Fractional cycle decompositions in hypergraphs

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Abstract
We prove that for any integer $k \geq 2$ and $\epsilon > 0$, there is an integer $\ell_0 \geq 1$ such that any $k$-uniform hypergraph on $n$ vertices with minimum codegree at least $(1/2 + \epsilon)n$ has a fractional decomposition into (tight) cycles of length $\ell$ ($\ell$-cycles for short) whenever $\ell \geq \ell_0$ and $n$ is large in terms of $\ell$. This is essentially tight. This immediately yields also approximate integral decompositions for these hypergraphs into $\ell$-cycles. Moreover, for graphs this even guarantees integral decompositions into $\ell$-cycles and solves a problem posed by Glock, Kühn, and Osthus. For our proof, we introduce a new method for finding a set of $\ell$-cycles such that every edge is contained in roughly the same number of $\ell$-cycles from this set by exploiting that certain Markov chains are rapidly mixing.

KEYWORDS
cycles, hypergraph decompositions, random walk

1 | INTRODUCTION

The results in this paper are motivated by questions triggered from the famous conjecture of Nash–Williams, which remains unsolved despite much research activity in the area. It is concerned with the question whether a given graph $G$ admits a decomposition of its edge set into edge-disjoint triangles. In order for this to be achievable basic divisibility conditions need to be satisfied, namely the number of edges of $G$ needs do be divisible by 3 and all vertices of $G$ need to have even degree. Let us call a graph triangle-divisible if it satisfies these two divisibility conditions.

Conjecture 1.1 (Nash–Williams). There exists an $n_0$ such that every triangle-divisible graph $G$ on $n$ vertices with $n \geq n_0$ and $\delta(G) \geq 3n/4$ admits a decomposition of its edge set into triangles.

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Although Conjecture 1.1 remains open, we already have some understanding regarding decompositions of the edge set into triangles (triangle decompositions). Arguably, the biggest step was done by Barber, Kühn, Lo, and Osthus [1] by proving that the problem can be relaxed to fractional triangle decompositions if we increase the lower bound on the minimum degree by a minor order term. Delcourt and Postle [2] recently showed the existence of a fractional triangle decomposition in a graph $G$ on $n$ vertices whenever $\delta(G) \geq 0.828n$ and $n$ is large. In turn this implies with the above result from [1] that every large triangle-divisible graph $G$ on $n$ vertices with $\delta(G) \geq 0.83n$ admits a triangle decomposition.

We say a hypergraph is $k$-uniform if all its edges contain exactly $k$ vertices and we also refer to $k$-uniform hypergraphs as $k$-graphs; 2-graphs are as usual called graphs. Given a $k$-graph $F$, an $F$-decomposition of a $k$-graph $H$ is a decomposition of the edge set of $H$ into edge-disjoint copies of $F$. The topic of $k$-graph decompositions has experienced major breakthroughs in the last decade; nevertheless many problems remain unsolved [4, 8, 9].

Let the degree $d(x)$ of a $(k-1)$-set $x$ of vertices in a $k$-graph $H$ be the number of edges that contain $x$ (observe that this coincides with the usual degree for graphs). Let $\delta(H)$ be the minimum degree of $H$ defined by \( \min \{d(x) : x \in \binom{V(H)}{k-1} \} \). Note that in order to guarantee an $F$-decomposition, $H$ has to satisfy certain divisibility conditions, including $e(F)|e(H)$; for graphs, we also need that $\gcd(d_F(v)) : v \in V(F)$ divides $\gcd(d_H(v)) : v \in V(H)$ and whenever $H$ satisfies these conditions, we say that $H$ is $F$-divisible. For $k \geq 3$, the concept of $F$-divisibility is similar but more complex and we refer the reader to [4] for (many) more details.

In this paper, we consider Dirac-type (minimum degree) conditions for $k$-graphs that enforce $F$-decompositions. In particular, we are interested in the $F$-decomposition threshold $\delta_F$ which is defined as the least number $\delta$ such that for every $\mu > 0$, there exists an integer $n_0$ such that every $F$-divisible $k$-graph $H$ on $n \geq n_0$ vertices with $\delta(H) \geq (\delta + \mu)n$ admits an $F$-decomposition.

The $F$-decomposition threshold is very closely linked to the threshold for approximate decompositions and fractional decompositions, respectively. For $\eta \geq 0$, we say that an $\eta$-approximate $F$-decomposition of $H$ is an edge-disjoint collection of copies of $F$ that cover all but at most $\eta n k$ edges of $H$. Let $\delta_F^\eta$ be the least number $\delta$ such that for every $\mu > 0$, there exists an integer $n_0$ such that every $k$-graph $H$ on $n \geq n_0$ vertices with $\delta(H) \geq (\delta + \mu)n$ admits an $\eta$-approximate $F$-decomposition. Clearly, $\delta_F^\eta$ is monotonically increasing as $\eta$ tends to 0. Let $\delta_F^{\eta+} := \sup_{\eta>0} \delta_F^\eta$ be the approximate $F$-decomposition threshold.

We say that an assignment of weight to copies of $F$ in $H$ that places total weight 1 on every edge of $H$ is a fractional $F$-decomposition of $H$. Formally, using $\mathcal{F}(H)$ to denote the set of all copies of $F$ in $H$, a fractional $F$-decomposition of $H$ is a function $\omega : \mathcal{F}(H) \rightarrow [0, 1]$ with $\sum_{F \in \mathcal{F}(H) : e \in E(F)} \omega(F) = 1$ for all $e \in E(H)$. We define the fractional $F$-decomposition threshold $\delta_F^\omega$ as the least number $\delta$ such that for every $\mu > 0$, there exists an integer $n_0$ such that every $k$-graph $H$ on $n \geq n_0$ vertices with $\delta(H) \geq (\delta + \mu)n$ admits a fractional $F$-decomposition.

Haxell and Rödl [6] showed that $\delta_F^\omega \geq \delta_F^{\eta+}$ for graphs and later this was extended to $k$-graphs for arbitrary $k$ in [13] using the (hypergraph) regularity lemma. Therefore, $\delta_F^\omega \geq \delta_F^\omega \geq \delta_F^{\eta+}$ . However, crucially for many $F$ it is known that all three values coincide. This reduces the complexity of finding $F$-decompositions.

Let us discuss now some results that relate these different decomposition thresholds.

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1For the first inequality, we ignore the fact that for integral decompositions we in addition assume that $H$ is $F$-divisible. To avoid this, we could define the fractional/approximate decomposition threshold also only for $F$-divisible $k$-graphs. Since this technicality plays no role in this paper, we stick with the (natural) definitions as above.
Theorem 1.2. (Barber, Kühn, Lo, and Osthus [1]). Let $F$ be an $r$-regular graph. Then $\delta_F \leq \max\{\delta^{0+}_F, 1 - 1/(3r)\}$. Moreover, $\delta_{C_4} = 2/3$ and $\delta_{C_5} = 1/2$ for even $\ell \geq 6$. For odd $\ell$, we have $\delta_{C_\ell} = \delta^{0+}_{C_\ell}$.

Further significant progress on determining $\delta_F$ is due to Glock, Kühn, Lo, Montgomery, and Osthus [3]. They calculated the decomposition threshold for all bipartite graphs $F$; in particular, their results imply $\delta_F \in \{0, 1/2, 2/3\}$ in this case. For graphs $F$ with chromatic number $\chi \geq 5$, they proved that $\delta_F \in \{\delta^{*+}_F, 1 - 1/\chi, 1 - 1/(\chi + 1)\}$.

When considering $\delta_F$ for $k$-graphs $F$ with $k \geq 3$ we are immediately in deep waters. Only recently Keevash [8] verified the existence conjecture, that is, he showed that whenever the complete $k$-graph on $n$ vertices $K^k_n$ is $K^k$-divisible and $n$ is large enough, then $K^k_n$ has a $K^k$-decomposition. Glock, Kühn, Lo, and Osthus extended this by showing that $K^k$ can be replaced by any other $k$-graph $F$ [4]. In addition, they worked in a framework that yields bounds on $\delta_F$ that can be explicitly calculated.

Our contribution concerns (tight) cycles. A cycle $C^k_\ell$ of length $\ell$ is a $k$-graph on $\ell$ vertices whose vertex set can be cyclically ordered such that a $k$-set forms an edge of $C^k_\ell$ if and only if its elements appear consecutively in this ordering (when $k = 2$, we also write $C_\ell$ instead of $C^2_\ell$). Our main result is as follows.

Theorem 1.3. For every integer $k \geq 2$ and $\epsilon > 0$, there exists an $\ell_0$ such that $\delta^{*+}_{C_\ell} \leq 1/2 + \epsilon$ for all $\ell \geq \ell_0$.

Note that for graphs, by Theorem 1.2 and as $\delta_F \geq \delta^{*+}_F \geq \delta^{0+}_F$, we can replace $\delta^{*+}_{C_\ell}$ by $\delta^{0+}_{C_\ell}$ in Theorem 1.3. Our main result also answers a question (in a strong form) by Glock, Kühn, and Osthus [5] who asked whether the statement of the theorem holds for $k = 2$.

As we mentioned above, for $k \geq 3$ decomposition problems are much more complicated than for graphs and thus is it not surprising that beside the fact that $\delta^{0+}_F = 0$ whenever $F$ is $k$-partite almost nothing is known about the exact values of $\delta^{0+}_F$, $\delta^{*+}_F$, $\delta_F$. Our main result implies almost tight bounds on $\delta^{*+}_{C_\ell}$ and $\delta^{0+}_{C_\ell}$ because whenever $\ell$ is not divisible by $k$, we have $\delta^{*+}_{C_\ell} \geq 1/2 + \frac{1}{(k+1)(\ell-1)}$ (see Example 2.2 in Section 2). Observe that previously it was only known that $\delta^{*+}_F$ is at most a number very close to $1/2$.

In view of further applications we prove a more general statement than in Theorem 1.3. In many scenarios, we would like to find a (fractional/approximate) $F$-decomposition of a typical random thin edge slice of a $k$-graph that admits such an $F$-decomposition. Hence we need an assumption on the $k$-graph that is robust with respect to taking random edge slices and covers the minimum degree condition in Theorem 1.3. A property which has this ability is $(\alpha, \ell)$-connectedness, which we introduce below.

In addition, our fractional decompositions have the property that all weights on the $\ell$-cycles are equal up to constant factors. This is another useful property for applications.

