Extended Reaction Rate Integral as Solutions of Some General Differential Equations

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Abstract. Here an extended form of the reaction rate probability integral, in the case of nonresonant thermonuclear reactions with the depleted tail and the right tail cut off, is considered. The reaction rate integral then can be looked upon as the inverse of the convolution of the Mellin transforms of Tsallis type statistics of nonextensive statistical mechanics and stretched exponential as well as that of superstatistics and stretched exponentials. The differential equations satisfied by the extended probability integrals are derived. The idea used is a novel one of evaluating the extended integrals in terms of some special functions and then by invoking the differential equations satisfied by these special functions. Some special cases of limiting situations are also discussed.

Keywords: Reaction rate probability integrals, extended probability integrals, pathway model, H-function, G-function, differential equations.

1 Introduction

Nuclear reactions govern major aspects of the chemical evolution of the universe. A proper understanding of the nuclear reactions that are going on in hot cosmic plasma, and those in the laboratories as well, requires a sound theory of nuclear-reaction dynamics. The reaction probability integral is the probability per unit time that two particles, confined to a unit volume, will react with each other (Haubold and Mathai, 1998). In the present article we will show that one can obtain the reaction rate probability integral as a solution of a certain differential equation, the technique used is a novel one. First we evaluate the integral and represent it as a special function (Mathai, 1993; Luke, 1969). Then we can invoke the differential equation satisfied by this special function, thereby establishing the differential equation for the reaction rate integral. A study of this integral as a solution of a differential equation is made because the behavior of physical systems is usually studied with the help of differential equations and hence the differential equations derived in the present paper may become useful one day.

Here we will consider the integrals of the following form:

Nonresonant case with depleted tail: (for details of the integrals see Mathai and Haubold, 1988)

\[ I_1^{(\delta)} = \int_0^\infty x^{\alpha-1} e^{-ax^d-bx^{-\beta}} \, dx, \quad a > 0, \ b > 0, \ \delta > 0, \ \rho > 0. \tag{1} \]

Nonresonant case with depleted tail and high energy cut-off:

\[ I_2^{(\delta)} = \int_0^d x^{\alpha-1} e^{-ax^d-bx^{-\beta}} \, dx, \quad a > 0, \ b > 0, \ \delta > 0, \ \rho > 0, \ d < \infty. \tag{2} \]
Do these integrals, as functions of $b$, satisfy some differential equations? No such differential equations can be easily seen from the integrals. We will show that these integrals will satisfy certain differential equations. The main results will be stated as follows. We will establish the following theorems for $I_1^{(\delta)}$ and extended $I_{1\beta}^{(\delta)}$ and $I_{2\beta}^{(\delta)}$.

**Theorem 1**

A constant multiple of the reaction rate integral in (1), under the condition $\frac{1}{\rho} = m, m = 1, 2, \cdots$, can be obtained as a solution of the differential equation,

$$\left[(-1)^{m+1} z - \eta \left(\frac{1}{m} \right) \cdots \left(\frac{m-1}{m} \right) \left(\eta - \frac{1}{m} \right) \left(\eta - \frac{\alpha}{m \rho}\right) \right] f(z) = 0,$$

where, $z = \frac{ab^m}{m^m}, \eta = z \frac{d}{dz}, f(z) = \frac{a^m m^m}{\alpha \beta} I_1^{(m \rho)}(z)$ and

$$I_1^{(m \rho)}(z) = \frac{(2\pi)^{\frac{1-m}{2}}}{\rho a^m m^m} G_{0, m+1}^0 \left[ \frac{ab^m}{m^m} \right], a > 0, b > 0, \alpha > 0, \rho > 0. \tag{4}$$

**Corollary 1**

When $\delta = 1$, a constant multiple of the reaction rate integral in (1), under the condition $\frac{1}{\rho} = m, m = 1, 2, \cdots$, can be obtained as a solution of the differential equation,

$$\left[(-1)^{m+1} z - \eta \left(\frac{1}{m} \right) \cdots \left(\frac{m-1}{m} \right) \left(\eta - \frac{1}{m} \right) \left(\eta - \alpha\right) \right] f(z) = 0,$$

where, $z = \frac{ab^m}{m^m}, \eta = z \frac{d}{dz}, f(z) = \frac{a^m m^m}{\alpha \beta} I_1(z)$ and

$$I_1(z) = \frac{(2\pi)^{\frac{1-m}{2}}}{\rho a^m m^m} G_{0, m+1}^0 \left[ \frac{ab^m}{m^m} \right], a > 0, b > 0, \alpha > 0, \rho > 0. \tag{6}$$

The proof of the theorem will be given later.

