CLASSIFICATION OF FINITE DYNAMICAL SYSTEMS

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ABSTRACT. This paper is motivated by the theory of sequential dynamical systems, developed as a basis for a mathematical theory of computer simulation. It contains a classification of finite dynamical systems on binary strings, which are obtained by composing functions defined on the coordinates. The classification is in terms of the dependency relations among the coordinate functions. It suggests a natural notion of the linearization of a system. Furthermore, it contains a sharp upper bound on the number of systems in terms of the dependencies among the coordinate functions. This upper bound generalizes an upper bound for sequential dynamical systems.

1. Introduction

In this paper a finite dynamical system will mean a mapping from the set of binary strings of a certain length to itself. Such mappings occur in a variety of contexts, in particular in the theoretical study of computer simulation. Representing a computer simulation as a particular finite dynamical system is one possible approach to its mathematical analysis, carried out in [1], [2], [3]. The finite dynamical systems considered there, so-called sequential dynamical systems (SDS), incorporate the essential features of a computer simulation. Local variables $v_1, \ldots, v_n$ take on binary states which evolve in discrete time, based on a local update function $f^i$ attached to each variable $v_i$, and which depends on the states of certain other variables, encoded by the edges of a dependency graph $Y$ on the vertices $v_1, \ldots, v_n$. Finally, an update schedule prescribes how these local update functions are to be composed in order to generate a global update function

$$f : \{0,1\}^n \to \{0,1\}^n$$

of the system. An important question, which can be answered in this setting, is how many different systems one can generate simply by varying the update schedule. The upper bound is in terms of invariants of the dependency graph $Y$.

In applications the dependency graph $Y$ frequently varies over time, however. The need for a framework that allows for such a change inspired the investigation of properties of tuples of “local” functions in [4] and certain equivalence relations on them. That paper also contains a Galois correspondence between sets of tuples of local functions and certain graphs. Tuples of local functions can be interpreted as parallel systems $f : \{0,1\}^n \to \{0,1\}^n$, so that the results pertain to the study of parallel systems as well. The present paper makes the connection between tuples of local functions (parallel systems) and sequential systems, in particular SDS, explicit by exploiting this Galois correspondence. In order to describe our main results we need to recall some definitions and results from [4].

Let $\mathbb{K} = \{0,1\}$, and let $\mathbb{K}^n$ be the $n$-fold cartesian product of $\mathbb{K}$. When convenient we will view $\mathbb{K}$ as the field with two elements.

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Definition 1.1. Let \( n \) be a positive integer, let \( d \) be a nonnegative integer, and let \( Y \) be a graph with vertex set \( \{1, \ldots, n\} \).

1. A function \( f : \mathbb{K}^n \rightarrow \mathbb{K}^n \) is \( d \)-local on \( Y \) if, for any \( 1 \leq j \leq n \), the \( j \)-th coordinate of the value of \( f \) on \( x \in \mathbb{K}^n \) depends only on the value of those coordinates of \( x \) that have distance less than or equal to \( d \) from vertex \( j \in Y \). In other words, if \( f(x) = (f_1(x), \ldots, f_n(x)) \), then \( f_j : \mathbb{K}^n \rightarrow \mathbb{K} \) depends only on those coordinates that have distance less than or equal to \( d \) from \( j \).

2. For \( 1 \leq d < n \) and \( 1 \leq j \leq n \), let \( L^j_d(Y) \) be the set of all functions \( f : \mathbb{K}^n \rightarrow \mathbb{K}^n \) such that
   \[
   f(x_1, \ldots, x_n) = (x_1, \ldots, x_{j-1}, f_j(x), x_{j+1}, \ldots, x_n),
   \]
   and \( f_j : \mathbb{K}^n \rightarrow \mathbb{K} \) depends only on the values of those coordinates of \( x \) which have distance at most \( d \) from \( j \) in \( Y \). Hence \( L^j_d(Y) \) consists of \( d \)-local functions on \( \mathbb{K}^n \), which are the identity on all but possibly the \( j \)-th coordinate.

3. For \( d = n \), define \( L^i_n(Y) \) to be the set of all functions on \( \mathbb{K}^n \), which are the identity on all but possibly the \( j \)-th coordinate. Observe that if \( Y \) is connected, then this definition of \( L^i_n(Y) \) directly extends the definition in (2).

Observe that \( L^i_0(Y) = L^i_1 \) does not depend on the graph \( Y \) and neither does \( L^i_n(Y) \). Furthermore, \( L^i_0 \) is isomorphic to Map(\( \mathbb{K}, \mathbb{K} \)), that is, it contains all four possible functions, namely the identity on \( \mathbb{K} \), the two projections to one element in \( \mathbb{K} \), and the inversion.

