Elementary aspects of the geometry of metric spaces

Stephen Semmes
Rice University

Abstract

The setting of metric spaces is very natural for numerous questions concerning manifolds, norms, and fractal sets, and a few of the main ingredients are surveyed here.

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A very basic introduction to the geometry of metric spaces can be found in [33], with more information in [34, 35].
1 Metric spaces

A metric space is a set $M$ equipped with a function $d(x, y)$ defined for $x, y \in M$ such that $d(x, y)$ is a nonnegative real number that is equal to 0 exactly when $x = y$,

$$d(y, x) = d(x, y)$$

for every $x, y \in M$, and

$$d(x, z) \leq d(x, y) + d(y, z)$$

for every $x, y, z \in M$, which is known as the triangle inequality.

Remember that the absolute value of a real number $r$ is denoted $|r|$ and equal to $r$ when $r \geq 0$ and to $-r$ when $r \leq 0$. It is easy to check that

$$|r + t| \leq |r| + |t|$$

and

$$|rt| = |r||t|$$

for any pair of real numbers $r, t$. The standard metric on the real line $\mathbb{R}$ is given by $|r - t|$, which is the first main example of a metric space.

If $(M, d(x, y))$ is a metric space, then $d(x, y)$ is called the distance function or metric on $M$. For each $x \in M$ and $r > 0$, the open ball in $M$ with center $x$ and radius $r$ is

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$  

Similarly, the closed ball with center $x$ and radius $r \geq 0$ is

$$\overline{B}(x, r) = \{y \in M : d(x, y) \leq r\}.$$  

Thus

$$B(x, r) \subseteq \overline{B}(x, r) \subseteq B(x, t)$$

when $r < t$. 

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Let \( a, b \) be real numbers with \( a < b \). The *open interval* \((a, b)\) in \( \mathbb{R} \) is defined by
\[
(a, b) = \{ r \in \mathbb{R} : a < r < b \},
\]
and the *closed interval* \([a, b]\) is defined by
\[
[a, b] = \{ r \in \mathbb{R} : a \leq r \leq b \}.
\]
One may also allow \( a = b \) for the latter. The *length* of these intervals is \( b - a \).

2 \hspace{1cm} \text{A little calculus}

Suppose that \( a, b \) are real numbers with \( a < b \), and that \( f(x) \) is a continuous real-valued function on the closed interval \([a, b]\) in the real line. The *extreme value theorem* states that there are elements \( p, q \) of \([a, b]\) at which \( f \) attains its maximum and minimum, which is to say that
\[
f(q) \leq f(x) \leq f(p)
\]
for every \( x \in [a, b] \). This works as well for continuous real-valued functions on compact subsets of metric spaces, or even topological spaces. If \( p \) or \( q \) is in the open interval \((a, b)\) and \( f \) is differentiable there, then the derivative \( f'(p) \) or \( f'(q) \) is equal to 0.

Suppose that \( f(x) \) is differentiable at every point in \((a, b)\). If \( f(a) = f(b) = 0 \), then *Rolle’s theorem* states that \( f'(x) = 0 \) for some \( x \in (a, b) \). This is because the maximum or minimum of \( f \) on \([a, b]\) is attained on \((a, b)\), or \( f(x) = 0 \) for every \( x \in [a, b] \). No matter the values of \( f(a), f(b) \), the *mean value theorem* says that there is an \( x \in (a, b) \) such that
\[
f'(x) = \frac{f(b) - f(a)}{b - a}.
\]
This follows from Rolle’s theorem applied to \( f - \phi \), where \( \phi(x) = \alpha x + \beta \) and \( \alpha, \beta \in \mathbb{R} \) are chosen so that \( \phi(a) = f(a), \phi(b) = f(b) \).

Of course, the derivative of a constant function is 0, and the mean value theorem implies that a continuous function \( f \) on \([a, b]\) is constant if the derivative of \( f \) exists and is equal to 0 at every point in \((a, b)\). If \( f \) is monotone increasing on \([a, b]\), in the sense that \( f(x) \leq f(y) \) when \( a \leq x \leq y \leq b \), then \( f'(x) \geq 0 \) for every \( x \in (a, b) \) at which \( f \) is differentiable. Conversely, if \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\), and \( f'(x) \geq 0 \) for each \( x \in (a, b) \), then \( f \) is monotone increasing on \([a, b]\), by the mean value theorem. If \( f'(x) > 0 \) for every \( x \in (a, b) \), then \( f \) is strictly increasing on \([a, b]\), in the sense that \( f(w) < f(y) \) when \( a \leq w < y \leq b \). However, the derivative of a strictly increasing function may be equal to 0, as when \( f(x) = x^3 \).
3 Norms on $\mathbb{R}^n$

Let $n$ be a positive integer, and let $\mathbb{R}^n$ be the space of $n$-tuples of real numbers. This means that an element $x$ of $\mathbb{R}^n$ is of the form $x = (x_1, \ldots, x_n)$, where the coordinates $x_1, \ldots, x_n$ of $x$ are real numbers. Addition and scalar multiplication on $\mathbb{R}^n$ are defined coordinatewise in the usual way, so that $\mathbb{R}^n$ becomes a finite-dimensional vector space over the real numbers.

A norm on $\mathbb{R}^n$ is a function $N(x)$ such that $N(x)$ is a nonnegative real number for every $x \in \mathbb{R}^n$ which is equal to 0 exactly when $x = 0$,

$$N(rx) = |r| N(x) \quad (3.1)$$

for every $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and

$$N(x + y) \leq N(x) + N(y) \quad (3.2)$$

for every $x, y \in \mathbb{R}^n$. If $N$ is a norm on $\mathbb{R}^n$, then

$$d_N(x, y) = N(x - y) \quad (3.3)$$

is a metric on $\mathbb{R}^n$.

For example, the absolute value function is a norm on $\mathbb{R}$, for which the corresponding metric is the standard metric on the real line. The standard Euclidean norm on $\mathbb{R}^n$ is defined by

$$|x| = \left( \sum_{j=1}^n x_j^2 \right)^{1/2} \quad (3.4)$$

and the corresponding metric is the standard Euclidean metric on $\mathbb{R}^n$. It is not so obvious that this satisfies the triangle inequality, and hence is a norm, and we shall discuss a proof of this fact in Section 5.

One can check directly that

$$\|x\|_1 = \sum_{j=1}^n |x_j| \quad (3.5)$$

and

$$\|x\|_\infty = \max(|x_1|, \ldots, |x_n|) \quad (3.6)$$

are norms on $\mathbb{R}^n$. We shall see in Section 5 that

$$\|x\|_p = \left( \sum_{j=1}^n |x_j|^p \right)^{1/p} \quad (3.7)$$

is a norm when $p \geq 1$, which includes the Euclidean norm as a special case.
4 Convex functions

A real-valued function \( f(x) \) on the real line is said to be convex if

\[
(4.1) \quad f(t x + (1-t) y) \leq t f(x) + (1-t) f(y)
\]

for every \( x, y \in \mathbb{R} \) and \( t \in [0,1] \). This is equivalent to

\[
(4.2) \quad \frac{f(w) - f(x)}{w - x} \leq \frac{f(y) - f(w)}{y - w}
\]

for every \( x, w, y \in \mathbb{R} \) such that \( x < w < y \). Applying this condition twice, we get that

\[
(4.3) \quad \frac{f(w) - f(x)}{w - x} \leq \frac{f(z) - f(y)}{z - y}
\]

when \( x < w < y < z \). As another refinement of (4.2), one can use (4.1) to show that

\[
(4.4) \quad \frac{f(w) - f(x)}{w - x} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(w)}{y - w}
\]

when \( x < w < y \).