Now we will consider a wider class of integrals as extended forms of (1) and (2). Consider the integrals

$$I_1^{(\delta)} = \int_0^\infty x^{\alpha-1} \left[1 + a(\beta - 1)x^\delta \right]^{-\frac{1}{\beta}} e^{-bx^-\rho} dx, a > 0, b > 0, \delta > 0, \rho > 0, \beta > 1, \tag{7}$$

$$I_2^{(\delta)} = \int_0^d x^{\alpha-1} \left[1 - a(1-\beta)x^\delta \right]^{-\frac{1}{\beta}} e^{-bx^-\rho} dx, a > 0, b > 0, \delta > 0, \rho > 0, \beta < 1. \tag{8}$$

These are obtained by replacing $e^{-ax^\delta}$ by $[1-a(1-\beta)x^\delta]^{-\frac{1}{\beta}}$ so that when $\beta \to 1$ we have the integrals in (1) and (2). Thus $I_1^{(\delta)}$ and $I_2^{(\delta)}$ are the extended families of integrals in $I_1^{(\delta)}$ and $I_2^{(\delta)}$ respectively. Here we obtain the following main results for the extended forms of the reaction probability integrals.

**Theorem 2**

A constant multiple of the extended reaction rate integral in (7), under the condition $\frac{1}{\rho} = m, m = 1, 2, \cdots$, can be obtained as a solution of the differential equation,

$$\left[(-1)^{m+1} z \left(\eta + \frac{1}{\beta - 1} \right) - \eta \left(\frac{1}{m} \right) \cdots \left(\frac{m-1}{m} \right) \left(\eta - \frac{1}{m} \right) \left(\eta - \frac{\alpha}{m \rho}\right) \right] f(z) = 0, \tag{9}$$
where, \( z = \frac{a(\beta - 1)b^m}{m^m} \), \( \eta = \frac{d}{dz} \), \( f(z) = \frac{\rho[a(\beta - 1)]^{\frac{\beta - m}{m}} \Gamma \left( \frac{1}{\beta} \right) m^\frac{m}{2} I_{1\beta}^{m\rho}(z) \) and

\[
I_{1\beta}^{m\rho}(z) = \frac{(2\pi)^{\frac{1-m}{2}}}{\rho[a(\beta - 1)]^{\frac{\beta - m}{m}} \Gamma \left( \frac{1}{\beta} \right) m^\frac{m}{2} G_{1,m+1}^m \left[ \frac{a(\beta - 1)b^m}{m^m} \frac{\frac{\beta - m}{m} + \alpha}{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, \alpha} \right],
\]

(10)

\( a > 0, \ b > 0, \ \alpha > 0, \ \rho > 0, \ \beta > 1. \)

**Corollary 2**

When \( \delta = 1 \), a constant multiple of the extended reaction rate integral in (7), under the condition \( \frac{1}{\rho} = m, m = 1, 2, \ldots \), can be obtained as a solution of the differential equation,

\[
\left[ (-1)^{m+1} z(\eta + \frac{1}{\beta - 1} - \alpha) - \eta(\eta - \frac{1}{m}) \cdots (\eta - \frac{m-1}{m}) (\eta - \alpha) \right] f(z) = 0,
\]

