Research Article

A Stability Result for a Viscoelastic Wave Equation in the Presence of Finite and Infinite Memories

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In this study, we consider a viscoelastic wave equation in the presence of finite and infinite memories. Under suitable conditions on the variable coefficients and for a wide class of relaxation functions, we establish an explicit and general decay result. Moreover, we use a weaker boundedness condition, on the initial data than the one used in the literature. The proof of our decay result is based on the multiplier method together with some convexity arguments. This study generalizes and improves previous literature outcomes.

1. Introduction

Let us consider an $n$-dimensional body occupies with a bounded open set $\Omega \subseteq \mathbb{R}^n$ ($n \geq 1$) with smooth boundary $\partial \Omega$. Let $x \mapsto u(x, t)$ be the position of the material particle $x$ at time $t$ so that $\forall x \in \Omega$ and $\forall t > 0$; the corresponding motion equation is

$$\begin{align*}
    u_{tt} - \Delta u + \int_0^t g_1(t-s)\text{div}(a_1(x)\nabla u(s))\,ds \\
    + \int_0^{\infty} g_2(s)\text{div}(a_2(x)\nabla u(t-s))\,ds &= 0, \\
    u(x, t) &= 0, \\
    u(x, -t) &= u_0(x, t), \\
    u_t(x, 0) &= u_1(x).
\end{align*}$$

(1)

The functions $g_1$ and $g_2$ are called the relaxation (kernels) functions, and they are positive nonincreasing and defined on $\mathbb{R}^+$. The functions $a_1$ and $a_2$ are essentially bounded nonnegative defined on $\Omega$. Here, $u_0$ and $u_1$ are the history and initial data. This model of materials consist of an elastic part (without memory) and a viscoelastic part, where the dissipation given by the memory is effective.

In this study, we are concerned with the above viscoelastic wave problem (1) and mainly interested in the asymptotic behavior of the solution $u$ when $t$ tends to infinity. In fact, we prove that the solution of the corresponding viscoelastic model decays to zero and no matter how small is the viscoelastic part of the material. Note that the above model is dissipative, and the dissipation is given by the memory term only and the memory is effective only in a part of the body. For materials with memory, the stress depends not only on the present values but also on the entire temporal history of the motion. Therefore, we have to prescribe the history of $u$ before 0. Here, we assume that $u(x, -t) = u_0(x, t)$, $\forall x \in \Omega$, $\forall t > 0$. Let us mention some other papers related to the problem we address. We start our literature review with the pioneer work of Dafermos [1], in 1970, where the author discussed a certain one-dimensional viscoelastic problem, established some existence results, and then proved that, for smooth monotone decreasing
relaxation functions, the solutions go to zero as $t$ goes to infinity. However, no rate of decay has been specified. In [2], a similar result, under a convexity condition on the kernel, has been established. After that, a great deal of attention has been devoted to the study of viscoelastic problems and many existence and long-time behavior results have been established. For example, Hrusa [3] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)\left(\psi(u_x(x,s))\right)_x ds = f(x,t),$$

and proved several global existence results for large data. He also proved an exponential decay for strong solutions when $m(s) = e^{-s}$ and $\psi$ satisfies certain conditions. Dassios and Zafiropoulos [4] studied a viscoelastic problem in $\mathbb{R}^3$ and proved a polynomial decay results for exponentially decaying kernels. After that, a very important contribution by Rivera was introduced. In 1994, Rivera [5] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^d$ in the bounded domain case, and for exponentially decaying memory kernel and regular solutions, he showed that the sum of the first and the second energy decays exponentially. For the whole-space case and for exponentially decaying memory kernels, he showed that the rate of decay of energy is of algebraic type and depends on the regularity of the solution. This result was later generalized to a situation, where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [6]. In [6], the authors considered the case of bounded domains as well as the case when the material is occupying the entire space and showed that the decay of solutions is also algebraic, at a rate which can be determined by the rate of decay of the relaxation function. This latter result was later improved by Baretto et al. [7], where equations related to linear viscoelastic plates were treated. Precisely, they showed that the solution energy decays at the same rate as the decay of the relaxation function.

Decay results for arbitrary growth of the damping term have been considered for the first time in the work of Lasiecka and Tataru [8]. They showed that the energy decays as fast as the solution of an associated differential equation whose coefficients depend on the damping term. Following the method of Lasiecka and Tataru [8], Liu and Zuazua [9] established decay rates for a nonlinear globally distributed damping for the wave equation with no growth assumptions at the origin. The proof of their result is based on constructing a convex function which captures the nonlinearity of the feedback at the origin and combines it, using Jensen’s and Young’s inequalities, with the use of equivalent energy which satisfies a differential equation, which however is not a priori dissipative. Alabau-Boussouira [10] used weighted integral inequalities and some convexity arguments to establish a semiexplicit formula which leads to decay rates of the energy in terms of the behavior of the nonlinear feedback close to the origin, for which the optimal exponential and polynomial decay rate estimates are only special cases. Alabau-Boussouira [11] presented a general approach based on convexity arguments to establish sharp optimal or quasi-optimal upper energy decay rates for finite and infinite dimensional vibrating damped systems.

Inspired by the experience with frictional damping initiated in the work of Lasiecka and Tataru [8], another step forward was done by Alabau-Boussouira and Cannarsa [12] when they considered relaxation functions satisfying

$$g'(t) \leq -H(g(t)).$$

They established an explicit rate of decay under some constraints imposed on the function $H$.

