ASSOCIATIVITY RELATIONS IN QUANTUM
COHOMOLOGY

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Abstract. We describe interdependencies among the quantum
cohomology associativity relations. We strengthen the first re-
construction theorem of Kontsevich and Manin by identifying a
subcollection of the associativity relations which implies the full
system of WDVV equations. This provides a tool for identifying
non-geometric solutions to WDVV.

1. Introduction

The geometry of moduli spaces of stable maps of genus 0 curves
into a complex projective manifold $X$ leads to a system of quadratic
equations in the tree-level (genus 0) Gromov-Witten numbers of $X$.

These quadratic equations were written down by physicists before the
geometry was worked out rigorously. In all the worked examples, the
equations were seen to determine a solution, uniquely and consistently,
from starting data. The beautiful paper of Di Francesco and Itzykson,
[DF-I], presents a number of examples in this context.

In one of the foundational papers in the area of quantum cohomology,
[K-M], the authors remark on the overdeterminedness of the system
of equations, and pose the question whether the seemingly redundant
equations follow algebraically from the useful ones.

This same paper presents the first reconstruction theorem, which
applies to manifolds $X$ such that $H^*(X, \mathbb{Q})$ is generated by $H^2(X)$.
This result gives an effective procedure for solving for genus 0 Gromov-
Witten numbers from starting data using the quadratic relations (since
this entire paper is concerned only with the genus 0 invariants, we omit
explicit mention of genus from now on).

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This paper continues in the spirit of these early investigations into the structure of these relations. After a review in which notation is introduced and the basic problem is set up (section 2), some results are presented (sections 4 and 5) explicitly showing interdependencies among these relations.

Then follows a generalization of the first reconstruction theorem. In this, we are forced to keep the hypothesis that the cohomology ring of $X$ be generated by divisors. We then show that an initial collection of numbers and relations determines, purely algebraically, the entire system of relations \emph{(strong reconstruction)} as well as the Gromov-Witten numbers.

Finally, some examples are worked out. The structure of the associativity relations in the case $X$ has dimension 2 is particularly easy to describe. Section 6 gives an explicit picture of strong reconstruction in this case. Section 9 presents some higher-dimensional examples.

Focusing on the algebra, rather than the geometry, of quantum cohomology leads to some generalizations. First, we can do away with $X$ entirely and work just with its cohomology ring, or more generally any \emph{positively graded Gorenstein} $\mathbb{Q}$-\emph{algebra} with socle in degree $n \geq 2$ ($n$ plays the role of the dimension of $X$, and we also must assume that the zeroth graded piece is isomorphic to $\mathbb{Q}$). The canonical class $K_X$ plays an important role in geometry since it determines the expected dimension of the moduli spaces, but is less important to us: we can change $K_X$ at will, or drop it entirely from the discussion. Finally, the very equations lead us to solutions which do not come from geometry. Some of the examples exhibit that the geometric solution, with its family of obvious rescalings, may live in an even larger solution space.

In an appendix (section 11), we sketch some of the geometry that the rest of the paper ignores. This is used to motivate the result in section 3, although this result is purely algebraic. This is in no sense a substitute for a survey of quantum cohomology; we recommend [F-P] as starting point to reader unacquainted with the subject.

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2. The basic problem

Setting up and solving the system of quadratic equations in the Gromov-Witten numbers requires a ring which, in an appropriate sense, looks like a cohomology ring of a manifold.

Precisely, let $A$ be a positively graded Gorenstein $\mathbb{Q}$-algebra with socle in degree $n \geq 2$ such that the zeroth graded piece of $A$ is isomorphic to $\mathbb{Q}$, and assume a nonzero choice of $\varphi \in \text{Hom}_A(\mathbb{Q}, A) \simeq \mathbb{Q}$ is made. We call the degree of an element its codimension, and denote the $k^{\text{th}}$ graded piece by $A^k$. Then $\varphi$ is an isomorphism of $\mathbb{Q}$ with $A^n$. Denote by $\int$ the composite of projection with this isomorphism:

$$A \to A^n \xrightarrow{\varphi^{-1}} \mathbb{Q}.$$ 

If $\{T_i | i \in I\}$ is a basis for $A$ as a $\mathbb{Q}$-module, where $I$ is a (necessarily finite) indexing set, let $g_{ij} = \int T_i \cdot T_j$. This is a nondegenerate pairing; we denote by $(g^{ij})$ the inverse matrix. We make the further assumption that $A^1$ is nonzero.

In order to get a well-defined system of equations, we need two more pieces of data: a maximal-rank integral lattice $\Lambda$ and a strongly convex polyhedral cone $\Theta$, both contained in $(A^1)^* \otimes \mathbb{R}$.

Let $B = \{T_1, T_{\sigma_1}, \ldots, T_{\sigma_r}, T_{\tau_1}, \ldots, T_{\tau_s}\}$ be a basis of $A$ as a $\mathbb{Q}$-module, consisting of homogeneous elements, with $T_1 = 1$, $T_{\sigma_i} \in A^1$ for each $i$, codim $T_{\tau_j} \geq 2$ for each $j$, and codim $T_{\tau_j} \leq$ codim $T_{\tau_k}$ whenever $j \leq k$. This defines $r$ and $s$ as the ranks of the first and higher graded pieces of $A$, respectively; recall also that $n$ is the top grading, necessarily equal to codim $T_{\tau_s}$. Assume the choice of basis is made so that $\Lambda$ is dual to the integral span of $T_{\sigma_1}, \ldots, T_{\sigma_r}$.

Geometry provides many examples of such data: $A^* = H^{2*}(X, \mathbb{Q})$, where $X$ is a complex projective manifold with homology only in even dimensions, $\int = \int_X$, $\Lambda = H_2(X, \mathbb{Z})$/torsion, $\Theta =$ dual to ample cone.

Let $C = \Theta \cap \Lambda \setminus \{0\}$. We define the set of unknown numbers to be the collection of all $N(\beta; d_1, \ldots, d_s)$ with $\beta \in C$ and $d_j \geq 0$ for all $j$. For any $\beta \in C$, denote by $c_i$ the pairing of $\beta$ with $T_{\sigma_i}$. We now define a system of equations in these unknowns, one for each 4-tuple $(i, j, k, l)$ of elements of the basis indexing set $I = \{1, \sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s\}$ and each degree $(\beta; d_1, \ldots, d_s)$.
To do this, introduce the potential function, a formal Laurent series
\[ \Phi = \Phi^{\text{cl}} + \Gamma \]
in variables \( y_i, i \in I \), composed of the classical part
\[ \Phi^{\text{cl}} = \frac{1}{6} \sum_{i,j,k \in I} \left( \int T_i \cdot T_j \cdot T_k \right) y_i y_j y_k \]
and the quantum correction
\[ \Gamma = \sum_{\beta \in C, d_1, \ldots, d_s \geq 0} N(\beta; d_1, \ldots, d_s) e^{c \gamma y_1} \ldots e^{c \gamma y_s} \frac{y_{\tau_1}^{d_1}}{d_1!} \ldots \frac{y_{\tau_s}^{d_s}}{d_s!}. \]
Then any \((i, j, k, l)\) determines a differential equation
\[ \sum_{e, f} \Phi_{ij} g^{ef} \Phi_{kl} = \sum_{e, f} \Phi_{jk} g^{ef} \Phi_{il}, \quad (1) \]
where we have used subscripts to denote partial differentiation. Isolating the coefficient of \( e^{c \gamma y_1} \ldots e^{c \gamma y_s} \frac{y_{\tau_1}^{d_1}}{d_1!} \ldots \frac{y_{\tau_s}^{d_s}}{d_s!} \) on each side produces a quadratic equation in \( N \)’s, which we call an associativity relation (they imply associativity of the so-called quantum product; see [F-P]). We follow Dubrovin in calling the system of equations \((1)\) the WDVV equations (after E. Witten, R. Dijkgraaf, H. Verlinde and E. Verlinde). A particular WDVV equation is represented symbolically by an equivalence of Feynman diagrams
\[ \left( \begin{array}{c} i \\ j \end{array} \right) \sim \left( \begin{array}{c} l \\ k \end{array} \right) \]
We adopt the notation \( \left( \begin{array}{c} i \\ j \end{array} \right) \left( \begin{array}{c} l \\ k \end{array} \right) \) to refer to the WDVV equation \((1)\)
and \( \left( \begin{array}{c} i \\ j \end{array} \right) \left( \begin{array}{c} l \\ k \end{array} \right)^{\beta; d_1, \ldots, d_s} \) to refer to a particular associativity relation.