For a $k$-graph $H$ with vertex set $V$ and edge set $E$ on $n$ vertices, we write $\tilde{E}(H)$ for the set of tuples $(v_1, \ldots, v_k) \in V^k$ such that $(v_1, \ldots, v_k) \in E$ and we define $\tilde{e}(H) := |\tilde{E}(H)|$. A walk $W$ of length $\ell$ in $H$ is a sequence $\tilde{e}_1 \ldots \tilde{e}_\ell$ of elements of $\tilde{E}(H)$ such that there is a sequence $v_1 \ldots v_{\ell+k-1}$ of vertices of $H$ with $\tilde{e}_i = (v_i, \ldots, v_{i+k-1})$ for all $i \in [\ell]$. We call $W$ a walk from $\tilde{e}_1$ to $\tilde{e}_\ell$. We say that $H$ is $(\alpha, \ell)$-connected if for all $\tilde{e}, \tilde{f} \in \tilde{E}(H)$, there are at least $an^{\ell-1}/\ell \tilde{e}(H)$ walks from $\tilde{e}$ to $\tilde{f}$ in $H$ that have length $\ell$.

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2Observe that decompositions into tight cycles directly yield decompositions into other types of $k$-graph cycles, including so-called loose cycles.
Theorem 1.4. For all $\alpha \in (0, 1)$, $\mu \in (0, 1/3)$ and integers $\ell, k \geq 2$, there is an $n_0$ such that the following holds for all $n \geq n_0$. Suppose $H$ is an $(\alpha, \ell_0)$-connected $k$-graph on $n$ vertices for some $\ell_0 \geq k + 1$ with $180k^\ell \alpha \log \frac{\ell_0}{a} \log \frac{1}{\mu} \leq \ell$. Then there is a fractional $C^k_\ell$-decomposition $\omega$ of $H$ with

$$(1 - \mu) \frac{2e(H)}{\Delta(H)^\ell} \leq \omega(C) \leq (1 + \mu) \frac{2e(H)}{\delta(H)^\ell}$$

for all $\ell$-cycles $C$ in $H$.

The fact that Theorem 1.4 indeed implies Theorem 1.3 is an immediate consequence of Lemma 2.3 which implies that $\delta(H) \geq \frac{1 + a}2 n$ yields $(a^{3\ell_1}, k^2)$-connectedness.

In another article together with Schülke [7], we employ Theorem 1.4 to prove a strong generalization of the well-known result due to Rödl, Ruciński, and Szemerédi [12]. We show that every $k$-graph $H$ on $n$ vertices with minimum degree $\delta(H) \geq (1/2 + o(1))n$ not only contains one Hamilton cycle but essentially as many edge-disjoint Hamilton cycles as $H$ may potentially have, namely, the largest $p$ for which $H$ has a spanning subgraph where every vertex is contained in $kp$ edges.

Next we give an overview of the proof of Theorem 1.4. When dealing with fractional decompositions many approaches so far have considered the strategy to start with the (appropriately scaled) uniform distribution on $F(H)$ and then turn this into a fractional decomposition by shifting some weight around the graph. In this case, the crucial question is in how many copies of $F$ does a particular edge $e$ lie because the total weight on $e$ will be proportional to this quantity. (From now on $F$ is a cycle of length $\ell'$.) However, when dealing with $k$-graphs of minimum degree close to $n/2$ this may vary significantly and starting with the uniform distribution on $F(H)$ seems hopeless to us.

Our main contribution is a novel approach to overcome this problem. In Section 3, we first find a set of $\ell$-cycles $C \subseteq F(H)$ such that every edge of $H$ lies in roughly the same number of elements of $C$. Consequently, a scaled uniform distribution on $C$ almost yields a fractional $F$-decomposition. In a final step (see Section 4), we adjust the weight distribution slightly and find the desired fractional decomposition. So how do we find $C$? This can be done as follows. Fix an ordered $(k - 1)$-set $x$. Among all edges that contain $x$, we introduce certain restrictions; that is, for some $u, v$ that form each together with $x$ an edge, we forbid that a cycle contains the vertices $u, x, v$ (or vice versa) in this order. We proceed similarly for each $x \in \binom{V(H)}{k - 1}$. Let $C$ be the set of all cycles that respect all these imposed restrictions.

The restrictions will be constructed in such way that for each $u, x$, there is always the same number of choices $r$, say, for the next vertex of the cycle. Consequently, starting with a $(k - 1)$-set $x$, there are almost exactly $r^m$ choices (we have to avoid repetitions) for the subsequent $m$ vertices.

Construct an auxiliary graph $J$ as follows; the vertex set of $J$ is the set of all ordered edges of $H$ and two vertices $\tilde{e}, \tilde{f}$ are adjacent if there is an ordering of $k + 1$ vertices of $H$ such that $\tilde{e}$ coincides with the first $k$ vertices and $\tilde{f}$ with the last $k$ vertices and our restrictions allow to transition from $\tilde{e}$ to $\tilde{f}$.

Now consider a simple random walk on $J$. Then the $m$-step transition probability from $\tilde{e}$ to $\tilde{f}$ multiplied by $r^m$ essentially equals the number of paths from $\tilde{e}$ to $\tilde{f}$. One of the key steps in our argument is that this random walk mixes rapidly. From this it is not hard to deduce that every edge of $H$ is contained in roughly the same number of cycles in $C$.

Let us also mention that we avoid the application of the arguably very technical hypergraph regularity lemma, which usually is applied when considering decomposition questions, and hence our proofs are comparably short and without much clutter.
2 | PRELIMINARIES

2.1 | Notation

For an integer $n \geq 1$, we define $[n] := \{1, \ldots, n\}$ and $[n]_0 := [n] \cup \{0\}$. For a set $A$, we say that $A$ is a $k$-set if $|A| = k$ and we define $\binom{A}{k} := \{A' \subseteq A : |A'| = k\}$. We often use $x, y$ to refer to sets and $\bar{x}, \bar{y}$ when considering tuples; however, if the tuple arises from ordering the vertices of an edge, then we use $\bar{e}, \bar{f}, \bar{g}, \bar{h}, \bar{s}, \bar{t}$. We may subsequently drop the arrow to denote the underlying set of an ordered set, so that for an ordered set $\bar{x} = (x_1, \ldots, x_k)$, we have $x = \{x_1, \ldots, x_k\}$. We generally identify a sequence $x_1 \ldots x_n$ with the tuple $(x_1, \ldots, x_n)$. An ordering of a $k$-set $x = \{x_1, \ldots, x_k\}$ is a sequence $x_1 \ldots x_k$ without repetitions.

For real numbers $\alpha, \beta, \delta, \delta'$, we write $\alpha = (1 + \delta)\beta$ whenever $(1 - \delta)\beta \leq \alpha \leq (1 + \delta)\beta$ and $(1 + \delta)\alpha = (1 + \delta')\beta$ whenever $(1 - \delta')\beta \leq (1 + \delta)\alpha \leq (1 + \delta')\beta$. We write $\alpha \ll \beta$ to mean that there is a nondecreasing function $\alpha_0 : (0, 1] \to (0, 1]$ such that for any $\beta \in (0, 1]$, the subsequent statement holds for $\alpha \in (0, \alpha_0(\beta)]$. Hierarchies with more constants are defined similarly and should be read from right to left. Constants in hierarchies will always be real numbers in $(0, 1]$. Moreover, if $1/n$ appears in a hierarchy, this implicitly means that $n$ is a natural number. We ignore rounding issues when they do not affect the argument.

Let $H$ be a $k$-graph on $n$ vertices, where $k \geq 2$. Whenever we use $k$ to refer to the uniformity of a hypergraph we tacitly assume that $k \geq 2$. We write $V(H)$ for the vertex set and $E(H)$ for the edge set of $H$.

A walk $W$ of length $\ell$, or simply an $\ell$-walk, in $H$ is a sequence $\bar{e}_1 \ldots \bar{e}_\ell$ of elements of $\bar{E}(H)$ such that there is a sequence $v_1 \ldots v_{\ell+(k-1)}$ of vertices of $H$ with $\bar{e}_i = (v_i, \ldots, v_{i+k-1})$ for all $i \in [\ell]$; we call this sequence of vertices the vertex sequence of $W$ and define $V(W) := \{v_1, \ldots, v_{\ell+(k-1)}\}$ and $E(W) := \{e_i : i \in [\ell]\}$. We say that $W$ is self-avoiding if $v_i \neq v_j$ for all distinct $i, j \in [\ell+(k-1)]$ and we say that $W$ is internally self-avoiding if $e_1 \ldots e_{\ell-k}$ and $e_{k+1} \ldots e_\ell$ are self-avoiding. We call $W$ closed if $\bar{e}_1 = \bar{e}_\ell$. We say a cycle $C$ is given by an internally self-avoiding closed walk $\bar{e}_1 \ldots \bar{e}_\ell$ if $E(C) = \{e_1, \ldots, e_\ell\}$. We write $C_\ell(H)$ for the set of $\ell$-cycles in $H$.

2.2 | Chernoff’s inequality

We will also need the following standard concentration inequality, which can be found in for example [10].

**Lemma 2.1.** Let $X_1, \ldots, X_n$ be independent Bernoulli random variables with parameter $p$ and let $X := \sum_{i \in [n]} X_i$. Then for any $\delta \in (0, 1]$,

$$\Pr[|X - E[X]| \geq \delta E[X]] \leq 2 \exp \left( -\frac{\delta^2}{3} E[X] \right).$$

2.3 | A lower bound on the (fractional) cycle decomposition threshold

The following example shows that the bound in Theorem 1.3 cannot be improved by much.

