New Black Hole Solutions with Axial Symmetry in Einstein-Yang-Mills Theory

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Abstract

We construct new black hole solutions in Einstein-Yang-Mills theory. They are static, axially symmetric and asymptotically flat. They are characterized by their horizon radius and a pair of integers \((k, n)\), where \(k\) is related to the polar angle and \(n\) to the azimuthal angle. The known spherically and axially symmetric EYM black holes have \(k = 1\). For \(k > 1\), pairs of new black hole solutions appear above a minimal value of \(n\), that increases with \(k\). Emerging from globally regular solutions, they form two branches, which merge and end at a maximal value of the horizon radius. The difference of their mass and their horizon mass equals the mass of the corresponding regular solution, as expected from the isolated horizon framework.
1 Introduction

The well-known regular Bartnik-McKinnon (BM) solutions \[1\] and the corresponding non-Abelian black hole solutions \[2\], are asymptotically flat, static spherically symmetric solutions of SU(2) Einstein-Yang-Mills (EYM) theory. They are unstable solutions, sphalerons \[3\], and are characterized by the number of nodes of the gauge field. Besides these spherically symmetric solutions there are also asymptotically flat, static regular and black hole solutions, which possess only axial symmetry \[4\]. These are characterized by two integers, the node number of their gauge field function(s), and the winding number with respect to the azimuthal angle, denoted \(n\). The spherically symmetric solutions have winding number \(n = 1\), while the only axially symmetric solutions have winding number \(n > 1\). The \(n > 1\) black hole solutions possess an event horizon with a slight elongation along the symmetry axis \[4\]. All these EYM black hole solutions exist for arbitrarily large horizon size.

In SU(2) Einstein-Yang-Mills-Higgs (EYMH) theory with a triplet Higgs field, gravitating monopole solutions and black holes with monopole hair arise \[5, 6\]. As in EYM theory, the regular EYMH solutions are characterized by two integers, the node number of the gauge field and the azimuthal winding number \(n\), which corresponds to the topological charge of the monopoles. The black hole solutions are characterized in addition by their horizon size. These EYMH black hole solutions exist only up to a maximal value of the horizon size \[5, 6\], unlike the known EYM black hole solutions.

EYMH theory allows for further static axially symmetric solutions, representing gravitating monopole-antimonopole pair, chain and vortex solutions \[7, 8, 9\], which can be characterized by the azimuthal winding number \(n\), and by a second integer \(m\), related to the polar angle. For the monopole-antimonopole chains, which arise in flat space for \(n = 1\) and 2, the integer \(m\) corresponds to the number of nodes of the Higgs field (and thus the number of poles on the symmetry axis), while in vortex solutions, which arise in flat space for winding number \(n > 2\), the Higgs field vanishes (for even \(m\)) on \(m/2\) rings centered around the symmetry axis \[8\]. To all these regular solutions associated black hole solutions should exist, obtained so far only for a monopole-antimonopole pair \[10\].

In the limit of vanishing Higgs expectation value, EYMH solutions approach (after rescaling) EYM solutions \[5, 7, 9\]. Interestingly, when \(n \geq 4\) and \(m \geq 4\), new regular EYM solutions appear as limiting solutions \[11, 9\]. These static axially symmetric solutions have been characterized by the integers \((k, n)\), where \(2k = m\). They have been constructed numerically for \(k = 2, n \geq 4\), and \(k = 3, n \geq 6\) \[11\]. Unlike the \(k = 1\) EYM solutions, the \(k > 1\) solutions always appear in pairs. In this letter we construct the corresponding EYM black hole solutions. As expected, a branch of black hole solutions is associated with each regular solution. Intriguingly, the two branches of black hole solutions from a pair of regular solutions merge and end at a maximal value of the horizon size.

In the isolated horizon framework \[12, 13, 14\], non-Abelian black hole solutions can be interpreted as bound states of regular solutions and Schwarzschild black holes \[14\]. Furthermore, the isolated horizon framework yields a relation for the mass of non-Abelian black hole solutions, representing it as the sum of the mass of the regular solution and the horizon mass of the black hole solutions \[13\]. We here show that this relation is also valid
for the new black holes. In particular, the two regular solutions of a given $k$ and $n$ are connected via this mass formula.

In section II we present the EYM action, the axially symmetric ansatz and the boundary conditions. In section III we address their asymptotic and horizon properties. We present our numerical results in section IV, and we give our conclusions in section V.

