An efficient nonnegativity preserving algorithm for multilinear systems with nonsingular $M$-tensors

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Abstract This paper addresses multilinear systems of equations which arise in various applications such as data mining and numerical partial differential equations. When the multilinear system under consideration involves a nonsingular $M$-tensor and a nonnegative right-hand side vector, it may have multiple nonnegative solutions. In this paper, we propose an efficient algorithm which can always preserve the nonnegativity of solutions. Theoretically, we show that the sequence generated by the proposed algorithm is a nonnegative decreasing sequence and converges to a nonnegative solution of the system. Numerical results further support the novelty of the proposed method. Particularly, when some elements of the right-hand side vector are zeros, the proposed algorithm works well while existing state-of-the-art solvers may not produce a nonnegative solution.

Keywords Multilinear systems · Nonsingular $M$-tensor · Nonnegative solution · Newton-type method.

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1 Introduction

Let \( \mathbb{R} \) be the real field. A multidimensional array consisting of \( n^m \) entries is called a real \( m \)-th order \( n \)-dimensional square tensor if we define it by

\[
A = (a_{i_1i_2 \cdots i_m}), \quad a_{i_1i_2 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, i_2, \ldots, i_m \leq n.
\]

In what follows, we denote the set of all real tensors of order \( m \) and dimension \( n \) by \( \mathbb{T}_{m,n} \). Let \( [n] := \{1, 2, \cdots, n\} \). For a tensor \( A = (a_{i_1i_2 \cdots i_m}) \in \mathbb{T}_{m,n} \) and a vector \( x = (x_1, x_2, \cdots, x_n)^\top \in \mathbb{R}^n \), we define \( A x^{m-1} \in \mathbb{R}^n \), whose \( i \)-th element is given by

\[
(A x^{m-1})_i := \sum_{i_2, \ldots, i_m=1}^{n} a_{i_1i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \quad \forall i \in [n],
\]

and \( A x^{m-2} \in \mathbb{R}^{n \times n} \), whose \((i, j)\)-th element is given by

\[
(A x^{m-2})_{ij} := \sum_{i_3, \ldots, i_m=1}^{n} a_{i_1i_2i_3 \cdots i_m} x_{i_3} \cdots x_{i_m}, \quad \forall i, j \in [n].
\]

Based upon the definition of tensor-vector product given in (1.1), the so-called multilinear system (a.k.a., tensor equations) refers to the task of finding a vector \( x \in \mathbb{R}^n \) such that

\[
(A x^{m-1}) = b,
\]

where \( A \in \mathbb{T}_{m,n} \) and \( b \in \mathbb{R}^n \). It has been verified that multilinear systems have many applications in data mining and numerical partial differential equations, e.g., see [3,5,10,12,22] for applications of this topic. Recently, many results in both theory and algorithms for (1.3) have been developed in the recent literature, e.g., [5,8,9,10,11,13,14,17,19,22]. In particular, some works are mainly contributed to a special case of (1.3) where the coefficient tensor is an \( M \)-tensor due to its widespread applications and promising properties (see [4,24]). For example, Ding and Wei [5] first showed that the multilinear system has a unique solution when the coefficient tensor \( A \) of (1.3) is a nonsingular \( M \)-tensor and \( b \) is a positive vector. To find a solution to the underlying system (1.3) with \( M \)-tensors, some state-of-the-art algorithms, including the Jacobi, Gauss-Seidel, Newton methods [5], homotopy method (denoted by ‘HM’) [8], Newton-Gauss-Seidel method [10] and tensor methods [22] for symmetric \( M \)-tensors, tensor splitting methods [11,17], and the locally and quadratically convergent Newton-type algorithm (denoted by ‘QCA’) for asymmetric tensors [9], are proposed. Besides, there are some recent papers devoted to (1.3) with other structured tensors, e.g., see [13,14,19], and extended models of (1.3), e.g., [6,16,23].

Indeed, most of papers mentioned above paid attention to the case where (1.3) has a positive vector \( b \), and some algorithms are developed under the assumption
that the coefficient tensor $A$ is symmetric. However, some real-world problems may not possess such positivity on $b$ and symmetry property on $A$, thereby possibly limiting the applicability of some algorithms. Naturally, the right-hand side vector $b$ being nonnegative (not necessarily positive) may be more general in some real-world problems or more useful for promoting a sparse solution (e.g., $[12, 18]$) than the fully positive case. When $b$ is nonnegative, a good news from [7] is that the system (1.3) with a nonsingular $M$-tensor has a nonnegative solution, but its solution may not be unique. Below, we take an example from [7] to illustrate the nonuniqueness of solutions if $b$ is nonnegative but not positive.

Example 1.1 Let $A = (a_{i_1i_2i_3i_4}) \in T_{4,2}$, where $a_{1111} = 1$, $a_{2222} = 1$, $a_{1112} = -2$ and all other $a_{i_1i_2i_3i_4} = 0$. It has been proved in [7] that $A$ is a nonsingular $M$-tensor. By the definition given in (1.1), we immediately have

$$A x^3 = \begin{pmatrix} x_3^3 - 2x_1^2x_2 \\ x_2^3 \end{pmatrix}.$$ 

Let $b = (0, 1)^\top \geq 0$, then it is easy to see that both $x^* = (0, 1)^\top$ and $x^\star = (2, 1)^\top$ are solutions of the system $A x^3 = b$.

Actually, we observe that, a common feature of the numerical experiments presented in most of the existing tensor equations papers is that, they only consider the case where $b$ is a fully positive vector. Therefore, a natural question is that do these algorithms still work for the case where $b$ is a nonnegative but not positive vector? If not, can we design a new algorithm to handle such a case? Particularly, we can see that both HM [8] and QCA [9] take $b^{1/(m-1)}$ as their starting point so that they have a perfect iterative sequence. So, we are also concerned with another question, that is, whether both HM and QCA are still valid or efficient when taking a nonnegative but not positive $b^{1/(m-1)}$ as their starting point? If not, can we propose a method such that its starting point could be allowed to be a nonnegative vector with zeros?

