RECONSTRUCTING ELECTROMAGNETIC OBSTACLES BY
THE ENCLOSURE METHOD

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ABSTRACT. We present a reconstruction algorithm for recovering both ”magnetic-hard” and ”magnetic-soft” obstacles in a background domain with known isotropic medium from the boundary impedance map. We use in our algorithm complex geometric optics solutions constructed for Maxwell’s equation.

1. Introduction

In this paper we study an inverse boundary value problem for Maxwell’s equation. Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary, filled with isotropic electromagnetic medium, characterized by three parameters: the permittivity \( \varepsilon(x) \), conductivity \( \sigma(x) \) and permeability \( \mu(x) \). A ”magnetic-hard” obstacle is a subset \( D \) of \( \Omega \), with smooth boundary, such that the electric-magnetic field \( (E, H) \) satisfies the following BVP for Maxwell’s equation

\[
\begin{align*}
\nabla \wedge E &= i\omega \mu H, \quad \nabla \wedge H = -i\omega(\varepsilon + i\frac{\sigma}{\omega})E & \text{in } \Omega \setminus D, \\
\nu \wedge E|_{\partial \Omega} &= f, \\
\nu \wedge H|_{\partial D} &= 0
\end{align*}
\]

(1.1)

where \( \nu \) is the unit outer normal vector to the boundary \( \partial \Omega \cup \partial D \). The boundary condition \( \nu \wedge H|_{\partial D} = 0 \) physically appears when an active object is presented. Another type of obstacle, what we call a ”magnetic-soft” obstacle, is a subdomain \( D \) such that the tangential component of the electric field \( \nu \wedge E \) vanishes on the interface \( \partial D \), when a passive object is presented. Define the impedance map by taking the tangential component of the electric field \( \nu \wedge E|_{\partial \Omega} \) to the tangential component of the magnetic field \( \nu \wedge H|_{\partial \Omega} \). Then our purpose is to retrieve information of the shape of \( D \) from the impedance map.

The well-known Calderón’s problem [2] is to determine the conductivity of a medium by making voltage and current measurements of the boundary. The information is encoded in the Dirichlet-to-Neumann map for the conductivity equation \( \nabla \cdot (\gamma \nabla u) = 0 \). In [16], Sylvester and Uhlmann constructed complex geometric optics (CGO) solutions for Schrödinger operator \( \Delta - q \) and proved the uniqueness of \( C^2 \) isotropic conductivity in dimensions \( n \geq 3 \). Further developments including improved regularity assumption, 2D problems and partial
Another application of CGO solutions, the enclosure method was first introduced by Ikehata [5, 6] to identify obstacles, cavities and inclusions embedded in conductive or acoustic medium. Geometrically, using the property of CGO solutions that decay on one side and grow on the other side of a hyperplane, one can enclosing obstacles by those hyperplanes. This idea was generalized to identify non-convex obstacles by Ide et al. [7] for isotropic conductivity equations (conductive medium) and by Nakamura and Yoshida [11] for Helmholtz equations (acoustic medium), by utilizing the so-called complex spherical waves (CSW), namely, CGO solutions with nonlinear Carleman limiting weights. In [19], Uhlmann and Wang constructed generalized complex geometric optics solutions for several systems with Laplacian as the leading order term, e.g., the isotropic elasticity system, and implemented them to reconstruct inclusions.

As for Maxwell’s equation, [15], [12] and [13] answered the uniqueness question for parameters with suitable regularity from the impedance map in a domain Ω. In [13], the Maxwell’s operator was reduced into a matrix Schrödinger operator and vector CGO solutions were constructed to recover electromagnetic parameters.

To address the inverse problem of determining an electromagnetic obstacle, we observe that solutions of a non-dissipative Maxwell’s equation (σ = 0) share similar asymptotical behavior (a key equality in Lemma 4.2) to those of Helmholtz equations (a key inequality in Lemma 4.1 in [5]). Therefore, with CGO solutions at hand, the enclosure method is applicable: one can define an indicator function \( I_ρ(τ, t) \) for each direction \( ρ ∈ S^2 \); by adjusting \( t \), the hyperplane moves along \( ρ \); for each \( ρ \) and \( t \), the asymptotical behavior of \( I_ρ(τ, t) \) as \( τ \gg 1 \) produces the support function of the convex hull of \( D \). However, unlike the Schrödinger equation, the CGO solution for Maxwell’s equation doesn’t behave as small perturbation (w.r.t. \( τ \)) of Calderón’s solutions. We overcome this by carefully choosing relatively large ”incoming” fields (w.r.t. \( τ \)) compared to the perturbation.

The rest of the paper is organized as following. In Section 2, we formulate the forward problem for a ”magnetic-hard” obstacle and define rigorously the impedance map. Then we construct the CGO solution of interests in Section 3. The main reconstruction algorithm for ”magnetic-hard” obstacles is introduced and proved in Section 4. Finally, we remark in Section 5 that the scheme also applies to ”magnetic-soft” obstacles. Through the whole paper, well-posedness of a mixed boundary value problem for Maxwell’s equation plays an important role. Hence, for completeness, we include a proof in Appendix A.
Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with smooth boundary and its complement \( \mathbb{R}^3 \setminus \overline{\Omega} \) is connected. We consider electric permittivity \( \varepsilon(x) \), conductivity \( \sigma(x) \) and magnetic permeability \( \mu(x) \) of the background medium as globally defined functions with following properties: there are positive constants \( \varepsilon_m, \varepsilon_M, \mu_m, \mu_M, \sigma_M, \varepsilon_0 \) and \( \mu_0 \) such that for all \( x \in \Omega \)

\[
\varepsilon_m \leq \varepsilon(x) \leq \varepsilon_M, \quad \mu_m \leq \mu(x) \leq \mu_M, \quad 0 \leq \sigma(x) \leq \sigma_M
\]

and \( \varepsilon - \varepsilon_0, \sigma, \mu - \mu_0 \in C^3(\Omega) \).

A "magnetic-hard" obstacle \( D \) (corresponding to the sound-hard obstacle for Helmholtz equations) is a subset of \( \Omega \) such that \( \Omega \setminus D \) is connected. Moreover, the electric field \( \mathbf{E} \) and the magnetic field \( \mathbf{H} \) in \( \Omega \setminus D \) satisfy the following boundary value problem of the time-harmonic Maxwell’s equation

\[
\begin{cases}
\nabla \wedge \mathbf{E} = i\omega \mu \mathbf{H}, \quad \nabla \wedge \mathbf{H} = -i\gamma \mathbf{E} & \text{in } \Omega \setminus \overline{D}, \\
\nu \wedge \mathbf{E} = f \in TH^{1/2}_{\text{Div}}(\partial \Omega) & \text{on } \partial \Omega,
\end{cases}
\tag{2.1}
\]

where \( \gamma = \varepsilon + i\frac{\sigma}{\omega} \), and the "magnetic-hard" boundary condition on the interface \( \partial D \)

\[
(\nu \wedge \mathbf{H})|_{\partial D} = 0,
\tag{2.2}
\]

where \( \nu \) is the unit outer normal vector on \( \partial \Omega \cup \partial D \). Through out this note, we assume the non-dissipative case \( \sigma = 0 \). Then \( \gamma = \varepsilon \) is a real function.

