Oscillations of rigid block on supported base with coulomb friction model

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Abstract. The analytical solution of the problem of forced oscillation of the solid parallelepiped on a horizontal base is presented. It is assumed that the dry friction force acts on the contact line between the body and the base and the base moves harmonically in a horizontal direction. The height of the parallelepiped is assumed to be much greater than the width. Dissipation of impact is taken into account in the framework of the hypothesis of Newton. The forced oscillation modes of parallelepiped corresponding to the main resonances are found by using the averaging method. The results are shown in the form of amplitude-frequency characteristics.

1. Introduction
The dynamic of a solid body in the form of parallelepiped on a vibrating base was first considered in [1]. The author noticed that the elongate tall free-standing structures sometimes much better suffer an earthquake than seemingly steady constructions. This phenomenon is due to the fact that such constructions may oscillate alternately resting on one of the support ribs. A lot of publications are dedicated to this task. A large number of combination resonances and modes of chaotic oscillations are found in [2]. Similar problems with slippage between the body and the support surface are considered in [3-6]. Most of results are obtained by direct numerical integration. Fluctuations are considered in [7], the boundary of stability of the body in the plane of the frequency – amplitude excitations with zero initial conditions – is found. Some particular problems of the motion of a solid block on a moving base were considered in [8].

2. Formulation of the problem
The motion of a solid block in the form of parallelepiped on a horizontal supporting plane is considered. The friction between the body and the base is taken into account by the Coulomb model. The lower base of the block can either be fully supported on a plane or deviate from it at an angle of $\alpha$ (Figure 1). We confine the box motion in the plane of the figure and assume that the supporting plane oscillates horizontally.

As the shock model we use Newton's hypothesis that the relation $\dot{\alpha}_+ = R\dot{\alpha}_-$, where $\dot{\alpha}_\pm = \dot{\alpha}(t_0 \pm 0), t_0$ – the time of the shock ($\alpha(t_0) = 0$), $R$ – the coefficient of restitution.
Using the theorems of the motion of the mass center and change in angular momentum of the body relative to the reference ribs, we have:

\[
\begin{align*}
    mv_x &= F, \\
    mv_y &= N - mg, \\
    J\ddot{\alpha} + rJ\dot{\alpha}\dot{\alpha}(\alpha) - \mu Nd \sin(\beta - \mu \alpha) + Fd \cos(\beta - \mu \alpha) &= 0,
\end{align*}
\]

where \(v_x, v_y\) are the projections of the velocity of the centre of mass on the axes \(x, y\); \(J\) is the axial moment of inertia of the parallelepiped relative to the axis passing through the centre of mass, perpendicular to the plane of oscillations; \(m\) is the mass; \(d\) is the half of the diagonal; \(\beta\) is the angle between the vertical rib and diagonal; \(g\) is the acceleration of gravity; \(F\) is the frictional force; \(r = 1 - R\), \(\delta(\alpha)\) is the normalized unit delta function; the dot denotes differentiation with respect to time.

For brevity, a notation \(\mu(\alpha) = \text{sign}(\alpha), \alpha \neq 0\) is introduced here. For a value \(\alpha = 0\), when the lower base of the block is fully supported on a plane, it can take any values in the range \(-1 < \mu < 1\).

The system of equations (1) contains four unknown functions, and therefore it must be supplemented by the coupling equation. Assuming the base to be absolutely rigid, the projection of the speed of the point of contact on the vertical, obtained by the theorem of addition of velocities, must be equated to zero:

\[
v_y - \mu \dot{\alpha} d \sin(\beta - \mu \alpha) = 0.
\]

After substituting the coupling equation (2) in the second equation of motion (1), the value of the reaction can be obtained and excluded from the remaining equations.

