SHARP STABILITY OF LOG-SOBOLEV AND MOSER-ONOFRI INEQUALITIES ON THE SPHERE

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ABSTRACT. In this paper, we are concerned with the stability problem for endpoint conformally invariant cases of the Sobolev inequality on the sphere $S^n$. Namely, we will establish the stability for Beckner’s log-Sobolev inequality and Beckner’s Moser-Onofri inequality on the sphere. We also prove that the sharp constant of global stability for the log-Sobolev inequality on the sphere $S^n$ must be strictly smaller than the sharp constant of local stability for the same inequality. Furthermore, we also derive the non-existence of the global stability for Moser-Onofri inequality on the sphere $S^n$.

1. INTRODUCTION

In the past decades, there have been extensive works done in studying the stability of geometric and functional inequalities. The stability problem essentially investigates the deficit of a geometric and functional inequality in terms of a certain appropriate distance function from the class of the maximizers for which the equality holds. Simply put, we like to know how close the function is to the manifold consisting of all the maximizers if a function makes the geometric or functional inequality almost an equality. Sharp stability further studies the best constant in front of the distance function.

The main purpose of this paper is concerned with the problems of establishing the stability for the log-Sobolev and Moser-Onofri inequalities on the sphere $S^n$. Since the celebrated work of L. Gross on the log-Sobolev inequality in $\mathbb{R}^n$ with respect to the Gaussian measure [25], the log-Sobolev inequality on the sphere was established on the sphere $S^n$ by Beckner [4] by using Fourier analysis and the spherical harmonics techniques. Furthermore, Beckner [4] also established the sharp Moser-Onofri inequality on the sphere $S^n$ in higher dimensions using Fourier analysis techniques, among other things. (See also Carlen and Loss for another proof of Beckner’s Moser-Onofri inequality on the sphere $S^n$ using the competing symmetry and logarithmic Hardy-Littlewood-Sobolev inequality [10].)

In this paper, we will establish the sharp local stability for the log-Sobolev inequality on the sphere $S^n$ by choosing suitable $L^2$ distance function and applying the spectrum estimate of spherical harmonics. (see Theorem 1.2.) We further prove that the best constant for the global stability of the log-Sobolev inequality on the sphere $S^n$ must be strictly smaller than the best...
constant for the local stability of the same inequality. (see Theorem 1.3.) The non-existence of the local stability of the log-Sobolev inequality on the sphere $S^n$ with respect to another natural distance function is also established. (see Theorem 1.4.) Furthermore, we will also prove the local stability of the Moser-Onofri inequality on the sphere $S^n$ and the non-existence of the global stability of the Moser-Onofri inequality on the sphere $S^n$. (see Theorem 1.5.) Another interesting result in this paper is a relationship between the sharp local stability and global stability for some general geometric inequality satisfying certain homogeneity conditions. (See Theorem 1.1.) We will show that when the remainder term of the little $o(d^p(u, M))$ in the local stability inequality holds for all the functions $u$ and the deficit of the geometric or functional inequality has the same homogeneity as the power of the distance function, then we can conclude the global stability inequality with the same best constant. On the other hand, when the deficit of the geometric or functional inequality has a different power of the homogeneity from that of the distance function on the right hand side, then there does not exist global stability of the geometric or functional inequality. Indeed, as an application of Theorem 1.1, we show the nonexistence of the global stability of the Moser-Onofri inequality on the sphere $S^n$.

Stability of geometric and functional inequalities is a classical question in the study of geometric and functional inequalities and its history can date back to Brezis and Lieb’s work in [8]. It measures the distance to the set of maximizers in terms of the deficit functional. Sharp geometrical inequalities and their stability have been a matter of intensive research due to the importance of these inequalities in applications to geometric analysis, partial differential equations, convex geometry, mathematical physics and problems in spectral theory and stability of matter, etc. While many mathematicians have made contributions in this direction, it is impossible to review all the results. We will begin with presenting a brief history of the main results on these problems only closely related to the questions under consideration in this paper.

1.1. Functional inequalities on the sphere $S^n$.

The classical sharp Sobolev inequality in $\mathbb{R}^n$ for $0 < s < \frac{n}{2}$ states

$$\left\|(-\Delta)^{s/2} U\right\|_2^2 \geq S_{s,n}\|U\|_{2n-2s}^2$$

for all $U \in \dot{H}^s(\mathbb{R}^n)$ (1.1)

with

$$S_{s,n} = (4\pi)^n \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)} \left(\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{2s/n} = \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)} |S^n|^{2s/n} ,$$

(1.2)

where $\dot{H}^s(\mathbb{R}^n)$ denotes the $s$-order homogenous Sobolev space in $\mathbb{R}^n$: the completion of $C^\infty_c(\mathbb{R}^n)$ under the norm $\left(\int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} U\right|^2 dx\right)^{1/2}$. The sharp constant of inequality (1.1) has been computed first by Rosen [33] in the case $s = 2$, $n = 3$ and then independently by Aubin [1] and Talenti [35] in the case $s = 2$.
and general $n$. For general $s$, the sharp inequality was proved by Lieb in [28] as an equivalent reformulation of sharp Hardy-Littlewood-Sobolev inequality in the case of conformal index. Furthermore, he also showed that the equality of Sobolev inequality (1.1) holds if and only if

$$U \in M_s := \left\{ cV\left(\frac{-x_0}{a}\right) : c \in \mathbb{R}, \ x_0 \in \mathbb{R}^n, \ a > 0 \right\},$$

where $V(x) = (1 + |x|^2)^{-\frac{n+2s}{2}}$.

By the stereographic projection, we know that $\mathbb{R}^n$ (or rather $\mathbb{R}^n \cup \{\infty\}$) and $\mathbb{S}^n (\subset \mathbb{R}^{n+1})$ are conformally equivalent. Thus, there exists an equivalent version of (1.1) on $\mathbb{S}^n$. This form was found explicitly by Beckner in [4, Eq. (19)], namely,

$$\left\| A_{2s}^{1/2} u \right\|_2^2 \geq S_{s,n} \left\| u \right\|_{2s,n}^2 \quad \text{for all } u \in H^s(\mathbb{S}^n) \quad (1.3)$$

with

$$A_{2s} = \frac{\Gamma(B + \frac{1}{2} + s)}{\Gamma(B + \frac{1}{2} - s)} \quad \text{and} \quad B = \sqrt{-\Delta_{\mathbb{S}^n} + \frac{(n-1)^2}{4}}, \quad (1.4)$$

where $-\Delta_{\mathbb{S}^n}$ denotes the Laplace-Beltrami operator on the sphere $\mathbb{S}^n$ and the operator $B$ and $A_{2s}$ acting on spherical harmonics $Y_{l,m}$ of degree $l$ satisfy

$$BY_{l,m} = (l + \frac{n-1}{2})Y_{l,m}, \quad A_{2s}Y_{l,m} = \frac{\Gamma(l + n/2 + s)}{\Gamma(l + n/2 - s)}Y_{l,m}.$$  

(We refer the reader to the books [31] and [34] for detailed exposition of spherical harmonics, and also Subsection 2.1 for a brief account.) In fact, the operators $A_{2s}$ is a 2$s$-order conformally invariant differential operator and can be written as $(-\Delta_{\mathbb{S}^n})^s + \text{lower order terms}$. For the integer $s$, they are related to the GJMS operators in conformal geometry [18, 24]. Beckner also proved that the equality holds in (1.3) if and only if

$$u \in M_s := \left\{ c(1 - \xi \cdot \omega)^{\frac{2s-n}{2}} : \xi \in B^{n+1}, \ c \in \mathbb{R} \right\}. \quad (1.5)$$

Inequality (1.3) becomes an equality as $s \to 0$. Differentiating at $s = 0$ and using the Funk-Hecke formula, Beckner [3, 5] proved the following invariant logarithmic Sobolev inequality on $\mathbb{S}^n$.

