ in the sense of [72] is just equal $|D_3| = (D_3^2)^{1/2}$ in our notation; note also that $(D_3)^2$ in the notation of [72] is not in general equal to our $D_3$, but $((D_3)^2)^2 = D_3^2$.

8Several competitive proposals have been proposed for the spectral construction of the pseudo-Riemann manifold, e. g. Connes and Marcolli [11] proposed to consider operator $D$ which is not selfadjoint but with self-adjoint $D^2$. Indeed one can recover $\mathcal{J}$ from such $D$, but we need more structured situation like that presented here in order to have the reconstruction theorem of Connes.
crucial in order to have the reconstruction theorem of Connes applicable (a theorem conjectured in [11] and proved in [10]). Of course passing to the non compact case will involve knew difficulties, like that concerned with the appropriate choice of the unitization, however these difficulties seem to be within our reach as for the reconstruction theorem, compare e.g. [28] and [64], where the analogue of the first five conditions of [10], §2, are proposed for the non compact case. Any way we may assume for our needs that we have the preferred unitization \( \tilde{A} \) of \( A \) at hand with the reconstruction problem reduced to the unital case. Indeed we are in the situation where \( \text{Spec } \mathcal{A} \) has the natural \( \mathbb{R}^4 \)-manifold structure with the standard Lorentzian pseudo-metric tensor and with \( A \) equal to the algebra of complex smooth Schwartz functions, so that we are at the non compact analogue of the \((\Rightarrow)\)-half situation of the Theorem 11.4 of [10] and we do not need the full reconstruction theorem in order to realize our hint of [74] for the construction of the undeformed \((A, D, \mathcal{H})\). It should be stressed that the non compact version of the reconstruction theorem is important as only under its validity we have the operator-algebraic characterisation of space-time justified, and on the other hand the operator-algebra format is capable of the deformation indicated to above. But we would like to stress here that proving the reconstruction theorem is a practically independent problem and do not intervene into the operator-algebra construction of \((A, D, \mathcal{H})\) as proposed in [74] (we have to assume at this stage that the reconstruction theorem holds true for the non compact case with the first five conditions of Connes replaced appropriately with the conditions suggested in particular in [28] and [64]).

The commutative algebra \( A \) of space-time coordinates constructed within the Fock space (actually in the subspace orthogonal to the vacuum and to the one-particle states) of free fields is not a sufficiently reach algebraic structure as for the solution of the problem (b). Even having a representation of its algebraic deformation coming from the perturbation series one cannot claim that the necessity and sufficiency for the existence of a solution of the problem (b) is thus reached. We are interesting rather with the construction of the representation of the deformation of observable algebra or the Doplicher ideal contained in it, or any other subalgebra sufficient for the reconstruction of the asymptotic particle states. But instead of \( A \) we can consider a wider algebra \( \mathcal{A} \) of operators in \( \mathcal{H} \), say a maximal one, which together with the cocycle \((\mathcal{H}, D_3)\) fulfils the first five conditions of Connes for spectral manifold. In particular the example of Jaffe-Lesniewski-Weitsman spectral triple ([12], IV.9β, Proposition 14 b) ) suggest that the algebra of functions of local filed operators whose commutator with \( D_3 \) is bounded is large enough, as in our case \( D \) and \( D_3 \) are closely related to the four-momentum field operators of free fields. Indeed comparing [33] with the abstract symbol calculus of Connes and Moscovici [15] we should expect that the construction of deformation as well as the Fedosov-type conditions should extend for obtaining the necessity and sufficiency conditions for existence of the representation of the deformation of the more general algebras \( \mathcal{A} \) containing \( A \). Thus in the plan of the proposal for solving the problem (b), as proposed in [74], the construction of undeformed spectral triple \((A, D, \mathcal{H})\) in the Fock space of free fields is of fundamental importance and perhaps is the most difficult task.
in this proposal.

Let us remind the hint of [24] for the construction of (undeformed) \((\mathcal{A}, D, \mathcal{H})\). It will be convenient to recall some rudiments of harmonic analysis on smooth manifold \(\mathcal{M}\) symmetric for a regular action under a classical semi simple Lie group \(G\) as the construction is in fact an application of harmonic analysis. Suppose we have a symmetric ((uniform) smooth Riemann (or pseudo-Riemann) manifold \(\mathcal{M}\) of dimension \(n\), acted on by a Lie group \(G\) with a (pseudo-) metric tensor \(g\) invariant under \(G\). Then we consider the Hilbert space \(\mathcal{H} = L^2(\mathcal{M}, d\nu)\) of square summable functions with respect to the invariant volume form \(d\nu\). It will be convenient to recall some rudiments of harmonic analysis on smooth manifold \(\mathcal{M}\) symmetric for a regular action under a classical semi simple Lie group \(G\) as the construction is in fact an application of harmonic analysis. Suppose we have a symmetric ((uniform) smooth Riemann (or pseudo-Riemann) manifold \(\mathcal{M}\) of dimension \(n\), acted on by a Lie group \(G\) with a (pseudo-) metric tensor \(g\) invariant under \(G\). Then we consider the Hilbert space \(\mathcal{H} = L^2(\mathcal{M}, d\nu)\) of square summable functions with respect to the invariant volume form \(d\nu\). We consider then the unitary regular right representation \(T\) of \(G\) acting in \(\mathcal{H}\) and an appropriate algebra \(\mathcal{A} = \mathcal{S}(\mathcal{M})\) of functions of fast decrease with nuclear Fréchet topology (just \(\mathcal{A} = C^\infty(\mathcal{M})\) for compact \(\mathcal{M}\)). We can consider the algebra \(\mathcal{S}(\mathcal{M})\) as acting in \(\mathcal{H}\) as a multiplication algebra with point wise multiplication. The regular representation \(T\) induces the transformation \(a \mapsto T_g a T_g^{-1}\) which coincides with the ordinary group action \(T_g a T_g^{-1}(x) = a(xg)\) for functions \(a \in \mathcal{S}(\mathcal{M})\). Harmonic analysis ("Fourier transform" on \(\mathcal{M}\)) corresponds to a decomposition of the regular right representation \(T\) acting in \(\mathcal{H}\) into direct integral of irreducible subrepresentations. To this decomposition there corresponds a decomposition of every element \(f \in \mathcal{H}\) into direct integral of its components belonging to the irreducible generalized proper subspaces of the Laplacian — the "inverse Fourier integral of \(f\)". For example Gelfand, Graev and Vilenkin [31] has done it for the Lobachevsky space \(\mathcal{M} = L^3\) acted on by the \(G = SL(2, \mathbb{C})\) and have constructed the appropriate algebra \(\mathcal{S}(\mathcal{M})\). It is important for us that in general the construction of harmonic analysis together with \(\mathcal{S}(\mathcal{M})\) can be given a purely operator-spectral shape. Namely we consider a maximal commutative algebra \(\hat{\mathcal{A}}\) generated by representors of one parameter subgroups (or their appropriate functions). Let \(\hat{\mathcal{A}}\) be generated by \(P_1, P_2, \ldots, P_n\). Let \(\text{sp}(P_1, P_2, \ldots, P_n)\) be their joint spectrum. In particular for the Lobachevsky plane \(\mathcal{M} = L^2\) acted on by the \(G = SL(2, \mathbb{R})\) group we may chose \(P_1\) to be the Casimir operator equal to the Laplacian on the Lobachevsky plane, and for the \(P_2\) we may chose the generator of a one parameter boost subgroup. In this case the inverse Fourier transform and the Fourier transform relating \(f \in \mathcal{H}\) and its Fourier transform \(\mathcal{F}f\) may be written as

\[
\int_{\text{sp}(P_1, P_2, \ldots, P_n)} \mathcal{F}f(s) \Theta(x, s) d\nu(s); \quad \mathcal{F}f = \int_{\mathcal{M}} f(x) \Theta(x, s) d\nu(x),
\]

where \(\Theta(\cdot, s)\) is a complete set of common generalized proper functions of the operators \(P_1, P_2, \ldots, P_n\) corresponding to the point \(s\) of their joint spectrum \(\text{sp}(P_1, P_2, \ldots, P_n)\). In fact the Fourier transform of [31] does not have this spectral form because the full (maximal set of commutative) generators \(P_1, P_2, \ldots\) (or their functions) have not been explicitly constructed (besides the Laplacian),
which are simultaneously diagonalized by the Fourier transform constructed there. However existence of Fourier transforms diagonalizing say the Laplacian on the Lobachevsky plane $L^2$ and the generator of a one parameter boost subgroup of $SL(2, \mathbb{R})$ follows from the general theory presented in [32], [31] (as well as from the papers of Harish-Chandra on harmonic analysis). Thus the Fourier transform diagonalizes the algebra of operators $\hat{A}$ and the inverse Fourier transform diagonalizes the algebra $A = S(M)$. In this sense the algebras $A$ and $\hat{A}$ are dual to each other. Note in passing that whenever the commutative algebra $\hat{A}$ is not maximal commutative in the algebra generated by generators of one parametric subgroups, then the subrepresentations in the direct integral decomposition of $T$ need not be irreducible, as is the case for example for the double covering of the Poincaré group with $P_1, \ldots, P_4$ equal to the translation generators, where we have two Casimir generators and one of which is not a function of $P_1, \ldots, P_n$.

Now suppose that $T$ is the (Krein-)isometric representation $T$ of the double covering of the Poincaré group $G$ acting in the Krein-Fock space $\mathcal{H}$ of the free theory underlying QED or more general gauge field. In this case we may repeat the above construction of (space-time) algebra $A = S(M)$ of functions now understood as operators in the appropriate subspace of the Fock space provided the algebra $\hat{A}$ generated by Schwartz functions of generators $P_1, \ldots, P_4$ acts with uniform multiplicity in the subspace. It is the case for the subspace of the Fock space orthogonal to the vacuum and to the one particle states. Indeed one can prove that in this case the joint spectral measure on the joint spectrum $\text{sp}(P_1, \ldots, P_4) \cong \text{Spec} \hat{A}$ is the Lebesgue measure in case of free fields but moreover the theory of quantum fields, and especially the theory of free quantum fields, is accompanied with a much stronger assumption (which is not always explicitly stated), that the joint spectrum of the translation generators is a subset of the smooth Minkowski space with the pseudo-Riemann (Lorentzian) structure. In case of the free fields underlying QED it is equal to the closed forward cone. In fact the "positivity of energy" assumption of the Wightmann axioms [71] would be meaningless if one would not be able to introduce the Lorentzian manifold structure with the $\text{sp}(P_1, \ldots, P_4)$ embedded into the manifold. This assumption is of much profound character then may apparently seem at first sight. In particular it enters non trivially into the definition of normal ordering of operator valued distributions, compare e.g. the Bogoliubov analysis of the Wick product theorem as well as the recurrent construction of the so called chronological products as operator valued distributions. Thus in order to construct the algebra $A$ we need to know the "Fourier transform $\mathcal{F}$" connecting it with the algebra of functions of the generators $P_1, \ldots, P_4$ on the subspace orthogonal to the vacuum and one particle states. Now it is important that we have fairly explicitly given the "Fourier transform $\mathcal{F}$" and it is suggested by the relation between the one particle states in the momentum and position representations. We extract subspaces of the Fock space where $\hat{A}$ acts with uniform multiplicity and invariant for $T$. Then $\mathcal{F}$ is achieved in two steps: i) using the uniformity of the algebra $\hat{A}$ we construct the Foldy-Wouthuysen transformation $\mathcal{F}_1$ which after
the second step ii) namely the construction of the von Neumann-Stone representation of the canonical pairs \((P_i, Q_i)\) giving the construction of the ordinary Fourier transform \(F_2\) (with uniform multiplicity), allows us to apply \(F_2 \circ F_1\), and gives a local transformation rule \(T_g a T_g^{-1}(x) = a(xg)\) for \(x \in \text{Spec } A\). We thus construct \(A\) as \(U_{\mathcal{F}} \hat{A} U_{\mathcal{F}}^{-1}\), where \(U_{\mathcal{F}}\) is the unitary transform defined by the composition \(F_2 \circ F_1\).

Let us remind that the irreducible representations (and their direct sums) acting in the subspaces of one particle states compose the so called Mackey’s systems of imprimitivity over the corresponding orbits in \(\text{sp}(P_1, \ldots, P_4)\) and the representations \(T\) are the respective sums of their symmetrized/antisymmetrized tensor products, which likewise compose systems of imprimitivity (but no longer corresponding to ergodic orbits). The vector states under the ordinary Fourier transform do not transform locally but formerly need to be transformed appropriately in order to transform locally after the application of the ordinary Fourier transform. This additional transformation (explicitly known for all irreducible positive mass and arbitrary spin representations) is sometimes called Foldy-Wouthuysen transformation. We adopt this terminology here. Perhaps we should remind that for zero-mass representations which act in one particle states of the Fock-Krein space (Gupta-Bleuler space in physics parlance) of the free photon field are not localized in the sense that they do not allow position measurement operator such as for one particle states in non relativistic quantum mechanics. But here we are talking about the full four dimensional Fourier transformation involving integration regions in \(\text{sp}(P_1, \ldots, P_4)\) intersecting many independent ergodic orbits corresponding to irreducible representations with the only requirement of locality for the transformed \(U_{\mathcal{F}} w\) vector states \(w \in \mathcal{H}\), i.e. we require the rule \(T_g a T_g^{-1}(x) = a(xg)\) for \(x \in \text{Spec } A\) to be fulfilled, compare the discussion in \([74]\).

Thus we need to know the action of \(T\) in the Fock space as explicitly as possible. In particular we may transform the Minkowski manifold structure expressed in the algebra-operator format from the joint spectrum \(\text{sp}(P_1, \ldots, P_4)\) over to spectrum \(\text{Spec } A = \text{Spec } U_{\mathcal{F}} \hat{A} U_{\mathcal{F}}^{-1}\) with the help of the unitary transformation \(U_{\mathcal{F}}\).

Two basic difficulties arise in the realization of this plan (proposed in \([74]\)). Namely

\(\alpha\) The joint spectrum \(\text{sp}(P_1, \ldots, P_4)\) restricted to the subspace orthogonal to the vacuum and to the one particle states, where \(\text{sp}(P_1, \ldots, P_4)\) is of uniform infinite multiplicity, is not the Minkowski manifold \(\mathbb{R}^4\) but a subset of it equal to the forward cone (e. g. for free gauge fields underlying QED).

\(\beta\) One needs to have an effective theory of decompositions of Krein-isometric representations \(T\) in Krein spaces encountered here, which as yet has not been developed. In fact one needs to have a generalization of Mackey theory of induced representations in Krein spaces in order to have decomposition theory of their tensor products, and thus to have decompositions
of representations $T$ of the double covering of the Poincaré group acting
in the Fock-Krein space.

In order to resolve the problem $\alpha$) let us recall the following observation
(in fact of a rather folkloric character among the community of mathematical
physicists working in QFT, but frequently erroneously understood by those
not working in mathematical aspects of QFT): in construction of the free QFT
(underlying say QED) and its causal perturbation method of introducing in-
teraction one may replace the “positive energy axiom” by the “negative energy
axiom” with the consequently replaced signs in both the lagrangians of free
fields and in the interaction lagrangian, in the commutators and with replace-
ment of advanced into retarded functions and vice versa. That in this case the
consequent Wick product theorem and the perturbation series may still be
constructed on essentially the same grounds as in the “positive energy theory”
has already been noted e.g. by Bogoliubov and Shirkov. In particular that
the supports of the distributions in the commutators of free fields still allow in
this case the construction of the Wick product has been noted in §II.16.9

Checking that the signs may consequently be changed involves a rather tedious
but essentially simple inspection, which we leave to the reader. Let us denote
the Fock space of the underlying free fields with the “positive energy axiom”
by $\mathcal{H}_+$ and respectively $\mathcal{H}_-$ for the free theory with the “negative energy
axiom”. Now we treat the different sign fields as independent fields which do not
interact and are represented by operator valued distributions ( operator fields
or operator algebras $\mathcal{F}(O)_+$ or $\mathcal{F}(O)_-$ resp.) acting in the tensor product
space $\mathcal{H}_+ \otimes \mathcal{H}_-$. Then we apply the deformation to each version (positive and
negative) separately in order to obtain the perturbation series and deformation
of the composed system $\mathcal{F}(O)_+ \otimes \mathcal{F}(O)_-$ (we may apply for simplicity just the
trivial deformation to the negative energy fields). At the end we may restrict
the allowed states to the positive energy states. From the physical point of view
this introduces nothing essentially new as the positive energy fields do not in-
teract with the negative ones (even the negative energy fields may have to be
chosen free), but for us this gives a considerable profit. It comes from the fact
that $sp(P_1, \ldots, P_4)$ in the subspace orthogonal to the vacuum and one particle
states in $\mathcal{H}_+ \otimes \mathcal{H}_-$ is not only of uniform constant multiplicity but it is the
Minkowski manifold. In fact the last assertion need to be proved and is in fact
reduced to the problem of decomposition of tensor product of ordinary unitary
induced representations of the double covering $T_4 \otimes SL(2, \mathbb{C})$ of the Poincaré
group, compare Remark [S of Sect. IK]

Concerning the problem $\beta$) to the solution of which this paper is largely de-
voted we need to know how to decompose the tensor products of representations
acting in the one particle states subspace of $\mathcal{H}_+ \otimes \mathcal{H}_-$. In particular for the

\footnote{Frequently repeated claim that in QFT positivity of energy is a necessary condition is
not strictly true. Taking into account the perturbative microlocal method in the standard
gauge field theory we may only say that “definiteness of energy sign axiom” is needed in order
to build the theory, but instead of “positive energy” one may equally use the opposite sign
version. Of course from the physical point of view the difference is rather of unimportant and
nomenclatural character.}
free field underlying QED (in both versions of the energy sign) we need to consider the two conjugate Majorana irreducible representations, both in positive and negative energy versions, whose direct sum acts in the one particle Hilbert space of the free electron-positron field. Besides, and this is is the whole point, we need to consider the Lopuszański representation acting in one-particle states of the free photon field in the Krein space, i.e. in the Hilbert space with the fundamental symmetry (Gupta-Bleuler) operator. The last representation is not unitary in the respective Hilbert space but only Krein-isometric. Here the whole problem arises, as no effective theory of tensor product construction and decomposition of tensor products of such representations has so far been developed. Similarly for the ghost fields (redundant for QED) the representations acting in particle states are not unitary but Krein-isometric. In order to solve the problem of decompositions of tensor products of such representations we observe first that the translation operators (or the normal translation subgroup of the double covering of Poincaré group) commutes with the Gupta-Bleuler operator, so that \( P_1, \ldots, P_4 \) are self-adjoint in the usual sense and the Lopuszański-type representations encountered in QFT compose a generalized systems of imprimitivity which are Krein-unitary equivalent to induced Krein-isometric representations in Krein spaces. Next we observe that the main theorems of the Mackey theory, namely the Subgroup Theorem and the Kronecker Product Theorem may as well be proved to hold (with unimportant changes) for the induced representations in Krein spaces encountered in QFT of gauge fields. So in this paper we present a rather detailed proofs of the Subgroup Theorem and the Kronecker Product Theorem for the induced representations in Krein spaces, and thereby generalize the known and classical theory of Mackey. These two theorems (in both versions – the orginal Mackey version and in the version presented in this paper) are the fundamental tools used repeatedly in proving that the algebra \( A \) generated by Schwartz-type functions of generators of translations \( P_1, \ldots, P_4 \) is of uniform (infinite) multiplicity in the subspace of \( \mathcal{H}_+ \otimes \mathcal{H}_- \) orthogonal to the vacuum and to the one particle states, and in decomposition of the Krein-isometric representation \( T \) acting in \( \mathcal{H}_+ \otimes \mathcal{H}_- \) into indecomposable components, i.e. in the construction of the undeformed \((A, \mathcal{H} = \mathcal{H}_+ \otimes \mathcal{H}_-, D_3)\). Thus decomposition of tensor product of induced representations acting in one particle subspaces as well as decomposition of \( T \) is largely reduced, via a generalized Mackey theory, to geometry of cosets and double cosets in the (double cover of) Poincaré group and to the Fubini-type theorem. The detailed proof of uniform multiplicity theorem we hope to present in a separate paper.

We have tried to be careful and have presented all details of the proof for several reasons. Although conceptually the general line of the proof resemblance that of Mackey [47] (and it was in fact the inspiration for our proof) there are several and important differences. The Krein-isometric induced representations which are Krein-unitary equivalent to the representations (imprimitivity systems) encountered in QFT are unbounded, and require a special care in the definition of the Kronecker product and moreover has to contain analytic subtleties which could have been omitted in the original Mackey theory. The other
difference is that we exploit (and prove) a decomposition/disintegration theorem for measures which are not finite (Mackey could have confined himself to finite ones, which is not the case for our representations), which makes the proof much longer in comparison to Mackey’s proof. We insisted here to stay with the level of generality the same as the original Mackey theory and generalize his theory with exactly the same assumptions posed on the group which Mackey did. However confining the theorems to the Poincaré group would not be very economical and perhaps this restriction would simplify our proof only in its part concerning decomposition of measures (in that case we could use a modification of the decomposition presented in the Bourbaki course – but still with some important modifications which still took a considerable place). Thus our paper reaches in this way an independent character as a generalization of the Mackey theory of induced representations in Krein spaces, which need not be tied to the representation problem in algebraic perturbative QFT outlined in this introduction.

Several remarks are in order:

1) That the $K$-theory may be extended to the case of algebraic deformations of the algebra of smooth (or Schwartz) functions on symplectic (or Poisson) manifolds have been noted by several authors and extensively studied e.g. by Fedosov [25]. That the Chern character may be extended to the case of algebraic deformations have been noted by Connes, Flato and Sternheimer [13]. It was also recognized by several authors (compare e.g. [12], IV.D) that this construction is not confined to deformations of a commutative $\mathcal{A}$.

2) The operators $D$ and $D_J$ are closely related to the four-momentum field operators. Thus comparing to the Jaffe-Lesniewski-Weitsman (JLW) spectral triple $(\mathcal{A}(\mathcal{O}), D, \mathcal{H})$ corresponding to the free Wess-Zumino model (compare [38], or [12], IV.9 β) one perhaps will be surprised with our construction of the space-time spectral triple. But two points has to be noted. First the cycle $(\mathcal{H}, D)$ of the JLW spectral triple indeed is not finitely summable (it has infinite spectral dimension), with $D$ being equal to the square root of the free Hamiltonian (time-component of the four-momentum of the free field), but we do not expect finite summability of our $(\mathcal{H}, D_3)$ but only the “local version” of the finite summability, i.e. the condition 1 of Def 3.1 of [28]. Second: it is true that this argument does not change the fact that we cannot expect the cohomological dimension of the algebra of local free fields to be finite, as suggest the same example of JLW spectral triple $(\mathcal{A}(\mathcal{O}), D, \mathcal{H})$: the index map given by $D$ cannot be represented by a cyclic cocycle (compare [12], IV.9β for detailed formulation). Here as well as in the spectral dimensionality, ”locality” of the condition of finite cohomological dimensionality intervenes: an algebra $\mathcal{A}$ smaller then the algebra generated by functions of local field operators remains within our reach. But on the other hand the Haag-Swieca compactness condition (or a much stronger Buchholz-Wichmann nuclearity condition, [34], V.5.1.1, V.5.2 fulfilled e.g. by free field theories) shows that a substantially smaller type I algebra then the one $\mathcal{A}(\mathcal{O})$ generated by functions of local field
operators will be sufficient for the construction of the asymptotic particle states provided some non-zero inaccuracy in the detection of the total energy in the detector region is allowed, compare [35], V (the “split property”) and VI.1.

Thus the representation problem (b) have a chance to be solved within our method but only in a somewhat weakened form. We cannot likewise exclude today that the solution of the problem (b) proposed in [74] depends on the energy scale. But this would be not very surprising in view of the running coupling constant phenomenon depending on the energy scale and the renormalization group related to it. Note in passing that the Buchholz-Wichmann nuclearity is a necessary and sufficient condition for the existence of physically important states in algebraic QFT, in particular for the existence of KMS-states for all positive inverse temperatures $\beta$, compare [35], V.5.1 and references there in.

3) Recall that although the Krein-isometric representation $T$ of $T_4 \otimes SL(2, \mathbb{C})$ is unbounded, it behaves regularly with respect to the inner product of the physical Hilbert space, equal to the kernel of the Kugo-Ojima operator modulo its image, [18], as the representation $T$ preserves the indefinite Krein-inner-product in the Fock-Krein space.

2 Preliminaries

It should be stressed that the analysis we give here is inapplicable for general linear spaces with indefinite inner product. We are concerned with non-degenerate, decomposable and complete inner product spaces in the terminology of [4], which have been called Krein spaces in [18], [72], [4] and [74] for the reasons we explain below. They emerged naturally in solving physical problems concerned with quantum mechanics ([16], [61]) and quantum field theory ([34], [3]) in quantization of electromagnetic field and turned up generally to be very important (and even seem indispensable) in construction of quantum fields with non-trivial gauge freedom. Similarly we have to emphasize that we are not dealing with general unitary (i. e. preserving the indefinite inner product in Krein space) representations of the double cover $\mathcal{G} = T_4 \otimes SL(2, \mathbb{C})$ of the Poincaré group, but only with the exceptional representations of Lopuszański-type, which naturally emerge in construction of the free photon field, which have a rather exceptional structure of induced representations, and allow non-trivial analytic constructions of tensoring and decomposing, which is truly exceptional among Krein-unitary (preserving the indefinite product) representations in Krein spaces.

The non-degenerate, decomposable and complete indefinite inner product space $\mathcal{H}$, hereafter called Krein space, may equivalently be described as an ordinary Hilbert space $\mathcal{H}$ with an ordinary strictly positive inner product $\langle \cdot, \cdot \rangle$, together with a distinguished self-adjoint (in the ordinary Hilbert space sense) fundamental symmetry (Gupta-Bleuler operator) $\mathcal{J} = P_+ - P_-$, where $P_+$ and $P_-$ are ordinary self-adjoint (with respect to the Hilbert space inner product $\langle \cdot, \cdot \rangle$) projections such that their sum is the identity operator: $P_+ + P_- = I$.

The indefinite inner product is given by $\langle \cdot, \cdot \rangle_\mathcal{J} = \langle \mathcal{J} \cdot, \mathcal{J} \cdot \rangle = \langle \cdot, \mathcal{J} \cdot \rangle$. Recall that in our previous paper [74] the indefinite product was designated by $\langle \cdot, \cdot \rangle$ and the
ordinary Hilbert space inner product associated with the fundamental symmetry \( J \) was designated by \((\cdot, \cdot)_J\). The indefinite and the associated definite inner product play symmetric roles in the sense that one may start with a fixed indefinite inner product in the Krein space and construct the Hilbert space associated with an admissible fundamental symmetry, or vice versa: one can start with a fixed Hilbert space and for every fundamental symmetry construct the indefinite inner product in it, both approaches are completely equivalent provided the fundamental symmetry being admissible (in the sense of [72]) and fixed. We hope the slight change of notation will not cause any serious misunderstandings and is introduced because our analytical arguments will be based on the ordinary Hilbert space properties, so will frequently refer to the standard literature on the subject, so we designated the ordinary strictly definite inner product by \((\cdot, \cdot)\) which is customary.

Let an operator \( A \) in \( \mathcal{H} \) be given. The operator \( A^\dagger \) in \( \mathcal{H} \) is called Krein-adjoint of the operator \( A \) in \( \mathcal{H} \) in case it is adjoint in the sense of the indefinite inner product:
\[
(Ax, y)_J = (JAx, y) = (Jx, A^\dagger y) = (x, A^\dagger y)_J
\]
for all \( x, y \in \mathcal{H} \), or equivalently
\[
A^\dagger = JA^*J,
\]
where \( A^* \) is the ordinary adjoint operator with respect to the definite inner product \((\cdot, \cdot)_J\), e. g. \((Ux, y)_J = (x, y)_J\) for all \( x, y \in \mathcal{H} \) (same for \( U^{-1} \)), equivalently \( UU^\dagger = U^\dagger U = I \), will also be called unitary (sometimes 3-unitary or Krein-unitary) trusting to the context or explanatory remarks to make clear what is meant in each instance: unitarity for the indefinite inner product or the ordinary unitarity for the strictly definite Hilbert space inner product.

In particular we may consider \( J \)-symmetric representations \( x \mapsto A_x \) of involutive algebras, i. e. such that \( x^* \mapsto A_x^\dagger \), where \( (\cdot) \mapsto (\cdot)^* \) is the involution in the algebra in question. A fundamental role for the spectral analysis in Krein spaces is likewise played by commutative (Krein) self-adjoint, or \( J \)-symmetric weakly closed subalgebras. However their structure is far from being completely described, with the exception of the special case when the rank of \( P_+ \) or \( P_- \) is finite dimensional (here the analysis is complete and was done by Neumark). Even in this particular case a unitary representation of a separable locally compact group in the Krein space, although reducible, may not in general be decomposable, compare [53, 54, 55, 56].

In case the dimension of the rank \( \mathcal{H}_+ = P_+ \mathcal{H} \) or \( \mathcal{H}_- = P_- \mathcal{H} \) of \( P_+ \) or \( P_- \) is finite we get the spaces analysed by Pontrjagin, Krein and Neumark, compare e. g. [62, 40] and the literature in [4].

The circumstance that the Krein space may be defined as an ordinary Hilbert space with a distinguished non-degenerate fundamental symmetry (or Gupta-Bleuler operator) \( J = J^*, J^2 = I \) in it, say a pair \((\mathcal{H}, J)\), allows us to extend the fundamental analytical constructions on a wide class of induced Krein-isometric representations of \( \mathfrak{g} = T_4 \otimes SL(2, \mathbb{C}) \) in Krein spaces. In particular we may define a Krein-isometric representation of \( T_4 \otimes SL(2, \mathbb{C}) \) induced by a Krein-unitary representation of a subgroup \( H \) corresponding to a particular class of \( SL_2, \mathbb{C} \)-orbits in on the dual group \( \hat{T}_4 \) of \( T_4 \) (in our case we consider the class corresponding to the representation with the spectrum of the four-momenta con-
centrated on the “light cone”) word for word as in the ordinary Hilbert space by replacing the representation of the subgroup $H$ by a Krein-unitary representation $L$ in a Krein space $(\mathcal{H}_L, \mathcal{J}_L)$. This leads to a Krein-isometric representation $U^L$ in a Krein space $(\mathcal{H}_L, \mathcal{J}_L)$ (see Sect. 5). Application of Lemma 7 of Section 5 leads to the ordinary direct integral $\mathcal{H} = \int_{\mathfrak{g}/H} \mathcal{H}_q \, d\mu_{\mathfrak{g}/H}(q)$ of Hilbert spaces $\mathcal{H}_q = \mathcal{H}_L$ over the coset measure space $\mathfrak{g}/H$ with the measure induced by the Haar measure on $\mathfrak{g}$. One obtains in this manner the Krein space $(\mathcal{H}, U^{-1}U^L)$ given by the ordinary Hilbert space $\mathcal{H}$ equal to the above mentioned direct integral of the fundamental symmetry $\mathcal{J} = U^{-1}U^L$ equal to the ordinary direct integral $\int_{\mathfrak{g}/H} \mathcal{J}_q \, d\mu_{\mathfrak{g}/H}(q)$ of fundamental symmetries $\mathcal{J}_q = \mathcal{J}_L$ as operators in $\mathcal{H}_q = \mathcal{H}_L$ and with the representation $U^{-1}U^L$ of $\mathfrak{g}$ in the Krein space $(\mathcal{H}, \mathcal{J})$ (and $U$ given by a completely analogous formula as that in Lemma 7 of Section 5) of Wigner’s form \cite{78} (imprimitivity system).

This is the case for the indecomposable (although reducible) representation of $\mathfrak{g} = T_4 \otimes SL(2, \mathbb{C})$ constructed by Lopuszański with $H = T_4 \cdot G_X$, with the "small" subgroup $G_X \cong \tilde{E}_2$ of $SL(2, \mathbb{C})$ corresponding to the “light-cone” orbit in the spectrum of four-momenta operators. One may give to it the form of representation $U^{-1}U^L$ equivalent to an induced representation $U^L$, because the representors of the normal factor (that is of the translation subgroup $T_4$) of the semidirect product $T_4 \times SL(2, \mathbb{C})$ as well as their generators, i.e. four-momentum operators $P_0, \ldots, P_3$, commute with the fundamental symmetry $\mathcal{J} = \int_{\mathfrak{g}/H} \mathcal{J}_q \, d\mu_{\mathfrak{g}/H}(q)$, so that all of them are not only $\mathcal{J}$-unitary but unitary with respect to the ordinary Hilbert space inner product (and their generators $P_0, \ldots, P_3$ are not only Krein-self-adjoint but also self-adjoint in the ordinary sense with respect to the ordinary definite inner product of the Hilbert space $\mathcal{H}$), so the algebra generated by $P_0, \ldots, P_3$ leads to the ordinary direct integral decomposition with the decomposition corresponding to the ordinary spectral measure, contrary to what happens for general Krein-selfadjoint commuting operators in Krein space $(\mathcal{H}, \mathcal{J})$ (for details see Sect. 5). This in case of $\mathfrak{g} = T_4 \otimes SL(2, \mathbb{C})$, gives to the representation $U^{-1}U^L$ of $\mathfrak{g}$ the form of Wigner \cite{78} (viz. a system of imprimitivity in mathematicians’ parlance) with the only difference that $L$ is not unitary but Krein-unitary in $(\mathcal{H}_L, \mathcal{J}_L)$.

Another gain we have thanks to the above mentioned circumstance is that we can construct tensor product $(\mathcal{H}_1, \mathcal{J}_1) \otimes (\mathcal{H}_2, \mathcal{J}_2)$ of Krein spaces $(\mathcal{H}_1, \mathcal{J}_1)$ and $(\mathcal{H}_2, \mathcal{J}_2)$ as $(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{J}_1 \otimes \mathcal{J}_2)$ where in the last expression we have the ordinary tensor products of Hilbert spaces and operators in Hilbert spaces (compare Sect. 5).

Similarly having any two such $(\mathcal{J}_1$- and $\mathcal{J}_2$-)isometric representations $U^L$ and $U^M$ induced by $(\mathcal{J}_1$ and $\mathcal{J}_M$-unitary) representations $L$ and $M$ of subgroups $G_1$ and $G_2$ in Krein spaces $(\mathcal{H}_L, \mathcal{J}_L)$ and $(\mathcal{H}_M, \mathcal{J}_M)$ respectively we may construct the tensor product $U^L \otimes U^M$ of Krein-isometric representations in the tensor product Krein space $(\mathcal{H}_1, \mathcal{J}_1) \otimes (\mathcal{H}_2, \mathcal{J}_2)$, which is likewise $(\mathcal{J}_1 \otimes \mathcal{J}_2)$-isometric. It turns out that the Kronecker product $U^L \times U^M$ and $U^L \times U^M$ are (Krein-)unitary equivalent (see Sect. 5) as representations of $\mathfrak{g} \otimes \mathfrak{g}$. Because the tensor product
$U^L \otimes U^M$ as a representation of $\mathcal{G}$ is the restriction of the Kronecker product $U^L \times U^M$ to the diagonal subgroup of $\mathcal{G} \times \mathcal{G}$ we may analyse the representation $U^L \otimes U^M$ by analysing the restriction of the induced representation $U^{L \times M}$ to the diagonal subgroup exactly as in the Mackey theory of induced representations in Hilbert spaces. Although in general for $J_1 \otimes J_2$-unitary representations in Krein space $(\mathcal{H}_1, J_1) \otimes (\mathcal{H}_2, J_2)$ ordinary decomposability breaks down, we can nonetheless still decompose the representation $U^{L \times M}$ restricted to the diagonal into induced representations which, by the above mentioned Krein-unitary equivalence, gives us a decomposition of the tensor product representation $U^L \otimes U^M$ of $\mathcal{G}$. Indeed, in turns out that the whole argument of Mackey [47] preserves its validity and effectiveness in the construction of decomposition of tensor product of induced representations for the case in which the representations $L$ and $M$ of the subgroups $G_1$ and $G_2$ are replaced with (specific) unitary (or $J_L$- and $J_M$-unitary) representations in Krein spaces $(\mathcal{H}_L, J_L)$ and $(\mathcal{H}_M, J_M)$ respectively. We give details on the subject below in Section 9. Because the Lopuszański representation is (Krein-unitary equivalent to) an induced representation in a Krein space (Sect. 5), we can decompose the tensor product of Lopuszański representations. The specific property of the group $T_4 \circ SL(2, \mathbb{C})$ is that this decomposition may be performed explicitly into indecomposable sub-representations.

**REMARK 1.** Let us emphasize that in this paper “continuity”, “density”, “boundedness”, and other standard analytic notions, as the “closure of a densely defined operator” or “weak” or “strong” topologies in the algebra of bounded operators, refer to the ordinary Hilbert space norm and definite Hilbert space inner product in $\mathcal{H}$ of the Krein space $(\mathcal{H}, J)$ in question. We are mainly concerned with the Lie group $\mathcal{G} = T_4 \circ SL(2, \mathbb{C})$ but the general theory of induced representations in Krein spaces presented here is valid for general separable locally compact topological groups $\mathcal{G}$. Thus separability and local compactness of $\mathcal{G}$ is assumed to be valid throughout the whole paper whenever the identification $\mathcal{G} = T_4 \circ SL(2, \mathbb{C})$ is not explicitly stated.

3 Definition of the induced representation $U^L$ in Krein space $(\mathcal{H}^L, J^L)$

Here by a Krein-unitary and strongly continuous representation $L : G \ni x \mapsto L_x$ of a separable locally compact group $G$ we shall mean a homomorphism of $G$ into the group of all (Krein-)unitary transformations of some separable Krein space $(\mathcal{H}_L, J_L)$ (i. e. with separable Hilbert space $\mathcal{H}_L$) onto itself which is:

(a) Strongly continuous: for each $v \in \mathcal{H}_L$ the function $x \mapsto L_xv$ is continuous with respect to the ordinary strictly definite Hilbert space norm $\|v\| = \sqrt{(v, v)}$ in $\mathcal{H}_L$.

(b) Almost uniformly bounded: there exist a compact neighbourhood $V$ of unity $e \in G$ such that the set $\|L_x\|, x \in V \subset G$ is bounded or, what is
the same thing, that the set $\| L_x \|$ with $x$ ranging over a compact set $K$ is bounded for every compact subset $K$ of $G$.

Because the strong operator topology in $B(H_L)$ is stronger than the weak operator topology then for each $v, \varphi \in H_L$ the function $x \mapsto (L_x v, \varphi)$ is continuous on $G$. One point has to be noted: because the range and domain of each $L_x$ equals $H_L$, which as a Krein space $(H_U, \mathfrak{J}_U)$ is closed and non-degenerate, then by Theorem 3.10 of [4] each $L_x$ is continuous i.e. bounded with respect to the Hilbert space norm $\| \cdot \|$ in $H_L$, and each $L_x$ indeed belongs to the algebra $B(H_L)$ of bounded operators in the Hilbert space $H_L$, which is non-trivial as an $\mathfrak{J}_L$-isometric densely defined operator in the Krein space $(H_L, \mathfrak{J}_L)$ may be discontinuous, as we will see in this Section, compare also [4]). We also could immediately refer to a theorem which says that Krein-unitary operator is continuous, i.e. Hilbert-space-norm bounded (compare Theorem 4.1 in [4]).

Besides in this paper will be considered a very specific class of Krein-isometric representations $U$ of $\mathfrak{G}$ in Krein spaces, to which the induced representations of $\mathfrak{G}$ in Krein spaces, hereby defined, belong. Namely here by a Krein-isometric and strongly continuous representation of a separable locally compact group $\mathfrak{G}$ we shall mean a homomorphism $U : \mathfrak{G} \ni x \mapsto U_x$ of $\mathfrak{G}$ into a group of Krein-isometric and closable operators of some separable Krein space $(H, \mathfrak{J})$ with dense common domain $\mathfrak{D}$ equal to their common range in $H$ and such that

$U$ is strongly continuous on the common domain $\mathfrak{D}$: for each $f \in \mathfrak{D} \subset H$ the function $x \mapsto U_x f$ is continuous with respect to the ordinary strictly definite Hilbert space norm $\| f \| = \sqrt{\langle f, f \rangle}$ in $H$.

Let $H$ be a closed subgroup of a separable locally compact group $\mathfrak{G}$. In the applications we have in view\footnote{E. g. in decomposing tensor products of the representations of the double cover $\mathfrak{G}$ of the Poincaré group in Krein spaces encountered in QFT} the right $H$-cosets, i.e. elements of $\mathfrak{G}/H$, are exceptionally regular, and have a “measure product property”. Namely every element (with a possible exception of a subset of $\mathfrak{G}$ of Haar measure zero) $g \in \mathfrak{G}$ can be uniquely represented as a product $g = h \cdot q$, where $h \in H$ and $q \in Q \cong \mathfrak{G}/H$ with a subset $Q$ of $\mathfrak{G}$ which is not only measurable but, outside a null set, is a sub-manifold of $\mathfrak{G}$, such that $\mathfrak{G}$ is the product $H \times \mathfrak{G}/H$ measure space, with the regular Baire measure space structure on $\mathfrak{G}/H$ associated to the canonical locally compact topology on $\mathfrak{G}/H$ induced by the natural projection $\pi : \mathfrak{G} \mapsto \mathfrak{G}/H$ and with the ordinary right Haar measure space structure $(H, \mathcal{B}_H, \mu_H)$ on $H$, which is known to be regular with the ring $\mathcal{B}_H$ of Baire sets\footnote{We will need the complete measure spaces on $\mathfrak{G}, \mathfrak{G}/H$ but the Baire measures are sufficient to generate them by the Carathéodory method, because we have assumed the topology on $\mathfrak{G}$ to fulfil the second axiom of countability.}. In short $(\mathfrak{G}, \mathcal{B}_\mathfrak{G}, \mu) = (H \times \mathfrak{G}/H, \mathcal{B}_H \times \mathfrak{J}_H, \mu_H \times \mu/\mathfrak{J}_H)$. In our applications we are dealing with pairs $H \subset \mathfrak{G}$ of Lie subgroups of the double cover $T_4 \odot SL(2, \mathbb{C})$ of the Poincaré group $\mathfrak{P}$ including the group $T_4 \odot SL(2, \mathbb{C})$ itself, with a sub-manifold structure of $H$ and $Q \cong \mathfrak{G}/H$. This opportunities allow us to reduce the analysis of the induced representation $U^L$ in the Krein
space defined in this Section to an application of the Fubini theorem and to the von Neumann analysis of the direct integral of ordinary Hilbert spaces. (The same assumption together with its analogue for the double cosets in \(G\) simplifies also the problem of decomposition of tensor products of induced representations of \(G\) and reduces it mostly to an application of the Fubini theorem and harmonic analysis on the "small" subgroups: namely at the initial stage we reduce the problem to the geometry of right cosets and double cosets with the observation that Mackey’s theorem on Kronecker product and subgroup theorem of induced representations likewise work for the induced representations in Krein spaces defined here, and then apply the Fubini theorem and harmonic analysis on the "small" subgroups.) Driving by the physical examples we assume for a while that the “measure product property” is fulfilled by the right \(H\)-cosets in \(G\). (We abandon soon this assumption so that our results, namely the subgroup theorem and the Kronecker product theorem, hold true for induced representations in Krein spaces, without this assumption.)