2 Action and Ansatz

We consider the SU(2) EYM action

$$S = \int \left( \frac{R}{16\pi G} - \frac{1}{2} \text{Tr}(F_{\mu\nu}F^{\mu\nu}) \right) \sqrt{-g} d^4x$$

with Ricci scalar $R$, field strength tensor

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} + ie [A_{\mu}, A_{\nu}] ,$$

gauge potential $A_{\mu} = \tau^a A_{\mu}^a/2$, and gravitational and Yang-Mills coupling constants $G$ and $e$, respectively. Variation of the action (1) with respect to the metric $g_{\mu\nu}$ leads to the Einstein equations, variation with respect to the gauge potential $A_{\mu}$ to the gauge field equations.

In isotropic coordinates the static axially symmetric metric reads

$$ds^2 = -f dt^2 + \frac{m}{f} dr^2 + \frac{mr^2}{f} d\theta^2 + \frac{lr^2 \sin^2 \theta}{f} d\phi^2 ,$$

where the metric functions $f$, $m$ and $l$ are functions of the coordinates $r$ and $\theta$, only. The z-axis ($\theta = 0, \pi$) represents the symmetry axis. Regularity on the z-axis requires $m = l$ there.

For the gauge field we employ the ansatz

$$A_{\mu}dx^\mu = \frac{1}{2er} \left[ \tau^r_{\varphi} (H_1 dr + (1 - H_2) r d\theta) - n \left( \tau^r_{\theta} H_3 + \tau^n_{\theta} H_4 \right) r \sin \theta d\phi \right] .$$

Here the symbols $\tau^r_{r}$, $\tau^n_{\theta}$ and $\tau^n_{\varphi}$ denote the dot products of the cartesian vector of Pauli matrices, $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$, with the spatial unit vectors

$$\vec{e}_{r}^{n,k} = (\sin k\theta \cos n\varphi, \sin k\theta \sin n\varphi, \cos k\theta) ,$$

$$\vec{e}_{\theta}^{n,k} = (\cos k\theta \cos n\varphi, \cos k\theta \sin n\varphi, -\sin k\theta) ,$$

$$\vec{e}_{\varphi}^{n} = (-\sin n\varphi, \cos n\varphi, 0) ,$$

respectively. The gauge field functions $H_i$, $i = 1 - 4$, depend on the coordinates $r$ and $\theta$, only. For $k = n = 1$ and $H_1 = H_3 = 0$, $H_2 = 1 - H_4 = w(r)$ the spherically symmetric black hole solutions [2] are recovered, while for $k = 1$, $n > 1$, one obtains the axially symmetric solutions of [4]. The new black hole solutions reported here are obtained for
In the limit of vanishing horizon radius they converge pointwise to the globally regular solutions with the same integers \(k\) and \(n\). The globally regular solutions are related to EYMH solutions with \(m = 2k\) in the limit of vanishing Higgs field \[11, 9\].

The ansatz is form-invariant under the Abelian gauge transformation \[4\]

\[ U = \exp \left( i \frac{r^2 r_0^a}{2} \Gamma(r, \theta) \right). \]  

We fix the gauge by choosing the gauge condition \[4, 8, 9, 11\]

\[ r \partial_r H_1 - \partial_\theta H_2 = 0. \]  

To obtain asymptotically flat solutions which are regular at the horizon and possess the proper symmetries, we need to impose appropriate boundary conditions \[4, 8, 9, 11\]. The horizon of the non-Abelian black hole solutions resides at a surface of constant radial coordinate \(r = r_H\). At the horizon we impose the boundary conditions

\[ f = m = l = 0, \quad H_1 = 0, \quad \partial_r H_2 = \partial_r H_3 = \partial_r H_4 = 0, \]  

at infinity we impose

\[ f = m = l = 1, \quad H_1 = H_3 = 0, \quad H_2 = 1 - 2k, \quad H_4 = 2 \sin(k\theta)/\sin \theta, \]  

and on the z-axis we impose

\[ \partial_\theta f = \partial_\theta m = \partial_\theta l = 0, \quad H_1 = H_3 = 0, \quad \partial_\theta H_2 = \partial_\theta H_4 = 0. \]

### 3 Properties

We introduce the dimensionless coordinate \(x = e^{\sqrt{\frac{6}{4\pi G}}} r\) and horizon radius \(x_H = e^{\sqrt{\frac{6}{4\pi G}}} r_H\). Defining the mass \(M\) of the black hole solutions via the Komar integral, the dimensionless mass \(\mu = \frac{eG}{\sqrt{4\pi G}} M\), is determined by the derivative of the metric function \(f\) at infinity

\[ \mu = \frac{1}{2} \lim_{x \to \infty} x^2 \partial_x f. \]  

