On the Classes of Interval Graphs of Limited Nesting and Count of Lengths

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Abstract

In 1969, Roberts introduced proper and unit interval graphs and proved that these classes are equal. Natural generalizations of unit interval graphs called $k$-length interval graphs were considered in which the number of different lengths of intervals is limited by $k$. Even after decades of research, no insight into their structure is known and the complexity of recognition is open even for $k = 2$. We propose generalizations of proper interval graphs called $k$-nested interval graphs in which there are no chains of $k + 1$ intervals nested in each other. It is easy to see that $k$-nested interval graphs are a superclass of $k$-length interval graphs. We give a linear-time recognition algorithm for $k$-nested interval graphs. This algorithm adds a missing piece to Gajarský et al. [FOCS 2015] to show that testing FO properties on interval graphs is FPT with respect to the nesting $k$ and the length of the formula, while the problem is W[2]-hard when parameterized just by the length of the formula. We show that a generalization of recognition called partial representation extension is NP-hard for $k$-length interval graphs, even when $k = 2$, while Klavík et al. show that it is polynomial-time solvable for $k$-nested interval graphs.

Keywords Interval graphs · Proper and unit interval graphs · Recognition · Partial representation extension

1 Introduction

For a graph $G$, we denote by $V(G)$ its vertices and $E(G)$ its edges. An interval representation $\mathcal{R}$ of a graph $G$ is a collection $\{\langle u \rangle : u \in V(G)\}$ of intervals of the real
line such that \( uv \in E(G) \) if and only if \( \langle u \rangle \cap \langle v \rangle \neq \emptyset \). A graph is an \textit{interval graph} if it has an interval representation, and we denote the class of interval graphs by \( \text{INT} \).

An interval representation is called \textit{proper} if \( \langle u \rangle \subseteq \langle v \rangle \) implies \( \langle u \rangle = \langle v \rangle \), and \textit{unit} if the length of all intervals \( \langle u \rangle \) is one. The classes of \textit{proper} and \textit{unit interval graphs} (denoted \( \text{PROPER INT} \) and \( \text{UNIT INT} \)) consist of all interval graphs which have proper and unit interval representations, respectively. Roberts [30] proved that \( \text{PROPER INT} = \text{UNIT INT} \).

\textbf{The Studied Classes} In this paper, we consider two hierarchies of subclasses of interval graphs which generalize proper and unit interval graphs. The class \( k \)-Nested\( \text{INT} \) consists of all interval graphs which have representations with no \( k + 1 \) intervals \( \langle u_0 \rangle, \ldots, \langle u_k \rangle \) such that \( \langle u_0 \rangle \subset \langle u_1 \rangle \subset \ldots \subset \langle u_k \rangle \); see Fig. 1a. The class \( k \)-Length\( \text{INT} \) consists of all interval graphs which have representations having at most \( k \) different lengths of intervals; see Fig. 1b. We know by [30] that \( 1 \)-Nested\( \text{INT} = \text{PROPER INT} = \text{UNIT INT} = 1 \)-Length\( \text{INT} \).

For an interval graph \( G \), we denote the minimum nesting over all interval representations by \( \nu(G) \), and the minimum number of interval lengths by \( \lambda(G) \). Since nested intervals have different lengths, we know that \( \nu(G) \leq \lambda(G) \) and this inequality may be strict (as in Fig. 1b). For each \( k \geq 2 \),

\[
(k - 1)\text{-LengthINT} \subsetneq k\text{-LengthINT} \subsetneq k\text{-NestedINT} \subsetneq (k + 1)\text{-NestedINT}.
\]

Fishburn [10, Theorem 5, p. 177] shows that graphs \( G \) in \( 2 \)-Nested\( \text{INT} \) have unbounded \( \lambda(G) \). Therefore, \( 2 \)-Nested\( \text{INT} \not\subset k \)-Length\( \text{INT} \) for each \( k \). Figure 2a depicts inclusions of considered classes.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(a) An interval representation with the nesting three. (b) The disjoint union of two components with the minimum nesting two requiring three different lengths of intervals. On the left, the shorter intervals are shorter than \( \frac{1}{4} \) of the longer ones. On the right, they are longer than \( \frac{1}{3} \).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{(a) The Hasse diagram of proper inclusions of the considered classes. (b) We can label each interval by the length of a maximal chain of nested intervals ending in it. We code the graph by the left-to-right sequence of left endpoints \( \ell \) and right endpoints \( r \) together with their labels.}
\end{figure}
Recognition For a subclass \( C \) of interval graphs, the following classical computational problem is studied:

| Problem: | Recognition – \text{RECOG}(C) |
|----------|--------------------------------|
| Input:   | A graph \( G \).              |
| Question:| Is there a \( C \)-interval representation of \( G \)? |

The problem \( \text{RECOG}(1\text{-NestedINT}) = \text{RECOG}(1\text{-LengthINT}) \) can be solved in linear time [7].

Partial Representation Extension These problems generalizing recognition were introduced by Klavík et al. [21]. A partial representation \( \mathcal{R}' \) of \( G \) is an interval representation \( \{\langle x \rangle' : x \in V(G')\} \) of an induced subgraph \( G' \) of \( G \). The vertices of \( G' \) and the intervals of \( \mathcal{R}' \) are called pre-drawn. A representation \( \mathcal{R} \) of \( G \) extends \( \mathcal{R}' \) if and only if it assigns the same intervals to the vertices of \( G' \): \( \langle x \rangle = \langle x \rangle' \) for every \( x \in V(G') \).

| Problem: | Partial Representation Extension – \text{REPExt}(C) |
|----------|-----------------------------------------------|
| Input:   | A graph \( G \) and a partial representation \( \mathcal{R}' \) of an induced subgraph \( G' \). |
| Question:| Is there a \( C \)-interval representation of \( G \) extending \( \mathcal{R}' \) ? |

An \( O(n^2) \)-time algorithm for \( \text{REPExt} \) was given in [21]. There are two different linear-time algorithms [1,20] for this problem. Minimal obstructions making partial representations non-extendible are described in [24]. A linear-time algorithm for proper interval graphs [18] and a quadratic-time algorithm for unit interval graphs [33,34] are known.

The partial representation extension problems were considered for other graph classes. Polynomial-time algorithms are known for circle graphs [6], and permutation and function graphs [17], and recently for trapezoid graphs [26]. The problems are NP-hard for chordal graphs [19], contact representations of planar graphs [5], and unit circular-arc graphs [35]. The complexity remains open for circular-arc graphs.

Previous Results and Motivation The classes \( k\text{-LengthINT} \) were introduced by Graham as a natural hierarchy between unit interval graphs and interval graphs; see Fig. 2a. Unfortunately, even after decades, the only results known are curiosities that illustrate the incredibly complex structure of \( k\text{-LengthINT} \), very different from the case of unit interval graphs. For instance, \( k\text{-LengthINT} \) is not closed under disjoint unions; see Fig. 1b. Timeline of results is depicted in Fig. 3.

Leibowitz et al. [27] show that the class \( 2\text{-LengthINT} \) contains caterpillars, threshold graphs, and unit interval graphs with one additional vertex. Further, interval graphs \( G \) with \( \lambda(G) > 2 \) such that \( \lambda(G \setminus x) \leq \lambda(G) - 2 \) for some \( x \in V(G) \) are constructed in [27]. Fishburn [9] shows that there are infinitely many forbidden induced subgraphs for \( 2\text{-LengthINT} \), while \( 1\text{-LengthINT} \) are interval graphs just without \( K_{1,3} \) [30]. It is also known [8] that there are interval graphs in \( 2\text{-LengthINT} \) such that, when the shorter length is fixed to 1, the longer one can be one of the real numbers belonging to arbitrary many distinct intervals of the real line, arbitrary far from each other.
Not much is known about the computational complexity of problems involving $k$-$\text{LengthINT}$, even recognition is open for $k = 2$. In [4], a polynomial-time algorithm is given for computing $\lambda(G)$ for interval graphs $G$ which are extended-bull-free or generalized threshold (which highly restricts them). Skrien [32] characterized 2-$\text{LengthINT}$ which can be realized by lengths zero (points) and one (unit intervals), leading to a linear-time recognition algorithm. As most of the efficient algorithms for intersection graph classes require representations, very little is known how to algorithmically use that a given interval graph can be represented by $k$ lengths. In this paper, we show that partial representation extension is $\text{NP}$-hard already for 2-$\text{LengthINT}$.

