Self-Similar Potentials and the 
q-Oscillator Algebra at Roots of Unity

Sergei Skorik  
Department of Physics, University of Southern California,  
Los Angeles, CA 90089-0484, USA  

Vyacheslav Spiridonov  
Laboratoire de Physique Nucléaire, Université de Montréal,  
C.P. 6128, succ. A, Montréal, Québec, H3C 3J7, Canada

Abstract

Properties of the simplest class of self-similar potentials are analyzed. Wave functions of 
the corresponding Schrödinger equation provide bases of representations of the q-deformed  
Heisenberg-Weyl algebra. When the parameter q is a root of unity the functional form of the  
potentials can be found explicitly. The general $q^3 = 1$ and the particular $q^4 = 1$ potentials 
are given by the equianharmonic and (pseudo)lemniscatic Weierstrass functions respectively. 

Mathematics Subject Classification (1991). 34L40, 17B37, 33D80, 81R50

\footnote{To appear in shortened form in Lett.Math.Phys. 27 (1993)}
\footnote{On leave from the Moscow Institute of Physics and Technology, Moscow, Russia}  
\footnote{On leave from the Institute for Nuclear Research, Moscow, Russia}  
\footnote{International NSERC Fellow}
1. Introduction

The one-dimensional Schrödinger equation

\[ L\psi(x) \equiv -\psi''(x) + u(x)\psi(x) = \lambda\psi(x) \]  \hspace{1cm} (1)

plays an important role in the theory of nonlinear evolution equations, where it serves as an auxiliary spectral problem allowing to integrate the KdV-equation. From this point of view the KdV-evolution of \( u(x) \) in time yields the isospectral deformations of (1). In the quantum mechanical context, solutions of (1) subjected to some boundary conditions describe physical states of a particle moving on the line. In both applications a special attention is drawn to the spectra of the operators \( L \) subjected to some boundary conditions describe physical states of a particle moving on the line. In both applications a special attention is drawn to the spectra of the corresponding Hamiltonians \( L_j \). So, the spectra of the Hamiltonians \( L_{j+1} \) and \( L_j \) will have close spectral properties if we set \( L_{j+1} \equiv A_j^+A_{j+1}^- + \lambda_{j+1} = A_j^-A_j^+ + \lambda_j \). This constraint is equivalent to the following differential equation relating potentials \( u_j \) and \( u_{j+1} \):

\[ f_j'(x) + f_{j+1}'(x) + f_j^2(x) - f_{j+1}^2(x) = \lambda_{j+1} - \lambda_j. \]  \hspace{1cm} (2)

The chain of such equations is called the dressing chain in the theory of solitons. It is quite useful for studying symmetries of the Schrödinger equation. For particular solutions of the dressing chain parameters \( \lambda_j \) coincide with ordered discrete spectra of the corresponding Hamiltonians \( L_j \). So, the spectra of harmonic oscillator, Coulomb, and other “old” solvable potentials are easily found after the plugging in (2) the ansatz: \( f_j(x) = a(x)+b(x)+c(x)/j \).

In this Letter we discuss the particular class of the self-similar potentials related to each other by the scaling of argument: \( u_{j+1}(x) = q^2u_j(qx) \), which emerges as a result of the constraints \( f_{j+1}(x) = qf_j(qx), \lambda_{j+1} = q^2\lambda_j \) imposed on (2). This class has been studied by A.Shabat for the real \( x \) and \( q, \) \( 0 < q < 1 \), when it corresponds to “infinite-soliton” systems, i.e. when the potentials are reflectionless and decrease slowly at space infinities. The corresponding discrete spectra are purely exponential; they are generated by the \( q \)-deformed Heisenberg-Weyl, or \( q \)-oscillator algebra. In the two limiting cases, \( q \to 0 \) and \( q \to 1 \), such potentials are reduced to the one-soliton and harmonic oscillator potentials respectively. In the present paper we consider this self-similar system in the complex domain of the coordinate \( x \) and parameter \( q \). First we analyze the existence and uniqueness of the corresponding functions \( f_j(x) \). Then we characterize the qualitative structure of singularities exhibited by \( f_j(x) \). When \( q \) is a primitive \( n \)-th root of unity the situation is simplified. The odd \( n \) cases are shown to be related to the finite-gap potentials. The even \( n \) cases may contain a functional non-uniqueness. The general \( q^3 = 1 \) solution and a special \( q^4 = 1 \) solution are expressed through the equianharmonic and (pseudo)lemniscatic Weierstrass functions respectively. Algebraically, all these root-of-unity potentials are naturally related to the representations of the \( q \)-oscillator algebra at \( q^n = 1 \). A more wide class of the self-similar potentials leading to more complicated \( q \)-deformed algebras has been described in [4].

1
Though some of our results may be reformulated easily for the latter systems, they will not be discussed here.

Appearance of the $q$-analogs of the harmonic oscillator and other spectrum generating algebras raises special interest to the self-similar potentials. This is inspired by the recent intensive discussion of quantum algebras, or $q$-deformations of Lie algebras [4], which bear universal character due to the large number of applications. The $q$-oscillators themselves were reinvented as the related objects [5]. It was found that a natural group-theoretical setting for the basic, or $q$-hypergeometric functions [6] and corresponding orthogonal polynomials is provided by the quantum algebras when they are realized with the help of purely finite-difference operators [8]. However, the $q$-hypergeometric functions are defined in general only at $|q| < 1$ and we were not able to find in the literature an example of $q$-special function for $q^n = 1$ — the cases when the representation theory of quantum algebras essentially differs from that for the standard Lie algebras [9]. The self-similar potentials provide a realization of the creation and annihilation operators of the $q$-oscillator algebra in terms of both differential and finite-difference operators. It is the presence of the differential part that principally differs our approach from the mentioned above and leads to the well-defined $q$-special functions at $q^n = 1$.

The paper is organized as follows. In Sect. 2 we describe the needed facts about the periodic dressing chain. In Sect. 3 we present the class of self-similar potentials and discuss its analytical properties. The central result here is the theorem on the existence and uniqueness of solutions for the basic differential equation with deviating argument (14) when $|q| < 1$, or $q^n = 1$. The $q^3 = 1$ and $q^4 = 1$ cases are considered in detail in Sects. 4 and 5 respectively. Sect. 6 is devoted to the description of $q$-oscillator algebra representations related to the taken $q$-transcendent. In the Appendix we present exact solution of the spectral problem associated with the $q^3 = 1$ system.

The results obtained in this paper were partially presented at the Canadian Mathematical Society Meeting (Montréal, December 1992).

2. The Periodic Dressing Chain

We discuss here some results of the theory of nonlinear evolution equations on the basis of the dressing chain.

