BLOW-UP SOLUTIONS FOR A SYSTEM OF SCHRÖDINGER EQUATIONS WITH GENERAL QUADRATIC-TYPE NONLINEARITIES IN DIMENSIONS FIVE AND SIX

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ABSTRACT. In this work, we show the existence of ground state solutions for an \( l \)-component system of non-linear Schrödinger equations with quadratic-type growth interactions in the energy-critical case. They are obtained analyzing a critical Sobolev-type inequality and using the concentration-compactness method. As an application, we prove the existence of blow-up solutions of the system without the mass-resonance condition in dimension six (and five), when the initial data is radial.

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1. INTRODUCTION

This paper is concerned with the study of the following initial-value problem

\[
\begin{cases}
  i\alpha_k \partial_t u_k + \gamma_k \Delta u_k - \beta_k u_k + f_k(u_1, \ldots, u_l) = 0, \\
  (u_1(x, 0), \ldots, u_l(x, 0)) = (u_{10}, \ldots, u_{l0}), \quad k = 1, \ldots, l,
\end{cases}
\]

(1.1)

where \( u_k : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C} \), \((x, t) \in \mathbb{R}^n \times \mathbb{R}\), \( \Delta \) is the Laplacian operator, \( \alpha_k, \gamma_k > 0 \), \( \beta_k \geq 0 \) are real constants and the nonlinearities \( f_k \) have a quadratic-type growth.

Multi-component Schrödinger systems with quadratic interactions arise in many physical situations, for instance, in fiber and waveguide nonlinear optics (see [23] for a review and applications). Such systems may be obtained, for instance, by using the so-called multistep cascading mechanism. In particular, multistep cascading can be achieved by second-order
nonlinear processes such as second harmonic generation (SHG) and sum-frequency mixing (SFM) (see [29]). An example of a three-step cascading model is

\[
\begin{align*}
2i\partial_t w + \Delta w - \beta w &= -\frac{1}{2}(u^2 + v^2), \\
i\partial_t v + \Delta v - \beta_1 v &= -\overline{w}v, \\
i\partial_t u + \Delta u - u &= -\overline{w}w,
\end{align*}
\]

which represents, in dimensionless variables, the reduced amplitude equations of a fundamental beam with frequency \( \omega \) entering a nonlinear medium with a quadratic response, derived in a slowly varying envelope approximation with the assumption of zero absorption of all interacting waves. Here \( \beta, \beta_1 \geq 0 \) are real constants and functions, \( u, v, \) and \( w \) represent the associated polarizations. Another example is given by

\[
\begin{align*}
i\partial_t w + \Delta w - w &= -(wv + \overline{w}u), \\
2i\partial_t v + \Delta v - \beta v &= -\left(\frac{1}{2}w^2 + \overline{w}u\right), \\
3i\partial_t u + \Delta u - \beta_1 u &= -vw,
\end{align*}
\]

where \( w, v, \) and \( u \) represent, in dimensionless variables, the complex electric fields envelopes of the fundamental harmonic, second harmonic, and third harmonic, respectively (see [28] for details). A model formally appearing as a non-relativistic version of some Klein-Gordon system, when the speed of light tends to infinity is given, in dimensionless variables, by (see [18])

\[
\begin{align*}
i\partial_t u + \Delta u &= -2\overline{w}v, \\
i\partial_t v + \kappa \Delta v &= -u^2,
\end{align*}
\]

where \( \kappa \) is a real constant. Similar systems can also be rigorously derived as a model in \( \chi^{(2)} \) media (see [8]).

From the mathematical point of view the interest in nonlinear Schrödinger systems with quadratic interactions has been increased in the past few years (see [7], [8], [15], [17], [18], [23], [24], [25], [33], [34], [36], [39] and references therein). So, in [41] we initiated the study of system (1.1) with general quadratic-type nonlinearities. More precisely, we assumed the following (with a slight modification in [H4]).

(H1). \( f_k(0, \ldots, 0) = 0, \quad k = 1, \ldots, l. \)

(H2). There exists a constant \( C > 0 \) such that for \( (z_1, \ldots, z_l), (z'_1, \ldots, z'_l) \in \mathbb{C}^l \) we have

\[
\left| \frac{\partial}{\partial z_m} [f_k(z_1, \ldots, z_l) - f_k(z'_1, \ldots, z'_l)] \right| \leq C \sum_{j=1}^{l} |z_j - z'_j|, \quad k, m = 1, \ldots, l;
\]

\[
\left| \frac{\partial}{\partial \overline{z}_m} [f_k(z_1, \ldots, z_l) - f_k(z'_1, \ldots, z'_l)] \right| \leq C \sum_{j=1}^{l} |z_j - z'_j|, \quad k, m = 1, \ldots, l.
\]

(H3). There exists a function \( F: \mathbb{C}^l \to \mathbb{C} \) such that

\[
f_k(z_1, \ldots, z_l) = \frac{\partial F}{\partial \overline{z}_k}(z_1, \ldots, z_l) + \overline{\frac{\partial F}{\partial z_k}(z_1, \ldots, z_l)}, \quad k = 1, \ldots, l.
\]

(H4). There exist positive constants \( \sigma_1, \ldots, \sigma_l \) such that for any \( (z_1, \ldots, z_l) \in \mathbb{C}^l \)

\[
\text{Im} \sum_{k=1}^{l} \sigma_k f_k(z_1, \ldots, z_l)\overline{z}_k = 0.
\]
(H5). Function $F$ is homogeneous of degree 3, that is, for any $\lambda > 0$ and $(z_1, \ldots, z_l) \in \mathbb{C}^l$,

$$F(\lambda z_1, \ldots, \lambda z_l) = \lambda^3 F(z_1, \ldots, z_l).$$

(H6). There holds

$$\left| \text{Re} \int_{\mathbb{R}^n} F(u_1, \ldots, u_l) \, dx \right| \leq \int_{\mathbb{R}^n} F(|u_1|, \ldots, |u_l|) \, dx.$$

(H7). Function $F$ is real valued on $\mathbb{R}^l$, that is, if $(y_1, \ldots, y_l) \in \mathbb{R}^l$ then

$$F(y_1, \ldots, y_l) \in \mathbb{R}.$$ 

Moreover, functions $f_k$ are non-negative on the positive cone in $\mathbb{R}^l$, that is, for $y_i \geq 0$, $i = 1, \ldots, l$,

$$f_k(y_1, \ldots, y_l) \geq 0.$$

(H8). Function $F$ can be written as the sum $F_1 + \cdots + F_m$, where $F_s$, $s = 1, \ldots, m$ is supermodular on $\mathbb{R}^d_+$, $1 \leq d \leq l$, and vanishes on hyperplanes, that is, for any $i, j \in \{1, \ldots, d\}$, $i \neq j$ and $k, h > 0$, we have

$$F_s(y + he_i + ke_j) + F_s(y) \geq F_s(y + he_i) + F_s(y + ke_j), \quad y \in \mathbb{R}^d_+,$$

and $F_s(y_1, \ldots, y_d) = 0$ if $y_j = 0$ for some $j \in \{1, \ldots, d\}$.

It is easy to see that functions $F$ associated to systems (1.2), (1.3), and (1.4) are given, respectively, by

$$F(z_1, z_2, z_3) = \frac{1}{2}(z_1^2 + z_2^2), \quad F(z_1, z_2, z_3) = \frac{1}{2}(z_1^2z_2 + z_1z_2z_3), \quad F(z_1, z_2) = z_1^2z_2. \quad (1.5)$$

In addition assumptions (H1)-(H8) hold in these particular examples.

The results established in [34] include local and global well posedness in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, $1 \leq n \leq 6$, existence and stability/instability of ground state solutions, and the dichotomy global existence versus blow up in finite time. In particular, assumptions (H1) and (H2) are enough to prove the existence of a local solution by using the contraction mapping principle in a suitable space based on the well-known Strichartz estimates. Assumptions (H3)-(H5) guarantee that (1.1) conserves the charge

$$Q(u(t)) := \sum_{k=1}^l \frac{\alpha_k \sigma_k}{2} \|u_k(t)\|^2_{L^2}, \quad (1.6)$$

and the energy

$$E(u(t)) := \sum_{k=1}^l \gamma_k \|\nabla u_k(t)\|^2_{L^2} + \sum_{k=1}^l \beta_k \|u_k(t)\|^2_{L^2} - 2 \text{Re} \int F(u(t)) \, dx \quad (1.7)$$

where we are using the notation $u(t) = (u_1(t), \ldots, u_l(t))$ (see notations below). By using the above conserved quantities one can show the existence of global solutions in $L^2(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$, $1 \leq n \leq 3$. Also, if the $H^1$-norm of the initial data is sufficiently small one can also show the global existence in $H^1(\mathbb{R}^n)$ if $n = 4$ or $n = 5$. Moreover, when (H6)-(H8) are assumed we proved the existence and stability/instability of ground state solutions. Using this special kind of solutions we were able to provide a sharp vectorial Gagliardo-Nirenberg-type inequality to give a sharp criterion for the existence of global solutions in dimensions $n = 4$ and $n = 5$. Some of the above results are summarized in Section 2.3.

Before presenting the main goal of this paper, we recall that by a standard scaling argument and the fact that $f_k$ are homogeneous functions of degree 2 (see (2.5)) we deduce that $\dot{H}^{n/2-2}(\mathbb{R}^n)$ is the critical (scaling invariant) Sobolev space for (1.1) (with $\beta_k = 0$). In
particular, \( L^2 \) and \( H^1 \) are critical in dimensions \( n = 4 \) and \( n = 6 \), respectively. As usual, we then adopt the following convention: we will say that system (1.1) is

\[
\begin{aligned}
L^2 &= \begin{cases} 
\text{subcritical,} & \text{if } 1 \leq n \leq 3, \\
\text{critical,} & \text{if } n = 4, \\
\text{supercritical,} & \text{if } n \geq 5
\end{cases}
\quad \text{and} \\
H^1 &= \begin{cases} 
\text{subcritical,} & \text{if } 1 \leq n \leq 5, \\
\text{critical,} & \text{if } n = 6, \\
\text{supercritical,} & \text{if } n \geq 7.
\end{cases}
\end{aligned}
\]

To proceed we introduce the following definition

**Definition 1.1.** We say that (1.1) satisfies the mass-resonance condition if

\[
\text{Im} \sum_{k=1}^{l} m_k f_k(z) z_k = 0, \quad z \in \mathbb{C}^l,
\]

where \( m_k := \frac{\alpha_k}{\gamma_k} \).

Let us illustrate Definition 1.1 using our examples above. We first check that (1.4) satisfies the mass-resonance condition if and only if \( \kappa = \frac{1}{2} \), which is in agreement with the terminology in the current literature. Indeed, as we already said, the function \( F \) associated to (1.4) is \( F(z_1, z_2) = \overline{z}_1 z_2 \) and (RC) is equivalent to

\[
\left(1 - \frac{1}{2\kappa}\right) \text{Im}(\overline{z}_1 z_2) = 0,
\]

which means that mass-resonance occurs only if \( \kappa = \frac{1}{2} \). On the other hand, using (1.5), it is easy to see that systems (1.2) and (1.3) both satisfy (RC).

We point out that instead of (H4) in [34] we have assumed

\[
\text{Re} \left( e^{i \frac{\alpha_1 \theta}{\gamma_1}} z_1, \ldots, e^{i \frac{\alpha_l \theta}{\gamma_l}} z_l \right) = \text{Re} F(z), \quad \theta \in \mathbb{R}, \ z = (z_1, \ldots, z_l),
\]

which, together with (H3), implies that (RC) holds (see Lemma 2.9 in [34]). Thus, all results obtained in [34] are under the assumption of mass-resonance. It is our goal in the present paper to study (1.1) without the assumption of mass-resonance.

Mass-resonance appears as a special relation between the masses of the system and it is closely related with the large time behavior of solutions. As pointed out in [42], it was first considered in Klein-Gordon systems. When considering system (1.4), for instance, it is well known that the value of the parameter \( \kappa \) influences the large-time behavior of its solutions, see [25]. In [13] the authors, among other things, proved the existence of ground state solutions for (1.4), when \( \kappa > 0 \), and used these solutions to give a sharp criterion for the existence of global \( H^1 \) solutions in the mass-resonance case and \( n = 4 \). This kind of result was extended to the non-mass-resonance case (\( \kappa \neq \frac{1}{2} \)) in [23], where the authors showed a blow-up result when the initial data is radial in dimensions \( n = 5 \) and \( n = 6 \). Some very recent works without mass-resonance condition have been appeared. In [24], the authors showed scattering in the \( L^2 \)-critical case with and without the mass-resonance condition. Moreover, scattering in the case \( n = 5 \) was proved in [10] and in [11].

From the mathematical point of view, the phenomenon of mass-resonance can be seen in the virial-type identity satisfied by solutions of system (1.1). Indeed, for \( 1 \leq n \leq 6 \), set \( \Sigma = \{ u \in H^1; xu \in L^2 \} \), where \( xu \) means \((xu_1, \ldots, xu_l)\), and define the function

\[
V(t) = \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} \int |x|^2 |u_k(x, t)|^2 \, dx,
\]

where \( \Sigma = \{ u \in H^1; xu \in L^2 \} \), with \( xu \) meaning \((xu_1, \ldots, xu_l)\), and define the function

\[
V(t) = \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} \int |x|^2 |u_k(x, t)|^2 \, dx.
\]
where \( u(t) \) is the corresponding solution of (1.1) with initial data \( u_0 \in \Sigma \). Then, if \( I \) is the maximal existence interval, a straightforward computation leads to

\[
V''(t) = 2nE(u_0) - 2n \sum_{k=1}^{l} \beta_k \| u_k \|_{L^2}^2 + 2(4 - n) \sum_{k=1}^{l} \gamma_k \| \nabla u_k \|_{L^2}^2
- \frac{d}{dt} \int |x|^2 \text{Im} \sum_{k=1}^{l} m_k f_k(u) \overline{u_k} \, dx,
\]

(1.10)

for all \( t \in I \). Assuming that (RC) holds, the last term in (1.10) disappears. In that case, in [34] it was shown using an argument due to Glassey that if \( E(u_0) < 0 \) (or \( E(u_0) = 0 \) and \( u_0 \) has negative momentum), the local solution blows-up in finite time in dimensions 4 \( \leq n \leq 6 \). The mass-resonance assumption has also been appeared in various works involving two and three-component Schrödinger systems (see for instance [38], [42], [39], [19], [20], [21] and [22] and references therein). When (RC) does not hold a more careful analysis must be performed and we do not know if solutions in \( \Sigma \) with negative energy, for instance, blow-up or not.

