SOME REPRESENTATIONS OF MOORE-PENROSE INVERSE
FOR THE SUM OF TWO OPERATORS AND THE EXTENSION
OF THE FILL-FISHKIND FORMULA

ABDESSALAM KARA AND SAID GUEDJIBA
University of Batna 2, Faculty of Mathematics and Computer Sciences
Department of Mathematics, Algeria
(Communicated by Nan-Jing Huang)

Abstract. In the setting of arbitrary Hilbert spaces, we give a representation
of M-P inverse of the sum of linear operators \( A + B \) under suitable conditions.
Based on the full-rank decomposition of an operator, we prove that the ex-
tension of the Fill-Fishkind formula for \( A \) and \( B \) with closed ranges, remains
valid, keeping the same conditions of Fill-Fishkind formula for two matrices,
also we obtain an analogous formula under the Fill-Fishkind conditions, be-
yond we derive some representations of M-P inverse of a 2-by-2 block operator
with disjoint ranges.

1. Introduction and preliminaries. In this work, \( H \) and \( K \) are infinite dimen-
sional complex Hilbert spaces with inner products \( \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle_K \), \( B(H, K) \) denotes
the set of all linear bounded operators from \( H \) to \( K \). Let \( A, B \in B(H, K) \), the
Moore-Penrose inverse (for short M-P inverse) of a closed range operator
\( A \) is the unique operator \( A^\dagger \in B(K, H) \) satisfying the following four Penrose equations:

\[
\begin{align*}
(i) & \quad AA^\dagger A = A, \\
(ii) & \quad A^\dagger AA^\dagger = A^\dagger, \\
(iii) & \quad (AA^\dagger)^* = AA^\dagger, \\
(iv) & \quad (A^\dagger A)^* = A^\dagger A.
\end{align*}
\]

We denote by \( A^*, R(A), N(A) \), respectively, the adjoint, the range and the null-
space of \( A \), it is well known that \( A^\dagger \) exists for a given \( A \in B(H, K) \) if and only
if \( R(A) \) is closed; in this case \( R(A^\dagger A) = R(A^\dagger) = R(A^*) \), \( R(AA^\dagger) = R(A) \) and
\( N(A^\dagger A) = N(A), N(AA^\dagger) = N(A^\dagger) = N(A^*) \).

In [10]: Fill and Fishkind exhibit a neat relationship between the M-P inverse of
a sum of two square matrices \( A \) and \( B \) and the M-P inverse of the individual terms,
this is the Fill–Fishkind formula:

\[
(A + B)^\dagger = (I - S)A^\dagger(I - T) + SB^\dagger T, \tag{1}
\]

Provided that

\[
R(A) \cap R(B) = \{0\} \text{ and } R(A^*) \cap R(B^*) = \{0\}, \tag{2}
\]

where: \( S = (P_{N(B)}P_{N(A)})^\dagger \) and \( T = (P_{N(A^*)}P_{N(B^*)})^\dagger \).

Recently, in the setting of Hilbert spaces, Arias, Corach and Maestripieri in [[1],
Theorem 5.2] extend the Fill- fishkind formula to \( A \) and \( B \) with closed ranges,
satisfying the assumptions: (2), \( R(A + B) = R(A) + R(B), R(A^* + B^*) = R(A^*) +

\[
\text{2010 Mathematics Subject Classification. Primary: 15A09, Secondary: 47A05.}
\]

Key words and phrases. Moore-Penrose inverse; Closed range operator. Sum of operators,
Disjoint ranges.
Proof. We know that \( R(\lambda^*) \cap R(B^*) = \{0\} \) and \( R(\lambda + B) \) is closed, or these: \( \lambda \) and \( B \) coincide on \( R(\lambda^*) \cap R(B^*) \), \( R(\lambda) \cap R(B) = \{0\} \), and \( R(\lambda + B) = R(\lambda) + R(B) \), \( R(\lambda^* + B^*) = R(\lambda^*) + R(B^*) \). We will use the notion of full rank decomposition of operator to prove that if \( \lambda \) and \( B \) have closed ranges and \( (2) \) holds, then we have \( R(\lambda + B) = R(\lambda) + R(B), R(\lambda^* + B^*) = R(\lambda^*) + R(B^*) \) and the subspaces \( R(\lambda + B), R(\lambda) + R(B) \) and \( R(\lambda^* + B^*) \) are closed, also that the extension of the Fill-Fishkind formula for \( \lambda \) and \( B \) with closed ranges is valid keeping the conditions of Fill-Fishkind formula for two matrices, which are \( (2) \). On the other hand we get an analogous formula under \( (2) \) to Fill-Fishkind formula for \( \lambda \) and \( B \) having closed ranges and derive certain cases where operator ranges are orthogonal.

From the idea that \([8]\) the closed range operator admits matrix form with respect to the orthogonal sum of subspaces of \( H \) and \( K \), we obtain a representation of the \( M \)-\( P \) inverse of the sum of two operators \( \lambda \) and \( B \) satisfying: \( R(\lambda) \perp R(B) \) and \( R(\lambda) + R(B) \) is closed, hence under suitable conditions, we give a general representation of the \( M \)-\( P \) inverse of the sum \( \lambda + B \). Beyond, we consider a 2-by-2 block operator \( M \) as sum of two operators \( M = \begin{bmatrix} 0 & A_1 & A_2 \\ 0 & 0 & A_3 \\ A_4 & 0 \\ \end{bmatrix} \) and then, we give some representations of \( M \)-\( P \) inverse of \( M \) under the condition \( R(M_1^*) \cap R(M_2^*) = \{0\} \).

