On the Solution of the Travelling Salesman Problem for Nonlinear Salesman Dynamics using Symbolic Optimal Control

Alexander Weber and Alexander Knoll

Abstract—This paper proposes an algorithmic method to heuristically solve the famous Travelling Salesman Problem (TSP) when the salesman’s path evolves in continuous state space and discrete time but with otherwise arbitrary (nonlinear) dynamics. The presented method is based on the framework of Symbolic Control. In this way, our method returns a provably correct state-feedback controller for the underlying coverage specification, which is the TSP leaving out the requirement for optimality on the route. In addition, we utilize the Lin-Kernighan-Helsgaun TSP solver to heuristically optimize the cost for the overall taken route. Two examples, an urban parcel delivery task and a UAV reconnaissance mission, greatly illustrate the powerfulness of the proposed heuristic.

I. INTRODUCTION

One of the most prominent problems in combinatorial optimization is the Travelling Salesman Problem (TSP), which R. Bellman formulates as: “A salesman is required to visit once and only once each of \( N \) different cities starting from a base city, and returning to this city. What path minimizes the total distance travelled by the salesman?” [1]. In this paper we aim at a fully-automated solution technique for the TSP when it is posed in the \( n \)-dimensional real space with the salesman’s path following nonlinear dynamics. To be specific, the salesman dynamics are assumed to be given by the time-discrete continuous-state control system

\[
x(t+1) \in F(x(t), u(t)),
\]

where \( x \) is the time-discrete state signal (“salesman”) taking values in \( \mathbb{R}^n \), \( u \) is the input signal and \( F \) is a strict set-valued map. In this dynamic setup we interpret the \( N \) cities of the original problem as \( N \) target sets in the state space \( \mathbb{R}^n \). Optimality on the route is understood in terms of minimizing a prescribed cost functional, which is not subject to smoothness restrictions and may impose hard constraints.

This generalization of the TSP is not an academic playground but possible applications are divers. For example, a UAV reconnaissance mission, on the one hand, asks for the most efficient sequence to visit the areas of interest. On the other hand, the dynamical model of the vehicle is more complicated than reducing its motion to straight lines, not to mention obstacles or wind in the mission area. Fig. 1 illustrates such a mission in simulation for which the state-feedback controller will result from the contributions of the present work. Before we outline them we would like to give a brief literature overview to the numerous works on the TSP and its variants. Existing literature can be grouped in basically four categories.

Early works – TSP on networks: The first category includes works which consider the original TSP on networks (more precisely, on ordinary directed or undirected graphs) and investigate algorithms or implementations for exactly or approximately solving the problem. The earliest works appear around 1955-65, e.g. utilizing linear programming [2] or dynamic programming [1], [3]. The Lin-Kernighan heuristic [4] is still considered a milestone for solving the TSP. A maintained software library for it, the Lin-Kernighan-Helsgaun solver, is online available [5], [6]. For a comprehensive survey on works of this decade see [7]. Simple variations of the TSP from this time period should also be mentioned, e.g. the multiple travelling salesman problem [8], [9] or the vehicle routing problem [10].

Advanced variations of the TSP on networks: Another group of works considers networks like the previous works but studies solution methods for more complex variations of the original problem statement. There is the Multi-depot Vehicle Routing Problem with fixed distribution [11], the Heterogeneous Multi-depot Multiple-TSP [12] or the Flying Sidekick TSP [13], just to mention a few. Others can be found in [14]–[17].

The TSP for vehicle dynamics: A series of other works abandons the framework of networks and poses the TSP for the case of vehicle dynamics. Therefore, classical solvers for the TSP cannot be directly applied to obtain an optimal route. More concretely, targets are not connected by straight lines but connecting paths follow nonholonomic planar vehicle dynamics like Dubins vehicle [18]–[21] or the Reed-Shepp vehicle [22]. In [23] a 3-DOF aircraft model is considered and a solution method for the multiple-depot-multiple TSP is given, where also spatial obstacles are taken into account. All these works consider indeed nontrivial dynamics, yet typically tailor their solutions for the particular motion characteristics they assume.

