Laser planar trapping

We investigate the phenomenon of the transverse breaking of symmetry of Gaussian lasers transmitted through dielectrics, showing for which incidence conditions and beam parameters it can be observed in optical experiments. The numerical analysis, done by using the integral form of the transmitted beam at the lower interface of a dielectric prism, shows, for incidence approaching the critical region, a laser planar trapping. By using an analytic approximation for the intensity of lower transmitted beam, we model the wave front curvature by a closed expression which depends on the refractive index, on the geometrical properties of the dielectric block, and, finally, on the incidence angle of the incoming beam. The analytic formula shows an excellent agreement with the numerical analysis and it could be useful in future experimental implementations.

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I. Introduction

In free space, the wavelength, the position of the beam waist and the waist diameter completely describe the behavior of a Gaussian optical beam [1–3]. When a beam passes through a dielectric interface, the wave front curvature is changed, resulting in new values of the waist diameter on the output side of the interface. In this paper, we aim to analyse the incidence conditions and beam parameters which maximize the breaking of the transverse symmetry of a Gaussian optical beam transmitted at the lower (dielectric/air) interface of a dielectric prism, see Fig. 1(a). The analysis, first done by numerically integrating the transmitted beam, will then be completed by giving an analytic expression for the modulating factor of the waist diameter parallel to the incidence plane. For the purpose of this article, it is convenient to write the incident electric field in its integral form [4,5],

\[ E_{\text{INC}}(r) = E_0 \int k_x k_y g(k_x, k_y) e^{ik \cdot r}, \]

where

\[ g(k_x, k_y) = \frac{w_0^2}{4\pi} \exp \left[ - \left( k_x^2 + k_y^2 \right) \frac{w_0^2}{4} \right] \]

is the Gaussian wave number distribution, \( w_0 \) the laser beam waist, \( |k| = 2\pi/\lambda \) and \( \lambda \) the wavelength. By using the paraxial approximation [3],

\[ k_z \approx |k| - \frac{1}{2} \frac{w_0^2}{|k|}, \]

the previous integral can analytically be solved leading to a Gaussian beam,

\[ E_{\text{INC}}(r) = \frac{E_0 e^{ik|z|}}{1 + i z/z_0} \exp \left[ - \frac{x^2 + y^2}{w_0^2 (1 + i z/z_0)} \right], \]

where \( z_0 = \pi w_0^2/\lambda \) is the Rayleigh axial range. The intensity,

\[ I_{\text{INC}} = |E_{\text{INC}}|^2, \]

\[ I_{\text{INC}} = I_0 \left[ \frac{w_0}{w(z)} \right]^2 \exp \left[ -2 \frac{x^2 + y^2}{w^2(z)} \right], \]

where \( I_0 = |E_0|^2 \) and

\[ w(z) = w_0 \sqrt{1 + \left( z/z_0 \right)^2}, \]

shows the well-known transverse symmetry of Gaussian beams.

This paper was prepared with the aim to show when this transverse symmetry is broken, to understand the reason of such a breaking of symmetry and, once numerically confirmed, to give an analytic closed expression to be used in future experimental implementations. As we shall see in detail later, this breaking of symmetry acts on the transverse component positioned in the plane of incidence, and, for critical incidence, generates a planar trapping of the beam in the plane perpendicular to the incidence one. It is important to observe that the phenomenon presented in this article can also be seen in a reversed order. Indeed, by an appropriate choice of the geometry and refractive index of the dielectric
When a plane electromagnetic wave encounters a plane interface between two transparent media, reflected and transmitted waves are generated at such an interface. The angles which these waves form with the normal to such an interface are determined by the reflection and Snell laws and the amplitudes described by the Fresnel coefficients. For Transverse Electric (TE) and Transverse Magnetic (TM) waves, the Fresnel coefficients at the left (air/dielectric) interface, \( \hat{x}-\hat{z} \) system in Fig. 1(b), can be calculated by solving the matrix continuity equation:

