STRATIFIED ROTATING BOUSSINESQ EQUATIONS IN GEOPHYSICAL FLUID DYNAMICS: DYNAMIC BIFURCATION AND PERIODIC SOLUTIONS

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Abstract. The main objective of this article is to study the dynamics of the stratified rotating Boussinesq equations, which are a basic model in geophysical fluid dynamics. First, for the case where the Prandtl number is greater than one, a complete stability and bifurcation analysis near the first critical Rayleigh number is carried out. Second, for the case where the Prandtl number is smaller than one, the onset of the Hopf bifurcation near the first critical Rayleigh number is established, leading to the existence of nontrivial periodic solutions. The analysis is based on a newly developed bifurcation and stability theory for nonlinear dynamical systems (both finite and infinite dimensional) by two of the authors [16].

1. Introduction

The phenomena of the atmosphere and ocean are extremely rich in its organization and complexity, and a lot of them cannot be produced by experiments. These phenomena involve a broad range of temporal and spatial scales. As we know, both the atmospheric and oceanic flows are flows under the rotation of the earth. In fact, fast rotation and small aspect ratio are two main characteristics of the large scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to the primitive equations, and the fast rotation leads to the quasi-geostrophic equations. These are fundamental equations in the study of atmospheric and oceanic flows; see Ghil and Childress [6], Lions, Temam and Wang [12, 13], and Pedlosky [23]. Furthermore, convection occurs in many regimes of the atmospheric and oceanic flows.

A key problem in the study of climate dynamics and in geophysical fluid dynamics is to understand and predict the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows. Stability/bifurcation theory enables one to determine how different flow regimes appear and disappear as control parameters, such as the Reynolds number, vary. It, therefore, provides one with a powerful tool to explore the theoretical capability in the predictability problem. Most studies so far have only considered systems of ordinary differential equations (ODEs) that are obtained by projecting the PDEs onto a finite-dimensional solution space, either by finite differencing or by truncating a Fourier expansion (see Ghil and Childress [6] and further references there).
These were pioneered by Lorenz [14, 15], Stommel [25], and Veronis [27, 28] among others, who explored the bifurcation structure of low-order models of atmospheric and oceanic flows. More recently, pseudo-arclength continuation methods have been applied to atmospheric (Legras and Ghil [11]) and oceanic (Speich et al. [24] and Dijkstra [5]) models with increasing horizontal resolution. These numerical bifurcation studies have produced so far fairly reliable results for two classes of geophysical flows: (i) atmospheric flows in a periodic mid-latitude channel, in the presence of bottom topography and a forcing jet; and (ii) oceanic flows in a rectangular mid-latitude basin, subject to wind stress on its upper surface; see among others Charney and DeVore [2], Pedlosky [22], Legras and Ghil [11] and Jin and Ghil [10] for saddle-node and Hopf bifurcations in the atmospheric channel, and [20, 18, 19, 24] for saddle-node, pitchfork or Hopf in the oceanic basin.

The main objective of this article is to conduct bifurcation and stability analysis for the original partial differential equations (PDEs) that govern geophysical flows. This approach should allow us to overcome some of the inherent limitations of the numerical bifurcation results that dominate the climate dynamics literature up to this point, and to capture the essential dynamics of the governing PDE systems.

The present article addresses the stability and transitions of basic flows for the stratified rotating Boussinesq equations. These equations are fundamental equations in the geophysical fluid dynamics; see among others Pedlosky [23]. We obtain two main results in this article. The first is to conduct a rigorous and complete bifurcation and stability analysis near the first eigenvalue of the linearized problem. The second is the onset of the Hopf bifurcation, leading to the existence of periodic solutions of the model.

The detailed analysis is carried out in two steps. The first is a detailed study of the eigenvalue problem for the linearized problem around the basic state. In comparison to the classical Bénard convection problem, the linearized problem here is non-selfadjoint, leading to much more complicated spectrum, and more complicated dynamics. We derive in particular two critical Rayleigh numbers $R_{c_1}$ and $R_{c_2}$. Here $R_{c_1}$ is the first critical Rayleigh number for the case where the Prandtl number is greater than one, and $R_{c_2}$ is the first critical Rayleigh number for the case where the Prandtl number is less than one. Moreover, $R_{c_1}$ leads to the onset of the steady state bifurcation while $R_{c_2}$ leads to the onset of the Hopf bifurcation. Both parameters are explicitly given in terms of the physical parameters. The crucial issues here include 1) a complete understanding of the spectrum, 2) identification of the critical Rayleigh numbers, and most importantly 3) the verification of the Principle of Exchange of Stabilities near these critical Rayleigh numbers.

The second step is to conduct a rigorous nonlinear analysis to derive the bifurcations at both the critical Rayleigh numbers based on the classical Hopf bifurcation theory and a newly developed dynamic bifurcation theory by two of the authors. This new dynamic bifurcation theory is centered at a new notion of bifurcation, called attractor bifurcation for dynamical systems, both finite dimensional and infinite dimensional, together with new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures. The bifurcation theory has been applied to various problems from science and engineering, including, in particular, the Kuramoto-Sivashinsky equation, the Cahn-Hillard equation, the Ginzburg-Landau equation, Reaction-Diffusion equations in Biology and Chemistry, and the Bénard convection problem, the Taylor problem; see [16, 17] and the references therein.
We remark that the non-selfadjointness of the linearized problem gives rise to the Hopf bifurcation. We prove that the Hopf bifurcation appears at the Rayleigh number \( R_{c2} \). As mentioned earlier, the understanding and prediction of the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows are key issues in the study of climate dynamics and in geophysical fluid dynamics. It is hoped that the study carried out in this article will provide some insights into these important issues.

Also, we would like to mention that rigorous proof of the existence of periodic solutions for a fluid system is normally a very difficult task from the mathematical point of view. For instance, with a highly involved analysis, Chen et al. [3] proved the existence of a Hopf bifurcation in an idealized Fourier space.

The paper is organized as follows. Section 2 gives the basic setting of the problem. Section 3 states the main results. The proofs of the main results occupies the remaining part of the paper: Section 4 recapitulates the essentials of the attractor bifurcation theory, Section 5 is on the eigenanalysis, and Section 6 is on the central manifold reduction and the completion of the proofs.

2. Stratified Rotating Boussinesq Equations in Geophysical Fluid Dynamics

The stratified rotating Boussinesq equations are basic equations in the geophysical fluid dynamics, and their non-dimensional form is given by

\[
\begin{align*}
\frac{\partial U}{\partial t} &= \sigma (\Delta U - \nabla p) + \sigma RT e - \frac{1}{Ro} e \times U - (U \cdot \nabla) U, \\
\frac{\partial T}{\partial t} &= \Delta T + w - (U \cdot \nabla) T, \\
\text{div} U &= 0,
\end{align*}
\]

for \((x, y, z)\) in the non-dimensional domain \( \Omega = \mathbb{R}^2 \times (0, 1) \), where \( U = (u, v, w) \) is the velocity field, \( e = (0, 0, 1) \) is the unit vector in the \( z \)-direction, \( \sigma \) is the Prandtl number, \( R \) is the thermal Rayleigh number, \( Ro \) is the Rossby number, \( T \) is the temperature function and \( p \) is the pressure function. We refer the interested readers to Pedlosky [23], Lions, Temam and Wang [13] for the derivation of this model and the related parameters. In particular, the term \( \frac{1}{Ro} e \times U \) represents the Coriolis force, the \( w \) term in the temperature equation is derived using the stratification, and the definition of the Rayleigh number \( R \) as follows:

\[
R = \frac{g \alpha \beta}{\kappa \nu} h^4.
\]

We consider the periodic boundary condition in the \( x \) and \( y \) directions

\[
(U, T)(x, y, z, t) = (U, T)(x + 2j\pi/\alpha_1, y, z, t) = (U, T)(x, y + 2k\pi/\alpha_2, z, t),
\]

for any \( j, k \in \mathbb{Z} \). At the top and bottom boundaries, we impose the free-free boundary conditions:

\[
(T, w) = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad \text{at} \quad z = 0, 1.
\]

