Motivic integrals and functional equations.

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Abstract

A functional equation for the motivic integral corresponding to the Milnor number of an arc is derived using the Denef-Loeser formula for the change of variables. Its solution is a function of five auxiliary parameters, it is unique up to multiplication by a constant and there is a simple recursive algorithm to find its coefficients. The method is universal enough and gives, for example, equations for the integral corresponding to the intersection number over the space of pairs of arcs and over the space of unordered tuples of arcs.

1 Introduction

Motivic integration, introduced by M. Kontsevich, is a powerful tool for exploring the space of formal arcs on a given variety. Motivic integrals provide the generating series for motivic measures of level sets of some arc invariants. There are examples of such integrals which can be calculated explicitly (see e.g. [2]). In more general situation the values of such integrals are unknown, however some of them satisfy functional equations if some auxiliary variables are introduced.

In this paper a functional equation for the motivic integral which gives the generating series corresponding to the Milnor number of a plane curve is derived using the Denef-Loeser formula for the change of variables ([2]). Its solution is unique up to multiplication by a constant and there is a simple algorithm to express its coefficients via the initial ones. For example it implies partial differential equations for the solution. This equation gives a method to compute the motivic measure (and, consequently, the Hodge-Deligne polynomial) of the stratum \( \{ \mu = \text{const} \} \) in the space of the plane curves. Some examples are considered in Section 4.

A similar idea gives some other equations, for example, an equation for the integral corresponding to the intersection number over the space of pairs of arcs. Moreover, using the notion of the power structure on the Grothendieck ring ([3]) we introduce a motivic measure on the space of unordered tuples of arcs. A curious equation for the integral corresponding to the intersection number in this case is derived as well.

Some generating series with coefficients in the Grothendieck ring of varieties \( K_0(Var_C) \) (or in the Grothendieck ring of Chow motives) satisfy functional equations similar to the functional equation for Hasse-Weil zeta function. These equations, obtained by M. Kapranov ([5]) and F. Heinloth ([1]),

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follow from the duality theory on curves and Abelian varieties. Notice that they have origin different from our approach.

2 Motivic measure

Let $\mathcal{L} = \mathcal{L}_{\mathbb{C}^2,0}$ be the space of arcs at the origin on the plane. It is the set of pairs $(x(t), y(t))$ of formal power series (without degree 0 terms).

Let $K_0(Var_\mathbb{C})$ be the Grothendieck ring of quasiprojective complex algebraic varieties. It is generated by the isomorphism classes of complex quasiprojective algebraic varieties modulo relations $[X] = [Y] + [X \setminus Y]$, where $Y$ is a Zariski closed subset of $X$. Multiplication is given by the formula $[X] \cdot [Y] = [X \times Y]$. Let $\mathbb{L} \in K_0(Var_\mathbb{C})$ be the class of the complex line.

Consider the ring $K_0(Var_\mathbb{C})[\mathbb{L}^{-1}]$ with the following filtration: $F_k$ is generated by the elements of type $[X] : [\mathbb{L}^{-n}]$ with $n - \dim X \geq k$. Let $\mathcal{M}$ be the completion of $K_0(Var_\mathbb{C})[\mathbb{L}^{-1}]$ corresponding to this filtration.

On an algebra of subsets of the space $\mathcal{L}$ J. Denef and F. Loeser (\cite{2}) (after M. Kontsevich) have constructed a measure $\chi_g$ with values in the ring $\mathcal{M}$. According to this measure, one can naturally define the (motivic) integral for simple functions on $\mathcal{L}$ (\cite{2}).

We will use the simple functions $v_x = \text{Ord}_0 x(t), v_y = \text{Ord}_0 y(t)$ and $v = \min\{v_x, v_y\}$, defined for an arc $\gamma(t) = (x(t), y(t))$.

Let $h : Y \to X$ be a proper birational morphism of smooth manifolds of dimension $d$ and $J = h^* K_X - K_Y$ be the relative canonical divisor on $Y$ (locally it is defined by the Jacobi determinant). It defines a function $\text{ord}_J$ on the space of arcs on $Y$ – the intersection number between the arc and the divisor. Then one has the following change of variables formula in the motivic integral:

**Theorem 1** (\cite{2}) Let $A$ be a measurable subset in the space of arcs on $X$, let $\alpha$ be a simple function. Then

$$\int_A \alpha d\chi_g = \int_{h^{-1}(A)} (h^* \alpha) \mathbb{L}^{-\text{ord}_J} d\chi_g.$$

If $h$ is a blow-up of the origin in the plane, the relative canonical divisor coincides with the exceptional line, so the function $\text{ord}_J$ coincides with the intersection number with this line.

3 Functional equation for Milnor number

The Milnor number of the plane curve given by the equation $\{f = 0\}$ can be defined as the codimension of the ideal generated by the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$.

We shall use the following statement.

**Lemma 1** Suppose that upon after blowing up the origin the Milnor number of an irreducible curve is equal to $\mu$, and the intersection number with the
exceptional divisor is equal to \( p \). Then the Milnor number of the initial curve is equal to
\[
\mu + p(p - 1).
\]

Let
\[
I(t, a, b, c, d, f) = \int t^\mu a^{v_x} b^{v_y} c^{v_z^2} d^{v_x v_y} f^{v^2_y} d\chi_g.
\]

**Theorem 2** This function satisfies the functional equation
\[
I(t, a, b, c, d, f) = I(t, t^{-1} ab \mathbb{L}^{-1}, b, t cd, df^2, f) + I(t, t^{-1} ab \mathbb{L}^{-1}, a, t cd, dc^2, c) + I(t, t^{-1} ab \mathbb{L}^{-1}, 1, t cd, 1, 1) \cdot (\mathbb{L} - 1).
\]

**Proof.** Let
\[
A(t, a, b, c, d, f) = \int \{v_y > v_x\} t^\mu a^{v_x} b^{v_y} c^{v_z^2} d^{v_x v_y} f^{v^2_y} d\chi_g,
\]
and note that
\[
\int \{v_x > v_y\} t^\mu a^{v_x} b^{v_y} c^{v_z^2} d^{v_x v_y} f^{v^2_y} d\chi_g = A(t, b, a, f, d, c).
\]

Let us compute the analogous integral over \( \{v_x = v_y\} \). For \( v_x = v_y \)
\[
y = \lambda x + \tilde{y}, \lambda \neq 0, v(\tilde{y}) > v(y).
\]

For \( \lambda \) fixed \( \mu(x(t), y(t)) = \mu(x(t), \tilde{y}(t)) \), hence
\[
\int \{v_x = v_y\} t^\mu a^{v_x} b^{v_y} c^{v_z^2} d^{v_x v_y} f^{v^2_y} d\chi_g = \int \{v_x = v_y\} t^\mu (ab)^{v_x} (cdf)^{v^2_y} d\chi_g =
\]
\[
(\mathbb{L} - 1) \int \{v_x < v_y\} t^\mu (ab)^{v_x} (cdf)^{v^2_y} d\chi_g = A(t, ab, 1, cdf, 1, 1) \cdot (\mathbb{L} - 1),
\]
and consequently,
\[
I(t, a, b, c, d, f) = A(t, a, b, c, d, f) + A(t, b, a, f, d, c) + A(t, ab, 1, cdf, 1, 1) \cdot (\mathbb{L} - 1).
\]

