On the Modified $q$-Euler Polynomials with Weight and Weak Weights

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Abstract

In this paper, we construct new $q$-extension of Euler numbers and polynomials with weight $\alpha$ and weak weight $\beta$ related to fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, and give new explicit formulas related to these numbers and polynomials. Also, we give another definition of the Euler polynomials of higher-order with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$.

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1. Introduction

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$.

In this paper, we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < p^{-\frac{1}{p-1}}$. So that $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. The $q$-number of $x$ is denoted by $[x]_q = \frac{1-q^x}{1-q}$. Note that $\lim_{q \to 1} [x]_q = x$. Let $d$ be a fixed integer bigger than 0 and let $p$ be
a fixed prime number and \((d, p) = 1\). We set
\[ X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z}, \quad X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp\mathbb{Z}_p), \]

\[ a + dp^N\mathbb{Z}_p = \{ x \in X | x \equiv a \pmod{dp^N} \}, \]

where \(a \in \mathbb{Z}\) lies in \(0 \leq a < dp^N\), (see [2-11]).

Let \(C(\mathbb{Z}_p)\) be the space of continuous functions on \(\mathbb{Z}_p\). For \(f \in C(\mathbb{Z}_p)\), the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\) is defined by Kim by
\[
I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{p^N - q} \sum_{x=0}^{p^N - 1} f(x)(-q)^x, \quad \text{(see [7, 8, 9]).}
\]

As is well known, Euler polynomials are defined by the generating function to be
\[
2e^t + 1 = e^{E(x)}t = e^{E(x)}t = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad \text{(see [1-13]).}
\]

with the usual convention about replacing \(E^n(x)\) by \(E_n(x)\). In the special case, \(x = 0\), \(E_n(0) = E_n\) are called the \(n\)-th Euler numbers.

In [7, 8], T. Kim defined the \(q\)-Euler numbers as follows:
\[
E_{0,q} = 1, \quad q(qE + 1)^n + E_{n,q} = \begin{cases} [2]_q, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}
\]

with the usual convection of replacing \(E^n(x)\) by \(E_{n,q}\). From (1.2), we also derive
\[
E_{n,q} = \frac{[2]_q}{(1 - q)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1 + q^{l+1}}, \quad \text{(see [7, 11]).}
\]

By using an invariant \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\), a \(q\)-extension of ordinary Euler polynomials which are called \(q\)-Euler polynomials are considered and investigated by Kim [4, 5, 6, 7, 9]. For \(x \in \mathbb{Z}_p\), \(q\)-Euler polynomials are defined as follows:
\[
E_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_{-q}(y). \quad \text{(1.3)}
\]

By (1.3), the following relation is hold:
\[
E_{n,q}(x) = \sum_{k=0}^{n} \binom{n}{k} [x]_q^{n-k} q^{nk} E_{k,q}.
\]

Recently, Ryoo considered the weighted \(q\)-Euler polynomials which are a slightly different Kim’s weighted \(q\)-Euler polynomials as follows:
\[
E_{n,q}^{(\alpha)}(x) = \int_{\mathbb{Z}_p} [x + y]_q^n d\mu_{-q}(y), \quad \text{for } n \in \mathbb{N} \text{ and } \alpha \in \mathbb{Z}, \quad \text{(see [12])}.
\]
On the modified $q$-Euler polynomials

In the special case, $x = 0$, $E^{(\alpha)}_{n,q}(0) = E^{(\alpha)}_{n,q}$ are called the $n$-th $q$-Euler numbers with weight $\alpha$, and showed that

$$E^{(\alpha)}_{n,q} = \frac{[2]_q}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^{\alpha l + 1}}$$

$$= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m]_{q^\alpha}^n. \quad (1.4)$$

And in [10], C. S. Ryoo defied $q$-Euler polynomials $\epsilon^{(\alpha)}_{n,q}(x)$ as follows:

$$\epsilon^{(\alpha)}_{n,q}(x) = \int_{\mathbb{Z}_p} [x + y]^n d\mu_{-q\alpha}(y). \quad (1.5)$$

