TWO NONRELATED FINSLER STRUCTURES ON A MANIFOLD

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Dedicated to Professor Radu Miron
on the occasion of his 70th birthday

Introduction

It is a standard and commonly used approach to consider two different differentiable structures on the same base manifold, which are related a priori in some sort (projectively, conformally,...etc.) and to study the relationships between the corresponding geometric objects (connection, geodesic, curvature tensors,...etc.) associated to those structures. In this direction, the reader may refer for example to [1], [3] and [7]. In the present paper, we proceed differently: We consider two different Finsler structures $L$ and $L^*$ on the same base manifold $M$, with no relation preassumed between them.

Introducing the $\pi$-tensor field representing the difference between the Cartan’s connections associated with $L$ and $L^*$, we investigate the (necessary and sufficient) conditions, to be satisfied by this $\pi$-tensor field, for the geometric objects associated with $L$ and $L^*$ to have the same properties. Among various items investigated in the paper, we consider the properties of being a geodesic, a Jacobi field, a Berwald manifold, a locally Minkowskian manifold and a Landsberg manifold.

It should be noticed that our approach is a global one. That is, it does not make use of local coordinate techniques.

1. Notations and Preliminaries

In this section we give a brief account of the basic concepts necessary for this work. For more details, refer to [2] or [5]. We make the general assumption that all geometric objects we consider are of class $C^\infty$. The following notations will be used throughout the paper:

- $M$: a differentiable manifold of finite dimension and of class $C^\infty$.
- $\pi_M : TM \longrightarrow M$: the tangent bundle of $M$.
- $\pi : TM \longrightarrow M$: the subbundle of nonzero vectors tangent to $M$.
- $P : \pi^{-1}(TM) \longrightarrow TM$: the bundle, with base space $TM$, induced by $\pi$ and $TM$.
- $\mathcal{F}(M)$: the $\mathbb{R}$-algebra of differentiable functions on $M$. 

\( \mathfrak{X}(M) \): the \( \mathfrak{F}(M) \)-module of vector fields on \( M \).

\( \mathfrak{X}(\pi(M)) \): the \( \mathfrak{F}(TM) \)-module of differentiable sections of \( \pi^{-1}(TM) \).

Elements of \( \mathfrak{X}(\pi(M)) \) will be called \( \pi \)-vector fields and will be denoted by barred letters \( \overline{X} \). Tensor fields on \( \pi^{-1}(TM) \) will be called \( \pi \)-tensor fields. The fundamental vector field is the \( \pi \)-vector field \( \overline{\eta} \) defined by \( \overline{\eta}(u) = (u, u) \) for all \( u \in TM \). The lift to \( \pi^{-1}(TM) \) of a vector field \( X \) on \( M \) is the \( \pi \)-vector field \( \overline{X} \) defined by \( \overline{X}(u) = (u, X(\pi(u))) \).

The tangent bundle \( T(TM) \) is related to the vector bundle \( \pi^{-1}(TM) \) by the exact sequence:

\[
0 \rightarrow \pi^{-1}(TM) \xrightarrow{\gamma} T(TM) \xrightarrow{\rho} \pi^{-1}(TM) \rightarrow 0,
\]

where the vector bundle morphisms are defined by \( \rho = (\pi_{TM}, d\pi) \) and \( \gamma(u, v) = j_u(v) \), where \( j_u \) is the natural isomorphism \( j_u : T_{\pi_M(u)}M \rightarrow T_u(T_{\pi_M(u)}M) \).

Let \( \nabla \) be an affine connection (or simply a connection) in the vector bundle \( \pi^{-1}(TM) \). We associate to \( \nabla \) the map

\[
K : TM \rightarrow \pi^{-1}(TM) : X \mapsto \nabla_X \overline{\eta},
\]
called the connection map of \( \nabla \). A tangent vector \( X \in T_u(TM) \) is said to be horizontal if \( K(X) = 0 \). The connection \( \nabla \) is said to be regular if

\[
T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall u \in TM,
\]

where \( V_u(TM) \) and \( H_u(TM) \) are respectively the vertical and horizontal spaces at \( u \). If \( M \) is endowed with a regular connection, we can define a section \( \beta \) of the morphism \( \rho \) by \( \beta = (\rho \circ H(TM))^{-1} \). It is clear that \( \rho \circ \beta \) is the identity map on \( \pi^{-1}(TM) \) and \( \beta \circ \rho \) is the identity map on \( H(TM) \).

