I'-CURVATURES IN HIGHER DIMENSIONS AND THE HIRACHI CONJECTURE

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ABSTRACT. We construct higher-dimensional analogues of the I'-curvature of Case and Gover in all CR dimensions \( n \geq 2 \). Our I'-curvatures all transform by a first-order linear differential operator under a change of contact form and their total integrals are independent of the choice of pseudo-Einstein contact form on a closed CR manifold. We exhibit examples where these total integrals depend on the choice of general contact form, and thereby produce counterexamples to the Hirachi conjecture in all CR dimensions \( n \geq 2 \).

1. INTRODUCTION

The \( Q' \)-curvature of a pseudo-Einstein manifold \([10, 18]\) has many formal similarities to the (critical) \( Q \)-curvature in conformal geometry \([5]\). These similarities begin with how the \( Q' \)- and \( Q \)-curvatures transform under a conformal rescaling of the contact form and the metric, respectively. If \( \theta \) and \( \hat{\theta} = e^Y \theta \) are pseudo-Einstein contact forms on a \((2n + 1)\)-dimensional CR manifold, then

\[
e^{(n+1)Y} \hat{Q}' = Q' + P'(Y) + \frac{1}{2} P(Y^2) \equiv Q' + P'(Y) \mod P'_{\perp},
\]

where \( P' \) is the \( P' \)-operator \([10, 18]\), \( P \) is the (critical) CR GJMS operator \([14]\), and \( P'_{\perp} \) is the \( L^2 \)-orthogonal complement to the space \( P \) of CR pluriharmonic functions. Similarly, if \( g \) and \( \hat{g} = e^{2Y} g \) are Riemannian metrics on a \( 2n \)-dimensional manifold, then

\[
e^{2nY} \hat{Q} = Q + P(Y),
\]

where \( P \) is the (critical) GJMS operator \([16]\). Importantly, the operators appearing in Equations (1.1) and (1.2) are formally self-adjoint and annihilate constants. In particular, the total \( Q' \)-curvature is a global secondary CR invariant — that is, it is independent of the choice of pseudo-Einstein contact form, if one exists, on a closed CR manifold — and the total \( Q \)-curvature is a global conformal invariant. Moreover, explicit formulae for the \( Q' \)-curvature of the round CR sphere \([9, 30]\) and the \( Q \)-curvature of the round sphere \([5]\) imply that these global invariants are nontrivial.

For \((2n + 1)\)-dimensional CR manifolds which can be realized as the boundary of a bounded strictly pseudoconvex domain in \( \mathbb{C}^{n+1} \), the total \( Q' \)-curvature is a global biholomorphic invariant of the domain. The Burns–Epstein invariant \([7, 8]\) is also a global biholomorphic invariant of such a domain. Marugame \([21]\) gave an
alternative realization of the Burns–Epstein invariant as the boundary term in a Gauss–Bonnet–Chern formula for the domain. When \( n = 1 \), the total \( Q' \)-curvature agrees, up to a multiplicative constant, with the Burns–Epstein invariant \[10\] \[13\]. When \( n = 2 \), the total \( Q' \)-curvature and the Burns–Epstein invariant are linearly independent, but an explicit relationship in terms of global secondary CR invariants is known \[9\] \[19\].

The analogue of the above paragraph in conformal geometry is the relationship between the total \( Q' \)-curvature and the Euler characteristic. It is well-known that the Gauss–Bonnet formula identifies the Euler characteristic of a closed surface with the total \( Q' \)-curvature, up to multiplicative constant. The Gauss–Bonnet–Chern formula in dimension four gives an explicit identity relating the Euler characteristic, the total \( Q' \)-curvature, and the \( L^2 \)-norm of the Weyl tensor \[6\]. Similarly, the Gauss–Bonnet–Chern formula in dimension six gives an explicit identity relating the Euler characteristic, the total \( Q' \)-curvature, and total integrals of local conformal invariants \[15\]. More generally, Alexakis \[1\] proved that if \( I \) is any natural Riemannian scalar invariant whose total integral is a conformal invariant on any closed \( 2n \)-dimensional manifold, then there is a constant \( c \in \mathbb{R} \) such that

\[
I = cQ' + \text{(local conformal invariant)} + \text{(divergence)}.
\]

Together with the close relationship between the \( Q' \) and \( Q \)-curvatures, Alexakis’ result motivated Hirachi \[18\] to pose the following conjecture:

**Conjecture 1.1** (Hirachi conjecture). Let \( I \) be a natural pseudohermitian scalar invariant whose total integral is a secondary CR invariant. Then there is a constant \( c \in \mathbb{R} \) such that

\[
(1.3) \quad I = cQ' + \text{(local CR invariant)} + \text{(divergence)}.
\]

Conjecture \[1.1\] is true \[18\] in CR dimension \( n = 1 \); i.e. if \( I \) is a natural pseudohermitian scalar invariant whose total integral is a secondary CR invariant on all closed CR three-manifolds, then \( I \) is of the form of Equation \[1.3\]. However, Conjecture \[1.1\] is false \[9\] \[26\] in CR dimension \( n = 2 \). The purpose of this article is to show that it is false in all CR dimensions \( n \geq 2 \) by producing a large collection of counterexamples. To motivate our results, we first describe in more detail what is known when \( n = 2 \).

Let \( (M^5, T^{1,0}, \theta) \) be a pseudohermitian manifold of CR dimension \( n = 2 \). Case and Gover \[9\] studied two invariants. First, they proved that

\[
X_{\alpha} := -iS_{\alpha\beta\gamma\bar{\sigma}}V^{\beta\gamma\bar{\sigma}} + \frac{1}{4} \nabla_\alpha |S_{\gamma\delta\beta\rho}|^2
\]

is a CR invariant \((1, 0)\)-form of weight \(-2\), where \( S_{\alpha\beta\gamma\bar{\sigma}} \) is the Chern tensor and, in general CR dimension \( n \),

\[
V_{\alpha\beta\gamma} := \frac{i}{n} \nabla^{\bar{\sigma}} S_{\alpha\beta\gamma\bar{\sigma}}.
\]

Case and Gover further showed that if \( (M^5, T^{1,0}) \) admits a pseudo-Einstein contact form, then \([\xi] = 4\pi^2 c_2(T^{1,0})\), where

\[
\xi := 2 \text{Re} \, X_{\alpha} \theta \wedge \theta^\alpha \wedge d\theta.
\]
By observing \([9, 31]\) that \(c_2(T^{1,0}) = 0\) in \(H^4(M; \mathbb{R})\), they conclude that \(\text{Re} \nabla^\alpha X_\alpha\) is orthogonal to \(\mathcal{P}\). Second, they proved that the \(T'\)-curvature,
\[
T' := -\frac{1}{8} \Delta_b [S_{\alpha \beta \gamma \delta}]^2 + [V_{\alpha \beta \gamma}]^2 + \frac{1}{2} P [S_{\alpha \beta \gamma \delta}]^2,
\]
where \(P := \frac{1}{2(n+1)} R\) is a constant multiple of the pseudohermitian scalar curvature, is such that
\[
e^{3\gamma} \hat{T}' = T' + 2 \text{Re} X_\alpha \Upsilon^\alpha
\]
for any \(\hat{\theta} = e^{\gamma} \theta\), where \(\hat{T}'\) is defined in terms of \(\hat{\theta}\). These facts imply that the total \(T'\)-curvature is a global secondary CR invariant; in fact, the Burns–Epstein invariant is a linear combination of the total \(Q'\) and \(T'\)-curvatures \([9]\). By computing on nonspherical real ellipsoids, Reiter and Son \([26]\) then showed that the \(T'\)-curvature is not a linear combination of a local CR invariant and a divergence, thereby disproving Conjecture \([1]\) in CR dimension two.

In this article we construct analogues of \(X_\alpha\) and \(T'\) in all CR dimensions \(n \geq 2\). To that end, let \(\delta^{\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_n}\) denote the generalized Kronecker delta and let \(\Phi^{\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_n}\) be an invariant polynomial of degree \(n\); in particular,
\[
\Phi^{\beta_{e(1)} \cdots \beta_{e(n)}}_{\alpha_{e(1)} \cdots \alpha_{e(n)}} = \Phi^{\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_n}
\]
for all elements \(\sigma \in S_n\) of the symmetric group on \(n\) elements. Define
\[
X_\alpha^\Phi := i (S^\Phi)_{\alpha}^{\beta} \mu V_\beta^\mu \nu - \frac{1}{n^2} \nabla_\alpha c_\Phi(S),
\]
where
\[
(S^\Phi)_{\alpha}^{\beta} \mu := \delta^{\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_n} \Phi^{\mu \mu_2 \cdots \mu_n}_{\beta_1 \cdots \beta_2 \cdots \beta_n} S^{\alpha_2 \alpha_1} \cdots S^{\alpha_n \alpha_2},
\]
\[
c_\Phi(S) := (S^\Phi)_{\alpha}^{\beta} \nu S^{\beta \alpha} \mu
\].

Taking \(\Phi_{\alpha_1 \alpha_2} = \delta^{\beta_3 \delta_{\beta_1 \beta_2}}\) recovers the definitions of Case and Gover \([9]\).

Our first result is that \(X_\alpha^\Phi\) is CR invariant:

**Theorem 1.2.** Let \((M^{2n+1}, T^{1,0}, \theta)\) be a pseudohermitian manifold, let \(\Phi\) be an invariant polynomial of degree \(n\), and let \(X_\alpha^\Phi\) be given by Equation \((1.4)\). Then \(X_\alpha^\Phi\) is a CR invariant \((1, 0)\)-form of weight \(-n\); i.e.
\[
e^{n\gamma} \hat{X}_\alpha^\Phi = X_\alpha^\Phi
\]
for all \(\hat{\theta} = e^{\gamma} \theta\), where \(\hat{X}_\alpha^\Phi\) is defined in terms of \(\hat{\theta}\). In particular, \(\text{Re} \nabla^\alpha X_\alpha^\Phi\) is a local CR invariant of weight \(-n-1\).

This follows by a direct computation using the CR invariance of the Chern tensor and the simple transformation formula for \(V_{\alpha \beta \gamma}\); see Section \([4]\) for details.

Now define the \(T'_\Phi\)-curvature of \((M^{2n+1}, T^{1,0}, \theta)\) by
\[
T'_\Phi := \frac{1}{n^2} \Delta_b c_\Phi(S) - \frac{2}{n^2} P c_\Phi(S)
\]
\[
+ (T_\Phi)_{\alpha}^{\beta} \mu_1 \nu_1 \mu_2 \nu_2 ((n-1)V_{\beta})_{\mu_1 \nu_1} V_\alpha^\mu_2 V_\mu_1^\nu_2 - S_{\beta}^{\alpha} \nu_1 \mu_1 U_{\nu_2}^\mu_2
\]
where
\[
(T_\Phi)_{\alpha}^{\beta} \mu_1 \nu_1 \mu_2 \nu_2 := \delta^{\beta_1 \cdots \beta_n}_{\alpha_1 \cdots \alpha_n} \Phi^{\mu_1 \mu_2 \cdots \mu_n}_{\beta_1 \cdots \beta_2 \cdots \beta_n} S^{\alpha_2 \alpha_1} \cdots S^{\alpha_n \alpha_2} \nu_1 \mu_2 \nu_2
\]
and \(U_{\alpha \beta}\) is related to \(\nabla^\alpha V_{\alpha \beta \gamma}\); see Section \([2]\) for the precise definition. Our second result is that the transformation formula for \(T'_\Phi\) is given by the first-order linear differential operator \(\text{Re} X_\alpha^\Phi \nabla^\alpha\).
Theorem 1.3. Let \((M^{2n+1}, T^{1,0}, \theta)\) be a pseudohermitian manifold, let \(\Phi\) be an invariant polynomial of degree \(n\), and let \(\mathcal{T}_{\Phi}\) be given by Equation (1.7). For any \(\Upsilon \in C^{\infty}(M)\), it holds that

\[
e^{(n+1)\Upsilon} \mathcal{T}_{\Phi} = \mathcal{T}_{\Phi} + 2 \text{Re} X^{\Phi}_{\alpha} \Upsilon^\alpha,
\]

where \(\mathcal{T}_{\Phi}\) is defined by \(\hat{\theta} := e^\Upsilon \theta\) and \(X^{\Phi}_{\alpha}\) is given by Equation (1.4).

This follows by a direct computation using the CR invariance of the Chern tensor and the simple transformation formulae for \(V_{\alpha\beta\gamma}\) and \(U_{\alpha\beta}\); see Section 4 for details.

Our last result is that there is a large variety of choices of invariant polynomials \(\Phi\) for which the total \(\mathcal{T}_{\Phi}\)-curvature is a secondary CR invariant.

Theorem 1.4. Let \((M^{2n+1}, T^{1,0})\) be a closed CR manifold which admits a pseudo-Einstein contact form \(\theta\) and let \(\Phi\) be an invariant polynomial of degree \(n\). If \(\hat{\theta}\) is also a pseudo-Einstein contact form, then

\[
\int_M \mathcal{T}_{\Phi} \hat{\theta} \wedge d\hat{\theta}^n = \int_M \mathcal{T}_{\Phi} \theta \wedge d\theta^n.
\]

Recall that if \(\theta\) is pseudo-Einstein, then \(e^\theta \hat{\theta}\) is pseudo-Einstein if and only if \(\Upsilon\) is a CR pluriharmonic function [22]. Thus Theorem 1.4 is equivalent to the claim that \(\text{Re} \int X^{\Phi}_{\alpha} \Upsilon^\alpha = 0\) for all CR pluriharmonic functions \(\Upsilon\). We prove this in the same spirit as the proof of Case and Gover [9] in the case \(n = 2\):

The CR invariance of \(X^{\Phi}_{\alpha}\) implies that \(\xi^{\Phi} := 2 \text{Re} X^{\Phi}_{\alpha} \theta \wedge \theta^\alpha \wedge d\theta^{n-1}\) is a CR invariant 2n-form of weight 0. A straightforward consequence of Lee’s Bianchi identities [22] implies that \(\xi^{\Phi}\) is closed. We show that if \((M^{2n+1}, T^{1,0})\) admits a pseudo-Einstein contact form, then \(\xi^{\Phi}\) is proportional to the characteristic class \(c_{\Phi}(T^{1,0})\) determined by \(\Phi\); see Proposition 1.3. An observation of Takeuchi [31] implies that \(c_{\Phi}(T^{1,0}) = 0\). Theorem 1.4 then follows from the fact that \(\text{Re} \int X^{\Phi}_{\alpha} \Upsilon^\alpha\) equals, up to a multiplicative constant, the evaluation of the cup product \(\xi^{\Phi} \cup d_{\theta} \Upsilon\) on the fundamental class of \(M\) whenever \(\Upsilon \in \mathcal{P}\). Note that Marugame [25] showed that one can relax the assumption that \((M^{2n+1}, T^{1,0})\) admits a pseudo-Einstein contact form to \(c_1(T^{1,0}) = 0\) in \(H^2(M; \mathbb{R})\).

Our last result is that there is a large variety of choices of invariant polynomials \(\Phi\) for which the total \(\mathcal{T}_{\Phi}\)-curvature gives a counterexample to Conjecture 1.1. Our strategy is as follows:

Suppose Conjecture 1.1 holds. Let \(\Phi\) be an invariant polynomial of degree \(n\). On the one hand, there exists a constant \(c\), depending only on \(\Phi\), such that

\[
\mathcal{T}_{\Phi} = cQ' + (\text{local CR invariant}) + (\text{divergence}).
\]

Consider the round CR sphere \((S^{2n+1}, T^{1,0}, \theta)\). In this case, \(\mathcal{T}_{\Phi}\) and any local CR invariant are identically zero, but \(Q'\) is a nonzero constant [30]. Integrating implies that \(c = 0\), and hence \(\mathcal{T}_{\Phi}\) can be written as the sum of a local CR invariant and a divergence. In particular, the total \(\mathcal{T}_{\Phi}\)-curvature is a global CR invariant. On the other hand, under a general conformal change \(\hat{\theta} = e^\theta \theta\), Theorem 1.3 implies that

\[
\int_M \mathcal{T}_{\Phi} \hat{\theta} \wedge d\hat{\theta}^n = \int_M \mathcal{T}_{\Phi} \theta \wedge d\theta^n - 2 \int_M (\text{Re} \nabla^\alpha X^{\Phi}_{\alpha}) \Upsilon \theta \wedge d\theta^n.
\]

One arrives at a contradiction by finding an example of \(\Phi\) and \((M, T^{1,0})\) such that \(\text{Re} \nabla^\alpha X^{\Phi}_{\alpha} \neq 0\); see Lemma 5.1.
Let $\varsigma = (\varsigma_1, \ldots, \varsigma_n) \in \mathbb{N}^n$ be such that $\varsigma_1 + 2\varsigma_2 + \cdots + n\varsigma_n = n$ and let $\Phi(\varsigma)$ be the invariant polynomial of degree $n$ defined by
\[
\Phi(\varsigma) = \beta_1^{\alpha_1} \cdots \alpha_n^{\beta_n} A_{\beta_1}^{\alpha_1} \cdots A_{\alpha_n}^{\beta_n} = \prod_{k=1}^{n} (\text{tr} A^k)^{\varsigma_k}
\]
for $\text{tr} A^k := A_{\gamma_1} \gamma_2 A_{\gamma_2} \gamma_3 \cdots A_{\gamma_k} \gamma_1$. Our first counterexamples come from considering $\Phi(\varsigma)$ on perturbations of the round CR sphere.