It is natural to put the constraint

\[
\int_{\Omega} udxdydz = \int_{\Omega} v dxdydz = 0.
\]
The initial value conditions are given by
\begin{equation}
\label{initial-condition}
(U, T) = (\tilde{U}, \tilde{T}) \quad \text{at} \quad t = 0.
\end{equation}

Let
\begin{align*}
H &= \{ (U, T) \in L^2(\Omega)^4 \mid \text{div} \, U = 0, w \mid_{z=0,1} = 0, (u, v) \text{ satisfies } (2.3) \text{ and } (2.5) \}, \\
H_1 &= \{ (U, T) \in H^2(\Omega)^4 \cap H \mid (U, T) \text{ satisfies } (2.3) - (2.5) \}, \\
\tilde{H} &= \{ (U, T) \in H \mid (u, v, w, T)(-x, -y, z) = (-u, -v, w, T)(x, y, z) \}, \\
\tilde{H}_1 &= H_1 \cap \tilde{H}.
\end{align*}

Let $L_R = -A - B_R : H_1 \to H$ (resp., $\tilde{H}_1 \to \tilde{H}$) and $G : H_1 \to H$ (resp., $\tilde{H}_1 \to \tilde{H}$) be defined by
\begin{align*}
A \psi &= (-P[\sigma \Delta U - \frac{1}{R_0} \sigma \times U], -\Delta T), \\
B_R \psi &= (-P[\sigma RT \epsilon], -w), \\
G(\psi) &= G(\psi, \psi),
\end{align*}
for any $\psi = (U, T) \in H_1$ (resp., $\tilde{H}_1$), where
\begin{align*}
G(\psi_1, \psi_2) &= (-P[(U_1 \cdot \nabla)U_2], -(U_1 \cdot \nabla)T_2),
\end{align*}
for any $\psi_1 = (U_1, T_1), \psi_2 = (U_2, T_2) \in H_1$. Here $P$ is the Leray projection to $L^2$ fields, and for a detailed account of the function spaces; see among many others [26].

\begin{remark}
Note that $\tilde{H}_1$ and $\tilde{H}$ are invariant under the bilinear operator $G$ in the sense that
\begin{align*}
G(\psi_1, \psi_2) &\in \tilde{H}, \quad \text{for } \psi_1, \psi_2 \in \tilde{H}_1.
\end{align*}
Hence, $\tilde{H}_1$ and $\tilde{H}$ are invariant under the operator $L_R + G$.
\end{remark}

Then the Boussinesq equations (2.1)-(2.5) can be written in the following operator form
\begin{equation}
\frac{d\psi}{dt} = L_R \psi + G(\psi), \quad \psi = (U, T).
\end{equation}

\section{Main Results}

\subsection{Definition of attractor bifurcation.}
To state the main theorems of this article, we proceed with the definition of attractor bifurcation, first introduced by T. Ma and S. Wang in [16, 17].

Let $H$ and $H_1$ be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equations
\begin{equation}
\label{general-equation}
\begin{cases}
\frac{du}{dt} = L_\lambda u + G(u, \lambda), \\
u(0) = u_0,
\end{cases}
\end{equation}
where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields depending
continuously on \( \lambda \in \mathbb{R}^1 \), which satisfy
\[
\begin{cases}
  -L_\lambda = A + B_\lambda & \text{a sectorial operator,} \\
  A : H_1 \to H & \text{a linear homeomorphism,} \\
  B_\lambda : H_1 \to H & \text{parameterized linear compact operators.}
\end{cases}
\]

It is easy to see \([7]\) that \( L_\lambda \) generates an analytic semi-group \( \{e^{tL_\lambda}\}_{t \geq 0} \). Then we can define fractional power operators \( (-L_\lambda)^\mu \) for any \( 0 \leq \mu \leq 1 \) with domain \( H_\mu = D((-L_\lambda)^\mu) \) such that \( H_\mu_1 \subset H_\mu_2 \) if \( \mu_1 > \mu_2 \), and \( H_0 = H \).

Furthermore, we assume that the nonlinear terms \( G(\cdot, \lambda) : H_\mu \to H \) for some \( 1 > \mu \geq 0 \) are a family of parameterized \( C^r \) bounded operators \( (r \geq 1) \) continuously depending on the parameter \( \lambda \in \mathbb{R}^1 \), such that
\[
G(u, \lambda) = o(||u||_{H_\mu}), \quad \forall \lambda \in \mathbb{R}^1.
\]

In this paper, we are interested in the sectorial operator \( -L_\lambda = A + B_\lambda \) such that there exist an eigenvalue sequence \( \{\rho_k\} \subset \mathbb{C}^1 \) and an eigenvector sequence \( \{e_k, h_k\} \subset H_1 \) of \( A \):
\[
\begin{cases}
  Az_k = \rho_k z_k, & z_k = \epsilon_k + ih_k, \\
  \text{Re} \rho_k \to \infty (k \to \infty), \\
  |\text{Im} \rho_k/(a + \text{Re} \rho_k)| \leq c,
\end{cases}
\]

for some \( a, c > 0 \), such that \( \{\epsilon_k, h_k\} \) is a basis of \( H \). Also we assume that there is a constant \( 0 < \theta < 1 \) such that
\[
B_\lambda : H_\theta \to H \text{ bounded, } \forall \lambda \in \mathbb{R}^1.
\]

Under conditions (3.4) and (3.5), the operator \( -L_\lambda = A + B_\lambda \) is a sectorial operator.

Let \( \{S_\lambda(t)\}_{t \geq 0} \) be an operator semi-group generated by the equation (3.1). Then the solution of (3.1) can be expressed as \( \psi(t, \psi_0) = S_\lambda(t)\psi_0 \), for any \( t \geq 0 \).

**Definition 3.1.** A set \( \Sigma \subset H \) is called an invariant set of (3.1) if \( S(t)\Sigma = \Sigma \) for any \( t \geq 0 \). An invariant set \( \Sigma \subset H \) of (3.1) is called an attractor if \( \Sigma \) is compact, and there exists a neighborhood \( W \subset H \) of \( \Sigma \) such that for any \( \psi_0 \in W \) we have
\[
\lim_{t \to \infty} \text{dist}_H(\psi(t, \psi_0), \Sigma) = 0.
\]

**Definition 3.2.**
1. We say that the solution of (3.1) bifurcates from \((\psi, \lambda) = (0, \lambda_0)\) to an invariant set \( \Omega_\lambda \), if there exists a sequence of invariant sets \( \{\Omega_{\lambda_n}\} \) of (3.1) such that \( 0 \not\in \Omega_{\lambda_n} \), \( \lim_{n \to \infty} \lambda_n = \lambda_0 \), and
   \[
   \lim_{n \to \infty} \max_{x \in \Omega_{\lambda_n}} |x| = 0.
   \]
2. If the invariant sets \( \Omega_\lambda \) are attractors of (3.1), then the bifurcation is called attractor bifurcation.

**3.2. Main theorems.** In this article, we consider two cases:

(3.6) \( \sigma > 1 \) and \( R_{c_1} \) is obtained only at \((j, k, l) = (j_1, 0, 1)\),

(3.7) \( \sigma < 1 \) and \( R_{c_2} \) is obtained only at \((j, k, l) = (j_2, 0, 1)\),
for some $j_1, j_2 \in \mathbb{N}$, where $R_{c_1}$ and $R_{c_2}$ are defined in (5.18) and (5.22) respectively. In the above cases, $R_{c_1}$ and $R_{c_2}$ are given by the following formulas:

$$R_{c_1} = \frac{(j_2^2 \alpha_1^2 + \pi^2)^3}{j_2^2 \alpha_1^2} + \frac{\pi^2}{\sigma^2 R_0 \alpha_1^2}$$

$$R_{c_2} = 2(\sigma + 1) \left(\frac{j_2^2 \alpha_1^2 + \pi^2}{\sigma + 1}\right) + 2 \pi^2 \left(\frac{j_2^2 \alpha_1^2 + \pi^2}{\sigma + 1}\right).$$

**Remark 3.3.**

1. Condition (3.6) guarantees that for $R \approx R_{c_1}$, the first eigenvalue of $L_R \mid_{H_1}$ (resp., $L_R \mid_{\tilde{H}_1}$) is real and of multiplicity two (resp., one); see Remark 5.3.