Other related works: Approaching the state of the art from the side of motion planning and algorithmic controller synthesis several works must also be mentioned. In these contexts the optimality on the overall route has not been studied yet but the underlying coverage specification, i.e. the requirement to visit \( N \) target sets in any order. As an LTL formula, a coverage specification reads in its simplest form as

\[
\phi \pi_1 \land \phi \pi_2 \land \ldots \land \phi \pi_N,
\]

where \( \pi_i \) is the proposition that is true if the salesman is in target \( i \) [24]. This class of specifications has been
investigated in [24]–[26] for second order robot models. The special case of two target sets for quite general sampled-data control systems is investigated in [27]–[29].

The contribution of this paper is a constructive method for synthesizing state-feedback controllers enforcing the said coverage specification on plants possessing dynamics (1). Unlike in existing literature, we are allowing for quite general dynamics including uncertainties, hard state constraints and measurement errors in the closed loop. Moreover, the synthesis of the controller is performed in a fully automated fashion. The presented method is based on Symbolic Controller Synthesis as established in [27] and the extension of Symbolic Optimal Control [30]. In addition, we utilize the Lin-Kernighan-Helsgaun solver [5] on top of previous synthesis method to heuristically determine the cheapest sequence for visiting the target sets. In our experiments we solve two Travelling Salesman Problems on sampled-data control systems whose continuous-time dynamics are governed by a differential inclusion.

The rest of this paper is organized as detailed below. Section II contains basic notation. Section III provides the basic formalism of Symbolic Optimal Control. Section IV is devoted to the rigorous definition of the TSP as considered herein. The main results are presented in Section V. Lastly, Section VI includes the simulation results and Section VII contains conclusions.

II. NOTATION

The field of real numbers is denoted by \( \mathbb{R} \). The subset of non-negative real numbers, of integers and of non-negative integers is denoted by \( \mathbb{R}_+ \), \( \mathbb{Z} \) and \( \mathbb{Z}_+ \), respectively. E.g. \( \mathbb{Z}_+ = \{0, 1, 2, \ldots \} \). For \( a, b \in \mathbb{R} \) the open, half-open and closed intervals with endpoints \( a, b \) are denoted by \( ]a, b[ \), \( ]a, b] \) and \( [a, b[ \) respectively, and the discrete versions by \( ]a; b[ \), \( ]a; b] \) and \( [a; b[ \). E.g. \( ]a; b[ = ]a, b[ \cap \mathbb{Z}, \{1; 2\} = \{2\} \). The symbol \( \emptyset \) stands for the empty set. The difference of two sets \( A \) and \( B \) is written as \( A \setminus B \). The restriction of a function \( f: A \to B \) to \( C \subseteq A \) is denoted by \( f|_C \).

III. PRELIMINARIES

The purpose of this section is to define the notion of control loop and optimality on it in the version that is considered in this work. To be specific, the considered class of optimal control problems is defined in Section III-A and its solution in Section III-B. To a large extend, the used concepts are adopted from [27], [29], [30].

A. System dynamics and optimal control problem

Plants with time-discrete dynamics of the form (1) are considered, where \( F: X \times U \to X \) is strict and \( X \) and \( U \) are non-empty sets. The triple

\[
(X, U, F)
\]

is called (transition) system with state and input space \( X \) and \( U \), respectively. Let \( S \) denote (2). The dynamics (1) induces a behaviour initialized at a state \( p \in X \), which is the set of all signal pairs \((u, x) \in (U \times X)^{\mathbb{Z}^+} \) such that \( x(0) = p \) and (1) holds for all \( t \in \mathbb{Z}_+ \). It is denoted by \( B_p(S) \). In this work, a strict set-valued map

\[
\mu: \bigcup_{T \in \mathbb{Z}_+} X^{[0; T]} \to U \times \{0, 1\}
\]