**Theorem 1.5.** For $n \geq 2$, there exists a perturbation of the round CR sphere in $\mathbb{C}^{n+1}$ such that $\Re \nabla^\alpha X_\alpha(\varsigma)$ is not identically zero for any $\varsigma$ with $\varsigma_1 = 0$. In particular, the $I'_{\Phi(\varsigma)}$-curvature gives a counterexample to the Hirachi conjecture.

This result follows from Theorem 6.3, where we compute variations of $\Re \nabla^\alpha X_\alpha(\varsigma)$ for a deformation of the round CR $(2n+1)$-sphere. This deformation is in the direction of a real ellipsoid, and gives a local (in the space of CR structures on $S^{2n+1}$) analogue of the computation of $\Re \nabla^\alpha X_\alpha^{(0,1)}$ on 5-dimensional ellipsoids by Reiter and Son [26].

Second, we consider the case that $\Phi = (n)$ is the generalized Kronecker delta on $n$ variables.

**Theorem 1.6 (= Theorem 7.1).** For $n \geq 2$, there exists a closed $(2n+1)$-dimensional pseudo-Einstein manifold $(M, T^{1,0}, \theta)$ such that $R_{\alpha \beta} = 0$, $A_{\alpha \beta} = 0$, $\Re \nabla^\alpha X_\alpha^{(n)} \neq 0$. In particular, the $I'_{(n)}$-curvature gives a counterexample to the Hirachi conjecture.

This is a consequence of degenerations of Ricci-flat Kähler metrics. There exists a smooth family of Ricci-flat Kähler metrics on a certain Calabi–Yau manifold whose curvature concentrates along some complex submanifolds. Together with the Gauss–Bonnet–Chern formula, this implies that for many members of this family, there is a circle bundle which is a Ricci-flat Sasakian manifold with $\Re \nabla^\alpha X_\alpha^{(n)} \neq 0$. These examples have the benefit of being significantly easier to compute.

Theorems 1.5 and 1.6 imply that the total $I'_{\Phi(\varsigma)}$-curvatures are nontrivial on general pseudohermitian manifolds. In fact, the total $I'_{\Phi(\varsigma)}$-curvature are nontrivial secondary CR invariants. We prove this by computing the total $I'_{\Phi(\varsigma)}$-curvatures of the boundaries of locally homogeneous Reinhardt domains.

**Theorem 1.7 (= Theorem 8.1).** For $r > 0$, let $M_r$ be the boundary of the bounded Reinhardt domain
\[
\Omega_r = \left\{ w = (w^0, \ldots, w^n) \in \mathbb{C}^{n+1} \left| \sum_{j=0}^{n} (\log |w^j|)^2 < r^2 \right. \right\}.
\]
The total $I'_{\Phi(\varsigma)}$-curvature $I'_{\Phi(\varsigma)}$ of $M_r$ is given by
\[
I'_{\Phi(\varsigma)} = -(n!)^2 \text{Vol}(S^n(1)) \left( \frac{\pi}{(n+1)r} \right)^{n+1} \prod_{k=1}^{n} \left[ \frac{(n+2)(1 - (n+2)^{k-1})}{k} \right]^{\varsigma_k}.
\]
where $\text{Vol}(S^n(1))$ is the volume of the unit sphere in $\mathbb{R}^{n+1}$. 
If \( \zeta_1 = 0 \), then the total \( I_{\Phi(\zeta)} \)-curvature of \( M_r \) is of the form \( Cr^{-n-1} \) for \( C \) a nonzero constant depending only on \( n \) and \( \zeta \). In particular, the total \( I_{\Phi(\zeta)} \)-curvature is a nontrivial secondary CR invariant when \( \zeta_1 = 0 \). Since two bounded strictly pseudoconvex domains in \( \mathbb{C}^{n+1} \) are biholomorphic if and only if their boundaries are CR equivalent [11], we obtain the following corollary.

**Corollary 1.8.** The domains \( \Omega_r \) and \( \Omega_{r'} \) are biholomorphic if and only if \( r = r' \).

This corollary was proven using different global CR invariants by Burns and Epstein [7] for \( n = 1 \), Marugame [24] for \( n = 2 \), and Reiter and Son [26] for any dimension. In other words, we give another proof of the result of Reiter–Son by using \( T' \)-curvatures. Note that this corollary also follows from a result by Sunada [29] for general bounded Reinhardt domains.

Finally, we note that Marugame [25] has independently established Theorems 1.2–1.4 in the same generality that we consider, and also discussed the nontriviality of the total \( I_{\Phi} \)-curvatures. His proof of the CR invariance of \( X_\Phi^R \) uses the tractor calculus in a way analogous to the work of Case and Gover [9], while his proof that the total \( I_{\Phi} \)-curvature is a secondary CR invariant uses a tractor-based proof that \( \xi^R \) represents a multiple of \( c_\Phi(T^{1,0}) \). His work produces other global secondary CR invariants, but their explicit realization as total integrals of local pseudohermitian invariants remains unknown. His work does not determine whether the invariants constructed give counterexamples to Conjecture [11].

This article is organized as follows. In Section 2 we collect some necessary background material. In Section 3 we give some equivalent realizations of characteristic classes in terms of various \( \text{End}(T^{1,0}) \)-valued two-forms. In Section 4 we prove Theorems 1.2–1.4. In Section 5 we further discuss our strategy to disprove Conjecture [11]. In Section 6 we prove Theorem 1.5. In Section 7 we prove Theorem 1.6. In Section 8 we prove Theorem 1.7. In Section 9 we propose a weaker version of Conjecture 1.1 and discuss it in the context of the \( T' \)-curvature.

2. **Background**

In this section we collect necessary background material.

2.1. **CR and pseudohermitian manifolds.** A **CR manifold** \((M^{2n+1}, T^{1,0})\) is a real \((2n+1)\)-dimensional manifold \(M^{2n+1}\) together with a rank \( n \) distribution \( T^{1,0} \subset TM \otimes \mathbb{C} \) such that \([T^{1,0}, T^{1,0}] \subset T^{1,0}\) and \( T^{1,0} \cap T^{0,1} = \{0\} \) for \( T^{0,1} := \overline{T^{1,0}} \).

We assume throughout that \( M \) is orientable. We say that \((M^{2n+1}, T^{1,0})\) is **strictly pseudoconvex** if there exists a real one-form \( \theta \) on \( M \) such that \( \ker \theta = \text{Re} T^{1,0} \) and \(-i \, d\theta(Z, \overline{W})\) defines a positive definite Hermitian form on \( T^{1,0} \). We call such a \( \theta \) a **contact form**. Note that contact forms are determined up to multiplication by a positive function.

Given a CR manifold \((M^{2n+1}, T^{1,0})\) and a smooth (complex-valued) function \( f \in C^\infty(M; \mathbb{C}) \), we denote by \( \partial_b f \) the restriction of \( df \) to \( T^{1,0} \); likewise \( \partial_{\overline{b}} f := df|_{T^{0,1}} \). A **CR function** is a function \( f \in C^\infty(M; \mathbb{C}) \) such that \( \partial_b f = 0 \). A **CR pluriharmonic function** is a (real-valued) function \( u \in C^\infty(M) \) such that locally \( u = \text{Re} f \) for some CR function \( f \); i.e. for every \( p \in M \), there is a neighborhood \( U \) of \( p \) and a CR function \( f \in C^\infty(U; \mathbb{C}) \) such that \( u|_U = \text{Re} f \).

Denote by \( \mathcal{P} \) the space of CR pluriharmonic functions. We emphasize that the notion of a CR pluriharmonic function is defined without reference to a choice of contact form. An infinitesimal
characterization of CR pluriharmonic functions via differential operators has been given by Lee [22 Propositions 3.3 and 3.4].

A pseudohermitian manifold \((M^{2n+1}, T^{1,0}, \theta)\) is a triplex consisting of a strictly pseudoconvex CR manifold \((M^{2n+1}, T^{1,0})\) and a choice of contact form. The Reeb vector field \(T\) is the unique vector field such that \(\theta(T) = 1\) and \(d\theta(T, \cdot) = 0\). Denote by \(T^{\ast(1,0)}\) the subbundle of \(T^\ast M \otimes \mathbb{C}\) which annihilates \(T^{0,1}\) and \(T\). Set \(T^{\ast(0,1)} := T^{\ast(1,0)}\). The Tanaka–Webster connection of \((M^{2n+1}, T^{1,0}, \theta)\) is defined as follows: Let \(\{\theta^\alpha\}_{\alpha=1}^n\) be an admissible coframe of \(T^{\ast(1,0)}\); i.e. \(\theta^\alpha \in T^{\ast(1,0)}\) for all \(\alpha = 1, \ldots, n\) and \(\{\theta^1, \ldots, \theta^n, \bar{\theta}^1, \ldots, \bar{\theta}^n, \theta\}\) forms a basis for \(T^\ast M \otimes \mathbb{C}\), where \(\theta^{\bar{\beta}} := \bar{\theta}^\beta\). It follows that there is a positive definite Hermitian matrix \(h_{\alpha\beta}\) such that

\[
d\theta = i h_{\alpha\bar{\beta}} \theta^\alpha \wedge \bar{\theta}^{\bar{\beta}}.
\]

We use \(h_{\alpha\beta}\) and its inverse \(h^{\alpha\bar{\beta}}\) to lower and raise indices as needed. The connection one-forms \(\omega^\alpha_{\beta}\) associated to \(\{\theta^\alpha\}\) are uniquely determined by

\[
\begin{align*}
d\theta^\alpha &= \theta^\beta \wedge \omega^\beta_{\alpha} + \theta \wedge \tau^\alpha, \\
dh_{\alpha\bar{\beta}} &= \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha}, \quad A_{\alpha\beta} = A_{\beta\alpha}.
\end{align*}
\]

The tensor \(A_{\alpha\beta}\) is the pseudohermitian torsion. Note that

\[
(2.1) \quad \theta^\gamma \wedge \tau^\alpha = 0.
\]

The connection one-forms determine the Tanaka–Webster connection by \(\nabla \theta = 0\) and \(\nabla \theta^\alpha = -\omega^\alpha \wedge \theta^\gamma\). The curvature two-forms \(\Pi_{\alpha\beta}\) are the End\((T^{1,0})\)-valued two-forms

\[
(2.2) \quad \Pi_{\alpha\beta} := d\omega^\beta_{\alpha} - \omega^\gamma_{\alpha} \wedge \omega^\beta_{\gamma}.
\]

The pseudohermitian curvature \(R_{\alpha\beta\gamma}\) is the coefficient of the \((1, 1)\)-part of \(\Pi_{\alpha\beta}\); i.e.

\[
\Pi_{\alpha\beta} \equiv R_{\alpha\beta\gamma\delta} \equiv \theta^\gamma \wedge \theta^\delta \quad \text{mod} \, \theta, \theta^\alpha \wedge \theta^\beta \wedge \theta^\gamma \wedge \theta^\delta.
\]

The pseudohermitian Ricci tensor \(R_{\alpha\beta}\) and pseudohermitian scalar curvature \(R\) are defined by taking traces in the usual way; i.e. \(R_{\alpha\beta} := R_{\alpha\beta\gamma}\) and \(R := R_{\gamma}\). We say that \((M^{2n+1}, T^{1,0}, \theta)\), \(n \geq 2\), is pseudo-Einstein if \(R_{\alpha\beta} = \frac{1}{n} Rh_{\alpha\beta}\). If \((M^{2n+1}, T^{1,0}, \theta)\) is pseudo-Einstein, then \(c_1(T^{1,0})\) vanishes in \(H^2(M; \mathbb{R}) [22\text{ Proposition D}]).

The pseudohermitian torsion, pseudohermitian curvature, and covariant derivatives are all tensorial. We may thus use abstract index notation to denote tensors. Specifically, unbarred Greek superscripts denote factors of \(T^{1,0}\), barred Greek superscripts denote factors of \(T^{0,1}\), unbarred Greek subscripts denote factors of \(T^{\ast(1,0)}M\), and barred Greek subscripts denote factors of \(T^{\ast(0,1)}M\). For example, \(C_{\alpha\beta\gamma}\) denotes a section of \(T^{\ast(1,0)} \otimes T^{\ast(0,1)} \otimes T^{1,0}\). We keep the notation \(\nabla\) to denote covariant derivatives. For example, \(\nabla_{\rho} C_{\alpha\beta\gamma}\) denotes the \((1, 0)\)-part of the covariant derivative of \(C_{\alpha\beta\gamma}\). When clear by context, we use subscripts to denote covariant derivatives of a function \(u \in C^\infty(M; \mathbb{C})\); e.g. \(u_{\alpha\beta} := \nabla_{\beta} \nabla_{\alpha} u\). We use \(\nabla_0\) to denote covariant derivatives in the direction of the Reeb vector field.

The sublaplacian \(\Delta_{\theta}\) of a pseudohermitian manifold is the operator

\[
\Delta_{\theta} := u^\gamma \wedge u^\gamma
\]

for all \(u \in C^\infty(M; \mathbb{C})\). Recall that if \((M^{2n+1}, T^{1,0}, \theta)\) is closed, then \(\ker \Delta_{\theta}\) equals the space of locally constant functions.
We require three curvature tensors naturally associated to a pseudohermitian manifold \((M^{2n+1}, T^{1,0}, \theta)\), all of which appear as components of the CR tractor curvature \([14]\).

The first curvature tensor we need is the Chern tensor
\[
S_{\alpha\beta\gamma\delta} := R_{\alpha\beta\gamma\delta} - P_{\alpha\beta} h_{\gamma\delta} - P_{\alpha\delta} h_{\gamma\beta} - P_{\gamma\delta} h_{\alpha\beta} - P_{\gamma\beta} h_{\alpha\delta},
\]
where \(P_{\alpha\beta} := \frac{1}{n+2} (R_{\alpha\beta} - Ph_{\alpha\beta})\) is the CR Schouten tensor and \(P := \frac{1}{2(n+1)}R\) is its trace. The relevance of the Chern tensor to CR geometry is that if \(n \geq 2\), then \(S_{\alpha\beta\gamma\delta} = 0\) if and only if \((M^{2n+1}, T^{1,0})\) is locally CR equivalent to the round CR \((2n+1)\)-sphere. Importantly, the Chern tensor is symmetric and trace-free:
\[
S_{\alpha\beta\gamma\delta} = S_{\alpha\alpha\gamma\delta} = S_{\gamma\beta\alpha\delta};
\]
\[
S_{\alpha\beta\gamma} = 0.
\]

The second curvature tensor we need is
\[
V_{\alpha\beta\gamma} := \nabla_\beta A_{\alpha\gamma} + i\nabla_\gamma P_{\alpha\beta} - iT_\alpha h_{\alpha\beta} - 2iT_\alpha h_{\gamma\beta},
\]
where \(T_\alpha := \frac{1}{n+2} (\nabla_\alpha P - i\nabla^\gamma A_{\alpha\gamma})\). This tensor is a divergence of the Chern tensor:
\[
\nabla^\gamma S_{\alpha\beta\gamma\delta} = -niV_{\alpha\beta\gamma};
\]
see \([9]\) Lemma 2.2. Importantly, \(V_{\alpha\beta\gamma}\) is symmetric and trace-free:
\[
V_{\alpha\beta\gamma} = V_{\gamma\delta\alpha};
\]
\[
V_{\alpha\gamma\gamma} = 0.
\]

The third curvature tensor we need is
\[
U_{\alpha\beta} := \nabla_\beta T_\alpha + \nabla_\alpha T_\beta + P_{\alpha\rho} P_{\beta\rho} - A_{\alpha\rho} A^\rho_\beta + Sh_{\alpha\beta},
\]
where \(S \in C^\infty(M; \mathbb{C})\) is such that \(U_{\gamma\gamma} = 0\). This tensor is closely related to a divergence of \(V_{\alpha\beta\gamma}\):
\[
\nabla^\gamma V_{\alpha\beta\gamma} = niU_{\alpha\beta} - iS_{\alpha\beta\gamma\delta} P^{\gamma\delta};
\]
see \([9]\) Lemma 2.2.

