Some Inequalities for Nilpotent Multipliers of Powerful \( p \)-Groups

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Abstract

In this paper we present some inequalities for the order, the exponent, and the number of generators of the \( c \)-nilpotent multiplier (the Baer invariant with respect to the variety of nilpotent groups of class at most \( c \geq 1 \)) of a powerful \( p \)-group. Our results extend some of Lubotzky and Mann’s (Journal of Algebra, 105 (1987), 484-505.) to nilpotent multipliers. Also, we give some explicit examples showing the tightness of our results and improvement some of the previous inequalities.

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1. Introduction and Motivation

Let $G$ be a group with a free presentation $F/R$. The abelian group

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]}$$

is said to be the $c$-nilpotent multiplier of $G$ (the Baer invariant of $G$, after R. Baer [1], with respect to the variety of nilpotent groups of class at most $c \geq 1$). The group $M(G) = M^{(1)}(G)$ is more known as the Schur multiplier of $G$. When $G$ is finite, $M(G)$ is isomorphic to the second cohomology group $H^2(G, C^*)$ [8].

It was conjectured for some time that the exponent of the Schur multiplier of a finite $p$-group is a divisor of the exponent of the group itself. I. D. Macdonald, J. W. Wamsley, and others [2] have constructed an example of a group of exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. In 2007 P. Moravec [15] proved that if $G$ is a group of exponent 4, then exp($M(G)$) divides 8. In 1973 Jones [7] proved that the exponent of the Schur multiplier of a finite $p$-group of class $c \geq 2$ and exponent $p^e$ is at most $p^{e(c-1)}$. A result of G. Ellis [4] shows that if $G$ is a $p$-group of class $k \geq 2$ and exponent $p^e$, then \(\text{exp}(M^{(c)}(G)) \leq p^{e[k/2]}\), where $[k/2]$ denotes the smallest integer $n$ such that $n \geq k/2$. For $c = 1$ P. Moravec [15] showed that $[k/2]$ can be replaced by $2[\log_2 k]$ which is an improvement if $k \geq 11$. Also he proved that if $G$ is a metabelian group of exponent $p$, then exp($M(G)$) divides $p$. S. Kayvanfar and M.A. Sanati [9] proved that \(\text{exp}(M(G)) \leq \text{exp}(G)\) when $G$ is a finite $p$-group of class 3, 4 or 5 under some arithmetical conditions on $p$ and the exponent of $G$. On the other hand, the authors in a joint paper [13] proved that if $G$ is a finite $p$-group of class $k$ with $p > k$, then \(\text{exp}(M^{(c)}(G))\) | $\text{exp}(G)$. In 1972 Jones [6] showed that the order of the Schur multiplier of a finite $p$-group of order $p^n$ with center of exponent $p^k$ is bounded by $p^{(n-k)(n+k-1)/2}$. In particular,
\[ |G'||M(G)| \leq p^{\frac{n(n-1)}{2}}. \] In 1973 Jones [7] gave a bound for the number of generators of the Schur multiplier of a finite \( p \)-group of class \( c \) and special rank \( r \). Recently the authors in a joint paper [13] have extended this result to the \( c \)-nilpotent multipliers. In 1987 Lubotzky and Mann [10] presented some inequalities for the Schur multiplier of a powerful \( p \)-group. They gave a bound for the order, the exponent and the number of generators of the Schur multiplier of a powerful \( p \)-group. Their results improve the previous inequalities for powerful \( p \)-groups. In this paper we will extend some results of Lubotzky and Mann [10] to the nilpotent multipliers and give some upper bounds for the order, the exponent and the number of generators of the \( c \)-nilpotent multiplier of a \( d \)-generator powerful \( p \)-group \( G \) as follows:

\[
d(M^{(c)}(G)) \leq \chi_{c+1}(d), \quad \exp(M^{(c)}(G))|\exp(G),
\]

and

\[
|M^{(c)}(G)| \leq p^{\chi_{c+1}(d)}\exp(G),
\]

where \( \chi_{c+1}(d) \) is the number of basic commutators of weight \( c+1 \) on \( d \) letters [5]. Our method is similar to that of [10]. Finally, by giving some examples of groups and computing the number of generators, the order and the exponent of their \( c \)-nilpotent multipliers explicitly, we compare these numbers with the bounds obtained and show that our results improve some of the previously mentioned inequalities.

