Extended weak convergence and utility maximisation with proportional transaction costs

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Abstract In this paper, we study utility maximisation with proportional transaction costs. Assuming extended weak convergence of the underlying processes, we prove the convergence of the time-0 values of the corresponding utility maximisation problems. Moreover, we establish a limit theorem for the optimal trading strategies. The proofs are based on the extended weak convergence theory developed in Aldous (Weak Convergence of Stochastic Processes for Processes Viewed in the Strasbourg Manner, 1981) and on the Meyer–Zheng topology introduced in Meyer and Zheng (Ann. Inst. Henri Poincaré Probab. Stat. 20:353–372, 1984).

Keywords Utility maximisation · Proportional transaction costs · Extended weak convergence · Meyer–Zheng topology

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1 Introduction

We deal with the continuity of the utility maximisation problem in the presence of proportional transaction costs under convergence in distribution of the financial markets. More specifically, we focus on utility maximisation from terminal wealth under an admissibility condition on the wealth processes.

In the presence of transaction costs, the problem of utility maximisation goes back to Magill and Constantinides [22], where the authors considered maximising the expected consumption in the Black–Scholes model. Analysing this as a stochastic control problem, Davis and Norman [12] gave a rigorous proof for the heuristic derivation of Magill and Constantinides. Later, Shreve and Soner [27] removed technical conditions needed in [12] and provided a complete solution under the assumption that the value function is finite. The more tractable problems of maximising the asymptotic growth rate for logarithmic or power utility in the Black–Scholes model under transaction costs have been studied by Taksar et al. [29] and Dumas and Luciano [16]. In [21], Kallsen and Muhle-Karbe applied the shadow price approach and solved the Merton problem with logarithmic utility and proportional transaction costs. Later this approach was extended to the power utility case; see Gerhold et al. [17]. Cvitanić and Karatzas [10] were the first to establish continuous-time duality results for hedging with proportional transaction costs and apply them for the utility maximisation problem. In the sequel, their seminal paper was extended and studied in much more generality. For multivariate financial markets, Deelstra et al. [13], Bouchard [7] and Campi and Owen [8] established duality and existence results for portfolio optimisation with proportional transaction costs. It is important to mention the work of Guasoni [18], who provided general existence results for utility maximisation (with proportional transaction costs) without assuming any semimartingale structure on the risky assets. For more recent results on duality theory with proportional transaction costs, see Czichowsky and Schachermayer [11] and Bayraktar and Yu [5].

Although utility maximisation with proportional transaction costs was widely studied, to the best of our knowledge, continuity under weak convergence was not considered before. Clearly, the problem of utility maximisation depends on the flow of information (the filtration). Hence one should not expect that convergence of asset prices alone will imply convergence of the time-0 values of the corresponding control problems. In particular, there are many examples (see for instance Aldous [1, Sect. 11]) of processes which are “close” to each other in distribution, but where the behaviour of the corresponding filtrations is completely different. This brings us to a stronger notion of convergence.

In his important unpublished manuscript [1], Aldous introduced the notion of extended weak convergence and showed that for optimal stopping, this is the right notion of convergence. Extended weak convergence is defined via weak convergence of the corresponding prediction processes. The prediction process is a measure-valued process representing the regular conditional distributions of the original stochastic process, and so it captures the structure of the information flow. It is important to mention the recent papers Backhoff-Veraguas et al. [2, 3] which provide a novel approach in the discrete-time setup to weak convergence topologies that take the information flow into account.
In this work, we consider a sequence of continuous-time financial markets with continuous asset prices which converge to a limit market. To ease notation, we focus on the case of one risky asset. Our main assumptions are extended weak convergence of the underlying processes and a strong version of absence of arbitrage (Assumption 2.3). Under these natural conditions, we prove that for a continuous and concave utility function, there is a convergence of maximal expected utilities; see Theorem 3.1. Moreover, we obtain a limit theorem for the optimal trading strategies; see Theorem 3.3. To the best of our knowledge, this paper is the first work that applies extended weak convergence to continuous-time portfolio optimisation problems.

In addition to the extended weak convergence, we also apply the Meyer–Zheng topology which was introduced in Meyer and Zheng [23]. As we shall see, this topology perfectly fits for hedging with proportional transaction costs. More precisely, the admissibility condition will imply tightness of the trading strategies in the Meyer–Zheng topology.

The current work is a continuation of Bayraktar et al. [4] where a similar problem was studied in a frictionless setup (i.e., with no transaction costs). Surprisingly, for the frictionless case, extended weak convergence is not a sufficient condition — in order to have convergence for the maximal expected utilities, one needs to require convergence of the equivalent martingale measures (see [4, Assumption 2.5]). These objects can be viewed as consistent price systems for the frictionless case. It is important to emphasise that in the presence of proportional transaction costs, there is no need to assume any convergence structure on the consistent price systems. In fact, the presence of proportional transaction costs provides the needed compactness.

The rest of the paper is organised as follows. In the next section, we introduce the setup and list our assumptions. In Sect. 3, we formulate our main results, which are proved in Sect. 4. In Sect. 5, we provide a specific example for financial markets which converge to a stochastic volatility model. We show that in the presence of proportional transaction costs, the maximal expected utilities converge, while there is no convergence in the frictionless setup.

2 Preliminaries and assumptions

2.1 Hedging with proportional transaction costs

We consider a model with one risky asset which we denote by \( S = (S_t)_{0 \leq t \leq T} \), where \( T < \infty \) is the time horizon. We assume that the investor has a bank account that for simplicity bears no interest. The process \( S \) is assumed to be an adapted, strictly positive and continuous process (not necessarily a semimartingale) defined on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), where the filtration \( \mathbb{F} := (\mathcal{F}_t)_{0 \leq t \leq T} \) satisfies the usual assumptions (right-continuity and completeness).

Let \( \kappa \in (0, 1) \) be a constant. Consider a model in which every purchase or sale of the risky asset at a time \( t \in [0, T] \) is subject to a proportional transaction cost at a rate \( \kappa \). A trading strategy is an adapted process \( \gamma = (\gamma_t)_{0 \leq t \leq T} \) of bounded variation with right-continuous paths; note that it automatically has left limits and hence is RCLL (right-continuous with left limits). The random variable \( \gamma_t \) denotes the number
of shares held at time \( t \). We use the convention \( \gamma_0^- = 0 \). Moreover, we require that \( \gamma_T = 0 \) which means that we liquidate the portfolio at the maturity date.

Let \( \gamma_t := \gamma_t^+ - \gamma_t^- \), \( t \in [0, T] \), be the Jordan decomposition into processes \( (\gamma_t^+)_{0 \leq t \leq T} \) of positive variation and \( (\gamma_t^-)_{0 \leq t \leq T} \) of negative variation. Since the bid price process is \( (1 - \kappa)S \) and the ask price process is \( (1 + \kappa)S \), the portfolio value of a trading strategy \( \gamma \) at time \( t \) is given by

\[
V_t^\gamma := (1 - \kappa) \int_0^t S_u d\gamma_u^- - (1 + \kappa) \int_0^t S_u d\gamma_u^+ + (1 - \kappa) S_t (\gamma_t^+) - (1 + \kappa) S_t (\gamma_t^-),
\]

where \( (\gamma_t^+) := \max(0, \gamma_t) \) and \( (\gamma_t^-) := \max(0, -\gamma_t) \). (Beware that these are not the same variables as \( \gamma_t^+, \gamma_t^- \) above.) We note that the integrals take into account a possible transaction at \( t = 0 \). Namely, we define

\[
\int_0^t S_u d\gamma_u^- := S_0 \gamma_0^- + \int_{(0,t]} S_u d\gamma_u^-, \quad \int_0^t S_u d\gamma_u^+ := S_0 \gamma_0^+ + \int_{(0,t]} S_u d\gamma_u^+.
\]

By rearranging terms, we get

\[
V_t^\gamma = \gamma_t S_t - \int_0^t S_u d\gamma_u^- - \kappa |\gamma_t| S_t - \kappa \int_0^t S_u |d\gamma_u|, \quad t \in [0, T]. \tag{2.1}
\]

Observe that the wealth process \( (V_t^\gamma)_{0 \leq t \leq T} \) is RCLL like \( \gamma \) and that \( \gamma_T = 0 \) implies \( V_T^- = V_T^\gamma \). For any initial capital \( x > 0 \), we denote by \( \mathcal{A}(x) \) the set of all trading strategies \( \gamma \) which satisfy the admissibility condition \( x + V_0^\gamma \geq 0 \) for all \( t \in [0, T] \).

