Independence in Uniform Linear Triangle-free Hypergraphs

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Abstract

The independence number \(\alpha(H)\) of a hypergraph \(H\) is the maximum cardinality of a set of vertices of \(H\) that does not contain an edge of \(H\). Generalizing Shearer's classical lower bound on the independence number of triangle-free graphs (J. Comb. Theory, Ser. B 53 (1991) 300-307), and considerably improving recent results of Li and Zang (SIAM J. Discrete Math. 20 (2006) 96-104) and Chishti et al. (Acta Univ. Sapientiae, Informatica 6 (2014) 132-158), we show that

\[
\alpha(H) \geq \sum_{u \in V(H)} f_r(d_H(u))
\]

for an \(r\)-uniform linear triangle-free hypergraph \(H\) with \(r \geq 2\), where

\[
\begin{align*}
  f_r(0) &= 1, \\
  f_r(d) &= \frac{1 + ((r-1)d^2 - d) f_r(d-1)}{1 + (r-1)d^2} \quad \text{for } d \geq 1.
\end{align*}
\]

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1 Introduction

We consider finite hypergraphs \(H\), which are ordered pairs \((V(H), E(H))\) of two sets, where \(V(H)\) is the finite set of vertices of \(H\) and \(E(H)\) is the set of edges of \(H\), which are subsets of \(V(H)\). The order \(n(H)\) of \(H\) is the cardinality of \(V(H)\). The degree \(d_H(u)\) of a vertex \(u\) of \(H\) is the number of edges of \(H\) that contain \(u\). The average degree \(d(H)\) of \(H\) is the arithmetic mean of the degrees of its vertices. Two distinct vertices of \(H\) are adjacent or neighbors if some edge of \(H\) contains both. The neighborhood \(N_H(u)\) of a vertex \(u\) of \(H\) is the set of vertices of \(H\) that are adjacent to \(u\). For a set \(X\) of vertices of \(H\), the hypergraph \(H - X\) arises from \(H\) by removing from \(V(H)\) all vertices in \(X\) and removing from \(E(H)\) all edges that intersect \(X\). If every two distinct edges of \(H\) share at most one vertex, then \(H\) is linear. If \(H\) is linear and for every two distinct non-adjacent vertices \(u\) and \(v\) of \(H\), every edge of \(H\) that contains \(u\) contains at most one neighbor of \(v\), then \(H\) is double linear. If there are not three distinct vertices \(u_1, u_2,\) and \(u_3\) of \(H\) and three distinct edges \(e_1, e_2,\) and \(e_3\) of \(H\) such that \(\{u_1, u_2, u_3\} \subseteq e_i\) for \(i \in \{1, 2, 3\}\), then \(H\) is triangle-free.
A set $I$ of vertices of $H$ is a (weak) independent set of $H$ if no edge of $H$ is contained in $I$. The (weak) independence number $\alpha(H)$ of $H$ is the maximum cardinality of an independent set of $H$. If all edges of $H$ have cardinality $r$, then $H$ is $r$-uniform. If $H$ is 2-uniform, then $H$ is referred to as a graph.

The independence number of (hyper)graphs is a well studied computationally hard parameter. Caro \[4\] and Wei \[14\] proved a classical lower bound on the independence number of graphs, which was extended to hypergraphs by Caro and Tuza \[5\]. Specifically, for an $r$-uniform hypergraph $H$, Caro and Tuza \[5\] proved

$$\alpha(H) \geq \sum_{u \in V(H)} f_{CT(r)}(d_H(u)),$$

where $f_{CT(r)}(d) = \left(\frac{d + \frac{1}{d} - 1}{d} \right)$. Thiele \[13\] generalized Caro and Tuza’s bound to general hypergraphs; see \[3\] for a very simple probabilistic proof of Thiele’s bound. Originally motivated by Ramsey theory, Ajtai et al. \[2\] showed that $\alpha(G) = \Omega\left(\frac{\ln d(G)}{d(G)} n(G)\right)$ for every triangle-free graph $G$. Confirming a conjecture from \[2\] concerning the implicit constant, Shearer \[11\] improved this bound to $\alpha(H) \geq f_{S_1}(d(G)) n(G)$, where $f_{S_1}(d) = \frac{\ln d - d + 1}{(d-1)^2}$. In \[11\] the function $f_{S_1}$ arises as a solution of the differential equation