Based on the above background, the main goal of this paper is to prove that blow-up in finite time also holds if mass-resonance does not occur, but the initial data is radial. We will instead of \( n \) be particularly interested in the cases \( n = 5 \) and \( n = 6 \). The “threshold” for the existence of blow-up solutions will be given in terms of the ground states associated with (1.1). See Theorem 4.1 below.

This work is organized as follows. In section 2 we first introduce some notations and give preliminaries results that will be used along the paper. We also list some consequences of our assumptions and give a review of some previous results about system (1.1). In section 3 we use the concentration-compactness method to prove the existence of ground state solutions in the \( H^1 \)-critical case. We also establish the optimal constant in a critical Sobolev-type inequality. Finally, section 4 is devoted to show that in dimensions \( n = 5 \) and \( n = 6 \) if the initial data is radially symmetric then the corresponding solution of (1.1) blows-up in finite time.

2. Preliminaries

In this section we introduce some notations and give some consequences of our assumptions.

2.1. Notation. We use \( C \) to denote several constants that may vary line-by-line. \( B(x, r) \) denotes the ball of radius \( r \) and center at \( x \in \mathbb{R}^n \). To simplify writing, given any set \( A \), by \( A^l \) (or \( A^i \)) we denote the product \( A \times \cdots \times A \) (\( l \) times). If \( A \) is a Banach space with norm \( \| \cdot \| \) then \( A \) is a Banach space with norm given by the sum. Thus, in \( \mathbb{C}^l \) we use frequently \( z \) instead of \((z_1, \ldots, z_l)\). Given any complex number \( z \in \mathbb{C} \), \( \text{Re} \) and \( \text{Im} \) represents its real and imaginary parts. Also, \( \overline{z} \) denotes its complex conjugate. We set \([z]\) for the vector \((|z_1|, \ldots, |z_l|)\). This is not to be confused with \(|z|\) which stands for the standard norm of the vector \( z \) in \( \mathbb{C}^l \). If \( w \) is a vector with non-negative real components, we write \( w \geq 0 \). Given \( z = (z_1, \ldots, z_l) \in \mathbb{C}^l \), we write \( z_m = x_m + iy_m \), where \( x_m = \text{Re} z_m \) and \( y_m = \text{Im} z_m \). The differential operators \( \partial/\partial z_m \) and \( \partial/\partial \overline{z}_m \) are defined by

\[
\frac{\partial}{\partial z_m} = \frac{1}{2} \left( \frac{\partial}{\partial x_m} - i \frac{\partial}{\partial y_m} \right), \quad \frac{\partial}{\partial \overline{z}_m} = \frac{1}{2} \left( \frac{\partial}{\partial x_m} + i \frac{\partial}{\partial y_m} \right).
\]

Let \( \Omega \subset \mathbb{R}^n \) be an open set. To simplify notation, if no confusion is caused we use \( \int f \, dx \) instead of \( \int_\Omega f \, dx \). The spaces \( L^p = L^p(\Omega) \), \( 1 \leq p \leq \infty \), and \( W^s_p = W^s_p(\Omega) \) denote the standard Lebesgue and Sobolev spaces. In the case \( p = 2 \), we use the notation \( H^s = W^s_2 \). We use \( \dot{H}^1 = \dot{H}^1(\mathbb{R}^n) \) to denote the homogeneous Sobolev spaces of order 1. For \( n \geq 3 \), \( s^* = \frac{2n}{n-2} \) denotes the critical Sobolev exponent. Recall that \( D^{1,2}(\Omega) = \{ u \in L^{2^*}(\Omega); \nabla u \in L^2(\Omega) \} \) and \( D^{1,2}_0(\Omega) \) is the completion of \( C_0^\infty(\Omega) \) with respect to the norm \( \left( \int \| \nabla u \|^2 \, dx \right)^{\frac{1}{2}} \), or equivalently, the closure of \( C_0^\infty(\Omega) \) in \( D^{1,2}(\Omega) \). By the Sobolev inequality we have \( D^{1,2}(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n) \) (with
equivalent norms). Since $D^{1,2}(\mathbb{R}^n) = D_0^{1,2}(\mathbb{R}^n)$ (see for instance [3] Lemma 1.2), we then see that

$$\tilde{H}^{1}(\mathbb{R}^n) = D^{1,2}(\mathbb{R}^n) = D_0^{1,2}(\mathbb{R}^n).$$

Thus we can use each one of these spaces in our arguments to follow.

Given a time interval $I$, the mixed spaces $L^p(I;L^q(\mathbb{R}^n))$ are endowed with the norm

$$\|f\|_{L^p(I;L^q)} = \left( \int_I \left( \int_{\mathbb{R}^n} |f(x,t)|^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{1}{p}},$$

with the obvious modification if either $p = \infty$ or $q = \infty$. When the interval $I$ is implicit and no confusion will be caused we denote $L^p(I;L^q(\mathbb{R}^n))$ simply by $L^p(I^q)$ and its norm by $\| \cdot \|_{L^p(I;L^q)}$. More generally, if $X$ is a Banach space, $L^p(I;X)$ represents the $L^p$ space of $X$-valued functions defined on $I$.

With $C_b(X)$ we denote the set of bounded continuous functions on $X$. Also, $C_c(X)$ stands for the set of continuous functions on $X$ with compact support. The set $\mathcal{M}_+(X)$ denotes the Banach space of non-negative Radon measures on $X$. Similarly, $\mathcal{M}_{b+}(X)$ and $\mathcal{M}_{b-}(X)$ represent the spaces of bounded (or finite) and probability Radon measures, respectively. By $\mathcal{B}(X)$ we denote the Borel $\sigma$-algebra on $X$. We write $\nu \ll \mu$ if the measure $\nu$ is absolutely continuous with respect to the measure $\mu$. For any $\mu \in \mathcal{M}_{b+}(X)$, $\|\mu\| := \mu(X)$ is called the total mass of $\mu$.

2.2. Weak convergence of measures. Here we introduce some notions of convergence of Radon measures. We refer the reader to [1] Chapter 4, Sections 30-31 for a more complete discussion about this topic. Let $X$ be a locally compact space. A sequence $(\mu_m) \subset \mathcal{M}_+(X)$ is said to converge vaguely to $\mu$ in $\mathcal{M}_+(X)$, written $\mu_m \xrightarrow{\ast} \mu$, provided $\int_X f \, d\mu_m \rightarrow \int_X f \, d\mu$, for all $f \in C_c(X)$. We say that a sequence $(\mu_m) \subset \mathcal{M}_{b+}(X)$ converges weakly to a measure $\mu$ in $\mathcal{M}_{b+}(X)$, written $\mu_m \rightharpoonup \mu$, if $\int_X f \, d\mu_m \rightarrow \int_X f \, d\mu$, for all $f \in C_b(X)$. A sequence $(\mu_m) \subset \mathcal{M}_{b+}(X)$ is said to be uniformly tight if, for every $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset X$ such that $\mu_m(X \setminus K_\epsilon) < \epsilon$, for all $m$.

We finish this paragraph with an useful result that guarantees the existence of vaguely convergent subsequences. We say that a set $\mathcal{H} \subset \mathcal{M}_+(X)$ is vaguely bounded if $\sup_{\mu \in \mathcal{H}} \left| \int_X f \, d\mu \right| < \infty$ for all $f \in C_c(X)$.

**Lemma 2.1.** Let $X$ be a locally compact space. Then,

(i) every vaguely bounded sequence in $\mathcal{M}_+(X)$ contains a vaguely convergent subsequence;

(ii) If $\mu_m \xrightarrow{\ast} \mu$ in $\mathcal{M}_+(X)$ and $(\|\mu_m\|)$ is bounded, then $\mu$ is finite.

**Proof.** See Theorems 31.2 and 30.6 in [1].

2.3. Some consequences of our assumptions. Here, we will present some consequences of our assumptions \([\text{H1}] - [\text{H8}]\). We start with the following.

**Lemma 2.2.** Assume that \([\text{H1}] - [\text{H7}]\) hold.

(i) We have

$$|\text{Re} F(z) - \text{Re} F(z')| \leq C \sum_{m=1}^{l} \sum_{j=1}^{t} (|z_j|^2 + |z_j'|^2)|z_m - z_m'|.$$  \hspace{1cm} (2.1)

In particular,

$$|\text{Re} F(z)| \leq C \sum_{j=1}^{t} |z_j|^3.$$
(ii) Let \( u \) be a complex-valued function defined on \( \mathbb{R}^n \). Then,

\[
\text{Re} \sum_{k=1}^{l} f_k(u) \overline{u}_k = \text{Re}[3F(u)].
\]

(iii) We have

\[
f_k(x) = \frac{\partial F}{\partial x_k}(x).
\]

In addition, \( F \) is positive on the positive cone of \( \mathbb{R}^l \).

**Proof.** The reader will find the details in [34]. More precisely, see Lemmas 2.10, 2.11, and 2.13. \( \Box \)

Now by using assumptions [H3] and [H4] we are able to derive a Gauge invariant condition satisfied by the non-linear interaction terms in (1.1). We start with the following invariant property of \( \text{Re} F \).

**Lemma 2.3.** Assume that [H3] and [H4] hold. Let \( \theta \in \mathbb{R} \) and \( z \in \mathbb{C}^l \). Then,

\[
\text{Re} F \left( e^{i\frac{\sigma_1}{2}\theta} z_1, \ldots, e^{i\frac{\sigma_l}{2}\theta} z_l \right) = \text{Re} F(z).
\]

**Proof.** Denote by \( w \) the vector \( (w_1, \ldots, w_l) := \left( e^{i\frac{\sigma_1}{2}\theta} z_1, \ldots, e^{i\frac{\sigma_l}{2}\theta} z_l \right) \) and let \( h(\theta) := F(w) \).

By the chain rule,

\[
\frac{dh}{d\theta} = \sum_{k=1}^{l} \frac{\partial F}{\partial w_k}(w) \frac{\partial w_k}{\partial \theta} + \sum_{k=1}^{l} \frac{\partial F}{\partial \overline{w}_k}(w) \frac{\partial \overline{w}_k}{\partial \theta} = \sum_{k=1}^{l} \frac{\partial F}{\partial w_k}(w) \left( \frac{\sigma_k}{2} \right) e^{i\frac{\sigma_k}{2}\theta} z_k + \sum_{k=1}^{l} \frac{\partial F}{\partial \overline{w}_k}(w) \left( -\frac{\sigma_k}{2} \right) e^{i\frac{\sigma_k}{2}\theta} \overline{z}_k.
\]

Taking the real part on both sides of (2.3) and using (H3) we obtain

\[
\text{Re} \frac{dh}{d\theta} = \frac{1}{2} \frac{1}{\text{Im}} \sum_{k=1}^{l} \sigma_k f_k(w) \overline{w}_k = 0,
\]

which implies the desired. \( \Box \)

With the previous result in hand we prove the following.

**Lemma 2.4.** Under the assumptions of Lemma 2.3. The functions \( f_k, k = 1, \ldots, l, \) satisfy the following Gauge condition

\[
f_k \left( e^{i\frac{\sigma_1}{2}\theta} z_1, \ldots, e^{i\frac{\sigma_l}{2}\theta} z_l \right) = e^{i\frac{\sigma_k}{2}\theta} f_k(z).
\]

**Proof.** First of all note that from the definition of the complex differential operators we may write

\[
f_k(z) = 2 \frac{\partial}{\partial \overline{z}_k} \text{Re} F(z).
\]

Now, as in the proof of Lemma 2.3 let \( w = \left( e^{i\frac{\sigma_1}{2}\theta} z_1, \ldots, e^{i\frac{\sigma_l}{2}\theta} z_l \right) \). Clearly,

\[
\frac{\partial}{\partial \overline{w}_k} \text{Re} F(z) = e^{-i\frac{\sigma_k}{2}\theta} \frac{\partial}{\partial \overline{w}_k} \text{Re} F(w).
\]

Hence,

\[
f_k(z) = 2 e^{-i\frac{\sigma_k}{2}\theta} \frac{\partial}{\partial \overline{w}_k} \text{Re} F(w) = e^{-i\frac{\sigma_k}{2}\theta} f_k(w) = e^{-i\frac{\sigma_k}{2}\theta} f_k \left( e^{i\frac{\sigma_1}{2}\theta} z_1, \ldots, e^{i\frac{\sigma_l}{2}\theta} z_l \right),
\]
which completes the proof. □

The next fact is a natural consequence of assumptions \([\text{H3}]\) and \([\text{H5}]\). The nonlinearities \(f_k\) are homogeneous functions of degree 2, i.e., for any \(z \in \mathbb{C}^l\),

\[
f_k(\lambda z) = \lambda^2 f_k(z), \quad \forall k = 1, \ldots, l, \quad \lambda > 0.
\] (2.5)

We finish this section by presenting an adapted vectorial version of the generalized Brezis-Lieb’s lemma (see [5, Theorem 2]). We start recalling that by assumption \([\text{H5}]\) \(F(0) = 0\) and, for all \(a, b \in \mathbb{C}^l\),

\[
|F(a + b) - F(b)| \leq C \sum_{k=1}^{l} \sum_{j=1}^{l} (|a_j| + |b_j|) |b_k|.
\]

In particular \(F\) is continuous and, by Young’s inequality, for any \(\varepsilon > 0\),

\[
|F(a + b) - F(b)| \leq \varepsilon \varphi(a) + \psi_\varepsilon(b),
\] (2.6)

where \(\varphi\) and \(\psi_\varepsilon\) are given by the non-negative functions \(\varphi(a) = \sum_{j=1}^{l} |a_j|^3\) and \(\psi_\varepsilon(b) = C_\varepsilon \sum_{j=1}^{l} |b_j|^3\), for some positive constant \(C_\varepsilon\).

