On certain submodules of Weyl modules for $\text{SO}(2n + 1, \mathbb{F})$ with $\text{char}(\mathbb{F}) = 2$

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Abstract

For $k = 1, 2, \ldots, n-1$ let $V_k = V(\lambda_k)$ be the Weyl module for the special orthogonal group $G = \text{SO}(2n + 1, \mathbb{F})$ with respect to the $k$-th fundamental dominant weight $\lambda_k$ of the root system of type $B_n$ and put $V_n = V(2\lambda_n)$. It is well known that all of these modules are irreducible when $\text{char}(\mathbb{F}) \neq 2$ while when $\text{char}(\mathbb{F}) = 2$ they admit many proper submodules. In this paper, assuming that $\text{char}(\mathbb{F}) = 2$, we prove that $V_k$ admits a chain of submodules $V_k = M_k \supset M_{k-1} \supset \ldots \supset M_1 \supset M_0 = 0$ where $M_i \cong V_i$ for $1, \ldots, k-1$ and $M_0$ is the trivial 1-dimensional module. We also show that for $i = 1, 2, \ldots, k$ the quotient $M_i/M_{i-2}$ is isomorphic to the so called $i$-th Grassmann module for $G$. Resting on this fact we can give a geometric description of $M_{i-1}/M_{i-2}$ as a submodule of the $i$-th Grassmann module. When $\mathbb{F}$ is perfect $G \cong \text{Sp}(2n, \mathbb{F})$ and $M_i/M_{i-1}$ is isomorphic to the Weyl module for $\text{Sp}(2n, \mathbb{F})$ relative to the $i$-th fundamental dominant weight of the root system of type $C_n$. All irreducible sections of the latter modules are known. Thus, when $\mathbb{F}$ is perfect, all irreducible sections of $V_k$ are known as well.

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1 Introduction

Let $V := V(2n + 1, \mathbb{F})$ be a $(2n + 1)$-dimensional vector space over a field $\mathbb{F}$ and let $\eta$ be a non singular quadratic form of $V$ of Witt index $n$. Let $\Delta$ be the building of type $B_n$ where the elements of type $k = 1, 2, \ldots, n$ are the $k$-dimensional subspaces of $V$ totally singular for $\eta$.

For $1 \leq k \leq n$, let $G_k$ be the $k$-grassmannian of $\text{PG}(V)$, that is $G_k$ is the point-line geometry where the points are the $k$-dimensional subspaces of $V$ and the lines are the sets $l_{X,Y} = \{Z \mid X \subset Z \subset Y, \ \dim(Z) = k\}$ for a given pair $(X,Y)$ of subspaces of $V$ where $\dim(X) = k-1$ and $\dim(Y) = k+1$. (Note that $X = 0$ when $k = 1$.)
Let $\Delta_k$ be the $k$-grassmannian of $\Delta$. The point-line geometry $\Delta_k$ is the proper subgeometry of $G_k$ defined as follows. The points of $\Delta_k$ are the $k$-elements of $\Delta$. When $1 \leq k < n$ the lines of $\Delta_k$ are the lines $l_{X,Y}$ of $G_k$ with $Y$ totally singular. Note that $\Delta_1$ is the polar space associated to $\eta$.

Let $k = n$. Then the lines of $\Delta_n$ are the sets
\[ l_X = \{ Z \mid X \subset Z \subset X^\perp, \dim(Z) = n, Z \text{ totally singular} \} \tag{1} \]
where $X$ is a given $(n - 1)$-dimensional totally singular subspace of $V$ and $X^\perp$ is the orthogonal of $X$ with respect to (the linearization of) $\eta$. The vector space $X^\perp/X$ is 3-dimensional and $l_X$ is a non-singular conic in the projective plane $\mathrm{PG}(X^\perp/X)$. The geometry $\Delta_n$ is called the dual polar space of type $B_n$.

Let $1 \leq k \leq n$ and $W_k := \bigwedge^k V$. Then $\dim(W_k) = \binom{2n+1}{k}$. Let $\iota_k : G_k \to W_k$ be the embedding of $G_k$ in $\mathrm{PG}(W_k)$ mapping every point of $G_k$, i.e. every $k$-dimensional subspace $\langle v_1, v_2, \ldots, v_k \rangle$ of $V$, onto the point $\langle v_1 \wedge v_2 \wedge \ldots \wedge v_k \rangle$ of $\mathrm{PG}(W_k)$. Let $\varepsilon_k := \iota_k|\Delta_k$ be the restriction of $\iota_k$ to $\Delta_k$. The map $\varepsilon_k$ is called the grassmann embedding of $\Delta_k$. If $k < n$ then $\varepsilon_k$ maps lines of $\Delta_k$ onto projective lines. If $k = n$ then $\varepsilon_k$ maps lines of $\Delta_k$ onto non-singular conics of $W_k$.

Put $W_k^o := \langle \varepsilon_k(\Delta_k) \rangle$. As proved in Cardinali and Pasini \cite{[1]},
\[ \dim(W_k^o) = \begin{cases} \binom{2n+1}{k} & \text{if } \operatorname{char}(\mathbb{F}) \neq 2, \\ \binom{2n+1}{k} - \binom{2n+1}{k-2} & \text{if } \operatorname{char}(\mathbb{F}) = 2. \end{cases} \tag{2} \]
So, if either $\operatorname{char}(\mathbb{F}) \neq 2$ or $k = 1$ then $W_k^o = W_k$ (recall that $\binom{2n+1}{-1} := 0$) while if $\operatorname{char}(\mathbb{F}) = 2$ and $k > 1$ then $W_k^o$ is a proper subspace of $W_k$ of codimension $\binom{2n+1}{k-2}$.

Let $G := \mathrm{SO}(2n+1, \mathbb{F})$ be the stabilizer of the form $\eta$ in the special linear group $\mathrm{SL}(V) = \mathrm{SL}(2n+1, \mathbb{F})$. The group $G$ also acts on $W_k$, according to the following rule:
\[ g(v_1 \wedge \ldots \wedge v_k) = g(v_1) \wedge \ldots \wedge g(v_k) \text{ for } g \in G \text{ and } v_1, \ldots, v_k \in V. \]

Note that $\mathrm{SO}(2n+1, \mathbb{F}) = \mathrm{PSO}(2n+1, \mathbb{F})$, namely $G$ is the adjoint Chevalley group of type $B_n$ defined over $\mathbb{F}$. The universal Chevalley group of type $B_n$ is the spin group $\tilde{G} = \mathrm{Spin}(2n+1, \mathbb{F})$. If $\operatorname{char}(\mathbb{F}) = 2$ then $\tilde{G} = G$. If $\operatorname{char}(\mathbb{F}) \neq 2$ then $\tilde{G} = 2G$, a non-split central extension of $G$ by a group of order two.

Clearly, $W_k^o$ is a $G$-module. It is called the $k$-grassmann orthogonal module for $G$ (also grassmann orthogonal module if $k$ is clear from the context).

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the fundamental dominant weights for the root system of type $B_n$, the nodes of the $B_n$-diagram being numbered in the usual way (see the picture at the beginning of this introduction). For a positive integral combination $\lambda$ of $\lambda_1, \lambda_2, \ldots, \lambda_n$ we denote by $V(\lambda)$ the Weyl module for $\tilde{G}$ with $\lambda$ as the highest weight. If $\tilde{G}$ acts unfaithfully on $V(\lambda)$ then it induces $G$ on $V(\lambda)$, namely $V(\lambda)$ is also a $G$-module. For instance, $V(\lambda_k)$ is a $G$-module for every $k < n$ as well as $V(2\lambda_n)$ while if $\operatorname{char}(\mathbb{F}) \neq 2$ then $V(\lambda_n)$ is a $G$-module but not a $G$-module.
We introduce the following notation not to be forced to split our statements in two distinct cases, namely $k < n$ and $k = n$. We put $\lambda_k^0 = \lambda_k$ for $k < n$ and $\lambda_n^0 = 2\lambda_n$. We also put $\lambda_0^0 = 0$ (the null weight). Thus $V(\lambda_0^0)$ is the trivial (1-dimensional) $G$-module. Note that
\[
\dim(V(\lambda_k^0)) = \binom{2n + 1}{k}
\]
for every $k = 0, 1, \ldots, n$, as one can see by applying the Weyl dimension formula (see [6] 24.3, for instance). The following is well known (see e.g. Blok [2]; see also Theorem 2.1 of Section 2.1 of this paper).

**Theorem 1.1** For $k = 1, 2, \ldots, n$ the grassmann orthognal module $W_k^\circ$ is a homomorphic image of the Weyl module $V(\lambda_k^0)$.

Therefore, in view of (2) and (3),

**Corollary 1.2** If either $\text{char}(\mathbb{F}) \neq 2$ or $k = 1$ then $W_k^\circ \cong V(\lambda_k^0)$. If $\text{char}(\mathbb{F}) = 2$ and $k > 1$ then $W_k^\circ$ is a proper homomorphic image of $V(\lambda_k^0)$.

It is worth spending a few more words on the statement of Theorem 1.1. We recall that $V(\lambda_k^0)$ is primarily defined as a module for the enveloping algebra of the Lie algebra of $G$. A choice of a $BN$-pair of $G$, equivalently an apartment $\Sigma$ of $\Delta$ and a chamber $C$ of $\Sigma$, is implicit in this way of defining $V(\lambda_k^0)$ in such a way that if $V^+$ is the (1-dimensional) weight space of $V(\lambda_k^0)$ of weight $\lambda_k^0$ and $X_k := \langle e_1, \ldots, e_k \rangle$ is the $k$-element of $C$ then the parabolic subgroup of $G$ stabilizing $X_k$ is the stabilizer of $V^+$ in $G$. The homomorphism $\varphi_k$ from $V(\lambda_k^0)$ to $W_k^\circ$ maps $V^+$ onto $\langle e_1 \wedge e_2 \wedge \ldots \wedge e_k \rangle$. This condition uniquely determines $\varphi_k$ modulo scalars. We denote the kernel of $\varphi_k$ by $K_k$. Thus, $W_k^\circ \cong V(\lambda_k^0)/K_k$.

In view of Corollary 1.2 we have $K_k = 0$ when either $\text{char}(\mathbb{F}) \neq 2$ or $k = 1$.

Let $\text{char}(\mathbb{F}) = 2$. Then the bilinear form associated to $\eta$ is degenerate with 1-dimensional radical $N_0$. The subspace $N_0$ is called the *nucleus* of the quadratic form $\eta$. It is the unique 1-dimensional subspace of $V$ stabilized by $G$ and it is non-singular.

Suppose firstly that $k > 1$. Given an element $X$ of $\Delta$ of type $k - 1$, let $\text{St}(X)$ be its upper residue, formed by the elements of $\Delta$ of type $k, k + 1, \ldots, n$ that contain $X$. We call $\text{St}(X)$ the *star* of $X$. Clearly, $\text{St}(X)$ is (the building of) an orthogonal polar space of rank $n - k + 1$ defined in $X^\perp/X$. Let $n_X$ be the nucleus of a quadratic form of $X^\perp/X$ associated to the polar space $\text{St}(X)$. Then $n_X = N_{X}/X$ where $N_X = \langle X, N_0 \rangle$. Hence $N_X$ is a point of $G_k$ and, since $n_X$ belongs to $X^\perp/X$, which is spanned by the 1-dimensional subspaces $Y/X$ for $Y$ ranging in the set of points of $\text{St}(X)$, the point $\nu_k(n_X)$ of $\text{PG}(W_k)$ belongs to $\text{PG}(W_k^\circ)$. We define the *$k$-nucleus subspace* $N_k$ of $W_k^\circ$ as the subspace of $W_k^\circ$ spanned by the 1-dimensional subspaces $\nu_k(n_X)$ for $X \in \Delta_{k-1}$ and we denote its preimage in $V(\lambda_k^0)$ by the symbol $N_k$. Thus, $N_k/K_k \cong N_k$. Note that, since the homomorphism $\varphi_k : V(\lambda_k^0) \rightarrow W_k^\circ$ is uniquely determined modulo scalars and $N_k = \varphi_k^{-1}(N_k)$, the submodule $N_k$ of $V(\lambda_k^0)$ is uniquely determined as well.
When \( k = 1 \) we put \( N_1 = N_0 \), regarded as a subspace of \( V(\lambda_1^2) \) (as we may, since \( K_1 = 0 \)). Clearly, \( N_k \) is stabilized by \( G \), whence it is a submodule of \( V(\lambda_k^2) \). The following is proved in Cardinali and Pasini [4].

**Theorem 1.3** Let \( \operatorname{char}(F) = 2 \). Then \( \dim(N_k) = \dim(V(\lambda_k^2)) = \binom{2n+1}{k-1} \) for every \( k = 1, 2, \ldots, n \), namely \( N_k \) has codimension \( \frac{\binom{2n+1}{k-1} - \binom{2n+1}{k-2}}{\binom{2n}{k-1}} = \frac{\binom{2n}{k}}{\binom{2n}{k-2}} \) in \( V(\lambda_k^2) \).