In addition to the well-known CR invariance of the Chern tensor, we need to know how the tensors \(V_{\alpha\beta\gamma}\) and \(U_{\alpha\beta}\) transform under change of contact form. To that end, given a natural pseudohermitian tensor \(B\) on \((M^{2n+1}, T^{1,0}, \theta)\) which is homogeneous of degree \(k\) in \(\theta\) — that is, \(B_{c\theta} = c^k B_\theta\) for all constants \(c > 0\) — define the conformal linearization \(D_\theta B\) of \(B\) at \(\theta\) by
\[
D_\theta B(Y) := \left. \frac{\partial}{\partial t} \right|_{t=0} e^{-ktY} B_{e^{t\theta}}
\]
for all \(Y \in C^\infty(M)\). It is clear that \(D_\theta B(1) = 0\). One easily checks that \(D_\theta\) extends to a derivation on the space of natural homogeneous pseudohermitian tensors. By a simple integration argument (cf. \([14]\)), the tensor \(B\) is a local CR invariant of weight \(k\) — that is,
\[
e^{-kY} B_{\hat{\theta}} = B_\theta
\]
for all contact forms \(\theta\) and \(\hat{\theta} = e^Y \theta\) — if and only if \(D_\theta B \equiv 0\).

The following lemma collects the well-known \([14]\) \([22]\) conformal linearizations of the CR Schouten tensor, the Chern tensor, and the Tanaka–Webster connection, as
well as the needed conformal linearizations of $V_{\alpha\bar{\beta}\gamma}$ and $U_{\alpha\bar{\beta}}$. Note that these conformal linearizations can also be deduced from the CR invariance of the curvature of the CR tractor connection [14].

**Lemma 2.1.** Let $(M^{2n+1}, T^{1,0}, \theta)$ be a pseudohermitian manifold and let $\Upsilon \in C^\infty(M)$. Then
\[
D_\theta P_{\alpha\bar{\beta}}(\Upsilon) = -\frac{1}{2} (\Upsilon_{\alpha\bar{\beta}} + \Upsilon_{\bar{\beta}\alpha}) ,
\]
\[
D_\theta S_{\alpha\bar{\beta}\gamma\bar{\rho}}(\Upsilon) = 0 ,
\]
\[
D_\theta V_{\alpha\bar{\beta}\gamma}(\Upsilon) = iS_{\alpha\bar{\beta}\gamma}^\rho \Upsilon_\rho ,
\]
\[
D_\theta U_{\alpha\bar{\beta}}(\Upsilon) = iV_{\bar{\alpha}\beta}^\gamma \Upsilon_\gamma - iV_{\alpha\beta}^\gamma \Upsilon_\gamma .
\]

If $f$ is a local scalar CR invariant of weight $k$, then
\[
D_\theta \nabla_\alpha f(\Upsilon) = kf\nabla_\alpha .
\]

If $\omega_\alpha$ is a natural pseudohermitian $(1, 0)$-form which is homogeneous of degree $k$ in $\theta$, then
\[
D_\theta \nabla_\alpha \omega_\alpha(\Upsilon) = (k - 1)\omega_\alpha \Upsilon_\alpha - \Upsilon_\alpha \omega_\alpha + \nabla_\al D_\theta \omega_\alpha(\Upsilon) ,
\]
\[
D_\theta \nabla_{\bar{\beta}} \omega_\alpha = k\omega_\alpha \Upsilon_{\bar{\beta}} + \Upsilon^\rho \omega_{\alpha\bar{\beta}} + \nabla_\beta D_\theta \omega_\alpha(\Upsilon) .
\]

**Proof.** All but the formulae for $D_\theta V_{\alpha\bar{\beta}\gamma}$ and $D_\theta U_{\alpha\bar{\beta}}$ follow from [14, Proposition 2.3, Equation (2.7), Equation (2.8)]. Computing the conformal linearization of both sides of Equations (2.3) and (2.4) yields the claimed formulae for $D_\theta V_{\alpha\bar{\beta}\gamma}$ and $D_\theta U_{\alpha\bar{\beta}}$, respectively. \qed

The following consequences of the Bianchi identities are useful in studying $X_\alpha^\Phi$ and related objects.

**Lemma 2.2.** Let $(M^{2n+1}, T^{1,0}, \theta)$ be a pseudohermitian manifold. Then
\[
\nabla_{[\alpha} S_{\beta]}^\rho \gamma^\sigma = iV_\rho^\gamma [\alpha\delta_\beta] + iV_\gamma^\sigma [\alpha\delta_\beta] ,
\]
\[
\nabla_\rho V_\gamma^\sigma = -S^\gamma_\rho_\sigma^\delta [\alpha\delta_\beta] + iQ_\rho_\sigma^\delta [\alpha\delta_\beta] ,
\]
\[
\nabla_0 S_{\alpha\bar{\beta}\gamma\bar{\rho}} = \nabla_\sigma V_{\alpha\bar{\beta}\gamma} + \nabla_\gamma V_{\beta\alpha\bar{\rho}} - iS_{\gamma\sigma\alpha}^\rho P_{\rho\bar{\beta}} - iS_{\gamma\sigma\beta}^\rho P_{\alpha\rho} + iU_{\alpha\beta} h_{\gamma\bar{\rho}} - iU_{\gamma\bar{\beta}} h_{\alpha\rho} .
\]

where $T_{[\alpha\gamma]} := \frac{1}{2}(T_{\alpha\gamma} - T_{\gamma\alpha})$ and $Q_{\alpha\gamma} := i\nabla_\alpha A_{\gamma} - 2i\nabla_\gamma A_\alpha + 2P_\alpha^\rho A_{\rho\gamma} .

**Proof.** Equation (2.5) follows from [22, Equation (2.7)]. Equation (2.6) follows from [22, Equations (2.9) and (2.14)]. Equation (2.7) follows from [22, Equation (2.8)]. \qed

2.2. **Sasakian manifolds.** We recall some facts about Sasakian manifolds; see [1] for a comprehensive introduction. A **Sasakian manifold** is a pseudohermitian manifold $(M, T^{1,0}, \theta)$ with pseudohermitian torsion identically zero, or equivalently, the Reeb vector field $T$ preserves the CR structure $T^{1,0}$.

A typical example of a Sasakian manifold is the circle bundle associated with a negative holomorphic line bundle. Let $Y$ be an $n$-dimensional complex manifold and $(L, h)$ a Hermitian holomorphic line bundle over $Y$ such that $\omega = -i\Theta_h = 2^{-1}d\log h$ defines a Kähler metric on $Y$, where $d\epsilon = i(\partial - \bar{\partial})$. Now we consider the circle bundle
\[
M := \{ v \in L \mid h(v, v) = 1 \} ,
\]
which is a real hypersurface in $L$. The one-form $\theta := 2^{-1}d^c \log h|_M$ is a connection one-form of the principal $S^1$-bundle $p: M \to Y$ and satisfies $d\theta = p^*\omega$. Moreover, the natural CR structure $T^{1,0}$ on $M$ coincides with the horizontal lift of the holomorphic tangent bundle $T^{1,0}Y$ of $Y$ with respect to $\theta$. Since $\omega$ defines a Kähler metric, we have

$$-id\theta(Z,\overline{Z}) = -i\omega(p_*Z, p_*\overline{Z}) > 0$$

for all nonzero $Z \in T^{1,0}$. Hence $(M, T^{1,0}, \theta)$ is a pseudohermitian manifold of dimension $2n + 1$. We call this triple the circle bundle associated with $(Y, L, h)$. Note that the Reeb vector field $T$ with respect to $\theta$ is a generator of the $S^1$-action on $M$.

Next, consider the Tanaka–Webster connection with respect to $\theta$. Take a local coordinate $(z^1, \ldots, z^n)$ of $Y$. The Kähler form $\omega$ is written as

$$\omega = ig_{\alpha\overline{\beta}}dz^\alpha \wedge d\overline{z}^\beta,$$

where $(g_{\alpha\overline{\beta}})$ is a positive definite Hermitian matrix. An admissible coframe is given by $(\theta, \theta^\alpha := p^*(dz^\alpha), \theta^\overline{\beta} := p^*(d\overline{z}^\beta))$. Since $d\theta = p^*\omega$, we have

$$d\theta = i(p^* g_{\alpha\overline{\beta}})\theta^\alpha \wedge \theta^\overline{\beta},$$

which implies that $h_{\alpha\overline{\beta}} = p^*g_{\alpha\overline{\beta}}$. The connection form $\psi_{\alpha\beta}$ of the Kähler metric with respect to the frame $(\partial/\partial z^\alpha)$ satisfies

$$(2.8) \quad 0 = d(dz^\beta) = dz^\alpha \wedge \psi_{\alpha\beta}, \quad dg_{\alpha\overline{\beta}} = \psi_{\alpha\gamma}g_{\gamma\overline{\beta}} + g_{\alpha\gamma}\overline{\psi}_{\gamma\overline{\beta}}.$$  

We write $\Psi_{\alpha\beta}$ the curvature form of the Kähler metric. Pulling back Equation (2.8) by $p$ gives

$$d\theta^\beta = \theta^\alpha \wedge (p^*\psi_{\alpha\beta}), \quad dh_{\alpha\overline{\beta}} = (p^*\psi_{\alpha\gamma})h_{\gamma\overline{\beta}} + h_{\alpha\gamma}(p^*\overline{\psi}_{\gamma\overline{\beta}}).$$

This yields $\omega_{\alpha\beta} = p^*\psi_{\alpha\beta}$. In particular, the pseudohermitian torsion vanishes identically; that is, $(M, T^{1,0}, \theta)$ is a Sasakian manifold. Moreover, the curvature form $\Pi_{\alpha\beta}$ of the Tanaka–Webster connection is given by $\Pi_{\alpha\beta} = p^*\Psi_{\alpha\beta}$.

### 3. Representatives for characteristic classes

In this section we give some equivalent representatives for the characteristic classes of a CR manifold. Given an invariant polynomial $\Phi$ of degree $k$ and a matrix $Y_{\alpha\beta}$ of two-forms, we define the characteristic form $c_\Phi(Y_{\alpha\beta})$ by

$$c_\Phi(Y_{\alpha\beta}) := \Phi_{\alpha_1\cdots\alpha_k}^\beta Y_{\beta_1}^{\alpha_1} \cdots Y_{\beta_k}^{\alpha_k},$$

throughout this section we multiply forms using the exterior product. The characteristic class of $(M^{2n+1}, T^{1,0})$ determined by $\Phi$ is

$$c_\Phi(T^{1,0}) := \left[ c_\Phi \left( \frac{i}{2\pi} \Pi_{\alpha\beta} \right) \right].$$

It is well-known $c_\Phi(T^{1,0})$ is independent of the choice of contact form.

We are interested in two other $\text{End}(T^{1,0})$-valued two-forms on a pseudohermitian manifold $(M^{2n+1}, T^{1,0}, \theta)$, namely

$$(3.1) \quad \Omega_{\alpha\beta} := R_{\alpha\beta}^{\mu\nu} \theta^\mu \theta^\nu - \nabla^\beta A_{\alpha\mu} \theta^\mu + \nabla_\alpha A^{\beta\nu} \theta^\nu,$$

$$(3.2) \quad \Xi_{\alpha\beta} := S_{\alpha\beta}^{\mu\nu} \theta^\mu \theta^\nu - V_{\alpha\beta}^{\mu\nu} \theta^\mu + V_{\beta\alpha}^{\mu\nu} \theta^\nu.$$
It is known [22] Equations (2.2) and (2.4)] that
\begin{equation}
\Omega_\alpha^\beta = \Pi_\alpha^\beta - i\theta_\alpha^\tau_\beta + i\tau_\alpha^\theta_\beta.
\end{equation}

The main results of this section are that $c\Phi(\Omega_\alpha^\beta)$ is closed and the induced element in $H^{2k}(M; \mathbb{R})$ agrees with $[c\Phi(\Pi_\alpha^\beta)]$, and moreover the same is true for $c\Phi(\Xi_\alpha^\beta)$ on pseudo-Einstein manifolds. This requires three observations.

Our first observation is that $[c\Phi(\Pi_\alpha^\beta)] = [c\Phi(\Omega_\alpha^\beta)]$.

**Proposition 3.1.** Let $(M^{2n+1}, T^{1,0}, \theta)$ be a pseudohermitian manifold and let $\Phi$ be an invariant polynomial of degree $k$. Then $c\Phi(\Omega_\alpha^\beta)$ is closed and
\[ [c\Phi(\Pi_\alpha^\beta)] = [c\Phi(\Omega_\alpha^\beta)]. \]

**Proof.** Denote
\[ T_k(\Pi_\alpha^\beta) := \text{tr} \Pi^k := \Pi_{\alpha_1, \gamma_1} \Pi_{\alpha_2, \gamma_2} \cdots \Pi_{\alpha_k, \gamma_1}; \]

note that $T_k(\Pi_\alpha^\beta) = k! \text{ch}_k(\Pi_\alpha^\beta)$ is proportional to the $k$-th Chern character form. Since \{T_k\}_{k=1}^\infty generates the algebra of invariant polynomials, it suffices to prove the result for all $T_k$.

Denote
\[ \Theta_\alpha^\beta := i\theta_\alpha^\tau_\beta - i\tau_\alpha^\theta_\beta, \]

so that $\Pi_\alpha^\beta = \Omega_\alpha^\beta + \Theta_\alpha^\beta$. Denote $(\Theta^s)_\alpha^\beta := \Theta_{\alpha_1}^{\gamma_1} \Theta_{\gamma_1}^{\alpha_2} \cdots \Theta_{\gamma_{s-1}}^{\alpha_s}$. We compute that
\begin{align*}
(\Theta^{2s+1})_\alpha^\beta &= (-i\tau_\gamma^\tau_\gamma d\theta)^s \Theta_\alpha^\beta, \\
(\Theta^{2s})_\alpha^\beta &= (-i\tau_\gamma^\tau_\gamma d\theta)^s (-i\tau_\gamma^\tau_\gamma d\theta)^s \Theta_\alpha^\beta = (\tau_\rho^\theta \theta_\rho^\tau_\theta - i\tau_\rho^\theta \theta_\rho^\tau_\theta - i\tau_\rho^\theta \theta_\rho^\tau_\theta d\theta)
\end{align*}

for all $s \in \mathbb{N}$. A direct computation using Equation (2.2) and the definition of $\omega_\alpha^\beta$ yields
\begin{align*}
d\Pi_\alpha^\beta &= \omega_\alpha^\gamma \Pi_\gamma^\beta - \Pi_\alpha^\gamma \omega_\gamma^\beta, \\
\Pi_\alpha^\gamma \theta_\gamma &= -d(\theta_\alpha^\tau_\gamma) - \theta_\alpha^\gamma \tau_\gamma, \\
\theta^\gamma \Pi_\gamma^\beta &= d(\theta_\beta^\gamma) + \theta_\gamma \omega_\gamma^\beta.
\end{align*}

It follows from these equations that
\begin{align*}
d\Omega_\alpha^\beta &= \omega_\alpha^\gamma \Omega_\gamma^\beta - \Omega_\alpha^\gamma \omega_\gamma^\beta + i\theta_\alpha^\tau_\beta (d\tau_\beta^\gamma - \tau_\gamma^\tau_\beta) + i(d\tau_\alpha^\gamma - \omega_\alpha^\gamma \tau_\gamma) \theta_\beta, \\
\Omega_\alpha^\gamma \theta_\gamma &= \theta(d\tau_\alpha^\gamma - \omega_\alpha^\gamma \tau_\gamma), \\
\theta^\gamma \Omega_\gamma^\beta &= -\theta(d\tau_\beta^\gamma - \tau_\gamma^\tau_\beta).
\end{align*}

We deduce that
\begin{align*}
d(\tau_\gamma^\theta(\Omega^\gamma)\tau_\beta^\theta_\gamma) &= (d\tau_\gamma^\theta - \tau_\gamma^\theta \omega_\rho^\gamma)(\Omega^\gamma)\tau_\beta^\theta_\gamma - \tau_\gamma^\theta(\Omega^\gamma)\tau_\beta^\theta_\gamma (d\tau_\beta^\gamma - \omega_\beta^\gamma \tau_\rho), \\
\Omega_\alpha^\beta \Theta_\gamma^\beta \Omega_\gamma^\rho &= i\theta \left( (d\tau_\alpha^\gamma - \omega_\alpha^\gamma \tau_\beta) \tau_\gamma^\tau_\beta - \Omega_\alpha^\beta \tau_\beta (d\tau_\rho^\gamma - \tau_\gamma \omega_\rho^\gamma) \right), \\
\Omega_\alpha^\beta (\Theta^2)_\beta^\gamma \Omega_\gamma^\rho &= -i\Omega_\alpha^\beta \tau_\beta \tau_\gamma \Omega_\gamma^\rho d\theta
\end{align*}

for all integers $s \geq 0$.

