A new scheme for the Klein-Gordon and Dirac fields on the lattice with axial anomaly

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Abstract
Using the method of finite differences a scheme is proposed to solve exactly the Klein-Gordon and Dirac free field equations, in a (1 + 1)-dimensional lattice. The hamiltonian of the Dirac field is translational invariant, hermitian, avoids fermion doubling, and, for the massless case, preserves global chiral symmetry. Coupling the fermion field to the electromagnetic vector potential we construct a gauge invariant vector current leading to the correct axial anomaly. PACS number: 02.20.+b, 11.30.-j

1 Introduction
The method of finite elements has become very powerful to solve evolution equations in quantum field theories. In particular Bender and his collaborators [1] have applied the method to the Klein-Gordon and Dirac quantum field equations, consistent with the equal time commutation relations, which preserves global chiral symmetries in the massless case and avoids fermion doubling. Matsuyama [2] has constructed an abelian gauge-coupled Dirac equation without species doubling. Milton [3] extended this method to the non-abelian gauge theories. Using the method of finite differences, Vázquez [4] also introduced an explicit scheme for the Dirac fields, such that the associated action is hermitian, the chiral symmetry is preserved and the fermion doubling is absent, but the equal time commutation relations are not satisfied. Nevertheless in all these models the solution is given for only one time step in the transfer matrix.

Lattice field theories have become very popular to analyze perturbation theories in the standard model [5]. With the lattice regularization comes the problem of fermion doubling if one imposes to the hamiltonian translational invariance, hermiticity and locality. Under these assumptions fermion doubling is unavoidable, as Nielsen and Ninomiya have proved [6].

We present a simple model for the Klein-Gordon and Dirac free field using the method of finite differences. The novelty of our approach lies, first, on the exact solutions for any time step of the difference equations, making a good contact with the covariant continuous theory (we have already worked out exact solutions for the Heisenberg difference equations of motions in a time lattice [7]).
Secondly, we try to escape the no-go theorem of Nielsen and Ninomiya because our Hamiltonian is non-local, although avoids fermion doubling.

To check our model we apply it to current algebra, namely, we implement the vector and axial current of fermion massless fields with abelian gauge fields. The axial current is then gauge invariant, but exhibit non-local point separation structure, which in the continuous limit leads to the correct axial anomaly.

2 A set of orthonormal functions on the lattice

Let us define the function

\[ f_j = \left( \frac{1 + \frac{1}{2}ik\varepsilon}{1 - \frac{1}{2}ik\varepsilon} \right)^j, \quad j = 0, \pm 1, \pm 2, \ldots \]  

(1)

where \( k \) is an arbitrary constant and \( \varepsilon \) some small quantity.

This function satisfies the difference equations

\[ f_{j+1} - f_j = ik\varepsilon \left( f_{j+1} + f_j \right), \quad f_j - f_{j-1} = ik\varepsilon \left( f_j + f_{j-1} \right) \]  

(2)

\[ f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}} = ik\varepsilon \left( f_{j+\frac{1}{2}} + f_{j-\frac{1}{2}} \right) \]  

(3)

Using the notation for the difference operators

\[ \Delta f_j \equiv f_{j+1} - f_j, \quad \nabla f_j = f_j - f_{j-1}, \quad \delta f_j \equiv f_{j+\frac{1}{2}} - f_{j-\frac{1}{2}} \]

and the average operators

\[ \tilde{\Delta} f_j \equiv \frac{1}{2} (f_{j+1} + f_j), \quad \tilde{\nabla} f_j = \frac{1}{2} (f_j + f_{j-1}), \quad \mu f_j \equiv \frac{1}{2} (f_{j+\frac{1}{2}} + f_{j-\frac{1}{2}}) \]

the above equations become

\[ \frac{1}{\varepsilon} \Delta f_j = ik\tilde{\Delta} f_j, \quad \frac{1}{\varepsilon} \nabla f_j = i\mu \tilde{\nabla} f_j \]  

(4)

\[ \frac{1}{\varepsilon} \delta f_j = ik\mu f_j \]  

(5)

These difference equations and their solutions are approximation of the differential equation

\[ \frac{df}{dx} = ikf \]  

(6)

with truncation error of second order in \( \varepsilon \) (the truncation error is the difference between the Taylor expansion of the finite difference equation and the differential equation).

On the other hand the function \( f_j \) converges to the solution of the differential equation (6):

\[ f_j = \left( \frac{1 + \frac{1}{2}ikx}{1 - \frac{1}{2}ikx} \right)^j \rightarrow e^{ikx} = f(x) \]  

(7)

when \( j \to \infty, \quad j\varepsilon \to x, \quad \varepsilon \to 0 \) with order of convergence \( O(\varepsilon^2) \).
If we impose the boundary conditions \( f_0 = f_N = 1 \) for some fixed number \( N \), we find \( N \) different values of \( k \)

\[
k_m = \frac{2\varepsilon}{\pi \frac{m}{N}} \tan \frac{\pi m}{N}, \quad m = 0, 1, \ldots, N - 1
\]

with properties \( k_{m+N} = k_m \) and \( k_{-m} = -k_m \). Therefore

\[
f_j(k_m) = \left(1 + \frac{\varepsilon}{2k_m}\right)^j, \quad m = 0, 1, \ldots, N - 1
\]

Notice that if \( m = N/2 \) for \( N \) even, then \( f_j(k_m) = (-1)^j \). Obviously,

\[
f_j(k_m) = \exp \left(i \frac{2\pi}{N} m j \right)
\]

These functions satisfy the orthogonality relations

\[
\frac{1}{N} \sum_{j=0}^{N-1} f_j^*(k_m) f_j(k_{m'}) = \delta_{mm'}
\]

with respect to the scalar product in the Hilbert space \( \mathcal{V}([0,N]) \). With the help of this orthogonality relation we can express any function \( F_j \) of discrete variable in terms of the \( N \) orthogonal functions \( f_j(k_m) \), namely,

\[
F_j = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} a_m f_j(k_m)
\]

with

\[
a_m = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f_j^*(k_m) F_j, \quad m = 0, 1, \ldots, N - 1
\]

The orthonormal set

\[
\left\{ \frac{1}{\sqrt{N}} f_j(k_m) \right\}_{m=0}^{N-1}
\]

can be chosen as a basis for the Hilbert space \( \mathcal{V}([0,N]) \), hence, a completeness relation can be constructed

\[
\frac{1}{N\varepsilon} \sum_{m=0}^{N-1} f_j^*(k_m) F_j(k_{m'}) = \frac{1}{\varepsilon} \delta_{jj'}
\]

the right hand side being the discrete analog of the Dirac Delta function.