(11)

where, \( z = \frac{a(\beta - 1)b^m}{m^m} \), \( \eta = \frac{d}{dz} \), \( f(z) = \frac{\rho[a(\beta - 1)]^{\frac{\beta - m}{m}} \Gamma \left( \frac{1}{\beta} \right) m^\frac{m}{2} I_{1\beta}(z) \) and

\[
I_{1\beta}(z) = \frac{(2\pi)^{\frac{1-m}{2}}}{\rho[a(\beta - 1)]^{\frac{\beta - m}{m}} \Gamma \left( \frac{1}{\beta} \right) m^\frac{m}{2} G_{1,m+1}^m \left[ \frac{a(\beta - 1)b^m}{m^m} \frac{\frac{\beta - m}{m} + \alpha}{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, \alpha} \right],
\]

(12)

\( a > 0, \ b > 0, \ \alpha > 0, \ \rho > 0, \ \beta > 1. \)

**Theorem 3**

A constant multiple of the extended reaction rate integral in (8), under the condition \( \frac{1}{\rho} = m, m = 1, 2, \ldots \), can be obtained as a solution of the differential equation,

\[
\left[ (-1)^m z(\eta - \frac{1}{1 - \beta} - \frac{\alpha}{m\rho}) - \eta(\eta - \frac{1}{m}) \cdots (\eta - \frac{m-1}{m}) (\eta - \frac{\alpha}{m\rho}) \right] f(z) = 0,
\]

(13)

where, \( z = \frac{a(1-\beta)b^m}{m^m} \), \( \eta = \frac{d}{dz} \), \( f(z) = \frac{\rho[a(1-\beta)]^{\frac{1-m}{m}} m^\frac{m}{2} I_{2\beta}^{m\rho}(z) \) and

\[
I_{2\beta}^{m\rho}(z) = \frac{\Gamma \left( \frac{2-\beta}{2} \right) (2\pi)^{\frac{1-m}{2}}}{\rho[a(1-\beta)]^{\frac{1-m}{m}} m^\frac{m}{2} G_{1,m+1}^{1,0-m} \left[ \frac{a(1-\beta)b^m}{m^m} \frac{\frac{2-\beta}{2} + \alpha}{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, \alpha} \right],
\]

(14)

\( a > 0, \ b > 0, \ \alpha > 0, \ \rho > 0, \ \beta < 1. \)

**Corollary 3**

When \( \delta = 1 \) a constant multiple of the extended reaction rate integral in (8), under the condition \( \frac{1}{\rho} = m, m = 1, 2, \ldots \), can be obtained as a solution of the differential equation,

\[
\left[ (-1)^m z(\eta - \frac{1}{1 - \beta} - \alpha) - \eta(\eta - \frac{1}{m}) \cdots (\eta - \frac{m-1}{m}) (\eta - \alpha) \right] f(z) = 0,
\]

(15)

where, \( z = \frac{a(1-\beta)b^m}{m^m} \), \( \eta = \frac{d}{dz} \), \( f(z) = \frac{\rho[a(1-\beta)]^{\frac{1-m}{m}} m^\frac{m}{2} I_{2\beta}(z) \) and

\[
I_{2\beta}(z) = \frac{\Gamma \left( \frac{2-\beta}{2} \right) (2\pi)^{\frac{1-m}{2}}}{\rho[a(1-\beta)]^{\frac{1-m}{m}} m^\frac{m}{2} G_{1,m+1}^{1,0-m} \left[ \frac{a(1-\beta)b^m}{m^m} \frac{\frac{2-\beta}{2} + \alpha}{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, \alpha} \right],
\]

(16)
The proofs of the above theorems will be given after evaluating the integrals first.

