The combined effect in one space dimension beyond the general theory for nonlinear wave equations

Dedicated to Professor Tohru Ozawa on his sixtieth birthday

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Abstract

In this paper, we show the so-called “combined effect” of two different kinds of nonlinear terms for semilinear wave equations in one space dimension. Such a special phenomenon appears only in case the total integral of the initial speed is zero. It is remarkable that, including the combined effect case, our results on the lifespan estimates are partially better than those of the general theory for nonlinear wave equations.

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1 Introduction

We consider the initial value problems;

\[
\begin{aligned}
&\begin{cases}
  u_{tt} - u_{xx} = A|u_t|^p + B|u|^q, & \text{in } \mathbb{R} \times (0, \infty), \\
  u(x,0) = \varepsilon f(x), & \text{and } u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R},
\end{cases}
\end{aligned}
\]  

(1.1)

where \( p, q > 1, A, B \geq 0, \) \( f \) and \( g \) are given smooth functions of compact support and a parameter \( \varepsilon > 0 \) is “small enough”. We are interested in the lifespan \( T(\varepsilon) \), the maximal existence time, of classical solutions of (1.1). Our results in this paper are the following estimates for \( A > 0 \) and \( B > 0 \):

\[
T(\varepsilon) \sim \min\{C\varepsilon^{-(p-1)}, C\varepsilon^{-(q-1)/2}\} \quad \text{if } \int_\mathbb{R} g(x)dx \neq 0
\]  

(1.2)

and

\[
T(\varepsilon) \sim \begin{cases}
  C\varepsilon^{-(p-1)/(q+1)} & \text{for } \frac{q+1}{2} \leq p \leq q, \\
  \min\{C\varepsilon^{-(p-1)}, C\varepsilon^{-q(q-1)/(q+1)}\} & \text{otherwise}
\end{cases} \quad \text{if } \int_\mathbb{R} g(x)dx = 0.
\]  

(1.3)

Here we denote the fact that there are positive constants, \( C_1 \) and \( C_2 \), independent of \( \varepsilon \) satisfying \( A(\varepsilon, C_1) \leq T(\varepsilon) \leq A(\varepsilon, C_2) \) by \( T(\varepsilon) \sim A(\varepsilon, C) \).

Recall that we have

\[
T(\varepsilon) \sim C\varepsilon^{-(p-1)} \quad \text{for } A > 0 \text{ and } B = 0.
\]

This result was verified by Zhou [18] for the upper bound, and by Li,Yu and Zhou [9, 10] for the lower bound with integer \( p \geq 2 \) including more general nonlinear term. We call [9, 10] “general theory” for nonlinear wave equations in one dimension. Recently, Kitamura, Morisawa and Takamura [8] have verified the lower bound for all \( p \geq 1 \) including the case that nonlinear term has spatial weights, in which only the \( C^1 \) solution of the associated integral equation is considered for \( 1 < p < 2 \). But it can be also the classical solution by trivial modifications on estimating the nonlinear term with Hölder continuity. See Remark 2.1 below. On the other hand, Zhou [17] obtained

\[
T(\varepsilon) \sim \begin{cases}
  C\varepsilon^{-(q-1)/2} & \text{if } \int_\mathbb{R} g(x)dx \neq 0, \\
  C\varepsilon^{-q(q-1)/(q+1)} & \text{if } \int_\mathbb{R} g(x)dx = 0
\end{cases} \quad \text{for } A = 0 \text{ and } B > 0.
\]
Therefore (1.2) and (1.3) are quite natural as taking the minimum of both results except for the first case in (1.3), in which, we have

$$C\varepsilon^{-p(q-1)/(q+1)} \leq \min\{C\varepsilon^{-(p-1)}, C\varepsilon^{-q(q-1)/(q+1)}\} \quad \text{for} \quad \frac{q+1}{2} \leq p \leq q.$$  

We call this special phenomenon by “combined effect” of two nonlinearities. The combined effect was first observed by Han and Zhou in [2], which targets to show the optimality of the result of Katayama [6] on the lower bound of the lifespan of classical solutions of nonlinear wave equations with a nonlinear term $u^3_t + u^4$ in two space dimensions including more general nonlinear terms. It is known that $T(\varepsilon) \sim \exp(C\varepsilon^{-2})$ for the nonlinear term $u^3_t$ and $T(\varepsilon) = \infty$ for the nonlinear term $u^4$, but Katayama [6] obtained only a much worse estimate than their minimum as $T(\varepsilon) \geq C\varepsilon^{-18}$. Surprisingly, more than ten years later, Han and Zhou [2] showed that this result is optimal as $T(\varepsilon) \leq C\varepsilon^{-18}$. They also considered (1.1) for all space dimensions $n$ bigger than 1 and obtain the upper bound of the lifespan. Its counterpart, the lower bound of the lifespan, was obtained by Hidano, Wang and Yokoyama [3] for $n = 2, 3$. See the introduction of [3] for the precise results and references. We note that the first case in (1.3) coincides with the lifespan estimate for the combined effect in [2, 3] if one sets $n = 1$ formally. Later, Dai, Fang and Wang [1] improved the lower bound of lifespan for the critical case in Hidano, Wang and Yokoyama [3]. They also showed that $T(\varepsilon) < \infty$ for all $p, q > 1$ in case of $n = 1$, i.e. (1.1).

Finally we strongly remark that our estimates in (1.2) and (1.3) are better than those of the general theory by Li, Yu and Zhou [9, 10]:

$$T(\varepsilon) \geq \begin{cases} C\varepsilon^{-(p-1)/2} & \text{if } \int_{\mathbb{R}} g(x)dx \neq 0, \\ C\varepsilon^{-p(p-1)/(p+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0 \end{cases}$$

in case of

$$\frac{q+1}{2} < p < q$$

with integer $p, q \geq 2$. Indeed, our result on the lower bound of the lifespan can be established also for the indefinite sign terms as $u_{tt} - u_{xx} = u^p + u^q$. The typical example is $(p, q) = (4, 5)$ in such a case. This fact shows a possibility to improve the general theory. For details, see the last half of the next section. Of course, some special structure of the nonlinear terms such as “null condition” guarantees the global-in-time existence. See Nakamura [13], Luli, Yang and Yu [12], Zha [15, 16] for examples in this direction. But
we are interested in the optimality of the general theory. The details are discussed at the end of Section 2 below.

This paper is organized as follows. In the next section, the preliminaries are introduced. Moreover, (1.2) and (1.3) are divided into four theorems, and we compare our results with those of the general theory. Sections 3 and 4 are devoted to the proof of the existence part of (1.2). Sections 5, 6 and 7 are devoted to the proof of the existence part of (1.3). Their main strategy is the iteration method in the weighted $L^\infty$ space due to Kitamura, Morisawa and Takamura [7, 8], which is originally introduced by John [4]. Finally, we prove the blow-up part of (1.2) and (1.3), which is different from the functional method by Han and Zhou [2].

2 Preliminaries and main results

Throughout this paper, we assume that the initial data $(f, g) \in C_0^2(\mathbb{R}) \times C_0^1(\mathbb{R})$ satisfies

$$\text{supp } f, \text{ supp } g \subset \{ x \in \mathbb{R} : |x| \leq R \}, \quad R \geq 1. \quad (2.1)$$

Let $u$ be a classical solution of (1.1) in the time interval $[0, T]$. Then the support condition of the initial data, (2.1), implies that

$$\text{supp } u(x, t) \subset \{ (x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R \}. \quad (2.2)$$

For example, see Appendix of John [5] for this fact.

It is well-known that $u$ satisfies the following integral equation.

$$u(x, t) = \varepsilon u_0(x, t) + L(A|u_t|^p + B|u|^q)(x, t), \quad (2.3)$$

where $u_0$ is a solution of the free wave equation with the same initial data,

$$u_0(x, t) := \frac{1}{2} \{ f(x + t) + f(x - t) \} + \frac{1}{2} \int_{x-t}^{x+t} g(y)dy, \quad (2.4)$$

and a linear integral operator $L$ for a function $v = v(x, t)$ in Duhamel’s term is defined by

$$L(v)(x, t) := \frac{1}{2} \int_0^t ds \int_{x-t+s}^{x+t-s} v(y, s)dy. \quad (2.5)$$

Then, one can apply the time-derivative to (2.3) to obtain

$$u_t(x, t) = \varepsilon u_0(x, t) + L'(A|u_t|^p + B|u|^q)(x, t) \quad (2.6)$$
and
\[ u_0^0(x,t) = \frac{1}{2}(f'(x + t) - f'(x - t) + g(x + t) + g(x - t)), \]  
(2.7)
where \( L' \) for a function \( v = v(x,t) \) is defined by
\[ L'(v)(x,t) := \frac{1}{2} \int_0^t \{v(x + t - s, s) + v(x - t + s, s)\} ds. \]  
(2.8)

On the other hand, applying the space-derivative to (2.3), we have
\[ u_x(x,t) = \varepsilon u_0^0 x(x,t) + L'(A|u_t|^p + B|u|^q)(x,t) \]  
(2.9)
and
\[ u_0^0 x(x,t) = \frac{1}{2}\{f'(x + t) + f'(x - t) + g(x + t) - g(x - t)\}, \]  
(2.10)
where \( L' \) for a function \( v = v(x,t) \) is defined by
\[ L'(v)(x,t) := \frac{1}{2} \int_0^t \{v(x + t - s, s) - v(x - t + s, s)\} ds. \]  
(2.11)

Therefore, \( u_x \) is expressed by \( u \) and \( u_t \). Moreover, applying one more space-derivative to (2.6) yields that
\[ u_{tx}(x,t) = \varepsilon u_0^0 tx(x,t) + L'(Ap|u_t|^{p-2}u_t u_{tx} + Bq|u|^{q-2}uu_x)(x,t) \]  
(2.12)
and
\[ u_0^0 tx(x,t) = \frac{1}{2}\{f''(x + t) - f''(x - t) + g'(x + t) + g'(x - t)\}. \]  
(2.13)

Similarly, we have that
\[ u_{tt}(x,t) = \varepsilon u_0^0 tt(x,t) + A|u_t(x,t)|^p + B|u(x,t)|^q \]  
\[ + L'(Ap|u_t|^{p-2}u_t u_{tx} + Bq|u|^{q-2}uu_x)(x,t) \]
and
\[ u_0^0 tt(x,t) = \frac{1}{2}\{f''(x + t) + f''(x - t) + g'(x + t) - g'(x - t)\}. \]

Therefore, \( u_{tt} \) is expressed by \( u, u_t, u_x, u_{tx} \) and so is \( u_{xx} \) because of
\[ u_{xx}(x,t) = \varepsilon u_0^0 xx(x,t) \]  
\[ + L'(Ap|u_t|^{p-2}u_t u_{tx} + Bq|u|^{q-2}uu_x)(x,t) \]
and
\[ u_0^0 xx(x,t) = u_0^0 tt(x,t). \]
Remark 2.1 In view of (2.12), it is sufficient to employ Hölder continuity of the nonlinear term, i.e.

$$||a|^{p-2}a - |b|^{p-2}b| \leq 2|a - b|^{p-1} \ (a, b \in \mathbb{R}, \ 1 < p < 2),$$

in estimating the difference of the approximation sequence to construct a classical solution for $p, q \in (1, 2)$. This fact was overlooked in Kitamura, Morisawa and Takamura [8] as stated in Introduction.

First, we note the following fact.

**Proposition 2.1** Assume that $(f, g) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$. Let $(u, w)$ be a $C^1$ solution of a system of integral equations

$$\begin{cases}
  u = \varepsilon u^0 + L(A|w|^p + B|u|^q), \\
  w = \varepsilon u_t^0 + L'(A|w|^p + B|u|^q)
\end{cases} \quad \text{in } \mathbb{R} \times [0, T] \quad (2.14)$$

with some $T > 0$. Then, $w \equiv u_t$ in $\mathbb{R} \times [0, T]$ holds and $u$ is a classical solution of (1.1) in $\mathbb{R} \times [0, T]$.

**Proof.** It is trivial that $w \equiv u_t$ by differentiating the first equation with respect to $t$. The rest part is easy along with the computation above in this section. \[\square\]

Our results in (1.2) and (1.3) are divided into the following four theorems.

**Theorem 2.1** Let $A > 0$ and $B > 0$. Assume (2.1) and

$$\int_{\mathbb{R}} g(x)dx \neq 0. \quad (2.15)$$

Then, there exists a positive constant $\varepsilon_1 = \varepsilon_1(f, g, p, q, A, B, \mathbb{R}) > 0$ such that a classical solution $u \in C^2(\mathbb{R} \times [0, T])$ of (1.1) exists as far as $T$ satisfies

$$T \leq \begin{cases}
  c\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2}, \\
  c\varepsilon^{-(q-1)/2} & \text{for } \frac{q+1}{2} \leq p,
\end{cases} \quad (2.16)$$

where $0 < \varepsilon \leq \varepsilon_1$ and $c$ is a positive constant independent of $\varepsilon$.

**Theorem 2.2** Let $A > 0$ and $B > 0$. Assume (2.1) and

$$\int_{\mathbb{R}} g(x)dx = 0. \quad (2.17)$$
Then, there exists a positive constant $\varepsilon_2 = \varepsilon_2(f, g, p, q, A, B, R) > 0$ such that a classical solution $u \in C^2(\mathbb{R} \times [0, T])$ of (1.1) exists as far as $T$ satisfies

$$T \leq \begin{cases} c\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2}, \\ c\varepsilon^{-p(q-1)/(q+1)} & \text{for } \frac{q+1}{2} \leq p \leq q, \\ c\varepsilon^{-q(q-1)/(q+1)} & \text{for } p \geq q, \end{cases}$$

(2.18)

where $0 < \varepsilon \leq \varepsilon_2$ and $c$ is a positive constant independent of $\varepsilon$.

**Theorem 2.3** Let $A > 0$ and $B > 0$. Assume (2.1) and

$$\int_{\mathbb{R}} g(x) dx > 0.$$  

(2.19)

Then, there exists a positive constant $\varepsilon_3 = \varepsilon_3(f, g, p, q, A, B, R) > 0$ such that a classical solution of (1.1) cannot exist as far as $T$ satisfies

$$T \geq \begin{cases} C\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2}, \\ C\varepsilon^{-(q-1)/2} & \text{for } \frac{q+1}{2} \leq p \leq q, \end{cases}$$

(2.20)

where $0 < \varepsilon \leq \varepsilon_3$ and $C$ is a positive constant independent of $\varepsilon$.