Let \(L\) be any \((\mathcal{J}_L,\mathcal{H}_L)\)-unitary strongly continuous and almost uniformly bounded representation of \(H\) in a Krein space \((\mathcal{H}_L,\mathcal{J}_L)\). Let \(\mu_H\) and \(\mu_{G/H}\) be (quasi) invariant measures on \(H\) and on the homogeneous space \(G/H\) of right \(H\)-cosets in \(G\) induced by the (right) Haar measure \(\mu\) on \(G\) by the “unique factorization”. Let us denote \(\mathcal{H}_H^L\) by \(\mathcal{H}_L^G\) the set of all functions \(f : G \ni x \mapsto f_x\) from \(G\) to \(\mathcal{H}_L\) such that

(i) \((f_x, v)\) is measurable function of \(x \in \mathcal{G}\) for all \(v \in \mathcal{H}_L\).

(ii) \(f_{hx} = L_h(f_x)\) for all \(h \in H\) and \(x \in \mathcal{G}\).

(iii) Into the linear space of functions \(f\) fulfilling (i) and (ii) let us introduce the operator \(\mathcal{J}_L^H\) by the formula \((\mathcal{J}_L^H f)_x = L_h \mathcal{J}_L L_{h^{-1}}(f_x)\), where \(x = h \cdot q\) is the unique decomposition of \(x \in \mathcal{G}\). Besides (i) and (ii) we require

\[
\int (\mathcal{J}_L((\mathcal{J}_L^H f)_{x}), f_x) d\mu_{E/H} < \infty,
\]

where the meaning of the integral is to be found in the fact that the integrand is constant on the right \(H\)-cosets and hence defines a function on the coset space \(G/H\).

Because every \(x \in \mathcal{G}\) has a unique factorization \(x = h \cdot q\) with \(h \in H\) and \(q \in Q \cong G/H\), then by “unique factorization” the functions \(f \in \mathcal{H}_L\) as well as the functions \(x \mapsto (f_x, v)\) with \(v \in \mathcal{H}_L\), on \(G\), may be treated as functions on the Cartesian product \(H \times Q \cong H \times \mathcal{G}/H \cong \mathcal{G}\). The axiom (i) means that the functions \((h, q) \mapsto (f_{h \cdot q}, v)\) for \(v \in \mathcal{H}_L\) are measurable on the product measure space \((H \times \mathcal{G}/H, \mathcal{M}_{H \times \mathcal{G}/H}, \mu_H \times \mu_{E/H}) \cong (H \times Q, \mathcal{M}_{H \times Q}, \mu_H \times \mu_{E/H})\). In particular let \(W : q \mapsto W_q \in \mathcal{H}_L\) be a function on \(Q\) such that \(q \mapsto (W_q, v)\) is measurable with respect to the standard measure space \((Q, \mathcal{M}_Q, dq)\) for all

\footnote{\(L\) in superscript! The \(\mathcal{H}_L\) with the lower case of the index \(L\) is reserved for the space of the representation \(L\) of the subgroup \(H \subset \mathcal{G}\).}
that \( \int (W_q W_q) d\mu_{\mathcal{F}/H}(q) < \infty \). Then by the analysis of \( \mathcal{L} \) (compare also \( \mathcal{L} \), §26.5) which is by now standard, the set of such functions \( W \) (when functions equal almost everywhere are identified) compose the direct integral \( \int \mathcal{H}_L d\mu_{\mathcal{F}/H}(q) \) Hilbert space with the inner product \( (W, F) = \int (W_q, F_q) d\mu_{\mathcal{F}/H}(q) \). For every such \( W \in \int \mathcal{H}_L d\mu_{\mathcal{F}/H}(q) \) the function \((h, q) \mapsto f_{h-q} = L_h W_q\) fulfils (i) and (ii). (ii) is trivial. For each \( h \in \mathcal{H}_L \) the function \((h, q) \mapsto (f_{h-q}, v) = (L_h W_q, v)\) is measurable on the product measure space \((H \times Q, \mathcal{R}_H \times Q, \mu_H \times \mu_{\mathcal{F}/H}) \cong (\mathcal{G}, \mathcal{R}_\mathcal{G}, \mu)\) because for any orthonormal basis \( \{e_n\}_{n \in \mathbb{N}} \) of the Hilbert space \( \mathcal{H}_L \) we have:

\[
(f_{h-q}, v) = (f_{h-q}, \mathcal{J}_L \mathcal{J}_L v) = (L_h W_q, \mathcal{J}_L \mathcal{J}_L v) = (\mathcal{J}_L L_h W_q, \mathcal{J}_L v) = (\mathcal{J}_L W_q, L_{h^{-1}} \mathcal{J}_L v) = \sum_{n \in \mathbb{N}} (\mathcal{J}_L W_q, e_n)(e_n, L_{h^{-1}} \mathcal{J}_L v)
\]

where each summand gives a measurable function \((h, q) \mapsto (\mathcal{J}_L W_q, e_n)(e_n, L_{h^{-1}} \mathcal{J}_L v)\) on the product measure space \((H \times Q, \mathcal{R}_H \times Q, \mu_H \times \mu_{\mathcal{F}/H})\) by Scholium 3.9 of \( \mathcal{L} \). On the other hand for every function \((h, q) \mapsto (f_{h-q}, v)\) measurable on the product measure space the restricted functions \( q \mapsto (f_{h-q}, v) \) and \( h \mapsto (f_{h-q}, v) \), i.e. with one of the arguments \( h \) and \( q \) fixed, are measurable, which follows from the Fubini theorem (compare e.g. \( \mathcal{L} \), Theorem 3.4) and thus \( q \mapsto (f_q, v) \) is measurable (i.e. with the argument \( h \) fixed and equal \( e \) in \((h, q) \mapsto (f_{h-q}, v)\)). Because a simple verification shows that

\[
\int (\mathcal{J}_L((\mathcal{J}_L f)_x), f_x) d\mu_{\mathcal{F}/H} = \int (\mathcal{J}_L((\mathcal{J}_L f)_{h-q}), f_{h-q}) d\mu_{\mathcal{F}/H}(q)
\]

one can see that when functions equal almost everywhere are identified \( \mathcal{H}_L \) becomes a Hilbert space with the inner product

\[
(f, g) = \int (\mathcal{J}_L((\mathcal{J}_L f)_x), g_x) d\mu_{\mathcal{F}/H}.
\]

(1)

(In fact because the values of \( f \in \mathcal{H}_L \) are in the fixed Hilbert space \( \mathcal{H}_L \) we do not have to tangle into the the whole machinery of direct integral Hilbert spaces of von Neumann. It suffices to make obvious modifications in the corresponding proof that \( L^2(\mathcal{G}/H) \) is a Hilbert space, compare \( \mathcal{L} \), §26.5.)

A simple verification shows that \( \mathcal{J}_L \) is a bounded self-adjoint operator in the Hilbert space \( \mathcal{H}_L \) with respect to the definite inner product \( \mathcal{P} \) and that \( (\mathcal{J}_L)^2 = I \). Therefore \((\mathcal{H}_L, \mathcal{J}_L)\) is a Krein space with the indefinite product

\[
(f, g)_{\mathcal{J}_L} = (\mathcal{J}_L f, g) = \int (\mathcal{J}_L(f_x), g_x) d\mu_{\mathcal{F}/H}
\]

(2)

which is meaningful because the integrand is constant on the right \( H \)-cosets, i.e. it is a function of \( q \in Q \cong \mathcal{G}/H \).
Let the function \( [x] \mapsto \lambda([x], g) \) on \( \mathcal{G}/H \) be the Radon-Nikodym derivative 
\[ \lambda(x, g) = \frac{dR_g\mu}{d\mu}(x), \]
where \([x]\) stands for the right \( H \)-coset \( Hx \) of \( x \in \mathcal{G} \) \( (\mu \) stands for the (quasi) invariant measure \( \mu_{\mathcal{G}/H} \) on \( \mathcal{G}/H \) induced by the assumed “factorization” property from the Haar measure \( \mu \) on \( \mathcal{G} \) and \( R_g\mu \) stands for the right translation of the measure \( \mu \): \( R_g\mu(E) = \mu(EG) \)).

For every \( g_0 \in \mathcal{G} \) let us consider a densely defined operator \( U^L_{g_0} \). Its domain \( \mathcal{D}(U^L_{g_0}) \) is equal to the set of all those \( f \in \mathcal{H}^L \) for which the function

\[ x \mapsto f'_x = \sqrt{\lambda([x], g_0)} f_{xg_0} \]

has finite Hilbert space norm (i.e. ordinary norm with respect to the ordinary definite inner product \( \langle \rangle \)) in \( \mathcal{H}^L \):

\[
(f', f') = \int \left( \mathfrak{J}_L((\mathfrak{J}_L^{-1} f')_x), f'_x \right) d\mu_{\mathcal{G}/H} \\
= \int \left( \mathfrak{J}_Lh(x)\mathfrak{J}_L^{-1} \sqrt{\lambda([x], g_0)} f_{xg_0}, \sqrt{\lambda([x], g_0)} f_{xg_0} \right) d\mu_{\mathcal{G}/H}(x) < \infty,
\]

where \( h(x) \in H \) is the unique element corresponding to \( x \) such that \( h(x)^{-1}x \in Q \); and whenever \( f \in \mathcal{D}(U^L_{g_0}) \) we put

\[ (U^L_{g_0}f)_x = \sqrt{\lambda([x], g_0)} f_{xg_0}. \]

\( U^L \), after restriction to a suitable sub-domain, becomes a group homomorphism of \( \mathcal{G} \) into a group of densely defined \( \mathfrak{J}^L \)-isometries of the Krein space \( (\mathcal{H}^L, \mathfrak{J}^L) \). Let us formulate this statement more precisely in a form of a Theorem:

**THEOREM 1.** The operators \( U^L_{g_0}, g_0 \in \mathcal{G} \), are closed and \( \mathfrak{J}^L \)-isometric with dense domains \( \mathcal{D}(U^L_{g_0}) \), dense ranges \( \mathcal{R}(U^L_{g_0}) \) and dense intersection \( \bigcap_{g_0 \in \mathcal{G}} \mathcal{D}(U^L_{g_0}) = \mathcal{R}(U^L_{g_0}) \). The set \( \mathcal{D}(U^L_{g_0}) \) of the restrictions \( U^L_{g_0} \) of the operators \( U^L_{g_0} \) to the domain \( \bigcap_{g_0 \in \mathcal{G}} \mathcal{D}(U^L_{g_0}) \) is a Krein-isometric representation of \( \mathcal{G} \). There exists a dense sub-domain \( \tilde{\mathcal{D}} \subset \bigcap_{g_0 \in \mathcal{G}} \mathcal{D}(U^L_{g_0}) \) such that \( U^L_{g_0} \mathcal{D} = \mathcal{D}, U^L_{g_0} \) is the closure of the restriction \( U^L_{g_0} \mathcal{D} \) of \( U^L_{g_0} \) to the sub-domain \( \tilde{\mathcal{D}} \), and \( g_0 \mapsto U^L_{g_0} \) is strongly continuous Krein-isometric representation of \( \mathcal{G} \) on its domain \( \mathcal{D} \).

Let us introduce the class \( C^{\mathcal{D}_{g_0}}_L \subset \mathcal{H}^L \) of functions \( h \cdot q \mapsto f_{h \cdot q} = L_h W_q \) with \( q \mapsto W_q \in \mathcal{H}_L \) continuous and compact support on \( Q \cong \mathcal{G}/H \). One easily verifies that all the conditions of Lemma 3 of (the next) Sect. 3 are true for the class \( C^{\mathcal{D}_{g_0}}_L \). Therefore \( C^{\mathcal{D}_{g_0}}_L \) is dense in \( \mathcal{H}^L \). Let \( h \cdot q \mapsto f_{h \cdot q} = L_h W_q \) be an element of \( C^{\mathcal{D}_{g_0}}_L \) and let \( K \) be the compact support of the function \( W \). Using the “unique factorization” let us introduce the functions \( (q, h_0, g_0) \mapsto h'_{q, h_0, g_0} \in H \) and \( (q, h_0, g_0) \mapsto q'_{q, h_0, g_0} \in Q \cong \mathcal{G}/H \) in the following way. Let
\( g_0 = q_0 \cdot h_0 \). We define \( h'_{q,h_0,q_0} \in H \) and \( q'_{q,h_0,q_0} \in Q \subset \mathcal{G} \) to be the elements, uniquely corresponding to \((q,h_0,q_0)\), such that

\[
q \cdot h_0 \cdot q_0 = h'_{q,h_0,q_0} \cdot q'_{q,h_0,q_0}.
\]

Finally let \( c_{K,g_0} = \sup_{q \in K} \| L_{h'_{q,h_0,q_0}} \| \), which is finite outside a null set, on account of the almost uniform boundedness of the representation \( L \), and because \( q \mapsto h'_{q,h_0,q_0} \) is continuous outside a \( \mu_{\mathcal{G}/H} \)-null set ("measure product property"

\[
\| U_{h_0,q_0,f}^L \|^2 = (U_{h_0,q_0,f}, U_{h_0,q_0,f}^L)
= \int (\mathcal{J}_L((\mathcal{J}^L U_{h_0,q_0,f}^L)_{h,q}), (U_{h_0,q_0,f}^L)_{h,q}) d\mu_{\mathcal{G}/H}(q)
= \int (L_{h'_{q,h_0,q_0},f} q'_{q,h_0,q_0}, L_{h'_{q,h_0,q_0},f} q'_{q,h_0,q_0}) d\mu_{\mathcal{G}/H}(q'_{q,h_0,q_0})
\leq c_{K,g_0} \int (f q'_{q,h_0,q_0}, f q'_{q,h_0,q_0}) d\mu_{\mathcal{G}/H}(q'_{q,h_0,q_0})
= c_{K,g_0} \| f \|^2,
\]

Thus it follows that \( C_{g_0}^L \subset \mathcal{D}(U_{g_0}^L) \) for every \( g_0 \in \mathcal{G} \). Similarly it is easily verifiable that \( C_{g_0}^L \subset \mathcal{R}(U_{g_0}^L) \) whenever the Radon-Nikodym derivative \( \lambda([x], g_0) \) is continuous in \([x]\). It follows from definition that for \( f \in \mathcal{H}^L \) being a member of \( \cap_{g_0 \in \mathcal{G}} \mathcal{D}(U_{g_0}^L) \) is equivalent to being a member of \( \cap_{g_0 \in \mathcal{G}} \mathcal{R}(U_{g_0}^L) \).

We shall show that \((U_{g_0}^L)^\dagger = U_{g_0}^L, \) where \( T^\dagger \) stands for the adjoint of the operator \( T \) in the sense of Krein \([4]\), page 121: for any linear operator \( T \) with dense domain \( \mathcal{D}(T) \) the vector \( g \in \mathcal{H}^L \) belongs to \( \mathcal{D}(T^\dagger) \) if and only if there exists a \( k \in \mathcal{H}^L \) such that

\[
(\mathcal{J}^L T f, g) = (\mathcal{J}^L f, k), \text{ for all } f \in \mathcal{D}(T),
\]

and in this case we put \( T^\dagger g = k \), with the unique \( k \) as \( \mathcal{D}(T) \) is dense (i. e. same definition as for the ordinary adjoint with the definite Hilbert space inner product \( \langle \cdot, \cdot \rangle \) given by \([1]\) replaced with the indefinite one \( \langle \cdot, \cdot \rangle \), given by \([2]\)).

Now let \( g \) be arbitrary in \( \mathcal{D}(U_{g_0}^L)^\dagger \), and let \((U_{g_0}^L)^\dagger g = k \). The inclusion \((U_{g_0}^L)^\dagger \subset U_{g_0}^L \) is equivalent to the equation \( U_{g_0}^L g = k \). By the definition of the Krein adjoint of an operator, for any \( f \in \mathcal{D}(U_{g_0}^L) \) we have

\[
(\mathcal{J}^L U_{g_0}^L f, g) = (\mathcal{J}^L f, k),
\]

\[
i. \ e. \int (\mathcal{J}_L(U_{g_0}^L f, x), g_x) d\mu_{\mathcal{G}/H}(x) = \int (\mathcal{J}_L f, k_x) d\mu_{\mathcal{G}/H}(x);
\]

\[\text{It holds true even if the "measure product property is not assumed" – compare the comments below in this Section.}\]
which by the definition of $U^L_{g_0}$ and quasi invariance of the measure $\mu_{\phi/H}$ means that

$$\int \left( \mathfrak{H}_f, \sqrt{\frac{d\mu_{\phi/H}(xg_0^{-1})}{d\mu_{\phi/H}(x)}} g_{xg_0^{-1}} \right) d\mu_{\phi/H}(x)$$

$$= \int \left( \mathfrak{H}_f, k_x \right) d\mu_{\phi/H}(x) \text{ for all } f \in \mathcal{D}(U^L_{g_0}),$$

i.e. the function $u$

$$x \mapsto u_x = \sqrt{\frac{d\mu_{\phi/H}(xg_0^{-1})}{d\mu_{\phi/H}(x)}} g_{xg_0^{-1}} - k_x$$

is $\mathfrak{H}^L$-orthogonal to all elements of $\mathcal{D}(\{ U^L_{g_0} \})$: ($\mathfrak{H}^L f, u) = 0$ for all $f \in \mathcal{D}(U^L_{g_0})$. Because $\mathcal{D}(U^L_{g_0})$ is dense in $\mathcal{H}^L$, and $\mathfrak{H}^L$ is unitary with respect to the ordinary Hilbert space inner product (1) in $\mathcal{H}^L$ it follows that $\mathfrak{H}^L \mathcal{D}(U^L_{g_0})$ is dense in $\mathcal{H}^L$. Therefore $u$ must be zero as a vector orthogonal to $\mathfrak{H}^L \mathcal{D}(U^L_{g_0})$ in the sense of the Hilbert space inner product (1). Thus

$$\sqrt{\frac{d\mu_{\phi/H}(xg_0^{-1})}{d\mu_{\phi/H}(x)}} g_{xg_0^{-1}} = k_x$$

almost everywhere, and because by definition $(k, k) < \infty$, we have shown that $U^L_{g_0^{-1}} g = k$.

Next we show that $(U^L_{g_0})^\dagger \supset U^L_{g_0^{-1}}$. Let $g$ be arbitrary in $\mathcal{D}(U^L_{g_0^{-1}})$ and let $U^L_{g_0^{-1}} g = k$. It must be shown that for any $f \in \mathcal{D}(U^L_{g_0})$, $(\mathfrak{H}^L U^L_{g_0} f, g) = (\mathfrak{H}^L f, k)$. This is the same as showing that

$$\int \left( \mathfrak{H}(U^L_{g_0} f), g \right) d\mu_{\phi/H}(x) = \int \left( \mathfrak{H} f, k \right) d\mu_{\phi/H}(x),$$

which again easily follows from definition of $U^L_{g_0}$ and quasi invariance of the measure $d\mu_{\phi/H}(x)$:

$$\int \left( \mathfrak{H}(U^L_{g_0} f), g \right) d\mu_{\phi/H}(x) = \int \sqrt{\frac{d\mu_{\phi/H}(xg_0)}{d\mu_{\phi/H}(x)}} \left( \mathfrak{H} f, g \right) d\mu_{\phi/H}(x)$$

$$= \int \sqrt{\frac{d\mu_{\phi/H}(xg_0^{-1}g_0)}{d\mu_{\phi/H}(xg_0^{-1})}} \left( \mathfrak{H} f, g \right) d\mu_{\phi/H}(x)$$

$$= \int \left( \mathfrak{H} f, \sqrt{\frac{d\mu_{\phi/H}(xg_0^{-1})}{d\mu_{\phi/H}(x)}} g_{xg_0^{-1}} \right) d\mu_{\phi/H}(x)$$

$$= \int \left( \mathfrak{H} f, (U^L_{g_0^{-1}} g) \right) d\mu_{\phi/H}(x) = \int \left( \mathfrak{H} f, k \right) d\mu_{\phi/H}(x).$$

Thus we have shown that $(U^L_{g_0})^\dagger = U^L_{g_0^{-1}}$. 25
Because $C_{00}^{L} \subset \mathfrak{D}(U_{g_{0}}^{L})$ then $\mathfrak{D}(U_{g_{0}}^{L})$ is dense, thus $U_{g_{0}}^{L}$, equal to $(U_{g_{0}}^{L})^{\dagger}$, is closed by Theorem 2.2 of [3] (Krein adjoint $T^{\dagger}$ is always closed, as it is equal $\mathfrak{J}^{L}T^{*}\mathfrak{J}^{L}$ with the ordinary adjoint $T^{*}$ operator, and because the fundamental symmetry $\mathfrak{J}^{L}$ is unitary in the associated Hilbert space $\mathcal{H}^{L}$, compare Lemma 2.1 in [3]).

In order to prove the second statement it will be sufficient to show that $(\tilde{U}_{g_{0}}^{L})^{\dagger} = U_{g_{0}}^{L-1}$ because the homomorphism property of the map $g_{0} \mapsto U_{g_{0}}^{L}$ restricted to $\bigcap_{g \in \mathfrak{G}} \mathfrak{D}(U_{g}^{L})$ is a simple consequence of the definition of $U_{g_{0}}^{L}$. But the proof of the equality $(\tilde{U}_{g_{0}}^{L})^{\dagger} = U_{g_{0}}^{L-1}$ runs exactly the same way as the proof of the equality $(U_{g_{0}}^{L})^{\dagger} = U_{g_{0}}^{L-1}$, with the trivial replacement of $\mathfrak{D}(U_{g_{0}}^{L})$ by $\mathfrak{D}$, as it is valid for any dense sub-domain $\mathfrak{D}$ contained in $\mathfrak{D}(U_{g_{0}}^{L})$ instead of $\mathfrak{D}(U_{g_{0}}^{L})$. Then by Theorem 2.5 of [3] it follows that $(\tilde{U}_{g_{0}}^{L})^{\dagger} = (U_{g_{0}}^{L-1})^{\dagger}$ is equal to the closure $\tilde{U}_{g_{0}}^{L}$ of the operator $U_{g_{0}}^{L-1}$. Because $(U_{g_{0}}^{L-1})^{\dagger} = U_{g_{0}}^{L}$, we get $U_{g_{0}}^{L} = U_{g_{0}}^{L}$. 

By the above remark we also have $U_{g_{0}}^{L} = \tilde{U}_{g_{0}}^{L}$ for any restriction $\tilde{U}_{g_{0}}^{L}$ of $U_{g_{0}}^{L}$ to a dense sub-domain $\mathfrak{D} \subset \mathfrak{D}(U_{g_{0}}^{L})$.

In order to prove the third statement, let us introduce a dense sub-domain $C_{0}^{L} \subset C_{0}^{L}$ of continuous functions with compact support on $\mathfrak{G}/H$. Its full definition and properties are given in the next Section. In particular $U_{g_{0}}^{L}C_{0}^{L} = C_{0}^{L}$ whenever the Radon-Nikodym derivative $\lambda([x],g_{0})$ is continuous in $[x]$. For each element $f_{0}$ of $C_{0}^{L}$ we have the inequality shown to be valid in the course of proof of Lemma 1 Sect. 5.

$$\|f_{x_{1}}^{0} - f_{x_{2}}^{0}\|^{2} \leq \sup_{h \in H} \|f_{(h,e),\cdot}(e,x_{1}) - f_{(h,e),\cdot}(e,x_{2})\|^{2} \sup_{x \in \mathfrak{G}} \mu_{H}(K^{-1} \cap H)$$

where $f_{(h,e),\cdot}$ is a function depending on $f^{0}$, continuous on the direct product group $H \times \mathfrak{G}$ and with compact support $K_{h} \times V$ with $V$ being a compact neighbourhood of the two points $x_{1}$ and $x_{2}$. Because any such function $f_{(h,e),\cdot}$ must be uniformly continuous, the strong continuity of $U_{g_{0}}^{L}$ on the sub-domain $C_{0}^{L}$ follows. Because $U_{g_{0}}^{L}C_{0}^{L} = C_{0}^{L}$, the third statement is proved with $\mathfrak{D} = C_{0}^{L}$ (In case the Radon-Nikodym derivative was not continuous and “measure product property” not satisfied it would be sufficient to use all finite sums $U_{g_{0}}^{L}f_{1} + \ldots + U_{g_{0}}^{L}f_{n}$, $f_{k} \in C_{0}^{L}$ as the common sub-domain $\mathfrak{D}$ instead of $C_{0}^{L}$).

**REMARK 2.** By definition of the Krein-adjoint operator and the properties: 1) $U_{g}^{L}\mathfrak{D} = \mathfrak{D}$, $g \in \mathfrak{G}$, 2) $(U_{g}^{L})^{\dagger} = U_{g_{-1}}^{L}$, $g \in \mathfrak{G}$, it easily follows that for each $g \in \mathfrak{G}$

$$\left(\begin{array}{c}
(U_{g}^{L})^{\dagger} U_{g}^{L} = I \\
U_{g}^{L} (U_{g}^{L})^{\dagger} = I
\end{array}\right)$$

(5) on the domain $\mathfrak{D}$. We may easily modify the common domain $\mathfrak{D}$ so as to achieve the additional property: 3) $\mathfrak{J}^{L}\mathfrak{D} = \mathfrak{D}$ together with 1) and 2) and thus with [3]. Indeed, to achieve this one may define $\mathfrak{D}$ to be the linear span of the set function.
\[ \left\{ \left( \lambda^L \right)^{m_k} U_{g_1}^L \ldots U_{g_n}^L \left( \lambda^L \right)^{n_k+1} \right\} f \]: with \( g_k \) ranging over \( G \), \( f \in C_0^L \), \( n \in \mathbb{N} \) and \( k \mapsto m_k \) over the sequences with \( m_k \) equal 0 or 1. In case the Radon-Nikodym derivative \( \lambda \) is continuous and the “measure product property” fulfilled, \( D = C_0^L \) meets all the requirements.

**Corollary 1.** For every \( U_{g_0}^L \) there exists a unique unitary (with respect to the definite inner product) operator \( U_{g_0} \) in \( \mathcal{H}^L \) and unique selfadjoint (with respect to (1)) positive operator \( H_{g_0} \), with dense domain \( \mathcal{D}(U_{g_0}^L) \) and dense range such that \( U_{g_0}^L = U_{g_0} H_{g_0} \).

\[ \blacksquare \]

Immediate consequence of the von Neumann polar decomposition theorem and closedness of \( U_{g_0}^L \).

\[ \blacksquare \]

Of course the ordinary unitary operators \( U_{g_0} \) of the Corollary do not compose any representation in general as the operators \( U_{g_0} \) and \( H_{g_0} \) of the polar decomposition do not commute if \( U_{g_0} \) is non normal.

**Theorem 2.** \( L \) and \( J_L \) commute if and only if \( U_L \) and \( J_L \) commute. If \( U_L \) and \( J_L \) commute, then \( L \) is not only \( J_L \)-unitary but also unitary in the ordinary sense for the definite inner product in the Hilbert space \( \mathcal{H}_L \). If \( U_L \) and \( J_L \) commute then \( U_L \) is not only \( J_L \)-isometric but unitary with respect to the ordinary Hilbert space norm (1) in \( \mathcal{H}_L \). The representation \( L \) is uniformly bounded if and only if the induced representation \( U_L \) is Krein-unitary (with each \( U_{g_0} \) bounded) and uniformly bounded.

\[ \blacksquare \]

Using the functions \((q, h_0, q_0) \mapsto h'_{q, h_0, q_0} \in H \) and \((q, h_0, q_0) \mapsto q'_{q, h_0, q_0} \in Q \cong G/H \) defined by (3), one easily verifies that \( L \) and \( J_L \) commute (and thus \( L \) is not only \( J_L \)-unitary but also unitary in the ordinary sense for the definite inner product in the Hilbert space \( \mathcal{H}_L \)) if and only if \( U_L \) and \( J_L \) commute (i.e. when \( U_L \) is not only \( J_L \)-isometric but unitary with respect to the ordinary Hilbert space norm (1) in \( \mathcal{H}_L \)). To this end we utilize the fact that for each fixed \( x \), \( f_x \) with \( f \) ranging over \( C_0^L \) has \( \mathcal{H}_L \) as their closed linear span. We leave details to the reader.

\[ \blacksquare \]

**Corollary 2.** If \( N \subset H \subset G \) is a normal subgroup of \( G \) such that the restriction of \( L \) to \( N \) is uniformly bounded (or commutes with \( J_L \)) then the restriction of \( U_L \) to the subgroup \( N \) is a Krein-unitary representation of the subgroup with each \( U_{g_0}^L \), \( n \in N \) bounded uniformly in \( n \) (or \( U_L \) restricted to \( N \) commutes with \( J_L \) and is an ordinary unitary representation of \( N \) in the Hilbert space \( \mathcal{H}_L \)).

\[ \blacksquare \]
the classical proof of strong continuity of the right regular representation of $\mathfrak{G}$ in $L^2(\mathfrak{G})$ or in $L^2(\mathfrak{G}/H)$ (of course with the obvious Radon-Nikodym factor in the latter case), with the necessary modifications required for the Krein space. In our proof of strong continuity on $\mathfrak{D}$ the strong continuity of the representation $L$ plays a much more profound role in comparison to the original Mackey’s theory.

The additional assumption posed on right $H$-cosets, i.e. “measure product property” is unnecessary. In order to give to this paper a more independent character we point out that the above construction of the induced representation in Krein space is possible without this assumption which may be of use for spectral analysis for (unnecessary elliptic) operators on manifolds uniform for more general semi-direct product Lie groups preserving indefinite pseudo-riemann structures. Namely for any closed subgroup $H \subset \mathfrak{G}$ (with the “measure product property” unnecessary fulfilled) the right action of $H$ on $\mathfrak{G}$ is proper and both $\mathfrak{G}$ and $\mathfrak{G}/H$ are metrizable so that a theorem of Federer and Morse [24] can be applied (with the regular Baire (or Borel) Haar measure space structure $(\mathfrak{G}, \mathscr{B}_\mathfrak{G}, \mu)$ on $\mathfrak{G}$) in proving that there exists a Borel subset $B \subset \mathfrak{G}$ such that:

- $B$ intersects each right $H$-coset in exactly one point
- Each compact subset $K$ of $\mathfrak{G}$, $\pi^{-1}(\pi(K)) \cap B$ has a compact closure (compare Lemma 1.1 of [47]). In short $B$ is a “regular Borel section of $\mathfrak{G}$ with respect to $H$”. In particular it follows that any $g \in \mathfrak{G}$ has unique factorization $g = h \cdot b$, $h \in H, b \in B$.

Using the Lemma and extending a technique of A. Weil in studying relatively invariant measures Mackey gave in [47] a general construction of quasi invariant measures in $\mathfrak{G}/H$ (all being equivalent).

The general construction of quasi invariant (standard) Baire (or Borel) Haar measure space structure $(\mathfrak{G}, \mathscr{B}_\mathfrak{G}, \mu)$ on $\mathfrak{G}$) in proving that there exists a Borel subset $B \subset \mathfrak{G}$ such that:

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Using the Lemma and extending a technique of A. Weil in studying relatively invariant measures Mackey gave in [47] a general construction of quasi invariant measures in $\mathfrak{G}/H$ (all being equivalent).

**Theorem.** a) Any two non zero quasi invariant measures on $\mathfrak{G}/H$ are equivalent.
b) If \( \mu \) and \( \mu' \) are two non zero quasi invariant measures on \( \mathfrak{G}/H \) and \( d(R_s \mu)/d\mu = d(R_s \mu')/d\mu' \) almost \( \mu \)-everywhere (and thus almost \( \mu' \)-everywhere), then \( \mu' = c \cdot \mu \), where \( c \) is a positive number.

**LEMMA.** Measure \( \rho \cdot \mu_0 \) has the form \( \mu^1 \) if and only if for each \( h \in H \) the equality

\[
\rho(hx) = \frac{\Delta_H(h)}{\Delta_\mathfrak{G}(h)} \rho(x)
\]

is fulfilled almost \( \mu_0 \)-everywhere on \( \mathfrak{G} \).

**THEOREM.** a) There exist functions \( \rho \) fulfilling the conditions of the preceding Lemma, for example

\[
\rho(x) = \frac{\Delta_H(h(x))}{\Delta_\mathfrak{G}(h(x))}
\]

where \( h(x) \in H \) is the only element of \( H \) corresponding to \( x \in \mathfrak{G} \) such that \( h(x)^{-1}x \in B \).

b) \( \rho \) can be chosen to be continuous.

c) One may chose the regular section \( B \) to be continuous outside a discrete countable set in \( \mathfrak{G}/H \) whenever \( \mathfrak{G} \) is a topological manifold with \( H \) as closed topological sub-manifold; thus \( x \mapsto h(x) \) becomes continuous outside a set of measure zero in \( \mathfrak{G} \).

d) Given such a function \( \rho \) one can construct a quasi invariant measure \( \mu \) on \( \mathfrak{G}/H \) such that \( \mu^1 = \rho \cdot \mu_0 \).

e) \( \rho(xs)/\rho(x) \) with \( s, x \in \mathfrak{G} \) does not depend on \( x \) within the class \( \pi(x) \) and determinates a function \( (\pi(x), s) \mapsto \lambda(\pi(x), s) \) on \( \mathfrak{G}/H \times \mathfrak{G} \) equal to the Radon-Nikodym derivative \( d(R_s \mu)/d\mu(\pi(x)) \).

f) Given any Baire (or Borel) function \( \lambda(\cdot, \cdot) \) on \( \mathfrak{G}/H \times \mathfrak{G} \) fulfilling the general properties of Radon-Nikodym derivative: (i) for all \( x, s, z \in \mathfrak{G} \), \( \lambda(\pi(z), xs) = \lambda(\pi(z), s)\lambda(\pi(z), x) \), (ii) for all \( h \in H \), \( \lambda(\pi(h), h) = \Delta_H(h)/\Delta_\mathfrak{G}(h) \), (iii) \( \lambda(\pi(e), s) \) is bounded on compact sets as a function of \( s \), one can construct a quasi invariant measure \( \mu \) on \( \mathfrak{G}/H \) such that \( d(R_s \mu)/d\mu(\pi(x)) = \lambda(\pi(x), s) \), almost \( \mu \)-everywhere with respect to \( s, x \in \mathfrak{G} \).

Thus any non zero quasi invariant measure \( \mu \) on \( \mathfrak{G}/H \) gives rise to a \( \rho \)-function and \( \lambda \)-function and vice versa every “abstract Radon-Nikodym derivative” i.e. \( \lambda \)-function (or equivalently every \( \rho \)-function) gives rise to a quasi invariant measure \( \mu \) on \( \mathfrak{G}/H \) determined up to a non zero constant factor. Every quasi invariant measure \( \mu \) on \( \mathfrak{G}/H \) is thus a pseudo-image of the right Haar measure \( \mu \) on \( \mathfrak{G} \) under the canonical projection \( \pi \) in the terminology of [3]. In particular if the groups \( \mathfrak{G} \) and \( H \) are unimodular (i.e. \( \Delta_\mathfrak{G} = 1 \) and \( \Delta_H = 1_H \)) then among quasi invariant measures on \( \mathfrak{G}/H \) there exists a strictly invariant measure.

The measure space structure of \( \mathfrak{G}/H \) uniform for the group \( \mathfrak{G} \) may be transferred to \( B \) together with the uniform structure, such that \( (\mathfrak{G}/H, \mathcal{B}_\mathfrak{G}/H, \mu_{\mathfrak{G}/H}) \cong (B, \mathcal{B}, \mu_B) \). The set \( B \) plays the role of the sub-manifold \( Q \) in the “measure product property”. This however would be insufficient, and we have to prove a kind of regularity of right \( H \)-cosets instead of “measure product property”. Namely let us define \( h(x) \in H \), which corresponds uniquely to \( x \in \mathfrak{G} \), such that \( h(x)^{-1}x \in B \). We have to prove that the functions \( x \mapsto h(x) \) and \( x \mapsto h(x)^{-1}x \) are Borel (thus in particular measurable), which however was carried through in the proof of Lemma 1.4 of [17]. Now the only point which has to be changed is the definition of the fundamental symmetry operator \( \mathfrak{J}^L \) in \( \mathcal{H}^L \). We put

\[
(\mathfrak{J}^L f)_x = L_{h(x)} \mathfrak{J}_L L_{h(x)^{-1}} f_x.
\]
We define \( \mathcal{H}^L \) as the set of functions \( \mathcal{G} \rightarrow \mathcal{H}_L \) fulfilling the conditions (i), (ii) and such that

\[
\int_B (\mathfrak{F}((\mathfrak{F} f)_x), f_x) \, d\mu < \infty.
\]

The proof that \( \mathcal{H}^L \) is a Hilbert space with the inner product

\[
(f, g) = \int_B (\mathfrak{F}((\mathfrak{F} f)_x), g_x) \, d\mu = \int_B (f_b, g_b) \, d\mu_B(b), \quad \text{where } b \in B,
\]

is the same in this case with the only difference that the regularity of \( H \)-cosets is used instead of the Fubini theorem in reducing the problem to the von Neumann’s direct integral Hilbert space construction. Namely we define a unitary map \( V : f \mapsto W^f = f|_B \) from the space \( \mathcal{H}^L \) to the direct integral Hilbert space \( \int \mathcal{H}_L \, d\mu_B \) of functions \( b \mapsto W_b \in \mathcal{H}_L \) by a simple restriction to \( B \) which is “onto” in consequence of the regularity of \( H \)-cosets. Its isometric character is trivial. \( V \) has the inverse \( W \mapsto f^W \) with \( f^W_x = L_{h(x)} W_{h(x)^{-1} x} \). In particular \( f^W \) is measurable on \( \mathcal{G} \) as for an orthonormal basis \( \{e_n\}_{n \in \mathbb{N}} \) of the Hilbert space \( \mathcal{H}_L \) and any \( v \in \mathcal{H}_L \) we have:

\[
(f^W_x, v) = (J^W_x, \mathfrak{F} \mathfrak{F} \mathfrak{F} v) = (L_{h(x)} W_{h(x)^{-1} x}, \mathfrak{F} \mathfrak{F} \mathfrak{F} v) = (\mathfrak{F} \mathfrak{F} \mathfrak{F} L_{h(x)} W_{h(x)^{-1} x}, \mathfrak{F} \mathfrak{F} \mathfrak{F} v) = (\mathfrak{F} \mathfrak{F} \mathfrak{F} W_{h(x)^{-1} x}, \mathfrak{F} \mathfrak{F} \mathfrak{F} \mathfrak{F} v) = \sum_{n \in \mathbb{N}} (\mathfrak{F} \mathfrak{F} \mathfrak{F} W_{h(x)^{-1} x}, e_n)(e_n, L_{h(x)^{-1} x} \mathfrak{F} \mathfrak{F} \mathfrak{F} v)
\]

which, as a point-wise convergent series of measurable (again by Scholium 3.9 of [99]) functions in \( x \) is measurable in \( x \). We have to prove in addition that the induced representations \( U^L \) in Krein spaces \( (\mathcal{H}^L, \mathfrak{F} \mathfrak{F} \mathfrak{F}) \) corresponding to different choices of regular Borel sections \( B \) are (Krein-)unitary equivalent. Namely let \( B_1 \) and \( B_2 \) be two Borel sections in question. The Krein-unitary operator \( U_{12} : (U_{12} f)_x = L_{h_{12}(x)} f_x \), where \( h_{12}(x) \in H \) transforms the intersection point of the right \( H \)-coset \( Hx \) with the section \( B_1 \) into the intersection point of the same coset \( Hx \) with the Borel section \( B_2 \), gives the Krein-unitary equivalence. The proof is similar to the proof of Lemma 7 of Sect. 5.

Therefore from now on everything which concerns induced representations in Krein spaces, with the group \( \mathcal{G} \) not explicitly assumed to be equal \( T_{\mathcal{G}} \mathbb{S}L(2, \mathbb{C}) \), does not assume “measure product property”. Also Theorems 1 and 2 and Corollaries 1 and 2 remain true without the “measure product property” for any locally compact and separable \( \mathcal{G} \) and its closed subgroup \( H \). Indeed using the regular Borel section \( B \) of \( \mathcal{G} \) the functions \( \mathfrak{F} \mathfrak{F} : (q, h_0, q_0) \mapsto h'_{q, h_0, q_0} \) and \( (q, h_0, q_0) \mapsto q'_{q, h_0, q_0} \) may likewise be defined in this more general situation. Moreover, by Lemma 1.1 and the proof of Lemma 1.4 of [17], \( h'_{q, h_0, q_0} \) ranges within a compact subset of \( H \), whenever \( q \) ranges within in a compact subset of \( \mathcal{G} \), so that the proofs remain unchanged.

The construction of the induced representation in Krein space has also another invariance property: it does not depend on the choice of a quasi invariant measure \( \mu \) on \( \mathcal{G}/H \) in the unique equivalence class, provided the Radon-Nikodym derivative \( \frac{d\mu'}{d\mu} \) corresponding to measures \( \mu' \) and \( \mu \) in the class is
essentially “upper” and “lower” bounded: there exist two positive numbers \( \delta \) and \( \Delta \) such that
\[
\text{ess sup } \frac{d\mu'}{d\mu} < \Delta \quad \text{and} \quad \text{ess sup } \frac{d\mu}{d\mu'} < \delta.
\]

This boundedness condition of Radon-Nikodym derivative is unnecessary in case of the original Mackey’s theory of induced representations, which are unitary in the ordinary sense. Introducing the left-handed-superscript \( \mu \) in \( \nu^H \) and \( \nu^U \) for indicating the measure used in the construction of \( H^L \) and \( U^L \), we may formulate a Theorem:

**THEOREM 3.** Let \( \mu' \) and \( \mu \) be quasi invariant measures in \( \mathcal{G}/H \) with Radon-Nikodym derivative \( \psi = \frac{d\mu'}{d\mu} \) essentially “upper” and “lower” bounded. Then there exists a Krein-unitary transformation \( V \) from \( \nu^H \) onto \( \nu^L \) such that \( V(\nu^H_y) V^{-1} = \nu^L_y \) for all \( y \in \mathcal{G} \); that is the representations \( \nu^H \) and \( \nu^L \) are Krein-unitary equivalent.

\[ \square \]

Let \( f \) be any element of \( \nu^H \) and let \( \pi \) be the canonical map \( \mathcal{G} \to \mathcal{G}/H \). Boundedness condition of the Radon-Nikodym derivative \( \psi \) ensures \( (\sqrt{\psi} \circ \pi f, \sqrt{\psi} \circ \pi f) \) to be finite, i. e. ensures \( \sqrt{\psi} \circ \pi f \) to be a member of \( \nu^H \) as \( \sqrt{\psi} \circ \pi \) is measurable; and moreover the Krein-square-inner product \( (\sqrt{\psi} \circ \pi f, \sqrt{\psi} \circ \pi f)_{\mathcal{H}^L} \) in \( \nu^H \) is equal to that \( (f, f)_{\mathcal{H}^L} \) in \( \nu^H \). Moreover boundedness of \( \psi \) guarantees that every \( g \) in \( \nu^H \) is evidently of the form \( \sqrt{\psi} \circ \pi f \) for some \( f \in \nu^H \). Let \( V \) be the operator of multiplication by \( \sqrt{\psi} \circ \pi \). Then \( V \) defines a Krein-unitary map of \( \nu^H \) onto \( \nu^L \). The verification that \( V(\nu^H_y) V^{-1} = \nu^L_y \) is immediate.

Finally we mention the following easy but useful

**THEOREM 4.** Let \( L \) and \( L' \) be Krein-unitary representations in \( (\mathcal{H}_L, \mathcal{J}_L) \), which are Krein-unitary and unitary equivalent, then the induced representations \( U^L \) and \( U^{L'} \) are Krein-unitary equivalent.