**Notations.** If \( F \) is a function space on \( \partial \Omega \), the subspace of all those \( f \in F^3 \) which are tangent to \( \partial \Omega \) is denoted by \( TF \). For example, for \( u \in (H^s(\partial \Omega))^3 \), we have decomposition \( u = u_t + u_\nu \nu \), where the tangential component \( u_t = -\nu \wedge (\nu \wedge u) \in TH^s(\partial \Omega) \) and the normal component \( u_\nu = u \cdot \nu \in H^s(\partial \Omega) \). For a bounded domain \( \Omega \) in \( \mathbb{R}^3 \), we denote

\[
TH^s_{\text{Div}}(\partial \Omega) = \{ f \in TH^s(\partial \Omega) \mid \text{Div}(f) \in H^s(\partial \Omega) \},
\]

\[
H^k_{\text{Div}}(\Omega) = \{ u \in (H^k(\Omega))^3 \mid \text{Div}(\nu \wedge u|_{\partial \Omega}) \in H^{k-1/2}(\partial \Omega) \},
\]

with norms

\[
\| f \|^2_{TH^s_{\text{Div}}(\partial \Omega)} = \| f \|^2_{H^s(\partial \Omega)} + \| \text{Div}(f) \|^2_{H^s(\partial \Omega)},
\]

\[
\| u \|^2_{H^k_{\text{Div}}(\partial \Omega)} = \| u \|^2_{H^k(\Omega)} + \| \text{Div}(\nu \wedge u|_{\partial \Omega}) \|^2_{H^{k-1/2}(\partial \Omega)},
\]

where \( \text{Div} \) is the surface divergence. There are natural inner products making them Hilbert spaces (see [15]). In addition, we define the weighted \( L^2 \) space in \( \mathbb{R}^3 \):

\[
L^2_\delta = \left\{ f \in L^2_{\text{loc}}(\mathbb{R}^3) : \| f \|^2_{L^2_\delta} = \int (1 + |x|^2)^{\delta} |f(x)|^2 dx < \infty \right\}.
\]
Admissibility. It can be shown (see Appendix A.) that for \( f \in TH^{1/2}_{\text{Div}}(\partial \Omega) \) and \( g \in TH^{1/2}_{\text{Div}}(\partial D) \), the boundary value problem for Maxwell’s equation
\[
\begin{align*}
\nabla \wedge E &= i \omega \mu H, \quad \nabla \wedge H = -i \omega \gamma E \quad \text{in} \quad \Omega \setminus \bar{D}, \\
\nu \wedge E|_{\partial \Omega} &= f \\
\nu \wedge H|_{\partial D} &= g,
\end{align*}
\]
has a unique solution \((E, H) \in H^1_{\text{Div}}(\Omega \setminus D) \times H^1_{\text{Div}}(\Omega \setminus D)\), except for a discrete set of magnetic resonance frequencies \(\{\omega_n\}\). It satisfies
\[
\|E\|_{H^1_{\text{Div}}(\Omega \setminus D)} + \|H\|_{H^1_{\text{Div}}(\Omega \setminus D)} \leq C \left( \|f\|_{TH^{1/2}_{\text{Div}}(\partial \Omega)} + \|g\|_{TH^{1/2}_{\text{Div}}(\partial D)} \right). \quad (2.4)
\]

Main result. Now we are in the position to define the impedance map for non-resonant frequencies, \(\Lambda_D(\nu \wedge E|_{\partial \Omega}) = \nu \wedge H|_{\partial \Omega}\), and it can be shown that \(\Lambda_D : TH^{1/2}_{\text{Div}}(\partial \Omega) \to TH^{1/2}_{\text{Div}}(\partial \Omega)\) is bounded. If \(\omega\) is a resonance frequency, one can replace the impedance map by the Cauchy data set
\[
\mathcal{C}_\omega = \{(\nu \wedge E|_{\partial \Omega}, \nu \wedge E|_{\partial \Omega}) \mid (E, H) \text{ satisfies (2.1) and (2.2)}\} \subset TH^{1/2}_{\text{Div}}(\partial \Omega) \times TH^{1/2}_{\text{Div}}(\partial \Omega).
\]

Denote by \(\Lambda_\emptyset\) the impedance map for the domain without an obstacle. Then the main result of the presenting work is to show

**Theorem 2.1.** For non-dissipative background medium \((\sigma = 0)\), there exists a reconstruction scheme for the obstacle \(D\) from the impedance map \(\Lambda_D\).

3. Complex geometric optics solutions

In [13], the Maxwell’s operator was reduced to an 8\times8 second order Schrödinger matrix operator by introducing the generalized Sommerfeld potential. A vector CGO-solution was constructed for the Schrödinger operator, simplifying the proof in [12]. Similar techniques also appeared in [3] when dealing with the inverse boundary value problems for Maxwell’s equations with partial data. For completeness, we include the construction of the solution in this work (see [13]
for more details) and provides a special choice of the "incoming" fields.

Define the scalar fields $\Phi$ and $\Psi$ as

$\Phi = \frac{i}{\omega} \nabla \cdot (\gamma \mathbf{E})$, $\Psi = \frac{i}{\omega} \nabla \cdot (\mu \mathbf{H})$. \hfill (3.1)

Under appropriate assumptions on $\Phi$ and $\Psi$, Maxwell’s equation is equivalent to

$\nabla \wedge \mathbf{E} - \frac{1}{\gamma} \nabla \left( \frac{1}{\mu} \Psi \right) - i\omega \mu \mathbf{H} = 0$, $\nabla \wedge \mathbf{H} + \frac{1}{\mu} \nabla \left( \frac{1}{\gamma} \Phi \right) + i\omega \gamma \mathbf{E} = 0$. \hfill (3.2)

Moreover, in this case, $\Phi$ and $\Psi$ vanish, leading to a solution of Maxwell’s equation. Let

$X = (\varphi, e, h, \psi)^T \in (\mathcal{D}')^8$ with

$e = \gamma^{1/2} \mathbf{E}$, $h = \mu^{1/2} \mathbf{H}$,

$\varphi = \frac{1}{\gamma \mu^{1/2}} \Phi$, $\psi = \frac{1}{\gamma^{1/2} \mu} \Psi$.

Then (3.1) and (3.2) read

$(P(i\nabla) - k + V)X = 0$, in $\Omega$ \hfill (3.3)

where

$P(i\nabla) = \begin{pmatrix}
0 & \nabla \cdot & 0 & 0 \\
\nabla & 0 & \nabla \wedge & 0 \\
0 & -\nabla \wedge & 0 & \nabla \\
0 & 0 & \nabla \cdot & 0
\end{pmatrix}$,

$V = (k - \kappa)1_8 + \begin{pmatrix}
0 & \nabla \cdot & 0 & 0 \\
\nabla & 0 & -\nabla \wedge & 0 \\
0 & \nabla \wedge & 0 & \nabla \\
0 & 0 & \nabla \cdot & 0
\end{pmatrix} D D^{-1}$

are matrix operators and

$D = \text{diag}(\mu^{1/2}, \gamma^{1/2} 1_3, \mu^{1/2} 1_3, \gamma^{1/2})$, $\kappa = \omega (\gamma \mu)^{1/2}$, $k = \omega (\varepsilon_0 \mu_0)^{1/2}$.