**Theorem 2.4** Let $A > 0$ and $B > 0$. Assume (2.1) and

$$f(x) \geq 0(\neq 0), \quad f'(x) < 0 \text{ for } x \in (0, R) \text{ and } g(x) \equiv 0.$$  

(2.21)

Then, there exists a positive constant $\varepsilon_4 = \varepsilon_4(f, p, q, A, B, R) > 0$ such that a classical solution of (1.1) cannot exist as far as $T$ satisfies

$$T \geq \begin{cases} C\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2}, \\ C\varepsilon^{-p(q-1)/(q+1)} & \text{for } \frac{q+1}{2} \leq p \leq q, \\ C\varepsilon^{-q(q-1)/(q+1)} & \text{for } p \geq q, \end{cases}$$

(2.22)

where $0 < \varepsilon \leq \varepsilon_4$ and $C$ is a positive constant independent of $\varepsilon$.

**Remark 2.2** It is trivial that Theorem 2.1 and Theorem 2.3 imply (1.2). On the other hand, we have that

$$p - 1 = \frac{q(q-1)}{q+1} \iff p = \frac{q^2 + 1}{q + 1}$$
and
\[ \frac{q+1}{2} < \frac{q^2+1}{q+1} < q. \]

Moreover, we see that
\[ p - 1 \leq \frac{p(q-1)}{q+1} \iff p \leq \frac{q+1}{2}. \]

Therefore Theorem 2.2 and Theorem 2.4 imply (1.3).

The proofs of four theorems above appear in the following sections. Form now on, we shall compare our results with those of the general theory by Li, Yu and Zhou [9, 10], in which the following problem of general form is considered:

\[
\begin{align*}
   & u_{tt} - u_{xx} = F(u, Du, \partial_x Du) \quad \text{in} \quad \mathbb{R} \times (0, \infty), \\
   & u(x, 0) = \varepsilon f(x), \quad u_t(x, 0) = \varepsilon g(x), \quad x \in \mathbb{R},
\end{align*}
\]

where we denote \( D := (\partial_t, \partial_x) \) and \( F \in C^\infty(\mathbb{R}^5) \) satisfies

\[ F(\lambda) = O(|\lambda|^{1+\alpha}) \quad \text{with} \quad \alpha \in \mathbb{N} \text{ near } \lambda = 0. \]

(2.23) requires \( f, g \in C^\infty_0(\mathbb{R}) \). Then, the lifespan of the classical solution of (2.23) defined by \( \tilde{T}(\varepsilon) \) has estimates from below as

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
   c\varepsilon^{-\alpha/2} & \text{in general,} \\
   c\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0, \\
   c\varepsilon^{-\alpha} & \text{if } \partial_\beta^\alpha F(0) = 0 \text{ for } 1 + \alpha \leq \forall \beta \leq 2\alpha.
\end{cases}
\]

(2.24)

This is the result of the general theory. If one applies it to our problem (1.1) with

\[ F(u, Du, \partial_x Du) = u_p^p + u_q^q \quad \text{with} \quad 2 \leq p, q \in \mathbb{N}, \]

one has the following estimates in each cases.

- When \( p < q \),

then, we have to set \( \alpha = p - 1 \) which yields that

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
   c\varepsilon^{-(p-1)/2} & \text{in general,} \\
   c\varepsilon^{-(p-1)/(p+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0, \\
   c\varepsilon^{-(p-1)} & \text{if } \partial_\beta^\alpha F(0) = 0 \text{ for } p \leq \forall \beta \leq 2(p - 1).
\end{cases}
\]
For $p \leq (q + 1)/2$, i.e. $2p - 1 \leq q$, we see that
\[
\partial_u^\beta F(u, Du, \partial_x Du) = O(u^{q-\beta}) \quad \text{and} \quad 1 \leq 2p - 1 - \beta \leq q - \beta
\]
because of $p + 1 \leq 2p - 1$, which yields
\[
\partial_u^\beta F(0) = 0 \quad \text{for} \quad p \leq \forall \beta \leq 2(p - 1).
\]
Therefore the third case holds and we obtain
\[
\tilde{T}(\varepsilon) \geq c\varepsilon^{-(p-1)}
\]
whatever the value of $\int_R g(x)dx$ is. On the other hand, for $(q+1)/2 < p$, i.e. $(p <) q < 2p - 1$, we see that
\[
\exists \beta \in \{p + 1, \ldots, 2p - 2\} \text{ s.t. } \partial_u^\beta F(0) \neq 0,
\]
so that the third case does not hold and we obtain
\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\frac{c\varepsilon^{-(p-1)/2}}{\sqrt{R}} & \text{if } \int_R g(x)dx \neq 0, \\
\frac{c\varepsilon^{-p(p-1)/(p+1)}}{\sqrt{R}} & \text{if } \int_R g(x)dx = 0.
\end{cases}
\]

- When $p \geq q$,

then, similarly to the case above, we have to set $\alpha = q - 1$, which yields that
\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\frac{c\varepsilon^{-(q-1)/2}}{\sqrt{R}} & \text{in general,} \\
\frac{c\varepsilon^{-q(q-1)/(q+1)}}{\sqrt{R}} & \text{if } \int_R g(x)dx = 0, \\
\frac{c\varepsilon^{-(q-1)}}{\sqrt{R}} & \text{if } \partial_u^\beta F(0) = 0 \text{ for } q \leq \forall \beta \leq 2(q - 1).
\end{cases}
\]

We note that the third case does not hold by $\partial_u^\beta F(0) \neq 0$.

In conclusion, for the special nonlinear term in (2.25), the result of the general theory is
\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\frac{c\varepsilon^{-(p-1)}}{2} & \text{for } p \leq \frac{q + 1}{2}, \\
\frac{c\varepsilon^{-(p-1)/2}}{\sqrt{R}} & \text{for } \frac{q + 1}{2} \leq p \leq q, \\
\frac{c\varepsilon^{-(q-1)/2}}{\sqrt{R}} & \text{for } q \leq p.
\end{cases}
\]
and
\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2}, \\
\varepsilon^{-(p-1)/(p+1)} & \text{for } \frac{q+1}{2} \leq p \leq q, \\
\varepsilon^{-(q-1)/(q+1)} & \text{for } q \leq p 
\end{cases}
\]
if \( \int_{\mathbb{R}} g(x)dx = 0 \).

Therefore a part of our results,
\[
T(\varepsilon) \sim \begin{cases} 
C\varepsilon^{-(q-1)/2} & \text{if } \int_{\mathbb{R}} g(x) \neq 0, \\
C\varepsilon^{-(p-1)/(q+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0 
\end{cases}
\quad \text{for } \frac{q+1}{2} \leq p \leq q,
\tag{2.26}
\]
is better than the lower bound of \( \tilde{T}(\varepsilon) \). If one follows the proof in the following sections, one can find that it is easy to see that our results on the lower bounds also hold for a special term (2.25) by estimating the difference of nonlinear terms from above after employing the mean value theorem. We note that we have infinitely many examples of \((p, q) = (m, m+1)\) as the inequality
\[
\frac{q+1}{2} = \frac{m+2}{2} < p = m < q = m+1
\]
holds for \( m = 3, 4, 5, \ldots \). This fact indicates that we still have a possibility to improve the general theory in the sense that the optimal results in (2.26) should be included at least.

3 Proof of Theorem 2.1

According to Proposition 2.1, we shall construct a \( C^1 \) solution of (2.14). Let \( \{(u_j, w_j)\}_{j \in \mathbb{N}} \) be a sequence of \( \{C^1(\mathbb{R} \times [0, T])\}^2 \) defined by
\[
\begin{align*}
&u_{j+1} = \varepsilon u_0 + L(A|w_j|^p + B|u_j|^q), \quad u_1 = \varepsilon u_0, \\
&w_{j+1} = \varepsilon u_0 + L'(A|w_j|^p + B|u_j|^q), \quad w_1 = \varepsilon u_0.
\end{align*}
\tag{3.1}
\]
Then, in view of (2.9) and (2.12), \( ((u_j)_x, (w_j)_x) \) has to satisfy
\[
\begin{align*}
&(u_{j+1})_x = \varepsilon u_0 + L(A|w_j|^p + B|u_j|^q), \\
&(u_1)_x = \varepsilon u_0, \\
&(w_{j+1})_x = \varepsilon u_0 + L'(A|w_j|^p + B|u_j|^q), \\
&(w_1)_x = \varepsilon u_0,
\end{align*}
\tag{3.2}
\]
so that the function space in which \( \{(u_j, w_j)\} \) converges is
\[
X := \{(u, w) \in \{C^1(\mathbb{R} \times [0, T])\}^2 : \|\!(u, w)\|_X < \infty, \quad \text{supp } (u, w) \subset \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\}\}.
\]
which is equipped with a norm

\[ \|(u, w)\|_X := \|u\|_1 + \|w_x\|_1 + \|w\|_2 + \|w_x\|_2, \]

where

\[ \|u\|_1 := \sup_{(x,t) \in \mathbb{R} \times [0,T]} |u(x, t)|, \]
\[ \|w\|_2 := \sup_{(x,t) \in \mathbb{R} \times [0,T]} |(t - |x| + 2R) w(x, t)|. \]

First we note that supp \((u_j, w_j)\) \(\subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\}\) implies supp \((u_{j+1}, w_{j+1})\) \(\subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\}\). It is easy to check this fact by assumption on the initial data (2.1) and the definitions of \(L, \overline{L}, L', \overline{L}'\) in the previous section.

The following lemma contains some useful a priori estimates.

**Proposition 3.1** Let \((u, w) \in \{C(\mathbb{R} \times [0,T])\}^2\) and supp \((u, w)\) \(\subset \{(x,t) \in \mathbb{R} \times [0,T] : |x| \leq t + R\}\). Then there exists a positive constant \(C\) independent of \(T\) and \(\varepsilon\) such that

\[ \begin{align*}
    \|L(|w|^p)|1 \leq C\|w\|_2^p(T + R), & \quad \|L(|w|^q)|1 \leq C\|w\|_2^q(T + R)^2, \\
    \|L'(|w|^p)|2 \leq C\|w\|_2^p(T + R), & \quad \|L'(|w|^q)|2 \leq C\|w\|_2^q(T + R)^2, \\
    \|L'(|w|^p)|1 \leq C\|w\|_2^p(T + R), & \quad \|L'(|w|^q)|1 \leq C\|w\|_2^q(T + R)^2.
\end{align*} \]

(3.3)

The proof of Proposition 3.1 is established in the next section. Set

\[ M := \sum_{\alpha=0}^{2} \|f^{(\alpha)}\|_{L^\infty(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})} + \sum_{\beta=0}^{1} \|g^{(\beta)}\|_{L^\infty(\mathbb{R})}. \]

The convergence of the sequence \(\{(u_j, w_j)\}\).

First we note that \(\|u_1\|_1, \|w_1\|_2 \leq M\varepsilon \) by (2.4) and (2.7). Since (3.1) and (3.3) yield that

\[ \begin{align*}
    \|u_{j+1}\|_1 & \leq M\varepsilon + A\|L(|w_j|^p)|1 + B\|L(|u_j|^q)|1 \\
    & \leq M\varepsilon + AC\|w_j\|_2^p(T + R) + BC\|u_j\|_2^q(T + R)^2; \\
    \|w_{j+1}\|_2 & \leq M\varepsilon + A\|L'(|w_j|^p)|2 + B\|L'(|u_j|^q)|2 \\
    & \leq M\varepsilon + AC\|w_j\|_2^p(T + R) + CB\|u_j\|_2^q(T + R)^2;
\end{align*} \]

the boundedness of \(\{(u_j, w_j)\}\), i.e.

\[ \|u_j\|_1, \|w_j\|_2 \leq 3M\varepsilon \quad (j \in \mathbb{N}), \]

(3.4)

follows from

\[ AC(3M\varepsilon)^p(T + R), BC(3M\varepsilon)^q(T + R)^2 \leq M\varepsilon. \]

(3.5)
Assuming (3.5), one can estimate \( (u_{j+1} - u_j) \) and \( (w_{j+1} - w_j) \) as follows.

\[
\|u_{j+1} - u_j\|_1 \leq \|L(A)|w_j|^p - A|w_{j-1}|^p + B|u_j|^q - B|u_{j-1}|^q\|_1 \\
\leq 2^{p-1}pA\|L\left(\left(|w_j|^{p-1} + |w_{j-1}|^{p-1}\right)|w_j - w_{j-1}|\right)\|_1 \\
+ 2^{q-1}qB\|L\left(\left(|u_j|^q - |u_{j-1}|^q\right)|u_j - u_{j-1}|\right)\|_1 \\
\leq 2^{p-1}pAC(T + R)\|\|w_j\|_2^{p-1} + \|w_{j-1}\|_2^{p-1}\|w_j - w_{j-1}\|_2 \\
+ 2^{q-1}qBC(T + R)^2\|\|u_j\|_q^{q-1} + \|u_{j-1}\|_q^{q-1}\|u_j - u_{j-1}\|_1 \\
\leq 2^p pAC(3M\varepsilon)^{p-1}(T + R)\|w_j - w_{j-1}\|_2 \\
+ 2^q qBC(3M\varepsilon)^{q-1}(T + R)^2\|u_j - u_{j-1}\|_1
\]

and

\[
\|w_{j+1} - w_j\|_2 \leq \|L'(A)|w_j|^p - A|w_{j-1}|^p + B|u_j|^q - B|u_{j-1}|^q\|_2 \\
\leq 2^{p-1}pA\|L'\left(\left(|w_j|^{p-1} + |w_{j-1}|^{p-1}\right)|w_j - w_{j-1}|\right)\|_2 \\
+ 2^{q-1}qB\|L'\left(\left(|u_j|^q - |u_{j-1}|^q\right)|u_j - u_{j-1}|\right)\|_2 \\
\leq 2^{p-1}pAC(T + R)\|\|w_j\|_2^{p-1} + \|w_{j-1}\|_2^{p-1}\|w_j - w_{j-1}\|_2 \\
+ 2^{q-1}qBC(T + R)^2\|\|u_j\|_q^{q-1} + \|u_{j-1}\|_q^{q-1}\|u_j - u_{j-1}\|_1 \\
\leq 2^p pAC(3M\varepsilon)^{p-1}(T + R)\|w_j - w_{j-1}\|_2 \\
+ 2^q qBC(3M\varepsilon)^{q-1}(T + R)^2\|u_j - u_{j-1}\|_1
\]

Here we employ Hölder’s inequality to obtain

\[
\|L(|w_j|^{p-1}|w_j - w_{j-1}|)\|_1 = \|L\left(\left(|w_j|^{(p-1)/p}|w_j - w_{j-1}|\right)\right)\|_1 \\
\leq C(T + R)\|\|w_j\|_1^{(p-1)/p} - \|w_{j-1}\|_1^{1/p}\|_p \\
\leq C(T + R)\|\|w_j\|_2^{(p-1)} - \|w_{j-1}\|_2\|_2
\]

and so on. Therefore the convergence of \( \{u_j\} \) follows from

\[
\|u_{j+1} - u_j\|_1 + \|w_{j+1} - w_j\|_2 \\
\leq \frac{1}{2} (\|u_j - u_{j-1}\|_1 + \|w_j - w_{j-1}\|_2) \quad (j \geq 2) \quad (3.6)
\]

provided (3.5) and

\[
2^p pAC(3M\varepsilon)^{p-1}(T + R), 2^q qBC(3M\varepsilon)^{q-1}(T + R)^2 \leq \frac{1}{4} \quad (3.7)
\]

are fulfilled.