If \( f \) is differentiable and \( f' \) is monotone increasing, then the mean value theorem implies (4.2) and hence that \( f \) is convex. Conversely, (4.3) implies that the derivative of \( f \) is monotone increasing when \( f \) is differentiable. Actually, one can show that the right and left derivatives \( f'_+(x), f'_-(x) \) exist for each \( x \in \mathbb{R} \) when \( f \) is convex, and satisfy

\[
(4.5) \quad f'_-(x) \leq f'_+(x)
\]

and

\[
(4.6) \quad f'_+(x) \leq f'_-(y)
\]

when \( x < y \). One can also show that these conditions characterize convexity, using analogues of Rolle’s theorem and the mean value theorem for functions with one-sided derivatives.

A function \( f : \mathbb{R} \to \mathbb{R} \) is strictly convex if

\[
(4.7) \quad f(t x + (1-t) y) < t f(x) + (1-t) f(y)
\]

when \( x \neq y \) and \( 0 < t < 1 \). This corresponds to strict inequality in (4.2), (4.3), and (4.4) as well. If \( f \) is differentiable on \( \mathbb{R} \), then \( f \) is strictly convex if and only if \( f' \) is strictly increasing. Otherwise, strict convexity can be characterized in terms of one-sided derivatives by the requirement that

\[
(4.8) \quad f'_+(x) < f'_-(y)
\]

when \( x < y \). Alternatively, if a convex function \( f \) on \( \mathbb{R} \) is not strictly convex, then \( f \) is equal to an affine function on an interval of positive length.

For example, consider \( f(r) = |r|^p \), \( p > 0 \). If \( p = 1 \), then \( f(r) = |r| \) is convex but not strictly convex on \( \mathbb{R} \). If \( p = 2 \), then \( f(r) = r^2 \) is twice-differentiable,
\[ f''(r) = 2, \text{ and } f \text{ is strictly convex. If } p > 2, \text{ then } f \text{ is twice-differentiable on } \mathbb{R}, \]
\[ f''(r) > 0 \text{ when } r \neq 0, \text{ and } f \text{ is strictly convex because } f' \text{ is strictly increasing. If } 1 < p < 2, \text{ then } f \text{ is differentiable on } \mathbb{R}, \text{ twice-differentiable on } \mathbb{R} \setminus \{0\}, \]
\[ f''(r) > 0 \text{ when } r \neq 0, \text{ and again } f \text{ is strictly convex since } f' \text{ is strictly increasing. If } 0 < p < 1, \text{ then } f \text{ is twice-differentiable on } \mathbb{R} \setminus \{0\}, \]
\[ f''(r) < 0 \text{ when } r \neq 0, \text{ and } f \text{ is not convex.} \]

5 Convex sets

A set \( E \subseteq \mathbb{R}^n \) is said to be convex if

\[ t x + (1 - t) y \in E \quad (5.1) \]

for every \( x, y \in E \) and \( t \in (0, 1) \). For example, open and closed balls associated to metrics defined by norms on \( \mathbb{R}^n \) are convex.

Conversely, suppose that \( N(x) \) is a nonnegative real-valued function on \( \mathbb{R}^n \) such that \( N(x) > 0 \) when \( x \neq 0 \) and the homogeneity condition (3.1) holds for all \( x \in \mathbb{R}^n \) and \( r \in \mathbb{R} \). If the closed unit ball

\[ B_N = \{ x \in \mathbb{R}^n : N(x) \leq 1 \} \quad (5.2) \]

is convex, then \( N \) satisfies the triangle inequality (3.2) and hence is a norm. Let \( x, y \in \mathbb{R}^n \) be given, and let us check (3.2). We may suppose that \( x, y \neq 0 \), since the inequality is trivial when \( x = 0 \) or \( y = 0 \). Put

\[ x' = \frac{x}{N(x)}, \quad y' = \frac{y}{N(y)} \quad (5.3) \]

so that \( N(x') = N(y') = 1 \). By hypothesis,

\[ N(t x' + (1 - t) y') \leq 1 \quad (5.4) \]

when \( 0 \leq t \leq 1 \). Applying this with

\[ t = \frac{N(x)}{N(x) + N(y)} \quad (5.5) \]

we get (3.2), as desired.

For example, suppose that \( N(x) = ||x||_p, 1 < p < \infty \). Let \( x, y \in \mathbb{R}^n \) with \( ||x||_p, ||y||_p \leq 1 \) be given, so that

\[ \sum_{j=1}^{n} |x_j|^p, \quad \sum_{j=1}^{n} |y_j|^p \leq 1. \quad (5.6) \]

We would like to show that

\[ ||t x + (1 - t) y||_p \leq 1 \quad (5.7) \]
when $0 \leq t \leq 1$, which is the same as

\begin{equation}
\sum_{j=1}^{n} |t x_j + (1 - t) y_j|^p \leq 1.
\end{equation}

The convexity of $|r|^p$ on $\mathbb{R}$ implies that

\begin{equation}
|t x_j + (1 - t) y_j|^p \leq t|x_j|^p + (1 - t)|y_j|^p
\end{equation}

for each $j$, and the desired inequality follows by summing this over $j$.

A norm $N$ on $\mathbb{R}^n$ is said to be strictly convex if the unit ball $B_N$ is strictly convex in the sense that

\begin{equation}
N(t x + (1 - t) y) < 1
\end{equation}

when $x, y \in \mathbb{R}^n$, $N(x) = N(y) = 1$, $x \neq y$, and $0 < t < 1$. It is easy to see that the absolute value function is strictly convex as a norm on $\mathbb{R}$, if not as a general function as in the previous section. One can also check that $\|x\|_p$ is a strictly convex norm on $\mathbb{R}^n$ when $p > 1$, using the strict convexity of $|r|^p$ on $\mathbb{R}$ and computations as in the preceding paragraph. However, $\|x\|_1$ and $\|x\|_\infty$ are not strictly convex norms on $\mathbb{R}^n$ when $n \geq 2$.

## 6 A little more calculus

Let $f$ be a continuous real-valued function on a closed interval $[a, b]$, $a < b$. The integral

\begin{equation}
\int_{a}^{b} f(t) \, dt
\end{equation}

can be defined in the usual way as a limit of finite sums. The convergence of the finite sums to the integral uses the fact that continuous functions on $[a, b]$ are actually uniformly continuous. It is well known that continuous functions on compact subsets of any metric space are uniformly continuous.

Consider the indefinite integral

\begin{equation}
F(x) = \int_{a}^{x} f(t) \, dt.
\end{equation}

This defines a continuous function on $[a, b]$ which is differentiable on $(a, b)$ and satisfies $F'(x) = f(x)$. Similarly, $F$ has one-sided derivatives at the endpoints $a, b$ that satisfy the same condition. If another differentiable function on $(a, b)$ has derivative $f$, then the difference of $F$ and this function is constant, by the mean value theorem.

