Critical Properties of Quantum Many-Body Systems

with $1/r^2$ Interaction

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We review recent results obtained for a class of one-dimensional quantum models with $1/r^2$ long-range interaction. Based on the asymptotic Bethe-ansatz solution and conformal field theory, we study critical properties of the continuum boson model, the $SU(\nu)$ spin chain, the $OSp(\nu,1)$ supersymmetric $t$-$J$ model, and a new hierarchy of models related to the fractional quantum Hall effect. We further investigate the class of $1/r^2$ models with harmonic confinement by means of a newly proposed method of the renormalized-harmonic oscillator solution.

PACS numbers:
One-dimensional (1D) many-body systems with long-range interaction of $1/r^2$ type \cite{1,3} have been studied extensively in connection with fundamental notions in condensed matter and statistical physics. Even in classical models \cite{4-7}, one can find remarkable results. For example, a path-integral formulation of the Kondo problem results in the ferromagnetic Ising model with $1/r^2$ interaction \cite{5}. Also, the antiferromagnetic Ising model shows a typical example of the devil staircase \cite{6}.

Quantum models with $1/r^2$ interaction have attracted renewed interest considerably \cite{1-3,8-42}, revealing new interesting aspects of integrable systems, and thus providing us with a paradigm of fundamental ideas actively discussed in condensed matter physics, such as the random matrix \cite{2,3}, the Gutzwiller state for correlated electrons \cite{8-10}, the fractional quantum Hall effect (FQHE) \cite{17,22}, the level statistics for disordered systems \cite{23,25}, etc.

A common feature in quantum $1/r^2$ systems is that the ground state is exactly given by a Jastrow-type wavefunction \cite{1-3,8-22}, namely a product of two-body functions. This characteristic nature of the wavefunction should be closely related to the integrability of $1/r^2$ models because it implies that the two-body scattering is essential in spite of the long-range interaction. More recently, the integrability of the class of quantum Hamiltonians has been shown, and algebraic structures for the underlying symmetry have been clarified systematically \cite{26-37}.

The asymptotic Bethe-ansatz solution (ABA) provides us with a systematic way to construct the energy spectrum \cite{2,3}, which has been known to give the exact solution to the continuum boson model and also to the SU(2) spin chain \cite{17}. More recently, the ABA solution has been systematically generalized to multicomponent quantum systems with $1/r^2$ interaction, such as the SU($\nu$) spin chain, the OSp($\nu$,1) supersymmetric $t$-$J$ model, and a new hierarchy of the models related to the FQHE \cite{12,14,18,21}. Furthermore, it has been found that a family of confined models with harmonic potential can be solved by the
renormalized harmonic oscillator (RHO) method [38], which is a variant of the ABA.

In this paper we wish to give a brief review of our recent studies on the critical properties of quantum $1/r^2$ models based on the ABA solution and conformal field theory (CFT). This paper is organized as follows. In the next section we outline the ABA method by taking Sutherland’s boson model as a simple example [2,3], and then discuss its critical properties based on CFT [11]. In §3 the ABA solution is further extended to multicomponent systems with more complicated internal symmetry, the SU($\nu$) Haldane-Shastry spin chain [14], and then in §4 the effects of hole-doping are discussed by the OSp($\nu,1$) supersymmetric $t$-$J$ model [12,14] which was introduced first by Kuramoto and Yokoyama for the OSp(2,1) case [10]. We further introduce in §5 a new family of $1/r^2$ models which are closely related to a certain hierarchy of the FQHE [18,21]. In §6, we propose a new approach based on the RHO method [38] in order to systematically construct the energy spectrum for the models with harmonic confinement [38–42]. We then prove, by explicitly constructing the eigenfunctions for the ground state [40] as well as the excited states [41], that the RHO indeed provides the exact spectrum for the above systems. Some applications to mesoscopic systems are also mentioned.

§2. Critical Properties of Continuum Boson Model

There are several variants of the integrable quantum models with $1/r^2$ interaction. In this paper we will systematically investigate the systems with periodic boundary conditions and also the confined systems with harmonic potential. We note that quantum $1/r^2$ models in continuum space was introduced by Calogero many years ago [1], and then have been actively studied by Sutherland [2,3]. In this section, we wish to mention characteristic properties of the quantum $1/r^2$ systems by taking the boson model with periodic boundary conditions, and outline how to apply the ABA method.

Let us introduce $N$ interacting bosons in a 1D chain of circumference $L$,
\[ \mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial r_j^2} + \sum_{i<j} U(r_i - r_j), \]

with \(1/r^2\) interaction \(U(r_i - r_j)\). In order to consider the model with periodic boundary conditions, the following form of the interaction should be used \[2\],

\[ U(r) = V \sum_{n=-\infty}^{\infty} (r + nL)^{-2} = \frac{V\pi^2}{L^2} \sin^{-2}\left(\frac{\pi r}{L}\right) \equiv Vd_{ij}^{-2}, \]

which leads to \(V/r^2\) in the limit of \(L \to \infty\). Here \(d_{ij}\) corresponds to the chord distance for the ring. It is known that the ground state for the model is given by the Jastrow wavefunction, i.e. the product of two-body functions \[1,2\],

\[ \psi = \prod_{j<l} |\sin \frac{\pi(r_j - r_l)}{L}|^\lambda \]

with the Jastrow parameter,

\[ \lambda = [\sqrt{1 + 2V} + 1]/2. \]

We will be concerned with the repulsive case \(V \geq 0\) hereafter. It should be noted that all the integrable \(1/r^2\) models take the Jastrow wavefunction as the ground state wavefunction. The two-body nature of the ground state is related to the integrability of the system, and implies that the two-body scattering may be essential for the many-body scattering. Based on this observation, Sutherland proposed the ABA method to exactly construct the excitation spectrum and full thermodynamics of the above boson system \[2\].

2.1. Asymptotic Bethe-ansatz

An asymptotic Bethe-ansatz (ABA) solution provides an elegant method to construct the excitation spectrum for the class of the above Hamiltonians \[2\]. The essence of the idea is that although ordinary Bethe-ansatz (BA) methods are not applicable to the above systems with long-range interactions, the eigenfunctions can be written down consistently like those for the BA,

\[ \psi = \sum_{P} A(Q; P) \exp[i \sum_{j} k_{P}x_{Qj}], \]
in the asymptotic region, $x_{Q1} \ll x_{Q2} \ll \cdots \ll x_{QN}$, where $Q$ ($P$) expresses one of $N!$ permutations for the coordinate (momentum) configurations. This form of the wavefunction implies that the many-body $S$-matrix in the asymptotic region can be factorized into two-body matrices consistently. If this is the case, one can diagonalize the many body scattering based on the factorized $S$-matrix, and then obtain the spectrum of the system. The factorization of the $S$-matrix may be somehow suspected from the two-body nature of the Jastrow wavefunction for the ground state. At first glance the ABA solution seems to work only for low-energy excitations of continuum systems with the low density of particles. Remarkably enough, however, it turns out that the ABA solution exactly reproduces the whole energy spectrum for boson systems of any density \[2\]. Furthermore, this method has been found to be also applicable for the lattice models with high density of particles \[17\].

The two-body scattering in the Sutherland model yields the $S$-matrix, $S_{ij} = -\exp[-i\theta(k_i - k_j)]$ with the phase shift function $\theta(k) = \pi(\lambda - 1)\text{sgn}(k)$. Imposing periodic boundary conditions, one can now diagonalize the many-body $S$-matrix using the factorized $S$ matrices, and then deduce the ABA equation for the rapidity $k_j \[2\],

\[ k_jL = 2\pi I_j + (\lambda - 1) \sum_{l=1}^{N} \Phi(k_j - k_l) \] (6)

with $\Phi(k) = \pi \text{sgn}(k_j - k_i)$, where $I_j$ is the quantum number which satisfies the selection rule $I_j = (N + 1)/2 \mod 1$. The interaction effects are now incorporated into $k_j$ via the phase shift function, and the energy is expressed simply in the noninteracting form, $E = \sum k_j^2$. Sutherland studied full thermodynamics based on the above ABA equations \[2\]. More recently, the critical behavior of correlation functions have been clarified \[11\] with the help of CFT \[43\]–\[46\].

2.2. Conformal properties

By combining the ABA solution with the finite-size scaling in CFT, we now study critical properties of the model, and then evaluate the critical exponents of correlation functions. We briefly review the results obtained in ref. \[11\]. Let us start with low-temperature properties
of the free energy \[2\],

\[
F(T) \simeq F(T = 0) - \frac{\pi T^2}{6v},
\]

(7)

where the velocity of elementary excitations is \(v = 2\pi \lambda n\). According to the known formula for finite-size scaling in CFT \[11\], we can read the central charge of the underlying Virasoro algebra as \(c = 1\) \[11\]. Therefore, the critical behavior of the present model is expected to be described by \(c = 1\) Gaussian CFT. To confirm this, we next calculate the finite-size corrections to excited states. Elementary excitations can be specified by the deviation of quantum numbers from the ground-state distribution. If we take \(I_j = (2j - N - m - 1)/2 + d\), the quantum number \(m\) labels the excitation which changes the particle number, whereas \(d\) denotes the excitation which carries the large momentum \(2k_Fd\) with \(k_F = \pi n\). We then classify the excitation as \[11\],

\[
\Delta E \simeq \frac{2\pi v}{L} x(m; d; n^\pm),
\]

(8)

where the chemical potential term has been omitted. According to the finite size scaling in CFT \[15\], we can deduce the scaling dimension,

\[
x(m; d; n^\pm) = \frac{\lambda}{4} m^2 + \frac{1}{\lambda} d^2 + n^+ + n^-,
\]

(9)

where particle-hole excitations are denoted by non-negative integers \(n^\pm\). The selection rule for the quantum number reads: \(d=\text{integer for bosons.}\) The momentum carried by the above excitation is

\[
P = 2\pi k_Fd + \frac{2\pi}{L}[md + n^+ - n^-].
\]

(10)

From (9) and (10) we can determine conformal weights \(\Delta^\pm\) for the holomorphic (antiholomorphic) piece, which characterize the operator content of the underlying Virasoro algebra in CFT \[13\]. Recalling that CFT predicts the energy and the momentum to be universally related to conformal weights as \(x = \Delta^+ + \Delta^-\) and \(P = (2\pi/L)(\Delta^+ - \Delta^-)\) for the \(1/L\) sector \[15\], we obtain conformal weights \[11\],
\[ \Delta^\pm(m; d; n^\pm) = \frac{1}{2} \left( \frac{m}{2R} \pm dR \right)^2 + n^\pm \]  

with \( R = 1/\sqrt{\lambda} \). We now confirm that the above expression for conformal weights is inherent in \( c = 1 \) Gaussian CFT realized by free bosons with periodicity \( R \), in which non-negative integers \( n^\pm \) feature the conformal tower [46]. From the above analysis [11], we now conclude that the present model with \( 1/r^2 \) interaction is a typical example of Luttinger liquids [47–49], i.e. its critical behavior is controlled by \( c = 1 \) CFT.