In this paper we study the set
   \[
   L_n \times \cdots \times L_n = \{(f^1, \ldots, f^n) \mid f^i \in L^i_n\},
   \]
that is, the set of \( n \)-tuples of functions \( f^i : \mathbb{K}^n \rightarrow \mathbb{K}^n \) which only change the \( i \)-th coordinate.

To be precise, \( f^i(x) = (x_1, \ldots, x_{i-1}, f^i_j(x), x_{i+1}, \ldots, x_n) \), with arbitrary functions \( f^i_j : \mathbb{K}^n \rightarrow \mathbb{K} \). We denote by \( \mathcal{F} \) the power set of this set without the empty set. The following theorem is one of the main results in [3].

Theorem 1.2. There is a Galois correspondence between \( \mathcal{F} \) and the set \( \mathcal{G} \) of subgraphs of the complete graph \( K_n \) on the vertex set \( \{1, \ldots, n\} \).

For the convenience of the reader we recall the construction of this Galois correspondence. Let \( F \in \mathcal{F} \). Define a subgraph \( \Phi(F) \) of \( K_n \) as follows. First construct the set \( \tilde{F} \) of all \( n \)-tuples \( \tilde{f} = (\tilde{f}^1, \ldots, \tilde{f}^n) \), which either are in \( F \) or arise from an element in \( F \) by replacing one of the coordinates by a \( 0 \)-local function, that is, by a function from \( L^i_0 \) for some \( i \). Now define the graph \( \Phi(F) \) as follows. An edge \((i, j)\) of \( K_n \) is in \( \Phi(F) \) if and only if \( \tilde{f}^i \circ \tilde{f}^j = \tilde{f}^j \circ \tilde{f}^i \) for all \( \tilde{f} = (\tilde{f}^1, \ldots, \tilde{f}^n) \in \tilde{F} \).

Conversely, let \( G \subset K_n \) be a subgraph. We define a set \( \Psi(G) \) of \( n \)-tuples of functions on \( \mathbb{K}^n \) by
   \[
   \Psi(G) = L^1_1(G) \times \cdots \times L^n_1(G),
   \]
where \( \overline{G} \) is the complement of \( G \) in \( K_n \). Then \( \Phi \) and \( \Psi \) together form the desired Galois correspondence.

In particular, if \( F \in \mathcal{F} \) consists of one element \( f = (f^1, \ldots, f^n) \), then the graph \( \Phi(\{f\}) = \Phi(f) \) encodes the dependency relations among the local functions \( f^i \). Conversely, for a subgraph \( G \subset K_n \), the set \( \Psi(G) \) contains all \( n \)-tuples of local functions whose dependency
relations are modeled by $G$. This observation provides the paradigm for the results in this paper.

First we encode the local functions as polynomials, which allows us to give an algebraic criterion to compute $\Phi(f)$. More importantly, it suggests a natural choice for the linearization of a system, which we define in the next section. Using this notion of linearization we explore a natural equivalence relation on tuples, setting two equivalent if they have the same dependency graph. We show that systems that are equivalent in this sense have the same linearization.

Finally, we consider systems $f: \mathbb{K}^n \rightarrow \mathbb{K}^n$ which are obtained by composing local functions from an $n$-tuple $f = (f^1, \ldots, f^n)$. We show that, if $t \geq 1$ is an integer and $W_t$ is the set of all words in the integers $1, \ldots, n$ of length $t$, allowing for repetitions and for the case that $t < n$, then we obtain an upper bound for the number of different systems $f^\pi : \mathbb{K}^n \rightarrow \mathbb{K}^n$ one can construct by forming

$$f^\pi = f^{i_t} \circ \cdots \circ f^{i_1},$$

where $\pi = (i_1, \ldots, i_t)$ ranges over all elements in $W_t$.

This upper bound generalizes one for SDS, derived in [7]. It suggests that a part of the theory of SDS can be derived for systems that are SDS-“like,” but have fewer restrictions on the local functions. In particular, it is not necessary to make the dependency graph an explicit part of the data defining an SDS. This approach is explored further in [6], in which a more general notion of SDS is introduced, and morphisms of SDS are defined, forming a category with interesting properties, that contains “classical” SDS as a subcategory. A morphism between two SDS can be viewed as a simulation of one system by the other.

2. Computation of $\Phi$

In this section we give a method for computing the graph $\Phi(f)$ for an $n$-tuple of local functions $f$. It relies on the representation of local functions as polynomials, for which we now give an elementary proof.

**Lemma 2.1.** Let $f: \mathbb{K}^n \rightarrow \mathbb{K}$ be a function. Then $f$ can be represented as a polynomial. That is, there is a polynomial $p \in \mathbb{K}[x_1, \ldots, x_n]$ such that

$$f(a_1, \ldots, a_n) = p(a_1, \ldots, a_n),$$

for all $(a_1, \ldots, a_n) \in \mathbb{K}^n$.