2. Condition (3.7) guarantees that, for $R \approx R_{c_2}$, there exists only one simple pair of conjugate complex eigenvalues of $L_R \mid_{\tilde{H}_1}$ crossing the imaginary axis; see Lemma 5.6.

3. Condition (3.6) or (3.7) can be satisfied easily; see Lemmas 5.4 and 5.5.

**Theorem 3.4.** Assume (3.7). Then the following assertions for Problem (2.1)-(2.5) defined in $H$ hold true.

1. If $R \leq R_{c_1}$, the steady state $(U, T) = 0$ is locally asymptotically stable.

2. For $R > R_{c_1}$, the problem bifurcates from $((U, T), R) = (0, R_{c_1})$ to an attractor $\Sigma_R = S^1$, consisting of only steady state solutions.

![Figure 3.1. Bifurcation from $(0, R_{c_1})$ to an attractor $\Sigma_R$ for $R > R_{c_1}$.](image)

**Theorem 3.5.** Assume (3.7) and

$$R_0 \alpha^2 < \frac{(1 - \sigma) \pi^2}{\sigma^2 (1 + \sigma) (j_2^2 \alpha_1^2 + \pi^2)^3}.$$

The following statements are true.

1. For Problem (2.1)-(2.5) defined in $H$, the steady state $(U, T) = 0$ is locally asymptotically stable if $R < R_{c_2}$.

2. For Problem (2.1)-(2.5) defined in $\tilde{H}$, a Hopf bifurcation occurs generically when $R$ crosses $R_{c_2}$. 


4. Preliminaries

4.1. Attractor bifurcation theory. Consider (3.1) satisfying (3.2) and (3.3). We start with the Principle of Exchange of Stabilities (PES). Let the eigenvalues (counting the multiplicity) of $L_\lambda$ be given by $\beta_1(\lambda), \beta_2(\lambda), \cdots$. Suppose that

$$\text{Re}\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases}$$

(4.1)

$$\text{Re}\beta_j(\lambda_0) < 0, \quad \text{if } m + 1 \leq j.$$  

(4.2)

Let the eigenspace of $L_\lambda$ at $\lambda_0$ be

$$E_0 = \bigcup_{1 \leq j \leq m} \bigcup_{k=1}^\infty \{ u, v \in H_1 \mid (L_{\lambda_0} - \beta_j(\lambda_0))^k w = 0, w = u + iv \}.$$  

It is known that dim $E_0 = m$.

**Theorem 4.1** (T. Ma and S. Wang [16, 17]). Assume that the conditions (3.2)-(3.3) and (4.1)-(4.2) hold true, and $u = 0$ is locally asymptotically stable for (3.1) at $\lambda = \lambda_0$. Then the following assertions hold true.

1. For $\lambda \neq \lambda_0$, (3.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to attractors $\Sigma_\lambda$, having the same homology as $S^{m-1}$, where $m - 1 \leq \dim \Sigma_\lambda \leq m$, which is connected if $m > 1$;
2. For any $u_\lambda \in \Sigma_\lambda$, $u_\lambda$ can be expressed as
   $$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \quad \forall v_\lambda \in E_0;$$
3. There is an open set $U \subset H$ with $0 \in U$ such that the attractor $\Sigma_\lambda$ bifurcated from $(0, \lambda_0)$ attracts $U \setminus \Gamma$ in $H$, where $\Gamma$ is the stable manifold of $u = 0$ with co-dimension $m$.

4.2. Center manifold theory. A crucial ingredient for the proof of the main theorems using the above attractor bifurcation theorem is an approximation formula for center manifold functions; see [16].

Let $H_1$ and $H$ be decomposed into

$$H_1 = E_1^\lambda \oplus E_2^\lambda, \quad H = \bar{E}_1^\lambda \oplus \bar{E}_2^\lambda,$$

(4.3)

for $\lambda$ near $\lambda_0 \in \mathbb{R}^1$, where $E_1^\lambda, E_2^\lambda$ are invariant subspaces of $L_\lambda$, such that dim $E_1^\lambda < \infty$, $\bar{E}_1^\lambda = E_1^\lambda$, $\bar{E}_2^\lambda = \text{closure of } E_2^\lambda$ in $H$. In addition, $L_\lambda$ can be decomposed into $L_\lambda = L_1^\lambda \oplus L_2^\lambda$ such that for any $\lambda$ near $\lambda_0$,

$$\begin{cases} L_1^\lambda = L_{\lambda|E_1^\lambda} : E_1^\lambda \rightarrow \bar{E}_1^\lambda, \\ L_2^\lambda = L_{\lambda|E_2^\lambda} : E_2^\lambda \rightarrow \bar{E}_2^\lambda, \end{cases}$$

(4.4)

where all eigenvalues of $L_2^\lambda$ possess negative real parts, and the eigenvalues of $L_1^\lambda$ possess nonnegative real parts at $\lambda = \lambda_0$. Furthermore, with $\mu < 1$ given by (3.3), let

$$E_2^\lambda(\mu) = \text{closure of } E_2^\lambda \text{ in } H_\mu.$$  

By the classical center manifold theorem (see among others [21, 22]), there exists a neighborhood of $\lambda_0$ given by $|\lambda - \lambda_0| < \delta$ for some $\delta > 0$, a neighborhood $B_\lambda \subset E_1^\lambda$ of $x = 0$, and a $C^1$ center manifold function $\Phi(\cdot, \lambda) : B_\lambda \rightarrow E_2^\lambda(\theta)$, called the center
manifold function, depending continuously on \( \lambda \). Then to investigate the dynamic bifurcation of (3.1) it suffices to consider the finite dimensional system as follows

\[
\frac{dx}{dt} = L^\lambda_1 x + g_1(x, \Phi_\lambda(x), \lambda), \quad x \in B^\lambda_1 \subset E^\lambda_1.
\]

Hence, an approximation formula for the center manifold function \( \Phi_\lambda \) is crucial for the bifurcation and stability study.

Let the nonlinear operator \( G \) be in the following form

\[
G(u, \lambda) = G_n(u, \lambda) + o(\|u\|^n),
\]

for some integer \( n \geq 2 \). Here \( G_n : H_1 \times \cdots \times H_1 \rightarrow H \) is a \( n \)-multilinear operator, and \( G_n(u, \lambda) = G_n(u, \cdots, u, \lambda) \).

**Theorem 4.2.** [10] Under the conditions (4.3), (4.4) and (4.6), the center manifold function \( \Phi(x, \lambda) \) can be expressed as

\[
\Phi(x, \lambda) = (-L^\lambda_2)^{-1} P^2 G_n(x, \lambda) + o(\|x\|^n) + O(\|\text{Re} \beta\| \|x\|^n),
\]

where \( L^\lambda_2 \) is as in (4.4), \( P^2 : H \rightarrow \tilde{E}_2 \) the canonical projection, \( x \in E^\lambda_1 \), and \( \beta = (\beta_1(\lambda), \cdots, \beta_m(\lambda)) \) the eigenvectors of \( L^\lambda_1 \).

