INTEGRAL FORMULAS FOR WAVE
FUNCTIONS OF QUANTUM MANY-BODY
PROBLEMS AND REPRESENTATIONS OF $\mathfrak{gl}_n$

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Abstract

We derive explicit integral formulas for eigenfunctions of quantum integrals of the Calogero-Sutherland-Moser operator with trigonometric interaction potential. In particular, we derive explicit formulas for Jack’s symmetric functions. To obtain such formulas, we use the representation of these eigenfunctions by means of traces of intertwining operators between certain modules over the Lie algebra $\mathfrak{gl}_n$, and the realization of these modules on functions of many variables.

Introduction

Integrable many body problems were considered by F.Calogero [C] and B.Sutherland [S] in early seventies. Later more general Hamiltonian systems of this type were described (see [OP] for references).

An important progress in the theory of integrable many-body problems was the discovery of their relationship to the theory of Lie groups and symmetric spaces. In [KKS], it was found that the classical Calogero and Sutherland systems can be obtained by Hamiltonian reduction from simpler looking Hamiltonian systems (but having more degrees of freedom) arising from Lie groups. Recently this approach was generalized to infinite-dimensional groups [GN1],[GN2], which allows to obtain the elliptic deformation of the Sutherland system (see [OP]), and its $q$-deformation [Ru].

Quantum integrable systems of the Calogero-Sutherland type first appeared in the theory of radial parts of Laplace’s operators on symmetric spaces (see [H],[W]). In the sixties Harish-Chandra computed the radial part of the second order Laplace’s operator acting on generalized spherical functions on a homogeneous space for a semisimple Lie group $G$ (see [W]). One can show that Sutherland’s quantum Hamiltonian can be obtained as a special case of such radial part. This
implies complete integrability of Sutherland’s Hamiltonian: its integrals are just the radial parts of the higher Laplacians, obtained from the center of the universal enveloping algebra of Lie $G$ (which happens to be $\mathfrak{gl}_n$ in this case). Comparing this to the classical situation, one can say that the procedure of computation of the radial part is a quantum counterpart of Hamiltonian reduction.

A proof of integrability of the Sutherland Hamiltonian and a construction of its wave functions based on these ideas (but using a more algebraic method) is given in [E]. The approach of [E] is to represent the wave functions as weighted traces of intertwining operators between certain representations of $G$. The goal of the present paper is to use this approach to give an explicit integral formula for wave functions.

The paper is organized as follows. In Section 1 we describe the construction of the wave functions as traces of intertwiners. In Section 2 we introduce the realization of representations of $\mathfrak{gl}_n$ by differential operators, and use this realization to represent intertwiners as functions of many variables. This allows to write down an integral formula for the trace of such intertwiner, which is our main result (Theorem 2.2). As a corollary, we get an explicit integral representation of Jack’s symmetric functions. It follows from Theorem 2.2 and the results of [EK1],[EK2].

1. Trace representation of wave functions

The Hamiltonian of the quantum $n$-body problem on the line is

\begin{equation}
H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - C \sum_{i<j} U(x_i - x_j),
\end{equation}

where $U$ is some potential function, and $C$ is a constant. We will consider the case $U(x) = \frac{1}{2 \sinh^2(x/2)}$. This case was first considered by Sutherland [S]; so we call the operator $H$ the Sutherland operator. It is a trigonometric deformation of the Calogero operator [C], which is given by (1.1) with $U(x) = x^{-2}/2$.

It is known (cf. [OP]) that the Sutherland operator defines a completely integrable quantum system. This means, it has $n$ pairwise commutative and algebraically independent quantum integrals

\begin{equation}
L_1 = \sum_i \frac{\partial}{\partial x_i}, \quad L_2 = H + \text{const}, \quad L_m = \sum_i \frac{\partial^m}{\partial x_i^m} + \text{lower order terms}, 3 \leq m \leq n
\end{equation}

(the choice of $L_m$ is not unique). Wave functions of this system are, by definition, joint eigenfunctions of $L_1, \ldots, L_n$, i.e. solutions of the differential equations

\begin{equation}
L_i \psi = \Lambda_i \psi, \quad 1 \leq i \leq n,
\end{equation}

where $\Lambda_i$ are some complex numbers. The space of such functions is $n!$-dimensional for every set of $\{\Lambda_i\}$.

In this paper we present an explicit integral formula for wave functions. It is given by Theorem 2.2.