The convergence of the sequence \( \{(u_j)_x, (w_j)_x\} \).

First we note that \( \|(u_1)_x\|_1, \|(w_1)_x\|_2 \leq M\varepsilon \) by (2.10) and (2.13). Assume that (3.5) and (3.7) are fulfilled. Since (3.2) and (3.3) yield that

\[
\|(u_{j+1})_x\|_1 \leq M\varepsilon + A\|T(|w_j|^p)\|_1 + B\|T(|u_j|^q)\|_1 \\
\leq M\varepsilon + AC(T + R)\|\|w_j\|_p^p + BC(T + R)^2\|u_j\|_1^q \\
\leq M\varepsilon + AC(3M\varepsilon)^p(T + R) + BC(3M\varepsilon)^q(T + R)^2
\]

and

\[
\|(w_{j+1})_x\|_2 \leq M\varepsilon + pAC(T + R)\|\|w_j\|_2^{p-1} + BC(T + R)^2\|u_j\|_1^q \\
\leq M\varepsilon + AC(3M\varepsilon)^{p-1}(T + R) + BC(3M\varepsilon)^q(T + R)^2
\]

and so on. Therefore the convergence of \( \{u_j\} \) follows from

\[
\|u_{j+1} - u_j\|_1 + \|w_{j+1} - w_j\|_2 \\
\leq \frac{1}{2} (\|u_j - u_{j-1}\|_1 + \|w_j - w_{j-1}\|_2) \quad (j \geq 2) \quad (3.6)
\]

provided (3.5) and

\[
2^p pAC(3M\varepsilon)^{p-1}(T + R), 2^q qBC(3M\varepsilon)^{q-1}(T + R)^2 \leq \frac{1}{4} \quad (3.7)
\]

are fulfilled.
because of a trivial property $|L(v)| \leq L'(|v|)$ and

$$
\|(w_{j+1})_x\|_2 \leq M\varepsilon + pA\|L'(|w_j|^{p-1}(w_j)_x)| \|_2
+ qB\|L'(|w_j|^{p-1}(w_j)_x)| \|_2
\leq M\varepsilon + pAC(T + R)\|w_j\|_2^{p-1}\|(w_j)_x\|_2
+ qBC(T + R)^2\|u_j\|_1^{r-1}\|(w_j)_x\|_1
\leq M\varepsilon + pAC(3M\varepsilon)^{p-1}(T + R)\|(w_j)_x\|_2
+ qBC(3M\varepsilon)^{r-1}(T + R)^2\|(u_j)_x\|_1.
$$

The boundedness of $\{(u_j)_x, (w_j)_x\}$, i.e.

$$
\|(u_j)_x\|_2, \|(w_j)_x\|_2 \leq 3M\varepsilon \quad (j \in \mathbb{N}),
$$

follows from

$$
pAC(3M\varepsilon)^p(T + R), qBC(3M\varepsilon)^q(T + R)^2 \leq M\varepsilon.
$$

Assuming (3.9), one can estimate $\{(u_{j+1})_x - (u_j)_x\}$ and $\{(w_{j+1})_x - (w_j)_x\}$ as follows. It is easy to see that

$$
\|(u_{j+1})_x - (u_j)_x\|_1 \leq A\|\overline{L}(|w_j|^p - |w_{j-1}|^p)| \|_1
+ B\|\overline{L}(|w_j|^q - |w_{j-1}|^q)| \|_1,
$$

which can be handled like $(w_{j+1} - w_j)$ as before, so that we have that

$$
\|(u_{j+1})_x - (u_j)_x\|_1 \leq 2pAC(3M\varepsilon)^{p-1}(T + R)\|w_j - w_{j-1}\|_2
+ 2qBC(3M\varepsilon)^{q-1}(T + R)^2\|u_j - u_{j-1}\|_1
$$

because of $|\overline{L}(v)| \leq L'(|v|)$, which implies that

$$
\|(u_{j+1})_x - (u_j)_x\|_1 = O\left(\frac{1}{2^j}\right)
$$

as $j \to \infty$ due to (3.6).

On the other hand, we have that

$$
\|(w_{j+1})_x - (w_j)_x\|_2 \leq pA\|L'(|w_j|^{p-2}w_j(w_j)_x - |w_{j-1}|^{p-2}w_{j-1}(w_{j-1})_x)| \|_2
+ qB\|L'(|w_j|^{q-2}u_j(w_j)_x - |u_{j-1}|^{q-2}u_{j-1}(w_{j-1})_x)| \|_2.
$$

The first term on the right hand side of this inequality is divided into two pieces according to

$$
|w_j|^{p-2}w_j(w_j)_x - |w_{j-1}|^{p-2}w_{j-1}(w_{j-1})_x
= (|w_j|^{p-2}w_j - |w_{j-1}|^{p-2}w_{j-1})(w_j)_x
+ |w_{j-1}|^{p-2}w_{j-1}(w_j)_x - (w_{j-1})_x.
$$
Since one can employ the estimate
\[
(\|w_j\|^p - w_j - |w_{j-1}|^p w_{j-1}| \leq \left\{ \begin{array}{ll}
(p - 1)2^{p-2}(|w_j|^p + |w_{j-1}|^p) |w_j - w_{j-1}| & \text{for } p \geq 2,
2|w_j - w_{j-1}|^p & \text{for } 1 < p < 2,
\end{array} \right.
\]
and the same one in which \(w\) is replaced with \(u\), we obtain that
\[
\|\{(w_{j+1})_x - (w_j)_x\|_2 \leq pAC(T + R)\|\{(w_j)_x\|_2 \times
\begin{cases}
(p - 1)2^{p-2}\|w_j\|_2^{p-2} + \|w_{j-1}\|_2^{p-2} & \text{for } p \geq 2,
0 & \text{for } 1 < p < 2
\end{cases}
+qBC(T + R)^2\|\{(u_j)_x\|_1 \times
\begin{cases}
(q - 1)2^{q-2}\|u_j\|_1^{q-2} + \|u_{j-1}\|_1^{q-2} & \text{for } q \geq 2,
0 & \text{for } 1 < q < 2
\end{cases}
+qBC(T + R)^2\|\{(u_j)_x - (u_{j-1})_x\|_1.
\]
Hence it follows from (3.6) and (3.10) that
\[
\|\|(w_{j+1})_x - (w_j)_x\|_2 \leq pAC(3M\varepsilon)^{p-1}(T + R)\|\{(w_j)_x - (w_{j-1})_x\|_2
+O\left(\frac{1}{j\min\{p-1,q-1,1\}}\right)
\]
as \(j \to \infty\). Therefore we obtain the convergence of \{((u_j)_x, (w_j)_x)\} provided
\[
pAC(3M\varepsilon)^{p-1}(T + R) \leq \frac{1}{2}, \quad (3.11)
\]
Continuation of the proof.

The convergence of the sequence \{\{(u_j, w_j)\} to \((u, w)\) in the closed subspace of \(X\) satisfying
\[
\|u\|_1, \|(u_x)\|_1, \|w\|_2, \|(w)_x\|_2 \leq 3M\varepsilon
\]
is established by (3.5), (3.7), (3.9), and (3.11), which follow from
\[
C_0\varepsilon^{p-1}(T + R) \leq 1 \quad \text{and} \quad C_0\varepsilon^{q-1}(T + R)^2 \leq 1,
\]
where
\[
C_0 := \max \{3^pACM^{p-1}, 3^qBCM^{q-1}, 2^{p+q}ACM, 2^{q+3}qBCM^{q-1}, 3^{p+1}ACM^{p-1}, 3^qBCM^{q-1}, 2 \cdot 3^{p-1}ACM^{p-1}\}.
\]
Therefore the statement of Theorem 2.1 is established with
\[
\begin{align*}
\varepsilon_1 := \min\{ (2C_0R)^{-1/(p-1)}, (2^2C_0R^2)^{-2/(q-1)} \}
\end{align*}
\]
because
\[
R \leq \frac{1}{2} \min\{ C_0^{-1} \varepsilon^{-(p-1)}, C_0^{-1/2} \varepsilon^{-(q-1)/2} \}
\]
holds for \(0 < \varepsilon \leq \varepsilon_1\).

\[\square\]

4 Proof of Proposition 3.1

In this section, we prove a priori estimate (3.3). Recall the definition of \(L\) in (2.5) and \(L'\) in (2.8). From now on, a positive constant \(C\) independent of \(T\) and \(\varepsilon\) may change from line to line.

It follows from the assumption on the supports and the definition of \(L\) that
\[
|L(|w|^p)(x, t)| \leq C\|w\|^pL_1(x, t) \quad \text{for } |x| \leq t + R,
\]
where we set
\[
I_1(x, t) := \int_0^t ds \int_{x-t+s}^{x+t-s} (s-|y| + 2R)^{-p} \chi_w(y, s) dy.
\]
Here, \(\chi_w\) denotes a characteristic function of supp \(w\). First, we consider the case of \(x \geq 0\). From now on, we employ the change of variables by
\[
\alpha = s + y, \quad \beta = s - y.
\]
For \(t + x \geq R\) and \(t - x \geq R\), making use of the symmetry of the weight in \(y\) and extending the domain of the integral, we have that
\[
I_1(x, t) \leq C \int_{t-x}^t \int_{t-x}^{t+x} d\alpha \int_{t-x}^{t+x} d\beta \int_{t-x}^{t+x} (\alpha + 2R)^{-p} d\alpha
+ C \int_{t-x}^t \int_{t-x}^{t+x} d\alpha \int_{t-x}^{t+x} (\beta + 2R)^{-p} d\beta
+ C \int_{t-x}^t \int_{t-x}^{t+x} \int_{t-x}^{t+x} \int_{t-x}^{t+x} d\alpha \int_{t-x}^{t+x} d\beta.
\]
For \(t + x \geq R\) and \(|t-x| \leq R\), we also have that
\[
I_1(x, t) \leq C \int_{t-x}^t \int_{t-x}^{t+x} d\alpha \int_{t-x}^{t+x} d\beta \leq C(T + R).
\]
For $t + x \leq R$, it is trivial that
\[ I_1(x, t) \leq C. \]

Summing up, we obtain that
\[ |L(|w|^p)(x, t)| \leq C\|w\|_2^p(T + R) \quad \text{for } 0 \leq x \leq t + R. \]

The case of $x \leq 0$ is similar to the one above, so we omit the details. Therefore we obtain the first inequality of the first line of (3.3).

The second inequality of the first line of (3.3) follows from
\[ |L(|u|^q)(x, t)| \leq C\|u\|_q^q I_2(x, t) \quad \text{for } |x| \leq t + R, \]

where
\[ I_2(x, t) := \int_0^t ds \int_{x-t+s}^{x+t-s} \chi_u(y, s) dy. \]

Here, $\chi_u$ denotes a characteristic function of supp $u$. Indeed, it is trivial that
\[ I_2(x, t) \leq C(T + R)^2 \quad \text{for } |x| \leq t + R. \]

Next, we shall show the second line in (3.3). It follows from the assumption on the supports and the definition of $L'$ that
\[ |L'(|w|^p)(x, t)| \leq C\|w\|_2^p \{I_+(x, t) + I_-(x, t)\} \quad \text{for } |x| \leq t + R, \]

where the integrals $I_+$ and $I_-$ are defined by
\[ I_\pm(x, t) := \int_0^t (s - |t - s \pm x| + 2R)^{-p} \chi_\pm(x, t; s) ds \]

and the characteristic functions $\chi_+$ and $\chi_-$ are defined by
\[ \chi_\pm(x, t; s) := \chi\{s; |t - s \pm x| \leq s \pm R\}, \]

respectively. First we note that it is sufficient to estimate $I_+$ for $x \geq 0$ due to its symmetry,
\[ I_+(−x, t) = I_−(x, t). \]

Hence it follows from $0 \leq x \leq t + R$ as well as
\[ |t - s + x| \leq s + R \quad \text{and} \quad 0 \leq s \leq t \]

that
\[ \frac{t + x - R}{2} \leq s \leq t. \]
Therefore we obtain

\[ I_+(x, t) \leq \int_{(t+R)/2}^{t} \left( s - (t - s + x) + 2R \right)^{-p} ds \leq C \quad \text{for } 0 \leq x \leq t + R. \]

On the other hand, the estimate for \( I_- \) is divided into two cases. If \( t - x \geq 0 \), then \( |t - s - x| \leq s + R \) yields that

\[
\begin{align*}
& \quad \left\{ \begin{array}{l}
t - s - x \leq s + R \quad \text{for } 0 \leq s \leq t - x, \\
-t + s + x \leq s + R \quad \text{for } t - x \leq s \leq t,
\end{array} \right.
\end{align*}
\]

so that

\[
I_-(x, t) \leq \int_{t-x}^{t} \left( s - (t - s - x) + 2R \right)^{-p} ds
\]

\[
+ \int_{t-x}^{t} \left( s + (t - s - x) + 2R \right)^{-p} ds 
\]

\[
\leq C \left( 1 + (t - x + 2R)^{-p} \right)
\]

follows. Therefore, neglecting \( (t - x + 2R)^{1-p} \), we obtain

\[
I_-(x, t) \leq C(t - x + 2R)^{-1}(T + R) \quad \text{for } t - x \geq 0.
\]

If \( (-R \leq) t - x \leq 0 \), \( |t - s - x| \leq s + R \) yields

\[
s - t + x \leq s + R \quad \text{for } 0 \leq s \leq t,
\]

so that

\[
I_-(x, t) \leq \int_{0}^{t} \left( s - (s - t + x) + 2R \right)^{-p} ds.
\]

Therefore we obtain

\[
I_-(x, t) \leq C(t - x + 2R)^{-1}(T + R) \quad \text{for } -R \leq t - x \leq 0.
\]

Summing up all the estimates for \( I_+ \) and \( I_- \), we have

\[
|L'(|w|^p)(x, t)| \leq C\|w\|_2^p (t - |x| + 2R)^{-1}(T + R) \quad \text{for } |x| \leq t + R.
\]

This yields the first inequality of the second and third lines in (3.3).

