Some three-point correlation functions in the 
\( \eta \)-deformed \( AdS_5 \times S^5 \)

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Abstract

We compute some normalized structure constants in the \( \eta \)-deformed \( AdS_5 \times S^5 \) in the framework of the semiclassical approach. This is done for the cases when the “heavy” string states are finite-size giant magnons carrying one angular momentum and for three different choices of the “light” state: primary scalar operators, dilaton operator with nonzero momentum, singlet scalar operators on higher string levels.
1 Introduction

The AdS/CFT duality \[1\] between string theories on curved space-times with Anti-de Sitter subspaces and conformal field theories in different dimensions has been actively investigated in the last years. A lot of impressive progresses have been made in this field of research based mainly on the integrability structures discovered on both sides of the correspondence. The most studied example is the correspondence between type IIB string theory on $\text{AdS}_5 \times S^5$ target space and the $\mathcal{N} = 4$ super Yang-Mills theory (SYM) in four space-time dimensions. However, many other cases are also of interest, and have been investigated intensively (for recent review on the AdS/CFT duality, see \[2\]).

Different classical string solutions play important role in checking and understanding the AdS/CFT correspondence \[3\]. To establish relations with the dual gauge theory, one has to take the semiclassical limit of large conserved charges \[4\]. An important example of such string solution is the so called ”giant magnon” living in the $R_t \times S^2$ subspace of $\text{AdS}_5 \times S^5$, discovered by Hofman and Maldacena \[5\]. It gave a strong support for the conjectured all-loop $SU(2)$ spin chain, arising in the dual $\mathcal{N} = 4$ SYM, and made it possible to get a deeper insight in the AdS/CFT duality. Characteristic feature of this solution is that the string energy $E$ and the angular momentum $J_1$ go to infinity, but the difference $E - J_1$ remains finite and it is related to the momentum of the magnon excitations in the dual spin chain in $\mathcal{N} = 4$ SYM. This string configuration have been extended to the case of dyonic giant magnon, being solution for a string moving on $R_t \times S^3$ and having second nonzero angular momentum $J_2$ \[6\]. Further extension to $R_t \times S^5$ have been also worked out in \[7\]. It was also shown there that such type of string solutions can be obtained by reduction of the string dynamics to the Neumann-Rosochatius integrable system, by using a specific ansatz.

An interesting issue to solve is to find the finite-size effect, i.e. $J_1$ large, but finite, related to the wrapping interactions in the dual field theory \[8\]. For (dyonic) giant magnons living in $\text{AdS}_5 \times S^5$ this was done in \[9\], \[10\]. The corresponding string solutions, along with the (leading) finite-size corrections to their dispersion relations have been found.

Another issue is to go beyond the spectral problem by computing different correlation functions. For two-, three- and four-point correlators a lot of interesting results have been obtained \[1\]. Further investigations on the problem include the finite-size effects on some of them \[16\]-\[21\].

Here we are going to consider particular three-point correlation functions in the newly discovered $\eta$-deformed $\text{AdS}_5 \times S^5$ string theory background \[22\]. To this end, we will need to use our knowledge about the properties of the finite-size giant magnon solutions on this target space \[25\].

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\[1\]The recent activity in computing the semiclassical correlation functions in the framework of the AdS/CFT duality was initiated in \[11\]-\[15\].

\[2\]For the recent investigations related to this integrable deformation of $\text{AdS}_5 \times S^5$ see \[23\]-\[39\].
The paper is organized as follows. In Sec. 2, we derive the exact semiclassical structure constants for the case when the heavy string states are finite-size giant magnons, carrying one angular momentum, for three different choices of the light state: primary scalar operators, dilaton operator with nonzero momentum, singlet scalar operators on higher string levels. Sec. 3 is devoted to our concluding remarks.