2 Evaluation of the integral $I_1^{(\delta)}$

The following procedure is available in Mathai and Haubold (1988). For the sake of completeness a brief outline of the derivation will be given here.

$$I_1^{(\delta)} = \int_0^\infty x^{\alpha-1}e^{-ax^\delta - bx^{\delta-\rho}}dx, \quad a > 0, \quad b > 0, \quad \delta > 0, \quad \rho > 0.$$  

Here the integrand can be taken as a product of positive integrable functions and then we can apply statistical distribution theory to evaluate this integral. Let $x_1$ and $x_2$ be real scalar independent random variables having densities

$$f_1(x_1) = \begin{cases} \ c_1 x_1^{\alpha-1}e^{-ax_1^\delta}, & 0 < x_1 < \infty, \quad a > 0, \quad \delta > 0 \\ 0, & \text{elsewhere} \end{cases} \quad (17)$$

and

$$f_2(x_2) = \begin{cases} \ c_2 e^{-x_2^{\rho}}, & 0 < x_2 < \infty, \quad \rho > 0 \\ 0, & \text{elsewhere} \end{cases} \quad (18)$$

where $c_1$ and $c_2$ are normalizing constants. Let us transform $x_1$ and $x_2$ to $u = x_1 x_2$ and $v = x_1$. Then the marginal density of $u$ is given by

$$g_1(u) = \int_0^\infty \frac{1}{u} f_1(v) f_2 \left( \frac{u}{v} \right) dv$$

$$= c_1 c_2 \int_0^\infty v^{\alpha-1}e^{-av^\delta - bv^{\delta-\rho}}dv, \quad \text{where} \quad b = u^\rho, \quad \delta > 0, \quad \rho > 0.$$

Let us evaluate the density through expected values or moments.

$$E(u^{s-1}) = E(x_1^{s-1}) E(x_2^{s-1})$$  

due to statistical independence of $x_1$ and $x_2$.

$$E(x_1^{s-1}) = c_1 \int_0^\infty x_1^{s-1}x_1^{\alpha-1}e^{-ax_1^\delta}dx_1.$$  

Putting $y = ax_1^\delta$ and evaluating the integral as a gamma integral, one has,

$$E(x_1^{s-1}) = \frac{c_1}{\delta a^{\frac{\alpha}{\delta} + \frac{s}{\delta}}} \Gamma \left( \frac{\alpha}{\delta} + \frac{s}{\delta} \right), \quad \Re(\alpha + s) > 0,$$  

where $\Re(.)$ denotes the real part of $(.)$.

$$E(x_2^{s-1}) = c_2 \int_0^\infty x_2^{s-1}e^{-x_2^{\rho}}dx_2.$$  

Putting $y = x_2^{\rho}$, we get

$$E(x_2^{s-1}) = \frac{c_2}{\rho} \Gamma \left( \frac{s}{\rho} \right), \quad \Re(s) > 0.$$

From (22) and (23)

$$E(u^{s-1}) = \frac{c_1 c_2}{\delta a^{\frac{\alpha}{\delta} + \frac{s}{\delta}}} \Gamma \left( \frac{\alpha}{\delta} + \frac{s}{\delta} \right) \Gamma \left( \frac{s}{\rho} \right), \quad \Re(s) > 0, \quad \Re(\alpha + s) > 0.$$  

From (22) and (23)
Looking at the $(s - 1)^{th}$ moment as the Mellin transform of the corresponding density and then taking the inverse Mellin transform we get the density of $u,$

$$g_1(u) = \frac{c_1c_2}{\delta \rho a^{\frac{\delta}{\rho}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{\alpha}{\delta} + \frac{s}{\delta}\right)\left(ab^\frac{\rho}{\delta}\right)^{-s} ds. \quad (25)$$

Comparing (20) and (25)

$$I_1^{(\delta)} = \int_0^\infty x^{a-1}e^{-ax-bx^{-\rho}} dx = \frac{1}{\delta \rho a^{\frac{\delta}{\rho}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{s}{\rho}\right)\Gamma\left(\frac{\alpha}{\delta} + \frac{s}{\delta}\right)\left(ab^\frac{\rho}{\delta}\right)^{-s} ds. \quad (26)$$

This contour integral can be written as an $H$-function (Mathai and Saxena, 1978). That is,

$$I_1^{(\delta)} = \frac{1}{\delta \rho a^{\frac{\delta}{\rho}}} H_{0,2}^{2,0}\left[\left(ab^\frac{\rho}{\delta}\right)^\alpha \left|_{(0,\frac{1}{\rho})} \left(\frac{s}{\delta}\right)\right.\right], \quad a > 0, \ b > 0, \ \alpha > 0, \ \delta > 0, \ \rho > 0. \quad (27)$$

