ASYMPTOTICS OF THE WEIL–PETERSSON METRIC

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Abstract. We consider the Riemann moduli space \( M_\gamma \) of conformal structures on a compact surface of genus \( \gamma > 1 \) together with its Weil-Petersson metric \( g_{\text{WP}} \). Our main result is that \( g_{\text{WP}} \) admits a complete polyhomogeneous expansion in powers of the lengths of the short geodesics up to the singular divisors of the Deligne-Mumford compactification of \( M_\gamma \).

1. Introduction

The Riemann moduli space \( M_\gamma \) of conformal structures on a compact surface of genus \( \gamma > 1 \) is an object of key importance in several branches of mathematics and mathematical physics. Part of its fascination is that it is endowed with numerous natural geometric structures. We focus here on one of these, the Weil-Petersson metric \( g_{\text{WP}} \). We recall that \( (M_\gamma, g_{\text{WP}}) \) is an incomplete Riemannian space, and a quasi-projective variety of (complex) dimension \( 3\gamma - 3 \). Its Deligne-Mumford compactification contains a collection of immersed divisors \( D_1 \cup \ldots \cup D_N, N = 2\gamma - 2 \), which meet with simple normal crossings, and along which \( \overline{M}_\gamma \) has orbifold singularities, cf. [9]. We denote by \( \mathbb{D} \) this entire divisor, i.e., the union of the \( D_j \). The Weil-Petersson metric is not fully compatible with this compactification in the sense that the local asymptotic behaviour of \( g_{\text{WP}} \) near these divisors is somewhat complicated: normal to each divisor it has cusp-like behavior, but at intersections of the divisors, these normal cusps do not interact. Our goal in this paper is to sharpen the work of Masur [13], Yamada [24] and Wolpert [21, 22], and in a slightly different direction, Liu-Sun-Yau [11, 12], each of whom provided successively finer estimates. This work also refines Wolpert’s very recent paper [23], which proves a certain uniformity of derivatives for this metric. We prove here that \( g_{\text{WP}} \) has a complete polyhomogeneous asymptotic expansion at \( \mathbb{D} \), with product type expansions at the intersections of the \( D_j \), see Theorem 1 below for the precise statement. In general terms, a polyhomogeneous expansion is an asymptotic series, which remains valid even after differentiation, but which may involve possibly fractional powers of the distance function to the boundary. We obtain, in fact, that the metric coefficients are essentially ‘log-smooth’, so in other words, involve only nonnegative integer powers of a natural boundary defining function \( \rho \), where

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each term $\rho^k$ is multiplied by a polynomial in $\log \rho$. (In fact, $\rho$ is the square root of the length of the degenerating geodesic.) As we explain later, this is the sharpest type of regularity one might hope to obtain, and in particular provides more information than the ‘stable regularity’ estimates of [23]. What makes this somewhat different than analogous regularity results, cf. [14], [7], [5], is that $g_{WP}$ does not satisfy an elliptic equation, but instead is the induced $L^2$ metric in the gauge-theoretic construction of $\mathcal{M}_\gamma$ – or in other words, is the restriction of the $L^2$ metric on the space of all symmetric 2-tensors to the finite dimensional subspace of transverse-traceless tensors.

Our work leaves open the precise identification of the terms in the expansion of $g_{WP}$. That is a very different sort of task, but one which becomes possible only once one has established that an expansion exists at all! The first few terms are computed in [18]. The result here is also consistent with (and relies on!) the recent paper of Melrose-Zhu [17], who obtain a similar type of expansion for the family of hyperbolic metrics on a degenerating hyperbolic surface; indeed, their result is one of the key ingredients here.

The broader context of this paper is that the necessity of determining higher asymptotics of the Weil-Petersson metric became apparent in the work that led up to [8]. The goal enunciated there is to study the natural elliptic operators on $(\mathcal{M}_\gamma, g_{WP})$, for example, the Hodge Laplacian, twisted Dirac operators, etc. Because of the singularities of $\mathcal{M}_\gamma$, the first step in any such study is to come to terms with the effect on these operators of the singular structure of the metric along the divisors, and in particular to determine whether these make it necessary to introduce new boundary conditions. It was shown in [8] that such boundary conditions are unnecessary for the scalar Laplacian, i.e., the scalar Laplacian is essentially self-adjoint. In current work by the second author here and Gell-Redman, it is proved that the Hodge Laplacian on differential forms is also essentially self-adjoint; in other words, the natural action of these operators on $C^\infty_0(\mathcal{M}_\gamma)$ has a unique self-adjoint extension in $L^2$. From these results one can go on to develop the spectral geometry and index theory for $(\mathcal{M}_\gamma, g_{WP})$, and indeed this is an area of ongoing investigation by the authors, Gell-Redman and others. One particularly interesting goal is to prove a signature theorem relative to the Weil-Petersson metric; this would be a counterpart to the Gauss-Bonnet theorem of [10].

Numerous people have been very helpful in teaching us about the Riemann moduli space, and in discussing various parts of the geometry below. We mention in particular Dan Freed, Lizhen Ji, Maryam Mirzakhani, András Vasy, Mike Wolf, Sumio Yamada, and in particular Scott Wolpert. We also thank Richard Melrose and Xuwen Zhu; their paper [17] appeared in the later stages of this research and clarified one part of our analysis substantially. Our paper was initiated during a several month visit to the Stanford Math Department by the second author, funded by DFG through grant Sw 161/1-1. R.M. was partially funded by the NSF Grant DMS-1105050. J.S.
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2. Preliminaries on the Riemann moduli space and \( g_{WP} \)

In this section we recall a number of well-known facts about the Riemann moduli space. We begin with the more topological aspects, following the monograph by Farb and Margalit [6].

Let \( S \) be the model surface, i.e. an oriented, closed smooth surface of genus \( \gamma \geq 2 \). The Teichmüller space \( T_\gamma \) of surfaces of genus \( \gamma \) is the set of equivalence classes \( [\Sigma, \varphi] \), where \( \Sigma \) is a Riemann surface (the Riemannian metric, or conformal or complex structure, is suppressed from the notation), \( \varphi: S \to \Sigma \) is a diffeomorphism, called a marking, and

\[
(\Sigma_1, \varphi_1) \sim (\Sigma_2, \varphi_2)
\]

if there is an isometry \( I: \Sigma_1 \to \Sigma_2 \) such that the maps \( I \) and \( \varphi_2 \circ \varphi_1^{-1} \) are isotopic. The set \( T_\gamma \) is in bijective correspondence with the representation variety \( \text{DF}(\pi_1(S), \text{PSL}(2, \mathbb{R}))/\text{PSL}(2, \mathbb{R}) \), i.e., the space of conjugacy classes of discrete faithful representations of the fundamental group \( \pi_1(S) \) into \( \text{PSL}(2, \mathbb{R}) \). The latter space carries a natural topology, induced by the compact-open topology of \( \text{Hom}(\pi_1(S), \text{PSL}(2, \mathbb{R})) \), in terms of which \( T_\gamma \) is Hausdorff.

There is a geometrically defined atlas of charts which make \( T_\gamma \) a smooth manifold of dimension \( 6\gamma - 6 \). To describe this, fix a maximal set of pairwise disjoint, oriented simple closed curves \( \{c_1, \ldots, c_{3\gamma-3}\} \) on \( S \). These decompose \( S \) into a collection \( \{P_1, \ldots, P_{2\gamma-2}\} \) of pairs of pants, i.e., spheres with three open disks removed. A hyperbolic metric on each \( P_j \) is determined up to isometry by specifying an unordered triple of positive numbers, corresponding to the lengths of the three boundary curves; these boundary curves are then geodesics for this hyperbolic metric. We can then attach pairs of pants to one another along a common boundary component of the same length. There is a twist parameter \( \omega \) when we attach any two boundary curves which takes values in \( \mathbb{R} \). Using all of this, one shows that an element of \( T_\gamma \) is determined by the pair of \((3\gamma-3)\)-tuples

\[
(\ell_1, \ldots, \ell_{3\gamma-3}) = (\ell_\Sigma(c_1), \ldots, \ell_\Sigma(c_{3\gamma-3})) \in \mathbb{R}_{+}^{3\gamma-3},
\]

and \( (\omega_1, \ldots, \omega_{3\gamma-3}) \in \mathbb{R}^{3\gamma-3} \).

By a result of Fricke [6, Theorem 9.5], the map which assigns to a point \([\Sigma, \varphi] \in T_\gamma\) its Fenchel-Nielsen coordinates \((\ell_1, \ldots, \ell_{3\gamma-3}, \omega_1, \ldots, \omega_{3\gamma-3})\) is a homeomorphism to \( \mathbb{R}_{+}^{3\gamma-3} \times \mathbb{R}^{3\gamma-3} \cong \mathbb{R}^{6\gamma-6} \). Fenchel-Nielsen coordinates provide global coordinates for \( T_\gamma \) which depend on the chosen collection of curves.

We are interested here in the Riemann moduli space, which is the quotient of \( T_\gamma \) by the action of the mapping class group \( \text{Map}(S) \) of the surface \( S \). This is the infinite discrete group of isotopy classes of orientation preserving
diffeomorphisms of $S$. It is isomorphic to the group of outer automorphisms of $\pi_1(S)$. The action of $\text{Map}(S)$ on $T_\gamma$ is given by

$$f : [(\Sigma, \varphi)] = [(\Sigma, \varphi \circ \psi^{-1})],$$

where $\psi : S \to S$ is any diffeomorphism representing $f$. We define the Riemann moduli space $M_\gamma$ as the quotient $T_\gamma / \text{Map}(S)$. This action is properly discontinuous [6, Sect. 11.3], hence $M_\gamma$ is an orbifold. Away from orbifold singularities, Fenchel-Nielsen parameters provide a local coordinate system on $M_\gamma$.

The space $M_\gamma$ is not compact. Indeed, letting any one length parameter $\ell_j$ tend to zero gives a sequence of points in $M_\gamma$ which leaves every compact set. The converse to this statement is provided by

**Mumford's compactness criterion** For any $\varepsilon > 0$, define the $\varepsilon$-thick part of $M_\gamma$

$$M_\gamma^\varepsilon = \{(\Sigma, \varphi) \in M_\gamma \mid \min_{1 \leq j \leq 3\gamma-3} \ell_\Sigma(c_j) \geq \varepsilon\};$$

Then for every $\varepsilon > 0$, $M_\gamma^\varepsilon$ is compact.

If $\varepsilon$ is sufficiently small, $M_\gamma \setminus M_\gamma^\varepsilon$ is connected, so $M_\gamma$ has precisely one end.

For the purposes of this paper, we recall an alternate approach to defining $T_\gamma$ and $M_\gamma$, following [19], which leads directly to the Weil-Petersson metric. Regarding $\Sigma$ as a smooth surface, consider the space $C^\infty(\Sigma; \text{Sym}^2 T^*\Sigma)$ of symmetric 2-tensors on $\Sigma$, and the open subset $C^\infty(\Sigma; \text{Sym}^2_+ T^*\Sigma)$ of sections which are everywhere positive definite. A Riemannian metric $g$ on $\Sigma$ is an element of this latter space, and we write $\text{Met}$ (or $\text{Met}(\Sigma)$) the space of all such metrics. The group of orientation preserving diffeomorphisms $\text{Diff}(\Sigma)$ acts on $\text{Met}$ by pullback. The subgroup $\text{Diff}_0(\Sigma)$, consisting of all orientation preserving diffeomorphisms isotopic to the identity, is the connected component of the identity in $\text{Diff}(\Sigma)$, and $\text{Diff}(\Sigma)/\text{Diff}_0(\Sigma) = \text{Map}(\Sigma)$. Next, to each Riemannian metric we associate its Gauss curvature function $K_g \in C^\infty(\Sigma)$. The space $\text{Met}^{-1}$ is the space of metrics with $K_g \equiv -1$ (this is nonempty because $\gamma \geq 2$). There is a smooth action of $\text{Diff}(\Sigma)$ on $\text{Met}(\Sigma)$, and the alternate characterizations of the Teichmüller and Riemann moduli spaces are as the quotients

$$T_\gamma = \text{Met}^{-1}(\Sigma)/\text{Diff}_0(\Sigma), \quad \text{and} \quad M_\gamma = \text{Met}^{-1}(\Sigma)/\text{Diff}(\Sigma).$$

Points in either of these space are denoted as equivalence classes $[g]$. In practice, one must actually introduce a finite-regularity topology on all of these objects; we have used smooth metrics and diffeomorphisms for simplicity of statement since these quotients are independent of which Banach topology we use. We discuss this more carefully in [19] below.