We would like to rewrite \((1)\) in which we split \( \Phi \) into its classical and quantum parts. If \( T_i \cdot T_j = \sum_h q_h T_h \) then denote \( \sum_h q_h \Gamma_{hkl} \) by \( \Gamma_{(ij)kl} \). Then by rewriting \((1)\) we get
\[ \Gamma_{ij(lk)} + \Gamma_{(ij)kl} - \Gamma_{jk(il)} - \Gamma_{(jk)il} = \sum_{e, f} \Gamma_{jke} g^{ef} \Gamma_{fil} - \sum_{e, f} \Gamma_{ij} g^{ef} \Gamma_{fkl}. \quad (2) \]
Thus we have split the WDVV equation into the linear contribution (left-hand side) and quadratic contribution (right-hand side).
We emphasize that \((i_j)(l_k) \beta; d_1, . . . , d_s\) is an equation; it can be scaled or combined linearly with other equations, and it can imply other equations. We will sometimes need to distinguish the linear and quadratic contributions to a particular associativity relation, each being the polynomial appearing on the appropriate side of (2).

With this set-up, we can now state

**Problem 1.** Given \(A, f, \Lambda, \Theta\) as above, find solutions in rational numbers \(N(\beta; d_1, . . . , d_s)\) to the full set of WDVV equations (3).

For the purposes of ordering elements of \(C\), we let \(\omega\) be an element of the interior of the dual cone to \(\Theta\) (which must be nonempty since \(\Theta\) is strongly convex). Then \(\langle \beta, \omega \rangle > 0\) for all \(\beta \in C\), and for each \(k\), the set \(\{\beta \in \mathbb{Z}, \sigma^*_1, . . . , \sigma^*_r\mid \langle \beta, \omega \rangle < k\}\) is finite.

We must remark on four distinct uses of the Feynman symbols. First, with \(i, j, k, l \in I\), \((i_j)(l_k)\) is as above. Second, for \(\xi, \pi, \rho, \sigma \in A\), \(\left(\xi_{\pi}^\sigma_{\rho}\right)\) refers to the equation obtained by writing each element in terms of the basis and summing in a multilinear fashion. Third, for subsets \(\Xi, \Pi, P, \Sigma\) of \(A\), \(\left(\Xi_{\Pi}^\Sigma_P\right)\) refers to the collection of all \(\left(\xi_{\pi}^\sigma_{\rho}\right)\) with \(\xi \in \Xi\), etc. Finally, as a special case, for integers \(w, x, y, z\), \((w_{x}^z)\) refers to \(\left(A^w_x\right)^z\). Also, if \(i, j, k, l, m \in I\), we write \((i_jk_l)^{m}\) as shorthand for \(\left(T_i^1 T_j^m T_k^1 T_l^m\right)\).

### 3. How many relations?

As we have seen, to each choice of \(i, j, k, l \in I\) there corresponds a WDVV equation. Thus, if the vector space \(A\) has dimension \(r\) then the number of distinct WDVV equations is of the order of magnitude \(r^4\).

It is an exercise in combinatorics to provide a precise count. First, if any of \(i, j, k, l\) is 1 (i.e., indexes the identity element of \(A\)), then the corresponding WDVV equation is a trivial identity. If \(k = i\) or \(l = j\) then the equation is trivial.

There are symmetries. If we swap \(i\) and \(j\) and swap \(k\) and \(l\) then the WDVV equation remains unchanged. Swapping two symbols on a
diagonal of the Feynman diagram only changes the WDVV equation by a sign. If \( i, j, k, \) and \( l \) are all distinct, then the three relations one obtains by cyclically permuting \( i, j, k \) are linearly related:

\[
\left( \begin{array}{c} i \\ j \\ k \end{array} \right) \left( \begin{array}{c} l \\ i \\ j \end{array} \right) + \left( \begin{array}{c} j \\ k \\ i \end{array} \right) \left( \begin{array}{c} l \\ k \\ i \end{array} \right) + \left( \begin{array}{c} k \\ i \\ j \end{array} \right) \left( \begin{array}{c} l \\ i \\ j \end{array} \right) = 0.
\]

A tally shows that the number of distinct nontrivial WDVV equations (modulo sign) is

\[
\frac{r^4 - 6r^3 + 15r^2 - 18r + 8}{8},
\]

while if we count only two out of the three distinct WDVV equations involving four distinct symbols — since by (3) any two imply the third, a fact we refer to as the two-out-of-three implication — the count is

\[
\frac{r^4 - 4r^3 + 5r^2 - 2r}{12}.
\]

Since there is only (at most) one unknown number in each degree, the count above is, na"ively, the factor of overdeterminedness in the system of associativity relations.

4. The Three Symbols Relation

Let \( i, j, k, l, m \in I \). Let \( \Phi = \Phi^{cl} + \Gamma \) be the potential function. The following algebraic identity, called the three symbols identity, holds:

\[
\frac{\partial}{\partial y_m} \left( \sum_{e, f} \Phi_{ije} g^{ef} \Phi_{fkl} - \sum_{e, f} \Phi_{jke} g^{ef} \Phi_{fil} \right) +
\]

\[
\frac{\partial}{\partial y_j} \left( \sum_{e, f} \Phi_{ile} g^{ef} \Phi_{fkm} - \sum_{e, f} \Phi_{lke} g^{ef} \Phi_{fim} \right) +
\]

\[
\frac{\partial}{\partial y_l} \left( \sum_{e, f} \Phi_{ime} g^{ef} \Phi_{fjk} - \sum_{e, f} \Phi_{mke} g^{ef} \Phi_{fij} \right) = 0.
\]

Let \( (\beta, d) \) be a degree. Define \( e_{\sigma_i} = 0 \) and \( e_{\sigma_j} = (0, \ldots, 1, \ldots, 0) \) with the 1 in the \( j \)th place. Then (4) gives us

**Proposition 1.** Suppose \( i, j, k, l, m \in I \) with \( \text{codim} T_m \geq 2 \), and let \( (\beta, d) \) be a degree with \( d_m \geq 1 \). Then relations \( \left( \begin{array}{c} i \\ l \\ m \end{array} \right)^{(\beta d + e_j - e_m)} \) and \( \left( \begin{array}{c} i \\ j \\ k \end{array} \right)^{(\beta d + e_l - e_m)} \) together imply \( \left( \begin{array}{c} i \\ j \\ k \end{array} \right)^{(\beta d)} \).
This we call the three symbols relation (3SR), and denote by the diagram \( (i) \langle j, l; m \rangle_k^{(\beta,d)} \).

