ON THE OPTIMAL MEAN PHOTON NUMBER FOR QUANTUM CRYPTOGRAPHY

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The optimal mean photon number ($\mu$) for quantum cryptography is the average photon number per transmitted pulse that results in the highest delivery rate of distilled cryptographic key bits, given a specific system scenario and set of assumptions about Eve’s capabilities. Although many experimental systems have employed a mean photon number ($\mu$) of 0.1 in practice, several research teams have pointed out that this value is somewhat arbitrary. In fact, various optimal values for $\mu$ have been described in the literature.

In this paper we offer a detailed analytic model for an experimental, fiber-based quantum cryptographic system, and an explicit set of reasonable assumptions about Eve’s current technical capabilities. We explicitly model total system behavior ranging from physical effects to the results of quantum cryptographic protocols such as error correction and privacy amplification. We then derive the optimal photon number ($\mu$) for this system in a range of scenarios. One interesting result is that $\mu \approx 1.1$ is optimal for a wide range of realistic, fiber-based QKD systems; in fact, it provides nearly 10 times the distilled throughput of systems that employ a more conventional $\mu = 0.1$, without any adverse affect on system security, as judged against a set of reasonable assumptions about Eve’s current capabilities.

Keywords: Quantum cryptography, Quantum key distribution, Mean photon number

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1 Background and Problem Statement

It now seems likely that Quantum Key Distribution (QKD) techniques can provide practical building blocks for highly secure networks, and in fact may offer valuable cryptographic services, such as unbounded secrecy lifetimes, that can be difficult to achieve by other techniques. Accordingly, a number of commercial and research organizations have begun to build and operate complete QKD systems. As quantum cryptography has started the transition from laboratory demonstrations to working systems in the field, questions of operating efficiency
and realistic levels of security have taken on a heightened importance.\footnote{The opinions expressed in this article are those of the authors alone, and do not necessarily reflect the views of the United States Department of Defense, DARPA, or the United States Air Force.}

A wide range of techniques has been proposed for quantum cryptography, and many have been experimentally demonstrated; see Gisin et al.\footref{1} for a superb overview. However, in realistic settings, such as operation through the atmosphere or through tens of kilometers of telecommunications fiber, even the most efficient of these techniques currently provide no more than roughly 1,000 distilled\footnote{In our terminology, a “distilled” key has been sifted, error corrected, and privacy amplified, and is thus ready to use as key material.} bits per second, depending on channel losses, choice of eavesdropping threat model, and a large number of technical parameters. While this key generation rate is more than sufficient for rapid rekeying of conventional cryptographic algorithms—for example it allows rekeying of an AES algorithm with fresh 256-bit keys 4 times per second—a faster key generation rate would allow a large number of cryptographically protected traffic flows to be rekeyed at a given rate. In addition, it is far too low for most uses of one-time pads. An important practical question, therefore, is how to increase this rate.

A number of different approaches may contribute to improved distilled key generation rates: detector efficiencies may be improved, e.g., by novel forms of detectors; pulse rates may be increased; and entropy estimates may be refined so that less privacy amplification is required for a given observed level of noise. Promising efforts are underway in all these areas (e.g.\footref{2,3,4}). This paper explores yet another avenue to increasing the key generation rate, namely, by finding an optimal value for the mean number of photons emitted in each pulse, i.e., that which maximizes the distilled key generation rate for a given scenario and set of eavesdropping assumptions. This mean photon number is often designated $\mu$ in the QKD literature.

This paper provides a detailed, quantitative analysis of the interaction between $\mu$, channel attenuation, and privacy-amplified key generation rates, and compares the results with prior research on optimal mean photon number. We specifically consider a phase-modulated system, with attenuated laser source and cooled InGaAs APDs, designed for telecommunications fiber; however the results can be readily generalized to other systems.

One interesting result is that $\mu \approx 1.1$ is optimal for a wide range of realistic, fiber-based QKD systems under a reasonable eavesdropping threat model; in fact, it provides nearly 10 times the distilled throughput of systems that employ a more conventional $\mu = 0.1$, without any adverse affect on system security. For many steeped in the field, it may seem counter-intuitive—even downright false—that a mean photon number as large as 1, let alone greater than 1, may be possible without sacrificing all security. However, as Prof. Gisin et al. have noted in their magisterial survey of quantum cryptography\footref{1}, “multiphoton pulses do not necessarily constitute a threat to key security, but they limit the key creation rate because they imply that more bits must be discarded during key distillation.” This paper may be viewed as an elaboration, and preliminary quantification, of that important remark.