2. Some lemmas.

Lemma 2.1. Let \( P \in B(K) \) and \( Q \in B(H) \) be projectors, then
1) \( PA = A \Leftrightarrow R(A) \subset R(P) \),
2) \( AQ = A \Leftrightarrow N(Q) \subset N(A) \),
3) If \( P \) is an orthogonal projector and \( PA \) has a closed range, then \( (PA) = (PA)^{P} PA \).
4) If \( K = H \), then
   \[ P = Q \Leftrightarrow R(P) \subset R(Q) \text{ and } N(P) \subset N(Q) \).

Definition 2.2. We say that \( \lambda \) and \( B \) are disjoint ranges if \( R(\lambda) \cap R(B) = \{0\} \), we denote by \( DR \) the set of all these pairs \( (\lambda, B) \); i.e.,
\[ DR := \{(\lambda, B) : \lambda, B \in B(H, K) \text{ and } R(\lambda) \cap R(B) = \{0\}\} \].

Lemma 2.3. Let \( \lambda \) has closed range, then the next statements are equivalent
1) \( (\lambda, B) \in DR \),
2) \( \bar{R}(\lambda^*) = \bar{R}(\lambda^*) P_{\lambda}(\lambda) \),
3) \( N(B) = N(P_{\lambda}(\lambda) B) \).

Proof. We know that \( \bar{R}(\lambda^*) = \bar{N}(B)^{P} \) and \( \bar{R}(\lambda^*) P_{\lambda}(\lambda) = N(P_{\lambda}(\lambda) B) \), then \( 2) \Leftrightarrow 3) \). Using absurd reasoning to proof both implications of the equivalence \( 1) \Leftrightarrow 3) \), first, \( \Rightarrow \): Let \( x \in H \) satisfies \( P_{\lambda}(\lambda) Bx = 0 \) and \( Bx \neq 0 \), which implies that \( \lambda^* Bx = Bx \) and \( Bx \neq 0 \), it follows that \( Ax = Bx \neq 0 \); where \( x' = \lambda Bx \), therefore contradiction with the assertion \( 1) \). Second, \( \Leftrightarrow \): Let \( y \in R(\lambda) \cap R(B) \neq \{0\} \), there exist \( x_1 \neq 0 \) and \( x_2 \neq 0 \) such that \( Ax_1 = Bx_2 \neq 0 \), form the equation \( (i) \) of Penrose, we obtain \( Ax_1 = Bx_2 \neq 0 \), then \( P_{\lambda}(\lambda) Bx = 0 \) and \( Bx \neq 0 \); hence contradiction. \( \square \)
Lemma 2.4. ([6], Theorem 22) If A and B have closed ranges, then the following statements are equivalent
1) \( AB \) has closed range,
2) \( N(A) + R(B) \) is closed.
3) \( N(B^*) + R(A^*) \) is closed.

We assume that \( Z \) of lemma 1, we find that (to see that equality (4) on the right by \( Z \) hence, the multiplication of the equality (4) on the left by \( M \), we suppose that \( P \) is injective if and only if \( N(A^*) \) is closed, which is equivalent, by lemma 2.3 to at each one of these equalities.

\[
\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in B(H \oplus L, K \oplus F).
\] (3)

Lemma 2.5. We assume that \( A_1 \) and \( A_2 \) are injective, then \( M = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} \) is injective if and only if \( R(A_1) \cap R(A_2) = \{0\} \).

Proof. We suppose that \( R(A_1) \cap R(A_2) \neq \{0\} \). This means that there exist \( x \notin N(A_1) \) and \( x' \notin N(A_2) \) suth that \( A_1 x = A_2(x') \neq 0 \) or \( A_1 x + A_2(-x') = 0 \) Which is equivalent to the existence of \( \left( x; -x' \right) \neq (0; 0) \) with \( M(\cdot, x') = (0; 0) \), that is to say, \( M \) is not injective.

Lemma 2.6. Let \( M = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix} \) be a 2-by-2 block row operator suth that \( R(A_1), R(A_2) \) are closed, then
\[
R(M) \text{ is closed } \iff R(P_{N(A_2)}A_1) \text{ is closed } \iff R(P_{N(A_2^*)}A_2) \text{ is closed.}
\]

If \( (A_1, A_2) \in DR \), we get
\[
M^\dagger = \begin{pmatrix} (P_{N(A_2^*)}A_1)^\dagger & 0 \\ (P_{N(A_2^*)}A_2)^\dagger & 0 \end{pmatrix} := Z.
\]

Proof. We have \( R(M) = R(A_1) + R(A_2) \oplus \{0\} \), so \( R(M) \) is closed if and only if \( R(A_1) + R(A_2) \) is closed, which is equivalent by the lemma 2.4 to \( P_{N(A_2)}A_1 \) (resp, \( P_{N(A_2^*)}A_2 \)) have closed ranges. Now we will see that \( Z \) satisfies the equations of M-P inverse of \( M \), firstly, applying the item 3 of lemma 2.1 we get that \( Z \) satisfies the equation (iv):

\[
ZM = \begin{pmatrix} (P_{N(A_2^*)}A_1)^\dagger P_{N(A_2^*)}A_1 & 0 \\ 0 & (P_{N(A_2^*)}A_2)^\dagger P_{N(A_2^*)}A_2 \end{pmatrix}.
\] (4)

Remark that \( (A_1, A_2) \in DR \) is equivalent, by lemma 2.3 to at each one of these equalities
\[
N((P_{N(A_2^*)}A_1)^\dagger P_{N(A_2^*)}A_1) = N(A_1) \quad \text{and} \quad N((P_{N(A_2^*)}A_2)^\dagger P_{N(A_2^*)}A_2) = N(A_2).
\]

