Boundary regularity of mixed local-nonlocal operators and its application

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Abstract
Let Ω be a bounded $C^2$ domain in $\mathbb{R}^n$ and $u \in C(\mathbb{R}^n)$ solves

$$\Delta u + alu + C_0|Du| \geq -K \quad \text{in } \Omega, \quad \Delta u + alu - C_0|Du| \leq K \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c,$$

in the viscosity sense, where $0 \leq a \leq A_0$, $C_0, K \geq 0$, and $I$ is a suitable nonlocal operator. We show that $u/\delta$ is in $C^\kappa(\bar{\Omega})$ for some $\kappa \in (0, 1)$, where $\delta(x) = \text{dist}(x, \Omega^c)$. Using this result, we also establish that $u \in C^{1,\gamma}(\bar{\Omega})$. Finally, we apply these results to study an overdetermined problem for mixed local-nonlocal operators.

Keywords Operators of mixed order ⋅ Semilinear equation ⋅ Overdetermined problems ⋅ Gradient estimate

Mathematics Subject Classification Primary: 35D40 ⋅ 47G20 ⋅ 35J61 ⋅ 35B65

1 Introduction and results

We are interested in the integro-differential operator $L$ of the form

$$Lu = \Delta u + alu,$$

where $0 \leq a \leq A_0$, and $I$ is a nonlocal operator given by

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for some nonnegative, symmetric kernel $k$, that is, $k(y) = k(-y) \geq 0$ for all $y$. Operator $L$ appears as a generator of a Lévy process which is obtained by superimposing a Brownian motion, running twice as fast as standard $n$-dimensional Brownian motion, and an independent pure-jump Lévy process corresponding to the nonlocal operator $aI$. Throughout this article, we impose the following assumptions on the kernel $k$.

**Assumption 1.1**

(a) For some $\alpha \in (0, 2)$ and a kernel $\widehat{k}$, we have that, for all $r \in (0, 1]$,

$$r^{\alpha+2} k(ry) \leq \widehat{k}(y) \quad \text{for } y \in \mathbb{R}^n,$$

and

$$\widehat{k}(y) = \frac{\Lambda}{|y|^{n+a}} 1_{B_r}(y) + J(y) 1_{B_r}(y),$$

where $\Lambda > 0$ and

$$\int_{B_r} J(y) dy < \infty.$$

(b) There exists a $\beta > 0$ such that for any $r \in (0, 1]$ and $x_0 \in \mathbb{R}^n$ the following holds: for all $x, y \in B_r(x_0)$ and $z \in B_r(x_0)$ we have

$$k(x - z) \leq \rho k(y - z) \quad \text{for } |y - z| < \beta,$$

for some $\rho > 1$.

Assumption 1.1(a) will be used to study certain scaled operators and to find the exact behaviour of $I\delta(x)$ near the boundary where $\delta(x)$ denotes the distance function from the complement $\Omega^c$. Assumption 1.1(b) will be used to apply the Harnack estimate from [24]. It should be noted that [24] uses a stronger hypothesis compared to Assumption 1.1(b). Assumption 1.1 is satisfied by a large class of nonlocal kernels as shown in the examples as follows:

**Example 1.1** The following class of nonlocal kernels satisfy Assumption 1.1.

(i) $k(y) = \frac{1}{|y|^{n+a}}$ for some $\alpha \in (0, 2)$. More generally, we may take $k(y) = \frac{1}{|y|^{n+a}} 1_B(y)$ for some ball $B$ centred at the origin.

(ii) $k(y) \asymp \frac{\Psi(|y|^{-2})}{|y|^n}$ where $\Psi$ is Bernstein function vanishing at 0. In particular, $\Psi$ is strictly increasing and concave. These class of nonlocal kernels correspond to a special class of Lévy processes, known as subordinate Brownian motions (see [37]). Assume that $\Psi$ satisfies a global weak scaling property with parameters $\mu_1, \mu_2 \in (0, 1)$, that is,

$$\lambda^{\mu_1} \Psi(s) \leq \Psi(\lambda s) \leq \lambda^{\mu_2} \Psi(s) \quad \text{for } s > 0, \lambda \geq 1.$$

Then, it is easily seen that
\[ r^{n+2\mu_j} k(r) \asymp r^{2\mu_j} \frac{\Psi(|r|^2)}{|y|^n} \leq \frac{\Psi(|y|^2)}{|y|^n} \leq \mathbf{1}_{B_1}(y) \frac{\Psi(1)}{|y|^{n+2\mu_j}} + \mathbf{1}_{B_1}(y) \frac{\Psi(1)}{|y|^{n+2\mu_j}} \]

for all \( r \in (0, 1) \). Thus, Assumption 1.1(a) holds. Using the weak scaling property, we can also check that Assumption 1.1(b) holds.

Let \( \Omega \) be a bounded \( C^2 \) domain in \( \mathbb{R}^n \). Let \( u \in C(\mathbb{R}^n) \) be a viscosity solution to

\[
Lu + C_0 |Du| \geq -K \quad \text{in } \Omega,
\]

\[
Lu - C_0 |Du| \leq K \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{in } \Omega^c,
\]

for some nonnegative constants \( C_0, K \). Though the results in this article are obtained for viscosity solutions, the results are also applied for weak solutions, see Assumption 1.2 below for more details. Our equations in (1.1) are motivated by the operators of the form

\[
Lu + H(Du, x) := Lu + \inf_{\mu, v} \{ b_{\mu, v}(x) \cdot Du(x) + f_{\mu, v}(x) \} = 0 \quad \text{in } \Omega, \quad \nu = 0 \quad \text{in } \Omega^c.
\]

Such equations arise in the study of stochastic control problems where the control can influence the dynamics only through the drift \( b_{\mu, v} \).

Integro-differential operators involving both local and nonlocal operators have gained interest very recently. See for instance, [1–6, 21, 25, 26, 32, 33]. For linear equations, one may recover up to the boundary \( C^{1,\gamma} \) regularity of \( u \) from the \( W^{2,p} \) regularity (cf. [25, Theorem 3.1.22]). Recently, inspired by [12, 13], interior regularity of the solutions of (1.2) are studied in [32, 33]. Let us also mention the recent works [17, 27] where interior Hölder regularity of the gradient is established for the weak solutions of degenerate elliptic equations of mixed type. In this article, we are interested in up to the boundary regularity of the solutions. It should also be noted that we are dealing with inequalities.

Our first result deals with the Lipschitz regularity of the solution.

**Theorem 1.1** Suppose that Assumption 1.1(a) holds and \( u \in C(\mathbb{R}^n) \) is a viscosity solution to (1.1). Then, for some constant \( C \), dependent only on \( n, \Omega, C_0, A_0, \hat{k} \), we have

\[
\|u\|_{C^{1}(\mathbb{R}^n)} \leq CK.
\]

Proof of Theorem 1.1 is based on two standard ingredients, interior estimates from [33] and barrier function. It can be easily shown that \( \text{dist}(\cdot, \Omega^c) \) gives a barrier function at the boundary (cf. lemma 2.1). Next we investigate finer regularity property of \( u \) near \( \partial \Omega \). Let \( \delta(x) = \text{dist}(x, \Omega^c) \) be the distance function from the boundary. Modifying \( \delta(x) \) inside \( \Omega \), if required, we may assume that \( \delta \in C^2(\hat{\Omega}) \) (cf. [16, Theorem 5.4.3]). Our next result establishes Hölder regularity of \( u/\delta \) up to the boundary.

**Theorem 1.2** Suppose that Assumption 1.1 holds. Let \( u \) be a viscosity solution to (1.1). Then, there exists \( \kappa \in (0, (2 - \alpha) \wedge 1) \) such that

\[
\|u/\delta\|_{C^{\kappa}(\hat{\Omega})} \leq C_1 K,
\]

for some constant \( C_1 \), where \( \kappa, C_1 \) depend on \( n, C_0, A_0, \hat{k}, \Omega \).
The regularity of \( \partial \Omega \) in the above result can be relaxed to \( C^{1,1} \). See Remark 2.3 for more detail. For elliptic operators similar estimate is obtained by Krylov [30]. Boundary estimate for fractional Laplacian operators are studied by Ros-Oton and Serra in [34–36]. Result of [34] has been extended for nonlocal operators with kernel of variable orders by Kim et al. [29] whereas extension to the fractional \( p \)-Laplacian operator can be found in [31]. For the proof of Theorem 1.2, we follow the approach of [34] which is inspired by a method of Caffarelli [28, p. 39]. A key step in this analysis is the oscillation lemma (see Proposition 2.1) for \( u/\delta \) which involves computation of \( L((u - \kappa \delta)^+) \) for some suitable constant \( \kappa \). Note that, by Theorem 1.1, \( Iu \) is bounded in \( \Omega \) for \( \alpha \in (0, 1) \), and therefore, in this case, we can follow standard approach of local operators to get the estimate (1.4). But for \( \alpha \in [1, 2), I\delta \) becomes singular near \( \partial \Omega \). So we have to do several careful estimates to apply the method of [34].

Using Theorem 1.2, we establish boundary regularity of \( Du \) (compare it with Fall-Jarohs [18]). This is the content of our next result.

**Theorem 1.3** Let Assumption 1.1 hold. There exist constants \( \gamma, C, \) dependent on \( \Omega, C_0, A_0, n, \hat{k} \), such that for any solution \( u \) of (1.1), we have

\[
\|u\|_{C^{1,\gamma}(\hat{k})} \leq CK.
\]

**Remark 1.1** By the dependency of the constants in Theorem 1.1 to 1.3 on \( \hat{k} \), we mean the dependency on \( \alpha, \Lambda \) and \( \int_{|y| \geq 1} J(y)dy \).

To cite a specific application of the above results, let us consider \( u, v \in C(\mathbb{R}^n) \) satisfying

\[
Lu + H_1(Du, x) = 0, \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c,
\]

\[
Lv + H_2(Dv, x) = 0, \quad \text{in } \Omega, \quad v = 0 \quad \text{in } \Omega^c,
\]

respectively. If \( |H_1(p, x) - H_2(q, x)| \leq C_0 |p - q| + K \) for all \( p, q \in \mathbb{R}^n \) and \( x \in \Omega \), then using the interior regularity of \( u, v \) from [33] and the coupling result [10, Theorem 5.1] it can be easily seen that \( w = u - v \) satisfies (1.1). Our result Theorem 1.3 then gives a \( C^{1,\gamma} \) estimate of \( w \) up to the boundary. The above results can also be used to establish anti-maximum principle for the generalized principal eigenvalues of nonlinear operators of the form (1.2) (cf. [7, 9, 15]).

**Remark 1.2** Though the above results are mentioned for viscosity solutions, Theorem 1.1-1.3 can also be applied for weak solutions (at least for equations). To see this, let us assume that \( \Omega \) be a \( C^{2,\kappa} \) domain, \( \kappa \in (0, 1) \). Suppose that for some given Lipschitz function \( f : \mathbb{R}^n \to \mathbb{R} \), there exists a unique weak solution \( u \in H^1_0(\Omega) \) to

\[
Lu + f(Du) = g \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \Omega^c,
\]

for every \( g \in L^\infty(\Omega) \). Now consider a sequence of smooth mollifications \( g_\epsilon \) of \( g \) such that \( \sup_{\Omega} |g_\epsilon - g| \to 0 \), as \( \epsilon \to 0 \). Let \( u_\epsilon \) be the unique weak solution to (1.6) corresponding to \( g_\epsilon \). Since, by Sobolev embedding \( u_\epsilon \in L^p(\Omega) \) for \( p \in [1, \frac{2n}{n-2}] \), applying [25, Theorem 3.1.22] and a bootstrapping argument we obtain that for some \( p > n \)

\[
\|u_\epsilon\|_{W^{2,p}(\Omega)} \leq C(1 + \|Du_\epsilon\|_{L^2(\Omega)} + \|g_\epsilon\|_{L^p(\Omega)}), \tag{1.7}
\]

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for some constant $C$ independent of $u_\varepsilon$. This, of course, implies $u_\varepsilon \in C^{1,\gamma} (\tilde{\Omega})$. Applying [25, Theorem 3.1.12], we have $u_\varepsilon \in C^{2,\gamma} (\tilde{\Omega})$, and therefore, $u_\varepsilon$ is a viscosity solution to (1.6) when $g$ is replaced by $g_\varepsilon$. Hence, we can apply Theorem 1.1-1.3 on $u_\varepsilon$. In particular,

$$\sup_{\varepsilon \in (0,1)} \| Du_\varepsilon \|_{L^\infty(\Omega)} < \infty.$$ 

Now, using the stability estimate (1.7), we can pass the limit, as $\varepsilon \to 0$, to show that $u_\varepsilon \to u$ where $u$ is the weak solution to (1.6) with data $g$ and $u$ also satisfies the estimates in Theorem 1.1-1.3.

Next we apply Theorem 1.2 and 1.3 to study an overdetermined problem. More precisely, we consider a solution $u$ to the problem

$$
Lu + H(|Du|) = f(u) \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{in } \Omega^c, \quad u > 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = c \quad \text{on } \partial \Omega,
$$

(1.8)

where $n$ is the unit inward normal and $H : \mathbb{R} \to \mathbb{R}, f : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz. In Theorem 3.1, we show that $\Omega$ must be a ball, provided the nonlocal kernel $k$ satisfies certain conditions. Overdetermined problem was first studied in the celebrated work of Serrin [38] where it was shown that if there exists a positive solution to

$$
-\Delta u = 1 \quad \text{in } \Omega, \quad u = 0, \quad \frac{\partial u}{\partial n} = c \quad \text{on } \partial \Omega,
$$

then $\Omega$ must be a ball. Serrin’s work has been generalized for a vast class of operators, see for instance, [8, 9, 14, 19, 20, 22, 23, 39]. In this article, we follow the method of [8, 19] to establish our result on overdetermined problem concerning (1.8).