## 5. Eigenvalue Problem

The eigenvalue problem of the linearized problem of (2.1)-(2.4) is given by

\[
\begin{align*}
\sigma(\Delta U - \nabla p) + \sigma RT e - \frac{1}{Ro} e \times U &= \beta U, \\
\Delta T + w &= \beta T, \\
\text{div} U &= 0,
\end{align*}
\]

supplemented with (2.3) and (2.4). For \( \psi = (U, T) \) satisfying (2.3) and (2.4), we expand the field \( \psi \) in Fourier series

\[
\psi(x, y, z) = \sum_{j,k=-\infty}^{\infty} \psi_{jk}(z) e^{i(j \alpha_1 x + k \alpha_2 y)}.
\]

Plugging (5.2) into (5.1), we obtain the following system of ordinary differential equations

\[
\begin{align*}
\sigma(D_{jk} u_{jk} - i j \alpha_1 p_{jk}) + \frac{1}{Ro} v_{jk} &= \beta u_{jk}, \\
\sigma(D_{jk} v_{jk} - i k \alpha_2 p_{jk}) - \frac{1}{Ro} u_{jk} &= \beta v_{jk}, \\
D_{jk} w_{jk} - p'_{jk} + RT_{jk} &= \sigma^{-1} \beta w_{jk}, \\
D_{jk} T_{jk} + w_{jk} &= \beta T_{jk}, \\
j \alpha_1 u_{jk} + i k \alpha_2 v_{jk} + w'_{jk} &= 0, \\
u'_{jk} \big|_{z=0.1} = v'_{jk} \big|_{z=0.1} = w_{jk} \big|_{z=0.1} = T_{jk} \big|_{z=0.1} = 0,
\end{align*}
\]
for \( j, k \in \mathbb{Z} \), where \( \gamma = d/dz \), \( D_{jk} = d^2/dz^2 - \alpha^2_{jk} \) and \( \alpha^2_{jk} = j^2\alpha^2_1 + k^2\alpha^2_2 \). If \( w_{jk} \neq 0 \), \((5.3)\) can be reduced to a single equation for \( w_{jk}(z) \):

\[
(D_{jk} - \beta)(\sigma D_{jk} - \beta)^2 D_{jk} + \frac{1}{Ro^2}(D_{jk} - \beta)(D_{jk} + \alpha^2_{jk}) + \sigma Ra^2_{jk}(\sigma D_{jk} - \beta)w_{jk} = 0,
\]

\[
w_{jk} = w_{jk} = w_{jk}^{(4)} = w_{jk}^{(6)} = 0 \quad \text{at} \quad z = 0, 1,
\]

for \( j, k \in \mathbb{Z} \). Thanks to \((5.5)\), \( w_{jk} \) can be expanded in a Fourier sine series

\[
w_{jk}(z) = \sum_{l=1}^{\infty} w_{jk,l} \sin l\pi z,
\]

for \((j, k) \in \mathbb{Z} \times \mathbb{Z}\). Substituting \((6.6)\) into \((5.4)\), we see that the eigenvalues \( \beta \) of the problem \((5.1)\) satisfy the cubic equations

\[
\beta^3 + (2\sigma + 1)\gamma^2_{jk} \beta^2 + [(\sigma^2 + 2\sigma)\gamma^4_{jk} + \frac{l^2 \pi^2}{Ro^2 \gamma^2_{jk}} - \sigma R \alpha^2_{jk}] \beta + \sigma^2 \gamma^6_{jk} - \sigma^2 Ra^2_{jk} + \frac{l^2 \pi^2}{Ro^2} = 0,
\]

for \( j, k \in \mathbb{Z} \) and \( l \in \mathbb{N} \), where \( \gamma^2_{jk} = \alpha^2_{jk} + l^2\pi^2 \). In the following discussions, we let

\[
g_{jk}(\beta) = (\beta + \gamma^2_{jk})[(\beta + \sigma \gamma^2_{jk})^2 + l^2 \pi^2 Ro^{-2} \gamma^2_{jk}],
\]

\[
h_{jk}(\beta) = \sigma Ra^2_{jk} \gamma^2_{jk} + \sigma \gamma^4_{jk},
\]

\[
f_{jk}(\beta) = g_{jk}(\beta) - h_{jk}(\beta),
\]

and \( \beta_{jk1}(R), \beta_{jk2}(R) \) and \( \beta_{jk3}(R) \) be the zeros of \( f_{jk} \) with \( \text{Re}(\beta_{jk1}) \geq \text{Re}(\beta_{jk2}) \geq \text{Re}(\beta_{jk3}) \).

5.1. **Eigenvectors.** In the following discussions, we consider the following index sets:

\[
\Lambda_1 = \{(j, k, l) \in \mathbb{Z}^2 \times \mathbb{N} \mid j \geq 0, (j, k) \neq (0, 0)\},
\]

\[
\Lambda_2 = \{(j, k, l) \in \mathbb{Z}^2 \times \mathbb{N} \mid j \geq 0, (j, k) \neq (0, 0)\},
\]

\[
\Lambda_3 = \{(j, k, l) \in \{(0, 0)\} \times \mathbb{N}\},
\]

\[
\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.
\]

1. For \((j, k, 0) \in \Lambda_2\), we define

\[
\psi^{\beta_{jk0}}_1 = (k\alpha_2 \sin(j\alpha_1 x + k\alpha_2 y), -j\alpha_1 \sin(j\alpha_1 x + k\alpha_2 y), 0, 0)^t,
\]

\[
\psi^{\beta_{jk0}}_2 = (-k\alpha_2 \cos(j\alpha_1 x + k\alpha_2 y), j\alpha_1 \cos(j\alpha_1 x + k\alpha_2 y), 0, 0)^t,
\]

\[
E_{jk0} = \text{span}\{\psi^{\beta_{jk0}}_1, \psi^{\beta_{jk0}}_2\},
\]

\[
\beta_{k0} = \cup_{(j, k, 0) \in \Lambda_2} \{\beta_{jk0}\},
\]

where \( \beta_{jk0} = -\sigma \gamma^2_{jk0} = -\sigma \alpha^2_{jk0} = -\sigma (j^2\alpha^2_1 + k^2\alpha^2_2) \). It is not hard to see that \( L_R(\psi^{\beta_{jk0}}_1) = \beta_{jk0} \psi^{\beta_{jk0}}_1 \) and \( L_R(\psi^{\beta_{jk0}}_2) = \beta_{jk0} \psi^{\beta_{jk0}}_2 \).
2. For \((0, 0, l) \in \Lambda_3\), we define
\[
\psi_{\beta_{001}} = (0, 0, 0, \sin l \pi z)^t, \quad \psi_{\beta_{001}^2} = (\cos l \pi z, 0, 0, 0)^t, \\
\psi_{\beta_{002}} = (0, \cos l \pi z, 0, 0)^t, \quad E_{\beta_{001}} = \text{span}\{\psi_{\beta_{001}}, \psi_{\beta_{001}^2}, \psi_{\beta_{002}}\}, \\
\beta_\lambda = \bigcup_{q=1}^3 \{\beta_{00q}\}, \quad \beta_\lambda = \bigcup_{q=1}^3 \{\beta_{00q}\},
\]
where \(\beta_{001} = -\gamma_{001}^2 = -l^2 \pi^2, \beta_{001}^2 = -\gamma_{001}^2 - \frac{1}{R_0} i \) and \(\beta_{002} = -\gamma_{002}^2 + \frac{1}{R_0} i\). It is easy to check that
\[
L_R(\psi_{\beta_{001}}) = \beta_{001} \psi_{\beta_{001}}, \\
L_R(\psi_{\beta_{001}^2}) = -\sigma_{001}^2 \psi_{\beta_{001}^2} - \frac{1}{R_0} \psi_{\beta_{001}}, \\
L_R(\psi_{\beta_{002}}) = \frac{1}{R_0} \psi_{\beta_{002}^2} - \sigma_{002}^2 \psi_{\beta_{002}}.
\]