2 \hspace{1em} Review of the Current Art

Although most practitioners of quantum cryptography have now converged upon a mean photon number ($\mu$) of 0.1 as a good benchmark value, “contrary to a frequent misconception, there is nothing special about a $\mu$ value of 0.1, even though it has been selected by most
experimentalists. The optimal value—i.e., the value that yields the highest key exchange rate after distillation—depends on the optical losses in the channel and on assumptions about Eve’s technology.” In fact, in recent years, at least three leading research teams have carefully investigated the optimal mean photon rate, and have come to differing conclusions. Accordingly, this section recapitulates both the widespread rationale for $\mu = 0.1$, and the previous research on the relationship of $\mu$ to distilled key rate.

As will be seen, for some years the QKD community has held, in effect, an ongoing discussion of the optimal mean photon number for various contexts, but generally as side comments within papers devoted to other topics. As a result, it has been difficult to find a detailed and explicit linkage between eavesdropping threat models and optimal mean photon numbers.

The origin of the value 0.1 for the mean photon number was the very first experimental realization of QKD by Bennett et al. in 1992 [5]. This early work analyzed various kinds of attacks on the small number of multi-photon pulses produced, including one version of unambiguous state discrimination, and concluded that unambiguous state discrimination was impossible for such a small $\mu$ without significantly biasing the detector statistics at Bob. Later researchers have shown that this conclusion is incorrect [6]. However, the number of bits that Eve can discover is very small, and Bennett et al. left a significant safety margin in their estimate. The attacks they considered feasible involved intercepting one photon from each multi-photon pulse and measuring it. For each such pulse that reaches Bob, they assume Eve gains one bit of information, thus implicitly allowing Eve to have a quantum memory and to measure the photon only when its basis is disclosed.

Many other experimental systems, including ours, borrowed from [5] the value of 0.1 for $\mu$ as well as the estimate of Eve’s advantage from photon-number splitting (PNS) attacks. Two experimental teams, from Los Alamos [7] and IBM Almaden [8], then calculated optimal numerical values for $\mu$ in their systems, based on this estimate. For the free-space system used by Los Alamos, this value of $\mu$ was 0.4, while for IBM’s “plug-and-play” fiber-based system it was 0.3. The IBM Almaden group also examined the throughput vs. mean photon number for a number of different eavesdropping models.

On the theoretical side, Gilbert and Hamrick [6] performed an extensive analysis of possible attacks on multi-photon pulses, including splitting, unambiguous state discrimination, and surreptitiously replacing the channel to Bob with a perfectly transparent one. In short, they selected a more formidable eavesdropping model than posited in the analyses of Bennett, Los Alamos, or IBM Almaden. Granting Eve such powers, they produced a much more conservative estimate of the amount of information Eve might gain. They also analyzed the optimum mean photon number in one specific scenario, an aircraft to a LEO satellite, and found it to be 0.455, although in this case they allow Eve less power than in a fiber link—specifically, she is not able to replace the channel with a lossless one.

These differing estimates are further compared in section 5.

3 Our Analytical Approach

Our analysis, in the following section, is derived from a moderately detailed mathematical model of a full QKD system for use in telecommunications fiber, including both physical effects and the outcomes of higher level protocols, validated against two working systems in
the laboratory. This section briefly describes our working systems (the concrete subjects of
analysis), then discusses the major elements in our analytic model. Appendix A contains the
full text of the model.

3.1 Functioning Systems for Quantum Cryptography
BBN, Boston University, and Harvard University are currently building a large-scale quantum
cryptography system, the DARPA Quantum Network, and fielding it into dark fiber in the
Cambridge, Massachusetts metropolitan area. See for example [9, 10] for details on this
network and its design goals. Two interoperable QKD systems in the DARPA Quantum
Network started 24x7 duty in October 2003; we call these 'Mark 2' systems because they
replaced our first-generation link, which started continuous operation in December 2002.
These systems were inspired by a pioneering Los Alamos system [11] and designed to run
through telecommunications fiber as widely deployed today.

Each Mark 2 system employs a highly attenuated telecommunications laser at 1550.12
nm, phase modulation via unbalanced Mach-Zehnder interferometers, and thermo-electrically
cooled InGaAs avalanche photo detectors (APDs). Most Mark 2 electronics are implemented
by discrete components such as pulse generators. At present, incoming dim pulses are de-
tected by Epitaxx EPM 239 AA APDs cooled to approximately –40 degrees Centigrade and
gated during a pulse arrival period. Since custom cooling and electronics are required, we
designed and built our own cooler package to maintain the APDs at the requisite operating
temperatures. Even with this special treatment, they suffer considerably from low Quantum
Efficiency (QE), relatively high dark noise, and serious after-pulsing problems. These cooled
detectors form one of the most important bottlenecks in the overall system performance, as
they require on the order of 10 $\mu$s to recover between detection events. The overall link has
been designed to run at up to 5 Mb/s transmit rate but with a dead-time circuit to disable
the APD after a detection event in order to accommodate this recovery interval and suppress
detector after-pulsing.

