BERNSTEIN-SATO IDEALS AND HYPERPLANE ARRANGEMENTS

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Abstract. We study the relation between zero loci of Bernstein-Sato ideals and roots of $b$-functions and obtain a criterion to guarantee that roots of $b$-functions of a reducible polynomial are determined by the zero locus of the associated Bernstein-Sato ideal. Applying the criterion together with a result of Maisonobe we prove that the set of roots of the $b$-function of a free hyperplane arrangement is determined by its intersection lattice.

We also study the zero loci of Bernstein-Sato ideals and the associated relative characteristic cycles for arbitrary central hyperplane arrangements. We prove the multivariable $n/d$ conjecture of Budur for complete factorizations of arbitrary hyperplane arrangements, which in turn proves the strong monodromy conjecture for the associated multivariable topological zeta functions.

1. Introduction

1.1. Bernstein-Sato ideals and diagonal specialization. Let $X$ be a smooth algebraic variety over $\mathbb{C}$ (or a complex manifold) and let $\mathbf{f} = (f_1, f_2, \ldots, f_r)$ be an $r$-tuple of regular functions (resp. holomorphic functions) on $X$. Then the Bernstein-Sato ideal of $\mathbf{f}$, denoted by $\mathcal{B}_f$, is the ideal in $\mathbb{C}[s_1, \ldots, s_r]$ generated by $b(s)$ satisfying

$$b(s)f^s = P \cdot f^{s+1},$$

for $P \in \mathcal{D}_X[s] = \mathcal{D}_X \otimes \mathbb{C}[s]$, where $s = (s_1, s_2, \ldots, s_r)$, $f^s = \prod_{i=1}^{r} f_i^{s_i}$ and $1_r = (1, 1, \ldots, 1)$. Or equivalently, the Bernstein-Sato ideal of $\mathbf{f}$ is the ideal of $\mathbb{C}[s]$-module annihilators for $\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}$ (see §3). When $r = 1$, the generator of the Bernstein-Sato ideal in $\mathbb{C}[s]$, denoted by $b_f(s)$, is called the Bernstein-Sato polynomial (or $b$-function) of the regular function (since $\mathbb{C}[s]$ is a PID).

We consider the diagonal embedding

$$\delta : \mathbb{C} \hookrightarrow \mathbb{C}^r \quad s \mapsto (s, s, \ldots, s).$$

Using the equation (1), we have that if $b(s) \in \mathcal{B}_f$, then $b(s, s, \ldots, s) \in \mathcal{B}_f$, where $f = \prod_{i=1}^{r} f_i$, which gives an inclusion of zero loci,

$$Z(\mathcal{B}_f) \subseteq \delta^{-1}(Z(\mathcal{B}_f)).$$

This process is called the specialization of Bernstein-Sato ideals, see Budur 7, §4.21 for more information.

Our main theorem is a criterion to ensure (2) being an equality.

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Theorem 1.1. With notations as above, if $\mathcal{O}_X[s]f^s$ is Cohen-Macaulay over $\mathcal{O}_X[s]$ (cf. Definition 2.3), then
\[ \delta^{-1}(Z(B_f)) = Z(B_f). \]

1.2. Hyperplane arrangements. A hyperplane arrangement $D$ is a finite collection of hyperplanes in $X = \mathbb{C}^n$, that is,
\[ D = \{D_1, D_2, \ldots, D_p\} \]
where $D_j$ are hyperplanes in $X$. Let $f_j$ be the linear polynomial defining $D_j$ and $f = \prod_{j=1}^p f_j^{m_j}$ with $m_j \geq 1$ so that the support of the divisor $(f = 0)$ is $D$. We also call $f$ a hyperplane arrangement. We set $f_D = \prod_j f_j$. We write by $L(D)$ the intersection lattice of a hyperplane arrangement $D$, that is,
\[ L(H) = \{ \bigcap_{H \in B} H \mid B \subseteq D \}. \]
Elements in $L(D)$ are called edges. For $W \in L(D)$, we write
\[ D_W = \{ H \in D \mid W \subseteq H \}, D^W = \{ H/W \mid H \in D_W \} \]
and $J(W) = \{ j \in \{1, 2, \ldots, p\} \mid W \subset D_j \}$. By definition, $D^W$ is a hyperplane arrangement in the quotient space $X/W$. The rank of $W$, denoted by $\text{rank}(W)$, is the codimension of $W$ in $X$. We set
\[ R_W = \left\{ -\frac{\text{rank}(D_W)}{|J(W)|}, -\frac{\text{rank}(D_W) + 1}{|J(W)|}, \ldots, -\frac{2J(W) - \text{rank}(D_W)}{|J(W)|} \right\}. \]

A hyperplane arrangement $D$ is central if $f_D$ is homogeneous and it is essential if $\{0\} \in L(D)$. A central hyperplane arrangement is irreducible if there is no linear change of coordinates on $\mathbb{C}^n$ such that $f$ can be written as the product of two non-constant polynomials in disjoint sets of variables. An edge $W \in L(D)$ is called dense if $D_W$ is irreducible. A central hyperplane arrangement $D$ is free if the underlying divisor is free in the sense of K. Saito [Sai80] (cf. Definition 5.1).

It is proved by Walther [Wal17] that the $b$-function of a hyperplane arrangement is not governed by its intersection lattice. By using Theorem 1.1 as well as Maisonobe’s formula for the generator of the Bernstein-Sato ideal of a free hyperplane arrangement (see Theorem 5.3), we obtain:

Theorem 1.2. With notations as above, if $D$ is a free hyperplane arrangement, then
\[ Z(B_{f_D}) = \bigcup_{\text{dense } W \in L(D)} R_W, \]
where the union goes over all dense edges in $L(D)$.

Recently, Bath obtained the same formula in [Bat19 Theorem 1.4] to calculate roots of $b$-functions for free hyperplane arrangements $D$ but from a different method. In loc. cit. Bath also generalized Maisonobe’s formula to possibly non-reduced free hyperplane arrangements. One can also generalize Theorem 1.2 for possibly non-reduced free hyperplane arrangements by using Theorem 1.1 see [Bat19 Eq.(4.24)].

The above theorem says nothing about the multiplicities. However, M. Saito [Sai16] proved that the multiplicity of the root $s = -1$ of the $b$-function for a central essential hyperplane arrangement is $n$. Furthermore, Terao’s famous conjecture predicts that the freeness of a hyperplane arrangement is determined from its intersection lattice. Motivated by Theorem 1.2, we conjecture the same phenomenon happening for $b$-functions of free hyperplane arrangements.
Conjecture 1.3. The $b$-function of a free hyperplane arrangement is determined from its intersection lattice.

Suppose that $f$ is a central hyperplane arrangement, not necessarily reduced. An $r$-tuple $f = (f_1, f_2, \ldots, f_r)$ is called a factorization of $f$ if each $f_j$ is a non-empty central hyperplane arrangement and $\prod_{j=1}^{r} f_j = f$. It is called a complete factorization if moreover each $f_j$ is linear (hence $\deg f = r$). For a complete factorization $f = (f_1, f_2, \ldots, f_r)$, we set

$$J(W,f) = \{ j \in \{1,2,\ldots,r\} \mid W \subset (f_j = 0) \}.$$  

Our next result is about the zero loci of Bernstein-Sato ideals for arbitrary central hyperplane arrangements.

Theorem 1.4. Suppose that $f = (f_1, f_2, \ldots, f_r)$ is a complete factorization of a central hyperplane arrangement $f$ (possibly nonreduced). If $W$ is a dense edge in its intersection lattice, then

$$(\sum_{j \in J(W,f)} s_j + \text{rank}(W) + k = 0) \subseteq Z(B_r).$$

for all $k = 0,1,\ldots,|J(W,f)| - 1$

The above theorem particularly proves the multivariable $n/d$-conjecture (a natural generalization of the $n/d$-Conjecture of Budur, Mustaţă and Teitler [BMT11]) for complete factorizations of central essential irreducible hyperplane arrangements (see [Bud15, Conjecture 1.13]).

Example. Take $f = (x,y,z,x+y+z)$ and $f = xyz(x+y+z)$ in $\mathbb{C}[x,y,z]$ (notice that $f$ is not free). One can compute with dmodideal.lib [DGPS19] and get that $B_r$ is principal and generated by

$$\prod_{i=1}^{4}(s_i + 1) \prod_{j=0}^{3}(s_1 + s_2 + s_3 + s_4 + 3 + j).$$

Then Theorem 1.4 is optimally verified.

For an $r$-tuple $f$, the modules $\mathscr{D}_X[\mathbb{C}[s]^r]$ and $\mathscr{D}_X[\mathbb{C}[s]^r]/\mathscr{D}_X[\mathbb{C}[s]^r+1]$ are both relative $\mathscr{D}$-modules over $\mathbb{C}[s]$. Maisonobe [Mai13a] proved that they are indeed relative holonomic (see §2.3 for definition). For relative holonomic $\mathscr{D}$-modules, their zero loci of Bernstein-Sato ideals are related to their relative characteristic varieties (cf. Lemma 2.2). When $f$ is a complete factorization of a central hyperplane arrangement, we obtain the following:

Theorem 1.5. Suppose that $f = (f_1, f_2, \ldots, f_r)$ is a complete factorization of a central hyperplane arrangement $f$ (possibly nonreduced) and that $W$ is a dense edge in its intersection lattice. Then for every $l \in \mathbb{Z}$, the subvariety of $T^*X \times \mathbb{C}^r$,

$$T^*_W(X \times (\sum_{j \in J(W,f)} s_j + l = 0))$$

is a component of $\text{Ch}^{\text{rel}}(\mathscr{D}_X[\mathbb{C}[s]^r]/\mathscr{D}_X[\mathbb{C}[s]^r+1])$ for every $k \gg l$, where $k_r = (k,k,\ldots,k)$ and $T^*_W X$ is the conormal bundle of $W$ in the cotangent bundle $T^*X$. 

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Moreover, the multiplicity of $T^*_W X \times (\sum_{j \in J(W; f)} s_j + l = 0)$ is

$$(-1)^{\text{rank}(W) - 1} \cdot \chi(\mathbb{P}(X/W) \setminus \bigcup_{H \in D^W} \mathbb{P}(H)),$$

where $\chi(\bullet)$ denotes the topological Euler characteristic. In particular,

$$(-1)^{\text{rank}(W) - 1} \cdot \chi(\mathbb{P}(X/W) \setminus \bigcup_{H \in D^W} \mathbb{P}(H)) > 0.$$

It is well known that $W$ is a dense edge if and only if

$$\chi(\mathbb{P}(X/W) \setminus \bigcup_{H \in D^W} \mathbb{P}(H)) \neq 0,$$

see [STV95, Proposition 2.6]. The above theorem gives a geometric interpretation of this nonzero number. But, the positivity result in the above Theorem can also be deduced directly from [FK00, Corollary 1.4] (see also [WZ19, Theorem 1.7]), as $D^W$ is a central essential hyperplane arrangement in $X/W$.

1.3. Multivariable Monodromy Conjecture. We recall the construction of the topological zeta function. See [Bud15] for the discussion of related topics.