3. For \((j, k, l) \in \Lambda_1\), we define
\[
\phi_{jkl} = (-\frac{j \alpha_1 l \pi}{\alpha_{jk}^2}, \sin(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, -\frac{k \alpha_2 l \pi}{\alpha_{jk}^2}, \sin(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, \\
\cos(j \alpha_1 x + k \alpha_2 y) \sin l \pi z, 0)^t, \\
\phi_{jkl}^2 = \frac{k \alpha_2 l \pi}{\alpha_{jk}^2} \sin(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, -\frac{j \alpha_1 l \pi}{\alpha_{jk}^2} \sin(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, 0, 0), \\
\phi_{jkl}^3 = (0, 0, 0, \cos(j \alpha_1 x + k \alpha_2 y) \sin l \pi z)^t, \\
\phi_{jkl}^4 = (\frac{j \alpha_1 l \pi}{\alpha_{jk}^2}, \cos(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, -\frac{k \alpha_2 l \pi}{\alpha_{jk}^2}, \cos(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, \\
\sin(j \alpha_1 x + k \alpha_2 y) \sin l \pi z, 0)^t, \\
\phi_{jkl}^5 = (-\frac{k \alpha_2 l \pi}{\alpha_{jk}^2}, \cos(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, \frac{j \alpha_1 l \pi}{\alpha_{jk}^2}, \cos(j \alpha_1 x + k \alpha_2 y) \cos l \pi z, 0, 0)^t, \\
\phi_{jkl}^6 = (0, 0, 0, \sin(j \alpha_1 x + k \alpha_2 y) \sin l \pi z)^t, \\
E_{jkl}^1 = \text{span}\{\phi_{jkl}^1, \phi_{jkl}^2, \phi_{jkl}^3\}, \quad E_{jkl}^2 = \text{span}\{\phi_{jkl}^4, \phi_{jkl}^5, \phi_{jkl}^6\}, \\
E_{jkl} = E_{jkl}^1 \oplus E_{jkl}^2, \quad \beta_\Lambda = \bigcup_{(j, k, l) \in \Lambda_1} \bigcup_{q=1}^3 \{\beta_{jklq}\}.
\]

It is easy to check that \(E_{jkl}^1\) and \(E_{jkl}^2\) are invariant subspaces of the linear operator \(L_R\) respectively, i.e., \(L_R(E_{jkl}^1) \subset E_{jkl}^1\) and \(L_R(E_{jkl}^2) \subset E_{jkl}^2\). The characteristic polynomial of \(L_R |_{E_{jkl}^1}\) (resp., \(L_R |_{E_{jkl}^2}\)) is given by \(f_{jkl}\) as defined in (5.8). Since \(E_{jkl}^1\) (resp., \(E_{jkl}^2\)) is of dimension three, the (generalized) eigenvectors of \(L_R |_{E_{jkl}^1\cup E_{jkl}^2} = \bigcup_{q=1}^3 \{\psi_{jklq}\}\) (resp., \(E_{jkl}^2\)), i.e., span\{\bigcup_{q=1}^3 \{\psi_{jklq}\}\} = E_{jkl}^1\) (resp., span\{\bigcup_{q=1}^3 \{\psi_{jklq}\}\} = E_{jkl}^2\). If \(\beta_{jklq}\) is a real zero of \(f_{jkl}\), the eigenvector corresponding to \(\beta_{jklq}\) in \(E_{jkl}^1\) (resp., \(E_{jkl}^2\)) is given by
\[
\psi_{jklq}^1 = A_1(\beta_{jklq}) \phi_{jklq} + A_2(\beta_{jklq}) \phi_{jklq}^2, \\
\psi_{jklq}^2 = \phi_{jklq}^3 + A_1(\beta_{jklq}) \phi_{jklq}^5 + A_2(\beta_{jklq}) \phi_{jklq}^6,
\]
where
\[
A_1(\beta) = \frac{-1}{R_0(\beta + \sigma_{jklq}^2)}, \quad A_2(\beta) = \frac{1}{\beta + \gamma_{jklq}^2}.
\]
If \( \beta_{jklq} = \beta_{jklq_2} \) (imaginary numbers) are zeros of \( f_{jkl} \), the (generalized) eigenvectors corresponding to \( \beta_{jklq} \) and \( \beta_{jklq_2} \) in \( E_{jkl}^1 \) (resp., \( E_{jkl}^2 \)) are given by

\[
\psi_{jklq_1}^{\beta_{jklq}} = \phi_{jkl}^1 + R_1(\beta_{jklq_1}) \phi_{jkl}^2 + R_2(\beta_{jklq_1}) \phi_{jkl}^3,
\]
\[
\psi_{jklq_2}^{\beta_{jklq}} = I_1(\beta_{jklq_1}) \phi_{jkl}^2 + I_2(\beta_{jklq_1}) \phi_{jkl}^3,
\]
where

\[
(\psi_{jklq_1}^{\beta_{jklq}}, \psi_{jklq_2}^{\beta_{jklq}}) = (\phi_{jkl}^1, I_1(\beta_{jklq_1}) \phi_{jkl}^2 + I_2(\beta_{jklq_1}) \phi_{jkl}^3).
\]

The dual vector corresponding to \( \psi_{jklq_1}^{\beta_{jklq}} \) (resp., \( \psi_{jklq_2}^{\beta_{jklq}} \)) is given by

\[
\psi_{jklq_1}^{\beta_{jklq}} = \phi_{jkl}^4 + R_1(\beta_{jklq_1}) \phi_{jkl}^5 + R_2(\beta_{jklq_1}) \phi_{jkl}^6,
\]
\[
(\psi_{jklq_1}^{\beta_{jklq}}, \psi_{jklq_2}^{\beta_{jklq}}) = (\phi_{jkl}^4, I_1(\beta_{jklq_1}) \phi_{jkl}^5 + I_2(\beta_{jklq_1}) \phi_{jkl}^6),
\]
where

\[
C_1(\beta) = \frac{1}{Ro(\beta + \sigma^2_{jkl})}, \quad C_2(\beta) = \frac{\sigma R}{\beta + \gamma^2_{jkl}}.
\]

The dual vector \( \psi_{jklq_1}^{\beta_{jklq}} \) (resp., \( \psi_{jklq_2}^{\beta_{jklq}} \)) satisfies

\[
< \psi_{jklq_1}^{\beta_{jklq}}, \psi_{jklq_2}^{\beta_{jklq}} >_{H} = 0 \quad ( < \psi_{jklq_2}^{\beta_{jklq}}, \psi_{jklq_2}^{\beta_{jklq}} >_{H} = 0),
\]
for \( q^* \neq q \).

We note that \( E_{jkl} \) is orthogonal to \( E_{jkl}^1 \) for \( (j_1, k_1, l_1) \neq (j_2, k_2, l_2) \) and \( E_{jkl}^1 \) is orthogonal to \( E_{jkl}^2 \) for \( (j, k, l) \in \Lambda_1 \). Hence the dual vector \( \psi_{jklq_1}^{\beta_{jklq}} \) (resp., \( \psi_{jklq_2}^{\beta_{jklq}} \)) satisfies

\[
< \psi, \psi_{jklq_1}^{\beta_{jklq}} >_{H} = 0 \quad \text{for} \quad \psi \in (\cup_{(j^*, k^*, l^*) \neq (j, k, l)} E_{j^*, k^*, l^*}) \cup E_{jkl}^2
\]
\[
( < \psi, \psi_{jklq_2}^{\beta_{jklq}} >_{H} = 0 \quad \text{for} \quad \psi \in (\cup_{(j^*, k^*, l^*) \neq (j, k, l)} E_{j^*, k^*, l^*}) \cup E_{jkl}^1).
\]

In view of the Fourier expansion, we see that \( \cup_{(j,k,l) \in \Lambda} E_{jkl}^1 \cup (\cup_{(j,k,l) \in \Lambda} E_{jkl}^2) \cup (\cup_{(j,k,0) \in \Lambda_2} \{\psi_1^{\beta_{jklq_1}}\}) \cup (\cup_{(0,0,0) \in \Lambda_3} \{\psi_0^{\beta_{jklq_2}}\}) \) is a basis of \( \bar{H}_1 \). Hence, by the discussion above, we have the following conclusions.

a) The set \( \beta_{H_1} = \beta_{\Lambda_1} \cup \beta_{\Lambda_2} \cup \beta_{\Lambda_3} \) consists of all eigenvalues of \( L_R |_{H_1} \), and the (generalized) eigenvectors of \( L_R |_{H_1} \) form a basis of \( H_1 \).

b) The set \( \beta_{\bar{H}_1} = \beta_{\Lambda_1} \cup \beta_{\Lambda_2} \cup \beta_{\Lambda_3} \) consists of all eigenvalues of \( L_R |_{\bar{H}_1} \), and the (generalized) eigenvectors of \( L_R |_{\bar{H}_1} \) form a basis of \( \bar{H}_1 \).

c) \( Re(\beta) < 0 \) for each \( \beta \in \beta_{\Lambda_2} \cup \beta_{\Lambda_3} \).