Let us blow-up the origin. If \( v_y > v_x \), then \( y(t) = x(t) \theta(t), \theta(0) = 0 \), and therefore the corresponding modifications of the curves pass through the fixed point \( p_0 \) of the exceptional divisor related to the \( x \)-axis. Thus
\[
\sigma^* \mu = \mu + v_x (v_x - 1), \sigma^* v_x = v_x, \sigma^* v_x^2 = v_x^2,
\]
\[
\sigma^* v_y = v_y + v_\theta, \sigma^* v_x v_y = v_x^2 + v_x v_\theta, \sigma^* v_y^2 = v_x^2 + 2 v_x v_\theta + v_\theta^2.
\]

Using the Denef-Loeser change of variables formula, we obtain:
\[
A(t, a, b, c, d, f) = \int_{v_y > v_x} t^\mu a^{v_x} b^{v_y} c^{v_z^2} d^{v_x v_y} f^{v^2_y} d\chi_g =
\]
\[
\int_{v_y > v_x} t^{\mu + v_x(v_x - 1)} a^{v_x} b^{v_y + v_\theta} c^{v_z^2} d^{v_x v_y} f^{v^2_y + 2 v_x v_\theta + v_\theta^2} \mathbb{L}^{-v_x} d\chi_g =
\]

3
\[
\int_{\mathcal{L}} t^\mu (t^{-1}abL^{-1})^{v_2} b^{v_0} (tcdf)^{v_2} (df^2)^{v_2} v_0 f v_0 d\chi_g = I(t, t^{-1}abL^{-1}, b, tcdf, df^2, f),
\]
therefore
\[
I(t, a, b, c, d, f) = I(t, t^{-1}abL^{-1}, b, tcdf, df^2, f) + I(t, t^{-1}abL^{-1}, a, tcdf, dc^2, c) + I(t, t^{-1}abL^{-1}, 1, tcdf, 1, 1) \cdot (L - 1).
\]

\[\square\]

It is clear that
\[
I(t, a, b, c, d, f) = I(t, b, a, f, d, c) \tag{2}
\]
and differentiating under the integral we get
\[
-\frac{c}{\partial c} = \left[-a \frac{\partial}{\partial a}\right]^2 I, -\frac{d}{\partial d} = \left[a \frac{\partial}{\partial a}\right] \circ b \frac{\partial}{\partial b} I, -f \frac{\partial I}{\partial f} = \left[-b \frac{\partial}{\partial b}\right]^2 I. \tag{3}
\]

**Theorem 3** A function \(I(t, a, b, c, d, f)\), divisible by \(abcdf\), satisfying the functional equation (1) and the symmetry condition (2), is unique up to multiplication by a constant.

Before proving Theorem 3 let us consider the following simpler example of an analogous functional equation for the motivic integral. Let
\[
f(a, b) = \int_{\mathcal{L}} a^{v_2} b^{v_0} d\chi_g = \frac{abL^{-2}(L - 1)^2}{(1 - aL^{-1})(1 - bL^{-1})}.
\]

Similarly to the calculations above using the change of variables formula one can obtain the functional equation
\[
f(a, b) = f(abL^{-1}, a) + f(abL^{-1}, b) + f(abL^{-1}, 1) \cdot (L - 1). \tag{4}
\]

Let us describe its solutions. Note that from the definition of \(f(a, b)\) it follows that \(f(a, b) = f(b, a), f(0, b) = 0\).

Let \(f(a, b) = \sum_{i,j} f_{ij} a^i b^j\). Then the functional equation (4) can be rewritten in the form
\[
\sum_{i,j} f_{ij} a^i b^j = \sum_{i,j} f_{ij} L^{-i} a^{i+j} b^j + \sum_{i,j} f_{ij} L^{-i} a^i b^{i+j} + (L - 1) \sum_{i,j} f_{ij} L^{-i} a^i b^j =
\]
\[
\sum_{i \ge j} f_{j,i-j} L^{-j} a^i b^j + \sum_{i \le j} f_{i,j-i} L^{-i} a^i b^j + (L - 1) \sum_{i,j} f_{ij} L^{-i} a^i b^j.
\]

Using the relations \(f_{ij} = f_{ji}, f_{i0} = 0\), we obtain the system of recurrence relations on the coefficients:
\[
\begin{cases}
  f_{ij} = L^{-j} f_{i-j,j}, i > j \\
  f_{ii} = L^{-i} (L - 1) \sum_{j=1}^{\infty} f_{ij}.
\end{cases} \tag{5}
\]

Below we will use a more general system of equations:
\[
\begin{cases}
  f_{ij} = \varepsilon_j f_{i-j,j}, i > j, \\
  f_{ii} = C \varepsilon_i \sum_{j=1}^{\infty} f_{ij}.
\end{cases} \tag{6}
\]
Lemma 2  Let $1 - \varepsilon_i - C\varepsilon_i \neq 0, C \neq 0, \varepsilon_i \neq 0, \varepsilon_i \neq 1$ for all $i$. Then the system (6) has a non-zero solution, which is defined uniquely up to multiplication by a constant, such that $f_{i,j} = f_{j,i}$ and $f_{i,0} = 0$.

Proof. From the first equation of (6) one has

$$\sum_{j > i} f_{ij} = \varepsilon_i \sum_{j > i} f_{ij-i} = \varepsilon_i \sum_{j > 0} f_{ij}.$$ Moreover, $f_{ii} = C\varepsilon_i \sum_{j > 0} f_{ij}$, so $f_{ii} = C \sum_{j > i} f_{ij}$, thus

$$\sum_{0 < j < i} f_{ij} = \sum_{j > 0} f_{ij} - \sum_{j > i} f_{ij} = f_{ii}(\frac{1}{C\varepsilon_i} - 1 - \frac{1}{C}),$$

so

$$f_{ii} = \frac{C\varepsilon_i}{1 - \varepsilon_i - C\varepsilon_i} \sum_{0 < j < i} f_{ij}.$$ Let $f_{11}$ be an arbitrary non-zero number. Let us compute $f_{ij}$. If $i \neq j$, we can use the first equation of (6), and if $i = j$ we can use the previous equality. In any case $f_{ij}$ will be expressed via $f_{k,l}$ with $k + l < i + j$, so this process will stop and $f_{ij}$ will be expressed via $f_{11}$. Therefore the solution is unique.

It is easy to prove that the described algorithm defines the solution to (6).

