TOPOLOGICAL DEFECTS IN THE ABELIAN HIGGS MODEL

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Abstract. We give a rigorous description of the dynamics of the Nielsen-Olesen vortex line. In particular, given a worldsheet of a string, we construct initial data such that the corresponding solution of the abelian Higgs model will concentrate near the evolution of the string. Moreover, the constructed solution stays close to the Nielsen-Olesen vortex solution.

1. Introduction

In 1973 Nielsen and Olesen [20] conjectured a relationship between the abelian Higgs model in \( \mathbb{R}^{1+3} \) and the Nambu-Goto action. In this paper we show that their conjecture follows from a conjecture of Jaffe and Taubes about the 2-dimensional Euclidean abelian Higgs model. In particular, since the Jaffe-Taubes conjecture is known to hold for a range of values of a coupling parameter appearing in the abelian Higgs model, our results show that the Nielsen-Olesen scenario holds in these situations.

The abelian Higgs model (see (1.1) below) arises in various branches of physics: in high-energy physics, as perhaps the simplest Yang-Mills-Higgs theory; in solid-state physics, in connection with superconductivity; and in cosmology where, for reasons stemming from its relevance to high-energy physics, it provides a basis for studies of the possible behavior of cosmic strings, should any such objects exist.

The Nambu-Goto action of a (1+1)-dimensional string in (1+3)-dimensional Minkowski space is (proportional to) the Minkowski area of its worldsheet, see (1.5) below for a precise formulation. The associated equations of motion are exactly the condition that the Minkowski mean curvature of the worldsheet vanishes. This is the simplest natural model for the relativistic dynamics of a string in Minkowski space. We refer to a solution of the equations of motion as a “timelike Minkowski minimal surface”, by analogy with ordinary (Euclidean) minimal surfaces. The action is due to Nambu [18] and Goto [6], and it has its origins in the early days of string theory, as a description of the evolution of a closed (dual) string. See [5] for a nice historical perspective.

The relationship between these two models proposed in [20] is that solutions of the abelian Higgs model exhibit, for suitable initial data, features known as vortex lines that, Nielsen and Olesen argued, should sweep out worldsheets that are approximately governed by the Nambu-Goto action. This proposal has subsequently been investigated particularly intensively by cosmologists interested in possible cosmic strings, starting with work of Kibble [15]. Models for cosmic strings assume that some form of Yang-Mills-Higgs (YMH) equation, perhaps arising from some yet-unknown grand unified theory, is relevant to descriptions of the distribution of matter in the universe. One can associate to a YMH model an object called the “vacuum manifold”, and it is believed that qualitative features of solutions known

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as topological defects are determined by the topology of the vacuum manifold. In particular, string-like defects are expected to form when the vacuum manifold has a nontrivial fundamental group. The abelian Higgs model, for which the vacuum manifold is given by $S^1$, provides the simplest case of this scenario, and it is thus studied as a useful prototype for more general models whose vacuum manifold is not simply connected.

There is a large body of mathematics describing strings and other defects in solutions of elliptic and parabolic equations with vacuum manifolds that are either disconnected or non-simply-connected. References and a more detailed discussion may be found in [12].

On the other hand, there is not a great deal of rigorous mathematical work describing dynamics of topological defects in nonlinear hyperbolic equations, and most of it deals with defects that can be thought of as point particles or 0-dimensional defects, see for example [23, 11, 17, 8]. Higher-dimensional defects are however treated in [3] and [12]. In particular, the latter work proves that topological defects in certain semilinear hyperbolic equations, including a non-gauged analog of the abelian Higgs model, do indeed approximately sweep out timelike minimal surfaces for suitable initial data. This covers the case of the domain wall, one of the basic examples of topological defects considered by cosmologists, associated to the real scalar equation $\square u + \epsilon^{-2}(u^2 - 1)u = 0$. Cosmic strings, as in the abelian Higgs model, have a much richer mathematical structure, and are also considered more likely to be present in our universe than domain walls.

The basic scheme we use here draws on that developed in [12]. To show that this scheme works for a gauge theory such as the abelian Higgs model we must, among other things, formulate and establish suitable stability estimates, relating energy and vorticity for the 2-dimensional Euclidean abelian Higgs model, and a large part of our work is devoted to these tasks.

We next present some necessary background about the abelian Higgs model, the Nambu-Goto action, the 2d Euclidean abelian Higgs model and the Jaffe-Taubes conjecture, and normal coordinates around a string. With this done, we will finally state our main result.

1.1. The abelian Higgs model. We will write the Lagrangian for the abelian Higgs model in the form

$$L(\varphi, A) = \frac{1}{2} \langle D_\alpha \varphi, D^\alpha \varphi \rangle + \frac{\epsilon^2}{4} F_{\alpha \beta} F^{\alpha \beta} + \frac{\lambda}{8 \epsilon^2} (|\varphi|^2 - 1)^2.$$  

Here

$$\varphi : \mathbb{R}^{1+3} \to \mathbb{C}, \quad A \text{ a 1-form with components } A_\alpha : \mathbb{R}^{3+1} \to \mathbb{R}, \quad \alpha \in 0, \ldots, 3,$$

$D_\alpha$ denotes the covariant derivative $D_\alpha \varphi := (\partial_\alpha - i A_\alpha) \varphi$, and $F := dA$, so that

$$F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha, \quad \alpha, \beta \in \{0, \ldots, 3\}.$$

One may regard $A$ as a $U(1)$ connection and $F$ as the associated curvature. We write $\langle f, g \rangle$ to denote the real inner product

$$\langle f, g \rangle = \text{Re}(fg).$$

In (1.1) we sum over repeated upper and lower indices, and we raise and lower indices with the Minkowski metric $(\eta^{\alpha \beta}) = (\eta_{\alpha \beta}) = \text{diag}(-1, 1, 1, 1)$ so that

$$\langle D_\alpha \varphi, D^\alpha \varphi \rangle = \eta^{\alpha \beta} \langle D_\alpha \varphi, D_\beta \varphi \rangle, \quad F_{\alpha \beta} F^{\alpha \beta} = \eta^{\alpha \gamma} \eta^{\beta \delta} F_{\alpha \beta} F_{\gamma \delta}.$$}

We will consider the scaling $0 < \epsilon \ll 1$, which is relevant to models describing cosmic strings, where typically $\epsilon \sim 10^{-16}$ in the units we have (implicitly) chosen.
We remark that the Lagrangian (1.1) is invariant under action of the $U(1)$ group, so for any sufficiently smooth function $X : \mathbb{R}^{1+3} \to \mathbb{R}$ we have
\[
\mathcal{L}(\varphi, A) = \mathcal{L}(e^{iX} \varphi, A + dX).
\]
The Euler-Lagrange equations associated to the action functional $\int_{\mathbb{R}^{1+3}} \mathcal{L}(\varphi, A)$ are
\begin{align*}
(1.2) \quad -D_\alpha D^\alpha \varphi + \frac{\lambda}{4e^2}(|\varphi|^2 - 1)\varphi &= 0, \\
(1.3) \quad -e^2 \partial_\alpha F^{\alpha\beta} - \eta^{\alpha\beta} \{i\varphi, D_\alpha \varphi\} &= 0.
\end{align*}
Our main theorem describes the behavior of certain solutions of this system, for well-chosen initial data.

1.2. the Nambu-Goto action: timelike minimal surfaces. The worldsheet of a closed string may be described by a function $H : (-T, T) \times S^1 \to (-T, T) \times \mathbb{R}^3$ of the form
\[
H(y^0, y^1) = (y^0, h(y^0, y^1)) \quad \text{for some } h : (-T, T) \times S^1 \to \mathbb{R}^3.
\]
Here and throughout this paper, $S^1$ denotes $\mathbb{R}/L\mathbb{Z}$ for some $L > 0$, so that $S^1$ is a circle of arbitrary positive length $L$. We write $\Gamma$ for the image of such a map $H$, and the induced metric on $\Gamma$ is denoted by
\[
\gamma_{ab} = \eta_{\alpha\beta} \partial_\alpha H^\alpha \partial_\beta H^\beta, \quad a, b \in \{0, 1\},
\]
where we implicitly sum over repeated indices $\alpha, \beta = 0, \ldots, 3$. A surface $\Gamma$ is said to be timelike if $\det(\gamma_{ab}) < 0$ at every point in $(-T, T) \times S^1$. The Nambu-Goto action is proportional to
\[
\mathcal{N}_G(H) := \int \sqrt{-\gamma} \quad \text{with } \gamma := \det(\gamma_{ab}).
\]
A timelike surface $\Gamma = \text{Image}(H)$ is called a minimal surface if $H$ is a critical point of $\mathcal{N}_G$.

A timelike minimal surface may be written in conformal coordinates, in which case
\[
\gamma_{01} = \gamma_{10} = 0, \quad -\gamma_{00} = \gamma_{11}.
\]
This is well-known in the physics literature, see for example [24, Section 6.2], and is proved in [2]. We will always assume $H$ is a smooth timelike embedding on $(-T, T) \times S^1$ and that (1.4), (1.6) hold. With a conformal parametrization, (1.6), $h(y^0, y^1)$ may be written in the form $\frac{1}{2}(a(y^0 + y^1) + b(y^0 - y^1))$ for functions $a, b : S^1 \to \mathbb{R}^3$ such that $|a'| = |b'| = 1$, and conversely every map of this form parametrizes a minimal surface. In particular, if $h_0 : S^1 \to \mathbb{R}^3$ is an arclength parametrization of a smooth embedded curve $\Gamma_0 = \text{image}(h_0)$, then the timelike minimal surface that agrees with $\Gamma_0$ at time $t = 0$ and with zero initial velocity can be written in the form (1.4), with
\[
h(y^0, y^1) = \frac{1}{2}(h_0(y^1 + y^0) + h_0(y^1 - y^0)).
\]
This is the situation that we will always consider, although we will rarely need the explicit formula (1.7).

Since $H$ is smooth and $\det \gamma < 0$ in $(-T, T) \times S^1$, it is clear that for every $T_1 < T$, there exists some $c_0 = c_0(T_1) > 0$ such that
\[
\gamma_{11} = -\gamma_{00} \geq c_0 \quad \text{for all } (y^0, y^1) \in (-T_1, T_1) \times S^1.
\]

1 It follows from results in [19], [13], or by inspection of (1.4), that $T < L$ if the image of $h_0'$ contains antipodal points in $S^2$. 
1.3. The Euclidean abelian Higgs model in 2 dimensions. We will write the 2d abelian Higgs energy density in the form

\[ e_{\epsilon,\lambda}(U) := \frac{1}{2} (|D_1 \phi|^2 + |D_2 \phi|^2) + \frac{\epsilon^2}{4} (F_{12})^2 + \frac{\lambda}{8\epsilon^2} (|\phi|^2 - 1)^2. \]

Here \( U = (\phi, A) \), where \( \phi \in H^1_{\text{loc}}(\mathbb{R}^2; \mathbb{C}) \) and \( A = A_1 dy^1 + A_2 dy^2 \) is a 1-form with components in \( H^1_{\text{loc}} \). We write as usual \( F_{12} = \partial_1 A_2 - \partial_2 A_1 \), so that \( dA = F_{12} dy^1 \land dy^2 \). A finite-energy configuration is a pair \( U = (\phi, A) \) such that \( e_{\epsilon,\lambda}(U) \in L^1(\mathbb{R}^2) \).