Hence, the multiplication of the equality (4) on the left by \( M \) and using the item 2 of lemma 1, we find that \( Z \) satisfies the equation (i), and the multiplication of the equality (4) on the right by \( Z \) we find that \( Z \) satisfies the equation (ii). It remains to see that \( Z \) satisfies the equation (iii), it results from the equations (i) and (ii) that \( MZ \) which has the matrix form below is a projection
\[
MZ = \begin{pmatrix} A_1(P_{N(A_2^*)}A_1)^\dagger + A_2(P_{N(A_2^*)}A_2)^\dagger & 0 \\ 0 & 0 \end{pmatrix}.
\]
And we have

\[ R(MZ) = R(M), \]
\[ N(MZ) = N(Z) = N(A_1(P_{N(A_2^*)}A_1)^\dagger + A_2(P_{N(A_3^*)}A_2)^\dagger) \oplus F. \]

We consider the orthogonal projection

\[ Q = \begin{bmatrix} P_{R(A_1) + R(A_2), N(A_2^*) \cap N(A_2^*)} & 0 \\ 0 & 0 \end{bmatrix} \in B(K \oplus F, K \oplus F). \]

From where

\[ R(Q) = R(M) \text{ and } N(Q) = N(A_1^*) \cap N(A_2^*) \oplus F, \]

We will see that \( MZ = Q \). These inclusions are easy to check

\[ N(A_1^*) \cap N(A_2^*) \subset N(A_1^*P_{N(A_2^*)}A_1) \subset N(A_1(P_{N(A_2^*)}A_1)^\dagger). \]

And

\[ N(A_1^*) \cap N(A_2^*) \subset N(A_2^*P_{N(A_1^*)}A_2) \subset N(A_2(P_{N(A_1^*)}A_2)^\dagger), \]

Hence

\[ N(A_1^*) \cap N(A_2^*) \subset N(A_1(P_{N(A_2^*)}A_1)^\dagger + A_2(P_{N(A_1^*)}A_2)^\dagger). \]

which implies that

\[ N(A_1^*) \cap N(A_2^*) \oplus F \subset N(MZ). \]

Consequently, \( N(Q) \subset N(MZ), R(Q) = R(M) \) it follows from the item 4 of lemma 2.1 that \( MZ = Q \).

\[ \square \]

**Lemma 2.7.** In the 2-by-2 block operator \( M = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix} \), we assume that \( A_1 \) is invertible and \( A_4 \) has closed range, then

\[ M^\dagger = \begin{bmatrix} G^{-1}A_1^* & G^{-1}D^* \\ -A_2^*A_3G^{-1}A_1^* & A_1^* - A_2^*A_3G^{-1}D^* \end{bmatrix}, \]

where: \( D = P_{N(A_3^*)}A_3 \), \( G = A_1^*A_1 + D^*D \).

**Proof.** Simple verification. \( \square \)

3. Some properties of full-rank decomposition of operators.

**Definition 3.1.** If there exist a Hilbert space \( H_A \) and operators \( G_A \in B(H, H_A) \); \( F_A \in B(H, K) \), such that \( G_A \) is right invertible, \( F_A \) is left invertible and

\[ A = F_AG_A \quad (5) \]

Then we say that (5) is a full-rank decomposition of \( A \).

The full-rank decomposition plays an important role in the theory of the generalized inverses, in particular for determining the expressions of M-P inverse of an operator; for more see [2], [4]. We recall that in [4], Caradas has proved that an operator \( A \in B(H, K) \) admits a full-rank decomposition if and only if there exists an operator \( X \in B(K, H) \) satisfying the equation \((i)\), means that the operator \( A \in B(H, K) \) admits a full-rank decomposition if and only if \( R(A) \) is closed, Dordević and Stanimirović mentioned in [9]; Theorem 2.1, (d) \( \), if \( F_AG_A \) is a full-rank decomposition of \( A \), then \( G_A^*F_A^\dagger \) is the M-P inverse of \( A \); (i.e., \( A^\dagger = G_A^*F_A^\dagger \)). Using below the concept of full-rank decomposition to collect a set of results:
Lemma 3.2. A has a full-rank decomposition if and only if $A^\dagger$ exists, and if $F_A G_A$ is a full-rank decomposition of $A$, then
1) $F_A^\dagger F_A$ and $G_A G_A^\dagger$ are invertible.
2) $F_A^\dagger$ is a left inverse of $F_A$, also $G_A^\dagger$ is a right inverse of $G_A$.
3) We have $R(A) = R(F_A)$, $N(A) = N(G_A)$. $R(A^*) = R(G_A^*)$ and $N(A^*) = N(F_A^*)$.
4) $A^\dagger A = G_A^\dagger, A$ and $A A^\dagger = F_A F_A^\dagger$.

We use this lemma below in the proof of theorem 5.3, to prove the identity (18).

Lemma 3.3. Let $F_A G_A$, $F_B G_B$ be full-rank decompositions of $A$ and $B$, respectively, then
a) we have

\[ R(P_{(B^*)}^A) = R(P_{(B^*)}^F_A), R(P_{(A^*)}^B) = R(P_{(A^*)}^F_B) \]

And

\[ R(P_{(A^*)}^B) = R(P_{(A^*)}^G_B), R(P_{(B^*)}^A) = R(P_{(B^*)}^G_A) \]

b) We suppose that $P_{(B^*)}^A$ has a closed range and $(A, B) \in DR$, then we have

\[ \langle P_{(B^*)}^A F_A \rangle = G_A \langle P_{(B^*)}^A \rangle, \langle P_{(A^*)}^F_B \rangle = G_B \langle P_{(A^*)}^B \rangle \]

c) We suppose that $BP_{(A^*)}$ has a closed range and $(A^*, B^*) \in DR$, then we have

\[ \langle G_B P_{(A^*)} \rangle = (BP_{(A^*)})^\dagger F_B, \langle G_A P_{(B^*)} \rangle = (AP_{(B^*)})^\dagger F_A \]