### 2. Notation and Preliminaries

Here we will give some definitions and theorems that will be used in our work. Throughout this paper \( \mathcal{U}_i(G) \) denotes the subgroup of \( G \) generated by all \( p^i \)'th powers, \( P_i(G) \) is defined by: \( P_1(G) = G \), and \( P_{i+1}(G) = [P_i(G), G]\mathcal{U}_1(P_i(G)) \). Finally \( d(G), cl(G), l(G), sr(G) \) denote respectively, the minimal number of generators, the nilpotency class, the derived length.
and the special rank of $G$, while $e(G)$ is defined by $\exp(G) = p^{e(G)}$.

**Theorem 2.1** (M. Hall [5]). Let $F$ be a free group on $\{x_1, x_2, ..., x_d\}$. Then for all $1 \leq i \leq n$,

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)}$$

is a free abelian group freely generated by the basic commutators of weights $n, n+1, ..., n+i-1$ on the letters $\{x_1, x_2, ..., x_d\}$ (for a definition of basic commutators see [5]).

**Lemma 2.2** (R. R. Struik [16]). Let $\alpha$ be a fixed integer and $G$ be a nilpotent group of class at most $n$. If $b_j \in G$ and $r < n$, then

$$[b_1, ..., b_{i-1}, b_i^\alpha, b_{i+1}, ..., b_r] = [b_1, ..., b_r]^{\alpha f_1(\alpha)} c_2^{f_2(\alpha)} ..., $$

where the $c_k$ are commutators in $b_1, ..., b_r$ of weight strictly greater than $r$, and every $b_j$, $1 \leq j \leq r$, appears in each commutator $c_k$, the $c_k$ listed in ascending order. The $f_i$ are of the following form:

$$f_i(n) = a_1 \binom{n}{1} + a_2 \binom{n}{2} + ... + a_{w_i} \binom{n}{w_i},$$

with $a_j \in \mathbb{Z}$, and $w_i$ is the weight of $c_i$ (in the $b_i$) minus $(r - 1)$.

Powerful $p$-groups were formally introduced in [10]. They have played a role in the proofs of many important results in $p$-groups. We will discuss some of them in this section. A $p$-group $G$ is called powerful if $p$ is odd and $G' \leq \mathcal{U}_1(G)$ or $p = 2$, and $G' \leq \mathcal{U}_2(G)$. There is a related notion that is often used to find properties of powerful $p$-groups. If $G$ is a $p$-group and $H \leq G$, then $H$ is said to be powerfully embedded in $G$ if $[G, H] \leq \mathcal{U}_1(H)$ ($[G, H] \leq \mathcal{U}_2(H)$ for $p = 2$). Any powerfully embedded subgroup is itself a powerful $p$-group and must be normal in the whole group. Also a $p$-group is powerful exactly when it is powerfully embedded in itself. While it is obvious that factor groups and direct products of powerful $p$-groups are powerful, this property is not subgroup-inherited [10].
We will require some standard properties of powerful $p$-groups. For the sake of convenience we collect them here.

**Theorem 2.3** ([10]). The following statements hold for a powerful $p$-group $G$.

i) $\gamma_i(G), G^i, \Phi(G)$ are powerfully embedded in $G$.

ii) $P_{i+1}(G) = \mathcal{U}_i(G)$ and $\Phi(\mathcal{U}_i(G)) = \mathcal{U}_{i+1}(G)$.

iii) Each element of $\mathcal{U}_i(G)$ can be written as $a^{p^i}$, for some $a \in G$ and hence $\mathcal{U}_i(G) = \{ g^{p^i} : g \in G \}$.

iv) If $G = \langle a_1, a_2, ..., a_d \rangle$, then $\mathcal{U}_i(G) = \langle a_1^{p^i}, a_2^{p^i}, ..., a_d^{p^i} \rangle$.

v) If $H \subseteq G$, then $d(H) \leq d(G)$.