An important property of this operator is that it allows to reduce Maxwell’s equation to the Schrödinger matrix equation by noticing

$(P(i\nabla) - k + V)(P(i\nabla) + k - V^T) = -(\Delta + k^2)1_8 + Q$, \hfill (3.4)

where

$Q = VP(i\nabla) - P(i\nabla)V^T + k(V + V^T) - VV^T$ is a zeroth-order matrix multiplier. Hence, by writing an ansatz for $X$, one can define the generalized Sommerfeld potential $Y$

$X = (P(i\nabla) + k - V^T)Y$.

So it satisfies the Schrödinger equation

$(-\Delta - k^2 + Q)Y = 0$. \hfill (3.5)
The following CGO-solution is constructed using Faddeev’s Kernel. Let $\zeta \in \mathbb{C}^3$ be a vector with $\zeta \cdot \zeta = k^2$. Suppose $y_{0,\zeta} \in \mathbb{C}^8$ is a constant vector with respect to $x$ and bounded with respect to $\zeta$. We refer $e^{ix \cdot \zeta}y_{0,\zeta}$ as the "incoming" field. Then there exists a unique solution of (3.5) of the form

$$Y_\zeta(x) = e^{ix \cdot \zeta}(y_{0,\zeta} - v_\zeta(x)),$$

where $v_\zeta(x) \in (L^2_{\delta+1})^8$ satisfying

$$\|v_\zeta\|_{L^2_{\delta+1}} \leq C/|\zeta|$$

for $\delta \in (-1,0)$. Moreover, one can show that $v_\zeta \in (H^s(\Omega))^8$ for $0 \leq s \leq 2$, e.g., see [1], and

$$\|v_\zeta(x)\|_{H^s(\Omega)} \leq C|\zeta|^{s-1}. \tag{3.6}$$

Lemma 3.1 in [13] states that if we choose $y_{0,\zeta}$ such that the first and the last components of $(P(\zeta) - k)y_{0,\zeta}$ vanish, where $P(\zeta)$ is the matrix obtained by replacing $i\nabla$ by $\zeta$ in $P(i\nabla)$, then for large $|\zeta|$, $X_\zeta$ provides the solution of the original Maxwell’s equation. We proceed to provide a more specific choice of $y_{0,\zeta}$ such that the CGO solution for Maxwell’s equation has special properties.

As in [13], choose

$$y_{0,\zeta} = \frac{1}{|\zeta|}(\zeta \cdot a, ka, kb, \zeta \cdot b)^T,$$

where

$$\zeta = -i\tau \rho + \sqrt{\tau^2 + k^2} \rho^\perp,$$

with $\rho, \rho^\perp \in \mathbb{S}^2$ and $\rho \cdot \rho^\perp = 0$. $\tau > 0$ is used to control the size of $|\zeta| = \sqrt{2\tau^2 + k^2}$. Then we obtain

$$x_{0,\zeta} := (P(-\zeta) + k)y_{0,\zeta} = \frac{1}{|\zeta|} \begin{pmatrix} 0 \\ - (\zeta \cdot a)\zeta - k\zeta \land b + k^2 a \\ k\zeta \land a - (\zeta \cdot b)\zeta + k^2 b \\ 0 \end{pmatrix}$$

satisfying the condition in Lemma 3.1 in [13].

Taking $\tau \to \infty$, we have

$$\frac{\zeta}{|\zeta|} \to \hat{\zeta} = \frac{1}{\sqrt{2}}(-i\rho + \rho^\perp).$$

We choose $a$ and $b$ such that

$$\hat{\zeta} \cdot b = 1, \quad \hat{\zeta} \cdot a = 0.$$
This is satisfied, for example, by taking \( a \in \mathbb{R}^3 \) and \( b \in \mathbb{C}^3 \) satisfying \( a \perp \rho \), \( a \perp \rho^\perp \) and \( b = \overline{\zeta} \). Given these choices, It’s easy to see that

\[
\eta := (x_0, \zeta)_2 \to -k\zeta \land b = ik\rho \land \rho^\perp \quad (\sim \mathcal{O}(1)) \quad \text{as} \quad \tau \to \infty,
\]

\[
\theta := (x_0, \zeta)_3 \sim \mathcal{O}(\tau) \quad \text{as} \quad \tau \to \infty.
\]

Then \( X_\zeta = (P(i\nabla) + k - V^T)Y_\zeta \) is written in the form

\[
X_\zeta = e^{\tau(x \cdot \rho)} + i\sqrt{\tau^2 + k^2} x \rho^\perp (x_0, \zeta + r_\zeta(x))
\]

where

\[
r_\zeta = P(-\zeta)v_\zeta + P(i\nabla)v_\zeta - V^T y_{0, \zeta} + kv_\zeta - V^Tv_\zeta \quad \text{(3.7)}
\]

satisfying for \( C > 0 \) independent of \( \zeta \)

\[
\|r_\zeta\|_{L^2(\Omega)} \leq C.
\]

Summing up, we obtain the following

**Proposition 3.1.** Let \( \rho, \rho^\perp \in \mathbb{S}^2 \) with \( \rho \cdot \rho^\perp = 0 \). Assume \( \omega \) is not a resonant frequency. Given \( \theta, \eta \in \mathbb{C}^3 \) as above, then for \( \tau > 0 \) large enough, there exists a unique complex geometric optics solution \( (E_0, H_0) \) of Maxwell’s equation

\[
\nabla \land E_0 = i\omega\mu H_0 \quad \nabla \land H_0 = -i\omega\varepsilon E_0 \quad \text{in} \quad \Omega
\]

of the form

\[
E_0 = \varepsilon^{-1/2}e^{\tau(x \cdot \rho)} + i\sqrt{\tau^2 + k^2} x \rho^\perp (\eta + R)
\]

\[
H_0 = \mu^{-1/2}e^{\tau(x \cdot \rho)} + i\sqrt{\tau^2 + k^2} x \rho^\perp (\theta + Q).
\]

Moreover, we have

\[
\eta = \mathcal{O}(1), \quad \theta = \mathcal{O}(\tau) \quad \text{for} \quad \tau \gg 1,
\]

and \( R = (r_\zeta)_2, Q = (r_\zeta)_3 \) are bounded in \( (L^2(\Omega))^3 \) for \( \tau \gg 1 \).

For reconstruction, one needs to compute the boundary tangential CGO-fields \( (\nu \land E_0|_{\partial \Omega}, \nu \land H_0|_{\partial \Omega}) \). In \([12, 13]\), by solving a boundary integral equation, one can recover them from the impedance map \( \Lambda_{\emptyset} \) if the background parameters are unknown. In our case with known medium, CGO-fields are known.