2.3. Correlation exponents

It is now easy to determine the critical exponents of correlation functions. For instance, let us begin with long-distance behavior of the density correlation function,

\[ < n(r)n(0) > \simeq \text{const.} + a_0 r^{-2} + a_2 r^{-\alpha} \cos 2k_F r, \]  

where \( n(r) \) is the density operator. By taking \((m, d, n^\pm) = (0, 1, 0)\) as the quantum numbers, which carry the \( 2k_F \) momentum transfer, we obtain the critical exponent \((\alpha = 2x)\),

\[ \alpha = 2/\lambda \]  

for \( 2k_F \) oscillation piece [11]. Note that there is no logarithmic correction to the correlation function. On the other hand, the field correlator for bosons

\[ < \phi^\dagger_b(r)\phi_b(0) > \simeq r^{-\beta_b}, \]

has the leading non-oscillation term. By taking the set of quantum numbers \((m, d, n^\pm) = (1, 0, 0)\) for the primary field, we obtain the corresponding critical exponent \([11]\),

\[ \beta_b = \lambda/2. \]  

Fourier transform of this correlator yields the momentum distribution function around the origin, \( n_b(k) \simeq |k|^{\theta_b}, \) with the corresponding critical exponent \( \theta_b = \beta_b - 1 = \lambda/2 - 1. \)

As the interaction strength \( V \) increases, the critical exponents vary continuously, characterizing the U(1) CFT critical line. For the special values such as \( V = 0 \) and 4, the present
exponents \([11]\) agree with those obtained by Sutherland using the random matrix theory \([2]\). We note that the system exhibits interesting properties at \(V = 4\), at which the periodicity of bosons becomes \(R = 1/\sqrt{2}\). This implies that symmetry of the model is enhanced to SU(2), and is described by the level-1 SU(2) Wess-Zumino-Witten model \([46]\). Therefore the effective theory for this special point is equivalent to that for the SU(2) Haldane-Shastry spin chain \([8], [9]\).

The above critical exponents satisfy the universal scaling relations for the Luttinger liquid, \(\beta_b = 1/\alpha \ [17]\). It is instructive to note that in contrast to the known models with short-range interaction, the above critical exponents do not depend on the density of particles \([8], [11]\), but only on the interaction strength \(V\). This peculiar property is characteristic of \(1/r^2\) quantum models. We will encounter similar examples in the following chapters.

We have been concerned with the boson case so far. The CFT analysis for the fermion case can be performed similarly \([11]\). For fermions, the selection rule should read \(d = m/2 \mod 1\), reflecting antisymmetry nature of the wavefunction. Hence the field correlator for fermions has the leading \(k_F\) oscillation term and the corresponding exponent is given as \(\beta_f = [\lambda + 1/\lambda]/2\) by choosing the quantum numbers as \((m, d, n^\pm) = (1, 1/2, 0)\). This leads to the critical exponent of the momentum distribution around the Fermi point, \(\theta_f = [\lambda + 1/\lambda - 1]/2\).

2.4. Haldane-Shastry spin chain

The quantum \(1/r^2\) model was extended independently by Haldane and Shastry to the lattice case, namely the antiferromagnetic \(S = 1/2\) spin chain with \(1/r^2\) exchange interaction \([8], [9]\). The Hamiltonian for a periodic ring with \(L\) sites reads,

\[
\mathcal{H} = \sum_{i<j} J_{ij} \left[ S_i^x S_j^x + S_i^y S_j^y + \frac{1}{2} p(p - 1) S_i^z S_j^z \right],
\]

where \(J_{ij}\) is chosen to satisfy periodic boundary conditions, \(J_{ij} = Jd(x_i - x_j)^{-2}\) with the chord distance \(d(x) = (L/\pi) \sin(\pi x/L)\) (the \(i\)-th site is denoted by \(x_i\)). Here \(p\) represents the anisotropy of the model which is assumed to be \(p \geq 2\) in what follows. The isotropic
model \((p = 2)\) has been quite well investigated by Haldane and Shastry \([8], [9]\). Particularly in this case the ground-state wavefunction is given by the completely-projected Gutzwiller wavefunction at half filling \([8], [9]\),

\[
\Psi_G(\{x_i\}) = \exp(-i\pi \sum_i x_i \prod_{i<j} d(x_i - x_j)^2),
\]

in terms of coordinates \(\{x_i\}\) for down spins. As for the anisotropic case, the ground-state was also obtained in the Jastrow form for a positive even integer \(p\) \([8]\). Here we briefly summarize CFT analysis of the anisotropic Haldane-Shastry model \([11]\).

It is known that the ABA solution is applicable to the above model although the asymptotic region is not realized for a lattice system with high densities of particles \([17]\). The two-body \(S\)-matrix for the model is obtained as \(S_{ij} = -\exp[-i\theta(k_i - k_j)]\) with the phase shift function \(\theta(k) = (p - 1)\pi \text{sgn}(k)\). Hence the ABA equation is essentially the same as the continuum boson case \([17]\),

\[
k_j L = 2\pi I_j + (p - 1) \sum_{i=1}^{N} \Phi(k_j - k_i),
\]

where \(N\) is the number of down spins. In contrast to the continuum model, however, the available range of \(k_j\) is restricted to \([-\pi, \pi]\), reflecting the periodicity of the lattice. One of the crucial consequences due to this restriction is that there exists the lower bound for the magnetization in the liquid phase, \(s_z = 1/2 - 1/p\). Therefore the massless phase is realized for magnetic fields, \(H_{c1} \leq H \leq H_{c2}\), where the upper critical field is given by

\[
H_{c2} = \frac{\pi^2 J}{12} [p(p - 1) + 1],
\]

at which the system is fully polarized, and the lower critical field is

\[
H_{c1} = \frac{\pi^2 J}{6} \left[\frac{1}{2} p(p - 1) - 1\right],
\]

at which the magnetization takes its minimum value \(s_z = 1/2 - 1/p\) in the liquid phase \([11]\). The ABA solution is applicable only for this range of magnetic fields. We would like to mention that the supermultiplet structure discovered by Haldane \([17]\), which is now
classified by the Yangian algebra [36], made it possible to construct the full thermodynamics for the isotropic case \((p = 2)\).

Similarly to the Sutherland model, the low temperature free energy in the liquid phase takes the form, \(F(T) \simeq F(T = 0) - \pi T^2/(6v)\) with the spin velocity \(v = \pi p(1 - 2s_z)/(4J)\). This implies that the spin-liquid behavior of the present model is described by CFT with \(c = 1\). Also, the excitation spectrum turns out to be classified in the same form as in (9). Hence, we can conclude that the spin-liquid phase for \(H_{c1} \leq H \leq H_{c2}\) is classified as that of Luttinger liquids [11].

Following the method outlined for boson case, it is straightforward to deduce correlation exponents. For instance, the asymptotic form of the spin correlation function is written as

\[
< S^z(x)S^z(0) > \simeq c_0 x^{-2} + c_2 x^{-\alpha} \cos 2k_F x .
\]

(21)

with the exponent \(\alpha = 2/p\). On the other hand, the transverse spin correlation has the leading non-oscillation term

\[
< S^+(x)S^-(0) > \simeq x^{-\beta} .
\]

(22)

with the exponent \(\beta = p/2\). These exponents indeed satisfy the universal scaling relation for the Luttinger liquid, \(\alpha = 1/\beta\), as is the case for the continuum case. Note that conformal properties for the isotropic case \((p = 2)\) has already been discussed by Haldane in detail [17].

We have been concerned so far with the liquid phase with \(s_z \geq 1/2 - 1/p\) under applied magnetic fields \(H_{c1} \leq H \leq H_{c2}\). We then ask what will happen for \(H < H_{c1}\). Unfortunately the ABA solution is not efficient to answer this question. We can say that the system shows a singularity at \(H = H_{c1}\), but it is not clear whether the ordered phase is realized below \(H_{c1}\) or not. We close this section by mentioning key points to solve the above question. We first point out that when the system approaches the boundary \(H_{c1}\) from the liquid phase, the spin correlation with the period \(p\) is enhanced at \(H = H_{c1}\) [11]. So, it may be interesting to check what kind of phase would be stabilized just below \(H_{c1}\). It is also instructive to
ask what happens for the magnetization when $H$ decreases further below $H_{c1}$. According to the results in the Ising limit [6], it may be possible that magnetization would show stair structures as a function of $H$. These interesting issues are to be clarified in the future study.

§3. SU(ν) Spin Chain

Now we wish to extend the quantum $1/r^2$ models to multicomponent cases. Such a generalization was first made by Kuramoto and Yokoyama using the supersymmetric $t$-$J$ model, who found the Gutzwiller state as the ground state and discussed low-energy excitations [10]. Subsequently, this model was solved by the ABA method [12], which has been later proven to give the exact spectrum of the model [13]. Further generalization to the SU(ν) spin chain and to the multicomponent $t$-$J$ model with OSp(ν, 1) supersymmetry has been done independently in refs. [14,15]. Before discussing the supersymmetric $t$-$J$ model we first study a SU(ν) generalization of the Haldane-Shastry model ($\nu \geq 2$), and then discuss the effects of hole-doping using the supersymmetric $t$-$J$ model in the next section.

We introduce the Hamiltonian of the SU(ν) spin chain with $1/r^2$ interaction by the following general form [14],

$$\mathcal{H} = \frac{1}{2} \sum_{i<j,\alpha,\beta} (-1)^{F(\beta)} J_{ij} X_i^{\alpha\beta} X_j^{\beta\alpha}, \quad (23)$$

where the Hubbard operator $X_i^{\alpha\beta} = |i\alpha > < i\beta|$ interchanges states at $i$-th site from $\beta$ to $\alpha$, and the exchange coupling $J_{ij}$ is given by inverse-square interaction (16). Here we have assumed that the system consists of $\nu$ components of spins (or colors) with $\alpha, \beta = 1, 2, \cdots, \nu$, and the fermion numbers take $F(\beta) = (0, 0, \cdots, 0)$ for $\beta = (1, 2, \cdots, \nu)$ in case of the SU(ν) spin chain. Hence all the particles obey the same statistics. This Hamiltonian, which is a SU(ν) generalization of the Haldane-Shastry model [14,15], is indeed invariant under the global SU(ν) transformation. If we add another state ($\alpha = \nu + 1$) with the different fermion number $F(\nu+1) = 1$, this Hamiltonian describes the multicomponent $t$-$J$ model with
OSp(\(\nu, 1\)) supersymmetry which is realized by doping holes into the SU(\(\nu\)) Haldane-Shastry model.