For every \( X, Y \in \mathfrak{X}(TM) \), the torsion form \( T \) and the curvature transformation \( R \) of the connection \( \nabla \) are defined by:

\[
T(X, Y) = \nabla_X \rho Y - \nabla_Y \rho X - \rho [X, Y], \quad R(X, Y) = [\nabla_Y, \nabla_X] + \nabla_{[X,Y]}.
\]

The horizontal and mixed torsion tensors, denoted respectively by \( S \) and \( T \), are defined, for all \( \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)) \), by:

\[
S(\overline{X}, \overline{Y}) = T(\beta \overline{X}, \beta \overline{Y}), \quad T(\overline{X}, \overline{Y}) = T(\gamma \overline{X}, \beta \overline{Y}).
\]

The horizontal, mixed and vertical curvature tensors, denoted respectively by \( R \), \( P \) and \( Q \), are defined, for all \( \overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M)) \), by:

\[
R(\overline{X}, \overline{Y}) \overline{Z} = R(\beta \overline{X}, \beta \overline{Y}) \overline{Z}, \quad P(\overline{X}, \overline{Y}) \overline{Z} = R(\gamma \overline{X}, \beta \overline{Y}) \overline{Z}, \quad Q(\overline{X}, \overline{Y}) \overline{Z} = R(\gamma \overline{X}, \gamma \overline{Y}) \overline{Z}.
\]

If \( c : I \rightarrow M \) is a regular curve in \( M \), its canonical lift to \( TM \) is the curve \( \tilde{c} \) defined by \( \tilde{c} : t \mapsto dc/dt \). The lift of a vector field \( X \) along \( c \) is the \( \pi \)-vector field along \( \tilde{c} \) defined by \( \overline{X} : \tilde{c}(t) \mapsto (\tilde{c}(t), X(\tilde{c}(t))) \). In particular, the velocity vector field \( dc/dt \) along \( c \) is lifted to the \( \pi \)-vector field \( \overline{dc/dt} = (dc/dt, dc/dt) \) along \( \tilde{c} \). Clearly, \( \rho(\overline{dc/dt}) = \overline{dc/dt} = \overline{\eta} \). A vector field \( X \) along a regular curve \( c \) in \( M \) is parallel along \( c \) with respect to the connection \( \nabla \) if \( D \overline{X}/dt = 0 \), where \( D/dt \) is the covariant derivative operator, associated with \( \nabla \), along \( \tilde{c} \). A regular curve \( c \) in \( M \) is a geodesic if the \( \pi \)-vector field \( D(\overline{dc/dt})/dt \) vanishes identically. In this case, the vector field \( dc/dt \) along \( \tilde{c} \) is horizontal.
2. Connections

Let \( L \) and \( L^* \) be two Finsler structures on a differentiable manifold \( M \), with no relation assumed a priori between them. Throughout this work, entities of \((M, L^*)\) will be marked by an asterisk “*”.

Let \( \nabla \) and \( \nabla^* \) be the Cartan’s connections associated respectively with the Finsler manifolds \((M, L)\) and \((M, L^*)\). For every \( X \in \mathfrak{X}(TM) \) and \( Y \in \mathfrak{X}(\pi(M)) \), let us write

\[

\nabla^*_X Y = \nabla_X Y + U(X, Y),
\]

(2.1)

where \( U \) is an \( \mathfrak{X}(TM) \)-bilinear mapping \( \mathfrak{X}(TM) \times \mathfrak{X}(\pi(M)) \rightarrow \mathfrak{X}(\pi(M)) \) representing the difference between the two connections \( \nabla^* \) and \( \nabla \).

For every \( X, Y \in \mathfrak{X}(\pi(M)) \), we set

\[

A(X, Y) := U(\gamma X, Y), \quad B(X, Y) := U(\beta X, Y).
\]

(2.2)

As a vector field \( X \) on \( TM \) can be represented by

\[

X = \gamma K X + \beta \rho X,
\]

(2.3)

it follows from (2.2) and (2.3) that

\[

U(X, Y) = A(K(X), Y) + B(\rho X, Y), \quad \forall X \in \mathfrak{X}(TM), \; Y \in \mathfrak{X}(\pi(M)).
\]

(2.4)

By (2.1) and (2.2), taking the regularity of the connections \( \nabla \) and \( \nabla^* \) into account, we get

**Lemma 2.1.** For every \( \pi \)-vector field \( \overline{X} \), we have

\[

A(\overline{X}, \overline{\eta}) = 0.
\]

Using Lemma 2.1, equations (2.1), (2.2) and (2.4) imply

**Lemma 2.2.** The relation between the connection maps \( K \) and \( K^* \) is given by

\[

K^* = K + N \circ \rho.
\]

The following result follows directly from Lemma 2.2.