Clearly $F$ is monotone increasing on $[a, b]$ if $f \geq 0$ on the whole interval. If $f > 0$ on $[a, b]$, then $F$ is strictly increasing. The same conclusion holds if $f \geq 0$ on $[a, b]$ and $f > 0$ at some point in any nontrivial subinterval of $[a, b]$. Equivalently, if $f \geq 0$ on $[a, b]$, and if $F$ is not strictly increasing on $[a, b]$, then $f = 0$ at every point in a nontrivial subinterval.
Suppose that \( f' \geq 0 \) on \((a, b)\), or simply that \( f \) is monotone increasing on \([a, b]\). This implies that

\[
f(x) \leq \frac{F(y) - F(x)}{y - x} \leq f(y)
\]

when \( a \leq x < y \leq b \). In particular,

\[
\frac{F(w) - F(x)}{w - x} \leq \frac{F(y) - F(w)}{y - w}
\]

when \( a \leq x < w < y \leq b \). If \( f \) is strictly increasing, then these inequalities are strict as well.

### 7 Supremum and infimum

A real number \( b \) is said to be an upper bound for a set \( A \subseteq \mathbb{R} \) if \( a \leq b \) for every \( a \in A \). We say that \( b_1 \in \mathbb{R} \) is the least upper bound or supremum of \( A \) if \( b_1 \) is an upper bound for \( A \) and \( b_1 \leq b \) for every upper bound \( b \) of \( A \). If \( b_2 \in \mathbb{R} \) also satisfies these two conditions, then \( b_1 \leq b_2 \) and \( b_2 \leq b_1 \), and hence \( b_1 = b_2 \). Thus the supremum of \( A \) is unique when it exists, in which case it is denoted \( \sup A \). The completeness property of the real line states that every nonempty set with an upper bound has a least upper bound.

More precisely, this is completeness with respect to the ordering on the real line, which can be defined for other ordered sets. There is also completeness for metric spaces, which means that every Cauchy sequence converges. Both forms of completeness hold on the real line, and are basically equivalent to each other in this particular situation. However, the two notions are distinct, because they can be applied in different circumstances. There are completeness conditions concerning the existence of solutions of ordinary differential equations as well, which may be related to completeness for an associated metric space.

Similarly, a real number \( c \) is said to be a lower bound for \( A \subseteq \mathbb{R} \) if \( c \leq a \) for every \( a \in A \), and \( c_1 \in \mathbb{R} \) is the greatest lower bound or infimum of \( A \) if \( c_1 \) is a lower bound for \( A \) and \( c \leq c_1 \) for every lower bound \( c \) of \( A \). This is unique when it exists for the same reasons as before, and is denoted \( \inf A \). It follows from completeness that a nonempty set \( A \subseteq \mathbb{R} \) with a lower bound has a greatest lower bound, which can be characterized as the supremum of the set of lower bounds of \( A \). Alternatively, the infimum of \( A \) is equal to the negative of the supremum of \( -A = \{-a : a \in A\} \).

### 8 Bounded sets

Let \((M, d(x, y))\) be a metric space. A set \( E \subseteq M \) is said to be bounded if there is a \( p \in M \) and an \( r \geq 0 \) such that

\[
d(p, x) \leq r
\]
for every \( x \in E \). This implies that for every \( q \in M \) there is a \( t \geq 0 \) such that 
\[ d(q, x) \leq t \]
for every \( x \in E \), by taking \( t = r + d(p, q) \).

Equivalently, \( E \subseteq M \) is bounded if the set of distances \( d(x, y) \) for \( x, y \in E \) has an upper bound in \( \mathbb{R} \). If \( E \) is nonempty and bounded, then the diameter of \( E \) is defined by
\[ (8.2) \quad \text{diam} E = \sup \{ d(x, y) : x, y \in E \}. \]
The diameter of the empty set may be interpreted as 0.

If \( E_1 \subseteq E_2 \subseteq M \) and \( E_2 \) is bounded, then \( E_1 \) is bounded, and
\[ (8.3) \quad \text{diam} E_1 \leq \text{diam} E_2. \]
The union of two bounded subsets of \( M \) is also bounded, but the diameter of the union may be much larger than the sum of the diameters of the two subsets.

Suppose that \( M \) is \( \mathbb{R}^n \) equipped with a norm \( N \) and its associated metric \( d_N(x, y) \). The convex hull \( \hat{E} \) of a set \( E \subseteq \mathbb{R}^n \) consists of all convex combinations of elements of \( E \). More precisely, \( \hat{E} \) is the set of all finite sums of the form
\[ (8.4) \quad \sum_{i=1}^{k} r_i x(i), \]
where \( k \) is a positive integer, \( r_1, \ldots, r_k \) are nonnegative real numbers such that
\[ (8.5) \quad \sum_{i=1}^{k} r_k = 1, \]
and \( x(1), \ldots, x(k) \) are elements of \( E \). It is well known that one can take \( k = n+1 \) here, but we shall not need this fact. By construction, \( \hat{E} \) is a convex set in \( \mathbb{R}^n \) that contains \( E \). Moreover, \( \hat{E} \) is the smallest such set, in the sense that \( \hat{E} \) is contained in any convex set in \( \mathbb{R}^n \) that contains \( E \). If \( E \) is bounded, so that \( E \) is contained in a ball, then \( \hat{E} \) is contained in the same ball, and hence \( \hat{E} \) is bounded. Let us check that
\[ (8.6) \quad \text{diam}_N \hat{E} \leq \text{diam}_N E, \]
where the subscript \( N \) indicates that the diameter uses the norm \( N \). Let
\[ (8.7) \quad \xi = \sum_{i=1}^{k} r_i x(i), \quad \eta = \sum_{j=1}^{l} t_j y(j) \]
be arbitrary elements of \( \hat{E} \), as before. Thus
\[ (8.8) \quad \xi - \eta = \sum_{i=1}^{k} \sum_{j=1}^{l} r_i t_j (x(i) - y(j)), \]
and therefore
\[ (8.9) \quad N(\xi - \eta) \leq \sum_{i=1}^{k} \sum_{j=1}^{l} r_i t_j N(x(i) - y(j)), \]
where \( N \) is a norm on \( \mathbb{R}^n \) and \( x(i) \) and \( y(j) \) are elements of \( E \).
by the properties of norms. This implies that
\[(8.10) \quad N(\xi - \eta) \leq \max\{N(x(i) - y(j)) : 1 \leq i \leq k, 1 \leq j \leq l\},\]
and consequently \(N(\xi - \eta) \leq \text{diam}_N E\), as desired.

9 Lipschitz mappings

Let \((M_1, d_1(x, y))\) and \((M_2, d_2(u, v))\) be metric spaces. A mapping \(f : M_1 \to M_2\) is said to be Lipschitz if
\[(9.1) \quad d_2(f(x), f(y)) \leq C d_1(x, y)\]
for some \(C \geq 0\) and all \(x, y \in M\). More precisely, this means that \(f\) is Lipschitz of order 1, and we shall discuss other Lipschitz conditions later. One can also say that \(f\) is \(C\)-Lipschitz or \(C\)-Lipschitz of order 1 to mention the constant \(C\) explicitly.

Thus \(f\) is \(C\)-Lipschitz with \(C = 0\) if and only if \(f\) is constant. Note that Lipschitz mappings are uniformly continuous. Suppose that \((M_3, d_3(w, z))\) is another metric space, and that \(f_1 : M_1 \to M_2\) and \(f_2 : M_2 \to M_3\) are Lipschitz mappings with constants \(C_1, C_2\), respectively. The composition \(f_2 \circ f_1\) is the mapping from \(M_1\) to \(M_2\) defined by
\[(9.2) \quad (f_2 \circ f_1)(x) = f_2(f_1(x)),\]
and it is easy to check that this is Lipschitz with constant equal to the product of \(C_1\) and \(C_2\).