The space $\text{Met}(\Sigma)$ carries a natural $L^2$ (or Ebin) metric, defined as follows. Elements of $T_\gamma \text{Met}$ are sections of $C^\infty(\Sigma; \text{Sym}^2(\Sigma))$ (with no positivity
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conditions), and then, for \( h_1, h_2 \in T_g \text{Met} \),

\[
\langle h_1, h_2 \rangle_{L^2} = \int_{\Sigma} \langle h_1, h_2 \rangle_g dA_g.
\]

This metric restricts to \( T_g \text{Met}^{-1} \). Now, if \( g \in \text{Met}^{-1} \), and if we consider the local diffeomorphism orbit through \( g \), \( O_g := \{ F^*g : F \in \text{Diff}(\Sigma), F \approx \text{id} \} \), then

\[ T_g O_g = \{ L_X g : X \in C^\infty(\Sigma; T\Sigma) \}. \]

The orthogonal complement of this tangent space with respect to \( \langle , \rangle_{L^2} \) is the space

\[
S_{tt}(g) = \{ \kappa \in C^\infty(\Sigma; \text{Sym}^2(\Sigma)) : \delta_g \kappa = 0, \ tr^g \kappa = 0 \}.
\]

Its elements are the so-called transverse-traceless tensors; these are naturally identified with holomorphic quadratic differentials on \((\Sigma, g)\) (i.e., holomorphic with respect to the complex structure compatible with \( g \)), and \( \dim S_{tt}(g) = 6\gamma - 6 \). An open neighborhood of 0 in \( S_{tt}(g) \) provides a local chart in (a finite orbifold cover) of \([g]\) in either \( T_\gamma \) or \( M_\gamma \), see [19], [15]. However, more simply, there is a canonical identification

\[ \text{for any } \text{Met}^{-1} \ni g \in [g]. \]

We may finally define the Weil-Petersson metric on either of these spaces by declaring that at any point \([g]\),

\[
g_{WP}\big|_{[g]}(\kappa_1, \kappa_2) = \langle \kappa_1, \kappa_2 \rangle_{L^2(\Sigma,g)},
\]

where the metric on the right is calculated on the Riemannian surface \((\Sigma, g)\).

The Weil-Petersson metric is Kähler and everywhere negatively curved. Furthermore, the Riemann moduli space has finite diameter with respect to \( g_{WP} \), and it is a remarkable fact (closely related to Mumford’s compactness criterion) that the corresponding metric completion is naturally identified with the algebro-geometric Deligne-Mumford compactification:

\[ \overline{M}_\gamma^{WP} \cong M_\gamma^{DM}. \]

This is the starting point for our work here. We are interested in the precise behavior of \( g_{WP} \) on approach to the points of \( \overline{M}_\gamma \). We now record the previously known results about \( g_{WP} \) in Fenchel-Nielsen coordinates near the divisors. Any point \([g_0]\) in a \( k \)-fold intersection \( D_J = D_{i_1} \cap \ldots \cap D_{i_k} \) defines a complete hyperbolic metric on a noded surface; the nearby points \([g] \in M_\gamma \) are hyperbolic surfaces on the compact surface \( \Sigma \) where the corresponding lengths \( \ell(c_{j_i}) \), \( i = 1, \ldots, k \), are all small. We are assuming for simplicity that the \( D_i \) are all distinct here, i.e., \([g]\) does not lie on a double-point of one of the immersed divisors; furthermore, we also relabel indices so that \( j_i = i \), \( i \leq k \). We now quote the result we intend to refine: in Fenchel-Nielsen coordinates near \( D_1 \cap \ldots \cap D_k \),

\[
g_{WP} = \pi \sum_{j=1}^{k} \left( \frac{d\ell_j^2}{\ell_j} + \frac{\ell_j^3}{4\pi^4} d\omega_j^2 \right) + g_D + \eta,
\]
where $g_D$ is the Weil-Petersson metric on the noded surface corresponding to that intersection of divisors, and $\eta$ is a lower order error term. The original version of this formula [13], by Masur, gave only a quasi-isometric equivalence between the two sides; substantial sharpenings were obtained by Wolpert [21] and even further by Yamada [24]. The net result of these papers, particularly the last, is that the remainder term $\eta$ contains no leading order terms, so that the $d\ell_j$ and $d\omega_j$ directions are orthogonal at $D_j$ in an appropriately rescaled sense, and that these length and twist directions corresponding to two different curves $c_i$ and $c_j$ are orthogonal to leading order at $D_i \cap D_j$. Later work by Liu, Sun and Yau [10] provided estimates of up to four derivatives of the metric coefficients; their motivation was to compute the curvature of the so-called Ricci metric, $g_{\text{Ric}} = -\text{Ric}(g_{\text{WP}})$. Quite recently, Wolpert has proved in [23] a certain uniform differentiability of the metric coefficients; in the language introduced below, this is equivalent to the conormality of the metric.

In much of this paper we shall work in local coordinates in $M_\gamma$ near a point $[g] \in D_1 \cap \ldots \cap D_k$, which we write as $w = (\ell_1, \omega_1, \ell_2, \omega_2, \ldots, \ell_k, \omega_k, y)$. Here each $(\ell_j, \omega_j)$ is the Fenchel-Nielsen length and twist pair and $y$ is any choice of smooth local coordinate on $D_1 \cap \ldots \cap D_k$. Thus $w_{2j-1} = \ell_j$ and $w_{2j} = \omega_j$, $j = 1, \ldots, k$ and $w_i = y_i$, $i > 2k$. We are implicitly using that $M_\gamma$ has a natural smooth (in fact, analytic) structure near $[g]$.

We can now state our main theorem more precisely:

**Theorem 1.** Writing $g_{\text{WP}}$ as in (4), then every coefficient $\eta_{pq}$ has a complete asymptotic expansion

$$\eta_{pq} \sim \sum_\alpha \sum_{j=0}^{N_\alpha} \ell_\alpha^{\alpha/2} (\log \ell)^j \eta_{\alpha,j}(\omega, y).$$

Each $\alpha$ and $j$ is a multi-index of nonnegative integers, so $\ell_\alpha^{\alpha/2} = \ell_1^{\alpha_1/2} \ldots \ell_k^{\alpha_k/2}$, $(\log \ell)^j = (\log \ell_1)^{j_1} \ldots (\log \ell_k)^{j_k}$, and furthermore each $\eta_{\alpha,j} \in C^\infty$.

**Remark 2.** As we describe carefully below, this result relies on the polynomiality of a certain uniformizing conformal factor. This has recently been proved by Melrose and Zhu [17] on $D^\text{reg}$, i.e., away from the intersections of the divisors. They anticipate that the extension of their result to all of $D$ should be true and will follow by an elaboration of their same technique. Since that result has not appeared yet, our theorem above is only claimed at present in that region of the compactified moduli space. However, all other parts of the argument here are valid even near $D^\text{sing}$.

Let us give a very brief sketch of the proof. We first introduce the universal family $\mathcal{M}_{\gamma,1}$, which is the space of Riemann surfaces with one marked point. This marked point traces out a copy of $\Sigma$ over each point $[g] \in \mathcal{M}_\gamma$, and there is a forgetful map

$$\Pi : \mathcal{M}_{\gamma,1} \to \mathcal{M}_\gamma;$$

(5)
the fibre $\Pi^{-1}([g])$ is a copy of that Riemann surface, i.e., of $\Sigma$ with an equivalence class of hyperbolic metrics. This is an orbifold fibration: up to finite covers of the domain and range, $\Pi$ is a fibration. Since these finite covers are irrelevant in our analysis, we often refer to (5) as a fibration, with this understanding. There is a corresponding map of Deligne-Mumford compactifications

$$\Pi : \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_{g},$$

which (again up to finite covers) is a singular fibration; in other words, it is a standard fibration away from the divisors in the base, while if $[g_0] \in \overline{\mathcal{M}}_{g} \setminus \mathcal{M}_{g}$, then $\Pi^{-1}([g_0])$ is a noded surface obtained by inserting a thrice-punctured (or thrice-marked) copy of $S^2$ between each of the nodes of the degenerated surface $\Sigma_{[g_0]}$.

The next step is to introduce a family of vertical hyperbolic metrics on $\overline{\mathcal{M}}_{g,1}$, or in other words a choice of representative hyperbolic metric $g$ on the fiber $\Sigma_{[g]}$ for each $[g] \in \mathcal{M}_{g}$. Thus when $[g]$ lies in the interior, $g$ is an element in $\text{Met}^{-1}$ representing $[g] \in \text{Met}^{-1}/\text{Diff}$, while if $[g] \in \overline{\mathcal{M}}_{g} \setminus \mathcal{M}_{g}$, then $g$ is a complete hyperbolic metric on the corresponding noded surface. There is probably not a completely natural choice of this representative, but the important feature here is that this choice depends smoothly on $[g]$ in the interior, and is polyhomogeneous (say in the Fenchel-Nielsen coordinates) on the compactification. In other words, in terms of one of the coordinate systems $w = (\ell, \omega, y)$ defined above, we seek a choice of representative $g$ depending smoothly on $w$ in the region where all $\ell_j > 0$ and which is polyhomogeneous as the $\ell_j \to 0$. This has an obvious meaning away from collection of curves on $\Sigma$ which are degenerating at these divisors, and we refer to the next section for a full description. This local choice of metric representative is a slice of the diffeomorphism action, and we refer to any slice with these regularity properties as a polyhomogeneous slice.

As we show later, it is in fact sufficient for most of the work below to work with a local polyhomogeneous slice where the individual metrics are approximately hyperbolic, namely they are required to be hyperbolic on a neighborhood of the degeneration locus but have variable curvature elsewhere on $\Sigma$. These are easy to construct. Only at the final step do we appeal to the existence of a family of fibre-wise conformal factors $\varphi_g$ which relate the approximately hyperbolic slice to an exactly hyperbolic one. The fact that these conformal factors depend in a polyhomogeneous way on $[g]$ at $D$ was proved recently by Melrose and Zhu [17]. As already noted in Remark 2 we record the caveat that their paper proves this fact only away from the intersections of $D$; however, they expect their results to extend to the general case and have communicated that details should be forthcoming shortly.

Having specified such a polyhomogeneous slice, we may then consider the bundle $\mathcal{S}_{tt}$ over the coordinate neighborhood in $\overline{\mathcal{M}}_{g}$ whose fiber at $[g]$ consists of the transverse-traceless tensors $\mathcal{S}_{tt}(g)$. Notice that the space
\[ S_{tt}(g) \text{ only depends on the conformal class, i.e., is the same for } g \text{ and } e^{2\varphi}g. \]

This is obviously a smooth bundle away from \( M_\gamma \), but we shall prove that there is a local frame of sections \( \kappa_i \) which are polyhomogeneous at the singular divisor in the base variables. Since the area form \( dA_g \) is also polyhomogeneous at this singular divisor, we see that every term in the integral on the right in (3), cf. (1), is polyhomogeneous, when evaluated on this special frame. Thus the matrix coefficients of \( g_{WP} \) in this special frame are polyhomogeneous. The key point in finding a polyhomogeneous frame \( \{ \kappa_i \} \) is as follows. It is clear from the existence of a polyhomogeneous slice \( g(w) \) that any partial derivative \( \partial w_i g \) is polyhomogeneous, and each such infinitesimal variation of metrics is an element of \( T_{g(w)}\text{Met}^{-1} \). However, we must then project these to elements of \( S_{tt}(g(w)) \). The main issue then is to prove that the family of orthogonal projections \( T^g: T_g\text{Met}^{-1} \to T_{[g]}M_\gamma \) has a polyhomogeneous extension, in an appropriate sense, up to \( \mathbb{D} \).

We conclude by saying more about why we can work with only an approximately hyperbolic slice until the last step. We have already remarked that \( S_{tt} \) is conformally invariant, and the inner product and area form transform simply under conformal changes. Thus if \( g(w) \) is an approximately hyperbolic polyhomogeneous slice, and if \( \varphi(g(w)) \) is the associated family of conformal factors so that \( e^{2\varphi(g(w))}g(w) \) is hyperbolic, then we can rewrite (1) and (3) in terms of \( g(w) \) as

\[
(\kappa_1, \kappa_2)_{WP} = \int S(\kappa_1, \kappa_2)_{g(w)} e^{-2\varphi(w)} dA_{g(w)},
\]

since the \( \kappa_i \) are the same for either \( g(w) \) or \( e^{2\varphi(g(w))}g(w) \). We then prove the main result as follows: each of the terms in this expression are polyhomogeneous, and the integral is a pushforward by a \( b \)-fibration, so by Melrose’s pushforward theorem, the corresponding matrix coefficients of \( g_{WP} \) are polyhomogeneous in \( w \). We explain this step later.

3. Blowups and Polyhomogeneous Slices

To describe and carry out the steps of the proof in more detail, we now introduce two auxiliary spaces which play a key role in this paper. These are resolutions of \( \overline{M}_\gamma, M_{\gamma,1} \), and are the spaces on which the metrics, frames, etc., described at the end of \( \mathbb{2} \) are actually polyhomogeneous.