is called controller, where the second component of the image, called stopping signal, indicates if the controller is in operation (‘0’) or is disabled (‘1’) [33]. The closed loop of a system (2) interconnected with a controller (3) is formalized by the closed-loop behaviour initialized at \( p \in X \), which is the set of all signals \((u, v, x) \in (U \times \{0, 1\} \times X)^{\mathbb{Z}^+} \) satisfying

\[
(u, x) \in B_p(S) \quad \text{and} \quad \forall t \in \mathbb{Z}_+ : (u(t), v(t)) \in \mu(x|_{[0; t]}).
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\[
(u, x) \in B_p(S) \quad \text{and} \quad \forall t \in \mathbb{Z}_+ : (u(t), v(t)) \in \mu(x|_{[0; t]}).
\]
rates the full trajectory until stopping.

- The running cost

\[ g: X \times X \times U \rightarrow \mathbb{R}_+ \cup \{ \infty \} \]  

occurs in between two consecutive points in time.

The function (4) is called total cost. Altogether, the following compact form for an optimal control problem can be given [29], [30].

**III.1 Definition.** Consider system (2) and let \( G \) and \( g \) be as in (6) and (7), respectively. The quintuple

\[ (X, U, F, G, g) \]  

is called optimal control problem.

We are interested in finding a controller so that the cost for operating the closed loop is finite and ideally minimized for the worst case.

Returning back to the context of the TSP, in Section IV we are going to define \( G \) in (6) such that it takes the value \( \infty \) if one target set is missed and otherwise 0. Additionally, we wish to minimize (4) along the route by appropriately selecting the order of visiting the target sets.

Next, we formalize the aforementioned notion of “worst case” and the value function of an optimal control problem.

**B. Suboptimal and optimal solutions**

Let us relate the optimal control problem (8) to a suboptimal and optimal solution.

**III.2 Definition.** Let \( \Pi \) be an optimal control problem of the form (8) and let \( J \) be the total cost (4) as defined for \( \Pi \). The map \( L \) assigning \( (p, \mu) \in X \times \mathcal{F}(X, U) \) to

\[ L(p, \mu) := \sup_{(u, v, x) \in \mathcal{B}_p(\mu \times S)} J(u, v, x) \]  

is called performance function of \( \Pi \). The map \( L(\cdot, \mu) \) is called closed-loop performance of \( \mu \).

Now, the notion of optimality can be defined [30, Sect. III.A.VLC].

**III.3 Definition.** Let \( \Pi \) be as in Definition III.2. The value function of \( \Pi \) is the map \( V: X \rightarrow \mathbb{R}_+ \cup \{ \infty \} \) defined by

\[ V(p) = \inf_{\mu \in \mathcal{F}(X, U)} L(p, \mu). \]

A controller \( \mu \in \mathcal{F}(X, U) \) is called optimal if \( V = L(\cdot, \mu) \). An optimal solution of \( \Pi \) is such a pair \( (V, \mu) \).

Suboptimal solutions as defined next will also turn out to be satisfactory in applications. Loosely speaking, a controller is suboptimal if its closed-loop performance is finite at every state at which the value function is finite.

**III.4 Definition.** Let \( \Pi, L \) and \( V \) be as in Definition III.3. Let \( A \subseteq X \). A controller \( \mu \in \mathcal{F}(X, U) \) solves \( \Pi \) suboptimally on \( A \) if for all \( p \in A \) it holds that \( V(p) < \infty \Rightarrow L(p, \mu) < \infty \).

IV. TRAVELLING SALESMAN PROBLEM

We proceed with the rigorous definition of the Travelling Salesman Problem as it is considered in this work. Before that we recall the rigorous formulation of the classical TSP on digraphs and its solution [7].

**A. The classical TSP formulation**

Firstly, we define a tour, which formalizes the order of visiting the \( N \) cities together with the base city. In the next definitions, the integers \( 1, \ldots, N \) represent the cities to visit. We fix index 1 for the base city.

