Combinatorial formulas for the moments of the Brownian motion on classical compact Lie groups are obtained. These expressions are deformations of formulas of B. Collins and P. Śniady for moments of the Haar measure.

1 Introduction

Let $G$ be a compact group that belongs to one of the classical series $\text{U}(N)$, $\text{O}(N)$ and $\text{Sp}(N)$. For any of these groups, let us denote by $V$ their fundamental representation, that is $\mathbb{C}^N$ for $\text{U}(N)$ and $\text{O}(N)$, and $\mathbb{C}^{2N}$ for $\text{Sp}(N)$. In the following, we are concerned in finding an explicit expression for

$$\int_{G} f(g) \mu(dg)$$

where $f$ is the composition of a polynomial function on $\text{End}(V)$ with the fundamental representation and $\mu$ is the measure associated to the law of a Brownian motion that we shall define in Section 4.2. When $\mu$ is the Haar measure, such quantities have been studied by B. Collins in [1] and later with P. Śniady in [4], see also [3] and [11] for a recent point of view. The subject of this article is to give formulas for (1) that are deformations of the ones obtained in [4]. Besides, our work completes the article [8] of T. Lévy who computes such integrals but for a smaller class of polynomials.

Our computations are motivated by the fact that the law of the Brownian motion is invariant by conjugation so that integrating with respect to it defines invariant quantities for the group $G$. For any space $E$ on which $G$ acts, we will denote by $E^G$ the set of points in $E$ fixed by the action of $G$. If $W$ is a representation of $G$, we denote by $\text{End}_G(W)$ the space of endomorphisms commuting with $G$. We will consider here the representations $W = V^\otimes n \otimes V^* \otimes m$. For such representations, $W^G$ is described by the first theorem of invariant theory and $\text{End}_G(W)$ by the Schur-Weyl duality (see [6]). For instance, for $G = \text{U}(N)$, $\text{End}_G(V^\otimes n)$ is the image of $\mathbb{C}[S_n]$ by the classical representation on $V^\otimes n$ given by permutation of the tensors. These two equivalent
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For any positive integer \( N \), let us denote by \( V \) the vector space \( \mathbb{C}^N \), endowed with its canonical hermitian product denoted by \( \langle \cdot, \cdot \rangle \). Let \( \beta \) be the canonical \( \mathbb{C} \)-bilinear form on \( V \) and if \( N \) is even let us denote by \( \omega \) the skew-symmetric bilinear form on \( V \) whose matrix in the canonical basis is \( J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \). We shall consider the following groups of matrices: the unitary group \( U(N) = \{ U \in M_N(\mathbb{C}) : U^* U = \text{Id} \} \), the orthogonal group \( O(N) = \{ O \in M_N(\mathbb{R}) : ^t OO = \text{Id} \} \) and the unitary symplectic group \( \text{Sp}(N) = \{ S \in U(2N) : ^t SJS = J \} \). We shall denote their Lie algebra by small gothic letters: \( u(N) = \{ X \in M_N(\mathbb{C}) : X + X^* = 0 \} \), \( o(N) = \{ X \in M_N(\mathbb{R}) : ^t X + X = 0 \} \) and \( \mathfrak{sp}(N) = \{ X \in u(2N) : ^t X J + J X = 0 \} \). For any \( x, y \in u(N) \), let us define

\[
\langle x, y \rangle = -\dim(V) \text{Tr}_V(xy).
\]

The restriction of the real bilinear form \( \langle \cdot, \cdot \rangle \) defines an invariant scalar product on the three series of Lie algebras we are considering. Let \( G \) be one of the above mentioned compact Lie group with Lie algebra \( \mathfrak{g} \). Let us denote by \( (K_t)_{t \geq 0} \) the classical Brownian motion on the Euclidean space \( (\mathfrak{g}, \langle \cdot, \cdot \rangle) \) that is the unique Gaussian process with covariance given for all \( x, y \in \mathfrak{g} \) and \( t, s \geq 0 \), by

\[
\mathbb{E}[\langle x, K_t \rangle \langle y, K_s \rangle] = \min(t, s) \langle x, y \rangle.
\]

The quadratic variation \( \langle dK_t, dK_t \rangle \) is equal to \( M dt \), where \( M \) is a constant matrix. Furthermore, \( M \) is invariant by conjugation by \( G \) and is therefore proportional to the identity. We denote by \( C_\mathfrak{g} \) the constant such that \( \langle dK_t, dK_t \rangle = C_\mathfrak{g} dt \). Let \( S \) be an element of the group \( G \) and define \( (G_t)_{t \geq 0} \) as the stochastic process solution of the following stochastic differential equation

\[
dG_t = G_t dK_t + \frac{C_\mathfrak{g}}{2} G_t dt, G_0 = S.
\]

For any bilinear form \( \psi \) preserved by \( G \), almost surely, for all \( t \geq 0 \), \( \psi \) is preserved by \( G_t \). Indeed, for any \( u, v \in V \), using Itô formula,

\[
d\psi(G_t u, G_t v) = \psi(dK_t u, v) + \psi(u, dK_t v) + C_\mathfrak{g} \psi(u, v) dt + \langle \psi(dK_t u, dK_t v) \rangle
\]

\[
= \psi(\langle dK_t dK_t \rangle u, v) + C_\mathfrak{g} \psi(u, v) dt = 0.
\]
The group $G$ is characterized by the invariance of some bilinear forms, hence, almost surely, the process $(G_t)_{t \geq 0}$ belongs to $G$. This Markovian process is called the Brownian motion on $G$ issued from $S$. Let us describe its generator. For any $x \in \mathfrak{g}$, let $\mathcal{L}_x$ be the left-invariant first order differential operator defined by setting for any differentiable function $f$ and any $g \in G$,

$$\mathcal{L}_x f(g) = \frac{d}{dt} f(g \exp(tx))|_{t=0}.$$  

Let $\mathcal{U}(\mathfrak{g})$ be the real enveloping algebra of $\mathfrak{g}$. The application $\mathcal{L}$ extends to an isomorphism of associative algebra between $\mathcal{U}(\mathfrak{g})$ and the algebra of left-invariant differential operators on $G$. Note that for any smooth function $F$ on $M_N(\mathbb{C})$, and any $x \in \mathfrak{g}, M \in G$,

$$\mathcal{L}_x \circ \mathcal{L}_x(F)(M) = d_M F(Mx^2) + d^2_M F(Mx, Mx).$$ (2)

For any orthonormal basis $(x_i)_{1 \leq i \leq d}$ of $\mathfrak{g}$ with respect to the scalar product $\langle \cdot, \cdot \rangle$, let $c_\mathfrak{g} = \sum_{i=1}^d x_i^2 \in \mathcal{U}(\mathfrak{g})$. The element $c_\mathfrak{g}$ does not depend on the orthonormal basis $(x_i)_{1 \leq i \leq d}$ and is called the Casimir element. The image of the Casimir element $c_\mathfrak{g}$ is called the Laplacian and is denoted by $\Delta_G$:

$$\Delta_G = \mathcal{L}_{c_\mathfrak{g}} = \sum_{i=1}^d \mathcal{L}_{x_i} \circ \mathcal{L}_{x_i}.$$  

Note that the image of $c_\mathfrak{g}$ by the classical representation on $V$ is $C_g$. The Itô formula combined with the equation $[2]$ implies that the Brownian motion on $G$ is the diffusion process with generator $\frac{1}{2} \Delta_G$.

Let $(G_t)_{t \geq 0}$ be a Brownian motion issued from $\text{Id}$ and $H$ a random variable distributed according to the Haar measure on $G$. Our main result is a combinatorial formula for the moments of the entries of the matrix $G_t$. The Theorem $[4,9]$ gives, for any $i_1, \ldots, i_n \in \{1, \ldots, N\}$, an expression for $\mathbb{E}[(G_t)_{i_1,j_1} \cdot \ldots \cdot (G_t)_{i_n,j_n}]$ that is a deformation of the expression of $[4]$ for $\mathbb{E}[H_{i_1,j_1} \cdot \ldots \cdot H_{i_n,j_n}]$. In order to use the multiplicative nature of Brownian motion, we get a joint expression for the moments of the same order as a morphism $\mathbb{E}[G_t^{\otimes n}] \in \text{End}(V^{\otimes n})$.

### 3 Integration formulas for the Haar measure

We give here a proof of the integration formula with respect to the Haar measure on the symplectic group $\text{Sp}(N)$ and recall the formulas for $\text{U}(N)$ and $\text{O}(N)$ (a proof for the compact symplectic group is also given in $[2]$). Let us denote by $dO, dU$ and $dS$ the Haar measure on the groups $\text{U}(N), \text{O}(N)$ and $\text{Sp}(N)$. In the following we shall compute the following integral:

$$\int_{\text{Sp}(2N)} S_{i_1,j_1} S_{i_2,j_2} \cdots S_{i_n,j_n} dS.$$ (3)

Observe that for any $S \in \text{Sp}(N)$, $\overline{S} = -JSJ$, therefore we can deduce from our computation formulas for integrals of polynomials in entries of $S$ and $\overline{S}$. Besides, since the Haar measure is invariant by multiplication by $-\text{Id}$, this integral is zero for odd $n$. Let us define a hermitian scalar product $\langle \cdot, \cdot \rangle$ on $V^{\otimes n}$ extending the termwise product of the canonical hermitian scalar product on $V$. The computation of $[3]$ relies on the following fact. Using invariance of the Haar
measure, it is easily seen that $\Phi$ is an orthogonal projection onto the vector space $(V \otimes 2p)^{\text{Sp}(N)}$ and computing $[3]$ amounts to computing

$$\langle e_1 \otimes \cdots e_{2p}, \Phi(e_{j_1} \otimes \cdots e_{j_{2p}}) \rangle.$$  

Let $(w_{l})_{l \in L}$ be any generating family of $(V \otimes 2p)^{\text{Sp}(N)}$, $G = ((w_{l}, w_{j}))_{l, j \in L}$ its Gram matrix and $\tilde{G}$ a matrix satisfying $GG\tilde{G} = G$. The orthogonal projection onto $(V \otimes 2p)^{\text{Sp}(N)}$ can be reformulated as

$$\Phi = \sum_{l, j \in L} \tilde{G}_{l, j} \langle w_{j}, \cdot \rangle w_{l}.$$  

For any pair of operators $f$ and $\tilde{f}$ such that $f \tilde{f} f = f$ and $\tilde{f} f \tilde{f} = \tilde{f}$, $\tilde{f}$ is called a pseudo-inverse of $f$. As soon as $f$ is self-adjoint for a fixed hermitian product, it admits a unique self-adjoint pseudo-inverse.