### 3.1. Ground-state wavefunction

We start by writing down the completely projected SU(\(\nu\)) Gutzwiller wavefunction as the exact ground state for the above SU(\(\nu\)) spin chain \([14],[15]\),

\[
|\Psi_G > = P^{(1)}_G \prod_{\alpha=1}^{\nu} \prod_{k_{\alpha}} a^{(\alpha)\dagger}_{k_{\alpha}} |0 >,
\]

where \(a^{(\alpha)\dagger}_{k_{\alpha}}\) is the creation operator of electrons with spin \(\alpha\) (1 \(\leq\) \(\alpha\) \(\leq\) \(\nu\)) and momentum \(k_{\alpha}\). We have assumed here that \(P^{(1)}_G\) selects configurations for which every cite is occupied by only a single electron with spin \(\alpha\). The projection \(P^{(1)}_G\) is easily done by taking the reference state \(|F >\) full of particles with \(\nu\)-th spin \([50]\). In this representation, we rewrite the Gutzwiller state as,

\[
|\Psi_G > = \sum_{(a,i)} \Psi_G (\{x^{(a)}_i\}) \prod_{a,i} b^{(a)\dagger}_i |F >,
\]

where \(b^{(a)\dagger}_i = a^{(a)\dagger}_i a^{(\nu)}_i\) a creation operator for spin particles (1 \(\leq\) \(\alpha\) \(\leq\) \(\nu - 1\)). The Gutzwiller wavefunction is given in terms of coordinates of spin particles \(\{x^{(a)}_i\}\),

\[
\Psi_G (\{x^{(a)}_i\}) = \exp(-i\pi \sum_{a,i} x^{(a)}_i \prod_{a,i<j} d(x^{(a)}_i - x^{(a)}_j)^2 \prod_{a<\beta,i,j} d(x^{(a)}_i - x^{(\beta)}_j),
\]

where \(d(x) = (L/\pi) \sin(\pi x/L)\) is the chord distance and \(L\) is the number of lattice sites. Following techniques developed for the SU(2) spin chain \([8],[9]\) and the OSp(2,1) supersymmetric model \([10]\), it is straightforward to show that the above SU(\(\nu\)) Gutzwiller state without holes gives the ground state for the SU(\(\nu\)) spin chain \([14],[15]\). The corresponding ground-state energy is thus computed as

\[
E/L = \frac{\pi^2}{12} \left( \frac{2 - \nu}{\nu} + \frac{1 - 2\nu}{L^2} \right),
\]

which will be shown to coincide exactly with the result deduced from the ABA solution.
3.2. ABA solution

We now derive the ABA solution to the SU($\nu$) spin chain, following the calculation outlined in ref. [14]. Let us begin with the two-body scattering. In the asymptotic region in the coordinate space, the two-body scattering matrix for the above model is known to take the simple form,

$$S_{ij}^{\alpha\beta} = \lim_{\eta \to 0} \frac{k_i - k_j + i\eta P_{\alpha\beta}}{k_i - k_j - i\eta},$$

(28)

in terms the permutation operator $P_{\alpha\beta}$ which interchanges the coordinates $x_\alpha$ and $x_\beta$. Internal SU($\nu$) symmetry is now simply taken into account via the operator $P_{\alpha\beta}$. It is quite remarkable that this form of the $S$-matrix is essentially the same as that for noninteracting electrons. It should be noted, however, that this model actually describes a non-trivial spin system with long-range interaction. We note that $S_{ij}^{\alpha\beta}$ satisfies the Yang-Baxter factorization equation,

$$S_{jk}^{\alpha\gamma}S_{ij}^{\beta\gamma} = S_{ij}^{\beta\gamma}S_{ik}^{\alpha\beta}S_{jk}^{\beta\gamma}.$$

Consider now the many-body scattering among particles with $\nu - 1$ different spins by taking the $\nu$-th species as the background. As mentioned above all the particles obey the same statistics for the present SU($\nu$) spin chain. Imposing periodic boundary conditions for the ring system with $L$ sites, we have to diagonalize the scattering problem

$$e^{ik_jL}\Psi = S_{(j+1)j}S_{(j+2)j} \cdots S_{(j-1)j}\Psi,$$

(29)

where $S_{ij} = S^{ij}_{\nu}$. Introducing $\nu - 1$ kinds of rapidities $k_j^{(\alpha)}$ for $\alpha = 1, 2, \cdots, \nu - 1$, one can solve this problem by the nested BA method [51][52]. We finally arrive at the nested ABA equations for the rapidities [14],

$$k_j^{(1)}L = 2\pi I_j^{(1)} + \sum_m \Phi(k_m^{(2)} - k_j^{(1)}) + \sum_l \Phi(k_l^{(1)} - k_j^{(1)}),$$

(30)

$$\sum_l \Phi(k_m^{(2)} - k_l^{(2)}) + 2\pi I_m^{(2)} = \sum_j \Phi(k_m^{(2)} - k_j^{(1)}) + \sum_s \Phi(k_m^{(2)} - k_s^{(3)}),$$

(31)

$$\cdots$$

$$\sum_l \Phi(k_s^{(\nu-1)} - k_l^{(\nu-1)}) + 2\pi I_s^{(\nu-1)} = \sum_j \Phi(k_s^{(\nu-1)} - k_j^{(\nu-2)}),$$

(32)
with \( \Phi(k) = \pi \text{sgn}(k) \), where \( I_j^{(a)} \) classifies \( \nu - 1 \) kinds of spin excitations,

\[
I_j^{(a)} = \frac{1}{2}(M_{\nu-1} + M_\nu + M_{\nu+1}) \mod 1,
\]

with \( M_0 = M_\nu = 0 \). Here have introduced the quantity

\[
M_\alpha = \sum_{\beta=\alpha}^{\nu-1} N_\beta,
\]

in terms of the number of particles \( N_\alpha \) with spin \( \alpha = 1, 2, \cdots, \nu \). Henceforth \( \nu - 1 \) kinds of spin excitations are referred to as spinons for simplicity. The energy is given in a non-interacting form,

\[
E = \sum_{j=1}^{M_1} \frac{1}{4}[(k_j^{(1)})^2 - \pi^2] + C_\epsilon
\]

with the energy shift

\[
C_\epsilon = \frac{\pi^2}{12}(1 - \frac{1}{L^2})(L - 2M_1) + \frac{\pi^2}{6}(1 - \frac{1}{L^2}).
\]

3.3. Bulk properties

We now introduce the density function \( \rho_\alpha(k) \) for the spin rapidity \( k_j^{(a)} \) in the thermodynamic limit \([14]\). Note that the density function \( \rho_\alpha(k) \) has constant values in the region \([R_\alpha : B_{\alpha+1} < |k| < B_\alpha]\) with the condition \( B_1 \geq B_2 \geq \cdots \geq B_{\nu-1} \) and \( B_\nu = 0 \). These density functions are easily evaluated as,

\[
\rho_\alpha(k) = \begin{cases} 
\frac{1}{2\pi} \frac{(\beta - \alpha + 1) / (\beta + 1)}{\beta \geq \alpha}, \\
0 & \beta < \alpha 
\end{cases}
\]

for the region \( R_\beta \). We note that \( \rho_\alpha \) has a discontinuity at every boundary \( B_\alpha \). The density of particles with spin \( \alpha \) is expressed in terms of \( \rho_\alpha(k) \) as

\[
n_\alpha = N_\alpha/L = \int_{-B_\alpha}^{B_\alpha} \rho_\alpha(k) dk - \int_{-B_{\alpha+1}}^{B_{\alpha+1}} \rho_{\alpha+1}(k) dk
\]

for \( \alpha = 1, 2, \cdots, \nu - 1 \) and that for the \( \nu \)-th spin is determined by the sum rule: \( \sum_{\alpha=1}^{\nu} n_\alpha = 1 \). The cut-off parameter \( B_\alpha \) is controlled by the applied magnetic field through the relation,
$B_{\alpha} = [\pi^2 - 2\alpha(\alpha + 1)H]^{1/2}$, $\alpha = 1, 2, \cdots, \nu - 1$, \hspace{1cm} (39)

if we assume that $\nu$ kinds of states have magnetic moment $S, S-1, \cdots, -S$ with $\nu = 2S+1$. Note that $B_{\alpha} = \pi$ corresponds to the SU($\nu$) singlet state while $B_{\alpha} = 0$ to the fully polarized state.

It is now straightforward to compute bulk static quantities [14]. For example, the energy for SU($\nu$) singlet is given as

$$E/L = \frac{\pi^2(2 - \nu)}{12\nu},$$

which agrees with the result (27) obtained by the SU($\nu$) Gutzwiller wavefunction. Also, the magnetization is evaluated as a function of magnetic fields,

$$s_z = \frac{\nu - 1}{2} - \frac{1}{2\pi} \sum_{\alpha=1}^{\nu-1} [\pi^2 - 2\alpha(\alpha + 1)H]^{1/2}\theta(H^{(\alpha)}_c - H),$$

where $\theta(x)$ is a step function and we introduced the critical field by $H^{(\alpha)}_c = \pi^2/[2\alpha(\alpha + 1)]$. This yields the uniform spin susceptibility at low fields,

$$\chi_s = \frac{1}{2\pi^2} \sum_{\alpha=1}^{\nu-1} \alpha(\alpha + 1).$$

It is seen from (41) that the contribution to the magnetization from $\alpha$-th spinons is saturated at $H = H^{(\alpha)}_c$, and the effective degrees of freedom for spinons are reduced successively as the magnetic field is further increased. Therefore spin-liquid phases realized in magnetic fields are classified into $\nu - 1$ sectors for which each phase boundary is determined by $H^{(\alpha)}_c$. It is also easy to evaluate the velocity of each spinon excitation, which is given by the formula,

$$v_{\alpha} = \frac{1}{2}[\pi^2 - 2\alpha(\alpha + 1)H]^{1/2}.$$

This velocity determines the specific-heat coefficient,

$$C/T = \frac{\pi}{3} \sum_{\alpha=1}^{\nu-1} \frac{1}{v_{\alpha}} \theta(H^{(\alpha)}_c - H).$$

According to the finite-size scaling in CFT [14], the above expression for $C/T$ implies that $\nu - 1$ kinds of spinon excitations are described by independent $c = 1$ CFT. This point will be discussed in detail below.
3.4. Conformal properties

We now classify low-energy spinon excitations in order to observe conformal properties \[14\]. Introduce first two vectors \(\vec{m}\) and \(\vec{d}\) out of quantum numbers associated with \(\nu - 1\) kinds of spinon excitations. A quantum number \(m_\alpha (\alpha = 1, 2, \cdots, \nu - 1)\) labels the change of the number of \(\alpha\)-th spinons, i.e. \(m_\alpha = \Delta M_\alpha\). On the other hand, \(d_\alpha\) denotes an excitation which carries a momentum \(2\pi M_\alpha/N\). Hence the total momentum is expressed as,

\[
\Delta K = \sum_{\alpha=1}^{\nu-1} 2(\pi - \alpha k_F)d_\alpha, \tag{45}
\]

where \(k_F = \pi/\nu\) for \(H = 0\), and \(1/L\) corrections to the momentum transfer have been neglected. According to boson nature of spinon excitations, the quantum numbers \(d_\alpha\) satisfy the selection rule

\[
d_\alpha = \frac{1}{2}(m_{\alpha-1} + m_{\alpha+1}) \mod 1, \tag{46}
\]

for \(1 \leq \alpha \leq \nu - 1\) with the condition \(m_0 = m_\nu = 0\), which directly follows from eq. (33).