Given $s \in \mathbb{N}$ and $N \in \mathbb{N}$, define $O_{s,N} \in \Lambda^{2N+4s-2}T^*M$ by
\begin{equation}
O_{s,N} := \sum_{j_1, \ldots, j_s \geq 1, j_1 + \cdots + j_s = N} \Theta_{\gamma_1}^{\beta_1} (\Omega^{j_1})_{\beta_1}^{\beta_1} (\Theta^2)_{\gamma_1}^{\gamma_2} (\Omega^{j_2})_{\beta_2}^{\gamma_2} \cdots (\Theta^2)_{\gamma_{s-1}}^{\gamma_s} (\Omega^{j_s})_{\beta_s}^{\gamma_s},
\end{equation}
with the convention $O_{s,N} = 0$ if $N < s$. It follows from Equations 3.6–3.8 that

$$O_{s,N} = -(-i)^s \theta d\theta^{s-1} \sum_{j_1, \ldots, j_s \geq 1, j_1 + \cdots + j_s = N} \Psi^{(j_1)} \cdots \Psi^{(j_s-1)} d\Psi^{(-1+j_s)},$$

where

$$\Psi^{(j)} := \tau^\alpha (\Omega_j)_{\alpha}^\beta \tau^\beta.$$

Given $s \in \mathbb{N}$ and $N \in \mathbb{N}$, define $E_{s,N} \in \Lambda^{2N+4s} T^* M$ by

$$E_{s,N} := \sum_{j_1, \ldots, j_s \geq 1, j_1 + \cdots + j_s = N} (\Theta^2)^{\gamma_1,\beta_1} (\Omega^{(j_1)})_{\beta_1}^{\gamma_1} \cdots (\Theta^2)^{\gamma_{s-1},\beta_s} (\Omega^{(j_s)})_{\beta_s}^{\gamma_s},$$

with the convention $E_{s,N} = 0$ if $N < s$. It follows from Equation 3.8 that

$$E_{s,N} = -(-i)^s d\theta^s \sum_{j_1, \ldots, j_s \geq 1, j_1 + \cdots + j_s = N} \Psi^{(j_1)} \cdots \Psi^{(j_s)}.$$

Combining Equations (3.9) and (3.11) yields

$$O_{s,N} = -\frac{1}{s} E_{s,N-1} - i \theta d(\tau^\gamma \tau^\gamma) E_{s-1,N-1}$$

$$+ \left( \frac{-i}{s} \right)^s d \left( \sum_{j_1, \ldots, j_s \geq 1, j_1 + \cdots + j_s = N-1} \Psi^{(j_1)} \cdots \Psi^{(j_s)} \theta d\theta^{s-1} \right),$$

with the convention $E_{0,0} = -1$ and $E_{0,N} = 0$ if $N \geq 1$.

Now consider $T_k(\Pi_{\alpha}^\beta) = T_k(\Omega_{\alpha}^\beta + \Theta_{\alpha}^\beta)$. Write

$$T_k(\Pi_{\alpha}^\beta) = \sum_{s=0}^{k} f_s,$$

where $f_s$ is the term obtained by expanding $T_k(\Omega_{\alpha}^\beta + \Theta_{\alpha}^\beta)$ as a polynomial in $\Omega_{\alpha}^\beta$ and $\Theta_{\alpha}^\beta$, and keeping only those terms which are homogeneous of degree $s$ in $\Theta_{\alpha}^\beta$. First note that, for $s \geq 0$ and $2s + 2 < k$,

$$f_{2s+2} = k \sum_{j=1}^{s+1} \binom{s}{j-1} \left( -i \tau^\gamma \tau^\gamma d\theta \right)^{s+1-j} E_{j,k-2-2s}. \tag{3.12}$$

To obtain this formula, first note that Equations 3.6 and 3.7 imply that all products with at least two factors of odd powers $(\Theta^{2l+1})_{\alpha}^\beta$, $\ell \geq 0$, of $\Theta_{\alpha}^\beta$ which are separated by powers of $\Omega$ must vanish; e.g. $\Omega_{\alpha}^\beta \Theta^\gamma \Theta^\gamma \Omega_{\alpha}^\beta = 0$. Therefore $f_{2s+2}$ can be written as a polynomial in $(\Theta^2)_{\alpha}^\beta$ and $\Omega_{\alpha}^\beta$. Group the summands according to how many times a positive power of $(\Theta^2)_{\alpha}^\beta$ is multiplied on the left and the right by $\Omega_{\alpha}^\beta$. Using Equation 3.5A, we see that the sum of all possible terms where this happens $j$ times is a multiple $c_j$ of

$$(-i \tau^\gamma \tau^\gamma d\theta)^{s+1-j} E_{j,k-2-2s}.$$

To compute the multiple, note that in the definition of $E_{j,k-2-2s}$, there are $j$ positions — corresponding to each of the factors of $(\Theta^2)_{\alpha}^\beta$ — where the extra $s+1-j$ copies of $(\Theta^2)_{\alpha}^\beta$ can be multiplied. There are $(s+1)$ ways these products appear in the expansion of $T_k(\Pi_{\alpha}^\beta)$. Since $E_{j,k-2-2s}$ is symmetric in the ordering of the
factors of \((\Theta^2)_{\alpha}^\beta\) and there are \(k\) different ways to cyclically permute the terms of \(\mathcal{E}_{j,k-2-2s}\), we conclude that \(c_j = \frac{j}{s} \left(\begin{array}{c}s \\ j-1\end{array}\right)\). This yields Equation \((3.12)\).

Equation \((3.5)\) implies that if \(k\) is even, then

\[
f_k = -2(-i\tau^j\tau_j d\theta)^{k/2}.
\]

Combining Equation \((3.12)\) and our conventions that \(\mathcal{E}_{0,0} = -1\) and \(\mathcal{E}_{j,0} = 0\) if \(j \geq 1\) implies that

\[
(3.13) \quad f_{2s+2} = k \sum_{j=0}^{s+1} \frac{1}{j} \left(\begin{array}{c}s \\ j\end{array}\right) (-i\tau^j\tau_j d\theta)^{s+1-j} \mathcal{E}_{j,k-2-2s}
\]

for all \(s \geq 0\), where we recall that \(\frac{j}{s} \left(\begin{array}{c}s \\ j\end{array}\right) = \frac{1}{j+s+1} (\begin{array}{c}s+j \\cdots \\cdots \\ j\end{array})\) to make sense of the coefficient when \(j = 0\).

Second note that, for \(s \geq 0\),

\[
(3.14) \quad f_{2s+1} = k \sum_{j=1}^{s+1} \left(\begin{array}{c}s \\ j\end{array}\right) (-i\tau^j\tau_j d\theta)^{s+1-j} \mathcal{O}_{j,k-1-2s}.
\]

We obtain this formula by following the same procedure as above, except that now there must be a single factor of an odd power of \(\Theta_{\alpha}^\beta\), and the location of this factor specifies a preferred ordering of the terms of the expansion, up to cyclic permutation.

Finally, it follows from Equations \((3.11)\), \((3.13)\) and \((3.14)\) that

\[
f_{2s+1} + f_{2s+2} = k \sum_{j=1}^{s+1} \left(\begin{array}{c}s \\ j\end{array}\right) (-i\tau^j\tau_j d\theta)^{s+1-j} \left[ -i\theta d(\tau^j\tau_j) \mathcal{E}_{j-1,k-2-2s} 
\right.
\]

\[+ \left. \frac{(-i)^j}{j} d \left( \sum_{\ell_1,\ldots,\ell_j \geq 1} \Psi^{(\ell_1)} \cdots \Psi^{(\ell_j)} \theta d\theta^{j-1} \right) \right]
\]

\[= k d \sum_{j=1}^{s+1} \left(\begin{array}{c}s \\ j\end{array}\right) \theta \tau^j \tau_j (-i\tau^j\tau_j d\theta)^{s-j} \mathcal{E}_{j,k-2-2s}. \]

In particular, \(f_{2s+1} + f_{2s+2}\) is exact for all integers \(s \geq 0\). Adopting the convention that \(f_\ell = 0\) for all \(\ell \geq k+1\), we may write

\[
T_k(\Pi_{\alpha}^\beta) = f_0 + \sum_{s=0}^{\infty} (f_{2s+1} + f_{2s+2}).
\]

Since \(f_0 = T_k(\Omega_{\alpha}^\beta)\) and \(f_{2s+1} + f_{2s+2}\) is exact, we conclude that \(T_k(\Omega_{\alpha}^\beta)\) is closed and \([T_k(\Pi_{\alpha}^\beta)] = [T_k(\Omega_{\alpha}^\beta)]\). □

Our second observation is that the form \(c_k(\Xi_{\alpha}^\beta)\) is always closed.

**Lemma 3.2.** Let \((\mathcal{M}^{2n+1}, T^{1,0}, \theta)\) be a pseudohermitian manifold and let \(\Phi\) be an invariant polynomial of degree \(k\). Then \(c_k(\Xi_{\alpha}^\beta)\) is closed.

**Proof.** It follows from Lemma \((2.2)\) that

\[
d\Xi_{\alpha}^\beta = -iV_{\alpha \rho}^\sigma \theta^\rho \theta^\sigma + iV_{\beta \mu}^\sigma \theta_{\sigma} \theta^\mu - i \left( S_{\mu \nu}^\rho P_{\rho}^\beta - S_{\mu \rho}^\nu P_{\nu}^\beta \right) \theta^\rho \theta^\nu
\]

\[+ iU_{\alpha}^\nu \theta^\nu \theta^\rho - iU_{\beta}^\rho \theta^\rho \theta^\alpha - iQ_{\alpha}^\gamma \theta^\gamma \theta^\beta - iQ_{\beta}^\gamma \theta^\gamma \theta^\alpha.
\]

Therefore, \(\Xi_{\alpha}^\beta\) is closed.
Using the facts that $S_{\alpha \beta \gamma \delta}$, $V_{\alpha \beta \gamma}$, and $Q_{\alpha \gamma}$ are all symmetric [9, Section 2.3], we readily verify from the above display that $dc_\Phi(\Xi_{\alpha \beta}) = 0$. □

Our third observation is that if $(M^{2n+1}, T^{1,0}, \theta)$ is pseudo-Einstein, then the cohomology classes $[c_\Phi(\Omega_{\alpha \beta})]$ and $[c_\Phi(\Xi_{\alpha \beta})]$ agree.

**Proposition 3.3.** Let $(M^{2n+1}, T^{1,0}, \theta)$ be a pseudo-Einstein manifold and let $\Phi$ be an invariant polynomial of degree $k$. Then

$$[c_\Phi(\Omega_{\alpha \beta})] = [c_\Phi(\Xi_{\alpha \beta})].$$

**Proof.** Since $(M^{2n+1}, T^{1,0}, \theta)$ is pseudo-Einstein,

$$R_{\alpha \beta \mu \nu} = S_{\alpha \beta \mu \nu} + \frac{2}{n} P_{\alpha \beta \mu} \delta_{\mu}^{\nu} + \frac{2}{n} P_{\mu \nu} \delta_{\beta}^{\alpha},$$

$$\nabla A_{\alpha \mu} = V_{\alpha \beta \mu} - \frac{2}{n} \delta_{\alpha}^{\beta} \nabla P - \frac{2}{n} \delta_{\mu}^{\beta} \nabla P.$$ It follows that

$$(3.15) \quad \Omega_{\alpha \beta} = \Xi_{\alpha \beta} - \frac{2}{n} \delta_{\alpha}^{\beta} d(P\theta) - \frac{2}{n} \left( P\theta \delta_{\alpha}^{\beta} \theta + i\nabla P \theta \theta + i\nabla P \theta \theta \right).$$

On the one hand, since $\Phi$ is an invariant polynomial of degree $k$, its trace $\Phi_{\alpha \beta}^{\gamma \delta \epsilon \cdots \beta_k}$ is an invariant polynomial of degree $k - 1$. Also, by Proposition [3.1], $c_\Phi(\Omega_{\alpha \beta})$ is closed. It follows immediately that $c_\Phi(\Omega_{\alpha \beta} + \frac{2}{n} \delta_{\alpha}^{\beta} d(P\theta))$ is closed and

$$(3.16) \quad [c_\Phi(\Omega_{\alpha \beta} + \frac{2}{n} \delta_{\alpha}^{\beta} d(P\theta))] = [c_\Phi(\Xi_{\alpha \beta})].$$

On the other hand, set

$$\Gamma_{\alpha \beta} := P\theta \delta_{\alpha}^{\beta} \theta + i\nabla P \theta \theta + i\nabla P \theta \theta.$$

Note that $\Omega_{\alpha \beta} + \frac{2}{n} \delta_{\alpha}^{\beta} d(P\theta) = \Xi_{\alpha \beta} + \frac{2}{n} \Gamma_{\alpha \beta}$. A straightforward induction argument yields

$$(\Gamma^m)_{\alpha \beta} := \Gamma_{\alpha \gamma_{m+1}} \Gamma_{\gamma_{m+2}} \cdots \Gamma_{\gamma_{2m}}$$

$$= (-1)^{m+1} i P^{m-1} \left( \Gamma_{\alpha \beta} d\theta^{m-1} + (m-1)(dP) \theta \delta_{\alpha \beta} d\theta^{m-2} \right)$$

for all $m \in \mathbb{N}$. In particular, we deduce that

$$\Xi_{\alpha \beta} \Gamma_{\alpha \beta}^{\gamma \delta \rho \varepsilon} = 0,$$

$$\Xi_{\alpha \gamma} (\Gamma^m)_{\gamma \rho \varepsilon} = 0$$

for all $m \in \mathbb{N}$. Combining this with Lemma [3.2] yields

$$(3.17) \quad [c_\Phi(\Xi_{\alpha \beta} + \frac{2}{n} \Gamma_{\alpha \beta})] = [c_\Phi(\Xi_{\alpha \beta})].$$

The conclusion follows immediately from Equations (3.15)–(3.17). □

**Remark 3.4.** In his proof of [25, Proposition 5.4], Marugame showed that the conclusion of Proposition [3.3] is true if the assumption on $(M^{2n+1}, T^{1,0}, \theta)$ is weakened to only assume that $c_1(T^{1,0}) = 0$ in $H^2(M; \mathbb{R})$. 

4. The Invariance of $X^\Phi_\alpha$ and the Total $I_\Phi'$-Curvature

In this section we prove that $X^\Phi_\alpha$ and $\nabla^\alpha X^\Phi_\alpha$ are CR invariant, derive the transformation formula for $I_\Phi'$ under change of contact form, and conclude that the total $I_\Phi'$-curvature is a secondary CR invariant.

First we prove that $X^\Phi_\alpha$ and $\nabla^\alpha X^\Phi_\alpha$ are CR invariant.

**Proof of Theorem 1.2.** On the one hand, since $c_\Phi(S)$ is a scalar CR invariant of weight $-n$, we conclude from Lemma 2.1 that

$$D_\theta \nabla_\alpha c_\Phi(S) \left( \Upsilon \right) = -nc_\Phi(S) Y_\alpha.$$

(4.1) On the other hand, since $D_\theta$ is a derivation and $\mathcal{S}^\Phi$ is a local CR invariant, we conclude from Lemma 2.1 that

$$D_\theta \left( i (\mathcal{S}^\Phi)_{\alpha \mu} \nu V^\mu_{\nu} \right) \left( \Upsilon \right) = - (\mathcal{S}^\Phi)_{\alpha \mu} \nu S^\beta_\nu \nu Y_\beta.$$

(4.2) Since $M$ has CR dimension $n$, it holds that

$$(\mathcal{U}^\Phi)^{\alpha \beta} := \delta^{\beta \beta_1 \cdots \beta_n} \mu_1 \cdots \mu_n \delta_\beta_1 \alpha_1 \nu_1 \cdots \delta_\beta_n \alpha_n \nu_n = 0.$$

(4.3) In particular,

$$0 = (\mathcal{U}^\Phi)_{\alpha \beta} \Upsilon_{\beta} = c_\Phi(S) Y_\alpha - n(\mathcal{S}^\Phi)_{\alpha \mu} \nu S^\beta_\nu \nu Y_\beta.$$

Combining Equations (4.1)–(4.3) implies that $X^\Phi_\alpha$ is a CR invariant $(1, 0)$-form of weight $-n$. It follows immediately from Lemma 2.1 that $\nabla^\alpha X^\Phi_\alpha$ is a CR invariant of weight $-n - 1$. \qed

Next we derive the transformation formula for $I_\Phi'$ under change of contact form.

