Introduction

The motivation for the paper is the following representation-theoretic interpretation of the center $Z(G)$ of a compact group $G$ discovered in [25]:
• On the one hand, one can impose a “universal grading” on the category $\text{Rep}(G)$ of unitary $G$-representations by assigning each irreducible $\rho \in \hat{G}$ an abstract generator $g_\rho$ and imposing the relation

$$g_\rho = g_\rho' g_\rho''$$

whenever we have a direct-summand inclusion

$$\rho \leq \rho' \otimes \rho''.$$  

This is the chain group $C(G)$ of [25, Proposition 2.3].

• On the other, there is an obvious grading of $\text{Rep}(G)$ by the Pontryagin dual $\hat{\mathbb{Z}}(G)$: the center acts by scalars on any irreducible representation $\rho \in \hat{G}$ by Schur’s Lemma [29, (8.6)], so $\rho$ naturally gets assigned a character $\chi_\rho \in \hat{\mathbb{Z}}(G)$; this assignment then satisfies the same type of tensor-product-compatibility relation:

$$\rho \leq \rho' \otimes \rho'' \Rightarrow \chi_\rho = \chi_\rho' \chi_\rho''.$$  

The main result of [25] is that these two procedures are equivalent. Formally, [25, Theorem 3.1] says that

$$C(G) \ni g_\rho \mapsto \chi_\rho \in \hat{\mathbb{Z}}(G) \text{ for } \rho \in \hat{G} \quad (0-1)$$

is an isomorphism between the chain group $C(G)$ and the dual $\hat{\mathbb{Z}}(G)$ of the center: the chain-center duality of this paper’s title. As that same title suggests, the focus here is on locally compact groups. Extending the framework for considering such duality results is not difficult, though some modifications are needed.

One problem is that in general, for locally compact groups, appearing as a direct summand (or not) is a poor indicator of “containment”. A much more reasonable notion is that of weak containment ([15, Chapter 3, §4.5, Definition 3] or [5, Theorem 3.4.4]), denoted throughout by ‘≺’ or ‘≤’. In view of this, our reworking of [25, Definition 2.1, Proposition 2.3] reads

**Definition 0.1** Let $G$ be a locally compact group. The chain group $C(G)$ is group defined

(a) by one generator $g_\rho$ for each irreducible unitary representation $\rho \in \hat{G}$;

(b) subject to $g_\rho = g_\rho' g_\rho''$ for any weak containment relation

$$\rho \preceq \rho' \otimes \rho'';$$

(c) as well as $g_\rho^* = g_\rho h_{\rho^{-1}}$ for the contragredient representation $\rho^*$ of $\rho$ [1, Definition A.1.10].

The chain group has a natural topology: since

• the unitary dual $\hat{G}$ surjects onto the set of generators $\{g_\rho\}$;

• and the relations ensure that the image of that surjection in fact encompasses all of $C(G)$,

we can equip the latter with the quotient of the Fell topology [1, Definition F.2.1] resulting from said surjection $\hat{G} \to C(G)$.   ♦

Schur’s Lemma and the canonical map (0-1) make just as much sense as for compact groups, so one can define the locally compact group $G$ to be chain-center dual or cc-dual for short if (0-1) is an isomorphism (see Definition 2.2). The obvious question flows naturally:
Question 0.2 Are all locally compact groups cc-dual?

I do not know the answer, but the results below identify several fairly large classes. Paraphrasing and summarizing:

Theorem 0.3 Let $G$ be a locally compact group. $G$ is cc-dual in each of the following cases:

(a) $G$ is a compact-by-abelian extension (i.e. has a closed, abelian normal subgroup with corresponding compact quotient): Theorem 3.2;

(b) $G$ is connected and nilpotent: Theorem 3.5;

(c) $G$ is countable discrete, and its non-trivial conjugacy classes are infinite (i.e. $G$ is icc): Theorem 3.7;

(d) $G$ is a connected semisimple Lie group: Theorem 4.1.

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1 Preliminaries

Topological groups are always assumed Hausdorff unless specified otherwise, and ‘representation’ always means unitary (for groups) or $*$-representation on a Hilbert space for a $C^*$-algebra.

For a locally compact group $G$ the symbol $\hat{G}$ denotes the set of isomorphism classes of irreducible unitary representations (as is standard in the literature: [5, §13.1.4], [14, Definition 1.46], etc.). More generally (though this will not happen often), $\hat{A}$ denotes the set of (isomorphism classes of) irreducible $*$-representations of the $C^*$-algebra $A$ (cf. [5, §2.3.2] or [27, §4.1.1]).

Some background on type-I $C^*$-algebras is needed, as covered, say, in [5, Chapter 9] (the term postliminal [5, Definition 4.3.1] is also in use) or [27, Chapter 6]. Locally compact groups are type-I if their universal $C^*$-algebras are (as in [5, §13.9.4], for instance). Numerous equivalent characterizations of the type-I property are known when the $C^*$-algebra in question is separable: see for instance [5, Theorem 9.1], [27, Theorem 6.8.7] or [3, Theorems IV.1.5.7 and IV.1.5.12]. Complications ensue in the non-separable case [27, §6.9]; for that reason, the following convention is in place.

Convention 1.1 All discussion of type-I algebras / groups will be limited to the separable case (for groups this means $G$ is assumed to be second-countable, i.e. have a countable basis of open sets [23, §1.3]).

We will, at some point, have to consider groups that are also topological spaces, but for which it is unclear whether the multiplication is continuous. We refer to these as groups-with-topology (cf. Remark 2.1) and denote the corresponding category (with continuous group morphisms) by $\text{Gr}_{\text{top}}$. By contrast, $\text{TopGp}$ denotes the category of topological groups.

Proposition 1.2 The inclusion functor $\iota : \text{TopGr} \subset \text{Gr}_{\text{top}}$ from topological groups to groups-with-topology is a right adjoint.
Proof We construct the left adjoint

\[ F : \text{Gr}_{\text{top}} \to \text{TopGr}. \]

On the level of sets and functions between them \( F \) is simply the identity; what it does to a group-with-topology is alter that topology, weakening it so as to ensure compatibility with the group operations.

Concretely, let \( G \) be a group-with-topology and denote by \( \tau \) that topology. Now consider the weaker topology defined as follows:

- keep only those open sets of \( \tau \) whose preimage through the multiplication \( G \times G \to G \) is open in the product topology \( \tau \times \tau \);
- and whose preimage through the inverse \( \bullet^{-1} : G \to G \) is also \( \tau \)-open.

Then repeat the procedure with \( \tau' \) in place of \( \tau \) if necessary, etc. Doing this countably infinitely many times will suffice to produce a topology \( \tau' \) weaker than \( \tau \) making the original group structure of \( G \) into a topological one.

By the very definition of \( \tau' \), any morphism \((G, \tau) \to H\) of groups-with-topology for a topological group \( H \) will in fact factor (uniquely) through the canonical continuous map \( \text{id} : (G, \tau) \to (G, \tau') \).

This universality is what makes \( F : (G, \tau) \mapsto (G, \tau') \) the left adjoint to \( \iota \). ■

2 Chain groups and centers

2.1 Generalities

The chain group \( C(G) \) attached to a locally compact group \( G \) was defined above (see Definition 0.1).

Remark 2.1 Note that at this stage \( C(G) \) has not been shown to be a topological group: the topology it is equipped with might, in principle, not be compatible with the multiplication.

While \( C(G) \) will be a topological group in the cases of interest (e.g. this is an implicit requirement of Definition 2.2), I do not know whether this is so in general. For that reason, we occasionally use the awkward phrase ‘groups-with-topology’ to refer to the category of groups equipped with a perhaps-incompatible topology, with continuous group morphisms.

The chain group relates to the center \( Z(G) \) as follows (cf. [25, Proposition 2.5]).

Definition 2.2 Since every central element \( g \in Z(G) \) acts as a scalar in each irreducible representation \( X \in \hat{G} \) (by Schur’s Lemma: e.g. [29, (8.6)]), we have a canonical map \( \hat{G} \to \hat{Z(G)} \) onto the Pontryagin dual of the center of \( G \). It is easily checked that that map extends to a continuous surjective (Lemma 2.5) morphism.

\[ \text{CAN} : C(G) \to \hat{Z(G)} \]

\((2\text{-}1)\)

\( G \) satisfies chain-center duality or is chain-center or cc-dual if the map \((2\text{-}1)\) is an isomorphism of topological groups.

Example 2.3 In the language of Definition 2.2, [25, Theorem 3.1] says that compact groups are cc-dual (note that in that case the topologies on the groups in question are discrete).

On the other hand, we also have
Lemma 2.4 Abelian locally compact groups are cc-dual in the sense of Definition 2.2.

**Proof** In the abelian case $Z(G) = G$. On the other hand, because irreducible unitary representations are 1-dimensional (e.g. by Schur’s lemma again, [29, (8.6)]), the dual $\hat{G}$ is simply the Pontryagin dual group with its standard topology [1, Example F.2.5]. That (2-1) is an identification is now clear.

We make note of the following simple fact, for future reference.