The following theorem, to be proved in Section 3, is the main result of this paper.

**Theorem 1.4** Let \( \operatorname{char}(F) = 2 \). Then \( N_k \cong V(\lambda_{k-1}^2) \) (isomorphism of \( G \)-modules) for every \( k = 1, 2, \ldots, n \). Moreover, if \( k > 1 \) then the isomorphism from \( N_k \) to \( V(\lambda_{k-1}^2) \) can be chosen in such a way that it maps \( K_k \) onto the submodule \( N_{k-1} \) of \( V(\lambda_{k-1}^2) \).

The next corollaries immediately follow from the previous theorem.

**Corollary 1.5** Let \( \operatorname{char}(F) = 2 \). Then \( K_k \cong V(\lambda_{k-2}^2) \) for every \( k = 2, 3, \ldots, n \).

**Corollary 1.6** Let \( \operatorname{char}(F) = 2 \) and \( 1 \leq k \leq n \). Then \( V(\lambda_k^2) \) admits a chain of submodules \( 0 \subset M_0 \subset M_1 \subset \cdots \subset M_{k-1} \subset M_k = V(\lambda_k^2) \) with \( M_i \cong V(\lambda_i^2) \) for \( i = 0, 1, \ldots, k-1 \). Moreover, \( M_i/M_{i-2} \cong \mathcal{W}_i^k \) for every \( i = 2, 3, \ldots, k \).

The statements of Theorem 1.3 and Corollary 1.6 can be made sharper when \( F \) is a perfect field of characteristic 2. We shall discuss this case in a few lines, but firstly we recall a few well known facts regarding symplectic polar grassmannians and their natural embeddings.

Put \( \mathfrak{V} := V(2n, F) \). Let \( \Delta \) be the building of type \( C_n \) associated with the symplectic group \( \mathfrak{G} := \operatorname{Sp}(2n, F) \) in its natural action on \( \mathfrak{V} \) and, for \( k = 1, 2, \ldots, n \), let \( \Delta_k \) be the \( k \)-grassmannian of \( \Delta \). Then \( \Delta_k \) is a subgeometry of the \( k \)-grassmannian \( \mathfrak{G}_k \) of \( \operatorname{PG}(\mathfrak{V}) \). Put \( \mathfrak{W}_k := \bigwedge^k \mathfrak{V} \) and let \( \iota_k : \mathfrak{G}_k \to \operatorname{PG}(\mathfrak{W}_k) \) be the embedding of \( \mathfrak{G}_k \) mapping every \( k \)-dimensional subspace \( \langle v_1, \ldots, v_k \rangle \) of \( \mathfrak{V} \) onto the point \( \langle v_1 \wedge \cdots \wedge v_k \rangle \) of \( \operatorname{PG}(\mathfrak{W}_k) \). Let \( \varepsilon_k \) be the restriction of \( \iota_k \) to \( \Delta_k \) and let \( \mathfrak{W}_k^{\mathfrak{G}} \) be the subspace of \( \mathfrak{W}_k \) spanned by \( \varepsilon_k(\Delta_k) \). Then \( \varepsilon_k \) is a projective embedding of \( \Delta_k \) in \( \operatorname{PG}(\mathfrak{W}_k^{\mathfrak{G}}) \), called the natural or Grassmann embedding of \( \Delta_k \). The subspace \( \mathfrak{W}_k^{\mathfrak{G}} \) of \( \mathfrak{W}_k \) is stabilized by \( \mathfrak{G} \) in its natural action on \( \mathfrak{W}_k \). Thus, \( \mathfrak{W}_k^{\mathfrak{G}} \) is a \( \mathfrak{G} \)-module. It is well known that \( \mathfrak{W}_k^{\mathfrak{G}} \) has codimension \( \binom{2n}{k-2} \) in \( \mathfrak{W}_k \) (see e.g. [3]). Moreover, \( \mathfrak{W}_k^{\mathfrak{G}} \) is isomorphic to the Weyl module for \( \mathfrak{G} \) associated to the \( k \)-th fundamental dominant weight of the root system of type \( C_n \). For the sake of completeness we denote by \( \mathfrak{W}_0 \) the trivial (1-dimensional) \( \mathfrak{G} \)-module.

Suppose now that \( F \) is a perfect field of characteristic 2. Then \( \mathfrak{G} \cong \mathfrak{G} \) and \( \Delta \cong \Delta \). Accordingly, \( \mathfrak{W}_k^{\mathfrak{G}} \) can be regarded as a \( \mathfrak{G} \)-module. The following is proved in Cardinali and Pasini [4]:
Theorem 1.7 Let $F$ be a perfect field of characteristic 2. Then $V(\lambda_k^p) / N_k \cong W_k^p$ and $N_k / K_k \cong W_k^{p-1}$, for every $k = 1, 2, ..., n$.

Consequently,

Corollary 1.8 Let $F$ be a perfect field of characteristic 2 and $1 \leq k \leq n$. Then $V(\lambda_k^p)$ admits a chain of submodules $0 \subset M_0 \subset M_1 \subset ... \subset M_{k-1} \subset M_k = V(\lambda_k^p)$ with $M_i \cong V(\lambda_i^p)$ and $M_i / M_{i-1} \cong W_i^p$ for $i = 1, 2, ..., k$. In other words, when $F$ is perfect the irreducible sections of $V(\lambda_k^p)$ are the irreducible sections of $W_i^p$ for $i = 1, 2, ..., k$. The latter are known (Baranov and Suprunenko [1]; also Premet and Suprunenko [8]). Therefore:

Corollary 1.9 Let $F$ be a perfect field of characteristic 2 and $1 \leq k \leq n$. Then all irreducible sections of $V(\lambda_k^p)$ are known.

It is likely that $V(\lambda_k^p)$ admits a unique series of submodules $M_0, M_1, ..., M_k$ satisfying the properties of Corollary 1.8. In Section 4 assuming that $F$ is perfect, we shall prove that this is indeed the case for $k \leq 4$.

2 Preliminaries

In this section we recall a number of known facts on Lie algebras, Chevalley groups and their Weyl modules, either assumed as well known in Section 1 or to be exploited in the proof of Theorem 1.4. We firstly recapitulate the construction of Chevalley groups from simple Lie algebras (Subsection 2.1). We rely on Humphreys [6] and Steinberg [9] for our exposition. We shall give some details, for the commodity of the reader, but not so many. The reader is referred to [6] and [9] for all we shall miss. In the second part of this section (Subsections 2.2 and 2.3) we shall give more detailed informations on orthogonal and symplectic Lie algebras.

2.1 General setting

Let $L = H \oplus (\oplus_{\alpha \in \Phi} X_\alpha)$ be a simple Lie algebra of rank $n$ over the complex field $\mathbb{C}$, where $H$ is the Cartan subalgebra, $\Phi$ is the set of all roots and $X_\alpha$ is the 1-dimensional subalgebra of $L$ corresponding to the root $\alpha$. Let $\Pi = \{\alpha_1, ..., \alpha_n\}$ be a set of simple roots of $L$ and $\Phi^+$ and $\Phi^-$ the set of roots that are positive or respectively negative with respect to $\Pi$. Let $\{H_1, ..., H_n\} \cup \{X_\alpha, Y_\alpha\}_{\alpha \in \Phi^+}$ be a Chevalley basis of $L$, where $H_i \in H$, $X_\alpha \in X_\alpha$ and $Y_\alpha \in X_{-\alpha}$. Denote by $\mathfrak{A}_C$
the universal enveloping (associative) algebra of \( \mathfrak{g}_\mathbb{C} \). Chosen an ordering on the set \( \Phi^+ \), put

\[
A^- := \{ A^-_t \}_{t \in T}, \quad A^-_t := \prod_{\alpha \in \Phi^+} \frac{Y_{\alpha}^t}{c_\alpha^t},
\]

\[
A^0 := \{ A^0_s \}_{s \in S}, \quad A^0_s := \prod_{i=1}^{n} \left( \frac{H_i}{s_i} \right),
\]

\[
A^+ := \{ A^+_t \}_{t \in T}, \quad A^+_t := \prod_{\alpha \in \Phi^+} \frac{X_{\alpha}^t}{r_\alpha^t}
\]

where

\[
\left( \frac{H_i}{s_i} \right) := \frac{H_i (H_i - I) ... (H_i - (s_i - 1) I)}{s_i!}
\]

(\( I \) being the identity element of \( \mathfrak{g}_\mathbb{C} \)), the factors \( Y_{\alpha}^t / c_\alpha^t \) and \( X_{\alpha}^t / r_\alpha^t \) in \( A^-_t \) and \( A^+_t \) are ordered according to the ordering previously chosen on \( \Phi^+ \), the letters \( T \) and \( S \) denote the set of mappings from \( \Phi^+ \) and \( \{ 1, 2, ..., n \} \) respectively to \( \mathbb{N} \) and, given \( t, r \in T \) and \( s \in S \), we write \( t_\alpha, r_\alpha \) and \( s_i \) for \( t(\alpha), r(\alpha) \) and \( s(i) \).

The set \( \mathcal{A} := A^- \cdot A^0 \cdot A^+ \cdot \{ A^-_t \}_{t \in T} \cdot \{ A^+_t \}_{t \in T} \) is a linear basis of the algebra \( \mathfrak{g}_\mathbb{C} \) (Humphreys [6] Chapter 26)). We call the elements of \( \mathcal{A}^-, \mathcal{A}^0 \) and \( \mathcal{A}^+ \) monomials of (-)-, (0)- and (+)-type respectively and the elements of \( \mathcal{A} \) just monomials. The identity element \( I \) of \( \mathfrak{g}_\mathbb{C} \) is regarded as a monomial of (-)-type as well as (0)- and (+)-type.

Every positive root \( \alpha \in \Phi^+ \) is a sum \( \alpha = \sum_{i=1}^{n} k_{\alpha,i} \alpha_i \) where the coefficients \( k_{\alpha,i} \) are nonnegative integers. For \( A^+_t \in \mathcal{A}^+ \) and \( i = 1, 2, ..., n \) we set \( d_i(A^+_t) = \sum_{\alpha \in \Phi^+} t_\alpha k_{\alpha,i} \) and we call \( d_i(A^+_t) \) the \( i \)-degree of \( A^+_t \). The nonnegative integer \( d(A^+_t) := \sum_{i=1}^{n} d_i(A^+_t) \) is the total degree of \( A^+_t \). Similarly, we put \( d_i(A^-_t) = -\sum_{\alpha \in \Phi^+} t_\alpha k_{\alpha,i} \) (i-degree of \( A^-_t \)) and \( d(A^-_t) := \sum_{i=1}^{n} d_i(A^-_t) \) (total degree of \( A^-_t \)). The i-degree (the total degree) of a monomial \( A^- A^0 A^+ \) is \( d(A^-_t A^0 A^+_s) := d_i(A^-_t) + d_i(A^+_s) \) (respectively \( d(A^-_t A^0 A^+_s) := d(A^-_t) + d(A^+_s) \)).

Regarding \( \mathfrak{g}_\mathbb{C} \) as a ring, we denote by \( \mathfrak{g}_\mathbb{Z} \) and \( \mathfrak{g}_Z \) the subrings of \( \mathfrak{g}_\mathbb{C} \) generated by the monomials of type (-) and all monomials respectively. The ring \( \mathfrak{g}_\mathbb{Z} \), regarded as a \( \mathbb{Z} \)-module, is free over the basis \( \mathcal{A}^- \) while \( \mathfrak{g}_Z \) is free over \( \mathcal{A} \).

We also put \( \mathfrak{g}_\mathbb{Z} := \mathfrak{g}_\mathbb{C} \cap \mathfrak{g}_\mathbb{Z} = \{ \{ H_i \}_{i=1}^{n} \cup \{ X_\alpha \}_{\alpha \in \Phi} \} \mathbb{Z} \).

Let \( \Lambda \) be the set of weights of the root system \( \Phi \) and \( \Lambda^+ \) the set of dominant weights (relative to the choice of \( \Pi = \{ \alpha_1, ..., \alpha_n \} \) as the set of simple roots). As in Humphreys [6], for two weights \( \lambda, \mu \in \Lambda \) we write \( \mu \prec \lambda \) if \( \lambda - \mu \) is a sum of positive roots.

Given a dominant weight \( \lambda \in \Lambda^+ \) let \( \mathbf{v}^+ \) be a vector of highest weight \( \lambda \) and \( Z_\mathbb{C}(\lambda) := \mathfrak{g}_\mathbb{C} \mathbf{v}^+ \) the cyclic \( \mathfrak{g}_\mathbb{C} \)-module associated to \( \lambda \) (called Verma module in Humphreys [6]). We recall that \( \mathbf{v}^+ \) is uniquely determined modulo scalars by the following conditions: \( H \mathbf{v}^+ = \lambda(H) \mathbf{v}^+ \) for every \( H \in \mathfrak{h} \) and \( X_\alpha \mathbf{v}^+ = 0 \) for every \( \alpha \in \Phi^+ \).