What is more, the theory of invariants gives an explicit generating family of $(V \otimes 2p)^{\text{Sp}(N)}$. Let $\mathcal{M}(2p)$ be the set of pair partitions of $\{1, \ldots, 2p\}$. Let us denote by $\pi_{0} \in \mathcal{M}(2p)$ the partition $\{\{1, 2\}, \{3, 4\}, \ldots, \{2p - 1, 2p\}\}$ and

$$w_{\pi_{0}} = \sum_{1 \leq i_{1}, i_{2}, \ldots, i_{2p} \leq N} (\prod_{k=1}^{p} J_{i_{2k-1}i_{2k}}) e_{i_{1}} \otimes e_{i_{2}} \otimes \cdots \otimes e_{i_{2p}}.$$  

The element $w_{\pi_{0}}$ is fixed by the action of $\text{Sp}(N)$. Let us denote by $\varepsilon : \mathcal{S}_{2p} \rightarrow \{-1, 1\}$ the signature homomorphism and $\rho_{\varepsilon} : \mathcal{S}_{2p} \rightarrow \text{GL}(V \otimes 2p)$ the homomorphism defined by setting for any $\sigma \in \mathcal{S}_{2p}$ and $v_{1}, \ldots, v_{2p} \in V$,

$$\rho_{\varepsilon}(\sigma).(v_{1} \otimes \cdots \otimes v_{2p}) = \varepsilon(\sigma)v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(2p)}.$$  

The stabilizer of $\pi_{0}$ under the natural action of $\mathcal{S}_{2p}$ on $\mathcal{M}(2p)$ is called the hyperoctahedral group, we denote it by $\mathcal{H}_{p}$. The group $\mathcal{H}_{p}$ stabilizes the element $w_{\pi_{0}}$ under the action of $\mathcal{S}_{2p}$ through $\rho_{\varepsilon}$, therefore for any $\pi \in \mathcal{M}(2p)$ and $\sigma \in \mathcal{S}_{2p}$ such that $\sigma \pi_{0} = \pi$, the vector

$$w_{\pi} = \rho_{\varepsilon}(\sigma).w_{\pi_{0}}$$  

is well defined. Furthermore, the action of $\text{Sp}(N)$ on $V \otimes 2p$ commutes with the one of $\mathcal{S}_{2p}$, therefore for any $\pi \in \mathcal{M}(2p)$, $w_{\pi} \in (V \otimes 2p)^{\text{Sp}(N)}$. The first geometric Theorem of invariant theory claims that all invariants are linear combinations of these ones.

**Theorem 3.1** ([6], Thm. 5.2.2 ). The family $(w_{\pi})_{\pi \in \mathcal{M}(2p)}$ generates $(V \otimes 2p)^{\text{Sp}(N)}$.

For any pair of partitions $\pi, \eta$ of $\{1, \ldots, 2p\}$, let us denote by $\pi \setminus \eta$ the finest partition coarser than $\pi$ and $\eta$. Let $\# \pi$ the number of blocks of $\pi$ and for any $i, j \in \{1, \ldots, n\}$, let us write $i \sim_{\pi} j$ if $i$ and $j$ belong to the same block of $\pi$. For any permutation $\sigma \in \mathcal{S}_{2p}$ and any list of $2p$ integers $I = (i_{1}, \ldots, i_{2p})$, let us set

$$\text{inv}_{2}(\sigma, I) = \# \{k \in \{1, \ldots, p\} : i_{\sigma(2k)} < i_{\sigma(2k-1)}\}.$$  

\footnote{Note that without the hermitian condition and if $f$ is not invertible, it admits several pseudo-inverse.}
Lemma 3.2. i) For any $\pi, \eta \in \mathcal{M}(2p)$,
\[
\langle w_\pi, w_\eta \rangle = (-1)^p(-2N)^{\#\pi \vee \eta}.
\]

ii) Let $\pi$ be an element of $\mathcal{M}(2p)$, $\sigma \in \mathfrak{S}_{2p}$ such $\sigma \pi_0 = \pi$ and $1 \leq i_1, \ldots, i_{2p} \leq N$. If $i_l = i_k + N$ mod $(2N)$, for all $l, k$ such that $l \sim \pi k$, then
\[
\langle w_\pi, e_{i_1} \otimes \cdots \otimes e_{i_{2p}} \rangle = \varepsilon(\sigma)(-1)^{\text{inv}_2(\sigma, l)},
\]
otherwise this term vanishes.

Proof. i) For any permutations $\alpha$ and $\beta$ such that $\alpha(\pi_0) = \pi$ and $\beta(\pi_0) = \eta$, the scalar product equals
\[
\sum_{i_1, \ldots, i_{2p}} \prod_{k=1}^p J_{i_{2k-1}, i_{2k}} J_{j_{2k-1}, j_{2k}} \langle e_{i_{\alpha^{-1}(1)}} \otimes \cdots \otimes e_{i_{\alpha^{-1}(2p)}}, e_{j_{\beta^{-1}(1)}} \otimes \cdots \otimes e_{j_{\beta^{-1}(2p)}} \rangle \varepsilon(\alpha \beta)
\]
\[
= \varepsilon(\alpha^{-1} \beta) \sum_{1 \leq i_1, \ldots, i_{2p} \leq 2N} \prod_{k=1}^p J_{i_{2k-1}, i_{2k}} J_{j_{2k-1}, j_{2k}} J_{\alpha^{-1}(2k-1), i_{\alpha^{-1}(2k)}}, \varepsilon(\alpha^{-1} \beta).
\]

Let us now choose specific permutations $\alpha$ and $\beta$. Let $\{A, B\}$ be the unique partition of $\{1, \ldots, 2p\}$ into two sets such that its blocks do not contain any block of $\pi$ and $\eta$. Let us choose a pair of permutations $\alpha$ and $\beta$ such that $\alpha(\{1, 3, \ldots, 2p-1\}) = A = \beta(\{1, 3, \ldots, 2p-1\})$ and $\alpha(2k-1) = \beta(2k-1)$, for any $k \in \{1, \ldots, p\}$. Then the product in the right-hand side of the last equality writes down $\prod_{k=1}^p J_{i_{2k-1}, i_{2k}} J_{j_{2k-1}, j_{2k}}$. This term is equal to 1 whenever $i_{\alpha^{-1}(2k)} = j_{2k}$, for any $k \in \{1, \ldots, p\}$ and vanishes otherwise. Observe further that the number of cycles of $\alpha^{-1} \beta(2, 4, \ldots, 2p)$ is $\#\pi \vee \eta$. Therefore the sum is equal to $(2N)^{\#\pi \vee \eta}$ and $\varepsilon(\alpha^{-1} \beta) = (-1)^{n + \#\pi \vee \eta}$.

ii) This second scalar product is equal to $\varepsilon(\sigma) \prod_{k=1}^p J_{i_{\sigma^{-1}(2k-1)}, i_{\sigma^{-1}(2k)}}$. This equality implies the second assertion. \qed

For any $z \in \mathbb{C}$ and any $\pi, \eta \in \mathcal{M}(2p)$, we set
\[
G_{\pi, \eta}(z) = z^{\#(\pi \vee \eta)}.
\]

Thanks to Lemma 3.2, the matrix $G(-2N)$ is the Gram matrix we were looking for. This matrix is real and symmetric, we denote its pseudo-inverse by $W(-2N)$. We shall show later in Lemma 4.5 that for any $z \in \mathbb{C}$, the matrix $G(z)$ has a pseudo-inverse that we shall denote by $W(z)$. Equation 1 and Lemma 3.2 yield the following: for $I = (i_1, \ldots, i_{2p}), J = (j_1, \ldots, j_{2p}) \in \{1, \ldots, 2N\}^{2p}$,
\[
\int_{Sp(2N)} S_{i_1, j_1} S_{i_2, j_2} \cdots S_{i_{2p}, j_{2p}} dS = (-1)^p \sum_{\pi, \eta \in \mathcal{M}(2p)} \langle w_\pi, w_\eta \rangle W_{\pi, \eta}(-2N) \tag{5}
\]
\[
= \sum_{\pi, \eta \in \mathcal{M}(2p)} \varepsilon(\sigma_\pi \sigma_\eta)(-1)^{\text{inv}_2(\sigma_\pi, I) + \text{inv}_2(\sigma_\eta, J) + p} W_{\pi, \eta}(-2N) \tag{6}
\]

\footnote{We shall see an expression of the matrix $G(z)$ in terms of Jucys-Murphys elements, which shows that $G(z)$ is diagonalizable so that $G(z)$ always has a pseudo-inverse. What is more, this expression implies that the family of matrices $\{G(z), z \in \mathbb{C}\}$ is commutative.}
Let us write 4.1 Brauer algebra as $G$ and $\sigma \mod (2)$ for any measure $\mu$ that is invariant by adjunction, the endomorphism $\int_G g^* \mu (dg)$ of $V^{\otimes n}$ commutes with the action of $G$. We shall specify the element of $\text{End}_G(V^{\otimes n})$ when $\mu$ is the Haar measure or the law of the Brownian motion. To begin with, let us describe the algebra $\text{End}_G(V^{\otimes n})$ as $G$ belongs to one of the classical series of compact Lie groups.

4 Reformulation in the Brauer algebra.

For any measure $\mu$ that is invariant by adjunction, the endomorphism $\int_G g^* \mu (dg)$ of $V^{\otimes n}$ commutes with the action of $G$. We shall specify the element of $\text{End}_G(V^{\otimes n})$ when $\mu$ is the Haar measure or the law of the Brownian motion. To begin with, let us describe the algebra $\text{End}_G(V^{\otimes n})$ as $G$ belongs to one of the classical series of compact Lie groups.

4.1 Brauer algebra

Let us write $I_n$ for the vector space with a basis indexed by $\mathcal{M}(2n)$. We give $I_n$ the structure of an algebra as follows. For every $\pi \in \mathcal{M}(2n)$, let $\pi^+ = \{1, \ldots, \{n\}\} \cup (\pi + n)$ and $\pi^- = \pi \cup \{2n + 1, \ldots, 3n\}$ be partitions of $\{1, 2, \ldots, 3n\}$. For any $\pi, \eta$, we denote by $b(\pi, \eta)$ the number of blocks of $(\pi + \eta^-)$ included in $\{n + 1, 2n\}$ and $\pi \circ \eta \in \mathcal{M}(2n)$ the partition obtained from $\pi + \eta^- \cap (\{1, \ldots, n\} \cup \{2n + 1, \ldots, 3n\})$ by shifting by $\{2n + 1, \ldots, 3n\}$ to $\{n + 1, \ldots, 2n\}$. We define

$$\pi \eta = z^b(\pi, \eta) \pi \circ \eta.$$  

Lemma 4.1. [Thm 4.4] For any $z \in \mathbb{C}$, the bilinear map $(\pi, \eta) \in I_n^2 \mapsto \pi \eta$ endows $I_n$ with a structure of an associative unitary algebra. We denote this algebra $B_n(z)$.

For any $\pi \in \mathcal{M}(2n)$, we shall represent the element $\pi \in B_n(z)$ as a diagram with points of $\{1, \ldots, 2n\}$ set on two lines as in the figure [4] so that multiplication in $B_n(z)$ amounts to concatenation of diagrams weighted by $z$ powered to the number of inside loops. For any integers $1 \leq a < b \leq n$, set

$$\tau_{a,b} = \{\{a, b\}, \{a + n, b + n\}\} \cup \{\{k, k + n\}, k \neq a, b\}$$
and
\[ s_{a,b} = \{a, b + n\}, \{b, a + n\} \cup \{k, k + n\}, k \neq a, b \].

These two kinds of partitions generates \( B_n(z) \) as an algebra.

Lemma 4.2 ([5] §7). The algebra \( B_n(z) \) is isomorphic to the quotient of the free unital associative algebra on \( \{\tau_{a,b}, s_{a,b} : 1 \leq a < b \leq n\} \) by the two-sided ideal generated by the following relations: for distincts integers \( a, b, c, d \) between 1 and \( n \),

(i) \( \overline{\tau}_{a,b}^2 = z \overline{\tau}_{a,b} \),
(ii) \( \overline{s}_{a,b}^2 = 1 \),
(iii) \( \overline{s}_{a,b} \overline{\tau}_{a,b} = \overline{\tau}_{a,b} \overline{s}_{a,b} \),
(iv) \( \overline{\tau}_{a,b} \overline{\tau}_{c,d} = \overline{\tau}_{c,d} \overline{\tau}_{a,b} \),
(v) \( \overline{s}_{a,b} \overline{s}_{c,d} = \overline{s}_{c,d} \overline{s}_{a,b} \),
(vi) \( \overline{s}_{a,b} \overline{s}_{b,c} \overline{s}_{a,b} = \overline{s}_{b,c} \overline{s}_{a,b} \overline{s}_{b,c} \).

