Asymptotic behavior of the wave equation with nonlocal weak damping, anti-damping and critical nonlinearity

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Abstract
In this paper, we first prove an abstract theorem on the existence of polynomial attractors and the concrete estimate of their attractive velocity for infinite-dimensional dynamical systems, then apply this theorem to a class of wave equations with nonlocal weak damping and anti-damping in case that the nonlinear term \( f \) is of subcritical growth.

Keywords: Wave equation, nonlocal weak damping, nonlocal weak anti-damping, critical nonlinearity, global attractor.

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1. Introduction

In this paper, we investigate the existence of the global attractor for the wave equation with nonlocal weak damping, nonlocal weak anti-damping and critical nonlinear source term

\[
\begin{align*}
    u_{tt} - \Delta u + k||u_t||_{L^2(\Omega)}^p u_t + f(u) &= \int_{\Omega} K(x,y)u_t(y)dy + h(x) \quad \text{in } [0, \infty) \times \Omega, \quad (1.1) \\
    u &= 0 \text{ on } [0, \infty) \times \partial \Omega, \quad (1.2) \\
    u(x,0) &= u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \quad (1.3)
\end{align*}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 3) \) with smooth boundary \( \partial \Omega \) and the following assumption holds:

\textbf{Assumption 1.1.} \hspace{0.5cm} (i) \hspace{0.5cm} \( k \) and \( p \) are positive constants, \( K \in L^2(\Omega \times \Omega) \), \( h \in L^2(\Omega) \); \\
(ii) \hspace{0.5cm} \( f \in C^1(\mathbb{R}) \) satisfies

\[
\begin{align*}
    |f'(s)| &\leq M(|s|^\frac{p-2}{2} + 1), \quad (1.4) \\
    \liminf_{|s| \to \infty} f'(s) &\equiv \mu > -\lambda_1, \quad (1.5)
\end{align*}
\]

where \( M \geq 0 \) and \( \lambda_1 \) is the first eigenvalue of the operator \(-\Delta\) equipped with Dirichlet boundary condition.

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Since the pioneering work of J.K. Hale et al. [26] on the dynamical behavior of dissipative wave equations in the 1970s, there has been a wealth of literature on the asymptotic dimension and existence of exponential attractors) of solutions of wave equations with various damping. Among them, we refer to [1, 2, 4, 7, 9, 24, 25, 27, 34, 45, 49, 51] for the wave equation with weak damping $k u$ which models the oscillation process occurring in many physical systems, including electrodynamics, quantum mechanics, nonlinear elasticity, etc. Wave equations with strong damping $-k\Delta u_t$ (see [42] for their physical background) were studied in [3, 11, 19]. Literatures [20–22, 28, 30, 33, 44, 50] were devoted to wave equations with nonlinear damping $g(u_t)$. The damping $(-\Delta)^\alpha u_t$ ($\alpha \in (0, 1)$) is called fractional damping. In particular, it is referred to as structural damping when $1/2 \leq \alpha < 1$ and to moderate damping when $0 \leq \alpha < 1/2$. Studies related to wave equations with fractional damping can be found in [6, 52, 53] and references therein.

On the other hand, the long-time behavior of hyperbolic equations with nonlocal damping has received great attention. For example, we refer to [11, 19] for the study of the Kirchhoff equation with the damping $M(||\nabla u||^2_{L^2(\Omega)}) \Delta u_t$, to [8, 35] for the case of nonlocal weak damping $M(||\nabla u||^2_{L^2(\Omega)}) u_t$, to [17] for the case of the damping $M(||\nabla u||^2_{L^2(\Omega)}) g(u_t)$, and to [10, 16] for the case of the damping $M(||\nabla u||^2_{L^2(\Omega)}) (-\Delta)^\theta u_t$. The damping terms involved in the references listed above all have Kirchhoff type coefficients $M(||\nabla u||^2_{L^2(\Omega)})$. In addition, Lazo [38] proved the existence of a global solution to the equation

$$u_{tt} + M(||A^{1/2} u||^2)Au + N(||A^\alpha u||^2)A^\alpha u_t = f,$$

where $A$ is a positive self-adjoint operator defined in Hilbert space $H$, $\alpha \in (0, 1]$ and the functions $M, N$ satisfy the nondegenerate condition.

While, to the best of our knowledge, only very few results are available for damped hyperbolic equations whose nonlocal damping coefficient depends on $u_t$. Among them we highlight that in 1989 Balakrishnan and Taylor [3] presented some extensible beam equations with nonlocal energy damping $\left[\int_{\Omega} (|\Delta u|^2 + |u_t|^2)dx\right]^q \Delta u_t$ to model the damping phenomena in flight structures. Recently Silva, Narciso and Vicente [31] have proved the global well-posedness, polynomial stability of the following beam model with the nonlocal energy damping

$$u_{tt} - \kappa \Delta u + \Delta^2 u - \gamma \left[\int_{\Omega} (|\Delta u|^2 + |u_t|^2)dx\right]^q \Delta u_t + f(u) = 0.$$

Lazo [37] considered the local solvability of the wave equation

$$u_{tt} - M(||\nabla u||^2_{L^2(\Omega)}) \Delta u + N(||u_t||^2_{L^2(\Omega)})u_t = b|u|^{p-1}u.$$

We are motivated by the literature mentioned above to study problem (1.1)–(1.3) in our last work [55]. As far as we know, this constituted the first result on the long-time behavior of wave equation with nonlocal damping $k||u_t||^p_{L^2(\Omega)} u_t$. In [55], we have proved via the method of Condition (C) that the system possesses a global attractor in the case that $f$ satisfies the subcritical growth condition. However, we did not solve the problem of
the existence of the global attractor for the critical case, which is the aim of the present paper.

Problem (1.1)-(1.3) is a weakly damped model, in which the nonlocal coefficient \( k ||u_t||^p \) reflects the effect of kinetic energy on damping in physics. The term \( \int_\Omega K(x,y)u_t(y)dy \) is an anti-damping because it may provide energy. The difficulty of this problem lies first in the nondegenerate, nonlocal coefficient of damping and the arbitrariness of the exponent \( p > 0 \). Due to the influence of nonlocal coefficient \( ||u_t||^p \), when the velocity \( u_t \) is very small, the nonlocal damping is weaker than the linear damping. Furthermore, as the velocity \( u_t \) is smaller and \( p \) is larger, the damping is weaker and thus energy dissipation is slower. In addition, the presence of the anti-damping term leads to the energy not decreasing along the orbit, and moreover, the effect of energy supplement brought by the anti-damping term needs to be overcome by the damping. All these factors cause difficulties in studying the long-term behavior of this model. At the same time, since \( f \) is of critical growth, the corresponding Sobolev embedding is no longer compact, which makes all the methods based on compactness, including Condition (C), no longer available to prove the existence of the global attractor.

In this paper, to overcome the difficulty of lack of compactness in the critical case, we employ the criterion of asymptotic smoothness relying on the repeated inferior limit (see Lemma 2.5 below) to prove the existence of the global attractor. Chueshov and Lasiecka [14, 15] proposed this criterion based on the idea of Khanmamdov [32]. To handle the difficulty that nonlocal damping coefficient \( ||u_t||^p \) brings in energy estimate, we use the strong monotone inequality for the general inner product space (see Lemma 2.6 below). The proof of compactness borrows many ideas from [14].

As for the dissipativity, A. Haraux [29] obtained via barrier’s method the uniform bound of the energy in terms of the initial energy for a dissipative wave equation with anti-damping. The key element of the method is that the dissipative term in the inequality for Lyapunov’s function has a coefficient sublinearly dependent on the energy. I. Chueshov and I. Lasiecka [14] further proved that systems whose Lyapunov’s functions satisfy such inequalities are ultimately dissipative. Their strategy was to select the perturbation parameter \( \epsilon \) in the energy inequality as a suitable function of the initial energy according to the sublinear dependence of the coefficient of the dissipation term on the energy; and thus they deduced that the energy is ultimately bounded by a constant independent on the initial data. Following the method in [14], we prove the dissipativity for problem (1.1)-(1.3).

The establishment of the global well-posedness follows the idea in [13, 14].

This paper is organized as follows. In Section 2, we present some notations and lemmas which will be needed later. In Section 3 and Section 4, we prove the global well-posedness and dissipativity of the dynamical system generated by problem (1.1)-(1.3), respectively. In Section 5, we establish the existence of the global attractor for this system.