Proof. a) The equality $R(P_{(B^*)}^A) = R(P_{(B^*)}^F_A)$ is proved as follows

\[ R(P_{(B^*)}^A) = P_{(B^*)}^A R(F_A) = P_{(B^*)}^A R(F_A F_A^\dagger) = P_{(B^*)}^A R(A A^\dagger) = P_{(B^*)}^A R(A) = R(P_{(B^*)}^A) \]

Similarly, we can have the other equality.

b) Let $U = P_{(B^*)} A$ and $V = G_A^\dagger$, we have

\[ R((P_{(B^*)}^A)^* P_{(B^*)}^A) U V^* U^\dagger \subset R(A^* P_{(B^*)}^A) \subset R(A^*) = R(G_A^\dagger) \]

So, we deduce that

\[ R(U^* V V^* U^\dagger) \subset R(V) \]

Now, note that $R(G_A^\dagger G_A^\dagger (P_{(B^*)}^A)^*) \subset R(G_A^\dagger) = R(A^*)$ and by the item 3 of lemma 3.2 we get $R( G_A^\dagger G_A^\dagger (P_{(B^*)}^A)^*) \subset R(A^*)$ On the other hand since $(A, B) \in DR$, it follows from the item 2 of lemma 2.3 that $R(G_A^\dagger G_A^\dagger (P_{(B^*)}^A)^*) \subset R((P_{(B^*)}^A)^*)$ that is

\[ R(V V^* U^*) \subset R(U^*) \]

According to (6) and (7) and [[8], item (4) of Theorem 2.2] then, $U$ and $V$ satisfy the reverse order law $(UV)^\dagger = V^\dagger U^\dagger$, that is $(P_{(B^*)}^A AG_A^\dagger)^\dagger = G_A (P_{(B^*)}^A)^\dagger$ While $P_{(B^*)}^A F_A = P_{(B^*)}^A G_A^\dagger$, so the equality $(P_{(B^*)}^A)^\dagger = G_A (P_{(B^*)}^A)^\dagger$ holds. In the same way we get that $(P_{(A^*)}^B)^\dagger = G_B (P_{(A^*)}^B)^\dagger$. Taking the adjoint on both sides of the equalities of item c) and we use the item b) we obtain

\[ (P_{(A^*)}^B)^\dagger = F_B (P_{(A^*)}^B)^\dagger \] and \[ (P_{(B^*)}^A)^\dagger = G_A^\dagger (P_{(B^*)}^A)^\dagger. \]

We take again the adjoints on both sides of two last equalities, obtaining the item c).
4. Representations of M-P inverse of the sum of two operators. In this section we assume that the operator $A$ has a closed range, the operator $A$ has the following matrix form with respect to the orthogonal sums $K = R(A) \oplus N(A^*)$ and $H = R(A^*) \oplus N(A)$:

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \rightarrow \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix},$$

(8)

where $A_{11}$ is invertible. Moreover,

$$A^\dagger = \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix} \rightarrow \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix}.$$ \hspace{1cm} (9)

To obtain the identity (10), we will use the matrix forms of $A$ and $B$ with respect to the orthogonal sums above of $K$ and $H$, to transform the sum $A + B$ into a 2-by-2 block operator block, which is the 11, hence by the lemma 2.7 we get 13 which is equivalent by identification to (10).

**Theorem 4.1.** If $R(A) \perp R(B)$; then $(A + B)^\dagger$ exists if and only if $\Omega_A^\dagger$ exists, and $(A + B)^\dagger$ can be expressed as:

$$(A + B)^\dagger = \Omega_A^\dagger + (I - \Omega_A^\dagger B)J_A^\dagger(\Delta_A^* + A^*),$$

(10)

where: $\Omega_A = BP_{N(A)}$, $\Delta_A = (I - \Omega_A^\dagger J_A^\dagger B)A$, $J_A = A^*A + \Delta_A^*\Delta_A$.

**Proof.** Under the assumption $R(A) \perp R(B)$, then $B$ has the matrix form:

$$B = \begin{bmatrix} 0 & 0 \\ B_{13} & B_{14} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \rightarrow \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}.$$ By the addition of $A$ and $B$ we have the matrix form of $A + B$

$$A + B = \begin{bmatrix} A_{11} & 0 \\ B_{13} & B_{14} \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \rightarrow \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix}.$$ \hspace{1cm} (11)

Hence,

$$\Omega_A = BP_{N(A)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} : \begin{pmatrix} R(A^*) \\ N(A) \end{pmatrix} \rightarrow \begin{pmatrix} R(A) \\ N(A^*) \end{pmatrix},$$

$$\Delta_A = (I - \Omega_A^\dagger B)A = \begin{bmatrix} 0 & 0 \\ P_{N(B_{14})}B_{13} & 0 \end{bmatrix}.$$ And $J_A = A^*A + \Delta_A^*\Delta_A = \begin{bmatrix} A_{11}^*A_{11} + (P_{N(B_{14})}B_{13})^*P_{N(B_{14})}B_{13} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}.$

