A stochastic epidemic model for the dynamics and control of maize streak disease
Obiora Cornelius Collins and Kevin Jan Duffy
Institute of Systems Science, Durban University of Technology, Durban 4000, South Africa

ABSTRACT
Maize is an essential staple food crop and is vulnerable to diseases. Maize streak disease is one of the serious illnesses that affect maize, especially in sub-Saharan Africa where the disease is endemic. We developed a stochastic epidemic model with three control measure types (mechanical, chemical and preventative) that mitigate the disease. A dynamical system analysis of the deterministic version of the model is provided. The probability of maize streak disease extinction or persistence is determined using the theory of a multi-type branching process. Using these results, the degrees to which these control measures are effective in reducing maize streak disease are considered. It is shown that chemical and mechanical control measures, preferably together, are better than preventive controls in reducing disease prevalence. However, considering the possible negative effects of a chemical control, it is shown that sufficient mechanical control combined with a small degree of each of preventative and chemical control could be the most viable strategy to limit maize streak disease.

Introduction
Maize (Zea mays L.) is an essential food crop. Maize, grown worldwide, is one of the most produced cereal crops and the crop most adapted to different ecosystems (Serna-Saldivar 2016). As such, it is a source of food for both humans and animals with an estimated annual production of 1 billion tons in 2013 and this amount is increasing annually (Serna-Saldivar 2016; Serna-Saldivar and Carrillo 2019). Possible reasons for this steady increase could be its adaptation to various ecological systems combined with increasing human demands (Serna-Saldivar 2016; Serna-Saldivar and Carrillo 2019). According to Mmboyi et al. (2010), maize is an essential food source and cash crop for approximately 100 million individuals in Africa. It is one of the crops cultivated by subsistence farmers for basic food and income (Mesfin et al. 1991).

Despite all these benefits of maize to humans, maize like many other crops is vulnerable to diseases. One of the major effects of these diseases is that they reduce the maize yield and consequently lead to food shortage and financial losses. These diseases are one of the major threats to global food security and nutrition. Maize streak disease (MSD) is caused by the maize streak virus (genus Mastrevirus, family Geminiviridae). This disease is regarded as one of the most destructive diseases for maize plants in sub-Saharan Africa (Bosque-Perez et al. 1998; Martin and Shepherd 2009; Mazenga 2016). The disease is hosted and transmitted by a vector (pest) called the leafhopper (Bosque-Perez et al. 1998; Martin and Shepherd 2009). The infected leafhopper transmits the disease to susceptible maize plants when it feeds on the leaves. Uninfected leafhoppers are infected with the disease when they feed on infected maize plants. Importantly, MSD cannot be transmitted from one maize plant to another maize plant nor from one leafhopper to another leafhopper (Martin and Shepherd 2009). Maize streak disease can lead to 100% loss of a maize yield if the crop is infected at an early stage (Bosque-Perez et al. 1998; Martin et al. 2009; Martin and Shepherd 2009). Therefore, control measures that can mitigate maize streak disease are crucial to avoid increasing food insecurity.

Control measures are available that mitigate MSD infections (Magenya et al. 2008). Preventive control measures target the susceptible maize plants and aim to prevent them from contracting the disease from the infected leafhoppers. These measures included the use of maize streak disease resistance varieties, crop rotation, proper clearing of bushes before planting, planting early, and any other control mechanism applied to the susceptible maize plants to prevent them from getting infected (Alemneh et al. 2020).
Another method of controlling maize streak disease is the use of mechanical control measures. These control measures target the infected plants by making sure they do not spread to other uninfected maize plants. Mechanical control measures include uprooting and removing, burying, burning infected maize plants so that the disease does not spread to other uninfected maize plants (Alemneh et al. 2020). A further control measure is chemical control. This control targets the leafhopper so that they do not spread the disease to the maize plants. Effective implementation of each of these control measures has been shown to successfully reduce maize streak disease to a certain level (Martin and Shepherd 2009). This study uses theoretical methods to investigate the effects of these control measures in mitigating maize streak disease.

The use of mathematical models is one of the primary theoretical models used successfully to study the dynamics of infectious diseases (van den Driessche and Watmough 2002; Castillo-Chavez et al. 2002; Castillo-Chavez and Song 2004; Tien and Earn 2010; Collins and Govinder 2014; Alemneh et al. 2020, 2021). Several studies have been carried out on the dynamics and control of maize streak disease and other related maize diseases (Vandermeer 1990; Van Maanen and Xu 2003; Collins and Duffy 2016, 2018; Alemneh et al. 2020, 2021). These studies have contributed to an understanding of the dynamics and control of maize streak disease. However, most of these studies use deterministic models to analyze the dynamics. If the initial number of infected individuals is very small, the deterministic model is less capable of a reliable prediction of the endemic dynamics of the disease (Allen and Lahodny Jr 2012; Lahodny Jr et al. 2015; Mbogo et al. 2018; Malyiyon et al. 2019; Mugabi et al. 2020). For such problems, a stochastic model is considered to give better predictions (Allen and Lahodny Jr 2012; Lahodny Jr et al. 2015; Mbogo et al. 2018; Malyiyon et al. 2019; Mugabi et al. 2020). Therefore, in this study, we consider a stochastic epidemic model to analyze the endemic dynamics and control of maize streak disease. The deterministic version of the stochastic model is also considered to support our analyses.