2. Preliminaries

Throughout this paper, we will denote the inner product and the norm on \( L^2(\Omega) \) by \( (\cdot,\cdot) \) and \( \| \cdot \| \), respectively, and the norm on \( L^p(\Omega) \) by \( \| \cdot \|_p \). The symbol \( \mathcal{A} \) denotes the strictly positive operator on \( L^2(\Omega) \) defined by \( \mathcal{A} = -\Delta \) with domain \( D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega) \). The symbols \( \hookrightarrow \) and \( \hookrightarrow \hookrightarrow \) stand for continuous embedding and compact embedding, respectively.
The capital letter “C” with a (possibly empty) set of subscripts will denote a positive constant depending only on its subscripts and may vary from one occurrence to another. And we write

\[ \Psi(u(t, x)) = \int_{\Omega} K(x, y)u(t, y) dy. \]

In later sections, we will use the following Sobolev embeddings:

\[ H^1_0(\Omega) \hookrightarrow L^{2N}(\Omega), \quad H^s(\Omega) \hookrightarrow L^{2N}(\Omega) \quad (s \in (0, 1)). \]

We then present some preliminaries.

**Definition 2.1.** [14] Let \( \{S(t)\}_{t \geq 0} \) be a semigroup on a complete metric space \((X, d)\). A closed set \( B \subseteq X \) is said to be absorbing for \( \{S(t)\}_{t \geq 0} \) iff for any bounded set \( B \subseteq X \) there exists \( t_0(B) \) such that \( S(t)B \subseteq B \) for all \( t > t_0(B) \). The semigroup \( \{S(t)\}_{t \geq 0} \) is said to be dissipative iff it possesses a bounded absorbing set.

**Definition 2.2.** [14] A compact set \( A \subseteq X \) is said to be a global attractor of the dynamical system \((X, \{S(t)\}_{t \geq 0})\) iff

(i) \( A \subseteq X \) is an invariant set, i.e., \( S(t)A = A \) for all \( t \geq 0 \);

(ii) \( A \subseteq X \) is uniformly attracting, i.e., for all bounded set \( B \subseteq X \) we have

\[ \lim_{t \to +\infty} \text{dist}(S(t)B, A) = 0. \]

Here and below \( \text{dist}(A, B) := \sup_{x \in A} \text{dist}_X(x, B) \) is the Hausdorff semi-distance.

**Definition 2.3.** [14] A dynamical system \((X, \{S(t)\}_{t \geq 0})\) is said to be asymptotically smooth iff for any bounded set \( B \) such that \( S(t)B \subseteq B \) for \( t > 0 \) there exists a compact set \( K \) in the closure \( \overline{B} \) of \( B \), such that

\[ \lim_{t \to +\infty} \text{dist}(S(t)B, K) = 0. \]

**Lemma 2.4.** [14] Let \((X, \{S(t)\}_{t \geq 0})\) be a dissipative dynamical system, where the phase space \( X \) is a complete metric space. Then \((X, \{S(t)\}_{t \geq 0})\) possesses a global attractor if and only if \((X, \{S(t)\}_{t \geq 0})\) is asymptotically smooth.

**Lemma 2.5.** [15] Let \((X, \{S(t)\}_{t \geq 0})\) be a dynamical system, where the phase space \( X \) is a complete metric space. Assume that for any bounded positively invariant set \( B \) in \( X \) and any \( \epsilon > 0 \) there exists \( T \equiv T(\epsilon, B) \) such that

\[ \liminf_{m \to +\infty} \liminf_{n \to +\infty} \text{dist}(S(T)y_n, S(T)y_m) \leq \epsilon \quad \text{for every sequence } \{y_n\} \subseteq B. \] (2.1)

Then \((X, \{S(t)\}_{t \geq 0})\) is asymptotically smooth.
Lemma 2.6. Let $(H, (\cdot, \cdot)_H)$ be an inner product space with the induced norm $\| \cdot \|_H$ and constant $p > 1$. Then there exists some positive constant $C_p$ such that for any $x, y \in H$ satisfying $(x, y) \neq (0, 0)$, we have

$$\left( \|x\|^{-2}_H x - \|y\|^{-2}_H y, x - y \right)_H \geq \begin{cases} C_p \|x - y\|_H^p, & p \geq 2; \\ C_p \frac{\|x - y\|_H^2}{\|x\|_H + \|y\|_H^{2-p}}, & 1 < p < 2. \end{cases}$$

(2.2)

Inequality (2.2), which was verified for $\mathbb{R}^N$ in [43, 47] and then for a general inner product space in [54], will play a crucial role in our estimate.

Lemma 2.7. Assume $X \leftrightarrow B \leftrightarrow Y$ where $X$, $B$, $Y$ are Banach spaces. The following statements hold.

(i) Let $F$ be bounded in $L^p(0,T;X)$ where $1 \leq p < \infty$, and $\partial F/\partial t = \{ \partial f/\partial t : f \in F \}$ be bounded in $L^1(0,T;Y)$, where $\partial/\partial t$ is the weak time derivative. Then $F$ is relatively compact in $L^p(0,T;B)$.

(ii) Let $F$ be bounded in $L^\infty(0,T;X)$ and $\partial F/\partial t$ be bounded in $L^r(0,T;Y)$ where $r > 1$.

Then $F$ is relatively compact in $C(0,T;B)$.

3. Global well-posedness

In this section we discuss the global well-posedness of problem (1.1)-(1.3). We will use the following definitions of solutions.

Definition 3.1. A function $u(t) \in C([0,T]; H^1_0(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ with $u(0) = u_0$ and $u_t(0) = u_1$ is said to be

(i) strong solution to problem (1.1)-(1.3) on the interval $[0,T]$, iff

- $u \in W^{1,1}(a,b; H^1_0(\Omega))$ and $u_t \in W^{1,1}(a,b; L^2(\Omega))$ for any $0 < a < b < T$;
- $-\Delta u(t) + k\|u_t(t)\|_{L^2(\Omega)} u_t(t) \in L^2(\Omega)$ for almost all $t \in [0,T]$;
- equation (1.1) is satisfied in $L^2(\Omega)$ for almost all $t \in [0,T]$;

(ii) generalized solution to problem (1.1)-(1.3) on the interval $[0,T]$, iff there exists a sequence of strong solutions $\{u^j(t)\}$ to problem (1.1)-(1.3) with initial data $(u_0^j, u_1^j)$ instead of $(u_0, u_1)$ such that $(u^j, u_1^j) \to (u, u_t)$ in $C([0,T]; H^1_0(\Omega) \times L^2(\Omega))$ as $j \to +\infty$;

(iii) weak solution to problem (1.1)-(1.3) on the interval $[0,T]$, iff

$$\int_\Omega u_t(t,x)\psi(x)dx = \int_\Omega u_1 \psi dx + \int_0^t \left[ \int_{\Omega \times \Omega} K(x,y) u_t(\tau,y) \psi(x) dxdy \\
+ \int_\Omega h(x) \psi(x) dx - \int_\Omega \nabla u(\tau,x) \nabla \psi(x) dx \\
- k\|u_t(\tau)\|^p \int_\Omega u_t(\tau,x) \psi(x) dx - \int_\Omega f(u(\tau,x)) \psi(x) dx \right] d\tau$$

(3.1)

holds for every $\psi \in H^1_0(\Omega)$ and for almost all $t \in [0,T]$. 


In order to prove the global well-posedness for (1.1)-(1.3), we need the following definitions and lemmas.

**Definition 3.2.** Let $X$ be a real Banach space with dual space $X^*$. $F : X \to X^*$ is said to be monotone if $\langle F(u) - F(v), u - v \rangle \geq 0$, $\forall u, v \in X$, and hemicontinuous if for each $u, v \in X$ the real-valued function $\lambda \to \langle F(u + \lambda v), v \rangle$ is continuous.

**Definition 3.3.** Let $X$ and $Y$ be Banach spaces. $F : X \to Y$ is called demicontinuous if it is continuous from $X$ with norm convergence to $Y$ with weak convergence.

**Lemma 3.4.** Let $X$ be a real reflexive Banach space, $F : X \to X^*$ hemicontinuous, monotone and coercive, i.e. $\frac{(F(x),x)}{|x|} \to \infty$ as $|x| \to \infty$. Then $F$ is onto $X^*$.