**Proof.** We will prove the lemma by showing that there are exactly as many different polynomial functions as there are functions. Since $|\mathbb{K}^n| = 2^n$, there are $2^{2n}$ functions from $\mathbb{K}^n$ to $\mathbb{K}$. Now let $V \subset \mathbb{K}[x_1, \ldots, x_n]$ be the subspace with the basis containing the $2^n$ monomials

$$B = \{1, m_1, m_2, \ldots, m_{2^n-1}\},$$

where $m_i = x_1^{b_1} \cdots x_n^{b_n}$ is given by the binary expansion $b_1 \ldots b_n$ of $i$. Then $V$ contains $2^{2n}$ elements, which is equal to the number of functions $\mathbb{K}^n \rightarrow \mathbb{K}$. Thus it suffices to show that different polynomials give rise to different functions on $\mathbb{K}^n$.

Suppose that $f, g \in V$, with $f \neq g$. Let $m = x_1^{b_1} \cdots x_n^{b_n}$ be a monomial of smallest total degree which is in $f$ but not in $g$, and let $h = f - g = m + q$. Let $a = (a_1, \ldots, a_n)$. Then $h(a) = 1 + q(a)$. Now observe that any monomial of $q$ which involves a variable that does not appear in $m$ evaluates to 0 at $a$. But since $m$ was chosen to have minimal degree, there
cannot be any monomials in $q$ that involve only variables appearing in $m$. Hence $q(a) = 0$, and $h(a) = 1$. This shows that $f \neq g : \mathbb{K}^n \rightarrow \mathbb{K}$, and the proof is complete. 

Therefore, any function in $L_n^d$ can be written as an $n$-tuple of polynomials in $n$ variables $x_1, \ldots, x_n$. Also observe that $(x_j)^m = x_j$ for all $1 \leq j \leq n$ and for all $m > 0$. Furthermore, $x_j^m = 1 + x_j$. Thus, we can represent an element of $L_n^1 \times \cdots \times L_n^d$ as an $n \times (2^n)$-matrix in which the $i$th row corresponds to the $i$th local function $f_i$.

**Example 2.2.** Let $f = (f^1, f^2, f^3)$ where

$$f^1 = (1 + x_3 + x_1x_2, x_2, x_3),$$
$$f^2 = (x_1, x_2 + x_1x_3 + x_2x_3, x_3),$$
$$f^3 = (x_1, x_2, 1 + x_1x_2x_3).$$

Then the $3 \times 8$ matrix, denoted by $M_f$, associated to $f$ with respect to the ordered basis

$$\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3, x_1x_2x_3\},$$

is equal to

$$\begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$
On the other hand, \( \tilde{f}^i(a) = (a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) \). So the \( j \)th coordinate of \( f^j \circ \tilde{f}^i(a) \) is

\[
(f^j \circ \tilde{f}^i(a))_j = f^j_j(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = (m_1 + \cdots + m_t)(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_n) = 0.
\]

Thus \( (f^j \circ \tilde{f}^i(a))_j \neq (f^j \circ \tilde{f}^i(a))_j \). Therefore, \( \tilde{f}^i \circ f^j \neq f^j \circ \tilde{f}^i \), and hence there cannot be an edge between vertices \( i, j \) in \( \Phi(f) \).

This proposition provides an easy algorithm to compute \( \Phi(f) \) for any \( f \), or, more generally, any set of such functions. (A C++ implementation is available from the authors.) It also suggests a natural notion of a linear system, which will be explored in the next section.

### 3. Linearization of Systems

For classical dynamical systems, a standard technique is to linearize the system, and then study the linearization. Proposition 2.3 suggests a natural definition of linearity for a finite system

\[ f = (f^1, \ldots, f^n) : \mathbb{K}^n \rightarrow \mathbb{K}^n. \]

One can then define the notion of linearization for a finite system and study the relationship between the original system and its linearization. In this section we study the notion of linearization with respect to the dependency relations among the local functions.

Each \( f^i : \mathbb{K}^n \rightarrow \mathbb{K}^n \) can be represented as a polynomial in the variables \( x_1, \ldots, x_n \), which changes only the \( i \)th coordinate. That is, \( f^i = (f^i_1, \ldots, f^i_n) \), with each \( f^i_j \in \mathbb{K}[x_1, \ldots, x_n] \).

**Definition 3.1.** A system \( f = (f^1, \ldots, f^n) : \mathbb{K}^n \rightarrow \mathbb{K}^n \) is called linear if all functions \( f^i \) are \( \mathbb{K} \)-linear polynomials.