For partially viscoelastic material, Rivera and Salviatierra [13] showed that the energy decays exponentially and provided the relaxation function decays in a similar fashion and the dissipation is acting on a part of the domain near to the boundary. See also, in this direction, the work of Rivera and Oquendo [14]. The uniform decay of solutions for the viscoelastic wave equation,

$$u_{tt} - k_0 \Delta u + \int_0^t \text{div}[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0,$$

was investigated by Cavalcanti and Oquendo [15] where they considered the condition $a(x) + b(x) \geq \delta > 0$. They established exponential and polynomial stability results based on some conditions on $g$ and the linearity of the function $h$. After that, Guesmia and Messaoudi [16] extended the work of [15], and they established a general decay result for (1) under the same conditions on $a_1$ and $a_2$ used in [15] and for some other conditions for the relaxation functions $g_1$ and $g_2$, from which the usual exponential and polynomial decay rates are only special cases. More precisely, they used the conditions $g_1^1(t) \leq -\xi_1(t)g_1^1(t)$ for all $t \geq 0$, and

$$\int_0^\infty \frac{g_1^1(s)}{G^{-1}(-g_1^1(s))} ds + \sup_{\tau \in \mathbb{R}_+} \frac{g_1^1(s)}{G^{-1}(-g_1^1(s))} < +\infty,$$

such that

$$G(0) = G'(0) = 0,$$

$$\lim_{t \to \infty} G'(t) = +\infty,$$

where $G: \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing strictly convex function.

In the present work, we consider (1) and under the same conditions on $a_1$ and $a_2$ used in [15], and for a large class of the relaxation functions, we prove that problem (1) is stable. Under this class of relaxation functions $g_1$ at infinity, we establish a relation between the decay rate of the solution and the growth of $g_1$ at infinity. In fact, we extend the works of [15, 16]. More precisely, we assume that the relaxations functions $g_1$ and $g_2$ satisfy

$$g_1'(t) \leq -\xi_1(t)H(g_1(t)), \quad \forall t \geq 0.$$

In addition, we use a weaker boundedness condition on the history data used in earlier papers such as [17–20].

The paper is organized as follows. In Section 2, we present some material needed for our work. We establish
some essential lemmas in Section 3. Section 4 contains the statement and the proof of our main result. We give some examples in Sections 5 to illustrate our energy decay result. Finally, we introduce conclusions and comparison study in Section 6.

2. Preliminaries

In this section, we present some material needed in the proof of our main result. Through this study, we use $c$ to denote a positive generic constant. Now, we start with the following assumptions:

(A1) $g_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ are differentiable nonincreasing functions such that, for any constant $\beta_0 > 0$ and $i = 1, 2$,

$$-\beta_0 g_i(s) \leq g_i'(s), \quad g_i(0) > 0,$$

$$1 - \left\| a_1 \right\|_\infty \int_0^{\infty} g_i(s)ds - \left\| a_2 \right\|_\infty \int_0^{\infty} g_2(s)ds = \ell > 0.$$

(8)

and there exist $C^1$ functions $H_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which are linear or strictly increasing and strictly convex $C^2$ functions on $(0, r]$ for some $r > 0$ with $H_i(0) = H_i'(0) = 0$, $\lim_{s \rightarrow +\infty} H_i(s) = +\infty$, $s \rightarrow H_i'(s)$, and $s \rightarrow s(H_i')^{-1}(s)$ are convex on $(0, r]$. Moreover, there exist positive nonincreasing differentiable functions $\xi_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that

$$g_i'(t) \leq -\xi_i(t)H_i(g_i(t)), \quad \forall t \geq 0. \quad (9)$$

(A2) $a_i : \mathbb{R} \rightarrow \mathbb{R}^+$ are $C^1(\mathbb{R})$ functions such that, for positive constants $\delta$ and $a_0$ and for $\Gamma_1, \Gamma_2 \subset \mathbb{R} \cap \mathbb{R}$ with meas $(\Gamma_i) > 0, i = 1, 2$, $\inf_{x \in \Omega} (a_1(x) + a_2(x)) \geq \delta$ and

$$a_i = 0, \quad \text{or} \quad \inf_{\Gamma_i} a_i(x) \geq 2a_0, \quad i = 1, 2. \quad (10)$$

(A3) We assume that

$$\int_0^{\infty} g_2(s)\|\nabla u_0(s)\|^2 ds < +\infty. \quad (11)$$

Remark 1. The class of relaxation functions satisfying (A1) in the present study is larger than the ones satisfying (5) and (6) used in [16]. In fact, the boundedness of the sup in (5), used in [16], can be interpreted as the inequality in (A1) in the present paper (with $\xi = 1$). Conditions (5) and (6) used in [16] ask also the boundedness of the integral. Moreover, the classical of relaxation functions $g$ that we consider in our example $g(t) = e^{-\beta t}t^\beta, \beta > 0$, satisfy both conditions (A1) in our paper and the one in [16]. So, it is better to consider (A1) used in the present study than the one used in [16].

Remark 2. Hypothesis (A3) is needed for proving the existence and stability results. For the stability, if (A3) holds, then $h_0$ which is defined in (25) is well defined, so inequality (53) makes sense. Moreover, Hypothesis (A3) is weaker than the one used in [16, 18]; that is, there exits a positive constant $M$ such that

$$\|\nabla u_0(s)\|^2 \leq M, \quad (12)$$

and our decay rate depends on the size of $\|\nabla u_0\|_\infty$.