All these difficulties lead us to introduce another hierarchy of $k$-$\text{NestedINT}$ which generalizes proper interval graphs; see Fig. 2. We illustrate the nice structure of $k$-$\text{NestedINT}$ by describing a relatively simple linear-time recognition algorithm based on MPQ-trees.

Fishburn’s book [10] introduces the parameter $\nu(G)$ called depth and links it to $k$-$\text{LengthINT}$. In [4], nesting is studied in the language of interval orders. It is proved that extended-bull-free interval graphs $G$ satisfy $\lambda(G) = \nu(G)$. A polynomial-time algorithm computing $\nu(G)$ of an arbitrary interval graph $G$ is presented, running in time $O(n^2m^2)$ where $n$ is the number of vertices and $m$ is the number of edges. There are some different generalizations of proper interval graphs [29], which are less rich and not linked to $k$-$\text{LengthINT}$. The paper [2] studies a related general graph parameter proper thinness and prove that $k$-$\text{NestedINT}$ have proper thinness at most $k$ while interval graphs of proper thinness two might have arbitrary large nesting.

Since $k$-$\text{NestedINT}$ seem to share many properties with proper interval graphs, several future directions of research are immediately offered. Using our results, it is possible to describe minimal forbidden induced subgraphs [15]. For the computational problems which are tractable for proper interval graphs and hard for interval graphs, the complexity of the intermediate problems for $k$-$\text{NestedINT}$ can be studied. (One such problem is FO property checking, discussed below.) In Lemma 3.2, we show that $k$-$\text{NestedINT}$ can be efficiently encoded, similarly to proper interval graphs.

**Our Results** In [16], a polynomial-time algorithm is given for recognizing 2-$\text{LengthINT}$ when intervals are partitioned into two subsets $A$ and $B$, each of one length, and both $G[A]$ and $G[B]$ are connected. This approach might be generalized for partial representation extension, but we show that removing the connectedness condition makes it hard:
Theorem 1.1 The problem $\text{RepExt}(2\text{-LengthINT})$ is NP-hard when every pre-drawn interval is of one length $a$. It remains NP-hard even when (i) the input prescribes two lengths $a = 1$ and $b$, and (ii) for every interval, the input assigns one of the lengths $a$ or $b$. Also, it is $W[1]$-hard when parameterized by the number of pre-drawn intervals.

We describe a dynamic programming algorithm for recognizing $k\text{-NestedINT}$, based on a data structure called an MPQ-tree. We show that we can optimize nesting greedily from the bottom to the top. We compute a so-called minimal representation for each subtree and we show how to combine them.

Theorem 1.2 The minimum nesting number $\nu(G)$ can be computed in time $O(n + m)$ where $n$ is the number of vertices and $m$ is the number of edges. Therefore, the problem $\text{RECOG}(k\text{-NestedINT})$ can be solved in linear time.

Since that extended-bull-free interval graphs $G$ have $\nu(G) = \lambda(G)$ [4, Corollary 12], we get the following:

Corollary 1.3 The minimum number of interval lengths $\lambda(G)$ of an extended-bull-free interval graph $G$ can be computed in time $O(n + m)$ where $n$ is the number of vertices and $m$ is the number of edges. \hfill \square

Theorem 1.2 has the following application in the computational complexity of deciding logic formulas over graphs. Let $\varphi$ be the length of a first-order logic formula for graphs. By the locality, this formula can be decided in $G$ in time $n^{O(\varphi)}$. Since it is $W[2]$-hard to decide it for general graphs when parameterized by $\varphi$, it is natural to ask for which graph classes there exists an FPT algorithm running in time $O(n^c \cdot f(\varphi))$ for some computable function $f$.

In [13], it is shown that the problem above is $W[2]$-hard even for interval graphs. On the other hand, if an interval graph is given together with a $k$-length interval representation, [13] gives an FPT algorithm with respect to the parameters $\varphi$ and the particular lengths of the intervals. It was not clear whether such an algorithm is inherently geometrical. Recently, Gajarský et al. [12] give a different FPT algorithm for FO property testing for interval graphs parameterized by $\varphi$ and the nesting $k$, assuming that a $k$-nested interval representation is given by the input. By our result, this assumption can be removed since we can compute an interval representation of the optimal nesting in linear time.

The problem $\text{RepExt}(k\text{-NestedINT})$ is more involved since a straightforward greedy optimization from the bottom to the top does not work. The described recognition algorithm can be generalized to solve $\text{RepExt}(k\text{-NestedINT})$ in polynomial time [23]. It contrasts with Theorem 1.1. The partial representation extension problems for $k\text{-NestedINT}$ and $k\text{-LengthINT}$ are problems for which the geometrical version (at most $k$ lengths) is much harder than the corresponding topological problem (the left-to-right ordering of endpoints of intervals).

2 Extending Partial Representations with Two Lengths

The complexity of recognizing $k\text{-LengthINT}$ is a long-standing open problem, even for $k = 2$. In this section, we show that $\text{RepExt}(k\text{-LengthINT})$ is NP-hard even
when $k = 2$. We adapt the reduction from 3-\textsc{Partition} used in [18,19] which is the following computational problem:

| Problem:         | 3-\textsc{Partition} |
|------------------|-----------------------|
| Input:           | Integers $A_1, \ldots, A_3$, and $M$ such that $\frac{M}{2} < A_i < \frac{M}{4}$ and $\sum A_i = Ms$. |
| Question:        | Can $A_i$’s be split into $s$ triples, each summing to exactly $M$? |

This problem is strongly \textsc{NP}-complete [14], which means that it is \textsc{NP}-complete even when all integers are coded in unary, i.e., they are bounded by a polynomial in $s$.

**Proof (Theorem 1.1)** Assume (i) and (ii). For an instance of 3-\textsc{Partition}, the reduction constructs an interval graph $G$ and a partial representation $\mathcal{R}'$ as depicted in Fig. 4. We claim that $\mathcal{R}'$ can be extended using two lengths of intervals if and only if the instance of 3-\textsc{Partition} is solvable. We set $a = 1$ and $b = s \cdot (M + 2) - 1$. The partial representation $\mathcal{R}'$ consists of $s + 1$ disjoint pre-drawn intervals $\langle v_0 \rangle', \ldots, \langle v_s \rangle'$ of length $a$ such that $\langle v_i \rangle' = \left[ i \cdot (M + 2), i \cdot (M + 2) + 1 \right]$. So they split the real line into $s$ equal gaps of size $M + 1$ and two infinite regions.

Aside $v_0, \ldots, v_s$, the graph $G$ contains a vertex $w$ represented by an interval of length $b$, adjacent to every vertex in $G$. Further, for each $A_i$, the graph $G \setminus w$ contains $P_{2A_i}$ (a path with $2A_i$ vertices) as one component, with each vertex represented by an interval of length $a$. Notice that $G$ has polynomial size with respect to the size of the input.

The described reduction clearly runs in polynomial time. It remains to show that $\mathcal{R}'$ is extendible if and only if the instance of 3-\textsc{Partition} is solvable. The length of $b$ implies that every extending representation has $\langle w \rangle = [1, s \cdot (M + 2)]$ to intersect both $\langle v_0 \rangle'$ and $\langle v_s \rangle'$. Therefore, each of the paths $P_{2A_i}$ has to be placed in exactly one of the $s$ gaps. In every representation of $P_{2A_i}$, it requires the space at least $A_i + \varepsilon$ for some $\varepsilon > 0$. Three paths can be packed into the same gap if and only if their three integers sum to at most $M$. Therefore, an extending representation $\mathcal{R}'$ gives a solution to 3-\textsc{Partition}, and vice versa. A similar reduction from \textsc{BinPacking} implies \textsc{W}[1]-hardness when parameterized by the number of pre-drawn intervals; see [19] for details.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Suppose that we have the following input for 3-\textsc{Partition}: $s = 2$, $M = 7$, $A_1 = A_2 = A_3 = A_4 = 2$ and $A_5 = A_6 = 3$. The associated graph $G$ is depicted on top, and at the bottom we find one of its extending representations, giving the 3-partitioning \{A_1, A_3, A_6\} and \{A_2, A_4, A_5\}.}
\end{figure}
This reduction can be easily modified when (i) and (ii) are avoided. We add two extra vertices: $w_0$ adjacent to $v_0$ and $w_s$ adjacent to $v_s$, both non-adjacent to $w$. It forces the length of $w$ to be in $[s \cdot (M + 2) - 1, s \cdot (M + 2) + 1]$, so the length $b$ does not have to be prescribed. Also, this reduction works even when non-predrawn intervals do not have lengths assigned.

