The power conjugacy problem in Higman-Thompson groups  

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Abstract

An introduction to the universal algebra approach to Higman-Thompson groups, including Thompson’s group \(V\), is given, following a series of lectures by Graham Higman in 1973. The algorithm for the conjugacy problem, as outlined in the lectures, is shown to be incomplete, and a revised and complete version of the algorithm is written out explicitly. An algorithm for the power conjugacy problem in these groups is constructed based on the conjugacy algorithm. Python implementations of these algorithms can be found at [D15].

1 Introduction

Thompson introduced the group now called “Thompson’s group \(V\)”, and its subgroups \(F < T\), in 1965, and so gave the first examples, namely \(V\) and \(T\), of finitely presented, infinite, simple groups (see \[\text{CFP96, Tho}\]). McKenzie and Thompson \[\text{MT73}\] also used \(V\) to construct finitely presented groups with unsolvable word problem. Subsequently, Galvin and Thompson (unpublished) identified \(V\) with the automorphism group of an algebra, \(V_{2,1}\), studied by Jónsson and Tarski \[\text{JT61}\]. Higman \[\text{Hig74}\] generalised this construction, defining \(G_{n,r}\) to be automorphism group of a generalisation \(V_{n,r}\) of \(V_{2,1}\), for \(n \geq 2\) and \(r \geq 1\), and showed the commutator subgroup of \(G_{n,r}\) to be a finitely generated, infinite, simple group, for all \(n \geq 2\). (\(G_{n,r}\) is perfect when \(n\) is even, and its commutator subgroup has index 2 when \(n\) is odd.)

The groups \(G_{n,r}\) are the “Higman-Thompson” groups of the title. There are many isomorphic groups in this set: in fact the algebras \(V_{n,r}\) and \(V_{n',r'}\) are isomorphic if and only if \(n = n'\) and \(r \equiv r' \mod n - 1\); so \(G_{n,r} \cong G_{n',r'}\) if \(n = n'\) and \(r \equiv r' \mod n - 1\). Higman \[\text{Hig74}\] showed that there are infinitely many non-isomorphic groups \(G_{n,r}\) and gave necessary conditions for such groups to be isomorphic. Recently Pardo \[\text{Pardo11}\] completed the isomorphism classification, showing that Higman’s necessary conditions are also sufficient: that is \(G_{n,r} \cong G_{n',r'}\) if and only if \(n = n'\) and

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\[ \gcd(n - 1, r) = \gcd(n' - 1, r'). \] Higman-Thompson groups have been much studied and further generalised: we refer to [CFP96, Brin04, BGG11, MPN13, DMP14, BCR] for example.

In this paper we consider the conjugacy and power-conjugacy problems in Higman-Thompson groups. We use Higman’s method, describing the groups \( G_{n,r} \) in terms of universal algebra. This allows us to give a detailed description of the algorithm for the conjugacy problem; and to uncover a gap in the original algorithm proposed by Higman. To be precise, Lemma 9.6 of [Hig74] is false, and consequently the “orbit sharing” algorithm in [Hig74] does not always detect elements in the same orbit of an automorphism. The orbit sharing algorithm is crucial to the algorithm for conjugacy given in [Hig74], which may fail to recognise that a pair of elements of \( G_{n,r} \) are conjugate. Fortunately it is not difficult to complete the algorithm. We then extend these results to construct an algorithm for the power conjugacy problem. The third author has implemented the algorithms described in this paper in Python [D15]. In fact it was the process of testing this implementation which uncovered the existence of an orbit unrecognised in [Hig74], and it became evident that the algorithms of [Hig74] are incomplete.

Note that other approaches to algorithmic problems in \( G_{n,r} \) have been developed. For example [SD10] gives a different algorithm for the conjugacy problem in \( G_{2,1} \), using the revealing tree pairs of Brin [Brin04]. In [BGG11] the same methods are used to study the centralisers of elements of \( G_{n,1} \) for \( n \geq 2 \). Again Belk and Matucci [BM07] gave a solution to the conjugacy problem in \( G_{2,1} \) based on strand diagrams. On the other hand, Higman’s methods were used by Brown [B87] to show that all the Higman-Thompson groups are of type \( FP_\infty \), and have been extended to generalisations of Higman-Thompson groups, to prove finiteness properties, by Martinez-Perez and Nucinkis [MPN13].

In detail the contents of the paper are as follows. In order to make this account self-contained we begin the paper with an introduction to universal algebra. Section 2 outlines the universal algebra required, following Cohn’s [Cohn91]. In Section 2.1 we introduce \( \Omega \)-algebras; that is universal algebras with signature \( \Omega \). Sections 2.2 and 2.3 cover quotients of \( \Omega \)-algebras, varieties of \( \Omega \)-algebras and free \( \Omega \)-algebras. We use this machinery in Section 3 to define the algebras \( V_{n,r} \) and establish their basic properties, following the exposition of [Hig74].

The groups \( G_{n,r} \) are defined in Section 4 as the automorphism groups of \( V_{n,r} \). Following [Hig74] we represent elements of \( G_{n,r} \) as bijections between carefully chosen generating sets of the algebras \( V_{n,r} \). This is done in two stages beginning with the semi-normal forms of Section 4.1. There are many ways of representing a given automorphism in semi-normal form, but in Section 4.2 it is shown that this representation may be refined to a unique quasi-normal form. Furthermore, an algorithm is given which takes an automorphism and produces a quasi-normal form representation.

The solution to the conjugacy problem is based on an analysis of certain orbits of automorphisms in quasi-normal form, and we give a full account of this analysis in Sections 4.1 and 4.2. Here we follow [Hig74] except that, as pointed out above, there exist orbits of types not recognised there, which give automorphisms in quasi-normal a richer structure, as described here.

Section 5 contains the algorithm for the conjugacy problem. This involves breaking an automorphism down into well-behaved parts. It is shown that every element of \( G_{n,r} \) decomposes into factors which are called periodic and regular infinite parts. The conjugacy problem for periodic and regular infinite components are solved separately and then the results recombined. The decomposition into these parts is the subject of Section 5.1 and here we give the main algorithm for the conjugacy problem, Algorithm 5.6. This algorithm depends on algorithms for periodic and regular infinite automorphisms: namely Algorithm 5.13 in Section 5.3 and Algorithm 5.27 in Section 5.4.

In Section 6 we turn to the power conjugacy problem. That is, given elements \( g, h \in G_{n,r} \) the
The problem is to find all pairs of non-zero integers \((a, b)\) such that \(g^a\) is conjugate to \(h^b\). Again the problem splits into the periodic and regular infinite parts. The periodic part is straightforward, and reduces to the conjugacy problem; see Section 6.4. The algorithm for power conjugacy of regular infinite elements is Algorithm 6.13 in Section 6.3 and gives the main result of the paper Theorem 6.14: that the power conjugacy problem is solvable. On input \(g, h \in G_{n,r}\) the algorithm returns a (possibly empty) set \(S\) consisting of all pairs of integers \((a, b)\) such that \(g^a\) and \(h^b\) are conjugate; as well as a conjugator, for each pair.

The examples given throughout the text are used as examples in [D15], from where these and other examples may be run through the third author’s implementations of the algorithms. (To find Example \(x,y\) in [D15], follow the instructions in the documentation to install the program; then run

from thompson.examples import example\_x\_y

in a Python session.)

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## 2 Universal Algebra

### 2.1 \(\Omega\)-algebra

In this section we review enough universal algebra to underpin the construction of the Higman-Thompson groups in later sections. We follow [Colm91].

**Definition 2.1.** An operator domain consists of a set \(\Omega\) and a mapping \(a : \Omega \to \mathbb{N}_0\). The elements of \(\Omega\) are called operators. If \(\omega \in \Omega\), then \(a(\omega)\) is called the arity of \(\omega\). We shall write \(\Omega(n) = \{\omega \in \Omega | a(\omega) = n\}\), and refer to the members of \(\Omega(n)\) as \(n\)-ary operations.

An algebra with operator domain (or signature) \(\Omega\) consists of a set \(S\), called the carrier of the algebra, and a family of maps \(\{\varphi_\omega\}_{\omega \in \Omega}\), indexed by \(\Omega\), such that for \(\omega \in \Omega(n)\), \(\varphi_\omega\) is a map from \(S^n\) to \(S\).

Following [Colm91] we suppress all mention of the maps \(\varphi_\omega\), identifying \(\varphi_\omega\) with \(\omega\), and referring to any algebra with carrier \(S\) and operator domain \(\Omega\) as an \(\Omega\)-algebra, which we denote by \((S, \Omega)\).

For example, a group \((G, \cdot, \cdot^{-1}, 1)\) is a \(\Omega\)-algebra with operator domain \(\{\cdot, \cdot^{-1}, 1\}\) and carrier \(G\), where \(\cdot\) is binary, \(\cdot^{-1}\) is unary and 1 is a constant; satisfying certain laws.

Given an \(\Omega\)-algebra \((S, \Omega)\) and \(f \in \Omega(n)\), we write \(s_1 \cdots s_nf\) for the image of the \(n\)-tuple \((s_1, \ldots, s_n) \in S^n\) under \(f\). We say that a subset \(T \subseteq S\) is closed under the operations of \(\Omega\) (or that \(T\) is \(\Omega\)-closed) if, for all \(n \geq 0\), for all \(f \in \Omega(n)\) and for all \(s_1, \ldots, s_n \in T\) the element \(s_1 \cdots s_nf\) is also an element of \(T\). Indeed, if \(T\) is a subset of \(S\) then \(T\) is \(\Omega\)-closed if and only if \((T, \Omega)\) is an \(\Omega\)-algebra: which brings us to the next definition.
Definition 2.2. Given an \( \Omega \)-algebra \((S, \Omega)\), an \( \Omega \)-subalgebra is an \( \Omega \)-algebra \((T, \Omega)\) whose carrier \( T \) is a subset of \( S \).

The intersection of any family of subalgebras is again a subalgebra. Hence, for any subset \( X \) of the set \( S \) we may define the subalgebra \( \langle X \rangle \) generated by \( X \) to be the intersection of all subalgebras containing \( X \). The subalgebra \( \langle X \rangle \) may also be defined recursively: that is \( \langle X \rangle \) is the subset of \( S \) such that (i) \( X \subseteq \langle X \rangle \), (ii) if \( y_1, \ldots, y_n \in \langle X \rangle \) then \( y_1 \cdots y_n \in \langle X \rangle \), for all \( f \in \Omega(n) \) and (iii) if \( s \) does not satisfy (i) or (ii) then \( s \) does not belong to \( \langle X \rangle \). Loosely speaking we might say that \( \langle X \rangle \) is obtained from \( X \) by applying a finite sequence of operations of \( \Omega \). If the subalgebra generated by \( X \) is the whole of \( S \), then \( X \) is called a generating set for \((S, \Omega)\).

A mapping \( g : A \to B \) between two \( \Omega \)-algebras \( A = (S, \Omega), B = (S', \Omega) \) is said to be compatible with \( f \in \Omega(n) \) if, for all \( s_1, \ldots, s_n \in S \),

\[
(s_1 g) \cdots (s_n g) f = (s_1 \cdots s_n f) g.
\]

If \( g \) is compatible with each \( f \in \Omega \), it is called a homomorphism from \( A = (S, \Omega) \) to \( B = (S', \Omega) \).

If a homomorphism \( g \) from \( A \) to \( B \) has an inverse \( g^{-1} \) which is again a homomorphism, \( g \) is called an isomorphism and then the \( \Omega \)-algebras \( A = (S, \Omega), B = (S', \Omega) \) are said to be isomorphic. An isomorphism of an algebra \( A = (S, \Omega) \) with itself is called an automorphism and a homomorphism of an algebra into itself is called an endomorphism. A homomorphism is determined once the images of a generating set are fixed.

Proposition 2.3 ([Cohn91 Proposition 1.1]). Let \( g, h : A \to B \) be two homomorphisms between \( \Omega \)-algebras \( A = (S, \Omega), B = (S', \Omega) \). If \( g \) and \( h \) agree on a generating set for \( A \), then they are equal.

From a family \( \{A_i\}_{i=1}^n \) \( (A_i = (S_i, \Omega)) \) of \( \Omega \)-algebras we can form the direct product \( P = \prod_{i=1}^n A_i \) of \( \Omega \)-algebras. Its set is the Cartesian product of the \( S_i \), and the operations are carried out component wise. Thus, if \( \pi_i : S \to S_i \) are the projections from the product to the factors then any \( f \in \Omega \) of arity \( n \) is defined on \( S^n \) by the equation

\[
(p_1 \cdots p_n f) \pi_i = (p_1 \pi_i) \cdots (p_n \pi_i) f,
\]

where \( p_i \in S \).

Let \( C \) be a class of \( \Omega \)-algebras, whose elements we will call \( C \)-algebras. By a free \( C \)-algebra on a set \( X \) we mean a \( C \)-algebra \( F \) with the following universal property.

There is a mapping \( \mu : X \to F \) such that every mapping \( f : X \to A \) into a \( C \)-algebra \( A \) can be factored uniquely by \( \mu \) to give a homomorphism from \( F \) to \( A \), i.e. there exists a unique homomorphism \( f' : F \to A \) such that \( \mu f' = f \).

In this case we say that \( X \) is a free generating set or a basis for \( F \). If \( X \) is a subset of \( F \) then we shall always assume that \( \mu \) is the inclusion map. Not every class has free algebras, but they do exist in the class under consideration here (see Proposition 2.16).

A free product is defined similarly, replacing the set \( X \) by a collection of \( C \) algebras. Given an indexing set \( I \) and for each \( i \in I \) an \( \Omega \) algebra \( A_i \) from \( C \) the free product \( A \) of \( \{A_i\}_{i \in I} \), written \( A = *_{i \in I} A_i \), is an \( \Omega \)-algebra in \( C \) satisfying the following property.

There exist homomorphisms \( \mu_i : A_i \to A \), for all \( i \in I \), such that for any \( \Omega \)-algebra \( B \) and homomorphisms \( f_i : A_i \to B \), for all \( i \in I \), there exists a unique homomorphism \( f' : A \to B \) such that \( \mu_i f' = f_i \), for all \( i \).

Given collections \( \{A_i\}_{i \in I} \) and \( \{B_i\}_{i \in I} \) of \( \Omega \)-algebras such that there exist free products \( A = *_{i \in I} A_i \) and \( B = *_{i \in I} B_i \), then, by definition, there exist homomorphisms \( \mu_i : A_i \to A \) and \( \mu'_i : B_i \to
Given a set \( S \subset \Omega \), the smallest equivalence \( \Gamma \) is defined as \( \Gamma = \{(s,s') \mid s,s' \in S\} \) if \( \Gamma \) is a subalgebra of \( A \times B \) notation and say \( \Gamma \) is a subalgebra of \( \Omega \), as defined in Lemma 2.4 ([Cohn91, Lemma 2.1, Chapter 1]).

2.2 Congruence on an \( \Omega \)-algebra

A relation between two sets \( S \) and \( R \) is defined to be a subset of the Cartesian product \( S \times R \). A mapping \( f : S \rightarrow R \) is a relation \( \Gamma_f \subset S \times R \) with the properties that for each \( s \in S \) there exists \( r \in R \) such that \((s,r) \in \Gamma_f \) (everywhere defined) and if \((s,r),(s,r') \in \Gamma_f \) then \( r = r' \) (single valued). A relation \( \Gamma \subset S \times R \) has an inverse \( \Gamma^{-1} \), defined by

\[
\Gamma^{-1} = \{(r,s) \in R \times S \mid (s,r) \in \Gamma\};
\]

and if \( \Delta \subset R \times T \) is a relation then the composition \( \Gamma \circ \Delta \) of \( \Gamma \) and \( \Delta \) is defined by

\[
\Gamma \circ \Delta = \{(s,t) \in S \times T \mid (s,x) \in \Gamma \text{ and } (x,t) \in \Delta \text{ for some } x \in R\}.
\]

If \( \Gamma \subset S \times R \) and \( S' \subset S \) we define

\[
S\Gamma = \{r \in R \mid (s,r) \in \Gamma \text{ for some } s \in S'\}.
\]

Given a set \( S \) the identity relation \( 1_S = \{(s,s) \mid s \in S\} \) and the universal relation \( S^2 = \{(s,s') \mid s,s' \in S\} \) always exist.

An equivalence on a set \( S \) is a subset \( \Gamma \) of \( S^2 \) with the properties \( \Gamma \circ \Gamma \subset \Gamma \) (transitivity): \( \Gamma^{-1} = \Gamma \) (symmetry) and \( 1_S \subseteq \Gamma \) (reflexivity). The equivalence class of \( s \in S \) is \( \{s' \in S \mid (s,s') \in \Gamma\} = \{s\}\Gamma \).

Given any subset \( U \) of \( S \times S \), the equivalence generated by \( U \) is

\[
E = \bigcap \{V \subseteq S \times S \mid V \text{ is an equivalence and } U \subseteq V\};
\]

that is, the smallest equivalence \( E \) on \( S \) containing \( U \). It follows that \( E \) is

\[
\{(a,b) \in S \times S \mid \text{there exists } a_0, \ldots, a_n \text{ such that } a_0 = a, a_n = b \text{ and } (a_i, a_{i+1}) \in U\}.
\]

Of particular interest in the study of \( \Omega \)-algebras are relations which are also subalgebras. Firstly, if \( A = (S,\Omega) \) and \( B = (R,\Omega) \) are \( \Omega \)-algebras and \( \Gamma \subset S \times R \) is a relation which is closed under the operations of \( \Omega \), as defined in \( A \times B \), then \( (\Gamma,\Omega) \) is a subalgebra of \( A \times B \). In this case we abuse notation and say \( \Gamma \) is a subalgebra of \( A \times B \).

**Lemma 2.4 ([Cohn91, Lemma 2.1, Chapter 1]).** Let \( A,B,C \) be \( \Omega \)-algebras and let \( \Delta \) be subalgebras of \( A \times B, B \times C \) respectively. Then \( \Gamma^{-1} \) is a subalgebra of \( B \times A \), \( \Gamma \circ \Delta \) is a subalgebra of \( A \times C \), and if \( \mathcal{A}' \) is a subalgebra of \( \mathcal{A} \), with carrier \( S' \subseteq S \), then \((S'\Gamma,\Omega)\) is a subalgebra of \( B \).

Let \( S \) and \( T \) be sets and \( f : S \rightarrow T \) a mapping between them. The image of \( f \) is defined as \( S\Gamma_f \), and the kernel of \( f \) is defined as

\[
\ker f = \{(x,y) \in S^2 \mid xf = yf\}.
\]

The latter is an equivalence on \( S \); the equivalence classes are the inverse images of elements in the image (sometimes called the fibres of \( f \)).
Example 2.5 (Groups). Given a group homomorphism \( f : G \to H \), the (group-theoretic) kernel of \( f \) is a normal subgroup \( N \); and the different cosets of \( N \) in \( G \) are the fibres of \( f \). So, the equivalence classes of \( \ker f \), in the definition above, are the cosets of \( N \) in \( G \).

A congruence on an \( \Omega \)-algebra \( A = (S, \Omega) \) is an equivalence on \( S \) which is also a subalgebra of \( A^2 \) i.e. an equivalence \( \Gamma \subseteq S \times S \) which is \( \Omega \)-closed. From the above, \( 1_A \) and \( A^2 \) are congruences on \( A \). Given any subset \( U \subseteq S \times S \) the congruence generated by \( U \) is

\[
C = \bigcap \{ V \subseteq S \times S | \text{\( V \) is a congruence and \( U \subseteq V \)} \}.
\]

It follows that \( C \) is the smallest congruence on \( A \) containing \( U \).

Let \( A \) be an \( \Omega \)-algebra. By definition a congruence is an equivalence which admits the operations \( \omega \) (\( \omega \in \Omega \)). Now each \( n \)-ary operator \( \omega \) defines an \( n \)-ary operation on \( A \):

\[
(a_1, \ldots, a_n) \mapsto a_1 \cdots a_n\omega \quad \text{for} \quad a_1, \ldots, a_n \in A.
\]

By giving fixed values in \( A \) to some of the arguments, we obtain \( r \)-ary operations for \( r \leq n \). In particular, if we fix all the \( a_j \) except one, say the \( i \)th, we obtain, for any \( n - 1 \) fixed elements \( a_1, \ldots, a_{n-1} \in A \), a unary operation

\[
x \mapsto a_1 \cdots a_{i-1}xa_i \cdots a_{n-1}\omega;
\]

and this applies for all \( i \in \{1, \ldots, n\} \). We say that the operation (2) is an elementary translation (derived from \( \Omega \) by specialisation in \( A \)). Given a finite sequence \( \tau_1, \ldots, \tau_n \) of elementary transformations the composition \( \tau = \tau_1 \circ \cdots \circ \tau_n \) is also a unary operation on \( A \), which we call a translation. (In particular we allow \( n = 0 \) in this definition, so the identity map on \( A \) is a translation.)

Proposition 2.6 ([Cohn81 Proposition 6.1, Chapter 6]). An equivalence \( \eta \) on an \( \Omega \)-algebra \( A \) is a congruence if and only if it is closed under all translations. More precisely, a congruence is closed under all translations, while any equivalence which is closed under all elementary translations is a congruence.

Remark 2.7. If \( U \subseteq S \times S \), then the congruence generated by \( U \) can be seen to consist of pairs \( (a, b) \in S \times S \) such that there exist \( m \geq 0 \), \( a_0, \ldots, a_m \in S \), and a translation \( \tau \) with

- \( a_0 = a \), \( a_m = b \) and
- \( (a_i, a_{i+1}) = (u_i\tau, u_{i+1}\tau) \)

where either \( (u_i, u_{i+1}) \in U \), \( (u_{i+1}, u_i) \in U \) or \( u_i = u_{i+1} \). That is, there exist \( s_1, \ldots, s_{n-1} \in S \), \( u_0, \ldots, u_m \in S \), and \( \omega \in \Omega(n) \) such that \( (u_i, u_{i+1}) \in U \cup U^{-1} \cup 1_S \) and setting

\[
a_i = (s_1, \ldots, s_{j-1}, u_i, s_j, \ldots, s_{n-1})\omega,
\]

for \( 0 \leq i \leq m \), we have \( a = a_0 \) and \( b = a_m \).

The next two theorems explain the significance of congruences for \( \Omega \)-algebras and will be used in the following section on free algebras and varieties.

Theorem 2.8 ([Cohn91 Theorem 2.2, Chapter 1]). Let \( g : A \to B \) be a homomorphism of \( \Omega \)-algebras. Then the image of \( g \) is a subalgebra of \( B \) and the kernel of \( g \) is a congruence on \( A \).
\textbf{Theorem 2.9} ([Cohn91, Theorem 2.3, Chapter 1]). \textit{Let }$A$\textit{ be an }$\Omega$\textit{-algebra and }$q$\textit{ a congruence on }$A$. \textit{Then, there exists a unique }$\Omega$\textit{-algebra, denoted }$A/q$\textit{, with carrier the set of all }$q$\textit{-classes such that the natural mapping }$\nu : A \rightarrow A/q$\textit{ is a homomorphism.}

The homomorphism $\nu$ in the previous theorem, which maps an element $s$ of the carrier of $A$ to its $q$-equivalence class, is called the \textit{natural homomorphism} from $A$ to $A/q$. The algebra $A/q$ is called the \textit{quotient algebra} of $A$ by $q$.

\textbf{Example 2.10.} Given a group $G$ and a normal subgroup $N$ of $G$, the natural mapping $G \rightarrow G/N$ is a homomorphism.

\section{2.3 Free algebras and varieties}

Let $X = \{x_1, x_2, \ldots\}$ be a non-empty, finite or countably enumerable set, called an \textit{alphabet}, and $\Omega$ an operator domain, with $\Omega \cap X = \emptyset$. We define an $\Omega$-algebra as follows. An $\Omega$-row in $X$ is a finite sequence of elements of $\Omega \cup X$. The set of all $\Omega$-rows in $X$ is denoted $W(\Omega; X)$. The length of the $\Omega$-row $w = w_1 \cdots w_m$ where $w_i \in \Omega \cup X$, is defined to be $m$ and is written $|w|$. The carrier of our $\Omega$-algebra is $W(\Omega; X)$, the set of $\Omega$-rows. We define the action of elements $\Omega$ on $W(\Omega; X)$ by concatenation. First observe that if $u$ and $v$ are $\Omega$-rows then the concatenation $uv$ of $u$ with $v$ is also an $\Omega$-row, and this may be extended to the concatenation of arbitrarily many $\Omega$-rows in the obvious way. Now, if $f \in \Omega(n)$, and $u_1, \ldots, u_n \in W(\Omega; X)$, then the the image of the $n$-tuple $(u_1, \ldots, u_n) \in W(\Omega, X)^n$ under the operation $f$ is the $\Omega$-row $u_1 \cdots u_nf$. By abuse of notation we will refer to $W(\Omega; X)$ as an $\Omega$-algebra.

It is clear that $X \subset W(\Omega; X)$ and we call the subalgebra generated by $X$ the $\Omega$-\textit{word algebra} on $X$, denoted $W_\Omega(X)$. Its elements are called $\Omega$-\textit{words} in the alphabet $X$. There is a clear distinction between $\Omega$-rows which are $\Omega$-words and those that are not. For example, if there is one binary operation $f$, then 

$$x_1x_2x_3x_4ff = (x_1, ((x_2, x_3)f, x_4)f)f$$

is a $\Omega$-row which is also an $\Omega$-word while $x_1fxfxf$ is an $\Omega$-row which is not an $\Omega$-word.

\textbf{Definition 2.11} ([Cohn91, Chapter 1]). We define the \textit{valency} of an $\Omega$-row $w = w_1 \cdots w_m$ ($w_i \in \Omega \cup X$) as $v(w) = \sum_{i=1}^{m} v(w_i)$ where

$$v(w_i) = \begin{cases} 1, & \text{if } w_i \in X, \\ 1 - n_i, & \text{if } w_i \in \Omega, \text{ of arity } n_i. \end{cases}$$

\textbf{Proposition 2.12} ([Cohn91, Proposition 3.1, Chapter 1]). An $\Omega$-row $w = w_1 \cdots w_m$ in $W(\Omega; X)$ is an $\Omega$-word if and only if every left-hand factor $u_i = w_1 \cdots w_i$ of $w$ satisfies

$$v(u_i) > 0 \text{ for } i = 1, \ldots, m,$$

and

$$v(w) = 1.$$

Moreover, each $\Omega$-word can be obtained in precisely one way by applying a finite sequence of operations of $\Omega$ to elements of $X$. 

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Let $\mathcal{A}$ be an $\Omega$-algebra. If in an element $w$ of $W_\Omega(X)$ we replace each element of $X$ by an element of $\mathcal{A}$ we obtain a unique element of $\mathcal{A}$. For $|w| = 1$, this is clear, so assume $|w| > 1$ and we will use induction on the length of $w$. We have $w = u_1 \cdots u_nf \ (f \in \Omega(n), \ u_i \in W_\Omega(X))$, where, by Proposition 2.12, the $u_i$ are uniquely determined once $w$ is given. By induction each $u_i$ becomes a unique element $a_i \in \mathcal{A}$, when we replace the elements of $X$ by elements of $\mathcal{A}$. Hence $w$ becomes $a_1 \cdots a_nf$; a uniquely determined element of $\mathcal{A}$.

This establishes the next theorem.

**Theorem 2.13** ([Cohn91] Theorem 3.2, Chapter 1). Let $\mathcal{A}$ be an $\Omega$-algebra and let $X$ be a set. Then any injective mapping $\theta : X \rightarrow \mathcal{A}$ extends, in just one way, to a homomorphism $\theta : W_\Omega(X) \rightarrow \mathcal{A}$. That is, $W_\Omega(X)$ is a free $\Omega$-algebra, freely generated by $X$.

**Corollary 2.14** ([Cohn91] Corollary 3.3, Chapter 1). Any $\Omega$-algebra $\mathcal{A}$ can be expressed as a homomorphic image of an $\Omega$-word algebra $W_\Omega(X)$ for a suitable set $X$. Here $X$ can be taken to be any set mapping onto a generating set of $\mathcal{A}$.

By an identity or law over $\Omega$ in $X$ we mean a pair $(u, v) \in W_\Omega(X) \times W_\Omega(X)$ or an equation $u = v$ formed from such a pair. We say that the law $(u, v)$ holds in the $\Omega$-algebra $\mathcal{A}$ or that $\mathcal{A}$ satisfies the equation $u = v$ if every homomorphism $W_\Omega(X) \rightarrow \mathcal{A}$ maps $u$ and $v$ to the same element of $\mathcal{A}$. This correspondence between sets of laws and classes of algebras establishes a pair of maps, with the following definitions.

- Given a set $\Sigma$ of laws over $\Omega$ in $X$, form $\mathcal{V}_\Omega(\Sigma)$, the class of all $\Omega$-algebras satisfying all the laws in $\Sigma$. This class $\mathcal{V}_\Omega(\Sigma)$ is called the variety generated by $\Sigma$.
- Given a class $\mathcal{C}$ of $\Omega$-algebras we can form the set $q = q(\mathcal{C})$ of all laws over $\Omega$ in $X$ which hold in all algebras of $\mathcal{C}$.

Thus we have a pair of maps $\mathcal{V}_\Omega$ and $q$: relating each variety of $\Omega$-algebras to a relation $q$ on $W_\Omega(X)$ and vice-versa. We shall see below that $q(\mathcal{C})$ is a congruence, but first we make a further definition.

A subalgebra of an $\Omega$-algebra $\mathcal{A}$ is called fully invariant if it is mapped into itself by all endomorphisms of $\mathcal{A}$. A congruence $\Gamma$ on $\mathcal{A}$ is said to be fully invariant if $(u, v) \in \Gamma$ implies $(u\theta, v\theta) \in \Gamma$, for all endomorphisms $\theta$ of $\mathcal{A}$. The fully invariant congruence generated by $\Gamma$ is

$$I = \bigcap\{V \mid V \text{ is a fully invariant congruence and } \Gamma \subseteq V\}.$$

It follows that $I$ is the smallest invariant congruence on $\mathcal{A}$ generated by $\Gamma$.

We claim that if $\mathcal{C}$ is a class of $\Omega$-algebras then $q(\mathcal{C})$ is a fully invariant congruence on $W_\Omega(X)$. To see that $q(\mathcal{C})$ is a congruence note first that in every class $\mathcal{C}$ of $\Omega$-algebras: $u = u$ for all $u \in W_\Omega(X)$; if $u = v$ holds then so does $v = u$; and if $u = v$ and $v = w$ then also $u = w$. Further, if $u_i = v_i$ for $i = 1, \ldots, n$, are laws holding in $\mathcal{A}$ and $\omega \in \Omega(n)$, then $u_1 \cdots u_n\omega = v_1 \cdots v_n\omega$ holds in $\mathcal{A}$. Hence $q(\mathcal{C})$ is indeed a congruence.

To see that $q(\mathcal{C})$ is a fully invariant congruence, let $(u, v) \in q(\mathcal{C})$ and let $\theta$ be any endomorphism of $W_\Omega(X)$. If $\alpha : W_\Omega(X) \rightarrow \mathcal{A}$, where $\mathcal{A} \in \mathcal{C}$, is any homomorphism, then so is $\theta\alpha$, hence $u\theta\alpha = v\theta\alpha$. Thus the law $u\theta = v\theta$ holds in $\mathcal{A}$, so $(u\theta, v\theta) \in q(\mathcal{C})$ and thus $q(\mathcal{C})$ is a fully invariant congruence.