\[
\begin{pmatrix}
    \frac{1}{k_x} & 1 \\
    -k_z & \frac{1}{k_z}
\end{pmatrix}
\begin{pmatrix}
    \frac{1}{R^{(e)}}
\end{pmatrix}
= \begin{pmatrix}
    \frac{a_p}{q_x/q_s}
\end{pmatrix}
\tilde{T}^{(e)}(k_x, k_z),
\]

where \( \{ a_{TE}, a_{TM} \} = \{ 1, n \} \), \( k_z = -k_x \sin \theta + k_z \cos \theta \), and \( q_x = \sqrt{(n^2 - 1)|k|^2 + q_z^2} \). The Fresnel coefficients depend on the wave number components appearing in the Gaussian distribution:

\[
\tilde{T}^{(e)}(k_x, k_z) = 2k_x / ( \alpha_s k_x + q_x / a_s),
\]

and reproduce the well known angular Fresnel coefficients when calculated at the center of the wave number distribution,

\[
\tilde{T}^{(e)}(0, 0) = 2 \cos \theta / ( a_s \cos \theta + n \cos \psi / a_s ).
\]

In (b) the system of coordinates of the incident, lower transmitted, and upper transmitted beams.

The angles \( \psi \) and \( \phi \) contain an implicit dependence on the incidence angles at the left (\( \theta \)) and lower (\( \varphi \)) interface, by the Snell law, i.e. \( \sin \theta = n \sin \psi \) and \( n \sin \varphi = \sin \phi \). Finally, the angle \( \varphi \) contains the information of the geometrical properties of the dielectric, \( \varphi = \psi + \alpha \).

II. The lower transmitted beam

When a plane electromagnetic wave encounters a plane interface between two transparent media, reflected and transmitted waves are generated at such an interface. The angles which these waves form with the normal to such an interface are determined by the reflection and Snell laws and the amplitudes described by the Fresnel coefficients. For Transverse Electric (TE) and Transverse Magnetic (TM) waves, the Fresnel coefficients at the left (air/dielectric) interface, \( \hat{x}-\hat{z} \) system in Fig. 1(b), can be calculated by solving the matrix continuity equation [1]-[3].

\[
\begin{pmatrix}
    a_s & q_x/n_s \\
    q_x/q_s & a_s
\end{pmatrix}
\begin{pmatrix}
    e^{i q_s h} \\
    e^{-i q_s h}
\end{pmatrix}
= \begin{pmatrix}
    a_s & q_x/n_s \\
    q_x/q_s & a_s
\end{pmatrix}
\begin{pmatrix}
    e^{i k_s h} \\
    e^{-i k_s h}
\end{pmatrix}
\]

where \( q_x = -q_x \sin \alpha + q_x \cos \alpha \) (note that \( q_x = k_x \)), \( k_s = \sqrt{(1-n^2)|k|^2 + q_z^2} \), and \( h \) is the distance along the \( z \) axis between the lower (dielectric/air) discontinuity and the origin of the axes fixed, in our analysis, at the point in which the incident beam touches the left (air/dielectric) interface, see Fig. 1(a).

This choice implies for an experimental implementation to set the position of the beam waist at the point in which the incident beam crosses the left (air/dielectric) interface. Solving Eqs. (7), we find

\[
T_s^{(e)}(k_x, k_z) = 2q_x e^{i(q_x - k_s)h} / (a_s k_x + q_x / a_s).
\]
where, at the center of the Gaussian distribution, become
\[ T_\alpha(x,0,0) = \frac{2n \cos \phi \exp\left[ i \left( n \cos \phi - \cos \phi \right) |k|h \right]}{a_x \cos \phi + n \cos \phi / a_x}. \tag{9} \]

Finally, the integral form of the lower transmitted beam is given by
\[ E_\alpha(x,y,z) = E_o \int k_x k_y T_\alpha(k_x,k_y) g(k_x,k_y) \times \exp\left[ i \left( k_{x_0} x + k_y y + k_{z_0} z \right) \right], \tag{10} \]

where \( T_\alpha = \tilde{T}_\alpha T_{\alpha}^{*}. \)

In order to check the transverse symmetry of the lower transmitted beam, it is convenient to rewrite the optical spatial phase in terms of its proper coordinates \( x-z \) system, drawn in Fig. 1(b),
\[ (k_{x_0} \cos \phi - k_{z_0} \sin \phi)x + k_y y + (k_{z_0} \sin \phi + k_{z_0} \cos \phi)z, \]
and then numerically calculate the intensity.