### 4. Reconstruction Scheme

Adding a parameter \( t > 0 \) into the CGO-solution in Proposition 3.1, we use

\[
E_0 = \varepsilon^{-1/2}e^{\tau(x \cdot \rho - t)} + i\sqrt{\tau^2 + k^2} x \rho^\perp (\eta + R)
\]

\[
H_0 = \mu^{-1/2}e^{\tau(x \cdot \rho - t)} + i\sqrt{\tau^2 + k^2} x \rho^\perp (\theta + Q) \quad \text{(4.1)}
\]

to define an indicator function which physically measures the differences between the energies required to keep the same boundary CGO electric field for the domain \( \Omega \) with and without the obstacle \( D \).
Definition 1. For $\rho \in S^2$, $\tau > 0$ and $t > 0$ we define the indicator function

$$I_\rho(\tau,t) := \int_{\partial\Omega} (\nu \wedge E_0) \cdot ((\Lambda_D - \Lambda_{\emptyset})(\nu \wedge E_0) \wedge \nu) \, dS$$

where $E_0$ is a CGO solution of Maxwell’s equation given by (4.1).

The enclosure method’s aim is to recover the convex hull $\text{ch}(D)$ of $D$ by reconstructing the following support function.

Definition 2. For $\rho \in S^2$, we define the support function of $D$ by

$$h_D(\rho) := \sup_{x \in D} x \cdot \rho.$$

Then, the reconstruction scheme in Theorem 2.1 is

Theorem 4.1. We assume that the set $\{x \in \mathbb{R}^3 \mid x \cdot \rho = h_D(\rho)\} \cap \partial D$ consists of one point and the Gaussian curvature of $\partial D$ is not vanishing at that point. Then, we can recover $h_D(\rho)$ by

$$h_D(\rho) = \inf \{ t \in \mathbb{R} \mid \lim_{\tau \to \infty} I_\rho(\tau,t) = 0 \}.$$<ref>
Moreover, if $D$ is strictly convex (the Gaussian curvature is everywhere positive), then we can reconstruct $D$.
</ref>

Remark 1. The proof of Theorem 4.1 mainly consists of showing the following limits:

$$\lim_{\tau \to \infty} I_\rho(\tau,t) = 0, \quad \text{when} \quad t > h_D(\rho);$$

$$\liminf_{\tau \to \infty} I_\rho(\tau,h_D(\rho)) = C > 0.$$<ref>
Remark 2. Gaussian curvature at a point on a surface is defined to be the product of two principal curvatures, which measure how the surface bends by different amounts in different directions at that point. A surface with positive Gaussian curvature at a point is locally convex. Note that the non-vanishing assumption on the Gaussian curvature in the theorem is not crucial since only finitely many directions $\rho$ violate the condition.
</ref>

4.1. A key integral equality. To show the limits in Remark 1, we need the following equality for non-dissipative Maxwell’s equation

Lemma 4.2. Let $\sigma = 0$. Assume $(E, H)$ is a solution of

$$\nabla \wedge E = i \omega \mu H, \quad \nabla \wedge H = -i \omega \varepsilon E,$$

in $\Omega \setminus \overline{D}$

satisfying the boundary condition

$$\nu \wedge H |_{\partial D} = 0 \quad \text{and} \quad \nu \wedge E |_{\partial \Omega} = \nu \wedge E_0 |_{\partial \Omega}.$$
Then we have
\[
\begin{align*}
  & i\omega \int_{\partial \Omega} (\nu \wedge E_0) \cdot \left[ (\nu \wedge H - \nu \wedge H_0) \wedge \nu \right] dS \\
  &= \int_{\Omega \setminus \overline{D}} \mu^{-1} |\nabla \wedge E - \nabla \wedge E_0|^2 - \omega^2 \varepsilon |E - E_0|^2 dx \\
  &+ \int_D \mu^{-1} |\nabla \wedge E_0|^2 - \omega^2 \varepsilon |E_0|^2 dx.
\end{align*}
\]
(4.2)

**Proof:** Denote
\[
I := i\omega \int_{\partial \Omega} (\nu \wedge E_0) \cdot \left[ (\nu \wedge H - \nu \wedge H_0) \wedge \nu \right] dS = i\omega \int_{\partial \Omega} (\nu \wedge E_0) \cdot (H - H_0)dS.
\]
First by integration by parts, we have
\[
\begin{align*}
  &\int_{\Omega \setminus \overline{D}} \mu^{-1} (\nabla \wedge E) \cdot (\nabla \wedge E - \nabla \wedge E_0) - \omega^2 \varepsilon E \cdot (E - E_0) dx \\
  &= -\left( \int_{\partial \Omega} - \int_{\partial D} \right) (\nu \wedge \mu^{-1} (\nabla \wedge E)) \cdot (E - E_0)dS = 0
\end{align*}
\]
by boundary conditions. Adding this to the following equality
\[
I = \int_{\partial \Omega} (\nu \wedge E_0) \cdot (\nabla \wedge E) dS - \int_{\partial D} (\nu \wedge \mu^{-1} (\nabla \wedge E)) \cdot (E - E_0)dS = 0
\]
with the last term vanishing due to the zero-boundary condition on the interface,
\[
\int_{\partial D} (\nu \wedge E_0) \cdot (-i\omega H) dS = \int_{\partial D} (\nu \wedge E_0) \cdot (\nabla (\nu \wedge H) \wedge \nu) dS = 0,
\]
we obtain (4.2).

4.2. **Proof of Theorem 4.1.** We proceed to show the first limit in Remark 1
\[
\lim_{\tau \to \infty} I_{\rho}(\tau, t) = 0 \quad \text{if} \quad t > h_D(\rho)
\]
by proposing an upper bound of the indicator function.