**Lemma 2.5.** Let \(v_m = u_m - u\) be a sequence of measurable functions from \(\mathbb{R}^n\) to \(\mathbb{C}\) such that

(i) \(v_m \to 0\) a.e.;
(ii) \(F(u) \in L^1(\mathbb{R}^n)\);
(iii) \(\int \varphi(v_m)(x) \, dx \leq M < \infty\), for some constant \(M\), independent of \(\varepsilon\) and \(m\);
(iv) \(\int \psi_\varepsilon(u)(x) \, dx < \infty\), for any \(\varepsilon > 0\).

Then, as \(m \to \infty\),

\[
\int |F(u_m) - F(v_m) - F(u)| \, dx \to 0.
\]

**Proof.** The proof is similar to that of Theorem 2 in [5]. So we omit the details. □

2.4. Local and global well-posedness. In [34] we studied \([11]\) by assuming \([\text{H1}]\) but with \([\text{H4}]\) replaced by \([\text{H8}]\). From the point of view of well-posedness in \(H^1\) the same results can also be obtained here in dimension \(1 \leq n \leq 6\). Indeed, to give a precise statement we introduce the space

\[
Y(I) := \begin{cases} \left( C \cap L^\infty(I; H^1) \cap L^{12/n}(I; W^{1,2}_2(\Omega)) \right), & 1 \leq n \leq 3, \\ \left( C \cap L^\infty(I; H^1) \cap L^2(I; W^{1,2}_2(\Omega)) \right), & n \geq 4, \end{cases}
\]

where \(I \subset \mathbb{R}\) is an interval. We have the following results.

**Theorem 2.6.** Let \(1 \leq n \leq 5\). Assume that \([\text{H1}]\) and \([\text{H2}]\) hold. Then for any \(r > 0\) there exists \(T(r) > 0\) such that for any \(u_0 \in H^1\) with \(\|u_0\|_{H^1} \leq r\), system \([1.1]\) has a unique solution \(u \in Y(I)\) with \(I = [-T(r), T(r)]\).

In addition, a blow up alternative also holds, that is, there exist \(T_*, T^* \in [0, \infty]\) such that the local solutions can extend to \((-T_*, T^*)\). Moreover if \(T_* < \infty\) (respect. \(T^* < \infty\)), then

\[
\lim_{t \to -T_*} \|u(t)\|_{H^1} = \infty, \quad \text{respect.} \quad \lim_{t \to T^*} \|u(t)\|_{H^1} = \infty.
\]

**Proof.** See [34], Theorem 3.9]. □

**Theorem 2.7.** Let \(n = 6\). Assume that \([\text{H1}]\) and \([\text{H2}]\) hold. Then for any \(u_0 \in H^1\) there exists \(T(u_0) > 0\) such that system \([1.1]\) has a unique solution \(u \in Y(I)\) with \(I = [-T(u_0), T(u_0)]\). In addition, a blow up alternative also holds, that is, there exist \(T_*, T^* \in (0, \infty]\) such that the local solution can extend to \((-T_*, T^*)\). Moreover if \(T_* < \infty\) (respect. \(T^* < \infty\)), then

\[
\lim_{t \to -T_*} \|u(t)\|_{L^6(W_2^1)} = \infty, \quad \text{respect.} \quad \lim_{t \to T^*} \|u(t)\|_{L^6(W_2^1)} = \infty.
\]
for any (Schrödinger admissible) pair \((q, r)\) satisfying

\[
\frac{2}{q} = 6 \left( \frac{1}{2} - \frac{1}{r} \right), \quad 2 \leq r \leq 3.
\]

**Proof.** See [34, Theorem 3.10] for the local well-posedness. The blow-up alternative can be established by extending the arguments in [6, Theorem 4.5.1]. \(\Box\)

Note that both results above hold only under assumptions (H1) and (H2). To extend the local solutions to global ones we need (H3) and (H4) to guarantee that the quantities (1.6) and (1.7) are conserved by the flow of (1.1). At this point, it is to be noted that in order to establish the conservation of \(Q\), (1.8) may indeed be replaced by (H4) (see [34, Lemma 3.11] for details). Actually (H4) is a vectorial extension for the well known assumptions for the scalar Schrödinger equation with a general nonlinearity (see [6, Chapter 3]).

Using the above mentioned conserved quantities combined with the Gagliardo-Nirenberg inequality it is possible to get an *a priori* bound for the \(L^2\) and \(H^1\)-norm of a solution. In particular, for any initial data in \(H^1\) local solutions can be extended to global ones, when \(1 \leq n \leq 3\). For \(n = 4\) and \(n = 5\) global solutions are obtained if the initial data is sufficiently small. To give a precise description of how small the initial data must be, the ground states solutions take a singular place. In fact, a standing wave solution for (1.1) is a special solution having the form

\[
u_k(x,t) = e^{i\frac{\sigma_k}{2}\omega t} \psi_k(x), \quad k = 1, \ldots, l,
\]

where \(\omega \in \mathbb{R}\) and \(\psi_k\) are real-valued functions decaying to zero at infinity, which by Lemma 2.4 satisfy the following semilinear elliptic system

\[
-\gamma_k \Delta \psi_k + \left( \frac{\sigma_k \alpha_k}{2} \omega + \beta_k \right) \psi_k = f_k(\psi), \quad k = 1, \ldots, l.
\]

The action functional associated to (2.8) is

\[
I(\psi) = \frac{1}{2} \left[ \sum_{k=1}^{l} \gamma_k \| \nabla \psi_k \|_{L^2}^2 + \sum_{k=1}^{l} \left( \frac{\sigma_k \alpha_k}{2} \omega + \beta_k \right) \| \psi_k \|_{L^2}^2 \right] - \int F(\psi) \, dx.
\]

A *ground state solution* for (2.8) is a nontrivial solution that is a minimum of \(I\) among all solutions of (2.8). Before proceeding, it is convenient to introduce the functionals

\[
Q(\psi) = \sum_{k=1}^{l} \left( \frac{\sigma_k \alpha_k}{2} \omega + \beta_k \right) \| \psi_k \|_{L^2}^2,
\]

\[
K(\psi) = \sum_{k=1}^{l} \gamma_k \| \nabla \psi_k \|_{L^2}^2, \quad P(\psi) = \int F(\psi) \, dx.
\]

and the set

\[
\mathcal{P} := \{ \psi \in H^1; P(\psi) > 0 \}.
\]

These functionals satisfy some useful identities.

**Lemma 2.8.** Assume \(1 \leq n \leq 5\) and let \(\psi\) be a (weak) solution of (2.8). Then,

\[
P(\psi) = 2I(\psi),
\]

\[
K(\psi) = nI(\psi),
\]

\[
Q(\psi) = (6 - n)I(\psi).
\]

**Proof.** See [34, Lemma 4.5]. \(\Box\)
Under our assumptions, ground states for (2.8) do exist if the coefficients $\sigma_k \alpha_k \omega + \beta_k$ are positive, $k = 1, \ldots, l$ and $1 \leq n \leq 5$. Thus, if we denote by $\mathcal{G}_n(\omega, \beta)$ the set of ground state solutions of (2.8), we have that $\mathcal{G}_n(\omega, \beta) \neq \emptyset$ if $1 \leq n \leq 5$. Moreover, the Gagliardo-Nirenberg-type inequality

$$P(u) \leq C_n^{opt} Q(u)^{\frac{n-4}{4}} K(u)^{\frac{n}{4}}, \quad u \in P,$$

holds with the optimal constant $C_n^{opt}$ given by

$$C_n^{opt} := 2(6-n)^{\frac{n-4}{4}} \frac{1}{Q(\psi)^{\frac{1}{2}}},$$

where $\psi \in \mathcal{G}_n(\omega, \beta)$, $1 \leq n \leq 5$ (see [34, Corollary 4.12]).

We summarize our global well-posedness results in the following theorem.

**Theorem 2.9.** Assume that (H1)-(H8) hold and let $\psi$ be a ground state solution of (2.8) in $\mathcal{G}_n(1, 0)$.

(i) If $1 \leq n \leq 3$ then for any $u_0 \in H^1$, system (1.1) has a unique solution $u \in Y(R)$.

(ii) Assume $n = 4$. Then for any $u_0 \in H^1$ satisfying

$$Q(u_0) < Q(\psi),$$

system (1.1) has a unique solution $u \in Y(R)$.

(iii) Assume $n = 5$. Suppose that $u_0 \in H^1$ satisfies

$$Q(u_0) E(u_0) < Q(\psi) E(\psi),$$

and

$$Q(u_0) K(u_0) < Q(\psi) K(\psi),$$

where $E$ is the energy defined in (1.7) with $\beta_k = 0$, $k = 1, \ldots, l$. Then system (1.1) has a unique solution $u \in Y(R)$.

**Proof.** See [34, Theorems 3.16 and 5.2].

**Remark 2.10.**

(i) In Section 4 we will show that under assumption (2.18), condition (2.19) is sharp in the sense if inequality has been reversed and the initial data is radial then the solution must blow up in finite time.

(ii) In dimension $n = 6$, since the existence time in Theorem 2.7 depends on the initial data itself, an a priori bound of the local solution is not enough to extend it globally in time.

### 3. Existence of ground states in the $H^1$-critical case

In this section we are interested in showing the existence of ground state solutions for (1.1) in the $H^1$-critical case. The section can be seen of independent interest since it purely deals with semilinear elliptic equations. In particular assumption (H8) can be dropped here.

For the scalar case, existence of ground-state solutions in the critical case is closely related with the optimal constant in the critical Sobolev inequality: (see for instance [30, Theorem 8.3] or [43, Corollary 1.3])

$$||f||^2_{L^3} \leq C||\nabla f||^2_{L^2}, \quad f \in \dot{H}^1(R^6).$$

(3.1)

This was addressed, for instance, in [1] (see also [32] and [40, Chapter I, Section 4]) where optimal constant and extremal functions were obtained.

In our case, we first note from (2.14) we must expect non-trivial solutions of (2.8) only if

$$\frac{\sigma_k \alpha_k}{2} \omega + \beta_k = 0,$$

(3.2)

which is fulfilled, for instance, if $\omega = 0$ and $\beta = 0$. Thus, system (2.8) and the action functional $I$ reduce to

$$-\gamma_k \Delta \psi_k = f_k(\psi) \quad k = 1, \ldots, l$$

(3.3)
and
\[ I(\psi) = \frac{1}{2} \sum_{k=1}^{l} \gamma_k \| \nabla \psi_k \|_{L^2}^2 - \int F(\psi) \, dx. \] (3.4)

Solutions of (3.3) can now be seen as critical points of the action (3.4). More precisely, we have the following.

**Definition 3.1.** A function \( \psi \in \dot{H}^1(\mathbb{R}^6) \) is called a solution (weak solution) of (3.3) if for any \( g \in \dot{H}^1(\mathbb{R}^6) \),
\[ \gamma_k \int \nabla \psi_k \cdot \nabla g_k \, dx = \int f_k(\psi) g_k \, dx, \quad k = 1, \ldots, l. \] (3.5)

Among all solution of (3.3) we single out the ones that minimizes \( I \).

**Definition 3.2.** A solution \( \psi \in \dot{H}^1(\mathbb{R}^6) \) is called a ground state of (3.3) if
\[ I(\psi) = \inf \{ I(\phi); \phi \in C \}, \]
where \( C \) denotes the set of all non-trivial solutions of (3.3). We denote by \( G_6 \) the set of all ground states of (3.3).

Let us start by observing if \( \psi \) is a non-trivial solution of (3.3) then the functional \( P \) must be positive at \( \psi \).

**Lemma 3.1.** Define \( \mathcal{D} := \{ \psi \in \dot{H}^1(\mathbb{R}^6); P(\psi) > 0 \} \). Then, \( \mathcal{C} \subset \mathcal{D} \).

**Proof.** Let \( \psi \in \mathcal{C} \). By taking \( g = \psi \) in (3.3) and using Lemma 2.2-(ii),
\[ 3P(\psi) = K(\psi), \] (3.6)
from which we deduce the desired. \( \square \)

It is convenient to introduce the following functionals:
\[ J(\psi) := \frac{K(\psi)^{\frac{3}{2}}}{P(\psi)}, \quad \psi \in \mathcal{D}, \] (3.7)
and
\[ E(\psi) := K(\psi) - 2P(\psi), \quad \psi \in \dot{H}^1(\mathbb{R}^6). \] (3.8)

**Remark 3.2.** Let \( \psi \) be a non-trivial solution of (3.3). Then, clearly
\[ E(\psi) = 2I(\psi) \]
and using (3.6),
\[ J(\psi) = \frac{6^{\frac{3}{2}}}{2} I(\psi)^{\frac{3}{2}}. \]
In particular, a non-trivial solution of (3.3) is a ground state if and only if its has least energy among all non-trivial solutions of (3.3) if only if it minimizes \( J \).

With this in mind, one of the main results of this paper reads as follows.

**Theorem 3.3.** There exists a ground state solution \( \psi_0 \) for system (3.3), that is, \( \mathcal{G}_6 \) is not empty.

In order to prove Theorem 3.3, we shall use the concentration-compactness method to obtain a solution to a constrained minimization problem deduced from a general critical Sobolev-type inequality, which turns out to be a ground state.
3.1. General critical Sobolev-type inequality. From now on we assume that all components of the vector \( u \) are real-valued functions. Hence, using Lemma 2.2 we obtain the following general critical Sobolev-type inequality:

\[ P(u) \leq CK(u)\frac{3}{2}, \quad \forall u \in D. \]  

(3.9)

In particular, this shows that functional \( J \) is bounded from below by a positive constant. Then, the infimum of \( J \) on \( D \) is positive and the best constant we can place in (3.9) is given by

\[ C_6^{-1} := \inf \{ J(u); \ u \in D \}. \]  

(3.10)

The subscript in the definition of \( C_6 \) is motivated by the dimension \( n = 6 \).

We will prove that the infimum (3.10) is attained. To this end, we consider the following normalized version

\[ S := \inf \{ K(u); \ u \in D, \ P(u) = 1 \}. \]  

(3.11)

Minimization problems as (3.11) was studied by Lions in [32, page 166, equation (30)]. However, since we are not assuming that \( F \) is strictly positive outside the origin his approach need to be slightly modified. This is why our minimization problem (3.11) is posed on \( D \) and not on \( H^1(\mathbb{R}^6) \).