The second inequality in the second and third lines in (3.3) readily follows from

\[
|L'(|u|^q)(x, t)| \leq C\|u\|_q^q \left\{ K_+(x, t) + K_-(x, t) \right\} \quad \text{for } |x| \leq t + R,
\]

where the integrals \( K_+ \) and \( K_- \) are defined by

\[
K_\pm(x, t) := \int_{0}^{t} \chi_\pm(x, t; s) ds.
\]

Indeed, it is easy to obtain that

\[
K_\pm(x, t) \leq C(t - |x| + 2R)^{-1}(T + R)^2 \quad \text{for } |x| \leq t + R.
\]

The proof of Proposition 3.1 is now completed. \( \square \)
5 Proof of Theorem 2.2

First we note that the strong Huygens’ principle

\[ u^0(x, t) \equiv 0 \quad \text{in } D \]  

(5.1)

holds in this case of (2.17), where

\[ D := \{(x, t) \in \mathbb{R} \times [0, \infty) : t - |x| \geq R\}. \]

This is almost trivial if one takes a look on the representation of \( u^0 \) in (2.4) and the support condition on the data in (2.1). But one can see also Proposition 2.2 in Kitamura, Morisawa and Takamura [7] for the details. So, our unknown functions are \( U := u - \varepsilon u^0 \) and \( W := w - \varepsilon u^0 \) in (2.14). Let \( \{(U_j, W_j)\}_{j \in \mathbb{N}} \) be a sequence of \( \{C^1(\mathbb{R} \times [0, T])\}^2 \) defined by

\[
\begin{aligned}
U_{j+1} &= L(A|W_j + \varepsilon u^0|^p + B|U_j + \varepsilon u^0|^q), \quad U_1 = 0, \\
W_{j+1} &= L'(A|W_j + \varepsilon u^0|^p + B|U_j + \varepsilon u^0|^q), \quad W_1 = 0.
\end{aligned}
\]

(5.2)

Then, \( \{(U_j, W_j)\} \) has to satisfy

\[
\begin{aligned}
(U_{j+1})_x &= L'(A|W_j + \varepsilon u^0|^p + B|U_j + \varepsilon u^0|^q), \\
(U_1)_x &= 0, \\
(W_{j+1})_x &= L'(Ap|W_j + \varepsilon u^0|^p - 2(W_j + \varepsilon u^0)(W_j)_x + \varepsilon u^0_x) + L'(Bq|U_j + \varepsilon u^0|^q - 2(U_j + \varepsilon u^0)(U_j)_x + \varepsilon u^0_x), \\
(W_1)_x &= 0,
\end{aligned}
\]

(5.3)

so that the function space in which \( \{(U_j, W_j)\} \) converges is

\[ Y := \{(U, W) \in \{C^1(\mathbb{R} \times [0, T])\}^2 : \|(U, W)\|_Y < \infty, \quad \text{supp } (U, W) \subset \{|x| \leq t + R\}\} \]

which is equipped with a norm

\[ \|(U, W)\|_Y := \|U\|_3 + \|U_x\|_3 + \|W\|_4 + \|W_x\|_4, \]

where

\[
\begin{aligned}
\|U\|_3 &:= \sup_{(x, t) \in \mathbb{R} \times [0, T]} (t + |x| + R)^{-1}|U(x, t)|, \\
\|W\|_4 &:= \sup_{(x, t) \in \mathbb{R} \times [0, T]} \|\chi_D(x, t) + (1 - \chi_D(x, t))(t + |x| + R)^{-1}\|W(x, t)|,
\end{aligned}
\]

and \( \chi_D \) is a characteristic function of \( D \). Similarly to the proof of Theorem 2.1, we note that \( \text{supp } (U_j, W_j) \in \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\} \) implies \( \text{supp } (U_{j+1}, W_{j+1}) \in \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\} \).

The following lemmas are a priori estimates in this case.
Proposition 5.1 Let \((U, W) \in \{\mathcal{C}(\mathbb{R} \times [0, T])\}^2\) with
\[
\text{supp} \ (U, W) \subset \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\}
\]
and \(U^0 \in \mathcal{C}(\mathbb{R} \times [0, T])\) with
\[
\text{supp} \ U^0 \subset \{(x, t) \in \mathbb{R} \times [0, T] : (t - |x|)_+ \leq |x| \leq t + R\}.
\]
Then there exists a positive constant \(E\) independent of \(T\) and \(\varepsilon\) such that
\[
\begin{align*}
\|L^0(U^0)^{p-m}W^m\|_3 &\leq E\|U^0\|^{p-m}_{q}(T + R)^{p}, \\
\|L^0(U^0)^{q-m}U^m\|_3 &\leq E\|U^0\|^{q-m}_{q}(T + R)^{q + 1}, \\
\|L'(U^0)^{p-m}W^m\|_4 &\leq E\|U^0\|^{p-m}_{q}(T + R)^{p}, \\
\|L'(U^0)^{q-m}U^m\|_4 &\leq E\|U^0\|^{q-m}_{q}(T + R)^{q + 1}, \\
\|L^0(U^0)^{p-m}W^m\|_3 &\leq E\|U^0\|^{p-m}_{q}(T + R)^{p}, \\
\|L'(U^0)^{q-m}U^m\|_3 &\leq E\|U^0\|^{q-m}_{q}(T + R)^{q + 1}.
\end{align*}
\]
(5.4)
where \(p - m, q - m > 0\) \((m = 0, 1, 2)\) and the norm \(\| \cdot \|_{\infty}\) is defined by
\[
\|U^0\|_{\infty} := \sup \ {U^0(x, t)}.
\]

Proposition 5.2 Let \((U, W) \in \{\mathcal{C}(\mathbb{R} \times [0, T])\}^2\) with
\[
\text{supp} \ (U, W) \subset \{(x, t) \in \mathbb{R} \times [0, T] : |x| \leq t + R\}.
\]
Then there exists a positive constant \(C\) independent of \(T\) and \(\varepsilon\) such that
\[
\begin{align*}
\|L(W^p)\|_3 &\leq C\|W\|_2^p(T + R)^p, \\
\|L(U^q)\|_3 &\leq C\|U\|_3^q(T + R)^{q+1}, \\
\|L'(W^p)\|_4 &\leq C\|W\|_2^p(T + R)^p, \\
\|L'(U^q)\|_4 &\leq C\|U\|_3^q(T + R)^{q+1}, \\
\|L(W^p)\|_3 &\leq C\|W\|_2^p(T + R)^p, \\
\|L'(U^q)\|_3 &\leq C\|U\|_3^q(T + R)^{q+1}.
\end{align*}
\]
(5.5)
The proofs of Proposition 5.1 and 5.2 are established in the next section and after the next section respectively. Set
\[
N := (2^n p A + 2^n q B) E \left[ \sum_{\gamma = 0}^{1} \left\{ \sum_{\alpha = 0}^{2} \left( \|f(\alpha)\|_{L^\infty(R)}^{p-\gamma} + \|f(\alpha)\|_{L^\infty(R)}^{q-\gamma} \right) \right\} + \|g\|_{L^\infty(R)}^{p-\gamma} + \|g\|_{L^\infty(R)}^{q-\gamma} + \sum_{\beta = 0}^{1} \left( \|g(\beta)\|_{L^\infty(R)}^{p-\gamma} + \|g(\beta)\|_{L^\infty(R)}^{q-\gamma} \right) \right] \\
+ \sum_{\alpha = 0}^{2} \|f(\alpha)\|_{L^\infty(R)} + \|g\|_{L^1(R)} + \sum_{\beta = 0}^{1} \|g(\beta)\|_{L^\infty(R)} \right],
\]
19
where $E$ is the one in (5.4). Since $u^0, u_1^0, u_2^0, u_{tx}^0$ satisfy
\[
\|u^0\|_\infty \leq \|f\|_{L^\infty(R)} + \|g\|_{L^1(R)}, \\
\|u_1^0\|_\infty, \|u_2^0\|_\infty \leq \|f'\|_{L^\infty(R)} + \|g\|_{L^\infty(R)}, \\
\|u_{tx}^0\|_\infty \leq \|f''\|_{L^\infty(R)} + \|g'\|_{L^\infty(R)},
\]
we have that
\[
\|u^0\|_\infty^p \leq 2^p \left( \|f\|_{L^\infty}^p + \|g\|_{L^1(R)}^p \right), \\
\|u_1^0\|_\infty^p, \|u_2^0\|_\infty^p \leq 2^p \left( \|f'\|_{L^\infty(R)}^p + \|g\|_{L^\infty(R)}^p \right), \\
\|u_{tx}^0\|_\infty^p \leq 2^p \left( \|f''\|_{L^\infty(R)}^p + \|g'\|_{L^\infty(R)}^p \right).
\]
Let us assume that
\[
0 < \varepsilon \leq 1
\]
and define $\varepsilon_i, (i = 1, 2, 3, 4)$ respectively by
\[
\varepsilon_{21} := \min \left\{ \left(2^{2p}3^pACN^{p-1}RP_1\right)^{-1/(\min\{p,q\}(p-1))}, \\
(2^{2q+1}qBNq^{-1}RP_1)^{-1/(\min\{p,q\}(q-1))} \right\}, \\
\varepsilon_{22} := \min \left\{ (2^33^{2(p-1)}ACN^{p-1}RP_1)^{-1/(\min\{p,q\}(p-1))}, \\
(2^{3p-1}NR_1)^{-1/(p-1)}, \\
(2^{3q-1}BNq^{-1}RP_1)^{-1/(q-1)} \right\}, \\
\varepsilon_{23} := \min \left\{ \varepsilon_{21}, (2^{2p-1}3^{p+1}ACN^{p-1}RP_1)^{-1/\min\{p,q\}(p-1)} \\
(2^{p-1}3^{p-1}N^{p-1}RP_1)^{-1/(p-1)} \right\}, \\
\varepsilon_{24} := \min \left\{ (2^{2p+1}3^{p-1}ACN^{p-1}RP_1)^{-1/(\min\{p,q\}(p-1))}, \\
(2^{3}NR_1)^{-1/(p-1)} \right\}. \tag{5.6}
\]

The convergence of the sequence $\{(U_j, W_j)\}$.
It follows from (5.2), Proposition 5.1 and 5.2 that
\[ \|U_{j+1}\|_3 \leq A\|L(|W_j + \varepsilon u_0^p|)|_3 + B\|L(|U_j + \varepsilon u_0^q|)|_3 \]
\[ \leq 2^p A \left\{ \|L(|W_j|^p)|_3 + \|L(\varepsilon |u_0^p|)|_3 \right\} + 2^q B \left\{ \|L(|U_j|^q)|_3 + \|L(\varepsilon |u_0^q|)|_3 \right\} \]
\[ \leq 2^p A \left\{ C\|W_j\|_t^p(T + R)^p + E\varepsilon^p\|u_0^p\|_\infty^p \right\} + 2^q B \left\{ C\|U_j\|_t^q(T + R)^q + E\varepsilon^q\|u_0^q\|_\infty^q \right\} \]
\[ \leq 2^p A C\|W_j\|_t^p(T + R)^p + 2^q B C\|U_j\|_t^q(T + R)^q + N\varepsilon^{\min\{p,q\}}. \]

and
\[ \|W_{j+1}\|_4 \leq A\|L'(|W_j + \varepsilon u_0^p|)|_4 + B\|L'(|U_j + \varepsilon u_0^q|)|_4 \]
\[ \leq 2^p A \left\{ \|L'|(|W_j|^p)|_4 + \|L'|(\varepsilon |u_0^p|)|_4 \right\} + 2^q B \left\{ \|L'|(|U_j|^q)|_4 + \|L'|(\varepsilon |u_0^q|)|_4 \right\} \]
\[ \leq 2^p A \left\{ C\|W_j\|_t^p(T + R)^p + E\varepsilon^p\|u_0^p\|_\infty^p \right\} + 2^q B \left\{ C\|U_j\|_t^q(T + R)^q + E\varepsilon^q\|u_0^q\|_\infty^q \right\} \]
\[ \leq 2^p A C\|W_j\|_t^p(T + R)^p + 2^q B C\|U_j\|_t^q(T + R)^q + N\varepsilon^{\min\{p,q\}}. \]

Hence the boundedness of \{(U_j, W_j)\}, i.e.
\[ \|U_j\|_3, \|W_j\|_4 \leq 3N\varepsilon^{\min\{p,q\}} \quad (j \in \mathbb{N}), \]

follows from
\[
\begin{aligned}
2^p A C(3N\varepsilon)^{p\min\{p,q\}}(T + R)^{p} & \leq N\varepsilon^{\min\{p,q\}}, \\
2^q B C(3N\varepsilon)^{q\min\{p,q\}}(T + R)^{q+1} & \leq N\varepsilon^{\min\{p,q\}}.
\end{aligned}
\]
(5.8)

Since (5.6) yields that
\[ R \leq C_1 \min \left\{ \varepsilon^{\min\{p,q\}(p-1)/p}, \varepsilon^{\min\{p,q\}(q-1)/(q+1)} \right\} \]
for \(0 < \varepsilon \leq \varepsilon_{21}\), where
\[ C_1 := \frac{1}{2} \min \left\{ (2^p 3^p ACN^{p-1})^{-1/p}, (2^q 3^q BCN^{q-1})^{-1/(q+1)} \right\}, \]
we find that (5.8) as well as (5.7) follows from
\[ T \leq C_1 \min \left\{ \varepsilon^{\min\{p,q\}(p-1)/p}, \varepsilon^{\min\{p,q\}(q-1)/(q+1)} \right\} \]
(5.9)
for \(0 < \varepsilon \leq \varepsilon_{21}\).

Let us write down this inequality in each cases. It follows from
\[
\frac{p - 1}{p} - \frac{q - 1}{q + 1} = \frac{2p - q - 1}{p(q + 1)}
\]
that

\[ T \leq C_1 \varepsilon^{-p(p-1)/p} = C_1 \varepsilon^{-(p-1)} \] for \( p \leq \frac{q+1}{2} < q \)

because of \( \min\{p,q\} = p \),

\[ T \leq C_1 \varepsilon^{-p(q-1)/(q+1)} \] for \( \frac{q+1}{2} \leq p \leq q \)

because of \( \min\{p,q\} = p \), and

\[ T \leq C_1 \varepsilon^{-q(q-1)/(q+1)} \] for \( p \geq q \)

because of \( \min\{p,q\} = q \).