3 Preliminaries and Basic Properties of $k$-Nested Interval Graphs

In this section, we describe basic definitions and properties about nesting in interval representations and about $k$-NestedINT.

Definitions For an interval representation $R$, the nesting defines a partial ordering $\subseteq$ of intervals. Intervals $\langle u_1 \rangle, \ldots, \langle u_k \rangle$ form a chain of nested intervals of length $k$ if $\langle u_1 \rangle \subseteq \langle u_2 \rangle \subseteq \ldots \subseteq \langle u_k \rangle$. By $\nu(u)$, we denote the length of a longest chain of nested intervals ending with $\langle u \rangle$. We denote $\nu(R)$ the length of a longest chain of nested intervals in $R$, i.e.,

$$\nu(R) = \max_{u \in V(G)} \nu(u) \quad \text{and} \quad \nu(G) = \min_{R} \max_{R} \nu(u).$$

For $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of $G$ induced by $A$. For a representation $R$ of $G$, let $R[A]$ be the representation of $G[A]$ created by restricting $R$ to the intervals of $A$. And for an induced subgraph $H$ of $G$, let $R[H] = R[V(H)]$.

Pruning Twins Two vertices $x$ and $y$ are twins if and only if $N[x] = N[y]$. The standard observation is that twins can be ignored since they can be represented by identical intervals, and notice that this does not increase nesting and the number of lengths. We can locate all twins in time $O(n + m)$ [31] and we can prune the graph by keeping one vertex per equivalence class of twins. An interval graph belongs to $k$-NestedINT if and only if the pruned graph belongs to $k$-NestedINT.

Decomposition into Proper Interval Representations The following equivalent definition of $k$-NestedINT is used by Gajarský et al. [12]:

Lemma 3.1 An interval graph belongs to $k$-NestedINT if and only if it has an interval representation which can be partitioned into $k$ proper interval representations.

Proof Let $R$ be an interval representation partitioned into proper interval representations $R_1, \ldots, R_k$. No chain of nested intervals contains two intervals from some $R_i$, so the nesting is at most $k$. On the other hand, let $R$ be a $k$-nested interval representation. We label each interval $\langle u \rangle$ by $\nu(u)$; see Fig. 2b. Notice that the intervals of each label $l \in \{1, \ldots, k\}$ form a proper interval representation $R_l$. ⊓⊔

Efficient encoding An interval graph can be encoded by $2n \lceil \log n \rceil$ bits by labeling the vertices $1, \ldots, n$ and listing the left-to-right ordering of labels of the endpoints. Proper interval graphs can be encoded more efficiently using only $2n$ bits: the sequence of endpoints ($\ell$ for left one, $r$ for right one), as they appear from left to right. We generalize it for $k$-NestedINT.

Lemma 3.2 A graph in $k$-NestedINT can be encoded by $2n \lceil \log k + 1 \rceil$ bits where $n$ is the number of vertices.
Proof  See Fig. 2b for an example. Let $R_1, \ldots, R_k$ be the labeling from the proof of Lemma 3.1. From left to right, we output $\ell$ or $r$ for each endpoint together with its labels. This encoding takes $\lceil \log k + 1 \rceil$ bits per endpoint. \hfill $\Box$

Minimal Forbidden Induced Subgraphs Interval graphs and the subclasses $k$-NestedINT and $k$-LengthINT are closed under induced subgraphs, so they can be characterized by minimal forbidden induced subgraphs. Lekkerkerker and Boland [28] describe them for interval graphs, and Roberts [30] proved that 1-NestedINT $= 1$-LengthINT are claw-free interval graphs. On the other hand, 2-LengthINT have infinitely many minimal forbidden induced subgraphs [9] which are interval graphs. In [15], our results in Sect. 5 are used to describe minimal forbidden induced subgraphs for $k$-NestedINT.

4 Maximal Cliques and MPQ-Trees

In this section, we review well-known properties of interval graphs. First, we describe their characterization in terms of orderings of maximal cliques. Then we introduce a data structure called an MPQ-tree which stores all feasible orderings.

Consecutive Orderings Fulkerson and Gross [11] proved the following characterization of interval graphs; see Fig. 5:

**Lemma 4.1** (Fulkerson and Gross [11]) A graph is an interval graph if and only if there exists a linear ordering $<$ of its maximal cliques such that, for each vertex, the maximal cliques containing this vertex appear consecutively.

An ordering of the maximal cliques satisfying the statement of Lemma 4.1 is called a consecutive ordering. Each interval graph has $O(n)$ maximal cliques of total size $O(n + m)$ which can be found in linear time [31].

Cleaned Representations For a given consecutive ordering of maximal cliques, it is easy to construct a representation minimizing the number of all nestings called a cleaned representation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{An interval graph $G$ and two of its representations with different left-to-right orderings of the maximal cliques, with choices of clique-points}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig6.png}
\caption{Two equivalent MPQ-trees with denoted sections. In all figures, we denote P-nodes by circles and Q-nodes by rectangles}
\end{figure}
Lemma 4.2 For a given consecutive ordering $< \text{ of maximal cliques, there exists a cleaned representation such that if } \langle u \rangle \subseteq \langle v \rangle, \text{ then } \langle u \rangle \text{ is nested in } \langle v \rangle \text{ in every interval representation with this consecutive ordering } <. \text{ We can construct it in time } O(n)\).

Proof We place maximal cliques on the real line according to $<$. For each $v \in V(G)$, we place $\langle v \rangle$ on top of the maximal cliques containing $v$. Let $v^<$ be the left-most clique containing $v$ and $v^>$ be the right-most clique containing $v$. We place $\langle v \rangle$ on the left of $v^<$ and on the right of $v^>$. For a maximal clique $C$, let $u_1, \ldots, u_\ell$ be all vertices having $u_i^< = C$, i.e., all intervals $\langle u_i \rangle$ start at $C$. Since there are no twins, we have $u_i^< \neq u_j^<$ for all $i \neq j$. We order the left endpoints of $\langle u_1 \rangle, \ldots, \langle u_\ell \rangle$ exactly as the maximal cliques $u_1^<, \ldots, u_\ell^<$ are ordered in $<$. Similarly, let $v_1, \ldots, v_{\ell'}$ be all vertices having $v_i^> = C$. We order the right endpoints of $\langle v_1 \rangle, \ldots, \langle v_{\ell'} \rangle$ exactly as the maximal cliques $v_1^>, \ldots, v_{\ell'}^>$ are ordered in $<$. The constructed interval representation avoids all unnecessary nesting. We get that $\langle u \rangle \subseteq \langle v \rangle$ if and only if $v^< < u^< \leq u^> < v^>$ in which case the nesting is clearly forced by the consecutive ordering $<$. The construction clearly runs in time $O(n + m)$.

$PQ$-Trees A $PQ$-tree $T$ is a rooted tree, introduced by Booth and Lueker [3]. Its leaves are in one-to-one correspondence with the maximal cliques. Its inner nodes are of two types: $P$-nodes and $Q$-nodes. Each $P$-node has at least two children, each $Q$-node at least three. The orderings of the children of inner nodes are given. The $PQ$-tree $T$ represents one consecutive ordering $<_T$ called the frontier of $T$ which is the ordering of the leaves from left to right.

The $PQ$-tree $T$ represents all consecutive orderings of $G$ as frontiers of equivalent $PQ$-trees which can be constructed from $T$ by sequences of equivalent transformations of two types: (i) an arbitrary reordering of the children of a $P$-node, and (ii) a reversal of the order of the children of a $Q$-node; see Fig. 6.

A subtree $T'$ of the $PQ$-tree $T$ consists of a node and all its descendants. For a node $N$, we denote the subtree having $N$ as the root by $T[N]$ and its subtrees are the subtrees which have the children of $N$ as the roots.