Next, we introduce our utility maximisation problem. Let \( C[0, T] \) be the space of all continuous functions \( f : [0, T] \to \mathbb{R} \) with the uniform topology. Consider a continuous function \( U : (0, \infty) \times C[0, T] \to \mathbb{R} \) such that for any \( s \in C[0, T] \), the function \( U(\cdot, s) \) is nondecreasing and concave. We extend \( U \) to \( \mathbb{R}_+ \times C[0, T] \) by \( U(0, s) := \lim_{v \downarrow 0} U(v, s) \) (the limit might be \( -\infty \)).

**Assumption 2.1** (i) For any \( x > 0 \), we have \( \mathbb{E}_P[U(x, S)] > -\infty \).

(ii) There exist continuous functions \( m_1, m_2 : [0, 1) \to \mathbb{R}_+ \) (moduli of continuity) with \( m_1(0) = m_2(0) = 0 \) and a nonnegative random variable \( \zeta \in L^1(\Omega, \mathcal{F}, \mathbb{P}) \) such that for any \( \lambda \in (0, 1) \) and \( v > 0 \),

\[
U((1 - \lambda)v, S) \geq (1 - m_1(\lambda)) U(v, S) - m_2(\lambda) \zeta.
\]

For a given initial capital \( x > 0 \), consider the optimisation problem

\[
u(x) := \sup_{\gamma \in \mathcal{A}(x)} \mathbb{E}_P[U(x + V_T^\gamma, S)],
\]

where we set \( \mathbb{E}_P[X] := -\infty \) for any random variable \( X \) with \( \mathbb{E}_P[\max(-X, 0)] = \infty \). Let us notice that Assumption 2.1 (i) implies \( \nu(x) > -\infty \).

**Remark 2.2** We should note that power and log utility satisfy Assumption 2.1. Moreover, for a continuous function \( g : \mathbb{R} \to \mathbb{R}_+ \), the utility \( U(v, s) := -(g(s_T) - v)^+ \).
which corresponds to shortfall risk minimisation for a vanilla option with the payoff $g(S_T)$ also satisfies Assumption 2.1 (provided that $\mathbb{E}_P[g(S_T)] < \infty$). Indeed, in this case, if $v \geq \frac{g(S_T)}{1-\lambda}$, then $U((1-\lambda)v, S) = U(v, S) = 0$. If $v < \frac{g(S_T)}{1-\lambda}$, then

$$|U((1-\lambda)v, S) - U(v, S)| \leq \lambda v \leq \frac{\lambda}{1-\lambda} g(S_T).$$

Thus for $m_1(\lambda) \equiv 0$, $m_2(\lambda) := \frac{\lambda}{1-\lambda}$ and $\zeta \equiv g(S_T)$, Assumption 2.1 holds true.

### 2.2 Approximating sequence of models

For any $n$, let $S^n = (S^n_t)_{0 \leq t \leq T}$ be a strictly positive continuous process defined on some filtered probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n, \mathbb{I}^n)$. The filtration $\mathbb{F}^n := (\mathcal{F}^n_t)_{0 \leq t \leq T}$ satisfies the usual assumptions under $\mathbb{P}^n$. For the $n$th model, a trading strategy is a right-continuous adapted processes $\gamma^n = (\gamma^n_t)_{0 \leq t \leq T}$ of bounded variation which satisfies $\gamma^n_T = 0$. As before, we use the convention that $\gamma^n_0 = 0$. The corresponding portfolio value is given by

$$V^{\gamma^n}_t := \gamma^n_t S^n_t - \int_0^t S^n_u d\gamma^n_u - \kappa |\gamma^n_t| S^n_t - \kappa \int_0^t S^n_u |d\gamma^n_u|, \quad t \in [0, T].$$

For any $x > 0$, we denote by $\mathcal{A}^n(x)$ the set of all trading strategies $\gamma^n$ which satisfy $x + V^{\gamma^n}_T \geq 0$ for all $t \in [0, T]$. Set

$$u^n(x) := \sup_{\gamma^n \in \mathcal{A}^n(x)} \mathbb{E}_{\mathbb{P}^n}[U(x + V^{\gamma^n}_T, S^n)].$$

The following assumption will be essential for proving tightness and some integrability properties of the admissible trading strategies.

**Assumption 2.3** There exist $\varepsilon \in (0, \kappa)$ and probability measures $Q \approx P$, $Q^n \approx P^n$, $n \in \mathbb{N}$, with the following properties:

1. There exist a local $Q$-martingale $(M_t)_{0 \leq t \leq T}$ and for any $n \in \mathbb{N}$ a local $Q^n$-martingale $(M^n_t)_{0 \leq t \leq T}$ such that

$$|M_t - S_t| \leq (\kappa - \varepsilon) S_t \quad \mathbb{P}$-a.s., $\forall t \in [0, T],

$$|M^n_t - S^n_t| \leq (\kappa - \varepsilon) S^n_t \quad \mathbb{P}^n$-a.s., $\forall t \in [0, T]$, for any $n$.

2. The sequence $(P^n)_{n \in \mathbb{N}}$ of probability measures is contiguous with respect to the sequence $(Q^n)_{n \in \mathbb{N}}$, i.e., for any sequence $A^n \in \mathcal{F}^n$, $n \in \mathbb{N}$, with $\lim_{n \to \infty} Q^n[A^n] = 0$, we have $\lim_{n \to \infty} P^n[A^n] = 0$.

**Remark 2.4** Assumption 2.3 can be viewed as a uniform version of the existence of strictly consistent price systems. For a given model, the existence of a strictly consistent price system is equivalent to a robust no-free-lunch-with-vanishing-risk condition for simple strategies; see e.g. Guasoni et al. [19].

Next, we assume the following uniform integrability assumptions.
Assumption 2.5  
(i) For any \( x > 0 \), the set \( \{ U^{-}(x, S^n) : n \in \mathbb{N} \} \) is uniformly integrable, where \( U^{-} := \max(-U, 0) \).

(ii) For any \( x > 0 \), the set \( \{ U^{+}(x + V^\gamma^n, S^n) : n \in \mathbb{N}, \gamma^n \in \mathcal{A}^n(x) \} \) is uniformly integrable, where \( U^{+} := \max(U, 0) \).

Remark 2.6 Since the markets are not necessarily defined on the same probability space, it is important to clarify the notion of uniform integrability. We say that a family of random variables (not necessarily defined on the same probability space) \( Z_i : \hat{\Omega}^i \rightarrow \mathbb{R}, i \in I \), is uniformly integrable if

\[
\lim_{N \rightarrow \infty} \sup_{i \in I} E_i \left[ |Z_i| I_{\{|Z_i| > N\}} \right] = 0,
\]

where \( E_i \) denotes the expectation on the corresponding probability space. Observe that uniform integrability implies \( \sup_{i \in I} E_i \left[ |Z_i| \right] < \infty \).

In this paper, we use the following known facts (see Billingsley [6, Sect. 3]). If we have the weak convergence \( Z_n \Rightarrow Z \) and \( Z_n, n \in \mathbb{N} \), are nonnegative, then the following hold:

(I) (Fatou’s lemma) \( E[Z] \leq \lim inf_{n \rightarrow \infty} E[Z^n] \).

(II) The uniform integrability of \( (Z^n)_{n \in \mathbb{N}} \) is equivalent to the convergence of the expectations, \( E[Z] = \lim_{n \rightarrow \infty} E[Z^n] \).

For not necessarily nonnegative random variables, we use the following result which follows immediately from the above (I) and (II) and from the decomposition \( Z = Z^+ - Z^- \), \( Z^n = Z^{n,+} - Z^{n,-} \) into positive and negative parts:

(III) If we have the weak convergence \( Z_n \Rightarrow Z \) and the negative parts \( (Z^n,-)_{n \in \mathbb{N}} \) are uniformly integrable, then \( Z^- \) is integrable and \( E[Z] \leq \lim inf_{n \rightarrow \infty} E[Z^n] \).

Notice that from Assumption 2.5,

\[
-\infty < - \sup_{n \in \mathbb{N}} E_{P^n} [U^{-}(x, S^n)] \leq \lim inf_{n \rightarrow \infty} u^n(x) \leq \lim sup_{n \rightarrow \infty} u^n(x)
\]

\[
\leq \sup_{n \in \mathbb{N}} \sup_{\gamma^n \in \mathcal{A}^n(x)} E_{P^n} [U^{+}(x + V^\gamma^n, S^n)] < \infty, \quad \forall x > 0.
\]

In general, if \( U \) is not bounded from above, the verification of Assumption 2.5 (ii) can be a difficult task. The following result gives quite general and easily verifiable conditions which are sufficient for Assumption 2.5 to hold true.

Proposition 2.7 Suppose there exist constants \( L > 0, 0 < \alpha < 1 \) and \( q > \frac{1}{1-\alpha} \) which satisfy the following:

(I) For all \( (v, s) \in (0, \infty) \times C[0, T] \), \( U(v, s) \leq L(1 + v^\alpha) \).