Following [10], one can rewrite (1) as a matrix equation and introduce a chain of associated spectral problems

$$\Psi_j'(x) = U_j \Psi_j(x), \quad U_j = \begin{bmatrix} 0 & 1 \\ u_j - \lambda & 0 \end{bmatrix}, \quad \Psi_j = \begin{pmatrix} \psi_j \\ \psi_j' \end{pmatrix}, \quad j \in \mathbb{Z}$$

solutions of which are related to each other by the transformations

$$\Psi_{j+1}(x) = B_j \Psi_j(x),$$

where $B_j$ is a $2 \times 2$ matrix whose entries are polynomials of the spectral parameter $\lambda$. The compatibility condition for (3)-(4) leads to the equation

$$B_j' = U_{j+1}B_j - B_j U_j.$$
When the map $\psi_j \to \psi_{j+1}$ does not depend on $\lambda$ the solution of (5) looks as follows

$$B_j = \begin{bmatrix} f_j(x) \\ f'_j(x) \end{bmatrix} + \lambda_j - \frac{1}{f_j(x)},$$

(6)

$$u_j(x) = f'_j(x) - f'_j + \lambda_j, \quad u_{j+1} - u_j = 2f'_j,$$

(7)

where $\lambda_j$ are some constants of integration. Substituting the first of relations (7) into the second one we get the dressing chain (2).

Denote by $A_j = B_{j+N-1} \ldots B_{j+1}B_j$ the product of Darboux transformation matrices (5). From (5) it follows that

$$A'_j = U_{j+N}A_j - A_jU_j.$$  

(8)

This equation allows to integrate the chain (2) for the special class of potentials defined by the periodicity condition

$$U_{j+N} = U_j, \quad \text{or} \quad f_{j+N}(x) = f_j(x), \quad \lambda_{j+N} = \lambda_j,$$

(9)

which describes a finite-dimensional dynamical system. For the odd $N$ this leads to the finite-gap potentials [10].

For convenience we fix the index $j$ in (8), (9), $j = 0$, and set $L \equiv L_0, U \equiv U_0, A \equiv A_0$. By its definition the matrix $A$ maps solutions of the corresponding Schrödinger equation onto each other and, hence, satisfies the matrix Lax equation $A' = [U, A]$. From the latter, one can derive the following relation for the matrix element $a_{12} \equiv \beta(x)$:

$$\frac{1}{2}\beta'' + (\lambda - u_0)\beta - \frac{1}{4}\beta^2 = \det A - \frac{1}{4}(Tr A)^2.$$  

(10)

Both $Tr A$ and $det A$ are easily seen to be integrals of motion; at $N = 2k + 1$ one has

$$det A = \prod_{i=0}^{2k} (\lambda - \lambda_i), \quad Tr A = (-1)^k (I_0 \lambda^k + I_1 \lambda^{k-1} + \cdots + I_k).$$

Equation (11) can be rewritten now in the form

$$u_0 - \lambda = F^2 - F', \quad F(x) \equiv \frac{\pm w - \beta'}{2\beta},$$

(11)

where

$$w^2 = (TrA)^2 - 4detA \equiv -4 \prod_{i=0}^{2k} (\lambda - E_i).$$

If all the constants $E_i$ are real and mutually different then the spectrum of non-singular potentials $u_j(x)$ has zonal structure and $E_i$ coincide with the boundaries of gaps. It is well-known that such potentials can be expressed in terms of the Riemann $\Theta$-function [11]:

$$u(x) = -2 \frac{d^2}{dx^2} \ln \Theta(\bar{l}(x)) + \text{const},$$

(12)

$$\Theta(\bar{l}) = \sum_{m_1, \ldots, m_k \in \mathbb{Z}} \exp \left\{ \sum_{p,s=1}^{k} m_p b_{ps} m_s + \sum_{p=1}^{k} l_p m_p \right\},$$

where
where \( l_p(x) \) are linear functions of \( x \) whose coefficients together with the parameters \( b_{ps} \) are determined by \( E_i \).

We shall not go further in the presentation of general results. For a detailed account of integrability of the dressing chain at even period \( N \) and in the case of a more general than (3) closure condition \( u_{j+N} = u_j + \alpha \), where \( \alpha \) is a constant, we refer to the recent work [12].

3. Self-Similar Potentials

The particular solution for the dynamical system (3), which we investigate in the present paper, is fixed by the following self-similarity condition

\[
  f_j(x) = q^j f(q^j x), \quad \mu_j = q^{2j} \mu, \quad \mu_j \equiv \lambda_{j+1} - \lambda_j.
\]  

(13)

In this case an infinite number of relations (3) is reduced to one differential equation with the deviating argument which was introduced by A.Shabat in [2]5:

\[
  \frac{d}{dx} (f(x) + qf(qx)) + f^2(x) - q^2 f^2(qx) = \mu.
\]  

(14)

Consider the initial value problem for the equation (14):

\[
  f(0) = \tilde{a}_0 < \infty.
\]  

(15)

We fix the initial condition at the point \( x = 0 \) because it is a fixed point of scaling \( x \to qx \) (there is another such point \( x = \infty \) which was considered in [14]). The parameter \( \mu \) is redundant – it may be removed by rescaling of the coordinate and \( f(x) \), but we prefer to keep it as a unique dimensional parameter.

A more general class of the self-similar potentials arises after the following \( q \)-periodic closure of the dressing chain [4]:

\[
  f_{j+N}(x) = q f_j(qx), \quad \mu_{j+N} = q^2 \mu_j.
\]  

(16)

Here we limit our discussion to the \( N = 1 \) case which coincides with (13).

Let \( f(z, q) \) denotes a function of two complex variables that satisfies the equation (14). Plugging into (14) the power series expansion

\[
  f(z, q) = \sum_{n=0}^{\infty} \tilde{a}_n(q) z^n,
\]  

(17)

one obtains the recursion formula for the coefficients \( \tilde{a}_n \):

\[
  (1 + q^{n+2}) \tilde{a}_{n+1} = \frac{q^{n+2} - 1}{n+1} \sum_{s=0}^{n} \tilde{a}_s \tilde{a}_{n-s}, \quad n \geq 1,
\]  

(18)

\[
  \tilde{a}_1 (1 + q^2) = \mu + (q^2 - 1) \tilde{a}_0^2.
\]

5 According to the classification of equations with a deviating argument [13], equation (14) is of the neutral type.
When \( f(0, q) = 0 \), all \( a_k \) with even \( k \) vanish, i.e. the \( f(z, q) \) is an odd function of \( z \) which corresponds to symmetric potentials. For this case it is convenient to rewrite formulae (17), (18) as follows:

\[
f(z, q) = \sum_{n=1}^{\infty} a_n(q)z^{2n-1},
\]

(19)

\[
(1 + q^{2n})a_n = \frac{q^{2n} - 1}{2n - 1} \sum_{s=1}^{n-1} a_s a_{n-s}, \quad a_1 = \frac{\mu}{1 + q^2}.
\]

(20)

**Lemma.** Series (19), (20) converges in the disc \( |z| < R_q \), for every fixed value of \( q, |q| < 1. \) For the radius of convergence \( R_q \) the following estimate holds: \( R_q \geq \frac{\pi}{2\sqrt{|a_1|\alpha}} \), where \( \alpha \equiv \frac{1 + |q|^2}{1 - |q|^2} \).