**Lemma 3.5.** Let $X$ be a reflexive Banach space. If $F : X \to X^*$ hemicontinuous, monotone and bounded, then it is demicontinuous.

**Lemma 3.6.** Let $A : D(A) \subseteq H \to H$ be a maximal accretive operator on a Hilbert space $H$, i.e., $(Ax_1 - Ax_2, x_1 - x_2)_H \geq 0$ for any $x_1, x_2 \in D(A)$ and $\text{Rg}(I+A) = H$; besides, assume that $0 \in A0$. Let $B : H \to H$ be a locally Lipschitz. If $u_0 \in D(A)$, $f \in W^{1,1}(0,t;H)$ for all $t > 0$, then there exists $t_{\max} \leq +\infty$ such that the initial value problem

$$u_t + Au + Bu \ni f \quad \text{and} \quad u = u_0 \in H$$

has a unique strong solution $u$ on the interval $[0,t_{\max})$.

Whereas, if $u_0 \in D(A)$, $f \in L^1(0,t;H)$ for all $t > 0$, then problem (3.2) has a unique generalized solution $u \in C([0,t_{\max});H)$.

Moreover, in both cases we have $\lim_{t \to t_{\max}} \|u(t)\|_H = \infty$ provided $t_{\max} < \infty$.

We are now ready to establish the global well-posedness for (1.1)-(1.3).

**Theorem 3.7.** Let $T > 0$ be arbitrary. Under Assumption 1.1, we have the following statements.

(i) For any $(u_0, u_1) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that $-\Delta u_0 + k\|u_1\|^pu_1 \in L^2(\Omega)$, there exists a unique strong solution $u(t)$ to problem (1.1)-(1.3) on $[0,T]$.

(ii) For every $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ there exists a unique generalized solution, which is also the weak solution to problem (1.1)-(1.3).

**Proof.** We divide our proof into three steps.

**Step 1.** We first prove local well-posedness of problem (1.1)-(1.3).

Let $U = (u, v)^T$ with $v = u_t$. We rewrite (1.1)-(1.3) as

$$\begin{cases}
U_t + A(U) = B(U), \quad t > 0,
U(0) = U_0,
\end{cases}$$

where $A(U) = (Au, Bu)^T$ and $B(U) = (Bu, Bv)^T$.

...
where $U_0 = (u_0, u_1)^T$, $A : D(A) \subseteq H^1_0(\Omega) \times L^2(\Omega) \rightarrow H^1_0(\Omega) \times L^2(\Omega)$ is given by

$$A(U) = \begin{pmatrix} -v \\ -\Delta u + k\|v\|^p v \end{pmatrix},$$

$U \in D(A) = \{(u, v)^T \in H^1_0(\Omega) \times H^1_0(\Omega) | -\Delta u + k\|v\|^p v \in L^2(\Omega)\}$

and $B : H^1_0(\Omega) \times L^2(\Omega) \rightarrow H^1_0(\Omega) \times L^2(\Omega)$ is given by

$$B(U) = \begin{pmatrix} 0 \\ \Psi(v) + h(x) - f(u) \end{pmatrix}, \quad U = (u, v)^T \in H^1_0(\Omega) \times L^2(\Omega).$$

We note that

$$\overline{D(A)} = H^1_0(\Omega) \times L^2(\Omega), \quad (3.4)$$

because

$$\left(H^1_0(\Omega) \cap H^2(\Omega)\right) \times \left(H^1_0(\Omega) \cap H^2(\Omega)\right) \subseteq D(A) \subseteq H^1_0(\Omega) \times L^2(\Omega)$$

and $(H^1_0(\Omega) \cap H^2(\Omega)) \times (H^1_0(\Omega) \cap H^2(\Omega))$ is dense in $H^1_0(\Omega) \times L^2(\Omega)$.

By Lemma 2.3 for every $v_1, v_2$ in $L^2(\Omega)$ we have

$$(\|v_1\|^p v_1 - \|v_2\|^p v_2, v_1 - v_2) \geq 0. \quad (3.5)$$

Consequently, we have

$$(A(U_1) - A(U_2), U_1 - U_2)_{H^1_0(\Omega) \times L^2(\Omega)}$$

$$= (\nabla (v_2 - v_1), \nabla (u_1 - u_2)) + (\nabla (u_1 - u_2), \nabla (v_1 - v_2))$$

$$+ (k\|v_1\|^p v_1 - k\|v_2\|^p v_2, v_1 - v_2)$$

$$\geq 0,$$

for all $U_1, U_2 \in D(A)$, where $U_1 = (u_1, v_1)^T, U_2 = (u_2, v_2)^T$.

We proceed to show that

$$\text{Rg}(I + A) = H^1_0(\Omega) \times L^2(\Omega), \quad (3.7)$$

i.e., for $\forall (f_0, f_1)^T \in H^1_0(\Omega) \times L^2(\Omega)$, the equation

$$(A + I)(U) = \begin{pmatrix} -v + u \\ -\Delta u + k\|v\|^p v + v \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \quad (3.8)$$

has a solution.

Eliminating $u$ from (3.8) gives

$$-\Delta v + k\|v\|^p v + v = f_1 + \Delta f_0 \in H^{-1}(\Omega). \quad (3.9)$$

Define $G : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)$ by $G(v) = -\Delta v + k\|v\|^p v + v$ for each $v \in H^1_0(\Omega)$. Obviously,
for each \( v_1, v_2 \in H^1_0(\Omega) \),
\[
\langle G(v_1 + \lambda v_2), v_2 \rangle = (\nabla (v_1 + \lambda v_2), \nabla v_2) + (1 + k\|v_1 + \lambda v_2\|^p)(v_1 + \lambda v_2, v_2)
\]
is a continuous function of real variable \( \lambda \).

It follows from (3.5) that
\[
\langle G(v_1) - G(v_2), v_1 - v_2 \rangle = \|
abla (v_1 - v_2)\|^2 + \|v_1 - v_2\|^2 + k(\|v_1\|^p v_1 - \|v_2\|^p v_2, v_1 - v_2) \\
\geq 0
\]
for all \( v_1, v_2 \in H^1_0(\Omega) \).

Moreover, we have
\[
\frac{\langle G(v), v \rangle}{\|
abla v\|} = \frac{\|
abla v\|^2 + \|v\|^2 + k\|v\|^{p+2}}{\|
abla v\|} \rightarrow +\infty
\]
as \( \|
abla v\| \rightarrow \infty \).

In summary, \( G \) is hemicontinuous, monotone and coercive. Thus, by Lemma 3.4, \( G \) is onto \( H^{-1}(\Omega) \) and (3.7) follows immediately. Combining (3.6) and (3.7) means \( A \) is m-accretive.

For any \( u_1, u_2 \in H^1_0(\Omega) \), by (1.4) we have
\[
\|f(u_1) - f(u_2)\| \\
= \left\{ \int_0^1 \left[ \int_0^1 f'(u_2 + \theta(u_1 - u_2))(u_1 - u_2) d\theta \right] dx \right\}^{\frac{1}{2}} \\
\leq C \left\{ \int_0^1 \left( |u_1|^\frac{n}{n-2} + |u_2|^\frac{n}{n-2} + 1 \right) |u_1 - u_2|^2 dx \right\}^{\frac{1}{2}} \\
\leq C \left( \|u_1\|^\frac{n}{n-2} + \|u_2\|^\frac{n}{n-2} + 1 \right) \|u_1 - u_2\|_2^{\frac{n}{n-2}} \\
\leq C \left( \|
abla u_1\|^\frac{2}{n-2} + \|
abla u_2\|^\frac{2}{n-2} + 1 \right) \|
abla (u_1 - u_2)\|.
\]

Moreover, using the Hölder inequality, we have
\[
\left\| \int_\Omega K(x,y)(v_1(y) - v_2(y)) dy \right\| \leq \|K\|_{L^2(\Omega \times \Omega)} \|v_1 - v_2\| \\
\]
for any \( v_1, v_2 \in L^2(\Omega) \).

It follows from (3.11) and (3.12) that \( B \) is locally Lipschitz.