Next, assuming (5.9), one can estimate \((U_{j+1} - U_j)\) and \((W_{j+1} - W_j)\) as follows. The inequalities

\[
\begin{align*}
\|U_{j+1} - U_j\|_3 &= \|L(A|W_j + \varepsilon u_0^p - A|W_{j-1} + \varepsilon u_0^p|p
+ B|U_j + \varepsilon u_0^q - B|U_{j-1} + \varepsilon u_0^q)\|_3 \\
&\leq pA\|L(|W_j - W_{j-1}|)|p-1|W_j - W_{j-1})\|_3 \\
&+ qB\|L((|U_j - U_{j-1}|)|q-1|U_j - U_{j-1})\|_3 \\
&\leq 3^{p-1}pA\|L(|W_j - W_{j-1}|)|p-1|W_j - W_{j-1})\|_3 \\
&+ 3^{q-1}qB\|L((|U_j - U_{j-1}|)|q-1|U_j - U_{j-1})\|_3 \\
&\leq 3^{p-1}pAC(|W_j - W_{j-1}|)|p-1|W_j - W_{j-1})\|_4(T + R)^p \\
&+ 3^{q-1}qBC(|U_j - U_{j-1}|)|q-1|U_j - U_{j-1})\|_3(T + R)^q+1 \\
&\leq 3^{p-1}\{2pAC(3N\varepsilon^{\min\{p,q\}})|p-1|W_j - W_{j-1})\|_4(T + R)^p + N\varepsilon^{p-1}(T + R)\} \\
&\times \|W_j - W_{j-1}\|_4 \\
&+ 3^{q-1}\{2qBC(3N\varepsilon^{\min\{p,q\}})|q-1|U_j - U_{j-1})\|_3(T + R)^q+1 + N\varepsilon^{q-1}(T + R)\} \\
&\times \|U_j - U_{j-1}\|_3
\end{align*}
\]
we find that (5.11) as well as (5.10) follows from
are fulfilled. Since (5.6) yields that
and
\[
\|W_{j+1} - W_j\|_4 = \|L'(A)|W_j + \varepsilon u_0^2|^p - A|W_{j-1} + \varepsilon u_0^2|^p \\
+ B|U_j + \varepsilon u_0|^q - B|U_{j-1} + \varepsilon u_0|^q\|_4 \\
\leq 3^{p-1}pA\|L'(|W_{j-1}|^{p-1} + |W_j|^{p-1} + \varepsilon^{p-1}|u_0|^p^{p-1})|W_j - W_{j-1}|\|_4 \\
+ 3^{q-1}qB\|L'(|U_{j-1}|^{q-1} + |U_j|^{q-1} + \varepsilon^{q-1}|u_0|^q^{q-1})|U_j - U_{j-1}|\|_4 \\
\leq 3^{p-1}pAC(\|W_{j-1}\|_3^{p-1} + \|W_j\|_3^{p-1})\|W_j - W_{j-1}\|_4(T + R)^p \\
+ 3^{q-1}qBC(\|U_{j-1}\|_3^{q-1} + \|U_j\|_3^{q-1})\|U_j - U_{j-1}\|_3(T + R)^q \\
\leq 3^{p-1}\{2pAC(3N_0\varepsilon^{\min(p,q)}p-1)(T + R)^p + N_0\varepsilon^{p-1}(T + R)\} \\
\times \|W_j - W_{j-1}\|_4 \\
+ 3^{q-1}\{2qBC(3N_0\varepsilon^{\min(p,q)}q-1)(T + R)^q + N_0\varepsilon^{q-1}(T + R)\} \\
\times \|U_j - U_{j-1}\|_3
\]
hold with some \(\theta \in (0,1)\). Here we employ Hölder’s inequality like the one
in the proof of Theorem 2.1. Therefore the convergence of \(\{U_j\}\) follows from
\[
\|U_{j+1} - U_j\|_3 + \|W_{j+1} - W_j\|_4 \\
\leq \frac{1}{2}(\|U_j - U_{j-1}\|_3 + \|W_j - W_{j-1}\|_4) \quad (j \geq 2) \quad (5.10)
\]
provided (5.8) and
\[
\begin{cases}
2pAC(3N_0\varepsilon^{\min(p,q)}p-1)(T + R)^p + N_0\varepsilon^{p-1}(T + R) & \leq \frac{1}{3^{p-1}4}, \\
2qBC(3N_0\varepsilon^{\min(p,q)}q-1)(T + R)^q + N_0\varepsilon^{q-1}(T + R) & \leq \frac{1}{3^{q-1}4}
\end{cases} \quad (5.11)
\]
are fulfilled. Since (5.6) yields that
\[
R \leq C_2 \min \left\{ \varepsilon^{-\min(p,q)(p-1)/p}, \varepsilon^{-\min(p,q)(q-1)/(q+1)} \right\},
\]
for \(0 < \varepsilon \leq \varepsilon_{22}\), where
\[
C_2 := \frac{1}{2} \min \left\{ (2^32^{2(p-1)}pACN^{p-1})^{-1/p}, (2^{2}3^{p-1}pAN)^{-1}, \\
(2^32^{q-1})qBCN^{q-1})^{-1/(q+1)}, (2^{2}3^{q-1}qBN)^{-1} \right\},
\]
we find that (5.11) as well as (5.10) follows from
\[
T \leq C_2 \min \left\{ \varepsilon^{-\min(p,q)(p-1)/p}, \varepsilon^{-(p-1)}, \varepsilon^{-\min(p,q)(q-1)/(q+1)}, \varepsilon^{-(q-1)} \right\} \quad (5.12)
\]
for $0 < \varepsilon \leq \varepsilon_{22}$.

It is trivial that
\[
\varepsilon - \min(p,q)(p-1)/p = \varepsilon - (p-1) \leq \varepsilon - (q-1) \quad \text{for } p \leq q,
\]
\[
\varepsilon - \min(p,q)(q-1)/(q+1) = \varepsilon - q(q-1)/(q+1) \leq \varepsilon - (q-1) \quad \text{for } p \geq q.
\]

Therefore the computations after (5.9) implies that (5.12) is equivalent to
\[
T \leq \begin{cases} 
C_2\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2} (< q), \\
C_2\varepsilon^{-p(q-1)/(q+1)} & \text{for } \frac{q+1}{2} \leq p \leq q, \\
C_2\varepsilon^{-q(q-1)/(q+1)} & \text{for } p \geq q.
\end{cases}
\]

The convergence of the sequence $\{((U_j)_x, (W_j)_x)\}$.

Assume (5.9) and (5.12). Then we have (5.7) and (5.10). Since it follows from (5.3) that
\[
\begin{align*}
|\langle U_{j+1} \rangle_x | & \leq 2^p A \{ L'(|W_j|^p) + \varepsilon^p L'(|u_0^p|) \} + 2^q B \{ L'(|U_j|^q) + \varepsilon^q L'(|u_0^q|) \}, \\
|\langle W_{j+1} \rangle_x | & \leq 2^{p-1} p A \{ L'(|W_j|^{p-1})(|W_j|_x) + \varepsilon L'(|W_j|^{p-1}|u_0^p|) \\
& \quad + \varepsilon^{p-1} L'(|u_0^{p-1}|(|W_j|_x) + \varepsilon^p L'(|u_0^p|)} \} + 2^{q-1} q B \{ L'(|U_j|^{q-1})(|U_j|_x) + \varepsilon L'(|U_j|^{q-1}|u_0^q|) \\
& \quad + \varepsilon^{q-1} L'(|u_0^{q-1}|(|U_j|_x) + \varepsilon^q L'(|u_0^q|)} \},
\end{align*}
\]

Proposition 5.1 and 5.2 yield that
\[
\begin{align*}
|\langle U_{j+1} \rangle_x |_3 & \leq 2^p A \{ L'(|W_j|^p) \|_3 + \varepsilon^p \| L'(|u_0^p|) \|_3 \} + 2^q B \{ L'(|U_j|^q) \|_3 + \varepsilon^q \| L'(|u_0^q|) \|_3 \} \\
& \leq 2^p A \{ C \| W_j \|_3^p (T + R)^p + \varepsilon^p \| u_0^p \|_\infty \} + 2^q B \{ C \| U_j \|_3^q (T + R)^q + \varepsilon^q \| u_0^q \|_\infty \} \\
& \leq N \varepsilon^{\min(p,q)} + 2^p AC \| W_j \|_3^p (T + R)^p + 2^q BC \| U_j \|_3^q (T + R)^q + 1.
\end{align*}
\]
and
\[
\|((W_{j+1})_x)\|_4 \leq 2^{p-1} p A \left\{ \|L'(|W_j|^{p-1} |W_j|_x)\|_4 + \varepsilon \|L'(|W_j|^{p-1} |u_{0x}|_x)\|_3 \\
+ \varepsilon^2 \|L'(|u_{0x}|^{p-1} |W_j|_x)\|_4 + \varepsilon^p \|L'(|u_{0x}^{p-1} |u_{0x}|_x)\|_4 \\
+ 2^{q-1} q B \left\{ \|L'(|U_j|^{q-1} |U_j|_x)\|_4 + \varepsilon \|L'(|U_j|^{q-1} |u_{0x}^q|)\|_4 \\
+ \varepsilon^q \|L'(|u_{0x}^q|^{q-1} |U_j|_x)\|_4 + \varepsilon^p \|L'(|u_{0x}^q|^{q-1} |u_{0x}^q|_x)\|_4 \right\} \right.
\]
\[
\leq 2^{p-1} p A \left\{ C \|W_j\|_4^{-1} \|W_j\|_4 (T + R)^p \\
+ \varepsilon E \|u_0^p\|_\infty \|W_j\|_3^{-1} (T + R)^{p-1} \\
+ \varepsilon^{p-1} E \|u_0^p\|_\infty^{-1} \|W_j\|_3 (T + R) + \varepsilon^p E \|u_0^p\|_\infty^{-1} \|u_{0x}^q\|_\infty \right\} \\
+ 2^{q-1} q B \left\{ C \|U_j\|_3^{-1} \|U_j\|_3 (T + R)^{q+1} \\
+ \varepsilon E \|u_0^q\|_\infty \|U_j\|_3^{-1} (T + R)^{q-1} \\
+ \varepsilon^{q-1} E \|u_0^q\|_\infty^{-1} \|U_j\|_3 (T + R) + \varepsilon^q E \|u_0^q\|_\infty^{-1} \|u_{0x}^q\|_\infty \right\} \\
\leq N \varepsilon^{-\min(p,q)} + 2^{p-1} p AC (3N \varepsilon^{-\min(p,q)}) (T + R)^p \|W_j\|_4 \\
+ N \varepsilon (3N \varepsilon^{-\min(p,q)}) (T + R)^{p-1} + N \varepsilon^{-\min(p,q)} (T + R) \|W_j\|_4 \\
+ 2^{q-1} q BC (3N \varepsilon^{-\min(p,q)}) (T + R)^{q+1} \|U_j\|_3 \\
+ N \varepsilon (3N \varepsilon^{-\min(p,q)}) (T + R)^{q-1} + N \varepsilon^{-\min(p,q)} \|U_j\|_3. \tag{5.13}
\]

Hence the boundedness of \{((U_j)_x, (W_j)_x)\}, i.e.
\[
\|((U_j)_x)\|_3, \|((W_j)_x)\|_4 \leq 3N \varepsilon^{-\min(p,q)} \quad (j \in \mathbb{N}), \tag{5.14}
\]
follows from
\[
\left\{ \begin{array}{l}
2^{p} AC (3N \varepsilon^{-\min(p,q)}) (T + R)^p \\
2^{q} BC (3N \varepsilon^{-\min(p,q)}) (T + R)^{q+1}
\end{array} \right. \leq N \varepsilon^{-\min(p,q)}, \tag{5.15}
\]
and
\[
\left\{ \begin{array}{l}
3 \cdot 2^{p-1} p AC (3N \varepsilon^{-\min(p,q)}) (T + R)^p \\
3N \varepsilon (3N \varepsilon^{-\min(p,q)}) (T + R)^{p-1} \\
3N \varepsilon^{-\min(p,q)} (T + R) \\
3 \cdot 2^{q-1} q BC (3N \varepsilon^{-\min(p,q)}) (T + R)^{q+1} \\
3N \varepsilon (3N \varepsilon^{-\min(p,q)}) (T + R)^{q-1} \\
3N \varepsilon^{-\min(p,q)} (T + R)
\end{array} \right. \leq N \varepsilon^{-\min(p,q)}.
\tag{5.16}
\]

Since (5.6) yields that
\[
R \leq C_3 \min \left\{ \varepsilon^{-\min(p,q)(p-1)/p}, \varepsilon^{-\min(p,q)(p-2)/p}, \varepsilon^{-\min(p,q)(p-1)}, \varepsilon^{-\min(p,q)(q-1)/q}, \varepsilon^{-\min(p,q)(q-2)/q}, \varepsilon^{-\min(p,q)(q-1)}, \varepsilon^{-\min(p,q)(q-1)} \right\}
\]
for \(0 < \varepsilon \leq \varepsilon_{23},\)
where
\[
C_3 := \frac{1}{2} \min \left\{ \left(2^{p-1} p AC N^{p-1}\right)^{-1/p}, \left(2^{q} BC N^{q-1}\right)^{-1/(q+1)} \right\}
\]
\[
\left(2^{p-1} p AC N^{p-1}\right)^{-1/p}, \left(2^{q} BC N^{q-1}\right)^{-1/(q+1)}, \left(3^2 N\right)^{-1}, \left(2^{q-1} q BC N^{q-1}\right)^{-1/(q+1)}, \left(3^2 N^{q-1}\right)^{-1/(q-1)}, \left(3^2 N^{-1}\right) \right\}, 25
we find that (5.14) and (5.15) as well as (5.13) follow from

\[ T \leq C_3 \min \{ \epsilon^{-\min\{p,q\}(p-1)/p}, \epsilon^{-(\min\{p,q\}(p-2)+1)/(p-1)}, \]

\[ \epsilon^{-(p-1)}, \epsilon^{-\min\{p,q\}(q-1)/(q+1)}, \]

\[ \epsilon^{-\min\{p,q\}(q-2)+1/(q-1)}, \epsilon^{-(q-1)} \} \]  

(5.16)

for 0 < \epsilon \leq \epsilon_{23}.