Figure 1: Multiplication \( \pi . \eta \) with \( \pi = \{\{1, 2\}, \{3, 4\}, \{5, 14\}, \{6, 7\}, \{8, 9\}, \{10, 11\}, \{12, 13\}\} \) and \( \eta = \{\{1, 3\}, \{25\}, \{4, 6\}, \{7, 12\}, \{8, 11\}, \{9, 10\}, \{13, 14\}\} \).

Figure 2: The two elements \( \tau_{a,b} \) et \( s_{a,b} \) of \( B_n(z) \).

Let us describe the standard representations of \( B_n(N) \) and \( B_n(-2N) \). For any \( \pi \in \mathcal{M}(2n) \), let us define the element of \( \text{End}(V^\otimes n) \)
\[ \rho_\pi(\pi) = \sum E_{i_{n+1}, i_1} \otimes \cdots \otimes E_{i_{2n}, i_n}, \]
the sum being over all \( i_1, \ldots, i_{2n} \in \{1, \ldots, N\} \) such that \( i_k = i_l \) for any \( k, l \) satisfying \( k \sim \pi l \).

For any \( 1 \leq a < b \leq n \), let us denote by \( (a b) = \rho_\pi(\tau_{a,b}) \) and \( (a b) = \rho_\pi(s_{a,b}) \).

Lemma 4.3. The application \( \rho_\pi \) extends linearly to an algebra homomorphism from \( B_n(N) \) to \( \text{End}(V^\otimes n) \).
Proof. One checks that the nine relations given in Lemma 4.2 are satisfied.

Lemma 4.4. The application $\rho_S$ extends to an algebra homomorphism from $B_n(-2N)$ to $\text{End}(V^\otimes n)$. 

Proof. One checks that the nine relations given in Lemma 4.2 are satisfied. \hfill $\square$

4.2 Expectation of tensors as elements of the Brauer algebra.

We will give here an expression for the orthogonal projection $\Phi$ of $V^\otimes 2p$ onto $(V^\otimes 2p)^{\text{Sp}(2N)}$ as the representation of an element of the Brauer algebra $B_{2p}(-2N)$. Let $\sigma(\pi_0) = \pi$ and for any $z \in \mathbb{C}$, set

$$G(z) = \sum_{\pi \in \mathcal{M}(2p)} z^\#(\pi \cup \pi_0) \sigma_\pi \in \mathbb{C}[\mathfrak{S}_{2p}].$$

Recall that $H_p$ is the subgroup of $\mathfrak{S}_{2p}$ fixing $\pi_0$, and let $\mathcal{P}_{H_p}$ denote the idempotent $\frac{1}{2p!} \sum_{h \in H_p} h$.

Lemma 4.5. i) For any $z \in \mathbb{C}$,

$$\mathcal{P}_{H_p} G(z) = G(z) \mathcal{P}_{H_p}.$$

ii) Let $R_{G(z)} : I_{2p} \rightarrow I_{2p}, \sigma \mapsto \sigma G(z) \sigma$. The endomorphism $R_{G(z)}$ is well defined, does not depend on the choice of $(\sigma_\pi)_{\pi \in \mathcal{M}(2p)}$ and its matrix in the canonical basis of $I_{2p}$ is $(G(\mu, \nu)(z))_{\mu, \nu \in \mathcal{M}(2n)}$.

Proof. i) Indeed, the element $G(z) \mathcal{P}_{H_p} = \frac{1}{2p!} \sum_{\sigma \in \mathfrak{S}_{2p}} z^\#(\pi_0 \cup \pi_0) \sigma$ is invariant by multiplication on the left and on the right by $H_p$.

ii) Thanks to the identity of point i), the homomorphism $R_{G(z)}$ is well define. Besides, it does not depend on the family $(\sigma_\pi)_{\pi \in \mathcal{M}(2p)}$. For any $\eta, \mu \in \mathcal{M}(2p)$, there is a unique $\pi \in \mathcal{M}(2p)$ with $\sigma_\eta \sigma_\pi \pi_0 = \mu$, what is more $\nu \cup \mu = \sigma_\eta(\pi_0 \cup \pi)$ and $\#(\pi \cup \pi_0) = \#(\mu \cup \nu)$. Hence

$$\sigma_\eta G(z) \pi_0 = \sum_{\pi \in \mathcal{M}(2p)} z^\#(\pi \cup \pi_0) \sigma_\eta \sigma_\pi \pi_0 = \sum_{\mu \in \mathcal{M}(2p)} G(z)_{\mu, \eta} \mu.$$

\hfill $\square$

For any integer $i \geq 2$, we let

$$X_i = (1 \; i) + (2 \; i) + \cdots + (i-1 \; i)$$

be the $i$-th Jucys-Murphy element. We set $X_1 = 0$. Let us point out the following fact proved in [11].

Proposition 4.6. [11, Proposition 3] There exists a family of permutations $(\sigma_\pi)_{\pi \in \mathcal{M}(2p)}$ such that $\sigma_\pi(\pi_0) = \pi$ for any $\pi \in \mathcal{M}(2p)$ and

$$G(z) = \prod_{k=1}^p (z + X_{2k-1}).$$
The Jucys-Murphy elements are jointly diagonalizable (see [10]) so that for any $z \in \mathbb{C}$, the element $G(z)$ has a pseudo-inverse $W(z)$ that is a polynomial of $G(z)$. Thus, the element $W(z)$ satisfies $W(z)P_{H_p} = P_{H_p}W(z)$ and it defines an endomorphism $R_{W(z)}$ which is a pseudo-inverse of $R_G(z)$. We shall denote by $W(z)$ its matrix in the canonical basis of $I_p$. Besides, observe that for any $z, z' \in \mathbb{C}$, the element $G(z)G(z') = G(z')G(z)$ induces the endomorphism $R_G(z)R_G(z')$. Hence, the family of matrices $\{G(z) : z \in \mathbb{C}\}$ is commutative.

We can now give another formulation of the results of the last section. For any $r, s, n \in \mathbb{N}$ such that $r \leq s \leq n$ and $s - r$ is odd, let us set $\tau_{[r,s]} = \prod_{1 \leq 2i+1 \leq s-r} \tau_{r+2i,r+2i+1} \in B_n(z)$.

**Proposition 4.7.** Let $I_{2p}(z) = \frac{1}{p!} \sum_{\sigma \in S_{2p}} \sigma W(z)\tau_{[1,2p]}^{-1}$,

$$\int_{Sp(2N)} S_{\otimes 2p} dS = \rho_S(I_{2p}(-2N))$$

and

$$\int_{O(N)} O_{\otimes 2p} dO = \rho_O(I_{2p}(N)).$$

**Proof.** Let us prove the first formula for the symplectic group. Let us recall that for any $\pi \in \mathcal{M}(2p)$, $w_\pi = \rho_S(\sigma_\pi)w_{\pi_0}$ and note that the following equality holds in $\text{End}(V_{\otimes 2p})$

$$\langle w_\eta, \cdot \rangle w_\pi = (-1)^p \langle \sigma_\pi \tau_{[1,2p]}^2 \eta^{-1} \rangle.$$

What is more, Lemma [3.2] implies

$$\langle w_\pi, w_\eta \rangle = (-1)^p G_{\pi,\eta}(-2N).$$

By definition of the matrix $W(z)$, $W(z)$ is a pseudo-inverse of $G(z)$ so that, thanks to formula [5], the left-hand-side of the formula of the Proposition equals to

$$\sum_{\pi, \eta \in \mathcal{M}(2p)} (-1)^p W_{\pi,\eta}(-2N) \langle w_\eta, \cdot \rangle w_\pi = (-1)^p \sum_{\eta \in \mathcal{M}(2p)} \langle w_\eta, \cdot \rangle \rho_S(\sigma_\eta W(-2N))w_{\pi_0}$$

$$= \sum_{\eta \in \mathcal{M}(2p)} \rho_S(\sigma_\eta W(-2N)\tau_{[1,2p]}^2 \eta^{-1}).$$

Recall that for any $z \in \mathbb{C}$, $W(z)P_{H_p} = P_{H_p}W(z)$ and note that $P_{H_p}\tau_{[1,2p]}P_{H_p} = \tau_{[1,2p]}$, so that the right-hand side of the last equation equals the right-hand side of the equation of the Proposition.

**4.3 Statement of the Theorem for orthogonal and symplectic matrices**

We shall now state our main result which extends Proposition 4.7 when the integration is not with respect to the Haar measure but with respect to the law of a Brownian motion at a fixed time. We shall denote $(O_t)_{t \geq 0}$ (resp. $(S_t)_{t \geq 0}$ and $(U_t)_{t \geq 0}$) the Brownian motion issued from the identity on $O(N)$ (resp. on $Sp(N)$ and on $U(N)$). Observe that almost surely the orthogonal Brownian motion takes its value in the connected component of the identity that is the special orthogonal group $SO(N) = \{O \in O(N) : \det(O) = 1\}$. 

...
For any \( C \subset \{1, \ldots, n\} \) and \( z \in \mathbb{C} \), let us define

\[
Z_C(z) = \frac{(1 - z^{-1})|C|}{2} + z^{-1} \sum_{a < b; \ a, b \in C} s_{a, b} \in B_n(z)
\]

and for \( 1 \leq i \leq n \),

\[Z_i = Z_{\{1, \ldots, i\}}(z).
\]

Let us set \( Z_0 = 0 \). For \( t \in \mathbb{R} \) and an integer \( k \geq 2 \), let \( s_t(z_1, \ldots, z_k) \) be the symmetric function defined by:

\[
s_t(z_1, \ldots, z_k) = \prod_{1 \leq i < j \leq k} (z_j - z_i)^{-1} \det \begin{pmatrix}
z_1^{k-2} & z_1^{k-3} & \cdots & 1 & e^{-t z_1} \\
z_2^{k-2} & z_2^{k-3} & \cdots & 1 & e^{-t z_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
z_k^{k-2} & z_k^{k-3} & \cdots & 1 & e^{-t z_k}
\end{pmatrix}
\]

For any integers \( r \) and \( k \), let us denote by

\[
h_{r}^{(k)}(z_1, \ldots, z_k) = \sum_{l_1, \ldots, l_k \geq 0 \atop l_1 + \ldots + l_k = r} z_1^{l_1} \ldots z_k^{l_k},
\]

the complete symmetric polynomial of degree \( r \). Let us recall the following classical fact that complete polynomials agree with Schur functions of the partitions \((r, 0)\).