Now that we have proved that \( A \) is m-accretive, \( B \) is locally Lipschitz and \( \overline{D(A)} = H^1_0(\Omega) \times L^2(\Omega) \), by Lemma 3.6 we conclude that there exists \( t_{max} \leq +\infty \) such that problem (1.1)–(1.3) has a unique strong solution \( u \) on \([0, t_{max})\) for every \((u_0, u_1) \in D(A)\) and it has a unique generalized solution \( u \) on \([0, t_{max})\) for every \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\), moreover \([0, t_{max})\) is the maximal interval on which the solution exists. Furthermore, for both the
strong solution and the generalized solution we have
\[
\lim_{t \to t_{\max}} \|(u, u_t)\|_{H_0^1(\Omega) \times L^2(\Omega)} = \infty, \text{ provided } t_{\max} < +\infty.
\] (3.13)

**Step 2.** Next, we will prove the global well-posedness of problem (1.1)-(1.3).
We denote
\[
\mathcal{E}(u(t), u_t(t)) = \frac{1}{2} \left( \|u_t\|^2 + \|\nabla u\|^2 \right) + \int_{\Omega} F(u) dx - \int_{\Omega} hudx.
\]
Choose \( \mu_0 \in \mathbb{R}^+ \cap (-\mu, \lambda_1) \). By (1.5), there exists \( M > 0 \) such that
\[
f'(s) > -\mu_0, \ |s| > M.
\]
It follows that
\[
\begin{aligned}
F(s) &\geq -\frac{\lambda_1 + \mu_0}{4} s^2 - C, \ |s| > M, \\
|F(s)| &\leq C, \ |s| \leq M.
\end{aligned}
\]
Consequently,
\[
\int_{\Omega} F(u) dx \geq \int_{\Omega_1} \left( -\frac{\lambda_1 + \mu_0}{4} u^2 - C \right) dx + \int_{\Omega_2} F(u) dx
\]
\[
\geq -\frac{\lambda_1 + \mu_0}{4} \int_{\Omega} u^2 dx - C_1,
\] (3.14)
where \( \Omega_1 = \{ x \in \Omega : |u(x)| > M \} \), \( \Omega_2 = \{ x \in \Omega : |u(x)| \leq M \} \) and \( C_1 \) is some positive constant.

It is easy to get
\[
\left| \int_{\Omega} hudx \right| \leq \frac{1}{16} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \|\nabla u\|^2 + C. \] (3.15)

By Poincaré’s inequality we have
\[
\|\nabla u\|^2 \geq \lambda_1 \|u\|^2. \] (3.16)

By (3.14)-(3.16) we have
\[
\mathcal{E}(u(t), u_t(t)) \geq \frac{1}{2} \left( \|u_t\|^2 + \|\nabla u\|^2 \right) - \frac{\lambda_1 + \mu_0}{4} \int_{\Omega} u^2 dx - C_1 - \frac{1}{8} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \|\nabla u\|^2 - C
\]
\[
\geq \frac{1}{8} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \left( \|u_t\|^2 + \|\nabla u\|^2 \right) - C. \] (3.17)
We deduce from (1.4) that
\[
\int_\Omega F(u)dx \leq \int_\Omega C(1 + |u|^\frac{2N}{N-2})dx \\
\leq C \left( \|\nabla u\|^\frac{2N}{N-2} + 1 \right).
\] (3.18)

Using (3.15) and (3.18) we obtain
\[
\mathcal{E}(u(t), u_t(t)) \leq C \left( \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^\frac{2N}{N-2} + 1 \right).
\] (3.19)

Multiplying (1.1) by \(u_t\) and integrating on \(\Omega\) yields
\[
\frac{d}{dt}\mathcal{E}(u(t), u_t(t)) = -k\|u_t\|^{p+2} + \int_\Omega \Psi(u_t)u_t dx
\] (3.20)
for \(t \in [0, t_{\text{max}}]\).

Using Young’s inequality with \(\epsilon\), we deduce from (3.20) that
\[
\frac{d}{dt}\mathcal{E}(u(t), u_t(t)) \leq -k\|u_t\|^{p+2} + \|K\|_{L^2(\Omega \times \Omega)}\|u_t\|^2
\leq -k\|u_t\|^{p+2} + \frac{k}{2}\|u_t\|^{p+2} + C
\leq C
\] (3.21)
for \(t \in [0, t_{\text{max}}]\).

Integrating (3.21), we have
\[
\mathcal{E}(u(t), u_t(t)) \leq \mathcal{E}(u_0, u_1) + Ct.
\] (3.22)

If \(t_{\text{max}} < +\infty\), we deduce from (3.17), (3.19) and (3.22) that
\[
\| (u(t), u_t(t)) \|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq C \left( \|u_1\|^2 + \|\nabla u_0\|^2 + \|\nabla u_0\|^\frac{2N}{N-2} + 1 + t_{\text{max}} \right) < +\infty.
\] (3.23)

By the definition of the generalized solution, it is easy to verify that (3.23) is also true for the generalized solution. Thus by (3.13), we have proved the global existence and uniqueness of the strong solution as well as the generalized solution.

**Step 3.** Finally, we will verify that the generalized solution to problem (1.1)-(1.3) is also weak.

Obviously, the strong solution to problem (1.1)-(1.3) is also weak.

Let \(u(t)\) be the generalized solution to problem (1.1)-(1.3), then by definition there exists a sequence of strong solutions \(\{u_j(t)\}\) to problem (1.1)-(1.3) with initial data \((u_0^j, u_1^j)\) instead of \((u_0, u_1)\) such that
\[
(u_j^j, u_t^j) \to (u, u_t) \text{ in } C([0, T]; H_0^1(\Omega) \times L^2(\Omega)) \text{ as } j \to +\infty.
\] (3.24)
By the Lebesgue convergence theorem, it follows from (3.24) that

\[
\begin{align*}
\lim_{j \rightarrow +\infty} \int_{\Omega} u_j^2(t, x) \psi(x) dx &= \int_{\Omega} u^2 \psi dx + \int_0^t \left[ \int_{\Omega \times \Omega} K(x, y) u_j^2(\tau, y \psi(x) dx dy \\
+ \int_{\Omega} h(\tau, x) \psi(x) dx - \int_{\Omega} \nabla u_j(\tau, x) \nabla \psi(x) dx \\
- k ||u_j(\tau)||^p \int_{\Omega} u_j^2(\tau, x) \psi(x) dx - \int_{\Omega} f(u_j(\tau, x)) \psi(x) dx \right] d\tau
\end{align*}
\]

holds for every \( \psi \in H^1_0(\Omega) \) and for almost all \( t \in [0, T] \).

Define \( D : L^2(\Omega) \rightarrow L^2(\Omega) \) by \( G(v) = ||v||^p v \) for each \( v \in L^2(\Omega) \). Inequality (3.25) indicates that \( D \) is accretive. Besides, it is apparent that \( D \) is hemicontinuous and bounded. Consequently, due to Lemma 3.5, \( D \) is demicontinuous. Thus, we have

\[
||u^j(\tau)||^p \int_{\Omega} u_j^2(\tau, x) \psi(x) dx \rightarrow ||u(\tau)||^p \int_{\Omega} u(\tau, x) \psi(x) dx \quad \text{as} \quad j \rightarrow +\infty.
\]

(3.26)

Since by (3.24) there exists \( J \in \mathbb{N} \) such that \( \max_{\tau \in [0, T]} ||u_j^2(\tau)|| \leq \max_{\tau \in [0, T]} ||u(\tau)|| + 1 \) for every \( j \geq J \), we have

\[
\left| ||u^j(\tau)||^p \int_{\Omega} u_j^2(\tau, x) \psi(x) dx \right| \leq \left( \max_{\tau \in [0, T]} ||u(\tau)|| + 1 \right)^{p+1} \||\psi|| \leq C.
\]

(3.27)

By the Lebesgue convergence theorem, it follows from (3.26) and (3.27) that

\[
\lim_{j \rightarrow +\infty} \int_0^t \left[ ||u^j(\tau)||^p \int_{\Omega} u_j^2(\tau, x) \psi(x) dx \right] d\tau = \int_0^t \left[ ||u(\tau)||^p \int_{\Omega} u^2(\tau, x) \psi(x) dx \right] d\tau.
\]

(3.28)

Letting \( j \rightarrow +\infty \) in (3.26) and using (3.24) and (3.28), we see that \( u(t) \) satisfies (3.11), which completes the proof.