Cohn shows in addition that the map $\mathcal{V}_\Omega$ is a bijection with inverse $q$, and deduces the following theorem.
Given sets $S$ and $T$ and a relation $\Gamma$ from $S$ to $T$, we may use $\Gamma$ to define a system of subsets of $S$, $T$, as follows. For any subset $X$ of $S$ we define a subset $X^*$ of $T$ by

$$X^* = \{ y \in T | (x, y) \in \Gamma \} \text{ for all } x \in X = \cap_{x \in X} \{ x \} \Gamma,$$

and similarly, for any subset $Y$ of $T$ we define a subset $Y^*$ of $S$ by

$$Y^* = \{ x \in S | (x, y) \in \Gamma \} \text{ for all } y \in Y = \cap_{y \in Y} \{ y \} \Gamma^{-1}.$$

We thus have mappings $X \mapsto X^*$ and $Y \mapsto Y^*$ of the power sets of $S$ and $T$ with the following properties:

$$X_1 \subseteq X_2 \Rightarrow X_1^* \supseteq X_2^*, \quad Y_1 \subseteq Y_2 \Rightarrow Y_1^* \supseteq Y_2^*, \quad (3)$$

$$X \subseteq X^{**}, \quad Y \subseteq Y^{**}, \quad (4)$$

$$X^{****} = X^*, \quad Y^{****} = Y^* \quad (5)$$

A pair of maps $X \mapsto X^*$, from the power set $2^S$ of $S$ to the power set $2^T$ of $T$, and $Y \mapsto Y^*$, from $2^T$ to $2^S$, satisfying (3) is called a Galois connection.

**Theorem 2.15** ([Cohn91] Theorem 3.5, Chapter 1). Let $W = W_\Omega(X)$ be the $\Omega$-word algebra on the alphabet $X$. The pair of maps $\Sigma \mapsto V_\Omega(\Sigma)$ and $C \mapsto q(C)$ forms a Galois connection giving a bijection between varieties of $\Omega$-algebras and fully invariant congruences $q$ on $W_\Omega(X)$.

**Proposition 2.16** ([Cohn91] Proposition 3.6, Chapter 1). Let $V$ be a variety of $\Omega$-algebras and $q$ the congruence on $W_\Omega(X)$ (the $\Omega$-word algebra generated by $X$) consisting of all the laws on $V$ i.e. the fully invariant congruence $q(V)$. Then $W_\Omega(X)/q$ is the free $V$-algebra on $X$.

Suppose $\Sigma$ is a set of laws over $\Omega$ in $X$ and let $V = V_\Omega(\Sigma)$ and $q = q(V)$. Then $\Sigma \subseteq q$ and, from Proposition 2.16, $q$ is a fully invariant congruence and $W_\Omega(X)/q$ is the free $V$-algebra.

Now let $p$ be the fully invariant congruence generated by $\Sigma$. Then, as $\Sigma \subseteq q$ and $q$ is a fully invariant congruence, we have $p \subseteq q$. Let $A = W_\Omega(X)/p$. Then $A$ is an $\Omega$-algebra, in which every law of $\Sigma$ holds (as $\Sigma \subseteq p$). Thus $A$ is a $V$-algebra. Then, from Proposition 2.16, the natural map $X \to A$ extends to a homomorphism $W_\Omega(X)/q \to A$. It follows that $q \subseteq p$. Therefore $p = q = q(V)$. We record this as a corollary which we shall use in Section 3 to construct Higman’s algebras $V_{n,r}$.

**Corollary 2.17.** Let $\Sigma$ be a set of laws over $\Omega$ in $X$, let $V = V_\Omega(\Sigma)$ and $q = q(V)$. Then $q$ is the fully invariant congruence generated by $\Sigma$.

3 The Higman Algebras $V_{n,r}$

In this section we define the algebras which Higman called $V_{n,r}$. Let $n$ be an integer, $n \geq 2$ and let $A$ be an $\Omega$-algebra, with carrier $S$ and operator domain $\Omega = \{ \lambda, \alpha_1, \ldots, \alpha_n \}$, such that $a(\alpha_i) = 1$, for $i = 1, \ldots, n$ and $a(\lambda) = n$. We call the $n$-ary operation $\lambda : S^n \to S$ a contraction and the unary operations $\alpha_i : S \to S$ descending operations. We define a map $\alpha : S \to S^n$, which we shall call an expansion, by

$$v\alpha = (v\alpha_1, \ldots, v\alpha_n),$$

for all $v \in S$. For any subset $Y$ of $S$, a simple expansion of $Y$ consists of substituting some element $y$ of $Y$ by the $n$ elements of the tuple $y\alpha$. A sequence of $d$ simple expansions of $Y$ is called a $d$-fold
expansion of \( Y \). A set obtained from \( Y \) by a \( d \)-fold expansion, \( d \geq 0 \), is called an expansion of \( Y \). For example, if \( x \in S \) then \( \{ x_{\alpha_1}, \ldots, x_{\alpha_n} \} \) is the unique simple expansion of \( \{ x \} \) and the 2-fold expansions of \( \{ x \} \) are the sets \( \{ x_{\alpha_1}, x_{\alpha_{i-1}}, x_{\alpha_{i+1}}, \ldots, x_{\alpha_n} \} \), for \( 1 \leq i \leq n \). Every \( d \)-fold expansion of \( Y \) has \( |Y| + (n - 1)d \) elements. Similarly, a simple contraction of \( Y \) consists of substituting \( n \) distinct elements \( \{ y_1, \ldots, y_n \} \in Y \) by the single element \( (y_1, \ldots, y_n) \lambda \). A set obtained from \( Y \) by applying a finite number of simple contractions is called a contraction of \( Y \).

From now on in this paper, \( \Omega \) is fixed as above. Let \( X \) be a non-empty set and recall that the \( \Omega \)-word algebra \( W_{\Omega}(X) \) is the free \( \Omega \)-algebra on \( X \).

**Definition 3.1.** Let \( \Sigma_n \) be the set of laws over \( \Omega \) in \( X \):

1. for all \( w \in W_{\Omega}(X) \),
   \[ w_{\alpha \lambda} = w, \]
   (or explicitly \( w_{\alpha_1} \cdots w_{\alpha_n} \lambda = w \)).
2. for all \( (w_1, \ldots, w_n) \in W_{\Omega}(X)^n \) and \( i \in \{1, \ldots, n\} \),
   \[ w_1 \cdots w_n \lambda_{\alpha_i} = w_i. \]

That is, \( \Sigma_n = \{(w_{\alpha_1} \cdots w_{\alpha_n} \lambda, w) | w \in W_{\Omega}(X)\} \cup \bigcup_{i=1}^{n} \{(w_1 \cdots w_n \lambda_{\alpha_i}, w_i) | w_i \in W_{\Omega}(X)\}. \)

Let \( \mathcal{V}_n = \mathcal{V}_{\Omega}(\Sigma_n) \) the variety of \( \Omega \)-algebras which satisfy \( \Sigma_n \) and let \( q = q(\mathcal{V}_n) \).

From Proposition 2.16 and Corollary 2.17, it follows that \( q \) is the fully invariant congruence on \( W_{\Omega}(X) \) generated by \( \Sigma_n \) and \( W_{\Omega}(X)/q \) is the free \( \mathcal{V}_n \)-algebra on \( X \).

**Definition 3.2.** Let \( X \) be a non-empty, finite or countably enumerable, set of cardinality \( r \) and \( n \geq 2 \) an integer. Then \( \mathcal{V}_{n,r}(X) \) is the the free \( \mathcal{V}_n \)-algebra \( W_{\Omega}(X)/q \), where \( q = q(\mathcal{V}_n) \) and \( \mathcal{V}_n = \mathcal{V}_{\Omega}(\Sigma_n) \).

When no ambiguity arises we refer to \( \mathcal{V}_{n,r}(X) \) as \( \mathcal{V}_{n,r} \).

**Remark 3.3.** In [Hig74, Section 2] Higman defines a standard form over \( X \) to be one of the finite sequences of elements of \( X \cup \{\alpha_1, \ldots, \alpha_n, \lambda\} \) specified by the following rules.

(i) \( x_{\alpha_{i_1}} \cdots x_{\alpha_{i_k}} \) is a standard form whenever \( k \geq 0 \) and \( 1 \leq i_j \leq n \) for \( j = 1, \ldots, k \).

(ii) If \( w_1, \ldots, w_n \) are standard forms then so is \( w_1 \cdots w_n \lambda \), unless there is a standard form \( u \) such that \( w_i = u_{\alpha_{i_j}} \) for \( i = 1, \ldots, n \).

(iii) No sequence is a standard form unless this follows from (i) and (ii).
The set of standard forms is made into an $\Omega$-algebra by defining the operations $\alpha_1, \ldots, \alpha_n, \lambda$ as follows.
\[
(x\alpha_i \cdots \alpha_{ik})\alpha_i = x\alpha_i \cdots \alpha_{ik}\alpha_i,
\]
\[
(w_1 \cdots w_n\lambda)\alpha_i = w_i
\]
for $i \in \{1, \ldots, n\}$ and
\[
(w_1, \ldots, w_n)\lambda = w_1 \cdots w_n\lambda
\]
unless there is a standard form $u$ such that $w_i = u\alpha_i$ for $i = 1, \ldots, n$ in which case
\[
(w_1, \ldots, w_n)\lambda = (u\alpha_1, \ldots, u\alpha_n)\lambda = u.
\]

Higman then goes on to prove that the algebra of standard forms is a free $\mathcal{V}_n$-algebra, freely generated by $X$ ([Hig74, Lemma 2.1]). This follows in our case from the definition above, and the remarks following it, together with Lemma 3.4 below.

**Lemma 3.4.** Let $\{w\}^*$ be an equivalence class of the congruence $\theta$ on $W_\mathcal{V}(X)$. Then there exists a unique minimal length element $u$ in $\{w\}^*$. The unique minimal length elements of equivalence classes are precisely the standard forms of Higman.

To prove Lemma 3.4 one of several standard arguments, to prove statements of this form in algebras of various types, may be followed: and we omit the details. Let $y$ be the minimal length representative of its equivalence class in $V_{n,r}$ i.e. $y$ is a standard form. Then the length of the equivalence class of $y$ is the length of $y$, denoted $|y|$, and the $\lambda$-length of the equivalence class of $y$ is the number of times the symbol $\lambda$ occurs in $y$.

Now that we have a concrete description of the free algebra $V_{n,r}$ in the variety $\mathcal{V}_n$, we recall those results of [Hig74, Section 2], required in the sequel.

**Lemma 3.5 (cf. [Hig74, Lemma 2.3]).** Let $B$ be a basis of $V_{n,r}(X)$.

1. Every expansion of $B$ is a basis of $V_{n,r}(X)$.
2. Every contraction of $B$ is a basis of $V_{n,r}(X)$.

**Proof.** 1. Let $Y$ be a $d$-fold expansion of $B$, where $d \geq 0$. Arguing by induction, we assume that every $d$-fold expansion of $B$ is a basis of $V_{n,r}$, and show that any simple expansion of $Y$ is also a basis. Let $y \in Y$ and let $Y'$ be the simple expansion
\[
Y' = (Y \setminus \{y\}) \cup \{y\alpha_1, \ldots, y\alpha_n\}.
\]
Since $y = y\alpha_1 \cdots y\alpha_n\lambda$, the set $Y'$ generates $V_{n,r}$. It remains to show that $Y'$ is a basis for $V_{n,r}$.

Given $A \in \mathcal{V}_n$ and a map $\theta : Y' \to A$, we shall show that there is a unique homomorphism $\bar{\theta} : V_{n,r} \to A$ extending $\theta$. Firstly, define $\theta^*$ from $Y$ to $A$ by $y\theta^* = y\theta$, for $y' \in Y \setminus \{y\}$, and $y\theta^* = y\alpha_1\theta \cdots y\alpha_n\theta\lambda$. As $Y$ is a basis, there is a unique homomorphism $\bar{\theta}^*$ from $V_{n,r}$ to $A$ extending $\theta^*$. Now
\[
(y\alpha_i)\bar{\theta}^* = (y\theta^*)\alpha_i = (y\theta^*\alpha_i = (y\alpha_1\theta \cdots y\alpha_n\theta\lambda)\alpha_i = y\alpha_i\theta.
\]
Hence $\bar{\theta}^*$ also extends $\theta$. Furthermore, any other homomorphism which extends $\theta$ must equal $\bar{\theta}^*$, since any such map must be defined on $Y$ in the same way as $\theta^*$.
2. This is proved in the same way as \[\square\]

The final statement of Corollary 3.13 forms a partial converse to this lemma, for finite bases.

Mostly we work with bases for \(V_{n,r}(X)\) which are expansions of \(X\), so we make the following definition.

**Definition 3.6.** Let \(A = \{\alpha_1, \ldots, \alpha_n\} \subset \Omega\). An \(A\)-basis of \(V_{n,r}(X)\) is an expansion of \(X\).

**Lemma 3.7.** Let \(B\) be an \(A\)-basis and \(Y\) a finite basis for \(V_{n,r}(X)\). If \(B \subseteq Y\langle A\rangle\) then \(B\) is an expansion of \(Y\).

**Proof.** Since \(Y\) is finite, there exists an expansion of \(Y\) contained in \(B\langle A\rangle\). Let \(d\) be minimal such that a \(d\)-fold expansion of \(Y\) is contained in \(B\langle A\rangle\) and let \(W\) be a \(d\)-fold expansion of \(Y\) which is contained in \(B\langle A\rangle\). Let \(w \in W\); so \(w = b\Gamma\), for some \(b \in B\) and \(\Gamma \in A^*\). As \(B \subseteq Y\langle A\rangle\), we have \(b = y\Delta\), for some \(y \in Y\) and \(\Delta \in A^*\); so \(w = y\Delta\Gamma\). Also, as \(w \in W\), there exists \(y' \in Y\) such that \(w = y'\Gamma'\), as part of an expansion of \(Y\). As \(Y\) is a basis it follows that \(y = y'\) and \(\Delta\Gamma = \Gamma'\). If \(\Gamma \neq 1\) then \(\Gamma = \Gamma_0\alpha_i\), for some \(\alpha_i \in A\) and \(\Gamma_0 \in A^*\), and as \(W\) is an expansion of \(Y\) it follows that \(y\Delta\Gamma_0\alpha_i \in W\), for all \(i \in \{1, \ldots, n\}\). Furthermore \(y\Delta\Gamma_0 \in B\langle A\rangle\), so \((W \setminus \{y\Delta\Gamma_0\alpha_i \mid 1 \leq i \leq n\}) \cup \{y\Delta\Gamma_0\} \subseteq B\langle A\rangle\). This contradicts the choice of \(d\) and \(W\), so \(\Gamma = 1\) and \(w \in B\). Hence \(W \subseteq B\).

Conversely, if \(b \in B\) then \(b = y\Gamma\), for some \(y \in Y\) and \(\Gamma \in A^*\), so either \(b\Delta = y\Gamma\Delta \in W\), for some \(\Delta \in A^*\), or \(y\Gamma_0 = w \in W\), where \(\Gamma = \Gamma_0\Gamma_1\). In the first case, \(b\Delta = w \in B\) implies \(w = b\) and \(\Delta = 1\). In the second case, \(b = y\Gamma = y\Gamma_1\Gamma_0 = w\Gamma_0\), with \(w \in B\), so again \(w = b\) and \(\Gamma_0 = 1\). Thus \(B \subseteq W\). \(\square\)

If \(A = (S, \Omega)\) is an \(\Omega\)-algebra with carrier \(S\) then we may form the \(A\)-algebra \((S, A)\) and the \(\{\lambda\}\)-algebra \((S, \{\lambda\})\), where the elements of \(A\) and \(\{\lambda\}\) have actions inherited from \(A\). We call these, respectively, the \(A\)-algebra and \(\{\lambda\}\)-algebra of \(A\). A subset \(U\) of \(V_{n,r}\) is said to be \(A\)-closed if \(u\alpha_i \in U\), for all \(\alpha_i \in A\), and an \(A\)-closed subset is called an \(A\)-subalgebra of (the \(A\)-algebra of) \(V_{n,r}\). Similarly \(W \subseteq V_{n,r}\) is called a \(\{\lambda\}\)-subalgebra of (the \(\{\lambda\}\)-algebra of \(V_{n,r}\)) if it is \(\{\lambda\}\)-closed: that is if \(w\lambda \in W\), for all \(w \in W\).

**Definition 3.8.** Let \(Y\) be a subset of \(V_{n,r}\). The \(A\)-subalgebra generated by \(Y\) is denoted \(Y\langle A\rangle\). The \(\{\lambda\}\)-subalgebra generated by \(Y\) is denoted \(Y\langle \{\lambda\}\rangle\).

The free monoid on a set \(L\) is denoted \(L^*\). If \(Y\) is a subset of \(V_{n,r}(X)\) then \(YA^* = \{y\Gamma \mid y \in Y, \Gamma \in A^*\}\) is \(A\)-closed, and it follows that \(Y\langle A\rangle = YA^*\). If, in addition, \(Y \subseteq X\langle A\rangle\) then \(y\Gamma\) is a standard form, for all \(y \in Y\) and \(\Gamma \in A^*\). In the sequel we write \(Y\langle A\rangle\langle \{\lambda\}\rangle\) for \((Y\langle A\rangle\langle \{\lambda\}\rangle)\langle \{\lambda\}\rangle\).

A word \(\Gamma \in A^*\) is called primitive if it is not a proper power of another word; that is, if \(\Gamma\) is non-trivial and \(\Gamma \in \{\Delta\}^*\), for some \(\Delta \in A^*\), then \(\Gamma = \Delta\).

**Proposition 3.9** ([Lot83], Proposition 1.3.1, Chapter 1). If \(\Gamma^n = \Delta^m\) with \(\Gamma, \Delta \in A^*\) and \(n, m \geq 0\), there exists a word \(\Lambda\) such that \(\Gamma, \Delta \in \{\Lambda\}^*\).

In particular, for each word \(\Gamma \in A^*\), there exists a unique primitive word \(\Lambda\) such that \(\Gamma \in \{\Lambda\}^*\).

**Proposition 3.10** ([Lot83], Proposition 1.3.2, Chapter 1). Two words \(\Gamma, \Delta \in A^*\) commute if and only if they are powers of the same word. More precisely, the set of words commuting with a word \(\Gamma \in A^*\) is a monoid generated by a single primitive word.
Lemma 3.11 ([Hig74, Section 2, Lemma 2.2]). Let $Y$ be a subset of $V_{n,r}$ and let $W$ be the $\Omega$-subalgebra of $V_{n,r}$ generated by $Y$. Then

1. $W = Y\langle A\rangle\langle \lambda \rangle$ and
2. for all $w \in W$, the set $w\langle A\rangle \setminus Y\langle A\rangle$ is finite.

Proof. 1. Let $w \in W$. Then there exists a finite subset $Y_0$ of $Y$ such that $w$ belongs to the subalgebra $W_0$ of $V_{n,r}$ generated by $Y_0$. Let $Z$ be an expansion of $X$ such that $|Z| \geq |Y_0|$. Choose a surjection $\beta$ of $Z$ onto $Y_0$. As $V_{n,r}$ is freely generated by $Z$ we may extend $\beta$ to a homomorphism from $V_{n,r}$ to $W_0$. Let $w_0$ be the preimage of $w$ under this homomorphism and let $l$ be the $\lambda$-length of the standard form of $w_0$ over $Z$. By a straightforward induction on $l$ it is apparent that $w_0 \in Z\langle A\rangle\langle \lambda \rangle$. Hence the image $w$ of $w_0$ in $W_0$ belongs to $Y_0\langle A\rangle\langle \lambda \rangle \subseteq Y\langle A\rangle\langle \lambda \rangle$, as required.

2. As in the previous part of the proof, we may assume that $W$ is freely generated by $Y$. Let $w \in W$ and let $l$ be the $\lambda$-length of the standard form of $w$ over $Y$. Then $w\alpha_n \cdots \alpha_1 \in Y\langle A\rangle$, whenever $r \geq l$. Hence, the only elements of the set difference $w\langle A\rangle \setminus Y\langle A\rangle$ are those of the form $w\alpha_n \cdots \alpha_1$ with $r < l$, and there are only finitely many of these since we only have $n$ choices for each $\alpha_i$.

Lemma 3.12 ([Hig74, Section 2, Lemma 2.4]). Let $x$ be a set of size $r \geq 1$ and let $X \subseteq V_{n,r}(x)$ be an expansion of $x$. If $U$ is a subset of $V_{n,r}(x)$ contained in $X\langle A\rangle$, then the following are equivalent:

1. $U = X\langle A\rangle \cap Y\langle A\rangle$, for some arbitrary generating set $Y$ of $V_{n,r}$,
2. $U$ is $A$-closed and $X\langle A\rangle \setminus U$ is finite, 
3. $U = Z\langle A\rangle$ for some expansion $Z$ of $X$.

Moreover, if $Y$ in $\boxempty$ is a basis for $V_{n,r}$ then $Z$ in $\boxcirc$ is an expansion of $Y$.

Proof. Firstly, let $U = X\langle A\rangle \cap Y\langle A\rangle$. Since $U$ is the intersection of $A$-closed sets, it is also $A$-closed. By lemma 3.11, $X\langle A\rangle \setminus Y\langle A\rangle$ is finite and therefore $X\langle A\rangle \setminus U$ is finite. So $\boxempty$ implies $\boxtwo$.

Secondly, assume that $U$ is $A$-closed and $X\langle A\rangle \setminus U$ is finite. We will prove $\boxthree$ and the final statement of the Lemma, by induction on the size of $|X\langle A\rangle \setminus U|$. If $|X\langle A\rangle \setminus U| = 0$, then $\boxthree$ holds with $Z = X$. Moreover, in this case it follows from Lemma 3.7 that $Z$ is an expansion of $Y$. Otherwise, $|X\langle A\rangle \setminus U| > 0$ and we choose an element $w \in X\langle A\rangle \setminus U$ of greatest length $|w|$ is maximal). Then the set $U^* = U \cup \{w\}$ is $A$-closed and $|X\langle A\rangle \setminus U^*| = |X\langle A\rangle \setminus U| - 1$. By induction, there is an expansion $Z^*$ of $X$ such that $U^* = Z^*\langle A\rangle$. If $Y$ is a basis then, in addition, $Z^*$ is an expansion of $Y$. The element $w$ belongs to $Z^*$, otherwise $w$ would have the form $w = z\alpha_n \cdots \alpha_1$, where $z \in Z^*$ and $r > 0$, and hence $z \in U^* \setminus \{w\} = U$. However, $U$ is $A$-closed and so this would imply that $w \in U$, a contradiction. If we take $Z = (Z^* \setminus \{w\}) \cup \{w\alpha_i\}_{1 \leq i \leq n}$, then this is again an expansion of $X$ (and of $Y$ in the case that $Y$ is a basis) and by the choice of $w$ we have $w\alpha_i \in U$, for all $i$. Therefore $U = Z\langle A\rangle$, $\boxthree$ implies $\boxtwo$ and the final statement holds if $Y$ is a basis.
Finally, if \( U = Z(A) \) for some expansion \( Z \) of \( X \), then \( U = X(A) \cap Y(A) \), with \( Y = Z \), and so 3 implies 1.

**Corollary 3.13** (cf. [Hig74, Corollary 1, page 12]). Let \( B \) and \( C \) be finite free generating sets for \( V_{n,r}(X) \). Then \( B \) and \( C \) have a common expansion \( Z \), which may be chosen such that \( Z(A) = B(A) \cap C(A) \). In particular, every finite basis of \( V_{n,r}(X) \) may be obtained from \( X \) by an expansion followed by a contraction.

**Proof.** Let \( f \) be an isomorphism from \( V_{n,r}(X) \) to \( V_{n,r}(B) \) mapping \( b \in B \subseteq V_{n,r}(X) \) to \( b \in V_{n,r}(B) \), for all \( b \in B \). Let \( C' = Cf \), so \( C' \) is a basis for \( V_{n,r}(B) \). From Lemma 3.12, \( B \) and \( C' \) have a common expansion \( Z' \) such that \( B(A) \cap C'(A) = Z'(A) \). Then \( B \) and \( C \) have common expansion \( Z = Z'f^{-1} \), and the remainder of the first statement of the lemma follows. The final statement follows on taking \( B \) to be an arbitrary finite free generating set and \( C = X \).

**Corollary 3.14** ([Hig74, Corollary 2, page 12]). \( V_{n,rs} \cong V_{n,s} \) if and only if \( r \equiv s \mod n - 1 \).

**Proof.** If \( r \equiv s \mod n - 1 \) then it follows from Lemma 3.13 that \( V_{n,rs} \cong V_{n,s} \). Conversely, let \( \theta \) be an isomorphism from \( V_{n,r}(X) \) to \( V_{n,s}(Y) \), where \( X \) and \( Y \) are sets of size \( r \) and \( s \), respectively. Then \( X\theta \) is a basis of \( V_{n,s}(Y) \) of size \( r \). From Corollary 3.13, there is a common expansion \( Z \) of \( X\theta \) and \( Y \). If \( Z \) is a \( d \)-fold expansion of \( X\theta \) and an \( e \)-fold expansion of \( Y \) then \( r + (n - 1)d = |Z| = s + (n - 1)e \), so \( r \equiv s \mod (n - 1) \), as claimed.

We could henceforth restrict to \( V_{n,r} \), where \( 1 \leq r \leq n - 1 \). However, we do not need to do this for what follows here, and it is convenient to allow arbitrary positive values of \( r \), and multiple instances of the same algebra.

**Definition 3.15.** Let \( u, v \) be elements of \( V_{n,r} \). Then, \( u \) is said to be a **proper initial segment** of \( v \) if \( v = u\Gamma \) for some non-trivial \( \Gamma \in A^+ \). If \( u = v \) or \( u \) is a proper initial segment of \( v \) then \( u \) is called an **initial segment** of \( v \).

**Lemma 3.16** ([Hig74, Section 2, Lemma 2.5(i)-(iii)]). Let \( B \) be an \( A \)-basis of \( V_{n,r} \) and \( V \) a subset of \( B(A) \).

1. If \( B \) and \( V \) are finite, then \( V \) is contained in an expansion of \( B \) if and only if the following condition is satisfied:

   
   no element of \( V \) is a proper initial segment of another. \ (†)

2. If \( B \) and \( V \) are finite, then \( V \) is an expansion of \( B \) if and only if \ (†) is satisfied and for each \( u \in B(A) \) there exists \( v \in V \) such that one of \( u, v \) is an initial segment of the other.

3. \( V \) is a set of free generators for the subalgebra it generates if and only if \ (†) is satisfied.

**Proof.** 1. If \( V \) is contained in an expansion of \( B \) then, using Lemma 3.16, \ (†) is satisfied.

Suppose \( V \) satisfies \ (†) and write

\[ U = B(A) \setminus \{ \text{proper initial segments of elements of } V \}. \]

Then \ (†) implies that \( V \subseteq U \). Also, \( U \) is \( A \)-closed and \( B(A) \setminus U \) consists of initial segments of the elements of the finite set \( V \), so it is finite. Thus, by Lemma 3.12 there is an expansion \( Z \)
of $B$ such that $U = Z(A)$. Therefore, $U \subseteq Z(A)$, and this implies that $V \subseteq Z$ (for an element of $Z(A) \setminus Z$ has a proper initial segment in $Z \subseteq U$ so it can not be in $V$ by the definition of $U$). Hence, $V$ is contained in an expansion of $B$.

2. If $V$ is an expansion of $B$ then $\text{(i)}$ is satisfied and for each $u \in B(A)$ there exists $v \in V$ such that one of $u, v$ is an initial segment of the other.

Suppose $V$ satisfies $\text{(i)}$ and for each $u \in B(A)$ there exists $v \in V$ such that one of $u, v$ is an initial segment of the other. By Part 1, $V$ is contained in an expansion $Z$ of $B$. If $V \not\subseteq Z$ then there is an element $z \in Z \setminus V$ and hence by the hypothesis there exists $v \in V$ such that one of $v$ or $z$ is an initial segment of the other. But no element of $Z$ can be an initial segment of another, so this is a contradiction and hence $V = Z$.

3. If $V$ is a set of free generators for the subalgebra it generates then $\text{(i)}$ is satisfied.

Suppose $\text{(i)}$ is satisfied. If $V$ is not a free generating set then the same is true of some finite subset $V_0$ and clearly $\text{(i)}$ is also satisfied with $V$ replaced by $V_0$. Then $V_0 \subseteq B(A)$ for some finite subset $B_0$ of $B$. As $\text{(i)}$ holds, it follows from Part 1 that $V_0$ is a subset of an expansion $Z_0$ of $B_0$. However, this means that $V_0$ is a subset of a basis of $V_{n,r}$, a contradiction.

\begin{proof}

For $m = 2$, from Corollary 3.13 we have a common expansion $Z$ of $Y_1$ and $Y_2$ such that $Z(A) = Y_1(A) \cap Y_2(A)$. Furthermore, if $W$ is a common expansion of $Y_1$ and $Y_2$ then, from Lemma 3.16 $W \subseteq Z(A)$, which implies that $W$ is an expansion of $Z$.

For $m > 2$, let $Z(A) = \cap_{i=1}^{m-1} (Y_i(A))$ and $V = Z(A) \cap Y_m(A)$, where we assume inductively that $Z$ is the unique minimal expansion of $Y_1, \ldots, Y_{m-1}$. From the previous paragraph there exists a unique minimal expansion $W$ of $Z$ and $Y_m$ such that $W(A) = V$. It follows that the result holds for $Y_1, \ldots, Y_m$ and hence by induction for all $m$.

\end{proof}

\begin{corollary}

Let $Y_i$ be a finite basis for $V_{n,r}$, for $i = 1, \ldots, m$. Then there is a unique minimal common expansion $Z$ of all the $Y_i$, and $Z$ satisfies $Z(A) = \cap_{i=1}^{m} (Y_i(A))$.

\end{corollary}

\begin{proof}

As $Y \subseteq B(A)$ and $Y$ is a basis, $Y$ satisfied $\text{(i)}$ from Lemma 3.16. If $u \in B(A)$ then $u \in Y(A)(\lambda)$, so for some $\Gamma, \Delta \in A^*$, $y \in Y$, we have $u\Gamma = y\Delta$. As $u \in B(A)$ and $y \in Y \subseteq B(A)$ there exist $b, b' \in B$ and $\Lambda, \Lambda' \in A^*$ such that $b\Lambda = u$ and $b'\Lambda' = y$, so $b\Lambda y = b'\Lambda' y$, and therefore $u = b\Lambda y$ and $\Lambda = \Lambda' y$. Now, $b\Lambda y = b\Lambda y$ so either $u = b\Lambda y$ is an initial segment of $y = \beta\Lambda y$, or vice-versa. Hence, from Lemma 3.16.2 $Y$ is an expansion of $B$.

\end{proof}

\begin{lemma} \textbf{[Hig74, Section 2, Lemma 2.5(iv)]} \end{lemma}

Let $B$ be an $A$-basis of $V_{n,r}$. Let $Y$ and $Z$ be $d$-fold expansions of $B$, for $d \geq 1$. If $Y \not\subseteq Z$ then some element of $Y$ is a proper initial segment of an element of $Z$.

\begin{proof}

If no element of $Y$ is a proper initial segment of an element of $Z$ then, from Corollary 3.13 $Y \subseteq Z(A)$. Then Lemma 3.16 implies that $Y$ is an expansion of $Z$. However, $Y$ and $Z$ are both $d$-fold expansions of $B$ and thus $Y = Z$. This competes the proof.

\end{proof}

\begin{lemma}

Let $u \in V_{n,r}$ and let $d$ be a non-negative integer.

\end{lemma}
1. If $v \in V_{n,r}$ then $u = v$ if and only if $u\Gamma = v\Gamma$, for all $\Gamma \in A^*$ of length $d$.

2. If $S$ is a subalgebra of $V_{n,r}$ then $u \in S$ if and only if $u\Gamma \in S$, for all $\Gamma \in A^*$ of length $d$.

Proof. 1. If $u = v$ then $u\Gamma = v\Gamma$ for all $\Gamma \in A^*$ of length $d$.

We shall show that given $d \geq 0$, 

if $u, v \in V_{n,r}$ and satisfy $u\Gamma = v\Gamma$ for all $\Gamma \in A^*$ of length $d$ then $u = v.$ (\textdagger)

If $d = 0$ this holds trivially. We will use induction on $d$. Assume that $d > 0$ and that for all $d'$ such that $0 \leq d' < d$ (\textdagger) holds, with $d'$ instead of $d$. Suppose then that $u, v \in V_{n,r}$ and $u\Gamma = v\Gamma$ for all $\Gamma$ of length $d$. In this case we will show that for any $\Delta \in A^a$ of length $d-1$ we have $u\Delta = v\Delta$. In fact, if $\Delta$ has length $d-1$ then $\Delta a_i$ has length $d$, for $i = 1, \ldots , n$. Therefore, $u(\Delta a_i) = v(\Delta a_i)$ and we obtain $u\Delta = (u\Delta)a_1 \cdots (u\Delta)a_n = (v\Delta)a_1 \cdots (v\Delta)a_n = v\Delta$. This applies to all $\Delta$ of length $d-1$, as required. From the inductive hypothesis $u = v$.