Observe that a linear system \( f = (f^1, \ldots, f^n) \) can be represented by an \((n \times n)\)-matrix with entries in \( \mathbb{K} \), since the constant term and all nonlinear terms of all functions \( f^i \) are equal to zero. Conversely, any \((n \times n)\)-matrix \( M \) over \( \mathbb{K} \) can be interpreted as a linear system

\[ l_M : \mathbb{K}^n \rightarrow \mathbb{K}^n. \]

**Lemma 3.2.** Let \( G \) be a graph on \( n \) vertices. Then there exists a linear system \( l_G = (l^1, \ldots, l^n) : \mathbb{K}^n \rightarrow \mathbb{K}^n \) such that \( \Phi(l_G) = G \).

**Proof.** We construct a linear system \( l_G \) by providing the associated \( n \times n \)-matrix \( M_{l_G} \) of \( l_G \). For \( 1 \leq i < j \leq n \), let

\[
(M_{l_G})_{ij} = (M_{l_G})_{ji} = \begin{cases} 1 & \text{if the edge } (i,j) \notin E(G), \\ 0 & \text{otherwise.} \end{cases}
\]

And let \( (M_{l_G})_{ii} = 0 \), for all \( 1 \leq i \leq n \). It is clear that \( l_G \) is a linear system. Moreover, by Proposition 2.3, \( \Phi(l_G) = G \).

**Remark 3.3.** The matrix \( M_{l_G} \) is the adjacency matrix of the complement of the graph \( \Phi(l_G) = G \). Observe that \( l_G \) is not the only linear system which gives the graph \( G \). The matrix \( M_{l_G} \) is symmetric. Consider a pair \( i \neq j \) for which the \((i,j)\)-entry (and the \((j,i)\)-entry) is equal to 1. If we change exactly one of these two entries to a 0, then \( l_G \) and \( l'_G \) have
the same graph $G$ as their image under $\Phi$. We will see later (Theorem 4.6), however, that the incidence matrix of the complement of $G$ is not just a canonical, but a natural choice of linear system in the inverse image of a graph $G$ under $\Phi$.

**Theorem 3.4.** Let $f = (f^1, \ldots, f^n) : \mathbb{K}^n \rightarrow \mathbb{K}^n$ be a system and let $G = \Phi(f)$. There exists a linear system $l_G = (l^1, \ldots, l^n) : \mathbb{K}^n \rightarrow \mathbb{K}^n$ such that $\Phi(l_G) = G = \Phi(f)$.

**Proof.** By Lemma 3.2, there exists a linear system $l_G$ such that $\Phi(l_G) = G = \Phi(f)$. □

**Definition 3.5.** The linear system $l_G$ corresponding to the adjacency matrix of the complement of $G$ is called the linearization of the system $f$.

The linearization of a system should ideally bear a certain relationship to the original system. Our focus in this paper is the structure of the dependencies among the component functions. The linearization of a system as defined here, has the same dependency structure among the component functions. Its dynamics will in general be very different, however.

### 4. Graph equivalence

Using the Galois correspondence we now define an equivalence relation on $n$-tuples of local functions, which captures equivalence of dependency relations among the entries in the tuples.

**Definition 4.1.** We say that $(f^1, \ldots, f^n), (g^1, \ldots, g^n) \in L_n^1 \times \cdots \times L_n^n$ are graph equivalent if and only if $\Phi((f^1, \ldots, f^n))$ is isomorphic to $\Phi((g^1, \ldots, g^n))$. This relation is denoted by $(f^1, \ldots, f^n) \sim (g^1, \ldots, g^n)$.

**Example 4.2.** Let $f = (f^1, f^2, f^3)$, where
\[
\begin{align*}
f^1(x_1, x_2, x_3) &= (x_2 + x_3, x_2, x_3), \\
f^2(x_1, x_2, x_3) &= (x_2, 1 + x_2, x_3), \\
f^3(x_1, x_2, x_3) &= (x_1, x_2, x_1).
\end{align*}
\]

We will usually denote such a function by $f = (x_2 + x_3, 1 + x_2, x_1)$. Let $g = (x_2, x_3, 0)$. Then $f \sim g$, since their graphs both have three vertices $1, 2, 3$, with edge $(2, 3)$ in $\Phi(f)$, and edge $(1, 3)$ in $\Phi(g)$.

The main result in this section is a characterization of graph equivalence of systems using a relationship on the matrices of the corresponding linearizations. We first define an action of the symmetric group on the set of matrices.

**Definition 4.3.** Let $S_n$ be the group of permutations of $n$ elements. We define an $S_n$-action on the set of $n \times n$-matrices as follows. For any $(n \times n)$-matrix $M$ and $\pi \in S_n$, $\pi M$ is the $(n \times n)$-matrix such that
\[
(\pi M)_{ij} = M_{\pi^{-1}(i)\pi^{-1}(j)}.
\]
That is, $\pi$ acts on $M$ by permuting rows and columns.