Note that \( \epsilon \) is just a scaling parameter in \((1.9)\), and one can easily change variables to set \( \epsilon = 1 \). That is, given a configuration \( U = (\phi, A) \), if we define \( U' = (\phi', A') \) by \( \phi'(y) := \phi(\frac{y}{\epsilon}), A'(y) := \frac{1}{\epsilon} A(\frac{y}{\epsilon}) \), then

\[ e_{\epsilon,\lambda}(U') = \frac{1}{\epsilon^2} e_{1,\lambda}(U; \frac{y}{\epsilon}), \quad \text{and thus} \quad \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U') = \int_{\mathbb{R}^2} e_{1,\lambda}(U). \]

However, we find it convenient to include the scaling parameter \( \epsilon \) in the energy.

We define the (2-dimensional) current \( j(U) \) and vorticity \( \omega(U) \), given by

\[ j(U) := \langle (i\phi, D_1 \phi), (i\phi, D_2 \phi) \rangle, \]
\[ \omega(U) = \frac{1}{2} \nabla \times (j(U) + A) = \frac{1}{2} \left[ \partial_1 j_2(U) - \partial_2 j_1(U) + F_{12} \right]. \]

As we will recall in slightly more detail in Section 2, if \( U \) is a finite-energy configuration then \( \omega(U) \in L^1(\mathbb{R}^2) \), and moreover

\[ \int_{\mathbb{R}^2} \omega(U) \, dy \in \pi \mathbb{Z} \quad \text{for every finite energy } U. \]

It follows that every finite-energy \( U \) belongs to exactly one of the sets

\[ H_n := \{ U = (\phi, A) \in H^1_{\text{loc}} \times H^1_{\text{loc}} : \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U) < \infty, \int_{\mathbb{R}^2} \omega(U) = \pi n \}. \]

(These sets are called weak homotopy classes by Rivièr [21], who establishes a slightly different description of them.) Note also that, while \( \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U) \) certainly depends on \( \lambda \), the condition \( \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U) < \infty \) is independent of \( \lambda \), and hence the homotopy classes \( H_n \) are also independent of \( \epsilon \) and \( \lambda \). We will use the notation

\[ \mathcal{E}_n^\lambda := \inf \{ \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U) : U \in H_n \}. \]

Our main results describe solutions of the 1 + 3-dimensional abelian Higgs model in terms of solutions, when they exist, of the 2d minimization problem:

\[ \text{find } U^m = U^m_{\epsilon,\lambda} \in H_m \text{ such that } \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U^m) = \mathcal{E}_m^\lambda. \]

For \( 0 < \epsilon \ll 1 \), the regime that interests us, we always assume that a minimizer \( U^m_{\epsilon,\lambda} \) is obtained by starting from a fixed minimizer of the \( \epsilon = 1 \) problem and scaling as in \((1.10)\), so that the energy and vorticity concentrate near the origin. 

Remark 1.1. For every \( n \in \mathbb{Z} \) and \( \lambda > 0 \), there exists an equivariant \( U^{(n)}(\nu) \in H_n \) solving the Euler-Lagrange equations associated to the minimization problem \((1.16)\), see [4]. Here “equivariant” implies for example that \( \phi^{(n)} \) can be written in the form \( f(r)e^{i n \theta} \). The equivariant solution is known to be linearly stable if \( |n| = 1 \) or \( \lambda \leq 1 \), and linearly unstable
(and hence not even a local energy minimizer) if \( \lambda > 1 \) and \( |n| \geq 2 \), see Gustafson and Sigal [7].

**Remark 1.2.** The conjecture of Jaffe and Taubes [9] Chapter III.1, Conjectures 1 and 2] mentioned earlier holds that the equivariant solution solves problem (1.16) for all parameter values for which it is linearly stable, and that no minimizer exists whenever the equivariant solution is linearly unstable. This is known to be true in the case \( \lambda = 1 \), which has a special structure that will be recalled in Section 2] and for all sufficiently large \( \lambda \), due to work of Rivière [21]. Otherwise it is open, as far as we know.

**Remark 1.3.** In Theorem 1.1 we establish a general sufficient condition, involving the behavior of the map \( m \mapsto E^\lambda_m \), for existence of solutions of problem (1.16). In particular we deduce from this that a minimizer exists for \( |n| = 1 \) and \( \frac{1}{5} < \lambda < 5 \).

Theorem 1.1 implies in particular that if \( \lambda \) satisfies
\[
\lambda_n > \lambda^1_n \quad \text{for} \ n > 1,
\]
then problem (1.16) has a solution for \( |n| = 1 \). Condition (1.17) is known to hold for large \( \lambda \), see [21], and we note in Lemma 2.1 that it is easily verified for \( \frac{1}{2} \leq \lambda \leq 2 \). It is expected that (1.17) holds for all \( \lambda > 0 \), and more generally that \( n \mapsto E^\lambda_n \) is increasing for \( n \in \mathbb{N} \). (It is easy to check that \( E^\lambda_n = E^\lambda_n \) for all \( n \).) A statement similar to (1.17) is proved for certain non-gauged generalized Ginzburg-Landau-type models by Almog et al in [1], but adapting their arguments to the gauged case seems not to be easy.

1.4. **Normal coordinates.** Next we describe a useful coordinate system, which we will refer to as normal coordinates, for a neighbourhood of a minimal surface \( \Gamma \). A key point in our analysis (as in [12]) will be to obtain estimates in these coordinates.

Given a minimal surface \( \Gamma \), always assumed to be represented via a conformal parametrization \( H : (-T, T) \times S^1 \to \mathbb{R}^{1+3} \), see (1.4), (1.6), we will parametrize a neighborhood of \( \Gamma \) by \( (-T, T) \times S^1 \times \mathbb{R}^2 \), and we will write points in this set as
\[
y = (y^1, y^\nu), \quad \text{with} \ y^1 = (y^0, y^1) \in (-T, T) \times S^1 \text{ and } y^\nu = (y^2, y^3) \in \mathbb{R}^2,
\]
where the superscripts stand for “tangential” and “normal” respectively. We will also sometimes write
\[
(y^\nu_1, y^\nu_2) = (y^2, y^3).
\]
We will arrange that \( y^0 \) is a timelike coordinate and \( y^1, \ldots, y^3 \) spacelike.

To define these coordinates, we first fix maps \( \tilde{\nu}_i : (-T, T) \times S^1 \to \mathbb{R}^{1+3}, i = 1, 2 \) such that
\[
\gamma_{ij} \tilde{\nu}_i \tilde{\nu}_j = \delta_{ij}, \quad \gamma_{ij} \tilde{\nu}_i \partial_0 H^\beta = \eta_{ij} \tilde{\nu}_i \partial_1 H^3 = 0
\]
for \( i, j = 1, 2 \). In other words, \( \{\tilde{\nu}_i(y^\nu)\}_{i=1}^2 \) is an orthonormal frame (with respect to the Minkowski metric) for the normal bundle to \( \Gamma \) at \( H(y^\nu) \). We then define \( \psi : (-T, T) \times S^1 \times \mathbb{R}^2 \to \mathbb{R}^{1+3} \) by
\[
\psi(y) = H(y^\nu) + \tilde{\nu}_1(y^\nu) y^2 + \tilde{\nu}_2(y^\nu) y^3 = H(y^\nu) + \tilde{\nu}_1(y^\nu) y^{\nu_1} + \tilde{\nu}_2(y^\nu) y^{\nu_2}.
\]
Writing \( B_\nu(\rho) = \{y^\nu = (y^\nu_1, y^\nu_2) : |y^\nu| < \rho \} \), we will restrict \( \psi \) to a set of the form \( (-T_1, T_1) \times S^1 \times B_\nu(\rho_0) \) on which \( \psi \) is injective and satisfies other useful properties; see Section 5 for details.

Since our argument will rely heavily on specific properties of the abelian Higgs model when written with respect to the new coordinates, we find it useful to distinguish between the Higgs field and connection when written in terms of the original, standard coordinates.
for Minkowski spacetime, which we will write \((\varphi, A)\), and the same objects written in terms of the new coordinates, which we will denote \((\phi, A)\). These are related by
\[
\phi = \varphi \circ \psi, \quad A = \psi^* A, \quad \text{so that } A_\alpha = \partial_\alpha \psi^\beta A_\beta \circ \psi
\]
on domain(\(\phi\)) = \((-T_1, T_1) \times S^1 \times B_\rho(\rho_0)\). The components of the curvature in the two coordinate systems will be denoted \(F_{\alpha\beta}\) and \(F_{\alpha\beta}\) respectively. We will also write \(U\) to denote a pair \((\varphi, A)\) and similarly \(U\) for a pair \((\phi, A)\), and we will write
\[
U = \psi^* U \quad \text{when } U \text{ and } U \text{ are related as in (1.22)}.
\]
We will also write \(U = (\psi^{-1})^* U\) to indicate that (1.22) holds.

1.5. **Main Theorem.** Our main result, stated below, asserts the existence of a solution whose energy concentrates around a minimal surface \(\Gamma\), and that in a neighborhood of \(\Gamma\) is close to a configuration that in the \(y\) coordinates takes the form \(U^{\text{NO}} = (\phi^{\text{NO}}, A^{\text{NO}})\), with
\[
\phi^{\text{NO}}(y^\tau, y^\nu) := \phi^m(y^\nu), \quad A^{\text{NO}}(y^\tau, y^\nu) := A^m_\tau(y^\nu)dy^\nu + A^m_\nu(y^\tau)dy^\tau,
\]
where \(U^m = (\phi^m, A^m) = U^{m}_{c,\lambda}\) is a ground state of the 2d minimization problem (1.16).

Thus, in standard coordinates this configuration can be written
\[
U^{\text{NO}} = (\phi^{\text{NO}}, A^{\text{NO}}) = (\psi^{-1})^* U^{\text{NO}}.
\]
Note that \(U^{\text{NO}}\) is only defined in the domain of \(\psi^{-1}\), which is a neighborhood of \(\Gamma\).

**Theorem 1.4.** Let \(\Gamma\) be a codimension 2 timelike minimal surface, given as the image of a conformal parametrization \(H\) (so that \(H\) satisfies (1.4), (1.6)) that is a smooth embedding in \((-T, T) \times S^1\). Assume also that the initial velocity of \(\Gamma\) at \(t = 0\) is everywhere 0.