A minimizing sequence for (3.10) is a sequence \((u_m)\) in \( D \) such that \( J(u_m) \to C_6^{-1} \). In the same way, a minimizing sequence for (3.11) is a sequence \((u_m)\) in \( D \) such that \( P(u_m) = 1 \) and \( K(u_m) \to S \). Since \( K(\|u\|) \leq K(u) \), assumption (H6) implies that \( J(\|u\|) \leq J(u) \). Thus, if \((u_m)\) is a minimizing sequence of (3.10) (or (3.11)) so is \((|u_m|)\). In particular, without loss of generality, we can (and will) assume that minimizing sequences are always non-negative.

**Remark 3.4.** Since the functionals \( K \) and \( P \) are homogeneous of degree 2 and 3, respectively, we have

(i) \( C_6 = S^{-\frac{2}{3}}, \) which means that (3.9) becomes

\[ P(u) \leq S^{-\frac{2}{3}}K(u)^{\frac{3}{2}}, \quad \forall u \in D. \]  

(3.12)

Moreover, if \( v \) is a minimizer for (3.11) it also is a minimizer for (3.10). In fact,

\[ J(v) = \frac{K(v)^{\frac{3}{2}}}{P(v)} = K(v)^{\frac{3}{2}} = S^{\frac{2}{3}} = C_6^{-1}. \]

(ii) The functionals \( K \) and \( P \) are invariant under the transformation

\[ u \mapsto v^{Ry}(x) = R^{-2}u \left(R^{-1}(x-y)\right), \]  

(3.13)

where \( R > 0 \) and \( y \in \mathbb{R}^6 \). In particular, if \((u_m)\) is a minimizing sequence for (3.10) (or (3.11)), so is the sequence \((v_m)\) with \( v_m(x) = R^{-2}u_m \left(R^{-1}(x-y)\right) \).

3.2. Concentration Compactness principle. To obtain that (3.11) has a minimizer we will use the concentration-compactness method. The first result in this direction is based on [31, Lemma I.1].

**Lemma 3.5 (Concentration-Compactness I).** Suppose that \( (\nu_m) \) is a sequence in \( \mathcal{M}^1_+(\mathbb{R}^n) \). Then, there is a subsequence, still denoted by \( (\nu_m) \), such that one of the following three conditions hold:

(i) (Vanishing) For all \( R > 0 \) there holds

\[ \lim_{m \to \infty} \left( \sup_{x \in \mathbb{R}^n} \nu_m(B(x, R)) \right) = 0. \]

(ii) (Dichotomy) There exists a number \( \lambda, 0 < \lambda < 1 \), such that for any \( \epsilon > 0 \) there exist a number \( R > 0 \) and a sequence \((x_m)\) with the following property: given \( R' > R \)

\[ \nu_m(B(x_m, R)) \geq \lambda - \epsilon, \]

\[ \nu_m(\mathbb{R}^n \setminus B(x_m, R')) \geq 1 - \lambda - \epsilon, \]
for \( m \) sufficiently large.

(iii) (Compactness) There exists a sequence \( (x_m) \subset \mathbb{R}^n \) such that for any \( \epsilon > 0 \) there is a radius \( R > 0 \) with the property that

\[
\nu_m(B(x_m, R)) \geq 1 - \epsilon,
\]

for all \( m \).

Proof. See for instance [40, Chapter I, Lemma 4.3] and [13, Lemma 23]. \( \square \)

The next lemma is inspired by the concentration-compactness principle in the limiting case (see [32]). For its proof we follow closely the ideas presented in [12, Theorem 1.4.2] (see also [40, Lemma 4.8]).

**Lemma 3.6 (Concentration-compactness II).** Let \((u_m) \subset \dot{H}^1(\mathbb{R}^6)\) be any sequence such that \( u_m \geq 0 \) and

\[
\begin{cases}
  u_m \rightharpoonup u, & \text{in } \dot{H}^1(\mathbb{R}^6), \\
  \mu_m := \sum_{k=1}^{l} \gamma_k |\nabla u_{km}|^2 \rightarrow \mu, & \text{in } \mathcal{M}_+^b(\mathbb{R}^6), \\
  \nu_m := \int F(u_m) \, dx \rightarrow \nu, & \text{in } \mathcal{M}_+^b(\mathbb{R}^6). 
\end{cases}
\]

(3.14)

Then,

(i) There exist an at most countable set \( J \), a family of distinct points \( \{x_j \in \mathbb{R}^6 : j \in J\} \), and a family of non-negative numbers \( \{\nu_j : j \in J\} \) such that

\[
\nu = F(u) \, dx + \sum_{j \in J} \nu_j \delta_{x_j},
\]

(3.15)

(ii) In addition, we have

\[
\mu \geq \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx + \sum_{j \in J} \mu_j \delta_{x_j},
\]

(3.16)

for some family \( \{\mu_j : j \in J\}, \mu_j > 0 \), such that

\[
\nu_j \leq S^{-\frac{d}{2}} \mu_j^\frac{d}{2}, \quad \forall j \in J.
\]

(3.17)

In particular, \( \sum_{j \in J} \nu_j^2 < \infty \).

**Remark 3.7.** Since \( u_m \geq 0 \), Lemma 2.2 (iii) implies that \( F(u_m) \geq 0 \). Hence, \( \nu_m \) is indeed a positive measure. Moreover, the weak convergence \( u_m \rightharpoonup u \) implies that, up to a subsequence, \( u_m \rightarrow u \) a.e. in \( \mathbb{R}^6 \) (see for instance [30, Corollary 8.7]). As a consequence, \( u \geq 0 \).

**Proof of Lemma 3.6.** We divide the proof into the cases \( u = 0 \) and \( u \neq 0 \).

**Step 1.** Assume first that \( u = 0 \).

Let \( \xi \in \mathcal{C}_c^\infty(\mathbb{R}^6) \). From the vague convergence of \((\nu_m)\) in (3.14) and assumption (H5) we have

\[
\int |\xi|^3 \, d\nu = \lim_{m \to \infty} \int |\xi|^3 F(u_m) \, dx = \lim_{m \to \infty} \int F(|\xi|u_m) \, dx \leq S^{-\frac{d}{2}} \liminf_{m \to \infty} K(\xi u_m)^\frac{d}{2},
\]

(3.18)

where we have used the critical Sobolev-type inequality (3.12) in the last inequality. Since \( u_m \rightharpoonup 0 \) in \( \dot{H}^1(\mathbb{R}^6) \) we know that (see [30, Theorem 8.6]), for any \( A \subset \mathbb{R}^6 \) with finite measure and \( k = 1, \ldots, l \) we have

\[
\chi_A u_{km} \rightarrow 0, \quad \text{strongly in } L^2(\mathbb{R}^6).
\]

(3.19)
Thus, using the triangular inequality and taking $A$ as $\text{supp}(|\nabla \xi|)$ in (3.19) we get
\[
\left| \left( \sum_{k=1}^{l} \gamma_k \| \nabla [\xi u_{km}] \|_{L^2}^2 \right)^{\frac{1}{2}} - \left( \sum_{k=1}^{l} \gamma_k \| \nabla u_{km} \|_{L^2}^2 \right)^{\frac{1}{2}} \right| \leq \left( \sum_{k=1}^{l} \gamma_k \| \nabla [\xi u_{km}] - \xi \nabla u_{km} \|_{L^2}^2 \right)^{\frac{1}{2}}
\]
\[
= \left( \sum_{k=1}^{l} \gamma_k \| u_{km} \nabla \xi \|_{L^2}^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \sum_{k=1}^{l} \int |\chi_A u_{km}|^2 \, dx \right)^{\frac{1}{2}} \to 0, \quad \text{as } m \to \infty.
\]

Combining this with the vague convergence of $(\mu_m)$ we obtain
\[
\liminf_{m \to \infty} K(\xi u_m)^{\frac{3}{2}} = \liminf_{m \to \infty} \left( \int |\xi|^2 \sum_{k=1}^{l} \gamma_k |\nabla u_{km}|^2 \, dx \right)^{\frac{3}{2}}
\]
\[
= \liminf_{m \to \infty} \left( \int |\xi|^2 \, d\mu_m \right)^{\frac{3}{2}}
\]
\[
= \left( \int |\xi|^2 \, d\mu \right)^{\frac{3}{2}}.
\]

Therefore, from (3.18) we deduce that
\[
\int |\xi|^3 \, d\nu \leq S^{-\frac{3}{2}} \left( \int |\xi|^2 \, d\mu \right)^{\frac{3}{2}}, \quad \xi \in C_c^\infty(\mathbb{R}^6).
\] (3.20)

We claim that inequality (3.20) actually implies that
\[
\nu(E) \leq S^{-\frac{3}{2}} \mu(E)^{\frac{3}{2}}, \quad \text{for any } E \in \mathcal{B}(\mathbb{R}^6).
\] (3.21)

In fact, since $\nu$ and $\mu$ are Radon measures, they are inner regular on open sets and outer regular on Borel sets, respectively. Let $U \subset \mathbb{R}^6$ be an open set and take any compact set $A$, with $A \subset U$. By $C^\infty$ Urysohn’s lemma (see for instance [14, Lemma 8.18]) there exists $f \in C_c^\infty(\mathbb{R}^6)$ such that $0 \leq f \leq 1$, $f = 1$ on $A$ and $\text{supp}(f) \subset U$. Then, from (3.20),
\[
\nu(A) = \int_A f^3 \, d\nu \leq \int f^3 \, d\nu \leq S^{-\frac{3}{2}} \left( \int f^2 \, d\mu \right)^{\frac{3}{2}} \leq S^{-\frac{3}{2}} \left( \int_{\text{supp}(f)} f^2 \, d\mu \right)^{\frac{3}{2}} \leq S^{-\frac{3}{2}} \left( \int_U d\mu \right)^{\frac{3}{2}}.
\]

Hence, $\nu(A) \leq S^{-\frac{3}{2}} \mu(U)^{\frac{3}{2}}$ for all $A \subset U$, $A$ compact. Thus, from the inner regularity of the measure $\nu$ we conclude that
\[
\nu(U) \leq S^{-\frac{3}{2}} \mu(U)^{\frac{3}{2}}, \quad \text{for any } U \subset \mathbb{R}^6, \quad U \text{ open.}
\] (3.22)

Now, consider any $E \in \mathcal{B}(\mathbb{R}^6)$ and let $U$ be an open subset with $E \subset U$. Then, from (3.22) we have $\nu(E) \leq \nu(U) \leq S^{-\frac{3}{2}} \mu(U)^{\frac{3}{2}}$. It follows from the outer regularity of the measure $\mu$ that $\nu(E) \leq S^{-\frac{3}{2}} \mu(E)^{\frac{3}{2}}$.

Next let $D$ be the set of atoms of the measure $\mu$, i.e., $D = \{ x \in \mathbb{R}^6 : \mu(\{x\}) > 0 \}$. Note that $D = \bigcup_{k=1}^{\infty} D_k$ with $D_k = \{ x \in \mathbb{R}^6 : \mu(\{x\}) > \frac{1}{k} \}$. Since $\mu$ is finite it follows that $D_k = \{ x \in \mathbb{R}^6 : \mu(\{x\}) > \frac{1}{k} \}$ is finite for all $k$, from which we deduce that $D$ is at most countable. Thus, we can write $D = \{ x_j : j \in J \}$, where $J$ is a countable subset of $\mathbb{N}$. 


Define $\mu_j := \mu(\{x_j\}), \ j \in J$. For any $E \in \mathcal{B}(\mathbb{R}^6)$ we have

$$\sum_{j \in J} \mu_j \delta_{x_j}(E) = \sum_{j \in J} \mu_j = \sum_{j \in J} \mu(\{x_j\}) \leq \mu(E).$$  \tag{3.23}$$

which proves (3.16) in the case $u = 0$.

Now, we will prove that (3.15) also holds. From (3.21) we have $\nu \ll \mu$, then by the Radon-Nikodym Theorem (see for instance [11, Section 1.6]) there exists a non-negative function $h \in L^1(\mathbb{R}^6, \mu)$ such that

$$\nu(E) = \int_E h(x)d\mu(x), \quad \text{for any } E \in \mathcal{B}(\mathbb{R}^6).$$  \tag{3.24}$$

In addition, $h$ satisfies

$$h(x) = \lim_{r \to 0} \frac{\nu(B(x,r))}{\mu(B(x,r))}, \quad \mu \text{ a.e. } x \in \mathbb{R}^6.$$  \tag{3.25}$$

Combining (3.25) and (3.21) we get $0 \leq h(x) \leq S^{-\frac{3}{2}} \mu(\{x\})^{\frac{1}{2}}$. This shows that $h(x) = 0$, $\mu$ a.e. on $\mathbb{R}^6 \setminus D$. In particular, $h$ assumes countable many values and, consequently, the integral in (3.24) can be represented (see for instance [4, Example 2.5.8]) by

$$\int_E h(x)d\mu(x) = \sum_{j \in J} h(x_j)\mu(\{x_j\}).$$  \tag{3.26}$$

Define $\nu_j := \nu(\{x_j\}), \ j \in J$. We see from (3.24) and (3.26) that in fact $\nu_j = h(x_j)\mu_j$, for all $j \in J$. Therefore, for any $E \in \mathcal{B}(\mathbb{R}^6)$,

$$\nu(E) = \sum_{j \in J} h(x_j)\mu(\{x_j\}) = \sum_{j \in J} \nu_j = \sum_{j \in J} \nu_j \delta_{x_j}(E),$$

which is (3.15) with $u = 0$.

Finally, inequality (3.17) follows immediately from the definitions of $\mu_j$, $\nu_j$ and (3.21). Note also that by taking $E = \mathbb{R}^n$ in (3.23) we deduce that $\sum_{j \in J} \mu_j$ is convergent. Therefore, the convergence of the series $\sum_{j \in J} \nu_j$ follows from (3.21).

**Step 2.** Assume now $u \neq 0$. First note that Lemma 2.2 (iii) implies $F(u) \geq 0$, so $F(u) \ dx$ defines a positive measure.