**Proposition 4.4.** The $S_n$-action in Definition 4.3 is a group action of $S_n$ on the set of $(n \times n)$-matrices over $\mathbb{K}$. 

Proof. Clearly \( eM = M \), where \( e \) is the identity permutation. Let \( \pi, \sigma \in S_n \), then
\[
(\pi(\sigma M))_{ij} = (\sigma M)_{\pi^{-1}(i)\pi^{-1}(j)} = M_{\pi^{-1}(i)\pi^{-1}(j)} = M_{(\pi\sigma)^{-1}(i)(\pi\sigma)^{-1}(j)} = (\pi\sigma)_{ij}.
\]

A graph automorphism can be represented by a permutation of the vertices of the graph which preserves adjacency. So we can represent automorphisms of a graph with \( n \) vertices as permutations in \( S_n \), and obtain in this way an \( S_n \)-action on the set of subgraphs of \( K_n \).

**Theorem 4.5.** Let \( G_1 \) and \( G_2 \) be two graphs with \((n \times n)\)-adjacency matrices \( M_1 \) and \( M_2 \), respectively. Then \( G_1 \) is isomorphic to \( G_2 \) via an automorphism \( \pi \in S_n \) (i.e., \( \pi(G_1) = G_2 \)) if and only if \( \pi M_1 = M_2 \).

**Proof.** Suppose first that \( \pi M_1 = M_2 \). For any two vertices \( i \) and \( j \) of \( G_2 \), \((i, j) \in E(G_2)\) if and only if
\[
(M_2)_{ij} = (M_2)_{ji} = 1.
\]
This happens if and only if
\[
(M_1)_{\pi^{-1}(i)\pi^{-1}(j)} = (M_1)_{\pi^{-1}(j)\pi^{-1}(i)} = 1,
\]
which is the case if and only if the edge \((\pi^{-1}(i), \pi^{-1}(j)) \in E(G_1)\). Thus \( \pi(G_1) = G_2 \).

Conversely, if \( \pi(G_1) = G_2 \), then \( (M_2)_{ij} = 1 \) if and only if \((i, j) \in E(G_2)\), that is, if and only if \((\pi^{-1}(i), \pi^{-1}(j)) \in E(G_1)\). This is the case if and only if \( (M_1)_{\pi^{-1}(i)\pi^{-1}(j)} = (M_1)_{\pi^{-1}(j)\pi^{-1}(i)} = 1 \). Therefore, \( \pi \cdot M_1 = M_2 \).

**Theorem 4.6.** Let \( f \) and \( g \) be two systems on \( \mathbb{K}^n \). Then \( f \) is graph equivalent to \( g \) (i.e., \( \pi(\Phi(f)) = \Phi(g) \) for some \( \pi \in S_n \)) if and only if \( \pi \cdot M_{\Phi(f)} = M_{\Phi(g)} \).

**Proof.** First observe that if \( G_1 \) and \( G_2 \) are graphs, then \( G_1 \) is isomorphic to \( G_2 \) via \( \pi \) if and only if \( \Phi(G_1) \) is isomorphic to \( \Phi(G_2) \) via \( \pi \). Therefore, \( \pi(\Phi(f)) = \Phi(g) \) if and only if \( \pi(\Phi(f)) = \Phi(g) \).

By Remark [3.3], \( M_{\Phi(f)} \) (resp. \( M_{\Phi(g)} \)) is the adjacency matrix of \( \Phi(f) \) (resp. \( \Phi(g) \)). Now, by Theorem [4.3], \( \pi(\Phi(f)) = \Phi(g) \) if and only if \( \pi \cdot M_{\Phi(f)} = M_{\Phi(g)} \). Therefore, \( \pi(\Phi(f)) = \Phi(g) \) if and only if \( \pi \cdot M_{\Phi(f)} = M_{\Phi(g)} \).

This theorem suggests that our choice of the adjacency matrix of \( \Phi(f) \) as the linearization of a system \( f \) is a natural one, since it preserves graph equivalence and makes the construction of \( l_G \) from \( G \) equivariant with respect to the \( S_n \)-action on both.

5. **An Upper Bound for Sequential Systems**

The real object of interest in many cases are composed systems \( f : \mathbb{K}^n \longrightarrow \mathbb{K}^n \) rather than merely tuples of local functions. Such systems are obtained by composing the local functions in some order. The order often corresponds to a choice of update schedule of the variables in a system, such as a simulation. A theoretical question which has important practical consequences is how many different systems one can obtain by simply varying the update schedule of the variables, that is, by composing the local functions in a different order. In this section we derive an upper bound for this number.
Definition 5.1. Let \( f = (f^1, \ldots, f^n) : \mathbb{K}^n \to \mathbb{K}^n \), and let \( W_t \) be the set of all words on \( \{1, \ldots, n\} \) of length \( t \), for some \( t \geq 1 \), allowing for repetitions. For \( \pi = (i_1, \ldots, i_t) \in W_t \), we denote by \( f^\pi \) the finite dynamical system given by
\[
f^{i_t} \circ \cdots \circ f^{i_1} : \mathbb{K}^n \to \mathbb{K}^n.
\]

Let \( F_{W_t}(f) = \{ f^\pi \mid \pi \in W_t \} \), the collection of all systems \( \mathbb{K}^n \to \mathbb{K}^n \) that can be obtained by composing the coordinate functions of \( f \) in all possible ways, using up to \( t \) of them.