**Lemma 2.5** For an arbitrary locally compact group $G$ the canonical morphism (2-1) is onto.

**Proof** Regard an arbitrary character $\hat{Z}(G) \ni \chi : Z(G) \to S^1 \subset \mathbb{C}$ as a 1-dimensional unitary representation of the center $Z := Z(G)$, and consider the induced representation $\rho := \text{Ind}_Z^G \chi$ [1, Definition E.1.6]. It is easy to see from the definition of the induction procedure that the restriction $\rho|_Z$ back to the center is a sum of copies of $\chi$ (i.e. $Z$ acts in $\rho$ via $\chi$). But then it follows from the Fell-continuity of the restriction operation $	ext{Rep}(G) \ni \sigma \mapsto \sigma|_Z \in \text{Rep}(Z)$ [1, Proposition F.3.4] that all irreducible representations $\sigma \in \hat{G}$ weakly contained in $\rho$ (of which there are plenty [1, Proposition F.2.7]) have the same property that $Z$ acts therein via $\chi$. The class of any such $\sigma$ will be mapped by (2-1) onto $\chi$, finishing the proof.

The canonical morphisms (2-1) are also functorial, with the appropriate caveats.

**Lemma 2.6** Both $C(G)$ and $\hat{Z}(G)$ are contravariant functors from the category $\mathcal{LCG}_{\text{dense}}$ of locally compact groups with dense-image morphisms to that of abelian groups-with-topology, and the morphisms (2-1) constitute a natural transformation.

**Proof** The naturality claim is easy to check once we have functoriality. Consider a dense-image morphism $f : G \to H$, and note first that

$$f(Z(G)) \subseteq Z(H)$$

(because of the image density) and hence, upon dualizing, $f$ induces a map $\hat{Z}(H)\to \hat{Z}(G)$ in the opposite direction.

As to chain groups, the dense-image condition now ensures that unitary-representation restriction along $f$ preserves irreducibility. The definitions of $C(G)$ and $C(H)$ by generators and relations starting with the irreducibles then makes it clear that we have an induced map $C(H) \to C(G)$.

Remark 2.1 notwithstanding, the topologies on $C(G)$ and $\hat{Z}(G)$ will not play much of a role in assessing cc-duality:

**Lemma 2.7** A locally compact group $G$ is cc-dual in the sense of Definition 2.2 if an only if the canonical map (2-1) is a bijection.
Proof One implication is obvious, so we handle the other. Assume (2-1) is bijective. It fits into a commutative diagram

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{C(G)} & \hat{Z}(G) \\
\downarrow & & \downarrow \\
\hat{G} & \xrightarrow{\text{can}} & \\
\end{array}
\]

(2-2)

where the top arrow is the quotient map giving \(C(G)\) its topology (by definition), while the bottom map is

\[
\hat{G} \ni \rho \mapsto \chi \in \hat{Z}(G) \text{ with } \rho|_{Z(G)} \cong \chi^{\oplus S}
\]

(for some index set \(S\)). This map too makes \(\hat{Z}(G)\) into a quotient topological space of \(\hat{G}\): it is continuous [1, Proposition F.3.4], surjective (Lemma 2.5), and closed (Lemma 2.9); it must thus be a quotient map by [26, §22, discussion preceding Example 1]. To conclude, observe that the bijectivity of \(\text{can}\) identifies the two quotient topologies. 

As a consequence, the central requirement for cc-duality is the injectivity of (2-1).

Corollary 2.8 A locally compact group \(G\) is cc-dual if and only if the canonical map (2-1) is one-to-one.

Proof Apply Lemma 2.5 and Lemma 2.7 jointly.

Lemma 2.9 Let \(A\) be a \(C^*\)-algebra and \(Z \subseteq A\) its center. The map \(\hat{A} \to \hat{Z}\) defined by

\[
\hat{A} \ni \rho \mapsto \chi \in \hat{Z}\text{ such that } \rho|_Z \cong \chi^{\oplus S}
\]

is closed with respect to the Fell topologies.

Proof It follows from [5, §3.2.2] that the closed subsets of \(\hat{A}\) are in bijection with the closed two-sided ideals, via

\[
\text{ideal } I \mapsto \hat{A}/I.
\]

Consider such a closed subset \(\hat{A}/I \subseteq \hat{A}\), consisting of precisely those irreducible representations whose kernels contain an ideal \(I\). The kernels of the restrictions

\[
\rho|_Z, \ \rho \in \hat{A}/I
\]

contain \(Z \cap I\), and on the other hand, because \(Z/Z \cap I\) embeds into \(A/I\), every character of \(Z/Z \cap I\) can be obtained in this manner [5, Proposition 2.10.2]. In short, the image of the arbitrary closed subset \(\hat{A}/I \subseteq \hat{A}\) through the map \(\hat{A} \to \hat{Z}\) in the statement is nothing but

\[
\hat{Z}/Z \cap I \subseteq \hat{Z}.
\]

This is a closed set, finishing the proof.

According to the already-cited [26, §22, discussion preceding Example 1], open (as opposed to closed) continuous surjections are also quotient maps. Unlike closure though, openness is not automatic for the maps \(\hat{A} \to \hat{Z}\) discussed in Lemma 2.9 (even for groups).
Example 2.10 For an example of a locally compact group \( G \) with \( \hat{G} \to \hat{Z}(G) \) not open it will suffice to consider \( G \):

- with Kazhdan’s property \((T)\) [1, Definition 1.1.3], so that the trivial representation constitutes an open point of \( \hat{G} \);

- and with discrete infinite center, ensuring that the trivial character is not open in \( \hat{Z}(G) \).

A concrete example is the universal cover \( \widetilde{Sp}(4, \mathbb{R}) \) of the \( 4 \times 4 \) symplectic group: it has property \((T)\) by [7, Theorem 6.8] (where \( Sp(4) \) is denoted by \( Sp(2) \)), and its center is easily computed as \( \mathbb{Z} \).

Lemma 2.11 Let \( G \) be a connected locally compact group and \( N \trianglelefteq G \) a closed, normal, discrete subgroup. The quotient \( \pi : G \to G/N \) then induces a surjection between the centers of the two groups.

Proof That \( \pi \) maps center to center follows from surjectivity, so the substance of the claim is that the restriction

\[
\pi|_{Z := Z(G)} : Z \to Z(G/N)
\]

is again onto.

Connected locally compact groups are the inverse limit of their Lie-group quotients by compact normal subgroups contained in arbitrarily small open neighborhoods of 1 [23, §4.6, Theorem]. In particular, they are all pro-Lie groups in the sense of [11, pp.161-162, equivalent definitions A, B and C].

Because \( G \) and \( G/N \) are connected pro-Lie groups their Lie algebras ([11, Definition 2.6])

\[
\text{Lie}(G) := \text{Hom}(\mathbb{R}, \mathbb{R}_+, G)
\]

(with the compact-open topology)

(and similarly for \( G/N \)) generate dense subgroups via the exponential maps [11, Corollary 4.22 (i)]. It follows that an element of \( G \) or \( G/N \) is central if and only if it acts trivially on the respective Lie algebra via the adjoint action ([11, Definition 2.27 and subsequent discussion]), supplying the two outer equivalences in (2-3) below.

Because \( \pi : G \to G/N \) has discrete kernel it identifies the two Lie algebras, securing the middle equivalence; all in all, for \( g \in \hat{G} \) we have

\[
\pi(g) \text{ is central } \iff \text{Ad}_{\pi(g)} = \text{id on Lie}(G/N) \\
\iff \text{Ad}_g = \text{id on Lie}(G) \\
\iff g \text{ is central in } G.
\]

This concludes the proof.

In general, neither the connectedness of \( G \) nor the discreteness of \( N \) can be left out in Lemma 2.11:

Example 2.12 Two incarnations of the Heisenberg group will illustrate both claims. First, in its real version

\[
H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right| x, y, z \in \mathbb{R} \right\}
\]
it shows that the discreteness of $N$ is necessary, as we can take $G = H$ and
\[
N := Z(G) = \left\{ \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.
\] (2-5)

The quotient $G/N$ is then (isomorphic to) the abelian group $(\mathbb{R}^2, +)$, so $\pi$ does not induce a surjection on centers.

On the other hand, consider the discrete version of $H$, defined as (2-4) but with entries ranging only over the integers. Taking $G = H$ and the center (2-5) for $N$ again we once more have an abelian quotient $G/N$.

In a variation of this last example we can even take $G$ finite, by allowing the elements $x, y$ and $z$ of (2-4) to range only over $\mathbb{Z}/n$ for some $n$.

A number of the preceding results now allow us to lift up cc-duality from quotients.

**Corollary 2.13** Let $G$ be a connected locally compact group and $N \trianglelefteq G$ a closed, normal, discrete subgroup. If $G/N$ is cc-dual then so is $G$.

**Proof** Suppose $G/N$ is cc-dual. According to Corollary 2.8, it will be enough to show that an irreducible representation $\rho \in \widehat{G}$ is trivial in the chain group $C(G)$ as soon as it is trivial on the center. Now, because $N$ is a discrete normal subgroup of a connected group, it must be central. It follows that $\rho$ is trivial on $N$ and hence
\[
\rho \in \widehat{G/N} \subseteq \widehat{G}.
\]

But then, by the cc-duality assumption, $\rho$ is trivial in $C(G/N)$, which in turn maps to $C(G)$ via the contravariant functoriality noted in Lemma 2.6. This finishes the proof. ■

It should come as no surprise that The passage from $\text{Rep}(G)$ to $C(G)$ loses much information, as the cc-duality of compact groups [25, Theorem 3.1] suggests. Another incarnation of this information-loss phenomenon is that $C(G)$ is only aware of the reduced dual of $G$; following [5, Definition 18.3.1] or [4, Appendix H]:

**Definition 2.14** For a locally compact group $G$ the reduced dual $\widehat{G}_{\text{red}}$ is the set of isomorphism classes of irreducible unitary representations weakly contained in the regular representation $L^2(G)$ with respect to a Haar measure.