Consider now the following function on the lattice

\[
u_j = \frac{2}{N\varepsilon i} \sum_{m=1}^{s} \frac{1}{k_m} \left\{ f_j(k_{m}) - 1 \right\}_{s=N}
\]

Using the difference equations (2) we can calculate \( u_j \), namely,

\[
u_j = \left. \frac{2}{N\varepsilon i} \sum_{m=1}^{s} \frac{ik_m \varepsilon}{2k_m} \left\{ f_j(k_{m}) + f_j(k_{m}) + f_j(k_{m}) + \ldots + f_2 + f_1 + f_0 \right\} \right|_{s=N} = 1, \quad \forall j > 0
\]
Similarly, \( u_j = -1, \quad \forall j < 0 \)

Hence we can have the discrete analog of the Heaviside function,
\[
\vartheta_j \equiv \frac{1}{2} (u_j + 1) = \begin{cases} 
1 & , \quad j > 0 \\
1/2 & , \quad j = 0 \\
0 & , \quad j < 0
\end{cases}
\] (17)

It is easy to prove the difference equations
\[
\frac{1}{\varepsilon} \Delta \vartheta_j = \frac{1}{2\varepsilon} (\delta_{j+1,0} + \delta_{j,0}), \quad \frac{1}{\varepsilon} \nabla \vartheta_j = \frac{1}{2\varepsilon} (\delta_{j,0} + \delta_{j-1,0})
\] (18)
\[
\frac{1}{\varepsilon} \delta \vartheta_j = \frac{1}{2\varepsilon} (\delta_{j+1/2,0} + \delta_{j-1/2,0})
\] (19)

conecting the discrete versions of the Heaviside and Dirac functions.

We have also the useful relations:
\[
\Delta (f_jg_j) = (\Delta f_j) (\tilde{\Delta} g_j) + (\tilde{\Delta} f_j) (\Delta g_j)
\] (20)
\[
\tilde{\Delta} (f_jg_j) = (\tilde{\Delta} f_j) (\tilde{\Delta} g_j) + \frac{1}{4} (\Delta f_j) (\Delta g_j)
\] (21)

3 Hamiltonian formalism of the Klein-Gordon field on a \((1 + 1)\)-dimensional Minkowski lattice

We introduce the method of finite differences for the Klein-Gordon scalar field. An explicit scheme for the wave equation consistent with the continuous case (the truncation error is of second order with respect to space and time variables) can be constructed as follows:
\[
\left( \frac{1}{\tau^2} \nabla_n \Delta_n \tilde{\nabla}_j \tilde{\Delta}_j - \frac{1}{\varepsilon^2} \nabla_j \Delta_j \tilde{\nabla}_n \tilde{\Delta}_n + M^2 \tilde{\nabla}_n \Delta_n \tilde{\nabla}_j \tilde{\Delta}_j \right) \phi^n_j = 0
\] (22)

where the field is defined in the grid points of the \((1 + 1)\)-dimensional lattice \( \phi^n_j \equiv \phi (j\varepsilon, n\tau) \), \( \varepsilon, \tau \) being the space and time fundamental intervals, \( j, n \) integer numbers and \( \Delta_j (\nabla_j) \) are the forward (backward) differences with respect to the space index, \( \tilde{\Delta}_j (\tilde{\nabla}_j) \) the forward (backward) averages, and similarly for the time index.

Using the method of separation of variables it can easily be proved that the following functions of discrete variables are solutions of the wave equation (22):
\[
f^n_j (k, \omega) = \left( 1 + \frac{1}{2} i\varepsilon k \right)^j \left( \frac{1 - \frac{1}{2} i\tau \omega}{1 + \frac{1}{2} i\tau \omega} \right)^n
\] (23)

provided the “dispersion relation” is satisfied:
\[
\omega^2 - k^2 = M^2
\] (24)

\( M \), being the mass of the particle.

We have also the solutions of (22)
\[
\phi^n_j = (-1)^{j+n}
\] (25)

In the limit, \( j \to \infty, \quad n \to \infty, \quad j\varepsilon \to x, \quad n\tau \to t \) the functions (23) become plane wave solutions
\[
f^n_j (k, \omega) \to \exp i (kx - \omega t)
\] (26)
For the positive energy solutions we define

\[ \omega = \pm \left( k_m^2 + M^2 \right)^{1/2} \]  

(29)

For the positive energy solutions we define

\[ \omega_m = + \left( k_m^2 + M^2 \right)^{1/2} \]  

(30)

The solutions (25) are a particular case of \( f_j^n (k_m, \omega_m) \) with \( m = N/2 \).

Starting from the wave equation (22) we can construct a current vector. Multiplying (22) by \( \tilde{\nabla}_n \tilde{\Delta}_n \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \) from the left, and multiplying the complex conjugate of the wave equation by \( \tilde{\nabla}_n \tilde{\Delta}_n \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \) from the right, substracting both results and using (21) we obtain the “conservation law”

\[ \frac{1}{\varepsilon} \nabla_j j_1 - i \frac{1}{\tau} \nabla_n j_4 = 0 \]  

(31)

where

\[ j_1 = i \left[ \frac{1}{\varepsilon} \Delta_j \left( \tilde{\nabla}_n \tilde{\Delta}_n \phi_j^n \right) \tilde{\Delta}_j \left( \tilde{\nabla}_n \tilde{\Delta}_n \phi_j^n \right) - \Delta_j \left( \tilde{\nabla}_n \tilde{\Delta}_n \phi_j^n \right) \frac{1}{\varepsilon} \Delta_j \left( \tilde{\nabla}_n \tilde{\Delta}_n \phi_j^n \right) \right] \]  

(32)

\[ j_4 = i \rho \equiv \left[ \frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \tilde{\Delta}_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) - \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \right] \]  

(33)

are the spatial and time component, respectively, of the charge vector current on the lattice.

The charge density \( \rho \) suggest that we can substitute the scalar field \( \phi (x, t) \) by the promediated quantities \( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \) and \( \phi^* (x, t) \) by \( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \).

Now we address ourselves to the real Klein-Gordon field. A suitable Hamiltonian for the field \( \phi_j^n \) and its conjugate momentum \( \pi_j^n \) can be defined as follows

\[ H_n = \varepsilon \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right)^2 + \frac{1}{\varepsilon^2} \left( \nabla_j \tilde{\nabla}_j \phi_j^n \right)^2 + M^2 \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right)^2 \right\} \]  

(34)

which is obviously hermitian. Using the periodicity condition of the fields and the identity (21) we can integrate by parts the second term on the right with the result:

\[ H_n = \varepsilon \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right)^2 + \frac{1}{\varepsilon^2} \left( \nabla_j \tilde{\nabla}_j \phi_j^n \right) \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) + M^2 \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right)^2 \right\} \equiv \varepsilon \sum_{j=0}^{N-1} \mathcal{H}_j^n \]  

(35)

As in the continuous case, we can derived the Hamilton equations of motions, varying the Hamiltonian density \( \mathcal{H}_j^n \) first with respecto to the promediate momentum and secondly with respect to scalar field:

\[ \frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) = \frac{\partial \mathcal{H}_j^n}{\partial \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right)} = \tilde{\Delta}_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \]  

(36)

\[ \frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right) = - \frac{\partial \mathcal{H}_j^n}{\partial \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right)} = \tilde{\Delta}_n \left( \frac{1}{\varepsilon^2} \nabla_j \tilde{\Delta}_j \phi_j^n - M^2 \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \]  

(37)
(The same result was obtained by Bender and Milton (1986) if one eliminates the auxiliary field Γ in their formula (2.5) of Reference 8).