**Proposition 2.4** ([10]). Let $N$ be a powerfully embedded subgroup of $G$. If $N$ is the normal closure of some subset of $G$, then $N$ is actually generated by this subset.

**Lemma 2.5.** Let $H, K$ be normal subgroups of $G$ and $H \leq K[H, G]$. Then $H \leq K[H, l G]$ for any $l \geq 1$. In particular, if $G$ is nilpotent, then $H \leq K$.

*Proof.* An easy exercise. $\square$

**Lemma 2.6.** Let $G$ be a finite $p$-group and $N \leq G$. Then $N$ is powerfully embedded in $G$ if and only if $N/[N, G, G]$ is powerfully embedded in $G/[N, G, G]$.

*Proof.* See a remark in the proof of Theorem 1.1 in [10]. $\square$

**Remark 2.7.** To prove that a normal subgroup $N$ is powerfully embedded in $G$ we can assume that

i) $[N, G, G] = 1$ by the above lemma.

ii) $\mathcal{U}_1(N) = 1$ ( $\mathcal{U}_2(N) = 1$ for $p = 2$ ) and try to show that $[N, G] = 1$.

iii) $[N, G]^2 = 1$ whenever $p = 2$, since if we assume that $N/[N, G]^2$ is powerfully embedded in $G/[N, G]^2$, then $N$ is powerfully embedded in $G$. This follows from the proof of Theorem 4.1.1 in [10].
3. Main Results

In order to prove the main results we need the following theorem.

**Theorem 3.1.** Let $F/R$ be a free presentation of a powerful $d$-generator $p$-group $G$. Let $Z = R/[R, cF]$ and $H = F/[R, cF]$, so that $G \cong H/Z$. Then $\gamma_{c+1}(H)$ is powerfully embedded in $H$ and $d(\gamma_{c+1}(H)) \leq \chi_{c+1}(d)$.

*Proof.* First let $p$ an odd prime. We may assume that $0 \leq (\gamma_{c+1}(H)) = 1$ and try to show that $[\gamma_{c+1}(H), H] = 1$ by Remark 2.7(ii). Also we may assume that $\gamma_{c+3}(H) = 1$ by Remark 2.7(i). Let $a, b_1, b_2, ..., b_c \in H$. Then by Lemma 2.2,

$$[a^p, b_1, ..., b_c] = [a, b_1, ..., b_c]^p c_1^{f_1(p)} c_2^{f_2(p)} ... .$$

Since $\gamma_{c+3}(H) = 1$ and $\mathcal{U}_1(\gamma_{c+1}(H)) = 1$ we have $[a, b_1, ..., b_c]^p = 1$, $c_1^{f_1(p)} = 1$, for all $i \geq 2$. Also $p > 2$ implies that $p|f_1(p)$, and hence $c_1^{f_1(p)} = 1$ so $a^p \in Z_c(H)$ and $\mathcal{U}_1(H) \subseteq Z_c(H)$. The powerfulness of $G$ yields $H' \leq \mathcal{U}_1(H)Z \leq Z_c(H)$. Therefore $[H', cH] = 1$, as desired. Since $H/Z$ is generated by $d$ elements and $Z \leq Z_c(H)$, $\gamma_{c+1}(H)$ is the normal closure of the commutators of weight $c + 1$ on $d$ elements. Hence Proposition 2.4 completes the proof, for $p > 2$.