**Proof.** For any natural number \( p \) and \( |q| < 1 \) one has: \( |1 - \frac{q^{2p}}{1 + q^{2p}}| \leq \alpha. \) As a result, \( |a_n| \leq c_n \), where \( c_n \)-coefficients are defined by the same recursion relations (20) with \( q^{2n} \) in place of \( q^n \) and \( \alpha \) being replaced by the \( \alpha \) for all \( n \). But the series \( \sum_{n=1}^{\infty} c_n z^{2n-1} \) is proportional to a scaled tangent function of \( z \). Hence, our series is majorized by

\[
|f(z, q)| \leq \sqrt{|a_1|} \tan \sqrt{|a_1|\alpha} |z|
\]

which asserts the statement of the lemma. At \( q = 0 \) this gives an exact answer for \( R_q \). Using analogous method one can prove convergence of the series (17) near \( z = 0 \) as well.

Solutions of (14) for \( |q| > 1 \) can be obtained from those for \( |q| < 1 \) by the following relation: \( f(z, q^{-1}) = iqf(-iqz, q) \); therefore it is sufficient to consider \( q \) from the unit disk: \( |q| \leq 1. \)

**Theorem.** For any \( q \) such that \( |q| < 1, \) or \( q^{2k+1} = 1, \) \( k = 0, 1, 2, \ldots \) in some neighbourhood of \( z = 0 \) there exists the unique solution \( f(z, q) \) of (14) satisfying the initial condition (15). When \( q \) is a primitive root of unity of even degree, \( q^{2k} = 1, \) existence of analytic solutions depends on the value of \( \alpha_0 \); depending on \( k \) the solution satisfying properly chosen initial condition may be non-unique.

**Proof.** Consider separately the following cases:

1. Suppose that \( |q| < 1. \) Then, by the given \( f(0, q) \) one can determine uniquely \( f'(0, q), f''(0, q), \) etc by taking successive derivatives of (14). Therefore the formal series (17) is defined uniquely, and by the lemma it converges near the \( z = 0 \) point. So, solution exists and it is unique.

2. Let \( q \) be a primitive root of unity of odd degree: \( q^{2k+1} = 1. \) In this case recursion relations (18), (20) define uniquely series coefficients. Their convergence in some neighborhood of \( z = 0 \) follows from the lemma because the lower bound for \( R_q \) can be estimated by replacing \( \alpha \rightarrow \alpha_k \), where \( \alpha_k \) is defined by the inequality \( |1 - q^{2k}| \leq \alpha_k \) for any natural \( p \) and \( q = \exp \frac{2\pi i}{2k+1}. \) Since \( q^a \neq -1 \) such finite \( \alpha_k \) does exist but its value depends on \( k. \)

\(^6\)At least for \( |q| < 1, \) as it will be seen.

\(^7\)For real \( q, 0 < q < 1, \) this statement follows from a general theorem on existence and uniqueness of solutions of differential equations with deviating argument of the neutral type proved by G.A. Kamenskii [3].
3. Let $q$ be a primitive root of unity of even degree: $q^{2k} = 1$. Now the left hand side of the recursion formula \([18]\) vanishes at $n + 2 = k(2l + 1)$, $l = 0, 1, \ldots$, but the right hand side does so only for special values of the parameter $\tilde{a}_0$. For these specific initial conditions the coefficients $\tilde{a}_{k(2l+1)-1}$ of the series \([17]\) can be defined in a non-unique way (i.e. some, or all of them may take arbitrary values), any other choice of $\tilde{a}_0$ leads to the mismatch in recursion relations for $\tilde{a}_n$ and therefore \([14]\) has no solutions analytical at $z = 0$. For example, when $q = -1$, $\mu \neq 0$, the analytic solution is unique and exists only for $\tilde{a}_0 = 0$ (it is $f(z, -1) = \mu z/2$). If $\mu = 0$ then there is no restriction on $\tilde{a}_0$ and any even function $f(z) = f(-z)$ is a solution; the recursion formula \([15]\) gives in this case an ambiguity in each $\tilde{a}_{2n}$. When $q^2 = 1$, an analytic solution exists iff $2\tilde{a}_0^2 = \mu$. Then, the derivative $f'(0, q)$ is not determined by the equation \([14]\) and it can take any value. In particular, one has $f(z, \pm i)_{\tilde{a}_0=\mu=0} = \tilde{a}_1z + \tilde{a}_5z^5 + \tilde{a}_9z^9 + \ldots$, where $\tilde{a}_1, \tilde{a}_5, \ldots$ are arbitrary coefficients such that the series converges. In section 5 below we give an exact dependence of the solutions on an arbitrary function for this case. Similarly, when $q^3 = -1$ one should put $\tilde{a}_0 = 0$, or $\tilde{a}_0^2 = \frac{\mu}{1-q^2}$ and ambiguous coefficients are $\tilde{a}_2, \tilde{a}_8, \tilde{a}_{14}, \ldots$.

Let us make two remarks. First, for the general system of equations \([2], [3]\) with $N = 2n$ the uniqueness of its solutions may be violated if $I_0 = 2 \sum_{i=1}^N f_i$ = 0 (see \([12]\)). This condition is obviously satisfied in the self-similar case when $q^N = 1$, but there are also other situations exhibiting non-uniqueness. Second, analysis of the existence of analytical solutions $f(z, q)$ when $q = e^{2\pi i \phi}$, $\phi$ – an irrational number, is essentially a number theory problem. The nature of irrationality of $\phi$ plays a crucial role in the determination of the asymptotic growth of the series coefficients, in this respect our problem is close to the question on the convergence of $q$-hypergeometric series at such $q$.

Since at $|q| < 1$ the function $f(z, q)$ is analytic in the neighbourhood of $z = 0$, one can in principle construct $f(z, q)$ on the whole complex plane as follows. Let us rewrite equation \([14]\) in the form $f'(z) + f^2(z) = v(qz)$, where $v(z) = q^2(f^2(z) - f'(z)) + \mu$, and fix some number $R$, $0 < R < R_q$. In the coordinate region $R \leq |z| \leq R/|q|$ this is just the Riccati equation because $v(qz)$ is the fixed function determined by the values $f(z, q)$ at $|z| \leq R$. The general solution of this equation is: $f(z, q) = d/dz \ln (w_1 + cw_2)$, where $w_{1,2}$ are independent solutions of the auxiliary Schrödinger equation $-w''(z) + v(qz)w(z) = 0$. The constant $c$ is fixed by some boundary condition. Continuing this procedure iteratively in the rings $R/|q|^n \leq |z| \leq R/|q|^{n+1}$, we recover $f(z, q)$ on the open complex $z$-plane. Since all singularities of $f(z, q)$ appear from zeros and singularities of the solutions of a linear differential equation, one can find their qualitative structure.