**Example 2.2.** Let $1/n \ll \zeta \ll 1/k, 1/\ell$, where $\ell \neq 0 \mod k$ and $n$ is even. Suppose $\varepsilon = \frac{1}{(k-1)+2/k(\ell-1)} - \delta$, let $A$ and $B$ be disjoint sets of size $n/2$, define $V := A \cup B$ and for $i \in [k]_0$, let $E_i$ be the set of edges $e$ of the complete $k$-graph on $V$ with $|e \cap A| = i$. Let $H_{0,2}$ denote the random $k$-graph
with vertex set $V$ whose edge set we obtain by including every edge in $E_0 \cup E_2$ independently at random with probability $2(1 + \zeta)\epsilon$. For all $x \in \left( \frac{V}{k-1} \right)$ with $|x \cap A| \leq 2$, we have

$$\mathbb{E}[d_{H_{0,2}}(x)] \geq 2(1 + \zeta)\epsilon \left( \frac{n}{2} - k \right) \geq \frac{\epsilon n + k}{1 - \zeta^2}$$

and

$$\zeta n^k \leq \mathbb{E}[e(H_{0,2})] = 2(1 + \zeta)\epsilon \left( \left( \frac{n/2}{k} \right) + \left( \frac{n/2}{k-2} \right) \right) \leq \frac{1 + 2\zeta}{1 + \zeta^2} \epsilon \left( k - 1 + \frac{2}{k} \right) \frac{n^k}{2^{k}(k-1)!} \leq \frac{|E_1| - \delta^2 n^k}{(1 + \zeta^2)(\ell - 1)}.$$  

Thus, Lemma 2.1 yields

$$\mathbb{P}[d_{H_{0,2}}(x) < \epsilon n + k] \leq \exp(-\zeta^5n) \quad \text{and} \quad \mathbb{P} \left[ e(H_{0,2}) \geq \frac{|E_1| - \delta^2 n^k}{\ell - 1} \right] \leq \exp(-\zeta^6n)$$

and a union bound over all $x \in \left( \frac{V}{k-1} \right)$ with $|x \cap A| \leq 2$ shows that with positive probability, we have $d(x) \geq \epsilon n + k$ for all $x \in \left( \frac{V}{k-1} \right)$ with $|x \cap A| \leq 2$ as well as $e(H_{0,2}) < \frac{|E_1| - \delta^2 n^k}{\ell - 1}$. Now suppose $H_{0,2}$ is a $k$-graph with these two properties and let $H := (V, E_{H_{0,2}}) \cup \bigcup_{i \in [k]\setminus[2]} E_i$. For all $x \in \left( \frac{V}{k-1} \right)$ with $|x \cap A| \leq 2$, there are at least $\frac{n}{2} - k$ edges $e \in E(H) \setminus E(H_{0,2})$ with $x \subseteq e$ and for all $x \in \left( \frac{V}{k-1} \right)$ with $|x \cap A| \geq 3$, there are $n - k$ edges $e \in E(H) \setminus E(H_{0,2})$ with $x \subseteq e$. Hence $\delta(H) \geq \left( \frac{1}{2} + \epsilon \right)n$.

Note the following: Suppose $v_1 \ldots v_\ell$ is the vertex sequence of a walk in $(V, E_1)$, then $v_j \in A$ implies $v_j \not\in A$ if $i < j < i + k$ as well as $v_j \in A$ if $j = i + k$. Thus, since $\ell \not\equiv 0 \mod k$, every $\ell$-cycle in $H$ that has an edge in $E_1$ must have at least one edge that is an edge of $H_{0,2}$. For every weight assignment $\omega : C_{\ell}(H) \to \{0, 1\}$ with $\sum_{C \in C_{\ell}(H)}:e\in E(C) \omega(C) \leq 1$ for all $e \in E(H)$, this implies

$$\sum_{e \in E_{1}} \sum_{\substack{C \in C_{\ell}(H): \{e\} \not= \emptyset}} \omega(C) \leq (\ell - 1) \sum_{C \in C_{\ell}(H): E(C) \cap E_{1} \not= \emptyset} \omega(C) \leq (\ell - 1) \sum_{e \in E(H_{0,2})} \sum_{\substack{C \in C_{\ell}(H): \{e\} \not= \emptyset}} \omega(C) \leq (\ell - 1) e(H_{0,2}) < |E_1| - \delta^2 n^k.$$  

Thus, for every such weight assignment $\omega$, there is an edge $e \in E_1 \subseteq E(H)$ with total weight $\sum_{C \in C_{\ell}(H)}:e\in E(C) \omega(C)$ less than 1 which shows that $H$ does not admit a fractional $C^k_{\ell}$-decomposition. Furthermore, if additionally $\omega$ only takes values in $\{0, 1\}$, there are more than $\delta^2 n^k$ edges $e \in E_1 \subseteq E(H)$ with $\sum_{C \in C_{\ell}(H)}:e\in E(C) \omega(C) = 0$, so $H$ does not admit a $\delta^2$-approximate decomposition either.
2.4 Intersecting hypergraphs

For \( x \in \left( \frac{V(H)}{k-1} \right) \), we define \( N_H(x) := \{ v \in V(H) : x \cup \{ v \} \in E(H) \} \). Let us call a \( k \)-graph \( H \) on \( n \) vertices \( \alpha \)-intersecting if \( |N_H(x) \cap N_H(y)| \geq \alpha n \) holds for all \( x, y \in \left( \frac{V(H)}{k-1} \right) \); in particular \( H \) is \( \alpha \)-intersecting if \( \delta(H) \geq \frac{1+\alpha}{2} n \). Theorem 1.4 applies for (\( \alpha, \ell' \))-connected \( k \)-graphs, a property which is sometimes slightly complicated to verify. We show in Lemma 2.3 that all \( \alpha \)-intersecting \( k \)-graphs are \( (\alpha^{3k}, k^2 - k + 2) \)-connected. The statement is in fact already proved by Rödl, Ruciński, and Szemerédi in [12] with a weaker dependence of the parameters and a fairly long proof. The idea of the short proof below is due to Reiher (unpublished) (B. Schülke [personal communication]) and we give it here to make it publicly available.

**Lemma 2.3.** Let \( 1/n \ll \alpha, 1/k \). Suppose \( H \) is an \( \alpha \)-intersecting \( k \)-graph on \( n \) vertices. Then \( H \) is \((\alpha^{3k}, k^2 - k + 2)\)-connected.

**Proof.** We need to prove that for all \( \tilde{s}, \tilde{t} \in \widetilde{E}(H) \), there are at least \( \alpha^{3k}n^{k^2-k+2}/\ell(H) \) walks from \( \tilde{s} \) to \( \tilde{t} \) in \( H \). To this end, we use induction on \( k \) to show the following.

Suppose \( a_k = (k-1)!\sum_{i=0}^{k-2}\frac{i+1}{n} \), \( \ell'_k = k^2 - k + 2 \) and \( H \) is an \( \alpha \)-intersecting \( k \)-graph on \( n \) vertices. Then the number of \( \ell'_k \)-walks from \( \tilde{s} \) to \( \tilde{t} \) in \( H \) is at least \( \alpha a_k n^{\ell'_k-k-1} \) for all \( \tilde{s}, \tilde{t} \in \widetilde{E}(H) \).

This suffices because for all \( k \geq 2 \) and every \( \alpha \)-intersecting \( k \)-graph \( H \) on \( n \) vertices (where \( n \) is sufficiently large), we have \( \ell(H) \geq \delta(H) \prod_{i \in [k-2]}(n-i) \geq \alpha^2 n^k \) and \( a_k \leq k!(\sum_{i=0}^{\infty}1/i!) - 2 \leq 3k! - 2 \) and hence \( \alpha^{3k}n^{k^2-k-1}/\ell(H) \leq \alpha a_k n^{\ell'_k-k-1} \). With some foresight we remark that \( a_k = (k-1)(a_{k-1} + k-1) \) holds for all \( k \geq 3 \).

For \( k = 2 \), we have \( \ell'_2 = 4 \) and \( a_k = 1 \) and thus the desired statement is a direct consequence of the fact that \( H \) is \( \alpha \)-intersecting as this guarantees that any two vertices of \( H \) have at least \( \alpha n \) common neighbors.

Suppose now that \( k \geq 3 \). Let \( V := V(H) \) and \((s_1, \ldots, s_k), (t_1, \ldots, t_k) \in \widetilde{E}(H) \). Let \( \tilde{s} := (s_1, \ldots, s_k) \) and \( \tilde{t} := (t_1, \ldots, t_k) \). For \( z \in V \), consider the indicator function \( 1_z \) defined on \( V^{(k-1)(k-2)} \) where \( 1_z(\tilde{x}) = 1 \) if and only if \( z \subseteq \tilde{x} \) and \( \tilde{x} \) is the vertex sequence of a \((k-1)^2 \)-walk in the link \( L_z \) of \( z \), that is the \((k-1)\)-graph \( L_z \) with vertex set \( V \setminus \{ z \} \) where \( (k-1)\)-set \( e \subseteq V \setminus \{ z \} \) is an edge of \( L_z \) if and only if \( e \cup \{ z \} \) is an edge of \( H \). Observe that for \( z \in V \) and \( \tilde{x} \in V^{(k-1)(k-2)} \) with \( 1_z(\tilde{x}) = 1 \), we have \( z \not\in \tilde{x} \). Let

\[
N := N_H(s_2, \ldots, s_k) \cap N_H(t_1, \ldots, t_{k-1}).
\]

Note that for all \( \tilde{x} = (x_1, \ldots, x_{k-1}(k-2)) \in V^{(k-1)(k-2)} \), we can obtain the vertex sequence \( \tilde{xy}^{-1} \) of an \( \ell'_k \)-walk in \( H \) by inserting vertices \( z \in N \) with \( 1_z(\tilde{x}) = 1 \) into \( \tilde{xy}^{-1} \) every \((k-1)\) vertices, that is by choosing not necessarily distinct \( z_1, \ldots, z_{k-1} \in N \) with \( 1_z(\tilde{x}) = 1 \) for all \( i \in [k-1] \) and defining \( y = y_1 \ldots y_{k-1} \) as the sequence with \( y_{k-1-i} = z_{k-1-i} \) for all \( i \in [k-1] \) that has \( \tilde{x} \) as a subsequence. For every \( \tilde{x} \in V^{(k-1)(k-2)} \), the number of \( \ell'_k \)-walks from \( \tilde{s} \) to \( \tilde{t} \) in \( H \) given by such insertion constructions is \((\sum_{z \in N} 1_z(\tilde{x}))^{k-1} \). For all \( z \in V \), the link \( L_z \) of \( z \) is \( \alpha \)-intersecting as this has minimum degree \( \delta(L_z) \geq \alpha n \). Hence, the number of \((k-1)\)-walks in \( L_z \) starting at \( s_2, \ldots, s_k \) is at least \( \alpha^{k-2}n^{k-2} \) and the induction hypothesis implies that for all \((k-1)\)-tuples \( s' \) such a walk ends at, the number of \( \ell'_{k-1} \)-walks from \( \tilde{s}' \) to \((t_1, \ldots, t_{k-1}) \) in \( L_z \) at least \( \alpha^2 a_{k-1} n^{\ell'_{k-1}-k} \), which shows that the number of \((\ell'_{k-1}+k-2)\)-walks from \( s_2, \ldots, s_k \) to \((t_1, \ldots, t_{k-1}) \) in \( L_z \), coincides with \( \sum_{\tilde{x} \in V^{(k-1)(k-2)}} 1_z(\tilde{x}) \),...
is at least \(d^{-2}n^{-2} \cdot a^{\alpha_{k-1} + k-2} n^{(k-1)(k-2)}\). Since \(x \mapsto x^{k-1}\) is convex, we conclude with Jensen’s inequality that the number of \(\ell\)-walks starting from \(\vec{s}\) to \(\vec{t}\) in \(H\) is at least

\[
\sum_{\vec{x} \in \mathcal{V}(\vec{s})} \left( \sum_{\vec{v} \in \mathcal{N}} 1_{\vec{v}}(\vec{x}) \right)^{k-1} \geq \sum_{\vec{x} \in \mathcal{V}(\vec{s})} n^{(k-1)(k-2)} \left( \sum_{\vec{v} \in \mathcal{N}} 1_{\vec{v}}(\vec{x}) \right)^{k-1} \geq n^{(k-1)(k-2)} \cdot \alpha n \cdot a^{\alpha_{k-1} + k-2} n^{(k-1)(k-2)} = a^{\alpha} n^{\ell-1}.
\]

which completes the proof.

\[\blacksquare\]

3 | ROUGH WEIGHT ASSIGNMENT

The key insight for the proof of Theorem 1.4 is the application of Markov chain limit theory with the purpose of estimating the number of cycles that contain a fixed edge and obey well-chosen conditions. In particular, we exploit that certain Markov chains are rapidly mixing; that is, the speed of convergence to the limit distribution is exponential. In order to easily relate a Markov chain with the number of walks starting at an edge we restrict our set of admissible walks by what we call transition systems.