If \(f : M_1 \to M_2\) is \(C\)-Lipschitz and \(E \subseteq M_1\) is bounded, then
\[(9.3) \quad f(E) = \{f(x) : x \in E\}\]
is bounded in \(M_2\), and
\[(9.4) \quad \text{diam}_2 f(E) \leq C \text{diam}_1 E.\]
Here the subscripts indicate in which metric space the diameter is taken. This is easy to verify, directly from the definitions, and suggests another way to look at the composition of Lipschitz mappings, as in the previous paragraph.

10 Real-valued functions

Let \(f\) be a real-valued function on an open interval \((a, b)\) in the real line. If \(f\) is \(C\)-Lipschitz with respect to the standard metric on the domain and range, then
\[(10.1) \quad |f'(x)| \leq C\]
at every point \(x \in (a, b)\) at which \(f\) is differentiable, by definition of the derivative. Conversely, if \(f\) is differentiable and satisfies this condition everywhere on \((a, b)\), then \(f\) is \(C\)-Lipschitz, by the mean value theorem.
Now let \((M, d(x, y))\) be a metric space. A function \(f : M \to \mathbb{R}\) is \(C\)-Lipschitz with respect to the standard metric on \(\mathbb{R}\) if and only if
\[
f(x) \leq f(y) + C \cdot d(x, y)
\]
for every \(x, y \in M\). This follows easily from the definitions. In particular, \(f_p(x) = d(p, x)\) is 1-Lipschitz for every \(p \in M\).

If \(A \subseteq M, A \neq \emptyset\), and \(x \in M\), then put
\[
\text{dist}(x, A) = \inf \{d(x, a) : a \in A\}.
\]
For each \(x, y \in M\) and \(a \in A\),
\[
\text{dist}(x, A) \leq d(x, a) \leq d(x, y) + d(y, a),
\]
and therefore
\[
\text{dist}(x, A) \leq \text{dist}(y, A) + d(x, y).
\]
This shows that \(\text{dist}(x, A)\) is 1-Lipschitz on \(M\).

Suppose that \(f_1, f_2 : M \to \mathbb{R}\) are Lipschitz with constants \(C_1, C_2\), respectively. For any \(r_1, r_2 \in \mathbb{R}\), \(r_1 f_1 + r_2 f_2\) is Lipschitz with constant \(|r_1| C_1 + |r_2| C_2\). Suppose also that \(f_1, f_2\) are bounded on \(M\), with
\[
|f_1(x)| \leq k_1, \quad |f_2(x)| \leq k_2
\]
for some \(k_1, k_2 \geq 0\) and every \(x \in M\). Because
\[
f_1(x) f_2(x) - f_1(y) f_2(y) = (f_1(x) - f_1(y)) f_2(x) + f_1(y) (f_2(x) - f_2(y))
\]
for every \(x, y \in M\), \(f_1 f_2\) is Lipschitz on \(M\) with constant \(k_2 C_1 + k_1 C_2\).

\section{R\(^n\)-valued functions}

Let \(N\) be a norm on \(\mathbb{R}^n\). Thus \(N\) is 1-Lipschitz as a real-valued function on \(\mathbb{R}^n\) with the metric \(d_N(x, y)\) associated to \(N\), as in the previous section. One can also show that \(N\) is bounded by a constant multiple of the standard Euclidean norm on \(\mathbb{R}^n\). This uses the finite-dimensionality of \(\mathbb{R}^n\) in an essential way, and it implies that \(N\) is Lipschitz with respect to the standard metric on \(\mathbb{R}^n\).

Suppose that \(f\) is a continuous \(\mathbb{R}^n\)-valued function on a closed interval \([a, b]\) in the real line. As an extension of the triangle inequality for \(N\),
\[
N\left(\int_a^b f(t) \, dt\right) \leq \int_a^b N(f(t)) \, dt.
\]
Indeed, the analogous statement for the finite sums follows from the triangle inequality for \(N\). The integral of \(f\) can be approximated by finite sums, and continuity of \(N\) as in the preceding paragraph can be employed to pass to the
limit. Alternatively, one can use duality, as follows. For any linear functional \( \phi : \mathbb{R}^n \to \mathbb{R} \),

(11.2) \[
\phi \left( \int_a^b f(t) \, dt \right) = \int_a^b \phi(f(t)) \, dt.
\]

If \( |\phi(w)| \leq N(w) \) for every \( w \in \mathbb{R}^n \), then we get that

(11.3) \[
\left| \int_a^b \phi(f(t)) \, dt \right| \leq \int_a^b N(f(t)) \, dt.
\]

A famous theorem states that for each \( v \in \mathbb{R}^n \) there is such a \( \phi \) with \( \phi(v) = N(v) \), which permits one to estimate the norm of the integral. We shall not discuss the proof of this here, but one can take \( \phi(w) \) to be the standard inner product of \( w \) with \( v/|v| \) when \( v \neq 0 \) and \( N \) is the Euclidean norm on \( \mathbb{R}^n \), and there are also explicit expressions for \( \phi \) when \( N(w) = \|w\|_p \), \( 1 \leq p \leq \infty \).

Suppose now that \( F : [a, b] \to \mathbb{R}^n \) is \( C \)-Lipschitz with respect to the standard metric on \( \mathbb{R} \) and the metric \( d_\mathbb{R}^n \) on \( \mathbb{R}^n \). If \( F \) is differentiable at a point \( x \in (a, b) \), then \( N(F'(x)) \leq C \). This follows from the definition of the derivative, as in the real-valued case. Conversely, if \( F \) is continuously differentiable on \( [a, b] \) and \( N(F') \leq C \), then one can use the fundamental theorem of calculus and the integral form of the triangle inequality to show that that \( F \) is \( C \)-Lipschitz with respect to \( N \). One can use duality to get the same conclusion when \( F \) is continuous on \( [a, b] \) and differentiable on \( (a, b) \) with \( N(F') \leq C \), by applying the mean value theorem to \( \phi \circ F \) for linear functionals \( \phi : \mathbb{R}^n \to \mathbb{R} \).

Let \( (M, d(x, y)) \) be a metric space, and let \( F = (F_1, \ldots, F_n) \) be a mapping from \( M \) into \( \mathbb{R}^n \). If \( \mathbb{R}^n \) is equipped with the norm \( \|w\|_\infty \), then it is easy to see that \( F \) is \( C \)-Lipschitz if and only if \( F_1, \ldots, F_n \) are \( C \)-Lipschitz as real-valued functions on \( M \). Of course, one can estimate Lipschitz conditions for \( F \) in terms of Lipschitz conditions for \( F_1, \ldots, F_n \) for other norms on \( \mathbb{R}^n \), and vice-versa, but the relationship between the constants is normally not quite as simple as for the norm \( \|w\|_\infty \).

## 12 Bounded variation

Let \( f \) be a real-valued function on a closed interval \([a, b] \). A partition of \([a, b] \) is a finite sequence \( \{t_j\}_{j=0}^n \) of real numbers such that

(12.1) \[
a = t_0 < t_1 < \cdots < t_n = b.
\]

For each partition \( \mathcal{P} = \{t_j\}_{j=0}^n \) of \([a, b] \), consider

(12.2) \[
V_\mathcal{P}(f) = \sum_{j=1}^n |f(t_j) - f(t_{j-1})|.
\]

This measures the variation of \( f \) on the partition \( \mathcal{P} \). We say that \( f \) has bounded variation on \([a, b] \) if there is an upper bound for \( V_\mathcal{P}(f) \) over all partitions \( \mathcal{P} \) of
\[ [a, b]. \text{In this case, the total variation } \mathcal{V}_b^a(f) \text{ of } f \text{ on } [a, b] \text{ is defined by} \]
\[
(12.3) \quad \mathcal{V}_b^a(f) = \sup \{ \mathcal{V}_P : \text{P is a partition of } [a, b] \}.
\]
Thus \( \mathcal{V}_b^a(f) = 0 \) if and only if \( f \) is constant on \([a, b] \).