**Lemma 5.1.** If \( R \) is small, then \( Re(\beta_{jklq}(R)) < 0 \) for each \( \beta_{jklq} \in \beta_{\Lambda_1} \).
Proof. Plugging $\beta = \gamma_{jkl}^2 \beta^*$ into $f_{jkl}$, we get $f_{jkl}(\beta) = \gamma_{jkl}^6 \tilde{f}_{jkl}(\beta^*)$, where

$$
\tilde{f}_{jkl}(\beta^*) = (\beta^* + 1)(\beta^* + \sigma)^2 + \frac{l^2 \pi^2}{\gamma_{jkl}^2 R_0 \alpha_{jkl}^2} (\beta^* + 1) - \sigma R \alpha_{jkl}^2 (\beta^* + \sigma).
$$

Hence, we only need to show that the real part of each zero of $\tilde{f}_{jkl}$ is strictly negative when $R$ is small. We observe that $\tilde{f}_{jkl}(\beta^*) > 0$ for all $\beta^* \geq 0$ provided $R < 1 + \sigma^{-1}$. Therefore, if all zeros of $\tilde{f}_{jkl}$ are real numbers, we are done.

For the case where only one of the zeros of $\tilde{f}_{jkl}$ is real, this real zero, $\beta^*_1$, is a perturbation of $-1$. There exists an $\epsilon$ (depending on $\sigma$ only) such that $-(1+2\sigma) < \beta^*_1 < 0$ provided $R < \epsilon$. This makes the real part of the other two zeros of $\tilde{f}_{jkl}$ strictly negative and the proof is complete. □

5.2. Characterization of Critical Rayleigh Numbers. Based on the above discussion, we know that only the eigenvalues in $\beta_{\Lambda_1}$ depend on the Rayleigh number $R$. Hence, to study the Principle of Exchange of Stabilities for problem (5.1), it suffices to focus the problem on the set $\beta_{\Lambda_1}$. We proceed with the following two cases.

Case 1. $\beta = 0$ is a zero of $f_{jkl}$ if and only if the constant term of the polynomial $f_{jkl}$ is 0. In this case, we have

$$
R = \frac{\gamma_{jkl}^6}{\alpha_{jkl}^2} + \frac{l^2 \pi^2}{\alpha_{jkl}^2 R_0^2 \alpha_{jkl}^2} \geq \frac{(\alpha_{jkl}^2 + \pi^2)^3}{\sigma^2 R_0^2 \alpha_{jkl}^2} + \frac{\pi^2}{\sigma^2 R_0^2 \alpha_{jkl}^2}.
$$

Hence the critical Rayleigh number $R_{c_1}$ is given by

$$
R_{c_1} = \min_{(j,k,l) \in \Lambda_1} \left\{ \frac{\gamma_{jkl}^6}{\alpha_{jkl}^2} + \frac{l^2 \pi^2}{\alpha_{jkl}^2 R_0^2 \alpha_{jkl}^2} \right\} = \frac{\gamma_{jkl}^6}{\alpha_{jkl}^2 R_0^2 \alpha_{jkl}^2} + \frac{\pi^2}{\sigma^2 R_0^2 \alpha_{jkl}^2},
$$

for some $(j_1, k_1, l_1) \in \Lambda_1$.

Case 2. A careful analysis on (5.18) shows that $\beta = ai \ (a \neq 0)$, a purely imaginary number, is a zero of $f_{jkl}$ if and only if the following two equations hold true:

$$
(\sigma^2 + 2\sigma) \gamma_{jkl}^4 + \frac{l^2 \pi^2}{R_0^2 \gamma_{jkl}^2} - \sigma R \gamma_{jkl}^2 > 0,
$$

$$
(2\sigma + 1) \gamma_{jkl}^2 [(\sigma^2 + 2\sigma) \gamma_{jkl}^4 + \frac{l^2 \pi^2}{R_0^2 \gamma_{jkl}^2} - \sigma R \gamma_{jkl}^2] = \sigma^2 \gamma_{jkl}^6 - \sigma^2 R_0 \alpha_{jkl}^2 + \frac{l^2 \pi^2}{R_0^2}.
$$

In this case, we have

$$
R = \frac{2(\sigma + 1) \gamma_{jkl}^6}{\alpha_{jkl}^2} + \frac{2l^2 \pi^2}{(\sigma + 1) R_0^2 \alpha_{jkl}^2},
$$

$$
R < \frac{(\sigma + 2) \gamma_{jkl}^6}{\alpha_{jkl}^2} + \frac{l^2 \pi^2}{\sigma R_0^2 \alpha_{jkl}^2}.
$$

Plugging (5.20) into (5.19), we derive an upper bound for $R_0^2$:

$$
R_0^2 < \frac{(1 - \sigma) l^2 \pi^2}{\sigma^2 (1 + \sigma) \gamma_{jkl}^6},
$$
which could only hold true when \( \sigma < 1 \).

As in Case 1, the minimum of the right hand side of (5.19) is always obtain at \( l = 1 \). Hence the critical Rayleigh number \( R_{c2} \) is given by

\[
R_{c2} = \min_{(j,k,l) \in \Lambda} \left\{ \frac{2(\sigma + 1)\gamma_{jk}^6}{\alpha_{jk}^2} + \frac{2l^2 \pi^2}{(\sigma + 1)Ro \alpha_{jk}^2} \right\}
\]

\[
= \frac{2(\sigma + 1)\gamma_{jk}^6}{\alpha_{jk}^2} + \frac{2\pi^2}{(\sigma + 1)Ro \alpha_{jk}^2},
\]

for some \((j_2, k_2, 1) \in \Lambda_1\). In the case of \( \sigma < 1 \), (5.21) with \( l = 1 \) implies \( R_{c2} \) is smaller than \( R_{c1} \). Hence, for Problem (2.1)-(2.5), \( R_{c1} \) is the first critical Rayleigh number if \( \sigma > 1 \) and \( R_{c2} \) is the first critical Rayleigh number if \( \sigma < 1 \). Therefore, the Principle of Exchange of Stabilities is given by Lemma 5.2 and Lemma 5.6.

**Lemma 5.2.** For fixed \( \sigma > 1 \) and \( Ro > 0 \), suppose that \((\alpha_{jk}^2, l) = (\alpha_{jk1}^2, 1)\) minimizes the right hand side of (5.17), then

\[
\beta_{j_1 k_1 1}(R) = \begin{cases} 
< 0 & \text{if } R < R_{c1} \\
0 & \text{if } R = R_{c1} \\
> 0 & \text{if } R > R_{c1}
\end{cases}
\]

\[
(5.24) \quad Re\beta_{jklq}(R) < 0 \quad \text{for} \quad (\alpha_{jk}^2, l) \neq (\alpha_{jk1}^2, 1), \quad q = 1, 2, 3, \quad R \text{ near } R_{c1}.
\]

**Proof.** By the above discussion, we only need to show that the first eigenvalue crosses the imaginary axis. We note that \( f_{j_1 k_1 1}(\beta) = 0 \) is equivalent to \( g_{j_1 k_1 1}(\beta) = h_{j_1 k_1 1}(\beta) \), i.e.,

\[
(5.25) \quad (\beta + \gamma_{j_1 k_1 1}^2)[(\beta + \sigma \gamma_{j_1 k_1 1}^2 + l^2 \pi^2 Ro \gamma_{j_1 k_1 1}^2 = \sigma R \alpha_{j_1 k_1 1}^2 \gamma_{j_1 k_1 1}^2 + \sigma \gamma_{j_1 k_1 1}^2).
\]