**Theorem A.** Assume $u \in L^2(\mathbb{S}^n)$. Then

$$\int_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|u(\omega) - u(\eta)|^2}{|\omega - \eta|^n} \, d\omega d\eta \geq C_n \int_{\mathbb{S}^n} |u(\omega)|^2 \ln \left(\frac{|u(\omega)|^2}{\|u\|^2}\right) d\omega \quad (1.6)$$

with sharp constant

$$C_n = \frac{4}{n} \frac{\pi^{n/2}}{\Gamma(n/2)}. \quad (1.7)$$
The equality holds if and only if
\[ u(\omega) = c \left( 1 - \xi \cdot \omega \right)^{-n/2} \] (1.8)
for some \( \xi \in B^{n+1} := \{ \xi \in \mathbb{R}^{n+1}, |\xi| < 1 \} \) and some \( c \in \mathbb{R} \).

Note that as \( s \nearrow \frac{n}{2} \), the integrability exponent \( \frac{2n}{n-2s} \) in (1.1) and (1.3) tends to \( +\infty \). In [32], Onofri [32] derived an endpoint inequality on the 2 dimensional sphere, which is known as the Moser-Onofri inequality. He proved that
\[
\ln \int_{S^2} e^f d\omega \leq \int_{S^2} f d\omega + \frac{1}{4} \int_{S^2} |\nabla f|^2 d\omega, \text{ for } f \in H^1(S^2).
\]
We also refer the reader to an independent proof by Hong [26]. In 1993, using the endpoint differentiation argument, spherical harmonics techniques and Lieb’s [28] Hardy-Littlewood-Sobolev inequality on the sphere, Beckner established in [4] the Moser-Onofri inequality on the higher dimensional sphere for all \( n \geq 1 \) and \( s = n/2 \).

**Theorem B.** Let \( f \in H^\frac{n}{2}(S^n) \). Then
\[
\ln \int_{S^n} e^f d\omega \leq \int_{S^n} f d\omega + \frac{1}{2n!} \int_{S^n} f(A_n f) d\omega. \tag{1.9}
\]
This inequality is conformal invariant under the transformation
\[ f(\omega) \rightarrow f(\Phi(\omega)) + \ln J_\Phi(\omega), \]
where \( \Phi \) is a conformal map on \( S^n \). The equality holds if and only if
\[ f(\omega) = -n \ln(1 - \xi \cdot \omega) + C \]
for some \( \xi \in B^{n+1} \) and some \( C \in \mathbb{R} \).

Here we need to point out that \( A_n \) acting on spherical harmonics \( Y_{l,m} \) satisfies
\[ A_n Y_{l,m} = \frac{\Gamma(l+n)}{\Gamma(l)} Y_{l,m} \text{ for } l \neq 0 \]
and \( A_n Y_{0,1} = 0 \). For the range \( s > n/2 \), a ”reverse” type of Sobolev inequality was considered. We refer the interested readers to the recent paper [21] and references therein.

1.2. Stability of Geometric and Functional inequalities.

Assume that the following functional inequality holds:
\[
\mathcal{E}(x) \leq \mathcal{F}(x) \text{ for all } x \in X, \tag{1.10}
\]
where \( \mathcal{E} \) and \( \mathcal{F} \) are two functionals defined on the linear space \( X \). The functional inequality (1.10) is said to be optimal if \( \mathcal{E}(x) \leq \mathcal{F}(x) \) for all \( x \in X \) and for any \( \lambda < 1 \), there exists some \( x \in X \) such that \( \mathcal{E}(x) \geq \lambda \mathcal{F}(x) \). Let
\[ X_0 = \{ x \in X : \mathcal{E}(x) = \mathcal{F}(x) < \infty \} \]
be the set of optimizers, we say that the functional inequality (1.10) is sharp if \( X_0 \neq \emptyset \). It should be noted that a sharp functional inequality is necessarily optimal, but not vice-versa.

Once a sharp functional inequality (1.10) in a normed linear space \( X \) was established, we are interested in the stability of the inequality, which means finding a suitable metric \( d \) in \( X \) (not necessarily being the metric induced by the norm) and a function \( \Phi : [0, \infty) \to [0, \infty) \) such that

\[
\mathcal{F}(x) - \mathcal{E}(x) \geq \Phi(d(x, X_0)) \text{ for all } x \in X.
\]  

(1.11)

In this paper, we will call the inequality (1.11) the global stable inequality compared with the following local stable inequality:

\[
\mathcal{F}(x) - \mathcal{E}(x) \geq \Phi(d(x, X_0)) + o(d(x, X_0)).
\]  

(1.12)

Obviously, the global stability implies the local stability but not vice-versa. However, we will reveal the fact that the local stability and global stability of functional inequalities are actually equivalent under some mild conditions.

In fact, assume that the sharp functional inequality

\[
\mathcal{E}(x) \leq \mathcal{F}(x) \text{ for all } x \in X
\]

holds and the functional \( \mathcal{E}(x) \) is \( p \)-homogeneous, which means \( \mathcal{E}(\lambda x) = \lambda^p \mathcal{E}(x) \) for all \( \lambda > 0 \) and the functional \( \mathcal{F}(x) \) is also \( p \)-homogeneous. Let \( x_0 \) be an element of the optimizer set \( X_0 \). Obviously, \( \lambda x_0 \in X_0 \) for any \( \lambda > 0 \). We also assume that the distance \( d \) satisfies \( d(x, X_0) \neq 0 \) and

\[
d(\lambda x, \lambda y) = \lambda d(x, y), \ \forall \ \lambda > 0, \ x, y \in X.
\]

Then we have the following theorem.

**Theorem 1.1.** Let \( \mathcal{E}, \mathcal{F}, X_0, d \) satisfy the above assumptions. Then

(i) If \( q \neq p \), there is no global stable inequality of the following form

\[
\mathcal{F}(x) - \mathcal{E}(x) \geq cd^q(x, X_0) \text{ for all } x \in X.
\]

If \( q < p \), for \( x \in X \) satisfy \( 0 < d(x, X_0) \) small enough, there exists no local stable inequality of the following form

\[
\mathcal{F}(x) - \mathcal{E}(x) \geq cd^p(x, X_0) + o(d^p(x, X_0)).
\]

(ii) The local stable inequality

\[
\mathcal{F}(x) - \mathcal{E}(x) \geq cd^p(x, X_0) + o(d^p(x, X_0))
\]  

(1.13)

implies the global stable inequality

\[
\mathcal{F}(x) - \mathcal{E}(x) \geq cd^p(x, X_0).
\]  

(1.14)

Moreover, if \( c_0 \) is the optimal constant of the local stable inequality (1.13) then it is also the optimal constant of the global stable inequality (1.14).
1.3. **Stability of Sobolev inequalities.** Research on the stability of Sobolev inequality started from the work by Brezis and Lieb. In [8] they asked if the following refined first order Sobolev inequality ($s = 1$ in (1.1)) holds for some distance function $d$:

$$\|(-\Delta)^{1/2}U\|_2^2 - S_{1,n}\|U\|_{\frac{2n}{n-2}}^2 \geq cd^2(U, M_1).$$

This question was answered affirmatively in the case $s = 2$ in a pioneering work by Bianchi and Egnell [6], in the case $s = 4$ by the second author and Wei [30] and in the case of any positive even integer $s < n/2$ by Bartsch, Weth and Willem [2]. In 2013, Chen, Frank and Weth [13] established the stability of Sobolev inequality for all $0 < s < n/2$.

**Theorem C.** [13] There exists a positive constant $c$ depending only on the dimension $n$ and $s \in (0, n/2)$ such that

$$\|(-\Delta)^{s/2}U\|_2^2 - S_{s,n}\|U\|_{\frac{2n}{n-2s}}^2 \geq cd^2(U, M_s),$$

(1.15)

for all $U \in \dot{H}^s(\mathbb{R}^n)$, where $d(U, M_s) = \min\{\|(-\Delta)^{s/2}(U - \phi)\|_{L^2} : \phi \in M_s\}$.