**Proposition 2.3.** A horizontal vector field with respect to \( \nabla \) (resp. \( \nabla^* \)) is horizontal with respect \( \nabla^* \) (resp. \( \nabla \)) if, and only if, \( N \circ \rho = 0 \).

The proof of the following result is not difficult.

**Proposition 2.4.** The relation between \( \beta \) and \( \beta^* \) is given by

\[

\beta^* = \beta - \gamma \circ N.
\]

**Proposition 2.5.** For every \( \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)) \), we have

(a) \( T^*(\overline{X}, \overline{Y}) = T(\overline{X}, \overline{Y}) + A(\overline{X}, \overline{Y}) \).

(b) \( T^*(N(\overline{X}), \overline{Y}) - T^*(N(\overline{Y}), \overline{X}) = B(\overline{X}, \overline{Y}) - B(\overline{Y}, \overline{X}) \).
Proof. One can easily show, for every $X, Y \in \mathfrak{X}(TM)$, that
\[ T^*(X, Y) = T(X, Y) + U(X, \rho Y) - U(Y, \rho X). \] (2.5)

(a) Setting $X = \rho \overline{X}$ and $Y = \beta \overline{Y}$ in (2.5), using Proposition 2.4 and taking the
definition of $T$ into account, (a) follows.
(b) Setting $X = \beta \overline{X}$ and $Y = \beta \overline{Y}$ in (2.5), using Proposition 2.4 and the fact that
$S^*(\overline{X}, \overline{Y}) = 0 = S(\overline{X}, \overline{Y})$, (b) follows. \[ \square \]

3. Curvature Tensors

Let $R$ and $R^*$ be the curvature transformations of the connections $\nabla$ and $\nabla^*$ respectively.

Lemma 3.1. For every $X, Y \in \mathfrak{X}(TM)$, $\overline{Z} \in \mathfrak{X}(\pi(M))$, we have
\[ R^*(X, Y)\overline{Z} = R(X, Y)\overline{Z} + \Omega(X, Y)\overline{Z}, \]
where
\[ \Omega(X, Y)\overline{Z} = (\nabla_Y B)(\rho X, \overline{Z}) - (\nabla_X B)(\rho Y, \overline{Z}) + (\nabla_Y A)(K(X), \overline{Z}) \]
- $(\nabla_X A)(K(Y), \overline{Z}) + A(R(X, Y)\overline{\eta}, \overline{Z}) - B(T(X, Y), \overline{Z})$
+ $U(Y, U(X, \overline{Z})) - U(X, U(Y, \overline{Z})).$

The following proposition gives some useful technical formulas which will be used
frequently in the sequel.

Proposition 3.2. For every $\overline{X}, \overline{Y}, \overline{Z} \in \mathfrak{X}(\pi(M))$, we have
(a) $R^*(\overline{X}, \overline{Y})\overline{Z} + P^*(N(\overline{X}), \overline{Y})\overline{Z} - P^*(N(\overline{Y}), \overline{X})\overline{Z} + Q^*(N(\overline{X}), N(\overline{Y}))\overline{Z}$
\[ = R(\overline{X}, \overline{Y})\overline{Z} + \Omega(\beta \overline{X}, \beta \overline{Y})\overline{Z}, \]
where
\[ \Omega(\beta \overline{X}, \beta \overline{Y})\overline{Z} = (\nabla_{\beta \overline{Y}} B)(\overline{X}, \overline{Z}) - (\nabla_{\beta \overline{X}} B)(\overline{Y}, \overline{Z}) + A(R(\overline{X}, \overline{Y})\overline{\eta}, \overline{Z}) \]
+ $B(\overline{Y}, B(\overline{X}, \overline{Z})) + B(\overline{X}, B(\overline{Y}, \overline{Z})).$
(b) $P^*(\overline{X}, \overline{Y})\overline{Z} + Q^*(\overline{X}, N(\overline{Y}))\overline{Z} = P(\overline{X}, \overline{Y})\overline{Z} + \Omega(\gamma \overline{X}, \beta \overline{Y})\overline{Z},$
where
\[ \Omega(\gamma \overline{X}, \beta \overline{Y})\overline{Z} = -(\nabla_{\gamma \overline{X}} B)(\overline{Y}, \overline{Z}) + (\nabla_{\gamma \overline{Y}} A)(\overline{X}, \overline{Z}) + A(P(\overline{X}, \overline{Y})\overline{\eta}, \overline{Z}) \]
- $B(T(\overline{X}, \overline{Y}), \overline{Z}) + B(\overline{Y}, A(\overline{X}, \overline{Z})) - A(\overline{X}, B(\overline{Y}, \overline{Z})).$
(c) $Q^*(\overline{X}, \overline{Y})\overline{Z} = Q(\overline{X}, \overline{Y})\overline{Z} + \Omega(\gamma \overline{X}, \gamma \overline{Y})\overline{Z},$
where
\[ \Omega(\gamma \overline{X}, \gamma \overline{Y})\overline{Z} = (\nabla_{\gamma \overline{X}} A)(\overline{X}, \overline{Z}) - (\nabla_{\gamma \overline{Y}} A)(\overline{Y}, \overline{Z}) + A(\overline{Y}, A(\overline{X}, \overline{Z})) - A(\overline{X}, A(\overline{Y}, \overline{Z})).$