Taking the aforementioned questions, in this paper, we are interested in the multilinear system (1.3) with a nonsingular (but not necessarily symmetric) $M$-tensor $A$ and a nonnegative (possibly with many zero components) vector $b$. Although it has been proved theoretically that such a system has one nonnegative solution (possibly not unique), there leaves an algorithmic gap. To our knowledge, it seems that no algorithm is designed for the system (1.3) with a nonnegative $b$. Therefore, we aim at introducing an efficient algorithm to solve (1.3) with a nonsingular $M$-tensor $A$ and a nonnegative vector $b$, thereby filling the gap from algorithmic perspective. It is noteworthy that the proposed algorithm is well-defined in the sense that, its iterative sequence is a nonnegative decreasing sequence and
converges to a nonnegative solution of the system. Numerical experiments tell us that the algorithm is efficient and can successfully find a nonnegative, as long as the problem under consideration has a nonnegative solution and an appropriate starting point is taken. However, the state-of-the-art solvers, e.g., HM and QCA, tailored for (1.3) with a positive vector $b$ may not produce a desired nonnegative solution with zeros in some situations.

The rest of this paper is organized as follows. In Section 2, we briefly review some basic definitions and properties about structured tensors. In Section 3, we present a nonnegativity preserving algorithm for solving (1.3), and analyze the convergence of the proposed algorithm. In Section 4, we report our numerical results to show the efficiency and novelty of the proposed algorithm. Finally, we conclude the paper with some remarks in Section 5.

We conclude this section with some notation and terminology. Throughout this paper, we use lowercases $x, y, z, \cdots$ for vectors, capital letters $A, B, C, \cdots$ for matrices and calligraphic letters $A, B, C, \cdots$ for tensors. We denote $\mathbb{R}^n := \{x = (x_1, x_2, \cdots, x_n)^T : x_i \in \mathbb{R}, \forall i \in [n]\}$, $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x \geq 0\}$, where $x \geq 0$ denotes $x_i \geq 0$ for any $i \in [n]$, and $\mathbb{R}_+^{n} := \{x \in \mathbb{R}^n : x > 0\}$ where $x > 0$ means $x_i > 0$ for any $i \in [n]$. Suppose that $\theta$ is a subset of $[n]$, then $x_{\theta} \in \mathbb{R}^{[\theta]}$ represents the corresponding sub-vector of $x \in \mathbb{R}^n$, where $[\theta]$ denotes the cardinality of the set $\theta$, and $A_{\theta \theta} \in \mathbb{R}^{[\theta] \times [\theta]}$ represents the corresponding principal sub-matrix of $A \in \mathbb{R}^{n \times n}$. Besides, $I = (\varrho_{i_1 \cdots i_m}) \in \mathbb{T}_{m,n}$ denotes the identity tensor, where $\varrho_{i_1 \cdots i_m}$ is the Kronecker symbol

$$\varrho_{i_1 \cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\
0, & \text{otherwise,} \end{cases} \quad 1 \leq i_1, \cdots, i_m \leq n,$$

and $A \geq 0$ denotes a nonnegative tensor, which means that all of its entries are nonnegative.

2 Preliminaries

In this section, we briefly recall some definitions and properties about structured matrices and tensors, which will be used throughout this paper.

A tensor $A \in \mathbb{T}_{m,n}$ is called a symmetric tensor if all its elements are invariant under arbitrary permutation of their indices, and it is called a semi-symmetric tensor with respect to the indices $\{i_2, \cdots, i_m\}$ if for any index $i \in [n]$, the $(m-1)$-order $n$-dimensional tensor $A_i := (a_{i i_2 \cdots i_m})_{1 \leq i_2, \cdots, i_m \leq n}$ is symmetric. Thus, from [8] we know that, for any tensor $A = (a_{i_1 i_2 \cdots i_m})$, there always exists a semi-symmetric tensor $\bar{A} = (\bar{a}_{i_1 i_2 \cdots i_m})$, denoted by

$$\bar{a}_{i_1 i_2 \cdots i_m} = \frac{1}{(m-1)!} \sum_{\pi} a_{i_1 \pi(i_2 \cdots i_m)},$$

(2.1)
such that $Ax^{m-1} = Ax^{m-1}$ and $(Ax^{m-1})' = (m-1)Ax^{m-2}$ for any $x \in \mathbb{R}^n$, where the sum is over all the permutations $\pi(i_2 \cdots i_m)$.

As defined independently in Qi [20] and Lim [15], we call $\lambda \in \mathbb{R}$ an eigenvalue and $x \in \mathbb{R}^n \setminus \{0\}$ the corresponding eigenvector of $A$ if they satisfy the following equality:

$$Ax^{m-1} = \lambda x^{[m-1]},$$

where $x^{[m-1]} \in \mathbb{R}^n$ is given by $(x^{[m-1]})_i := (x_i)^{m-1}, i = 1, 2, \ldots, n$. The spectral radius $\rho(A)$ of $A$ is the maximum modulus of its eigenvalues, which is given by

$$\rho(A) = \max \{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Here, we refer the reader to the recent monograph [21] for more details of spectral theory on tensors.

Now we present some definitions and properties about $M$-tensors.

**Definition 2.1 ([4, 24])** A tensor $A \in T_{m,n}$ is called

(i) a $Z$-tensor if all of its off-diagonal entries are non-positive;

(ii) an $M$-tensor if it can be written as $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B)$;

(iii) a nonsingular $M$-tensor if it is an $M$-tensor with $s > \rho(B)$.

Obviously, (nonsingular) $M$-tensor is a generalization of (nonsingular) $M$-matrix.

Below, we recall some properties of nonsingular $M$-matrices, which will be used in the later analysis.