If $p = 2$, then the proof is similar, so we leave out the details, but note that in this case

$$[a^4, b_1, ..., b_c] = [a, b_1, ..., b_c]^4 c_1^{f_1(4)} c_2^{f_2(4)} ... .$$

By Remark 2.7 we can assume $\gamma_{c+3}(H) = \mathcal{U}_2(\gamma_{c+1}(H)) = ([\gamma_{c+1}(H), H])^2 = 1$. Hence we have $[a^4, b_1, ..., b_c] = 1$ ($c_1^{f_1(4)} = 1$, since $2|f_1(4)$) so $\mathcal{U}_2(H) \subseteq Z_c(H)$. \(\Box\)

An interesting corollary of this theorem is as follows.

**Corollary 3.2.** Let $G$ be powerful $p$-group with $d(G) = d$. Then $d(M^{(c)}(G)) \leq \chi_{c+1}(d)$. 6
Proof. Let $F/R$ be a free presentation of $G$ with $Z = R/[R, cF]$, so that $G \cong H/Z$, where $H = F/[R, cF]$. Then the above result and Theorem 2.3(v) implies that
\[
d\left(\frac{R \cap \gamma_{c+1}(F)}{[R, cF]}\right) \leq d(\gamma_{c+1}(F)) \leq \chi_{c+1}(d).
\]
Hence the result follows.

Note that by a similar method we can prove Corollary 2.2 of [10] without using the concept of covering group for $G$.

The authors in a joint paper [12] have proved that if $G$ is a finite d-generator $p$-group of special rank $r$ and nilpotency class $t$, then
\[
ed(M^{(c)}(G)) \leq \chi_{c+1}(d) + r^{c+1}(t-1).\]
Clearly Corollary 3.2 improves this bound for nonabelian powerful $p$-groups.

**Theorem 3.3.** Let $G$ be powerful $p$-group. Then $e(M^{(c)}(G)) \leq e(G)$.

*Proof.* Let $p > 2$ and $F/R$ be a free presentation of $G$ with $Z = R/[R, cF]$ and $H = F/[R, cF]$, so that $G \cong H/Z$. Since $e(R \cap \gamma_{c+1}(F)/[R, cF]) \leq e(\gamma_{c+1}(H))$ and $e(H/Z_c(H)) \leq e(G)$ it is enough to show that $e(\gamma_{c+1}(H)) = e(H/Z_c(H))$.

We will establish by induction on $k$ the equality
\[
\mathcal{U}_k(\gamma_{c+1}(H)) = [\mathcal{U}_k(H), cH],
\]
which implies the above claim.

If $k = 0$, then (1) holds. Now Assume that (1) holds for some $k$. Since $\gamma_{c+1}(H)$ is powerfully embedded in $H$ by Theorem 3.1, we have $\mathcal{U}_{k+1}(\gamma_{c+1}(H)) = \mathcal{U}_1(\mathcal{U}_k(\gamma_{c+1}(H)))$, by Theorem 2.3(ii). Similarly $\mathcal{U}_{k+1}(G) = \mathcal{U}_1(\mathcal{U}_k(G))$.

Since $G \cong H/Z$ we have $\mathcal{U}_{k+1}(H)Z/Z = \mathcal{U}_1(\mathcal{U}_k(H)Z)Z/Z$. Therefore
\[
[\mathcal{U}_{k+1}(H), cH] = [\mathcal{U}_{k+1}(H)Z, cH] = [\mathcal{U}_1(\mathcal{U}_k(H)Z)Z, cH]
\]
\[
= [\mathcal{U}_1(\mathcal{U}_k(H)Z), cH].
\]
This implies that
\[
[\mathcal{U}_{k+1}(H), cH] = [\mathcal{U}_1(\mathcal{U}_k(H)Z), cH].
\]
Thus (1) for \( k + 1 \) is equivalent to \( \mathcal{U}_1(\mathcal{U}_k(\gamma_{c+1}(H))) = [\mathcal{U}_1(\mathcal{U}_k(H)Z), cH] \).