In the special case, $x = 0$, $\epsilon^{(\alpha)}_{n,q}(0) = \epsilon^{(\alpha)}_{n,q}$ are called the $n$-th $q$-Euler numbers with weak weight $\alpha$, and show that

$$\epsilon^{(\alpha)}_{n,q} = [2]_q \alpha \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1 + q^{\alpha l + 1}}$$

$$= [2]_q \alpha \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} [m]_{q^\alpha}^n. \quad (1.6)$$

A systemic study of some families of the modified $q$-Euler polynomials with weight is presented by using the multivariate fermionic $p$-adic integral on $\mathbb{Z}_p$. The study of these higher-order $q$-Euler numbers and polynomials yields an interesting $q$-analogue of identities for Stirling numbers.

In recent years, many mathematicians and physicists have investigated zeta functions, multiple zeta functions, $L$-functions, and multiple $q$-Bernoulli numbers and polynomials, mainly because of their interest and importance. These functions and polynomials are used not only in complex analysis and mathematical physics, but also in $p$-adic analysis and other areas. In particular, multiple zeta functions and multiple $L$-functions occur within the context of knot theory, quantum field theory, applied analysis, and number theory (see [1-13]).

In this paper, we construct new $q$-extension of Euler numbers and polynomials with weight $\alpha$ and weak weight $\beta$ related to fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, and give new explicit formulas related to these numbers and polynomials. Also, we give another definition of the Euler polynomials of higher-order with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$.
2. A NEW APPROACH OF $q$-EULER POLYNOMIALS WITH WEIGHT $\alpha$ AND WEAK WEIGHT $\beta$

Consider the $q$-Euler number with weight $\alpha$ and weak weight $\beta$ $E_{n,q}^{(\alpha,\beta)}$. Note that, by (1.1),

$$E_{n,q}^{(\alpha,\beta)} = \int_{z_p} [x]^n_{q^\alpha} d\mu_{-q^\beta}(x) = \frac{[2]_{q^\beta}}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1 + q^{\alpha l + \beta}}. \quad (2.1)$$

As a new $q$-extension of Euler polynomials, we define the modified $q$-Euler polynomials with weight $\alpha$ and weak weight $\beta$ $\tilde{E}_{ij}$ which are defined by fermionic $q$-integral to be

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(\alpha,\beta)}(x)(1 - q^\alpha)^n \frac{t^n}{n!} = \int_{z_p} e^{(x + [y]_{q^\alpha})(1 - q^\alpha)t} d\mu_{-q^\beta}(y). \quad (2.2)$$

From (2.1) and (2.2), we obtain the following equation:

$$\int_{z_p} e^{(x + [y]_{q^\alpha})(1 - q^\alpha)t} d\mu_{-q^\beta}(y)$$

$$= e^{x(1 - q^\alpha)t}[2]_{q^\beta} \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1 + q^{\alpha l + \beta}} t^n$$

$$= e^{((1 - q^\alpha)x + 1)t} \sum_{l=0}^{\infty} [2]_{q^\beta} \frac{(-1)^l}{1 + q^{\alpha l + \beta}}$$

$$= \left( \sum_{m=0}^{\infty} ((1 - q^\alpha)x + 1)^m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} [2]_{q^\beta} \frac{(-1)^l}{1 + q^{\alpha l + \beta}} \right)$$

$$= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l [2]_{q^\beta}}{1 + q^{\alpha l + \beta}} ((1 - q^\alpha)x + 1)^{n-l} \frac{t^n}{n!}. \quad (2.3)$$

Thus, by (2.2) and (2.3), we get

$$\tilde{E}_{n,q}^{(\alpha,\beta)}(x) = \frac{1}{(1 - q^\alpha)^n} \sum_{l=0}^{\infty} \binom{n}{l} \frac{(-1)^l [2]_{q^\beta}}{1 + q^{\alpha l + \beta}} ((1 - q^\alpha)x + 1)^{n-l}$$

$$= \frac{1}{(1 - q^\alpha)^n} \sum_{l=0}^{\infty} \sum_{j=0}^{n-l} \binom{n}{l} \frac{(-1)^l [2]_{q^\beta}}{1 + q^{\alpha l + \beta}} (1 - q^\alpha)^j x^j. \quad (2.4)$$