Low-energy excitations are now classified in the matrix formula,

\[
\Delta E = \frac{2\pi}{L} \sum_{\alpha=1}^{\nu-1} v_\alpha x_\alpha, \tag{47}
\]

where \(x_\alpha\) is the scaling dimension for each spinon with spin \(\alpha\),

\[
x_\alpha = \frac{1}{4}(Z^{-1}\vec{m})^2_\alpha + (Z^t\vec{d})^2_\alpha + n^+_\alpha + n^-_\alpha, \tag{48}
\]

and \(n^\pm_\alpha\) labels particle-hole type excitations for the \(\alpha\)-th spinons. The \((\nu - 1) \times (\nu - 1)\) matrix \(Z\), which is referred to as the dressed charge matrix \[53\], is given as \[14\]

\[
Z_{\alpha \beta} = \begin{cases} 
\alpha[\beta(\beta + 1)]^{-1/2}, & \alpha \leq \beta \leq \nu - 1, \\
0, & \alpha > \beta.
\end{cases} \tag{49}
\]

One can see that the above scaling dimension has the typical form inherent in \(c = 1\) CFT \[46\] and that non-negative integers \(n^\pm_\alpha\) form conformal towers which characterize the representation of Virasoro algebra \[53\].
Let us now consider the total scaling dimension \([14]\),

\[
x = \sum_{\alpha=1}^{\nu-1} x_{\alpha},
\]

which determines the critical exponents \(\eta = 2x\) for correlation functions. It turns out that \(x\) is expressed in terms of the Cartan matrix \(C\) for SU(\(\nu\)) Lie algebra,

\[
x = \frac{1}{4} \bar{m}' C \bar{m} + \bar{d}' C^{-1} \bar{d} + \sum_{\alpha=1}^{\nu-1} (n^+_\alpha + n^-_\alpha),
\]

where

\[
C = \begin{pmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& \ddots & \ddots & \ddots \\
& & -1 & 2
\end{pmatrix}.
\]

A remarkable point is that this expression for \(x\) holds even in magnetic fields although there does not exist SU(\(\nu\)) symmetry and the velocities of spinons are strongly modified. These characteristic properties are in contrast to those for the SU(\(\nu\)) antiferromagnetic Heisenberg chain with nearest neighbor interaction for which the scaling dimension changes continuously and the Cartan matrix can specify excitations only at \(H = 0\) \([54]\). Therefore the correlation exponents are not affected by magnetic fields in the present model, and are determined by those of \(c = \nu - 1\) CFT.

3.5. Correlation exponents

Following standard methods in CFT \([13, 45]\), we deduce all the critical exponents of various correlation functions \([14]\). We apply the assignment of quantum numbers to a given correlation function, which was used for the Hubbard model \([48]\) and subsequently applied to the t-J model with nearest-neighbor hopping \([49]\). For example, let us consider the long-distance behavior of the spin correlation functions characterized by the exponent \(\beta_j\),

\[
<S_z(r)S_z(0) > \approx \sum_{j=1}^{\nu-1} A_j \cos(2j k_F r) r^{-\beta_j}.
\]

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This correlation function conserves the number of particles, so that we set \( \vec{m} = (0, \ldots, 0) \).

There remain several choices for the quantum numbers \( \vec{d} \). A choice of \( d_\alpha = \delta_{\alpha j} \) \((1 \leq j \leq \nu - 1)\) results in the \( 2jk_F \) oscillation piece of the spin correlation functions. We thus obtain spin-correlation exponents for the \( 2jk_F \) oscillation part \([14]\),

\[
\beta_j = \frac{2j(\nu - j)}{\nu}, \quad 1 \leq j \leq \nu - 1.
\]

For the SU(2) case, this result agrees with that of the Gutzwiller wavefunction at half filling \([59]\). We stress here again that the above critical exponents do not depend on magnetic fields.

\[\text{§4. OSp(} \nu, 1) \text{ Supersymmetric } t-J \text{ Model}\]

Now we wish to observe what is modified when we dope holes into the SU(\( \nu \)) Haldane-Shastry model. This problem seems interesting since highly correlated electron systems have attracted particular attention recently. Such an attempt was firstly done for the OSp(2,1) supersymmetric model \([10]\), and subsequently the model was extended to more generic supersymmetric model with OSp(\( \nu,1 \)) symmetry \([14],[15]\). We give a brief review of ref. \([14]\) here.

\[4.1. \text{Symmetry properties}\]

We first note that if the fermion numbers are taken as \( F(\beta) = (0, \ldots, 0, 1) \) for \( \beta = (1, 2, \ldots, \nu + 1) \) in the Hamiltonian \((23)\), this model is equivalent to the multicomponent \( t-J \) model \([14],[15]\),

\[
\mathcal{H} = - \sum_{i<j}^{\nu} \sum_{\alpha=1}^{\nu} t_{ij} c_{i\alpha}^\dagger c_{j\alpha} + \sum_{i<j}^{\nu} J_{ij} \frac{1}{2} \left[ \sum_{\alpha,\beta \leq \nu} X_{i}^{\alpha \beta} X_{j}^{\beta \alpha} - (1 - n_{i})(1 - n_{j}) \right],
\]

with the so-called \textit{supersymmetric condition} \( t_{ij} = J_{ij} \), where \( n_i \) is the electron number at the \( i \)-th site. Here the double occupation of every site is strictly prohibited. We have
used the fact that the term like $X_i^{\alpha(\nu+1)} X_j^{(\nu+1)\alpha}$ with $\alpha = 1, 2, \ldots, \nu$ corresponds to the hopping term of electrons with spin $\alpha$. For a special case with three-component particles with $F = (0, 0, 1)$, the model reduces to the OSp(2,1) supersymmetric $t$-$J$ model (apart from the chemical potential term) \cite{10},

$$
\mathcal{H} = - \sum_{i<j,\sigma} t_{ij} c_i^{\dagger \sigma} c_{j\sigma} + \sum_{i<j} J_{ij} \left[ \mathbf{S}_i \cdot \mathbf{S}_j - \frac{1}{4} n_i n_j \right],
$$

(56)

which is obtained by doping holes into the SU(2) Haldane-Shastry model.

Here, we briefly mention supersymmetry properties of the above $t$-$J$ model. For the graded model (23) or (55) with bosons and fermions, it is known that the Hubbard operator should satisfy the following commutation relation \cite{55,56},

$$
[X_i^{\alpha\beta}, X_j^{\gamma\eta}]_\pm = \delta_{ij} (\delta_{\beta\gamma} X_i^{\alpha\eta} \pm \delta_{\alpha\eta} X_i^{\gamma\beta}),
$$

(57)

where the anti-commutator (+) should be used only for the fermion operators. This algebra is called as the doubly-graded Lie superalgebra with OSp($\nu, 1$) supersymmetry. Therefore the model (55) is invariant under the global OSp($\nu, 1$) transformation \cite{55,56}. Thanks to the supersymmetry of the Hamiltonian, we can successfully apply the ABA method to the above $t$-$J$ model \cite{57}.

4.2. Ground-state wavefunction

Before proceeding with the ABA calculation, we mention the ground-state wavefunction. The ground state of the OSp($\nu, 1$) model is given by the SU($\nu$) Gutzwiller state with holes, as was demonstrated for OSp(2,1) case \cite{10}. In this case the Gutzwiller projection operator in (24) projects out configurations with more than one electron on each lattice site, which implies that there can be cites without any electrons (assigned as holes). Taking the fully polarized state $|F>$ as the reference state, the Gutzwiller state with holes is written as \cite{10,14,15},

$$
|\Psi_G >= \sum_{(\alpha,i),j} \Psi_G(\{x_i^{(a)}\}, \{s_j\}) \prod_{\alpha,i} b_i^{(a)\dagger} \prod_j h_j^\dagger |F>,
$$

(58)
where $b_i^{(\alpha)^\dagger}$ is a creation operator for spin particles and $h_j^{\dagger} = a_j^{(\nu)}$ is that for doped holes. The Gutzwiller wavefunction is now expressed in terms of coordinates of spin particles ($x_i^{(\alpha)}$) and holes ($s_m$) as,

$$
\Psi_G\{\{x_i^{(\alpha)}\}, \{s_j\}\} = \exp[-i\pi(\sum_{\alpha,i} x_i^{(\alpha)} + \sum_j s_j)] \prod_{\alpha,i<j} d(x_i^{(\alpha)} - x_j^{(\alpha)})^2 
\times \prod_{\alpha<\beta,i,j} d(x_i^{(\alpha)} - x_j^{(\beta)}) \prod_{\alpha,i,m} d(x_i^{(\alpha)} - s_m) \prod_{m<n} d(s_m - s_n)
$$

(59)

One can show easily that this gives the ground-state wavefunction of the OSp($\nu,1$) supersymmetric $t$-$J$ model, the detail for which can be found in refs. [10,14,15].

4.3. ABA solution

We shortly outline how to deduce the ABA solution to the Osp($\nu,1$) supersymmetric $t$-$J$ model [14]. A remarkable point is that the two-body $S$-matrix for this model is given by the same formula (28) as in the SU($\nu$) spin chain, and symmetry properties can be taken into account solely via the permutation operator $P_{\alpha\beta}$. Hence, using the $S$-matrix (28) for the two-body scattering, it is now straightforward to deduce the ABA solution to the OSp($\nu,1$) $t$-$J$ model. Taking the reference state full of particles with the $\nu$-th spin, consider now the scattering problem among $\nu - 1$ kinds of spinons and holons (holes). In order to diagonalize this problem, therefore, besides $\nu - 1$ kinds of spin rapidities $k_j^{(\alpha)}$ ($\alpha = 1, 2, \cdots, \nu - 1$), it is necessary to introduce the charge rapidity $k_j^{(\nu)}$ [22]. As a result the last line of the ABA equations in (32) should be modified by grading holes [14],

$$
k_j^{(1)}L = 2\pi I_j^{(1)} + \sum_m \Phi(k_m^{(2)} - k_j^{(1)}) + \sum_l \Phi(k_j^{(1)} - k_l^{(1)}),
$$

(60)

$$
\sum_l \Phi(k_m^{(2)} - k_l^{(2)}) + 2\pi I_m^{(2)} = \sum_j \Phi(k_m^{(2)} - k_j^{(1)}) + \sum_s \Phi(k_m^{(2)} - s^{(3)}),
$$

(61)

$$
\cdots
$$

$$
2\pi I_s^{(\nu)} = \sum_j \Phi(k_s^{(\nu)} - k_j^{(\nu-1)}), \quad s = 1, 2, \cdots, M_\nu,
$$

(62)
where $I^{(\nu)}_s$ is a quantum number which labels the degrees of freedom for doped holes. According to the antisymmetric nature of electrons, the selection rule for this quantum number reads,

$$I^{(\nu)}_s = \frac{1}{2} M_\nu \mod 1. \quad (63)$$

### 4.4. Bulk properties

We calculate bulk quantities following techniques outlined for the spin chain. It is remarkable that the shape of the the calculated magnetization curve does not depend on electron concentrations, which is given by the same formula as (41), but only the critical field

$$H^{(\alpha)}_c = \frac{2\nu n(n-2)}{\pi^2 2\alpha(\alpha + 1)} \quad (64)$$

is modified by hole-doping ($n$: electron density) \[14\]. Also, other bulk quantities are easily calculated. By the second derivative of the ground state energy with respect to $n$, we obtain the charge susceptibility (compressibility) as a function of electron concentrations,

$$\chi_c = \frac{2\nu}{\pi^2 (1-n)} \quad (65)$$

which shows a divergent behavior near insulating phase $n = 1$ \[10,14,15\]. The coefficient of the $T$-linear heat capacity is given in terms of the velocities of spinons and holons \[10,14,15\],

$$\gamma = \frac{\pi}{3} \left[ \frac{1}{v_c} + \sum_{\alpha=1}^{\nu-1} \frac{1}{v_\alpha} \theta(H^{(\alpha)}_c - H) \right] \quad (66)$$

where the velocity of spinon excitations, $v_\alpha$, takes the same formula as in (43) while that for holon excitations depends on the electron concentration,

$$v_c = \pi (1-n)/2. \quad (67)$$
The above expression for the specific heat implies that the critical behavior of the present model is described by $c = 1$ CFT. We can see that there are the Luttinger-liquid relation among bulk quantities, $\pi \chi_c v_c = \nu$.