Remark 3. If $a_i \neq 0, i = 1, 2$, there exist neighborhoods $\omega_i$ of $\Gamma_i, i = 1, 2$, such that

$$\inf_{\omega_i} a_i(x) \geq a_0 > 0, \quad i = 1, 2. \quad (13)$$

As in [15, 16], let $d = \min\{a_0, \delta\}$ and let $a_i \in C^1(\mathbb{R})$, $i = 1, 2$, be such that

$$0 \leq a_i(x) \leq a_i(x), \quad a_i(x) = 0, \quad \text{if} \quad a_i(x) \leq \frac{d}{4}, \quad (14)$$

$$a_i(x) = a_i(x), \quad \text{if} \quad a_i(x) \geq \frac{d}{2} \quad (15)$$

Lemma 1. The functions $a_i, i = 1, 2$, are not identically zero and satisfy

$$a_1(x) + a_2(x) \geq \frac{d}{2}. \quad (16)$$

Proof

(1) For $x \in \Omega \cap \omega_i$, we have $a_i(x) \geq a_0 \geq d$, which implies, by (14), that $a_i(x) = a_i(x) \geq d$. Thus, $a_i$ is not identically zero.

(2) If $a_i(x) \leq (d/2)$ then $a_i(x) = a_i(x)$. Consequently, $a_i(x) + a_i(x) \geq a_i(x) \geq (d/2)$. If $a_i(x) < (d/2)$, then $a_i(x) > (d/2)$ which implies, by (14), $a_i(x) = a_i(x) > (d/2)$. Consequently, $a_i(x) + a_i(x) > (d/2)$. This completes the proof.

The existence and uniqueness of the solution of problem (1) is given by the following theorem.

Theorem 1. If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and hypotheses (A1) − (A3) hold, then problem (1) has a unique solution in the class

$$u \in C^0([0, \infty); H_0^1(\Omega)) \cap C^1([0, \infty); L^2(\Omega)). \quad (17)$$

The proof of the above theorem can be obtained, making use, for instance, of the Faedo−Galerkin method and considering standard arguments of density (see [15]).

We define the “modified” energy functional of the weak solution of problem (1) by
where

\[
\begin{align*}
    (g_1 \circ \nabla u)(t) &= \int_0^1 a_1(x) \int_0^t g_1(t-s)\nabla u(t) - \nabla u(s) \, ds \, dx, \\
    (g_2 \circ \nabla u)(t) &= \int_0^1 a_2(x) \int_0^t g_2(t-s)\nabla u(t) - \nabla u(s) \, ds \, dx.
\end{align*}
\]  

\tag{18}

**Lemma 2.** The “modified” energy functional satisfies, along the solution of problem (1), the following:

\[
E'(t) = -\frac{1}{2} g_1(t) \int_0^1 \nabla u(t)^2 \, dx + \frac{1}{2} (g_1' \circ \nabla u)(t) \leq 0,
\]

\tag{19}

where

\[
\begin{align*}
    (g_1' \circ \nabla u)(t) &= \int_0^1 a_1(x) \int_0^t g_1'(t-s)\nabla u(t) - \nabla u(s) \, ds \, dx, \\
    (g_1' \circ \nabla u)(t) &= \int_0^1 a_2(x) \int_0^t g_1'(t-s)\nabla u(t) - \nabla u(s) \, ds \, dx.
\end{align*}
\]  

\tag{20}

**Proof.** By multiplying the first equation in problem (1) by \(u\) and integrating over \(\Omega\), using integration by parts, hypotheses (A1) and (A2) and some manipulations as in [6,21] and others, we obtain (19) for regular solutions. This inequality remains valid for weak solutions by a simple density argument. \(\Box\)

3. Technical Lemmas

In this section, we present and prove additional lemmas that are pivotal for the principal result.

**Lemma 3** (see [22]). For \(i = 1, 2\), we have

\[
\int_0^t \left( \int_0^t g_i(t-s)(\nabla u(s) - \nabla u(t-s)) \, ds \right)^2 \, ds \leq C_\alpha (h_i \circ \nabla u)(t),
\]

where, for any \(0 < \alpha < 1\),

\[
C_\alpha = \int_0^\alpha \frac{g_i^2(s)}{\alpha g_i(s) - g_i'(s)} \, ds,
\]

\tag{22}

and recalling the Lebesgue dominated convergence theorem, one can deduce that

\[
a C_\alpha = \int_0^\alpha \frac{g_i^2(s)}{\alpha g_i(s) - g_i'(s)} \, ds \to 0 \text{ as } \alpha \to 0.
\]

**Lemma 4.** Assume that (A1) – (A3) hold. There exists a positive constant \(M_1\) such that

\[
\int_\Omega a_2(x) \int_0^t g_2(t-s)\nabla u(t) - \nabla u(t-s) \, ds \leq M_1 h_0(t),
\]

where \(h_0(t) = \int_0^t g_2(t+s)(1 + \|\nabla u_0(s)\|^2) \, ds\).

**Proof.** The proof is identical to the one in [23]. Indeed, we have

\[
\begin{align*}
\int_\Omega a_2(x) \int_0^t g_2(t-s)\nabla u(t) - \nabla u(t-s) \, ds &
\leq 2 \|a_2\|_\infty \|\nabla u(t)\|^2 \int_0^t g_2(s) \, ds + 2 \|a_2\|_\infty \int_0^t g_2(s) \|\nabla u(t-s)\|^2 \, ds \\
&\leq 2 \|a_2\|_\infty \sup_{s \geq t} \|\nabla u(s)\|^2 \int_0^t g_2(t+s) \, ds + 2 \|a_2\|_\infty \int_0^t g_2(t+s) \|\nabla u_0(s)\|^2 \, ds \\
&\leq \frac{4}{\epsilon} \|a_2\|_\infty E(s) \int_0^t g_2(t+s) \, ds + 2 \|a_2\|_\infty \int_0^t g_2(t+s) \|\nabla u_0(s)\|^2 \, ds \\
&\leq \frac{4}{\epsilon} \|a_2\|_\infty E(0) \int_0^t g_2(t+s) \, ds + 2 \|a_2\|_\infty \int_0^t g_2(t+s) \|\nabla u_0(s)\|^2 \, ds \\
&\leq M_1 \int_0^t g_2(t+s)(1 + \|\nabla u_0(s)\|^2) \, ds.
\end{align*}
\]  

\tag{26}
where $M_1 = \max\{2\|a_2\|_{\infty}, (4\|a_2\|_{\infty}E(0)\ell)\}$.