BBN’s QKD protocol stack is an industrial-strength implementation written in the C
programming language for ready portability to embedded real-time systems. At present all
protocol control messages are conveyed in IP datagrams so that control traffic can be conveyed
via an internet. Two aspects of BBN’s QKD protocol stack deserve special mention. First, it
implements a complete suite of QKD protocols. In fact, it implements multiple “plug compat-
able” versions of some functions, e.g., it provides both the traditional BB84 sifting protocol
and the newer “Geneva” style sifting [12]. It also provides a choice of entropy estimation
functions including the well-known BBBSS92 estimates [5], Slutsky’s defense frontier analy-
sis [13], and the newer Myers-Pearson estimate [4]. We expect to add additional options and
variants as they are developed. Second, BBN’s QKD protocols have been carefully designed
to make it as easy as possible to plug in other QKD systems, i.e., to facilitate the introduction
of QKD links from other research teams into the overall DARPA Quantum Network.

3.2 Analytic Tools used in this Paper
Over the past two years, we have developed a Matlab / Octave model to analyze the ex-
pected efficiency of current and projected fiber-based QKD systems in the DARPA Quantum
Network. The complete model is provided in Appendix A. Some aspects of the model have
been derived from the QKD literature, but most have been developed from first principles.
Dr. John Myers of Harvard University has provided many of the equations in this model; the authors have provided the remainder. Of course the authors are solely responsible for any flaws in this published model.

This model provides for a wide range of input parameters such as pulse rate, mean photon number at Alice, attenuation, detector efficiency, dark count, and after-pulsing characteristics, residual phase error in the Mach-Zehnder interferometers, and so forth. It also provides input parameters for higher layers of the QKD protocol stack, such as the sifting protocol employed, information revealed during error detection and correction, entropy estimation technique, etc. We briefly discuss these inputs, and the associated calculations, in the following paragraphs. Although the model provides basic estimates for a range of physical and protocol phenomena, it is by no means complete. For example, it does not include any characterization of stray light, of chromatic or polarization mode dispersion, and so forth. However, the current version of this model has been validated against our QKD systems running both through a fiber spool in the laboratory and through a 17km fiber loop between BBN and Harvard University, and its results agree well with experimental measurement. Thus it appears to capture at least the most important drivers for realistic system behavior.

As shown in Appendix A, the model inputs represent a fiber-based system with a 5 Mb/s pulse rate, 0.1 mean photon number (µ), operating through 10.55 km of telecommunications fiber with an overall fiber attenuation of 2.5 dB. The average receiver loss factor is 10.4 dB, with a residual phase error in the Mach-Zehnder interferometers of 3 degrees after both passive and active path length stabilization. The path length stabilization and framing overhead results in a duty cycle of 80% for usable QKD bits. Detector efficiency is 13%, with mis-steered light occurring in 0.9% of the detections, and a dark count probability of $2 \times 10^{-5}$ per pulse. At higher layers of the QKD protocol stack, the traditional BB84 sifting algorithm is modeled, with the BBN variant of the Cascade error detection and correction protocol using a block size of 4,096 bits with 64 sets, the traditional BBBS92 entropy estimate, and a residual confidence level (the probability that Eve has more information than estimated) of $10^{-6}$. These values capture the current state of our QKD systems as of January 2004.

It should be apparent from inspection of Appendix A that these parameters can be readily adjusted to model other fiber-based systems, e.g., different detector characteristics, protocol behavior, and so forth. One could also extend the model to free-space systems, or systems based on pairs of entangled photons, but this would require that additional equations be added to the model rather than mere adjustment of input parameters.

## 4 Eavesdropping Model and Defense Function

The most critical factor driving an optimal choice of mean photon number is determining what sort of attacks Eve can employ. For intercept-resend attacks on the single-photon pulses, there is a fairly well-developed theory about how much privacy amplification is necessary [14]. For multi-photon pulses, a number of possible attacks have been proposed and analyzed [6, 14, 15], but it is by no means clear that the list of possible attacks is complete yet [16]. Many of the theoretically possible attacks are very far from practical implementation with current technology.

Note that these assumptions about Eve’s abilities must be built into the privacy amplification margin used in any working QKD system, so they are by no means idle questions. If
one wishes to deploy QKD securely, one must choose these assumptions carefully. Once we have chosen these assumptions and the privacy amplification formula, numerical optimization techniques can determine the optimal multi-photon probability. Therefore it is useful to explicitly list a set of assumed capabilities for Eve for a given scenario, as the rates vary greatly depending on the assumptions.

We must decide, for example, whether we wish to guard against an Eve possessing the capabilities listed in Table 1. Many research results assume that Eve possesses all these capabilities; for some papers it is difficult to determine exactly which capabilities are assumed.