$MPQ$-Trees An $MPQ$-tree [25] is an augmentation of a $PQ$-tree $T$ in which the nodes of $T$ have assigned subsets of $V(G)$ called sections. To a leaf representing a clique $C$, we assign one section $s(C)$. Similarly, to each $P$-node $P$, we assign one section $s(P)$. For a $Q$-node $Q$ with subtrees $T_1, \ldots, T_q$, we have $q$ sections $s_1(Q), \ldots, s_q(Q)$ ordered from left to right, each corresponding to one subtree, and let $s(Q) = s_1(Q) \cup \ldots \cup s_q(Q)$. Examples of sections are depicted in Fig. 6. The section $s(C)$ has all vertices contained in the maximal clique $C$ and no other maximal clique. The section $s(P)$ of a $P$-node $P$ has all vertices that are contained in all maximal cliques of the subtree rooted at $P$ and in no other maximal clique. Let $Q$ be a $Q$-node with subtrees $T_1, \ldots, T_q$. Let $x$ be a vertex contained only in maximal cliques of the subtree rooted at $Q$, contained in maximal cliques of at least two subtrees. Then $x$ is contained in every section $s_i(Q)$ such that some maximal clique of $T_i$ contains $x$.

Every vertex $x$ is in sections of exactly one node of $T$. In the case of a $Q$-node, it is placed in consecutive sections of this node. For a $Q$-node $Q$, if $x$ is placed in a section
s_i(Q), then x is contained in all cliques of T_i. Every section of a Q-node is non-empty, and two consecutive sections have a non-empty intersection. After pruning twins, no two vertices belong to exactly same sections of the MPQ-tree.

Let G[T] be the interval graph induced by the vertices of the sections of T. By G[N], we denote G[T][N]. For a representation R, we have R[T] = R[G[T]] and R[N] = R[T[N]].

**Forced Nestings** Let Q be a Q-node with sections s_1(Q),...,s_q(Q) and u be a vertex.

- If u does not belong to sections of T[Q], let s_u^−(Q) = s_1(Q) and s_u^+(Q) = s_q(Q).
- If u ∈ s(Q), let s_u^−(Q) be the leftmost section of Q containing u and s_u^+(Q) be the rightmost one.
- If u belongs to sections of a subtree T_i of Q, we put s_u^−(Q) = s_u^+(Q) = s_i(Q).

We study under which conditions is ⟨u⟩ forced to be nested in ⟨v⟩ in every interval representation, and we represent this by a partial ordering ⊑_F. We have u ⊑_F v if and only if there there exists a Q-node Q such that s_u^−(Q) is on the left of s_u^+(Q) and s_v^−(Q) is on the right of s_u^+(Q).

**Lemma 4.3** We have ⟨u⟩ ⊑ ⟨v⟩ for every interval representation if and only if u ⊑_F v.

**Proof** If u ⊑_F v, then every consecutive ordering contains at least one maximal clique containing v on the left of all maximal cliques containing u and at least one on the right, so necessarily ⟨u⟩ ⊑ ⟨v⟩.

Suppose that there exists a cleaned representation with ⟨u⟩ ⊑ ⟨v⟩. Therefore, every maximal clique containing u also contains v, so u and v appear in sections of a path from a leaf to the root of the MPQ-tree, and v appears at least as high as u. Suppose that u ⊑_F v. Both u and v do not belong to a same Q-node, otherwise they could not be nested in a cleaned representation. There is no Q-node on the path between u and v, above u; possibly u belongs to all sections of one Q-node. Therefore, we can reorder all these P-nodes to place the subtrees containing u on the side, and the obtained cleaned representation has ⟨u⟩ ⊑ ⟨v⟩. □

5 Recognizing k-Nested Interval Graphs

In this section, we describe a linear-time algorithm for computing minimal nesting of interval graphs. By Lemma 4.2, the problem reduces to finding a consecutive ordering of maximal cliques which minimizes the nesting of a cleaned representation. So we want to reorder the MPQ-tree to minimize the nesting, which is done by dynamic programming from the bottom to the top.

**Intuition** We process the MPQ-tree from the bottom to the top, and we optimize the nesting. Let N be a node of the MPQ-tree and let T_1,...,T_ℓ be its subtrees. Suppose that we know ν(G[T_1]),...,ν(G[T_ℓ]) from the dynamic programming. Is ν(G[N]) determined? The answer is that almost. Let R_1,...,R_ℓ be interval representations of G[T_1],...,G[T_ℓ] minimizing the nesting. We consider two model situations, depicted in Fig. 7:
(i) Suppose that \( N \) is a P-node with \( s(N) = \{w\} \). Then \( G[N] \) is the disjoint union of \( G[T_1], \ldots, G[T_\ell] \) together with the universal vertex \( w \). Every interval representation of \( G[N] \) looks as depicted in Fig. 7a. We have two representations \( R_s \) and \( R_t \) placed on the left and right sides of \( \langle w \rangle \), respectively, while the remaining \( R_i \), for \( i \neq s, t \), are placed inside \( \langle w \rangle \). Therefore, their nestings \( v(G[T_i]) \) are increased by one with \( \langle w \rangle \). On the other hand, some intervals of \( R_s \) and \( R_t \) may stretch out of \( \langle w \rangle \), so the nestings \( v(G[T_s]) \) and \( v(G[T_t]) \) is not necessarily increased by one. More precisely, the intervals of \( R_s \) contained in the left-most clique and the intervals of \( R_t \) contained in the right-most clique are not nested in \( \langle w \rangle \) in a cleaned representation.

(ii) Suppose that \( N \) is a Q-node and we consider the following simplified situation depicted in Fig. 7b. The graph \( G[N] \) consists of \( G[T_1] \) together with two universal vertices \( u \) and \( v \), each attached some other part of \( G[N] \) non-adjacent to all vertices of \( G[T_i] \). Then \( R_i \) is covered from, say, left by \( \langle u \rangle \) and from right by \( \langle v \rangle \). The nesting of \( v(G[T_i]) \) is not necessarily increased by one with \( \langle u \rangle \) or \( \langle v \rangle \). More precisely, in a cleaned representation, the intervals of \( R_i \) contained in the left-most clique are not nested in \( \langle v \rangle \) and those contained in the right-most clique are not nested in \( \langle u \rangle \). It is possible that both sides cannot be optimized simultaneously.

Therefore, the dynamic programming computes three values for each subtree \( T \), denoted as a triple \( (\alpha, \beta, \gamma) \), which we define formally in the next subsection. We have \( \alpha = v(G[T]) \). The value \( \beta \) is the increase in the nesting when \( T \) is placed on the side, as in (i); so either \( \beta = \alpha \), or \( \beta = \alpha - 1 \). The value \( \gamma \) is the increase in the nesting of one side, subject to the other side being optimized according to \( \beta \), as in (ii). So always \( \beta \leq \gamma \) and either \( \gamma = \alpha \) or \( \gamma = \alpha - 1 \).

5.1 Triples \( (\alpha, \beta, \gamma) \)

For an interval graph \( G \), we define the triple \( (\alpha, \beta, \gamma) \) as follows. Let \( G_\alpha, G_\beta \) and \( G_\gamma \) be the graphs constructed from \( G \) as in Fig. 8. Let

\[
\alpha = v(G_\alpha) - 1, \quad \beta = v(G_\beta) - 1, \quad \text{and} \quad \gamma = v(G_\gamma) - 1.
\]

---

**Fig. 7** a The nesting \( v(G[T_i]) \), for \( i \neq s, t \), is always increased by one with \( \langle w \rangle \), but the nestings \( v(G[T_s]) \) and \( v(G[T_t]) \) may or may not be increased by one. b The nesting \( v(G[T_i]) \) may be increased by one with \( \langle u \rangle \) or \( \langle v \rangle \). It might not be possible to preserve nesting on both sides, for instance when \( G[T_i] \) is the disjoint union of \( K_{1,3} \) and \( K_1 \).
Fig. 8 The graphs $G_\alpha$, $G_\beta$ and $G_\gamma$ with representations, defining the triple $(\alpha, \beta, \gamma)$ of $T$. The vertices of $G$ are adjacent to the added vertices $u_\alpha$, $u_\beta$, $u_\gamma$, and $v_\gamma$, and not to the others. In bottom, we depict the structure of their representations with $\mathcal{R}$ being a representation of $G$.

Similarly, for a subtree $T$ of the MPQ-tree, we define its triple as the triple of $G[T]$. The dynamic algorithm computes triples of all subtrees from the leaves to the root, and outputs $a$ of the root as $v(G)$.

**Lemma 5.1** For every interval graph $G$, its triple $(\alpha, \beta, \gamma)$ satisfies $\alpha - 1 \leq \beta \leq \gamma \leq \alpha$.

**Proof** We prove equivalently that $v(G_\alpha) - 1 \leq v(G_\beta) \leq v(G_\gamma) \leq v(G_\alpha)$. We trivially know that $v(G_\beta) \leq v(G_\gamma)$ since $G_\beta$ is an induced subgraph of $G_\gamma$.