The first step in deriving this formula is a representation of wave functions as traces of intertwining operators between certain representations of the Lie algebra $\mathfrak{gl}_n$. 

Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_n$, $\mathfrak{h} = \mathbb{C}^n$ denote the Cartan subalgebra of diagonal matrices, $M_\lambda$ denote the Verma module over $\mathfrak{g}$ with highest weight $\lambda$, and let $U_k \ (k \in \mathbb{C})$ be the space of functions

$$U_k = \{(z_1 \ldots z_n)^{k-1} p(z), p(z) \in \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}], \deg p = 0\}$$

with the action of $\mathfrak{g}$ given by

$$E_{ij} \to z_i \frac{\partial}{\partial z_j} - (k-1)\delta_{ij},$$

where $E_{ij}$ are the elementary matrices: $(E_{ij})_{lm} = \delta_{il}\delta_{jm}$.

**Lemma 1.1.** If $M_\lambda$ is irreducible then there exists a unique up to a factor $\mathfrak{g}$-intertwining operator $\Phi_\lambda : M_\lambda \to M_\lambda \hat{\otimes} U_k$, where $\hat{\otimes}$ denotes a completed tensor product.

Consider the function

$$\Psi(x, \lambda) = \text{Tr}|_{M_\lambda}(\Phi_\lambda e^x), \ x \in \mathfrak{h}.$$ This is a series defining an analytic function in the region $\text{Re}x_1 > \ldots > \text{Re}x_n$, where $x_i =< x, h_i >$, $h_i = E_{ii}$. This function takes values in the zero-weight subspace $U_k[0]$ of $U_k$, which is one-dimensional. Thus, we will regard it as a scalar function.

**Theorem 1.2.** [E] Let $Z$ be any element of the center of $U(\mathfrak{g})$. Then there exists a unique differential operator $D_Z$ on functions of $n$ variables $x_1, \ldots, x_n$ dependent on $k$ but not on $\lambda$ such that for a generic $\lambda$

$$\text{Tr}|_{M_\lambda}(\Phi_\lambda Ze^x) = D_Z \text{Tr}|_{M_\lambda}(\Phi_\lambda e^x).$$

Consider the free generators of the center of $U(\mathfrak{g})$: $Z_m = \sum_{i_1, \ldots, i_m=1}^n E_{i_1i_2} \ldots E_{i_{m-1}i_m} E_{i_m i_1}$, $1 \leq m \leq n$. Set $\hat{L}_m = D_{Z_m}$.

Also, introduce the Weyl denominator

$$\phi(x) = 2^{n(n-1)/2} \prod_{1 \leq i<j \leq n} \sinh\left(\frac{x_i - x_j}{2}\right).$$

Define the constant $C$ in (1.1) by

$$C = k(k-1).$$

**Theorem 1.3.** [E] The operators $\hat{L}_1, \ldots, \hat{L}_m$ are pairwise commutative and simultaneously conjugate to quantum integrals of the Sutherland operator: if one defines $L_m$ by the formula $L_m f = \phi \hat{L}_m \left(\frac{f}{\phi}\right)$ then $L_2 = H - < \rho, \rho >$, $\rho = (\frac{n-1}{2}, \ldots, \frac{1-n}{2})$, and $L_m$ are quantum integrals of the Sutherland operator defined by (1.2).

From now on we will assume that the choice of $L_m$ is made according to Theorem 1.3.

Let $P_i(\lambda)$ be the scalar by which $Z_i$ acts in $M_\lambda$ (it is a symmetric polynomial of $\lambda + \rho$). Consider the function

$$\psi(x, \lambda) = \phi(x) \Psi(x, \lambda).$$

Let us say that $\lambda$ is generic if $\lambda_i - \lambda_j + i - j$ is not a positive integer when $i > j$. A weight $\lambda$ is generic in this sense iff the corresponding Verma module $M_\lambda$ is irreducible.
Corollary 1.4. [E] (i) The function $\psi$ is an eigenfunction of $L_1, \ldots, L_n$ (a wave function) for a generic $\lambda$:

$$L_i \psi(x, \lambda) = P_i(\lambda)\psi(x, \lambda).$$

(ii) If the weights $w(\lambda + \rho) - \rho$ are generic for any $w \in S_n$ ($S_n$ is the symmetric group) then the functions $\psi(x, w(\lambda + \rho) - \rho), w \in S_n$ form a basis in the space of solutions of (1.11).