Equivalently [5, §18.3.2], these are the irreducible unitary representations of the reduced (as opposed to full or maximal) group $C^*$-algebra $C^*_{\text{red}}(G)$ (for full/reduced group $C^*$-algebras see [4, Definition 6] or [1, Definition F.4.6]).

The observation alluded to above is

**Proposition 2.15** For any locally compact group $G$, the canonical map
\[
\widehat{G}_{\text{red}} \subseteq \widehat{G} \to C(G) \quad (2-6)
\]
is onto.
Proof This follows, essentially, from the absorption property of the left regular representation $\Lambda_G$ of $G$ with respect to a left Haar measure [1, Corollary E.2.6 (ii)]: for any unitary representation $\rho$,\[
\rho \otimes \lambda_G \cong \lambda_G^\oplus \dim \rho. \tag{2-7}
\]
Fix an arbitrary $\rho \in \hat{G}$ and also an irreducible $\lambda \in \hat{G}_{\text{red}}$ (i.e. weakly contained in $\lambda_G$). We then have\[
\rho \otimes \lambda \preceq \rho \otimes \lambda_G \preceq \lambda_G, \tag{2-8}
\]
where the first weak containment follows from the Fell continuity of tensor products [1, Proposition F.3.2] and the second is a consequence of (2-7). In particular, any irreducible weakly contained in the left-hand side $\rho \otimes \lambda$ of (2-8) will belong to $\hat{G}_{\text{red}}$. Denote, temporarily, the image of (2-6) by $\rho \cdot \text{C}G_{\text{red}}$; the preceding discussion then amounts to\[
\rho \cdot \text{C}G_{\text{red}} \subseteq \text{C}G_{\text{red}}. \tag{2-9}
\]
Next, observe that $\text{C}G_{\text{red}}$ is\begin{itemize}
  \item closed under inverses, by condition (c) of Definition 0.1, because $\lambda_G$ is self-contragredient and hence $\hat{G}_{\text{red}}$ is invariant under the contragredient operation;
  \item closed under products, because the tensor product descends to representations of $C^*_\text{red}(G)$ and hence $\hat{G}_{\text{red}}$ is closed under taking simple constituents of (i.e. irreducible representations weakly contained in) tensor products.
\end{itemize}
$\text{C}G_{\text{red}} \subseteq \text{C}G$ is, in other words, a subgroup, so (2-9) shows that the arbitrary $\rho \in \text{C}G$ is a member thereof. □

2.2 Almost-connected groups and their Lie quotients

We will see that as far as connected locally compact groups $G$ go, the study of the chain group $\text{C}G$ can be reduced to Lie groups. In fact, this can be extended slightly; consider the following class of groups (following, for instance, [12, p. viii]).

Definition 2.16 A topological group $G$ is almost-connected if its quotient $G/G_0$ by the connected identity component is compact. ♦

To continue, recall from [23, §4.6, Theorem] that an almost-connected locally compact group $G$ is pro-lie in the sense of [11, Theorem 3.39]:\begin{itemize}
  \item every neighborhood $U \subset G$ of the identity
  \item contains a compact normal subgroup $K \trianglelefteq G$
  \item such that $G/K$ is Lie;
  \item and furthermore, as a consequence, $G$ is the limit (in the category of topological spaces) of the filtered system of Lie quotients $G/K$ consisting of the surjective connecting maps $G/K_1 \to G/K_2$
\end{itemize}
resulting from inclusions $K_1 \subset K_2$. 9
We write $\text{LIEQ}(G)$ for the set of compact normal subgroups $K \triangleleft G$ with corresponding Lie quotients $G/K$. Dualizing this inverse-limit expression

$$G \cong \underset{K \in \text{LIEQ}(G)}{\text{lim}} G/K,$$

we have

**Theorem 2.17** Let $G$ be an almost-connected locally compact group. The canonical inclusions

$$\hat{G}/K \subseteq \hat{G}, \ K \in \text{LIEQ}(G)$$

realize $\hat{G}$ as a filtered colimit

$$\hat{G} \cong \underset{K \in \text{LIEQ}(G)}{\text{lim}} \hat{G}/K$$

(2-10)

in the category of topological spaces.

**Proof** The goal is twofold:

(a) proving (2-10) set-theoretically, i.e. showing that $\hat{G}$ is the union of its subsets $\hat{G}/K$ as $K$ ranges over $\text{LIEQ}(G)$;

(b) and showing that the Fell topology of $\hat{G}$ is generated by the underlying Fell topologies of its subspaces $\hat{G}/K$ in the sense, say, of [9, discussion following Proposition A.5]: a subset $U \subseteq \hat{G}$ is open if and only if all intersections $U \cap \hat{G}/K$ are.

**Proof** The goal is twofold:

(a) **$\hat{G}$ is the union of $\hat{G}/K$.** Consider an irreducible representation $\rho \in \hat{G}$ and fix some $K \in \text{LIEQ}(G)$. By Lemma 2.19 we have a decomposition

$$\rho|_K \cong \bigoplus_{\pi \in S} \pi \otimes n_{\pi}$$

for a finite set $S \subseteq \hat{K}$. Since irreducible unitary representations of compact groups are finite-dimensional [5, Theorem 15.1.3], $\pi$ factors through some Lie group quotient

$$K \to \bar{K} \subseteq U(n)$$

of $K$.

The kernel $N$ of $K \to \bar{K}$ is normal in the compact group $K$. Now, the proof of [13, Theorem 2] shows that the connected component $G_0$ is contained in the product $KZ_G(K)$, so any normal subgroup of $K$ will automatically be normalized by $G_0$. On the other hand, the quotient

$$G/G_0 \to G/G_0 K \cong (G_0 K/G_0)/(G/G_0)$$

is both profinite (as a quotient of a profinite group [12, Chapter 1, Exercise E1.13]) and Lie, so it must be finite. This means that $K$ maps to a finite-index subgroup of the profinite group $G/G_0$; if $s_i \in G$, $1 \leq i \leq n$ is a system of representatives for the cosets then

$$N_{\text{small}} := \bigcap_{i=1}^n s_i N s_i^{-1} \triangleleft K$$

is a normal subgroup.
whose corresponding quotient $K/N_{\text{small}}$ is Lie [23, Lemma 4.7.1];

which is normalized by the connected component $G_0$, as noted above;

whose image in $G/G_0$ is normal, by construction;

and hence normal in $G$.

We now have an extension

$$1 \to K/N_{\text{small}} \to G/N_{\text{small}} \to G/K \to 1$$

where the outer terms Lie, meaning that the middle term $G/N_{\text{small}}$ must be Lie as well [13, Theorem 7]. To summarize, $N_{\text{small}}$ is a member of $\text{LIEQ}(G)$. Since the arbitrary $\rho \in \hat{G}$ is by construction trivial on $N_{\text{small}}$, we are done with (a).

(b): The Fell topology on $\hat{G}$ is the colimit of the Fell topologies of $\hat{G}/N$. We have already established that we have a union

$$\hat{G} = \bigcup_{K \in \text{LIEQ}(G)} \hat{G}/K.$$

It now remains to observe that the $\hat{G}/K$ are all open in $\hat{G}$ (in addition to being closed). To see this, recall first [1, Proposition F.3.4] that restricting $G$- to $K$-representations is a Fell-continuous operation; we also saw in the course of the proof of Lemma 2.19 that [14, Proposition 4.68] applies, and hence every restriction

$$\rho|_K, \rho \in \hat{G}$$

lives on a (finite) $G$-orbit of the discrete set $\hat{K}$. But we then have a continuous map

$$\hat{G} \ni \rho \mapsto (\text{orbit where } \rho|_K \text{ lives}) \in \hat{K}/G,$$

to a discrete set, and $\hat{G}/K$ is nothing but the preimage of the open singleton $\{\text{trivial orbit}\}$. We conclude via the general (and trivial) remark that a filtered union of open subspaces is automatically a colimit topology. ■

Remark 2.18 Under certain circumstances, a “transitivity of normality” obtains for topological groups: according to [13, Theorem 3],

(closed normal) in (compact normal) in (connected) ⇒ normal.

The construction of the globally-normal subgroup $N_{\text{small}} \trianglelefteq G$ in the proof of Theorem 2.17 runs along the same conceptual lines; in fact, were $G$ connected (rather than almost-connected), we could have simply skipped constructing $N_{\text{small}}$ altogether and worked directly with the (already normal, by [13, Theorem 3]) group $N$. ♦

Lemma 2.19 Let $G$ be an almost-connected locally compact group and $K \trianglelefteq G$ a compact normal subgroup.

For an irreducible representation $\rho \in \hat{G}$ the restriction $\rho|_K$ is a sum

$$\rho|_K \cong \bigoplus_{\pi \in S} \pi \otimes n_\pi$$

for a finite subset $S \subseteq \hat{K}$ and multiplicities $n_\pi$. Furthermore, $S$ is a singleton if $G$ is connected.
Proof In the language of [14, Definition 4.59], we have to argue that the restriction $\rho|_K$ lives on a finite subset $S \subseteq \hat{K}$, and that that set is a singleton if $G$ is connected.