Therefore, by (2.4), we obtain the following theorem.
Theorem 2.1. For \( n \geq 0 \),

\[
\tilde{E}_{n,q}^{(\alpha,\beta)}(x) = \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l [2]_q^\alpha}{1 + q^\alpha l + \beta} ((1-q^\alpha)x + 1)^{n-l}
\]

\[
= \frac{1}{(1-q^\alpha)^n} \sum_{l=0}^{n} \sum_{j=0}^{n-l} \binom{n}{l} \binom{n-l}{j} (-1)^l [2]_q^\beta (1-q^\alpha)^j x^j.
\] (2.5)

Note that, by (2.2), we get

\[
\sum_{n=0}^{\infty} \tilde{E}_{n,q}^{(\alpha,\beta)}(x)(1-q^\alpha)^n \frac{t^n}{n!} = \int_{\mathbb{Z}_p} e^{(x+[y]_q^\alpha)(1-q^\alpha)t} d\mu_{-q^\beta}(y)
\]

\[
= \sum_{n=0}^{\infty} (1-q^\alpha)^n \int_{\mathbb{Z}_p} (x+[y]_q^\alpha)^n d\mu_{-q^\beta}(y) \frac{t^n}{n!}.
\] (2.6)

Thus, by (2.6), we obtain the following theorem.

Theorem 2.2. For \( n \geq 0 \),

\[
\tilde{E}_{n,q}^{(\alpha,\beta)}(x) = \int_{\mathbb{Z}_p} (x+[y]_q^\alpha)^n d\mu_{-q^\beta}(y).
\]

When \( x = 0 \), \( \tilde{E}_{n,q}^{(\alpha,\beta)}(0) = \tilde{E}_{n,q} \) are \( n \)-th modified \( q \)-Euler numbers, and in the special case, \( \alpha = \beta = 1 \), \( \tilde{E}_{n,q}^{(1,1)}(0) \) is \( q \)-Euler numbers. Note that

\[
\int_{\mathbb{Z}_p} (x+[y]_q^\alpha)^n d\mu_{-q^\beta}(y) = \sum_{l=0}^{n} \binom{n}{l} x^l \int_{\mathbb{Z}_p} [y]_{q^\alpha}^{n-l} d\mu_{-q^\beta}(y)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} x^l E_{n-l,q}^{(\alpha,\beta)}.
\] (2.7)

Therefore, by (2.7), we obtain the following corollary.

Corollary 2.3. For \( n \geq 0 \), we have

\[
\tilde{E}_{n,q}^{(\alpha,\beta)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{n-l,q}^{(\alpha,\beta)} x^l = \sum_{l=0}^{n} \binom{n}{l} E_{l,q}^{(\alpha,\beta)} x^{n-l},
\]

where \( E_{n,q} \) are the \( n \)-th \( q \)-Euler numbers with weight \( \alpha \) and weak weight \( \beta \).
3. Multiple of modified $q$-Euler polynomials with weight $\alpha$ and weak weights $\beta_1, \ldots, \beta_k$

Let us consider the following multiple $q$-Euler number with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$ as follow:

$$\epsilon_{n,q}^{(\alpha,\beta_1,\ldots,\beta_k)}$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_1 + \cdots + x_k]_q^n d\mu_{-q^{\beta_1}}(x_1) \cdots d\mu_{-q^{\beta_k}}(x_k)$$

$$= \prod_{j=1}^{N_1} \cdots \prod_{k=1}^{N_k} \frac{1}{[p^{N_1}]_{q^{\beta_1}} \cdots [p^{N_k}]_{q^{\beta_k}}}$$

$$\times \sum_{j_1=0}^{p^{N_1}-1} \cdots \sum_{j_k=0}^{p^{N_k}-1} \left( \frac{1 - q^{\beta_1(j_1 + \cdots + j_k)}}{1 - q^{\alpha}} \right)^n (-q^{\beta_1})^{j_1} \cdots (-q^{\beta_k})^{j_k}$$

$$= \lim_{N_1 \to \infty} \cdots \lim_{N_k \to \infty} \frac{\prod_{j=1}^{k}[2]_{q^{\beta_j}}}{2^k (1 - q^\alpha)^n}$$