The rest of the article is organized as follows: In the next Sect. 2, we provide the proofs of Theorem 1.1 to 1.3 and Sect. 3 discusses the overdetermined problems.

### 2 Proofs of Theorems 1.1 to 1.3

We begin by showing that $\delta$ is a barrier function to $u$ at the boundary.

**Lemma 2.1** Suppose that Assumption 1.1(a) holds and $u$ be a viscosity solution to (1.1). Then, there exists a constant $C$, dependent only on $n, A_0, C_0, \text{diam}(\Omega), \text{radius of exterior sphere}$ and $\int_{\mathbb{R}^n}(|y|^2 \wedge 1)\hat{k}(y)dy$, such that

$$
|u(x)| \leq CK\delta(x) \quad \text{for all } x \in \Omega,
$$

(2.1)

where $\delta(x) = \text{dist}(x, \Omega^c)$.

**Proof** We first show that

$$
|u(x)| \leq \kappa K \quad x \in \mathbb{R}^n,
$$

(2.2)

for some constant $\kappa$. From [32, Lemma 5.5], we can find a non-negative function $\chi \in C^2(\tilde{\Omega}) \cap C_b(\mathbb{R}^n)$, with $\inf_{\mathbb{R}^n} \chi > 0$, satisfying

$$
\chi \leq \frac{1}{\kappa} K \quad \text{on } \Omega.
$$

For any $x \in \Omega$, there exists a $y \in \mathbb{R}^n$, such that $|x-y| = \text{dist}(x, \Omega^c) > 0$. Let $z = x + \delta(x) (y-x)$. Then, $z \in \Omega$ and

$$
|z| = |x| - |x-y| = |x| - \text{dist}(x, \Omega^c) > 0.
$$

Since $u$ is a viscosity solution to (1.1), we have

$$
Lu(z) + H(|Du(z)|) \leq f(u(z)) \quad \text{in } \Omega,
$$

and

$$
Lu(z) + H(|Du(z)|) \geq f(u(z)) \quad \text{in } \Omega.
$$

Combining these inequalities, we get

$$
Lu(z) + H(|Du(z)|) = f(u(z)) \quad \text{in } \Omega.
$$

This, together with the fact that $\chi \leq \frac{1}{\kappa} K$, implies

$$
|u(z)| \leq \frac{1}{\kappa} K \quad \text{in } \Omega.
$$

Since $z = x + \delta(x) (y-x)$, we have

$$
|u(x)| = |u(z)| \leq \frac{1}{\kappa} K \quad \text{in } \Omega.
$$

This completes the proof of Lemma 2.1.
\[ L\chi + C_0 |D\chi| \leq -1 \quad \text{in} \quad \Omega. \]

Note that, since \( \chi \in C^2(\bar{\Omega}) \), the above equation holds in the classical sense. Defining \( \psi = 2(K + \epsilon)\chi, \epsilon > 0 \), we have that \( \inf_{\bar{\Omega}} \psi > 0 \) and

\[ L\psi + C_0 |D\psi| \leq -2(K + \epsilon) \quad \text{in} \quad \Omega. \tag{2.3} \]

We claim that \( u \leq \psi \) in \( \mathbb{R}^n \). Suppose, on the contrary, that \( (u - \psi)(z) > 0 \) at some point in \( z \in \Omega \). Define

\[ \theta = \inf \{ t : u \leq t + \psi \text{ in } \mathbb{R}^n \}. \]

Since \( (u - \psi)(z) > 0 \), we must have \( \theta \in (0, \infty) \). Again, since \( u = 0 \) in \( \Omega^c \), there must be a point \( x_0 \in \Omega \) such that \( u(x_0) = \theta + \psi(x_0) \) and \( u \leq \theta + \psi \) in \( \mathbb{R}^n \). Since \( \psi \) is \( C^2 \) in \( \Omega \), we get from the definition of viscosity subsolution that

\[ -K \leq L(\theta + \psi)(x_0) + C_0 |D\psi(x_0)| = L\psi(x_0) + C_0 |D\psi(x_0)| \leq -2(K + \epsilon), \]

using (2.3). But this is a contradiction. This proves the claim that \( u \leq \psi \) in \( \mathbb{R}^n \). Similar calculation using \( -u \) will also give us \( -u \leq \psi \) in \( \mathbb{R}^n \). Thus

\[ |u| \leq 2 \sup_{\mathbb{R}^n} |\chi|(K + \epsilon) \quad \text{in} \quad \mathbb{R}^n. \]

Since \( \epsilon \) is arbitrary, we get (2.2).

Now we can prove (2.1). In view of (2.2), it is enough to consider the case \( K > 0 \). Since \( \Omega \) belongs to the class \( C^2 \), it satisfies a uniform exterior sphere condition from outside. Let \( r_0 \) be a radius satisfying uniform exterior condition. From [32, Lemma 5.4], there exists a bounded, Lipschitz continuous function \( \varphi \), Lipschitz constant being \( r_0^{-1} \), satisfying

\[
\begin{align*}
\varphi &= 0 \quad \text{in} \quad B_{r_0}, \\
\varphi &> 0 \quad \text{in} \quad \bar{B}_{r_0}, \\
\varphi &\geq \epsilon \quad \text{in} \quad B_{(1+\delta)r_0}, \\
L\varphi + C_0 |D\varphi| &\leq -1 \quad \text{in} \quad B_{(1+\delta)r_0} \setminus \bar{B}_{r_0},
\end{align*}
\]

for some constants \( \epsilon, \delta \), dependent on \( C_0, A_0 \) and \( \int_{\mathbb{R}^n} (|y|^2 + 1)\tilde{k}(y)dy \). Furthermore, \( \varphi \) is \( C^2 \) in \( B_{(1+\delta)r_0} \setminus \bar{B}_{r_0} \). For any point \( y \in \partial \Omega \), we can find another point \( z \in \Omega \) such that \( \bar{B}_{r_0}(z) \) touches \( \partial \Omega \) at \( y \). Let \( w(x) = \epsilon^{-1} \kappa K \varphi(x - z) \). Also \( L(w) + C_0 |Dw| \leq -K \). Then,

\[ L(u - w) + C_0 |D(u - w)| \geq 0 \quad \text{in} \quad B_{(1+\delta)r_0}(z) \cap \Omega. \]

Since, by (2.2), \( u - w \leq 0 \) in \( (B_{(1+\delta)r_0}(z) \cap \Omega)^c \), from comparison principle [10, Theorem 5.2] it follows that \( u(x) \leq w(x) \) in \( \mathbb{R}^n \). Note that the operator in [10, Theorem 5.2] does not have any gradient term, but the same proof (using the supersolution in [32, Lemma 5.5]) also gives a comparison principle for the above operator. Repeating a similar calculation for \( -u \), we can conclude that \( -u(x) \leq w(x) \) in \( \mathbb{R}^n \). This relation holds for any \( y \in \partial \Omega \). For any point \( x \in \Omega \) with \( \text{dist}(x, \partial \Omega) < r_0 \), we can find \( y \in \partial \Omega \) satisfying \( \text{dist}(x, \partial \Omega) = |x - y| < r_0 \). By previous estimate, we then obtain

\[ |u(x)| \leq \epsilon^{-1} \kappa K \varphi(x - z) \leq \epsilon^{-1} \kappa K (\varphi(x - z) - \varphi(y - z)) \leq \epsilon^{-1} \kappa K r_0^{-1} \text{dist}(x, \partial \Omega). \]

This gives us (2.1). \( \square \)

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Now we are ready to prove that \( u \in C^{0,1}(\mathbb{R}^n) \).

**Proof of Theorem 1.1** Let \( x \in \Omega \) and \( r \in (0,1) \) be such that \( 4r = \text{dist}(x, \partial \Omega) \wedge 1 \). Without loss of any generality, we assume \( x = 0 \). Define \( v(y) = u(ry) \) in \( \mathbb{R}^n \). From Lemma 2.1, we then have

\[
|v(y)| \leq C_1 K \min\{r^{n/2}(1 + |y|^{n/2}), r(1 + |y|)\} \quad y \in \mathbb{R}^n,
\]

for some constant \( C_1 \). We let

\[
I_r f(x) = r^n \frac{1}{2} \int_{\mathbb{R}^n} (f(x + y) + f(x - y) - f(x))k(ry)r^ny dy,
\]

and \( L_0 f = \Delta f + r^{2-n} a I_r f \). Let us compute \( L_0 v(x) + C_0 r|Dv| \) in \( B_2 \). Clearly, we have \( \Delta v(x) = r^2 \Delta u(rx) \) and \( Dv(x) = rDu(rx) \). Also

\[
 r^{2-a} I_r v(x) = r^2 \frac{1}{2} \int_{\mathbb{R}^n} (u(rx + ry) + u(rx - ry) - 2u(rx))k(ry)r^ny dy = r^2 I_v(rx).
\]

Thus, it follows from (1.1) that

\[
\begin{align*}
L_0 v + C_0 r|Dv| & \geq -K r^2 \quad \text{in } B_2, \\
L_0 v - C_0 r|Dv| & \leq K r^2 \quad \text{in } B_2.
\end{align*}
\]

Now consider a smooth cut-off function \( \varphi, 0 \leq \varphi \leq 1 \), satisfying

\[
\varphi = \begin{cases} 1 & \text{in } B_{3/2}, \\
0 & \text{in } B'_2.
\end{cases}
\]

Let \( w = \varphi v \). Clearly, \((\varphi - 1)v)(y) = 0\) for all \( y \in B_{3/2} \), which gives \( D((\varphi - 1)v) = 0 \) and \( \Delta((\varphi - 1)v) = 0 \) in \( x \in B_{3/2} \). Since \( w = v + (\varphi - 1)v \), we obtain

\[
\begin{align*}
L_0 w + C_0 r|Dw| & \geq -K r^2 - A_0 r^{2-a}|I_r((\varphi - 1)v)| \quad \text{in } B_1, \\
L_0 w - C_0 r|Dw| & \leq K r^2 + A_0 r^{2-a}|I_r((\varphi - 1)v)| \quad \text{in } B_1,
\end{align*}
\]

from (2.5). Again, since \((\varphi - 1)v = 0 \) in \( B_{3/2} \), we have in \( B_1 \) that

\[
| r^{2-a} I_r((\varphi - 1)v)(x)| = r^{n+2} \frac{1}{2} \int_{|y| \geq 1/2} |((\varphi - 1)v)(x + y) + ((\varphi - 1)v)(x - y)k(ry)dy| \\
\leq r^{n+2} \int_{|y| \geq 1/2} |v(x + y)|k(ry)dy \\
\leq r^{n+2} \int_{\frac{1}{2} \leq |y| \leq \frac{1}{r}} |v(x + y)|k(ry)dy + r^{n+2} \int_{|y| > 1} |v(x + y)|k(ry)dy, \\
: = I_{r,1} + I_{r,2}.
\]

By Assumption 1.1 (a)
\[ I_{r,1} = r^{n+2} \int_{\frac{1}{r} \leq |y| \leq \frac{1}{r}} |v(x+y)|k(ry)dy \leq r^{2-a} \int_{\frac{1}{r} \leq |y| \leq \frac{1}{r}} |v(x+y)|\widehat{k}(y)dy \]
\[ = \Lambda r^{2-a} \int_{\frac{1}{r} \leq |y| \leq \frac{1}{r}} |v(x+y)| \frac{1}{|y|^{n+a}}dy \leq r^{2-a} \Lambda 3^{n+a} \int_{|y| \geq 1/2} \frac{|v(x+y)|}{1 + |x+y|^{n/2}} \frac{1 + |y|^{n/2}}{1 + |y|^{n+a}}dy \leq \kappa_2 Kr^{2-a/2} \int_{|y| \geq 1/2} \frac{1 + |x+y|^{n/2}}{1 + |x+y|^{n+a}}dy \leq \kappa_3 Kr^{2-a/2}, \]
for some constants \( \kappa_2, \kappa_3 \), and in the fifth line we use (2.4). Again, by (2.2), we have
\[ I_{r,2} \leq \kappa r^{2} K \int_{|y| > 1} r^n k(ry)dy = \kappa r^{2} K \int_{|y| > 1} k(y)dy \leq \kappa r^{2} K \int_{|y| > 1} \widehat{k}(y)dy \leq \kappa r^{2} K \int_{|y| > 1} J(y)dy \leq \kappa_4 r^{2} K, \]
for some constant \( \kappa_4 \). Therefore, putting the estimates of \( I_{r,1} \) and \( I_{r,2} \) in (2.6), we obtain
\[ L_r v + C_0 r |Dv| \geq -\kappa_5 Kr^{2-a/2} \text{ in } B_1, \]
\[ L_r v - C_0 r |Dv| \geq \kappa_5 Kr^{2-a/2} \text{ in } B_1, \] (2.7)
for some constant \( \kappa_5 \). Applying [33, Theorem 4.1], we obtain from (2.7)
\[ \|v\|_{C^1(B_{\frac{1}{2}})} \leq \kappa_6 \left( \|v\|_{L^n(B_{\frac{1}{2}})} + r^{2-a/2} K \right) \] (2.8)
for some constant \( \kappa_6 \). The proof in [33, Theorem 4.1] is stated for equations but it is easily seen that the same proof also works for a system of inequalities as in (2.7). From (2.4) and (2.8), we then obtain
\[ \sup_{y \in B_{r/2}(x) \setminus y \neq x} \frac{|u(x) - u(y)|}{|x-y|} \leq \kappa_7 K, \] (2.9)
for some constant \( \kappa_7 \).