2 Exact semiclassical three-point correlation functions

It is known that the correlation functions of any conformal field theory can be determined in principle in terms of the basic conformal data \( \{ \Delta_i, C_{ijk} \} \), where \( \Delta_i \) are the conformal dimensions defined by the two-point correlation functions

\[
\left\langle O_i(x_1)O_j(x_2) \right\rangle = \frac{C_{ij}}{|x_1 - x_2|^{2\Delta_i}}
\]

and \( C_{ijk} \) are the structure constants in the operator product expansion

\[
\left\langle O_i(x_1)O_j(x_2)O_k(x_3) \right\rangle = \frac{C_{ijk}}{|x_1 - x_2|^{\Delta_i + \Delta_j - \Delta_k}|x_2 - x_3|^{\Delta_i + \Delta_k - \Delta_j}|x_3 - x_1|^{\Delta_j + \Delta_k - \Delta_i}}.
\]

Therefore, the determination of the initial conformal data for a given conformal field theory is the most important step in the conformal bootstrap approach.

The three-point functions of two “heavy” operators and a “light” operator can be approximated by a supergravity vertex operator evaluated at the “heavy” classical string configuration [15, 40]:

\[
\left\langle V_H(x_1)V_H(x_2)V_L(x_3) \right\rangle = V_L(x_3)_{\text{classical}}.
\]

For \( |x_1| = |x_2| = 1, x_3 = 0 \), the correlation function reduces to

\[
\left\langle V_H(x_1)V_H(x_2)V_L(0) \right\rangle = \frac{C_{123}}{|x_1 - x_2|^{2\Delta_H}}.
\]

Then, the normalized structure constants

\[
\mathcal{C} = \frac{C_{123}}{C_{12}}
\]

can be found from

\[
\mathcal{C} = c_{\Delta} V_L(0)_{\text{classical}}, \tag{2.1}
\]

were \( c_{\Delta} \) is the normalized constant of the corresponding “light” vertex operator. Actually, we are going to compute the normalized structure constants (2.1).
2.1 Finite-size giant magnons and primary scalar operators

According to [18], the normalized structure constants for the undeformed case can be written as

\[ C_{pr,j} = c_{pr,j} \Delta \left[ \frac{2}{\cosh^2(\kappa \tau_e)} - 1 \right] \int_{-L}^{L} d\sigma \chi_j^2 \]  

\[ - \int_{-\infty}^{\infty} d\tau_e \frac{1}{\cosh^2(\kappa \tau_e)} \int_{-L}^{L} d\sigma \chi_j^2 \partial X_K \bar{\partial} X_K \]  

(2.2)

where \( t = \kappa \tau_e \) is the Euclidean AdS time, the term \( \partial X_K \bar{\partial} X_K \) is proportional to the string Lagrangian on \( S^2 \) computed on the finite-size giant magnon solution living in the \( R_t \times S^2 \) subspace, and \( \chi = \cos^2 \theta \) (\( \theta \) is the non-isometric angle on \( S^2 \)). The parameter \( L \) gives the size of the giant magnon.

For giant magnon solution on the \( \eta \)-deformed background the contribution from the AdS subspace is the same. So, the integration over \( \tau_e \) leads to

\[ C_{pr,j}^{\eta} = c_{pr,j}^{\eta} \Delta \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\frac{j}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} \left[ \frac{1}{\kappa} \int_{-L}^{L} d\sigma \chi_j^2 - \frac{1}{\kappa} \int_{-L}^{L} d\sigma \chi_j^2 \partial X_K \bar{\partial} X_K \right]. \]  

(2.3)

To take into account the \( \eta \)-deformation of the two-sphere, we should compute \( \partial X_K \bar{\partial} X_K \) on \( S^2_\eta \). By using some of the results obtained in [25], one can show that

\[ \partial X_K \bar{\partial} X_K = \frac{2 - 2(1 + v^2)\kappa^2 - 2\chi}{1 - v^2}, \]  

(2.4)

where \( \nu \) is the worldsheet velocity.

The integration over \( \sigma \) can be changed in the following way

\[ \int_{-L}^{L} d\sigma = 2 \int_{\chi_m}^{\chi_p} \frac{d\chi}{\chi'}, \]  

(2.5)

where

\[ \chi_m = \chi_{min}, \quad \chi_p = \chi_{max}, \]

and according to [25]

\[ \chi' = \frac{2\tilde{\eta}}{1 - v^2} \sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)\chi}, \]  

(2.6)

\[ \tilde{\eta} = \frac{2\eta}{1 - \eta^2}, \quad \chi_\eta = 1 + \frac{1}{\eta^2}, \quad \chi_p = 1 - v^2\kappa^2, \quad \chi_m = 1 - \kappa^2. \]  

(2.7)