2. The proof is similar to that of part \textdagger

\hfill \Box

4 The Higman-Thompson groups $G_{n,r}$

In this section we define the groups which form the object of study in this paper. Throughout the remainder of the paper, we assume that $n \geq 2$, and that $V_{n,r} = V_{n,r}(x) = W_1(x)/q$, where $x = \{x_1, \ldots , x_r\}$. When $r = 1$ we let $x = \{x\}$.

When we discuss automorphisms of $V_{n,r}$ we assume that they are given by listing the images of a free generating set of $V_{n,r}$. Assume $\psi$ is an automorphism of $V_{n,r}$ defined by the map $\psi : Y \to Z$, where $Y$ and $Z$ freely generate $V_{n,r}$. If we expand $y \in Y$ and form the free generating set $Y' = Y \setminus \{y\} \cup \{ya_1, \ldots , ya_n\}$, then $ya_i \psi = y\psi a_i = za_i$ for $i = 1, \ldots , n$. Thus, when we expand $Y$ the automorphism $\psi$ induces an expansion $Z'$ of $Z$ such that $Y'\psi = Z'$. Hence, if $Y$ and $Z$ are not expansions of $x$ then by taking suitable expansions we may replace them by bases $Y'$ and $Z'$ contained in $x\langle A \rangle$, and define the automorphism by a map from $Y'$ to $Z'$. From Lemma 3.7 it follows that we may always describe an automorphism by a bijection between $A$-bases.

As bijections between bases are not particularly easy to read we represent automorphisms using pairs of rooted forests; as follows. An $n$-ary rooted tree is a tree with a single distinguished root vertex of degree $n$, such that all other vertices have degree $n + 1$ or 1. If a vertex $v$ is at distance $d \geq 1$ from the root then the $n$ vertices incident to $v$ and not on the path to the root are its children. Vertices of degree 1 are called leaves. An $n$-ary rooted tree is said to be $A$-labelled if the edges joining a vertex $v$ to its $n$ children are labelled with the elements $a_i \in A$, so that two edges joining $v$ to different children are labelled differently. An $A$-labelled, $r$-rooted, $n$-ary forest is a disjoint union of $r$ rooted, $A$-labelled, $n$-ary trees.

Let $T$ be a finite, $A$-labelled, $r$-rooted, $n$-ary forest, and let $x = \{x_1, \ldots , x_r\}$ be the set of roots of the $r$ trees which make it up. We may now identify elements of the subtree $T_i$ with root $x_i$, recursively, with elements of $\{x_i\}\langle A \rangle \subseteq V_{n,r}$. We identify the root vertex $x_i$ with the corresponding element of $V_{n,r}(x)$. If a vertex $v$ of $T_i$, of degree $n + 1$, corresponds to an element $x_i\Gamma$, for some $\Gamma \in A^a$, the child joined to $v$ by the edge labelled $a_j$ corresponds to $x_i\Gamma a_j$, $j = 1, \ldots , n$. Carrying out this correspondence for each subtree $T_i$, we identify each node of $T$ with a uniquely determined element of $x\langle A \rangle$. Furthermore, by construction, the leaves of $T$ correspond to an expansion of $x$. We use such trees to represent automorphisms as in the following example.
Example 4.1. Let \( n = 2, r = 1, x = \{ x \} \) and let \( \psi \) be the element of \( G_{2,1} \) corresponding to the bijective map between \( A \)-bases \( Y = \{ x_0^1, x_0^1x_1^1, x_0^1x_2^1 \} \) and \( Z = Y \psi = \{ x_0^1, x_0^1x_1^1, x_0^1x_2^1 \} \) given by \( x_0^1 \psi = x_0^1x_1^1, x_0^1x_1^1 \psi = x_0^1x_2^1, x_0^1x_2^1 \psi = x_0^1x_1^1 \).

The \( A \)-labelled binary trees corresponding to these bases are shown below. The labelling of edges is not shown, but edges from a vertex to its children are always ordered from left to right in the order \( \alpha_1, \ldots, \alpha_n \). Thus the leaves of the left hand tree correspond to \( Y \) and the leaves of the right hand tree to \( Z \). The numbering below the leaves determines the mapping \( \psi \); by taking leaf labelled \( j \) on the left to leaf labelled \( j \) on the right.

\[
\psi : 1 \ 2 \ 3 \ \rightarrow \ 1 \ 2 \ 3
\]

For an arbitrary automorphism, described as a bijection between \( A \)-bases, we generalise this example in the obvious way.

Definition 4.2 ([Hig74]). The Higman-Thompson group \( G_{n,r} \) is the group of \( \Omega \)-algebra automorphisms of \( V_{n,r} \).

Lemma 4.3 ([Hig74] Lemma 4.1). If \( \{ \psi_1, \ldots, \psi_k \} \) is a finite subset of \( G_{n,r} \), and \( X \) is an \( A \)-basis of \( V_{n,r} \), then there is a unique minimal expansion \( Y \) of \( X \) such that \( Y \psi_i \subseteq X(A) \), for \( i = 1, \ldots, k \). That is, any other expansion of \( X \) with this property is an expansion of \( Y \).

Proof. For each \( i \), \( X \psi_i^{-1} \) is a generating set for \( V_{n,r} \), but may not be a subset of \( X(A) \). Let \( U_i = X(A) \cap X \psi_i^{-1}(A) \). Then, by Lemma 3.12, \( U_i \) is \( A \)-closed and there exists an expansion \( Y_i \) of \( X \) such that \( U_i \subseteq Y_i(A) \). Now, Corollary 3.17 gives a unique minimal common expansion \( Y \), of the \( Y_i \)'s, and \( Y(A) = \bigcap_{i=1}^k (Y_i(A)) \). Then, for all \( i \), \( Y \subseteq Y_i(A) = U_i \subseteq X \psi_i^{-1}(A) \), so \( Y \psi_i \subseteq X(A) \).

Let \( Z \) be an expansion of \( X \). If \( Z \psi_i \subseteq X(A) \), for all \( i \), then (by the definition of \( U_i \)) \( Z \subseteq U_i = Y_i(A) \), so \( Z \subseteq \bigcap_{i=1}^k (Y_i(A)) = Y(A) \). Hence, from Lemma 3.12, \( Z \) is an expansion of \( Y \).

Definition 4.4. Let \( \{ \psi_1, \ldots, \psi_k \} \) be a finite subset of \( G_{n,r} \) and let \( X \) be an \( A \)-basis of \( V_{n,r} \). The expansion \( Y \) of \( X \) given by Lemma 4.3 is called the minimal expansion of \( X \) associated to \( \{ \psi_1, \ldots, \psi_k \} \).

4.1 Semi-normal forms

Let \( \psi \in G_{n,r} \), let \( X \) be an \( A \)-basis of \( V_{n,r} \), and \( y \in V_{n,r} \). The \( \psi \)-orbit of \( y \) is the set \( \{ y \psi^n | n \in \mathbb{Z} \} \).

We consider how \( \psi \)-orbits intersect the \( A \)-subalgebra \( X(A) \). To this end an \( X \)-component of the \( \psi \)-orbit of \( y \) is a maximal subsequence \( C \) of the sequence \( (y \psi^i)_{i=-\infty} \) such that all elements of \( C \) are in \( X(A) \). More precisely, \( C \) must satisfy

1. if \( y \psi^p \) and \( y \psi^q \) belong to \( C \), where \( p < q \) then \( y \psi^k \) belongs to \( X(A) \), for all \( k \) such that \( p \leq k \leq q \); and
2. \( C \) is a maximal subset of the \( \psi \)-orbit of \( y \) for which \( \bullet \) holds.

Note: \( X \)-components are what Higman, in [Hig74], refers to as “orbits in \( X(A) \)”.

First we distinguish the five possible types of \( X \)-component of \( \psi \) by giving them names.
1. **Complete infinite X-components.** For any \( y \) in such an X-component, \( y\psi^i \) belongs to \( X(A) \) for all \( i \in \mathbb{Z} \), and the elements \( y\psi^i \) are all different.

2. **Complete finite X-components.** For any \( y \) in such an X-component, \( y\psi^i = y \) for some positive integer \( i \), and \( y, y\psi, \ldots, y\psi^{i-1} \) all belong to \( X(A) \).

3. **Right semi-infinite X-components.** For some \( y \) in the X-component, \( y\psi^i \) belongs to \( X(A) \) for all \( i \geq 0 \), but \( y\psi^{-1} \) does not. The elements \( y\psi^i, i \geq 0 \), are then, of course, necessarily all different.

4. **Left semi-infinite X-components.** For some \( y \) in the X-component, \( y\psi^{-i} \) belongs to \( X(A) \) for all \( i \geq 0 \), but \( y\psi \) does not. The elements \( y\psi^{-i}, i \geq 0 \), are then, of course, necessarily all different.

5. **Incomplete finite X-components.** For some \( y \) in the X-component and some non-negative integer \( i \) we have \( y, y\psi, \ldots, y\psi^i \) belonging to \( X(A) \) but \( y\psi^{-1} \) and \( y\psi^{i+1} \) do not.

**Example 4.5.** Let \( n = 2 \), \( r = 1 \) and \( X = \{ x \} \) and let
\[
Y = \{ x\alpha_1, x\alpha_2, x\alpha_1\alpha_2, x\alpha_2\alpha_1, x\alpha_2^2 \}
\]
and
\[
Z = \{ x\alpha_1, x\alpha_1\alpha_2, x\alpha_1\alpha_2, x\alpha_2^2, x\alpha_2\alpha_1 \}.
\]
Let \( \psi \) be the automorphism defined by \( Y\psi = Z \), such that \( y\psi = z_i \) for \( i = 1, \ldots, 5 \) with the ordering given above.

\[
\psi:
\begin{align*}
1 & \rightarrow 2 \\
2 & \rightarrow 3 \\
3 & \rightarrow 4 \\
4 & \rightarrow 5 \\
5 & \rightarrow 1
\end{align*}
\]

Then \( Y \) is the minimal expansion of \( X \) associated to \( \psi \). The X-component of \( x\alpha_1^3 \) is left semi-infinite:
\[
\cdots \rightarrow x\alpha_1^4 \rightarrow x\alpha_1^3 \rightarrow x\alpha_1^2
\]
and the X-component of \( x\alpha_1\alpha_2 \) is right semi-infinite:
\[
x\alpha_1\alpha_2 \rightarrow x\alpha_1\alpha_2^2 \rightarrow x\alpha_1\alpha_2^3 \rightarrow \cdots
\]
The X-component of \( x\alpha_1^2\alpha_2 \) is complete infinite:
\[
\cdots \rightarrow x\alpha_1^4\alpha_2 \rightarrow x\alpha_1^3\alpha_2 \rightarrow x\alpha_1^2\alpha_2 \rightarrow x\alpha_1\alpha_2\alpha_1 \rightarrow x\alpha_1\alpha_2^2\alpha_1 \rightarrow \cdots
\]
and \( (x\alpha_2\alpha_1, x\alpha_2^2) \) is a complete finite X-component. We have \( x\alpha_2 = x\alpha_2\alpha_1 x\alpha_2^2 \lambda \), \( x\alpha_2 \psi = x\alpha_2^2 x\alpha_2\alpha_1 \lambda \) and \( x\alpha_2 \psi^2 = x\alpha_2 \). Therefore \( (x\alpha_2) \) is an incomplete finite X-component.

Let \( \psi \in G_{n,r} \) be an \( A \)-basis of \( V_{n,r} \), let \( Y \) be the minimal expansion of \( X(A) \) associated to \( \psi \) and let \( Z = Y\psi \). Then, as discussed above, \( Y \) and \( Z \) are both expansions of \( X \). From Lemma 3.12 both \( X(A) \setminus Z(A) \) and \( X(A) \setminus Y(A) \) are finite. Furthermore, as \( |Y| = |Z| \), both \( X \) and \( Y \) are \( d \)-fold expansions, for some \( d \), so \( |X(A) \setminus Z(A)| = |X(A) \setminus Y(A)| \).
By definition \( Y(A) = X(A) \cap X(A)\psi^{-1}, \) and moreover \( \psi \) maps no proper contraction of \( Y \) into \( X(A). \) Hence
\[
Z(A) = Y(A)\psi = X(A)\psi \cap X(A).
\]

Thus, if \( u \in X(A) \setminus Z(A) \) then \( u \not\in X(A)\psi, \) so \( u\psi^{-1} \not\in X(A) \) and hence \( u \) is an initial element either of an incomplete finite \( X \)-component or of a right semi-infinite \( X \)-component i.e. in an \( X \)-component of type (3) or (5). Similarly, if \( v \in X(A) \setminus Y(A) \) then \( v \not\in X(A)\psi^{-1}, \) so \( v\psi \not\in X(A) \) and hence \( v \) is a terminal element either of an incomplete finite \( X \)-component or of a left semi-infinite \( X \)-component i.e. in an \( X \)-component of type (4) or (5).

If \( O \) is an \( X \)-component of type (3) or (5), then by definition \( O \) has an initial element \( u: \) that is \( u\psi^{-1} \not\in X(A). \) Then \( u \not\in X(A)\psi, \) and so \( u \in X(A) \setminus Z(A). \) Similarly, if \( O \) is an \( X \)-component of type (4) or (5), then \( O \) has a terminal element \( v: \) that is \( v\psi \not\in X(A). \) Again, \( v \not\in X(A)\psi^{-1} \) and so \( v \in X(A) \setminus Y(A). \)

Let \( u \) be an initial element of an incomplete finite \( X \)-component \( O. \) By the above, \( u \in X(A) \setminus Z(A) \) and by definition of an incomplete finite \( X \)-component, there is some non-negative integer \( k \) such that \( u, u\psi, \ldots, u\psi^k \) all belong to \( X(A) \) but \( u\psi^{k+1} \) does not. Since \( u\psi^k \) is the terminal element of the incomplete finite \( X \)-component \( O, \) we have \( u\psi^k \in X(A) \setminus Y(A). \) Therefore, the initial elements of incomplete finite \( X \)-components in \( X(A) \setminus Z(A) \) and terminal elements of incomplete finite \( X \)-components in \( X(A) \setminus Y(A) \) pair up.

Given that the initial and terminal elements of the incomplete finite \( X \)-components must be in one-to-one correspondence, all other elements of \( |X(A) \setminus Z(A)| \) (respectively \( |X(A) \setminus Y(A)| \)) are initial (respectively terminal) elements in right (respectively left) semi-infinite \( X \)-components. Hence there are as many right semi-infinite \( X \)-components as left semi-infinite \( X \)-components.

The above is summarised in a lemma.

**Lemma 4.6 ([Hig74] Lemma 9.1).** Let \( \psi \) be an element of \( G_{n,r} \) and let \( X \) be an \( A \)-basis of \( V_{n,r}. \) Then there are only finitely many \( X \)-components of \( \psi \) of type (3), (4) and (5) and there are as many of type (3) as of type (4). If \( Y \) is the minimal expansion of \( X(A) \) associated to \( \psi \) and \( Z = Y\psi \) then
\[
Y(A) = X(A) \cap X(A)\psi^{-1} \quad \text{and} \quad Z(A) = X(A)\psi \cap X(A)
\]
and
\begin{itemize}
  \item \( u \) is an initial element of an orbit of type (3) or (5) if and only if \( u \in X(A) \setminus Z(A) \) and
  \item \( u \) is a terminal element of an orbit of type (4) or (5) if and only if \( u \in X(A) \setminus Y(A). \)
\end{itemize}

**Example 4.7.** In Example [4.5] we have \( X(A) \setminus Z(A) = \{x, x\alpha_1, x\alpha_1\alpha_2, x\alpha_2\} \) and \( X(A) \setminus Y(A) = \{x, x\alpha_1, x\alpha_1, x\alpha_2\}. \) The incomplete finite \( X \)-components are \( (x), (x\alpha_1) \) and \( (x\alpha_2), \) while \( x\alpha_1\alpha_2 \) is an initial element of a right semi-infinite \( X \)-component and \( x\alpha_1 \) is a terminal element of a left semi-infinite \( X \)-component. All other \( X \)-components of elements of \( X(A) \) are complete.

**Definition 4.8 ([Hig74] Section 9]).** An element \( \psi \) of \( G_{n,r} \) is in semi-normal form with respect to the \( A \)-basis \( X \) if no element of \( X(A) \) is in an incomplete finite \( X \)-component of \( \psi. \)

**Lemma 4.9 ([Hig74] Lemma 9.2]).** Let \( \psi \in G_{n,r} \) and let \( X \) be an \( A \)-basis of \( V_{n,r}. \) There exists an expansion of \( X \) with respect to which \( \psi \) is in semi-normal form.
Proof. Let $\psi \in G_{n,r}$. We prove the lemma by induction on the number of elements in $X(A)$ which belong to an incomplete finite $X$-component. Note first that from Lemma 4.6 it follows that there are only finitely many elements of $X(A)$, which belong to incomplete finite $X$-components.

If there are no such elements then we are done. Suppose then that there exists an element $u$ in $X(A)$ which belongs to an incomplete finite $X$-component. Thus, there exist $y \in X$ and $\Gamma \in A^*$ such that $u = y\Gamma$ and some minimal $m, k \in N_0$ such that $u\psi^{-(m+1)}, u\psi^{k+1} \notin X(A)$. It follows that $y\psi^{-m}, y\psi^{k+1} \notin X(A)$, so that $y$ is also in an incomplete finite $X$-component. Let $X' = X \setminus \{y\}$ and let $X'' = X' \cup \{y\alpha_1, \ldots, y\alpha_n\}$. Then $X''$ is an expansion of $X$ and $X(A) \setminus X''(A) = \{y\}$ so the number of elements of $X''(A)$ in an incomplete finite $X''$-component is one less than the number of elements of $X(A)$ in an incomplete finite $X$-component. Hence, by induction, there exists an expansion of $X$ with respect to which $\psi$ is in semi-normal form.

Remark 4.10. Continuing the discussion above Lemma 4.6 observe that if $u \in X(A)$ and $u \notin Y(A) \cup Z(A)$ then $u$ is both the initial an terminal element of an $X$-component of $\psi$; so $\{u\}$ constitutes an incomplete finite $X$-component. Therefore, in implementing the argument of Lemma 4.9 to find a semi-normal form for $\psi$, we may pass immediately to a minimal expansion containing all elements of $X(A) \setminus (Y(A) \cup Z(A))$.

Example 4.11. Let $n = 2, r = 1, x = \{x\}$ and let $\psi$ be the automorphism of Example 4.1. Here $Y = \{x\alpha_1^2, x\alpha_1\alpha_2, x\alpha_2\}$ is the minimal expansion of $x$ associated to $\psi$ and $Z = Y\psi = \{x\alpha_1, x\alpha_2\alpha_1, x\alpha_2^2\}$. In this example, $\langle x(A) \setminus (Y(A) \cup Z(A)) = \{x\} \rangle$ and the minimal expansion of $x$ not containing $x$ is $X = \{x\alpha_1, x\alpha_2\}$. Then $Y$ remains the minimal expansion of $X$ associated to $\psi$, $X(A) \setminus Z(A) = \{x\alpha_2\}$ and $X(A) \setminus Y(A) = \{x, x\alpha_1\}$. As $x\alpha_1$ is the terminal element of a left semi-infinite $X$-component, while $x\alpha_2$ is the initial element of a right semi-infinite $X$-component it follows that $\psi$ is in semi-normal form with respect to $X$.

Example 4.12. Let $n = 2, r = 1, x = \{x\}$ and let $\psi$ be the element of $G_{2,1}$ corresponding to the bijective map:

$$x\alpha_1^2\psi = x\alpha_2^2, x\alpha_1\alpha_2\psi = x\alpha_2\alpha_1, x\alpha_2\psi = x\alpha_1.$$ 

Again, $Y = \{x\alpha_1^2, x\alpha_1\alpha_2, x\alpha_2\}$ is the minimal expansion of $x$ associated to $\psi$ and setting $Z = Y\psi = \{x\alpha_1, x\alpha_2\alpha_1, x\alpha_2^2\}$, the minimal expansion of $x$ not containing any element of $x(A) \setminus (Y(A) \cup Z(A))$ is $X_1 = \{x\alpha_1, x\alpha_2\}$; and $Y$ is still the minimal expansion of $X_1$ associated to $\psi$. However $(x\alpha_2, x\alpha_1)$ is in an incomplete finite $X_1$-component, so $\psi$ is not in semi-normal form with respect to $X_1$. As $x\alpha_1$ is in an incomplete finite $X_1$-component, first take the simple expansion of $X_1$ at $x\alpha_1$, giving $X_2 = Y$. As $x\alpha_2\psi = x\alpha_1 \notin X_2(A)$, $(x\alpha_2)$ is now an incomplete finite $X_2$-component, so $\psi$ is not in semi-normal form with respect to $X_2$. We take a further simple expansion of $X_2$ at $x\alpha_2$, to obtain a new $A$-basis $X_3 = \{x\alpha_1^2, x\alpha_1\alpha_2, x\alpha_2\alpha_1, x\alpha_2^2\}$. Then $\psi$ maps $X_3$ to itself:

$$x\alpha_1^2\psi = x\alpha_2^2, x\alpha_1\alpha_2\psi = x\alpha_2\alpha_1, x\alpha_2\alpha_1\psi = x\alpha_1^2, x\alpha_2^2 = x\alpha_1\alpha_2.$$ 

As all elements of $X_3$ are in complete finite $X_3$-components, $\psi$ is in semi-normal form with respect to $X_3$. The minimal expansion of $X_3$ associated to $\psi$ is of course $X_3$. 

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Example 4.13. The automorphism $\psi$ of Example 4.7 is not in semi-normal form with respect to $X$ or $X_1 = \{x_1, x_2\}$, as both $x_1$ and $x_2$ are in incomplete finite $X$-components. However, $\psi$ is in semi-normal form with respect to $X_2 = \{x_1^3, x_1 x_2, x_2^3, x_1^2 x_2, x_1 x_2, x_1 x_2^2\}$. The minimal expansion of $X_2$ associated to $\psi$ is the $A$-basis $Y$ of Example 4.5.

The following, which follows directly from the definitions, summarises the possibilities for the intersection with $X(A)$ of the orbit of an element under an automorphism in semi-normal form.

Corollary 4.14. Let $\psi$ be an element of $G_{n,r}$ in semi-normal form with respect to the $A$-basis $X$, let $v \in V_{n,r}$ and let $O_v$ be the $\psi$-orbit of $v$. Then $O_v$ has one of the following six types.

1. $O_v \cap X(A) = \emptyset$.
2. $O_v$ is finite and $O_v \subseteq X(A)$, so $O_v$ is a complete finite $X$-component.
3. $O_v$ is infinite and $O_v \subseteq X(A)$, so $O_v$ is a complete infinite $X$-component.
4. $O_v \cap X(A)$ consists of a unique left semi-infinite $X$-component.
5. $O_v \cap X(A)$ consists of a unique right semi-infinite $X$-component.
6. $O_v \cap X(A)$ is the disjoint union of a left semi-infinite $X$-component and a right semi-infinite $X$-component.

Remark 4.15. As can be seen from Example 4.17 below, there are automorphisms for which orbits of the final type in this list exist. In fact we shall show in Section 4.2 that there exist automorphisms which have such orbits with respect to every semi-normal form. This means that [Hig74, Lemma 9.6] is false and that the algorithms described in [Hig74] for determining if two elements belong to a single orbit, and for conjugacy of automorphisms, [Hig74, Lemma 9.7 and Theorem 9.3], are incomplete.

Definition 4.16. Let $\psi$ be an element of $G_{n,r}$ in semi-normal form with respect to the $A$-basis $X$, and let $O$ be a $\psi$-orbit of the type given in Corollary 4.14 type 6. Then $O$ is called a pond orbit with respect to $X$. The sub-sequence of $O$ consisting of elements not in $X(A) \cup (Y(A) \cup Z(A))$ is $X = \{x_1^3, x_1 x_2, x_2^3, x_1 x_2, x_1 x_2^2\}$. Two
of these elements are endpoints of semi-infinite $X$-components, whereas the other two belong to complete infinite $X$-components.

\[
\cdots \mapsto x\alpha_1^1 \mapsto x\alpha_1^2 \mapsto x\alpha_1^3 \mapsto \cdots
\]

Thus \(\psi\) is in semi-normal form with respect to \(X\). Now let us compute the \(\psi\)-orbit of the element \(x\alpha_1^2\).

\[
\cdots \mapsto x\alpha_1^1 \alpha_2 \mapsto x\alpha_1^2 \alpha_2 \mapsto x\alpha_1^3 \alpha_2 \mapsto x\alpha_1^4 \alpha_2 \mapsto \cdots
\]

We see that this orbit \(\mathcal{O}\) consists of two semi-infinite \(X\)-components and a single element \(x\alpha_2\).

Lemma 4.18 ([Hig74] Lemma 9.3). Let \(\psi\) be an element of \(G_{n,r}\) in semi-normal form with respect to the \(A\)-basis \(X\). Suppose that \(x\) is an element of \(X\). Then one of the following holds.

(A) There exists \(\Gamma \in A^*\) such that \(x\Gamma\) is in a complete finite \(X\)-component. In this case \(x\) belongs to a complete finite \(X\)-component, which consists of elements of \(X\), and we say \(x\) is of type (A).

(B) There exist \(\Gamma, \Delta \in A^*\), with \(\Gamma \neq \Delta\), such that \(x\Gamma\) and \(x\Delta\) belong to the same \(X\)-component. In this case there exists \(\Lambda \in A^*, m \in \mathbb{Z}, m \neq 0, \) with \(|m|\) minimal, such that \(x\psi^m = x\Lambda\) and we say \(x\) is of type (B). If \(m > 0\) then the \(X\)-component containing \(x\) is right semi-infinite; if \(m < 0\) then the \(X\)-component containing \(x\) is left semi-infinite.

(C) \(x\) is not of type (A) or (B) above and there exists some \(z \in X\) of type (B) and non-trivial \(\Delta \in (A)\) such that \(x\psi^z = z\Delta\). In this case the \(X\)-component of \(x\) is infinite; and we say \(x\) is of type (C).

Proof. (A) If \(x\) belongs to an infinite \(X\)-component of \(\psi\) (of types (1), (3) or (4) that is), then so does \(x\Gamma\), a contradiction. As \(\psi\) is in semi-normal form with respect to \(X\) it follows that \(x\) is in a complete finite \(X\)-component. Let \(d\) be the smallest positive integer such that \(x\psi^d = x\), let \(1 \leq i \leq d - 1\) and let \(z \in X\) and \(\Delta \in A^*\) be such that \(x\psi^i = z\Delta\). Then \(z\) is also in a complete finite \(X\)-component, so there exists \(y \in X\) and \(\Gamma \in A^*\) such that \(z\psi^{d-i} = y\Gamma\). Hence \(x = x\psi^d = z\Delta\psi^{d-i} = z\psi^{d-i}z\Delta = y\Gamma\Delta\). From Lemma 3.11, we have \(y = x\) and \(\Gamma = \Delta = \varepsilon\), so \(x\psi^z = z \in X\), as claimed.

(B) If \(x\) belongs to a finite \(X\)-component then, from (A), the \(X\)-component of \(x\Gamma\) consists of elements \(z\Gamma\), where \(z \in X\), contrary to the hypotheses of (B). Therefore \(x\) belongs to an infinite \(X\)-component of \(\psi\). Without loss of generality we may assume that there is \(i > 0\) such that \(x\Gamma\psi^i = x\Delta\). Suppose first that \(x\psi^k \in X(A)\), for all \(k \geq 0\). Then \(x\psi^k = v\Lambda\), for some \(v \in X\) and \(\Lambda \in A^*\), and thus \(x\Delta = x\Gamma\psi^i = v\Lambda\Gamma\); so \(v = x\) and \(\Delta = \Lambda\Gamma\), and we obtain \(x\psi^z = x\Lambda\).

Similarly, if \(x\psi^{-k} \in X(A)\), for all \(k \geq 0\), then \(x\psi^{-i} = x\Lambda'\), for some \(\Lambda' \in A^*\), with \(\Gamma = \Lambda'\Delta\). Note that if \(x\psi^k \in X(A)\) for all \(k\), then \(x = x\Lambda\Lambda'\), which forces \(\Lambda = \Lambda' = \varepsilon\), so \(\Gamma = \Delta\), a contradiction. Hence the final statement of (B) holds.
(C) In this case $x$ must belong to an infinite $X$-component, as (A) does not hold. As $X$ is finite there is $z \in X$ such that $z\Gamma$ and $z\Delta$ belong to the $X$-component of $x$, for distinct $\Gamma$ and $\Delta$ in $A^*$; and then $z$ is of type (B), as required.

\[\square\]

**Definition 4.19.** Let $u \in V_{n,r}$ and $\psi \in G_{n,r}$. If $u \psi^m = u \Gamma$, for some $d \in \mathbb{Z} \setminus \{0\}$ and some $\Gamma \in A^* \setminus \{1\}$, then $u$ is a **characteristic element for $\psi$**. If $u$ is a characteristic element for $\psi$ then the **characteristic** of $u$ is the pair $(m, \Gamma)$ such that $m \in \mathbb{Z} \setminus \{0\}, \Gamma \in A^*$ with

- $u \psi^m = u \Gamma$ and
- for all $i$ such that $0 < |i| < |m|$, $u \psi^i \notin u(A)$.

In this case $\Gamma$ is called the **characteristic multiplier** and $m$ is the **characteristic power** for $u$, with respect to $\psi$.

From the definition, if $\psi$ is in semi-normal form with respect to $X$ then an element $x \in X$ is of type (B) if and only if $x$ is a characteristic element: in which case it follows from Lemma 4.24 below that the $\psi$-orbit of $x$ is of type 4 or 5 in Corollary 4.14. On the other hand, if $x \in X$ has type (C) then the $\psi$-orbit of $x$ may be of types 3, 4, 5 or 6, in Corollary 4.14.

**Example 4.20.** In Example 4.13, the automorphism $\psi$ is in semi-normal form with respect to an $A$-basis $X$. The elements $x \alpha_2 \alpha_1$ and $x \alpha_2^2$ of $X$ are of type (A). The element $x \alpha_2 \alpha_1 \in X$ is of type (B) with characteristic $(-1, \alpha_1)$, while $x \alpha_1 \alpha_2 \in X$ is of type (B) with characteristic $(1, \alpha_2)$; and both of these elements are extremal in the unique semi-infinite $X$-components contained in their $\psi$-orbits.

In Example 4.17 elements $x \alpha_2 \alpha_1$ and $x \alpha_2^2$ of $X$ are of type (C), are not characteristic and belong to complete infinite $X$-components. An example of an element of type (C) in a semi-infinite $X$-component can be found at [D15]; follow the link to "Examples" and refer to the example named "semi_inf.c".

**Lemma 4.21.** If $u \in V_{n,r}$ is a characteristic element for $\psi \in G_{n,r}$ then

1. $(m, \Gamma)$ is uniquely determined and

2. if $v$ is in the same $\psi$-orbit as $u$ then $v$ is a characteristic element with the same characteristic as $u$.

**Proof.** To see [1] suppose that $u$ is a characteristic element and with characteristic $(m, \Gamma)$. If $u \psi^m = u \Delta$ and, for all $k$ such that $0 < |k| < |m|$, we have $u \psi^k \notin u(A)$ then $|m| \leq |m|$, by Definition 4.19, so $m = \pm m'$. If $u \psi^{-m} = u \Delta$ then $u = u \psi^m \Delta = u \Gamma \Delta$, which cannot happen.