**IV.1 Definition.** Let \( N \in \mathbb{Z} \), \( N \geq 2 \). A finite sequence

\[ (1, t_2, \ldots, t_N, 1) \]

where \( t_i \in [2; N] \) and \( t_i \neq t_j \) for all \( i, j \in [2; N], i \neq j \) is called tour (of length \( N \)).

For example, for \( N = 3 \) the tour \((1, 3, 2, 1)\) means to visit city 3, then city 2 and then returning to the base city. In general, there are \((N - 1)!\) possible tours. Below, the entry \((i, j)\) of the matrix \( C \) is the cost to travel from city \( i \) to city \( j \).

**IV.2 Definition.** Let \( N \) be as in Definition IV.1. A classical travelling salesman problem on a weighted directed graph is a tuple \((N, C)\), where \( C \in \mathbb{R}^{N \times N} \). An optimal solution of \((N, C)\) is an element of the set

\[ \arg \min \left\{ \sum_{i=1}^N C_{t(i), t(i+1)} \mid t:\text{tour of length } N \right\}. \]

Next, we transfer previous problem formulation to the continuous and dynamic setup that we are considering.

**B. Travelling Salesman Problem formulation in this work**

We would like to address some technical details about the problem specification to be defined next.

Firstly, the classical problem formulation prohibits to visit a city twice except for the base city. We will not adopt this requirement in favour of a clearer presentation. However, that restriction can be easily added to our definition. Secondly, we require the salesman not only to visit the targets but also to avoid obstacles in the (continuous) state space during travelling. This requirement can be encoded in the running cost (7) by letting \( g \) satisfy \( g(x, y, u) = \infty \) whenever \( x \) is an element of the obstacle set [30, Ex. III.5].

In the following problem definition the sets \( A_1, \ldots, A_N \) take the role of the \( N \) cities, where \( A_1 \) corresponds to the base city (depot). Unlike in Definition IV.2 the cost to travel from \( A_i \) to \( A_j \) is naturally not explicitly given but is determined by summing up the running cost \( g \) (see (5)).

**IV.3 Definition.** Let \( \Pi \) be an optimal control problem of the form (8) such that \( G \) is defined by

\[ G(x|_{[0; t]}) = \begin{cases} 0 & \text{if condition } (*) \text{ holds} \\ \infty & \text{otherwise} \end{cases} \]
where the involved condition is
\[(\forall s \in (0, t) : x(s) \in A_1) \land (\forall t \in [1; N] \exists s \in (0, t) : x(s) \in A_1) \quad (*),\]
with non-empty sets \(A_1, \ldots, A_N \subseteq X\). Then \(\Pi\) is called Travelling Salesman Problem with target sets \(A_1, \ldots, A_N\) and depot \(A_1\).

Subsequently, if \(S\) denotes the system \((X, U, F)\) then
\[
\text{TSP}_{S, \theta}(A_1, \ldots, A_N)
\]
stands for the Travelling Salesman Problem \((X, U, F, G, g)\) with target sets \(A_1, \ldots, A_N\) and depot \(A_1\).

V. CONTROLLER SYNTHESIS ALGORITHM

In this section, our main contributions are presented, whose core is given in Fig. 3. The algorithm is presented first and then some remarks on implementation and application to sampled-data control systems are discussed.

A. Statement and properties of the algorithm

Before discussing the algorithm, we recall quantitative reach-avoid problems as they play an important role in the algorithm. Roughly speaking, the key idea is to split the TSP into a sequence of special quantitative reach-avoid problems and to use an optimality result in [29].

V.1 Definition ([29]). Let \(\Pi\) be of the form (8) such that
\[
G(x|\pi_{01}) = \begin{cases} G_0(x(t)), & \text{if } x(t) \in A \\
\infty, & \text{otherwise} \end{cases}
\]
defines \(G\), where \(G_0 : X \to \mathbb{R}_+ \cup \{\infty\}\) and \(A \subseteq X\) is a non-empty set. Then \(\Pi\) is called (quantitative) reach-avoid problem associated with \(A\) and \(G_0\).