Let \( \tilde{E} = E - E_0 \) be the reflected solution in \( \Omega \setminus \overline{D} \). It satisfies
\[
\begin{cases}
  \nabla \wedge (\mu^{-1} \nabla \wedge \tilde{E}) - \varepsilon^{-1} \nabla (\varepsilon \tilde{E}) - \omega^2 \varepsilon \tilde{E} = 0 & \text{in } \Omega \setminus \overline{D}, \\
  \nu \wedge \tilde{E}_{|\partial \Omega} = 0, \\
  \nu \wedge (\mu^{-1} \nabla \wedge \tilde{E})_{|\partial D} = -i\omega \nu \wedge H_0_{|\partial D} \in TH_{\text{Div}}^{1/2}(\partial D).
\end{cases}
\]
(4.4)
By (A.10) in Appendix A., we have
\[ \| \mathbf{E} \|_{L^2(\Omega \setminus \mathcal{D})}^2 \leq C \| \nabla \times \mathbf{H}_0 \|_{H^{-1/2}(\partial D)}^2 \leq C(\| \nabla \times \mathbf{E}_0 \|_{L^2(D)}^2 + \| \mathbf{E}_0 \|_{L^2(D)}^2), \]  
(4.5)
where the second inequality is valid since
\[ i \omega (\nabla \times \mathbf{H}_0, \mathbf{F})_{\partial D} = (\mu^{-1} \nabla \times \mathbf{E}_0, \nabla \times \mathbf{F})_D - (i \omega \mathbf{E}_0, \mathbf{F})_D \quad \text{for} \quad F \in (H^1(D))^3. \]
Therefore, by (4.2) and (4.5), we have
\[ I_\rho(\tau, t) \leq C(\| \mathbf{E}_0 \|_{L^2(D)}^2 + \| \nabla \times \mathbf{E}_0 \|_{L^2(D)}^2) \leq C(\| \mathbf{E}_0 \|_{L^2(D)}^2 + \| \mathbf{H}_0 \|_{L^2(D)}^2). \]  
(4.6)
Plugging in the CGO-solution (4.1), we obtain the following estimates:
\[ \| \mathbf{E}_0 \|_{L^2(D)}^2 \leq C e^{2\tau (h_D(\rho) - t)} \| \eta + R \|_{L^2(D)}^3 \sim e^{2\tau (h_D(\rho) - t)} \quad \tau \gg 1, \]
\[ \| \mathbf{H}_0 \|_{L^2(D)}^2 \leq C e^{2\tau (h_D(\rho) - t)} \| \theta + Q \|_{L^2(D)}^3 \sim \tau^2 e^{2\tau (h_D(\rho) - t)} \quad \tau \gg 1. \]  
(4.7)
Therefore, we obtain
\[ I_\rho(\tau, t) \leq C \tau^2 e^{2\tau (h_D(\rho) - t)} \]
for \( \tau \) large enough, proving the first limit (4.3).

To show the second limit
\[ \liminf_{\tau \to -\infty} I_\rho(\tau, h_D(\rho)) = C > 0, \]  
(4.8)
it suffices to prove the following two lemmas.

**Lemma 4.3.** If \( t = h_D(\rho) \) in CGO-solution (4.1), then
\[ \liminf_{\tau \to -\infty} \int_D \mu^{-1} |\nabla \times \mathbf{E}_0|^2 \, dx = C, \]
with some constant \( C > 0 \).

**Lemma 4.4.** If \( t = h_D(\rho) \), then there exists a positive number \( c \) such that
\[ \frac{\omega^2 \left( \int_{\Omega \setminus \mathcal{D}} |\mathbf{E} - \mathbf{E}_0|^2 \, dx + \int_D \varepsilon |\mathbf{E}_0|^2 \, dx \right)}{\int_D \mu^{-1} |\nabla \times \mathbf{E}_0|^2 \, dx} \leq c < 1, \]  
(4.9)
for \( \tau \) large enough.

**Proof of Lemma 4.3:** This is obtained by noticing, in Proposition 3.1, that the first order growth of the constant vector \( \theta \) in \( \mathbf{H}_0 \) with respect to \( \tau \). Then the left hand side integral
\[ \int_D \mu^{-1} |\nabla \times \mathbf{E}_0|^2 \, dx \geq C \| \mathbf{H}_0 \|_{L^2(D)^3}^2 \geq C \int_D \tau^2 e^{2\tau (x \cdot \rho - h_D(\rho))} \, dx \geq C \quad \text{for} \quad \tau \gg 1. \]
To show the last inequality, we denote by \( x_0 \) the point in \( \{ x \in \mathbb{R}^3 \mid x \cdot \rho = h_D(\rho) \} \cap \partial D \). It’s not hard to see that there exist \( C_\rho > 0 \) and \( \delta_\rho > 0 \) such that
\[ \mu_2(D_\rho(\delta_\rho, s)) \geq C_\rho s. \]
where $\mu_2$ denotes the two dimensional Lebesgue measure and
\[
D_\rho(\delta, s) = \{ x \in D \mid x \cdot \rho = h_D(\rho) - s \}.
\]

Then we decompose $D$ into
\[
D_\rho(\delta) = \{ x \in D \mid h_D(\rho) - \delta < x \cdot \rho \leq h_D(\rho) \}
\]
and $D \setminus D_\rho(\delta)$. The integral of $\tau^2 e^{2\tau(x \cdot \rho - h_D(\rho))}$ on $D \setminus D_\rho(\delta)$ vanishes as $\tau \to \infty$.

On $D_\rho(\delta)$, we have
\[
\int_{D_\rho(\delta)} \tau^2 e^{2\tau(x \cdot \rho - h_D(\rho))} = \tau^2 \int_0^{\delta} ds \int_{D_\rho(\delta, s)} e^{-2\tau s} dS \geq C_\rho \tau^2 \int_0^{\delta} s e^{-2\tau s} ds = C_\rho \int_0^{\tau \delta} s e^{-2\tau s} ds \to \frac{1}{4} C_\rho \text{ as } \tau \to \infty.
\]

Proof of Lemma 4.4: The proof follows a similar scheme to [5] for Helmholtz equations.

Noticing that
\[
\omega^2 \int_D \varepsilon |E_0|^2 dx = \int_D \mu^{-1} |\nabla \wedge E_0|^2 dx \sim \mathcal{O}(\tau^{-2}) \quad \text{for } \tau \gg 1,
\]

it suffices to show
\[
\lim_{\tau \to \infty} \omega^2 \int_{\Omega \setminus \mathcal{D}} \varepsilon |E - E_0|^2 dx = \lim_{\tau \to \infty} \frac{\int_{\Omega \setminus \mathcal{D}} \varepsilon |E - E_0|^2 dx}{\int_D \mu |H_0|^2 dx} = 0.
\]

To estimate the numerator, we consider the boundary value problem:
\begin{align*}
\begin{cases}
\nabla \wedge P = i\omega \mu Q, & \nabla \wedge Q = -i\omega \varepsilon P - \frac{i\varepsilon}{\omega} \bar{E} \quad \text{in } \Omega \setminus \mathcal{D},
\nu \wedge P|_{\partial \Omega} = 0,
\nu \wedge Q|_{\partial D} = 0.
\end{cases}
\end{align*}
\tag{4.10}

or equivalently
\begin{align*}
\begin{cases}
\nabla \wedge (\mu^{-1} \nabla \wedge P) - s \varepsilon^{-1} \nabla (\nabla \cdot \varepsilon P) - \omega^2 \varepsilon P = e \bar{E} \quad \text{in } \Omega \setminus \mathcal{D},
\nu \wedge P|_{\partial \Omega} = 0,
\nu \wedge (\nabla \wedge P)|_{\partial D} = 0.
\end{cases}
\end{align*}
\tag{4.11}