For [2], let $u \psi^r = v$. For all $k$ such that $u \psi^k = u \Delta, \Delta \in A^*$, we have

\[v \psi^k = u \psi^r \psi^k = u \psi^k \psi^r = u \Delta \psi^r = u \psi^r \Delta = v \Delta.\]

Interchanging $u$ and $v$ we see also that whenever $v \psi^k = v \Delta$ then $u \psi^k = u \Delta$. \[\square\]

From Lemma 4.21 if a $\psi$-orbit has a characteristic element, then every $X$-component of this $\psi$-orbit contains a characteristic element, and all these elements have the same characteristic. Bearing this in mind we make the following definition.
Definition 4.22. Let $\psi$ be an element of $G_{n,r}$ and $O$ an $X$-component of $\psi$ containing a characteristic element $u$. Then we define the characteristic of $O$ to be equal to the characteristic of $u$.

Theorem 4.23 ([Hig74 Theorem 9.4]). Let $\psi \in G_{n,r}$ be in semi-normal form with respect to $X$. Then $\psi$ is of infinite order if and only if it has a characteristic element. Moreover, if $\psi$ is of infinite order then this characteristic element may be taken to belong to $X$.

Proof. If $u$ is a characteristic element for $\psi$ with characteristic $(m, \Gamma)$ then $u\psi^m = u\Gamma$ and $u\psi^{mq} = u\Gamma^q$ and, for sufficiently large $q$, $u\psi^{mq} \in X(A)$. Then $u\psi^{mq}$ is a characteristic element and $u\psi^{mq} = x\Delta$, for some $x \in X$ and $\Delta \in A^*$. Now $x\Delta \Gamma = u\psi^{mq}\Gamma = u\psi^{m(q+1)} = x\Delta \psi^m$, so from Lemma 4.18, $x$ has type (B). Thus we may assume $u \in X$. Now,

$$u\psi^{mj} = u\psi^m\psi^{m(j-1)} = u\Gamma \psi^{m(j-1)} = u\Gamma^m \psi^{m(j-2)} = \cdots = u\Gamma^j,$$

for $j \in \mathbb{N}$. Since $\Gamma$ is a characteristic multiplier, the elements $u\Gamma^j$ are all different for $j \in \mathbb{N}$, so $\psi$ has infinite order.

Conversely, if $\psi$ has no characteristic element, then certainly there are none in $X$, so $X$ has no elements of type (B) nor type (C). Thus all elements of $X$ are of type (A), as $\psi$ is in semi-normal form; whence $\psi$ is a permutation of $X$ and has finite order. \qed

Lemma 4.24. Let $\psi$ be in semi-normal form with respect to an $A$-basis $X$ and let $u \in V_{n,r}$. If $u$ has characteristic $(m, \Gamma)$ then the $\psi$-orbit of $u$ has precisely one $X$-component, which is semi-infinite (right semi-infinite if $m > 0$ and left semi-infinite if $m < 0$) and consists of elements of the form $x\Lambda$, where $x \in X$, $x$ is of type (B), and $\Lambda \in A^*$. Furthermore, if $x\Lambda$ belongs to the $X$-component of the $\psi$-orbit of $u$, where $x \in X$ and $\Lambda \in A^*$, then $x$ has characteristic $(m, \Gamma_1 \Gamma_0)$, where $\Gamma = \Gamma_0 \Gamma_1$, $\Lambda = (\Gamma_1 \Gamma_0)^p \Gamma_1 = \Gamma_1 \Gamma^p$, $p \geq 0$, and $\Gamma_0$ is non-trivial.

Proof. As $u\psi^m = u\Gamma$ we have $u\psi^{mq} = u\Gamma^q$, for all integers $q$, and choosing $q$ sufficiently large $u\Gamma^q \in X(A)$. Thus we may assume that $u \in X(A)$. Let $u = x\Lambda$, where $x \in X$ and $\Lambda \in A^*$ and assume first that $m > 0$. As $u$ has characteristic $(m, \Gamma)$, both $x\Lambda$ and $x\Sigma \Lambda$ belong to $S$, so from Lemma 4.18, $x$ is of type (B). Suppose there is an integer $K \geq 0$ such that $u\psi^{-k} \in X(A)$, for all $k \geq K$. Let $\Lambda = \Lambda_0 \Gamma^t$, where $\Lambda_0$ has no terminal segment equal to $\Gamma$. Then, for $j$ such that $m(j+1) \geq K$ and $j \geq t$, $u\psi^{-m(j+1)} \in X(A)$, so for some $z \in X$ and $\Xi \in A^*$, $u\psi^{-m(j+1)} = z\Xi$ and, from Lemma 4.21, $z\Xi$ has characteristic $(m, \Gamma)$. Hence

$$z\Xi \Gamma^j = z\Xi \psi^m(j+1) = u = x\Lambda_0 \Gamma^t,$$

which implies $z = x$ and $\Xi \Gamma^{j-t+1} = \Lambda_0$, a contradiction. As $\psi$ is in semi-normal form with respect to $X$ and $u$ is not in a complete $X$-component, the $X$-component $C$ of $u$ must be in a right semi-infinite. Moreover, as we have just shown the $\psi$-orbit of $u$ contains no left semi-infinite $X$-component, so $C$ is the unique $X(A)$-component of this $\psi$-orbit.

Suppose $x$ has characteristic $(k, \Omega)$. If the $X$-component of $x$ is left semi-infinite then $x\Lambda\psi^{-j} \in X(A)$, for all $k \geq 0$, so $C$ is not right semi-infinite. Hence $x$ is in a right semi-infinite $X$-component and $k > 0$. If $\Lambda = \Omega \Lambda_1$ then $x\Lambda_1 \psi^k = x\Omega \Lambda_1 = u$ and so $C$ contains $x\Lambda_1$; and it suffices to prove the Lemma under the assumption that that $\Lambda$ has no initial segment equal to $\Omega$. Suppose that $m = kp + r$, where $0 \leq r < k$. Then $x\Lambda \psi^{kp} = x\Omega \Lambda$ and $x\Omega^p \Lambda \psi^r = x\Lambda \psi^{kp+r} = x\Lambda \psi^m = x\Lambda$.
However, as \( x \) is in a right semi-infinite \( X \)-component, \( x\psi^r = z\Xi \), for some \( z \in X \) and \( \Xi \in A^* \). Thus \( x\Lambda = x\Omega^p \Lambda \psi^r = x\psi^r \Omega^p \Lambda = z\Xi \Omega^p \Lambda \), which implies that \( z = x \) and \( \Lambda \Gamma = \Xi \Omega^p \Lambda \). Now, as \( x\psi^r = z\Xi \), with \( 0 \leq r < k \), and \( x \) has characteristic \((k, \Omega)\), it must be that \( r = 0 \), \( m = kp \) and \( \Xi = \varepsilon \). We have now \( \Lambda \Gamma = \Omega^p \Lambda \) and, as \( \Lambda \) has no initial segment equal to \( \Omega \), consequently \( \Omega = \Lambda \Omega_1 \).

Now \( u\psi^k = x\Lambda \psi^k = x\psi^k \Lambda = x\Lambda \Omega_1 \Lambda = u\Omega_1 \Lambda \), so \( k \geq m \), by definition of characteristic. Therefore \( k = m \) and \( \Gamma = \Omega_1 \Lambda \), completing the proof in the case \( m > 0 \).

In the case when \( m < 0 \) the result follows from the above on replacing \( \psi \) by \( \psi^{-1} \).

**Lemma 4.25.** Let \( \theta \in G_{n,r} \) and \( u \in V_{n,r} \) such that \( u\theta^k = u\Delta \), where \( \Delta \neq \varepsilon \). Then \( u \) has characteristic \((m, \Gamma)\) with respect to \( \theta \), where \( k = mq \) and \( \Delta = \Gamma^q \), for some positive integer \( q \).

**Proof.** Let \( \theta \) be in semi-normal form with respect to \( X \). Suppose first that \( k > 0 \). Let \((m, \Gamma)\) be the characteristic of \( u \). As in the proof of Lemma 4.24 we may assume that \( u \in X(A) \), the \( X \)-component of \( u \) is right semi-infinite and that there exist \( x \in X \) and \( \Gamma_1 \in A^* \) such that \( u = x\Gamma_1 \) and \( x \) has characteristic power \( m \). Then \( k \geq m \), so \( k = mq + s \), where \( 0 \leq s < m \) and \( q \geq 1 \). Let \( x\theta^s = y\Lambda' \), where \( y \in X \) and \( \Lambda' \in A^* \).

Now \( x\Gamma_1 \Delta = u\Delta = u\theta^k = u\theta^{mq+s} = u\Gamma^q \theta^s = x\theta^s \Gamma_1 \Gamma^q = y\Lambda' \Gamma_1 \Gamma^q \).

Hence \( x = y \) and \( s = 0 \) and \( m = kq \). Moreover \( x\Lambda \Delta = u\Delta = u\theta^k = u\theta^{mq} = u\Gamma^q = x\Lambda \Gamma^q \), so \( \Lambda \Delta = \Lambda \Gamma^q \), from which \( \Delta = \Gamma^q \), as required.

If \( k < 0 \) then let \( \psi = \theta^{-1} \). We have \( u\psi^{-k} = u\Delta \), so from the previous part of the proof, \( u \) has characteristic \((m, \Gamma)\), with respect to \( \psi \), where \( -k = mq, q > 0 \), and \( \Delta = \Gamma^q \). If follows that \( u \) has characteristic \((−m, \Gamma)\), with respect to \( \theta \), and \( −m = kq \), completing the proof.

**Corollary 4.26.** Let \( \psi \) be in semi-normal form with respect to an \( A \)-basis \( X \) and let \( u \in V_{n,r} \). Then there exists an element \( \Lambda \in A^* \) such that \( u\Lambda \) belongs to a complete \( X \)-component of \( \psi \).

**Proof.** Multiplying by a sufficiently long element of \( A^* \) we may, as usual, assume that \( u \in X(A) \), so \( u \) belongs to either a complete or a semi-infinite \( X \)-component of \( \psi \). There are finitely many semi-infinite \( X \)-components. If \( S \) is a characteristic semi-infinite \( X \)-component with characteristic \((m, \Gamma)\) then, from Lemma 4.24 elements of \( S \) have the form \( x\Lambda \) where \( x \in X \), \( \Lambda \in A^* \) and, for all but finitely many elements of \( S \), \( \Lambda \) is periodic of period \( m \). Let \( F_S \) be the finite subset of elements of \( A^* \) such that \( \Lambda \in F_S \) only if \( x\Lambda \in S \) and \( \Lambda \) is not periodic of period \( m \). Let \( F_0 \) be the union of the \( F_S \) over all characteristic semi-infinite \( X \)-components. If \( S \) is non-characteristic then, from Lemma 4.24, \( S \) contains an element \( z\Delta \), where \( z \in X \) of type (B), with characteristic \((m', \Gamma')\), say. It follows, from Lemma 4.24 again, that all but finitely many elements of \( S \) have the form \( x\Lambda \Delta \) where \( x \in X \), \( \Lambda \in A^* \) and \( \Lambda \) is periodic of period \( m' \). This time, let \( F_S \) be the finite subset of elements of \( A^* \) such that \( \Lambda \Delta \in F_S \) only if \( x\Lambda \Delta \in S \) and \( \Lambda \) is not periodic of period \( m' \). Let \( F_1 \) be the union of the \( F_S \) over all non-characteristic semi-infinite \( X \)-components. Let \( M \) be the maximum of lengths of elements of \( F_0 \cup F_1 \) and assume \( u = x\Gamma \), where \( x \in X \), \( \Gamma \in A^* \). Choose element \( \Xi \) of \( A^* \) such that \( \Gamma \Xi \) has length greater than \( M \), is not periodic and does not factor as \( \Lambda \Delta \), where \( \Lambda \) is periodic and \( \Delta \in F_1 \). Then \( u\Xi = x\Gamma \Xi \) cannot belong to a semi-infinite \( X \)-component, so must belong to a complete \( X \)-component.

4.2 Quasi-normal forms

Quasi-normal forms, introduced in [Hig74, Section 9], are particular semi-normal forms which give representations of automorphisms minimising the number of elements in pond orbits. In [Hig74] it is claimed that if an automorphism is given with respect to a quasi-normal form, then it has no pond orbits. In this section we shall see that this is not the case.
Proof. Assume \( \psi \) is given by listing the images of elements of \( X \), where \( X \) is an \( A \)-basis of \( V_{n,r} \). We modify \( X \) to find an \( A \)-basis \( X' \) with respect to which \( \psi \) is in semi-normal form. For each \( y \in X \) we list elements of the \( \psi \)-orbit of \( y \):
\[
\ldots, y\psi^{-3}, y\psi^{-2}, y\psi^{-1}, y, y\psi, y\psi^2, y\psi^3, \ldots
\]
We begin with \( y \) and go forward in the sequence \( y\psi^i \), for \( i > 1 \), until we reach \( i = m \geq 0 \) such that,

(1F) either \( y\psi^m \in X\langle A \rangle \) with \( y\psi^{m+1} \notin X\langle A \rangle \) or,

(2F) for some \( l \) with \( 0 \leq l < m \) and for some \( y \in X \) and \( \Gamma, \Delta \in A^* \), \( y\varphi^l = \hat{y}\Gamma \) and \( y\varphi^m = \hat{y}\Delta \).

Similarly, we go backwards in the sequence, from \( y \), until we reach \( i = -k, k \geq 0 \) such that,

(1B) either \( y\psi^{-k} \in X\langle A \rangle \) with \( y\psi^{-(k+1)} \notin X\langle A \rangle \) or,

(2B) for some \( l \) with \( 0 \leq l < k \) and for some \( y \in X \) and \( \Gamma, \Delta \in A^* \), \( y\varphi^{-l} = \hat{y}\Gamma \) and \( y\varphi^{-k} = \hat{y}\Delta \).

Given \( y \in X \), the forward part of the process above produces a sequence of elements of \( X\langle A \rangle \), until it halts. As \( X \) is finite, if it does not halt at step (1F) then it halts at step (2F); so always halts. Similarly, the backward part of the process always halts.

If some \( y \) satisfies (1F) and (1B), then \( \psi \) is not in semi-normal form with respect to \( X \). In this case we take a simple expansion \( X' \) of \( X \) at the element \( y \) and start again, replacing \( X \) with \( X' \). That is, we implement the process described in the proof of Lemma 4.9 to find a basis with respect to which \( \psi \) is in semi-normal form. It follows from that proof that, eventually we shall find \( X \) such that no \( y \in X \) satisfies both (1F) and (1B), and then \( \psi \) is in semi-normal form with respect to \( X \), by Lemma 4.18. We can now assume \( \psi \) is in semi-normal form with respect to \( X \). We can thus test all the contractions of the \( A \)-basis \( X \) to find an expansion of \( x \) with respect to which \( \psi \) is in a quasi-normal form.

For uniqueness, we will argue by contradiction. Let \( \psi \) be in quasi-normal form with respect to \( X_1 \) and \( X_2 \), with \( X_1 \neq X_2 \). Since \( X_1 \neq X_2 \) and \( X_1, X_2 \) are expansions of \( x \), (without loss of generality) there exists a contraction \( X'_1 \) of \( X_1 \) which contains an element \( y \) of \( X_2 \setminus X_1 \). Then \( X'_1\langle A \rangle = X_1\langle A \rangle \cup \{ y \}\langle A \rangle \) and, as \( \psi \) is in semi-normal form with respect to \( X_2 \), it is also in semi-normal form with respect to \( X'_1 \), contrary to the definition of quasi-normal form.

Remark 4.29. Let \( \psi \in G_{n,r} \) be in quasi-normal form with respect to \( X \). The proof of this lemma illustrates that if \( \psi \) is in semi-normal form with respect to \( X' \), then \( X' \) is an expansion of \( X \). The converse is false: it is not true in general that \( \psi \) is in semi-normal form with respect to all expansions of \( X \).
Lemma 4.30. Let $\psi \in G_{n,r}$ be in semi-normal form with respect to an $A$-basis $X$ and let $u, v \in X(A)$. Then we can effectively decide whether or not $u, v$ are in the same $X$-component, and if so, find the integers $m$ for which $u\psi^m = v$.

Proof. As $u \in X(A)$, we have $u = y\Lambda$, where $y \in X$ and $\Lambda \in A^*$. We now run the process of Lemma 4.28 on $y$. If the process halts with $y\psi^m = y$, for some $m$ then we may list the elements $u\psi^i = y\psi^i\Lambda$, $i = 0, \ldots, m - 1$, of the (complete finite) $\psi$-orbit of $u$. In this case $v$ is in the same $\psi$-orbit as $u$ if and only if it appears in the list, so we are done. Otherwise the process halts at (1F) and (2B), at (2F) and (1B) or at (2F) and (2B). In all cases we obtain $\tilde{y} \in X$ such that, for some $k \neq l$ and $\Lambda_1 \neq \Lambda_2 \in A^*$, we have $y\varphi^k = \tilde{y}\Lambda_1$ and $y\varphi^l = \tilde{y}\Lambda_2$. It follows from Lemma 4.18 that $\tilde{y}$ is of type (B). As $u\varphi^k = y\Lambda\varphi^k = \tilde{y}\Lambda_1 \Lambda$ we may replace $u = y\Lambda$ with $\tilde{u} = \tilde{y}\Lambda_1 \Lambda$. Therefore we may assume that $u = y\Lambda$, where $y$ is of type (B). Now, when we run the process of Lemma 4.28 on $y$ it halts at (2F) and (1B) or at (1F) and (2B). Suppose first the forward part halts at (2F). Then $y$ is in a right semi-infinite $X$-component and there is a minimal positive integer $m$ such that $y\psi^m = y\Gamma$, with $\Gamma \neq 1$. That is $y$ has characteristic $(m, \Gamma)$, with $m > 0$.

If $\Lambda = \Gamma^i \Lambda_0$, where $\Lambda_0$ has no initial segment $\Gamma$, and we set $u_0 = y\Lambda_0$ then,

$$u_0\psi^m = y\Lambda_0\psi^m = y\psi^m\Lambda_0 = y\Gamma^i\Lambda_0 = y\Lambda = u,$$

so $u_0$ is in the same $\psi$-orbit as $u$. Hence we may replace $u = y\Lambda$ by $u_0 = y\Lambda_0$. Once we have done this we may suppose $\Lambda$ has no initial segment equal to the characteristic multiplier $\Gamma$ of $y$.

Next we run the process of Lemma 4.28 on $u$ instead of $y$. As $y$ is in a right semi-infinite $X$-component the forward part of the process halts at (2F). We obtain a list of elements of the $X$-component of $u$ of the form

$$z_r \Phi_r, \ldots, z_1 \Phi_1, u = y\Lambda, y_1 \Gamma_1 \Lambda, \ldots, y_{m-1} \Gamma_{m-1} \Lambda, y \Gamma \Lambda,$$

where $y_j, z_j \in X$, $\Gamma_j', \Phi_j \in A^*$. $z_j \Phi_j = u\psi^{-j}$, for $1 \leq j \leq r$ and for some $r \geq 0$, and $y\psi^s = y_i \Gamma_i$, for $0 < s < m$. (The $y_j$’s must be distinct otherwise $u$ would have characteristic power less than $m$.)

If the backward part of the process halts at (1B) then $z_r \Phi_r \psi^{-1} = u\psi^{-r-1} \notin X(A)$. In this case, the entire $X$-component of $u$ consists of the elements on this list together with elements

$$y_i \Gamma_i \Gamma^q \Lambda, q > 0, 0 < i \leq m,$$

where we set $y_0 = y$, $\Gamma_0' = \Gamma$.

As $v \in X(A)$ we also have $z \in X$ and $\Delta$ in $A^*$ such that $v = z\Delta$. If $z$ is in a finite $X$-component then $v$ cannot belong to the same $X$-component as $u$, so we assume $z$ is in an infinite $X$-component. As in the case of $u$, we may adjust $v$ so that $z$ is of type (B). As before we find a characteristic multiplier $\Phi$ for $z$ and, replacing $\Delta$ with a shorter element if necessary, we may assume that $\Delta$ has no initial segment equal to $\Phi$.

If $v = u\psi^d$, where $d \geq 0$, then $v = y_i \Gamma_i \Gamma^q \Lambda$, for some $q \geq 0$ and $i$ with $0 \leq i < m$. In this case, $z = y_i$ and by Lemma 4.24 and our assumption on $v$ we have $q = 0$, so $v = y_i \Gamma_i \Lambda$, which appears on list (6). Assume then that $v = u\psi^d$, where $d < 0$. As the backward part of the enumeration of the $\psi$-orbit of $u$ halts at (1B), the $X$-component of $u$ has initial element $z_r \Phi_r$, and $v$ must appear on list (6).

On the other hand, if the backward part of the process stops at (2B) then $u$ is in a complete infinite $X$-component and, for some $s$ with $0 \leq s \leq r$, we have $z_r = z_s$ (and $r$ is minimal with this property). It follows that $z_r$ is of type (B) and in a left semi-infinite $X$-component. Again, we
may assume that \( v = z\Delta \), where \( \Delta \in A^* \), \( z \in X \) is of type (B) and has characteristic multiplier \( \Phi \), such that \( \Delta \) has no initial segment equal to \( \Phi \). As before if \( v = u\psi^d \), where \( d \geq 0 \), then \( v \) appears on list (6). Assume then that \( v = u\psi^d \), where \( d < 0 \). Repeating the argument above, using the left semi-infinite \( X \)-component of \( z \), instead of the right semi-infinite \( X \)-component of \( y \), it follows again that \( v \) appears on list (6).

Therefore, in the case where \( y \) is in a right semi-infinite \( X \)-component we have \( v \) in the \( X \)-component of \( u \) if and only if \( v \) lies on the list (6), and we may compute \( m \) such that \( u\psi^m = v \), if this is the case. Finally, if the enumeration of the \( X \)-component of \( y \) halts at steps (1F) and (2B) then the process is essentially the same, except that we deal with a left, rather than a right, semi-infinite \( X \)-component of \( y \). \[ \square \]

This procedure allows us to decide if two given words belong to the same \( X \)-component so, if there are no pond orbits, we may decide if two words belong to the same \( \psi \)-orbit. On the other hand, as the enumeration of components always stops once we fall outside \( X(A) \), we cannot detect when a pair of elements lie in the same \( \psi \)-orbit but on opposite sides of a pond. We demonstrate below that in some cases there are no semi-normal forms which are free of ponds; and therefore we require a strategy to deal with ponds.

**Lemma 4.31.** Let \( \psi \in G_{n,r} \) be in semi-normal form with respect to \( X \), and suppose that some \( \psi \)-orbit \( O \) contains a pond with respect to \( X \). If \( \psi \) is in semi-normal form with respect to an expansion \( X' \) of \( X \), then \( O \) is also a pond-orbit with respect to \( X' \).

**Proof.** Let us write \( O \) as

\[
O: \ldots l\psi^{-k}, \ldots l\psi^{-1}, l, p_1, \ldots, p_k, r, r\varphi, \ldots, r\varphi^s, \ldots
\]

where \( l, r \in X(A) \) are endpoints of semi-infinite \( X \)-components and the \( p_i \notin X(A) \) form a pond of length \( m \). To begin with we claim that, for sufficiently large \( s \geq 0 \), we have \( r\psi^s \in X'(A) \). Indeed, because \( r \) belongs to a semi-infinite \( X \)-component, Lemma 3.18 implies that there is some \( s' \geq 0 \) for which \( r\psi^{s'} = r'\Delta \), where \( \Delta \in A^* \) and \( r' \in X \) has characteristic \((m, \Gamma)\). Therefore, for all \( q \geq 0 \),

\[
r\psi^{s'+mq} = r\psi^{s'}\psi^{mq} = r'\Delta\psi^{mq} = r'\psi^{mq}\Delta = r'\Gamma^q\Delta.
\]

By taking \( q \) sufficiently large, we can ensure that \( r\psi^{s'+mq} \in X'(A) \), since the difference \( X(A) \setminus X'(A) \) is finite. That is, we can find \( s \geq 0 \) such that \( r\psi^s \in X'(A) \). Similarly, there is some \( t \geq 0 \) for which \( l\psi^{-t} \in X'(A) \).

Since \( X'(A) \subset X(A) \), it follows that each \( p_i \notin X'(A) \). Appealing to Corollary 4.14 the only possibility is that \( O \) is a pond-orbit with respect to \( X' \). \[ \square \]

This proof demonstrates that the pond width with respect to \( X' \) is at least the previous width \( k \) with respect to \( X \). Additionally, if \( \psi \) was in quasi-normal form with respect to \( X \), this lemma shows that every semi-normal form \( X' \) for \( \psi \) contains the pond given above.

**Lemma 4.32.** Given an element \( \psi \in G_{n,r} \) in semi-normal form with respect to an \( A \)-basis \( X \) we may effectively construct a list of all initial and terminal elements of semi-infinite \( X \)-components of \( \psi \) and the set \( P(\psi) \) of the pairs \((l, r)\) such that \( l \) and \( r \) are initial and terminal elements of \( X \)-components of a pond orbit, and for each such pair the integer \( k \) such that \( r = l\psi^k \).
Proof. Let \( Y \) be the minimal basis for \( \psi \) and \( Z = Y \psi \). Then the set of initial elements of right semi-infinite \( X \)-components is \( R = X(A) \setminus Z(A) \), which we may enumerate effectively. An analogous statement applies to the set \( L = X(A) \setminus Z(A) \) of terminal elements of left semi-infinite \( X \)-components.

To enumerate the required set \( P(\psi) \), for each \((l, r) \in L \times R\) note that \((l, r) \in P(\psi)\) only if \( r = l\psi^k \), for some \( k \); in which case \( r\Gamma = l\Gamma\psi^k \), for all \( \Gamma \in A^* \). With this in mind, first find \( \Gamma \in A^* \) such that \( l\Gamma \) is in a complete orbit. For a given \( \Gamma \) this may be done using the process of Lemma 4.28 we iterate this until we find \( \Gamma \) such that, on input \( l\Gamma \), the process halts at (2F) and (2B). Now check, using Lemma 4.30, whether \( r\Gamma \) and \( l\Gamma \) are in the same \( X \)-component. If not, then \((l, r)\) cannot be in \( P(\psi) \).

Assume then that \( r\Gamma = l\Gamma \psi^m \), for some \( m \). If \((l, r) \in P(\psi)\), with \( r = l\psi^k \), then \( l\psi^k \Gamma = r\Gamma = l\Gamma \psi^m \), so \( l\Gamma \psi^{k-m} = l\Gamma \), and as \( l\Gamma \) belongs to a complete infinite \( X \)-component, this implies \( k = m \).

Hence, once we have found \( m \) such that \( r\Gamma = l\Gamma \psi^m \), we conclude that \((l, r)\) is in \( P(\psi) \) if and only if \( r = l\psi^m \).

In practice, in enumerating the sets \( L \) and \( R \) in the proof above, we need consider only non-characteristic elements, as Lemma 4.28 implies that no characteristic element belongs to a pond orbit.

Example 4.33. Let \( \psi \) and \( X \) be the automorphism and basis described in Example 4.17. As noted above \( \psi \) is in quasi-normal form with respect to \( X \) and we have seen that \( \psi \)-orbit of \( x\alpha_1^2\alpha_2 \) is a pond-orbit. We claim that this \( \psi \)-orbit is the only pond orbit with respect to \( X \).

The endpoints of semi-infinite \( X \)-components are precisely

\[
X(A) \setminus Y(A) = \{x\alpha_1^2, x\alpha_1^3, x\alpha_1^2\alpha_2\}
\]

and \( X(A) \setminus Z(A) = \{x\alpha_1\alpha_2, x\alpha_1\alpha_2\alpha_1, x\alpha_1\alpha_2^3\} \).

The four endpoints \( x\alpha_1^2, x\alpha_1^3, x\alpha_1\alpha_2 \) and \( x\alpha_1\alpha_2\alpha_1 \) have characteristics \((-1, \alpha_1^2), (-1, \alpha_1^3), (1, \alpha_1\alpha_2)\) and \((1, \alpha_1\alpha_2)\) respectively. The two remaining endpoints are separated by the pond we identified in Example 4.17. Hence \( P(\psi) = \{(x\alpha_1^2\alpha_2, x\alpha_1\alpha_2^3)\} \).

Lemma 4.34 (cf. [Hig74] Lemma 9.7). Let \( \psi \in G_{n,r} \) and \( u, v \in V_{n,r} \). Then we can effectively decide whether or not \( u, v \) are in the same \( \psi \)-orbit, and if so, find the integers \( m \) for which \( u\psi^m = v \).

Proof. For a fixed integer \( s \geq 0 \) we have \( u\psi^s = v \) if and only if

\[
(u\Gamma)\psi^s = u\psi^s\Gamma = v\Gamma
\]

for all \( \Gamma \in A^* \) of length \( s \) (using Lemma 4.20). Now, suppose that we have an algorithm \( A \) to decide whether \( v' = u'\psi^m \), for some \( m \), for elements \( u', v' \) of \( X(A) \) (and to return \( m \), if so). Then if \( u, v \) are arbitrary elements of \( V_{n,r} \) we may choose \( s \) such that \( u\Gamma \) and \( v\Gamma \) belong to \( X(A) \), for all \( \Gamma \in A^* \) of length \( s \), and input all these elements to the algorithm \( A \) in turn. In the light of the previous remark, this allows us to determine whether or not \( u \) and \( v \) belong to the same \( \psi \)-orbit (and to return appropriate \( m \), if so). Hence we may assume \( u, v \in X(A) \).

By Corollary 4.14 \( u \) and \( v \) belong to the same \( \psi \)-orbit if and only if either they belong to the same \( X(A) \)-component of a \( \psi \)-orbit, or they belong to different \( X \)-components of a single pond orbit. We may use Lemma 4.30 to decide whether or not \( u \) and \( v \) both belong to the same \( X \)-component. If so we are finished. If not, and both belong to semi-infinite orbits, then for each pair
Example 4.35. Let \( \psi \) be the automorphism of Examples 4.17 and 4.33, which is in quasi-normal form with respect to \( X = \{ q_1 = x_0^2, q_2 = x_0^2 q_2, q_3 = x_0^2 q_4, q_4 = x_0^2 \} \). The elements \( q_1 \) and \( q_2 \) have characteristics \((-1, \alpha^2_1\) and \((1, \alpha_1 \alpha_2)\) respectively, whereas \( q_3 \) and \( q_4 \) belong to complete infinite \( X \)-components such that \( q_3 \psi = q_2^2 \) and \( q_4 \psi^{-1} = q_1 \alpha_2 \alpha_1 \).

1. We wish to test if \( u = x_0^2 \alpha_0^2 \alpha_1^2 \alpha_2 = q_2^2 \alpha_0^2 \alpha_2^2 \) and \( v = x_0^2 \alpha_2^2 \) belong to the same \( \psi \)-orbit. Because \( q_3 \) is not characteristic, the algorithm first replaces \( v = q_3 \alpha_1 \) with \( \psi = q_3 \psi^{-1} \psi = q_2^2 \alpha_0^2 \), which begins with the characteristic element \( q_2 \) of \( X \). Enumerating the \( X \)-component containing \( u \) gives us a specific instance of list (6)

\[
\begin{align*}
  x_0^4 \alpha_0^2 \alpha_1^2 \alpha_2 &\mapsto x_0^2 \alpha_2^2 \alpha_0^2 \alpha_2 \mapsto x_0^2 \alpha_2^2 \alpha_1 \alpha_2 \mapsto x_0^2 \alpha_2^2 \alpha_2^2 \alpha_0^2 \mapsto x_0^2 \alpha_0^2 \alpha_1^2 \alpha_2 \mapsto &\ldots
\end{align*}
\]

once the enumeration has halted at stages (2F) and (2B). Since \( \psi \) does not lie on this list, we conclude that \( \psi \) does not belong to the \( X \)-component of \( u \), so neither does \( v \).

We now need to check if \( u \) and \( v \) are separated by a pond. In Example 4.33 we showed that \( \psi \) has only one pond-orbit, and referring to the enumeration given in Example 4.17 we see that neither \( u \) nor \( v \) belong to this orbit. Hence \( u \) and \( v \) do not share a \( \psi \)-orbit.

Now let us test if \( u = x_0^4 \alpha_0^2 \alpha_2 \) and \( w = x_0^2 \alpha_2^2 \alpha_1 \) share a \( \psi \)-orbit. We remove the characteristic multiplier \( \alpha_2 \) of \( q_1 \) from \( w \), obtaining \( w' = q_1 \alpha_2 \alpha_2 \) where \( w' \psi^{-1} = w \). From the list (6) we notice that \( w \psi^{-2} = w' \), so \( w \psi^{-2} = w \).