Using the partition that consists of only \( a, b \), we get that
\[
(12.4) \quad |f(b) - f(a)| \leq \mathcal{V}_b^a(f).
\]
If \( f \) is monotone increasing on \([a, b] \), then
\[
(12.5) \quad \mathcal{V}_P(f) = f(b) - f(a)
\]
for every partition \( \mathcal{P} \) of \([a, b] \). Hence \( f \) has bounded variation on \([a, b] \), and
\[
(12.6) \quad \mathcal{V}_b^a(f) = f(b) - f(a).
\]
Conversely, if \( f \) has bounded variation on \([a, b] \) and
\[
(12.7) \quad \mathcal{V}_b^a(f) = |f(b) - f(a)|,
\]
then \( f \) is either monotone increasing or decreasing on \([a, b] \).

If \( f \) is \( C \)-Lipschitz on \([a, b] \), then
\[
(12.8) \quad \mathcal{V}_P(f) \leq C (b - a)
\]
for every partition \( \mathcal{P} \) of \([a, b] \). Hence \( f \) has bounded variation on \([a, b] \), and
\[
(12.9) \quad \mathcal{V}_b^a(f) \leq C (b - a).
\]
If \( \phi : \mathbb{R} \to \mathbb{R} \) is \( C \)-Lipschitz, then
\[
(12.10) \quad \mathcal{V}_P(\phi \circ f) \leq C \mathcal{V}_P(f)
\]
for every \( f : [a, b] \to \mathbb{R} \) and partition \( \mathcal{P} \) of \([a, b] \). If \( f \) has bounded variation on \([a, b] \), then it follows that \( \phi \circ f \) has bounded variation on \([a, b] \), and
\[
(12.11) \quad \mathcal{V}_b^a(\phi \circ f) \leq C \mathcal{V}_b^a(f).
\]

Let \( f_1, f_2 : [a, b] \to \mathbb{R} \) and \( r_1, r_2 \in \mathbb{R} \) be given. For any partition \( \mathcal{P} \) of \([a, b] \),
\[
(12.12) \quad \mathcal{V}_P(r_1 f_1 + r_2 f_2) \leq |r_1| \mathcal{V}_P(f_1) + |r_2| \mathcal{V}_P(f_2).
\]
If \( f_1, f_2 \) have bounded variation on \([a, b] \), then it follows that \( r_1 f_1 + r_2 f_2 \) also has bounded variation, with
\[
(12.13) \quad \mathcal{V}_b^a(r_1 f_1 + r_2 f_2) \leq |r_1| \mathcal{V}_b^a(f_1) + |r_2| \mathcal{V}_b^a(f_2).
\]
Suppose that \( f_1, f_2 \) are bounded on \([a, b] \), so that
\[
(12.14) \quad |f_1(x)| \leq k_1, \quad |f_2(x)| \leq k_2
\]
for some $k_1, k_2 \geq 0$ and every $x \in [a, b]$. It is easy to check that

$$V_P(f_1 f_2) \leq k_2 V_P(f_1) + k_1 V_P(f_2)$$

for every partition $P$ of $[a, b]$. If $f_1, f_2$ have bounded variation on $[a, b]$, then $f_1 f_2$ has bounded variation, and

$$V_a^b(f_1 f_2) \leq k_2 V_a^b(f_1) + k_1 V_a^b(f_2).$$

This is analogous to the earlier estimate for the Lipschitz constant of the product of bounded Lipschitz functions, and to the Leibniz rule for differentiating the product of two functions.

Suppose that $a_1, b_1$ are real numbers such that $a \leq a_1 \leq b_1 \leq b$. If $f$ has bounded variation on $[a, b]$, then $f$ has bounded variation on $[a_1, b_1]$, and

$$V_{a_1}^{b_1}(f) \leq V_a^b(f).$$

This is because every partition of $[a_1, b_1]$ can be extended to a partition of $[a, b]$. In particular, $f$ is bounded on $[a, b]$ when it has bounded variation.

A partition $P'$ of $[a, b]$ is said to be a refinement of a partition $P$ of $[a, b]$ if $P'$ contains all of the terms in $P$. In this case, one can check that

$$V_P(f) \leq V_{P'}(f)$$

for every $f : [a, b] \rightarrow \mathbb{R}$, using the triangle inequality. Also, any finite collection of partitions of $[a, b]$ has a common refinement.

Suppose that $f$ has bounded variation on $[a, b]$, and that $x \in (a, b)$. Thus the restrictions of $f$ to $[a, x]$ and to $[x, b]$ have bounded variation, and moreover

$$V_a^x(f) + V_x^b(f) = V_a^b(f).$$

Indeed, any partitions $P_1, P_2$ of $[a, x], [x, b]$, respectively, can be combined to get a partition $P_3$ of $[a, b]$ for which

$$V_{P_1}(f) + V_{P_2}(f) = V_{P_3}(f),$$

which implies that $V_a^x(f) + V_x^b(f) \leq V_a^b(f)$. To get the opposite inequality, note that every partition of $[a, b]$ can be refined if necessary to contain $x$, and hence to be a combination of partitions of $[a, x]$ and $[x, b]$. The same argument shows that $f$ has bounded variation on $[a, b]$ if it has bounded variation on $[a, x]$ and on $[x, b]$.

Suppose that $f$ is continuously differentiable on $[a, b]$. If $a \leq r \leq t \leq b$, then

$$|f(t) - f(r)| = \left|\int_r^t f'(\xi) \, d\xi\right| \leq \int_r^t |f'(\xi)| \, d\xi.$$

This implies that

$$V_P(f) \leq \int_a^b |f'(\xi)| \, d\xi$$
for every partition \( \mathcal{P} \) of \([a, b]\). One can show that

\[
V_b^a(f) = \int_a^b |f'(\xi)| \, d\xi,
\]

using very fine partitions \( \mathcal{P} \) of \([a, b]\).

For each \( r \in \mathbb{R} \), put \( r_+ = r \) when \( r \geq 0 \) and \( r_+ = 0 \) when \( r \leq 0 \), and \( r_- = -r \) when \( r \leq 0 \) and \( r_- = 0 \) when \( r \geq 0 \), so that

\[
r_+ - r_- = r, \quad r_+ + r_- = |r|.
\]

Given \( f : [a, b] \to \mathbb{R} \) and a partition \( \mathcal{P} = \{t_j\}_{j=0}^n \) of \([a, b]\), put

\[
P_\mathcal{P}(f) = \sum_{j=1}^{n} (f(t_j) - f(t_{j-1}))^+
\]

and

\[
N_\mathcal{P}(f) = \sum_{j=1}^{n} (f(t_j) - f(t_{j-1}))^-.
\]

Thus

\[
P_\mathcal{P}(f) + N_\mathcal{P}(f) = V_\mathcal{P}(f)
\]

and

\[
P_\mathcal{P}(f) - N_\mathcal{P}(f) = f(b) - f(a).
\]

Suppose that \( f \) has bounded variation on \([a, b]\), and put

\[
P_a^b(f) = \sup\{P_\mathcal{P}(f) : \mathcal{P} \text{ is a partition of } [a, b]\}
\]

and

\[
N_a^b(f) = \sup\{N_\mathcal{P}(f) : \mathcal{P} \text{ is a partition of } [a, b]\}.
\]

One can check that

\[
P_a^b(f) + N_a^b(f) = V_a^b(f)
\]

and

\[
P_a^b(f) - N_a^b(f) = f(b) - f(a).
\]

Similarly,

\[
P_a^x(f) - N_a^x(f) = f(x) - f(a)
\]

when \( a \leq x \leq b \). This implies that \( f \) can be expressed as the difference of two monotone increasing functions on \([a, b]\), since \( P_a^x(f), N_a^x(f) \) are monotone increasing in \( x \).