**Lemma 4.8.** i) For any integers \( r \) and \( k \),

\[
h_r(z_1, \ldots, z_k) = \prod_{1 \leq i < j \leq k} (z_j - z_i)^{-1} \det \begin{pmatrix}
z_1^{r+k-1} & z_1^{k-2} & z_1^{k-3} & \cdots & 1 \\
z_2^{r+k-1} & z_2^{k-2} & z_2^{k-3} & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
z_k^{r+k-1} & z_k^{k-2} & z_k^{k-3} & \cdots & 1
\end{pmatrix}.
\]

ii) For any \( t \in \mathbb{R} \),

\[
s_t(z_1, \ldots, z_{k+1}) = (-1)^k \sum_{r \geq 0} \frac{(-t)^{r+k}}{(r+k)!} h_r(z_1, \ldots, z_{k+1}).
\]

iii) For any pairwise distinct non-zero complex numbers \( z_1, \ldots, z_l \),

\[
s_t(z_1, \ldots, z_k, 0) = \prod_{1 \leq i < j \leq k} (z_j - z_i)^{-1} \det \begin{pmatrix}
z_1^{k-2} & z_1^{k-3} & \cdots & 1 & z_1^{-1}(1 - e^{-t z_1}) \\
z_2^{k-2} & z_2^{k-3} & \cdots & 1 & z_2^{-1}(1 - e^{-t z_2}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
z_k^{k-2} & z_k^{k-3} & \cdots & 1 & z_k^{-1}(1 - e^{-t z_k})
\end{pmatrix}.
\]

**Proof.** i) A proof lies in [9].

ii) Decompose the exponential into its power series. Note that the first \( l - 1 \) terms vanish.

iii) The third point is left to the reader.

\(\square\)
As a corollary, observe that the function $s_t$ is well defined on $\mathbb{C}^k$ and holomorphic. For any integers $n$ and $k$, such that $0 \leq 2k \leq n$, let

$$I_{n,t}^k(z) = \frac{1}{2^k z^k (n - 2k)!} \sum_{\sigma \in \mathfrak{S}_n} \sigma s_t(Z_{n-2k}, Z_{n-2k+2}, \ldots, Z_n) \tau_{[n-2k+1, n]} \sigma^{-1}$$

and $I_{n,t}(z) = \sum_{0 \leq 2k \leq n} I_{n,t}^k(z)$. Our main result is the following pair of formulas.

**Theorem 4.9.** For any $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$,

$$\mathbb{E}[O_t^{\otimes n}] = \rho_O(I_{n,t}(N))$$

and

$$\mathbb{E}[S_t^{\otimes n}] = \rho_S(I_{n,t}(-2N)).$$

### 4.4 Walled Brauer algebra

For two integers $n$ and $m$, let $\mathcal{M}(n,m)$ be the set of pair partitions of $\{1, 2, \ldots, 2(n + m)\}$, which does not connect $\{1, \ldots, n\}$ with $\{2n + m + 1, \ldots, 2(n + m)\}$ nor $\{n + 1, \ldots, n + m\}$ with $\{n + m + 1, \ldots, 2n + m\}$. Let us denote by $\mathcal{B}_{n,m}(z)$ the vector space spanned by $\mathcal{M}(n,m)$.

Observe that $\mathcal{B}_{n,m}(z)$ is a subalgebra of $\mathcal{B}_{n+m}(z)$.

![Figure 3: An element of $\mathcal{B}_{n,m}(z)$.](image)

Let us denote by $\tilde{\pi}_0 \in \mathcal{M}(n,n)$ the partition $\{\{1, n+1\}, \ldots, \{n, 2n\}\}$. For any $\pi \in \mathcal{M}(n,n)$, let $\tilde{\sigma}_\pi \in \mathfrak{S}_n$ be the permutation such that $\tilde{\sigma}_\pi(\tilde{\pi}_0) = \pi$. For any permutation $\sigma \in \mathfrak{S}_n$, let us write $\# \sigma$ for the number of cycles of $\sigma$ and set

$$\Omega(z) = \sum_{\sigma \in \mathfrak{S}_n} z^{\# \sigma} \sigma \in \mathbb{C}[\mathfrak{S}_n].$$

For any $\pi, \eta \in \mathcal{M}(n,n)$, let us observe that $\# \sigma_\pi^{-1} \sigma_\eta = \#(\sigma_\pi^{-1} \sigma_\eta \times \text{Id}_n(\tilde{\pi}_0) \vee \tilde{\pi}_0) = \# \pi \vee \eta$, hence

$$G(z)_{\pi,\eta} = \Omega(z)(\sigma_\pi^{-1} \sigma_\eta).$$

The following formula was first proved in [7].

**Proposition 4.10.** [11, Prop. 1] For any $z \in \mathbb{C}$,

$$\Omega(z) = \prod_{i=1}^n (z + X_i).$$
Proposition 4.11. Let $\tau_{k}$ be such that $1 \leq k \leq n \leq m$, let
\[
\tilde{\tau}_{[r,s]} = \prod_{s \leq k \leq r} \tau_{k,k+n}.
\]
We can now give another formulation of expression (8).

Theorem 4.9. For any nonnegative integers $n$, $m$, $s$, $r$ such that $1 \leq r \leq s \leq n \leq m$, let
\[
\Omega(z) = \int_{U(N)} U_{[r,s]} \otimes U_{[0,0]} dU = \rho_{O}(I_{n,n}(N)).
\]

4.5 Statement of the Theorem for unitary matrices

We shall state the counterpart of Proposition 4.11 for the Brownian motion. Let $n$, $m$ be two integers such that $n \leq m$. Let us denote by $p^{w}$ the projection from $B_{n+m}(z)$ to $B_{n,m}(z)$ such that $p^{w}(\pi) = 0$ for any $\pi \in M(2(n + m)) \setminus M(n, m)$. Set for any $C \subset \{1, \ldots, n + m\}$, $Y_{C} = p^{w}(Z_{C}) + \frac{|C|^{-1}}{2} \in B_{n,m}(z)$ and for any $1 \leq i \leq n$,
\[
Y_{i} = Y_{\{1, \ldots, i\} \cup (n+1, \ldots, m+i)}.
\]
Let $Y_{0}$ be the element $Y_{\{n+1, \ldots, m\}}$ if $n < m$, and zero if $n = m$. For any integer $k$ such that $0 \leq k \leq n$, let
\[
\Omega_{n,m,t}(z) = \frac{1}{z^{k(n-k)!}} \sum_{\sigma \in B_{n} \times B_{m}} \sigma_{t}(Y_{n-k}, Y_{n-k+1}, \ldots, Y_{n}) \tilde{\tau}_{\{n-k+1, n\}} \sigma^{-1}
\]
and $I_{n,m,t}(z) = \sum_{k=0}^{n} \Omega_{n,m,t}(z)$. The formula of Proposition 4.11 has the following deformation.

Theorem 4.9. For any nonnegative integers $n \leq m$ and $t \geq 0$,
\[
\mathbb{E}[U_{t}^{\otimes n} \otimes U_{t}^{\otimes m}] = \rho_{O}(I_{n,m,t}(N)).
\]

4.6 Limit as time goes to infinity

For any compact connected Lie group, the law at a time $t$ of the Brownian motion issued from the identity converges to the Haar measure as $t$ goes to infinity. In this section, we shall deduce formulas of integration against the Haar measure from formulas for the Brownian motion obtained in Theorem 4.9. Note that $O(N)$ is not connected and that the law of a Brownian motion at time $t$ issued from the identity converges to the Haar measure on $SO(N)$ as $t$ goes to infinity. If $A$ is any matrix of $O(N) \setminus SO(N)$, let $(g_{t})_{t \geq 0}$ a Brownian motion on $O(N)$ whose initial condition $g_{0}$ is equal to $A$ or $Id$ with probability $\frac{1}{2}$. The law of $g_{t}$ converges to the Haar measure as $t \rightarrow \infty$. Using the formulas of B. Collins and P. Śniady as quoted in Propositions 4.7 and 4.11 the convergence of the law of Brownian motion towards the Haar measure yields the first assertion of the following lemma.
Lemma 4.12. i) Let \( n, m \) and \( N \) be fixed integers with \( n \leq m \). Let us set \( \mathcal{I}_n(z) = 0 \) if \( n \) is odd and \( \mathcal{I}_{n,m}(z) = 0 \) if \( n < m \). Let \( A \in SO(N) \) such that \( \det(A) = -1 \). As \( t \) goes to infinity,

\[
\frac{1}{2} \rho_O(\mathcal{I}_{n,t}(N))(1 + A^\otimes n) \rightarrow \rho_O(\mathcal{I}_n(N)),
\]

\[
\rho_S(\mathcal{I}_{n,t}(-2N)) \rightarrow \rho_S(\mathcal{I}_{n}(-2N))
\]

and

\[
\rho_O(\mathcal{I}_{n,m,t}(N)) \rightarrow \rho_O(\mathcal{I}_{n,m}(N)).
\]

ii) If \( n - N \) is a non-negative even integer, let

\[
\mathcal{I}'_n(N) = \frac{1}{(2N)^{\frac{n}{2}}N!^2} \sum_{\sigma \in S_n} \sigma \tau_{n+1,n+1} \varepsilon_N Z_{N+2}(N)^{-1} \cdots Z_{n-2}(N)^{-1} Z_n(N)^{-1} \sigma^{-1}.
\]

Otherwise, set \( \mathcal{I}'_n(N) = 0 \). For any integers \( n \) and \( N \), as \( t \to \infty \),

\[
\rho_O(\mathcal{I}_{n,t}(N)) \to \rho_O(\mathcal{I}_n(N) + \mathcal{I}'_n(N)).
\]

We shall give here a direct proof of that lemma without using the Propositions 4.7 and 4.11 and thus get another proof of the formulas for the Haar measure. Besides, this Lemma yields an expression for integration over special orthogonal group. For any vector space \( E \) that is acted on by \( O(N) \) and that is endowed with an invariant scalar product, let us denote by \( E^{SO(N),\varepsilon} \) the orthogonal complement of \( E^{O(N)} \) in \( E^{SO(N)} \). The Lemma implies that the orthogonal projection on \( (V \otimes n)^{SO(N)} \) (resp. \( (V \otimes n)^{O(N)} \)) is \( \rho_O(\mathcal{I}_n(N) + \mathcal{I}'_n(N)) \) (resp. \( \rho_O(\mathcal{I}_n(N)) \)), hence the orthogonal projection on \( (V \otimes n)^{SO(N),\varepsilon} \) is \( \rho_O(\mathcal{I}'_n(N)) \). Furthermore,

\[
\int_{SO(N)} O^{\otimes n} dO = \int_{O(N)} O^{\otimes n}(1 + \det(O))dO
\]

and the orthogonal projection on \( (V \otimes n)^{SO(N),\varepsilon} \) is therefore

\[
\int_{O(N)} O^{\otimes n} \det(O)dO = \rho_O(\mathcal{I}'_n(N)). \tag{11}
\]

Sketch of a proof of Lemma 4.12: Let us first give a sketch of a proof for the orthogonal group \( O(N) \), as \( N \) is large and \( n \) is an even integer \( 2p \). In that case, the elements \( Z_i(N) - Z_j(N) \), for \( 1 \leq i < j \leq 2p \) have inverses in the group algebra \( \mathbb{C}[S_{2p}] \) and for any \( i \in \{1, \ldots, 2p\} \), the spectrum of \( \rho_O(Z_i(N)) \) is positive. Therefore, as \( t \) goes to infinity, for any \( i \in \{1, \ldots, 2p\} \), \( \rho_O(e^{-tZ_i(N)}) \to 0 \). Recall the definition of \( s_t \) given in (9), for any \( k \in \{1, \ldots, p\} \), \( \rho_O(s_t(Z_{2k}(N), Z_{2k+2}(N), \ldots, Z_{2p}(N))) \to 0 \) and \( \rho_O(s_t(0, Z_2, Z_4, \ldots, Z_{2p})) \to \rho_O(Z_2(N)^{-1} Z_4(N)^{-1} \cdots Z_{2p}(N)^{-1}) \). Using theorem 4.9 we get the following asymptotic formula: as \( t \) goes to infinity,

\[
\rho_O(\mathcal{I}_{2p,t}(N)) \to \frac{1}{2^{2p}} \sum_{\sigma \in S_{2p}} \rho_O(\sigma N^{-p} Z_2(N)^{-1} Z_4(N)^{-1} \cdots Z_{2p}(N)^{-1} \tau_{1,2p} \sigma^{-1}).
\]

We shall prove that this formula agrees with Proposition 4.7. Let \( \mathcal{P}_{2p} \) denote the sum \( \frac{1}{2^{2p}} \sum_{h \in \mathcal{H}_{2p}} h \), this relation follows from the following equality.
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\[ p!P_{H_2p}N^{-p}Z_2^{-1}(N)Z_4^{-1}(N) \cdots Z_{2p}^{-1}(N)P_{H_2p} = G(N)^{-1}P_{H_2p}. \]  

For \( N \) large enough, for any \( i \in \{1, \ldots, p\} \), \( N + X_{2i-1} \) have an inverse in \( \mathbb{C}[S_{2p}] \) and using the factorization formula of Proposition 4.6, this equality can be proved inductively (see Lemma 4.17 below).