This corollary shows that in order to compute joint eigenfunctions of $L_i$ for generic eigenvalues, it is enough to calculate the trace function $\Psi(x, \lambda)$. In the subsequent sections we will do it using the Borel-Weil realization ("bosonization") of the Lie algebra $\mathfrak{gl}_n$.

2. The Borel-Weil realization of $\mathfrak{gl}_n$ and integral formulas

By the Borel-Weil realization we mean the realization of an irreducible Verma module $M_\lambda$ over $\mathfrak{gl}_n$ in the space of regular functions on the big cell of the flag variety; the action of the Lie algebra in this space is given by first order differential operators.

Let $B^+$ be the Borel subgroup in $G = GL_n$ consisting of upper triangular matrices. Let $C_0$ be the big cell of the Bruhat decomposition of $G$. Let $\tilde{B}^+$, $\tilde{C}_0$ be the universal covers of $B^+$, $C_0$. We have a left action of $\mathfrak{g}$ on $\tilde{C}_0$ which commutes with the (free) right action of $\tilde{B}^+$. Consider the induced action of $\mathfrak{g}$ on the space $M_\lambda$ of regular functions $f : \tilde{C}_0 \rightarrow \mathbb{C}$ satisfying the condition $f(xb) = \chi_\lambda(b)f(x)$, $b \in \tilde{B}^+$, where $\chi_\lambda : \tilde{B}^+ \rightarrow \mathbb{C}^*$ is the character of $\tilde{B}^+$ obtained by extension of $\lambda \in \mathfrak{h}^*$. Clearly, the space $M_\lambda$ can be realized as the space of regular functions on the big Schubert cell $F_0 = \tilde{C}_0/\tilde{B}^+$. This cell can be naturally realized as the set of all strictly upper triangular matrices. Thus, we have a representation of $\mathfrak{g}$ in the space of functions of strictly upper triangular matrices.

Explicitly, this representation looks as follows.

Set $M_\lambda = \mathbb{C}[\{y_{ij}\}]$ (as a vector space), where $\{y_{ij}, 1 \leq i < j \leq n\}$ is a collection of independent variables. Define the action of $\mathfrak{g}$ on $M_\lambda$ as follows:

$$E_{ii} - E_{i+1,i+1} = 2y_{ii+1} \frac{\partial}{\partial y_{ii+1}} + \sum_{j=1}^{i-1} (y_{j+1, j+1} \frac{\partial}{\partial y_{j+1, j+1}} - y_{ji} \frac{\partial}{\partial y_{ji}}) + \lambda_i - \lambda_{i+1}, 1 \leq i \leq n - 1,$$

$$E_{i+1i} = \frac{\partial}{\partial y_{i+1i}} + \sum_{j=i+1}^{n-1} y_{j+1, j+1} \frac{\partial}{\partial y_{j+1, j+1}}, 1 \leq i \leq n - 1,$$

$$E_{ii+1} = \sum_{j=1}^{i-1} y_{ii+1} y_{ij} \frac{\partial}{\partial y_{ij}} - \sum_{j=1}^{i} y_{ii+1} y_{ji+1} \frac{\partial}{\partial y_{ji+1}} - (\lambda_i - \lambda_{i+1}) y_{ii+1} + \sum_{j=i+1}^{n-1} y_{j+1} \frac{\partial}{\partial y_{j+1, j+1}} - \sum_{j=1}^{i-1} y_{ji} \frac{\partial}{\partial y_{ji}}, 1 \leq i \leq n - 1,$$

$$\sum_i E_{ii} = \sum_i \lambda_i.$$
Let us now present a functional realization of the restricted dual of a Verma module, $M_\lambda^*$. We will realize it in the space $\mathbb{C}[\{t_{ij}^{-1}\}]$, where $t_{ij}$, $i < j$, are independent variables. Geometrically this space is the space of regular functions on the image of the big cell $F_0$ under the action of the longest element in the Weyl group. Define the pairing between $M_\lambda$ and $M_\lambda^*$ by the formula

$$(f, g) = (2\pi \sqrt{-1})^{-n(n-1)/2} \int |y_{ij}| = 1 f(y) g(y) \frac{dy}{y}, \quad f \in M_\lambda, \; g \in M_\lambda^*, \tag{2.4}$$

where by definition $\frac{dy}{y} = \wedge_{i=1}^{n} \wedge_{j=i+1}^{n} \frac{dy_{ij}}{y_{ij}}$.