Suppose \(H\) is a \(k\)-graph with vertex set \(V\). For \(\vec{x} \in \left( \begin{array}{c} V \\ k-1 \end{array} \right)\), a transition graph of \(H\) at \(\vec{x}\) is a graph with vertex set \(\{ e \in E(H) : \vec{x} \subseteq e \}\). A transition system of \(H\) is a family \(\mathcal{T} = (T_{\vec{x}})_{\vec{x} \in \left( \begin{array}{c} V \\ k-1 \end{array} \right)}\) where \(T_{\vec{x}}\) is a transition graph of \(H\) at \(\vec{x}\). A walk in \(H\) with vertex sequence \(v_1 \ldots v_{\ell+k-1}\) is \(\mathcal{T}\)-compatible if \(\{v_i, \ldots, v_{i+k-1}\}\) and \(\{v_{i+1}, \ldots, v_{i+k}\}\) are adjacent in \(T_{\{v_{i+1}, \ldots, v_{i+k}\}}\) for all \(i \in [\ell-1]\) and if \(C\) is a cycle, then for \(C\) to be \(\mathcal{T}\)-compatible we require that \(C\) is given by a \(\mathcal{T}\)-compatible internally self-avoiding closed walk. We say that \(\mathcal{T}\) is \(r\)-regular if \(T_{\vec{x}}\) is \(r\)-regular for all \(\vec{x} \in \left( \begin{array}{c} V \\ k-1 \end{array} \right)\) and we say that \(\mathcal{T}\) is regular if \(\mathcal{T}\) is \(r\)-regular for some \(r\). We will often choose \(\mathcal{T}\) randomly; to be more precise, we say that \(\mathcal{T}\) is a random \(r\)-transition system if for \(\vec{x} \in \left( \begin{array}{c} V \\ k-1 \end{array} \right)\), the graph \(T_{\vec{x}}\) is an \(r\)-regular transition graph of \(H\) at \(\vec{x}\) chosen independently and uniformly at random among all \(r\)-regular subgraphs of the complete graph with vertex set \(\{ e \in E(H) : \vec{x} \subseteq e \}\). Note that graphs with transition systems already have been studied, often with an algorithmic perspective (see, e.g., [14]).

For a given \((\alpha, \ell)\)-connected \(k\)-graph \(H\) on \(n\) vertices, the first step in the proof of Theorem 1.4 is the choice of a transition system \(\mathcal{T}\) such that every edge of \(H\) is contained in approximately equally many \(\mathcal{T}\)-compatible cycles in \(H\). Lemmas 3.1 and 3.3 imply that this happens with high probability when \(\mathcal{T}\) is a random \(r\)-transition system. We prove this as follows. Lemma 3.1 verifies that \((\alpha, \ell)\)-connectedness is maintained (with high probability) when moving to a random transition system provided the parameter \(\alpha\) is appropriately lowered. In Lemma 3.2, we prove that certain Markov chains are rapidly mixing and apply this in Lemma 3.3 to prove the desired result about cycle counts.

We say that a \(k\)-graph \(H\) on \(n\) vertices with \(r\)-regular transition system \(\mathcal{T}\) is \(\mathcal{T}\)-compatibly \((\alpha, \ell)\)-connected if the number of \(\mathcal{T}\)-compatible \(\ell\)-walks from \(\vec{s}\) to \(\vec{t}\) is at least \(ar^{\ell-1}/\ell!\) for all \(\vec{s}, \vec{t} \in \vec{E}(H)\).

**Lemma 3.1.** Let \(1/n \ll \alpha, \ell, 1/k, 1/\ell\). Let \(H\) be an \((\alpha, \ell)\)-connected \(k\)-graph on \(n\) vertices, \(r\) an even integer with \(\epsilon n \leq r \leq \delta n\) and suppose \(\mathcal{T}\) is a random \(r\)-transition system of \(H\). Then \(H\) is \(\mathcal{T}\)-compatibly \((\alpha/2, \ell)\)-connected with probability at least \(1 - \exp(-\sqrt{n})\).
Proof. We will show that for sufficiently many suitably chosen disjoint sets of $\ell$-walks in $H$ of size $n^{3/4}$, the number of $T$-compatible walks in these sets is not much lower than their expected value with very high probability. A union bound then finishes the argument.

To this end, fix tuples $\vec{s} = (s_1, \ldots, s_k)$ and $\vec{t} = (t_1, \ldots, t_k)$ in $\vec{E}(H)$. Let $\mathcal{W}$ denote the set of the internally self-avoiding $\ell$-walks from $\vec{s}$ to $\vec{t}$ in $H$. The number of $\ell$-walks from $\vec{s}$ to $\vec{t}$ in $H$ that are not internally self-avoiding is at most $\ell^{2n^{3/4}/\alpha_2} + \frac{1}{2} \alpha n^{\ell-1}/\alpha_2$ and thus $|\mathcal{W}| \leq \frac{3}{4} \alpha n^{\ell-1}/\alpha_2$ holds.

Let $\mathcal{V}_1 = \{ \mathcal{W} : \mathcal{V} \in \mathcal{W} \}$ and $\mathcal{V}_2 = \{ \mathcal{W} : \mathcal{V} \notin \mathcal{W} \}$. We wish to obtain pairwise disjoint $\mathcal{V}_1, \ldots, \mathcal{V}_p$ of internally self-avoiding $\ell$-walks from $\vec{s}$ to $\vec{t}$ in $H$ such that for all $i \in [p]$ and $\mathcal{W}, \mathcal{W}' \in \mathcal{V}_i$, the internally self-avoiding walks $\mathcal{W}$ and $\mathcal{W}'$ share only the vertices given by their first and last edge, that is we have $V(\mathcal{W}) \cap V(\mathcal{W}') = s \cup t$. Observe that $|\mathcal{W}| - pq \geq \frac{1}{12} \alpha n^{\ell-1}/\alpha_2$. Note that for all subsets $\mathcal{V}' \subseteq \mathcal{V}$ of size at most $q$, there are at most $\ell^{2q n^{3/4}/\alpha_2} \leq n^{\ell-9/8}/\alpha_2 < \frac{1}{12} \alpha n^{\ell-1}/\alpha_2$ walks $\mathcal{W} \in \mathcal{V}$ with $V(\mathcal{W}) \cap V(\mathcal{W}') \neq s \cup t$ for at least one walk $\mathcal{W}' \in \mathcal{V}'$. These two observations imply that there are sets $\mathcal{V}_1, \ldots, \mathcal{V}_p$ as described above (we may simply greedily construct them).

A union bound over these sets shows that it suffices to obtain

$$\mathbb{P}\left[ \{|\mathcal{W}' \in \mathcal{V} : \mathcal{W}' \text{ is } T\text{-compatible}\}| \leq \frac{3}{4} \left( \frac{r}{n} \right)^{\ell-1} q \right] \leq \exp(-n^{2/3}) \quad (3.1)$$

for all $\mathcal{V}' \in \{ \mathcal{V}_1, \ldots, \mathcal{V}_p \}$. To this end, fix $\mathcal{V}' \in \{ \mathcal{V}_1, \ldots, \mathcal{V}_p \}$. For $\mathcal{W} = \vec{e}_1 \ldots \vec{e}_\ell \in \mathcal{V}'$ and $i \in [\ell - 1]$, let $X_{W,i}$ be the indicator random variable of the event that $\vec{e}_i \vec{e}_{i+1}$ is $T$-compatible. From the fact that the walks in $\mathcal{V}'$ are internally self-avoiding, the symmetry of the complete graphs whose vertices are the edges of $H$ that contain $\{s_2, \ldots, s_k\}$ and $\{t_2, \ldots, t_k\}$, respectively, $q \leq 2n^{3/4}$ and $r \geq \varepsilon n$, we obtain

$$\mathbb{P}\left[ X_{W,1} = X_{W,\ell-1} = 1 \left| (X_{W,i})_{W' \in W' \setminus W_i} \right. \right] = \mathbb{P}\left[ X_{W,1} = X_{W,\ell-1} = 1 \left| (X_{W,i})_{W' \in W' \setminus W_i} \right. \right] \geq \frac{5}{6} \left( \frac{r}{n} \right)^{\ell-1}$$

for all $\mathcal{W} \in \mathcal{V}'$. For $\mathcal{W} \in \mathcal{V}'$, let $X_{W}$ be the indicator random variable of the event that $\mathcal{W}$ is $T$-compatible. Since the walks in $\mathcal{V}'$ are internally self-avoiding and since we have $V(W_1) \cap V(W_2) \subseteq s \cup t$ for all $W_1, W_2 \in \mathcal{W}$, we conclude that

$$\mathbb{P}[X_W = 1 | (X_{W'})_{W' \in W' \setminus W}] \geq \frac{5}{6} \left( \frac{r}{n} \right)^{\ell-1}$$

holds for all $\mathcal{W} \in \mathcal{V}'$. This shows that $\sum_{W \in \mathcal{W}} X_W$ stochastically dominates a binomial random variable with parameters $q$ and $\frac{4}{5} \left( \frac{r}{n} \right)^{\ell-1}$ and implies (3.1) by using Chernoff’s inequality (see Lemma 2.1). □

For the next statement, we introduce some very basic terminology from Markov chain theory. For further explanations, see [10]. For a set $S = \{s_1, \ldots, s_n\}$ of size $n$ and a real matrix $P = (p_{ij})_{i,j \in [n]}$, we say that a sequence of random variables $X = (X_t)_{t \in N_0}$ is a Markov chain with state space $S$ and transition matrix $P$ if

$$\mathbb{P}[X_{t+1} = s_{i_{t+1}} | X_0 = s_{i_0}, \ldots, X_t = s_{i_t}] = \mathbb{P}[X_{t+1} = s_{i_{t+1}} | X_t = s_{i_t}] = p_{i_t,i_{t+1}}$$

holds for all $t \in N_0$ and $i_0, \ldots, i_{t+1} \in [n]$. We say that the stationary distribution of $X$ is given by $(\sigma_i)_{i \in [n]} \in \mathbb{R}^n$ if we have $\sum_{i \in [n]} \sigma_i = 1$ and $\sum_{i \in [n]} p_{ij} \sigma_i = \sigma_j \geq 0$ for all $j \in [n]$. 
Lemma 3.2. Let \( (X_t)_{t \in \mathbb{N}} \) be a Markov chain with state space \( \{s_1, \ldots, s_n\} \), transition matrix \( (p_{ij})_{i,j \in [n]} \) with \( p_{ij} \neq 0 \) for all \( i,j \in [n] \) and (unique) stationary distribution given by \( (\sigma_i)_{i \in [n]} \) with \( \sigma_i \neq 0 \) for all \( i \in [n] \). Let \( \alpha := \min_{i,j,k \in [n]} \frac{p_{ij}}{\sigma_k} \) and \( \beta := \max_{i,j,k \in [n]} \frac{p_{ij}}{\sigma_k} \). Then

\[
\mathbb{P}[X_t = s_i] = (1 \pm (1 - \alpha/2)^t)\sigma_i
\]

holds for \( t \geq 2 + 2\alpha^{-1} \log \beta \).