Since \( \mathcal{U}_k(\gamma_{c+1}(H)) \) is powerfully embedded in \( H \) by Theorem 2.3(i), this implies, by (1) and Lemma 2.2,

\[
[\mathcal{U}_1(\mathcal{U}_k(H)Z), cH] \leq \mathcal{U}_1([\mathcal{U}_1(\mathcal{U}_k(H)Z), cH]] [\mathcal{U}_k(H)Z, cH, H] \\
\leq \mathcal{U}_1(\mathcal{U}_k(H)Z, H) [\mathcal{U}_k(H), cH, H] \\
\leq \mathcal{U}_1(\mathcal{U}_k(\gamma_{c+1}(H))) [\mathcal{U}_k(\gamma_{c+1}(H)), H] \\
\leq \mathcal{U}_1(\mathcal{U}_k(\gamma_{c+1}(H))).
\]

For the reverse inclusion note that since \( \mathcal{U}_1(\mathcal{U}_k(\gamma_{c+1}(H))) = [\mathcal{U}_1(\mathcal{U}_k(H), cH)] \) it is enough to show that

\[
\mathcal{U}_1([\mathcal{U}_1(\mathcal{U}_k(H), cH)] \equiv 1 \pmod{[\mathcal{U}_1(\mathcal{U}_k(H)Z), cH]}.
\]

By Theorem 2.3(i), \( \mathcal{U}_k(H/Z) \) is powerfully embedded in \( H/Z \) so that

\[
\left[ \frac{\mathcal{U}_k(H)Z}{Z}, H \right] \leq \mathcal{U}_1(\mathcal{U}_k(H)Z)Z \tag{3}
\]

Also (2) implies that \( \mathcal{U}_1(\mathcal{U}_k(H)Z) \leq Z_c(H) \pmod{[\mathcal{U}_{k+1}(H), cH]} \). Now (2), (3) and the last inequality imply that

\[
[\mathcal{U}_k(H)Z, H] \leq \mathcal{U}_1(\mathcal{U}_k(H)Z)Z \leq Z_c(H) \pmod{[\mathcal{U}_{k+1}(H), cH]}.
\]

Hence by Lemma 2.2

\[
\mathcal{U}_1([\mathcal{U}_k(H), cH]) \equiv \mathcal{U}_1([\mathcal{U}_k(H)Z, cH]) \\
\equiv [\mathcal{U}_1(\mathcal{U}_k(H)Z), cH] \\
\equiv 1 \pmod{[\mathcal{U}_1(\mathcal{U}_k(H)Z), cH]},
\]

as desired.

If \( p = 2 \), then the proof is similar to the previous case. This completes the proof. \( \square \)
Note that G. Ellis [4], using the nonabelian tensor products of groups, showed that \( \exp(M^{(c)}(G)) \) divides \( \exp(G) \) for all \( c \geq 1 \) and all \( p \)-groups satisfying \( [(G^{p^{i-1}}, G)] \subseteq G^{p^i} \) for \( 1 \leq i \leq e \), where \( \exp(G) = p^e \). Note that the results of [10] imply that every powerful \( p \)-group \( G \) satisfies the latter commutator condition.

Lubotzky and Mann [10] found bounds for \( cl(G), l(G), |G| \) and \( |M(G)| \) of a powerful \( d \)-generator \( p \)-group \( G \) of exponent \( p^e \) as follows:

\[
cl(G) \leq e, \quad l(G) \leq \log_2 e + 1, \quad |G| \leq p^{de} \quad \text{and} \quad |M(G)| \leq p^{(d(d-1)/2)e}.
\]

In the following proposition we find an upper bound for the order of \( c \)-nilpotent multiplier of \( G \).

**Proposition 3.4.** Let \( G \) be a powerful \( p \)-group, with \( d(G) = d \) and \( e(G) = e \). Then \( |M^{(c)}(G)| \leq p^{c(d+1)e} \).

**Proof.** It is obtained by combining Corollary 3.2 and Theorem 3.3. \( \square \)

### 4. Some Examples

In this final section we are going to give some explicit examples of \( p \)-groups and calculate their \( c \)-nilpotent multipliers in order to compare our new bounds with the exact values. This will show tightness of our results and improvement some of the previously mentioned inequalities.