Let us write down this inequality in each case. For \( p \leq (q+1)/2 (\leq q) \), we have that

\[
\min\{p, q\}(p - 1) - (p - 1) = 0, \\
\frac{p}{p-1} - (p - 1) = \frac{p(p-2) + 1}{p-1} - (p - 1) = 0, \\
\frac{q+1}{p-1} - (p - 1) = \frac{p(q-1)}{q+1} - (p - 1) = 0, \\
\frac{q+1}{q-1} - (p - 1) = \frac{p(q-2) + 1}{q-1} - (p - 1) = 0,
\]

\( (q - 1) - (p - 1) > 0 \)

which implies that (5.16) is equivalent to

\[ T \leq C_3 \epsilon^{-(p-1)} \quad \text{for } p \leq \frac{q+1}{2}. \]

For \( (q+1)/2 \leq p \leq q \), we have that

\[
\frac{p}{p-1} - (p - 1) = \frac{p(p-2) + 1}{p-1} - (p - 1) = 0, \\
\frac{q+1}{q-1} - (p - 1) = \frac{p(q-2) + 1}{q-1} - (p - 1) = 0,
\]

\( (q - 1) - (p - 1) > 0 \)

(26)
which implies that (5.16) is equivalent to

$$T \leq C \varepsilon^{-p(q-1)/(q+1)} \quad \text{for } \frac{q+1}{2} \leq p \leq q.$$ 

For $p \geq q$, we have that

$$\min\{p, q\}(p-1) - \frac{q(q-1)}{q+1} = \frac{q(p-1)}{p} - \frac{q(q-1)}{q+1}.$$

Next, assuming (5.9), (5.12) and (5.16), we shall estimate \{$(U_{j+1}) - (U_j)$\} and \{(W_{j+1}) - (W_j)$\}. It is easy to see that

$$\begin{align*}
|[(U_{j+1}) - (U_j)]| &\leq L(|A[W_j + \varepsilon u_0^p] - [W_{j-1} + \varepsilon u_0^p]) \\
&+ L(|B[U_j + \varepsilon u_0^q] - [U_{j-1} + \varepsilon u_0^q]) \\
&\leq pAL'(|W_{j-1} + \varepsilon u_0^p + \theta(W_{j+1} - W_j)|^{p-1}|W_j - W_{j-1}) \\
&+ qBL'(|U_{j-1} + \varepsilon u_0^q + \theta(U_{j+1} - U_j)|^{q-1}|U_j - U_{j-1}) \\
&\leq 3^{p-1}pAL'(|[W_{j-1}]^{p-1} + |W_j|^{p-1} + \varepsilon^{p-1}|u_0^0|^{p-1})|W_j - W_{j-1}| \\
&+ 3^{q-1}qBL'(|[U_{j-1}]^{q-1} + |U_j|^{q-1} + \varepsilon^{q-1}|u_0^0|^{q-1})|U_j - U_{j-1}| \\
&\quad \text{for some } \theta \in (0, 1). 
\end{align*}$$

Moreover we have that

$$\begin{align*}
|[(W_{j+1}) - (W_j)]| &\leq L(|pA[W_j + \varepsilon u_0^p]^{p-2}(W_j + \varepsilon u_0^p)((W_j) + \varepsilon u_0^p) \\
&- pA[W_{j-1} + \varepsilon u_0^p]^{p-2}(W_{j-1} + \varepsilon u_0^p)((W_{j-1}) + \varepsilon u_0^p))| \\
&+ L(|qB[U_j + \varepsilon u_0^q]^{q-2}(U_j + \varepsilon u_0^q)((U_j) + \varepsilon u_0^q) \\
&- qB[U_{j-1} + \varepsilon u_0^q]^{q-2}(U_{j-1} + \varepsilon u_0^q)((U_{j-1}) + \varepsilon u_0^q))|.
\end{align*}$$
In order to estimate the quantities in the right hand side of this inequality, we employ

\[
\begin{align*}
|W_j + \varepsilon u^0_j|^{p-2}(W_j + \varepsilon u^0_j)(W_j + \varepsilon u^0_j)_x \\
- |W_{j-1} + \varepsilon u^0_j|^{p-2}(W_{j-1} + \varepsilon u^0_j)(W_{j-1} + \varepsilon u^0_j)_x \\
= (|W_j + \varepsilon u^0_j|^{p-2}(W_j + \varepsilon u^0_j) - |W_{j-1} + \varepsilon u^0_j|^{p-2}(W_{j-1} + \varepsilon u^0_j))(W_j + \varepsilon u^0_j)_x \\
+ |W_{j-1} + \varepsilon u^0_j|^{p-2}(W_{j-1} + \varepsilon u^0_j)((W_j + \varepsilon u^0_j)_x - (W_{j-1} + \varepsilon u^0_j)_x)
\end{align*}
\]

and

\[
\begin{align*}
||W_j + \varepsilon u^0_j|^{p-2}(W_j + \varepsilon u^0_j) - |W_{j-1} + \varepsilon u^0_j|^{p-2}(W_{j-1} + \varepsilon u^0_j)|| \\
\leq \begin{cases} 
(p - 1)|W_j + \varepsilon u^0_j + \theta(W_j - W_{j-1})|^{p-2}|W_j - W_{j-1}| & \text{for } p \geq 2, \\
2|W_j - W_{j-1}|^{p-1} & \text{for } 1 < p < 2,
\end{cases}
\end{align*}
\]

with some \( \theta \in (0, 1) \). Hence it follows from Propositions 5.1 and 5.2 that

\[
\begin{align*}
\| & (U_{j+1})_x - (U_j)_x \|_3 \\
\leq & 3^{p-1}pA \{ C(||W_{j-1}||^{p-1} + ||W_j||^{p-1})||W_j - W_{j-1}||_4(T + R)^p \\
& + \varepsilon^{p-1}E||u^0_j||^{p-1}||W_j - W_{j-1}||_4(T + R) \}
\end{align*}
\]

and

\[
\begin{align*}
\| & (W_j)_x - (W_{j-1})_x \|_4 \\
\leq & \begin{cases} 
3^{p-2}p(p - 1)A \{ C(||W_{j-1}||^{p-2} + ||W_j||^{p-2}) \\
\times ||W_j - W_{j-1}||_4||W_j + \varepsilon u^0_j||_4(T + R)^p \\
+ \varepsilon^{p-2}E||u^0_j||^{p-2}||W_j - W_{j-1}||_4 \\
\times ||(W_j)_x + \varepsilon u^0_j||_4(T + R)^2 \} & \text{for } p \geq 2, \\
2pAC||W_j - W_{j-1}||^{p-1} \\
\times ||(W_j)_x + \varepsilon u^0_j||_4(T + R)^p & \text{for } 1 < p < 2,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\| & L'(W_{j-1} + \varepsilon u^0_j)^{p-1}(W_j)_x - (W_{j-1})_x \|_4 \\
\leq & \begin{cases} 
3^{\sqrt{q}(q - 1)B} \{ C(||U_{j-1}||^{\sqrt{q}-2} + ||U_j||^{\sqrt{q}-2}) \\
\times ||U_j - U_{j-1}||_3||U_j + \varepsilon u^0_j||_3(T + R)^{q+1} \\
+ \varepsilon^{\sqrt{q}-2}E||u^0_j||^{\sqrt{q}-2}||U_j - U_{j-1}||_3 \\
\times ||(U_j)_x + \varepsilon u^0_j||_3(T + R)^2 \} & \text{for } q \geq 2, \\
2qBC||U_j - U_{j-1}||^{q-1} \\
\times ||(U_j)_x + \varepsilon u^0_j||_3(T + R)^{q+1} & \text{for } 1 < q < 2,
\end{cases}
\end{align*}
\]

Since (5.9) and (5.12) yield (5.10), we have that

\[
\|U_{j+1} - U_j\|_3 + ||W_{j+1} - W_j||_3 \leq O \left( \frac{1}{2^j} \right) \text{ as } j \to \infty.
\]
This fact implies that
\[ \| (U_{j+1})_x - (U_j)_x \|_3 = O \left( \frac{1}{2^j} \right) \quad \text{as } j \to \infty. \] (5.17)

Moreover it follows from (5.9) as well as (5.7) that
\[
P_A \| L'([W_{j-1} + \varepsilon u_0^{p-1}](W_j)_x - (W_{j-1})_x) \|_4 \\
\leq 2^{p-1} P_A \| L'([W_{j-1} + \varepsilon^{p-1} u_0^{p-1}](W_j)_x - (W_{j-1})_x) \|_4 \\
\leq 2^{p-1} P_A \| W_j \|_4^{p-1} \| (W_j)_x - (W_{j-1})_x \|_4 (T + R)^p \\
+ 2^{p-1} P_A \| \varepsilon^{p-1} u_0^{p-1} \|_\infty \| (W_j)_x - (W_{j-1})_x \|_4 (T + R) \\
\leq 2^{p-1} P_A C N_{\varepsilon_{\min(p,q)}} (T + R)^p \| (W_j)_x - (W_{j-1})_x \|_4 \\
+ N^p (T + R) \| (W_j)_x - (W_{j-1})_x \|_4 + O \left( \frac{1}{2^j \min(p-1,q-1,1)} \right).
\]

Hence we have that
\[
\| (W_{j+1})_x - (W_j)_x \|_4 \\
\leq 2^{p-1} 3^p - 1 P_A C N_{\varepsilon_{\min(p,q)}} (T + R)^p \| (W_j)_x - (W_{j-1})_x \|_4 \\
+ N^p (T + R) \| (W_j)_x - (W_{j-1})_x \|_4 + O \left( \frac{1}{2^j \min(p-1,q-1,1)} \right),
\]
as \( j \to \infty \). Therefore we obtain that
\[ \| (W_{j+1})_x - (W_j)_x \|_4 \leq O \left( \frac{1}{2^j} \right) \quad \text{as } j \to \infty \] (5.18)

provided
\[
\left\{\begin{array}{l}
2^{p-1} 3^p - 1 P_A C N_{\varepsilon_{\min(p,q)}} (T + R)^p \\
N^p (T + R)
\end{array}\right\} \leq \frac{1}{4}. \] (5.19)

holds. Since (5.6) yields that
\[ R \leq C_4 \min\{ \varepsilon_{\min(p,q)(p-1)/p}, \varepsilon^{-(p-1)} \} \]
for \( 0 < \varepsilon \leq \varepsilon_{21} \), where
\[ C_4 := \frac{1}{2} \min \left\{ (2^{p+1} 3^p - 1 P_A C N_{p-1})^{1/p}, (2^2 N)^{-1} \right\}, \]
we find that (5.19) as well as (5.17) and (5.18) follows from
\[ T \leq C_4 \min\{ \varepsilon_{\min(p,q)(p-1)/p}, \varepsilon^{-(p-1)} \} \] (5.20)
for \(0 < \varepsilon \leq \varepsilon_{24}\). We note that (5.20) is equivalent to
\[
T \leq \begin{cases} 
C_4 \varepsilon^{-(p-1)} & \text{for } p \leq q, \\
C_4 \varepsilon^{-q(p-1)/p} & \text{for } q \leq p.
\end{cases}
\]
Due to the computations after (5.16), we have that
\[
\varepsilon^{-(p-1)/(q+1)} \leq \varepsilon^{-(p-1)} \quad \text{for } \frac{q + 1}{2} \leq p \leq q
\]
and
\[
\varepsilon^{-(q-1)/(q+1)} \leq \varepsilon^{-q(p-1)/p} \quad \text{for } q \leq p.
\]

Continuation of the proof.

The convergence of the sequence \(\{(U_j, W_j)\}\) to \((U, W)\) in the closed subspace of \(Y\) satisfying
\[
\|U\|_3, \|(U_x)\|_3, \|W\|_4, \|(W)_x\|_4 \leq 3N\varepsilon^{\min\{p,q\}}
\]
is established by (5.6), (5.9), (5.12), (5.16) and (5.20). Therefore the statement of Theorem 2.2 is established with
\[
\left\{ c = \min\{C_1, C_2, C_3, C_4\}, \quad \varepsilon_2 = \min\{1, \varepsilon_{21}, \varepsilon_{22}, \varepsilon_{23}, \varepsilon_{24}\}\right. 
\]

\[ \square \]

6 Proof of Proposition 5.1

In this section, we prove a priori estimate (5.4) in the similar manner to the one for (3.3). A positive constant \(C\) independent of \(T\) and \(\varepsilon\) may change from line to line.

It follows from the assumption on the supports that
\[
\left| L((U^0)^{p-m}|W|^m)(x, t) \right| \leq C\|U^0\|^{p-m}_\infty \|W\|_4^m(T+R)^m J_0(x, t) \quad \text{for } |x| \leq t + R,
\]
where we set
\[
J_0(x, t) := \int_0^t ds \int_{x-t+s}^{x+t-s} \chi_{\text{supp } U^0(y, s)} dy.
\]
First, we consider the case of \(x \geq 0\). From now on, we employ the change of variables
\[
\alpha = s + y, \beta = s - y.
\]
For \( t + x \geq R \) and \( t - x \geq R \), extending the domain of the integral, we have that
\[
J_0(x, t) \leq C \int_{-R}^{R} \beta \, d\beta \int_{-R}^{t+x} \alpha \, d\alpha + C \int_{-R}^{t-x} \beta \, d\beta \int_{-R}^{R} \alpha \, d\alpha \\
\leq C(t + x + R).
\]

For \( t + x \geq R \) and \( |t - x| \leq R \), we also have that
\[
J_0(x, t) \leq C \int_{-R}^{R} \beta \, d\beta \int_{-R}^{t+x} \alpha \, d\alpha \leq C(t + x + R).
\]

For \( t + x \leq R \), it is trivial that \( J_0(x, t) \leq C \). Summing up, we obtain that
\[
|L(U_0|p-m|W|m)(x, t)| \leq C\|U_0\|_{p-m}\|W\|_m(T + R)^m(t + x + R) \quad \text{for } 0 \leq x \leq t + R.
\]

The case of \( x \leq 0 \) is similar to the one above, so we omit the details. Therefore we obtain the first inequality in (5.4). The second inequality in (5.4) follows from the computation above because we have that
\[
|L(U_0|q-m|U|m)(x, t)| \leq C\|U_0\|_{q-m}\|U\|_m(T + R)^mJ_0(x, t) \quad \text{for } |x| \leq t + R.
\]

Next, we shall show the third inequality in (5.4). It follows from the assumption on the support and the definition of \( L' \) that
\[
|L(U_0|p-m|W|m)(x, t)| \leq C\|U_0\|_{p-m}\|W\|_m(T + R)^m\{J_{1+}(x, t) + J_{1-}(x, t)\} \quad \text{for } |x| \leq t + R,
\]
where the integrals \( J_{1±} \) and \( J_{1±} \) are defined by
\[
J_{1±}(x, t) := \int_{0}^{t} \chi_{1±}(x, t; s)ds
\]
and the characteristic functions \( \chi_+ \) and \( \chi_- \) are defined by
\[
\chi_{1±}(x, t; s) := \chi(s; |s - t - s ± x| \leq R),
\]
respectively. First we note that it is sufficient to estimate \( J_{1±} \) for \( x \geq 0 \) due to its symmetry,
\[
J_{1+}(-x, t) = J_{1-}(x, t).
\]
For \( (x, t) \in D \cap \{x \geq 0\} \), we have that
\[
J_{1±}(x, t) = \int_{(t±x-R)/2}^{(t±x+R)/2} ds \leq C.
\]
On the other hand, for \( t + x \geq R, x \geq 0 \) and \(|t - x| \leq R\), we have that
\[
J_{1+}(x, t) = \int_{(t+x-R)/2}^{t} ds \leq C
\]
and
\[
J_{1-}(x, t) = \int_{(t-x-R)/2}^{t} ds \leq C(t + x + R).
\]
It is trivial that, for \( t + x \leq R, x \geq 0 \), we have \( J_{1\pm}(x, t) \leq C \). Summing up all the estimates above, we obtain that
\[
|L'(|U^0|^{p-m}|W|^m)(x, t)| \leq C\|U^0\|_{p-m}^{p-m}\|W\|_{m}^{m}(T + R)^{m}
\times \{\chi_D(x, t) + (1 - \chi_D(x, t))(t + |x| + R)\}
\]
for \(|x| \leq t + R\).