We now record one application of 3SR.

**Lemma 1.** With the notation of Problem 4, suppose \((\beta;d)\) is a degree with \(d \neq 0\). Then the collection of all relations in degrees \((\beta;d')\) with \(\sum_i d_i' = (\sum_i d_i) - 1\) implies \( (i) \langle j, 1; m \rangle \) yields \( (i) \langle 1; j \rangle \) for any \(i, j \in I\).

### 5. The five symbols relation

Given \(i, j, k, l, m \in I\), the following algebraic identity holds:

\[
\sum_{e,f} \Gamma_{ij(me)} g^{ef} \Gamma_{fkl} = \sum_{e,f} \Gamma_{kl(me)} g^{ef} \Gamma_{fij}.
\]

Recall, if \(T_m \cdot T_e = \sum_q t_q T_q\) then by \(\Gamma_{ij(me)}\) we mean \(\sum_q t_q \Gamma_{ijq}\). Now (3) follows by observing that with \(g_{abc} = \int T_a \cdot T_b \cdot T_c\) we have \(t_q = \sum_p g_{mep} g^{pq}\), and now the coefficient \(\sum_{e,p} g^{ef} g_{mep} g^{pq}\) of \(\Gamma_{ijq} \Gamma_{fkl}\) on the left-hand side is symmetric in \(f\) and \(q\).

The appendix gives geometric motivation for (3).

We write the expression \(\sum \Gamma_{ij(me)} g^{ef} \Gamma_{fkl} - \sum \Gamma_{kl(me)} g^{ef} \Gamma_{fij}\) and add to it the four additional expressions obtained by permuting the variables \(i, j, k, l, m\) cyclically. We use the identity coming from the associativity relation \( (e) \langle m; j \rangle_i \) and its cyclic translates to obtain

\[
\begin{align*}
0 &= \Gamma_{ij(me)} \Gamma_{fkl} + \Gamma_{jk(ie)} \Gamma_{fim} + \Gamma_{km(ie)} \Gamma_{fji} + \Gamma_{mi(ie)} \Gamma_{fjk} \\
&\quad - \Gamma_{mi(ve)} \Gamma_{fkl} - \Gamma_{ij(ke)} \Gamma_{fim} - \Gamma_{jk(ve)} \Gamma_{fmi} - \Gamma_{kl(ve)} \Gamma_{fji} - \Gamma_{lm(ve)} \Gamma_{fjk} \\
&\quad + \Gamma_{mi(ve)} \Gamma_{fkl} + \Gamma_{kj(ve)} \Gamma_{fim} + \Gamma_{jk(ve)} \Gamma_{fmi} + \Gamma_{kl(ve)} \Gamma_{fji} + \Gamma_{lm(ve)} \Gamma_{fjk} \\
&\quad - \Gamma_{ij(me)} \Gamma_{fkl} - \Gamma_{ji(ke)} \Gamma_{fim} - \Gamma_{jk(me)} \Gamma_{fmi} - \Gamma_{kl(me)} \Gamma_{fji} - \Gamma_{lm(me)} \Gamma_{fjk}.
\end{align*}
\]

We have omitted summations symbols and \(g^{ef}\)'s to save space. We have also omitted the (cubic) terms obtained by substituting the quadratic contributions of the associativity relations, but the key point is that these cancel.

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The final expression above is the quadratic contribution of a sum of associativity relations, conveniently written

\[(6) \quad (m_i \langle i) \langle j \rangle^a (l_k) - (m_i \langle i) \langle j \rangle^a (l_k) + (m_i \langle i) \langle j \rangle^a (l_k) - (m_i \langle i) \langle j \rangle^a (l_k) + (l m_i \langle i) \langle j \rangle^a (l_k). \]

Since the linear contribution of (6) vanishes, as may be checked, we have, at least formally, that the indicated associativity relations imply the vanishing of (6).

We turn this into a precise, practical statement by grading the terms in \( \Gamma \) by degree. We use the notation of Problem 1. Rewrite the above, isolate the coefficient of some degree \((\beta; d)\), and note that then every quadratic term is a sum over \( \beta_1 + \beta_2 = \beta \) with \( \langle \beta_1, \omega \rangle < \langle \beta, \omega \rangle \) for \( i = 1, 2 \). This establishes

**Proposition 2.** Suppose \( i, j, k, l, m \in I \), and let \((\beta; d)\) be a degree. The collection of relations consisting of \((i_j^e_k)^{(\beta', d')}\) and its cyclic translates through \(\{i, j, k, l, m\}\), for all \(e \in I\) and all degrees \((\beta', d')\) with \(\langle \beta', \omega \rangle < \langle \beta, \omega \rangle\) and \(d' \leq d\) (componentwise), implies the relation

\[
(m_i \langle i) \langle j \rangle^a (l_k) - (m_i \langle i) \langle j \rangle^a (l_k) + (m_i \langle i) \langle j \rangle^a (l_k)
- (m_i \langle i) \langle j \rangle^a (l_k) + (l m_i \langle i) \langle j \rangle^a (l_k). \]

We call this the **five symbols relation** (5SR). We employ the notation \((m_i \langle i) \langle j \rangle^a (l_k)^{(\beta; d)}\) to describe the above relation.

6. **Strong reconstruction for \( n = 2 \)**

To illustrate an application of the three and five symbols relations, we work out a strong reconstruction theorem for \( n = 2 \), where the associativity relations are simple to organize. When \( n = 2 \), there is only one \( \tau \), and there are three types of associativity relations:

(i) \( \tau \langle \sigma_i \sigma_1 \rangle; \)

(ii) \( \tau \langle \sigma_1 \sigma_k \rangle; \)

(iii) \( \sigma_i \langle \sigma_1 \sigma_k \rangle. \)
Assume for simplicity that, as in the geometric situation, there is a canonical class $K$ which dictates that there is at most one nontrivial number $N(\beta; d)$ in each curve class $\beta$, namely when $d = \langle \beta, -K \rangle - 1 \geq 0$ (for the general case, see the next section). The potential function is composed of

$$
\Phi^{cl} = \frac{1}{2} y_\tau^2 y_\tau + \frac{1}{2} \sum_{e,f=1}^r g_{ef} y_{\sigma_e} y_{\sigma_f},
$$

$$
\Gamma = \sum_{\langle \beta, -K \rangle \geq 1} N(\beta; d) e^{c_1 y_{\sigma_1} \ldots e^{c_r y_{\sigma_r}} \frac{y_\tau^d}{d!}}
$$

(we write $g_{ef}$ for $g_{\sigma_e \sigma_f}$).

Suppose we are given all $N(\beta; d)$ with $d \leq 2$, and suppose that these satisfy

$$
\left( \frac{\tau}{\sigma_i} \right)_{\sigma_k}^{(\beta;0)} \text{ and } \left( \frac{\sigma_l}{\sigma_i} \right)_{\sigma_k}^{(\beta;0)}
$$

for all $i, j, k, l \in \{1, \ldots, r\}$ and all $\beta$. We claim that relations of type (i) allow us to solve for all further $N(\beta; d)$ (reconstruction) and that the numbers thus obtained satisfy the full system of WDVV equations (strong reconstruction).