The definition of $G_\alpha$ implies that $v(G_\alpha) = v(G) + 1$, since in every interval representation of $G_\alpha$, both endpoints of $\langle u_\alpha \rangle$ are covered by attached paths, and therefore a representation $\mathcal{R}$ of $G$ is nested in $\langle u_\alpha \rangle$. Since $G$ is an induced subgraph of $G_\beta$, we have $v(G) \leq v(G_\beta)$, so the inequality $v(G_\alpha) - 1 \leq v(G_\beta)$ follows. For an alternative proof, consider a representation of $G_\beta$ minimizing nesting. We modify it to a representation of $G_\alpha$ by stretching $\langle u_\beta \rangle$ into $\langle u_\alpha \rangle$, which increases nesting by at most one, and by adding the second path attached to $\langle u_\alpha \rangle$. So $v(G_\alpha) \leq v(G_\beta) + 1$.

It remains to show the last inequality that $v(G_\gamma) \leq v(G_\alpha)$. Consider a representation of $G_\alpha$ with minimal nesting, we have $G$ strictly contained inside $\langle u_\alpha \rangle$. By shifting $r(u_\alpha)$ to the left, we get $\langle u_\gamma \rangle$. By adding $\langle v_\gamma \rangle$, we do not increase the nesting and we get a representation of $G_\gamma$. So $v(G_\gamma) \leq v(G_\alpha)$. $\square$

Therefore, the triples classify interval graphs into three types; see Fig. 9 for examples.

**Corollary 5.2** Interval graphs $G$ with $v(G) = k$ have triples of three types: $(k, k - 1, k - 1)$, $(k, k - 1, k)$ and $(k, k, k)$.

**Interpreting Triples** Let $(\alpha, \beta, \gamma)$ be the triple for $G$. We want to argue how the formal definition relates to the description in the last paragraph of Intuition. We can interpret the triple of $G$ as increase in the nesting, depending how $G$ is represented with respect to the rest of the graph. Since $\alpha = v(G)$, it is just the nesting of $G$. Next, we describe an interpretation for the value $\beta$. 
Lemma 5.3 For every representation of $G_\beta$, we have $v(u_\beta) \geq \beta + 1$.

Proof Without loss of generality, we assume that a representation $R_\beta$ of $G_\beta$ is cleaned. By the definition of $\beta$, there exists a maximal chain of nested intervals of length at least $\beta + 1$. Suppose that its length is at least two. Let $R = R_\beta[G]$, and we assume that $\langle u_\beta \rangle$ covers $R$ from the left. If this chain does not end with $\langle u_\beta \rangle$, it ends with an interval of $R$ placed in the right-most maximal clique. Since every other interval of the chain is nested in $\langle u_\beta \rangle$, we replace this end with $\langle u_\beta \rangle$, and obtain a chain of nested intervals of length at least $\beta + 1$ ending with $\langle u_\beta \rangle$. 

In other words, in every representation of $G$, there exists a chain of length at least $\beta$ which is nested in any interval in the rest of the graph which plays the role of $\langle u_\beta \rangle$. In Lemma 5.5, we show that there exists a representation for which the lengths of longest such chains are exactly $\beta$ and $\gamma$, respectively. This links the value $\beta$ to Fig. 7a.

Last, we describe an interpretation for the value $\gamma$.

Lemma 5.4 For every representation of $G_\gamma$, we have

$$\min\{v(u_\gamma), v(v_\gamma)\} \geq \beta + 1 \quad \text{and} \quad \max\{v(u_\gamma), v(v_\gamma)\} \geq \gamma + 1.$$  

Proof We prove this similarly as in Lemma 5.3. Consider a cleaned representation $R_\gamma$ of $G_\gamma$. It contains a maximal chain of length at least $\gamma + 1$ ending with $\langle x \rangle$. If $x \neq u_\gamma$ and $x \neq v_\gamma$, we can replace $\langle x \rangle$ with both $\langle u_\gamma \rangle$ and $\langle v_\gamma \rangle$, so both $v(u_\gamma) \geq \gamma + 1$ and $v(v_\gamma) \geq \gamma + 1$. Otherwise, suppose that, say, $x = v_\gamma$. Then $v(v_\gamma) \geq \gamma + 1$ and by removing $\langle v_\gamma \rangle$ and the added intervals, we obtain a representation of $R_\beta$ with $u_\beta = u_\gamma$. By Lemma 5.3, $v(u_\beta) \geq \beta + 1$. Therefore, in every representation of $G$, there exists a chain of length at least $\gamma$ which is nested in any interval in the rest of the graph which plays the role of either $\langle u_\gamma \rangle$ or $\langle v_\gamma \rangle$, while there is a chain of length at least $\beta$ which in nested in any interval playing the role of the other one. In Lemma 5.5, we show that there exists a representation for which the lengths of longest such chains are exactly $\beta$ and $\gamma$, respectively. This links the value $\gamma$ to Fig. 7b.

Minimal Representations Let $(\alpha, \beta, \gamma)$ be a triple of an interval graph $G$ and let $R$ be a cleaned representation of $G$ with $C^{\leftarrow}$ and $C^{\rightarrow}$ being the leftmost and the rightmost maximal cliques in its consecutive ordering of maximal cliques. We define:

$$v^{\rightarrow}(R) = \max_{x \in V(G) \setminus C^{\leftarrow}} v(x), \quad \text{and} \quad v^{\leftarrow}(R) = \max_{x \in V(G) \setminus C^{\rightarrow}} v(x).$$
The representation $\mathcal{R}$ of $G$ is minimal if $v(\mathcal{R}) = \alpha$, $v^\rightarrow(\mathcal{R}) = \beta$ and $v^\leftarrow(\mathcal{R}) = \gamma$. So a minimal representation $\mathcal{R}$ can be used simultaneously in representations of $G_\alpha$, $G_\beta$ and $G_\gamma$ to get nestings $\alpha + 1$, $\beta + 1$ and $\gamma + 1$, respectively. For instance, all representations in Fig. 9 are minimal.

**Lemma 5.5** For every interval graph $G$, there exists a minimal representation $\mathcal{R}$.

**Proof** We argue according to the type of the triple of $G$. The triple $(k, k - 1, k - 1)$. Let $\mathcal{R}_\gamma$ be a cleaned representation of $G_\gamma$ minimizing the nesting, we have $v(\mathcal{R}_\gamma) = k$. Since $v(G) = k$, we have $v(\mathcal{R}_\gamma[G]) = k$ as well. By Lemma 5.4, $v(u_\gamma) \geq k$ and $v(v_\gamma) \geq k$, so we get equalities. The representation $\mathcal{R} = \mathcal{R}_\gamma[G]$ has $v^\leftarrow(\mathcal{R}) = v^\rightarrow(\mathcal{R}) = k - 1$ and $\mathcal{R}$ is minimal.

The triple $(k, k - 1, k)$. Let $\mathcal{R}_\beta$ be a cleaned representation of $G_\beta$ minimizing the nesting such that $\{u_\beta\}$ intersects $\mathcal{R}_\beta[G]$ from left, we have $v(\mathcal{R}_\beta) = k$. Let $\mathcal{R} = \mathcal{R}_\beta[G]$. Since $v(G) = k$, we have $v(\mathcal{R}) = k$ as well. Similarly as in the proof of Lemma 5.3, we get $v^\leftarrow(\mathcal{R}) = k - 1$. If $u_\beta = u_\beta$ and we add $\{v_\beta\}$ with the attached path, we obtain a representation of $G_\gamma$ with nesting at least $k + 1$. Therefore, $v^\leftarrow(\mathcal{R}) = k$ and $\mathcal{R}$ is minimal.

The triple $(k, k, k)$. Let $\mathcal{R}$ be a cleaned representation of $G$ minimizing the nesting, so $v(\mathcal{R}) = k$. If $v^\rightarrow(\mathcal{R}) < k$ or $v^\leftarrow(\mathcal{R}) < k$, we can add $\{u_\beta\}$ and the attached intervals to obtain a representation of $G_\beta$ of nesting $k$, so $\beta = k - 1$; a contradiction. So $v^\rightarrow(\mathcal{R}) = v^\leftarrow(\mathcal{R}) = k$ and $\mathcal{R}$ is minimal. \hfill $\Box$

For a representation $\mathcal{R}$, the flipped representation $\mathcal{R}^{\leftarrow}$ is created by reversing the left-to-right order of endpoints of intervals. Notice that $v(\mathcal{R}) = v(\mathcal{R}^{\leftarrow})$, $v^\rightarrow(\mathcal{R}) = v^\leftarrow(\mathcal{R}^{\leftarrow})$ and $v^\leftarrow(\mathcal{R}) = v^\rightarrow(\mathcal{R}^{\leftarrow})$.