Applying the difference operator $\frac{\partial}{\partial n}$ on both sides of (36) and substituting (37) in the result we recover the wave equation (23).

Using (36) and (37) it can easily be proved that the Hamiltonian (35) is independent of the time index $n$, namely:

$$\nabla_n H_n = \Delta_n H_n = 0$$  \hspace{1cm} (38)

hence

$$\tilde{\nabla}_n H_n = \tilde{\Delta}_n H_n = H_n$$  \hspace{1cm} (39)

Since the plane wave solutions $f^n_j (k_m, \omega_m)$ \((m = 0, 1 \ldots N - 1)\) form a complete set of orthogonal functions, we can expand the wave field and its conjugate momentum as

$$\phi^n_j = \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \left( 1 + \frac{i}{4\varepsilon^2 k_m^2} \right) \left( a_m f^n_j (k_m, \omega_m) + a_m^* f^{*n}_j (k_m, \omega_m) \right)$$  \hspace{1cm} (40)

$$\pi^n_j = -\frac{i}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\omega_m/2} \left( 1 + \frac{i}{4\varepsilon^2 k_m^2} \right) \left( a_m f^n_j (k_m, \omega_m) - a_m^* f^{*n}_j (k_m, \omega_m) \right)$$  \hspace{1cm} (41)

Applying the average operator on both sides we get

$$\tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j = \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \frac{1}{\sqrt{2\omega_m}} \left( a_m f^n_j (k_m, \omega_m) + a_m^* f^{*n}_j (k_m, \omega_m) \right)$$  \hspace{1cm} (42)

$$\tilde{\nabla}_j \tilde{\Delta}_j \pi^n_j = -\frac{i}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\omega_m/2} \left( a_m f^n_j (k_m, \omega_m) - a_m^* f^{*n}_j (k_m, \omega_m) \right)$$  \hspace{1cm} (43)

Using the completeness relation (15) we can calculate the coefficients in the Fourier expansion

$$a_m = \frac{\varepsilon}{\sqrt{N\varepsilon}} \sum_{j=0}^{N-1} \left\{ \sqrt{\omega_m/2} f^{*n}_j (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j + \frac{i}{\sqrt{2\omega_m}} f^n_j (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \pi^n_j \right\}$$  \hspace{1cm} (44)

$$a_m^* = \frac{\varepsilon}{\sqrt{N\varepsilon}} \sum_{j=0}^{N-1} \left\{ \sqrt{\omega_m/2} f^n_j (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j - \frac{i}{\sqrt{2\omega_m}} f^{*n}_j (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \pi^n_j \right\}$$  \hspace{1cm} (45)

Using (21), the periodicity condition and Hamilton equations of motions (36, 37), one can easily prove that this coefficients are independent of $n$, namely,

$$\nabla_n (a_m) = \Delta_n (a_m) = \nabla_n (a_m^*) = \Delta_n (a_m^*) = 0$$  \hspace{1cm} (46)

Turning to the complex Klein-Gordon field we can define a Hamiltonian as follows:

$$H_n = \varepsilon \sum_{j=0}^{N-1} \left\{ \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi^n_j \right) \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j \right) - \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi^{*n}_j \right) \frac{1}{\varepsilon^2} \nabla_j \Delta_j \phi^n_j + M^2 \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi^{*n}_j \right) \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j \right\}$$  \hspace{1cm} (47)

from which the Hamilton equations of motion can be derived:

$$\frac{1}{\tau^2} \Delta_n \tilde{\nabla}_j \tilde{\Delta}_j \phi^n_j = \tilde{\Delta}_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi^{*n}_j \right)$$  \hspace{1cm} (48)

$$\frac{1}{\tau^2} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi^n_j \right) = \tilde{\Delta}_n \left( \frac{1}{\varepsilon^2} \nabla_j \Delta_j \phi^n_j - M^2 \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi^{*n}_j \right) \right)$$  \hspace{1cm} (49)
leading to the wave equation (22). The solution of the wave equation can be expanded in Fourier series as before:

\[
\phi_j^n = \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{2\omega_m} \left( 1 + \frac{1}{4} \varepsilon^2 k_m^2 \right) \left( a_m f_j^n (k_m, \omega_m) + b_m^* f_j^{*n} (k_m, \omega_m) \right)
\]

\[
\pi_j^n = -i \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\omega_m} \left( 1 + \frac{1}{4} \varepsilon^2 k_m^2 \right) \left( b_m f_j^n (k_m, \omega_m) - a_m^* f_j^{*n} (k_m, \omega_m) \right)
\]

and similar expressions for the complex fields.

In order to make connection of our scheme with the Einstein-de Broglie relations \( E = \hbar \omega, \ p = \hbar k \) we take for the period \( T \) and wave length \( \lambda \) of the discrete plane waves functions (23) and (27)

\[
T = N\tau, \quad \lambda = N\varepsilon
\]

and for the phase velocity

\[
v_p = \frac{\lambda}{T} = \frac{\varepsilon}{\tau}
\]

We have define the wave number and the angular frequency of the wave functions as:

\[
k_m = \frac{2}{\varepsilon} \tan \frac{\pi m}{N}, \quad \omega_m = \frac{2}{\tau} \tan \frac{\pi m}{N}, \quad m = 0, 1, \ldots, N - 1
\]

substituting the Einstein-de Broglie relations in the relativistic expresion \( E^2 - p^2 = M^2 \) (we use natural units \( \hbar = c = 1 \)), we obtain

\[
\omega_m^2 - k_m^2 = \omega_m^2 \left( 1 - \frac{\pi^2}{\varepsilon^2} \right) = \omega_m^2 \left( 1 - \frac{1}{v_p^2} \right) = M^2
\]

Since the phase velocity and group velocity satisfy \( v_p v_g = 1 \), we have finally

\[
\omega_m^2 = \frac{M^2}{1 - v_g^2}
\]

4 Quantitation of the Klein-Gordon field

Let us start form the Hamilton equation of motion (36, 37) where the real fields \( \phi_j^n \) and \( \pi_j^n \) now are, in general, non commuting operators:

\[
\frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) = \tilde{\Delta}_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right)
\]

\[
\frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right) = \tilde{\Delta}_n \left( \frac{1}{\varepsilon^2} \nabla_j \Delta_j \phi_j^n - M^2 \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \right)
\]

Eliminating the field \( \pi_j^n \) we obtain again the wave equation on the lattice (22) the solutions of which can be expanded in terms of the “plane wave functions”

\[
\tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n = \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \frac{1}{\sqrt{2\omega_m}} \left( a_m f_j^n (k_m, \omega_m) + a_m^* f_j^{*n} (k_m, \omega_m) \right)
\]

\[
\tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n = -i \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\omega_m} \left( a_m f_j^n (k_m, \omega_m) - a_m^* f_j^{*n} (k_m, \omega_m) \right)
\]
where the coefficients \( a_m, \ a^1_m \), are, in general, non-commuting operators.