Lemma 5 (see [16]). Assume that (A1) – (A3) hold; then, the functionals,

$$
\psi_1(t) = \int_\Omega uu_t dx,
$$

$$
\psi_2(t) = -\int_\Omega a_1(x)u_t \int_0^t g_1(t-s)(u(t)-u(s))ds dx,
$$

$$
\psi_3(t) = -\int_\Omega a_2(x)u_t \int_0^\infty g_2(s)(u(t)-u(t-s))ds dx,
$$

(27)

satisfy, along the solution of (1) and for any $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ and $\forall t \geq 0$, the following estimates

$$
\psi_1'(t) \leq \int_\Omega u_t^2 dx - \left[1 - \epsilon_1 - \|a_1\|_{\infty}^2 \int_0^\infty g_1(s)ds - \|a_2\|_{\infty} \int_0^\infty g_2(s)ds \right] \int_\Omega |\nabla u|^2 dx
$$

$$
+ \frac{cC_a}{\epsilon_1} (h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t),
$$

(28)

$$
\psi_2'(t) \leq -\left[\int_0^\infty g_2(s)ds - \epsilon_2\right] \int_\Omega a_2(x)u_t^2 dx + \frac{\epsilon_2}{2} \int_\Omega |\nabla u|^2 dx - \frac{c}{\epsilon_2} (g_1' \circ \nabla u)(t)
$$

$$
+ \frac{cC_a}{\epsilon_3} (h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t),
$$

(29)

and

$$
\psi_3'(t) \leq -\left[\int_0^\infty g_2(s)ds - \epsilon_3\right] \int_\Omega a_2(x)u_t^2 dx
$$

$$
+ \frac{\epsilon_3}{2} \int_\Omega |\nabla u|^2 dx - \frac{c}{\epsilon_2} (g_1' \circ \nabla u)(t)
$$

$$
+ \frac{cC_a}{\epsilon_3} (h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t),
$$

(30)

Lemma 6. Assume that (A1) – (A3) hold; then, for a suitable choice of $N, M, m > 0$ and for all $t \geq 0$, the functional,

$$
\mathcal{L}(t) := NE(t) + M\psi_1(t) + \psi_2(t) + \psi_3(t),
$$

(31)

satisfies, for any $t > 0$

$$
\mathcal{L}'(t) \leq -mE(t) + \frac{1}{4} (g_1 \circ \nabla u + g_2 \circ \nabla u)(t).
$$

(32)

Proof. Let, for $t_0 > 0$ fixed, $g_0 := \min\{\int_0^h g_1(s) ds, \int_0^\infty g_2(s) ds\}$. Then, direct differentiation of $\mathcal{L}$ and using (19), (28), (29), and (30) leads to

Recalling that $g_i = \alpha g_1 - h_i$, where $i = 1, 2$. We obtain

$$
\mathcal{L}'(t) \leq a \left(\frac{N - \epsilon}{\epsilon_2}\right) (g_1 \circ \nabla u + g_2 \circ \nabla u)(t)
$$

$$
- \int_\Omega [(g_0 - \epsilon)(\alpha_1 + \alpha_2) - M]u_t^2 dx
$$

$$
+ \frac{cC_a}{\epsilon_3} + \frac{McC_a}{\epsilon_1} (h_1 \circ \nabla u)(t) + (h_2 \circ \nabla u)(t)
$$

$$
- [(t - \epsilon_1)M - \epsilon_3] \int_\Omega |\nabla u|^2 dx, \ \forall t \geq t_0.
$$

(33)

Simplifying the above estimate, we arrive at...
Lemma 7. Assume that (A1) – (A3) hold. Then, for all $t > 0$ and fixed positive constant $m_0$, we have the following estimates:

$$
\int_0^t E(s)ds < m_0 \left(1 + \int_0^t h_0(s)ds\right),
$$

where $h_0$ is defined in (25).

Proof. The proof of this lemma can be established by following the same arguments in [23, 24].

Lemma 8. If (A1) – (A3) are satisfied, then we have, for all $t > 0$ and for $i = 1, 2$, the following estimates:

$$
\int_0^t a_i(x)\int_0^t g_i(s)|\nabla u(t) - \nabla u(t - s)|^2 dsdx 
\leq \frac{1}{\Lambda_i(t)}H_i^{-1}\left(\Lambda_i(t)\mu_i(t)\right),
$$

where $\Lambda_i(t) = (\lambda_i/1 + \int_0^t h_0(s)ds), \lambda_i \in (0, 1), H_i$ and $\xi_i$ are introduced in (A1), and

$$
\mu_i(t) = -\int_0^t a_i(x)\int_0^t g_i(s)|\nabla u(t) - \nabla u(t - s)|^2 dsdx \leq -cE'(t).
$$

Proof. To establish (42), we introduce the following functional:

$$
\chi_i(t) := \Lambda_i(t)\int_0^t a_i(x)\int_0^t |\nabla u(t) - \nabla u(t - s)|^2 dsdx.
$$