**Model Development**

The two major factors in the dynamics of maize streak disease are maize plants and leafhoppers. Thus, we consider the total population $N_1(t)$ of maize plants and the total population $N_2(t)$ of leafhoppers, each at time $t$. The maize population is made up of two sub-populations: susceptible maize plants and infected maize plants denoted respectively by $S(t)$ and $I(t)$. Similarly, the leafhopper population comprises two sub-populations: susceptible leafhoppers and infected leafhoppers denoted respectively by $X(t)$ and $Y(t)$. For the transmission dynamics of MSD, it is important, as already stated, that the disease is not transmitted horizontally (i.e. from maize plant to maize plant or from leafhopper to leafhopper). Also, the disease is not genetically inherited. In the absence of infected leafhoppers, the maize population grows logistically with an intrinsic growth rate $r$ to a carrying capacity $K$. The susceptible maize plants become infected from infected leafhoppers with a contact rate $\beta$. Infected maize plants die from the disease at a rate $\mu$. The susceptible leafhoppers are recruited through birth or migration at a rate $\Lambda$. The susceptible leafhoppers contract the virus from infected maize plants at a rate $\alpha$ when they feed on the plants. The leafhoppers have a natural death rate $\delta$ (the virus has little or no effect on the leafhoppers themselves). The control measures are incorporated into the model as follows. The preventive control ($d_1$) reduces the rate that susceptible maize contracts the disease. This control involves those practices that reduce contact between susceptible maize and infected leafhoppers. The mechanical control ($d_2$) reduces the rate that susceptible leafhoppers contract the disease. This control involves uprooting, removing or burning infectious maize plants and in this way reduces the contact rate between infected maize and susceptible leafhoppers. The chemical control ($d_3$) is the rate that leafhoppers are removed from the system (die) by the application of insecticide. These three control measures are independent and hence can be implemented simultaneously. Based on these assumptions and formulation, we obtain the model

$$\frac{dS(t)}{dt} = rS(t) \left(1 - \frac{S(t) + I(t)}{K}\right) - (1 - d_1)\beta S(t)Y(t),$$

$$\frac{dI(t)}{dt} = (1 - d_1)\beta S(t)Y(t) - (d_2 + \mu)I(t),$$

$$\frac{dX(t)}{dt} = \Lambda - (1 - d_2)\alpha X(t)I(t) - (d_3 + \delta)X(t),$$

$$\frac{dY(t)}{dt} = (1 - d_2)\alpha X(t)I(t) - (d_3 + \delta)Y(t).$$

The meaning of variables and parameters can be found in Table 1.

**Model Analysis**

The basic mathematical features of the deterministic model (1) are determined and analyzed in this section.
The epidemiological implications of these mathematical features assist in understanding the dynamics of the MSD.

Model (1) has a disease-free equilibrium (DFE)

\[
(S^0, I^0, X^0, Y^0) = \left( K, 0, \frac{\Lambda}{d_3 + \delta}, 0 \right).
\]

(2)

The basic reproduction number \( R_0 \) for model (1) can be defined as the expected number of secondary infections produced by an infected maize plant or leafhopper in an entirely susceptible population in the presence of control measures. Using the next generation matrix method (van den Driessche and Watmough 2002), the \( R_0 \) of model (1) is

\[
R_0 = \sqrt{\frac{(1 - d_1)(1 - d_2)\alpha \beta \Lambda K}{(d_2 + \mu)(d_3 + \delta)^2}}.
\]

(3)

The value of \( R_0 \) indicates if an outbreak will be eradicated or persist in the system. For instance, if \( R_0 < 1 \), we expect the outbreak to be eradicated effectively, i.e. in a reasonable time frame. On the contrary, if \( R_0 > 1 \), the outbreak is likely to persist.

Model (1) has an endemic equilibrium (EE) when \( R_0 \geq 1 \) and this is

\[
(S^*, I^*, X^*, Y^*) = \left( \frac{\xi}{b^* + \sqrt{b^* + 2a^*}}, \frac{a \xi}{\alpha + \psi}, \frac{\alpha \xi I^*}{\alpha I^* + \psi} \right),
\]

(4)

where \( \eta = d_2 + \mu, \psi = d_3 + \delta, a^* = -r \xi (1 + \xi), b^* = r \xi K - 2r \rho \xi - r \rho - \eta K, c^* = r \rho (K - \rho), \xi = \frac{\eta}{(1 - d_1)(1 - d_2)}, \rho = \frac{\psi}{(1 - d_1)(1 - d_2)} \). Clearly, \( a^* < 0 \) and \( c^* > 0 \). Thus, a unique EE exists for model (1) irrespective of the value of \( b^* \).

**Stability Analysis**

The stability of model (1) about the DFE (2) describes the short-term dynamics of MSD (Liao and Wang 2011). Thus, the local and global stability analyses of model (1) about the DFE are presented in the following theorems.

**Theorem 1.** The DFE (2) is locally stable if \( R_0 < 1 \).

**Proof:** The eigenvalues of the Jacobian of model (1) at the DFE are:

\[
\lambda_1 = -r, \quad \lambda_2 = -(d_3 + \delta), \quad \lambda_3 = \frac{-(\eta + \psi) - \sqrt{(\eta + \psi)^2 + 4\eta\psi(R_0 + 1)(R_0 - 1)}}{2}, \quad \lambda_4 = \frac{-(\eta + \psi) + \sqrt{(\eta + \psi)^2 + 4\eta\psi(R_0 + 1)(R_0 - 1)}}{2},
\]

where \( \eta = d_2 + \mu \) and \( \psi = d_3 + \delta \). Obviously, \( \lambda_1 < 0, \lambda_2 < 0 \) and \( \lambda_3 < 0 \). Mathematically, \( \lambda_4 < 0 \) if \( R_0 < 1 \). Hence, the DFE (2) is locally stable if \( R_0 < 1 \).

Epidemiologically, this shows that maize streak disease could be eliminated if the initial population sizes of infected maize plants and leafhoppers at outbreak are within some neighbourhood of the DFE (2). On the contrary, if the population of the infected maize plants and leafhoppers at the beginning of the outbreak are outside this neighbourhood of the DFE (2), globally stability analysis of the DFE (2) is required to determine whether the disease can be eradicated or not. Mathematical proof of the global stability is presented using a Lemma by Castillo-Chavez et al. (2002).