Make the transformation $\frac{s}{\delta} = s_1$ in (26)

$$I_1^{(\delta)} = \frac{1}{\rho a^{\frac{\delta}{\rho}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{\delta s_1}{\rho}\right)\Gamma\left(\frac{\alpha}{\delta} + s_1\right)\left(ab^\frac{\rho}{\delta}\right)^{-s_1} ds_1. \quad (28)$$

Let us consider the special case where $\frac{s}{\rho} = m, \ m = 1, 2, \ldots$. In physics problems $\rho = \frac{1}{2}$ and $\delta$ an integer. Then this assumption of $\frac{s}{\rho} = m, \ m = 1, 2, \ldots$ is meaningful at least in some physical problems. That is,

$$I_1^{(m\rho)} = \frac{1}{\rho a^{\frac{m}{\rho}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(ms_1)\Gamma\left(\frac{\alpha}{m\rho} + s_1\right)\left(ab^m\right)^{-s_1} ds_1. \quad (29)$$

This can be reduced to a $G$-function by using the multiplication formula for gamma functions, namely,

$$\Gamma(mz) = (2\pi)^{\frac{1}{2m}} m^{\frac{m}{2} - \frac{1}{2}} \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \cdots \Gamma\left(z + \frac{m - 1}{m}\right), \ m = 1, 2, \ldots \quad (30)$$

Then we have,

$$I_1^{(m\rho)} = \frac{(2\pi)^{\frac{1}{2m}}}{\rho a^{\frac{m}{\rho}} m^{\frac{m}{2}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s_1)\Gamma\left(s_1 + \frac{1}{m}\right) \cdots \Gamma\left(s_1 + \frac{m - 1}{m}\right) \Gamma\left(s_1 + \frac{\alpha}{m\rho}\right)\left(ab^m\right)^{-s_1} ds_1. \quad (31)$$

On evaluating (30), we get (4). (see Mathai and Haubold, 1988; Saxena, 1960; Haubold and Mathai, 1984).

**Particular case, $\delta = 1$**

When $\delta = 1,$ we get the reaction rate integral in (1), under the condition $\frac{1}{\rho} = m, \ m = 1, 2, \ldots$ as,

$$I_1 = \int_0^\infty x^{a-1}e^{-ax-bx^{-\rho}} dx, \ a > 0, \ b > 0, \ \rho > 0. \quad (31)$$

On evaluating this integral, we get (6). Observe that $m = 2$ is a real physical situation, see for example, Mathai and Haubold (1988).
3 Evaluation of extended integrals

We have,

\[ I^{(\delta)}_{2\beta} = \int_0^\infty x^{\alpha-1} [1 + a(1 - \beta)x^\delta] \frac{1}{1 + \beta} e^{-bx^\rho} \, dx, \quad a > 0, \ b > 0, \ \delta > 0, \ \rho > 0, \ \beta > 1. \]

As \( \beta \to 1 \), \( [1 - a(1 - \beta)x^\delta] \frac{1}{1 + \beta} \) becomes \( e^{-ax^\delta} \) so that we can extend the reaction rate integrals in (1) and (2) using the pathway parameter \( \beta \). Then here arise two cases: (i) \( \beta < 1 \), (ii) \( \beta > 1 \).

Case(i), \( \beta < 1 \)

\[ I^{(\delta)}_{2\beta} = \int_0^d x^{\alpha-1} [1 - a(1 - \beta)x^\delta] \frac{1}{1 + \beta} e^{-bx^\rho} \, dx, \quad a > 0, \ b > 0, \ \delta > 0, \ \beta < 1. \]

Here let us take

\[ f_1(x_1) = \begin{cases} c_1 x_1^\alpha [1 - a(1 - \beta)x_1^\delta] \frac{1}{1 + \beta}, & 0 < x_1 < \left[ \frac{1}{a(1 - \beta)} \right]^{\frac{1}{\delta}}, \ a > 0, \ \delta > 0, \ \beta < 1 \quad (32) \\
0, & \text{elsewhere} \end{cases} \]

and

\[ f_2(x_2) = \begin{cases} c_2 e^{-x_2^\delta}, & 0 < x_2 < \infty, \ \rho > 0 \quad (33) \\
0, & \text{elsewhere} \end{cases} \]

where \( c_1 \) and \( c_2 \) are the normalizing constants. Let us consider the case where \( d = \left[ \frac{1}{a(1 - \beta)} \right]^{\frac{1}{\delta}} \) in (8).