$$\times \sum_{j_1=0}^{p^{N_1}-1} \cdots \sum_{j_k=0}^{p^{N_k}-1} n \left( -1 \right)^l q^{\alpha l(j_1 + \cdots + j_k)} (-q^{\beta_1})^{j_1} \cdots (-q^{\beta_k})^{j_k}$$

$$= \lim_{N_2 \to \infty} \cdots \lim_{N_k \to \infty} \frac{\prod_{j=1}^{k}[2]_{q^{\beta_j}}}{2^k (1 - q^\alpha)^n}$$

$$\sum_{l=0}^{n} \left( -1 \right)^l \frac{2}{1 + q^{\alpha l + \beta_1}}$$

Thus, we define the multiple modified $q$-Euler polynomials with weight $\alpha$ and weak weight $\beta_1, \ldots, \beta_k$ as follow:

$$\epsilon_{n,q}^{(\alpha,\beta_1,\ldots,\beta_k)}(x)$$

$$= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x + [x_1 + \cdots + x_k]_q^n) d\mu_{-q^{\beta_1}}(x_1) \cdots d\mu_{-q^{\beta_k}}(x_k)$$

$$= \sum_{l=0}^{n} \binom{n}{l} \epsilon_{l,q}^{(\alpha,\beta_1,\ldots,\beta_k)} x^{n-l}$$

where

$$\epsilon_{n,q}^{(\alpha)}(0) = \epsilon_{n,q}^{(\alpha)} = \frac{\prod_{j=1}^{k}[2]_{q^{\beta_j}}}{(1 - q^\alpha)^n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{\alpha l + \beta_j})}.$$
Thus, by (3.2),

\[
(1 - q^\alpha)^n \epsilon_{n,q}^{(\alpha, \beta_1, \ldots, \beta_k)} = \prod_{j=1}^{k} (1 + q^{\beta_j}) \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l}{\prod_{j=1}^{k} (1 + q^{\alpha_l + \beta_j})}.
\]

Consider the equation

\[
\sum_{n=0}^{\infty} (1 - q^\alpha)^n \epsilon_{n,q}^{(\alpha, \beta_1, \ldots, \beta_k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \prod_{j=1}^{k} [2]_{q^{\beta_j}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{\prod_{j=1}^{k} (1 + q^{\alpha_l + \beta_j})} \frac{t^n}{n!}.
\]

Since

\[
e^{(1-q^\alpha)xt} \sum_{n=0}^{\infty} (1 - q^\alpha)^n \epsilon_{n,q}^{(\alpha, \beta_1, \ldots, \beta_k)} \frac{t^n}{n!} = \left( \sum_{m=0}^{\infty} (1 - q^\alpha)^m x^m \frac{t^m}{m!} \right) \left( \sum_{n=0}^{\infty} (1 - q^\alpha)^n \epsilon_{n,q}^{(\alpha, \beta_1, \ldots, \beta_k)} \frac{t^n}{n!} \right)
\]

\[
= \sum_{l=0}^{\infty} (1 - q^\alpha)^l \sum_{n=0}^{l} \binom{l}{n} \epsilon_{l,q}^{(\alpha, \beta_1, \ldots, \beta_k)} x^{l-n} \frac{t^l}{l!}
\]

\[
= \sum_{l=0}^{\infty} (1 - q^\alpha)^l \epsilon_{l,q}^{(\alpha, \beta_1, \ldots, \beta_k)} (x) \frac{t^l}{l!}
\]

(3.3)
and

\[ e^{(1-q^α)x} \left( \prod_{j=1}^{k} [2]_q^{α_j} e^{t (\sum_{l=0}^{\infty} \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{α+l}) l!} t^l)} \right) \]

\[ = \prod_{j=1}^{k} [2]_q^{α_j} e^{((1-q^α)x+1)t} \left( \sum_{l=0}^{\infty} \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{α+l}) l!} t^l \right) \]

\[ = \prod_{j=1}^{k} [2]_q^{α_j} \left( \sum_{m=0}^{\infty} ((1-q^α)x + 1)^m \frac{t^m}{m!} \right) \left( \sum_{k=0}^{\infty} \frac{1}{1 + q^{α+k}} \frac{t^k}{k!} \right) \]

\[ = \prod_{j=1}^{k} [2]_q^{α_j} \left( \sum_{m=0}^{\infty} ((1-q^α)x + 1)^m \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{α+l}) l!} t^l \right) \]

\[ = \prod_{j=1}^{k} [2]_q^{α_j} \sum_{n=1}^{\infty} \sum_{l=0}^{n-l} (((1-q^α)x + 1)^n t^n \ln^{n-l} (1 + q^{α+l}) l! \prod_{j=1}^{k} (1 + q^{α+l}) x^j. \]