4.5. Conformal properties

We now classify low-energy spin and charge excitations in order to study conformal properties [14]. Let us introduce $\nu$-component vectors $\vec{m}$ and $\vec{d}$ for quantum numbers which specify excitations. We assume the $\nu$-th component of vectors to be related to the holon degrees of freedom. According to the antisymmetry properties of electron wavefunction, the selection rule for quantum numbers reads

$$d_\alpha = \frac{1}{2}(m_{\alpha-1} + m_{\alpha+1}) \mod 1,$$

for spinon excitations ($\alpha = 1, 2, \cdots, \nu - 1$), whereas that for holon excitations is

$$d_\nu = \frac{1}{2}(m_{\nu-1} + m_{\nu}) \mod 1.$$  (69)

Note that quantum numbers $d_\alpha$ carry the large momentum transfer $2(\pi - \alpha k_F)d_\alpha$ for $1 \leq \alpha \leq \nu$ with Fermi momentum $k_F = \pi n/\nu$. The excitation spectrum is now classified as

$$\Delta E = \frac{2\pi}{L} \sum_{\alpha=1}^{\nu} v_\alpha x_\alpha,$$  (70)

from which we can read the scaling dimension $x_\alpha$. The resulting scaling dimension is expressed in the same formula as (48) by extending the dressed charge matrix [33] to the $\nu \times \nu$ matrix $Z$,

$$Z_{\alpha\beta} = \begin{cases} \alpha[\beta(\beta+1)]^{-1/2}, & \alpha \leq \beta \leq \nu - 1, \\ \alpha/\sqrt{\nu}, & \beta = \nu, \ 1 \leq \alpha \leq \nu \end{cases}$$  (71)

and $Z_{\alpha\beta} = 0$ otherwise [14]. Matrix elements of $\nu$-th row and the $\nu$-th column are related to the holon degrees of freedom. From the expression (70), one can see that that holon excitations as well as spinon excitations are described by independent $c = 1$ CFT. Namely
this liquid is classified as the Luttinger liquid. The total scaling dimension \( x = \sum_{\alpha=1}^{\nu} x_{\alpha} \) is reduced to the simple expression (50) where the \( \nu \times \nu \) matrix \( C \) in this case is

\[
C = \begin{pmatrix}
2 & -1 & & & \\
-1 & \ddots & -1 & & \\
& \ddots & 2 & \ddots & \\
& & -1 & 1 & \\
& & & & &
\end{pmatrix}
\] (72)

which is nothing but the Cartan matrix for the OSp(\( \nu, 1 \)) Lie superalgebra \([14]\). We wish to emphasize here again that the above scaling dimension depends neither on magnetic fields nor on electron concentrations, characterizing the universality class of quantum \( 1/r^2 \) models.

4.6. Correlation exponents

Let us now evaluate critical exponents for various correlation functions with the use of CFT. First we compute the critical exponents of the spin correlation function. By choosing the quantum numbers \( d_{\alpha} = \delta_{\alpha j} \) (1 \( \leq \) \( j \) \( \leq \) \( \nu - 1 \)) for the spin correlation function \([48],[49],[57],[58]\), we obtain the spin correlation exponents for the \( 2j k_F \) oscillation part,

\[
\beta_j = 2j, \quad 1 \leq j \leq \nu - 1.
\] (73)

Remarkably enough, all the exponents are given by the canonical (integer) values. Also, critical exponents for the charge correlation functions are obtained similarly, which are given by the same formula as for the spin exponents, but the \( 2\nu k_F \) oscillation piece appears with \( \beta_\nu = 2\nu \). It is instructive to note that the spin correlation exponents discontinuously change when the holes are doped into the spin chain. This can be checked by comparing the expressions (54) and (73). This property of the discontinuity is known to be common in the correlated electron systems close to the insulator.

We now evaluate the correlation exponent \( \eta \) of the field correlator of electrons, \( \langle c_\alpha^\dagger (r) c_\alpha (0) \rangle \propto \cos(k_F r)^{-\eta} \). A simple excitation relevant to this correlator is given by the set of quantum numbers \( (m_1, m_2, \cdots, m_\nu) = (1, 1, \cdots, 1) \) and \( (d_1, d_2, \cdots, d_\nu) = (1/2, 0, \cdots, 0) \) \([48],[49],[57],[58]\). The momentum carried by this excitation is \( \pi(1 - k_F) \). The resultant
critical exponent turns out to be $\eta = 1$ \cite{14}. Fourier transformation of this correlation functions gives the momentum distribution

$$n_k = n_{k_F} - \text{const.}|k - k_F|^\theta \text{sgn}(k - k_F),$$

with the corresponding exponent $\theta = \eta - 1 = 0$. By taking account the fact that there is no logarithmic correction in the present case, $\theta = 0$ implies that there should be a discontinuity in the momentum distribution at the Fermi point, as firstly pointed out for the OSp(2, 1) model \cite{10}. Recall again that all the critical exponents of correlation functions are given by the canonical values as well known for the Gutzwiller wavefunction \cite{59}. At first glance these canonical exponents seem to be contradicted to Luttinger liquid theory \cite{17}, but we should keep in mind that this fixed point is indeed on the critical line of the Luttinger liquid ($c = 1$ Gaussian CFT). The present model is, therefore, classified as a specific example of the Luttinger liquid which has canonical exponents \cite{10}.

We have not been concerned here with thermodynamic properties at arbitrary temperatures. In order to describe the full thermodynamics of the lattice models, it is crucial to study the degeneracy of excited states. Concerning this problem, the free-spinon picture of Haldane \cite{17}, Yangian symmetry \cite{36} and related methods \cite{13,32} have been successfully used to construct correct thermodynamics.

§5. Hierarchical Models Related to FQHE

One of the most interesting aspects of the $1/r^2$ quantum systems is the intimate relationship \cite{17,22} to the fractional quantum Hall effect (FQHE) \cite{60,61,62}. We have indeed seen that the construction of eigenstates for the $1/r^2$ models is quite analogous to that for the FQHE: the ground state is given by the Jastrow wavefunction, and the excited states are constructed by multiplying polynomials to the ground state wavefunction. Therefore, besides much interest in the integrability, these quantum models should exhibit interesting
phenomena related to the FQHE. We have recently proposed a novel hierarchy of the 1D quantum models with $1/r^2$ interaction [18], the construction of which is essentially same as that of a hierarchical FQHE with the filling fraction [61,62],

$$f_\nu = \frac{1}{p_1 - \frac{1}{p_2 \cdots \frac{1}{p_\nu}}}.$$  \hspace{1cm} (75)

In particular the matrix deduced from the energy spectrum has been shown to coincide with the topological-order matrix which characterizes the internal structure of the FQHE state [63]–[65]. In this section we briefly review the hierarchical models related to the FQHE [18,21].

5.1. Continuum Models and ABA solution

Let us introduce a family of $\nu$-component electron models ($\alpha = 1, 2, \cdots, \nu$) with $1/r^2$ interaction in the periodic ring of length $L$ [18],

$$H = -\frac{1}{2} \sum_i \frac{\partial^2}{\partial x_i^2} + \sum_{\alpha \leq \beta} \sum_{i<j} d_{ij}^{-2} V_{\alpha\beta} (V_{\alpha\beta} + P_{ij}^{\alpha\beta}),$$  \hspace{1cm} (76)

with chord distance $d_{ij} = (L/\pi) \sin[\pi(x_i - x_j)/L]$, where $P_{ij}^{\alpha\beta}$ is the spin exchange operator with spin indices $\alpha, \beta = 1, 2, \cdots \nu$, and $V_{\alpha\beta}$ is the interaction parameters dependent on spin indices. Starting from noninteracting SU($\nu$) electrons (zero-th stage), let us specify the stage of the hierarchy in terms of the matrix formula of the interaction parameter. The first stage is defined by introducing interaction $\lambda_1$ uniformly to all the electrons, i.e. $V_{\alpha\beta} = \lambda_1$. The resultant model is the SU($\nu$) Sutherland model for which only the charge excitation is affected by the interaction [15],[27],[28]. The second family is introduced by turning on the uniform interaction $V_{\alpha\beta} = \lambda_2$ among electrons except for the $\alpha = 1$ species. By iterating this procedure step by step, the $\nu$-th stage of the model is characterized by the interaction matrix [18],

25
\[
\mathbf{V} = \left( \begin{array}{ccc}
\lambda_1 & \cdots & \lambda_1 \\
\vdots & \ddots & \vdots \\
\lambda_1 & \cdots & \lambda_1
\end{array} \right) + \left( \begin{array}{ccc}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{array} \right) + \cdots + \left( \begin{array}{ccc}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{array} \right). \tag{77}
\]

Note that the interaction \( \lambda_m \) acts on particles with spin indices \( \alpha = m, m + 1, \ldots, \nu \), hence it is regarded as spin-dependent interaction. The construction of the above hierarchy is quite analogous to that for the FQHE [61], in particular, to Jain’s composite-fermion construction of the hierarchical FQHE [62]. In the FQHE case, the interaction matrix introduced here serves as a topological order matrix which specifies the internal structure of the FQHE [63]–65]. We shall see that the interaction matrix completely specifies the low energy excitations, and that the present model can describe essential properties for edge states of the hierarchical FQHE with filling fraction \( f_\nu \) [63].