**Claim.** The measures

$$\mu - \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \ dx \quad \text{and} \quad \nu - F(u) \ dx,$$  \tag{3.27}$$

are non-negative.

To prove this, define $v_m = u_m - u$ and consider the sequences of measures

$$\tilde{\mu}_m := \sum_{k=1}^{l} \gamma_k |\nabla v_{km}|^2 \ dx \quad \text{and} \quad \tilde{\nu}_m := F(\{v_m\}) \ dx.$$

Recall that $|v_m|$ denotes the vector $(|v_{1m}|, \ldots, |v_{km}|)$. Since $v_m \to 0$ in $H^1(\mathbb{R}^6)$, the sequence $(K(v_m))$ is uniformly bounded. In view of

$$\left| \int f \ d\tilde{\mu}_m \right| \leq \|f\|_{L^\infty} K(v_m), \quad f \in C_c(\mathbb{R}^6),$$

it follows that $(\tilde{\mu}_m)$ is a vaguely bounded sequence in $\mathcal{M}^6_+(\mathbb{R}^6)$. Hence, by Lemma 2.4 there exists a subsequence, still denoted by $(\tilde{\mu}_m)$, and $\tilde{\mu} \in \mathcal{M}^6_+(\mathbb{R}^6)$ such that

$$\tilde{\mu}_m \rightharpoonup^* \tilde{\mu}, \quad \text{in } \mathcal{M}^6_+(\mathbb{R}^6).$$  \tag{3.28}$$
We claim that

$$\mu_m \overset{a}{\to} \tilde{\mu} + \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx \quad \text{in} \quad \mathcal{M}_+^b (\mathbb{R}^6). \quad (3.29)$$

If this is the case, since the vague limit is unique, we have

$$\mu = \tilde{\mu} + \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx.$$ 

Since all the measures involved are finite, it follows that the first difference in (3.27) is a non-negative measure.

Let us prove (3.29). Taking into account that \( \partial_{x_i} v_{km} \to 0 \) in \( L^2 (\mathbb{R}^6) \) and \( f \partial_{x_i} u_k \in L^2 (\mathbb{R}^6) \), for any \( f \in C_c (\mathbb{R}^6) \), we deduce

$$\lim_{m \to \infty} \int f \nabla v_{km} \cdot \nabla u_k \, dx = 0, \quad k = 1, \ldots, l. \quad (3.30)$$

Thus, for any \( f \in C_c (\mathbb{R}^6) \) we get

$$0 \leq \left| \int f \, d\mu_m - \int f \left[ d\tilde{\mu} + \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx \right] \right| = \left| \int f \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx - \int f \left[ d\tilde{\mu} + \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx \right] \right|$$

$$= \left| \int f \sum_{k=1}^{l} \gamma_k \left( |\nabla v_{km}|^2 + 2 \nabla v_{km} \cdot \nabla u_k + |\nabla u_k|^2 \right) \, dx - \int f \, d\tilde{\mu} - \int f \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \, dx \right|$$

$$\leq \left| \int f \, d\tilde{\mu}_m - \int f \, d\tilde{\mu} \right| + 2 \sum_{k=1}^{l} \gamma_k \left| \int f \nabla v_{km} \cdot \nabla u_k \, dx \right|$$

The first term goes to zero by the vague convergence in (3.28), the second one goes to zero by (3.30). This establishes (3.29).

Next, we are going to prove that \((\tilde{v}_m)\) is also vaguely bounded in \( \mathcal{M}_+^b (\mathbb{R}^6) \). As we point out before, \((K(v_m))\) is uniformly bounded. Hence, from the critical Sobolev inequality (3.1) \((v_m)\) is uniformly bounded in \( L^6 (\mathbb{R}^6) \). Thus, for any \( f \in C_c (\mathbb{R}^6) \),

$$\left| \int f \, d\tilde{v}_m \right| \leq \|f\|_{L^\infty} \int F(|v_m|) \, dx \leq C \int \sum_{k=1}^{l} |v_{km}|^3 \, dx \leq M_2,$$

for some constant \( M_2 \). From Lemma 2.1 again there exist a subsequence, still denoted by \((\tilde{v}_m)\), and a measure \( \tilde{\nu} \in \mathcal{M}_+^b (\mathbb{R}^6) \) such that

$$\tilde{v}_m \overset{a}{\to} \tilde{\nu}, \quad \text{in} \quad \mathcal{M}_+^b (\mathbb{R}^6). \quad (3.31)$$

We claim that

$$\nu_m \overset{a}{\to} \tilde{\nu} + F(u) \, dx \quad \text{in} \quad \mathcal{M}_+^b (\mathbb{R}^6), \quad (3.32)$$

which implies that \( \nu = \tilde{\nu} + F(u) \, dx \) and, therefore, the measure \( \nu - F(u) \, dx \) is non-negative.

We prove (3.32) by using the generalized version of Brezis-Lieb’s lemma stated in Lemma 2.5 with \( F(|x|) \) instead of \( F(x) \). Indeed, first we may assume that \( v_m \to 0 \) a.e. on \( \mathbb{R}^6 \) (see Remark 3.7). Now, since \( u \in L^3 (\mathbb{R}^6) \), Lemma 2.2 implies that \( F(|u|) \in L^1 (\mathbb{R}^6) \). Moreover, the sequence \((v_m)\) is uniformly bounded in \( L^3 (\mathbb{R}^6) \). Hence, if \( \varphi \) and \( \psi \) are the functions defined in (2.6) we have

$$\int \varphi(v_m) \, dx \leq M, \quad \text{and} \quad \int \psi(u) \, dx < \infty,$$
for some constant $M$ independent of $\epsilon > 0$ and $m$. Lemma \[\text{Lemma 2.5}\] then yields
\[
\lim_{m \to \infty} \int \left| F\left(\|u_m\|\right) - F\left(\|v_m\|\right) - F\left(\|u\|\right) \right| \, dx = 0, \tag{3.33}
\]
where we have used that $\|u_m\| = u_m$ and $\|u\| = u$. Thus, for any $f \in C_c(\mathbb{R}^6)$,
\[
0 \leq \left| \int f \, dv_m - \int f \left[ \tilde{d} + F(u) \right] \, dx \right| = \left| \int fF(u_m) \, dx - \int fF(v_m) \, dx + \int fF(v_m) \, dx - \int f \left[ \tilde{d} + F(u) \right] \, dx \right| \\
\leq \|f\|_{L^\infty} \int \left| F\left(\|u_m\|\right) - F\left(\|v_m\|\right) - F\left(\|u\|\right) \right| \, dx + \left| \int f \tilde{d}_m - \int f \tilde{d} \right|.
\]
The first term goes to zero by (3.33) and the second one goes to zero by the vague convergence (3.31). This proves that the second measure in (3.27) is also non-negative and the proof of the claim is completed.

Finally, from the proof of the above claim
\[
\left\{ \begin{array}{l}
\sum_{k=1}^l \gamma_k |\nabla v_{km}|^2 \, dx \xrightarrow{*} \mu - \sum_{k=1}^l \gamma_k |\nabla u_k|^2 \, dx, \quad \text{in} \quad \mathcal{M}_1^b(\mathbb{R}^6), \\
F\left(\|v_m\|\right) \, dx \xrightarrow{*} \nu - F(u) \, dx, \quad \text{in} \quad \mathcal{M}_1^b(\mathbb{R}^6).
\end{array} \right.
\]
So the proof of the lemma is completed if now we apply Step 1. It must be observed that Step 1 holds if we do not have $u_m \geq 0$ but replace the sequence $(\nu_m)$ in (3.14) by $\nu_m := F(\|u_m\|) \, dx$. \[\square\]

The next result is useful to construct a localized Sobolev-type inequality.

**Lemma 3.8.** For every $\delta > 0$ there is a constant $C(\delta) > 0$ with the following property: if $0 < r < R$ with $r/R \leq C(\delta)$ and $x \in \mathbb{R}^6$, then there is a cut-off function $\chi^r_R \in W^1_\infty(\mathbb{R}^6)$ such that $\chi^r_R = 1$ on $B(x, R)$, $\chi^r_R = 0$ outside $B(x, R)$,
\[
K(\chi^r_Ru) \leq \sum_{k=1}^l \gamma_k \int_{B(x,R)} |\nabla u_k|^2 \, dy + \delta K(u), \tag{3.34}
\]
and
\[
K((1 - \chi^r_R)u) \leq \sum_{k=1}^l \gamma_k \int_{\mathbb{R}^6 \setminus B(x, r)} |\nabla u_k|^2 \, dy + \delta K(u), \tag{3.35}
\]
for any $u \in \dot{H}^1(\mathbb{R}^6)$.

**Proof.** This result was essentially proved in [13, Lemma 8]. Without loss of generality assume $x = 0$. The function $\chi^r_R$ is given by
\[
\chi^r_R(y) = \begin{cases}
1, & |y| \leq r, \\
\log(|y|/R), & r \leq |y| \leq R, \\
0, & |y| \geq R.
\end{cases}
\]
It is easy to see that $\chi^r_R \in W^1_\infty(\mathbb{R}^6)$ and
\[
\int_{B(0, R)} |\nabla \chi^r_R|^6 = \frac{\omega_6}{(\log(R/r))^6}, \tag{3.36}
\]
where $\omega_6$ is the measure of the unit sphere in $\mathbb{R}^6$. 

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Next observe that Young and Hölder’s inequalities, (3.1), and (3.36) imply, for any $\varepsilon > 0$,
$$
\int_{B(0,R)} |\nabla [\chi_R u_k]|^2 \, dy \leq (1 + \varepsilon) \int_{B(0,R)} |\chi_R|^2 |\nabla u_k|^2 \, dy + \left(1 + \frac{1}{\varepsilon}\right) \int_{B(0,R)} |u_k|^2 |\nabla \chi_R|^2 \, dy
$$
$$
\leq (1 + \varepsilon) \int_{B(0,R)} |\chi_R|^2 |\nabla u_k|^2 \, dy + \left(1 + \frac{1}{\varepsilon}\right) \|u_k\|_{L^2}^2 \left(\int_{B(0,R)} |\nabla \chi_R|^6 \, dy\right)^{\frac{1}{6}}
$$
$$
\leq (1 + \varepsilon) \int_{B(0,R)} |\chi_R|^2 |\nabla u_k|^2 \, dy + \left(1 + \frac{1}{\varepsilon}\right) \frac{C\omega_6^{\frac{1}{3}}}{(\log(R/r))^{\frac{1}{2}}} \int_{R^6} |\nabla u_k|^2 \, dy
$$
Multiplying the above expression by $\gamma_k$ and summing up we obtain
$$
K(\chi_R u) \leq \sum_{k=1}^l \gamma_k \int_{B(0,R)} |\nabla u_k|^2 \, dy + \left[\varepsilon + \left(1 + \frac{1}{\varepsilon}\right) \frac{\zeta^2}{(\log(R/r))^{\frac{1}{2}}}\right] K(u),
$$
where $\zeta = \sqrt{C\omega_6^{\frac{1}{3}}}$. By taking $\varepsilon = \sqrt{\delta + 1} - 1$ and
$$
C(\delta) := \exp \left[-\left(\frac{\zeta}{\sqrt{\delta + 1} - 1}\right)^\frac{6}{5}\right],
$$
we see that if $r/R \leq C(\delta)$ then
$$
\varepsilon + \left(1 + \frac{1}{\varepsilon}\right) \frac{\zeta^2}{(\log(R/r))^{\frac{1}{2}}} \leq \delta
$$
and (3.34) follows. For (3.35) note that
$$
\int_{R^6 \setminus B(0,r)} |\nabla [(1 - \chi_R^r)u_k]|^2 \, dy \leq (1 + \varepsilon) \int_{R^6 \setminus B(0,r)} |1 - \chi_R^r|^2 |\nabla u_k|^2 \, dy
$$
$$
\quad + \left(1 + \frac{1}{\varepsilon}\right) \int_{R^6 \setminus B(0,r)} |u_k|^2 |\nabla (1 - \chi_R^r)|^2 \, dy.
$$
So, since $|\nabla (1 - \chi_R^r)|^2 = |\nabla \chi_R^r|^2$ and $\chi_R^r = 0$ outside $B(0,R)$ we see that (3.35) follows as in
(3.34).

Now we are able to establish the following localized version of the Sobolev inequality. It will be used to rule out dichotomy in the the concentration-compactness lemma below.