Now we can complete the proof. Not that if \( |x-y| \geq \frac{1}{8} \), then
\[ \frac{|u(x) - u(y)|}{|x-y|} \leq 2\kappa K, \]
by (2.2). So we consider \( |x-y| < \frac{1}{8} \). If \( |x-y| \geq 8^{-1}(\delta(x) \lor \delta(y)) \), then using Lemma 2.1 we get
\[ \frac{|u(x) - u(y)|}{|x-y|} \leq 4CK(\delta(x) + \delta(y))(\delta(x) \lor \delta(y))^{-1} \leq 8CK. \]
Now let \( |x-y| < 8^{-1} \min\{\delta(x) \lor \delta(y), 1\} \). Then, either \( y \in B_{\frac{\delta(x)}{8}}(x) \) or \( x \in B_{\frac{\delta(y)}{8}}(y) \). Without loss of generality, we suppose \( y \in B_{\frac{\delta(x)}{8}}(x) \). From (2.9), we get
\[
\frac{|u(x) - u(y)|}{|x - y|} \leq \kappa \gamma K.
\]

This completes the proof. \[ \square \]

With the help of Theorem 1.1, we may choose \( \beta = \infty \) in Assumption 1.1(b).

**Remark 2.1** Since \( u \) is globally Lipschitz, choosing \( \mathcal{I}(y) = |y|^{-n-\zeta}, \zeta \in (0, 1 \land a) \), we see from Theorem 1.1 that

\[
|\int_{\mathbb{R}^n} (u(x + y) + u(x - y) - 2u(x)) \mathcal{I}(y)dy| \leq \kappa \|u\|_{C^{0,1}(\mathbb{R}^d)} \leq \kappa CK (2.10)
\]

for some constant \( \kappa \). Let \( \tilde{k}(y) = k(y)\mathbb{1}_{|y| \leq \beta'} + \mathcal{I}(y) \), where \( \beta' < \frac{2}{3}\beta \). It is easy to see that \( \tilde{k} \) satisfies Assumption 1.1(a).

We now show that Assumption 1.1(b) also holds for this kernel with \( \beta = \infty \). Fix \( r \in (0, 1] \) and \( x_0 \in \mathbb{R}^n \) and choose \( x, y \in B_r(x_0) \) and \( z \in B_r^c(x_0) \). Without any loss of generality, we may assume that given \( \beta \) is in \( (0, 1/2) \). If \( \max \{|x - z|, |y - z|\} \leq \beta' \), we have from Assumption 1.1(b) that

\[
\tilde{k}(x - z) = k(x - z) + \mathcal{I}(x - z) \leq \rho k(y - z) + 3^{n+\zeta} \mathcal{I}(y - z) \leq (\rho \lor 3^{n+\zeta})\tilde{k}(y - z),
\]

using the fact

\[
\frac{1}{3}|x - z| \leq |y - z| \leq 3|x - z|.
\]

Also, if \( |x - z| > \beta' \), then

\[
\tilde{k}(x - z) = \mathcal{I}(x - z) \leq 3^{n+\zeta}\tilde{k}(y - z).
\]

Suppose \( |x - z| \leq \beta' \) and \( |y - z| > \beta' \). Note that \( |y - z| \leq \frac{3}{2}\beta' < \beta \). Using Assumption 1.1(a), we find that

\[
\tilde{k}(x - z) \leq \rho k(y - z) + 3^{n+\zeta} \mathcal{I}(y - z) \leq \rho \Lambda |y - z|^{-n-a} + 3^{n+\zeta} \mathcal{I}(y - z) \\
\leq (\rho \Lambda (\beta')^{-a+\zeta} + 3^{n+\zeta}) \mathcal{I}(y - z) \\
= (\rho \Lambda (\beta')^{-a+\zeta} + 3^{n+\zeta})\tilde{k}(y - z).
\]

Thus, the kernel \( \tilde{k} \) satisfies Assumption 1.1(b) for \( \beta = \infty \).

On the other hand, replacing the kernel \( k \) by \( \tilde{k} \) and using Theorem 1.1, (2.10), we obtain from (1.1) that

\[
\Delta u + aIu + C_0|Du| \geq -C_1 K \quad \text{in } \Omega,
\]
\[
\Delta u + aIu - C_0|Du| \leq C_1 K \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{in } \Omega^c,
\]

for some constant \( C_1 \), dependent on \( \hat{k}, \zeta, A_0 \). This modification of nonlocal kernel would be useful apply the Harnack inequality from [24].
2.1 Fine boundary regularity

In this section, we study the regularity of \( u/\delta \) in \( \Omega \). Since \( u \) is Lipschitz, using the estimate (1.3) we may write (1.1) as follows:

\[
|Lu| = |\Delta u + alu| \leq CK \quad \text{in } \Omega, \quad \text{and} \quad u = 0 \quad \text{in } \Omega^c,
\]

where \( C \) is a constant depending on \( \hat{k}, A_0, C_0 \). Also, in view of Remark 2.1, we can assume that \( k \) satisfies Assumption 1.1(b) for \( \beta = \infty \). The rest of the section is devoted to the proofs of Theorem 1.2 and 1.3. Towards the proof of Theorem 1.2, our first goal is to get the oscillation estimate Proposition 2.1. To obtain this result, we need Lemma 2.2 to 2.6.

**Lemma 2.2** There exists a constant \( \bar{k} \), dependent on \( n, A_0, \hat{k} \), such that for any \( r \in (0, 1] \), we have a bounded radial function \( \phi_r \), satisfying

\[
\begin{aligned}
L\phi_r &\geq 0 &\text{in } B_{4r} \setminus \tilde{B}_r, \\
0 &\leq \phi_r &\leq \bar{k}r &\text{in } B_r, \\
\phi_r &\geq \frac{1}{8}(4r - |x|) &\text{in } B_{4r} \setminus B_r, \\
\phi_r &\leq 0 &\text{in } \mathbb{R}^n \setminus B_{4r}.
\end{aligned}
\]

Moreover, \( \phi_r \in C^2(B_{4r} \setminus \tilde{B}_r) \).

**Proof** Fix \( r \in (0, 1] \) and define \( v_r(x) = e^{-\eta q(x)} - e^{-\eta (4r)^2} \), where \( q(x) = |x|^2 \wedge 2(4r)^2 \) and \( \eta > 0 \). Clearly, \( 1 \geq v_r(0) \geq v_r(x) \) for all \( x \in \mathbb{R}^n \). Thus

\[
v_r(x) \leq 1 - e^{-\eta (4r)^2} \leq \eta (4r)^2,
\]

using the fact that \( 1 - e^{-s} \leq s \) for all \( s \geq 0 \). Again, for \( x \in B_{4r} \setminus B_r \), we have

\[
v_r(x) = e^{-\eta (4r)^2} (e^{\eta((4r)^2-q(x))} - 1) \geq \eta e^{-\eta (4r)^2} ((4r)^2 - |x|^2) \\
= \eta e^{-\eta (4r)^2} (4r^2 + |x|)(4r - |x|) \geq 5\eta r e^{-\eta (4r)^2} (4r - |x|).
\]

Now we estimate \( Lv_r \) in \( B_{4r} \setminus \tilde{B}_r \). Fix \( x \in B_{4r} \setminus \tilde{B}_r \). Then,

\[
\Delta v_r = \eta e^{-\eta |x|^2} (4\eta |x|^2 - 2n),
\]

and, since \( Lv_r = I(v_r + e^{-\eta (4r)^2}) \), using the convexity of exponential map obtain

\[
I(e^{-\eta q(x)}) \geq -\eta e^{-\eta |x|^2} \int_{\mathbb{R}^n} (q(x + y) + q(x - y) - 2q(x)) k(y) dy \\
\geq -\eta e^{-\eta |x|^2} \left[ \int_{|y| \leq 1} (q(x + y) + q(x - y) - 2q(x)) k(y) dy + \int_{|y| > 1} (8r)^2 k(y) dy \right] \\
\geq -\eta e^{-\eta |x|^2} \left[ \int_{|y| \leq r} \frac{2|y|^2}{|y|^{n+a}} dy + \int_{r < |y| < 1} \frac{(8r)^2}{|y|^{n+a}} dy + (8r)^2 \int_{|y| > 1} J(y) dy \right] \\
\geq -\eta e^{-\eta |x|^2} \kappa r^{2-a},
\]

for some constant \( \kappa \), independent of \( \eta \). Combining the above estimates we see that, for \( x \in B_{4r} \setminus \tilde{B}_r \).
Thus, letting \( \eta = \frac{1}{r^2}(n + A_0\kappa) \), we obtain
\[
Lv_r > 0 \quad \text{in} \ B_{4r} \setminus \tilde{B}_r.
\]
We set \( \phi_r = \frac{y}{r} \) and the result follows from (2.13)-(2.14).

Let us now define the sets that we use for our oscillation estimates. We borrow the notations of [34].

**Definition 2.1** Let \( \kappa \in (0, \frac{1}{16}) \) be a fixed small constant and let \( \kappa' = \frac{1}{2} + 2\kappa \). Given a point \( x_0 \in \partial \Omega \) and \( R > 0 \), we define
\[
D_R = D_R(x_0) = B_R(x_0) \cap \Omega,
\]
and
\[
D_{\kappa R} = D_{\kappa R}(x_0) = B_{\kappa R}(x_0) \cap \{ x \in \Omega : (x - x_0) \cdot n(x_0) \geq 2\kappa R \},
\]
where \( n(x_0) \) is the unit inward normal at \( x_0 \). Using the \( C^2 \) regularity of the domain, there exists \( \rho > 0 \), depending on \( \Omega \), such that the following inclusions hold for each \( x_0 \in \partial \Omega \) and \( R \leq \rho \):
\[
B_{\kappa R}(y) \subset D_R(x_0) \quad \text{for all} \ y \in D_{\kappa R}(x_0),
\]
and
\[
B_{4\kappa R}(y^* + 4\kappa Rn(y^*)) \subset D_R(x_0), \quad \text{and} \quad B_{\kappa R}(y^* + 4\kappa Rn(y^*)) \subset D_{\kappa R}(x_0)
\]
for all \( y \in D_R/2 \), where \( y^* \in \partial \Omega \) is the unique boundary point satisfying \( |y - y^*| = \text{dist}(y, \partial \Omega) \). Note that, since \( R \leq \rho \), \( y \in D_R/2 \) is close enough to \( \partial \Omega \) and hence the point \( y^* + 4\kappa Rn(y^*) \) belongs to the line joining \( y \) and \( y^* \).

**Remark 2.2** In the remaining part of this section, we fix \( \rho > 0 \) to be a small constant depending only on \( \Omega \), so that (2.15)-(2.16) hold whenever \( R \leq \rho \) and \( x_0 \in \partial \Omega \). Also, every point on \( \partial \Omega \) can be touched from both inside and outside \( \Omega \) by balls of radius \( \rho \). We also fix \( \gamma > 0 \) small enough so that for \( 0 < r \leq \rho \) and \( x_0 \in \partial \Omega \), we have
\[
B_{\eta r}(x_0) \cap \Omega \subset B_{(1 + \sigma)r}(z) \setminus \tilde{B}_r(z) \quad \text{for} \quad \eta = \sigma/8, \ \sigma \in (0, \gamma),
\]
for any \( x' \in \partial \Omega \cap B_{\eta r}(x_0) \), where \( B_r(z) \) is a ball contained in \( \mathbb{R}^n \setminus \Omega \) that touches \( \partial \Omega \) at point \( x' \).

We first treat the case \( \alpha \in (0, 1) \). Note that in this situation, \( Iu \) can be defined in the classical sense and is bounded in \( \Omega \), by Theorem 1.1.

**Lemma 2.3** Let \( \alpha \in (0, 1) \) and \( \Omega \) be a bounded \( C^2 \) domain. Let \( u \) be such that \( u \geq 0 \) in \( \mathbb{R}^n \), and \( |Lu| \leq C_2 \) in \( D_R \), for some constant \( C_2 \). Then, there exists a positive constant \( C \), depending only on \( n, \Omega, A_0, \kappa \), such that
\[
\inf_{D_{\gamma \delta}^+} \left( \frac{u}{\delta} \right) \leq C \inf_{D_{\delta}^+} \left( \frac{u}{\delta} + C_2 R \right)
\]  
(2.17)

for all \( R \leq \rho \), where the constant \( \rho_0 \) depends only on \( n, \Omega, A_0 \) and \( \int_{\mathbb{R}^n} (|y|^2 \wedge 1) \tilde{k}(y) dy \).

**Proof** We split the proof in two steps.

Step 1. Suppose \( C_2 = 0 \) and \( R \leq \rho \), where \( \rho \) is given by Remark 2.2. Define
\[
m = \inf_{D_{\kappa \rho}^+} u/\delta \geq 0. \quad \text{By (2.15),}
\]
\[
u \geq m \delta \geq m (\kappa R) \quad \text{in} \quad D_{\kappa \rho}^+.
\]  
(2.18)

Again, by (2.16), for any \( y \in D_{\kappa \rho}/2 \), we have either \( y \in D_{\kappa \rho}^+ \) or \( \delta(y) < 4\kappa R \). If \( y \in D_{\kappa \rho}^+ \), it follows from the definition of \( m \) that \( m \leq u(y)/\delta(y) \). Now let \( \delta(y) < 4\kappa R \). Let \( y^\ast \) be the nearest point to \( y \) on \( \partial \Omega \) and \( \tilde{y} = y^\ast + 4\kappa R n(y^\ast) \). Again by (2.16), we have \( B_{4\kappa R}(\tilde{y}) \subset D_{\rho} \) and \( B_{\kappa R}(\tilde{y}) \subset D_{\kappa \rho}^+ \). Recall that \( L u = 0 \) in \( D_{\rho} \) and \( u \geq 0 \) in \( \mathbb{R}^n \).