We have \( Z_\mathbb{C}(\lambda) = \bigoplus_{\mu \leq \lambda} \mathbf{V}_\mu \) where \( \mathbf{V}_\mu \) is the weight space with weight \( \mu \) (Humphreys [6] Chapter 20]). More explicitly, \( \mu = \lambda - \sum_{i=1}^{n} m_i \alpha_i \) for nonnegative integers \( m_1, ..., m_n \) and \( \mathbf{V}_\mu \) is the direct sum of the 1-dimensional subspaces
\[ \langle A_i^+ v^+ \rangle \] for \( A_i^- \) such that \( d_i(A_i^-) = -m_i \) for every \( i = 1, 2, \ldots, n \). Moreover, \( V_\mu \) is the \( \mu \)-eigenspace of \( \delta_\mu \), namely its vectors are the vectors \( x \in ZC(\lambda) \langle \) such that \( H(x) = \mu(H)x \) for every \( H \in \mathfrak{h} \). In particular, \( V_\lambda = \langle v^+ \rangle \). Note also that, for every weight \( \mu \leq \lambda \) and every element \( X \in A \) we have \( XV_\mu \subseteq \lambda \mu + \sum_{i=1}^n d_i(X)\alpha_i \), with the convention that \( V_{\mu + \sum_{i=1}^n d_i(X)\alpha_i} = 0 \) if \( \lambda < \mu + \sum_{i=1}^n d_i(X)\alpha_i \).

The module \( ZC(\lambda) \) admits a unique maximal proper submodule \( JC(\lambda) \). The \textit{Weyl module} relative to \( \lambda \) is defined as the quotient \( V_\lambda = ZC(\lambda)/JC(\lambda) \). The Weyl module \( V_\lambda \) is finite dimensional. In particular, \( JC(\lambda) \) contains all weight spaces \( V_\mu \) with \( \mu < 0 \). It follows that for every \( \alpha \in \Phi \) both elements \( X_\alpha \) and \( Y_\alpha \) act on \( V_\lambda \) (by left multiplication) as nilpotent mappings. Thus, the exponential map \( e^{X_\alpha t} := \sum_{i=0}^{\infty} \frac{X_\alpha^i t^i}{i!} \) can be defined on \( V_\lambda \) for every \( \alpha \in \Phi \) and every \( t \in \mathbb{C} \). This map is (linear and) invertible. If \( X \) denotes the Dynkin type of the root system \( \Phi \), the group \( G_C(\lambda) = \langle e^{X_\alpha t} \rangle_{\alpha \in \Phi, t \in \mathbb{C} \text{ is the complex Chevalley group of type } X} \) associated to the weight \( \lambda \).

Before to go on we fix a notation which allows us to immediately recognize if an element is regarded as an element of \( V_\lambda \) or \( ZC(\lambda) \). We denote vectors and subspaces of \( ZC(\lambda) \) by bold characters, as we have done so far, keeping normal font letters for the corresponding vectors and subspaces of \( V_\lambda \). Thus, a letter as \( v \) stands for a vector of \( ZC(\lambda) \) while \( \nu \) denotes the vector \( \pi_C(v) \), where \( \pi_C \) is the canonical projection of \( ZC(\lambda) \) onto \( V_\lambda \). Similarly, if \( V_\mu \) is a weight space of \( ZC(\lambda) \) then \( V_\mu = \pi_C(V_\mu) \).

We shall now define the analogues \( G_F(\lambda) \) and \( V_F(\lambda) \) of \( G_C(\lambda) \) and \( V_\lambda \) over an arbitrary field \( F \). To that goal we must preliminary define suitable \( \mathbb{Z} \)-analouges \( Z_\lambda \), \( J_\lambda \) and \( V_\lambda \) of \( ZC(\lambda) \), \( J_C(\lambda) \) and \( V_\lambda \) respectively.

The definition of \( J_\lambda \) is obvious: we put \( Z_\lambda := \mathfrak{g}_F v^+ = \mathfrak{g}_F v^+ \) in view of the previous description of \( ZC(\lambda) \) and its weight spaces). In order to define \( J_\lambda \) we must firstly define \( V_\lambda \).

According to Steinberg [9 page 17], there exists a basis \( B \) of \( V_\lambda \) such that \( \langle B \rangle_\lambda = \mathfrak{g}_F v^+ \). The equality \( \langle B \rangle_\lambda = \mathfrak{g}_F v^+ \) implies that we can choose the basis \( \lambda \) of \( \mathfrak{g}_F v^+ \) in such a way that its elements are monomial vectors of the form \( A_i^+ v^+ \), namely \( B \subseteq A^+ v^+ \). So, we assume that \( B \) has been chosen in that way. We put \( V_\lambda := \langle B \rangle_\lambda = \mathfrak{g}_F v^+ = \mathfrak{g}_F v^+ \), since \( \mathfrak{g}_F v^+ = \mathfrak{g}_F v^+ \).

We have \( V_\lambda = \oplus_{\mu \leq \lambda} V_\mu, Z_\lambda, \) where \( V_\mu, Z_\lambda = \pi_C(V_\mu, Z) \) and \( V_\mu, Z = V_\mu \cap Z_\lambda \). Denoted by \( \mathfrak{B}_\mu \) the monomial basis of \( V_\mu \) contained in \( A^+ v^+ \), the set \( \mathfrak{B}_\mu \) contains a subset \( \mathfrak{B}_\mu \) such that \( \pi_C \) induces a bijection from \( \mathfrak{B}_\mu \) to a basis \( B_\mu = \pi_C(B_\mu) \) of \( V_\mu \) contained in \( B \). (We warn that \( B_\mu = \emptyset \) is allowed; this is certainly the case when \( \mu < 0 \), since \( V_\mu = 0 \) in this case.)

We shall now replace the set \( \mathfrak{B}_\mu \) \( \mathfrak{B}_\mu \) with an independent set of sets vectors \( \mathfrak{B}_\mu' \) such that \( \mathfrak{B}_\mu' \subseteq J_C(\lambda) \cap (\mathfrak{B}_\mu)_{\mathbb{Z}} \) (where \( (\mathfrak{B}_\mu)_{\mathbb{Z}} \) stands for the set of integral combinations of vectors of \( \mathfrak{B}_\mu \) such that \( (\mathfrak{B}_\mu \cup \mathfrak{B}_\mu')_{\mathbb{Z}} = (\mathfrak{B}_\mu)_{\mathbb{Z}} \) (whence \( \mathfrak{B}_\mu \cup \mathfrak{B}_\mu' \) is a basis of \( V_\mu \)). Having done this we will be able to define \( J_\lambda \).

Let \( b'_1, b'_2, \ldots, b' \) be the elements of \( \mathfrak{B}_\mu \setminus \mathfrak{B}_\mu \), taken in some given order. We construct \( \mathfrak{B}_\mu' \) recursively as follows. We put \( \mathfrak{B}_\mu' := \emptyset \). Assuming to have already defined \( \mathfrak{B}_\mu' \) for \( 0 \leq k < r \), if \( \pi_C(b_{k+1}^') = 0 \) then \( \mathfrak{B}_\mu'_{k+1} := \mathfrak{B}_\mu'_{k} \cup \{b'_{k+1}\} \). Otherwise \( b'_{k+1} := \pi_C(b_{k+1}^') \in V_\mu \cap A^+ v^+ \subseteq V_\mu, Z \) is an integral
combination of the vectors \( b_1, b_2, \ldots \) of \( B_\mu \), say \( b'_k = \sum_i z_i b_i \). Clearly \( b'_k - \sum_i z_i b_i < J_C(\lambda) \). We put \( B'_\mu, k := B'_{\mu, k} \cup \{ b'_{k+1} - \sum_i z_i b_i \} \). Finally, \( B'_{\mu} := B'_{\mu, r} \).

Put \( J(\lambda) := \langle \cup_{\mu \leq \lambda} B'_{\mu} \rangle \). Then \( \cup_{\mu \leq \lambda} B'_{\mu} \) is a basis of \( Z(\lambda) \) and \( J(\lambda) = Z(\lambda) \cap J_C(\lambda) \) (as it follows from the definition of \( B'_{\mu} \) and the properties of the sets \( B'_{\mu} \) defined above). We can now interpret \( V(\lambda) \) (previously defined as \( V(\lambda) := \langle B \rangle = V_\lambda v^+ = V_\lambda v^+ \) as the quotient \( V(\lambda) := Z(\lambda)/J(\lambda) \).

The projection \( \pi : Z(\lambda) \to V(\lambda) \) induces a homomorphism \( \pi_\lambda \) from \( Z(\lambda) \) to \( V(\lambda) \) and \( J(\lambda) = \text{ker}(\pi_\lambda) \).

Let now \( \mathbb{F} \) be an arbitrary field. Then we put \( \mathfrak{A}_\mathbb{F} := \mathbb{F} \otimes \mathfrak{A}_\mathbb{Z}, \mathfrak{A}_\mathbb{F} := \mathfrak{A}_\mathbb{F} \otimes \mathfrak{A}_\mathbb{F} \), \( Z_\mathbb{F}(\lambda) = \mathbb{F} \otimes \mathfrak{A}_\mathbb{Z}(\lambda), V_\mathbb{F}(\lambda) = V_\mathbb{F} \otimes \mathfrak{A}_\mathbb{F}(\lambda) \) and \( J_\mathbb{F}(\lambda) = \mathbb{F} \otimes \mathfrak{A}_\mathbb{F}(\lambda) \). Since \( V_\mathbb{F}(\lambda) = Z_\mathbb{F}(\lambda)/J_\mathbb{F}(\lambda) \), we also have \( V_\mathbb{F}(\lambda) = Z_\mathbb{F}(\lambda)/J_\mathbb{F}(\lambda) \). The Chevalley group \( G_\mathbb{F}(\lambda) \) is defined as a group of linear transformations of \( V_\mathbb{F}(\lambda) \), by means of the exponentials \( e^{X_\mu} \) just as in the complex case. The Lie algebra \( \mathfrak{L}_\mathbb{F} = \mathbb{F} \otimes \mathfrak{L}_\mathbb{Z} \) is the Lie algebra of \( G_\mathbb{F}(\lambda) \).

We finish this subsection with a result on homomorphic images of Weyl modules, already but implicitly used in Section 2.

**Theorem 2.1** Let \( U \) be a finite dimensional \( G_\mathbb{F}(\lambda) \)-module over \( \mathbb{F} \), satisfying the following properties.

1. \( U \) is also an \( \mathfrak{A}_\mathbb{F} \)-module and the action of \( G_\mathbb{F}(\lambda) \) on \( U \) is induced by the action of \( \mathfrak{A}_\mathbb{F} \). Explicitly, for every root \( \alpha \in \Phi \) the element \( X_\alpha \) acts on \( U \) as a nilpotent endomorphism and the element \( X_\alpha(t) = e^{X_\mu} \) of \( G_\mathbb{F}(\lambda) \) maps \( v \) onto \( \sum_{n=0}^{\infty} t^n X_\alpha^n(v) \), for every vector \( v \in U \).

2. \( U \) admits a highest weight vector of weight \( \lambda \), namely a vector \( \bar{v}^+ \) such that \( X_\alpha(\bar{v}^+) = 0 \) for every \( \alpha \in \Phi^+ \) and \( H_i(\bar{v}^+) = \lambda(H_i) \bar{v}^+ \) for every \( i = 1, 2, \ldots, n \).

Then there exists a unique homomorphism of \( \mathfrak{A}_\mathbb{F} \)-modules from \( V_\mathbb{F}(\lambda) \) to \( U \) mapping \( v^+ \) onto \( \bar{v}^+ \) (whence \( g(v^+) \) onto \( g(\bar{v}^+) \) for every element \( g \in \mathfrak{A}_\mathbb{F} \)).

This theorem is implicit in the final result of Humphreys [7, 3.3]. In [7, 3.3] the field \( \mathbb{F} \) is assumed to be algebraically closed, but in the hypotheses of Theorem 2.1 that assumption can be dropped. Indeed we can firstly replace \( \mathbb{F} \) with its algebraic closure \( \overline{\mathbb{F}} \) and apply the theorem of [7, 3.3] to \( G_{\overline{\mathbb{F}}}(\lambda) \) acting on \( V_{\overline{\mathbb{F}}}(\lambda) \) and \( \overline{\mathbb{F}} \otimes \overline{\mathbb{F}} \).

Next we turn back to \( G_\mathbb{F}(\lambda) \) acting on \( V_\mathbb{F}(\lambda) \) and \( U \).