**Proof of Theorem 1.3.** It follows from Lemma 2.1 that

$$D_\theta \left( \Delta_{\theta} c_\Phi(S) - 2n P c_\Phi(S) \right) \left( \Upsilon \right) = -2n \mathrm{Re} \, \Upsilon^\gamma \nabla_\gamma c_\Phi(S),$$

(4.4) $$D_\theta \left( V_{\beta}^{\mu_1 \nu_1} V^\alpha_{\nu_2 \mu_2} \right) \left( \Upsilon \right) = -2 \mathrm{Re} \, i V_{\beta}^{\mu_1 \nu_1} S^\alpha_{\nu_2 \mu_2} Y_\beta.$$

(4.5) Since $\mathcal{T}^\Phi$ is a local CR invariant, we conclude from Lemma 2.1 and Equation (4.5) that

$$D_\theta \mathcal{V} \left( \Upsilon \right) = 2 \mathrm{Re} \, i (\mathcal{S}^\Phi)_{\alpha \mu} \nu V^\mu_{\nu} Y_\alpha,$$

(4.6) where

$$\mathcal{V} := (\mathcal{T}^\Phi)_{\alpha \mu_1 \nu_1 \mu_2 \nu_2} \left( ((n - 1)V_{\beta}^{\mu_1 \nu_1} V^\alpha_{\nu_2 \mu_2} - S^\alpha_{\nu_2 \mu_1} U_{\nu_2} \mu_2) \right).$$

Combining Equations (4.4) and (4.6) and Theorem 1.2 yields

$$\frac{\partial}{\partial t} e^{(n+1)t} \mathcal{T}_\Phi' e^{i t \theta} = e^{(n+1)t} \mathcal{T}_\Phi' \left( 2 \mathrm{Re} \, X^\Phi_\alpha \Upsilon_\alpha \right) e^{i t \theta} = 2 \mathrm{Re} \, (X^\Phi_\alpha \Upsilon_\alpha)^\theta.$$

Integrating this equation in $t \in [0, 1]$ yields the desired result. \qed

The rest of this section is devoted to the proof that the total $I_\Phi'$-curvature is a secondary CR invariant. The main task is to relate $X^\Phi_\alpha$ to the characteristic class $c_\Phi(T^{1,0})$.

**Proposition 4.1.** Let $(M^{2n+1}, T^{1,0}, \theta)$ be a pseudo-Einstein manifold and let $\Phi$ be an invariant polynomial of degree $n$. Set

$$\xi^\Phi := X^\Phi_\alpha \theta \wedge \theta^n \wedge d\theta^{n-1} + X^\Phi_\beta \theta \wedge \theta^\beta \wedge d\theta^{n-1}.$$

Then $\xi^\Phi$ is closed. Moreover, $n[\xi^\Phi] = (2\pi)^n (n-1) c_\Phi(T^{1,0})$ in $H^{2n}(M; \mathbb{R})$. 

Proof. Combining Propositions 3.1 and 3.3 yields

\[(2\pi)^n c_\Phi (T^{1,0}) = [c_\Phi (i\Xi_\alpha^\beta)], \]

where $\Xi_\alpha^\beta$ is defined by Equation (4.8). An easy computation yields

\[c_\Phi (i\Xi_\alpha^\beta) = \frac{1}{n!} (S^\Phi)_{\alpha}^\beta \alpha \mu \nu V_\beta^\mu \theta^\alpha \theta^n d\theta - \frac{n^2}{(n-1)!} d_\Phi \left( c_\Phi (S) \theta d\theta^{n-1} \right) \]

In particular,

\[c_\Phi (i\Xi_\alpha^\beta) = \frac{1}{n!} d_\Phi \left( c_\Phi (S) \theta d\theta^{n-1} \right) - \frac{n}{(n-1)!} c_\Phi. \]

We conclude that $\xi_\Phi$ is closed and $[c_\Phi (i\Xi_\alpha^\beta)] = -\frac{n}{(n-1)!} [\xi_\Phi]$. The conclusion now follows from Equation (4.7). \qed

We now conclude that the total $\mathcal{I}_\rho'$-curvature is a secondary invariant.

Proof of Theorem 4.4. Let $\xi_\Phi$ be as in Proposition 4.1. We may thus consider the cohomology class $[\xi_\Phi] \in H^{2n}(M; \mathbb{R})$. Recall [22, Lemma 3.1] that $\mathcal{Y} \in C^\infty (M)$ is CR pluriharmonic if and only if $d_\mathcal{Y} := i(\mathcal{Y}_\beta \theta^\beta - \mathcal{Y}_\alpha \theta^\alpha) \in \Gamma (T^*M/\langle \theta \rangle)$ is closed in the sense of Rumin [28]. In particular, for any $\mathcal{Y} \in \mathcal{P}$, the cup product $[\xi_\Phi] \cup [d_\mathcal{Y}] := [\xi_\Phi \land d_\mathcal{Y}]$ is well-defined in $H^{2n+1}(M; \mathbb{R})$. A straightforward computation implies that

\[\langle [\xi_\Phi] \cup [d_\mathcal{Y}], [\mathcal{M}] \rangle = \frac{2}{n} \text{Re} \int_M X_\alpha^\Phi \mathcal{Y}^\alpha \theta \land d\theta^n, \]

where $[\mathcal{M}]$ is the fundamental class of $M$.

Next, a result of Takeuchi [31, Theorem 1.1] implies that $c_\Phi (T^{1,0}) = 0$. Combining Proposition 4.1 and Equation (4.8) yields $\text{Re} \int_M X_\alpha^\Phi \mathcal{Y}^\alpha \theta \land d\theta^n = 0$ for all $\mathcal{Y} \in \mathcal{P}$. Combining this with Equation (1.8) yields the desired result. \qed

5. Counterexamples to the Hirachi Conjecture

As noted in the introduction, if Conjecture 1.1 holds, then the $\mathcal{I}_\rho'$-curvature must be a linear combination of a local CR invariant and a divergence. However, there is a local CR invariant whose vanishing is necessary for $\mathcal{I}_\rho'$ to be a linear combination of a local CR invariant and a divergence.

Lemma 5.1. Let $\Phi$ be an invariant polynomial of degree $n$. If there exists a pseudo-Hermitian manifold $(M^{2n+1}, T^{1,0}, \theta)$ such that $\text{Re} \nabla^\alpha X_\alpha^\Phi$ is not identically zero, then $\mathcal{I}_\rho'$ is not the linear combination of a local CR invariant and a divergence.
We shall present two ways to find such a manifold. Theorem 1.3 implies that \( \int_M I_n^\alpha X_\alpha^\Phi \) is not identically zero in order to conclude that \( I_n^\alpha X_\alpha^\Phi \) gives a counterexample to Conjecture [11]. Indeed, since \( \nabla X_\alpha^\Phi \) is CR invariant, it suffices to find a pseudohermitian manifold \((M, \partial_t \theta_0)\) which admits a pseudo-Einstein contact form and is such that \( \nabla X_\alpha^\Phi \) is not identically zero. We shall present two ways to find such a manifold.

First, in Section 6, we compute the change of \( \nabla X_\alpha^\Phi \) along a particular perturbation of the round CR \((2n+1)\)-sphere. This approach is computationally challenging and can be regarded as a local (in the space of CR structures on \( S^{2n+1} \)) generalization of computations of Reiter and Son [26] for five-dimensional real ellipsoids.

Second, in Section 7, we compute \( \nabla X_\alpha^\Phi \) on circle bundles over a Calabi–Yau manifold in the case when \( \Phi \) is the generalized Kronecker delta. This approach is computationally simple and relies on explicit examples of degenerating sequences of Calabi–Yau manifolds in complex dimensions two and three.

6. Counterexample via perturbations of \( S^{2n+1} \)

The purpose of this section is to prove Theorem 1.5 by considering the \( I_n^\alpha \)-curvatures on perturbations of the round CR \((2n+1)\)-sphere. To that end, we need to know the first variation of the Chern tensor \( S_{\alpha\beta\gamma\delta}^\alpha \) along a suitable deformation. This formula is known [17], but since we cannot find a proof in the literature, we provide one here.

**Lemma 6.1.** Let \( \rho_t : \mathbb{C}^{n+1} \to \mathbb{R} \) be a one-parameter family of smooth functions such that \( \rho_0(z) = 1 - |z|^2 \). Set \( M_t := \rho_t^{-1}(0) \), \( T_t^{1,0} := T^{1,0} \mathbb{C}^{n+1} \cap (TM_t \otimes \mathbb{C}) \), and \( \theta_t := \text{Im} \partial_t \rho_t |_{M_t} \). Let \( F_t : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} \) be a one-parameter family of diffeomorphisms such that \( F_0 = \text{Id} \), \( \rho_t \circ F_t = \rho_0 \), and \( F_t \ker \theta_t = \ker \theta_0 \). Denote by \( S^t := F_t^* S^\theta_0 \) the pullback of the Chern tensor of \( \theta_t \) by \( F_t \). Then

\[
S_{\alpha\beta\gamma\delta} := \left. \frac{\partial}{\partial t} \right|_{t=0} S^t_{\alpha\beta\gamma\delta} = \text{tf}(\hat{\mu})_{\alpha\beta\gamma\delta},
\]

where \( \text{tf} u_{\alpha\beta\gamma\delta} \) denotes the totally trace-free part of \( u_{\alpha\beta\gamma\delta} \),

\[
\text{tf} u_{\alpha\beta\gamma\delta} := u_{\alpha\beta\gamma\delta} - \frac{1}{n+2} (u_{\alpha\beta\mu} h_{\gamma\delta} + u_{\gamma\beta\mu} h_{\alpha\delta} + u_{\alpha\gamma\mu} h_{\beta\delta} + u_{\gamma\alpha\mu} h_{\beta\delta} + h_{\alpha\beta} + h_{\alpha\gamma} + h_{\beta\delta})
\]

\[
+ \frac{1}{(n+1)(n+2)} \delta_{\mu\nu} (h_{\alpha\beta} h_{\gamma\delta} + h_{\alpha\gamma} h_{\beta\delta}),
\]

and \( h_{\alpha\beta} \) is the Levi form of the round CR \((2n+1)\)-sphere \((M_0, T_0^{1,0})\).
Remark 6.2. The existence of diffeomorphisms $F_t: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ as in the statement of Lemma 6.1 is guaranteed by [19, Lemma 4.1]. Note that the restriction $F_t: M_0 \to M_t$ is a contact diffeomorphism.

Proof. Fix $p \in M_0$. By permuting coordinates if necessary, we may assume that $(w, z) \in \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}$ are such that $(\rho_p)_w := \frac{\partial \rho}{\partial w}$ is nowhere zero in a neighborhood of $(0, p)$ in $\mathbb{R} \times \mathbb{C}^{n+1}$. Consider the frame $Z^\alpha_t := \partial_{\alpha} - \frac{(\rho_p)_w}{(\rho_p)_w} \partial_w$ of $T_t^{1,0}$ near $F_t(p)$. Applying [26, Theorem 3.1] yields

\begin{equation}
S^\alpha_t = tf(R_{\alpha\beta\gamma\delta}(\rho_t) + h_{\beta\delta}^{\xi} D_{\alpha\gamma}(\rho_{\bar{\delta}}) D_{\beta\delta}(\rho_j) + h_{\beta\delta}^{\xi} D_{\alpha\gamma}(\rho_k) + h_{\alpha\gamma}^{\xi\eta} D_{\beta\delta}(\rho_j) - |\xi|^2 h_{\alpha\gamma} h_{\beta\delta}),
\end{equation}

where $\xi = \xi(t)$ is the unique $(1,0)$-vector field in $\mathbb{C}^{n+1}$ such that $\partial \rho_1(\xi) = 1$ and $\xi, i \partial \bar{\rho}_1 \equiv 0 \mod \bar{\rho}_1$.

We prove Theorem 6.3 by applying Lemma 6.1 to the specific family of defining functions, where $(\theta^\alpha_0)$ is the admissible coframe of $T_t^{1(1,0)}$ dual to $\{Z^\alpha_t\}$ of $T_t^{1,0}$ and its conjugate. By definition,

$S^t = ((S^\alpha_t)_{\alpha\beta\gamma\delta} \circ F_t)((F_t^* \theta^\alpha_0) \wedge (F_t^* \theta^\beta_0)) \otimes ((F_t^* \theta^\gamma_0) \wedge (F_t^* \theta^\delta_0))$,

where $\{\theta^\alpha_0\}$ is the admissible coframe of $T_t^{1(1,0)}$ dual to $\{Z^\alpha_t\}$. Since $S^\theta_0 = 0$, we see that

$\frac{\partial}{\partial t} \bigg|_{t=0} S^t = \frac{\partial}{\partial t} \bigg|_{t=0} ((S^\alpha_t)_{\alpha\beta\gamma\delta} \circ F_t)((\theta^\alpha_0 \wedge \theta^\beta_0) \otimes (\theta^\gamma_0 \wedge \theta^\delta_0))$.

Combining this with Equation (6.2) and the facts $h_{\alpha\gamma} = h_{\beta\delta} = 0$ and $D_{\alpha\gamma}(\rho_k) = D_{\beta\delta}(\rho_j) = 0$ at $t = 0$ yields Equation (6.1).

We prove Theorem 6.3 by applying Lemma 6.1 to the specific family

$\rho_t := 1 - |z|^2 - |w|^2 + \frac{t}{4}|w|^4$

of defining functions, where $(w, z) \in \mathbb{C} \times \mathbb{C}^n$. In fact, we prove the following sharper result:

Theorem 6.3. Let $\zeta = (\zeta_1, \zeta_2, \zeta_3, \ldots, \zeta_n) \in \mathbb{N}^n$ be such that $\zeta_1 = 0$ and $n = \sum_{k=1}^n k \zeta_k$. For $t$ sufficiently close to zero, consider the pseudohermitian manifolds $(M_t, T_t^{1,0}, \theta_t)$ and contact diffeomorphisms $F_t: M_0 \to M_t$ as in Lemma 6.1. Let $\Phi = \Phi(\zeta)$ be as in Equation (13). Then

$\frac{\partial^k}{\partial t^k} \bigg|_{t=0} \text{Re} F_t^* \left(\nabla^\alpha X^\alpha_{\theta_t}\right) = 0$
for all nonnegative integers $k < n$ and

\[
\frac{\partial^n}{\partial t^n} \left|_{t=0} \right. \Re F_t^* \left( \nabla^\alpha X^\Phi_\alpha \right)^{\theta_t} \neq 0.
\]

In particular, for all $t \neq 0$ sufficiently close to zero, it holds that $\Re(\nabla^\alpha X^\Phi_\alpha)^{\theta_t} \neq 0$.

The proof of Theorem 6.3 only requires that $\rho_t$ is the defining function of the round $(2n+1)$-sphere and the formula for $\frac{\partial}{\partial t} \left|_{t=0} \right. \rho_t$. In particular, the conclusion of Theorem 6.3 also holds for some of the ellipsoids considered by Reiter and Son [26]; see Remark 6.4 for further discussion.

**Proof of Theorem 6.3.** Let $\rho_t$ be given by Equation (6.3). Let $(S^t)^{\alpha\bar{\beta}\gamma\bar{\sigma}}$ denote the pullback of the Chern tensor of $(M_t, T_{1,0}, \theta_t)$ by $F_t$. Since $\rho_0$ is the defining function of the round CR $(2n+1)$-sphere, it holds that $(S_0)^{\alpha\bar{\beta}\gamma\bar{\sigma}} = 0$. This yields Equation (6.4).

Recall that, on the round CR $(2n+1)$-sphere,

\[
w_{\alpha\bar{\beta}} = 0, \quad w_{\alpha\bar{\beta}} = -w h_{\alpha\bar{\beta}}, \quad w_{\bar{\beta}} = 0, \quad w^\gamma w_\gamma = 1 - |w|^2.
\]

(One can deduce these formulae using the fact that $\frac{1}{|1+w|^2} \theta$ on $S^{2n+1} \setminus \{w = -1\}$ equals the pullback of the standard contact form on the Heisenberg group under Cayley transform [20] and the transformation laws [22] for the pseudohermitian curvature and torsion.) Since $\dot{\rho}_t = \frac{1}{4} |w|^4$, we conclude from Lemma 6.1 that

\[
\dot{S}_{\alpha\bar{\beta}\gamma\bar{\sigma}} = tf w_{\alpha\bar{\beta}} w_{\gamma\bar{\sigma}}.
\]

(We emphasize that $\alpha, \bar{\beta}, \gamma, \bar{\sigma}$ are abstract indices in this formula.) Equation (2.3) then implies that

\[
\dot{V}_{\alpha\bar{\beta}} := \left( \frac{\partial}{\partial t} \right)_{t=0} V^t_{\alpha\bar{\beta}} = -\frac{n+3}{n+2} i \dot{w} tf w_{\alpha\bar{\beta}} w_\gamma,
\]

where $V^t := F_t^* V^\theta_t$ and

\[
\dot{u}_{\alpha\bar{\beta}} := u_{\alpha\bar{\beta}} - \frac{1}{n+1} \left( u_{\mu\nu} h_{\gamma\bar{\sigma}} + u_{\mu\nu} h_{\alpha\bar{\beta}} \right).
\]

Define

\[
\dot{C}_\alpha^\beta := i \dot{S}_{\alpha^\beta}^{\mu\nu} \theta^\mu \theta^\nu, \quad \dot{V}_\alpha^\beta := \dot{V}_{\alpha^\beta \mu} \theta^\mu,
\]

where products are taken in the exterior algebra $\Lambda^* S^{2n+1}$. It follows from Equation (6.7) and Equation (6.8) that

\[
\dot{C}_\alpha^\beta = W_{\alpha^\beta}^\gamma + \frac{n+1}{n+2} c \left( \Psi_\alpha^\beta + M_\alpha^\beta + \delta_\alpha^\beta (i(\partial_\gamma w)(\overline{\partial_\bar{\beta} w}) + c d\theta) \right),
\]

\[
\dot{V}_\alpha^\beta = -\frac{n+3}{n+2} \theta^\alpha \left( (w_{\alpha\bar{\beta}} + c \delta_\alpha^\beta) \partial_\gamma w + cw_{\alpha\bar{\beta}} \right).
\]
respectively, where
\[ W_\alpha^\beta := iw_\alpha w^\beta (\partial_b w)(\overline{\partial_b w}), \]
\[ \Psi_\alpha^\beta := w_\alpha w^\beta d\theta + iw_\alpha \theta^\beta (\overline{\partial_b w}) + iW^\beta (\partial_b w)\theta_\alpha, \]
\[ M_\alpha^\beta := ic\theta^\beta \theta_\alpha, \]
\[ c := -\frac{1}{n+1}w_\alpha w^\gamma. \]

We break the computation into four steps.