In fact, via the stereographic projection, they first proved the following equivalent local stable Sobolev inequality on the sphere

$$\left\|\gamma^{1/2}s\right\|_2^2 - S_{s,n}\|u\|_{\frac{2n}{n-2s}}^2 \geq \frac{4s}{n+2s}d^2(u, M_s) + o(d^2(u, M_s)),$$

(1.16)

where $M_s$ is the optimizer set and $d(u, M_s) = \min\{\|\gamma^{1/2}s(u - v)\|_2^2 : v \in M_s\}$.

To obtain the global stability, the concentration compactness type argument plays an important role in showing that $\|(-\Delta)^{s/2}U_n\|_2^2 - S_{s,n}\|U_n\|_{\frac{2n}{n-2s}}^2 \rightarrow 0$ implies $d(U_n, M_s) \rightarrow 0$ for $U_n \in H^s(\mathbb{R}^n)$. It is notable that they pointed out that the constant $\frac{4s}{n+2s+2}$ in (1.16) is sharp (at the end of their first section).

Because of using the concentration-compactness argument, they could not give any lower bound information about the sharp constant of stable fractional Sobolev inequality. In fact, to our knowledge, the sharp stability for Sobolev inequalities with the best constants still remain open (see [9, 14, 6]). We also note that Dolbeault et al. [15] recently gave the lower bound estimate for sharp constant of stable global Sobolev inequalities.

Though the local stability of the Sobolev inequality has been established in Theorem C, this local stability does not satisfy the local stability assumption required in Theorem 1.1. Therefore, we cannot conclude that the global stability of the Sobolev inequalities with the same best constants can follow from the local stability stated in Theorem C. In fact, König has recently showed that the best constant for the global stability of the Sobolev inequality must be strictly smaller than $\frac{4s}{n+2s+2}$. (see [27]).
All the above stability of Sobolev inequalities were established in a Hilbert space. In fact, for some other type functional inequalities such as Gagliardo-Nirenberg, weighted-Hardy-Sobolev and Caffarelli-Kohn-Nirenberg inequalities, their stability can also been done in the framework of Hilbert space, one can refer to [11, 12, 17, 36] for details. For the study of stability of Sobolev inequalities in non-Hilbert space, we refer the interested readers to the papers [9, 14, 19, 20] and references therein.

1.4. Stability of the Log-Sobolev and Moser-Onofri inequalities on the sphere $\mathbb{S}^n$. In this paper, our main goals are to establish the stability of the endpoint conformally invariant inequalities on the sphere $\mathbb{S}^n$, namely the log-Sobolev inequality (1.6) and the Moser-Onofri inequality (1.9). We will also explore the optimal constants of these stable inequalities.

Let
$$ M = \left\{ v_{c,\xi} (\omega) = c (1 - \xi \cdot w)^{-n/2} : \xi \in B^{n+1}, c \in \mathbb{R} \right\} $$
be the optimizer set of the log-Sobolev inequality on the sphere $\mathbb{S}^n$. Denote the log-Sobolev functional by
$$ LS(u) = \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|u(\omega) - u(\eta)|^2}{|\omega - \eta|^n} \, d\omega d\eta - C_n \int_{\mathbb{S}^n} |u(\omega)|^2 \ln \frac{|u(\omega)|^2 |\mathbb{S}^n|}{\|u\|^2_2} \, d\omega. $$
Our first goal in the paper is to explore the stability of the log-Sobolev inequality on the sphere $\mathbb{S}^n$. First of all, we need to find a suitable distance function. It is natural to adopt the distance induced by the double integral in the inequality, namely to establish a type of stability inequality in the form of
$$ LS(u) \geq c d_0^2(u, M), $$
for some $c > 0$, where
$$ d_0^2(u, v) = \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|u(\omega) - v(\omega) - u(\eta) + v(\eta)|^2}{|\omega - \eta|^n} \, d\omega d\eta. $$
Nevertheless, we can’t expect that this stability inequality holds (see Theorem 1.4).

Instead, we will choose the $L^2$ distance since the log-Sobolev inequality is $L^2$ conformally invariant. Denote
$$ \mathcal{D}_L = \left\{ v \in L^2(\mathbb{S}^n) : \iint_{\mathbb{S}^n \times \mathbb{S}^n} \frac{|v(\omega) - v(\eta)|^2}{|\omega - \eta|^n} \, d\omega d\eta < \infty \right\}. $$
For any functions $u, v \in \mathcal{D}_L$, we define the inner product
$$ \langle u, v \rangle = \int_{\mathbb{S}^n} u(\omega) \overline{v(\omega)} \, d\omega. $$
We also define
\[ d^2(u, M) = \inf \{ \| u - \phi \|_{L^2}^2 : \phi \in M \} = \inf_{(c, \xi) \in \mathbb{R} \times B^{n+1}} \langle u - v_{c,\xi}, u - v_{c,\xi} \rangle, \]
where \( B^{n+1} = \{ x \in \mathbb{R}^{n+1} : |x| < 1 \} \).

In this paper we first establish the local stability of the log-Sobolev inequality (1.6) in the setting of \( L^2 \) distance.

**Theorem 1.2.** Let \( u \in \mathcal{D}_L \) with \( \| u \|_{L^2} = 1 \). Then
\[ LS(u) \geq \frac{8\pi^{n/2}}{\Gamma(n/2)(n+2)} d^2(u, M) + o(d^2(u, M)), \]
where \( o(d^2(u, M)) \) is only dependent on \( d(u, M) \) but independent of \( u \). Moreover, the constant \( \frac{8\pi^{n/2}}{\Gamma(n/2)(n+2)} \) is sharp.

Moreover, motivated by the recent work of the global stability of the Sobolev inequalities by König [27], we can also establish the following result.

**Theorem 1.3.** Assume that there exists \( C > 0 \) independent of \( u \) such that
\[ LS(u) \geq Cd^2(u, M) \]
for any \( u \in \mathcal{D}_L \). Then \( C \) is strictly smaller than the optimal constant \( \frac{8\pi^{n/2}}{\Gamma(n/2)(n+2)} \) of locally stable log-Sobolev inequality.

Next, we will explain why we can’t expect the type of log-Sobolev stability inequality with the distance \( d_0 \) to hold.

**Theorem 1.4.** There exists a sequence of functions \( \{u_m\} \subset \mathcal{D}_L \) such that
\[ \frac{LS(u_m)}{d^2_0(u_m, M)} \to 0. \]

Our second goal in this paper is to try to set up the stability of the sharp Moser-Onofri inequality on the sphere \( \mathbb{S}^n \). The main difficulty in solving this problem lies in finding a suitable distance function between a function \( u \) and the set of optimizer. Although the authors in [16] employed the entropy and flow method to give the improved Moser-Onofri inequality, the remainder term is quite complicated and cannot be regarded as some distance function. In order to overcome this difficulty, we will first change the form of the Moser-Onofri inequality.

Let \( u = e^{f/2} \). Then the sharp inequality (1.9) becomes the following equivalent inequality:
\[ \ln \int_{\mathbb{S}^n} u^2 d\omega \leq \int_{\mathbb{S}^n} \ln u^2 d\omega + \frac{1}{2n!} \int_{\mathbb{S}^n} \ln u^2 (A_n(\ln u^2)) d\omega \quad \text{for} \ u \in Q_M, \quad (1.17) \]
where

\[ Q_M = \{ u \in L^2(S^n) : \ln u^2 \in H^2 \} \].

