In this communication, one shows that there exists in the literature a certain form of deformed derivative that can here be identified as the dual of conformable derivative. The deformed subtraction is used here, together with the duality concept, as the basic definitions and starting points in order to obtain the connected dual operators. The q-exponential, in the context of generalized statistical mechanics, is the eigenfunction of this dual conformable derivative. The basic properties of the dual deformed-derivatives and also some perspective of applications and simple models are presented. The importance of this deformed derivative for position-dependent models is highlighted. An outlook of potential applications and developments is presented.

Keywords: Dual Conformable derivatives, Generalized Statistical Mechanics, Deformed Operators.

1. INTRODUCTION

Over the last decades, several researchers have been pursuing alternative formalisms and mathematical tools in order to describe complex systems in a more reliable way. Complex systems are known to be composed by a large number of simple members that mutually interact and can also interact with their environment; for that reason they are considered as open systems. They have the potential to give rise to macroscopic new collective behavior and properties, including the manifestation of new structures due to self-organization and internal-structure interactions, new dynamics and possible macroscopic emergent properties. Complex sys-
tems most likely involve nonlinear interactions between sub-units with enhanced behavior of coherence or order that extends far away from what could be reached to any individual sub-unit. That is, there are long-range interactions.

The use of deformed derivative is justified here based on our proposition that there exists an intimate relationship between dissipation, coarse-grained media and some limit scale of energy for the interactions, as already explained in Refs. [1–5]. In this context, deformed derivatives have emerged as a proposal to deal with complex problems with the mathematical tools of local operators.

One of the most recent definitions of deformed derivative appeared in 2014, with Khalil [6]; it was called conformable derivative. Another important kind of deformed derivative is the $q$-derived, in the context of the nonextensive statistical mechanics [7] and also fractal derivative [8], also called Hausdorff derivative [9, 10].

Here, one consider that physical basis involved in the justification for the use of deformed derivatives, the mapping into the fractal continuum [1, 11–13]. We are not talking about including classical definitions of fractional derivatives nor including operators of integer order acting on a $d$-dimensional space, but one considers a mapping from a fractal coarse-grained (fractal porous) space, which is essentially discontinuous in the embedding Euclidean space, to a continuous one [2]. A mapping into a continuous fractal space naturally yields the need for modifications in the derivatives and, with connection with the metric, the modifications of the derivatives leads to a change in the algebra involved, which, in turn may, lead arrive at a generalized statistical mechanics with some suitable definition of entropy [3].

Another explanation for deformed derivatives can be thought in terms of a canonical transformation from one Euclidean space to a deformed space [14]. A number of discussions on the physical interpretation of deformed derivatives in terms of the Gateaux extended derivative can be found in Ref. [15].

To develop the dual conformable derivative, some concepts are necessary. One of these is the concept of duality for derivative operators. The basic ideas of duality, for $q$-deformed derivative operators, are originally found in Ref. [16, 17]. But there, in terms of a property instead of a basic definition. Also in the reference 16, appears a derivative of form $\tilde{D}_x^\alpha f = f^{\alpha - 1} \frac{df}{dx}$, where there was indicated that it has been first appeared, in this form, in Ref. [18]. The latter reference does not make clear the real origin of this derivative and was inserted it in a certain ad hoc mode. Also in Ref. 16, the author shows that the dual derivative, related to its dual
\( \tilde{D}_x^\alpha f \), is \( D_x^\alpha f = x^{1-\alpha} \frac{df}{dx} \), indicating the origin of the latter operator in the article of one of the authors here, Weberszpil [1], but not making clear the origin of both derivatives, \( D_x^\alpha f \) and its dual \( \tilde{D}_x^\alpha f \) and also, unfortunately, erroneously associated with fractional calculus, that is a non-local formalism. The deformed chain rule was also indicated there. The deformed algebra in those references is also different from the one presented here.