We now define an equivalence relation on \( W_t \).

Definition 5.2. Let \( G \) be a graph on the \( n \) vertices \( 1, \ldots, n \). Let \( \sim_G \) be the equivalence relation on \( W_t \) generated by the following relation. Let \( \pi = (i_1, \ldots, i_t) \in W_t \). For \( 1 \leq k < t \), if \( i_k = i_{k+1} \) or there is no edge between \( i_k \) and \( i_{k+1} \) in \( G \), then
\[
\pi \sim_G \pi',
\]
where \( \pi' = (i_1, \ldots, i_{k+1}, i_k, i_{k+2}, \ldots, i_t) \).

Remark 5.3. If \( f = (f^1, \ldots, f^n) : \mathbb{K}^n \to \mathbb{K}^n \) is such that \( \Phi(f) = G \), and \( \pi \sim_G \pi' \), then
\[
f^{i_t} \circ \cdots \circ f^{i_k} \circ f^{i_{k+1}} \circ \cdots \circ f^1 = f^{i_t} \circ \cdots \circ f^{i_k} \circ f^{i_{k+1}} \circ \cdots \circ f^1.
\]

We now derive an upper bound on the size of the set \( F_{W_t}(f) \), that is, on the number of different systems one obtains by composing the coordinate functions of \( f \) in all possible orders, with up to \( t \) of them at a time.

Definition 5.4. Let \( f = (f^1, \ldots, f^n) : \mathbb{K}^n \to \mathbb{K}^n \), and let \( G(f) = G = \Phi(f) \). Let \( \pi = (i_1, \ldots, i_t) \in W_t \). Let \( H_\pi(f) = H_\pi \) be the graph on \( t \) vertices \( v_1, \ldots, v_t \), corresponding to \( i_1, \ldots, i_t \) (with \( v_a \neq v_b \) even in the case that \( i_a = i_b \)), with an edge between \( v_a \) and \( v_b \) if and only if the following two conditions hold:
1. \( i_a \neq i_b \),
2. the edge \( (i_a, i_b) \) is not in \( G \).

Remark 5.5. Observe that if \( \pi \in S_n \), that is, \( t = n \) and \( \pi \) contains no repetitions, then \( H_\pi = \overline{G} \).

Let \( \text{Acyc}(H) \) be the set of all acyclic orientations of a graph \( H \). Given \( \pi = (i_1, \ldots, i_t) \in W_t \), we construct an acyclic orientation of \( H_\pi \) by orienting an edge \((v_i, v_j)\) toward the vertex whose label occurs first in \( \pi \). If all entries of \( \pi \) are distinct, then this clearly produces an acyclic orientation. But even if an entry is repeated we cannot produce an oriented cycle, since there is no edge between the vertices corresponding to the repetitions. Denote this acyclic orientation by \( O_\pi(f) \).

Lemma 5.6. If \( \pi \sim_G \pi' \), then \( H_\pi(f) = H_{\pi'}(f) \) and \( O_\pi(f) = O_{\pi'}(f) \).

Proof. If \( \pi \sim_G \pi' \), then they differ by a sequence of transpositions of adjacent letters, which are either equal, or for which the corresponding vertices in \( G \) are connected by an edge. Hence \( H_\pi(f) \) and \( H_{\pi'}(f) \) have the same vertex set. Furthermore, an edge \((a, b)\) is in \( H_\pi(f) \) if and only if \( i_a \neq i_b \) and \((i_a, i_b)\) is an edge in \( G \). Similarly for \( H_{\pi'}(f) \). Observe that the transposition in \( \pi \) of adjacent letters which are connected by an edge in \( G \) does not change the resulting acyclic orientation, because, by construction, the vertices \( v_a \) and \( v_b \) are not connected by an edge in \( H_\pi(f) \). Hence the proof of the lemma is complete. \( \Box \)
Theorem 5.9. Let $f$ be a system and $G = \Phi(f)$. There is a one-to-one correspondence

\[ \psi_G : W_t / \sim_G \rightarrow \{ \text{Acyc}(H_\pi(f)) | \pi \in W_t \}. \]