Table 1. Eavesdropping model used in this analysis.

| Eve Has? | Potential Technological Capabilities for Eve |
|----------|-----------------------------------------------|
| ✓        | Perfect detectors                             |
| ✓        | A perfect long-term quantum memory            |
| ✓        | Adaptive beam-splitters, which split at most one photon from the signal [16] |
| ✓        | Reliable quantum non-demolition measurement of the total number of photons |
|          | The ability to perform unambiguous state discrimination on pulses with 3 or more photons |
|          | The ability to discriminate multi-photon pulses in intercept/resend attacks [1] |
|          | The ability to substitute low or zero-loss fiber, or to perform quantum teleportation with small loss |

It is our belief, following Gisin, et al. [1], that it is reasonable to guard against eavesdropping that is currently feasible, or may be in the not-too-distant future, rather than make deployment infeasible by attempting to guard against theoretical attacks that may never be possible. Note, in particular, that near-perfect detectors, particularly if they can resolve the number of photons in a pulse, adaptive beam-splitters, or quantum non-demolition (QND) measurements can all give us a reliable way to build a true single-photon source, which would, in turn, render PNS attacks harmless. QKD is very likely to shift to true single-photon emitters long before we need to worry about an eavesdropper with a long-term quantum memory. It is one of the greatest virtues of QKD that, unlike classical cryptography, there is no risk that a future powerful adversary endangers our communications in the present.

Accordingly, the check marks in Table 1 indicate which technology we assume Eve has for the purposes of this analysis, and for the current operation of our working QKD systems. We believe that these assumptions are reasonable for current scenarios, since many of the postulated technologies appear to be beyond today’s current state of the art.

Finally, given this explicit set of assumptions about Eve’s current capabilities, one must select an entropy estimate used as input for privacy amplification. This entropy estimate includes Eve’s information from intercept-resend attacks, called by Slutsky et al. the “defense function” [13]. Here we use results based on the original entropy estimate in BBBSS92, but our
analytic model explicitly calculates three different entropy estimates (BBBSS92 [5], Slutsky [13], Myers-Pearson [4]). The choice of optimal mean photon number is very similar for all choices of entropy estimate.

5 Results and Discussions

Given all these assumptions, we can employ an analytic model (Appendix A) to calculate the optimal mean photon number ($\mu$) over a range of scenarios. Recall that the “optimal” value is that which maximizes the delivery rate of distilled bits / second, i.e., optimizes across the system-wide effects of multi-photon emission probabilities, attenuation, dark noise, sifting, bits revealed during error detection and correction, and the necessary amount of privacy amplification.

The model allows us to extrapolate system performance in a number of scenarios, e.g. if we had longer fibers, a faster pulse rate, or better detectors. In particular, we can analyze the effects of changing the mean photon number. In Figure 1 we vary only the mean photon number $\mu$, with all other parameters derived from one of our current QKD systems (with 10.55 km of optical fiber between Alice and Bob). It is very apparent that the current mean photon number $\mu$, approximately 0.1 photon, is far from optimal in this setting. Instead the mean photon number $\mu$ should be slightly more than 1 (about 1.15) to achieve the optimal distilled key rate.

![Fig. 1. Distilled Key Rate as a Function of Mean Photon Number (\(\mu\)) for a 10.55 km fiber link with 2.5 dB loss.](image)

Another major objective in optimizing $\mu$ is to maximize the distance available for practical QKD over metropolitan fiber. Figure 2 shows how the distilled key rate varies with both fiber
length and $\mu$, again given specific system characteristics (Appendix A) and the eavesdropping model of Section 4.

As can be seen, the distilled key rate falls off dramatically with distance, and requires high values of $\mu$ for long distances. These specific results are driven by the relatively low quantum efficiency, and relatively high dark count, of our current InGaAs detector suite, but the phenomenon is more general. Larger $\mu$ naturally leads to more photons at the receiver, and correspondingly more raw key bits per second, but more importantly it keeps the valid detect rate high compared to receiver dark noise. Dark noise with a highly attenuating channel decreases the distilled rate in a very dramatic way because it translates directly into a higher error rate. The error detection and correction protocol, such as Cascade, then must reveal a substantial amount of information to correct the errors. Since it must be assumed, conservatively, that all these errors are due to eavesdropping, the estimate of the remaining entropy in the bits drops sharply.

![Distilled Key Rate](image)

**Fig. 2. Distilled Key Rate as a Function of Distance and Mean Photon Number ($\mu$).**

Since many factors affect the distilled key rate, it is not surprising that there is not a single optimum value of $\mu$ to employ in all scenarios. However, for our systems, the optimum value does not vary by much. Figure 3 shows the optimum $\mu$ for distances from zero to 50 km. The optimum varies by less than 20%, from about 1 to 1.2. The peak of the key rate curve (Figure 1) is rather broad, so choosing a value of 1.0, say, for $\mu$ seems to be applicable for a broad range of operating conditions.

Since our estimates of the optimal mean photon number are quite different from conventional wisdom, careful review of the assumptions and calculations is in order. We believe that Bennett, Los Alamos, IBM Almaden, and this paper all employ similar eavesdropping models. This is important, because the eavesdropping threat model drives the calculations of optimal mean photon number.