The conjugation action by $G$ on $K$ induces one on the discrete space $\hat{K}$, as in [14, §4.8] (the focus there is on the space $\text{Prim}(K)$ of primitive ideals of $C^*(K)$, but this coincides with $\hat{K}$ for compact groups). The connected component $G_0$ acts trivially (as it must on any discrete set), while the compact quotient $G/G_0$ acts with finite orbits.

According to [14, Proposition 4.68] the restriction $\rho|_K$ lives on a $G$-orbit of $\hat{K}$; since we have just established that orbits are finite and singletons when $G = G_0$, we will be done once we show that the first hypothesis of [14, Proposition 4.68] is satisfied.

Specifically, that hypothesis requires that $K \leq G$ be regularly embedded in the sense of [14, paragraph following Lemma 4.67]: the quotient space $\hat{K}/G$ is almost Hausdorff. Whatever the latter condition (defined in [14, paragraph before Remark 4.19]) means, it will certainly be satisfied by the discrete space $\hat{K}/G$; this finishes the proof. ■

Switching back to chain groups, this implies

**Corollary 2.20** The chain group of a connected locally compact group $G$ can be realized as a filtered colimit

$$C(G) \cong \lim_{K \in \text{Lieq}(G)} C(G/K)$$

in the category of groups-with-topology via the contravariant functoriality provided by Lemma 2.6.

**Proof** This is a consequence of Theorem 2.17, since all constructions involved in passing from $\hat{G}$ to $C(G)$ (the generators-and-relations definition of the group structure and the quotient topology) can be cast as colimits and hence commute with $\lim_{\leftarrow} K$ [19, Chapter IX, ending paragraph of §8]. ■

The analogous claim holds on the right-hand, center side of Definition 2.2.

**Proposition 2.21** For a connected locally compact group $G$ the duals of the canonical maps $Z(G) \to Z(G/K)$ express the dual center as the filtered colimit

$$\hat{Z}(G) \cong \lim_{K \in \text{Lieq}(G)} \hat{Z}(G/K)$$

in the category $\text{TopGp}$ of topological groups.

**Proof** The center $Z(G/K)$ of each $G/K$ can be recovered as the equalizer [19, Chapter III, §4] of all of the automorphisms of $G/K$ induced by elements $g \in G$. It follows, from the general principle that limits commute with limits ([19, Chapter IX, §2, dual to (2)]), that

$$Z(G) \cong \lim_{\leftarrow} Z(G/K)$$

as topological groups. The conclusion follows by taking duals and using the fact that duality $\bullet \mapsto \hat{\bullet}$ is a contravariant, involutive autoequivalence of the category of locally compact abelian groups [12, Proposition 7.11, Theorem 7.63]. ■

Aggregating a number of these results, we have

**Corollary 2.22** A connected locally compact group $G$ is cc-dual in the sense of Definition 2.2 if all of its Lie quotients $G/K$ by compact normal subgroups are.

**Proof** This is a direct consequence of Lemma 2.6, Corollary 2.20 and Proposition 2.21. ■
2.3 Reciprocity for type-I locally compact groups

Proposition 2.23 is not used directly in the sequel, but it seems pertinent to understanding and working with the chain group.

Let $G$ be a type-I, second-countable locally compact group. As explained in [32, Theorem 14.11.2] (or [5, Theorem 18.8.1]), we have a well-defined (up to equivalence) Borel measure $\mu_G$ on $\hat{G}$ (the Plancherel measure of $G$) such that the regular representation $L^2(G,\lambda)$ with respect to a left-invariant Haar measure decomposes as a direct integral [32, §14.9]

$$L^2(G,\lambda) \cong \int_{\hat{G}} \rho \oplus \dim \rho \, d\mu_G(\rho).$$

By [5, Proposition 8.6.8], those $\rho$ contained in the support $\text{supp}(\mu_G)$ (i.e. the smallest closed subset of $\hat{G}$ whose complement has $\mu_G$-measure zero) are precisely the elements of the reduced dual (Definition 2.14).

The main reciprocity result of [24] can be recast as follows ('*' superscripts denoting contragredient representations, as before).

**Proposition 2.23** Let $G$ be a type-I, second-countable locally compact group and

$$\pi_i \in \hat{G}_{\text{red}}, \ 1 \leq i \leq 3.$$

If $\pi_3^* \leq \pi_1 \otimes \pi_2$ then

$$\pi_i^* \leq \pi_j \otimes \pi_k, \text{ for all permutations } (i,j,k) \text{ of } (1,2,3).$$

**Proof** Consider the direct-integral decomposition

$$\pi_1 \otimes \pi_2 \cong \int_{\hat{G}} (\pi_3^*)^{\otimes n(\pi_1,\pi_2,\pi_3)} \, d\mu_{\pi_1,\pi_2}(\pi_3) \quad (2-11)$$

of [24, p.362], with the slight caveat that our $\mu_{\pi_1,\pi_2}$ is that source’s $\pi_{\pi_1,\pi_2}$. Here $n(\cdot,\cdot,\cdot)$ is the relevant multiplicity function (positive-valued; see Remark 2.24). According to [5, Proposition 8.6.8] we have $\pi_3^* \leq \pi_1 \otimes \pi_2$ precisely when $\pi_3$ is contained in the support of $\mu_{\pi_1,\pi_2}$:

$$\mu_{\pi_1,\pi_2}(U) > 0 \text{ for all open neighborhoods } \pi_3 \in U \subseteq \hat{G}. \quad (2-12)$$

Next note that this implies

$$\mu_{\pi_1',\pi_2'}(U) > 0 \text{ for } (\pi_1',\pi_2') \text{ in a neighborhood of } (\pi_1,\pi_2).$$

Indeed, otherwise we could find Fell-convergent nets

$$\pi_{i,\alpha} \to \pi_i, \ i = 1,2$$

such that all $\mu_{\pi_{1,\alpha},\pi_{2,\alpha}}$ are carried by the closed set $F := \hat{G}_{\text{red}} \setminus U$ and hence all $\pi_{1,\alpha} \otimes \pi_{2,\alpha}$ are representations of the quotient of $C^*(G)_{\text{red}}$ corresponding to that closed subset. By the Fell-continuity of the tensor product [1, Proposition F.3.2], this contradicts (2-12).

What all of this allows us to conclude is that $\pi_3^* \leq \pi_1 \otimes \pi_2$ precisely when

$$(\pi_1,\pi_2,\pi_3) \in \text{supp}(\mu_{\pi_1,\pi_2} \, d\mu_G(\pi_1) \, d\mu_G(\pi_2)), \quad (2-13)$$

where $\mu_G$ is, as before, the Plancherel measure of $G$. Now, the (class of the) measure on the right-hand side of (2-13) is proven in [24, §2, Theorem] (where that measure is denoted by $\lambda$) to be invariant under permuting the $\pi_i$, hence the conclusion. ■
Remark 2.24 Although this does not seem to be spelled out in [24], the multiplicity function \( n(\cdot, \cdot, \cdot) \) appearing there and in the proof of Proposition 2.23 is positive rather than non-negative. That is, the failure of a representation \( \pi_3 \) to appear in the decomposition (2-11) is encoded in the vanishing of \( \mu_{\pi_1, \pi_2} \) rather than that of \( n(\pi_1, \pi_2, \pi_3) \).

This is clear from the claim made in [24, §2, following the first equation display] that \( \mu_{\pi_1, \pi_2} \) is uniquely defined up to equivalence: if we were to allow vanishing \( n(\cdot, \cdot, \cdot) \) then \( \mu_{\pi_1, \pi_2} \) would be arbitrary over those \( \pi_3 \) where \( n(\pi_1, \pi_2, \pi_3) \) does vanish.

Remark 2.25 Because the measure on the right-hand side of (2-13) (i.e. the \( \lambda \) of [24, §2]) is assembled by integrating \( \mu_{\pi_1, \pi_2} \) against the Cartesian square of the Plancherel measure \( \mu_G \) supported on \( \hat{G}_{\text{red}} \), Proposition 2.23 is only valid for members of the reduced dual.

There is thus no contradiction between Proposition 2.23 and the fact that even though we always have \( \pi \preceq 1 \otimes \pi \), we rarely have \( 1 \preceq \pi \otimes \pi^* \) (with that containment closely linked to amenability; cf. Remark 3.6).

\[ \Box \]

3 Chain-center duality for various classes of groups

3.1 Compact-by-abelian groups

We extend the previously-mentioned results minimally, so as to house both Example 2.3 and Lemma 2.4 under a common umbrella.

Definition 3.1 A locally compact group \( G \) is an extension of a compact group by an abelian group, or compact-by-abelian for short, if it fits into an exact sequence

\[ 1 \to N \to G \to K \to 1 \quad (3-1) \]

with \( N \leq G \) closed and abelian and \( K \cong G/N \) compact.

\[ \Box \]

Theorem 3.2 Compact-by-abelian locally compact groups are cc-dual.

This will require some background, including analyzing the representations of \( G \) in terms of those of its closed normal (in this case abelian) subgroup \( N \leq G \). The general theory dealing with this is sometimes referred to as the “Mackey machine” for G.W. Mackey (e.g. [20, §3.8]); we follow the treatment in [14, Chapter 4, especially §4.3] (where the phrase is “Mackey analysis” instead).