$$(d + 1) f(d) = 1 + (d - d^2) f'(d) \text{ and } f(0) = 1.$$ 

In \[12\] Shearer showed that

$$\alpha(G) \geq \sum_{u \in V(G)} f_{S_2}(d_G(u))$$

for every triangle-free graph $G$, where $f_{S_2}$ solves the difference equation

$$(d + 1) f(d) = 1 + (d - d^2) \left(f(d) - f(d - 1)\right) \text{ and } f(0) = 1.$$ 

Since $f_{S_1}(d) \leq f_{S_2}(d)$ for every non-negative integer $d$, and $f_{S_1}$ is convex, Shearer’s bound from \[12\] is stronger than his bound from \[11\].

Li and Zang \[10\] adapted Shearer’s approach to hypergraphs and obtained the following.

**Theorem 1 (Li and Zang \[10\]):** Let $r$ and $m$ be positive integers with $r \geq 2$.

If $H$ is an $r$-uniform double linear hypergraph such that the maximum degree of every subhypergraph of $H$ induced by the neighborhood of a vertex of $H$ is less than $m$, then

$$\alpha(H) \geq \sum_{u \in V(H)} f_{LZ(r,m)}(d_H(u)),$$

where

$$f_{LZ(r,m)}(x) = \frac{m}{B} \int_0^1 \frac{(1 - t)^{\frac{a}{m}}}{t^b(m - (x - m)t)} dt,$$

$$a = \frac{1}{(r-1)^2}, \quad b = \frac{r-2}{r-1}, \quad \text{and} \quad B = \int_0^1 (1 - t)^{\left(\frac{a}{m} - 1\right)} t^{-b} dt.$$ 

Note that for $r \geq 2$, an $r$-uniform linear hypergraph $H$ is triangle-free if and only if it is double linear and the maximum degree of every subhypergraph of $H$ induced by the neighborhood of a vertex of $H$ is less than 1. Therefore, since $f_{S_1} = f_{LZ(2,1)}$ and $f_{S_1}$ is convex, Theorem 1 implies Shearer’s bound from \[11\]. Nevertheless, since $f_{S_1}(d) < f_{S_2}(d)$ for every integer $d$ with $d \geq 2$, Shearer’s bound from \[11\] does not quite follow from Theorem 1.

In \[6\] Chishti et al. presented another version of Shearer’s bound from \[11\] for hypergraphs.
Theorem 2 (Chishti et al. \[6\]) Let $r$ be an integer with $r \geq 2$.
If $H$ is an $r$-uniform linear triangle-free hypergraph, then
\[
\alpha(H) \geq f_{CZPI}(d(H))n(H),
\]
where
\[
f_{CZPI}(x) = \frac{1}{r-1} \int_0^1 \frac{1-t}{t^b(1-(r-1)x-1)t)dt}
\]
and $b = \frac{r-2}{r-1}$.

Since $f_{S_1} = f_{CZPI(2)}$, for $r = 2$, the last result coincides with Shearer’s bound from \[11\].
A drawback of the bounds in Theorem 1 and Theorem 2 is that they are very often weaker than Caro and Tuza’s bound \[5\], which holds for a more general class of hypergraphs. See Figure 1 for an illustration.