**Lemma 1** Castillo-Chavez et al. (2002): Consider a model system written in the form

\[
\frac{dZ_1}{dt} = F(Z_1, Z_2)
\]

\[
\frac{dZ_2}{dt} = G(Z_1, Z_2), \quad G(Z_1, 0) = 0,
\]

(5)

where \( Z_1 \in \mathbb{R}^m \) and \( Z_2 \in \mathbb{R}^n \). \( Z_0 = (Z_1^*, 0) \) denotes the disease-free equilibrium of the system. Assume that

(H1) For \( \frac{dZ_1}{dt} = F(Z_1, 0) \), \( Z_1^* \) is globally asymptotically stable;

(H2) \( G(Z_1, Z_2) = AZ_1 - \hat{G}(Z_1, Z_2), \quad \hat{G}(Z_1, Z_2) \geq 0 \) for \( (Z_1, Z_2) \in \Omega \), where the Jacobian \( A = \frac{\partial \hat{G}(Z_1, Z_2)}{\partial Z_1} \) is an M-matrix (the off diagonal elements of A are non-negative) and \( \Omega \) is the region where the model makes biological sense.
Thus, \( Z_0 \) is globally asymptotically stable provided that \( R_0 < 1 \) (Castillo-Chavez et al. 2002).

**Theorem 2:** The DFE (2) is globally stable if \( R_0 < 1 \).

**Proof:** Using Lemma 1, we only need to show that conditions (H1) and (H2) hold when \( R_0 < 1 \). From model Equation (1), \( S(t) = S(t), X(t), Z_1 = (l(t), Y(t)) \). The disease free model

\[
\frac{dZ_1}{dt} = F(Z_1, 0) = \left( rS(t)(1 - \frac{S(t+\delta t)}{K}) - (d_1 + \delta)X(t) \right)
\]

(6)

is a set of linear ordinary differential equations and its exact solution can be calculated as \( S(t) = \frac{c}{1 + c \cdot S_0^t}, X(t) = X^0 + (X(0) - X^0)e^{-(d_1 + \delta)t} \), where \( c = \frac{S_0}{K - S_0} \). Clearly, \( S(t) \to S^0, X(t) \to X^0 \) as \( t \to \infty \).

Thus, \( Z_1^* \) is globally asymptotically stable, confirming that condition (H1) holds. To show that condition (H2) holds, consider

\[
\frac{dZ_2}{dt} = G(Z_1, Z_2)
\]

(7)

By computing the Jacobian of (7) about the DFE, we obtain

\[
A = \left( \begin{array}{cc}
-(d_2 + \mu)(1 - d_1)BS^0 & (1 - d_2)\alpha X^0
\end{array} \right)
\]

(8)

Obviously, \( A \) is an M--matrix with all its off diagonal elements non-negative. From Equations (7) and (8), we obtain

\[
\hat{G}(Z_1, Z_2) = \left( \begin{array}{cc}
(1 - d_1)\beta Y(S^0 - S) & (1 - d_2)\alpha Y(X^0 - X)
\end{array} \right)
\]

(9)

Clearly, \( \hat{G}(Z_1, Z_2) \geq 0 \), since \( S^0 \geq S \) and \( X^0 \geq X \). This shows that condition (H2) holds and hence completes the proof.

Epidemiologically, this results indicate that maize streak disease could possibly be eradicated irrespective of the size of population of the infected maize plants and leafhoppers at the beginning of the outbreak, provided the control measures are effective to keep \( R_0 \) below one.

**Bifurcation Analysis**

A bifurcation can be regarded as a qualitative change in the solution trajectories as a result of a change in the parameter. The parameter that leads to this change is called the bifurcation parameter and the point at which the bifurcation occurs is called the bifurcation point. Mathematically, the number of equilibrium points, or their stability conditions, or both, changes at the bifurcation point. Epidemiologically, the quantity \( R_0 \) is often a bifurcation parameter. The possible direction of bifurcation at \( R_0 = 1 \) for model (1) is determined using center manifold theory (Castillo-Chavez and Song 2004).

**Theorem 3** Castillo-Chavez and Song (2004): Let us consider a general system of ordinary differential equations with a parameter \( \phi \):

\[
\frac{dx}{dt} = f(x, \phi), \quad f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n, \quad \phi \in C^2(\mathbb{R}^n \times \mathbb{R}).
\]

(10)

Let \( x = 0 \) be an equilibrium point for the system in Equation (10). That is \( f(0, \phi) = 0 \) for all values of the parameter \( \phi \). Assume the following:

(i) \( A = D_x f(0, 0) = (\frac{\partial^2 f}{\partial x_i \partial x_j}(0, 0)) \) is the linearization matrix of System (10) around the equilibrium 0 with \( \phi \) evaluated at 0. Zero is a simple eigenvalue of \( A \) and all other eigenvalues of \( A \) have negative real parts.

(ii) Matrix \( A \) has a nonnegative right eigenvector \( w \) and a left eigenvector \( v \) corresponding to the zero eigenvalue.

Let \( f_k \) be the kth component of \( f \) and

\[
a = \sum_{k,j=1}^{n} v_k w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0, 0), \quad \phi \in C^2(\mathbb{R}^n \times \mathbb{R}).
\]

(11)

\[
b = \sum_{k,i=1}^{n} v_i w_j \frac{\partial^2 f_k}{\partial x_i \partial \phi}(0, 0).
\]

(12)

The local dynamics of (10) around 0 are totally determined by \( a \) and \( b \).