**Proposition 1.** The function $f(z, q)$ is meromorphic in the open complex $z$-plane for any fixed $q$, $|q| < 1$.

**Proof.** Because $v(z)$ is analytic in the disc $|z| \leq R$, all permitted singularities of $f(z, q)$ in the first ring $R \leq |z| \leq R/|q|$ are simple poles with the residues $r_1 = 1$. These poles originate from simple zeros, $w(z_p) = 0$, of the wave function $w(z)$ fixed by the boundary condition $w'/w|_{z=R} = c_0$. In the ring $R \leq |z| \leq R/|q|$ the function $v(z)$ may contain double poles at the points $z = z_p/|q|$ with the residues equal to $r_1(r_1 + 1) = 2$. There may be also accompanying simple poles, but they are irrelevant. In the vicinity of these singularities solutions of the auxiliary Schrödinger equation have the form $w(z) \propto (z - z_p/|q|)^r$ where $r$ is found from the indicial equation $r(r - 1) = r_1(r_1 + 1)$. 
This equation has two solutions: \( r = -r_1 \) and \( r = r_1 + 1 \) corresponding to linearly independent functions \( w_{1,2} \). The boundary condition \( f(R/|q|, q) = c_1 \) fixes their relevant combination. Since \( w_{1,2} \) may have only singularities (or zeros) of the form \( (z - z_p/q)^r \) and simple zeros, we conclude that the function \( f(z, q) \) in the second ring can have only pole-type singularities with the integer residues \( r_2 = r \), or 1. Continuing this procedure iteratively from one ring to another we prove the assertion. It is convenient to call the poles with the residues equal to +1, which arise due to the simple zeros of \( w(z) \) and are located at \( z = z_p \), as the primary ones. The poles arising from them as the descendents in the points \( z = z_p/q^n \) with the residues equal to one of the possible values of \( r \in Z, r \neq 1 \), then may be called as the secondary ones. Some of the series of these secondary poles may be truncated due to the particular sequence of the residues: \( 1, \ldots, n, -n, \ldots, -1, 0 \) \( \square \)

The “method of steps” described above does not work when \( |q| = 1 \).

When \( |q| \ll 1 \) it is possible to construct the solution of (14) by expansion of \( f(z, q) \) into a perturbation series over \( q \). Plugging \( f(z, q) = \sum_{i=0}^{\infty} h_i(z)q^{2i}, f(0, q) = 0, \) into (14) and solving the resulting differential equations for \( h_i(z) \), one obtains the function \( f \). For the first non-vanishing correction we have (\( \mu = 1 \)):

\[
\begin{align*}
  f(z, 0) &= h_0(z) = \tanh z, \\
  h_1'(z) + 2h_1(z)\tanh z + 1 &= 0, \quad h_1(0) = 0, \\
  f(z, q) &= \tanh z - q^2z + \frac{1}{2}\sinh 2z \left( \frac{2}{\cosh^2 z} \right) + \ldots .
\end{align*}
\]

As it was shown above, the properties of the self-similar potentials differ for \( q \) being a root of unity of odd and even degree. We study below the first nontrivial cases, \( q^3 = 1 \) and \( q^4 = 1 \), in detail.

4. Exact Solution for \( q^3 = 1 \)

Let us first derive a general solution of the dressing chain at \( N = 3 \)-periodic closure, which evidently contains as a particular case the \( q^3 = 1 \) self-similar solution. System (2) consisting of three equations has two invariants of motion generated by \( TrA = -\lambda I_0 - I_1 \):

\[
\begin{align*}
  I_0 &= 2(f_0(x) + f_1(x) + f_2(x)), \quad (21) \\
  I_1 &= \frac{1}{3} \sum_{i=1}^{3} f_i^3 - \sum_{i \neq j} \lambda_i f_j - \frac{1}{24} f_0^3. \quad (22)
\end{align*}
\]

Deriving explicitly the matrix \( A \) one finds \( \beta(x) = \frac{1}{2}I_0 f_1 + f_0 f_2 + \lambda_1 - \lambda \equiv \gamma(x) - \lambda \). Substituting \( \beta(x) \) in this form into (14) and setting the spectral parameter \( \lambda \) equal to \( \gamma(x) \) we obtain:

\[
\frac{1}{4}(\gamma')^2 = (\gamma - E_0)(\gamma - E_1)(\gamma - E_2). \quad (23)
\]

This is the differential equation for a Weierstrass \( \wp \)-function (15):

\[
\begin{align*}
  (\wp')^2 &= 4\wp^3 - g_2\wp - g_3, \quad (24) \\
  \wp(x + 2\omega) &= \wp(x + 2\omega') = \wp(x), \quad \wp(x) = \frac{1}{x^2} + \frac{g_2}{20}x^2 + \frac{g_3}{28}x^4 + \ldots ,
\end{align*}
\]

7
where ω and ω’ denote complex semiperiods.

The expression for the potential \(u_0\) is obtained by setting equal to zero an expression in front of \(\lambda^2\) in (10):

\[
u_0(x) = -2\gamma(x) + \sum_{i=0}^{2} E_i = 2\varphi(x + x_0) + \frac{1}{2} \sum_{i=0}^{2} E_i.
\]

(25)

In the self-similar case one has \(I_0 = 0\) independently on the value of \(\tilde{a}_0 = f_0(0)\). Without loss of generality we can set \(\tilde{a}_0 = 0\), which fixes \(I_1 = 0\), i.e. \(Tr A = 0\). Constants \(E_i\) are just equal to \(\lambda_i\): \(E_i = \lambda_i\). Setting \(\lambda_j = \frac{\mu}{q^2 - 1} q^{2j}\), and choosing \(\mu = q - 1\) we get real \(u_0(x)\) for real \(x\):

\[
u_0(x) = f_0^2 - f_0' + \lambda_0 = 2\varphi(x + \omega_2), \quad u_0(0) = -2\lambda_1 = 2,
\]

(26)

where \(\varphi(x)\) is the equianharmonic Weierstrass function and \(\omega_2\) is the corresponding real semiperiod, \(\omega_2 = \omega + \omega'\). This \(\varphi\)-function is defined by the conditions \(g_2 = 0, g_3 = 4\) and it has definite ratio of semiperiods leading to very simple transformation rules under the scaling of its argument by \(q\):

\[\omega' = e^{2\pi i/3} \omega, \quad \Rightarrow \varphi(qx) = q\varphi(x).
\]

(27)

The \(f_i(x)\) can be determined by solving either the differential equation (26), or the system of three algebraic equations (21), (22), and (25):

\[
f_i(x) = -\frac{\varphi'(x + \omega_{i-1})}{2(\varphi(x + \omega_{i-1}) + \lambda_i)},
\]

(28)

where we put \(\omega_1 \equiv q\omega_2, \omega_3 \equiv q^2\omega_2, \omega_{i+3} \equiv \omega_i\).