**Example 4.1.** Let \( G \) be a finite abelian \( p \)-group. Clearly \( G \) is a powerful \( p \)-group and by the fundamental theorem of finitely generated abelian groups \( G \) has the following structure

\[
G \cong Z_{p^{\alpha_1}} \oplus Z_{p^{\alpha_2}} \oplus \ldots \oplus Z_{p^{\alpha_d}}
\]

for some positive integers \( \alpha_1, \alpha_2, \ldots, \alpha_d \), where \( \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_d \). By [11] the \( c \)-nilpotent multiplier of \( G \) can be calculated explicitly as follows:

\[
M^{(c)}(G) \cong Z_{p^{\alpha_2}} \oplus Z_{p^{\alpha_3}} \oplus \ldots \oplus Z_{p^{\alpha_d}}
\]
where \( b_i = \chi_{c+1}(i) \) and \( Z_n^{(m)} \) denotes the direct sum of \( m \) copies of the cyclic group \( Z_n \). Now it is easy to see that

(i) \( d(M^{(c)}(G)) = \chi_{c+1}(d) \), where \( d = d(G) \). Hence the bound of Corollary 3.2 is attained and the best one in the abelian case.

(ii) \( e(M^{(c)}(G)) = \alpha_2 \), whereas \( e(G) = \alpha_1 \). Hence the bound of Theorem 3.3 is attained when \( \alpha_1 = \alpha_2 \) and it is the best one in the abelian case.

(iii) \( |M^{(c)}(G)| = p^{\alpha_2 b_2 + \sum_{i=3}^d \alpha_i (b_i - b_{i-1})} \leq p^{\alpha_1 \chi_{c+1}(d)} \). Hence the bound of Proposition 3.4 is attained if and only if \( \alpha_1 = \alpha_2 = \ldots = \alpha_d \).

**Example 4.2.** Let \( p \) be any odd prime number and \( s, t \) be positive integers with \( s \geq t \). Consider the following finite \( d \)-generator \( p \)-group with nilpotency class 2:

\[
P_{s,t} = \langle y_1, \ldots, y_d : y_i^{p^s} = [y_j, y_k]^{p^t} = [[y_j, y_k], y_i] = 1, 1 \leq i, j, k \leq d, j \neq k \rangle.
\]

One can see that \( P_{s,t} \) is not a powerful \( p \)-group (clearly \( \mathcal{U}_1(P_{1,1}) = 1 \)). By [14] the \( c \)-nilpotent multiplier of \( P_{s,t} \) is as follows:

\[
M^{(c)}(P_{s,t}) \cong Z_{p^s}^{(\chi_{c+1}(d))} \oplus Z_{p^t}^{(\chi_{c+2}(d))}.
\]

Therefore we have

(i) \( d(M^{(c)}(P_{s,t})) = \chi_{c+1}(d) + \chi_{c+2}(d) > \chi_{c+1}(d) \). Hence the condition of being powerful cannot be omitted from Corollary 3.2.

(ii) \( |M^{(c)}(P_{s,t})| = p^{s \chi_{c+1}(d) + t \chi_{c+2}(d)} > p^{s \chi_{c+1}(d)} \). Hence powerfulness is also a necessary condition for the bound of Proposition 3.4. Note that here we have \( e(M^{(c)}(P_{s,t})) = se(P_{s,t}) \).

The authors in a joint paper [13] have proved that \( \exp(M^{(c)}(G)) \mid \exp(G) \), when \( G \) is a nilpotent \( p \)-group of class \( k \), and \( k < p \). In the following example we find a powerful \( p \)-group of class \( k \geq p \) such that \( \exp(M^{(c)}(G)) \) divides \( \exp(G) \).