Below, if \(S\) denotes the system \((X, U, F)\) then
\[
\text{Reach}_{S, \theta}(A, G_0)
\]
stands for the reach-avoid problem \((X, U, F, G, g)\) associated with \(A\) and \(G_0\). An optimal controller for (10) can be represented as a strict set-valued map \(X \Rightarrow U \times \{0, 1\}\) [30].

Before rigorously formulating the properties of the algorithm in Fig. 3a, a rough description is given.

First part (lines 1–15). This part of the algorithm is a fixed-point iteration. In case of success, i.e. in case line 16 is reached, non-empty subsets \(A'_i\) of the target sets \(A_i\) are found for each \(i \in [1; N]\) such that the following holds: For every \(i, j \in [1; N], i \neq j\) an optimal controller for \(\text{Reach}_{S, \theta}(A'_j, 0)\) successfully steers any state in \(A'_i\) to \(A'_j\). In other words, the underlying coverage specification is solved.

Second part (line 16). This part heuristically optimizes the order of visiting the target sets as follows. The cost for reaching \(A'_j\) starting from \(A'_i\) is optimistically estimated using the previously calculated value function \(V_j\) and stored in the entry \((i,j)\) of the matrix \(C\). The resulting classical TSP is then solved.

Third part (lines 17–19). In line 18 controllers that are optimal for visiting two target sets in succession are calculated by utilizing [29, Th. III.1]. More concretely, assuming,

Input: TSP\(_{S, \theta}(A_1, \ldots, A_N)\)
1: \(Q \leftarrow \{1, \ldots, N\}\) \(\quad\) \(\#\) Initialize a queue
2: \((A'_1, \ldots, A'_N) \leftarrow (A_1, \ldots, A_N)\) \(\#\) Subsets of the targets
3: \(\text{while } Q \neq \emptyset \text{ do} \)
4: \(\text{Pick } i \in Q\)
5: \(Q \leftarrow Q \setminus \{i\}\)
6: \(V_i \leftarrow \text{value function of } \text{Reach}_{S, \theta}(A'_i, 0)\)
7: \(\text{for all } j \in Q \text{ do} \)
8: \(\text{if } A'_j \setminus V_j^{-1}(\infty) = \emptyset \text{ then} \)
9: \(\text{return } \text{"Problem can't be solved"} \)
10: \(\text{else if } A'_j \neq A'_j \setminus V_j^{-1}(\infty) \text{ then} \)
11: \(A'_j \leftarrow A'_j \setminus V_j^{-1}(\infty)\) \(\quad\) \(\#\) Shrink \(A'_j\)
12: \(Q \leftarrow Q \cup \{j\}\)
13: \(\text{end if} \)
14: \(\text{end for} \)
15: \(\text{end while} \)
16: \(\text{Tour} \leftarrow \text{solution of classical TSP } (N, C), \text{ where} \ C \in \mathbb{R}^{N \times N}_+\) s. that \(v_{i,j} : C_{i,j} = \min\{V_j(p) \mid p \in A'_j\}\)
17: \(\text{for all } i \in [1; N] \text{ do} \)
18: \(\mu_i \leftarrow \text{optim. controller of } \text{Reach}_{S, \theta}(A'_{\text{Tour}(i)}, V_{\text{Tour}(i+1)})\)
19: \(\text{end for} \)
20: \(\text{return } \text{Tour and } \mu_1, \ldots, \mu_N\) and \(A'_1\)

(a) Controller synthesis algorithm. The highlighted lines heuristically optimize the closed-loop performance of the resulting controller in Fig. 3b.