The first such space, \( \overline{M}_\gamma \), is the real blowup of \( \overline{M}_\gamma \) along the divisor \( \mathbb{D} \). Assume, as before, that \([g] \in D_J\), where \( J = \{1, \ldots, k\} \). Choose local holomorphic coordinates \((z_1, \ldots, z_N)\) in a neighborhood \( U \) of \([g]\) so that each \( D_j \cap U = \{z_j = 0\} \), \( j = 1, \ldots, k \), and that \((z_{k+1}, \ldots, z_N)\) is a local holomorphic chart on \( D_J \).

Suppose first that \(|J| = 1\), so \( D_J = D_1 \), and \([g]\) lies in a single divisor. The blowup of \( D_1 \cap U \) in \( U \) is obtained by replacing each point \( q \in D_1 \cap U \) by its normal circle bundle. This yields a manifold with boundary, where the
boundary is the total space of a circle bundle over $D_1 \cap U$. More generally, when $|J| > 1$, this process is carried out along each one of the $D_j \cap U$, and it is clear from the fact that the $D_j$ meet with simple normal crossings that the blowups in the different divisors are independent from one another. The part of the blowup over each intersection locus $D_j$ is a $(S^1)^{|J|}$ bundle. This construction is well-defined globally and defines a manifold with corners which we call $\tilde{M}_\gamma$. To be in accord with current usage, this space is actually not quite a manifold with corners for the simple reason that some of the boundary hypersurfaces intersect themselves. Since our considerations are local on $\tilde{M}_\gamma$, we may overlook this point, and for simplicity we may assume that we are working on a manifold with corners.

The interiors of the boundary hypersurfaces of $\tilde{M}_\gamma$ are $S^1$ bundles over $\mathbb{D}^{\text{reg}}$, the set of points $q \in \mathbb{D}$ which lie on only one divisor, and away from double-points. The codimension $k$ corners are $(S^1)^k$ bundles over the appropriate intersections of divisors $D_j$. Altogether, we write

$$\tilde{M}_\gamma = [M_\gamma; \mathbb{D}].$$

This space has boundary hypersurfaces $H_j, j = 1, \ldots, N$, each corresponding to one of the divisors $D_j$, as well as corners corresponding to the various intersections of the $D_j$. We denote by $\beta$ the blowdown map $\tilde{M}_\gamma \to M_\gamma$.

It is worth noting that the local holomorphic coordinates used here are not directly comparable to Fenchel-Nielsen coordinates, i.e., while we may set $y = (z_{k+1}, \ldots, z_N)$, it is not the case that each $(\ell_j, \omega_j)$ is a function of $\Re z_j, \Im z_j$. Nonetheless, the fibres $\beta^{-1}([g])$ for $[g] \in \mathbb{D}$ are well-defined. In fact, defining the polar coordinates $|z_j|, \arg z_j$, then both of the sets of coordinates $(|z_1|, \arg z_1, \ldots, |z_k|, \arg z_k, y)$ and $(\ell_1, \omega_1, \ldots, \ell_k, \omega_k, y)$ lift to smooth (actually, real analytic) coordinate systems on $\tilde{M}_\gamma$, and these are smoothly (real analytically) equivalent.

We next consider the analogous construction for the compactified universal family $\overline{M}_\gamma$. As noted earlier, up to finite covers, there is a singular fibration $\bar{\Pi} : \overline{M}_{\gamma,1} \to \overline{M}_\gamma$, and we wish to define a manifold with corners obtained by blowing up certain submanifolds of $\overline{M}_{\gamma,1}$ so that $\bar{\Pi}$ lifts to a map

$$\bar{\Pi} : \overline{M}_{\gamma,1} \to \overline{M}_\gamma$$

which is a $b$-fibration between manifolds with corners. To carry out this construction, we first blow down the noded $S^2$ components of the singular fibres $\bar{\Pi}^{-1}([g_0])$ in $\overline{M}_{\gamma,1}$; thus we replace these singular fibres by the union of noded Riemann surfaces which correspond to the complete hyperbolic metric $g_0$. The resulting space $(\overline{M}_{\gamma,1})'$ is the true universal family of $\overline{M}_\gamma$. (We could, of course, have bypassed the Deligne-Mumford compactification of $\overline{M}_{\gamma,1}$ and proceeded directly to this smaller compactification.) The next step is to lift this singular fibration via the blowdown map $\beta : (\overline{M}_{\gamma,1})' \to \overline{M}_\gamma$; the final step is to blow up the lifts of the nodes of these singular fibres in the total space. The resulting space is denoted $\overline{M}_{\gamma,1}$.
It is more transparent to describe all of this in local coordinates. For simplicity, consider this first near $D_{\text{reg}}$. Use the (lifted) Fenchel-Nielsen coordinates $(\ell, \omega, y)$ on $\tilde{M}_\gamma$ as well as local coordinates $(\tau, \theta)$ on a neighborhood $C$ of the appropriate curve $c \subset \Sigma$ which degenerates at $[g_0]$. We suppose that the local choice of approximately hyperbolic metrics $g$ on each fiber of $\tilde{\Pi}$ is adapted to these coordinates in the sense that $C$ is the open cylinder $C = (-1, 1)_\tau \times S^1_\theta$ and

\begin{equation}
 g|_C = \frac{d\tau^2}{\tau^2 + \ell^2} + (\tau^2 + \ell^2) d\theta^2.
\end{equation}

Note that (8) is indeed a hyperbolic metric and can be reduced to the more familiar form

\begin{equation}
 dt^2 + \ell^2 \cosh^2(t) d\theta^2
\end{equation}

by the change of coordinates $t = \text{arcsinh} \, \tau / \ell$. These cylindrical coordinates extend to the fibers of $(\tilde{M}_{\gamma,1})'$ over $\mathbb{D}$, but become dependent at the preimage of the nodal set $\{\ell = \tau = 0\}$. This preimage has coordinates $y, \omega, \theta$, and its blowup reduces to the blowup of $(0, 0)$ in $[0, \ell_0) \times (-1, 1)_\tau$, with the remaining coordinates $(y, \omega, \theta)$ as parameters. Let us use polar coordinates $\ell = \rho \sin \chi, \tau = \rho \cos \chi$, with $\rho \geq 0, \chi \in [0, \pi]$. Altogether then, $(\rho, \chi, \omega, \theta, y)$ is a coordinate system on $\tilde{M}_{\gamma,1}$. The new face $\rho = 0$ is called the front face.

It is useful to consider a reduced version of this construction where we replace the product $C \times [0, \ell_0)_\ell$ by its blowup

\begin{equation}
 \tilde{C} = [C \times [0, \ell_0); \{\ell = \tau = 0\}].
\end{equation}

(In other words, we suppress the coordinates $y$ and $\omega$ along the hypersurface boundary $H$ of $\tilde{M}_\gamma$. As before, the new face is called the front face of $\tilde{C}$. A neighborhood of the corresponding region in $\tilde{M}_{\gamma,1}$ is a product $\tilde{C} \times S^1_\omega \times \mathcal{W}_y$, where $\mathcal{W}_y$ is a neighborhood of $[g_0]$ in $\mathbb{D}$. More generally, near crossing points of $\mathbb{D}$, this construction may be carried out independently near each of the degenerating curves $c_j$.

The space $\tilde{M}_{\gamma,1}$ has two types of boundary hypersurfaces. The first are closures of the hypersurfaces $H_j^{\text{reg}} \times (\Sigma \setminus c_j)$, where the $H_j^{\text{reg}}$ are the hypersurfaces in $\tilde{M}_\gamma$ which cover the components of $\mathbb{D}^{\text{reg}}$, and the second are the faces $F_j$ obtained by blowing up $H_j^{\text{reg}} \times c_j$. Because of the self-intersections of the irreducible components of $\mathbb{D}$, these boundary hypersurfaces of $\tilde{M}_{\gamma,1}$ may self-intersect at their boundaries, and for this reason, $\tilde{M}_{\gamma,1}$ is slightly more general than a manifold with corners. However, our considerations are sufficiently local that this does not affect anything here, and we shall think of this space as a manifold with corners.

The fibration $\Pi : M_{\gamma,1} \to M_\gamma$ extends to a $b$-fibration

\begin{equation}
 \tilde{\Pi} : \tilde{M}_{\gamma,1} \to \tilde{M}_\gamma.
\end{equation}
Briefly, a $b$-fibration is the most useful analogue of a submersion in the setting of maps between manifolds with corners. We refer to [16] for the precise definition of $b$-fibrations. We appeal to this structure only once again, at the very end of this paper. The preimage of a point $q$ in the interior of $M_\gamma$ is the compact surface $\Sigma$. If $q \in H^\text{reg}_j$, then $\tilde{\Pi}^{-1}(q)$ is a union of the bordered surface obtained by adding the boundary curves to $\Sigma \setminus c_j$ and the cylinder $[-1, 1] \times S^1$, which is the new face created by the final blowup. We also define the vertical tangent bundle $T^\text{ver} \tilde{M}_\gamma,1$. Its fibres are the tangent planes to the fibres $\tilde{\Pi}^{-1}(q)$ for $q$ in the interior; over the boundary, however, these fibres are ‘broken’, so the vertical tangent space is either the tangent plane to the noded degeneration of $\Sigma$, or else to the tangent space of $\tilde{C}$.

One of the key properties of this fibration is that the family of hyperbolic metrics $g_q$ on the vertical tangent bundle extends naturally to a (degenerate) fibrewise metric on $\tilde{M}_\gamma,1$. Over the boundaries of $\tilde{M}_\gamma,1$, these vertical hyperbolic metrics are either the complete finite area hyperbolic metrics, over the noded degenerations of $\Sigma$, or else the complete (infinite area) hyperbolic metric

$$\frac{dT^2}{1 + T^2} + (1 + T^2)d\theta^2$$

on the front face of $\tilde{C}$. Melrose and Zhu [17] prove the following

**Proposition 3** ([17]). *The family of metrics $g(w)$ on $\Sigma$ over the interior of $M_\gamma$ extends to a polyhomogeneous section of the symmetric second power of the dual of the vertical tangent bundle of $\tilde{M}_\gamma,1$.*

**Remark 4.** As we have already noted, [17] only establishes this polyhomogeneity near points of $D^\text{reg}$, but not at the intersection locus of the divisors. They expect to complete the proof in that case soon as well.

We conclude this section with a brief explanation for how to make the translation between the results in [17] and what is needed here. We begin with a polyhomogeneous family of approximately hyperbolic metrics $\hat{g}(w)$ and then find the conformal factor $\varphi(w)$ such that $e^{2\varphi(w)}\hat{g}(w)$ is hyperbolic. The family $\hat{g}(w)$ has been constructed to be polyhomogeneous. We indicate now why the main theorem of [17] shows that $\varphi(w)$ is as well.

The notation in [17] is as follows. The annulus $A$ is identified with the quadric $\{(z, w) : zw = t\}$; for simplicity here we assume that $t \in \mathbb{R}$, $0 < t < 1/4$; this parameter is equivalent to the length parameter $\ell$, see below. We parametrize half of this region by the annulus $\{\sqrt{t} \leq |z| \leq 1/2\}$. The family of hyperbolic metrics here is given as

$$\left(\frac{\pi \log |z|}{\log t} \csc \frac{\pi \log |z|}{\log t}\right)^2 \frac{|dz|^2}{|z|^2(\log |z|)^2}.$$
This agrees with (8) upon making the substitutions

\[ \ell = \frac{\pi}{|\log t|}, \quad \ell = \frac{\pi}{|\log |z||}. \]

In [17], the radial variable $|z|$ is replaced by $1/|\log |z||$, so this change of variables shows that polyhomogeneity (and indeed log smoothness) in this new logarithmic variable, as proved in [17], is equivalent to polyhomogeneity (log smoothness) in $\tau$.

4. Global analysis and $\mathcal{M}_\gamma$

We now recall some standard facts about deformations of hyperbolic metrics on surfaces. The point of view adopted here is the one promoted by Tromba [19], and is the specialization to this low dimension of the deformation theory of Einstein metrics.

4.1. Curvature equations and Bianchi gauge. Consider the operator which assigns to a metric its Gauss curvature: $g \mapsto K_g$. This is a second order nonlinear differential operator, and since $\text{Ric}^g = K^g g$ in dimension 2, metrics of constant curvature are the same as Einstein metrics in this setting.

Let $(\Sigma^2, g_0)$ be a closed surface where $K_{g_0} \equiv -1$. Nearby hyperbolic metrics correspond to solutions of

\[ S^2(\Sigma, T^*\Sigma) \ni h \mapsto E^{g_0}(h) := (K_{g_0} + h + 1) \cdot (g_0 + h) = 0 \]

with $h$ suitably small. The nonlinear operator $E^{g_0}$ is called the Einstein operator. Since $\Sigma$ is compact, we can let $E^{g_0}$ act between appropriate Sobolev or Hölder spaces, but for simplicity we do not specify the function spaces precisely until necessary.