Let \(\lambda > 0\) and \(m \in \mathbb{Z}\) be such that the 2d minimization problem (1.16) has a solution, and in addition assume that \(E^\lambda_m \leq E^\lambda_n\) whenever \(|m| \leq |n|\).

Then, given \(T_0 < T\), there exists an neighborhood \(\mathcal{N}_1 \subset \text{Image}(\psi)\) of \(\Gamma\) in \((-T_0, T_0) \times \mathbb{R}^3\), and a constant \(C\), both independent of \(\epsilon\), such that given \(\epsilon \in (0, 1]\), there exists a solution \(U\) of the abelian Higgs model (1.2), (1.3) satisfying the following estimates. First,
\[
\int_{\mathcal{N}_1} |\varphi - \phi^{\text{NO}}|^2 + \epsilon^2 |A - A^{\text{NO}}|^2 \leq C \epsilon^2
\]
in a suitable gauge, for \(U^{\text{NO}}\) defined in (1.25). Second,
\[
\int_{\mathcal{N}_1} (d^\nu)^2 \left( |D\varphi|^2 + \epsilon^2 |F|^2 + \frac{\lambda}{8 \epsilon^2} (|\varphi|^2 - 1)^2 \right) \ dx \ dt \leq C \epsilon^2,
\]
where \(d^\nu : N_1 \to \mathbb{R}\) is the distance in normal coordinates to \(\Gamma\), so that \(d^\nu \circ \psi(y) := |y^\nu|\). And finally,
\[
\int_{[-T_1, T_1] \times \mathbb{R}^3 \setminus \mathcal{N}_1} |D\varphi|^2 + \epsilon^2 |F|^2 + \frac{\lambda}{8 \epsilon^2} (|\varphi|^2 - 1)^2 \ dx \ dt \leq C \epsilon^2.
\]

**Remark 1.5.** From the construction of the initial data and conservation of energy, we will have \(\int_t \mathbb{R}^3 |D\varphi|^2 + \epsilon^2 |F|^2 \geq C > 0\) for every \(t\). Thus (1.28), (1.27) contain highly nontrivial information about energy concentration around \(\Gamma = \{d^\nu = 0\}\).

**Remark 1.6.** In fact we prove a more general stability result, giving estimates for any solution that at \(t = 0\) has a vortex filament near \(\Gamma_0\) and satisfies certain smallness conditions. For details see Proposition 6.1, from which one can also extract further estimates satisfied by the particular solution \(U\) of Theorem 1.4.
Remark 1.7. The hypotheses on $\lambda$ and $m$ are known to be satisfied for

- $|m| = 1$ and $\lambda \in \left[ \frac{1}{2}, 2 \right]$. This follows from Lemma 2.1 and Theorem 4.1 below.
- $|m| = 1$ and all $\lambda$ larger than some $\lambda_0$, see [21].
- $\lambda = 1$ and all $m \in \mathbb{Z}$, see [9].
- any $\lambda > 0$, and $m$ minimizing $n \mapsto \mathcal{E}_n^\lambda$ among nonzero integers. This again follows from Theorem 4.1.

They are believed to hold for all $\lambda > 0$ when $|m| = 1$, and for all $m \in \mathbb{Z}$ when $\lambda \in (0, 1)$. (See the next remark and remarks 1.2 and 1.3).

Remark 1.8. Nielsen and Olesen [20] and authors in the subsequent physics literature have in mind solutions of the form $U \approx U_{NO}$, where $U_{NO}$ satisfies (1.24), and $U_m$ is an equivariant $m = 1$ solution of the 2d abelian Higgs model. Thus, their exact scenario is established in Theorem 1.4 in the cases ($\lambda = 1$ or $\lambda$ large) when the $m = 1$ minimizer is known to (exist and) coincide with the equivariant solution. A full proof of their conjecture would involve showing that the $m = 1$ equivariant solution is a minimizer of the 2d Euclidean energy in the weak homotopy class $H_1$ for every $\lambda$. This is exactly the $m = 1$ case of the conjecture of Jaffe and Taubes mentioned in Remark 1.2.

Remark 1.9. The estimates obtained in [12, Theorem 2] for the non-gauged analog of the abelian Higgs model are similar to (1.27), (1.28) but much weaker. Indeed, they show that the total energy diverges like $|\ln \epsilon|$, whereas the weighted energy (as in (1.27), (1.28)) is bounded as $\epsilon \to 0$. Thus energy concentrates very weakly around the manifold $\Gamma$. In addition, no useful estimate along the lines of (1.26) is obtained in [12].

Remark 1.10. The assumption on $\Gamma$ to have zero initial velocity is there to avoid some technicalities of [12]. We also use it in the construction of the initial data. In principle, one could repeat the steps and assume nonzero velocity.

1.6. global well-posedness for the abelian Higgs model. The abelian Higgs model is a $U(1)$ version of the Yang-Mills-Higgs (YMH), where the gauge group is in general nonabelian. For the purposes of this article, we need the 1 + 3 dimensional abelian Higgs model to be well posed for $H^1_{1, loc} \times H^1_{1, loc}$ data (this is made precise in Section 1.7 below). In addition, since we are going to rescale, we need the well-posedness for large data and for all time (we would like the analysis to hold at least for the existence of the time-like minimal surface). Finally, we are interested in the topological behavior at infinity, $|\phi| \to 1$ instead of having $|\phi| \to 0$.

The strongest well-posedness result in the literature for YMH is due to Keel [14]. It shows global well-posedness of the 1 + 3 solution in the energy class for any size data in the temporal gauge. Moreover, the Higgs potential is taken to be energy critical, $V(\varphi) = |\varphi|^p$ with $p = 6$; the power six is the highest power that can be controlled using the Sobolev embedding by the kinetic part of the energy. A power $p < 6$ is called subcritical, and is significantly easier to handle. Since we have a quartic potential, the global well-posedness for the abelian Higgs model we need, in the temporal gauge, is implied by [14]. The only detail left is then addressing $|\phi| \to 1$ (see Section 1.7 below).

On the other hand, the proof in [14] is more sophisticated than what we need, and not only because of the critical power of the potential. An intermediate step leading to the global result is changing to a Coulomb gauge. In the nonabelian case, the Coulomb gauge can be constructed only locally in space, and hence the nonabelian case is much more technical than the abelian one, where the global Coulomb gauge can be constructed.
Therefore, due to having subcritical potential and the abelian problem, heuristically speaking, the global well-posedness we need can also follow from the work done on the Maxwell-Klein-Gordon problem [16, 22].

1.7. initial data. The solution \( \mathcal{U} \) that we find in Theorem 1.4 will be obtained by invoking the results in [14, Theorem 1.2]. To do this we will impose the temporal gauge

\[
A_0(0) = 0,
\]

and we require initial data such that

\[
\varphi(0), A_i(0) \in H^1_{\text{loc}}(\mathbb{R}^3),
\]

\[
\partial_t \varphi(0), \partial_t A_i(0) \in L^2_{\text{loc}}(\mathbb{R}^3), \quad i = 1, 2, 3,
\]

with the compatibility condition (stated in the temporal gauge)

\[
e^2 \partial_t \varphi(t) + (i \varphi(0), \partial_t \varphi(0)) = 0.
\]

Note in particular that (1.31), (1.32) hold if \( \partial_t \varphi(0) = \partial_t A_i(0) = 0, \quad i = 1, 2, 3 \), which will be the case for us. The initial data \( \mathcal{U}_{|t=0} \) is carefully constructed in the proof of Theorem 1.4 in Section 7. It has a rather explicit description near \( \Gamma \cap \{ t = 0 \} \), in terms of the minimizer \( U^m \) from (1.16) and the diffeomorphism \( \psi \) from (1.21), and away from \( \Gamma \cap \{ t = 0 \} \) it has the form

\[
\varphi(0) = e^{iq}, \quad A_i(0) = \frac{\partial q}{\partial x^i}, \quad i = 1, 2, 3.
\]

for some smooth \( q \). From these facts it follows that (1.30) holds, and from [14] we then obtain a global solution in the temporal gauge.

We note that in particular we will consider data such that (1.33) holds in \( \mathbb{R}^3 \setminus B_R \) for some \( R \). By finite propagation speed and an easy explicit calculation, \( \varphi(t) = e^{iq}, A(t) = dq \) is a solution on \( |x| > R + t \). By uniqueness, it must agree with the solution obtained using [14].

1.8. some notation. As mentioned above, we implicitly sum over repeated upper and lower indices. We use the convention that greek indices \( \alpha, \beta, \mu, \nu \ldots \) run from 0 to 3, and latin indices \( i, j, k \ldots \) run from 1 to 3.

For the convenience of the reader, we include the following summary of the different solutions we work with

- In normal coordinates: \( U = (\phi, A) \)
- In the standard Minkowski coordinates: \( U = (\varphi, A) \), and when applicable we use (1.23) to relate \( U \) and \( \mathcal{U} \).
- \( U^m = U_{c, \lambda}^m \) solution of the 2D minimization problem (1.16)
- \( U^{(m)} \) equivariant solution of the 2D minimization problem
- \( U^{\text{Nielsen-Olesen}} \) Nielsen-Olesen solution in the normal coordinates given by (1.24)
- \( U^{\text{Nielsen-Olesen}} \) Nielsen-Olesen solution in the standard Minkowski coordinates given by (1.25)

1.9. organization of this paper. Sections 2 - 4 deal with aspects of the 2d Euclidean abelian Higgs model needed for our main dynamical results. We start in Section 2 with some general background material. Section 3 introduces, and establishes some basic properties of, what we call a vorticity confinement functional. This functional plays an important role in the proof of Theorem 1.4. In Section 4 we prove Theorem 4.1 giving a criterion for existence of solutions of the minimization problem (1.16). As mentioned in Remark 1.7.
above, this result shows that the hypotheses of Theorem 1.4 are satisfied for a range of values of the parameters \(m, \lambda\).

Sections 5 and 6 consider the abelian Higgs model in 1 + 3-dimensional Minkowski space. A basic ingredient in our analysis, as in [12], is supplied by weighted energy estimates in the normal coordinate system, introduced in Section 1.4. These estimates are proved in Section 5, using results about the vorticity confinement functional from Section 3. Finally, section 6 is devoted to the proof of Theorem 1.4.

2. Energy and vorticity in 2 dimensions

In the next three sections, we focus on Euclidean abelian Higgs model in 2 dimensions. In this section we record some facts, mostly well-known, relating the energy \(e_{\epsilon,\lambda}(U)\) and the vorticity \(\omega(U)\), defined in (1.9) and (1.12) respectively. We recall that the parameter \(\epsilon\) is just a scaling parameter, see (1.10), so that all results in this section reduce to the case \(\epsilon = 1\). However, due to the role it plays elsewhere in this paper, it seems useful to formulate things here for general \(\epsilon > 0\).