Then the inequality (1.17) is conformally invariant under the transformation

\[ u(\omega) \rightarrow J^{1/2}_{\Phi}(\omega)u(\Phi(\omega)), \]

where \( \Phi \) is a conformal map on \( S^n \) and the equality holds if and only if \( u \) belongs to

\[ M = \left\{ v_{c,\xi} = c(1 - \xi \cdot w)^{-n/2} : \xi \in B^{n+1}, c \in \mathbb{R} \right\}. \]

Once we have observed the \( L^2 \) conformal invariance of the inequality (1.17), it is natural for us to adopt the \( L^2 \) distance to study the stability of the new form of Moser-Onofri inequality on the sphere \( S^n \). Now we define the Moser-Onofri functional \( MO(u) \) by

\[
MO(u) = \frac{1}{2n!} \int_{S^n} \ln u^2 \left(A_n(\ln u^2)\right) d\omega + \int_{S^n} \ln u^2 d\omega - \ln \int_{S^n} u^2 d\omega.
\]

We will establish the existence of the local stability for the Moser-Onofri inequality and the nonexistence of the global stability for the Moser-Onofri inequality on the sphere \( S^n \).

**Theorem 1.5.** For \( u \in Q_M \) with \( \|u\|_{L^2} = 1 \), we have

\[
MO(u) \geq \frac{\Gamma(n+1)}{2n-1\pi^{n/2}\Gamma(n/2)} d^2(u, M) + o(d^2(u, M)),
\]

where \( o(d^2(u, M)) \) is only dependent on \( d(u, M) \) but independent of \( u \). Furthermore, there exists a sequence of functions \( \{u_m\} \subset Q_M \) such that

\[
\frac{MO(u_m)}{d^2(u_m, M)} \rightarrow 0
\]

which means that the global stability for the Moser-Onofri inequality on the sphere does not hold.

This paper is organized as follows. In Section 2, we will prove Theorem 1.1 which reveals the relation between the local and global stability of functional inequalities in a fairly general framework. We also characterize the tangent space of the manifold of the optimizers of the log-Sobolev inequality on the sphere \( S^n \). Section 3 is devoted to giving the sharp stability of the log-Sobolev inequality with \( L^2 \) distance (Theorem 1.2) and that the log-Sobolev inequality is unstable with respect to the distance \( d_0 \) (Theorem 1.4). In Section 4, we will prove the sharp local stability for the Moser-Onofri inequality and the nonexistence of the global stability for the Moser-Onofri inequality on the sphere \( S^n \) (Theorem 1.5).

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2. Preliminary

2.1. Spherical harmonics.
In this subsection, we will recall some facts about spherical harmonics and a detailed discussion on spherical harmonics can be found in, e.g., [31] and [34]. In brief, spherical harmonics are restrictions to the unit sphere $S^n$ of polynomial $Y(x)$, which satisfy $\Delta x Y = 0$, where $\Delta x$ is the Laplacian operator in $\mathbb{R}^{n+1}$. The space of all spherical harmonics of degree $l$ on $S^n$, denoted by $H^l_n$, has an orthonormal basis 

$$\{Y_{l,m} : m = 1, \ldots, N(n, l), l = 1, 2, \ldots\},$$

where $N(n, 0) = 1$ and $N(n, l) = \frac{(2l+n-1)\Gamma(l+n-1)}{\Gamma(l+1)\Gamma(l)}$.

Every $u \in L^2(S^n)$ can be expanded in terms of spherical harmonics

$$u = \sum_{l=0}^{+\infty} \sum_{m=1}^{N(n, l)} u_{l,m} Y_{l,m},$$

where $u_{l,m} = \int_{S^n} u Y_{l,m} d\omega$, and by Parseval’s identity,

$$\|u\|_{L^2(S^n)}^2 = \int_{S^n} |u|^2 d\omega = \sum_{l=0}^{+\infty} \sum_{m=1}^{N(n, l)} |u_{l,m}|^2.$$

And the Sobolev space $H^s(S^n)$ on the sphere is defined by

$$H^s(S^n) := \{u \in L^2(S^n) : \|u\|_{H^s}^2 = \sum_{l=0}^{+\infty} (1 + l(l + n - 1))^{s} \sum_{m=1}^{N(n, l)} |u_{l,m}|^2 < +\infty\}.$$

2.2. Local stability implies global stability: Proof of Theorem 1.1.
In this subsection, we will reveal the relation between the local and global stability of functional inequalities. Namely, we shall give the proof of Theorem 1.1.

Proof. When $q \neq p$, we will first prove the conclusion (i) by contradiction. Assume that the global stability inequality

$$F(x) - E(x) \geq cd^q(x, X_0) \text{ for all } x \in X$$

holds. Fix $x \in X$ such that $d(x, X_0) \neq 0$ and $F(x) < \infty$, and define $x_\lambda = \lambda x$ for $\lambda > 0$. Then it is clear that $d(x_\lambda, X_0) = \lambda d(x, X_0)$. Thus

$$\frac{F(x_\lambda) - E(x_\lambda)}{d^q(x_\lambda, X_0)} = \lambda^{p-q} \frac{F(x) - E(x)}{d^q(x, X_0)},$$
which will go to 0 as \( \lambda \to 0 \) when \( p > q \) and as \( \lambda \to \infty \) as \( p < q \). That is a contradiction. Thus, part (i) of Theorem 1.1 is proved. Using the same method, we can prove that the local stable inequality does not hold if \( q < p \).

Next, we will prove part (ii) of Theorem 1.1. Namely, if \( p = q \), then the local stability implies the global stability. Indeed, for any fixed \( x \in X \) such that \( d(x, X_0) \neq 0 \), choose \( x_m = \frac{x}{d(x, X_0)} \), a simple calculation gives \( d(x_m, X_0) = \frac{1}{m}d(x, X_0) \to 0 \). Applying the local stable inequality, we obtain

\[
\mathcal{F}(x_m) - \mathcal{E}(x_m) \geq cd^p(x_m, X_0) + o(d^p(x_m, X_0)).
\]

Since \( \mathcal{E}(x) \) and \( \mathcal{F}(x) \) are \( p \)-homogeneous, then

\[
\frac{\mathcal{F}(x) - \mathcal{E}(x)}{d^p(x, X_0)} = \frac{\mathcal{F}(x_m) - \mathcal{E}(x_m)}{d^p(x_m, X_0)} \geq c + o(1),
\]

which implies the global stability inequality (1.14).

Finally, we will show that the sharp constant of global stable inequality is consistent with that of the local stable inequality. Assume that \( c_0 \) and \( c_1 \) are the optimal constants of the local and global stable inequalities respectively. Obviously \( c_1 \leq c_0 \). If \( c_1 < c_0 \), then there exist a sequence \( \{x_i\} \) such that

\[
\frac{\mathcal{F}(x_i) - \mathcal{E}(x_i)}{d^p(x_i, X_0)} \to c_1 < c_0.
\]

Without loss of generality, we may assume \( d(x_i, X_0) \) is bounded and has a positive lower bound. Otherwise, we can choose \( x'_i = \frac{x_i}{d(x_i, X_0)} \). Consider the sequence \( \{x'_i\} \), then \( d(x_i/i, X_0) \to 0 \) and

\[
\frac{\mathcal{F}(x_i/i) - \mathcal{E}(x_i/i)}{d^p(x_i/i, X_0)} \to c_1 < c_0,
\]

which is a contradiction with the fact that \( c_0 \) is the optimal constant of the local stable inequality (1.13). Therefore, we must have \( c_1 = c_0 \), which completes the proof. \( \square \)

2.3. The tangent space.

In this subsection, we will characterize the tangent space of the optimizer set of log-Sobolev inequality (1.6) and Moser-Onofri inequality (1.17). Set

\[
M = \left\{ v_{c,\xi} = c \left( \frac{\sqrt{1 - |\xi|^2}}{1 - \xi \cdot w} \right)^{n/2} : \xi \in \mathbb{R}^{n+1}, c \in \mathbb{R}, |\xi| < 1 \right\}.
\]