(i) \( a > 0, b > 0 \). When \( \phi < 0 \) with \( |\phi| \ll 1 \), 0 is locally asymptotically stable, and there exists a positive unstable equilibrium; when \( 0 < \phi \ll 1 \), 0 is unstable and there exists a negative and locally asymptotically stable equilibrium;

(ii) \( a < 0, b > 0 \). When \( \phi < 0 \) with \( |\phi| \ll 1 \), 0 is unstable, and there exists a positive unstable equilibrium; when \( 0 < \phi \ll 1 \), 0 is locally asymptotically stable, and there exists a positive unstable equilibrium;

(iii) \( a > 0, b < 0 \). When \( \phi < 0 \) with \( |\phi| \ll 1 \), 0 is unstable, and there exists a locally asymptotically stable negative equilibrium; when \( 0 < \phi \ll 1 \), 0
is stable, and a positive unstable equilibrium appears;

(iv) $a < 0$, $b > 0$. When $\phi$ changes from negative to positive, 0 changes its stability from stable to unstable. Correspondingly a negative unstable equilibrium becomes positive and locally asymptotically stable.

To determine the direction of bifurcation of model (1) about $R_0 = 1$, let $\phi = \beta$ be a bifurcation parameter. By denoting $x_1 = S(t)$, $x_2 = L(t)$, $x_3 = X(t)$, $x_4 = Y(t)$, model (1) becomes

$$\begin{align*}
\frac{dx_1}{dt} &= \nu_1(1 - \frac{x_1 + x_2}{K}) - (1 - d_1)\phi x_1 x_4 := f_1, \\
\frac{dx_2}{dt} &= (1 - d_1)\phi x_1 x_4 - (d_2 + \mu)x_2 := f_2, \\
\frac{dx_3}{dt} &= \lambda - (1 - d_2)\alpha x_3 x_4 - (d_3 + \delta)x_3 := f_3, \\
\frac{dx_4}{dt} &= (1 - d_2)\alpha x_3 x_4 - (d_3 + \delta)x_4 := f_4,
\end{align*}$$

with $R_0 = 1$ corresponding to $\phi = \phi^* = \frac{(d_2 + \mu)(d_3 + \delta)}{(1 - d_1)(1 - d_2)\lambda K}$.

The DFE (2) is $x^*_1, x^*_2, x^*_3, x^*_4 = (K, 0, \frac{1}{\lambda + \mu}, 0)$. For $\phi = \phi^*$, the Jacobian of model (13) at the DFE becomes

$$J = \begin{pmatrix}
-r & -r & 0 & -(1 - d_1)\phi^* K \\
0 & -(d_2 + \mu) & 0 & (1 - d_1)\phi^* K \\
0 & 0 & -(d_3 + \delta) & 0 \\
0 & 0 & 0 & -(d_3 + \delta)
\end{pmatrix}.$$  

Clearly, 0 is a simple zero eigenvalue for $J$. A right eigenvector $w = (w_1, w_2, w_3, w_4)^T$ associated with simple zero eigenvalue is

$$w = \left(1 + \frac{d_2 + \mu}{r}\right)w_2, w_2, \frac{(1 - d_2)\alpha \lambda}{(d_3 + \delta)^2}w_2, \frac{(1 - d_2)\alpha \lambda}{(d_3 + \delta)^2}w_2,$$

where $w_2 = w_2 > 0$ and $T$ represent the transpose. Similarly, a left eigenvector $v = (v_1, v_2, v_3, v_4)^T$ associated with simple zero eigenvalue is

$$v = \left(0, \frac{(d_3 + \delta)^3}{((d_3 + \delta)^{1/2} + (1 - d_2)^2 \alpha \lambda K \phi^*)w_2}, 0, \frac{(1 - d_2)\phi^* K v_2}{d_3 + \delta} \right),$$

where $v_2$ is calculated such that the condition $v.w = 1$ is satisfied by the eigenvectors. Observe that the first term and third term of the eigenvector $v$ are zero. Consequently, the partial derivatives of $f_1$ and $f_3$ will all vanish in the computation of $a$ and $b$. Algebraic calculations of the remaining derivatives of $f_2$ and $f_4$ gives

$$\begin{align*}
\frac{\partial^2 f_2}{\partial x_1 \partial x_4} &= \frac{\partial^2 f_2}{\partial x_4 \partial x_1} = (1 - d_1)\phi^*, \\
\frac{\partial^2 f_2}{\partial x_3 \partial x_4} &= \frac{\partial^2 f_2}{\partial x_4 \partial x_3} = (1 - d_2)\alpha, \\
\frac{\partial^2 f_2}{\partial x_4 \partial \phi} &= (1 - d_1)x^*_1.
\end{align*}$$

Note that the remaining second partial derivatives in the Equations (11) and (12) for $a$ and $b$ respectively are all zeros.

From the above partial derivatives, we obtain

$$a = v_2 w_1 w_4 (1 - d_1)\phi^* + v_2 w_4 w_1 (1 - d_1)\phi^* > 0, \quad (15)$$

$$b = v_2 w_4 (1 - d_1)x^*_1 < 0, \quad (16)$$

since $v_2 > 0$, $w_1 < 0$ and $w_4 < 0$. Thus, using (iv) of Theorem 3 above, we obtain the result summarized in the following theorem:

**Theorem 4:** Model (1) undergoes a forward bifurcation at $R_0 = 1$.

The epidemiological implication of these results is that reducing $R_0$ below unity is sufficient to eliminate maize streak disease irrespective of the population of infected maize plants and leafhoppers at the onset of an outbreak (Castillo-Chavez et al. 2002; Castillo-Chavez and Song 2004; Blayneh et al. 2010). Thus, effective control measures that keep $R_0$ below 1 are required for complete eradication of maize streak disease.

When the infected population at the beginning of an outbreak is very small a deterministic epidemic model does not give accurate predictions of the endemic dynamics of the disease progression. For better predictions of the endemic dynamics, when the initial infected population is small, a stochastic epidemic model is necessary. In the subsequent sections, we consider a stochastic epidemic model version of model (1) to analyse the endemic dynamics of maize streak disease.