![Figure 1: The values of $f_{LZ(r,1)}(d)$ (line), $f_{CZPI(r)}(d)$ (dashed line), $f_{CT(r)}(d)$ (empty circles), and $f_r(d)$ (solid circles) for $0 \leq d \leq 40$ and $r = 3$ (left) and $r = 4$ (right).](image)

In the present paper we extend Shearer’s approach from \[12\] and establish a lower bound on the independence number of a uniform linear triangle-free hypergraph that considerably improves Theorem 1 and Theorem 2 and is systematically better than Caro and Tuza’s bound.
For further related results we refer to Ajtai et al. \[1\], Duke et al. \[7\], Dutta et al. \[8\] and Kostochka et al. \[9\]. Note that our main result provides explicit values when applied to a specific hypergraph but that we do not completely understand its asymptotics. In contrast to that, results as in \[1\] \[7\] \[8\] are essentially asymptotic statements but are of limited value when applied to a specific hypergraph.
2 Results

For an integer \( r \) with \( r \geq 2 \), let \( f_r : \mathbb{N}_0 \to \mathbb{R}_0 \) be such that

\[
\begin{align*}
  f_r(0) &= 1 \\
  f_r(d) &= 1 + \frac{(r-1)d^2 - d}{1 + (r-1)d^2} f_r(d-1)
\end{align*}
\]

for every positive integer \( d \).

Lemma 3 If \( r \) and \( d \) are integers with \( r \geq 2 \) and \( d \geq 0 \), then \( f_r(d) - f_r(d+1) \geq f_r(d+1) - f_r(d+2) \).

Proof: Substituting within the inequality \( f_r(d) - 2f_r(d+1) + f_r(d+2) \geq 0 \) first \( f_r(d+2) \) with

\[
1 + \frac{(r-1)(d+2)^2 - (d+2)}{1 + (r-1)(d+2)^2} f_r(d+1)
\]

and then \( f_r(d+1) \) with

\[
1 + \frac{(r-1)(d+1)^2 - (d+1)}{1 + (r-1)(d+1)^2} f_r(d)
\]

and solving it for \( f_r(d) \), it is straightforward but tedious to verify that it is equivalent to \( f_r(d) \geq L(r,d) \) where

\[
L(r,d) = \frac{(2r-1)d + 3r}{r(d^2 + 5d + 5)}.
\]

Therefore, in order to complete the proof, it suffices to show \( f_r(d) \geq L(r,d) \). For \( d = 0 \), we have \( f_r(0) = 1 > \frac{3}{5} = L(r,0) \). Now, let \( f(d) \geq L(r,d) \) for some non-negative integer \( d \). Since \((r-1)(d+1)^2 - (d+1) \geq 0\), we obtain by a straightforward yet tedious calculation

\[
f(d+1) - L(r,d+1) = \frac{1 + \left( (r-1)(d+1)^2 - (d+1) \right) f(d)}{1 + (r-1)(d+1)^2} - L(r,d+1)
\]

\[
\geq \frac{(1 + \left( (r-1)(d+1)^2 - (d+1) \right) L(r,d))}{1 + (r-1)(d+1)^2} - L(r,d+1)
\]

\[
= \frac{2(1 + (r-1)(d+2)^2)}{r(d^2 + 7d + 11)(d^2 + 5d + 5)},
\]

which is positive for \( r \geq 2 \). Therefore, \( f(d+1) \geq L(r,d+1) \), which completes the proof by an inductive argument. \( \square \)

The following is our main result.

Theorem 4 Let \( r \) be an integer with \( r \geq 2 \).

If \( H \) is an \( r \)-uniform linear triangle-free hypergraph, then

\[
\alpha(H) \geq \sum_{u \in V(H)} f_r(d_H(u)).
\]
Before we proceed to the proof, we compare our bound to the bounds of Caro and Tuza [5], Li and Zang [10], and Chishti et al. [6]. Figure 1 illustrates some specific values. An inspection of Li and Zang’s proof in [10] reveals that they actually prove a lower bound on the so-called strong independence number, which is defined as the maximum cardinality of a set of vertices that does not contain two adjacent vertices. Therefore, especially for large values of $r$, Theorem 1 is much weaker than Theorem 2. In fact, it is quite natural that it is worse by a factor of about $r - 1$.

As we show now, our bound is systematically better than Caro and Tuza’s bound [5].
Lemma 5 If \( r \) and \( d \) are integers with \( r \geq 3 \) and \( d \geq 2 \), then \( f_r(d) > f_{CT(r)}(d) \).