(II) For any \( n \in \mathbb{N} \), there exist a probability measure \( \hat{Q}^n \approx P^n \), a local \( \hat{Q}^n \)-martingale \( (\hat{M}_t^n)_{0 \leq t \leq T} \) such that for all \( t \in [0, T] \), we have \( |\hat{M}_t^n - S_t^n| \leq \kappa S_t^n P^n\text{-a.s.} \) (i.e., \( (\hat{M}_t^n, \hat{Q}^n) \) is a consistent price system), and

\[
\sup_{n \in \mathbb{N}} E_{\hat{Q}^n} \left[ \left( \frac{dP^n}{d\hat{Q}^n} \right)^q \right] < \infty.
\]

Then Assumption 2.5 (ii) holds true.
Proof This is done by using similar arguments as in Bayraktar et al. [4, Lemma 2.2]. The only needed property is that for any $n \in \mathbb{N}$ and $\gamma^n \in A^n(x)$, we have $\mathbb{E}_{Q^n} [V^n] \leq 0$. This is a well-known property of consistent price systems (see e.g. Schachermayer [25, Proposition 1.6]). □

We end this section with an example of a sequence of market models which satisfy Assumptions 2.3 and 2.5.

**Example 2.8** Assume that for any $n$, the process $(S^n_t)_{0 \leq t \leq T}$ is given by the SDE

$$dS^n_t = S^n_t (\mu^n_t dt + \sigma^n_t dW^n_t), \quad t \in [0, T],$$

where $W^n$ is a Brownian motion and the processes $\mu^n, \sigma^n$ are predictable with respect to $\mathbb{F}^n$. Similarly, assume that

$$dS_t = S_t (\mu_t dt + \sigma_t dW_t), \quad t \in [0, T],$$

where $W$ is a Brownian motion and the processes $\mu, \sigma$ are predictable with respect to $\mathbb{F}$. Moreover, we assume that $\sigma, \sigma^n$ are strictly positive and there exists a constant $C$ which does not depend on $n$ such that

$$\sup_{0 \leq t \leq T} \frac{\mu_t}{\sigma_t}, \sup_{0 \leq t \leq T} \left| \frac{\mu^n_t}{\sigma^n_t} \right| \leq C \quad \text{a.s., } \forall n \in \mathbb{N}. \quad (2.3)$$

Then the Girsanov theorem implies that Assumption 2.3 (1) holds true for $M := S, M^n := S^n, n \in \mathbb{N}$, and

$$\frac{dQ}{dP} := \exp \left( - \int_0^T \frac{\mu_t}{\sigma_t} dW_t - \int_0^T \frac{1}{2} \left( \frac{\mu^n_t}{\sigma^n_t} \right)^2 dt \right),$$

$$\frac{dQ^n}{dP^n} := \exp \left( - \int_0^T \frac{\mu^n_t}{\sigma^n_t} dW^n_t - \int_0^T \frac{1}{2} \left( \frac{\mu^n_t}{\sigma^n_t} \right)^2 dt \right), \quad n \in \mathbb{N}.$$  

From (2.3), we have

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{Q^n} \left[ (\frac{dP^n}{dQ^n})^q \right] < \infty, \quad \forall q \in \mathbb{R},$$

and so Assumption 2.3 (2) holds true. Moreover, in view of Proposition 2.7, it follows that Assumption 2.5 holds true provided that there exist $L, \alpha > 0$ such that

$$U(v,s) \leq L(1 + v^\alpha), \quad \forall (v,s) \in (0, \infty) \times C[0, T].$$

Another type of example can be obtained by linearly interpolating discrete-time processes which are discrete-time analogues of (2.2). In Sect. 5, we provide a detailed analysis of a particular example of this type.
2.3 Extended weak convergence

We start with formulating our convergence assumptions.

**Assumption 2.9** For any $k \in \mathbb{N}$, let $\mathcal{D}([0, T]; \mathbb{R}^k)$ be the space of all RCLL functions $f : [0, T] \to \mathbb{R}^k$ equipped with the Skorokhod topology (for details see [6, Sect. 12]). We assume that there exist $m \in \mathbb{N}$ and stochastic processes $X^n : \Omega \to \mathcal{D}([0, T]; \mathbb{R}^m)$, $n \in \mathbb{N}$, $X : \Omega \to C([0, T]; \mathbb{R}^m)$ (i.e., $X$ is continuous) which satisfy the following:

(i) The filtrations $\mathcal{F}^n$, $n \in \mathbb{N}$, and $\mathbb{P}$ are the usual filtrations (right-continuous and completed by the corresponding nullsets) generated by $X^n$, $n \in \mathbb{N}$, and $X$, respectively.

(ii) We have the weak convergence $(S^n, X^n, \mathbb{P}^n) \Rightarrow (S, X, \mathbb{P})$ on $\mathcal{D}([0, T]; \mathbb{R}^{m+1})$.

The above relation means that the joint distribution of $(S^n, X^n)$ under $\mathbb{P}^n$ converges to the joint distribution of $(S, X)$ under $\mathbb{P}$.

(iii) We have the extended weak convergence $(X^n, \mathbb{P}^n) \Rightarrow (X, \mathbb{P})$. This means that for any $k$ and any continuous bounded function $\psi : \mathcal{D}([0, T]; \mathbb{R}^m) \to \mathbb{R}^k$, we have

$$(X^n, Y^n, \mathbb{P}^n) \Rightarrow (X, Y, \mathbb{P}) \quad \text{on} \quad \mathcal{D}([0, T]; \mathbb{R}^{m+k}),$$

where

$$Y^n_t := \mathbb{E}_{\mathbb{P}^n}[\psi(X^n)|\mathcal{F}^n_t] \quad \text{and} \quad Y_t := \mathbb{E}_{\mathbb{P}}[\psi(X)|\mathcal{F}_t], \quad t \in [0, T].$$

**Remark 2.10** Aldous [1] introduced the notion of “extended weak convergence” via prediction processes. He proved that extended weak convergence is equivalent to a more elementary condition which does not require the use of prediction processes (see [1, Proposition 16.15]). This is the definition we use above.

The verification of extended weak convergence was studied in [1] and Jakubowski and Słomiński [20]. Citing [1, Sect. 21], “any weak convergence result proved by the martingale technique can be improved to extended weak convergence”. In particular, if the processes $X^n$, $n \in \mathbb{N}$, have independent increments (with respect to $\mathbb{P}^n$) and $X$ is continuous in probability (with respect to $\mathbb{P}$), then the weak convergence $(X^n, \mathbb{P}^n) \Rightarrow (X, \mathbb{P})$ implies the extended weak convergence $(X^n, \mathbb{P}^n) \Rightarrow (X, \mathbb{P})$ (see [1, Proposition 20.18] and [20, Corollary 2]). For more results related to extended weak convergence, see [20].

3 Main results

We are ready to state our first limit theorem.

**Theorem 3.1** Under Assumptions 2.1, 2.3, 2.5 and 2.9, we have that

$$u(x) = \lim_{n \to \infty} u^n(x) \quad \text{for any} \quad x > 0.$$
A natural question is whether we have some kind of convergence for the optimal trading strategies (optimal controls) as well. In order to formulate such a limit theorem, we need some preparations.

Any function \( f \in \mathbb{D}([0, T]) := \mathbb{D}([0, T]; \mathbb{R}) \) can be extended to a function \( f : \mathbb{R}_+ \to \mathbb{R} \) by \( f(t) := f(T) \) for all \( t \geq T \). The Meyer–Zheng topology introduced in Meyer and Zheng [23] is the relative topology on the image measures on graphs \((t, f(t))\) of trajectories \( t \to f(t), t \in \mathbb{R}_+\), under the measure \( \lambda(dt) := e^{-t}dt \) (called pseudo-paths) induced by the weak topology on probability laws on the compactified space \([0, \infty) \times \tilde{\mathbb{F}}\). From [23, Lemma 1], it follows that the Meyer–Zheng topology on \( \mathbb{D}([0, T]) \) is given by the metric

\[
d_{MZ}(f, g) := \int_0^T \min \left( 1, |f(t) - g(t)| \right) dt + |f(T) - g(T)|, \quad f, g \in \mathbb{D}([0, T]).
\]

We denote the corresponding space by \( \mathbb{D}_{MZ}([0, T]) \).

**Remark 3.2** [23, Lemma 1] proves that convergence for the Meyer–Zheng topology on \( \mathbb{D}([0, \infty)) \) is equivalent to convergence in measure for \( \lambda(dt) := e^{-t}dt \). Since in our setup, the functions are constant on the time interval \([T, \infty)\), convergence in measure is given by the above metric \( d_{MZ} \).

Now we are ready to formulate our second limit theorem.