Proof. For \( t \geq 0 \) and \( j \in [n] \), let \( \delta^j_t := \mathbb{P}[X_t = s_j]/\sigma_j \). We start with finding a lower and an upper bound for \( \delta^j_{t+1} \) in terms of \( \alpha, \beta \) and \( \delta^1_t, \ldots, \delta^n_t \). Then we use these bounds and induction on \( t \) to prove the desired statement.

For \( t \geq 0 \), let \( \delta^i_{t,\min} := \min_{j \in [n]} \delta^j_t \). For all \( j \in [n] \), we may write

\[
\mathbb{P}[X_{t+1} = s_j] = \sum_{i \in [n]} p_{ij}(\delta^i_{t,\min} \sigma_i + \mathbb{P}[X_t = s_i] - \delta^i_{t,\min} \sigma_i) = \delta^i_{t,\min} \mathbb{P}[X_t = s_i] + \sum_{i \in [n]} p_{ij}(\mathbb{P}[X_t = s_i] - \delta^i_{t,\min} \sigma_i).
\]

This yields

\[
\delta^j_{t+1} \geq \delta^i_{t,\min} + \sum_{i \in [n]} \alpha(\mathbb{P}[X_t = s_i] - \delta^i_{t,\min} \sigma_i) = \delta^i_{t,\min} + \alpha - \alpha \delta^i_{t,\min} = 1 - (1 - \alpha)(1 - \delta^i_{t,\min})
\]

(3.2)

and

\[
\delta^j_{t+1} \leq \delta^i_{t,\min} + \sum_{i \in [n]} \beta(\mathbb{P}[X_t = s_i] - \delta^i_{t,\min} \sigma_i) = \delta^i_{t,\min} + \beta - \beta \delta^i_{t,\min} = 1 + (\beta - 1)(1 - \delta^i_{t,\min}).
\]

(3.3)

We claim that

\[
\delta^i_t \geq 1 - (1 - \alpha)^t.
\]

(3.4)

holds for all \( t \geq 0 \) and \( i \in [n] \). Indeed, it holds for \( t = 0 \). Suppose (3.4) holds for some \( t_0 \geq 0 \) (and all \( i \in [n] \)). Then this implies \( \delta^i_{t_0,\min} \geq 1 - (1 - \alpha)^0 = 1 - \alpha \) and thus (3.2) implies that (3.4) holds for \( t = t_0 + 1 \) and all \( i \in [n] \). By induction, (3.4) holds for all \( t \geq 0 \) and \( i \in [n] \).

Now (3.3), (3.4) and the observation that \( \beta \geq 1 \) imply

\[
\delta^i_t \leq 1 + (\beta - 1)(1 - \delta^i_{t-1,\min}) \leq 1 + (\beta - 1)(1 - \alpha)^{t-1}
\]

for all \( t \geq 1 \) and \( i \in [n] \). If \( t \geq 2 + 2\alpha^{-1} \log \beta \geq 2 + \log(1/\beta)/\log(1 - \alpha/2) \), then \( (\beta - 1)(1 - \alpha)^{t-1} \leq \beta(1 - \alpha/2)^{2t-2} \leq (1 - \alpha/2)^t \).

\[\blacksquare\]

For \( \zeta \in (0, 1) \), we say that a \( k \)-graph \( H \) on \( n \) vertices with \( r \)-regular transition system \( \mathcal{T} \) is \( \zeta \)-exactly \( \mathcal{T} \)-compatibly \( \ell \)-connected if for all \( \bar{s}, \bar{t} \in \bar{E}(H) \), the number of \( \mathcal{T} \)-compatible \( \ell \)-walks from \( \bar{s} \) to \( \bar{t} \) in \( H \) is \( (1 \pm \zeta)\ell^{\ell - 1}/e(H) \).

Lemma 3.3. Let \( 1/n \ll \epsilon, 1/k \). Suppose \( H \) is a \( k \)-graph on \( n \) vertices and \( \mathcal{T} \) is an \( r \)-regular transition system of \( H \) with \( r \geq \epsilon n \) such that \( H \) is \( \mathcal{T} \)-compatibly \( (\alpha, \ell_0) \)-connected. Then \( H \) is \( (1 - \frac{\alpha}{3\ell_0})^\epsilon \)-exactly \( \mathcal{T} \)-compatibly \( \ell \)-connected for all \( \ell \geq 3k\ell_0 \log(2/\epsilon)/\alpha \).
Proof. This is essentially a special case of Lemma 3.2. Fix \( \ell' \) as above and let \( \bar{s}, \bar{t} \in \bar{E}(H) \) and \( w \) be the number of \( \mathcal{T} \)-compatible \( \ell' \)-walks from \( \bar{s} \) to \( \bar{t} \) in \( H \). First, we argue why we essentially may assume that \( \ell' - 1 \) is a multiple of \( \ell_0 - 1 \). Let \( \ell'' \) be the remainder in the division of \( \ell' - 1 \) by \( \ell_0 - 1 \). The number of \( \mathcal{T} \)-compatible \((\ell'' + 1)\)-walks starting at \( \bar{s} \) is \( r^{\ell''} \). Thus it suffices to show that for every \( \bar{s}' \) that is the last element of such a walk, or more generally every \( \bar{s}' \in \bar{E}(H) \), the number of \((\ell' - \ell'')\)-walks from \( \bar{s}' \) to \( \bar{t} \) is \((1 \pm (1 - \frac{a}{2\ell_0})^{\ell' - \ell''}) r^{\ell' - \ell'' - 1}/\bar{e}(H) \) (note that \((1 - \frac{a}{2\ell_0})^{\ell' - \ell''} \leq (1 - \frac{a}{2\ell_0})^{\ell'} \)). Therefore, from now on we assume \((\ell_0 - 1)(\ell' - 1) \) and show that \( w = (1 \pm (1 - \frac{a}{2\ell_0})^{\ell'} r^{\ell' - 1}/\bar{e}(H) \) holds.

Consider a \( \mathcal{T} \)-compatible simple random walk \((\bar{E}_t)_{t \in \mathbb{N}_0} \) on \( \bar{E}(H) \) starting at \( \bar{s} \), that is we have \( \bar{E}_0 = \bar{s} \) and \( \bar{E}_t \) with \( t \geq 1 \) is iteratively chosen uniformly at random among all elements \( \bar{e} \in \bar{E}(H) \) for which \( \bar{E}_0 \ldots \bar{E}_{t-1}\bar{e} \) is a \( \mathcal{T} \)-compatible walk in \( H \). For all \( \bar{e}, \bar{f} \in \bar{E}(H) \), the number of \( \mathcal{T} \)-compatible \( \ell_0 \)-walks from \( \bar{e} \) to \( \bar{f} \) is at least \( ar^{\ell_0-1}/\bar{e}(H) \) and since the number of \( \mathcal{T} \)-compatible \( \ell_0 \)-walks starting at \( \bar{e} \) is \( r^{\ell_0-1} \), we conclude that for all \( t \geq 0 \), we have

\[
P[\bar{E}_{t+\ell_0-1} = \bar{f}|\bar{E}_t = \bar{e}] \geq \frac{ar^{\ell_0-1}}{\bar{e}(H)r^{\ell_0-1}} \geq \frac{a}{\bar{e}(H)}
\]

and similarly, since the number of \( \mathcal{T} \)-compatible \( \ell_0 \)-walks from \( \bar{e} \) to \( \bar{f} \) in \( H \) is at most \( r^{\ell_0-1-k} \), we obtain

\[
P[\bar{E}_{t+\ell_0-1} = \bar{f}|\bar{E}_t = \bar{e}] \leq \frac{r^{\ell_0-1-k}}{r^{\ell_0-1}} \leq \frac{1}{\varepsilon h^k} \leq \frac{1}{\varepsilon^k \bar{e}(H)}.
\]

Since \((\bar{E}_t)_{t \in \mathbb{N}_0} \) is a Markov chain, \((\bar{E}_{(\ell_0-1)t})_{t \in \mathbb{N}_0} \) is also a Markov chain. Moreover, they have the same stationary distribution and since \( \mathcal{T} \) is regular, this stationary distribution is given by \((1/\bar{e}(H))_{\bar{e} \in \bar{E}(H)} \). Note that the given lower bound for \( \ell \) implies \( (\ell_0 - 1)/\ell_0 \geq \frac{2\log(a^4)}{a} + 2 \). Thus applying Lemma 3.2 to \((\bar{E}_{(\ell_0-1)t})_{t \in \mathbb{N}_0} \) yields

\[
w = P[\bar{E}_{\ell_0-1} = \bar{t}]r^{\ell_0-1} = (1 \pm (1 - \alpha/2)^{\ell_0-1}) (1 - \frac{\alpha}{2\ell_0})^{\ell_0} \leq (1 - \frac{\alpha}{2\ell_0})^{\ell'}.
\]

which implies the desired bounds because \((1 - \frac{\alpha}{2\ell_0})^{\ell_0} \leq (1 - \frac{\alpha}{2\ell_0})^{\ell'} \).