We introduce the quantization procedure imposing the commutation relations of the fields for \( n = 0 \)

\[
\begin{align*}
[\tilde{\nabla}_j \tilde{\Delta}_j \phi_j^0, \tilde{\nabla}_j' \tilde{\Delta}_j' \phi_j'^0] &= 0 \quad (54) \\
[\tilde{\nabla}_j \tilde{\Delta}_j \pi_j^0, \tilde{\nabla}_j' \tilde{\Delta}_j' \pi_j'^0] &= 0 \quad (55) \\
[\tilde{\nabla}_j \tilde{\Delta}_j \phi_j^0, \tilde{\nabla}_j' \tilde{\Delta}_j' \pi_j'^0] &= i \frac{\epsilon}{\sqrt{2\Delta_n}} \delta_{jj'} \quad (56)
\end{align*}
\]

In order to prove unitarity we must prove that these relations hold also for any \( n \). Since (52) and (53) are valid for \( n = 0 \), we solve them for \( a_m \) and \( a^1_m \)

\[
(a_m^0) = \frac{\epsilon}{\sqrt{N\epsilon}} \sum_{j=0}^{N-1} \left\{ \frac{\omega_m}{2} f_j^0 (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^0 + i \frac{\epsilon}{\sqrt{2\omega_m}} f_j^0 (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^0 \right\} \quad (57)
\]

\[
(a^1_m) = \frac{\epsilon}{\sqrt{N\epsilon}} \sum_{j=0}^{N-1} \left\{ \frac{\omega_m}{2} f_j^0 (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^0 - i \frac{\epsilon}{\sqrt{2\omega_m}} f_j^0 (k_m, \omega_m) \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^0 \right\} \quad (58)
\]

Using (54–56) it is straight forward to prove that

\[
\begin{align*}
\begin{bmatrix} (a_m^0) \ 0 \end{bmatrix}^0 &= 0, \quad \begin{bmatrix} (a_m^0) \ (a_m^0) \end{bmatrix}^0 = 0 \quad (59) \\
\begin{bmatrix} (a_m^0) \ (a_m^0) \end{bmatrix}^0 &= \delta_{mm'} \quad (60)
\end{align*}
\]

From (54, 56) and (57, 58) one can prove (remember 41)

\[
(a_m^0) = (a_m^1)^0, \quad (a^1_m) = (a^1_m)^0 \quad (61)
\]

hence (59, 60) is also true for \( n = 1 \), and from (52, 53) we derive

\[
\begin{align*}
[\tilde{\nabla}_j \tilde{\Delta}_j \phi_j^1, \tilde{\nabla}_j' \tilde{\Delta}_j' \phi_j'^1] &= 0 \quad (62) \\
[\tilde{\nabla}_j \tilde{\Delta}_j \pi_j^1, \tilde{\nabla}_j' \tilde{\Delta}_j' \pi_j'^1] &= 0 \quad (63)
\end{align*}
\]

and the proof is completed by induction.

The Hamiltonian for the real scalar field reads:

\[
H_n = \epsilon \sum_{j=0}^{N-1} \frac{1}{2} \left\{ \left( \nabla_j \nabla_j \pi_j^0 \right)^2 - \frac{1}{\epsilon^2} \left( \nabla_j \nabla_j \phi_j^0 \right) \left( \nabla_j \nabla_j \phi_j^0 \right) + M^2 \left( \nabla_j \nabla_j \phi_j^0 \right)^2 \right\} \quad (64)
\]

A convenient scheme for the Heisenberg equation is:

\[
\begin{align*}
\frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) &= \frac{1}{i} \left[ \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right), H_n \right] \\
\frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right) &= \frac{1}{i} \left[ \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right), H_n \right]
\end{align*}
\]

Using (54) and (60) and (50, 51) for any \( n \), we get for the first equation

\[
\begin{align*}
\frac{1}{\tau} \Delta_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) &= \frac{1}{i} \Delta_n \left[ \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n, H_n \right] = \tilde{\Delta}_n \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) \quad (65)
\end{align*}
\]
and similarly for the second equation
\[
\frac{1}{\tau} \Delta_n \left( \nabla_j \Delta_j \pi_j^n \right) = \tilde{\Delta}_n \left\{ \frac{1}{\varepsilon^2} \nabla_j \Delta_j \phi_j^n - M^2 \nabla_j \Delta_j \phi_j^n \right\} \tag{66}
\]
consistent with the Hamilton equation of motions \((50, 51)\).

Substituting \((57, 58)\) in \((64)\) we find
\[
H = \frac{1}{2} \sum_{m=-N/2}^{N/2-1} \omega_m \left( a_m a_m^\dagger + a_m^\dagger a_m \right) \tag{67}
\]

If we define the linear momentum on the lattice as
\[
P = \varepsilon \sum \frac{1}{i\varepsilon} \left( \nabla_j \Delta_j \pi_j^n \right) \left( \nabla_j \Delta_j \phi_j^n \right) \]
we find
\[
P = \sum_{m=-N/2}^{N/2-1} k_m \left( a_m a_m^\dagger + a_m^\dagger a_m \right) = \sum_{m=-N/2}^{N/2-1} k_m a_m^\dagger a_m
\]
because the zero point momentum vanish by cancellation of \(k_m\) with \(k_{-m}\).

For the complex Klein-Gordon field we have the hermitian Hamiltonian
\[
H_n = \varepsilon \sum_{j=0}^{N-1} \left\{ \left( \nabla_j \Delta_j \pi_j^n \right) \left( \nabla_j \Delta_j \phi_j^n \right) - \left( \nabla_j \Delta_j \phi_j^n \right) \frac{1}{\varepsilon^2} \left( \nabla_j \Delta_j \phi_j^n \right) + M^2 \left( \nabla_j \Delta_j \phi_j^n \right) \left( \nabla_j \Delta_j \phi_j^n \right) \right\}
\tag{68}
\]
The equal time commutation relations now read:
\[
\left[ \nabla_j \Delta_j \phi_j^n, \nabla_{j'} \Delta_{j'} \pi_{j'}^n \right] = \frac{1}{\varepsilon} \delta_{jj'}, \quad \left[ \nabla_j \Delta_j \phi_j^n, \nabla_{j'} \Delta_{j'} \phi_{j'}^n \right] = \frac{1}{\varepsilon} \delta_{jj'} \tag{69}
\]
with all other equal time commutators vanishing.

The Heisenberg equations of motion lead, as in the real case, to the discrete wave equation for the fields \(\phi_j^n, \pi_j^n\), and their adjoints. The Fourier expansion in terms of the plane wave solutions are
\[
\phi_j^n = \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \left( \frac{1 + \frac{i}{4}\varepsilon^2 k_m}{\sqrt{2\omega_m}} \right) \left( a_m f_j^n (k_m, \omega_m) + b_m^\dagger f_{j'}^n (k_m, \omega_m) \right) \tag{70}
\]
\[
\pi_j^n = -\frac{i}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\frac{\omega_m}{2}} \left( 1 + \frac{1}{4}\varepsilon^2 k_m \right) \left( b_m f_j^n (k_m, \omega_m) - a_m^\dagger f_{j'}^n (k_m, \omega_m) \right) \tag{71}
\]
Inverting this equations we can calculate with the help of \((69)\),
\[
\left[ a_m, a_m^\dagger \right] = \delta_{mm'}, \quad \left[ b_m, b_m^\dagger \right] = \delta_{mm'} \tag{72}
\]
with all other commutators vanishing.