The operator $E^{g_0}$ is not elliptic because it is invariant under the infinite-dimensional group $\text{Diff}(\Sigma)$ of diffeomorphisms of $\Sigma$. For any metric $g$, the tangent space of the $\text{Diff}(\Sigma)$ orbit through $g$ consists of all symmetric 2-tensors of the form $L_X g$, where $X$ is a vector field on $\Sigma$, or equivalently, as $(\delta g)^* \omega$, where $\omega$ is the 1-form metrically dual to $X$. Here $\delta^g : S^2(\Sigma, T^*\Sigma) \to \Omega^1(\Sigma)$ is the divergence operator and $(\delta g)^*$ is its adjoint; in local coordinates,

\[ ((\delta g)^* \omega)_{ij} = \frac{1}{2} (\omega_{i;j} + \omega_{j;i}). \]

Note that $\text{tr}^g (\delta g)^* = -\delta^g : \Omega^1 \to \Omega^0$, where $\delta^g : \Omega^1(\Sigma) \to \Omega^0(\Sigma)$ is the standard codifferential. The conformal Killing operator is the projection of $(\delta g)^*$ onto its trace-free part:

\[ D^g \omega := (\delta g)^* \omega + \frac{1}{2} \delta^g (\omega) g : \Omega^1(\Sigma) \to S^2_0(\Sigma, T^*\Sigma). \]

This is the adjoint of $\delta^g : S^2_0(\Sigma, T^*\Sigma) \to \Omega^1(\Sigma)$. It follows from all this that the nullspace of $\delta^g$ on $S^2(\Sigma, T^*\Sigma)$ equals the $L^2$-orthogonal complement.
of the tangent space of the diffeomorphism orbit passing through \( g \), and furthermore that the system
\[
h \mapsto (E^g(h), \delta^g(h))
\]
is elliptic. It is convenient to consider instead the single operator
\[
N^g(h) := E^g(h) + (\delta^{g+h})^* B^g(h) = (K^{g+h} + 1)(g + h) + (\delta^{g+h})^* B^g(h),
\]
where
\[
h \mapsto B^g(h) := \delta^g(h) + \frac{1}{2} \mathrm{dtr}^g h
\]
is the Bianchi operator. We say that \( h \) is in Bianchi gauge if \( B^g(h) = 0 \). Clearly, if \( E^g(h) = 0 \) and \( B^g(h) = 0 \), then \( N^g(h) = 0 \). The converse, which is due to Biquard, is true as well.

**Proposition 5.** Suppose that \( g_0 \) is hyperbolic. If \( h \in S^2(\Sigma, T^* \Sigma) \) is sufficiently small and \( N^{g_0}(h) = 0 \), then \( g_0 + h \) has constant Gauss curvature \(-1\) and \( B^{g_0}(h) = 0 \).

Before recalling Biquard’s proof, recall that if \( h = fg \) is pure trace, then
\[
B^g(fg) = \delta^g(fg) + \frac{1}{2} \mathrm{dtr}^g(fg) = -df + df = 0.
\]
Thus applying \( B^{g+h} \) to \( N^g(h) \) yields
\[
B^{g+h}(E^g(h)) = B^{g+h}(\delta^{g+h})^* B^g(h) = 0.
\]
For any metric, write
\[
P^g := B^g \circ (\delta^g)^*: \Omega^1(\Sigma) \to \Omega^1(\Sigma);
\]
by a standard Weitzenböck identity,
\[
P^g = \frac{1}{2}(\Delta^g - 2K^g),
\]
where \( \Delta^g \) is the Hodge Laplacian on 1-forms.

**Proof.** It suffices to establish that, if \( g = g_0 \) is hyperbolic, then \( \omega = B^{g_0}(h) = 0 \). By (14), \( P^{g_0+h} \omega = 0 \), or equivalently, by (16),
\[
\langle 2P^{g_0+h} \omega, \omega \rangle = \| \nabla^{g_0+h} \omega \|^2 - 2K^{g_0+h} \| \omega \|^2 = 0.
\]
However, since \( K^{g_0} = -1 \), then if \( h \) is sufficiently small, \( K^{g_0+h} < 0 \). We conclude that \( \omega = 0 \), and thus \( E^{g_0}(h) = 0 \), as desired. \( \square \)
4.2. Linearized curvature operators. The linearizations of the curvature operators above are not hard to compute. As before, assume throughout that \( K^{g_0} \equiv -1 \). If \( h = h^0 + fg_0 \) is the decomposition into trace-free and pure-trace parts, then

\[
DK^{g_0}(h) = \left( \frac{1}{2} \Delta^{g_0} + 1 \right) f + \frac{1}{2} \delta^{g_0} \delta^{g_0} h^0.
\]

We see directly from this that

\[
DE^{g_0}(k) = \left( \frac{1}{2} \Delta^{g_0} + 1 \right) f + \frac{1}{2} \delta^{g_0} \delta^{g_0} h^0 \circ g_0,
\]

and furthermore,

\[
DN^{g_0}(h) := L^{g_0}(h) = \frac{1}{2} (\nabla^* \nabla - 2) h^0 + \left( \frac{1}{2} (\Delta^{g_0} + 2) f \right) g.
\]

We call \( L^{g_0} \) the linearized Bianchi-gauged Einstein operator. Note finally that by differentiating the identity \( B^{g_0} h N^{g_0}(h) = P^{g_0} h B^{g_0}(h) \) at \( h = 0 \), we obtain

\[
B^{g_0} L = P^{g} B^{g}.
\]

4.3. Transverse-traceless tensors. A key object in this paper is the space \( S_{tt} = S_{tt}(g_0) \) of transverse-traceless tensors on the surface \( \Sigma \) with respect to the metric \( g_0 \); by definition, this is the nullspace of \( L^{g_0} \). This space represents the tangent space of \( \mathcal{M}_c \) at \( g_0 \) and depends only on the conformal class \([g_0]\), cf. Proposition 6 below.

Using (18), \( L^{g_0}(h^0 + fg_0) = 0 \) if and only if \( f = 0 \) and \( (\nabla^* \nabla - 2) h^0 = 0 \); by (16), the second condition is equivalent to \( \delta^{g_0} h^0 = 0 \). Therefore

\[
\ker L^{g_0} = S_{tt}(g_0) = \{ h \in S^2(\Sigma, T^*\Sigma) \mid \delta^{g_0} h = 0, \quad \text{tr}^{g_0} h = 0 \}
\]

and hence \( S_{tt} \) is the tangent space to the submanifold of metrics with constant curvature \( K \equiv -1 \) in Bianchi gauge with respect to \( g_0 \). In particular, \( S_{tt} \) is orthogonal to the diffeomorphism orbit through \( g_0 \). It is straightforward to check that since \( \dim \Sigma = 2 \), \( \delta^{g_0} \) is elliptic as a map between sections of \( S^1_0(\Sigma, T^*\Sigma) \) and \( \Omega^1(\Sigma) \), hence \( \dim S_{tt} < \infty \) and consists of smooth elements. In fact

\[
\dim S_{tt}^g = 6(\gamma - 1);
\]

this holds because there is a canonical identification of \( S_{tt} \) with the space of holomorphic quadratic differentials on \( \Sigma \).

**Proposition 6.** When \( \dim \Sigma = 2 \), \( S_{tt} \) is conformally invariant; in other words, if \( g_1 = e^{2u} g \) then

\[
\text{tr}^{g_1} h = 0, \quad \delta^{g_1} h = 0 \quad \iff \quad \text{tr}^g h = 0, \quad \delta^g h = 0.
\]

**Proof.** The first part of the condition is obvious since \( \text{tr}^{g_1} h = e^{-2u} \text{tr}^g h \). The rest follows from the general identity

\[
\delta^g h = e^{-2u}(\delta^g h + (\text{tr}^g h) du + (2 - n) \epsilon(\nabla^g u) h),
\]

so since \( n = 2 \), if \( \text{tr}^g h = 0 \) and \( \delta^g h = 0 \), then \( \delta^{g_1} h = 0 \), as claimed. \( \Box \)
4.4. Local deformation theory. It is now standard to deduce some features of the local deformation theory of hyperbolic metrics. We refer to [15] and [19] for complete proofs.

If \((\Sigma, g_0)\) is a closed hyperbolic surface, as above, we describe the Banach space structure of all nearby Riemannian metrics \(g\) (in the \(C^{2,\alpha}\) topology) with \(K_g = -1\), and identify those metrics in this Banach submanifold which are in a local slice with respect to the action of \(\text{Diff}(\Sigma)\).

Write \(M^{\text{hyp}}\) for the set of \(C^{2,\alpha}\) Riemannian metrics with curvature \(-1\).

**Lemma 7.** The space \(M^{\text{hyp}}\) is a Banach submanifold in \(C^{2,\alpha}\) \(S^2(\Sigma, T^*\Sigma)\).

By the discussion above,
\[
T_{g_0}M^{\text{hyp}} = \{ h \in C^{2,\alpha}S^2(\Sigma, T^*\Sigma) \mid DK_{g_0}h = 0 \} = \{ h = h_0 + f g_0 \in C^{2,\alpha}S^2(\Sigma, T^*\Sigma) \mid tr^{g_0}h = 0, (\Delta^{g_0} + 2)f + \delta^{g_0}h = 0 \}.
\]
We now impose the gauge condition.

**Lemma 8.** There is a constant \(\varepsilon\) depending on \(g_0\) such that the intersection of
\[
S_{g_0,\varepsilon} := M^{\text{hyp}} \cap \{ g_0 + h : B^{g_0}h = 0, \| h \|_{2,\alpha} < \varepsilon \}
\]
with the orbit of \(\text{Diff}^{3,\alpha}(\Sigma)\) is transverse at \(g_0\). The space \(S_{g_0,\varepsilon}\) is identified with the space of solutions \(h\) to \(N^{g_0}(h) = 0\) with \(\| h \|_{2,\alpha} < \varepsilon\), and its tangent space at \(g_0\) equals \(S_{tt}(g_0)\).

There is also a local slice theorem.

**Theorem 9.** There is a positive constant \(\varepsilon\) depending on \(g_0\) and a neighborhood \(U\) of \(\text{id}\) in \(\text{Diff}^{3,\alpha}(\Sigma)\) such that the map
\[
U \times S_{g_0,\varepsilon} \ni (F, h) \mapsto F^*(g_0 + h)
\]
is a diffeomorphism onto some neighborhood of \(g_0\) in \(M^{\text{hyp}}\).

In fact, since the genus \(\gamma > 1\), one can show that the action of the entire connected component of the identity of the diffeomorphism group acts properly.

4.5. The Weil-Petersson metric. We now use this formalism to describe the Weil-Petersson metric.

Let \(t \mapsto g_t, |t| < \varepsilon\), be a smooth path of hyperbolic Riemannian metrics through \(g_0\). For \(\varepsilon\) small, then Theorem 9 shows that there exists a unique \(h_t \in S^2(\Sigma, T^*\Sigma)\) such that \(N^{g_0}(h_t) = 0\) and \(h_t = F_t^*g_t\) for some diffeomorphism \(F_t\), so in particular the differential \(\kappa := \frac{d}{dt}\big|_{t=0}h_t\) lies in \(S_{tt}(g_0)\). We now compute \(\kappa\) in terms of \(\dot{g} = \frac{d}{dt}\big|_{t=0}g_t\). Indeed, there exists a smooth family of diffeomorphisms \(F_t\) and a smooth family of scalar functions \(u_t\) such that
\[
g_0 + h_t = F_t^*e^{2u_t}g_t.
\]
Write \( X = \frac{d}{dt} |_{t=0} F_t \) and \( \dot{u} = \frac{d}{dt} |_{t=0} u_t \). Differentiating (19) with respect to \( t \) at \( t = 0 \) yields

\[
(20) \quad \kappa = L_X g_0 + 2 \dot{u} g_0 + \dot{\gamma} = (\delta g_0)^* \omega + 2 \dot{u} g_0 + \dot{\gamma},
\]

where \( X^\flat = \omega \in \Omega^1(\Sigma) \) is the 1-form dual to \( X \) with respect to \( g_0 \).

We first determine \( \omega \) as follows. Apply \( B g_0 \) to (20) and recall (15), to get

\[
P g_0 \omega = -B g_0 \dot{\gamma},
\]

since \( B g_0 \kappa = 0 \). By (16), \( P g_0 \) is invertible, so denoting its inverse by \( G g_0 \), we can write

\[
\omega = -G g_0 B g_0 \dot{\gamma}.
\]

Finally, with \( \pi g_0 : S^2(\Sigma, T^*\Sigma) \to S^2_0(\Sigma, T^*\Sigma) \) the orthogonal projection onto trace-free tensors, we obtain that

\[
\kappa = -\pi g_0 (\delta g_0)^* G g_0 B g_0 \dot{\gamma} + \pi g_0 \dot{\gamma}.
\]

In other words, \( \kappa = T g_0 \dot{\gamma} \), where the \( L^2 \)-orthogonal projection \( L^2 S^2_0(\Sigma, T^*\Sigma) \to S_{tt} \) is given by

\[
(21) \quad T g_0 = \pi g_0 \circ (\text{id} - (\delta g_0)^* G g_0 B g_0).
\]

This now gives the formulae for the Weil-Petersson norm and inner product:

\[
(22) \quad \| \kappa \|^2_{WP} = \| T g_0 \dot{\gamma} \|^2_{L^2(\Sigma, dA_{g_0})}, \quad \langle \kappa_1, \kappa_2 \rangle_{WP} = \langle T g_0 \dot{\gamma}_1, T g_0 \dot{\gamma}_2 \rangle_{L^2(\Sigma, dA_{g_0})}.
\]

Our goal in the remainder of this paper is to analyze these expressions near the singular divisors of \( M_\gamma \). Our main result is that there is a basis of sections of \( S_{tt} \) which depends in a polyhomogeneous manner on \( g_0 \). The main issue is to prove that the operator \( G g_0 \) is polyhomogeneous in \( g_0 \).