By Theorem 3.7, problem (1.1)-(1.3) generates an evolution semigroup \( \{S(t)\}_{t \geq 0} \) in the space \( H^1_0(\Omega) \times L^2(\Omega) \) by the formula \( S(t)(u_0, u_1) = (u(t), u_1(t)) \), where \( (u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega) \) and \( u(t) \) is the weak solution to problem (1.1)-(1.3).

4. Dissipativity

In this section, we will prove the dissipativity of the dynamical system generated by the weak solution to problem (1.1)-(1.3), which is a necessary condition for the existence of the global attractor.

**Theorem 4.1.** Under Assumption 1.1, the dynamical system \( (H^1_0(\Omega) \times L^2(\Omega), \{S(t)\}_{t \geq 0}) \) generated by the weak solution of problem (1.1)-(1.3) is dissipative, i.e., there exists \( R > 0 \) satisfying the property: for any bounded set \( B \) in \( H^1_0(\Omega) \times L^2(\Omega) \), there exists \( t_0(B) \) such that \( ||S(t)y||_{H^1_0(\Omega) \times L^2(\Omega)} \leq R \) for all \( y \in B \) and \( t \geq t_0(B) \).
Proof. Choose $\mu_0 \in \mathbb{R}^+ \cap (-\mu, \lambda_1)$. By (1.5), there exists $M > 0$ such that
\[ f'(s) > -\mu_0, \ |s| > M. \tag{4.1} \]
It follows that
\[
\begin{cases}
F(s) \geq -\frac{\lambda_1 + \mu_0}{4} s^2 - C, & |s| > M; \\
|F(s)| \leq C, & |s| \leq M.
\end{cases}
\]
Consequently,
\[
\int_{\Omega} F(u) dx \geq \int_{\Omega_1} \left( -\frac{\lambda_1 + \mu_0}{4} u^2 - C \right) dx + \int_{\Omega_2} F(u) dx 
\]
\[
\geq -\frac{\lambda_1 + \mu_0}{4} \int_{\Omega} u^2 dx - C_1, \tag{4.2}
\]
where $\Omega_1 = \{ x \in \Omega : |u(x)| > M \}$, $\Omega_2 = \{ x \in \Omega : |u(x)| \leq M \}$ and $C_1$ is some positive constant.
Let
\[
V_{\epsilon}(t) = \frac{1}{2} \left( \|u_t\|^2 + \|\nabla u\|^2 \right) + \int_{\Omega} F(u) dx - \int_{\Omega} hudx + \epsilon \int_{\Omega} u_t u dx.
\]
Since by Poincaré’s inequality we have
\[
\|\nabla u\|^2 \geq \lambda_1 \|u\|^2, \tag{4.3}
\]
there exists $\epsilon_0 > 0$ such that
\[
\left| \epsilon \int_{\Omega} u_t u dx \right| \leq \frac{1}{16} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \left( \|u_t\|^2 + \|\nabla u\|^2 \right), \tag{4.4}
\]
holds for all $\epsilon \leq \epsilon_0$.
Hereafter we assume $\epsilon \in (0, \epsilon_0)$.
We also have
\[
\left| \int_{\Omega} hudx \right| \leq \frac{1}{16} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \|\nabla u\|^2 + C \tag{4.5}
\]
We deduce from (1.4) that
\[
\left| \int_{\Omega} F(u) dx \right| \leq C \left( \|\nabla u\|^{\frac{2N}{N-2}} + 1 \right). \tag{4.6}
\]
We deduce from (4.2)-(4.6) that
\[
\frac{1}{8} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \left( \|u_t\|^2 + \|\nabla u\|^2 \right) - C \leq V_{\epsilon}(t) \leq C \left( \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{\frac{2N}{N-2}} + 1 \right). \tag{4.7}
\]
Multiplying (4.1) by \( u_t + \epsilon u \) and integrating on \( \Omega \) yields

\[
\frac{d}{dt} V_\epsilon(t) = -k\|u_t\|^{p+2} + \int_\Omega \Psi(u_t)u_t \, dx + \epsilon \left[ -\|\nabla u\|^2 + \|u_t\|^2 \right]
\]

\[
- k\|u_t\|^p \int_\Omega u_t \, dx - \int_\Omega f(u)u \, dx + \int_\Omega \Psi(u_t)u_t \, dx + \int_\Omega h u \, dx. \tag{4.8}
\]

We estimate the terms on the right hand side of identity (4.8) as follows:

\[
\left| -k\|u_t\|^p \int_\Omega u_t \, dx \right| \leq kC \|u_t\|^{p+1}\|\nabla u\|
\]

\[
\leq C k \left( \|u_t\|^{p+1}\|\nabla u\|^{\frac{p}{p+2}} \right)^{\frac{p+2}{p+1}} + \frac{1}{12} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \|\nabla u\|^{\frac{2}{p+2}(p+2)} \tag{4.9}
\]

\[
= C k \|u_t\|^{p+2}\|\nabla u\|^{\frac{p}{p+1}} + \frac{1}{12} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \|\nabla u\|^2;
\]

\[
\left| \int_\Omega \Psi(u_t)u_t \, dx \right| \leq \frac{1}{12} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \|\nabla u\|^2 + C \|u_t\|^2; \tag{4.10}
\]

\[
\left| \int_\Omega \Psi(u_t)u_t \, dx \right| \leq \|K\|_{L^2(\Omega \times \Omega)} \|u_t\|^2. \tag{4.11}
\]

We infer from (4.1) that

\[
F(s) \leq f(s)s + \frac{\mu_0}{2} s^2 + C, \quad |s| > M. \tag{4.12}
\]

By (4.3) and (4.12) we have

\[
- \int_\Omega f(u)u \, dx \leq - \left( \int_\Omega F(u) \, dx + \frac{\lambda_1 + \mu_0}{4} \int_\Omega u^2 \, dx + C_1 \right)
\]

\[
+ \frac{1}{4} \left( 3\mu_0 \lambda_1 + 1 \right) \|\nabla u\|^2 + C. \tag{4.13}
\]

Using (4.2), (4.5), (4.7), (4.9), (4.10), (4.11), (4.13) and Young’s inequality with \( \epsilon \), we deduce from (4.8) that

\[
\frac{d}{dt} V_\epsilon(t)
\]

\[
\leq - k\|u_t\|^{p+2} \left( 1 - C\epsilon\|\nabla u\|^{\frac{p}{p+1}} \right) + \frac{k}{2} \|u_t\|^{p+2} + C
\]

\[
- \epsilon \left[ \frac{1}{2} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \left( \|u_t\|^2 + \|\nabla u\|^2 \right) + \left( \int_\Omega F(u) \, dx + \frac{\lambda_1 + \mu_0}{4} \int_\Omega u^2 \, dx + C_1 \right) \right] \tag{4.14}
\]

\[
\leq - k\|u_t\|^{p+2} \left[ \frac{1}{2} - C\epsilon \left( V_\epsilon(t) + C \right)^{\frac{p}{p+1}} \right] - \frac{2}{3} \left( 1 - \frac{\mu_0}{\lambda_1} \right) \epsilon V_\epsilon(t) + C.
\]

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Integrating (4.14) from $s$ to $t$ and rescaling $\epsilon$, we have
\[
V_t(s) \leq e^{-\epsilon(t-s)}V_t(s) + \frac{C}{\epsilon} - \int_s^t e^{-\epsilon(t-\tau)}k\|u_t\|^{p+2} \left[ \frac{1}{2} - C\epsilon(V_\tau + C') \right] d\tau
\] (4.15)
for all $t \geq s \geq 0$.

Inequality (4.15) is exactly formula (3.44) in Theorem 3.15 in [14] with $b(\cdot) = C$ and $\gamma = \frac{p}{2(p+1)}$, and thus Theorem 3.15 in [14] can be directly applied to gain the ultimate dissipativity of the dynamical system generated by the problem (1.1)-(1.3).

5. The existence of the global attractor

Having verified the dissipativity, by Lemma [24] in order to establish the existence of the global attractor, we only need to prove that the system is asymptotically smooth. Further, by Lemma [25] it is sufficient to verify inequality (2.1). This is exactly what we do when proving the following theorem.

**Theorem 5.1.** Under Assumption [11] the dynamical system generated by the weak solution of problem (1.1)-(1.3) possesses a global attractor.

