Irregular Hodge theory

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The Riemann existence theorem

- \( P(z, \partial_z) = \sum_{0}^{d} a_k(z) \left( \frac{d}{dz} \right)^k, \quad a_k \in \mathbb{C}[z], \quad a_d \neq 0 \)

- \( S = \{ z \mid a_d(z) = 0 \} \) sing. set (assumed \( \neq \emptyset \))

- Associated linear system

\[
(*) \quad \frac{d}{dz} \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} = A(z) \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix}, \quad A(z) \in \text{End}(\mathbb{C}(z)^d)
\]

- \( \rightsquigarrow \) **Monodromy** representation of the solution vectors by analytic continuation

\[
\rho : \pi_1(\mathbb{C} \setminus S, z_0) \longrightarrow \text{GL}_d(\mathbb{C})
\]
The Riemann existence theorem

\[ \rho \iff (T_s \in \text{GL}_d(\mathbb{C}))_{s \in S} \text{ (and } T_\infty := (\prod_s T_s)^{-1}) \]
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Conversely, any \( \rho \) (any finite \( S \)) comes from a system (*) s.t., \( \forall s \in S \cup \{\infty\}, \exists \text{ formal merom. gauge transf.} \rightarrow \text{at most simple pole} \) (i.e., reg. sing.):

\[ \exists M(z - s) \in \text{GL}_d(\mathbb{C}((z - s))) \text{ s.t.} \]

\[ (z - s) \cdot [M^{-1}AM + M^{-1}M'_z] \in \text{End}(\mathbb{C}[z - s]). \]
The Riemann existence theorem

\[ \rho \iff (T_s \in \text{GL}_d(\mathbb{C}))_{s \in S} \text{ (and } T_\infty := (\prod_s T_s)^{-1}) \]

Conversely, any \( \rho \) (any finite \( S \)) comes from a system \((*)\) s.t., \( \forall s \in S \cup \infty, \exists \text{ formal merom. gauge transf.} \rightarrow \text{at most simple pole} \) (i.e., \text{reg. sing.}):

\[ \exists M(z - s) \in \text{GL}_d(C((z - s))) \text{ s.t.} \]

\[ (z - s) \cdot [M^{-1}A + M^{-1}M'_z] \in \text{End}(C[z - s]). \]

Proof: Near \( s \in S \), this amounts to finding \( C_s \in \text{End}(\mathbb{C}^d) \) s.t. \( T_s = e^{-2\pi i C_s} \). Then \( A(z) := C_s / (z - s) \) has monodromy \( T_s \) around \( s \).

Globalization: non-explicit procedure.
Rigid irreducible representations

Assume $\rho$ is *irreducible*:

**cannot put all $T_s$ in a upper block-triang. form simultaneously**

and *rigid*:

\[
\text{if } T'_s \sim T_s \ \forall s \in S \cup \infty, \text{ then } \rho' \sim \rho
\]

and assume $\forall s \in S \cup \infty$,

**\[ \forall \lambda \text{ eigenvalue of } T_s, \ |\lambda| = 1 \]**

\[ \Rightarrow \textbf{More structure} \text{ on the solution to the Riemann existence th.} \]
Variations of pol. Hodge structure

**Theorem (Deligne 1987, Simpson 1990):**

\[ \exists \text{ var. of polarized Hodge structure (wt. } = 0) \text{ adapted to } \rho \]
Variations of pol. Hodge structure

**THEOREM** (Deligne 1987, Simpson 1990):

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- \( G_z \): *pos. def. Herm. \( d \times d \) matrix, \( C^\infty \) w.r.t. \( z \in \mathbb{C} \setminus S \)
Variations of pol. Hodge structure

**Theorem** (Deligne 1987, Simpson 1990):

∃! var. of polarized Hodge structure (wt. \(= 0\)) adapted to \(\rho\)