First, by a direct computation we have the following identity, due to Bogomol’nyi:

\[
e_{\epsilon,\lambda}(U) = \pm \omega(U) + \frac{1}{2} |(D_1 \pm iD_2)\phi|^2 + \frac{1}{2} \left( \epsilon F_{12} \pm \frac{1}{2\epsilon} (|\phi|^2 - 1) \right)^2 + \frac{\lambda - 1}{8\epsilon^2} (|\phi|^2 - 1)^2.
\]

We emphasize that the identity holds pointwise. Note that (2.1) implies that

\[
|\omega(U)| \leq \max\{1, \lambda^{-1}\} e_{\epsilon,\lambda}(U)
\]

pointwise. This follows immediately from (2.1) if \(\lambda \geq 1\), and if \(\lambda \leq 1\) it follows by noting that \(|\omega| \leq e_{\epsilon,1}(U) \leq \lambda^{-1} e_{\epsilon,\lambda}(U)\).

We immediately deduce from (2.2) that \(\omega(U)\) is integrable for any finite-action \(U = (\phi, A)\), and it is known (and follows rather easily from Lemma 2.4 below) that \(\int_{\mathbb{R}^2} \omega(U) \in \pi\mathbb{Z}\).

Let \(U^{(m)}\) denote the \(\lambda = 1\), equivariant solution in the weak homotopy class \(H_m\), discussed in Section 1.3. It is well-known that \(U^{(m)}\) satisfies

\[
(D_1 + \sigma iD_2)\phi = 0, \quad F_{12} + \sigma \frac{1}{2\epsilon} (|\phi|^2 - 1) = 0, \quad \sigma := \text{sign}(m),
\]

see for example [9]. By combining these with (2.1) we see that

\[
E_{m}^\lambda = \pi|m| = \int_{\mathbb{R}^2} |\omega(U^{(m)})|.
\]

From this we easily deduce the following

Lemma 2.1. If \(|m| \leq \lambda \leq \frac{|m|+1}{|m|}\), then

\[
E_{m}^\lambda < E_{n}^\lambda \text{ whenever } |m| < |n|.
\]

In particular, \(E_{1}^\lambda < E_{n}^\lambda\) for all \(|n| \geq 2\) if \(\frac{1}{2} \leq \lambda \leq 2\).

Proof. It is clear from (2.3) that the lemma holds for \(\lambda = 1\). If \(1 < \lambda \leq \frac{|m|+1}{|m|}\), then \(e_{\epsilon,1}(U) \leq e_{\epsilon,\lambda}(U) \leq \lambda e_{\epsilon,1}(U)\) pointwise for every \(U\). In addition, \(\lambda|m| \leq |n|\) if \(|n| > |m|\). From these we deduce that

\[
E_{m}^\lambda \leq \lambda E_{m}^1 = \lambda|m|\pi \leq |n|\pi = E_{n}^1 \leq E_{n}^\lambda \quad \text{if } |m| < |n|,
\]
and it is not hard to check that at least one inequality is strict. If $\frac{|m|}{|m|+1} \leq \lambda < 1$ and $|m| < |n|$, then $\lambda e_{\epsilon,1}^U(U) \leq e_{\epsilon,\lambda}^U(U) \leq e_{\epsilon,1}^U(U)$ and $|m| \leq \lambda |n|$, and the conclusion follows very much as above. \hfill \Box

**Remark 2.2.** With a little more work one can prove by similar arguments that (2.4) holds for a slightly larger range of $\lambda$, but these sorts of simple arguments have no hope of proving the natural conjecture, which is that it is valid for all $\lambda > 0$.

We also remark that it is known from [21] that if $\lambda$ is sufficiently large then (2.4) is true for all $m$ and $n$.

We conclude this section by proving the lemma mentioned above, which shows that the vorticity is approximately quantized on a set on which the boundary energy is not too large. For this we need

**Lemma 2.3.** There exists constant $C$ such that if $S \subset \mathbb{R}^2$ is a bounded, connected, and simply connected set, and $\partial S$ is Lipschitz with $|\partial S| \geq \epsilon$, and if $\rho$ is a smooth nonnegative function on a neighborhood of $\partial S$, then

$$\int_{\partial S} \frac{1}{2} |\nabla \tau \rho|^2 + \frac{\lambda}{8\epsilon^2} (1 - \rho^2)^2 dH^1 \geq \frac{\sqrt{\lambda}}{C\epsilon} \|1 - \rho\|_{L^\infty(\partial S)},$$

where $\nabla \tau$ denotes the tangential derivative along $\partial S$.

This is proved in [10, lemma 2.3] when $S$ is a ball, and exactly the same argument applies here, since the proof only involves integrating along $\partial S$, which is isometric to a circle.

**Lemma 2.4.** Assume that $S \subset \mathbb{R}^2$ is connected and simply connected with Lipschitz boundary. Let $\lambda > 0$. There exists a constant $C$, depending on $\lambda$, such that if $|\partial S| \geq \epsilon$ then

$$\left| \int_S \omega - \pi n \right| \leq C\epsilon \int_{\partial S} \left[ \frac{|\nabla A \phi|^2}{2} + \frac{\lambda (|\phi|^2 - 1)^2}{8\epsilon^2} \right] dH^1$$

for some $n \in \mathbb{Z}$. Moreover, if $\int_{\partial S} e_{\epsilon,\lambda}^\nu(U) \leq \frac{1}{C\epsilon}$, then in fact $n = \text{deg}(\frac{\phi}{(|\phi|}; \partial S)$.

**Proof.** **Case 1:** If $\inf_{\partial S} |\phi| < \frac{1}{2}$, then since $|\nabla \tau \phi| \leq |\nabla A \phi|$, we can apply Lemma 2.3 to $\rho = |\phi|$ to find that

$$\epsilon \int_{\partial S} \left[ \frac{|\nabla A \phi|^2}{2} + \frac{\lambda (|\phi|^2 - 1)^2}{8\epsilon^2} \right] dH^1 \geq \frac{\sqrt{\lambda}}{4C}.$$

Since $\min_{n \in \mathbb{Z}} |a - \pi n| \leq \frac{\pi}{2}$ for every $a \in \mathbb{R}$, this implies (2.5).

**Case 2:** We assume that

$$|\phi| \geq \frac{1}{2} \quad \text{on} \quad \partial S.$$  \hfill (2.6)

Note in particular that this occurs if $\int_{\partial S} e_{\epsilon,\lambda}^\nu(U) \leq \frac{1}{C\epsilon}$, due to Lemma 2.3. Because of (2.6) we can then write $\phi = \rho e^{i\eta}$ on $\partial S$, and in this notation,

$$j(U) = \rho^2 (\nabla \eta - A) \quad \text{on} \quad \partial S,$$
so that \( j + A = \nabla \eta + (\rho^2 - 1)(\nabla \eta - A) \) on \( \partial S \). Hence
\[
\int_S \omega = \frac{1}{2} \int_S \nabla \times (j + A) \\
= \frac{1}{2} \int_{\partial S} (j + A) \cdot \tau \\
= \frac{1}{2} \int_{\partial S} \nabla \eta \cdot \tau + \frac{1}{2} \int_{\partial S} (\rho^2 - 1)(\nabla \eta - A) \cdot \tau.
\]
And the conclusion now follows by noting that
\[
\int_{\partial S} \nabla \eta \cdot \tau = 2 \pi \deg (\phi ; \partial S) \in 2\pi \mathbb{Z},
\]
and, recalling (2.6),
\[
\left| \frac{1}{2} \int_{\partial S} (\rho^2 - 1)(\nabla \eta - A) \cdot \tau \right| \leq \frac{1}{2} \int_{\partial S} \left| \frac{\rho^2 - 1}{\rho} \right| |\rho(\nabla \eta - A)| \\
\leq \int_{\partial S} \left| \phi \right|^2 - 1 \left| \nabla A \phi \right| \\
\leq \frac{2c}{\lambda^{1/2}} \int_{\partial S} \left| \nabla A \phi \right|^2 + \frac{\lambda}{8c^2} (|\phi|^2 - 1)^2.
\]
\[\square\]

3. 2D Vorticity confinement functional

Let \( m \) be a positive integer. For a configuration \( U = (\phi, A) \) on \( B_{\nu}(R) \subset \mathbb{R}^2 \), we define\(^2\)
\[
\mathcal{D}_m^\nu(U; R) := \pi m - \int_{B_{\nu}(R)} f(|y'|) \omega(U)(y') dy'
\]
where \( f : [0, R] \to [0, 1] \) is a fixed smooth function satisfying
\[
f(0) = 1, \quad f(R) = 0, \quad 0 \geq f'(r) \geq -Cr^2 \text{ for all } r
\]
where of course \( C \) depends on \( R \). We expect \( \mathcal{D}_m^\nu(U; R) \) to be small (or negative) if (at least) \( m \) quanta of vorticity are concentrated near the center of the ball \( B_{\nu}(R) \).

The main results of this section are the following two propositions, both of which relate \( c_{\epsilon, \lambda}^\nu \) and \( D_m^\nu \). They together yield stability properties that are used in a crucial way in our proof of Theorem 1.4.

Our first proposition will allow us to control changes in \( \mathcal{D}_m^\nu \). In its statement and proof, we write points in \((0, T) \times B_{\nu}(R)\) in the form \((y', y^0)\).

**Proposition 3.1.** Let \( U = (\phi, A) \) be a configuration on \((0, T) \times B_{\nu}(R)\), for some \( T > 0 \) and \( B_{\nu}(R) \subset \mathbb{R}^2 \), so that \( A \) has components \( A_i \in H^1((0, T) \times B_{\nu}(R)) \) for \( i = 0, 1, 2 \). Then for every \( \lambda > 0 \), integer \( m \), there exists a constant \( C = C(\lambda, m, R) \) such that
\[
|\mathcal{D}_m^\nu(U(t); R) - \mathcal{D}_m^\nu(U(0); R)|
\leq C \int_{(0, t) \times B_{\nu}(R)} \left( |D_0 \phi|^2 + c^2(F_{01}^2 + F_{02}^2) + |y'|^2 c_{\epsilon, \lambda}^\nu(U) \right) dy' dy^0
\]
\(^2\)In this section, we write \( y'^\nu \) to denote a point in \((y', y^0) \subset \mathbb{R}^2 \). This variable plays the same role as the \( y^\nu \) in Sections 5 and 6 where however \( y^\nu = (y^1, y^0) \).
for every $t \in (0, T)$.

We will in fact prove something stronger than (3.3), but here we have recorded only the conclusion that is needed for the proof of Theorem 1.4.

The other main result of the section shows that control over $D^\nu_m$ implies good lower energy bounds.