Therefore, the normalized structure constant can be represented as

\[ C_{pr,j}^{\eta} = \frac{2c_{pr,j}^{\eta}}{\tilde{\eta} \kappa} \frac{1}{\Gamma\left(\frac{j+1}{2}\right)} \left[ \frac{1 - \kappa^2 + j(1 - v^2\kappa^2)}{1 + j} J_j - J_{jp} \right]. \]  

(2.8)
where
\[ J_j = \int_{\chi_m}^{\chi_p} \frac{\chi^{\frac{j}{2}}}{\sqrt{(\chi - \chi_m)(\chi_p - \chi)(\chi - \chi_m)}} d\chi, \quad (2.9) \]
\[ J_{jp} = \int_{\chi_m}^{\chi_p} \frac{\chi^{\frac{j+1}{2}}}{\sqrt{(\chi - \chi_m)(\chi_p - \chi)(\chi - \chi_m)}} d\chi. \quad (2.10) \]

To compute the above two integrals, we introduce the variable
\[ x = \frac{\chi - \chi_m}{\chi_p - \chi_m} \in (0, 1). \]

Then \( J_j \) becomes
\[ J_j = \frac{1}{\chi_m^j} (\chi_\eta - \chi_m)^{-\frac{j}{2}} \int_0^1 x^{-\frac{j}{2}} (1 - x)^{-\frac{j}{2}} \left( 1 - \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m} x \right)^{-\frac{j}{2}} \left( 1 + \frac{\chi_p - \chi_m}{\chi_m} x \right)^{\frac{j+1}{2}} dx. \quad (2.11) \]

Comparing the above expression with the integral representation for the hypergeometric function of two variables \( F_1(a, b_1, b_2; c, z_1, z_2) \) \[41\]
\[ F_1(a, b_1, b_2; c, z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 x^{a-1} (1 - x)^{c-a-1} (1 - z_1 x)^{-b_1} (1 - z_2 x)^{-b_2}, \]
\[ Re(a) > 0, \quad Re(c-a) > 0, \]
one finds
\[ J_j = \pi \chi_m^{-\frac{j+1}{2}} (\chi_\eta - \chi_m)^{-\frac{j}{2}} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-j-1}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, \frac{\chi_p - \chi_m}{\chi_m} \right). \quad (2.12) \]

In order to compute \( J_{jp} \), we have to replace \( j \) with \( j + 2 \). Doing this, we obtain
\[ J_{jp} = \pi \chi_m^{-\frac{j+1}{2}} (\chi_\eta - \chi_m)^{-\frac{j}{2}} F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-j+1}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, \frac{\chi_p - \chi_m}{\chi_m} \right). \quad (2.13) \]

The replacement of (2.12) and (2.13) into (2.8) gives
\[ C_{\eta}^{pr,j} = \frac{2\pi^2 \varepsilon_\Delta^{pr,j}}{\eta \kappa^2} \frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{j+2}{2})} \chi_m^{-\frac{j+1}{2}} \sqrt{\chi_\eta - \chi_m} \left\{ \left[ 1 - \frac{(1 + j v^2)\kappa^2}{1 + j} \right] \times \right. \]
\[ F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1 - j}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, \frac{\chi_p - \chi_m}{\chi_m} \right) \]
\[ - \chi_m F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1 + j}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, \frac{\chi_p - \chi_m}{\chi_m} \right) \right\}. \quad (2.14) \]
Knowing that according to (2.7)
\[ \chi_\eta = 1 + \frac{1}{\eta^2} \quad \chi_p = 1 - v^2 \kappa^2, \quad \chi_m = 1 - \kappa^2, \]
and using the relation [41]
\[ F_1(a, b_1, b_2; c; z_1, z_2) = (1 - z_1)^{a-b_1} (1 - z_2)^{-b_2} F_1 \left(c - a, c - b_1 - b_2, b_2; z_1, \frac{z_1 - z_2}{1 - z_2} \right), \]
we can rewrite (2.14) in the following form
\[ C^{pr,j}_{\eta} = \frac{2\pi^2 \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2}, \frac{1}{2} \right) \sqrt{\kappa^2 + \eta^2 \kappa^2}} \left[ 1 - \frac{(1 + j v^2 \kappa^2)}{1 + j} \right] \times \]
\[ F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; \frac{\eta^2}{1 + \eta^2 \kappa^2}, \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right) \]
\[ -(1 - v^2 \kappa^2) F_1 \left( \frac{1}{2}, \frac{2 + j}{2}, \frac{1 - j}{2}; 1; \frac{\eta^2}{1 + \eta^2 \kappa^2}, \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right). \]
This is our final exact semiclassical result for this type of three-point correlation functions.