4 | ADJUSTMENTS TO THE INITIAL WEIGHT DISTRIBUTION

Suppose \( H \) is an \((a, \ell_0)\)-connected \( k \)-graph on \( n \) vertices. Lemmas 3.1 and 3.3 ensure the existence of a transition system \( \mathcal{T} \) of \( H \) such that every edge of \( H \) is contained in approximately the same number of \( \mathcal{T} \)-compatible \( \ell \)-cycles in \( H \) whenever \( \ell \) is large in terms of \( \ell_0 \). Placing the same weight on every such cycle yields an initial weight distribution \( \omega_0 \) on the edges of \( H \) such that the weight \( \sum \in\mathcal{C}(H) \in\mathcal{E}(H) \omega_0(e) \) on the edge \( e \) is roughly the same for all \( e \in \mathcal{E}(H) \), say after suitable normalization, approximately 1. This section comprises the description of the refinement of \( \omega_0 \) by incremental modifications until we obtain a fractional \( C^\mathcal{T} \)-decomposition. These modifications mimic sending small fractions of weight from one edge to another and are realized by manipulating the weights assigned to cycles that form certain substructures of \( H \), which we call transporters.

For a \( k \)-tuple \( \bar{e} = (e_1, \ldots, e_k) \) and an integer \( i \geq 0 \), we define \( \bar{e}^{(i)} := (e_{i+1}, \ldots, e_{i+k}) \) with indices considered modulo \( k \). Let \( \ell \geq 4 \). For \( \bar{s}, \bar{t} \in \bar{E}(H) \) with \( s \cap t = \emptyset \), a closed \((k\ell + 1)\)-walk \( \bar{e}_0 \ldots \bar{e}_{k\ell} \) in \( H \) is an \((\ell - 1, \bar{s}, \bar{t})\)-transporter in \( H \) if
We say that a closed walk \( T \) in \( H \) is an \((\ell' - 1)\)-transporter in \( H \) if there are \( s, t \in \tilde{E}(H) \) such that \( T \) is an \((\ell' - 1, s, t)\)-transporter in \( H \). For these and the following definitions it is notationally more convenient to consider \((\ell' - 1)\)-transporters, however in the proof of Lemma 4.2 we will use \( \ell' \)-transporters instead.

Let \( T = \tilde{e}_0 \ldots \tilde{e}_{k} \) be an \((\ell' - 1, (s_1, \ldots, s_k), (t_1, \ldots, t_k))\)-transporter in \( H \). Observe that property (i) implies that for all \( i \in [k - 1]_0 \), there are \( u, v, u', v' \in V(H) \) such that

\[
\begin{align*}
\tilde{e}_{i+1} = (s_i, t_{i+1}, \ldots, s_{i+k}) & \quad \text{and} \quad \tilde{e}_{i+1} + \ell' = (s_{i+1}, \ldots, s_{i+k}, v') \\
\tilde{e}_{i+1} = (u', s_{i+2}, \ldots, s_{i+k}) & \quad \text{and} \quad \tilde{e}_{i+1} + \ell' = (s_{i+2}, \ldots, s_{i+k}, v')
\end{align*}
\]

with indices of \( \tilde{e} \) considered modulo \( k\ell' \) and indices of \( s \) and \( t \) considered modulo \( k \). This allows the following definition of two crucial sets of cycles associated with \( T \) as illustrated in Figure 1: The sending cycles of \( T \) are the \((\ell' - 1)\)-cycles given by the internally self-avoiding closed walks

\[
\tilde{e}_{i+1} \ldots \tilde{e}_{i+1+\ell'/2} \ldots \tilde{e}_{i+1+\ell'/2+1} \ldots \tilde{e}_{i+1}
\]

and the receiving cycles of \( T \) are the \((\ell' - 1)\)-cycles given by the internally self-avoiding closed walks

\[
\tilde{e}_{i+1+\ell'/2} \ldots \tilde{e}_{i+1+\ell'/2+1} \ldots \tilde{e}_{i+1+\ell'/2+2}
\]

with indices considered modulo \( k\ell' \) and \( i \in [k - 1]_0 \). Observe that every edge \( e \in E(T) \setminus \{s, t\} \) is an edge of exactly one sending and one receiving cycle while \( s \) is an edge of exactly \( k \) sending and zero receiving cycles and \( t \) an edge of exactly zero sending and \( k \) receiving cycles of \( T \).

![Figure 1](https://example.com/figure1.png) For \( \ell' \geq 4 \), a \( k \)-graph \( H \) and \( \tilde{s}, \tilde{t} \in \tilde{E}(H) \) with \( s \cap t = \emptyset \) \((\ell' - 1, \tilde{s}, \tilde{t})\)-transporter \( T = \tilde{e}_0 \ldots \tilde{e}_{k'} \) in \( H \) visualized as a directed cycle in the directed graph \( D \) with vertex set \( \tilde{E}(H) \) whose edges are the pairs \((\tilde{e}, \tilde{f}) \in \tilde{E}(H)^2 \) for which \( \tilde{e} \tilde{f} \) is a walk in \( H \). Vertical zigzag arrows represent directed \((|\ell'/2| - 2)\)-paths in \( D \) and diagonal zigzag arrows represent directed \((|\ell'/2| - 2)\)-paths in \( D \). Dashed arrows indicate the shortcuts taken for the sending cycles (in red) and receiving cycles (in blue).
Let \( \omega : C_{\ell - 1}(H) \rightarrow \mathbb{R} \) and \( w \geq 0 \). We say the function obtained from \( \omega \) by using \( T \) with weight \( w \) is the function \( \omega' : C_{\ell - 1}(H) \rightarrow \mathbb{R} \) with

\[
\omega'(C) = \begin{cases} 
\omega(C) - \frac{w}{k} & \text{if } C \text{ is a sending cycle of } T; \\
\omega(C) + \frac{w}{k} & \text{if } C \text{ is a receiving cycle of } T; \\
\omega(C) & \text{otherwise}
\end{cases}
\]

for all \( C \in C_{\ell - 1}(H) \). Note that as a consequence of the observation above, using \( T \) with weight \( w \) shifts weight \( w \) from \( s \) to \( t \) while the weight on all other edges of \( H \) remains unchanged.

In the following we use transporters to shift weight between edges. In order to obtain in the end a fractional decomposition, we have to ensure that all cycle weights remain nonnegative. Thus our use of a transporter for weight transportation is constrained by the weights on its sending cycles.

The goal of the refinement of \( \omega \) is a redistribution of the deviations of the weights on the edges from (their target value) 1. Consider these deviations as vertex weights given by \( \xi : V \rightarrow \mathbb{R} \) in an auxiliary directed graph \( D \) with vertex set \( V \) in which possible weight transportations through the use of transporters are represented by arcs. We adjust our weights as follows and as detailed in the proof of Lemma 4.1. For all distinct \( x, y \in V \), send weight \( \xi(x)/|V| \) from \( x \) to \( y \) in \( D \) spread equally among directed paths from \( x \) to \( y \) that have length \( \ell \) (we will apply the statement with \( \ell = 2 \)).

**Lemma 4.1.** Suppose \( \ell \geq 1 \) is an integer, \( \alpha > 0 \) and \( D \) is a directed graph on \( n \) vertices with vertex set \( V \) and arc set \( A \) with \( (v, u) \in A \) for all \( (u, v) \in A \). Suppose that for all distinct \( s, t \in V \), there are at least \( an^{\ell - 1} \) directed \( \ell \)-paths from \( s \) to \( t \) in \( D \). Let \( \beta > 0 \) and \( \xi : V \rightarrow [-\beta, \beta] \) with \( \sum_{v \in V} \xi(v) = 0 \). Then there is a function \( \eta : A \rightarrow \left[0, \frac{2\beta\ell}{an} \right] \) such that

\[
\xi(v) + \sum_{u: (u, v) \in A} \eta(u, v) - \sum_{u: (v, u) \in A} \eta(v, u) = 0
\]

holds for all \( v \in V \).

**Proof.** We may think of \( \xi(v) \) as a (potentially negative) weight located at \( v \) and we want to find a (nonnegative) flow \( \eta \) that distributes these weights such that afterwards the weight located at each vertex is 0. To this end, for all distinct vertices \( u \) and \( v \), let \( P_{u,v} \) be a set of \([an^{\ell - 1}] \) (directed) \( \ell \)-paths from \( u \) to \( v \) in \( D \). As a first step, we define a flow \( \eta' : A \rightarrow \mathbb{R} \) which may also send negative weight through arcs. Our strategy is as follows; every vertex \( v \) sends weight \( \frac{\xi(v)}{n} \) to each vertex \( u \neq v \) by sending weight \( \frac{\xi(v)}{n[\gamma_{P_{v,u}}]} \) along every path in \( P_{v,u} \). Thus every vertex \( v \) sends weight \( \left( 1 - \frac{1}{n} \right) \xi(v) \) to other vertices. How much weight does \( v \) receive? From each vertex \( u \neq v \), it receives \( \xi(u)/n \) and so in total \( \sum_{u \in V \setminus \{v\}} \xi(u)/n = -\xi(v)/n \). Therefore, this achieves the desired weight distribution.

As each arc of \( D \) is contained in at most \( \ell n^{\ell - 1} \) paths of length \( \ell \) in \( D \) and as we send weight of absolute value at most \( \frac{\beta}{an} \) along each path, we observe that \( |\eta'(x, y)| \leq \frac{\beta\ell}{an} \) holds for all \( (x, y) \in A \).

Now, we simply transform a negative flow \( \eta'(x, y) \) on an arc \((x, y)\) into a positive flow in the opposite direction by setting \( \eta(x, y) = \max\{\eta'(x, y), 0\} \) for all \((x, y) \in A \). As \( |\eta(x, y)| \leq |\eta'(x, y)| + |\eta'(y, x)| \leq \frac{2\beta\ell}{an} \), this completes the proof.

We are now ready to prove our key lemma which guarantees the existence of a well-behaved fractional \( C_\ell \)-decomposition for sufficiently large \( \ell \) whenever there is a suitable transition system.
Lemma 4.2. Let $1/n \ll \epsilon, \zeta, 1/k, 1/\ell$. Suppose $H$ is a $k$-graph on $n$ vertices with $\delta(H) \geq \delta n$ and $\mathcal{T}$ is an $r$-regular transition system of $H$ with $r \geq \epsilon n$ such that $H$ is $(1 - \zeta)\ell'$-exactly $\mathcal{T}$-compatibly $\ell'$-connected for all integers $\ell'$ with $\ell'/3 \leq \ell' \leq 2\ell$. Let $\mu \in (2^{-\ell'}, 1)$ and suppose $\ell'(1 - \zeta/2)^{\ell'+1} \leq \frac{\delta^{k+1} \mu}{400k}$ holds. Then there is a fractional $C_k^\ell$-decomposition $\omega$ of $H$ with $\omega(C) = (1 \pm \mu)\frac{2e(H)}{\ell'}$ for all $\mathcal{T}$-compatible $C \subseteq C_r(H)$ and $\omega(C) \leq \mu \frac{2e(H)}{r}$ for all $C \subseteq C_r(H)$ that are not $\mathcal{T}$-compatible.