Now take \( r = \kappa R \) and let \( \phi^r \) be the subsolution in Lemma 2.2. Define \( \tilde{\phi}^r(x) = \frac{1}{\kappa} \phi^r(x - \tilde{y}) \).

Using (2.18) and the comparison principle [10, Theorem 5.2] in \( B_{4r}(\tilde{y}) \backslash \tilde{B}_r(\tilde{y}) \), it follows that \( u(x) \geq m \tilde{\phi}^r(x) \) in all of \( \mathbb{R}^n \). In particular, we have \( u/\delta \geq (1/\kappa^2) m \) on the segment joining \( y^\ast \) and \( \tilde{y} \), that contains \( y \). Hence,
\[
\inf_{D_{\gamma \delta}^+} \left( \frac{u}{\delta} \right) \leq C \inf_{D_{\delta}^+} \left( \frac{u}{\delta} \right).
\]

Step 2. Suppose \( C_2 > 0 \). Define \( r' = \eta r \) for \( r \leq \rho \) and \( \eta \leq 1 \) to be chosen later. Let \( \tilde{u} \) to be the solution of (cf. [10, Theorem 1.1])
\[
\begin{align*}
L \tilde{u} &= 0 \quad \text{in} \quad D_{r'}, \\
\tilde{u} &= u \quad \text{in} \quad \mathbb{R}^n \backslash D_{r'}.
\end{align*}
\]

From step 1, we see that \( \tilde{u} \) satisfies (2.17). Define \( w = \tilde{u} - u \). Applying [10, Theorem 5.1], we obtain that \( |Lw| \leq C_2 \) in \( D_{r'} \) and \( w = 0 \) in \( \mathbb{R}^n \backslash D_{r'} \). Since \( r \leq \rho \), points of \( \partial \Omega \) can be touched by exterior ball of radius \( r \). Thus, for any point \( y \in \partial \Omega \), we can find another point \( z \in \Omega^c \) such that \( B_r(z) \) touches \( \partial \Omega \) at \( y \). From the proof of [32, Lemma 5.4], there exists a bounded, Lipschitz continuous function \( \varphi_r \), with Lipschitz constant \( r^{-1} \), that satisfies
\[
\begin{align*}
\varphi_r &= 0, \quad \text{in} \quad \tilde{B}_r, \\
\varphi_r &> 0, \quad \text{in} \quad \tilde{B}_r^c, \\
L \varphi_r &\leq -\frac{1}{r}, \quad \text{in} \quad B_{(1+\sigma)r} \backslash \tilde{B}_r,
\end{align*}
\]
for some constant \( \sigma \), independent of \( r \). Without any loss of generality, we may assume \( \sigma \leq \gamma \) (see Remark 2.2). We set \( \eta = \frac{\sigma}{\gamma} \). Then, \( D_{r'} \subset B_{(1+\sigma)r}(z) \backslash \tilde{B}_r(z) \). Letting \( v(x) = C_2 r^2 \varphi_r(x - z) \) will give us a desired supersolution, and therefore, by comparison principle, we get \( |w| \leq v \) in \( \mathbb{R}^n \). For any point \( x \in D_{r'} \), we can find \( y \in \partial \Omega \) satisfying \( \text{dist}(x, \partial \Omega) = |x - y| \). By above estimate, we obtain
\[
|w(x)| \leq C_2 r^2 \varphi_r(x - z) \leq C_2 r^2 (\varphi_r(x - z) - \varphi_r(y - z)) \leq C_2 r \text{dist}(x, \partial \Omega) = C_2 r \delta(x).
\]

Thus, we obtain
Combining with step 1, we have

$$\inf_{D_{r'}^{\sigma}} \left( \frac{u}{\delta} \right) \leq \frac{C}{\eta} \left( \inf_{D_{r'}^{\sigma}} \frac{u}{\delta} + C_2 R' \right).$$

Setting $\rho_0 = \eta \rho$ and $R = r'$ we have the desired result. \qed

Next we obtain a similar estimate when $\alpha \in [1, 2)$.

**Lemma 2.4** Let $\Omega$ be a bounded $C^2$ domain and $u$ be such that $u \geq 0$ in all of $\mathbb{R}^n$ and $|Lu| \leq C_2g$ in $D_R$ for some positive constant $C_2$ and $g$ is given by

$$g(x) = \begin{cases} 
(\delta(x))^{1-\alpha} & \text{if } \alpha > 1, \\
-\log(\delta(x)) + C_3 & \text{if } \alpha = 1,
\end{cases}$$

for some constant $C_3$. Set $\hat{\alpha} = 2 - \alpha$ for $\alpha \in (1, 2)$ and for $\alpha = 1$, $\hat{\alpha}$ is any number in $(0, 1)$. Then, there exists a positive constant $C$, depending on $\Omega, n$ and $\hat{\alpha}$, such that

$$\inf_{D_{r/2}^R} \frac{u}{\delta} \leq C \left( \inf_{D_{r/2}^R} \frac{u}{\delta} + C_2 R^3 \right)$$

for all $R < \rho_0$, where $\rho_0$ is a positive constant depending only on $\Omega, n, \hat{\alpha}, A_0$ and $\int_{\mathbb{R}^n} (|y|^2 \wedge 1) \hat{\kappa}(y) dy$.

**Proof** When $C_2 = 0$, the proof follows from Step 1 of Lemma 2.3. So we let $C_2 > 0$. As before, we consider $\tilde{u}$ to be the solution of

$$L\tilde{u} = 0 \quad \text{in } D_R,$$

$$\tilde{u} = u \quad \text{in } \mathbb{R}^n \setminus D_R.$$

Then

$$\inf_{D_{r/2}^R} \frac{\tilde{u}}{\delta} \leq C \inf_{D_{r/2}^R} \frac{\tilde{u}}{\delta}$$

holds, by step 1 of Lemma 2.3. Defining $w = \tilde{u} - u$, we get $|Lw| \leq C_2g$ in $D_R$ by using [10, Theorem 5.1] and $w = 0$ in $D_R^c$. As before, we would consider an appropriate supersolution and then apply comparison principle to establish (2.19).

For this construction of supersolution, we take inspiration from [32, Lemma 5.8]. We set

$$\tilde{\psi}(s) = \int_0^s 2e^{-ql-t} \int_0^t \Theta(\tau) d\tau dt - s,$$

where $q > 0$ is to be chosen later and $\Theta$ is given by

$$\Theta(s) = \int_{|z| > s} \min\{1, |z|\} \hat{\kappa}(z) dz.$$
Since $\Theta$ is integrable in a neighbourhood of 0, there exists a sufficiently small constant $s(q) > 0$ such that, for $0 < s < s(q)$, 
$$\psi''(s) = 2e^{-\psi(s) - q} \int_0^s \Theta(r) dr - 1 \geq 1.$$
Set $\sigma_1 = \min\left(\frac{\min\{\eta_1, \eta_2\}}{8}, 1, \gamma\right)$. For any $r \in (0, 1)$, we define
$$\psi_{r,z}(x) = \begin{cases} \psi\left(\frac{d_{B_r(z)}(x)}{r}\right) & \text{if } d_{B_r(z)}(x) < r\sigma_1, \\
\psi(\sigma_1) & \text{if } d_{B_r(z)}(x) \geq r\sigma_1,
\end{cases} \quad (2.20)$$
where $d_{B_r(z)}(x) = \text{dist}(x, B_r(z))$. Let $\eta = \frac{\av}{8}, 0 < r \leq \rho$ and $B_{\eta_\rho}(x_0) \cap \Omega = \intr\eta$. We define
$$\Phi_r(x) = \begin{cases} \psi\left(\frac{\delta(x)}{r}\right) & \text{if } \delta(x) < r\sigma_1, \\
\psi(\sigma_1) & \text{if } \delta(x) \geq r\sigma_1.
\end{cases}$$
For $x \in \intr\eta$, then we have $x^* \in \partial\Omega$ such that $\delta(x) = |x - x^*|$. Let $z^*_x = z$ be a point in $\Omega^z$ such that $B_r(z)$ touches $\partial\Omega$ at $x^*$. From Remark 2.2, we have that
$$\intr\eta \subset B_{(1+\sigma_1)r}(z) \setminus B_r(z).$$
Since $\psi'' < 0$ and $|D\delta(x)| \geq \kappa > 0$ for $\delta(x) \in (0, \rho_1)$, $\rho_1$ sufficiently small, it follows that
$$\Delta \Phi_r(x) \leq \frac{C}{r} + \psi''\left(\frac{\delta(x)}{r}\right) \kappa^2 \frac{1}{r^2}. \quad (2.21)$$
Consider $\psi_{r,z}$ from (2.20) and notice that $\psi_{r,z}(x) = \Phi_r(x)$ and $\delta(x + y) \leq d_{B_r(z)}(x + y)$ for all $y \in \R^n$. Hence,
$$\psi_{r,z}(x + y) + \psi_{r,z}(x - y) - 2\psi_{r,z}(x) \geq \Phi_r(x + y) + \Phi_r(x - y) - 2\Phi_r(x).$$
This readily gives (see [32, Lemma 5.8])
$$I\Phi_r \leq I\psi_{r,z}(x) \leq \frac{C}{r} \left(1 + \Theta\left(\frac{d_{B_r(z)}(x)}{r}\right)\right) = \frac{C}{r} \left(1 + \Theta\left(\frac{\delta(x)}{r}\right)\right), \quad (2.22)$$
using the fact $\delta(x) = |x - x^*| = d_{B_r(z)}(x)$. Combining (2.21) and (2.22) we have
$$L\Phi_r \leq \frac{CA}{r} \left(1 + \Theta\left(\frac{\delta(x)}{r}\right)\right) - \frac{2k^2q}{r^2} \left(1 + \Theta\left(\frac{\delta(x)}{r}\right)\right).$$
for all $x \in \intr\eta$. Now choose $q = \frac{1}{2}\min\{CA_0 + 1\}$ in the expression of $\psi$ we obtain
$$L\Phi_r \leq -\frac{1}{r^2} \left(1 + \Theta\left(\frac{\delta(x)}{r}\right)\right) \leq -\frac{1}{r^2} \Theta\left(\frac{\delta(x)}{r}\right). \quad (2.23)$$
for all $x \in \intr\eta$.

Next we estimate the function $\Theta$ in $\intr\eta$. For $\xi \in (0, 1]$, we see that
\[
\Theta(\xi) = \int_{|z| > \xi} \min\{1, |z|\} \lambda(z) dz = \lambda \int_{|z| \leq 1} \frac{|z|}{|z|^{n+\alpha}} dz + \int_{|z| \geq 1} J(z) dz
\]

\[
= \lambda \omega_n \int_{\xi}^{1} \frac{r^n}{r^{n+\alpha}} dr + \kappa_1
\]

\[
= \left\{ \begin{array}{ll}
\frac{\omega_n \Lambda}{\alpha - 1} \left[ \xi^{1-\alpha} - 1 \right] + \kappa_1 & \text{for } \alpha \in (1, 2), \\
\omega_n \Lambda (-\log \xi) + \kappa_1 & \text{for } \alpha = 1,
\end{array} \right.
\]  

(2.24)

for some positive constant \(\kappa_1\). Here, \(\omega_n\) denotes the surface area of the unit sphere in \(\mathbb{R}^n\). Since \(\frac{\delta(x)}{r} < \frac{1}{2}\) in \(D_{\eta r}\), we get from the above estimate that

\[
\Theta\left(\frac{\delta(x)}{r}\right) = \frac{\omega_n \Lambda}{\alpha - 1} \left[ \left(\frac{\delta(x)}{r}\right)^{1-\alpha} - 1 \right] + \kappa_1 \geq \Lambda \omega_n \left(\frac{\delta(x)}{r}\right)^{1-\alpha} \left(\frac{2^{\alpha-1} - 1}{2^{\alpha-1}(\alpha - 1)} \right) \geq \kappa_2 \left(\frac{\delta(x)}{r}\right)^{1-\alpha}.
\]

for \(\alpha \in (1, 2)\), where the constant \(\kappa_2\) is independent of \(\alpha\). Again, for \(\alpha = 1\), we have

\[
\Theta\left(\frac{\delta(x)}{r}\right) = \Lambda \omega_n \log\left(\frac{r}{\delta(x)}\right) + \kappa_1
\]

(2.25)

for \(x \in D_{\eta r}\). We claim that for any \(0 < \xi < 1\), there exists a \(r_\theta < 1\) such that for all \(r < r_\theta\)

\[
\log(rz) \geq r^\zeta \log(z)
\]

(2.26)

for all \(z \geq \frac{1}{\theta r}\), where \(0 < \theta < 1\) is a fixed positive constant. To prove the claim, we let

\[
h(z) = \frac{\log(rz)}{\log(z)}.
\]