Proof: Note that \( f_r(0) = f_{CT(r)}(0) = 1 \), \( f_r(1) = f_{CT(r)}(1) = \frac{r-1}{r} \), and \( f_{CT(r)}(d) = \frac{d}{d+r-1} f_{CT(r)}(d-1) \) for \( d \in \mathbb{N} \), which immediately implies that \( f_{CT(r)}(d) < \frac{r-1}{r} \) for \( d \geq 2 \). Now, if \( f_r(d-1) \geq f_{CT(r)}(d-1) \) for some \( d \geq 2 \), then the statement follows by induction, because every \( u \) belongs to exactly one of the sets in \( R \) and \( \sum_{v \in V(H) \setminus \{x\}} f_r(d_{H-x}(u)) = \sum_{u \in V(H)} f_r(d_H(u)) \).

Hence we may assume that \( H \) has no vertex of degree 0.

Since \( H \) is \( r \)-uniform and linear, for every two edges \( e_1 \) and \( e_2 \) with \( e_1 \cap e_2 = \{u\} \) for some vertex \( u \) of \( H \), the sets \( e_1 \setminus \{u\} \) and \( e_2 \setminus \{u\} \) are disjoint and of order \( r-1 \). Therefore, for every vertex \( u \) of \( H \), there is a set \( R(u) \) of \( r-1 \) sets of neighbors of \( u \) such that every neighbor of \( u \) belongs to exactly one of the sets in \( R(u) \), and \( |e \cap R| = 1 \) for every edge \( e \) of \( H \) with \( u \in e \) and every \( R \in R(u) \).

If \( x \) is a vertex of \( H \) and \( R \in \mathcal{R}(x) \) is such that

\[
1 + \sum_{u \in V(H) \setminus \{x\} \cup R} f_r(d_{H-(\{x\} \cup R)}(u)) \geq \sum_{u \in V(H)} f_r(d_H(u)),
\]

then the statement follows by induction, because \( \alpha(H) \geq 1 + \alpha(H - (\{x\} \cup R)) \). Therefore, in order to complete the proof, it suffices to show that the following term is non-negative:

\[
P = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \left( 1 + \sum_{u \in V(H) \setminus \{x\} \cup R} f_r(d_{H-(\{x\} \cup R)}(u)) - \sum_{u \in V(H)} f_r(d_H(u)) \right).
\]
Since $H$ is linear and triangle-free, we have $d_{H-(\{x\} \cup R)}(z) = d_H(z) - |N_H(z) \cap R|$ for every vertex $z$ in $V(H) \setminus (\{x\} \cup R)$. Trivially, $d_{H-(\{x\} \cup R)}(z) = d_H(z)$ for $z \not\in N_H(R)$, and hence $P$ equals $P_1 + P_2$, where

$$P_1 = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \left( 1 - f_r(d_H(x)) - \sum_{y \in R} f_r(d_H(y)) \right)$$

and

$$P_2 = \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_H(R) \setminus \{x\}} \left( f_r(d_H(z) - |N_H(z) \cap R|) - f_r(d_H(z)) \right)$$

Since for every vertex $u$ of $H$, there are exactly $(r-1)d_H(u)$ many vertices $v$ of $H$ such that $u$ belongs to exactly one of the sets in $\mathcal{R}(v)$, we have

$$P_1 = \sum_{x \in V(H)} \left( (r-1) - (r-1)(d_H(x) + 1)f_r(d_H(x)) \right).$$

Since $f_r(d-1) - f_r(d)$ is decreasing by Lemma 3, we have $f_r(d-n) - f_r(d) \geq n(f_r(d-1) - f_r(d))$ for all positive integers $d$ and $n$ with $n < d$. Therefore,

$$P_2 \geq \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{z \in N_H(R) \setminus \{x\}} |N_H(z) \cap R| \left( f_r(d_H(z) - 1) - f_r(d_H(z)) \right)$$

$$= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_H(R) \setminus \{x\}} |N_H(z) \cap \{y\}| \left( f_r(d_H(z) - 1) - f_r(d_H(z)) \right)$$

$$= \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_H(R) \setminus \{x\}} \sum_{y \in R} \left( f_r(d_H(z) - 1) - f_r(d_H(z)) \right)$$