**Lemma 5.6** For every interval graph $G$, there exists a cleaned representation $\mathcal{R}$ of $G$ minimizing the nesting such that for every subtree $T$ of its MPQ-tree, $\mathcal{R}[T]$ is minimal or $\mathcal{R}^{\leftarrow}[T]$ is minimal.

**Proof** Let $\mathcal{R}$ be a cleaned representation of $G$ minimizing the nesting, and consider a maximal subtree $T$ for which $\mathcal{R}[T]$ and $\mathcal{R}^{\leftarrow}[T]$ are not minimal. By Lemma 5.5, there exists a minimal representation $\mathcal{R}_T$ of $G[T]$. If $v^\rightarrow(\mathcal{R}[T]) \leq v^\leftarrow(\mathcal{R}[T])$, let $\mathcal{R}_T^{\leftarrow} = \mathcal{R}_T$, otherwise let $\mathcal{R}_T^\rightarrow = \mathcal{R}_T^{\leftarrow}$. Since $\mathcal{R}[T]$ appears consecutively in $\mathcal{R}$, we replace it by $\mathcal{R}_T$, and construct a modified cleaned representation $\hat{\mathcal{R}}$ of $G$. It remains to argue that for every subtree $T'$ containing $T$, the representation $\hat{\mathcal{R}}[T']$ remains minimal; the lemmas then follows by induction.

We know that $\mathcal{R}[T']$ is minimal. The modification only changed chains which start in $\mathcal{R}_T^\leftarrow$. By Lemmas 5.4, 5.5, we get that $v(\mathcal{R}_T^\leftarrow) \leq v(\mathcal{R}[T])$, $v^\rightarrow(\mathcal{R}_T^\leftarrow) \leq v^\rightarrow(\mathcal{R}[T])$ and $v^\leftarrow(\mathcal{R}_T^\leftarrow) \leq v^\leftarrow(\mathcal{R}[T])$. Therefore, every chain above $\mathcal{R}[T]$ extends only chains with lengths equal or shorter, so $\hat{\mathcal{R}}[T']$ remains minimal. \hfill $\Box$

**Triples for Leaves** Recall that we have no twins. For a leaf $L$ of the MPQ-tree, we have either $G[L]$ having no vertices (when $s(L) = \emptyset$), or $G[L] \cong K_1$ (when $s(L) = \{v\}$). In the former case, the triple of $L$ is equal $(0, 0, 0)$. In the latter case, it is equal $(1, 0, 0).$
5.2 Triples for P-Nodes

Let $T_1, \ldots, T_p$ be the children of a P-node $P$, with $p \geq 2$, with the computed triple $(\alpha_i, \beta_i, \gamma_i)$ for each subtree $T_i$. We compute the triple $(\alpha, \beta, \gamma)$ of the subtree $T = T[P]$ using the following formulas; see Fig. 10 for an example:

$$\alpha = \begin{cases} \max\{\alpha_1, \ldots, \alpha_p\}, & \text{if } s(P) = \emptyset, \\ \min_{s \neq t} \max\{\beta_s + 1, \beta_t + 1, \alpha_i + 1 : i \neq s, t\}, & \text{if } s(P) = \{w\}. \end{cases}$$

$$\beta = \min_s \max \{\beta_s, \alpha_i : i \neq s\}.$$

$$\gamma = \max\{\alpha_1, \ldots, \alpha_p\}.$$

Lemma 5.7 The formulas compute the triple $(\alpha, \beta, \gamma)$ of $T[P]$ correctly.

Proof This proof also explains how these formulas are formed.

The value $\alpha$ is computed correctly. We know that $\alpha = \nu(G[T])$. If $s(P) = \emptyset$, then $G[T]$ is the disjoint union of $G[T_1], \ldots, G[T_p]$ with $\alpha_i = \nu(G[T_i])$, so $\alpha = \max\{\alpha_1, \ldots, \alpha_p\}$. Otherwise, we get the situation from Fig. 7a.

First, we argue that $G[T]$ has a representation $R$ of nesting $\alpha$ from the formula. Intervals of all subtrees except for the leftmost subtree $T_s$ and the rightmost one $T_t$ are completely nested inside $\langle w \rangle$; and we minimize over all possible choices of $s \neq t$. Let $R_i$ be a minimal representation for $G[T_i]$ from Lemma 5.5, and we use $R_s^\leftrightarrow$ for $G[T_s]$. For every $i \neq s, t$, we get that $\nu(R_i) = \alpha_i$ is increased by one with $\langle w \rangle$. For $R_s$ and $R_t$, only $\nu^\leftarrow(R_s^\leftrightarrow) = \beta_s$ and $\nu^\rightarrow(R_t) = \beta_t$ are increased by one with $\langle w \rangle$. We get

$$\nu(R) = \nu(w) = \min_{s \neq t} \max\{\beta_s + 1, \beta_t + 1, \alpha_i + 1 : i \neq s, t\} = \alpha.$$

On the other hand, consider a representation $R$ of $G[T]$. There is no chain of nested intervals containing intervals from two different subtrees $T_i$ and $T_j$. Let $R_i = R[T_i]$ and let $R_s$ and $R_t$ be the leftmost and the rightmost of these representations, respectively. For every $i \neq s, t$, the representation $R_i$ has the nesting at least $\alpha_i$, so $\nu(R) \geq \alpha_i + 1$. By Lemma 5.4, we know that $\nu^\leftarrow(R_s) \geq \beta_s$ and $\nu^\rightarrow(R_t) \geq \beta_t$ and these chains are nested in $\langle w \rangle$, so $\nu(R) \geq \max\{\beta_s + 1, \beta_t + 1\}$. So $\nu(R) \geq \alpha$ from the formula.

![Fig. 10](image)

Fig. 10 An MPQ-tree representing $G$ with the computed triples $(\alpha, \beta, \gamma)$ (equal on each level) and a cleaned representation minimizing nesting. We have that $\nu(G) = 3$
The value $\beta$ is computed correctly. First, we construct a representation $R_\beta$ of $G_\beta$ with nesting $\beta + 1$. If $s(P) = \{w\}$, in every cleaned representation, $\langle w \rangle \not\subseteq \langle u_\beta \rangle$, so that every other interval is either nested in both, or in neither. So we can assume that $s(P) = \emptyset$.

When the added intervals are placed on the right of $\langle u_\beta \rangle$, intervals of all subtrees except for a left-most one $T_s$ are completely nested inside $\langle u_\beta \rangle$; and we again minimize over all possible choices of $s$. Let $R_i$ be a minimal representation of $G[T_i]$, we use $R_{s}^{\leftrightarrow}$ for $G[T_s]$. For every $i \neq s$, we get that the nesting $v(R_i) = \alpha_i$ is increased by one by $\langle u_\beta \rangle$. For $T_s$, the nesting $v(\langle u_\beta \rangle) = \beta_s$ is increased by one with $\langle u_\beta \rangle$. We get

$$v(R_\beta) = v(\langle u_\beta \rangle) = \min_{s} \max \{\beta_s + 1, \alpha_i + 1 : i \neq s\} = \beta + 1.$$ 

For the other implication, consider a cleaned representation of $G_\beta$. Similarly, as above, we get that the nesting is at least $\beta + 1$.

The value $\gamma$ is computed correctly. We just sketch the argument, it is similar as above. Let $R_i$ be a minimal representation of $G[T_i]$. We may choose $T_s$ and $T_i$, and use $R_{s}^{\leftrightarrow}$ for $T_s$. Then only the nesting $\gamma = v(R_{s}^{\leftrightarrow}) = \beta_s$ is increased by one with $\langle u_\gamma \rangle$ and only the nesting $v(\langle u_\gamma \rangle) = \beta_i$ is increased by one with $\langle u_\gamma \rangle$. But since $R_{s}^{\leftrightarrow}$ is nested inside $\langle u_\gamma \rangle$ and $R_i$ is nested inside $\langle u_\gamma \rangle$, it does not matter and the nestings $v(R_{s}^{\leftrightarrow})$ and $v(\langle u_\gamma \rangle)$ are both increased by one anyway. Therefore, this choice of $T_s$ and $T_i$ is useless and a constructed representation of $G_\gamma$ has the nesting $\max \{\alpha_1, \ldots, \alpha_p\} + 1$. The other implication is proved similarly as before.