The main difference in our calculation from those of Los Alamos is as follows. They used, following Bennett et al., a fairly rough estimate for the fraction of detected pulses that are
vulnerable to splitting. Bennett’s (intentionally conservative) estimate was that Eve could learn a fraction $\mu$ of the bits through beamsplitting. This obviously would never allow $\mu = 1$, since then Eve would learn all of the bits. But this is quite conservative indeed, since the fraction $m$ of non-empty pulses that contain multiple photons (all of which we want to assume Eve intercepts) may be more precisely estimated by the Poisson distribution,

$$m = 1 - \frac{\mu e^{-\mu}}{1 - e^{-\mu}}$$

(1)

This fraction $m$ is close to $\mu/2$ when $\mu$ is small, but diverges farther from $\mu$ at higher values. Figure 4(a) shows the effect of this difference between the estimates on distilled key rate for the specific scenario depicted in Figure 1. The estimate we use throughout this paper (“revised Bennett”) is $mN + \sqrt{2}\text{erf}^{-1}(c)\sqrt{Nm(1-m)}$ where $m$ is defined above, and $c = 10^{-6}$ is a confidence parameter, the residual probability that Eve might gain more information from multi-photon pulses. The original BBBSS92 estimate (“original Bennett”) is identical except for using $m = \mu$. We are not the first to employ this revised estimate. Both Lütkenhaus [17] and Gilbert and Hamrick [6] derive their results with the correct multiphoton Poisson statistics, and indeed predict that for low loss and high efficiency detectors, the optimum efficiency is achieved for mean photon numbers greater than 1. Without much discussion, the IBM Almaden results [8] included curves for the “revised Bennett” estimate as well as the “original Bennett” estimate for a range of detector efficiencies and channel losses, in a most interesting graph of the effect of $\mu$ on distilled key rate in other eavesdropping models, including those of [6] and [17]. These graphs showed that under some circumstances $\mu > 1$ is optimal in these other models.

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**Fig. 3.** Optimal Mean Photon Number ($\mu$) as a Function of Fiber Length.
Figure 4(b) shows the effect of using Gilbert and Hamrick’s estimate [6], based on a more severe eavesdropping model. Since they allow Eve perfect unambiguous state discrimination attacks and zero-loss fiber, it is not surprising that their estimate results in a far lower key rate.

Fig. 4. The effect of different eavesdropping estimates on the distilled key rate as a function of $\mu$ for a 10.55 km fiber link with 2.5 dB loss.

The threat model treated in this paper has been implicitly assumed in the eavesdropping estimates for multi-photon pulses provided by other research teams [5, 7, 8]. We believe it is a plausible threat model, given current technology. It is, however, important to realize that with larger values of $\mu$ we are moving out of the “comfort zone” of these assumptions. Certain attacks that aren’t readily feasible at small $\mu$ become easier at $\mu = 1$. For example, Bennett et al. considered a special case of unambiguous state discrimination in [5], splitting incoming pulses and measuring one portion in each basis. In some cases of 3 or more photon pulses, the measurement would result in both detectors firing in one basis and one firing in the other. When this happens, Eve can generate a new signal (close to Bob) without introducing any errors. For small values of $\mu$, Bennett et al. concluded this attack was harmless. However, when $\mu = 1$ and with perfect detectors for Eve, this attack becomes feasible with a fiber loss of about 18 dB, corresponding to approximately 90km of fiber at 0.2 dB/km attenuation. Another attack examined by Gisin et al. [1] involves improving the odds of intercept/resend attacks by splitting the beam, measuring each half in a different basis, and using detectors that can determine the number of photons in the signal. In certain operating regimes (small $\mu$ or short fiber length) this attack is no better than traditional intercept/resend, and we may use the same defense function. However by changing the defense function appropriately (i.e. granting Eve more information for each error bit received), one can in fact operate safely with a larger mean photon number. For the operating configuration analyzed in this paper, the result is still an optimal value of $\mu \approx 1.1$.

For this analysis, we assume, following Gilbert and Hamrick [6], that Bob is watching for anomalously high numbers of double detections (when both detectors click). Without this precaution, Eve would be able to send more than single-photon signals to Bob after successfully determining the state, and the attack would be feasible if the total attenuation, including Bob’s receiver, was 18dB.
6 Conclusions and Future Work

In this paper we offer a detailed analytic model for an experimental, fiber-based quantum cryptographic system, and a set of reasonable assumptions about Eve’s current technical capabilities. We explicitly model total system behavior ranging from physical effects to the results of quantum cryptographic protocols such as error correction and privacy amplification. We then derive the optimal photon number (µ) for this system in a range of scenarios. One interesting result is that µ ≈ 1.1 is optimal for a wide range of realistic, fiber-based QKD systems; in fact, it provides about 10 times the distilled throughput of systems that employ a more conventional µ = 0.1, without any adverse affect on system security, given an explicit set of reasonable assumptions about Eve’s current capabilities.