The \( x/y \) transverse profiles of the normalized lower transmitted beam intensity
\[ \left| E_\alpha^{(x)}(x,y,z) / E_\alpha^{(x)}(x_0,0,z) \right|^2, \]

where \( x_0 = h \sin(\varphi - \phi) / \cos \phi \) is the center of the beam predicted by the Snell law \( \sin \theta = n \sin \psi, \varphi = \psi + \alpha, n \sin \varphi = \sin \phi \), are plotted in Fig. 2 for a laser, with wave length of 633 nm and \( w_0 = 100 \mu \text{m} \), whose axial direction forms incidence angles of \( -60^\circ, -30^\circ \), and \( -10^\circ \) with the normal to the left (air/dielectric) interface of a BK7 \( (n = 1.5151) \) dielectric prism with \( \alpha = \pi/4 \). For each incidence angle, the intensity of the beam is plotted for three axial values, i.e. \( z = 1, 5, 10 \text{ cm} \). The transverse profiles appearing in Fig. 3 refer to a laser with \( w_0 = 1 \text{ mm} \).

The numerical analysis shows that the \( y \)-profiles of the lower transmitted beam are practically unchanged with respect to the incident one. So, the first conclusion is that, for short propagation in the dielectric, the wave front of the beam does not change in the direction perpendicular to the plane of incidence. For what concerns the \( x \)-profiles, the data plotted in Figs. 2 and 3 clearly manifest a breaking of symmetry in the transverse component parallel to the plane of incidence.

To quantify this breaking of symmetry, we first use the
numerical data of the profiles plotted in Fig. 3. In this case, due to the fact that the axial dependence does not play a significant role, by taking, from the numerical data, the ratio between the $x$ and the $y$ profile, we immediately obtain

$$\begin{bmatrix} \tilde{w}_{0,-60} \\ \tilde{w}_{0,-30} \\ \tilde{w}_{0,-10} \end{bmatrix} = \begin{bmatrix} 1.60 \\ 0.91 \\ 0.43 \end{bmatrix} \text{mm}. \quad (11)$$

The modulation factors 1.60, 0.91, and 0.43 are in excellent agreement with the axial behaviour showed for the $x$-profiles in Fig. 2, where $w_0 = 100 \mu m$. This clearly suggests that the breaking of symmetry in the transverse profiles parallel to the incidence plane does not depend on the choice of $w_0$. One point we should stress is that the planar trapping in the plane perpendicular to the incidence plane is found for incidence angles approaching the critical angle, $\varphi_{\text{cri}} = \arcsin(1/n)$ which implies $\theta_{\text{cri}} = -5.6^\circ$, and, due to the increase of the beam divergence, it is detectable for axial position of the camera near to the lower interface. The contour plots shown in Fig. 4 refer to an incident Gaussian beam with $w_0 = 1 \text{ mm}$ and incidence angle $\theta = -6^\circ$. The plots manifest a planar trapping at an axial distance $z \leq 10 \text{ cm}$. The axial behaviour of $x$ profile of the lower transmitted beam is compatible with

$$\tilde{w}_{0,-6} = 0.13 \text{ mm}. \quad (12)$$

III. The modulation factor

In order to obtain an analytic expression for the beam divergence of the lower transmitted beam in the plane parallel to the incidence plane, we have to calculate the intensity of the lower transmitted beam by approximating the transmission coefficient, which appears in the modulated wave number distribution, by $T^{(0,0)}(0,0)$ and by using the Taylor expansion of the optical phase up to the second order in $k_x$ and $k_y$.