**Theorem 3.3** Assume that Assumptions 2.1, 2.3, 2.5 and 2.9 hold. Fix \( x > 0 \) and let \( \hat{\gamma}^n \in \mathcal{A}^n(x), n \in \mathbb{N} \), be a sequence of asymptotically optimal portfolios, i.e.,

\[
\lim_{n \to \infty} \left( u^n(x) - \mathbb{E}_{\mathbb{P}^n}[U(x + V_T^{\hat{\gamma}^n}, S^n)] \right) = 0. \tag{3.1}
\]

Then the sequence of laws \((S^n, X^n, \hat{\gamma}^n), \mathbb{P}^n)_{n \in \mathbb{N}} \) on \( \mathbb{D}([0, T]; \mathbb{R}^{1+m}) \times \mathbb{D}_{MZ}([0, T]) \) is tight, and thanks to Prohorov’s theorem (see [6, Sect. 5]), it is relatively compact. Moreover, any cluster point of the sequence \((S^n, X^n, \hat{\gamma}^n), \mathbb{P}^n)_{n \in \mathbb{N}} \) (there is at least one) is of the form \((S, X, \hat{\gamma}, \mathbb{P})\), where \( \hat{\gamma} = (\hat{\gamma}_t)_{0 \leq t \leq T} \) is a process of bounded variation with right-continuous paths. In particular, the joint distribution of the first two components equals the original joint distribution of \((S, X)\) under \( \mathbb{P} \). Thus, although the cluster point is not necessarily defined on the original probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), we still (with an abuse of notation) denote the corresponding probability measure by \( \mathbb{P} \), the first two components by \((S, X)\) and by \( \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \) the usual filtration (right-continuous and \( \mathbb{P} \)-completed) generated by \( X \). Finally, the optional projection \( \hat{\gamma}^F := \circ \hat{\gamma} \) of \( \hat{\gamma} \) with respect to \( \mathbb{F} \) is an optimal portfolio, i.e., it is well defined, satisfies \( \hat{\gamma}^F \in \mathcal{A}(x) \) and

\[
u(x) = \mathbb{E}_F[U(x + V_T^{\hat{\gamma}^F}, S)].
\]

We end this section with the following remark about Theorem 3.3.

**Remark 3.4** In view of Assumption 2.9 (ii), any cluster point of the sequence of laws \((S^n, X^n, \hat{\gamma}^n), \mathbb{P}^n)_{n \in \mathbb{N}} \) must be of the form \((S, X, \hat{\gamma}), \mathbb{P})\). We can show that for such
a cluster point, \( \hat{\gamma} \) attains the maximal expected utility. However, \( \hat{\gamma} \) is not necessarily adapted to the filtration generated by \( X \).

One possible way to treat this issue is to use the weak formulation setup. Roughly speaking, a weak formulation allows the investor to randomise from the start, and so the filtration is rich enough in the sense that the law of any cluster point \( ((S, X, \hat{\gamma}), \mathbb{P}) \) can be represented with an adapted process \( \hat{\gamma} \). On the other hand, since the enlarged filtration does not provide any additional information about the future (in comparison with the original filtration), the value of the utility maximisation problem remains the same as in the original setup. For more details, see Chau and Rásonyi [9].

In this paper, we do not consider the weak formulation approach; instead we solve the measurability issue by projecting the limit portfolio \( \hat{\gamma} \) on the investor’s filtration \( F \). We prove below that the projection is well defined and gives an optimal trading strategy.

Let us remark that if we had uniqueness results for the optimal trading strategy, then we could prove that the sequence \( ((S^n, X^n, \hat{\gamma}^n), \mathbb{P}^n)_{n \in \mathbb{N}} \) converges to \( ((S, X, \hat{\gamma}), \mathbb{P}) \), where \( \hat{\gamma} \) is the unique optimal control and in particular is adapted to \( F \). Surprisingly, up to date, there are no results related to the uniqueness of the optimal trading strategy. Of course, for strictly concave utility functions, we can prove the uniqueness of the optimal terminal wealth, but this does not imply the uniqueness of the optimal trading strategy (see [26, Remark 6.9]). The latter is an interesting question which is left for future research.

For the frictionless case, if the risky asset is a semimartingale with non-singular volatility matrix, then the Itô isometry and the uniqueness of the optimal terminal wealth imply the uniqueness of the optimal trading strategy.

4 Proof of the main results

4.1 Three crucial lemmas

We start with the following result. Recall that by Assumption 2.1 (i), \( u(x) > -\infty \).

**Lemma 4.1** The function \( u : (0, \infty) \to \mathbb{R} \cup \{\infty\} \) is continuous: For any \( x > 0 \), we have \( u(x) = \lim_{y \to x} u(y) \), where a priori the joint value can be equal to \( \infty \).

**Proof** This is done by similar arguments as in Bayraktar et al. [4, Lemma 2.1] for the frictionless case. The only needed property is that for any \( \lambda > 0 \) and trading strategy \( \gamma \), we have the equality \( V_t^{\lambda \gamma} = \lambda V_t^{\gamma} \), \( t \in [0, T] \). Trivially, this property holds true in our setup. \( \square \)

Now we prove the lower bound part in Theorem 3.1.

**Lemma 4.2** For any \( x > 0 \), we have

\[
\lim_{n \to \infty} u^n(x) \leq u(x).
\]
Proof 1) Fix $x > 0$. Without loss of generality (by passing to a subsequence), we assume that $\lim_{n \to \infty} u^n(x)$ exists. In view of Lemma 4.1, to prove the statement, it is sufficient to prove that for any $\varepsilon \in (0, x/3)$ and $\gamma \in \mathcal{A}(x - 3\varepsilon)$, we have

$$
\mathbb{E}_p[U(x + V_T^\gamma - 3\varepsilon, S)] \leq \lim_{n \to \infty} u^n(x). \quad (4.1)
$$

From the Skorohod representation theorem (Dudley [15, Theorem 3]) and Assumption 2.9 (ii), we can redefine the stochastic processes $(S^n, X^n), n \in \mathbb{N}$, and $(S, X)$ on the same probability space such that

$$(S^n, X^n) \longrightarrow (S, X) \quad \text{a.s. in } \mathbb{D}([0, T]; \mathbb{R}^{m+1}). \quad (4.2)$$

Choose $\varepsilon \in (0, x/3)$ and $\gamma \in \mathcal{A}(x - 3\varepsilon)$. We aim to prove (4.1).

2) For any $n \in \mathbb{N}$, let $\Gamma^n$ be the set of all trading strategies (they need not satisfy the admissibility condition) in the $n$th model. First we show that there exists a subsequence $\gamma^n \in \Gamma^n$, $n \in \mathbb{N}$, (for simplicity still indexed by $n$) such that

$$V_T^\gamma \leq \liminf_{n \to \infty} V_T^\gamma^n \quad (4.3)$$

and

$$\liminf_{n \to \infty} \inf_{0 \leq t \leq T} V_T^\gamma^n \geq 2\varepsilon - x. \quad (4.4)$$

To that end, define for any $n \in \mathbb{N}$ the $\mathbb{R}$-predictable process $\tilde{\gamma}^n = (\tilde{\gamma}_t^n)_{0 \leq t \leq T}$ by

$$\tilde{\gamma}_t^n := \sum_{i=1}^{n-1} \gamma \left(\frac{-iT}{n} \leq t < \frac{(i+1)T}{n}\right), \quad t \in [0, T].$$

We made a small shift in time in order to make $\tilde{\gamma}^n$ predictable. Since $S$ is continuous and $\gamma_T = 0$, we have

$$V_T^\gamma = -\int_0^T S_t d\gamma_t - \kappa \int_0^T S_t |d\gamma_t|$$

$$= \lim_{n \to \infty} \left( -\int_0^T S_t d\tilde{\gamma}_t^n - \kappa \int_0^T S_t |d\tilde{\gamma}_t^n| \right)$$

$$= \lim_{n \to \infty} V_T^{\tilde{\gamma}_t^n}. \quad (4.5)$$

Fix $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n - 1\}$ and $t \in \left[\frac{kT}{n}, \frac{(k+1)T}{n}\right)$. From (2.1) and the simple relations