Note that \( f_1(x_1) \) in (32) with \( \alpha = 0 \) and \( \delta = 1 \) is Tsallis statistics leading to nonextensive statistical mechanics. Also observe that the functional part of (32) for \( \alpha = 0 \) and \( \delta = 1 \) gives the power law as well.

\[ \frac{d}{dx} \left( \frac{f_1(x_1)}{c_1} \right) = - \left( \frac{f_1(x_1)}{c_1} \right)^{\beta} \quad (34) \]

Note that \( f_2(x_2) \) in (33) is what is known as stretched exponential in physics literature. Thus the extended reaction rate integral in (8) is the Mellin convolution of Tsallis nonextensive statistics and stretched exponentials. The starting publication of nonextensive statistical mechanics may be seen from Tsallis (1988). Then proceeding as before, we get,

\[ I^{(\delta)}_{2\beta} = \int_0^\left[ \frac{1}{a(1 - \beta)} \right]^{\frac{1}{\delta}} x^{\alpha-1} [1 - a(1 - \beta)x^\delta] \frac{1}{1 + \beta} e^{-bx^\rho} \, dx \]

\[ = \frac{\Gamma(1 + \frac{1}{\beta})}{\rho \delta[a(1 - \beta)]^{\frac{1}{\delta}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left( \frac{\alpha}{\delta} + \frac{1}{\beta} \right) \Gamma\left( \frac{2\alpha}{\delta} + \frac{\rho}{\beta} \right)}{\Gamma\left( \frac{\alpha}{\delta} + 1 + \frac{1}{\beta} + \frac{s}{\rho} \right)} \, ds \]

\[ = \frac{\Gamma(1 + \frac{1}{\beta})}{\rho \delta[a(1 - \beta)]^{\frac{1}{\delta}}} H_{1,2}^2 \left[ a^{\frac{\alpha}{\delta}} (1 - \beta)^{\frac{\rho}{\beta}} b^{\frac{\rho s}{\beta}} \left| \left( \frac{\alpha}{\delta}, \frac{\rho}{\beta} \right), \left( \frac{\alpha}{\delta}, \frac{\rho}{\beta} \right) \right| \right] \quad (37) \]

\[ a > 0, \ b > 0, \ \alpha > 0, \ \delta > 0, \ \rho > 0, \ \beta < 1, \ \Re(s) > 0, \ \Re(\alpha + s) > 0. \]

Make the transformation \( \frac{s}{\rho} = s_1 \) in (36)

\[ I^{(\delta)}_{2\beta} = \frac{\Gamma(1 + \frac{1}{\beta})}{\rho [a(1 - \beta)]^{\frac{1}{\delta}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left( \frac{\rho s_1}{\beta} \right) \Gamma\left( \frac{\alpha}{\delta} + s_1 + \frac{\rho}{\beta} \right)}{\Gamma\left( \frac{\alpha}{\delta} + 1 + \frac{1}{\beta} + s_1 \right)} \, ds_1 \]

\[ = \frac{\Gamma(1 + \frac{1}{\beta})}{\rho [a(1 - \beta)]^{\frac{1}{\delta}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left( \frac{\rho s_1}{\beta} \right) \Gamma\left( \frac{\alpha}{\delta} + s_1 + \frac{\rho}{\beta} \right)}{\Gamma\left( \frac{\alpha}{\delta} + 1 + \frac{1}{\beta} + s_1 \right)} \, ds_1 \]

(38)

Let us consider the case where \( \frac{s}{\rho} = m, \ m = 1, 2, \ldots \) Then we get (14).
Lemma 1: As $\beta \to 1$, $I_{2\beta}^{(\delta)}$ becomes $I_2^{(\delta)}$.