Next, we would like to compare (2.15) with the known expression for the undeformed case [18]. To this end, we take the limit \( \tilde{\eta} \rightarrow 0 \) and by using that [41]
\[ F_1(a, b_1, b_2; c; 0, z_2) = 2F_1(a, b_2; c; z_2), \]
where \( 2F_1(a, b_2; c; z_2) \) is the Gauss’ hypergeometric function, we find
\[ C^{pr,j}_{\eta} = \frac{2\pi^2 \Gamma \left( \frac{2}{2} \right)}{\kappa \Gamma \left( \frac{5}{2} \right)} \left[ (1 - v^2 \kappa^2) 2F_1 \left( \frac{1}{2}, \frac{-1 + j}{2}; \frac{1}{1 - v^2 \kappa^2} \right) \right] \]
\[ -(1 - \kappa^2) 2F_1 \left( \frac{1}{2}, \frac{1 - j}{2}; \frac{1}{1 - v^2 \kappa^2} \right). \]
This is exactly the same result found in [18] for \( u = 0 \) (finite-size giant magnons with one nonzero angular momentum) as it should be [3].

Let us also give an example for the simplest case when \( j = 1 \). In that case (2.15) reduces to
\[ C^{pr,1}_{\eta} = \frac{2\pi \Gamma \left( \frac{2}{2} \right)}{\tilde{\eta}^2 \sqrt{\kappa^2 + \eta^2 \kappa^2}} \left[ 2(1 + \eta^2 \kappa^2) E \left( \eta^2 \frac{(1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2} \right) \right] \]
\[ -(2 + (1 + v^2 \eta^2 \kappa^2)) K \left( \eta^2 \frac{(1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2} \right), \]
where \( K \) and \( E \) are the complete elliptic integrals of first and second kind. In the limit \( \tilde{\eta} \rightarrow 0 \), \( C^{pr,1}_{\eta} \rightarrow 0 \).

\[ ^3 \text{In our notations} \ W = \kappa^2 \text{ and (2.7) is taken into account.} \]
2.2 Finite-size giant magnons and dilaton with nonzero momentum

The case of finite-size giant magnons and dilaton with zero momentum \((j = 0)\) have been considered in [37]. Here we will be interested in the case when \(j > 0\).

According to [18] the semiclassical normalized structure constants for the undeformed case are given by

\[
C_{d,j}^\Delta = c_{d,j}^\Delta \frac{\Delta}{\Gamma(\frac{5}{2})} \int_{-\infty}^{\infty} \frac{d\tau_e}{\cosh^{4+j}(\kappa \tau_e)} \int_{-L}^{L} d\sigma \chi^\frac{j}{2} \left( \kappa^2 + \partial X_K \bar{\partial} X_K \right)
\]

\[
= c_{d,j}^\Delta \sqrt{\pi} \Gamma(2 + \frac{j}{2}) \frac{\Delta}{\kappa \Gamma(\frac{5}{2} + \frac{j}{2})} \int_{-L}^{L} d\sigma \chi^\frac{j}{2} \left( \kappa^2 + \partial X_K \bar{\partial} X_K \right).
\]

In order to take into account the \(\eta\)-deformation, we must use (2.4)-(2.7). Working in the same way as in the previous subsection, one obtains the following result for \(C_{d,j}^\eta\):

\[
C_{d,j}^\eta = \frac{\pi^{\frac{j}{2}} c_{d,j}^\Delta \Gamma \left( 2 + \frac{j}{2} \right) (1 - v^2)}{\Gamma \left( \frac{5+j}{2} \right) \tilde{\eta} \kappa} \int_{\chi_m}^{\chi_p} d\chi \chi^{\frac{j+1}{2}} \left[ \kappa^2 \frac{2 - (1 + v^2)\kappa^2 - 2\chi}{1 - v^2} \right] \sqrt{(\chi_\eta - \chi)(\chi_p - \chi)(\chi - \chi_m)} \chi_m^{\frac{j+1}{2}} (1 - v^2) \kappa^2
\]