**2.2 A Chevalley basis of \( \mathfrak{o}(2n+1, \mathbb{F}) \)**

Given a field \( \mathbb{F} \), let \( \mathfrak{L}_\mathbb{F} = \mathfrak{o}(2n+1, \mathbb{F}) = \mathfrak{H} \oplus (\oplus_{\alpha \in \Phi} \mathfrak{X}_\alpha) \) be the Lie algebra of type \( B_n \) over \( \mathbb{F} \), where \( \mathfrak{H} \), \( \Phi \) and \( \mathfrak{X}_\alpha \) have the same meaning as in Section 2.1. Let \( \mathfrak{A}_\mathbb{F} = \mathbb{F} \otimes \mathfrak{L}_\mathbb{Z} \) be the enveloping algebra of \( \mathfrak{L}_\mathbb{F} \), \( \Pi = \{ \alpha_1, \ldots, \alpha_n \} \) a given basis of simple roots of \( \Phi \) and \( \{ X_\alpha, Y_\alpha \}_{\alpha \in \Phi} + \{ H_i \}_{i=1}^{n} \) a Chevalley basis of \( \mathfrak{L}_\mathbb{F} \) where \( Y_\alpha := X_\alpha, \alpha \), as in Section 2.1.
The roots $\alpha \in \Phi$ are represented by the following vectors of $\mathbb{R}^n$, where $(u_i)_{i=1}^n$ is the natural orthonormal basis of $\mathbb{R}^n$ (see Humphreys [6, 12.1]):

\begin{equation}
(\Pi) \quad \left\{ \begin{array}{ll}
\alpha_i = u_i - u_{i+1} & \text{for } 1 \leq i \leq n - 1 \quad (\text{long simple roots}) \\
\alpha_n = u_n & \text{(short simple root)}
\end{array} \right.
\end{equation}

\begin{equation}
(\Phi^+) \quad \left\{ \begin{array}{ll}
u_i - u_j = \sum_{r=i}^{j-1} \alpha_r & \text{for } 1 \leq i < j \leq n \quad (\text{long}) \\
u_i + u_j = \sum_{r=i}^{j-1} \alpha_r + 2 \sum_{r=j}^{n} \alpha_r & \text{for } 1 \leq i < j \leq n \quad (\text{long})
\end{array} \right.
\end{equation}

\begin{equation}
(\Phi^-) \quad \left\{ \begin{array}{ll}
-u_i = -\sum_{r=i}^{n} \alpha_r & \text{for } 1 \leq i < j \leq n \quad (\text{short}) \\
-u_i - u_j = -\sum_{r=i}^{j-1} \alpha_r - 2 \sum_{r=j}^{n} \alpha_r & \text{for } 1 \leq i < j \leq n \quad (\text{long})
\end{array} \right.
\end{equation}

As in Section II let $V = V(2n+1, \mathbb{F})$ and let $\eta$ be a non-singular quadratic form on $V$. Up to rescaling the form $\eta$, we can assume to have chosen an ordered basis $E = (e_1, e_2, \ldots, e_{2n+1})$ of $V$ with respect to which

$$\eta(x_1, \ldots, x_{2n+1}) = \sum_{i=1}^{n} x_i x_{n+i} + x_{2n+1}^2. \quad (4)$$

Regarding $G = \text{SO}(2n+1, \mathbb{F})$ as the subgroup of $\text{SL}(V)$ preserving $\eta$ and $\Sigma_\mathbb{F}$ as the Lie algebra of $G$, the algebra $\mathfrak{so}(\mathbb{F})$ acts on $V$ as a subalgebra of $\text{End}(V)$.

We can assume that the chosen Chevally basis $\{X_\alpha, Y_\alpha\}_{\alpha \in \Phi^+} \cup \{H_i\}_{i=1}^n$ of $\mathfrak{so}(\mathbb{F})$ acts as follows on the elements of $E$, where $\delta_{i,j}$ is the Kronecker symbol:

\begin{align*}
X_{u_i-u_j}(e_l) &= \delta_{i,j} e_i - \delta_{l,n+i} e_{n+j}, & (1 \leq i < j \leq n), \\
Y_{u_i-u_j}(e_l) &= \delta_{i,j} e_j - \delta_{l,n+j} e_{n+i}, & (1 \leq i < j \leq n), \\
X_{u_i+u_j}(e_l) &= \delta_{l,n+i} e_j - \delta_{l,n+j} e_i, & (1 \leq i < j \leq n), \\
Y_{u_i+u_j}(e_l) &= -\delta_{i,j} e_{n+j} + \delta_{l,n+i} e_{n+i}, & (1 \leq i < j \leq n), \\
X_{u_i}(e_l) &= \delta_{l,n+i} e_{2n+1} - 2\delta_{l,2n+1+i} e_i, & (1 \leq i \leq n), \\
Y_{u_i}(e_l) &= -\delta_{i,j} e_{2n+1} + 2\delta_{l,2n+1+i} e_{n+i}, & (1 \leq i \leq n),
\end{align*}

$$H_i(e_l) = \delta_{i,j} e_i - \delta_{i,i+1} e_{i+1} - \delta_{l,n+i} e_{n+i} + \delta_{l,n+i+1} e_{n+i+1}, \quad (1 \leq i < n),$$

$$H_n(e_l) = 2\delta_{l,n} e_n - 2\delta_{l,2n} e_{2n}.$$

(Recall that $H_i = H_{\alpha_i} = H_{u_i-u_{i+1}}$ if $i < n$ and $H_n = H_{\alpha_n} = H_{u_n}$.) Moreover,

$$H_{u_i-u_j} = \sum_{r=i}^{j-1} H_r = -H_{u_j-u_i}, \quad (1 \leq i < j \leq n),$$

$$H_{u_i} = 2\sum_{r=i}^{n-1} H_r + H_n = -H_{-u_i}, \quad (1 \leq i \leq n)$$

$$H_{u_i+u_j} = \sum_{r=i}^{j-1} H_r + 2\sum_{r=j}^{n-1} H_r + H_n = -H_{-u_i-u_j}, \quad (1 \leq i < j \leq n).$$
The elements $X_\alpha$ and $Y_\alpha$ act on $V$ as nilpotent endomorphisms of exponent 2 or 3, according to whether $\alpha$ is long or short. Moreover,
\[
\frac{X^2}{2}(e_i) = -\delta_{i,n+i} e_i \quad (1 \leq i \leq n),
\]
\[
\frac{Y^2}{2}(e_i) = -\delta_{i,n} e_{n+i} \quad (1 \leq i \leq n).
\]

We now turn to $W_k = \bigwedge_k V$. We firstly fix some notation. We set $I := \{1,2,...,2n+1\}$ and $E_\lambda := (e_J)_{J \in \binom{I}{k}}$, where $\binom{I}{k}$ stands for the set of subsets of $I$ of size $k$ and $e_J = e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_k}$ for every $k$-subset $J = \{j_1,\ldots,j_k\}$ of $I$, with the convention that $j_1 < j_2 < \ldots < j_k$. Thus, $E_\lambda$ is the basis of $W_k$ associated to $E$.

The action of $\Sigma_k^\circ$ on $V$ induces an action of $\Sigma_k^\circ$ on $W_k$ defined as follows: if $g \in \Sigma_k^\circ$ and $v = v_1 \wedge v_2 \wedge \ldots \wedge v_k$ then
\[
g(v) = \sum_{i=1}^k v_1 \wedge \ldots \wedge v_{i-1} \wedge g(v_i) \wedge v_{i+1} \wedge \ldots \wedge v_k.
\] (5)

The action of the elements $X_\alpha/t$ and $Y_\alpha/t$ on $W_k$ can be computed with the help of formula (5). We leave these computations for the reader.

Let $\bar{v}^+: = e_1 \wedge e_2 \wedge \ldots \wedge e_k$. Then $\bar{v}^+$ behaves as the maximal vector $v^+$ of $V(\lambda_k^\circ)$ (where $\lambda_k^\circ$ has the meaning stated in Section 1), namely $X_\alpha(\bar{v}^+) = 0$ for any $\alpha \in \Phi^+$ and $H(\bar{v}^+) = \lambda_k^\circ(H)\bar{v}^+$ for every $H \in H$. In particular,
\[
H_{u,-u+1}(\bar{v}^+) = \begin{cases} \bar{v}^+ & \text{if } k = n, \\ 0 & \text{otherwise} \end{cases}
\]
\[
H_{u,n}(\bar{v}^+) = \begin{cases} 2\bar{v}^+ & \text{if } k = n, \\ 0 & \text{otherwise} \end{cases}
\]

Note that
\[
\lambda_k^\circ = \sum_{i=1}^{k-1} i \alpha_i + k \cdot \sum_{i=k}^n \alpha_i.
\] (Humphreys [H 13.2]). Thus,
\[
\lambda_k^\circ(H_i) = \langle \lambda_k^\circ, \alpha_i \rangle = \begin{cases} -(i-1) + 2i - (i+1) = 0 & \text{if } i < k < n, \\ -(k-1) + 2k - k = 1 & \text{if } i = k < n, \\ -k + 2k - k = 0 & \text{if } k < i < n, \\ 2k - 2k = 0 & \text{if } k < i = n, \\ -2(n-1) + 2n = 2 & \text{if } k = i = n, \\ -(n-1) + 2n = 2 & \text{if } k < n, \end{cases}
\]

which fits with the above values of $H_{u,-u+1}(\bar{v}^+)$ and $H_{u,n}(\bar{v}^+)$ (recall that $H_i = H_{u,-u+1}$ if $i < n$ and $H_n = H_{u,n}$).

Let $W_k^\circ(F) := \mathcal{V}_F(\bar{v}^+)$. Thus $W_k^\circ(F)$ is the module called $W_k^\circ$ in Section 1 but now we keep a record of the field $F$ in our notation. In view of the above, the homomorphism $\varphi_{k,F} : V(\lambda_k^\circ) \to W_k^\circ(F)$ (Theorem 2.1) can be chosen in such a way that $\varphi_{k,F}(v^+) = \bar{v}^+$. This condition uniquely determines $\varphi_{k,F}$.

As recalled in Section 1, the homomorphism $\varphi_{k,F}$ is an isomorphism if and only if either $\text{char}(F) \neq 2$ or $k = 1$. If $\text{char}(F) = 2$ and $k > 1$ then the kernel $K_k$ of $\varphi_{k,F}$ has dimension $\dim(K_k) = \binom{2n+1}{k-1}$. 10
2.3 A Chevalley basis of $\mathfrak{sp}(2n, F)$

Let $\mathfrak{sp}^p_n = \mathfrak{sp}(2n, F) = \mathfrak{c} \oplus (\oplus_{\beta \in \Phi} \mathfrak{U}_{\beta})$ be the Lie algebra of type $C_n$, where $\mathfrak{c}$ is the Cartan subalgebra, $\Phi$ is the root system of type $C_n$ and $\mathfrak{U}_{\beta}$ is the 1-dimensional subalgebra of $\mathfrak{sp}^p_n$ corresponding to the root $\beta$. Let $\mathfrak{A}^p_n$ be the enveloping algebra of $\mathfrak{sp}^p_n$. Chosen a basis $\Pi = \{\beta_1, ..., \beta_n\}$ of simple roots for $\Phi$ let $\Phi^+$ (respectively $\Phi^-$) be the set of roots that are positive (negative) with respect to $\Pi$. Let $\{C_i\}_{i=1}^n \cup \{U_\beta, V_\beta\}_{\beta \in \Phi}$ be a Chevalley basis of $\mathfrak{sp}^p_n$, where $C_i \in \mathfrak{c}$ for $i = 1, ..., n$ and $U_\beta \in \mathfrak{U}_\beta$ and $V_\beta = U_{-\beta} \in \mathfrak{U}_{-\beta}$ for $\beta \in \Phi^+$.

The positive roots $\beta \in \Phi^+$ are represented by the following vectors of $\mathbb{R}^n$ where $(u_i)_{i=1}^n$ is the natural orthonormal basis of $\mathbb{R}^n$, as in the previous subsection (see Humphreys [6, 12.1]):

\[
\beta_i = u_i - u_{i+1} \text{ for } 1 \leq i \leq n - 1 \quad \text{(short simple)}
\]

\[
\beta_n = 2u_n \quad \text{(long simple)}
\]

\[
u_i - u_j = \sum_{r=1}^{j-1} \beta_r \text{ for } 1 \leq i < j \leq n \quad \text{(short)}
\]

\[
2u_i = 2 \sum_{r=1}^{n-1} \beta_r + \beta_n \text{ for } i = 1, 2, ..., n \quad \text{(long)}
\]

\[
u_i + u_j = \sum_{r=1}^{j-1} \beta_r + 2 \sum_{r=j}^{n-1} \beta_r + \beta_n \text{ for } 1 \leq i < j \leq n \quad \text{(short)}
\]

The negative roots are the opposite of these. We omit to list them. Note that the short roots of $\Phi$ are just the long roots of the root system $\Phi$ of type $B_n$ while the long roots of $\Phi$ are the short roots of $\Phi$ multiplied by 2.

With $V = V(2n+1, F)$ and $E = \{e_1, ..., e_{2n+1}\}$ as in the previous subsection, let $V$ be the hyperplane of $V$ spanned by the subset $E = \{e_1, e_2, ..., e_{2n}\}$ of $E$. Let $V$ be endowed with the non-degenerate alternating form $\zeta$ represented as follows with respect to the basis $E$ of $V$:

\[
\zeta((x_1, x_2, ..., x_{2n}), (y_1, y_2, ..., y_{2n})) = \sum_{i=1}^{n} x_i y_{n+i} - \sum_{i=1}^{n} x_{n+i} y_i. \quad (7)
\]

(Note that if $\text{char}(F) = 2$ then, according to [4], $\zeta$ is the form induced on $V$ by the bilinearization of the quadratic form $\eta$ of $V$.) Regarded $G := \text{Sp}(2n, F)$ as the subgroup of $\text{SL}(V)$ preserving $\zeta$ and $\mathfrak{sp}^p_n$ as the Lie algebra of $G$, the algebra $\mathfrak{sp}^p_n$ acts on $V$ as a subalgebra of $\text{End}(V)$.