**Proposition 3.2.** Suppose that $\lambda > 0$ and $m \in \mathbb{N}$ satisfy

\begin{equation}
(3.4) \quad \mathcal{E}^\lambda_m \leq \mathcal{E}^\lambda_n \quad \text{whenever } n \geq m.
\end{equation}

Then for every $R > 0$, there exist constants $\kappa_1$ and $C$, depending on $R, \lambda, m$, such that if $U = (\phi, A)$ is a finite-energy configuration satisfying

$$D^\nu_m(U; R) < \kappa_1,$$

then

\begin{equation}
(3.5) \quad \int_{B^\nu(R)} e^\nu_{\epsilon,\lambda}(U) \geq \mathcal{E}^\lambda_m - C \epsilon^2
\end{equation}

for all $\epsilon \in (0, 1]$.

We remark that, although one could extract from our arguments estimates of how various constants depend on $\lambda$, we have not made any effort to optimize this dependence, and indeed we appeal several times to (2.2), which is far from sharp when $0 < \lambda \ll 1$ or $\lambda \gg 1$.

### 3.1. Proof of Proposition 3.1

We will use the following lemma

**Lemma 3.3.** Assume the hypotheses of Proposition 3.1. Then for every $r \in (0, R)$ and $t \in (0, T)$

\begin{equation}
(3.6) \quad \left| \int_{B^\nu(r)} \omega(U(t)) - \int_{B^\nu(r)} \omega(U(0)) \right| \leq \max \{1, \lambda^{-1}\} \int_{(0,t) \times \partial B^\nu(r)} e^{3d}_{\epsilon,\lambda}(U) \ d\mathcal{H}^2
\end{equation}

where $\mathcal{H}^2$ is 2-dimensional Hausdorff measure and $e^{3d}_{\epsilon,\lambda}(U)$ denotes the 3-dimensional energy density

$$e^{3d}_{\epsilon,\lambda}(U) := \frac{1}{2} \sum_{a=0}^{2} |D_a \phi|^2 + \frac{\epsilon^2}{2} \sum_{0 \leq a < b \leq 2} F_{ab}^2 + \frac{\lambda}{8 \epsilon^2} (|\phi|^2 - 1)^2.$$

Note that although $y^0$ is naturally identified with time, we think of and write $e^{3d}_{\epsilon,\lambda}$ as 3-dimensional rather than 1 + 2 dimensional, since $\frac{1}{2} \sum_{a=0}^{2} |D_a \phi|^2$ is a sort of Euclidean (rather than Minkowski) norm-squared of the covariant derivative.

**Proof.** For this proof only, we will write $\omega$ to denote the 3-dimensional vorticity, which we identify with the 2-form $d(j + A)$ in $B^\nu(r) \times [0, T]$. Here $j = \sum_{a=0}^{2} (i \phi, D_a \phi) dy^a$. Since $\omega$ is exact,

$$\int_{\partial((0,t) \times B^\nu(r))} \omega = \int_{(0,t) \times B^\nu(r)} d\omega = 0.$$

Breaking $\partial((0,t) \times B^\nu(r))$ into pieces, we deduce that

$$\left| \int_{\{t\} \times B^\nu(r)} \omega - \int_{\{0\} \times B^\nu(r)} \omega \right| = \left| \int_{(0,t) \times \partial B^\nu(r)} \omega \right|$$

Note that although $y^0$ is naturally identified with time, we think of and write $e^{3d}_{\epsilon,\lambda}$ as 3-dimensional rather than 1 + 2 dimensional, since $\frac{1}{2} \sum_{a=0}^{2} |D_a \phi|^2$ is a sort of Euclidean (rather than Minkowski) norm-squared of the covariant derivative.
where \( \{ s \} \times B_\nu(r) \) is understood to have the standard orientation for \( s = 0, t \) (rather than the orientation inherited as part of \( \partial((0, t) \times B_\nu(r)) \)). The left-hand side of this identity is just the left-hand side of (3.6) in slightly different notation, so it suffices to estimate the right-hand side, which can be written more explicitly as

\[
\left| \int_{(0,t) \times \partial B_\nu(r)} \omega(U)(\tau_0, \tau_1) \, d\mathcal{H}^2 \right|
\]

where \( \tau_0(y), \tau_1(y) \) is a properly oriented orthonormal basis for \( T_y((0,t) \times \partial B_\nu(r)) \), and \( \omega(U)(\tau_0, \tau_1) \) at a point \( y \) denotes the number obtained by the two-form \( \omega(U(y)) \) acting on the vectors \( \tau_0(y), \tau_1(y) \). Thus it suffices in fact to show that

\[
|\omega(U)(\tau_0, \tau_1)| \leq \max\{1, \lambda^{-1}\} e_{\epsilon, \lambda}^{3d}(U) \quad \text{for any orthonormal vectors } \tau_0, \tau_1.
\]

To prove this, note that \( \omega(\tau_0, \tau_1) \) at a point \( y \) is just the two-dimensional vorticity of the restriction of \( U \) to the (suitably oriented) plane through \( y \) spanned by \( \tau_0, \tau_1 \), and so (2.2) implies that \( |\omega(U)(\tau_0, \tau_1)| \) is bounded by \( \max\{1, \lambda^{-1}\} \) times the 2-dimensional energy \( e_{\epsilon, \lambda}^{\nu} \) of the restriction of \( U \) to the same plane, and this clearly implies (3.7).

\[\square\]

Using the above lemma, we complete the

**Proof of proposition 3.1.** Let \( g = -f' \), where \( f \) is the function appearing in the definition of \( D_\nu^\nu \). Then the choice (3.2) of \( f \) implies that

\[
0 \leq g(r) \leq Cr^2, \quad \int_0^R g(r) \, dr = 1.
\]

Then

\[
\int_{B_\nu(R)} \omega(y') f(|y'|) \, dy' = \int_{B_\nu(R)} \omega(y') \left( \int_0^R g(s) \, ds \right) \, dy' \]

\[
= \int_{B_\nu(R)} \int_0^R \omega(y') \chi_{|y'| < s} g(s) \, ds \, dy' \]

\[
= \int_0^R g(s) \left( \int_{B_\nu(s)} \omega(y') \, dy' \right) \, ds.
\]

(3.9)

It follows from this and the definition (3.1) of \( D_\nu^\nu \) that

\[
|D_\nu(U(t)) - D_\nu(U(0))| = \left| \int_{B_\nu(R)} f(r) \left( \omega(U(r)) - \omega(U(0)) \right) \right|
\]

\[
= \left| \int_0^R g(r) \left( \int_{B_\nu(r)} \omega(U(t)) - \omega(U(0)) \right) \, dr \right|.
\]

(3.8)

This is easily verified by fixing an orthonormal basis \( \tau_0, \tau_1, \tau_2 \) for \( \mathbb{R}^3 \), and then noting that \( \omega(\tau_0, \tau_1) = \partial_0(j(U) + A_1) - \partial_1(j(U) + A_0) \), where \( \partial_i, j(U)_i \), and \( A_i \) are all written with respect to the basis \( \{ \tau_i \} \).
Then by (3.8) and Lemma 3.3, and recalling the definition of $e_{\epsilon,\lambda}(U),$

$$|D_{\nu}(U(t)) - D_{\nu}(U(0))| \leq C \int_0^R r^2 \left( \int_{(0,t) \times \partial B_{\nu}(r)} e_{\epsilon,\lambda}(U) \ d\mathcal{H}^2 \right) dr$$

$$= C \int_{(0,t) \times B_{\nu}(R)} |y|^2 \left[ \frac{|D_{\phi}|^2}{2} + \frac{\epsilon^2}{2} (\mathcal{F}^2_0 + \mathcal{F}^2_0) + e_{\epsilon,\lambda}(U) \right] dy^\nu dy^0,$$

and this immediately implies (3.3). □

3.2. Lower Energy Bounds. A large part of the proof of Proposition 3.2 is contained in the following lemma.

Lemma 3.4. Given a smooth configuration $U = (\phi, A)$ on $B_{\nu}(R) \subset \mathbb{R}^2,$ for every $C_1 > \frac{2}{R}$ there exists $C_2, \epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$ and

$$\int_{\partial B_{\nu}(s)} e_{\epsilon,\lambda}(U) \leq C_1 \quad \text{for some } s \in (\epsilon, R - \frac{1}{C_1}),$$

then $n(s) := \deg(\frac{\phi}{|\phi|}; \partial B_{\nu}(s)) \in \mathbb{Z}$ is well-defined, and

$$\int_{B_{\nu}(R)} e_{\epsilon,\lambda}(U) \geq \mathcal{E}_{n(s)} - C_2 \epsilon^2.$$  

We defer the proof of this until the end of this section. We note however that when $\lambda = 1,$ a weaker version of the conclusion (with an error term of order $\epsilon$ rather than $\epsilon^2$) follows immediately from Lemma 2.4 and the fact that $\mathcal{E}_n^1 = \pi |n|.$

Proof of proposition 3.2 1. We may assume that

$$\int_{B_{\nu}(R)} e_{\epsilon,\lambda}(U) \leq \mathcal{E}_m^\lambda$$

as otherwise the conclusion of the proposition is immediate. Also, standard density arguments allow us to assume that $U$ is smooth.

Consider balls $B_{\nu}(s) = B_s, \epsilon \leq s \leq R.$ We say that $s$ is good if it satisfies the hypotheses of Lemma 3.4 for some $C_1 > \frac{2}{R}$ to be chosen below, so that $\epsilon < s < R - \frac{1}{C_1}$ and

$$\int_{\partial B_s} e_{\epsilon,\lambda}(U) d\mathcal{H}^1 \leq C_1.$$  

If $s$ is not good, then it is said to be bad. Lemma 3.4 implies that there exists some $C_2$ (depending on $C_1$) such that if $s$ is good, then

$$\int_{B_s} e_{\epsilon,\lambda}(U) \geq \mathcal{E}_{n(s)}^\lambda - C_2 \epsilon^2$$

for $n(s) = \deg(\frac{\phi}{|\phi|}; \partial B_s).$

Because $n \mapsto \mathcal{E}_n^\lambda$ is increasing for $n \geq 0$ by hypothesis, see (3.4), the proposition follows if

$$\int_{B_s} e_{\epsilon,\lambda}(U) \geq \mathcal{E}_{n(s)}^\lambda - C_2 \epsilon^2$$

for $n(s) = \deg(\frac{\phi}{|\phi|}; \partial B_s).$

We therefore assume that (3.14) does not hold, which in view of Lemma 2.4 and the definition of good $s$ implies that