Then, using (17) and the fact that $E$ is nonincreasing to get,

$$
\chi_i(t) \leq 2\Lambda_i(t)\left(\int_0^t a_i(x)\int_0^t |\nabla u(t)|^2 + \int_0^t a_i(x)\int_0^t |\nabla u(t - s)|^2 dsdx\right)
\leq \frac{4\Lambda_i(t)\|a_i\|_{\infty}}{t} \int_0^t (E(t) + E(t - s))ds
\leq \frac{8\Lambda_i(t)\|a_i\|_{\infty}}{t} \int_0^t E(s)ds.
$$

Using (41) and the definition of $\Lambda_i(t)$, (45) becomes

$$
\chi_i(t) \leq \frac{8m_0h_0\|a_i\|_{\infty}}{t(1 + \int_0^t h_0(s)ds)}\left(1 + \int_0^t h_0(s)ds\right).
$$

Therefore, we can choose $0 < \lambda_i < (\ell/8m_0\|a_i\|_{\infty})$ such that, for all $t > 0$,

$$
\chi_i(t) < 1.
$$

Without loss of the generality, for all $t > 0$, we assume that $\chi_i(t) > 0$; otherwise, we get an exponential decay from (42). The use of Jensen’s inequality and using (43) and (47) give
\[ \mu_1(t) = \frac{1}{\Lambda(t)} \int_0^t \Lambda_1(t)(-g_1'(s)) \int_\Omega a_i(x) |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \]

\[ \geq \frac{1}{\Lambda(t)} \int_0^t \Lambda_1(t) \xi(t)(s) H_i(g_i(s)) \int_\Omega a_i(x) |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \]

\[ \geq \frac{\xi(t)}{\Lambda(t)} H_i \left( \Lambda_1(t) \int_0^t g_i(s) \int_\Omega a_i(x) |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right) \]

\[ = \frac{\xi(t)}{\Lambda(t)} H_i \left( \Lambda_1(t) \int_0^t g_i(s) \int_\Omega a_i(x) |\nabla u(t) - \nabla u(t-s)|^2 \, dx \, ds \right). \]

Hence, (42) is established.

4. Decay Result

In this section, we state and prove our main result and provide some examples to illustrate our decay results. Let us start introducing some functions and then establishing several lemmas needed for the proof of our main result. Now, for \( i = 1, 2 \), let us take

\[ \xi = \min \{ \xi_1 \}, \]
\[ \mu = \max \{ \mu_1 \}, \]
\[ \lambda = \max \{ \lambda_1 \}, \]
\[ G = \left( H_1^{-1} + H_2^{-1} \right)^{-1}. \]

As in [23], we introduce the following functions:

\[ G_1(t) = \int_0^t \frac{1}{sG'(s)} \, ds, \]

\[ G_5(t) = G_1^{-1} \left( c_1 \int_0^t \xi(s) ds \right), \]

\[ G_2(t) = tG'(t), \]

\[ G_3(t) = t(G')^{-1}(t), \]

\[ G_4(t) = G_3^*(t), \]

where \( G_3^* \) is the convex conjugate of \( G_3 \) in the sense of Young (see [25]). One can easily verify that \( G_1 \) is decreasing function over \((0, r]\), and \( G_2, G_3, G_4 \) are convex and increasing functions on \((0, r]\). Furthermore, we introduce the class \( \mathcal{S} \) of functions \( \mathcal{S} : \mathbb{R}_+ \rightarrow (0, r] \) satisfying for fixed \( c_1, c_2 > 0 \):

\[ \mathcal{S} \in C^1(\mathbb{R}_+), \quad \mathcal{S} \leq 1, \quad \mathcal{S}' \leq 0, \]

and

\[ c_2 G_4 \left[ \frac{c}{d} \Lambda(t) h_0(t) \right] \leq c_1 \left( G_2 \left( \frac{G_3(t)}{\mathcal{S}(t)} \right) - G_2 \left( G_3(t) \right) \right), \]

where \( h_0 \) is defined in (25) and \( d > 0 \).

\[ \text{Theorem 2. Assume that (A1) – (A3) hold; then, there exists a strictly positive constant } C \text{ such that the solution of (1) satisfies, for all } t \geq 0, \]

\[ E(t) \leq \frac{CG_5(t)}{\vartheta(t) \Lambda(t)}, \]

where \( G_5 \) and \( \vartheta \) are defined in (50) and (52), respectively.

\[ L'(t) \leq -mE(t) + \frac{c}{\Lambda(t)} G^{-1}(\Lambda(t) \mu(t)) + ch_0(t). \]

Without loss of generality, one can assume that \( E(0) > 0 \). For \( \varepsilon_0 < r \), let the functional \( \mathcal{F} \) be defined by

\[ \mathcal{F}(t) = G \left( \varepsilon_0 \frac{E(t) \Lambda(t)}{E(0)} \right) L(t), \]

which satisfies \( \mathcal{F} \sim E \). By noting that \( G'' \geq 0, \Lambda' \leq 0, \) and \( E' \leq 0, \) we obtain

\[ \mathcal{F}'(t) = \varepsilon_0 \left( \frac{E(t)}{E(0)} \right) G'' \left( \varepsilon_0 \frac{E(t) \Lambda(t)}{E(0)} \right) L(t) \]

\[ + G' \left( \varepsilon_0 \frac{E(t) \Lambda(t)}{E(0)} \right) L'(t) \]

\[ \leq -mE(t) G' \left( \frac{E(t) \Lambda(t)}{E(0)} \right) \]

\[ + \frac{c}{\Lambda(t)} G' \left( \frac{E(t) \Lambda(t)}{E(0)} \right) G^{-1}(\Lambda(t) \mu(t)) \]

\[ + ch_0(t) G' \left( \frac{E(t) \Lambda(t)}{E(0)} \right). \]