### 4 Certain dense subspaces of \( \mathcal{H}^L \)

We present here some lemmas of analytic character which we shall need later and which we have used in the proof of Thm. 3 of Sect. 3. Let \( \mu_H \) be the right invariant Haar measure on \( H \). Let \( C^L \) denote the set of all functions \( f : \mathcal{G} \ni x \mapsto f_x \in H_L \), which are continuous with respect to the Hilbert space norm \( \| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle} \) in the Hilbert space \( H_L \), and with compact support. Let us denote the support of \( f \) by \( K_f \).

**LEMMA 1.** For each \( f \in C^L \) there is a unique function \( f^0 \) from \( \mathcal{G} \) to \( H_L \)

\[
\text{such that } \int (\mathcal{J}_L L_h^{-1} f_x, v) \, d\mu_H(h) = \langle f^0_x, v \rangle \quad \text{for all } x \in \mathcal{G} \text{ and all } v \in H_L.
\]

This function is continuous and it is a member of \( H^L \). The function \( \mathcal{G}/H \ni [x] \mapsto (\mathcal{J}_L (f^0)_x, f^0_x) \) as well as the function \( \mathcal{G}/H \ni [x] \mapsto f^0_x \) has a compact support. Finally \( \sup_{x \in \mathcal{G}} (\mathcal{J}_L (f^0)_x, f^0_x) = \sup_{x \in B} (f^0_x, f^0_x) < \infty \), where \( B \) is a regular Borel section of \( \mathcal{G} \) with respect to \( H \) of Sect. 3.
Let $f \in C^L$. For each fixed $x \in \mathcal{G}$ consider the anti-linear functional
\[ v \mapsto F_x(v) = \int (\overline{3}_L L_{h^{-1}} f_{hx}, v) d\mu_H(h) \]
on $\mathcal{H}_L$. From the Cauchy-Schwarz inequality for the Hilbert space inner product $(\cdot, \cdot)$ in the Hilbert space $\mathcal{H}_L$ and unitarity of $\overline{3}_L$ with respect to the inner product $(\cdot, \cdot)$ in $\mathcal{H}_L$, one gets
\[
|F_x(v)| \leq \int |(\overline{3}_L L_{h^{-1}} f_{hx}, v)| d\mu_H(h) \leq \int \|\overline{3}_L L_{h^{-1}} f_{hx}\| \|v\| d\mu_H(h) \\
= \left( \int \|\overline{3}_L L_{h^{-1}} f_{hx}\| d\mu_H(h) \right) \|v\| = \left( \int \|L_{h^{-1}} f_{hx}\| d\mu_H(h) \right) \|v\|;
\]
where the integrand in the last expression is a compactly supported continuous function of $h$ as a consequence of the strong continuity of the representation $L$ and because $f$ is compactly supported norm continuous. Therefore the integral in the last expression is finite, so that the functional $F_x$ is continuous. Thus by Riesz’s theorem (in the conjugate version) there exists a unique element $g_x$ of $\mathcal{H}_L$ (depending of course on $x$) such that for all $v \in \mathcal{H}_L : F_x(v) = (g_x, v)$. We put $f^0_x = \overline{3}_L g_x$ so that $F_x(v) = (\overline{3}_L f^0_x, v)$, $v \in \mathcal{H}_L$. We have to show that $f^0 : x \mapsto f^0_x$ has the desired properties.

That $f^0_{h'x} = L_{h'} f^0_x$ for all $h' \in H$ and $x \in \mathcal{G}$ follows from right invariance of the Haar measure $\mu_H$ on $H$:
\[
(\overline{3}_L L_{h'} f^0_x, v) = (\overline{3}_L f^0_x, L_{h'^{-1}} v) = \int (\overline{3}_L L_{h^{-1}} f_{hx}, L_{h'^{-1}} v) d\mu_H(h) \\
= \int (\overline{3}_L L_{h'} L_{h^{-1}} f_{hx}, v) d\mu_H(h) = \int (\overline{3}_L L_{(h'h'^{-1})^{-1}} f_{hx}, v) d\mu_H(h) \\
= \int (\overline{3}_L L_{(hh' x)^{-1}} f_{hx}, v) d\mu_H(h') = \int (\overline{3}_L L_{(hh' x)^{-1}} f_{hx}, v) d\mu_H(h) \\
= \int (3_L L_{h} f_{hx'}, v) d\mu_H(h) = (3_L f^0_x, v),
\]
for all $v \in \mathcal{H}_L$, $h' \in H$, $x \in \mathcal{G}$.

Denote the compact support of $f$ by $K$. From the strong continuity of the representation $L$ it follows immediately that the function
\[
(h, x) \mapsto f^L_{(h, x)} = L_{h^{-1}} f_{hx}
\]
is a norm continuous function on the direct product group $H \times \mathcal{G}$ and compactly supported with respect to the first variable, i.e. for every $x \in \mathcal{G}$ the function $h \mapsto f^L_{(h, x)}$ has compact support equal $K x^{-1} \cap H$. It is therefore uniformly norm continuous on the direct product group $H \times \mathcal{G}$ with respect to the first variable. For any compact subset $V$ of $\mathcal{G}$ let $\phi_V$ be a real continuous function on $\mathcal{G}$ with compact support equal 1 everywhere on $V$ (there exists such a function because $\mathcal{G}$ as a topological space is normal). For $f \in C^L$ and any compact $V \subset \mathcal{G}$ we
introduce a norm continuous function on the direct product group $H \times \mathfrak{G}$ as a product $f^L \phi_V$:

$$(h, x) \mapsto f^{L,V}_{(h,x)} = f^L_{(h,x)}\phi_V(x),$$

which in addition is compactly supported and has the property that

$$f^{L,V}_{(h,x)} = L_{h^{-1}}f_{hx}$$

for $(h, x) \in H \times V \subset H \times \mathfrak{G}$. In particular $f^{L,V}$ as compactly supported is not only norm continuous but uniformly continuous on the direct product group $H \times \mathfrak{G}$ (i. e. uniformly in both variables jointly). Let $\{e_n\}_{n \in \mathbb{N}}$ be an orthonormal basis in the Hilbert space $\mathcal{H}_L$ and let $\mathcal{O} \subset \mathfrak{G}$ be any open set containing $x_1, x_2 \in \mathfrak{G}$ with compact closure $V$. From the definition of $f^0$ it follows that

$$\|f^0_{x_1} - f^0_{x_2}\|^2 = \|\mathfrak{J}_L(f^0_{x_1} - f^0_{x_2})\|^2 = \sum_{n \in \mathbb{N}} \left| (\mathfrak{J}_L(f^0_{x_1} - f^0_{x_2}), e_n) \right|^2$$

$$= \sum_{n \in \mathbb{N}} \left| \int (\mathfrak{J}_L h^{-1}(f_{hx_1} - f_{hx_2}), e_n) \, d\mu_H(h) \right|^2$$

$$\leq \sum_{n \in \mathbb{N}} \int \left| (\mathfrak{J}_L h^{-1}(f_{hx_1} - f_{hx_2}), e_n) \right|^2 \, d\mu_H(h)$$

$$= \sum_{n \in \mathbb{N}} \int \left| (\mathfrak{J}_L h^{-1}(f_{hx_1} - f_{hx_2}), e_n) \right|^2 \, d\mu_H(h)$$

$$= \int \|\mathfrak{J}_L h^{-1}(f_{hx_1} - f_{hx_2})\|^2 \, d\mu_H(h) \leq \int \|L_{h^{-1}}f_{hx_1} - L_{h^{-1}}f_{hx_2}\|^2 \, d\mu_H(h)$$

$$\leq \sup_{h \in H} \|L_{h^{-1}}f_{hx_1} - L_{h^{-1}}f_{hx_2}\|^2 \mu_H((Kx_1^{-1} \cap H) \cup (Kx_2^{-1} \cap H))$$

$$G_2 \cap (x_0^{-1}G_1x_0) \leq \sup_{x \in \mathfrak{G}} \|f^{L,V}_{(h,e^{-1},x_1)} - f^{L,V}_{(h,e^{-1},x_2)}\|^2 \sup_{x \in \mathfrak{G}} \mu_H(Kx^{-1} \cap H).$$

Because the function $f^{L,V}$ is norm continuous on $H \times \mathfrak{G}$ and the continuity is uniform and $\sup_{x \in \mathfrak{G}} \mu_H(Kx^{-1} \cap H) < \infty$ (37, proof of Lemma 3.1) the norm continuity of $f^0$ is proved.

Similarly we get

$$\|f^0_x\|^2 \leq \sup_{h \in H} \|L_{h^{-1}}f_{hx}\|^2 \mu_H(Kx^{-1} \cap H)$$

$$= \sup_{h \in H} \|f^{L,V}_{(h,e^{-1},x)}\|^2 \mu_H(Kx^{-1} \cap H) < \infty,$$

because $K$ is compact and $f^{L,V}$ is norm continuous on $H \times \mathfrak{G}$ and compactly supported, where $V^x$ is a compact neighbourhood of $x \in \mathfrak{G}$. Therefore $\|f^0_x\| = 0$ for all $x \notin HK$. Thus as a function on $\mathfrak{G}/H$: $[x] \mapsto ([L^0]_x, f^0_x)$ and a fortiori the function $[x] \mapsto (\mathfrak{J}_L [L^0]_x, f^0_x)$ vanishes outside the compact canonical image of $HK$ in $\mathfrak{G}/H$. 33
Finally let us note that if \( h(x) \) is the element of \( H \) defined in Sect. 8 corresponding to \( x \in \mathfrak{S} \), then

\[
(\mathfrak{J}_L(\mathfrak{J}^L f^0)_x, f^0_x) = (f^0_b, f^0_b)
\]

with \( b = h(x)^{-1} x \) – the unique intersection point of the coset \( Hx \) with the Borel section \( B \). Because \( f \) is continuous with compact support, then the last assertion of the Lemma follows from Lemma 1.1 of [47]. ■

We shall denote the class of functions \( f^0 \) for \( f \in C^L \) of Lemma 11 by \( C^L_0 \).

**Lemma 2.** For each fixed \( x \in \mathfrak{S} \) the vectors \( f^0_x \) for \( f^0 \in C^L_0 \) form a dense linear subspace of \( \mathcal{H}_L \).

Note that if \( f^0 \in C^L_0 \) and \( R_s f \) is defined by the equation \( (R_s f)_x = f_{xs} \) for all \( x \) and \( s \in \mathfrak{S} \) then \( R_s f^0 = (R_s f)^0 \) so that for all \( f \in C^L \) and \( s \in \mathfrak{S} \), \( R_s f^0 \in C^L_0 \). Therefore the set \( \mathcal{H}^L_0 \) of vectors \( f^0_x \) for \( f^0 \in C^L_0 \) and \( x \) fixed is independent of \( x \). Let \( \mathcal{H}_L^0 \) be the \( \mathfrak{J}_L \)-orthogonal complement of \( \mathcal{H}^L_0 \), i.e. the set of all \( v \in \mathcal{H}_L \) such that \( (\mathfrak{J}Lg, v) = 0 \) for all \( g \in \mathcal{H}^L_0 \). Then if \( v \in \mathcal{H}_L^0 \) we have \( (f^0_x,v) = 0 \) for all \( f^0 \in C^L_0 \) and all \( x \in \mathfrak{S} \). Therefore \( (\mathfrak{J}L f^0_{hx}, v) = (\mathfrak{J}L f^0_{x}, h_{-1} v) = 0 \) for all \( f^0 \) in \( C^L_0 \), all \( x \) in \( \mathfrak{S} \) and all \( h \in H \). Hence \( \mathcal{H}_L' \) is invariant under the representation, as \( L \) is \( \mathfrak{J}_L \)-unitary. Let \( L' \) be the restriction of \( L \) to \( \mathcal{H}_L' \). Suppose that there exists a non zero member \( f^0 \) of \( C^L_0 \). Thus a fortiori \( f^0 \in C^L_0 \) and we have a contradiction since the values of \( f^0 \) are all in \( \mathcal{H}_L' \), so that we would have in \( (\mathfrak{J}L, \mathcal{H}L) \):

\[
(f^0, g)_{\mathfrak{J}L} = (\mathfrak{J}L f^0, g)
\]

and

\[
= \int (\mathfrak{J}L f^0_x, g_x) d\mu_{\mathfrak{S}/H} = 0
\]

for all \( g \in \mathcal{H}_L \), which would give us \( f^0 = 0 \), because the Krein space \( (\mathfrak{J}L, \mathcal{H}L) \) of the induced representation \( U^L \) is non degenerate (or \( \mathfrak{J}L \) invertible). Thus in order to show that \( \mathcal{H}_L = 0 \) we need only show that when \( \mathcal{H}_L' \neq 0 \) there exists a non zero member \( f^0 \) of \( C^L_0 \). But if none existed then

\[
\int (\mathfrak{J}L L_{h^{-1}} f_{hx}, v) d\mu_H(h)
\]

would be zero for all \( x \), all \( v \) in \( \mathcal{H}_L \) and all \( f \) in \( C^L \). In particular the integral would be zero for \( f = uv' \), for all continuous complex functions \( u \) on \( \mathfrak{S} \) of compact support and all \( v' \in \mathcal{H}_L' \), i.e.

\[
\int u(hx)(\mathfrak{J}L L_{h^{-1}} u', v') d\mu_H(h)
\]

would be zero for all \( x \), all \( v \) in \( \mathcal{H}_L \), all \( v' \in \mathcal{H}_L' \) and all complex continuous \( u \) of compact support on \( \mathfrak{S} \), which, because \( L \) (and thus \( L' \)) is strongly continuous, would imply that

\[
(\mathfrak{J}L L_{h^{-1}} u', v') = 0
\]
for all $\nu$ in $H$, all $\nu'$ in $H'$ and all $h \in H$. This is impossible because the Krein space $(JL, H_L)$ of the representation $L$ is non degenerate and $L_{h^{-1}}$ non-singular as a Krein-unitary operator. Thus we have proved that $H'_L = 0$. This means that $JLH''_L$ is dense in the Hilbert space $H_L$, and because $JL$ is unitary in $H_L$ with respect to the ordinary definite inner product $(\cdot, \cdot)$, this means that $H''_L$ is dense in the Hilbert space $H_L$. □

**Lemma 3.** Let $C$ be any family of functions from $G$ to $H_L$ such that:

(a) $C \subset H_L$.

(b) For each $s \in G$ there exists a positive Borel function $\rho_s$ such that for all $f \in C$, $\rho_s R_s f \in C$ where $(R_s f)_x = f_{xs}$.

(c) If $f \in C$ then $gf \in C$ for all bounded continuous complex valued functions $g$ on $G$ which are constant on the right $H$-cosets.

(d) There exists a sequence $f^1, f^2, \ldots$ of members of $C$ and a subset $P$ of $G$ of positive Haar measure such that for each $x \in P$ the members $f^1_x, f^2_x, \ldots$ of $H_L$ have $H_L$ as their closed linear span.

Then the members of $C$ have $H_L$ as their closed linear span.

Choose $f^1, f^2, \ldots$ as in the condition (d). Let $u$ be any member of $H_L$ which is $JL$-orthogonal to all members of $C$:

$$(f, u)_{JL} = (J^L f, u) = \int (JL(f_x), u_x) d\mu_{\Phi/H} = 0$$

for all $f \in C$. Then

$$(J^L(\rho_s g)(R_s f^j), u) = \int (JL((\rho_s g)(R_s f^j)_x), u_x) d\mu_{\Phi/H} = 0$$

for every $j \in \mathbb{N}$, all $s$ and every bounded continuous $g$ on $G$ which is constant on the right $H$-cosets. It follows at once that for all $s$ and all $j \in \mathbb{N}$ $\langle J_L f^j_x, u_x \rangle = 0$ for almost all $x \in G$. Since $x \mapsto \langle J_L f^j_x, u_x \rangle$ is a Borel function on $G$ the function

$$(x, s) \mapsto \langle J_L f^j_x, u_x \rangle = \sum_{n \in \mathbb{N}} \langle J_L f^j_x, e_n \rangle (e_n, u_x)$$

is Borel on the product measure space $G \times G$ on repeating the argument of Sect. 3 (Scholium 3.9 of [69]) and joining it with the fact that composition of a measurable (Borel) function on $G$ with the continuous function $G \times G \ni (x, s) \mapsto xs \in G$ is measurable (Borel) on the product measure space $G \times G$ (compare e.g. [69]). Thus we may apply the Fubini theorem (Thm. 3.4 in [69]) and conclude that for almost all $x$, $\langle J_L f^j_x, u_x \rangle$ is zero for almost all $s$. Since $j$ runs over a countable class we may select a single null set $N \subset G$ such that for each $x \notin N$, $\langle J_L f^j_x, u_x \rangle$ is, for almost all $s$, zero for all $j \in \mathbb{N}$. It follows that for
each \( x \notin N \) there exists \( s \in x^{-1}P \) such that \( (3_L f^j_{xs}, u_x) = 0 \) for \( j \in \mathbb{N} \) and hence that \( u_x = 0 \) because \( 3_L \) is unitary with respect to the ordinary definite Hilbert space inner product in the Hilbert space \( \mathcal{H}_L \). Thus \( u \) is almost everywhere zero and \( 3^L C \) must be dense in \( \mathcal{H}_L \). Because \( 3^L \) is unitary in the ordinary sense with respect to the definite inner product (eq. 11 of Sect. 3) in \( \mathcal{H}_L \), \( C \) must be dense in \( \mathcal{H}_L \).

**LEMMA 4.** Let \( C^1 \) be any family of functions from \( \mathcal{G} \) to \( \mathcal{H}_L \) such that:

(a) For each \( f \in C^1 \) there exists a positive Borel function \( \rho \) on \( \mathcal{G} \) such that

\[
\left( 3_L \frac{1}{\rho(x)} f_x, v \right) = \left( \frac{1}{\rho(x)} 3_L f_x, v \right) = \frac{1}{\rho(x)} \left( 3_L f_x, v \right)
\]

is continuous as a function of \( x \) for all \( v \in \mathcal{H}_L \).

(b) \( C^1 \subset \mathcal{H}^L \).

(c) For each \( s \in \mathcal{G} \) there exists a positive Borel function \( \rho_s \) such that for all \( f \in C^1 \), \( \rho_s R_s f \in C^1 \) where \( (R_s f)_x = f_{xs} \).

(d) If \( f \in C^1 \) then \( g f \in C^1 \) for all bounded continuous complex valued functions \( g \) on \( \mathcal{G} \) which are constant on the right \( H \)-cosets and vanish outside of \( \pi^{-1}(K) \) for some compact subset \( K \) of \( \mathcal{G} / H \).

(e) For some (and hence all) \( x \in \mathcal{G} \) the members \( f_x \) of \( \mathcal{H}_L \) for \( f \in C^1 \) have \( \mathcal{H}_L \) as their closed linear span.

Then the members of \( C^1 \) have \( \mathcal{H}^L \) as their closed linear span.

Choose \( f^1, f^2, \ldots \) in \( C^1 \) so that \( f^j_1, f^j_2, \ldots \) have \( \mathcal{H}_L \) as their closed linear span; \( e \) being the identity of \( \mathcal{G} \). Let \( u \) be any member of \( \mathcal{H}^L \) which is \( 3^L \)-orthogonal to all members of \( C^1 \). Then

\[
(3^L (\rho, g)(R_s f^j), u) = \int (3_L ((\rho_s g)(x)(R_s f^j)_x), u_x) d\mu_{\mathcal{G} / H} = 0
\]

for every \( j \in \mathbb{N} \), all \( s \) and every bounded continuous \( g \) on \( \mathcal{G} \) which is constant on the right \( H \)-cosets. It follows at once that for all \( s \) and all \( j \in \mathbb{N} \) \( (3_L f^j_{xs}, u_x) = 0 \) for almost all \( x \in \mathcal{G} \). Since \((x, s) \mapsto (3_L f^j_{xs}, u_x)\) is a Borel function on the product measure space \( \mathcal{G} \times \mathcal{G} \) (compare the proof of Lemma 3) we may apply the Fubini theorem as in the preceding Lemma and conclude that for almost all \( x \), \( (3_L f^j_{xs}, u_x) \) is zero for almost all \( s \). Since \( j \) runs over a countable class we may select a single null set \( N \) in \( \mathcal{G} \) such that for each \( x \notin N \), \( (3_L f^j_{xs}, u_x) \) is for almost all \( s \) zero for all \( j \). Suppose that \( u_{x_1} \neq 0 \) for some \( x_1 \notin N \). Then \( (3_L f^j_{x_1}, u_{x_1}) \neq 0 \) for some \( j \) as \( 3_L \) is unitary with respect to the ordinary Hilbert space inner product \((\cdot, \cdot)\) in \( \mathcal{H}_L \) (as in the proof of the preceding Lemma). But for some positive Borel function \( \rho \), \( (3_L f^j_{x_1}, u_{x_1}) / \rho(x_1) \) is continuous in \( x \). Hence \( (3_L f^j_{xs}, u_{x_1}) / \rho(x_1) \neq 0 \) for \( s \) in some neighbourhood of \( x_1^{-1} \). Thus
\((3_L f_{x, s}, u_{x_1}) \neq 0\) for \(s\) in some neighbourhood of \(x_1^{-1}\). But this contradicts the fact that \((3_L f_{x, s}, u_{x_1})\) is zero for almost all \(s \in \mathcal{G}\). Therefore \(u_x\) is zero almost everywhere. Thus only the zero element is orthogonal (in the ordinary positive inner product space in \(H^L\)) to all members of \(3^L C^1\) and it follows that \(3^L C^1\) must be dense in \(H^L\). Because \(3^L\) is unitary with respect to the ordinary definite inner product \((\cdot, \cdot)\) in \(H^L\), it follows that \(C^1\) is dense in \(H^L\). ■

**Lemma 5.** \(C^L_0\) is dense in \(H^L\).

The Lemma is an immediate consequence of Lemmas 2 and 4. ■

**Lemma 6.** There exists a sequence \(f^1, f^2, \ldots\) of elements \(C^L_0 \subset H^L\) such that for each fixed \(x \in \mathcal{G}\) the vectors \(f^k_x, k = 1, 2, \ldots\) form a dense linear subspace of \(H^L\).

We have seen in the previous Sect. that as a Hilbert space \(H^L\) is unitary equivalent to the direct integral Hilbert space \(\int H^L d\mu_{\mathcal{G}/H}\) over the \(\sigma\)-finite and regular Baire (or Borel) measure space \((\mathcal{G}/H, \mathcal{R}_{\mathcal{G}/H}, \mu_{\mathcal{G}/H})\) with separable \(H^L\). Because \(\mathcal{G}/H = \mathcal{X}\) is locally compact metrizable and fulfills the second axiom of countability its minimal (one point or Alexandroff) compactification \(\mathcal{X}_+\) is likewise metrizable (compare e. g. [22], Corollary 7.5.43). Thus the Banach algebra \(C(\mathcal{X}_+)\) is separable, compare e. g. [40], Thm. 2 or [90]). Because \(C(\mathcal{X}_+)\) is equal to the minimal unitization \(C_0(\mathcal{X})^+\) of the Banach algebra \(C_0(\mathcal{X})\) of continuous functions on \(\mathcal{X}\) vanishing at infinity (compare [57]), thus by the construction of minimal unitization it follows that \(C_0(\mathcal{X})\) is separable (of course with respect to the supremum norm in \(C_0(\mathcal{X})\)) as a closed ideal in \(C_0(\mathcal{X})^+\) of codimension one. Because the measure space \((\mathcal{G}/H, \mathcal{R}_{\mathcal{G}/H}, \mu_{\mathcal{G}/H})\) is the regular Baire measure space, induced by the integration lattice \(C_K(\mathcal{X}) \subset C_0(\mathcal{X})\) of continuous functions with compact support (compare [93]), it follows from Corollary 4.4.2 of [93] that the Hilbert space \(L^2(\mathcal{G}/H, \mu_{\mathcal{G}/H})\) of square summable functions over \(\mathcal{X} = \mathcal{G}/H\) is separable[14]. Let \(\{e_n\}_{n \in \mathbb{N}}\) be an orthonormal basis in \(H^L\). Using standard – by now – Hilbert space ([57]) and measure space (e. g. Fubini

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[14]: For the reasons explained in Sect. 8 we are interesting in complete measure spaces on \(\mathcal{G}/H\) and on all other quotient spaces encountered later in this paper. But the Baire or Borel measure is pretty sufficient in the investigation of the associated Hilbert spaces \(L^2(\mathcal{G}/H, \mu_{\mathcal{G}/H})\) or \(H^L\) as all measurable sets differ from the Borel sets just by null sets, and the space of equivalence classes of Borel square summable functions in \(L^2(\mathcal{G}/H, \mu_{\mathcal{G}/H})\) is the same as the space of equivalence classes of square summable measurable functions. Recall that the Baire measure space may be completed to a Lebesgue-type measure space, e. g. using the Carathéodory method. In other words the Baire or Borel (the same in this case) measure space may be completed such that any subset of measurable null set will be measurable.
theorem\(^\text{15}\) techniques and the results of \[52\] one can prove that
\[
\int \mathcal{H}_L d\mu_{\mathfrak{G}/H} = \bigoplus_{n \in \mathbb{N}} \int C_n d\mu_{\mathfrak{G}/H} = \bigoplus_{n \in \mathbb{N}} H_n \text{ where } H_n \sim L^2(G/H, \mu_{G/H}).
\]
Thus \(\mathcal{H}_L d\mu_{\mathfrak{G}/H}\) itself must be separable and therefore \(\mathcal{H}_L\) is separable. Thus we may choose a sequence \(f_1, f_2, \ldots\) of elements \(C_L \subset H_L\) such that for each \(f \in C_L\) there exists a subsequence \(f_{n_1}, f_{n_2}, \ldots\) which converges in norm \(\| \cdot \|\) of \(\mathcal{H}_L\) to \(f\). Then a slight and obvious modification of the standard proof of the Riesz-Fischer theorem (e. g. \[69\], Thm. 4.2) gives a sub-subsequence \(f_{n_{m_1}}, f_{n_{m_2}}, \ldots\) which, after restriction to the regular Borel section \(B \sim G/H\) converges almost uniformly to the restriction of \(f\) to \(B\) (where \(B \leq \mathfrak{G}/H\) is locally compact with the natural topology induced by the canonical projection \(\pi\), with the Baire measure space structure \((\mathfrak{G}/H, \mathcal{B}_{\mathfrak{G}/H}, \mu_{\mathfrak{G}/H}) \cong (B, \mathcal{B}_B, \mu_B)\) obtained by Mackey’s technique of quotienting the measure space \(\mathfrak{G}\) by the group \(H\) recapitulated shortly in Sect. 3). As \(f^k, f\) are continuous and compactly supported as functions on \(B \cong \mathfrak{G}/H\), the convergence is uniform on \(B\). The Lemma now, for \(x \in B\), is an immediate consequence of Lemma 2. Because for each \(x \in \mathfrak{G}\) we have \(f_x^k = L_{h(x)} f_{h(x)^{-1} x}^k, f_x = L_{h(x)} f_{h(x)^{-1} x}\) with \(h(x)^{-1} x \in B\) and because \(L_h\) is invertible (and bounded) for every \(h \in H\), the Lemma is proved. \(\blacksquare\)

5 Lopuszański representation as an induced representation

Let \(\mathfrak{G}\) be a separable locally compact group and \(H\) its closed subgroup. In this section we shall need Lemma 7 (below), which we prove assuming the “measure product property”, because it is sufficient for the analysis of the Lopuszański representation of the double covering of the Poincaré group. However it can be proved without this assumption, as the reader will easily see by recalling the respective remarks of Sect. 3.

Thus we assume (for simplicity) that the right Haar measure space \((\mathfrak{G}, \mathcal{B}_G, \mu_{G})\) be equal to the product measure space \((H \times \mathfrak{G}/H, \mathcal{B}_H \times \mathcal{B}_{\mathfrak{G}/H}, \mu_H \times \mu_{\mathfrak{G}/H})\) with \((H, \mathcal{B}_H, \mu_{H})\) equal to the right Haar measure space on \(H\) and with the Mackey quotient measure space \((\mathfrak{G}/H, \mathcal{B}_{\mathfrak{G}/H}, \mu_{\mathfrak{G}/H})\) on \(\mathfrak{G}/H\) (described briefly in Sect 3). In most cases of physical applications both \(\mathfrak{G}\) and \(H\) are unimodular. Let \(g = h \cdot q\) be the corresponding unique factorization of \(g \in \mathfrak{G}\)

\(^{15}\)Compare eq. 24 of Sect 3.
with \( h \in H \) and \( q \in Q \subset \mathfrak{G} \) representing the class \([g] \in \mathfrak{G}/H\). Uniqueness of the factorization allows us to introduce the following functions (already mentioned in Sect. 3) \((q,h_0,q_0) \mapsto h'_{q,h_0,q_0} \in H\) and \((q,h_0,q_0) \mapsto q'_{q,h_0,q_0} \in Q \subset \mathfrak{G}\), where for any \( g_0 = q_0 \cdot h_0 \in \mathfrak{G} \) we define \( h'_{q,h_0,q_0} \in H \) and \( q'_{q,h_0,q_0} \in Q \sim G \), such that

\[
q \cdot h_0 \cdot q_0 = h'_{q,h_0,q_0} \cdot q'_{q,h_0,q_0}.
\]

In particular if \( g = hq \), then \( q \) represents \([g] \in \mathfrak{G}/H\), and \( q'_{q,h_0,q_0} \) represents \([gg_0]\), i.e. the right action of \( \mathfrak{G} \) on \( \mathfrak{G}/H \). It is easily verifiable that \((q,h_0,q_0) \mapsto h'_{q,h_0,q_0} \) behaves like a multiplier, i.e. denoting \( h'_{q,h_0,q_0} \) and \( q'_{q,h_0,q_0} \) just by \( h'_{q,g_0} \) and \( q'_{q,g_0} \) we have

\[
\begin{align*}
&h'_{q,g_0} \cdot h'_{q_1,g_1} = h'_{q \cdot g_0 \cdot g_1}.
\end{align*}
\]

Let \( U^L \) be the Krein isometric representation of \( \mathfrak{G} \) induced by an almost uniformly bounded Krein-unitary representation of \( H \) in the Krein space \((H_L, J_L)\), defined as in Sect. 3. Let us introduce the Hilbert space

\[
\mathcal{H} = \int_{\mathfrak{G}/H} H_L \, d\mu_{\mathfrak{G}/H}
\]

and the fundamental symmetry \( J \)

\[
J = \int_{\mathfrak{G}/H} J_L \, d\mu_{\mathfrak{G}/H}
\]

in \( \mathcal{H} \), i.e. operator decomposable with respect to the decomposition \((1)\) whose all components in its decomposition are equal \( J_L \). Because \( J_L^2 = J_L \) and \( J_L J_L^* = J_L J_L^* = I \), then by \( (2) \) the same holds true of the operator \( J \), i.e. it is unitary and selfadjoint, i.e. \( J^* = J \) and \( J^* J = J J^* = I \), so that \( J^2 = I \) and \( J \) is a fundamental symmetry. We may therefore introduce the Krein space \((\mathcal{H}, J)\).

**Lemma 7.** Let \( \mathfrak{G} \) be a separable locally compact group and \( H \) its closed subgroup. Assume (for simplicity) that the ”measure product property” is fulfilled by \( \mathfrak{G} \) and \( H \). Then the operators

\[
U : \mathcal{H} \to \mathcal{H}^L, \quad \text{and} \quad S : \mathcal{H}^L \to \mathcal{H},
\]

declared as follows

\[
(UW)_{h,q} = L_h W_q, \quad \text{and} \quad (Sf)_q = L_{h^{-1}} f_{h \cdot q},
\]

for all \( W \in \mathcal{H} \) and \( f \in \mathcal{H}^L \), are well defined operators, both are isometric and Krein-isometric between \((\mathcal{H}, J)\) and \((\mathcal{H}^L, J^L)\) and moreover \( US = I \) and \( SU = I \) and moreover

\[
U^{-1} J^L U = J.
\]

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In short: $U$ is unitary and Krein-unitary. We have

$$
(V_{g_0} W)_q = (U^{-1} U_{g_0}^L U W)_q = \sqrt{\lambda(g_0)} L_{q,g_0} \ W_{q,g_0};
$$
or equivalently

$$
(V_{g_0} W)_q = (U^{-1} U_{g_0}^L U W)_q = \sqrt{\lambda(\varphi^1 g_0)} L_{\varphi^1 q,\varphi^1 g_0} \ W_{\varphi^1 q,\varphi^1 g_0}.
$$

In short: $U^L$ is unitary and Krein unitary equivalent to the Krein-isometric representation $V$ of $\mathfrak{G}$ in $(H, \mathfrak{J})$.

That the functions $UW, W \in H$, and $U^{-1} f, f \in H^L$ fulfill the required measurability conditions has been already shown in Sect. 3. Verification of the isometric and Krein-isometric character of both $U$ and $S$ is easy, and we leave it to the reader. Checking $US = I$ and $SU = I$ as well as the last equality is likewise simple.

Now let us turn our attention to the construction of semi-direct product groups and their specific class of Krein-isometric representations to which the Lopuszański representation belong together with the related systems of imprimitivity in the Krein space $(H, \mathfrak{J})$, say of Lemma 4. Let $G_1$ and $G_2$ be separable locally compact groups and let $G_1$ be abelian $(G_1$ plays the role of four translations subgroup $T_1$ and $G_2$ plays the role of the $SL(2,\mathbb{C})$ subgroup of the double covering $\mathfrak{G} = T_0 \mathbb{SL}(2,\mathbb{C})$ of the Poincaré group). Let there be given a homomorphism of $G_2$ into the group of automorphisms of $G_1$ and let $y[x] \in G_1$ be the action of the automorphism corresponding to $x \in G_1$. We assume that $(x, y) \mapsto y[x]$ is jointly continuous in both variables. We define the semi-direct product $\mathfrak{G} = G_1 \mathbb{S} G_2$ as the topological product $G_1 \times G_2$ with the multiplication rule $(x_1, y_1) (x_2, y_2) = (x_1 y_1 x_2, y_1 y_2)$. $\mathfrak{G} = G_1 \mathbb{S} G_2$ under this operation is a separable locally compact group. Recall that the subset of elements $(x, e)$ with $x \in G_1$ and $e$ being the identity is a closed subgroup of the semi direct product $\mathfrak{G}$ naturally isomorphic to $G_1$ and similarly the set of elements $(e, y)$, $y \in G_2$ is a closed subgroup of $\mathfrak{G} = G_1 \mathbb{S} G_2$ naturally isomorphic to $G_2$. Let us identify those subgroups with $G_1$ and $G_2$ respectively. Since $(x, e) (e, y) = (x, y)$ it follows at once that any Krein-isometric representation $(x, y) \mapsto V_{(x,y)}$ of $\mathfrak{G} = G_1 \mathbb{S} G_2$ in the Krein space $(H, \mathfrak{J})$ is determined by its restrictions $N$ and $U$ to the subgroups $G_1$ and $G_2$ respectively: $V_{(x,y)} = N_x U_y$. Conversely if $N$ and $U$ are Krein-isometric representations of $G_1$ and $G_2$ which act in the same Krein space $(H, \mathfrak{J})$ and with the same core invariant domain $\mathfrak{D}$, and moreover if the representation $N$ commutes with the fundamental symmetry $\mathfrak{J}$ and is therefore unitary, then one easily checks that $(x, y) \mapsto N_x U_y$ defines a Krein-isometric representation if and only if $U_y N_x U^{-1} = N_y[x]$. Indeed the “if” part is easy. Assume then that $V_{(x,y)} = N_x U_y$ is a representation. Then for any $(x, y), (x', y') \in G_1 \mathbb{S} G_2$ one has $N_x U_y N_{x'} U_{y'} = N_{x} N_{y[x]} U_{y[y']}$ on the core dense set $\mathfrak{D}$. Because $N_x$ is unitary it follows that $U_y N_{x'} U_{y'-1} U_{y'y'} = N_{y[x']} U_{y[y']}$. Because $U_y \mathfrak{D} = \mathfrak{D}$ for all $y \in G_2$ and $U_y U^{-1} = I$ on $\mathfrak{D}$, then it follows that $U_y N_{x'} U_{y'-1} = N_{y[x']} \mathfrak{D}$ for all $x, x' \in G_1$ and all $y \in G_2$. 

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Because the right hand side is unitary, then $U_y N_x U_{y^{-1}}$ can be extended to a unitary operator, although $U$ is in general unbounded. Now assume (which is the case for representations of translations acting in one particle states in QFT, for example this is the case for the restriction of the Lopuszański representation to the translation subgroup) that the representation $N$ of the abelian subgroup $G_1$ commutes with the fundamental symmetry $\mathfrak{F}$ in $\mathcal{H}$, and thus it is not only Krein-isometric but unitary in $\mathcal{H}$ in the usual sense. Moreover the restrictions $N$ of representations acting in one particle states are in fact of uniform (even finite) multiplicity. Because $N$ is a unitary representation of a separable locally compact abelian group $G_1$ in the Hilbert space the Neumak’s theorem is applicable, which says that $N$ is determined by a projection valued (spectral) measure $S \mapsto E_S$ (which as we will see may be associated with the direct integral decomposition (6) with the appropriate subgroup $H$), defined on the Borel (or Baire) sets $S$ of the character group $\hat{G}_1$ of $G_1$:

$$N_x = \int_{\hat{G}_1} \chi(x) \, dE(\chi).$$

It is readily verified that $N$ and $U$ satisfy the above identity if and only if the spectral measure $E$ and the representation $U$ satisfy $U_y E_S U_{y^{-1}} = E_{S[y]}$, for all $y \in G_2$ and all Borel sets $S \subset \hat{G}_1$; where the action $[\chi]y$ of $y \in G_2$ on $\chi \in \hat{G}_1$ is defined by the equation $([\chi]y, x) = (\chi, y^{-1}[x])$ (with $[\chi, x]$ denoting the value of the character $\chi \in \hat{G}_1$ on the element $x \in G_1$). Indeed:

$$U_y N_x U_{y^{-1}} = \int_{\hat{G}_1} \chi(x) \, d(U_y E(\chi) U_{y^{-1}}) = N_{y|x} = \int_{\hat{G}_1} \chi(y|x) \, dE(\chi)$$

$$= \int_{\hat{G}_1} ([\chi]y^{-1})(x) \, dE(\chi) = \int_{\hat{G}_1} \chi(x) \, dE([\chi]y). \quad (8)$$

We call such $E, N$, and $U$ a system of imprimitivity in the Krein space $(\mathcal{H}, \mathfrak{F})$, after Mackey [14] who defined the structure for representations $N$ and $U$ in Hilbert space $\mathcal{H}$ which are both unitary in the ordinary sense.

Consider now the action of $G_2$ on $\hat{G}_1$. If the spectral measure $E$ is concentrated in one of the orbits of $\hat{G}_1$ under $G_2$ let $\chi_0$ be any member of this orbit $\mathcal{O}_{\chi_0}$ and let $G_{\chi_0}$ be the subgroup of all $y \in G_2$ for which $[\chi_0]y = \chi_0$. Then $y \mapsto [\chi_0]y$ defines a one-to-one Borel set preserving map between the points of this orbit $\mathcal{O}_{\chi_0}$ and the points of the homogeneous space $G_2/G_{\chi_0} = \mathfrak{F}/H$, where $H = G_1 \cdot G_{\chi_0}$. In this way $E, N, U$, becomes a system of imprimitivity based on the homogeneous space $\mathfrak{F}/H$. Now when $E$ is concentrated on a single orbit the assumption of uniform multiplicity of $N$ would be unnecessary, but instead we may require $U$ to be “locally bounded”: $||Uf|| < c_\Delta ||f||$ for all $f \in \mathcal{H}$ whose spectral support (in their decomposition with respect to $E$) is contained within compact subset $\Delta \subset G_2/G_{\chi_0} = \mathfrak{F}/H$, with a positive constant $c_\Delta$ depending on $\Delta$. (In fact we have implicitly used the “local boundedness” in the first equality

\[ \text{FIGURE 41} \]
of \((\mathfrak{g})\). Then using ergodicity of the action of \(G_2\) (resp. \(\mathfrak{g}/H\)) one can prove uniform multiplicity of the spectral measure \(E\). A computation similar to that performed by Mackey in \[48\] (compare also \[49\], \S 6 or \[50\], \S 3.7) shows that the representation \(V(x,y) = N_x \cdot U_y\) defined by the system

\[
V(x,y) = N_x \cdot U_y
\]

is just equal to the Krein-isometric representation \(V = G_1 \circledast G_2\) in the Krein space \((\mathcal{H}, J)\) of the Lemma \[7\] with a representation \(L\) of the subgroup \(H\), which is easily checked to be Krein-unitary in case the multiplicity of \(N\) is assumed to be finite. Thus it follows the following theorem

**Theorem 5.** Let \(E, N, U\) be a system of imprimitivity giving a Krein-isometric representation \(V(x,y) = N_x \cdot U_y\) of a semi direct product \(\mathfrak{g} = G_1 \circledast G_2\) of separable locally compact groups \(G_1\) and \(G_2\) with \(G_1\) abelian in a Krein space \((\mathcal{H}, J)\) and with the representation \(N\) commuting with \(J\) and thus being unitary in \(\mathcal{H}\), for which the following assumptions are satisfied:

1) The spectral measure is concentrated on a single orbit \(\mathcal{O}_{x_0}\) in \(\mathcal{G}_1\) under \(G_2\).

2) The representation \(U\) (equivalently the representation \(V\)) is “locally bounded” with respect to \(E\).

Then the representation \(N\) (and equivalently the spectral measure \(E\)) is of uniform multiplicity. The fundamental symmetry \(J\) is decomposable with respect to the decomposition of \(\mathcal{H}\) associated (in the sense of \[52\]) to the spectral measure \(E\) of the system, and has a decomposition of the form \(7\). Assume moreover that:

3) The representation \(N\) has finite multiplicity.

Then \(V\) is unitary and Krein-unitary equivalent to a Krein-isometric representation \(U^L\) induced by a Krein unitary representation \(L\) of the subgroup \(H = G_1 \cdot G_{x_0}\) associated to the orbit.

This theorem may be given a more general form by discarding 3), but the given version is sufficient for the representations acting in one particle states of free fields with non trivial gauge freedom, and thus acting in Krein spaces (with the fundamental symmetry operator \(J\) called Gupta-Bleuler operator in physicists parlance), where the representations \(L\) act in Krein spaces \((\mathcal{H}_L, J_L)\) of finite dimension.