Let us prove Theorem 3. Consider the equation (11). Let

$$I(t, a, b, c, d, f) = \sum_{k_1, k_2, k_3, k_4, k_5} g_{k_1, k_2, k_3, k_4, k_5} a^{k_1} b^{k_2} c^{k_3} d^{k_4} f^{k_5},$$

then

$$\sum_k g_k a^{k_1} b^{k_2} c^{k_3} d^{k_4} f^{k_5} = \sum_{k_1, k_2, k_3, k_4, k_5} g_{k_1, k_2, k_3, k_4, k_5} (t^{-1}L^{-1}ab)^{k_1} b^{k_2} (tcdf)^{k_3} (df^2)^{k_4} f^{k_5} +$$

$$\sum_{k_1, k_2, k_3, k_4, k_5} g_{k_1, k_2, k_3, k_4, k_5} (t^{-1}L^{-1}ab)^{k_1} a^{k_2} (tcdf)^{k_3} (dc^2)^{k_4} c^{k_5} +$$

$$(L - 1) \cdot \sum_{k_1, k_2, k_3, k_4, k_5} g_{k_1, k_2, k_3, k_4, k_5} (t^{-1}L^{-1}ab)^{k_1} (tcdf)^{k_5}.$$ We obtain the following system of equations

$$\begin{cases}
g_{k_1, k_2, k_3, k_4, k_5} = t^{k_3-k_1} L^{-k_1} g_{k_1, k_2 - k_1, k_3 - k_4 - k_5 - 2k_4 + k_5}, & \text{if } k_2 < k_1, k_4 > k_3, k_5 > 2k_4 - k_3; \\
g_{k_1, k_3, k_4, k_5} = t^{k_3-k_1} L^{-k_1} (L - 1) \sum_{k_2, k_4, k_5} g_{k_1, k_2, k_3, k_4, k_5}; \\
g_{k_1, k_2, k_3, k_4, k_5} = g_{k_2, k_1, k_3, k_4, k_5}. 
\end{cases}$$

Moreover, $g_{k_1, k_2, k_3, k_4, k_5} = 0$, if the collection $(k_1, k_2, k_3, k_4, k_5)$ does not satisfy the inequalities

$$k_2 \geq k_1, k_4 \geq k_3, k_5 \geq 2k_4 - k_3,$$

or

$$k_1 \geq k_2, k_4 \geq k_5, k_3 \geq 2k_4 - k_5.$$
Similarly to the proof of Lemma 2, one can check that any coefficient
could be expressed via $g_{1,1,1,1}$ and the solution is unique.

Since by Theorem 2 $\int L t^{m} a^{x} b^{y} c^{z} d^{x} e^{y} f^{x} d\mu$ satisfies the equation (1),
every solution of this equation is proportional to it.

Therefore

$$I(t, a, b, c, d, f) = \sum G_{i,j}(t) a^{i} b^{j} c^{i} d^{i} e^{j} f^{j}.$$  

Thus (7) yields the partial differential equations (3).

Let us compute $g_{1,1,1,1,1}$. If $v_x = v_y = 1$, then $\mu = 0$, so

$$g_{1,1,1,1,1} = \chi_y \{v_x = v_y = 1\} = (L - 1)^2 L^{-2}.$$

4 Examples

Recall that

$$G_{i,j}(t) = \int_{v_x = i, v_y = j} t^{\mu} d\mu.$$  

One can check, that the system (7) gives a system of equations for $G_{i,j}(t)$ of
the form (6) with with

$$\varepsilon_k = (\varepsilon - k) L^{-k}, C = (L - 1).$$

Therefore one has (from the proof of lemma 2)

$$\begin{cases}
G_{i,j}(t) = t^{\varepsilon - j} L^{-j} G_{i-j,j}, j < i \\
G_{i,i}(t) = (\varepsilon - 1) t^{\varepsilon - i} L^{-1} \sum_{j<i} G_{i,j}(t) \\
G_{1,1}(t) = (L - 1)^2 L^{-2}.
\end{cases}$$

Then $G_{1,n}(t) = L^{-1} G_{1,n-1}$, hence

$$G_{1,n}(t) = (L - 1)^2 L^{-1-n}.$$  

This corresponds to that fact that every arc with $v_x = 1$ is smooth, so $\mu = 0$.

Moreover, $G_{2,n}(t) = t^2 L^{-2} G_{2,n-2}(t)$ for $n > 2$, so

$$G_{2,2n-1} = t^{2n-2} L^{2-2n} G_{2,1} = (L - 1)^2 t^{2n-2} L^{-1-2n}.$$  

Every arc with $v_x = 2$ and $v_y$ odd has the Milnor number equal to $v_y - 1$,
because the singularity is of $A_{v_y-1}$ type.

The result for the case when $v_y$ is even is more interesting.

$$G_{2,2n}(t) = t^{2n-2} L^{2-2n} G_{2,2} = t^{2n-2} L^{-2n} \frac{(L - 1)}{1 - t^2 L^{-1}} G_{1,2}(t) =$$

$$\frac{t^{2n-2} L^{-2n-3} (L - 1)^3}{1 - t^2 L^{-1}} = \sum_{k=0}^{\infty} t^{2(n+k)} L^{-2n-3-k} (L - 1)^3.$$  

Let us explain this answer. Making a change of variables one can obtain

$$x(t) = t^2.$$  

Let

$$y(t) = a_{2n} t^{2n} + \ldots + a_{2m+1} t^{2m+1} + \ldots,$$
where \( a_{2m+1} \) is the first non-zero coefficient with odd index. Then the equation of this curve is

\[
F(x, y) = x^{2m+1} + (y - a_{2n}x^n - \ldots - a_{2m}x^m)^2 + \ldots = 0,
\]

so the Milnor number equals to \( 2m \). Consider \( k = m - n \). For fixed \( x(t) \) the measure of series \( y(t) \) with given \( m \) is equal to

\[
(\mathbb{L} - 1)^2\mathbb{L}^{-2n-(m-n+1)} = (\mathbb{L} - 1)^2\mathbb{L}^{-2n-k-1},
\]

and the measure of series \( x(t) \) with order 2 is equal to \((\mathbb{L} - 1)\mathbb{L}^{-2}\). Multiplying these expressions, we obtain the above formula.

**Lemma 3** Let \( a \) be the greatest common divisor of \( i \) and \( j \). Then

\[
G_{i,j}(t) = t^{(i-1)(j-1)-(a-1)^2}\mathbb{L}^{2a-i-j}G_{a,a}(t).
\]

**Proof.** For \( i = j \) the formula is tautological. Suppose that it is true for \( i - j \) and \( j \) (\( i > j \)). Then

\[
G_{i,j}(t) = t^{j^2-j}\mathbb{L}^{-j}G_{i-j,j} = t^{(i-j-1)(j-1)-(a-1)^2+j(j-1)}\mathbb{L}^{2a-(i-j)-j}G_{a,a}(t).
\]

Therefore the proposition follows from the Euclid algorithm. \( \square \)

Similarly to the discussion above one can obtain the following answers:

| \( a \) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| \( G_{a,a}(t) \) | \((\mathbb{L} - 1)\mathbb{L}^2\) | \((\mathbb{L} - 1)^4t^2\mathbb{L}^{-5}1-t^4\mathbb{L}^{-3}\) | \((\mathbb{L} - 1)^4t^6\mathbb{L}^{-7}(1+t^2\mathbb{L}^{-1})1-t^6\mathbb{L}^{-2}\) | \((\mathbb{L} - 1)^3t^{12}\mathbb{L}^{-9}(1-t^2\mathbb{L}^{-1}+t^4\mathbb{L}^{-1}-t^6\mathbb{L}^{-3})1-t^{12}\mathbb{L}^{-3}(1-t^2\mathbb{L}^{-1})\) |

This table together with Lemma 3 provides \( G_{i,j}(t) \) with \( \gcd(i, j) \leq 4 \).