We can assume that the chosen Chevalley basis $\{U_\beta, V_\beta\}_{\beta \in \Phi^+} \cup \{C_i\}_{i=1}^n$ of $\mathfrak{sp}^p_n$ acts as follows on the elements of $E$:

\[
U_{u_i-u_j}(e_l) = \delta_{l,j} e_i - \delta_{l,n+j} e_{n+j}, \quad (1 \leq i < j \leq n),
\]

\[
V_{u_i-u_j}(e_l) = \delta_{l,i} e_j - \delta_{l,n+j} e_{n+j}, \quad (1 \leq i < j \leq n),
\]

\[
U_{u_i+u_j}(e_l) = \delta_{l,n+i} e_j + \delta_{l,n+j} e_i, \quad (1 \leq i < j \leq n),
\]

\[
V_{u_i+u_j}(e_l) = \delta_{l,i} e_{n+j} + \delta_{l,j} e_{n+i}, \quad (1 \leq i < j \leq n),
\]

\[
U_{2u_i}(e_l) = \delta_{l,n+i} e_i, \quad (1 \leq i \leq n),
\]

\[
V_{2u_i}(e_l) = \delta_{l,i} e_{n+i}, \quad (1 \leq i \leq n).
\]
\[ C_i(e_i) = \delta_{i,i}e_i - \delta_{i,i+1}e_{i+1} + \delta_{i,n+i}e_{n+i} + \delta_{i,n+i+1}e_{n+i+1}, \quad (1 \leq i < n), \]
\[ C_n(e_i) = \delta_{i,n}e_i - \delta_{i,2n}e_{2n}. \]

Note that \( U_{\beta} \) and \( V_{\beta} \) act on \( \mathcal{V} \) as nilpotent endomorphisms of exponent 2 for every \( \beta \in \mathfrak{F}^+. \) Moreover, the elements \( X_{u_i-u_j}, Y_{u_i-u_j}, \frac{X^2}{2} \), \( \frac{Y^2}{2} \), \( X_{u_i+u_j} \) and \( Y_{u_i+u_j} \) of \( \mathfrak{f}_p^\circ \) stabilize the subspace \( \mathcal{V} \) of \( \mathcal{V} \) and the following hold for every vector \( v \in \mathcal{V} \)

\[
\begin{align*}
U_{u_i-u_j}(v) = X_{u_i-u_j}(v), & \quad V_{u_i-u_j}(v) = Y_{u_i-u_j}(v), \\
U_{2u_i}(v) = -\frac{X^2}{2}(v), & \quad V_{2u_i}(v) = -\frac{Y^2}{2}(v), \\
U_{u_i+u_j}(v) = X_{u_i+u_j}(v) - 2X_{-e_i-e_j}X^2(v) = X_{u_i+u_j}(v) + 2U_{e_i-e_j}U_{2u_i}(v), & \\
V_{u_i+u_j}(v) = Y_{u_i+u_j}(v) - 2\frac{Y^2}{2}Y_{-e_i-e_j}(v) = Y_{u_i+u_j}(v) + 2V_{2u_i}V_{e_i-e_j}. 
\end{align*}
\]

In particular, when \( \text{char}(\mathcal{F}) = 2 \)

\[ U_{u_i+u_j}(v) = X_{u_i+u_j}(v), \quad V_{u_i+u_j}(v) = Y_{u_i+u_j}(v). \]  

Moreover, \( U_{u_i+u_j} = [U_{u_i-u_j}, U_{2u_i}] \) and \( V_{u_i+u_j} = [V_{2u_i}, V_{u_i-u_j}] \). Consequently,

\[
\begin{align*}
U_{u_i+u_j}(v) = \frac{X^2}{2}X_{u_i-u_j}(v) - X_{u_i-u_j}\frac{X^2}{2}(v), & \\
V_{u_i+u_j} = Y_{u_i-u_j}\frac{Y^2}{2}(v) - \frac{Y^2}{2}Y_{u_i-u_j}(v). 
\end{align*}
\]

We now turn to \( \mathcal{W}_k = \bigwedge^+ \mathcal{V} \). We take \( \mathcal{E}_\wedge := \langle e_j \rangle_{j \in \{1, 2, \ldots, 2n\}} \) as a basis for \( \mathcal{W}_k \), where \( \bar{I} := \{1, 2, \ldots, 2n\} \). The action of \( \mathfrak{f}_p^{\circ} \) on \( \mathcal{V} \) induces an action of \( \mathfrak{f}_p^{\circ} \) on \( \mathcal{W}_k \), defined according to the rule [8].

Note firstly that \( U_{\beta} \) and \( V_{\beta} \) act on \( \mathcal{W}_k \) as nilpotent endomorphisms of exponent 3 or 2 according to whether \( \beta \) is short or long. In view of [8] the elements \( U_{u_i-u_j}, \frac{U^2}{2-u_i}, V_{u_i-u_j}, \frac{V^2}{2-u_i}, U_{2u_i} \) and \( V_{2u_i} \) act on \( \mathcal{W}_k \) as \( X_{u_i-u_j}, \frac{X^2}{2-u_i}, Y_{u_i-u_j}, \frac{Y^2}{2-u_i}, -\frac{X^2}{2} \) and \( -\frac{Y^2}{2} \) respectively. According to (10), denoted by \( \langle \mathcal{E}_\wedge \rangle_{22} \) the set of integral combination of vectors of \( \mathcal{E}_\wedge \) with even coefficients,

\[
\{U_{u_i+u_j}(e_J) - X_{u_i+u_j}(e_J), V_{u_i+u_j}(e_J) - Y_{u_i+u_j}(e_J)\} \subset \langle \mathcal{E}_\wedge \rangle_{22}
\]

for every \( e_J \in \mathcal{E}_\wedge \). In particular, if \( \text{char}(\mathcal{F}) = 2 \) then \( U_{u_i+u_j} \) and \( V_{u_i+u_j} \) act on \( \mathcal{W}_k \) in the same way as \( X_{u_i+u_j} \) and \( Y_{u_i+u_j} \) respectively.

Let \( \bar{v}^+ := e_1 \wedge e_2 \wedge \ldots \wedge e_k \), as in the previous subsection and let \( \mathcal{W}_k^{\circ}(\mathcal{F}) = \mathfrak{f}_p^{\circ}(\bar{v}^+) \). Thus, \( \mathcal{W}_k^{\circ}(\mathcal{F}) \) is the module called \( \mathcal{W}_k^{\circ} \) in Section [1] By [8], [9] and [10], we see that \( \mathcal{W}_k^{\circ}(\mathcal{F}) \subseteq W_k^0(\mathcal{F}) \).
Let $\lambda_1^{sp}, \lambda_2^{sp}, ..., \lambda_n^{sp}$ be the fundamental dominant weights for the root system of type $G$. It is well known that $\mathbf{W}_k^{sp}(\mathbb{F})$ is isomorphic to the Weyl module $V_\varphi(\lambda_k^{sp})$ and that, if $v^+$ is the highest weight vector chosen in $V_\varphi(\lambda_k^{sp})$, we can assume that the isomorphism from $V_\varphi(\lambda_k^{sp})$ to $\mathbf{W}_k^{sp}$ maps $v^+$ onto $\bar{v}^+$ (see e.g. Premet and Suprunenko [8]).

3 Proof of Theorem 1.4

Throughout this section char($\mathbb{F}$) = 2. If $k = 1$ then $K_k = 0$ and $\mathcal{N}_k$ is 1-dimensional. In fact $\mathcal{N}_1$ is the nucleus of the quadratic form $\eta$ chosen to define $G$. Thus, in this case the first claim of Theorem 1.4 is obvious. The second claim is empty.

Assume $k > 1$. We keep the notation of Subsections 2.2 and 2.3. In particular, if $v^+$ is the highest weight vector chosen in $V_\varphi(\lambda_1^k)$ and $\bar{v}^+ = e_1 \wedge ... \wedge e_k$, we can assume to have chosen the homomorphism $\varphi_{k,\mathbb{F}} : V_\varphi(\lambda_k^k) \to W_k^k(\mathbb{F})$ in such a way that it maps $v^+$ onto $\bar{v}^+$. We put $K_k := \ker(\varphi_{k,\mathbb{F}})$. Clearly, $K_k$ is an $\mathfrak{A}_k^\mathbb{F}$-submodule of $V_\varphi(\lambda_k^k)$, namely

Lemma 3.1 We have $\mathfrak{A}_k^\mathbb{F}(K_k) = K_k$.

Let $v_1^+ := Y_{u_k}(v^+)$, $V_1(\mathbb{F}) := \mathfrak{A}_k^\mathbb{F}(v_1^+)$ and

$$\bar{v}_1^+ := \varphi_{k,\mathbb{F}}(v_1^+) = Y_{u_k}(\bar{v}^+) = Y_{u_k}(e_1 \wedge ... \wedge e_k) = -e_1 \wedge ... \wedge e_{k-1} \wedge e_{2n+1}.$$ 

Our first goal is to prove the following proposition:

Proposition 3.2 We have $K_k \subseteq V_1(\mathbb{F})$ and $V_1(\mathbb{F}) \cong V_\varphi(\lambda_k^{n-1})$, with $v_1^+ \in V_1(\mathbb{F})$ corresponding to a highest weight vector of $V_\varphi(\lambda_k^{n-1})$.

3.1 Proof of Proposition 3.2

In the sequel we shall often go back and forth between $V_\varphi(\lambda_k^k)$ and $V_C(\lambda_k^k)$. By a little abuse, we will use the same symbols for vectors of $V_\varphi(\lambda_k^k)$ and $V_C(\lambda_k^k)$, thus using $v^+$ to denote both the highest weight vector chosen in $V_\varphi(\lambda_k^k)$ and the one of $V_C(\lambda_k^k)$. Accordingly, the symbol $v_1^+$, which we have introduced to denote the vector $Y_{u_k}(v^+)$ of $V_\varphi(\lambda_k^k)$ also denotes the vector $Y_{u_k}(v^+)$ of $V_C(\lambda_k^k)$. We do the same in $W_k(\mathbb{C})$. Thus, the symbol $\bar{v}_1^+ = e_1 \wedge ... \wedge e_{k-1} \wedge e_{2n+1}$ denotes $\varphi_{k,\mathbb{C}}(v_1^+)$ as well as $\varphi_{k,\mathbb{C}}(v_1^+)$, according to the context. We need one more convention. We put

$$2 \cdot \mathfrak{A}_k^\mathbb{F} := \{ 2g \mid g \in \mathfrak{A}_k^\mathbb{F} \},$$ 

$$2 \cdot V_2(\lambda_k^k) := \{ 2v \mid v \in V_2(\lambda_k^k) \} = 2 \cdot \mathfrak{A}_k^\mathbb{F}(v^+).$$

We say that two vectors $v, w \in V_2(\lambda_k^k)$ are congruent modulo 2 and we write $v \equiv_w w$ if $v - w \in 2 \cdot V_2(\lambda_k^k)$. Clearly, the canonical mapping from $V_2(\lambda_k^k)$ to $V_\varphi(\lambda_k^k)$ maps $2 \cdot V_2(\lambda_k^k)$ onto 0. Thus, if $v \equiv_w w$ in $V_2(\lambda_k^k)$ then $v$ and $w$ are equal when regarded as vectors of $V_\varphi(\lambda_k^k)$. 

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**Lemma 3.3** The vector $v_1^+$ belongs to the weight space $V_\mu$ of $V_\nu(\lambda_k^n)$ with weight 
$\mu = \lambda_k^n - \sum_{i=k}^{n} \alpha_i$.

**Proof.** We have $u_k = \sum_{i=k}^{n} \alpha_i$. Hence $v_1^+ \in V_\mu$ with $\mu = \lambda_k^n - \sum_{i=k}^{n} \alpha_i$.