$$\int_{B_s} \omega \leq \pi (m - 1) + C \epsilon$$

for all good $s,$
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(For $C$ depending on $C_1$). We will show that this implies the desired lower bound \((3.5)\). Toward this end, first note that, owing to (3.12),
\[
E_{\lambda}^\epsilon m \geq \int_{\partial B_s} e_{\epsilon,\lambda}^\nu(U) dH^1 ds \geq \int_{\partial B_s} (\int_{\partial B_s} e_{\epsilon,\lambda}^\nu(U) dH^1 ) ds \\
\geq C_1 |\{ s \in (\epsilon, R) : \text{(3.13) fails} \} |.
\]
Hence
\[
|\{ s \in (\epsilon, R) : \text{(3.13) fails} \} | \leq \frac{E_{\lambda}^\epsilon m}{C_1}.
\]
As a result, if $0 \leq \epsilon < \epsilon_0$ and $\epsilon_0 \leq \frac{1}{C_1}$, then
\[(3.16) \ |\{ s \in (0, R) : s \text{ is bad} \}| \leq \epsilon + \frac{1}{C_1} + \frac{C_1}{C_1} \leq \frac{(2 + E_{\lambda}^\epsilon m)}{C_1}.
\]
2. It follows from (3.9) and the definition (3.1) of $D_{\nu}^\epsilon$ that if $D_{\nu}^\epsilon(U; R) \leq \kappa_1$, then
\[
\pi m - \kappa_1 \leq \int_0^R g(s) \left( \int_{B_s} \omega dy^\nu \right) ds.
\]
Then we deduce from (3.15) that
\[
\pi m - \kappa_1 \leq \int_{\text{good } s} g(s) \left( \int_{B_s} \omega dy^\nu \right) ds + \int_{\text{bad } s} g(s) \left( \int_{B_s} \omega dy^\nu \right) ds \\
\leq \pi (m - 1) + C_\epsilon + |\{ \text{bad } s \}| \| g \|_\infty \sup_{\text{bad } s} \left( \int_{B_s} \omega dy^\nu \right).
\]
Rearranging and using (3.16), we find that
\[(3.17) \sup_{\text{bad } s} \left( \int_{B_s} \omega dy^\nu \right) \geq \frac{\pi - \kappa_1 - C_\epsilon}{\| g \|_\infty |\{ \text{bad } s \}|} \geq \frac{C_1 (\pi - \kappa_1 - C_\epsilon)}{\| g \|_\infty (2 + E_{\lambda}^\epsilon m)}.
\]
Also, it follows from (2.2) that
\[
\int_{B_{\nu}(R)} e_{\epsilon,\lambda}^\nu(U) \geq \min\{1, \lambda\} \sup_{\text{bad } s} \left( \int_{B_s} \omega(U) dy^\nu \right).
\]
Then (3.5) follows from (3.17) for all sufficiently small $\epsilon$, if we choose $\kappa_1 = \frac{\pi}{2}$ and $C_1 \geq \frac{R}{2}$ such that $C_1 \geq \max\{1, \lambda^{-1}\} \| g \|_\infty E_{\lambda}^\epsilon m (2 + E_{\lambda}^\epsilon m)$ for example.

We now prove the lemma that was used above to guarantee a nearly sharp lower energy bound for a ball bounded by a “good radius”.

Proof of Lemma 3.4. We have assumed that $U$ satisfies
\[(3.18) \int_{\partial B_{\nu}(s)} e_{\epsilon,\lambda}^\nu(U) dH^1 \leq C_1.
\]
It follows from this and Lemma (2.3) that for $\epsilon$ small enough,
\[(3.19) \ |1 - |u|| \leq \frac{1}{2} \text{ on } \partial B_{\nu}(s)
\]
and hence that $n = n(s) = \deg_-_\varphi(\partial B_{\nu}(s))$ is well-defined. (In fact $\int_{B_{\nu}(s)} \omega = n + O(\epsilon)$, by Lemma (2.4).
1. We first claim that there is a configuration $\tilde{U}$ on $\mathbb{R}^2$ such that $\tilde{U} = U$ in $B_\nu(s)$,

$$
\tilde{U} \in H_n, \quad \text{and} \quad \int_{\mathbb{R}^2 \setminus B_\nu(s)} e^{\nu}_{\epsilon, \lambda}(\tilde{U}) < C \epsilon \int_{\partial B_\nu(s)} e^{\nu}_{\epsilon, \lambda}(U)d\mathcal{H}^1.
$$

Although our definition of $H_n$ requires that $\tilde{A} \in H^1_{loc}$, it suffices to construct $\tilde{U}$ such that $\tilde{A} \in L^1_{loc}$ and the distributional exterior derivative satisfies $d\tilde{A} = F_{12}dy^1 \wedge dy^2$, with $F_{12} \in L^2$, since any such $\tilde{U}$ can be approximated arbitrarily well by smooth (hence $H^1_{loc}$) functions, via a standard mollification procedure for example.

**Definition of $\tilde{U}$**. We will write $\tilde{U}$ on $\mathbb{R}^2 \setminus B_\nu(s)$ in polar coordinates $(r, \theta), r \geq s, \theta \in \mathbb{R}/2\pi\mathbb{Z}$. First we write $U$ on $\partial B_\nu(s)$ in the form

$$
\phi(s, \theta) = \rho(\theta)e^{iq(\theta)}
$$

$$
A(s, \theta) = A_r(\theta)dr + A_\theta(\theta)d\theta
$$

for certain smooth functions $\rho, A_r, A_\theta : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ and $q : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$. Note that $\rho(\theta) \geq \frac{1}{2}$ for every $\theta$, and that

$$
\int_0^{2\pi} q'(\theta) d\theta = 2\pi \deg\left(\frac{\phi}{|\phi|}, \partial B_\nu(s)\right) = 2\pi n.
$$

We define $\tilde{U} = (\tilde{\phi}, \tilde{A})$ as follows:

- $\tilde{U} = U$ in $B_\nu(s)$.
- If $s < r < s + \epsilon$, then

$$
\tilde{\phi}(r, \theta) = \left[\rho + \frac{r - s}{\epsilon}(1 - \rho)\right] e^{i\tilde{q}}, \quad \tilde{A}(r, \theta) = \left[A_\theta + \frac{r - s}{\epsilon}(q' - A_\theta)\right] d\theta.
$$

- If $r \geq s + \epsilon$, then $\tilde{\phi}(r, \theta) = e^{iq(\theta)}$, $\tilde{A}(r, \theta) = q'(\theta)d\theta$.

It is standard, and easy to check, that the distributional exterior derivative of $\tilde{A}$ satisfies $d\tilde{A} = d(\tilde{A}_r dr + \tilde{A}_\theta d\theta) = (\partial_r \tilde{A}_\theta - \partial_\theta \tilde{A}_r)dr \wedge d\theta$ in $\mathbb{R}^2$, despite the possible discontinuity of $\tilde{A}_r$ across $\{(r, \theta) : r = s\}$. In particular $d\tilde{A} \in L^2_{loc}(\mathbb{R}^2)$. Note also that $e^{\nu}_{\epsilon, \lambda}(\tilde{U}) = 0$ outside of $B_{s+\epsilon}$, so $\tilde{U}$ is a finite-energy configuration. It then follows from (3.21) that $\tilde{U} \in H_n$.

**Energy of $\tilde{U}$**. Since as noted above $e^{\nu}_{\epsilon, \lambda}(\tilde{U}) = 0$ outside $B_{s+\epsilon}$, to complete the proof of (3.20) it suffices to estimate the energy of $\tilde{U}$ in the annulus $s < r < s + \epsilon$. So we henceforth restrict our attention to this set.

Writing $\tilde{\phi} = \tilde{\rho} e^{i\tilde{q}}$ and noting from (3.19) that $\frac{1}{2} \leq \tilde{\rho} \leq \frac{3}{2}$, we estimate

$$
|D_\tilde{A} \tilde{\phi}|^2 = \frac{1}{r^2}(\partial_\theta \tilde{\rho})^2 + (\partial_r \tilde{\rho})^2 + \tilde{\rho}^2 \left[\frac{(\partial_\theta \tilde{q} - \tilde{A}_\theta)^2}{r^2} + (\partial_r \tilde{q} - \tilde{A}_r)^2\right]
$$

$$
\leq \frac{1}{r^2} \rho'(\theta)^2 + \frac{(1 - \rho(\theta))^2}{\epsilon^2} + C \left(q'(\theta) - A_\theta(\theta)\right)^2
$$

and similarly

$$
e^2 |d\tilde{A}|^2 = |(q' - A_\theta)dr \wedge d\theta|^2 = \frac{1}{r^2}(q' - A_\theta)^2
$$

for certain smooth functions $\tilde{\rho}, \tilde{q}, \tilde{A}_r, \tilde{A}_\theta$.
and clearly \((\rho^2 - 1)^2 \leq (\rho(\theta))^2 - 1)^2\). We combine these and conclude, after noting that \((\rho - 1)^2 \leq (\rho^2 - 1)^2\) for \(\rho \geq 0\) and again using (3.19), that for \(s \leq r \leq s + \epsilon\),

\[
e_{\epsilon, \lambda}(\bar{U})(r, \theta) \leq C(\lambda) \left( \frac{1}{2s^2} \rho'(\theta)^2 + \frac{\rho'(\theta)(q' - A_{\theta})^2}{s^2} + \frac{\lambda}{8s^2}(1 - \rho^2(\theta))^2 \right)
\]

\[
\leq C(\lambda) e_{\epsilon, \lambda}(U)(s, \theta).
\]

Thus

\[
\int_0^{2\pi} \int_s^{s+\epsilon} e_{\epsilon, \lambda}(\bar{U})(r, \theta) r \, dr \, d\theta \leq C(\lambda) \int_s^{s+\epsilon} \frac{r}{s} \int_0^{2\pi} e_{\epsilon, \lambda}(U)(s, \theta) s \, d\theta
\]

\[
\leq C(\lambda) \epsilon \int_{\partial B_{\nu}(s)} e_{\epsilon, \lambda}(U) dH^1
\]

which completes the proof of (3.20).

2. Now we compute using the definition of \(\mathcal{E}_n^\lambda\) and (3.18),

\[
\mathcal{E}_n^\lambda \leq \int_{\mathbb{R}^2} e_{\epsilon, \lambda}(\bar{U}) = \int_{B_{\nu}(s)} e_{\epsilon, \lambda}(U) + \int_{\mathbb{R}^2 \setminus B_{\nu}(s)} e_{\epsilon, \lambda}(\bar{U}) = \int_{B_{\nu}(R)} e_{\epsilon, \lambda}(U) - \int_{B_{\nu}(R) \setminus B_{\nu}(s)} e_{\epsilon, \lambda}(U) + C\epsilon.
\]

As a result, the conclusion (3.11) follows unless

\[
C\epsilon \geq \int_{B_{\nu}(R) \setminus B_{\nu}(s)} e_{\epsilon, \lambda}(U) = \int_s^{R} \int_{\partial B_{\nu}(\sigma)} e_{\epsilon, \lambda}(U) dH^1.
\]

And if this holds, we can find some \(\sigma \in (s, R)\) such that

\[
\int_{\partial B_{\nu}(\sigma)} e_{\epsilon, \lambda}(U) dH^1 \leq \frac{C\epsilon}{R - s} \leq C_1 \epsilon, \quad \text{since} \ s < R - \frac{1}{C_1} \text{by hypothesis.}
\]

Then by exactly the construction of Step 1, we can find some \(\hat{U}\) that equals \(U\) in \(\partial B_{\nu}(\sigma)\), and such that

\[
\hat{U} \in H_{n(\sigma)}, \quad \int_{\mathbb{R}^2 \setminus B_{\nu}(s)} e_{\epsilon, \lambda}(\hat{U}) < C\epsilon \int_{\partial B_{\nu}(\sigma)} e_{\epsilon, \lambda}(U) dH^1 \leq C\epsilon^2
\]

Note also that it follows from (3.23), the fact that \(|\omega(U)| \leq C e_{\epsilon, \lambda}(U)\) and Lemma 2.4 that \(n(\sigma) = n(s) = n\). So by arguing exactly as in (3.22) we find that

\[
\mathcal{E}_n^\lambda \leq \int_{\mathbb{R}^2} e_{\epsilon, \lambda}(\hat{U}) \leq C\epsilon^2 + \int_{B_{\nu}(s)} e_{\epsilon, \lambda}(U) \leq C\epsilon^2 + \int_{B_{\nu}(R)} e_{\epsilon, \lambda}(U),
\]

completing the proof of the lemma.