Proof. We assign to a word $\pi \in W_t$ the associated acyclic orientation $O_\pi(f)$ on $H_\pi(f)$. By Lemma 5.5, this induces a mapping $\psi_G$ on $W_t / \sim_G$. There is an obvious inverse mapping, assigning to an acyclic orientation on $H_\pi$ the corresponding $\pi'$, equivalent to $\pi$, such that $i_a$ appears before $i_b$ in $\pi'$ if there is an edge $(a, b)$ in $H_\pi(f)$, oriented from $a$ to $b$. \hfill \Box

Example 5.8. We illustrate this correspondence with the following example. Let $G$ be a 4-cycle with vertices $1, \ldots, 4$, and let $\pi = (1, 2, 1, 3)$. Then $H_\pi$ has the four vertices $1, 11, 2, 3$, where $11$ represents the vertex corresponding to the second 1 in $\pi$. There is an edge $3 \rightarrow 1$, which becomes oriented toward 1 in the acyclic orientation $O_\pi$.

The next theorem provides an upper bound on the number of different systems $f^\pi : \mathbb{K}^n \rightarrow \mathbb{K}^n$ one can obtain from composing the coordinate functions of an $n$-tuple $f = (f^1, \ldots, f^n)$, up to $t$ of them at a time.

Theorem 5.9. Let $f = (f^1, \ldots, f^n)$ be a system of local functions on $\mathbb{K}^n$, and let $F_{W_t}(f) = \{ f^\pi | \pi \in W_t \}$. Then

\[ |F_{W_t}(f)| \leq \{|\text{Acyc}(H_\pi(f))| \pi \in W_t\| = \sum_{\pi \in W_t} |\text{Acyc}(H_\pi(f))|. \]

Proof. By Proposition 5.7, $|W_t / \sim_G| \leq |\{ \text{Acyc}(H_\pi(f)) : \pi \in W_t \}|$. But we have seen that if $\pi \sim_G \pi'$ then $f^\pi = f^{\pi'}$. Hence

\[ |F_{W_t}(f)| \leq |W_t / \sim_G| \leq |\{ \text{Acyc}(H_\pi(f)) : \pi \in W_t \}|. \]

This result shows in particular that if $\pi \sim_G \pi'$, then the two systems $f^\pi$ and $f^{\pi'}$ are equal. The following example shows that the upper bound in the theorem is not attained in general.

Example 5.10. Let $f = (x_3x_3, x_1x_0) : \mathbb{K}^3 \rightarrow \mathbb{K}^3$. Then $\Phi(f)$ does not contain any edges. Let $\pi = (3, 2, 1), \pi' = (3, 1, 2)$. Then $\pi \not\sim_G \pi'$. However,

\[ f^\pi = f^1 \circ f^2 \circ f^3 = 0 = f^2 \circ f^1 \circ f^3 = f^{\pi'}. \]

Corollary 5.11. If $\pi \in S_n$, then $\{ \text{Acyc}(H_\pi(f)) : \pi \in W_n \} = \{ \text{Acyc}(\bar{G}) \}$. Thus in this case we recover the upper bound for the number of different SDS obtained in [2].

If we restrict ourselves to SDS, this bound is known to be sharp. For general systems this seems to be a substantially more difficult question.

6. Finite Systems and SDS

The approach to the study of finite systems taken in this paper was originated in [5], motivated by the desire to better understand sequential dynamical systems.

Recall that an SDS $f : \mathbb{K}^n \rightarrow \mathbb{K}^n$ is given by a graph $Y$ with $n$ vertices, functions $f^i : \mathbb{K}^n \rightarrow \mathbb{K}^n$, which change only the $i$th coordinate and take as input those coordinates
connected to $i$ in the graph $Y$. These functions are then composed according to an update schedule given by a permutation $\pi \in S_n$. That is,

$$f = f^{\pi(n)} \circ \cdots \circ f^{\pi(1)}.$$  

The functions $f^i$ are required to be symmetric in their inputs, that is, permuting the inputs does not change the value of the function.

In this paper we study $n$-tuples of functions $f^i : \mathbb{K}^n \rightarrow \mathbb{K}^n$, which change only the $i$th coordinate, without any further restrictions. In particular, we do not suppose the a priori existence of a graph $Y$, that governs the dependencies among these functions. The Galois correspondence $\Psi$, constructed in [5], provides such a graph when needed. And, as for SDS, it is the invariants of this graph that determine many properties of the $n$-tuple and finite systems derived from it. It shows that even for SDS it is not necessary to explicitly include the dependency graph $Y$ in the data defining an SDS. The Galois correspondence also shows that in general there will be more than one system whose dependency relations are modeled by a given graph. There is, however, a natural choice, which also provides a definition for the linearization of a system.