**Theorem 2.1 ([2])** Let $A \in \mathbb{R}^{n \times n}$ be a $Z$-matrix, then the following statements are equivalent:

(i) $A$ is a nonsingular $M$-matrix;

(ii) There exists an $x \in \mathbb{R}_{++}^n$ satisfying $Ax \in \mathbb{R}_{++}^n$;

(iii) $A^{-1}$ exists and $A^{-1}$ is a nonnegative matrix.

Similarly, some properties of nonsingular $M$-tensors are shown below.

**Theorem 2.2 ([4])** Let $A \in T_{m,n}$ be a $Z$-tensor, then the following statements are equivalent:

(i) $A$ is a nonsingular $M$-tensor;

(ii) There exists an $x \in \mathbb{R}_{++}^n$ satisfying $Ax^{m-1} \in \mathbb{R}_{++}^n$;

(iii) All diagonal entries of $A$ are positive and there exists a positive diagonal matrix $D \in \mathbb{R}^{n \times n}$ such that $AD^{m-1}$ is strictly diagonally dominated, where

$$AD^{m-1} = A \times_2 D \times_3 \cdots \times_m D,$$

and

$$(A \times_k D)_{i_1 \cdots i_{k-1} j_k i_{k+1} \cdots i_m} = \sum_{i_k = 1}^n a_{i_1 \cdots i_k x_{i_k j_k}}, \quad \forall k \in [m].$$
Let $\mathcal{A} \in \mathbb{T}_{m,n}$ and $\bar{\mathcal{A}} \in \mathbb{T}_{m,n}$ be the corresponding semi-symmetric tensor of $\mathcal{A}$ which satisfies (2.1). Then, we can obtain the following relationship between these two tensors.

**Lemma 2.1 (8)** Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be a nonsingular $M$-tensor. Then, $\bar{\mathcal{A}}$ is also a nonsingular $M$-tensor.

Based on the above lemma, we can further conclude the following result about the Jacobian of $\bar{\mathcal{A}}^{m-1}$.

**Lemma 2.2** Suppose that $\bar{\mathcal{A}} \in \mathbb{T}_{m,n}$ is a semi-symmetric nonsingular $M$-tensor. Then, for any $\hat{x} \in \mathbb{R}_n^+$ satisfying $\bar{\mathcal{A}} \hat{x}^{m-1} \in \mathbb{R}_n^+$, the Jacobian $(m-1)\bar{\mathcal{A}}\hat{x}^{m-2}$ of $\bar{\mathcal{A}}^{m-1}$ is a nonsingular $M$-matrix.

**Proof** Let $\bar{\mathcal{A}}$ be a semi-symmetric nonsingular $M$-tensor, then, there must exist an $\hat{x} \in \mathbb{R}_n^+$ with $\bar{\mathcal{A}} \hat{x}^{m-1} \in \mathbb{R}_n^+$. Thus the $(i, j)$-th entry of the Jacobian matrix $(m-1)\bar{\mathcal{A}}\hat{x}^{m-2}$ is given by

$$(m-1)\bar{\mathcal{A}}\hat{x}^{m-2} \hat{x} = (m-1) \sum_{i_3, \cdots, i_m = 1}^{n} \bar{a}_{ij,i_3 \cdots i_m} \hat{x}_{i_3} \cdots \hat{x}_{i_m}, \quad \forall i, j \in [n].$$

Since $\bar{\mathcal{A}}$ is a nonsingular $M$-tensor, when $i \neq j$, $\bar{a}_{ij,i_3 \cdots i_m} \leq 0$. Thus, $(m-1)\bar{\mathcal{A}}\hat{x}^{m-2}$ is a Z-matrix. Recall that

$$(m-1)\bar{\mathcal{A}}\hat{x}^{m-2} \cdot \hat{x} = (m-1)\bar{\mathcal{A}}\hat{x}^{m-1} > 0,$$

where we use ‘·’ to represent the matrix-vector product throughout this paper when the matrix is induced by the definition of (1.2). Then, it immediately follows from (2.2) and Theorem 2.1 that $(m-1)\bar{\mathcal{A}}\hat{x}^{m-2}$ is a nonsingular $M$-matrix. \qed

At the end of this section, we recall two important results on (1.3), which guarantee that the solution set of the problem under consideration is nonempty.

**Theorem 2.3 (5)** Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be a nonsingular $M$-tensor and $b \in \mathbb{R}_n^+$. Then, the system $\mathcal{A} x^{m-1} = b$ has a unique positive solution.

**Theorem 2.4 (7)** Let $\mathcal{A} \in \mathbb{T}_{m,n}$ be a nonsingular $M$-tensor and $b \in \mathbb{R}_n^+$. Then, the system $\mathcal{A} x^{m-1} = b$ has a nonnegative solution.

### 3 Algorithm and Convergence Analysis

In this section, we are going to present a Newton-type method, who can always preserve the nonnegativity of the iterative sequence for the system (1.3) with a
nonsingular \( M \)-tensor and a nonnegative right-hand side vector. We will also state that this algorithm is well-defined and converges to a solution of the system.

For notational convenience, we first define \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \) by
\[
F(x) = Ax^{m-1} - b. \tag{3.1}
\]
Then, the system (1.3) is equivalent to \( F(x) = 0 \). Furthermore, we have
\[
F'(x) = (m - 1)\tilde{A}x^{m-2},
\]
where \( \tilde{A} \) is the corresponding semi-symmetric tensor, which satisfies (2.1), of the tensor \( A \). Moreover, when \( x \) and \( A x^{m-1} \) are both positive, from Lemma 2.2, \( F'(x) \) is a nonsingular \( M \)-matrix.

Hereafter, we describe details of the new algorithm for solving the system (1.3) in Algorithm 1.