5. Hyperbolic cylinders

The first step in this analysis is to study this problem for a model family of finite hyperbolic cylinders \((C, g_\ell)\) where the length \( \ell \) of the central geodesic decreases to 0. We have already described these metrics in (5) above, and we note here that either component of the boundary \( \partial C = \{ \tau = \pm 1 \} \) is at distance \( \arcsin(1/\ell) \sim |\log(\frac{1}{2})| \) from the central geodesic \( \{ \tau = 0 \} \).

We consider here the family of inverses \( G g_\ell \) to the operators \( P g_\ell \) on \( C \) and shall prove that it is polyhomogeneous in a particular sense as \( \ell \searrow 0 \). This analysis is explicit and we use it later to construct a parametrix for \( P g_\ell \) on families of degenerating compact hyperbolic surfaces.

It is here that the blowup \( \hat{C} \) of \( C \times [0, \ell_0) \), which we introduced in (5), becomes important. Indeed, the family \( G g_\ell \) has a rather complicated structure at \( \ell = 0 \); the most precise description of this behaviour, which was studied carefully in (1), is that it is polyhomogeneous on a space obtained by a sequence of blowups of \( C \times C \times [0, \ell_0) \). Fortunately we do not need all of this, and shall only require a much weaker form of this, which we can prove directly. Namely, we show that if the family of 1-forms \( h_\ell \) on \( C \times [0, \ell_0) \)
vanishes in a neighbourhood $|\tau| \leq c$ for all $\ell$ and is polyhomogeneous on this product space at $\ell = 0$, then the family of solutions $\omega_\ell = G^{\eta_\ell} h_\ell$ lifts to be polyhomogeneous on $\hat{C}$. We say in either case that $h_\ell$ and $\omega_\ell$ are polyhomogeneous in $\ell$, the passage to $\hat{C}$ being implied. We can sidestep the full machinery of [1] precisely because $h_\ell$ vanishes near $\tau = 0$, since this means that $G^{\eta_\ell} h_\ell$ does not depend on the behaviour of the Schwartz kernel $G^{\eta_\ell}(\tau, \tilde{\tau}, \ell)$ near $\tau = \tilde{\tau} = \ell = 0$, where its structure is particularly complicated.

5.1. **The operator** $P^{\eta_\ell}$. We now compute the action of $P^{\eta_\ell}$ on 1-forms. This is simplest if we express any such form as a combination $u_{\rho_1} + v_{\rho_2}$, where $\rho_1 = \sigma_1 - i\sigma_2$, $\rho_2 = \sigma_1 + i\sigma_2$, and

\begin{align*}
\sigma_1 &= \frac{d\tau}{\sqrt{\tau^2 + \ell^2}}, & \sigma_2 &= \sqrt{\tau^2 + \ell^2} d\theta.
\end{align*}

Introducing the Fourier decompositions $u = \sum u_k e^{i k \theta}$, $v = \sum v_k e^{i k \theta}$, then a short computation gives that the operator induced by $P^{\eta_\ell}$ on the $k$th Fourier mode of the column vector $(u_k, v_k)^T$ is

\begin{align*}
P_{\ell,k} &= \frac{1}{2} \begin{pmatrix} P^{\pm}_{\ell,k} & 0 \\ 0 & P^{\pm}_{\ell,k} \end{pmatrix},
\end{align*}

where

\[ P^\pm_{\ell,k} = -(\tau^2 + \ell^2) \frac{d^2}{d\tau^2} - 2\tau \frac{d}{d\tau} + 1 + \frac{2(\tau \mp k)^2}{\tau^2 + \ell^2}. \]

We are only interested in solving $P^{\eta_\ell} \omega = h$, and understanding the precise asymptotics of this solution as $\ell \to 0$, when $h$ is polyhomogeneous and also vanishes near $\tau = 0$, and this provides a substantial simplification of the discussion below. This analysis separates into one for the eigenmode $k = 0$ and another for the sum of all the other eigenmodes together.

The latter case is slightly simpler.

**Proposition 10.** Suppose that $h$ is polyhomogeneous on $\hat{C}$ and vanishes for $|\tau| \leq c < 1$. Suppose too that the eigenmode $h_0 = 0$. Then the unique solution to $P^{\eta_\ell} \omega = h$ with $\omega(\pm 1, \theta) = \eta_{\pm}$ specified (and polyhomogeneous in $\ell$) is polyhomogeneous on $\hat{C}$ and vanishes rapidly at the front face of $\hat{C}$.

**Proof.** We have reduced the problem to a scalar one, involving the operators $P^\pm_{\ell,k}$, and so we first prove the result for any one of these. Let

\[ P^\pm_{\ell,k} \omega_k = h_k, \]

where $\omega_k(\pm 1) = \eta_{\pm k}$ and $h_k$ vanishes for $|\tau| \leq c < 1$. Choose a constant $C > 0$, independent of $k$, such that

\[ C \geq \max \{ \|h_k\|_{L^\infty, \omega(\pm 1)} \}. \]

It follows that the function $\tilde{\omega}_k = \omega_k - C$ satisfies $\tilde{\omega}_k(\pm 1) \leq 0$ and

\[ P^+_{\ell,k} \tilde{\omega}_k = h_k - C \left( 1 + \frac{(\tau + k)^2}{\tau^2 + \ell^2} \right) \leq h_k - C \leq 0. \]
Thus by the maximum principle, $\tilde{\omega}_k \leq 0$, or equivalently $\omega_k \leq C$. Similarly, $\omega_k \geq -C$. Now we find a sharper barrier function. Set

$$\zeta_k = C e^{\alpha |k| \left( \frac{1}{\epsilon} - \frac{1}{|\tau|} \right)},$$

where $C$ is the same constant as above, and compute that

$$P_{\ell,k}^+ \zeta_k = \left( \frac{1}{\tau^2 + \ell^2} - \frac{\alpha^2 (\tau^2 + \ell^2)}{\tau^4} \right) k^2 + \left( \frac{2\alpha \ell^2}{\tau^2} + \frac{2\tau}{\tau^2 + \ell^2} \right) |k| \tau + 1 + \frac{\tau^2}{\tau^2 + \ell^2} \zeta_k \geq 0$$

for all $k \neq 0$ and $|\tau| \leq 1$ provided $\alpha \in (0, 1)$ is chosen sufficiently small. Consider the homogeneous equation $P_{\ell,k}^+ \omega_k = 0$ on $|\tau| \leq c$. Then

$$\zeta_k(\pm c) = C \geq |\omega_k| \quad \text{and} \quad P_{\ell,k}^+ (\zeta_k - \omega_k) \geq 0, \quad P_{\ell,k}^+ (\omega_k - \zeta_k) \leq 0,$$

hence

$$|\omega_k| \leq \zeta_k.$$

This same estimate holds also for $P_{\ell,k}^-$.

Altogether, summing over all nonzero $k \in \mathbb{Z}$, and using that $\omega_0 = 0$, we see that the solution to the original problem $P^g \omega = h$ satisfies

$$|\omega(\tau, \theta)| \leq \sum_{k \neq 0} |\omega_k(\tau)| \leq \sum_{k \neq 0} C e^{\alpha |k| \left( \frac{1}{\epsilon} - \frac{1}{|\tau|} \right)} \leq C e^{\alpha \left( \frac{1}{\epsilon} - \frac{1}{|\tau|} \right)} \left( 1 - e^{\alpha \left( \frac{1}{\epsilon} - \frac{1}{|\tau|} \right)} \right),$$

and this decays rapidly as $\tau \to 0$, uniformly in $\ell$. $\square$

We now discuss the case $k = 0$ and analyze the family of operators

$$P_{\ell,0}^+ = P_{\ell,0}^- = - (\tau^2 + \ell^2) \frac{d^2}{d\tau^2} - 2\tau \frac{d}{d\tau} + 1 + \frac{\tau^2}{\tau^2 + \ell^2}.$$

The change of variables $T = \frac{\tau}{\ell}$ transforms this to

$$P_0 = -(T^2 + 1) \frac{d^2}{dT^2} - 2T \frac{d}{dT} + 1 + \frac{T^2}{T^2 + 1}.$$

This has two linearly independent homogeneous solutions

$$u_\pm = \sqrt{T^2 + 1}, \quad v_\pm = \frac{1}{\sqrt{T^2 + 1}} \left( T + \arctan(T) + T^2 \arctan(T) \right),$$

hence $u_\ell(\tau) = u_\pm (\frac{\tau}{\ell})$ and $v_\ell(\tau) = v_\pm (\frac{\tau}{\ell})$ are solutions of $P_{\ell,0}^\pm u = 0$. By direct inspection, these functions are polyhomogeneous on $\tilde{C}$.

We now write the unique solution to the inhomogeneous problem $P_{\ell,0}^\pm \omega = h$ with boundary values $\omega(\pm 1) = \eta^\pm$ as

$$\omega(\tau) = Au_\ell + Bv_\ell +$$

$$\left( \frac{u_\ell(\tau)}{2\ell} \int_{-1}^\tau v_\ell(\sigma) \frac{h(\sigma)}{\sigma^2 + \ell^2} d\sigma - \frac{v_\ell(\tau)}{2\ell} \int_{-1}^\tau u_\ell(\sigma) \frac{h(\sigma)}{\sigma^2 + \ell^2} d\sigma \right).$$

(25)
The constants $A = A(\ell, h, \eta^\pm)$ and $B = B(\ell, h, \eta^\pm)$ are given by
\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \frac{1}{D} \begin{pmatrix}
v_{\ell}(-1) & -v_{\ell}(1) \\
-u_{\ell}(-1) & u_{\ell}(1)
\end{pmatrix} \begin{pmatrix}
\eta^+ + u_{\ell}(1)I_1 - v_{\ell}(1)I_2 \\
\eta^- + u_{\ell}(-1)I_1 - v_{\ell}(-1), I_2
\end{pmatrix},
\]
where
\[
I_1 = \frac{1}{2\ell} \int_{-1}^{1} v_{\ell}(\tau)h(\tau) d\tau, \quad I_2 = \frac{1}{2\ell} \int_{-1}^{1} u_{\ell}(\tau)h(\tau) d\tau,
\]
and $D = u_{\ell}(1)v_{\ell}(-1) - u_{\ell}(-1)v_{\ell}(1)$.

**Proposition 11.** Suppose that $h$ is polyhomogeneous on $\hat{C}$ and vanishes for $|\tau| \leq c < 1$. Then the unique solution to $D_{,6}^\pm \omega = h$ with given boundary values $\omega(\pm 1) = \eta^\pm$ (which may also be polyhomogeneous in $\ell$) lifts to a polyhomogeneous function on $\hat{C}$.

*Proof.* We have already noted that $u_{\ell}$ and $v_{\ell}$ are polyhomogeneous, so we may concentrate on the other terms in the solution. By assumption, $h$ vanishes for $|\tau| \leq C$, it is also clear that each of the integrals, including the ones in the definitions of $I_1$, $I_2$, also has an expansion in $\ell$ as $\ell \searrow 0$. This proves the claim. □

As a final remark, let us compute
\[
\pi_{\ell} \circ (\delta^{\eta^\pm})(\omega(\tau)\sigma_1)
\]
in the region $|\tau| \leq c$, for $\omega = u_{\ell}$ or $v_{\ell}$; observe that this is the expression in the projection formula \((21)\). We see that
\[
\pi_{\ell} \circ (\delta^{\eta^\pm}) u_{\ell}\sigma_1 = 0 \quad \text{and} \quad \pi_{\ell} \circ (\delta^{\eta^\pm}) v_{\ell}\sigma_1 = -\frac{2\ell^2}{(\tau^2 + \ell^2)^2} d\tau^2 + 2\ell^2 d\theta^2;
\]
note that these are scalar multiples of the transverse-traceless tensor $\kappa_{\ell,0}$ which appears in \((27)\) below. Similarly, with $\omega = u_{\ell}, v_{\ell}$, then $\omega(\tau)\sigma_2$ is mapped to a multiple of the transverse-traceless tensor $\nu_{\ell,0}$.