The potential \(u_0(x)\) is periodic with minima at the points \(x_l = 2l\omega_2\) and double poles at the points \(x_l = (2l+1)\omega_2, l \in \mathbb{Z}\). The other two potentials are: \(u_1(x) = 2\varphi(x + \omega_3), u_2(x) = 2\varphi(x + \omega_1)\). Note that all \(f_i\) are complex and only \(u_0\) is real. In the Appendix we present exact solution of the spectral problem for \(u_0\) defined by the zero boundary conditions at singular points.

5. The Case \(q^4 = 1\)

When \(q^4 = 1, q = \pm i\), the general solution of the equation (14) looks as follows:

\[2f(z) = g(z) + \frac{\mu}{g(z)} - \frac{g'(z)}{g(z)},\]

(29)

where \(g(z)\) is an arbitrary function subjected to simple constraints:

\[g(qz)g(z) = \mu q, \quad g(0) = (1 + q) f(0).
\]

(30)

The conditions (30) for \(g(z)\) are compatible when \(f(0) = \pm \sqrt{\mu q}/(1 + q)\), i.e. a solution exists only for a definite value of \(f(0)\). In particular, there is no function \(g(z)\) such that \(g(0) = 0\) when \(\mu \neq 0\).

The family of functions \(g(z)\) which satisfy the functional equation (30) is rather rich. We consider here two of the possible representations. The first one is:

\[g(z) = \pm \sqrt{\mu q} \exp\{F(z^2)\},
\]

(31)
where \( F(y) \) is an arbitrary odd function of its argument \((y \equiv z^2)\). Corresponding potentials in the Schrödinger equation (3) are real when \( \sqrt{\mu q} \) and \( F \) are real:

\[
u(z) = \frac{\partial F(y)}{\partial y} + y \left( \frac{\partial F(y)}{\partial y} \right)^2 + 2y \frac{\partial^2 F}{\partial y^2} + \frac{\mu q}{2} \sinh 2F(y) \mp 2\sqrt{\mu q y} \frac{\partial F}{\partial y} e^{F(y)}.
\] (32)

When \( F(y) \to +\infty \) at \( z \to \pm\infty \) and \( \mu \neq 0 \), potentials (32) grow exponentially and, hence, have a purely discrete spectrum. In the limiting case \( \mu \to 0 \) the choice \( F(y) \equiv y \) yields the potential of a harmonic oscillator.

The second solution is related to the (pseudo)lemniscatic Weierstrass function \([15]\) which corresponds to the invariants \( g_3 = 0 \), \( g_2 = \pm 1 \). Explicitly,

\[
f(z) = -\frac{1}{2} \frac{\varphi'(z + \frac{q-1}{2} \zeta) - \varphi'(\zeta)}{\varphi(z + \frac{q-1}{2} \zeta) - \varphi(\zeta)}, \quad \mu = 2\varphi(\zeta),
\] (33)

\[
\varphi'^2 = 4\varphi^3 \pm \varphi, \quad \omega' = i\omega, \quad \varphi(\pm i z) = -\varphi(z),
\] (34)

\[
u(z) = 2\varphi(z + z_0), \quad z_0 = \frac{q-1}{2} \zeta.
\]

Taking \( z, z_0 \) to be real we obtain again the simplest Lamé equation with the real potential. The corresponding spectral problem is exactly solvable (see Appendix).

Note that for a general \( \varphi \)-function and arbitrary \( q \) one has \( \varphi(qz; q\omega, q\omega') = q^{-2} \varphi(z; \omega, \omega') \). The \( \varphi \)-function constructed by the periods \( q\omega, q\omega' \) is equal to that built up by \( \omega, \omega' \) if the lattices of periods \( \Gamma = \{\omega, \omega'\} \) and \( \Gamma' = \{q\omega, q\omega'\} \) are equivalent. But it is known that for two-dimensional lattices the only possible groups of rotations preserving the lattice are \( C_n \) with \( n = 1, 2, 3, 4, 6 \). So, the transformation properties (37) and (38) of the Weierstrass \( \varphi \)-function obtained for the cases \( q^3 = 1 \) and \( q^4 = 1 \) can take place in such a form only for one more non-trivial case when \( q^6 = 1 \). It is possible to extend the analysis to the higher roots of unity when the potentials are given by (12), using the appropriate transformation properties of the Riemann \( \Theta \)-function.

6. Realization of the \( q \)-Oscillator Algebra

Group-theoretical content of the self-similar potentials discussed in the previous sections is described by the \( q \)-deformed Heisenberg-Weyl, or \( q \)-oscillator algebra [3]:

\[
a^- a^+ - q^2 a^+ a^- = \mu, \quad [a^\pm, \mu] = 0,
\] (35)

where \( a^\pm \) are \( q \)-analogs of the creation and annihilation operators. The explicit realization of (35) utilizes the scaling operator \( T_q \):

\[
T_q f(z) = f(qz), \quad T_q \frac{d}{dz} = \frac{1}{q} \frac{d}{dz} T_q,
\] (36)

\[
T_q T_r = T_{qr}, \quad T_q^{-1} = T_{q^{-1}}, \quad T_1 = 1,
\]

and looks as follows [3]:

\[
a^+ = \sqrt{q}(-\frac{d}{dz} + f(z, q)) T_q, \quad a^- = \frac{1}{\sqrt{q}} T_q^{-1}(\frac{d}{dz} + f(z, q)),
\] (37)
where \( f(z, q) \) is a solution for the equation (14) at \( x = z/q \). For simplicity we keep in this section \( \mu > 0 \).

When \( q^2 \to 1 \) the relations (33) are formally reduced to the standard bosonic oscillator commutation relations. As it was shown above, our system has a well-defined limit \( q \to 1 \) for arbitrary initial condition (13); however, the \( q \to -1 \) limit exists only when \( f(0, q) = 0 \). Although in both cases we get a harmonic oscillator potential, the second case corresponds to the non-standard realization of the Heisenberg-Weyl algebra:

\[
\begin{align*}
b^+ &= \pm i \left( -\frac{d}{dz} + \frac{1}{2} \mu z \right) P, \\
b^- &= \mp i P \left( \frac{d}{dz} + \frac{1}{2} \mu z \right), \\
[b^-, b^+] &= \mu,
\end{align*}
\]

(38)

where \( P \) is the parity operator, \( Pf(z) = f(-z) \), \( P^2 = 1 \). This observation displays the ability of the \( q \)-deformation procedure to connect continuously different realizations of the original undeformed algebra.