From the equations of motion it follows \[4\], that the Kretschmann scalar is finite at the horizon, and that the surface gravity \(\kappa\) \[15\],

\[ \kappa^2 = -(1/4) g^{tt} g^{ij} (\partial_i g_{tt})(\partial_j g_{tt}) \],

is constant, as required by the zeroth law of black hole physics. Expansion of the metric functions near the horizon in the form

\[ f(x, \theta) = \bar{x}^2 f_2(\theta) + O(\bar{x}^3), \quad m(x, \theta) = \bar{x}^2 m_2(\theta) + O(\bar{x}^3), \]
where $\tilde{x} = (x/x_H - 1)$, yields the dimensionless surface gravity $\tilde{\kappa} = \frac{f_2(\theta)}{x_H \sqrt{m_2(\theta)}}$ related to $\kappa$ by $\kappa = \tilde{\kappa} e / \sqrt{4\pi G}$.

We introduce the area parameter $x_\Delta$ [12, 13], defined via the dimensionless area of the black hole horizon $A$,

$$A = 2\pi x_H^2 \int_0^\pi d\theta \sin \theta \sqrt{\frac{lf}{m}} \bigg|_{x=x_\Delta} = 4\pi x_\Delta^2 . \quad (13)$$

The deformation of the horizon is revealed, when the circumference of the horizon along the equator, $L_e$, is compared to the circumference of the horizon along the poles, $L_p$,

$$L_e = \int_0^{2\pi} d\varphi \sqrt{\frac{l}{f}} x \sin \theta \bigg|_{x=x_H, \theta=\pi/2} , \quad L_p = 2 \int_0^\pi d\theta \sqrt{\frac{m}{f}} x \bigg|_{x=x_H, \varphi=\text{const.}} , \quad (14)$$

since the black hole solutions have $L_p \neq L_e$ (in general).

The isolated horizon framework [12, 13] yields an intriguing relation between the ADM mass $\mu$ of a black hole with area parameter $x_\Delta$ and the mass $\mu_{\text{reg}}$ of the corresponding globally regular solution [12],

$$\mu = \mu_\Delta + \mu_{\text{reg}} , \quad (15)$$

where the (dimensionless) horizon mass $\mu_\Delta$ is defined via

$$\mu_\Delta = \int_0^{x_\Delta} \tilde{\kappa}(x'_\Delta) x'_\Delta dx'_\Delta . \quad (16)$$

The isolated horizon formalism further suggests to interpret a non-Abelian black hole as a bound state of a regular solution and a Schwarzschild black hole [14],

$$\mu = \mu_{\text{reg}} + \mu_S + \mu_{\text{bind}} , \quad (17)$$

where $\mu_S = x_\Delta/2$ is the ADM mass of the Schwarzschild black hole with area parameter $x_\Delta$, and $\mu_{\text{bind}}$ represents the binding energy of the system,

$$\mu_{\text{bind}} = \mu_\Delta - \mu_S . \quad (18)$$

4 Results

Subject to the above boundary conditions, we solve the system of seven coupled non-linear partial differential equations numerically. To map spatial infinity to the finite value $\bar{x} = 1$, we employ the radial coordinate

$$\bar{x} = \frac{x - x_H}{1 + x} . \quad (19)$$

The numerical calculations are based on the Newton-Raphson method, and are performed with help of the program FIDISOL [16]. The equations are discretized on a non-equidistant
grid in $\bar{x}$ and $\theta$. Typical grids used have sizes $70 \times 30$, covering the integration region $0 \leq \bar{x} \leq 1$ and $0 \leq \theta \leq \pi/2$.

We construct numerically the black hole solutions for $(k = 1, n = 1 - 8)$, $(k = 2, n = 4 - 8)$ and $(k = 3, n = 6 - 8)$. For $k = 1$ and fixed $n$ a branch of black hole solutions emerges from the single globally regular solution, when the horizon radius $x_H$ is increased from zero \cite{4}. This branch of black hole solutions exists for arbitrarily large horizon size. When $k \geq 2$ and $n \geq 2k$, however, a pair of globally regular solutions exists for a given set $(k, n)$. Thus we find two branches of black hole solutions emerging from the corresponding globally regular solutions, when the horizon radius is increased from zero. Surprisingly, the two branches do not exist for arbitrarily large horizon size. Instead they merge and end at a maximal value of the horizon size, i.e., at the maximal value of the area parameter $x_{\Delta,\text{max}}$, which depends on $k$ and $n$.