For the Coulomb model, the frictional force is defined as \(F = -k N \text{sign}(u_0)\) if the slip velocity between the body and the base in the contact area \(u_0 \neq 0\). Here \(k\) is the coefficient of dry friction. The slipage velocity is determined by the formula:

\[
u_0 = v_x + \dot{\alpha} d \cos(\beta - \mu \alpha) - \frac{A_{00}}{\theta_0} \cos\theta_0 t.
\]

Here \(A_{00}\) and \(\theta_0\) are acceleration amplitude and frequency of supported base vibrations, respectively. In the case \(u_0 = 0\), the frictional force can take any values from the range \(-kN < F < kN\).

For parallelepiped whose height is substantially greater than the width \((\beta << 1)\), a small parameter \(\varepsilon\) is introduced, so that \(\cos(\beta) = h/d, \sin(\beta) = b/d\), where \(h, b\) are the half of the height and the half of the width of the block, respectively.

Further, a dimensionless time \(\tau = t\omega_0\) and dimensionless velocity \(v = v_x/(h\omega_0), u = u_0/(h\omega_0)\) and acceleration amplitude \(A_0 = A_{00}/(h\omega_0^2)\) are introduced, where \(\omega_0 = \sqrt{mg/hJ}\). Assuming the shock is
close to absolutely elastic, a small dissipation parameter \( r = \varepsilon(1 - R) \) is introduced. The dimensionless coefficient of dry friction \( \eta = \varepsilon^2 k g / (h \omega^2) \) is assumed to be of the second order of smallness.

The dimensionless equations of oscillations has a form:

\[
\dot{v} = \varepsilon^2 \eta f, \tag{4}
\]

\[
\dot{\alpha} - \alpha + \mu c = q(\alpha, \dot{\alpha}, \tau), \quad q(\alpha, \dot{\alpha}, \tau) = - r \dot{\alpha} \delta(\alpha) + \eta f, \tag{5}
\]

and the dimensionless slippage velocity is determined by the formula:

\[
u = v + \varepsilon \dot{\alpha} - \frac{A_c}{\tau} \cos \theta \tag{6}\]

where the terms not higher than the third order of smallness are left, the notations \( f = F / g, \ c = tg \beta, \ \theta = \theta_0 / \omega_0 \) are introduced. The function of dry friction has a form:

\[
f(u) = -\text{sign}(u), \quad u \neq 0, \quad -1 < f(u) < 1, \quad u = 0. \tag{7}
\]

The system of equations (4, 5) with (6) and (7) is closed with respect to the variables \( v \) and \( \alpha \). Next, we consider solutions for the case of slipping at the point of contact over the whole period of oscillations. The case of no slippage over the whole period of oscillations was investigated in [8].

3. The case of slipping over the whole period of oscillations.

At \( \varepsilon = 0 \) equations (4, 5) describe the rigid block oscillations on the absolutely smooth fixed base. For the initial conditions \( \alpha(0) = A, \dot{\alpha}(0) = 0 \), the solution has the form \( v = 0, \ \alpha(\tau) = c - (c - A) \tau \), which is valid on a quarter of the oscillation period. From the condition \( \alpha(\pi / 2\omega) = 0 \) we obtain the dependence of the oscillation amplitude \( A \) on the frequency \( \omega \):

\[
A = ce^{1 - c h^{-1} \left( \frac{\pi}{2\omega} \right)}. \tag{8}
\]

The solution at one period of oscillation has the form:

\[
\alpha(\tau) = c - (c - A) \tau, \quad - \frac{\pi}{2\omega} \leq \tau \leq \frac{\pi}{2\omega},
\]

\[
\alpha(\tau) = -c + (c - A) \tau \left( - \frac{\pi}{2\omega} \leq \tau \leq \frac{3\pi}{2\omega}. \tag{9}\right.
\]

The solution of equation (5) will be sought in the form:

\[
\alpha = \mu \left[ c - (c - A) ch \left( \frac{B(\psi)}{\omega} \right) \right], \quad \dot{\alpha} = -\mu(c - A) sh \left( \frac{B(\psi)}{\omega} \right). \tag{10}\]

Here the notations \( B(\psi) = \psi - (1 - \mu) \frac{\pi}{2} \) and \( \psi = \theta \tau + \phi_0 \) are introduced. Thus the parameter \( \mu = 1 \) in the case \(- \frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}\), and \( \mu = -1 \) in the case \( \frac{\pi}{2} \leq \psi \leq \frac{3\pi}{2} \).