**Step 1. Compute powers of \( \hat{\mathcal{C}}_\alpha^\beta \).**

Observe that
\[ \Psi_\alpha^\gamma \Psi_\gamma^\beta = W_\alpha^\beta d\theta - (n+1)c\Psi_\alpha^\beta d\theta + (n+1)iM_\alpha^\beta (\partial_b w)(\overline{\partial_b w}), \]
\[ \Psi_\alpha^\gamma M_\gamma^\beta = M_\alpha^\gamma \Psi_\gamma^\beta = -iM_\alpha^\beta (\partial_b w)(\overline{\partial_b w}), \]
\[ M_\alpha^\gamma M_\gamma^\beta = -cM_\alpha^\beta d\theta. \]

Combining this with the identities
\[ W_\alpha^\gamma W_\gamma^\beta = 0, \]
\[ W_\alpha^\gamma \Psi_\gamma^\beta = \Psi_\alpha^\gamma W_\gamma^\beta = -(n+1)cA_\alpha^\beta d\theta, \]
\[ W_\alpha^\gamma M_\gamma^\beta = M_\alpha^\gamma W_\gamma^\beta = 0, \]
\[ W_\alpha^\beta (\partial_b w) = W_\alpha^\beta (\overline{\partial_b w}) = 0 \]

yields
\[ (\Psi^k)_\alpha^\beta = (-c)^{k-2} [(k-1)(n+1)^{k-2}W_\alpha^\beta d\theta \]
\[ - (n+1)^{k-1}c\Psi_\alpha^\beta d\theta + (n+1)^{k-1}iM_\alpha^\beta (\partial_b w)(\overline{\partial_b w})] d\theta^{k-2}, \]
\[ (M^k)_\alpha^\beta = (-c)^{k-1}M_\alpha^\beta d\theta^{k-1} \]
for all \( k \geq 2 \), where \( (\Psi^k)_\alpha^\beta := \Psi_\alpha^\gamma \Psi_\gamma^\kappa \cdots \Psi_\kappa^\beta \). It follows that
\[ (M^1)_\alpha^\gamma (\Psi^k)_\gamma^\beta = 0 \]
for all \( j \geq 1 \) and \( k \geq 2 \). In particular,
\[ ((\Psi + M)^k)_\alpha^\beta = (\Psi^k)_\alpha^\beta + k\Psi_\alpha^\gamma (M^{k-1})_\gamma^\beta + (M^k)_\alpha^\beta \]
\[ = (-c)^{k-2} [(k-1)(n+1)^{k-2}W_\alpha^\beta d\theta - (n+1)^{k-1}c\Psi_\alpha^\beta d\theta \]
\[ - cM_\alpha^\beta d\theta + (n+1)^{k-1}W_\alpha^\beta (\partial_b w)(\overline{\partial_b w})] d\theta^{k-2} \]
for all \( k \geq 1 \). Define
\[ \mathcal{P}_\alpha^\beta := \Psi_\alpha^\beta + M_\alpha^\beta + \delta_\alpha^\beta (i(\partial_b w)(\overline{\partial_b w}) + c d\theta). \]

Observe that
\[ (\mathcal{P}^k)_\alpha^\beta = \sum_{j=0}^{k} \binom{k}{j} c^{k-j-1} ((\Psi + M)^j)_\alpha^\beta (c d\theta + (k-j)i(\partial_b w)(\overline{\partial_b w})) d\theta^{k-j-1}; \]
as \( (c d\theta + (k-j)i(\partial_b w)(\overline{\partial_b w})) d\theta^{k-j-1} = c d\theta^{k-j} + (k-j)i(\partial_b w)(\overline{\partial_b w}) d\theta^{k-j-1} \), we interpret this factor as multiplication by the scalar function \( c \) when \( j = k \) in the
We conclude that (6.11) yields
\[(P^k)_{\alpha\beta} = c^{k-1} \delta_{\alpha}^\beta (c\,d\theta + ki(\partial_\theta w)(\bar{\partial}_b \bar{w}))\,d\theta^{k-1}
\]
\[+ \sum_{j=1}^k (-1)^{j-1} \binom{k}{j} c^{k-1} [(n+1)^{j-1} - k \, M_{\alpha}^{\beta} (\partial_\theta w)(\bar{\partial}_b \bar{w})] d\theta^{k-1}
\]
\[+ i \sum_{j=1}^k (-1)^{j} \binom{k}{j} c^{k-2} \left((n+1)^{j-1} - k \right) M_{\alpha}^{\beta} (\partial_\theta w)(\bar{\partial}_b \bar{w}) d\theta^{k-2}
\]
\[+ \sum_{j=1}^k (-1)^{j} \binom{k}{j} (n+1)^{j-2} c^{k-2} \left((n+2)j - 1 - k(n+1)\right) W_{\alpha}^{\beta} d\theta^{k-1}
\]
for all \(k \geq 1\), where we adopt the convention \(d\theta^{-1} := 0\). Evaluating the summations yields
\[(P^k)_{\alpha\beta} = c^{k-2} \left[c^2 \delta_{\alpha}^\beta d\theta^2 + ic k c \delta_{\alpha}^\beta (\partial_\theta w)(\bar{\partial}_b \bar{w}) d\theta - \frac{(-n)^k - 1}{n+1} c \Psi_{\alpha}^{\beta} d\theta\right]
\[+ c M_{\alpha}^{\beta} d\theta + \frac{(-n)^k - 1 + k(n+1)}{n+1} i M_{\alpha}^{\beta} (\partial_\theta w)(\bar{\partial}_b \bar{w})
\]
\[+ \frac{k(1 - 2(-n)^{k-1})}{n+1} + \frac{1 - (-n)^k}{(n+1)^2} W_{\alpha}^{\beta} d\theta\] d\theta^{k-2}
\]
for all \(k \geq 1\), where we distribute the multiplication by \(d\theta^{k-2}\) and use our convention \(d\theta^{-1} = 0\) to make sense of the case \(k = 1\).

Finally, since \(W_{\alpha}^{\beta}\) and \(P_{\alpha}^{\beta}\) commute, we have that
\[(C^k)_{\alpha\beta} = \left(\frac{n+1}{n+2} c\right)^{k-1} \left[c^{n+1} (P^k)_{\alpha\beta} + kW_{\gamma} (P^{k-1})_{\gamma\beta} \right].
\]
Combining this with Equation (6.12) yields
\[(C^k)_{\alpha\beta} = \left(\frac{n+1}{n+2} c\right)^{k-1} \left[c^{n+1} (P^k)_{\alpha\beta} + kW_{\gamma} (P^{k-1})_{\gamma\beta} \right] + c \delta_{\alpha}^\beta d\theta
\]
\[+ \frac{(-n)^k - 1 + k(n+1)}{n+2} i M_{\alpha}^{\beta} (\partial_\theta w)(\bar{\partial}_b \bar{w})
\]
\[+ \frac{k(n+1)+1}{(n+1)(n+2)} W_{\alpha}^{\beta} d\theta\] d\theta^{k-2}
\]
for all \(k \geq 1\). Using the facts
\[W_{\gamma}^{\gamma} = -(n+1) ic (\partial_\theta w)(\bar{\partial}_b \bar{w}),
\]
\[M_{\gamma}^{\gamma} = c\,d\theta,
\]
\[\Psi_{\gamma}^{\gamma} = -(n+1) c\,d\theta + 2i(\partial_\theta w)(\bar{\partial}_b \bar{w}),
\]
we conclude that
\[(C^k)_{\alpha\beta} = \left(\frac{n+1}{n+2} c\right)^{k} \left[n + (-n)^k \right] c^{2k-1} \left(c\,d\theta + ki(\partial_\theta w)(\bar{\partial}_b \bar{w})\right) d\theta^{k-1}
\]
for all \( k \geq 1 \).

**Step 2. Compute derivatives of \( c_\Phi(S) \) and \( \nabla_\alpha c_\Phi(S) \).**

Recall that
\[
\frac{1}{n!} c_\Phi(S^t) d\theta^n = c_\Phi \left( i(S^t)_\alpha^\beta \mu_\theta \theta^\nu \right).
\]
It follows that
\[
\frac{1}{(n!)^2} \frac{\partial^n}{\partial t^n} c_\Phi(S^t) d\theta^n = c_\Phi(\dot{C}_\alpha^\beta).
\]
Using our assumption \( \Phi = \Phi(\varsigma) \), \( \varsigma_1 = 0 \), we have that
\[
c_\Phi(\dot{C}_\alpha^\beta) = \prod_{k=2}^n (\text{tr } \dot{C}_k^\beta)\varsigma_k.
\]
Using (6.14) and the fact
\[
\int \partial_\beta w(\partial_\alpha w) d\theta^n = -\left( n + 1 \right) c_\Phi d\theta^n,
\]
we deduce that
\[
c_\Phi(\dot{C}_\alpha^\beta) = -n \left( \frac{n+1}{n+2} \right) p(\varsigma) c_\Phi(\dot{C}_\alpha^\beta).
\]
where
\[
p(\varsigma) := \prod_{k=2}^n (n + (-n)^k)^{\varsigma_k}.
\]

Since \( \nabla_\alpha c_\Phi(S^t) = \frac{1}{n+1} c_\Phi(S^t) \nabla_\alpha d\theta^n \), we deduce that
\[
\nabla_\alpha c_\Phi(S^t) = -\left( \frac{n+1}{n+2} \right)^n p(\varsigma) c_\Phi(\dot{C}_\alpha^\beta).
\]

**Step 3. Compute \( i(\dot{C}_\Phi)_{\alpha}^{\beta} \mu_\nu V^\mu_{\beta} \nu \).**

Observe that
\[
\dot{V}_\alpha^\alpha = 0,
\]
\[
W_\alpha^\beta \dot{V}_\beta^\alpha = 0,
\]
\[
M_\alpha^\beta \dot{V}_\beta^\alpha = 0,
\]
\[
\Psi_\alpha^\beta \dot{V}_\beta^\alpha = -\left( n + 1 \right) c_\Phi \left( \int \partial_\alpha w(\partial_\beta w) d\theta^n + 3 \right) \frac{n}{n+2} ic_\Phi(\dot{C}_k^\beta) \dot{V}_\beta^\alpha.
\]
Combining this with Equation (6.13) yields
\[
\text{tr } \dot{C}^{k-1} \dot{V} := (\dot{C}^{k-1})_{\alpha}^\beta \dot{V}_\alpha^\beta
\]
\[
= -\left( \frac{n+1}{n+2} \right)^k \frac{(n+3)(n+(-n)^k)}{n+1} ic_\Phi(\dot{C}_k^\beta) \dot{V}_\beta^\alpha
\]
for all \( k \geq 1 \).

We now compute the derivatives in \( t \) of \( i(S_\Phi)_{\alpha}^\beta \mu_\nu V^\mu_{\beta} \nu \) using the fact that
\[
\frac{1}{(n-1)!} \left( i(S_\Phi)_{\alpha}^\beta \mu_\nu V^\mu_{\beta} \nu \right) d\theta^{n-1} = ic_\Phi(\dot{C}^\alpha_{\nu_{\mu \theta}} \theta^\beta V_{\beta}^\mu_{\nu \theta})
\]
for all \( t \), where
\[
c_\Phi(\dot{C})_{\alpha_{n-1}}(Y^\beta_{\alpha_1} \cdots Y^\beta_{n-1} Z^\alpha_{\beta_1} \cdots Z^\alpha_{\beta_n}) := \Phi_{\alpha_1 \cdots \alpha_n}^\beta \beta_1 \cdots \beta_n \}
\]
for all invariant polynomials $\Phi$ of degree $n$, all $\text{End}(T^{1,0})$-valued two-forms $Y_\alpha^\beta$, and all $\text{End}(T^{1,0})$-valued one-forms $Z_\alpha^\beta$. Note that if $\Phi = \Phi(\varsigma)$, then

$$c_{\Phi,n-1}(Y,Z) = \frac{1}{n} \sum_{k=2}^{n} k\varsigma_k(\text{tr} Y^k)^{n-k-1}(\text{tr} Y^{k-1} Z) \prod_{j\neq k} (\text{tr} Y^j)^{\varsigma_j}.$$  

Using Equations (6.14) and (6.17) and our assumption $\Phi = \Phi(\varsigma)$, we compute that

$$ic_{\Phi,n-1}(\dot{\varsigma}_\alpha^\beta, \dot{\varsigma}_\beta^\alpha) = \frac{n+3}{n+1} \left( \frac{n+1}{n+2} \right)^n p(\varsigma)c^{2n-1}w(\partial_\varsigma w) d\theta^{n-1}.$$  

In particular,

$$(6.18) \quad \frac{i}{(n-1)!n!} \frac{\partial^n}{\partial s^n} (S_\varsigma^\Phi)_{\alpha}^\beta (V^t)_{\mu}^\nu = \frac{n+3}{n+1} \left( \frac{n+1}{n+2} \right)^n p(\varsigma)c^{2n-1}w_{\alpha}. $$

**Step 4.** Compute derivatives of $X_\alpha^\Phi$ and $\text{Re} \nabla^\alpha X_\alpha^\Phi$.

We now compute $X_\alpha^\Phi$ for $\Phi = \Phi(\varsigma)$. Equations (6.16) and (6.18) imply that

$$\frac{1}{(n)!^2} \frac{\partial^n}{\partial s^n} F_\varsigma^*(V^\alpha X_\alpha^\Phi)_{\theta}^\gamma = \frac{3}{n} \left( \frac{n+1}{n+2} \right)^n p(\varsigma)c^{2n-1}(3n(n+1)c+3n-1).$$

Since $p(\varsigma) \neq 0$, we conclude that Equation (6.5) holds. 

**Remark 6.4.** Reiter and Son [26] Equation (4.4)] computed the Chern tensor of the real ellipsoids $\Omega_s = \{(w,z) \in \mathbb{C} \times \mathbb{C}^n \mid 1 - |z|^2 - |w|^2 - s \text{Re} w^2 > 0\}$ with respect to the unique pseudohermitian structure which is volume-normalized with respect to $dw \wedge dz^1 \wedge \cdots \wedge dz^n|_{\Omega_s}$. Their computation shows that

$$\frac{\partial}{\partial s} F_\varsigma^*(S_\varsigma^\Phi)_{\alpha \beta \gamma \delta} = 0,$$

and

$$\frac{\partial^2}{\partial s^2} F_\varsigma^*(S_\varsigma^\Phi)_{\alpha \beta \gamma \delta} = 2 tf w_\alpha \overline{w_\beta} w_\gamma \overline{w_\delta},$$

where $F_\varsigma : \partial \Omega_0 \to \partial \Omega_s$ is a one-parameter family of contact diffeomorphisms with $F_0 = \text{Id}$. (Recall that we denote $w_\alpha = Z_\alpha w$, whereas Reiter and Son write their computation in terms of $Z_\alpha := \partial_\alpha - \frac{\partial_\alpha}{\rho_\alpha} \partial w$, where $\rho_\alpha$ is the given defining function for $\partial \Omega_s$.) In particular, our proof of Theorem 6.3 shows that, for $s$ close to zero, the invariants $I^s_\Phi(\varsigma)$ on the real ellipsoids $\Omega_s$ give counterexamples to the Hirachi conjecture when $\varsigma_1 = 0$.