- \(G_z\): *pos. def. Herm.* \(d \times d\) matrix, \(C^\infty\) w.r.t. \(z \in \mathbb{C}\setminus \mathcal{S}\)
- *Hodge decomp.* \(\forall z \in \mathbb{C}\setminus \mathcal{S}:\)

\[
\mathbb{C}^d = \bigoplus_p H_z^p, \quad H_z^{-p} = \overline{H_z^p}
\]
Variations of pol. Hodge structure

**THEOREM** (Deligne 1987, Simpson 1990):

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- **\( G_z \): pos. def. Herm. \( d \times d \) matrix, \( C^\infty \) w.r.t. \( z \in \mathbb{C} \setminus S \)

- **Hodge decomp.** \( \forall z \in \mathbb{C} \setminus S \):
  
  \[ \mathbb{C}^d = \bigoplus_p H^p_z, \quad H^{-p}_z = \overline{H^p_z} \]

- **\( z \mapsto H^p_z \): \( C^\infty \) \& possibly *not hol.* but

  \[ z \mapsto F^p H_z := \bigoplus_{p' \geq p} H^{p'}_z \text{ holomorphic} \text{ and} \]

  \[ \left( \frac{d}{dz} + A \right) \cdot F^p H_z \subset F^{p-1} H_z \]
Variations of pol. Hodge structure

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- $G_z$: **pos. def. Herm.** $d \times d$ matrix, $C^\infty$ w.r.t. $z \in \mathbb{C}\setminus S$

- **Hodge decomp.** \(\forall z \in \mathbb{C}\setminus S:\)

  \[
  \mathbb{C}^d = \bigoplus_p H^p_z, \quad H^{-p}_z = \overline{H^p_z}
  \]

- \(z \mapsto H^p_z: C^\infty\) & possibly **not hol.** but

- \(z \mapsto F^p H_z := \bigoplus_{p' \geq p} H^p_z\) **holomorphic** and

  \[
  \left(\frac{d}{dz} + A\right) \cdot F^p H_z \subset F^{p-1} H_z
  \]

- \(\tilde{G}_z\) s. t. \(\tilde{G}_z|_{H^p} := (-1)^p G|_{H^p}\), then

  \[
  \partial_z \tilde{G}_z \cdot \tilde{G}_z^{-1} = tA(z).
  \]
Variations of pol. Hodge structure

**Theorem** (Deligne 1987, Simpson 1990):

∃! var. of polarized Hodge structure (wt. = 0) adapted to $\rho$

- $\Rightarrow$ Numbers $f^p = \text{rk } F^p H_z$ attached to $\rho$.

- Moreover (Griffiths),

$$\mathbb{C}[z, (z - s)^{-1}_{s \in S}]^d = \mathcal{O}(\mathbb{C} \setminus S)^d_{G\text{-mod. growth}}.$$
Hypergeom. differential eqns

Given \( 0 \leq \alpha_1 \leq \cdots \leq \alpha_d < 1, \)
\( 0 \leq \beta_1 \leq \cdots \leq \beta_d < 1, \quad \alpha_i \neq \beta_j \forall i, j. \)

\[
P(z, \partial_z) := \prod_{i=1}^{d} \left( z \frac{d}{dz} - \alpha_i \right) - z \prod_{j=1}^{d} \left( z \frac{d}{dz} - \beta_j \right)
\]

\( S = \{0, 1\}. \)

Beukers & Heckman: \( \rho \) is irreducible rigid, with \( \lambda = e^{-2\pi i \alpha} \) or \( e^{2\pi i \beta} \).

Set \( \ell_j = \# \{ i \mid \alpha_i \leq \beta_j \} - j \)
Hypergeom. differential eqns

Given \( \begin{cases} 0 \leq \alpha_1 \leq \cdots \leq \alpha_d < 1, \\ 0 \leq \beta_1 \leq \cdots \leq \beta_d < 1, \end{cases} \) \( \alpha_i \neq \beta_j \ \forall i, j. \)

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Set \( \ell_j = \# \{ i \mid \alpha_i \leq \beta_j \} - j \), e.g.

