Giant vortices in small mesoscopic disks: an approximate description

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We present an approximate description of the giant vortex state in a thin mesoscopic superconducting disk within the phenomenological Ginzburg-Landau approach. Analytical asymptotic expressions for the energies of the states with fixed vorticity are obtained when a small magnetic flux is accumulated in the disk. The spectrum of the lowest Landau levels of such a disk is also discussed.

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I. INTRODUCTION

Progress in microfabrication technology has made mesoscopic superconductors (mesoscopic disks, in particular) a very popular subject of study (see, for example, Refs. [1,2]). Mesoscopic samples have sizes comparable to the coherence length $\xi$ and the magnetic field penetration length $\lambda$. The behaviour of such structures in an external magnetic field $H$ is strongly influenced by the sample shape [3] and may lead to various superconducting states [4,5].

The vortex structure in mesoscopic disks with finite thickness was studied early using a numerical solution of the system of two coupled non-linear Ginzburg-Landau (GL) equations [6]. For thin disks it is possible to simplify the solution of the problem by averaging the order parameter over the disk thickness [5] and by representing the superconducting order parameter as a superposition of eigenfunctions of the linearized GL equation [2,5,7]. But, even then the problem has still to be solved numerically because of the presence of confluent hypergeometric functions.

It was shown previously that in some limiting cases the description of the vortex structure can be simplified. Buzdin and Brisson [8] described certain vortex configurations in a small superconducting disk within the London approximation. Akkermans et al. calculated the vortex structure of mesoscopic disks at the vicinity of the dual point $\kappa^2 = 1/2$ [9] and in the London limit [10]. Recently, the structure of a giant vortex in an infinite plane was analyzed analytically within the GL theory for arbitrary values of $\kappa$ in the limits of small and large values of the vorticity [11].

In the present paper we obtain analytical results for the free energy of small disks within the approach of Refs. [5,7], using the magnetic flux trapped by the disk as an expansion parameter.

II. GENERAL THEORETICAL FORMALISM

We consider a mesoscopic superconducting disk with radius $R$ and thickness $d << \lambda, \xi$ magnetized by the external magnetic field $\vec{H} = (0, 0, H)$ which is uniform and directed normal to the disk plane. The theoretical model was already described in detail in Refs. [5,7] and therefore we sketch only those steps which are necessary for our analytical approach. For a thin disk, to a first approximation, the magnetic field is uniform inside the disk and equal to the external one. As a result of this approximation, the distribution of the superconducting order parameter in the disk plane $\psi(\rho)$ is described by the first GL equation

$$
\left(-i\nabla_{2D} - \vec{A}\right)^2 \psi = \psi \left(1 - |\psi|^2\right),
$$

with $\vec{A}(\rho) = (0, H\rho/2, 0)$ and with the boundary condition at the sample surface

$$
\left(-i\nabla_{2D} - \vec{A}\right) \psi \bigg|_{\rho=R} = 0.
$$

The index 2D refers to the two-dimensional operator. Due to the circular symmetry of the sample we use cylindrical coordinates: $\vec{\rho} = (\rho, \theta)$ ($\rho$ is the radial distance from the disk center, $\theta$ is the azimuthal angle). All distances are measured in units of the coherence length $\xi$, the magnetic field in $H_{c2} = 2^{1/2}/\kappa H_c$, where $H_c$ is the thermodynamical critical field.

In the giant vortex state [4] the order parameter can be expressed as

$$
\psi(\rho, \theta) = \left(-\frac{I_1}{I_2}\right)^{1/2} f_L(\rho) \exp(iL\theta),
$$
where

\[ f_L(\rho) = \left(\frac{H\rho^2}{2}\right)^{L/2} \exp\left(-\frac{H\rho^2}{4}\right) M\left(-\nu, L + 1, \frac{H\rho^2}{2}\right), \]  

(4)

\[ I_m(L, \Phi) = \int_0^\Phi t^m \exp\left(-mt\right) \left[M\left(-\nu, L + 1, t\right)\right]^{2m} dt, \]  

(5)

are the eigenfunctions of the linearized Eq. (1) and

\[ \Lambda = -1 + \frac{2\Phi}{R^2} (1 + 2\nu) \]  

(6)

determines the spectrum of the lowest Landau levels,

\[ M(a, c, z) = 1 + \sum_{k=1}^{\infty} \frac{\Gamma(a + k) \Gamma(c) z^k}{\Gamma(a) \Gamma(c + k) k!} \]  

(7)

is the Kummer function \([12]\). The value of \(\nu\) is determined by a non-linear equation, which results from the boundary condition \([6]\)

\[ (L - \Phi) M(-\nu, L + 1, \Phi) - \frac{2\nu\Phi}{L + 1} M(-\nu + 1, L + 2, \Phi) = 0, \]  

(8)

where \(\Phi = HR^2/2\) (measured in units of \(\Phi_0 = \pi\hbar c/e\)) is the magnetic flux through the disk in the absence of any flux expulsion. The free energy, measured in \(F_0 = H^2V/8\pi\) units, is

\[ F = -\frac{\Lambda^2 I_2^2(L)}{\Phi I_2(L)}. \]  

(9)

**III. APPROXIMATE RESULTS**

First, we have to solve the non-linear Eq. 8. In the case of small \(\Phi\) we can restrict the series \([7]\) for the Kummer function by the first four terms. The solution of Eq. 8 can be written as an infinite series of which the first four terms are

\[ \nu = \frac{(L + 1)(L - \Phi)}{2\Phi} \left[1 - \frac{L + \Phi}{2(L + 2)} + \frac{L(L + \Phi)}{2(L + 2)^2(L + 3)} + \frac{(L + \Phi)\left(\Phi^2 - L^2 + 2L^3\right)}{4(L + 2)^3(L + 3)(L + 4)}\right] + \ldots. \]  