Consider for example the double covering \(\mathfrak{g} = T_4 \circledast SL(2, \mathbb{C})\) of the Poincaré group with the semi direct product structure defined by the following homomorphism: \(\alpha[t_z] = \alpha x \alpha^*\), where the translation \(t_{x} : (a_0, a_1, a_2, a_3) \mapsto (a_0, a_1, a_2, a_3) + (x_0, x_1, x_2, x_3)\) is written as a hermitian matrix

\[
x = \begin{pmatrix}
x_0 + x_3 & x_1 - ix_2 \\
x_1 + ix_2 & x_0 - x_3
\end{pmatrix}
\]

in the formula \(\alpha x \alpha^*\) giving \(\alpha[t_z]\) and \(\alpha^*\) is the hermitian adjoint of \(\alpha \in SL(2, \mathbb{C})\).
Characters $\chi_p \in \widehat{T}_4$ of the group $T_4$ have the following form

$$\chi_p(t_x) = e^{i(p_0 a_0 + p_1 x_1 + p_2 x_2 + p_3 x_3)},$$

for $p = (p_0, p_1, p_2, p_3)$ ranging over $\mathbb{R}^4$. For each character $\chi_p \in \widehat{T}_4$ let us consider the orbit $\mathcal{O}_x$, passing through $\chi_p$, under the action $\chi_p \mapsto [\chi_p]\alpha$, $\alpha \in SL(2, \mathbb{C})$, where $[\chi_p]\alpha$ is the character given by the formula

$$T_4 \ni t_x \xrightarrow{[\chi_p]\alpha} ([\chi_p]\alpha)(t_x) = [\chi_p](\alpha^{-1}[t_x]) = [\chi_p](\alpha^{-1}x\alpha^*) = [\chi_{\alpha\alpha}^*](x) = [\chi_{\alpha\alpha}^*](t_x),$$

where in the formulas $\alpha\alpha^*$ and $\alpha^{-1}x\alpha^*$, $x$ and $p$ are regarded as hermitian $2 \times 2$ matrices:

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} \quad \text{and} \quad p = \begin{pmatrix} p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & p_0 - p_3 \end{pmatrix}.$$

Let $G_{\chi_p}$ be the stationary subgroup of the point $\chi_p \in \widehat{T}_4$. Let $H = H_{\chi_p} = T_4 \cdot G_{\chi_p}$, and let $L'$ be a Krein-unitary representation of the stationary group $G_{\chi_p}$. Then $L$ given by

$$L_{t_x, g} = [\chi_p(t_x)]L_{g}, \quad t_x \in T_4, g \in G_{\chi_p},$$

is a well defined Krein-unitary representation of $H_{\chi_p} = T_4 \cdot G_{\chi_p}$ because $G_{\chi_p}$ is the stationary subgroup for the point $\chi_p$. The functions $(q, h_0, q_0) \mapsto h'_{q, h_0, q_0} \in H$ and $(q, h_0, q_0) \mapsto q'_{x, h_0, q_0} \in Q \cong \mathfrak{N}/H_{\chi_p}$ corresponding to the respective $H = H_{\chi_p}$ or the respective orbits $\mathcal{O}_x$, are known for all orbits in $\widehat{T}_4$ under $SL(2, \mathbb{C})$ and may be explicitly computed.

For example for $p = (1, 0, 0, 1)$ lying on the light cone in the joint spectrum $\text{sp}(P_0, \ldots, P_3)$ of the canonical generators of one parameter subgroups of translations, the stationary subgroup $G_{\chi_p} = G_{\chi_{(1,0,0,1)}}$ is equal to the group of matrices

$$\begin{pmatrix} e^{i\phi/2} & e^{i\phi/2}z \\ 0 & e^{-i\phi/2} \end{pmatrix}, \quad 0 \leq \phi < 4\pi, \quad z \in \mathbb{C}$$

isomorphic to (the double covering of) the symmetry group $E_2$ of the Euclidean plane and with the orbit $\mathcal{O}_{\chi_{(1,0,0,1)}}$ equal to the forward cone with the apex removed.

Consider then the Hilbert space $\mathcal{H}_L$ to be equal $\mathbb{C}^4$ with the standard inner product and with the fundamental symmetry equal

$$J_L = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally let $L'$ be the following Krein-unitary representation

$$\begin{pmatrix} e^{i\phi/2} & e^{i\phi/2}z \\ 0 & e^{-i\phi/2} \end{pmatrix} L'_{z, \phi} \rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} |z|^2 & \frac{1}{\sqrt{2}} z & \frac{1}{\sqrt{2}} \frac{1}{2} |z|^2 & -\frac{1}{\sqrt{2}} |z|^2 \\ \frac{1}{\sqrt{2}} e^{-i\phi} z & e^{-i\phi} & 0 & \frac{1}{\sqrt{2}} e^{-i\phi} \frac{1}{2} |z|^2 \\ \frac{1}{\sqrt{2}} e^{i\phi} z & 0 & e^{i\phi} & -\frac{1}{\sqrt{2}} e^{i\phi} \frac{1}{2} |z|^2 \\ \frac{1}{\sqrt{2}} |z|^2 & \frac{1}{\sqrt{2}} z & \frac{1}{\sqrt{2}} \frac{1}{2} |z|^2 & 1 - \frac{1}{2} |z|^2 \end{pmatrix}.$$ (9)
of \( G_{x(1,0,0,1)} \cong \widetilde{E}_2 \) in the Krein space \((\mathcal{H}_L, \mathfrak{J}_L)\) and define the Krein-unitary representation \( L: H = T_4 \cdot G_{x(1,0,0,1)} \ni t_x \cdot (z, \phi) \xrightarrow{L_{tx(z,\phi)}} \chi_{x(1,0,0,1)} (t_x) L'_{(z,\phi)} \) corresponding to the Krein-unitary representation \( L' \) of \( G_{x(1,0,0,1)} \). Then one obtains in this way the system of imprimitivity with the representation \( V \) of the Lemma equal to the Lopuszański representation acting in the one particle states of the free photon field in the momentum representation, having exactly Wigner’s form \( [78] \) with the only difference that \( L \) is not unitary but Krein-unitary.

Several remarks are in order.

1) In case of \( \mathfrak{G} = T_4 \otimes SL(2, \mathbb{C}), \widetilde{T}_1 = \mathbb{R}^4 \) with the natural smooth action of \( SL(2, \mathbb{C}) \) giving it the Lorentz structure. The possible orbits \( \mathcal{O}_x \subset \widetilde{T}_1 = \mathbb{R}^4 \) are: the single point \((0, 0, 0, 0)\) – the apex of “the light-cone”, the upper/lower half of the light cone (without the apex), the upper/lower sheet of the paraboloid, and the one-sheet hyperboloid. Thus all of them are smooth manifolds (with the exclusion of the apex, of course). Joining this with the Mackey analysis of quasi invariant measures on homogeneous \( \mathfrak{G}/H \) spaces one can see that the spectral measures of the translation generators (for representations with the joint spectrum \( \text{sp}(P_0, \ldots, P_3) \) concentrated on single orbits) are equivalent to measures induced by the Lebesgue measure on \( \mathbb{R}^4 = \widetilde{T}_1 \) (of course with the exclusion of the representations corresponding the the apex – the single point orbit, with the zero \((0, 0, 0, 0)\) as the only value of the joint spectrum \( \text{sp}(P_0, \ldots, P_3) \).

2) Note that for the system of imprimitivity \( E, N, U \) in the Krein space the condition:

\[
V_{(x,y)} E_S V_{(x,y)}^{-1} = N_x U_y E_S U_y^{-1} N_x^{-1},
\]

\[= N_x E_{[S]y} N_x^{-1} = E_{[S]y} \quad \text{for all } (x,y) \in G_1 \otimes G_2 \text{ and all Borel sets } S \subset \widetilde{G}_1 \]
holds, and is essentially equivalent to the condition:

\[ U_y E_S U_y^{-1} = E_{[S]y}, \quad \text{for all } y \in G_2, \text{ and all Borel sets } S \subset \widetilde{G}_1. \]

We may write it as \( V_{(x,y)} E_S V_{(x,y)}^{-1} = E_{[S|x,y]} \), with the trivial action \([\chi]|[x,e] = \chi, \ x \in G_1 \) and \([\chi]|(e,y) = [\chi]|y \). It is more convenient to relate the system of imprimitivity immediately to \( V \) and inspired by Mackey put the following more general definition.

Let \( V \) be a Krein-isometric representation of a separable locally compact group \( \mathfrak{G} \) in a Krein space \((\mathcal{H}, \mathfrak{J})\). By a system of imprimitivity for \( V \), we mean the system \( E, B, \varphi \) consisting of

a) an analytic Borel set \( B \);

b) an anti-homomorphism \( \varphi \) of \( \mathfrak{G} \) into the group of all Borel automorphisms of \( B \) such that \((y,b) \mapsto (y, [b]y)\) is a Borel automorphism of \( \mathfrak{G} \times B \); here we have written \([b]y\) for the action of the automorphism \( \varphi(y) \) on \( b \in B \).
c) The spectral measure $E$ consists of selfadjoint and Krein selfadjoint projections commuting with $\mathfrak{J}$ in $(\mathcal{H}, \mathfrak{J})$, and is such that $V_y E S V_y^{-1} = E[S] y^{-1}$.

d) The representation $V$ is “locally bounded” with respect to $E$.

Any induced Krein-isometric representation $\mu U_L$ possesses a canonical system of imprimitivity in $(\mathcal{H}^L, \mathfrak{J}^L)$ related to it. Namely let $S$ be a Borel set on $\mathcal{O}/H$, and let $S'$ be its inverse under the quotient map $\mathcal{O} \to \mathcal{O}/H$. Let $1_{S'}$ be the characteristic function of $S'$. Then $f \mapsto E_{S'} 1_{S'} f$, $f \in \mathcal{H}$ is a self adjoint and Krein self adjoint projection, which commutes with $\mathfrak{J}^L$. Thus $S \mapsto E_{S}$ is a spectral measure based on the analytic Borel space $\mathcal{O}/H$. By the inequality (4) in the proof of Theorem 1 the representation $\mu U_L$ is “locally bounded”, i.e. fulfills condition 3) of Theorem 5 or condition d).

The representation $V$ of Lemma 7 in the Krein space $(\mathcal{H}, \mathfrak{J})$ together with the spectral measure $E'$ on $B = \mathcal{O}/H$ associated with the decomposition (6) is a system of imprimitivity in Krein space which by Lemma 8 is Krein-unitary and unitary equivalent to the canonical system of imprimitivity $U_L$, $E$, $\varphi$ defined above. That $V, E'$ of Lemma 8 with $\varphi_{g_0}(q) = q'_{g_0}$ composes a system of imprimitivity can be checked directly using the multiplier property of the function $(q, g_0) \mapsto h'_{q, g_0}$.

3) The plan for further computations is the following. First we start with the systems of imprimitivity fulfilling the conditions 1)-3) of Theorem 5 sufficient for accounting for the representations acting in one particle states of free fields. Then we prove the “subgroup” and “Kronecker product theorems” for the induced representations in order to achieve decompositions of tensor products of these representations into direct integrals of representations connected with imprimitivity systems concentrated on single orbits (using Mackey double-coset-type technics). The component representations of the decomposition will not in general have the standard form of induced representations (contrary to what happens for tensor products of induced representations of Mackey which are unitary in ordinary sense). But then we back to Theorem 5 applied again to each of the component representations in order to restore the standard form of induced representation in Krein space to each of them separately. In this way we may repeat the procedure of decomposing tensor product of the component representations (now in the standard form) and continue it potentially in infinitum. It turns out that the condition 3) of finite multiplicity will have to be abandoned in further stages of this process, but we have all the grounds for the condition 2) of “local boundedness” to be preserved in all cases at all levels of the decomposition. Indeed recall that the spectral values $(p_0, \ldots, p_3)$ of the translation generators (four-momentum operators) in the tensor product of representations corresponding to imprimitivity systems concentrated on single orbits $O', O'' \subset \hat{T}_4$, are the sums $(p_0', \ldots, p_3') + (p_0'', \ldots, p_3'')$, with the spectral values $(p_0', \ldots, p_3')$ and $(p_0'', \ldots, p_3'')$ ranging over $O'$ and $O''$ respectively. Now the geometry of the orbits in case of $\mathcal{O} = \hat{T}_4 \otimes SL(2, \mathbb{C})$ is such that the sets of all values $(p_0, \ldots, p_3)$ and $(p_0', \ldots, p_3')$ for which $(p_0, \ldots, p_3)$ ranges over a compact set, are compact (discarding irrelevant null sets of $(p_0, \ldots, p_3)$ not belonging to
the joint spectrum of momentum operators of the tensor product representation – the light cones – in the only case of tensoring representation corresponding to the positive energy light cone orbit with the representation corresponding to the negative energy light cone).

4) In fact the representation of one particle states in the Fock space (with the Gupta-Bleuler or fundamental symmetry operator) is induced by the following representation $L''$ in the above defined Krein space $(\mathcal{H}_L, J_L)$ of the double covering of the symmetry group of the Euclidean plane:

$$L''(z, \phi) = \begin{pmatrix}
1 + \frac{1}{2}|z|^2 & \frac{1}{2}(\overline{z} + z) & \frac{1}{2}(z - \overline{z}) & -\frac{1}{2}|z|^2 \\
\frac{1}{2}(e^{-i\phi}z + e^{i\phi}z) & \cos \phi & \sin \phi & -\frac{1}{2}(e^{-i\phi}z + e^{i\phi}z) \\
\frac{1}{2}(e^{i\phi}z - e^{-i\phi}z) & -\sin \phi & \cos \phi & -\frac{1}{2}(e^{i\phi}z - e^{-i\phi}z) \\
\frac{1}{2}|z|^2 & \frac{1}{2}(\overline{z} + z) & \frac{1}{2}(z - \overline{z}) & 1 - \frac{1}{2}|z|^2
\end{pmatrix}, \quad (10)$$

compare e.g. [76, 77], or [45, 46]. But the operator

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1/\sqrt{2} & i/\sqrt{2} & 0 \\
0 & 1/\sqrt{2} & -i/\sqrt{2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

which is Krein-unitary and unitary in $(\mathcal{H}_L, J_L)$ sets up Krein-unitary and unitary equivalence between the representation $L'$ of (9) and the representation $L''$ of (10) as well as between the associated representations $L$. By Theorem 4 it makes no difference which one we use, but for some technical reasons we prefer the representation $L$ associated with (9).

5) The representation which we have called by the name of Lopuszański have appeared in physics rather very early, compare [79], and then in relation to the Gupta-Bleuler quantization of the free photon field: [76, 77], [36], [42]. But it was Lopuszański [45, 46] who initiated a systematic study of the relation of the representation with the Gupta-Bleuler formalism. That’s why we call the representation after him.

6 Knorkecker product of induced representations in Krein spaces

In this Section we define the outer Kronecker product and inner Kronecker product of Krein isometric (and Krein unitary) representations and give an important theorem concerning Krein isometric representation induced by a Kronecker product of Krein-unitary representations.

The whole construction is based on the ordinary tensor product of the associated Hilbert spaces and operators in the Hilbert spaces. We recapitulate shortly a specific realization of the tensor product of Hilbert spaces as trace class conjugate-linear operators, in short we realize it by the Hilbert-Schmidt
class of conjugate-linear operators\textsuperscript{16} with the standard operator $L^2$-norm, for details we refer the reader to the original paper by Murray and von Neumann \textsuperscript{51}.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two separable Hilbert spaces over $\mathbb{C}$ (recall that by the proof of Lemma \textsuperscript{6} the Hilbert space $\mathcal{H}^L$ of the Krein-isometric representation $U^L$ of a separable locally compact group $\mathfrak{G}$ induced by a Krein-unitary representation $L$ of a closed subgroup $G_1 \subset \mathfrak{G}$ is separable). A mapping $T$ of $\mathcal{H}_2$ to $\mathcal{H}_1$ is conjugate-linear iff $T(\alpha f + \beta g) = \overline{\alpha} T(f) + \overline{\beta} T(g)$ for all $f, g \in \mathcal{H}_2$ and all complex numbers $\alpha$ and $\beta$, with the “over-line” sign standing for complex conjugation. For any such conjugate-linear operator $T$ we define the conjugate version of its adjoint $T^*$, namely this is the operator fulfilling $(Tg, f) = (T^* f, g)$ for all $f \in \mathcal{H}_1$ and all $g \in \mathcal{H}_2$. In particular if $T$ is bounded, conjugate-linear, finite-rank operator so is its conjugate adjoint $T^*$. If $U_1$ and $U_2$ are bounded operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively then $U_1TU_2$ is a finite rank operator from $\mathcal{H}_2$ into $\mathcal{H}_1$. One easily verifies that $(ATB)^* = B^*T^*A^*$, where $A$ and $B$ are linear operators in $\mathcal{H}_1^1$ and $\mathcal{H}_2^2$ with $A^*$ and $B^*$ equal to their ordinary adjoint operators. If $U_1$ and $U_2$ are densely defined operators in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively on linear domains $\mathcal{D}_1 \subset \mathcal{H}_1$ and $\mathcal{D}_2 \subset \mathcal{H}_2$ and $T$ is finite rank operator with the rank contained in $\mathcal{D}_1$ and supported in $\mathcal{D}_2$, then $U_1TU_2$ is a well defined finite rank operator. Let $\mathcal{H}' = \mathcal{H}_1 \otimes \mathcal{H}_2$ be the linear space of finite rank conjugate-linear operators $T$ of $\mathcal{H}_2$ into $\mathcal{H}_1$. For any two such operators $T$ and $S$ the operator $TS^*$ is linear from $\mathcal{H}_1$ into $\mathcal{H}_1$ and of finite rank (similarly $T^*S$ is linear and finite rank from $\mathcal{H}_2$ into $\mathcal{H}_2$). We may therefore introduce the following inner product in $\mathcal{H}'$:

$$\langle T, S \rangle = \text{Tr}[TS^*] = \sum_n (TS^*e_n, e_n)$$

$$= \sum_n (T^*e_n, S^*e_n) = \sum_m (T\varepsilon_m, S\varepsilon_m)$$

$$= \sum_m (T^*S\varepsilon_m, \varepsilon_m) = \text{Tr}[T^*S],$$

where $\{e_n\}_{n \in \mathbb{N}}$ and $\{\varepsilon_m\}_{m \in \mathbb{N}}$ are orthonormal bases in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The completion of $\mathcal{H}'$ with respect to this inner product composes the tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $A$ and $B$ be bounded operators in $\mathcal{H}_1$ and $\mathcal{H}_2$. Their tensor product $A \otimes B$ acting in $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined as the operator $T \mapsto ATB^*$, for $T \in \mathcal{H}_1 \otimes \mathcal{H}_2$. In particular if for any $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ we define the finite rank conjugate-linear operator $T_{f,g} : w \mapsto f \cdot (g, w)$ supported on the linear subspace generated by $g$ with the range generated by $f$, then $T_{f,g} \in \mathcal{H}_1 \otimes \mathcal{H}_2$ is written as $f \otimes g$ and we have $(f_1 \otimes g_1, f_2 \otimes g_2) = \text{Tr} \left[ T_{f_1,g_1} \left( T_{f_2,g_2} \right)^* \right] = \text{Tr} \left[ T_{f_1,g_1} T_{g_2,f_2}^* \right] = (f_1, f_2) \cdot (g_1, g_2)$ because $(T_{g_2,f_2})^* = T_{g_2,f_2}$.

\textsuperscript{16}Alternatively one may consider linear Hilbert-Schmidt class operators, but replace one of the Hilbert spaces in question by its conjugate space, compare \textsuperscript{47}, §5.
If \((\mathcal{H}_1, \mathcal{J}_1)\) and \((\mathcal{H}_2, \mathcal{J}_2)\) are two Krein spaces, then we define their tensor product as the Krein space \(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{J}_1 \otimes \mathcal{J}_2\); verification of the self-adjointness of \(\mathcal{J}_1 \otimes \mathcal{J}_2\) and the property \((\mathcal{J}_1 \otimes \mathcal{J}_2)^2 = I\) is immediate.

We say an operator \(T\) from \(\mathcal{H}_2\) into \(\mathcal{H}_1\) is supported by finite dimensional (or more generally: closed) linear subspace \(\mathfrak{M} \subset \mathcal{H}_2\) or by the projection \(P_{\mathfrak{M}}\), in case \(T = TP_{\mathfrak{M}}\), where \(P_{\mathfrak{M}}\) is the self adjoint projection with range \(\mathfrak{M}\). Similarly we say an operator \(T\) from \(\mathcal{H}_2\) into \(\mathcal{H}_1\) has range in a finite dimensional (or more generally: closed) linear subspace \(\mathfrak{N} \subset \mathcal{H}_1\), in case \(T = P_{\mathfrak{N}}T\), where \(P_{\mathfrak{N}}\) is the self adjoint projection with range \(\mathfrak{N}\). One easily verifies the following tracial property. Let \(B\) be any finite rank and linear operator from \(\mathcal{H}_1\) into \(\mathcal{H}_1\) supported on a finite dimensional linear subspace of the domain \(\mathcal{D}_1\) and with the range also finite dimensional and lying in \(\mathcal{D}_1\). Then for any linear operator defined on the dense domain \(\mathcal{D}_1 \subset \mathcal{H}_1\) and preserving it, i.e. with \(\mathcal{D}_1\) contained in the common domain of \(A\) and its adjoint \(A^*\), we have the tracial property

\[
\text{Tr } [BA] = \text{Tr } [AB].
\]

Indeed any such linear \(B\) is a finite linear combination of the operators \(T_{f,j}^{(j')}\), defined as follows: \(T_{f,j}^{(j')}(w) = (w,f) \cdot j\). By linearity it will be sufficient to establish the tracial property for the linear operator \(B\) of the form \(B = T_{f_1,f_2} + T_{f_3,f_4}\) with \(f_i \in \mathcal{D}_1\), \(i = 1, 2, 3, 4\). Using the Gram-Schmidt orthogonalization we construct an orthonormal basis \(\{e_n\}_{n \in \mathbb{N}}\) of \(\mathcal{H}_1\) with \(e_n \in \mathcal{D}_1\). We have in this case

\[
\text{Tr } [BA] = \text{Tr } \left[ (T_{f_1,f_2} + T_{f_3,f_4})A \right] = \text{Tr } [T_{f_1,f_2}A] + \text{Tr } [T_{f_3,f_4}A] = \sum_n (T_{f_1,f_2}Ae_n, e_n) + \sum_n (T_{f_3,f_4}Ae_n, e_n) = \sum_n (Ae_n, f_1) \cdot (f_2, e_n) + \sum_n (Ae_n, f_3) \cdot (f_4, e_n) = \sum_n (e_n, A^*f_1) \cdot (f_2, e_n) + \sum_n (e_n, A^*f_3) \cdot (f_4, e_n) = (Af_2, f_1) + (Af_4, f_3) < \infty, \quad (11)
\]

because by the assumed properties of the operator \(A\) the vectors \(f_1, f_3 \in \mathcal{D}_1\) are contained in the domain of \(A^*\) and likewise the vectors \(f_2, f_4 \in \mathcal{D}_1\) lie in the domain of \(A\). Similarly we have:

\[
\text{Tr } [AB] = \text{Tr } \left[ (AT_{f_1,f_2} + AT_{f_3,f_4}) \right] = \text{Tr } [AT_{f_1,f_2}] + \text{Tr } [AT_{f_3,f_4}] = \sum_n (AT_{f_1,f_2}e_n, e_n) + \sum_n (AT_{f_3,f_4}e_n, e_n) = \sum_n (e_n, f_1) \cdot (Af_2, e_n) + \sum_n (e_n, f_3) \cdot (Af_4, e_n) = (Af_2, f_1) + (Af_4, f_3) < \infty. \quad (12)
\]
Comparing (11) and (12) we obtain the tracial property.

Now let $U_1 = U^L_1$ and $U_2 = U^M_2$ be densely defined and closable Krein isometric operators of the respective Krein isometric induced representations of the groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ in $\mathcal{H}_1 = \mathcal{H}^L$ and $\mathcal{H}_2 = \mathcal{H}^M$ respectively with linear domains $\mathfrak{D}_i \subset \mathcal{H}_i$, $i = 1, 2$, equal to the corresponding domains $\mathfrak{D}$ of Theorem 1 and Remark 2 and with the respective fundamental symmetries $\mathfrak{J}_1 = \mathfrak{J}^L$, $\mathfrak{J}_2 = \mathfrak{J}^M$. Therefore by Theorem 1 and Remark 2 $U^*(\mathfrak{D}_i) = \mathfrak{D}_i$ and $\mathfrak{J}_i(\mathfrak{D}_i) = \mathfrak{D}_i$, $i = 1, 2$, so that $\mathfrak{D}_i$ is contained in the domain of $U^*_i$ and $U^*_i(\mathfrak{D}_1) = \mathfrak{D}_1$. Finally let $T, S$ be any finite rank operators in the linear subspace $\mathfrak{D}_{12} = \text{linear span}\{T_{f,g}, f \in \mathfrak{D}_1, g \in \mathfrak{D}_2\}$ of finite rank operators supported in $\mathfrak{D}_2$ and with ranges in $\mathfrak{D}_1$. In particular for each $S \in \mathfrak{D}_{12}$, $S^*$ is supported in $\mathfrak{D}_1$ and has rank in $\mathfrak{D}_2$. By the known property of Hilbert Schmidt operators $\mathfrak{D}_1 \otimes \mathfrak{D}_2 = \mathfrak{D}_{12}$ is dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. We claim that $U_1 \otimes U_2$ is well defined on $\mathfrak{D}_1 \otimes \mathfrak{D}_2 = \mathfrak{D}_{12}$. Indeed, by the Gram-Schmidt orthonormalization we may construct an orthonormal base $\{e_n\}_{n \in \mathbb{N}}$ of $\mathcal{H}_1$ with each $e_n$ being an element of the linear dense domain $\mathfrak{D}_1$. For any $f_1, f_2 \in \mathfrak{D}_1$ and $g_1, g_2 \in \mathfrak{D}_2$ we have

$$
\left\| (U_1 \otimes U_2)(f_1 \otimes g_1 + f_2 \otimes g_2) \right\|^2
\begin{aligned}
&= \left\langle U_1(T_{f_1,g_1} + T_{f_2,g_2})U_2^*, \ U_1(T_{f_1,g_1} + T_{f_2,g_2})U_2^* \right\rangle \\
&= \text{Tr} \left[ U_1(T_{f_1,g_1})U_2^*(U_1(T_{f_1,g_1})U_2^*)^* \right] + \text{Tr} \left[ U_1(T_{f_1,g_1})U_2^*(U_1(T_{f_2,g_2})U_2^*)^* \right] \\
&+ \text{Tr} \left[ U_1(T_{f_2,g_2})U_2^*(U_1(T_{f_1,g_1})U_2^*)^* \right] + \text{Tr} \left[ U_1(T_{f_2,g_2})U_2^*(U_1(T_{f_2,g_2})U_2^*)^* \right] \\
&= \sum_n (U_1 f_1, e_n) \cdot (g_1, U_2^* U_2 g_1) \cdot (U_1^* e_n, f_1) \\
&+ \sum_n (U_1 f_1, e_n) \cdot (g_1, U_2^* U_2 g_2) \cdot (U_1^* e_n, f_2) \\
&+ \sum_n (U_1 f_2, e_n) \cdot (g_2, U_2^* U_2 g_1) \cdot (U_1^* e_n, f_1) \\
&+ \sum_n (U_1 f_2, e_n) \cdot (g_2, U_2^* U_2 g_2) \cdot (U_1^* e_n, f_2).
\end{aligned}
$$

Because $\mathfrak{D}_1$ is in the domain of $U_1^*$ and $U_1^*(\mathfrak{D}_1) = U_1(\mathfrak{D}_1) = \mathfrak{D}_1$ and similarly
for $U_2$, the last expression is equal to

$$\sum_n (U_1 f_1, e_n) \cdot (U_2 g_1, U_2 g_1) \cdot (e_n, U_1 f_1)$$

$$+ \sum_n (U_1 f_1, e_n) \cdot (U_2 g_1, U_2 g_2) \cdot (e_n, U_1 f_2)$$

$$+ \sum_n (U_1 f_2, e_n) \cdot (U_2 g_2, U_2 g_1) \cdot (e_n, U_1 f_1)$$

$$+ \sum_n (U_1 f_2, e_n) \cdot (U_2 g_2, U_2 g_2) \cdot (e_n, U_1 f_2)$$

$$= (U_1 f_1, U_1 f_1) \cdot (U_2 g_1, U_2 g_1) + (U_1 f_1, U_1 f_2) \cdot (U_2 g_1, U_2 g_2)$$

$$+ (U_1 f_2, U_1 f_1) \cdot (U_2 g_2, U_2 g_1) + (U_1 f_2, U_1 f_2) \cdot (U_2 g_2, U_2 g_2) < \infty,$$

so that

$$\left\| (U_1 \otimes U_2)(f_1 \otimes g_1 + f_2 \otimes g_2) \right\|^2$$

$$= \left\langle U_1(T_{1, g_1} + T_{2, g_2})U_2^* , U_1(T_{1, g_1} + T_{2, g_2})U_2^* \right\rangle < \infty$$

and $(U_1 \otimes U_2)(f_1 \otimes g_1 + f_2 \otimes g_2)$ is well defined. By induction for each $T \in \mathcal{D}_{12}$, $(U_1 \otimes U_2)(T) = U_1 T U_2^*$ is well defined conjugate-linear operator of Hilbert-Schmidt class, so that $U_1 \otimes U_2$ is well defined on the linear domain $\mathcal{D}_{12}$ dense in $\mathcal{H}_1 \otimes \mathcal{H}_2$. By the Proposition of Chap. VIII.10, page 298 of [63] it follows that $U_1 \otimes U_2$ is closable. Next, let $T, S \in \mathcal{D}_{12}$, then by Theorem 1 and Remark 2

$$\mathcal{J}_i(\mathcal{D}_i) = \mathcal{D}_1 \text{ and } U_i(\mathcal{D}_i) = \mathcal{D}_i \text{ and and } U_i^*(\mathcal{D}_i) = \mathcal{D}_i$$

$$(U_i)^\dagger U_i = \mathcal{J}_i U_i^* \mathcal{J}_i U_i = I \text{ and } U_i \mathcal{J}_i U_i^* \mathcal{J}_i = I \text{ on } \mathcal{D}_i.$$ 

Thus for each $T, S \in \mathcal{D}_{12}$ the following expressions are well defined and (e. g. for $T = T_{1, g_1}$ and $S = T_{2, g_2}$)

$$\left\langle (\mathcal{J}_1 \otimes \mathcal{J}_2)(U_1 \otimes U_2)(f_1 \otimes g_1) , (U_1 \otimes U_2)(f_2 \otimes g_2) \right\rangle$$

$$= \left\langle \mathcal{J}_1 U_1 T U_2^* \mathcal{J}_2, U_1 S U_2^* \right\rangle = \text{Tr} \left[ \mathcal{J}_1 U_1 T U_2^* \mathcal{J}_2 (U_1 S U_2^*)^* \right]$$

$$= \text{Tr} \left[ \mathcal{J}_1 U_1 T U_2^* \mathcal{J}_2 U_2 S^* U_1^* \right] = \text{Tr} \left[ \mathcal{J}_1 U_1 T \mathcal{J}_2 U_2 S^* U_1^* \right]$$

$$= \text{Tr} \left[ \mathcal{J}_1 U_1 T \mathcal{J}_2 \mathcal{J}_2 U_2 S^* U_1^* \right] = \text{Tr} \left[ \mathcal{J}_1 U_1 T \mathcal{J}_2 S^* U_1^* \right]$$

$$= \text{Tr} \left[ \mathcal{J}_1 U_1 T \mathcal{J}_2 \mathcal{J}_1 \{ \mathcal{J}_1 U_1^* \mathcal{J}_1 U_1 \} \right]$$

$$= \text{Tr} \left[ \mathcal{J}_1 T \mathcal{J}_2 S^* \mathcal{J}_1 \right] = \text{Tr} \left[ \mathcal{J}_1 T \mathcal{J}_2 S^* \mathcal{J}_1 \right]$$

$$= \left( (\mathcal{J}_1 \otimes \mathcal{J}_2)(f_1 \otimes g_1) , f_2 \otimes g_2 \right).$$

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because the tracial property is applicable to the pair of operators

\[ B = T\mathcal{J}_2 S^* U_1^* \quad \text{and} \quad A = \mathcal{J}_1 U_1 \]

as well as to the pair of operators

\[ B = T\mathcal{J}_2 S^* \quad \text{and} \quad A = \mathcal{J}_1, \]

as both the operators \( B \) are linear finite rank operators supported on finite dimensional subspaces contained in \( \mathcal{D}_1 \) and with finite dimensional ranges contained in \( \mathcal{D}_1 \) and for the operators \( A \) indicated above the linear domain \( \mathcal{D}_1 \) is contained in the common domain of \( A \) and \( A^* \); and moreover \( \mathcal{J}_1(U_1)^* \mathcal{J}_1 U_1 \) and \( \mathcal{J}_2 U_1^* \mathcal{J}_1 U_2 \) are well defined unit operators on the domains \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \), respectively. Therefore \( U_1 \otimes U_2 = U^L \otimes U^M \) is Krein-isometric on its domain \( \mathcal{D}_{12} \) which holds by continuity for its closure.

We may therefore define the outer Kronecker product Krein-isometric representation \( U^{L \times M} : \mathcal{G}_1 \times \mathcal{G}_2 \ni (x, y) \mapsto U^L_x \otimes U^M_y \) of the product group \( \mathcal{G}_1 \times \mathcal{G}_2 \), which is Krein isometric in the Krein space \( (\mathcal{H}^L \otimes \mathcal{H}^M, \mathcal{J}_L \otimes \mathcal{J}_M) \). All the more, if \( U_1 \) and \( U_2 \) are Krein-unitary representations of \( G_1 \) and \( G_2 \), respectively in \((\mathcal{H}_1, \mathcal{J}_1)\) and \((\mathcal{H}_2, \mathcal{J}_2)\), so is \( U_1 \times U_2 \) in the Krein space \((\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{J}_1 \otimes \mathcal{J}_2)\). Similarly one easily verifies that \( U_1 \times U_2 \) is almost uniformly bounded whenever \( U_1 \) and \( U_2 \) are. In particular if \( G_1 \) and \( G_2 \) are two closed subgroups of the separable locally compact groups \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) respectively and \( L \) and \( M \) their Krein unitary and uniformly bounded representations, then we may define the outer Kronecker product representation \( L \times M \) of the product group \( G_1 \times G_2 \) by the ordinary formula \( \mathcal{G}_1 \times \mathcal{G}_2 \ni (\xi, \eta) \mapsto L_\xi \otimes M_\eta \), which is Krein unitary and almost uniformly bounded in the Krein space \((\mathcal{H}_L \otimes \mathcal{H}_M, \mathcal{J}_L \otimes \mathcal{J}_M)\) whenever \( L \) and \( M \) are in the respective Krein spaces \((\mathcal{H}_L, \mathcal{J}_L)\) and \((\mathcal{H}_M, \mathcal{J}_M)\). We may therefore define the Krein-isometric representation \( \mu_1 \times \mu_2 U^{L \times M} \) of the group \( \mathcal{G}_1 \times \mathcal{G}_2 \) in the Krein space \( \mathcal{H}^{L \times M} \) induced by the representation \( L \times M \) of the closed subgroup \( G_1 \times G_2 \), where \( \mu_i \) are the respective quasi invariant measures in \( \mathcal{G}_i/G_i \).

Let us make an observation used in the proof of the Theorem of this Section. Let \( B_1 \) be a Borel section of \( \mathcal{G}_1 \) with respect to \( G_1 \) and respectively \( B_2 \) a Borel section of \( \mathcal{G}_2 \) with respect to \( G_2 \) defined as in Section 3 with the associated Borel functions \( h_1 : \mathcal{G}_1 \ni x \mapsto h_1(x) \in G_1 \) such that \( h_1(x)^{-1} x \in B_1 \) and \( h_2 : \mathcal{G}_2 \ni y \mapsto h_2(y) \in G_2 \) such that \( h_2(y)^{-1} y \in B_2 \). Then \( B_1 \times B_2 \) is a Borel section of \( \mathcal{G}_1 \times \mathcal{G}_2 \) with respect to the closed subgroup \( G_1 \times G_2 \) with the associated Borel function \( h : (x, y) \mapsto h(x, y) \in G_1 \times G_2 \) such that \( h(x, y)^{-1}(x, y) \in B_1 \times B_2 \), equal to \( h(x, y) = (h_1(x), h_2(y)) = h_1(x) \times h_2(y) \). Let \( w \in \mathcal{H}^{L \times M} \). Thus the corresponding operator \( \mathcal{J}^{L \times M} \) acts as follows

\[
(\mathcal{J}^{L \times M} w)_{(x, y)} = (L \times M)_{h_1(x) \times h_2(y)} \circ \mathcal{J}^{L \times M} \circ (L \times M)_{h_1(x)^{-1} \times h_2(y)^{-1}} w_{(x, y)}
\]

\[
= \left( L_{h_1(x)} \otimes M_{h_2(y)} \right) \circ \mathcal{J}^{L \times M} \circ \left( L_{h_1(x)^{-1}} \otimes M_{h_2(y)^{-1}} \right) w_{(x, y)}
\]

\[
= \left( L_{h_1(x)} \mathcal{J}_L L_{h_1(x)^{-1}} \right) \otimes \left( M_{h_2(y)} \mathcal{J}_M M_{h_2(y)^{-1}} \right) w_{(x, y)}.
\]
Thus the vector $\mathfrak{J}_{L \times M}(\mathfrak{J}^{L \times M}w)_{(x,y)}$ in the integrand in the formula for the inner product in $\mathcal{H}^{L \times M}$

$$(w, u) = \int_{\mathfrak{S}_1 \times \mathfrak{S}_2} \mathfrak{J}_{L \times M}(\mathfrak{J}^{L \times M}w)_{(x,y)}(u)_{(x,y)} \, d(\mu_1 \times \mu_2)([(x, y)])$$

may be written as follows

$$\mathfrak{J}_{L \times M}(\mathfrak{J}^{L \times M}w)_{(x,y)} = (\mathfrak{J}_L \otimes \mathfrak{J}_M) \circ (\mathfrak{J}^{L \times M}w)_{(x,y)}$$

$$= (\mathfrak{J}_L L_{h_1(x)} \mathfrak{J}_L L_{h_1^{-1}(x)}) \otimes (\mathfrak{J}_M M_{h_2(y)} \mathfrak{J}_M M_{h_2^{-1}(y)}) w_{(x,y)} = (\mathfrak{J}_L \otimes \mathfrak{J}_M) w_{(x,y)},$$

where we have introduced the following self-adjoint operators

$$x \mathfrak{J}^L = \mathfrak{J}_L L_{h_1(x)} \mathfrak{J}_L L_{h_1^{-1}(x)} \quad \text{and} \quad y \mathfrak{J}^M = \mathfrak{J}_M M_{h_2(y)} \mathfrak{J}_M M_{h_2^{-1}(y)}^{-1}$$

acting in $\mathcal{H}_L$ and $\mathcal{H}_M$, respectively, with the ordinary tensor product operator $\mathfrak{J}^L \otimes \mathfrak{J}^M$ acting in the tensor product $\mathcal{H}_L \otimes \mathcal{H}_M$ Hilbert space.

Checking their self-adjointness is immediate. Indeed, because $L$ is Krein unitary in $(\mathcal{H}_L, \mathfrak{J}_L)$ we have (and similarly for the rep. $M$):

$$L_{h_1(x)}^{-1} = (L_{h_1(x)})^\dagger = \mathfrak{J}_L (L_{h_1(x)})^\ast \mathfrak{J}_L.$$

Therefore

$$x \mathfrak{J}^L = \mathfrak{J}_L L_{h_1(x)} (L_{h_1(x)})^\ast \mathfrak{J}_L,$$

because $(\mathfrak{J}_L)^2 = I$. Because $\mathfrak{J}_L$ is self-adjoint, self-adjointness of $x \mathfrak{J}^L$ is now immediate (self-adjointness of $y \mathfrak{J}^M$ follows similarly).

We are ready now to formulate the main goal of this Section:

**THEOREM 6.** Let $L$ and $M$ be Krein-unitary strongly continuous and almost uniformly bounded representations of the closed subgroups $G_1$ and $G_2$ of the separable locally compact groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$, respectively, in the Krein spaces $(\mathcal{H}_L, \mathfrak{J}_L)$ and $(\mathcal{H}_M, \mathfrak{J}_M)$. Then the Krein isometric representation $\mu_1 \times \mu_2 U^L \times M$ of the group $\mathfrak{G}_1 \times \mathfrak{G}_2$ with the representation space equal to the Krein space $(\mathcal{H}^{L \times M}, \mathfrak{J}^{L \times M})$ is unitary and Krein-unitary equivalent to the Krein-isometric representation $\mu_1 U^L \times U^M$ of the group $\mathfrak{G}_1 \times \mathfrak{G}_2$ with the representation space equal to the Krein space $(\mathcal{H}^{L \times M}, \mathfrak{J}^{L \times M})$. More precisely: there exists a map $V : \mathcal{H}_L \otimes \mathcal{H}_M \mapsto \mathcal{H}^{L \times M}$ which is unitary between the indicated Hilbert spaces and Krein-unitary between the Krein spaces $(\mathcal{H}_L \otimes \mathcal{H}_M, \mathfrak{J}_L \otimes \mathfrak{J}_M)$ and $(\mathcal{H}^{L \times M}, \mathfrak{J}^{L \times M})$ and such that

$$V^{-1} \left( \mu_1 \times \mu_2 U^L \times M \right) V = \mu_1 U^L \times \mu_2 U^M. \quad (13)$$
\[ ||T||^2 = \left( \sum_{i,j=1}^{n} \int_{\mathcal{G}_1 \times \mathcal{G}_2} \left( \int_{\mathcal{H}_1} (\mathcal{J}^L f_i)_x \cdot (f_j)_y \, d\mu_1([x]) \right) \left( \int_{\mathcal{H}_2} \left( \mathcal{J}^M g_i)_y \cdot (g_j)_y \, d\mu_2([y]) \right) \right) \right) \]

\[ = \sum_{i,j=1}^{n} \int_{\mathcal{G}_1 \times \mathcal{G}_2} \int_{\mathcal{H}_1} \left( \sum_{k \in \mathbb{N}} (z^{\mathcal{J}^L f_i)_x} \cdot e_k \right) \cdot (e_{k}, (f_j)_x) \cdot (e_{k}, (g_j)_y) \cdot (e_{k}, (g_j)_y) \, d\mu_1([x]) \]

\[ = \sum_{i,j=1}^{n} \int_{\mathcal{G}_1 \times \mathcal{G}_2} \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} \left( \sum_{k \in \mathbb{N}} (z^{\mathcal{J}^L f_i)_x} \cdot (f_j)_x \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \right) \, d\mu_1([x]) \]

\[ = \int_{\mathcal{G}_1 \times \mathcal{G}_2} \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} \left( \sum_{k \in \mathbb{N}} (e_{k}, (f_j)_x) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \right) \, d\mu_1([x]) \]

\[ = \int_{\mathcal{G}_1 \times \mathcal{G}_2} \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} \left( \sum_{k \in \mathbb{N}} (e_{k}, (f_j)_x) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \right) \, d\mu_1([x]) \]

\[ = \int_{\mathcal{G}_1 \times \mathcal{G}_2} \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} \left( \sum_{k \in \mathbb{N}} (e_{k}, (f_j)_x) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \right) \, d\mu_1([x]) \]

\[ = \int_{\mathcal{G}_1 \times \mathcal{G}_2} \int_{\mathcal{H}_1} \int_{\mathcal{H}_2} \left( \sum_{k \in \mathbb{N}} (e_{k}, (f_j)_x) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \cdot (e_{k}, (g_i)_y) \right) \, d\mu_1([x]) \]
\[
\int \left( \sum_{i,j=1}^{n} \sum_{k \in \mathbb{N}} \left( x^{g} \circ T_{(f_{i})_{x},(g_{j})_{y}} \circ y^{M} \circ T_{(g_{j})_{x},(f_{i})_{y}} (e_{k}), e_{k} \right) \right) \, d(\mu_{1} \times \mu_{2})([(x,y)])
\]
\[
= \int \left( \sum_{i,j=1}^{n} \text{Tr} \left[ x^{g} \circ T_{(f_{i})_{x},(g_{j})_{y}} \circ y^{M} \circ T_{(g_{j})_{x},(f_{i})_{y}} \right] \right) \, d(\mu_{1} \times \mu_{2})([(x,y)])
\]
\[
= \int \left( \sum_{i,j=1}^{n} \text{Tr} \left[ x^{g} \circ T_{(f_{i})_{x},(g_{j})_{y}} \circ y^{M} \circ (T_{(f_{i})_{x},(g_{j})_{y}})^{\ast} \right] \right) \, d(\mu_{1} \times \mu_{2})([(x,y)])
\]
\[
\int_{\mathfrak{G}_{1} \times \mathfrak{G}_{2}} \left( \mathfrak{J} \otimes \mathfrak{J} \right) (V(T))_{(x,y)}, (V(T))_{(x,y)} \, d(\mu_{1} \times \mu_{2})([(x,y)]) = \|V(T)\|^2.
\]

(The unspecified domain of integration in the above formulas is of course equal \( \mathfrak{G}_{1} \times \mathfrak{G}_{2} \). Therefore \( V \) is isometric and \( V(T) \in \mathcal{H}^{L \times M} \) for the indicated \( T \), as the required measurability conditions again easily follow from Scholium 3.9 of [69].