Let us now consider the formulas of Theorem 4.9 for all \( N \). We shall use the decomposition of the \( \mathfrak{S}_n \)-module \( V^\otimes n \) into irreducible components and the explicit knowledge of the spectrum of Jucys-Murphy elements. Let us recall some classical notations. A partition of an integer \( n \) is a non-increasing sequence of integers \( \lambda = (\lambda_1, \lambda_2, \ldots) \) such that \( \lambda_1 + \lambda_2 + \cdots = n \). We shall write \( \lambda \vdash n \). If \( \lambda \) is a partition, we denote the number of its non-zero terms by \( l(\lambda) \). The integer \( l(\lambda) \) is called the length of the partition \( \lambda \). Recall that irreducible representations of \( \mathfrak{S}_n \) are indexed by partitions of the integer \( n \), whereas finite dimensional irreducible representations of \( \text{GL}_N(\mathbb{C}) \) are indexed by partitions of length less than \( N \). For any partition \( \lambda \) such that \( \lambda \vdash n \) (resp. \( l(\lambda) \leq N \)), let \((\rho^\lambda, V^\lambda)\) (resp. \((\rho_\lambda, V_\lambda)\)) denote the irreducible representation of \( \mathfrak{S}_n \) (resp. \( \text{GL}_N(\mathbb{C}) \)) indexed by \( \lambda \). We need the following Theorem due to Hermann Weyl:

**Theorem 4.13.** [3 Thm 9.1.2] The \( \text{GL}_N(\mathbb{C}) \times \mathfrak{S}_n \)-module \( V^\otimes n \) is multiplicity-free and is isomorphic to

\[ \bigoplus_{\lambda \vdash n, l(\lambda) \leq N} V_\lambda \otimes V^\lambda. \]

Let us now describe the spectrum of Jucys-Murphy elements. For any partition \( \lambda \), let \( D_\lambda = \{(i, j) \in \mathbb{N}^{\times 2} : j \leq \lambda_i \} \). The subset \( D_\lambda \) is called the diagram of \( \lambda \). For any \( (i, j) \in D_\lambda \), the integer \( c((i, j)) = j - i \) is called the content of \( (i, j) \). A tableau of a partition \( \lambda \vdash n \) is a bijection \( T : D_\lambda \to \{1, \ldots, n\} \) such that \( T \) is increasing in each variable. For any partition \( \lambda \), let us write \( T_\lambda \) the set of tableaux of the partition \( \lambda \), \( T_n = \bigcup_{\lambda \vdash n} T_\lambda \) and \( T_{n,N} = \bigcup_{\lambda \vdash n, l(\lambda) \leq N} T_\lambda \).

**Theorem 4.14 ([10]).** i) For any partition \( \lambda \vdash n \), there exists a basis \( (e_T)_{T \in T_\lambda} \) of \( V^\lambda \) that diagonalizes the family \( (\rho^\lambda(X_1), \rho^\lambda(X_2), \ldots, \rho^\lambda(X_n)) \) such that for any tableau \( T \), \( e_T \) has eigenvalue \( (c(T^{-1}(1)), c(T^{-1}(2)), \ldots, c(T^{-1}(n))) \).

For any partition \( \lambda \vdash n \), let \((P_T)_{T \in T_\lambda} \) denote the elements of the group algebra \( \mathbb{C}[\mathfrak{S}_n] \) whose image via the canonical isomorphism \( \mathbb{C}[\mathfrak{S}_n] = \bigoplus_{\lambda \vdash n} \text{End}(V_\lambda) \) is the family of projections corresponding to the basis \( (e_T)_{T \in T_\lambda} \). Let us denote by \( P_\lambda = \sum_{T \in T_\lambda} P_T \) the projection on the isotypic component \( \lambda \). Let \( \lambda' \) be the partition such that \( D_{\lambda'} = \{(i, j) \in \mathbb{N}^{\times 2} : (j, i) \in D_\lambda \} \) and for any tableau \( T \in T_\lambda \), let \( T' \in T_{\lambda'} \) be the tableau symmetric to \( T \). For any integer \( l \), let us denote by \( \varepsilon_l \in \mathbb{C}[\mathfrak{S}_l] \) the signature of \( \mathfrak{S}_l \). The isotypic projection \( P_{(\lambda l)} \) is equal to \( \frac{1}{\varepsilon_l} P_\lambda \).

For any \( \sigma \in \mathfrak{S}_n \), let \( \sigma^\varepsilon \) denote the element of the group algebra \( \varepsilon(\sigma) \sigma \in \mathbb{C}[\mathfrak{S}_n] \) and extend linearly this definition to \( \mathbb{C}[\mathfrak{S}_n] \). Note that for any \( i \in \{1, \ldots, n\} \), \( X_i^\varepsilon = -X_i \) and that for any \( \lambda \vdash n \), \( P_\lambda = P_{\lambda^\varepsilon} \). Observe further that for any permutation \( \sigma \), \( \rho_S(\sigma) = \rho_0(\sigma^\varepsilon) \). We can now describe the spectrum of the elements contributing to the formula of Theorem 4.9. For any self-adjoint operators \( L \) and \( J \), let us write \( L \geq J \) whenever the eigenvalues of \( L - J \) are non-negative.

**Lemma 4.15.** Let \( N \) be an integer greater than 1. For any \( i \in \{1, \ldots, n\} \), \( \rho_0(Y_i(N)) \geq \text{Id} \) and \( \rho_S(Z_i(-2N)) \geq \text{Id} \). For any \( i \in \{1, \ldots, n\} \setminus \{N\} \), \( \rho_O(Z_i(N)) \geq \frac{N-1}{2N} \). For any \( n \geq N \), \( \rho_O(Z(N-N)) \geq \rho_0(1 - \frac{1}{N}\varepsilon_N) \) whereas \( \varepsilon_N Z_N(N) = 0 \).

\(^3\)Besides, it is also proved in [10] that \( P_T \) is a polynomial in \( X_1, X_2, \ldots, X_n \). We shall not use this fact.
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Proof. For any complex \( z \), let us write

\[
z Z_i(z) = \sum_{k=1}^{i} \left( \frac{z-1}{2} + X_k \right)
\]

and

\[
z Y_i(z) = \sum_{k=1}^{i} \left( \frac{z}{2} + X_k \right) + \alpha \sum_{k=1}^{m-n+i} \left( \frac{z}{2} + X_k \right) \alpha^{-1},
\]

where \( \alpha \) is a permutation of \( \{1, \ldots, m+n\} \) such that for any \( k \leq m \), \( \alpha(k) = k + n \). Observe that the two terms of the right-hand side of the last equality commute and can be jointly diagonalized in a basis indexed by pairs of tableaux of partitions of length less than \( N \). For any \( i \in \{1, \ldots, n\} \), let \( d, r \) (resp. \( r', d' \)) be two integers such that \( r < 2N \) and \( i = 2Nd + r \) (resp. \( r' < N \) and \( i = Nd' + r' \)). The smallest eigenvalue of \( \rho_0(Z_i(-2N)) \) (resp. \( \rho_0(Z_i(N)) \)) is the one indexed by the tableaux \( T \in \mathcal{T}_{n,2N} \) (resp. \( \mathcal{T}_{n,N} \)) such that \( T^{-1}(\{1, \ldots, i\})' = D_{(2N,i)} \) (resp. \( D_{(N',r')} \)) that is \( \frac{d(d+1)}{2} + \frac{(2N+2d-r)r}{4N} \) (resp. \( \frac{d'(d'-1)}{2} + \frac{(2d+N-r')r'}{2N} \)). Note that all this eigenvalues are larger than 1 (resp. \( \frac{1}{2} - \frac{1}{2N} \)), except the one of \( \rho_0(Z_N(N)) \), which is zero. The second eigenvalue of \( \rho_0(Z_N(N)) \) is indexed by the tableaux \( T \in \mathcal{T}_{n,N} \) such that \( T^{-1}(\{1, \ldots, N\}) = D_{(2,1)} \) and equals 1. To conclude the proof for the orthogonal group, recall the following fact proved in [10]: for any \( N \leq n \), if \( \mathfrak{S}_N \) is canonically embedded into \( \mathfrak{S}_n \), for any \( \mu \vdash N \),

\[
\mathcal{P}_\mu = \sum_{T \in \mathcal{T}_n : T^{-1}(\{1, \ldots, N\}) = D_\mu} \mathcal{P}_T \in \mathbb{C}[\mathfrak{S}_n]. \tag{13}
\]

The above analysis of the spectrum of \( \rho_0(Z_N(N)) \) and equality (13) yield the assertion. Let us check the unitary case. For any \( i \in \{1, \ldots, n\} \), let \( d, r, d', r' \), the integers such that \( r, r' < N \), \( i = dN + r \) and \( i + m - n = Nd' + r' \). The smallest eigenvalue of \( Y_i(N) \) is indexed by pairs of tableaux \( (T_1, T_2) \in \mathcal{T}_{n,N} \times \mathcal{T}_{m,N} \), such that \( T_1^{-1}(\{1, \ldots, i\})' = D_{(N',r)} \) and \( T_2^{-1}(\{1, \ldots, i\})' = D_{(N',r')} \), that is \( \frac{d(d+1)}{2} + \frac{(2d+N-r-1)+r'(2d+N-r'-1)}{2N} \). For any \( i \in \{1, \ldots, n\} \), the spectrum of \( \rho_0(Y_i(N)) \) satisfies the above inequality.  

To complete the proof of Lemma 4.12, we shall check that for any \( x \in \mathbb{R}^n_+ \), \( s_t(x) \rightarrow 0 \) and \( g_t(x) \rightarrow \prod_{i=1}^{n} x_i^{-1} \). Let us denote \( D = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \} \) and \( P = \bigcup_{1 \leq i < j \leq n} \{ z \in \mathbb{C}^n : z_i = z_j \} \). Note that \( s_t(z) \rightarrow 0 \) and \( g_t(z) \rightarrow \prod_{i=1}^{n} z_i^{-1} \) uniformly on any compact subset of \( D^n \setminus P \). For any \( z \in D^n \cap P \) with \( z_i = z_j \), using Cauchy formula in an affine plane including \( z \) tranverse to \( P \), implies that the convergence still holds for \( z \). Let us give another proof. To that purpose, we shall decompose \( s_t \) as a sum of symmetric polynomials as in Lemma 4.8.