In particular, if $\overline{Z} = \overline{\eta}$, we get
(a)$' R^*(\overline{X}, \overline{Y})\overline{\eta} + P^*(N(\overline{X}), \overline{Y})\overline{\eta} - P^*(N(\overline{Y}), \overline{X})\overline{\eta} = R(\overline{X}, \overline{Y})\overline{\eta} + \Omega(\beta \overline{X}, \beta \overline{Y})\overline{\eta},$
where
\[ \Omega(\beta \overline{X}, \beta \overline{Y})\overline{\eta} = (\nabla_{\beta \overline{Y}} N)(\overline{X}) - (\nabla_{\beta \overline{X}} N)(\overline{Y}) + B(\overline{Y}, N(\overline{X})) - B(\overline{X}, N(\overline{Y})).$
(b)$' P^*(\overline{X}, \overline{Y})\overline{\eta} = P(\overline{X}, \overline{Y})\overline{\eta} + \Omega(\gamma \overline{X}, \beta \overline{Y})\overline{\eta},$
where
\[ \Omega(\gamma \overline{X}, \beta \overline{Y})\overline{\eta} = -(\nabla_{\gamma \overline{X}} N)(\overline{Y}) + B(\overline{Y}, \overline{X}) - N(T(\overline{X}, \overline{Y})) - A(\overline{X}, N(\overline{Y})).$
(c)$' A(\overline{X}, \overline{Y}) = A(\overline{Y}, \overline{X}),$
that is, the $\pi$-tensor field $A$ is symmetric.
Proof. (a) follows from Lemma 3.1 for $X = \beta X, Y = \beta Y$ and from Proposition 2.4. (b) follows from Lemma 3.1 for $X = \gamma X, Y = \beta Y$ and from Proposition 2.4. (c) follows from Lemma 3.1 for $X = \gamma X, Y = \gamma Y$. □

Proposition 3.3. Assume that the $\pi$-tensor field $B$ vanishes. Then, $R^* = 0$ if, and only if, $R = 0$.

Proof. Since $B = 0$, it follows from Proposition 3.2(a) that

$$R^*(\bar{X}, \bar{Y})\bar{Z} = R(\bar{X}, \bar{Y})\bar{Z} + A(R(\bar{X}, \bar{Y})\bar{\eta}, \bar{Z}).$$

It is thus clear that if $R = 0$, then $R^* = 0$.

Conversely, if $R^* = 0$, then

$$R(\bar{X}, \bar{Y})\bar{Z} = -A(R(\bar{X}, \bar{Y})\bar{\eta}, \bar{Z}).$$

(3.1)

Setting $\bar{Z} = \bar{\eta}$ in (3.1) and using Lemma 2.1, we get $R(\bar{X}, \bar{Y})\bar{\eta} = 0$. Hence, it follows again from (3.1) that $R = 0$. □

Proposition 3.4. The $\pi$-tensor field $N$ vanishes if, and only if, $N_0$ vanishes.

Proof. Firstly, it is clear that $N = 0$ implies $N_0 = 0$.