New variables \( \dot{A} = A(\tau) \) and \( \psi = \psi(\tau) \) are functions of time, and \( \omega = \omega(A) \) is a dependence of the oscillation frequency on the amplitude for parallelepiped in a motionless base, defined by equation (8). When \( \psi = \omega \tau \), solution (10) corresponds to the solution of the unperturbed equation (5).

In the new variables the equation of oscillations of a rigid block has the form:

\[
\dot{A} = -\varepsilon \mu q sh \left( \frac{B(\psi)}{\omega} \right), \quad \dot{\psi} = \omega - \varepsilon \mu q \left[ \frac{ch \left( \frac{B(\psi)}{\omega} \right)}{c - A} + \frac{d\omega}{dA} \frac{sh \left( \frac{B(\psi)}{\omega} \right)}{\omega} B(\psi) \right]. \tag{11}\]
In the small neighbourhood of the main resonance, \( \psi = \theta - \phi_0 = \omega + \varepsilon \lambda - \phi_0 \), where \( \varepsilon \lambda = \theta - \omega \) is a small frequency difference. The system of equations for the slow variables \( A \) and \( \phi_0 \) has a form:

\[
\dot{A} = -\varepsilon \mu \sqrt{B(\psi)} \left( \frac{B(\psi)}{\omega} \right), \quad \dot{\phi}_0 = \frac{\varepsilon \lambda}{m} + \varepsilon \mu \left[ \frac{\text{ch} \left( \frac{B(\psi)}{\omega} \right)}{A} + \frac{\text{sh} \left( \frac{B(\psi)}{\omega} \right)}{A} \right].
\]

(12)

In equations (11,12), it is necessary to substitute the dependence which is obtained from equation (8):

\[
\frac{d\omega}{dA} = \frac{2\omega^2}{c\pi} \text{th} \left( \frac{\pi}{2\omega} \right).
\]

(13)

Now we consider equation (4). When \( \alpha = 0 \), it describes the motion of a material point on a vibrating surface with dry friction. The equation admits periodic solutions with the period equal to \( 2\pi \) only of two types. The motion of the first type is a motion with two instantaneous stops. The point oscillates in the neighbourhood of the equilibrium position so that during the whole cycle, slipping occurs between the body and the supporting surface. The motion of the second type is a motion with two long stops on the oscillation period, at which a part of the cycle occurs with slipping, and another part – without slipping.

In the case of slipping between the supporting edge of the block and the base during the entire oscillation period, the stationary solution of equation (4) is a piecewise linear function. For one period \( \phi_1 - \pi < \psi < \phi_1 + \pi \) it has a form:

\[
v(\psi) = -\frac{e^2 \eta}{\theta} (\psi + \frac{\pi}{2} - \phi_1), \quad f = -1, \quad \phi_1 - \pi < \psi < \phi_1, \quad \phi_1 - \pi < \psi < \phi_1 + \pi,
\]

(14)

the phase \( \psi_1 \) is determined from the condition \( u(\psi = \phi_1) = 0 \). When \( \psi = \phi_1 \), there is a stop at the point of contact and the speed of slipping, and with it the force of friction, reverses the sign. Taking into account (6), this condition has the form:

\[
-\frac{e^2 \eta \pi}{\theta} - \varepsilon \mu (c - A) \text{sh} \left( \frac{B(\phi_1)}{\omega} \right) - \frac{A_0}{\theta} \cos (\phi_1 - \phi_0) = 0.
\]

(15)