**Proposition.** Let \( a = \gcd(i, j) \). \( G_{i,j}(t) \) is a power series of \( t \), which coefficients are Laurent polynomials in \( \mathbb{L} \). If \( a = 1 \), then

\[
G_{i,j}(t) = (\mathbb{L} - 1)^2t^{(i-1)(j-1)}\mathbb{L}^{-i-j},
\]

and

\[
G_{i,j}(t) = (\mathbb{L} - 1)^3t^{(i-1)(j-1)+a-1}\mathbb{L}^{-i-j-1} + \text{terms of higher degree in } t.
\]

**Proof.** The first statement can be easily checked by induction. The case \( a = 1 \) follows from Lemma 3. Let us prove the formula for the case \( a > 1 \). One has

\[
G_{a,a}(t) = \frac{(\mathbb{L} - 1)ta^{2-a}\mathbb{L}^{-a}}{1-ta^{2-a}\mathbb{L}^{-a}}(G_{a,1}(t) + O(t)) = (\mathbb{L} - 1)t^{a^2-a}\mathbb{L}^{-a}(1+O(t))(\mathbb{L} - 1)^2\mathbb{L}^{-a+1}+O(t) = (\mathbb{L} - 1)^3t^{a^2-a}\mathbb{L}^{-2a-1}+O(t^{a^2-a+1}).
\]

Now the statement follows from Lemma 3. \( \square \)
5 Functional equation for the intersection number

Let

\[ J(t,a,b,c,d,p,q,r,s) = \int_{\mathcal{L}(1) \times \mathcal{L}(2)} t^{1+\alpha_2} a^{(1)} b^{(2)} c^{(1)} d^{(2)} p^{(1)} q^{(2)} r^{(1)} s^{(2)} \, d\chi_g, \]

\[ B(t,a,b,c,d,p,q,r,s) = \int_{\{v_y^{(1)} > v_x^{(1)}, v_y^{(2)} > v_x^{(2)}\}} t^{1+\alpha_2} a^{(1)} b^{(2)} c^{(1)} d^{(2)} p^{(1)} q^{(2)} r^{(1)} s^{(2)} \, d\chi_g, \]

Note that

\[ \sigma(\gamma_1) \circ \sigma(\gamma_2) = \gamma_1 \circ \gamma_2 + v_1 v_2, \]

hence if after blowing up the origin arcs intersect the exceptional divisor at different points, their intersection number equals to the product of multiplicities.

Let us decompose \( \mathcal{L}(1) \times \mathcal{L}(2) \) into components with respect to the inequalities between \( v_y^{(1)} \) and \( v_x^{(1)} \), and also \( v_y^{(2)} \) and \( v_x^{(2)} \).

1) \( v_y^{(1)} > v_x^{(1)}, v_y^{(2)} > v_x^{(2)} \). By definition, the integral equals to

\[ B(t,a,b,c,d,p,q,r,s). \]

2) \( v_y^{(1)} > v_x^{(1)}, v_y^{(2)} < v_x^{(2)} \). After blow-up arcs intersect the exceptional divisor at different points, and therefore the integral equals to

\[ \int_{\{v_y^{(1)} > v_x^{(1)}, v_y^{(2)} < v_x^{(2)}\}} t^{1+\alpha_2} a^{(1)} b^{(2)} c^{(1)} d^{(2)} p^{(1)} q^{(2)} r^{(1)} s^{(2)} \, d\chi_g = \]

\[ \left( v^{(1)}_y = i, v^{(1)}_x = j, v^{(2)}_y = k, v^{(2)}_x = l \right) = \]

\[ \sum_{i<j,k<l} a^{i} b^{j} c^{k} d^{l} p^{i} q^{j} r^{k} s^{l} (L-1)^{4} L^{-i-j-k-l} = \]

\[ (L-1)^{4} \Phi(bt,a,d,c;L^{-1}p,L^{-1}q,L^{-1}s,L^{-1}r), \]

where \( \Phi \) is defined by the equation

\[ \Phi(\alpha, \beta, \gamma, \delta, \pi, \kappa, \rho, \sigma) = \sum_{i<j,k<l} \alpha^{ik} \beta^{jl} \gamma^{jk} \delta^{il} \pi^{lj} \kappa^{j} \rho^{k} \sigma^{l}. \]

3) \( v_y^{(1)} > v_x^{(1)}, v_y^{(2)} = v_x^{(2)} \). After blowing up arcs intersect the exceptional divisor at different points. Therefore the integral equals to

\[ \int_{\{v_y^{(1)} > v_x^{(1)}, v_y^{(2)} = v_x^{(2)}\}} t^{1+\alpha_2} a^{(1)} b^{(2)} c^{(1)} d^{(2)} p^{(1)} q^{(2)} r^{(1)} s^{(2)} \, d\chi_g = \]

\[ \sum_{i<j,k} (tab)^{ik} (cd)^{jk} p^{i} q^{j} (rs)^{k} (L-1)^{4} L^{-i-j-2k} = (L-1)^{4} \Psi(tab, cd, pL^{-1}, qL^{-1}, rsL^{-2}), \]

where \( \Psi \) is defined by the equation

\[ \Psi(\alpha, \beta, \pi, \kappa, \rho) = \sum_{i<j,k} \alpha^{ik} \beta^{jk} \pi^{i} \kappa^{j} \rho^{k}. \]
4) \( v_y^{(1)} < v_x^{(1)}, v_y^{(2)} > v_x^{(2)} \).

\[ \int_{\{v_y^{(1)} < v_x^{(1)}, v_y^{(2)} > v_x^{(2)}\}} t^{v_y^{(1)} v_x^{(2)}} a^{v_x^{(1)} v_y^{(2)}} b^{v_x^{(1)} v_y^{(2)}} c^{v_x^{(1)} v_y^{(2)}} d^{v_y^{(1)} v_x^{(2)}} p^{v_y^{(1)} q_x^{(1)}} r^{v_y^{(2)} s_y^{(2)}} d\chi_g = \]

\[ \sum_{i<j, k<l} a^{i k} b^{i l} (ct)^{i k} d^{i t} q^{i j} r^{k j} s^{i l} (L - 1)^{4 L_{-i-j-k-l}} = \]

\((L - 1)^{4 \Phi(ct, d, a, b, q L^{-1}, p L^{-1}, r L^{-1}, s L^{-1})}\).

5) \( v_y^{(1)} < v_x^{(1)}, v_y^{(2)} < v_x^{(2)} \). From the symmetry it is clear that the integral equals to

\[ \int_{\{v_y^{(1)} > v_x^{(1)}, v_y^{(2)} > v_x^{(2)}\}} t^{v_y^{(1)} v_x^{(2)}} a^{v_x^{(1)} v_y^{(2)}} b^{v_x^{(1)} v_y^{(2)}} c^{v_x^{(1)} v_y^{(2)}} d^{v_y^{(1)} v_x^{(2)}} p^{v_y^{(1)} q_x^{(1)}} r^{v_y^{(2)} s_y^{(2)}} d\chi_g = B(t, d, c, a, q, p, s, r). \]

6) \( v_y^{(1)} < v_x^{(1)}, v_y^{(2)} = v_x^{(2)} \).

\[ \int_{\{v_y^{(1)} < v_x^{(1)}, v_y^{(2)} = v_x^{(2)}\}} t^{v_y^{(1)} v_x^{(2)}} a^{v_x^{(1)} v_y^{(2)}} b^{v_x^{(1)} v_y^{(2)}} c^{v_x^{(1)} v_y^{(2)}} d^{v_y^{(1)} v_x^{(2)}} p^{v_y^{(1)} q_x^{(1)}} r^{v_y^{(2)} s_y^{(2)}} d\chi_g = \]

\[ \sum_{i<j, k} (tc d)^{i k} (ab)^{i k} p^{j q} r^{j s} (L - 1)^{4 L_{-i-j-2k}} = (L - 1)^{4 \Phi(tc d, a b, q L^{-1}, p L^{-1}, r s L^{-2})}. \]