Since \( M \) can be viewed as an \( n + 2 \) dimensional smooth manifold embedded in \( L^2(\mathbb{S}^n) \) via the mapping

\[
\mathbb{R} \times B^{n+1} \to L^2(\mathbb{S}^n), (c, \xi) \to v_{c,\xi},
\]

then the tangent space of \( M \) at \( v_{c_0,\xi_0} \) is

\[
TM_{v_{c_0,\xi_0}} = \text{span}\{v_{1,\xi_0}, \partial_\xi v_{c,\xi}|_{(c_0,\xi_0)}\}.
\]
For any element \( v_{c_0, \xi_0} \) in \( M \), by the classification of conformal maps of \( S^n \), there exists a conformal transformation \( \Phi \) (depending on \( \xi_0 \), for convenience we omit \( \xi_0 \)) such that
\[
v_{c_0, \xi_0}(\omega) = c_0 J_{\Phi}^{1/2} = c_0 \left( \frac{\sqrt{1 - |\xi_0|^2}}{1 - \xi_0 \cdot w} \right)^{n/2}.
\]

We will characterize \( TM_{v_{c_0, \xi_0}} \) by spherical harmonics of degrees 0 and 1. But at first we need the following lemma, which was proved by Frank, König and the third author [22] in order to establish the conformal invariance of the Euler-Lagrange equation related to the log-Sobolev inequality (1.6). For any \( v \in L^2(S^n) \), let us denote \( v_{\Phi^{-1}} \) by
\[
v_{\Phi^{-1}} = J_{\Phi^{-1}}^{1/2} v \circ \Phi^{-1}.
\]

Lemma 2.1. Let \( u, v \in D_L \) and \( \Phi \) be a conformal map on \( S^n \). Then \( u, v \in D_L \) and
\[
H(u_\Phi) = (Hu)_\Phi + C_n u_\Phi \ln J_{\Phi}^{1/2},
\]
where
\[
H(u)(\omega) = P.V. \int_{S^n} \frac{u(\omega) - u(\eta)}{|\omega - \eta|^n} d\eta
\]
is the principal value of the integral.

Now we are in the position to characterize the tangent space \( TM_{v_{c_0, \xi_0}} \).

Lemma 2.2.
\[
TM_{v_{c_0, \xi_0}} = \text{span}\{J_{\Phi}^{1/2} Y_0, J_{\Phi}^{1/2} Y_{1, i} \circ \Phi; \ i = 1, \cdots, n + 1\}.
\]

Proof. In order to prove this lemma, we will take advantage of the fact that \( v_{c_0, \xi_0} \) is the element of the optimizer set of log-Sobolev inequality (1.6). Since \( v_{c, \xi} \) is the extremal function of the log-Sobolev inequality, then it must satisfy the following Euler-Lagrange equation
\[
H(u) = \frac{C_n}{2} u \ln \frac{u^2 |S^n|}{\|u\|^2},
\]
where \( C_n = \frac{4}{n+1} \frac{e^{n/2}}{\Gamma(n/2)} \) is the sharp constant of log-Sobolev inequality (1.6). Differentiating at \( (c_0, \xi_0) \), we know that \( v_{1, \xi_0} \) and \( \partial_{\xi_i} v_{c, \xi} |_{c_0, \xi_0} \) for \( i = 1, 2, \cdots, n + 1 \) satisfying the following equation
\[
H(u) = \frac{C_n}{2} u \ln \frac{v_{c_0, \xi_0}^2 |S^n|}{\|v_{c_0, \xi_0}\|^2} + C_n u - C_n \int_{S^n} v_{c_0, \xi_0} v_{c_0, \xi_0} u d\omega
\]
\[
= C_n u - C_n \frac{\int_{S^n} v_{c_0, \xi_0} u d\omega}{\|v_{c_0, \xi_0}\|^2} v_{c_0, \xi_0}, \quad (2.1)
\]
which implies that the tangent space belongs to the solution space of the equation (2.1).

Next, we will show that in fact, the tangent space is equal to the solution space of the equation (2.1) by the dimensions of the two spaces. We already know that \( v_{c_0,\xi_0} = c_0 J_{\Phi}^{1/2} \) for some conformal transformation \( \Phi \) on \( S^n \). For any solution \( u \) of the equation (2.1), let \( u_{\Phi^{-1}} = J_{\Phi^{-1}}^{1/2} u \circ \Phi^{-1} \). Then by Lemma 2.1 we know that

\[
H(u_{\Phi^{-1}}) = (H(u))_{\Phi^{-1}} + C_n u_{\Phi^{-1}} \ln J_{\Phi^{-1}}^{1/2}.
\]

(2.2)

Using (2.1) and \( v_{c_0,\xi_0} = c_0 J_{\Phi}^{1/2} \) we can obtain

\[
(H(u))_{\Phi^{-1}} = \frac{C_n}{2} u_{\Phi^{-1}} \ln(J_{\Phi} \circ \Phi^{-1}) + C_n u_{\Phi^{-1}} - C_n \int_{S^n} \frac{J_{\Phi}^{1/2} u d\omega}{|S^n|} J_{\Phi^{-1}}(J_{\Phi} \circ \Phi^{-1})
\]

\[
= -\frac{C_n}{2} u_{\Phi^{-1}} \ln J_{\Phi^{-1}} + C_n u_{\Phi^{-1}} - C_n \int_{S^n} u_{\Phi^{-1}} d\omega \frac{1}{|S^n|}.
\]

(2.3)

Then it follows from (2.2) and (2.3) that

\[
H(u_{\Phi^{-1}}) = C_n \left[ u_{\Phi^{-1}} - \int_{S^n} u_{\Phi^{-1}} d\omega \frac{1}{|S^n|} \right].
\]

Then by the eigenvalue problem (3.1) which we will solve in the next section, we know that \( u_{\Phi^{-1}} \) must be of the form

\[
u_{\Phi^{-1}} = cY_{0,1} + \sum_{i=1}^{n+1} c_i Y_{1,i}.
\]

So the dimension of the solution space is \( n+1 \). At the same time, the dimension of the tangent space is also \( n+1 \), which completes the conclusion

\[
\text{span}\{v_{1,\xi_0}, \partial_\xi v_{c,\xi}|_{(c,\xi_0)}\} = \text{span}\{J_{\Phi}^{1/2} Y_{0,1}, J_{\Phi}^{1/2} Y_{1,i} \circ \Phi, \ i = 1, \ldots, n+1\}.
\]

\[
\square
\]

3. STABILITY OF THE LOG-SOBOLEV INEQUALITY ON THE SPHERE \( S^n \)

In this section, we will give the proof of the stability of the log-Sobolev inequality on the sphere \( S^n \). First, we will solve the following eigenvalue problem

\[
H(u) = \lambda C_n \left[ u - \int_{S^n} u d\omega \frac{1}{|S^n|} \right], \ u \in \mathcal{L}. \quad (3.1)
\]

We have the following

**Lemma 3.1.** The eigenvalues of (3.1) are given by

\[
\lambda_l = \frac{n}{2} \left( \frac{\Gamma'(n/2 + l)}{\Gamma(n/2 + l)} - \frac{\Gamma'(n/2)}{\Gamma(n/2)} \right), \ l = 1, 2, \ldots.
\]