It is clear that $\Omega_A^\dagger$ exists if and only if $B_{14}^\dagger$ exists, on the other hand as $A_{11}$ is invertible, we have

$$A + B = \begin{bmatrix} I & 0 \\ B_{13}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & B_{14} \end{bmatrix},$$ \hspace{1cm} (12)

it follows from the (12) that $(A + B)^\dagger$ exists if and only if $B_{14}^\dagger$ exists, then it is automatically $(A + B)^\dagger$ exists if and only if $\Omega_A^\dagger$ exists. We will find the expression (10), applying the lemma 2.7, we get

$$(A + B)^\dagger = \begin{bmatrix} \Sigma^\dagger A_{11}^* & \Sigma^\dagger(P_{N(B_{14})}B_{13})^* \\ -B_{14}^\dagger B_{13}^\dagger \Sigma^\dagger A_{11}^* & B_{14}^\dagger - B_{14}^\dagger B_{13}^\dagger \Sigma^\dagger(P_{N(B_{14})}B_{13})^* \end{bmatrix}.$$ \hspace{1cm} (13)
We say that

\[ \text{Definition 5.1.} \]

follows we need the following definition:

\[
\begin{bmatrix}
0 & 0 \\
B_{14} & R
\end{bmatrix} + \begin{bmatrix}
\Sigma^\dagger A_{11}^* & 0 \\
- B_{14}^* B_{13} \Sigma A_{11}^* & 0
\end{bmatrix} + \begin{bmatrix}
0 & \Sigma^\dagger (P_{N(B_{14}^*)} B_{13})^* \\
0 & - B_{14}^* B_{13} \Sigma^\dagger (P_{N(B_{14}^*)} B_{13})^*
\end{bmatrix}.
\]

By identification

\[
(A + B)^\dagger = \Omega^\dagger_A + (I - \Omega^\dagger_A B) J^\dagger_A A^* + (I - \Omega^\dagger_A B) J^\dagger_A (\Delta^* A + A^*).
\]

\[
\Omega_A = \bar{A} P_{N(\bar{A})} \Delta_A = (I - \Omega^\dagger_A \bar{A}) J^\dagger_A = \bar{A}^* \bar{A} + \Delta^* A\Delta_A.
\]

5. **Representations of M-P inverse of the sum of two operators with disjoint ranges.** We assume in this section that \( A \) and \( B \) have closed ranges, in what follows we need the following definition:

**Definition 5.1.** We say that \( A \) and \( B \) have the range additivity property if \( R(A + B) = R(A) + R(B) \). We denote by \( R \) the set of all these pairs \((A, B)\), i.e.,

\[
R := \{(A, B) : A, B \in L(H, K) \text{ and } R(A + B) = R(A) + R(B)\}.
\]

**Theorem 5.2.** We have

1) If \((A, B) \in DR\), then \((A^*, B^*) \in R\), and

\( R(A + B) \) is closed if and only if \( R(A^*) + R(B^*) \) is closed.

2) If \((A^*, B^*) \in DR\), then \((A, B) \in R\), and

\( R(A + B) \) is closed if and only if \( R(A) + R(B) \) is closed.

3) If \((A, B) \in DR\) and \((A^*, B^*) \in DR\), then

\( (A, B) \in R\), \((A^*, B^*) \in R\),

In addition \( R(A + B) = R(A) + R(B) = R(A^*) + R(B^*) \) are closed.

**Proof.** Let \( F_A G_A \) and \( F_B G_B \) be full-rank decomposition of \( A \) and \( B \) with \( H_A = R(A) \) and \( H_B = R(B) \), we consider the operator

\[
M_0 = \begin{bmatrix}
A + B & 0 \\
0 & 0
\end{bmatrix} \in B(H \oplus L \oplus K \oplus F).
\]

We have

\[
M_0 = \begin{bmatrix}
F_A & F_B \\
G_A & G_B
\end{bmatrix} \begin{bmatrix}
G_A & 0 \\
G_B & 0
\end{bmatrix} = A_0 B_0,
\]

where

\[
A_0 = \begin{bmatrix}
0 & 0 \\
0 & B_{14}
\end{bmatrix}
\]

and

\[
B_0 = \begin{bmatrix}
\Sigma^\dagger A_{11} & 0 \\
0 & (P_{N(B_{14}^*)} B_{13})^*
\end{bmatrix}.
\]
Corollary 2. If \( \mathbf{A} \in \mathbb{R} \) is such that the item 3 of lemma 3.2 that \( A_0 \) is injective, is equivalent to \( A_0^* \) is surjective; i.e. \( R(A_0^*) = R(A) \oplus R(B) \), so \( A_0 \) has a closed range, now remark that

\[
R(M_0^*) = R(B_0^* A_0^*) = B_0^* R(A_0^*) = B_0^* R(A_0^* A_0) = B_0^* R(A_0^* A_0) = B_0^* R((A_0^* A_0)^*).
\]

And by the item 3 of lemma 3.2 that

\[
R(B_0^*) = R(G_0^*) + R(G_0^*) \oplus \{0\} = R(A^*) + R(B^*) \oplus \{0\}.
\]

Hence,

\[
R(M_0^*) = R(A^*) + R(B^*) \oplus \{0\}.
\]

As \( R(M_0^*) = R(A^* + B^*) \oplus \{0\} \), so

\[
R(A^* + B^*) \oplus \{0\} = R(A^*) + R(B^*) \oplus \{0\}.
\]

which implies that

\[
R(A^* + B^*) = R(A^*) + R(B^*).
\]

We know that \( R(A^* + B^*) \) is closed means that \( R(A + B) \) is closed, then from the last equality we deduce that \( R(A + B) \) is closed if and only if \( R(A^*) + R(B^*) \) is closed. 2): To prove the item 2, taking the adjoint on both sides of (15) and applying the item 1. 3): we already showed in items 1 and 2 that the equalities below are satisfied

\[
R(A + B) = R(A) + R(B), R(A^* + B^*) = R(A^*) + R(B^*).
\]

Note that \( B_0 \) is surjective because, by the lemma 2.5, \( B_0^* \) is injective, on the other hand we showed that \( A_0 \) is injective, it follows from the of lemma 3.2 that \( A_0 B_0 \) is full-rank decomposition of \( A + B \), which means that \( A + B \) has a closed range, of course it results from the two last equalities that \( R(A) + R(B) \) and \( R(A^*) + R(B^*) \) are closed.