Note that
\[
\nabla \cdot \varepsilon \bar{E} = 0, \quad \text{in } \Omega \setminus \mathcal{D}
\]
because $\nabla \cdot \varepsilon \bar{E} = 0$ in $\Omega \setminus \mathcal{D}$.
Since \( \omega \) is admissible, the boundary value problem (4.11) is well-posed for
\[ \varepsilon \tilde{E} = \varepsilon (E - E_0) \] which is in \( H^1_\text{Div}(\Omega \setminus \overline{D}) \). Moreover, by Proposition A.3, one has
\[ P \in (H^2(\Omega \setminus \overline{D}))^3 \] satisfying
\[ \|P\|_{H^2(\Omega \setminus \overline{D})} \leq C\|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})}. \]
By the Sobolev embedding theorem, we have
\[ |P(x) - P(y)| \leq C|x - y|^{1/2}\|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})} \quad \text{for } x, y \in \Omega \setminus \overline{D}, \]
\[ \sup_{x \in \Omega \setminus \overline{D}} |P(x)| \leq C\|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})}. \]
Since
\[ \nabla \wedge (\mu^{-1} \nabla \wedge P) = \omega^2 \varepsilon P = \varepsilon \tilde{E}, \]
integration by parts gives
\[
\int_{\Omega \setminus \overline{D}} \varepsilon |\tilde{E}|^2 \, dx = \int_{\Omega \setminus \overline{D}} \tilde{E} \cdot (\nabla \wedge (\mu^{-1} \nabla \wedge P) - \omega^2 \varepsilon P) \, dx
\]
\[
= \int_{\Omega \setminus \overline{D}} \mu^{-1} (\nabla \wedge \tilde{E}) \cdot (\nabla \wedge P) - \omega^2 \varepsilon \tilde{E} \cdot P \, dx
\]
\[+ \left( \int_{\partial \Omega} - \int_{\partial D} \right) \tilde{E} \cdot (\nu \wedge (\mu^{-1} \nabla \wedge P)) \, dS
\]
\[= \int_{\Omega \setminus \overline{D}} \nabla \wedge (\mu^{-1} \nabla \wedge \tilde{E}) \cdot P - \omega^2 \varepsilon \tilde{E} \cdot P \, dx
\]
\[ - \left( \int_{\partial \Omega} - \int_{\partial D} \right) \nu \wedge (\mu^{-1} \nabla \wedge \tilde{E}) \cdot P \, dS
\]
\[= - \int_{\partial D} \nu \wedge (\mu^{-1} \nabla \wedge E_0) \cdot P \, dS. \]
Expanding the RHS of the last equality at \( x_0 \), one has
\[
\int_{\Omega \setminus \overline{D}} \varepsilon |\tilde{E}|^2 \, dx = \int_{\partial D} (P(x_0) - P(x)) \cdot \nu \wedge (\mu^{-1} \nabla \wedge E_0) \, dS - \int_{\partial D} \omega^2 \varepsilon E_0 \cdot P(x_0) \, dx
\]
\[\leq C \left\{ \int_{\partial D} |x - x_0|^{1/2} |\nu \wedge H_0| \, dS + \int_{\partial D} |E_0| \, dx \right\} \|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})}
\]
\[\leq C \left\{ \int_{\partial D} \tau |x - x_0|^{1/2} \varepsilon^{(x, \rho - D)} \, dS + \int_{\partial D} \varepsilon^{(x, \rho - D)} \, dx \right\} \|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})}
\]
This yields
\[
\|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})}^2 \leq C \left( \frac{\int_{\Omega \setminus \overline{D}} \varepsilon |\tilde{E}|^2 \, dx}{\|\tilde{E}\|_{L^2(\Omega \setminus \overline{D})}} \right)^2
\]
\[\leq C \left\{ \tau^2 \left( \int_{\partial D} |x - x_0|^{1/2} \varepsilon^{(x, \rho - D)} \, dS \right)^2 + \left( \int_{\partial D} \varepsilon^{(x, \rho - D)} \, dx \right)^2 \right\}. \]
It’s easy to see that the second term \( \left( \int_D e^{\tau(x \cdot \rho - h_D(\rho))}dx \right)^2 \) can be absorbed by the denominator \( \int_D \mu^{-1} |\nabla \wedge E_0|^2 dx \) in (4.9) for large \( \tau \). Therefore, it’s sufficient to show
\[
\lim_{\tau \to \infty} \tau \int_{\partial D} |x - x_0|^{1/2} e^{\tau(x \cdot \rho - h_D(\rho))} dS = 0.
\]
This is shown in [5], where the assumption that the Gaussian curvature of \( \partial D \) at \( x_0 \) is non-vanishing was used. This completes the proof of the lemma, hence proves the theorem. ■

5. Enclosing ”magnetic-soft” obstacles and inclusions

In [6], the reconstruction procedure for sound-hard obstacles also works for sound-soft obstacles. Inspired by this, our method also applies to enclosing a ”magnetic-soft” obstacle. Suppose our domain \( \Omega \), obstacle \( D \) and all the electromagnetic parameters in the background satisfy the same hypothesis in the ”magnetic-hard” case, except that the fields \( (E, H) \) satisfy Maxwell’s equation (2.1) with the boundary condition
\[
\nu \wedge E|_{\partial D} = 0.
\]
Then the reconstruction scheme Theorem 4.1 applies simply by noticing the following key equality.

Lemma 5.1. Let \( \sigma = 0 \). Assuming Maxwell’s equation with the boundary conditions (5.1) and \( \nu \wedge E|_{\partial \Omega} = \nu \wedge E_0|_{\partial \Omega} \), we have \( (E, H) \) satisfying
\[
-I = \int_{\Omega \setminus D} \mu^{-1} |\nabla \wedge E - \nabla \wedge E_0|^2 - \omega^2 \varepsilon |E - E_0|^2 dx + \int_D \mu^{-1} |\nabla \wedge E_0|^2 - \omega^2 \varepsilon |E_0|^2 dx
\]
where
\[
I := i\omega \int_{\partial \Omega} (\nu \wedge E_0) \cdot \left[ (\nu \wedge H - \nu \wedge H_0) \wedge \nu \right] dS.
\]
Therefore, the proof essentially follows the ”magnetic-hard” obstacle case, except that to show
\[
\lim_{\tau \to \infty} \int_{\Omega \setminus D} \varepsilon |E - E_0|^2 dx = 0,
\]
we implement the regularity of the solution for an auxiliary boundary value problem similar to (4.11)
\[
\begin{cases}
\nabla \wedge P = i\omega \mu Q, & \nabla \wedge Q = -i\omega \varepsilon P - \frac{i}{\omega} \overline{E} \quad \text{in } \Omega \setminus \overline{D}, \\
\nu \wedge P|_{\partial \Omega} = 0, & \nu \wedge Q|_{\partial D} = 0,
\end{cases}
\]
see [8], and the Sobolev embedding theorem.
A. WELL-POSEDNESS OF A MIXED BOUNDARY VALUE PROBLEM FOR MAXWELL’S EQUATIONS

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with smooth boundary and \( \overline{D} \subset \Omega \), consider the boundary value problem of the Maxwell equation

\[
\begin{cases}
\nabla \wedge E = i \omega \mu H & \text{in } \Omega \setminus \overline{D}, \\
\nu \wedge E|_{\partial \Omega} = f, \\
\nu \wedge (\mu^{-1} \nabla \wedge E)|_{\partial D} = i \omega \nu \wedge H|_{\partial D} = g.
\end{cases}
\]  

(A.1)

Here \( \varepsilon \) and \( \gamma \) are complex-valued functions in \( C^k(\Omega \setminus \overline{D}) \) with positive real parts and \( \omega \in \mathbb{C} \).