Consider the limit \( q \to 0 \). Then, one has formally \( a^- a^+ = \mu \). This algebra admits large variety of realizations, e.g. \( a^- \) may be a \( l \times m \) rectangular matrix. In our case this limit is singular; however, the relation \( a^- a^+ - q^2 a^- a^- = \mu \) still may be meaningful. In particular, if we substitute \( z = qx \) into (37), then \( \lim_{q \to 0} q^2 a^+ a^- = -d^2/dx^2 \neq 0 \) and \( \lim_{q \to 0} a^- a^+ = \mu - d^2/dx^2 \) (if \( z \) is kept finite, then \( q \to 0 \) corresponds to the limit \( x \to \infty \) which is not analyzed in this paper). Although in this case the self-similar potential is reduced to the known exactly-solvable one, we exclude the point \( q = 0 \) from the further discussion because operators (37) are not well-suited for the algebraic treatment of corresponding systems.

In the \( q \)-deformed case the notion of number operator is not universal. In order to show this, let us consider the following combination of the ladder operators \( a^\pm \):

\[
L = a^+ a^- - \nu
\]

(39)

\[
= -\frac{d^2}{dz^2} + f^2(z) - f'(z) - \nu, \quad \nu = \frac{\mu}{1 - q^2},
\]

where we assume that \( q^2 \neq 1 \). Operator \( L \) satisfies the relations

\[
La^\pm = q^\pm 2 a^\pm L,
\]

(40)

which form a \( q \)-analog of the spectrum generating algebra of a harmonic oscillator problem, \( [N, b^\pm] = \pm b^\pm, N = b^+ b^- \). However, equations (40) do not imply the existence of a well-defined number operator \( N \) with the properties \( [N, a^\pm] = \pm a^\pm \). Indeed, the algebra (33), (40) has finite-dimensional representations for which any power of \( a^\pm \) does not vanish (see below). The latter means that the spectrum of \( N \) would be unbounded neither from below nor from above, which in turn would contradict the finite-dimensionality of these cyclic representations. Note that usually the relations (40) are used for the definition of the formal operator \( L \) itself. Sometimes the \( q \)-oscillator algebra is defined as a set of identities involving \( a^\pm, \mu, L \), and the inverse \( L^{-1} \) as well. We do not assume invertibility of \( L \), and, in a sense, utilize the minimal version of the \( q \)-oscillator algebra (33), when the operator \( L \) is expressed through \( a^\pm \) as it is given in (39).

Let us discuss briefly the representation theory of (33), (40) paying most attention to the unitarity and finite-dimensionality of modules. Denote by \( \psi^{(r)}_\lambda \) the eigenstates of the abstract operator \( L \),

\[
L \psi^{(r)}_\lambda = \lambda \psi^{(r)}_\lambda, \quad r = 1, \ldots, d,
\]

(41)
where $\lambda$ is some (complex) number. On the algebraic level nothing can be said about the degree of degeneracy $d$. For our explicit realization \([41]\) is a standard Schrödinger equation which has two independent solutions, i.e. $d = 2$. We suppose that the states $\psi^{(r)}_\lambda$ are uniquely fixed by some (boundary) conditions for all $\lambda$. Relations \([4\text{I}]\) allow to write

$$a^+ \psi^{(r)}_\lambda = \alpha^+_r(\lambda q^2) \psi^{(s)}_{\lambda q^2}, \quad a^- \psi^{(r)}_\lambda = \alpha^-_r(\lambda) \psi^{(s)}_{\lambda q^{-2}}. \quad (42)$$

Substituting this into \([35]\) and using definition \([39]\) we deduce

$$\alpha^\pm_r(\lambda) = \alpha^+_r(\lambda) M^\pm_\lambda(\lambda), \quad \alpha^-_r(\lambda) = \nu + \lambda, \quad (43)$$

where $M^\pm_\lambda$ is some non-degenerate matrix. The exact form of $M^\pm_\lambda$ and $\alpha^\pm_r(\lambda)$ depends on the definition of the basis vectors $\psi^{(r)}_\lambda$ and can not be found algebraically.

**Proposition 2.** There are only five types of essentially different from each other unitary modules of the algebra \([33]\) $(q \neq 0)$ arising at: 1. $\lambda < 0$, $0 < q^2 \leq 1$; 2. $\lambda > 0$, $0 < q^2 < 1$; 3. $\lambda = 0$, $q \neq \pm 1$; 4. $\lambda \neq 0$, $-1 < q^2 < 0$; 5. $\lambda \neq 0$, $q^2 = -1$.

**Proof.** We call the module unitary when the operators $a^\pm$ are hermitian conjugates of each other in the corresponding basis of states. The hermiticity conditions are ensured by $\lambda$, $q^2$ being real and the choice $\alpha^+_r(\lambda) = (\alpha^-_r(\lambda))^* = \sqrt{\nu + \lambda}$, $M^\dagger = M^{-1}$. Let us consider first the interval $0 < q^2 \leq 1$. For negative eigenvalues $\lambda$ the representation should be truncated from below; otherwise $\nu + \lambda$ will start to be negative. So, we get the highest-weight infinite-dimensional representation:

$$L|j\rangle = -\nu q^{2j}|j\rangle, \quad a^+|j\rangle = \sqrt{\nu(1 - q^{2(j+1)})}|j + 1\rangle, \quad a^-|l\rangle = \sqrt{\nu(1 - q^{2l})}|l - 1\rangle, \quad a^-|0\rangle = 0, \quad (44)$$

which is generated by the vacuum state $|0\rangle$ corresponding to the particularly chosen $\psi^{(r)}_\nu$. Note that we absorbed the matrix $M_{rs}$ \([13]\) into the definition of states $|j\rangle$. In our case vacuum state is unique because it is defined by the first-order differential equation. The eigenvalues $\lambda_j = -\nu q^{2j}$ provide the physical discrete spectrum of the infinite-soliton system described by our self-similar potential at real $z$, $0 < q^2 < 1$, $\mu > 0$ \([2,3]\). The module \([14]\) is well-defined when $q^2 \to 1$.

For positive $\lambda$ we get a non-highest-weight infinite-dimensional unitary module

$$L|j\rangle_\lambda = \lambda q^{2j}|j\rangle_\lambda, \quad j \in \mathbb{Z}$$

$$a^+|j\rangle_\lambda = \sqrt{\nu + \lambda q^{2(j+1)}}|j + 1\rangle_\lambda$$

$$a^-|j\rangle_\lambda = \sqrt{\nu + \lambda q^{2j}}|j - 1\rangle_\lambda. \quad (45)$$

Here we assume that the module is generated by some particularly chosen eigenstate $|0\rangle_\lambda$ of the Hamiltonian $L$ with eigenvalue $\lambda$, and we absorbed again the matrix $M_{rs}$ into the definition of states $|j\rangle_\lambda$. Note that in this case $a^+$ lowers the energy. For our specific realization of the $q$-oscillator algebra wave functions $|j\rangle_\lambda$ form a particular subset of the scattering states of the infinite-solitonic potential. The limit $q^2 \to 1$ is not defined for \([43]\).
A peculiar situation is described by the subspace of states corresponding to zero eigenvalue of $L$. In this case $a^+$ and $a^-$ represent the integrals of motion, $[a^+, a^-] = [L, a^\pm] = 0$, and they can be diagonalized simultaneously with $L$:

$$L \psi^{(r)} = 0, \quad a^\pm \psi^{(r)} = \sqrt{\nu} e^{\pm i \theta} \psi^{(r)}.$$  \hspace{1cm} (46)

This is a $d$-dimensional (for any real $q^2$, $q \neq \pm 1$) reducible representations. The irreducible one-dimensional components are the simplest cyclic representations for which the number operator can not be defined. In our case $d = 2$ and in principle all angles $\theta_i$ may be found explicitly. The wave functions $\psi^{(r)}$ provide the simplest coherent states of the $q$-oscillator algebra.