**Corollary 1.** If \( (A^*, B^*) \in DR \) and \( R(A) \perp R(B) \), then we have:

\[
(A + B)^\dagger = (BP_{N(A)})^\dagger + (I - (BP_{N(A)})^\dagger B)A^\dagger.
\]

**Proof.** From the item 3 of theorem 5.2, \( (A + B)^\dagger \) exists and by lemma 2.4 \( (BP_{N(A)})^\dagger \) exists, it follows from the lemma 2.3 and lemma 2.1 that \( \Omega_A \Omega_A^\dagger = BB^\dagger \) consequently, \( \Delta_A = (I - \Omega_A \Omega_A^\dagger)B = 0 \), so the substitution of \( \Delta_A \) by the null operator in (10), we obtain (16).

Similarly, we can prove this corollary:

**Corollary 2.** If \( (A, B) \in DR \) and \( R(A^*) \perp R(B^*) \), we have:

\[
(A + B)^\dagger = (P_{N(A^*)}B)^\dagger + (I - (P_{N(A^*)}B)^\dagger B)A^\dagger.
\]

**Theorem 5.3.** If \( (A, B) \in DR \) and \( (A^*, B^*) \in DR \), then

\[
(A + B)^\dagger = (BP_{N(A)})^\dagger B(P_{N(A^*)}B)^\dagger + (AP_{N(B)})^\dagger A(P_{N(B^*)}A)^\dagger.
\]
Proof. The subspaces $R(A + B), R(A) + R(B)$ and $R(A^*) + R(B^*)$ are closed by the theorem 5.2, it follows from the lemma 2.3 that the M-P inverses that appear in the identity (18) exist. Let $M_0$ be as in (15), it results from the lemma 2.5 that \[
abla = \begin{bmatrix}
A & F
B & 0
\end{bmatrix}
\begin{bmatrix}
G_A & G_B
0 & 0
\end{bmatrix}
\] are injective, so \[
\begin{bmatrix}
G_A & 0
G_B & 0
\end{bmatrix}
\] is surjective, then $A_0B_0$ is a full-rank decomposition of $M_0$, in this case by the [[9], Theorem 2.1;(d)] we have
\[
M_0 = \begin{bmatrix} G_A & 0 \\
G_B & 0 
\end{bmatrix}
\begin{bmatrix} F_A & F_B \\
0 & 0 
\end{bmatrix} = B_0^1A_0^1.
\]
Now from the item a) of lemma 3.3 and lemma 2.4 and the theorem 5.2; $(P_{N(B^*)}F_A)^\dagger$, $(P_{N(A^*)}F_B)^\dagger, (G_BP_{N(A)})^\dagger, (G_AP_{N(B)})^\dagger$ Exist, hence from $B_0^1 = (B_0^\dagger)^*$ and using the lemma 2.6 we get $M$
\[
M_0 = \begin{bmatrix} (G_A P_{N(G)})^\dagger & (G_B P_{N(G)})^\dagger \\
0 & 0 
\end{bmatrix}
\begin{bmatrix} (P_{N(F_A)}F_A)^\dagger & 0 \\
(P_{N(F_B)}F_B)^\dagger & 0 
\end{bmatrix}
\]
\[
= \begin{bmatrix} (G_A P_{N(B)})^\dagger (P_{N(B^*)}F_A)^\dagger + (G_B P_{N(A)})^\dagger (P_{N(A^*)}F_B)^\dagger \\
0 & 0 
\end{bmatrix}.
\]
Using the equality of item b) and c) of lemma 3.3, we get
\[
M_0 = \begin{bmatrix} (A P_{N(B)})^\dagger A (P_{N(B^*)})^\dagger + (B P_{N(A)})^\dagger B (P_{N(A^*)})^\dagger \\
0 & 0 
\end{bmatrix}.
\]
Then by identification with $M_0^1$, we obtain (18). \(\square\)

**Corollary 3.** In the previous theorem, if $R(A) \perp R(B^*)$ then we obtain the identity (19) and also if $R(A) \perp R(B)$ then we obtain the identity (20)
\[
(A + B)^\dagger = B^1B (P_{N(A^*)})^\dagger + A^1A (P_{N(B^*)})^\dagger
\]
\[
(A + B)^\dagger = (BP_{N(A^*)})^\dagger BB^1 + (AP_{N(B^*)})^\dagger AA^1.
\]

**Proof.** We have $R(A^*) \perp R(B^*) \iff (AP_{N(B)})^\dagger = A^\dagger$ Also $(BP_{N(A)})^\dagger = B^\dagger$ And we replace $(AP_{N(B)})^\dagger$ and $(BP_{N(A)})^\dagger$ by $A^\dagger$ and $B^\dagger$ in (18) we obtain (19) By the same way we can prove (20). \(\square\)

In section 5 of the article [1], Arias, Corach and Maestripieri. extended the formula of Fill-Fishkind to the infinite Hilbert space case, by adding two other conditions to the property of the additivity of ranges.

From the theorem below, we see that the Fill-Fishkind formula remains valid in infinite dimensional Hilbert spaces under the same conditions of the case of matrices.

**Theorem 5.4.** If $(A, B) \in DR$ and $(A^*, B^*) \in DR$, then
\[
(A + B)^\dagger = (I - S)A^\dagger(I - T) + SB^\dagger T,
\]
where: $S = (P_{N(B^*)}P_{N(A)})^\dagger$ and $T = (P_{N(A^*)}P_{N(B^*)})^\dagger$.