2. Let \( u = x_0^4 \alpha_0^2 \), \( v = x_0^4 \alpha_2 \alpha_1 \), and \( w = x_0^2 \alpha_2 \alpha_2 \). In terms of \( X \), these are \( u = q_1 \alpha_0^2 \alpha_2 \), \( v = q_1 \alpha_0^2 \alpha_2 \alpha_1 \), and \( w = q_2 (\alpha_0 \alpha_1 \alpha_2^2 \alpha_2 \). Since \( q_1 \) and \( q_2 \) are characteristic, we remove copies of the characteristic multipliers. We obtain \( u' = q_1 \alpha_2 = w^{-3} \), \( v' = q_1 \alpha_2 \alpha_1 = w \psi^2 \) and \( w' = q_2 \alpha_2 = w \psi^{-2} \). Enumerating the \( X \)-component of \( u' \) gives us

\[
\ldots \mapsto x_0^4 \alpha_2 \mapsto x_0^2 \alpha_2 \alpha_2 = u',
\]

(halting at stages (1F) and (2B)) and we see that neither \( u' \) nor \( w' \) are in this list. However, \( u' \) is adjacent to a pond. Referring once more to Example 4.17 we see that the corresponding endpoint is \( \bar{u} = u' \psi^2 = x_0^4 \alpha_2 \). Its \( X \)-component begins

\[
\begin{align*}
\bar{u} = x_0^4 \alpha_2 &\mapsto x_0^2 \alpha_0 \alpha_2 \mapsto \ldots
\end{align*}
\]

Since this list does not contain \( u' \), we conclude that \( u \) and \( v \) do not share a \( \psi \)-orbit. On the other hand, we note that \( w' = \bar{u} \) belongs to the list. Hence \( u \) and \( w \) belong to the same \( \psi \)-orbit, and having kept track of the various powers, we calculate that

\[
\begin{align*}
w \psi^{-2} = w' = \bar{u} = u' \psi^2 = w \psi^3 \psi^2 &\implies w = w \psi^7.
\end{align*}
\]
5 The Conjugacy problem

For a group with presentation \((X \mid R)\), the conjugacy problem is to determine, given words \(g, h \in \mathbb{F}(X)\) whether or not \(g\) is conjugate to \(h\) in \(G\); denoted \(g \sim h\). The strong form, which we consider here, requires in addition that, if \(g\) is conjugate to \(h\), then an element \(c \in \mathbb{F}(X)\) is found, such that \(c^{-1}gc = h\). We say the conjugacy problem is decidable if there is an algorithm which, given \(g\) and \(h\) outputs “yes” if they’re conjugate and “no” otherwise. The stronger form entails the obvious rejoinder. The word problem is the special case of the conjugacy problem where \(h = 1\).

As pointed out at the beginning of Section 4, an element \(u\) of \(V_{n,r}\) may be uniquely represented by the triple \((Y, Z, \psi_0)\), where \(Y\) is the minimal expansion of \(\psi\), \(Z = Y\psi\) and \(\psi_0\) is a bijection between \(Y\) and \(Z\), namely \(\psi_0 = \psi|_Y\). This triple is called a symbol for \(\psi\). In [Hig74] Section 4|a finite presentation of \(G_{n,r}\) is given, with generators the symbols \((Y, Z, \psi_0)\) such that \(Y\) is a \(d\)-fold expansion of \(x\), for \(d \leq 3\). As we may effectively enumerate symbols and effectively construct the symbol for \(\psi_1\psi_2\), from the symbols for \(\psi_1\) and \(\psi_2\), words in Higman’s generators effectively determine symbols and vice-versa. Therefore when we consider algorithmic problems in \(G_{n,r}\) we may work with symbols for automorphisms, and leave the presentation in the background. That is, we always assume that automorphisms are given as maps between bases of \(V_{n,r}\) (from which a symbol may be computed). As minimal expansions are unique it follows immediately that the word problem is solvable in \(G_{n,r}\). In this section we give an algorithm for the conjugacy problem in \(G_{n,r}\), based on (a complete version of) Higman’s solution.

5.1 Higman’s \(\psi\)-invariant subalgebras \(V_P\) and \(V_{RI}\)

Let \(\psi\) be an element of \(G_{n,r}\). Higman defined two subalgebras of \(V_{n,r}\), determined by \(\psi\); namely

- the subalgebra \(V_{P,\psi}\) generated by the set of elements of \(V_{n,r}\) which belong to finite \(\psi\)-orbits.
- the subalgebra \(V_{RI,\psi}\) generated by the set of characteristic elements for \(\psi\).

Where there is no ambiguity, we will write \(V_P\) for \(V_{P,\psi}\) and \(V_{RI}\) for \(V_{RI,\psi}\).

If \(u \in V_{n,r}\) then the \(\psi\)-orbit of \(u\) is identical to the \(\psi\)-orbit of \(uw\); so \(u\) is in a finite \(\psi\)-orbit if and only if \(uw\) is in a finite \(\psi\)-orbit. From Lemma 4.21 an element \(u\) is a characteristic element for \(\psi\) if and only if \(uw\) is a characteristic element for \(\psi\). Therefore \(V_{P,\psi}\) and \(V_{RI,\psi}\) are \(\psi\)-invariant subalgebras of \(V_{n,r}\). (A subalgebra \(S\) is \(\psi\)-invariant if \(S\psi = S\).) Hence \(\psi_P = \psi|_{V_P}\) is an automorphism of \(V_{P,\psi}\) and \(\psi_{RI} = \psi|_{V_{RI,\psi}}\) is an automorphism of \(V_{RI,\psi}\).

If \(\psi\) and \(\varphi\) are conjugate elements of \(G_{n,r}\) and \(\rho^{-1}\psi\rho = \varphi\), for some \(\rho \in G_{n,r}\), then, for all \(\Gamma \in A^*\) we have \(u\varphi^m = u\Gamma\) if and only if \((u\rho^{-1})\psi^m\rho = u\Gamma\) if and only if \((u\rho^{-1})\psi^m = (u\rho^{-1})\Gamma\). Thus \(u\) is in a finite \(\varphi\)-orbit if and only if \(u\rho^{-1}\) is in a finite \(\psi\)-orbit (taking \(\Gamma = \varepsilon\)) and \(u\) is a characteristic element for \(\varphi\) if and only if \(u\rho^{-1}\) is a characteristic element for \(\psi\) \((\Gamma \neq \varepsilon)\). It follows that the restriction \(\rho|_{V_{P,\psi}}\) of \(\rho\) to \(V_{P,\psi}\) maps \(V_{P,\psi}\) isomorphically to \(V_{P,\varphi}\) and similarly, \(\rho|_{V_{RI,\psi}}\) is an isomorphism from \(V_{RI,\psi}\) to \(V_{RI,\varphi}\).

Now suppose that \(\psi\) is in semi-normal form with respect to an \(A\)-basis \(X\). Partition \(X\) into

\[
X_P = X_{P,\psi} = \{ y \in X \mid y \text{ is of type (A)} \},
\]

and

\[
X_{RI} = X_{RI,\psi} = \{ y \in X \mid y \text{ is of type (B) or (C)} \}.
\]
Theorem 5.1 ([Hig74 Theorem 9.5]). Let $\psi$ be an element of $G_{n,r}$, in semi-normal form with respect to $A$-basis $X$. Then, with the notation above, the following hold.

1. $V_{n,r} = V_P \ast V_{RI}$, the free product of the $\psi$-invariant subalgebras $V_P$ and $V_{RI}$.

2. 

\[ V_P = X_P(A)\langle \lambda \rangle \]

and

\[ V_{RI} = X_{RI}(A)\langle \lambda \rangle \]

3. Let $\psi$ and $\varphi$ be elements of $G_{n,r}$ and write $\psi_P = \psi_{V_P,\varphi}$, $\varphi_P = \varphi_{V_P,\varphi}$, $\psi_{RI} = \psi_{V_{RI},\varphi}$ and $\varphi_{RI} = \varphi_{V_{RI},\varphi}$.

Then $\rho^{-1}\psi\rho = \varphi$, where $\rho \in G_{n,r}$, if and only if writing $\rho_P = \rho|_{V_P,\varphi}$ and $\rho_{RI} = \rho|_{V_{RI},\varphi}$ we have

\[ \rho_P^{-1}\psi_P \rho_P = \varphi_P \text{ and } \rho_{RI}^{-1}\psi_{RI} \rho_{RI} = \varphi_{RI}. \]

Proof. Let $W_P = X_P(A)\langle \lambda \rangle$ and $W_{RI} = X_{RI}(A)\langle \lambda \rangle$. As $X$ is the disjoint union of $X_P$ and $X_{RI}$, we have $V_{n,r} = W_P \ast W_{RI}$, using Lemma 3.11. We shall show that $V_P = W_P$ and $V_{RI} = W_{RI}$. By definition, $W_P \subseteq V_P$. If $x \in X_{RI}$ is of type (B) then $x \in V_{RI}$, by definition. If $x \in X_{RI}$ is of type (C) then there exists $z \in X_{RI}$, of type (B), and $\Delta \in A^*$, such that $x\psi^i = z\Delta$. As $z \in V_{RI}$, so is $z\Delta$, and as $V_{RI}$ is $\psi$-invariant we have $x = z\Delta\psi^{-1} \in V_{RI}$. Hence $W_{RI} \subseteq V_{RI}$.

To see that $V_P \subseteq W_P$, let $u \in V_{n,r}$ have a finite $\psi$-orbit. Choose $d \in \mathbb{N}$ such that, for all $\Gamma \in A^*$ of length $d$, we have $u\Gamma \in X(A)$, for all $\Gamma \in A^*$ of length $d$. Let $\Gamma$ be a length $d$ element of $A^*$, so $u\Gamma = x\Delta$, for some $x \in X$, $\Delta \in A^*$. As $u$ is in a finite $\psi$-orbit so is $u\Gamma$, so $x \in X_P$ and hence $u\Gamma = x\Delta \in W_P$. This holds for all $\Gamma$ in $A^*$ of length $d$, we have $u \in W_P$, by Lemma 3.11. Hence $V_P \subseteq W_P$.

To see that $V_{RI} \subseteq W_{RI}$, we first show that $W_{RI}$ is $\psi$-invariant. Let $Y$ be the minimal expansion of $X$ associated to $\psi$ and let $x \in X_{RI}$. Then choose $d$ such that $x\Gamma \in Y(A)$, for all $\Gamma \in A^*$ of length $d$. Given $\Gamma \in A^*$ of length $d$, let $y \in Y$ and $\Delta \in A^*$ such that $x\Gamma = y\Delta$. Then $x\Gamma\psi = y\psi\Delta \in X(A)$, so $x\Gamma\psi = z\Lambda$, for some $z \in X$ and $\Lambda \in A^*$. Moreover, $z$ must have type (B) or (C), as $x$ does, so $x\Gamma\psi \in X_{RI}(A) \subseteq W_{RI}$. This holds for all $\Gamma$ of length $d$, so again $x\psi \in W_{RI}$. It follows that $W_{RI}\psi \subseteq W_{RI}$. Repeating the same argument, using $Z = Y\psi$ instead of $Y$ and $\psi^{-1}$ instead of $\psi$ gives $W_{RI}\psi^{-1} \subseteq W_{RI}$; so $W_{RI}$ is $\psi$-invariant as claimed. Now let $u \in V_{n,r}$ be a characteristic element for $\psi$. Then, from Lemma 4.24 we have $u\psi^i = x\Lambda$, for some integer $i$, $x \in X_{RI}$ and $\Lambda \in A^*$. Thus $u = x\Lambda\psi^{-1} \in W_{RI}$, as $W_{RI}$ is $\psi$-invariant; and we have $V_{RI} \subseteq W_{RI}$. This proves 1 and 2 of the Theorem, and 3 then follows from the discussion preceding the statement of the Theorem.

Note that in the case that $\rho^{-1}\psi\rho = \varphi$ in the theorem above we have $\rho = \rho_P \ast \rho_{RI}$ an isomorphism from $V_{P,\varphi} \ast V_{RI,\varphi}$ to $V_{P,\psi} \ast V_{RI,\psi}$, both of which are isomorphic to $V_{n,r}$.

Example 5.2. Let $\psi$ be as in Example 4.3. Then $X_P = \{x\alpha_2\alpha_1, x\alpha_2^2\}$ and $X_{RI} = \{x\alpha_1^2, x\alpha_1\alpha_2\}$. Thus $\psi_P$ is the automorphism of $V_P = X_P(A)\langle \lambda \rangle$ defined by

\[ x\alpha_2\alpha_1 \mapsto x\alpha_2^2, x\alpha_2 \mapsto x\alpha_2\alpha_1. \]

Let $Y_{RI} = \{x\alpha_3^2, x\alpha_1^2\alpha_2, x\alpha_1\alpha_2\}$ and $Z_{RI} = \{x\alpha_2^3, x\alpha_1\alpha_2\alpha_1, x\alpha_1\alpha_2^2\}$, both of which are expansions of $X_{RI}$. Then $\psi_{RI}$ is the automorphism of $V_{RI} = X_{RI}(A)\langle \lambda \rangle$ defined by

\[ x\alpha_3^2 \mapsto x\alpha_2^3, x\alpha_1^2\alpha_2 \mapsto x\alpha_1\alpha_2\alpha_1, x\alpha_1\alpha_2 \mapsto x\alpha_1\alpha_2^2. \]
Theorem 5.1 allows us to decompose the conjugacy problem for ψ and φ into conjugacy problems for ψ_p and φ_p for ψ_R and φ_R. Indeed, if |X_p| = d_p and |X_R| = d_R then V_p ≅ V_{n,d_p} and V_R ≅ V_{n,d_R} and we regard ψ_p and ψ_R as automorphisms of V_{n,d_p} and V_{n,d_R}, respectively. It turns out that ψ_p and ψ_R are each of particularly simple types; so if we can solve the conjugacy problem for these simple types of automorphism, then we can solve it in general. In the remainder of this subsection we describe in detail how this decomposition works.

First consider a single automorphism ψ ∈ G_{n,r}, where ψ is in semi-normal form with respect to an A-basis X. Here we assume that V_{n,r} is the free V_n algebra on a set x of size r, and that X is an expansion of x. Let X_p and X_R be defined as above, let Y be the minimal expansion of X associated to ψ and let Z = Y ψ. As Y is an expansion of X, for all x ∈ X the set Y_x = Y ∩ {x}⟨A⟩ is an expansion of {x}, by Lemma 3.10. Therefore Y_p = Y ∩ X_p⟨A⟩ is an expansion of X_p, and Y_R = Y ∩ X_R⟩(A) is an expansion of X_R. Similarly, Z_p = Z ∩ X_p⟨A⟩ and Z_R = Z ∩ X_R⟩(A) are expansions of X_p and X_R, respectively. In fact, as ψ permutes the elements of X of type (A), ψ_p permutes the elements of X_p, so X_p = Y_p = Z_p. Therefore ψ_p is an automorphism of V_p = V_p⟨A⟩⟨λ⟩, which permutes the elements of X_p. For all y ∈ X_R we have ψ y = z ∈ Z, and z is in X_R⟩(A), since V_R is ψ-invariant, so Y_R ψ = Z_R. Now ψ_R is an automorphism of V_R, where V_R is freely generated by X_R and Y_R is the minimal expansion of X_R associated to ψ_R (as Y is the minimal expansion of X associated to ψ). Furthermore Y_R ψ_R = Z_R and if u is an element of X_R⟨A⟩ such that ψ_u ∈ X(A) then ψ_u ∈ X⟨A⟩ ∩ V_R = X_R⟨A⟩: so no element of X_R⟨A⟩ is in an incomplete finite X_R-component of ψ_R. Now let |X_p| = a and |X_R| = b and let X_p = {x_1, ..., x_a} and X_R = {x_{a+1}, ..., x_{a+b}}, where x_i ∈ X(A). Then, regarding the x_i as new symbols, we may view V_p as V_{n,a}, the free V_n algebra on {x_1, ..., x_a}, and V_R as V_{n,b}, the free V_n algebra on {x_{a+1}, ..., x_{a+b}}. We may thus regard ψ_p and ψ_R as elements of G_{n,a} and G_{n,b}, respectively. In this case, ψ_p is in quasi-normal form with respect to the A-basis X_p = {x_1, ..., x_a} and ψ_R is in quasi-normal form with respect to X_R = {x_{a+1}, ..., x_{a+b}}. Moreover, the minimal expansion of X_R associated to ψ_R is Y_R. (Here we write all elements of Y and Z in terms of the x_i, rather than as expansions of elements of x.)

**Example 5.3.** Let n = 2, r = 1 and V_{2,1} be free on {x}. Let

\[ Y = \{x_0, x_0^2, x_0^3, x_0^4, x_0^5, x_0^6, x_0^7, x_0^8\} \]

and

\[ Z = \{x_0, x_0^2, x_0^3, x_0^4, x_0^5, x_0^6, x_0^7, x_0^8\} \]

and let ψ be the element of G_{n,r} determined by the bijection from Y to Z given by x_0^4 ψ = x_0^1, x_0^2 ψ = x_0^7, x_0^3 ψ = x_0^2, x_0^5 ψ = x_0^5, x_0^6 ψ = x_0^2, x_0^7 ψ = x_0^2, and x_0^8 ψ = x_0^2.

Then Y is the minimal expansion of {x} associated to ψ. The minimal expansion of {x} not containing any element of of {x}⟨A⟩ \ (Y⟨A⟩ ∪ Z⟨A⟩) is

\[ X = \{x_0^3, x_0^4, x_0^5, x_0^6, x_0^7, x_0^8\} \]
Then $X(A) \setminus (Y(A) \cap Z(A)) = \{x\alpha_1^3, x\alpha_1^2\alpha_2, x\alpha_2\alpha_1, x\alpha_2^3\}$. The $X$-components of these elements are
\[
\cdots \mapsto x\alpha_1^4 \mapsto x\alpha_1^3 \\
\phantom{x}x\alpha_1^2\alpha_2 \mapsto x\alpha_1^2\alpha_2^2 \mapsto \cdots \\
\phantom{xx}x\alpha_2\alpha_1 \mapsto x\alpha_2\alpha_1^2 \mapsto \cdots \\
\cdots \mapsto x\alpha_2^3 \mapsto x\alpha_2^2
\]
so $\psi$ is in quasi-normal form with respect to $X$. Define $x_1 = x\alpha_1^3$, $x_2 = x\alpha_1^2\alpha_2$, $x_3 = x\alpha_1\alpha_2\alpha_1$, $x_4 = x\alpha_1\alpha_2^2$, $x_5 = x\alpha_2\alpha_1$ and $x_6 = x\alpha_2^2$. Then $X_P = \{x_1, x_4\}$ and $X_{RI} = \{x_1, x_2, x_5, x_6\}$.

Let $V_{2,2}$ be free on $\{x_3, x_4\}$. Then, as an element of $G_{2,2}$ the map $\psi_P$ is the map sending $x_3$ to $x_4$ and $x_4$ to $x_3$. Let $V_{2,4}$ be free on $\{x_1, x_2, x_5, x_6\}$. We have
\[
Y_{RI} = \{x\alpha_1^4, x\alpha_1^2\alpha_2, x\alpha_1^3\alpha_2, x\alpha_2\alpha_1, x\alpha_2^2\alpha_1, x\alpha_2^3\} = \{x_1\alpha_1, x_1\alpha_2, x_2, x_5, x_6\alpha_1, x_6\alpha_2\}
\]
and
\[
Z_{RI} = \{x\alpha_1^3, x\alpha_1^2\alpha_2\alpha_1, x\alpha_1^2\alpha_2^2, x\alpha_2\alpha_1\alpha_2, x\alpha_2^2\alpha_2\} = \{x_1, x_2\alpha_1, x_2\alpha_2, x_5\alpha_1, x_5\alpha_2, x_6\}
\]
so as an element of $G_{2,4}$ the map $\psi_{RI}$ is given by

\[
\psi_{RI} : \begin{array}{cccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\hline
x_1 & x_2 & x_5 & x_6 & \mapsto & x_1 & x_2 & x_5 & x_6
\end{array}
\]

Next we need definitions of the simple types of automorphism alluded to above.

**Definition 5.4.** Let $\psi$ be an element of $G_{n,r}$. Then $\psi$ is called **periodic** if $V_{RI} = \emptyset$ and $\psi$ is called **regular infinite** if $V_P = \emptyset$.

**Lemma 5.5.** Let $\psi$ be an element of $G_{n,r}$ in semi-normal form with respect to an $A$-basis $X$.

1. $\psi$ is periodic if and only if $\psi$ permutes the elements of $X$.
2. $\psi$ is regular infinite if and only if no element of $X$ is of type (A).

**Proof.** 1. If $\psi$ permutes the elements of $X$ then $X$ contains no element of type (B) or (C); so $X = X_P$ and $V_{n,r} = V_P$, by Theorem 5.1. As $V_{n,r}$ is the free product of $V_P$ and $V_{RI}$ it follows that $V_{RI} = \emptyset$ so $\psi$ is periodic.

   If $\psi$ is periodic then $X_{RI} \subseteq V_{RI} = \emptyset$, so $X = X_P$. Thus $X$ consists of elements of type (A), which are permuted by $\psi$, by Lemma 4.18.

2. If $\psi$ is regular infinite then $V_P = \emptyset$, so $X_P = \emptyset$; i.e. no element of $X$ is of type (A). If $X$ contains no element of type (A) then $X_P = \emptyset$ and therefore $V_P = \emptyset$, from Theorem 5.1 and $\psi$ is regular infinite.

$\square$
Algorithm 5.6. Let \( \psi \) and \( \varphi \) be an elements of \( G_{n,r} \).

Step 1: Find \( A \)-bases \( X_\psi \) and \( X_\varphi \) such that \( \psi \) and \( \varphi \) are in quasi-normal form with respect to \( X_\psi \) and \( X_\varphi \), respectively; as in Lemma 4.28. The sets \( X_{P,\psi} \), \( X_{RL,\psi} \), \( X_{P,\varphi} \) and \( X_{RL,\varphi} \) are obtained as part of this process.

If \( |X_{P,\psi}| \equiv |X_{P,\varphi}| \mod n - 1 \) and \( |X_{RL,\psi}| \equiv |X_{RL,\varphi}| \mod n - 1 \); continue. Otherwise output “No” and stop.

It follows that, in the notation established above Example 5.3 the automorphism \( \psi_P \in G_{n,a} \) is periodic and \( \psi_{RI} \in G_{n,b} \) is regular infinite. Thus, the decomposition of Theorem 5.1.1 may be viewed as factoring \( \psi \) into a product of a periodic and a regular infinite automorphism. It remains to see how to regard a pair of automorphisms in this way, simultaneously in the same algebra.

To this end suppose that \( \psi_i \in G_{n,a_i} \) is in semi-normal form with respect to an \( A \)-basis \( X_i \), where \( |X_i| = a_i \), for \( i = 1, 2 \). If there exists an isomorphism \( \rho : V_{n,a_1} \rightarrow V_{n,a_2} \) with the property that \( \rho^{-1}\psi_1\rho = \psi_2 \) then, from Corollary 3.14 \( a_1 \equiv a_2 \mod n - 1 \). Also, if \( a_1 \equiv a_2 \mod n - 1 \) then \( V_{n,a_i} \) is isomorphic to \( V_{n,s} \) where \( 1 \leq s \leq n - 1 \) and \( s \equiv a_i \). If this is the case then we may take an \( A \)-basis \( X_s \) of \( s \) elements of \( V_{n,s} \) and choose expansions \( X'_1 \) and \( X'_2 \) of \( X_s \) of \( a_1 \) and \( a_2 \) elements respectively.

Now let \( f_i \) be the map taking \( X_i \) to \( X'_i \). Then there exists an isomorphism \( \rho : V_{n,a_1} \rightarrow V_{n,a_2} \) such that \( \rho^{-1}\psi_1\rho = \psi_2 \) if and only if \( a_1 \equiv a_2 \mod n - 1 \) and, setting \( \tilde{\psi}_i = f_i^{-1}\psi_if_i \in G_{n,s} \), we have \( \rho^{-1}f_1\tilde{\psi}_1f_1^{-1}\rho = f_2\tilde{\psi}_2f_2^{-1} \): that is \( \theta^{-1}\psi_1\theta = \psi_2 \), where \( \theta = f_1^{-1}\rho f_2 \in G_{n,s} \). (See Figure 5.1.1)

Combining this with Theorem 5.1.1 gives a decomposition of the conjugacy problem into the conjugacy problem for periodic and for regular infinite elements, separately. Let \( \psi \) and \( \varphi \) be elements of \( G_{n,r} \), write \( V_{n,a_1} = V_{RL,\psi} \), \( \psi_1 = \psi_{RI} \), \( V_{n,a_2} = V_{RL,\varphi} \) and \( \psi_2 = \varphi_{RI} \). Using the procedure above, if \( \rho_{RI} \) exists (in the notation of Theorem 5.1) then we may regard \( \tilde{\psi}_i = \psi_i \), \( i = 1, 2 \), as a regular infinite element of \( G_{n,s} \), namely \( \tilde{\psi}_1 \), for appropriate \( s \). Similarly, we may regard \( \psi_P \) and \( \varphi_P \) as periodic automorphisms of a single algebra.

We can now outline the algorithm for the conjugacy problem.

### 5.2 The conjugacy algorithm

**Algorithm 5.6.** Let \( \psi \) and \( \varphi \) be an elements of \( G_{n,r} \).

**Step 1:** Find \( A \)-bases \( X_\psi \) and \( X_\varphi \) such that \( \psi \) and \( \varphi \) are in quasi-normal form with respect to \( X_\psi \) and \( X_\varphi \), respectively; as in Lemma 4.28. The sets \( X_{P,\psi} \), \( X_{RL,\psi} \), \( X_{P,\varphi} \) and \( X_{RL,\varphi} \) are obtained as part of this process.

If \( |X_{P,\psi}| \equiv |X_{P,\varphi}| \mod n - 1 \) and \( |X_{RL,\psi}| \equiv |X_{RL,\varphi}| \mod n - 1 \); continue. Otherwise output “No” and stop.
Step 2: Find the minimal expansion $Y_\psi$ of $X_\psi$ associated to $\psi$ and the minimal expansion $Y_\varphi$ of $X_\varphi$ associated to $\varphi$. (See Lemma 4.3.) Construct $Y_{RI,\psi}$ and $Y_{RI,\varphi}$: the sets elements of $Y_\psi$ and $Y_\varphi$ which are not in finite orbits (as in the the discussion following Theorem 5.1). Construct $Z_{RI,\psi} = Y_{RI,\psi}$ and $Z_{RI,\varphi} = Y_{RI,\varphi}$.

Step 3: For $T = P$ and for $T = RI$ carry out the following. Find the integer $s_T$ such that $1 \leq s_T \leq n - 1$ and $s_T \equiv |X_{T,\psi}|$. Let $x_T$ be a set of $s_T$ elements, let $V_{n,s_T}$ be free on $x_T$ and find expansions $W_{T,\psi}$ and $W_{T,\varphi}$ of $x_T$ of sizes $|X_{T,\psi}|$ and $|X_{T,\varphi}|$, respectively. Construct a map $f_{T,\psi}$ mapping $X_{T,\psi}$ bijectively to $W_{T,\psi}$ and $f_{T,\varphi}$ mapping $X_{T,\varphi}$ bijectively to $W_{T,\varphi}$. Write $\psi_T$ and $\varphi_T$ as elements of $G_{n,s_T}$, using these maps.

Step 4: Input $\psi_P$ and $\varphi_P$ into Algorithm 5.13 for conjugacy of periodic elements of $G_{n,r}$, below. If $\psi_P$ and $\varphi_P$ are not conjugate, return “No” and stop. Otherwise return a conjugating element $\rho_P$.

Step 5: Input $\psi_{RI}$ and $\varphi_{RI}$ into Algorithm 5.27 for conjugacy of regular infinite elements of $G_{n,s_{RI}}$ in Section 5.4 below. If $\psi_{RI}$ and $\varphi_{RI}$ are not conjugate, return “No” and stop. Otherwise return a conjugating element $\rho_{RI}$.

Step 6: Return the conjugating element $\rho_P * \rho_{RI}$.

Given this algorithm we have the following theorem.

Theorem 5.7. [Hig74, part of Theorem 9.3] The conjugacy problem is solvable in $G_{n,r}$.

Proof. Apply Algorithm 5.6.

5.3 Conjugacy of periodic elements

Let $\psi \in G_{n,r}$ be a periodic element. For $u \in V_{n,r}$ the size of the $\psi$-orbit of $u$ is the least positive integer $d$ such that $u \psi^d = u$.

Definition 5.8. Let $\psi$ be a periodic element of $G_{n,r}$ in semi-normal form with respect to the $A$-basis $X$. The cycle type of $\psi$ is the set

$$T_\psi = T_\psi(X) = \{d \in \mathbb{N} | x \text{ has } \psi \text{-orbit of size } d \text{ for some } x \in X\}.$$  

For $d \in \mathbb{N}$, define the $\psi$-multiplicity of $d$ to be $m_\psi(d, X) = D/d$, where $D$ is the number of elements of $X$ which belong to a $\psi$-orbit of size $d$.

Note that, as $\psi$ is periodic and in semi-normal form with respect to $X$, all $X$-components of $\psi$ are (ordered) $\psi$-orbits and all $\psi$-orbits of elements of $X(A)$ are $X$-components (once ordered appropriately). Also, $d \in T_\psi(X)$ if and only if $m_\psi(d, X) \neq 0$; the size of the set $X$ is $|X| = \sum_{d \in T_\psi} dm_\psi(d, X)$; if $d \in T_\psi$ then $X$ contains $m_\psi(d, X)$ disjoint $\psi$-orbits of size $d$; and $\psi$ is a torsion element of order equal to the least common multiple of elements of $T_\psi$.

Example 5.9. Let $n = 2$, $r = 1$ and $V_{2,1}$ be free on \{x\}. Let

$$Y = \{x a_1^3, x a_1^2 a_2, x a_1 a_2, x a_2 a_1^2, x a_2 a_1 a_2, x a_2^2 a_1, x a_2^3\}$$

and let $\psi$ be the periodic element of $G_{2,1}$ defined by

$$x a_1^3 \mapsto x a_1^2 a_2, x a_1^2 a_2 \mapsto x a_1 a_2, x a_2 a_1 \mapsto x a_1^3,$$

$$x a_2 a_1^2 \mapsto x a_2 a_1 a_2, x a_2 a_1 a_2 \mapsto x a_2 a_1, x a_2^2 a_1 \mapsto x a_2^3, x a_2^3 \mapsto x a_2^2 a_1.$$
\[ \psi : 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \]

Then the cycle type of \( \psi \) is \( \{2, 3\} \), \( m_\psi(2, Y) = 2 \) and \( m_\psi(3, Y) = 1 \).

**Lemma 5.10.** Let \( \psi \) be a periodic element of \( G_{n,r} \) in semi-normal form with respect to the A-basis \( X \) and the A-basis \( Z \), where \( Z \) is a q-fold expansion of \( X \). Then \( T_\psi(X) = T_\psi(Z) \) and \( m_\psi(d, X) \equiv m_\psi(d, Z) \mod n-1 \), for all \( d \in T_\psi \).

**Proof.** Let \( z \in Z \), say \( z = x\Gamma \), where \( x \in X \) and \( \Gamma \in A^* \), and let \( x \) have \( \psi \)-orbit of size \( d \). As \( \psi \) is periodic it permutes the elements of \( \psi \) so the \( \psi \)-orbit of \( x \) is \( \{x_0, \ldots, x_{d-1}\} \), where \( x_1 \in X \), \( x_{i+1} = x_i\psi \), subscripts modulo \( n-1 \), and \( x_0 = x \). Then, for \( i \in \{0, \ldots, d-1\} \), we have \( z\psi^i = x\Gamma\psi^i = x\psi^i\Gamma = x_i\Gamma \). If \( x_i\Gamma = z = x\Gamma \) then \( x = x_i \), a contradiction. Also \( z\psi^d = x\psi^d\Gamma = x\Gamma = z \) and thus the \( \psi \)-orbit of \( z \) has size \( d \). It follows that \( T_\psi(X) = T_\psi(Z) \).