The Hamiltonian can be expressed in terms of these creation and annihilation operators:
\[
H = \frac{1}{2} \sum_{m=-N/2}^{N/2-1} \omega_m \left( a_m^\dagger a_m + b_m^\dagger b_m \right) \tag{73}
\]
The charge of the complex field is given by

\[
Q = -i\varepsilon \sum_{j=0}^{N-1} \left( \tilde{\nabla}_j \tilde{\Delta}_j \pi_j^n \right) \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) - \left( \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n \right) = \sum_{m=-N/2}^{N/2-1} \left( a_m^\dagger a_m + b_m^\dagger b_m \right) \tag{74}
\]

Finally, two very important quantities can be evaluated on the lattice: the general commutations rules and the Feynman propagator.

First of all, using (70) and (72) in order to calculate the commutator of the fields for two different space-time points, we find:

\[
\left[ \tilde{\nabla}_j \tilde{\Delta}_j \phi_j^n, \tilde{\nabla}_j' \tilde{\Delta}_j' \phi_j'^{n'} \right] = \frac{1}{N\varepsilon} \sum_{m=-N/2}^{N/2-1} \frac{1}{2\omega_m} \left( f^{n-n'}_j (k_m, \omega_m) - f^{n-n'}_j' (k_m, \omega_m) \right) \tag{75}
\]

which becomes zero for equal time \((n = n')\).

Secondly, the Feynman propagator function is defined to be the vacuum expectation value

\[
\Delta_F \equiv \left\langle 0 \left| T\phi_j^n \phi_j'^{n'} \right| 0 \right\rangle \tag{76}
\]

of the time ordered product

\[
T\phi_j^n \phi_j'^{n'} = \begin{cases} 
\phi_j^n \phi_j'^{n'} & \text{for } n > n' \\
\phi_j'^{n'} \phi_j^n & \text{for } n' > n
\end{cases} \tag{77}
\]

Notice that there is no ambiguity of the time ordered product for equal times, since \(\phi_j^n\) and \(\phi_j'^{n'}\) commute for \(n = n'\) because of (75).

The discrete analog of the Feynman propagator is from (70) and (77)

\[
\Delta_F = \frac{1}{N\varepsilon} \sum_{m=-N/2}^{N/2-1} \left( 1 + \frac{1}{4} \varepsilon^2 k_m^2 \right) \frac{1}{2\omega_m} \left\{ \vartheta_n f^n_j (k_m, \omega_m) + \vartheta_{-n} f^{*n}_{j'} (k_m, \omega_m) \right\} \tag{78}
\]

\(\vartheta_n\) being the discrete realization of the Heaviside functions (17).

In order to see the analogy with the continuous case we could take the discrete Fourier transform of (78). Instead, we take the Fourier transform of the wave equation (22), namely,

\[
\left( -\omega_r^2 + k_r^2 + M^2 \right) \frac{1}{1 + \frac{1}{4} \varepsilon^2 k_m^2} \frac{1}{1 + \frac{1}{4} \tau^2 \omega_m^2} \tilde{\phi}_m = 0
\]

where

\[
\tilde{\phi}_m = \sum_{j=0}^{N-1} \sum_{n=0}^{N-1} f_j^n (k_m, \omega_r) \phi_j^n
\]

is the Fourier transform of the Klein-Gordon field.

Therefore the Feynman propagator is defined in the momentum space

\[
\tilde{\Delta}_F = -i \frac{\left( 1 + \frac{1}{4} \varepsilon^2 k_m^2 \right) \left( 1 + \frac{1}{4} \tau^2 \omega_m^2 \right)}{k_m^2 - \omega_r^2 + M^2}
\]
and in the configuration space

\[ G_{j-j'}^{n-n'} = \frac{1}{(N\varepsilon)(N\tau)} \sum_{m=-N/2}^{N/2-1} \sum_{r=-N/2}^{N/2-1} \hat{G}_{m}^{r}f_{j-j'}^{n-n'}(k_{m},\omega_{r}) \]

(Notice that now \( \omega_{r} = \frac{2}{\tau} \tan \frac{\pi r}{N} \), can take negative values)

It can be proved that this propagator is the Green function for the discrete wave equation, namely,

\[ \left( \frac{1}{\tau^2} \nabla_{n} \Delta_{n} \nabla_{n} \Delta_{n} - \frac{1}{\varepsilon} \nabla_{j} \Delta_{j} \nabla_{n} \Delta_{n} + M^2 \nabla_{n} \Delta_{n} \nabla_{j} \Delta_{j} \right) G_{j-j'}^{n-n'} = i \delta_{nn'} \varepsilon \delta_{jj'} \]

With the aid of the identities:

\[ \nabla_{n} \Delta_{n} \left[ \partial_{n} f_{j}^{n} \right] = \partial_{n} \nabla_{n} \Delta_{n} f_{j}^{n} + (\Delta_{n} \partial_{n}) \left( \Delta_{n} f_{j}^{n} \right) + (\Delta_{n} \partial_{n-1}) \left( \Delta_{n} f_{j}^{n-1} \right) + (\nabla_{n} \Delta_{n} \partial_{n}) f_{j}^{n} \]

\[ \tilde{\nabla}_{n} \tilde{\Delta}_{n} \left[ \partial_{n} f_{j}^{n} \right] = \partial_{n} \tilde{\nabla}_{n} \tilde{\Delta}_{n} f_{j}^{n} + \frac{1}{4} \left[ (\Delta_{n} \partial_{n}) \left( \Delta_{n} f_{j}^{n} \right) + (\Delta_{n} \partial_{n-1}) \left( \Delta_{n} f_{j}^{n-1} \right) \right] + \frac{1}{4} (\nabla_{n} \Delta_{n} \partial_{n}) f_{j}^{n} \]

and the equations (18) and (24) one can prove that \( \Delta_{F} \) defined by (78) satisfies also the same condition, therefore \( \Delta_{F} \) and \( G_{m}^{n} \) are equal up to a constant.

