Application of the Hertz equation for calculating the parameters of the elastic-plastic central impact of two solid deformable bodies

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Abstract. The study of the dynamics of mechanical impact of solid deformable bodies is an urgent problem in mechanics. This problem is associated with the strength calculations of machine parts and mechanisms. The most difficult case is the strength calculation under shock and vibration loads. Exact mathematical equations are obtained only for simple structural elements of machines: balls, rods, plates, beams, etc. The mathematical description of the dynamics of elements of machines of complex shapes under impact conditions is seriously difficult. A method of dynamic strength analysis of machine parts using the Hertzian contact theory is proposed. For the cases under consideration, the impact velocity of two bodies does not exceed 100 m/s.

1. Introduction
Currently, a large number of machines and mechanisms are used in various industries. During operation, the parts of these machines experience various loads that affect their performance. Practice has shown that under shock loads, elastic and elastoplastic deformations are formed in the contact zone of machine parts.

With such deformations, residual deformation occurs at the point of contact of the bodies, and plastic deformations are concentrated in a small neighborhood around this point. This neighborhood is called the contact area. Outside the contact zone, only elastic deformations propagate in the form of compression and tension shock waves. Under such loads, machine elements can maintain their performance for long periods of time. Let's consider in detail the process of formation of shock loads.

The physical process of impact of two solid deformable bodies differs significantly from the process of their static loading. The impact force arising at the point of contact of bodies acts for very short periods of time (tens and hundreds of microseconds). As a result of this phenomenon, various stress waves are formed in bodies, which can decay over long periods of time in comparison with the time of the impact forces. As a result of the collision at the point of contact, the process of introducing one body into another is observed. This process depends on many factors: the shape and type of the materials of the bodies, the relative speed of their collision, as well as other factors.

In general, analytical solutions to mathematical equations that accurately describe the impact behavior of a material are difficult to obtain. Therefore, in engineering practice, simplified mathematical models are used that describe the behavior of the material of machine parts.

Impact dynamics are currently well studied theoretically. Among the large number of works, it is necessary to highlight the following main publications. Johnson K. L. [1] considers the contact strength of materials when they are pressed against each other in static and dynamic conditions. Goldsmith W. [2], Stronge W. J. [3], Aleksandrov E. V. et al. [4], Popov V. L. [5] study the main
sections of impact theory: central impact of solid deformable bodies; impact of solid deformable bodies in the plane and in space; numerical simulation of the impact process, impact of bodies of variable stiffness and other theories.

The foundations of the contact theory of bodies were laid by G. Hertz. [6]. He studied the static pressure of two elastic spheres on each other, as well as the indentation of a sphere into an elastic half-space. The pressure of the sphere was carried out along the normal to the surface of the second body without taking into account the friction forces. Hertz’s theory became the foundation for the mathematical description of the impact of various bodies.

Consider publications based on Hertz’s theory, and devoted to various options for the impact of two bodies. Goldsmith W. et al. [7] carried out experiments on the impact of spheres on the ends of rods made of various materials. Wu, C. et al. [8] developed a model for calculating the parameters of the oblique impact of a sphere on an elastic half-space. The model is capable of calculating the impact parameters for elastic and elastoplastic spheres during oblique impact. Thornton C. [9] derives an analytical solution for the coefficient of recovery in terms of the ratio of the velocities of the impact of bodies. Vu-Quoc L. et al. [10] presents an elastoplastic model of the dependence of the normal force and displacement for the collision of two spheres. Li L. et al. [11] consider a theoretical model of the normal contact between a rigid sphere and a plastic half-space, as well as the impact of a plastic ball on a solid half-space. Equations for the relationship between the force and displacement of bodies for static loading and equations for calculating the coefficient of recovery for the dynamic case are obtained. Labous L. et al. [12] study the collision of two spheres using high-speed video analysis. The authors calculate the values of the recovery factor for different cases. Gunes, R. et al. [13] study the impact of a sphere on a plate made of cermet. The article contains various graphs linking different parameters of the impact with each other. Christoforou R. et al. [14] investigate the impact of a sphere on a plate and shell. The article contains various graphs connecting the impact force and time, as well as other parameters of the impact.