**Corollary 3.9.** Let $u \in \dot{H}^1(R^6)$ with $u \geq 0$. Fix $\delta > 0$ and $r/R \leq C(\delta)$ with $C(\delta)$ as in Lemma 3.8, then
$$
\int_{B(x,r)} F(u) \, dy \leq S^{-\frac{1}{2}} \left[ \int_{B(x,r)} \sum_{k=1}^l \gamma_k |\nabla u_k|^2 \, dy + \delta K(u) \right]^{\frac{1}{2}}, \quad (3.37)
$$
$$
\int_{R^6 \setminus B(x,r)} F(u) \, dy \leq S^{-\frac{1}{2}} \left[ \int_{R^6 \setminus B(x,r)} \sum_{k=1}^l \gamma_k |\nabla u_k|^2 \, dy + (2\delta + \delta^2) K(u) \right]^{\frac{1}{2}}. \quad (3.38)
$$

**Proof.** Without loss of generality we may assume $x = 0$. Note that $\chi_R^r = 1$ on $B(0,r)$ and $\text{supp}(\chi_R^r) = B(0,R)$. Then, (3.12) and (3.34) give
$$
\int_{B(0,r)} F(u) \, dx \leq \int_{R^6} F(\chi_R^r u) \, dx
$$
$$
\quad \leq S^{-\frac{1}{2}} K(\chi_R^r u)^{\frac{1}{2}}
$$
$$
\quad \leq S^{-\frac{1}{2}} \left[ \sum_{k=1}^l \int_{B(0,R)} \gamma_k |\nabla u_k|^2 \, dx + \delta K(u) \right]^{\frac{1}{2}},
$$
which is \([3.37]\). To prove \([3.38]\) we use the cut-off function \((1 - \chi_R^r)\chi_{R_2}, r < R < R_1 < R_2\) and \(R_1/R_2 \leq C(\delta)\). Indeed, since \((1 - \chi_R^r)\chi_{R_2} = 1\) on \(B(0, R_1) \setminus B(0, R)\) we have

\[
\int_{B(0,R_1)\setminus B(0,R)} F(u) \, dx = \int_{B(0,R_1)\setminus B(0,R)} F\left(\chi_{R_2}^{R_1}(1 - \chi_R^r) u \right) \, dx
\]

\[
\leq \int_{B(0,R_1)} F\left(\chi_{R_2}^{R_1}(1 - \chi_R^r) u \right) \, dx
\]

\[
\leq S^{-\frac{3}{2}} \left[ \sum_{k=1}^{l} \int_{B(0,R_2)} \gamma_k |\nabla[(1 - \chi_R^r)u_k]|^2 \, dx + \delta K((1 - \chi_R^r)u) \right]^{\frac{3}{2}},
\]

where we have used \([3.37]\) in the last inequality. Since \(R_1\) and \(R_2\) can be taken arbitrarily large satisfying \(R_1/R_2 \leq C(\delta)\), the above inequality implies that

\[
\int_{\mathbb{R}^6\setminus B(0,R)} F(u) \, dx \leq S^{-\frac{3}{2}} \left[ K((1 - \chi_R^r)u) + \delta K((1 - \chi_R^r)u) \right]^{\frac{3}{2}}.
\]

Finally, \([3.35]\) yields

\[
\int_{\mathbb{R}^6\setminus B(0,R)} F(u) \, dx \leq S^{-\frac{3}{2}} \left[ \sum_{k=1}^{l} \int_{\mathbb{R}^6\setminus B(x,r)} |\nabla u_k|^2 \, dy + \delta K(u) + \delta(1 + \delta)K(u) \right]^{\frac{3}{2}},
\]

which is the desired. \(\square\)

The following result is an adapted version of Lemma 1.7.4 in \([9]\).

**Lemma 3.10.** Let \((u_m) \subset L^3(\mathbb{R}^6)\) be such that \(u_m \geq 0\) and \(\int F(u_m) \, dx = 1\), for all \(m\). Consider the concentration function \(Q_m(R)\) of \(F(u_m)\), i.e.,

\[
Q_m(R) = \sup_{y \in \mathbb{R}^6} \int_{B(y,R)} F(u_m) \, dx, \quad R > 0.
\]

Then, for each \(m\) there exists \(y = y(m, R) \in \mathbb{R}^6\) such that

\[
Q_m(R) = \int_{B(y,R)} F(u_m) \, dx.
\]

**Proof.** Fix \(m \in \mathbb{N}\). Given \(R > 0\), from the definition of \(Q_m\) there exists a sequence \((y_j) \subset \mathbb{R}^6\) such that

\[
Q_m(R) = \lim_{j \to \infty} \int_{B(y_j,R)} F(u_m) \, dx > 0.
\]

Thus, there is \(j_0\) such that if \(j \geq j_0\) then \(\int_{B(y_j,R)} F(u_m) \, dx \geq \varepsilon\), where \(\varepsilon\) is a positive constant.

We claim that \((y_j)\) is bounded. Otherwise, there exists an infinite subsequence, still denoted by \((y_j)\), such that \(B(y_j, R) \cap B(y_i, R) = \emptyset\), for all \(i \neq j\). Then,

\[
1 = \int F(u_m) \, dx \geq \sum_{j \geq j_0} \int_{B(y_j,R)} F(u_m) \, dx = +\infty,
\]

which is a contradiction. Hence, \((y_j)\) has a convergent subsequence \((y_{j_*})\), with limit \(y = y(m, R)\). An application of the dominated convergence theorem gives

\[
Q_m(R) = \lim_{j_* \to \infty} \int_{B(y_{j_*},R)} F(u_m) \, dx = \int_{B(y,R)} F(u_m) \, dx,
\]

and the proof is completed. \(\square\)
3.3. Proof of Theorem 3.3

With the results introduced in last section we are able to prove Theorem 3.3. We start with a consequence of the results presented in the previous subsection.

**Theorem 3.11.** Suppose that \((u_m)\) is any minimizing sequence for \((3.11)\) with \(u_m \geq 0\). Then, up to translation and dilation, it is relatively compact in \(D\); i.e., there is a subsequence \((u_{m_j})\) and sequences \((R_j) \subset \mathbb{R}\) and \((y_j) \subset \mathbb{R}^6\) such that

\[
v_j(x) := R_j^{-2}u_{m_j}\left(R_j^{-1}(x - y_j)\right),
\]

converges strongly in \(D\) to some \(v\), which is a minimizer for \((3.11)\).

**Proof.** Let \((u_m) \subset D\) be any minimizing sequence of \((3.11)\) with \(u_m \geq 0\), that is,

\[
\lim_{m \to \infty} K(u_m) = S \quad \text{and} \quad P(u_m) = \int F(u_m) \, dx = 1. \tag{3.39}
\]

**Claim 1.** There are sequences \((R_m) \subset \mathbb{R}\) and \((y_m) \subset \mathbb{R}^6\) such that

\[
v_m(x) := R_m^{-2}u_m\left(R^{-1}_m(x - y_m)\right), \tag{3.40}
\]

satisfies

\[
\sup_{y \in \mathbb{R}^6} \int_{B(y,1)} F(v_m(x)) \, dx = \int_{B(0,1)} F(v_m) \, dx = \frac{1}{2}. \tag{3.41}
\]

To prove this let us consider the following scaling

\[
v_{m,R}^w(x) := R^{-2}u_m\left(R^{-1}_m(x - w)\right), \quad R > 0, \; w \in \mathbb{R}^6.
\]

From Remark 3.3 we have \(K(v_{m,R}^w) = K(u_m)\) and \(P(v_{m,R}^w) = P(u_m) = 1\). Denote by \(Q_{m,R}^w(t)\) the concentration function corresponding to \(F(v_m)\), i.e.,

\[
Q_{m,R}^w(t) := \sup_{y \in \mathbb{R}^6} \int_{B(y,t)} F(v_{m,R}^w(x)) \, dx.
\]

A change of variable gives that \(Q_m(t/R) = Q_{m,R}^w(t)\), for all \(t \geq 0\) and \(w \in \mathbb{R}^6\), where, as in Lemma 3.10,

\[
Q_m(t) = \sup_{y \in \mathbb{R}^6} \int_{B(y,t)} F(u_m) \, dx.
\]

In particular, \(Q_m(1/R) = Q_{m,R}^w(1)\) for all \(m\). Since for each \(m\), \(Q_m\) is a non-decreasing function with \(Q_m(0) = 0\) and \(\lim_{t \to \infty} Q_m(t) = 1\) we have

\[
\lim_{R \to 0^+} Q_{m,R}^w(1) = \lim_{R \to 0^+} Q_m(1/R) = 1.
\]

Hence, for each \(m\) we may choose a number \(R_m > 0\) such that

\[
Q_{m,R_m}^w(1) = Q_m(1/R_m) = \frac{1}{2}, \quad \text{for any } w \in \mathbb{R}^6, \tag{3.42}
\]

that is,

\[
\sup_{y \in \mathbb{R}^6} \int_{B(y,1)} F(v_{m,R_m}^w(x)) \, dx = Q_{m,R_m}^w(1) = \frac{1}{2}, \quad \text{for any } w \in \mathbb{R}^6. \tag{3.43}
\]

On the other hand, since \(\int F(v_{m,R}^w) = 1\) and \(v_{m,R}^w \geq 0\), Lemma 3.10 implies that there is \(y_m \in \mathbb{R}^6\) such that

\[
\sup_{y \in \mathbb{R}^6} \int_{B(y,1)} F\left(R_m^{-2}u_m\left(R_m^{-1}(x - w)\right)\right) \, dx = \sup_{y \in \mathbb{R}^6} \int_{B(y,1)} F(v_{m,R_m}^w(x)) \, dx
\]

\[
= \int_{B(y_m,1)} F(v_{m,R_m}^w(x)) \, dx
\]

\[
= \int_{B(0,1)} F\left(R_m^{-2}u_m\left(R_m^{-1}(z + y_m - w)\right)\right) \, dz,
\]
where we have used the change of variables $x = z + y_m$. Thus, taking $w = 2y_m$ in the above equality and using (3.43) we obtain

$$
\int_{B(0,1)} F \left( R_m^{-2} u_m \left( R_m^{-1} (z - y_m) \right) \right) \, dz = \sup_{y \in \mathbb{R}^6} \int_{B(y,1)} F \left( R_m^2 u_m \left( R_m^{-1} (x - 2y_m) \right) \right) \, dx
$$

$$
= Q_{R_m,2y_m}(1)
$$

$$
= \frac{1}{2},
$$

which is the second equality in (3.41). The first one also follows in view of (3.43).

Next, from Remark 3.7 and Claim 1 we have that $(v_m)$ is a minimizing sequence for (3.11) with $v_m \geq 0$, which means that

$$
\lim_{m \to \infty} K(v_m) = S \quad \text{and} \quad P(v_m) = \int F(v_m) \, dx = 1, \text{ for all } m \in \mathbb{N}. \quad (3.44)
$$

In particular, $(v_m)$ is uniformly bounded in $D$. Then, there exist a subsequence, still denoted by $(v_m)$, and $v \in \dot{H}^1(\mathbb{R}^6)$ such that

$$
v_m \rightharpoonup v, \quad \text{in } \dot{H}^1(\mathbb{R}^6). \quad (3.45)
$$

It follows from Remark 3.7 that $v \geq 0$.

Define the sequences of measures $(\mu_m)$ and $(\nu_m)$ by

$$
\mu_m = \sum_{k=1}^{l} \gamma_k \left( \nabla v_{km} \right)^2 \, dx, \quad \text{and} \quad \nu_m = F(v_m) \, dx. \quad (3.46)
$$

From (3.44) we have that $(\nu_m)$ is a probability sequence of measures. Then by Lemma 3.5 we know that, up to a subsequence, one of the three cases occur: vanishing, dichotomy, or compactness. We will show that neither vanishing nor dichotomy occur.

**Claim 2.** Vanishing does not occur.

This follows immediately from (3.41) because

$$
\lim_{m \to \infty} \sup_{y \in \mathbb{R}^6} \nu_m(B(y,1)) \geq \frac{1}{2}.
$$

**Claim 3.** Dichotomy does not occur.

In fact, suppose by contradiction that dichotomy occurs. Then, there is $\lambda \in (0,1)$ such that for any $\epsilon > 0$ there exist a number $R > 0$ and a sequence $(x_m)$ with the property: given $R' > R$ and $m$ sufficiently large,

$$
\nu_m(B(x_m,R)) \geq \lambda - \epsilon, \quad \nu_m(\mathbb{R}^6 \setminus B(x_m,R')) \geq 1 - \lambda - \epsilon. \quad (3.47)
$$

For $m$ (large) fixed and a given $\delta > 0$, Corollary 3.9 implies that choosing $\rho$ such that $R < \rho < R'$ with $\frac{\rho}{R} \leq C(\delta)$ and $\frac{R}{\rho} \leq C(\delta)$ we obtain

$$
\int_{B(x_m,R)} F(v_m) \, dx \leq S^{-\frac{3}{2}} \left[ \sum_{k=1}^{l} \int_{B(x_m,\rho)} \gamma_k \left( \nabla v_{km} \right)^2 \, dx \right]^{\frac{3}{2}},
$$

$$
\int_{\mathbb{R}^6 \setminus B(x_m,R')} F(v_m) \, dx \leq S^{-\frac{3}{2}} \left[ \sum_{k=1}^{l} \int_{\mathbb{R}^6 \setminus B(x_m,\rho)} \gamma_k \left( \nabla v_{km} \right)^2 \, dx + (2\delta + \delta^2) K(v_m) \right]^{\frac{3}{2}},
$$

These inequalities combined with (3.47) lead to

$$
S \left[ (\lambda - \epsilon) \frac{3}{2} + (1 - \lambda - \epsilon) \frac{3}{2} \right] \leq K(v_m) + (3\delta + \delta^2) K(v_m). \quad (3.48)
$$
According to (3.44), the right-hand side of (3.48) is bounded by $K(v_m) + (3\delta + \delta^2)M$, for some positive constant $M$ independent of $m$. Thus, as $\epsilon, \delta \to 0$ and $m \to \infty$ we obtain

$$S \left[ \lambda \frac{2}{3} + (1 - \lambda) \frac{2}{3} \right] \leq S,$$

that is, $\lambda \frac{2}{3} + (1 - \lambda) \frac{2}{3} \leq 1$. This is a contradiction with the fact that $\lambda \frac{2}{3} + (1 - \lambda) \frac{2}{3} > 1$ for $\lambda \in (0, 1)$. Hence, dichotomy does not occur.

As a consequence of Lemma 3.5 there is a sequence $(x_m) \subset \mathbb{R}^6$, such that for any $\epsilon > 0$ there exists a positive number $R$ with

$$\nu_m(B(x_m, R)) \geq 1 - \epsilon, \quad \text{for all } m.$$

**Claim 4.** The sequence $(\nu_m)$ is uniformly tight.

In fact, we start claiming that $B(x_m, R) \cap B(0, 1) \neq \emptyset$, for all $m$. Suppose the contrary, that is, there exists $m_0$ such that $B(x_{m_0}, R) \cap B(0, 1) = \emptyset$. Taking $0 < \epsilon < \frac{1}{2}$ in (3.50) we have

$$\int_{B(x_{m_0}, R)} F(v_{m_0}) \, dx > \frac{1}{2}.$$  

This combined with the normalization condition (3.41) lead to

$$\int F(v_{m_0}) \, dx \geq \int_{B(x_{m_0}, R)} F(v_{m_0}) \, dx + \int_{B(0, 1)} F(v_{m_0}) \, dx > \frac{1}{2} + \frac{1}{2} = 1,$$

which contradicts (3.41). Hence, the claim follows.

Next, because $B(x_m, R) \subset B(0, 2R + 1)$, for all $m$, (3.50) yields

$$\nu_m(B(0, 2R + 1)) \geq 1 - \epsilon, \quad \forall m.$$  

Consequently, since $(\nu_m)$ is a sequence of probability measures,

$$\nu_m \left( \mathbb{R}^6 \setminus B(0, 2R + 1) \right) = 1 - \nu_m(B(0, 2R + 1)) \leq \epsilon, \quad \text{for all } m,$$

that is, $(\nu_m)$ is a uniformly tight sequence, as claimed.