Let $f = (f_1, f_2, \ldots, f_r)$ be an $r$-tuple of polynomials in $\mathbb{C}[x_1, x_2, \ldots, x_n]$. Denote $X = \mathbb{C}^n$, $D_i = (f_i = 0)$ and $D = \sum_{i=1}^r D_i$. Take a log resolution

$$\mu: (Y, F) \rightarrow (X, D),$$

i.e. $F$ has support of normal crossing divisors, with irreducible components $E_i$ for $i \in S$. Define integers $a_{ij}, k_i$ by

$$\mu^* D_j = \sum_i a_{ij} E_i, \quad K_{Y/X} = \sum_i k_i E_i,$$

where $K_{Y/X}$ is the canonical divisor of $\mu$. The topological zeta function of $f$ is

$$Z_f(s) := \sum_{I \subseteq S} \chi(E^o_I) \prod_{i \in I} \frac{1}{\sum_{j=1} a_{ij} s_j + k_i + 1},$$

where $E^o_I = \cap_{i \in I} E_i \setminus \cup_{i \in S \setminus I} E_i$. The rational function $Z_f(s)$ is independent of the choice of log resolutions. We denote by $\text{PL}(Z_f(s))$ the pole locus of $Z_f(s)$.

Conjecture 1.6. [Bud15, Conjecture 1.17]

$$\text{PL}(Z_f(s)) \subseteq Z(B_f).$$

When $r = 1$, the above conjecture is the Strong Monodromy Conjecture of Igusa-Denef-Loeser. It (even in the case $r = 1$) is widely open.

As a main application, we obtain:

Theorem 1.7. Suppose that $f = (f_1, f_2, \ldots, f_r)$ is a complete factorization of a hyperplane arrangement $f$ (possibly nonreduced). Then Conjecture 1.6 holds for $f$.

Since successive blowups along dense edges give a log resolution of a hyperplane arrangement [STV95, Theorem 3.1], the above theorem is a direct consequence of Theorem 1.3 (or one can apply [Bud15, Theorem 1.18]).

When $f$ is a tame hyperplane arrangement, Walther [Wal17] proved Conjecture 1.6 for $f$ by proving the $n/d$-Conjecture of Budur, Mustaţă and Teitler; based on Walther’s idea, Bath [Bat19] further proved Conjecture 1.6 for certain factorizations of $f$. 
At this moment, we cannot conclude Conjecture 1.6 for the hyperplane arrangement $f$ (namely $r = 1$) from Theorem 1.7 because we cannot exclude the possibility

$$\delta^{-1}(Z(B_f)) \supseteq Z(B_f)$$

where $\delta$ is the diagonal embedding as in Theorem 1.1. Conversely, it is not known whether Conjecture 1.6 for $f$ implies Theorem 1.7.

1.4. In Section 2, we recall the general theory of relative $\mathcal{D}$-modules. Section 3 is about Bernstein-Sato ideals and their relations with relative characteristic varieties. In Section 4, we discuss the diagonal specialization and prove Theorem 1.1. Section 5 is devoted to applications for hyperplane arrangements.

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2. Relative $\mathcal{D}$-modules and characteristic cycles

2.1. Suppose that $X$ is a smooth algebraic variety over $\mathbb{C}$ (or a complex manifold) of dimension $n$ and let $\mathcal{D}_X$ be the sheaf of rings of the algebraic (or analytic) differential operators on $X$. We let $R$ be a commutative noetherian $\mathbb{C}$-algebra integral domain such that the localization at every prime ideal is a regular local ring. We always assume that the Krull dimension of $R$ is finite. We then set

$$\mathcal{A}_R = \mathcal{D}_X \otimes \mathbb{C} R.$$

For instance, if $R = \mathbb{C}[s] = \mathbb{C}[s_1, \ldots, s_r]$ for some integer $r \geq 0$, then

$$\mathcal{A}_{\mathbb{C}[s]} = \mathcal{D}_X[s] := \mathcal{D}_X \otimes \mathbb{C}[s_1, \ldots, s_r].$$

A (left or right) $\mathcal{A}_R$-module $M$ is called coherent over $\mathcal{A}_R$ or relative coherent over $R$ (or Spec $R$) if $M$ is locally finite presented over $\mathcal{A}_R$. The definition of relative characteristic cycles is similar to that of $\mathcal{D}$-modules, that is, the case when $R = \mathbb{C}$; see [HTT08, §2] for this case. We recall the construction in general for completeness; see also [BVWZ19, §3].

The order filtration $F^\bullet$ on $\mathcal{D}_X$ induces a relative filtration on $\mathcal{A}_R$ by

$$F^\bullet \mathcal{A}_R = F^\bullet \mathcal{D}_X \otimes \mathbb{C} R,$$

in other words, elements in $R$ have degree zero. Now we assume that $M$ is a coherent $\mathcal{A}_R$-module. Then a relative good filtration of $M$ is a filtration $F^\rel_\bullet$ on $M$ compatible with $F^\rel_\bullet \mathcal{A}_R$ so that

$$\gr^\rel_\bullet M$$

is coherent over $\gr^\rel_\bullet \mathcal{A}_R \simeq \gr^\rel_\bullet \mathcal{D}_X \otimes \mathbb{C} R$.

Similar to the case for coherent $\mathcal{D}$-modules, good filtrations always exist in the algebraic case and locally in the analytic case for relative coherent $\mathcal{D}$-modules.

Since Spec($\gr^\rel_\bullet \mathcal{D}_X(s)$) $\simeq T^*X \times$ Spec $R$, the $\sim$-functor gives that $\gr^\rel_\bullet M$ a coherent $\mathcal{T}^*_X \times \mathbb{C}$-module. We then define the relative characteristic cycle of $M$ inside $T^*X \times$ Spec $R$ to be the cycle of the sum of irreducible components of the support of $\gr^\rel_\bullet M$ with multiplicities, denoted by $\text{CC}^\rel(M)$. We also write the relative characteristic variety of $M$, that is the support of $\text{CC}^\rel(M)$, by $\text{Ch}^\rel(M)$. Similar to the case for $\mathcal{D}$-modules (namely $R = \mathbb{C}$), one can check that $\text{Ch}^\rel(M)$
and $CC^{\text{rel}}(\mathcal{M})$ are independent of the choices of good filtration. In particular, $m_p(\mathcal{M})$ is independent of such choices, where $p$ is the generic point of an irreducible component of $\text{Ch}^{\text{rel}}(\mathcal{M})$ and $m_p$ is the multiplicity of $\text{gr}^{\mathcal{M}}$ at $p$. In the analytic case, $CC^{\text{rel}}(\mathcal{M})$ is a locally finite sum.

We denote by $CC^{\text{rel}}_k(\mathcal{M})$ the part of $CC^{\text{rel}}(\mathcal{M})$ with pure dimension $k$ and by $\text{Ch}^{\text{rel}}_k(\mathcal{M})$ the part of $\text{Ch}^{\text{rel}}(\mathcal{M})$ with pure dimension $k$. If $R$ is $\mathbb{C}$ or a field extension of $\mathbb{C}$, then we use $\text{Ch}$ and $CC$ to denote the characteristic variety and the characteristic cycle respectively for $\mathcal{D}$-modules on smooth varieties over the base field $R$.

It is worth mentioning that $\text{supp}(\text{gr}^{\mathcal{M}})$ might has embedded associated primes for some good filtration and the embedded associated primes depend on filtration. However, for pure modules, there at least exists one good filtration so that $\text{supp}(\text{gr}^{\mathcal{M}})$ is equi-dimensional without embedded associated primes; see [Bj93, Theorem A:IV 4.11].

2.2. Localization of relative $\mathcal{D}$-modules. We discuss localization of relative $\mathcal{D}$-modules as $\mathcal{A}_R$-modules and localization of relative characteristic cycles.

Suppose that $\mathcal{M}$ is a coherent $\mathcal{A}_R$-module. Let $q \subseteq \mathcal{A}_R$ be a prime ideal. Then the localization of $\mathcal{M}$ at $q$ is defined as

$$\mathcal{M}_q = \mathcal{M} \otimes R q$$

where $R_q$ is the localization of $R$ at $q$. Then $\mathcal{M}_q$ is a coherent $\mathcal{A}_R q$-module.

Define

$$B_M := \text{Ann}_R(\mathcal{M}) \subseteq R$$

the ideal of annihilators of $\mathcal{M}$ as an $R$-module, called the Bernstein-Sato ideal of $\mathcal{M}$ over $R$. By definition, $B_M$ localizes, that is, if $S \subseteq R$ is a multiplicative set, then

$$S^{-1}B_M = \text{Ann}_{S^{-1}R}(S^{-1}\mathcal{M}) \subseteq S^{-1}R.$$ 

In particular, if $q \subseteq R$ a prime ideal, then

$$(3) \quad B_{M,q} = \text{Ann}_{R_q}(\mathcal{M}_q) \subseteq R_q.$$ 

Recall that the support of $\mathcal{M}$ as a $R$-module is

$$\text{supp}_R(\mathcal{M}) = \{ \text{prime ideals } m \in \text{Spec } R | \mathcal{M}_m \neq 0 \}.$$ 

By abuse of notations, we also use $\text{supp}_R(\mathcal{M})$ to denote the set of closed points of maximal ideal in $\text{supp}_R(\mathcal{M})$ in Spec $R$. Since $B_M$ localizes, we immediately have

$$Z(B_M) \subseteq \text{supp}_R(\mathcal{M}),$$

where $Z(B_M)$ denotes the zero locus of the ideal $B_M$. However, since $\mathcal{M}$ might not be finite generated over $R$, in general we do not have $Z(B_M) = \text{supp}_R(\mathcal{M})$.

We write the relative characteristic cycle of $\mathcal{M}$ by

$$CC^{\text{rel}}(\mathcal{M}) = \sum_p m_p \bar{p},$$

where $p$ goes over the generic points of the irreducible components of the support of $\text{gr}^{\mathcal{M}}$ and $\bar{p}$ is its closure in $T^*X \times \text{Spec } R$. We also define the localization of the characteristic cycle by

$$CC^{\text{rel}}(\mathcal{M})_q = \sum_{p \subseteq p'} m_p \bar{p}$$
where \( p_2 : T^*X \times \text{Spec } R \to \text{Spec } R \) and \( \bar{p} \) is the closure of \( p \) inside \( T^*X \times \text{Spec } R \).

We take a good filtration \( F_\bullet M \) and then define
\[
F_\bullet M_q := (F_\bullet M)_q := (F_\bullet M) \otimes_R R_q,
\]
which is a good filtration on \( M_p \) as the localization functor is exact. We then immediately have
\[
\text{gr}^{rel}(M_q) = \text{gr}^{rel}(M)_q := \text{gr}^{rel}(M) \otimes_R R_q.
\]
As a consequence, we have
\[
(4) \quad \text{CC}^{rel}(M_q) = \text{CC}^{rel}(M)_q.
\]

2.3. Relative holonomicity. We recall the following definition from [BVWZ19], which is motivated by [Mai16a, Définition 1 and Proposition 8] and will play a key role in understanding Bernstein-Sato ideals.