Therefore the third inequality in (5.4) is established.

Due to the assumptions on the supports as well as the definitions of the norms, it is clear that the proofs of the three remaining inequalities in (5.4) easily follow from the computations above. The proof of Proposition 5.1 is now completed. \( \square \).

7 Proof of Proposition 5.2

In this section, we prove a priori estimate (5.5). Again, a positive constant \( C \) independent of \( T \) and \( \varepsilon \) may change from line to line.

It follows from the assumption on the supports and the definition of \( L \) that
\[
|L(|W|^p)(x, t)| \leq C\|W\|_{4}^{p}J_1(x, t) \quad \text{for } |x| \leq t + R,
\]
where we set
\[
J_1(x, t) := \int_{0}^{t} ds \int_{x-t+s}^{x+s-t} \{\chi_D(s, y) + (1 - \chi_D(s, y))(s + |y| + R)^{p}\} \chi_{\text{supp } w(s, y)} dy.
\]

First, we consider the case of \( x \geq 0 \). From now on, we employ the change of variables
\[
\alpha = s + y, \beta = s - y.
\]
For \((x, t) \in D\), extending the domain of the integral, we have that
\[
J_1(x, t) \leq C \int_{-R}^{R} d\beta \int_{-R}^{x+t} (\alpha + R)^p d\alpha \\
+ C \int_{R}^{t-x} (\beta + R)^p d\beta \int_{R}^{t+x} d\alpha \\
+ C R \int_{R}^{t-x} d\beta \int_{R}^{t+x} d\alpha \\
\leq C(T + R)^p(t + x + R).
\]

For \(t + x \geq R\) and \(|t - x| \leq R\), we also have that
\[
J_1(x, t) \leq C + C \int_{-R}^{t-x} R (\beta + R)^p d\beta \int_{R}^{t+x} d\alpha \\
\leq C(T + R)^p(t + x + R).
\]

For \(t + x \leq R\), it is trivial that
\[
J_1(x, t) \leq C.
\]

Summing up, we obtain that
\[
|L(|W|^p)(x, t)| \leq C \|W\|_p^p(T + R)^p(t + x + R) \quad \text{for } 0 \leq x \leq t + R.
\]

The case of \(x \leq 0\) is similar to the one above, so we omit the details. Therefore we obtain the first inequality of the first line of (5.5).

The second inequality in (5.5) follows from
\[
|L(|U|^q)(x, t)| \leq C \|U\|_q^q J_2(x, t) \quad \text{for } |x| \leq t + R,
\]
where
\[
J_2(x, t) := \int_{0}^{t} ds \int_{-t+s}^{x+t-s} (s + |y| + R)^q \chi_{\text{supp } U}(y, s) dy.
\]

It is trivial that
\[
J_2(x, t) \leq C(T + R)^{q+1}(t + |x| + R) \quad \text{for } |x| \leq t + R.
\]

Next, we shall show the third inequality in (5.5). It follows from the assumption on the supports and the definition of \(L'\) that
\[
|L'(|W|^p)(x, t)| \leq C \|W\|_4^p \{J_+(x, t) + J_-(x, t)\} \quad \text{for } |x| \leq t + R,
\]
where the integrals \(J_+\) and \(J_-\) are defined by
\[
J_\pm(x, t) := \int_{0}^{t} \left\{ \chi_{2\pm}(x, t; s) \\
+ (1 - \chi_{2\pm}(x, t; s))(s + |t - s \pm x| + R)^p \right\} \chi_\pm(x, t; s) ds
\]

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and the characteristic functions $\chi_+ , \chi_- , \chi_{2+}$ and $\chi_{2-}$ are defined by

$$
\chi_{\pm}(x, t; s) := \chi\{s: |t-s \pm x| \leq s \pm R\}, \\
\chi_{2\pm}(x, t; s) := \chi\{s: |t-s \pm x| \geq R\}
$$

respectively. First we note that it is sufficient to estimate $J_{\pm}$ for $x \geq 0$ due to its symmetry,

$$
J_{\pm}(-x, t) = J_{\pm}(x, t).
$$

For $(x, t) \in D \cap \{x \geq 0\}$, we have

$$
J_{+}(x, t) \leq C \int_{(t+x-R)/2}^{(t+x+R)/2} (t + x + R) ds + C \int_{(t+x-R)/2}^{t} ds \\
\leq C(t + x + R) + C(t + x + R) \\
\leq C(T + R)^p
$$

and

$$
J_{-}(x, t) \leq C \int_{(t-x-R)/2}^{(t-x+R)/2} (s + |t - s - x| + R) ds + C \int_{(t-x-R)/2}^{t} ds \\
\leq C(t + x + R) + C(t + x + R) \\
\leq C(T + R)^p.
$$

For $t + x \geq R$ and $|t - x| \leq R$, we have

$$
J_{+}(x, t) \leq C \int_{(t+x-R)/2}^{t} (t + x + R) ds \leq C(t + x + R)(T + R)^p
$$

and

$$
J_{-}(x, t) \leq C \int_{0}^{t-x} (s + |t - s - x| + R) ds \\
\leq C \int_{0}^{t-x} (t - x + R) ds + C \int_{t-x}^{t} (2s - t + x + R) ds \\
\leq C(t - x + R)^{p+1} + C(t + x + R)^{p+1} \\
\leq C(t + x + R)(T + R)^p.
$$

It is trivial that $J_{\pm}(x, t) \leq C$ for $t + x \leq R$. Therefore we obtain the third inequality in (5.5).

The fourth, fifth and sixth inequalities in (5.5) readily follow from the computations above. The proof of Proposition 5.2 is now completed. \qed

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8 Proofs of Theorem 2.3 and Theorem 2.4

This section is devoted to the blow-up part of (1.2) and (1.3). The proof can be established by a contradiction argument to the assumption that \((u, w)\) is a continuous solution in the time interval \([0, T]\) of the associated system of integral equations (2.14).

**Proof of Theorem 2.3.**

Recall that our assumption on the initial data is (2.19). If one neglects \(A|u_t|^p\) in (1.1), one immediately obtains the contradiction if \(T\) satisfies

\[ T \geq C \varepsilon^{-(q-1)/2} \]

with suitably small \(\varepsilon\), where \(C\) is a positive constant independent of \(\varepsilon\), by Theorem 1.1 in Zhou [17] or Theorem 5.1 in Takamura [14] for the classical solution of (1.1). This estimate is also available for the continuous solution of (2.14) if one employs the proof of Theorem 2.3 with \(a = -1\) in Kitamura, Morisawa and Takamura [7], which is applied to

\[ u \geq \varepsilon u^0 + BL(|u|^q). \]

On the other hand, if one neglects \(B|u|^q\) in (1.1), one immediately obtains the contradiction if \(T\) satisfies

\[ T \geq C \varepsilon^{-(p-1)} \]

with suitably small \(\varepsilon\), where \(C\) is a positive constant independent of \(\varepsilon\), by Theorem 1.1 in Zhou [18] for the classical solution of (1.1). This estimate is also available for the continuous solution of (2.14) if one employs the proof of Theorem 2.2 with \(a = -1\) in Kitamura, Morisawa and Takamura [8], which is applied to

\[ w \geq \varepsilon u^0 + AL'(|w|^p). \]

Therefore we obtain the contradiction if \(T\) satisfies

\[ T \geq \min\{C \varepsilon^{-(p-1)}, C \varepsilon^{-(q-1)/2}\} \]

with suitably small \(\varepsilon\), where \(C\) is a positive constant independent of \(\varepsilon\), which asserts the statement of Theorem 2.3. \(\square\)

**Proof of Theorem 2.4.**

Recall that our assumption on the initial data is (2.21). If one neglects \(A|u_t|^p\) in (1.1), one immediately obtains the contradiction if \(T\) satisfies

\[ T \geq C \varepsilon^{-(q-1)/(q+1)} \]

(8.1)
with suitably small $\varepsilon$, where $C$ is a positive constant independent of $\varepsilon$, by Theorem 1.2 in Zhou [17] or Theorem 5.1 in Takamura [14] for the classical solution of (1.1). This estimate is also available for the continuous solution of (2.14) if one employs the proof of Theorem 2.4 with $a = -1$ in Kitamura, Morisawa and Takamura [7], which is applied to

$$u \geq \varepsilon u^0 + BL(|u|^q).$$

On the other hand, one can obtain the contradiction if $T$ satisfies

$$T \geq C\varepsilon^{-(p-1)}$$

(8.2)

with suitably small $\varepsilon$, where $C$ is a positive constant independent of $\varepsilon$, for the continuous solution of (2.14). To see this, one has to employ

$$w \geq \varepsilon u^0 + AL'(|w|^p)$$

for which the definition of $L'$ implies

$$w(x, t) \geq \varepsilon u^0(x, t) + A \int_0^t |w(x - t + s, s)|^p ds.$$

Setting $x - t = R/2$, we have $x + t = 2t + R/2$ so that

$$w \left( t + \frac{R}{2}, t \right) \geq C_5 \varepsilon + A \int_{R/4}^t \left| w \left( s + \frac{R}{2}, s \right) \right|^p ds \quad \text{for } t \geq \frac{R}{4}$$

due to (2.7), where we set

$$C_5 := -\frac{1}{4} f'' \left( \frac{R}{2} \right) > 0.$$

Hence the comparison argument with a function $z$ satisfying

$$z(t) = C_5 \varepsilon + A \int_{R/4}^t |z(s)|^p ds \quad \text{for } t \geq \frac{R}{4}$$

yields that

$$w \left( t + \frac{R}{2}, t \right) > z(t) \quad \text{for } t \geq \frac{R}{4},$$

so that the desired contradiction can be obtained by the same arguments as in the proof of Theorem 2.2 in Kitamura, Morisawa and Takamura [8]. In fact, if we assume that there exists a point

$$t_0 := \inf \left\{ t > \frac{R}{4} : w \left( t + \frac{R}{2}, t \right) = z(t) \right\},$$

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then we obtain a contradiction,

\[ 0 = w \left( t_0 + \frac{R}{2}, t_0 \right) - z(t_0) = A \int_{R/4}^{t_0} \left\{ \left| w \left( s + \frac{R}{2}, s \right) \right|^p - |z(s)|^p \right\} ds > 0, \]

by \( w(R/4 + R/2, R/4) > z(R/4) \) and the continuity of \( w, z \). Taking into account of the fact that \( z \) satisfies

\[
\begin{cases}
  z' = A|z|^p & \text{in } [R/4, \infty), \\
  z(R/4) = C_5 \varepsilon,
\end{cases}
\]

we have a contradiction if \( T \) satisfies

\[ T \geq C \varepsilon^{-(p-1)} \geq \{ \text{The blow-up time of } z \} \quad \text{for } 0 < \varepsilon \leq \varepsilon_4, \]

where \( C \) is a positive constant independent of \( \varepsilon \) and \( \varepsilon_4 \) is a constant depending on \( A, C_5, R, p \). In fact, the representation of \( z \) is

\[ z(t) = \left( C_5 \varepsilon \right)^{-(p-1)} - (p-1)A \left( t - \frac{R}{4} \right) \right)^{-1/(p-1)} \]

so that \( z(t) \to \infty \) as

\[ t \to \frac{C_5^{-(p-1)}}{(p-1)A} \varepsilon^{-(p-1)} + \frac{R}{4}. \]

If one sets

\[ \varepsilon_4 = \left( \frac{4C_5^{-(p-1)}}{(p-1)AR} \right)^{1/(p-1)}, \]

then one has a contradiction, \( w(T + R/2, T) = \infty \), provided

\[ T \geq \frac{2C_5^{-(p-1)}}{(p-1)A} \varepsilon^{-(p-1)} \geq \frac{C_5^{-(p-1)}}{(p-1)A} \varepsilon^{-(p-1)} + \frac{R}{4} \]

for \( 0 < \varepsilon \leq \varepsilon_4 \). Therefore it follows from (8.1) and (8.2) that we obtain the contradiction if \( T \) satisfies

\[ T \geq \min \{ C \varepsilon^{-(p-1)}, C \varepsilon^{-(q-1)/(q+1)} \} \]

with suitably small \( \varepsilon \), where \( C \) is a positive constant independent of \( \varepsilon \). This estimate asserts Theorem 2.4 for \( p \leq (q + 1)/2 \), or \( p \geq q \). See (1.3) and Remark 2.2.
Hence we have to improve this estimate in the case of \((q + 1)/2 \leq p \leq q\). To do this, we start with

\[ u = \varepsilon u^0 + L(A|w|^p + B|u|^q). \]

Set

\[ D_+ := \{(x,t) : t - x \geq R, \ x \geq 0\}. \]