Now by the properties of Hilbert-Schmidt operators, the finite rank conjugate-linear operators \( T : \mu_{2} \mathcal{H}^{M} \rightarrow \mu_{1} \mathcal{H}^{L} \) are dense in \( \mu_{1} \mathcal{H}^{L} \otimes \mu_{2} \mathcal{H}^{M} \) (compare e. g. [51], Chap. II or [69], Chap. 14.2 or [67]). Thus the domain of the operator \( V \) is dense.

In order to show that the range of \( V \) is likewise dense, consider the closure \( C^{1} \) under the norm in \( \mathcal{H}^{L \times M} \) of the linear set of all functions \( V(T) \), where \( T = T_{f_{1},g_{1}} + \ldots + T_{f_{n},g_{n}} \) with \( f_{i} \) ranging over \( C_{0}^{L} \subset \mathcal{H}^{L} \) and \( g_{j} \) over the corresponding set \( C_{0}^{M} \subset \mathcal{H}^{M} \). Because \( V \) is isometric it can be uniquely extended so that \( C^{1} \) lies in the range of this unique extension. Let us denote the extension likewise by \( V \). (For a densely defined Krein-isometric map this would in general be impossible because \( V \) could be discontinuous, this is the reason why we need to know if \( V \) is continuous, i. e. bounded for the ordinary positive definite inner products.)

Now by the property of Hilbert-Schmidt operators (mentioned above) the linear span of operators \( T_{u,x} : \mathcal{H}_{M} \rightarrow \mathcal{H}_{L} \) with \( u \) and \( v \) ranging over dense subsets of \( \mathcal{H}_{L} \) and \( \mathcal{H}_{M} \), respectively, is dense in \( \mathcal{H}_{L} \otimes \mathcal{H}_{M} \). This property of Hilbert-Schmidt operators together with a repeated application of Lemma [2] and [3] of Sect. [1] and Scholium 3.9 and 5.3 of [69] will show that all the conditions, (a)-(e), of Lemma [4] are satisfied for \( C^{1} \subset \mathcal{H}^{L \times M} \).

In particular if \( \psi \) is a complex valued continuous function on \( \mathfrak{G}_{1} \times \mathfrak{G}_{2} \) which is constant on the right \( G_{1} \times G_{2} \) cosets and vanish outside of \( \mathfrak{G}_{1} \times \mathfrak{G}_{2} \) for some compact subset \( K \) of \( \mathfrak{G}_{1} \times \mathfrak{G}_{2} \), then it is measurable and \( \psi \in L^{2}(\mathfrak{G}_{1} \times \mathfrak{G}_{2})/(G_{1} \times G_{2}), \mu_{1} \times \mu_{2} \) and by Scholim 3.9 and 5.3 of [69] it is an \( L^{2} \)-limit of continuous such functions of “product form” \( \phi \cdot \varphi : \mathfrak{G}_{1} \times \mathfrak{G}_{2} \) (\( x,y \rightarrow \phi(x) \cdot \varphi(y) \)). Thus the condition (d) of Lemma [4] follows. The above mentioned property of Hilbert-Schmidt operators and Lemma [2] applied

\( \pi^{\text{v}} \) denotes here the canonical quotient map \( \mathfrak{G}_{1} \times \mathfrak{G}_{2} \rightarrow (\mathfrak{G}_{1} \times \mathfrak{G}_{2})/(G_{1} \times G_{2}) \).
to $C_0^L \subset \mathcal{H}_L$ and to $C_0^M \subset \mathcal{H}_M$, proves condition (e) of Lemma 4. Condition (b) follows from the fact that $V(T) \in \mathcal{H}^{L \times M}$ for finite rank operators $T$, proved in the first part of the proof. An application of the Lusin Theorem (Corollary 5.2.2 of [69], together with an obvious adaptation of the standard proof of the Riesz-Fischer theorem already used in the proof of Lemma 5) proves condition (a) of Lemma 4. By the remark opening the proof of Lemma 2 the linear sets $C_0^L$ and $C_0^M$ of functions are closed with respect to right $\mathcal{G}_1$ and $\mathcal{G}_2$-translations, respectively. Thus it easily follows that the linear set of functions $V(T_{j_1, s_1} + \ldots + T_{j_n, s_n})$ with $f_j \in C_0^L$, $g_j \in C_0^M$ is closed under the right $\mathcal{G}_1 \times \mathcal{G}_2$-translations. Then, a simple continuity argument shows that $C^1$ is closed under right $\mathcal{G}_1 \times \mathcal{G}_2$-translations. Thus condition (c) of Lemma 4 is satisfied with trivial functions $\rho$, all equal identically to the constant unit function.

Thus Lemma 4 may be applied to $C^1$ lying in the range of $V$, so that the range is dense in $\mathcal{H}^{L \times M}$. Therefore $C^1 = \mathcal{H}^{L \times M}$ and $V$ is unitary.

We shall show that $V$ is Krein-unitary. By the unitarity of $V$, it will be sufficient by continuity to show that $V$ is Krein-isometric on finite rank operators $T \in \mathcal{H}^L \otimes \mathcal{H}_M$. By self-adjointness of $\mathcal{J}^L$ and $\mathcal{J}^M$ we have the following equalities for $T$ of the form indicated to above:

$$
\left(\|T\|_{\mathcal{J}^L \otimes \mathcal{J}^M}\right)^2 = \left(\mathcal{J}^L \otimes \mathcal{J}^M \left( f_1 \otimes g_1 + \ldots + f_n \otimes g_n \right) \right)^2
= \text{Tr} \left[ \mathcal{J}^L \left( T_{j_1, s_1} + \ldots + T_{j_n, s_n} \right) \mathcal{J}^M \left( T_{j_1, s_1} + \ldots + T_{j_n, s_n} \right)^* \right]
= \sum_{i,j=1}^n \left( \mathcal{J}^L f_i, f_j \right) \cdot \left( \mathcal{J}^M g_i, g_j \right)
= \int_{\mathcal{G}_1 \times \mathcal{G}_2} \sum_{i,j=1}^n \left( \mathcal{J}^L (f_i)_{x,s_1} \cdot (f_j)_{x,s_1} \right) \cdot \left( \mathcal{J}^M (g_i)_{y,s_2} \cdot (g_j)_{y,s_2} \right) \ d(\mu_1 \times \mu_2)([(x,y)])
= \int_{\mathcal{G}_1 \times \mathcal{G}_2} \text{Tr} \left[ \mathcal{J}^L \left( T_{(j_1)_{x,s_1}} \otimes \ldots + T_{(j_n)_{x,s_1}} \otimes \ldots \right) \mathcal{J}^M \left( T_{(j_1)_{x,s_1}} \otimes \ldots \right)^* \right] \ d(\mu_1 \times \mu_2)([(x,y)])
= \int_{\mathcal{G}_1 \times \mathcal{G}_2} \left( \mathcal{J}^L \otimes \mathcal{J}^M \left( V(T) \right) \left( V(T) \right)^* \right) \ d(\mu_1 \times \mu_2)([(x,y)])
= \text{Tr} \left[ \mathcal{J}^L_x \cdot \mathcal{J}^M_y \left( V(T) \right) \left( V(T) \right)^* \right] \ d(\mu_1 \times \mu_2)([(x,y)])
= \left( \|V(T)\|_{\mathcal{J}^L \otimes \mathcal{J}^M} \right)^2.
$$

Recall that the domain $\mathcal{D}_{12}$ (common for all $(x,y) \in \mathcal{G}_1 \times \mathcal{G}_2$) of the
operators \( U_1 \otimes U_2 = \mu_1 U_x^L \otimes \mu_2 U_y^M = (\mu_1 U_x^L \times \mu_2 U_y^M)_{(x,y)} \) representing \((x,y) \in \mathfrak{G}_1 \times \mathfrak{G}_2, \) is invariant for the operators \( U_1 \otimes U_2 = \mu_1 U_x^L \otimes \mu_2 U_y^M = (\mu_1 U_x^L \times \mu_2 U_y^M)_{(x,y)} \). For each \((x,y)\) let us denote the closure of \( \mu_1 U_x^L \times \mu_2 U_y^M = \mu_1 U_x^L \otimes \mu_2 U_y^M \) likewise by \( \mu_1 U_x^L \times \mu_2 U_y^M \). Note that \( V(T), T \in \mathfrak{D}_{12} \) compose an invariant domain of the representation \( \mu_1 \times \mu_2 U_x^L \times U_y^M \). Denote the closures of the operators \( \mu_1 \times \mu_2 U_x^L \times U_y^M \) with the common invariant domain \( V(\mathfrak{D}_{12}) \) likewise by \( \mu_1 \times \mu_2 U_x^L \times U_y^M \).

The equality (13) is regarded as equality for the closures of the operators \( \mu_1 \times \mu_2 U^L \times U^M \) and \( \mu_1 U_x^L \times \mu_2 U_y^M \).

By Theorem 1 and its proof the closures of \( \mu_1 \times \mu_2 U^L \times U^M \) do not depend on the choice of the dense common invariant domain. Therefore in order to show the equality (13) it is sufficient that the respective closed operators in (13) coincide on the domain of all finite rank operators \( T \in \mathfrak{D}_{12} \). This however is immediate. Indeed, let \( T = T_{f_1, g_1} + \ldots + T_{f_n, g_n} \) with \( f_i \in \mathfrak{D}_1 \) and \( g_j \in \mathfrak{D}_2 \). Then

\[
\begin{align*}
(\mu_1 U^L \times \mu_2 U^M)_{(x_0, y_0)}(T) &= (\mu_1 U^L_{x_0} \otimes \mu_2 U^M_{y_0})(T) = \mu_1 U^L_{x_0} \cdot (\mu_2 U^M_{y_0})^* \\
&= \sqrt{\lambda_1(x_0)} \sqrt{\lambda_2(y_0)} \left( T_{R_{x_0} f_1, R_{y_0} g_1} + \ldots + T_{R_{x_0} f_n, R_{y_0} g_n} \right),
\end{align*}
\]

On the other hand we have:

\[
\begin{align*}
(\mu_1 \times \mu_2 U^L \times U^M)_{(x_0, y_0)} V(T)_{(x, y)}(T) &= \sqrt{\lambda_1(x_0)} \sqrt{\lambda_2(y_0)} \left( V(T)_{(x_0, y_0)} \right) \\
&= \sqrt{\lambda_1(x_0)} \sqrt{\lambda_2(y_0)} \left( T_{R_{x_0} f_1, R_{y_0} g_1} + \ldots + T_{R_{x_0} f_n, R_{y_0} g_n} \right),
\end{align*}
\]

so that

\[
V^{-1} \left( \mu_1 \times \mu_2 U^L \times U^M \right) V(T) = \sqrt{\lambda_1(x_0)} \sqrt{\lambda_2(y_0)} \left( T_{R_{x_0} f_1, R_{y_0} g_1} + \ldots + T_{R_{x_0} f_n, R_{y_0} g_n} \right).
\]

Comparing it with (14) one can see that (13) holds on \( \mathfrak{D}_{12} \). Thus the proof of (13) is complete now. The Theorem is hereby proved completely.

Presented proof of Theorem 6 is an extended and modified version of the Mackey’s proof of Theorem 5.2 in [47].

Note, please, that the equality (13) for the closures of the operators \( \mu_1 \times \mu_2 U^L \times U^M \) and \( \mu_1 U_x^L \times \mu_2 U_y^M \) is non trivial. Indeed, recall that in general almost all kinds of pathology not excluded by general theorems can be shown to exist for unbounded operators. In particular two distinct and closed operators may still coincide on a dense domain. This is why we need to be careful in proving (13).

This in particular shows that the fundamental theorems of the original Mackey theory by no means are automatic for the induced Krein-isometric representations, where the representors are in general densely defined and unbounded. Here we saw it for the Theorem 6. But differences in the proofs arise likewise in the latter part of the theory. In particular if we want to prove the subgroup theorem and the so-called Kronecker product theorem for the induced Krein-isometric
representations with precisely the same assumptions posed on the group as in Mackey’s theory, then some additional analysis will have to be made in treating decompositions of non finite quasi invariant measures. Compare Sect. 8.

7 Subgroup theorem in Krein spaces. Preliminaries

This Section is a word for word repetition of the argument of §6 of [47]. That the general Mackey’s argument may be applied to induced representations in Krein spaces is the whole point. Although it is rather clear that his general argumentation is applicable in the Krein space, we restate it here because it lies at the very heart of the presented method of decomposition of tensor product of induced representations, and will make the paper self contained. It should be noted however that it requires some additional analysis in decomposing non finite quasi invariant measures, which makes a difference in proving the existence of the corresponding direct integral decompositions.

The circumstance that the Lopuszański representation of $G$ is equivalent to an induced representation in a Krein space greatly simplifies the problem of decomposing tensor product of Lopuszański representations and reduces it largely to the geometry of right cosets and double cosets in the group $G$ and to a “Fubini-like” theorem, just like for the ordinary induced representations of Mackey. Similar decomposition method of quotiening by a subgroup in construction of complete sets of unitary representations of semi simple Lie groups was applied by Gelfand and Neumark, and by several authors in constructing harmonic analysis on classical Lie groups. The main gain is that the subtle analytic properties of the Lopuszański representation (unboundedness) does not intervene dramatically after this reduction to geometry of cosets and double cosets.

Our main theorem asserts the existence of a certain useful direct integral decomposition of the tensor product $U^L \otimes U^M$ of two induced representations of a group $G$ in a Krein space, whose construction is completely analogous to that of Mackey for ordinary unitary representations, compare [47]. By definition $U^L \otimes U^M$ is obtained from the outer Kronecker product representation $U^L \times U^M$ of $G \times G$ by restricting $U^L \times U^M$ to the diagonal subgroup $G \cong G$ of all $(x,y) \in G \times G$ with $x = y$. By the Theorem of Sect. 6 $U^L \times U^M$ is Krein-unitary equivalent to $U^{L \times M}$. Thus $U^L \otimes U^M$ can be analysed by analysing the restriction of $U^{L \times M}$ to the diagonal subgroup $G \cong G$. Our theorem on tensor product decomposition follows (just as in [47]) from these remarks and a theorem on restriction to a subgroup of an induced representation in a Krein space, say a subgroup theorem. Subgroup theorem gives a decomposition of the restriction of an induced representation (in a Krein space) to a closed subgroup, with the component representations in the decomposition themselves Krein-unitary equivalent to induced representations. Namely, let $L$ be strongly continuous almost uniformly bounded Krein-unitary representation of the closed subgroup.
$G_1$ of $\mathfrak{G}$ and consider the restriction $\mathcal{G}_2 U^L$ of $U^L$ to a second closed subgroup $G_2$. While $\mathfrak{G}$ acts transitively on the homogeneous space $\mathfrak{G}/G_1$ of right $G_1$-cosets this will not be true in general of $G_2$. Moreover, and this is the main advantage of induced representations, any division of $\mathfrak{G}/G_1$ into two parts $S_1$ and $S_2$, each a Baire (or Borel) set which is not a null set (with respect to any, and hence every quasi invariant measure on $\mathfrak{G}/G_1$), and each invariant under $G_2$ leads to a corresponding direct sum decomposition of $\mathcal{G}_2 U^L$. Indeed the closed subspaces $\mathcal{H}_1^L$ and $\mathcal{H}_2^L$ of all $f \in \mathcal{H}^L$ which vanish respectively outside of $\pi^{-1}(S_1)$ and $\pi^{-1}(S_2)$ are invariant and are orthogonal complements of each other with respect to the ordinary (as well as the Krein) inner product on $\mathcal{H}^L$.

Assume for a while, just for illustrative purposes, that there is a null set $N$ in $\mathfrak{G}/G_1$ whose complement is the union of countably many non null orbits $C_1, C_2, \ldots$ of $\mathfrak{G}/G_1$ under $G_2$. Then by the above remarks we obtain a direct sum decomposition of $\mathcal{G}_2 U^L$ into as many parts as there are non null orbits. Our analysis reaches its goal after analysing the nature of these parts. Analysis of these parts is our goal of the rest of this Section.

In our paper we shall consider a more general case in which all of the orbits can be null sets and the sum becomes an integral and we have to use the von Neumann theory of direct integral Hilbert spaces [52]. Of course according to the definition given above (with $S_1$ or $S_2$ equal to a $G_2$ orbit $C$ in $\mathfrak{G}/G_1$), $\mathcal{H}_C^L$ will be zero dimensional whenever the orbit $C$ is a null set. However it is possible to reword the definition so that it always gives a non zero Hilbert space (with the respective Krein structure) and so that when $C$ is not a null set this definition is essentially the same as that already given, compare [47], §6. Indeed note that when $C$ is a non null set then $\mathcal{H}_C^L$ may be equivalently defined as follows. Let $x_c$ be any member of $\mathfrak{G}$ such that $\pi(x_c) \in C$ and consider the set $\mathcal{H}_C^L$ of all functions $f$ from the double coset $G_1 x_c G_2$ to $\mathcal{H}_L$ such that: (i) $x \mapsto (f_x, v)$ is a Borel function for all $v \in \mathcal{H}_L$, (ii) $f_{x} = L_\xi(f_x)$ for all $\xi \in G_1$ and all $x \in G_1 x_c G_2$ and (iii):

$$
\|f\|_C = \int_C (\mathcal{J}_L((\mathcal{J}_L f)_x), f_x) d\mu_{\mathfrak{G}/G_1} = \int_{(G_1 x_c G_2) \cap B} (f_b, f_b) d\mu_B(b) < \infty,
$$

where $B$ is the regular Borel section of $\mathfrak{G}$ with respect to $G_1$ of Sect. 3 (we could use as well the sub-manifold $Q$ of Sect. 6 but we prefer to proceed generally and independently of the “factorization” assumption). The operator $\mathcal{J}_L^c$ in $\mathcal{H}_C^L$ is given by simple restriction, and its definition on $\mathcal{H}_C^{L'}$ is obvious:

$$(\mathcal{J}_L^c f)_x = L_{h(x)} \mathcal{J}_L^c L_{h(x)^{-1}} f_x;
$$

with the obvious definition of the Krein inner product in $\mathcal{H}_C^{L'}$

$$(f, g)_{\mathcal{J}_L^c} = (\mathcal{J}_L^c f, g) = \int_C (\mathcal{J}_L(f_x), g_x) d\mu_{\mathfrak{G}/G_1}, \ f, g \in \mathcal{H}_C^{L'}.
$$

Similarly we define the operator $U^L_\xi$ in $\mathcal{H}_C^L$ for $\xi \in G_2$ as the restriction of $U^L_\xi$ to $\mathcal{H}_C^L$, i.e. to the functions supported by the orbit $C$, and its definition giving
an equivalent representation on $\mathcal{H}_C^U$ is likewise obvious:

$$(U_t^{C, \xi} f)_x = \sqrt{\lambda(x, \xi)} f_x$$

with the $\lambda$-function of the quasi invariant measure $\mu$ restricted to $C \times G_2$.

Moreover, and this is the whole point, the measure in $C$ need not be defined by restricting $\mu = \mu_\Theta/G_1$ to $C$. There exists a non zero measure $\mu_C$ on $C$ quasi invariant with respect to $G_2$ determined up to a constant factor, whose Radon-Nikodym function $d(R_\eta \mu_C)/d\mu_C$, $\eta \in G_2$ (i. e. the associated $\lambda_C$-function) is equal to the restriction to the subspace $C \times G_2$ of the $\lambda$-function, i. e. Radon-Nikodym derivative $d(R_\eta \mu)/d\mu$, associated with $\mu = \mu_\Theta/G_1$. Indeed, although $\mu$ does not have the form of a quotient of a group by its closed subgroup, it follows from Theorem 3, page 253 of [43] that the map $x \mapsto \pi(x, x)$ induces a Borel isomorphism $\psi$ of the quotient space $G_2/G_{x_0}$ onto $C$, where $G_{x_0} = G_2 \cap (x_0^{-1}G_1x_0)$ is the closed subgroup of all $x \in G_2$ such that $\pi(x_0, x) = \pi(x, x)$. Thus $C \times G_2 \cong G_2/G_{x_0} \times G_2$ as Borel spaces under the indicated isomorphism and moreover if $[x] \in G_2/G_{x_0}$ and $[z] = \pi(x, x)$ correspond under this isomorphism and $\eta \in G_2$ then $[x]\eta$ and $[z]\eta$ do also,

where $[x]\eta = [x\eta]$ and $[z]\eta = [z\eta]$ denote the action of $\eta \in G_2$ on $[x] \in G_2/G_{x_0}$ and $[z] \in C$ respectively. Thus the existence of the quasi invariant measure $\mu_C$ on $C$ follows from the general Mackey classification of quasi invariant measures on the quotient of a locally compact group by a closed subgroup, compare the respective Theorem of Sect. 3. Using the quasi invariant measure $\mu_C$ on $C$ gives a non trivial space $\mathcal{H}_C^U$ for every orbit $C$, which in case of a non null orbit $C$ is trivially equivalent to $\mathcal{H}_C^L$.

We are now in a position to formulate the main goal of this Section:

**Lemma 8.** Let $C$ be any orbit in $\Theta/G_1$ under $G_2$ and let $x_0$ be such that $\pi(x_0, x) \in C$. Let $\mathcal{H}_C^L$ be defined as above. Let $\mu^{x_0} U^{x_0}$ be the representation of $G_2$ induced by the strongly continuous almost uniformly bounded Krein-unitary representation $L^{x_0} : \eta \mapsto L_{x_0, \eta} : G_2 \cap (x_0^{-1}G_1x_0)$ with the representation space of $L^{x_0}$ equal to $\mathcal{H}_{L^{x_0}} = \mathcal{H}_L$ and the fundamental symmetry $\mathfrak{Z}_{L^{x_0}} = \mathfrak{Z}_L$; and with the quasi invariant measure $\mu^{x_0}$ in the homogeneous space $G_2/(G_2 \cap (x_0^{-1}G_1x_0))$ equal to the transfer of the measure $\mu_C$ in $C$ over to the homogeneous space by the map $\psi$. Let $\mu^{x_0} \mathcal{H}_{L^{x_0}}$ be the Krein space of the induced representation $\mu^{x_0} U^{x_0}$. We assume the fundamental symmetry $\mathfrak{Z}_{x_0}$ in $\mu^{x_0} \mathcal{H}_{L^{x_0}}$ to be defined by the equation $(\mathfrak{Z}_{x_0}, g) = L_{h(x_0), t}J L_{h(x_0), t}^{-1}$ and the Krein inner product given by the ordinary formula

$$\int_{G_2/(G_2 \cap (x_0^{-1}G_1x_0))} (\mathfrak{Z}_L f, f) d\mu^{x_0}([t]), \quad t \in G_2.$$
Then there is a Krein-unitary map $V_{x_c}$ of $H_c^{L'}$ onto $\mu^x H^{L_c}$ such that if $g \in \mu^x H^{L_c}$ corresponds to $f \in H_c^{L'}$ then $\mu^x U_s f$ corresponds to $U_s^{L_c} g$ where $(U_s^{L_c} f)_x = f_{x s} \sqrt{\lambda C(x)}$ for all $x \in C$ and all $s \in G_2$.

For each function $f$ on $G_1 x_c G_2$ satisfying the conditions (i) and (ii) of the definition of $H_c^{L'}$ let $f$ be defined by $\tilde{f}_t = f_{x, t}$ for all $t \in G_2$. Then $(\tilde{f}_t, \nu)$ is a Borel function of $t$ on $G_2$ for all $\nu \in H_L$. If $\eta \in G_x = G_2 \cap (x_c^{-1} G_1 x_c)$ then if $\xi = x_c \eta x_c^{-1}$ we have $\tilde{f}_\eta(t) = \tilde{f}_{\xi, \nu(t)} = f_{\xi, \nu(t), \eta} = L_{\nu(t), \eta} \tilde{f}_t$; that is

$$\tilde{f}_\eta(t) = L_{x_c \eta x_c^{-1}, \eta} \tilde{f}_t$$

for all $t \in G_2$ and all $\eta \in G_2 \cap (x_c^{-1} G_1 x_c)$. Conversely let $g$ be any function from $G_2$ to $H_L$ which is Borel in the sense that $x \mapsto (g_{x, \nu})$ is a Borel function on $G_2$ for all $\nu \in H_L$ and which satisfies (15). We define the corresponding function $f$ by the equation $f_{\xi, \nu(t)} = L_{\xi}(g_{\eta})$ for all $\xi \in G_1$ and $t \in G_2$. If $\xi_1 x_1 t_1 = \xi_2 x_2 t_2$ then $\xi_2^{-1} \xi_1 = x_1 t_1 x_2^{-1} t_2$ so that $g_{\xi_2^{-1} \xi_1 t_1} = L_{\xi_2^{-1} \xi_1}(g_{\eta})$. Therefore $L_{\xi_2}(g_{\eta}) = L_{\xi_1}(g_{\eta})$ and $f$ is well defined. Next we show that $(f_{\xi, \nu})$ is Borel function of $x$ on $G_1 x_c G_2$ for all $\nu \in H_L$. Let $f'$ be the function on $G_1 \times G_2$ defined by $f'(\xi, \eta) = L_{\xi}(g_{\eta})$ for all $(\xi, \eta) \in G_1 \times G_2$. Choose now an orthonormal basis $\{\varphi_i\}_{i \in N}$ in $H_L$. Then we have $(f'(\xi, \eta), \varphi_i) = (f'(\xi, \eta), \varphi_i) = (\xi \varphi_i, \varphi_i) = (\xi \varphi_i, \varphi_i) = \sum_{i=1}^\infty (\xi \varphi_i, \varphi_i) = (\xi \varphi_i, \varphi_i)$. By Scholium 3.9 of (16) $f'(\xi, \eta, \nu)$ is a Borel function of $(\xi, \eta)$ on $G_1 \times G_2$ regarded as the product measure space, for all $\nu \in H_L$. Let us introduce after Mackey a new group operation in $G_1 \times G_2$ putting $(\xi_1, \eta_1)(\xi_2, \eta_2) = (\xi_1 \xi_2, \eta_1 \eta_2)$ and call the resulting group $G_3$. Then $\xi_1 x_1 \eta_1 = \xi_2 x_2 \eta_2$ if and only if $(\xi_2, \eta_2)^{-1}(\xi_1, \eta_1) = (\xi_2^{-1} \xi_1, \eta_2^{-1} \eta_1)$ has the form $(\xi, x_c^{-1} \xi_1 x_c)$. The set of all $(\xi, x_c^{-1} \xi_1 x_c)$, $\xi \in G_1$ is a subgroup of $G_4$ of $G_3$. Thus the map $(\xi, \eta) \mapsto \xi \eta x_c$ sets up a one-to-one correspondence between the points of the homogeneous space $G_3 / G_4$ of left $G_4$-cosets and the points of the double coset $G_1 x_c G_2$. The map is continuous and on account of the assumed separability it follows again from Theorem 3, page 253 of (13) that the map sets up a Borel isomorphism. Moreover the function $(\xi, \eta) \mapsto f'(\xi, \eta, \nu)$ is constant on left $G_4$-cosets in $G_3$, as an easy computation shows that $(f'(\xi, \eta, \nu)) = (f'(\xi, \eta, \nu))$ for all $\nu = (\xi, x_c^{-1} \xi_0^{-1} x_c) \in G_4$. Therefore $(\xi, \eta) \mapsto f'(\xi, \eta, \nu)$ defines a function on $G_3 / G_4$ which by Lemma 1.2 of (17) must be Borel because $(\xi, \eta) \mapsto f'(\xi, \eta, \nu)$ is Borel and $G_3$ onto $G_3$. That $(f_{\xi, \nu})$ is a Borel function of $x \in G_1 x_c G_2$ now follows from the fact that the mapping of $G_3 / G_4$ onto $G_1 x_c G_2$ is a Borel isomorphism and preserves $G_4$ sets. Finally observe that $f = g$. Therefore $f \mapsto f$ is a one-to-one map of functions satisfying (i) and (ii) of the definition of $H_c^{L'}$ onto Borel functions satisfying (15). Consider the mapping $t \mapsto \pi(x, t)$ of $G_2$ onto $C$. It defines one-to-one and Borel set preserving map $\psi$ from $G_2 / (G_2 \cap (x_c^{-1} G_1 x_c))$ onto $C$ and such that if $[t] = \pi'(t)$ and $[z] = \pi(z)$ correspond under the map $\psi$ and $\eta \in G_2$ then $[x \eta \pi']$ and $[z \eta]$ do also $(\pi' \ast \pi'(t)$ stands for the canonical projection $G_2 / (G_2 \cap (x_c^{-1} G_1 x_c)) \to G_2$). Finally $z \mapsto (3_L f_z, \tilde{f}_z)$ and $t \mapsto (3_L f_t, \tilde{f}_t)$ define functions $\pi(z) \mapsto (3_L f_{\pi(z)}, \tilde{f}_{\pi(z)})$ and $\pi'(t) \mapsto (3_L f_{\pi'(t)}, \tilde{f}_{\pi'(t)})$ on $C$ and

$$60$$
$G_2/(G_2 \cap (x_c^{-1}G_1x_c))$ respectively which correspond under the same map $\psi$: 
$(\mathcal{J}_L f_{\psi((x_c^{-1}G_1x_c))}, f_{\psi((x_c^{-1}G_1x_c))}) = (\mathcal{J}_L f_{\pi(x_c)}, f_{\pi(x_c)}) = (\mathcal{J}_L f_{\pi(x_c)}, f_{\pi(x_c)}) = (\mathcal{J}_L f_{\pi(x_c)}, f_{\pi(x_c)}).$

If we use this same map $\psi$ to transfer the measure $\mu_C$ on $C$ over to the homogeneous space $G_2/(G_2 \cap (x_c^{-1}G_1x_c))$ we will get a quasi invariant measure $\mu_{xc}$ there such that 

$$
\int_C (\mathcal{J}_L f_z, f_z) \, d\mu_C([z]) = \int_C (\mathcal{J}_L f_z, f_z) \, d\mu_C([z])
$$

$$
\int_C (\mathcal{J}_L f_{\psi([t])}, f_{\psi([t])}) \, d\mu_C([t]) = \int_{G_2/(G_2 \cap (x_c^{-1}G_1x_c))} (\mathcal{J}_L f_{\psi([t])}, f_{\psi([t])}) \, d\mu_{xc}([t])
$$

Thus by the polarization identity (compare e. g. [69], §8.3, page 222 or [4], page 4) the map $f \mapsto \tilde{f}$ sets up the Krein-unitary transformation $V_{xc}$ demanded by the Lemma as the verification of $V_{xc} U^L_s C V_{xc}^{-1} = \mu_{xc} U^L_s$, $s \in G_2$, and $V_{xc} \mathcal{J}_L C V_{xc}^{-1} = \mathcal{J}_{xc}$ is almost immediate as $V_{xc}$ is bounded, which we show below in Lemma 9. Similarly verification that $\mathcal{J}_{xc} \mathcal{J}_{xc} = I$ and that $\mathcal{J}_{xc}$ is self adjoint with respect to the definite inner product

$$
(\tilde{f}, \tilde{g})_{x_c} = \int_{G_2/(G_2 \cap (x_c^{-1}G_1x_c))} (\mathcal{J}_L (\mathcal{J}_{xc} \tilde{f}_t), \tilde{g}_t) \, d\mu_{xc}([t]) \quad (16)
$$

in the Hilbert space $\mu_{xc} H^{L_{xc}}$, is likewise immediate. $\blacksquare$

Note that in general the norm and topology induced by the inner product (16) defined by $\mathcal{J}_{xc}$ is not equivalent to the norm

$$
\|f\|^2 = (\tilde{f}, \tilde{f}) = \int_{G_2/(G_2 \cap (x_c^{-1}G_1x_c))} (\mathcal{J}_L (\mathcal{J}_{xc} \tilde{f}_t), \tilde{f}_t) \, d\mu_{xc}([t])
$$

and topology defined by the ordinary fundamental symmetry $\mathcal{J}^{L_{xc}}$ of Sect. 3 (of course with $\mathcal{G}$ and $H$ replaced with $G_2$ and $G_2 \cap (x_c^{-1}G_1x_c)$):

$$
\mathcal{J}^{L_{xc}} \tilde{f}_t = L_{h_{x_c}(t)}^{x_c} \mathcal{J}_{h_{x_c}(t)}^{L_{xc}} L_{h_{x_c}(t)}^{x_c} \tilde{f}_t,
$$

where $h_{x_c}(t) \in G_2 \cap (x_c^{-1}G_1x_c)$ is defined as in Remark 2 by a regular Borel section $B_{xc}$ of $G_2$ with respect to the subgroup $G_2 \cap (x_c^{-1}G_1x_c)$. However if for each $t \in G_2$, $h(x_c,t) \in G_{xc}$, then the two topologies coincide. Similarly whenever the homogeneous space $G_2/(G_2 \cap (x_c^{-1}G_1x_c))$ is compact then the two topologies coincide (but this case is not interesting).
LEMMA 9. The operators $V_{x_\mu}$ of the preceding Lemma are also isometric with respect to the norms $\| \cdot \|_C$ in $\mathcal{H}^L_C$ and $\| \cdot \|_{x_\mu} = \sqrt{\langle \cdot, \cdot \rangle_{x_\mu}}$ in $\mu^{x_\mu}\mathcal{H}^{L^{x_\mu}}$, where $(\cdot, \cdot)_{x_\mu}$ is defined as by (16), giving the norm in $\mu^{x_\mu}\mathcal{H}^{L^{x_\mu}}$ induced by $\mathfrak{J}_{x_\mu}$. In particular we have $\| V_{x_\mu} \| = 1$ for all $x_\mu$.

Denote the subgroup $G_2 \cap (x^{-1}_\mu G_1 x_\mu)$ by $G_{x_\mu}$. The Lemma is an immediate consequence of definitions of $\| \cdot \|_C$, $V_{x_\mu}$, and (16) giving the norm $\| \cdot \|$ in $\mu^{x_\mu}\mathcal{H}^{L^{x_\mu}}$:

$$\| V_{x_\mu} f \|_{x_\mu}^2 = (\tilde{f}, \tilde{f})_{x_\mu} = \int_{G_2/G_{x_\mu}} (\mathfrak{J}_L(\mathfrak{J}_{x_\mu} \tilde{f}), \tilde{f}) \, d\mu^{x_\mu}([t])$$

$$= \int_{G_2/G_{x_\mu}} (\mathfrak{J}_L(V_{x_\mu}^{-1} \mathfrak{J}^{C} V_{x_\mu} V_{x_\mu}^{-1} f), (V_{x_\mu}^{-1} f)) \, d\mu^{x_\mu}([t])$$

and because $V_{x_\mu}$ is Krein-unitary, i.e. isometric for the Krein inner products

$$\int_C (\mathfrak{J}_L(\cdot), (\cdot))_C \, d\mu_C([z]) \text{ and } \int_{G_2/G_{x_\mu}} (\mathfrak{J}_L(\cdot), (\cdot))_C \, d\mu^{x_\mu}([t])$$

the last integral in (17) is equal to

$$\int_C (\mathfrak{J}_L(\mathfrak{J}^{C} f)_2, f_2) \, d\mu_C([z]) = \| f \|_C^2.$$

Note, please, that the Lemmas of Sect. 4 i.e. Lemmas 10 - 6 are equally applicable to the Krein space $(\mathcal{H}^{L^{x_\mu}}, \mathfrak{J}_{x_\mu})$, with $\mathfrak{J}^{L^{x_\mu}}$ replaced by $\mathfrak{J}_{x_\mu}$, and with the section $B_{x_\mu}$ replaced with the image of $G_2/(G_2 \cap \langle x^{-1}_\mu G_1 x_\mu \rangle)$ under the inverse of the map $t \mapsto x_t t$. We formulate this remark as a separate

LEMMA 10. The Lemmas 10 - 6 are true for the Hilbert space $\mathcal{H}^{L^{x_\mu}}$ of the Krein space $(\mathcal{H}^{L^{x_\mu}}, \mathfrak{J}_{x_\mu})$, i.e. with $L$ replaced by $L^{x_\mu}$, $\mathfrak{J}_L$ replaced by $\mathfrak{J}_{L^{x_\mu}}$, $\mathcal{H}_L$ replaced with $\mathcal{H}^{L^{x_\mu}}$, $\mathfrak{J}^{L^{x_\mu}}$ replaced by $\mathfrak{J}_{x_\mu}$ and finally with the section $B_{x_\mu}$ replaced with the image of $G_2/(G_2 \cap \langle x^{-1}_\mu G_1 x_\mu \rangle)$ under the inverse of the map $t \mapsto x_t t$.

The proofs remain unchanged.

In Subsection 9 we explain why we are using $\mathfrak{J}_{x_\mu}$ in $\mu^{x_\mu}\mathcal{H}^{L^{x_\mu}}$ instead of $\mathfrak{J}^{L^{x_\mu}}$.  

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8 Decomposition (disintegration) of measures

Let $G$, $G_1$ and $G_2$ be such as in Sect. 4. Because the base of the system of neighbourhoods of unity in $G$ is countable, the uniform space $X = \mathfrak{G}/G_1$ is metrizable (compare e.g. [75], §2) for any closed subgroup $G_1 \subset G$. The right action of $G_1$ on $\mathfrak{G}$ is proper and the quotient map $\pi : \mathfrak{G} \rightarrow \mathfrak{G}/G_1$ is open, so that the space $X = \mathfrak{G}/G_1$ of right $G_1$ orbits ($G_1$ cosets) automatically has the required regularity: measurability of the equivalence relation defined by the $G_1$ orbits. In particular the quotient space $X$ is Hausdorff, separable and locally compact and the measure $\rho \cdot \mu_0$ (with the $\rho$-function of Sect. 6 and right Haar measure $\mu_0$ on $\mathfrak{G}$) is decomposable into a direct integral of measures $\rho \cdot \mu_0 = \int_{\mathfrak{G}/G_1} \beta[x] \, d\mu([x])$ with the component measures $\beta[x]$ of the decomposition concentrated in the $G_1$ orbit (right coset) $[x]$ and with Radon-Nikodym derivative associated with the action of the subgroup $G_1$ (i.e. $\lambda_{[x]}$-function) corresponding to $\beta[x]$ equal to the restriction to the orbit $[x]$ and to the subgroup $G_1$ of the Radon-Nikodym (i.e. $\lambda$-function) corresponding to the measure $\rho \cdot \mu_0$. This in particular gives us the quasi invariant regular Baire (or Borel) measure $\mu = \mu_{\mathfrak{G}/G_1}$ on the uniform space $X$ corresponding to $\rho$, i.e. the factor measure of $\rho \cdot \mu_0$ (Mackey’s method of constructing general regular quasi invariant measure on the quotient space $X = \mathfrak{G}/G_1$).

This is not the case if we replace $\mathfrak{G}$ with $X = \mathfrak{G}/G_1$ acted on by a second closed subgroup $G_2 \subset \mathfrak{G}$. The quotient space $X/G_2$ is in general a badly behaved non Hausdorff space with non measurable equivalence relation defined in $X$ with the $G_2$ orbits as equivalence classes. We require a regularity condition in order to achieve an effective tool for constructing effectively a dual of the group $\mathfrak{G}$ in question with the help of decomposition of tensor product of induced representations.

Let $X$, for example $X = \mathfrak{G}/G_1$, be any separable locally compact metrizable space with an equivalence relation $R$ in $X$, for example with the equivalence classes given by right $G_2$-orbits in $X = \mathfrak{G}/G_1$ under the right action of a second closed subgroup $G_2 \subset \mathfrak{G}$. Let the equivalence classes form a set $\mathcal{C}$ and for each $x \in X$ let $\pi_x(x) \in \mathcal{C}$ denote the equivalence class of $x$. Let $X$ be endowed with a regular measure $\mu$ (quasi invariant in case $X = \mathfrak{G}/G_1$). We define following [65] the relation $R$ to be measurable if there exists a countable family $E_0, E_1, E_2, \ldots$ of subsets of $\mathcal{C}$ such that $\pi_x^{-1}(E_i)$ is a Baire (or Borel) set for each $i$ and such that $\mu(\pi_x^{-1}(E_0)) = 0$, and such that each point $C$ of $\mathcal{C}$ not belonging to $E_0$ is the intersection of the $E_i$ which contain it. Under this assumption of measurability $\mu$ may be decomposed (disintegrated) as an integral $\mu = \int \mu_C \, d\nu(C)$ over $\mathcal{C}$ of measures $\mu_C$, with each $\mu_C$ concentrated on the corresponding equivalence class $C$, i.e. $G_2$ orbit in case $X = \mathfrak{G}/G_1$, with a regular measure $\nu = \mu_{\mathfrak{G}/G_2}$ on $\mathcal{C} = \mathfrak{G}/G_2$ i.e. the factor measure of $\mu$, which we may

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19Strictly speaking in Rohlin’s definition of measurability of $R$, accepted by Mackey in [17], the set $E_0$ is empty and $\pi_x^{-1}(E_k)$, $k \geq 1$, are just $\mu$-measurable and not necessary Borel. But the difference is unessential as we explain below in this Sect..
call the “double factor measure” \( \mu_{(G/G_1)/G_2} \) of \( \mu_0 = \mu_\varnothing \) in case \( \mathfrak{X} = \mathfrak{S}/G_1 \); and moreover in this case when \( \mathfrak{X} = \mathfrak{S}/G_1 \) the Radon-Nikodym derivative (i. e. \( \lambda_C \)-function) corresponding to \( \mu_C \) and associated with action of the subgroup \( G_2 \) is equal to the restriction to the orbit \( C \) and to the subgroup \( G_2 \) of the Radon-Nikodym derivative (\( \lambda \)-function) corresponding to \( \mu \). In this case we say after Mackey that the subgroups \( G_1 \) and \( G_2 \) are \textit{regularly related}. In short: the orbits in \( \mathfrak{S}/G_1 \) under the right action of \( G_2 \) form the equivalence classes of a measurable equivalence relation \( \mathcal{E} \).