According to Lemma 3.8, $\lambda_k^n - \sum_{i=k}^{n} \alpha_i = \sum_{i=1}^{k-2} i \alpha_i + (k-1) \sum_{i=k-1}^{n} \alpha_i = \lambda_k^{n-1}$. $\square$

**Lemma 3.4** We have $X_\alpha(v_1^+) = 0$ for every positive root $\alpha \in \Phi^+$.

**Proof.** We switch from $V_\nu(\lambda_k^n)$ to $B_\nu(\lambda_k^n)$. Since $B_\nu(\lambda_k^n) \cong W_\Phi^n(C)$, we can make our computations inside $W_\Phi^n(C)$. It is easy to see that $X_{u_i-u_j}(\bar{v}_1^+) = 0$ for every choice of $1 \leq i < j \leq n$ while $X_{u_i}(\bar{v}_1^+) = -2e_1 \wedge \ldots \wedge e_{k-1} \wedge e_i$ for every $i = 1, 2, \ldots, n$. Therefore $X_{u_i}(v_1^+) = 0$ for every $i = 1, 2, \ldots, n$. It follows that in $V_\nu(\lambda_k^n)$ we have $X_\alpha(v_1^+) = 0$ for every $\alpha \in \Phi^+$. $\square$

**Corollary 3.5** The $\mathfrak{g}_F$-module $V_1(F)$ is a homomorphic image of $V_\nu(\lambda_{k-1})$ and we can choose the homomorphism $\psi_{k-1,F} : V_\nu(\lambda_{k-1}) \to V_1(F)$ in such a way that $\psi_{k-1,F}$ maps the highest weight vector chosen in $V_\nu(\lambda_{k-1})$ onto $v_1^+$.

**Proof.** This follows from Theorem 2.1 and Lemmas 3.3 and 3.4. $\square$

Our next step is to prove that $K_k \subseteq V_1(F)$. We firstly compute $Y_{u_i}(v^+)$ and $Y_{u_i}Y_{u_j}(v^+)$ for $i < j$. Recall that $Y_{u_k}(v^+) = v_1^+$ by definition.

**Lemma 3.6** If $i > k$ then $Y_{u_i}(v^+) = 0$. If $i < k$ then $Y_{u_i}(v^+) = Y_{u_i-u_k}(v_1^+)$. 

**Proof.** We switch to $B_\nu(\lambda_k^n)$. As $W_k(C) = W_\Phi^n(C) \cong V_\nu(\lambda_k^n)$, we can replace $v^+$ with $\bar{v}_1^+ = e_1 \wedge \ldots \wedge e_k$. Computing in $W_k(C)$ we see that if $i > k$ then $Y_{u_i}(\bar{v}_1^+) = 0$ while if $i < k$ then $Y_{u_i}(\bar{v}_1^+) = Y_{u_i-u_k}Y_{u_k}(\bar{v}_1^+) = Y_{u_i-u_k}(\bar{v}_1^+)$. Hence in $B_\nu(\lambda_k^n)$ we have $Y_{u_i}(v^+) = 0$ if $i > k$ and $Y_{u_i}(v^+) = Y_{u_i-u_k}(v_1^+) = 0$ if $i < k$. Clearly, the same holds in $V_\nu(\lambda_k^n)$. $\square$

**Corollary 3.7** We have $Y_{u_i}(v^+) \in V_1(F)$ for every $i = 1, 2, \ldots, n$.

**Lemma 3.8** If $i < j$ then $Y_{u_i}Y_{u_j}(v^+) \in K_k$.

**Proof.** If $j > k$ then $Y_{u_i}Y_{u_j}(v^+) = 0$ by Lemma 3.6. Let $j = k$. Then $Y_{u_i}Y_{u_j}(\bar{v}_1^+) = Y_{u_i}Y_{u_k}(\bar{v}_1^+) = Y_{u_i}(\bar{v}_1^+) = 2 \cdot e_1 \wedge \ldots \wedge e_{k-1} \wedge e_{n+i}$, which is $0$ in $W_k(F)$. Therefore $Y_{u_i}Y_{u_j}(v^+) \in K_k$. Finally, let $j < k$. Then $Y_{u_i}Y_{u_j}(v^+) = Y_{u_i}Y_{u_j}(v^+) = 0$ by Lemma 3.6. However $Y_{u_i}$ and $Y_{u_j}$ commute, since $-u_i - u_j + u_k$ is not a root. Therefore $Y_{u_i}Y_{u_j}(\bar{v}_1^+) = Y_{u_i-u_k}Y_{u_j}(\bar{v}_1^+) = 0$ since $Y_{u_i}Y_{u_k}(\bar{v}_1^+) = 0$, as shown above. $\square$

We need to fix a few more conventions. The first convention has to do with the order in which we take the factors $Y_i$ of a monomial element of type $(-)$ of $\mathfrak{g}_F$ (or $\mathfrak{g}_C^+$. Recall that we only need these elements to produce $V_\nu(\lambda_k^n)$ (respectively $V_\nu(\lambda_k^n)$) starting from $v^+$. We assume to write each such element $Y$ as a product $Y = \prod_{i=1}^{r} Y_i$ where $Y_i = Y_{u_{i_1}}Y_{u_{i_2}}\ldots Y_{u_{i_r}}$ with $i_1 < i_2 < \ldots < i_r$ (and
possibly \( r = 0 \), namely \( Y' = 1 \) and \( \overline{Y} \) is a product of elements as \( \frac{Y^2}{2}, Y_{u_i + u_j}, \\frac{Y^2}{2} \), \( Y_{u_i - u_j} \) and \( \frac{Y^2}{2} \) (and possibly \( \overline{Y} = 1 \)), taken in some given order. Note that there is no need to consider elements as \( \frac{Y^2}{2} \) with \( r > 2 \), since \( Y_\alpha \) acts on \( V_2(\lambda_j^\alpha) \) as a nilpotent endomorphisms of exponent 3. (Indeed, as noticed in Subsection 2.2, this is true for the action of \( Y_\alpha \) on \( W_k(C) \) and \( W_k(C) \cong V_2(\lambda_j^\alpha) \).)

With \( Y = \overline{Y}Y' \) as above, we call \( \overline{Y} \) (respectively \( Y' \)) the pseudo-symplectic part (non-symplectic part) of \( Y \). If \( Y' = Y_{u_{i_1}}Y_{u_{i_2}}...Y_{u_{i_r}} \) we put \( \delta(Y) := r \) and we call \( \delta(Y) \) the non-symplectic degree of \( Y \). If \( \delta(Y) = 0 \), namely \( Y' = 1 \), then we say that \( Y \) is pseudo-symplectic, otherwise we say that \( Y \) is non-symplectic.

The next corollary immediately follows from Corollary 3.7 and Lemma 3.8.

**Corollary 3.9** With \( Y \) as above, if \( \delta(Y) > 0 \) then \( Y(v^+) \in V_1(\mathbb{F}) \). If \( \delta(Y) > 1 \) then \( Y(v^+) \in K_k \).

When \( \delta(Y) = 1 \) a sharper statement can be proved. As in Subsection 2.3 let \( \overline{V} \) be the hyperplane of \( V \) spanned by \( \overline{E} = \{ e_1, ..., e_{2n} \} \) and put \( \overline{W}_{k-1} := \bigwedge^{k-1}_k \overline{V} \). It follows from the information gathered in Subsections 2.2 and 2.3 that \( \overline{V} \) is stabilized by all monomials of pseudo-symplectic type. Hence these monomials stabilize \( \overline{W}_{k-1} \) too.

By Lemma 3.6 we immediately obtain the following:

**Corollary 3.10** Let \( Y \) be a monomial element of \((-)\)-type with \( \delta(Y) = 1 \). Thus \( Y = \overline{Y}Y_{u_1} \), where \( \overline{Y} \) is the pseudo-symplectic part of \( Y \). Define \( \overline{w} \in \overline{W}_{k-1} \) as follows: \( \overline{w} = 0 \) if \( i > k \), \( \overline{w} = e_1 \wedge ... \wedge e_{k-1} \) if \( i = k \) and

\[
\overline{w} = Y_{u_i - u_k}(e_1 \wedge ... \wedge e_{k-1}) = e_1 \wedge ... \wedge e_{i-1} \wedge e_k \wedge e_{i+1} \wedge ... \wedge e_{k-1}
\]

if \( i < k \). Then \( Y(\overline{w}^+) = \overline{Y}(\overline{w}) \wedge e_{2n+1} \in \overline{W}_{k-1} \wedge e_{2n+1} \).

Suppose now that for some monomials \( Y_1, ..., Y_m \) of \((-)\)-type and scalars \( t_1, ..., t_m \in \mathbb{F} \) we have

\[
\sum_{i=1}^{m} t_i Y_i(\overline{w}^+) = 0.
\]

(13)

Regarded \( \mathbb{F} \) as a vector space over its prime subfield \( \mathbb{F}_2 \), let \( \tau_0, \tau_1, \tau_2, \ldots \) form a basis of \( \mathbb{F} \) over \( \mathbb{F}_2 \). Thus, \( t_i = \tau_0 t_{0,i} + \tau_1 t_{1,i} + \tau_2 t_{2,i} + ... \) for suitable scalars \( t_{0,i}, t_{1,i}, t_{2,i}, \ldots \in \mathbb{F}_2 \). Accordingly, (13) is equivalent to a (possibly infinite) system of equations

\[
\sum_{i=1}^{m} t_j Y_i(\overline{w}^+) = 0, \quad (j = 0, 1, 2, ...).
\]

Thus there is no loss in assuming that in (13) the coefficients \( t_1, t_2, ..., t_m \) are taken from \( \mathbb{F}_2 \).

**Lemma 3.11** Let \( g = \sum_{i=1}^{m} t_i Y_i \) be a linear combination of monomials of type \((-) \) with \( t_1, t_2, ..., t_m \in \mathbb{F}_2 \). Suppose that \( g(\overline{w}^+) = 0 \). Then \( g(v^+) \in V_1(\mathbb{F}) \).
Proof. We can split $g$ as a sum $g = g_0 + g_1 + g_2$ where $g_0$ is a sum of pseudo-symplectic monomials, $g_1$ is a sum of non-symplectic monomials of degree 1 and $g_2$ is a sum of non-symplectic monomials of degree at least 2. We have $(g_1 + g_2)(v^+) \in V_1(F)$ by Lemma 3.6. It remains to prove that $g_0(v^+) \in V_1(F)$. We shall prove more than this. Indeed we shall show that $g_0(v^+) = 0$.

We have $g_2(\bar{v}^+) = 0$ by Lemma 3.8. Moreover, $g_0(\bar{v}^+) \in \bar{W}_k$ (since all monomial elements of pseudo-symplectic type stabilize $V$). Let $g_1 = \tau_r Y_r + t_{r+1} Y_{r+1} + ... + t_s Y_s$ where $Y_j = \perp_j^\perp Y_{u_{i_j}}$ for $j = r, r+1,..., s$, $\perp_j^\perp$ being the pseudo-symplectic part of $Y_j$. Put $\bar{Y}_j = \perp_j^\perp T_j$ where $T_j = 0$ if $i_j > k$, $T_j = 1$ if $i_j = k$ and $T_j = Y_{u_{i_j} - u_k}$ if $i_j < k$. Let $\hat{g}_1 = \sum_{j=r}^s t_j \bar{Y}_j$. By Corollary 3.10 we have $g_1(\bar{v}^+) = (\hat{g}_1(e_1 \wedge ... \wedge e_{k-1})) \wedge e_{2n+1}$. Therefore

$$g(\bar{v}^+) = g_0(\bar{v}^+) + (\hat{g}_1(e_1 \wedge ... \wedge e_{k-1})) \wedge e_{2n+1} = 0.$$ 

It follows that $g(\bar{v}^+) = 0$ if and only if $g_0(\bar{v}^+) = 0$ and $\hat{g}_1(e_1 \wedge ... \wedge e_{k-1}) = 0$.

Thus $g_0(\bar{v}^+) = 0$ (as $g(\bar{v}^+) = 0$ by assumption) and, without loss of generality, we can assume that $g_1 = g_2 = 0$. So, $g_0 = g = \tau_1 Y_1 + t_2 Y_2 + ... + t_m Y_m$ with $\delta(Y) = 0$ for $i = 1, 2, ..., m$. The scalars $t_1, t_2, ..., t_m$ belong to $F$. So, $g$ can also be regarded as an element of $\mathbb{A}_Z^p$.

In the sequel of this proof we will freely go back and forth between $F$ and $\mathbb{C}$ or $\bar{F}$ and $\mathbb{Z}$. In order to avoid any confusion, we slightly modify our notation, writing $v^+$ and $\bar{v}^+$ instead of $v^+$ and $\bar{v}^+$ if we work in $V_\perp(\lambda^p_\perp)$ and $W_\perp^p(F)$ and using the symbols $v^+$ and $\bar{v}^+$ when we deal with $V_\perp(\lambda^p)_{\perp}$ or $V_\perp(\lambda^p)$ and $W_\perp^p(C)$. However we will do so only for the duration of this proof. Afterwards we will turn back to our earlier simpler notation.