Various technical conditions have to be met in order for the theory to go through in full force, but they are in our case by [14, Remark 4.26 (3)] because the quotient \( K := G/N \) is assumed compact. For this reason, the various results from [14, Section 4.3] cited below hold in the present setup. The main such result is [14, Theorem 4.27], which we can summarize as follows:

- Every irreducible representation of \( G \), when restricted to \( N \), lives on ([14, Definition 4.59]) an orbit \( O \) of the conjugation action ‘\( \triangleright \)’ of \( G \) on \( \hat{N} \), defined as usual by

  \[ g \triangleright \chi := \chi(g^{-1} \cdot g) \quad \text{for} \quad g \in G, \chi \in \hat{N}. \quad (3-2) \]

- For a character \( \chi \in \hat{N} \) with isotropy group

  \[ G_\chi := \{ g \in G \mid g \triangleright \chi = \chi \} \]
with respect to the conjugation action the induction

\[ \text{Ind}_{G}^{\hat{G}}: \text{Rep}(G_{\chi}) \to \text{Rep}(G) \]

gives a bijection between the irreducible representations of \( G_{\chi} \) whose restrictions to \( N \) live on \( \{\chi\} \) and those of \( G \) which live on the orbit \( O := G \triangleright \chi \).

In first instance, we can essentially reduce Theorem 3.2 to the case when the abelian normal subgroup \( N \trianglelefteq G \) is in fact central.

**Lemma 3.3** Let \( G \) be a compact-by-abelian group fitting into an exact sequence (3-1) and \( Z := Z_{G}(N) \) the centralizer of \( N \) in \( G \).

The irreducible representations of \( G \) that restrict trivially to \( Z \) are trivial in the chain group \( C(G) \).

**Proof** We will work with representations that are trivial on \( N \) (as all of those trivial on \( Z \) must be, since \( N \leq Z \)), identified in the obvious manner with representations of the quotient \( K := G/N \).

Consider a character \( \chi \in \hat{N} \), its isotropy group \( K_{\chi} \leq K \), and a representation \( \rho \in \hat{K} \) which contains \( K_{\chi} \)-invariant vectors; or in other words, one for which the restriction

\[ \rho|_{K_{\chi}} \in \text{Rep}(K_{\chi}) \]

contains the trivial representation (of \( K_{\chi} \)). Let also

\[ \text{Ind}_{G_{\chi}}^{\hat{G}} \pi \in \hat{G} \]

be a representation whose restriction to \( N \) lives on \( G \triangleright \chi \) (so that it is indeed induced from a \( G_{\chi} \)-representation whose restriction to \( N \) lives on \( \{\chi\} \), by the discussion preceding the present result). Then, by the induction-restriction formula

\[ \bullet \otimes \text{Ind}_{A}^{B}(\square) \cong \text{Ind}_{A}^{B}(\bullet|_{A} \otimes \square) \]

([14, Theorem 2.58]), we have

\[ \rho \otimes \text{Ind}_{G_{\chi}}^{\hat{G}} \pi \cong \text{Ind}_{G_{\chi}}^{\hat{G}}(\rho|_{G_{\chi}} \otimes \pi). \tag{3-3} \]

Since \( \rho|_{G_{\chi}} \) contains the trivial representation by assumption, (3-3) will contain \( \text{Ind}_{G_{\chi}}^{\hat{G}}(\pi) \) as a summand. In the chain group, then, we have

\[ \rho \cdot \text{Ind}_{G_{\chi}}^{\hat{G}} \pi = \text{Ind}_{G_{\chi}}^{\hat{G}} \pi \]

and \( \rho \) is trivial.

The \( \rho \in \hat{K} \subseteq \hat{G} \) as above, containing \( G_{\chi} \)-invariant vectors for some \( \chi \), generate a full, summand-closed rigid monoidal subcategory \( \text{Rep}_{0,N}(K) \) of \( \text{Rep}(K) \) all of whose members are trivial in the chain group \( C(G) \). It remains to

\[ \bullet \text{ recall (25, (2.3))} \text{ the Galois correspondence between normal closed subgroups of the compact group } K \text{ and full, summand-closed rigid monoidal subcategories of } \text{Rep}(K) \text{ given by} \]

\[ (H \trianglelefteq K) \mapsto (\text{representations trivial on } H) \simeq \text{Rep}(K/H) \subseteq \text{Rep}(K); \tag{3-4} \]
• observe that via this correspondence the monoidal category $\text{Rep}_{0,N}(K)$ can be identified with $\text{Rep}(K/H)$, where $H$ is the intersection of all conjugates

$$gK \chi g^{-1}, \quad g \in G, \quad \chi \in \hat{N}.$$  

That intersection is nothing but the image through $G \to K = G/N$ of the centralizer $Z = Z_G(N)$, hence the conclusion.

**Lemma 3.4** Let $G$ be a compact-by-abelian group fitting into an exact sequence (3-1), with $N$ maximal among normal abelian subgroups.

Any $\rho \in \hat{K} \subseteq \hat{G}$ is trivial in the chain group $C(G)$.

**Proof** The maximality assumption has a number of consequences:

• First, $N$ contains the center $Z(G)$.

• Additionally, in the quotient $K := G/N$ we have

$$Z(K) \cap Z_K(N) = \{1\}; \quad (3-5)$$

or: the center of $K$ and the centralizer of $K$-action on $N$ induced by conjugation in $G$ intersect trivially.

The first observation is obvious, as we can always adjoin the center of $G$ to $N$ thus enlarging the latter. As for the second remark, note that for any $k$ in the intersection (3-5) the subgroup of $G$ generated by $N$ and the preimage of $k$ through $G \to K$ is abelian normal, so we again conclude by maximality.

Now, on the one hand, we know from Lemma 3.3 that irreducible $K$-representations (also regarded as $G$-representations) trivial on $Z_K(N)$ are trivial in $C(G)$. On the other hand, Theorem 3.2 for compact groups (which is nothing but [25, Theorem 3.1]), applied to $K$, ensures that irreducible $K$-representations trivial on $Z(K)$ are trivial in $C(G)$.

We can now conjoin these two observations: by and the Galois correspondence (3-4) the rigid monoidal subcategory of $\text{Rep}(K)$ generated by all irreducibles that are trivial on either $Z(K)$ or $Z_K(N)$ is precisely

$$\text{Rep}(K/Z(K) \cap Z_K(N)) = \text{Rep}(K),$$

where the equality appeals to (3-5). But this is precisely the sought-after conclusion: $\rho \in \hat{K} \subseteq \hat{G}$ are all trivial in $C(G)$. 

**Proof of Theorem 3.2** We can assume without loss of generality (via an application of Zorn’s lemma) that the abelian normal subgroup $N \leq G$ is maximal among abelian normal subgroups, so that Lemma 3.4 applies and $\hat{K} \subseteq \hat{G}$ is annihilated upon passing to the chain group $C(G)$.

As noted in the discussion preceding Lemma 3.3, each irreducible representation $\rho \in \hat{G}$, when restricted to $N$, lives on an orbit $O = G \triangleright \chi$ of the conjugation action (3-2) on $\hat{N}$. Now, if $\rho_i \in \hat{G}$, $i = 1, 2$ live, respectively, on the orbits $O_i \subseteq \hat{N}$, the irreducible constituents of the tensor product $\rho_1 \otimes \rho_2$ live on the orbits contained in

$$O_1 + O_2 \subset \hat{N}.$$ 

It follows that the quotient map $\hat{G} \to C(G)$ factors through the quotient group

$$\hat{N}/\{\chi \in \hat{N} \mid g \triangleright \chi, \quad g \in G, \quad \chi \in \hat{N}\}.$$ 

Pontryagin duality identifies that group with the dual of

$$N^G := \{n \in N \mid gng^{-1} = n, \quad \forall g \in G\} = Z(G),$$

and we thus have the desired identification of $C(G)$ with $\widehat{Z(G)}$. 

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3.2 Nilpotent groups

**Theorem 3.5** Connected, nilpotent locally compact groups are chain-center dual in the sense of Definition 2.2.

**Proof** According to Corollary 2.22 it will be enough to handle the Lie-group quotients of such a group by compact normal subgroups. For that reason, we specialize the rest of the proof to Lie (rather than arbitrary, locally compact) nilpotent connected groups.

Consider first *simply-connected* (nilpotent, connected) Lie groups. That is precisely the class of groups to which the orbit method of [15] applies most straightforwardly (see the discussion on [15, p.xviii]). Per the summary in [15, p.xix, User’s guide],

- The unitary dual \( \hat{G} \) is homeomorphic to the space \( \mathcal{O}(G) \) of orbits in the dual Lie algebra \( \mathfrak{g}^* \) under the coadjoint action (i.e. the space of *coadjoint orbits*).
- Having identified \( X \in \hat{G} \) with an orbit \( \Omega_X \) and similarly for \( Y \) and \( Z \), the weak containment
  \[
  X \preceq Y \otimes Z
  \]
  simply says that
  \[
  \Omega_X \subseteq \Omega_Y + \Omega_Z
  \]
  (with the ‘+’ denoting addition of subsets in the vector space \( \mathfrak{g}^* \)).
- The center \( \mathbb{Z}(G) \) is connected, as are all centers of connected nilpotent Lie groups [22, Lemma 3]. In particular, a generic element of \( \mathbb{Z}(G) \) is of the form \( \exp(x) \), \( x \in \mathfrak{z} := \text{Lie}(\mathbb{Z}(G)) \).