Proof:

\[
\lim_{\beta \to 1} I_{2\beta}^{(\delta)} = \lim_{\beta \to 1} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\alpha}{\beta} \right) \Gamma\left(\frac{\beta}{\delta} + \frac{s}{\beta}\right) (a + b)^{s - \delta}}{\rho \delta a^{s + \frac{\alpha}{\beta}} \Gamma\left(\frac{\beta}{\delta} + \frac{s}{\beta} + 1 + \frac{1}{1-\beta}\right)} \, ds. \tag{39}
\]

Now apply the asymptotic formula for gamma functions, namely, for $|z| \to \infty$ and $a$ is bounded (Mathai, 1993),

\[
\Gamma(z + a) \to (2\pi)^\frac{1}{2} z^{\frac{1}{2} a - \frac{1}{2}} e^{-z}. \tag{40}
\]

Apply this to the gamma ratios in (39) by taking $z$ as $\frac{1}{1-\beta}$ and $a$ as $\frac{\alpha}{\beta} + \frac{s}{\beta}$ + 1 respectively. Then,

\[
\lim_{\beta \to 1} \frac{\Gamma\left(1 + \frac{1}{1-\beta}\right) (1-\beta)^{-\left(\frac{\alpha}{\beta} + \frac{s}{\beta}\right)}}{\Gamma\left(\frac{\alpha}{\beta} + \frac{s}{\beta} + 1 + \frac{1}{1-\beta}\right)} = 1 \tag{41}
\]

Hence

\[
\lim_{\beta \to 1} I_{2\beta}^{(\delta)} = I_2^{(\delta)} \tag{42}
\]

which establishes the result.

Case(ii), $\beta > 1$

\[
I_{13}^{(\delta)} = \int_{0}^{\infty} x^{a-1} [1 + a(\beta - 1)x^\delta]^{-\frac{1}{x+\delta}} e^{-\beta x} \, dx, \quad a > 0, \quad b > 0, \quad \delta > 0, \quad \rho > 0, \quad \beta > 1. \tag{43}
\]

Here let us take

\[
f_1(x_1) = \begin{cases} 
    c_1 x_1^\alpha [1 + a(\beta - 1)x_1^\delta]^{-\frac{1}{x_1+\delta}}, & 0 < x_1 < \infty, \quad a > 0, \quad \delta > 0, \quad \beta > 1 \\
    0, & \text{elsewhere} \end{cases}
\]

and

\[
f_2(x_2) = \begin{cases} 
    c_2 e^{-x_2^\rho}, & 0 < x_2 < \infty, \quad \rho > 0 \\
    0, & \text{elsewhere}, \end{cases}
\]

where $c_1$ and $c_2$ are the normalizing constants. Observe that $f_1(x_1)$ of (43) is nothing but the superstatistics of Beck and Cohen (2003), and the density in (44) is the stretched exponential. Hence (7) can be looked upon as the Mellin convolution of superstatistics and stretched exponentials. A large number of published articles are there on superstatistics. Then proceeding as before, we get,

\[
I_{13}^{(\delta)} = \int_{0}^{\infty} x^{a-1} [1 + a(\beta - 1)x^\delta]^{-\frac{1}{x+\delta}} e^{-bx} \, dx
\]

where $K = \frac{1}{\Gamma\left(\frac{\alpha}{\beta} \right) \rho \delta (a - 1)^{\delta}}$, $a > 0$, $b > 0$, $\alpha > 0$, $\rho > 0$, $\delta > 0$, $\beta > 1$, $\Re(s) > 0$, $\Re(a + s) > 0$. Make the transformation $\delta = s_1$ and let $\frac{\alpha}{\beta} = m$, $m = 1, 2, \cdots$ in (45). Then we get (10).

Lemma 2: As $\beta \to 1$, $I_{13}^{(\delta)}$ becomes $I_1^{(\delta)}$. 