**Streak Stochastic Epidemic Model**

The stochastic version of the deterministic model (1) will be derived by considering a continuous-time Markov chain (CTMC) model with a discrete population size of maize plants and leafhoppers. Multitype branching process theory will be considered to estimate the probability of maize streak disease extinction or persistence.
Stochastic Epidemic Model Formulation

Let \(S(t), I(t), X(t)\) and \(Y(t)\), respectively, be discrete-valued random variables for the number of susceptible maize plants, infected maize plants, susceptible leafhoppers and infected leafhoppers, each with a finite state space and where time \(t \in [0, \infty)\) is continuous. For simplicity, the same symbols used for the deterministic model (1) are used for the stochastic model. Details of the state transitions as well as rates for the CTMC model can be found in Table 2.

Multitype Galton-Watson Branching Process

Multitype branching process theory can be used to estimate the probability of disease eradication or persistence (Allen and Lahodny Jr 2012). The theory is only applied to infectious populations and assumes that non-infectious populations are at the disease-free equilibrium (Allen and Lahodny Jr 2012). Also, the theory assumes that the transition rates are linear with regard to the infectious variables \(I(t)\) and \(Y(t)\). A probability generating function (pgf) for the birth/growth and death/removal of each of the infectious maize plants \(I(t)\) and infectious leafhoppers \(Y(t)\) can be determined (Lahodny Jr et al. 2015). These pgfs are used to estimate the probability of maize streak disease persistence or extinction (Lahodny Jr et al. 2015). The pgf for \(n\) variables is given by

\[
g_1(u_1, u_2, \ldots, u_n) = \sum_{k_0=0}^{\infty} \sum_{k_0-k_1=0}^{\infty} \sum_{k_0-k_1-k_2=0}^{\infty} \cdots \sum_{k_0-k_1-k_2-\cdots-k_n=0}^{\infty} P_1(k_1, k_2, \ldots, k_n)u_1^{k_1}u_2^{k_2} \cdots u_n^{k_n},
\]

(17)

where

\[
P_1(k_1, k_2, \ldots, k_n) = \text{Prob}(Z_1 = k_1, Z_2 = k_2, \ldots, Z_n = k_n)
\]

is the probability that an infected individual of type \(i\) produces \(k_j\) individuals of type \(j\).

Using this definition, the offspring pgf for an infected maize plant (i.e, \(I(0) = 1\) and \(Y(0) = 0\)) is

\[
g_1(u_1, u_2) = \sum_{k_0=0}^{\infty} \sum_{k_0-k_1=0}^{\infty} P_1(k_1, k_2)u_1^{k_1}u_2^{k_2},
\]

(18)

where \(P_1(k_1, k_2)u_1^{k_1}u_2^{k_2}\) is the probability that an infected maize plant produces another infected maize plant \(k_1\) or an infectious leafhopper \(k_2\).

Similarly, the offspring pgf for an infected leafhopper (i.e, \(I(0) = 0\) and \(Y(0) = 1\)) is

\[
g_2(u_1, u_2) = \sum_{k_0=0}^{\infty} \sum_{k_0-k_1=0}^{\infty} P_2(k_1, k_2)u_1^{k_1}u_2^{k_2},
\]

(19)

where \(P_2(k_1, k_2)u_1^{k_1}u_2^{k_2}\) is the probability that an infected leafhopper produces another infected maize plant \(k_1\) or an infectious leafhopper \(k_2\).

When the initial populations of maize plants and leafhoppers are close to the disease-free equilibrium \((S(0) = S^0\) and \(X(0) = X^0\)), the specific offspring pgfs for \(I(t)\) and \(Y(t)\) could possibly be determined by the use of the transition rates in Table 2.

Using the transition rates in Table 2, the offspring pgf for an infected maize plant is

\[
g_1(u_1, u_2) = \frac{(1 - d_2)ax^0u_1u_2 + d_2 + \mu}{(1 - d_2)ax^0 + d_2 + \mu}.\]  

(20)

The term \(\frac{(1 - d_2)ax^0}{(1 - d_2)ax^0 + d_2 + \mu}\) can be defined as the probability that an infected maize plant infects a susceptible leafhopper which leads to one infected maize plant and one infected leafhopper. The term \(\frac{d_2 + \mu}{(1 - d_2)ax^0 + d_2 + \mu}\) denotes the probability that an infected maize plant is removed (through control measures or death due to the infection) before transmitting the disease, resulting in zero infected maize plants and leafhoppers.

Similarly, the offspring pgf for an infected leafhopper is

\[
g_2(u_1, u_2) = \frac{(1 - d_1)b\beta u_1u_2 + d_3 + \delta}{(1 - d_1)b\beta u_1u_2 + d_3 + \delta}.\]  

(21)

The term \(\frac{(1 - d_1)b\beta}{(1 - d_1)b\beta + d_3 + \delta}\) can be interpreted as the probability that an infected leafhopper infects a susceptible maize plant which leads to one infected maize plant and one infected leafhopper. The term \(\frac{d_3 + \delta}{(1 - d_1)b\beta + d_3 + \delta}\) can also be interpreted as the probability that an infected leafhopper is removed (through control measures or death due to the infection) before transmitting the disease resulting in zero infected maize plants and leafhoppers.

| Event                  | Next state | Transition rate |
|------------------------|------------|-----------------|
| Birth/growth of \(S(t)\) | \((S(t) + 1, X(t), Y(t))\) | \(rs\) \((1 - \frac{d_0}{2})\) |
| Infection of \(S(t)\)   | \((S(t) - 1, I(t) + 1, X(t), Y(t))\) | \((1 - d_1)\beta S I Y(0)\) |
| Death/removal of \(I(t)\) | \((S(t), I(t) - 1, X(t), Y(t))\) | \(d_1\) |
| Recruitment of \(X(t)\) | \((S(t), I(t), X(t) + 1, Y(t))\) | \(\Lambda\) |
| Infection of \(X(t)\)   | \((S(t), I(t), X(t) - 1, Y(t) + 1)\) | \((1 - d_2)X I Y(0)\) |
| Death of \(Y(t)\)       | \((S(t), I(t), X(t), Y(t) - 1)\) | \((d_3 + \delta) Y(0)\) |
The Stochastic Threshold