Proof. We will start with a scaled uniform distribution $\omega_0$ on all $\mathcal{T}$-compatible $\ell'$-cycles in $H$ and show that $\omega_0$ is already almost a weight function $\omega$ as in the statement (see (4.3) and (4.4) below). In a further step we will slightly adjust $\omega_0$ using transporters to obtain a fractional $C_k^\ell$-decomposition of $H$.

Let us observe that for all integers $\ell' \geq 1$ with $\ell'/3 \leq \ell' \leq 2\ell$, $U \subset V(H)$ with $|U| \leq k\ell$ and $\bar{s}, \bar{t} \in \mathcal{E}(H-U)$, the number of internally self-avoiding $\mathcal{T}$-compatible $\ell'$-walks from $\bar{s}$ to $\bar{t}$ in $H-U$ is

$$\left(1 \pm (1 - \zeta/2)^{\ell'}\right)^{\frac{r^{\ell'-1}}{\bar{e}(H)}}. \quad (4.1)$$

Indeed, the number of $\ell'$-walks from $\bar{s}$ to $\bar{t}$ in $H$ that are not internally self-avoiding is at most

$$4\ell^2 r^{\ell'-k-2} \leq \frac{4\ell^2}{\epsilon^{k+1} n} \frac{r^{\ell'-1}}{n^k} \leq \frac{1}{\sqrt{n}} \frac{r^{\ell'-1}}{\bar{e}(H)}$$

and the number of $\ell'$-walks from $\bar{s}$ to $\bar{t}$ in $H$ that are not walks in $H-U$ is at most

$$2k\ell^2 r^{\ell'-k-2} \leq \frac{2k\ell^2}{\epsilon^{k+1} n} \frac{r^{\ell'-1}}{n^k} \leq \frac{1}{\sqrt{n}} \frac{r^{\ell'-1}}{\bar{e}(H)}.$$

Note that for every fractional $C_k^\ell$-decomposition $\omega$ of $H$, we have $\sum_{C \in C_r(H)} \omega(C) = e(H)/\ell$. Let $c_{\mathcal{T}, \ell'}$ be the number of $\mathcal{T}$-compatible $\ell'$-cycles in $H$ and let $\omega_0 : C_r(H) \to [0, \infty)$ such that

$$\omega_0(C) = \begin{cases} \frac{e(H)}{\ell' c_{\mathcal{T}, \ell'}} & \text{if } C \text{ is } \mathcal{T}\text{-compatible}; \\ 0 & \text{otherwise} \end{cases}$$

holds for all $C \in C_r(H)$. When counting the cycles given by the internally self-avoiding closed $\mathcal{T}$-compatible $(\ell' + 1)$-walks from $\bar{s}$ to $\bar{t}$ in $H$ for all $\bar{e} \in \mathcal{E}(H)$, we count every $\mathcal{T}$-compatible $\ell'$-cycle in $H$ exactly $2\ell'$ times. Together with (4.1) this shows

$$c_{\mathcal{T}, \ell'} = \left(1 \pm (1 - \zeta/2)^{\ell'+1}\right) \frac{r^\ell}{\bar{e}(H)} \cdot \frac{1}{2\ell'} = \left(1 \pm (1 - \zeta/2)^{\ell'+1}\right) \frac{r^\ell}{2\ell'}. \quad (4.2)$$

As $(1 - \zeta/2)^{\ell'+1} \leq \mu/4$, this implies

$$\frac{e(H)}{\ell' c_{\mathcal{T}, \ell'}} = \left(1 \pm \frac{\mu}{2}\right) \frac{2e(H)}{r^\ell}. \quad (4.3)$$
For \( e \in E(H) \), let \( \xi(e) \) be the weight by which the sum of the weights of the \( \mathcal{T} \)-compatible \( \ell \)-cycles \( C \) in \( H \) with \( e \in E(C) \) deviates from 1, that is, let \( \xi : E(H) \to \mathbb{R} \) such that

\[
\xi(e) = \left( \sum_{C \in \mathcal{C}_\ell(H) : e \in E(C)} \omega_0(C) \right) - 1
\]

holds for all \( e \in E(H) \). When counting the cycles given by the internally self-avoiding closed \( \mathcal{T} \)-compatible \((\ell + 1)\)-walks from \( \tilde{f} \) to \( \tilde{f} \) in \( H \) for all \( \tilde{f} \in \tilde{E}(H) \) with \( f = e \), we count every \( \mathcal{T} \)-compatible \( \ell \)-cycle \( C \) in \( H \) with \( e \in E(C) \) exactly twice. Consequently, since \((1 - \zeta/2)^{\ell+1} \leq 1/2\), exploiting \((4.1)\) and \((4.2)\) yields

\[
\xi(e) = (1 \pm (1 - \zeta/2)^{\ell+1}) \frac{r^\ell}{e(H)} \cdot k! \cdot \frac{e(H)}{\ell^\mathcal{T}_\ell} - 1 = 0 \pm 4(1 - \zeta/2)^{\ell+1}. 
\] (4.4)

To obtain from Lemma 4.1 the existence of a flow describing adjustments of \( \omega_0 \) that transform it into a function \( \omega \) as desired, consider the directed graph \( D \) with vertex set \( E(H) \) where a pair \((e,f)\) of edges of \( H \) is an arc in \( D \) if and only if \( e \cap f = \emptyset \). Let \( \xi_{\max} := \max_{e \in E(H)} |\xi(e)| \). Observe that \( e(H) \geq \frac{\delta n^2}{2k!} \). For all distinct \( s,t \in E(H) \), there are at least \( e(H)/2 \) edges in \( H \) disjoint from \( s \) and \( t \) and thus there are at least \( e(H)/2 \) (directed) paths from \( s \) to \( t \) in \( D \) that have length 2. Consequently, Lemma 4.1 yields a function \( \eta : A(D) \to \left[ 0, \frac{16e_{\max}}{\delta n} \right] \) with

\[
\sum_{C \in \mathcal{C}_\ell(H) : e \in E(C)} \omega_0(C) + \sum_{f : (f,e) \in A} \eta(f) - \sum_{f : (e,f) \in A} \eta(e,f) = 1 
\] (4.5)

for all \( e \in E(H) \). (Note that we now consider \( \ell \)-transporters, not \((\ell - 1)\)-transporters as in the discussion above.) Observe that we can build an \((\ell', \tilde{s}, \tilde{t})\)-transporter \( \tilde{e}_0 \cdots \tilde{e}_{k(\ell+1)} \) whose sending cycles are \( \mathcal{T} \)-compatible by choosing \( \tilde{e}_i(\ell+1)+\lfloor \frac{\ell+1}{2} \rfloor + 1 \), and concatenating \( \mathcal{T} \)-compatible self-avoiding \( \lfloor \frac{\ell+1}{2} \rfloor \)-walks from \( \tilde{e}_i(\ell+1)+\lfloor \frac{\ell+1}{2} \rfloor + 1 \) to \( \tilde{e}_i(\ell+1)+\lfloor \frac{\ell+1}{2} \rfloor + 1 \) (with indices considered modulo \( k(\ell+1) \)) for all \( i \in [k-1]_0 \). As we have a lower bound for the number of such walks, this implies that for all \( \tilde{s}, \tilde{t} \in \tilde{E}(H) \) with \( s \cap t = \emptyset \), the number of \((\ell', \tilde{s}, \tilde{t})\)-transporters in \( H \) whose sending cycles are \( \mathcal{T} \)-compatible is at least

\[
m := \left( (\delta n - k\ell) \cdot \left( 1 - \left( 1 - \zeta/2 \right)^{\lfloor \frac{\ell+1}{2} \rfloor} \right) \frac{r^{\lfloor \frac{\ell+1}{2} \rfloor} - 1}{e(H)} \cdot \left( 1 - \left( 1 - \zeta/2 \right)^{\lfloor \frac{\ell+1}{2} \rfloor} \right) \frac{r^{\lfloor \frac{\ell+1}{2} \rfloor} - 1}{e(H)} \right)^k 
\]

\[
\geq \frac{1}{2} \delta^k n^k \frac{r^k}{e(H)^{3k}}.
\]

For \( \tilde{s}, \tilde{t} \in \tilde{E}(H) \) with \( s \cap t = \emptyset \), let \( \mathcal{F}_{\tilde{s}, \tilde{t}} \) be a set of \( m \) distinct \((\ell', \tilde{s}, \tilde{t})\)-transporters in \( H \) whose sending cycles are \( \mathcal{T} \)-compatible. Let \( T_1 \ldots T_p \) be an ordering of

\[
\bigcup_{\tilde{s}, \tilde{t} \in \tilde{E}(H) : s \cap t = \emptyset} \mathcal{F}_{\tilde{s}, \tilde{t}}
\]
and for \( i \in [p] \), let \( \omega_i \) be the function obtained from \( \omega_{j-1} \) by using the \((\ell', \tilde{s}_i, \tilde{t}_i)\)-transporter \( T_i \) with weight \( \frac{\eta(s, t)}{(k!)^{m}} \). (Note that for all disjoint \( s, t \in E(H) \), the number of pairs \((s', t')\) with \( s', t' \in E(H), s' = s \) and \( t' = t \) is \( (k!)^2 \).) Let \( \omega := \omega_p \). By (4.5), the function \( \omega \) satisfies

\[
\sum_{c \in C, \gamma, \epsilon \in E(c)} \omega(C) = 1
\]

for all \( e \in E(H) \) and it remains to show \( \omega(C) \geq 0 \) for all \( C \in C_\ell(H) \) and \( \omega(C) = (1 + \mu) \frac{2e(H)}{r^\ell} \) for all \( C \in C_\ell(H) \) that are \( \mathcal{T} \)-compatible as well as \( \omega(C) \leq \mu \frac{2e(H)}{r^\ell} \) for all \( C \in C_\ell(H) \) that are not \( \mathcal{T} \)-compatible.

For all \( C \in C_\ell(H) \), there are at most \( 2k\ell \cdot n \) pairs \( (\tilde{s}, \tilde{t}) \in \tilde{E}(H)^2 \) such that \( C \) is a sending cycle of an \((\ell, \tilde{s}, \tilde{t})\)-transporter in \( H \) whose sending cycles are \( \mathcal{T} \)-compatible (given \( C \), we have \( 2k\ell \) choices for \( \tilde{s} \) and then at most \( n \) choices for \( \tilde{t} \)). Now we argue similarly as above and conclude that for all \( C \in C_\ell(H) \), the number of \( \ell \)-transporters in \( H \) whose sending cycles are \( \mathcal{T} \)-compatible that have \( C \) as one of their sending cycles is bounded from above by

\[
m' := 2k\ell n \left( n \cdot \left( 1 + (1 - \zeta/2)^{(k\ell)} \right) \cdot \left( 1 + (1 - \zeta/2)^{(k\ell)} \right) \right)^{k-1} 
\leq 2k\ell n^k \frac{3}{2} \ell^{k-\ell}.
\]

Furthermore, similar considerations show that \( m' \) is also an upper bound for the number of \( \ell \)-transporters in \( H \) whose sending cycles are \( \mathcal{T} \)-compatible that have \( C \) as one of their receiving cycles.