- \( \alpha_1 \leq \beta_1 \leq \alpha_2 \leq \cdots \leq \beta_d \Rightarrow \ell_j = 0 \ \forall j \)
- \( \alpha_d \leq \beta_1 \Rightarrow \ell_j = d - j \).
Hypergeom. differential eqns

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**Theorem** (R. Fedorov, 2015):

\[ f^p = \# \{ j \mid \ell_j \geq p \} \]
Hypergeom. differential eqns

Given \[ \begin{cases} 0 \leq \alpha_1 \leq \cdots \leq \alpha_d < 1, \\ 0 \leq \beta_1 \leq \cdots \leq \beta_d < 1, \end{cases} \quad \alpha_i \neq \beta_j \quad \forall i, j. \]

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Beukers & Heckman: \( \rho \) is \textit{irreducible rigid}, with \( \lambda = e^{-2\pi i\alpha} \) or \( e^{2\pi i\beta} \).

Set \( \ell_j = \#\{i \mid \alpha_i \leq \beta_j\} - j \)

\textbf{THEOREM} (R. Fedorov, 2015):

\[ f^p = \#\{j \mid \ell_j \geq p\} \]

- mixed: \( F^1 = 0, F^0 = \mathcal{O}(\mathbb{C} \setminus S)^d \Rightarrow \text{unitary conn.} \)
- unmixed: \( 0 = F^d \subset \cdots \subset F^0 = \mathcal{O}(\mathbb{C} \setminus S)^d \).
Confluent hypergeom. diff. eqns

\[ P(z, \partial_z) := \prod_{i=1}^{d'} \left( z \frac{d}{dz} - \alpha_i \right) - z \prod_{j=1}^{d} \left( z \frac{d}{dz} - \beta_j \right) \]

with \( d' < d \) \( \Rightarrow \) \( S = 0 \) and 0 is an **irreg. sing.**
\( (\infty = \text{reg. sing}) \).

- Riemann existence th. breaks down for irreg. sing.
- Need **Stokes data** to reconstruct the differential eqn from sols.
- \( \rightsquigarrow \) Riemann-Hilbert-Birkhoff correspondence.
Confluent hypergeom. diff. eqns

\[ P(z, \partial_z) := \prod_{i=1}^{d'} \left( z \frac{d}{dz} - \alpha_i \right) - z \prod_{j=1}^{d} \left( z \frac{d}{dz} - \beta_j \right) \]

with \( d' < d \).

- Same condition on \( \alpha, \beta \)'s \( \Rightarrow \) **irreducible** and **rigid**.
Confluent hypergeom. diff. eqns

\[ P(z, \partial_z) := \prod_{i=1}^{d'} \left( z \frac{d}{dz} - \alpha_i \right) - z \prod_{j=1}^{d} \left( z \frac{d}{dz} - \beta_j \right) \]

with \( d' < d \).

- Same condition on \( \alpha, \beta \)'s \( \Rightarrow \) \textit{irreducible} and \textit{rigid}:
  - \textit{irreducible}: Cannot split
    \[ P(z, \partial_z) = P_1(z, \partial_z) \cdot P_2(z, \partial_z) \]
    in \( \mathbb{C}(z)\langle \partial_z \rangle \) with \( \deg P_1, \deg P_2 \geq 1 \).
  - \textit{rigid}: Any other linear diff. syst. (sings at \( S \cup \infty \))
    which is \textit{gauge-equiv. over} \( \mathbb{C}((z - s)) \) at each \( s \in S \cup \infty \)
    to the given system is \textit{gauge-equiv. over} \( \mathbb{C}(z) \) to the given system.
Confluent hypergeom. diff. eqns

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with \( d' < d \).