The expectation matrix of the offspring pgfs is defined by

$$\mathbb{M} = \begin{pmatrix} \frac{\alpha_1 g_1}{u_1} & \frac{\alpha_2 g_2}{u_2} \\ \frac{\alpha_1 g_1}{u_1} & \frac{\alpha_2 g_2}{u_2} \end{pmatrix} \text{ evaluated at } (u_1, u_2) = (1,1). \tag{22}$$

From Equation (22), we obtain

$$\mathbb{M} = \begin{pmatrix} \frac{(1-d)_1 g_1^0}{(1-d)_1 g_1^0 + d + \mu} & \frac{(1-d)_2 g_2^0}{(1-d)_2 g_2^0 + d + \mu} \\ \frac{(1-d)_1 g_1^0}{(1-d)_1 g_1^0 + d + \mu} & \frac{(1-d)_2 g_2^0}{(1-d)_2 g_2^0 + d + \mu} \end{pmatrix}. \tag{23}$$

By setting $R_1 = \sqrt{\frac{(1-d)_1 g_1^0}{d + \mu}}$, $R_2 = \sqrt{\frac{(1-d)_2 g_2^0}{d + \mu}}$ such that $R_1 R_2 = R_0$, and simplifying, the expectation matrix $\mathbb{M}$ becomes

$$\mathbb{M} = \begin{pmatrix} \frac{R_1^2}{1 + R_1^2} & \frac{R_2^2}{1 + R_2^2} \\ \frac{R_1^2}{1 + R_1^2} & \frac{R_2^2}{1 + R_2^2} \end{pmatrix}. \tag{24}$$

The spectral radius of the expectation matrix $\mathbb{M}$ can be determined as

$$\rho(\mathbb{M}) = \frac{R_1^2}{1 + R_1^2} + \frac{R_2^2}{1 + R_2^2}, \tag{25}$$

where

$$R_1 = \sqrt{\frac{(1-d)_1 g_1^0}{d + \mu}} \quad \text{and} \quad R_2 = \sqrt{\frac{(1-d)_2 g_2^0}{d + \mu}}.$$

The expression $\rho(\mathbb{M})$ is a threshold quantity of the stochastic model that indicates the possibility of disease persistence or eradication. When $\rho(\mathbb{M}) < 1$ there is a probability one that the disease will be completely eradicated. On the contrary, if $\rho(\mathbb{M}) > 1$, there is a non-zero probability that the disease will persist. Therefore, the value of $\rho(\mathbb{M})$ (stochastic threshold) is equivalent to the basic reproduction number $R_0$ (deterministic threshold) (Allen 2013). For instance, it is straightforward to verify that $\rho(\mathbb{M}) < 1$ iff $R_0 < 1$ and $\rho(\mathbb{M}) > 1$ iff $R_0 > 1$.

Probability of Disease Extinction

The magnitude of the stochastic threshold quantity $\rho(\mathbb{M})$ indicates whether the probability of disease eradication is less than one or equal to one (Allen and Lahodny Jr 2012). For instance, if $\rho(\mathbb{M}) < 1$ (subcritical) or $\rho(\mathbb{M}) = 1$ (critical), the probability of disease eradication is one, that is,

$$\lim_{t \to \infty} \text{Prob}(l(t) = 0, Y(t) = 0) = 1. \tag{26}$$

For this case, $(u_1, u_2) = (1,1)$ is the only possible stable fixed point (Allen and Lahodny Jr 2012; Lahodny Jr et al. 2015).

On the other hand, if $\rho(\mathbb{M}) > 1$ (supercritical), there exists another fixed point $(u_1, u_2) \in (0,1)^2$ of the offspring pgfs, $g_i(u_1, u_2) = u_i$ for $i = 1,2$ such that the probability of disease eradication is

$$P_0 = \lim_{t \to \infty} \text{Prob}(l(t) = 0, Y(t) = 0) = u_1^0 u_2^0, \tag{27}$$

where $i_0 = l(0)$ and $y_0 = Y(0)$ (Allen and Lahodny Jr 2012; Lahodny Jr et al. 2015). The value of $u_1$ and $u_2$, are the probabilities of disease eradication for $l(t)$ and $Y(t)$, respectively. These values can be obtained by solving the fixed points $g_i(u_1, u_2) = u_i$ for $i = 1,2$ in Equations (20) and (21) and taking into account that $R_1 = \sqrt{\frac{(1-d)_1 g_1^0}{d + \mu}}$, $R_2 = \sqrt{\frac{(1-d)_2 g_2^0}{d + \mu}}$ and $R_0 = R_1 R_2$, we obtain

$$u_1 = \frac{1 + R_2^2 + 2R_1^2 \pm \sqrt{(1 + R_2^2 + 2R_1^2)^2 - 4(R_0^2 + R_1^2)(1 + R_1^2)}}{2(R_0^2 + R_1^2)}, \tag{28}$$

$$u_2 = \frac{1}{1 + (1 - u_1)R_1^2}. \tag{29}$$

Consequently, the probability of a major outbreak becomes

$$P_m = 1 - P_0. \tag{30}$$

From the above Equations (27) and (30), it is obvious that the probability of disease eradication or persistence can determine if the parameter values in the model and the initial number of $l(t)$ and $Y(t)$ are known. In the next section, we will determine the probability of maize streak disease eradication or persistence with respect to the control measures $d_1, d_2$ and $d_3$.

**Numerical Results**

In this section, we use numerical simulations to investigate the effects of the control measures $d_1, d_2, d_3$ on the probability of maize streak disease eradication or persistence. The parameter values used for the numerical simulations can be found in Table 3.