Indeed, by the three symbols relation,

$$
\left( \frac{\tau}{\sigma_i} \right)_{\sigma_j}^{(\beta;0)} \text{ and } \left( \frac{\sigma_l}{\sigma_j} \right)_{\sigma_k}^{(\beta;0)} \Rightarrow \left( \frac{\sigma_i}{\sigma_j} \right)_{\sigma_k}^{(\beta;1)},
$$

and thus the hypothesis implies relations (ii) $(\beta; 1)$. Similarly, relations (i) $(\beta; d)$ imply (ii) $(\beta; d + 1)$ and (iii) $(\beta; d + 2)$.

Inductively on $d$, assume all $N(\beta; d')$ known and all relations satisfied for $\langle \beta, -K \rangle < d + 4$. Relation $\left( \frac{\tau}{\sigma_i} \right)_{\sigma_j}^{(\beta; d)}$ reads $g_{ij} \Gamma^{(\beta; d)}_{zzz} = Q_{ij}^{(\beta; d)}$, where $Q_{ij}^{(\beta; d)}$ is a quadratic expression in known quantities. Now $\left( \frac{\tau}{\sigma_k} \right)_{\sigma_l}^{(\sigma_i)}$, tells us $g_{kl} Q_{ij}^{(\beta; d)} = g_{ij} Q_{kl}^{(\beta; d)}$, which says we can solve for $\Gamma^{(\beta; d)}_{zzz}$ (that is, $N(\beta; d + 3)$) satisfying (i), and we have just seen that the relations of type (i) imply all the relations in degree $\beta$. 

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7. STRONG RECONSTRUCTION THEOREM

The main result here is a generalization of the first reconstruction theorem of Kontsevich and Manin [K-M]. Working in the same class of cohomology rings, namely those generated by elements in codimension 1, we prove that an identifiable collection of numbers satisfying an identifiable collection of relations gives us strong reconstruction, i.e. allows us to solve uniquely for numbers satisfying the complete system of associativity relations. In case $-K$ is ample, we need only a finite collection of numbers and relations as starting data.

The result is

**Theorem 1.** With the notation of Problem 4, suppose $A$ is generated by $A^1$. Then the collection of $N(\beta; d)$ with $\sum_{i=1}^{s} d_i \leq 2$ extends to a solution to WDVV if and only if $\left( \begin{array}{c} A \\ A^1 \end{array} \right)^{(\beta;0)}_{A^1} \left( \begin{array}{c} A^1 \\ A \end{array} \right)$ is satisfied for all $\beta$.

We begin by organizing notation. By hypothesis, we may assume the basis $B$ chosen such that for each $j$, $1 \leq j \leq s$, there exists $i_j \in \{1, \ldots, r\}$ and $\mu_j \in I$ such that $T_{\tau_j} = T_{\sigma_i} \cdot T_{\mu_j}$.

We wish to impose a partial order on the collection of degrees $d = (d_1, \ldots, d_s)$ with fixed $|d| := \sum_{i=1}^{s} d_i$, such that $(d_1, \ldots, d_s)$ precedes $(d_1, \ldots, d_i + 1, \ldots, d_j - 1, \ldots, d_s)$ for any $i < j$. A convenient way is to order by $\sum id_i$.

Let us give an outline of the proof of the theorem. Inducting on $\langle \beta; \omega \rangle$, then on $|d|$, then downwards on $\sum id_i$, we verify all associativity relations in degree $(\beta; d)$, showing that those of the form

$\left( \begin{array}{c} \mu_j \\ \sigma_i \end{array} \right)^{(\beta; d)}_{(\tau_l; \tau_k)}$

with codim $T_{\tau_j} \leq \text{codim } T_{\tau_k} \leq \text{codim } T_{\tau_l}$ and max$(j, k, l) \leq \min\{m \mid d_m \neq 0\}$ determine the numbers $N(\beta; d+e_j+e_k+e_l)$ (here $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the $i$th position).

8. PROOF OF THE STRONG RECONSTRUCTION THEOREM

The induction breaks up into an outer induction on degrees and an inner induction within each degree. The outer induction proceeds with respect to the partial order: $(\beta', d') < (\beta, d)$ if $\langle \beta', \omega \rangle < \langle \beta, \omega \rangle$ and...
\[\sum id' > \sum id,\]

The inner induction is on \((u, c, a, b)\) with \(u\) (corresponding to codim \(T_a\), above) up from 1, \(c = \text{codim } T_{c-1} + \text{codim } T_{c}\) up from 2\((u + 1)\), and \(a = \text{codim } T_{c-1}\) up from \(u + 1\) to \([c/2]\). Define \(b = c - a\); then we always have \(a \leq b\).

The induction hypothesis, at a given step \((\beta, d, u, c, a, b)\), consists of all relations in previous degrees plus all numbers they refer to (i.e., all \(N(\beta'; d' + e')\) with \((\beta', d') < (\beta, d), |e'| \leq 3\), plus, in the current degree, all \((z, y \rangle u, x)\) with \(\min(x, y, z) < u\), all \((u, y \rangle x)\) and \((x, y \rangle u)\) with \(x \geq u + 1, y \geq u + 1\), and either \(x + y < c\) or \(x + y = c\) with \(\min(x, y) < a\), together with all numbers these relations refer to.

In any degree, for any integers \(x\) and \(y\), \((x, 1 \rangle y)\) follows either by hypothesis \((d = 0)\) or by the induction hypothesis and Lemma \(\llbracket\) \((|d| \geq 1)\). When \(u \geq 2\) we obtain \((x, 1 \rangle u, y)\) for \(x \geq u\) and \(y \geq u\) from \((x, 1 \rangle u - 1, y)\), and now \((u, 1 \rangle x)\) for \(x > u\) from \((u - 1, 1 \rangle x, u)\).

The main step is to deduce \((u, 1 \rangle b, y)\). Here the linear terms coming from the associativity relation possibly involve new \(N\)'s. We divide this into two steps.

First, we show it suffices to prove a distinguished set of \((u, 1 \rangle b, a)\). Let \(S\) be the set of relations \((\mu_j, \tau_l - \tau_k)\) with codim \(T_{\tau_j} = u + 1\), codim \(T_{\tau_k} = a\), and codim \(T_{\tau_j} = b\). We claim that \(S\) (and the new \(N\)'s referred to) implies \((u, 1 \rangle a, b)\). Indeed, if codim \(T_{\tau} = u\) and codim \(T_{\sigma} = 1\) with \(T_{\mu} = \sum \lambda_j T_{\tau_j}\), then comparing \((\mu, \tau_l - \tau_k)\) with \(\sum \lambda_j (\mu_j, \tau_l - \tau_k)\) establishes \((\mu, \tau_l - \tau_k)\) from the relations in \(S\).

For the second step, we establish all relations in \(S\). Each \((\mu_j, \tau_l - \tau_k)\) in \(S\) involves the variable \(N(\beta; d + e_j + e_k + e_l)\). For \(a, b, u + 1\) distinct, there is a one-to-one correspondence between elements of \(S\) and such variables. In other cases, we shall need symmetrizing arguments to show any two relations in \(S\) sharing a common such variable
are equivalent. In case $a = b$, the two-out-of-three implication gives
\[ (\mu_j \sigma_{ij} \tau_i) \iff (\mu_j \sigma_{ij} \tau_k) \]. In case $a = u + 1$, we get
\[ (\mu_j \sigma_{ij} \tau_i) \iff (\mu_j \sigma_{ik} \mu_k) \].