We would like this pairing to be $g$-invariant, which uniquely determines the action of $g$ in $M_\lambda^*$:

$$E_{ij} = \sum_{l<m} A_{ijlm}(t, \lambda) \frac{\partial}{\partial t_{lm}} + b_{ij}^*(t, \lambda), \tag{2.5}$$

where

$$b_{ij}^*(t, \lambda) = \sum_{l<m} \left( \frac{\partial A_{ijlm}}{\partial t_{lm}} - \frac{A_{ijlm}}{t_{lm}} \right) - b_{ij}(t, \lambda).$$

Now we can give a functional realization of the intertwiner $\Phi_\lambda$ which was defined in Section 1. We can regard $\Phi_\lambda$ as an element of the space $M_\lambda \otimes M_\lambda^* \otimes U_k$, i.e. as a function of $y_{ij}$, $t_{ij}$, $z_i$: $\Phi_\lambda = \Phi_\lambda(y,t,z)$. This function is invariant under the action of the Cartan subalgebra, which implies that it can be written as

$$\Phi_\lambda(y,t,z) = (z_1...z_n)^{k-1} \Theta_\lambda(\xi, \eta), \tag{2.6}$$

where $\xi_{ij}$, $\eta_{ij}$ are the new variables: $\xi_{ij} = y_{ij} z_j / z_i$, $\eta_{ij} = t_{ij} z_j / z_i$. Also, (2.6) satisfies the system of differential equations $E_{ij} \Phi_\lambda = 0$, which can be rewritten in the new coordinates as follows:

$$\sum_{l<m} A_{ijlm}(\xi, \lambda) \frac{\partial \Theta_\lambda}{\partial \xi_{lm}} + \sum_{l<m} A_{ijlm}(\eta, \lambda) \frac{\partial \Theta_\lambda}{\partial \eta_{lm}} +$$

$$\sum_{p<j} (\xi_{pj} \frac{\partial \Theta_\lambda}{\partial \xi_{pj}} - \eta_{pj} \frac{\partial \Theta_\lambda}{\partial \eta_{pj}}) - \sum_{p>j} (\xi_{jp} \frac{\partial \Theta_\lambda}{\partial \xi_{jp}} - \eta_{jp} \frac{\partial \Theta_\lambda}{\partial \eta_{jp}}) +$$

$$(b_{ij}(\xi, \lambda) + b_{ij}^*(\eta, \lambda) + (k-1)) \Theta_\lambda = 0, \quad 1 \leq i, j \leq n, \; i \neq j \tag{2.7}$$

System (2.7) can be regarded as a system of $n^2 - n$ nonhomogeneous linear equations with respect to the $n^2 - n$ variables $\frac{\partial \Theta_\lambda}{\partial \xi_{ij}}, \frac{\partial \Theta_\lambda}{\partial \eta_{ij}}$. Denote the matrix of this system by $N(\xi, \eta)$. It is a square matrix.
Lemma 2.1. The matrix $N(\xi, \eta)$ is invertible over the field of rational functions of $\xi, \eta$. The inverse matrix is regular in the neighborhood of the point $\xi = 0, \eta = \infty$.

Proof. Let us show that the vector fields by which the $n^2 - n$ elements $E_{ij}, i \neq j$ act on the space $F \times F$, where $F$ is the flag variety, are linearly independent at the point $(B^+, B^-)$, where $B^+, B^-$ are the Borel subgroups in $G = GL_n(\mathbb{C})$ consisting of upper (respectively lower) triangular matrices. Indeed, the stabilizer of $B^\pm$ is $B^\pm$ itself, so the stabilizer of both $B^+, B^-$ is $B^+ \cap B^- = H$ – the maximal torus consisting of diagonal matrices. Thus, the orbit of $(B^+, B^-)$ is isomorphic to $G/H$, and its tangent space at $(B^+, B^-)$ is $\mathfrak{g}/\mathfrak{h}$. The elements $E_{ij}$ project onto a basis of $\mathfrak{g}/\mathfrak{h}$, so they produce linearly independent vectors on the tangent space. □

Lemma 2.1 implies that system of differential equations (2.7) can be written in the form

$$\frac{\partial \Theta_\lambda}{\partial \xi_{ij}} = \alpha_{ij}(\xi, \eta) \Theta_\lambda,$$

$$\frac{\partial \Theta_\lambda}{\partial \eta_{ij}} = \beta_{ij}(\xi, \eta) \Theta_\lambda,$$