This paper takes one more step in the ongoing exploration of optimal mean photon number for a realistic system. Looking ahead, careful specification of a whole range of eavesdropping threats, and necessary countermeasures, and of the quantitative effects of each potential threat model, will be required before QKD can be trusted in practice. Broadly accepted analysis of a wider range of eavesdropping techniques, under a range of technologies available to Alice, Bob, and Eve, is thus desirable.

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Appendix A: Analytical Model

%% File: model.m
%%
%% Description: Analytic model of QKD throughput
%%
%% Copyright (c) 2004 by BBN Technologies
%%
%% This is a Matlab / Octave model of the QKD throughput of a weak-coherent
%% source using BB84 through fiber, given various parameters of the system.
%% Key parameters include:
%%
%% pulseRate -- the repetition rate of the source, in Hz
%% dutyCycle -- portion of pulses for payload (vs. header, training)
%% mpn -- the mean photon number per pulse at Alice
%% fiberLength -- the length of fiber, in km
%% fiberLoss -- the attenuation of the fiber, in dB/km
%% rxLoss -- receiver loss, in dB (Myers eta_rec, as dB)
%% detEff{0,1} -- detector efficiency, for each detector (eta_det)
% detLeak{0,1} -- leakage of other's light into this detector (epsilon)
% pDark{0,1} -- probability that detector fires w/ no light
% pAfter{0,1} -- probability pulse results in unsuppressed afterpulse
% residPhase -- residual phase error, RMS, in radians
%
% blockSize -- number of bits in block for EDAC / privacy amplify
% nEdacSets -- number of subsets for EDAC
% estType -- entropy estimate type ('Bennett', 'Slutsky', 'Myers')
% confidence -- probability Eve has more information than estimated
% siftType -- type of sifting ('BB84', 'SARG')
%
% This file defines typical values for these variables (which are all
% global variables), and functions which use them to compute the rate
% of detects, errors, and finished bits. To try different scenarios,
% you can simply modify the global parameters and re-execute the function.

pulseRate = 5e6; % Alice-Bob link runs at 5MHz
dutyCycle = .8; % Measured duty cycle
mpn = .1; % Target value (was calibrated recently)
fiberLength = 10.55; % Length of fiber spool, in km
fiberLoss = .237; % dB/km for spool, if total = 2.5dB
rxLoss = 10.4; % measured loss (dB, average over all paths)
residPhase = 3 * pi/180; % not measured recently
detEff0 = .117; % from analysis of data
detEff1 = detEff0;
detLeak0 = .009;
detLeak1 = detLeak0;
pDark0 = 2.8e-5;
pDark1 = pDark0;
pAfter0 = .001; % SW/HW suppression should keep this quite low
pAfter1 = pAfter0;

blockSize = 4096; % Configured min (average slightly higher)
nEdacSets = 64; % Configured
estType = 'Bennett'; % Configured
confidence = 1e-6; % Hard-wired
siftType = 'BB84'; % Configured

eveChan = 0; % Assume Eve has perfect fiber

% sourceRate -- the raw rate of symbols at the source (not counting
% attenuation)

function r = sourceRate
    global pulseRate dutyCycle
    r = pulseRate * dutyCycle;
function p = probOr(varargin)
    p = 1;
    for i = (1:nargin)
        p = p * (1-varargin{i});
    endfor
    p = 1 - p;
endfunction

% Here we estimate the probability of the different kinds of detections, and
% turn those probabilities into the sifted rate and QBER.
% pmCorr = probability that correct detector fires when bases match
% pmIncorr = probability that incorrect detector fires when bases match
% pwDetect = prob that detector fires when bases wrong (same for both D0 & D1)

function [rate, qber] = siftedRate
    global mpn fiberLength fiberLoss rxLoss residPhase
global detEff0 detEff1 detLeak0 detLeak1 pDark0 pDark1 pAfter0 pAfter1
    pDark = (pDark0 + pDark1) / 2;
    e = (detEff0*detLeak0 + detEff1*detLeak1) / (detEff0 + detEff1);
    atten = .1^(.1*(fiberLength*fiberLoss + rxLoss));
    c = (detEff0+detEff1)/2 * mpn * atten / (1+detLeak0+detLeak1);
    pwDetect = probOr (pDark, 1-exp(-c*(e + .5)));   % probability that detector fires when bases wrong
    pAfter = pwDetect * (pAfter0 + pAfter1) / 2;
    pwDetect = probOr (pwDetect, pAfter);
    pmCorr = probOr (pDark, pAfter, 1-exp(-c*(e + cos(residPhase/2)^2))) ;% probability that correct detector fires when bases match
    pmIncorr = probOr (pDark, pAfter, 1-exp(-c*(e + sin(residPhase/2)^2)));
    pmValid = probOr (pmCorr, pmIncorr);
    rate = pmValid / 2 * sourceRate;
    qber = (pmIncorr - pmCorr*pmIncorr) / pmValid;
endfunction