**Algorithm 1** (An Efficient Nonnegativity Preserving Algorithm for (1.3)).

1: Let \( x^0 \in \mathbb{R}_+^n \) satisfying \( F(x^0) \geq 0 \) be a starting point.
2: while \( \|F(x^k)\| \neq 0 \) do
3: Let \( I_k = \{i : (A(x^k))^{m-1} > b_i\} \) and \( I_k^c = [n] \setminus I_k \).
4: Let \( d^k \in \mathbb{R}^n \), where \( d^k_{I_k} = 0 \) and
\[
d^k_{I_k^c} = -\left[ F'(x^k) \right]^{-1}_{I_k^c I_k} \left[ F(x^k) \right]_{I_k}.
\tag{3.2}
\]
5: Find \( \lambda^k \in (0, 1] \) such that
\[
x^{k+1} := x^k + \lambda^k d^k \geq 0 \quad \text{and} \quad F(x^{k+1}) \geq 0. \tag{3.3}
\]
6: end while

**Remark 3.1** Notice that Algorithm 1 needs an initial point \( x^0 \) satisfying \( x^0 \in \mathbb{R}_+^n \) and \( A(x^0)^{m-1} \geq b \), which can not be guaranteed for an arbitrary \( x^0 \in \mathbb{R}_+^n \). Accordingly, we here introduce a simple method to find a suitable initial point for Algorithm 1. Since we already know that, the nonsingular \( M \)-tensor \( A \) can be written as \( A = sI - B \) where \( B \geq 0 \) and \( s > \rho(B) \), then we can use the fixed point iteration to find a starting point for Algorithm 1. The algorithm for finding initial point is stated in Algorithm 2.

Since the starting point of Algorithm 2 is nonnegative, \( B \) is a nonnegative tensor and \( b_k \) is positive, the sequence \( \{\tilde{x}^k\} \) generated by this algorithm is a positive sequence. Moreover, according to the positive homogeneity of \( A x^{m-1} \) and the termination condition \( A(\tilde{x}^k)^{m-1} > 0 \) of Algorithm 2, we can use the output \( \tilde{x}^k \)
Algorithm 2 (Select initial point for Algorithm 1).

1: Choose a constant vector $\epsilon := (10^{-3}, 10^{-3}, \ldots, 10^{-3})^\top \in \mathbb{R}^n$. Let $b_\epsilon = b + \epsilon$.
2: Let $\bar{x}^0 \in \mathbb{R}_+^n$ be a starting point.
3: Let $J := \{i : (A(\bar{x}^0)^{m-1})_i \leq 0\}$.
4: while $J \neq \emptyset$ do
5: Compute $\bar{x}^{k+1} := \left(\frac{1}{t}B(\bar{x}^k)^{m-1} + \frac{1}{t}b_\epsilon\right)^{\frac{1}{m-1}}$.
6: Update $J = \{i : (A(\bar{x}^{k+1})^{m-1})_i \leq 0\}$.
7: end while

of Algorithm 2 to construct an $x^0$ satisfying $x^0 \in \mathbb{R}^n_+$ and $A(x^0)^{m-1} \geq b$. Then, $x^0$ can be a starting point of Algorithm 1. Empirically, we suggest taking $\bar{x}^0 = tb$ with $t \geq 1$ as an initial point of Algorithm 2 in accordance with our experiments.

Remark 3.2 Notice that Step 4 in Algorithm 1 is indeed a Newton step. It can be easily seen from the definition of the index set $I_k$ that the subproblem (3.2) is always a nonsingular matrix (see Lemma 3.1). If the scale of $I_k$ is relatively small, we can directly compute the inverse of $[F'(x^k)]_{I_kI_k}$ and obtain the accurate solution of (3.2). Otherwise, we can alternatively solve the linear system (3.2) inexactly by well-developed solvers, e.g., (preconditioned or bi-) conjugate gradient method.

Next, we will present a convergence analysis of Algorithm 1, in addition to showing that the proposed algorithm has some promising theoretical properties.

Lemma 3.1 In Algorithm 1, for any $k$, $[F'(x^k)]_{I_kI_k}$ is a nonsingular M-matrix.

Proof It follows from the notation given in (3.1) that

$$[F'(x^k)]_{I_kI_k} = (m-1) \left(\bar{A}(x^k)^{m-2}\right)_{I_kI_k},$$

which is a Z-matrix. From the definition of $I_k$ in Algorithm 1, we have $x^k_{I_k} > 0$ and

$$\left(\bar{A}(x^k)^{m-1}\right)_{I_k} > b_{I_k} \geq 0.$$

In this way, we obtain $x^k_{I_k} > 0$ with

$$[F'(x^k)]_{I_kI_k} \cdot x^k_{I_k} = (m-1) \left(\bar{A}(x^k)^{m-2}\right)_{I_kI_k} \cdot x^k_{I_k} \geq (m-1) \left(\bar{A}(x^k)^{m-1}\right)_{I_k} > 0.$$

Thus, it follows from Theorem 2.1 and Lemma 2.2 that $[F'(x^k)]_{I_kI_k}$ is a nonsingular M-matrix. \qed
Lemma 3.2 Based on the condition of Algorithm 1, at the k-th iteration, there exists a positive number $\lambda^k \in (0, 1]$ such that $x^{k+1} = x^k + \lambda^k d^k \geq 0$ and $F(x^{k+1}) \geq 0$.

Proof First, Lemma 3.1 tells us that $\left[F'(x^k)\right]_{J_k I_k}$ is a nonsingular $M$-matrix, which together with Theorem 2.1 implies that $\left[F'(x^k)\right]^{-1}_{J_k I_k}$ is a nonnegative matrix. Thus, it immediately follows from (3.2) that $d^k_{J_k} < 0$.

Let $A := A(x^k)^{m-2}$ and $y := A(x^k)^{m-1}$ for notational simplicity. By Definition 2.1, since $\left[F'(x^k)\right]_{J_k I_k}$ is a nonsingular $M$-matrix, its diagonal entries are positive and off-diagonal entries are nonnegative. Thus, for any $\lambda \in (0, 1]$, $x^{k+1}_{J_k} = x^k_{J_k} + \lambda d^k_{J_k} = x^k_{J_k} - \frac{\lambda}{m-1} (A_{J_k I_k})^{-1}(y - b)_{I_k}$

$= (A_{J_k I_k})^{-1} \left( A_{I_k I_k} x^k_{I_k} - \frac{\lambda}{m-1} y_{I_k} + \frac{\lambda}{m-1} b_{I_k} \right)$

$\geq (A_{J_k I_k})^{-1} \left( y_{I_k} - \frac{\lambda}{m-1} y_{I_k} + \frac{\lambda}{m-1} b_{I_k} \right)$

$> 0.$

Besides, $d^k_{J_k} = 0$ leads to the truth that $x^{k+1}_{J_k} = x^k_{J_k} \geq 0$. Then, for any number $\lambda \in (0, 1]$, we have $x^{k+1} = x^k + \lambda d^k \geq 0$.