Let us now find the solution to the problem by the ABA method [18]. We first consider the scattering of two particles by the interaction \( d_{ij}^2 V_{\alpha\beta} (V_{\alpha\beta} + 1) \). As mentioned in the previous sections, this interaction yields the \( S \)-matrix of \( \exp[-i\phi_{\alpha\beta}(k_i - k_j)] \) with the phase shift function,

\[
\phi_{\alpha\beta}(k) = V_{\alpha\beta} \pi \text{sgn}(k), \tag{78}
\]

in the asymptotic region (\(|x_i - x_j| >> 1\)). The step-like form of the phase shift \( \phi_{\alpha\beta}(k) \) is characteristic of the \( 1/r^2 \) interaction. Taking into account the spin degrees of freedom, we then find the two-body \( S \)-matrix for the above hierarchical models as [18],

\[
S_{ij} = \lim_{\epsilon \to 0} \frac{k_i - k_j - i\epsilon P_{ij}}{k_i - k_j - i\epsilon} e^{-i\phi_{\alpha\beta}(k_i - k_j)}, \tag{79}
\]

where the first factor arises from non-interacting SU(\( \nu \)) electrons, while the second is due to the \( 1/r^2 \) interaction. The key feature that the \( S \)-matrix (79) is a simple superposition of two \( S \)-matrices makes it possible to treat the scattering problem consistently by nested Bethe ansatz techniques.

As mentioned in the previous section, the essence of the ABA method is that the many-body \( S \)-matrix in this family can be decomposed into two-body matrices in spite of long-
range nature of interaction. For a simple case of SU(ν) symmetry, we have already checked that this solution gives the exact spectrum \[14,21\]. If this is the case in general, we can diagonalize the many body S-matrix by the standard nested Bethe ansatz. Consequently we arrive at the ABA equations for ν-kinds of rapidities \(k_j^{(α)}\) \[13,18\],

\[
\begin{align*}
    k_j^{(1)} L &= 2\pi I_j^{(1)} + \sum_{m=1}^{M_2} \Phi(k_m^{(2)} - k_j^{(1)}) + \lambda_1 \sum_{l=1}^{M_1} \Phi(k_j^{(1)} - k_l^{(1)}), \\
    (\lambda_α + 1) \sum_{l=1}^{M_α} \Phi(k_m^{(α)} - k_l^{(α)}) + 2\pi I_m^{(α)} &= \sum_{s=±1} \sum_{j=1}^{M_{α+s}} \Phi(k_m^{(α)} - k_j^{(α+s)}),
\end{align*}
\]

(80)

(81)

for \(2 ≤ α ≤ ν\), where \(\Phi(k) = π\text{sgn}(k)\) and \(I_j^{(α)}\) is an integer or a half integer which classifies the charge and spin excitations. In the above equations we have introduced the quantity \(M_α = \sum_{β=α}^μ N_β\) where \(N_β\) is the number of electrons with spin \(β\). The energy is written in the noninteracting form \(E = (1/2) \sum_j (k_j^{(1)})^2\).

5.2. Bulk properties

Let us calculate bulk quantities \[18\]. It is remarkable that all the bulk quantities are expressed solely by the parameter \(f_ν\) introduced in (75) if we assign the set of parameters \(p_i\) in terms of interaction parameters \(λ_i\) as,

\[
p_i = λ_i + 2 - δ_{i1},
\]

(82)

for \(α = 1, 2, \cdots, ν\). For instance, the ground-state energy is evaluated as

\[
E_g / L = π^2 n^3 / (6f_ν^2),
\]

(83)

where \(n\) is the electron density. The second derivative of \(E_g(n)\) yields the charge susceptibility (compressibility) in terms of \(f_ν\),

\[
χ_c = (f_ν / π)^2 n^{-1}.
\]

(84)

Roughly speaking, the quantity \(1/f_ν\) corresponds to the size of the exchange-correlation hole due to the \(1/r^2\) interaction: we can regard the effective volume of electrons to become \(1/f_ν\) times as large as that of free electrons.
The free energy at finite temperatures \((T)\) is formulated as,

\[
F = -\mu - \left(\frac{T}{2\pi}\right) \int_{-\infty}^{\infty} \log[1 + \exp(-\epsilon_1(k))]dk
\]  

(85)

in terms of the dressed energy,

\[
\epsilon_1(k)/T = \frac{1}{2}k^2 - \mu - \log[1 + \exp(-\epsilon_2(k))] + \lambda_1 \log[1 + \exp(-\epsilon_1(k)/T)],
\]  

(86)

\[
\epsilon_\alpha(k)/T = \sum_{s=-1,0,1} (-1)^s \epsilon_\alpha + s(k) \log[1 + \exp(-\epsilon_\alpha + s(k))/T],
\]  

(87)

for \(\alpha = 2, 3, \cdots, \nu\), with \(\epsilon_{\nu+1} = \infty\). The low-temperature expansion of the free energy yields

the coefficient of the \(T\)-linear specific heat as,

\[
C/T = \frac{\pi}{3v}
\]  

(88)

with the velocity \(v = \pi n/(2f_\nu)\). Note that all the velocities for \(\nu\) kinds of elementary excitations have the same value \(v\) although there is not SU(\(\nu\)) symmetry in the model. The Luttinger-liquid relation between the charge susceptibility and the velocity takes the form,

\[
\pi \chi_c v = f_\nu/2,
\]

which can determine the critical behavior of charge excitations.

### 5.3. Excitations

Let us now turn to the excitation spectrum \([18]\). Using the ABA equations, low-energy excitations are classified in the matrix formula,

\[
\epsilon = \frac{2\pi v}{L} \left[ \frac{1}{4} \bar{m}^T \bar{T} \bar{m} + \bar{d}^T (\bar{T})^{-1} \bar{d} \right],
\]  

(89)

where the \(\nu \times \nu\) matrix \(\bar{T}\) is evaluated as,

\[
\bar{T} = \begin{pmatrix}
 p_1 & -1 \\
 -1 & p_2 \\
 & \ddots \\
 & & \ddots & -1 \\
 -1 & p_\nu
\end{pmatrix}.
\]  

(90)

Here the \(\nu\)-component vector \(\bar{m}\) is out of quantum numbers which classify the charge and spin excitations, where we assumed that \(m_1\) and \(d_1\) label charge excitations. It is remarkable
that the above matrix $T$ deduced from the energy spectrum is nothing but the topological-order matrix for the FQHE with filling fraction $f_\nu$. One can see that there is $\nu$ free parameters, $p_i$, in the above matrix. This implies that the critical behavior of the present model is described by the corrections of $\nu$ independent CFT with the central charge $c = 1$, where $p_i$ can make $c = 1$ critical lines.

5.4. $t$-$J$ model and lattice effects

To see the relationship to the composite fermion theory in the FQHE more explicitly, let us observe what happens for the lattice case [18]. We introduce a hierarchical family of $t$-$J$ models with $1/r^2$ interaction,

$$\mathcal{H} = \sum_{\alpha,i \neq j} d_{ij}^{-2} c_i^{\dagger} c_j + \sum_{\alpha \leq \beta, i < j} d_{ij}^{-2} V_\beta (V_\beta + P_{ij}^{\alpha \beta}), \quad (91)$$

with interaction parameters $V_{\alpha \beta}$ defined in (77). Here configurations with more than one electron at each site are assumed to be prohibited. The first family of the hierarchy coincides with the $\mathrm{SU}(\nu)\ t$-$J$ model introduced in [15]. This class of lattice models can be solved by the ABA method, and the resulting nested ABA equations are given by the same formula as in (80) and (81). Hence, the bulk quantities are given by the same expressions as for the continuum case. In the lattice $t$-$J$ model, however, we encounter two crucial constraints [18]: (a) parameter $V_{\alpha \beta}$ should be an even positive integer, and (b) the rapidity $k_{ij}^{(\alpha)}$ should be in the region $[-\pi, \pi]$ as is the case for ordinary lattice models. According to (a), the parameter $f_\nu$ should be a fraction with the odd denominator, which demonstrates the analogy to FQHE explicitly. Furthermore, the constraint (b) brings about a remarkable property, i.e. a singular property at the electron density $n = f_\nu$ above which the Luttinger-liquid state breaks down ($n > f_\nu$) [18]. This means that a band-edge singularity for non-interacting $\mathrm{SU}(\nu)$ lattice electrons at the density $n = \nu$ is modified into the singularity at the fractional filling $n = f_\nu$ in the presence of the $1/r^2$ interaction. We note that this phenomenon is essentially the same as that for the composite fermion theory of the FQHE, i.e. the hierarchical FQHE with the filling $n = f_\nu$ can be realized by starting from the integer (noninteracting) QHE...
5.5. Chiral constraint and FQHE edge states

In order to observe how remarkably the present model reproduces essential properties expected for edge states of the FQHE, let us think of what will happen if we consider only right (or left)-moving electrons in the 1D system \[18\),21\]. This is referred to as chiral constraint which is essential for edge states of the FQHE in disk geometry \[63\). In order to deal with chiral constraint, we make use of a trick valid for the \(1/r^2\) systems. We first add two electrons at the left and right Fermi points, in order to suppress the current \(\vec{d}\) which does not exist in the chiral model. This results in the energy increment both for the right and left branches. Dividing the energy increase into two parts and discarding the right-going piece, we obtain the excitation spectrum for left-going piece as \[18\),21\],

\[
\epsilon = \frac{\pi v}{L} \vec{m}^T \mathbf{m}.
\]

This formula reproduces the spectrum described by the holomorphic piece of \(\nu\) independent \(c = 1\) CFT. Hence we can determine the critical exponents for various correlation functions. For example, the critical exponent \(\theta_\alpha\) for the momentum distribution function

\[
n_k^{(\alpha)} \simeq \text{const.} + a_0 \text{sgn}(k - k_F)|k - k_F|^{\theta_\alpha}
\]

is obtained as

\[
\theta_\alpha = \sum_{j=1}^{\alpha} \lambda_j.
\]

All the other exponents can be similarly obtained. We stress here that all the critical exponents agree with those of effective field theory (chiral Luttinger liquids) for the edge states of FQHE with the fraction \(f_\nu\) \[63\]. In particular the key matrix \(\mathbf{T}\) deduced from the excitation spectrum coincides exactly with the topological order matrix which characterizes the internal structure of the FQHE with the filling fraction \(f_\nu\) \[64,65\).

We have observed that the hierarchical models considered here describe characteristic properties for the edge states of the FQHE with filling fraction \(f_\nu\) so remarkably. The
correspondence is not accidental, and we can indeed see a clear reason for it in the role played by the phase shift function \( \lambda_\alpha \pi \text{sgn}(k) \). Recall that in the composite fermion theory for the FQHE [62], flux quanta are attached to electrons in order to evolve the hierarchy starting from noninteracting electron systems [62]–[65]. A crucial point is that attaching \( \lambda_\alpha \) flux quanta in the FQHE corresponds to introducing the phase shift function \( \lambda_\alpha \pi \text{sgn}(k) \) in the present 1D system, which can be given by the \( 1/r^2 \) interaction [18]. Hence, one can clearly see from this correspondence why the family of \( 1/r^2 \) models can describe characteristic properties of the hierarchical FQHE remarkably.