Then, changing variables by \(\alpha = s + y, \beta = s - y\), we have that

\[ L(|w|^p)(x, t) \geq \frac{1}{4} \int_{-R}^{0} d\beta \int_{x}^{t+x} |w(s, y)|^p d\alpha \quad \text{for} \ (x, t) \in D_. \]

Hence it follows from

\[ w(y, s) \geq \varepsilon u^0_t(y, s) = -\frac{\varepsilon}{2} f'(y - s) \quad \text{for} \ s + y \geq R, \ 0 < y - s < R \]

that

\[
\begin{aligned}
\left\{
\begin{array}{l}
u(x, t) \geq \frac{B}{4} \int_{x}^{t+x} d\beta \int_{R}^{t-x} |u(s, y)|^q d\alpha, \quad \text{for} \ (x, t) \in D_+, \\
u(x, t) \geq AC_6\varepsilon^p(t + x - R),
\end{array}
\right.
\end{aligned}
\]

where we set

\[ C_6 := \frac{1}{2p+2} \int_{-R}^{0} |f'(-\beta)|^p d\beta > 0. \]

From now on, we employ a routine iteration procedure. Assume an estimate

\[ u(x, t) \geq M_n(t + x - R)^{a_n}(t - x - R)^{b_n} \quad \text{for} \ (x, t) \in D_+ \quad (8.4) \]

holds, where \(a_n, b_n \geq 0\) and \(M_n > 0\). The sequences \(\{a_n\}, \{b_n\}\) and \(\{M_n\}\) are defined later. Then it follows from the first line in (8.3) that

\[
\begin{aligned}
u(x, t) &\geq \frac{BM_n^q}{4} \int_{R}^{t-x} (\beta - R)^{b_n} d\beta \int_{x}^{t+x} (\alpha - R)^{a_n} d\alpha \\
&\geq \frac{BM_n^q}{4(qa_n + 1)(qb_n + 1)}(t + x - R)^{qa_n+1}(t - x - R)^{qb_n+1}
\end{aligned}
\]

for \((x, t) \in D_+.\) Hence (8.4) holds for all \(n \in \mathbb{N}\) provided

\[
\left\{
\begin{array}{l}
a_{n+1} = qa_n + 1, \quad a_1 = 1, \\
b_{n+1} = qb_n + 1, \quad b_1 = 0
\end{array}
\right.
\]

and

\[ M_{n+1} \leq \frac{BM_n^q}{4(qa_n + 1)(qb_n + 1)}, \quad M_1 = AC_6\varepsilon^p. \]
It is easy to see that
\[ a_n = \frac{q^n - 1}{q - 1}, \quad b_n = \frac{q^{n-1} - 1}{q - 1} \quad (n \in \mathbb{N}), \]
which implies
\[ (qa_n + 1)(qb_n + 1) \leq (qa_n + 1)^2 = a_{n+1}^2 \leq \frac{q^{2(n+1)}}{(q - 1)^2}. \]

Therefore \( M_n \) should be defined by
\[ M_{n+1} = BC_7q^{-2(n+1)}M_n^2, \quad M_1 = AC_6\varepsilon^p, \]
where we set
\[ C_7 := \frac{(q - 1)^2}{4} > 0, \]
so that (8.4) implies that
\[ u(x, t) \geq C_8\{(t + x - R)(t - x - R)\}^{-1/(q-1)} \exp \left\{ Z(x, t)q^{n-1} \right\} \quad (8.5) \]
for \((x, t) \in D_+\), where
\[ Z(x, t) := \frac{1}{q - 1} \log\{(t + x - R)^q(t - x - R)\} + \frac{1}{q - 1} \log(BC_7) - 2S_q \log q + \log(AC_6\varepsilon^p), \]
\[ C_8 := \exp \left( -\frac{1}{q - 1} \log(BC_7) \right) > 0. \]

Indeed, \( M_n \) satisfies
\[ \log M_{n+1} = \log(BC_7) - 2(n + 1) \log q + q \log M_n, \]
which implies
\[
\log M_{n+1} = (1 + q + \cdots + q^{n-1}) \log(BC_7) \leq 2\{n + 1 + qn + \cdots + q^{n-1}(n + 1 - n + 1)\} \log q + q^n \log M_1 \\
= \frac{q^n - 1}{q - 1} \log(BC_7) - 2q^{n-1} \log q \sum_{j=0}^{n} \frac{j + 2}{q^j} + q^n \log M_1 \\
\geq -\frac{1}{q - 1} \log(BC_7) + q^n \left\{ \frac{1}{q - 1} \log(BC_7) - 2S_q \log q + \log M_1 \right\},
\]
where we set
\[ S_q := \sum_{j=0}^{\infty} \frac{j+2}{q^{j+1}} < \infty. \]

In view of (8.5), if there exists a point \((x_0, t_0) \in D_+\) such that \(Z(x_0, t_0) > 0\), we have a contradiction \(u(x_0, t_0) = \infty\) to the fact that \((u, w)\) is a continuous solution on the time interval \([0, T]\) with \(T \geq t_0\) of (2.14) by letting \(n \to \infty\). Let us set \(t_0 = 2x_0\) and \(t_0 \geq 4R\). Then, since we have
\[ (t_0 + x_0 - R)^q (t_0 - x_0 - R) \geq \frac{1}{4} \left( \frac{5}{4} \right)^q t_0^{q+1}, \]
\(Z(x_0, t_0) > 0\) follows from
\[ t_0^{q+1} > 4 \left( \frac{4}{5} \right)^q \frac{q^{2(q-1)}S_q}{A^{(q-1)}BC_6^{(q-1)}C_7} \varepsilon^{-p(q-1)}. \]

Therefore this inequality completes the proof of Theorem 2.4 for \((q+1)/2 \leq p \leq q\) with \(\varepsilon_4 > 0\) satisfying
\[ \left\{ 4 \left( \frac{4}{5} \right)^q \frac{q^{2(q-1)}S_q}{A^{(q-1)}BC_6^{(q-1)}C_7} \right\}^{1/(q+1)} \varepsilon_4^{-p(q-1)/(q+1)} = 4R. \]

\[ \square \]

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References

[1] W. Dai, D. Fang and C. Wang, *Global existence and lifespan for semilinear wave equations with mixed nonlinear terms*, J. Differential Equations, **267** (2019), no. 5, 3228-3354.

[2] W. Han and Y. Zhou, *Blow up for some semilinear wave equations in multi-space dimensions*, Comm. Partial Differential Equations, **39** (2014), no. 4, 651-665.

[3] K. Hidano, C. Wang and K. Yokoyama, *Combined effects of two nonlinearities in lifespan of small solutions to semi-linear wave equations*, Math. Ann., **366** (2016), no. 1-2, 667-694.

[4] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., **28** (1979), no. 1-3, 235-268.

[5] F. John, “Nonlinear Wave Equations, Formation of Singularities”, ULS Pitcher Lectures in Mathematical Science, Lehigh University, American Mathematical Society, Providence, RI, 1990.

[6] S. Katayama, *Lifespan of solutions for two space dimensional wave equations with cubic nonlinearity*, Comm. Partial Differential Equations, **26** (2001), no. 1-2, 205-232.

[7] S. Kitamura, K. Morisawa and H. Takamura, *The lifespan of classical solutions of semilinear wave equations with spatial weights and compactly supported data in one space dimension*, J. Differential Equations, **307** (2022), 486-516.

[8] S. Kitamura, K. Morisawa and H. Takamura, *Semilinear wave equations of derivative type with spatial weights in one space dimension*, arXiv:2112.01015, to appear in Nonlinear Analysis, RWA.

[9] T.-T. Li (D.-Q. Li), X. Yu and Y. Zhou, *Durée de vie des solutions régulières pour les équations des ondes non linéaires unidimensionnelles* (French), C. R. Acad. Sci. Paris Sér. I Math., **312** (1991), no. 1, 103-105.

[10] T.-T. Li, X. Yu and Y. Zhou, *Life-span of classical solutions to one-dimensional nonlinear wave equations*, Chinese Ann. Math., Ser. B, **13** (1992), no. 3, 266-279.

[11] M. Liu and C. Wang, *Blow up for small-amplitude semilinear wave equations with mixed nonlinearities on asymptotically Euclidean manifolds*, J. Differential Equations, **269** (2020), no. 10, 8573-8596.
[12] G. K. Luli, S. Yang, P. Yu, *On one-dimension semi-linear wave equations with null conditions*, Adv. Math., 329 (2018), 174-188.

[13] M. Nakamura, *Remarks on a weighted energy estimate and its application to nonlinear wave equations in one space dimension*, J. Differential Equations, 256, no. 2, (2014), 389-406.

[14] H. Takamura, *Improved Kato’s lemma on ordinary differential inequality and its applications to semilinear wave equations*, Nonlinear Anal., 125 (2015), 227-240.

[15] D. Zha, *On one-dimension quasilinear wave equations with null conditions*, Calc. Var. Partial Differential Equations, 59 (2020), no. 3, Paper No. 94, 19 pp.

[16] D. Zha, *Global stability of solutions to two-dimension and one-dimension systems of semilinear wave equations*, J. Funct. Anal., 282 (2022), no. 1, Paper No. 1092219, 26 pp.

[17] Y. Zhou, *Life span of classical solutions to $u_{tt} - u_{xx} = |u|^{1+\alpha}$*, Chinese Ann. Math. Ser.B, 13 (1992), no. 2, 230-243.

[18] Y. Zhou, *Blow up of solutions to the Cauchy problem for nonlinear wave equations*, Chinese Ann. Math. Ser. B, 22 (2001), no. 3, 275-280.

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Erratum to “The combined effect in one space dimension beyond the general theory for nonlinear wave equations”

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Abstract

Our previous paper has a wrong citation of the result on the general theory for nonlinear wave equations which causes a slightly different conjecture on a possibility to improve the theory, related to the title of the paper.

In Morisawa, Sasaki and Takamura [3], there is a wrong citation of the result on the general theory for nonlinear wave equations in one space dimension by [1, 2] at the last case in (2.24). This error causes a slightly different conjecture on a possibility to improve the general theory. So that, the descriptions from (2.24) to (2.26) in [3] on pp. 1634-1635 should be modified as follows.

In fact, (2.24) in [3] should be replaced with

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
 c\varepsilon^{-\alpha/2} & \text{in general,} \\
 c\varepsilon^{-\alpha(1+\alpha)/(2+\alpha)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0, \\
 c\varepsilon^{-\min(\beta_0/2, \alpha)} & \text{if } \partial^R u F(0) = 0 \text{ for } 1 + \alpha \leq \forall \beta \leq \beta_0.
\end{cases}
\]

If one applies it to (2.25) in [3], one has the following estimates in each cases.

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• When \( p < q \),
then, we have to set \( \alpha = p - 1 \) and \( \beta_0 = q - 1 \) which yield that

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\varepsilon^{-(p-1)/2} & \text{in general,} \\
\varepsilon^{-p(p-1)/(p+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0, \\
\varepsilon^{-\min\{(q-1)/2, p-1\}} & \text{if } \partial^q_u F(0) = 0 \\
\text{for } p \leq \forall \beta \leq q - 1.
\end{cases}
\]

We note that the third case is available for (2.25) in [3]. Therefore, for \( p \leq (q + 1)/2 \), we obtain that

\[
\tilde{T}(\varepsilon) \geq \varepsilon^{-(p-1)}
\]

whatever the value of \( \int_{\mathbb{R}} g(x)dx \) is. On the other hand, for \((q + 1)/2 < p\), i.e.

\[
\frac{q - 1}{2} < p - 1,
\]
we obtain

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\varepsilon^{-(q-1)/2} & \text{if } \int_{\mathbb{R}} g(x)dx \neq 0, \\
\varepsilon^{-\max\{(q-1)/2, p(p-1)/(p+1)\}} & \text{if } \int_{\mathbb{R}} g(x)dx = 0.
\end{cases}
\]

• When \( p \geq q \),
then, similarly to the case above, we have to set \( \alpha = q - 1 \), which yields that

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\varepsilon^{-(q-1)/2} & \text{in general,} \\
\varepsilon^{-q(q-1)/(q+1)} & \text{if } \int_{\mathbb{R}} g(x)dx = 0, \\
\varepsilon^{-\min\{\beta_0/2, (q-1)\}} & \text{if } \partial^q_u F(0) = 0 \text{ for } q \leq \forall \beta \leq \beta_0.
\end{cases}
\]

We note that the third case does not hold for (2.25) in [3] by \( \partial^q_u F(0) \neq 0 \).

In conclusion, for the special nonlinear term in (2.25) in [3], the result of the general theory is

\[
\tilde{T}(\varepsilon) \geq \begin{cases} 
\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q + 1}{2}, \\
\varepsilon^{-(q-1)/2} & \text{for } \frac{q + 1}{2} \leq p \text{ if } \int_{\mathbb{R}} g(x)dx \neq 0.
\end{cases}
\]
and
\[ \tilde{T}(\varepsilon) \geq \begin{cases} 
    c\varepsilon^{-(p-1)} & \text{for } p \leq \frac{q+1}{2}, \\
    c\varepsilon^{-\max\{(q-1)/2,p(p-1)/(p+1)\}} & \text{for } \frac{q+1}{2} \leq p \leq q, \\
    c\varepsilon^{-(q-1)/(q+1)} & \text{for } q \leq p \\
\end{cases} \]
if \( \int_{\mathbb{R}} g(x) \, dx = 0 \).

Therefore a part of our results in [3],

\[ T(\varepsilon) \sim C\varepsilon^{-p(q-1)/(q+1)} \]
if \( \int_{\mathbb{R}} g(x) \, dx = 0 \) and \( \frac{q+1}{2} < p < q \),

is better than the lower bound of \( \tilde{T}(\varepsilon) \). So we have a possibility to improve the general theory at least to include the case above.

References

[1] T.-T. Li (D.-Q. Li), X. Yu and Y. Zhou, Durée de vie des solutions régulières pour les équations des ondes non linéaires unidimensionnelles (French), C. R. Acad. Sci. Paris Sér. I Math., 312 (1991), no. 1, 103-105.

[2] T.-T. Li, X. Yu and Y. Zhou, Life-span of classical solutions to one-dimensional nonlinear wave equations, Chinese Ann. Math., Ser. B, 13 (1992), no. 3, 266-279.

[3] K. Morisawa, T. Sasaki and H. Takamura, The combined effect in one space dimension beyond the general theory for nonlinear wave equations, Comm. Pure Appl. Anal. 22 (2023), no. 5, 1629-1658.