By our choice of parameters \(z, r, \theta\), we have \(h(z) > 0\). Since \(\log z \geq \log(rz)\), we have

\[
h'(z) = \frac{(\log z - \log(rz))}{z(\log z)^2} > 0
\]

for \(z > \frac{1}{\theta r}\). Thus, \(h\) is strictly increasing in \(\{(\theta r)^{-1}, \infty\}\), and therefore,

\[
h(z) \geq h((\theta r)^{-1}) = \frac{\log \left(\frac{1}{\theta r}\right)}{\log \left(\frac{1}{\theta r}\right)} = \frac{\log r}{\log(r\theta)} \geq r^\zeta,
\]

for all \(r \in (0, r_\theta)\), where \(r_\theta\) depends only on \(\theta\) and \(\zeta\). The gives us (2.26). Putting (2.26) in (2.25), we have

\[
\Theta\left(\frac{\delta(x)}{r}\right) \geq \Lambda \omega_n r^\zeta \log \left(\frac{1}{\delta(x)}\right) + \kappa_1 \geq C r^\zeta \left(\log \left(\frac{1}{\delta(x)}\right) + \kappa_1\right),
\]

in \(D_{\eta r}\), for all \(r \leq r_\theta\). Using the above estimate and (2.23), we define the supersolutions as

\[
v(x) = \mu r^{1+\delta} \Phi_\alpha(x),
\]

where the constant \(\mu\) is chosen suitably so that \(Lv \leq -g\) in \(D_{\eta r}\), for all \(r \leq r_\theta\). Thus, in \(x \in D_{\eta r}\), we have \(L(C_2 v)(x) \leq -C_2 g(x)\) and \(C_2 v(x) \geq 0\) in \(\mathbb{R}^n\). Using comparison principle [10, Theorem 5.2], we then obtain \(C_2 v(x) \geq w(x)\) in \(\mathbb{R}^n\). Repeating the same argument with
-w, we get |w(x)| \leq C_2 v(x) in D_\eta^R. Now we can complete the proof repeating the same argument as in Lemma 2.3.

\[ \text{Lemma 2.5 Let } \Omega \text{ be a bounded } C^2 \text{ domain, and } u \text{ be a bounded continuous function such that } u \geq 0 \text{ in all of } \mathbb{R}^n, \text{ and } |Lu| \leq C_2(1 + \frac{1}{|x|}\dot{a} + g) \text{ in } D_R^0 \text{ for some constant } C_2. \text{ Let } \dot{a} = 1 \wedge (2 - \alpha) \text{ for } \alpha \neq 1, \text{ and for } \alpha = 1, \dot{a} \text{ be any number in } (0, 1). \text{ Then, there exists a positive constant } C_0 \text{ depending only on } n, \Omega, A_0 \text{ and } k, \text{ such that}
\]
\[
\sup_{D^+_\delta R^+} \left( \frac{u}{\delta} \right) \leq C \left( \inf_{D^+_\delta R^+} \frac{u}{\delta} + C_2 R^{\dot{a}} \right)
\]
\[\text{(2.27)}\]

for all \( R \leq \rho_0 \), where constant \( \rho_0 \) depends only on \( \Omega, n, A_0, \dot{a} \) and \( \int_\mathbb{R}^n (|y|^2 \wedge 1) \hat{k}(y)dy \).

**Proof** Recall from Remark 2.1 that we may take \( \beta = \infty \) in Assumption 1.1(b). This property will be useful to apply the Harnack inequality from [24]. We split the proof in two steps.

**Step 1.** Let \( C_2 = 0 \). In this case, (2.27) follows from the Harnack inequality for \( L \). Let \( R \leq \rho \). Then for each \( y \in D^+_\delta R \) we have \( B_\kappa y \subset D_R \). Hence, we have \( Lu = 0 \) in \( B_\kappa y \). Without loss of generality, we may assume \( y = 0 \). Let \( r = \kappa R \) and define \( v(x) = u(rx) \) for all \( x \in \mathbb{R}^n \). Then, it can be easily seen that
\[
r^2 Lu(rx) = L_r v(x) := \Delta v(x) + r^2 \int_{\mathbb{R}^n} (v(x + y) + v(x - y) - 2v(x)) \hat{k}(y) r^\alpha dy \quad \text{for all } x \in B_1.
\]

This gives \( L_r v(x) = 0 \) in \( B_1 \) and \( v \geq 0 \) in whole \( \mathbb{R}^n \). From the stochastic representation of \( v \) [10, Theorem 1.1], it follows that \( v \) is also a harmonic function in the probabilistic sense as considered in [24]. Hence, by the Harnack inequality [24, Theorem 2.4], we obtain
\[
\sup_{B_1} v \leq C \inf_{B_1} v,
\]
where constant \( C \) does not depend on \( r \). This, of course, implies
\[
\sup_{B_{\delta R}} u \leq C \inf_{B_{\delta R}} u.
\]

Now cover \( D^+_\delta R \) by a finite number of balls \( B_{\kappa R/2} \), independent of \( R \), to obtain
\[
\sup_{D^+_\delta R} u \leq C \inf_{D^+_\delta R} u.
\]

(2.27) follows since \( \kappa R/2 \leq \delta \leq 3\kappa R/2 \) in \( D^+_\delta R \).

**Step 2.** Let \( C_2 > 0 \). The proof follows by combining Step 1 above and Step 2 of Lemma 2.3 and 2.4.

Next we compute \( L\delta \) in \( \Omega \).

**Lemma 2.6** We have \(|L\delta(x)| \leq C g(x)\), where \( g \) is given by

\[
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\]
\[ g(x) = \begin{cases} 
(\delta(x) \wedge 1)^{1-\alpha} & \text{for } \alpha > 1, \\
\log(\frac{1}{\delta(x)\wedge 1}) + 1 & \text{for } \alpha = 1, \\
1 & \text{for } \alpha \in (0, 1). 
\end{cases} \] (2.28)

**Proof** Since \( \delta \in C^{0,1}(\mathbb{R}^n) \cap C^2(\overline{\Omega}) [16, \text{Theorem 5.4.3}] \), (2.28) easily follows for the case \( \alpha \in (0, 1) \). Let \( \Omega_{\rho_0} = \Omega \cap \{ \delta < \rho_0 \} \) where \( \rho_0 < 1 \). It is enough to show that
\[ |L\delta(x)| \leq C \Theta(\delta(x)) \quad \text{for } x \in \Omega_{\rho_0}, \] (2.29)
where \( \Theta \) is defined as before
\[ \Theta(\xi) = \int_{|z| > \xi} \min\{1, |z|\} \hat{k}(z)dz. \]

First of all
\[ |L\delta(x)| \leq |\Delta \delta(x)| + A_0 |I\delta(x)| \leq \kappa + A_0 |I\delta(x)|, \] (2.30)
for some constant \( \kappa \), depending on \( \Omega \). Again,
\[ I\delta(x) = \int_{\mathbb{R}^n} \left( \delta(x+z) + \delta(x-z) - 2\delta(x) \right) k(z)dz \]
\[ = \int_{|z| \leq \delta(x)/2} + \int_{|z| > \delta(x)/2}. \]

Since \( \delta(x+z) + \delta(x-z) - 2\delta(x) \leq \kappa_2 |z|^2 \) for \( |z| \leq \delta(x)/2 \), we have
\[ \int_{|z| \leq \delta(x)/2} \left( \delta(x+z) + \delta(x-z) - 2\delta(x) \right) k(z)dz \leq \kappa_2 \int_{|z| \leq \delta(x)/2} |z|^2 \hat{k}(z)dz \leq \kappa_3, \]
for some constant \( \kappa_3 \). Since \( \delta \) is Lipschitz, it follows that
\[ \delta(x+z) + \delta(x-z) - 2\delta(x) \leq 2(\text{diam}(\Omega) \lor 1) \min\{|z|, 1\}. \]

Thus
\[ \int_{|z| > \delta(x)/2} \left( \delta(x+z) + \delta(x-z) - 2\delta(x) \right) k(z)dz \leq \kappa_4 \int_{|z| > \delta(x)/2} \min\{|z|, 1\} \hat{k}(z)dz = \kappa_4 \Theta(\delta(x)/2), \]
for some constant \( \kappa_4 \). Inserting these estimates in (2.30), we obtain
\[ |L\delta(x)| \leq \kappa_5 (1 + \Theta(\delta(x)/2)) \quad \text{for all } x \in \Omega_{\rho_0}, \]
for some constant \( \kappa_5 \). Choosing \( \rho_0 \) sufficiently small, (2.29) follows from (2.24). \( \square \)

Now we are ready to prove our key estimate towards the regularity of \( u/\delta \).

**Proposition 2.1** Let \( u \) be a bounded continuous function such that \( |Lu| \leq K \) in \( \Omega \), for some constant \( K \), and \( u = 0 \) in \( \Omega^c \). Given any \( x_0 \in \partial \Omega \), let \( DR \) be as in the Definition 2.1. Then, for some \( \tau \in (0, 1 \wedge (2 - \alpha)) \), there exists \( C \), dependent on \( \Omega, n, A_0, \alpha \) and \( \hat{k} \) but not on \( x_0 \), such that

\[ \tag{2.31} \]
\[
\sup_{D_R} \frac{u}{\delta} - \inf_{D_R} \frac{u}{\delta} \leq CKR^\tau
\]  
(2.31)

for all \( R \leq \rho_0 \), where \( \rho_0 > 0 \) is a constant depending only on \( \Omega, n, \hat{\alpha}, A_0 \) and \( \int_{\mathbb{R}^n} (|y|^2 + 1) \hat{k}(y) dy \).

**Proof** For the proof, we follow a standard method, similar to [34], with the help of Lemma 2.4 to 2.6. Fix \( x_0 \in \partial \Omega \) and consider \( \rho_0 > 0 \) to be chosen later. With no loss of generality, we assume \( x_0 = 0 \). In view of (2.2), we only consider the case \( K = 0 \). By considering \( u/K \) instead of \( u \), we may assume that \( K = 1 \), that is, \( |Lu| \leq 1 \) in \( \Omega \). From Lemma 2.1, we note that \( |u|_{C^{1/(\mathbb{R}^n)}} \leq C_1 \). For \( \alpha \in (0, 1) \), we can calculate \( Lu \) classically and \( |Lu| \leq \hat{C} \) in \( \Omega \). We can combine the nonlocal term on the rhs and only deal with \( \Delta u \). In this case, the proof is simpler and can be done following the same method as for the local case (the proof below also works with minor modifications). Therefore, we only deal with \( \alpha \in [1, 2) \).

We show that there exists \( G > 0 \), \( \rho_1 \in (0, \rho_0) \) and \( \tau \in (0, 1) \), dependent only on \( \Omega, n, A_0 \) and \( \hat{k} \), and monotone sequences \( \{ M_k \} \) and \( \{ m_k \} \) such that, for all \( k \geq 0 \),

\[
M_k - m_k = \frac{1}{4^k \tau}, \quad -1 \leq m_k \leq m_{k+1} < M_{k+1} \leq M_k \leq 1,
\]  
(2.32)
and

\[
m_k \leq G^{-1} \frac{u}{\delta} \leq M_k \quad \text{in} \quad D_{R_k} = D_{R_k}(x_0), \quad \text{where} \quad R_k = \frac{\rho_1}{4^k}.
\]  
(2.33)

Note that (2.33) is equivalent to the following

\[
m_k \delta \leq G^{-1} u \leq M_k \delta, \quad \text{in} \quad B_{R_k} = B_{R_k}(x_0), \quad \text{where} \quad R_k = \frac{\rho_1}{4^k}.
\]  
(2.34)

Next we construct monotone sequences \( \{ M_k \} \) and \( \{ m_k \} \) by induction.

The existence of \( M_0 \) and \( m_0 \) such that (2.32) and (2.34) hold for \( k = 0 \) is guaranteed by Lemma 2.1. Assume that we have the sequences up to \( M_k \) and \( m_k \). We want to show the existence of \( M_{k+1} \) and \( m_{k+1} \) such that (2.32)-(2.34) hold. We set

\[
u_k = \frac{1}{G} u - m_k \delta.
\]  
(2.35)

Note that to apply Lemma 2.5, we need \( u_k \) to be nonnegative in \( \mathbb{R}^n \). Therefore, we work with \( u^+_k \), the positive part of \( u_k \). Let \( u_k = u^+_k - u^-_k \) and by the induction hypothesis,

\[
u^+_k = u^+_k \quad \text{and} \quad u^-_k = 0 \quad \text{in} \quad B_{R_k}.
\]  
(2.36)

We need a lower bound on \( u_k \). Since \( u_k \geq 0 \) in \( B_{R_k} \), we get for \( x \in B_{R_k}^c \) that

\[
u_k(x) = u_k(R_k x_0) + u_k(x) - u_k(R_k x_0) \geq -C_L |x - R_k x_0|,
\]  
(2.37)

where \( z_u = \frac{1}{|z|} z \) for \( z \neq 0 \) and \( C_L \) denotes a Lipschitz constant of \( u_k \) which can be chosen independent of \( k \). Using Lemma 2.1, we also have \( |u_k| \leq G^{-1} + \text{diam}(\Omega) = C_1 \) for all \( x \in \mathbb{R}^n \). Thus, using (2.36) and (2.37), we calculate \( Lu_k \) in \( D_{R_k/\tau} \). Let \( x \in D_{R_k/2}(x_0) \). By (2.36), \( \Delta u^-_k(x) = 0 \). Denote by
\[ \bar{g}(r) = \begin{cases} 
|\log(r)| & \text{for } r > 0, \quad \alpha = 1, \\
 r^{1-\alpha} & \text{for } r > 0, \quad \alpha \in (1, 2). \end{cases} \]

Then
\[
0 \leq Lu_k^-(x) = \int \{ \{ |y| \geq \frac{\delta}{2}, \ x+y \neq 0 \} \} u_k^-(x+y)k(y)dy
\leq \int \{ \|y\| \geq \frac{\delta}{2}, \ x+y \neq 0 \} u_k^-(x+y)k(y)dy
\leq C_L \int_{\frac{\delta}{2} \leq |y| \leq 1, \ x+y \neq 0} \left| (x+y) - R_k(x+y) \right| \tilde{k}(y)dy + C_1 \int_{|y| \geq 1} J(y)dy
\leq C_L \int_{\frac{\delta}{2} \leq |y| \leq 1} \left| (x+y) - R_k \right| \frac{1}{|y|^n} dy + C_L \int_{|y| \geq 1} \frac{1}{|y|^n} dy + \kappa_1 C_1
\leq \kappa_3 \left( R_k^{1-\alpha} + \bar{g}(R_k/2) + 1 \right)
\leq \kappa_4 \bar{g}(R_k)
\]

for some constants \( \kappa_1, \kappa_3, \kappa_4 \), independent of \( k \).