Let $T$ be the set of all 4-tupels $(x, R, y, z)$ with $x \in V(H)$, $R \in \mathcal{R}(x)$, $y \in R$, and $z \in N_H(y) \setminus \{x\}$. Note that $y \in N_H(z)$ for every $(x, R, y, z)$ in $T$. Since $H$ is linear, for a given vertex $z$ of $H$ and a given neighbor $y$ of $z$, there are $(r-1)d_H(y) - 1$ many vertices $x$ of $H$ with $y \in R$ for some $R$ in $\mathcal{R}(x)$ and $z \in N_H(y) \setminus \{x\}$. Furthermore, by the properties of $\mathcal{R}(x)$, given $x$ and $y$, the set $R$ in $\mathcal{R}(x)$ with $y \in R$ is unique. Therefore,

$$P_2 \geq \sum_{x \in V(H)} \sum_{R \in \mathcal{R}(x)} \sum_{y \in R} \sum_{z \in N_H(y)} \sum_{y \in R} \left( f_r(d_H(z) - 1) - f_r(d_H(z)) \right)$$

$$= \sum_{z \in V(H)} \sum_{y \in N_H(z)} \left( (r-1)d_H(y) - 1 \right) \left( f_r(d_H(z) - 1) - f_r(d_H(z)) \right).$$

Let $\mathcal{E}$ be the edge set of the graph that arises from $H$ by replacing every edge of $H$ by a clique, that is, $\mathcal{E}$ is the set of all sets containing exactly two adjacent vertices of $H$.

We obtain

$$P_2 \geq \sum_{z \in V(H)} \sum_{y \in N_H(z)} \left( (r-1)d_H(y) - 1 \right) \left( f_r(d_H(z) - 1) - f_r(d_H(z)) \right)$$

$$= \sum_{\{y, z\} \in \mathcal{E}} \left( h_1(y)h_2(z) + h_1(z)h_2(y) \right), \text{ where}$$

$$h_1(x) = (r-1)d_H(x) - 1 \quad \text{and} \quad h_2(x) = f_r(d_H(x) - 1) - f_r(d_H(x)).$$

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If \( d_H(y) \geq d_H(z) \), then \( h_1(y) \geq h_1(z) \) and, by Lemma 3, \( h_2(z) \geq h_2(y) \), which implies
\[
\left( h_1(y) - h_1(z) \right) \left( h_2(z) - h_2(y) \right) \geq 0.
\]
Therefore, \( h_1(y)h_2(z) + h_1(z)h_2(y) \geq h_1(y)h_2(y) + h_1(z)h_2(z) \).

Since, for every vertex \( y \) of \( H \), there are exactly \( (r - 1)d_H(y) \) many vertices \( z \) of \( H \) with \( \{y, z\} \in E \), we obtain
\[
P_2 \geq \sum_{\{y, z\} \in E} \left( h_1(y)h_2(z) + h_1(z)h_2(y) \right)
\geq \sum_{\{y, z\} \in E} \left( h_1(y)h_2(y) + h_1(z)h_2(z) \right)
= \sum_{x \in V(H)} (r - 1)d_H(x)h_1(x)h_2(x)
= \sum_{x \in V(H)} (r - 1)d_H(x) \left( (r - 1)d_H(x) - 1 \right) \left( f_r(d_H(x) - 1) - f_r(d_H(x)) \right).
\]
Combining these estimates, we see that
\[
P = P_1 + P_2 \geq \sum_{x \in V(H)} \left( (r - 1) - (r - 1)(d_H(x) + 1)f_r(d_H(x)) \right.
+ (r - 1)d_H(x) \left( (r - 1)d_H(x) - 1 \right) \left( f_r(d_H(x) - 1) - f_r(d_H(x)) \right),
\]
which is 0 by the definition of \( f_r \). This completes the proof. \( \square \)

It seems a challenging task to extend the presented results to non-uniform and/or non-linear triangle-free hypergraphs.

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