And the eigenfunctions corresponding to the eigenvalue \( \lambda_l \) are the spherical harmonics of degree \( l \) and constant functions.
Proof. For \( s > 0 \), we define the operator \( \mathcal{P}_{2s} \) given by
\[
\mathcal{P}_{2s}(u)(\xi) := \Gamma\left(\frac{n-2s}{2}\right) \int_{\mathbb{S}^n} \frac{u(\eta)}{|\xi - \eta|^{n-2s}} d\eta.
\] (3.2)
By the Funk-Hecke formula (see [4, Eq. (17)] and also [23, Corollary 4.3]), we have
\[
\mathcal{P}_{2s}(Y_{l,m}) = \frac{\Gamma(l + n/2 - s)}{\Gamma(l + n/2 + s)} Y_{l,m}.
\]
Expanding \( u \in L^2(\mathbb{S}^n) \) in terms of the spherical harmonics,
\[
u = \sum_{l=0}^{\infty} \sum_{m=1}^{N(n,l)} u_{l,m} Y_{l,m}
\] with \( u_{l,m} = \int_{\mathbb{S}^n} u Y_{l,m} d\omega \), (3.3)
then we have the representation
\[
\mathcal{P}_{2s} u = \sum_{l=0}^{\infty} \sum_{m=1}^{N(n,l)} u_{l,m} \frac{\Gamma(l + n/2 - s)}{\Gamma(l + n/2 + s)} Y_{l,m}.
\] (3.4)
In passing, we note that the right hand side of (3.4) is equal to \( A_{2s}^{-1} u \) with the operator \( A_{2s} \) defined in (1.4). Now, we are in the position to give the action of \( H \) on \( L^2 \) functions with spherical harmonics expansion (3.3), which follows from the identity
\[
H(u)(\xi) = \lim_{s \to 0} \int_{\mathbb{S}^n} \frac{u(\xi) - u(\eta)}{|\xi - \eta|^{n-2s}} d\eta
\]
\[
= \lim_{s \to 0} \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{2s} \left( \frac{\Gamma(n/2 - s)}{\Gamma(n/2 + s)} - \mathcal{P}_{2s} \right) u
\]
\[
= \frac{2\pi^{n/2}}{\Gamma(n/2)} \sum_{l=0}^{\infty} \sum_{m=1}^{N(n,l)} u_{l,m} \left( \frac{\Gamma'(n/2 + l)}{\Gamma(n/2 + l)} - \frac{\Gamma'(n/2)}{\Gamma(n/2)} \right) Y_{l,m}.
\] (3.5)
On the other hand, a direct calculation gives
\[
\lambda C_n \left[ u - \int_{\mathbb{S}^n} u d\omega \right] = \lambda \frac{4\pi^{n/2}}{n\Gamma(n/2)} \left[ \sum_{l=0}^{\infty} \sum_{m=1}^{N(n,l)} u_{l,m} Y_{l,m} - u_{0,1} Y_{0,1} \right]
\]
\[
= \lambda \frac{4\pi^{n/2}}{n\Gamma(n/2)} \left[ \sum_{l=1}^{\infty} \sum_{m=1}^{N(n,l)} u_{l,m} Y_{l,m} \right].
\]
By the orthogonality of spherical harmonics of different degree, the eigenvalue \( \lambda_l \) must satisfy
\[
\lambda_l = \frac{n}{2} \left( \frac{\Gamma'(n/2 + l)}{\Gamma(n/2 + l)} - \frac{\Gamma'(n/2)}{\Gamma(n/2)} \right), \ l = 1, 2, \cdots,
\]
and the corresponding eigenfunctions are spherical harmonics \( Y_{l,m} \). This concludes the proof of our lemma. \( \square \)

We are now ready to give the proof of our Theorem 1.2 and Theorem 1.3.
Proof of Theorem 1.2. First we will prove the sharp local stability of the log-Sobolev inequality on the sphere $\mathbb{S}^n$. Namely,

$$LS(u) \geq \frac{8\pi^{n/2}}{\Gamma(n/2)(n+2)}d^2(u, M) + o(d^2(u, M)) \text{ for all } u \in \mathcal{D}_L. \quad (3.6)$$

Take $u \in \mathcal{D}_L$ such that $d(u, M)$ is small enough. Since

$$d^2(u, M) = \inf_{c, \xi} \langle u - v_{c, \xi}, u - v_{c, \xi} \rangle,$$

it is easy to prove that the infimum is attained at a point $(c_0, \xi_0) \in (\mathbb{R}^+, B^{n+1})$. Then it follows that $u - v_{c_0, \xi_0} \perp TM_{v_{c_0, \xi_0}}$ and we can rewrite $u$ as $u = v_{c_0, \xi_0} + dv$ with $\langle v, v \rangle = 1$ and $d = d(u, M)$. Since the optimizer $v_{c_0, \xi_0}$ satisfies

$$v_{c_0, \xi_0} = c_0 J^{1/2}_{\Phi}$$

for some conformal transformation $\Phi$, then

$$u_{\Phi^{-1}} = c_0 + dv_{\Phi^{-1}}$$

and $\langle v_{\Phi^{-1}}, v_{\Phi^{-1}} \rangle = 1$. Due to the functional $LS(u)$ being conformal invariant under the transformation $u \rightarrow u_{\Phi^{-1}}$ and of class $C^2$ on $\mathcal{D}_L \setminus \{0\}$, then

$$LS(u) = LS(u_{\Phi^{-1}})$$

$$= LS(c_0) + \frac{d}{dt} \bigg|_{t=0} LS(c_0 + tdv_{\Phi^{-1}}) + \frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} LS(c_0 + tdv_{\Phi^{-1}}) + o(d^2). \quad (3.7)$$

Careful computations yield

$$LS(c_0) = \frac{d}{dt} \bigg|_{t=0} LS(c_0 + tdv_{\Phi^{-1}}) = 0, \quad (3.8)$$

$$\frac{d^2}{dt^2} \bigg|_{t=0} LS(c_0 + tdv_{\Phi^{-1}}) = 4d^2 \left( \langle v_{\Phi^{-1}}, H(v_{\Phi^{-1}}) - C_n v_{\Phi^{-1}} + \frac{C_n}{|S^n|} \int_{S^n} v_{\Phi^{-1}} d\omega \rangle \right). \quad (3.9)$$

Recall that we have already proved in Lemma 2.2 that the tangent space

$$TM_{v_{c_0, \xi_0}} = \text{span} \{v_{1, \xi_0}, \partial_\xi v_{c_0, \xi_0}\}$$

is equal to the space which is generated by the conformal transformation of $Y_{0,1}$ and $Y_{1,i}$, i.e. span$\{J^{1/2}_{\Phi^{-1}}, J^{1/2}_{\Phi^{-1}} Y_{1,i} \circ \Phi^{-1}, i = 1, \ldots, n + 1\}$. Since

$$v \perp TM_{v_{c_0, \xi_0}} = \text{span} \{J^{1/2}_{\Phi^{-1}}, J^{1/2}_{\Phi^{-1}} Y_{1,i} \circ \Phi^{-1}, i = 1, \ldots, n + 1\},$$

then $v_{\Phi^{-1}} \perp \text{span} \{Y_{0,1}, Y_{1,i}, i = 1, \ldots, n + 1\}$. Thus, it follows that $v_{\Phi^{-1}}$ must have the spherical harmonics expansion

$$v_{\Phi^{-1}} = \sum_{l=2}^{\infty} \sum_{m=1}^{N(n,l)} c_{l,m} Y_{l,m}. \quad (3.10)$$
Therefore, by (3.7), (3.8), (3.9) and (3.10) we can obtain

\[ LS(u) = 2d^2 \left\langle v_{\Phi^{-1}}, H(v_{\Phi^{-1}}) - C_n v_{\Phi^{-1}} + \frac{C_n}{\|S_n\|} \int_{S_n} v_{\Phi^{-1}} d\omega \right\rangle + o(d^2) \]

\[ = 2d^2 \frac{4\pi^{n/2}}{n\Gamma(n/2)} \sum_{l=2}^{\infty} \sum_{m=1}^{N(n,l)} \left( \frac{n}{2} \left( \frac{\Gamma'(n/2 + l)}{\Gamma(n/2 + l)} - \frac{\Gamma'(n/2)}{\Gamma(n/2)} - 1 \right) c_{l,m}^2 + o(d^2) \right) \]

\[ \geq d^2 \frac{8\pi^{n/2}}{n\Gamma(n/2)} \left( \frac{n}{2} \left( \frac{\Gamma'(n/2 + 2)}{\Gamma(n/2 + 2)} - \frac{\Gamma'(n/2)}{\Gamma(n/2)} - 1 \right) \sum_{l=2}^{\infty} \sum_{m=1}^{N(n,l)} c_{l,m}^2 + o(d^2) \right) \]

\[ = d^2 \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} + o(d^2). \quad (3.11) \]