Equation (15) has two roots, one of which corresponds to a negative coefficient of friction. For a root corresponding to a positive one, if it exists, two conditions must be satisfied simultaneously:

\[
\frac{du}{d\psi} (\psi = \phi_1 - \pi) = -\frac{e^2 \eta}{\theta} - \frac{\varepsilon \mu (c - A)}{\omega} \text{ch} \left( \frac{B(\phi_1 - \pi)}{\omega} \right) - \frac{A_0}{\theta} \sin (\phi_1 - \phi_0) > 0,
\]

\[
\frac{du}{d\psi} (\psi = \phi_1) = -\frac{e^2 \eta}{\theta} - \frac{\varepsilon \mu (c - A)}{\omega} \text{ch} \left( \frac{B(\phi_1)}{\omega} \right) + \frac{A_0}{\theta} \sin (\phi_1 - \phi_0) < 0
\]

(16)

For some values of the problem parameters, the roots of equation (15) exist, but conditions (16) are not satisfied. This means that a part of the cycle of oscillations occurs with slippage, and part without slippage. This case is not considered in the present paper.

Since both \( A \) and \( \phi_0 \) are a slow function of time, let us replace the equation (12) by the time-averaged equations. The equations averaged up to terms of the first order have the form:
Here it is taken into account that the function \( f(\psi) \) occurring in the notation \( q(\alpha, \dot{\alpha}, \psi) \) has the form (14). Therefore, after averaging, the equations include an unknown phase \( \varphi_1 \).

We will further consider steady oscillations, which correspond to zero left-hand sides of equations (17). These equations together with equations (8) and (15) relate the amplitude and phases of oscillation \( A, \varphi_0, \varphi_1 \), the frequency of oscillations \( \omega \) and frequency difference \( \lambda \). The frequency of the excitation and the frequency of oscillations are associated dependency \( \omega = \theta - \omega \) that allows one to build a dependence of the oscillation amplitude on the excitation frequency.

For investigating the stability of the solutions, the system of equations (17) is linearized in the neighbourhood of the steady state. For asymptotic stability of stationary regimes it is necessary and sufficient that the eigenvalues of the right side of the equations have negative real parts. These conditions can be obtained in an explicit form, but are not mentioned here because of their bulkiness.

Figure 2 shows the dependence \( A(\theta) \) for the parameters \( c = 0.1, r = 0.1, \gamma = 3, A_b = 0.25 \) and two values of friction coefficient \( f = 0.002; 0.004 \) (curves 2 and 3 respectively). Curve 1 indicates the amplitude-frequency dependence for block oscillations in the motionless base (equation (2)). The thick lines represent stable solutions. For small excitation amplitude \( f = 0.002 \) the two solutions form a loop, the upper part of which corresponds to the stable solution. As follows from equation (17), frequency deferens increases with excitation frequency and at its small values close to zero (Figure 4), so Figures 2, 3 show the results only in the range of \( 1 < \theta < 3 \). For \( f = 0.004 \) (curve 3), even in this setting the frequency deferens \( \lambda \) is not small, so the results should be regarded as qualitative.

Figure 2. The dependence of the oscillation amplitude on the excitation frequency.

The dependences of the phases of the oscillations \( \varphi_0, \varphi_1 \) on the excitation frequency \( \theta \) for the same parameters of the problem are shown in Figure 3. Curves 1 and 2 are the phases \( \varphi_0, \varphi_1 \) in case \( f = 0.002 \) respectively, curves 3 and 4 are the same phases in case \( f = 0.004 \).
4. Conclusion
The oscillations of rigid block on the supported base with slipping over the whole period are possible only in the case of small values of the friction coefficient. The amplitude of oscillation $A$ and phase $\varphi_1$ are independent from the acceleration amplitude of supported base vibrations $A_0$, but phase $\varphi_0$ is a function from $A_0$. For enough small value $A_0$ equation (15) has no roots and conditions (16) are not satisfied. For example, in the case of the coefficient of dry friction $f = 0.002$, the solution is obtained for $A_0 > 0.18$. In the paper, solutions for the cases of small values of the friction coefficient in the Coulomb model are obtained. The case, when a part of the cycle of oscillations occurs with slipping and a part - without slipping, remains unexplored.

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Figure 3. The dependence of phases on the excitation frequency.

Figure 4. The dependence of frequency deferens on the excitation frequency.