(2.8)

where $\alpha_{ij}, \beta_{ij}$ are rational functions of $\xi, \eta$ found from (2.7) (all functions from (2.7) implicitly depend on $k$). This determines $\Theta_\lambda$ uniquely up to a constant. If we normalize $\Theta_\lambda$ by the condition $\Theta_\lambda(0, \infty) = 1$ then $\Theta_\lambda$ is given by the formula

$$\Theta_\lambda(\xi^0, \eta^0) = \exp \left( \int_{(0, \infty)} \sum_{i < j} (\alpha_{ij}(\xi, \eta)d\xi_{ij} + \beta_{ij}(\xi, \eta)d\eta_{ij}) \right).$$

(2.9)

It follows from the above that this function is regular in the neighborhood of $(0, \infty)$.

Now we can deduce integral formulas for $\Psi(x, \lambda)$. The idea is that $\Psi$ is obtained from $\Phi_\lambda \in M_\lambda \otimes M_\lambda^* \otimes U_k$ by contracting $M_\lambda$ with $M_\lambda^*$. This contraction, in the functional realization, is expressed by a contour integral (see (2.4)). Therefore, from (1.6) and (2.6) we get

$$\Psi(x, \lambda) = (2\pi \sqrt{-1})^{-n(n-1)/2} e^{<\lambda, x>} \int_{|y_{ij}|=1} \Theta_\lambda(\{y_{ij}\}, \{e^{x_i-x_j} y_{ij}\}) \frac{dy}{y}.$$ 

(2.10)

Thus, we have proved the following theorem, which is our main result.

**Theorem 2.2.** For generic values of $\Lambda_i$ in (1.3) there exists $\mu \in \mathbb{C}^n$ such that the functions

$$\psi(x, \lambda) =$$

$$\phi(x)e^{<x, \lambda>} \int_{|y_{ij}|=1} \exp \left( \int_{(0, \infty)} \sum_{i < j} (\alpha_{ij}(\xi, \eta, \lambda, k)d\xi_{ij} + \beta_{ij}(\xi, \eta, \lambda, k)d\eta_{ij}) \right) \frac{dy}{y},$$

(2.11)

$$\lambda = w(\mu + \rho) - \rho, w \in S_n,$$

where $\alpha_{ij}, \beta_{ij}$ are the rational functions defined above, $C = k(k - 1)$, and $\phi$ is given by (1.8), form a basis of the space of solutions of system (1.3). The weight

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1In (2.11) we explicitly specify the dependence of $\alpha_{ij}, \beta_{ij}$ on $\lambda, k$, which we have omitted so far.
\( \mu \) is found from the equations \( P_i(\mu) = \Lambda_i, \ 1 \leq i \leq n \), where \( P_i(\lambda) \) is the scalar by which the central element \( Z_m \) acts in the Verma module \( M_\lambda \).

**Remark.** It is known that if \( n = 2 \) then eigenfunctions of \( L_1, L_2 \) express via Gauss hypergeometric function. It is easy to check that for \( n = 2 \) formula (2.11) becomes (after some transformations) a special case of the standard integral formula for the Gauss hypergeometric function.

Now we can apply Theorem 2.2 to derivation of integral formulas for Jack’s symmetric functions (see [M]). In [EK1],[EK2] it was shown that the Jack’s symmetric functions can be obtained as follows:

\[
J_k^\lambda(x) = \phi(x)^{1-k} \text{Tr}|M_{\lambda+(k-1)\rho}(\Phi_{\lambda+(k-1)\rho} e^x)
\]

This formula combined with Theorem 2.2 gives the desired integral representation of Jack’s functions.

**Remark.** For integer values of \( k \), the representation \( U_k \) contains a finite dimensional subrepresentation or quotient representation. Therefore, in this case common eigenfunctions of \( \hat{L}_i \) can be interpreted as generalized spherical functions on the symmetric space \( K \times K/K_{\text{diag}} \), where \( K = SU(n) \), and \( K_{\text{diag}} \) is the diagonal in \( K \times K \). Therefore, it is possible to write a Harish-Chandra type integral formula for these eigenfunctions, as described in [W], where integration is carried out over \( K \). It is an interesting question what is the connection of this formula with the one given by Theorem 2.2.

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