### 5.2. Symmetric transverse-traceless two-tensors.

We explicitly determine the space $\mathcal{S}_{tt}^{\eta_{\ell}}$ of symmetric transverse-traceless 2-tensors for the hyperbolic metric $g_{\ell}$ on $\mathcal{C}$ and study its limit as $\ell \searrow 0$.

We still work in the orthonormal frame of one-forms $\{\sigma_1, \sigma_2\}$ from \((5.1)\) and let $\{\sigma_1^2 - \sigma_2^2, \sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1\}$ be the induced frame of the bundle $\mathcal{S}_0^2(\mathcal{C}, T^* \mathcal{C})$ of symmetric traceless two-tensors. With respect to these frames the divergence operator $\delta^{\eta^\pm}: \mathcal{S}_0^2(\mathcal{C}, T^* \mathcal{C}) \to \Omega^1(\mathcal{C})$ takes the form
\[
\delta^{\eta^\pm} = \begin{pmatrix}
\nabla_1 + \frac{2\tau}{\sqrt{\tau^2 + \ell^2}} & \nabla_2 \\
-\nabla_2 & \nabla_1 + \frac{2\tau}{\sqrt{\tau^2 + \ell^2}}
\end{pmatrix}.
\]

With the Fourier decompositions $\Phi = \sum_{k \in \mathbb{Z}} \varphi_k e^{ik\theta}$, $\Psi = \sum_{k \in \mathbb{Z}} \psi_k e^{ik\theta}$ we obtain that $\delta^{\eta^\pm} = \oplus_{k \in \mathbb{Z}} \delta_{\ell,k}$. The same change of basis $\rho_1 = \sigma_1 - i\sigma_2$, $\rho_2 = \sigma_1 + i\sigma_2$.
\[ \sigma_1 + i\sigma_2 \] as before diagonalizes each \( \delta_{\ell,k} \), i.e.

\[
\delta_{\ell,k} = \begin{pmatrix}
\sqrt{\tau^2 + \ell^2} \frac{d}{d\tau} + \frac{2\tau-k}{\sqrt{\tau^2 + \ell^2}} & 0 \\
0 & \sqrt{\tau^2 + \ell^2} \frac{d}{d\tau} + \frac{2\tau+k}{\sqrt{\tau^2 + \ell^2}}
\end{pmatrix}.
\]

Solutions of the homogeneous equation \( \delta_{\ell,k}(\lambda_k,\mu_k)^T = 0 \) are given by

\[
(\lambda_k, \mu_k) = \begin{pmatrix}
ed^{\frac{k}{\ell} \arctan(\frac{\tau}{\ell})} & e^{-\frac{k}{\ell} \arctan(\frac{\tau}{\ell})} \\
ed^{\frac{-k}{\ell} \arctan(\frac{\tau}{\ell})} & \frac{\tau^2}{\tau^2 + \ell^2}
\end{pmatrix}.
\]

For the following it is convenient to transform back to the original basis and to introduce a normalization.

**Proposition 12.** Let \( \ell > 0 \). A basis of the Hilbert space \( L^2 S_{tt}^{g\ell} \) of square-integrable transverse-traceless tensors is given by

\[
\kappa_{\ell,0} = \frac{\ell^3}{(\arctan(\frac{1}{\ell}))^2} \begin{pmatrix} 1 \\ \ell^2 + \tau^2 \end{pmatrix}, \quad \nu_{\ell,0} = \frac{\ell^3}{(\arctan(\frac{1}{\ell}))^2} \begin{pmatrix} 0 \\ 1 \\ \ell^2 + \tau^2 \end{pmatrix},
\]

and

\[
\kappa_{\ell,k} = \frac{C_{\ell,k}}{\tau^2 + \ell^2} \left( \cos(k\theta) \cosh \left( \frac{k}{\ell} \arctan \left( \frac{\tau}{\ell} \right) \right), -\sin(k\theta) \sinh \left( \frac{k}{\ell} \arctan \left( \frac{\tau}{\ell} \right) \right) \right),
\]

\[
\nu_{\ell,k} = \frac{C_{\ell,k}}{\tau^2 + \ell^2} \left( \sin(k\theta) \cosh \left( \frac{k}{\ell} \arctan \left( \frac{\tau}{\ell} \right) \right), \cos(k\theta) \sinh \left( \frac{k}{\ell} \arctan \left( \frac{\tau}{\ell} \right) \right) \right),
\]

where \( k \geq 1 \) and

\[
C_{\ell,k} := \sqrt{k} e^{-\frac{k}{\ell} \arctan(\frac{1}{\ell})}.
\]

These constants are chosen to normalize the tensors \( \kappa_{\ell,k} \) and \( \nu_{\ell,k} \) in the sense that

\[
c_0 \leq \|\kappa_{\ell,k}\|_{L^2(S_{tt}^{g\ell})}, \|\nu_{\ell,k}\|_{L^2(S_{tt}^{g\ell})} \leq c_1
\]

where the constants \( c_0, c_1 > 0 \) are independent of \( k \) and \( \ell \).

**Proof.** The tensors \( \kappa_{\ell,k} \) and \( \nu_{\ell,k} \) are obtained from those in (26) by applying the linear transformation \( T \). Hence they are contained in \( S_{tt}^{g\ell} \) and form a
basis of the Hilbert space $L^2$. To verify inequality (28) for $k \geq 1$ we compute

$$
\left\| \frac{\kappa_{\ell,k}}{C_{\ell,k}} \right\|_{L^2(C,G)}^2 = \int_{-1}^{1} \int_{0}^{2\pi} \frac{1}{(\tau^2 + \ell^2)^2} \left( \cos^2(k\theta) \cosh^2 \left( \frac{k}{\ell} \arctan \left( \frac{\tau}{\ell} \right) \right) + \sin^2(k\theta) \sinh^2 \left( \frac{k}{\ell} \arctan \left( \frac{\tau}{\ell} \right) \right) \right) d\theta d\tau
$$

$$
= 2\pi \int_{0}^{1} \frac{\cosh \left( \frac{2k}{\ell} \arctan \left( \frac{T}{\ell} \right) \right)}{(\tau^2 + \ell^2)^2} d\tau
$$

$$
= \frac{2}{\ell^3} \int_{0}^{1} \frac{\cosh \left( \frac{2k}{\ell} \arctan(T) \right)}{(T^2 + 1)^2} dT
$$

$$
= \frac{\pi}{2k(k^2 + \ell^2)(1 + \ell^2)} \left( 2k \cosh \left( \frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right) \right) \right)
$$

$$
+ (2k^2 + 1 + \ell^2) \sinh \left( \frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right) \right).
$$

(29)

In the limit $\ell \searrow 0$, the last expression behaves asymptotically as

$$
C k^{-1} e^{\frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right)},
$$

and thus after multiplying by $C_{\ell,k}^2$ is uniformly bounded above and below. The corresponding statements for $\nu_{\ell,k}$ and in the case $k = 0$ follow similarly.

□

We now consider the limiting behavior of the tensors $\kappa_{\ell,k}$ and $\nu_{\ell,k}$ as $\ell \searrow 0$. We again discuss the cases $k = 0$ and $k \geq 1$ separately.

Case $k = 0$. Looking at the explicit formulæ in Proposition 12 we see that both $\kappa_{\ell,0}(\tau, \theta)$ and $\nu_{\ell,0}(\tau, \theta)$ concentrate along $\tau = 0$ as $\ell \searrow 0$. To understand this behavior, we lift these tensors to $\hat{C}$. Thus we set $T = \frac{\tau}{\ell}$, and calculate that

$$
\ell^2 \kappa_{\ell,0} = \frac{dT^2}{\arctan \left( \frac{1}{\ell} \right)^2 (1 + T^2)^2} - \frac{\ell^2}{\arctan \left( \frac{1}{\ell} \right)^2} d\theta^2,
$$

(30)

$$
\ell^2 \nu_{\ell,0} = \frac{\ell}{\arctan \left( \frac{1}{\ell} \right)^2 (1 + T^2)} (dT \otimes d\theta + d\theta \otimes dT).
$$

Both families of tensors here are normalized in $L^2$ with respect to the area form $\ell dT \wedge d\theta$; they converge uniformly as $\ell \searrow 0$ to $(\frac{2}{\pi})^{\frac{1}{2}} \frac{dT^2}{(1 + T^2)^2}$ and to 0, respectively.

Case $k \geq 1$. By contrast, the tensors $\kappa_{\ell,k}$ and $\nu_{\ell,k}$, $k \geq 1$, in Proposition 12 converge uniformly on $C$. We study this now.

First consider $\kappa_{\ell,k}$; we restrict to $\tau > 0$ since $\tau < 0$ is similar. By definition

$$
\kappa_{\ell,k}(\tau, \theta) = \frac{\sqrt{k}}{\tau^2 + \ell^2} \left( \cos(k\theta) \frac{\cosh \left( \frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right) \right)}{e^{\frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right)}} - \sin(k\theta) \frac{\sin \left( \frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right) \right)}{e^{\frac{2k}{\ell} \arctan \left( \frac{1}{\ell} \right)}} \right).
$$
Since
\[
\lim_{\ell \to 0} \arctan(\frac{\tau}{\ell}) - \arctan(\frac{1}{\ell}) = 1 - \frac{1}{\tau},
\]
it follows that
\[
\lim_{\ell \to 0} \frac{\cosh(\frac{k}{\ell} \arctan(\frac{\tau}{\ell}))}{e^{\frac{k}{\ell} \arctan(\frac{\tau}{\ell})}} = \frac{1}{2} \lim_{\ell \to 0} \frac{e^{\frac{k}{\ell} \arctan(\frac{\tau}{\ell})}}{e^{\frac{k}{\ell} \arctan(\frac{1}{\ell})}} + \frac{1}{2} \lim_{\ell \to 0} \frac{e^{-\frac{k}{\ell} \arctan(\frac{\tau}{\ell})}}{e^{\frac{k}{\ell} \arctan(\frac{1}{\ell})}} = \frac{1}{2} e^{(1 - \frac{1}{\tau})k},
\]
and similarly,
\[
\lim_{\ell \to 0} \frac{\sinh(\frac{k}{\ell} \arctan(\frac{\tau}{\ell}))}{e^{\frac{k}{\ell} \arctan(\frac{\tau}{\ell})}} = \frac{1}{2} e^{(1 - \frac{1}{\tau})k}.
\]
We obtain, finally, that for \(|\tau| \leq 1\),
\[
\kappa_{\ell,k}(\tau, \theta) \to \kappa_{0,k}(\tau, \theta) = \frac{\sqrt{k}}{2\tau^2} e^{(1 - \frac{1}{|\tau|})k}(\cos(k\theta), -\sgn(\tau) \sin(k\theta)),
\]
and similarly,
\[
\nu_{\ell,k}(\tau, \theta) \to \nu_{0,k}(\tau, \theta) = \frac{\sqrt{k}}{2\tau^2} e^{(1 - \frac{1}{|\tau|})k}(\sin(k\theta), \sgn(\tau) \cos(k\theta)).
\]
Furthermore, the \(L^2\) norms of these limits, with respect to the limiting metric \(g_0\), are uniformly bounded in \(k\). It can be verified by straightforward calculation that if \(k \neq 0\), then \(\kappa_{0,k}\) and \(\nu_{0,k}\) belong to \(S_{tt}(g_0)\).

We point out, finally, that the equation \(\delta g_0 \mu_k = 0\) admits the further family of solutions
\[
\mu_k(\tau, \theta) = \frac{1}{\tau^2} e^{\frac{k}{|\tau|}}(\cos(k\theta), \sgn(\tau) \sin(k\theta)),
\]
\[
\lambda_k(\tau, \theta) = \frac{1}{\tau^2} e^{\frac{k}{|\tau|}}(-\sin(k\theta), \sgn(\tau) \cos(k\theta)) \quad (k \in \mathbb{N}).
\]
However, these blow up exponentially at \(\tau = 0\), and do not enter our considerations further.

**Remark 13.** The \(tt\) tensors \(\kappa_{0,k}\) and \(\nu_{0,k}\) admit a geometric interpretation when \(k = 0\) and \(1\). The case \(k = 0\) corresponds to an infinitesimal change of length and Dehn twist coordinates, while for \(k = 1\) these tensors represent infinitesimal translations of the node \(\{p\}\) along the punctured surface \(\Sigma \setminus \{p\}\). Passing to local holomorphic coordinates and identifying transverse-traceless tensors with meromorphic quadratic differentials, it is not hard to see that \(k = 0\) corresponds to meromorphic quadratic differentials with poles of order 2, while the case \(k = 1\) corresponds to those with poles of order 1.
6. Parametrix construction

We now construct a parametrix for the operator $P^{g_\ell}$ by gluing together two local parametrices $G_{\ell,0}$ and $G_{\ell,1}$, the first defined on some long cylinder $(\mathcal{C}, g_\ell)$ and the second on the ‘thick’ part $(\Sigma \setminus \mathcal{C}, g_\ell)$ of the Riemann surface $(\Sigma, g_\ell)$. To simplify notation, we carry this out in the case of a single divisor, i.e. when $|J| = 1$ in the notation introduced in §3. The general case is proved in exactly the same way.