% EDAC overhead -- this is for the amount of extra information revealed,
% per bit, given the error rate. This is specifically for the BBN variant of
% Cascade, other protocols are likely to differ slightly. This also
% represents an average, over many blocks of slightly varying size and
% error rate. The estimate does not include the error bits themselves.

function ovhd = EDACoverhead (qber)
    global nEdacSets blockSize
    ovhd = qber*(1-log2(qber)) + nEdacSets / blockSize;
endfunction

% entropyEstimate -- this applies the specific entropy estimate chosen
% and then turns it into a fraction of the sifted bits. The entropy
% estimate here is the information Eve may be assumed to have derived
% from eavesdropping on the single-photon pulses, there is a separate
function est = entropyEstimate(qber)
    global estType blockSize confidence
    b = blockSize;
    e = qber*b;
    switch (estType)
        case 'Bennett'
            est = bennett(b,e,confidence);
        case 'Slutsky'
            est = slutsky(b,e,confidence);
        case 'Myers'
            est = myers(b,e,confidence);
        otherwise
            error('Unknown entropy estimate type %s',estType);
    end
    est = est/blockSize;
endfunction

function est = bennett(b,e,confidence)
    t = 2.828427*e;
    dev2 = 6.828427*e;
    conf1 = sqrt(2) * erfinv(1-confidence);
    est = b - e - t - conf1*sqrt(dev2);
    est = est + 2*log2(confidence);
endfunction

function est = slutsky(b,e,confidence)
    conf1 = erfinv(1-confidence);
    eprime = min(e / b + conf1 / sqrt(2*b), 1/3);
    t = (1 - 3*eprime) / (1 - eprime);
    t = (1 + 1.442695*log(1 - 0.5*t*t)) * (b-e);
    dev2 = (b-e)/2;
    est = b - e - t - conf1*sqrt(dev2);
    est = est + 2*1og2(confidence);
endfunction

% estimatePNSbits -- how many bits to discard because of "undetectable"
% eavesdropping, i.e. photon-number splitting attacks or unambiguous state
% discrimination (PNS or USD). This version is essentially Bennett's
% with a more accurate expression for multi-photon pulses. We assume
% that in all multi-photon pulses, one is captured by Eve and stored until % the bases are announced.

function mpdisc = estimatePNSbits(sift)
    global mpn detEff0 detEff1 rxLoss
    p0 = exp(-mpn);
    p1 = p0*mpn;
    p2x = 1-p0-p1;
    m = p2x / (p1+p2x);
    mpdisc = m * sift;
endfunction

% estimatePNSgh -- Gilbert & Hamrick's estimate of Eve's information from % "undetectable" eavesdropping

function mpdisc = estimatePNSgh(sift)
    global fiberLength fiberLoss mpn detEff0 detEff1 rxLoss eveChan
    p0 = exp(-mpn);
    p1 = p0*mpn;
    p2 = p1*mpn/2;
    p2x = 1-p0-p1;
    s2 = sqrt(2);
    y = .1^(.1*(fiberLength*fiberLoss*eveChan + rxLoss)) * (detEff0+detEff1)/2;
    m1 = p2x - 1/(1-y)*(exp(-mpn*y)-exp(-mpn)*(1+mpn*(1-y)));
    m2 = p2*y + 1 - exp(-mpn)*s2*sinh(mpn/s2)+2*cosh(mpn/s2)-1;
    m3 = p2*y + exp(-mpn)*sinh(mpn)-s2*sinh(mpn/s2);
    p2k = p2;
    for k = (2:20)
        p2k = p2k * mpn * mpn / (k*(4*k-2));
        m3 = m3 + p2k*max(1-(1-y)^(2*k-1),1-2^(1-k));
    endfor
    m = max([m1,m2,m3]);
    mpdisc = m * sift*sourceRate / 2;
endfunction

% estimatePNSb -- Bennett, et al.'s estimate for Eve's information from % "undetectable" eavesdropping (BBBSS92)

function mpdisc = estimatePNSb(sift)
    global mpn
    mpdisc = sift*mpn;
endfunction

%distilledRate -- this is the final answer, number of distilled bits per % second.

function rate = distilledRate
    global confidence
    [sift, qber] = siftedRate;
    ovhd = EDACoverhead(qber);
    ent = entropyEstimate(qber);
    mpd = estimatePNSbits(sift);
\[ \text{rate} = \max(\text{sift} \times (\text{ent} - \text{ovhd}) - \text{mpd}, 0) \]