By (3.3) and (3.4),

\[ (1-q^α)^n \tilde{\varepsilon}_{n,q}^{(α,β_1,\ldots,β_k)}(x) \]

\[ = \prod_{j=1}^{k} [2]_q^{α_j} \sum_{n=1}^{\infty} \sum_{l=0}^{n-l} \binom{n}{l} \left( (1-q^α)x + 1 \right)^{n-l} \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{α+l}) l!} \]

Thus, we have the following result.

**Theorem 3.1.** For \( n \geq 1 \),

\[ \tilde{\varepsilon}_{n,q}^{(α,β_1,\ldots,β_k)}(x) = \prod_{j=1}^{k} [2]_q^{α_j} \sum_{l=0}^{n} \binom{n}{l} \left( (1-q^α)x + 1 \right)^{n-l} \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{α+l}) l!} \]

\[ = \prod_{j=1}^{k} [2]_q^{α_j} \sum_{n=1}^{\infty} \sum_{l=0}^{n-l} \binom{n}{l} \left( (n-l) \frac{(-1)^l}{\prod_{j=1}^{k} (1 + q^{α+l}) l!} \right) (1-q^α)^{2-n} x^j. \]

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**References**

[1] S. Araci and M. Açıkoz, *A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials*, Adv. Stud. Contemp. Math., 22 (2012), no. 3, 399–406.

[2] L. Carlitz, *q-Bernoulli and Eulerian numbers*, Trans. Amer. Math. Soc., 76 (1954), 332-350.
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[3] D. S. Kim, N. Lee, J. Na and H. K. Pak, Identities of symmetry for higher-order Euler polynomials in three variables (I), Adv. Stud. Contemp. Math., 22 (2012), no. 1, 51-74.

[4] D. S. Kim, T. Kim, Y. H. Kim and S. H. Lee, Some arithmetic properties of Bernoulli and Euler numbers, Adv. Stud. Contemp. Math., 22 (2010), no. 4, 467-480.

[5] T. Kim, Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, Adv. Stud. Contemp. Math., 22 (2012), no. 1, 51-74.

[6] T. Kim, An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic $p$-adic invariant $q$-integrals on $\mathbb{Z}_p$, Rocky Mountain J. Math., 41 (2011), no. 1, 239-247.

[7] T. Kim, $q$-generalized Euler numbers and polynomials, Russ. J. Math. Phys., 13 (2006), no. 3, 293-298.

[8] T. Kim, $q$-Volkenborn integration, Russ. J. Math. Phys., 9 (2002), no. 3, 288-299.

[9] T. Kim, Symmetry of power sum polynomials and multivariated fermionic $p$-adic invariant integral on $\mathbb{Z}_p$, Russ. J. Math. Phys., 16 (2009), no. 1, 93-96.

[10] H. Y. Lee, N. S. Jung and C. S. Ryoo, A note on the $q$-Euler numbers and polynomials with weak weight $\alpha$ J. Appl. Math.2011, (2011), Article ID 497409, 14 pp.

[11] S. H. Rim and J. Jeong, On the modified $q$-Euler numbers of higher order with weight, Adv. Stud. Contemp. Math., 22 (2012), no. 1, 93-98.

[12] C. S. Ryoo, A note on the weighted $q$-Euler numbers and polynomials, Adv. Stud. Contemp. Math. 21 (2011), no. 1, 47-54.

[13] C. S. Ryoo, A note on the weighted $q$-Euler numbers and polynomials, Adv. Stud. Contemp. Math., 21 (2011), no. 1, 47-54.

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