- Same condition on \( \alpha, \beta \)'s \( \Rightarrow \) **irreducible** and **rigid**:
  - **irreducible**: Cannot split \( P(z, \partial_z) = P_1(z, \partial_z) \cdot P_2(z, \partial_z) \) in \( \mathbb{C}(z) \langle \partial_z \rangle \) with \( \deg P_1, \deg P_2 \geq 1 \).
  - **rigid**: Any other linear diff. syst. (sings at \( S \cup \infty \)) which is **gauge-equiv. over** \( \mathbb{C}((z - s)) \) at each \( s \in S \cup \infty \) to the given system is **gauge-equiv. over** \( \mathbb{C}(z) \) to the given system.

- But: **Cannot find** a var. of pol. Hodge struct. s.t. the sol. to R-H-B exist. th. given by \( \phi(\mathbb{C} \setminus S)^d_G \)-mod. growth.
Harmonic metrics

Given:
- A diff. system \( \frac{d}{dz} + A(z) \), \( A(z) \in \text{End}(\mathbb{C}(z)^d) \),
- Pole set \( S \subset \mathbb{C} \).
- \( G_z \): any pos. def. Herm. mtrx, \( C^\infty \) w.r.t. \( z \in \mathbb{C} \setminus S \).

Then \( \exists! A'_G \), \( A''_G \) \( d \times d \), \( C^\infty \) w.r.t. \( z \), s.t.

\[
\begin{align*}
\partial_z G_z &= tA'_G \cdot G_z + G_z \cdot \overline{A''_G} \\
\overline{\partial_z G_z} &= tA''_G \cdot G_z + G_z \cdot \overline{A'_G} \\
-A''_G &= (A - A'_G)^*.
\end{align*}
\]

(Compatibility with \( G \))

\( G \) is harmonic w.r.t. \( A \) if

\[
\overline{\partial_z \theta'_z} + [\theta'_z, \theta'_z^*] = 0
\]
Harmonic metrics

**Theorem** (Simpson 1990, CS 1998, Biquard-Boalch 2004, T. Mochizuki 2011):

- If \( A \) is **irreducible**, \( \exists! \) harmonic metric \( G \) w.r.t. \( A \) s.t.
  - Coefs of \( \text{Char} \ \theta' \) have **mod. growth** at \( S \cup \infty \),
  - \( \mathbb{C}[z, ((z - s)^{-1})_{s \in S}]^d = (\mathcal{O}(\mathbb{C} \setminus S)^d)_G\)-mod. growth.

- E.g., the Hodge metric of a var. pol. Hodge structure is harmonic w.r.t. the **reg. sing. conn.** \( A \).
- If \( A \) is **irreg.**, what about **rigid** irreducible \( A \)?
- Answer in the last slide of the talk.
The irregular Hodge filtration
The irregular Hodge filtration

Deligne (2007):
“The analogy between vector bundles with integrable connection having irregular singularities at infinity on a complex algebraic variety $U$ and $\ell$-adic sheaves with wild ramification at infinity on an algebraic variety of characteristic $p$, leads one to ask how such a vector bundle with integrable connection can be part of a system of realizations analogous to what furnishes a family of motives parametrized by $U$ ...

In the ‘motivic’ case, any de Rham cohomology group has a natural Hodge filtration. Can we hope for one on $H^i_{\text{dR}}(U, \nabla)$ for some classes of $(V, \nabla)$ with irregular singularities?”
The irregular Hodge filtration

“The reader may ask for the usefulness of a “Hodge filtration” not giving rise to a Hodge structure. I hope that it forces bounds to $p$-adic valuations of Frobenius eigenvalues. That the cohomology of $e^{-z}z^{\alpha}$ ($0 < \alpha < 1$) has Hodge degree $1 - \alpha$ is analogous to formulas giving the $p$-adic valuation of Gauss sums.”
The irregular Hodge filtration

Ex.: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = d + df + \alpha \frac{dz}{z}$
The irregular Hodge filtration

Ex.: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = d + df + \alpha dz/z$

$\nabla : \mathbb{C}[z, z^{-1}] \rightarrow \mathbb{C}[z, z^{-1}] \cdot \frac{dz}{z}$
The irregular Hodge filtration