In the transition from \( \omega_0 \) to \( \omega \), by employing transporters as suggested by \( \eta \), we used transporters with weight at most \( \frac{1}{(k!)^{m}} \cdot \frac{16k\ell_{\text{max}}}{\delta n^{k}} = \frac{16k\ell_{\text{max}}}{\delta k! mn^k} \). Recall that using a transporter with weight \( w \) changes the weights on its sending and receiving cycles by \( w/k \). Thus exploiting (4.3) yields that it suffices to show that

\[
m' \cdot \frac{1}{k} \cdot \frac{16\xi_{\text{max}}}{\delta k! mn^k} \leq \frac{\mu}{2} \frac{2e(H)}{r^\ell}.
\]

We obtain

\[
m' \cdot \frac{1}{k} \cdot \frac{16\xi_{\text{max}}}{\delta k! mn^k} \leq \frac{6\epsilon e(H)^2}{\delta^{k+2}} \cdot \frac{16\xi_{\text{max}}}{\delta k! mn^k} \leq \frac{192}{\delta^{k+1}} \cdot \ell (1 - \zeta/2)^{\ell+1} \cdot \frac{2e(H)}{r^\ell} \leq \frac{\mu}{2} \frac{2e(H)}{r^\ell},
\]

which completes the proof.

5  Obtaining an Almost Uniform Fractional Decomposition

Observe that we can immediately deduce from Lemmas 3.1, 3.3 and 4.2 that all sufficiently large \((\alpha, \ell_0)\)-connected \( k \)-graphs admit a fractional \( \ell \)-cycle decomposition whenever \( \ell \) is large in terms of \( \ell_0 \). However, in the following proof of Theorem 1.4, we additionally exploit that Lemma 3.1 does not simply guarantee the existence of a transition system suitable for the application of Lemma 3.3 and thus Lemma 4.2, but that a randomly chosen transition system is suitable with high probability.
Proof of Theorem 1.4. First we show that Lemma 3.1 and a probabilistic argument yield multiple suitable transition systems as a basis for an application of Lemmas 3.3 and 4.2. Then we use the fractional decompositions obtained from Lemma 4.2 to construct a fractional decomposition as desired.

Suppose $1/n \ll a, \mu, 1/\ell, 1/k$. Suppose $H$ is an $(a, \ell_0)$-connected $k$-graph as in Theorem 1.4. Let $\delta := \delta(H)/n$, $\Delta := \Delta(H)/n$, $\varepsilon := (1 - \mu/4)^{1/\ell} \delta$, $r \geq 2$ be an even integer with $\varepsilon n \leq r \leq \delta n$ and $\zeta := \frac{a}{6\ell_0}$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_n$ be independently chosen random $r$-transition systems of $H$. For $\vec{s} = (s_1, \ldots, s_k) \in \vec{E}(H)$ with $d_H([s_2, \ldots, s_k]) = \delta(H)$ and $\vec{t} \in \vec{E}(H)$, the number of $\ell_0$-walks from $\vec{s}$ to $\vec{t}$ in $H$ is at most $\delta(H)n^\ell k^{-2}$. Thus $\delta \geq a$ holds. The lower bound for $\ell$ implies

$$\frac{\ell}{3} \geq \frac{60k\ell_0}{a} \log \left( \frac{3}{\delta} \right) \geq \frac{60k\ell_0}{a} \log \left( \frac{3}{\delta} \right) = \frac{30k\ell_0}{a/2} \log \left( 3(1 - \mu/4)^{1/\ell} \frac{1}{\varepsilon} \right) \geq \frac{3k\ell_0 \log(2/\varepsilon)}{\alpha/2}.$$ 

Thus Lemmas 3.1 and 3.3 imply

$$\mathbb{P}[H \text{ is } (1 - \zeta)\ell'-\text{exactly } \mathcal{T}_i\text{-compatible} \text{ for all } \ell' \geq \ell/3] \geq 1 - \exp(-\sqrt{n})$$

for all $i \in [n]$. For all $C \in C_r(H)$ and $i \in [n]$, the cycle $C$ is $\mathcal{T}_i$-compatible with probability at least $(\frac{\varepsilon n}{\Delta(H)})^\ell = (1 - \mu/4)^{\delta/\Delta^\ell}$. Consequently, Lemma 2.1 and a suitable union bound show that it is possible to select $\mathcal{T}_1, \ldots, \mathcal{T}_n$ with

$$|\{i \in [n] : C \text{ is } \mathcal{T}_i\text{-compatible}\}| \geq (1 - \mu/2) \frac{\delta^\ell}{\Delta^\ell} n$$

for all $C \in C_r(H)$ such that $H$ is $(1 - \zeta)\ell'$-exactly $\mathcal{T}_i$-compatible $\ell'$-connected for all $\ell' \geq \ell/3$ and $i \in [n]$. The function $x \mapsto x \exp(-x)$ is monotonically decreasing on $[1, \infty)$. In addition, with room to spare, we will exploit that $\exp(-15kAB) \leq \exp(-11k) \exp(-3kA) \exp(-kB)$ for $A, B \geq 1$, where we set $A := \log \frac{\ell_0}{\alpha}$ and $B := \log \frac{1}{\mu}$. The lower bound for $\ell$ that is given in the statement yields

$$\frac{\zeta}{2} \leq \frac{180\zeta k \ell_0}{2} \log \left( \frac{\ell_0}{\alpha} \right) \log \left( \frac{1}{\mu} \right)$$

and hence we obtain

$$\ell(1 - \zeta/2)^{\ell+1} \leq \frac{2}{\zeta} \cdot \frac{\zeta}{2} \ell' \exp \left( -\frac{\zeta}{2} \ell' \right) \leq \frac{2}{\zeta} \cdot \frac{180\zeta k \ell_0}{2} \log \left( \frac{\ell_0}{\alpha} \right) \log \left( \frac{1}{\mu} \right) \exp \left( \frac{180k \ell_0}{2} \alpha \log \left( \frac{\ell_0}{\alpha} \right) \log \left( \frac{1}{\mu} \right) \right) \leq 180k \ell_0 \log \left( \frac{\ell_0}{\alpha} \right) \log \left( \frac{1}{\mu} \right) \exp \left( -15k \log \left( \frac{\ell_0}{\alpha} \right) \log \left( \frac{1}{\mu} \right) \right) \leq 180k \ell_0 \log \left( \frac{\ell_0}{\alpha} \right) \log \left( \frac{1}{\mu} \right) \cdot \exp(-11k) \left( \frac{\alpha}{\ell_0} \right)^{3k} \mu^k \leq \frac{180k \exp(11k) (\frac{\alpha}{\ell_0})^{k+1}}{400k} \mu.$$ 

Thus, Lemma 4.2 yields fractional $\ell^k$-decompositions $\omega_1, \ldots, \omega_n$ of $H$ with

$$\left| \left\{ i \in [n] : \omega_i(C) \geq (1 - \mu/2) \frac{2e(H)}{r^\ell} \right\} \right| \geq (1 - \mu/2) \frac{\delta^\ell}{\Delta^\ell} n \quad \text{and} \quad \omega_i(C) \leq (1 + \mu/2) \frac{2e(H)}{r^\ell}.$$
for all \( C \in C_\ell(H) \) and \( i \in [n] \). For \( \omega = \frac{1}{n} \sum_{i \in [n]} \omega_i \) and \( C \in C_\ell(H) \) the first of these two inequalities implies

\[
\omega(C) \geq (1 - \mu/2)^2 \frac{2e(H)}{\Delta} \geq (1 - \mu/2)^2 \frac{2\Delta(H)}{\Delta} \geq (1 - \mu) \frac{2\Delta(H)}{\Delta}
\]

and from the second we obtain

\[
\omega(C) \leq (1 + \mu/2)^2 \frac{2e(H)}{\Delta} \leq (1 + \mu/2) \frac{2e(H)}{\Delta} \leq (1 + \mu) \frac{2e(H)}{\Delta}.
\]

\[
\square
\]

6 | CONCLUDING REMARKS

In this paper, we prove the existence of fractional cycle decompositions in \( k \)-graphs \( H \) on \( n \) vertices under the very mild assumption that \( H \) is “well-connected.” This in particular includes the case that \( \delta(H) \geq (1/2 + o(1))n \), settles a question posed by Glock, Kühn, and Osthus for graphs [5], and improves significantly on previous results [4]. This is particularly interesting as very recently, Piga and Sanhueza-Matamala [11] showed that for 3-graphs the decomposition threshold for (long) cycles is in fact 2/3. It is therefore plausible that for hypergraphs of higher uniformity this threshold is also significantly higher than 1/2. In fact one of their key tools is Theorem 1.3 because it yields approximate decompositions into (long) cycles of the 3-graphs in their setting.

As we indicated earlier, Theorem 1.4 is also applied in a recent project of Schülke [7] and the authors concerning a generalization of the famous result in [12] where an analog of Dirac’s theorem for hypergraphs is proved. We expect that Theorem 1.4 will be useful for further results in the area.

The method presented here can be easily pushed further. Consider for example graphs. Here, a transition system for a graph \( G \) has a transition graph at every vertex whose vertices are the incident edges and for a length \( \ell \geq 4 \) and two ordered edges \( (s_1, s_2), (t_1, t_2) \in \mathcal{E}(G) \) with \( \{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset \), an \( (\ell, (s_1, s_2), (t_1, t_2)) \)-transporter is the union of self-avoiding walks from \( (s_1, s_2) \) to \( (t_1, t_2) \), from \( (t_1, t_2) \) to \( (s_2, s_1) \), from \( (s_2, s_1) \) to \( (t_2, t_1) \) and from \( (t_2, t_1) \) back to \( (s_1, s_2) \); that is, ignoring orientations, it is a particular subdivision of a complete graph on four vertices. With few modifications one can show that the fractional decomposition threshold for subdivisions of cliques (or even any connected graph) also tends to 1/2 when the length of the subdivided edges tends to infinity.

Another natural question arising from Theorem 1.3 is the best dependence between \( \varepsilon \) and \( \ell \) (for fixed \( k \)) where \( \varepsilon \) is defined by \( \delta_{C_\ell}^{\min} = 1/2 + \varepsilon \). Example 2.2 shows that \( \varepsilon \geq \frac{1}{(k-1+2/k)(\ell-1)} \) and our results imply, say, that \( \varepsilon = O(\varepsilon^{\ell-1/(4k)}) \). Any substantial improvement on these bounds would be interesting.

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