**Definition 2.1.** [BVWZ19, Definition 3.2.3] Suppose that \( M \) is a coherent \( \mathcal{A}_R \)-module. We say that \( M \) is relative holonomic over \( R \) (or \( \text{Spec } R \)) if each irreducible component of \( \text{Ch}^{rel}(M) \) has a decomposition as \( \Lambda \times S \), where \( \Lambda \) is an irreducible conic Lagrangian in \( T^*X \) and \( S \) is an algebraic irreducible subvariety of \( \text{Spec } R \).

It is worth mentioning that if \( M \) is relative holonomic, then its graded number \( j(M) \geq n \), by Theorem 2.4(1). The category of all relative holonomic modules over \( R \) is an abelian category (see [BVWZ19, 3.2.4(1)])

**Lemma 2.2.** [BVWZ19, Lemma 3.4.1] If \( M \) is a relative holonomic \( \mathcal{A}_R \)-module, then
\[
Z(B_M) = p_2(\text{Ch}^{rel}(M)) \quad \text{and} \quad Z(B_M) = \text{supp}_R(M)
\]
where \( p_2 : T^*X \times \text{Spec } R \to \text{Spec } R \) is the natural projection.

2.4. Duality. We now discuss duality of \( \mathcal{A}_R \)-modules in a way compatible with the duality for usual \( \mathcal{D} \)-modules.

**Definition 2.3.** Assume that \( M \) is a non-zero coherent (left) \( \mathcal{A}_R \)-module.

1. The duality functor is
\[
\mathcal{D}(M) := \mathfrak{D} \text{hom}_{\mathcal{A}_R}(M, \mathcal{A}_R) \otimes \omega^{-1}_X[n],
\]
where \( \omega_X \) is the dualizing sheaf of \( X \).

2. The graded number of \( M \), denoted by \( j(M) \), is defined by
\[
j(M) = j_{\mathcal{A}_R}(M) := \min\{k|\mathcal{E}xt^k_{\mathcal{A}_R}(M, \mathcal{A}_R) \neq 0\}.
\]

3. For some \( j \in \mathbb{Z}_{\geq 0} \), \( M \) is called \( j \)-pure if
\[
j(N) = j(M) = j
\]
for every non-zero submodule \( N \subseteq M \).

4. For some \( j \in \mathbb{Z}_{\geq 0} \), \( M \) is called \( j \)-Cohen-Macaulay over \( \mathcal{A}_R \) if
\[
\mathcal{E}xt^k_{\mathcal{A}_R}(M, \mathcal{A}_R) = 0, \quad \text{for } k \neq j.
\]

One can define purity and Cohen-Macaulayness for \( \text{gr}^{rel}_{\mathcal{A}_R} \)-modules in a similar way. See [Bj93] Appendix IV in general.

Similar to the case for coherent \( \mathcal{D} \)-modules, we have:
Theorem 2.4. [BVWZ19, Theorem 3.2.2] Suppose that $\mathcal{M}$ is a coherent $\mathcal{A}_R$-module. Then

1. $j(\mathcal{M}) + \dim(\text{Ch}_{\text{rel}}(\mathcal{M})) = 2n + \dim(\text{Spec } R)$;
2. if $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ is a short exact sequence of coherent $\mathcal{D}_X[s]$-modules, then

$$\text{Ch}_{\text{rel}}(\mathcal{M}) = \text{Ch}_{\text{rel}}(\mathcal{M}') \cup \text{Ch}_{\text{rel}}(\mathcal{M}'')$$

and if $p$ is the generic point of an irreducible component of $\text{Ch}_{\text{rel}}(\mathcal{M})$ then

$$m_p(\mathcal{M}) = m_p(\mathcal{M}') + m_p(\mathcal{M}'').$$

Using Part(1) of the above theorem, one can easily see that $j$-Cohen-Macaulayness implies $j$-purity.

Since $R$ is in the center of $\mathcal{A}_R$, by definition, it is obvious that localization and duality commute, that is,

$$\mathcal{D}(\mathcal{M})_q \cong \mathcal{D}(\mathcal{M}_q)$$

where $q$ is a prime ideal in $R$.

We collect the following lemma for later usage.

Lemma 2.5. Suppose that we have a short exact sequence of relative holonomic $\mathcal{A}_R$-modules:

$$0 \to \mathcal{M}_1 \to \mathcal{M} \to \mathcal{M}_2 \to 0.$$ 

If $\mathcal{M}_1$ and $\mathcal{M}$ are Cohen-Macaulay, $j(\mathcal{M}_1) = j(\mathcal{M})$ and $j(\mathcal{M}_2) = j(\mathcal{M}) + 1$, then $\mathcal{M}_2$ is Cohen-Macaulay.

Proof. We apply $\mathcal{R}\text{hom}_{\mathcal{A}_R}(\mathcal{M}, \mathcal{A}_R)$ to the short exact sequence. By consider the associated long exact sequence, the required statement follows immediately. $\square$

Proposition 2.6. Assume that $\mathcal{M}$ is a relative holonomic $\mathcal{A}_R$-module. If $\mathcal{M}$ is $(n+k)$-pure for some integer $k \geq 0$, then $Z(B_{\mathcal{M}})$ is equi-dimensional of codimension $k$.

Proof. By purity, we can find a good filtration $F_* \mathcal{M}$ over $F_*^{\text{rel}} \mathcal{A}_R$ so that $\text{gr}^F_* \mathcal{M}$ is also $(n+r)$-pure over $gr^F_* \mathcal{A}_R$, thanks to [Bj93, A:IV. Theorem 4.11]. Then

$$\text{supp}(\text{gr}^F_* \mathcal{M}) = \text{Ch}_{\text{rel}}(\mathcal{M})$$

is equi-dimensional by [Bj93, A:IV. Proposition 3.7]. Then the required statement follows from the definition of relative holonomicity and Lemma 2.2. $\square$

2.5. Direct image functor and base change. Let $\mu: X \to Y$ be a morphism between smooth complex varieties. If $\mathcal{M}$ be a left relative $\mathcal{D}$-module on $X$ over $R$ (or more generally a complex of left relative $\mathcal{D}$-module on $X$ over $R$), then since $\mathcal{M}$ is particularly a $\mathcal{D}_X$-module, we define the direct image of $\mathcal{M}$ under $\mu$ by

$$\mu_+ (\mathcal{M}) = R\mu_* (\mathcal{M} \otimes_{\mathcal{O}_X} \omega_X \otimes_{\mathcal{O}_Y} \mu^* (\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}))$$

where $\omega_X$ (resp. $\omega_Y$) is the dualizing sheaf of $X$ (resp. $Y$). Namely, the direct images of relative $\mathcal{D}$-modules are just their direct images as absolute $\mathcal{D}$-modules. It is obvious that $\mu_+ (\mathcal{M})$ is a complex of left relative $\mathcal{D}$-module on $Y$ over $R$. Similar to the absolute case (see [Bj93, Theorem 2.3.15]), one can check that $\mu_+$ preserves coherence.
Theorem 2.7. Suppose that \( \mathcal{M} \) is a coherent relative \( \mathcal{D} \)-module on \( X \) over \( R \), and that \( \mu : X \to Y \) is a proper morphism between smooth complex varieties. Then we have a canonical isomorphism

\[
\mu_+(\mathcal{D}\mathcal{M}) \simeq \mathcal{D}(\mu_+\mathcal{M}).
\]

Proof. We first prove the case \( \mathcal{M} = \mathcal{N} \otimes_\mathcal{C} R \) for \( \mathcal{N} \) a coherent \( \mathcal{D}_X \)-module. Then

\[
\mathcal{D}(\mathcal{M}) \simeq \mathcal{D}(\mathcal{N}) \otimes_\mathcal{C} R \quad \text{and} \quad \mu_+\mathcal{M} = (\mu_+\mathcal{N}) \otimes_\mathcal{C} R.
\]

Thus

\[
\mu_+(\mathcal{D}\mathcal{M}) \simeq \mu_+(\mathcal{D}\mathcal{N}) \otimes_\mathcal{C} R.
\]

By the commutativity between the direct image and the duality functor for absolute \( \mathcal{D} \)-modules (see for instance \cite[Theorem 2.7.2]{HT08}), we have

\[
\mu_+(\mathcal{D}\mathcal{N}) \simeq \mathcal{D}(\mu_+\mathcal{N}).
\]

Therefore,

\[
\mu_+(\mathcal{D}\mathcal{M}) \simeq \mu_+(\mathcal{D}\mathcal{N}) \otimes_\mathcal{C} R \simeq \mathcal{D}(\mu_+\mathcal{N}) \otimes_\mathcal{C} R \simeq \mathcal{D}(\mu_+\mathcal{M}).
\]

We then can apply the method in the proof of \cite[Theorem 1.5.8]{Bj93} to get a finite resolution

\[
\mathcal{P}^\bullet \to \mathcal{M}
\]

such that each \( \mathcal{P}^i = \mathcal{D}_X \otimes_\mathcal{C} \mathcal{L}^i \otimes_\mathcal{C} R \) and each \( \mathcal{L}^i \) is a coherent \( \mathcal{O}_X \)-module. We then consider the distinguish triangle

\[
\tau_{\leq j}(\mathcal{P}^\bullet) \to \mathcal{P}^\bullet \to \tau_{> j}(\mathcal{P}^\bullet) \xrightarrow{\pm 1}
\]

for each \( j \), where \( \tau \) is the canonical truncation functor. One then can do induction on the length of \( \mathcal{P}^\bullet \) and reduce to the case we have proved. \( \square \)

Suppose that \( R \to S \) is a homomorphism between commutative rings and that \( \iota : \text{Spec } S \to \text{Spec } R \) is the induced morphism of schemes. We have the derived pullback functor defined for a \( \mathcal{A}_R \)-module \( \mathcal{M} \) by

\[
\mathcal{L}\iota^*(\mathcal{M}) := \mathcal{M} \otimes_R^L S.
\]

Then \( \mathcal{L}\iota^*(\mathcal{M}) \) is a complex of relative \( \mathcal{D} \)-modules over \( S \).