On the other hand, we have $\left(F(x^k)\right)_i = a_{i \cdot \cdot \cdot m}(x^k)^{m-1} + \sum_{(i_2, \ldots, i_m) \neq (i, \ldots, i)} a_{i i_2 \cdot \cdot \cdot i_m} x^k_{i_2} \cdots x^k_{i_m} - b_i.$

Since $A$ is a nonsingular $M$-tensor, all of its diagonal entries are positive and off-diagonal entries are non-negative. Hence, we know that $a_{i \cdot \cdot \cdot m}(x^k)^{m-1} \geq 0$ and $\sum_{(i_2, \ldots, i_m) \neq (i, \ldots, i)} a_{i i_2 \cdot \cdot \cdot i_m} x^k_{i_2} \cdots x^k_{i_m} \leq 0.$

Clearly, when $(F(x^k))_i = 0$, that is $i \in \bar{I}_k$, then $d^k_i = 0$. Hence, $x^{k+1}_i = x^k_i$ and $\left(F(x^{k+1})\right)_i \geq (F(x^k))_i = 0$.

When $(F(x^k))_j > 0$, that is $j \in I_k$, then $x^{k+1}_j > x^k_j$, thus $a_{j \cdot \cdot \cdot j}(x^{k+1})^{m-1}_j < a_{j \cdot \cdot \cdot j}(x^k)^{m-1}_j \quad (3.4)$

and at the same time,

$\sum_{(i_2, \ldots, i_m) \neq (j, \ldots, j)} a_{j i_2 \cdot \cdot \cdot i_m} x^{k+1}_{i_2} \cdots x^{k+1}_{i_m} \geq \sum_{(i_2, \ldots, i_m) \neq (j, \ldots, j)} a_{j i_2 \cdot \cdot \cdot i_m} x^k_{i_2} \cdots x^k_{i_m} \quad (3.5)$

Hence, by adjusting the value of $\lambda \in (0, 1]$, we can guarantee that the sum of the left-hand sides of (3.4) and (3.5) is nonnegative, i.e., we can find a suitable number $\lambda^k \in (0, 1]$ satisfying $x^{k+1} = x^k + \lambda^k d^k \geq 0$ and $F(x^{k+1}) \geq 0$. \hspace{1cm} \Box
From Lemma 3.1 we obtain that, $F'(x^k)I_kI_k$ is a nonsingular $M$-matrix. From Lemma 3.2 we know that, $\lambda^k$ always exists. Thus, these two aspects together result in the fact that Algorithm 1 is well-defined at the $k$-th iteration.

**Theorem 3.1** The sequence \{x^k\} generated by Algorithm 1 is a decreasing sequence: $0 \leq x^{k+1} \leq x^k$ for all $k$, and $x^k \to x^*$ as $k \to \infty$, where $x^* \in \mathbb{R}^n_+$. In particular, if $b$ is positive, the limit point $x^*$ is also positive.

**Proof** From Lemma 3.2 we know that, $d^k$ is non-positive. Since $\lambda^k$ is a positive number, we always have $x^{k+1} = x^k + \lambda^kd^k \leq x^k$. Hence, the sequence \{x^k\} is a decreasing sequence. Moreover, because $x^k \geq 0$ for all $k$, the sequence \{x^k\} is lower bounded, which leads to the fact that, there exists a vector $x^* \in \mathbb{R}^n_+$ such that, as $k \to \infty$, $x^k \to x^*$ and $A(x^*)^m - b \geq 0$.

If $b$ is positive, now we will show that all the entries of $x^*$ are positive, that is $x^* \in \mathbb{R}^n_+$. Assume on the contrary, some entries of $x^*$ is 0. Without loss of generality, let $(x^*)_j = 0$ for some $j \in [n]$. Since $A$ is a nonsingular $M$-tensor, all of its off-diagonal entries are non-positive. Thus,

$$
(A(x^*)^{m-1} - b)_j = \sum_{i_2, \ldots, i_m=1}^n a_{j,i_2,\ldots,i_m}x^*_i x^*_i \cdots x^*_i - b_j < 0,
$$

which is a contradiction. Hence, the limit point $x^*$ is positive. $\square$

**Theorem 3.2** Let $x^*$ be the limit point of \{x^k\} generated by Algorithm 1. Then, $x^*$ is a solution of $F(x) = 0$.

**Proof** Suppose on the contrary, the limit point $x^*$ is not a solution of the equation $F(x) = 0$. Then, there exists at least one index $i \in [n]$ such that $(A(x^*)^{m-1})_i > b_i$. Thus the algorithm will continue to iterate and produce a new iteration point $\hat{x}$, which satisfies the inequality $\hat{x}_i < x^*_i$. This contradicts the fact that $x^*$ is the limit point of the decreasing sequence \{x^k\}. Hence, the limit point $x^*$ is a solution of $F(x) = 0$. $\square$

When $b$ is positive, combining the above two theorems together, we know that, the limit point $x^*$ of the sequence generated by Algorithm 1 is positive as well as the solution of $F(x) = 0$. Hence, the proposed algorithm is also valid to the case where the system (1.3) involves a nonsingular $M$-tensor $A$ and a positive vector $b$. Another remarkable contribution is that, for the system (1.3) equipped with a nonnegative but not positive vector $b$, our proposed algorithm is always globally convergent, however it is unclear, to our knowledge, whether the previous algorithms still have the same convergence property.
4 Numerical Experiments

In this section, we will show the numerical performance of Algorithm 1 (denoted by ‘ENPA’) on the multilinear system (1.3) by writing a code and implementing it in MATLAB. Apart from this, we will also compare our algorithm with the homotopy method (denoted by ‘HM’) proposed by Han in [8], whose code can be downloaded from Han’s homepage\(^1\), and the globally and quadratically convergent algorithm (denoted by ‘QCA’) proposed by He et al. in [9]. All numerical experiments are done in MATLAB R2014a on a laptop computer with Intel(R) Core(TM) CPU i7-8550U @ 1.80GHz and 16 GB memory running Microsoft Windows 10. Throughout, we employ the tensor toolbox [1] to compute tensor-vector products as well as semi-symmetric tensors.