Functions of bounded variation do not have to be continuous, but they can only have jump discontinuities. More precisely, if \( f \) has bounded variation on \([a, b]\), then \( f \) has one-sided limits from both sides at every point in \((a, b)\), and from the right and left sides at \( a, b \), respectively. This follows from the analogous statement for monotone functions and the fact that a function of bounded variation can be expressed in terms of monotone functions, and it can also be shown more directly.
13 Lengths of paths

Let \((M, d(x, y))\) be a metric space, let \(a, b\) be real numbers with \(a \leq b\), and let \(f\) be a function on \([a, b]\) with values in \(M\). For each partition \(\mathcal{P} = \{t_j\}_{j=0}^n\) of \(M\), consider

\[
\Lambda_{\mathcal{P}}(f) = \sum_{j=1}^n d(f(t_j), f(t_{j-1})).
\]

This is the same as the variation \(V_{\mathcal{P}}(f)\) of \(f\) on \(\mathcal{P}\) when \(M\) is the real line with the standard metric. If there is an upper bound for \(\Lambda_{\mathcal{P}}\) over all partitions \(\mathcal{P}\) of \([a, b]\), then we say that the path \(f : [a, b] \to M\) has finite length, and the length of the path is defined by

\[
\Lambda^b_a(f) = \sup\{\Lambda_{\mathcal{P}} : \mathcal{P} \text{ is a partition of } [a, b]\}.
\]

This is the same as the total variation \(V^b_a(f)\) of \(f\) when \(M = \mathbb{R}\). As in the previous case, \(\Lambda^b_a(f) = 0\) if and only if \(f\) is constant. If \(a \leq r \leq t \leq b\), then

\[
d(f(r), f(t)) \leq \Lambda_{\mathcal{P}}(f)
\]

for any partition \(\mathcal{P}\) of \([a, b]\) that contains \(r, t\). Hence \(f([a, b])\) is a bounded set in \(M\) when \(f : [a, b] \to M\) has finite length, with

\[
\text{diam } f([a, b]) \leq \Lambda^b_a(f).
\]

Of course, \(f([a, b])\) is a compact set in \(M\) when \(f\) is continuous, and therefore bounded. If \(f\) is continuous, then \(f([a, b])\) is also a connected set in \(M\).

If \(f : [a, b] \to M\) is \(C\)-Lipschitz, then \(f\) has finite length, and

\[
\Lambda^b_a(f) \leq C(b - a).
\]

Let \((\widetilde{M}, \widetilde{d}(u, v))\) be another metric space, and suppose that \(\phi : M \to \widetilde{M}\) is \(C\)-Lipschitz. For any \(f : [a, b] \to M\) and partition \(\mathcal{P}\) of \([a, b]\),

\[
\tilde{\Lambda}_{\mathcal{P}}(\phi \circ f) \leq C \Lambda_{\mathcal{P}}(f),
\]

where \(\tilde{\Lambda}\) is the analogous quantity for \(\widetilde{M}\). If \(f : [a, b] \to M\) has finite length, then \(\phi \circ f : [a, b] \to \widetilde{M}\) does too, and

\[
\tilde{\Lambda}^b_a(\phi \circ f) \leq C \Lambda^b_a(f).
\]

In particular, if \(f\) has finite length and \(\phi : M \to \mathbb{R}\) is Lipschitz, then \(\phi \circ f\) has bounded variation.

If \(f : [a, b] \to M\) has finite length and \(a \leq a_1 \leq b_1 \leq b\), then the restriction of \(f\) to \([a_1, b_1]\) has finite length, and

\[
\Lambda^b_{a_1}(f) \leq \Lambda^b_a(f).
\]
If $P, P'$ are partitions of $[a, b]$ and $P'$ is a refinement of $P$, then

$$\Lambda_P(f) \leq \Lambda_{P'}(f)$$

(13.9)

for any $f : [a, b] \to M$, as in the case of real-valued functions in the previous section. As before, one can use this to show that

$$\Lambda_x^a(f) + \Lambda_x^b(f) = \Lambda_x^b(f)$$

(13.10)

for every $x \in (a, b)$ when $f$ has finite length. If $a \leq x < b$, then

$$\lim_{y \to x^+} \Lambda_y^a(f)$$

exists, because $\Lambda_x^a(f)$ is monotone increasing in $x$, and hence

$$\lim_{y \to x^+} \sup \{ \Lambda_w^y(f) : x < w \leq y \} = 0.$$  

(13.12)

This implies that

$$\lim_{y \to x^+} \text{diam } f((x, y]) = 0,$$

since

$$\text{diam } f((x, y]) = \sup \{ \text{diam } f([w, y]) : x < w \leq y \} \leq \sup \{ \Lambda_w^y(f) : x < w \leq y \}.$$  

(13.14)

If $M$ is complete, then it follows that $f$ has a limit from the right at $x$, and similarly there is a limit from the left when $a < x \leq b$.

Suppose now that $M$ is $\mathbb{R}^n$, equipped with a norm $N$, and thus the metric $d_N(x, y)$ associated to $N$ too. If $f_1, f_2 : [a, b] \to \mathbb{R}^n$ have finite length and $r_1, r_2 \in \mathbb{R}$, then $r_1 f_1 + r_2 f_2$ has finite length, and

$$\Lambda_a^b(r_1 f_1 + r_2 f_2) \leq |r_1| \Lambda_a^b(f_1) + |r_2| \Lambda_a^b(f_2).$$

(13.15)

This is similar to the case of real-valued functions, and one can also treat the product of a real-valued function and an $\mathbb{R}^n$-valued function on $[a, b]$ in the same way as before. If $f : [a, b] \to \mathbb{R}^n$ is continuously differentiable, then one can show that $f$ has finite length and that

$$\Lambda_a^b(f) = \int_a^b N(f'(\xi)) \, d\xi,$$

(13.16)

in practically the same way as before. It can be interesting to consider integral norms

$$\left( \int_a^b N(f'(\xi))^p \, d\xi \right)^{1/p}$$

(13.17)

as well, $1 \leq p < \infty$. The $p = \infty$ case corresponds to the maximum of $N(f')$ on $[a, b]$. This integral norm is especially interesting when $p = 2$ and $N$ is the standard Euclidean norm on $\mathbb{R}^n$. For other $p$, there is some simplification when $N(v) = ||v||_p$. If $N(v) = ||v||_1$, then the length of any path of finite length in $\mathbb{R}^n$ is equal to the sum of the total variations of the coordinates of the path. This uses the fact that any finite collection of partitions of $[a, b]$ has a common refinement, so that independent partitions for the coordinate functions are equivalent to using the same partition for the whole path.
14 Snowflake metrics

Let $\alpha$ be a positive real number, with $\alpha < 1$. For any pair of nonnegative real numbers $u, v$,

\[(u + v)^\alpha \leq u^\alpha + v^\alpha.\]  

(14.1)

To see this, observe that

\[\max(u, v) \leq (u^\alpha + v^\alpha)^{1/\alpha},\]  

(14.2)

and hence

\[u + v \leq \max(u, v) \frac{1}{1 - \alpha} (u^\alpha + v^\alpha) \leq (u^\alpha + v^\alpha)^{1/\alpha}.\]  

(14.3)

Note that the inequality is strict in (14.1) when $u, v > 0$.