**Proof.** Let $B$ be a positively invariant bounded set in $H^1_0(\Omega) \times L^2(\Omega)$.

For any sequence $\{(u_0^{(n)}, u_1^{(n)})\}_{n=1}^{\infty}$ in $B$, we set $S(t)(u_0^{(n)}, u_1^{(n)}) = (u^{(n)}(t), u_t^{(n)}(t))$. It follows from the positive invariance property of $B$ that
\[
\|u^{(n)}(t), u_t^{(n)}(t)\|_{H^1_0(\Omega) \times L^2(\Omega)} \leq C_B, \ \forall t > 0, n \in \mathbb{N}.
\] (5.1)

Write
\[
E^{n,m}(t) = \frac{1}{2} \left[ \|\nabla(u^{(n)}(t) - u^{(m)}(t))\|^2 + \|u_t^{(n)}(t) - u_t^{(m)}(t)\|^2 \right].
\]

**Step 1.** We first estimate $E^{n,m}(T)$.

The difference $u^{(n)} - u^{(m)}$ satisfies
\[
u_t^{(n)} - u_t^{(m)} - \Delta(u^{(n)} - u^{(m)}) + k\|u_t^{(n)}\|^p v_t^{(n)} - k\|u_t^{(m)}\|^p v_t^{(m)} = -f(u^{(n)}) + f(u^{(m)}) + \Psi(u_t^{(n)} - u_t^{(m)}).
\] (5.2)

Multiplying (5.2) by $(u_t^{(n)}(t) - u_t^{(m)}(t))$ in $L^2(\Omega)$ and then integrating from $t$ to $T$, we obtain
\[
E^{n,m}(T) = E^{n,m}(t) + \int_t^T \int_{\Omega} \left[ \left( \Psi(u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) \right) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) \right.
\]
\[
- \left( f(u^{(n)}(\tau)) - f(u^{(m)}(\tau)) \right) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau))
\]
\[
- \left( k\|u_t^{(n)}(\tau)\|^p v_t^{(n)}(\tau) - k\|u_t^{(m)}(\tau)\|^p v_t^{(m)}(\tau) \right) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) \right] dxd\tau.
\] (5.3)
Integrating (5.3) with respect to $t$ between 0 and $T$ gives

$$T \cdot E^{n,m}(T)$$

$$= \int_0^T E^{n,m}(t) dt + \int_0^T \int_t^T \int_\Omega \left[ (\Psi(u_t^{(n)}(\tau) - u_t^{(m)}(\tau))) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) ight.$$

$$\left. - (f(u_t^{(n)}(\tau)) - f(u_t^{(m)}(\tau))) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) ight.$$  \(5.4\)

$$\left. - (k \parallel u_T^{(n)}(\tau) \parallel^p u_t^{(n)}(\tau) - k \parallel u_T^{(m)}(\tau) \parallel^p u_t^{(m)}(\tau)) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) \right] dx d\tau dt.$$  \(5.4\)

Multiplying (5.2) by $(u_t^{(n)}(t) - u_t^{(m)}(t))$ in $L^2(\Omega)$ and then integrating from 0 to $T$, we obtain

$$\int_0^T E^{n,m}(t) dt$$

$$= -\frac{1}{2} \left[ \int_\Omega (u_t^{(n)}(t) - u_t^{(m)}(t)) (u_t^{(n)}(t) - u_t^{(m)}(t)) dx \right]_0^T$$

$$+ \int_0^T \parallel u_t^{(n)}(t) - u_t^{(m)}(t) \parallel^2 dt$$  \(5.5\)

$$+ \frac{1}{2} \int_0^T \int_\Omega \left[ (\Psi(u_t^{(n)}(t) - u_t^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) ight.$$

$$\left. - (f(u_t^{(n)}(t)) - f(u_t^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) ight.$$  \(5.5\)

$$\left. - (k \parallel u_T^{(n)}(\tau) \parallel^p u_t^{(n)}(\tau) - k \parallel u_T^{(m)}(\tau) \parallel^p u_t^{(m)}(\tau)) (u_t^{(n)}(t) - u_t^{(m)}(t)) \right] dx dt.$$  \(5.5\)

By Lemma 2.6,

$$\int_0^T \int_\Omega \left[ (k \parallel u_T^{(n)}(\tau) \parallel^p u_t^{(n)}(\tau) - k \parallel u_T^{(m)}(\tau) \parallel^p u_t^{(m)}(\tau)) (u_t^{(n)}(t) - u_t^{(m)}(t)) \right] dx d\tau dt \geq 0. \hspace{1cm} \text{5.6}$$

Let $0 < s < 1$. We infer from (5.1) that

$$-\frac{1}{2} \left[ \int_\Omega (u_t^{(n)}(t) - u_t^{(m)}(t)) (u_t^{(n)}(t) - u_t^{(m)}(t)) dx \right]_0^T \leq C_B \hspace{1cm} \text{5.7}$$

and

$$\int_0^T \int_\Omega - (k \parallel u_T^{(n)}(\tau) \parallel^p u_t^{(n)}(\tau) - k \parallel u_T^{(m)}(\tau) \parallel^p u_t^{(m)}(\tau)) (u_t^{(n)}(t) - u_t^{(m)}(t)) dx dt$$

$$\leq k \int_0^T (\parallel u_t^{(n)}(t) \parallel^{p+1} + \parallel u_t^{(m)}(t) \parallel^{p+1}) \parallel u_t^{(n)}(t) - u_t^{(m)}(t) \parallel dt$$

$$\leq TC_B \sup_{t \in [0,T]} \parallel u_t^{(n)}(t) - u_t^{(m)}(t) \parallel$$

$$\leq TC_B \sup_{t \in [0,T]} \parallel u_t^{(n)}(t) - u_t^{(m)}(t) \parallel_{H^s(\Omega)}.$$  \(5.8\)
By (1.4) and (5.1), we have
\[
\|f(u^{(n)}(t)) - f(u^{(m)}(t))\| \\
= \left\{ \int_\Omega \left[ \int_0^1 f'(u^{(m)}(t)) + \theta(u^{(m)}(t) - u^{(m)}(t)) (u^{(n)}(t) - u^{(m)}(t)) \, d\theta \right]^2 \, dx \right\}^{1/2} \\
\leq C \left\{ \int_\Omega \left( |u^{(n)}(t)|^{N-2} + |u^{(m)}(t)|^{N-2} + 1 \right) |u^{(n)}(t) - u^{(m)}(t)|^2 \, dx \right\}^{1/2} \\
\leq C \left( \|u^{(n)}(t)\|^{\frac{2}{N-2}} + \|u^{(m)}(t)\|^{\frac{2}{N-2}} + 1 \right) \|u^{(n)}(t) - u^{(m)}(t)\|^{\frac{2}{N-2}} \\
\leq C \left( \|\nabla u^{(n)}(t)\|^{\frac{2}{N-2}} + \|\nabla u^{(m)}(t)\|^{\frac{2}{N-2}} + 1 \right) \|\nabla (u^{(n)}(t) - u^{(m)}(t))\| \\
\leq C_B.
\]
Consequently,
\[
\int_0^T \int_\Omega \left( f(u^{(n)}(t)) - f(u^{(m)}(t)) \right) (u^{(n)}(t) - u^{(m)}(t)) \, dx \, dt \\
\leq \int_0^T \|f(u^{(n)}(t)) - f(u^{(m)}(t))\| \cdot \|u^{(n)}(t) - u^{(m)}(t)\| \, dt \\
\leq TC_B \sup_{t \in [0,T]} \|u^{(n)}(t) - u^{(m)}(t)\| \\
\leq TC_B \sup_{t \in [0,T]} \|u^{(n)}(t) - u^{(m)}(t)\|_{H^s(\Omega)}.
\]
By Lemma 2.6 for any \( \epsilon > 0 \), we have
\[
\|u^{(n)}_t(t) - u^{(m)}_t(t)\|^2 \\
\leq \frac{\epsilon}{2} + C_\epsilon \|u^{(n)}_t(t) - u^{(m)}_t(t)\|^{p+2} \\
\leq \frac{\epsilon}{2} + C_\epsilon k \int_\Omega \left( \|u^{(n)}_t(t)\|^p u^{(n)}_t(t) - \|u^{(m)}_t(t)\|^p u^{(m)}_t(t) \right) (u^{(n)}_t(t) - u^{(m)}_t(t)) \, dx.
\]
We deduce from (5.1), (5.3) and (5.11) that

\[
\int_0^T \|u_t^{(n)}(t) - u_t^{(m)}(t)\|^2 dt \\
\leq \epsilon T + C_{\epsilon} \left\{ E^{m,m}(0) - E^{m,m}(T) \\
+ \int_0^T \int_\Omega \left[ (\Psi(u_t^{(n)}(t) - u_t^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) \\
- (f(u^{(n)}(t)) - f(u^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) \right] dx dt \right\} \\
\leq \frac{\epsilon}{2} T + C_{\epsilon,B} + C_{\epsilon} \int_0^T \int_\Omega \left[ (\Psi(u_t^{(n)}(t) - u_t^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) \\
- (f(u^{(n)}(t)) - f(u^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) \right] dx dt.
\]