Below, we divide the experiments into two parts. On the one hand, a special case of the problem under consideration that has been studied in the literature is the right-hand side vector \(b\) being fully positive. So, we will report some computational results to see the numerical behaviors of the proposed method for the well-tested problems in [5,8,9]. On the other hand, the main contribution of the paper is a nonnegativity preserving algorithm customized for the multilinear system with a nonnegative but not positive right-hand side \(b\). Correspondingly, we will show that Algorithm 1 is efficient for the case with a nonnegative vector \(b\), while both HM [8] and QCA [9] may not produce a nonnegative solution in some scenarios.

4.1 Case I: Positive \(b\) in (1.3)

In this subsection, we consider the case where the system (1.3) has a fully positive right-hand side vector \(b\).

The problem under experimentation comes from [8,9], where the pair of tensors and vectors \((A,b)\) are generated in a random way as follows. First, we give a random nonnegative tensor \(B = (b_{i_1i_2\cdots i_m}) \in T_{m,n}\), whose entries are uniformly distributed in \((0,1)\). Let \(\omega > 0\) and

\[
s = (1 + \omega) \cdot \max_{i \in [n]} \left( \sum_{i_2, \cdots, i_m = 1}^{n} b_{i_1i_2\cdots i_m} \right).
\]

(4.1)

Thus, we have

\[
\rho(B) \leq \max_{i \in [n]} \left( \sum_{i_2, \cdots, i_m = 1}^{n} b_{i_1i_2\cdots i_m} \right),
\]

\(^{1}\) http://homepages.umflint.edu/~lxhan/software.html
and \( s > \rho(B) \), which together lead to the fact that \( A := sI - B \) is an asymmetric nonsingular \( M \)-tensor. In this paper, we set \( \omega = 0.01 \), which is the same setting as used in [8,9]. Next, we give a random positive vector \( b \in \mathbb{R}^n \), whose entries are uniformly distributed in \((0, 1)\). Then, we obtain a pair of \((A, b)\), where \( A \) is a nonsingular \( M \)-tensor and \( b \) is a positive vector, and the resulting multilinear system has a unique positive solution.

The same as the algorithms proposed in [8] and [9], we solve the scaled system

\[
\tilde{A}x^{m-1} = \tilde{b}
\]

of the original one when comparing the three algorithms, where \( \tilde{A} = \frac{1}{\kappa}A \), \( \tilde{b} = \frac{1}{\kappa}b \) and \( \kappa \) is the largest absolute value among the values of the entries of \( A \) and \( b \). Besides, we also use the same starting point \( x^0 = (1, 1, \cdots, 1)^\top \) and termination condition for the three methods, where the stopping criterion is defined by

\[
\text{ReErr} := \|\tilde{A}(x^k)^{m-1} - \tilde{b}\|_2 \leq \text{Tol}, \tag{4.2}
\]

here \( \text{Tol} > 0 \) is a preset tolerance. For the parameters of QCA, we follow the settings as used in [9], i.e., \( \delta = 0.5, \gamma = 0.8, \sigma = 0.2 \), and \( \bar{t} = 2/(5\gamma) \). Additionally, we employ throughout the MATLAB script ‘bicg’ (i.e., ‘biconjugate gradients method’) to solve the subproblem of QCA in accordance with the results reported in [9]. We notice that Step 5 in Algorithm 1 (see (3.3)) indeed is a line search step. So, we employ the simplest Armijo search procedure to update the next iterate \( x^{k+1} \) by setting an initial \( \lambda^0 = 0.2 \). Additionally, the subproblem (3.2) is a dimensionality reduced linear system, where the coefficient matrix \( [F(x^k)]_{I_kI_k} \) is always a nonsingular \( M \)-matrix. So, we will solve such a subproblem directly by the ‘left matrix divide: \( \backslash \)’, which roughly refers to the multiplication of the inverse of a matrix and a vector.

Since the data of \( A \) and \( b \) is generated randomly, we conduct seven scenarios of \((m, n)\) and 10 groups random data sets for each scenario \((m, n)\). Besides, we investigate the sensitivity of the proposed method to the preset tolerance \( \text{Tol} \), which can implicitly demonstrate its convergence behavior for the problem under consideration. Tables 1 and 2 summarize the results corresponding to \( \text{Tol} = 10^{-6} \) and \( \text{Tol} = 10^{-12} \), respectively, where ‘iter’ denotes the number of iterations, ‘time’ represents the computing time in seconds, and ‘ReErr’ is the residue defined by (4.2).