**Lemma 5.8** For a P-node with $p$ children, the triple $(\alpha, \beta, \gamma)$ can be computed in $O(p)$.

**Proof** By Lemma 5.1, we always have either $\beta_i = \alpha_i - 1$, or $\beta_i = \alpha_i$. Only in the former case, we may improve the nesting by choosing $s = i$ or $t = i$. We call subtrees $T_i$ with $\beta_i = \alpha_i - 1$ as savable.

For $\gamma$ and for $\alpha$ with $s(P) = \emptyset$, we just find the maximum $\alpha_i$ which can be done in time $O(p)$. For $\alpha$ with $s(P) \neq \emptyset$ and $\beta$, we first locate all $T_j$ which maximize $\alpha_i$. If at least one of them is not savable, say $T_j$, then $\alpha = \alpha_j + 1$ and $\beta = \alpha_j$. Otherwise if all are savable, then the values $\alpha$ and $\beta$ depend on the number of these subtrees. If there are at most two, then $\alpha = \alpha_i$, otherwise $\alpha = \alpha_i + 1$. If there is exactly one, then $\beta = \beta_i$, otherwise $\beta = \beta_i + 1$.

### 5.3 Triples for Q-Nodes

The situation is more complex and the values $\gamma$ are also required. Let $Q$ be a Q-node with subtrees $T_1, \ldots, T_q$, where $q \geq 3$, each with a triple $(\alpha_i, \beta_i, \gamma_i)$. We want to compute the triple $(\alpha, \beta, \gamma)$ of the subtree $T = T[Q]$. Since lengths of chains are not changed by flipping $Q$, we can fix the left-to-right order of its subtrees as $T_1, \ldots, T_q$. See Fig. 11 for an example.

**Structure of Chains** Suppose that we choose some cleaned representations $R_1, \ldots, R_q$ of $G[T_1], \ldots, G[T_q]$. Then the corresponding cleaned representation of $G[T]$ is uniquely determined. What is the structure of chains of nested intervals? Each
chain starts in some subtree $T_i$ and then continues with intervals in $s(Q)$ as follows. If it contains $(x) \subseteq (y)$ for $x, y \in s(Q)$, then $x \subseteq_F y$, so $y$ starts more to the left and ends more to the right than $x$ in sections of $Q$. We represent the relation $\subseteq_F$ on $s(Q)$ by a DAG $D$, having an edge $(x, y)$ if and only if $x \subseteq_F y$; see Fig. 11 on the right.

Suppose that a chain of nested intervals of $s(Q)$ of length $\ell$ starts with $(x)$. Let $s_x^L(Q) = s_x(Q)$ and $s_x^R(Q) = s_t(Q)$, for some $s < t$. Then every chain of every $R_i$ such that $s < i < t$ is nested in $(x)$, so for each $R_i$, there exists a chain of nested intervals of length $\nu(R_i) + \ell$. But there might not be chains of lengths $\nu(R_s) + \ell$ and $\nu(R_t) + \ell$. The reason is that only chains in $R_s$ not ending with an interval contained in the leftmost maximal clique of $R_x$ are nested in $(x)$, and only those of $R_t$ avoiding the rightmost maximal clique of $R_y$. So there only exist chains of lengths $\nu^+(R_s) + \ell$ and $\nu^+(R_t) + \ell$.

By Lemma 5.5, there exists a minimal representation $R_i$ of $G[T_i]$ with $\nu(R_i) = \alpha_i$, $\nu^+(R_i) = \beta_i$ and $\nu^+(R_i) = \gamma_i$; and we can swap the last two nestings with $R_i^*$. Let $R_i^* \in \{R_i, R_i^+\}$. We denote $\bigcirc_i^- = \nu^+(R_i^*)$ and $\bigcirc_i^+ = \nu^-(R_i^*)$.

For each $T_i$, we choose either $R_i^* = R_i$: $\bigcirc_i^- = \beta_i$, $\bigcirc_i^+ = \gamma_i$, or $R_i^* = R_i^+$: $\bigcirc_i^- = \gamma_i$, $\bigcirc_i^+ = \beta_i$. (1)

By combining these chosen representations $R_i^*$ for all subtree $T_i$, we get one of $2^q$ possible representations $R[T]$. For each of them, we compute $\nu(x)$ for all $x \in s(Q)$ using the following formulas:

$$
\nu(x) = \max \{ \bigcirc_s^- + 1, \bigcirc_i^+ + 1, \alpha_i + 1, \nu(y) + 1 : s_x^L(Q) = s_s(Q), s_x^R(Q) = s_t(Q), s < i < t \text{ and } y \in \text{Pred}_D(x) \},
$$

where $\text{Pred}_D(x)$ denotes the set of all direct predecessors of $x$ in $D$. These values can be computed according to a topological sort of $D$.

![Diagram](image-url) Fig. 11 On the left, a Q-node $Q$ with eight subtrees. On the right, the DAG $D$ of forced nestings in $s(Q)$. For $R_5^* = R_5$ (depicted), we get $\nu(x_3) = 2, \nu(x_4) = 3, \nu(x_5) = 4$, and $\nu(x_2) = 5$. For $R_5^* = R_5^+$ (by flipping $T_5$), we get $\nu(x_3) = 3, \nu(x_5) = 2, \nu(x_4) = 3$, and $\nu(x_2) = 4$. The second option minimizes the nesting and it gives the triple $(4, 3, 4)$ for $T[Q]$.

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Formulas for Triples The triple \((\alpha, \beta, \gamma)\) is determined by minimal nestings of \(G_\alpha\), \(G_\beta\), and \(G_\gamma\). We study how chains in \(s(Q)\) are extended by the added intervals \(\langle u_\alpha \rangle\), \(\langle u_\beta \rangle\), \(\langle u_\gamma \rangle\) and \(\langle v_\gamma \rangle\). Further, we consider two copies \(\langle u_\beta \rangle\) and \(\langle u_\gamma \rangle\) of \(\langle u_\beta \rangle\). Recall that the left-to-right ordering of the subtrees of \(Q\) is fixed. Therefore, \(\langle u_\beta \rangle\) can intersect \(\mathcal{R}\) either from left (represented by \(\langle u_\beta \rangle\)), or from right (represented by \(\langle u_\beta \rangle\)). Similarly, we assume that \(\langle u_\gamma \rangle\) intersects \(\mathcal{R}\) from left while \(\langle v_\gamma \rangle\) from right.

We add auxiliary vertices \(u_\alpha\), \(u_\beta\), \(u_\gamma\) and \(v_\gamma\) into \(D\) and get the following extended DAG \(D'\):

\[
V(D') = V(D) \cup \{u_\alpha, u_\beta, u_\beta^-, u_\gamma, v_\gamma\}
\]

\[
E(D') = E(D) \cup \{(x, u_\alpha), (y, u_\beta^-), (y, v_\gamma), (z, u_\beta^-), (z, u_\gamma) : x \in s(Q), y \in s(Q) \setminus s_1(Q), z \in s(Q) \setminus s_q(Q)\}.
\]

In other words, \(u_\alpha\) extends every chain in \(s(Q)\), but \(u_\beta\) and \(v_\gamma\) extend only those not ending with an interval in \(s_1(Q)\), and \(u_\beta\) and \(u_\gamma\) only those not ending with an interval in \(s_q(Q)\).

We compute \((\alpha, \beta, \gamma)\) of \(T[Q]\) using the following formulas:

\[
\alpha = \min_{\forall R_i^\beta} \max \{\alpha_1, \ldots, \alpha_q, \nu(y) : y \in \text{Pred}_{D'}(u_\alpha)\},
\]

\[
\beta^- = \min_{\forall R_i^\beta} \max \{\beta_1, \alpha_2, \ldots, \alpha_q, \nu(y) : y \in \text{Pred}_{D'}(u_\beta^-)\},
\]

\[
\beta^- = \min_{\forall R_i^\beta} \max \{\alpha_1, \ldots, \alpha_{q-1}, \beta_q, \nu(y) : y \in \text{Pred}_{D'}(u_\beta^-)\},
\]

\[
\beta = \min_{\forall R_i^\beta} \{\beta^-, \beta^-\},
\]

\[
\gamma = \min_{\forall R_i^\beta} \max \{\alpha_1, \ldots, \alpha_q, \nu(y) : y \in \text{Pred}_{D'}(u_\gamma) \cup \text{Pred}_{D'}(v_\gamma)\}.
\]

Lemma 5.9 The formulas compute the triple \((\alpha, \beta, \gamma)\) of \(T[Q]\) correctly.