Let us explain the meaning of the regularity condition. Even if \( G_1 \) and \( G_2 \) were not regularly related we could of course find a countable set \( E_0, E_1, E_2, \ldots \) of Borel unions of orbits which generate the \( \sigma \)-ring of all measurable unions of orbits. The unique equivalence relation \( R \) such that \( \mathfrak{X} \subset \mathfrak{S}/G_1 \) and \( \mathfrak{Y} \subset \mathfrak{S}/G_1 \) are in the relation whenever \( \mathfrak{X} \) and \( \mathfrak{Y} \) are in the same sets \( E_j \) will be measurable. This equivalence relation gives us a decomposition of the quasi invariant measure \( \mu \) into quasi invariant component measures \( \mu_P \) concentrated on subsets \( P \subset \mathfrak{S} \), but in this general non regular situation the subsets \( P \) are unions of many orbits \( C \subset \mathfrak{S} \). This would give us decomposition of \( U^L \) restricted to \( G_2 \), but in this decomposition the component representations will not be associated with single orbits, i. e. with single double cosets \( G_1 x_0 G_2 \) and will not be identifiable as “induced representations” \( U^{L^c} \) of \( G_2 \) of Lemma \( \S \) of Sect. \( \S \). Little or nothing is known of such component representations related to non transitive systems of imprimitivity. In fact the regularity of the \( G_2 \)-orbits in \( \mathfrak{S}/G_1 \) is essentially equivalent \( \S \) for the group \( \mathfrak{S} \) to be of type I. Because of the biunique correspondence between \( G_2 \) orbits in \( \mathfrak{S}/G_1 \) and double cosets \( G_1 x G_2 \) in \( \mathfrak{S} \), and because of the relation between Borel structures on \( \mathfrak{X} = \mathfrak{S}/G_1 \) and on \( \mathfrak{X}/G_2 \), we may reformulate the regularity condition as follows. We assume that there exists a sequence \( E_0, E_1, E_2, \ldots \) of measurable subsets of \( \mathfrak{S} \) each of which is a union of double cosets such that \( E_0 \) has Haar measure zero and each double coset not in \( E_0 \) is the intersection of the \( E_j \) which contain it (compare Lemma \( \S \)).

\textbf{Example 1.} The equivalence relation on the two-torus \( \mathfrak{X} = \mathbb{R}^2/\mathbb{Z}^2 \) given by the leaves of the Kronecker foliation associated to an irrational number \( \theta \), i. e. given by the differential equation

\[ dy = \theta dx, \]

\footnote{Using literally Rohlin’s definition of measurability: almost all of the orbits in \( \mathfrak{S}/G_1 \) under the right action of \( G_2 \) form the equivalence classes of a Rohlin-measurable equivalence relation.}

\footnote{In fact the representations \( U^{L^c} \) of Lemma \( \S \) of Sect. \( \S \) do not have the standard form of induced representations defined in Sect. \( \S \) as \( J_{x_0} \neq J^{L^c} \), but in relevant cases of representations encountered in QFT they may be shown to be Krein-unitary equivalent to standard induced representations (in the sense of Sect. \( \S \)). Anyway they are concentrated in single orbits.}

\footnote{One may characterise the space of orbits by considering the respective group algebra or the associated universal enveloping \( C^* \)-algebra. Connes developed a general theory of crossproduct \( C^* \)-algebras and von Neumann algebras associated with foliations, strongly motivated by the Mackey theory of induced representations, compare \( \S \) and references there in.}
is not measurable. The leaves, i.e. equivalence classes, can be viewed as orbits of the additive group $\mathbb{R}$ on the two-torus $X = \mathbb{R}^2/\mathbb{Z}^2$.

In the original Mackey’s theory the induced representations $\mu U^L$ and $\mu' U^L$ are unitary equivalent whenever the quasi invariant measures $\mu$ and $\mu'$ on $\mathcal{G}/G_1$ are equivalent, which is always the case, as all such measures are equivalent. In this case we may assume all measures $\mu$ in the induced representations $\mu U^L$ to be finite without any lost of generality. In particular, and this simplifies matter, we may restrict ourself to finite measures $\mu$ on $\mathcal{G}/G_1$, as Mackey did in [17], in constructing decomposition (disintegration) $\mu = \int \mu_C d\nu(C)$ with each of the measures $\mu_C$ concentrated on the corresponding orbit $C$ and the corresponding Radon-Nikodym derivative associated with $\mu_C$ under the action of $G_2$ equal to the restriction to the orbit $C$ and to the subgroup $G_2$ of the Radon-Nikodym derivative associated with $\mu$ (this is proved in §11 of [17]). This is not the case for the induced representations $\mu U^L$ and $\mu' U^L$ in Krein spaces defined here, as already indicated by Theorem 1 and its proof: they are (Krein-unitary) inequivalent whenever the quotient space $\mathcal{G}/G_1$ is not compact and the Radon-Nikodym derivative $d\mu'/d\mu$ is not “lower” or “upper” bounded. Therefore we cannot restrict ourself to finite measures $\mu$ in construction of the decomposition $\mu = \int \mu_C d\nu(C)$ with the above mentioned properties. Because Mackey’s construction of decomposition of finite measure $\mu$ is sufficient for the theory of unitary group representations (as well as for the extension of the construction of induced representation to representations of $C^*$-algebras along the lines proposed by Rieffel) decomposition having the above mentioned properties of a quasi invariant measure $\mu$ which is not finite has not been constructed explicitly in the classical mathematical literature, at least the author was not able to find it (in the Bourbaki’s course on integration [8], Chap. 7.2.1-7.2.3 decomposition of this type is constructed but under stronger assumption than measurability of the equivalence relation given by right $G_2$ action on $X = \mathcal{G}/G_1$ where it is assumed instead that the action is proper and moreover where it is assumed that the measure $\mu$ is relatively invariant and not merely quasi invariant – assumptions too strong for us). Because the required decomposition of not necessary finite quasi invariant measure $\mu$ on $\mathcal{G}/G_1$ is important for the decomposition of the restriction of the induced representation $\mu U^L$ in a Krein space to a closed subgroup and a fortiori to a decomposition of tensor product of induced representations $\mu U^L$ and $\mu' U^M$ in Krein spaces we present here its construction explicitly only for the sake of completeness. The construction presented here uses a localization procedure in reducing the problem of decomposition to the Mackey-Godement decomposition ([17], §11) of a finite quasi invariant measure.

Whenever the action of $G_2$ on $X = \mathcal{G}/G_1$ is proper one can just replace the continuous homomorphism $\chi : G_2 \to \mathbb{R}_+$ in [5], Chap. 7.2.1-7.2.3, by the Radon-Nikodym derivative associated with the measure $\mu$ on $X = \mathcal{G}/G_1$ in this case. Using the Federer and Morse theorem [24] one constructs a regular Borel section of $X$ with respect to $G_2$ which enables the construction of the factor measure $\nu$ on the quotient $C = X/G_2$ of the space $X$ by the group $G_2$ with the method of [8] changed in
Let $X$ be the separable locally compact metrizable (in fact complete metric) space $\mathfrak{G}/G_1$ equipped with a regular quasi invariant measure $\mu$. Let $R$ be the equivalence relation in $X$ given by the right action of a second closed subgroup $G_2$ with the associated quotient map $\pi_X : X \mapsto X/R = \mathfrak{G}/G_2$, and let $K$ be a compact subset of $X$. There is canonically defined equivalence relation $R_K$ on $K$ induced by $R$ on $K$ with the associated quotient map $\pi_K : K \mapsto K/R_K$ equal to the restriction of $\pi_X$ to the subset $K$. Note please that for an equivalence relation $R$ in the separable locally compact and metrizable space $\mathfrak{G} = \mathfrak{G}/G_1$ the above mentioned (Rohlin’s [65]) condition of measurability of $R$ is equivalent to the following condition: the family $\mathcal{R}$ of those compact sets $K \subset \mathfrak{G}$ for which the quotient space $K/R_K$ is Hausdorff is $\mu$-dense, i. e. one of the following and equivalent conditions is fulfilled:

(I) For a subset $A \subset X$ to be locally $\mu$-negligible it is necessary and sufficient that $\mu(A \cap K) = 0$ for all $K \in \mathcal{R}$.

(II) For any compact subset $K_0$ of $X$ and for any $\epsilon > 0$ there exists a subset $K \in \mathcal{R}$ contained in $K_0$ and such that $\mu(K_0 - K) \leq \epsilon$.

(III) For each compact subset $A$ of $X$ there exists a partition of $A$ into a $\mu$-negligible subset $N$ and a sequence $\{K_n\}_{n \in \mathbb{N}}$ of compact subsets belonging to $\mathcal{R}$.

(IV) For each compact subset $K$ of $X$ there exists an increasing $H_1 \subseteq H_2 \subseteq \ldots$ sequence $\{H_n\}_{n \in \mathbb{N}}$ of compact sets belonging to $\mathcal{R}$ contained in $K$ and such that the set $Z = K - \bigcup_{n \in \mathbb{N}} H_n$ is $\mu$-negligible.

Indeed, because the the system of neighbourhoods of unity in $\mathfrak{G}$ is countable, the uniform space $X = \mathfrak{G}/G_1$ is completely metrizable and locally compact (compare e. g. [65], §2) for any closed subgroup $G_1 \subset \mathfrak{G}$. Therefore Proposition 3 of [7], Chap. VI, §3.4, is applicable. By this Proposition we need only show that using the family $\mathcal{R}$ one can construct the sets $E_0, E_1, \ldots$ of the Rohlin’s measurability condition of $R$, for which $\pi_X^{-1}(E_k)$, $k \geq 1$, are not only $\mu$-measurable but moreover Borel. This however follows from the fact that $X$ is countable at infinity: there exists a sequence of compact subsets $K_1 \subset K_2 \subset \ldots$ of $X$ such that $X = \cup_i K_i$ and moreover we may assume that they are regular closed sets: $\text{cl} \text{int} \ K_m = K_m$.

Indeed, let $\{O_k\}_{k \in \mathbb{N}}$ be a countable base of the topology in $X$, such that the closure $\overline{O_k}$ of each $O_k$ is compact (there exists such a base because $X$ is second countable and locally compact). For each $\overline{O_k}$ choose a sequence $\{K_{kl}\}_{l \in \mathbb{N}}$ of compact sets belonging to $\mathcal{R}$ and a $\mu$-negligible subset $M_k$ giving the partition $\overline{O_k} = M_k \cup K_{kl} \cup \ldots$ of $\overline{O_k}$, existence of which is assured by the condition (III). Define the $\mu$-negligible set $M = \cup_k M_k$ and a maximal subset $M_0$ of $M$ invariant under the action of $G_2$ on $X$.

By the condition (IV) we can construct for each $K_m$ a sequence $H_{m1} \subset H_{m2} \subset H_{m3} \subset \ldots$ of compact subsets of $K_m$ belonging to $\mathcal{R}$ and a $\mu$-negligible
subset $Z_m$ such that $K_m = Z_m \cup (\cup_n K_{m,n})$. Define the $\mu$-negligible set $Z = \bigcup \limits_{n \in \mathbb{N}} Z_m$ and the maximal subset $Z_0$ of $Z$ invariant under the action of $G_2$.

Let us define a countable family of sets $E_0 = \pi_X(Z_0 \cup M_0), E_{mn} = \pi_X(K_{m,n}) = \pi_{K_{m,n}}(K_{m,n}) = K_{m,n}/R_{K_{m,n}}, m, n \in \mathbb{N}$ in $\mathcal{X}/G_2 = \mathcal{X}/R$, where $K_{m,n}/R_{K_{m,n}}$ is Hausdorff by assumption.

Now let $\tau_1$ and $\tau_2$ be two elements of $\mathcal{X}$ not in $N_0 = Z_0 \cup M_0$ such that $\pi_X(\tau_1) \neq \pi_X(\tau_2)$. Then by construction there exists $H_{mn} \in \mathcal{R}$ containing $\tau_1$ and $\tau_2$ with $\pi_X(\tau_1) = \pi_X(\tau_2)$ and $\pi_X(\tau_2) = \pi_X(\tau_2)$.

$H_{mn}/R_{H_{mn}}$ containing $\pi_X(\tau_1) = \pi_X(\tau_1)$ and $\pi_X(\tau_2) = \pi_X(\tau_2)$ is Hausdorff by construction. Thus there exist two compact non intersecting neighbourhoods $\mathcal{O}_{\tau_1}$ and $\mathcal{O}_{\tau_2}$ of $\tau_1$ and $\tau_2$ respectively such that for $K_{\tau_1} = \mathcal{O}_{\tau_1}, \cap H_{mn}$ and $K_{\tau_2} = \mathcal{O}_{\tau_2} \cap H_{mn}$ we have $\pi_X^{-1}(K_{\tau_1}) \cap \pi_X^{-1}(K_{\tau_2}) = \emptyset$. By construction we may choose $K_{m_1,n_1} \subset K_{\tau_1}$ and $K_{m_2,n_2} \subset K_{\tau_2}$ in $\mathcal{R}$ such that $\tau_1 \in K_{m_1,n_1}$ and $\tau_2 \in K_{m_2,n_2}$. Of course we have $E_{m_1,n_1} \cap E_{m_2,n_2} = \pi_X^{-1}(K_{m_1,n_1}) \cap \pi_X^{-1}(K_{m_2,n_2}) = \emptyset$. Thus the intersection of all $E_{mn}$ in $\mathcal{R}$ containing $\pi_X(\tau_1) \in \mathcal{X}/G_2$ is equal to $\{\pi_X(\tau_1)\}$. We have to show that $\pi_X^{-1}(E_{mn}) = \pi_X^{-1}(\pi_X(K_{m,n}))$ are Baire (or Borel) sets. To this end observe please that $\pi_X^{-1}(\pi_X(K_{m,n}))$ is equal to the saturation of $K_{m,n}$, i.e. $\pi_X^{-1}(\pi_X(K_{m,n})) = K_{m,n} \cdot G_2$. Choose a compact neighbourhood $V$ of the unit in $G_2$ such that $V = V^{-1}$. Then if $G_2$ is connected then $G_2 = \bigcup \limits_{\eta \in \mathbb{N}} V_n$; if $G_2$ is not connected then it is still a countable sum of connected components of the form $\bigcup \limits_{\eta \in \mathbb{N}} V_n \eta_m$, with $\eta_m \in G_2$ chosen from $m$-th connected component $G_{2m}$ of $G_2$. Thus in each case $G_2$ is a countable sum $\bigcup \limits_{k,l \in \mathbb{N}} V_{kl}$ of compact sets $V_{kl}$. Therefore $\pi_X^{-1}(E_{mn}) = K_{m,n} \cdot G_2 = \bigcup \limits_{k,l \in \mathbb{N}} K_{m,n} \cdot V_{kl}$ being a countable sum of compact sets is contained in the $\sigma$-ring generated by the compact sets and all the more it is a Borel set contained in the $\sigma$-ring generated by the closed sets. Thus both definitions of measurability of the equivalence relation $R$ on $\mathcal{X}$ are equivalent.

**Lemma 11.** There exists a Borel set $B_0$ in $\mathcal{X} = \mathcal{G}/G_1$ and a $\mu$-negligible subset $N_0 \subset \mathcal{X}$ consisting of $G_2$ orbits in $\mathcal{X} = \mathcal{G}/G_1$ such that $B_0$ intersects each $G_2$ orbit not contained in $N_0$ in exactly one point.

For the proof compare e.g. [7], Chap. VI, §3.4, Thm. 3.

Adding to $B_0$ any section of the $\mu$-negligible set $N_0$ we obtain a measurable section $B_{00}$ for the whole space $\mathcal{X}$. For equivalence relations $R$ on smooth manifold $\mathcal{X}$ defined by foliations on $\mathcal{X}$ (i.e. smooth and integrable sub-bundles of $T\mathcal{X}$) existence of a measurable section is equivalent for the foliation to be of type I; i.e. the von Neumann algebra associated to the foliation is of type I iff the foliation admits a Lebesgue measurable section, compare [12], Chap. I.4.2, Proposition 5.

Because the Borel space $\mathcal{X} = \mathcal{G}/G_1$ is standard it follows by the second Theorem on page 74 of [30] that the quotient Borel structure on $\mathcal{X}/G_2 - N_0/G_2$ is

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likewise standard; i.e., there exists a Borel isomorphism \( \psi_0: (X - N_0)/G_2 \to S_0 \subset \) onto a Borel subset \( S_0 \) of a complete separable metric space \( S \).

The space \((X - N_0)/G_2\) however need not be locally compact and it is not if the action of \( G_2 \) on \( X = \mathfrak{G}/G_1 \) is not proper but only measurable, i.e. with measurable equivalence relation determined by the action of \( G_2 \). Similarly \( G_2 \)-orbit \( C \) in \( X \) as a subset of a locally compact space \( X \) need not be closed if the action of \( G_2 \) is not proper and thus need not be locally compact with the topology induced from the surrounding space \( X \).

**Lemma 12.** Let \( N_0 \) be as in Lemma 11. A necessary and sufficient condition that a subset \( E \) of \( X/G_2 - N_0/G_2 \) be a Borel set is that \( \pi^{-1}(E) \) be a Borel set in \( X - N_0 \). A necessary and sufficient condition that a function \( f \) on \( X/G_2 - N_0/G_2 \) be a Borel function is that \( f \circ \pi \) be a Borel function on \( X - N_0 \).

\[ \square \]

Let \( p_0 \) be the Borel function \( \psi_0 \circ \pi : X - N_0 \to S_0 \). Let \( E' \) be any subset of \( S_0 \) such that \( p_0^{-1}(E') \) is a Borel set. Let \( B_0 \) be the Borel section of \( X - N_0 \) with respect to \( G_2 \), existence of which has been proved in Lemma 11. Then \( p_0(p_0^{-1}(E') \cap B_0) = E' \), and thus \( E' \) is a Borel set by Theorem 3, page 253 of [43], compare likewise the Theorems on pages 72-73 of [50], because \( p_0 \) is one-to-one Borel function on \( B_0 \). Conversely: if \( E' \) is Borel in \( S_0 \) then because \( p_0 \) is a Borel function, so is the set \( p_0^{-1}(E') \). The first part of the Lemma follows now from this and from definition of the Borel structure induced on \( \psi_0((X - N_0)/G_2) \) and a fortiori on \( X/G_2 - N_0/G_2 \). The remaining part of the Lemma is an immediate consequence of the first part.

\[ \square \]

We have the following disintegration theorem for the (not necessarily finite) measure \( \mu \) and any of its pseudo image measures \( \nu \) on \( X/G_2 \) (for definition of pseudo image measure \( \nu \) compare e.g. [7], Chap. VI.3.2):

**Lemma 13.** For each orbit \( C = \pi^{-1}(d_0) \subset X \) with \( d_0 \in X/G_2 \) there exists a Borel measure \( \mu_C \) in \( X \) concentrated on the orbit \( C \), i.e. \( \mu_C(X - C) = 0 \). For any \( g \in L^1(X, \mu) \) the set of all those \( G_2 \) orbits \( C \) for which \( g \) is not \( \mu_C \)-integrable is \( \nu \)-negligible and the function

\[ C \mapsto \int g(x) d\mu_C(x) \]

is \( \nu \)-summable and \( \nu \)-measurable, and

\[ \int d\nu(C) \int g(x) d\mu_C(x) = \int g(x) d\mu(x). \]  

In short

\[ \mu = \int \mu_C(x) d\nu(C). \]
REMARK 3. For each orbit $C$ the measure $\mu_C$ may also be naturally viewed as a measure on the $\sigma$-ring $\mathcal{R}_C$ of measurable subsets of $C$ induced from the surrounding space $\mathfrak{X}$: $E \in \mathcal{R}_C$ iff $E = E' \cap C$ for some $E' \in \mathcal{R}_\mathfrak{X}$, i. e. with the subspace Borel structure.

■ For the proof we refer the reader e. g. to [7], Chap. VI, §3.5. ■

We shall show that for each $C$ the measure $\mu_C$ is quasi invariant and that for all $\eta \in G_2$ the Radon-Nikodym derivative $\lambda_C(\cdot, \eta) = \frac{d(R_{\eta} \mu_C) (\cdot)}{d\mu_C (\cdot)}$ is equal to the restriction of the Radon-Nikodym derivative $\lambda(\cdot, \eta) = \frac{d(R_{\eta} \mu)}{d\mu} (\cdot)$ to the orbit $C$. In doing so we prefer reducing the problem to the Mackey-Godement decomposition of a finite measure ([47], §11) using a localization of the measure space $(\mathfrak{X}, \mathcal{R}_\mathfrak{X}, \mu)$ and its disintegration. Toward this end we need some further Lemmas.

**Lemma 14.** Let $\mu$, $\mu_C$ and $\nu$ be as in the preceding Lemma. Let $K$ be a compact subset of $\mathfrak{X}$. Then $\pi_\mathfrak{X}(K)$ is measurable on $\mathfrak{X}/G_2$.

■ Let $K$ be any compact subset of $\mathfrak{X}$ and let $Z, K_n$ be the subsets of condition (IV), i. e. $K_n \in \mathfrak{R}$ is an increasing sequence of compact subsets of $K$, and $Z$ is $\mu$-negligible subset of $K$ such that $K = Z \cup (K_1 \cup K_2 \cup \ldots)$. Let us define the subset (if any) $Z_0 \subset Z$ consisting of intersections of full $G_2$-orbits with $K$, i. e. the maximal subset of $Z$ invariant under the action of $G_2$ on $\mathfrak{X}$.

Then $\pi_\mathfrak{X}(K-Z) = \pi_\mathfrak{X}(K-Z_0)$. We shall show that $\mu(Z_0 \cap G_2) = \mu(\pi_\mathfrak{X}^{-1} (\pi_\mathfrak{X}(Z_0))) = 0$. Toward this end observe that because $\mathfrak{X}$ is metrizable and separable we may assume the elements $O_m$, $m \in \mathbb{N}$, of basis of topology to be the balls with compact closure $\overline{O_m}$; and the $\sigma$-ring of Borel sets on $\mathfrak{X}$ generated by the open $O_m$ or closed $\overline{O_m}$ balls.
For each $\epsilon > 0$ there exists open $O_\epsilon \supset K$ with: $\mu(O_\epsilon - K) < \epsilon$

By regularity of $\mu$

By the regularity and quasi invariance of the measure $\mu$ it easily follows that the $\mu$-measure of the intersection of $Z_0 \cdot G_2$ with any open set in $X$ is equal zero, and thus again by the regularity of $\mu$ and second countability of $X$ it easily follows that $\mu(Z_0 \cdot G_2) = \mu(\pi_X^{-1}(\pi_X(Z_0))) = 0$. Thus $\pi_X(Z_0)$ is a subset of a measurable null set, and so must be a measurable set with $\nu(\pi_X(Z_0)) = 0$, because $\nu$ is a pseudo-image measure of $\mu$ under $\pi_X$. Moreover, we have:

$$\pi_X(K - Z) = \pi_X(K - Z_0) = \pi_X(K) - \pi_X(Z_0),$$

because $Z_0$ consists of intersections of $G_2$-orbits with $K$.

On the other hand

$$\psi_0 \circ \pi_X(K - Z)$$

is a Borel set in $S$, and thus $\pi_X(K - Z)$ is a Borel set in $X/G_2$ as $\psi_0$ is a Borel isomorphism. Indeed, because images preserve the set theoretic sum operation we have

$$\psi_0 \circ \pi_X(K - Z) = \bigcup_{n \in \mathbb{N}} \psi_0 \circ \pi_X(K_i).$$

Because $K_j \in \mathfrak{K}$ then $K_j/R_{K_j}$ is Hausdorff and the quotient map $\pi_{K_j}$ is closed and thus the quotient space $K_j/R_{K_j}$ is homeomorphic to the compact space $\pi_{K_j}(K_j)$, and moreover because $K_j$ is compact and metrizable (as a subspace of the metrizable space $X$) the quotient space $K_j/R_{K_j}$ is likewise metrizable ([22], Thm. 7.5.22). We can therefore apply the Federer and Morse Theorem 5.1 of [24] in order to prove the existence for each $j$ of a Borel subset $B_j \subset K_j$ such that $\pi_{K_j}(B_j) = \pi_{K_j}(K_j)(= \pi_X(K_j))$ and such that $\pi_{K_j}$ is one-to-one on $B_j$. Therefore $\psi_0 \circ \pi_X$ is one-to-one Borel function on a Borel subset $B_j$ of complete separable metric space $X$ to a complete separable metric space $S$. Therefore again by the Theorem on page 253 of [13] (compare likewise the Theorem on page 72 of [41]), it follows that $\psi_0 \circ \pi_X(B_j) = \psi_0 \circ \pi_X(K_j)$ is a Borel set. Because $\psi_0$ is a Borel isomorphism it follows that $\pi_X(K_j)$ is a Borel set in $X/G_2$. 

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Thus $\pi_X(K)$ differs from a Borel set $\pi_X(K - Z)$ by a measurable $\nu$-negligible subset $\pi_X(Z_0) \subset \pi_X(K)$; so we have shown that $\pi_X(K)$ is measurable. ■

Note that the Lemma 14 is non trivial. By the well known theorem of Suslin – continuous image of a Borel set is not always Borel, but it is always measurable, compare e. g. [39], Lemm. 11.6, page 142 and Thm. 11.18 , page 150, where the references to the original literature are provided. However this argument would be insufficient for $\pi_X(K)$ to be measurable in $X/G_2$ for any compact set $K \subset X$. Indeed it would in addition require to be shown that the quotient Borel structure on $X/G_2$ is equal to the $\sigma$-ring of Borel sets generated by the closed (open) sets of the quotient topology on $X/G_2$.

LEMMA 15. Let $\mu, \mu_C, \nu$ be as in Lemma 13 and let $K$ be a compact subset of $X$. Let $\eta \in G_2$ and let $R_K$ be the $\sigma$-ring of Borel subsets of $K$ induced form the surrounding measure space $X$. Let $(\mu)^\prime_K$ and $(\mu_C)^\prime_K$ denote the restrictions of $\mu$ and $\mu_C$ to $K$ defined on the $\sigma$-ring $R_K$ respectively, and let $R_\eta \mu, R_\eta \mu_C$ denote their right translations; and similarly let $(\nu)^\prime_{\pi_X(K)}$ be the restriction of the measure $\nu$ to the subset $\pi_X(K)$. Then

(a)  
$$(\mu)^\prime_K = \int (\mu_C)^\prime_K d(\nu)^\prime_{\pi_X(K)}(C)$$
with each $(\mu_C)^\prime_K$ concentrated on $C \cap K$.

(b)  
$$R_\eta \mu = \int R_\eta \mu_C d\nu(C).$$

■ Part (a) of the Lemma is an immediate consequence of Lemmas 13 and 14 with $1_K \cdot g$ inserted for $g$ in the formula (18), where $1_K$ is the characteristic function of the compact set $K$. The only non-trivial part of the proof lies in showing that $\pi_X(K)$ is measurable, which was proved in Lemma 14.

For (b) observe that if $R_{\eta^{-1}} g \in L^1(X, \mu) \iff g \in L^1(X, R_\eta \mu)$, then by Lemma 14

$$\int g(x) d(R_\eta \mu) = \int g(x \cdot \eta^{-1}) d\mu = \int d\nu(C) \int g(x \cdot \eta^{-1}) d\mu_C(x)$$
$$= \int d\nu(C) \int g(x) d(R_\eta \mu_C)(x),$$
thus
$$R_\eta \mu = \int R_\eta \mu_C d\nu(C).$$

23The $\sigma$-ring of Borel sets with a regular measure on this ring is sufficient to recover all measurable subsets and their measures obtained by the standard completion of the Borel measure space.
Note that the operations of restriction \( (\cdot)_k' \) to \( K \) and right translation \( R_\eta(\cdot) \) do not commute. Indeed if we write \( R_\eta \circ (\cdot)_K' \) for \( R_\eta((\cdot)_K') \), then \( R_\eta \circ (\cdot)_K' = (\cdot)_{K,\eta^{-1}}' \circ R_\eta = (R_\eta(\cdot))_{K,\eta^{-1}}' \). i. e. first restrict to \( K \) and then translate \( R_\eta \) is the same as first translate \( R_\eta \) and then restrict to \( K \cdot \eta^{-1} \) (and not to \( K \)).

**Remark 4.** Let \( Op(\mu) \) denote a repeated application of several restrictions to compact sets and translations: \( (\cdot)_{K_1}', R_{\eta_1}(\cdot), \ldots \) performed on the measure \( \mu \). Then the repeated application of Lemma 15 (a) and (b) gives

\[
Op(\mu) = \int Op(\mu') \, d\overline{Op}(\nu)(C),
\]

where \( \overline{Op}(\nu) \) denotes the restriction \( (\cdot)'_{\pi\chi(K)} \) with the compact set \( K \subset \chi \) which arises in the following way: \( (\cdot)'_K \) is the restriction which arises from \( Op \) by commuting all translations to the right (so as to be performed first) and all restrictions to the left (so as to be performed after all translations): \( Op = (\cdot)'_K \circ R_\eta(\cdot) \) or \( Op(\cdot) = (R_\eta(\cdot))_{K}' \).

**Lemma 16.** Let \( K, (\mu)'_K, (\mu_C)'_K, (\nu)'_{\pi\chi(K)} \) be as in the preceding Lemma. For any bounded and \( (\mu)'_K \)-measurable function \( g \) and for any \( f \in L^1(\pi^{-1}_\chi(K), (\nu)'_{\pi\chi(K)}) \) the set of all those \( G_2 \) orbits \( C \) having non empty intersection \( C \cap K \) for which \( g \) is not \( \mu_C \)-integrable is \( \nu \)-negligible and the the function

\[
C \mapsto \int g(x) d(\mu_C)'_K(x)
\]
on this set of orbits \( C \) is \( (\nu)'_{\pi\chi(K)} \)-summable and \( (\nu)'_{\pi\chi(K)} \)-measurable, and

\[
\int f(C) \int g(x) d(\mu_C)'_K(x) \, d(\nu)'_{\pi\chi(K)}(C) = \int f(\pi\chi(x)) g(x) \, d(\mu)'_K(x). \tag{19}
\]

The Lemma is an immediate consequence of the preceding Lemma. The only non-trivial part of the proof is to show that \( f \) is measurable on \( \chi/G_2 \) if and only if \( f \circ \pi\chi \) is measurable on \( \chi \). But this is an immediate consequence of Lemma 12.

In order to simplify notation let us denote the operation of restriction \( (\cdot)'_K \) to \( K \) just by \( (\cdot)' \) in the next Lemma and its proof. In all other restrictions \( (\cdot)'_D \) the sets \( D \) will be specified explicitly.

**Lemma 17.** Let \( \mu, \mu_C \) be as in Lemma 15 and let \( K \) be a compact subset of \( \chi \). Let \( \eta \in G_2 \) and let \( C \) be any \( G_2 \)-orbit having non empty intersection \( C \cap K \cdot \eta^{-1} \cap K \). Then for the respective measures obtained by right translations and restrictions performed on \( \mu \) and \( \mu_C \) respectively we have:
(a) The measures \(((\mu_C)'_K \eta^{-1})\) and \((R_\eta(\mu_C)'_K \eta^{-1})\), defined on measurable subsets of \(C \cap K \cap K \cdot \eta^{-1}\), are equivalent.

(b) \[
\lambda_C(\cdot, \eta) = \frac{d(R_\eta \mu_C)}{d\mu_C}(\cdot) = \frac{d((\mu_C)'_K \eta^{-1})}{d((\mu)'_K \eta^{-1})}(\cdot) = \frac{d R_\eta \mu}{d\mu}(\cdot) = \lambda(\cdot, \eta)
\]
on \(C \cap K \cap K \cdot \eta^{-1}\).

In addition to the operations of translation and restriction let us introduce after Mackey, \[17\], §11, one more operation \(\tilde{\cdot}\) defined on finite measures \(\mu\) on \(\mathcal{X}\), giving measures \(\tilde{\mu}\) on \(\mathcal{X}/G_2\). Namely we put \(\tilde{\mu}(E) = \mu(\pi_X^{-1}(E))\). \(\tilde{\mu}\) is well defined for any quasi invariant measure \(\mu\) on \(\mathcal{X}/G_2\) because \(\mu\) is finite. More precisely \(\tilde{\mu}\) is defined on the \(\sigma\)-ring of measurable subsets \(E\) of \(\pi_X(K)\) by the formula: \(\tilde{\mu}'(E) = \mu'(\pi_X^{-1}(E)) = \mu(K \cap \pi_X^{-1}(E))\). A simple verification of definitions shows that \(\tilde{\mu}\) is a pseudo image measure of the measure \(\mu\) under \(\pi_X\), so that

\[\mu' = \int \mu'_C \, d\tilde{\mu}'(C),\]
on measurable subsets of \(K\) and where the integral is over the orbits \(C\) having non void intersection with \(K\) and with \(\mu'_C\) concentrated on \(C \cap K\). Similarly we have for the pairs of measures

\[
\left( (\mu)'_{K \eta^{-1}}, (\mu')_{K \eta^{-1}} \right) \text{ and } \left( (R_\eta \mu)'_{K \eta^{-1}}, (R_\eta \mu)'_{K \eta^{-1}} \right) : (20)
\]

\[
(\mu)'_{K \eta^{-1}} = \int \left( (\mu)'_{K \eta^{-1}} \right)_C \, d(\tilde{\mu}'_{K \eta^{-1}})(C)
\]
and

\[
(R_\eta \mu)'_{K \eta^{-1}} = \int \left( R_\eta \mu)'_{K \eta^{-1}} \right)_C \, d(\tilde{\mu}'_{K \eta^{-1}})(C),
\]
both \((\mu)'_{K \eta^{-1}}\) and \((R_\eta \mu)'_{K \eta^{-1}}\) defined on measurable subsets of \(K \cdot \eta^{-1} \cap K\) (instead of \(K\)): with the measure \((R_\eta \mu)'_{K \eta^{-1}}\) equal to the measure \(R_\eta \mu\) restricted to \(K \cdot \eta^{-1} \cap K\), and \((\mu)'_{K \eta^{-1}}\) equal to the measure \(\mu\) restricted to the same compact subset \(K \cdot \eta^{-1} \cap K\); and with the corresponding tilde measures both defined on measurable subsets of the measurable (Lemma \[13\]) set \(\pi_X(K \cdot \eta^{-1} \cap K)\); namely

\[
\tilde{(R_\eta \mu)'}(E) = (R_\eta \mu)'(\pi_X^{-1}(E)) = R_\eta \mu'(K \cap \pi_X^{-1}(E)) = (R_\eta \mu)'_{K \eta^{-1}}(K \cap \pi_X^{-1}(E)) = R_\eta \mu(K \eta^{-1} \cap K \cap \pi_X^{-1}(E)) = R_\eta \mu(K \eta^{-1} \cap K \cap \pi_X^{-1}(E))
\]

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Now applying again the Lemma 11.4 of [47] to the pairs of measures:

\[(\widehat{\mu'})_{K_{\eta^{-1}}} (E) = (\mu')_{K_{\eta^{-1}} \cap K} (E) = \mu(K_{\eta^{-1}} \cap K \cap \pi^{-1}_{\mathcal{X}}(E)).\]

Note please that our Lemma 16 holds true for any pseudo-image measure \(\nu\) of \(\mu\). By Lemma 14 any pseudo-image measure of the restriction \(\mu'\) is a restriction \((\nu')_{\pi x(K)}\) of a pseudo-image measure of \(\mu\). It follows that the Lemma 16 is applicable to the pairs of measures \((20)\). Indeed it is sufficient to insert \(K \cdot \eta^{-1} \cap K\) instead of \(K\) in the Lemma 14 and apply it to \((\mu')_{K_{\eta^{-1}}} = (\mu')_{K_{\eta^{-1}} \cap K}\) (or to \((R_{\eta}\mu') = (R_{\eta}\mu')_{K_{\eta^{-1}} \cap K}\) instead of \(\mu'\), because for an appropriate \(\nu, (\nu')_{\pi x(K_{\eta^{-1}} \cap K)}\) gives the pseudo-image measure \((\mu')_{K_{\eta^{-1}}}(\pi x(K)\cap K)\) (or respectively \((R_{\eta}\mu')\) of \((\mu')_{K_{\eta^{-1}}}(\pi x(K)\cap K)\) (or respectively of \((R_{\eta}\mu')\)). We may thus apply Lemma 11.4 of [47], §11, to the pairs of measures \((20)\). Because \(\mu\) is quasi invariant, the measures \((\mu')_{K_{\eta^{-1}}} = (\mu')_{K_{\eta^{-1}} \cap K}\) and \((R_{\eta}\mu') = (R_{\eta}\mu')_{K_{\eta^{-1}} \cap K}\) are equivalent as measures on \(K \cdot \eta^{-1} \cap K\), and thus by Lemma 11.4 of [47] it follows that \((\mu')_{K_{\eta^{-1}}} = (\mu')_{K_{\eta^{-1}} \cap K}\) and \((R_{\eta}\mu') = (R_{\eta}\mu')_{K_{\eta^{-1}} \cap K}\) are equivalent as measures on \(\pi x(K \cdot \eta^{-1} \cap K)\). Introducing the corresponding measurable weight function \(f_1\) on \(\mathcal{X}/G_2\) which is non zero on \(\pi x(K \cdot \eta^{-1} \cap K)\), we have

\[f_1 \cdot d(\widehat{\mu'})_{K_{\eta^{-1}}} = d(\widehat{R_{\eta}\mu'})\]

and

\[(R_{\eta}\mu')' = \int f_1(C) \left( R_{\eta}\mu' \right)' \, d(\widehat{\mu'})_{K_{\eta^{-1}}} (C), \quad (21)\]

\[d(\mu')_{K_{\eta^{-1}}} = \int \left( (\mu')_{K_{\eta^{-1}}} \right)' \, d(\widehat{\mu'})_{K_{\eta^{-1}}} (C). \quad (22)\]

Now applying again the Lemma 11.4 of [47] to the pairs of measures:

\[\left( (\mu')_{K_{\eta^{-1}}} \cap (\mu')_{K_{\eta^{-1}}} \right) \text{ and } \left( (R_{\eta}\mu')' \cap (\mu')_{K_{\eta^{-1}}} \right)\]

with the respective decompositions \((22)\) and \((21)\) we prove that the measures \(\left( (\mu')_{K_{\eta^{-1}}} \right)' \) and \(\left( R_{\eta}\mu' \right)' \) are equivalent and

\[f_1(C) \cdot \frac{d \left( R_{\eta}\mu' \right)'}{d \left( (\mu')_{K_{\eta^{-1}}} \right)'} (\cdot) = \frac{d \left( R_{\eta}\mu' \right)'}{d \left( \mu' \right)_{K_{\eta^{-1}}}'} (\cdot) = \frac{d \left( R_{\eta}\mu \right)}{d \mu} (\cdot) = \lambda(\cdot, \eta),\]

on \(C \cap K \cdot \eta^{-1} \cap K\), where the last two equalities follow from definitions and where

\[f_1 = \frac{d \left( R_{\eta}\mu' \right)'}{d \left( (\mu')_{K_{\eta^{-1}}} \right)'} .\]
On the other hand it follows from Lemma 13 and Remark 3 that
\[ (R_\eta \mu')' = \int (R_\eta (\mu_C')')' \, d (\nu)_{\pi_X(K\eta^{-1} \cap K)}(C) \]
and
\[ (\mu')'_{K\eta^{-1}} = \int ((\mu_C')'_{K\eta^{-1}}) \, d (\nu)_{\pi_X(K\eta^{-1} \cap K)}(C). \]
Thus both \((R_\eta \mu')'\) and \((\nu)'_{\pi_X(K\eta^{-1} \cap K)}\) being pseudo-image measures of the measure \((R_\eta \mu')'\) under \(\pi_X\) (of course restricted to \(K\eta^{-1} \cap K\)) are equivalent. Introducing the respective measurable, non zero on \(\pi_X(K\eta^{-1} \cap K)\), weight function \(f_2\) we have
\[ f_2 \cdot d (\nu)'_{\pi_X(K\eta^{-1} \cap K)} = d (R_\eta \mu'), \]
so that
\[ d (R_\eta (\mu_C')')' = f_2(C) \cdot d \left( R_\eta (\mu_C')' \right)^{C}. \]
Similarly because \(\mu\) is quasi invariant, the measures \((\mu')'_{K\eta^{-1}}\) and \((R_\eta \mu')'\) are equivalent, and thus again by Lemma 11.4 of [47] the measures \((\mu')'_{K\eta^{-1}}\) and \((\nu)'_{\pi_X(K\eta^{-1} \cap K)}\) are likewise equivalent. Introducing the respective non zero on \(\pi_X(K\eta^{-1} \cap K)\) and measurable weight function \(f_3\) we have
\[ f_3 \cdot d (\nu)'_{\pi_X(K\eta^{-1} \cap K)} = d (\mu')_{K\eta^{-1}}, \]
so that
\[ d ((\mu_C')'_{K\eta^{-1}}) = f_3(C) \cdot d \left( (\mu')_{K\eta^{-1}} \right)^{C}. \]
Joining the above equalities we obtain (the last two equalities follows from definition of \(\lambda_C\) and from definition of Radon-Nikodym derivative, i.e. its local character)
\[ \lambda(\cdot, \eta) = f_1(C) \cdot \frac{d (R_\eta (\mu_C')')}{d (\mu')_{K\eta^{-1}}} = f_1(C) \cdot \frac{1}{f_2(C)} \cdot f_3(C) \cdot \frac{d (R_\eta (\mu_C')')}{d ((\mu_C')'_{K\eta^{-1}})} = \frac{d (R_\eta (\mu_C')')}{d (\mu_C')_{K\eta^{-1}}}(\cdot) = \lambda_C(\cdot, \eta) \]
on \(C \cap K \cap K \cdot \eta^{-1}\), because by the known property of Radon-Nikodym derivatives (compare e.g. Scholium 4.5 of [69])
\[ f_1 \cdot \frac{1}{f_2} \cdot f_3 = \frac{d (R_\eta (\mu_C')')}{d (\mu')_{K\eta^{-1}}} \cdot \frac{d (\nu)'_{\pi_X(K\eta^{-1} \cap K)}}{d (R_\eta \mu')'} \cdot \frac{d (\mu')_{K\eta^{-1}}}{d (\nu)'_{\pi_X(K\eta^{-1} \cap K)}} = 1, \]
on all orbits \(C\) with non void intersection \(C \cap K \cap K \cdot \eta^{-1}\).  

We are are now in a position to formulate the main goal of this Section.
LEMMA 18. Let \( \mu \) be any quasi invariant measure on \( X \) and let \( \nu \) be any pseudo image measure of \( \mu \). Then the measures \( \mu_C \) in the decomposition
\[
\mu = \int \mu_C(x) \, d\nu(C)
\]
of Lemma 13 are also quasi invariant and for each \( \eta \in G_2 \) the Radon-Nikodym derivative \( \lambda_C(\cdot, \eta) = \frac{d(R_\eta \mu_C)}{d\mu_C}(\cdot) \) is equal to the restriction of the Radon-Nikodym derivative \( \lambda(\cdot, \eta) = \frac{d(R_\eta \mu)}{d\mu}(\cdot) \) to the orbit \( C \).

Indeed, let \( x \) be any point in \( X \) and \( \eta \) any element of \( G_2 \). We show that on a neighbourhood of \( x \) the statement of the Theorem holds true. To this end let \( O_m \) be a neighbourhood of \( x \) chosen from the basis of topology constructed above. Then \( O_m \cdot \eta \) is a neighbourhood of \( x \cdot \eta \). Therefore the compact set \( K = \overline{O_m} \cup (\overline{O_m} \cdot \eta) \) has the property that \( K \cap (K \cdot \eta^{-1}) \) contains an open neighbourhood of \( x \). Now it is sufficient to apply Lemma 14 with this \( K \) in order to show that the equality of the Theorem holds true on some open neighbourhood of \( x \).