**Proof.** From the item 3) of the theorem 5.2, $(A + B)^\dagger$ exists and $R(A^*) + R(B^*)$ is closed (resp., $R(A) + R(B)$ is closed) which implies by the item 3 of lemma 2.4 (resp., by the item 2 of lemma 2.4) that $S$ exists (resp., $T$ exists). As $B$ has a closed range, it results that $P_{N(B^*)} = B^\dagger B$ and $P_{N(A^*)} = BB^\dagger$. It follows from the lemma 2.1 that $BS = B(P_{N(B^*)}P_{N(A)})^\dagger = B(P_{N(B^*)}P_{N(A)})(P_{N(B^*)}P_{N(A)})^\dagger = BS^\dagger S$, \(\square\)
on the other hand, Since \( R(A) \cap R(B) = \{0\} \), so by the item 3 of lemma 2.3 we obtain \( BS = B \), by the same way we get \( TB = B \), also by the lemma 2.1 we obtain \( AS = 0 \) and \( TA = 0 \). Now we will check that \( (I - S)A^\dagger(I - T) + SB^\dagger T \) satisfies the equations of M-P inverse of \( A + B \).

The equations (iii):

\[
(A + B)(I - S)A^\dagger(I - T) + SB^\dagger T = (A + B)(A^\dagger - AB^\dagger + BA^\dagger - BA^\dagger T + BA^\dagger T + BB^\dagger T) = (A + B)(A + B)(I + T)(A + B) = (I + T)(A + B).
\]

The equations (iv):

\[
((I - S)A^\dagger(I - T) + SB^\dagger T)(A + B) = (I + T)(A + B) = (I + T)(A + B).
\]

The equations (i):

\[
(A + B)(I - S)A^\dagger(I - T) + SB^\dagger T = (A + B)(I + T)(A + B) = (I + T)(A + B).
\]

The equations (ii):

\[
((I - S)A^\dagger(I - T) + SB^\dagger T)(A + B) = (I + T)(A + B) = (I + T)(A + B).
\]

6. **Some representations of M-P inverse of a 2-by-2 block operator.** In this section we obtain some representations of M-P inverse of a 2-by-2 block operator under condition

\[
\left( \begin{array}{c}
A_1^\dagger \\
A_2^\dagger \\
A_3^\dagger \\
A_4^\dagger
\end{array} \right) \in DR.
\]

**Theorem 6.1.** Let \( M \) be defined as in (3) with closed range such that \( R(A_1) + R(A_2) \) and \( R(A_3) + R(A_4) \) are closed, if \( \left( \begin{array}{c}
A_1^\dagger \\
A_2^\dagger \\
A_3^\dagger \\
A_4^\dagger
\end{array} \right) \in DR \), then

\[
M^\dagger = \begin{bmatrix}
A_1^\dagger S_1^\dagger - W_1^\dagger Z_1 S_1^\dagger & W_1^\dagger Y_1^\dagger \\
A_2^\dagger S_2^\dagger - W_2^\dagger Z_1 S_2^\dagger & W_2^\dagger Y_1^\dagger
\end{bmatrix},
\]

where: \( S_1 = A_1^\dagger A_1^\dagger + A_2^\dagger A_2^\dagger \), \( Z = A_3^\dagger A_1^\dagger + A_4^\dagger A_2^\dagger \), \( W_1 = A_3^\dagger - ZS_1^\dagger A_1^\dagger \), \( W_2 = A_4^\dagger - ZS_1^\dagger A_2^\dagger \), \( T_1 = W_1W_1^\dagger + W_2W_2^\dagger \).

**Proof.** We have

\[
M = \begin{bmatrix}
A_1 & A_2 \\
A_3 & A_4
\end{bmatrix} = \begin{bmatrix}
A_1 & A_2 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 \\
A_3 & A_4
\end{bmatrix} =: M_1 + M_2.
\]
Clearly that the assumptions of corollary (1) are satisfied for \( M_1 \) and \( M_2 \), we deduce from (16) that
\[
M^\dagger = (M_2 P_{N(M_1)})^\dagger + (I - (M_2 P_{N(M_1)})^\dagger M_2)M_1^\dagger.
\]

Next we know that, \( M_1^\dagger = M_1^\dagger (M_1 M_1^\dagger)^\dagger \), then we get
\[
M_1^\dagger = \begin{bmatrix} A_1^* S_1^\dagger & 0 \\ A_2^* S_1^\dagger & 0 \end{bmatrix},
\]
\[
P_{N(M_1)} = \begin{bmatrix} I - A_1^* S_1^\dagger A_1 & -A_1^* S_1^\dagger A_2 \\ -A_2^* S_1^\dagger A_1 & I - A_2^* S_1^\dagger A_2 \end{bmatrix},
\]
\[
M_2 P_{N(M_1)} = \begin{bmatrix} 0 & 0 \\ A_3 - Z S_1^\dagger A_1 & A_4 - Z S_1^\dagger A_2 \end{bmatrix} := \begin{bmatrix} 0 & 0 \\ W_1 & W_2 \end{bmatrix}.
\]

Applying \((M_2 P_{N(M_1)})^\dagger = (M_2 P_{N(M_1)})^\dagger ((M_2 P_{N(M_1)}) (M_2 P_{N(M_1)})^\ast)^\dagger\), we obtain
\[
(M_2 P_{N(M_1)})^\dagger = \begin{bmatrix} 0 & W_1 \Upsilon_1^\dagger \\ 0 & W_2 \Upsilon_1^\dagger \end{bmatrix}.
\]

On the other hand
\[
(I - (M_2 P_{N(M_1)})^\dagger M_2)M_1^\dagger = \begin{bmatrix} A_1^* S_1^\dagger - W_1 \Upsilon_1^\dagger Z S_1^\dagger & 0 \\ A_2^* S_1^\dagger - W_2 \Upsilon_1^\dagger Z S_1^\dagger & 0 \end{bmatrix}.
\]