Input: \(x \in X\)
Require: global integer \(i = 1\)
1: \(\text{if } \mu_{\text{Tour}(i)}(x) \neq \emptyset \text{ and } i \leq N \text{ then} \)
2: \(i \leftarrow i + 1\)
3: \(\text{end if} \)
4: \(\text{return } \mu_{\text{Tour}(i)}(x)\)

(b) Mapping rule of the proposed controller involving the outputs \(\text{Tour}\) and \(\mu_1, \ldots, \mu_N\) of the algorithm in Fig. 3a; The integer \(i\) is initialized to 1 and resides in memory throughout operation.

(c) Illustration of the closed loop. The block “switching logic” corresponds to lines 1–3 in Fig. 3b.

Fig. 3: Proposed controller synthesis algorithm and resulting controller for solving the Travelling Salesman Problem.

Fig. 4: Principle of symbolic controller synthesis [27], [29].
functions suboptimally on does not change the statement of the theorem. Denote by $V_1 \in F(X, U)$ obtained by the algorithm shown in Fig. 3a when applied to $\Pi := \text{TSP}_{S,g}(A_1, \ldots, A_N)$. The controller $\mu \in F(X, U)$ defined by Fig. 3b solves $\Pi$ suboptimally on $A_1$.

Proof. When the algorithm arrives at line 16 the value functions $V_i$ are finite on $A'_j$ for all $i, j \in [1; N]$. Then note that changing line 16 to the trivial tour

$$\text{Tour} \leftarrow (1, 2, \ldots, N, 1)$$

and line 18 to

$$\mu_i \leftarrow \text{optimal controller of Reach}_{S,g}(A'_{\text{Tour}(i)}, 0) \quad (11)$$

does not change the statement of the theorem. Denote by $\Pi_1$ the optimal control problem involved in (11). Note that $V_i = L_i(\cdot, \mu_i)$ for all $i \in [1; N]$, where $L_i$ is the performance function of $\Pi_i$. Let $p \in A'_1$ and $(u, v, x) \in B_p(\mu \times S)$. Then $1 \in \mu(x|_{0,t})_2 = \mu_2(x|_{0,t})_2$ for some $t \in \mathbb{Z}$ since $\mu_2$ solves $\Pi_2$ optimally. So $x(t) \in A'_2$. By induction, for all $i \in [2; N]$ there exists $s \in \mathbb{Z}$ such that $x(s) \in A'_i$. Since $\mu_1$ solves $\Pi_1$ optimally the proof is completed. \hfill $\square$

The controller $\mu$ and the resulting closed loop are illustrated in Fig. 3c.

B. Implementation and application of the algorithm

We would like to comment on implementing the algorithm in Fig. 3a. First, for solving the classical TSP in line 16 we propose the use of the Lin-Kernighan-Helsgaun solver [5]. Clearly, any other solver can also be used. Second, lines 6 and 18 require to solve quantitative reach-avoid problems on the system $S$. For the case that the state and input spaces of $S$ are finite such algorithms exist [28], [30], [34].

Consequently, in combination with the principle of symbolic controller synthesis [27], [30] our synthesis algorithm can be applied to sampled-data control systems, whose dynamics are of the form

$$\dot{x}(t) = f(x(t), u(t)) + W. \quad (12)$$

In (12), $x$ is the state signal, $u$ is the input signal taking values in $U \subseteq \mathbb{R}^m, f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ and $W \subseteq \mathbb{R}^n$ is a set accounting for disturbances. By means of sampling a formulation as a system (2) with $X = \mathbb{R}^n$ is possible under certain assumptions on $f$ and $W$ [27, Sect. VIII.A].

The approach to synthesize controllers for an optimal control problem associated with a sampled system is as follows [30]. The original optimal control problem $\Pi = (X, U, F, g)$ is transferred to an abstract optimal control problem $\Pi' = (X', U', F', g')$. The involved discrete abstraction $(X', U', F')$ of the sampled system associated with (12) has finite state and input space, $X'$ and $U'$. (Typically, $X'$ is a cover of $X$ such that most of its elements are translations of $[0, \eta_1] \times \cdots \times [0, \eta_n]$. The vector $\eta \in \mathbb{R}^n_+$ is called grid parameter, which will be mentioned later in the experimental results.) In the case that $\Pi'$ can be solved the obtained controller $\mu'$ for $\Pi'$ is refined to a controller for $\Pi$. This refinement step requires for the methodology of [27] only the interconnection with a simple quantizer. See Fig. 4.