Plugging (5.5), (5.6), (5.7), (5.8), (5.10) and (5.12) into (5.4), we obtain

\[
E^{m,m}(T) \\
\leq \frac{C_{\epsilon,B}}{T} + \epsilon + C_B \sup_{t \in [0,T]} \|u^{(n)}(t) - u^{(m)}(t)\|_{H^s(\Omega)} \\
+ C_{\epsilon,B} \frac{1 + T}{T} \int_0^T \|\Psi(u_t^{(n)}(t) - u_t^{(m)}(t))\| dt \\
+ C_{\epsilon} \left[ \left| \int_0^T \int_\Omega (f(u^{(n)}(t)) - f(u^{(m)}(t))) (u_t^{(n)}(t) - u_t^{(m)}(t)) dx dt \right| \\
+ \left| \int_0^T \int_\Omega (f(u^{(n)}(\tau)) - f(u^{(m)}(\tau))) (u_t^{(n)}(\tau) - u_t^{(m)}(\tau)) dx d\tau dt \right| \right].
\]  

**Step 2.** Next, we will investigate some convergence properties of the term on the right in (5.13).

By Alaoglu’s theorem and Lemma 2.7, we deduce from (5.1) and \( H_0^1(\Omega) \hookrightarrow H^s(\Omega) \hookrightarrow L^2(\Omega) \) that there exists a subsequence of \( \{(u^{(n)}, u_t^{(n)})\}_{n=1}^\infty \), still denoted by \( \{(u^{(n)}, u_t^{(n)})\}_{n=1}^\infty \), such that

\[
\begin{cases} 
(u^{(n)}, u_t^{(n)}) \rightharpoonup (u, v) \text{ in } L^\infty(0,T; H_0^1(\Omega) \times L^2(\Omega)), \\
u^{(n)} \to w \text{ in } C([0,T]; H^s(\Omega)), \\
\end{cases}
\]  

as \( n \to \infty \). (5.14)

Moreover, we can verify that \( v = u_t \) and \( w = u \). Indeed, by (5.14), for any \( \phi(s) \in \).
\( C^\infty_c[0, t] \) and any \( \psi_0(x) \in H^2(\Omega) \cap H^1_0(\Omega) \), we have

\[
\int_0^t \left( u_t^{(n)}(s), \phi(s) \Delta \psi_0(x) \right) ds = \int_0^t \phi(s) \frac{d}{dt} \left( u^{(n)}(s), \Delta \psi_0(x) \right) ds
\]

\[
= - \int_0^t \phi'(s) \left( u^{(n)}(s), \Delta \psi_0(x) \right) ds
\]

\[
= \int_0^t \left( \nabla u^{(n)}(s), \phi'(s) \nabla \psi_0(x) \right) ds
\]

\[
\rightarrow \int_0^t \left( \nabla u(s), \phi'(s) \nabla \psi_0(x) \right) ds
\]

\[
= \int_0^t \left( u_t(s), \phi(s) \Delta \psi_0(x) \right) ds
\]

and

\[
\int_0^t \left( u_t^{(n)}(s), \phi(s) \Delta \psi_0(x) \right) ds \rightarrow \int_0^t \left( v(s), \phi(s) \Delta \psi_0(x) \right) ds
\]

as \( n \to \infty \). It follows that \( v = u_t \).

Since

\[
\int_0^T \left( \nabla (u^{(n)}(t) - w), \nabla \varphi \right) dt = \int_0^T \left( A^{1-\frac{1}{2}} (u^{(n)}(t) - w), A^{1-\frac{1}{2}} \varphi \right) dt
\]

\[
\leq \sup_{t \in [0, T]} \| u^{(n)}(t) - w \|_{H^s(\Omega)} \| \varphi \|_{H^{2-s}(\Omega)} dt
\]

holds for any \( \varphi \in L^1(0, T; H^{2-s}(\Omega)) \), by (5.14), we have \( \int_0^T \left( \nabla (u^{(n)}(t) - w), \nabla \varphi \right) dt \rightarrow 0 \) as \( n \to \infty \), which together with (5.14) gives \( w = u \).

Let \( V \) be the completion of \( L^2(\Omega) \) with respect to the norm \( \| \cdot \|_V \) given by \( \| \cdot \|_V = \| \Psi(\cdot) \| + \| A^{-\frac{1}{2}} \cdot \| \) and \( W \) be the completion of \( L^2(\Omega) \) with respect to the norm \( \| \cdot \|_W \) given by \( \| \cdot \|_W = \| A^{-\frac{1}{2}} \cdot \| \). Since \( K \in L^2(\Omega \times \Omega) \), we have

\[
L^2(\Omega) \rightarrow V \rightarrow W. \tag{5.15}
\]

Replacing \( u^{(m)}(t) \) by 0 in (5.9) gives \( \| f(u^{(n)}(t)) - f(0) \| \leq C_B \), i.e., \( \| f(u^{(n)}(t)) \| \leq C_B \). In addition, it is easy to get

\[
\| \Psi(u^{(n)}_t(t)) \| \leq \| K \|_{L^2(\Omega \times \Omega)} \| u^{(n)}_t(t) \| \leq C_B.
\]
Therefore, from (1.1) we get
\[
\|A^{-\frac{1}{2}} u_{tt}^{(n)}(t)\| \\
\leq \|\nabla u^{(n)}(t)\| + k \|u_{t}^{(n)}(t)\|^p \|A^{-\frac{1}{2}} u_{t}^{(n)}(t)\| + \|A^{-\frac{1}{2}} (\Psi(u_{t}^{(n)}(t)) + h - f(u^{(n)}(t)))\| \\
\leq C_B.
\]
Consequently,
\[
\int_0^T \|A^{-\frac{1}{2}} u_{tt}^{(n)}(t)\| \, dt \leq C_{B,T}.
\tag{5.16}
\]
Besides, we have
\[
\int_0^T \|u_{t}^{(n)}(t)\| \, dt \leq C_{B,T}.
\tag{5.17}
\]
By Lemma 2.7, (5.15)-(5.17) imply that \(\{u_{t}^{(n)}(t)\}_{n=1}^{\infty}\) is relatively compact in \(L^1(0, T; V)\). Thus there exists a subsequence of \(\{(u^{(n)}, u_{t}^{(n)})\}_{n=1}^{\infty}\) (still denoted by itself) such that
\[
\lim_{n,m \to \infty} \int_0^T \|\Psi(u_{t}^{(n)}(t) - u_{t}^{(m)}(t))\| \, dt = 0.
\tag{5.18}
\]
In addition, it follows from (5.14) that
\[
\lim_{n,m \to \infty} \sup_{t \in [0,T]} \|u^{(n)}(t) - u^{(m)}(t)\|_{H^s(\Omega)} = 0,
\tag{5.19}
\]
which together with 5.18 gives
\[
I_1 \equiv \liminf_{n \to \infty} \liminf_{m \to \infty} \left[ C_B \sup_{t \in [0,T]} \|u^{(n)}(t) - u^{(m)}(t)\|_{H^s(\Omega)} \\
+ C_{\epsilon,B} \frac{1 + T}{T} \int_0^T \|\Psi(u_{t}^{(n)}(t) - u_{t}^{(m)}(t))\| \, dt \right] \\
= 0.
\tag{5.20}
\]
Let $F(\mu) = \int_0^\mu f(\tau)d\tau$. By (1.4) and (5.1),

$$
\left| \int_\Omega F(u^{(n)}(t))dx - \int_\Omega F(u(t))dx \right| \\
\leq \int_\Omega \left| \int_0^1 f(u(t) + \theta(u^{(n)}(t) - u(t))) \cdot (u^{(n)}(t) - u(t))d\theta \right| dx \\
\leq C \int_\Omega (|u^{(n)}(t)|^{\frac{N}{N-2}} + |u(t)|^{\frac{N}{N-2}} + 1) \cdot |u^{(n)}(t) - u(t)|dx \\
\leq C\|u^{(n)}(t) - u(t)\| \cdot (1 + \|u^{(n)}(t)\|^\frac{2}{N-2} + \|u(t)\|^\frac{2}{N-2}) \\
\leq C\|u^{(n)}(t) - u(t)\|_{H^s(\Omega)} (1 + \|\nabla u^{(n)}(t)\|^{\frac{N}{N-2}} + \|\nabla u(t)\|^{\frac{N}{N-2}}) \\
\leq C_B\|u^{(n)}(t) - u(t)\|_{H^s(\Omega)}
$$