Setting $\bar{Y} = \bar{\eta}$ in Proposition 3.2(b)' and using the properties of the torsion and curvature tensors, we get

$$\nabla_{\gamma X}N_0 = N(\bar{X}) + B(\bar{\eta}, \bar{X}) - A(\bar{X}, N_0).$$

On the other hand, we have from Proposition 2.5(b)

$$T^*(N_0, \bar{X}) = B(\bar{\eta}, \bar{X}) - N(\bar{X}).$$

It follows, from the above two identities, that

$$\nabla_{\gamma X}N_0 = 2N(\bar{X}) + T^*(N_0, \bar{X}) - A(\bar{X}, N_0).$$

Consequently, $N_0 = 0$ implies $N = 0$. □

Now, let us assume that the $\pi$-vector field $N_0$ vanishes. Then, by Proposition 3.4, the formula (a)' of Proposition 3.2 takes the form $R^*(\bar{X}, \bar{Y})\bar{\eta} = R(\bar{X}, \bar{Y})\bar{\eta}$. It is well-known that [6] the horizontal distribution with respect to $\nabla$ is completely integrable if, and only if, $R(\bar{X}, \bar{Y})\bar{\eta} = 0$. Therefore, we have

Theorem 3.5. Suppose that the $\pi$-vector field $N_0$ vanishes. The horizontal distribution with respect to $\nabla^*$ is completely integrable if, and only if, the horizontal distribution with respect to $\nabla$ is completely integrable.

4. Geodesics and Jacobi Fields

Let $D/dt$ and $D^*/dt$ be the covariant derivative operators, corresponding respectively to the Cartan’s connections $\nabla$ and $\nabla^*$, along a curve $\tilde{c}$ in $TM$. One can easily show that

$$D^*\bar{X}/dt = D\bar{X}/dt + U(d\tilde{c}/dt, \bar{X}),$$

(4.1)

for every $\pi$-vector field $\bar{X}$ along $\tilde{c}$. This formula gives directly
Lemma 4.1. Let $\tilde{c}$ be a curve in $TM$. A parallel $\pi$-vector field $\bar{X}$ along $\tilde{c}$ in $(M, L)$ (resp. $(M, L^*)$) is parallel along $\tilde{c}$ in $(M, L^*)$ (resp. $(M, L)$) if, and only if, $U(d\tilde{c}/dt, \bar{X}) = 0$.

Theorem 4.2. A necessary and sufficient condition for a geodesic $c$ in $(M, L)$ (resp. $(M, L^*)$) to be a geodesic in $(M, L^*)$ (resp. $(M, L)$) is that $B(\nabla, \nabla) = 0$, where $\nabla = \eta|_{\tilde{c}(t)}$.

In other words, the Finsler manifolds $(M, L)$ and $(M, L^*)$ are projectively related if, and only if, $B(\nabla, \nabla) = 0$ for every $\nabla$.

Proof. Let $c$ be a regular curve in $M$. The canonical lift $\tilde{c}$ of $c$ to $T^*M$ is such that $\rho(d\tilde{c}/dt) = \eta|_{\tilde{c}(t)} = \nabla$. Then, by (2.4) and Lemma 2.1, we have $D^*\nabla/\eta = B(\nabla, \nabla)$. It follows then from (4.1) that $D^*\nabla/\eta = D\nabla/\eta + B(\nabla, \nabla)$. The result follows from this relation and the fact that $c$ is a geodesic in $(M, L)$ (resp. $(M, L^*)$) if, and only if, $D\nabla/\eta = 0$ (resp. $D^*\nabla/\eta = 0$). □

A vector field $J$ along a geodesic $c$ in $M$ is called a Jacobi field with respect to $\nabla$ if it satisfies the Jacobi differential equation

$$\frac{D^2 J}{dt^2} + R(\nabla, J)\nabla = 0,$$

where $J$ and $\nabla$ are respectively the lifts of $J$ and $V = dc/dt$ along $\tilde{c}$.

Writing equation (4.1) for $X = J$ and using (2.4), we get

$$\frac{D^* J}{dt} = \frac{D J}{dt} + B(\nabla, J) + A(K(d\tilde{c}/dt), J). \quad (4.2)$$

Proposition 3.2(a) for $X = Z = V$ and $Y = J$ yields

$$R^*(\nabla, J)\nabla + P^*(N(\nabla), J)\nabla = R(\nabla, J)\nabla + \Omega(\beta\nabla, \beta J)\nabla, \quad (4.3)$$

where

$$\Omega(\beta\nabla, \beta J)\nabla = \nabla_{\beta J} B(\nabla, \nabla) - (\nabla_{\beta \nabla} B)(J, \nabla) + B(J, B(\nabla, \nabla)) - B(\nabla, B(J, \nabla)).$$