REMARK 5. It has been proved in Sect. 7 that for each orbit \( C \) there exists a measure \( \mu_C \), concentrated on \( C \), with the associated Radon-Nikodym derivative equal to the restriction to the orbit \( C \) of the Radon-Nikodym derivative associated with \( \mu \). This however would be insufficient because we need to know that the measures \( \mu_C \) conspire together so as to compose a decomposition of the measure \( \mu \). This is why we need Lemma 18. Although the Lemma was not explicitly formulated in [47], it easily follows for the case of finite \( \mu \) from the Lemmas of [47], §11.

Using Lemma 18 and the general properties of the integral and the algebra of measurable functions one can prove a slightly strengthened version of Lemma 14, which may be called a skew version of the Fubini theorem, because it extends the Fubini theorem to the case where we have a skew product measure \( \mu \) with only one projection, i.e. the quotient map \( \pi_X \):

LEMMA 19 (Skew Version of the Fubini Theorem). Let \( \mu \), \( \mu_C \) and \( \nu \) be such as in Lemma 13. Let \( g \) be a positive complex valued and measurable function on \( X \). Then
\[
C \mapsto \int g(x) \, d\mu_C(x)
\]
is measurable, and if any one of the following two integrals:
\[
\int d\nu(C) \int g(x) \, d\mu_C(x) \quad \text{and} \quad \int g(x) \, d\mu(x),
\]
does exist, then there exists the other and both are equal in this case.

In particular it follows that if \( g \) is integrable on \( (X, \mathcal{F}_X, \mu) \) then
\[
\int d\nu(C) \int g(x) \, d\mu_C(x) = \int g(x) \, d\mu(x).
\]
For the proof compare [7], Chap. VI, Remark of § 3.4. Here we give only few comments: The Lemma holds for positive and continuous \( g \) with compact support as a consequence of Lemma 13. Next we note that the class of functions which satisfy (23) and (24) is closed under sequential convergence of increasing sequences.

The Lemma follows by repeated application of the sequential continuity of the integral for increasing sequences; compare, please, the proof of Thm. 3.4 and Corollary 3.6.2 of [69].

Note that the integral

\[
\int g(x) d\mu_C(x)
\]

in (23) and (24) may be replaced with

\[
\int_C g^C(x) d\mu_C(x),
\]

where \( g^C \) is the restriction of \( g \) to the orbit \( C \), because \( \mu_C \) is concentrated on \( C \). However just like in the ordinary Fubini theorem the whole difficulty in application of the skew version of the Fubini Theorem lies in proving the measurability of \( g \) on the “skew product” \( \mathcal{X} \overset{\pi}{\longrightarrow} X/G_2 \) measure space \((\mathcal{X}, \mathcal{B}_X, \mu)\). Indeed even if the orbits \( C \) were nice closed subsets and \( g^C \) measurable on \( C \) (with respect to the measure structure induced from the surrounding space \( \mathcal{X} \)) the function \( g \) still could be non measurable on \((\mathcal{X}, \mathcal{B}_X, \mu)\); for simple examples we refer e. g. to [69] or to any other book on measure theory. More restrictive constrains are to be put on the separate \( g^C \) as functions on the orbits \( C \) in order to guarantee the measurability of \( g \) on the measure space \( \mathcal{X} \). We face the same problem with the ordinary Fubini theorem. If in addition \( g^C \in L^2(C, \mu_C) \) for each \( C \) (or \( \nu \)-almost all orbits \( C \)), the required additional requirement is just the von Neumann direct integral structure put on \( C \mapsto g^C \) which is the necessary and sufficient condition for the existence of a function \( f \in L^2(\mathcal{X}, \mu) \) such that \( f^C = g^C \) for \( \nu \)-almost all orbits \( C \). Namely, consider the space of functions \( C \mapsto g^C \in L^2(C, \mu_C) \), which composes

\[
\int_{X/G_2} L^2(C, \mu_C) \ d\nu(C),
\]

then for every element \( C \mapsto g^C \) of direct integral (25) there exists a function \( f \in L^2(\mathcal{X}, \mu) \) such that \( f^C = g^C \) for \( \nu \)-almost all orbits \( C \). In short

\[
\int_{X/G_2} L^2(C, \mu_C) \ d\nu(C) = L^2(\mathcal{X}, \mu).
\]

We skip proving the equality (26) because in the next Section we prove a more general version of (26) for vector valued functions \( g \in \mathcal{H}^L \) on \( \mathcal{X} = \mathcal{G}/G_1 \).
compare Lemma 22 (a). This strengthened version of the skew Fubini theorem lies behind harmonic analysis on classical Lie groups and provides also an effective tool for tensor product decompositions of induced representations in Krein spaces. In practice the classical groups with the harmonic analysis relatively complete on them, have the structure of cosets and double cosets (corresponding to the orbits $C$) much more nice in comparison to what we have actually assumed, so that a vector valued version of the strengthened version of the ordinary Fubini theorem:

$$ \int_X L^2(Y, \mu_Y) \, d\mu_X = L^2(X \times Y, \mu_X \times \mu_Y) $$

(27)

would be sufficient for our applications. Namely the “measure product property” holds also in our practical applications for the double coset space:

$$ (G, R_G, \mu_G) = (G_1 \times G_1 \times (G/G_1)/G_2, \mathcal{R}_{G_1 \times G_1 \times (G/G_1)/G_2}, \mu_{G_1} \times \mu_{G_1} \times \mu_{(G/G_1)/G_2}) $$

with the analogous functions and measure $\mu = \mu_{G_1}$ and the pseudo image measure $\nu = \mu_{(G/G_1)/G_2}$ effectively computable.

Note that (26) and (27) may be proved for more general measure spaces.

Our proof of (26) may be easily adopted to general non-separable case, provided that the assertion of Lemma 19 holds true for the measures $\mu$ and $\nu$. Here the measure spaces are not “too big”, so that the associated Hilbert spaces of square summable functions are separable.

At the end of this Section we transfer the measure structure on $X/G_2$ over to the the set $G_1 : G_2$ of all double cosets $G_1xG_2$, using the natural bi-unique correspondence $C \mapsto D_C = \pi^{-1}(C)$ between the orbits $C$ and double cosets $D$. Next we transfer it again to a measurable section $\mathcal{B}$ of $G$ cutting every double coset at exactly one point and give measurability criterion for a function on $\mathcal{B}$ with this measure structure inherited from $X/G_2$. We shall use it in Sections 9 and 10.

**Definition 1.** We put $d\nu_0(D) = d\nu(C_D)$ for the measure $\nu_0$ transferred over to measurable subsets of the set of all double cosets, where $C_D$ is the orbit corresponding to the double coset, i.e. $D = \pi^{-1}(C)$. Let $B_0$ be a measurable section of $G_2$, existence of which has been proved in Lemma 17. Let $B$ be a measurable (even Borel) section of $G$ with respect to $G_1$ (which exists by Lemma 1.1 of [47]). Next we define the set $\mathcal{B} = \pi^{-1}(B_0) \cap B$. We call $\mathcal{B}$ the section of $G$ with respect to double cosets.

$\mathcal{B}$ is measurable by Lemma 1.1 of [47] and by Lemmas 11, 12 of this Section. It has the property that every double coset intersects $\mathcal{B}$ at exactly one point. We may transfer the measure space structure $(X/G_2, \mathcal{R}_{X/G_2}, \nu)$ over to get $(\mathcal{B}, R_{\mathcal{B}}, \nu_\alpha)$.

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24Our proof of (26) may be easily adopted to general non-separable case, provided that the assertion of Lemma 19 holds true for the measures $\mu$ and $\nu$. 78
DEFINITION 2. For each double coset $D$ there exists exactly one element $x_D \in \mathcal{B} \cap D$. We define $d_{\nu_B}(x_D) = d_{\nu_0}(D)$. The same holds for orbits $C$: to each orbit $C$ there exists exactly one element $x_C \in \mathcal{B} \cap \pi^{-1}(C)$. We put respectively $d_{\nu_B}(x_C) = d_{\nu}(C)$. \text{Note that} $x_C = x_D$ \text{iff} $C$ and $D$ correspond.

LEMMA 20. A set $E$ of orbits $C$ is measurable iff the sum of the corresponding double cosets, regarded as subsets of $G$, is measurable in $G$. Thus in particular a function $g$ on $\mathcal{B}$ is measurable iff there exists a function $f$ measurable on $G$ and constant along each double coset, such that the restriction of $f$ to $\mathcal{B}$ is equal to $g$.

By Lemma 1.2 of [47] a set $F \subset X = G/G_1$ is measurable iff $A = \pi^{-1}(F)$ is measurable in $\mathcal{G}$ and by Lemma [12] a subset $E \subset X/G_2$ is measurable iff $F = \pi_X^{-1}(E)$ is measurable on $X$. Thus a set $E$ of orbits $C$ is measurable iff the sum of the corresponding double cosets, regarded as subsets of $G$, is measurable in $\mathcal{G}$, (as already claimed at the beginning of this Section). This proves the Lemma.

In particular if we define $s(x)$ to be the double coset containing $x$, then we transfer the measure $\nu$ over to the subsets of double cosets correctly if we define the set $E$ of double orbits to be measurable if and only if $s^{-1}(E)$ is measurable on $\mathcal{G}$.

Writing $x$ for the variable with values in $\mathcal{G}$, and writing $[x]$ for $\pi(x)$ varying over $X = G/G_1$ we have

LEMMA 21. Let $\mu, \mu_C$ and $\nu$ be as in Lemma [13]. Let $g$ be a positive complex valued and measurable function on $X$. Let $\mu_D = \mu_{x_D} = \mu_C D$ be the measure concentrated on the orbit $C_D$ corresponding to the double coset $D$. Then:

$$D \mapsto \int g([x]) \, d\mu_D([x]) \quad \text{and} \quad \mathcal{B} \ni x_D \mapsto \int g([x]) \, d\mu_{x_D}([x])$$

are measurable, and

1) if any one of the following two integrals:

$$\int d_{\nu_0}(D) \int g([x]) \, d\mu_D([x]) \quad \text{and} \quad \int g([x]) \, d\mu([x]),$$

does exist, then there exists the other and both are equal in this case.

In particular it follows that if $g$ is integrable on $(X, \mathcal{R}_X, \mu)$ then

$$\int d_{\nu_0}(D) \int g([x]) \, d\mu_D([x]) = \int g([x]) \, d\mu([x]).$$

2) Similarly if any one of the following two integrals:

$$\int d_{\nu_B}(x_D) \int g([x]) \, d\mu_{x_D} \quad \text{and} \quad \int g([x]) \, d\mu([x]),$$

does exist, then there exists the other and both are equal in this case.
In particular it follows that if \( g \) is integrable on \((\mathcal{X}, \mathcal{A}_\mathcal{X}, \mu)\) then

\[
\int d\nu_{x_0}(x_0) \int g([x]) \, d\mu_{x_0}([x]) = \int g([x]) \, d\mu([x]).
\]

Because by definition (with \( x \in \mathcal{X} = \mathcal{G}/G_1 \) and \( x \in \mathcal{G} \))

\[
\int_{C'} g(x) \, d\mu_C(x) = \int g([x]) \, d\mu_D([x]),
\]

the Lemma is an immediate consequence of definitions Def 1 and 2 and Lemma 19.

\[\blacksquare\]

9 Subgroup theorem in Krein spaces

Let \( G_1 \) and \( G_2 \) be regularly related closed subgroups of \( \mathcal{G} \) (for definition compare Sect. 8). Consider the restriction \( \sigma_2 U^L \) to the subgroup \( G_2 \subset \mathcal{G} \) of the representation \( \nu U^L \) of \( \mathcal{G} \) in the Krein space \( \nu \mathcal{H}^L \), induced from a representation \( L \) of the subgroup \( H = G_1 \), defined as in Sect 3. For each \( G_2 \)-orbit \( C \) in \( \mathcal{X} = \mathcal{G}/G_1 \) let us introduce the Krein-isometric representation \( U^L, C \), defined in Sect. 7, and acting in the Krein space \( \mathcal{H}^L_C \). Let \( \nu \) be any pseudo image measure of \( \mu \) on \( \mathcal{X}/G_2 \), for its definition compare [7], Chap. VI.3.2. For simplicity we drop the \( \mu \) superscript in \( \mu U^L \) and \( \mu \mathcal{H}^L \) and just write \( U^L \) and \( \mathcal{H}^L \).

Let us remind the definition of the direct integral of Hilbert spaces after [68], but compare also [52]:

**Definition 3** (Direct integral of Hilbert spaces). Let \( (\mathcal{X}/G_2, \mathcal{A}_{\mathcal{X}/G_2}, \nu) \) be a measure space \( M \), and suppose that for each point \( C \) of \( \mathcal{X}/G_2 \) there is a Hilbert space \( \mathcal{H}^L_C \). A Hilbert space \( \mathcal{H}^L \) is called a direct integral of the \( \mathcal{H}^L_C \) over \( M \), symbolically

\[
\mathcal{H}^L = \int \mathcal{H}^L_C \, d\nu(C),
\]

if for each \( g \in \mathcal{H}^L \) there is a function \( C \mapsto g^C \) on \( \mathcal{X}/G_2 \) to the disjoint union \( \bigcup_{C \in \mathcal{X}/G_2} \mathcal{H}^L_C \), such that \( g^C \in \mathcal{H}^L_C \) for all \( C \), and with the following properties 1) and 2):

1) If \( g \) and \( k \) are in \( \mathcal{H}^L \) and if \( u = \alpha g + \beta k \), and if \((\cdot, \cdot)_C \) is the inner product in \( \mathcal{H}^L_C \) then \( C \mapsto (g^C, k^C)_C \) is integrable on \( M \), and the inner product \( (g, k) \) on \( \mathcal{H}^L \) is equal to

\[
(g, k) = \int_{\mathcal{X}/G_2} (g^C, k^C)_C \, d\nu(C),
\]
and \( u^C = \alpha g^C + \beta k^C \) for almost all \( C \in \mathcal{X}/G_2 \), and all \( \alpha, \beta \in \mathbb{C} \).

2) If \( C \mapsto u^C \) is a function with \( u^C \in \mathcal{H}_L^L \) for all \( C \), if \( C \mapsto (g^C, u^C)_C \) is measurable for all \( g \in \mathcal{H}_L \), and if \( C \mapsto (u^C, u^C)_C \) is integrable on \( M \), then there exists an element \( u' \) of \( \mathcal{H}_L \) such that

\[
u^C = u^C \text{ almost everywhere on } M.
\]

The function \( C \mapsto g^C \) is called the decomposition of \( g \) and is symbolized by

\[
g = \int_{\mathcal{X}/G_2} g^C \, d\nu(C).
\]

A linear operator \( U \) on \( \mathcal{H}_L \) is said to be decomposable with respect to the direct integral Hilbert space decomposition (31) if there is a function \( C \mapsto U^C \) on \( \mathcal{X}/G_2 \) with \( U^C \) being a linear operator in \( \mathcal{H}_L^L \) for each \( C \), and

3) the property that for each \( g \) in its domain and all \( k \) in \( \mathcal{H}_L \), \( (Ug)^C = U^C g^C \) almost everywhere on \( M \) and the function \( C \mapsto (U^C g^C, k^C)_C \) is integrable on \( M \).

If \( U \) is densely defined the property 3) is equivalent to the following:

3') for all \( g, k \) in \( \mathcal{H}_L \) in the domain of \( U \), \( C \mapsto (U^C g^C, k^C)_C \) is integrable on \( M \) and

\[
\int_{\mathcal{X}/G_2} (U^C g^C, k^C)_C \, d\nu(C) = (Ug, k).
\]

The function \( C \mapsto U^C \) is then called the decomposition of \( U \) with respect to (31) and symbolized by

\[
U = \int_{\mathcal{X}/G_2} U^C \, d\nu(C).
\]

If \( C \mapsto U^C \) is almost everywhere a scalar operator, \( U \) is called diagonalizable with respect to (31). The totality of all bounded operators diagonalizable with respect to (31) composes the commutative von Neumann algebra \( \mathfrak{A}_{\mathcal{E}/G_2} \) associated with the decomposition (31), compare (52). A bounded operator \( U \) in \( \mathcal{H}_L \) is decomposable with respect to (31) if and only if it commutes with all elements of \( \mathfrak{A}_{\mathcal{E}/G_2} \Leftrightarrow U \in (\mathfrak{A}_{\mathcal{E}/G_2})' \). This condition may easily be extended on unbounded operators: e. g. closable \( U \) is decomposable with respect to (31) if the spectral projectors of both the factors in its polar decomposition commute with all elements of \( \mathfrak{A}_{\mathcal{E}/G_2} \); or still more generally: \( U \) is decomposable with respect to (31) if the spectral projectors of both the factors in its polar decomposition commute with all elements of \( \mathfrak{A}_{\mathcal{E}/G_2} \); or still more generally: \( U \) is decomposable with respect to (31) if and only if it commutes with every compact operator in the commutor \( (\mathfrak{A}_{\mathcal{E}/G_2})'' \) of \( \mathfrak{A}_{\mathcal{E}/G_2} \), i.e. iff it commutes with every unitary operator in the commutor \( (\mathfrak{A}_{\mathcal{E}/G_2})'' = \mathfrak{A}_{\mathcal{E}/G_2} \) of \( (\mathfrak{A}_{\mathcal{E}/G_2})'' \).
Note that the map $T$ which transforms $g$ into its decomposition $C \mapsto g^C$ may be viewed as a unitary operator decomposing $U$:

$$TUT^{-1} = \int_{X/G_2} U^C \, d\nu(C).$$

There are many possible realizations $T: f \mapsto T(f)$ of the Hilbert space $H^L$ as the direct integral \(\mathfrak{A}_{G_2}/G_2\). However the difference between any two $T: f \mapsto T(f) = \left( C \mapsto f^C \right)$ and $T': f \mapsto T'(f) = \left( C \mapsto (f^C)' \right)$ of them is irrelevant: there exists for them a map $C \mapsto U^C$ with each $U^C$ unitary in $H^L_C$ and such that:

1) $U^C f^C = (f^C)'$ for almost all $C$.

2) $C \mapsto (f^C, g^C)_C$ is measurable in realization $T \Leftrightarrow C \mapsto (U^C f^C, U^C f^C)_C$ is measurable in realization $T'$.

(Compare [52]).

For the reasons explained in the footnote to Lemma [6] it is sufficient to consider the $\sigma$-rings $\mathfrak{R}_{X/G_2}$ and $\mathfrak{R}_X$ of Borel sets, with the Borel structure on $X/G_2$ defined as in Sect. [8] in the investigation of the respective Hilbert and Krein spaces.

We shall need a

**LEMMA 22.** (a) $H^L \cong \int_{X/G_2} H^L_C \, d\nu(C)$.

(b) $e^L e^L \cong \int_{X/G_2} U^{L,C} \, d\nu(C)$.

(c) $J^L \cong \int_{X/G_2} J^{L,C} \, d\nu(C)$.

The equivalences $\cong$ are all under the same map (or realization) $T: H^L \mapsto \int_{X/G_2} H^L_C \, d\nu(C)$ giving the corresponding decomposition $T(f): C \mapsto f^C$ for each $f \in H^L$, in which $f^C$ is the restriction of $f$ to the double coset $D_C = G_1 x_c G_2 = \pi^{-1}(C)$ corresponding [25] to $C$.

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251 e. we chose $x_c \in B \subset \mathfrak{B}$ for which $\pi(x_c) \in C$, compare Def. [1] and [2]
In particular $T$ is unitary and Krein-unitary map between the Krein spaces $(H^L, \mathfrak{J}^L)$ and \( \int \mathcal{H}_{C}^{H_L} \, d\nu(C), \int \mathfrak{J}^{L,C} \, d\nu(C) \).

**REMARK 6.** The equivalences $\cong$ may be read in fact as ordinary equalities.

Let $(\cdot, \cdot)_C = \| \cdot \|_C^2 = \int_C (J_L(J_L g_C \cdot x, k_C x)) \, d\mu_C(x)$ be defined on $H^L_C$ as in Sect. \[.\] Recall that for any element $g$ of \( \int \mathcal{H}_{C}^{H_L} \, d\nu(C) \), i.e. a function $C \mapsto g_C$ from the set of $G_2$-orbits $X/G_2$ to the disjoint union $\bigcup_{C \in X/G_2} H^L_C$ such that $g_C \in H^L_C$ for all $C$, the function $C \mapsto \|g_C\|_C^2 = (g_C, g_C)_C$ is $\nu$-summable and $\nu$-measurable and defines inner product for any $g, k \in \int \mathcal{H}_{C}^{H_L} \, d\nu(C)$ by the formula

\[
(g, k) = \int_{X/G_2} d\nu(C) \int_C (J_L(J_L g_C \cdot x, k_C x)) \, d\mu_C(x) = \int_{X/G_2} (g_C, k_C)_C \, d\nu(C).
\]

We shall exhibit a natural unitary map $T$ from $H^L$ onto $\int \mathcal{H}_{C}^{H_L} \, d\nu(C)$ or, what is equivalent, we shall show that the decomposition $T(f) = (C \mapsto f_C)$ corresponding to each $f \in H^L$, with $f_C$ equal to the restriction of $f$ to the double coset $D_C = G_1 x G_2 = \pi^{-1}(C)$ corresponding to $C$, has all the properties required in Definition \[.\]

Let $f$ and $k$ be any functions in $H^L$. Then by Lemma \[.\] we have

\[
\int_{X/G_2} d\nu(C) \int_C (J_L(J_L f)_x, k_x) \, d\mu_C(x) = \int_{X} (J_L(J_L f)_x, k_x) \, d\mu(x) = \|f\|^2 < \infty,
\]

with the set of all $G_2$ orbits $C$ for which $x \mapsto (J_L(J_L f)_x, k_x)$ is not $\mu_C$-integrable being $\nu$-negligible and the function

\[
C \mapsto \int_C (J_L(J_L f)_x, k_x) \, d\mu_C(x)
\]

being $\nu$-summable and $\nu$-measurable. Moreover, because for each orbit $C$ the measure $\mu_C$ is concentrated on $C$ (Lemma \[.\]), the integral

\[
\int_C (J_L(J_L f)_x, k_x) \, d\mu_C(x)
\]

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is equal

\[ \int_C \mathfrak{A}(\mathfrak{A}(f))_x, (k^C)_x \, d\mu_C(x) = \int_C \mathfrak{A}(\mathfrak{A}(f^C)_x, (k^C)_x \, d\mu_C(x) \]

where \( f^C \) (and similarly for \( k^C \)) is the restriction of \( f \) to the double coset \( D_C = G_1 x G_2 = \pi^{-1}(C) \) corresponding to \( C \). i.e. with any \( x \) for which \( \pi(x) \in C \), say \( x = x_c \), with \( C \mapsto x_c \in \mathfrak{B} \) of Sect. \( \S \). Because \( f^C \in \mathcal{H}_L^L \) and likewise \( \mathfrak{A}^L \) are defined as the ordinary restrictions, \( \mathfrak{A}(f^C)_x = \mathfrak{A} f^C = \mathfrak{A}^L \) is the restriction of \( \mathfrak{A} f \) to the double coset \( D_C = G_1 x G_2 \) corresponding to \( C \).

We thus obtain

\[ \int_C \mathfrak{A}(\mathfrak{A}^L f)_x, (k^C)_x \, d\mu_C(x) = \int_C \mathfrak{A}(\mathfrak{A}^L f^C)_x, (k^C)_x \, d\mu_C(x). \]

Therefore it follows that the map \( T : f \mapsto (C \mapsto f^C) \), where \( f^C \) is the restriction of \( f \) to the double coset corresponding to the orbit \( C \), fulfils the requirements of Part 1) of Definition \( \S \), in particular \( \|T(f)\| = \|f\| \) and the range \( T(\mathcal{H}_L^L) \) is a Hilbert space with the inner product \( (32) \).

We shall verify Part 2) of the Definition \( \S \), i.e. that the decomposition map \( T(f) = (C \mapsto f^C) \) defined as above has the properties indicated in 2) of Definition \( \S \) on its whole range \( T(\mathcal{H}_L^L) \). Toward this end let \( C \mapsto u^C \) fulfill the conditions required in 2) of Def. \( \S \)

\[ C \mapsto \int_C \mathfrak{A}(\mathfrak{A}^L u^C)_x, (u^C)_x \, d\mu_C(x) = (u^C, k^C)_C \quad (33) \]

is measurable for each \( k \in \mathcal{H}_L^L \) and

\[ C \mapsto \int_C \mathfrak{A}(\mathfrak{A}^L u^C)_x, (u^C)_x \, d\mu_C(x) = (u^C, u^C)_C \quad (34) \]

is measurable and integrable. Consider the space \( \mathfrak{F} \) of all functions \( C \mapsto k^C \in \mathcal{H}_L^L \) fulfilling the following conditions:

\[ C \mapsto \int_C \mathfrak{A}(\mathfrak{A}^L g^C)_x, (g^C)_x \, d\mu_C(x) = (k^C, k^C)_C \]

is measurable and integrable. Let \( X \) be the maximal linear subspace of \( \mathfrak{F} \), where a subspace of \( \mathfrak{F} \) we have called linear, whenever it is closed under formation of finite linear combinations over \( C \). \( X \) is not empty as it contains the subspace \( T(\mathcal{H}_L) \) itself, which is a Hilbert space. Moreover if \( C \mapsto k^C, C \mapsto r^C \) are any

\[ ^{26}\text{We have chosen } x = x_c \text{ to belong to the measurable section } \mathfrak{B} \text{ of double cosets in } \mathfrak{G} \text{ constructed in Sect. } \S \text{ but this is unnecessary here.} \]
two functions belonging to \( X \) the formula

\[
\begin{align*}
    h \left( C \mapsto k^C, \ C \mapsto r^C \right) &= \int_{x/G_2} \left( k^C, r^C \right)_C \, d\nu(C) \\
    &= \int_{x/G_2} \left( \int_C \left( f_L \left( \mathfrak{H}_L \right) k^C \right)_x, \left( r^C \right)_x \, d\mu_C(x) \right) \, d\nu(C)
\end{align*}
\]

defines a hermitian form on \( X \). Thus by the Cauchy-Schwarz inequality we have:

\[
\left| \int_{x/G_2} (k^C, r^C)_C \, d\nu(C) \right|^2 \leq \left( \int_{x/G_2} (k^C, k^C)_C \, d\nu(C) \right) \cdot \left( \int_{x/G_2} (r^C, r^C)_C \, d\nu(C) \right). \tag{35}
\]

Now by the first part of the proof, \( T(\mathcal{H}_L) \) is a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle \) and in particular a linear subspace of \( \mathfrak{g} \). We may thus insert for \( C \mapsto k^C \) in (35) any decomposition \( C \mapsto f^C \) of \( f \in \mathcal{H}_L \), with \( f^C \) equal to the restriction of \( f \) to the double coset \( D_{\nu} = \pi^{-1}(C) \) corresponding to \( C \). Similarly we may insert the function \( C \mapsto u^C \) for the function \( C \mapsto r^C \) in (35). Indeed, because of the conditions (33) and (34), fulfilled by the function \( C \mapsto u^C \), the function

\[
C \mapsto (f^C + u^C, f^C + u^C)_C = (f^C, f^C)_C + (f^C, u^C)_C + (u^C, f^C)_C + (u^C, u^C)_C
\]

is measurable and by the Cauchy-Schwarz inequality integrable, for all \( f \in \mathcal{H}_L \). Therefore \( C \mapsto u^C \) and \( T(\mathcal{H}_L) \) are both contained in one linear subspace of \( \mathfrak{g} \), and thus by the maximality of \( X \) they are contained in \( X \), so that we can insert \( C \mapsto u^C \) for \( C \mapsto r^C \) in (35). Thus the indicated insertions in the inequality (35) lead us to the inequality

\[
\left| \int_{x/G_2} (f^C, u^C)_C \, d\nu(C) \right|^2 \leq \left( \int_{x/G_2} (f^C, f^C)_C \, d\nu(C) \right) \cdot \left( \int_{x/G_2} (u^C, u^C)_C \, d\nu(C) \right)
\]

for all \( C \mapsto f^C \) in \( T(\mathcal{H}_L) \). Therefore the linear functional

\[
T(f) \mapsto L(T(f)) = L\left( C \mapsto f^C \right) = h \left( C \mapsto f^C, \ C \mapsto u^C \right),
\]

on \( T(\mathcal{H}_L) \) is bounded by the last inequality. Because the range \( T(\mathcal{H}_L) \) of \( T \) is a Hilbert space it follows by the Riesz theorem ((e. g. Corollary 8.3.2. of [69]) applied to the linear functional \( L \) that there exists exactly one element \( T(f') \) in the range of \( T \) such that

\[
(f, f') = (T(f), T(f')) = \int_{x/G_2} (f^C, f'^C)_C \, d\nu(C) = h\left( C \mapsto f^C, \ C \mapsto u^C \right).
\]
for all \( f \in \mathcal{H}_L \). Therefore
\[
\int_{x/G_2} \left( \langle f^C, f'^C \rangle_C \right) \, d\nu(C) = \int_{x/G_2} \left( \langle f^C, u^C \rangle_C \right) \, d\nu(C)
\]
for all \( f \in \mathcal{H}_L \) and for a fixed \( f' \in \mathcal{H}_L \), or equivalently
\[
\int_{x/G_2} \left( \langle f^C, f'^C - u^C \rangle_C \right) \, d\nu(C) = 0,
\]
for all \( f \in \mathcal{H}_L \). Inserting the definition of \( \langle f^C, f'^C - u^C \rangle_C \) we get:
\[
\int_{x/G_2} \int_C \left( \mathfrak{L}(\mathcal{L}_\ast f^C)_x, (f'^C - u^C)_x \right) \, d\mu_C(x) \, d\nu(C)
\]
\[
= \int_{x/G_2} \int_C \left( \mathfrak{L}(\mathcal{L}_f)_x, (f'^C - u^C)_x \right) \, d\mu_C(x) \, d\nu(C) = 0,
\]
for all \( f \in \mathcal{H}_L \). By Lemma \( \text{[6]} \) there exists a sequence \( f^1, f^2, \ldots \) of elements \( C_0^L \subset \mathcal{H}_L \) such that for each fixed \( x \in \mathfrak{X} \) the vectors \( f^k_x \), \( k = 1, 2, \ldots \) form a dense linear subspace of \( \mathcal{H}_L \). By the proof of the same Lemma \( \text{[6]} \) there exists a sequence \( g_1, g_2, \ldots \) of continuous complex valued functions on \( \mathfrak{X} = \mathfrak{G}/G_1 \) with compact supports, dense in \( L^2(\mathfrak{X}, \mu) \) with respect to the \( L^2 \) norm \( \| \cdot \|_{L^2} \). For each \( g_j \) define the corresponding function \( g^j \) on \( \mathfrak{G} \) by the formula \( g^j_j(x) = g_j(\pi(x)) \), where \( \pi \) is the canonical quotient map \( \mathfrak{G} \mapsto \mathfrak{G}/G_1 = \mathfrak{X} \). Note, please, that
\[
\left( \mathfrak{L}(\mathcal{L}_\ast g^j)_x, (f'^C - u^C)_x \right) = (g^j)_x \cdot \left( \mathfrak{L}(\mathcal{L}_f)_x, (f'^C - u^C)_x \right)
\]
for all \( j \in \mathbb{N} \) and all \( f \in \mathcal{H}_L \). Inserting now \( g^j \cdot f^i \) for \( f \) in \( \text{[36]} \) we get
\[
\int_{x/G_2} \int_C g_j(C) \cdot \left( \mathfrak{L}(\mathcal{L}_\ast f^i)_x, (f'^C - u^C)_x \right) \, d\mu_C(x) \, d\nu(C) = 0,
\]
for all \( i, j \in \mathbb{N} \). Because \( \{g_j\}_{j \in \mathbb{N}} \) is dense in \( L^2(\mathfrak{X}, \mu) \) and the function
\[
C \mapsto \int_C \left( \mathfrak{L}(\mathcal{L}_\ast f^i)_x, (f'^C - u^C)_x \right) \, d\mu_C(x)
\]
by construction belongs to \( L^2(\mathfrak{X}, \mu) \), it follows that outside a \( \nu \)-negligible subset \( N \) of orbits \( C \)
\[
\int_C \left( \mathfrak{L}(\mathcal{L}_\ast f^i)_x, (f'^C - u^C)_x \right) \, d\mu_C(x) = 0,
\]
for all \( i \in \mathbb{N} \). Thus if \( C \notin N \), then
\[
\int_C \left( \mathfrak{L}(\mathcal{L}_\ast f^i)_x, (f'^C - u^C)_x \right) \, d\mu_C(x) = 0, \tag{37}
\]
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for all \(i \in \mathbb{N}\). Applying Lemma \(8\) to this orbit \(C\) and the associated \(H^L_C\) we get an isomorphism of it with a Krein space \(H^{L_c}\) of an induced representation (recall that \(x_c \in B \subset G\) with \(\pi(x_c) \in C\), compare Def. \(2\)). Then \(37\) together with Lemma \(10\) or \(4\) applied to \(H^{L_c}\) gives \(f^C - u^C = 0\). This shows that the decomposition \(T : f \mapsto (C \mapsto f^C)\) fulfils Part 2) of Definition \(3\).

We have thus proved Part (a) of the Lemma.

Then we have to prove that the operators \(T \circ_2 U^L T^{-1}\) and \(T \mathcal{J}^L T^{-1}\) are decomposable with respect to \((x_c, \eta) \mapsto \lambda(\rho^C, \eta)\) corresponding to the measure \(\mu\) and analogously writing \(\lambda_C(\eta)\) function \([x] \mapsto \lambda_C([x], \eta)\) corresponding to \(\mu_C\) we have:

\[
\left( T \circ_2 U^L T^{-1} \right)(C \mapsto f^C) = (T \circ_2 U^L \eta^C)(f) = (T \circ_2 U^L \eta^C \lambda(\rho^C, \eta)) R^C f^C,
\]

where \(\lambda(\eta)|_C\) denotes the restriction of \(\lambda(\eta)\) to the orbit \(C\). By Lemma \(18\) the restriction \(\lambda(\eta)|_C\) of \(\lambda(\eta)\) to the orbit \(C\) is equal to \(\lambda_C(\eta)\), so that

\[
\left( T \circ_2 U^L T^{-1} \right)(C \mapsto f^C) = (C \mapsto \sqrt{\lambda_C(\eta)} R^C f^C),
\]

which means that

\[
\circ_2 U^L \cong \int_{x/G_2} U^{L, C} \text{d} \nu(C),
\]

and proves (b). Similarly for the operator \(\mathcal{J}^L\):

\[
\left( T \mathcal{J}^L T^{-1} \right)(C \mapsto f^C) = (T \mathcal{J}^L \eta^C)(f) = (T \mathcal{J}^L \eta^C \lambda(\rho^C, \eta)) R^C f^C
\]

By definition of the operator \(\mathcal{J}^L\) we have \((\mathcal{J}^L f)^C = \mathcal{J}^L f^C = \mathcal{J}^{L, C} f^C\). Therefore

\[
\left( T \mathcal{J}^L T^{-1} \right)(C \mapsto f^C) = (C \mapsto \mathcal{J}^{L, C} f^C),
\]

which means that

\[
\mathcal{J}^L \cong \int_{x/G_2} \mathcal{J}^{L, C} \text{d} \nu(C),
\]

and proves (c).

Because for each \(C\), \(\mathcal{J}^{L, C}\) is unitary and self adjoint in \(H^{L'}_C\) and \((\mathcal{J}^{L, C})^2 = I\), then by \(52\), §14, the same holds true for the operator

\[
\int_{x/G_2} \mathcal{J}^{L, C} \text{d} \nu(C) \quad \text{in} \quad \int_{x/G_2} H^{L'}_C \text{d} \nu(C),
\]

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so that
\[ \left( \int_{\mathcal{H}^L \rightarrow C} \mathcal{H}^L \, d\nu(C), \int_{\mathcal{H}^L \rightarrow C} \mathcal{H}^L \, d\nu(C) \right), \]
is a Krein space.

Finally we have to show that \( T \) is Krein unitary. To this end observe that for each \( f, g \in \mathcal{H}^L \)
\[
\left( T(f), T(g) \right)_{\mathcal{H}^L \rightarrow C} = \int_{\mathcal{H}^L \rightarrow C} \left( \mathcal{H}^L \cdot f^C, g^C \right) \, d\nu(C)
\]
\[
= \int_{\mathcal{H}^L \rightarrow C} \left( \mathcal{H}^L \cdot (\mathcal{H}^L \cdot f^C)^C, (g^C)^C \right) \, d\mu_C(x) \, d\nu(C)
\]
\[
= \int_{\mathcal{H}^L \rightarrow C} \left( \mathcal{H}^L \cdot (f^C), (g^C) \right) \, d\mu_C(x) \, d\nu(C).
\]
Because \( f^C \) and \( g^C \) are the ordinary restrictions of \( f \) and \( g \) to \( G_1 \cdot x \cdot G_2 \) and
the measure \( \mu_C \) is concentrated on \( C \) (Lemma 13), the integrand in the last integral may be replaced with \( \left( \mathcal{H}^L \cdot (f^C), (g^C) \right) \). Because \( f, g \in \mathcal{H}^L \), the function \( x \mapsto \left( \mathcal{H}^L \cdot (f^C), (g^C) \right) \) is constant on the right \( G_1 \)-cosets and measurable and integrable on \( X = \mathfrak{S}/G_1 \) as a function of right \( G_1 \)-cosets. Thus by Lemma 19 the last integral is equal to
\[
\int \int_{\mathcal{H}^L \rightarrow C} \left( \mathcal{H}^L \cdot (f^C), (g^C) \right) \, d\mu_C(x) \, d\nu(C) = \int \left( \mathcal{H}^L \cdot (f^C), (g^C) \right) \, d\mu(x) = (f, g)_{\mathcal{H}^L},
\]
so that
\[
\left( T(f), T(g) \right)_{\mathcal{H}^L \rightarrow C} = (f, g)_{\mathcal{H}^L}.
\]

Actually we could merely use all \( g' \cdot f \), with \( g \in C_K(X) \) and \( f \in C^0_L \) instead of its denumerable subset \( g^j_i \cdot f^i \), \( i, j \in \mathbb{N} \) in the proof of Lemma 22. Its denumerability shows that \( \mathcal{H}^L \) is separable as the direct integral (a). This however is superfluous because separability of \( \mathcal{H}^L \) has been already shown within the proof of Lemma 6.

**Lemma 23.** Let \( \mathfrak{B} \) be the section of \( \mathfrak{S} \) with respect to double cosets of Def. 4 and let \( C \mapsto x_C \in \mathfrak{B} \), \( D \mapsto x_D \in \mathfrak{B} \) be the bi-unique maps of Def. 2. Let \( \nu_0 \) be the measure on the subsets of the set \( G_1 \cdot G_2 \) of all double cosets \( D \) equal to the transfer of the measure \( \nu \) on \( X/G_2 \) over to the set of double cosets by the natural bi-unique map \( C \mapsto D_C = \pi^{-1}(C) \). Let \( \nu_\mathfrak{B} \) be the measure on the section \( \mathfrak{B} \) equal to the transfer of \( \nu \) over to the section \( \mathfrak{B} \) by the map \( C \mapsto x_C \) (or equivalently equal to the transfer of \( \nu_0 \) by the map \( D \mapsto x_D \)). Let \( \mu_D = \mu_{C_D} \), where \( C_D \)}}
is the orbit corresponding to the double coset $D$, be the measure concentrated on $C_D$, where $\mu_C$ is the measure of Lemma 13. Let us denote the space of functions $\mathcal{H}_C^L$ of Sect. 14 defined on the double coset $D$ corresponding to $C$ just by $\mathcal{H}_D^L$ and similarly if $U^{L,C}$ and $J^{L,C}$ is the representation and the operator of Sect. 14, then we put $U^{L,D} = U^{L,C}_D$ and $J^{L,D} = J^{L,C}_D$; analogously we define $U^{L,x,D} = U^{L,C}_D$ and $J^{L,x,D} = J^{L,C}_D$. Then we have

\[(a) \quad \mathcal{H}^L \cong \int_{X/G_2} \mathcal{H}_C^L \, d\nu(C) = \int_{G_1:G_2} \mathcal{H}_D^L \, d\nu_0(D) = \int_{\mathfrak{B}} \mathcal{H}_{x,D}^L \, d\nu_\mathfrak{B}(x_D).\]

\[(b) \quad g_2 U^L \cong \int_{X/G_2} U^{L,C} \, d\nu(C) = \int_{G_1:G_2} U^{L,D} \, d\nu_0(D) = \int_{\mathfrak{B}} U^{L,x,D} \, d\nu_\mathfrak{B}(x_D).\]

\[(c) \quad J^L \cong \int_{X/G_2} J^{L,C} \, d\nu(C) = \int_{G_1:G_2} J^{L,D} \, d\nu_0(D) = \int_{\mathfrak{B}} J^{L,x,D} \, d\nu_\mathfrak{B}(x_D).\]

The equivalences $\cong$ are all under the same map $T : \mathcal{H}^L \mapsto \int_{G_1:G_2} \mathcal{H}_D^L \, d\nu(C)$ giving the corresponding decomposition $T(f) : D \mapsto f^{C_D}$ (or respectively $T(f) : x_D \mapsto f^{C_D}_D$) for each $f \in \mathcal{H}^L$, in which $f^{C_D}$ is the restriction of $f$ to the double coset $D = D_{C_D} = G_1 x_D G_2 = \pi^{-1}(C_D)$ corresponding to $C_D$. In particular $T$ is unitary and Krein-unitary map between the Krein spaces

\[(\mathcal{H}^L, J^L)\]

and

\[\left( \int_{G_1:G_2} \mathcal{H}_D^L \, d\nu(C), \int_{G_1:G_2} J^{L,D} \, d\nu_0(D) \right)\]

or respectively

\[\left( \int_{\mathfrak{B}} \mathcal{H}_{x,D}^L \, d\nu_\mathfrak{B}(x_D), \int_{\mathfrak{B}} J^{L,x,D} \, d\nu_\mathfrak{B}(x_D) \right).\]

\[\square\]

The Lemma follows from Lemma [22] by a mere renaming of the points of the measure space $X/G_2$ of $G_2$-orbits $C$ in $X$, with the preservation of the measure structure under the indicated renaming, which is guaranteed by Def. 1 and 2. \[\square\]
LEMMA 24. Let \((\mu^x \mathcal{H}^{L^c}, \mathfrak{J}_x)\) be the Krein space of the representation \(\mu^x \mathcal{U}^{L^c}\) of the subgroup \(G_2\) defined in Lemma \(8\) with the inner product \((\cdot, \cdot)_x\) in \(\mu^x \mathcal{H}^{L^c}\) defined by eq. \((16)\) in the proof of Lemma \(8\). For each \(x_D \in \mathfrak{B}\) we put \(\mu^x \mathcal{H}^{L^D} = \mu^x \mathcal{H}^{L^c}\), \(\mathfrak{J}_{x_D} = \mathfrak{J}_x\), \(G^x_D = G^x\) and \((\cdot, \cdot)_{x_D} = (\cdot, \cdot)_x\) with the orbit \(C\) corresponding to \(D\). For each fixed element \(f \in \mathcal{H}^L\) consider the following function \(\mathfrak{B} \ni x_D \mapsto \check{f}^D \in \mu^x \mathcal{H}^{L^D}\) where for each \(x_D\), \(\check{f}^D\) is defined as the function \(G_2 \ni t \mapsto (\check{f}^D)_{x_D,t} = (f^D)_{x_D,t}\) with \(f^D\) equal to the restriction of \(f\) to \(D\). The linear set \(\mathcal{H}\) of all such functions \(x_D \mapsto \check{f}^D\) with \(f\) ranging over the whole space \(\mathcal{H}^L\) and with the inner product

\[
(f, g) = \int_{\mathfrak{B}} (\check{f}^D, \check{g}^D)_{x_D} d\nu_{\mathfrak{B}}(x_D),
\]

is equal to

\[
\int_{\mathfrak{B}} \mu^x \mathcal{H}^{L^D} d\nu_{\mathfrak{B}}(x_D).
\]

Note, please, that by definition of the measures \(\mu^x \mathcal{H}\) and the operators \(\mathfrak{J}_x\)

\[
(f^D, g^D)_{x_D} = \int_{G_2/G^x_D} (\mathfrak{J}_x f^D)_{x_D} (g^D)_{x_D} d\mu^x (\mathfrak{J}_x)\]

\[
= \int_{G_2/G^x_D} (\mathfrak{J}_x h(x_D,t) \mathfrak{J}_x (h(x_D,t)^{-1} (f^D)_{x_D,t} (g^D)_{x_D,t}) d\mu^x (\mathfrak{J}_x)
\]

\[
= \int_{D} (\mathfrak{J}_x (3^L f^D)_{x}, (g^D)_{x}) d\mu_D([x]) = \int_{D} (\mathfrak{J}_x (3^L f)_{x}, g) d\mu_D([x])
\]

and because

\[
\mathfrak{G}/G_1 \ni [x] \mapsto (\mathfrak{J}_x (3^L f)_{[x]}, g) = (\mathfrak{J}_x (3^L f)_{[x]}, g)
\]

is measurable it follows from \((23)\) of Lemma \(21\) that the function

\[
x_D \mapsto (\check{f}^D, \check{g}^D)_{x_D}
\]

is measurable for all \(f, g \in \mathcal{H}^L\). Similarly by \((24)\) of part 2) of Lemma \(21\)
\[(\tilde{f}, \tilde{g}) = \int_{\mathcal{B}} \left( \tilde{f} \tilde{x}^D, \tilde{g} \tilde{x}^D \right)_{x^D} \, d\nu_{\mathbb{B}}(x^D)\]

\[= \int_{\mathcal{B}} \int_{G_2/G_{x^D}} \left( \tilde{J}_L \left( \tilde{3}_{x^D} \tilde{f} \right), \left( \tilde{g} \tilde{x}^D \right) \right)_{x^D} \, d\mu^{\tilde{x}^D}([t]) \, d\nu_{\mathbb{B}}(x^D)\]

\[= \int_{\mathcal{B}} \int_{D} \tilde{J}_L \left( \tilde{L} \tilde{f} \right) \tilde{g} \, d\mu_D([x]) \, d\nu_{\mathbb{B}}(x^D) = \int_{G_1} \tilde{J}_L \left( \tilde{L} \tilde{f} \right) \tilde{g} \, d\mu([x])\]

\[= (f, g).\]

Therefore \(\mathcal{H}\) is a Hilbert space with the inner product (38) as the isometric image of the Hilbert space \(\mathcal{H}^L\). We need only show Part 2) of Def. 3 to be fulfilled. Toward this end let \(x^D \mapsto u \in \mu^{\tilde{x}^D} \mathcal{H}^{L^D}\) be a function fulfilling the conditions of Part 2) of Def. 3 (of course with the obvious replacements of \(C\) with \(D\) and \(\mathcal{H}^{L^C}\) with \(\mu^{\tilde{x}^D} \mathcal{H}^{L^D}\)). We have to show existence of a function \(f' \in \mathcal{H}^L\) such that the function \(x^D \mapsto \tilde{f}'\) is equal almost everywhere to the function \(x^D \mapsto u\). We proceed exactly as in the proof of Part (a) of Lemma 22 by formation of the analogous maximal linear subspace \(X\) in the space \(\mathfrak{A}\) of all functions \(x^D \mapsto k\) for which

\[x^D \mapsto (k, x^D)\]

is measurable and integrable and then using Riesz theorem and Lemma 3 or 4 in proving the existence of \(f'\) (in this case the proof is even simpler because the Lemma 3 is not necessary in proving \(\tilde{f}' - u = 0\) from the analogue of (38)); indeed it is sufficient to apply Lemma 10 and Lemma 2 or 4).