Let us denote by $\mathbb{A}_Z^p$ the subring of $\mathbb{A}_Z^p$ generated by the monomials of $(-)$-type. As we have seen in Subsection 2.3 for every monomial $Y$ of pseudo-symplectic type there is a unique monomial $\sigma(Y) \in \mathbb{A}_Z^p$ such that

$$Y(\bar{v}^+) - \sigma(Y)(\bar{v}^+) \in 2 \cdot \mathbb{A}_Z^p(v^+).$$

Explicitly,

$$\sigma(Y_{u_i - u_j}) = V_{u_i - u_j}, \quad \sigma\left(\frac{Y^2}{2}\right) = V_{2u_i}, \quad \sigma(Y_{u_i + u_j}) = V_{u_i + u_j}.$$

We extend $\sigma$ to a product $Y_1 Y_2 ... Y_r$ of monomials $Y_1, Y_2, ..., Y_r$ as above by setting $\sigma(Y_1 Y_2 ... Y_r) = \sigma(Y_1) \sigma(Y_2) ... \sigma(Y_r)$ and finally we extend it to an integral combinations $t_1 Y_1 + t_2 Y_2 + ... + t_m Y_m$ of pseudo-symplectic monomials $Y_1, Y_2, ..., Y_m$ by setting $\sigma(\sum_{i=1}^m t_i Y_i) = \sum_{i=1}^m t_i \sigma(Y_i)$. Thus, $\sigma(g) = \sum_{i=1}^m t_i \sigma(Y_i)$ and

$$g(\bar{v}^+) - \sigma(g)(\bar{v}^+) \in 2 \cdot \mathbb{A}_Z^p(v^+).$$

(15)

Turning back to $F$, since by assumption $g(\bar{v}^+) = 0$ in $W_\perp^p(F)$ we also have $\sigma(g)(\bar{v}^+) = 0$ in $\bar{W}_k^p(F)$. However, $\bar{W}_k^p(F) \cong V_\perp(\lambda^p_\perp)$. Hence $\sigma(g)(u_\perp^p) = 0$ in $V_\perp(\lambda^p_\perp)$, where $u_\perp^p$ stands for the highest weight vector of $V_\perp(\lambda^p_\perp)$ corresponding to $\bar{v}^+$ in the isomorphism from $V_\perp(\lambda^p_\perp)$ to $\bar{W}_k^p(F)$. This implies that $\sigma(g)(u_\perp^p) \in$...
$2 \cdot V_2(\lambda_k^{\mathrm{sp}})$. Whence $\sigma(g)(\bar{v}_c^+) \in 2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+)$. However, for every element $f \in \mathfrak{A}_\mathbb{C}^p$ there is an integral combination $h$ of pseudo-symplectic monomial elements of type $(\bar{\cdot})$ of $\mathfrak{A}_\mathbb{C}^p$ such that $\sigma(h) = f$. We have $h(\bar{v}_c^+) - f(\bar{v}_c^+) \in 2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+)$ by (14). Therefore $2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+) \subseteq 2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+)$. It follows that

$$\sigma(g)(\bar{v}_c^+) \in 2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+).$$

(16)

By comparing (15) with (16) we obtain that

$$g(\bar{v}_c^+) \in 2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+).$$

(17)

Claim (17) holds in $W_k^\circ(\mathbb{C})$. However $W_k^\circ(\mathbb{C}) \cong V_\mathbb{C}^\circ(\lambda_k^\circ)$. Therefore

$$g(\bar{v}_c^+) \in 2 \cdot \mathfrak{A}_\mathbb{C}^p(\bar{v}_c^+) = 2 \cdot V_2(\lambda_k^\circ).$$

(18)

Namely $g(\bar{v}_c^+) \equiv 0$. Claim (18) implies that $g(\bar{v}_c^+) = 0$, as we wished to prove.

\[\square\]

**Corollary 3.12** We have $K_k \subseteq V_1(\mathbb{F})$.

**Proof.** This immediately follows from Lemma 3.11

Regarded $\mathfrak{A}_\mathbb{F}^p$ as an $\mathbb{F}$-vector space, let $\mathfrak{A}_\mathbb{F}^p$ be its subspace spanned by the set of pseudo-symplectic monomials of $(\bar{\cdot})$-type. Then $\mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+)$ is a subspace (but not a submodule) of $W_k^\circ(\mathbb{F})$. The following is implicit in the proof of Lemma 3.11

**Proposition 3.13** We have $W_k^\circ(\mathbb{F}) = \mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+) \oplus \mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+)$ (direct sum of vector spaces rather than submodules, since $\mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+)$ is a subspace but not a submodule of $W_k^\circ(\mathbb{F})$). Moreover $\mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+) = \mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+) = W_k^\circ(\mathbb{F})$ (which, as noticed above, is not a submodule of $W_k^\circ(\mathbb{F})$) and $\mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+) = \mathfrak{A}_\mathbb{F}^p(\bar{v}_c^+)$ (which in fact is a submodule of $W_k^\circ(\mathbb{F})$).

**Corollary 3.14** We have $\dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1}$.

**Proof.** Since $\dim(W_k^\circ(\mathbb{F})) = \binom{2n}{k} - \binom{2n}{k-2}$, Proposition 3.13 implies that $\mathfrak{A}_\mathbb{F}^p(\bar{v}_1^+)$ has codimension $\binom{2n}{k} - \binom{2n}{k-2}$ in $W_k^\circ(\mathbb{F})$. On the other hand, $K_k \subseteq V_1(\mathbb{F})$ by Corollary 3.12 and $V_1(\mathbb{F})/K_k \cong \mathfrak{A}_\mathbb{F}^p(\bar{v}_1^+)$. Hence $V_1(\mathbb{F})$ has codimension $\binom{2n}{k} - \binom{2n}{k-2}$ in $V_2(\lambda_k^\circ)$. Hence $\dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1}$, since $\dim(V_2(\lambda_k^\circ)) = \binom{2n+1}{k}$.

\[\square\]

The next corollary finishes the proof of Proposition 3.2

**Corollary 3.15** The homomorphism $\psi_{k-1,\mathbb{F}} : V_{\mathbb{F}}(\lambda_{k-1}^\circ) \to V_1(\mathbb{F})$ of Corollary 3.5 is indeed an isomorphism.
Proof. Clear from Corollary 3.14 since \( \dim(V_F(\lambda_{k-1}^o)) = \binom{2n+1}{k-1} \). \( \Box \)

Remark. We warn that \( \mathfrak{N}^o_F \) is not a subalgebra of \( \mathfrak{A}_k^o \). For instance, \([Y_{u_i-u_j}, \frac{Y_{u_i}^2}{2}] = Y_{u_i+u_j} - Y_{u_i}Y_{u_j} = -Y_{u_i+u_j} - Y_{u_i}Y_{u_j} \). Each of \( Y_{u_i-u_j}, \frac{Y_{u_i}^2}{2} \) and \( Y_{u_i+u_j} \) belongs to \( \mathfrak{N}_F \) but the sum \( Y_{u_i+u_j} - Y_{u_i}Y_{u_j} \) does not. Therefore \([Y_{u_i-u_j}, \frac{Y_{u_i}^2}{2}] \notin \mathfrak{N}_F \). In other words, only one of the products \( Y_{u_i-u_j} \frac{Y_{u_i}^2}{2} \) and \( \frac{Y_{u_i}^2}{2}Y_{u_i-u_j} \) belongs to \( \mathfrak{N}_F \), depending on the order chosen for the factors occurring in the pseudo-symplectic monomial elements.

On the other hand, \( Y_{u_i}Y_{u_j}(\bar{v}^+) = 0 \) by Corollary 3.19. Hence \([Y_{u_i-u_j}, \frac{Y_{u_i}^2}{2}])(\bar{v}^+) = -Y_{u_i+u_j}(\bar{v}^+) \). Thus, if for instance \( \mathfrak{N}_F \) contains the product \( Y_{u_i-u_j} \frac{Y_{u_i}^2}{2} \) (whence \( \frac{Y_{u_i}^2}{2}Y_{u_i-u_j} \notin \mathfrak{N}_F \)), then the space \( \mathfrak{N}_F(\bar{v}^+) \) also contains the vector \( \frac{Y_{u_i}^2}{2}Y_{u_i-u_j}(\bar{v}^+) \), but represented as a sum \( \frac{Y_{u_i}^2}{2}Y_{u_i-u_j}(\bar{v}^+) = Y_{u_i-u_j}(\bar{v}^+) + Y_{u_i+u_j}(\bar{v}^+) \).

3.2 End of the proof of Theorem 1.4

In order to finish the proof of Theorem 1.4 it remains to prove that \( V_1(\mathbb{F}) = \mathcal{N}_k \) (notation as in Section 1) and that the isomorphism \( \psi_{k-1,k} : V_F(\lambda_{k-1}^o) \rightarrow V_1(\mathbb{F}) \) maps the subspace \( \mathcal{N}_k-1 \) of \( V_F(\lambda_{k-1}^o) \) onto \( K_k \).

Proposition 3.16 We have \( V_1(\mathbb{F}) = \mathcal{N}_k \).

Proof. As \( K_k \subseteq V_1(\mathbb{F}) \), in order to prove that \( V_1(\mathbb{F}) = \mathcal{N}_k \) we only must prove that \( V_1(\mathbb{F})/K_k = \mathcal{N}_k/K_k \), namely the homomorphism \( \varphi_{k,k} : V_F(\lambda_k^o) \rightarrow W_k^o(\mathbb{F}) \) maps \( V_1(\mathbb{F}) \) onto the \( k \)-nucleus subspace \( \mathcal{N}_k \) of \( W_k^o(\mathbb{F}) \).

We recall that \( \mathcal{N}_k = \langle \epsilon_k(N_X) \rangle \chi \in \Delta_{k-1} \), where \( \Delta_{k-1} \) is the set of totally singular \( (k-1) \)-subspaces of \( V \), \( N_0 \) is the nucleus of the quadric described by \( \eta \) and \( N_X = \langle X, N_0 \rangle \) for \( X \in \Delta_{k-1} \).

According to the conventions stated at the beginning of Subsection 2.2 we have \( N_0 = \langle e_{2n+1} \rangle \). Put \( X_1 := \langle e_1, e_2, \ldots, e_{k-1} \rangle \). Then \( X_1 \in \Delta_{k-1} \) and \( N_{X_1} = \langle e_1, \ldots, e_{k-1}, e_{2n+1} \rangle \). Accordingly, \( \epsilon_k(N_X) = \langle e_1 \wedge \ldots \wedge e_{k-1} \wedge e_{2n+1} \rangle = \langle e_1^1 \rangle \). Therefore \( \varphi_{k,k}(V_1(\mathbb{F})) \) contains \( \epsilon_k(N_X) \). As \( \varphi_{k,k}(\mathbb{F}) \) is a \( G \)-module and \( G \) acts transitively on \( \Delta_{k-1} \), the module \( \varphi_{k,k}(V_1(\mathbb{F})) \) contains \( \mathcal{N}_k \).

On the other hand, both \( \mathcal{N}_k \) and \( \varphi_{k,k}(V_1(\mathbb{F})) \) have codimension \( \binom{2n+1}{k} - \binom{2n+1}{k-1} \) in \( W_k^o(\mathbb{F}) \) (see Theorem 1.3 and Corollary 3.14). Hence \( \varphi_{k,k}(V_1(\mathbb{F})) = \mathcal{N}_k \). \( \Box \)

Remark. The argument used in the proof of Proposition 3.16 can be exploited to prove that \( \dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1} \) (Corollary 3.14) avoiding Proposition 3.13. Indeed by that argument we obtain that \( \varphi_{k,k}(V_1(\mathbb{F})) \supseteq \mathcal{N}_k \). Therefore \( \dim(V_1(\mathbb{F})) \geq \binom{2n+1}{k-1} \), since \( \mathcal{N}_k \) has codimension \( \binom{2n+1}{k} - \binom{2n+1}{k-1} \) in \( W_k^o(\mathbb{F}) \). On the other hand, \( \dim(V_1(\mathbb{F})) \leq \binom{2n+1}{k-1} \) as \( V_1(\mathbb{F}) \) is a homomorphic image of \( V_F(\lambda_{k-1}^o) \). Hence \( \dim(V_1(\mathbb{F})) = \binom{2n+1}{k-1} \).
Proof. Put $v_1^+ := Y_{k-1}v_1^+$ and $V_2(F) := \mathfrak{K}(v_2^+)$. We must prove that $V_2(F) = K_k$. We have $Y_{k-1}(\check{v}_1^+) = Y_{k-1}V_k(\check{v}_2^+)$ and $0$ (by Corollary 3.9). Therefore $V_2(F) \subseteq K_k$. The pre-images $\psi_{k,1}(V_2(F))$ and $\psi_{k,1}(v_1^+)$ of $V_2(F)$ and $v_1^+$ are the analogues of $V_1(F)$ and $v_1^+$ respectively, but in $V_2(\lambda^o_{k-1})$ instead of $V_1(\lambda^o_{k-1})$. So, we can apply Proposition 5.2 to them, obtaining that $\psi_{k-1,1}(V_2(F)) \cong V_2(\lambda^o_{k-2})$, whence $V_2(F) \cong V_2(\lambda^o_{k-2})$. It follows that $\dim(V_2(F)) = \binom{2n+1}{k-2}$, hence $V_2(F) = K_k$, since $V_2(F) \subseteq K_k$. 

The proof of Theorem \ref{thm14} is complete.