**Claim 5.** Up to a subsequence, $(\nu_m)$ converges weakly to some $\nu \in \mathcal{M}_+^1(\mathbb{R}^6)$.

Indeed, first note that for any $f \in C_c(\mathbb{R}^6)$,

$$\left| \int f \, d\nu_m \right| \leq \|f\|_{L^\infty} \nu_m(\mathbb{R}^6) = \|f\|_{L^\infty} < \infty.$$

Thus, from Lemma 2.1 there is $\nu \in \mathcal{M}_+^b(\mathbb{R}^6)$ such that, up to a subsequence, $\nu_m \rightharpoonup \nu$ in $\mathcal{M}_+^b(\mathbb{R}^6)$. The uniform tightness of $(\nu_m)$ then implies that $\nu_m \rightharpoonup \nu$ weakly in $\mathcal{M}_+^b(\mathbb{R}^6)$ (see for instance [1] Theorem 30.8), i.e.,

$$\int f \, d\nu_m \to \int f \, d\nu, \quad \text{for any } f \in C_b(\mathbb{R}^6).$$

In particular, by taking $f \equiv 1$, we obtain

$$\nu(\mathbb{R}^6) = \lim_{m \to \infty} \nu_m(\mathbb{R}^6) = 1,$$

from which we deduce that $\nu \in \mathcal{M}_+^1(\mathbb{R}^6)$.

Next, because $(K(v_m))$ is uniformly bounded it follows that $(\mu_m)$ is also vaguely bounded. Then, up to a subsequence, there exists $\mu \in \mathcal{M}_+^b(\mathbb{R}^6)$ such that

$$\mu_m \rightharpoonup \mu \quad \text{in } \mathcal{M}_+^b(\mathbb{R}^6).$$

In particular we have $\mu(\mathbb{R}^6) \leq \liminf_{m \to \infty} \mu_m(\mathbb{R}^6)$. 

Now, (3.45), (3.51) and (3.53) allow us to invoke Lemma 3.6 to obtain
\[
\mu \geq \sum_{k=1}^{l} \gamma_k |\nabla v_k|^2 \, dx + \sum_{j \in J} \mu_j \delta_{x_j}, \quad \text{and} \quad \nu = F(v) \, dx + \sum_{j \in J} \nu_j \delta_{x_j},
\]
for some family \(\{x_j \in \mathbb{R}^6 : j \in J\}\), \(J\) countable, and \(\mu_j, \nu_j\) non-negative numbers satisfying
\[
\nu_j \leq S^{-\frac{2}{3}} \mu_j^\frac{2}{3}, \quad \text{for any } j \in J.
\]
with \(\sum_{j \in J} \nu_j^\frac{2}{3}\) convergent. Consequently, (3.52), (3.54) and (3.55) give
\[
S = \lim_{m \to \infty} \mu_m(\mathbb{R}^6) \geq \mu(\mathbb{R}^6) \geq K(v) + \sum_{j \in J} \mu_j \\
\geq S \left[ P(v) + \sum_{j \in J} \nu_j^\frac{2}{3} \right]^\frac{3}{2} \\
> S \left[ P(v) + \sum_{j \in J} \nu_j \right]^\frac{2}{3} \\
= S \left[ \nu(\mathbb{R}^6) \right]^\frac{2}{3} \\
= S,
\]
where we also have used that \(\lambda \mapsto \lambda^{2/3}\) is a strictly concave function. Thus, all inequalities in (3.56) are indeed equalities. But by the strictly concavity of the function \(\lambda \mapsto \lambda^{2/3}\), for (3.56) to be an equality at most one of the terms \(P(v)\) or \(\nu_j, j \in J\), must be different from zero.

**Claim 6.** We claim that \(\nu_j = 0\) for all \(j \in J\).

Otherwise, assume \(\nu_{j_0} \neq 0\) for some \(j_0 \in J\). Then from the above discussion, (3.52) and the decomposition (3.54) we obtain \(\nu = \nu_{j_0} \delta_{x_{j_0}}\), and then
\[
1 = \nu(\mathbb{R}^6) = \nu_{j_0}.
\]
The normalization condition (3.41) gives
\[
\frac{1}{2} \geq \int_{B(x_{j_0}, 1)} F(v_m) \, dx = \nu_m(B(x_{j_0}, 1)), \quad \text{for all } m,
\]
which leads to
\[
\frac{1}{2} \geq \lim_{m \to \infty} \nu_m(B(x_{j_0}, 1)) = \nu(B(x_{j_0}, 1)) = \int_{B(x_{j_0}, 1)} d\nu = \nu_{j_0},
\]
where the first equality is a consequence of the weak convergence (3.51) (see, for instance, [1, Theorem 30.12]). But, this contradicts (3.57) and the claim is proved.

Therefore, it must be the case that \(\nu = F(v) \, dx\) and from (3.52)
\[
P(v) = \int F(v) \, dx = 1,
\]
which means that \(v \in D\).

It remains to prove that \(K(v) = S\). From (3.58) and the definition of \(S\) we know that \(S \leq K(v)\). On the other hand, the lower semi-continuity of the weak convergence (3.45) gives
\[
K(v) \leq \liminf_{m \to \infty} K(v_m) = S. \quad \text{Hence, we conclude that } K(v) = S = \lim_{m \to \infty} K(v_m) \text{ and also that } v_m \to v \text{ strongly in } D. \quad \text{This finishes the proof.} \]
Note that actually we have proved the following:

**Corollary 3.12.** There exists \( v \in D \) satisfying \( P(v) = 1 \) and \( K(v) = C_6^{-\frac{2}{3}} \), where \( C_6 \) is the best constant in the general critical Sobolev-type inequality (3.9).

Finally we are now in a position to prove the existence of ground state solutions for (3.3).

**Proof of Theorem 3.3.** Let \( v \) be the minimizer of (3.11) obtained in Theorem 3.11. From the Lagrange multiplier theorem, there exists a constant \( \Lambda \) such that

\[
2\gamma_k \int \nabla v_k \cdot \nabla g_k \, dx = \Lambda \int f_k(v)g_k \, dx,
\]

for any \( g \in \dot{H}^1(\mathbb{R}^6) \). By taking \( g = v \) in (3.59) we promptly see that \( \Lambda \neq 0 \). Now, define \( \psi_0(x) := \frac{\Lambda}{2}v(x) \). By the above discussion \( \psi_0 \) is non-trivial. We will see that \( \psi_0 \) is a ground state solution for (3.3). First of all, note that \( \psi_0 \) is a solution (3.3). Indeed, from (3.59) we have, for any \( g \in \dot{H}^1(\mathbb{R}^6) \),

\[
\gamma_k \int \nabla \psi_0 \cdot \nabla g_k \, dx = \frac{\Lambda}{2} \int \nabla v \cdot \nabla g_k \, dx = \int \left( \frac{\Lambda}{2} \right)^2 f_k(v)g_k \, dx = \int f_k(\psi_0)g_k \, dx,
\]

where we have used (2.5) in the last equality.

Next, since \( \psi_0 \) is a solution, from Remark 3.2 it follows that \( J(\psi_0) = \frac{6^\frac{3}{2}}{\gamma} I(\psi_0)^\frac{1}{2} \). On the other hand, according to Remark 3.4 (i), \( v \) is a minimizer of \( J \) and since \( J(\psi_0) = J(v) \), so is \( \psi_0 \). Consequently, \( \psi_0 \) is a ground state, as we required.

**Remark 3.13.** We actually know the exact value of the Lagrange multiplier \( \Lambda \). Indeed, since \( \psi_0 \) is a solution of (3.3) we have

\[
\left( \frac{\Lambda}{2} \right)^2 K(v) = K(\psi_0) = 3P(\psi_0) = 3 \left( \frac{\Lambda}{2} \right)^3 P(v).
\]

Hence, recalling that \( K(v) = C_6^{-\frac{2}{3}} \) and \( P(v) = 1 \) we deduce that \( \Lambda = \frac{2}{3} C_6^{-\frac{2}{3}} \).

**Corollary 3.14.** The inequality

\[
P(u) \leq C_6^{opt} K(u)^\frac{3}{2},
\]

holds, for all \( u \in D \), with the optimal constant \( C_6^{opt} \) given by

\[
C_6^{opt} = \frac{1}{\frac{3}{2} \mathcal{E}(\psi)^\frac{1}{2}},
\]

where \( \psi \) is any ground state solution of (3.3).

**Proof.** In Remark 3.4 we saw that (3.60) holds with \( C_6^{-1} = C_6^{opt} = \inf \{ J(u); u \in D \} \). Now if \( \psi \) is any ground state of (3.3), Remark 3.2 implies that

\[
C_6^{-1} = J(\psi) = \frac{6^\frac{3}{2}}{2} I(\psi)^\frac{1}{2} = 3^\frac{3}{2} \mathcal{E}(\psi)^\frac{1}{2},
\]

which is the desired.

**Remark 3.15.** Note that all ground states of (3.3) have the same energy. Therefore, the constant \( C_6 \) does not depend on the choice of the ground state.
4. Blow-up results

This section aims to show the existence of blows-up solutions of (1.1). To give the precise statement of our result we set

\[ G := \begin{cases} \mathcal{G}_5(1,0), & \text{if } n = 5 \\ \mathcal{G}_6, & \text{if } n = 6. \end{cases} \]

Thus the main result of this section is the following.

**Theorem 4.1.** Assume that \( u_0 \in H^1(\mathbb{R}^n) \) and let \( u \) be the corresponding solution of (1.1) defined in the maximal time interval of existence, say \( I \).

(i) If \( n = 5 \) assume

\[ Q(u_0)E(u_0) < Q(\psi)E(\psi), \quad (4.1) \]

and

\[ Q(u_0)K(u_0) > Q(\psi)K(\psi). \quad (4.2) \]

(ii) If \( n = 6 \) assume

\[ E(u_0) < E(\psi) \quad (4.3) \]

and

\[ K(u_0) > K(\psi). \quad (4.4) \]

where \( E \) is the energy defined in (3.8) and \( \psi \in \mathcal{G} \).

Then, if \( u_0 \) is radially symmetric we have that \( I \) is finite.

As we already said, to prove Theorem 4.1 we follow closely the arguments in [23]. Let us start by introducing, for \( \varphi \in C^\infty_0(\mathbb{R}^n) \),

\[ V(t) = \int \varphi(x) \left( \sum_{k=1}^{l} \frac{\alpha_k^2}{\gamma_k} |u_k|^2 \right) dx. \]

Then, the solution \( u \) of system (1.1) satisfies

\[ V'(t) = 2 \sum_{k=1}^{l} \alpha_k \text{Im} \int \nabla \varphi \cdot \nabla u_k \overline{u}_k dx - 4 \int \varphi(x) \text{Im} \sum_{k=1}^{l} m_k f_k(u) \overline{u}_k dx \]

\[ =: \mathcal{R}(t) - 4 \int \varphi(x) \text{Im} \sum_{k=1}^{l} m_k f_k(u) \overline{u}_k dx. \quad (4.5) \]

If \( u_0 \) is a radially symmetric function, so is the corresponding solution \( u \). Hence, if in addition \( \varphi \) is radially symmetric, by a direct calculation (see for instance [26, Lemma 2.9] or [34, Theorem 5.5]) we can rewrite \( \mathcal{R}' \) as

\[ \mathcal{R}'(t) = 4 \int \varphi'' \left( \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \right) dx - \int \Delta^2 \varphi \left( \sum_{k=1}^{l} \gamma_k |u_k|^2 \right) dx - 2\text{Re} \int \Delta \varphi F(u) \ dx. \quad (4.6) \]

The approach used in [23] to prove the existence of blow-up solutions consists in getting a contradiction by working with \( \mathcal{R} \) and \( \mathcal{R}' \) instead of \( V \) and \( V'' \).

We start with two technical lemmas.

**Lemma 4.2.** Assume that \( n \geq 1 \). Let \( r = |x|, \ x \in \mathbb{R}^n \). Define, for a positive constant \( c \),

\[ \chi(r) = \begin{cases} r^c, & 0 \leq r \leq 1, \\ c, & r \geq 3. \end{cases} \quad (4.7) \]

Assume also that \( \chi''(r) \leq 2 \) and \( 0 \leq \chi'(r) \leq 2r \), for any \( r \geq 0 \). Let \( \chi_R(r) = R^2 \chi(r/R) \).

Then, if \( r \leq R \),

\[ \Delta \chi_R(r) = 2n \quad \text{and} \quad \Delta^2 \chi_R(r) = 0. \quad (4.8) \]
On the other hand, if \( r \geq R \), then
\[
\Delta \chi_{R}(r) \leq C \quad \text{and} \quad |\Delta^{2} \chi_{R}(r)| \leq \frac{C}{R^{2}}, \quad (4.9)
\]
where \( C \) is a constant independent of \( R \).

**Proof.** The proof is a straightforward computation. \( \square \)

**Lemma 4.3.** Let \( I \) be an open interval with \( 0 \in I \). Let \( a \in \mathbb{R} \), \( b > 0 \) and \( q > 1 \). Define 
\[
\gamma = (bq)^{-\frac{1}{q-1}} \quad \text{and} \quad f(r) = a - r + br^{q}, \quad \text{for} \quad r \geq 0.
\]
Let \( G(t) \) a non-negative continuous function such that \( f \circ G \geq 0 \) on \( I \). Assume that \( a < \left( 1 - \frac{1}{q} \right) \gamma \).

(i) If \( G(0) < \gamma \), then \( G(t) < \gamma \), \( \forall t \in I \).
(ii) If \( G(0) > \gamma \), then \( G(t) > \gamma \), \( \forall t \in I \).