\[
= 2\pi^{\frac{j}{2}} c_{d,j}^\Delta \Gamma \left( 2 + \frac{j}{2} \right) \tilde{\eta} \sqrt{(\chi_\eta - \chi_m)(1 - \chi_m)} \times
\]

\[
\left[ F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{j+1}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, -\frac{\chi_p - \chi_m}{\chi_m} \right) \right] - F_1 \left( \frac{1}{2}, \frac{1}{2}, \frac{j-1}{2}; 1; \frac{\chi_p - \chi_m}{\chi_\eta - \chi_m}, -\frac{\chi_p - \chi_m}{\chi_m} \right)
\]

\[
= 2\pi^{\frac{j}{2}} c_{d,j}^\Delta \Gamma \left( 2 + \frac{j}{2} \right) (1 - v^2) \kappa^2 \chi_m^{\frac{j+1}{2}} \kappa \sqrt{(1 - \tilde{\eta}^2 \kappa^2)} \times
\]

\[
\left[ (1 - v^2) \kappa^2 F_1 \left( \frac{1}{2}, \frac{2 + j}{2}, \frac{-1 + j}{2}; 1; \tilde{\eta}^2 (1 - v^2) \kappa^2, \frac{(1 + \tilde{\eta}^2)(1 - v^2) \kappa^2}{1 + \tilde{\eta}^2 \kappa^2} \right) \right] - (1 - \kappa^2) F_1 \left( \frac{1}{2}, \frac{j - 1}{2}, \frac{1 - j}{2}; 1; \tilde{\eta}^2 (1 - v^2) \kappa^2, \frac{(1 + \tilde{\eta}^2)(1 - v^2) \kappa^2}{1 + \tilde{\eta}^2 \kappa^2} \right) .
\]
Now we take the limit $\tilde{\eta} \to 0$ in (2.16) and obtain

$$C_{d,j} = \frac{2\pi^3 c_{\Delta}^d j \Gamma \left( 2 + \frac{j}{2} \right) (1 - v^2 \kappa^2)^{j/2}}{\kappa \Gamma \left( \frac{5+j}{2} \right)} \times \left[ \left( 1 - v^2 \kappa^2 \right)_2 F_1 \left( \begin{array}{c} \frac{1}{2}, -1 + \frac{j}{2} \\ \frac{1}{2} \end{array} ; 1; \frac{(1 - v^2 \kappa^2)}{1 - v^2 \kappa^2} \right) \right].$$

This is exactly what was found in [18] for $u = 0$, as it should be.

Let us also say that in the particular case when $j = 1$, (2.16) simplifies to

$$C_{d,1} = \frac{3\pi^3 c_{\Delta}^d}{2\sqrt{\pi} \kappa^2} \left[ K \left( \frac{\eta^2 (1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2} \right) - E \left( \frac{\eta^2 (1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2} \right) \right].$$

In the limit $\tilde{\eta} \to 0$, $C_{d,1}$ becomes

$$C_{d,1} = \frac{3}{8} \pi^3 c_{\Delta}^d \kappa (1 - v^2).$$

### 2.3 Finite-size giant magnons and singlet scalar operators on higher string levels

According to [19] the normalized structure constant for the undeformed case is given by

$$C^q = c_{\Delta}^q \int_{-\infty}^{\infty} \frac{d\tau_e}{\cosh^2(\Delta_q \kappa \tau_e)} \int_{-L}^{L} d\sigma \left( \partial X_K \partial X_K \right)^q \quad (2.18)$$

$$= c_{\Delta}^q \frac{\sqrt{\pi}}{\kappa} \frac{\Gamma \left( \frac{\Delta_q}{2} \right)}{\Gamma \left( \frac{\Delta_q + 1}{2} \right)} \int_{-L}^{L} d\sigma \left( \partial X_K \partial X_K \right)^q.$$