Let \( G^* \) be the convex conjugate of \( G \) in the sense of Young (see [25]); then,

\[ G^* \left( s \right) = s(G')^{-1} \left( s \right) - G \left[ (G')^{-1} \left( s \right) \right], \quad \text{ifs } \epsilon \in (0, \infty). \]
and $G^*$ satisfies the following generalized Young inequality:

$$AB \leq G^*(A) + G(B), \quad \text{if } A, B \in (0, \infty). \quad (59)$$

So, with $A = G' \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right)$ and $B = G^{-1} \left( \Lambda(t) \mu(t)/\xi(t) \right)$ and using (57)–(59), we arrive at

$$\mathcal{F}'(t) \leq -mE(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) \frac{E(t)\Lambda(t)}{E(0)} + \frac{c}{\Lambda(t)} G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right)$$

$$\leq -mE(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) \frac{E(t)\Lambda(t)}{E(0)} + \frac{c}{\Lambda(t)} G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) + \epsilon \mu(t)/\xi(t)$$

$$+ ch_0(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right). \quad (60)$$

So, multiplying (60) by $\xi(t)$ and using (43) and the fact that $\varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) < r$ give

$$\xi(t)\mathcal{F}'(t) \leq -m\xi(t)(t)E(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) + \xi(t)\varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right)$$

$$+ \xi(t)\mu(t)\Lambda(t) + \xi(t)h_0(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right)$$

$$\leq -\varepsilon_0 \left( mE(0)/\varepsilon_0 - c \right) \xi(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) - cE' + c\xi(t)h_0(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right). \quad (61)$$

Consequently, recalling the definition of $G_3$ and choosing $\varepsilon_0$ so that $k = (mE(0)/\varepsilon_0 - c) > 0$, we obtain, for all $t \in \mathbb{R}_+$,

$$\mathcal{F}_1'(t) \leq -k \xi(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right)$$

$$+ c\xi(t)h_0(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right)$$

$$\leq -k \xi(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) + c\xi(t)h_0(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right). \quad (62)$$

$$ch_0(t)G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) = \frac{d}{\Lambda(t)} \left[ \frac{c}{d} \Lambda(t)h_0(t) \right] \left( G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) \right)$$

$$\leq \frac{d}{\Lambda(t)} G_3 \left( G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) \right) + \frac{d}{\Lambda(t)} G_3^* \left[ \frac{c}{d} \Lambda(t)h_0(t) \right]$$

$$\leq \frac{d}{\Lambda(t)} \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) \left( G^* \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) \right) + \frac{d}{\Lambda(t)} G_4 \left[ \frac{c}{d} \Lambda(t)h_0(t) \right]$$

$$\leq \frac{d}{\Lambda(t)} G_2 \left( \varepsilon_0 \left( E(t) \Lambda(t)/E(0) \right) \right) + \frac{d}{\Lambda(t)} G_4 \left[ \frac{c}{d} \Lambda(t)h_0(t) \right]. \quad (64)$$
Now, combining (62) and (64) and choosing \(d\) small enough so that \(k_0 = (k - d) > 0\), we arrive at

\[
\mathcal{F}_1'(t) \leq -k_1 \frac{\xi(t)}{\Lambda(t)} G_2 \left( \frac{\xi_0(t) \Lambda(t)}{E(t)} \right) + \frac{d\xi(t)}{\Lambda(t)} G_2 \left( \frac{\xi_0(t) \Lambda(t)}{E(t)} \right) + \frac{d\xi(t)}{\Lambda(t)} G_4 \left[ \frac{c}{d} \Lambda(t) h_0(t) \right].
\]

(65)

Using the equivalent property in (63) and the increasing of \(G_2\), we have

\[
G_2 \left( \frac{\xi_0(t) \Lambda(t)}{E(t)} \right) \geq G_2 \left( d_0 \mathcal{F}_1(t) \Lambda(t) \right).
\]

(66)

Letting \(\mathcal{F}_2(t) = d_0 \mathcal{F}_1(t) \Lambda(t)\) and recalling \(\Lambda' \leq 0\), then we arrive at, for some \(c_1, c_2 > 0\),

\[
\mathcal{F}_1'(t) \leq -c_1 \xi(t) G_2(\mathcal{F}_2(t)) + c_2 \xi(t) G_4 \left[ \frac{c}{d} \Lambda(t) h_0(t) \right].
\]

(67)

Since \(d_0 \Lambda(t)\) is nonincreasing, using the equivalent property \(\mathcal{F}_1 \sim E\) implies that there exists \(b_0 > 0\) such that \(\mathcal{F}_2(t) \geq b_0 \Phi(t) \Lambda(t)\). Let \(t \in \mathbb{R}_+\) and \(\Phi(t)\) satisfying (52) and (53). If

\[
b_0 \Lambda(t) \Phi(t) \leq 2 \frac{G_2(t)}{\Phi(t)},
\]

(68)

then, we have

\[
G_2 \left( \xi \Phi(\mathcal{F}_2(s) - \xi G_5(s)) \right) \leq \xi \Phi(\mathcal{F}_2(s) - \xi G_5(s)) \geq G_2 \left( \frac{G_5(s)}{\Phi(s)} \right) - \xi \Phi(\mathcal{F}_2(s) - \xi G_5(s)) \Phi(\mathcal{F}_2(s) - \xi G_5(s)) - \Phi(\mathcal{F}_2(s) - \xi G_5(s)) G_4 \left( \frac{G_5(s)}{\Phi(s)} \right).
\]