$$\int_{(i-1)T/n}^{iT/n} d\gamma_t = \int_{(i-1)T/n}^{iT/n} d\tilde{\gamma}_t^n, \quad \int_{(i-1)T/n}^{iT/n} |d\gamma_t| \geq \int_{(i-1)T/n}^{iT/n} |d\tilde{\gamma}_t^n|, \quad i = 1, \ldots, k,$$
it follows that
\[
V_{kT/n}^\gamma - V_{(k+1)T/n}^\gamma^n \leq 2|\gamma_{kT/n}| |S_{(k+1)T/n} - S_{kT/n}|
+ 2 \int_0^{kT/n} |d\gamma_u| \sup_{0 \leq t_1, t_2 \leq (k+1)T/n, |t_2-t_1| \leq 2kT/n} |S_{t_2} - S_{t_1}|.
\]
Moreover, since \( \tilde{\gamma}^n \) is (a random) constant on the interval \([kT/n, t]\), (2.1) gives
\[
V_{kT/n}^\gamma - V_t^\gamma^n \leq 2|\tilde{\gamma}^n_{kT/n}| |S_t - S_{kT/n}|.
\]
Thus for any \( n \in \mathbb{N} \) (notice that \( V_t^\gamma^n = 0 \) for \( t < T/n \)),
\[
\inf_{0 \leq t \leq T} V_t^\gamma^n \geq \min_{0 \leq k \leq n} V_{kT/n}^\gamma - 6 \int_0^T |d\gamma_u| \sup_{|t_2-t_1| \leq 2kT/n} |S_{t_2} - S_{t_1}|.
\]
Since \( \gamma \in \mathcal{A}(x - 3\epsilon) \), we conclude that
\[
\lim \inf_{n \to \infty} \inf_{0 \leq t \leq T} V_t^\gamma^n \geq \inf_{0 \leq t \leq T} V_t^\gamma \geq 3\epsilon - x. \tag{4.6}
\]

Next, let \( \tilde{\Gamma} \) be the set of all simple integrands of the form
\[
\tilde{\gamma}_t = \sum_{i=1}^k \beta_i \mathbb{I}_{[t_i \leq t < t_i+1]}, \tag{4.7}
\]
where \( k \in \mathbb{N} \), \( 0 = t_1 < t_2 < \cdots < t_{k+1} = T \) is a deterministic partition and
\[
\beta_i = \phi_i(X_{a_{i,1}}, \ldots, X_{a_{i,m_i}}), \quad i = 1, \ldots, k, \tag{4.8}
\]
for a deterministic partition \( 0 = a_{i,1} < \cdots < a_{i,m_i} = t_i \) and a continuous bounded function \( \phi_i : (\mathbb{R}^m)^{m_i} \to \mathbb{R}^d \). Since the filtration \( \mathbb{F} \) is generated by \( X \), standard density arguments imply that any \( \mathcal{F}_t^- \)-measurable random variable can be approximated (with respect to convergence in probability) by random variables of the form \( \phi(X_{a_1}, \ldots, X_{a_k}) \), where \( 0 = a_1 < \cdots < a_k = t \) is a deterministic partition and \( \phi \) a continuous bounded function. Hence for any \( n \in \mathbb{N} \), the trading strategy \( \tilde{\gamma}^n \) can be approximated by trading strategies in \( \tilde{\Gamma} \). This together with (4.5) and (4.6) implies that for any \( \delta > 0 \), there exists \( \tilde{\gamma} \in \tilde{\Gamma} \) such that
\[
\mathbb{P}[V_T^\gamma > \delta + V_T^\tilde{\gamma}] < \delta \tag{4.9}
\]
and
\[
\mathbb{P}\left[ \inf_{0 \leq t \leq T} V_t^\tilde{\gamma} < 2\epsilon - x \right] < \delta. \tag{4.10}
\]
Let $\tilde{\gamma}$ be of the form (4.7) and (4.8). Define the strategies $\tilde{\gamma}_n \in \Gamma^n$, $n \in \mathbb{N}$, by

$$\tilde{\gamma}_t^n = \sum_{i=1}^{k} \phi_i(X_{a_i,1}^n, \ldots, X_{a_i,m_i}^n) 1_{\{t \leq t_i < t_{i+1}\}}.$$

From (4.2) and the fact that $\phi_i$, $i = 1, \ldots, k$, are continuous, we obtain

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T} |V_t^{\tilde{\gamma}} - V_t^{\tilde{\gamma}_n}| = 0 \quad \text{a.s.} \quad (4.11)$$

By applying the Borel–Cantelli lemma and (4.9)–(4.11), we obtain that there exists a subsequence $\gamma_n \in \Gamma_1^n$, $n \in \mathbb{N}$, which satisfies (4.3) and (4.4).

3) Now we modify the trading strategies $\gamma_n \in \Gamma^n$, $n \in \mathbb{N}$, in order to meet the admissibility requirements. For any $n \in \mathbb{N}$, define the stopping time

$$\tau_n := T \wedge \inf\{t : x + V_t^{\gamma_n} < \epsilon\}$$

and consider the trading strategy

$$\beta_t^n := \gamma_t^n 1_{\{t < \tau_n\}}, \quad t \in [0, T].$$

From (2.1) and the definition of $\tau_n$, it follows that for any $n \in \mathbb{N}$,

$$V_t^{\beta_n} = V_t^{\gamma_n} 1_{\{t < \tau_n\}} + V_{\tau_n}^{\gamma_n} 1_{\{t \geq \tau_n\}} \geq \epsilon - x, \quad t \in [0, T]. \quad (4.12)$$

Thus $\beta_n \in \mathcal{A}(x)$, $n \in \mathbb{N}$. From (4.4), we have

$$1_{\{\tau_n = T\}} \to 1 \quad \text{a.s.} \quad (4.13)$$

Recall (see the first paragraph after (2.1)) that due to the liquidation at the maturity date, the portfolio value processes are continuous at time $T$. Thus from (4.3), (4.12) and (4.13),

$$\liminf_{n \to \infty} V_T^{\beta_n} \geq V_T^{\gamma}. \quad (4.14)$$

In view of Assumption 2.5 (i) and the inequality $x + V_T^{\beta_n} \geq \epsilon$, $n \in \mathbb{N}$, we deduce the uniform integrability of $(U^-(x + V_T^{\beta_n}, S^n))_{n \in \mathbb{N}}$. Hence from the extended version of Fatou’s lemma (see (III) in Remark 2.6), (4.2) and (4.14), we obtain

$$\lim_{n \to \infty} u^n(x) \geq \liminf_{n \to \infty} \mathbb{E}[U(x + V_T^{\beta_n}, S^n)] \geq \mathbb{E}[U(x + V_T^{\gamma} - 3\epsilon, S)],$$

and (4.1) follows. \qed

Next, we prove the following key result.

**Lemma 4.3** Fix $x > 0$ and let $\gamma^n \in \mathcal{A}(x)$, $n \in \mathbb{N}$, be a sequence of admissible trading strategies. Then the sequence of laws $((X^n, S^n, \gamma^n), \mathbb{P}^n)$ on the product space $\mathcal{D}([0, T]; \mathbb{R}^{m+1}) \times \mathcal{D}_{MZ}([0, T])$ is tight, and so by Prohorov’s theorem (see [6, Sect. 12]), it is relatively compact. Moreover, any cluster point is
of the form \(((X, S, \gamma), \mathbb{P})\) and has the following conditional independence property: If \(\mathbb{F}^{X,\gamma} = (\mathcal{F}^{X,\gamma}_t)_{0 \leq t \leq T}\) denotes the usual filtration (right-continuous and \(\mathbb{P}\)-completed) generated by \(X\) and \(\gamma\), then for any \(t < T\), \(\mathcal{F}^{X,\gamma}_t\) and \(\mathcal{F}_T\) are conditionally independent given \(\mathcal{F}_t\). As before, \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) is the usual filtration generated by \(X\).