Ex.: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = d + df + \alpha dz/z$

\[ \mathbb{C}[z, z^{-1}] \xrightarrow{\nabla} \mathbb{C}[z, z^{-1}] \cdot \frac{dz}{z} \xrightarrow{} H^{1}_{dR}(U, \nabla) \]
The irregular Hodge filtration

Ex.: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = d + df + \alpha dz/z$

$$\begin{align*}
\mathbb{C}[z, z^{-1}] & \xrightarrow{\nabla} \mathbb{C}[z, z^{-1}] \cdot \frac{dz}{z} \xrightarrow{} H^1_{dR}(U, \nabla) \\
\mathbb{C}[z, z^{-1}] e^{-z} z^\alpha & \xrightarrow{d} \mathbb{C}[z, z^{-1}] \cdot e^{-z} z^\alpha \frac{dz}{z} \xrightarrow{} \mathbb{C} \cdot \left[ e^{-z} z^\alpha \frac{dz}{z} \right]
\end{align*}$$
The irregular Hodge filtration

Ex.: $U = \mathbb{C}^*$, $f : z \mapsto -z$, $\nabla = d + df + \alpha dz/z$

$$\mathbb{C}[z, z^{-1}] \xrightarrow{\nabla} \mathbb{C}[z, z^{-1}] \cdot \frac{dz}{z} \xrightarrow{\text{period}} \int_{0}^{\infty} e^{-z} z^{\alpha} \frac{dz}{z} = \Gamma(\alpha)$$

$\Rightarrow [e^{-z} z^{\alpha} dz/z] \in F^{1-\alpha} H^1_{dR}(U, \nabla)$. 

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The Hodge filtration in dim $\geq 1$

Setting:
- $U$: smooth cplx quasi-proj. var. (e.g. $U = (\mathbb{C}^*)^n$).
- Choose (according to Hironaka) any $X$ such that
  - $X$: smooth cplx proj. variety,
  - $D$: reduced divisor with normal crossings in $X$ locally, $D = \{x_1 \cdots x_\ell = 0\}$
  - $U = X \setminus D$. 
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- **Theorem** (Deligne 1972):
  $$H^k(U, \mathbb{C}) \simeq H^k(X, (\Omega_X^\bullet (\log D), d))$$
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- **Theorem** (Deligne 1972):
  $$H^k(U, \mathbb{C}) \simeq H^k(X, (\Omega_X^\bullet (\log D), d))$$

\[\begin{align*}
\Omega_X^1(\log D) &\overset{\text{loc.}}{=} \sum_{i=1}^{\ell} \mathcal{O}_X \frac{dx_i}{x_i} + \sum_{j>\ell} \mathcal{O}_X \ dx_j, \\
\Omega_X^k(\log D) &= \wedge^k \Omega_X^1(\log D)
\end{align*}\]
The Hodge filtration in dim $\geq 1$

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    - $X$: smooth cplx proj. variety,
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    - $U = X \setminus D$.

- **THEOREM** (Deligne 1972):
  \[ H^k(U, \mathbb{C}) \simeq H^k(X, (\Omega^\bullet_X(\log D), d)) \]
  and $\forall p$, (\textit{E}$_1$-degeneration)
  \[ H^k(X, \sigma^{\geq p}(\Omega^\bullet_X(\log D), d)) \longrightarrow H^k(X, (\Omega^\bullet_X(\log D), d)) \]
  is injective, its image defining the Hodge filtration $F^p H^k(U, \mathbb{C})$.

- $\leadsto$ Mixed Hodge structure on $H^k(U, \mathbb{C})$. 
Twisted de Rham cohomology

- Setting:
  - $U$: smth cplx quasi-proj. var., $f : U \to \mathbb{C}$ alg. fnct.
Twisted de Rham cohomology

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- $U$: smth cplx quasi-proj. var., $f : U \to \mathbb{C}$ alg. fnct.