In this paper, it is proposed to use the Hertz equation to describe the shock interaction taking into account elastoplastic deformations at average impact velocities (up to 100 m/s).

2. Formulation of the problem

The classical theory describing the static compression of two elastic bodies is the theory of Heinrich Hertz, obtained on the basis of an electrostatic analogy [6]. The profiles of the contact surfaces of two bodies must be smooth and continuous, i.e. described by mathematical equations of surfaces of the second degree. Due to the fulfilment of this condition, the stresses arising in the bodies always have finite values. In cases where the indenter has the shape of a wedge, cone or pyramid, this condition is not met. Consider the application of Hertz’s theory for a point contact of two bodies.

Hertz’s theory is based on the following assumptions.
1. Materials of interacting bodies are homogeneous and isotropic.
2. The forces applied to the bodies form in the contact zone only elastic deformations, depending on Hooke’s law.
3. The contact area and the volume of the contact zone are small compared to the dimensions of the contacting bodies.
4. Contact forces are normal to the contacting surfaces of bodies.

The Hertz equation has the following form [6]

\[ F = K_H \delta^{3/2} \]  

(1)

where \( F \) is the contact force, \( K_H \) is the coefficient, the value of which depends on the shape and properties of the materials of the contacting bodies, \( \delta \) is the magnitude of the approach of the contacting bodies.

The \( K_H \) value is determined by the expression:

\[ K_H = \frac{4}{3} \sqrt{R} \frac{E_1E_2}{(1-\mu_1^2)E_1+(1-\mu_2^2)E_2} \]  

(2)
where $E_1$, $E_2$ are the values of the elastic modulus of the materials of the first and second bodies (Young's modulus); $\mu_1$, $\mu_2$ - values of Poisson's ratios of materials of the first and second bodies; $R$ is the reduced radius of curvature.

For real contact interaction, the contact area of two bodies has the shape of an ellipse and the volume of the contact zone is close in shape to an ellipsoid. To simplify calculations, it is possible to determine the reduced curvature in the plane of the closest contact of bodies, i.e. make the assumption that the ellipse and ellipsoid transform into a circle and a sphere, respectively. Then for spherical contact surfaces we write the expression

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

(3)

where $R_1$, $R_2$ are the radii of curvature of the contact surfaces of the first and second bodies.

3. Theory

Hertz's expression (1) is used for static loading. Consider the use of the Hertz equation for dynamic impact interaction.

The magnitude of the approach $\delta$ is determined by the expression

$$\delta = \delta_1 + \delta_2$$

(4)

where $\delta_1$, $\delta_2$ are the values of the approach of the first and second bodies, figure 1.

![Figure 1. Zone of impact contact of two bodies](image)

During the elastic impact, the centers of mass of the first and second bodies approach each other by the amount of approach $\delta$. The speed of convergence of bodies is determined by the expression

$$V_1 - V_2 = \frac{d\delta}{dt}$$

(5)

where $V_1$, $V_2$ are the velocities of the first and second bodies before impact.

Contact force $F$ is determined by Newton's law
\[ F = m \frac{d^2 \delta}{dt^2} = -K \delta^{3/2} \]  

Integrating equation (6), we obtain

\[ \frac{1}{2} \left( \frac{d\delta}{dt} \right)^2 = -K \frac{2}{5m} \delta^{5/2} + \text{const} \]  

Let us define \( \text{const} \) based on the initial conditions

\[ t = 0, \quad \frac{d\delta}{dt} = V_0, \quad \delta = 0, \quad \frac{1}{2}(V_0)^2 = \text{const} \]  

\[ \left( \frac{d\delta}{dt} \right)^2 = -K \frac{4}{5m} \delta^{5/2} + (V_0)^2 \]  

\[ \frac{d\delta}{dt} = \sqrt{(V_0)^2 - K \frac{4}{5m} \delta^{5/2}} \]  

When both bodies come as close as possible to each other, i.e. \( \delta = \delta_{\text{max}} \), the speed of convergence of bodies will be equal to zero – \( d\delta/dt = 0 \). Expression (9) takes the form

\[ \delta_{\text{max}} = \left( \frac{5}{4K} m V_0^2 \right)^{2/5} \]  