Next we will prove that the constant \( \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} \) in (3.6) is optimal. To this end, we only need to construct a suitable test function sequence \( \{u_k\}_k \) such that

\[ \lim_{k \to +\infty} LS(u_k) = \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} \]

On the other hand, since \( Y_{2,m} \perp TM v_{1,0} (v_{1,0} = 1) \), then \( d(1 + \epsilon Y_{2,m}, M) = \epsilon \) for sufficiently small \( \epsilon \) and

\[ \frac{LS(1 + \epsilon Y_{2,m})}{d^2(1 + \epsilon Y_{2,m}, M)} \to \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} \] as \( \epsilon \to 0. \]

This proves that the constant \( \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} \) is optimal.

\[ \square \]

Next, we claim that the optimal constant of global stability of log-Sobolev inequality must be strictly smaller than \( \frac{8\pi^{n/2}}{\Gamma(n/2)(n + 2)} \). Namely, we will give the proof of Theorem 1.3.

\[ \text{Proof of Theorem 1.3.} \] We adopt the idea of Konig’s in [27] to do the third-order Taylor expansion to the functional \( LS \). For fixed \( y_2 \in Y_2 \) with \( \|y_2\|_{L^2} = 1 \), let \( U_\epsilon = 1 + \epsilon y_2 \). Direct computations yields

\[ LS(1 + \epsilon y_2) = LS(1) + \left. \frac{d}{dt} \right|_{t=0} LS(1 + t\epsilon y_2) + \frac{1}{2} \left. \frac{d^2}{dt^2} \right|_{t=0} LS(1 + t\epsilon y_2) \]

\[ + \left. \frac{1}{6} \frac{d^2}{dt^2} \right|_{t=0} LS(1 + t\epsilon y_2) + o(\epsilon^3), \quad (3.12) \]

where

\[ LS(1) = \left. \frac{d}{dt} \right|_{t=0} LS(1 + t\epsilon y_2) = 0, \quad (3.13) \]
\[
\frac{1}{2} \frac{d^2}{dt^2} \bigg|_{t=0} LS(1 + t \epsilon y_2) = 4 \epsilon^2 \left< y_2, H(y_2) - C_n y_2 + \frac{C_n}{|S^n|} \int_{S^n} y_2 d\omega \right> = \frac{8 \pi^{n/2}}{\Gamma(n/2)(n+2)} \epsilon^2, \tag{3.14}
\]

\[
\frac{1}{6} \frac{d^3}{dt^3} \bigg|_{t=0} LS(1 + t \epsilon y_2) = -\frac{2}{3} C_n \epsilon^3 \int_{S^n} y_2^3 d\sigma. \tag{3.15}
\]

According to the definition of the global stability of log-Sobolev inequality, we obtain
\[
\inf_{u \in D_L} \frac{LS(u)}{d^2(u, M)} \leq \frac{LS(1 + \epsilon y_2)}{d(1 + \epsilon y_2, M)} = \frac{8 \pi^{n/2}}{\Gamma(n/2)(n+2)} - \frac{2}{3} C_n \epsilon \int_{S^n} y_2^3 d\sigma + o(\epsilon). \tag{3.16}
\]

In order to show that
\[
\inf_{u \in D_L} \frac{LS(u)}{d^2(u, M)} < \frac{8 \pi^{n/2}}{\Gamma(n/2)(n+2)},
\]
we only need to find suitable \( y_2 \in Y_2 \) such that \( \int_{S^n} y_2^3 d\sigma > 0 \). In fact, pick \( \tilde{y}_2 = w_1 w_2 + w_2 w_3 + w_3 w_1 \), then we have
\[
\tilde{y}_2^2 = I_1(w) + I_2(w) + I_3(w),
\]
where
\[
I_1(w) = 6 w_1^2 w_2^2 w_3^2, \\
I_2(w) = 6(w_1 w_2^2 w_3^2 + w_2 w_1^2 w_3^2 + w_3 w_1^2 w_2^2), \\
I_3(w) = w_1^3 w_2^3 + w_2^3 w_3^3 + w_3^3 w_1^3.
\]

Observing that all the monomials in \( I_2 \) and \( I_3 \) are odd function of at least one coordinate \( w_i \) (i=1, 2, 3), hence
\[
\int_{S^n} I_2(w) d\sigma = \int_{S^n} I_3(w) d\sigma = 0.
\]

Picking \( y_2 = \frac{\tilde{y}_2}{\|\tilde{y}_2\|_{L^2}} \), we derive \( \int_{S^n} y_2^3 d\sigma = \|\tilde{y}_2\|_{L^2}^{-1} \int_{S^n} 6 w_1^2 w_2^2 w_3^2 d\sigma > 0 \). \( \square \)

Finally, we will explain why we cannot expect the following stable log-Sobolev inequality using double integral as distance: for any \( u \in D_L \), there holds
\[
LS(u) \geq c d_0^2(u, M)
\]
for some \( c > 0 \), where
\[
d_0^2(u, v) = \int_{S^n \times S^n} \frac{[u(\omega) - v(\omega) - u(\eta) + v(\eta)]^2}{|\omega - \eta|^n} d\omega d\eta.
\]
Proof of Theorem 1.4. First, we claim that for any $u \in \mathcal{D}_L$, there exist a $v_{c_0,\xi_0}$ in the optimizer set $M$ such that

$$d_0(u, v_{c_0,\xi_0}) = d_0(u, M).$$

Observing that $d_0(u, v_{c,\xi})$ is continuous with respect to the variables $(c, \xi) \in \mathbb{R} \times B^{n+1}_{\epsilon}$, in order to prove that $d_0(u, M)$ can be achieved by some function $v_{c_0,\xi_0}$, we only need to show that

$$\lim_{c \to +\infty} d_0(u, v_{c,\xi}) = +\infty, \quad \lim_{\xi \to \partial B^{n+1}_{\epsilon}} d_0(u, v_{c,\xi}) = +\infty.$$

Clearly,

$$\lim_{c \to +\infty} d_0(u, v_{c,\xi}) \geq \lim_{c \to +\infty} |d_0(v_{c,\xi}, 0) - d_0(u, 0)| = \lim_{c \to +\infty} |cd_0(1, \xi, 0) - d_0(u, 0)| = +\infty.$$

On the other hand, by (3.5) and Parseval’s identity, there holds

$$d_0(v_{c,\xi}, 0) \geq C_0 \frac{\|v_{c,\xi} - \overline{v_{c,\xi}}\|_{L^2(\mathbb{S}^n)}}{c},$$

for some constant $C_0 > 0$, where $\overline{v_{c,\xi}} = \int_{\mathbb{S}^n} v_{c,\xi} d\omega/|\mathbb{S}^n|$. A direct calculation gives that

$$\lim_{\xi \to \partial B^{n+1}_{\epsilon}} \int_{\mathbb{S}^n} v_{c,\xi} d\omega \lesssim \lim_{\xi \to \partial B^{n+1}_{\epsilon}} \int_{\mathbb{S}^n} \frac{1}{|\xi - w|} \frac{\partial}{\partial w} d\omega \lesssim 1.$$

Applying Fatou’s Lemma, we directly derive that

$$\lim_{\xi \to \partial B^{n+1}_{\epsilon}} \left\|v_{c,\xi} - \overline{v_{c,\xi}}\right\|^2_{L^2(\mathbb{S}^n)} \geq \int_{\mathbb{S}^n} \lim_{\xi \to \partial B^{n+1}_{\epsilon}} \left|v_{c,\xi} - \overline{v_{c,\xi}}\right|^2 d\omega = +\infty,$$

which together with $d_0(u, v_{c,\xi}) \geq |d_0(v_{c,\xi}, 0) - d_0(u, 0)|$ and $d_0(v_{c,\xi}, 0) \geq C_0 \frac{\|v_{c,\xi} - \overline{v_{c,\xi}}\|_{L^2(\mathbb{S}^n)}}{c}$ yields that

$$\lim_{\xi \to \partial B^{n+1}_{\epsilon}} d_0(u, v_{c,\xi}) = +\infty.$$

This accomplishes the proof of the existence of an optimizer.