Using the local coordinates $(\tau, \theta)$ as in §3 set $\Sigma_0 := \Sigma \setminus \left([-\frac{1}{2}, \frac{1}{2}] \times S^1\right)$ and $\mathcal{C} := (-\frac{3}{4}, \frac{3}{4}) \times S^1$; we sometimes refer to this last set as $\Sigma_1$. Thus $\{\Sigma_0, \Sigma_1\}$ is an open cover of $\Sigma$. Choose a partition of unity $\{\chi_0, \chi_1\}$ subordinate to this cover; also, choose functions $\tilde{\chi}_j \in C^\infty(\Sigma_j)$, such that $\tilde{\chi}_j \chi_j = \chi_j$. Let $G_{\ell,0}$ and $G_{\ell,1}$ denote the exact inverses of the operator $P^{g_\ell}$, say with Dirichlet conditions at the boundaries, acting on 1-forms on $\Sigma_0$ and $\Sigma_1$. Now define the parametrix

\begin{equation}
\tilde{G}_\ell = \tilde{\chi}_0 G_{\ell,0} \chi_0 + \tilde{\chi}_1 G_{\ell,1} \chi_1.
\end{equation}

It is immediate that $P^{g_\ell} \tilde{G}_\ell = \text{Id} + \sum_{j=0,1} [P^{g_\ell}, \tilde{\chi}_j] G_{\ell,j} \chi_j$.

The error term

\[ R_\ell := - \sum_{j=0,1} [P^{g_\ell}, \tilde{\chi}_j] G_{\ell,j} \chi_j \]

is smoothing. Indeed, the supports of $[P^{g_\ell}, \tilde{\chi}_j]$ and $\chi_j$ are disjoint and $G_{\ell,j}$ is a pseudodifferential operator, so its Schwartz kernel is singular only along the diagonal, so the Schwartz kernel of $R_\ell$ is $C^\infty$ and has support disjoint from the diagonal.

We adjust the parametrix $\tilde{G}_\ell$ slightly to make $R_\ell$ vanishes at $\ell = 0$. To do this, let $F_0(z, z') \in C^\infty(\Sigma \times \Sigma)$ be the solution of

\[ P^{g_\ell} F_0(z, z') = R_0(z, z'), \]

where $z'$ is regarded as a parameter. If we restrict $z$ to lie in $|\tau| \leq 1$, then $F_0$ decomposes as $F_0^0 + F_0^1$, where the first term is the zero Fourier mode (in $z$) and the other is the sum of all the other Fourier modes. Our explicit calculations above show that $F_0^1$ vanishes to all orders at $\tau = 0$, while $F_0^0$ is polyhomogeneous there. Now extend $F_0$ to a polyhomogeneous family $F_\ell$ on $[\Sigma \times \Sigma; \{\tau = \ell = 0\}]$. Note that we can assume that this vanishes for $|\tau'| \leq c$ and for all $\ell$ since $R_0$ vanishes in this region.

Next define

\[ \bar{G}_\ell := \tilde{G}_\ell + F_\ell \quad \text{and} \quad S_\ell := R_\ell - P^{g_\ell} F_\ell. \]

By construction, $S_\ell$ vanishes along the face $\ell = 0$, is polyhomogeneous, and its Schwartz kernel $S_\ell(z, z')$ has support in $\{ |\tau| \leq c \} \times \Sigma$. Observe finally that the family of operators $S_\ell$ is uniformly bounded on $L^2(\Sigma, dA_{g_\ell})$ and converges to 0 as $\ell \searrow 0$ in the operator norm topology. Since

\[ P^{g_\ell} \bar{G}_\ell = P^{g_\ell} \tilde{G}_\ell + P^{g_\ell} F_\ell = \text{Id} + \sum_{j=0,1} [P^{g_\ell}, \tilde{\chi}_j] G_{\ell,j} \chi_j + P^{g_\ell} F_\ell = \text{Id} - S_\ell, \]
we see that for \( \ell \) sufficiently small, the exact inverse of \( P^{g_\ell} \) is given by
\[
G^{g_\ell} = \bar{G}_\ell (\text{Id} - S_\ell)^{-1} : L^2(\Sigma, dA_{g_\ell}) \to L^2(\Sigma, dA_{g_\ell}).
\]
where
\[
(\text{Id} - S_\ell)^{-1} = \sum_{k=0}^{\infty} S^k_\ell
\]
is the norm-convergent Neumann series.

**Lemma 14.** Suppose that \( h_\ell \) is polyhomogeneous on \( \Sigma \times [0, \ell_0) \) and vanishes for \( |\tau| \leq c < 1 \). Then the unique solution to \( P^{g_\ell} \omega_\ell = h_\ell \) is polyhomogeneous on \( \Sigma \times [0, \ell_0) \). 

**Proof.** The solution \( \omega_\ell \) equals \( G^{g_\ell} h_\ell = \bar{G}_\ell k_\ell \) where \( k_\ell = (\text{Id} - S_\ell)^{-1} h_\ell \), or equivalently, \( k_\ell = h_\ell + Sk_\ell \). Notice that both terms on the right vanish for \( |\tau| \leq c \), the first term by hypothesis and the second because \( S_\ell(z, z') \) vanishes in this region. Therefore \( k_\ell \) itself vanishes near \( \tau = 0 \). Therefore, by Proposition 11, \( \omega_\ell = \bar{G}_\ell k_\ell \) is polyhomogeneous, as claimed. \( \square \)

7. **Proof of the main result**

Following the notation of §3, fix \( q_0 \in H_j \) and let \( V \subset \tilde{M}_\gamma \) be a neighborhood of \( q_0 \). Recall also from Proposition 3 the existence of a polyhomogeneous slice, i.e., family of symmetric 2-tensors \( h_\sigma \) on the vertical tangent bundle \( T^\text{ver} \tilde{M}_{\gamma,1} \), which restricts to a family of approximately hyperbolic metrics.

In this final section, we complete the main step of our main result. This is done by constructing a local frame for the subbundle \( S_\sigma(h_\sigma) \subseteq \text{Sym}^2(T^\text{ver} \tilde{M}_{\gamma,1}) \) of transverse-traceless two-tensors which depends in a polyhomogeneous way on the vertical tangent bundle \( T^\text{ver} \tilde{M}_{\gamma,1} \), which restricts to a family of approximately hyperbolic metrics.

In this section, we construct a polyhomogeneous \((6\gamma - 6)\)-frame whose elements are approximately transverse-traceless, and then correct these sections to be exactly transverse-traceless, preserving polyhomogeneity in the process. This construction relies crucially on the polyhomogeneity of solutions \( \omega_\ell \) to \( P^{g_\ell} \omega_\ell = h_\ell \) when \( h_\ell \) has support disjoint from the set of degenerating central geodesics, cf. Lemma 14. The fact that these sections remain independent is because the correction terms are uniformly small.

**Lemma 15.** Let \( g \) be a smooth metric on \( \Sigma \) and \( G \) the unique hyperbolic metric conformal to \( g \), i.e., \( G = e^{2u}g \) for some \( u \in C^\infty(\Sigma) \). Then there exists a constant \( C > 0 \), which only depends on \( \|u\|_{C^0} \), such that
\[
\|T^g \kappa - \kappa\|_{L^2(\Sigma, dA_g)} \leq C \|\delta^g \kappa\|_{L^2(\Sigma, dA_g)}
\]
for all \( \kappa \in S^2_0(\Sigma, T^*\Sigma) \).

**Proof.** We first prove this estimate when \( g = G \) is already hyperbolic. Set \( \sigma = T^g \kappa - \kappa \). Note that \( B^g \kappa = \delta^g \kappa \) since \( \text{tr}^g \kappa = 0 \); furthermore, by \( \|u\|_{C^0} \),
we have \( \sigma = -\pi^g(\delta^g)^*G^g\delta^gK \). The claim in this case then follows from the estimate

\[
\|\sigma\|_{L^2(\Sigma,dA_\sigma)}^2 = \|\pi^g(\delta^g)^*G^g\delta^gK, \pi^g(\delta^g)^*G^g\delta^gK\|_g = \|\pi^g(\delta^g)^*G^g\delta^gK, (\delta^g)^*G^g\delta^gK\|_g = \|\delta^g, G^g\delta^gK, G^g\delta^gK\|_g \leq \|\delta^gK\|_{L^2(\Sigma,dA_\sigma)}^2.
\]

(35)

The fourth identity again uses that \( B^g = \delta^g \) on trace-free tensors. The last inequality holds because \( P^g \geq 1 \) (as a self-adjoint operator on \( L^2 \)), cf. (16), since \( K^g = -1 \). This proves the claim in the case where \( g \) is hyperbolic.

Consider now the case of a general metric \( g \), where \( G = e^{2u}g \). We have \( \sigma = T^gK - \kappa \) as before, but now write \( \sigma_1 = T^GK - \kappa \). Since \( S_\text{tt}(g) = S_\text{tt}(G) \), and \( T^g \) is an orthogonal projector, it follows that

\[
\|\sigma\|_{L^2(\Sigma,dA_\sigma)} \leq \|\sigma_1\|_{L^2(\Sigma,dA_\sigma)}.
\]

Using the general identity \( \delta^Gh = e^{-2u}\delta^g h \) for traceless tensors \( h \), and taking norms with respect to \( g \), we can further estimate

\[
\|\sigma\|_{L^2(\Sigma,dA_\sigma)} \leq \|\sigma_1\|_{L^2(\Sigma,dA_\sigma)} \leq C\|\sigma_1\|_{L^2(\Sigma,dA_G)} \leq C\|\delta^G\kappa\|_{L^2(\Sigma,dA_G)} \leq C\|\delta^g\kappa\|_{L^2(\Sigma,dA_\sigma)},
\]

where the constant \( C > 0 \) depends only on \( \|u\|_{C^0(\Sigma)} \). Here the third inequality holds by (35). This proves the claim in the general case. \( \square \)

We shall apply Lemma 15 to the family \( g_\ell \) of approximately hyperbolic metrics. To do so, it is clearly important to show that the constant \( C \) appearing in this Lemma is uniform in \( \ell \) as \( \ell \searrow 0 \). In other words, we must prove that the conformal factor is uniformly bounded.

**Lemma 16.** Let \( g_\ell \) be the family of approximately hyperbolic metrics and let \( G_\ell = e^{2u_\ell}g_\ell \) be the unique hyperbolic metric conformally equivalent to \( g_\ell \). Then there are constants \( c > 0 \) and \( \ell_0 > 0 \) such that

\[
\|u_\ell\|_{C^0(\Sigma)} \leq c
\]

for \( 0 \leq \ell \leq \ell_0 \).

**Proof.** By construction, the metric \( g_0 \) is hyperbolic, and \( g_\ell \to g_0 \) uniformly in \( C^\infty \) on \( \Sigma \setminus A \), where \( A \) is the annulus \((-\tau_0, \tau_0) \times S^1\), as \( \ell \searrow 0 \). Thus \( K_{g_\ell} < 0 \) for \( \ell \) small enough. With \( \Delta_{G_\ell} \) the (negative semidefinite) scalar Laplacian, \( u_\ell \) satisfies

\[
\Delta_{G_\ell}u_\ell + K_{G_\ell} - e^{-2u_\ell}K_{g_\ell} = 0.
\]

(36)
Since $K_{G_\ell} = -1$ and $K_{g_\ell} < 0$, there are sub- and supersolutions
\[ u_{\text{sub}} \equiv -c, \quad u_{\text{sup}} \equiv c, \]
for some $c \gg 0$, and for all $\ell < \ell_0$. This means that $-c \leq u_\ell \leq c$ for all such $\ell$, as claimed.

Our discussion now splits naturally into two cases.

Transverse-traceless tensors concentrating at a central geodesic. For the rest of this section, we fix the following notation. Consider a neighborhood $\mathcal{V} = [0, \varepsilon) \times S^1 \times W$ where $W$ is a neighborhood in some divisor $D_j$. Let $q \in V$. Then we use the notation $g_\ell$ for a hyperbolic metric on $\Sigma$ (which is degenerate if $\ell = 0$) representing the point $q$. We suppress its dependence on the remaining Fenchel-Nielsen coordinates.

Let $\kappa_{\ell,0}$ and $\nu_{\ell,0} \in S_{tt}(g_\ell)$ be the symmetric transverse-traceless 2-tensors on the cylinder $(C, g_\ell)$. Fix a smooth cutoff function $\chi: [-1, 1] \to [0, 1]$ with $\text{supp}(\chi) \subseteq [-\frac{3}{4}, \frac{3}{4}]$ and $\chi \equiv 1$ for $|\tau| \leq \frac{1}{2}$. Now set
\[ \hat{\mu}^1_\ell := \chi \kappa_{\ell,0} \quad \text{and} \quad \hat{\mu}^2_\ell := \chi \nu_{\ell,0}, \]
which we extend by 0 to all of $\Sigma$, and then consider their projections
\[ \mu^1_\ell := T^{g_\ell} \hat{\mu}^1_\ell \quad \text{and} \quad \mu^2_\ell := T^{g_\ell} \hat{\mu}^2_\ell \]
to $S_{tt}(g_\ell)$.