Proof 1) In Meyer and Zheng [23, Lemma 8], the authors proved that for any \(c > 0\), the set
\[
\left\{ f : f \text{ is of bounded variation and } \int_0^T |f(t)| \leq c \right\} \subseteq \mathbb{D}_{MZ}([0, T])
\]
is compact in the Meyer–Zheng topology. Thus in order to prove tightness in the Meyer–Zheng topology of the sequence \((\gamma^n, \mathbb{P}^n) \in \mathscr{A}^n(x), n \in \mathbb{N}\), it is sufficient to prove that for any \(\delta > 0\), there exist \(c > 0\) and \(N \in \mathbb{N}\) such that
\[
\mathbb{P}^n\left[ \left| \int_0^T |d\gamma^n_t| > c \right| < 2\delta, \quad \forall n > N. \right. \quad (4.15)
\]
Choose \(\delta > 0\). Since \(S\) is strictly positive, the weak convergence \((S^n, \mathbb{P}^n) \Rightarrow (S, \mathbb{P})\) implies that there exist \(\delta > 0\) and \(N \in \mathbb{N}\) such that
\[
\mathbb{P}^n\left[ \inf_{0 \leq t \leq T} S^n_t < \delta \right] < \delta, \quad \forall n > N. \quad (4.16)
\]
Next, recall the processes \(M^n, n \in \mathbb{N}\), and the probability measures \(\mathbb{Q}^n, n \in \mathbb{N}\), given by Assumption 2.3. From the inequalities
\[
\max\{|M^n_t - S^n_t|, |M^n_{t-} - S^n_t|\} \leq (\kappa - \varepsilon)S^n_t \quad \mathbb{P}^n\text{-a.s., } \forall t \in [0, T],
\]
the admissibility property of \(\gamma^n \in \mathscr{A}^n(x), n \in \mathbb{N}\), and integration by parts, we get
\[
0 \leq x + \gamma^n_t S^n_t - \int_0^t S^n_u d\gamma^n_u - \kappa |\gamma^n_t| S^n_t - \kappa \int_0^t S^n_u |d\gamma^n_u| \\
\leq x + \gamma^n_t M^n_t + (\kappa - \varepsilon) |\gamma^n_t| S^n_t - \int_0^t M^n_u d\gamma^n_u + (\kappa - \varepsilon) \int_0^t S^n_u |d\gamma^n_u| \\
- \kappa |\gamma^n_t| S^n_t - \kappa \int_0^t S^n_u |d\gamma^n_u| \\
\leq x + \int_0^t \gamma^n_u dM^n_u - \varepsilon \int_0^t S^n_u |d\gamma^n_u|, \quad \forall t \in [0, T].
\]
Thus for any \(n \in \mathbb{N}\),
\[
\mathbb{E}_{\mathbb{Q}^n} \left[ \int_0^T S^n_u |d\gamma^n_u| \right] \leq \frac{x}{\varepsilon}. \quad (4.17)
\]
From the Markov inequality and the fact that the sequence \((\mathbb{P}_n)_{n \in \mathbb{N}}\) is contiguous with respect to the sequence \((\mathbb{Q}_n)_{n \in \mathbb{N}}\), we conclude that there exists \(\hat{c} > 0\) such that
\[
\mathbb{P}_n \left[ \int_0^T S_u^n |dY_u^n| > \hat{c} \right] < \delta, \quad \forall n \in \mathbb{N}.
\]
This together with (4.16) implies (4.15) for \(c := \frac{\hat{c}}{\delta}\), and tightness follows.

2) From Assumption 2.9 (ii), we conclude that \(((S_n, X_n, \gamma_n), \mathbb{P}_n)_{n \in \mathbb{N}}\) is tight on \(D([0, T]; \mathbb{R}^{m+1}) \times D_{MZ}([0, T])\), and so it is relatively compact by Prohorov’s theorem. Moreover, from Assumption 2.9 (ii), it follows that any cluster point is of the form \(((S, X, \gamma), \mathbb{P})\), and the process \(\gamma\) is of bounded variation with right-continuous paths because the set of these functions is closed in the Meyer–Zheng topology.

It remains to establish the conditional independence property. Choose \(t < T\). We need to show (see Dellacherie and Meyer [14, II.43]) that for any bounded random variable \(Z_1\) which is \(\mathcal{F}_T\)-measurable and any bounded random variable \(Z_3\) which is \(\mathcal{F}_t\)-measurable, we have the equality
\[
\mathbb{E}[Z_1 Z_3 | \mathcal{F}_t] = \mathbb{E}[Z_1 | \mathcal{F}_t] \mathbb{E}[Z_3 | \mathcal{F}_t].
\]
This is equivalent to proving that for any bounded random variable \(Z_2\) which is \(\mathcal{F}_t\)-measurable, we have
\[
\mathbb{E}[Z_1 Z_2 Z_3] = \mathbb{E}[\mathbb{E}[Z_1 | \mathcal{F}_u] Z_2 Z_3].
\]
Since the filtration \(\mathcal{F}\) is right-continuous, the above equality follows from the equality
\[
\mathbb{E}[Z_1 Z_2 Z_3] = \mathbb{E}[\mathbb{E}[Z_1 | \mathcal{F}_u] Z_2 Z_3], \quad \forall u > t. \tag{4.18}
\]
Standard density arguments yield that without loss of generality, we can assume that \(Z_1 = \psi_1(X)\) for a continuous bounded function \(\psi_1 : \mathbb{D}([0, T]; \mathbb{R}^m) \to \mathbb{R}\). Let
\[
Y^n_u := \mathbb{E}^{\mathbb{P}_n}[\psi_1(X^n) | \mathcal{F}^n_u] \quad \text{and} \quad Y_u := \mathbb{E}[\psi_1(X) | \mathcal{F}_u], \quad u \in [0, T].
\]
By passing to a subsequence (again indexed by \(n\)), we can assume without loss of generality that \(((S^n, X^n, \gamma^n), \mathbb{P}^n)_{n \in \mathbb{N}}\) converges weakly to \(((S, X, \gamma), \mathbb{P})\). From Assumption 2.9 (ii) and (iii), we obtain that the sequence \(((S^n, X^n, Y^n, \gamma^n), \mathbb{P}^n)_{n \in \mathbb{N}}\) is tight on the space
\[
\mathbb{D}([0, T]; \mathbb{R}^{m+1}) \times \mathbb{D}([0, T]; \mathbb{R}) \times D_{MZ}([0, T]),
\]
and so it is relatively compact by Prohorov’s theorem. Moreover, Assumption 2.9 (ii) and (iii) imply that any cluster point is of the form \(((S, X, Y, \gamma), \mathbb{P})\).

By the Skorohod representation theorem (Dudley [15, Theorem 3]), there exist a probability space and a subsequence (again labelled by \(n\)) \((S^n, X^n, Y^n, \gamma^n)_{n \in \mathbb{N}}\) and \((S, X, Y, \gamma)\) on the same probability space such that
\[
(S^n, X^n, Y^n, \gamma^n) \longrightarrow (S, X, Y, \gamma) \quad \text{a.s.} \quad \tag{4.19}
\]
on the space $\mathbb{D}([0, T]; \mathbb{R}^{m+1}) \times \mathbb{D}([0, T]; \mathbb{R}) \times \mathbb{D}_{MZ}([0, T])$. Next, bounded convergence yields

$$
\lim_{n \to \infty} \mathbb{E} \left[ \int_0^T \min(1, |\gamma_u^n - \gamma_u|) du \right] = 0,
$$

where $\mathbb{E}$ denotes the expectation on the common probability space. From Fubini’s theorem, we conclude that there exists a subset $I \subseteq [0, T]$ of full Lebesgue measure such that

$$
\gamma_u = \lim_{n \to \infty} \gamma_u^n, \quad \forall u \in I,
$$

where the limit is in probability. Choose $u > t$ in (4.18). Again, standard density arguments imply that without loss of generality, we can assume that $Z_2, Z_3$ in (4.18) are of the form $Z_2 = \psi_2(X_{t_1}, \ldots, X_{t_k})$ for some $k \in \mathbb{N}, t_1, \ldots, t_k \in I \cap [0, u]$ and a continuous bounded function $\psi_2 : (\mathbb{R}^m)^k \to \mathbb{R}$ and $Z_3 = \psi_3(X_{t_1}, \ldots, X_{t_k}, \gamma_{t_1}, \ldots, \gamma_{t_k})$ for a continuous bounded function $\psi_3 : (\mathbb{R}^m)^k \times \mathbb{R}^k \to \mathbb{R}$. From the bounded convergence theorem, the fact that $\gamma_u^n$ is $\mathcal{F}_n$-adapted and (4.19) and (4.20), we obtain

$$
\mathbb{E}_P[Z_1 Z_2 Z_3] = \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}^n}[\psi_1(X^n)\psi_2(X^n_{t_1}, \ldots, X^n_{t_k})\psi_3(X^n_{t_1}, \ldots, X^n_{t_k}, \gamma^n_{t_1}, \ldots, \gamma^n_{t_k})] = \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}^n}[\psi_1(X^n)\psi_2(X^n_{t_1}, \ldots, X^n_{t_k})\psi_3(X^n_{t_1}, \ldots, X^n_{t_k}, \gamma^n_{t_1}, \ldots, \gamma^n_{t_k})] = \mathbb{E}_P[Y_u Z_2 Z_3].
$$

This completes the proof of (4.18). □

Now we are ready to complete the proof of the main results.

### 4.2 Completion of the proof of Theorems 3.1 and 3.3

**Proof of Theorems 3.1 and 3.3** We prove these two results together.

1) Fix $x > 0$ and let $\hat{\gamma}^n \in \mathcal{A}^n(x), n \in \mathbb{N}$, be a sequence of portfolios which satisfy (3.1). By passing to a subsequence, we assume without loss of generality that $\lim_{n \to \infty} u^n(x)$ exists. From (3.1), $\lim_{n \to \infty} \mathbb{E}_{\mathbb{P}^n}[U(x + V_T^{\hat{\gamma}^n}, S^n)]$ exists as well.