Twisted de Rham cohomology $H^k_{dR}(U, d + df)$:
Cohomology of the alg. de Rham cplx. E.g. $U$ affine:

$$0 \to \mathcal{O}(U) \xrightarrow{d + df} \cdots \xrightarrow{d + df} \Omega^n(U) \to 0$$
Twisted de Rham cohomology

Setting:

- \( U \): smooth complex quasi-projective variety, \( f: U \to \mathbb{C} \) algebraic function.

**Twisted de Rham cohomology** \( H^k_{dR}(U, d + df) \): Cohomology of the algebraic de Rham complex. E.g. \( U = \mathbb{C}^n \):

\[
0 \to \mathbb{C}[x] \to \bigoplus_i \mathbb{C}[x]dx_i \to \cdots \to \bigoplus_i \mathbb{C}[x]dx_i \to \mathbb{C}[x]dx \to 0
\]
Twisted de Rham cohomology

- **Setting**:
  - \( U \): smth cplx quasi-proj. var., \( f : U \to \mathbb{C} \) alg. fnct.

- **Twisted de Rham cohomology** \( H^k_{dR}(U, d + df) \):
  Cohomology of the alg. de Rham cplx. E.g. \( U = \mathbb{C}^n \):

\[
0 \to \mathbb{C}[x] \to \bigoplus_i \mathbb{C}[x]dx_i \to \cdots \to \bigoplus_i \mathbb{C}[x]d\hat{x}_i \to \mathbb{C}[x]dx \to 0
\]

\[
g(x) \mapsto \sum_i (g'_{x_i} + g f'_{x_i})dx_i
\]

\[
\sum_i h_i d\hat{x}_i \mapsto \left[ \sum_i (-1)^{i-1}((h_i)'_{x_i} + h_i f'_{x_i}) \right]dx
\]
Choose (according to Hironaka) any $X$ such that

- $X$: smooth cplx proj. variety,
- $D$: reduced divisor with normal crossings in $X$ locally, $D = \{x_1 \cdots x_\ell = 0\}$
- $U = X \setminus D$.
- s.t. $f$ extends as an hol. map

\[
f : X \longrightarrow \mathbb{P}^1 = \mathbb{C} \cup \infty, \quad f^{-1}(\infty) \subset D. \quad P := f^*(\infty).
\]
The Kontsevich complex

For $\alpha \in [0, 1) \cap \mathbb{Q}$,

- $\Omega^k_X(\log D)([\alpha P])$: forms with log pole along $D - P$ and pole at most “$\log + [\alpha P]$” along $f^{-1}(\infty)$. (e.g. $df = f \cdot df/f \in \Omega^1_X(\log D)(P)$.)

- Define $\Omega^k_f(\alpha)$ as

$$\left\{ \omega \in \Omega^k_X(\log D)([\alpha P]) \mid df \wedge \omega \in \Omega^{k+1}_X(\log D)([\alpha P]) \right\}$$

- Significant $\alpha$'s: $\ell/m$, $m =$ mult. of a component of $P$, $\ell = 0, \ldots, m - 1$. 

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The Kontsevich complex

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- Significant $\alpha$'s: $\ell/m$, $m = \text{mult. of a component of } P$, $\ell = 0, \ldots, m - 1$.

- $\leadsto$ Kontsevich complex $(\Omega^\bullet_f(\alpha), d + df)$.

$$H^k(X, (\Omega^\bullet_f(\alpha), d + df)) \simeq H^k_{dR}(U, d + df)$$
The irreg. Hodge filtration in dim $\geq 1$

**Theorem** (Kontsevich, Esnault-CS-Yu 2014, M. Saito 2014, T. Mochizuki 2015):

- $\forall p$, ($E_1$-degeneration)

  \[ H^k(X, \sigma \geq p(\Omega_f^*(\alpha), d+df)) \longrightarrow H^k(X, (\Omega_f^*(\alpha), d+df)) \]

  is injective, its image defining the irregular Hodge filtration $F^{p-\alpha} H^k_{dR}(U, d + df)$.