**Proposition 17.** The families $\mu^1_\ell$, $\mu^2_\ell$ are polyhomogeneous.

**Proof.** Since the tensors $\kappa_{\ell,0}$ and $\nu_{\ell,0}$ are divergence-free with respect to $g_\ell$, then certainly $\delta^{g_\ell} \hat{\mu}^1_\ell$ vanishes except when $\frac{1}{2} \leq |\tau| \leq \frac{3}{4}$. Thus Lemma 14 can be applied, with $h_\ell = \delta^{g_\ell} \hat{\mu}^1_\ell$, and shows that the family of 1-forms $G^{g_\ell} \delta^{g_\ell} \hat{\mu}^1_\ell$ is polyhomogenous. The claim then follows from (21). \qed

We must also prove that $\mu^1_\ell$ and $\mu^2_\ell$ are linearly independent when $\ell$ is small. This is proved in the remainder of this section. By Lemma 15 it remains to estimate the divergences of $\hat{\mu}^1_\ell$ and $\hat{\mu}^2_\ell$.

**Proposition 18.** The divergence of $\hat{\mu}^j_\ell$ vanishes outside $A' := \{ \frac{1}{2} \leq |\tau| \leq \frac{3}{4} \}$ and satisfies
\[ \| \delta^{g_\ell} \hat{\mu}^j_\ell \|_{L^2(\Sigma, dA_{g_\ell})} \leq C \ell^{\frac{3}{2}} \quad (j = 1, 2). \]
Moreover, $\| \sigma^j_\ell \|_{L^2} = \| \mu^j_\ell - \hat{\mu}^j_\ell \|_{L^2} \to 0$ as $\ell \searrow 0$.

**Proof.** Since $\kappa_{\ell,0}$ is divergence-free, it follows from (27) that
\[ \delta^{g_\ell} \hat{\mu}^j_\ell = (\partial_\tau \chi) \frac{\ell^2}{\arctan(\frac{1}{4})} \frac{\ell^{\frac{3}{2}}}{(\ell^2 + \tau^2)^{\frac{3}{2}}}, \]
and
\[ \| \delta^{g_\ell} \hat{\mu}^j_\ell \|_{L^2(\Sigma, dA_{g_\ell})} \leq C \ell^{\frac{3}{2}} \quad (j = 1, 2). \]
which has support in $A'$. We estimate

$$\|\delta g^{\ell} \hat{\mu}_{j}\|_{L^2(S, dA_{g^{\ell}})}^2 = 2 \int_{\frac{3}{4}}^{\frac{3}{2}} \int_0^{2\pi} \frac{\ell^3 |\partial_\tau \chi(\tau)|^2}{\arctan(\frac{\ell}{\tau^2 + \ell^2})} d\theta d\tau \leq \frac{C \ell^3}{\arctan(\frac{3}{4\ell})} \int_{\frac{3}{4}}^{\frac{3}{2}} \frac{d\tau}{\tau^2 + \ell^2} = \frac{C \ell^2}{\arctan(\frac{3}{4\ell})} \left( \arctan(\frac{3}{4\ell}) - \arctan(\frac{1}{2\ell}) \right).$$

Observing that

$$\lim_{\ell \to 0} \frac{\arctan(\frac{3}{4\ell}) - \arctan(\frac{1}{2\ell})}{\ell} = \frac{2}{3},$$

the assertion on the decay rates of $\delta g^{\ell} \hat{\mu}_{j}$, $j = 1, 2$, is immediate.

For the second claim, let $G^{\ell} = e^{-2u^{\ell}} g^{\ell}$ be the hyperbolic metric, as before. By Lemma 16, the family of conformal exponents $u^{\ell}$ is $C^0$ bounded, so we can apply Lemma 15 to get the result. □

**Transverse-traceless tensors decaying on long cylinders.** We next construct a local $(6\gamma - 8)$-frame of transverse-traceless tensors which is polyhomogeneous and complements the two sections determined in the last subsection. Taking all these transverse-traceless tensors together, we will have obtained a local polyhomogeneous frame of the rank-$(6\gamma - 6)$ bundle $S_{tt}(g_q)$.

Fix $q_0 \in H_j$ and let $(\Sigma, g_0)$ be the complete surface representing $q_0$. The space of symmetric tensors which are transverse-traceless with respect to $g_0$ and decay along the cusps is denoted $S_{tt}(g_0)$. It is well-known that the real dimension of this space is $6\gamma - 8$. Fix an orthonormal basis $\{\kappa^3, \ldots, \kappa^{6\gamma-6}\}$ for this space. By (31), each $\kappa^j$ decays exponentially as $\tau \to 0$, so we obtain the approximately orthonormal local frame

$$\mu^j(q) := \chi \kappa^j \quad (q \in \mathcal{V}, j = 3, \ldots, 6\gamma - 6),$$

which vanishes outside the cylinder, is identically 1 near $\tau = 0$, and so $\partial_\tau \chi$ is supported away from $\{\tau = 0\}$.

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$$\mu^j(q) := \chi \kappa^j \quad (q \in \mathcal{V}, j = 3, \ldots, 6\gamma - 6),$$

which vanishes outside the cylinder, is identically 1 near $\tau = 0$, and so $\partial_\tau \chi$ is supported away from $\{\tau = 0\}$.

**Proposition 19.** The projected family $\{\mu^j_{\ell}\}$ is polyhomogeneous and for $\ell$ sufficiently small, these vectors are independent.

**Proof.** The first statement follows immediately from the polyhomogeneity of $(\delta g^{\ell})^*$ and of the family of 1-forms $G^{\ell} \delta g^{\ell} \mu^j$. Note that we use here that $\mu^j$ is supported away from $\{\tau = 0\}$.
To prove independence, we consider
\[ \delta^{\ell} \mu^j = \delta^{\ell} (\chi \kappa^j) = \chi \delta^{\ell} \kappa^j + \partial_\tau \chi \sqrt{\tau^2 + \ell^2 \kappa^j} (E_1) \]
Since \( \text{supp} \chi \subseteq \Sigma \setminus C_2 \) and \( \text{supp} \partial_\tau \chi \subseteq C_1 \), it follows that
\[ \frac{1}{2} \left\| \delta^{\ell} \mu^j \right\|_{L^2(\Sigma, dA_\ell)}^2 \leq \left\| \chi \delta^{\ell} \kappa^j \right\|_{L^2(\Sigma \setminus C_2, dA_\ell)}^2 + \left\| \partial_\tau \chi \sqrt{\tau^2 + \ell^2 \kappa^j} (E_1) \right\|_{L^2(C_1, dA_\ell)}^2. \]
Fix \( \varepsilon > 0 \). Choosing \( \tau_2 \ll 1 \) and \( \ell_0 > 0 \) sufficiently small, it follows that if \( \ell \leq \ell_0 \) and \( j = 3, \ldots, 6\gamma - 6 \), the right-hand side of (41) is bounded by \( C \varepsilon \). Indeed, on \( \Sigma \setminus C_2 \), \( \delta^{\ell} \kappa^j \) converges uniformly to 0. As for the second term on the right in (41), the equations (31) and (32) imply that \( \kappa^j(\tau, \theta) \) decays rapidly as \( \tau \to 0 \), hence this term is small also. To conclude, Lemma 15 gives finally that
\[ \left\| \mu^j - \mu^j \right\|_{L^2(\Sigma, dA_\ell)} \leq C \left\| \delta^{\ell} \mu^j \right\|_{L^2(\Sigma, dA_\ell)}, \]
where \( C \) does not depend on \( \ell \). Hence \( \{\mu^1_\ell, \ldots, \mu^{6\gamma-6}_\ell\} \) is independent when \( \varepsilon \) is small enough. \( \square \)

**Proof of the main result.** We have now constructed the frame \( \{\mu^1_\ell, \ldots, \mu^{6\gamma-6}_\ell\} \) of transverse-traceless tensors, and the remaining task is to show that these span the vector space \( S_{tt}(g_q) \), for each \( q \in V \).

**Lemma 20.** There is a constant \( \ell_0 > 0 \) such that for all \( 0 \leq \ell < \ell_0 \) the set
\[ \{\mu^1_\ell, \mu^2_\ell, \mu^3_\ell, \ldots, \mu^{6\gamma-6}_\ell\} \]
is a polyhomogeneous local frame over \( V \) of the vector bundle \( S_{tt}(g_q) \).

**Proof.** Recall the symmetric (but not necessarily divergence-free) tensors \( \tilde{\mu}_i^j \), \( i = 1, 2 \). These are supported on the cylinder \( C \subseteq \Sigma \) and their coefficients depend only on the variable \( \tau \). We then consider the family (39). Altogether, this gives another family \( \{\mu^1_\ell, \ldots, \mu^{6\gamma-6}_\ell\} \) of tensors whose restriction to \( C \) has vanishing zeroth Fourier mode,
\[ \langle \tilde{\mu}_i^j, \mu^j \rangle_{g_\ell} = 0 \]
for all \( i = 1, 2, j = 3, \ldots, 6\gamma - 6 \), and any \( \ell > 0 \). By Propositions 18 and 19 the estimates
\[ \left\| \mu^i_\ell - \tilde{\mu}_i^j \right\|_{L^2(\Sigma, dA_q)} < \varepsilon \quad (i = 1, 2) \]
and
\[ \left\| \mu^j_\ell - \mu^j \right\|_{L^2(\Sigma, dA_q)} < \varepsilon \quad (j = 3, \ldots, 6\gamma - 6) \]
hold for all $\varepsilon > 0$ and every sufficiently small $0 \leq \ell \leq \ell_0(\varepsilon)$. This implies that

$$|\langle \mu_1^i, \mu_2^j \rangle_{g_{\ell}}| \leq |\langle \mu_1^i - \hat{\mu}_1^i, \mu_2^j \rangle_{g_{\ell}}| + |\langle \mu_1^i - \hat{\mu}_1^i, \mu_2^j \rangle_{g_{\ell}}| + |\langle \mu_2^j, \mu_2^j \rangle_{g_{\ell}}| < \varepsilon^2 + \varepsilon(\|\hat{\mu}_1^i\|_{L^2(\Sigma, dA_{g_{\ell}})} + \|\mu_2^j\|_{L^2(\Sigma, dA_{g_{\ell}})}) \leq C\varepsilon.$$  

Hence if $\varepsilon$ is sufficiently small, the subspaces spanned by $\{\mu_1^1, \mu_1^2\}$ and $\{\mu_3^3, \ldots, \mu_6^6\}$ are transversal. This implies the claim. \(\square\)

We are now in position to prove our main result.

**Proof of Theorem 1** It suffices to establish our result in an open set $V$ around a point $q_0 \in \partial \tilde{M}_{\gamma,1}$. Let us choose a polyhomogeneous slice of approximately hyperbolic metrics $g(w)$ in $V$. By Lemma 20 yields the existence of a local polyhomogeneous frame $\kappa_1, \ldots, \kappa_{6\gamma-6}$ of sections for the bundle of transverse-traceless tensors over $V$. In addition, by [17], the family of conformal factors $e^{2\varphi(w)}$ relating the approximately hyperbolic metrics $g(w)$ to the exact hyperbolic metrics on each fibre is also polyhomogeneous on $\tilde{M}_{\gamma,1}$. The matrix coefficients of $g_{WP}$ are given by the expression

$$(g_{WP})_{ij} = \int_{\Sigma} \langle \kappa_i, \kappa_j \rangle_{g(w)} e^{-2\varphi(w)} \, dA_{g(w)}.$$  

Everything in this expression is polyhomogeneous. To finish, we observe that the integral over $\Sigma$ can be regarded as a pushforward with respect to the $b$-fibration $\tilde{\Pi} : \tilde{M}_{\gamma,1} \to \tilde{M}_{\gamma}$. We may therefore invoke the properties of polyhomogeneous functions with respect to pushforwards by $b$-fibrations, as proved in [16]. This theorem proves that each $(g_{WP})_{ij}$ is polyhomogeneous on $\tilde{M}_{\gamma}$. More precisely, the powers in the polyhomogeneous expansions of each quantity here are nonnegative integer powers of $\ell^{1/2}$ (or in half-integer powers of the appropriate boundary defining functions at each face of $\tilde{M}_{\gamma,1}$), and each term $\ell^{k/2}$ is possibly multiplied by a polynomial in $\log \ell$. The pushforward theorem then asserts that this pushforward has an expansion of exactly the same form.

As we have noted before, the result of Melrose and Zhu at present only asserts polyhomogeneity of $\varphi$ near $D^{reg}$, so our result only applies near this portion of $\tilde{M}_{\gamma,1}$. However, they expect their result to hold in general, and when that is complete, our result here will then extend. \(\square\)

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