\(\mathfrak{A}\) From now on we identify the Hilbert space \(\mathcal{H}^L\) with the direct integral:

\[\mathcal{H}^L = \int_{\mathcal{B}} \mathcal{H}^{L^D}_{x^D} \, d\nu_{\mathbb{B}}(x^D)\]

with the realization \(T \mapsto T(f)\) of the direct integral equal to \(T(f) : x^D \mapsto f^D\), where \(f^D\) is the ordinary restriction of \(f \in \mathcal{H}^L\) to the double coset \(D\). Similarly by

\[\int_{\mathcal{B}} \mu^{\tilde{x}^D} \mathcal{H}^{L^D} \, d\nu_{\mathbb{B}}(x^D),\]

we understand the direct integral with the realization of Lemma 24

**Lemma 25.** For each orbit \(C\) let \(V_{x_c}\) be the Krein-unitary map defined in Lemma 8. For each \(x^D \in \mathcal{B}\) (equivalently: each double coset \(D\)) let us put
\( V_{x_D} = V_{x_C} \) with \( C \) corresponding to \( D \). Then \( x_D \mapsto V_{x_D} \) is a decomposition of a well defined operator

\[
\mathcal{H}^L = \int_{\mathbb{B}} \mathcal{H}_{x_D}^L \, d\nu_g(x_D) \xrightarrow{\mathbb{B}} \int_{\mathbb{B}} \mu^{x_D} \mathcal{H}^{L,x_D} \, d\nu_g(x_D) : \]

\( (x_D \mapsto f^{x_D}) \mapsto (x_D \mapsto V_{x_D} f^{x_D}) \).

In short

\[
V = \int_{\mathbb{B}} V_{x_D} \, d\nu_g(x_D).
\]

The operator \( V \) is unitary and Krein-unitary between the Krein spaces

\[
\left( \int_{\mathbb{B}} \mu^{x_D} \mathcal{H}_{x_D}^{L,x_D} \, d\nu_g(x_D), \int_{\mathbb{B}} \mathcal{J}_{x_D} \, d\nu_g(x_D) \right)
\]

and

\[
\left( \int_{\mathbb{B}} \mathcal{H}_{x_D}^{L} \, d\nu_g(x_D), \int_{\mathbb{B}} \mathcal{J}_{x_D}^{L,x_D} \, d\nu_g(x_D) \right) = (\mathcal{H}^L, \mathcal{J}^L);
\]

and moreover:

\[
V \left( g^L \right) V^{-1} = V \left( \int_{\mathbb{B}} U^{L,x_D} \, d\nu_g(x_D) \right) V^{-1} = \int_{\mathbb{B}} \mu^{x_D} U^{L,x_D} \, d\nu_g(x_D)
\]

and

\[
V \left( \mathcal{J}^L \right) V^{-1} = V \left( \int_{\mathbb{B}} \mathcal{J}_{x_D}^{L,x_D} \, d\nu_g(x_D) \right) V^{-1} = \int_{\mathbb{B}} \mathcal{J}_{x_D} \, d\nu_g(x_D).
\]

Let \( f \) be any element of \( \mathcal{H}^L \) and \( t \in G_2 \). By definition we have

\[
(V_{x_D} f^{x_D})_t = (f^D)_{x_D,t} = (\tilde{f}^{x_D})_t,
\]

with \( \tilde{f}^{x_D} \) defined in Lemma 24. Thus by the realization of

\[
\int_{\mathbb{B}} \mu^{x_D} \mathcal{H}_{x_D}^{L,x_D} \, d\nu_g(x_D)
\]

given in Lemma 24 \( V \) is onto. Moreover, by the proof of Lemma 24

\[
x_D \mapsto (V_{x_D} f^{x_D}, \tilde{g}^{x_D})_{x_D} = (f^D, \tilde{g}^D)_{x_D}
\]

is measurable for all \( g \in \mathcal{H}^L \), and thus for all

\[
(x_D \mapsto \tilde{g}^{x_D}) \in \int_{\mathbb{B}} \mu^{x_D} \mathcal{H}_{x_D}^{L,x_D} \, d\nu_g(x_D);\]
therefore $V$ is a well defined operator. Moreover, by the proof of Lemma 24

$$(Vf, Vg) = \int_{\mathcal{B}} (V_{x_{D}} f_{x_{D}}^{x_{D}}, V_{x_{D}} g_{x_{D}}^{x_{D}})_{x_{D}} \, d\nu_{\mathfrak{m}}(x_{D})$$

$$= \int_{\mathcal{B}} (f_{x_{D}}^{x_{D}}, g_{x_{D}}^{x_{D}})_{x_{D}} \, d\nu_{\mathfrak{m}}(x_{D}) = (f, g),$$

so that $V$ is unitary (it likewise follows from Lemma 9).

Again by Lemma 21 we have:

$$(Vf, Vg)_{\mathcal{J}x_{D} d\nu_{\mathfrak{m}}(x_{D})} = \int_{\mathcal{B}} (\mathcal{J}x_{D} f_{x_{D}}^{x_{D}}, \mathcal{J}x_{D} g_{x_{D}}^{x_{D}})_{x_{D}} \, d\nu_{\mathfrak{m}}(x_{D})$$

$$= \int_{\mathcal{B}} \int_{\mathcal{D}} (\mathcal{J}x_{D} f_{x_{D}}^{x_{D}}, \mathcal{J}x_{D} g_{x_{D}}^{x_{D}}) \, d\mu_{D}(x) \, d\nu_{\mathfrak{m}}(x_{D})$$

$$= \int_{\mathcal{D}} \mathcal{J}x_{D} f_{x_{D}}^{x_{D}} \, d\mu_{D}(x) \, d\nu_{\mathfrak{m}}(x_{D}) = (f, g)_{\mathcal{J}x_{D}}$$

which shows that $V$ is Krein unitary.

Because by Lemma 8 $V_{x_{D}} U_{x_{D}}^{L,D} V_{x_{D}}^{-1} = \mu^{x_{D}} U_{x_{D}}^{L,D}$ and $V_{x_{D}} \mathcal{J}x_{D} V_{x_{D}}^{-1} = \mathcal{J}x_{D}$, the rest of the Lemma is thereby proved. 

REMARK 7. By a mere renaming of points associated to the isomorphisms $\mathfrak{B} \cong G_{1} : G_{2} \cong \mathfrak{X}/G_{2}$ of measure spaces, e.g introducing $V_{D} = V_{x_{D}}$, $\mu^{D} = \mu^{x_{D}}$ and the measure $\nu_{0}$ as in Def. 2 we may rephrase Lemma 25 as follows. $D \mapsto V_{D}$ is a decomposition of a well defined operator

$$V = \int_{G_{1} : G_{2}} V_{D} \, d\nu_{0}(D) :$$

$$\mathcal{H}^{L} = \int_{G_{1} : G_{2}} \mathcal{H}_{D}^{L} \, d\nu_{0}(D) \mapsto \int_{G_{1} : G_{2}} \mu^{D} \mathcal{H}_{D}^{L} \, d\nu_{0}(D) :$$

$$\left( D \mapsto f^{D} \right) \mapsto \left( D \mapsto V_{D} f^{D} \right).$$

The operator $V$ is unitary and Krein-unitary between the Krein spaces

$$\left( \int_{G_{1} : G_{2}} \mu^{D} \mathcal{H}_{D}^{L} \, d\nu_{0}(D), \int_{G_{1} : G_{2}} \mathcal{J}_{D} \, d\nu_{0}(D) \right)$$

and

$$\left( \int_{G_{1} : G_{2}} \mathcal{H}_{D}^{L} \, d\nu_{0}(D), \int_{G_{1} : G_{2}} \mathcal{J}^{L,D} \, d\nu_{0}(D) \right) = (\mathcal{H}^{L}, \mathcal{J}^{L});$$

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and moreover:

\[
V\left(g_2 U^L\right)V^{-1} = V\left(\int_{G_1:G_2} U^{L,D} \, dv_0(D)\right)V^{-1} = \int_{G_1:G_2} \mu^D U^{L,D} \, dv_0(D)
\]

and

\[
V\left(\mathfrak{H}^L\right)V^{-1} = V\left(\int_{G_1:G_2} \mathfrak{H}^{L,D} \, dv_0(x_D)\right)V^{-1} = \int_{G_1:G_2} \mathfrak{H}_D \, dv_0(D).
\]

**DEFINITION 4.** Let \(G_1\) and \(G_2\) be two closed subgroups of a separable locally compact group \(\mathfrak{G}\). Let \(B\) be any Borel section of \(\mathfrak{G}\) with respect to \(G_1\) and for each \(x \in \mathfrak{G}\) let \(h(x)\) be the unique element of \(G_1\) such that \(h(x)^{-1} \cdot x \in B\). Let \(\mu\) be any quasi-invariant measure \(\mu\) on \(\mathfrak{G}/G_1\) and let \(\nu\) be any pseudo-image measure on \((\mathfrak{G}/G_1)/G_2\) of the measure \(\mu\) under the quotient map \(\pi_{\mathfrak{G}/G_1} : \mathfrak{G}/G_1 \to (\mathfrak{G}/G_1)/G_2\); so that:

\[
\mu = \int_{(\mathfrak{G}/G_1)/G_2} \mu_C \, dv(C).
\]

Let us call any measure \(\nu_0\) on measurable subsets of the set \(G_1 : G_2\) of all double cosets admissible iff it is equal to the transfer of \(\nu\) over to \(G_1 : G_2\) by the natural map \((\mathfrak{G}/G_1)/G_2 \ni C \mapsto \pi^{-1}(C) \in G_1 : G_2\). Finally let \(x\) be any element of \(\mathfrak{G}\) with \(\pi(x) \in C\). We put \(\mu^x\) for the measure on \(G_2 : G_x\) equal to the transfer of the measure \(\mu_C\) over to \(G_2 : G_x\) by the map \(G_2 : G_x \ni [y] \mapsto [xy] \in C \subset \mathfrak{G}/G_1\), where \(G_x = G_2 \cap (x^{-1}G_1x)\) and where \([\cdot]\) denotes the respective equivalence classes.

Summing up we have just proved the following

**THEOREM 7 (Subgroup Theorem).** Let \(U^L\) be the isometric representation of the separable locally compact group \(\mathfrak{G}\) in the Krein space \((\mathcal{H}^L, \mathfrak{H}^L)\), induced by the Krein-unitary representation \(L\) of the closed subgroup \(G_1\) of \(\mathfrak{G}\) and the quasi invariant measure \(\mu\) on \(\mathfrak{G}/G_1\) and the Borel section \(B\) of \(\mathfrak{G}\) with respect to \(G_1\). Then \(U^L\) is independent to within Krein-unitary equivalence of the choice of \(B\). Let \(G_2\) be a second closed subgroup of \(\mathfrak{G}\) and suppose that \(G_1\) and \(G_2\) are regularly related. For each \(x \in \mathfrak{G}\) consider the closed subgroup \(G_x = G_2 \cap (x^{-1}G_1x)\) and let \(U^{L_x}\) denote the representation of \(G_2\) in the Krein space \((\mathcal{H}^{L_x}, \mathfrak{H}_x)\) induced by the Krein-unitary representation \(L^x : \eta \mapsto L_{x\eta x^{-1}}\) of the subgroup \(G_x\) in the Krein space \((\mathcal{H}_x, \mathfrak{H}_x)\), where \((\mathfrak{H}_x, g)\) = \(L_{h(x^{-1})} \mathfrak{H} L_{h(x^{-1})}^{-1}(g)\), and with the inner product in \(\mathcal{H}^{L_x}\) and Krein-inner product in \((\mathcal{H}^{L_x}, \mathfrak{H}_x)\) defined respectively by the formulas

\[
(f, g)_x = \int_{G_2/G_x} \left(\mathfrak{H}_x(f), (g)_x\right) \, d\mu^x([t])
\]

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and
\[(f,g)_{J_x} = \langle \mathfrak{J}_x f, g \rangle = \int_{G_2/G_x} \left( \mathfrak{J}_L(f), (g) \right) \text{d}\mu^x([t]);\]

and with the quasi invariant measure $\mu^x$ on $G_2/G_x$ given by Def. 4. Then $U^L$ is determined to within Krein-unitary and unitary equivalence by the double coset $G_1xG_2 = s(x)$ to which $x$ belongs and we may write $U^{L_D} = U^L$, where $D = s(x)$. Finally $U^L$ restricted to $G_2$ is a direct integral over $G_1$,\[G_2\]\[\text{with respect to any admissible (Def. 4) measure in } G_1: G_2, \text{ of the representations } U^{L_D}.\]

It may happen that all the component representations $\mu^D U^{L_D}$ are bounded and thus Krein-unitary, although $U^L$ is unbounded. In this case the norms $\|\mu^D U^{L_D}\|_{x_D}$ are unbounded functions of $x_D$ (resp. $D$). Unfortunately instead of $J_x$, we cannot use any standard fundamental symmetry in $\mu^D H^{L_D}$:

\[\left( J^{L_D} f \right)_x = L_{h x_D (t)} J L_{h x_D^{-1}} \left( f \right)_x,\]

where $h x_D(t) \in G_{x_D}$ is defined as in Sect. 3 by a regular Borel section $B_{x_D}$ of $G_2$ with respect to the subgroup $G_{x_D} = G_2 \cap (x_D^{-1} G_1 x_D)$. A difficulty will arise with this $J^{L_D}$. Namely in general the norms $\|\mu^D U^{L_D}\|_{x_D}$ are such that the operator $V$ would be unbounded with the standard fundamental symmetries in $\mu^D H^{L_D}$.

It is important that in practical computations, e.g. with $\mathfrak{G}$ equal to the double covering of the Poincaré group, much stronger regularity is preserved, e.g. the “measure product property” (see the end of Sect. 5), with the measurable sections $B$ and $\mathfrak{B}$ as differential sub-manifolds (if we discard unimportant null subset), so that the function $x \mapsto h(x)$ and all the remaining functions – analogue of 3 – associated to the measure product structure are effectively computable together with the measures $\mu$ and $\nu_0$. This is important because together with the theorem of the next Section give an effective tool for decomposing tensor product of induced representations of the double cover of the Poincaré group in Krein spaces. Moreover the operator $V$ of Lemma 25 and Remark 7 is likewise effectively computable in this case.

10 Kronecker product theorem in Krein spaces

Let $\mu^L$ and $\mu^M$ be Krein-isometric representations of the separable locally compact group $\mathfrak{G}$ induced from Krein-unitary representations of the closed subgroups $G_1 \subset \mathfrak{G}$ and $G_2 \subset \mathfrak{G}$ respectively. The Krein-isometric representation $\mu^L \otimes \mu^M$ of $\mathfrak{G}$ is obtained from the Krein-isometric representation $\mu^L \times \mu^M$ of $\mathfrak{G} \times \mathfrak{G}$ by restriction to the diagonal subgroup $\mathfrak{G}$ of all those $(x,y) \in \mathfrak{G} \times \mathfrak{G}$ for which $x = y$, which is naturally isomorphic to $\mathfrak{G}$ itself:
$\mathfrak{G} \cong \mathfrak{G}$, with the natural isomorphism $(x,x) \mapsto x$. Thus by the natural isomorphism the representation $\mu_1 U^L \otimes \mu_2 U^M$ of $\mathfrak{G}$ may be identified with the restriction of the representation $\mu_1 U^L \times \mu_2 U^M$ of the group $\mathfrak{G} \times \mathfrak{G}$ to the diagonal subgroup $\mathfrak{G}$. By Theorem $\mathfrak{G}$ $\mu_1 U^L \times \mu_2 U^M$ is Krein-unitary and unitary equivalent to the Krein-isometric representation $\mu_1 \times \mu_2 U^L \times U^M$ of $\mathfrak{G} \times \mathfrak{G}$ induced by the Krein-unitary representation $L \times M$ of the closed subgroup $G_1 \times G_2$.

Thus the Krein-isometric representation $U^L \otimes U^M$ of $\mathfrak{G}$ is naturally equivalent to the restriction of the Krein-isometric representation $\mu_1 \times \mu_2 U^L \times U^M$ of $\mathfrak{G} \times \mathfrak{G}$ to the closed diagonal subgroup $\mathfrak{G}$. Thus we are trying to apply the Subgroup Theorem $\mathfrak{G}$ inserting $\mathfrak{G} \times \mathfrak{G}$ for $\mathfrak{G}$, $\mathfrak{G}$ for $G_2$, and the subgroup $G_1 \times G_2 \subset \mathfrak{G} \times \mathfrak{G}$ for $G_1$ in the Subgroup Theorem. But the Subgroup Theorem is applicable in that way if the subgroups $G_1 \times G_2$ and $\mathfrak{G}$ are regularly related. Mackey recognized that they are indeed regularly related in $\mathfrak{G} \times \mathfrak{G}$ if and only if if $G_1$ and $G_2$ are in $\mathfrak{G}$, pointing out a natural measure isomorphism between the measure spaces $(G_1 \times G_2) : \mathfrak{G}$ and $G_1 : G_2$ of double cosets respectively in $\mathfrak{G} \times \mathfrak{G}$ and $\mathfrak{G}$. The isomorphism is induced by the map $\mathfrak{G} \times \mathfrak{G} \ni (x,y) \mapsto xy^{-1} \in \mathfrak{G}$. However his argumentation strongly depends on the finiteness of the quasi-invariant measures in the homogeneous spaces $(\mathfrak{G} \times \mathfrak{G})/\{G_1 \times G_2\}$ and $\mathfrak{G}/G_1$ which slightly simplifies the construction of the $\sigma$-rings of measurable subsets in the corresponding spaces of double cosets. Our proof that the map $(x,y) \mapsto xy^{-1}$ induces isomorphism of the respective spaces of double cosets must have been slightly changed at this point by addition of Lemma $\mathfrak{G}$. The rest of the proof of Theorem $\mathfrak{G}$ of this Section follows from the Subgroup Theorem $\mathfrak{G}$ in the same way as Theorem 7.2 from Theorem 7.1 in $\mathfrak{G}$.

By the above remarks we shall show that the measure spaces $(G_1 \times G_2) : \mathfrak{G}$ and $G_1 : G_2$ of double cosets constructed as in Sect. $\mathfrak{G}$ are isomorphic, with the isomorphism induced by the map $\mathfrak{G} \times \mathfrak{G} \ni (x,y) \mapsto xy^{-1} \in \mathfrak{G}$. Note first of all that the indicated map sets up a one-to-one correspondence between the double cosets in $(G_1 \times G_2) : \mathfrak{G}$ and double cosets in $G_1 : G_2$, in which the double coset $(G_1 \times G_2)(x,y)\mathfrak{G}$ corresponds to the double coset $G_1 xy^{-1} G_2$. Moreover in this mapping a set is measurable if and only if its image is measurable and vice versa, a set is measurable if and only if its inverse image is measurable. Thus it is an isomorphism of measure spaces. Indeed $(x_1,x_2)$ and $(x_2,y_2)$ go into the same point of $\mathfrak{G}$ under the indicated map if and only if they belong to the same left $\mathfrak{G}$ coset in $\mathfrak{G} \times \mathfrak{G}$. Now by Lemma $\mathfrak{G}$ of sect. $\mathfrak{G}$ and by Lemma 1.2 of $\mathfrak{G}$ (equally applicable to left coset spaces) the indicated one-to-one map of double coset spaces is an isomorphism of measure spaces. Thus the Subgroup Theorem $\mathfrak{G}$ is applicable to $\mu_1 \times \mu_2 U^L \times U^M$ with $L$ replaced by $L \times M$, $\mathfrak{G}$ replaced by $\mathfrak{G} \times \mathfrak{G}$, $G_1$ replaced by $G_1 \times G_2$ and $G_2$ replaced by $\mathfrak{G}$ and the function $\mathfrak{G} \ni x \mapsto h(x) \in G_1$ replaced by the function $\mathfrak{G} \times \mathfrak{G} \ni (x,y) \mapsto h(x,y) = (h_1(x), h_2(y)) \in G_1 \times G_2$, where the functions $\mathfrak{G} \ni x \mapsto h_1(x) \in G_1$ and $\mathfrak{G} \ni y \mapsto h_2(y) \in G_2$ correspond to the respective Borel sections of $\mathfrak{G}$ with respect to $G_1$ and $G_2$ respectively used in the construction of the representations $\mu_1 U^L$ and $\mu_2 U^M$ (compare Sect. $\mathfrak{G}$ and $\mathfrak{G}$).

In order to simplify formulation of the upcoming theorem let us give the following
DEFINITION 5. Let $L_0^{12}$ be the admissible measure on the set of double cosets $(G_1 \times G_2) : \mathfrak{G}$ in $\mathfrak{G} \times \mathfrak{G}$ given by Def. 4, where we have used the product quasi invariant measure $\mu = \mu_1 \times \mu_2$ on the homogeneous space $(\mathfrak{G} \times \mathfrak{G}) / (G_1 \times G_2)$.

Let us define the measure $\nu_{12}$ on the space $G_1 \times G_2$ of double cosets in $\mathfrak{G}$ to be equal to the transfer of $\nu_0^{12}$ by the map induced by $\mathfrak{G} \times \mathfrak{G} \ni (x, y) \mapsto xy^{-1} \in \mathfrak{G}$. If $(\mu_1 \times \mu_2)(x, y)$ is the measure $\nu_{12}$ equal to the transfer of $\nu_0^{12}$ by the map induced by $\mathfrak{G} \times \mathfrak{G}$ given by Def. 4 with $G(x, y) = G / \mathfrak{G}$ to $G_1 \times G_2$ then we define $\mu_{x, y}$ to be the transfer of the measure $(\mu_1 \times \mu_2)(x, y)$ over to the homogeneous space $\mathfrak{G} / (x^{-1} G_1 \times y^{-1} G_2)$ by the map $x \mapsto x$.

Now we are ready to formulate the main goal of this paper:

THEOREM 8 (Kronecker Product Theorem). Let $G_1$ and $G_2$ be regularly related closed subgroups of the separable locally compact group $\mathfrak{G}$. Let $L$ and $M$ be Krein-unitary representations of $G_1$ and $G_2$ respectively in the Krein spaces $(\mathcal{H}_L, \mathfrak{J}_L)$ and $(\mathcal{H}_M, \mathfrak{J}_M)$. For each $(x, y) \in \mathfrak{G} \times \mathfrak{G}$ consider the Krein-unitary representations $L^x : s \mapsto L_{x, x}^{-1}$ and $M^y : s \mapsto M_{y, y}^{-1}$ of the subgroup $(x^{-1} G_1 x) \cap (y^{-1} G_2 y)$ in the Krein spaces $(\mathcal{H}_L, \mathfrak{J}_L)$ and $(\mathcal{H}_M, \mathfrak{J}_M)$ respectively.

Let us denote the tensor product $L^x \otimes M^y$ Krein-unitary representation acting in the Krein space $(\mathcal{H}_L \otimes \mathfrak{J}_L, \mathfrak{J}_L \otimes \mathfrak{J}_M)$, by $N^{x, y}$. Let $U^{N^{x, y}}$ be the Krein-isometric representation of $\mathfrak{G}$ induced by $N^{x, y}$ acting in the Krein space $(\mathcal{H}^{N^{x, y}}, \mathfrak{J}_{x, y})$, where for each $w \in \mathcal{H}^{N^{x, y}}$

$$(\mathfrak{J}_{x, y} w)_x = (L_{h_1(x)} \mathfrak{J}_L L_{h_1(x)}^{-1} \otimes (M_{h_2(y)} \mathfrak{J}_M M_{h_2(y)}^{-1})) (w)_x,$$

and with the inner product in Hilbert space $\mathcal{H}^{N^{x, y}}$ and the Krein-inner product in the Krein space $(\mathcal{H}^{N^{x, y}}, \mathfrak{J}_{x, y})$ given by the formulas

$$(w, g)_{x, y} = \int_{\mathfrak{G} / (x^{-1} G_1 x \cap y^{-1} G_2 y)} (\mathfrak{J}_L \otimes \mathfrak{J}_M (\mathfrak{J}_{x, y} w)_x, (g)_x) \, d\mu^{x, y}(s),$$

and

$$(w, g)_{x, y} = (\mathfrak{J}_{x, y} w, g)_{x, y} = \int_{\mathfrak{G} / (x^{-1} G_1 x \cap y^{-1} G_2 y)} (\mathfrak{J}_L \otimes \mathfrak{J}_M (w)_x, (g)_x) \, d\mu^{x, y}(s),$$

with the quasi invariant measure $\mu^{x, y}$ given by Def. 4. Then $U^{N^{x, y}}$ is determined to within Krein-unitary equivalence by the double coset $D = G_1 x y^{-1} G_2$ to which $x y^{-1}$ belongs and we may write $U^{N^{x, y}} = U^D$. Finally $U_L \otimes U_M$ is Krein-unitary equivalent to the direct integral of $U^D$ with respect to the measure $\nu_{12}$ (Def. 4) on $G_1 : G_2$.

By the above remarks the Subgroup Theorem 7 is applicable to the restriction of the representation $\mu_1 \times \mu_2 U_L \times U_M$ of $\mathfrak{G} \times \mathfrak{G}$ to the subgroup $\mathfrak{G}$. By
this theorem, \( \mu_1 \times \mu_2 U^{L \times M} \) restricted to \( \mathfrak{G} \) is a direct integral over the space of double cosets\(^{27}\) \((G_1 \times G_2)(x, y)\mathfrak{G}\) with exactly one representant \((x, y)\) for each double coset, of the representations \(U^{L \times M(x, y)}\) of the subgroup \(\mathfrak{G}\). Each of the representations \(U^{L \times M(x, y)}\) of \(\mathfrak{G}\) is induced by the Krein-unitary representation \((L \times M)^{x, y} :(s, s) \mapsto (L \times M)^{x, y(x, y)} = L x y^{-1} \otimes M x y^{-1}\) of the subgroup \(G(x, y) = \mathfrak{G} \cap ((x, y)^{-1}(G_1 \times G_2)(x, y)) \subset \mathfrak{G}\) in the Krein space \((\mathcal{H}_L \otimes \mathcal{H}_M, \mathfrak{J}_L \otimes \mathfrak{J}_M)\). Moreover \(U^{L \times M(x, y)}\) acts in the Krein space \((\mathcal{H}^{L \times M(x, y)}, \mathfrak{J}^{(x, y)})\) where for each function \(w \in \mathcal{H}^{L \times M(x, y)}\) we have

\[
(\mathfrak{J}^{(x, y)} w(x, y) = (L \times M)h((x, y)-(s, s))\mathfrak{J}_L \mathfrak{J}_M (L \times M)h((x, y)-(s, s))^{-1}(w(x, y) = L \times M)h_1(x, y) \mathfrak{J}_L \mathfrak{J}_M (L \times M)(h_1(x, y)-1, h_2(y, x)-1)(w(x, y) = (L h_1(x, y) \mathfrak{J}_L h_2(y, x)-1) \otimes (M h_2(y, x) \mathfrak{J}_M M h_2(x, y)-1)(w(x, y).
\]

The inner product in \(\mathcal{H}^{L \times M(x, y)}\) and Krein-inner product in \((\mathcal{H}^{L \times M(x, y)}, \mathfrak{J}^{(x, y)})\) are defined by

\[
(w, g)_{(x, y)} = \int_{\mathfrak{G}/G(x, y)} \left( \mathfrak{J}_L \mathfrak{J}_M (\mathfrak{J}^{(x, y)} w(x, y), (g(x, y)) \right) d(\mu_1 \times \mu_2)(x, y) \left(\{(s, s)\}\right)
\]

and

\[
(w, g)_{(x, y)} = (\mathfrak{J}^{(x, y)} w(x, y), (g(x, y)) \right) d(\mu_1 \times \mu_2)(x, y) \left(\{(s, s)\}\right)
\]

with the quasi invariant measure \((\mu_1 \times \mu_2)(x, y)\) on \(\mathfrak{G}/G(x, y)\) given by Def. \(^{27}\)

Now under the natural isomorphism \((x, x) \mapsto x\) transferring \(\mathfrak{G}\) onto \(\mathfrak{G}\) the group \(G(x, y) = \mathfrak{G} \cap ((x, y)^{-1}(G_1 \times G_2)(x, y))\) is transferred onto the subgroup \(x^{-1} G_1 x \cap y^{-1} G_2 y\) of \(\mathfrak{G}\) and the homogeneous space \(\mathfrak{G}/G(x, y)\) with the quasi invariant measure \((\mu_1 \times \mu_2)(x, y)\) is transferred over to the homogeneous space \(\mathfrak{G}/(x^{-1} G_1 x \cap y^{-1} G_2 y)\) with the quasi invariant measure, which we denote by \(^{27}\) for 

\(^{27}\) I.e. with \((x, y)\) ranging over \(\mathfrak{B}_1 \times \mathfrak{B}_2\) — the corresponding section of \(\mathfrak{G} \times \mathfrak{G}\) with respect to double cosets \((G_1 \times G_2) : \mathfrak{G}\).
11 Krein-isometric representations induced by decomposable Krein-unitary representations

We say a family $\mathcal{S}$ of operators in a Hilbert space $\mathcal{H}$ is reducible by an idempotent $P$ (i.e., a bounded operator $P$ which satisfies the identity $P^2 = P$), or equivalently by a closed subspace equal to the range $PH$ of $P$, in case $PUP = UP$ for all $U \in \mathcal{S}$. We say the family $\mathcal{S}$ is decomposable in case $PU = UP$ for all $U \in \mathcal{S}$. In this case the Hilbert space $\mathcal{H}$ is the direct sum of closed subspaces $\mathcal{H}_1 = PH$ and $\mathcal{H}_2 = (I - P)\mathcal{H}$ and every operator in $\mathcal{S}$ is a direct sum of operators $U_1$ and $U_2$ with $U_i$ acting in $\mathcal{H}_i$, $i = 1, 2$. The closed subspaces $\mathcal{H}_i$, $i = 1, 2$, are orthogonal iff $P$ is self-adjoint. Moreover if $(\mathcal{H}, J)$ is a Krein space, the closed subspaces $\mathcal{H}_i$, $i = 1, 2$, are Krein-orthogonal iff the idempotent $P$ is Krein-self-adjoint: $P^\dagger = P$.

Now the Krein-isometric representations $U^L$ inherit decomposability from decomposability of $L$. Namely for each idempotent $P_L$ acting in the Krein space of the representation $L$ we may define a natural idempotent $P_L$ by the formula $(P_L f)_x = P_L f_x$ for $f \in H^L$ provided $P_L$ commutes with the representation $L$. Checking that $P^L$ is well defined (with measurable $x \mapsto \left( (P^L f)_x, v \right)$ for each $v \in H_L$ and $(P^L f)_x = L_h (P^L f)_x$) and that $P^L$ is a bounded idempotent is immediate. Moreover $P^L$ likewise commutes with $U^L$ and is self-adjoint whenever $P_L$ is.

Thus in particular for the standard Krein-isometric representation we have the following

**THEOREM 9.** Let $H$ be a closed subgroup of the separable locally compact group $\mathcal{G}$. Let $U^L$ be the Krein-isometric representation of $\mathcal{G}$ acting in the Krein space $(H^L, J^L)$, induced by the Krein-unitary representation $L$ of the subgroup $H$, acting in the Krein space $(H_L, J^L)$. Let for a measure space $(\mathbb{R}, \mathcal{A}, m)$ the operators of the representation $L$ and the fundamental symmetry $J_L$ be decomposable:

$$L = \int_\mathbb{R} \lambda^L \, dm(\lambda)$$

and

$$J_L = \int_\mathbb{R} J^L_\lambda \, dm(\lambda)$$

with respect to a direct integral decomposition

$$H_L = \int_\mathbb{R} H^L_\lambda \, dm(\lambda),$$

(39)

of the Hilbert space $H_L$. Then

$$H^L = \int_\mathbb{R} H^L_\lambda \, dm(\lambda)$$

(40)

and all operators of the representation $U^L$ and the fundamental symmetry $J^L$
are decomposable with respect to (40), i.e.
\[ U^L = \int_{\mathbb{R}} U^{\lambda^L} \, d\lambda \quad \text{and} \quad J^L = \int_{\mathbb{R}} J^{\lambda^L} \, d\lambda; \]

where \( U^{\lambda^L} \) is the Krein-isometric representation in the Krein space \( (\mathcal{H}^{\lambda^L}, J^{\lambda^L}) \)
induced by the Krein-unitary representation \( \lambda^L \) of the subgroup \( \mathcal{H} \), acting in the
Krein space \( (\mathcal{H}_{\lambda^L}, J_{\lambda^L}) \).

\[ [\text{Outline of the proof.}] \]

Let \( \lambda \mapsto E(\lambda)_L \) be the spectral measure associated
with the decomposition (39). Consider the direct integral decompositions
\[ \mathcal{H}^L = \int_{\mathbb{R}} \mathcal{H}^L(\lambda) \, d\lambda, \]
\[ J^L = \int_{\mathbb{R}} J^L(\lambda) \, d\lambda \quad \text{and} \quad U^L = \int_{\mathbb{R}} U^L(\lambda) \, d\lambda, \]
of \( \mathcal{H}^L, J^L \) and \( U^L \), associated with the corresponding spectral measure \( \lambda \mapsto E(\lambda)_L \)
and the same measure \( m \). Using the vector-valued version of (27) and
the Fubini theorem one shows that \( \mathcal{H}^L(\lambda) = \mathcal{H}_{\lambda^L} \) and the equalities of the
Radon-Nikodym derivatives
\[ \frac{d}{d\lambda} \left( E(\lambda)_L J^L f, E(\lambda)_L g \right) = \frac{d}{d\lambda} \left( E(\lambda)^L J^{\lambda^L} f, E(\lambda)^L g \right), \]
\[ \frac{d}{d\lambda} \left( E(\lambda)_LU^L f, E(\lambda)_L g \right) = \frac{d}{d\lambda} \left( E(\lambda)^LU^{\lambda^L} f, E(\lambda)^L g \right), \]
for all \( f, g \in \mathcal{H}^L \) in the domain of \( U^L \), which means that \( J^L(\lambda) = J^{\lambda^L} \) and
\( U^L(\lambda) = U^{\lambda^L} \).

Using the Dunford-Gelfand-Mackey [17] (or more general [26, 44]) spectral measures and corresponding decompositions, we could generalize the last
theorem keeping Krein self adjointness of the idempotents of the decomposition
of \( L \) just using Dunford or more general spectral measures), but abandoning
their commutativity with \( J_L \), and thus discarding their self-adjointness.

But in decomposition of the Krein-isometric induced representation restricted
to a closed subgroup as in the Subgroup Theorem (or respectively in decomposition
of the tensor product of Krein-isometric induced representations as in
the Kronecker Product Theorem) we have encountered Krein-isometric induced representations \( U^{\lambda_L} \) in the Krein space \( (\mathcal{H}^{\lambda_L}, J_{\lambda_L}) \)
with the non-standard fundamental symmetry \( J_{\lambda_L} \) instead of the standard one \( J^{\lambda_L} \) (respectively their tensor
product \( U^{\lambda_L} \) acting in the tensor product Krein space \( (\mathcal{H}^{\lambda_L \times \lambda_L}, J_{\lambda_L \times \lambda_L}) = (\mathcal{H}^{\lambda_L} \otimes \mathcal{H}^{\lambda_L}, J_{\lambda_L} \otimes J_{\lambda_L}) \)). In this case for each idempotent \( P_L \) acting in the representation
space \((\mathcal{H}_L, \mathfrak{m}_L)\) and commuting with \(L^x\) we could similarly define the corresponding operator \(P^L\): \((P^L f)_x = P_L f_x\) for \(f \in \mathcal{H}^{L^x}\). Similarly we can define \(P^N_{x,y}\) commuting with \(N^x,y\.) However in this case with non-standard fundamental symmetry \((\mathfrak{m} g)_x = L_{h_1(x)} \mathfrak{m} L_{h_1(x)^{-1}}(g), g \in \mathcal{H}^{L^x}\) (resp. \((\mathfrak{m} w)_x = (L_{h_1(x)} \mathfrak{m} L_{h_1(x)^{-1}}) \otimes (M_{h_2(y)} \mathfrak{m} M_{h_2(y)^{-1}})(w)_x, w \in \mathcal{H}^{N^x,y}\) the operator \(P^L\) (or \(P^N_{x,y}\)) is in general unbounded. Moreover \(P^L\) (resp. \(P^N_{x,y}\)) is non self-adjoint in this case even if \(P^L\) (resp. \(P^N_{x,y}\)) is self-adjoint. We hope the slightly misleading (unjustified) notation \(U_{N^x,y}\) will cause no serious troubles.

Thus in particular the Theorem 9 (and its generalizations with Dunford-Gelfand-Mackey spectral measure decompositions) cannot in general be immediately applied to the representations \(U^{N^x,y}\) standing in the Kronecker Product Theorem for the tensor product of Lopuszański representations of the double covering \(\mathfrak{G} = T_4 \otimes SL(2, \mathbb{C})\) of the Poincaré group. But \(U^{N^x,y}\) as a Krein-isometric representation of the semi-direct product \(\mathfrak{G} = T_4 \otimes SL(2, \mathbb{C})\) defines an imprimitivity system in Krein space (in the sense of Sect 5) which is concentrated on a single orbit. We then restore the form of the ordinary induced representation to \(U^{N^x,y}\) by applying Theorem 5 of Sect. 6 but we need the generalized version of this theorem with the finite multiplicity condition 3) discarded. We hope to present in a subsequent paper the full analysis of the component representations \(U^{N^x,y}\) in the decomposition of tensor product of Lopuszański representations.

The necessity of restoring the standard form of induced Krein-isometric representation to the component Krein-isometric representations in the decomposition of tensor product of standard induced Krein-isometric representations is the main difference in comparison to the Mackey theory of unitary induced representations. In case of the double covering \(\mathfrak{G}\) of the Poincaré group this “restoring” is quite elaborate, but effectively computable. The case of tensor products of ordinary unitary induced representations may be rather effectively reduced to the harmonic analysis on “small groups” \(G_{\chi_p} = SU(2, \mathbb{C})\) or \(SL(2, \mathbb{R})\) (see Sect. 6) and to the tensor products \(G_{\chi_p}\) of Gelfand-Neumark representations of \(SL(2, \mathbb{C})\), with the help of the original Mackey’s Subgroup and Kronecker Product Theorems for unitary induced representations. Indeed for tensor products of integer spin representations (for both versions of the energy sign) these decompositions have indeed been computed by Tatsuuma [73]. Unfortunately the paper [73] presents only the results without proofs, and some of the results presented there are not correct, namely those under X).

**REMARK 8.** Because the representation of the translation subgroup \(T_4 \subset T_4 \otimes SL(2, \mathbb{C})\) in Lopuszański-type representation is equivalent to the representation of \(T_4\) in direct sum of several (four in case of the Lopuszański representation) representations of, say helicity zero, ordinary unitary induced representations of \(T_4 \otimes SL(2, \mathbb{C})\), corresponding to the “light-cone orbit” in the momentum
space, and the representation of $T_4 \otimes SL(2, \mathbb{C})$ in the Fock space is the direct sum of symmetrized/anisymmetrized tensor products of one-particle representations, then investigation of the multiplicity of the representation of $T_4$ in the Fock space is reduced to the decomposition of tensor products of ordinary unitary induced representations of $T_4 \otimes SL(2, \mathbb{C})$.

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