% Myers/Pearson entropy estimate
%
% First we find the probability \( p \) for which the first \( k \) terms of the binomial
% distribution \( \text{binom}(n, i) \times p^i \times (1-p)^{(n-i)} \) sum up to ‘confidence’, the
% probability that we’re wrong.
%
% Then, given this probability, \( p \), the best conditional probability of Eve
% correctly guessing a bit is:
% \[
% \text{pe} = 0.5 + \sqrt{\frac{p}{(1-p)} \times \left(1 - \frac{p}{1-p}\right)}
% \]
%
% Then Eve’s least Renyi entropy (order \( R \)) for the \( n-k \) non-error bits is:
% \[
% h(R) = \frac{(n-k)}{(1-R)} \times \log2(\text{pe}^R + (1-\text{pe})^R)
% \]
%
% Now from Cachin’s paper (Smooth Entropy and Renyi Entropy), theorem 8,
% we know that the amount of smooth entropy (which we can feed into privacy
% amplification) is at least:
% \[
% h(R) - \log2(m+1) - \frac{r}{R-1} - t - 2
% \]
% where \( m = \log2(\log2(m+1)) = n+t \), and \( 2^{-(r)} + 2^{-(t)} = \text{confidence} \).
%
% If we ignore the negligible effect of \( t \) on the value of \( \log(m) \), the optimal
% values of \( r \) and \( t \) are:
% \[
% r = \log2(\frac{R}{\text{confidence}})
% \]
% \[
% t = \log2(\frac{R}{(R-1) \times \text{confidence}})
% \]
%
% and the value of \( m \) is approximately:
% \[
% m = n + t + \log2(n+t+1)
% \]
% or \( m = n + t + \log2(n+t+1+\log2(n+t+1+\log2(n+t+1))) \) etc.
%
% In our internal function, we negate this, so we can minimize.

function h = myers_neg_renyi_entropy (r)
    global myers_n myers_k myers_confidence myers_pe
    h = (myers_n - myers_k) / (1-r) \* log2(myers_pe\(^r\) + (1-myers_pe\(^r\));
    t = log2(r/(r-1)*myers_confidence));
    h = h - log2(myers_n+t+1+log2(myers_n+t+1+log2(myers_n+t+1)));
    h = h - log2(r/myers_confidence)/(r-1) - t - 2;
    h = -h;
endfunction

% Another internal function -- the sum of the first \( myers_k \) terms of the
% binomial distribution, minus \( myers_confidence \) (so we can find a zero)
function s = myers_binomtail (p)
    global myers_n myers_k myers_confidence
    k1 = myers_k;
    k2 = myers_n-myers_k;
    if (k1 > k2)
        k1 = k2;
        k2 = myers_k;
    endif
    % Compute the highest term, then go backwards
    if (k1*log(myers_n) < 200)
        % exact if < 10^86
        l = 1;
        for i = 1:k1
            l = l * (myers_n-i+1) / i;
        endfor
        t = l * p^myers_k * (1-p)^(myers_n-myers_k);
    else
        % otherwise use Stirling's approximation
        k1 = k1+1;
        k2 = k2+1;
        n1 = myers_n+1;
        l = 1 - .5*log(2*pi);
        l = l + (1/(n1) - 1/(k1) - 1/(k2)) / 12;
        l = l - (1/(n1)^3 - 1/(k1)^3 - 1/(k2)^3) / 360;
        l = l + (1/(n1)^5 - 1/(k1)^5 - 1/(k2)^5) / 1260;
        l = l + (n1-.5)*log(n1) - (k1-.5)*log(k1) - (k2-.5)*log(k2);
        t = exp(l + myers_k*log(p) + (myers_n-myers_k)*log(1-p));
    endif
    % Now loop back to the beginning, but exit if we stop changing sum
    s = t - myers_confidence;
    for k1 = (myers_k-1:-1:0)
        t = t * (k1+1) * (1-p) / (p * (myers_n-k1));
        s1 = s + t;
        if s1 == s
            break
        endif
        s = s1;
    endfor
endfunction

function entropy = myers(n,k,confidence)
    global myers_n myers_k myers_confidence myers_pe
    % Approximate starting point
    p = 1 - InvBetaApprox(n-k,k,confidence);
    myers_n = n;
    myers_k = k;
myers_confidence = confidence;

% Solve for probability p, and compute Eve's probability of guessing
p = fzero('myers_binomtail',p);
p = min(p,1/3);
myers_pe = .5 + sqrt( p/(1-p) * (1 - p/(1-p)) );

% Maximize entropy measure over Renyi order R
r = fminbnd('myers_neg_renyi_entropy',1.01,2);

% Return the maximized entropy
entropy = -myers_neg_renyi_entropy(r);
endfunction

% Abramowitz and Stegun approximation for the inverse of the incomplete Beta function
function v = InvBetaApprox(a,b,p)
    y = sqrt(2) * erfinv(1-2*p);
    l = y*y/6 - .5;
    a1 = 1/(2*a-1);
    b1 = 1/(2*b-1);
    h = 2/(a1+b1);
    w = y*sqrt(h+l)/h - (b1-a1)*l+5/6-2/(3*h);
    v = a/(a+b*exp(2*w));
endfunction