As \( Z \) is an expansion of \( X \), the set \( \{x\}\{A\} \subseteq Z \) is an expansion of \( \{x\} \), for all \( x \in X \). Let \( E = \{\Gamma \in A^*: x\Gamma \in X\} \), so \( \{x\}\{A\} \subseteq Z = xE \), and assume the \( \psi \)-orbit of \( x \) is \( O_x = \{x_0, \ldots, x_{d-1}\} \), as before. For \( \Gamma \in E \) we have shown that the \( \psi \)-orbit of \( z = x\Gamma \) is \( \{x_i\Gamma | 0 \leq i \leq d-1\} \) and as \( \psi \) is in semi-normal form with respect to \( Z \) it follows that \( x_i\Gamma \in Z \), for all \( i \). This holds for all \( \Gamma \in E \), so \( x_iE = \{x_i\Gamma | \Gamma \in E\} \) is contained in \( Z \). As \( x_iE \) is an expansion of \( \{x_i\} \) and \( Z \) is an expansion of \( X \) it follows that \( \{x_i\}\{A\} \subseteq Z = x_iE \), \( i = 0, \ldots, d \). Therefore \( O_x\{A\} \subseteq Z = \bigcup_{i=0}^{d-1} x_iE = Z.E \). By definition, \( xE \) is a q-fold expansion of \( \{x\} \), for some \( q \in Z \), so \( O_xE \) is a dq-fold expansion of \( O_x \). Also, each element \( z = x\Gamma \) of \( O_xE \) belongs to an \( \psi \)-orbit of size \( d \), as above. Now let \( m_\psi(d, X) = m \) and \( O_1, \ldots, O_m \) be the \( \psi \)-orbits of size \( d \) contained in \( X \). Then \( O_j\{A\} \subseteq Z \) is a gq-fold expansion of \( O_j \), for \( 1 \leq j \leq m \). Hence, setting \( p = \sum_{j=1}^{m} q_j \) and \( O = \bigcup_{j=1}^{m} O_j \), we see that \( O\{A\} \subseteq Z \) is a pd-fold expansion of \( O \). The set \( O \) is precisely the set of elements of \( X \) with \( \psi \)-orbits of size \( d \) and there are \( dm_\psi(d, X) \) such elements, that is \( |O| = dm_\psi(d, X) \). Moreover, from the above, \( z \in Z \) has \( \psi \)-orbit size \( d \) if and only if \( z \in O_j\{A\} \subseteq Z \), for some \( j \), if and only if \( z \in O\{A\} \subseteq Z \). A dp-fold expansion of a set of size \( M \) has \( M + dp(n-1) \) elements, so \( Z \) contains \( |O| + pd(n-1) = dm_\psi(d, X) + pd(n-1) \) elements with \( \psi \)-orbits of size \( d \). Therefore \( m_\psi(d, Z) = m_\psi(d, X) + p(n-1) \) \equiv m_\psi(d, X) \mod n-1 \), as required.

Note that it follows from this lemma that if \( \psi \) is in semi-normal form with respect to both \( X \) and \( X' \) then \( T_\psi(X) = T_\psi(X') \), since we may take a common expansion of both \( X \) and \( X' \) and then expand this to an A-basis \( Z \) with respect to which \( \psi \) is in semi-normal form. Hence, from now on, we refer to \( T_\psi \) as the cycle type of \( \psi \), without reference to the A-basis \( X \).

**Proposition 5.11.** Let \( \psi \) and \( \varphi \) be periodic elements of \( G_{n,r} \) in semi-normal form with respect to the A-bases \( X_\psi \) and \( X_\varphi \), respectively. Then \( \psi \) is conjugate to \( \varphi \) if and only if

1. \( T_\psi = T_\varphi \) and
2. \( m_\psi(d, X_\psi) \equiv m_\varphi(d, X_\varphi) \mod n-1 \), for all \( d \in \mathbb{N} \).

**Proof.** Assume that \( \psi \) and \( \varphi \) are conjugate and let \( \rho \in G_{n,r} \) be such that \( \rho^{-1}\psi\rho = \varphi \). Let \( \rho \) be in semi-normal form with respect to \( X_\varphi \), let \( Y \) be the minimal expansion of \( X_\varphi \) associated to \( \rho \) and let \( Z = Y\rho \). Let \( W \) be a common expansion of \( X_\psi \) and \( Y \) and let \( \psi \) be in semi-normal form with respect to an expansion \( X_\psi' \) of \( W \). (Such an expansion of \( W \) exists, by Lemma 4.9.) As \( \psi \) is periodic and in semi-normal form it permutes the elements of \( X_\psi' \), so for all \( x \in X_\psi' \) we have...
$x' \in X'_\psi$ such that $x\rho \varphi = x\psi \rho = x' \rho \in X\rho$. Therefore $\varphi$ permutes the elements of $X'_\psi \rho$. Hence $\varphi$ is in semi-normal form with respect to $X'_\psi = X'_{\psi, \rho}$. As $X'_\psi$ is an expansion of $Y$ and $Z = Y\rho$ it follows that $X'_\psi$ is an expansion of $Z$. Now if $x \in X'_\psi$ and $i \in \mathbb{Z}$ then $x\rho \varphi^i = x\psi^i \rho$, so if $x\psi^d = x$ then $x\rho \varphi^d = x\rho$, and vice-versa, so $x$ and $x\rho$ have orbits of equal size. This holds for all $x \in X'_\psi$, so $T\psi = T\varphi$ and both $X'_\psi$ and $X'_\varphi$ have the same number of elements with an orbit of size $d$. Therefore $m\psi(d, X'_\psi) = m\varphi(d, X'_\psi)$, for all $d \in T\psi = T\varphi$. Hence 2 follows, from Lemma 5.10 and the fact that $X'_\psi$ and $X'_\varphi$ are expansions of $X\psi$ and $X\varphi$, respectively.

Conversely, suppose that 1 and 2 hold. Let $T\psi = T\varphi = \{d_1, \ldots, d_k\}$ and write $m_j = m\psi(d_j, X\psi)$ and $m'_j = m\varphi(d_j, X\varphi)$. Fix $j \in \{1, \ldots, k\}$. Assume first that $m_j > m'_j$. Then, by hypothesis, $m_j = m'_j + q_j(n-1)$, for some positive integer $q_j$. Let $x \in X\varphi$ have $\varphi$-orbit $O$ of size $d_j$, say $O = \{x_0, \ldots, x_{d_j-1}\} \subseteq X\varphi$, where $x = x_0$ and $x_{i+1} = x_i \varphi$, subscripts modulo $d_j$. Let $Y_x$ be a $q_j$-fold expansion of $\{x\}$ and, as in the proof of Lemma 5.10 let $E = \{\Gamma \in A^* : x\Gamma \in Y_x\}$, so $X_x = xE$. Then $x_e E$ is a $q_j$-fold expansion of $\{x_i\}$, for $0 \leq i \leq d_j-1$. Moreover, as in the proof of Lemma 5.10 for given $\Gamma \in E$, the map $\varphi$ cyclically permutes the elements of the set $\{x_i\Gamma \mid 0 \leq i \leq d_j-1\}$. Hence the set $\mathcal{O}E = \{x_i\Gamma \mid \Gamma \in E, 0 \leq i \leq d_j-1\}$ is a $q_j d_j$-fold expansion of $\mathcal{O}$ consisting of elements with $\varphi$-orbit size $d_j$. Now $X_x$ has $m_j d_j$ elements with $\varphi$-orbit size $d_j$, so if $\mathcal{O}$ is expanded in this way then the resulting expansion of $X_x$ has exactly $m_j d_j + d_j q_j(n-1)$ elements with $\varphi$-orbits of size $d_j$.

For each $j$ such that $m_j > m'_j$ apply this process to a single element of $X\varphi$ with $\varphi$-orbit size $d_j$. Now, for each $j$ such that $m'_j > m_j$ apply the process to an element of $X\psi$ with $\psi$-orbit size $d_j$, interchanging the roles of $\varphi$ and $\psi$. The result is an expansion $X'_\psi$ of $X\psi$ and an expansion $X'_\varphi$ of $X\varphi$ such that, if $m_j > m'_j$ then $m\varphi(d_j, X'_\varphi) = (m'_j d_j + d_j q_j(n-1))/d_j = m_j d_j + q_j(n-1) = m_j = m\psi(d_j, X'_\psi)$ and similarly, if $m'_j > m_j$ then $m\psi(d_j, X'_\psi) = m'_j = m\varphi(d_j, X'_\varphi)$ (and this equality obviously also holds if $m_j = m'_j$). Now define $\rho : X'_\psi \rightarrow X'_\varphi$ by mapping orbits of size $d$ to each other in the obvious way. In detail, if $\mathcal{O}_1, \ldots, \mathcal{O}_m$ are the $\psi$-orbits of size $d$ of elements of $X'_\psi$, then there are precisely $m$ $\varphi$-orbits, $\mathcal{O}'_1, \ldots, \mathcal{O}'_m$, of size $d$ of elements of $X'_\varphi$. For each $j \in \{1, \ldots, m\}$, if $\mathcal{O}_j = \{x_0, \ldots, x_{d-1}\}$ and $\mathcal{O}'_j = \{y_0, \ldots, y_{d-1}\}$, where $x_i \psi = x_{i+1}$ and $y_i \varphi = y_{i+1}$ (subscripts modulo $d$), then we set $x_i \rho = y_i$, for $0 \leq i \leq d-1$. Then $x_i \rho \varphi = y_i \varphi = y_{i+1} = x_{i+1} \rho = x_{i} \rho \psi$, for all $x_i$, and it follows that $x \rho \varphi = x \psi \rho$, for all $x \in X'_\psi$. Hence $\rho^{-1} \psi \rho = \varphi$, as required. \hfill \Box

**Example 5.12.** Let $n = 2$, $r = 1$ and $V_{2, 1}$ be free on $\{x\}$. Let

$X = \{x_0^4, x_1^4, x_2^3, x_2 x_0, x_0 x_1, x_0 x_2, x_2 x_1, x_2 x_2\}
$ and let $\psi$ be the periodic element of $G_{2, 1}$ given by $x_0^4 \psi = x_1^4 x_2, x_1^4 x_2 \psi = x_0^4, x_2 x_0 \psi = x_2 x_1, x_2 x_1 \psi = x_2 x_2, x_0 x_2 \psi = x_2 x_2 x_2$.

Then $\psi$ has cycle type $T\psi = \{2\}$ and $m\psi(2, X) = 3$. In fact, the $\psi$-orbits of elements of $X$ are $\mathcal{O}_1 = \{x_0^4, x_1^4 x_2\}, \mathcal{O}_2 = \{x_0^4 x_2, x_0 x_2\}$ and $\mathcal{O}_3 = \{x_1 x_2, x_2^2\}$. Let $Y = \{x_0, x_1, x_2\}$ and let $\varphi$ be the periodic element of $G_{2, 1}$ given by $x_0 \varphi = x_2$ and $x_2 \varphi = x_0$.

\begin{center}
\begin{tikzpicture}[->, >=stealth, baseline=(current bounding box.center)]
  \node (1) at (0,0) {1};
  \node (2) at (1,0) {2};
  \node (3) at (2,0) {3};
  \node (4) at (3,0) {4};
  \node (5) at (4,0) {5};
  \node (6) at (5,0) {6};
  \node (2') at (1,1) {2'};
  \node (2'') at (1,2) {2''};
  \draw (1) to (2);
  \draw (2) to (3);
  \draw (3) to (4);
  \draw (4) to (5);
  \draw (5) to (6);
  \draw (6) to (1);
\end{tikzpicture}
\end{center}

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Then \( \varphi \) has cycle type \( T_\varphi = \{2\} \) and \( m_\varphi(2,Y) = 1 \). From Proposition 5.11 \( \psi \) is conjugate to \( \varphi \). We can construct a conjugator by applying the process of the proof. We take the same 2-fold expansion of both \( x\alpha_1 \) and \( x\alpha_2 \) to give a 4-fold expansion

\[
Y' = \{x\alpha_1^3, x\alpha_2^2\alpha_2, x\alpha_1\alpha_2, x\alpha_2\alpha_1\alpha_2, x\alpha_2^2\}
\]

of \( Y \) such that \( \varphi \) is in semi-normal form with respect to \( Y' \). In fact, for \( x\alpha_1\Gamma \in Y' \) we have \( x\alpha_1\Gamma \varphi = x\alpha_2\Gamma \) and \( x\alpha_2\Gamma \varphi = x\alpha_1\Gamma \). The \( \varphi \)-orbits of elements of \( Y' \) are \( \mathcal{O}_i' = \{x\alpha_1^i, x\alpha_2^i\} \), \( \mathcal{O}_2' = \{x\alpha_1^i\alpha_2, x\alpha_2\alpha_1\alpha_2\} \) and \( \mathcal{O}_3' = \{x\alpha_1\alpha_2, x\alpha_2^3\} \) and \( m_\varphi(2,Y') = 3 \). Choose \( \rho \) to be the element of \( G_{2,1} \) defined by mapping \( \mathcal{O}_1 \) to \( \mathcal{O}_2' \), \( i = 1, 2, 3 \), by \( x\alpha_1^i\rho = x\alpha_1^i \), \( x\alpha_2\varphi \alpha_2\rho = x\alpha_2\alpha_1\alpha_2 \), \( x\alpha_1\alpha_2\rho = x\alpha_2\alpha_1\alpha_2 \), \( x\alpha_2\alpha_1\rho = x\alpha_1\alpha_2 \) and \( x\alpha_2^2\rho = x\alpha_2^2 \).

\[
\rho : 1 \to 3, 2 \to 5, 3 \to 6, 4 \to 2, 5 \to 4, 6 \to 1
\]

Then \( \rho^{-1}\psi \rho = \varphi \).

From the proof of Theorem 5.11 we extract the following algorithm for the conjugacy of periodic elements of \( G_{n,r} \).

**Algorithm 5.13.** Let \( \psi \) and \( \varphi \) be periodic elements of \( G_{n,r} \).

**Step 1:** Construct \( A \)-bases \( X_\psi \) and \( X_\varphi \) with respect to which \( \psi \) and \( \varphi \) are in semi-normal form (Lemma 4.9).

**Step 2:** Compute the cycle types \( T_\psi \) and \( T_\varphi \). If \( T_\psi \neq T_\varphi \), output “No” and stop.

**Step 3:** Compute \( m_\psi(d,X_\psi) \) and \( m_\varphi(d,X_\varphi) \), for all \( d \in T_\psi \). If \( m_\psi(d,X_\psi) \neq m_\varphi(d,X_\varphi) \mod n-1 \), output “No” and stop.

**Step 4:** Construct \( A \)-bases \( X_\psi' \) and \( X_\varphi' \) as described in the proof of Theorem 5.11.

**Step 5:** Choose a map \( \rho \) sending \( \psi \)-orbits of elements of \( X_\psi' \) to \( \varphi \)-orbits of elements of \( X_\varphi' \), as in the proof of the theorem, and output \( \rho \).

### 5.4 Conjugacy of regular infinite elements

We begin with a necessary condition for two regular infinite elements to be conjugate. Let \( \psi \) be a regular infinite element of \( G_{n,r} \) in quasi-normal form with respect to \( X \). By Lemma 4.13 \( \psi \) has finitely many semi-infinite \( X \)-components, each of which has a unique characteristic (see Definition 4.22). Moreover, if \( \psi \) is in semi-normal form with respect to both \( X \) and \( Y \) and \( \mathcal{O} \) is a semi-infinite \( X \)-component of \( \psi \) with characteristic \((m,\Gamma)\) then \( \mathcal{O} \) contains an element \( u \) of characteristic \((m,\Gamma)\). From Lemma 4.22 the \( \psi \)-orbit of \( u \) has precisely one \( Y \)-component, which is again semi-infinite of characteristic \((m,\Gamma)\). Therefore, the set of pairs \((m,\Gamma)\) which are characteristics of semi-infinite \( X \)-components is independent of the choice of \( X \) (with respect to which \( \psi \) is in semi-normal form), and we may make the following definition.

**Definition 5.14.** Let \( \psi \) be a regular infinite element of \( G_{n,r} \) in semi-normal form with respect to \( X \). Define

\[
\mathcal{M}_\psi = \{(m,\Gamma)\mid (m,\Gamma) \text{ is the characteristic of a semi-infinite } X \text{-component of } \psi\}.
\]

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Example 5.15. We refer to the following example through the remainder of this section. Let
\( n = 2, r = 1, x = \{ x \} \) and \( \varphi \in G_{2,1} \) be determined by the bijection from \( A \)-basis
\[
Y = \{ x\alpha_1, x\alpha_2 \alpha_1, x\alpha_2^2 \alpha_1^2, x\alpha_2^2 \alpha_1 \alpha_2, x\alpha_2^3 \}
\]
to the \( A \)-basis
\[
Z = \{ x\alpha_1^3, x\alpha_1^2 \alpha_2, x\alpha_1 \alpha_2, x\alpha_2 \alpha_1, x\alpha_2^2 \}
\]
as illustrated below.
\[
\varphi : \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array} \rightarrow \begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
Then \( Y \) is the minimal expansion of \( x \) associated to \( \varphi \) and \( Z = Y \varphi \). The elements of \( x \langle A \rangle \setminus (Y \langle A \rangle \cup Z \langle A \rangle) \) are \( x \) and \( x \alpha_2 \), so we start the search for a quasi-normal form \( A \)-basis by taking the unique minimal expansion \( X = \{ x\alpha_1, x\alpha_2 \alpha_1, x\alpha_2^2 \} \) of \( x \) not containing either of these elements.

The \( X \)-component of \( x\alpha_1 \) is
\[
x\alpha_1 \mapsto x\alpha_1^3 \mapsto x\alpha_2^3 \mapsto \cdots,
\]
which is right semi-infinite of characteristic \((1, \alpha_1^2)\). Next, \( x\alpha_2 \alpha_1 \) belongs to a complete infinite \( X \)-component:
\[
\cdots x\alpha_2^3 \mapsto x\alpha_2^4 \mapsto x\alpha_2^5 \mapsto x\alpha_1^2 \alpha_2 \mapsto x\alpha_1^4 \alpha_2 \mapsto x\alpha_1^5 \alpha_2 \mapsto \cdots
\]
Finally, the \( X \)-component of \( x\alpha_2^2 \) is
\[
\cdots \mapsto x\alpha_2^3 \alpha_1 \mapsto x\alpha_2^4 \alpha_1 \mapsto x\alpha_2^5 \alpha_1 \mapsto x\alpha_2^6 \alpha_1 \mapsto x\alpha_2,
\]
which is left semi-infinite of characteristic \((-1, \alpha_1^2)\). Thus \( \varphi \) is in quasi-normal form with respect to \( X \).

To determine \( M_\varphi \), we compute the sets \( X \langle A \rangle \setminus Y \langle A \rangle = \{ x\alpha_2^2, x\alpha_2^2 \alpha_1 \} \) and \( X \langle A \rangle \setminus Z \langle A \rangle = \{ x\alpha_1, x\alpha_2^3 \} \). The \( X \)-components we have yet to calculate are those of \( x\alpha_2^2 \alpha_1 \) and \( x\alpha_1^2 \); these are the sets \( \{ x\alpha_2^2 \alpha_1^2 \alpha_i \ | \ i \geq 1 \} \) and \( \{ x\alpha_1^2, x\alpha_1^2 \ | \ i \geq 1 \} \) with characteristics \((1, \alpha_1^2)\) and \((-1, \alpha_1^2)\) respectively. Hence
\[
M_\varphi = \{(1, \alpha_1^2), (-1, \alpha_1^2)\}.
\]

Lemma 5.16. Suppose that \( \psi \) and \( \varphi \) are conjugate regular infinite elements of \( G_{n,r} \) in quasi-normal form with respect to \( A \)-bases \( X \) and \( Y \) respectively. Then the sets \( M_\psi \) and \( M_\varphi \) coincide. Moreover, if \( \rho \in G_{n,r} \) is such that \( \rho^{-1} \psi \rho = \varphi \) then \( \rho \) maps every semi-infinite \( X \)-component of \( \psi \), of characteristic \((m, \Gamma)\), to a \( \varphi \)-orbit which contains a \( \varphi \)-semi-infinite \( Y \)-component of characteristic \((m, \Gamma)\).

Proof. Let \( \psi \) and \( \varphi \) be in quasi-normal form with respect to the \( A \)-bases \( X \) and \( Y \) respectively and let \( \rho \) be such that \( \rho^{-1} \psi \rho = \varphi \). Thus, if \( u \) is an element of \( X \langle A \rangle \) such that \( \psi^m = u \Gamma \rho \), for some \( m \) and \( \Gamma \), then
\[
u \rho \psi^m = u \psi^m \rho = u \Gamma \rho = u \rho \Gamma.
\]
The same argument can be applied starting with an element \( v \in Y \langle A \rangle \) and interchanging \( \psi \) and \( \varphi \). Hence if \( u \) belongs to a \( \psi \)-orbit of characteristic \((m, \Gamma)\) then \( \rho \) belongs to an \( \varphi \)-orbit of characteristic \((m, \Gamma)\). Thus, from Lemma 4.24 a \( \psi \)-orbit that contains a semi-infinite \( X \)-component of characteristic \((m, \Gamma)\) is mapped by \( \rho \) to a \( \varphi \)-orbit which has a semi-infinite \( Y \)-component of the same characteristic. \( \square \)
Definition 5.17. Let $\psi$ be in semi-normal form with respect to $X$. The equivalence relation $\equiv$ on $X$, is that generated by the relation $x \equiv x'$, whenever $x\Gamma$ and $x'\Delta$ are in the same $\psi$-orbit, for some $\Gamma, \Delta \in A^*$. 

Example 5.18. Let $\varphi$ be as in Example 5.15. Then $x\alpha_2 \alpha_1 \varphi = (x\alpha_1) \alpha_1 \alpha_2$, so $x\alpha_2 \alpha_1 \equiv x\alpha_1$. Also, $x\alpha_2 \alpha_1 \varphi^{-1} = (x\alpha_2^2) \alpha_2$, so $x\alpha_2 \alpha_1 \equiv x\alpha_2^2$. Therefore all elements of $X$ are related by $\equiv$.

Proposition 5.19. Let $\psi$ be a regular infinite element in quasi-normal form with respect to $X$. Let $X = \bigcup_{i=1}^{m} X_i$ where the $X_i$ are the equivalence classes of $\equiv$ defined on $X$ under the action of $\psi$. Then $V_{n,r}$ is the free product of the $\psi$-invariant subalgebras $V_1, \ldots, V_m$, where $V_i$ is the subalgebra generated by $X_i$.

Proof. As $\psi$ is regular infinite, the sets $X_i$ partition $X$, so $V_{n,r}$ is the free product of the $V_i$'s. To show that $V_i$ is $\psi$-invariant it suffices to show that if $x \in X_i$, then $x\psi$ and $x\psi^{-1}$ are in $V_i$. To this end, choose $d \geq 0$ such that $x\psi\Gamma$ and $x\psi^{-1}\Gamma$ belong to $X(A)$, for all $\Gamma \in A^*$ of length $d$. Then, for $\Gamma$ of length $d$, we have $x\psi\Gamma = y\Delta$ and $x\psi^{-1}\Gamma = z\Delta$, for some $y, z \in X$ and $\Delta, \Lambda \in A^*$. By definition then $y \equiv x \equiv z$, so $y, z \in X_i$. This implies that $x\psi\Gamma = y\Delta \in V_i$ and $x\psi^{-1}\Gamma = z\Delta \in V_i$. This holds for all $\Gamma$ of length $d$, so from Lemma 5.19, $x\psi$ and $x\psi^{-1}$ belong to $V_i$, as required. Hence $V_i$ is $\psi$ invariant.

Lemma 5.20. Let $\psi$ be a regular infinite element in quasi-normal form with respect to $X$ and let $X_i, i = 1, \ldots, m$, be the equivalence classes of $\equiv$ defined on $X$ under the action of $\psi$. Then we may effectively construct the $X_i$.

Proof. From Lemmas 4.28 and 4.3 we may effectively construct $X$, the minimal expansion $Y$ of $\psi$ with respect to $X$, and the basis $Z = Y\psi$. For each $v \in X \cup Y \cup Z$ we may enumerate part of the $X$-component of $v$ using the procedure of Lemma 4.30. Denote by $O_v$ the part of the $\psi$-orbit of $v$ enumerated this way. Let $\equiv_0$ be the equivalence relation on $X$ generated by $y \equiv_0 z$ if $y\Gamma$ and $z\Delta$ belong to $O_v$, for some $v \in X \cup Y \cup Z$ and $\Gamma, \Delta \in A^*$. We claim that $\equiv_0 = \equiv$.

By definition, $\equiv_0 \subseteq \equiv$. To prove the opposite inclusion, we suppose that there exist $p \in Z, x, y \in X$ and $\Delta, \Phi \in A^*$ such that $x\Phi = y\Delta \psi^p$ and $x$ and $y$ are not related under the relation $\equiv_0$. In this case we may assume, interchanging $x$ and $y$ if necessary, that $p > 0$. Let $p$ be a minimal positive integer for which such $x, y$ exist. As $y\Delta \psi^p = x\Phi$ it follows that $y\Delta \psi^p \in X(A)$, for $p' = 1, \ldots, p - 1$. Let $y\Delta \psi = y'\Delta'$, so $y'\Delta' \psi^{-p-1} = x\Phi$. By minimality of $p$ we have $y' \equiv_0 x$. Let $\Delta_0$ be an initial subword of $\Delta$ of maximal length such that $y\Delta_0 \psi \in X(A)$, say $\Delta = \Delta_0 \Delta_1$. Then $y\Delta_0 \equiv_0 Y$ and $y\Delta_0 \psi = y'\Delta_0'\psi$, for some $y' \in X$ and $\Delta_0' \in A^*$. Now $y'\Delta' = y\Delta_0 \Delta_1 \psi = y'\Delta_0''\psi \Delta_1$, so $y'' = y'$ and $\Delta' = \Delta_0'' \Delta_1$. Thus $y\Delta_0 \psi = y'\Delta_0'' \psi$ and, as $y\Delta_0 \equiv_0 Y$, we have $y \equiv_0 y'$. Therefore $y \equiv_0 x$, a contradiction. We conclude that no such $p, x$ and $y$ exist and so $\equiv_0 \subseteq \equiv$, as required. Thus $\equiv_0 = \equiv$, and as we may effectively compute the sets $O_v$, it follows that we may compute the equivalence classes $X_i$.

Lemma 5.21. Let $\psi$ be a regular infinite element in quasi-normal form with respect to $X$ and let $X_i, i = 1, \ldots, m$, be the equivalence classes of $\equiv$ defined on $X$ under the action of $\psi$. Define

$$x\theta_i = \begin{cases} x\psi & \text{if } x \in X_i, \\ x & \text{if } x \in X_j \text{ for } i \neq j, \end{cases}$$

for $i = 1, \ldots, m$. Then $\theta_i$ extends to an element of $G_{n,r}$ which commutes with $\psi$ and with $\theta_j$, for all $j \in \{1, \ldots, m\}$. 

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Thus $\rho$ in a semi-infinite

Proof. Since $\theta_{1}$ is an element of type (B) in $X$, there exists a conjugator $\rho$ such that $\theta_{1}$ is a terminal or initial element in a semi-infinite $Y$-component of $\varphi$.

Lemma 5.22. Let $\psi$ and $\varphi$ be regular infinite elements of $G_{n,r}$, in quasi-normal form with respect to the $A$-bases $X$ and $Y$ respectively and let $X_{i}$, $i = 1, \ldots, m$, be the equivalence classes of $\equiv$ defined on $X$ under the action of $\psi$. If $\psi$ and $\varphi$ are conjugate then, given $x_{1}, \ldots, x_{m}$ such that $x_{i}$ is an element of type (B) in $X_{i}$, there exists a conjugator $\rho$ such that $x_{i}\rho$ is a terminal or initial element in a semi-infinite $Y$-component of $\varphi$.

Proof. Since $\psi$ and $\varphi$ are conjugate, by Lemma 5.16 the sets $M_{\psi}$ and $M_{\varphi}$, of characteristics for $\psi$ and $\varphi$, coincide and there exists an element $\rho^{\prime}$ such that $\rho^{\prime}^{-1}\psi\rho^{\prime} = \varphi$. Let $x_{i}$ be the given element of type (B) in $X_{i}$. Then, from Lemma 5.16 $x_{i}\rho^{\prime}$ belongs to a $\varphi$-orbit which contains a semi-infinite $Y$-component, with the same characteristic as $x_{i}$. Let $y_{i} \in Y(A)$ be an initial or terminal element of this $\varphi$-orbit. Then there exists $j_{i}$ such that

$$x_{i}\rho^{\prime} = y_{i}\varphi^{j_{i}}.$$

Thus

$$y_{i} = x_{i}\varphi^{j_{i}}\rho^{-j_{i}} = x_{i}\rho^{\prime}\varphi^{-j_{i}} = x_{i}\psi^{-j_{i}}\rho^{\prime}.$$

For each equivalence class $X_{i}$, define $\theta_{i}$ as in Lemma 5.21 and $\rho \in G_{n,r}$ by

$$\rho = (\prod_{i=1}^{n} \theta_{i}^{-j_{i}})\rho^{\prime}.$$

Then $\theta = \prod_{i=1}^{n} \theta_{i}^{-j_{i}}$ commutes with $\psi$, so $\rho^{-1}\psi\rho = \rho^{\prime}^{-1}\theta^{-1}\psi\theta^{\prime} = \rho^{\prime}^{-1}\psi\rho^{\prime} = \varphi$ and, for each chosen $x_{i} \in X_{i}$,

$$x_{i}\rho = x_{i}(\prod_{i=1}^{n} \theta_{i}^{-j_{i}})\rho^{\prime} = x_{i}\theta_{i}^{-j_{i}}\rho^{\prime} = x_{i}\psi^{-j_{i}}\rho^{\prime} = y_{i}.$$

Thus $\rho$ is the required conjugator.

Definition 5.23. Let $\psi$ and $\varphi$ be regular infinite elements in quasi-normal form with respect to $X$ and $Y$ and let $X_{i}$, $i = 1, \ldots, m$ be the equivalence classes of $\equiv$ defined on $X$ under the action of $\psi$. Let $R_{i}(\psi, \varphi)$ be the set of pairs $(x, y)$, where $x$ is of type (B) in $X_{i}$ and $y$ is an initial or terminal element of a semi-infinite $Y$-component of $\varphi$ with the same characteristic as $x$.

Given $(x_{i}, y_{i}) \in R_{i}(\psi, \varphi)$, $i = 1, \ldots, m$, let $\rho_{0}$ be the map from $\{x_{1}, \ldots, x_{m}\}$ to $\{y_{1}, \ldots, y_{m}\}$ given by

$$x_{i}\rho_{0} = y_{i}, \quad i = 1, \ldots, m.$$

Let $R(\psi; \varphi)$ be the set of all such maps.

The set $R_{i}(\psi, \varphi)$ is finite since the number of elements of type (B) in $X$ and the number of semi-infinite $Y$-components of $\varphi$ is finite, so $R(\psi; \varphi)$ is also finite.

Lemma 5.24. Given $\rho_{0} \in R(\psi, \varphi)$, there are finitely many ways of extending $\rho_{0}$ to an element $\rho$ of $G_{n,r}$ such that $\varphi = \rho^{-1}\psi\rho$. Moreover the existence of such an extension $\rho$ can be effectively determined, and if such $\rho$ exists then the images $y\rho$ can be effectively determined, for all $y \in X$. 42
Lemma 4.34 gives an effective procedure to determine whether an integer $x\vert\rho$ exists and satisfies $\varphi = \rho^{-1}\psi\rho$. From Lemma 5.20 we may effectively construct the equivalence classes $X_i$, and so also the sets $R_i(\psi, \varphi)$.