Proposition 2.8. Suppose that \( \mathcal{M} \) is a coherent relative \( \mathcal{D} \)-module over \( R \), that \( \mu : X \to Y \) is a proper morphism between smooth complex varieties and that \( \iota : \text{Spec } S \to \text{Spec } R \). Then we have a natural quasi-isomorphism

\[
\mathcal{L}\iota^*(\mu_+(\mathcal{M})) \overset{q.i.}{\cong} \mu_+(\mathcal{L}\iota^*(\mathcal{M})).
\]

Proof. By \cite[Proposition 2.5.13]{KS13}, we know

\[
\mathcal{L}\iota^*(\mu_+(\mathcal{M})) \overset{q.i.}{\simeq} \mathbb{R}\mu_*(\mathcal{M} \otimes_\mathcal{O} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mu^*(\mathcal{D}_Y \otimes_\mathcal{O} \mathcal{O}_Y^{-1}) \otimes_R^L S).
\]

Since \( R \) as well as \( S \) is in the center of \( \mathcal{A}_R \) (resp. \( \mathcal{A}_S \)),

\[
\mathbb{R}\mu_*(\mathcal{M} \otimes_\mathcal{O} \mathcal{O}_X \otimes_{\mathcal{O}_X} \mu^*(\mathcal{D}_Y \otimes_\mathcal{O} \mathcal{O}_Y^{-1}) \otimes_R^L S) \simeq \mu_+(\mathcal{M} \otimes_R^L S),
\]

and the proof is done. \( \square \)
3. The module $\mathscr{D}_X[s]f^s$

3.1. Relative characteristic cycles and Bernstein-Sato ideal. Suppose that $X$ is a smooth algebraic variety (or a complex manifold) of dimension $n$ and $f = (f_1, \ldots, f_r)$ is an $r$-tuple of regular functions (or germs of holomorphic functions in the analytic case) on $X$. We write by $D$ the divisor $(\prod_i f_i = 0)$ and $j: U = X \setminus D \hookrightarrow X$ the open embedding. We denote by

$$f^s = \prod_{i=1}^r f_i^{s_i}.$$ 

and write the $\mathcal{O}_X$-algebra by

$$j_*(\mathcal{O}_U[s]) := j_*(\mathcal{O}_U \otimes \mathbb{C}[s]) = j_*(\mathcal{O}_U \otimes \mathbb{C}[s]).$$

With the natural actions of differential operators, $j_*(\mathcal{O}_U[s]f^s)$, the free $j_*(\mathcal{O}_U[s])$-module generated by $f^s$, is a left $\mathscr{D}_X[s]$-module, but not necessarily a coherent $\mathscr{D}_X[s]$-module, that is, we assign

$$v(f^s) = \left( \sum_{i=1}^r v(f_i)/f_i \right) f^s$$

for vector fields $v$ on $X$.

We then consider the (left) coherent $\mathscr{D}_X[s]$-submodule generated by $f^s$,

$$\mathscr{D}_X[s]f^s \subseteq j_*(\mathcal{O}_U[s]f^s).$$

In general, for $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ we also consider the submodule generated by $f^{s+a} = \prod_i f_i^{s_i+a_i}$, denoted by $\mathscr{D}_X[s]f^{s+a}$.

**Remark 3.1.** In the analytic case, one can replace $j_*\mathcal{O}_U$ by

$$\mathcal{O}^an_{*D} := \mathcal{O}^an_{X}[1/ \prod_{i=1}^r f_i],$$

the algebraic localization of $\mathcal{O}^an$ along $D$, and construct $\mathcal{D}^an_{X}[s]f^{s+a}$ similarly.

If $a \geq b \in \mathbb{Z}^r$ (that is, $a_i \geq b_i$ for every $i \in \{1, 2, \ldots, r\}$), we obviously have

$$\mathscr{D}_X[s]f^{s-b} \subseteq \mathscr{D}_X[s]f^{s-a}.$$

We then write the quotient module by

$$\mathcal{M}_{\mathfrak{f}}^{a,b} := \frac{\mathscr{D}_X[s]f^{s-a}}{\mathscr{D}_X[s]f^{s-b}}.$$ 

As an $\mathcal{O}_X$-module, it is supported on $D_{a-b}$, where $D_a = (\prod_{i=0} f_i) = 0$.

The modules $\mathscr{D}_X[s]f^{s-a}$ and $\mathcal{M}_{\mathfrak{f}}^{a,b}$ are $\mathcal{O}$-modules over $\mathbb{C}[s]$. We denote the $\mathbb{C}[s]$-module annihilator of $\mathcal{M}_{\mathfrak{f}}^{a,b}$ by

$$B_{\mathfrak{f}}^{a,b} := B_{\mathcal{M}_{\mathfrak{f}}^{a,b}} = \text{Ann}_{\mathbb{C}[s]}(\mathcal{M}_{\mathfrak{f}}^{a,b}) \subseteq \mathbb{C}[s],$$

called the Bernstein-Sato ideals of $\mathfrak{f}$ with indices $a \geq b$.

For simplicity, we write $\mathcal{M}_{\mathfrak{f}}^{0,b}$ and $B_{\mathfrak{f}}^{0,b}$ by $B_{\mathfrak{f}}^b$ when $0 \geq b$, and $B_{\mathfrak{f}}^{-1}$ by $B_{\mathfrak{f}}$, where $1$ denotes the vector $(1, 1, \ldots, 1)$. We denote by $Z(B_{\mathfrak{f}}^{a,b})$ the zero locus of $B_{\mathfrak{f}}^{a,b}$ and by $Z_l(B_{\mathfrak{f}}^{a,b})$ the part of $Z(B_{\mathfrak{f}}^{a,b})$ with pure dimension $l$. 
When \( r = 1 \), \( B_f \) is a principal ideal in \( \mathbb{C}[s] \), we write the monic polynomial generating \( B_f \) by \( b_f(s) \); it is the usual Bernstein-Sato polynomial (or \( b \)-function) for \( f \) (see [Kas77]).

The following theorem is essentially due to Maisonobe with Part(2) improved in [BVWZ20]:

**Theorem 3.2.** We have

1. \( \mathcal{D}_X[s]^{r+\alpha} \) is relative holonomic and \( n \)-pure for every \( \alpha \in \mathbb{Z}^r \);
2. \( \text{CC}^r(\mathcal{D}_X[s]^{r+\alpha}) = \text{CC}(j_* \mathcal{O}_U) \times \mathbb{C}^r \);
3. \( \dim(\text{Ch}^r(\mathcal{M}^a_b)) = n + r - 1 \) for every pair \( a > b \in \mathbb{Z}^r \) and hence
   \[
   j(\mathcal{M}^a_b) = n + 1;
   \]
4. \( \mathcal{M}^a_b \) is relative holonomic for every pair \( a > b \in \mathbb{Z}^r \);
5. \( p_2(\text{Ch}^r(\mathcal{M}^a_b)) = Z(B^{a,b}_f) \) for every pair \( a > b \in \mathbb{Z}^r \), where
   \[
   p_2 : T^*X \times \mathbb{C}^r \to \mathbb{C}^r
   \]
   is the projection.

**Proof.** Part (1) and Part (2) for the relative characteristic varieties follow from [Mai16a, Résultat 1 and Proposition 14]. Part (3) follows from Résultat 2 in loc. cit. and Proposition 3.3 below, while Part (4) follows from Lemma 2.2. See [BVWZ20, Theorem 4.3.4] for Part (2) in cycles. □

**Proposition 3.3.** For \( a, b, c \in \mathbb{Z}^r \), if \( a \geq b \geq c \), then

\[
\text{CC}^r(\mathcal{M}^{a,b}) = \text{CC}^r(\mathcal{M}^{a,c}) + \text{CC}^r(\mathcal{M}^{b,c}),
\]
and

\[
Z_{r-1}(B^{a,c}_f) = Z_{r-1}(B^{a,b}_f) \cup Z_{r-1}(B^{b,c}_f).
\]

**Proof.** Since \( a \geq b \geq c \), we have a short exact sequence

\[
0 \to \mathcal{M}^{b,c}_f \to \mathcal{M}^{a,c}_f \to \mathcal{M}^{a,b}_f \to 0.
\]

Since \( \dim(\text{Ch}^r(\mathcal{M}^{a,b}_f)) = n + r - 1 \), the first statement follows from Theorem 2.4(2). The second then follows from Theorem 3.2(5). □

**Theorem 3.4.** (1) [Sab87] There exists a polynomial

\[
b(s) = \prod_{L, \alpha} (L \cdot s + \alpha) \in B_f
\]

where the product is over finite many (possibly repeated) \( L \in \mathbb{Z}_{\geq 0}^r \) and \( \alpha \in \mathbb{Q} \).

(2) [Gyo93] There exists a polynomial

\[
b(s) = \prod_{L, \alpha \in \mathbb{Q}_{>0}} (L \cdot s + \alpha) \in B_f
\]

where the product is over finite many \( L \in \mathbb{Z}_{\geq 0}^r \) and \( \alpha \in \mathbb{Q}_{>0} \).

By the above theorem and Theorem 3.2(3), (4) and (5) and Proposition 2.6, we immediately have:
Corollary 3.5. There exist decompositions
\[ \text{Ch}_{n+r-1}^{\text{rel}}(\mathcal{M}_f^{-1}) = \sum_{L \geq 0, \alpha > 0} \Lambda_{L, \alpha} \times (L \cdot s + \alpha = 0) \]
and
\[ \text{Ch}_{n+r-1}^{\text{rel}}(\mathcal{M}_f^{-k}) = \sum_{L \geq 0, \alpha \in \mathbb{Q}} \Lambda_{L, \alpha} \times (L \cdot s + \alpha = 0) \]
for all pairs \( a \geq b \) and \( \Lambda_{L, \alpha} \) are conic Lagrangian subvarieties in \( T^* X \), and
\[ Z_{r-1}(B_f) = \bigcup_{L \geq 0, \alpha > 0} (L \cdot s + \alpha = 0) \]
and
\[ Z_{r-1}(B_f^{a-b}) = \bigcup_{L \geq 0, \alpha \in \mathbb{Q}} (L \cdot s + \alpha = 0). \]

For \( i \in \{1, 2, \ldots, r\} \), we write by \( \mathbf{e}_i \) the unit vector with 1 at the position \( i \). Corollary \( \text{[Mai16a, Résultat 3]} \) says that:

Corollary 3.6. \( Z_{r-1}(B_f^{-\kappa_i}) \) is a finite union of divisors \((L \cdot s + \alpha = 0)\) with \( L > 0 \) and \( \alpha > 0 \in \mathbb{Q} \).

For higher codimensional parts, Maisonobe gave the following estimate (see [Mai16a, Résultat 3]):

Theorem 3.7 (Maisonobe). Every irreducible component of \( Z(B_f) \) of codimension \( > 1 \) can be translated by an element of \( Z^r \) inside a component of \( Z_{r-1}(B_f) \).

When \( \mathcal{M}_f^{-1} \) is pure, by Proposition 3.6 one can improve the result in the above theorem as follows.