We show the ADM mass $\mu$ as a function of the area parameter $x_\Delta$ in Fig. 1. The mass increases monotonically with increasing $x_\Delta$. For the new solutions with $k \geq 2$, the maximal horizon size increases with increasing $n$. At the same time, the mass difference of the two regular solutions increases, indicating a possible correlation between the soliton mass difference and the maximal horizon size.

Considering the shape of the horizon, we observe that the deviation from spherical symmetry is small, though. We observe a ratio of circumferences for $k = 1$ solutions of up to $L_e/L_p = 0.988$, and for $k = 2$ and $k = 3$ solutions of up to $L_e/L_p = 0.984$ and $L_e/L_p = 0.979$, respectively, as seen in Fig. 2. Thus the new black holes have a slightly more deformed horizon.

The inverse surface gravity $1/\kappa$ of the black hole solutions is exhibited in Fig. 3. For $k = 1$ black hole solutions the inverse surface gravity increases monotonically (except for $n = 1$) as a function of the area parameter $x_\Delta$. For $k = 2$ and $k = 3$ black holes it increases along the lower branch, and beyond the transition to the upper branch it decreases back to zero.

Our numerical results indicate, that the mass relation \cite{15}, obtained in the isolated horizon formalism, also holds for the new non-Abelian black hole solutions. (In Fig. 1 for each branch the corresponding regular solution is the reference point for the integration.) Consequently, the mass of the regular solution on the upper branch is related to the mass of the regular solution on the lower branch via the horizon mass integral, performed along both branches. A similar result was obtained previously for EYMH black holes with dipole hair \cite{10}.

In \cite{14} the mass formula $\mu_{\text{reg}} = 1/2 \int_0^\infty (1 - x_\Delta \hat{\kappa}) dx_\Delta$ was derived and shown to hold for spherically symmetric EYM solutions in \cite{17}. We checked that the same formula also holds for the $k = 1$ axially symmetric solutions.

In Fig. 4 we illustrate the binding energy of the non-Abelian black hole solutions. The binding energy is always calculated w.r.t. the corresponding regular solution. Consequently, for $k = 2$ and $k = 3$ black holes the binding energy along the upper branches is smaller than the binding energy along the lower branches. The difference in binding energy of the black hole solutions for a given set of $(k, n)$ at the maximal value of the area parameter represents the mass difference of the corresponding pair of regular solutions.
Fig. 1 (a) The ADM mass $\mu$ is shown as function of the area parameter $x_\Delta$ for $k = 1$, $n = 1 - 8$ (symbols ×). Also shown is $\mu_\Delta + \mu_{\text{reg}}$ (lines). (b) Same as (a) for $k = 2$, $n = 4 - 8$. (c) Same as (a) for $k = 3$, $n = 6 - 8$. Fig. 2 Same as Fig. 1 for the ratio $L_e/L_p$. 
Fig. 3 Same as Fig. 1 for the inverse surface gravity $1/\kappa$. Fig. 4 Same as Fig. 1 for the binding energy.

5 Conclusions

We have constructed numerically new static axially symmetric black hole solutions of EYM theory and investigated their properties. These solutions are characterized by the set of integers $(k, n)$, related to the polar and azimuthal angles, respectively. In particular, we
have obtained black hole solutions for $k = 2$, $n = 4 - 8$, and $k = 3$, $n = 6 - 8$.

The $(1, n)$ solutions (with one node) exist most likely for any integer $n \geq 1$ and for arbitrary horizon size. The new $(2, n)$ and $(3, n)$ solutions have lower bounds on $n$, $n = 2k$, and upper bounds on the horizon size. For each allowed set $(k, n)$ two branches of black hole solutions emerge from the two globally regular solutions and merge and end at the maximal value of the horizon size. The existence of an upper bound of the horizon radius is a surprising and interesting new feature for EYM solutions. Previously a maximal value of the horizon size was known only in theories with an in-built length scale already in Minkowski space, hence it was conjectured, that “there is no bound on the horizon radius of hairy, static EYM black holes” [12]. The underlying reason for the occurrence of this maximal horizon size for the new EYM black holes is yet to be understood.

We expect, that the $(2, n)$ and $(3, n)$ regular and black hole solutions represent only the first sequences of new solutions, and conjecture the existence of $(k, n)$ regular and black hole solutions also for higher values of $k$. The (spherically symmetric) $k = 1$ black holes are unstable [3], and there is all reason to believe, that the new $(k, n)$ solutions are unstable as well.