(73)

Now, we let

\[
\mathcal{F}_3(s) = \xi \Phi(\mathcal{F}_2(s) - \xi G_5(s)),
\]

(74)

where \(\xi\) small enough so that \(\mathcal{F}_3(0) \leq 1\). Then, (73) becomes, for any \(0 \leq s \leq 1\),

\[
G_2(\mathcal{F}_3(s)) \leq \xi \Phi(\mathcal{F}_2(s) - \xi G_5(s)) G_2 \left( \frac{G_5(s)}{\Phi(s)} \right).
\]

(75)

Furthermore, we have

\[
\mathcal{F}_3'(t) = \xi \Phi'(t) \mathcal{F}_2(t) + \xi \Phi(t) \mathcal{F}_2'(t) - \xi G_5'(t).
\]

(76)

Since \(\Phi' \leq 0\) and using (67), then, for any \(0 \leq s \leq t\), \(0 < \xi \leq 1\), we obtain

\[
E(t) \leq \frac{2}{b_0} \frac{G_2(t)}{\Phi(t)} \frac{G_5(t)}{\Phi(t)}.
\]

(69)

If

\[
b_0 \Lambda(t) \Phi(t) \geq 2 \frac{G_2(t)}{\Phi(t)},
\]

(70)

then, for any \(0 \leq s \leq t\), we obtain

\[
b_0 q(s) \Phi(s) > 2 \frac{G_2(t)}{\Phi(t)}.
\]

(71)

since \(\Lambda(t) \Phi(t)\) is a nonincreasing function. Therefore, for any \(0 \leq s \leq t\), we have

\[
\mathcal{F}_2(s) \geq 2 \frac{G_2(t)}{\Phi(t)}.
\]

(72)

Using (72), recalling the definition of \(G_2\), the fact that \(G_2\) is convex, and \(0 < \Phi \leq 1\), we have, for any \(0 \leq s \leq t\) and \(0 < \xi \leq 1\),

\[
\mathcal{F}_3'(t) \leq \xi \Phi'(t) \mathcal{F}_2(t) + \xi \Phi(t) \mathcal{F}_2'(t) - \xi G_5'(t).
\]

(77)

Then, using (53) and (75), we obtain

\[
\mathcal{F}_3'(t) \leq -c_1 \xi \Phi(t) \mathcal{F}_2(t) + c_2 \xi \Phi(t) G_4 \left[ \frac{c}{d} \Lambda(t) h_0(t) \right] - \xi G_5'(t).
\]

(78)

From the definition of \(G_1\) and \(G_5\), we have
\[ G_1(G_5(s)) = c_1 \int_0^t \xi(r)dr. \quad (79) \]

Hence,

\[ c_2 \varepsilon_1 \xi(t) \partial(t)G_2 \left[ \frac{c}{d} \Lambda(t)h_0(t) \right] - c_1 \varepsilon_1 \xi(t) \partial(t)G_2 \left( \frac{G_5(t)}{\theta(t)} \right) - \varepsilon_1 G'_2(t) \]

\[ = c_2 \varepsilon_1 \xi(t) \partial(t)G_4 \left[ \frac{c}{d} \Lambda(t)h_0(t) \right] - \varepsilon_1 \xi(t) \partial(t)G_2 \left( \frac{G_5(t)}{\theta(t)} \right) + c_2 \varepsilon_1 \xi(t)G_2 (G_5(t)) \]

\[ = \varepsilon_1 \xi(t) \partial(t) \left( c_2 G_4 \left[ \frac{c}{d} \Lambda(t)h_0(t) \right] - c_1 G_2 \left( \frac{G_5(t)}{\theta(t)} \right) + c_2 \xi G_5(t) \right). \quad (81) \]

Then, according to (53), we obtain

\[ \varepsilon_1 \xi(t) \partial(t) \left( c_2 G_4 \left[ \frac{c}{d} \Lambda(t)h_0(t) \right] - c_1 G_2 \left( \frac{G_5(t)}{\theta(t)} \right) - \varepsilon_1 \frac{G_5(t)}{\theta(t)} \right) \leq 0. \quad (82) \]

Then, (78) gives

\[ \mathcal{F}_3(t) \leq - c_1 \xi(t) G_2 (\mathcal{F}_3(t)). \quad (83) \]

Thus, from (83) and the definition of \( G_1 \) and \( G_2 \) in (50) and (51), we obtain

\[ (G_1(\mathcal{F}_3(t)))' \geq c_1 \xi(t). \quad (84) \]

Integrating (84) over \([0, t]\), we obtain

\[ G_1(\mathcal{F}_3(t)) \geq c_1 \int_0^t \xi(s) ds + G_1(\mathcal{F}_3(0)). \quad (85) \]

Since \( G_1 \) is decreasing, \( \mathcal{F}_3(0) \leq 1 \) and \( G_1(1) = 0 \), then

\[ \mathcal{F}_3(t) \leq G_1^{-1} \left( c_1 \int_0^t \xi(s) ds \right) = G_5(t). \quad (86) \]

Recalling that \( \mathcal{F}_3(t) = \varepsilon_1 \partial(t) \mathcal{F}_2(t) - \varepsilon_1 G_5(t) \), we have

\[ \mathcal{F}_2(t) \leq \frac{1 + \varepsilon_1}{\varepsilon_1} \frac{G_5(t)}{\theta(t)}. \quad (87) \]

Similarly, recall that \( \mathcal{F}_2(t) = d_0 \mathcal{F}_1(t) \Lambda(t); \) then,

\[ \mathcal{F}_1(t) \leq \frac{1 + \varepsilon_1}{d_0 \varepsilon_1} \frac{G_5(t)}{\theta(t) \Lambda(t)}. \quad (88) \]