Lemma 4.16. i) For any pairwise distinct complex numbers \( z_1, \ldots, z_k \),

\[
h_t^{(s+k-1)}(z_1, \ldots, z_k) = \frac{1}{(s-1)!} \frac{\partial^{s-1}}{\partial z_1^{s-1}} \frac{1}{z_1^{s-1}} h_t(z_1, z_2, \ldots, z_k).
\]

ii) For any \( z \in \mathbb{C}^k \), \( s_t^{(s+k-1)}(z_1, \ldots, z_k) \) is equal to
\[
\frac{1}{(s-1)!} \frac{\partial^{s-1}}{\partial z_1^{s-1} z_1^{s-1}} \prod_{1 \leq i < j \leq k} (z_j - z_i)^{-1} \det \begin{pmatrix}
  z_{k-2}^{k-2} & z_{k-3}^{k-3} & \cdots & 1 \\
  z_{k-2}^{k-3} & z_{k-3}^{k-3} & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  z_{k-2}^k & z_{k-3}^k & \cdots & 1 \\
\end{pmatrix} 
\]

and \( s_t^{(s+k)}(z_1, \ldots, z_1, z_2, \ldots, z_k, 0) \) equals

\[
\frac{1}{(s-1)!} \frac{\partial^{s-1}}{\partial z_1^{s-1} z_1^{s-1}} \prod_{1 \leq i < j \leq k} (z_j - z_i)^{-1} \det \begin{pmatrix}
  z_{k-2}^{k-2} & z_{k-3}^{k-3} & \cdots & 1 \\
  z_{k-2}^{k-3} & z_{k-3}^{k-3} & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  z_{k-2}^k & z_{k-3}^k & \cdots & 1 \\
\end{pmatrix} 
\]

iii) For any \( x \in \mathbb{R}_{+}^k \), as \( t \to \infty \),

\[
s_t(x_1, \ldots, x_k) \to 0
\]

and

\[
s_t(x_1, \ldots, x_k, 0) \to \prod_{i=1}^k x_i^{-1}.
\]

Proof. i) Indeed,

\[
h_t^{(s+k-1)}(z_1, \ldots, z_1, z_2, \ldots, z_k) = \sum_{l_1, l_2, \ldots, l_k \geq 0, \sum_{l_1+\cdots+l_k=r}} \binom{l_1+s-1}{s-1} z_1^{l_1} z_2^{l_2} \cdots z_k^{l_k} = \frac{1}{(s-1)!} \frac{\partial^{s-1}}{\partial z_1^{s-1} z_1^{s-1}} h_r^{(k)}(z_1, z_2, \ldots, z_k).
\]

ii) The decomposition of \( s_t \) as a sum of complete symmetric polynomials given in ii) of Lemma 4.8 and the identity for \( h_r \) proved in i) imply that \( s_t^{(s+k-1)}(z_1, \ldots, z_1, z_2, \ldots, z_k) \) is equal to

\[
\sum_{l \geq k+s-1} \frac{(-t)^l}{l!} \frac{1}{(s-1)!} \frac{\partial^{s-1}}{\partial z_1^{s-1} z_1^{s-1}} \prod_{1 \leq i < j \leq k} (z_j - z_i)^{-1} \det \begin{pmatrix}
  z_{k-2}^{l-k-2} & z_{k-3}^{l-k-3} & \cdots & 1 \\
  z_{k-2}^{l-k-3} & z_{k-3}^{l-k-3} & \cdots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  z_{k-2}^{l-k} & z_{k-3}^{l-k} & \cdots & 1 \\
\end{pmatrix} 
\]

Note that the terms in the above summand are zero for \( s-1 \leq l \leq k+s-2 \). We claim that they are also zero for \( 0 \leq l \leq s-2 \), so that the formula follows. Indeed, for \( 0 \leq l \leq s-2 \), the function derived in the summand is equal to \( z_1^{l-k-1} \cdots z_k^{l-k-1} s_{(s-l-2)^{k-1}, 0}(z_1, \ldots, z_k) \), where \( s_{(j^k, 0)}(z_1, \ldots, z_k) \) is the Schur function associated to the diagram \((j^{k-1}, 0)\). Recall that \( s_{(j^k, 0)}(z_1, \ldots, z_k) = (z_1 \cdots z_k)^j \sum_{i=1}^k z_i^{-j} \). Hence, for \( l \leq s-2 \),

\[
\frac{\partial^{s-1}}{\partial z_1^{s-1} z_1^{s-1} \cdots z_k^{s-1}} s_{(s-l-2)^{k-1}, 0}(z_1, \ldots, z_k) = 0.
\]
The same expansion yields the second formula.

iii) Thanks to the assertion ii), \( s_t(x_1, \ldots, x_k) \) and \( s_t(x_1, \ldots, x_k, 0) - \prod_{i=1}^{k} x_i^{-1} \) are rational functions in \( (z, t) \) multiplied by a polynomial in \((e^{-t x_1}, \ldots, e^{-t x_k})\) of valuation greater than 1. This fact yields the two claims. \( \square \)

Let us make a slight abuse of notation by denoting for any integer \( i \) by \( Z_i^{-1}(z) \), the element of \( \mathbb{C} [\mathfrak{S}_n] \) that is the pseudo-inverse of \( Z_i(z) \), such that for any partition \( \lambda \vdash n \), \( \rho^\lambda(Z_i(z)^{-1}) \) is diagonal in the basis \((e_T)_{T \in T_n} \). Let us set for any even integer \( n \),

\[
\tilde{\mathcal{I}}_n(z) = (2z)^{-\frac{n}{2}} \sum_{\sigma \in \mathfrak{S}_n} \sigma \tau_{[1,n]} Z_2(z)^{-1} Z_4(z)^{-1} \cdots Z_n(z)^{-1} \sigma^{-1} \tag{14}
\]

and \( \mathcal{I}_n(z) = 0 \), for any odd \( n \). Set, as well, for any integers \( n, m \),

\[
\tilde{\mathcal{I}}_{n,m} = z^{-m} \sum_{\sigma \in \mathfrak{S}_m \times \mathfrak{S}_n} \rho_O(\sigma \tau_{[1,m]} Y_1(z)^{-1} Y_2(z)^{-1} \cdots Y_m(z)^{-1} \sigma^{-1}), \tag{15}
\]

if \( n = m \), and 0 otherwise. Lemma \([4.3]\) combined with the Lemma \([4.15]\) yields, as \( t \to \infty \),

\[
\rho_S(\mathcal{I}_{n,t}(-2N)) \to \rho_S(\tilde{\mathcal{I}}_n(-2N)) \tag{16}
\]

and

\[
\rho_O(\mathcal{I}_{n,m,t}(N)) \to \rho_O(\tilde{\mathcal{I}}_{n,m}(N)). \tag{17}
\]

Let us now consider the element associated to the Brownian motion on \( \text{SO}(N) \). We shall care about the non-invertible element \( \rho_O(Z_N(N)) \). Note that whenever \( k \leq N - 2 \) is such that \( n - k \) is even, \( \tau_{[k+1,n]} \varepsilon_N = 0 \). Therefore, the Lemmas \([4.15]\) and \([4.8]\) yield that for any \( p \neq \frac{n-N}{2} \), with \( 2p < n \), \( \rho_O(\mathcal{I}_{n,t}^p(N)) \to 0 \), and if \( n = 2p \),

\[
\rho_O(\mathcal{I}_{n,t}^p(N)) \to (2N)^{-p} \sum_{\sigma \in \mathfrak{S}_n} \rho_O(\sigma \tau_{[1,n]}[1 - \frac{1}{N!} \varepsilon_N] Z_2(N)^{-1} Z_4(N)^{-1} \cdots Z_n(N)^{-1} \sigma^{-1}). \tag{18}
\]

Besides, if \( n - N = 2p \), then \( N^p N! \rho_O(\mathcal{I}_{n,t}^p(N)) \) equals to

\[
\sum_{\sigma \in \mathfrak{S}_n} \sigma \left( s_t(0, Z_{N+2}, \ldots, Z_{N+2p}) \frac{1}{N!} \varepsilon_N + s_t(Z_N, Z_{N+2}, \ldots, Z_{N+2p}) \frac{1}{N!} \varepsilon_N \right) \tau_{[N+1,N+2p]} \sigma^{-1}.
\]

Hence, using the notation of Lemma \([4.12]\) as \( t \) goes to infinity,

\[
\rho_O(\mathcal{I}_{n,t}(N)) \to \rho_O \left( \tilde{\mathcal{I}}_n(N) + \mathcal{I}_n'(N) \right). \tag{19}
\]

Let \( A \in O(N) \setminus \text{SO}(N) \), note that \( A^\otimes n \rho_O(\tilde{\mathcal{I}}_n(N)) = \rho_O(\tilde{\mathcal{I}}_n(N)) \) and \( A^\otimes n \rho_O(\mathcal{I}_n'(N)) = -\rho_O(\mathcal{I}_n'(N)) \). Therefore, as \( t \) goes to infinity, \( \frac{1}{2}(1 + A^\otimes n) \rho_O(\mathcal{I}_{n,t}(N)) \to \rho_O(\tilde{\mathcal{I}}_n(N)). \tag{20} \)

To complete the proof of the formulas, we shall now show that relation \([12]\) holds true for all integers \( N \), as well as its analog for the elements \( Y_1(N), \ldots, Y_i(N) \). We shall prove that \( \tilde{\mathcal{I}}_n(z) = \mathcal{I}_n(z) \) and \( \tilde{\mathcal{I}}_{n,m}(z) = \mathcal{I}_{n,m}(z) \), for any \( z \in \mathbb{C} \). For any finite group \( K \), let us write
Lemma 4.17. i) For any integer \(i \geq 1\),
\[
(X_1 + X_2 + \cdots + X_{2i}) \mathcal{P}_{H_{2i}} = \mathcal{P}_{H_{2i}} i(1 + X_{2i-1}) \mathcal{P}_{H_{2i}}
\]
and
\[
\sum_{1 \leq a < b \leq i \text{ or } i < a < b \leq 2i} (a \ b) \mathcal{P}_{D(\mathfrak{S}_i)} = \mathcal{P}_{D(\mathfrak{S}_i)} iX_i \mathcal{P}_{D(\mathfrak{S}_i)}.
\]

ii) Let us assume that \(G(z)\) and \(Z_2(z), Z_4(z), \ldots, Z_{2p}(z)\) have inverses in \(\mathbb{C}[\mathfrak{S}_p]\). Then
\[
z^p G(z)^{-1} \mathcal{P}_{H_{2p}} = p! \mathcal{P}_{H_{2p}} Z_2^{-1} Z_4^{-1} \cdots Z_{2p}^{-1}(z) \mathcal{P}_{H_{2p}}.
\]
Besides, if \(\Omega(z)\) and \(Y_1(z), Y_2(z), \ldots, Y_p(z)\) have inverses in \(\mathbb{C}[\mathfrak{S}_p]^2\), then
\[
z^p \Omega(z)^{-1} \mathcal{P}_{D(\mathfrak{S}_p)} = p! \mathcal{P}_{D(\mathfrak{S}_p)} Y_1^{-1} Y_2^{-1} \cdots Y_p^{-1}(z) \mathcal{P}_{D(\mathfrak{S}_p)}.
\]

iii) For any \(\lambda \vdash p\), set \(R_\lambda = \prod_{(i,j) \in \lambda} (z + j - i)^{-1}, R_{2,\lambda} = \prod_{(i,j) \in \lambda} (z + 2j - i)\),
\[
\mathbf{W}(z) = \sum_{\lambda \vdash p} R_{2,\lambda} \mathcal{P}_{2\lambda}
\]
and
\[
\mathbf{W}_g(z) = \sum_{\lambda \vdash p} R^{-1}_\lambda \mathcal{P}_{\lambda}.
\]