- $\lambda \geq \mu \in \mathbb{Q} \Rightarrow$

  \[ F^\lambda H^k_{dR}(U, d + df) \subset F^\mu H^k_{dR}(U, d + df) \]

  Jumps at most at $\lambda = \ell/m + p, p \in \mathbb{Z}$,

  $\ell = 0, \ldots, m - 1$, $m = $ mult. component of $P$. 

Irregular Hodge theory – p. 19/23
History of the result, dim. one

Deligne (1984, IHÉS seminar notes).

\( A \in \text{GL}_d(\mathbb{C}(z)) \) with \textit{reg sing}. on \( S \cup \infty \), and \textit{unitary}. \( f \in \mathbb{C}(z) \). Defines a filtr. \((\lambda \in \mathbb{R})\)

\[
F^\lambda \mathbb{C}[z, (z-s)_{s \in S}]^d \xrightarrow{\text{d} + A + df} F^{\lambda-1} \mathbb{C}[z, (z-s)_{s \in S}]^d dz
\]

an proves \( E_1 \)-degeneration.

Deligne (2006). Adds more explanations and publication in the volume “Correspondance Deligne-Malgrange-Ramis” (SMF 2007).

CS (2008). Same as Deligne, with \( A \) underlying a \textit{pol. var. of Hodge structure}. Uses harmonic metrics through the theory of var. of twistor structures (Simpson, Mochizuki, CS).
History of the result, dim \( \geq 1 \)

- **J.-D. Yu (2012):** defines \( F^\lambda H^k_{\text{dR}}(U, d + df) \) + many properties and \( E_1 \)-degeneration in some cases.
- **Esnault-CS-Yu (2013):** \( E_1 \)-degeneration by reducing to (CS, 2008) (push-forward by \( f \)).
- **Kontsevich (2012), letters to Katzarkov and Pantev, arXiv 2014:** defines the Kontsevich complex and proves \( E_1 \)-degeneration if \( P = P_{\text{red}} \), by the method of Deligne & Illusie (reduction to char. \( p \)). **Does not extend if** \( P \neq P_{\text{red}} \). Motivated by mirror symmetry of Fano manifolds.
- **M. Saito (2013):** \( E_1 \)-degeneration by comparing with limit mixed Hodge structure of \( f \) at \( \infty \).
- **T. Mochizuki (2015):** \( E_1 \)-degeneration by using the theory of mixed twistor \( \mathcal{D} \)-modules.
Rigid irreducible diff. eqns

- Given diff. operator \( \frac{d}{dz} + A(z) \), \( A(z) \in \text{End}(\mathbb{C}(z)^d) \),
- pole set = \( S \subset \mathbb{C} \).
- Assume it is **irreducible** and **rigid**.
- Assume eigenvalues \( \lambda \) of \( \hat{T}_s \) (\( s \in S \cup \infty \)) s.t. \( |\lambda| = 1 \).

**Theorem (CS 2015):** \( \exists \) **canonical** filtration

\[
F^\lambda \mathbb{C}[z, ((z - s)^{-1})_{s \in S}]^d \quad (\lambda \in \mathbb{R})
\]

by free \( \mathbb{C}[z, ((z - s)^{-1})_{s \in S}] \)-modules attached to \( A(z) \), s.t.

\[
\left( \frac{d}{dz} + A(z) \right) F^\lambda \subset F^{\lambda - 1}.
\]
Rigid irreducible diff. eqns

- Needs the construction of a category of *Irregular mixed Hodge modules* between the category of mixed Hodge modules (M. Saito) and that of mixed twistor $\mathcal{D}$-modules (T. Mochizuki). Use of the Arinkin-Deligne’s algorithm similar to Katz’ algorithm.

**QUESTION:** For confluent hypergeom. eqns, how to compute the *jumping indices* and the *rank* of the Hodge bundles?

- Recent work of Castaño Dominguez and Sevenheck on some confluent hypergeometric diff. eqns.

- Other interesting examples: rigid irregular connections of Gross-Frenkel.