7) \( v_y^{(1)} = v_x^{(1)}, v_y^{(2)} > v_x^{(2)} \).

\[ \int_{\{v_y^{(1)} = v_x^{(1)}, v_y^{(2)} > v_x^{(2)}\}} t^{v_y^{(1)} v_x^{(2)}} a^{v_x^{(1)} v_y^{(2)}} b^{v_x^{(1)} v_y^{(2)}} c^{v_x^{(1)} v_y^{(2)}} d^{v_y^{(1)} v_x^{(2)}} p^{v_y^{(1)} q_x^{(1)}} r^{v_y^{(2)} s_y^{(2)}} d\chi_g = \]

\[ \sum_{i<j, k} (tb d)^{i k} (ac)^{j k} s^{i r} (L - 1)^{4 L_{-i-j-2k}} = (L - 1)^{4 \Phi(tc b, a d, s L^{-1}, r L^{-1}, p q L^{-2})}. \]

8) \( v_y^{(1)} = v_x^{(1)}, v_y^{(2)} < v_x^{(2)} \).

\[ \int_{\{v_y^{(1)} = v_x^{(1)}, v_y^{(2)} < v_x^{(2)}\}} t^{v_y^{(1)} v_x^{(2)}} a^{v_x^{(1)} v_y^{(2)}} b^{v_x^{(1)} v_y^{(2)}} c^{v_x^{(1)} v_y^{(2)}} d^{v_y^{(1)} v_x^{(2)}} p^{v_y^{(1)} q_x^{(1)}} r^{v_y^{(2)} s_y^{(2)}} d\chi_g = \]

\[ \sum_{i<j, k} (tb d)^{i k} (ac)^{j k} r^{i s} (L - 1)^{4 L_{-i-j-2k}} = (L - 1)^{4 \Phi(tc b, a c, s L^{-1}, r L^{-1}, p q L^{-2})}. \]

9) \( v_y^{(1)} = v_x^{(1)}, v_y^{(2)} = v_x^{(2)} \). In this case

\[ \begin{cases} y^{(1)} = \lambda_1 x^{(1)} + \tilde{y}^{(1)}, v_y^{(1)} > v_x^{(1)}; \\ y^{(2)} = \lambda_2 x^{(2)} + \tilde{y}^{(2)}, v_y^{(2)} > v_x^{(2)}. \end{cases} \]

\[ \lambda_1 \neq 0, \lambda_2 \neq 0. \]

a) \( \lambda_1 \neq \lambda_2 \). In this case

\[ \{\lambda_1, \lambda_2 \in \mathbb{C}^*|\lambda_1 \neq \lambda_2\} = [\mathbb{C}^*]^2 - \{\lambda_1, \lambda_2 \in \mathbb{C}^*|\lambda_1 = \lambda_2\} = (L - 1)^2 - (L - 1) = (L - 1)(L - 2), \]

and for fixed \( (\lambda_1, \lambda_2) \) the integral equals to

\[ \int_{\{v_y^{(1)} > v_x^{(1)}, v_y^{(2)} > v_x^{(2)}\}} t^{v_y^{(1)} v_x^{(2)}} a^{v_x^{(1)} v_y^{(2)}} b^{v_x^{(1)} v_y^{(2)}} c^{v_x^{(1)} v_y^{(2)}} d^{v_y^{(1)} v_x^{(2)}} p^{v_y^{(1)} q_x^{(1)}} r^{v_y^{(2)} s_y^{(2)}} d\chi_g = \]

9
Therefore the contribution of the stratum \( \{ v_x^{(1)} = v_y^{(1)}, v_x^{(2)} = v_y^{(2)}, \lambda_1 \neq \lambda_2 \} \) equals to

\[
(\mathbb{L} - 1)^5 \Phi(tabcd, 1, 1, 1; \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}).
\]

b) \( \lambda_1 = \lambda_2 = \lambda \). Under the affine change of variables \( (x, y) \mapsto (x, y - \lambda x) \) intersection number does not change, so for fixed \( \lambda \) the integral equals to

\[
\int_{\{v_y^{(1)} > v_x^{(1)}, v_y^{(2)} > v_x^{(2)}\}} t_1^{\gamma_1 + \gamma_2} a_v^{(1)} b_v^{(1)} c_v^{(2)} d_v^{(1)} p_v^{(1)} q_v^{(1)} r_v^{(2)} s_v^{(2)} d\chi_g =
\]

\[
B(t, abcd, 1, 1, 1, pq, 1, rs, 1),
\]

hence the total contribution of this stratum equals to

\[
(\mathbb{L} - 1)B(t, abcd, 1, 1, 1, pq, 1, rs, 1).
\]

So we have:

\[
J(t, a, b, c, d, p, q, r, s) = B(t, a, b, c, d, p, q, r, s) + (\mathbb{L} - 1)^4 \Phi(bt, a, d, c; \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}) +
\]

\[
(\mathbb{L} - 1)^4 \Phi(tabcd, 1, 1, 1; \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}).
\]

Consider a blow-up. If \( v_y > v_x \), then \( y(t) = x(t)\theta(t). \) So

\[
\sigma^*(\gamma_1 \circ \gamma_2) = \gamma_1 \circ \gamma_2 + y_x^{(1)} y_x^{(2)}, \sigma^* v_x^{(1)} = v_x^{(1)},
\]

\[
\sigma^* v_x^{(2)} = y_x^{(2)}, \sigma^* v_y^{(1)} = v_y^{(1)} + y_y^{(1)}, \sigma^* v_y^{(2)} = v_y^{(2)} + y_y^{(2)}.
\]

Using the Denef-Loeser change of variables formula we have

\[
B(t, a, b, c, d, p, q, r, s) = \int_{\mathcal{L}^1 \times \mathcal{L}^2} t_1^{\gamma_1 + \gamma_2} a_v^{(1)} b_v^{(1)} c_v^{(2)} d_v^{(1)} p_v^{(1)} q_v^{(1)} r_v^{(2)} s_v^{(2)} \times
\]

\[
\times d_v^{(1)} v_v^{(2)} + v_v^{(2)} + v_v^{(1)} + v_v^{(2)} + v_v^{(1)} + v_v^{(2)} + v_v^{(1)} + v_v^{(2)} L - v_v^{(1)} - v_v^{(2)} d\chi_g =
\]

\[
J(t, tabcd, bd, cd, d, pq\mathbb{L}^{-1}, q, rs\mathbb{L}, s).
\]

Substituting these expressions for \( B(\cdot) \) into (8), we get the following statement.