It can be easily seen from Tables 1 and 2 that, when the right-hand side vector \( b \) is positive, all of these three methods are valid. Taking a close look at the data in the tables, it tells us that the three methods work equally well in terms of taking a short period of time with the acceptable residue when we set \( \text{Tol} = 10^{-6} \) and the dimensionality \( n \) is relatively small (e.g., \( n \leq 50 \)). For the case \((m, n) = (3, 100)\),
Table 1: Numerical results for positive $b$ by setting $\text{Tol} = 10^{-6}$.

| $(m, n)$ | HM iter / time / ReErr | QCA iter / time / ReErr | ENPA iter / time / ReErr |
|---------|------------------------|-------------------------|-------------------------|
| $(3, 50)$ | 9.0 / 0.11 / 6.53 $10^{-8}$ | 7.8 / 0.06 / 2.68 $10^{-7}$ | 47.4 / 0.20 / 9.18 $10^{-7}$ |
| $(3, 100)$ | 9.0 / 0.63 / 1.96 $10^{-7}$ | 9.1 / 0.45 / 2.31 $10^{-7}$ | 36.9 / 1.08 / 9.08 $10^{-7}$ |
| $(4, 10)$ | 9.0 / 0.14 / 3.03 $10^{-7}$ | 6.3 / 0.05 / 3.66 $10^{-7}$ | 48.1 / 0.19 / 8.91 $10^{-7}$ |
| $(4, 50)$ | 9.0 / 0.40 / 5.89 $10^{-7}$ | 9.6 / 0.25 / 1.83 $10^{-7}$ | 22.8 / 0.34 / 9.21 $10^{-7}$ |
| $(5, 10)$ | 9.0 / 0.10 / 2.69 $10^{-7}$ | 8.7 / 0.06 / 1.16 $10^{-7}$ | 30.8 / 0.12 / 9.16 $10^{-7}$ |
| $(5, 20)$ | 9.8 / 0.27 / 4.56 $10^{-8}$ | 9.9 / 0.16 / 3.50 $10^{-7}$ | 19.9 / 0.19 / 9.20 $10^{-7}$ |
| $(6, 10)$ | 9.6 / 0.15 / 3.25 $10^{-7}$ | 8.9 / 0.08 / 4.35 $10^{-7}$ | 17.4 / 0.09 / 9.06 $10^{-7}$ |

we can see that the proposed ENPA takes more iterations and computing time to get an approximate solution. The main cause is that we just simply employ the MATLAB script ‘left matrix divide: \’ to compute the accurate solution of the subproblem (3.2). In fact, some extra numerical experiments tell us that the employment of the MATLAB script ‘pcg’ (i.e., preconditioned conjugate gradient method) to the subproblem (3.2) still work for higher dimension cases. From the data in Table 2, it tells us that both HM [8] and QCA [9] outperform the proposed ENPA when we set a smaller tolerance ‘Tol’. What is more, all results in Tables 1 and 2 potentially imply that the ENPA (i.e., Algorithm 1) has a linear convergence rate (also shown in Fig. 1), which is also one of our future concerns. Briefly speaking, the numerical results demonstrate that the proposed ENPA is still a reliable solver to the system (1.3) with a positive right-hand side vector $b$, but HM [8] and QCA [9] are the better choices when requiring a solution with an extremely high precision.

4.2 Case II: Nonnegative $b$ in (1.3)

We have shown that the ENPA works for the case where the system (1.3) has a positive right-hand side vector $b$. However, ones may be concerned with the
performance of ENPA on the case of solving the system (1.3) with a nonnegative but not positive vector $b$. In this subsection, we will show the efficiency of ENPA for finding nonnegative solutions to (1.3) through experimentation with synthetic data.

Example 4.1 This example is a modified version of Example 1.1 in Introduction. Here, we use the same tensor $\mathcal{A}$ described in Example 1.1. Clearly, (i). when we take the right-hand side vector $b$ as $b = (0, p^3)^\top$, where $p$ is a nonnegative number, the resulting multilinear system has two solutions $(0, p)^\top$ and $(2p, p)^\top$; (ii). when we set the right-hand side vector $b$ as $b = (p^3, 0)^\top$, it then has only one solution $(p, 0)^\top$ to the multilinear system. In our experiments, we consider the aforementioned two scenarios and take $p = 2$.

As shown in Example 4.1, the multilinear system with a nonnegative but not positive right-hand side vector $b$ has an analytic nonnegative solution with zero components. We will use this example to show that our ENPA can successfully find a nonnegative but not positive solution when setting an appropriate initial point, while both HM and QCA may obtain a fully positive solution. Since this problem is extremely simple, we solve the original system without scaling technique and use the similar stopping criterion defined in (4.2) with Tol = $10^{-12}$ for all methods.

For the both scenarios, i.e., $b = (0, 2^3)^\top$ and $b = (2^3, 0)^\top$, we test three different initial points $x^0$ for the three methods, respectively. All results are listed in Tables 3 and 4, where ‘–’ means that the method fails to find a solution because either the subproblem approaches to a singular linear system subproblem or the number of iterations exceeds the preset maximum iteration 2000. The ‘solution’ corresponds to the approximate solution obtained by a method.

Table 3: Numerical results for Example 4.1: (i). $b = (0, 2^3)^\top$.

| Alg. | $x^0 = (0, 20)^\top$ | $x^0 = (20, 0)^\top$ | $x^0 = (20, 20)^\top$ |
|------|----------------------|----------------------|----------------------|
|      | iter / time / solution | iter / time / solution | iter / time / solution |
| HM   | – / – / –           | – / – / –           | 13 / 0.06 / (4.0, 2.0)^\top |
| QCA  | – / – / –           | – / – / –           | 42 / 0.33 / (4.0, 2.0)^\top |
| ENPA | 10 / 0.03 / (0.0, 2.0)^\top | 5 / 0.03 / (4.0, 2.0)^\top | 10 / 0.01 / (4.0, 2.0)^\top |

Notice that both HM [8] and QCA [9] took their starting points as $x^0 = b^{1/(m-1)}$ in their numerical experiments. In fact, such a positive initial point can ensure that their subproblems are nonsingular in the iterative procedure. However, if we take a nonnegative but not positive initial point $x^0$, it seems from Tables 3 and 4 that both HM and QCA are no longer valid, while ENPA still works...
and finds a nonnegative solution with zero components. If we take a fully positive initial point, it can be seen from Table 3 that the three methods find a fully positive solution when the multilinear system has multiple nonnegative solutions including at least one fully positive solution. Interestingly, we can observe from Table 4 that, both HM and ENPA can successfully obtain a nonnegative solution with zeros when taking a fully positive starting point. Thus, we guess empirically that HM is also available to find the nonnegative solution by setting an appropriate positive starting point when the multilinear system has a unique nonnegative solution. However, HM may fail to find a desired nonnegative solution if the system has one fully positive solution. Promisingly, the proposed ENPA works well with different initial points for Example 4.1.