Proposition 3.8. If the \( \mathcal{M}_f^{-1} \) is \((n+1)\)-pure, then
\[ Z(B_f) = Z_{r-1}(B_f). \]

3.2. Set of slopes and generalization of the log canonical threshold. We define the set of slopes
\[ S_{f} := \{ L \text{ primitive vectors in } \mathbb{Z}^r_{\geq 0} \mid (L \cdot s + \alpha = 0) \subseteq Z_{r-1}(B_f) \text{ for some } \alpha \in \mathbb{Q} \} \]
and define for \( i \in \{1, 2, \ldots, r\} \)
\[ S_{f,i} := \{ L \text{ primitive vectors in } \mathbb{Z}^r_{\geq 0} \mid (L \cdot s + \alpha = 0) \subseteq Z_{r-1}(B_f^{-\kappa_i}) \text{ for some } \alpha \in \mathbb{Q} \}. \]

Since \( L \in S_{f} \) are primitive in the lattice \( \mathbb{Z}^r \), \( S_{f} \) and \( S_{f,i} \) are finite sets. By Proposition 3.3, we have
\[ S_{f} = \bigcup_{i=1}^{r} S_{f,i}. \]

For \( i \in \{1, 2, \ldots, r\} \) and \( L > 0 \in \mathbb{Z}^r \), we define
\[ \kappa(L, i) = \kappa_{f}(L, i) := \begin{cases} \min\{ \alpha \mid (L \cdot s + \alpha = 0) \subseteq Z_{r-1}(B_f^{-\kappa_i}) \} & \text{if } L \in S_{f,i} \\ \infty & \text{if } L \notin S_{f,i} \end{cases} \]
and
\[ \kappa(L) = \kappa_{f}(L) := \min\{ \kappa(L, i) \mid i = 1, 2, \ldots, r \}. \]

By Eq. (5) and Corollary 3.6, if \( L \in S_{f} \), then for some \( i \)
\[ 0 < \kappa(L) = \kappa(L, i) < \infty. \]
Lemma 3.10. Suppose that \((L \cdot s + \alpha = 0)\) is a divisor in \(\mathbb{C}^r\) with \(L > 0 \in \mathbb{Z}^r\) and \(\alpha \in \mathbb{Q}\) and \(q\) is the prime ideal generated by \(L \cdot s + \alpha\). If \((L \cdot s + \alpha = 0) \subseteq \mathbb{Z}_{r-1} (B^{n_{L,\alpha}}_f)\)
then \(B^{n_{L,\alpha}}_{f,q} = ((L \cdot s + \alpha)^{n_{L,\alpha}}) \subseteq \mathbb{C}[s]_q\) for some integer \(n_{L,\alpha} > 0\). Moreover, if \((L \cdot s + \alpha = 0) \nsubseteq \mathbb{Z}_{r-1} (B^{n_{L,\alpha}}_f)\)
then \(B^{n_{L,\alpha}}_{f,q} = \mathbb{C}[s]_q\).

Proof. The second case is obvious by definition. In the first case, by definition we know \(B^{n_{L,\alpha}}_{f,q} \neq \mathbb{C}[s]_q\). Since \(\mathbb{C}[s]_q\) is a DVR and hence a PID, the required statement then follows. \(\square\)

Proposition 3.11. Let \(L \in S_f, \alpha \in \mathbb{Z}\), and \(q\) be the prime ideal in \(\mathbb{C}[s]\) generated by \(L \cdot s + \alpha\). Then for every \(a \in \mathbb{Z}^r\) such that \(L \cdot a < \kappa(L) - \alpha\), the modules
\[
\mathcal{D}_X[s]_q f^{\alpha-a}
\]
are all the same.

Proof. It suffices to prove that, for all \(a\) satisfying \(L \cdot a < \kappa(L) - \alpha\), and for every \(i \in \{1, \ldots, r\}\), we have
\[
\mathcal{D}_X[s]_q f^{\alpha-a} = \mathcal{D}_X[s]_q f^{\alpha-(a-e_i)}.
\]
Indeed, if so, then all lattice points with \(L \cdot a < \kappa(L) - \alpha\) are connected by these equalities.

The \(\supseteq\) direction is clear, we now prove \(\subseteq\). Note that
\[
\mathbb{Z}_{r-1}(B^{n_{L,\alpha}}_f) = \mathbb{Z}_{r-1}(B^{n_{L,\alpha}}_f - a),
\]
since one is replacing \(s\) by \(s - a\) in all the equations. Thus, the “first” hyperplane in the \(L\) direction that one encounters in \(\mathbb{Z}_{r-1}(B^{n_{L,\alpha}}_f)\) becomes \((L \cdot (s-a) + \kappa(L, i) = 0)\). Since
\[
\alpha < \kappa(L) - L \cdot a \leq \kappa(L, i) - L \cdot a,
\]
we have
\[
(L \cdot s + \alpha = 0) \nsubseteq \mathbb{Z}_{r-1} (B^{n_{L,\alpha}}_f).
\]
By Lemma 3.10 we obtain that \(M^{n_{L,\alpha}-e_i}_{f,q} = 0\) and hence
\[
\mathcal{D}_X[s]_q f^{\alpha-a-e_i} = \mathcal{D}_X[s]_q f^{\alpha-a}.
\]
\(\square\)

3.3. The maximal extension and the \(j_i\) extension. The module \(j_* \mathcal{O}_U[s] f^s\), called the maximal extension, is the ambient module where all \(\mathcal{D}_X[s]_q f^{s+a}\) live inside. Globally, it is not coherent over \(\mathcal{D}_X[s]\). However, after localization we have:

Theorem 3.12. Let \(q\) be a prime ideal in \(\mathbb{C}[s]\). Then \(j_* \mathcal{O}_U[s] f^s)_q\) is \(n\)-Cohen-Macaulay and
\[
j_*(\mathcal{O}_U[s] f^s)_q = \mathcal{D}_X[s]_q f^{s-k}
\]
for all \(k \gg 0\), where \(k = (k, k, \ldots, k)\).
Proof. For a maximal ideal \( m \), the equality
\[
j_*(\mathcal{O}_U[s]^r)_m = \mathcal{D}_X[s]^r_{m^{r-k}}
\]
is a special case of [WZ19, Theorem 5.3(ii)]. The \( n \)-Cohen-Macaulayness is contained in its proof in loc. cit. See also [BVWZ20, §5].

In general, we take a maximal ideal so that \( q \subseteq m \). Since duality and localization commute (cf. [24]), the proof is done. \( \square \)

For a prime ideal \( q \subseteq \mathbb{C}[s] \), we now define
\[
j_!(\mathcal{O}_U[s]^r)_q := \mathcal{D}_X(j_*(\mathcal{D}_U((\mathcal{O}_U[s]^r)_q)).
\]
One can easily see that \( \mathcal{O}_U[s]^r_q \) is \( n \)-Cohen-Macaulay (cf. [BVWZ20, Lemma 5.3.1]). Therefore, by Theorem 3.12, \( j_!(\mathcal{O}_U[s])_q \) is a sheaf (instead of a complex) and it is \( n \)-Cohen-Macaulay. Since \( \mathcal{D} \circ \mathcal{D} \) is identity, using the adjunction pair \((j^{-1}, j_*)\), we obtain a natural morphism
\[
j_!(\mathcal{O}_U[s]^r)_q \rightarrow j_*(\mathcal{O}_U[s]^r)_q.
\]

**Theorem 3.13.** Let \( q \) be a prime ideal in \( \mathbb{C}[s] \). Then \( j_!(\mathcal{O}_U[s]^r)_m \) is \( n \)-Cohen-Macaulay, the natural morphism
\[
j_!(\mathcal{O}_U[s]^r)_q \hookrightarrow j_*(\mathcal{O}_U[s]^r)_q
\]
is injective and
\[
j_!(\mathcal{O}_U[s]^r)_q = \mathcal{D}_X[s]^r_{q^{r+k}}
\]
for all \( k \gg 0 \), where \( k = (k, k, \ldots, k) \).

**Proof.** For a maximal ideal \( m \), the injectivity and the identity
\[
j_!(\mathcal{O}_U[s]^r)_m = \mathcal{D}_X[s]^r_{m^{r+k}}
\]
for all \( k \gg 0 \) follows from [WZ19, Theorem 5.4(iii)]. One then takes a maximal ideal \( m \) such that \( q \subseteq m \) and takes further localization. \( \square \)

### 4. Diagonal Specializations

We continue using notations introduced in [24] and suppose that \( f = (f_1, \ldots, f_r) \) is a \( r \)-tuple of regular functions (or holomorphic function) on a smooth complex variety (or a complex manifold) \( X \) for some \( r \geq 2 \).

Let us consider the following diagonal embedding:
\[
\Delta : \mathbb{C}^{r-1} \rightarrow \mathbb{C}^r, \ w \mapsto s, \ s_j = w_j \text{ for } j \leq r - 1 \text{ and } s_r = w_{r-1},
\]
where \( w = (w_1, \ldots, w_{r-1}) \) is the algebraic coordinates of the affine space \( \mathbb{C}^{r-1} \). We write by
\[
g = (f_1, \ldots, f_{r-2}, f_{r-1}f_r) \text{ and } g^w = \prod_{i=1}^{r-1} g_i^{w_i}.
\]

We assume that \( \mathcal{M} \) is a relative \( \mathcal{D} \)-module over \( \mathbb{C}^r \). Since it is particularly a \( \mathbb{C}[s] \)-module, we consider the pullback complex
\[
\mathbb{L}^\Delta^* \mathcal{M} := \mathcal{M} \otimes _{\mathbb{C}[s]} \mathbb{C}[w]
\]
is a complex of relative \( \mathcal{D} \)-module over \( \mathbb{C}^{r-1} \). We also write
\[
\Delta^* \mathcal{M} = \mathbb{L}^0 \Delta^* \mathcal{M} \simeq \mathcal{M} \otimes _{\mathbb{C}[s]} \mathbb{C}[w].
\]
Lemma 4.1. With notations as above, we have

\[ \Delta^* (j_*(\mathcal{O}_U[s]\mathfrak{f}^s)) \simeq j_*(\mathcal{O}_U[w]\mathfrak{g}^w) \]

as \( \mathcal{D}_X[w] \)-modules.

Proof. Since \( \mathcal{O}_U[s]\mathfrak{f}^s \) is a free \( \mathcal{O}_U[s] \)-module generated by \( \mathfrak{f}^s \) and the functor \( j_* \) is exact, \( \Delta^* (j_*(\mathcal{O}_U[s]\mathfrak{f}^s)) \) is a free \( j_*(\mathcal{O}_U[w]) \)-module of rank 1 and hence

\[ \Delta^* (j_*(\mathcal{O}_U[s]\mathfrak{f}^s)) \simeq j_*(\mathcal{O}_U[w]\mathfrak{g}^w). \]

It is obvious that the \( P \)-actions on both \( \Delta^* (j_*(\mathcal{O}_U[s]\mathfrak{f}^s)) \) and \( j_*(\mathcal{O}_U[w]\Delta^* \mathfrak{f}^s) \) are compatible with the above isomorphism for differential operators \( P \in \mathcal{D}_X \), and hence the above isomorphism of \( j_*(\mathcal{O}_U[w]) \)-modules is indeed an isomorphism of \( \mathcal{D}_X[w] \)-modules.