5.6. Dual bases: holon-spinon and electron

We would like to point out another instructive relationship to the FQHE [21]. So far, we have classified the excitation spectrum in terms of the matrix \( T \) and the corresponding quantum numbers out of charge and spin excitations. This basis, which is referred to as holon-spinon basis, always shows up when we use the Bethe ansatz method. We should recall that there is an alternative basis, i.e. electron basis. Although the electron basis may not classify the excitation spectrum when various excitations have different velocities, it still describes the critical behavior of correlation functions correctly [21]. Quantum numbers in the electron basis are obtained from those of holon-spinon basis via a linear transformation,

\[
\vec{N} = U \vec{m}, \quad \vec{J} = (U^t)^{-1} \vec{d},
\]

with the matrix

\[
U = \begin{pmatrix}
1 & -1 \\
1 & \ddots & \ddots \\
& \ddots & \ddots & -1 \\
& & \ddots & \ddots \\
& & & 1 \\
\end{pmatrix}.
\]

The matrix \( T \) is then transformed into the \( \nu \times \nu \) symmetric matrix [21],

\[
\tilde{T} = I + V,
\]
where $I$ is the $\nu \times \nu$ unit matrix. Remarkably, we encounter the matrix $V$ out of the interaction parameters (see (77)). Hence the matrix $V$ itself can classify the excitation spectrum in the electron basis [21]. We should emphasize here that there exists the exactly same matrix (97) in the 2D FQHE which also characterizes the internal structure of the hierarchical state [64,65]. The above two kinds of bases are called the symmetric basis and the hierarchical basis in the FQHE, both of which characterize the same topological order of the hierarchical FQHE [64,65].

\textbf{5.7. Example: SU($\nu$) Sutherland model}

As a simple example, we consider the first family of the hierarchical models (76), i.e. the SU($\nu$) Sutherland model characterized by the parameters $V_{\alpha\beta} = \lambda$ [15,27,28]. In this case the exact wavefunction has been obtained in the Jastrow form [15],

$$
\psi = \prod_{l>m} |z_l^{(\alpha)} - z_m^{(\beta)}|^\lambda \psi_0. 
$$

(98)

Here $\psi_0$ is the wavefunction for SU($\nu$) free electrons,

$$
\psi_0 = \prod_{\alpha,j} [z_j^{(\alpha)}]^{2d_{\alpha}} \prod_{\alpha,\beta,l>m} (z_l^{(\alpha)} - z_m^{(\beta)})^{\delta_{\alpha\beta}\delta_{lm}} \exp[\frac{i}{2\pi} \text{sgn}(\alpha_l - \alpha_m)]. 
$$

(99)

where $z_m^{(\alpha)} = \exp(2\pi i x_m^{(\alpha)}/L)$, $L$ is the length of the periodic system and $x_m^{(\alpha)}$ are spatial coordinates of electrons with spin $\alpha (= 1, 2, \cdots, \nu)$. The current $2d_{\alpha}$ carried by $\alpha$-spin electrons is assumed to take an integer value [15]. This form of the Jastrow wavefunction clearly demonstrates the analogy to Jain’s construction of the wavefunction for the FQHE with filling factor $f_{\nu} = \nu/(\nu m + 1)$ [62]. Namely, starting from the noninteracting wavefunction, interacting electrons can be described by introducing the Jastrow factor. In the electron basis, the excitation is classified by the symmetric matrix [15,21]

\[
\tilde{T} = \begin{pmatrix}
1 + \lambda & \cdots & 1 + \lambda \\
\vdots & \ddots & \vdots \\
1 + \lambda & \cdots & 1 + \lambda
\end{pmatrix}.
\]   

(100)
In order to observe the symmetry property, it is more convenient to make use of the holon-spinon representation, which is characterized by the following matrix \[ T \]

\[
T = \begin{pmatrix}
1 + \lambda & -1 \\
-1 & 2 & -1 \\
\vdots & \ddots & \ddots & \ddots \\
-1 & 2
\end{pmatrix}
\] (101)

From this expression we can see U(1) symmetry for the charge sector \((T_{11})\) and SU(\(\nu\)) symmetry for the spin sector \((T_{\alpha\beta} \text{ for } \alpha \geq 2)\). Particularly the \((\nu - 1) \times (\nu - 1)\) matrix out of \(T_{\alpha\beta}\) for \(\alpha, \beta \geq 2\) is the SU(\(\nu\)) Cartan matrix which characterizes the Lie algebra with SU(\(\nu\)) symmetry. Hence, the critical behavior of the spin sector is characterized by level-1 SU(\(\nu\)) Kac-Moody algebra with the central charge \(c = \nu - 1\), and the holon sector is governed by \(c = 1\) CFT for which the scaling dimensions vary continuously according to \(\lambda\).

It is now easy to obtain critical exponents for the correlation functions. For example, the critical exponent for the momentum distribution function is obtained as \[ \theta_\alpha = \frac{1}{2} \nu^2 \lambda^2 (1 + \nu \lambda)^{-1}, \] which is in contrast to the chiral case,

\[
\theta^{(c)}_\alpha = \lambda.
\] (103)

Note that the critical exponent for the chiral case exactly coincides with the result for chiral Luttinger liquids of the edge states with filling fraction \(f_\nu = \nu/(\nu \lambda + 1)\) \[ \[33\]. \]

§6. Confined Models and Renormalized Harmonic Oscillators

We have been concerned so far with quantum models with periodic boundary conditions. There is another class of the integrable \(1/r^2\) models with harmonic confinement \[ [1,2,29,34,35,38,11] \]. Besides much interest in the integrability of the confined models
there have also been several attempts to apply them to conductance oscillations in mesoscopic systems \[66, 67, 39, 40]\). In this section, we propose a systematic construction of the energy spectrum for the class of $1/r^2$ models with harmonic confinement \[38\]. This approach is referred to as the renormalized-harmonic oscillator (RHO) solution, since the essence of the idea is that all the interaction effects are incorporated in terms of the renormalized quantum numbers of oscillators. In this sense, this method is regarded as a variant of the ABA. We use this idea for a systematic construction of the spectrum for the confined $1/r^2$ models \[38\]. By constructing the eigenfunctions explicitly \[40, 41\], we then give a proof that the RHO solution indeed provides the exact spectrum of the models. In the final part of this section we briefly mention an application of the model to conductance-oscillation phenomena in narrow channels \[66, 67, 40\].

6.1. Renormalized-harmonic-oscillator solution

6.1.1. Calogero-Sutherland model

To depict the essence of the idea, we begin with the Calogero-Sutherland model which is given by interacting spinless fermions (or bosons) confined by harmonic potential \[1, 2\],

$$ H = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^{N} \omega^2 x_i^2 + \sum_{j>i} \frac{\lambda(\lambda + 1)}{(x_j - x_i)^2}, \quad (104) $$

where the interaction parameter is assumed to be $\lambda \geq 0$. Here, we recall the key property common to $1/r^2$ models, i.e. the interaction gives rise to the repulsion among energy levels, and enlarges the spacing of quantum numbers uniformly. As seen in the previous sections, the repulsion effect can be formulated by introducing the step-wise phase shift of the two-body $S$-matrix in the periodic case. This is the heart of the ABA in which the renormalized quantities correspond to the rapidities in the ABA equations.

For the case of harmonic confinement, the repulsion of energy levels can be taken into account by the renormalization of the quantum numbers for oscillators \[38\]. In the RHO approach, all the interaction effects are conjectured to be incorporated into the renormalized quantum number, $n_j$, and the energy is given in the expression for free harmonic oscillators,
\[ E = \omega \sum_{j=1}^{N} (n_j + \frac{1}{2}), \]  

We note that the renormalized quantum numbers should be related to the conserved charges which ensure the integrability of the model, although the explicit relation has not been derived yet. As is the case for the periodic case, the repulsion of energy levels are described by introducing the step function. Hence the quantum number \( n_j \) is to be determined by the equation

\[ n_j = I_j + \lambda \sum_{l=1}^{N} \theta(n_j - n_l), \quad (106) \]

where \( I_j = 0, 1, 2, \cdots \) is the bare quantum number, and the step function is introduced such that \( \theta(x) = 1 \) for \( x > 0 \) and \( \theta(x) = 0 \) for \( x \leq 0 \). Consequently we obtain the energy for the Calogero-Sutherland model as

\[ E = \omega \left[ \frac{1}{2} \lambda N (N - 1) + \sum_{j=1}^{N} (I_j + \frac{1}{2}) \right], \quad (107) \]

The ground state is given by the successive quantum numbers \( I_j = 0, 1, \cdots, N - 1 \), and particle-hole excitations are described by changing the quantum numbers \( I_j \) from those for the ground state. Remarkably enough, the above results deduced from the RHO reproduce the exact spectrum for the Calogero-Sutherland model \([1],[2],[29],[34]\).

6.1.2. \( SU(\nu) \) confined model

The RHO solution can be applied to more general multicomponent models \([38]\). As an example, let us study the \( SU(\nu) \) electron model with harmonic confinement, which is a variant of the \( SU(\nu) \) Sutherland model with periodic boundary conditions discussed in the previous sections. The Hamiltonian reads \([34],[35],[40]\),

\[ H = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i=1}^{N} \omega^2 x_i^2 + \sum_{j>i}^{N} \frac{\lambda(\lambda + P_{ij}^{\alpha\beta})}{(x_j - x_i)^2}, \quad (108) \]

with the spin-exchange operator \( P_{ij}^{\alpha\beta} \) of two particles \( (\alpha, \beta = 1, 2, \cdots, \nu) \). As usual, to complete the diagonalization for the \( SU(\nu) \) model, it is necessary to introduce the set of
renormalized quantum numbers $n_j^{(\alpha)} (\alpha = 1, 2, \cdots, \nu)$ which satisfy the nested algebraic equations [38],

$$n_j^{(1)} = I_j^{(1)} - \sum_m \theta(n_j^{(1)} - n_m^{(2)}) + \lambda \sum_l \theta(n_l^{(1)} - n_j^{(1)}),$$

$$\sum_l \theta(n_m^{(\alpha)} - n_l^{(\alpha)}) + I_m^{(\alpha)} = \sum_j \theta(n_m^{(\alpha)} - n_j^{(\alpha-1)}) + \sum_s \theta(n_m^{(\alpha)} - n_s^{(\alpha+1)}),$$

for $2 \leq \alpha \leq \nu$. Here the bare quantum numbers $I_j^{(\alpha)}$ are the non-negative integers ($= 0, 1, \cdots$) which specify $\nu$ kinds of elementary excitations. In the above equations, the quantity $M_\alpha = \sum_{\beta=\alpha}^{\nu} N_\beta$ was introduced, where $N_\beta$ is the number of electrons with $\beta$ spin ($M_1 = N = \sum_{\beta=1}^{\nu} N_\beta$).