An important theoretical result, proved in [7], gives a sharp upper bound on the number of different SDS that can be obtained by varying the update schedule over all of $S_n$. The proof of this result assumes that all vertices of $Y$ that have the same degree also have the same local function attached to them. Theorem 5.9 generalizes this upper bound by removing the restrictions on the local functions $f^i$ and on the graph $Y$. More importantly, it removes the restriction that the update schedule be given by a permutation. Thus, the upper bound holds for compositions of the coordinate functions, which allows for repetitions of the functions, and does not require that all functions are actually used.

These generalizations suggest that a more relaxed definition of SDS can still lead to a class of systems about which one can prove theorems like the above upper bound. Such a definition is proposed in [6], where a category of more general SDS is developed.

In [8] an upper bound for dynamically non-equivalent SDS is given. In general, two maps $f, g : \mathbb{K}^n \rightarrow \mathbb{K}^n$ are dynamically equivalent if there exists a bijection $\varphi : \mathbb{K}^n \rightarrow \mathbb{K}^n$ such that

$$g = \varphi \circ f \circ \varphi^{-1}.$$  

This upper bound relies on the fact that conjugacy yields an SDS with the same graph and local functions. This is not true for the general systems discussed in this paper as the following example shows. Thus, this upper bound holds exactly for the class of SDS.

**Example 6.1.** Let $f = (0, x_3, x_2) : \mathbb{K}^3 \rightarrow \mathbb{K}^3$. Then $\Phi(f)$ is the graph on three vertices 1, 2, 3 with edges (1, 2), (1, 3). Hence there are only two functionally non-equivalent systems which correspond to the permutations $id = (123)$ and (321), that is, the systems $f^3 \circ f^2 \circ f^1$ and $f^1 \circ f^2 \circ f^3$. These two systems have the state spaces in Figure 4.

Nevertheless, if we let $\varphi = (213)$, then the system $\varphi \circ f^{id} \circ \varphi^{-1}$ has the state space in Figure 4.

As expected, this state space is isomorphic to the state space of $f^{id}$ but it is not equal to any of the two possible state spaces given in Figure 4. In fact, it is not equal to the state space of any system obtained by composing the functions $f^1, f^2, f^3$ according to any word. This is easily seen because, for example, the state $(1, 0, 1)$ is sent to itself, but $f^1$ is the zero function. So $f^1$ cannot be involved. On the other hand, the first coordinate of other states changes, so there must be a function involved that changes the first coordinate.
7. Stably isomorphic systems

In this section we answer a question that suggests itself naturally from the Galois correspondence. Given two systems that have the same dependency graphs, are they stably isomorphic, that is, do they have the same limit cycle structure in their state spaces? Recall that the state space of a system is a directed graph, whose vertices are the states of the system, that is, all binary strings of a given length. Directed edges correspond to system transitions.

Definition 7.1. Let $f_1, f_2$ be systems with state spaces $S_{f_i}$ and subdigraphs of limit cycles $L_{f_i}$. We call $f_1$ and $f_2$ stably isomorphic if there exists a digraph isomorphism between $L_{f_1}$ and $L_{f_2}$.

Let $f = (f^1, \ldots, f^n)$ and $g = (g^1, \ldots, g^n)$ be two systems with the same dependency graphs, that is, $\Phi(f) = \Phi(g)$. One can now ask if $f^\pi(n) \circ \cdots \circ f^\pi(1)$ is stably isomorphic to $g^\pi(n) \circ \cdots \circ g^\pi(1)$ for all $\pi \in W_i$. In this section we show that this is not true, by providing a counterexample.

Example 7.2. Let $f = (f^1, f^2, f^3)$ and $g = (g^1, g^2, g^3)$ be two triples of functions with

$$
\begin{align*}
    f^1(x_1, x_2, x_3) &= (1 + x_2 x_3, x_2, x_3), \\
    f^2(x_1, x_2, x_3) &= (x_1, x_1, x_3), \\
    f^3(x_1, x_2, x_3) &= (x_1, x_2, 1 + x_1), \\
    g^1(x_1, x_2, x_3) &= (x_1 + x_2 + x_3, x_2, x_3), \\
    g^2(x_1, x_2, x_3) &= (x_1, 1 + x_1 + x_2, x_3), \\
    g^3(x_1, x_2, x_3) &= (x_1, x_2, x_1 + x_3).
\end{align*}
$$

By using Theorem 2.3, it is easy to see that $\Phi(f) = \Phi(g)$ is the graph on vertices 1, 2, 3, with the single edge (2, 3).

Let $f = f^3 \circ f^2 \circ f^1$ and $g = g^3 \circ g^2 \circ g^1$. The state spaces of $f$ and $g$ are given in Figure 3. From Figure 3 it is clear that $f$ and $g$ are not stably isomorphic.
Figure 3. The state spaces of $f$ and $g$.

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