First, we consider each control applied individually. To gain more insight on the possible effects of each of these controls, we include a small degree (10%) of each of the other controls. This is presented in this

| Table 3. Parameter values used for numerical simulations. |
|---------------------------------------------------------|
| Parameters symbols | Value          | Source       |
|---------------------|----------------|--------------|
| $r$                 | 0.014          | Duffy (2011) |
| $\Lambda$           | 250.0          | Estimate     |
| $\beta$             | 4.4998 $\times$10$^{-5}$ | Estimate |
| $a$                 | 4.4998 $\times$10$^{-5}$ | Estimate |
| $k$                 | 10000          | Alemneh et al. (2021) |
| $\mu$               | 0.008          | Alemneh et al. (2021) |
| $\delta$            | 0.0303         | Magenya et al. (2008) |
| $d_i, i = 1, 2, 3$  | 0.0 – 1.0      | Estimate     |
way because the effect was significantly altered when both of the other controls were minimally implemented. The actual amounts used are illustrative. Second, the effects of combinations of controls are considered.

Figure 1(a) is a graphical illustration of the effects of the prevention strategies ($d_1$) on the probability of disease eradication $P_0$. In the absence of mechanical and chemical control (i.e. $d_2 = d_3 = 0$) the probability of MSD extinction is zero. This suggests that the preventive measure $d_1$ alone is not sufficient to eliminate maize streak disease.

To gain more insight on the possible effects of $d_1$ on $P_0$, we consider a small degree of mechanical and chemical control ($d_2 = d_3 = 0.1$). For this case, the preventive measures $d_1$ increase the probability of MSD extinction and it is eradicated beyond a threshold (Figure 1). To illustrate this further, the corresponding stochastic threshold $\rho(M)$ and deterministic threshold

Figure 1. Plot illustrating the effects of $d_1$ on the probability of maize streak disease eradication.

Figure 2. Plot illustrating the effects of $d_2$ on the probability of maize streak disease eradication.
$R_0$ are presented in Figure 1(b). Figure 1(b) shows that as $d_1$ increases both $\rho(\mathbb{M})$ and $R_0$ decreases. When $0 \leq d_1 \leq 0.75$, $0 \leq \rho(\mathbb{M})$, $R_0 > 1$ and $0 \leq P_0 < 1$, there is a chance of a major MSD outbreak. When $0.75 \leq d_1 \leq 1.0$, $\rho(\mathbb{M}) < 1$, $R_0 < 1$ and $P_0 = 1$, the MSD will be eradicated.

Figure 2(a) is a graphical illustration of the effects of mechanical control ($d_2$) on the probability of disease eradication $P_0$. In the absence of preventive and chemical control (i.e. $d_1 = d_3 = 0$), the probability of MSD extinction is zero. Thus, mechanical control alone cannot eliminate the disease. If we introduce a small amount of the other control measures ($d_1 = d_3 = 0.1$) then mechanical control increases the probability of disease extinction. When $0 \leq d_2 \leq 0.3$ then $\rho(\mathbb{M}) > 1$ and $R_0 > 1$ and $0 < P_0 < 1$, results in the possibility of a MSD outbreak. However, when $0.3 \leq d_2 \leq 1.0$ then $\rho(\mathbb{M}) < 1$ and $R_0 < 1$ and $P_0 = 1$, then MSD will be eradicated.

Figure 3. Plot illustrating the effects of $d_3$ on the probability of maize streak disease eradication.

Figure 4. Plot illustrating the combined effects of $d_1$ and $d_2$ on the probability of maize streak disease eradication.
In Figure 3(a) when \( d_1 = d_2 = 0 \), the probability of disease eradication using chemical control \((d_3)\) is small. Again, we consider a small amount of the preventive and mechanical controls \((d_1 = d_2 = 0.1)\). For this case, when \( 0 \leq d_3 \leq 0.2 \) which corresponds to \( R(M) > 1 \) and \( R_0 > 1 \) and \( 0 < P_0 < 1 \), there is possibility for a MSD outbreak to occur. On the other hand, when \( 0.2 \leq d_3 \leq 1.0 \) which corresponds to \( R(M) < 1 \) and \( R_0 < 1 \) and \( P_0 = 1 \), the disease will be eradicated.

Figure 4 illustrates the combined effects of the preventive control \((d_1)\) and the mechanical control \((d_2)\) on the probability of disease eradication, without the chemical control \((d_3 = 0)\). From this figure, combined effects of the preventive and the mechanical control measures do not have significant effects in maize streak disease eradication. The minimal combined control measures for chances of maize streak disease eradication is about \( d_1 > 0.9 \) and \( d_2 > 0.9 \).

Figure 5 illustrates the combined effects of the mechanical control \((d_2)\) and the chemical control \((d_3)\) on the probability of disease eradication, in the absence of the preventive control \((d_1 = 0)\). From this figure, the combined effects of these two controls have a significant effect on maize streak disease eradication. The minimal combined control effort required to achieve maize streak disease eradication with a probability one is approximately \( d_2 > 0.2 \) and \( d_3 > 0.15 \).

Figure 6 illustrates the combined effects of the preventive control \((d_1)\) and the chemical control \((d_3)\) on the probability of disease eradication, without the mechanical control \((d_2 = 0)\). From this figure, combined effects of the preventive control and the chemical control are not very effective for maize streak disease elimination. The minimal combined control effort required to achieve disease eradication with a probability of one is about \( d_3 > 0.6 \) and \( d_1 > 0.9 \) which is difficult to achieve practically.