To close this section, we record for future reference the fact that Lemma 3.4 holds on domains more general than balls; this will be used in the proof of Theorem 4.1. Although we state the result for a square, which is what we need, it is clear that the proof remains valid for any domain that is bi-Lipschitz homeomorphic to a ball, with a constant depending on the domain. For simplicity, we prove the lemma with error terms of order \(\epsilon\) rather than \(\epsilon^2\), as this suffices for our later application.
Lemma 3.5. Given a configuration $U = (\phi, A)$ on an open set containing $Q_s = (-s, s)^2 \subset \mathbb{R}^2$, for every $C_1 > 0$ there exists a $C_2$ such that if

$$\int_{\partial Q_s} e_{\epsilon, \lambda}(U) \leq C_1 \quad (3.24)$$

then $n(s) := \text{deg}(\phi_{|Q_s}; \partial Q_s) \in \mathbb{Z}$ is well-defined and

$$\int_{Q_s} e^\nu_{\epsilon, \lambda}(U) \geq \mathcal{E}^\lambda_{n(s)} - C_2 \epsilon. \quad (3.25)$$

Proof. If $\Psi : B \to Q$ is a Lipschitz map between subsets of $\mathbb{R}^2$, and $U$ is a configuration on $Q$, we will write $\Psi^* U$ to denote the configuration $(\phi \circ \Psi, \Psi^* A)$ on $B$. Note that

$$e^\nu_{\epsilon, \lambda}(\Psi^* U)(y^\nu) \leq \max\{1, \|D\Psi\|^{-1}_\infty\} e^\nu_{\epsilon, \lambda}(U)(\Psi(y^\nu)) \quad (3.26)$$

for $y^\nu \in B$.

Define $\Psi(0) := 0$, and $\Psi(y^\nu) := \frac{|y^\nu||y^\nu|}{\max\{|y^\nu|^2, |y^\nu|\}}$, $|y^\nu| \neq 0$, where $|y^\nu|$ is the standard Euclidean norm. Then $\Psi(B_\nu(s)) = Q_s$, and $\|D\Psi\|_{\infty} \leq 1 \leq \|D(\Psi^{-1})\|_{\infty} \leq \sqrt{2}$. It follows from (3.24) and (3.26) and a change of variables that

$$\int_{\partial B_\nu(s)} e^\nu_{\epsilon, \lambda}(\Psi^* U) \leq C \int_{\partial Q_s} e^\nu_{\epsilon, \lambda}(U) \leq C_1.$$ 

Thus $\text{deg}(\phi_{|Q_s}; \partial Q_s) = \text{deg}(\phi_{|Q_s} \circ \Psi, \partial B_\nu(s)) =: n(s) \in \mathbb{Z}$ exists, and the proof of Lemma 3.4 shows that there exists a configuration $\tilde{U}$ on $\mathbb{R}^2$ such that $\tilde{U} = \Psi^* U$ on $B_\nu(s)$,

$$\tilde{U} \in H_n, \quad \int_{\mathbb{R}^2 \setminus B_\nu(s)} e^\nu_{\epsilon, \lambda}(\tilde{U}) < C \epsilon \int_{\partial B_\nu(s)} e^\nu_{\epsilon, \lambda}(\Psi^* U) \leq C_2 \epsilon.$$ 

Now define $\hat{U} := (\Psi^{-1})^* \tilde{U}$. Then $\hat{U} = U$ in $Q_s$, $\hat{U} \in H_n$. Moreover, again using (3.26), we have

$$\int_{\mathbb{R}^2 \setminus Q_s} e^\nu_{\epsilon, \lambda}(\hat{U}) \leq C \int_{\mathbb{R}^2 \setminus B_\nu(s)} e^\nu_{\epsilon, \lambda}(\tilde{U}) \leq C_2 \epsilon$$

and then (3.25) follows exactly as in the proof of Lemma 3.4 see (3.22). \qed

4. Minimizers of the 2D Euclidean Energy

The main result of this section gives a criterion for existence of solutions of the minimization problem (1.16):

Theorem 4.1. Assume that $\lambda$ and $N$ are such that

$$\mathcal{E}^\lambda_N < \min\{\mathcal{E}^\lambda_{n_1} + \cdots + \mathcal{E}^\lambda_{n_i} : n_1 + \cdots + n_i = N, \text{ at least two } n_j \text{ are nonzero}\}. \quad (4.1)$$

Then there exists $U \in H_N$ such that $\int_{\mathbb{R}^2} e^\nu_{\epsilon, \lambda}(U) = \mathcal{E}^\lambda_N$.

In particular, there exists $U \in H_{n_1}$ minimizing $\int_{\mathbb{R}^2} e^\nu_{\epsilon, \lambda}(\cdot)$ if $\frac{1}{2} < \lambda < 5$.

It is proved in [21] Theorem I.2] that $\int_{\mathbb{R}^2} e^\nu_{\epsilon, \lambda}(\cdot)$ attains its minimum in $H_n$ for $|n| = 1$ and for all $\lambda$ sufficiently large. Our argument is close in spirit to that of [21], and we omit some details that are either standard or can be found in [21].
Proof. We will show that there exists \( \epsilon_0 > 0 \), to be specified below, such that for all \( \epsilon \in (0, \epsilon_0) \), there exists \( U \in H_N \) such that \( \int_{\mathbb{R}^2} e_{\epsilon, \lambda}^\nu(U) = \mathcal{E}_N^\lambda \). In view of scale invariance, see \([1.10]\), this will establish the theorem.

1. We first remark that it follows from \([2.3]\) and \([2.2]\) that \( \mathcal{E}_m^\lambda \geq \min\{1, \lambda\} \pi|m| \), and hence that the minimum \((4.1)\) is indeed attained, and in fact there exists \( \delta_N^\lambda > 0 \) such that

\[
(4.2) \quad \text{if } n_1 + \cdots + n_i = N \text{ and at least } n_j \text{ are nonzero, then } \sum_{j=1}^i \mathcal{E}_{n_j}^\lambda > (1 + \delta_N^\lambda) \mathcal{E}_N^\lambda.
\]

We may also assume that \( \delta_N^\lambda \leq 1 \).

Assume that \( \epsilon < \epsilon_0 \), and let \( (U_k) \subset H_N \) be a minimizing sequence, so that \( \int_{\mathbb{R}^2} e_{\epsilon, \lambda}^\nu(U_k) \to \mathcal{E}_N^\lambda \) as \( k \to \infty \). We further assume, without loss of generality, that \( U_k \) is smooth and that \( \int_{\mathbb{R}^2} e_{\epsilon, \lambda}^\nu(U_k) < (1 + \frac{1}{2} \delta_N^\lambda) \mathcal{E}_N^\lambda \) for every \( k \).

Let \( \mathcal{L} := (\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) \), and for \( y \in \mathbb{R}^2 \), let \( \tau_y U_k(x) := U_k(x - y) \). Then it follows from Fubini’s Theorem that

\[
\int_{y \in (0,1)^2} \int_{\mathcal{L}} e_{\epsilon, \lambda}^\nu(\tau_y U_k) = \int_{y \in (0,1)^2} \int_{\tau_y \mathcal{L}} e_{\epsilon, \lambda}^\nu(U_k) = 2 \int_{\mathbb{R}^2} e_{\epsilon, \lambda}^\nu(U_k) \leq 3 \mathcal{E}_N^\lambda
\]

for every \( k \). Thus, replacing \( U_k \) by a suitable translation \( \tau_{y_k} U_k \) for every \( k \), we may arrange that

\[
(4.3) \quad \int_{\mathcal{L}} e_{\epsilon, \lambda}^\nu(U_k) \leq 3 \mathcal{E}_N^\lambda \quad \text{for every } k.
\]

2. Now for every \( p = (p_1, p_2) \in \mathbb{Z}^2 \), let \( Q^p := p + (0,1)^2 = \{(x + p_1, y + p_2) : (x, y) \in (0,1)^2\} \). It follows from \([4.3]\) that

\[
\int_{\partial Q^p} e_{\epsilon, \lambda}^\nu(U_k) \leq 3 \mathcal{E}_N^\lambda \quad \text{for every } k \in \mathbb{N} \text{ and } p \in \mathbb{Z}^2.
\]

We can thus apply Lemma \([3.5]\) on every square \( Q^p \) to find that if \( \epsilon \) is small enough, then \( n(p, k) := \deg(\frac{\partial}{\partial \nu}; \partial Q^p) \) is well-defined, and

\[
\int_{Q^p} e_{\epsilon, \lambda}^\nu(U_k) \geq \mathcal{E}_{n(p,k)}^\lambda - C \epsilon.
\]

Since \( \mathcal{E}_m^\lambda \geq \min\{1, \lambda\} |m| \) for all \( m \), we can fix \( \epsilon_0 = \epsilon_0(\lambda, N) \) so small that \( \mathcal{E}_m^\lambda - C \epsilon \geq \mathcal{E}_m^\lambda(1 - \frac{1}{4} \delta_N^\lambda) \) for all nonzero \( m \) and all \( \epsilon < \epsilon_0 \). Then

\[
\int_{\mathbb{R}^2} e_{\epsilon, \lambda}^\nu(U_k) \geq \sum_{p:n(p,k) \neq 0} \mathcal{E}_{n(p,k)}^\lambda - C \epsilon \geq (1 - \frac{1}{4} \delta_N^\lambda) \sum_{p:n(p,k) \neq 0} \mathcal{E}_{n(p,k)}^\lambda.
\]

Also, as noted in Section 3, \( \omega(U_k) \) is integrable, so by Lemma \([2.4]\) the definition of \( H_N \), and the additivity of degree,

\[
N = \lim_{t \to \infty} \int_{\cup |p| < t Q^p} \omega = \lim_{t \to \infty} \deg \left( \frac{\phi_k}{|\phi_k|}; \partial(\cup |p| < t Q^p) \right) = \sum_{p \in \mathbb{Z}^2} n(p, k).
\]

Thus the definition of \( \delta_N^\lambda \) implies that if at least two \( n(p, k) \) are nonzero, then

\[
\int_{\mathbb{R}^2} e_{\epsilon, \lambda}^\nu(U_k) \geq (1 - \frac{1}{4} \delta_N^\lambda)(1 + \delta_N^\lambda) \mathcal{E}_N^\lambda = (1 + \frac{3}{4} \Delta_N^\lambda - \frac{1}{4} (\delta_N^\lambda)^2) \mathcal{E}_N^\lambda \geq (1 + \frac{1}{2} \delta_N^\lambda) \mathcal{E}_N^\lambda
\]

as desired.
for all $k$, since we assumed that $\delta_k N \leq 1$. But this is not the case for any $k$, by our choice of the sequence $U_k$. We conclude for every $k$,

\[(4.4) \text{there exists } p_0 = p_0(k) \in \mathbb{Z}^2 \text{ such that } n(p_0, k) = N, \ n(p, k) = 0 \text{ for } p \neq p_0.\]

Replacing (again) $U_k$ by a suitable translation, we may assume that $p_0 = (0, 0)$ for every $k$.