The rational fonctions \(\mathbf{W}(z) \mathcal{P}_{H_p}\) and \(\mathbf{W}_g(z)\) are respectively pseudo-inverses of \(G(z) \mathcal{P}_{H_p}\) and \(\Omega(z)\) and
\[
p!(-2N)^{-p} \rho_S \left( \mathcal{P}_{H_{2p}} Z_2^{-1}(-2N) Z_4^{-1}(-2N) \cdots Z_{2p}^{-1}(-2N) \right) \mathcal{P}_{H_{2p}} = \rho_S(\mathbf{W}(-2N) \mathcal{P}_{H_{2p}})
\]
\[
p!N^{-p} \rho_O \left( Y_1^{-1}(N) Y_2^{-1}(N) \cdots Y_p^{-1}(N) \right) \mathcal{P}_{D(\mathfrak{S}_p)} = \rho_O(\mathbf{W}_g(N) \mathcal{P}_{D(\mathfrak{S}_p)})
\]
and
\[
p!N^{-p} \rho_O \left( \mathcal{P}_{H_{2p}} Z_2^{-1}(N) Z_4^{-1}(N) \cdots Z_{2p}^{-1}(N) \right) \mathcal{P}_{H_{2p}} = \rho_O(\mathbf{W}(N) \mathcal{P}_{H_{2p}}).
\]

Proof. i) Let us write the left-hand side of the first formula as \(\sum_{1 \leq a < b \leq 2i} (a \ b) \mathcal{P}_{H_{2i}}\). Observe that the transpositions \((2k-1 \ 2k)\) for \(1 \leq k \leq i\) belong to \(\mathcal{H}_{2i}\) and that the other transpositions with support in \(\{1, \ldots, 2i\}\) are conjugated to \((2i-2 \ 2i-1)\) by an element of \(\mathcal{H}_{2i}\). As \(X_1 + \cdots + X_{2i} = \sum_{1 \leq a < b \leq 2i} (a \ b)\) is in the center of \(\mathfrak{S}_{2i}\), we get the first equality conjugating by elements of \(\mathcal{H}_{2i}\) on both sides of the equation. A similar proof with \(D(\mathfrak{S}_i)\) in place of \(\mathcal{H}_{2i}\) yields the second formula.

ii) Let us prove this assertion inductively on \(p\). For \(p = 1\), the assertion is equivalent to \(zZ_2(z) \mathcal{P}_{H_2} = z \mathcal{P}_{H_2}\). Recall that \(zZ_2 = z - 1 + (1 \ 2)\), so assertion holds true for \(p = 1\). Assume
proved in Proposition 2.6 and 2.8 of [8].

Let us recall that the element $z$ is invertible in $\mathbb{C}[S_{2p+2}]$, for any $i \in \{1, \ldots, p+2\}$, $z + X_{2i-1}$ has an inverse. Using that $Z_{2p+2}$ is in the center of $\mathbb{C}[S_{2p+2}]$, the assumption yields

$$(p + 1)!z^{-p-1}P_{H_{2p+2}} Z_{2}^{-1} Z_{4}^{-1} \cdots Z_{2p+2}^{-1} P_{H_{2p+2}} = (p + 1)P_{H_{2p+2}} \prod_{i=1}^{p}(z + X_{2i-1})^{-1} z^{-1} Z_{2p+2}^{-1} P_{H_{2p+2}}$$

Recall that $zZ_{2p+2}(z) = (z - 1)(p + 1) + X_{1} + \cdots + X_{2p+2}$. The point i) implies the relation $(p + 1)P_{H_{2p+2}}(z + X_{2p+1}) P_{H_{2p+2}} = zZ_{2p+2}(z) P_{H_{2p+2}}$. What is more, thanks to Lemma 4.5, $P_{H_{2p+2}}G(z)^{-1} = G(z)^{-1} P_{H_{2p+2}}$. These two facts yield the formula for $p + 1$. Using that $Y_{\rho}$ and $\Omega(z)$ commute with $D(S_{2p})$, the same proof holds for the second formula.

iii) For any $\lambda \vdash p$, point ii) implies the following equality of $\mathbb{C}[S_{2p}]$-valued rational functions

$$G(z)^{-1} P_{2\lambda} P_{H_{2p}} = p! z^{-p} P_{H_{2p}} Z_{2}^{-1}(z) Z_{4}^{-1}(z) \cdots Z_{2p}^{-1}(z) P_{2\lambda} P_{H_{2p}}$$

and

$$\Omega(z)^{-1} P_{\lambda} P_{D(S_{2p})} = p! z^{-p} P_{D(S_{2p})} Y_{1}^{-1}(z) Y_{2}^{-1}(z) \cdots Y_{p}^{-1}(z) P_{\lambda} P_{D(S_{2p})}.$$  

Recall that $P_{\lambda}$ is a projector whose image is included in the kernel of the restriction of $\rho_{O}$ (resp. $\rho_{S}$) to $\mathbb{C}[S_{2p}]$ as $l(\lambda) > N$ (resp., because $\rho_{S}(P_{\lambda}) = \rho_{O}(P_{\lambda'})$, $l(\lambda') > 2N$). Thanks to Lemma 4.15 these equalities of rational functions can be evaluated to yield the formulas of point iii).

\section{5 Expectation of tensors with respect to Brownian motion’s law}

The rest of the paper is devoted to the proof of our main result Theorem 4.9.

\subsection{5.1 The Casimir element and its representations}

Let $(G_{t})_{t \geq 0}$ be a Brownian motion issued from $\text{Id}$ on a compact Lie group $G$ that belongs to one of the classical series $O(N)$, $U(N)$ or $Sp(N)$. For any finite dimensional representation $(V_{\rho}, \rho)$ and any $t \geq 0$,

$$\frac{d}{dt}\mathbb{E}[\rho(G_{t})] = \mathbb{E}[\frac{1}{2} \Delta_{G}(\rho)(G_{t})] = \mathbb{E}[\frac{1}{2} \rho(c_{g}) \rho(G_{t})] \in \text{End}(V_{\rho})$$

and

$$\mathbb{E}[\rho(G_{t})] = \exp(\frac{t}{2} \rho(c_{g})).$$  \hfill (21)

Let us recall that the element $c_{g}$ is invariant by adjunction, hence $\rho(c_{g}) \in \text{End}_{G}(V_{\rho})$. Our aim here is to give an explicit expression for $\exp(\frac{t}{2} \rho(c_{g}))$ as an element of $\text{End}_{G}(V_{\rho})$ when $(V_{\rho}, \rho)$ is a tensor power of the fundamental representation. All the following formulas but the last one are proved in Proposition 2.6 and 2.8 of [8].
Lemma 5.1. For any integers \( n \) and \( m \), let us write
\[
\Delta_{B_n}(z) = -\frac{(1 - z^{-1})n}{2} + z^{-1} \sum_{1 \leq a < b \leq n} \tau_{a,b} - s_{a,b} \in B_n(z)
\]
and
\[
\Delta_{B_{n,m}}(z) = p^m(\Delta_{B_{n+m}}(z)) - \frac{n + m}{2z} \in B_{n+m}(z).
\]

The Casimir elements of the classical groups have the following tensor representations:
\[
\rho_{V^{\otimes n}}(c_{O(N)}) = 2\rho_O(\Delta_{B_n}(N)),
\]
\[
\rho_{V^{\otimes n}}(c_{Sp(N)}) = 2\rho_S(\Delta_{B_n}(-2N))
\]
and
\[
\rho_{V^{\otimes n}} \otimes \overline{\rho}_{V^{\otimes m}}(c_{u(N)}) = 2\rho_O(\Delta_{B_{n+m}}(N)).
\]

Proof. Let us prove the third equality. Let us choose the orthonormal basis \( \{ \frac{1}{\sqrt{2N}}(E_{k,l} - E_{l,k}), \frac{i}{\sqrt{2N}}(E_{k,l} + E_{l,k}) : 1 \leq k < l \leq N \} \) of \((u(N), \langle \cdot, \cdot \rangle)\) and compute the Casimir as an element of the real algebra \( \mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))\):
\[
Nc_{u(N)} = \frac{1}{2} \sum_{1 \leq k, l \leq N} (1 + i \otimes i)E_{k,l} \otimes E_{k,l} + (i \otimes i - 1)E_{k,l} \otimes E_{l,k} \in \mathcal{U}(\mathfrak{gl}_N(\mathbb{C})).
\]

Considering \( \rho_{V^{\otimes n}} \otimes \overline{\rho}_{V^{\otimes m}} \) as a representation of the real enveloping algebra \( \mathcal{U}(\mathfrak{gl}_N(\mathbb{C})) \) implies that
\[
N\rho_{V^{\otimes n}} \otimes \overline{\rho}_{V^{\otimes m}}(c_{u(N)}) = (n + m)NId_{V^{\otimes n+m}} + 2 \sum_{1 \leq a \leq n < b \leq n+m} \langle a, b \rangle - 2 \sum_{1 \leq a < b \leq n+m \atop a, b \leq n \text{ or } a, b > n} (a, b)
\]
\[
= \rho_O \left( (n + m)NId_{B_{n+m}(N)} + 2p^w \left( \sum_{1 \leq a < b \leq n+m} \tau_{a,b} - s_{a,b} \right) \right).
\]
Dividing by \( N \) yields the result.

Let us check the formula for the symplectic group. Let us define \( \iota : M_N(\mathbb{C}) \times M_N(\mathbb{C}) \to M_{2N}(\mathbb{C}) : (A, B) \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \) and recall that \( \mathfrak{sp}(N) = \{ \iota(A, B) : A \in \mathfrak{u}(N), B \in M_N(\mathbb{C}) , iB = B \} \). Let us choose the basis of \( \mathfrak{sp}(N) \) formed by the unions of the following families:
\[
\frac{1}{2\sqrt{N}}\{\iota(E_{a,b} - E_{b,a}, 0), \iota(0, E_{a,b} - E_{b,a}), \iota(E_{a,b} + E_{b,a}, 0), \iota(0, i(E_{a,b} + E_{b,a})) : 1 \leq a < b \leq N\}
\]
and
\[
\frac{1}{\sqrt{2N}}\{\iota(iE_{a,a}, 0), \iota(0, E_{a,a}), \iota(0, iE_{a,a}) : 1 \leq a \leq N\}.
\]
Using this orthogonal basis, the Casimir element of $\mathfrak{sp}(N)$, viewed as an element of the complex envelopping algebra $\mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))$, has the following expression:

$$-2Nc_{\mathfrak{sp}(N)} = - \sum_{1 \leq a,b,c,d \leq 2N} J_{a,c}J_{b,d}E_{a,b} \otimes E_{c,d} + \sum_{1 \leq a,b \leq 2N} E_{a,b} \otimes E_{b,a} \in \mathcal{U}(\mathfrak{gl}_N(\mathbb{C})).$$

Considering $\rho_V \otimes \omega$ as a representation of the algebra $\mathcal{U}(\mathfrak{gl}_N(\mathbb{C}))$ yields

$$-2N\rho_V \otimes \omega(\Delta_{\mathfrak{sp}(2N)}) = 2(2N + 1)n + 2 \sum_{1 \leq a < b \leq n} (a \ b) - (a \ b)\omega$$

$$= 2\rho_{V} \left( (2N + 1)n + \sum_{1 \leq a < b \leq n} \tau_{a,b} - s_{a,b} \right).$$

Dividing by $-2N$ gives the announced formula. \(\square\)

In the following section, we shall give a reduction of the morphism of multiplication by $\Delta_{\mathcal{B}_n}(z)$ (resp. $\Delta_{\mathcal{B}_{n+m}}(z)$) in the algebra $\mathcal{B}_n(z)$ (resp. $\mathcal{B}_{n+m}(z)$).