Thus, it suffices to establish only those $(\mu_j \sigma_{ij} \tau_i) \in S$ such that
\[ j \leq k \leq l \]. In case $d_m \geq 1$ for some $m < l$, $(\mu_j \sigma_{ij} \tau_i; \tau_m)$ establishes
\[ (\mu_j \sigma_{ij} \tau_i) \]. Otherwise, $N(\beta; d + e_j + e_k + e_l)$ is actually an unknown,
so solving $(\mu_j \sigma_{ij} \tau_i)$ establishes simultaneously the number and the
relation. Finally, two-out-of-three establishes $(u_{a_1}^1 \tau_{a_2})$ from $(u_{a_1}^1 \tau_{a_2})$.

Having finished the inner induction, to establish general $(u_{a_1}^1 \tau_{a_2})$
is an easy induction on $\min(w, x, y, z)$, using 5SR by decomposing the
entry of lowest codimension.

9. Examples

Since we wish not to stray far from geometry, we focus mainly on
rings of the form $A = A_\mathbb{Q}^* X := A^* X \otimes \mathbb{Q}$, where $X$ is a projective
manifold, and with a dimension restriction on numbers coming from a
class $K \in A^1$: $N(\beta; d) = 0$ unless $\sum d_j (\text{codim } T_j - 1) = \langle \beta, -K \rangle + n - 3$, where $n = \dim X$. Unless otherwise stated, $K = K_X$, the canonical
class on $X$.

**Example 1.** $X = \mathbb{P}^n$. The ring $A = A_\mathbb{Q}^* \mathbb{P}^n$ has $\text{rk } A^1 = 1$, so strong
reconstruction dictates a vacuous set of relations (a Feynman diagram
with a symbol appearing twice, in opposite corners, determines a trivial
relation), and the solutions to WDVV correspond exactly to choices of
$N(\beta; d)$ with $|d| \leq 2$. With $K = K_X$, there is only one such $N$, so
there are no solutions beyond the geometric solution and its rescalings.

Let $h = c_1(\mathcal{O}(1))$. When $K = -bh$ with $1 \leq b \leq n - 1$, there
is more than one $N(\beta; d)$ with $|d| \leq 2$, and hence there is a family
of solutions to WDVV of dimension $> 1$. Some of these solutions
have geometric significance. If $V \hookrightarrow \mathbb{P}^{n+k}$ is a $(s_1, \ldots, s_k)$-complete
intersection with $b = n + k + 1 - \sum s_j \geq 1$, and $K = -bh$, then
$K_V = i^* K$, and we get a solution to WDVV for $A = A^*_Q$ with this “wrong” canonical class by setting $N(\beta; d)$ equal to the sum over $\beta' \in H_2 V$ with $i_* \beta' = \beta$ of the (geometric) Gromov-Witten number for $V$ corresponding to $\bigotimes (i^* T_j \otimes d_j)$ in curve class $\beta'$ (cf. [G], sec. 4).

**Example 2.** A toric manifold. In the integral lattice of rank 3, let $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, $v_3 = (0, 0, 1)$, $v_4 = (-2, -1, -1)$, $v_5 = (-1, 0, 0)$, and let the toric manifold $X$ be defined by the fan consisting of the cones $\langle v_1, v_2, v_3 \rangle$, $\langle v_1, v_2, v_4 \rangle$, $\langle v_1, v_3, v_4 \rangle$, $\langle v_2, v_3, v_5 \rangle$, $\langle v_2, v_4, v_5 \rangle$, $\langle v_3, v_4, v_5 \rangle$. If $D_i$ denotes the divisor corresponding to vector $v_i$, then $D_1, D_2$ span $A^1 X$ with the dual to the ample cone generated by the dual basis elements $D_1^*, D_2^*$. We have $A = A^*_Q X = \mathbb{Q}[D_1, D_2]/(D_1^2 - 2D_1D_2, D_2^2)$ with $\mathbb{Q}$-basis $\{1, D_1, D_2, D_1D_2, D_2^2, D_1D_2^2\}$ and $-K = 2D_1 + D_2$.

Given this setup, the strong reconstruction theorem dictates 21 relations involving 17 variables. When written out, this system of equations reduces to

\[
N(D_1^*; 2, 0, 0) = N(D_1^*; 0, 0, 1);
N(D_1^* + D_2^*; 0, 1, 1) = -N(D_1^*; 0, 0, 1)N(D_2^*; 0, 1, 0);
0 = N(D_1^*; 0, 0, 1) \left[N(2D_2^*; 0, 2, 0) + N(D_2^*; 0, 1, 0)^2\right]
\]

(with the rest of the variables all zero). Thus the solution space to WDVV has two irreducible components: one, containing the geometric solution, with its expected two-dimensional family of rescalings, and another, supported in curve classes lying along one edge of the dual to the ample cone, with a one-dimensional family of rescalings plus dependence on another free parameter.

**Example 3.** $X = G(2, 4)$, $X = \text{Sym}^2 \mathbb{P}^2$. Both have isomorphic cohomology rings (up to scale factors), not generated by divisors, so we are outside the scope of the strong reconstruction theorem. We illustrate here a technique which allows one to prove, subject to a genericity hypothesis on starting data, a strong reconstruction result in this case. The starting data according to the statement of strong reconstruction consists of just one number when $X = G(2, 4)$ and 3 numbers when $X = \text{Sym}^2 \mathbb{P}^2$. The result we seek is that modulo a genericity hypothesis (some quantity in the starting data being nonzero) any choice of starting data extends to a solution to WDVV. It is helpful to recall the dimension condition on relations. For $\left(\frac{\xi}{\rho}\right)^{(\beta, d)}$ to be nontrivial
\[ \langle \beta, -K \rangle - \sum_{j=1}^{s} d_j (\operatorname{codim} T_{r_j} - 1) = \]
\[ \operatorname{codim} \xi + \operatorname{codim} \pi + \operatorname{codim} \rho + \operatorname{codim} \sigma - n. \]

One may take as cohomology basis the powers of the ample generator \( h \) of \( A^1 X \), plus an extra codimension 2 element, chosen orthogonal to \( h \). So \( B = \{1, h, c, h^2, h^3, h^4\} \) with \( c \cdot h = 0 \), \( \int h^4 \neq 0 \), \( \int c^2 \neq 0 \). We have \( K = -4h \), \( K = -3h \) in the cases of the two respective varieties; set \( \kappa = 4 \), \( \kappa = 3 \) accordingly.

The relations in curve class \( \beta \) never involve the number \( N(\beta; \kappa \beta + 1, 0, 0, 0) \). We recover this exceptional number from a particular degree \( \beta + 1 \) relation in which it appears in a quadratic term, for which, to be able to solve, we must add the hypothesis that \( N(1; 0, 0, 1, 1) \neq 0 \) (resp. \( N(1; 1, 0, 1, 1) \neq 0 \)) when \( \kappa = 4 \) (resp. \( \kappa = 3 \)).

For each \( \beta \), there are 10 numbers \( N(\beta; d) \) which are not of the form \( N(\beta; t, u, v, w) \) with \( u + v + w \geq 3 \); these are the numbers unreachable by the proof of strong reconstruction applied to the subring of \( A \) generated by \( h \). We must show how to solve for 9 of these (all except \( N(\beta; \kappa \beta + 1, 0, 0, 0) \)) as well as the leftover degree \( \beta - 1 \) number. Table 1 outlines how to do this.