Finally, when $-1 \leq q^2 < 0$ we have the following possibilities. At $|q| < 1$ the infinite-dimensional representation with the defining relations as in (44) arises, but now the successive eigenvalues of $L$ have different signs. At $q = \pm i$ we have the following finite-dimensional representation:

$$L|0\rangle_\lambda = \lambda|0\rangle_\lambda, \quad L|1\rangle_\lambda = -\lambda|1\rangle_\lambda,$$

$$a^+|0\rangle_\lambda = \sqrt{\nu - \lambda}|1\rangle_\lambda, \quad a^+|1\rangle_\lambda = \sqrt{\nu + \lambda}|0\rangle_\lambda,$$

$$a^-|0\rangle_\lambda = \sqrt{\nu + \lambda}|0\rangle_\lambda, \quad a^-|1\rangle_\lambda = \sqrt{\nu - \lambda}|1\rangle_\lambda,$$

where $\lambda$ belongs to the interval $[-\nu, \nu]$. In this case the $q$-oscillator algebra is reduced to the following superalgebra:

$$\{a^-, a^+\} = \mu, \quad \{a^\pm, a^\mp\} = \mu \pm, \quad [\mu, a^\pm] = [\mu \pm, a^\mp] = \mu = 0.$$  \hspace{1cm} (48)

The operators $\mu_\pm$ may be set equal to zero, which corresponds to the boundary values of the parameter $\lambda$ in (47), $\lambda = \pm \nu$, and this gives the standard fermionic oscillator algebra. □

Discussion of the physical significance of the states (46), and analysis of the structure of modules provided by the self-similar potentials at $-1 \leq q^2 < 0$ lies beyond the scope of the present work.

It is easy to see that the representations can be finite-dimensional iff $\lambda(q^n - 1) = 0$, $q \neq \pm 1$. Indeed, the ordinary bosonic oscillator algebra arising at $q = \pm 1$ does not have finite-dimensional representations; the possibility $q = 0$ was already excluded due to the exoticity. The case $\lambda = 0$ was considered above and it corresponds to the $d$-dimensional, in general reducible, representation. At $\lambda \neq 0$ all modules are constructed from series of states $\psi^{(r)}_{\lambda q^2 j}$ for some range of $j$. Let $q$ be not a root of unity. Then the set $\{\psi^{(r)}_{\lambda q^2 j}\}$ can be finite-dimensional iff it is truncated from below and above due to the zeros of $a^\pm_r(\lambda)$. The equation $a^- \psi^{(r)}_\lambda = 0$ holds at $\lambda = -\nu$ and $a^+ \psi^{(r)}_\lambda = 0$ at $\lambda' = -\nu q^{-2}$. Evidently, these states belong to two different irreducible infinite-dimensional highest-weight representations. We thus conclude that $q^n = 1$, $q^2 \neq 1$ is a necessary condition for the existence of finite-dimensional representations at $\lambda \neq 0$.

Finite-dimensionality of the modules at $q^n = 1$ ($n$ is the lowest number satisfying this identity) emerges due to the existence of the non-trivial central elements of the algebra:

$$C^\pm = (\pm i a^\pm)^[n], \quad [C^\pm, a^\mp] = [C^\pm, L] = 0,$$  \hspace{1cm} (49)

where $[n] = n$ for odd $n$ and $[n] = n/2$ for even $n$. Due to the definition of $L$ the states $\psi_{\lambda q^2 j}$, $j = 0, \ldots, [n]$ form the $[n]$-dimensional irreducible module which can not be unitary at
The non-zero $C^\pm$ correspond to the cyclic representations (“cyclic” oscillators) for which the number operator does not exist. By purely algebraic means one finds that $C^\pm$ satisfy the following polynomial algebra:

$$C^\pm C^\mp = \nu^{[n]} - (-L)^{[n]},$$

which is the particular case of algebras, considered in [10, 12] from a different point of view. A subset of states, for which the right hand side of (50) vanishes, form the highest weight finite-dimensional representation. Note that the operator $L^{[n]}$ enters (50) with different signs for $n = 4k$ and $n = 4k + 2$.

Returning to our explicit realization of the $q$-oscillator algebra we note that for odd $n$ the $C^\pm$ are ordinary differential operators of the $n$-th order whose commutativity with the Hamiltonian defines the particular cases of hyperelliptic potentials (12). In general the operators $C^\pm$ are not equal – this depends on the choice of the integrals of motion. E.g., the $q^3 = 1$ system (28) gives $C^+ = C^- \equiv C$,

$$C = \nu \frac{d^3}{dz^3} - \frac{3}{2} \{\wp, \frac{d}{dz}\}, \quad C^2 = L^3 + 1. \quad (51)$$

For the spectral problem considered in the Appendix, the $L^3 + 1$ operator is self-adjoint, but it is easy to see that its square root $C$ does not preserve imposed boundary conditions. One cannot, however, exclude the possibility that for some of the spectral problems the operators $C^\pm$ represent real physical observables.

**Acknowledgements**

The authors are grateful to A. Shabat for a general guidance and participation in this research. We would like also to thank V. Kac and L. Vinet for stimulating discussions.

**Appendix**

Though the spectrum of a particle described by the Schrödinger equation

$$-\psi'' + 2\wp(x)\psi = \lambda\psi, \quad \psi(0) = \psi(2\omega_2) = 0, \quad (52)$$

is known (see, for example [17]), we find it to be useful to show here how it can be derived from the chain (4). This derivation serves as an application of the factorization method to the spectral problems associated with singular potentials given by the (hyper)elliptic functions which appear after the periodic closure of the dressing chain.