The energy is written in the expression for harmonic oscillators, $E = \omega \sum_j (n_j^{(1)} + 1/2)$, in the RHO method. By substituting the nested equations to this formula, and then iterating the substitutions, we obtain the final expression for the energy as [38],

$$E = \omega \left[ \frac{1}{2} \lambda N (N - 1) + \sum_{\alpha=1}^{\nu} (\frac{1}{2} N_\alpha^2 - \frac{1}{2} M_\alpha (M_\alpha - 1) + \sum_{j=1}^{M_\alpha} I_j^{(\alpha)}) \right].$$

The ground state is described by the successive non-negative quantum numbers $I_j^{(\alpha)} = 0, 1, \cdots, M_\alpha - 1$, resulting in the ground-state energy for SU($\nu$) singlet,

$$E_g = \omega \left[ \frac{1}{2} \lambda N (N - 1) + \frac{1}{2} \sum_{\alpha=1}^{\nu} N_\alpha^2 \right].$$

We note that all the interaction effects are incorporated via the first term of (112), and any effects of the interaction do not show up so far as the number of electrons are fixed. Hence, the excitation spectrum is described by free oscillators in case of the fixed number of electrons. We should note that this does not mean the system to be out of free oscillators, as is clearly seen from the level-repulsion effects in the RHO equations.

6.1.3. hierarchical models

It is straightforward to apply the RHO solution to the hierarchical family of the confined models analogous to (76) [42].
where the interaction parameters are the same as those of (77). Applying the RHO techniques described above, we obtain the energy as,

$$E/ = \omega \sum_{\alpha=1}^{\nu} \left[ \frac{1}{2} N^2_{\alpha} + \frac{1}{2} (\lambda_{\alpha} - 1) M_{\alpha} (M_{\alpha} - 1) + \sum_{j=1}^{M_{\alpha}} f_{j(\alpha)} \right].$$

(114)

The ground-state energy then takes the form,

$$E_g = \frac{\omega}{2} \sum_{\alpha=1}^{\nu} [N^2_{\alpha} + \lambda_{\alpha} M_{\alpha} (M_{\alpha} - 1)].$$

(115)

We can classify the excitation spectrum in the matrix formula, which leads to the expression similar to (92) in which we should replace $(2\pi v/L)$ by $\omega$. The detail of this part will be reported elsewhere [42].

6.2. Construction of eigenfunctions

We have studied so far the energy spectrum making use of the RHO solution which is deduced from the level-repulsion effects of $1/r^2$ interaction. Here we give the microscopic foundation of the RHO solution by constructing the eigenfunctions explicitly. We give a brief review of the results of Vacek et al. to construct the eigenfunctions of the ground-state [40] as well as the excited states [41].

6.2.1. ground state

We start by writing down a general form of Jastrow wavefunction which is to be the exact eigenstate of the family of $1/r^2$ models. It consists of two parts,

$$\Psi(x_1\alpha_1, \ldots, x_N\alpha_N) = \prod_{j>i} |x_j - x_i|^\lambda \Psi_0(x_1\alpha_1, \ldots, x_N\alpha_N),$$

(116)

where the first one is the Jastrow factor, and the second, $\Psi_0$, is the eigenfunction for the noninteracting SU($\nu$) electron model ($\alpha = 1, 2, \cdots, \nu$). For the above wavefunction, therefore, all the interaction effects are assumed to be taken into account solely by the Jastrow factor $|x_j - x_i|^\lambda$. We should like to emphasize that this expression is quite general for the
1/r^2 models. For example, the wavefunction for the SU(\nu) Sutherland model takes this form (99). We will see that this is also the case for excited states.

For the confined model of (108), the noninteracting wavefunctions is given by that for free electrons in the harmonic potential. Hence, we can expect that the ground-state wavefunction should take the form [34],[40],

$$\Psi_g = \prod_{j>i} |x_j - x_i|^{\lambda}(x_j - x_i)^{\delta_{\alpha_j\alpha_i}} \exp \left[ \frac{i}{2} \text{sgn}(\alpha_j - \alpha_i) \right] \prod_{i=1}^{N} \exp(-\frac{\omega}{2}x_i^2).$$  (117)

It has been shown that this wavefunction is indeed the exact eigenfunction of the confined model (108) [40]. Applying the kinetic term and the potential term on the wavefunction, we obtain

$$\frac{1}{\Psi_g} \left[ \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) \right] \Psi_g = \frac{\omega}{2} \left[ \lambda N(N-1) + \sum_{\alpha} N_{\alpha}^2 \right] - u, $$  (118)

where the term u is given by

$$u = \sum_{k<\ell} \frac{\lambda(\lambda - 1)}{(x_k - x_\ell)^2} + \sum_{k<\ell} \frac{2\lambda\delta_{\alpha_k\alpha_\ell}}{(x_k - x_\ell)^2} + \sum_{i\neq k\neq \ell} \frac{\lambda\delta_{\alpha_i\alpha_k}}{(x_i - x_k)(x_i - x_\ell)}. $$  (119)

On the other hand, the action of the interaction term yields [40]

$$\frac{1}{\Psi_g} \left[ \sum_{j>i} \frac{\lambda(\lambda + P_{ij}^{\alpha\beta})}{(x_j - x_i)^2} \right] \Psi_g = \sum_{k<\ell} \frac{\lambda(\lambda - 1)}{(x_k - x_\ell)^2} + \sum_{k<\ell} \frac{2\lambda\delta_{\alpha_k\alpha_\ell}}{(x_k - x_\ell)^2} $$

$$+ \sum_{k<\ell} \frac{\lambda}{(x_k - x_\ell)^2} \left[ 1 - \prod_{i\neq k\neq \ell} \frac{(x_i - x_\ell)(x_i - x_k)(x_i - x_\ell)}{(x_i - x_k)^2} \right] (1 - \delta_{\alpha_k\alpha_\ell}). $$  (120)

A remarkable point is that multiparticle terms in the above expressions (119) and (120) coming from the kinetic energy and the interaction energy cancel each other completely [40]. Consequently, it has been proven that the Jastrow wavefunction (117) is the exact eigenfunction of the Hamiltonian (108), and the corresponding energy is given by the expression (115). The detail of the calculation is given in ref. [40]. One can see evidence that the above eigenstate indeed corresponds to the ground state, though it is not easy to give a rigorous proof for it. For instance, in the case of spinless fermions, the wavefunction (117) reduces to the exact ground-state wavefunction [1],[27]. Also, in the limit of \( \omega \to 0, \)
we can show that (117) is the ground-state wavefunction. Based on these observations we believe that the eigenfunction generally describes the exact ground state of the Hamiltonian.

6.2.2. excited states

Concerning the eigenfunctions for excited states, we can also use a general form of the Jastrow wavefunction (116). In this case, it is necessary to introduce the exited wavefunctions of free electrons for \( \Psi_0 \). Therefore the following Jastrow wavefunction should be a candidate for the exact excited state \([41]\),

\[
\Psi = \prod_{j>i} |x_j - x_i|^\lambda \left[ \sum_{m_1+...+m_N=I} \prod_{i=1}^N \frac{1}{m_i!} H_{m_i}(\sqrt{\frac{m_i \omega}{\hbar}} x_i) \right] \Psi_g, \tag{121}
\]

where \( H_m \) is the Hermite polynomials which can produce the excited states systematically. It has been shown explicitly in \([41]\) that this wavefunction gives the exact excited state of the Hamiltonian (108) with the corresponding energy obtained by the RHO method (114). Therefore, the microscopic derivation of the eigenfunctions establishes that the RHO approach proposed here gives the exact solution to the confined \( 1/r^2 \) systems. We conclude this subsection by stressing that the general form of the wavefunction (116) may be quite helpful for constructing the excited states generally for the family of the \( 1/r^2 \) systems.

6.3. Application to conductance oscillations

Here, we briefly mention some attempts to apply the confined \( 1/r^2 \) model to the transport of electrons through a narrow channel of the semiconductor nanostructure. We first note that it is crucial to take into account the effects of mutual electron interactions to explain experiments of conductance oscillations \([68],[69]\). Also, it is known experimentally that one-dimensional electrons are confined to a finite segment by impurities or constrictions \([69]\). Hence it may be necessary to introduce a interacting electron model with certain confining potential. Motivated by the above experiments, the \( 1/r^2 \) models with harmonic confinement have been applied to conductance oscillations \([60],[61]\), and several characteristic properties have been explained. Subsequently, the effect of the internal spin degrees of freedom was taken into account correctly \([39],[40]\), and it was demonstrated that there can be two kinds
of periods in the conductance oscillations. We briefly summarize the results of refs. [39, 40]. We will be concerned with the SU(2) case of (108) which corresponds to the ordinary electron systems.

In order to consider the transport in narrow channels, a weak coupling is introduced between the segment given by the model Hamiltonian (108) and two reservoirs. Let us suppose that the conductance is controlled by a resonant tunneling between the one-dimensional segment and the reservoirs. Therefore a peak in conductance oscillations occurs when the chemical potential of the reservoirs satisfy the relation, \( \mu(N) = E_g(N + 1) - E_g(N) \). The spacing \( \delta \) of two successive peaks in the conductance oscillations is then given by \( \delta(N) = \mu(N + 1) - \mu(N) \). Using the formula (115) for the ground state energy for the SU(2) case (electrons), one can see that there appear two independent periods of the conductance oscillations [39, 40],

\[
\delta_1 = \omega \lambda, \quad \delta_2 = \omega(\lambda + 1),
\]

reflecting the exchange effect due to the internal spin degrees of freedom. We stress that this result improves those previously obtained by Tewari [66] and also by Johnson and Payne [67], who employed the spinless fermion models and concluded a single period for the conductance oscillations for any strength of the interaction.

For the parameters employed in ref. [67], two periods become \( \delta_1 = 7.5\omega \) and \( \delta_2 = 8.5\omega \) in the present model. Hence, the correction due to the exchange effect is small for these parameters (strong correlation regime). The exchange effect, however, becomes more conspicuous when the interaction becomes weaker (smaller \( \lambda \)), and then two periods becomes more distinct from each other. In the weak-coupling limit (\( \lambda \rightarrow 0 \)), the present model reproduces the results for free electrons in the harmonic well: \( \delta_1 \rightarrow 0, \delta_2 \rightarrow \omega \). It may be interesting to study experimentally whether two periods due to the exchange effect can be observed in the conductance oscillations.

§7. Summary

40
We have reviewed our recent works on the quantum $1/r^2$ models. We have successfully applied the ABA solution to the multicomponent models such as the SU($\nu$) spin chain, the Osp($\nu$, 1) supersymmetric $t$-$J$ model, and the hierarchical models related to the FQHE. Applying CFT techniques to the ABA equations, we have studied the critical behavior of this class of integral models. As for the confined models with harmonic potential, the RHO solution has been proposed, which enables us to construct the energy spectrum systematically. We have then proven, by explicitly constructing the eigenfunctions, that the RHO indeed gives the exact solution to the family of the confined $1/r^2$ models. The results have been used to discuss the exchange-correlation effects on the conductance oscillations in narrow channels.

Acknowledgements

We would like to express our sincere thanks to Y. Kuramoto, A. Okiji and K. Vacek and S.-K. Yang for useful discussions and fruitful collaborations. Valuable discussions with I. Affleck, H. Frahm, D. Haldane, P. Horsch, A. Klümper, V. Korepin, A. Schadschneider, S. Shastry and J. Zittartz are also acknowledged. This work was partly supported by Grant-in-Aid from the Ministry of Education, Science and Culture and also by Monbusho International Scientific Research Program.
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