First consider a single equivalence class $X_i$. We are given an element $x_i$ of type (B) and an element $y_i$ such that

$$x_i\rho_0 = y_i,$$

where $y_i$ is an initial or terminal element of a semi-infinite $Y$-component of $\varphi$ with the same characteristic multiplier and power as $x_i$.

Let $x \in X$ of type (B). Then, by definition of $\equiv$, we have $x \in X_i$ if and only if there exist elements $x_i = u_{i_0}, \ldots, u_{i_m} = x$ of $X$, elements $\Gamma_j, \Delta_j \in A^*$ and $k_j \in \mathbb{Z}$ with $u_{i_{j+1}}\Delta_{j+1} = u_j\Gamma_j\psi^{k_j}$, for $j = 0, \ldots, m - 1$. Before going any further we show that we may assume that $u_j$ is of type (B), for all $j$. Suppose not, say $u_j$ is an initial or terminal element of a semi-infinite $Y$-component of type (B) such that $u_j\psi^{k_j} = u_j'\Gamma_j'$. Now

$$u_{j-1}\Gamma_{j-1}\psi^{k_{j-1}+k_j} = u_j\Delta_j\psi^{k_j} = u_j'\Gamma_j'\Delta_j$$

and

$$u_j'\Gamma_j'\psi^{k_j-k_j} = u_j'\Gamma_j'\psi^{-k_j}\Gamma_j'\psi^{k_j} = u_j\Delta_j\psi^{k_j} = u_{j+1}\Delta_{j+1},$$

so we may replace $u_j$ by $u_j'$. Continuing this way, eventually all $u_j$ will be of type (B).

We show, by induction on $m$, that there are finitely many possible values of $x\rho$, for an element $\rho \in G_{n, r}$ such that $x\varphi = x\rho^{-1}\psi\rho$ (where $x_i\rho = x_i\rho_0 = y_i$) and describe an effective procedure to enumerate the set of all such elements. Suppose first that $m = 1$, so $x = u_1$ and we have $\Gamma = \Gamma_0$, $\Delta = \Delta_1$ and $k = k_0$ such that $x_1\Gamma\psi^k = x\Delta$. Given that $\rho$ exists, from Lemma 5.10 $x\rho$ belongs to a semi-infinite $Y$-component $O$ of $\varphi$ with the same characteristic as $x$. Therefore (if $\rho$ exists) there exists an element $(x, w) \in R_i(\psi, \varphi)$ such that $w$ is the initial or terminal element of $O$; and an integer $l$ such that $w\varphi^l = x\rho$. This implies that

$$w\Delta\varphi^l = (x\Delta)\rho = x_1\Gamma\psi^k\rho = x_1\Gamma\rho\varphi^k = x_i\rho_0\varphi^k\Gamma,$$

so

$$w\Delta\varphi^{l-k} = x_i\rho_0\Gamma = y_i\Gamma. \quad (7)$$

Lemma 5.33 gives an effective procedure to determine whether an integer $l$ satisfying (7) exists, and if so find it. Given $\rho_0$ and $x$, the integer $k$ and the elements $\Gamma$ and $\Delta$ are uniquely determined so, to decide whether an appropriate value $x\rho$ exists, we may check each pair $(x, w)$ in the set $R_i(\psi, \varphi)$ to see if (7) holds for some $l$ or not. For each such $w$ there is at most one $l$ such that (7) has a solution and, as $R_i(\psi, \varphi)$ is finite, we may effectively enumerate the values $w\Delta\varphi^{l-k}$ that could be assigned to $x\rho$. Hence the result holds if $m = 1$.

Now assume that $m > 1$ and the result holds for all $x$ related to $x_i$ by a chain of length at most $m - 1$. Then $u_{m-1}$ is of type (B) and by assumption $u_{m-1}\rho$ may be given one of finitely many values, and we have a procedure to enumerate these values. Suppose then that $u_{m-1}\rho = v$. Now $x = u_m$ and we have $\Gamma_{m-1}, \Delta_m \in A^*$ and $k_{m-1} \in \mathbb{Z}$ such that $u_{m-1}\Gamma_{m-1}\psi^{k_{m-1}} = x\Delta_m$. Applying the argument of the case $m = 1$ with $u_{n-1}, \Gamma_{n-1}, \Delta_n$ and $v$ in place of $x_i$, $\Gamma$, $\Delta$ and $y_i$, we see that a finite set of possible values for $x\rho$ may be effectively determined. Therefore, by induction, the result holds for all $x \in X_i$ of type (B).
Finally, if \( x \in X \) is of type (C), then by Lemma 4.18 there is a \( z \Sigma \) in the \( X \)-component of \( x \), for some \( z \) of type (B) and \( \Sigma \in A^* \), i.e. \( x \psi^p = z \Sigma \) for some integer \( p \). Since we have already determined the possible images of all the type (B) elements in \( X \), if \( \rho \) exists we have, for each choice of \( z \rho \),

\[
x \rho = z \Sigma \psi^{-p} \rho = z \rho \Sigma \psi^{-p}
\]

and this determines the image of the type (C) element under \( \rho \) (uniquely once we have made our initial choice for the image of \( z \rho \)).

We carry out this process on each equivalence class in turn. An extension of \( \rho_0 \) exists only if the process results in a at least one possible value for each element \( X \) of \( Y \) with respect to \( X \) and \( Y \). Then \( \rho \) is an extension of \( \rho_0 \) of the required type, if \( X \rho \) is an \( A \)-basis of \( V_{n,r} \); which may be verified effectively, using Lemma 3.10.

We are now able to state the main result of this section.

**Proposition 5.25.** Let \( \psi \) and \( \varphi \) be regular infinite elements of \( G_{n,r} \) in quasi-normal form with respect to \( X \) and \( Y \) respectively.

Then, \( \psi \) is conjugate to \( \varphi \) if and only if there exists a map \( \rho_0 \in \mathcal{R}(\psi; \varphi) \) such that \( \rho_0 \) extends to an element \( \rho \) of \( G_{n,r} \) with \( \rho^{-1} \psi \rho = \varphi \).

**Proof.** Obviously, if \( \rho_0 \) extends to an element of \( G_{n,r} \) such that \( \rho^{-1} \psi \rho = \varphi \), then \( \psi \) is conjugate to \( \varphi \) by \( \rho \).

Assume that \( \psi \) is conjugate to \( \varphi \). Lemma 5.22 tells us that there exists a conjugator \( \rho \) such that, for each equivalence class \( X_i \), there exists an element \( x_i \) of type (B) in \( X_i \) with \( y_i = x_i \rho \) an initial or terminal element of a semi-infinite \( Y \)-component of \( \varphi \).

We define \( \rho_0 \) to be the map \( x_1 \mapsto y_1, \ldots, x_m \mapsto y_m \), where \( y_i = x_i \rho \) for each \( i = 1, \ldots, m \). Thus, \( \rho_0 \) is an element of the finite set \( \mathcal{R}(\psi; \varphi) \). Now \( \rho_0 \) is the restriction of \( \rho \) to \( \{x_1, \ldots, x_m\} \), so it certainly extends to \( \rho \), as required.

**Example 5.26.** Let \( n = 2 \), \( r = 1 \) and \( V_{2,1} \) be free on \( \{x\} \). Let

\[
Y = \{x \alpha_1, x \alpha_2 \alpha_1^2, x \alpha_2 \alpha_1 \alpha_2, x \alpha_2^2\}
\]

and

\[
Z = \{x \alpha_1^3, x \alpha_1^2 \alpha_2, x \alpha_1 \alpha_2, x \alpha_2^2\}
\]

and let \( \psi \) be given by \( x \alpha_1 \psi = x \alpha_1^3 \), \( x \alpha_2 \alpha_1 \alpha_2 \psi = x \alpha_2 \), \( x \alpha_2 \alpha_1 \alpha_2 \psi = x \alpha_1 \alpha_2 \).

\[\psi : \begin{array}{ccc} 1 & 2 & 3 \\ & 4 & \end{array} \rightarrow \begin{array}{ccc} 1 & 3 & 4 \\ & 2 & \end{array}\]

Then \( Y \) is the minimal expansion of \( \{x\} \) associated to \( \psi \) and \( Z = Y \psi \). The only element of \( \{x\} \langle A \rangle \) not in \( Y \langle A \rangle \sqcup Z \langle A \rangle \) is \( x \). Thus we take \( X = \{x \alpha_1, x \alpha_2\} \). Then \( X \langle A \rangle \setminus Y \langle A \rangle = \{x \alpha_2, x \alpha_2 \alpha_1\} \) and \( X \langle A \rangle \setminus Z \langle A \rangle = \{x \alpha_1, x \alpha_1^2\} \). The \( X \)-component of \( x \alpha_1 \) is

\[O_{1,\psi} = \{x \alpha_1^{2k+1}\}_{k \in \mathbb{N}_0},\]

and the \( X \)-component of \( x \alpha_1^2 \) is

\[O_{2,\psi} = \{x \alpha_1^{2k}\}_{k \in \mathbb{N}},\]

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both of which are right semi-infinite with characteristic \((1, \alpha_1^2)\). The \(X\)-component of \(x\alpha_2\) is

\[O_{3, \psi} = \{x\alpha_2 \alpha_1^{2k}\}_{k \in \mathbb{N}_0}\]

and the \(X\)-component of \(x\alpha_2 \alpha_1\) is

\[O_{4, \psi} = \{x\alpha_2 \alpha_1^{2k} \alpha_1\}_{k \in \mathbb{N}_0},\]

both of which are left semi-infinite with characteristic \((-1, \alpha_1^2)\). Hence \(\psi\) is in quasi-normal form with respect to \(X\), both elements of \(X\) are of type (B) and \(M_\psi = \{(1, \alpha_1^2), (-1, \alpha_1^2)\}\). As \(x\alpha_2 \alpha_2 \psi = (x\alpha_1)\alpha_2\) there is one equivalence class of \(\equiv\), that is \(X_1 = X\).

Let \(\varphi\) be automorphism of Examples 5.15, 5.15 and 5.18. Then \(\varphi\) is in quasi-normal form with respect to the \(A\)-basis \(X_\varphi = \{x\alpha_1, x\alpha_2 \alpha_1, x\alpha_2^2\}\), \(M_\varphi = M_\psi\). The initial elements of right semi-infinite \(X_\varphi\)-components are \(x\alpha_1\) and \(x\alpha_1^2\) and the terminal elements of left semi-infinite \(X_\varphi\)-components are \(x\alpha_2^2\) and \(x\alpha_2^2 \alpha_1\).

The set \(R_1(\psi, \varphi)\) consists of the pairs \((x\alpha_1, x\alpha_1), (x\alpha_1, x\alpha_1^2), (x\alpha_2, x\alpha_2^2)\) and \((x\alpha_2, x\alpha_2^2 \alpha_1)\). Let us choose \(x\alpha_1\) as our initial choice of the type (B) element in \(X_1 = X\). We have two choices for the image of \(x\alpha_1\) under \(\varphi_0\), corresponding to the two pairs \((x\alpha_1, (x\alpha_1), (x\alpha_1, x\alpha_1^2) \in R_1\). Denote these by \(\rho_1\) and \(\rho_2\), where

\[x\alpha_1 \rho_1 = x\alpha_1\] and \(x\alpha_1 \rho_2 = x\alpha_1^2\).

Next we determine the images of the other type (B) element \(x\alpha_2\) of \(X\) under the action of \(\rho_1\) and \(\rho_2\). Following the proof of Lemma 5.24 we note first that \(x\alpha_1 \equiv x\alpha_2\) because \((x\alpha_1)\alpha_2 \psi^{-1} = (x\alpha_2)\alpha_2\), so in the notation of the proof of Lemma 5.24 we have \(\Gamma = \alpha_2\), \(\Delta = \alpha_2\) and \(k = -1\). Substituting these values into equation (7), we wish to find \(l\) such that

\[w \alpha_2 \varphi^{l+1} = (x\alpha_1) \rho \alpha_2,\]

where \(i = 1\) or \(2\), and \(w = x\alpha_2^2\) or \(x\alpha_2^2 \alpha_1\). Whenever we find such an \(l\) then we set \(x\alpha_2 \rho_i = w \varphi^l\) and check to see if \(\rho_i\) determines an automorphism and if so, if this automorphism is a conjugator.

**Case i = 1, x\alpha_1 \rho_1 = x\alpha_1**.

(i) When \(w = x\alpha_2^2\) we have,

\[x\alpha_2^3 \varphi^{l+1} = x\alpha_1 \alpha_2,\] if and only if

\[x\alpha_2 \alpha_1 \varphi = x\alpha_1 \alpha_2\]

which has no solutions, as may be verified using the process of Lemma 4.34.

(ii) When \(w = x\alpha_2^2 \alpha_1\) we have,

\[x\alpha_2^3 \alpha_1 \alpha_2 \varphi^{l+1} = x\alpha_1 \alpha_2,\] if and only if

\[x\alpha_1 \alpha_2 \varphi = x\alpha_1 \alpha_2\]

which has solution \(l = 0\). Therefore we set

\[x\alpha_2 \rho_1 = x\alpha_2^2 \alpha_1\]

The endomorphism determined by \(\rho_1\) now maps \(X\) to \(\{x\alpha_1, x\alpha_2^2 \alpha_1\}\), which is not a basis of \(V_{2,1}\) (see Lemma 3.10).
Neither value of \( w \) results in \( \rho_1 \) which could be a conguator.

**Case i = 2, \( x_{a_1} \rho_2 = x_{a_1^2} \).**

(i) When \( w = x_{a_2^2} \) we have,
\[
\begin{align*}
\alpha_2^3 \varphi^{l+1} & = x_{a_1^2} \alpha_2, \quad \text{if and only if} \\
\alpha_2 \alpha_1 \varphi^{l} & = x_{a_1^2} \alpha_2
\end{align*}
\]
which has solution \( l = 1 \). Therefore we set
\[
x_{a_2} \rho_2 = x_{a_2^2} \varphi.
\] (8)

In this case \( x_{a_2^2} \) is in \( X_{\varphi}(A) \setminus W(A) \), where \( W \) is the minimal expansion associated to \( \varphi \), and so to define \( \rho_2 \), in terms of \( X(A) \) we must take an expansion of \( X \) at \( x_{a_2} \). We take the minimal expansion which allows us to define the map into \( \{x\}(A) \); namely \( \{x_{a_2} \alpha_1^2, x_{a_2} \alpha_1 \alpha_2, x_{a_2^2}\} \). Then we obtain, from (8),
\[
\begin{align*}
x_{a_2} \alpha_1^2 \rho_2 & = (x_{a_2^2}) \alpha_1^2 \varphi = x_{a_2^2} \\
x_{a_2} \alpha_1 \alpha_2 \rho_2 & = (x_{a_2^2}) \alpha_1 \alpha_2 \varphi = x_{a_2} \alpha_2 \\
x_{a_2^2} \rho_2 & = (x_{a_2^2}) \alpha_2 \varphi = x_{a_2} \alpha_1.
\end{align*}
\]

In this case \( \rho_2 \) maps the expansion \( \{x_{\alpha_1}, x_{a_2} \alpha_1^2, x_{a_2} \alpha_1 \alpha_2, x_{a_2^2}\} \) of \( X \) to \( \{x_{a_1^2}, x_{a_2^2}, x_{\alpha_2} \alpha_1, x_{a_2} \alpha_1 \} \) which is a basis for \( V_{2,1} \); so \( \rho_2 \) determines an element of \( G_{2,1} \). Moreover, as can be verified \( \rho_2^{-1} \psi \rho_2 = \varphi \), so \( \rho_2 \) is a conguator. At this point we could stop but we give the final case for completeness.

(ii) When \( w = x_{a_2^2} \alpha_1 \) we have
\[
\begin{align*}
x_{a_2^2} \alpha_1 \alpha_2 \rho_2^{l+1} & = x_{a_1^2} \alpha_2, \quad \text{if and only if} \\
x_{\alpha_1} \alpha_2 & = x_{a_1^2} \alpha_2
\end{align*}
\]
which has no solutions.

Therefore, we find one conguating element \( \rho_2 \) and we see that \( \psi \) and \( \varphi \) are conguate.

The algorithm for the conguacy of regular infinite elements of \( G_{n,r} \) is as follows.

**Algorithm 5.27.** Let \( \psi \) and \( \varphi \) be regular infinite elements of \( G_{n,r} \).

**Step 1:** Construct \( A \)-bases \( X_\psi \) and \( X_\varphi \) with respect to which \( \psi \) and \( \varphi \) are in quasi-normal form (Lemma 4.25).

**Step 2:** Construct the equivalence classes \( X_i, \; i = 1, \ldots, m, \) of \( \equiv \) on \( X_\psi \) (Lemma 5.24).

**Step 3:** Find the initial and terminal elements of semi-infinite \( X_\varphi \)-components of \( \varphi \), by finding the minimal expansion of \( X_\varphi \) associated to \( \varphi \) (Lemma 4.9).

**Step 4:** Construct the sets \( R_i(\psi, \varphi) \).

**Step 5:** For each equivalence class \( X_i \) of \( \equiv \) on \( X_\psi \) choose an element \( x_i \in X_i \), of type (B).
Step 6: For each \( i \) and each pair \((x_i, y)\) of \( \mathcal{R}_i(\psi, \varphi) \), construct a map \( \rho_i : X_i \rightarrow X_i \), using equation (7), as in the proof of Lemma 5.24 if possible. In each case check that \( \rho_i \) is an automorphism.

Step 7: For each \( m \)-tuple \( \rho_1, \ldots, \rho_m \) of automorphisms, from the previous step, check whether the map \( \rho = \rho_1 \cdots \rho_m \) conjugates \( \psi \) to \( \varphi \).

6 The power conjugacy problem

For a group with presentation \( (X | R) \), the power conjugacy problem is to determine, given words \( g, h \in \mathbb{F}(X) \) whether or not there exist non-zero integers \( a \) and \( b \) such that \( g^a \) is conjugate to \( h^b \) in \( G \). We may in addition require that, if the answer to this question is “yes”, then integers \( a \) and \( b \), and an element \( c \in \mathbb{F}(X) \), are found, such that \( c^{-1}g^ac = G h^b \). We say the power conjugacy problem is decidable if there is an algorithm which, given \( g \) and \( h \) outputs “yes” if they’re conjugate and “no” otherwise. Again, the stronger form entails the obvious extra requirements. Again, in \( G_{n,r} \) we work entirely with symbols for automorphisms, ignoring the presentation.

As in the case of the conjugacy problem, we break the power conjugacy problem down into two cases; one for periodic elements and one for regular infinite elements. Then, we construct an algorithm that combines the two parts.

6.1 The power conjugacy for periodic elements

Let \( \psi \) and \( \varphi \) be periodic elements of \( G_{n,r} \), of order \( k \) and \( m \) respectively, in quasi-normal form with respect to the \( A \)-bases \( X \) and \( Y \). Then, to test whether \( \psi^a \) is conjugate to \( \varphi^b \) for \( a, b \in \mathbb{Z} \), we can apply Proposition 5.11 to the pair \( \psi^c, \varphi^d \) for all \( c \in \{1, \ldots, k\} \) and all \( d \in \{1, \ldots, m\} \).

6.2 Regular infinite elements

The first step is to compare the sets \( \mathcal{M}_\psi \) and \( \mathcal{M}_{\psi^a} \), \( a \in \mathbb{Z}, |a| > 1 \), for a regular infinite automorphism \( \psi \). First note that if \( \psi \) is in semi-normal form with respect to an \( A \)-basis \( X \), then as \( X \)-components of \( \psi \) are super-sets of \( X \)-components of \( \psi^a \), we have \( \psi^a \) in semi-normal form with respect to \( X \).

Lemma 6.1. Let \( \psi \) be a regular infinite element of \( G_{n,r} \) and let \( a \) be a non-negative integer. Then

\[
\mathcal{M}_{\psi^a} = \{(m/d, \Gamma^q) \mid (m, \Gamma) \in \mathcal{M}_\psi, \gcd(m, a) = d \text{ and } |a| = qa\}.
\]

Proof. Let \( \psi \) be in semi-normal form with respect to \( X \); so \( \psi^a \) is also in semi-normal form with respect to \( X \). First we show that the right hand side is contained in the left hand side. If \( (m, \Gamma) \in \mathcal{M}_\psi \) then there exists an element \( u \) of \( V_{n,r} \) in a semi-infinite \( X \)-component for \( \psi \) of characteristic \((m, \Gamma)\); and we may assume \( u \in X(A) \). Suppose first that \( a > 0 \). If \( d = \gcd(m, a) \), \( p = m/d \), \( q = a/d \) and \( k = ma/d \), then \( u(\psi^a)^p = u\psi^mq = u\Gamma^q \), (as \( mq \) has the same sign as \( m \)). If \( a < 0 \) then, from the above, with \( d = \gcd(m, -a) \), \( p = m/d \), \( q = -a/d \) and \( k = -ma/d \), we have \( u\psi^{-ap} = u\Gamma^q \).

In all cases therefore \( u \) is a characteristic element of \( \psi^a \). Furthermore, if \( u(\psi^a)^r = u\Delta \), with \( \Delta \neq 1 \) then, from Lemma 5.25 \( m|ar \), which we can rewrite as \( pd|qr \), so \( p|qr \). As \( \gcd(p, q) = 1 \), this implies \( p|r \), so that \( |m/d| = |p| \leq |r| \). Hence \( u \) has characteristic \((m/d, \Gamma^q)\), with respect to \( \psi^a \). As
$u$ belongs to a semi-infinite $X$-component for $\psi^a$, it follows that $(m/d, \Gamma^q)$ is in $\mathcal{M}_{\psi^a}$ and so we have

$$\mathcal{M}_{\psi^a} \supseteq \{(m, \Gamma^q) \mid (md, \Gamma) \in \mathcal{M}_\psi, d > 0, \gcd(m, q) = 1 \text{ and } |a| =qd\}.$$ 

On the other hand, suppose that $(r, \Delta) \in \mathcal{M}_{\psi^a}$. Assume first that $a > 0$. Then again, there exists $u \in X(A)$ such that $u$ is a characteristic element of $\psi^a$, so $u\psi^a = u\Delta$. Thus, from Lemma 4.2, $u$ is a characteristic element for $\psi$, with characteristic $(m, \Gamma) \in \mathcal{M}_\psi$, such that $m|ar$ and $\Delta = \Gamma^q$, where $ar = mt$, $t > 0$. Let $d = \gcd(a, m) = pd$ and $a = qd$. Then $dqr = pdr$, so $qr = pt$ and $\gcd(p, q) = 1$, so $r = pr'$ and $t = qt'$. However, we have $u(\psi^a)^p = u\psi^a^{dpr} = u\psi^{mq} = u\Gamma^q$, and so, by definition of $(r, \Delta) \in \mathcal{M}_{\psi^a}$, we see that $|p| \geq |r|$, so $r' = \pm 1$. Since $a > 0$, $r' = 1$. It now follows that $r = p = m/d$ and $\Delta = \Gamma^q$, and $(r, \Delta)$ belongs to the set on the right hand side of the equality in the lemma. That is

$$\mathcal{M}_{\psi^a} \subseteq \{(m, \Gamma^q) \mid (md, \Gamma) \in \mathcal{M}_\psi, d > 0, \gcd(m, q) = 1 \text{ and } |a| =qd\}.$$ 

If $a < 0$ then the lemma follows by applying the result above to $\mathcal{M}_{\psi^{-1}(-a)}$, as for all $\theta \in G_{n,r}$ we have $(m, \Gamma) \in \mathcal{M}_\theta$ if and only if $(-m, \Gamma) \in \mathcal{M}_{\theta^{-1}}$.

Example 6.2. Let $n = 2$ and $r = 1$ and let $V_{2,1}$ be free on \{x\}. Let $\varphi$ be the regular infinite element of $G_{2,1}$ defined by the bijection from

$$Y = \{x\alpha_1^3, x\alpha_1^2\alpha_2, x\alpha_1\alpha_2, x\alpha_2\},$$

and

$$Z = \{x\alpha_1^2, x\alpha_1\alpha_2, x\alpha_2, x\alpha_2\alpha_2\},$$

given by $x\alpha_1^3\varphi = x\alpha_1\alpha_2$, $x\alpha_1\alpha_2\varphi = x\alpha_1^2$, $x\alpha_1^2\alpha_2\varphi = x\alpha_2\alpha_2$ and $x\alpha_2\varphi = x\alpha_2^2$.

Then $Y$ is the minimal expansion of $\{x\}$ associated to $\varphi$. The minimal expansion of $\{x\}$ not containing any element of $\{x\}\langle A \rangle \setminus (Y(A) \cup Z(A))$ is $X = \{x\alpha_1^3, x\alpha_1\alpha_2, x\alpha_2\}$. $X(A) \setminus Y(A) = \{x\alpha_1^2\}$ and $X(A) \setminus Z(A) = \{x\alpha_2\}$. The $X$-components of these elements are

$$\cdots \mapsto x\alpha_1\alpha_2\alpha_1 \mapsto x\alpha_1^3 \mapsto x\alpha_1\alpha_2 \mapsto x\alpha_1^2$$

with characteristic $(-2, \alpha_1)$ and

$$x\alpha_2 \mapsto x\alpha_2^2 \mapsto x\alpha_2^3 \mapsto x\alpha_2^4 \mapsto \cdots$$

with characteristic $(1, \alpha_2)$. Hence $\varphi$ is in quasi-normal form with respect to $X$ and $\mathcal{M}_{\varphi} = \{(-2, \alpha_1), (1, \alpha_2)\}$.

The map $\varphi^2$ may be defined by the bijection from

$$U = \{x\alpha_1^3, x\alpha_1^2\alpha_2, x\alpha_1\alpha_2\alpha_1, x\alpha_1\alpha_2^2, x\alpha_2\},$$

to

$$V = \{x\alpha_1^2, x\alpha_1\alpha_2, x\alpha_2, x\alpha_2\alpha_1\alpha_1, x\alpha_2^3\},$$

given by $x\alpha_1^3\varphi^2 = x\alpha_1^2$, $x\alpha_1^2\alpha_2\varphi^2 = x\alpha_2\alpha_1\alpha_1\varphi^2 = x\alpha_1\alpha_2$, $x\alpha_1\alpha_2\alpha_1\varphi^2 = x\alpha_2\alpha_1$ and $x\alpha_2\varphi^2 = x\alpha_2^3$. 

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Then $U$ is the minimal expansion of $\{x\}$ associated to $\varphi^2$ and the minimal expansion of $\{x\}$ not containing any element of $\{x\} \setminus (U(A) \cup V(A))$ is $X$ again. $X(A) \setminus U(A) = \{xa_1^2, xa_1 a_2\}$ and $X(A) \setminus V(A) = \{xa_2, xa_2^3\}$ and these elements have $X$-components

$$\cdots \mapsto xa_1^3 \mapsto xa_1^2$$

and

$$\cdots \mapsto xa_1 a_2 a_1 \mapsto xa_1 a_2$$

with characteristic $(-1, \alpha_1)$ and

$$xa_2 \mapsto xa_2^3 \mapsto \cdots$$

and

$$xa_2^4 \mapsto \cdots$$

with characteristic $(1, \alpha_2^3)$. Hence $\varphi^2$ is in quasi-normal form with respect to $X$ and $M_{\varphi^2} = \{(-1, \alpha_1), (1, \alpha_2^3)\}$, as predicted by Lemma 6.1.

Lemma 6.3 and Proposition 6.6 will allow us to find “minimal” pairs $(a, b)$ such that $\psi^a$ and $\varphi^b$ are conjugate.

**Lemma 6.3.** Let $\psi$ and $\varphi$ be regular infinite elements of $G_{n, r}$ and let $c$ be an integer, such that $c$ is coprime to $m$, for all $m \in \mathbb{Z}$ such that $(m, \Gamma) \in M_{\psi} \cup M_{\varphi}$. Then $\psi^c \sim \varphi^c$ if and only if $\psi \sim \varphi$.

**Proof.** If $\psi \sim \varphi$ then it is immediate that $\psi^c \sim \varphi^c$. For the converse, observe that we may assume, without loss of generality, that $c > 0$. Suppose that $\psi$ and $\varphi$ are in quasi-normal form with respect to $A$-bases $X$ and $Y$, respectively. From Lemma 6.1, $M_{\psi^c} = \{(m, \Gamma^c) : (m, \Gamma) \in M_{\psi}\}$ and $M_{\varphi^c} = \{(m, \Delta^c) : (m, \Delta) \in M_{\varphi}\}$.

Let $u$ be an element of $X(A)$ which is characteristic for $\psi$, with $\psi$-characteristic $(m, \Gamma)$. Then, from Lemma 6.1 (and its proof), $u$ has $\psi^c$-characteristic $(m, \Gamma^c)$ and, as $\psi^c \sim \varphi^c$, its image $u \rho$ has $\varphi^c$-characteristic $(m, \Gamma^c)$. Hence, from Lemma 6.1 again, $u \rho$ has $\varphi$-characteristic $(m, \Gamma)$. As $\gcd(c, m) = 1$, there exist integers $s$ and $t$ such that $ms + ct = 1$. Since $\psi^c \rho = \psi \varphi^c$ we have, in the case where $s > 0$,

$$u \psi \rho = u \psi^{ms + ct} \rho = (u \psi^m)^s \psi^c \rho = u \Gamma^s \psi^c \rho = u \Gamma^s \rho \varphi^c = (u \rho) \varphi^c = (u \rho)^c \varphi^{ms + ct} = u \rho \varphi^c.$$

If $s < 0$ then we have $m(-s) + c(-t) = -1$, with $-s > 0$ and the argument above implies instead that $u \psi^{-1} \rho = u \rho \varphi^{-1}$. In this case, let $v = u \psi$, so $v$ also has $\psi$-characteristic $(m, \Gamma)$ and replacing $u$ by $v$ gives $v \psi^{-1} \rho = v \rho \varphi^{-1}$ from which it follows that $u \psi \rho = u \rho \varphi$. This applies in particular to all elements of $X$ of type (B). Let $y'$ be an element of type (C); so there exists an element $y \in X$ of type (B) such that $y' \psi^k = y \Omega$, for some $k \in \mathbb{Z}$. Then $y' = y \Omega \psi^{-k}$, and $y \psi^j$ has the same $\psi$-characteristic as $y$, for all $j$; and so is a characteristic element for $\psi$. From the above then $y \psi^j \rho = (y \rho)^{\varphi^j}$, for all $j$. Now

$$y' \psi \rho = y \Omega \psi^{-k} \rho = y \psi^{1-k} \rho \Omega = y \rho \varphi^{1-k} \Omega = y \rho \varphi^{-k} \varphi \Omega = y \psi^{-k} \rho \varphi \Omega = y \psi^{-k} \Omega \rho \varphi = y' \rho \varphi.$$
Therefore, \( y\psi\rho = yp\varphi \), for all \( y \in X \), so \( \psi \sim \varphi \). \( \square \)

**Definition 6.4.** Let \( \psi \) be a regular infinite element of \( G_{n,r} \) and let \( a \) be a positive integer. Define a map \( \hat{\psi}^a : M_\psi \to M_\psi^a \) by \( \hat{\psi}^a(m, \Gamma) = (p, \Gamma^a) \), where \( d = \gcd(m, a) \), \( p = m/d \) and \( \alpha = a/d \).

**Example 6.5.** For \( \varphi \) in Example 6.2 with \( a = 2 \), the map

\[
\hat{\varphi}^2 : M_\varphi \to M_{\varphi^2}
\]

is given by

\[
\hat{\varphi}^2(-2, \alpha_1) = (-1, \alpha_1) \text{ and } \hat{\varphi}^2(1, \alpha_2) = (1, \alpha_2^2).
\]

From Lemma 6.1 this is a well defined map, and is surjective. In general it is not injective. For instance if \( p, s \) and \( t \) are pairwise coprime positive integers and we have \( m_1 = ps \), \( m_2 = pt \) and \( a = st \), then \( d_1 = \gcd(m_1, a) = s \), \( d_2 = \gcd(m_2, a) = t \), \( \alpha_1 = a/d_1 = t \) and \( \alpha_2 = a/d_2 = s \). If, for some non-trivial \( \Lambda \in A^* \) we have \( (m_1, \Lambda^s) \) and \( (m_2, \Lambda^t) \) in \( M_\psi \), then both these elements are mapped by \( \hat{\psi}^a \) to \( (p, \Lambda^{at}) \).