4 Uniqueness conjectures

So far we have proved that $V_2(\lambda^o_k)$ admits a series of submodules

$$0 \subset M_0 \subset M_1 \subset ... \subset M_{k-1} \subset M_k = V_2(\lambda^o_k)$$

where

$$M_i = Y_{u_{i+1}}Y_{u_{i+2}}...Y_{u_{k-1}}Y_{u_k} (v^+ \equiv V_2(\lambda^o_i), \quad (i = 0, 1, 2, ..., k - 1). \tag{19}$$

Moreover $M_i$ corresponds to the submodule $\mathcal{N}_{i+1}$ of $V_2(\lambda^o_{i+1})$ as well as to the kernel $K_{i+2}$ of the homomorphism $\varphi_{i+2,1} : V_2(\lambda^o_{i+2}) \rightarrow W_i^{o}(1) (i < k - 1)$. Avoiding any mention of the elements $Y_{u_{i+1}}...Y_{u_k} (v^+ \equiv V_2(\lambda^o_i))$, we can rephrase the above conditions as follows:

$$M_i \cong V_2(\lambda^o_i) \quad (i = 0, 1, 2, ..., k), \tag{20}$$

$$M_i/M_{i-1} \cong W_i^{o}(1) \quad (i = 1, 2, ..., k). \tag{21}$$

Recall that $V_2(\lambda^o_0)$, to be considered in (20) for $i = 0$, is 1-dimensional, by convention. We allow $i = 1$ in (21). This forces us to define $M_{-1}$ too. We put $M_{-1} := 0$.

When $F$ is perfect we also know that

$$M_i/M_{i-1} \cong W_i^{o}(1). \tag{22}$$

(Theorem \ref{thm17} and Corollary \ref{cor18}) The following conjecture is quite natural.

Conjecture 4.1 The series of submodules defined as in (19) is the unique series of $G$-submodules of $V_2(\lambda^o_k)$ satisfying conditions (20) and (21).
If the $G$-module $V_F(\lambda_k^\pm)$ was rigid, namely all of its automorphisms as a $G$-module fix $(v^\pm)$, then Conjecture 4.1 would be not so difficult to prove, but we do not know if $V_F(\lambda_k^\pm)$ is rigid (although it is certainly rigid as an $\mathfrak{g}_F$-module). Recall that all irreducible Weyl modules are rigid (Humphreys [7, 2.2]), but when char($F$) = 2, as we are assuming here, $V_F(\lambda_k^\pm)$ is not irreducible.

When $F$ is perfect we can consider the following variation of Conjecture 4.1.

**Conjecture 4.2** Let $F$ be perfect. Then the series of submodules defined as in (19) is the unique series of $G$-submodules of $V_F(\lambda_k^\pm)$ satisfying conditions (20) and (22).

We shall prove Conjecture 4.2 for $2 \leq k \leq 4$ (when $k = 1$ both Conjectures 4.1 and 4.2 are trivial). We will rely on the information on the lattice of submodules of $\mathfrak{w}_F^p(F)$ provided in Baranov and Suprunenko [1] (see also Premet and Suprunenko [8]). As the reader will see, in principle our method can be applied to check Conjecture 4.2 for every particular value of $k$, but apparently there is no way to make a general argument out of it.

Let $F$ be a perfect field of characteristic 2. Let $(M_i)_{i=0}^k$ be the series defined as in (19) and let $(M_i')_{i=0}^k$ be another series satisfying conditions (20) and (22).

1) Let $k = 2$ and suppose by contradiction that $M_1' \neq M_1$. Then either $M_1 \cap M_1' = M_0$ or $M_1 \cap M_1' = 0$. Hence $(M_1 + M_1')/M_1$ occurs in $M_2/M_1 \cong \mathfrak{w}_2^p(F)$ as a submodule isomorphic to either $W_1^p(F)$ or $W_2^p(F)$. But according to [1] and [8] no such submodules occur in $\mathfrak{w}_2^p(F)$.

Therefore $M_1' = M_1$. As $M_1' = M_1 \cong V_F(\lambda_1^\pm)$ admits a unique 1-dimensional submodule, we also have $M_0' = M_0$.

2) Let $k = 3$. According to [1] and [8], if $n$ is odd then the module $M_2/M_1 \cong \mathfrak{w}_2^p(F)$ is irreducible while if $n$ is even then it admits a unique proper submodule $M_{2,1}$, of dimension 1. In this case let $M_{2,1} \supseteq M_1$ be such that $M_{2,1}/M_1 = M_{2,1}$. As for $M_1 \cong W_1^p(F)$, we know that $M_0$ is its unique proper submodule. The module $W_2^p(F)$ does not admit any 1-dimensional submodule, as one can check by direct computations. Hence $M_{2,1}/M_0$ does not split as a direct sum of $M_2/M_0$ and a 1-dimensional module.

It follows from the above that if $M_2' \neq M_2$ the following are the only possibilities for $M_2 \cap M_2'$:

$$
\begin{array}{c|c|c|c}
M_2 \cap M_2' & \dim(M_2 \cap M_2') & \dim((M_2 + M_2')/M_2) \\
\hline
0 & 0 & (2n+1)/2 \\
M_0 & 1 & (2n+1)/2 - 1 \\
M_1 & 2n+1 & (2n+1)/2 - 2n - 1 \\
M_{2,1} & 2n+2 & (2n+1)/2 - 2n - 2 & \text{(only if } n \text{ is even)} \\
\end{array}
$$

On the other hand, according to [1] and [8], if $n$ is even then the module $\mathfrak{w}_3^p(F)$ is irreducible while if $n$ is odd then it admits a unique proper submodule, isomorphic to $\mathfrak{w}_3^p(F)$. Therefore $(2n) - 2n$ and $2n$ are the only possibilities for
dim((M_2 + M'_2)/M_2). Neither of the numbers listed in the second column of the above table is equal to any of these two numbers, except for \( \binom{2n+1}{2} - 2n - 1 \), which is equal to \( \binom{2n-3}{2} - 2n \) for \( n = 3 \).

So, we are left with the case of \( n = 3 \), \( M'_2 \cap M_2 = M_1 \) and \( M'_2 + M_2 = M_3 = V_2(\lambda_3) \). However \( W_2^p(F) = V_2(\lambda_3)/K_3 = M_3/M_1 \). Whence \( W_2^p(F) \) now splits as the direct sum of two copies \( M_2/M_1 \) and \( M'_2/M_1 \) of \( W_{2,3}^p(F) \). (Note that, in view of the analysis made in the case \( k = 2 \), \( M_1 = M'_1 \) is the unique submodule of \( M'_2 \) of dimension \( 2n + 1 = 7 \).) Accordingly, the stabilizer in \( G \) of the 1-dimensional subspace spanned by \( \bar{v}_1 = Y_{v_3} \bar{v}^+ \in M_2/M_1 \) in \( G \) also stabilizes a 1-dimensional subspace of \( M'_2/M_1 \). It is not difficult to see by direct computations that this is not the case.

It follows that \( M'_2 = M_2 \). The equalities \( M'_1 = M_1 \) and \( M'_0 = M_0 \) now follow by the analysis of case \( k = 2 \) applied to \( M_2 = V_2(\lambda_2) \).

3) Let \( k = 4 \). As seen in the previous cases, if \( n \) is odd then \( M_0 \) and \( M_1 \) are the unique non-zero proper submodules of \( M_2 \) while if \( n \) is even then \( M_2 \) also admits a submodule \( M_{2,1} \supset M_1 \) with \( \dim(M_{2,1}/M_1) = 1 \). We have also seen that \( M_3/M_2 \) is irreducible if \( n \) is even while if \( n \) is odd then \( M_3/M_2 \) admits a unique proper submodule \( M_{3,1} \), isomorphic to \( W_{1,3}^p(F) \). Let \( M_{3,1} \supset M_2 \) be such that \( M_{3,1}/M_2 = M_{3,1} \). By considering the action of \( G \) on \( M_3/M_1 = W_3^p(F) \) it is not difficult to see that \( M_{3,1}/M_1 \) cannot split as a direct sum \( M_{3,1}/M_1 = M_2/M_1 \oplus M_{3,1} \) of \( G \)-submodules. Therefore the following are the unique non-zero proper \( G \)-submodules of \( M_3 \):

\[
\begin{array}{ll}
\text{submodule} & \text{dimension} \\
M_0 & 1 \\
M_1 & 2n + 1 \\
M_{2,1} & 2n + 2 \\
M_2 & \binom{2n+1}{3} \\
M_3 & \binom{2n+1}{3}
\end{array}
\quad
\begin{array}{ll}
\text{submodule} & \text{dimension} \\
M_0 & 1 \\
M_1 & 2n + 1 \\
M_2 & \binom{2n+1}{2} \\
M_{3,1} & \binom{2n+1}{3} + 2n \\
M_3 & \binom{2n+1}{3}
\end{array}
\]

Accordingly, the following are the only possibilities for \( M_3 \cap M'_3 \) if \( M'_3 \neq M_3 \):

\[
\begin{array}{ll}
M_3 \cap M'_3 & \dim((M_3 + M'_3)/M_3) \\
0 & \binom{2n+1}{3} \\
M_0 & \binom{2n+1}{3} - 1 \\
M_1 & \binom{2n+1}{3} - 2n - 1 \\
M_{2,1} & \binom{2n+1}{3} - 2n - 2 \\
M_2 & \binom{2n+1}{3} - \binom{2n+1}{2} \\
M_{3,1} & \binom{2n+1}{3} - \binom{2n+1}{2} - 2n
\end{array}
\]

(only if \( n \) is even)

According to [11] and [8], if \( n \equiv 3 \pmod{4} \) then the module \( W_{4}^p(F) \) is irreducible while if \( n \equiv 1 \pmod{4} \) then it admits a unique non-zero proper submodule, which is 1-dimensional. If \( n \equiv 0 \pmod{4} \) then \( W_{4}^p(F) \) admits a unique non-zero proper
submodule, which has dimension equal to \((\frac{2n}{2}) - 2\) and is isomorphic to the unique irreducible quotient of \(W_{2p}^p(F)\). Finally, if \(n \equiv 2 \pmod{4}\) then \(W_{2p}^p(F)\) admits just two proper submodules, namely either \(W_{2p}^p(F)\) and its unique 1-dimensional submodule or the dual of \(W_{2p}^p(F)\) and its unique submodule of codimension 1. Therefore the following are the only possibilities for \(\dim((M_3 + M'_3)/M_3)\):

| Expression | Condition |
|------------|-----------|
| \(1\)     | \(n \equiv 1 \pmod{4}\) or \(n \equiv 2 \pmod{4}\) |
| \((\frac{2n}{2}) - 2\) | only if \(n\) is even |
| \((\frac{2n}{2}) - 1\) | only if \(n \equiv 2 \pmod{4}\) |
| \((\frac{2n}{4})\) | |

Comparing these values with the feasible dimensions previously listed for \((M_3 + M'_3)/M_3\) we see that \(M_3 \cap M'_3 = 0\) with \(n = 5\) is the unique possibility. In this case \((M_3 + M'_3)/M_3 = M_4/M_3\). This implies that \(M_4/M_2 \cong W^4_2(F)\) splits as the direct sum of two \(G\)-submodules, one of which is \(M_3/M_2 \cong W^4_2(F)\) and the other one is \((M'_3 + M_2)/M_2 \cong M'_3 \cong V_F(\lambda^3_3)\). By an argument similar to the one used in a similar situation in case \(k = 3\), one can see that \(W^4_2(F)\) does not admit such a splitting.

Therefore \(M'_4 = M_3\). We can now apply to \(M_3 \cong V_F(\lambda^3_3)\) the analysis made in case \(k = 3\), obtaining that \(M'_2 = M_2\), \(M'_1 = M_1\) and \(M'_0 = M_0\).

**Remark.** An analysis as above has also been done in \([5]\), but only for \(k \leq n \leq 3\).

**References**

[1] A. A. Baranov and I. D. Suprunenko. Brancing rules for modular fundamental representations of symplectic groups. Bull. L. M. S. 32 (2000), 409-420.

[2] R. J. Blok. Highest weight modules and polarized embeddings of shadow spaces. J. Alg. Combin. 34, 1, (2011), 67–113.

[3] R. J. Blok and B. N. Cooperstein. Projective subgrassmannians of polar Grassmannian. Bull. Belg. Math. Soc. Simon Stevin 17 (2010), no. 4, 675–691.

[4] I. Cardinali and A. Pasini. Grassmann and Weyl embeddings of orthogonal grassmannians. J. Alg. Combin. To appear.

[5] I. Cardinali and A. Pasini. Veronesean embeddings of dual polar spaces of orthogonal type. J. Combin. Theory Ser. A. 120 (2013), 1328-1350.

[6] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, New York, 1972.

[7] J. E. Humphreys. *Modular Representations of Finite Groups of Lie Type*. Cambridge Univ. Press, Cambridge, 2006.
[8] A. A. Premet and I. D. Suprunenko. The Weyl modules and the irreducible representations of the symplectic group with the fundamental highest weights. *Comm. Algebra* **11** (1983), 1309-1342.

[9] R. Steinberg. *Lectures on Chevalley Groups*. Notes prepared by J. Faulkner and R. Wilson, Yale University, 1967.

[10] J. Tits. *Buildings of Spherical Type and Finite BN-pairs*. Lect. Notes in Math. **386**, Springer, Berlin, 1974.

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