Proof We assume that the left-to-right order of subtrees of \(Q\) is fixed, it does not change nesting. Recall that in a cleaned representation \(\mathcal{R}\) of \(G[T]\), the nesting \(\nu(\mathcal{R})\) is determined by representations \(\mathcal{R}[T_1], \ldots, \mathcal{R}[T_q]\).

The value \(\alpha\) is computed correctly. First, we construct a representation \(\mathcal{R}_\alpha\) of \(G_\alpha\) with \(\nu(\mathcal{R}_\alpha) = \nu(u_\alpha) = \alpha + 1\) for \(\alpha\) given by the above formula. We construct \(2^q\) representations for all choices of \(R_i^\beta\) using (1), and we use a representation minimizing the nesting, corresponding to the minimum \(\min_{\forall R_i^\beta} \mathcal{R}_\beta\) in the formula. The choices \(R_i^\beta\) determine a cleaned representation \(\mathcal{R}_\alpha\) of \(G_\alpha\). Nesting of the intervals of \(s(Q)\) is computed using (2) and \(\nu(u_\alpha)\) is equal the length of the longest chain in \(\mathcal{R}_\alpha[G[Q]]\) increased by one. The formula for \(\alpha\) maximizes over lengths of all chains in \(\mathcal{R}_\alpha[G[Q]]\).

On the other hand, consider a cleaned representation \(\mathcal{R}\) of \(G[T]\). We argue that \(\nu(\mathcal{R}) \geq \alpha\) for \(\alpha\) given by the above formula. By Lemma 5.4, the representation \(\mathcal{R}[T_i]\) has \(\nu(\mathcal{R}[T_i]) \geq \alpha_i\) and either \(\nu^-(\mathcal{R}[T_i]) \geq \beta_i\) and \(\nu^-(\mathcal{R}[T_i]) \geq \gamma_i\), or \(\nu^-(\mathcal{R}[T_i]) \geq \gamma_i\) and \(\nu^-(\mathcal{R}[T_i]) \geq \beta_i\). As in the proof of Lemma 5.6, by replacing \(\mathcal{R}[T_i]\) with \(R_i\) in the former case and with \(R_i^\alpha\) in the latter case, we do not increase the nesting. We obtain a representation of \(G[T]\) of nesting \(\alpha\), so \(\nu(\mathcal{R}) \geq \alpha\).
The value $\beta$ is computed correctly. Concerning $\beta$, in every cleaned representation $R_\beta$ of $G_\beta$, either intervals of $s_\beta(Q)$ are not nested in $\langle u_\beta \rangle$ (represented by $\langle u_\beta \rangle$), or intervals of $s_1(Q)$ are not nested in $\langle u_\beta \rangle$ (represented by $\langle u_\beta \rangle$). We compute both possibilities in $\beta^\leftarrow$ and $\beta^\rightarrow$, and use the minimum. The rest of the arguments is similar as above.

The value $\gamma$ is computed correctly. Again, the arguments are similar as for $\alpha$ above, the only difference is that $\langle u_\alpha \rangle$ is replaced by both $\langle u_\gamma \rangle$ and $\langle v_\gamma \rangle$.

Unfortunately, formulas do not directly lead to a polynomial-time algorithm since they minimize over $2^q$ possible choices of $R_i^*$. Next, we prove that these choices can be done greedily.

**Lemma 5.10** For each of $\alpha$, $\beta^\leftarrow$, $\beta^\rightarrow$ and $\gamma$, we can locally choose $R_i^*$ minimizing the value.  

**Proof** Notice that the choices of $R_i^*$ are independent of each other since each $R_i^*$ influences only leghts of chains starting in $R[T_i]$. We give a description for $\alpha$, and it works similarly for the others.

For each $x \in s(Q)$, we compute the length $\ell(x)$ of a longest chain in $s(Q) \cup \{u_\alpha\}$ starting with $\langle x \rangle$. Let

$$\ell_i^\leftarrow = \max_{x \in s(Q)} \ell(x), \text{ and } \ell_i^\rightarrow = \max_{x \in s(Q)} \ell(x),$$

and let $\ell_i^* = 0$ if no such $x \in s(Q)$ exists. We choose $R_i^* = R_i$ if and only if $\ell_i^\leftarrow \geq \ell_i^\rightarrow$. These choices minimize lengths of all chains in a representation of $G[T]$. For instance, in Fig. 11, we get $\ell_5^\leftarrow = 3$ and $\ell_5^\rightarrow = 2$, so we choose $R_5^* = R_5^\leftarrow$.

**Lemma 5.11** For a $Q$-node $Q$ with $q$ children, the triple $(\alpha, \beta, \gamma)$ of $T[s(Q)]$ can be computed in time $O(q + m_Q)$, where $m_Q$ is the number of edges of $G[s(Q)]$.

**Proof** Since $G[s(Q)]$ is connected, it contains at most $m_Q$ vertices. For every $x \in s(Q)$, we know $s_x^\leftarrow(Q)$ and $s_x^\rightarrow(Q)$ which we use to compute the DAG $D$. This can be done by considering all $m_Q$ edges, and testing for each whether the pair is nested. Then, we construct the extended DAG $D'$. For each $x \in V(D')$ and each $y \in \{u_\alpha, u_\beta^\leftarrow, u_\beta^\rightarrow, u_\gamma, v_\gamma\}$, we compute the length of a longest path from $x$ to $y$. This can be done in linear time for all vertices $x$ by processing $D'$ from the top to the bottom. For each $T_i$, we choose greedily $R_i^*$ as described in the proof of Lemma 5.10. We compute the triple $(\alpha, \beta, \gamma)$ using the above formulas. The total running time is $O(q + m_Q)$.

**5.4 Construction of Linear-Time Algorithm**

We use the above results to prove that $\nu(G)$ can be computed in linear time:

**Proof** (Theorem 1.2) For an interval graph $G$, we compute its MPQ-tree in time $O(n + m)$ [25]. Then we process the tree from the leaves to the root and compute $\nu(G)$. 

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triplies \((\alpha, \beta, \gamma)\) for every node, as described above. We output \(\alpha\) of the root which is the minimal nesting number \(\nu(G)\). By Lemmas 5.7 and 5.9, this value is computed correctly. By Lemmas 5.8 and 5.11, the running time of the algorithm is \(O(n + m)\). \(\Box\)

6 Conclusions

In this paper, we have introduced \(k\)-nested interval graphs which is a new hierarchy of graph classes between proper interval graphs and interval graphs. The presented understanding is already much greater than understanding of \(k\)-length interval graphs reached after more than 35 years of their research. We have presented a relatively simple recognition algorithm based on dynamic programming and minimal representations. Other research directions immediately open. In [15], our results are used to derive minimal forbidden induced subgraphs of \(k\)-NestedINT.

Problem 6.1 Which structural properties and characterizations of proper interval graphs generalize to \(k\)-nested interval graphs?

Problem 6.2 Which computational problems solvable efficiently for proper interval graphs can be solved efficiently for \(k\)-nested interval graphs as well?

The second problem is interesting for computational problems which are harder for general interval graphs. One example is deciding first-order logic properties which is \(W[2]\)-hard for interval graphs [13], but can be solved in \(FPT\) for \(k\)-nested interval graphs [12].

Inspired by [4], there is the following natural question:

Question 6.3 Which interval graphs \(G\) satisfy \(\nu(G) = \lambda(G)\)?

From [4,27,30], we know that these interval graphs include proper/unit interval graphs, graphs created from these by adding one vertex, generalized threshold graphs and extended-bull-free interval graphs; see [4, p. 535].

In [23], our results are used to attack the problem \(\text{REPExt}(k\text{-NestedINT})\). A polynomial-time algorithm for finding an extending interval representation of minimal nesting in derived, by a much more involved dynamic programming than in the recognition algorithm of Sect. 5. A partial representation \(\mathcal{R}'\) poses three restrictions:

(i) Some pre-drawn intervals can be nested in each other which increases the nesting.
(ii) The consecutive ordering has to extend \(<\) which restricts the possible shuffling of subtrees.
Some subtrees can be optimized differently depending on the side they are attached.

The problem is difficult since we have to deal with them simultaneously. For an example, see Fig. 12.

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