Below, we consider some slightly higher dimensional problems to further illustrate the novelty of the proposed method on preserving nonnegativity of solutions.

**Example 4.2** We generate a sparse 3-rd order 5-dimensional nonsingular $M$-tensor $A$, whose components are $a_{i_{1}i_{2}i_{3}} = 2.2845, (i_{1}, i_{2}, i_{3}) = (1, 2, 3), a_{444} = 2.1074, a_{555} = 1.6873, a_{455} = -0.9121, a_{544} = -0.9884, a_{455} = -0.1842, a_{544} = -0.6628, a_{554} = -0.1040, a_{554} = -0.5400$, and all others are zeros. The right-hand side vector $b = (0.0185, 0.0149, 0, 0, 0)^T$ is generated by $b = A(x^*)^2$, where $x^*$ is a preset solution given by $x^* = (0.0899, 0.0809, 0, 0, 0)^T$.

We can see that the system described in Example 4.2 has at least one nonnegative solution $x^* = (0.0899, 0.0809, 0, 0, 0)^T$. So, in the experiments, we test two starting points $x^0 = 2b$ and $x^0 = (1, \cdots, 1)^T$ and report numerical results in Table 5, where we also solve the original system without scaling and set Tol = $10^{-12}$.

The data listed in Table 5 shows that our ENPA can always successfully find a nonnegative solution, which is the same as the true solution of the system, while both HM and QCA do not work as shown in Tables 3 and 4 when setting a nonnegative but not positive starting point. Certainly, there is a potentially good news for HM, that is, such a method is probably applicable to the system (1.3) involving a nonnegative vector $b$ with zeros when we take a positive starting point. However, we need some more theoretical results on HM [8] to support the numerical performance.

### Table 4: Numerical results for Example 4.1: (ii). $b = (2^3, 0)^T$.

| Alg. | $x^0 = (0, 20)^T$ | $x^0 = (20, 0)^T$ | $x^0 = (20, 20)^T$ |
|------|------------------|------------------|------------------|
|      | iter / time / solution | iter / time / solution | iter / time / solution |
| HM   | - / - / -         | - / - / -         | 13 / 0.03 / (2,0,0) |
| QCA  | - / - / -         | - / - / -         | - / - / -         |
| ENPA | 265 / 2.52 / (2,0,0) | 10 / 0.03 / (2,0,0) | 89 / 0.55 / (2,0,0) |
Finally, we consider another two higher order sparse nonsingular $M$-tensors, which are constructed in a similar way used in Section 4.1. Specifically, we first randomly generate a sparse nonnegative tensor $B \in \mathbb{T}_{m,n}$, whose 80% components are zeros. Then, we set $\omega = 0.1$ in (4.1) and let $A = sI - B$. For the vector $b$, we first generate a sparse vector $x^* = \text{sprand}(n, 1, 0.4)$ by the MATLAB script ‘sprand’, where 0.4 controls the number of zero components, and then let $b = A(x^*)^{m-1} \in \mathbb{R}^n_+$. Therefore, we can always ensure that the resulting multilinear system has at least one nonnegative but not positive solution.

In this test, we use $2b$ and Tol = $10^{-12}$ to be the starting point and tolerance of ENPA, respectively. Here, we only show the performance of ENPA on this example but without comparison with both HM and QCA. To further support our conjecture (i.e., linear convergence rate behavior of ENPA) observed in Section 4.1, we graphically show in Fig. 1 the evolutions of the residue function defined by (4.2) with respect to the number of iterations. Moreover, we compare the approximate solution obtained by ENPA with the known true solution of the multilinear system in Fig. 2.

It can be easily seen from Figs. 1 and 2 that ENPA can get a sparse nonnegative solution to the multilinear system by taking a few of iterations. Moreover, the convergence curves in Fig. 1 show that ENPA seems to be a linearly convergent algorithm, and in particular, such a method performs a quadratic convergence behavior when the iterates are close enough to the true solution. Moreover, Fig. 2 show that ENPA can perfectly find a nonnegative sparse solution to the multilinear system under test. The promising nonnegativity preserving property might be helpful to algorithmic design for sparse nonnegative tensor equations studied in [12,18], which is also one of our future concerns.

According to the results reported in this section, we can draw the conclusion that, compared to HM [8] and QCA [9], the proposed ENPA (Algorithm 1) has its own advantages, i.e., it has high efficiency, meanwhile, it can be applied to a wider range of cases. In particular, when the multilinear system has multiple nonnegative solutions, and if our purpose is to get as more solutions as possible,
An efficient nonnegativity preserving algorithm for multilinear systems

Fig. 1: ENPA: Evolutions of the residue defined by ReErr in (4.2) with respect to the number of iterations.

Fig. 2: Comparison between the true solution and the one recovered by ENPA.

the proposed algorithm may be a better candidate solver to achieve this goal by choosing different starting points.
5 Conclusion

In this paper, we mainly studied the multilinear system in the form of (1.3). We showed that the multilinear system, whose coefficient tensor is a nonsingular $M$-tensor and right-hand side vector is nonnegative, always has a nonnegative solution, but the solution may not be unique. Aiming at this case, we proposed a Newton-type algorithm who can perfectly preserve the nonnegativity of the iterative sequence. Moreover, we can prove that a nonnegative decreasing sequence generated by our proposed algorithm converges to a nonnegative solution of the system under consideration. By numerical experiments, we stated that our method is efficient and it has advantages over other algorithms: when the right-hand side is nonnegative but not positive, our algorithm can still output a solution of the system, while the others produce a solution depending on the choice of starting points. In the future, we will try to analyze the convergence rate of the algorithm and apply it to real-life sparse problems.

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