In this contribution, inspired on the concepts of deformed operations, duality and the definition of the conformable derivative (CD), one shows that the deformed operator \( \tilde{D}_x^\alpha f \) is in fact a dual conformable derivative (DCD) operator. Also, by the deformed chain rule, one shows that eigenvalue problems with q-exponential, in the sense of generalizes statistical mechanics, can be more suitably treated with the DCD operator.

Our Letter is outlined as follows. Section 2 addresses some mathematical aspects to define the deformed subtraction and the \( \tilde{D}_x^\alpha f \) operator. In Section 3, we focus on the duality concept and identification of \( \tilde{D}_x^\alpha f \) as the DCD operator and the dual conformable integral. In Section 4, one obtains the deformed Leibniz product rule and the deformed chain rule. Section 5 is devoted to some eigenfunction and eigenvalue aspects and potential applications. Finally, in Section 6, we cast our general conclusions and possible paths for further investigations.

2. CONFORMABLE SUBTRACTION

In this section, motivated by the deformation on the argument of CD [6], one obtains a deformed subtraction operation, that one call Conformable Subtraction.

The CD is defined as [6]

\[
D_x^\alpha f(x) = \lim_{\epsilon \to 0} \frac{f(x + \epsilon x^{1-\alpha}) - f(x)}{\epsilon}.
\]  

(1)

Note that the deformation is placed in the independent variable.

For differentiable functions, the CD can be written as

\[
D_x^\alpha f = x^{1-\alpha} \frac{df}{dx}.
\]  

(2)

This can be obtained by a simple change of variable [6].

One now proceeds to briefly investigate two algebraic operations.

The argument of eq. (1) is

\[
y = x + \epsilon x^{1-\alpha}.
\]  

(3)

So, the infinitesimal parameter \( \epsilon \) is

\[
\epsilon = \frac{y - x}{x^{1-\alpha}}.
\]  

(4)

Motivated by the generalized statistical mechanics approach and its related algebra [7, 19], with the help eq.(1), one can define a
conformable subtraction as
\[ y \ominus_\alpha x \equiv \frac{y - x}{x^{1 - \alpha}}. \] (5)

With those simple algebraic definitions, the conformable derivative and its dual conformable derivative, respectively, can be expressed formally as
\[ D_\alpha x F \equiv \lim_{y \to x} \frac{F(y) - F(x)}{y \ominus_\alpha x}, \] (6)
\[ \tilde{D}_\alpha x F \equiv \lim_{y \to x} \frac{F(y) \ominus_\alpha F(x)}{y - x}. \] (7)

For the DCD, it can be explicitly written as
\[ \tilde{D}_\alpha x F = \lim_{y \to x} \frac{F(y) - F(x)}{y - x}[F(x)]^{\alpha-1} = F^{\alpha-1} \frac{dF}{dx}. \] (8)

This latter expression first appeared in Ref. [18], but without a clear origin.

Connections with Chen’s fractal derivative

It is worthy a short note here on a connection with Chen’s fractal derivative [8–10].

The Chen’s fractal derivative is defined as
\[ \frac{\partial g(x)}{\partial x^\alpha} = \lim_{x' \to x} \frac{g(x') - g(x)}{x'^{1 - \alpha} - x^\alpha}. \] (9)

Now, by a simple variable change,
\[ x' = x + \epsilon x^{1 - \alpha} \]
and consequently \( \epsilon = \frac{x' - x}{x^{1 - \alpha}} = x' \ominus_\alpha x \).

For \( \epsilon \ll 1 \), we can write, at first order in \( \epsilon \):
\[ x'^\alpha = x^\alpha \left[ 1 + \epsilon x^{-\alpha} \right]^{\alpha} \approx x^\alpha \left[ 1 + \alpha \epsilon x^{-\alpha} \right], \] (10)
\[ x'^\alpha - x^\alpha \approx \alpha \epsilon = \alpha \left[ \frac{x' - x}{x^{1 - \alpha}} \right] = \alpha (x' \ominus_\alpha x). \] (11)

So,
\[ \frac{\partial g(x)}{\partial x^\alpha} = \lim_{x' \to x} \frac{g(x') - g(x)}{x'^{1 - \alpha} - x^\alpha} \approx \lim_{x' \to x} \frac{F(x') - F(x)}{\alpha (x' \ominus_\alpha x)} = \frac{x^{1 - \alpha} \frac{dF}{dx}}{\alpha \frac{dF}{dx}}. \] (12)

The result above shows that the Chen’s fractal derivative is proportional to CD for differentiable functions, up to the first order in the deformation parameter \( \epsilon \).