We see that both \( g_{j_1 k_1 1} \) and \( h_{j_1 k_1 1} \) are strictly increasing for \( \beta > -\gamma_{j_1 k_1 1}^2 \) (since \( \sigma > 1 \)). Let \( \Gamma_1 \) be the graph of \( \eta = g_{j_1 k_1 1}(\beta) \) and \( \Gamma_2 \) be the graph of \( \eta = h_{j_1 k_1 1}(\beta) \) as shown in Figure 5.1. When \( R = R_{c1} \), Point \( S_0 \), the intersecting point of \( \Gamma_1 \) and \( \Gamma_2 \) corresponding to \( \beta_{j_1 k_1 1}(R) \) (i.e., the \( \beta \) coordinate of \( S_0 \) is \( \beta_{j_1 k_1 1}(R) \)), is on the \( \eta \) axis. When \( R \) increases (resp., decreases), \( S_0 \) becomes \( S_1 \) (resp., \( S_2 \)). This proves (5.23) and the proof is complete.
Remark 5.3. (1) In the proof of Lemma 5.2 as shown by (5.25) and Figure 5.1, we see that, for \( R \approx R_{c_1} \), the first eigenvalue \( \beta_{j,k_{11}} \) is a simple zero of \( f_{j,k_{11}}(\beta) \). We have seen in Section 5.1 that there are eigenvectors \( \psi_{1}^{\beta_{j,k_{11}}} \in E_{1}^{j,k_{11}} \) and \( \psi_{2}^{\beta_{j,k_{11}}} \in E_{2}^{j,k_{11}} \) corresponding to \( \beta_{j,k_{11}} \). Therefore, the multiplicity of the first eigenvalue of \( L_{|H_{1}} \) (resp., \( L_{|\tilde{H}_{1}} \)) is \( m_{H_{1}} = 2 \) (resp., \( m_{\tilde{H}_{1}} = m \)), where \( m \) is the number of \((j,k,1)'s \in \Lambda_{1}\) satisfying \( \alpha_{jk}^{2} = \alpha_{j,k_{11}}^{2} \). Hence, Condition (3.6) guarantees that, for \( R \approx R_{c_1} \), the first eigenvalue of \( L_{R_{|H_{1}}} \) (resp., \( L_{R_{|\tilde{H}_{1}}} \)) is real and of multiplicity two (resp., one).

(2) For the classical Bénard problem without rotation, the second term on the right hand side of (5.17), hence the second term on the right hand side of (5.18), is not presented. Therefore, the first critical Rayleigh number of the classical Bénard problem depends only on the aspect of ratio; while the first critical Rayleigh number of the rotating problem depends on the aspect of ratio, the Prandtl number and the Rossby number. And it is clear that the first critical Rayleigh number of fast rotating flows is remarkably larger than the first critical Rayleigh number of the classical Bénard problem. This indicates that the rotating flows are much more stable than the non-rotating flows.

(3) \( R_{c_1} \) is the first Critical Rayleigh number if the Prandtl number is greater than one. For the case where the Prandtl number is smaller than one, \( R_{c_2} \) is the first Critical Rayleigh number and, in general, there are a few critical values between \( R_{c_2} \) and \( R_{c_1} \).

For \( x > 0 \), \( b \geq 0 \), we define

\[
(5.26) \quad f_{b}(x) = \frac{(x + \pi^{2})^{3} + b}{x}.
\]

Let \( x = \alpha_{jk}^{2} \), then the right hand side of (5.18) could be expressed as \( f_{b_{1}}(x) \), where \( b_{1} = \frac{\pi^{2}}{\sigma_{Ro}^{2}} \); and the second line of (5.22) could be expressed as \( 2(\sigma + 1)f_{b_{2}}(x) \),
where \( b_2 = \frac{\pi^2}{(x+1)^2 R_{c_1}} \). Consider

\[
(5.27) \quad f_b'(x) = \frac{(2x - \pi)(x + \pi)}{x^2} - b.
\]

As shown in Figure 5.2, it is easy to see that

a) for \( x \in (0, \infty) \), \( f_b(x) \) has only one critical number \( x_b \),

b) \( f_b'(x) < 0 \) if \( x < x_b \),

c) \( f_b'(x) > 0 \) if \( x > x_b \),

d) \( f_b(x_b) \) is the global minimum of \( f_b(x) \), and

e) \( x_b \) is strictly increasing in \( b \), hence, \( x_{b_1} > x_{b_2} > \frac{\pi^2}{2} \).

\[ y = (2x - \pi)(x + \pi) \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

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\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

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\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]

\[ x = \frac{x_b}{2} \]

\[ x_b \]

\[ x = \frac{\pi}{2} \]

\[ y = \pi \]

\[ y = 0 \]

\[ x = 0 \]

\[ y = b \]

\[ y = -\pi \]
Lemma 5.6. Assume (3.7), the rest part of the proof is the same as the proof of Lemma 5.4. □

Proof. Consider Lemma 5.5. (1) \( \text{Condition } (3.7) \) holds true under the assumption (5.30).

Generically, Condition (3.7) holds true under the assumption (5.31).

Proof. We only need to prove (5.32). Under the assumptions of the lemma together with (5.11), (5.19) and (5.20), by the discussion in Case (2) at the beginning of this subsection, we know that \( \{\beta_{j_2011}(R), \beta_{j_2012}(R)\} \) (\( \beta_{j_2011}(R) = \beta_{j_2012}(R) \)) is the only simple pair of complex eigenvalues of the problem (5.7) in space \( \tilde{H}_1 \) satisfying

\[
\text{Re}(\beta_{j_2011}(R)) = \begin{cases} 
< 0 & \text{if } R < R_{c_2}, \\
= 0 & \text{if } R = R_{c_2}, \\
> 0 & \text{if } R > R_{c_2}.
\end{cases}
\]

(5.32) \( \text{Re}\beta_{j_{kl}q}(R) < 0 \) for \( (\alpha_{jk}^2, l) \neq (\alpha_{j_0}^2, 1) \), \( q = 1, 2, 3, R \text{ near } R_{c_2} \).

Proof. We only need to prove (5.32). Under the assumptions of the lemma together with (5.11), (5.19) and (5.20), by the discussion in Case (2) at the beginning of this subsection, we know that \( \{\beta_{j_2011}(R), \beta_{j_2012}(R)\} \) is the only simple pair of complex eigenvalues of \( L_R |_{\tilde{H}_1} \) with \( \text{Re}(\beta_{j_2011}(R_{c_2})) = \text{Re}(\beta_{j_2012}(R_{c_2})) = 0 \). Since \( \beta_{j_2013}(R) \) (real), \( \beta_{j_2011}(R) \) and \( \beta_{j_2012}(R) \) are zeros of \( f_{j_201} \), we know that

\[
\beta_{j_2013}(R) = -(\text{Re}(\beta_{j_2011}(R)) + \text{Re}(\beta_{j_2012}(R))) - (2\sigma + 1)\gamma_{j_201}^2.
\]

Hence (5.32) is equivalent to

\[
\beta_{j_2013}(R) = \begin{cases} 
-(2\sigma + 1)\gamma_{j_201}^2 & \text{if } R < R_{c_2}, \\
-(2\sigma + 1)\gamma_{j_201}^2 & \text{if } R = R_{c_2}, \\
-2(2\sigma + 1)\gamma_{j_201}^2 & \text{if } R > R_{c_2}.
\end{cases}
\]

(5.33) which is true as shown in Figure 5.3. This completes the proof.
Lemma 5.7. For fixed \( \alpha_1, \alpha_2 > 0 \) and \( \sigma > 1 \), \( R_{c_1} \to \infty \) as \( Ro \to 0 \). More precisely, \( R_{c_1} = O(Ro^{-\frac{4}{3}}) \).

Proof. Since \( b_1 = \frac{\pi^2}{\sigma^2 Ro^2} \), by (5.27), \( x_{b_1} = O(b_1^\frac{1}{6}) \) as \( Ro \to 0 \). Hence, \( R_{c_1} = O(f_{b_1}(x_{b_1})) = O(b_1^\frac{1}{6}) = O(Ro^{-\frac{4}{3}}) \).