The spatial phase of the lower transmitted beam can then be approximated by

$$|k| z + k_x \frac{x - x_0}{\rho} - \frac{k_x}{2 |k|} z^2 + k_y y - \frac{k_y}{2 |k|} z,$$

where $x_0 = h \sin(\varphi - \phi)/\cos \varphi$ is the geometrical shift pre-
dicted by the Snell law and
\[
\rho = \frac{\cos \psi \cos \phi}{\cos \theta \cos \varphi} \tag{13}
\]
is an angular ratio which is an implicit function of the incidence angle \( \theta \) and of the refraction index \( n \) (which also contains an implicit dependence on \( \lambda \)). From the approximated spatial phase, we immediately observe the breaking of symmetry between the the transverse components of the lower transmitted beam.

The approximations done for transmission coefficient and for the optical phase allow us to analytic integrate Eq. (10) and to obtain a closed expression for the lower transmitted beam and consequently for its intensity,
\[
I^{(\sigma)} = I_0 \rho \left| T^{(\sigma)}(0,0) \right|^2 \frac{w_0^2}{w(z)w(z/\rho)} \times \exp \left[ -2 \frac{(x-x_0)^2}{w^2(z)} \right] \exp \left[ -2 \frac{y^2}{w^2(z)} \right]. \tag{14}
\]

Introducing
\[
\tilde{w}(z) = \rho w(z/\rho^2) = \rho w_0 \sqrt{1 + (z/\rho^2 z_0)^2} = \tilde{w}_0 \sqrt{1 + (z/\tilde{z}_0)^2}, \tag{15}
\]
where \( \tilde{z}_0 = \pi \tilde{w}_0^2/\lambda \), we can rewrite the intensity in terms of the proper transverse waists of the beam, i.e. \( w_0 \) for the transverse one parallel to the plane of incidence and \( \tilde{w}_0 \) for the transverse component perpendicular to the plane of incidence,
\[
I^{(\sigma)} = I_0 \rho \left| T^{(\sigma)}(0,0) \right|^2 \frac{w_0 \tilde{w}_0}{w(z)\tilde{w}(z)} \times \exp \left[ -2 \frac{(x-x_0)^2}{w^2(z)} \right] \exp \left[ -2 \frac{y^2}{\tilde{w}^2(z)} \right]. \tag{16}
\]

The modulation factor \( \rho \) is plotted in Fig. 5 and shows an excellent agreement with the numerical data obtained in Section II.

Once obtained the analytical expression for the modulation factor \( \rho \), it is also possible to calculate the incidence angle for

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**Figure 4:** Contour plots of the lower transmitted beam for an incident Gaussian laser with wavelength of 633 nm and a beam waist of 1 mm, are plotted for and incidence angles of \(-6^\circ\) and different axial positions.
which we recover the transverse symmetry between. Indeed, by solving the equation

\[ \rho = 1 \]

which implies

\[ \cos^2 \psi_{\text{sym}} [1 - n^2 \sin^2 \varphi_{\text{sym}}] = (1 - n^2 \sin^2 \psi_{\text{sym}}) \cos^2 \varphi_{\text{sym}}, \]

we immediately get \( \cos^2 \psi_{\text{sym}} = \cos^2 \varphi_{\text{sym}} \) and consequently \( \psi_{\text{sym}} = -\alpha / 2 \). This means that the transverse symmetry is recovered for the following incidence angle

\[ \theta_{\text{sym}} = -\arcsin \left( n \sin \frac{\alpha}{2} \right). \]  

(17)

For Fused Silica, BK7, and LASF9 prisms, the incidence angles for which we recover the transverse symmetry, in the case of an incoming beam with a wavelength of 633 nm, are

\[
\begin{pmatrix}
\theta_{\text{sym}}^{\text{FUSED SILICA}} \\
\theta_{\text{sym}}^{\text{BK7}} \\
\theta_{\text{sym}}^{\text{LASF9}}
\end{pmatrix} = \begin{pmatrix} 22.15'' \pi/6, & 33.89'' \pi/4, & 46.76'' \pi/3 \\
23.09'' \pi/6, & 35.44'' \pi/4, & 49.25'' \pi/3 \\
28.52'' \pi/6, & 44.91'' \pi/4, & 67.29'' \pi/3 \end{pmatrix},
\]

where the lower script angles indicate the \( \alpha \) angle of the prism.