Here the parameter $q$ is related to the string level $n$ as $q = n + 1 \geq 1$ and

$$\Delta_q = 2 \left( 1 + \sqrt{(q - 1)\sqrt{\lambda} + \frac{1}{2}q(q - 1)} \right),$$

where $\lambda$ is the 't Hooft coupling constant in the dual gauge theory. Taking into account that the string tension $T$ is related to the 't Hooft coupling as

$$T = \frac{\sqrt{\lambda}}{2\pi},$$
\[ (2.19) \] can be rewritten as
\[
\Delta_q = 2 \left( 1 + \sqrt{2\pi T(q - 1) + 1 - \frac{1}{2} q(q - 1)} \right).
\] (2.20)

For the \( \eta \)-deformed case we have \([25]\)
\[
T = g \sqrt{1 + \eta^2}.
\]

So,
\[
\Delta^\eta_q = 2 \left( 1 + \sqrt{2\pi g \sqrt{1 + \eta^2(q - 1) + 1 - \frac{1}{2} q(q - 1)}} \right).
\] (2.21)

In addition, to compute the integral over \( \sigma \) in \((2.18)\), we have to use \((2.4)-(2.7)\). Thus
\[
\mathcal{C}_q^\eta = e^\Delta \frac{\sqrt{\pi}}{\kappa} \frac{\Gamma \left( \frac{\Delta_q}{2} \right)}{\Gamma \left( \frac{\Delta_q + 1}{2} \right)} \frac{(-1)^q}{\eta(1 - v^2)^{q-1}} \int_{\chi_m}^{\chi_p} \frac{[2 - (1 + v^2)\kappa^2 - 2\chi]^q}{\sqrt{(\chi_q - \chi)(\chi_p - \chi)(\chi - \chi_m)}} d\chi
\] (2.22)

\[
= e^\Delta \frac{\pi^{\frac{3}{2}}}{\kappa} \frac{\Gamma \left( \frac{\Delta_q}{2} \right)}{\Gamma \left( \frac{\Delta_q + 1}{2} \right)} \frac{(-1)^q [2 - (1 + v^2)\kappa^2]^q}{\eta(1 - v^2)^{q-1}\sqrt{\kappa^2(1 + \eta^2\kappa^2)(1 - v^2\kappa^2)}} \sum_{k=0}^{q} \frac{q!}{k!(q-k)!} \times
\]

\[
\sum_{k=0}^{q} \frac{q!}{k!(q-k)!} \left[ -\frac{1}{1 - \frac{1}{2}(1 + v^2)\kappa^2} \right]^k \chi_m^{k-\frac{1}{2}} F_1 \left( \frac{1}{2}, \frac{1}{2} - k; 1; \frac{X_p - \chi_m}{\chi_q - \chi_m}, -\frac{X_p - \chi_m}{\chi_m} \right)
\]

\[
= e^\Delta \frac{\pi^{\frac{3}{2}}}{\kappa} \frac{\Gamma \left( \frac{\Delta_q}{2} \right)}{\Gamma \left( \frac{\Delta_q + 1}{2} \right)} \frac{(-1)^q [2 - (1 + v^2)\kappa^2]^q}{\eta(1 - v^2)^{q-1}\sqrt{\kappa^2(1 + \eta^2\kappa^2)(1 - v^2\kappa^2)}} \times
\]

\[
\left[ -\frac{1}{1 - \frac{1}{2}(1 + v^2)\kappa^2} \right]^k F_1 \left( \frac{1}{2}, \frac{1}{2} - k; 1; \frac{\tilde{\eta}^2(1 - v^2)\kappa^2}{1 + \tilde{\eta}^2\kappa^2}, (1 + \tilde{\eta}^2)(1 - v^2)\kappa^2 \right).
\]

In order to compare with the undeformed case, we take the limit \( \tilde{\eta} \rightarrow 0 \) in \((2.22)\) and obtain
\[
\mathcal{C}_q^\eta = e^\Delta \frac{\pi^{\frac{3}{2}}}{\kappa} \frac{\Gamma \left( \frac{\Delta_q}{2} \right)}{\Gamma \left( \frac{\Delta_q + 1}{2} \right)} \frac{(-1)^q [2 - (1 + v^2)\kappa^2]^q}{(1 - v^2)^{q-1}\sqrt{\kappa^2(1 - v^2\kappa^2)}} \times
\]

\[
\sum_{k=0}^{q} \frac{q!}{k!(q-k)!} \times
\]

\[
\left[ -\frac{1}{1 - \frac{1}{2}(1 + v^2)\kappa^2} \right]^k F_1 \left( \frac{1}{2}, \frac{1}{2} - k; 1; (1 - v^2)\kappa^2 \right).
\]