**Lemma 4** The function \( J \) satisfies the functional equation:

\[
J(t, a, b, c, d, p, q, r, s) = J(t, tabcd, bd, cd, d, pq\mathbb{L}^{-1}, q, rs\mathbb{L}^{-1}, s) +
\]

\[
(\mathbb{L} - 1)^4 \Phi(bt, a, d, c, \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}) + (\mathbb{L} - 1)^4 \Phi(tabcd, 1, 1, 1; \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}) +
\]

\[
(\mathbb{L} - 1)^4 \Phi(acbd, 1, 1, 1; \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}) + (\mathbb{L} - 1)^4 \Phi(bt, bd, \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}pq, \mathbb{L}^{-1}) +
\]

\[
(\mathbb{L} - 1)^4 \Phi(tcbd, 1, 1, 1; \mathbb{L}^{-1}pq, \mathbb{L}^{-1}, \mathbb{L}^{-1}rs, \mathbb{L}^{-1}) + (\mathbb{L} - 1)^4 \Phi(t, tabd, 1, 1, 1, pq\mathbb{L}^{-1}, 1, rs\mathbb{L}^{-1}, 1).
\]
6 Power structures

The notion of the power structure over a (semi)ring was introduced by S. Gusein-Zade, I. Luengo and A. Melle-Hernandez in [3].

**Definition:** A power structure on the ring $\mathcal{R}$ is a map

$$(1 + t\mathcal{R}[[t]]) \times \mathcal{R} \to 1 + t\mathcal{R}[[t]] : (A(t), m) \mapsto (A(t))^m,$$

satisfying the following properties:

1. $(A(t))^0 = 1,$
2. $(A(t))^1 = A(t),$  
3. $((A(t) \cdot B(t))^m = ((A(t))^m \cdot (B(t))^m,$
4. $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n,$
5. $(A(t))^{mn} = ((A(t))^n)^m,$
6. $(1 + t)^m = 1 + mt + \text{terms with higher degree},$
7. $(A(t^k))^m = ((A(t))^m)|_{t \to t^k}.$

A power structure is called finitely determined if for every $N > 0$ there exists such $M > 0$ that the $N$-jet of the series $(A(t))^m$ is uniquely determined by the $M$-jet of the series $A(t).$

To fix a finitely determined power structure it is sufficient to define the series $(1 - t)^{-m}$ for every $m \in \mathcal{R}$ such that $(1 - t)^{-m-n} = (1 - t)^{-m} \cdot (1 - t)^{-n}.$

Over the Grothendieck ring of varieties there is the power structure, such that

$$(1 - t)^{-[X]} = 1 + [S^1 X]t + [S^2 X]t^2 + \ldots,$$

where $S^k X = X^k / S_k$ denotes the $k$-th symmetric power of $X.$ For example, for $j \geq 0$

$$(1 - t)^{-L^j} = \sum_{k=0}^{\infty} t^k L^kj = (1 - tL^j)^{-1}.$$

For $X \in K_0(Var_C), k > 0,$ let

$$(1 - t)^{-L^{-k}X} = (1 - u)^{-X}|_{u = L^{-k}t}.$$

The following statement defines the corresponding power structure over the ring $\mathcal{M}$

**Lemma 5** The map $K_0(Var_C)\mathcal{L}^{-1} \to 1 + tK_0(Var_C)\mathcal{L}^{-1}[[t]],$

$$Z \mapsto (1 - t)^{-Z}$$

is well defined. It transforms the sum into the product and is continuous with respect to the filtration $F_k.$

Let us construct a measure on the symmetric power $S^k \mathcal{L}.$ For a cylindric set $A = \pi_{\mathcal{L}}^{-1}(A_n)$ let $\mu(S^k A) = L^{-2nk}[S^k A_n].$ This construction corresponds to the power structure over the ring $\mathcal{M},$ so

$$\sum_k \mu(S^k A)t^k = (1 - t)^{-\mu(A)}.$$
Lemma 6 Let \( f \) be a simple function on \( \mathcal{L} \). Define a function \( F \) on \( \sqcup_k S^k \mathcal{L} \) by the formula \( F(\gamma_1, \ldots, \gamma_k) = \prod_i f(\gamma_i) \). Then

\[
\int_{\sqcup_k S^k \mathcal{L}} F d\chi_g = \int_{\mathcal{L}} (1 - f)^{-d\chi_g}.
\]

Here \( d\chi_g \) is in the exponent to emphasize that \( 1 - f \) is considered as an element of an abelian group with respect to multiplication.

Proof. Consider \( \mathcal{L} = \sqcup_j B_j \), \( f|_{B_j} = f_j \). Then

\[
S^k \mathcal{L} = \sqcup_{k_1 + k_2 + \ldots = k} S^{k_1} B_1 \times S^{k_2} B_2 \times \ldots,
\]
and

\[
F|_{S^{k_1} B_1 \times S^{k_2} B_2 \times \ldots} = f_1^{k_1} \cdot f_2^{k_2} \cdot \ldots.
\]

Therefore

\[
\int_{\sqcup_k S^k \mathcal{L}} = \sum_k \sum_{k_i = k} \mu(S^{k_1} B_1 \times S^{k_2} B_2 \times \ldots) f_1^{k_1} f_2^{k_2} \ldots =
\]

\[
\prod_j (1 + \mu(S^1 B_j) f_j + \mu(S^2 B_j) f_j^2 + \ldots) = \prod_j (1 - f_j)^{-\mu(B_j)} = \int_{\mathcal{L}} (1 - f)^{-d\chi_g}.
\]

\( \square \)

7 An equation for the intersection numbers on \( \mathcal{L} \times \sqcup_k S^k \mathcal{L} \)

Consider the generating function:

\[
I(t,a,b,c,d,p,q,r,s,u)=\int_{\mathcal{L} \times \sqcup_k S^k \mathcal{L}} t^{\gamma_1} v^{\gamma_2} p^{(1)} v^{(2)} r^{(1)} s^{(2)} u^{k} d\chi_g.
\]

In this section we will obtain a functional equation for the function \( I \).

Let

\[
\prod_{k \geq l} (1 - x^k y^l u)^{-(L-1)^2} = \sum_{k_1, k_2} e_{k_1, k_2}(u) x^{k_1} y^{k_2}
\]
and