Now we write \( u_k^+ = G^{-1}u - m_k \delta + u_k^- \) and applying the operator \( L \), we get
\[
|Lu_k^+| \leq G^{-1}|Lu| + m_k |L\delta| + |Lu_k^-|
\leq G^{-1} + m_k Cg(x) + \kappa_4 \bar{g}(R_k), \tag{2.38}
\]

using Lemma 2.6. Since \( \rho_1 \geq R_k \geq \delta \) in \( D_{R_k} \), for \( \alpha \geq 1 \), we have \( R_k^{1-\alpha} \leq \delta^{1-\alpha} \), and hence, from (2.38), we have
\[
|Lu_k^+| \leq \left[ G^{-1}[\bar{g}(\rho_1)]^{-1} + C + \kappa_4 \right] g(x) := \kappa_5 g(x) \quad \text{in } D_{R_k/2}.
\]

Now we are ready to apply Lemma 2.5. Recalling that
\[
u_k^+ = u_k = G^{-1}u - m_k \delta \quad \text{in } D_{R_k},
\]
we get from Lemma 2.4 and 2.5 that
\[
\sup_{D_{\rho_1/2}^+} \left( G^{-1} \frac{u}{\delta} - m_k \right) \leq C \left( \inf_{D_{\rho_1/2}^+} \left( G^{-1} \frac{u}{\delta} - m_k \right) + \kappa_5 \bar{g}(R_k) \right)
\leq C \left( \inf_{D_{R_k/4}} \left( G^{-1} \frac{u}{\delta} - m_k \right) + \kappa_5 \bar{g}(R_k) \right). \tag{2.39}
\]

Repeating a similar argument for the function \( \tilde{u}_k = M_k \delta - G^{-1}u \), we obtain
\[
\sup_{D_{\rho_1/2}^+} \left( M_k - G^{-1} \frac{u}{\delta} \right) \leq C \left( \inf_{D_{R_k/4}} \left( M_k - G^{-1} \frac{u}{\delta} \right) + \kappa_5 \bar{g}(R_k) \right) \tag{2.40}
\]

Combining (2.39) and (2.40), we obtain
Putting \( M_k - m_k \leq C \left( \inf_{D_{R_k/4}} \left( M_k - G^{-1}\frac{u}{\delta} \right) + \inf_{D_{R_k/4}} \left( G^{-1}\frac{u}{\delta} - m_k \right) + \kappa_5 R_k^2 \right) \)
\[
= C \left( \inf_{D_{R_k/4}} G^{-1}\frac{u}{\delta} - \sup_{D_{R_k+1}} G^{-1}\frac{u}{\delta} + M_k - m_k + \kappa_5 R_k^2 \right). \tag{2.41}
\]

Putting \( M_k - m_k = \frac{1}{4\tau_\delta} \) in (2.41), we have
\[
\sup_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} - \inf_{D_{R_{k+1}}} G^{-1}\frac{u}{\delta} \leq \left( \frac{C - 1}{C} + \frac{1}{4\tau_\delta} + \kappa_5 R_k^2 \right) \quad \text{(2.42)}
\]

Since \( R_k = \frac{\rho_1}{4^k} \) for \( \rho_1 \in (0, \rho_0) \), we can choose \( \rho_0 \) and \( \tau \) small so that
\[
\left( \frac{C - 1}{C} + \kappa_5 R_k^2 \right) \leq \frac{1}{4\tau}. \tag{2.43}
\]

Thus, we find \( m_{k+1} \) and \( M_{k+1} \) such that (2.32) and (2.33) hold. It is easy to prove (2.31) from (2.32)-(2.33). \( \square \)

Now we are ready to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2** As mentioned before, it is enough to consider (2.12). Replacing \( u \) by \( \frac{u}{CK} \), we may assume that \(|Lu| \leq 1 \) in \( \Omega \). Let \( v = u/\delta \). From Lemma 2.1, we then have
\[
\|v\|_{L^\infty(\Omega)} \leq C, \tag{2.43}
\]
for some constant \( C \). Also, from Theorem 1.1, we have
\[
\|u\|_{C^{0,1}(\mathbb{R}^n)} \leq C. \tag{2.44}
\]

It is also easily seen that for any \( x \in \Omega \) with \( R = \delta(x) \), we have
\[
\sup_{z_1,z_2 \in B_{R/2}(x)} \frac{|\delta^{-1}(z_1) - \delta^{-1}(z_2)|}{|z_1 - z_2|} \leq CR^{-2}. \tag{2.45}
\]

Combining it with (2.44) gives
\[
\sup_{z_1,z_2 \in B_{R/2}(x)} \frac{|v(z_1) - v(z_2)|}{|z_1 - z_2|} \leq C(1 + R^{-2}). \tag{2.46}
\]

Again, by Proposition 2.1, for each \( x_0 \in \partial \Omega \) and for all \( r > 0 \), we have
\[
\sup_{D_r(x_0)} v - \inf_{D_r(x_0)} v \leq Cr^\gamma. \tag{2.46}
\]

where \( D_r(x_0) = B_r(x_0) \cap \Omega \) as before. To complete the proof, it is enough to show that
for some \( \eta > 0 \). Consider \( x, y \in \Omega \) and let \( r = |x - y| \). We also suppose that \( \delta(x) \geq \delta(y) \). If \( r \geq 1/2 \), then

\[
\sup_{x,y \in \Omega, x \neq y} \frac{|v(x) - v(y)|}{|x - y|^\kappa} \leq C,
\]

(2.47)

by (2.43). So we suppose \( |x - y| = r < 1/2 \). Let \( R = \delta(x) \) and \( x_0, y_0 \in \partial \Omega \) satisfying \( \delta(x) = |x - x_0| \) and \( \delta(y) = |y - y_0| \). Fix \( p > 2 \). Set \( \kappa = [2 + \text{diam } \Omega]^{-p} \). If \( r \leq \kappa R^p \), then \( r < \frac{1}{2}R \). In this case, it follows from (2.45) that

\[
|v(x) - v(y)| \leq C(1 + R^{-2})r \leq C(r + \kappa^{2/p}r^{1-\gamma/p}) \leq C_1 r^{1-\gamma/p}.
\]

Again, if \( r \geq \kappa R^p \), we have \( R \leq [r/\kappa]^{1/p} \). Thus, \( y \in B_{\kappa r^{1/p}}(x_0) \) for \( \kappa_1 = 1 + \kappa^{-p} \). From (2.46), we then have

\[
|v(x) - v(y)| \leq C_2 r^{1/p}.
\]

Thus, (2.47) follows by fixing \( \kappa = \min\{ \frac{2}{p}, 1 - \frac{2}{p} \} \). This completes the proof. \( \square \)

Remark 2.3 The regularity of \( \partial \Omega \) in Theorem 1.2 can be relaxed to \( C^{1,1} \). In this case, \( \delta \) will be a \( C^{1,1} \) function. Therefore, \( I\delta \) is defined classically and \( L\delta \) can be interpreted in the viscosity sense (see [11]). The proof of Theorem 1.2 goes through due to the coupling result in [10, Lemma 5.2].

### 2.2 Boundary regularity of \( Du \)

Using Theorem 1.2, we show that \( Du \in C^{1,\gamma}(\Omega) \) for some \( \gamma > 0 \). Recall (1.1)

\[
Lu + C_0|Du| \geq -K \quad \text{in } \Omega,
\]

\[
Lu - C_0|Du| \leq K \quad \text{in } \Omega,
\]

(2.48)

\[
u = 0 \quad \text{in } \Omega^c.
\]

Let \( v = u/\delta \). From Theorem 1.2, we know that \( v \in C^\infty(\Omega) \). We extend \( v \) in all of \( \mathbb{R}^n \) as a \( C^\infty \) function without altering its \( C^\infty \) norm (cf. [34, Lemma 3.8]). Below we find the equations satisfied by \( v \).

**Lemma 2.7** If \( |Lu| \leq C \) in \( \Omega \) and \( u = 0 \) in \( \Omega^c \), then we have

\[
\frac{1}{\delta}[-C - vL\delta - Z[v, \delta]] \leq Lv + 2 \frac{D\delta}{\delta} \cdot Dv \leq \frac{1}{\delta}[C - vL\delta - Z[v, \delta]],
\]

(2.49)

in \( \Omega \), where

\[
Z[v, \delta](x) = \int_{\mathbb{R}^n} (v(y) - v(x))(\delta(y) - \delta(x))k(y - x)dy.
\]
\textbf{Proof} First of all, since \( u \in C^1(\Omega) \) \cite[Theorem 4.1]{33}, we have \( v \in C^1(\Omega) \). Therefore, \( Z[v, \delta] \) is continuous in \( \Omega \). Consider a test function \( \psi \in C^2(\Omega) \) that touches \( v \) from above at \( x \in \Omega \). Define

\[ \psi_r(z) = \begin{cases} \psi(z) & \text{in } B_r(x), \\ v(z) & \text{in } B'_r(x). \end{cases} \]

By our assertion, we must have \( \psi_r \geq v \) for all \( r \) small. To verify (2.49), we must show that

\[ L\psi_r(x) + 2 \frac{D\delta}{\delta \cdot D\psi_r(x)} \geq \frac{1}{\delta(x)} [-C + v(x)L\delta(x) - Z[v, \delta](x)], \tag{2.50} \]

for some \( r \) small. We define

\[ \tilde{\psi}_r(z) = \begin{cases} \delta(z)\psi(z) & \text{in } B_r(x), \\ u(z) & \text{in } B'_r(x). \end{cases} \]

Then, \( \tilde{\psi}_r \geq u \) for all small. Since \( |Lu| \leq C \) and \( \delta \psi_r = \tilde{\psi}_r \), we obtain

\[ -C \leq L\tilde{\psi}_r(x) = \delta(x)L\psi_r(x) + v(x)L\delta(x) + 2D\delta(x) \cdot D\psi_r(x) + Z[\psi_r, \delta](x) \]

for all \( r \) small. Rearranging the terms we have

\[ -C - v(x)L\delta(x) - Z[\psi_r, \delta](x) \leq \delta(x)L\psi_r(x) + 2D\delta(x) \cdot D\psi_r(x). \tag{2.51} \]

Let \( r_1 \leq r \). Since \( \psi_r \) is decreasing with \( r \), we get from (2.51) that

\[ \delta(x)L\psi_r(x) + 2D\delta(x) \cdot D\psi_r(x) \geq \delta(x)L\psi_{r_1}(x) + 2D\delta(x) \cdot D\psi_r(x) \]

\[ \geq \lim_{r_1 \rightarrow 0} [-C - v(x)L\delta(x) - Z[\psi_{r_1}, \delta](x)] \]

\[ = [-C - v(x)L\delta(x) - Z[v, \delta](x)], \]

by dominated convergence theorem. This gives (2.50). Similarly we can verify the other side of (2.49). \( \square \)

In order to prove Theorem 1.3, we also need the following estimate on \( v \). Define \( \Omega_\sigma = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \sigma \} \). Then, we have

\textbf{Lemma 2.8} For some constant \( C \), it holds that

\[ \|Dv\|_{L^\infty(\Omega_\sigma)} \leq CK\sigma^{k-1} \quad \text{for all } \sigma \in (0, 1). \tag{2.52} \]

Furthermore, there exists \( \eta \in (0, 1) \) such that for any \( x \in \Omega_\sigma \) and \( 0 < |x - y| \leq \sigma/8 \), we have

\[ \frac{|Dv(y) - Dv(x)|}{|x - y|^\eta} \leq CK\sigma^{k-1-\eta}, \]

for all \( \sigma \in (0, 1) \).