We induct first on curve class \( \beta \). The induction hypothesis consists of all relations and all numbers in degrees less than \( \beta - 1 \), and all relations and all numbers except \( N(\beta - 1; \kappa \beta - \kappa + 1, 0, 0, 0) \) in degree \( \beta - 1 \). The relations indicated in entries (a), (b) of Table 1 give us two new numbers, including \( N(\beta - 1; \kappa \beta - \kappa + 1, 0, 0, 0) \). Thus from now on we assume all relations and all numbers in degrees less than or equal to \( \beta - 1 \).

Now, inductively on \( d \) via the partial ordering \( d' = (t', u', v', w') \prec d = (t, u, v, w) \iff |d'| < |d| \text{ or } |d'| = |d|, t' > t \text{ or } |d'| = |d|, t' = t, u' + 2v' + 3w' > u + 2v + 3w \), we establish the relations indicated in (c)-(k) of the table, as applicable to the current degree. Step (c) is the inner induction of the proof of the strong reconstruction theorem, applied to the subring of \( A \) generated by \( h \).

Finally, it follows from 5sr (this takes a bit of checking) that in any degree, the (31 out of 55 total) relations indicated in Table 1 imply all the relations. The 21 encoded by step (c) follow by the proof of
strong reconstruction, so we are reduced to establishing the remaining 10. When \( u = v = w = 0 \), we are done by Table \[ \text{(each remaining relation solves for an unknown number).} \] Otherwise, since each of the Feynman diagrams indicated in (a), (b), (d)–(k) of the table has a diagonal containing \( c \) and \( h \), an application of 3SR reduces us to relations obtained in previous degrees (via the partial ordering above).

Summarizing, in a neighborhood of the geometric solution, the solution space to WDVV consists of only the expected rescalings for \( X = G(2, 4) \), but has a dependence on two extra free parameters in case \( X = \text{Sym}^2 \mathbb{P}^2 \) (where the geometric solution comes from viewing \( X \) as a global quotient of a homogeneous variety by a finite group). When \( N(1; 0, 0, 1, 1) = 0 \) (resp. \( N(1; 1, 0, 0, 1) = 0 \)) the situation is considerably more complicated. For instance, for \( X = G(2, 4) \), setting \( N(1; 5, 0, 0, 0) = 1 \) and all other \( N(1; t, u, v, w) = 0 \) yields a consistent
solution through $\beta = 4$ (with $N(\beta; 4\beta + 1, 0, 0, 0)$ indeterminate for
$\beta = 2, 3, 4$), but fails to satisfy the relations consistently starting in
degree 5.

Example 4. $X = G(2, 5)$. The same technique as that of Example
3 establishes strong reconstruction for $G(2, 5)$. The statement of strong
reconstruction dictates 3 starting numbers and no relationships, but as
we shall see, we need to assume one relationship. As in the previous
example, we will also have to make a genericity assumption. We shall
find that the geometric solution to WDVV lives in a two-parameter
family of solutions.

Thinking of $X$ as the space of rank 2 quotients of $\mathbb{C}^5$, let $Q$ be the
universal quotient bundle and $c_i = c_i(Q)$. We use the following basis
for $A^*_Q X$, suggested by T. Graber,

| codim 0: | $t_0 = 1$ |
| codim 1: | $t_1 = c_1$ |
| codim 2: | $t_2 = c_1^2$; $t_3 = 2c_1^2 - 5c_2$ |
| codim 3: | $t_4 = c_1^3$; $t_5 = 2c_1^3 - 5c_1c_2$ |
| codim 4: | $t_6 = c_1^4$; $t_7 = c_1^4 - 5c_2^2$ |
| codim 5: | $t_8 = c_1c_2^2$ |
| codim 6: | $t_9 = c_2^3$ (point class) |

with multiplication table given in Table 2.

We denote a typical unknown by $N(\beta; d_2, d_3, d_4, d_6, d_8, d_9, d_3, d_5, d_7)$
(note special order). Inductively on degree $\beta$, we show that degree $\beta$
relations solve consistently for all but 8 degree $\beta$ numbers, plus the

| $t_1$ | $t_2$ | $t_3$ | $t_4$ | $t_5$ | $t_6$ | $t_7$ | $t_8$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| $t_1$ | $t_2$ | $t_1$ | $t_5$ | $t_6$ | $(1/3)t_7$ | $5t_8$ | 0 | $t_9$ |
| $t_2$ | $t_4$ | $t_6$ | $(1/3)t_7$ | $5t_8$ | 0 | $5t_9$ | 0 | 0 |
| $t_3$ | $t_5$ | $(1/3)t_7$ | $t_6 - (11/3)t_7$ | 0 | $5t_8$ | 0 | $15t_9$ | 0 |
| $t_4$ | $t_6$ | $5t_8$ | 0 | $5t_9$ | 0 | 0 | 0 | 0 |
| $t_5$ | $(1/3)t_7$ | 0 | $5t_8$ | 0 | $5t_9$ | 0 | 0 | 0 |
| $t_6$ | $5t_8$ | $5t_9$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $t_7$ | 0 | 0 | $15t_9$ | 0 | 0 | 0 | 0 | 0 |
| $t_8$ | $t_9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2. Multiplication table for $A^*_Q G(2, 5)$ with respect to the basis of T. Graber.
5 linear expressions shown in Table 3. The 8 exceptions are the 7 numbers appearing in Table 3 as well as \( N(\beta; 0, 0, 0, 0, 5\beta + 3, 0, 0) \).

The genericity assumption is \( N(1; 0, 0, 0, 0, 1, 0, 1) \neq 0 \). Given the induction hypothesis, we solve for the remaining degree \((\beta - 1)\) numbers according to Table 3, first by the path shown with \((0, 0, 0, 0, 0, -4, 2, 0)\) added to all degrees, then by the path shown with \((0, 0, 0, 0, 0, -2, 1, 0)\) added to all degrees, and then by the path as shown. Only for \( \beta = 2 \), during the first pass, we must substitute \( \left( \begin{array}{c} t_9 \\ t_1 \\ t_8 \end{array} \right) \) for step (c).

Once we have all the numbers in degree \( \beta - 1 \), we then induct on \( d \) with respect to the partial ordering \( d' \prec d \) \iff\ :

(i) \(|d'| < |d|\), or
(ii) \(|d'| = |d|\) and \(d_2' + d_4' + d_6' + d_8' + d_9' < d_2 + d_4 + d_6 + d_8 + d_9\), or
(iii) \(|d'| = |d|\) and \(d_2' + d_4' + d_6' + d_8' + d_9' = d_2 + d_4 + d_6 + d_8 + d_9\), but \(d_2' + 2d_4' + 3d_6' + 4d_8' + 5d_9' > d_2 + 2d_4 + 3d_6 + 4d_8 + 5d_9\), or
(iv) \(|d'| = |d|\), \((d_2', d_4', d_6', d_8', d_9') = (d_2, d_4, d_6, d_8, d_9)\), and \(d_3' + 2d_5' + 3d_7' < d_3 + 2d_5 + 3d_7\).