\[
\prod_{k < l} (1 - (xy)^k z^l u)^{-(L-1)^2} \prod_{k > l} (1 - x^k (yz)^l u)^{-(L-1)^2} \prod_{l > k} (1 - (xyz)^k \mathbb{L}^{-l} u)^{-(L-2)(L-1)^2}
\]
so the first integral is equal to
\[ J_{\gamma_1}(t, a, b, c, d, r, s, u) = \int_{L(2)} t^{\alpha_1 \gamma_2} a^{(1)}(v_2) b^{(1)}(v_2) c^{(1)} y_g(2) d^{(1)} y_g(2) r^{(2)} s^{(2)} u^k d\chi_g = \]
\[ \int_{L(2)} (1 - t^{\alpha_1 \gamma_2} a^{(1)}(v_2) b^{(1)}(v_2) c^{(1)} y_g(2) d^{(1)} y_g(2) r^{(2)} s^{(2)} u^k) d\chi_g, \]
then
\[ I(t, a, b, c, d, p, q, r, s, u) = \int_{L(1)} p^{(1)} q^{(1)} J_{\gamma_1}(t, a, b, c, d, r, s, u) d\chi_g. \]
If \( v_y^{(1)} > v_2^{(1)} \), then
\[ J_{\gamma_1}(t, a, b, c, d, r, s, u) = \int_{\{v_y^{(2)} > v_2^{(2)}\}} (1 - t^{\alpha_1 \gamma_2} a^{(1)}(v_2) b^{(1)}(v_2) c^{(1)} y_g(2) d^{(1)} y_g(2) r^{(2)} s^{(2)} u^k) d\chi_g \]
\[ \int_{\{v_y^{(2)} \leq v_2^{(2)}\}} (1 - t^{\alpha_1 \gamma_2} a^{(1)}(v_2) b^{(1)}(v_2) c^{(1)} y_g(2) d^{(1)} y_g(2) r^{(2)} s^{(2)} u^k) d\chi_g. \]
Let us blow-up the origin. By the Denef-Loeser formula \( d\chi_g = L^{-v_2} d\chi_g \),
so the first integral is equal to
\[ \int_{L(2)} (1 - t^{\alpha_1 \gamma_2} (tabcd)^{(v_2)} \gamma_{v_2}^{(1)} (bd)^{v_2} \gamma_{v_2}^{(1)} (cd)^{v_2} \gamma_{v_2}^{(1)} (rs)^{v_2} \gamma_{v_2}^{(1)} (u)^{v_2} \gamma_{v_2}^{(1)} (v)^{v_2} \gamma_{v_2}^{(1)} (w)^{v_2} \gamma_{v_2}^{(1)} (x)^{v_2} \gamma_{v_2}^{(1)} (y)^{v_2} \gamma_{v_2}^{(1)} (z)^{v_2} \gamma_{v_2}^{(1)} d\chi_g = \]
\[ J_{\gamma_1}(t, tabcd, bd, cd, d, rsL^{-1}, s, u). \]
The second integral equals to
\[ \Pi_{k \leq 1} (1 - a^{k} b^{k} c^{k} d^{k} e^{k} f^{k} g^{k} h^{k} i^{k} j^{k} k^{k} l^{k} m^{k} n^{k} o^{k} p^{k} q^{k} r^{k} s^{k} t^{k}) = \]
\[ = \sum_{k_1, k_2} \varepsilon_{k_1, k_2}(u) (L^{-1} r)^{k_1} (L^{-1} s)^{k_2} a^{k_1 v_2^{(1)}} (b)^{k_2 v_2^{(1)}} (c)^{k_1 v_2^{(1)}} (d)^{k_2 v_2^{(1)}} (e)^{k_1 v_2^{(1)}} (f)^{k_2 v_2^{(1)}} (g)^{k_1 v_2^{(1)}} (h)^{k_2 v_2^{(1)}} (i)^{k_1 v_2^{(1)}} (j)^{k_2 v_2^{(1)}} (k)^{k_1 v_2^{(1)}} (l)^{k_2 v_2^{(1)}} (m)^{k_1 v_2^{(1)}} (n)^{k_2 v_2^{(1)}} (o)^{k_1 v_2^{(1)}} (p)^{k_2 v_2^{(1)}} (q)^{k_1 v_2^{(1)}} (r)^{k_2 v_2^{(1)}} (s)^{k_1 v_2^{(1)}} (t)^{k_2 v_2^{(1)}} (u)^{k_1 v_2^{(1)}} (v)^{k_2 v_2^{(1)}} (w)^{k_1 v_2^{(1)}} (x)^{k_2 v_2^{(1)}} (y)^{k_1 v_2^{(1)}} (z)^{k_2 v_2^{(1)}} d\chi_g. \]
Hence
\[ J_{\gamma_1}(t, a, b, c, d, r, s, u) = J_{\gamma_1}(t, tabcd, bd, cd, d, rsL^{-1}, s, u) \times \]
\[ (\sum_{k_1, k_2} \varepsilon_{k_1, k_2}(u) (L^{-1} r)^{k_1} (L^{-1} s)^{k_2} a^{k_1 v_2^{(1)}} (b)^{k_2 v_2^{(1)}} (c)^{k_1 v_2^{(1)}} (d)^{k_2 v_2^{(1)}} (e)^{k_1 v_2^{(1)}} (f)^{k_2 v_2^{(1)}} (g)^{k_1 v_2^{(1)}} (h)^{k_2 v_2^{(1)}} (i)^{k_1 v_2^{(1)}} (j)^{k_2 v_2^{(1)}} (k)^{k_1 v_2^{(1)}} (l)^{k_2 v_2^{(1)}} (m)^{k_1 v_2^{(1)}} (n)^{k_2 v_2^{(1)}} (o)^{k_1 v_2^{(1)}} (p)^{k_2 v_2^{(1)}} (q)^{k_1 v_2^{(1)}} (r)^{k_2 v_2^{(1)}} (s)^{k_1 v_2^{(1)}} (t)^{k_2 v_2^{(1)}} (u)^{k_1 v_2^{(1)}} (v)^{k_2 v_2^{(1)}} (w)^{k_1 v_2^{(1)}} (x)^{k_2 v_2^{(1)}} (y)^{k_1 v_2^{(1)}} (z)^{k_2 v_2^{(1)}} d\chi_g). \]
Therefore we get that
\[ E(t, a, b, c, d, p, r, s, u) = \int_{v_y^{(1)} > v_2^{(1)}} t^{\alpha_1 \gamma_2} a^{(1)}(v_2) b^{(1)}(v_2) c^{(1)} y_g(2) d^{(1)} y_g(2) r^{(2)} s^{(2)} u^k d\chi_g = \]
\[ \int_{v_y^{(1)} > v_2^{(1)}} J_{\gamma_1}(t, a, b, c, d, r, s, u) p^{(1)} q^{(1)} d\chi_g = \sum_{k_1, k_2} \varepsilon_{k_1, k_2}(u) (L^{-1} r)^{k_1} (L^{-1} s)^{k_2} \times \]
\[ \int_{v_y^{(1)} > v_2^{(1)}} J_{\gamma_1}(t, a, b, c, d, r, s, u) d\chi_g = \sum_{k_1, k_2} \varepsilon_{k_1, k_2}(u) (L^{-1} r)^{k_1} (L^{-1} s)^{k_2} \times \]
\[ \sum_{k_1, k_2} \varepsilon_{k_1, k_2}(u) (L^{-1} r)^{k_1} (L^{-1} s)^{k_2} \int_{v_y^{(1)} > v_2^{(1)}} J_{\gamma_1}(t, tabcd, bd, cd, d, rsL^{-1}, s, u) d\chi_g = \]
\[ \sum_{k_1, k_2} \varepsilon_{k_1, k_2}(u) (L^{-1} r)^{k_1} (L^{-1} s)^{k_2} \int_{v_y^{(1)} > v_2^{(1)}} J_{\gamma_1}(t, tabcd, bd, cd, d, rsL^{-1}, s, u) d\chi_g. \]
\[(a^k b d t^k b p) v_x^{(1)} (c^1 d^2 q)^{y_x^{(1)}} d x_y^{(1)} = \sum_{k_1, k_2} \epsilon_{k_1, k_2} (u)(L^{-1} r)^{k_1} (L^{-1} s)^{k_2} \times \]

\[\int_{L^1} J_{\gamma_1}(t, t a b c d, r s L^{-1}, s, u) ((a c)^{k_1} (b d t^k p q L^{-1}) v_x^{(1)} (c^1 d^2 q)^{y_x^{(1)}} d x_y^{(1)} = \]

\[\sum_{k_1, k_2} \epsilon_{k_1, k_2} (u)(L^{-1} r)^{k_1} (L^{-1} s)^{k_2} I(t, t a b c d, d, (a c)^{k_1} (b d t^k p q L^{-1}, c^1 d^2 q, r s L^{-1}, s, u). \]