Lemma 4.2. With notations in Lemma 4.1, we have a natural surjective morphism

\[ \Delta^* (\mathcal{D}_X[s]\mathfrak{f}^s + a) \to \mathcal{D}_X[w](\mathfrak{f}^a \cdot \mathfrak{g}^w). \]

Proof. Pulling back the inclusion

\[ \mathcal{D}_X[s]\mathfrak{f}^s + a \hookrightarrow j_*(\mathcal{O}_U[s]\mathfrak{f}^s), \]

by Lemma 4.1 we get a natural morphism of \( \mathcal{D}_X[w] \)-modules,

\[ \Delta^* (\mathcal{D}_X[s]\mathfrak{f}^s + a) \to j_*(\mathcal{O}_U[w]\mathfrak{g}^w) \simeq \Delta^* (j_*(\mathcal{O}_U[s]\mathfrak{f}^s))), \]

whose image is obviously generated by \( \mathfrak{f}^a \cdot \mathfrak{g}^w \).

\[ \square \]

Theorem 4.3. With notations in Lemma 4.1, we assume that \( \mathcal{D}_X[s]\mathfrak{f}^s \) is \( n \)-Cohen-Macaulay over \( \mathcal{D}_X[s] \). Then \( \mathcal{D}_X[w]\mathfrak{g}^w \) is \( n \)-Cohen-Macaulay over \( \mathcal{D}_X[w] \) and

\[ Z(B_g) = \Delta^{-1}(Z(B_f)). \]

Proof. By construction of \( \Delta, \Delta(C^r-1) \) is the smooth divisor

\[ (s_{r-1} - s_r = 0). \]

Then for \( \mathbb{C}[s] \)-module \( \mathcal{M} \), we have

\[ \Delta^* \mathcal{M} \simeq \mathcal{M} \otimes_{\mathbb{C}[s]} \frac{\mathbb{C}[s]}{(s_{r-1} - s_r)}. \]
For $k \geq 0$ and $k := (k, k, \ldots, k)$, we consider the following commutative diagram

$$
\begin{array}{cccc}
0 & \to & \mathcal{D}_X[s][f^{a-k+1}] & \to & \mathcal{D}_X[s][f^{a-k}] & \to & M^{k-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Delta^*(\mathcal{D}_X[s][f^{a-k+1}]) & \to & \Delta^*(\mathcal{D}_X[s][f^{a-k}]) & \to & \Delta^*(M^{k-1}) & \to & 0
\end{array}
$$

Since $\mathcal{D}_X[s][f^{a-k}] \subseteq j_*(\mathcal{O}_U[s])f^a$ and the later is free over $\mathbb{C}[s]$, $\mathcal{D}_X[s][f^{a-k}]$ is torsion free for every $k = (k, k, \ldots, k) \in \mathbb{Z}^r$ and hence the first two columns are exact. Since $\mathcal{D}_X[s][f^a]$ is $n$-Cohen-Macaulay, so are $\mathcal{D}_X[s][f^{a-k}]$ by substitution. Hence, by Lemma 2.3, $M^{k-1}_f$ are $(n+1)$-Cohen-Macaulay. We then can apply [BVWZ19, Lemma 3.4.2] and conclude that the third column is also exact. Therefore, by 3×3 Lemma, the third row is also exact. We hence obtained a direct system of inclusions

$$
\{\Delta^*(\mathcal{D}_X[s][f^{a-k+1}]) \hookrightarrow \Delta^*(\mathcal{D}_X[s][f^{a-k}])\}_{k \geq 0}.
$$

Since

$$
\lim_{k \to \infty} \mathcal{D}_X[s][f^{a-k}] = j_*(\mathcal{O}_U[s])f^a,
$$

using the fact that direct limit functor is exact and commute with tensor-product we have inclusions

$$
\Delta^*(\mathcal{D}_X[s][f^{a+1}]) \hookrightarrow \Delta^*(\mathcal{D}_X[s][f^a]) \hookrightarrow \Delta^*(j_*(\mathcal{O}_U[s])f^a).
$$

By Lemma 4.1 and Lemma 4.2 we thus obtain that

$$
\Delta^*(\mathcal{D}_X[s][f^{a+1}]) \simeq \mathcal{D}_X[w][g^{w+1-r-1}] \text{ and } \Delta^*(\mathcal{D}_X[s][f^a]) \simeq \mathcal{D}_X[w][g^w],
$$

where $1_r = (1, 1, \ldots, 1) \in \mathbb{Z}^r$. Hence,

$$
\Delta^*(M^{1-r}_f) \simeq M^{1-r}_g.
$$

Since $\Delta^*(\mathcal{D}_X[s][f^a])$ is annihilated by $(s_{r-1} - s_r)$, we can apply the Rees theorem in homological algebra (see for instance [Rot08, Theorem 8.34]) and conclude that

$$
\mathcal{E}xt^{l}_{\mathcal{D}_X[w]}(\Delta^*(\mathcal{D}_X[s][f^a]), \mathcal{D}_X[w]) \simeq \mathcal{E}xt^{l+1}_{\mathcal{D}_X[s]}(\Delta^*(\mathcal{D}_X[s][f^a]), \mathcal{D}_X[s]).
$$

We consider the first column (for $k = 1$) of the above 3×3 diagram and conclude that $\Delta^*(\mathcal{D}_X[s][f^a])$ is $(n+1)$-Cohen-Macaulay over $\mathcal{D}_X[s]$ by Lemma 2.3. Thus, $\Delta^*(\mathcal{D}_X[s][f^a])$ and hence $\mathcal{D}_X[w][g^w] are n-Cohen-Macaulay over $\mathcal{D}_X[w]$.

We now consider the third column of the 3×3-diagram above for $k = 0$. Since $M^{1-r}_f$ is $(n+1)$-Cohen-Macaulay (and hence $(n+1)$-pure), we can apply Proposition 5.8. We thus conclude that $s_{r-1} - s_r$ is not contained in any minimal prime ideal containing $B_f$ by Corollary 4.3. Then we consider the morphism by multiplication,

$$
M^{1-r}_f \xrightarrow{(s_{r-1} - s_r)} M^{1-r}_f.
$$
Thanks to [BVWZ19] Lemma 3.4.2 again, we have a relative good filtration $F_*(\mathcal{M}_{\mathcal{T}}^{-1-r})$ over $F_\bullet \mathcal{D}_X[s]$ so that
\[ \text{gr}^F_*(\mathcal{M}_{\mathcal{T}}^{-1-r}) \xrightarrow{-(s_{r-1}s_r)} \text{gr}^F_*(\mathcal{M}_{\mathcal{T}}^{-1-r}) \]
is also injective. Therefore,
\[ \text{Ch}^\text{rel}(\mathcal{M}_g^{-1-r}) = \text{Ch}^\text{rel}(\Delta^*(\mathcal{M}^{-1})) = \text{Ch}^\text{rel}(\mathcal{M}^{-1})|_{s_{r-1}=s_r}. \]
By Lemma 2.2 we thus have
\[ Z(B_g) = \Delta^{-1}(Z(B_T)). \]
\[ \square \]

Recently, Bath [Bat19] Proposition 2.29 studied a similar specialization problem but with the Cohen-Macaulay hypothesis replaced by some geometric requirement (but more restrictive).

**Proof of Theorem 1.1.** We inductively apply Theorem 4.3 until $r = 1$ and Theorem 1.1 follows. \( \square \)

5. **Hyperplane arrangements**

5.1. **A Cohen-Macaulay criterion.** We recall a Cohen-Macaulay criterion of $\mathcal{D}_X[s]|\mathbb{F}^n$ with the help of using free divisors (in the sense of K. Saito [Sai80]). Since freeness is an analytic notion, we suppose that $f = (f_1, \ldots, f_r)$ is an $r$-tuple of holomorphic functions on $X = \mathbb{C}^n$ for some $r \geq 1$. We keep the notation from §3 but under the analytic setting (see Remark 3.1).

**Definition 5.1.** Suppose that $h$ is a holomorphic function on $X$ and $D_h$ is the divisor ($h = 0$). Denote by $\mathcal{I}_{D_h}$ the ideal sheaf of $D_h$. We let $\mathcal{I}_X(-\log D_h)$ be the sheaf of holomorphic logarithmic vector fields along $D_h$, that is, the sheaf is generated by vector fields $v$ so that $v \cdot \mathcal{I}_{D_h} \subseteq \mathcal{I}_{D_h}$. Then $h$, as well as $D_h$, is called free if $\mathcal{I}_X(-\log D_h)$ is a locally free $\mathcal{O}_X$-module.

The following theorem is first observed by Narvaéz-Macarro [Mac15] when $r = 1$; Maisonobe [Mai16b] generalized it in general.

**Theorem 5.2** (Maisonobe). Suppose that $f = (f_1, \ldots, f_r)$ is an $r$-tuple of holomorphic functions on $X = \mathbb{C}^n$. If $\prod_{i=1}^r f_i$ is locally quasi-homogeneous and free, then $\mathcal{D}_X[s]|\mathbb{F}^n$ is $n$-Cohen-Macaulay.

**Proof.** This lemma indeed holds unconditionally when $r = 1$. We first prove this case. By Theorem 3.2(1), $\mathcal{D}_X[s]|\mathbb{F}^n$ is $n$-pure. When $r = 1$, by for instance [BVWZ19] Lemma 4.3.3(2) and Auslander regularity, we know that
\[ j(\mathcal{E}xt_{\mathcal{D}_X[s]}^{n+j}((\mathcal{D}_X[s]f^s, \mathcal{D}_X[s])) > n + j. \]
If $\mathcal{E}xt_{\mathcal{D}_X[s]}^{n+j}((\mathcal{D}_X[s]f^s, \mathcal{D}_X[s])) \neq 0$ for $j > 0$, then by Theorem 2.4(1),
\[ \text{dim}(\text{Ch}^\text{rel}(\mathcal{E}xt_{\mathcal{D}_X[s]}^{n+j}((\mathcal{D}_X[s]f^s, \mathcal{D}_X[s]))) < n - j + 1. \]
But by [BVWZ19] Lemma 3.2.4(2)], $\mathcal{E}xt_{\mathcal{D}_X[s]}^{n+j}((\mathcal{D}_X[s]f^s, \mathcal{D}_X[s])$ is relative holonomic over $\mathbb{C}[s]$ and hence
\[ \text{dim}(\text{Ch}^\text{rel}(\mathcal{E}xt_{\mathcal{D}_X[s]}^{n+j}((\mathcal{D}_X[s]f^s, \mathcal{D}_X[s]))) > n. \]
This is a contradiction and hence $\mathcal{E}xt^{n+j}_{\mathcal{O}_X[s]}(\mathcal{O}_X[s]f^s, \mathcal{O}_X[s]) = 0$ for $j \geq 0$, which implies that $\mathcal{O}_X[s]f^s$ is $n$-Cohen-Macaulay.

So the real content of this lemma is for the case $r \geq 2$. When $r \geq 2$, it is implied immediately by [Mai16b Proposition 5]. Indeed, the Spencer complex in [Mai16b Proposition 5] is a length-$n$ free resolution of $\mathcal{O}_X[s]f^s$ as $\mathcal{O}_X[s]$-modules. Hence,

$$\mathcal{E}xt^{n+j}_{\mathcal{O}_X[s]}(\mathcal{O}_X[s]f^s, \mathcal{O}_X[s]) = 0,$$

for $j > 0$.

Since $j(\mathcal{O}_X[s]f^s) = n$, $\mathcal{O}_X[s]f^s$ is $n$-Cohen-Macaulay.

5.2. **Proof of Theorem 1.2.** Maisonobe obtained the following formula to compute the Bernstein-Sato ideals for free hyperplane arrangements. We write by $f_D = (f_1, f_2, \ldots)$ a complete factorization of $f_D$ for a hyperplane arrangement $D$.