We write \( \mathfrak{g}_{\Gamma \to \mathbb{Z}}^* \) for the set of such \( f \in \mathfrak{g}^* \); to reiterate:

\[
\mathfrak{g}_{\Gamma \to \mathbb{Z}}^* := \{ f \in \mathfrak{g}^* \mid f(\exp^{-1}(\Gamma)) \in \mathbb{Z} \}.
\]

The map \( X \mapsto \Omega_X \) induces an identification of the chain group \( C(G) \) with the quotient abelian group

\[
\mathfrak{g}_{\Gamma \to \mathbb{Z}}^*/\text{span}\{x - Ad_gx \mid x \in \mathfrak{g}_{\Gamma \to \mathbb{Z}}^*, \ g \in G\};
\]

that is, the space \( \mathfrak{g}_{\Gamma \to \mathbb{Z}}^*/Ad \) of coinvariants under the (co)adjoint action. Vector-space duality identifies this with

\[
\{ f \in \mathfrak{s}^* = (\mathfrak{g}^Ad)^* \mid f(\exp^{-1}(\Gamma)) \subseteq \mathbb{Z} \}.
\]
(where the superscript $\bullet^{Ad}$ denotes invariants under the adjoint action). It follows that the map

$$\mathfrak{g}^*_\to\mathbb{Z}/Ad \ni \text{class of } f \mapsto (\exp(x) \mapsto e^{2\pi i f(x)}) \in \hat{Z(G)}$$

that implements (2-1) (cf. (3-6)) then identifies this space with the dual

$$\hat{Z(G)}/\Gamma \cong \hat{Z(G)}$$

as desired (this last isomorphism being a consequence of Lemma 2.11). ■

Remark 3.6 In the context of Theorem 3.5 condition (c) of Definition 0.1 is automatic: the trivial representation is weakly contained in $X \otimes X^*$ for any irreducible $X \in \hat{G}$, since in coadjoint-orbit language this reads as the trivial observation that

$$\{0\} \subseteq \Omega + (-\Omega)$$

for every $Ad$-orbit $\Omega \subset \mathfrak{g}^*$.

In fact, the property that the trivial representation be weakly contained in $X \otimes X^*$ for every $X \in \hat{G}$ characterizes the amenable locally compact groups: this follows, for instance, from [2, Theorem 5.1 and Corollary 5.5]. Amenability is not the only way to render condition (c) of Definition 0.1 redundant though: see Remark 3.10 below.

3.3 Discrete groups with infinite conjugacy classes

Before stating the next result, recall (e.g. [31, Chapter V, Definition 7.10]) that a group is icc (for infinite conjugacy class) if its only finite conjugacy class is $\{1\}$.

Theorem 3.7 Countable discrete icc groups are cc-dual in the sense of Definition 2.2.

Proof Let $G$ be such a group. The center is trivial by the icc condition, so the goal is to show that $C(G)$ is also trivial.

According to [4, Proposition 21] $G$ has an irreducible representation $\pi \in \hat{G}$ weakly equivalent (denoted below by ‘$\sim$’) to the regular representation $\lambda_G$. The Fell continuity of tensoring together with the absorption property (2-7) then shows that for arbitrary $\rho \in \hat{G}$ we have

$$\rho \otimes \pi \sim \rho \otimes \lambda_G \sim \lambda_G \sim \pi.$$ 

In the group $C(G)$ this reads $\rho \cdot \pi = \pi$, which is only possible if the arbitrary $\rho \in C(G)$ is trivial. ■

3.4 Some easily accessible semisimple examples

The present section, like Section 4, is concerned with connected semisimple Lie groups. These are all second-countable and type-I [32, Theorem 14.6.10]. Even better, in fact; that result says that the universal $C^*$-algebra $C^*(G)$ is CCR [32, §14.6.9], or liminal in the sense of [5, Definition 4.2.1]: all operators

$$\pi(a), \ a \in C^*(G) \text{ for an irreducible } \pi : C^*(G) \to B(H)$$

are compact.

Although Theorem 3.8 and Theorem 3.9 will be superseded by Section 4 below, the proofs are simpler because in the cases treated here more is known about tensor products of irreducible unitary representations.
Theorem 3.8 All connected Lie groups locally isomorphic to $SL(n, \mathbb{R})$, $n \geq 2$ are cc-dual.

Proof Fix a positive integer $n \geq 2$ throughout. All Lie groups as in the statement cover $PSL(n, \mathbb{R})$ with discrete kernel, so the result follows from Corollary 2.13 if we prove it for projective special linear groups. As the literature typically treats the representation theory of $SL$ rather than $PSL$ we take a detour, discussing the former in order to reach the latter.

The principal-series (e.g. [17, Chapter VII, §3]) representations $\pi_{\sigma, \chi}$ of $G := SL(n, \mathbb{R})$ are those of the form

$$\text{Ind}_{MAN}^G (\sigma \otimes \chi \otimes \text{triv})$$

where

- $M \subset G = SL(n, \mathbb{R})$ is the group of diagonal matrices with entries $\pm 1$ and $\sigma$ is a unitary irreducible representation thereof;
- $A \subset G$ is the group of positive-entry diagonal matrices and $\chi$ one of its unitary characters;
- triv is the trivial representation on the subgroup $N \subset G$ of upper triangular matrices with identity diagonal.

"Most" $\pi_{\sigma, \chi}$ are irreducible, in the sense that this happens for $\chi$ ranging over an open dense set ([32, Corollary 12.5.4]); for that reason, tensor products of principal-series representations are relevant to computing $C(G)$.

Case 1: $n \geq 3$. The direct-integral decompositions of tensor products $\pi_{\sigma, \chi} \otimes \pi_{\sigma', \chi'}$ are described in [21, p.210, Theorem]:

- When $n$ is odd the center $Z(G)$ is trivial and $SL(n, \mathbb{R}) = PSL(m, \mathbb{R})$. In that case the tensor product of any two principal-series representations is nothing but the regular representation. All $\rho \in \hat{G}_{\text{red}}$ are thus identified in the chain group $C(G)$, so the latter must be trivial by Proposition 2.15.
- For even $n$, when the center $Z(G)$ is $\pm 1$, the tensor product in question decomposes as

$$\pi_{\sigma, \chi} \otimes \pi_{\sigma', \chi'} \cong \int_{\hat{G}_\pm} \pi_{\oplus N_0} \, d\mu_G(\pi)$$

where $\mu_G$ is the Plancherel measure (in conformity with prior notation), and $\hat{G}_\pm$ is the set of irreducible unitary representations whose central character is trivial (for ‘+’) or not (for ‘−’). In passing to $PSL(n, \mathbb{R}) = G/Z(G)$ only the + portion survives, once again collapsing all of $\hat{G}_{\text{red}}/(G/Z(G))_{\text{red}}$ to a singleton in the chain group.

Case 2: $n = 2$. We can argue along the same lines as for $n \geq 3$, with the added complication that now the reduced dual, in addition to the principal series, also contains discrete-series representations [17, Proposition 9.6]: those that appear as summands of the regular representation $L^2(G)$ (rather than just being weakly contained therein).

Now, all tensor products of two irreducible representations of $SL(2, \mathbb{R})$ are described in [28, §4]. In that source’s notation, in passing to $PSL(2, \mathbb{R})$, the surviving representations of interest are

- $\pi_{r, \text{triv}}$ for $r \in i\mathbb{R}_{\geq 0}$ (principal series);
and $T_{x^n}, n \geq 2$ even (discrete series).

[28, §4, (d)] shows that the tensor product between a discrete- and a principal-series representation weakly contains the entire principal series of $PSL(2, \mathbb{R})$, so the chain group is exhausted by the classes of the principal-series elements.

Finally, to conclude, note that by [28, §4, (a)] (or, say, [21, §VI B., Theorem]) we have a direct summand

$$\pi_{r,\text{triv}} \otimes \pi_{r',\text{triv}} \cong \int (\text{entire principal series of } PSL);$$

in other words, the tensor product of two principal-series representations again weakly contains all of the latter. As desired, then, the chain group is trivial. ■

**Theorem 3.9** *Connected, complex semisimple Lie groups are cc-dual.*

**Proof** This is very similar in spirit to the proof of Theorem 3.8, this time using the decomposition of tensor products obtained in [34].

For complex semisimple Lie groups the principal-series representations $\pi_\lambda$ are those induced from 1-dimensional unitary representations $\lambda$ of a Borel subgroup $B \subset G$ (termed *non-degenerate* principal series in [34, Introduction]). By [34, Theorem 4.5.9] we have a decomposition

$$\pi_\lambda \otimes \pi_{\lambda'} \cong \int_{\theta} \pi_{\theta}^{\text{multiplicity}} \, d\mu(\theta)$$

for a measure $\mu$ supported on precisely those $\pi_\theta$ whose central character is the product of the central characters of $\pi_\lambda$ and $\pi_{\lambda'}$. Once more, then, (2-1) is one-to-one when restricted to the principal series.

To conclude, note that in the complex case the principal series comprises all of $\hat{G}_{\text{red}}$. This follows from the Plancherel formula for $G$, originally due to Harish-Chandra [8]; without expanding on such a broad topic here, the only thing to note is that the formula (e.g. [34, equation (4.2.7)]) involves integration over only the principal series, and the conclusion that the latter constitutes all of $\hat{G}_{\text{red}}$ then follows from the uniqueness of the Plancherel measure [32, Theorem 14.11.2 (3)]. ■

**Remark 3.10** In both Theorem 3.8 and Theorem 3.9 condition (c) of Definition 0.1 was redundant, despite these groups being non-amenable; cf. Remark 3.6 ♦

## 4 Arbitrary connected semisimple Lie groups

This section’s main result is

**Theorem 4.1** *Connected semisimple Lie groups are cc-dual.*

The proof requires some background.