It is clear that the same integral over \([v_x^{(1)} < v_y^{(1)}]\) equals to \(E(t, d, b, a, q, p, s, r, u). \)

Let us compute the integral over \([v_x^{(1)} = v_y^{(1)}]. \) Let \(y^{(1)} = \lambda_1 x^{(1)} + \tilde{y}^{(1)}. \) Then

\[J_{\gamma_1}(t, a, b, c, d, r, s, u) = \int_{\{v_x^{(2)} < v_y^{(2)}\}} (1 - t^{110^2} v_x^{(2)} v_y^{(2)} (b^2 v_{y}^{(2)} c^{y} v_{y}^{(2)} d^{y} v_{y}^{(2)} r v^{(2)} s v^{(2)} u) - d\lambda_1^{(2)} \times \]

\[\int_{\{v_x^{(2)} > v_y^{(2)}\}} (1 - t^{110^2} v_x^{(2)} b v_{y}^{(2)} c^{y} v_{y}^{(2)} d^{y} v_{y}^{(2)} r v^{(2)} s v^{(2)} u) - d\lambda_1^{(2)} = \]

\[\Pi_{k < l} (1 - \alpha k v_x^{(1)} (b d t v_x^{(1)} r k s u) - L^{-k-1} L^{-l-1} L^{-2} - \Pi_{k > l} (1 - \alpha c v_x^{(1)} (b d t v_x^{(1)} r k s u) - L^{-k-1} L^{-l-1} L^{-2} \times \]

\[\int_{\{v_x^{(2)} < v_y^{(2)}\}} (1 - t^{110^2} v_x^{(2)} b v_{y}^{(2)} c^{y} v_{y}^{(2)} d^{y} v_{y}^{(2)} r v^{(2)} s v^{(2)} u) - d\lambda_1^{(2)} \]

The last integral can be decomposed into the product of integrals over \([\lambda_2 \neq \lambda_1] \) and \([\lambda_2 = \lambda_1]. \) The integral over \([\lambda_2 = \lambda_1] \) equals to

\[\prod_{k < l} (1 - (t a b c d)^{v_x^{(1)}} (r s)^k u) - L^{-k-1} L^{-l-1} L^{-2}. \]

If \(\lambda_1 = \lambda_2 = \lambda \) one can make the affine change of variables \(A_\lambda : (x, y) \mapsto (x, y - \lambda x), \) and this integral equals to

\[\int_{\{v_x^{(2)} > v_y^{(2)}\}} (1 - t^{A_\lambda(\gamma_1)} o v_x^{(2)} (r s)^{v_x^{(2)} u) - d\lambda_1^{(2)} = J_{a-1(A_\lambda(\gamma_1))}(t, t a b c d, 1, 1, r s L^{-1}, 1, u). \]

The product of the remaining factors is equal to

\[\Pi_{k < l} (1 - (t^{v_x^{(1)}}, d a c)^{v_x^{(1)} r L^{-1}} (b d t^{v_x^{(1)}}) s L^{-1} u) - L^{-k-1} L^{-l-1} L^{-2} \times \]

\[\prod_{k < l} (1 - (t v_x^{(1)}, d a c)^{v_x^{(1)} r L^{-1} (b d t^{v_x^{(1)}}) s L^{-1} u) - L^{-k-1} L^{-l-1} L^{-2} = \]

\[\sum_{k_1, k_2, k_3} \alpha_{k_1, k_2, k_3} (u) t^{k_1 v_x^{(1)}} (a c)^{k_2 v_x^{(1)}} (r L^{-1})^{k_2} (b d)^{k_3 v_x^{(1)}} (s L^{-1})^{k_3}, \]

therefore

\[\int_{\{v_x^{(1)} = v_y^{(1)}\}} J_{\gamma_1}(t, a, b, c, d, r, s, u) p^{v_x^{(1)}} q^{v_y^{(1)}} d x_y^{(1)} = \]

\[\int_{\mathbb{C}^*} d\lambda_1(\lambda) \int_{\{y^{(1)} = \lambda x^{(1)} + \tilde{y}^{(1)}\}} \sum_{k_1, k_2, k_3} \alpha_{k_1, k_2, k_3} (u) (r L^{-1})^{k_2} (s L^{-1})^{k_3} \times \]

\[\times J_{a-1(A_\lambda(\gamma_1))}(t, t a b c d, 1, 1, 1, r s L^{-1}, 1, u) (t^{k_1} (a c)^{k_2} (b d)^{k_3} p q)^{v_x^{(1)}} d x_y^{(1)} = \]

\[(L - 1) \int_{\{v_y^{(1)} > v_x^{(1)}\}} \sum_{k_1, k_2, k_3} \alpha_{k_1, k_2, k_3} (u) (r L^{-1})^{k_2} (s L^{-1})^{k_3} \times \]

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\[ \times J_{\sigma^{-1}(\gamma_1)}(t, tabcd, 1, 1, 1, rs\mathbb{L}^{-1}, 1, u)(t^{k_1} (ac)^{k_2} (bd)^{k_3}pq)^{k_1} d\chi^{(1)}_2 = \]

\[ (\mathbb{L} - 1) \int_{L^{(1)}} \sum \alpha_{k_1, k_2, k_3} (u) (r\mathbb{L}^{-1})^{k_1} (s\mathbb{L}^{-1})^{k_3} \times \]

\[ \times J_{\gamma_1}(t, tabcd, 1, 1, 1, rs\mathbb{L}^{-1}, 1, u)(t^{k_1} (ac)^{k_2} (bd)^{k_3}pq\mathbb{L}^{-1})^{k_1} d\chi^{(1)}_2 = \]

\[ (\mathbb{L} - 1) \sum \alpha_{k_1, k_2, k_3} (u)(r\mathbb{L}^{-1})^{k_2} (s\mathbb{L}^{-1})^{k_3} I(t, tabcd, 1, 1, 1, t^{k_1} (ac)^{k_2} (bd)^{k_3}pq, 1, rs\mathbb{L}^{-1}, 1, u). \]

Combining these integrals, one obtains the following proposition.

**Theorem 4**

\[ I(t, a, b, c, d, p, q, r, s, u) = \]

\[ \sum k_1, k_2 \varepsilon_{k_1, k_2} (u)(\mathbb{L} - 1)^{k_1} (\mathbb{L} - 1)^{k_2} I(t, tabcd, bd, cd, d, (ac)^{k_1} (bd)^{k_2}pq\mathbb{L}^{-1}, c^{k_1} d^{k_2} q, r, s\mathbb{L}^{-1}, 1, u) + \]

\[ \sum k_1, k_2 \varepsilon_{k_1, k_2} (u)(\mathbb{L} - 1)^{k_1} (\mathbb{L} - 1)^{k_2} I(t, tabcd, ac, ab, a, (bd)^{k_1} (act)^{k_2}pq\mathbb{L}^{-1}, a^{k_1} b^{k_2} p, r, s\mathbb{L}^{-1}, 1, u) + \]

\[ (\mathbb{L} - 1) \sum \alpha_{k_1, k_2, k_3} (u)(\mathbb{L} - 1)^{k_2} (\mathbb{L} - 1)^{k_3} I(t, tabcd, 1, 1, 1, t^{k_1} (ac)^{k_2} (bd)^{k_3}pq, 1, rs\mathbb{L}^{-1}, 1, u). \]

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