\textbf{Proof} As earlier, we suppose \( K > 0 \). Diving \( u \) by \( K \) in (2.48), we may assume \( K = 1 \). Using Theorem 1.1, we can write \( |Lu| \leq C_1 \) in \( \Omega \), for some constant \( C_1 \). By Lemma 2.7, we then have

\( \square \) Springer
\[
\frac{1}{\delta}[-C_1 - vL\delta - Z[v, \delta]] \leq Lv + 2\frac{D\delta}{\delta} \cdot Dv \leq \frac{1}{\delta}[C_1 - vL\delta - Z[v, \delta]], \tag{2.53}
\]
in \(\Omega\). Fix \(x_0 \in \Omega_\sigma\) and define

\[
w(x) = v(x) - v(x_0).
\]
From (2.53), we then obtain

\[
-\frac{1}{\delta}C_1 - \ell \leq Lw + 2\frac{D\delta}{\delta} \cdot Dw \leq \frac{1}{\delta}C_1 - \ell, \tag{2.54}
\]
in \(\Omega\), where

\[
\ell(x) = \frac{1}{\delta(x)}[w(x)L\delta(x) + Z[v, \delta](x) + v(x_0)L\delta(x)].
\]
Set \(r = \frac{\sigma}{2}\). We claim that

\[
\|\ell\|_{L^\infty(B_r(x_0))} \leq \kappa_1 \sigma^{\kappa - 2}, \quad \text{for all } \sigma \in (0, 1), \tag{2.55}
\]
for some constant \(\kappa_1\). Let us denote by

\[
\xi_1 = \frac{wL\delta}{\delta}, \quad \xi_2 = \frac{1}{\delta}Z[v, \delta] \quad \text{and} \quad \xi_3 = \frac{v(x_0)}{\delta}L\delta.
\]
Recall that \(\kappa \in (0, (2 - \alpha) \wedge 1)\) from Theorem 1.2. Since

\[
\|\Delta \delta\|_{L^\infty(\Omega)} < \infty \quad \text{and} \quad \|I\delta\|_{L^\infty(\Omega_\sigma)} \lesssim \begin{cases} 
(\delta(x) \wedge 1)^{1-\alpha} & \text{for } \alpha > 1, \\
\log(\frac{1}{\delta(x)\lambda}) + 1 & \text{for } \alpha = 1, \\
1 & \text{for } \alpha \in (0, 1),
\end{cases}
\]
(cf. Lemma 2.6), and

\[
\|v\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \|w\|_{L^\infty(B_r(x_0))} \lesssim r^\kappa,
\]
it follows that

\[
\|\xi_3\|_{L^\infty(B_r(x_0))} \lesssim \begin{cases} 
\sigma^{-(\alpha\lambda)} & \text{for } \alpha \neq 1, \\
\sigma^{-1}|\log \sigma| & \text{for } \alpha = 1,
\end{cases} \lesssim \sigma^{-2+\kappa},
\]
and

\[
\|\xi_1\|_{L^\infty(B_r(x_0))} \lesssim \begin{cases} 
\sigma^{\kappa-1+1-(1+\alpha)} & \text{for } \alpha \neq 1, \\
\sigma^{\kappa-1}|\log \sigma| & \text{for } \alpha = 1,
\end{cases} \lesssim \sigma^{-2+\kappa}.
\]
So we are left to compute the bound for \(\xi_2\). Let \(x \in B_r(x_0)\). Denote by \(\hat{r} = \delta(x)/4\). Note that

\[
\delta(x) \geq \delta(x_0) - |x - x_0| \geq 2r - r = r \Rightarrow \hat{r} \geq r/4.
\]
Thus, since \(u \in C^1(\Omega)\) by [33, Theorem 4.1] (as mentioned before, the proof of [33] works for inequations),
using (2.1). Since \( \delta \) is Lipschitz and bounded in \( \mathbb{R}^n \), we obtain

\[
|Z[v, \delta](x)| \leq \Lambda \int_{y \in B_r(x)} \frac{|\delta(x) - \delta(y)||v(x) - v(y)|}{|x - y|^{n+a}} dy + \int_{y \in B_r(x)} |\delta(x) - \delta(y)||v(x) - v(y)| \tilde{\kappa}(y-x) dy
\]

\[
\leq [\delta(x)]^{-1} \int_{y \in B_r(x)} |x - y|^{2-n} dy + \int_{y \in B_r(x) \setminus B_r(x)} \frac{(\delta(x) - \delta(y))(v(x) - v(y))}{|x - y|^{n+a}} dy
\]

\[
+ \int_{|y| > 1} |\delta(x) - \delta(y + x)||v(x) - v(y + x)| J(y) dy
\]

\[
\leq [\delta(x)]^{-1-a} + \int_{y \in B_r(x) \setminus B_r(x)} \frac{|\delta(x) - \delta(y)||v(x) - v(y)|}{|x - y|^{n+a}} dy + \kappa_2,
\]

for some constant \( \kappa_2 \). The second integration on the right hand side can be computed as follows: for \( \alpha \leq 1 \) we write

\[
\int_{y \in B_r(x) \setminus B_{r_1}(x)} \frac{|\delta(x) - \delta(y)||v(x) - v(y)|}{|x - y|^{n+a}} dy
\]

\[
\leq \int_{y \in B_r(x) \setminus B_{r_1}(x)} |x - y|^{-n-a+1+\kappa} dy \lesssim (1 - r_1^{1-a+\kappa}) \lesssim \sigma^{-1+\kappa},
\]

whereas for \( \alpha \in (1, 2) \) we can compute it as

\[
\int_{y \in B_r(x) \setminus B_{r_1}(x)} \frac{|\delta(x) - \delta(y)||v(x) - v(y)|}{|x - y|^{n+a}} dy \lesssim \int_{y \in B_r(x) \setminus B_{r_1}(x)} |x - y|^{-n-a+1} \lesssim r_1^{-a+1} \lesssim \sigma^{-1+\kappa}.
\]

Combining the above estimates we obtain

\[
\|\xi_2\|_{L^1(B_r(x))} \lesssim \sigma^{-2+\kappa}.
\]

Thus, we have established the claim (2.55).

Let us now define \( \zeta(z) = w(z)z + x_0 \). Letting \( b(z) = 2\frac{D\delta(\zeta z + x_0)}{\delta(\zeta z + x_0)} \) and \( r_1 = \frac{\zeta}{2} \), it follows from (2.54) that

\[
r_1^2 \left(-\frac{C}{\delta} - \ell\right) (r_1 z + x_0) \leq \Delta \zeta + r_1^{2-a} I_{r_1} \zeta + r_1 b(z) \cdot D\zeta \leq r_1^2 \left(-\frac{C}{\delta} - \ell\right) (r_1 z + x_0)
\]

(2.56)

in \( B_{\frac{r_1}{2}}(0) \), where

\[
I_{r_1} f(x) = r_1^2 \frac{1}{2} \int_{B_{r_1}} (f(x+y) + f(x-y) - f(x)) k(r_1 y) r_1^d dy.
\]

Consider a cut-off function \( \varphi \) satisfying \( \varphi = 1 \) in \( B_{3/2}(0) \) and \( \varphi = 0 \) in \( B_z(0) \). Defining \( \tilde{\zeta} = \zeta \varphi \) we get from (2.56) that
\[
|\Delta \tilde{\zeta}(z) + r_1^{2-a}I_{r_1} \tilde{\zeta}(z) + r_1 b(z) \cdot D\tilde{\zeta}| \leq r_1^2 \left( C \frac{\zeta}{\delta} + |\ell'| \right) (r_1 z + x_0) + r_1^{2-a} |I_{r_1}((\varphi - 1)\zeta)|
\]
in \( B_1(0) \). Since
\[
\|r_1 b\|_{L^\infty(B_1(0))} \leq \kappa_3 \quad \text{for all } \rho \in (0, 1),
\]
applying [33, Theorem 4.1] (this result works for inequality) we obtain, for some constant \( \kappa_6 \) independent of \( \rho \in (0, 1) \),
\[
\|D\zeta\|_{C^0(B_1(0))} \leq \kappa_6 \left( \|\zeta\|_{L^\infty(\mathbb{R}^n)} + C r_1 + r_1^2 \|\ell'(r_1 \cdot + x_0)\|_{L^\infty(B_1)} + r_1^{2-a} \|I_{r_1}((\varphi - 1)\zeta)\|_{L^\infty(B_1)} \right),
\]
for some constant \( \kappa_6 \) independent of \( \rho \in (0, 1) \). Since \( v \) is in \( C^\kappa(\mathbb{R}^n) \), it follows that
\[
\|\zeta\|_{L^\infty(\mathbb{R}^n)} = \|\zeta\|_{L^\infty(B_2)} \leq \|\zeta\|_{L^\infty(B_1)} \leq \sigma^\kappa.
\]
Also, by (2.55),
\[
r_1^2 \|\ell'(r_1 \cdot + x_0)\|_{L^\infty(B_1)} \leq \sigma^\kappa.
\]
Note that, for \( z \in B_1(0) \),
\[
|I_{r_1}((\varphi - 1)\zeta)| \leq \int_{|y| \geq 1/2} |(\varphi(x + y) - 1)\zeta(x + y)| \hat{k}(y) dy 
\]
\[
\leq 2 \|v\|_{L^\infty(\mathbb{R}^n)} \int_{|y| \geq 1/2} \hat{k}(y) dy 
\]
\[
\leq \kappa_3
\]
for some constant \( \kappa_3 \). Putting these estimates in (2.57) and calculating the gradient at \( z = 0 \), we obtain
\[
|Dv(x_0)| \leq \kappa_4 \sigma^{-1+\kappa},
\]
for all \( \sigma \in (0, 1) \). This proves the first part.

For the second part, compute the Hölder ratio with \( D\zeta(0) - D\zeta(z) \) where \( z = \frac{2}{r}(y - x_0) \) for \( |x_0 - y| \leq \sigma/8 \). This completes the proof. \( \square \)

Now we can establish the Hölder regularity of the gradient up to the boundary.

**Proof of Theorem 1.3** Since \( u = v\delta \), it follows that
\[
Du = vD\delta + \delta Dv.
\]
Since \( \delta \in C^2(\Omega) \), it follows from Theorem 1.2 that \( vD\delta \in C^\kappa(\Omega) \). Thus, we only need to concentrate on \( \delta = Dv \). Consider \( \eta \) from Lemma 2.8 and with no loss of generality, we may fix \( \eta \in (0, \kappa) \).

For \( |x - y| \geq \frac{1}{8} (\delta(x) \lor \delta(y)) \), it follows from (2.52) that
\[
\frac{|\delta(x) - \delta(y)|}{|x - y|^\eta} \leq C(\delta^\kappa(x) + \delta^\kappa(y)) (\delta(x) \lor \delta(y))^{-\eta} \leq 2C.
\]
So consider the case \( |x - y| < \frac{1}{8} (\delta(x) \lor \delta(y)) \). Without loss of generality, we may assume that \( |x - y| < \frac{1}{8} \delta(x) \). Then
By Lemma 2.8, it follows
\[
\frac{9}{8} \delta(x) \geq |x - y| + \delta(x) \geq \delta(y) \geq \delta(x) - |x - y| \geq \frac{7}{8} \delta(x).
\]
This completes the proof by setting $\gamma = \eta$. \hfill \Box

3 Overdetermined problems

In this section, we solve an overdetermined problem. Let $H : \mathbb{R} \to \mathbb{R}$ be a given locally Lipschitz function. We also assume that $a \in (0, A_0)$. Our main result of this section is the following.

**Theorem 3.1** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded set with $C^2$ boundary. Suppose that Assumption 1.1 holds, $k = k(|y|)$ and $k : (0, \infty) \to (0, \infty)$ is strictly decreasing. Let $f : \mathbb{R} \to \mathbb{R}$ be locally Lipschitz and $u$ be a viscosity solution to
\[
Lu + H(|Du|) = f(u) \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{in } \Omega^c, \quad u > 0 \quad \text{in } \Omega,
\]
\[
\frac{\partial u}{\partial n} = c \quad \text{on } \partial\Omega,
\]
for some fixed $c > 0$, where $\nu$ is the unit inward normal on $\partial\Omega$. Then, $\Omega$ must be a ball. Furthermore, $u$ is radially symmetric and strictly decreasing in the radial direction.

Proof of Theorem 3.1 follows from the boundary estimates in Theorem 1.2 combined with the approach of [8, 19]. Also, note that we have taken $k$ to be positive valued. This is just for convenience and the proofs below can be easily modified to include kernel $k$ that is non-increasing but strictly decreasing in a neighbourhood of 0, provided we assume $\Omega$ to be connected. We provide a sketch for the proof of Theorem 3.1 and the finer details can be found in [8, 19]. From Theorem 1.1 and 1.3, we see that $u \in C^{0,1}(\mathbb{R}^n) \cap C^{1,\gamma}(\overline{\Omega})$, and therefore, $u \in C^{2,\gamma}(\Omega)$ by [33]. Therefore, we can assume that $u$ is a classical solution to (3.1).

Given a unit vector $e$, let us define the half space
\[
\mathcal{H} = \mathcal{H}_{\lambda,e} = \{x \in \mathbb{R}^n : x \cdot e > \lambda\},
\]
and let $\mathcal{R}_{\lambda,e}(x) = x - 2(x \cdot e)e + 2\lambda e$ be the reflection of $x$ along $\partial\mathcal{H} = \{x \cdot e = \lambda\}$. We say $v : \mathbb{R}^n \to \mathbb{R}$ is anti-symmetric if $v(x) = -v(\mathcal{R}_e(x))$ for all $x \in \mathbb{R}^n$. Now let $D \subset \mathcal{H}$ be a bounded open set and $u$ be a bounded anti-symmetric solution to
\[
Lu - \beta|Du| \leq g \quad \text{in } D
\]
\[
u \geq 0 \quad \text{in } \mathcal{H} \setminus D,
\]
where $\beta > 0$ is a fixed constant. Let

$$v = \begin{cases} -u & \text{in } \{u < 0\} \cap D, \\ 0 & \text{otherwise}. \end{cases}$$

Then, it can be easily seen that $v$ solves

$$Lv - \beta |Dv| \geq -g \quad \text{in } \Sigma := \{u < 0\} \cap D,$$

in the viscosity sense. To check (3.2), consider $x \in \Sigma$ and test function $\phi$ such that $\phi(x) = v(x)$ and $\phi(y) > v(y)$ for $y \in \mathbb{R}^n \setminus \{x\}$. Define $\psi := \phi + (-u - v)$. Then, $\psi(x) = -u(x)$ and $\psi(y) > -u(y)$ for $y \in \mathbb{R}^n \setminus \{x\}$. Furthermore, $\psi \in \Sigma$. Thus, we get $L\psi(x) + \beta |D\psi(x)| \geq -g(x)$. This implies $L\phi + \beta |D\phi(x)| + L(-u - v)(x) + |D\phi(x)| \geq -g(x)$. Since $k$ is radially decreasing and $u$ is anti-symmetric, it follows that $L(-u - v)(x) \leq 0$ (cf. [8, p. 11]). This gives us (3.2).