3. DUALITY FOR CONFORMABLE DERIVATIVE AND CONFORMABLE INTEGRAL

The concept of duality for derivative operators is already present in the standard calculus and it can be understood in a simple manner.

Consider the first order derivative of a real bijective continuous differentiable function \( y; y, x \in \mathbb{R} \).

\[ D_1^1 y = \frac{dy}{dx}, \] (13)
Now, considering an domain-image interchange as \( x = x(y) \), we can define the dual derivative of \( x \) relative to \( y \) as

\[
\tilde{D}_y^1 x = \frac{dx}{dy}.
\]  

(14)

The following property will be considered as the main definition of derivative duality:

\[
(\tilde{D}_y^1 x(y))(D_x^1 y(x)) = 1.
\]  

(15)

In this sense, this means that the first order derivative and its dual are self-dual [16]. That is,

\[
\frac{dy}{dx} \cdot \frac{dx}{dy} = 1,
\]  

(16)

or

\[
\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1}.
\]  

(17)

Now, one can follow in the direction to extend this concept of duality for the derivative operators and for a special case of deformed derivatives. The conformable derivative is one kind of deformed derivatives of a function \( f \).

Considering the defining property (15) extended to conformable derivative, the dual conformable-derivative can be defined, considering one independent variable \( x \) as function of \( y \), where \( x, y \in \mathbb{R} \), continuous and differentiable functions:

\[
\tilde{D}_y^\alpha x = \left[x^{1-\alpha} \frac{dy}{dx}\right]^{-1} = x^{\alpha-1} \frac{dx}{dy}.
\]  

(18)

For a continuous differentiable real valued function \( F \), with \( x \) as the independent variable, one can properly generalize the dual conformable-derivative as

\[
\tilde{D}_x^\alpha F = F^{\alpha-1} \frac{dF}{dx}.
\]  

(19)

that corresponds to the expression of dual conformable-derivative in eq. (8), for differentiable functions.

The expression in eq. (19) is identical to eq. (8).

All the operators cast above are local operators and not related to any version of non-integer fractional calculus; the latter is a non-local operator. This is emphasized here in order to avoid any possible misunderstanding.

In short, one can cast the expressions for conformable-derivative and its dual, for differentiable functions:

\[
\begin{align*}
D_x^\alpha F &= x^{1-\alpha} \frac{dF}{dx}, & (Conformable) \\
\tilde{D}_x^\alpha F &= F^{\alpha-1} \frac{dF}{dx}, & (Dual – conformable)
\end{align*}
\]  

(20)

An important point to be noticed here is that the deformations affect different functional spaces, depending on the problem under consideration. For the conformable derivative, the deformations are put in the independent variable, which can be a space coordinate, in the case of, e.g, mass position dependent problems, or even time or space-time variables, for temporal dependent pa-
rameter or relativistic problems. The physics will be main guide.

Deformed derivatives and deformed dual derivatives, in the context of generalized statistical mechanics are also present. There, the q-deformed derivative has also a dual derivative and a q-exponential related function. For details the reader can see the Ref. [14] and references therein. But here, the advantage for the use of dual-conformable-derivative will be evident, particularly for issues associated with eigenvalue problems and mass dependent problems.

The Dual Conformable-Integral

In order to define a dual-conformable-integral, one recalls that the conformable integral is already known to have the Riemann integral structure [2, 6], in such a way that

\[ D_α^x F = \int F(x) d^α x, \]  

(21)

Here, the differential element \( d^α x \) is \( d^α x = x^{α−1} dx \).