5 Quantization of the Dirac field

The Hamiltonian for the Dirac field \( \psi_{\alpha j}^{n} \) and its hermitian adjoint \( \psi_{\alpha j}^{\dagger n} \) can be defined as:

\[ H_{n} = \varepsilon \sum_{j=0}^{N-1} \tilde{\Delta}_{j} \psi_{\alpha j}^{\dagger n} \left\{ \gamma_{4} \gamma_{1} \frac{1}{\varepsilon} \tilde{\Delta}_{j} \psi_{\alpha j}^{n} + M \gamma_{4} \tilde{\Delta}_{j} \psi_{\alpha j}^{n} \right\} \]  

(79)

with

\[ \gamma_{1} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_{4} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\gamma_{1}\gamma_{4} = \gamma_{5} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \]  

(80)

In the next chapter it will be shown that the substitution \( \psi(x) \rightarrow \tilde{\Delta}_{j} \psi_{\alpha j}^{n} \) is consistent with the density function

\[ \rho = \tilde{\Delta}_{j} \psi_{\alpha j}^{\dagger n} \tilde{\Delta}_{j} \psi_{\alpha j}^{n} \]  

(81)

If we defined the conjugate momentum \( \pi_{\beta j}^{n} = i\psi_{\alpha j}^{\dagger n} \) the Hamilton equation of motion are obtained varying the Hamiltonian density \( H_{j}^{n} \) with respect to \( i\psi_{\alpha j}^{\dagger n} \), that is to say:

\[ \frac{1}{\tau} \Delta_{n} \tilde{\Delta}_{j} \psi_{\alpha j}^{n} = \frac{1}{\varepsilon} \tilde{\Delta}_{n} \left\{ \gamma_{4} \gamma_{1} \frac{1}{\varepsilon} \tilde{\Delta}_{j} \psi_{\alpha j}^{n} + M \gamma_{4} \tilde{\Delta}_{j} \psi_{\alpha j}^{n} \right\} \]  

(82)

from which it can be derived the Dirac equation on the lattice:

\[ \left( \gamma_{1} \frac{1}{\varepsilon} \Delta_{n} \tilde{\Delta}_{j} - i \gamma_{4} \frac{1}{\tau} \Delta_{n} \tilde{\Delta}_{j} + M \Delta_{n} \tilde{\Delta}_{j} \right) \psi_{\alpha j}^{n} = 0 \]  

(83)

The same result was obtained by Bender, Milton and Sharp (Reference 1, formula 18) applying the method of finite elements to the action.

From (24) and (83) one proves for the Dirac Hamiltonian (79)

\[ \Delta_{n} H_{n} = 0, \quad \text{hence} \quad \Delta_{n} H_{n} = H_{n} \]  

(84)
If we impose periodicity conditions in the fields we can prove by “integration by parts” of (79) that $H_n$ is an hermitian function.

Applying the operator
\[
\gamma_1 \frac{1}{\varepsilon} \nabla_j \nabla_n - i \gamma_4 \frac{1}{\varepsilon} \nabla_n \Delta_j - M \nabla_j \nabla_n
\]
on both sides of (82) we recover the wave equation (22) for $\psi_j^n$.

Let us construct solutions to (83) of the form
\[
\psi_j^n = w(k, E) f_j^n(k, E)
\]
where $f_j^n(k, E)$ is given by (23).

The four component spinors $w(k, E)$ must satisfy
\[
(i \gamma_1 k - \gamma_4 E + M) w(k, E) = 0
\]
Multiplying this equation from the left by $(i \gamma_1 k - \gamma_4 E - M)$ we obtain the “dispersion relation” for the Dirac equation
\[
E^2 - k^2 = M^2
\]
Following standard procedure we can construct a complete set of Dirac wave functions of the form
\[
\sum_{m=-N/2}^{N/2-1} \frac{1}{\sqrt{N\varepsilon}} \sqrt{\frac{M}{E_m}} u_m f_j^n(k_m, E_m)
\]
and the spinors $u_m$ and $v_m$ are defined as usual:
\[
\begin{align*}
u_m &= w(k_m, E_m) = \left(\frac{E_m + M}{2M}\right)^{1/2} \left(\frac{1}{E_m + M}\right) \\
v_m &= w(-k_m, -E_m) = \left(\frac{E_m + M}{2M}\right)^{1/2} \left(\frac{k_m}{E_m + M}\right)
\end{align*}
\]
With the aid of this orthonormal set we can expand the fields in the usual way (remember 12)
\[
\begin{align*}
\Delta_j \psi_j^n &= \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\frac{M}{E_m}} \left( u_m c_m f_j^n(k_m, E_m) + v_m d_m f_j^\dagger \gamma_4(k_m, E_m) \right) \\
\Delta_j \overline{\psi_j^n} &= \frac{1}{\sqrt{N\varepsilon}} \sum_{m=-N/2}^{N/2-1} \sqrt{\frac{M}{E_m}} \left( \overline{u}_m c_m f_j^n(k_m, E_m) + \overline{v}_m d_m f_j^\dagger \gamma_4(k_m, E_m) \right)
\end{align*}
\]
where $\overline{\psi_j^n} \equiv \psi_j^n \gamma_4$. Inverting this expressions, we find
\[
c_m = \frac{1}{\sqrt{N\varepsilon}} \sqrt{\frac{M}{E_m}} \sum_{j=0}^{N-1} f_j^\dagger \gamma_4(k_m, E_m) \overline{u}_m \Delta_j \psi_j^n
\]
\[
\begin{align*}
c_m^\dagger &= \frac{1}{\sqrt{N\varepsilon}} \sqrt{\frac{M}{E_m}} \sum_{j=0}^{N-1} \Delta_j \bar{\psi}_j^\dagger \gamma_4 u_m f_j^n (k_m, E_m) \\
d_m &= \frac{1}{\sqrt{N\varepsilon}} \sqrt{\frac{M}{E_m}} \sum_{j=0}^{N-1} \Delta_j \bar{\psi}_j \gamma_4 v_m f_j^n (k_m, E_m) \\
d_m^\dagger &= \frac{1}{\sqrt{N\varepsilon}} \sqrt{\frac{M}{E_m}} \sum_{j=0}^{N-1} f_j^n (k_m, E_m) \tau_m \gamma_4 \Delta_j \bar{\psi}_j^n
\end{align*}
\]

For the quantization of the Dirac field we start from the Hamilton equation of motion (82) where the time order operator acts as follows

\[
\tau = 1 + i \frac{\Delta_n (\bar{\psi}_j^n)}{\varepsilon} \frac{d}{dt} + \frac{i}{\varepsilon} \Delta_n (\bar{\psi}_j^n H_n) = 1 + i \frac{\Delta_n (\bar{\psi}_j^n H_n)}{\varepsilon} = 1 + i \Delta_n \left\{ \gamma_4 \gamma_1 \frac{1}{\varepsilon} \Delta_n (\bar{\psi}_j^n + M \gamma_4 \Delta_j \bar{\psi}_j^n) \right\}
\]

Using (99, 100) we can derived the anticommutation relations for the creation and annihilation operators

\[
\left[ \Delta_j \psi^{n,\dagger}_{\alpha j}, \bar{\Delta}_j \psi^n_{\beta j'} \right]_+ = \frac{1}{\varepsilon} \delta_{\alpha\beta} \delta_{jj'}
\]

\[
\left[ \Delta_j \psi^{n,\dagger}_{\alpha j}, \bar{\Delta}_j \psi^n_{\beta j'} \right]_+ = \left[ \bar{\Delta}_j \psi^n_{\alpha j}, \bar{\Delta}_j \psi^{n,\dagger}_{\beta j'} \right]_+ = 0
\]

provided it holds for \( n = 0 \).