### 5.2 A decomposition of the Brauer algebra

For any subset $A$ of $\{1, \ldots, n\}$, let us denote by $\mathcal{M}_A$ the set of partition in $\mathcal{M}(2n)$ such that $n + A$ is the maximal subset of $\{n + 1 \ldots, 2n\}$ not connected to $\{1, \ldots, n\}$. For any integer $p$ less than $n$, let $M_p$ be the vector space spanned by $\cup_{A \subset \{1, \ldots, n\} : |A| = 2p} \mathcal{M}_A$ and define $(P_{M_p})_{0 \leq 2p \leq n}$ the family of projectors induced by the decomposition $\mathcal{B}_n(z) = \bigoplus_{0 \leq 2p \leq n} M_p$.

**Lemma 5.2.** Let $n \leq m$ be two integers. Let $A \subset \{1, \ldots, n\}$ and $\pi \in \mathcal{M}_A$. For any $a \neq b \notin A$, $s_{a,b}\pi \in \mathcal{M}_A$ and $\tau_{a,b}\pi \in \mathcal{M}_{\mathcal{A} \cup \{a,b\}}$. What is more,

$$\Delta_{\mathcal{B}_n}(z)\pi = \left( \frac{(1 - z^{-1})(n - |A|)}{2} + z^{-1} \sum_{a < b, a \notin A} \tau_{a,b} - s_{a,b} \right) \pi$$

and if $\pi \in \mathcal{M}(n,m)$,

$$\Delta_{\mathcal{B}_{n+m}}(z)\pi = \left( \frac{(n + m - |A|)}{2} + z^{-1} \sum_{a < b, a \notin A} p_{\omega}(\tau_{a,b} - s_{a,b}) \right) \pi.$$

**Proof.** We shall prove the two formulas and leave the first point to the reader. For any $k \in \{1, \ldots, n\}$, set $k' = k + n$ and let $\pi(k') \in \{1, \ldots, 2n\}$ denote the two integers such that $\{k', \pi(k')\}$ is a block of $\pi$. For any $1 \leq k \neq l \leq n$, let us compute $\tau_{k,l}\pi$ taking into account whether $k'$ or $l'$ are in $A$ and whether they are in the same block of $\pi$ (see figure [4]):

a) If $k', l' \notin A$, $\tau_{k,l}\pi \in \mathcal{M}_{\mathcal{A} \cup \{k,l\}}$;

b) if $k', l' \in A$ and $l' \neq \pi(k')$, $\tau_{k,l}\pi = s_{k,\pi(l') - n}\pi$;

c) if $k', l' \in A$ and $l' = \pi(k')$, $\tau_{k,l}\pi = z s_{k,l}\pi = z\pi$;

d) if $k' \in A$ and $l' \notin A$, $\tau_{k,l}\pi = s_{\pi(k') - n,l}\pi$. 


This relations yield the following identity

\[
\left( \sum_{1 \leq k < l \leq n} \tau_{k,l} - s_{k,l} \right) \pi = \left( \frac{(z - 1)|A|}{2} + \sum_{1 \leq a < b \leq n, a,b \notin A} \tau_{a,b} - \sum_{1 \leq a < b \leq n, a,b \notin A} s_{a,b} \right) \pi \in B_n(z).
\]

Note that if \( \pi \in M(n,m) \), transpositions "crossing the wall" appearing in c) do not occur if we consider just elements in \( B_{n,m}(z) \) and

\[
p^w \left( \sum_{1 \leq k < l \leq n} \tau_{k,l} - s_{k,l} \right) \pi = \left( \frac{z|A|}{2} + \sum_{1 \leq a < b \leq n, a,b \notin A-n} \tau_{a,b} - \sum_{1 \leq a < b \leq n, a,b \notin A-n} s_{a,b} \right) \pi \in B_{n,m}(z).
\]

The result follows from these two formulas.

The former decomposition of the Brauer algebras yields explicit formulas for the power of the elements \( \Delta_{B_n}(z) \) and \( \Delta_{B_{n+m}}(z) \) through the following lemma.

**Lemma 5.3.** i) For any \( r < k \), \( P_{M_k} \Delta_{B_n}^r = P_{M_k} \Delta_{B_{n+m}}^r = 0 \).

ii) For \( r \geq k \), if \( 2p < n \),

\[
z^k P_{M_k} \Delta_{B_n}^r = \frac{(-1)^{r-k}}{2^k(n-2k)!} \sum_{\sigma \in S_n} \sigma h_{r-k}(Z_{n-2k}, Z_{n-2k+2}, \ldots, Z_n) \tau_{[n-2k+1,n]} \sigma^{-1}
\]
and if $k < n$ or $n < m$,

$$z^k P_m \Delta_{B_n,m}^r = \frac{(-1)^{r-k}}{(n-k)!(m-k)!} \sum_{\sigma \in \mathbb{S}_n \times \mathbb{S}_m} \sigma h_{r-k}(Y_{n-k}, \ldots, Y_{n-1}, Y_n) \tau_{[n-k+1,n]} \sigma^{-1}. $$

iii) For $r \geq p$, if $n = 2p$,

$$z^p P_{M_p} \Delta_{B_n}^r = \frac{(-1)^{r-p}}{2^p} \sum_{\sigma \in \mathbb{S}_n} \sigma h_{r-p}(Z_2, Z_4, \ldots, Z_n) \tau_{[1,n]} \sigma^{-1}$$

and if $n = m$

$$z^n P_m \Delta_{B_n,n}^r = (-1)^{r-n} \sum_{\sigma \in \mathbb{S}_n} \sigma h_{r-n}(Y_1, Y_2, \ldots, Y_n) \tau_{[1,n]} \sigma^{-1}. $$

**Proof.** The first assertion of Lemma 5.2 implies that for any $r \geq 0$, $\Delta_{B_n}(z) \in \bigoplus_{0 \leq k \leq r} M_k$ and the point i) follows. We shall prove points ii) and iii) only for $\Delta_{B_n}(z)$, the same proof applies to the computation of powers of $\Delta_{B_n,n}(z)$. For $q_1 < q_2 < \ldots < q_{2k-1} < q_{2k}$ distinct non-zero integers smaller than $n$, note $B = \{1, \ldots, n\} \setminus \{q_1, q_2, \ldots, q_{2k}\}$ and for $r \geq k$, let $F_r(q_1, q_2, \ldots, q_{2k-1}, q_{2k}) \in B_n(z)$ be the element

$$\sum_{l_0, l_1, \ldots, l_k \geq 0} Z_{B,q_1,q_2}^{l_0} Z_{B,q_2}^{l_1} \tau_{q_1,q_2} Z_{q_3,q_4} \cdots Z_{q_{2k-1},q_{2k}}^{l_{k-1}} Z_{q_{2k}} \in \mathbb{S}_n \setminus \{1, \ldots, n\},$$

if $B \neq \emptyset$, and

$$\sum_{l_1, \ldots, l_k \geq 0} \tau_{q_1,q_2} Z_{B,q_1,q_2}^{l_1} \tau_{q_3,q_4} \cdots Z_{q_{2k-1},q_{2k}}^{l_{k-1}} Z_{q_{2k}} \in \mathbb{S}_n \setminus \{1, \ldots, n\},$$

otherwise. For $r \geq k$, Lemma 5.2 yields

$$z^k P_m \Delta_{B_n}^r(z) = (-1)^{r-k} \sum_{q_1 < q_2 < \ldots < q_{2k-1} < q_{2k}} \# \{q_1, \ldots, q_{2k}\} = 2k \quad F_r(q_1, q_2, \ldots, q_{2k-1}, q_{2k}).$$

(22)

For $r \geq k$ and $2k < n$,

$$F_{r,k} = \sum_{l_0, l_1, \ldots, l_k \geq 0} Z_{n-2k}^{l_0} \tau_{n-2k+1,n-2k+2} Z_{n-2k+2}^{l_1} \tau_{n-2k+3,n-2k+4} \cdots Z_{n-2}^{l_{k-1}} \tau_{n-1,n} Z_n^{l_k} \in M_k,$$

if $n = 2k$, let

$$F_{r,k} = \sum_{l_1, \ldots, l_k \geq 0} \tau_{1,2} Z_{2}^{l_1} \tau_{3,4} \cdots Z_{n-2}^{l_{k-1}} \tau_{n-1,n} Z_n^{l_k}.$$ 

Using these notations, the formula (22) implies that

$$P_m \Delta_{B_n}^r = \frac{(-1)^{r-k}}{2^k z^k (n-2k)!} \sum_{\sigma \in \mathbb{S}_n} \sigma F_{r,k} \sigma^{-1}. $$
Using the fact that if \( 1 \leq i < j \leq n, \tau_{j,j+1} \) commutes with \( Z_i \) and that the elements \((Z_s)_{1 \leq s \leq n}\) commute with each other, allows to reorder the product occurring in the expression of \( F_{k,p} \). For \( 2k < n \), it yields

\[
F_{r,k} = \tau_{n-2k+1,n-2k+2} \cdots \tau_{n-1,n} h_{r-k}(Z_{n-2k}, Z_{n-2k+2}, \ldots, Z_n)
\]

and if \( n = 2k \),

\[
F_{r,k} = \tau_{1,2} \cdots \tau_{n-1,n} h_{r-k}(Z_2, Z_4, \ldots, Z_n).
\]

\[\square\]

**Proof of Theorem 4.9.** Expanding the exponential into power series and using the assertion ii) of Lemma 5.3 combined with Lemma 4.8 imply that for \( 2k < n \),

\[
P_{M_k} \exp[t \Delta B_n(z)] = I_{n,t}^k(z)
\]

and for \( k < n \) or \( k = n \) and \( n < m \),

\[
P_{M_k} \exp[t \Delta B_{n,m}(z)] = I_{n,m,t}^k(z).
\]

Using iii) of Lemma 5.3 implies that if \( n = 2p \),

\[
P_{M_p} \exp[t \Delta B_n(z)] = \frac{1}{2^p z^p} \sum_{\sigma \in S_{2p}} \sigma s_t(0, Z_2, Z_4, \ldots, Z_{2p}) \tau_{[1,2p]} \sigma^{-1}
\]

and for any integer \( n \),

\[
P_{M_n} \exp[t \Delta B_{n,n}(z)] = \frac{1}{z^n} \sum_{\sigma \in S_n \times S_n} \sigma s_t(0, Y_1, Y_2, \ldots, Y_n) \tilde{\tau}_{[1,n]} \sigma^{-1}.
\]

We can now conclude the proof using formula (21) and Lemma 5.1. \[\square\]

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