7. **Counterexample via Calabi–Yau manifolds**

In this section, we prove the following result:

**Theorem 7.1.** For $n \geq 2$, there exists a closed $(2n+1)$-dimensional pseudo-Einstein manifold $(M, T^{1,0}, \theta)$ such that

$$R_{\alpha \beta} = 0, \quad A_{\alpha \beta} = 0, \quad \text{Re} \nabla^\alpha X_\alpha^{(n)} \neq 0.$$


We construct such a CR manifold as a certain circle bundle over a Calabi–Yau manifold. Let \((Y, \omega)\) be an \(n\)-dimensional Kähler manifold. There exists a smooth function \(f_\omega\) on \(Y\) such that the \(n\)-th Chern form \(c_n(\omega)\) with respect to \(\omega\) coincides with \(f_\omega \cdot \omega^n\).

**Theorem 7.2.** For each positive integer \(n \geq 2\), there exists an \(n\)-dimensional closed, connected Ricci-flat Kähler manifold \((Y, \omega)\) such that \([\omega/2\pi] \in H^2(Y; \mathbb{Z})\) and \(f_\omega\) is non-constant.

**Proof of Theorem 7.2 assuming Theorem 7.1** Since \([\omega/2\pi] \in H^2(Y; \mathbb{Z})\) and \(Y\) is Kähler, there exists a holomorphic Hermitian line bundle \((L, h)\) over \(Y\) such that \(\omega = -i\Theta_h\). Consider the circle bundle \((M, T^{1,0}, \theta)\) associated with \((Y, L, h)\). Since \(\omega\) is Ricci-flat, \(\theta\) is a contact form satisfying

\[
R_{\alpha\beta\gamma\delta} = 0, \quad A_{\alpha\beta} = 0;
\]

in particular, \(V_{\alpha\beta\gamma} = 0\). Moreover, \(S_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}\), and hence \(c(n)(S)\) is a nonzero constant multiple of \(f_\omega\). In particular, \(c(n)(S)\) is non-constant. Therefore

\[
\text{Re} \nabla^\alpha X^{(n)}_{\alpha} = -\frac{1}{n} \text{Re} \nabla^\alpha \nabla_{\alpha} c(n)(S) = -\frac{1}{2n^2} \Delta_{\Theta} c(n)(S) \neq 0,
\]

which completes the proof. 

It remains to show Theorem 7.2.

**Proof of Theorem 7.2** As we will see in the following two subsections, such \((Y, \omega)\) exists in the cases of \(n = 2\) and \(3\). Since the conditions in Theorem 7.2 are closed under the product, we can construct \((Y, \omega)\) for any \(n \geq 2\).

### 7.1. Two-dimensional case.

Consider the two-dimensional complex torus \(T = \mathbb{C}^2/(\mathbb{Z} + i\mathbb{Z})^2\). Multiplication by \(-1\) on \(\mathbb{C}^2\) induces an involution \(\iota\) on \(T\) that has 16 fixed points \(p_1, \ldots, p_{16}\). Let \(\sigma: T' \to T\) be obtained from \(T\) by blowing up at \(p_1, \ldots, p_{16}\). The involution \(\iota\) lifts to an involution \(\iota'\) on \(T'\), and the quotient \( p: T' \to Y = T'/\langle \iota' \rangle\) is a closed \(K3\) surface; this is called the *Kummer surface associated to \(T\)* [2 Chapter V.16]. The space \(Y\) contains 16 complex projective curves \(E_1, \ldots, E_{16}\) corresponding to \(p_1, \ldots, p_{16}\). It is known that the Euler characteristic of any \(K3\) surface is equal to 24 [2 Chapter VIII.3].

Let \(\omega^{(0)}\) be the Kähler form on \(T\) induced by \(2\pi i \sum_{j=1}^2 dz^j \wedge d\bar{z}^j\) on \(\mathbb{C}^2\); the coefficient is chosen so that \([\omega^{(0)}/2\pi] \in H^2(T; \mathbb{Z})\). For \(0 < s \ll 1\), the cohomology class \(p_0\sigma^*[\omega^{(0)}] - s \sum_{k=1}^{16} c_1(O(E_k))\) contains a unique Ricci-flat Kähler metric \(\omega_s\) on \(Y\) such that \(\omega_s\) converges smoothly to a flat Kähler metric as \(s \to +0\) on any compact subset of \(Y \setminus \bigcup_{k=1}^{16} E_k\) [21 Chapter 2]. Note that

\[
0 < \int_Y \omega_s^2 = \int_Y (p_0\sigma^*[\omega^{(0)}])^2 - 32s^2 < \int_Y (p_0\sigma^*[\omega^{(0)}])^2.
\]

Suppose that \(f_s := f_\omega\) is constant for any \(0 < s \ll 1\). From the Gauss–Bonnet–Chern formula, it follows that

\[
24 = \int_Y c_2(\omega_s) = f_s \int_Y \omega_s^2 < f_s \int_Y (p_0\sigma^*[\omega^{(0)}])^2;
\]

that is,

\[
f_s > 24 \left[ \int_Y (p_0\sigma^*[\omega^{(0)}])^2 \right]^{-1}.
\]
However, since $\omega_s$ converges smoothly to a flat Kähler metric as $s \to +0$ on any compact subset of $Y \setminus \bigcup_{k=1}^{16} E_k$, we have $f_s \ll 1$ for sufficiently small $s$; this is a contradiction. Hence $f_s$ is non-constant for sufficiently small $s$. If we take a sufficiently large positive integer $N$, the Ricci-flat Kähler metric $\omega = N \cdot \omega_{2\pi/N}$ satisfies $[\omega/2\pi] \in H^2(Y; \mathbb{Z})$ and $f_\omega$ is non-constant.

7.2. Three-dimensional case. Let $\zeta$ be a primitive cubic root of one, and denote by $E_\zeta$ the elliptic curve $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\zeta)$. Multiplication by $\zeta$ on $\mathbb{C}^3$ induces a biholomorphism $\Phi_\zeta$ on $E_\zeta^3$. This map satisfies $\Phi_\zeta^3 = \text{Id}$ and has 27 fixed points $p_1, \ldots, p_{27}$. Let $\sigma: \bar{Y} \to E_\zeta^3$ be obtained from $E_\zeta^3$ by blowing up at $p_1, \ldots, p_{27}$. The biholomorphism $\Phi_\zeta$ lifts to a biholomorphism $\Phi$ on $\bar{Y}$ satisfying $\Phi^3 = \text{Id}$, and the quotient $p: \bar{Y} \to Y = \bar{Y}/(\Phi, \zeta)$ is a closed smooth Calabi–Yau threefold, called a Kummer threefold. The space $Y$ contains 27 complex projective planes $E_1, \ldots, E_{27}$ corresponding to $p_1, \ldots, p_{27}$. The Euler characteristic of $Y$ is 72 [23, Theorem 5(i)].

Let $\omega^{(0)}$ be the Kähler form on $E_\zeta^3$ induced by $(2\pi \sqrt{3})i \sum_{j=1}^3 dz_j \wedge d\overline{z}_j$ on $\mathbb{C}^3$; the coefficient is chosen so that $[\omega^{(0)}/2\pi] \in H^2(E_\zeta^3; \mathbb{Z})$. For $0 < s \ll 1$, the cohomology class $p_\ast \sigma^* [\omega^{(0)}] - s \sum_{k=1}^{27} c_1(\mathcal{O}(E_k))$ contains a unique Ricci-flat Kähler metric $\omega_s$ on $Y$ such that $\omega_s$ converges in $C^{4,\alpha}$ to a flat Kähler metric as $s \to +0$ on any compact subset of $Y \setminus \bigcup_{k=1}^{27} E_k$ [23, Section 3.1]. Note that

$$0 < \int_Y \omega_s^3 = \int_Y (p_\ast \sigma^* [\omega^{(0)}])^3 - 243s^3 < \int_Y (p_\ast \sigma^* [\omega^{(0)}])^3.$$

Suppose that $f_s := f_{\omega_s}$ is constant for any $0 < s \ll 1$. From the Gauss–Bonnet–Chern formula, it follows that

$$72 = \int_Y c_3(\omega_s) = f_s \int_Y \omega_s^3 < f_s \int_Y (p_\ast \sigma^* [\omega^{(0)}])^3;$$

that is,

$$f_s > 72 \left[ \int_Y (p_\ast \sigma^* [\omega^{(0)}])^3 \right]^{-1}.$$

However, since $\omega_s$ converges in $C^{4,\alpha}$ to a flat Kähler metric as $s \to +0$ on any compact subset of $Y \setminus \bigcup_{k=1}^{27} E_k$, we have $f_s \ll 1$ for sufficiently small $s$; this is a contradiction. Hence $f_s$ is non-constant for sufficiently small $s$. If we take a sufficiently large positive integer $N$, the Ricci-flat Kähler metric $\omega = N \cdot \omega_{2\pi/N}$ satisfies $[\omega/2\pi] \in H^2(Y; \mathbb{Z})$ and $f_\omega$ is non-constant.

8. The $T'$-curvature of the boundary of a Reinhardt domain

For $r > 0$, let $M_r$ be the boundary of the bounded Reinhardt domain

$$\Omega_r := \{ w = (w^0, \ldots, w^n) \in \mathbb{C}^{n+1} \mid \rho_r(w) > 0 \},$$

where

$$\rho_r(w) := \frac{1}{2} - \frac{1}{2r^2} \sum_{j=0}^n (\log |w^j|)^2.$$

We would like to compute the total $T'$-curvatures for $M_r$. To this end, consider the holomorphic map

$$\psi_r: \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}; (z^0, \ldots, z^n) \mapsto (\exp(2rz^0), \ldots, \exp(2rz^n)).$$
The pull-back $\psi_r^* \rho_r(z)$ coincides with 

$$\rho(z) := \frac{1}{2} - 2 \sum_{j=0}^{n} (\text{Re} \ z^j)^2,$$

and the pre-image of $\Omega_r$ by $\psi_r$ is the tube domain

$$\Omega = \left\{ z = (x^0 + iy^0, \ldots, x^n + iy^n) \in \mathbb{C}^{n+1} \middle| |x|^2 = \sum_{j=0}^{n} (x^j)^2 < \frac{1}{4} \right\}.$$ 

The holomorphic map $\psi_r$ induces also a pseudohermitian map

$$(M := \partial \Omega, T^{1,0}, \theta := \text{Im} \overline{\partial} \rho|_M) \rightarrow (M_r, T^{1,0}_r, \theta_r := \text{Im} \overline{\partial} \rho_r|_{M_r}).$$

where $T^{1,0} := T^{1,0}\mathbb{C}^{n+1} \cap (TM \otimes \mathbb{C})$ and $T^{1,0}_r := T^{1,0}\mathbb{C}^{n+1} \cap (TM_r \otimes \mathbb{C})$. The group $G = O(n+1) \ltimes (\mathbb{R})^{n+1}$ acts on $\mathbb{C}^{n+1}$ as a subgroup of the complex affine transformation group $GL(n+1, \mathbb{C}) \ltimes \mathbb{C}^{n+1}$, and its action preserves $\rho$. In particular, the pseudohermitian manifold $(M, T^{1,0}, \theta)$ is homogeneous with respect to the above $G$-action. Hence it suffices to consider a given point $p := (1/2, 0, \ldots, 0) \in M$ for computing pseudo-Hermitian invariants. We set $x' := (x^1, \ldots, x^n)$. Let

$$\xi := \frac{1}{2|x|^2} \sum_{j=0}^{n} x^j \frac{\partial}{\partial x^j} \in \Gamma(T^{1,0}\mathbb{C}^{n+1}|_M).$$

This vector field satisfies

$$\xi \rho = 1, \quad \xi \cdot \overline{\partial} \rho = -\frac{1}{4|x|^2} \overline{\partial} \rho.$$

For $\alpha \in \{1, \ldots, n\}$, the $(1, 0)$-forms

$$\theta^\alpha := dz^\alpha + \frac{1}{2|x|^2} x^\alpha \partial \rho$$

annihilate $\xi$ and their restriction to $M$ gives an admissible coframe. A calculation shows that the Levi form $h_{\alpha\beta}$ is given by

$$h_{\alpha\beta} = \delta_{\alpha\beta} + \frac{x^\alpha x^\beta}{(x^0)^2} = \delta_{\alpha\beta} + 4x^\alpha x^\beta + O(|x'|^4).$$

A similar computation to that in the proof of [21] Proposition 5.2] gives that

$$\omega_{\alpha}^\beta = -i\delta_{\alpha}^\beta \theta + 2x^\beta \theta^\alpha + 2x^\beta \theta^\alpha + O(|x'|^2),$$

$$A_{\alpha \beta} = -i\delta_{\alpha \beta} + O(|x'|^2).$$

At $p$, the pseudohermitian torsion $A_{\alpha \beta}$ satisfies

$$\nabla_{\gamma} A_{\alpha \beta} = 0, \quad \nabla_{\gamma} A_{\alpha \beta} = 0, \quad \nabla_{\alpha} A_{\alpha \beta} = 2iA_{\alpha \beta}, \quad A_{\alpha \beta} A^\beta_{\gamma} = h_{\alpha \gamma}.$$ 

Since both sides of these equalities are tensorial and $(M, T^{1,0}, \theta)$ is homogeneous, these in fact hold on the whole of $M$. Similarly, the curvature form $\Pi_{\alpha}^\beta$ at $p$ is given by

$$\Pi_{\alpha}^\beta = (\delta_{\alpha \beta} h_{\rho \bar{\sigma}} + \delta_{\rho} h_{\alpha \bar{\sigma}} - A_{\alpha \rho} A_{\beta \sigma}) \theta^\rho \wedge \theta^\sigma - i\gamma_{\alpha \gamma} \wedge \theta^\beta + i\theta_{\alpha \gamma} \wedge \tau^\beta.$$
The right hand side is tensorial, and so this equality holds on the whole of \( M \). Local pseudohermitian invariants can be calculated explicitly:

\[
P_{\alpha \bar{\beta}} = \frac{n}{2(n+1)} h_{\alpha \bar{\beta}},
\]

\[
S_{\alpha \bar{\beta} \gamma \bar{\sigma}} = \frac{1}{n+1} (h_{\alpha \bar{\beta}} h_{\gamma \bar{\sigma}} + h_{\alpha \bar{\sigma}} h_{\gamma \bar{\beta}}) - A_{\alpha \gamma} A_{\bar{\beta} \bar{\sigma}},
\]

\[
V_{\alpha \bar{\beta} \gamma} = 0,
\]

\[
U_{\alpha \bar{\beta}} = 0.
\]

In particular, \( \theta \) (or \( \theta_r \)) is a pseudo-Einstein contact form with constant scalar curvature but nonvanishing pseudohermitian torsion. Moreover, the Chern tensor is parallel:

\[
\nabla_{\rho} S_{\alpha \bar{\beta} \gamma \bar{\sigma}} = 0, \quad \nabla_{\bar{\rho}} S_{\alpha \bar{\beta} \gamma \bar{\sigma}} = 0, \quad \nabla_0 S_{\alpha \bar{\beta} \gamma \bar{\sigma}} = 0.
\]

**Theorem 8.1.** The total \( T_{\Phi(\iota)}' \)-curvature \( T_{\Phi(\iota)} \) for \( M_r \) is given by

\[
T_{\Phi(\iota)} = -(n!)^2 \text{Vol}(S^n(1)) \left( \frac{2\pi}{(n+1)r} \right)^{n+1} \prod_{k=1}^{n} [(n+2)(1-(n+2)^{k-1})] c_{\iota},
\]

where \( \text{Vol}(S^n(1)) \) is the volume of the unit sphere in \( \mathbb{R}^{n+1} \).