Following this reasoning, one can write an expression for the dual conformable integral, in such a manner that the fundamental theorem of calculus is also valid:

\[ \tilde{D}_α^x (\tilde{I}_x^α F) = F. \]  

(22)

Defining the dual conformable integral as

\[ \tilde{I}_x^α F = \int F^{1−α}(x) F(x) dx, \]  

(23)

can be easily verified that

\[ \tilde{D}_x^α (\tilde{I}_x^α F) = F^{α−1}(x) \frac{d}{dx} \left( \int F^{1−α}(x) F(x) dx \right) = F(x). \]  

(24)

In so doing, we are ensuring the validity of the fundamental theorem of calculus.

4. THE LEIBNIZ PRODUCT AND THE CHAIN RULES FOR DUAL CONFORMABLE DERIVATIVE

For conformable derivative, several properties are well known and it is clear that they share similarities with the standard calculus [6]. So, the Leibniz product property and the chain rule are, respectively,

\[ D_α^x (F G) = (D_α^x F) G + (D_α^x G) F, \]  

(25)

\[ D_α^x [F(G(x))] = x^{1−α} \frac{d[F(G(x))]}{dx} = x^{1−α} \frac{dF}{dG} \frac{dG}{dx} = \frac{dF}{dG} (D_α^x G). \]  

(26)

For the dual-conformable derivative, the rules are directly obtained as follows. For
Leibniz product rule one has

\[ \tilde{D}_x^\alpha (FG) = F^{\alpha-1} G^{\alpha-1} \frac{d(FG)}{dx} = \]

\[ = F^{\alpha-1} G^{\alpha-1} \left[ G \frac{dF}{dx} + F \frac{dG}{dx} \right], \tag{27} \]

that is symbolized as

\[ \tilde{D}_x^\alpha (FG) = G^{\alpha} (\tilde{D}_x^\alpha F) + F^{\alpha} (\tilde{D}_x^\alpha G). \tag{28} \]

The chain rule is determined as

\[ \tilde{D}_x^\alpha [F(G(x))] = F^{\alpha-1} \frac{dF(G(x))}{dx} = \]

\[ = F^{\alpha-1} \frac{dF}{dG} \frac{dG}{dx} = \frac{dG}{dx} (\tilde{D}_x^\alpha F). \tag{29} \]

That is,

\[ \tilde{D}_x^\alpha [F(G(x))] = \frac{dG}{dx} (\tilde{D}_x^\alpha F). \tag{30} \]

This important result is of great valuable for eigenvalue problems. Particularly with generalized statistical mechanics and with the so called q-exponential functions.

Also important to note is that DCD is a nonlinear operator and in this sense,

\[ \tilde{D}_x^\alpha [F + G] = \left( \frac{F + G}{f + g} \right)^{\alpha-1} \left[ G^{\alpha-1} \tilde{D}_x^\alpha F + F^{\alpha-1} \tilde{D}_x^\alpha G \right] \]

\[ \neq \tilde{D}_x^\alpha F + \tilde{D}_x^\alpha G. \tag{31} \]

5. DUAL CONFORMABLE DERIVATIVE EIGENFUNCTION AND EIGENVALUE PROBLEMS

The stretched exponential is an eigenfunction of the \( D_x^\alpha \) operator \[1\], since the simple solution of the equation

\[ x^{1-\alpha} \frac{dy}{dx} = y, \tag{32} \]

along with the the initial condition \( y(0) = 1 \),

leads to the solution as an stretched exponential of form

\[ y = \exp \left[ \frac{x^\alpha}{\alpha} \right]. \tag{33} \]

Analogously, one may think of an equation for the dual-conformable-derivative as

\[ [F(x)]^{\alpha-1} \frac{dF(x)}{dx} = F(x). \tag{34} \]

Solving, along with the condition \( F(0) = 1 \),

lead to the following important function

\[ F(x) = [1 + (\alpha - 1)x]^{1/(\alpha - 1)}. \tag{35} \]

This function is nothing but the reparametrized q-exponential, ubiquitous in the Tsallis version of generalized statistical mechanics \[20\]. This can clearly seen by redefining the relevant parameter \( \alpha \) in term of the entropic parameter \( q \), as \( \alpha - 1 = 1 - q \), or \( \alpha = 2 - q \). By this reparametrization, the solution of eq. \[34\] becomes
\[ F(x) = [1 + (1 - q)x]^{1/(1-q)}, \quad (36) \]
that is exactly the q-exponential \[ e^{xq}(x), \] in the Tsallis generalized statistical mechanics.