From Lemma 4.3, we obtain that $((S^n, X^n, \hat{\gamma}^n), \mathbb{P}^n)_{n \in \mathbb{N}}$ is tight and any cluster point is of the form $((S, X, \hat{\gamma}), \mathbb{P})$. As before, $\hat{\gamma}$ is a right-continuous process of bounded variation. Moreover, since $\hat{\gamma}_T^{(n)} = 0$ for all $n$, $\hat{\gamma}_T = 0$ as well. Thus we define $(V_t^{\hat{\gamma}})_{0 \leq t \leq T}$ by (2.1).

Let us prove the admissibility condition

$$
x + V_t^{\hat{\gamma}} \geq 0, \quad \forall t \in [0, T],
$$

and the inequality

$$
\mathbb{E}_P[U(x + V_T^{\hat{\gamma}}, S)] \geq \lim_{n \to \infty} u^n(x).
$$
From (4.15), it follows that the sequence
\[
\left( \left( S^n, X^n, \hat{\gamma}^n, \int_0^T |d\hat{\gamma}^n_t| \right), \mathbb{P}^n \right)_{n \in \mathbb{N}}
\]
is tight on the space \( \mathbb{D}([0, T] ; \mathbb{R}^{m+1}) \times \mathbb{D}_{M_2}([0, T]) \times \mathbb{R} \). Thus by passing to a further subsequence and applying the Skorohod representation theorem, we obtain that there exists a common probability space where we have the almost sure convergence
\[
\left( S^n, X^n, \hat{\gamma}^n, \int_0^T |d\hat{\gamma}^n_t| \right) \longrightarrow (S, X, \hat{\gamma}, \eta) \quad \text{a.s.} \quad (4.23)
\]
for some random variable \( \eta < \infty \). We conclude that
\[
\sup_{n \in \mathbb{N}} \int_0^T |d\hat{\gamma}^n_t| < \infty \quad \text{a.s.} \quad (4.24)
\]
Next, using the same arguments as before (4.20) gives that there exists a subset \( I \subseteq [0, T] \) of full Lebesgue measure such that \( \hat{\gamma}^n_u \rightarrow \hat{\gamma}_u \) in probability for all \( u \in I \).

From a diagonalisation argument, it follows that there exist a countable dense set \( \hat{I} \) and a subsequence \((S(n), X^{(n)}, \hat{\gamma}^n)_{n \in \mathbb{N}}\) such that
\[
\hat{\gamma}_u = \lim_{n \to \infty} \hat{\gamma}^n_u \quad \text{a.s., } \forall u \in \hat{I}. \quad (4.25)
\]
Since \( \hat{\gamma}_T = 0 \) and \( \hat{\gamma}_T^n = 0 \), \( n \in \mathbb{N} \), we assume without loss of generality that \( T \in \hat{I} \).

Let \( t \in \hat{I} \). From Morrey and Protter [24, Theorem 12.16] and (4.23)–(4.25),
\[
\int_0^t S_u d\hat{\gamma}_u = \lim_{n \to \infty} \int_0^t S^n_u d\hat{\gamma}^n_u \quad \text{a.s.} \quad (4.26)
\]
Next, choose a partition \( \{0 = a_0 < a_1 < \cdots < a_k = t\} \subseteq \hat{I} \). From (4.23)–(4.25),
\[
\liminf_{n \to \infty} \int_0^t S^n_u |d\hat{\gamma}^n_u| = \liminf_{n \to \infty} \int_0^t S_u |d\hat{\gamma}^n_u| \\
\geq \liminf_{n \to \infty} \sum_{i=0}^{k-1} \min_{a_i \leq u \leq a_{i+1}} S_u |\hat{\gamma}^n_{a_{i+1}} - \hat{\gamma}^n_{a_i}| \\
= \sum_{i=0}^{k-1} \min_{a_i \leq u \leq a_{i+1}} S_u |\hat{\gamma}_{a_{i+1}} - \hat{\gamma}_{a_i}|.
\]
By taking the mesh of the partition to zero, we conclude that
\[
\liminf_{n \to \infty} \int_0^t S^n_u |d\hat{\gamma}^n_u| \geq \int_0^t S_u |d\hat{\gamma}_u| \quad \text{a.s.} \quad (4.27)
\]
From (4.25)–(4.27),

\[
V_{\hat{t}} \gamma \geq \limsup_{n \to \infty} V_{\hat{t}} \gamma^n \geq -\epsilon \quad \text{a.s.}, \quad \forall t \in \hat{I}.
\] (4.28)

Since \( \hat{I} \) is a dense set which contains \( T \) and \((V_{\hat{t}} \gamma)^{0 \leq t \leq T}\) is RCLL, we obtain (4.21). From Assumption 2.5 (ii), (3.1) and (4.28) (for \( t = T \)), we conclude (4.22).

2) From the conditional independence property proved in Lemma 4.3, it follows that any martingale with respect to the filtration \( \mathbb{F} \) is also a martingale with respect to the filtration \( \mathbb{F}^{\hat{X}, \hat{\gamma}} \). Thus we can redefine the probability measure \( \mathbb{Q} \) and the local \( \mathbb{Q} \)-martingale \( M \) from Assumption 2.3 on the common probability space. Since \( M \) is a local \( \mathbb{Q} \)-martingale with respect to \( \mathbb{F}^{\hat{X}, \hat{\gamma}} \), we obtain by the same arguments as before (4.17) that (4.21) implies \( \mathbb{E}_{\mathbb{Q}}[\int_0^T S_u d\hat{\gamma}_u] \leq \epsilon/\epsilon \). This together with the inequality \( \inf_{0 \leq u \leq T} S_u > 0 \) a.s. and the fact that \( \frac{d\mathbb{Q}}{d\mathbb{P}} \) is \( \mathbb{F}_T \)-measurable gives

\[
\mathbb{E}_{\mathbb{P}} \left[ \int_0^T |d\hat{\gamma}_u| \right] \mathbb{F}_T < \infty \quad \text{a.s.} \] (4.29)

Let \( \hat{\gamma}_t := \hat{\gamma}_t^+ - \hat{\gamma}_t^- \), \( t \in [0, T] \), be the Jordan decomposition into processes \((\hat{\gamma}_t^+)_{0 \leq t \leq T}\) of positive variation and \((\hat{\gamma}_t^-)_{0 \leq t \leq T}\) of negative variation. From (4.29), it follows that the optional projections \( \beta^+ := \circ \hat{\gamma}^+ \) and \( \beta^- := \circ \hat{\gamma}^- \) of the processes \( \hat{\gamma}^+ \) and \( \hat{\gamma}^- \) with respect to \( \mathbb{F} \) are well defined and right-continuous and nondecreasing processes. Hence the optional projection of the process \( \hat{\gamma} \) with respect to \( \mathbb{F} \) is well defined and given by \( \hat{\gamma}^\mathbb{F} := \beta^+ - \beta^- \). Clearly, the process \( \hat{\gamma}^\mathbb{F} \) is of bounded variation with right-continuous paths. Next, fix \( t \in [0, T] \) and \( n \in \mathbb{N} \). From Dellacherie and Meyer [14, Theorem II.45] and the conditional independence property proved in Lemma 4.3,

\[
\mathbb{E}_{\mathbb{P}}[\hat{\gamma}_t^+ \wedge n | \mathbb{F}_T] = \mathbb{E}_{\mathbb{P}}[\hat{\gamma}_t^+ \wedge n | \mathbb{F}_T].
\]

By letting \( n \to \infty \), we get

\[
\hat{\gamma}_t^+ = \mathbb{E}_{\mathbb{P}}[\hat{\gamma}_t^+ | \mathbb{F}_T], \quad \forall t \in [0, T]. \] (4.30)

Similarly,

\[
\hat{\gamma}_t^- = \mathbb{E}_{\mathbb{P}}[\hat{\gamma}_t^- | \mathbb{F}_T], \quad \forall t \in [0, T]. \] (4.31)

Thus

\[
\hat{\gamma}_t^\mathbb{F} = \hat{\gamma}_t^+ - \hat{\gamma}_t^- = \mathbb{E}_{\mathbb{P}}[\hat{\gamma}_t | \mathbb{F}_T], \quad \forall t \in [0, T]. \] (4.32)