**Example 4.3** ([17]). We work in \( \text{GL}(\mathbb{Z}_{p^{2}}) \), the \( 2 \times 2 \) invertible matrices
over the ring of integers modulo $p^{l+2}$. In this ring any integer not divisible by $p$ is invertible. Consider the matrices

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 - p \end{bmatrix}, \quad Y = \begin{bmatrix} 1/(1-p) & p/(1-p) \\ 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix}.$$ 

One quickly calculates that $[X, Y] = Z^p$, $[X, Z] = Z^p$, $[Y, Z] = Z^p$ and

$$[Z^p, kX] = \begin{bmatrix} 1 & (-1)^{k+2}p^{k+2} \\ 0 & 1 \end{bmatrix}.$$  \hspace{1cm} (4)

Notice also that $X^{p^{l+1}} = Y^{p^{l+1}} = Z^{p^{l+1}} = 1$. We claim that $P = \langle X, Y, Z \rangle$ is a powerful $p$-group. By the above relations we can express every word in $P$ as a product $X^a Y^b Z^c$ for some $0 \leq a, b, c < p^{l+1}$. Also

$$X^a Y^b Z^c = \begin{bmatrix} 1 & \frac{1+pe-(1-p)^b}{(1-p)^a} \\ 0 & (1-p)^a \end{bmatrix}$$

and hence all of these elements are distinct. Therefore the order of $P$ is $p^{3(l+1)}$ and hence $P$ is a $p$-group and the relations imply that $P' \leq U_1(P)$. Therefore $P$ is a powerful $p$-group. The exponent of $P$ is $p^{l+1}$, and (4) implies that $P$ has nilpotency class $l + 1$. By Theorem 3.3 $\exp(M^{(c)}(P))$ divides $\exp(P)$. Note that the nilpotency class of $P$ is $l + 1$ which is greater than or equal to $p$.

Let $G$ be a finite $d$-generator $p$-group of order $p^n$ where $p$ is any prime. By [11] we have

$$p^{\chi_{c+1}(d)} \leq |M^{(c)}(G)||\gamma_{c+1}(G)| \leq p^{\chi_{c+1}(n)}.$$ 

Now if we put $l = 2$ in the above example, then $P$ is 3-generator powerful $p$-group of order $p^9$ with nilpotency class 3. Thus by the above bounds we have

$$p^{18} = p^{\chi_{4}(3)} \leq |M^{(3)}(P)||\gamma_{4}(P)| = |M^{(3)}(P)| \leq p^{\chi_{4}(9)} = p^{1620}.$$
But by Proposition 3.4 \(|M^{(3)}(P)| \leq p^{3x^3(3)} = p^{54}\). Hence this example and also Example 4.1 show that Proposition 3.4 improves the above bound for powerful \(p\)-groups.

**Example 4.4.** Using the list of nonabelian groups of order at most 30 with their \(c\)-nilpotent multipliers for \(c = 1, 2\) in the table of Fig.2 in [3], we are going to give two nonabelian powerful \(p\)-groups in order to compute explicitly the number of generators, the order and the exponent of their 2-nilpotent multipliers and then compare these numbers with bounds obtained.

(i) Consider the finite 2-group \(G = \langle a, b : a^2 = 1, aba = b^{-3} \rangle\). It is easy to see that \(G\) is a powerful 2-group and \(|G| = 16, d(G) = 2, \exp(G) = 8\). By \([3, \text{Fig.2}, \# 13]\) \(M^{(2)}(G) \cong \mathbb{Z}_2(2)\) and hence \(|M^{(2)}(G)| = 4, d(M^{(2)}(G)) = 2, \exp(M^{(2)}(G)) = 2\). It is seen that the bound of Corollary 3.2 is attained.

(ii) Consider the finite 3-group \(G = \langle a, b : a^3 = 1, a^{-1}ba = b^{-2} \rangle\). It is easy to see that \(G\) is a powerful 3-group and \(|G| = 27, d(G) = 2, \exp(G) = 9\). By \([3, \text{Fig.2}, \# 40]\) \(M^{(2)}(G) \cong \mathbb{Z}_3(2)\) and hence \(|M^{(2)}(G)| = 9, d(M^{(2)}(G)) = 2, \exp(M^{(2)}(G)) = 3\). It is also seen that the bound of Corollary 3.2 is attained.

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