The following narrow domain maximum principle is a consequence of the ABP estimate in [32, Theorem A.4].

**Lemma 3.1** Let $\mathcal{H}$ be the half space and $D \subset \mathcal{H}$ be open and bounded. Also, assume $c$ to be bounded. Then, there exists a positive constant $C$, depending on $\text{diam} \, D, n, k$, such that if $u \in C_b(\mathbb{R}^n)$ is an anti-symmetric supersolution of

$$Lu - \beta |Du| - c(x)u = 0 \quad \text{in } D,$$

$$u \geq 0 \quad \text{in } \mathcal{H} \setminus D,$$

then we have

$$\sup_{\Sigma} u^- \leq C\|c^+\|_{L^\infty(D)}\|u^-\|_{L^\infty(D)}.$$

In particular, given $\kappa_\infty > 0$ and $c^+ \leq \kappa_\infty$ on $D$, there is a $\delta > 0$ such that if $|D| < \delta$, we must have $u \geq 0$ in $\mathcal{H}$.

**Proof** Set $\Sigma = \{u < 0\} \cap D$ and define $v$ as above. From (3.2), we see that

$$Lv + \beta |Dv| + c(x)v \geq 0 \quad \text{in } \Sigma.$$

Since $v \geq 0$ in $\Sigma$ and $c \leq c^+$, we get

$$Lv + \beta |Dv| + c^+v \geq 0 \quad \text{in } \Sigma.$$

Taking $f = -c^+v$ and using [32, Theorem A.4] we obtain, for some constant $C_1$, that

$$\sup_{\Sigma} u^- = \sup_{\Sigma} v \leq \sup_{\Sigma} |v| + C_1\|f\|_{L^\infty(D)} \leq C_1\|c^+\|_{L^\infty(D)}\|v\|_{L^\infty(\Sigma)} = C_1\|c^+\|_{L^\infty(D)}\|u^-\|_{L^\infty(D)}.$$

This completes the proof of the lemma. \hfill \Box

Next result is a Hopf’s lemma for anti-symmetric functions.

**Lemma 3.2** Let $\mathcal{H}$ be a half space, $D \subset \mathcal{H}$, and $c \in L^\infty(D)$. If $u \in C_b(\mathbb{R}^n)$ is an anti-symmetric supersolution of $Lu - \beta |Du| - c(x)u = 0$ in $D$ with $u \geq 0$ in $\mathcal{H}$, then either $u \equiv 0$ in $\mathbb{R}^n$ or $u > 0$ in $D$. Furthermore, if $u \not\equiv 0$ in $D$ and there exists a $x_0 \in \partial D \setminus \partial \mathcal{H}$ with $u(x_0) = 0$ such that there is a ball $B \subset D$ with $x_0 \in \partial B$, then there exists a $C > 0$ such that
\[
\liminf_{t \to 0} \frac{u(x_0 - tn)}{t} \geq C,
\]
where \(n\) is the inward normal at \(x_0\).

**Proof** With any loss of generality, we may assume that \(c \geq 0\). Suppose that \(u \not\equiv 0\) and \(u \not> 0\) in \(D\). Then, there exists a compact set \(K \subset D\) such that \(\inf_K u = \delta > 0\) and a point \(x_1 \in D\) such that \(u(x_1) = 0\). For \(\varepsilon\) small enough, we can choose the test function
\[
\phi(y) = \begin{cases} 
0 & \text{for } y \in B_{\varepsilon}(x_1), \\
u & \text{for } y \in B^c_{\varepsilon}(x_1).
\end{cases}
\]

Note that \(\Delta \phi(x_1) = 0, D\phi(x_1) = 0\). Since \(k\) is radially non-increasing and positive, from the proof of [8, Theorem 3.2] it follows that \(u \equiv 0\) in \(D\). This contradicts our assertion. Thus, either \(u \equiv 0\) in \(\mathbb{R}^n\) or \(u > 0\) in \(D\).

Now we prove the second part of the lemma. Assume that \(u > 0\) in \(D\). Let \(B\) be a ball in \(D\) that touches \(\partial D\) at \(x_0\) and \(B \Subset \mathcal{H}\). This is possible since \(x_0 \in \partial D \setminus \partial \mathcal{H}\). Let \(\theta\) be the positive solution to
\[
L\theta - |D\theta| = -1 \quad \text{in } B, \quad \text{and} \quad \theta = 0 \quad \text{in } B^c.
\]
Existence of \(\theta\) follows from Theorem 1.3 and Leray-Schauder fixed point theorem. Define \(w = \kappa(\theta - \theta \circ \mathcal{H})\). Then, we have \(Lw \geq -\kappa C\) in \(B\) for some positive constant \(C\). Now repeating the arguments of [8, Theorem 3.2], it follows that for some \(\kappa > 0\) we have \(u \geq w\) in \(B\). To complete the proof, we need to apply Hopf’s lemma on \(\theta\) at the point \(x_0\) (cf. [10, Theorem 2.2]).

Given \(\lambda \in \mathbb{R},\ e \in \partial B_1(0)\), define
\[
\nu(x) = v_{\lambda,e}(x) = u(x) - u(\tilde{x}) \quad x \in \mathbb{R}^n,
\]
where \(\tilde{x} = \mathcal{H}_{\lambda,e}(x)\) denotes the reflection of \(x\) by \(T_{\lambda,e} := \partial \mathcal{H}_{\lambda,e}\) and \(\mathcal{H}_{\lambda,e} = \{x \in \mathbb{R}^n : x \cdot e > \lambda\}\). We note that \(\mathbb{R}^n \setminus \mathcal{H}_{\lambda,e} = \mathcal{H}_{-\lambda,-e}\). Moreover, let \(\lambda < l := \sup_{x \in \Omega} x \cdot e\). Then, \(\mathcal{H} \cap \Omega\) is nonempty for all \(\lambda < l\) and we put \(D_\lambda := \mathcal{H}_{\lambda,e}(\Omega \cap \mathcal{H})\). Then, for all \(\lambda < l\), the function \(\nu\) satisfies
\[
\begin{align*}
Lv + \beta |Dv(x)| - c(x) &\geq 0 \quad \text{in } D_\lambda, \\
Lv - \beta |Dv(x)| - c(x) &\leq 0 \quad \text{in } D_\lambda, \\
\nu &\geq 0 \quad \text{in } \mathcal{H}_{-\lambda,-e} \setminus D_\lambda, \\
\nu(x) &\equiv -\nu(\tilde{x}) \quad \text{for all } x \in \mathbb{R}^n,
\end{align*}
\]
where \(\beta\) is the Lipschitz constant of \(H\) on the interval \([0, \sup |Du|]\) and
\[
c(x) = \frac{f(u(x)) - f(u(\tilde{x}))}{u(x) - u(\tilde{x})}.
\]
In view of Lemmas 3.1-3.2 and (3.4), we see that \(\nu = v_{\lambda,e}\) is either 0 in \(\mathbb{R}^n\) or positive in \(D_\lambda\) for \(\lambda\) close to \(l\). Now as we decrease \(\lambda\), one of following two situation may occur.

**Situation A:** there is a point \(p_0 \in \partial \Omega \cap \partial D_\lambda \setminus T_{\lambda,e}^c\).

**Situation B:** \(T_{\lambda,e}\) is orthogonal to \(\partial \Omega\) at some point \(p_0 \in \partial \Omega \cap T_{\lambda,e}^c\).
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Let \( \lambda_0 \) be the maximal value in \((-\infty, l)\) such that one of these situations occur. We show that \( \Omega \) is symmetric with respect to \( T_{\lambda_0, e} \). This would complete the proof of Theorem 3.1 since \( e \) is arbitrary. Also, note that, since \( u > 0 \) in \( \Omega \), to establish the symmetry of \( \Omega \) with respect to \( T_{\lambda_0, e} \), it is enough to show that \( v = 0 \) in \( \mathbb{R}^n \). Suppose, to the contrary, that \( v > 0 \) in \( D_\lambda_0 \).

**Situation A:** In this case, we have \( v(p_0) = 0 \) and therefore, by Theorem 1.3 and 3.2, we get \( \frac{\partial v}{\partial n}(p_0) > 0 \). But, by (3.1), we have

\[
\frac{\partial v}{\partial n}(p_0) = \frac{\partial u}{\partial n}(p_0) - \frac{\partial u}{\partial n}(\mathcal{R}(p_0)) = 0.
\]

This is a contradiction.

**Situation B:** This situation is a bit more complicated than the previous one. Set \( T = T_{\lambda_0, e}, \mathcal{H} = \mathcal{H}_{\lambda_0, e} \) and \( \mathcal{R} = \mathcal{R}_{\lambda_0, e} \). By rotation and translation, we may set \( \lambda_0 = 0, p_0 = 0, e = e_1 \) and \( e_2 \in T \) is the interior normal at \( \partial D \).

Next two lemmas are crucial to get a contradiction in Situation B.

**Lemma 3.3** We have

\[
v(t\eta) = o(t^2), \quad \text{as} \quad t \to 0^+,
\]

where \( \eta = e_2 - e_1 = (-1, 1, 0, ..., 0) \in \mathbb{R}^n \).

**Proof of Lemma 3.3** follows from Theorem 1.2 and [8, Lemma 3.2].

**Lemma 3.4** Let \( D \subset \mathbb{R}^n, n \geq 2, \) be an open bounded domain such that \( 0 \in \partial D \) and \( \{x_1 = 0\} \) is orthogonal and there is a ball \( B \subset D \) with \( \bar{B} \cap \partial D = \{0\} \). Denote

\[
D^* := D \cap \{x_1 < 0\},
\]

and assume that \( w \in C_b(\mathbb{R}^n) \) is an anti-symmetric supersolution of

\[
Lw - \beta|Dw| - c(x)w = 0 \quad \text{in} \quad D^*
\]

\[
w \geq 0 \quad \text{in} \quad \{x_1 < 0\}
\]

\[
w > 0 \quad \text{in} \quad D^*.
\]

Let \( \tilde{\eta} = e_2 - e_1 = (-1, 1, \cdots, 0) \in \mathbb{R}^n \), then there exist positive \( C, t_0, \) dependent on \( D^*, n, \) such that

\[
w(t\tilde{\eta}) \geq Ct^2
\]

for all \( t \in (0, t_0) \).

Clearly, Lemma 3.3 and 3.4 give a contradiction to the Situation B.

**Proof of Theorem 3.1** Proof follows from the above discussion and the arguments in [19, p. 11].

In the remaining part of this section, we provide a proof of Lemma 3.4.

**Proof of Lemma 3.4** We follow the arguments of [8, Lemma 3.3]. Fix a ball \( B = B_R(Re_2) \subset D \) for some \( R > 0 \) small enough with \( \partial B \cap \partial D = \{0\} \). Denote

\[ Springer \]
Let $M_1 \in D^*$ such that $\theta := \inf_{M_1} w > 0$ and $M_2 = \mathcal{R}(M_1)$, that is, reflection of $M_1$ with respect to $\{x_1 = 0\}$. Furthermore, we may assume that $M_1$ to be an open ball and taking $R$ smaller, we also assume that $\text{dist}(K, M_1) > 0$ and $|K| < \varepsilon$ for some small $\varepsilon > 0$. Now let $g$ be the unique positive viscosity solution to

$$Lg - \beta |Dg| = -1 \quad \text{in } B$$

$$g = 0 \quad \text{in } B^c.$$

From Theorem 1.1, we know that $\|g\|_{C^{0,1}(\mathbb{R}^n)} \leq C$. Let $\phi \in C^\infty_c(\mathbb{R}^d)$, support($\phi$) $\subset M_1$, $0 \leq \phi \leq 1$ and there exists a $U \subset M_1$ such that $\phi = 1$ in $U$, $|U| > 0$. Construct a barrier function $h$ of the following form:

$$h(x) = -\kappa x_1 g(x) + \theta \phi(x) - \theta \phi(\mathcal{R}(x)).$$

Choosing $\kappa > 0$ small enough, it can be easily checked that (see [8, Lemma 3.3])

$$Lh - \beta |Dh| + c(x)h \geq 0$$

in $K$. It is also standard to see that $g$ is radial about the point $Re_2$ (cf. [10, Theorem 4.1]). Thus, we have: (i) $w - h$ is anti-symmetric, (ii) $w - h \geq 0$ in $\{x_1 < 0\} \setminus K$, since because $\theta > 0$, and (iii) $L(w - h) - \beta |D(w - h)| - c(x)(w - h) \leq 0$. Applying Lemma 3.1, we obtain $w - h \geq 0$ in $\{x_1 < 0\}$. Hence,

$$w(t\bar{\eta}) \geq h(t\bar{\eta}) \geq Ct^2$$

for $t \in (0, t_0)$, where we used Hopf’s lemma on $g$. This completes the proof.

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