6. Proof of Main Theorems

6.1. Center manifold reduction. We are now in a position to reduce equations of (2.1)-(2.5) to the center manifold. For any \( \psi = (U,T) \in H_1 \), we have

\[
\psi = \sum_{(j,k,l) \in \Lambda_1} \sum_{q=1}^3 (x_{jklq} \psi_1^{\beta_{jklq}} + y_{jklq} \psi_2^{\beta_{jklq}}) + \sum_{(j,k,0) \in \Lambda_2} \sum_{q=1}^3 (x_{jk0q} \psi_1^{\beta_{jk0q}} + y_{jk0q} \psi_2^{\beta_{jk0q}}) + \sum_{l=1}^\infty \sum_{q=1}^3 x_{00lq} \psi_1^{\beta_{00lq}}.
\]

Under the assumption (3.6), the first critical Rayleigh number is given by

\[
R_{c_1} = \frac{\gamma_{j_01}^6}{\alpha_{j_10}^2} + \frac{\pi^2}{\sigma^2 Ro^2 \alpha_{j_10}^2}.
\]

In this case, the multiplicity of the first eigenvalue is two and the reduced equations of (2.1)-(2.5) are given by

\[
\begin{align*}
\frac{dx_{j_1,011}}{dt} &= \beta_{j_1,011}(R)x_{j_1,011} + \frac{1}{\psi_1^{\beta_{j_1,011}}, \Psi_1^{\beta_{j_1,011}}} < G(\psi, \psi), \Psi_1^{\beta_{j_1,011}>H,} \\
\frac{dy_{j_1,011}}{dt} &= \beta_{j_1,011}(R)y_{j_1,011} + \frac{1}{\psi_1^{\beta_{j_1,011}}, \Psi_1^{\beta_{j_1,011}}} < G(\psi, \psi), \Psi_1^{\beta_{j_1,011}} > H.
\end{align*}
\]
By Theorem 4.2 and (6.4)-(6.5), we obtain
\[ G(\psi_1, \psi_2) = -(P(U_1 \cdot \nabla)U_2, (U_1 \cdot \nabla)T_2)^t \]
and
\[ < G(\psi_1, \psi_2), \psi_3 >_H = - \int_0^1 \int_0^{2\pi/\alpha_2} \int_0^{2\pi/\alpha_1} \left[ < (U_1 \cdot \nabla)U_2, U_3 >_{\mathbb{R}^3} + (U_1 \cdot \nabla)T_2T_3 \right] dx dy dz, \]
where P is the Leray projection to \( L^2 \) fields. Let the center manifold function be

\[ \Phi = \sum_{\beta \neq \beta_{j_{101}}} (\Phi^\beta_1(x_{j_{101}}, y_{j_{101}})\psi_1^\beta + \Phi^\beta_2(x_{j_{101}}, y_{j_{101}})\psi_2^\beta). \]

The direct calculation shows that
\[ G(\psi_{1_{j_{101}}}, \psi_{1_{j_{101}}}) = - (0, \frac{A_1\pi^2}{2j_1\alpha_1} \sin 2j_1\alpha_1 x, 0, \frac{A_2\pi^2}{2} \sin 2\pi z)^t, \]
\[ G(\psi_{2_{j_{101}}}, \psi_{2_{j_{101}}}) = - \left( \frac{\pi^2}{2j_1\alpha_1} \cos 2\pi z, \frac{A_1\pi^2}{2j_1\alpha_1} (\cos 2\pi z - \cos 2j_1\alpha_1 x), 0, 0 \right)^t, \]
\[ G(\psi_{2_{j_{101}}}, \psi_{1_{j_{101}}}) = - \left( \frac{\pi^2}{2j_1\alpha_1} \cos 2\pi z, -\frac{A_1\pi^2}{2j_1\alpha_1} (\cos 2j_1\alpha_1 x + \cos 2\pi z), 0, 0 \right)^t, \]
\[ G(\psi_{2_{j_{101}}}, \psi_{2_{j_{101}}}) = - (0, -\frac{A_1\pi^2}{2j_1\alpha_1} \sin 2j_1\alpha_1 x, 0, \frac{A_2\pi^2}{2} \sin 2\pi z)^t. \]

where \( A_1 = A_1(\beta_{j_{101}}), A_2 = A_2(\beta_{j_{101}}) \) \( C_1 = C_1(\beta_{j_{101}}) \) and \( C_2 = C_2(\beta_{j_{101}}) \).

Hereafter, we make the following convention:
\[ o(2) = o(x^2_{j_{101}} + y^2_{j_{101}}) + O(\beta_{j_{101}}(R) \cdot (x^2_{j_{101}} + y^2_{j_{101}})), \]
\[ o(3) = o((x^2_{j_{101}} + y^2_{j_{101}})^{3/2}) + O(\beta_{j_{101}}(R) \cdot (x^2_{j_{101}} + y^2_{j_{101}})^{3/2}), \]
\[ o(4) = o(x^2_{j_{101}} + y^2_{j_{101}})^2) + O(\beta_{j_{101}}(R) \cdot (x^2_{j_{101}} + y^2_{j_{101}})^2). \]

By Theorem 4.2 and (6.4)-(6.5), we obtain
\[ \Phi = \Phi^\beta_{1j_{101}}\psi_1^\beta + \Phi^\beta_{2j_{101}}\psi_2^\beta + \Phi^\beta_{0021}\psi_1^\beta + o(2), \]
that, for a standard energy estimate on (6.11) together with the center manifold theory show
(6.11)
\[ \delta \]
where
\[ \delta \]
de derive that
(6.12)
\[ \delta \]
Note that for any \( \psi_i \in H_1 (i = 1, 2, 3), \)
(6.7)
\[ \delta \]
(6.8)
\[ \delta \]
and for any \( \psi_i \in E_{jkl} \ (i = 1, 2, 3), \)
(6.9)
\[ \delta \]
The direct calculation shows that
(6.10)
\[ \delta \]
Then by \( \psi = x_{j_111} \psi_1^{\beta_{j_110}} + y_{j_111} \psi_2^{\beta_{j_110}} + \Phi (x_{j_111}, y_{j_111}) \) and (6.4)-(6.10), we derive that
\[ \psi_1^{\beta_{j_110}}, \psi_2^{\beta_{j_110}}, \psi_1^{\beta_{j_110}} \]
Hence, the reduction equations are given by
(6.11)
\[ \delta \]
\[ \delta \]
where
(6.12)
\[ \delta \]
A standard energy estimate on (6.11) together with the center manifold theory show
that, for \( R \leq R_{11}, \ (U, T) = 0 \) is locally asymptotically stable for the problem (2.1)-(2.5). Hence by Theorem 4.1, the solutions to (2.1)-(2.5) bifurcate from (U, T, R) =
(0, \(R_{c1}\)) to an attractor \(\Sigma_R\). Moreover, by (0.11)-(0.12) together with Theorem 5.10 in \[18\], we conclude that \(\Sigma_R\) is homeomorphic to \(S^1\) in \(H\).

### 6.2. Completion of the proof of Theorem 3.4.

In this subsection, we prove that \(\Sigma_R\) consists of steady state solutions. It is clear that the first eigenvalue of \(L_R|_{\tilde{H}_1}\) is simple for \(R \approx R_{c1}\). By the Kransnoselski bifurcation theorem (see among others Chow and Hale \[4\] and Nirenberg \[21\]), when \(R\) crosses \(R_{c1}\), the equations bifurcate from the basic solution to a steady state solution in \(\tilde{H}\). Therefore the attractor \(\Sigma_R\) contains at least one steady state solution. Secondly, it’s easy to check that the equations (2.1)-(2.5) defined in \(H\) are translation invariant in the \(x\)-direction. Hence if \(\psi_0(x, y, z) = (U(x, y, z), T(x, y, z))\) is a steady state solution, then \(\psi_0(x + \rho, y, z)\) are steady state solutions as well. By the periodic condition in the \(x\)-direction, the set

\[
S_{\psi_0} = \{\psi_0(x + \rho, y, z)|\rho \in \mathbb{R}\}
\]

is a cycle homeomorphic to \(S^1\) in \(H\). Therefore the steady state of \(\Sigma_R\) generates a cycle of steady state solutions. Hence the bifurcated attractor \(\Sigma_R\) consists of steady state solutions. The proof of Theorem 3.4 is complete.

### 6.3. Proof of Theorem 3.5.

The proof follows directly from the classical Hopf bifurcation theorem and Lemma 5.6.

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