If $(M, d(x, y))$ is a metric space, then it follows from (14.1) that $d(x, y)^\alpha$ is also a metric on $M$. This does not change the topology of $M$, but it does change the geometry. Many familiar examples of snowflake curves in the plane have approximately this type of geometry, for instance.

Suppose that $f : [a, b] \rightarrow M$ is a continuous path with finite length with respect to $d(x, y)^\alpha$. This means that

\[\sum_{j=1}^{n} d(f(t_j), f(t_{j-1}))^\alpha \leq A\]  

(14.4)

for some $A \geq 0$ and every partition $\{t_j\}_{j=0}^{n}$ of $[a, b]$. Let $\epsilon > 0$ be given. By continuity and compactness, $f$ is uniformly continuous, and so there is a $\delta > 0$ such that

\[d(f(r), f(w)) < \epsilon\]  

(14.5)

for every $r, w \in [a, b]$ such that $|r - w| < \delta$. Hence

\[\sum_{j=1}^{n} d(f(t_j), f(t_{j-1})) \leq \epsilon^{1 - \alpha} A\]  

(14.6)

when $t_j - t_{j-1} < \delta$ for each $j = 1, \ldots, n$. Every partition of $[a, b]$ has a refinement with this property, which implies that the length $\Lambda_a^b(f)$ of $f$ with respect to $d(x, y)$ satisfies

\[\Lambda_a^b(f) \leq \epsilon^{1 - \alpha} A.\]  

(14.7)

Thus $\Lambda_a^b(f) = 0$, since $\epsilon > 0$ is arbitrary, and $f$ must be constant.

15 Hölder continuity

Let $(M_1, d_1(x, y))$ and $(M_2, d_2(w, z))$ be metric spaces. A mapping $f : M_1 \rightarrow M_2$ is said to be Hölder continuous of order $\alpha$, $0 < \alpha < 1$, if

\[d_2(f(x), f(y)) \leq C d_1(x, y)^\alpha\]  

(15.1)
for some $C \geq 0$ and every $x, y \in M_1$. One might also say that $f$ is Lipschitz of order $\alpha$ in this case, but it will be convenient to refer to this as a Lipschitz condition when $\alpha = 1$ and Hölder continuity when $0 < \alpha < 1$. Similar names are sometimes used for other related conditions as well.

As in the previous section, $d_1(x, y)^\alpha$ is a metric on $M_1$, and therefore $f$ is Hölder continuous of order $\alpha$ with respect to $d_1(x, y)$ if and only if $f$ is Lipschitz with respect to $d_1(x, y)^\alpha$. Thus many basic properties of Hölder continuous mappings follow from the corresponding statements for Lipschitz mappings. In particular,

$$f_p(x) = d_1(p, x)^\alpha$$

is a real-valued Hölder continuous function of order $\alpha$ on $M_1$ with $C = 1$ for each $p \in M_1$.

Let $(M, d(x, y))$ be a metric space, and consider the case of a continuous path $f : [a, b] \to M$. If $f$ is $C$-Lipschitz, then $f$ is Hölder continuous of order $\alpha$ for each $\alpha \in (0, 1)$ with constant $C (b - a)^{1-\alpha}$. Of course, $f$ also has finite length $\leq C (b - a)$ when $f$ is $C$-Lipschitz. However, continuous paths of finite length need not be Hölder continuous of any positive order, and there are counterexamples already for monotone increasing real-valued functions. Similarly, Hölder continuous paths may not have finite length.

Suppose that $f : [a, b] \to M$ is Hölder continuous of order $\alpha$ with constant $C > 0$. For each $\rho > 0$, $f([a, b])$ is contained in the union of $O(\rho^{-1/\alpha})$ subintervals of $M$ with diameter $\leq \rho$, because $[a, b]$ is the union of $O(\rho^{-1/\alpha})$ subintervals of length $\leq (\rho/C)^{1/\alpha}$. This implies that the Minkowski dimension of $f([a, b])$ is $\leq 1/\alpha$, and hence the Hausdorff dimension of $f([a, b])$ is $\leq 1/\alpha$ too. This is an analogue for Hölder continuous paths of the fact that Lipschitz paths have finite length.

## 16 Coverings

If $[a, b], [a_1, b_1], \ldots, [a_n, b_n]$ are closed intervals in the real line such that

$$[a, b] \subseteq \bigcup_{j=1}^n [a_j, b_j],$$

then

$$b - a \leq \sum_{j=1}^n (b_j - a_j).$$

More generally, if $E_1, \ldots, E_n$ are bounded subsets of $\mathbb{R}$ such that

$$[a, b] \subseteq \bigcup_{j=1}^n E_j,$$

then

$$b - a \leq \sum_{j=1}^n \text{diam } E_j.$$
Indeed, each $E_j$ is contained in a closed interval of the same diameter.

Let $(M, d(x,y))$ be a metric space, and let $A, E_1, \ldots, E_n$ be bounded subsets of $M$ such that
\[
A \subseteq \bigcup_{j=1}^{n} E_j.
\]
(16.5)

If $A$ is connected, then
\[
\operatorname{diam} A \leq \sum_{j=1}^{n} \operatorname{diam} E_j.
\]
(16.6)

To see this, remember first that continuous mappings send connected sets to connected sets. If $f : M \to \mathbb{R}$ is continuous, then $f(A)$ is an interval in the real line, which may be open, or closed, or half-open and half-closed. At any rate,
\[
\operatorname{diam}_{\mathbb{R}} f(A) \leq \sum_{j=1}^{n} \operatorname{diam}_{\mathbb{R}} f(E_j),
\]
(16.7)

where the subscripts indicate that these are diameters in $\mathbb{R}$. If $f$ is $C$-Lipschitz, then
\[
\operatorname{diam}_{\mathbb{R}} f(A) \leq C \sum_{j=1}^{n} \operatorname{diam} E_j.
\]
(16.8)

The desired estimate follows by applying this to 1-Lipschitz functions of the form $f_p(x) = d(p, x), p \in A$.

The hypothesis that $A$ be connected is essential here. If $A$ is a finite set with at least two elements, then $\operatorname{diam} A > 0$, but $A$ is contained in the union of finitely many sets with one element and thus diameter 0. Cantor’s middle-thirds set in the real line has diameter equal to 1, and is contained in the union of $2^\ell$ intervals of length $3^{-\ell}$ for each $\ell \geq 1$. A compact set $A \subseteq \mathbb{R}$ has Lebesgue measure 0 exactly if for each $\epsilon > 0$ there are finitely many bounded sets $E_1, \ldots, E_n \subseteq \mathbb{R}$ such that $A \subseteq \bigcup_{j=1}^{n} E_j$ and $\sum_{j=1}^{n} \operatorname{diam} E_j < \epsilon$.