**Theorem 5.3.** [Mai16b Théorème 1] Suppose that $D$ is a free hyperplane arrangement in $\mathbb{C}^n$. Then the Bernstein-Sato ideal $B_{f_D}$ is principal and generated by

$$\prod_{\text{dense } W \in L(D)} \prod_{j=0}^{2(|J(W)| - \text{rank}(W))} \left( \sum_{i \in J(W)} s_i + \text{rank}(W) + j \right).$$

Combining Theorem 1.1, Theorem 5.2 and Theorem 5.3 together, we conclude Theorem 1.2.

5.3. **Zero loci of Bernstein-Sato ideals for hyperplane arrangements.** The following lemma is a special case of [Bud15 Lemma 6.4].

**Lemma 5.4.** Let $f = (f_1, \ldots, f_r)$ be a complete factorization of an irreducible, essential, central hyperplane arrangement $f$. Then

$$\sum_{i=1}^{r} s_i + k = 0$$

defines an irreducible component of $Z(B_f)$ for some $k \in \mathbb{Z}_{>0}$.

We denote for a factorization $f = (f_1, \ldots, f_r)$ of a hyperplane arrangement $f$ and for $a \in \mathbb{Z}^r$

$$\mathcal{N}_a := \frac{\mathcal{O}_X[s]f^s - a}{\sum_{i=1}^{r} \mathcal{O}_X[s]f^s - a + e_i}.$$

**Lemma 5.5.** Let $f = (f_1, \ldots, f_r)$ be a complete factorization of a central hyperplane arrangement $f$. If a subset of all irreducible components of $f$ gives a coordinate system of $X = \mathbb{C}^n$, then

$$\sum_{i=1}^{r} (s_i - a_i) + n \in B_{\mathcal{N}_a}.$$

**Proof.** Without loss of generality we can assume that $(f_1, f_2, \ldots, f_n)$ give a coordinate system of $\mathbb{C}^n$, which we rename to $w_i$ to avoid confusion. One first observes that

$$0 = \sum_{i=1}^{r} s_i - \sum_{j=1}^{n} w_i \partial_{w_i} f^s = (\sum_{i=1}^{r} s_i + n - \sum_{j=1}^{n} \partial_{w_i} w_i) f^s.$$

Hence

$$\sum_{i=1}^{r} s_i + n f^s = \sum_{i=1}^{n} \partial_{w_i} f^{s+e_i}.$$
Thus,
\[ \sum_{i=1}^{r} s_i + n \in B_{N_0}. \]
For the general case, one substitutes \( s_i \) by \( s_i - a_i \).

**Theorem 5.6.** Let \( f = (f_1, \ldots, f_r) \) be a complete factorization of an irreducible, essential, central hyperplane arrangement \( f \). Then \( \kappa(1) = \kappa(1, i) = n \), where \( 1 = (1,1,\ldots,1) \in \mathbb{Z}^r \). In particular,
\[ \sum_{i=1}^{r} s_i + n = 0 \]
defines a component of \( Z_{r-1}(B_f) \).

**Proof.** We first prove that \( \kappa(1) = n \). By Lemma 5.4, we know \( 0 < \kappa(1) < \infty \). Suppose \( \kappa(1) \neq n \) and let \( q \) be the prime ideal generated by \( 1 \cdot s + \kappa(1) \). Then we have \( M_{f,q}^{-1} \neq 0 \). Since \( 1 \cdot s + n \notin q \), and it annihilates \( N_0 \) by Lemma 5.5, hence \( N_0,q = 0 \). However, thanks to Proposition 3.11, taking \( \alpha = \kappa(1) \), we have
\[ D_X[s]_{q} f^{s+e_i} = D_X[s]_{q} f^{s+1}, \quad \forall i = 1, \ldots, r. \]
Hence \( N_0,q = M_{f,q}^{-1} \), which is a contradiction. Hence \( \kappa(1) = n \).

By symmetry and Proposition 3.3, one can easily see \( \kappa(1) = \kappa(1, i) \) for all \( i \).

**Proof of Theorem 1.4.** We first deal with the case that \( f \) is central, essential and irreducible. For simplicity we write \( 1_l = (1,1,\ldots,1,0,0,\ldots,0) \in \mathbb{Z}^r \). For \( 0 < l \leq r \), we consider \( M_{f^{-1},1,-1}^{-1} \), which is a subquotient of \( M_f \). By Theorem 5.6 we know
\[ \left( \sum_{j=1}^{r} s_j + n = 0 \right) \subset Z(B_f^{n}). \]
By substitution, we hence know for \( 0 < l \leq r \)
\[ \left( \sum_{j=1}^{r} s_j + n + l - 1 = 0 \right) \subset Z(B_f^{-1,-1}). \]
By Proposition 3.3 we hence know
\[ \left( \sum_{j=1}^{r} s_j + n + l = 0 \right) \subset Z(B_f) \]
for \( 0 \leq l < r \). Since \( f \) is essential, one observes that the component \( \left( \sum_{j=1}^{r} s_j + n + l = 0 \right) \) is supported at the origin of \( \mathbb{C}^n \), that is, it is not a component of \( Z(B_f) \) over neighborhood away from the origin.
In general, we choose a dense edge and set
\[ f_W = (f_j)_{j \in J(W,f)} \text{ and } f_W = \prod_{j \in J(W,f)} f_j. \]
We then consider the complete factorization $f_W$ on $X/W$. Since $f_W$ is central, essential and irreducible on $X/W$,  
\[
\left( \sum_{j \in J(W,f)} s_j + \text{rank}(W) + l = 0 \right) \subset Z(B_{f_W})
\]
for $0 \leq l < |J(W,f)|$ and each component $\left( \sum_{j \in J(W,f)} s_j + \text{rank}(W) + l = 0 \right)$ is supported on $W$. Moreover, by definition one can see that $B_f$ and $B_{f_W}$ are the same over $X \setminus \cup_{j \geq W} D_j$. Since the component $\left( \sum_{j \in J(W,f)} s_j + \text{rank}(W) + l = 0 \right)$ of $Z(B_{f_W})$ is supported on $W$, we have  
\[
(8) \quad \left( \sum_{j \in J(W,f)} s_j + \text{rank}(W) + l = 0 \right) \subset Z(B_f)
\]
for $0 \leq l < |J(W,f)|$.  

5.4. Characteristic cycles for hyperplane arrangements. Suppose that $f$ is a complete factorization of a central essential irreducible hyperplane arrangement $f$ on $X = \mathbb{C}^n$ with $\text{supp}(f = 0) = D$. The canonical log resolution of $(X, D)$ is the morphism $\mu : Y \to X$ obtained by composition of the blowups along (the proper transform of) the union of dense edges in the increasing order of dimensions of dense edges. The canonical log resolution $\mu$ is a log resolution [STV95, Theorem 3.1]. We write  
\[
\tilde{f} = \mu^*f, \hat{f} = \mu^*f \text{ and } \tilde{j} : U = X \setminus D \hookrightarrow Y.
\]
We denote  
\[
\text{supp}(\mu^*D) = \sum_{i \in S} E_i \text{ and } E_I = \bigcap_{i \in I} E_i \text{ for } I \subseteq S.
\]
We set $q$ to be the prime ideal generated by $\sum_{j=1}^r s_j + 1$ in $\mathbb{C}[s]$.  

Now, we study the relative characteristic cycle $\text{CC}^\text{rel}(M_f)$.  

**Lemma 5.7.** With notations as above, we have:  
1. $j_*(\mathcal{O}_U[s]|_{\hat{f}^*\mathbb{C}}) \simeq \mathcal{O}_Y[s]|_{\tilde{f}^*\mathbb{C}}$;  
2. $j_!(\mathcal{O}_U[s]|_{\hat{f}^*\mathbb{C}}) \simeq \mathcal{O}_Y[s]|_{\tilde{f}^*\mathbb{C}^{+1}}$;  
3. for a general point $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in (\sum_{j=1}^r s_j + 1) \subset \mathbb{C}^r$, the $\mathcal{O}_Y$-module $M_{f,m}$ is supported on $E$ and the $\mathcal{O}_Y$-module $i^*(M_{f,m})$ are holonomic and supported on $E$ with  
\[
\text{CC}(i^*(M_{f,m})) = \sum_{E_i \subseteq E} T_{E_i} Y,
\]
where $E$ is the exceptional divisor over the origin $0 \in X = \mathbb{C}^n$, $m$ is the maximal ideal of $\alpha$ in $\mathbb{C}[s]$ and $i : \mathbb{C} \to \mathbb{C}^r$ the closed embedding defined by  
\[
s \mapsto (s, \alpha_2, \alpha_3, \ldots, \alpha_r);
\]
4. $\text{CC}^\text{rel}(M_{f,q}) = \sum_{E_i \subseteq E} T_{E_i} Y \times (\sum_{j=1}^r s_j + 1 = 0)$.

**Proof.** Since $\mu^*D$ is normal crossing, $\mathcal{O}_X[s]|_{\hat{f}^*\mathbb{C}}$ is Cohen-Macaulay and hence $M_{f}$ is also Cohen-Macaulay by Lemma 2.3 and we can calculate Bernstein-Sato ideals for $\hat{f}$ locally around any analytic neighborhood. Furthermore, by local computation, one can see that $B_f$ (on an analytic neighborhood) is principal, that its generator
is reduced, and that $\sum_{j=1}^{r} s_j + l$ is a factor of the generator only when $1 \leq l \leq r$. Therefore, by substitution

$$M_{f,q}^{k_r-k_r-1_r} = \frac{\mathcal{D}_{Y}[s]_{q}\tilde{f}^{\ast-k_r}}{\mathcal{D}_{Y}[s]_{q}\tilde{f}^{\ast+k_r+1_r}} = 0$$

for $k \neq 0$, where $k_r = (k, k, \ldots, k)$. Hence, Part (1) follows. Using the inclusion

$$\tilde{j}^\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_q \hookrightarrow \tilde{j}^\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_q$$

in Theorem 3.13 and duality, we see that $\tilde{j}^\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_q$ is the minimal extension of $\mathcal{O}_{U}[s]_{q}\tilde{f}^{\ast}$. By minimality and (9) for $k < 0$, Part (2) follows.