From (4.29), we have \( \mathbb{E}_{\mathbb{P}}[\sup_{0 \leq t \leq T} S_t \int_0^T |d\hat{\gamma}_u| | \mathbb{F}_T] < \infty \) a.s. Hence dominated convergence and (4.30)–(4.32) yield

\[
\int_0^t S_u d\hat{\gamma}_u^\mathbb{F} = \mathbb{E}_{\mathbb{P}} \left[ \int_0^t S_u d\hat{\gamma}_u^\mathbb{F} | \mathbb{F}_T \right], \quad \forall t \in [0, T].
\]
and
\[
\int_0^t S_u d\hat{\gamma}_u^+ \leq \int_0^t S_u d\hat{\beta}_u^+ + \int_0^t S_u d\hat{\beta}_u^-
\]
\[
= \mathbb{E}_\mathbb{P} \left[ \int_0^t S_u d\hat{\gamma}_u^+ | \mathcal{F}_T \right] + \mathbb{E}_\mathbb{P} \left[ \int_0^t S_u d\hat{\gamma}_u^- | \mathcal{F}_T \right]
\]
\[
= \mathbb{E}_\mathbb{P} \left[ \int_0^t S_u d\hat{\gamma}_u | \mathcal{F}_T \right], \quad \forall t \in [0, T].
\]

This together with (2.1) and the simple relations
\[
\hat{\gamma}_F t S_t \leq \mathbb{E}_\mathbb{P}[\hat{\gamma}_t S_t | \mathcal{F}_T], \quad \forall (v,s) \in [0, \infty) \times C[0,T]
\]
gives
\[
V^F_t \geq \mathbb{E}_\mathbb{P}[V^F_t | \mathcal{F}_T], \quad \forall t \in [0, T],
\]
and so from (4.21), \( \hat{\gamma}_F \in A(x) \). Thus from the Jensen inequality, Lemma 4.2, (4.22) and (4.33),
\[
\lim_{n \to \infty} u^n(x) \geq u(x) \geq \mathbb{E}_\mathbb{P}[U(x + V^F_T, S)] \geq \mathbb{E}_\mathbb{P}[U(x + \hat{\gamma}_T, S)] \geq \lim_{n \to \infty} u^n(x),
\]
and the proof is completed. \( \square \)

5 Example: transaction costs make things converge

Consider a random utility which corresponds to shortfall risk minimisation for a call option with strike price \( K > 0 \). More precisely, we set
\[
U(v,s) := -(s_T - K)^+ + v, \quad \forall (v,s) \in [0, \infty) \times C[0,T].
\]

We take \((\Omega^n, \mathcal{F}^n, \mathbb{P}^n) = (\Omega, \mathcal{F}, \mathbb{P})\) for all \( n \in \mathbb{N} \) and let \( \xi_i, i \in \mathbb{N} \), be i.i.d. and symmetric with values ±1. For any \( n \in \mathbb{N} \), define the scaled random walks (which do depend on \( n \)) \( X^{n,1}_t, X^{n,2}_t, t \in [0, T] \), by
\[
X^{n,1}_t := \sqrt{\frac{T}{n}} \sum_{i=1}^{\lfloor nt/T \rfloor} \xi_i,
\]
\[
X^{n,2}_t := \sqrt{\frac{T}{n}} \sum_{i=1}^{\lfloor nt/T \rfloor} \prod_{j=1}^{i} \xi_j,
\]
where \( \lfloor \cdot \rfloor \) is the integer part of \( \cdot \) and we set \( \sum_{i=1}^{0} := 0 \). Set \( X^n := (X^{n,1}, X^{n,2}) \) and let \( \mathbb{F}^n \) be the filtration generated by \( X^n \). Observe that each \( X^n \) is a \( \mathbb{P}^n \)-martingale. From Bayraktar et al. [4, Lemma 3.1], it follows that we have the weak convergence \((X^n, \mathbb{P}^n) \Rightarrow (X, \mathbb{P})\), where \( X = (X^1, X^2) \) is a standard two-dimensional Brownian
motion with respect to \( \mathbb{P} \). From Jakubowski and Slomiński [20, Corollary 6], we deduce the extended weak convergence
\[
(X^n, \mathbb{P}^n) \Rightarrow (X, \mathbb{P}).
\] (5.1)

We remark that although [20] only deals with real-valued processes, the extension of the results there to the multidimensional case is straightforward.

Now we introduce the financial markets. For any \( n \in \mathbb{N} \), define the discrete-time stochastic processes \((\tilde{\nu}^n_k)_{k=0,1,...,n}\) and \((\tilde{S}^n_k)_{k=0,1,...,n}\) by
\[
\tilde{\nu}^n_k := \prod_{i=1}^k \left( 1 + \sqrt{\frac{T}{n}} \xi_i \right),
\]
\[
\tilde{S}^n_k := \prod_{i=1}^k \left( 1 + \min \left( \frac{\nu^n_{(i-1)T}}{n}, \ln n \right) \sqrt{\frac{T}{n}} \prod_{j=1}^i \xi_j \right),
\]
where we set \( \prod_{i=1}^0 \xi_i := 1 \). Let \( S^n = (S^n_t)_{0 \leq t \leq T}, n \in \mathbb{N} \), be the linear interpolation
\[
S^n_t := \left( \lfloor nt/T \rfloor + 1 \right) T - nt \tilde{S}^n_{\lfloor nt/T \rfloor - 1} + (nt - \lfloor nt/T \rfloor T) \tilde{S}^n_{\lfloor nt/T \rfloor},
\] (5.2)
where we set \( \tilde{S}^n_{-1} := 1 \). We take a shift of one time period in order to make \( S^n \) adapted to \( \mathbb{P}^n \). For any \( n \), define the process \((\nu^n_t)_{0 \leq t \leq T}\) by \( \nu^n_t := \tilde{\nu}^n_{\lfloor nt/T \rfloor} \). Using the same arguments as in [4, Example 3.3], we obtain the weak convergence
\[
((\nu^n, S^n, X^n), \mathbb{P}^n) = \lim_{n \to \infty} ((\nu, S, X), \mathbb{P}),
\] (5.3)
where \((\nu, S)\) is the (unique strong) solution of the SDE (Hull and White model)
\[
d\nu_t = \nu_t dX^1_t, \quad \nu_0 = 1,
\]
\[
dS_t = \nu_t S_t dX^2_t, \quad S_0 = 1.
\] (5.4)

Note that although this is a martingale model for \( S \), the utility is state-dependent so that utility maximisation makes sense.

Next, we verify Assumptions 2.1, 2.3, 2.5 and 2.9. Clearly, \( \mathbb{E}_\mathbb{P}[S_T] \leq S_0 = 1 \). Thus Assumption 2.1 (i) holds true. From Remark 2.2, it follows that Assumption 2.1 (ii) holds true as well.

Let \( \kappa > 0 \) be the rate of transaction costs. Observe that \((\tilde{S}^n_k)_{k=0,1,...,n}\) is a \( \mathbb{P}^n \)-martingale. Thus for sufficiently large \( n \), the martingale \( M^n = (M^n_t)_{0 \leq t \leq T} \) given by \( M^n_t := \tilde{S}^n_{\lfloor nt/T \rfloor} \) satisfies Assumption 2.3 for \( \mathbb{Q}^n := \mathbb{P}^n \). For the limit model, we just take \( M := S \) and \( \mathbb{Q} := \mathbb{P} \).

It is well known that the process \( S \) given by (5.4) is a true martingale (see Sin [28, Theorem 3.3]). Thus we have
\[
\lim_{n \to \infty} \mathbb{E}_{\mathbb{P}^n}[S^n_T] = \lim_{n \to \infty} S^n_0 = S_0 = \mathbb{E}_\mathbb{P}[S_T].
\]
Remark 2.6 (II) yields that the random variables $(S^n_T)_{n \in \mathbb{N}}$ are uniformly integrable. This gives Assumption 2.5 (i). Since $U^+ = 0$, Assumption 2.5 (ii) holds trivially. Finally, Assumption 2.9 follows from (5.1) and (5.3).

From Theorem 3.1, it follows that in the presence of proportional transaction costs, the shortfall risks in the models given by (5.2) converge to the shortfall risk in the Hull–White stochastic volatility model given by (5.4).

On the other hand, for the frictionless case $\kappa = 0$, there is no convergence. In fact, the models given by (5.2) are not arbitrage-free. Observe that for any interval of the form $[kT/n, (k+1)T/n]$, $k \geq 1$, the investor knows immediately after time $kT/n$ all the stock prices in this interval. Thus we have an obvious arbitrage, which makes the shortfall risk equal to zero for any initial capital. Clearly, this is not the case for the limit model.

Moreover, assume that for model $n$, we restrict the trading times to the discrete set $\{0, T/n, 2T/n, \ldots, T\}$. We notice that for $k = 1, \ldots, n - 1$, the conditional support $\text{supp}(S^n_{(k+1)T/n} | S^n_T, \ldots, S^n_{kT/n})$ consists of exactly two points and the corresponding binomial model is complete and arbitrage-free. Still, even with this restriction, in the frictionless setup, the shortfall risk in the Hull–White model is strictly bigger than the limit of the shortfall risks in the models given by $S^n, n \in \mathbb{N}$. This fact was established in [4, Example 3.3].
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