Since \( \mathcal{F}_1 \sim E \), then, for some \( b > 0 \), we have \( E(t) \leq b \mathcal{F}_1 \), which gives

\[ E(t) \leq \frac{b(1 + \varepsilon_1)}{d_0 \varepsilon_1} \frac{G_5(t)}{\theta(t) \Lambda(t)}. \quad (89) \]

From (69) and (89), we obtain the following estimate:

\[ E(t) \leq c_3 \left( \frac{G_5(t)}{\theta(t) \Lambda(t)} \right). \quad (90) \]

where \( c_3 = \max \{ 2/\beta_0, \beta(1 + \varepsilon_1)/d_0 \varepsilon_1 \} \).

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\end{center}

5. Examples

Let \( g(t) = (a/(1 + t)^\beta) \), where \( \beta > 1 \) and \( 0 < a < \beta - 1 \) so that (A1) is satisfied. In this case \( \xi(t) = \beta t \), and \( G(t) = t^{\beta - 1} \). Then, there exist positive constants \( a_i (i = 0, 1, \ldots, 3) \) depending only on \( a, \beta \) such that

\begin{align*}
G_4(t) &= a_1 t^{\beta+1}, \\
G_2(t) &= a_2 t^{\beta+1}, \\
G_1(t) &= a_3 t + 1, \\
G_3(t) &= (a_3 t + 1)^{-\beta}.
\end{align*}

We will discuss two cases.

Case 1. If

\[ m_0 (1 + t)^{\gamma} \leq m_1 (1 + t)^{\gamma}, \quad (92) \]

where \( 0 < r < \beta - 1 \) and \( m_0, m_1 > 0 \), then we have, for some positive constants \( a_i (i = 4, 5, \ldots, 7) \), depending only on \( a, \beta, m_0, m_1, r \), the following:

\[ a_4 (1 + t)^{\beta+1+r} \leq h_0(t) \leq a_5 (1 + t)^{-\beta+1+r}, \quad (93) \]

\[ \frac{\lambda}{\Lambda(t)} \geq 2 \begin{cases}
1 + \ln(1 + t), & \beta - r = 2, \\
2, & \beta - r > 2, \\
(1 + t)^{-\beta+2}, & 1 < \beta - r < 2,
\end{cases} \quad (94) \]

\[ \frac{\lambda}{\Lambda(t)} \leq a_7 \begin{cases}
1 + \ln(1 + t), & \beta - r = 2, \\
2, & \beta - r > 2, \\
(1 + t)^{-\beta+2}, & 1 < \beta - r < 2.
\end{cases} \quad (95) \]

We notice that condition (53) is satisfied if

\[ (1 + t)^{\beta} \Lambda(t) h_0(t) \theta(t) \leq a_8 (1 - (\theta(t)^{\beta+1})) \quad (96) \]

where \( a_8 > 0 \) depending on \( a, \beta, c_1, \) and \( c_2 \). Choosing \( \theta(t) \) as the following,
\[ \vartheta(t) = \lambda \begin{cases} (1 + t)^{-\beta}, & \beta - r \geq 2, \\ (1 + t)^{-\beta}, & 1 < \beta - r < 2, \\ \end{cases} \tag{97} \]

so that (52) is valid. Moreover, using (93) and (94), we see that (96) is satisfied if 0 < \lambda \leq 1 is small enough, and then, (53) is satisfied. Hence, (54) and (95) imply that, for any \( t \in \mathbb{R}_+ \),

\[ E(t) \leq a_0 \begin{cases} (1 + \ln(1 + t))(1 + t)^{-(\beta - r - 1)}, & \beta - r = 2, \\ (1 + t)^{-(\beta - r - 1)}, & \beta - r > 2 \text{ or } 1 < \beta - r < 2. \end{cases} \tag{98} \]

Thus, estimate (98) gives \( \lim_{t \to \infty} E(t) = 0. \)

Case 2. If \( m_0 \leq 1 + \|Vu_0\|^2 \leq m_1 \). That is, \( r = 0 \) in (92), as it was assumed in many papers in the literature. So, it is clear that estimate (98) gives \( \lim_{t \to \infty} E(t) = 0 \) when \( r = 0. \)

### 6. Conclusion

Comparing our result with some earlier results in literature, we find that the result in this study improves and extends some earlier decay results in the literature in the following:

(i) Our decay result is obtained with more general relaxation function that is used in the literature such as the one in [15,16].

(ii) If \( m_0 (1 + t)^\gamma \leq 1 + \|Vu_0\|^2 \leq m_1 (1 + t)^\gamma \), where \( 0 < \gamma < \beta - 1 \) and \( m_0, m_1 > 0 \), our estimate (98) extends and improves the decay rate \((1 + t)^\gamma\) obtained in [17] where \( 0 < \beta \) is small enough and \( \beta > 2. \)

(iii) If \( m_0 \leq 1 + \|Vu_0\|^2 \leq m_1 \), our estimate (98) gives a better decay rate than the ones \((1 + t)^{\beta}\), where \( 0 < \beta < ((q - 1)/2) \) is obtained in [18], and also better than the decay rate \((1 + t)^{\beta}\), where \( 0 < \beta \) is small enough and obtained in [19]. Moreover, our estimate (98) provides a better decay than \((1 + t)^{-(q^2 - q - 1)/4}\) obtained in [20].

### Data Availability

No data were used to support this study.

### Disclosure

A version of this work has been presented as Arxiv (https://Arxiv.org) [26].

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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