For each \( d \), we perform the inner induction of the proof of strong reconstruction to obtain all relations involving only powers of \( t_1 \). Next, we obtain all of

\[
N(\beta; 0, 0, 0, 0, 2, u, v, w) \\
u + 2v + 3w = 5\beta - 7
\]

by \( \left( \begin{array}{c} t_9 \\ t_1 \\ t_8 \end{array} \right) \) \( (\beta; 0, 0, 0, 0, u, v, w-1) \)

or

by \( \left( \begin{array}{c} t_9 \\ t_5 \\ t_8 \end{array} \right) \) \( (\beta; 0, 0, 0, 0, u, v, w) \)

or

by \( \left( \begin{array}{c} t_9 \\ t_3 \\ t_8 \end{array} \right) \) \( (\beta; 0, 0, 0, 0, u-1, v, w) \)

\[
N(\beta; 0, 0, 0, 1, 1, u, v, w) \\
u + 2v + 3w = 5\beta - 6
\]

by \( \left( \begin{array}{c} t_9 \\ t_1 \\ t_6 \end{array} \right) \) \( (\beta; 0, 0, 0, 0, u, v, w-1) \)

or etc.

\[
N(\beta; 2, 0, 0, 0, 0, u, v, w) \\
u + 2v + 3w = 5\beta + 1
\]

by \( \left( \begin{array}{c} t_2 \\ t_1 \\ t_1 \end{array} \right) \) \( (\beta; 0, 0, 0, 0, u, v, w-1) \) etc.

coming from relations in degree \((\beta; d)\). The exception to be noted occurs in attempting to solve for \( N(\beta; 2, 0, 0, 0, 0, 5\beta + 1, 0, 0) \): we get a value for \((L5)\) of Table 3 rather than a single \( N \).
(L1) \(N(\beta; 0, 0, 0, 0, 5\beta, 0, 1) - 3N(\beta; 0, 0, 0, 0, 5\beta - 1, 2, 0)\)

(L2) \(N(\beta; 1, 0, 0, 0, 0, 5\beta, 1, 0) - N(\beta; 0, 1, 0, 0, 0, 5\beta + 1, 0, 0) + \beta N(\beta; 0, 0, 0, 0, 5\beta - 1, 2, 0)\)

(L3) \(N(\beta; 1, 0, 0, 0, 0, 5\beta, 1, 0) - 2\beta N(\beta; 0, 0, 0, 0, 5\beta - 1, 2, 0)\)

(L4) \(N(\beta; 1, 0, 0, 0, 0, 5\beta + 2, 0, 0) - 2\beta N(\beta; 0, 0, 0, 0, 5\beta + 1, 1, 0) - 11\beta^2 N(\beta; 0, 0, 0, 0, 5\beta - 1, 2, 0)\)

(L5) \(N(\beta; 2, 0, 0, 0, 0, 5\beta + 1, 0, 0) - 4\beta^2 N(\beta; 0, 0, 0, 0, 5\beta - 1, 2, 0)\)

Table 3. The linear expressions obtained by degree \(\beta\) relations.

Still in a particular degree, we obtain

\[
N(\beta; 0, 0, 0, 0, 1, u, v, w) = \frac{u + 2v + 3w = 5\beta - 2}{\begin{pmatrix} t_5 \\ t_1 \end{pmatrix}, \begin{pmatrix} t_7 \\ t_1 \end{pmatrix} (\beta; 0, 0, 0, 0, u, v, w - 2)} \text{ by } \begin{pmatrix} t_8 \\ t_1 \\ t_7 \end{pmatrix} (\beta; 0, 0, 0, 0, u, v, w - 1, w - 1)
\]

or

etc.

with exceptions noted below:

(L2) \(\begin{pmatrix} t_2 \\ t_1 \\ t_3 \end{pmatrix} (\beta; 0, 0, 0, 0, 5\beta - 1, 0, 0)\)

(L3) \(\begin{pmatrix} t_1 \\ t_1 \\ t_3 \\ t_5 \end{pmatrix} (\beta; 0, 0, 0, 0, 5\beta - 1, 0, 0)\)

(L4) \(\begin{pmatrix} t_1 \\ t_1 \\ t_3 \end{pmatrix} (\beta; 0, 0, 0, 0, 5\beta, 0, 0)\)

Lastly, we have numbers of the form \(N(\beta; 0, 0, 0, 0, u, v, w)\) and the relations that produce these:

| relation | for cases | relation | for cases |
|----------|-----------|----------|-----------|
| \(\begin{pmatrix} t_5 \\ t_1 \\ t_7 \end{pmatrix}\) | \(w \geq 3\) | \(\begin{pmatrix} t_3 \end{pmatrix}, \begin{pmatrix} t_3 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} t_1 \\ t_7 \end{pmatrix}\) | \(u \geq 1\) | \(v \geq 1\) | \(w \geq 1\) |
| \(\begin{pmatrix} t_5 \\ t_1 \\ t_7 \end{pmatrix}\) | \(v \geq 1\) | \(w \geq 2\) | \(\begin{pmatrix} t_5 \end{pmatrix}, \begin{pmatrix} t_3 \end{pmatrix}, \begin{pmatrix} t_1 \end{pmatrix}, \begin{pmatrix} t_3 \end{pmatrix}\) | \(v \geq 3\) |
| \(\begin{pmatrix} t_5 \end{pmatrix}, \begin{pmatrix} t_1 \end{pmatrix}, \begin{pmatrix} t_7 \end{pmatrix}\) | \(u \geq 1\) | \(w \geq 2\) | \(\begin{pmatrix} t_3 \end{pmatrix}, \begin{pmatrix} t_3 \end{pmatrix}, \begin{pmatrix} t_1 \end{pmatrix}, \begin{pmatrix} t_5 \end{pmatrix}\) | (only to get (L1)) |
| \(\begin{pmatrix} t_3 \end{pmatrix}, \begin{pmatrix} t_1 \end{pmatrix}, \begin{pmatrix} t_7 \end{pmatrix}\) | \(v \geq 3\) | \(w \geq 1\) |

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Finally, we obtain \( \left( \begin{array}{c} t_3 \\ t_1 \\ t_9 \end{array} \right) \), \( \left( \begin{array}{c} t_5 \\ t_1 \\ t_9 \end{array} \right) \), and \( \left( \begin{array}{c} t_7 \\ t_1 \\ t_9 \end{array} \right) \). When \( (d_2, d_4, d_6, d_8, d_9) = (0, 0, 0, 0, 0) \) and \( d_5 = d_7 = 0 \), these are established by the three passes through the path in Table 3. For the details see Table 3. Note that each pass through Table 3 is used to solve for an unknown when \( \beta \geq 2 \); when \( \beta = 1 \) we actually get a condition on starting data, described below.

When \( (d_2, d_4, d_6, d_8, d_9) = (0, 0, 0, 0, 0) \) but \( d_5 \neq 0 \) or \( d_7 \neq 0 \), then because of the induction order, some of the (85 total) relations listed as determining numbers will determine numbers that have already been solved for. But in each such case, 3SR allows us to deduce the relation in question. Finally, when \( (d_2, d_4, d_6, d_8, d_9) \neq (0, 0, 0, 0, 0) \) then all 85 relations follow by 3SR just as in Example 3.

As in Example 3, we now note that the 85 relations in the above lists plus the 120 only involving powers of \( t_1 \) imply the remaining 461 relations by 5SR. Because of the number of relations involved, the author used a computer to complete this verification.