Introducing the Hamiltonian (7) in the Heisenberg equation of motion and using the anticommutation relations (99, 100) we recover the Dirac equation, namely,

\[
\frac{1}{i\hbar} \Delta_n (\bar{\Delta}_j \psi_j^n) = \frac{1}{i} \Delta_n (\bar{\Delta}_j \psi_j^n, H_n) = \frac{1}{i} \Delta_n \left[ \bar{\Delta}_j \psi_j^n, H_n \right] = \frac{1}{i} \Delta_n \left\{ \gamma_4 \gamma_1 \frac{1}{\varepsilon} \Delta_j \bar{\psi}_j^n + M \gamma_4 \bar{\Delta}_j \psi_j^n \right\}
\]

Using (99, 100) we can derived the anticommutation relations for the creation and annihilation operators

\[
\left[ c_m, c_m^{\dagger} \right]_+ = \delta_{mm'} \quad \left[ d_m, d_m^{\dagger} \right]_+ = \delta_{mm'}
\]

with other anticommutation relations vanishing.

Finally, the Hamiltonian (79) can be written in terms of these operators

\[
H = \sum_{m=-N/2}^{N/2-1} E_m \left( c_m^{\dagger} c_m + d_m^{\dagger} d_m - 1 \right)
\]

Similarly the momentum and charge operators are written:

\[
P = -i \sum_{j=0}^{N-1} \left( \bar{\Delta}_j \psi_j^{n,\dagger} \right) \left( \bar{\Delta}_j \psi_j^n \right) = \sum_{m=-N/2}^{N/2-1} k_m \left( c_m^{\dagger} c_m + d_m^{\dagger} d_m \right)
\]

\[
Q = \sum_{j=0}^{N-1} \left( \Delta_j \psi_j^{n,\dagger} \right) \left( \Delta_j \psi_j^n \right) = \sum_{m=-N/2}^{N/2-1} \left( c_m^{\dagger} c_m - d_m^{\dagger} d_m \right) + 2N
\]

The discrete analog of the Feynman propagator for the Dirac field is defined as before

\[
S_{\alpha \beta, j' - j}^{n - n'} = \langle 0 \left| T \bar{\psi}_{\alpha j}^{n', \dagger} \psi_{\beta j'}^{n'} \right| 0 \rangle
\]

where the time order operator acts as follows

\[
T \bar{\psi}_{j'}^{n'} \psi_{j}^{n} = \begin{cases} \psi_{j'}^{n'} \bar{\psi}_j^n & \text{for } n > n' \\ -\psi_{j'}^{n'} \bar{\psi}_j^n & \text{for } n' > n \end{cases}
\]
Following standard methods we find:

\[
S^{n-n'}_{\alpha\beta,j-j'} = \partial_n \frac{1}{N\varepsilon} \sum_{m=-N/2}^{N/2-1} \left( 1 - \frac{i}{2} \varepsilon k_m \right)^2 \left( \gamma_1 k_m + i E_m + i M \right) \alpha\beta f^{n-n'}_{j-j'}(k_m, E_m)
\]

\[
- \partial_{-n} \frac{1}{N\varepsilon} \sum_{m=-N/2}^{N/2-1} \left( 1 - \frac{i}{2} \varepsilon k_m \right)^2 \left( \gamma_1 k_m + i E_m + i M \right) \alpha\beta f^{n-n'}_{j-j'}(k_m, E_m)
\]

We can also derive the Feynman propagator from the Dirac equation. Taking the Fourier transform of (83) we get

\[
\hat{G}_m = \left( 1 - \frac{1}{2} i \varepsilon k_m \right) \left( 1 + \frac{1}{2} i \tau E_r \right) \left( 1 + i \gamma_1 k_m - \gamma_4 E_r + M \right) \left( E_r^2 - k_m^2 - M^2 \right)
\]

hence

\[
G^{n-n'}_{j-j'} = \frac{1}{N\varepsilon N\tau} \sum_{m=-N/2}^{N/2-1} \sum_{r=-N/2}^{N/2-1} \hat{G}_m f^{n-n'}_{j-j'}(k_m, E_r)
\]

Both function (108) and (110) are the Green function for the Dirac equations, therefore they are equal up to a constant.

Our model for the fermion field satisfies the following conditions:

i) the hamiltonian (79) is translationally invariant with respect to the space indices.

ii) the hamiltonian is hermitian.

iii) for \( M = 0 \), the wave equation (83) is invariant under global chiral transformations.

iv) there is no “fermion doubling” as it can be seen in (90). In fact, \( E_m \) takes the value \( M \) at \( m = 0 \) and nowhere else.

v) the hamiltonian is non-local. Using the finite Fourier transform (13) the hamiltonian \( H(k_m) = i\gamma_4 \gamma_1 k_m + \gamma_4 M \), given by (86), and consequently the dispersion relations (90) are smooth functions of \( k_m \), except for \( m = N/2 \). Therefore our model escapes the no-go theorem of Nielsen and Ninomiya [6].

6 Anomalies in axial vector current

In order to construct vector and axial currents in terms of the fermions fields in the presence of external electromagnetic vector potential, we multiply (83) from the left by \( \tilde{\Delta}_j \tilde{\Delta}_n \psi^n_j \) and then we multiply the adjoint equation of (83) from the right by \( \Delta_j \Delta_n \psi^n_j \). Adding together both results and using (20) we find:

\[
\frac{1}{\varepsilon} \partial_j \left( \tilde{\Delta}_n \psi^n_j \gamma_1 \Delta_n \psi^n_j \right) - \frac{1}{\tau} \partial_n \left( \Delta_j \psi^n_j \gamma_4 \Delta_j \psi^n_j \right) = 0
\]

This equation can be considered the discrete version of the “conservation law” for the vector current

\[
j_1 = i \left( \Delta_n \psi^n_j \right) \gamma_1 \left( \Delta_n \psi^n_j \right), \quad j_4 = i \left( \Delta_j \psi^n_j \right) \gamma_4 \left( \Delta_j \psi^n_j \right)
\]
The same equation (111) can be applied to the axial current

\[ j_1^5 = i \left( \Delta_n \psi_j \right) \gamma_1 \gamma_5 \left( \Delta_n \psi_j^n \right), \quad j_4^5 = i \left( \Delta_j \psi_j \right) \gamma_4 \gamma_5 \left( \Delta_j \psi_j^n \right) \]  

(113)

Both currents are invariant under global chiral transformations but they are not invariant under \( U(1) \)-gauge transformations:

\[ \psi_j^n \rightarrow \Omega_j^n \psi_j^n, \quad \bar{\psi}_j^n \rightarrow \bar{\psi}_j^n \Omega_j^n \]  

(114)

with \( \Omega_j^n \) some unitary function of discrete variables. In order to have gauge invariance we define a gauge field on the lattice

\[ U_j^{nn'} = \frac{1 + i \frac{\epsilon}{4} \{ (A_0)_j^n + (A_0)_j^{n'} \}}{1 - i \frac{\epsilon}{4} \{ (A_0)_j^n + (A_0)_j^{n'} \}} \]  

(115)

and

\[ U_{jj'}^{nn'} = \frac{1 + i \epsilon \{ (A_1)_j^n + (A_1)_j^{n'} \}}{1 - i \epsilon \{ (A_1)_j^n + (A_1)_j^{n'} \}} \]  

(116)

where \( (A_1, iA_0) \) are the two component electromagnetic vector potential, each of them satisfying the wave equation (22) with \( M = 0 \) namely,

\[ \left( \frac{1}{\tau^2} \nabla_n \Delta_n \nabla_j \Delta_j - \frac{1}{\epsilon^2} \nabla_j \Delta_j \nabla_n \Delta_n \right) (A_1)_j^n = 0 \]  

(117)

and similarly for \( (A_0)_j^n \).