Fix a connected semisimple Lie group (we will specialize further soon). Recall, first, the *Iwasawa decomposition* ([18, Theorem 6.46] or [10, Chapter IX, Theorem 1.3]).

$$K \times A \times N \cong G,$$  \hspace{1cm} (4-1)

where

- $K \leq G$ contains the center $Z(G)$ [18, Theorem 6.31] and is maximal compact in $G$ if that center is finite (which we henceforth assume);
• $A \leq G$ is closed, abelian and simply-connected;
• $N \leq G$ is closed, nilpotent and simply-connected;
• and the isomorphism (4-1) is given by multiplication in $G$.

Per [10, Chapter IX, discussion preceding Theorem 1.1] (or [18, Chapter VII, §6]) the real rank of $G$ is $\dim A$.

Defining
\[ M := Z_K(a := \text{Lie}(A)) \]  
(4-2)

(the centralizer of the Lie algebra $a$ under the adjoint action by $K$ on $g := \text{Lie}(G)$) as in, say [10, Chapter IX, §1], $B := MAN \leq G$ is a minimal parabolic subgroup of $G$; see also [18, Chapter VII, §7].

The principal-series [17, Chapter VII, §3] unitary representations of $G$ are those of the form
\[ \pi_{\sigma, \chi} := \text{Ind}^G_{MAN}(\sigma \otimes \chi \otimes \text{triv}), \ \sigma \in \hat{M}, \ \chi \in \hat{A} \]
as in the proof of Theorem 3.8.

$\pi_{\sigma, \chi}$ need not be irreducible in full generality (there are reducible examples for $SL(2n, \mathbb{R})$, for instance [16, Example (1), preceding Theorem 3]), but for every $\sigma \in \hat{M}$ the set of those $\chi \in \hat{A}$ for which $\pi_{\sigma, \chi}$ is irreducible contains an open dense subset ([32, Corollary 12.5.4] or [33, Theorem 5.5.2.1]). Furthermore, all $\pi_{\sigma, \chi}$ decompose as finite sums of irreducible representations [33, Corollary 5.5.2.2].

Given that most of the tensor-product-decomposition literature seems to focus on the principal series, the following auxiliary result will come in handy in arguing that that is more or less enough for the purpose of studying the chain group $C(G)$.

**Theorem 4.2** Let $G$ be a connected semisimple Lie group with finite center.

If $\rho, \pi \in \hat{G}$ are irreducible unitary representations with $\pi$ principal-series the tensor product $\rho \otimes \pi$ weakly contains an irreducible summand of some principal-series representation.

**Proof** The usual notation is in place: $G = KAN$ and $\pi = \pi_{\sigma, \chi}$ is induced from the minimal parabolic subgroup $B := MAN$. We thus have
\[ \rho \otimes \pi = \rho \otimes \text{Ind}^G_{MAN}(\sigma \otimes \chi \otimes \text{triv}) \cong \text{Ind}^G_{MAN}(\rho|_{MAN} \otimes (\sigma \otimes \chi \otimes \text{triv})) \]
for some $\sigma \in \hat{M}$ and $\chi \in \hat{A}$, where the last isomorphism is the usual induction-restriction formula [14, Theorem 2.58].

The restriction of $\rho|_{MAN}$ to $N$ is a unitary representation of that simply-connected ([18, Theorem 6.46]) nilpotent Lie group, so its support [5, Definition 3.4.6] (i.e. the set of irreducible unitary $\hat{N}$-representations it weakly contains) can be identified, per [15, p.xix, User’s guide], with a set of orbits in the dual of
\[ n^* := \text{Lie}(N)^* \]
under the coadjoint action of $N$. Furthermore, because $\rho|_{N}$ is restricted from $MA$, that set of coadjoint orbits is in fact invariant under the coadjoint action of $MA$ on $n^*$.

The (non-trivial) adjoint action of $A$ on the Lie algebra $n := \text{Lie}(N)$ can be realized via diagonal matrices with positive real eigenvalues [10, Chapter VI, Lemma 3.5], so the same is true of the coadjoint action on the dual $n^*$. But this means that any $Ad(A)$-invariant subsets of $n^*$ contain $0 \in n^*$ in their closure, and hence $\rho|_{MAN}$ will contain in its Fell closure some irreducible unitary representation trivial on $N$. 

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Now, since irreducible unitary $MAN$-representations trivial on $N$ must be of the form
\[ \sigma' \otimes \chi' \otimes \text{triv}, \quad \sigma' \in \hat{M}, \chi' \in \hat{A}, \]
it follows from the Fell-continuity of the induction operation [1, Theorem F.3.5] that
\[ \rho \otimes \pi \cong \text{Ind}_{MAN}^G (\rho|_{MAN} \otimes (\sigma \otimes \chi \otimes \text{triv})) \rightarrow \text{Ind}_{MAN}^G ((\sigma \otimes \sigma') \otimes \chi\chi' \otimes \text{triv}) \]
(the arrow denoting Fell convergence). Since the right-hand limit contains principal-series representations and the classes of the latter’s irreducible constituents form a clopen subset of the reduced dual (as described, say, in [32, §14.12.3]), it follows that $\rho \otimes \pi$ must have weakly-contained an irreducible summand of some principal-series representation to begin with. \[\blacksquare\]

The following observation will allow us to ignore the continuous parameter $\chi$ in $\pi_{\sigma,\chi}$ when handling principal-series representations.

**Proposition 4.3** Let $G$ be a connected semisimple Lie group with finite center with Iwasawa decomposition $G = KAN$ and a corresponding minimal parabolic subgroup $B = MAN$.

The class in $C(G)$ of an irreducible summand $\rho \leq \pi_{\sigma,\chi}$ only depends on $\sigma$.

**Proof** [21, Theorem 1] says that
\[ \pi_{\sigma,\chi} \otimes \pi_{\sigma',\chi'} \cong \text{Ind}_{MA}^G ((\sigma \otimes \sigma') \otimes \chi\chi'), \quad (4-3) \]
which by [21, Theorem 2] does not depend on $\chi$ and $\chi'$.

Now, if $\rho \leq \pi_{\sigma,\chi}$ is a simple summand then $\rho \otimes \pi_{\sigma',\chi'}$ will in turn be a summand of (4-3). As noted before, by [33, Theorem 5.5.2.1] we can choose $\chi'$ and $\chi''$ so that $\pi_{\sigma',\chi'}$ and $\pi_{\sigma,\chi''}$ are irreducible, and every simple representation weakly contained in (4-3) will then also be weakly contained in
\[ \pi_{\sigma,\chi''} \otimes \pi_{\sigma',\chi'} \]
and hence have class
\[ \pi_{\sigma,\chi''} \cdot \pi_{\sigma',\chi'} \in C(G). \]
Since $\chi'$ and $\chi''$ can be chosen in terms of $\sigma'$ and $\sigma$ alone, independently of $\chi$, we are done. \[\blacksquare\]

As a consequence, as mentioned above, as far as $C(G)$ goes we can focus on the principal series only.

**Corollary 4.4** For a semisimple connected group with finite center the canonical map (2-1) restricted to classes of irreducible principal-series representations is onto.

**Proof** Theorem 4.2 shows that an arbitrary $\rho \in \hat{G}$ is, in the chain group, the product of two irreducible summands of principal series representations. The latter can then be replaced by irreducible principal-series $\pi_{\sigma,\chi}$ by Proposition 4.3. \[\blacksquare\]

Proposition 4.3 (together with the irreducibility of $\pi_{\sigma,\chi}$ for “most” $\chi$) will now allow us to define a map
\[ \hat{M} \ni \sigma \mapsto \text{class of } \pi_{\sigma,\chi} \in C(G)_{\text{princ}}. \quad (4-4) \]
Proposition 4.5  For a Lie group $G$ as in Proposition 4.3 the map (4-4) descends to a surjective group morphism

$$C(M) \to C(G).$$ (4-5)

Proof  Because of the independence on $\chi$ implicit in (4-4), we will typically mention generic characters $\chi$ that will not otherwise play much of a role.

The surjectivity claim is clear from Corollary 4.4, which ensures that $C(G)$ consists of the (classes of) $\pi_{\sigma,\chi}$. What is at issue here is showing that (4-4) does indeed factor through $C(M)$ and that that factorization is a group morphism. Since $M$ is compact the simpler definition of the chain group introduced in [25, Proposition 2.3] applies and we are left having to prove:

$$\sigma'' \leq \sigma \otimes \sigma' \Rightarrow \pi_{\sigma'',\chi''} = \pi_{\sigma,\chi} \pi_{\sigma',\chi'} \text{ in } C(G),$$ (4-6)

where the left-hand side of the implication means that $\sigma'' \in \hat{M}$ is a summand of the tensor product. Now, $\sigma'' \leq \sigma \otimes \sigma'$ together with (4-3) imply that

$$\pi_{\sigma,\chi} \otimes \pi_{\sigma',\chi'} \geq \text{Ind}_{MA}^{G}(\sigma'' \otimes \chi'')$$ (4-7)

(containment as a direct summand) for some character $\sigma''$. Since

$$\pi_{\sigma'',\chi''} = \text{Ind}_{MAN}^{G}(\sigma'' \otimes \chi'' \otimes \text{triv})$$ (4-8)

by definition and the right-hand side of (4-7) breaks up as

$$\text{Ind}_{MAN}^{G} \circ \text{Ind}_{MA}^{MAN}(\sigma'' \otimes \chi'')$$ (4-9)

by induction in stages [14, Theorem 2.47], it will be enough to observe the following.