**Theorem A.1.** There is a discrete subset \( \Sigma \) of \( \mathbb{C} \) such that for \( \omega \) not in \( \Sigma \), there exists a unique solution \((E, H) \in H^k_{\text{Div}}(\Omega \setminus \overline{D}) \times H^k_{\text{Div}}(\Omega \setminus \overline{D})\) of (A.1) given any \( f \in TH^k_{\text{Div}}(\partial \Omega) \) and \( g \in TH^k_{\text{Div}}(\partial D) \). The solution satisfies

\[
\|E\|_{H^k_{\text{Div}}(\Omega \setminus \overline{D})} + \|H\|_{H^k_{\text{Div}}(\Omega \setminus \overline{D})} \leq C(\|f\|_{TH^{k-1/2}_{\text{Div}}(\partial \Omega)} + \|g\|_{TH^{k-1/2}_{\text{Div}}(\partial D)}) \]  

(A.2)

with \( C > 0 \) independent of \( f \) and \( g \).

Here we proceed to prove the theorem by modifying the variational method in [8], [2] and [9]. From the Maxwell’s equation (A.1), the electric field \( E \) satisfies the second order equation

\[
\nabla \wedge (\mu^{-1} \nabla \wedge E) - \omega^2 \gamma E = 0 \quad \text{in } \Omega \setminus \overline{D},
\]  

and

\[
\nabla \cdot \gamma E = 0 \quad \text{in } \Omega \setminus \overline{D}.
\]  

(A.3)

Therefore, we consider

\[
\nabla \wedge (\mu^{-1} \nabla \wedge E) - s \gamma^{-1} \nabla(\nabla \cdot \gamma E) - \omega^2 \gamma E = 0 \quad \text{in } \Omega \setminus \overline{D}
\]  

(A.4)

where \( s \) is a positive real number. The equation (A.3) will follow from

\[
s \nabla \cdot (\gamma^{-1} \nabla(\nabla \cdot \gamma E)) + \omega^2 \nabla \cdot \gamma E = 0
\]

which is obtained by taking divergence of (A.4). Denoting the \( L^2(\Omega \setminus \overline{D}) \) inner product by \( \langle \cdot, \cdot \rangle \) and \( L^2(\Gamma) \) inner products by \( \langle \cdot, \cdot \rangle_\Gamma \) (where \( \Gamma = \partial \Omega \) or \( \partial D \)), we define the bilinear form associated with the elliptic system (A.4):

\[
B(E, F) := \langle \mu^{-1} \nabla \wedge E, \nabla \wedge E \rangle + s \langle \nabla \cdot \gamma E, \nabla \cdot \gamma E \rangle
\]  

(A.5)

for \( E, F \in X \) where

\[
X = \{ F \in (H^1(\Omega \setminus \overline{D}))^3 | \nu \wedge F|_{\partial \Omega} = 0, \nu \cdot \gamma F|_{\partial D} = 0 \}.
\]

By Green’s formulæ, we have that \( B \) is related to the differential operator

\[
P = \nabla \wedge (\mu^{-1} \nabla \wedge) - s \gamma^{-1} \nabla(\nabla \cdot \gamma)
\]
by
\[ B(\mathbf{E}, \mathbf{F}) = (\mathbf{PE}, \mathbf{F}) - \langle \nu \wedge (\mu^{-1} \nabla \wedge \mathbf{E}), \mathbf{F} \rangle_{\partial(\Omega \setminus \overline{D})} + \langle s \nabla \cdot \gamma \mathbf{E}, \nu \cdot \gamma \mathbf{F} \rangle_{\partial(\Omega \setminus \overline{D})} \] (A.6)

for \( \mathbf{E}, \mathbf{F} \in (H^1(\Omega \setminus \overline{D}))^3 \). Then for \( \tilde{f} \in X' \) the weak formulation of the mixed boundary value problem

\[
\begin{cases}
\mathbf{PE} = \tilde{f} \quad \text{in } \Omega \setminus \overline{D}, \\
\nu \wedge \mathbf{E} |_{\partial \Omega} = 0, \\
\nu \wedge (\mu^{-1} \nabla \wedge \mathbf{E}) |_{\partial D} = 0.
\end{cases}
\]

is: Find \( \mathbf{E} \in X \) such that

\[ B(\mathbf{E}, \mathbf{F}) = (\tilde{f}, \mathbf{F}) \quad \text{for all } \mathbf{F} \in X. \]

By (A.6), this implies the natural boundary condition

\[ \nu \wedge (\mu^{-1} \nabla \wedge \mathbf{E}) |_{\partial D} = 0, \quad \nabla \cdot \gamma \mathbf{E} |_{\partial \Omega} = 0. \] (A.7)

To show the theorem, one first has for the homogeneous boundary conditions,

**Proposition A.2.** Suppose \( \gamma \) and \( \mu \) are complex functions in \( C^1(\Omega \setminus \overline{D}) \) with positive real parts, and let \( s \) be a positive real number. There is a discrete set \( \Sigma_s \subset \mathbb{C} \) such that if \( \omega \) is outside this set, then for any \( f \in X' \) there exists a unique solution \( \mathbf{E} \in X \) of

\[ \nabla \wedge (\mu^{-1} \nabla \wedge \mathbf{E}) - s \gamma^{-1} \nabla(\gamma \mathbf{E}) - \omega^2 \gamma \mathbf{E} = \tilde{f} \] (A.8)

satisfying

\[ \| \mathbf{E} \|_{H^1(\Omega \setminus \overline{D})} \leq C \| \tilde{f} \|_{X'}. \]

**Proof.** It’s sufficient to show that \( B \) is bounded and coercive on \( X \). It’s clear that \( B \) is bounded,

\[ |B(\mathbf{E}, \mathbf{F})| \leq C \| \mathbf{E} \|_{H^1(\Omega \setminus \overline{D})} \| \mathbf{F} \|_{H^1(\Omega \setminus \overline{D})}. \]

To show the coercivity, first we have

\[ |B(\mathbf{E}, \mathbf{E})| \geq c \| \nabla \wedge \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})} + s \| \nabla \cdot \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})} - C \| \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})} \| \mathbf{E} \|_{H^1(\Omega \setminus \overline{D})}. \]

It can be shown that there is a Poincaré inequality for 1-forms in \( X \) similar to that in [17]

\[ \| \mathbf{E} \|_{H^1(\Omega \setminus \overline{D})} \leq C (\| \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})} + \| \nabla \wedge \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})} + \| \nabla \cdot \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})}). \quad \mathbf{E} \in X, \]

we have

\[ B(\mathbf{E}, \mathbf{E}) \geq c \| \mathbf{E} \|_{H^1(\Omega \setminus \overline{D})}^2 - C \| \mathbf{E} \|_{L^2(\Omega \setminus \overline{D})}^2. \]

\[ \blacksquare \]
To show the higher order regularity of solutions, we define for \( k \geq 2 \),
\[
X^k = \{ F \in (H^k(\Omega \setminus \mathcal{D}))^3 \mid \nu \wedge F|_{\partial \Omega} = \nabla \cdot F|_{\partial \Omega} = \nu \cdot F|_{\partial D} = \nu \wedge (\mu^{-1} \nabla \wedge F)|_{\partial D} = 0 \}.
\]
Then we have