**Proof.** See, for instance, [2, Lemma 5.2], [10, Lemma 4.2] or [35, Lemma 3.1]. \( \square \)

4.1. **Proof of Theorem 4.1**
In this section we will prove Theorem 4.1. Let us start by introducing the “Pohozaev” functional,
\[
\mathcal{T}_{n}(u(t)) = K(u(t)) - \frac{n}{2}P(u(t)), \quad n = 5, 6. \quad (4.10)
\]
From the definition of the energy functional we may write
\[
\mathcal{T}_{n}(u(t)) = \frac{n}{4}E(u(t)) - \left( \frac{n-4}{4} \right)K(u(t)) - \frac{n}{4}L(u(t)). \quad (4.11)
\]

Our first result establishes that under the assumptions of Theorem 4.1 the Pohozaev functional is strictly negative.

**Lemma 4.4.** Under the assumptions of Theorem 4.1 there exists \( \delta > 0 \) such that
\[
\mathcal{T}_{n}(u(t)) \leq -\delta < 0, \quad t \in I.
\]

**Proof.** We follow the ideas presented in the proof of Theorem 1.3 in [9]. We will give the proof only in the cases \( n = 5 \). The analysis for \( n = 6 \) follows exactly the same strategy using the results in Section 3.

We first note that by Lemma 2.8 and the definition of the energy functional we obtain
\[
K(\psi) = 5\mathcal{E}(\psi). \quad (4.12)
\]
Since \( \psi \in \mathcal{G}_{5}(1, 0) \) the functionals \( Q \) in (2.10) and \( Q \) are the same. Therefore, from [H6] and (2.15),
\[
K(u) = E(u_{0}) - L(u) + 2P(u) \leq E(u_{0}) + 2|P(u)| \leq E(u_{0}) + 2C_{G}^{opt}Q(u_{0})^{\frac{1}{2}}K(u)^{\frac{1}{2}}, \quad (4.13)
\]
Now, in the notation of Lemma 4.3 if we take \( G(t) = K(u(t)), \ a = E(u_{0}), \ b = 2C_{G}^{opt}Q(u_{0})^{\frac{1}{2}} \) and \( q = \frac{5}{4} \), then \( \gamma = 5Q(u_{0})^{\frac{1}{2}} \) and from (4.13) \( f \circ G \geq 0 \). Moreover, by using (4.12) a direct calculation gives
\[
a < \left( 1 - \frac{1}{q} \right) \gamma \iff Q(u_{0})E(u_{0}) < Q(\psi)\mathcal{E}(\psi), \quad G(0) > \gamma \iff Q(u_{0})K(u_{0}) > Q(\psi)K(\psi).
\]
Hence, an application of Lemma 4.3 yields
\[
Q(u_{0})K(u(t)) > Q(\psi)K(\psi), \quad t \in I. \quad (4.14)
\]
Thus, from (4.14), (4.12), and (4.14) we have
\[
\frac{5}{4}E(u(t)) = \frac{5}{4}E(u_{0}) < \frac{5}{4}\mathcal{E}(\psi) \frac{Q(\psi)}{Q(u_{0})} \leq \frac{1}{4}K(\psi) \frac{Q(\psi)}{Q(u_{0})} < \frac{1}{4}K(u(t)).
\]
This combined with (4.11) yields
\[
\mathcal{T}_{5}(u(t)) < 0, \quad t \in I. \quad (4.15)
\]
We claim that there exists $\sigma_0 > 0$ such that

$$\mathcal{T}_5(u(t)) < -\sigma_0 K(u(t)), \quad t \in I. \tag{4.16}$$

Indeed, if $E(u_0) \leq 0$ from (4.11) we can promptly take $\sigma_0 = \frac{1}{4}$. Now suppose $E(u_0) > 0$ and assume by contradiction that (4.16) does not hold. Then we can find sequences $(t_m) \subset I$ and $(\sigma_m) \subset \mathbb{R}_+$ with $\sigma_m \to 0$ such that

$$-\sigma_m \frac{1}{4} K(u(t_m)) \leq \mathcal{T}_5(u(t_m)) < 0.$$

Thus, the last inequality and (4.11) gives

$$E(u(t_m)) = \frac{4}{5} \mathcal{T}_5(u(t_m)) + \frac{1}{5} K(u(t_m)) + L(u(t_m))$$

$$\geq -\sigma_m \frac{1}{4} K(u(t_m)) + \frac{1}{5} K(u(t_m)) + L(u(t_m))$$

$$\geq (1 - \sigma_m) \frac{1}{5} K(u(t_m)).$$

From this, the conservation of the energy, (4.14) and (4.12) we get

$$Q(u_0) E(u_0) = Q(u_0) E(u(t_m))$$

$$\geq (1 - \sigma_m) \frac{1}{5} Q(u_0) K(u(t_m))$$

$$\geq (1 - \sigma_m) \frac{1}{5} Q(\psi) K(\psi)$$

$$= (1 - \sigma_m) Q(\psi) E(\psi),$$

Taking $m \to \infty$ in the last inequality we obtain a contradiction with (4.1), so the claim is proved.

Finally note that (4.14) gives $K(u(t)) > K(\psi) \frac{Q(\psi)}{Q(u_0)} =: \epsilon_0$. Therefore the result follows immediately from (4.16). \hfill \Box

We are now in a position to prove Theorem 4.1

**Proof of Theorem 4.1** Suppose that the maximal existence interval is $I = (-T_*, T^*)$. We proceed by contradiction. Without loss of generality assume that $T^* = +\infty$. Using $\varphi(x) = \chi_R(|x|)$ with $\chi_R$ defined by (4.7), from (4.5) and (4.6) we can write

$$\mathcal{R}(t) = 2 \sum_{k=1}^{l} \alpha_k \int \nabla \chi_R \cdot \nabla u_k \overline{u}_k \, dx$$

and

$$\mathcal{R}'(t) = 8 \mathcal{T}_n(u) + 4 \int (\chi''_R - 2) \left( \sum_{k=1}^{l} \gamma_k |\nabla u_k|^2 \right) \, dx$$

$$- \int \Delta^2 \chi_R \left( \sum_{k=1}^{l} \gamma_k |u_k|^2 \right) \, dx - 2 \text{Re} \int (\Delta \chi_R - 2n) F(u) \, dx$$

$$=: 8 \mathcal{T}_n(u) + \mathcal{R}_1(t) + \mathcal{R}_2(t) + \mathcal{R}_3(t).$$

Here $R$ is seen as a parameter that will be chosen later.

Since from Lemma 4.2 we have $\chi''_R(r) \leq 2$ for all $r \geq 0$, it follows that $\mathcal{R}_1 \leq 0$. From (4.8) and the conservation of the charge,

$$\mathcal{R}_2(t) \leq \int |\Delta^2 \chi_R| \left( \sum_{k=1}^{l} \gamma_k |u_k|^2 \right) \, dx \leq C \int_{\{x \geq R\}} R^{-2} \left( \sum_{k=1}^{l} \gamma_k |u_k|^2 \right) \, dx \leq CR^{-2} Q(u_0).$$
Also, (4.8) and Lemma 2.2 imply

\[ \mathcal{R}_3 = -2\text{Re} \int_{\{|x| \geq R\}} (\Delta \chi_R - 2n) F(u) \, dx \]

\[ \leq C \int_{\{|x| \geq R\}} |\text{Re} F(u)| \, dx \]

\[ \leq C \int_{\{|x| \geq R\}} \sum_{k=1}^{l} |u_k|^3 \, dx \]

\[ = C \sum_{k=1}^{l} \|u_k\|^3_{L^3(|x| \geq R)}. \]

Recall that for any radial function \( f \in H^1(\mathbb{R}^n) \) (see, for instance, [37, equation (3.7)])

\[ \|f\|^3_{L^3(|x| \geq R)} \leq CR^{-(\frac{n-1}{2})} \|f\|_{L^2(|x| \geq R)}^\frac{5}{2} \|\nabla f\|_{L^2(|x| \geq R)}^\frac{1}{2}. \]

Hence, from Young’s inequality we can write, for any \( \epsilon > 0 \),

\[ \sum_{k=1}^{l} \|u_k\|^3_{L^3(|x| \geq R)} \leq C \sum_{k=1}^{l} R^{-(\frac{n-1}{2})} \|u_k\|_{L^2(|x| \geq R)}^\frac{5}{2} \|\nabla u_k\|_{L^2(|x| \geq R)}^\frac{1}{2} \]

\[ \leq C \epsilon R^{-\frac{2(n-1)}{3}} Q(u_0)^{\frac{5}{2}} + 2(n-4)\epsilon K(u), \]

where \( C_\epsilon \) is a positive constant depending on \( \epsilon, \alpha_k, \gamma_k \), and \( \sigma_k \).

Gathering together the estimates for \( \mathcal{R}_1, \mathcal{R}_2 \) and \( \mathcal{R}_3 \) we obtain

\[ \mathcal{R}'(t) \leq 8T_n(u) + CR^{-2}Q(u_0) + 2(n-4)\epsilon K(u) + C \epsilon R^{-\frac{2(n-1)}{3}} Q(u_0)^{\frac{5}{2}}, \quad \epsilon > 0. \]  

(4.17)

Assume that \( 0 < \epsilon < 1 \). Using (4.11) and Lemma 4.4 we get

\[ \mathcal{R}'(t) \leq 8(1-\epsilon)T_n(u) + 2n\epsilon |E(u_0)| + CR^{-2}Q(u_0) + C \epsilon R^{-\frac{2(n-1)}{3}} Q(u_0)^{\frac{5}{2}} \]

\[ \leq -8(1-\epsilon)\delta + 2n\epsilon |E(u_0)| + CR^{-2}Q(u_0) + C \epsilon R^{-\frac{2(n-1)}{3}} Q(u_0)^{\frac{5}{2}}. \]

Now in the last inequality, we fix \( R \) sufficiently large and choose \( \epsilon \) sufficiently small such that \( \mathcal{R}'(t) \leq -2\delta \). Integrating this inequality on \([0, t]\) we obtain

\[ \mathcal{R}(t) \leq -2\delta t + \mathcal{R}(0). \]  

(4.18)

On the other hand, from Hölder’s inequality we deduce

\[ |\mathcal{R}(t)| \leq 2 \sum_{k=1}^{l} \alpha_k \int R|x|/R| \|\nabla u_k\||u_k| \, dx \]

\[ \leq CR \sum_{k=1}^{l} \alpha_k \|u_k\|_{L^2} \|\nabla u_k\|_{L^2} \]

\[ \leq CRQ(u_0)^{\frac{5}{2}} K(u)^{\frac{1}{2}}. \]

(4.19)

We now may choose \( T_0 > 0 \) sufficiently large such that \( \frac{\mathcal{R}(0)}{\delta} < T_0 \). From this and (4.18),

\[ \mathcal{R}(t) \leq -\delta t < 0, \quad t \geq T_0. \]  

(4.20)

Thus, (4.19) and (4.20) lead to \( \delta t \leq -|\mathcal{R}(t)| = |\mathcal{R}(t)| \leq CRQ(u_0)^{\frac{5}{2}} K(u)^{\frac{1}{2}}, \) or equivalently,

\[ K(u(t)) \geq C_0 t^2, \quad t \geq T_0 \]  

(4.21)

for some positive constant \( C_0 \).
Moreover, taking into account that $\varepsilon$ is sufficiently small (less than 1/2 is enough) by (4.17) and (4.11) we get

$$
R'(t) \leq 2nE(u) - 2(n - 4)K(u) + CR^{-2}Q(u_0) + (n - 4)K(u) + CR^{-\frac{2(n-1)}{3}}Q(u_0)^{\frac{3}{2}}
$$

\begin{equation}
\leq -(n - 4)K(u) + 2nE(u_0) + CR^{-2}Q(u_0) + CR^{-\frac{2(n-1)}{3}}Q(u_0)^{\frac{3}{2}},
\end{equation}

where we have used the conservation of the energy and the fact that $L(u) \geq 0$. We note that the last three terms in (4.22) do not depend on $t$. So, we may take $T_1 > T_0$ such that

$$
C_0 \frac{(n-4)}{2} T_1^2 \geq 2nE(u_0) + CR^{-2}Q(u_0) + CR^{-\frac{2(n-1)}{3}}Q(u_0)^{\frac{3}{2}},
$$

where $C_0$ is the constant appearing in (4.21). Thus, (4.21) and (4.22) give

$$
R'(t) \leq -\frac{(n-4)}{2} K(u(t)), \quad t > T_1.
$$

Now integrating the last inequality on $[T_1, t]$ gives

$$
R(t) \leq \frac{(n-4)}{2} \int_{T_1}^{t} K(u(s)) \, ds + R(T_1) \leq -\frac{(n-4)}{2} \int_{T_1}^{t} K(u(s)) \, ds.
$$

Combining this with (4.19) we get

$$
\frac{(n-4)}{2} \int_{T_1}^{t} K(u(s)) \, ds \leq -R(t) = |R(t)| \leq CRQ(u_0)^{\frac{3}{2}} K(u)^{\frac{1}{2}}.
$$

(4.23)

Define $\eta(t) := \int_{T_1}^{t} K(u(s)) \, ds$ and $A := \frac{(n-4)^2}{4CRC_0Q(u_0)}$. From (4.21), we have that $\eta(t) > 0$ for $t > T_1$. Thus (4.23) can be written as $A \leq \frac{\eta(t)}{\eta^2(t)}$. Finally, taking $T_1 < T'$ and integrating on $[T', t]$ we obtain

$$
A(t-T') \leq \int_{T'}^{t} \frac{\eta'(s)}{\eta^2(s)} \, ds = \frac{1}{\eta(T')} - \frac{1}{\eta(t)} \leq \frac{1}{\eta(T')},
$$

or equivalently,

$$
0 < \eta(T') \leq \frac{1}{A(t-T')}. 
$$

Letting $t \to \infty$ we arrive to a contradiction. Hence the proof of Theorem 4.1 is completed. \(\square\)

**Remark 4.5.** As we pointed out in Theorem 2.2 if the inequality in (4.2) is reversed then the solution exists globally in time. We believe if we reverse inequality (4.4) then the solution is also global. However, since this is the energy-critical case, much more efforts is needed. A possible technique to obtain the result is the one developed in [27].

**Acknowledgement**

N.N. is partially supported by Universidad de Costa Rica, through the OAICE.

A.P. is partially supported by CNPq/Brazil grants 402849/2016-7 and 303098/2016-3 and FAPESP/Brazil grant 2019/02512-5.

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