Now choose $1 + \epsilon Y_{1,m}$, by the same computation as done in (3.11) we have

$$LS(1 + \epsilon Y_{1,m}) = o(\epsilon^2).$$

On the other hand, by the early proved claim that for any $u \in \mathcal{D}_L$, there exist a $v_{c_0,\xi_0}$ in the optimizer set $M$ such that

$$d_0(u, v_{c_0,\xi_0}) = d_0(u, M),$$

there exist a $v_{c_0,\xi_0}$ such that $d_0(Y_{1,m}, v_{c_0,\xi_0}) = d_0(Y_{1,m}, M)$, which implies $d_0(Y_{1,m}, M) \neq 0$. Otherwise $Y_{1,m} = v_{c_0,\xi_0} + c$, which is impossible according to the definition of $Y_{1,m}$. Then

$$d_0^2(1 + \epsilon Y_{1,m}, M) = \epsilon^2 d_0^2(Y_{1,m}, M).$$

Hence, we derive that

$$\lim_{\epsilon \to 0} \frac{LS(1 + \epsilon Y_{1,m})}{d_0^2(1 + \epsilon Y_{1,m}, M)} = \lim_{\epsilon \to 0} \frac{\epsilon^2 d_0^2(Y_{1,m}, M)}{d_0^2(1 + \epsilon Y_{1,m}, M)} = 0,$$

which completes our proof. □
4. Stability of the Moser-Onofri Inequality on the Sphere $S^n$

In this section, we will prove the local stability of the Moser-Onofri inequality (1.17):

$$MO(u) \geq \frac{\Gamma(n+1)}{2^{n-1}\pi^{n/2}\Gamma(n/2)}d^2(u, M) + o(d^2(u, M))$$  \hspace{2cm} (4.1)

and show that the global version of this type of stable inequality does not hold.

Proof of Theorem 1.5. As we discussed in Section 3, for any $u \in Q_M$, there exists a function $v_{c_0, \xi_0}$ in $M$ satisfying $u = v_{c_0, \xi_0} + dv$, where $\|v\|_{L^2} = 1$, $v \perp TM_{v_{c_0, \xi_0}}$ and $v_{c_0, \xi_0} = c_0 J^{1/2}$ for some conformal map on $S^n$. Recall $u_{\Phi^{-1}} = J^{1/2} u \circ \Phi^{-1}$, thus

$$u_{\Phi^{-1}} = c_0 + d v_{\Phi^{-1}}.$$

Since the functional $MO(u)$ is conformal invariant under the transformation $u \to u_{\Phi^{-1}}$ and of class $C^2$ on $Q_M \setminus \{0\}$, then

$$MO(u) = MO(u_{\Phi^{-1}})$$

$$= MO(c_0) + \frac{d}{dt}\bigg|_{t=0} MO(c_0 + tdv_{\Phi^{-1}}) + \frac{1}{2} \frac{d^2}{dt^2}\bigg|_{t=0} MO(c_0 + tdv_{\Phi^{-1}}) + o(d^2).$$  \hspace{2cm} (4.2)

Direct computations give

$$MO(c_0) = \frac{d}{dt}\bigg|_{t=0} MO(c_0 + tdv_{\Phi^{-1}}) = 0,$$  \hspace{2cm} (4.3)

$$\frac{d^2}{dt^2}\bigg|_{t=0} MO(c_0 + tdv_{\Phi^{-1}})$$

$$= 4d^2 \left[ \frac{1}{n!} \int_{S^n} v_{\Phi^{-1}}(A_n(v_{\Phi^{-1}}))d\omega - \int_{S^n} v_{\Phi^{-1}}^2 d\omega + \left( \int_{S^n} v_{\Phi^{-1}} d\omega \right)^2 \right].$$  \hspace{2cm} (4.4)

Since we have already proved that

$$v_{\perp TM_{v_{c_0, \xi_0}}} = \text{span}\{J^{1/2} v_{\Phi^{-1}}, J^{1/2} Y_{1,i} \circ \Phi^{-1}, i = 1, \cdots, n + 1\},$$

then $v_{\Phi^{-1}} \perp \text{span}\{Y_0, Y_{1,i}, i = 1, \cdots, n + 1\}$. Thus $v_{\Phi^{-1}}$ has the spherical harmonics expansion

$$v_{\Phi^{-1}} = \sum_{l=2}^{\infty} \sum_{m=1}^{N(n,l)} c_{l,m} Y_{l,m}.$$  \hspace{2cm} (4.5)

Using this fact with (4.2),(4.3),(4.4) and $\|v_{\Phi^{-1}}\|_{L^2} = 1$ we can obtain

$$MO(u) = 2d^2 \left[ \frac{1}{n!} \int_{S^n} v_{\Phi^{-1}}(A_n v_{\Phi^{-1}})d\omega - \int_{S^n} v_{\Phi^{-1}}^2 d\omega + \left( \int_{S^n} v_{\Phi^{-1}} d\omega \right)^2 \right] + o(d^2)$$

$$= 2 \frac{d^2}{|S^n|} \sum_{l=2}^{\infty} \sum_{m=1}^{N(n,l)} \left( \frac{l(l+1) \cdots (l+n-1)}{n!} - 1 \right) c_{l,m}^2 + o(d^2)$$

$$= \frac{d^2}{|S^n|} \sum_{l=2}^{\infty} \sum_{m=1}^{N(n,l)} \left( \frac{l(l+1) \cdots (l+n-1)}{n!} - 1 \right) c_{l,m}^2 + o(d^2).$$  \hspace{2cm} (4.6)
\[
\frac{\Gamma(n+1)}{2^{n-1} \pi^{n/2} \Gamma(n/2)} = \frac{\Gamma(n+1)}{2^{n-1} \pi^{n/2} \Gamma(n/2)} d^2
\]

To prove that the constant \(\frac{\Gamma(n+1)}{2^{n-1} \pi^{n/2} \Gamma(n/2)}\) is optimal, we choose the test functions \(1 + \varepsilon Y_{2,m}\). By the same calculation as done in (4.5), we have
\[
MO(1 + \varepsilon Y_{2,m}) = \varepsilon^2 \frac{\Gamma(n+1)}{2^{n-1} \pi^{n/2} \Gamma(n/2)} + o(\varepsilon^2).
\]
On the other hand, since \(Y_{2,m} \perp TM_1\), then \(d(1 + \varepsilon Y_{2,m}, M) = \varepsilon\) for small \(\varepsilon\). Then it follows that
\[
MO(1 + \varepsilon Y_{2,m}) = \varepsilon^2 \frac{\Gamma(n+1)}{2^{n-1} \pi^{n/2} \Gamma(n/2)} d^2 (1 + \varepsilon Y_{2,m}, M) + o(d^2 (1 + \varepsilon Y_{2,m}, M)),
\]
which implies the sharpness of the constant \(\frac{\Gamma(n+1)}{2^{n-1} \pi^{n/2} \Gamma(n/2)}\).

Finally, since the functional \(MO(u)\) is 0-homogeneous, by Theorem (1.1), we deduce that there does not exist the global version of the local stable Moser-Onofri inequality (4.1) on the sphere \(S^n\).

\[\Box\]

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