**Proposition 6.6.** Let \( \psi \) and \( \varphi \) be regular infinite elements of \( G_{n,r} \), let \( a \) and \( b \) be positive integers and let the images of \( \hat{\psi}^a \) and \( \hat{\varphi}^b \) be

\[
M_{\psi^a} = \{(p_i, \Gamma^a_i)|i = 1, \ldots, M\} \text{ and } M_{\varphi^b} = \{(q_i, \Delta^b_i)|i = 1, \ldots, N\}.
\]

For \( i = 1, \ldots, M \), let

\[
(\hat{\psi}^a)^{-1}(p_i, \Gamma^a_i) = \{(m_{i,j}, \Gamma_{i,j})|1 \leq j \leq M_i\}
\]

and, for \( i = 1, \ldots, N \), let

\[
(\hat{\varphi}^b)^{-1}(q_i, \Delta^b_i) = \{(n_{i,j}, \Delta_{i,j})|1 \leq j \leq N_i\}.
\]

If \( \psi^a \sim \varphi^b \) then \( M = N \) and, after reordering if necessary, we have \( p_i = q_i \) and \( \Gamma^a_i = \Delta^b_i \). Moreover, there exist positive integers \( \alpha, \beta, g, d_{i,j}, e_{i,k}, s_{i,j,k}, t_{i,j,k}, f_{i,j,k} \), and \( \Lambda_{i,j,k} \in A^* \), for \( 1 \leq i \leq M, 1 \leq j \leq M_i \) and \( 1 \leq k \leq N_i \), such that

\[
\alpha = \frac{a}{g} = d_{i,j} f_{i,j,k} t_{i,j,k} \text{ and } \beta = \frac{b}{g} = e_{i,k} f_{i,j,k} s_{i,j,k}, \text{ for all } i, j, k,
\]

and

\[
\psi^a \sim \varphi^b,
\]

where \( d_{i,j} \) is a positive divisor of \( m_{i,j} \), \( e_{i,k} \) is a positive divisor of \( n_{i,k} \), \( \Gamma_{i,j} = \Lambda^a_{i,j,k} \) and \( \Delta_{i,j} = \Lambda^b_{i,j,k} \), and

\[
\prod_{i,j,k} (t_{i,j,k} d_{i,j}) | f_{i',j',k'} d_{i',j'},
\]

for all \( i', j', k' \).
Proof. Assume \( \psi^a \sim \psi^b \), with \( a, b > 0 \), and that \( \rho^{-1} \psi^a \rho = \psi^b \). From Lemma 5.10 \( \mathcal{M}_{\psi^a} \) and \( \mathcal{M}_{\psi^b} \) are equal, so \( M = N \), and we may order \( \mathcal{M}_{\psi^a} \) so that \( (p_i, \Gamma_i^a) = (q_i, \Delta_i^a) \), so \( p_i = q_i \) and \( \Gamma_i^a = \Delta_i^a \). With the notation for \( (\psi^a)^{-1} \) and \( (\psi^b)^{-1} \) given in the statement of the proposition, let \( d_{i,j} = \gcd(a, m_{i,j}) \) and \( e_{i,k} = \gcd(b, n_{i,k}) \), so

\[
m_{i,j}/d_{i,j} = p_i = q_i = n_{i,k}/e_{i,k}
\]

and let

\[
\alpha_{i,j} = a/d_{i,j}, \beta_{i,k} = b/e_{i,k},
\]

and

\[
\Gamma_{i,j}^{\alpha_{i,j}} = \Gamma_i^a = \Delta_i^a = \Delta_i^b = \Delta_{i,j}^{\beta_{i,k}}, \quad (9)
\]

by Definition 6.4 for \( 1 \leq i \leq M, 1 \leq j \leq M_i \) and \( 1 \leq k \leq N_i \). As \( \Gamma_{i,j}^{\alpha_{i,j}} = \Delta_{i,j}^{\beta_{i,k}} \), by Proposition 6.9 there exist \( \Lambda_{i,j,k} \in A^* \) and positive integers \( s_{i,j,k}, t_{i,j,k} \) such that \( \Gamma_{i,j} = \Lambda_{i,j,k}^{s_{i,j,k}} \) and \( \Delta_{i,j} = \Lambda_{i,j,k}^{t_{i,j,k}} \). Taking a power of \( \Lambda_{i,j,k} \) if necessary, we may assume that \( \gcd(s_{i,j,k}, t_{i,j,k}) = 1 \). Then

\[
\Lambda_{i,j,k}^{s_{i,j,k} \alpha_{i,j}} = \Gamma_{i,j}^{\alpha_{i,j}} = \Delta_{i,j}^{\beta_{i,k}} = \Lambda_{i,j,k}^{t_{i,j,k} \beta_{i,k}}, \quad (10)
\]

so \( s_{i,j,k} \alpha_{i,j} = t_{i,j,k} \beta_{i,k} \). As \( s_{i,j,k} \) and \( t_{i,j,k} \) are coprime this implies that \( \alpha_{i,j} = t_{i,j,k} / s_{i,j,k} = c_{i,j,k} \in \mathbb{Z} \), and \( \alpha_{i,j} = c_{i,j,k} t_{i,j,k} \) and \( \beta_{i,k} = c_{i,j,k} s_{i,j,k} \).

Let

\[
g = \gcd\{c_{i,j,k} | 1 \leq i \leq M, 1 \leq j \leq M_i, 1 \leq k \leq N_i\}.
\]

Then there exist integers \( f_{i,j,k} \) such that \( c_{i,j,k} = g f_{i,j,k} \), for all \( i, j, k \). From Lemma 6.1 \( \mathcal{M}_{\psi^a/g} \) consists of elements \( (m/p, \Gamma^a) \), where \( (m, \Gamma) \in \mathcal{M}_{\psi^a} \), \( p = \gcd(m, a/g) \) and \( \alpha = a/gp \). Similarly, elements of \( \mathcal{M}_{\psi^b/g} \) are of the form \( (n/q, \Delta^b) \), where \( (n, \Delta) \in \mathcal{M}_{\psi}, q = \gcd(n, b/g) \) and \( \beta = b/gq \). Now \( g | c_{i,j,k} \) and \( c_{i,j,k} | \alpha_{i,j} \) and \( c_{i,j,k} | \beta_{i,k} \). Therefore \( \gcd(m_{i,j}, a/g) = \gcd(m_{i,j}, a) = d_{i,j} \) and similarly \( \gcd(n_{i,k}, b/g) = e_{i,k} \). Thus \( g \) is coprime to

\[
p_i = \frac{m_{i,j}}{\gcd(m_{i,j}, a/g)} = \frac{n_{i,k}}{\gcd(n_{i,k}, b/g)},
\]

for all \( i, j, k \). From Lemma 6.3 it follows that \( \psi^a/g \sim \psi^b/g \).

Now

\[
a/g = \alpha_{i,j} d_{i,j}/g = c_{i,j,k} t_{i,j,k} d_{i,j}/g = f_{i,j,k} t_{i,j,k} d_{i,j}
\]

and similarly

\[
b/g = f_{i,j,k} s_{i,j,k} e_{i,k},
\]

for all \( i, j, k \). Also

\[
\gcd\{f_{i,j,k} | 1 \leq i \leq M, 1 \leq j \leq M_i, 1 \leq k \leq N_i\} = 1
\]

so, for fixed \( i', j', k' \),

\[
f_{i', j', k'} \left| \prod_{i,j,k} (t_{i,j,k} d_{i,j}) \right| t_{i', j', k'} d_{i', j'}.
\]

\( \square \)
Corollary 6.7. The power conjugacy problem for regular infinite elements of $G_{n,r}$ is solvable.

Proof. Let $\psi$ and $\varphi$ be regular infinite elements of $G_{n,r}$. Suppose that $\psi^a$ is conjugate to $\varphi^b$, for some non-zero $a, b$. Replacing either $\psi$ or $\varphi$ or both by their inverse, we may assume that $a, b > 0$. Then, in the notation of the proposition above, we have $\psi^α \sim \varphi^β$, where $α = f_{i,j,k}t_{i,j,k}d_{i,j}$ and $β = f_{i,j,k}s_{i,j,k}e_{i,k}$. From the conclusion of the theorem it is clear that there are finitely many choices for $f_{i,j,k}, s_{i,j,k}, t_{i,j,k}, d_{i,j}$ and $e_{i,k}$. Hence there are finitely many possible $α$ and $β$, and we may effectively construct a list of all possible pairs $(α, β)$. Having constructed this list we may check whether or not $\psi^a \sim \varphi^b$, using Algorithm 5.27. Hence we may decide whether or not there exist $a, b$ such that $\psi^a \sim \varphi^b$. \hfill \Box

The proof of Proposition 6.6 forms the basis for our algorithm for the power conjugacy problem. Given $\psi, \varphi \in G_{n,r}$ we construct bounds $\hat{a}$ and $\hat{b}$ such that if some (positive) power of $\psi$ is conjugate to a (positive) power of $\varphi$ then $\psi^c \sim \varphi^d$, for $0 < c \leq \hat{a}$ and $0 < d \leq \hat{b}$. Following the proof of the proposition, if $\psi^a \sim \varphi^b$, for some $a, b > 0$, then the inverse images $\hat{\psi}_a$ and $\hat{\varphi}_b$ partition $\mathcal{M}_\psi$ and $\mathcal{M}_\varphi$, so we have integers $L, M_1, N_1$ such that

$$\mathcal{M}_\psi = \bigcup_{i=1}^{L} \{(m_{i,j}, \Gamma_{i,j}) \mid 1 \leq j \leq M_i\}$$

and

$$\mathcal{M}_\varphi = \bigcup_{i=1}^{L} \{(n_{i,k}, \Delta_{i,k}) \mid 1 \leq k \leq N_i\}.$$  

Given any $\Gamma \in A^*$ there exists unique $\Lambda \in A^*$ and $r \in \mathbb{N}$ such that $\Gamma = \Lambda^r$ and if $\Gamma = \Lambda^s$ then $s \leq r$. We denote $\Lambda$ by $\sqrt{\Gamma}$ and $r$ by $m(\Gamma)$. From equations (9) and (10), it follows that

$$\sqrt{\Lambda_{i,j,k}} = \sqrt{T_{i,j}} = \sqrt{\Gamma_i} = \sqrt{\Delta_{i,k}}$$

and

$$s_{i,j,k} \leq m(\Gamma_{i,j}) \text{ and } t_{i,j,k} \leq m(\Delta_{i,k}),$$

for $1 \leq i \leq L$, $1 \leq j \leq M_i$ and $1 \leq k \leq N_i$.

From Proposition 6.6 we have $α = d_{1,1}f_{1,1,1}t_{1,1,1}$ and $f_{1,1,1} \leq \prod_{(i,j,k) \neq (1,1,1)} d_{i,j}t_{i,j,k}$. As $d_{i,j} \leq |m_{i,j}|$ and $t_{i,j,k} \leq m(\Delta_{i,k})$, this means that

$$\alpha \leq \prod_{i=1}^{L} \prod_{j=1}^{M_i} \prod_{k=1}^{N_i} d_{i,j}t_{i,j,k}$$

$$\leq \prod_{i=1}^{L} \prod_{j=1}^{M_i} \prod_{k=1}^{N_i} |m_{i,j}| m(\Delta_{i,k})$$

$$\leq \prod_{i=1}^{L} \prod_{j=1}^{M_i} \left( |m_{i,j}|^{N_i} \prod_{k=1}^{N_i} m(\Delta_{i,k}) \right)$$

$$\leq \prod_{i=1}^{L} \left( \prod_{j=1}^{M_i} |m_{i,j}| \right)^{N_i} \left( \prod_{k=1}^{N_i} m(\Delta_{i,k}) \right)^{M_i}. \quad (11)$$
Similarly
\[ \beta \leq \prod_{i=1}^{L} \left[ \prod_{k=1}^{N_i} |m| \prod_{n=1}^{M_i} m(\Delta_i) \right]. \tag{12} \]

Now suppose that a solution \( \psi' \sim \varphi' \) gives rise to a sub-partition of the partition above. Straightforward calculation shows that in this case, the bounds on \( \alpha \) and \( \beta \) obtained are again less than or equal to the right hand sides of (11) and (12) (calculated using the original partition). Thus, in computing (upper) bounds \( \hat{a} \) and \( \hat{b} \) we may take partitions of \( \mathcal{M}_\psi = P_1 \cup \cdots \cup P_L \) and \( \mathcal{M}_\varphi = Q_1 \cup \cdots \cup Q_L \) with \( L \) as small as possible, subject to the constraint that, for each \( i \) such that \( 1 \leq i \leq L \) we have \( \sqrt{\Gamma} = \sqrt{\Delta} \), for all \( (m, \Gamma) \in P_i \) and \( (n, \Delta) \in Q_i \). If these partitions satisfy these properties, and this does not hold for any partition of fewer than \( L \) subsets, (in other words the partitions are formed by gathering together characteristics with the same root) then the bounds \( \hat{a} \) and \( \hat{b} \) are given by
\[ \hat{a} = \prod_{i=1}^{L} \left[ \prod_{(m, \Gamma) \in P_i} |m| \prod_{(n, \Delta) \in Q_i} m(\Delta) \right], \tag{13} \]
and
\[ \hat{b} = \prod_{i=1}^{L} \left[ \prod_{(n, \Delta) \in Q_i} |n| \prod_{(m, \Gamma) \in P_i} m(\Gamma) \right]. \tag{14} \]

**Example 6.8.** Let \( n = 2 \) and \( r = 1 \) and \( V_{2,1} \) be free on \( \{x\} \). Let \( \psi \) be the regular infinite element of \( G_{2,1} \) of Example 4.11 which is in quasi-normal form with respect to the \( A \)-basis \( X = \{x_{a_1}, x_{a_2}\} \) and has \( \mathcal{M}_\psi = \{(1, \alpha_2), (1, \alpha_1)\} \).

Let \( \varphi \) be the regular infinite element of \( G_{2,1} \) defined by a bijective map from
\[ Y_\varphi = \{x_{a_1}, x_{a_2}a_1^3, x_{a_2}a_1^2a_2, x_{a_2}a_1a_2, x_{a_2}^2\} \]
to
\[ Z_\varphi = \{x_{a_1}^2, x_{a_1}a_2a_1, x_{a_1}a_2^2a_1, x_{a_1}a_1^2a_2, x_{a_2}\} \]
given by \( x_{a_1} \varphi = x_{a_1}a_2^3, x_{a_2} \varphi = x_{a_2}, x_{a_2}a_1^2 \varphi = x_{a_2}, x_{a_2}a_1 \varphi = x_{a_2}a_1a_2 \) and \( x_{a_2}^2 \varphi = x_{a_1}a_2a_1 \).

\[ \varphi : \quad \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \quad \rightarrow \quad \begin{array}{c} 3 \\ 4 \\ 5 \\ 1 \\ 2 \end{array} \]

Then \( Y_\varphi \) is the minimal expansion of \( \{x\} \) associated to \( \varphi \) and the minimal expansion of \( \{x\} \) not containing an element which lies outside \( Y_\varphi(A) \) and \( Z_\varphi(A) \) is \( X \). We have \( X(A) \setminus Y_\varphi(A) = \{x_{a_2}, x_{a_2}a_1, x_{a_2}a_1^2\} \) and \( X(A) \setminus Z_\varphi(A) = \{x_{a_1}, x_{a_1}a_2, x_{a_1}a_2^2\} \) The \( X \)-components of these elements are:
\[ x_{a_1} \mapsto x_{a_1}a_2^3 \mapsto x_{a_1}a_2^5 \mapsto \cdots \]
\[ x_{a_1}a_2 \mapsto x_{a_1}a_2^4 \mapsto x_{a_1}a_2^7 \mapsto \cdots \]
\[ x\alpha_1a_2^2 \mapsto x\alpha_1a_2^3 \mapsto x\alpha_1a_2^5 \mapsto \cdots \]
\[ \cdots \mapsto x\alpha_2a_1^6 \mapsto x\alpha_2a_1^3 \mapsto x\alpha_2 \]
\[ \cdots \mapsto x\alpha_2a_1^7 \mapsto x\alpha_2a_1^4 \mapsto x\alpha_2a_1 \]
\[ \cdots \mapsto x\alpha_2a_1^8 \mapsto x\alpha_2a_1^5 \mapsto x\alpha_2a_1^2 \]

so \( \varphi \) is in quasi-normal form with respect to \( X \) and \( M_\varphi = \{(1, \alpha_2^3), (-1, \alpha_1^3)\} \). In the notation above, we have partitions \( M_\psi = P_1 \cup P_2 \) and \( M_\varphi = Q_1 \cup Q_2 \) with \( P_1 = \{(1, \alpha_2),\} \), \( P_2 = \{(-1, \alpha_1)\} \), \( Q_1 = \{(1, \alpha_2)\} \) and \( Q_2 = \{(-1, \alpha_1)\} \), so we obtain bounds \( \hat{a} = 9 \) and \( \hat{b} = 1 \).

Assume there exists positive integers \( a, b \) such that \( \psi^a \sim \varphi^b \). We may now assume that \( a \leq 9 \) and \( b = 1 \). The map \( \hat{\psi}^a : M_\psi \rightarrow M_{\psi^a} \) is given by
\[ \hat{\psi}^a(1, \alpha_2) = (1/d_1, \alpha_2^{a/d_1}) \]
and here \( d_1 = \gcd(1, a) = 1 \) and \( d_2 = \gcd(-1, a) = 1 \). Thus
\[ M_{\psi^a} = \{(1, \alpha_2^3), (-1, \alpha_1^3)\}. \]

The only possible choice for \( a \) making \( M_{\psi^a} = M_\varphi = M_\varphi \) is \( a = 3 \). Applying Algorithm 5.27 to \( \psi^a \) and \( \varphi \) we find a conjugating element \( \rho \), given by \( x\alpha_1 \rho = x\alpha_2 \) and \( x\alpha_2 \rho = x\alpha_1 \).

**Remark 6.9.** In Corollary 6.7 we supposed that the powers \( a \) and \( b \) were positive, giving us upper bounds \( a \leq \hat{a} \) and \( b \leq \hat{b} \) for the minimal powers which solve the power conjugacy problem. Now suppose that \( a > 0 \) and \( b > 0 \). We may write \( \psi^a = (\psi^{-1})^{-a} \) and then \( -a > 0 \). If we apply Corollary 6.7 to \( \psi^{-1}, \varphi \), we would obtain a second pair of bounds \( -a \leq \hat{a} \) and \( b \leq \hat{b} \). Observing that \((m, \Gamma) \in M_\varphi \) if and only if \((-m, \Gamma) \in M_{\psi^{-1}} \), we note that this replacement \( \psi \rightarrow \psi^{-1} \) preserves the absolute value \( |m| \) of all characteristic multipliers. Thus each of the terms \(|m_i,j|, |n_i,k|, |m| \) and \(|n| \) in equations (1), (3) is unchanged. We conclude that \( \hat{a} = \hat{a} \) and \( \hat{b} = \hat{b} \).

The same argument applies equally well to the remaining two cases \( a > 0, b < 0 \) and \( a < 0, b > 0 \). Thus, once we have obtained \( \hat{a} \) and \( \hat{b} \), we need only to check the ranges \( 1 \leq |a| \leq \hat{a} \) and \( 1 \leq |b| \leq \hat{b} \) to find minimal conjugating powers.

**Example 6.10.** Let \( \psi \) be as in Example 6.8 and let \( \varphi \) be as in 6.2. Then \( M_\psi = \{(1, \alpha_2), (-1, \alpha_1)\} \) and \( M_\varphi = \{(-2, \alpha_1), (1, \alpha_2)\} \). In the notation above, we have partitions \( M_\psi = P_1 \cup P_2 \) and \( M_\varphi = Q_1 \cup Q_2 \) with \( P_1 = \{(1, \alpha_2)\}, P_2 = \{(-1, \alpha_1)\}, Q_1 = \{(1, \alpha_2)\} \) and \( Q_2 = \{(-2, \alpha_1)\} \), so we obtain bounds \( \hat{a} = 1 \) and \( \hat{b} = 2 \).

Assume there exists positive integers \( a, b \) such that \( \psi^a \sim \varphi^b \); with \( a = 1 \) and \( b \leq 2 \). The map \( \tilde{\varphi}^b : M_\varphi \rightarrow M_{\psi^b} \) is given by
\[ \tilde{\varphi}^b(1, \alpha_2) = (1/d_1, \alpha_2^{b/d_1}) \]
\[ \tilde{\varphi}^b(-2, \alpha_1) = (-2/d_2, \alpha_1^{b/d_2}) \]
where \( d_1 = \gcd(1, b) = 1 \) and \( d_2 = \gcd(-2, b) = b \). Thus,
\[ M_{\psi^b} = \{(1, \alpha_2), (-2, \alpha_1)\} \text{ or } \{(1, \alpha_2^2), (-1, \alpha_1)\}. \]

Therefore there is no pair of positive integers \( a, b \) such that \( \psi^a \sim \varphi^b \). The same argument applies replacing \( \varphi \) or \( \psi \) by \( \varphi^{-1} \) or \( \psi^{-1} \) respectively, so no nontrivial power of \( \varphi \) is conjugate to a power of \( \psi \).
In order to solve the power conjugacy problem for general elements of \( G_{n,r} \) we require an algorithm which finds all pairs \((a, b)\), within the bounds calculated, rather than merely deciding whether or not such a pair exists. This is the algorithm we describe here. It constructs a set \( \mathcal{PC}_{RI} \) consisting of triples \((a, b, \rho)\), such that \( \rho^{-1}\psi^a\rho = \varphi^b \).

**Algorithm 6.11.** Let \( \psi \) and \( \varphi \) be regular infinite elements of \( G_{n,r} \).

**Step 1:** Construct \( A \)-bases \( X_\psi \) and \( X_\varphi \) with respect to which \( \psi \) and \( \varphi \) are in quasi-normal form (Lemma 4.28).

**Step 2:** Construct the sets \( M_\psi \) and \( M_\varphi \) (see Definition 5.14).

**Step 3:** Calculate the bounds on \( \hat{a} \) and \( \hat{b} \), using equations (13) and (14).

**Step 4:** For all pairs \( a, b \) such that \( 1 \leq |a| \leq \hat{a} \) and \( 1 \leq |b| \leq \hat{b} \), input \( \psi^a \) and \( \varphi^b \) to Algorithm 5.27. If a conjugating automorphism \( \rho \) is returned, add \((a, b, \rho)\) to the set \( \mathcal{PC}_{RI} \).

**Step 5:** If \( \mathcal{PC}_{RI} = \emptyset \), output “No” and halt. Otherwise output \( \mathcal{PC}_{RI} \).

**Corollary 6.7** may be strengthened.

**Corollary 6.12.** Given regular infinite elements \( \psi, \varphi \in G_{2,1} \) there is a finite subset \( \mathcal{PC}_{RI} \) of \( \mathbb{Z} \times \mathbb{Z} \times G_{n,r} \), which may be effectively constructed, such that \( \psi^a \sim \varphi^b \) if and only if \( a = cg \) and \( b = dg \), for some \((c, d, \rho) \in \mathcal{PC}_{RI} \) and \( g \in \mathbb{Z} \). Moreover, for all \((c, d, \rho) \in \mathcal{PC}_{RI} \) and \( g \in \mathbb{Z} \), we have \( \rho^{-1}\psi^c\rho = \varphi^d \).

**Proof.** From Lemma 6.6 and the construction of Algorithm 6.11 it follows that if \( \psi^a \sim \psi^b \), for some positive \( a, b \in \mathbb{Z} \) then \((a/g, b/g, \rho) \in \mathcal{PC}_{RI} \), a finite set, and in this case \( \rho^{-1}\psi^a\rho = \varphi^b \). Replacing one or other, or both, of \( \psi \) and \( \varphi \) by their inverses the same holds, without the constraint that \( a, b \) be positive. On the other hand if \((c, d, \rho) \) is in \( \mathcal{PC}_{RI} \) then \( \psi^c \sim \varphi^d \), so \( \psi^c g \sim \varphi^d g \), for all \( g \in \mathbb{Z} \). \( \square \)

### 6.3 The power conjugacy algorithm

We combine the algorithms of Sections 6.1 and 6.2 to give an algorithm for the power conjugacy problem in \( G_{n,r} \). In fact in Sections 6.1 and 6.2 we find a description of all solutions of the power conjugacy problem for periodic and regular infinite automorphisms, respectively; and the algorithm in this section does the same for arbitrary elements of \( G_{n,r} \).

If we are only interested in the existence of a solution to the power conjugacy problem then we may essentially ignore the periodic part of automorphisms, as long as the regular infinite part is non-trivial. To see this, suppose \( \psi \) and \( \varphi \) are elements of \( G_{n,r} \) and we have decompositions \( \psi = \psi_P \ast \psi_{RI} \), \( \varphi = \varphi_P \ast \varphi_{RI} \). Assume that we have found that \( V_{RI, \psi} \) is non-trivial and \( \psi_{RI}^b \) is conjugate to \( \varphi_{RI}^a \), \( a, b \neq 0 \). In this case, \( \psi_P \) and \( \varphi_P \) have finite orders, \( m \) and \( k \) say, and so we immediately have a solution \( \psi^{amk} \sim \varphi^{bmk} \), \( amk, bmk \neq 0 \), of the power conjugacy problem. The algorithm described below allows this type of solution but also tries to find a solution to the power conjugacy problem corresponding to each pair \((c, d)\) such that \( \psi_P^c \sim \varphi_P^d \). Thus, in Theorem 6.14 we obtain a description of all solutions to the power conjugacy problem, for \( \psi \) and \( \varphi \).

**Algorithm 6.13.** Let \( \psi \) and \( \varphi \) be elements of \( G_{n,r} \).

**Step 1:** Run Steps 1, 2 and 3 of Algorithm 5.6.
Step 2: Input \( \psi_{RI} \) and \( \varphi_{RI} \) to Algorithm 6.11.

Step 3: If \( X_{RI,\psi} \) is non-empty (that is, \( V_{RI,\psi} \) is non-empty) and \( \mathcal{PC}_{RI} \) is empty, output “No” and stop.

Step 4: Compute the orders \( k \) and \( m \) of \( \psi_P \) and \( \varphi_P \). Input \( \psi^*_P \) and \( \varphi^*_P \) to Algorithm 5.13 for all \( c, d \) such that \( 1 \leq c \leq k \) and \( 1 \leq d \leq m \). Construct the set \( \mathcal{PC}_P \) of all triples \((c, d, \rho)\) found such that \( \rho^{-1}\psi^*_P \) is conjugate to \( \varphi^*_P \). If \( X_{RI,\psi} \) is non-empty, adjoin the triple \((0, 0, \theta_0)\) to \( \mathcal{PC}_P \), where \( \theta_0 \) is the identity map of the algebra \( V_{n,s_P} \), of Step 3 of Algorithm 5.6.

Step 5: If \( \mathcal{PC}_P \) is empty, output “No” and stop. If \( \mathcal{PC}_P \) is non-empty and \( X_{RI,\psi} \) is empty output \( \mathcal{PC}_P \) and stop.

Step 6: If this step is reached then both \( \mathcal{PC}_P \) and \( \mathcal{PC}_{RI} \) are non-empty. For all \((\alpha, \beta, \rho_{RI})\) in \( \mathcal{PC}_{RI} \) and all pairs \((c, d, \rho_P)\) in \( \mathcal{PC}_P \) consider the simultaneous congruences

\[
\alpha x \equiv c \mod k \text{ and } \beta x \equiv d \mod m,
\]

where \( k \) and \( m \) are the orders of \( \psi_P \) and \( \varphi_P \) found in Step 2. For each positive solution \( x = g \) (less than \( \text{lcm}(k, m) \)) add \((\alpha g, \beta g, g, \rho_P * \rho_{RI})\) to the set \( \mathcal{PC} \) (which is empty at the start).

We verify that this algorithm solves the power conjugacy problem in the proof of the following theorem.

**Theorem 6.14.** The power conjugacy problem for the Higman-Thompson group \( G_{n,r} \) is solvable. Furthermore, given elements \( \psi, \varphi \in G_{n,r} \), let \( \psi_P \) have order \( k \), let \( \varphi_P \) have order \( m \) and let \( l = \text{lcm}(k, m) \). There is a finite subset \( \mathcal{PC} \subseteq \mathbb{Z}^3 \times G_{n,r} \), which may be effectively constructed, such that \( \psi^a \sim \varphi^b \) if and only if \((ag/h, bg/h, g, \rho) \in \mathcal{PC} \), where \( \rho \in G_{n,r} \) and \( g, h \in \mathbb{Z} \) such that \( h \equiv g \mod l \), \( h|a \) and \( h|b \). In this case \( \rho^{-1}\psi^a \rho = \varphi^b \).

**Proof.** Apply Algorithm 6.13 to \( \psi \) and \( \varphi \). If there exist \( a, b \in \mathbb{Z} \) such that \( \psi^a \sim \varphi^b \) then \( \psi^*_P \sim \varphi^*_P \) and \( \psi^*_R \sim \varphi^*_R \). In this case let \( \psi_P \) and \( \varphi_P \) have orders \( k \) and \( m \), respectively and let \( a_1, b_1 \in \mathbb{Z} \) be such that \( 1 \leq a_1 < k \) and \( 1 \leq b_1 < m \) and \( a_1 \equiv a \) mod \( k \), \( b_1 \equiv b \) mod \( m \). Then there exists \( \rho_P \) such that \((a_1, b_1, \rho_P) \in \mathcal{PC}_P \). Furthermore, from Corollary 6.12 there exists \((a_2, b_2, \rho_{RI}) \in \mathcal{PC}_{RI} \) and \( h \in \mathbb{Z} \) such that \( a = a_2 h \) and \( b = b_2 h \). Let \( g \) be such that \( 1 \leq g < \text{lcm}(k, m) \), and \( g \equiv h \mod \text{lcm}(k, m) \) so \( g \equiv h \mod k \) and \( g \equiv h \mod m \). As \( h \) is a solution to the congruences \( a_2 x \equiv a_1 \) mod \( k \) and \( b_2 x \equiv b_1 \) mod \( m \), it follows that \( g \) is also a solution to these congruences. Therefore \((a_2 g, b_2 g, g, \rho_P \ast \rho_{RI}) \in \mathcal{PC} \). As \( a_2 = a/h \) and \( b_2 = b/h \), this is an element of \( \mathbb{Z}^3 \times G_{n,r} \) of the required form.

Conversely, assume \((u, v, g, \rho_P \ast \rho_{RI}) \in \mathcal{PC} \), where \( u = ag/h \) and \( v = bg/h \), for some \( a, b \in \mathbb{Z} \) satisfying the hypotheses of the theorem. Then there exist \((\alpha, \beta, \rho_{RI}) \) in \( \mathcal{PC}_{RI} \) and \((c, d, \rho_P) \) in \( \mathcal{PC}_P \) such that \( u = \alpha g \equiv c \mod k \) and \( v = \beta g \equiv d \mod m \). As \( g \equiv h \mod l \) this implies that \( a = (u/g)h = \alpha h \equiv c \mod k \) and \( b = (v/g)h = \beta h \equiv d \mod m \). Therefore \( \psi_P = \psi^*_P \sim \varphi^*_P = \varphi^*_R \), by definition of \( \mathcal{PC}_P \), and indeed \( \rho_P^{-1}\psi_P \rho_P = \varphi_P^b \). Also, \( a = \alpha h \) and \( b = \beta h \) implies \( \rho_{RI}^{-1}\psi_P \rho_{RI} = \varphi^*_R \), by Corollary 6.12 so

\[
\psi^a = (\psi_P \ast \psi_{RI})^a = \psi^*_P \ast \psi^*_R \sim \varphi^*_P \ast \varphi^*_R = (\varphi_P \ast \varphi_{RI})^b = \varphi^b
\]

and \( \rho_P \ast \rho_{RI} \) is a conjugating element. \( \square \)
Examples which illustrate how the algorithm works on automorphisms which are not necessarily periodic or regular infinite can be found at [D15]: follow the link to “Examples” and refer to the examples named “mixed_pconj_phi” and “mixed_pconj_psi”.

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