It was found that $F(x) = \frac{\pm w - \beta'(x)}{2\beta(x)}$ is the solution of the Riccati equation (11), related to the Schrödinger equation (52). Taking $x = 0$ to be a singular point of $\beta(x)$ and plugging in $\beta(x) = -\wp(x) - \lambda$, we obtain:

$$-F(x) \frac{\psi'}{\psi} = \pm w + \wp(x) \frac{\psi'}{\wp(x) + \lambda}, \quad (53)$$
where \( w^2 = (trA)^2 - 4\det A = -4(\lambda^3 + 1) \) for the \( q^3 = 1 \) system. Integrating (52), one obtains the general solution of (52):

\[
\psi(x) = c_1 \exp \left( -\int \frac{w - \wp'}{2(\wp + \lambda)} \, dt \right) + c_2 \exp \left( \int \frac{w + \wp'}{2(\wp + \lambda)} \, dt \right)
= c_\sqrt{\wp + \lambda} \sin \left( \sqrt{\lambda^3 + 1} \int_0^x \frac{dt}{\wp(t) + \lambda} + \delta \right).
\] (54)

The quantization condition \( \psi(0) = 0 \) yields \( \delta = 0 \). When \( x \to 0, \sqrt{\wp + \lambda} \sim x^{-1} \), but \( \sin \left( \int_0^x \frac{dt}{\wp(t) + \lambda} \right) \sim x^3 \), so that eigenfunction \( \psi(x) \sim x^2 \) as \( x \to 0 \). The condition \( \psi(2\omega_2) = 0 \) yields the transcendental equation for \( \lambda \) which determines the spectrum:

\[
\sqrt{\lambda^3 + 1} \int_0^{\omega_2} \frac{dt}{\wp(t) + \lambda} = \frac{1}{2} \pi n.
\] (55)

The \( \lambda \to \infty \) asymptotics is:

\[
\lambda^{(n)} = \frac{\pi^2 n^2}{4\omega_2^2}, \quad \omega_2 = \frac{1}{2} \int_1^{+\infty} \frac{dx}{\sqrt{x^3 - 1}} = \frac{1}{2} B \left( \frac{1}{2}, \frac{1}{6} \right) \approx 1.21.
\]

This is the spectrum of a particle in the infinitely deep well, as might be expected. For the first three eigenvalues numerical solution gives: \( \lambda^{(1)} = 2.34, \lambda^{(2)} = 5.39, \lambda^{(3)} = 9.18 \). For any \( \lambda^{(n)} \) one can find that \( \psi \sim (x - 2\omega_2)^2 \) when \( x \to 2\omega_2 \).

This method of derivation of spectrum is not restricted to the equianharmonic Weierstrass function (self-similar case); the straightforward generalization of (53) yields the spectrum for arbitrary \( \wp \)-function (24):

\[
\prod_{i=0}^{2} \sqrt{\lambda - E_i} \int_0^{\omega_2} \frac{dx}{\wp(x) + \lambda} = \frac{1}{2} \pi n, \quad \prod_{i=0}^{2} (\lambda - E_i) = \lambda^3 - \frac{1}{4}(g_2\lambda - g_3).
\] (56)

The procedure presented above is not applicable in general for the \( q^4 = 1 \) potentials, because in this case the matrix element \( a_{12} \) of the product of four Darboux transformations vanishes identically on the solutions of (2). However, the spectra of the particular cases corresponding to the (pseudo)lemniscatic Weierstrass function can be found from the formula (56).

The obtained purely discrete spectrum (53) originates from the requirement of the finiteness of wave functions in the singular points of the potential. It is possible to remove singularities from the real axis by shifting the argument of the \( \wp \)-function along the imaginary axis. This leads in general to a complex potential. In particular, one cannot get a real potential without singularities in the case of the equianharmonic and pseudolemniscatic \( \wp \)-functions (when two of the roots \( E_i \) of the polynomial \( 4x^3 - g_2 x - g_3 \) are complex). For the lemniscatic Weierstrass function the shift of the coordinate \( z \to z + \omega' \) leads to the real periodic bounded potential \( -\frac{1}{2} < \wp < 0 \) with one finite gap in the spectrum. So, for the latter case one has two physically different self-adjoint spectral problems.

\[\textsuperscript{8}\text{The choice of constants } E_i \text{ is restricted to the case when the corresponding potential is real and has singularities over the real axis.}\]
References

[1] Infeld, L. and Hull, T.D., *Rev.Mod.Phys.* **23**, 21 (1951).

[2] Shabat, A., *Inverse Prob.* **8**, 303 (1992).

[3] Spiridonov, V., *Phys.Rev.Lett.* **69**, 398 (1992).

[4] Spiridonov, V., in *Proc. of the XIXth ICGTMP*, Salamanca, 29 June - 4 July 1992, to appear; Preprint UdeM-LPN-TH-123, 1992, Comm.Theor.Phys. (Allahabad), to appear.

[5] Kulish, P.P. and Reshetikhin, N.Yu., *Zap. Nauchn. Sem. LOMI* Vol. **101** (Nauka, 1981) p.101 (in Russian); Sklyanin, E.K., *Funct.Anal.Appl.* **17**, 273 (1983); Drinfel’d, V.G., in *Proc. of the ICM*, Berkeley, 1986; Jimbo, M., *Lett.Math.Phys.* **11**, 247 (1986).

[6] Biedenharn, L.C., *J.Phys.* **A22**, L873 (1989); Macfarlane, A.J., *J.Phys.* **A22**, 4581 (1989); Hayashi, T., *Comm.Math.Phys.* **127**, 129 (1990).

[7] Exton, H., *q-Hypergeometric Functions and Applications*, Ellis Horwood, Chichester, 1983; Atakishiyev, N.M. and Suslov, S.K., in *Progress in Approximation Theory*, A.A.Gonchar and E.B.Saff (eds.), Springer, 1991.

[8] Vaksman, L.L. and Soibel’man, Ya.S., *Funct.Anal.Appl.* **22**, 170 (1988); Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M., Saburi, Y., and Ueno, K., *Lett.Math.Phys.* **19**, 187, 195 (1990); Floreanini, R. and Vinet, L., *Lett.Math.Phys.* **22**, 45 (1991); Kalnins, E.G., Manocha, H.L., and Miller, W., *J.Math.Phys.* **33**, 2365 (1992).

[9] Roche, P. and Arnaudon, D., *Lett.Math.Phys.* **17**, 295 (1989); De Concini, C. and Kac, V.G., *Progr.Math.* **92**, 471 (1990).

[10] Shabat, A.B. and Yamilov, R.I., *Leningrad Math.J.* **2**, 377 (1991).

[11] Its, A.R. and Matveev, V.B., *Sov.J.Theor.Math.Phys.* **23**, 343 (1975).

[12] Shabat, A.B. and Veselov, A.P., Preprint of Forschungsinstitut für Mathematik ETH Zürich, 1992, to appear in *Funct.Anal.Appl.*

[13] El’sgol’ts, L.E. and Norkin, S.B., *Introduction to the Theory and Application of Differential Equations with Deviating Arguments*, Academic Press, 1973; Kamenskii, G.A., *Mat.Sbornik* **55(97)**, 363 (1961).

[14] Novokshenov, V.Yu., Clarkson University preprint, INS-203, 1992.

[15] Abramowitz, M. and Stegun, I.A., *Handbook of Mathematical Functions*, Nat. Bur. of Stand., 1966.

[16] Olshanetsky, M.A. and Perelomov, A.M., *Phys.Rept.* **94**, 322 (1983).