The modulation factor \( \rho \) goes to zero when the incidence angle approaches the critical angle,

\[ \theta_{\text{cri}} = \arcsin \left( n \sin \left( \arcsin \frac{1}{n} - \alpha \right) \right). \]

For Fused Silica, BK7, and LASF9 prisms, the critical angle are

\[
\begin{pmatrix}
\theta_{\text{cri}}^{\text{FUSED SILICA}} \\
\theta_{\text{cri}}^{\text{BK7}} \\
\theta_{\text{cri}}^{\text{LASF9}}
\end{pmatrix} = \begin{pmatrix} 19.65'' \pi/6, & -2.42'' \pi/4, & -24.69'' \pi/3 \\
17.27'' \pi/6, & -5.61'' \pi/4, & -29.06'' \pi/3 \\
5.21'' \pi/6, & -22.90'' \pi/4, & -57.42'' \pi/3 \end{pmatrix}.
\]

Incidence near the critical angle generates the phenomenon of planar trapping. For a beam waist of 1 mm, the planar trapping is evident in the contour plots of Fig. 4(a). In Fig. 6, it is shown what happens when we reduce the beam waist up to 100 \( \mu \)m. The center of the transverse \( x \) profiles moves when we increase the axial distance \( z \) and this is a clear evidence of angular deviations [7–10].
IV. Conclusions

The main objective of the study presented in this paper was to obtain an analytical formula for the waist diameter modification of an optical Gaussian beam transmitted through dielectrics. The analysis was done for the transmission from an air/dielectric and a dielectric/air interface forming an angle $\alpha$. For experimental implementations, the prisms found in commerce could be used to test our prediction for $\alpha = \pi/4$.

The analytical formula for the modulation factor of the waist diameter

$$\tilde{w}_0 = \frac{\cos \psi \cos \phi}{\cos \theta \cos \varphi} w_0,$$

valid for $\theta \leq \theta_{\text{cri}}$, was found by using the Taylor expansion of the optical phase up to its second order and by using the Fresnel coefficients calculated at the center of the Gaussian distribution. It depends on the geometrical properties of the dielectric, parameter $\alpha$, on the incidence angle, parameter $\theta$, and on the refractive index, parameter $n$, which also depends on the wavelength $\lambda$ of the beam (Sellmeier equations) and shows an excellent agreement, see Fig. 5, with the numerical data extracted by the plots of Figs. 2, 3, and 4.

Once obtained the closed expression for the modulation factor, Eq. (18), it can be used in optical experiments to modulate Gaussian lasers or correct eventual transverse asymmetries of the beam. For example, by using a BK7 prism (with $\alpha = \pi/4$) depending of the wavelength of the incident laser we can easily obtain a reduction to one half or an amplification of a factor two by using the following incidence angle,

$$\begin{pmatrix} \theta_{\text{[532 nm]}} \\ \theta_{\text{[633 nm]}} \end{pmatrix} = - \begin{pmatrix} 1.116^\circ & 6.688^\circ \\ \pi/4 & \pi/4 \end{pmatrix} \begin{pmatrix} 11.85^\circ \\ 66.85^\circ \end{pmatrix}.$$

To obtain an optical beam with the same modulation factor in the transverse components, we can use a second prism rotated by an angle of $\pm \pi/2$ along the $z$-axis, see Fig. 1. The lower transmitted beam is then forced to pass trough the second prism with the same incidence angle used in the first prism. In this way, we compensate the asymmetry created by the first transmission. Finally, we observe that Eq. (18) is also useful to prove that there is no modulation in the upper transmitted beam. Eq. (18) contains the product of the ratio between the transmitted and incident wave num-
ber component along the normal to the air/dielectric and dielectric/air interfaces. For transmission trough the left (air/dielectric) and lower (dielectric/air) interfaces, we have found $$(\cos \psi/\cos \theta) \times (\cos \phi/\cos \varphi)$$. Consequently, for transmission through the left (air/dielectric) and upper (dielectric/air) interface, we expect $$(\cos \psi/\cos \theta) \times (\cos \theta/\cos \psi) = 1$$. We hope that the theoretical prediction of the waist modulation reported in this paper could find a confirmation in future experimental implementations.

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