(10)

In the limit of large \(\Phi \gg 1\) we use the asymptotic expression for the Kummer function

\[ M(a, c, z) = \frac{\Gamma(c)}{\Gamma(a)} z^{a-c} \exp(z) \left[1 + O\left(|z|^{-1}\right)\right]. \]  

(11)

Substituting it in Eq. 8 we obtain the equation \((L + \Phi)/\Gamma(-\nu) = 0\), which leads to the solution

\[ \nu_{inf} = 0. \]  

(12)
A. Lowest Landau levels

Substituting $\nu$ from Eq. (10) into Eq. (6) we obtain the eigenvalues $\Lambda$ in the $\Phi << 1$ limit:

$$\Lambda = -1 + \frac{2L(L+1)}{R^2} \left[ 1 - \frac{L}{2(L+2)} + \frac{L^2}{2(L+2)^2(L+3)} + \frac{L^3(2L-1)}{4(L+2)^3(L+3)(L+4)} \right]$$

$$-\frac{2L}{R^2}\Phi + \frac{(L+1)}{(L+2) R^2} \left[ 1 - \frac{L}{(L+2)(L+3)} - \frac{L^2(L-1)}{(L+2)^2(L+3)(L+4)} \right] \Phi^2$$

$$-\frac{L+1}{2(L+2)^3(L+3)(L+4) R^2} \Phi^4.$$  \hspace{1cm} (13)

Because series (10) is asymptotic, each new term gives a contribution to all the coefficients of the $\Phi$-series.

In the $\Phi >> 1$ limit we obtain for $\Lambda$ the asymptotic expression

$$\Lambda_{\text{inf}} = -1 + \frac{2\Phi}{R^2},$$  \hspace{1cm} (14)

which does not depend on the vorticity $L$. In Fig. 1 the $\Lambda(\Phi)$ dependences are shown for a disk with radius $R = 3\xi$. Only the curves with $L < 6$ are shown. The solid curves represent the dependences calculated from the numerical solution of expressions (10) and (11) and the dashed curves are our new approximate results as given by Eq. (13). Notice that for small values of the magnetic flux the approximate expression (13) describes quite well the lowest Landau level spectrum.

![Graph showing the magnetic flux dependence of the lowest eigenvalues of the linearized GL equation (1) for different vorticities $L$.](image)

**FIG. 1.** The magnetic flux dependence of the lowest eigenvalues of the linearized GL equation (1) for different vorticities $L$. The “exact” numerical results are shown by the solid curves. The approximate results are shown by the dashed curves ($\Phi << 1$ limit) and by the dash-dotted curve ($\Phi >> 1$ limit).

B. Free energy of giant vortices in the small $\Phi$ limit

The eigenvalues $\Lambda$, Eq. (13), determine the minimal free energy $F$, Eq. (9), of the giant vortices. We expand Eq. (9) in a series with respect to $\Phi$. We found that for small disks with radius $R = (2 \div 3)\xi$ it is sufficient to keep in the series (10) only the first four terms. Expanding the free energy up to order $\Phi^4$ we obtain the approximate expressions:
The free energy for the next $L$ states has the same form as $F_{L=1}$ but with different numeric coefficients and, therefore, will not be given here.

\begin{align*}
F_{L=0} &= -1 + \frac{\Phi^2}{R^2} - \left( \frac{557}{2880} + \frac{1}{96R^2} + \frac{1}{4R^4} \right) \Phi^4, \\
F_{L=1} &= -1.16 + \frac{7.87}{R^2} - \frac{13.34}{R^4} + \left( 3.10 - \frac{25.66}{R^2} + \frac{51.38}{R^4} \right) \Phi - \left( \frac{18.47 - 139.08}{R^2} + \frac{263.87}{R^4} \right) \Phi^2 \\
&\quad + \left( \frac{96.56 - 732.51}{R^2} + \frac{1388.80}{R^4} \right) \Phi^3 - \left( \frac{501.67 - 3810.91}{R^2} + \frac{7235.91}{R^4} \right) \Phi^4.
\end{align*}

The free energy for the next $L$ states has the same form as $F_{L=1}$ but with different numeric coefficients and, therefore, will not be given here.

FIG. 2. The free energy of the giant vortex states for a thin disk with radius $R/\xi = 2.0$. The solid curves are the numerical solution and the dashed curves are from our analytical approximation. The dash-dotted curves are from our analytical expansion up to the $\Phi^2$ term. The dotted curves are from the analytical expression of Ref. [13].

In Fig. 2 the free energy of the disk with $R = 2\xi$ is shown. Notice that limiting oneself to terms in $F$ up to $\Phi^2$ leads to a poor approximation, while including terms up to $\Phi^4$ results in an excellent agreement with the numerical results. The dotted curves in Fig. 2 present the free energy calculated with the expression

\[ F(L, \Phi) = -1 + L - \frac{\Phi^2}{R^2} + \frac{(L - \Phi)^2}{R} - \frac{(L - \Phi)^4}{2R^3}, \]

proposed in Ref. [13] (and rewritten in the notation of the present paper). Notice that expression (17) describes less accurately our numerical result (solid curves). It agrees for $L = 0$ with our $\Phi^2$ approximation, but for $L = 1$ there is a substantial deviation with our approximate result. This disagreement is even larger for $L = 2$ and is therefore not given.

**IV. CONCLUSION**

We calculated approximate expressions for the lowest Landau levels and the free energy of the giant vortex states in thin mesoscopic disk with small radius. We found that these approximate analytical expressions agree very well with the more involved numerical calculation [5,7] and are a substantial improvement to those found in Ref. [13].
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