Relation \( \left( \begin{array}{c} t_3 \\ t_1 \\ t_9 \end{array} \right) \) from (i) in Table 3 solves for an unknown number whenever \( \beta \geq 2 \), but still needs to be verified when \( \beta = 1 \). This is what imposes the one condition on starting data. When
Assumptions
\[
\begin{align*}
(t_8, t_3) &\rightarrow (t_3, t_9) \\
(t_1, t_7) &\rightarrow (t_3, t_9) \\
(t_5, t_7) &\rightarrow (t_3, t_9) \\
(t_6, t_7) &\rightarrow (t_3, t_9) \\
(t_8, t_7) &\rightarrow (t_3, t_9) \\
(t_1, t_5) &\rightarrow (t_3, t_9)
\end{align*}
\]

Implications
\[
\begin{align*}
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5)
\end{align*}
\]

+ 3SR \Rightarrow \quad \text{(5.3–8.1,0)}

\[
\begin{align*}
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5)
\end{align*}
\]

3SR \Rightarrow \quad \text{(5.3–7.0,0)}

\[
\begin{align*}
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5) \\
(t_3, t_9) &\rightarrow (t_3, t_5)
\end{align*}
\]

Table 5. How \((t_3, t_9)\) follows from 3SR and 5SR. We use \((u, v, w)\) as shorthand for degree \((\beta; 0, 0, 0, 0, 0, u, v, w)\). Each relation listed as an assumption solves for an unknown number and for that reason is satisfied. Note that \(t_3, t_9\) is just the first three steps above.

written out for \(\beta = 1\), (e)–(g) of Table 4 translate as

\[
11N(1; 0, 0, 0, 0, 1, 0, 0, 0) = 6N(1; 0, 0, 1, 0, 1, 0, 0, 0) + 15N(1; 0, 0, 0, 2, 0, 0, 0, 0).
\]

10. APPENDIX: GEOMETRIC GROUNDWORK

Let \(X\) be a complex projective manifold, and for simplicity, assume \(X\) has homology only in even dimensions. To avoid writing doubled indices, set \(A_{d}X = H_{2d}(X, \mathbb{Z})\) and \(A^dX = H^{2d}(X, \mathbb{Z})\). Set \(A^d_{\mathbb{Q}}X = A^dX \otimes \mathbb{Q}\). For \(\beta \in A_1X\), we denote by \(M_{0,n}(X, \beta)\) the moduli space of \(n\)-pointed rational curves \(f: \mathbb{P}^1 \rightarrow X\) such that \(f_*[\mathbb{P}^1] = \beta\). The Kontsevich space \(\overline{M}_{0,n}(X, \beta)\) is a compactification of \(M_{0,n}(X, \beta)\) whose points correspond to \(n\)-pointed trees of \(\mathbb{P}^1\)’s mapping into \(X\) satisfying a stability hypothesis. Evaluation maps \(\rho_i: \overline{M}_{0,n}(X, \beta) \rightarrow X\) send a given map \(f\) to the image under \(f\) of the \(i\)th marked point.
If $X$ is a homogeneous variety (a quotient of a complex reductive Lie group by a parabolic subgroup), then the tree-level system of Gromov-Witten numbers is the system of maps $I_\beta: \bigoplus \text{Sym}^n A^*_Q X \rightarrow \mathbb{Q}$ for each $\beta \in A_1 X$ given by the formula

$$I_\beta(\gamma_1 \cdots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n).$$

(7)

With the notation of Problem 1, the geometric solution to WDVV is given by $N(\beta; d) = I_\beta(\bigotimes (T_{\tau_j})^{\otimes d_j})$. These numbers have enumerative significance: $N(\beta; d)$ is the number of rational curves on $X$ in homology class $\beta$ meeting $d_j$ general translates of a cycle Poincaré dual to $T_{\tau_j}$, for each $j$.

For general $X$, there is still a geometric solution to WDVV, given by formula (7) but with the integration performed over a virtual fundamental cycle, although the enumerative significance of these numbers is less clear. It is still the case for any $\beta \neq 0$ that $I_\beta = 0$ unless $\int_\beta \omega > 0$ for every ample divisor $\omega$.

There exist forgetful maps forgetting $X$ and forgetting any subset of the marked points, so in particular to any $\{i,j,k,l\} \subset \{1, \ldots, n\}$ there corresponds a map $\overline{M}_{0,n}(X,\beta) \rightarrow \overline{M}_{0,\{i,j,k,l\}}$. Pulling back rational equivalences on $\overline{M}_{0,4} \cong \mathbb{P}^1$ leads to the associativity relations that the Gromov-Witten numbers must satisfy.

Since the differential equation (1) is invariant under translations $y_{\sigma_i} \mapsto y_{\sigma_i} + \alpha_i$, there is an $r$-dimensional family of rescalings acting on the geometric solution to WDVV, where $r = \text{rk} A^1 X$. Much of the contents of this paper was motivated by a search for solutions to WDVV besides the geometric solution and its translates under these rescalings.

The formal computation of section 5 is motivated by considering a typical component of the boundary of one of the Kontsevich spaces $\overline{M}_{0,n}(X,\beta)$. Say $A \cup B = \{1, \ldots, n\}$, $A \cap B = \emptyset$, $|A| \geq 2$, $|B| \geq 2$, and $\beta_1 + \beta_2 = \beta$. Then there is a component $D(A, B; \beta_1, \beta_2)$ of the boundary of $\overline{M}_{0,n}(X,\beta)$, which fits into a fiber diagram (see [F-P])

$$D(A, B; \beta_1, \beta_2) \longrightarrow X^n \times X$$

$$\overline{M}_{0,A \cup \{*\}}(X, \beta_1) \times \overline{M}_{0,B \cup \{*\}}(X, \beta_2) \longrightarrow \overline{M}_{0,n}(X, \beta) \xrightarrow{\delta} X^n \times X \times X \times X$$

where $\delta$ is given by the diagonal embedding of $X$ in $X \times X$. 21
Given cohomology classes \( \gamma_1, \ldots, \gamma_n, \xi \in A^*X \), we have
\[
\delta_*(\gamma_1 \times \cdots \times \gamma_n \times \xi) = \sum_{e,f} g^{ef} \gamma_1 \times \cdots \times \gamma_n \times (\xi \cup T_e) \times T_f = \sum_{e,f} g^{ef} \gamma_1 \times \cdots \times \gamma_n \times T_e \times (\xi \cup T_f)
\]
corresponding to two ways of writing the class \( \xi \) on the diagonal. Pulling back by \( \rho \) and integrating gives the identity
\[
\sum_{e,f} g^{ef} I_{\beta_1}(\prod_{a \in A} \gamma_a \cdot (\xi \cup T_e)) I_{\beta_2}(\prod_{b \in B} \gamma_b \cdot T_f) = \sum_{e,f} g^{ef} I_{\beta_1}(\prod_{a \in A} \gamma_a \cdot T_e) I_{\beta_2}(\prod_{b \in B} \gamma_b \cdot (\xi \cup T_f)).
\]
If \( \gamma_1 = T_i, \gamma_2 = T_j, \gamma_3 = T_k, \gamma_4 = T_l, \) and \( \xi = T_m \), and if we sum over partitions \( (\beta_1, \beta_2, A, B) \) such that \( A \) contains 1 and 2 and \( B \) contains 3 and 4, we get a special case of (5), namely the case where the potential function \( \Gamma \) is given by the geometric solution to WDVV.

**References**

[DF-I] P. Di Francesco and C. Itzykson, *Quantum intersection rings*, The moduli space of curves (Texel Island, 1994), 81–148, Progress in Mathematics 129, Birkhäuser, 1995.

[F-P] W. Fulton and R. Pandharipande, *Notes on Stable Maps and Quantum Cohomology* to appear in Algebraic Geometry (Santa Cruz, 1995).

[G] A. Givental, *Equivariant Gromov-Witten Invariants*, International Mathematics Research Notices, 1996, No. 13.

[K-M] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Commun. Math. Phys. 164 (1994) 525–562.

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