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Particular case, $\delta = 1$

When $\delta = 1$, we get the reaction rate integral under the condition $\frac{1}{\rho} = m$, $m = 1, 2, \cdots$ as,

$$I_{2\beta} = \int_0^1 x^{\alpha-1} [1 - a(1 - \beta)x]^{-\frac{1}{\rho}} e^{-bx^\beta} dx, \quad a > 0, b > 0, \rho > 0, \beta < 1,$$

(46)
on evaluating this integral we get (16), and

$$I_{1\beta} = \int_0^\infty x^{\alpha-1} [1 + a(\beta - 1)x]^{-\frac{1}{\rho}} e^{-bx^\beta} dx, \quad a > 0, b > 0, \rho > 0, \beta > 1,$$

(47)on evaluating this integral we get (12).

**Proof of Theorem 1**

For proving the theorem we will make use of the fact that the reaction probability integral as well as the extended reaction probability integrals, as given in (1),(2),(7),(8), under the condition $\delta = m\rho$, $m = 1, 2, \cdots$ can be written in terms of $G$-functions. Hence when this condition is satisfied we can invoke the properties of $G$-functions. It is well known that the $G$-function defined by

$$G_{m,n}^{p,q}[z|a_1,\ldots,a_p] = \frac{1}{2\pi i} \int_L \phi(s) z^{-s} ds,$$

(48)where

$$\phi(s) = \frac{\{\prod_{j=1}^m \Gamma(b_j + s)\} \{\prod_{j=1}^n \Gamma(1 - a_j - s)\}}{\{\prod_{j=m+1}^n \Gamma(1 - b_j - s)\} \{\prod_{j=n+1}^p \Gamma(a_j + s)\}},$$
a,, $j = 1, 2, \ldots, p$ and $b_j, j = 1, 2, \ldots, q$ are complex numbers, $L$ is a contour separating the poles of $\Gamma(b_j + s), j = 1, 2, \ldots, m$ from those of $\Gamma(1 - a_j - s), j = 1, 2, \ldots, n,$ satisfies the following differential equation.

$$[(-1)^{p-m-n} z \prod_{j=1}^p (\eta - a_j + 1) - \prod_{j=1}^q (\eta - b_j)] G(z) = 0, \quad \eta = \frac{d}{dz}.$$
This equation is intuitively evident from the following facts:

\[ \eta z^{-s} = z \frac{d}{dz}(z^{-s}) = (-s)z^{-s}; \]
\[ (\eta - a_j + 1)z^{-s} = (1 - a_j - s)z^{-s}; \]
\[ (1 - a_j - s)\Gamma(1 - a_j - s) = \Gamma(2 - a_j - s), \]

(see Mathai, 1993). The depleted case of the reaction rate integral

\[ I_1^{(s)} = \int_0^\infty x^{\alpha-1}e^{-ax^{-s}}dx, \quad a > 0, b > 0, \delta > 0, \rho > 0 \]  
(50)

can be expressed in terms of G-functions under the conditions \( \frac{\delta}{\rho} = m, m = 1, 2, \cdots \) shown in (4).

That is,

\[ (2\pi)^{m-1}m!\rho \frac{a}{\delta} \Gamma_1^{(m\rho)} = G_{0,m+1}^{m+1,0}\left[ \frac{ab^m}{m^{m+1}} \right] \left[ \frac{a}{\delta} \right] \left[ \frac{b}{\rho} \right] \]  
(51)

\[ a > 0, b > 0, \alpha > 0, \delta > 0, \rho > 0, \frac{\delta}{\rho} = m, m = 1, 2, \cdots. \]

The G-function in (51), satisfies the differential equation in (3). So the left hand side of (51) also satisfies the differential equation (3). Similar are the proofs of the other theorems and hence deleted.

**Acknowledgment**

The authors would like to thank the Department of Science and Technology, Government of India, New Delhi, for the financial assistance for this work under project-number SR/S4/MS:287/05 and the Centre for Mathematical Sciences for providing all facilities.

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