VI. EXPERIMENTAL RESULTS

Subsequently, two examples are presented, which greatly demonstrate the powerfulness of the proposed heuristic. Both examples are related to “vehicles” since Travelling Salesman Problems naturally are most intuitive when the target sets have a spatial component. Nevertheless it is worth pointing out once more that the presented method can be applied to any transition system with dynamics (1).

A. Reconnaissance mission

Firstly, the reconnaissance mission with an uninhabited aerial vehicle (UAV) that was mentioned as a motivation in Section I is investigated. The mission is illustrated in Fig. 1.

1) Control problem: The dynamics of Dubins vehicle [35] are assumed for the UAV with additional disturbances, i.e. dynamics (12) with $W = [-5, 5] \times [-2, 2] \times [-0.04, 0.04]$ and $f : \mathbb{R}^3 \times U \rightarrow \mathbb{R}^3$ defined by $U = [20, 50] \times [-0.5, 0.5], f(x, u) = (u_1 \cos(x_3), u_1 \sin(x_3), u_2)$.

Thus, the planar position of the UAV is described by $(x_1, x_2)$ and $x_3$ is its heading. The control inputs $u_1$ and $u_2$ are the velocity and the angular velocity, respectively. By theory, a time-discrete version of the continuous dynamics needs to be considered, which is the transition system $S = (\mathbb{R}^3, U, F)$ defined as the sampled system associated with (12) and sampling period $\tau = 0.65$ [27, Def. VIII.1].

The TSP of the form (8) with targets $A_1, A_2, \ldots, A_{41}$ and depot $A_1$ is to be solved on $S$, where $A_1 := A_{\text{wry}}$.

$A_{\text{wry}} = [300, 900] \times [80, 160] \times [-10^5, 10^5], \quad \text{("runway")}$

$A_2 = [375, 425] \times [775, 825] \times \mathbb{R}, \quad \text{("area of interest")}$

$A_3 = [525, 575] \times [775, 825] \times \mathbb{R}, \quad \text{("area of interest")}$

The other target sets are translations of $A_2$, which are positioned as depicted in Fig. 1. The running cost of the mission compromises between minimum time and small absolute angular velocities. Moreover, it includes the requirement of avoiding obstacles (cf. Section IV-B). Specifically,

$$g(x, y, u) = \begin{cases} \infty, & \text{if } x \in (\mathbb{R}^3 \setminus X_{\text{mis}}) \cup A_{\text{nofly}} \cup A_{\text{hill}} \\ \tau + u_2^2, & \text{otherwise (}u_2\text{ in radians)} \end{cases}$$

defines $g$ in (8), where $A_{\text{hill}} \subseteq \mathbb{R}^3$ is a spatial obstacle set as indicated in Fig. 1. $X_{\text{mis}} = [0, 2500] \times [0, 2200] \times \mathbb{R}$ is the mission area and $A_{\text{nofly}} = [320, 880] \times [100, 140] \times [12^\circ, 348^\circ]$ forces a proper approach to the airfield.

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Table I: Computational details to Section VI-A. The implementation is written in C. Computations are executed in parallel with 48 cores on x86-64 SuSE Linux (Intel Xeon E5-2697 v3, 2.6 GHz).

| Quantity | Value (description) |
|----------|---------------------|
| $[X]$ | $41.25 \cdot 10^6$ (grid parameter $(0.1, 0.1, 2\pi/75)$) |
| $[^\circ]$ | 49 (7.7 values of $U$) |
| Runtime lines 1-15 | 69 min. (using [28]) |
| Runtime line 16 | < 1 sec. (using LH−2.0, 0, 9 [5]) |
| Runtime lines 17-19 | 78 min. (using [28]) |
| Total runtime | 147 min. |
| Total RAM usage | 6.7 GB |

Fig. 5: Detail to the reconnaissance mission in Section VI-A. The black-coloured trajectory is the same as in Fig. 1. The blue-coloured trajectory is subject to the opposite disturbance acting on the black-coloured trajectory.