Combining expressions (10) and (1) we obtain

\[ F_{\text{max}} = K^{2/5} \left( \frac{5}{4} m V_0^2 \right)^{3/5} \]  

By analogy with expression (3), we introduce the equation

\[ \frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2} \]  

\[ m = \frac{m_1 m_2}{m_1 + m_2} \]  

Substitute expressions (5) and (12) into equation (11)

\[ F_{\text{max}} = K^{2/5} \left( \frac{5}{4} \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \right)^{3/5} \]  

For expression (10), we carry out the following transformation.

\[ \delta_{\text{max}} = \left( \frac{5}{4K} \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \right)^{2/5} \]  

Equation (13) shows the relationship between the force and the kinetic energy of the elastic impact of two bodies. Let us determine the possibility of using equation (13) for the elastoplastic impact of two bodies.

Consider the equations given in the work of Chernyavsky, D. et al. [15, 16]. The shock process can be described by the following equations

\[ \eta = \frac{m_1 U_1^2 + m_2 U_2^2}{m_1 V_1^2 + m_2 V_2^2} \]  

\[ \frac{m_1 U_1 + m_2 U_2}{m_1 V_1 + m_2 V_2} = 1 \]  

\[ k = \frac{U_2 - U_1}{V_1 - V_2} \]  

where \( U_1, U_2 \) are the velocities of the first and second bodies after impact, \( \eta \) is the efficiency of the transfer of kinetic energy during impact, \( k \) is the recovery factor.

Solving equations (1.15–1.17) together we obtain
Let's introduce the notation:

\[ \eta_1 = \frac{m_1^2 V_1^2 + m_2^2 V_2^2}{(m_1 V_1^2 + m_2 V_2^2)(m_1 + m_2)} \]

Equation (19) shows that the effectiveness of the shock process depends on three main factors.

1. Impact efficiency depends on the value of the product of the kinetic energy of the moving bodies by the values of the masses of these bodies. This conclusion is a consequence of the law of conservation of energy.

2. Impact efficiency depends on the product of mechanical impulses of the colliding bodies on each other. This conclusion is a consequence of the law of conservation of momentum.

3. Effectiveness of impact depends on material properties, shape, internal structure of colliding bodies, as well as other factors. This conclusion is a consequence of the formula for Newton's recovery factor.

4. Results of the theoretical studies

We transform the equation (18)

\[ k^2 \frac{m_1 m_2 (V_1 - V_2)^2}{(m_1 + m_2)} = \eta (m_1 V_1^2 + m_2 V_2^2) - \frac{(m_1 V_1 + m_2 V_2)^2}{(m_1 + m_2)} \]

Equation (23) describes elastic impact. Equation (24) can describe elastic, elastoplastic and plastic impact. The coefficient of recovery \( k \) shows the dissipation of energy during the impact. This factor takes into account the different properties of the first and second bodies. Similarly, we transform expression (14).

\[ \delta_{\text{max}} = \left( \frac{5}{4K_H^2} k^2 \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \right)^{3/5} \]

Equations (24, 25) have the general boundary condition

\[ k^2 \frac{m_1 m_2}{m_1 + m_2} (V_1 - V_2)^2 \geq 0 \]
The kinetic energy transfer efficiency has the following boundary conditions. Let's consider two such options.

1. \( \eta = 0 \), \( m_1 V_1 = -m_2 V_2 \). Under such conditions, all the kinetic energy of the impact of two bodies is transferred into the internal energy of these bodies.

2. \( \eta = 1 \), \( V_1 = V_2 \). Under these conditions of motion of bodies, the kinetic energy of the impact of two bodies remains constant, since there is no impact.

Due to the limited volume of the article, issues related to the analysis of expressions (24–27), as well as the development of a computer model, will be considered in future publications.

6. Conclusions

The modification of the well-known Hertz equation (1), which describes only elastic impact, proposed by the authors, makes it possible to determine the maximum impact force and convergence in the case of elastoplastic impact of solid deformable bodies. The impact velocity of the investigated bodies does not exceed 100 m/s. This method can be used for strength analysis of machine parts and other structures.

7. References

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