Now, if $B(\nabla, \nabla) = 0$, it follows from Proposition 3.4 that $B(\nabla, \nabla) = 0$ for every $\pi$-vector field $\bar{X}$. Then, by Proposition 2.5, we have $B(\nabla, \bar{X}) = B(\bar{X}, \nabla) = 0$, and consequently $\Omega(\beta\nabla, \beta J)\nabla = 0$. Moreover, $K(d\tilde{c}/dt) = 0$ since the vector field $d\tilde{c}/dt$ is horizontal with respect to $\nabla$. Therefore, equations (4.2) and (4.3) reduce respectively to

$$\frac{D^* J}{dt} = \frac{D J}{dt} \quad \text{and} \quad R^*(\nabla, J)\nabla = R(\nabla, J)\nabla.$$

These two identities imply:

$$\frac{D^2 J}{dt^2} + R(\nabla, J)\nabla = \frac{D^2 J}{dt^2} + R(\nabla, J)\nabla.$$

Hence, by the Jacobi equation, we obtain

Theorem 4.3. If $B(\nabla, \nabla) = 0$ for every geodesic $c$ in $M$, then the two Finsler manifolds $(M, L)$ and $(M, L^*)$ have the same Jacobi fields.
It is not difficult to show that if $B(\nabla, \nabla) = 0$ for every geodesic $c$ in $M$, then the geodesics of the two Finsler manifolds $(M, L)$ and $(M, L^*)$ possess the same conjugate points. Therefore, as a consequence of Theorem 4.3 and the well-known Morse index theorem [2], we get

**Theorem 4.4.** If $B(\nabla, \nabla) = 0$ for every geodesic $c$ in $M$, then the geodesics of the two Finsler manifolds $(M, L)$ and $(M, L^*)$ have the same Morse index.

5. Special Finsler Manifolds

A Finsler manifold $(M, L)$ is a Berwald manifold [3] if the torsion tensor $T$ satisfies the condition that $\nabla_{\beta X} T = 0$ for every $X \in \mathfrak{X}(\pi(M))$.

Suppose now that the $\pi$-tensor field $B$ vanishes. Then, by Proposition 2.4, we have $\beta^* = \beta$. Hence, $\nabla^*_{\beta^* X} \nabla Y = \nabla_{\beta X} Y$ $\forall X, Y \in \mathfrak{X}(\pi(M))$. From which, using Proposition 2.5, we get

$$\nabla^*_{\beta^* X} T^* = \nabla_{\beta X} T + \nabla_{\beta X} A.$$

This relation implies

**Theorem 5.1.** Assume that the $\pi$-tensor field $B$ vanishes. Let $(M, L)$ (resp. $(M, L^*)$) be a Berwald manifold. A necessary and sufficient condition for $(M, L^*)$ (resp. $(M, L)$) to be a Berwald manifold is that $\nabla_{\beta X} A = 0$ for all $X \in \mathfrak{X}(\pi(M))$.

A Finsler manifold $(M, L)$ is locally Minkowskian [3] if, and only if, $R = 0$ and $\nabla_{\beta X} T = 0$ for all $X \in \mathfrak{X}(\pi(M))$.

Combining Proposition 3.3 and Theorem 5.1, we get

**Theorem 5.2.** Assume that the $\pi$-tensor field $B$ vanishes. Let $(M, L)$ (resp. $(M, L^*)$) be a locally Minkowskian manifold. A necessary and sufficient condition for $(M, L^*)$ (resp. $(M, L)$) to be locally Minkowskian is that $\nabla_{\beta X} A = 0$ for all $X \in \mathfrak{X}(\pi(M))$.

A Finsler manifold $(M, L)$ is a Landsberg manifold [4] if it satisfies the condition that $P(X, Y)\eta = 0$ for all $X, Y \in \mathfrak{X}(\pi(M))$.

Using Proposition 3.2(b)', we obtain

**Theorem 5.3.** Suppose that the $\pi$-tensor field $B$ vanishes. The Finsler manifold $(M, L)$ is a Landsberg manifold if, and only if, the Finsler manifold $(M, L^*)$ is a Landsberg manifold.

**Concluding Remark.** The results obtained in this paper may be applied to treat any two related Finsler structures, where the $\pi$-tensor fields $A$ and $B$ take special forms depending on the type of relation between the two structures. In a forthcoming paper, we shall apply these results to "Generalized Randers Manifolds".
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