Now we prove Part (3) and Part (4). We pick a general point $\alpha$ and write $\mathcal{O}_{U}[\tilde{f}]$ explicitly and obtain

$$\tilde{j}^\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_q \hookrightarrow \tilde{j}^\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_q$$

by the local factorization at $\tilde{j}^\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_q$. Now we define

$$\nu: \mathcal{C} = \text{Spec } \mathbb{C}[s] \hookrightarrow \mathbb{C}^r$$

defined by

$$s \mapsto (s, \alpha_2, \alpha_3, \ldots, \alpha_r).$$

Since $M_{\tilde{f}}$ is Cohen-Macaulay, one can apply the method proving (7) in the proof of Theorem 4.3 (inductively using $s_i - \alpha_i = 0$ to cut $\mathbb{C}^r$ for $i \geq 2$) and obtain

$$\text{L}_{\nu}^\ast(M_{\tilde{f},m}) \cong \nu^\ast(M_{\tilde{f},m})$$

and

$$\text{CC}^\text{rel}(\nu^\ast(M_{\tilde{f},m})) = \nu^\ast(\text{CC}^\text{rel}(M_{\tilde{f},m})).$$

Since $\mu^\ast D$ is normal crossing, we can calculate the relative characteristic cycles of $M_{\tilde{f}}$ and $M_{\tilde{f},m}$ explicitly and obtain

$$\text{CC}^\text{rel}(M_{\tilde{f},m}) = \sum_{E_i \subseteq \mathbb{C}^r} T_{E_i} Y \times \left(\sum_{j=1}^{r} s_j + 1 = 0\right) \subseteq T^\ast X \times \text{Spec } \mathbb{C}[s]_{m}.$$  

In particular, Part (4) follows. We also obtain

$$\text{CC}^\text{rel}(\nu^\ast(M_{\tilde{f},m})) = \sum_{E_i \subseteq \mathbb{C}^r} T_{E_i} Y \times (s - \alpha_1) \subseteq T^\ast X \times \text{Spec } \mathbb{C}[s]_{\tilde{m}},$$

where $\tilde{m}$ is the maximal ideal of $\alpha_1 \in \mathbb{C}$. Thanks to Lemma 2.2 by considering its Bernstein-Sato ideal over $\mathbb{C}[s]_{\tilde{m}}$, $\nu^\ast(M_{\tilde{f},m})$ is annihilated by $(s - \alpha_1)^m$ for some $m \geq 1$. Hence, $\nu^\ast(M_{\tilde{f},m})$ is coherent over $\mathcal{D}_{Y}$ and thus a good filtration of $\nu^\ast(M_{\tilde{f},m})$ over $\mathcal{D}_{Y}$ is also good over $\mathcal{D}_{Y}[s]$. We further conclude

$$\text{CC}(\nu^\ast(M_{\tilde{f},m})) = \sum_{E_i \subseteq \mathbb{C}^r} T_{E_i} Y \subseteq T^\ast Y$$

and that $\nu^\ast(M_{\tilde{f},m})$ is a holonomic $\mathcal{D}_{Y}$-module (it is indeed regular holonomic).

\begin{lemma}
For every $\alpha \in \mathbb{C}^r$, we have

1. $\mu_+ (\tilde{j}_\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_m) \cong \tilde{j} \ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_m$

2. $\mu_+ (\tilde{j}_\ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_m) \cong \tilde{j} \ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_m$

3. $\mu_+ (M_{\tilde{f},m}) \cong \tilde{j} \ast(\mathcal{O}_{U}[s]\tilde{f}^{\ast})_m$, where $m$ is the maximal ideal of $\alpha$ in $\mathbb{C}[s]$.
\end{lemma}
Proof. Since \( \mu \) is identical over \( \mathcal{O} \), Part (1) is obvious.

By Theorem 2.7 we know that \( \mathbb{D} \) and \( \mu_+ \) commute. Thus, Part (2) follows from the definition of \( \overset{\sim}{j}_! \) (cf. §3.3). Since \( \mu_+ \) is an exact derived functor, by Lemma 5.7 (1) and (2) we have a distinguish triangle

\[
\mu_+(\overset{\sim}{j}_!(\mathcal{O}_U[s]\mathfrak{f}^*q)) \to \mu_+(\overset{\sim}{j}_!(\mathcal{O}_U[s]\mathfrak{f}^*q)) \to \mu_+(\mathcal{M}_{f,q}) \stackrel{+1}{\to}.
\]

Taking the associated long exact sequence of cohomology sheaves, by Part (1) and Part (2) that we have just proved, we obtain a long exact sequence

\[
0 \to \mathcal{H}^{-1}(\mu_+(\mathcal{M}_{f,q})) \to j_!(\mathcal{O}_U[s]\mathfrak{f}^*q) \to j_!(\mathcal{O}_U[s]\mathfrak{f}^*q) \to \mathcal{H}^0(\mu_+(\mathcal{M}_{f,q})) \to 0.
\]

By Theorem 3.13

\[
j_!(\mathcal{O}_U[s]\mathfrak{f}^*q) \to j_!(\mathcal{O}_U[s]\mathfrak{f}^*q)
\]

is injective. Thus, Part (3) follows. \( \square \)

Proof of Theorem 3.13. We first prove the case that \( f \) is central essential and irreducible. By substitution, it is enough to assume \( l = 1 \). We pick a general point \( \alpha \in (\sum_{j=1}^r s_j + 1 = 0) \), and write \( m \) the maximal ideal of \( \alpha \) in \( \mathbb{C}[s] \). By Lemma 5.4 \( (\sum_{j=1}^r s_j + 1 = 0) \) is a component of \( Z(B_f^{k_{-r}}) \) for all \( k \gg 1 \). By Theorem 3.12 and Theorem 3.13 we have

\[
\mathcal{M}_{f,q}^{k,-k_r} = \frac{\mathcal{O}_X[s]s^{f,-k_r}}{\mathcal{O}_X[s]s^{f,+k_r}} = \frac{j_!(\mathcal{O}_U[s]\mathfrak{f}^*q)}{j_!(\mathcal{O}_U[s]\mathfrak{f}^*q)}
\]

and

\[
\mathcal{M}_{f,m}^{k,-k_r} = \frac{\mathcal{O}_X[m]s^{f,-k_r}}{\mathcal{O}_X[m]s^{f,+k_r}} = \frac{j_!(\mathcal{O}_U[s]\mathfrak{f}^*m)}{j_!(\mathcal{O}_U[s]\mathfrak{f}^*m)}
\]

for all \( k \gg 1 \). Since \( \mathbb{C}[s]_q \) is a DVR and hence a PID, \( B_f^{k_{-r}} \) is generated by \( (\sum_{j=1}^r s_j + 1 = 0)m \) for some integer \( m \geq 1 \). Upstairs, \( \mathcal{M}_{f,m} \) is supported on \( E \) as a sheaf on \( Y \) by Lemma 5.7 (3). By Lemma 5.8 (3), \( j_!(\mathcal{O}_U[s]\mathfrak{f}^*m) \) is thus supported at the origin \( 0 \in X \) and hence so are \( \mathcal{M}_{f,m}^{k,-k_r} \) for all \( k \gg 1 \). Therefore, by Lemma 2.2 and relative holonomicity,

\[
T_{(0)}^r X \times (\sum_{j=1}^r s_j + 1 = 0)
\]

is a component of \( \text{Ch}^{rel}(\mathcal{M}_f^{k_{-r}}) \) for every \( k \gg 1 \). We assume that its multiplicity is \( \ell \). Then since relative characteristic cycles localize,

\[
\text{CC}^{rel}(\mathcal{M}_f^{k_{-r}}) = \text{CC}^{rel}(\frac{j_!(\mathcal{O}_U[s]\mathfrak{f}^*m)}{j_!(\mathcal{O}_U[s]\mathfrak{f}^*m)}) = \ell \cdot T_{(0)}^r X \times (\sum_{j=1}^r s_j + 1 = 0).
\]

Combining Theorem 3.12, Theorem 3.13 and Lemma 2.5 we see that \( j_!(\mathcal{O}_U[s]\mathfrak{f}^*m) \) is Cohen-Macaulay. Similar to (10), we obtain

\[
\mu_+(\mathcal{M}_{f,m}^{k,-k_r}) \overset{q}{\cong} \mu_+(\mathcal{M}_{f,m}^{k,-k_r}) \text{ and } \text{CC}^{rel}(\mu(\mathcal{M}_{f,m}^{k,-k_r})) = \mu(\text{CC}^{rel}(\mathcal{M}_f^{k_{-r}}))
\]
for $k \gg 1$. Similar to the way we prove that $\iota^*(M_{f,m})$ is coherent over $\mathcal{D}_Y$ in the proof of Lemma 5.7(3), we can also obtain that $\iota^*(M_{f,m}^k_r-k_r)$ is coherent over $\mathcal{D}_X$ for every $k \gg 1$. Therefore,

$$\text{CC}(\iota^*(M_{f,m}^k_r-k_r)) = \mu_+^E(\mathcal{N}_{E}) \approx \mathcal{N}_{(0)}.$$  

We now consider the following commutative diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\eta_E} & Y \\
\downarrow{\mu} & & \downarrow{\mu} \\
\{0\} & \xrightarrow{\eta} & X.
\end{array}
$$

By Lemma 5.7(3), $\iota^*(M_{f,m})$ is a holonomic $\mathcal{D}_Y$-module supported on $E$, by Kaishiwara’s equivalence (see for instance [HTT08, Theorem 1.6.1]),

$$\iota^*(M_{f,m}) \approx \eta_E_+ \mathcal{N}_E,$$

and $\text{CC}(\mathcal{N}_E) = \sum_{E \subseteq E_i} T_{E_i}^* E \subset T^* E$.

for some holonomic $\mathcal{D}_E$-module $\mathcal{N}_E$. Similarly,

$$\iota^*(M_{f,m}^k_r-k_r) \approx \eta_+ \mathcal{N}_{(0)}$$

for some $\mathbb{C}$-vector space $\mathcal{N}_{(0)}$ of dimension $\ell$. Since

$$\mathbb{L}_r^*(M_{f,m}^r-k_r) \approx \iota^*(M_{f,m}^r-k_r)$$

and $\mathbb{L}_r^*(M_{f,m}) \approx \iota^*(M_{f,m})$,

by Proposition 2.8 and Lemma 5.8(3), we have

$$\iota^*(M_{f,m}^k_r-k_r) \approx \mu_+(\iota^*(M_{f,m}))$$

for $k \gg 1$ and hence

$$\mu_+^E(\mathcal{N}_{E}) \approx \mathcal{N}_{(0)}.$$ 

We now apply the Dubson-Kashiwara index theorem (see for instance [Gin86, Theorem 9.1] and also [WZ19, Theorem 1.6] in the log case) and obtain that

$$\sum_i (-1)^i h^i(\mu_+^E(\mathcal{N}_{E})) = T_{E_1}^* E \cdot \sum_{E \subseteq E_i} T_{E_i}^* E$$

where $h^i$ denote the dimension of the $i$-th cohomology and the intersection number on the right hand side is the degree of the zero cycle in $T_{E_i}^* E \simeq E$. It is well known that

$$T_{E}^* E \cdot \sum_{E \subseteq E_i} T_{E_i}^* E = (-1)^{n-1} \chi(E^o),$$

where $E^o = E \setminus \bigcup_{E \subseteq E_i} E$. By the construction of $\mu$,

$$E^o = \mathbb{P}(X) \setminus \bigcup_{H \in D(0)} \mathbb{P}(H).$$

By (11), we then know the dimension of $\mathcal{N}_{(0)}$ and hence $\ell$ are both

$$(-1)^{n-1} \chi(\mathbb{P}(X) \setminus \bigcup_{H \in D(0)} \mathbb{P}(H)).$$

We thus have proved the case that $f$ is central essential and irreducible. In general, one can pick a dense edge $W$ and replace $f$ by $f_W$ and $X$ by $X/W$. Since
$X \to X/W$ is smooth, we get $T_w^*X$ by pulling back $T_{(0)}^*(X/W)$ and the general case follows. □

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