Considering the new static axially symmetric solutions from the isolated horizon formalism point of view, we have verified the mass relation between the regular and the black hole solutions, showing that the black hole mass is given by the sum of the mass of the regular solution and the horizon mass. In particular, the mass of the regular solution on the upper branch is related to the mass of the regular solution on the lower branch via the horizon mass integral, performed along both branches. Interpreting the non-Abelian black holes as bound states of regular solutions and Schwarzschild black holes [14], we have obtained the binding energy of these bound systems.

We note that the globally regular solutions were first obtained in EYMH theory as limiting solutions for vanishing Higgs field [9, 11]. Hence we expect that EYMH black holes can be found from the new EYM black holes by gradually switching on the Higgs field.

Rotating EYM black hole solutions based on the static $k = 1$ black hole solutions are known [15]. It appears straightforward to construct rotating EYM black hole solutions based on the static $k = 2$ and $k = 3$ black hole solutions. In contrast, rotating regular EYM solutions do not appear to exist [12], though recently the first rotating regular EYMH solutions have been constructed [19, 20].

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References

[1] R. Bartnik and J. McKinnon, Phys. Rev. Lett. 61 (1988) 141.
[2] M. S. Volkov and D. V. Galt’sov, Sov. J. Nucl. Phys. 51 (1990) 747;  
P. Bizon, Phys. Rev. Lett. 64 (1990) 2844;  
H. P. Künzle and A. K. M. Masoud-ul-Alam, J. Math. Phys. 31 (1990) 928.

[3] N. Straumann and Z. H. Zhou, Phys. Lett. B237 (1990) 353; B243 (1990) 33;  
M. S. Volkov and D. V. Gal’tsov, Phys. Lett. B341 (1995) 279;  
G. Lavrelashvili, and D. Maison, Phys. Lett. B343 (1995) 214;  
M. S. Volkov, O. Brodbeck, G. Lavrelashvili and N. Straumann, Phys. Lett. B349 (1995) 438.

[4] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 78 (1997) 2527; Phys. Rev. Lett. 79 (1997) 1595; Phys. Rev. D57 (1998) 834; Phys. Rev. D57 (1998) 6138.

[5] K. Lee, V.P. Nair and E.J. Weinberg, Phys. Rev. D45 (1992) 2751;  
P. Breitenlohner, P. Forgacs and D. Maison, Nucl. Phys. B383 (1992) 357; B442 (1995) 126.

[6] B. Hartmann, B. Kleihaus, and J. Kunz, Phys. Rev. Lett. 86 (2001) 1422; Phys. Rev. D65 (2002) 024027.

[7] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 85 (2000) 2430.

[8] B. Kleihaus, J. Kunz and Ya. Shnir, Phys. Lett. B570 (2003) 237; Phys. Rev. D68 (2003) 101701; Phys. Rev. D70 (2004) 065010.

[9] B. Kleihaus, J. Kunz, and Ya. Shnir, Phys. Rev. D71 (2005) 024013.

[10] B. Kleihaus, and J. Kunz, Phys. Lett. B494 (2000) 130.

[11] R. Ibadov, B. Kleihaus, J. Kunz and Ya. Shnir, Phys. Lett. B609 (2005) 150.

[12] A. Ashtekar, S. Fairhurst and B. Krishnan, Phys. Rev. D62 (2000) 104025.

[13] A. Corichi and D. Sudarsky, Phys. Rev. D61 (2000) 101501;  
A. Corichi, U. Nucamendi and D. Sudarsky, Phys. Rev. D62 (2000) 044046.

[14] A. Ashtekar, A. Corichi and D. Sudarsky, Class. Quantum Grav. 18 (2001) 919.

[15] R. M. Wald, General Relativity (University of Chicago Press, Chicago, 1984)

[16] W. Schönauer and R. Weiß, J. Comput. Appl. Math. 27 (1989) 279;  
M. Schauder, R. Weiß and W. Schönauer, The CADSOL Program Package, Universität Karlsruhe, Interner Bericht Nr. 46/92 (1992).

[17] A. Corichi, U. Nucamendi and D. Sudarsky, Phys. Rev. D64 (2001) 107501.
[18] B. Kleihaus and J. Kunz, Phys. Rev. Lett. 86 (2001) 3704; B. Kleihaus, J. Kunz, and F. Navarro-Lérida, Phys. Rev. D66 (2002) 104001; Phys. Rev. Lett. 90 (2003) 171101; Phys. Rev. D69 (2004) 064028; Phys. Lett. B599 (2004) 294.

[19] J. J. van der Bij and E. Radu, Int. J. Mod. Phys. A17 (2002) 1477; Int. J. Mod. Phys. A18 (2003) 2379.

[20] B. Kleihaus, J. Kunz, and U. Neemann, Phys. Lett. B in press (gr-qc/0507047).