- $MAN$ being amenable (as the semidirect product of a compact group $M$ and a solvable group $AN$ [30, Theorem 1.2.11]), we have the weak containment

  $$\sigma'' \otimes \chi'' \otimes \text{triv} \leq \text{Ind}_{MA}^{MAN}(\sigma'' \otimes \chi'') \cong \text{Ind}_{MA}^{MAN}(\sigma'' \otimes \chi'' \otimes \text{triv})|_{MA}$$

  by [6, p.296, Theorem];

- whence (4-8) is weakly contained in (4-9) by further inducing up to $G$, because induction is Fell-continuous [1, Theorem F.3.5].

The conclusion is that we have the weak containment relation

$$\pi_{\sigma'',\chi''} \preceq \pi_{\sigma,\chi} \otimes \pi_{\sigma',\chi'},$$

hence the right-hand side of the desired implication (4-6). \[\blacksquare\]

Proposition 4.5 can be recast in somewhat different terms.

Proposition 4.6  Let $G = KAN$ be the Iwasawa decomposition of a connected semisimple Lie group with finite center and $B = MAN$ the corresponding minimal parabolic.

(a) The map

$$\Psi_M : \hat{M} \ni \sigma \mapsto (\text{class of any irreducible } \pi \preceq \text{Ind}_{G}^{G}\sigma) \in C(G)$$

is well defined and induces the epimorphism $C(M) \to C(G)$ of Proposition 4.5, denoted abusively by the same symbol $\Psi_M$. 

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(b) Furthermore, $\Psi(\sigma_1) = \Psi(\sigma_2)$ provided $\text{Ind}_M^K \sigma_i \in \hat{K}$, $i = 1, 2$ are not disjoint.

**Proof** It follows from Proposition 4.5 that $\pi_{\text{triv}, \chi}$ are all in the trivial class of $C(G)$, whereupon (4-3) identifies the image of $\sigma \in \hat{M}$ through (4-5) with

$$\left(\text{class of any irreducible } \pi \preceq \text{Ind}_M^K \sigma \otimes \chi \right) \in C(G)$$

(for arbitrary $\chi \in \hat{A}$). This coincides with

$$\left(\text{class of any irreducible } \pi \preceq \text{Ind}_M^K \sigma \right) \in C(G)$$

because

$$\text{Ind}_M^K \sigma \cong \sigma \otimes (\text{regular representation } L^2(A)) \cong \sigma \otimes \int_A \chi \, d\mu_A(\chi),$$

proving part (a).

As for (b), it is a consequence of (a): if $\text{Ind}_M^K \sigma_i$ have summands in common then so do $\text{Ind}_M^K \sigma_i \cong \text{Ind}_M^K \sigma \bigcirc \text{Ind}_M^K \sigma \bigcirc \text{Ind}_M^K \sigma_i$, proving part (b).

The advantage of Proposition 4.6 is that it has a counterpart for $K$ rather than $M$.

**Proposition 4.7** Let $G = KAN$ be the Iwasawa decomposition of a connected semisimple Lie group with finite center and $B = MAN$ the corresponding minimal parabolic.

(a) The map

$$\Psi_K : \hat{K} \ni \sigma \mapsto \left(\text{class of any irreducible } \pi \preceq \text{Ind}_K^K \sigma \right) \in C(G)$$

is well defined and induces an epimorphism $C(K) \to C(G)$ denoted by the same symbol $\Psi_K$.

(b) Furthermore, $\Psi(\sigma_1) = \Psi(\sigma_2)$ provided $\sigma_i|_M \in \hat{M}$, $i = 1, 2$ are not disjoint.

**Proof** That $\psi_K : \hat{K} \to C(G)$ is well defined follows from Proposition 4.6: an arbitrary $\sigma \in \hat{K}$ is a summand of some

$$\text{Ind}_M^K \sigma', \sigma' \in \hat{M},$$

so that in turn

$$\text{Ind}_K^K \sigma \preceq \text{Ind}_M^K \sigma \bigcirc \text{Ind}_M^K \sigma \bigcirc \text{Ind}_M^K \sigma'.$$

By part Proposition 4.6 (a) all summands of the right-hand side are identified in $C(G)$, hence the conclusion (that $\Psi_K$ is well defined).

If now both $\sigma_i \in \hat{K}$, $i = 1, 2$ contain $\sigma' \in \hat{M}$ when restricted to $M$, then by Frobenius reciprocity for compact groups [29, (8.9)] we have

$$\sigma_i \preceq \text{Ind}_M^K \sigma', \quad i = 1, 2.$$

We can thus choose the same $\sigma'$ for both $\sigma_i$ in the argument above, proving part (b) of the statement.

It thus remains to prove the portion of part (a) claiming that $\Psi_K$ induces a group morphism $C(K) \to C(G)$ (rather than just a map):

$$\sigma \preceq \sigma' \otimes \sigma'' \Rightarrow \Psi_K(\sigma) = \Psi(\sigma') \Psi(\sigma'') \in C(G).$$
Once more using Frobenius reciprocity and the induction-restriction formula ([14, Theorem 2.58]) we have
\[
\sigma \leq \sigma' \otimes \text{Ind}_M^K(\sigma''|_M) \cong \text{Ind}_M^K(\sigma'|_M \otimes \sigma''|_M) \tag{4-10}
\]
We know from Proposition 4.6 that further inducing the right-hand side of (4-10) to \( G \) produces a direct integral of irreducible \( \rho \in \hat{G} \), all identified with
\[
\psi_M(\text{irreducible} \leq \sigma'|_M):\psi_M(\text{irreducible} \leq \sigma''|_M) \in C(G).
\]
As seen in the preceding discussion (in the present proof) this is precisely
\[
\Psi_K(\sigma')\Psi_K(\sigma'') \in C(G).
\]
Since at the representation level this object was obtained via \( \text{Ind}_G^K \) from a \( K \)-representation containing \( \sigma \in \hat{K} \) as a summand, it must also coincide with \( \Psi_K(\sigma) \).

**Proof of Theorem 4.1** It is enough to consider (connected, semisimple, Lie) groups \( G \) with finite center, since their cc-duality entails their covers' cc-duality by Corollary 2.13.

Consider the epimorphism
\[
\Psi_K : \hat{K} \cong C(K) \to C(G)
\]
of Proposition 4.7 (where the first isomorphism uses the cc-duality of the compact group \( K \) [25, Theorem 3.1]). According to Corollary 2.8, we have to argue that \( \Psi_K \) annihilates (classes of) irreducible representations \( \sigma \in \hat{K} \) trivial on \( Z(G) \subseteq K \).

Consider such a \( \sigma \in \hat{K} \), and an irreducible summand
\[
\widehat{M} \ni \sigma' \leq \sigma|_M
\]
which must also be trivial on \( Z(G) \leq M \). Because \( Z(K) \cap M = Z(G) \) (Lemma 4.8), \( \sigma' \) extends to an irreducible representation (also denoted by \( \sigma' \)) of \( MZ(K) \), trivial on \( Z(K) \). But then \( \sigma' \) further extends to an irreducible \( K \)-representation \( \overline{\sigma} \)
\[
\overline{\sigma} \, = \, \text{any irreducible summand of} \, \text{Ind}_{MZ(K)}^K \sigma'
\]
will do, by Frobenius reciprocity. Now, on the one hand \( \overline{\sigma} \) is trivial on \( Z(K) \) and hence trivial in \( C(K) \) (so annihilated by \( \Psi_K : C(K) \to C(G) \)) by the cc-duality of \( K \). On the other hand though,
\[
\Psi_K(\overline{\sigma}) = \Psi_K(\sigma)
\]
by Proposition 4.7 (b). This finishes the proof.

**Lemma 4.8** Let \( G = KAN \) be the Iwasawa decomposition of a connected semisimple Lie group with finite center and \( B = MAN \) the corresponding minimal parabolic. We then have
\[
Z(K) \cap M = Z(G).
\]

**Proof** The global center \( Z(G) \) is contained in \( K \) [18, Theorem 6.31] and hence also in \( M \) (given the latter’s definition as the centralizer of \( a = \text{Lie}(A) \) in \( K \): (4-2)), so the inclusion
\[
Z(K) \cap M \supseteq Z(G)
\]
is clear. To prove the opposite inclusion, consider first a Cartan decomposition
\[
\text{Lie}(G) =: g = \mathfrak{k} \oplus \mathfrak{p}, \, \mathfrak{k} := \text{Lie}(K)
\]
as in [18, (6.23)], compatible with the Iwasawa decomposition $G = KAN$ as in [18, Chapter VI, §4]; in particular, the Lie algebra $\mathfrak{a} = \text{Lie}(A)$ is a maximal abelian subspace of $\mathfrak{p}$.

The intersection of all conjugates $kMk^{-1}$ will centralize

$$\bigcup_k \text{Ad}_k(\mathfrak{a}) = \mathfrak{p},$$

where the last equality is [18, Theorem 6.51]. Since $Z(K)$ centralizes $\mathfrak{t}$,

$$Z(K) \cap M \subseteq Z(K) \cap \bigcap_{k \in K} kMk^{-1}$$

centralizes both $\mathfrak{t}$ and $\mathfrak{p}$ and hence $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$. In short, that intersection must be central in $G$. ■

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