In particular, if $A \subseteq M$ is a bounded connected set and $\rho > 0$, then $A$ is not covered by fewer than $\operatorname{diam} A / \rho$ bounded subsets of $M$ of diameter $\leq \rho$. Depending on the situation, many more of these subsets may be required. If $M$ is $\mathbb{R}^n$ equipped with the standard metric, for example, then a bounded set $A$ can be covered by $O(\rho^{-n})$ sets of diameter $\leq \rho$. One needs at least a positive multiple of $\rho^{-n}$ such sets when $A$ has nonempty interior, because otherwise the $n$-dimensional volume of $A$ would be too small.

17 Domains in $\mathbb{R}^n$

A set $U$ in a metric space $M$ is an open set if for every $p \in U$ there is an $r > 0$ such that $B(p, r) \subseteq U$. Any norm $N$ on $\mathbb{R}^n$ determines the same open sets as the standard metric. This is because $N$ is less than or equal to a constant times the standard norm, and conversely the standard norm is less than or equal to a constant times $N$. The first statement can be checked directly by
expressing any element of $\mathbb{R}^n$ as a linear combination of the standard basis for $\mathbb{R}^n$ and using the triangle inequality. As mentioned previously, this and the triangle inequality imply that $N$ is continuous with respect to the standard norm. Hence the minimum of $N$ is attained on the standard unit sphere in $\mathbb{R}^n$, since the latter is compact. The standard norm times the minimum of $N$ on the unit sphere is less than or equal to $N$ on all of $\mathbb{R}^n$, by homogeneity, which implies the second statement. For explicit norms like $\|\cdot\|_p$, $1 \leq p \leq \infty$, the comparison with the standard norm can be verified directly.

Suppose that $U$ is a connected open set in $\mathbb{R}^n$, which is to say that $U$ is not the union of two disjoint nonempty open sets. It is well known that $U$ is then pathwise-connected, so that for every $p, q \in U$ there is a continuous mapping $f : [a, b] \to U$ such that $f(a) = p$ and $f(b) = q$. More precisely, one can even take $f$ to be piecewise-affine on $[a, b]$. In particular, $f$ then has finite length.

However, it is not clear how small the length of $f$ can be. Of course, the length of $f$ is at least the distance between $p$ and $q$. If $U$ is convex, then one can take $f$ to be affine, and the length of $f$ is equal to the distance between $p$ and $q$. Otherwise, the length of $f$ may have to be quite large compared to the distance between $p$ and $q$. It is easy to give examples where this happens in the plane. For instance, there may be elements of $U$ on opposite sides of the boundary locally. The boundary of $U$ might also be complicated, with spirals or other obstacles.

Even if the boundary of $U$ is complicated, it may be that $U$ behaves well in terms of lengths of paths. For example, if $U$ is the region in the plane bounded by the von Koch snowflake, then every pair of elements of $U$ can be connected by a path of length bounded by a constant multiple of the distance between them. The main idea is for the path to avoid the boundary as much as possible, without going too far away. There can also be relatively small parts of the boundary that only cause minor detours for paths in the domain.

18 Lipschitz graphs

Let $k, l, n$ be positive integers such that $k + l = n$, and let us identify $\mathbb{R}^n$ with $\mathbb{R}^k \times \mathbb{R}^l$, so that an element $x$ of $\mathbb{R}^n$ may be expressed as $(x', x'')$, where $x' \in \mathbb{R}^k$ and $x'' \in \mathbb{R}^l$. Also let $A : \mathbb{R}^k \to \mathbb{R}^l$ be a continuous mapping, and consider its graph

$$(18.1) \quad \{(x', x'') \in \mathbb{R}^n : x'' = A(x')\}.$$ 

This is a nice $k$-dimensional topological submanifold of $\mathbb{R}^n$. If $k = n - 1$, then this hypersurface has two complementary components $U_+, U_-$ consisting of $(x', x'') \in \mathbb{R}^n$ such that $x'' > A(x')$ and $x'' < A(x')$, respectively. If $k < n$, then the complement of the graph in $\mathbb{R}^n$ is connected. For any $k$,

$$(18.2) \quad (x', x'') \mapsto (x', x'' + A(x'))$$

defines a homeomorphism on $\mathbb{R}^n$ that sends the $k$-plane $x'' = 0$ to the graph of $A$. If $f(t)$ is a continuous path in $\mathbb{R}^k$ parameterized by an interval $[a, b]$, then
\( \hat{f}(t) = (f(t), A(f(t))) \) is a continuous path in the graph of \( A \). The graph of \( A \) is itself a curve in \( \mathbb{R}^n \) when \( k = 1 \).

Suppose that \( A \) is Lipschitz. If \( f \) has finite length, then \( \hat{f} \) does too, and the length of \( \hat{f} \) is bounded by a constant multiple of the length of \( f \). If the Lipschitz constant of \( A \) is small, then this constant multiple is close to 1. Using affine paths in \( \mathbb{R}^k \), we get that every pair of elements of the graph of \( A \) can be connected by a continuous path in the graph of \( A \) of finite length bounded by a constant multiple of the distance between them, where the constant multiple is close to 1 when \( A \) has small Lipschitz constant.

A \( k \)-dimensional \( C^1 \) submanifold of \( \mathbb{R}^n \) is locally the same as the graph of a continuously-differentiable mapping on \( \mathbb{R}^k \) with respect to a suitable choice of coordinate axes. By rotating the axes so that \( \mathbb{R}^k \) is parallel to the tangent plane of the submanifold at a particular point, the submanifold can be represented near the point as the graph of a function with small Lipschitz constant. The Lipschitz constant tends to 0 as one approaches the point in question. Thus distances on \( C^1 \) submanifolds are approximately the same as the infimum of lengths of paths on the submanifold locally.

19 Real analysis

For the sake of simplicity, we have so far avoided referring to Lebesgue integrals and measure. However, this more sophisticated theory can be quite convenient in the present context. Let us mention some of the key points.

A basic fact is that a monotone real-valued function on an open interval in the real line is differentiable “almost everywhere”, which is to say on the complement of a set of Lebesgue measure 0. Thus additional hypotheses of differentiability are sometimes superfluous. Unfortunately, even continuous monotone functions cannot necessarily be recovered from their almost everywhere derivative, as in the fundamental theorem of calculus, without an extra condition of “absolute continuity”. Indeed, there are examples of nonconstant continuous monotone increasing functions with derivative equal to 0 almost everywhere.

It follows that a real-valued function of bounded variation on an interval in the real line is differentiable almost everywhere, since it can be expressed as the difference of two monotone increasing functions. In particular, a real-valued Lipschitz function on an interval is differentiable almost everywhere. Lipschitz functions are absolutely continuous, and so there is a version of the fundamental theorem of calculus for them. As corollaries of this fact, a Lipschitz function \( f \) on an interval is constant if \( f'(x) = 0 \) almost everywhere, \( f \) is monotone increasing if \( f'(x) \geq 0 \) almost everywhere, and \( f \) is \( C \)-Lipschitz if \( |f'(x)| \leq C \) almost everywhere.

At the same time, bounded variation and Lipschitz conditions have natural extensions involving metric spaces, as we have seen. The composition of a path of finite length in a metric space with a real-valued Lipschitz function on the metric space is a function of bounded variation, which is Lipschitz when the path is. There are also a lot of real-valued Lipschitz functions on any metric
space. Even on \( \mathbb{R}^n \), there are a lot of nice functions that are Lipschitz and not continuously differentiable, such as the distance to a point or to a set.

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