2) Heuristic solution: To solve the defined TSP, the algorithm in Fig. 3 is applied and the controller shown in Fig. 3b controls the UAV. To apply the algorithm, a discrete abstraction $(X', U', F')$ for $S$ is computed. The equations of motion of the truck is in (12) and sampling period $\tau = 0.1$. The delivery task is defined as the TSP (8) with targets sets $A_1, \ldots, A_5$ and depot $A_1$, where

$$A_1 = [12, 20] \times [12, 16] \times ([0, 2\pi] \setminus I_{\text{south}}) \times [0, 7],$$

$$I_{\text{south}} = 3\pi/2 + [−3\pi/8, 3\pi/8],$$

$$A_2 = [43, 47] \times [15, 19] \times [0, 2\pi] \times [0, 7].$$

Both $A_1$ and $A_2$ limit speed while $A_1$ additionally prohibits a truck orientation to the south. The other targets sets are similar to $A_2$ and can be identified from Fig. 6.

The running cost $g$ satisfies $g(x, y, u) = \infty$ in two cases: Firstly, if $x$ is in the obstacles set $(\mathbb{R}^3 \setminus \bar{X}) \cup O$, where $O$ is the union of the grey-coloured sets in Fig. 6 and

$$\bar{X} = [0, 80] \times [0, 30] \times [0, 2\pi] \times [0, 18].$$

Secondly, if $x$ violates the common right-hand traffic rules: the traffic rules are not violated if, e.g., $x$ is in

$$[0, 80] \times [26, 30] \times I_{\text{west}} \times [0, 18], I_{\text{west}} = \pi + \left[−\frac{3\pi}{8}, \frac{3\pi}{8}\right],$$

which is the northernmost lane, or if $x$ is in

$$[10, 22] \times [12, 30] \times [0, 2\pi] \times [0, 18],$$

which are states in proximity of the depot. In the finite case, $g$ balances minimum time and proper driving style, i.e.

$$g(x, y, u) = \tau + u_2^2 + \min_{m \in M} \| (y_1, y_2) - m \|_2.$$ 

Here, $M \subseteq \mathbb{R}^2$ describes the axes of the roadways, e.g., $[2, 78] \times [28] \subseteq M$, and the traffic guidance into the depot by

$$(\{12\} \times [14, 28]) \cup (\{20\} \times [14, 28]) \cup (\{12, 20\} \times [14]) \subseteq M.$$

2) Heuristic solution: A discrete abstraction $(X', U', F')$ is computed, where $X'$ possesses the grid parameter $(8/15, 30)/57, 2\pi/62, 9/25)$ and $U'$ consists of 8-10 values of $U$. The total runtime to solve the problem with the algorithm in Fig. 3a is 3 hours using 21 GB RAM. The total cost for the closed-loop trajectory shown in Fig. 6 is 94.2. Tab. II lists the costs for other possible tours and confirms that the heuristic we proposed returns the cheapest tour.

VII. CONCLUSIONS

We considered a generalization of the Travelling Salesman Problem, where the salesman’s path evolves subject to continuous-state discrete-time dynamics with possible uncertainties. By subdividing the problem into several special (quantitative) reach-avoid problems we succeeded in synthesizing controllers steering the salesman heuristically optimal to its targets. Formally, the obtained controllers are correct-by-design ensuring that the involved coverage specification is enforced, at least qualitatively, on the closed loop.

Finally, we would like to point out that our method can be easily extended to a similar generalization of the Multiple Travelling Salesman Problem [8]. This requires only to slightly generalize line 16 of the algorithm in Fig. 3a. Then, for example, also control policies for multi-UAV missions can be synthesized.

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