This is exactly what was found in \([19]\) for finite-size giant magnons with one nonzero angular momentum.
Let us consider two particular cases. From (2.22) it follows that the normalized structure constants for the first two string levels, for the case at hand, are given by

$$q = 1 \ (\text{level} \ n = 0)$$

$$C_1 = 2c_1^1 \pi \frac{1}{2} \frac{\Gamma \left( \frac{\Delta}{2} \right)}{\Gamma \left( \frac{\Delta + 1}{2} \right)} \frac{1}{\sqrt{\kappa^2 (1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)}} \times \left[ \pi (1 - v^2 \kappa^2) F_1 \left( \frac{1}{2}, 1, -\frac{1}{2}; 1; \frac{\eta^2 (1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2}, \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right) - (2 - (1 + v^2 \kappa^2) K \left( \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right) \right].$$

$$q = 2 \ (\text{level} \ n = 1)$$

$$C_2 = 2c_2^1 \pi \frac{1}{2} \frac{1}{\sqrt{\kappa^2 (1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)}} \times \left[ \left\{ \frac{1}{\pi} K \left( \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right) - \frac{2(1 - v^2 \kappa^2)}{(2 - (1 + v^2 \kappa^2))} \times \left[ (2 - (1 + v^2 \kappa^2) F_1 \left( \frac{1}{2}, 1, -\frac{3}{2}; 1; \frac{\eta^2 (1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2}, \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right) - (1 - v^2 \kappa^2) F_1 \left( \frac{1}{2}, 2, -\frac{3}{2}; 1; \frac{\eta^2 (1 - v^2 \kappa^2)}{1 + \eta^2 \kappa^2}, \frac{(1 + \eta^2)(1 - v^2 \kappa^2)}{(1 + \eta^2 \kappa^2)(1 - v^2 \kappa^2)} \right) \right] \right\}. \right]$$

In the limit \( \eta \to 0 \), the above two expressions simplify to

$$C^1 = 2c_1^1 \pi \frac{1}{2} \frac{1}{\sqrt{\kappa^2 (1 - v^2 \kappa^2)}} \times \left[ 2(1 - v^2 \kappa^2) E \left( \frac{(1 - v^2 \kappa^2)}{1 - v^2 \kappa^2} \right) - (2 - (1 + v^2 \kappa^2) K \left( \frac{(1 - v^2 \kappa^2)}{(1 - v^2 \kappa^2)} \right) \right].$$
and

\[ C^2 = 2c^2 \Delta \Gamma \left( \frac{\Delta}{2} \right) \frac{1}{(1 - v^2)^{1/2}(1 - v^2\kappa^2)} \times \]

\[ \left[ (2 - (1 + v^2)\kappa^2)^2 K \left( \frac{1 - v^2\kappa^2}{1 - v^2\kappa^2} \right) \right. \]

\[ -4 \left( 2 - (1 + v^2)\kappa^2 \right) (1 - v^2\kappa^2)E \left( \frac{1 - v^2\kappa^2}{1 - v^2\kappa^2} \right) \]

\[ +2\pi(1 - v^2\kappa^2)\right)_2 F_1 \left( \frac{1}{2}, -\frac{3}{2}; 1; \frac{1 - v^2\kappa^2}{1 - v^2\kappa^2} \right). \]

respectively.

3 Concluding Remarks

In this paper, in the framework of the semiclassical approach, we computed the normalized structure constants for some three-point correlation functions in \( \eta \)-deformed \( AdS_5 \times S^5 \). Namely, we found the normalized structure constants in several classes of three-point correlators. This was done for the cases when the “heavy” string states are finite-size giant magnons carrying one angular momentum and for three different choices of the “light” state:

1. Primary scalar operators
2. Dilaton operator with nonzero momentum
3. Singlet scalar operators on higher string levels

The results are given in terms of hypergeometric functions of two variables. In the limit \( \tilde{\eta} \to 0 \), when the deformation disappear, they reduce to the known results for the undeformed case.

A natural generalization will be to consider the case of dyonic giant magnons with two nonzero angular momenta. We hope to report on this soon.

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