(5.21)
holds for all $t \geq 0$.
Combining (5.14) and (5.21) gives

$$
\int_\Omega F(u^{(n)}(t))dx \Rightarrow \int_\Omega F(u(t))dx \text{ as } n \to \infty.
$$

(5.22)

It follows from $H^N(\Omega) \hookrightarrow L^\infty(\Omega)$ that $L^1(\Omega) \hookrightarrow (L^\infty(\Omega))^* \hookrightarrow H^{-N}(\Omega)$. Hence we deduce from (1.4) and (5.1) that

$$
\|A^{-\frac{N}{2}}f(u^{(n)}(t)) - A^{-\frac{N}{2}}f(u(t))\| \\
= \|f(u^{(n)}(t)) - f(u(t))\|_{H^{-N}(\Omega)} \\
\leq C\|f(u^{(n)}(t)) - f(u(t))\|_1 \\
\leq C \int_\Omega \left| \int_0^1 f'(\theta u^{(n)}(t) + (1 - \theta)u(t)) (u^{(n)}(t) - u(t))d\theta \right| dx \\
\leq C \int_\Omega (|u^{(n)}(t)|^{\frac{2}{N-2}} + |u(t)|^{\frac{2}{N-2}} + 1) \cdot |u^{(n)}(t) - u(t)|dx \\
\leq C\|u^{(n)}(t) - u(t)\| \cdot (1 + \|u^{(n)}(t)\|^\frac{2}{N-2} + \|u(t)\|^\frac{2}{N-2}) \\
\leq C\|u^{(n)}(t) - u(t)\|_{H^s(\Omega)} (1 + \|\nabla u^{(n)}(t)\|^{\frac{N}{N-2}} + \|\nabla u(t)\|^{\frac{N}{N-2}}) \\
\leq C_B\|u^{(n)}(t) - u(t)\|_{H^s(\Omega)}
$$

(5.23)
holds for all $t \geq 0$.
Combining (5.14) and (5.23) gives

$$
\sup_{t \in [0,T]} \|A^{-\frac{N}{2}}(f(u^{(n)}(t)) - f(u(t))\| \longrightarrow 0 \text{ as } n \to \infty.
$$

(5.24)
For each fixed \( t \in [0, T] \) and each \( \varphi \in L^1(0, T; H^N(\Omega) \cap H_0^1(\Omega)) \), we have

\[
\int_t^T (f(u^{(n)}(\tau)) - f(u(\tau)), \varphi) d\tau = \int_t^T (A^{n/2} (f(u^{(n)}(\tau))) - f(u(\tau))), A^{n/2} \varphi) d\tau \leq \sup_{\tau \in [0,T]} \|A^{n/2} (f(u^{(n)}(\tau))) - f(u(\tau)))\| \int_0^T \|\varphi\|_{H^N(\Omega)} d\tau,
\]

which, together with (5.24), gives

\[
\int_t^T (f(u^{(n)}(\tau)) - f(u(\tau)), \varphi) d\tau \longrightarrow 0 \text{ as } n \to \infty. \tag{5.25}
\]

Since \( L^1(t, T; H^N(\Omega) \cap H_0^1(\Omega)) \) is dense in \( L^1(t, T; L^2(\Omega)) \), (5.25) implies

\[
f(u^{(n)}) \rightharpoonup f(u) \text{ in } L^\infty(t, T; L^2(\Omega)) \text{ as } n \to \infty. \tag{5.26}
\]

By (5.14), we have

\[
(u^{(n)}, u_t^{(n)}) \rightharpoonup (u, u_t) \text{ in } L^\infty(t, T; H_0^1(\Omega) \times L^2(\Omega)) \text{ as } n \to \infty. \tag{5.27}
\]

From (5.26) and (5.27), we obtain

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_t^T \int_\Omega f(u^{(n)}(\tau))u_t^{(m)}(\tau) dxd\tau = \int_t^T \int_\Omega f(u^{(n)}(\tau))u_t(\tau) dxd\tau = \int_t^T \int_\Omega f(u(\tau))u_t dxd\tau = \int_\Omega F(u(t))dx - \int_\Omega F(u(T))dx
\]

and

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_t^T \int_\Omega f(u^{(m)}(\tau))u_t^{(n)}(\tau) dxd\tau = \int_\Omega F(u(T))dx - \int_\Omega F(u(t))dx. \tag{5.29}
\]
We deduce from (5.22), (5.28) and (5.29) that
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_t^T \int_\Omega \left( f(u^{(n)}(\tau)) - f(u^{(m)}(\tau)) \right) \cdot \left( u_t^{(n)}(\tau) - u_t^{(m)}(\tau) \right) dx d\tau = 0
\]
for all \( t \in [0, T] \).

Due to (5.1) and (5.9),
\[
\left| \int_t^T \int_\Omega \left( f(u^{(n)}(\tau)) - f(u^{(m)}(\tau)) \right) \cdot \left( u_t^{(n)}(\tau) - u_t^{(m)}(\tau) \right) dx d\tau \right| \leq C_{B,T}.
\]
(5.31)

By Lebesgue’s dominated convergence theorem, combining (5.30) and (5.31) yields
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_t^T \int_\Omega \left( f(u^{(n)}(\tau)) - f(u^{(m)}(\tau)) \right) \cdot \left( u_t^{(n)}(\tau) - u_t^{(m)}(\tau) \right) dx d\tau dt = 0.
\]
(5.32)

It follows from (5.30) and (5.32) that
\[
I_2 \equiv \lim_{n \to \infty} \lim_{m \to \infty} \left\{ \frac{C_{\epsilon,T}}{T} \left[ \left| \int_0^T \int_\Omega \left( f(u^{(n)}(t)) - f(u^{(m)}(t)) \right) \left( u_t^{(n)}(t) - u_t^{(m)}(t) \right) dx dt \right| 
+ \left| \int_t^T \int_\Omega \left( f(u^{(n)}(\tau)) - f(u^{(m)}(\tau)) \right) \left( u_t^{(n)}(\tau) - u_t^{(m)}(\tau) \right) dx d\tau dt \right| 
+ \frac{C_{\epsilon,B}}{T} + \frac{\epsilon}{2} \right\}
= \frac{C_{\epsilon,B}}{T} + \frac{\epsilon}{2}.
\]
(5.33)

We deduce from (5.13), (5.20) and (5.33) that
\[
\liminf_{m \to \infty} \liminf_{n \to \infty} E^{n,m}(T) \leq I_1 + I_2 = \frac{C_{\epsilon,B}}{T} + \frac{\epsilon}{2} \leq \epsilon
\]
for \( T \geq \frac{2C_{\epsilon,B}}{\epsilon} \), which implies
\[
\liminf_{m \to \infty} \liminf_{n \to \infty} \left\| (u^{(n)}(T), u_t^{(n)}(T)) - (u^{(m)}(T), u_t^{(m)}(T)) \right\|_{H^1_0(\Omega) \times L^2(\Omega)} \leq \sqrt{2\epsilon}.
\]
Consequently, by Lemma 2.3, the dynamical system generated by problem (1)-(3) is asymptotically smooth. In addition, Theorem 4.1 states that it is also dissipative. Thus by Lemma 2.4 it possesses a global attractor.

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