The gauge fields \( U_j^{nn'} \) are associated with the link between the points \( (j, n) \rightarrow (j, n') \) in the positive direction, and they transform under the gauge group as follows:

\[ U_j^{nn'} \rightarrow \Omega_j^n U_j^{nn'} \Omega_j^{n'} \]  

(118)

\[ U_{jj'}^{nn'} \rightarrow \Omega_j^n U_{jj'}^{nn'} \Omega_{j'}^{n'} \]  

(119)

Inserting these fields in the vector and axial currents between fermion fields at separated points, we get

\[ j_1 = \frac{i}{4} \left( \bar{\psi}_j^n \gamma_1 \psi_j^n + \bar{\psi}_j^n \gamma_1 U_j^{n,n+1} \psi_j^{n+1} + \bar{\psi}_j^{n+1} \gamma_1 U_j^{n+1,n} \psi_j^n + \bar{\psi}_j^{n+1} \gamma_1 \psi_j^{n+1} \right) \]  

(120)

\[ j_4 = \frac{i}{4} \left( \bar{\psi}_j^n \gamma_4 \psi_j^n + \bar{\psi}_j^n \gamma_4 U_{j,j+1}^{n,n} \psi_{j+1}^n + \bar{\psi}_{j+1}^n \gamma_4 U_{j+1,j}^{n,n} \psi_j^n + \bar{\psi}_{j+1}^n \gamma_4 \psi_{j+1}^n \right) \]  

(121)

and similarly for \( j_1^5 \) and \( j_4^5 \). Using (118) and (119) we can prove that all these expressions for the vector and axial currents are invariant under the gauge transformations (114).

Now we want to calculate the vacuum expectation value of the divergence of the vector and axial current. We assume that this vacuum expectation value approaches the corresponding non-interacting fields for vanishing spacial separation of the fields (83). Thus we use solution of the free massless fermion field (83) with \( M = 0 \), namely,

\[ \psi_j^n = \frac{1}{\sqrt{N}} \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} \left( u_m c_m f_j^n (k_m, \omega_m) + v_m d_m f_j^n (k_m, -\omega_m) \right) \]  

(122)
with \( u_m = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), and \( v_m = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), and the operators \( c_m \) and \( d_m \) satisfying the usual anticommutation relations:

\[
\left[ c_m, c_{m'}^\dagger \right]_+ = \delta_{mm'} , \quad \left[ d_m, d_{m'}^\dagger \right]_+ = \delta_{mm'}
\]

(123)

Applying these operators to the vacuum of the Fock space, we get

\[
\begin{align*}
\langle 0 | \frac{1}{2} \Delta_j j_1^5 - \frac{i}{2} \Delta_n j_4^5 | 0 \rangle &= \frac{i}{4\varepsilon} \langle 0 | \mathbf{v}_{j+1}^n \gamma \mathbf{v}_{j+1}^{n+1} | 0 \rangle S_{j+1}^{n+1} - \frac{i}{4\varepsilon} \langle 0 | \mathbf{v}_{j+1}^n \gamma \mathbf{v}_{j+1}^{n+1} | 0 \rangle S_{j+1}^{n+1} \\
&+ \frac{i^2}{4\tau} \langle 0 | \mathbf{v}_{j+1}^n \gamma \mathbf{v}_{j+1}^{n+1} | 0 \rangle S_{j+1}^{n+1} - \frac{i^2}{4\tau} \langle 0 | \mathbf{v}_{j+1}^n \gamma \mathbf{v}_{j+1}^{n+1} | 0 \rangle S_{j+1}^{n+1} + \text{c.c.} = \\
&- \frac{1}{N\varepsilon} \left\{ \sum_{m=0}^{N/2-1} \frac{1 - \frac{i}{2} \tau \omega_m}{1 + \frac{i}{2} \tau \omega_m} + \sum_{m=-N/2}^{-1} \frac{1 + \frac{i}{2} \tau \omega_m}{1 - \frac{i}{2} \tau \omega_m} \right\} \frac{1}{4\varepsilon} (S_{j+1}^{n+1} - S_{j+1}^n) \\
&+ \frac{1}{N\varepsilon} \left\{ \sum_{m=0}^{N/2-1} \frac{1 + \frac{i}{2} \varepsilon k_m}{1 - \frac{i}{2} \varepsilon k_m} + \sum_{m=-N/2}^{-1} \frac{1 - \frac{i}{2} \varepsilon k_m}{1 + \frac{i}{2} \varepsilon k_m} \right\} \frac{1}{4\tau} (S_{j+1}^{n+1} - S_{j+1}^n) + \text{c.c.} = \\
&- \frac{1}{N} \frac{1}{2\pi} \frac{1}{1 - i\varepsilon \Delta_j \Delta_n (A_0)_j^n - \frac{\varepsilon^2}{4} \Delta_j^2 (A_0)_j^{2n} + c.c.} + \frac{e^{-i\pi/N}}{2N\pi} \frac{1}{1 - i\varepsilon \Delta_j \Delta_n (A_1)_j^n - \frac{\varepsilon^2}{4} \Delta_n^2 (A_1)_j^{2n} + c.c.}
\end{align*}
\]

which in the limit, \( N \to \infty, \varepsilon \to 0, \tau \to 0 \), becomes

\[
\langle 0 | \partial_1 j_1^5 + \partial_4 j_4^5 | 0 \rangle = -\frac{1}{\pi} (\partial_1 A_0 - \partial_0 A_1) = -\frac{1}{\pi} F_{10}
\]

(126)

as required. Similarly we can prove

\[
\langle 0 | \frac{1}{2} \Delta_j j_1 + \frac{1}{2} i \Delta_n j_4 | 0 \rangle = 0
\]

We come to the conclusion that our model leads to an interaction which is \( U(1) \)-gauge invariant, but the divergence of the axial current gives in the continuous limit a photon mass, as expected by the axial anomaly.

For the sake of simplicity in the notation we have worked all the formulas in \((1+1)\)-dimension. The expressions in this paper can be easily generalized to \((3+1)\)-dimensional lattice by the use of the promodiate fields:

\[
\tilde{\nabla}_{j_1} \tilde{\Delta}_{j_1} \tilde{\nabla}_{j_2} \tilde{\Delta}_{j_2} \tilde{\nabla}_{j_3} \tilde{\Delta}_{j_3} \phi (\varepsilon_{1j_1}, \varepsilon_{2j_2}, \varepsilon_{3j_3}, \tau n) \rightarrow \phi (x_1, x_2, x_3, t)
\]

and similarly for space an time differences.

We have discussed symmetries and conservation laws for this scheme elsewhere[10].
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