Finally \( M^\dagger = (M_2 P_{N(M_1)})^\dagger + (I - (M_2 P_{N(M_1)})^\dagger M_2)M_1^\dagger = \)
\[
\begin{bmatrix} 0 & W_1 \Upsilon_1^\dagger \\ 0 & W_2 \Upsilon_1^\dagger \end{bmatrix} + \begin{bmatrix} A_1^* S_1^\dagger - W_1 \Upsilon_1^\dagger Z S_1^\dagger & 0 \\ A_2^* S_1^\dagger - W_2 \Upsilon_1^\dagger Z S_1^\dagger & 0 \end{bmatrix} =
\]
\[
\begin{bmatrix} A_1^* S_1^\dagger & W_1 \Upsilon_1^\dagger \\ A_2^* S_1^\dagger & W_2 \Upsilon_1^\dagger \end{bmatrix}.
\]

\[\square\]

**Corollary 4.** Let \( M \) be defined as in (3) with closed range such that \( R(A_1) + R(A_2) \)
and \( R(A_3) + R(A_4) \) are closed, if \( R \left( \begin{bmatrix} A_1^* \\ A_2^* \end{bmatrix} \right) \perp R \left( \begin{bmatrix} A_3^* \\ A_4^* \end{bmatrix} \right) \), then
\[
M^\dagger = \begin{bmatrix} A_1^* S_1^\dagger & A_2^* S_2^\dagger \\ A_2^* S_1^\dagger & A_4^* S_2^\dagger \end{bmatrix},
\]
where \( S_1 = A_1 A_1^* + A_2 A_2^* \), \( S_2 = A_3 A_3^* + A_4 A_4^* \).

**Theorem 6.2.** Let \( M \) be defined as in (3) with closed range such that \( R(A_1) + R(A_2) \)
and \( R(A_3) + R(A_4) \) are closed, if \( \left( \begin{bmatrix} A_1^* \\ A_2^* \end{bmatrix} \right) \in DR \), then
\[
M^\dagger = \begin{bmatrix} W_3 \Upsilon_1^\dagger S_1^\dagger & W_1 \Upsilon_1^\dagger S_2^\dagger \\ W_4 \Upsilon_1^\dagger S_1^\dagger & W_2 \Upsilon_1^\dagger S_2^\dagger \end{bmatrix},
\]
where: \( S_1 = A_1 A_1^* + A_2 A_2^*, S_2 = A_3 A_3^* + A_4 A_4^* \), \( Z = A_3 A_1^* + A_4 A_2^* \), \( W_3 = A_1 - Z^* S_2^\dagger A_3, W_4 = A_2 - Z^* S_2^\dagger A_4, \) \( \Upsilon_1 = W_1 W_1^* + W_2 W_2^* \), \( \Upsilon_2 = W_3 W_3^* + W_4 W_4^* \).

**Proof.** We use the identity (19) to obtain the M-P inverse of \( M \). \[\square\]
7. Conclusion. After having transformed the sum of two operators in the setting of Hilbert spaces into a 2-by-2 block operator by applying the orthogonal sum of the spaces, and in other hand by the full-rank decomposition of operators, we conclude some representations of the Moore-Penrose inverse of a sum of two operators, in the closedness conditions for ranges, and show that the extension of the Fill-Fishkind formula remains valid, only by keeping the conditions of the Fill-Fishkind formula for the matrices.

Acknowledgments. The authors would like to thank the reviewers for their helpful comments and suggestions.

REFERENCES

[1] M. L. Arias, G. Corach and A. Maestripieri, Range additivity, shorted operator and the Sherman-Morrison-Woodbury formula, Linear Algebra Appl., 467 (2015), 86–99.
[2] A. Ben-Israel and T. N. E Greville, Generalized Inverses, Theory and Applications, 2nd ed Berlin Springer, New York, 2003.
[3] S. L. Campbell and C. D Meyer, Generalized Inverses of Linear Transformations, Dover Publ., New York, 1979.
[4] S. R. Caradus, Generalized Inverses and Operator Theory, Queen’s paper in pure and applied mathematics, Queen’s University, Kingston, 1978.
[5] R. E. Cline, Representations for the generalized inverse of sum of matrices, SIAM J. Numer. Anal., 2 (1965), 99–114.
[6] F. Deutsch, The angle between subspaces of a Hilbert space, in Approximation Theory, Wavelets and Applications (ed. S. P. Singh), Kluwer Academic Publ., (1995), 107–130.
[7] M. S. Djikić, Extensions of the Fill–Fishkind formula and the infimum–parallel sum relation, Linear and Multilinear Algebra, 64 (2016), 2335–2349.
[8] D. S. Djordjević and N. Ć. Dinčić, Reverse order law for Moore-Penrose inverse, Journal Math. Anal. Appl., 361 (2010), 252–261.
[9] D. S. Dordević and P. S. Stanimirović, General representations of pseudoinverses, Matematicki vesnik, 51 (1999), 69–76.
[10] J. A. Fill and D. E. Fishkind, The Moore–Penrose generalized inverse for sums of matrices, SIAM J. Matrix Anal. Appl., 21 (1999), 629–635.
[11] J. Groß, On oblique projection, rank additivity and the Moore-Penrose inverse of the sum of two matrices, Linear and Multilinear Algebra, 46 (1999), 265–275.
[12] M. R. Hestenes, Relative hermitian matrices, Pacific Journal Math., 11 (1961), 225–245.
[13] S. Izumino, Product of operators with closed range and an extension of the reverse order law, Tôhoku. Math. J., 34 (1982), 43–52.

Received May 2020; 1st revision March 2021; Final revision April 2021.

E-mail address: abdessalam.kara@univ-batna.dz
E-mail address: s.guedjiba@univ-batna2.dz