**Discussion**

Maize is one of the most important staple food crops grown worldwide. It is an important source of income or employment to both small-scale and commercial farmers (Serna-Saldivar and Carrillo 2019). Unfortunately, maize is vulnerable to many diseases that reduce yields. Thus, disease poses a threat to global food security and income especially those engaged in maize farming. Maize streak disease is one of the most destructive maize diseases especially in sub-Saharan Africa where the disease is endemic. This disease can lead to 100% yield loss of a maize crop if the crop is infected with the disease at an early stage. Despite available control measures that can mitigate maize streak disease infections, the disease remains endemic, especially in sub-Saharan Africa. Therefore, understanding how to mitigate the disease is crucial to improve maize yields.

We developed a deterministic epidemic model for maize streak disease under the assumption that three control measures (i.e. preventive control, mechanical control and chemical control) can be implemented to eradicate the disease. The dynamical system analyses of the model help to gain initial insight into the
dynamics and influence of these control measures on the disease. For example, the basic reproduction number \( R_0 \) showed that the disease-free equilibrium point of the model is both locally and globally stable when the basic reproduction number is less than unity. Thus, maize streak disease will be completely eradicated irrespective of the population sizes of infected maize plants and leafhoppers at the onset of an outbreak provided that \( R_0 < 1 \). When \( R_0 = 1 \), using center manifold theory, we proved that the model undergoes a forward bifurcation. This indicates that reducing the basic reproduction number below unity is sufficient to eradicate maize streak disease. When \( R_0 > 1 \), a unique endemic equilibrium emerged in the deterministic model. The endemic dynamics of the maize streak disease can be determined by investigating the stability of this endemic equilibrium. However, for a small population of infected maize and leafhoppers, analyses using the deterministic model do not give accurate predictions of the endemic dynamics. Consequently, we developed a stochastic version of the deterministic model.

A stochastic model in the form of continuous-time Markov chain (CTMC) model (with a discrete population size of maize plants and leafhoppers) was developed. This model is a stochastic form of the deterministic model (1). Again, qualitative analyses of the model were carried out accordingly. A stochastic threshold \( \rho(M) \) for maize streak disease persistence or extinction was determined. The value of this stochastic threshold \( \rho(M) \) was shown to have a similar role to the deterministic threshold (the basic reproduction number \( R_0 \)) as expected (Allen 2013). For instance, if \( \rho(M) \leq 1 \) maize streak disease is eliminated with a probability of one. On the other hand, if \( \rho(M) > 1 \), maize streak disease persists with a non-zero probability. Next, multitype branching process theory was used to determine the probability of maize streak disease eradication \( P_0 \) or persistence \( (1 - P_0) \).

Numerical simulations were used to determine the effects of the control measures on the probability of maize streak disease eradication \( P_0 \) or persistence \( (1 - P_0) \). The results showed that no single control measure (preventive, mechanical or chemical) is sufficient to eliminate maize streak disease. However, when each of the control measures is supported with a 10% implementation of the other control measures combined, the probability of disease eradication increases significantly. Also, total eradication of maize streak disease is possible provided any one control is above a certain threshold and is supported by the other controls at 10%. For instance, a preventive control \( > 0.75 \) is required for total eradication of maize streak disease with the other control measures at 10%. Similarly, with the same assumptions for the other controls \( (d_1 = d_3 = 0.1) \) effective implementation of mechanical control \( > 0.3 \) leads to maize streak disease extinction. Also, with \( d_1 = d_2 = 0.1 \) a chemical control \( > 0.2 \) results in disease eradication.

These lower thresholds required for chemical or mechanical control to allow disease eradication, as
compared to prevention, should be easier and less costly to attain. Hence, mechanical or chemical control are likely to be more practical than prevention as an effective control of maize streak disease eradication provided they are complemented with a small amount of each of the other control measures.

It was expected that combining any two controls would be more effective than the single control alone (Alemneh et al. 2020). However, surprisingly this was not the case when combining the prevention and mechanical controls which required both to be $> 0.9$ to eradicate the disease. This would be very difficult to achieve. Combining the prevention and chemical controls required the first to be $> 0.9$ and the second to be $> 0.6$, which could also be difficult to achieve practically.

By contrast, combining mechanical control and chemical control presents a much more practical control strategy. For a mechanical control $> .1$ together with a chemical control $> 0.2$ eradication of the disease is possible and these thresholds should be far easier to attain. Hence, the combined strategy of mechanical and chemical control could be recommended for maize streak disease elimination whenever outbreaks occur. However, environmental impacts of chemical control should be considered first.

Integrated control management strategies to limit maize streak disease have been suggested by Alegbejo et al. (2002); Allen and Lahodny Jr (2012); Karavina (2014) but they did not provide the specific strategic combinations in an optimal approach. Our results, in agreement with Alemneh et al. (2020, 2021), show how combinations of different controls can effectively limit maize streak disease. First, any one control is unlikely to be effective. Second, a small amount of two of the three controls can improve the probability of limiting the disease if the third control is at a higher threshold. This strategy is more effective if the focused control is either mechanical or chemical. Third, when mechanical control and chemical control are combined the disease can be limited at low enough threshold values to be practical. However, the possible negative effects of chemical control should also be considered.

In summary, taking into account possible environmental concerns, it appears mechanical control with a combination of small degrees of the other two controls could be the best choice for limiting the disease. These results can be used to guide further research on the exact costings required to control maize streak disease.

**Disclosure statement**

The authors declare they have no conflict of interest.

**Funding**

This work was supported by the National Research Foundation (NRF) of South Africa [grant number 131604].

**Notes on contributors**

Obiora Cornelius Collins obtained PhD in applied mathematics. His research interest includes application of mathematical modeling in studying real life problems such as infectious disease dynamics, ecosystem dynamics etc.

Kevin Jan Duffy is the South African National Research Foundation Chair for Applying Mathematics to Human and Natural Systems and the Director of the Institute of Systems Science, Durban University of Technology, South Africa. He has a PhD from the University of Virginia, United States of America. His MSc. degree and BSc. degree (with a major in mathematics) are from the University of KwaZulu-Natal, South Africa (formerly University of Natal).

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