**Proof.** Set

\[
\Sigma_{\alpha \bar{\beta}} := \frac{1}{n+1} (-i \delta_{\alpha}^{\beta} d\theta + \theta^{\beta} \wedge \theta_{\alpha}), \quad L_{\alpha \bar{\beta}} := -\tau_{\alpha} \wedge \tau^{\beta},
\]

which satisfy \( \Xi_{\alpha \bar{\beta}} = \Sigma_{\alpha \bar{\beta}} + L_{\alpha \bar{\beta}} \), where \( \Xi_{\alpha \bar{\beta}} \) is defined by Equation (3.2). These \( \Sigma_{\alpha \bar{\beta}} \) and \( L_{\alpha \bar{\beta}} \) satisfy

\[
\text{tr } \Sigma = -i d\theta, \quad \text{tr } L = i d\theta,
\]

\[
\Sigma_{\alpha \gamma} \wedge \Sigma_{\gamma \beta} = -\frac{i}{n+1} d\theta \wedge \Sigma_{\alpha \beta},
\]

\[
L_{\alpha \gamma} \wedge \Sigma_{\gamma \beta} = \Sigma_{\alpha \gamma} \wedge L_{\gamma \beta} = -\frac{i}{n+1} d\theta \wedge L_{\alpha \beta},
\]

\[
L_{\alpha \gamma} \wedge L_{\gamma \beta} = -i d\theta \wedge L_{\alpha \beta}.
\]

Hence

\[
(\Xi^k)_{\alpha \bar{\beta}} = (\Sigma^k)_{\alpha \bar{\beta}} + \sum_{j=1}^{k} \binom{k}{j} (\Sigma^{k-j})_{\alpha \gamma} \wedge (L^j)_{\gamma \beta}
\]

\[
= \frac{1}{(n+1)^{k-1}} (-i d\theta)^{k-1} \wedge \Sigma_{\alpha \beta} + \sum_{j=1}^{k} \binom{k}{j} \left( \frac{1}{n+1} \right)^{k-j} (-i d\theta)^{k-1} \wedge L_{\alpha \beta}
\]

\[
= (-i d\theta)^{k-1} \wedge \left[ \frac{1}{(n+1)^{k-1}} \Sigma_{\alpha \beta} + \frac{(n+2)^{k-1}-1}{(n+1)^k} L_{\alpha \beta} \right],
\]

and so

\[
\text{tr } \Xi^k = \frac{(n+2)[1-(n+2)^{k-1}]}{(n+1)^k} (-i d\theta)^k.
\]

Since

\[
c_{\Phi(\iota)}(i \Xi_{\alpha \bar{\beta}}) = \frac{1}{n!} c_{\Phi(\iota)}(S) d\theta^n,
\]
we have
\[ c_{\Phi(\varsigma)}(S) = n! \prod_{k=1}^{n} \left( \frac{(n+2)(1-(n+2)^{k-1})}{(n+1)^{k}} \right)^{\varsigma_k} \]
\[ = \frac{n!}{(n+1)^{n+1}} \prod_{k=1}^{n} [(n+2)(1-(n+2)^{k-1})]^{\varsigma_k}. \]

Therefore the \( I'_{\Phi(\varsigma)} \)-curvature of \( M \) is given by
\[ I'_{\Phi(\varsigma)} = -\frac{n!}{(n+1)^{n+1}} \prod_{k=1}^{n} [(n+2)(1-(n+2)^{k-1})]^{\varsigma_k}. \]

In particular, \( I'_{\Phi(\varsigma)} \) is constant, and equal to zero if and only if \( \varsigma_1 \neq 0 \).

In light of Alexakis’ characterization of global conformal invariants \( \Pi \), it is natural to expect that a weaker version of Conjecture 1.1 is true. One way to weaken Conjecture 1.1 is to allow, in addition to local CR invariants, pseudohermitian scalar invariants \( I \) for which \( P \subset \ker D_{\theta}I \) for all pseudo-Einstein contact forms \( \theta \). We propose allowing an even weaker type of invariant.

9. Concluding remarks

In light of Alexakis’ characterization of global conformal invariants \( \Pi \), it is natural to expect that a weaker version of Conjecture 1.1 is true. One way to weaken Conjecture 1.1 is to allow, in addition to local CR invariants, pseudohermitian scalar invariants \( I \) for which \( P \subset \ker D_{\theta}I \) for all pseudo-Einstein contact forms \( \theta \). We propose allowing an even weaker type of invariant.
**Definition 9.1.** Fix $n \in \mathbb{N}$. A homogeneous pseudohermitian scalar invariant $I^\theta$ is a **local secondary invariant** if

$$
(9.1) \quad \int_M u I^\theta \bar{\theta} \wedge d\bar{\theta}^n = \int_M u I^{\bar{\theta}} \theta \wedge d\theta^n
$$

for any pseudo-Einstein contact forms $\theta$ and $\bar{\theta}$ on a closed CR manifold $(M^{2n+1}, T^{1,0})$ and any $u \in \mathcal{P}$.

Note that if $I$ is homogeneous of degree $-n-1$ in $\theta$ and if $\mathcal{P} \subset \ker D\theta$ for all pseudo-Einstein contact forms $\theta$, then it is a local secondary invariant. We propose the following weaker version of Conjecture 1.1.

**Conjecture 9.2.** Let $I$ be a natural pseudohermitian scalar invariant whose total integral is a secondary CR invariant. Then there is a constant $c \in \mathbb{R}$ such that

$$
I = c Q' + \text{(local secondary invariant)} + \text{(divergence)}.
$$

There are two motivations behind Definition 9.1, and hence Conjecture 9.2.

Our first motivation is in analogy with the $Q'$-curvature. Let $\mathcal{P}^\perp$ denote the space of smooth volume forms which annihilate $\mathcal{P}$; i.e. given a closed pseudohermitian manifold $(M^{2n+1}, T^{1,0}, \theta)$, we set

$$
\mathcal{P}^\perp := \left\{ \psi \theta \wedge d\theta^n \left| \int_M u \psi \theta \wedge d\theta^n = 0 \text{ for all } u \in \mathcal{P} \right. \right\}.
$$

Note that $\psi \theta \wedge d\theta^n \in \mathcal{P}^\perp$ if and only if $\psi$ is $L^2$-orthogonal to $\mathcal{P}$ with respect to $\theta$, so that this definition coincides with the definition of $\mathcal{P}^\perp$ given in the introduction. Since $\mathcal{P}^\perp$ is CR invariant, Definition 9.1 is equivalent to the requirement that $I^\theta \bar{\theta} \wedge d\bar{\theta}^n$ is independent of the choice of pseudo-Einstein contact form modulo $\mathcal{P}^\perp$. This is analogous to how one realizes the $Q'$-curvature as having a linear transformation law when working modulo $\mathcal{P}^\perp$; see Equation (1.1).

Our second motivation is speculation based on the compatibility of Definition 9.1 with the heuristic construction of “primed” invariants by analytic continuation in the dimension (cf. [9, 10, 25]). Suppose that $I$ is a family of local CR invariants of weight $-n-1$ defined on all CR manifolds of CR dimension $d \geq n$, and moreover suppose that $I^\theta = 0$ for any pseudo-Einstein contact form in CR dimension $n$. Suppose further that the formal limit

$$
(9.2) \quad I' = \lim_{d \to n} \frac{1}{d-n} I^\theta
$$

makes sense when restricted to pseudo-Einstein manifolds. The fact that $I$ is CR invariant implies that

$$
\int_{M^{2d+1}} u \bar{\theta} \wedge d\bar{\theta}^d = \int_{M^{2d+1}} u I^\theta \wedge d\theta^d
$$

for all closed CR manifolds $(M^{2d+1}, T^{1,0})$, all contact forms on $(M, T^{1,0})$, and all (real) densities $u$ of weight $n-d$; i.e. all equivalence classes $u = [u, \theta]$ subject to the relation $[u, \theta] = [e^{(n-d)T} u, e^T \theta]$. Dividing both sides by $d-n$, restricting to pseudo-Einstein contact forms, taking the limit $d \to n$, and restricting to CR pluriharmonic functions then implies that $I'$ is a local secondary invariant. The restriction to CR pluriharmonic functions is for symmetry reasons, as two pseudo-Einstein contact forms $\theta$ and $\bar{\theta} = e^T \theta$ are necessarily such that $\mathcal{T} \in \mathcal{P}$. 
Unfortunately, none of our nontrivial $\mathcal{I}^d_\Phi$-curvatures seem to be local secondary invariants in the sense of Definition 9.1. This observation arises from two heuristics.

First, the Case–Gover construction [9] of $\mathcal{I}$ in CR dimension two arises from analytic continuation in the dimension after working modulo divergences. Since working modulo divergences breaks CR invariance, we expect $\mathcal{I}$ to only be a local secondary invariant modulo a divergence. A similar interpretation to the higher-dimensional $\mathcal{I}_\Phi$-curvatures was given by Marugame [25].

Second, the $\mathcal{I}^d_\Phi$-curvatures can be realized via analytic continuation without working modulo divergences, but by starting with variational pseudohermitian scalar invariants:

Let $\Phi$ be an invariant polynomial of degree $n$ and let $(M^{2d+1}, T^{1,0}, \theta)$ be a pseudohermitian manifold of CR dimension $d$. Define

$$c_\Phi(S) := \delta^{\alpha_1 \cdots \alpha_n}_{\alpha n_1 \cdots \alpha n_n} \Phi^{\mu_1 \cdots \mu_n} \Sigma_{\beta_1} \alpha_1 \mu_1 \cdots S_{\beta_n} \alpha_n \mu_n,$$

$$\alpha_0 \hat{X}_{\alpha} := i(S^{\alpha})_{\beta} \mu V^{\beta} \nu - \frac{1}{d \theta} \nabla_{\alpha} c_\Phi(S),$$

$$\mathcal{I}_\Phi := \frac{2}{n} \mathcal{U}_{\alpha} \beta P_{\beta} \alpha + (d - n) \left[ \frac{1}{d \theta (2n - d)} (\Delta c_\Phi(S) - 2n P c_\Phi(S)) \right]$$

$$+ (T^\Phi)_{\alpha} \mu_1 \mu_2 \nu_1 \nu_2 ((n - 1) V^{\beta} \mu_1 \nu_1 V^{\mu_2} \nu_2 - S_{\beta} \alpha \mu_1 \nu_1 U^{\nu_2} \nu_2),$$

where

$$(S^\Phi)_{\alpha} \beta \mu := \delta^{\beta_1 \cdots \beta_n}_{\alpha n_1 \cdots \alpha n_n} \Phi^{\mu_1 \cdots \mu_n} S_{\beta_1} \alpha_1 \mu_1 \cdots S_{\beta_n} \alpha_n \mu_n,$$

$$(T^\Phi)_{\alpha} \mu_1 \mu_2 \nu_1 \nu_2 := \delta^{\beta_1 \cdots \beta_n}_{\alpha n_1 \cdots \alpha n_n} \Phi^{\mu_1 \cdots \mu_n} S_{\beta_1} \alpha_1 \mu_1 \cdots S_{\beta_n} \alpha_n \mu_n,$$

$$\mathcal{U}_{\alpha} \beta := \delta^{\beta_1 \cdots \beta_n}_{\alpha n_1 \cdots \alpha n_n} \Phi^{\mu_1 \cdots \mu_n} S_{\beta_1} \alpha_1 \mu_1 \cdots S_{\beta_n} \alpha_n \mu_n - \frac{d - n}{d} \delta_{\alpha} \beta c_\Phi(S).$$

Note that when $d = n$, each of $c_\Phi(S)$, $\hat{X}_{\alpha}$, $(S^\Phi)_{\alpha} \beta \mu$, and $(T^\Phi)_{\alpha} \mu_1 \mu_2 \nu_1 \nu_2$ recovers our original definitions given in the introduction. Moreover, note that $\mathcal{U}_{\alpha} \beta$ is trace-free for all $d$ and that $U_{\alpha} \beta = 0$ when $d = n$. These observations imply that $\mathcal{U}_{\alpha} \beta P_{\beta} \alpha = 0$ on all pseudo-Einstein manifolds. Indeed, by restricting $\mathcal{I}_\Phi$ to pseudo-Einstein manifolds and formally taking a dimensional limit, we have that

$$\lim_{d \to n} \frac{1}{d \theta(n + 1)} \mathcal{I}_\Phi = \mathcal{I}^d_\Phi,$$

that is, the $\mathcal{I}^d_\Phi$-curvature can be interpreted as the secondary invariant associated to $\mathcal{I}_\Phi$ via analytic continuation in the dimension, analogous to the heuristic interpretation of the $Q^d$-curve [10, 18].

One nice property of $\mathcal{I}_\Phi$ is that it is a variational pseudo-Einstein invariant. More precisely, using the identity

$$\nabla_{\nu} U_{\alpha} \beta = n(d - n)X^\Phi_{\alpha},$$

it is straightforward to compute that

$$e^{(n+1)\gamma} \mathcal{I}^\gamma_\Phi = \mathcal{I}_\Phi - \frac{2}{n} \text{Re} \nabla_{\gamma} (\mathcal{U}_{\alpha} \beta Y_{\beta})$$

for all pseudohermitian manifolds $(M^{2d+1}, T^{1,0}, \theta)$ and all $\hat{\theta} := e^{\gamma} \theta$, $\gamma \in C^\infty(M)$. It follows that

$$\frac{d}{dt} \bigg|_{t=0} \int_M (\mathcal{I}_\Phi)^\theta_{\hat{\theta}} \wedge d\theta^d = (d - n) \int_M \mathcal{I}_\Phi \theta \wedge d\theta^d$$
for all one-parameter families \( \theta_t = e^{tY} \theta \) of contact forms on \((M^{2d+1}, T^{1,0})\).

Together with the realization of \( T^* \) as the limit of Equation (6.3), the previous paragraph suggests that the \( T^*_\alpha \)-curvature should be variational in the space of pseudo-Einstein contact forms. More precisely, we expect that there is a trace-free Hermitian tensor \( \omega_{\alpha \bar{\beta}} \) such that

\[
e^{-2(n+1)} T^*_\alpha = T^*_\alpha + 2 \Re \nabla^\gamma (\omega_{\gamma \beta} \mathcal{Y}_\beta)
\]

for all pseudo-Einstein contact forms \( \theta \) and \( \widetilde{\theta} = e^Y \theta \) on \((M^{2n+1}, T^{1,0})\). By Equation (9.3), one may formally think of \( \omega_{\alpha \bar{\beta}} \) as the limit of \( \frac{1}{2n} d\mu_{\alpha \bar{\beta}} \) as \( d \to n \). By Theorem 1.3, the transformation formula of Equation (9.7) is equivalent to asking that the real \((2n-1)\)-form

\[
\omega := i \omega_{\alpha \beta} \theta \wedge \theta^\alpha \wedge \theta^\beta \wedge d\theta^{n-2}
\]

is such that

\[-(n-1) \partial_0 \omega = X_0 \theta \wedge \theta^\alpha \wedge d\theta^{n-1},\]

where \( \partial_0 \omega := i \nabla^\gamma \omega_{\alpha \bar{\beta}} \theta \wedge \theta^\gamma \wedge \theta^\alpha \wedge \theta^\beta \wedge d\theta^{n-2} \). This conclusion has an interpretation in terms of the bigraded Rumin complex \([12, 13]\) which is stronger than the fact, established in the proof of Theorem 1.4, that \( [\xi_0] = 0 \) in \( H^{2n}(M; \mathbb{R}) \).

Suppose that the real \((2n-1)\)-form \( \omega \) exists. If there is a natural \(2n\)-form \( \zeta := \zeta_0 \theta \wedge \theta^\alpha \wedge d\theta^{n-1} \) such that

\[
\tilde{\zeta} = \zeta + \partial_0 \mathcal{Y} \wedge \omega
\]

for all pseudo-Einstein contact forms \( \theta \) and \( \widetilde{\theta} = e^Y \theta \), then \( T^*_\alpha - 2(n-1) \Re \nabla^\gamma \zeta_\gamma \) is a local secondary invariant in the sense of Conjecture [14]. We do not expect that \( \omega \) and \( \zeta \), if they exist, are natural. Instead, we hope that they can be canonically defined in terms of a pseudo-Einstein contact form.

The previous two paragraphs are pure speculation, intended to suggest a path towards better understanding the \( T^*_\alpha \)-curvatures and Conjecture 9.2. We conclude by proving that the \( T^*_\alpha \)-curvatures are not local secondary invariants, and thus providing further justification for the speculations above.

**Proposition 9.3.** Let \((M, T^{1,0}, \theta)\) and \( \Phi \) be as in Theorem 6.3. Then \( T^*_\Phi \) is not a local secondary invariant in the sense of Definition 9.7.

**Proof.** Note that, since \( X^{\Phi}_\alpha \) is a CR invariant, it suffices to find a CR manifold \((M^{2n+1}, T^{1,0})\) which admits a pseudo-Einstein contact form and also admits functions \( u, v \in P \) such that \( \int u \Re X^{\Phi}_\alpha v^\alpha \neq 0 \). We accomplish this by computing

\[
D := \frac{d^n}{dt^n} \int_{S^{2n+1}} u \Re (X^{\Phi}_{\alpha}) u^\alpha \theta_t \wedge d\theta_t ,
\]

where \((S^{2n+1}, T^{1,0}, \theta_t)\) is as in Theorem 6.3 and \( u = 2 \Re w \). Note that \( u \) is a CR pluriharmonic function on \( S^{2n+1} \). A straightforward computation using Equation (6.19) yields

\[
\frac{1}{(n!)^2} D = \frac{3(n+1)}{2n} \left( \frac{n+1}{n+2} \right)^n p(s) \int_{S^{2n+1}} e^{2n} u^2 \theta \wedge d\theta^n \neq 0.
\]

Hence \( T^*_\Phi \) is not a local secondary invariant for any nonzero \( t \) sufficiently close to zero. \( \square \)
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