A very important point to stress here is that, in the case of dual-conformable-derivative operator, the chain rule formalized in eq. (30), allows eigenvalue problems. The q-derivative, a version used in the context of Tsallis formalism, does not allow this. It can directly be proven, with the help of eqs. (19, 30), that the dual conformable-derivative applied to an q-exponential function, with a dilatation on the independent variable, leads to an eigenvalue problem, as follows:

\[ \tilde{D}_{x}^{2-q}(e_{2-q}(\lambda x)) = \lambda e_{2-q}(\lambda x), \quad (37) \]
where \( e_{q}(\lambda x) \) is the q-exponential and \( \lambda \) is some real eigenvalue and we used \( \alpha = 2 - q \).

It can be verified that this not occur with the use of q-derivative, unless \( \lambda = 1 \) or if one use another kind of q-derivative, called scale-q-derivative [2, 4].

Another point to observe in eq. (37) is the appearance of \( \alpha \to 2 - q \) duality. In terms of generalized statistical mechanics, some aspects of this duality were studied in Refs. [21, 23].

One more point to highlight here is that, despite this is the first time that dual conformable-derivative is more clearly connected with conformable derivative operator, the appearance of analogous derivative is reported in Ref. [18], however, in this article, it was introduced in some an ad hoc manner.

Potential Applications

- Position-dependent Problems: A simple example is a model for a mass-dependent harmonic oscillator. Using the dual conformable derivative definition, one can deform Newtonian mechanics. For example, deforming the time derivative, as follows:

\[ x^{\alpha-1} \frac{d(D_{x}^{(\alpha)} x)}{dt} = -(\omega_{0,\alpha}^{2})x. \]

Here, \( \omega_{0,\alpha} \) is dimensionally consistent and related to spring elastic constant and the mass of the oscillating point mass. The solution to this equation can be obtained by proposing a q-exponential kind of solution and rewriting it in terms of deformed circular functions. It is worthy commenting that the solution to the present model is different from the result of Ref. [14], since here the circular functions are simultaneously compatible with the q-exponential and the dual conformable derivative. By other side, in the Ref. [14] the solutions are compatible with q-dual deformed derivative and, in this way, the models of de-
formed oscillators are different.

- Nonlinear Equations, Relativity and Quantum mechanics: Several possibilities may be studied, like nonlinear Fokker-Planck equations, heat transfer, and even nonlinear and position-dependent mass in quantum mechanics. Also, several different problems can be revisited, like classical fields \[24\], Bohmian mechanics \[25\], Zitterbewegung problem \[26\] and so on.

6. CONCLUSIONS AND OUTLOOK FOR FURTHER INVESTIGATIONS

In this work, motivated by the generalized statistical mechanics approach, one build up a conformable subtraction and also used the duality definition for derivative operators, in order to identify the deformed derivative

\[ \hat{D}_\alpha^a f = f^{a-1} \frac{df}{dx} \]

as the dual of conformable derivative. A number of properties have been discussed, like deformed Leibniz and chain rules. The q-exponential, in the context of generalized statistical mechanics, is the eigenfunction of this DCD. The relevance and potentiality of DCD for position-dependent and nonlinear problems have been put in evidence here. Inter-relations with Chen’s fractal derivative have also been treated in our contribution.

As a prospect for future research, the variational approach, similar to what was done in Ref. \[2\] but for \( \alpha \) near 1 (low nonlinearity limit), shall be re-assed and published elsewhere.

Acknowledgments: Thanks to J. A. Helayël-Neto for for reading and reviewing the manuscript.

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