**Proposition A.3.** Let \( \gamma \) and \( \mu \) be functions in \( C^k(\Omega \setminus \mathcal{D}) \), \( k \geq 2 \), with positive real parts, and let \( s > 0 \). Suppose \( \omega \notin \Sigma_{s} \), then for any \( \check{f} \in (H^{k-2}(\Omega \setminus \mathcal{D}))^3 \) the equation (A.8) has a unique solution \( E \in X^k \) and
\[
\|E\|_{H^k(\Omega \setminus \mathcal{D})} \leq C\|\check{f}\|_{H^{k-2}(\Omega \setminus \mathcal{D})}.
\]

This can be proved by the same techniques in Section 5.9 of [17] for the Hodge Laplacian.

**Proof of Theorem A.1:** As in [8], we take \( \Sigma \) to be the set \( \Sigma_1 \) in previous propositions, then \( s \) can be chosen such that \( \omega \notin \Sigma_{s} \) (for more details see [8]).

To show uniqueness, suppose \( (E, H) \in H^1_{\text{Div}}(\Omega \setminus \mathcal{D}) \times H^1_{\text{Div}}(\Omega \setminus \mathcal{D}) \) solves (A.1) with \( f = g = 0 \). One has
\[
\nabla \wedge (\mu^{-1} \nabla \wedge E) = \omega^2 \gamma E, \quad \nabla \cdot E = 0.
\]
It follows that \( E \in X \) (by the natural boundary conditions) is a solution of (A.8) with \( \check{f} = 0 \), which implies \( E = H = 0 \) by Proposition A.2.

For existence, given \( f \in TH^{k-1/2}_{\text{Div}}(\partial \Omega) \) and \( g \in TH^{k-1/2}_{\text{Div}}(\partial D) \subset TH^{k-3/2}_{\text{Div}}(\partial D) \), we can find \( E_0 \in H^k_{\text{Div}}(\Omega \setminus \mathcal{D}) \) with
\[
\nu \wedge E_0|_{\partial \Omega} = f, \quad \nu \wedge (\mu^{-1} \nabla \wedge E_0)|_{\partial D} = g
\]
such that the extension is bounded, namely
\[
\|E_0\|_{H^k(\Omega \setminus \mathcal{D})} \leq C(\|f\|_{H^{k-1/2}(\partial \Omega)} + \|g\|_{H^{k-3/2}(\partial D)})
\]
Suppose \( \check{E} \in X^k \) is a solution, given by Proposition A.3, of (A.8) with
\[
\check{f} = -\nabla \wedge (\mu^{-1} \nabla \wedge E_0) + s \gamma^{-1} \nabla (\nabla \cdot \gamma E_0) + \omega^2 \gamma E_0 \in (H^{k-2}(\Omega \setminus \mathcal{D}))^3.
\]
Notice that
\[
\|\check{f}\|_{H^{k-2}(\Omega \setminus \mathcal{D})} \leq C\|E_0\|_{H^k(\Omega \setminus \mathcal{D})}.
\]
Then \( E = E_0 + \check{E} \in (H^k(\Omega \setminus \mathcal{D}))^3 \) satisfies
\[
\nabla \wedge (\mu^{-1} \nabla \wedge E) - s \gamma^{-1} \nabla (\nabla \cdot \gamma E) - \omega^2 \gamma E = 0. \quad \tag{A.9}
\]
This implies \( \nabla \cdot \gamma E = 0 \) by an earlier argument and particular choice of \( s \). If we define \( H = \frac{1}{\omega \mu} \nabla \wedge E \in (H^{k-1}(\Omega \setminus \mathcal{D}))^3 \), then we have \( (E, H) \) the solution of Maxwell’s equation.

Applying the same argument to \( H \), which satisfies a second order elliptic system by eliminating \( E \) from the original Maxwell’s equation. By uniqueness, one has \( H \in (H^k(\Omega \setminus \mathcal{D}))^3 \).
The fact $\mathbf{E} \in H^k_{\text{Div}}(\Omega \setminus \overline{D})$ is obtained by
\[
\text{Div}(\nu \wedge \mathbf{E}|_{\partial(\Omega \setminus D)}) = -\nu \cdot \nabla \wedge \mathbf{E}|_{\partial(\Omega \setminus D)} = -i\omega \mu \nu \cdot \mathbf{H}|_{\partial(\Omega \setminus D)} \in (H^{k-1/2}(\partial(\Omega \setminus D)))^3.
\]

Finally, the estimate (A.2) is derived from
\[
\|\mathbf{E}\|_{H^k(\Omega \setminus D)} \leq \|\mathbf{E}_0\|_{H^k(\Omega \setminus D)} + \|\mathbf{E}\|_{H^k(\Omega \setminus D)} \\
\leq C(\|\mathbf{E}_0\|_{H^k(\Omega \setminus D)} + \|\mathbf{f}\|_{H^{k-2}(\Omega \setminus D)}) \\
\leq C\|\mathbf{E}_0\|_{H^k(\Omega \setminus D)} \\
\leq C(\|\mathbf{f}\|_{H^{k-1/2}(\partial\Omega)} + \|g\|_{H^{k-3/2}(\partial D)}) \quad \text{(A.10)}
\]

The same computation applies to $\mathbf{H}$,
\[
\|\mathbf{H}\|_{H^k(\Omega \setminus D)} \leq C(\|\mathbf{f}\|_{H^{k-3/2}(\partial\Omega)} + \|g\|_{H^{k-1/2}(\partial D)}) \\
\leq C(\|\mathbf{f}\|_{H^{k-1/2}(\partial\Omega)} + \|g\|_{H^{k-1/2}(\partial D)}).
\]

Then
\[
\|\text{Div}(\nu \wedge \mathbf{E}|_{\partial(\Omega \setminus D)})\|_{H^{k-1/2}(\partial(\Omega \setminus D))} \\
\leq C(\|\text{Div}(\mathbf{f})\|_{H^{k-1/2}(\partial\Omega)} + \|\nu \cdot \mathbf{H}\|_{H^{k-1/2}(\partial D)}) \\
\leq C(\|\text{Div}(\mathbf{f})\|_{H^{k-1/2}(\partial\Omega)} + \|\mathbf{H}\|_{H^k(\Omega \setminus D)}) \\
\leq C(\|\text{Div}(\mathbf{f})\|_{H^{k-1/2}(\partial\Omega)} + \|f\|_{H^{k-1/2}(\partial\Omega)} + \|g\|_{H^{k-1/2}(\partial D)}) \\
\leq C(\|f\|_{TH^{k-1/2}(\partial\Omega)} + \|g\|_{TH^{k-1/2}(\partial D)}).
\]

The same estimate can be obtained for $\mathbf{H}$.

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