3. The remainder of the existence proof is now standard, and very similar points are treated in detail in Rivièrë [21], so we summarize the arguments only briefly. First, if we impose the Coulomb gauge condition $\nabla \cdot t$ treated in detail in Rivièrë [21], we may assume that $p_0 = (0, 0)$ for every $k$.

Next, using (4.4) and (4.3) we can check that $n(p) = \deg(\frac{\partial}{|\partial|}, \partial Q^p)$ is well-defined and that $n(p_0) = N$ and that $n(p) = 0$ if $p \neq p_0 = (0, 0)$. It follows that $U \in H_N$ and hence that $U$ is an energy-minimizer in $H_N$.

4. Finally, since

\[
\min \{1, \lambda\} |n| \pi \leq E^\lambda_n \leq \max \{1, \lambda\} |n| \pi \]

it is easy check that (4.1) is satisfied for $N = 1$ as long as $\frac{1}{5} < \lambda < 5$. For example, if $1 < \lambda < 5$, then $E^\lambda_1 = E^\lambda_1 \leq \lambda \pi < 5\pi$. Now consider nonzero integers $n_1, \ldots, n_i$ such that $n_1 + \ldots + n_i = 1$, with at least two nonzero $n_i$. If $|n_j| = 1$ for any $j$, it is clear that $\sum E^\lambda_{n_j} = \sum E^\lambda_{|n_j|} > E^\lambda_1$, and if $|n_j| \geq 2$ for all $j$, then

\[
\sum E^\lambda_{n_j} \geq \pi \sum |n_j| \geq 5\pi,
\]

since $\sum |n_j|$ is odd and must be greater than 3. The case $\frac{1}{5} < \lambda < 1$ is similar. \hfill \Box

5. **Abelian Higgs Model: Energy Estimates in Normal Coordinates**

In this section we consider the abelian Higgs model in the coordinate system introduced in Section 1.4.

In particular, recall the map $\psi$ defined in (1.24), built around a timelike minimal surface parametrized by an embedding $H : (-T, T) \times S^1 \to \mathbb{R}^{1+3}$ as described in Section 1.4. Given $T_0 < T$, we henceforth restrict the domain of $\psi$ to a set of the form $(-T_1, T_1) \times S^1 \times B_\nu(\rho_0)$. We do this in such a way that

\[(5.1) \quad T_0 < T_1 < T, \quad \psi \text{ is injective on } (-T_1, T_1) \times S^1 \times B_\nu(\rho_0)\]

and

\[(5.2) \quad \psi^0(-T_1, y^1, y^\nu) < -T_0, \quad \psi^0(T_1, y^1, y^\nu) > T_0, \quad \text{for all } |y^\nu| \leq \rho_0, y^1 \in S^1,\]

where $\psi^0$ denotes the 0th component of $\psi$, corresponding to the $t$ variable. Given $T_0 < T$, and having fixed $T_1 \in (T_0, T)$ and $\rho_0$ as above, we will write

\[(5.3) \quad \mathcal{N} := \psi \left( (-T_1, T_1) \times S^1 \times B_\nu(\rho_0) \right) \cap \left( (-T_0, T_0) \times \mathbb{R}^3 \right).\]

We will write $(g_{\alpha\beta})$ to denote the metric tensor written in the normal coordinate system

\[
g_{\alpha\beta} := \eta_{\gamma\delta} \partial_\alpha \psi^\gamma \partial_\beta \psi^\delta, \quad (\eta_{\alpha\beta}) := \text{diag}(-1, 1, 1, 1)\]
and we also use the notation
\[(g^{\alpha\beta}) := (g_{\alpha\beta})^{-1}, \quad g := \det(g_{\alpha\beta}).\]

Thus, in the normal coordinate system the abelian Higgs model takes the form
\[\begin{align*}
-\frac{1}{\sqrt{-g}} D_\alpha (g^{\alpha\beta} D_\beta \phi \sqrt{-g}) + \frac{\lambda}{2\epsilon^2} (|\phi|^2 - 1) \phi = 0, \\
-\epsilon^2 \frac{1}{\sqrt{-g}} \partial_\alpha (F^{\alpha\beta} \sqrt{-g}) - g^{\beta\alpha} \langle i\phi, D_\alpha \phi \rangle = 0.
\end{align*}\]

Here \(\alpha, \beta\) run from 0 to 3, and we raise and lower indices with \((g^{\alpha\beta})\) and \((g_{\alpha\beta})\) respectively.

We find it useful to write the above system as
\[\begin{align*}
-\partial_\alpha (g^{\alpha\beta} D_\beta \phi) - b \cdot D\phi + V'(\phi) = 0, \\
-\epsilon^2 \left( \partial_\alpha F^{\alpha\beta} + g^{\beta\nu} b^\mu F_{\mu\nu} \right) - g^{\beta\alpha} \langle i\phi, D_\alpha \phi \rangle = 0,
\end{align*}\]
where
\[b^\beta = \frac{\partial_\alpha \sqrt{-g} g^{\alpha\beta}}{\sqrt{-g}} \quad V'(\phi) = \frac{\lambda}{8\epsilon^2} (|\phi|^2 - 1)^2.\]

5.1. properties of the metric. We will need a number of properties of the metric \((g_{\alpha\beta})\). These are mostly well-known and can be found in the proof of [12, Prop.1] for example. First,
\[(g_{\alpha\beta}) (y^\tau, y^\nu) = \begin{pmatrix} \gamma_{ab} (y^\tau) & 0 \\ 0 & \text{Id} \end{pmatrix} + \begin{pmatrix} O(|y^\nu|) & O(|y^\nu|) \\ O(|y^\nu|) & O(|y^\nu|^2) \end{pmatrix},\]
(in block \(2 \times 2\) form), where \((\gamma_{ab})\) was introduced in Section 1.2 and satisfies (1.6), (1.8).

Hence
\[(g^{\alpha\beta}) (y^\tau, y^\nu) = \begin{pmatrix} (\gamma^{ab}) (y^\tau) & 0 \\ 0 & \text{Id} \end{pmatrix} + \begin{pmatrix} O(|y^\nu|) & O(|y^\nu|) \\ O(|y^\nu|) & O(|y^\nu|^2) \end{pmatrix}.\]

In [12, Prop.1] it is further shown that
\[(\partial_0 g^{\alpha\beta}) (y^\tau, y^\nu) = \begin{pmatrix} O(1) & O(|y^\nu|) \\ O(|y^\nu|) & O(|y^\nu|^2) \end{pmatrix}.\]

and that the vector field \(b\) defined in (5.7) satisfies
\[|b^\nu| := \sqrt{(b^2)^2 + (b^3)^2} \leq C|y^\nu|, \quad |b^\tau| := \sqrt{(b^0)^2 + (b^1)^2} \leq C.\]

All the above estimates are uniform in \((-T_1, T_1) \times S^1 \times B_{\nu}(\rho_0).\)

We remark that the estimate \(|b^\nu| \leq C|y^\nu|\) follows from the condition that \(\Gamma\) is a minimal surface, and it is the only place in our argument that we directly invoke this assumption.

5.2. energy. The natural energy for (5.5)-(5.6) is obtained from the stress-energy tensor
\[T_{\alpha\beta} = \frac{1}{2} g_{\alpha\beta} \mathcal{L} - \frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} = \frac{1}{2} g_{\alpha\beta} \mathcal{L} - \frac{1}{2} \langle D_\alpha \phi, D_\beta \phi \rangle - \epsilon^2 g^{\mu\nu} F_{\alpha\mu} F_{\beta\nu}.\]
We will state estimates in terms of the energy density $e_{\epsilon,\lambda}(U) := 2T_0^0$. Explicitly,
\[
T_0^0 = \frac{1}{2} \mathcal{L} - \frac{1}{2} g^{\alpha\gamma} (D_\gamma \phi, D_\alpha \phi) - \frac{\epsilon^2}{2} F_{0\mu} F_{\mu
u}
\]
\[
= \left[ \frac{1}{4} g^{\mu\nu} (D_\mu \phi, D_\nu \phi) - \frac{1}{2} g^{\alpha\gamma} (D_\alpha \phi, D_\gamma \phi) \right] + \frac{\epsilon^2}{8} \left[ F_{\alpha\beta} F^{\alpha\beta} - 4 F^0\beta F_{0\beta} \right] + \frac{\lambda}{16\epsilon^2} (|\phi|^2 - 1)^2.
\]

We define $(a^{\alpha\beta})$ so that $a^{\alpha\beta} \xi_\alpha \xi_\beta = g^{\alpha\beta} \xi_\alpha \xi_\beta - g^{0\beta} \xi_0 \xi_\beta$, which leads to
\[
a^{00} = -g^{00}, \quad a^{i0} = a^{0i} = 0, \quad a^{ij} = g^{ij}, \quad i, j = 1, \ldots, 3.
\]

With this notation,
\[
e_{\epsilon,\lambda}(U) := 2T_0^0 = \frac{1}{2} g^{\alpha\mu} (D_\mu \phi, D_\nu \phi) + \frac{\epsilon^2}{4} \left( F_{\alpha\beta} F^{\alpha\beta} - 4 F^0\beta F_{0\beta} \right) + \frac{\lambda}{8\epsilon^2} (|\phi|^2 - 1)^2.
\]

We also remark that
\[
T_0^0 = -\frac{1}{2} g^{\gamma j} (D_\gamma \phi, D_0 \phi) - \frac{\epsilon^2}{2} F^{j\nu} F_{0\nu},
\]
where we remind the reader that repeated latin indices are summed from 1 to 3. We will need the following

**Lemma 5.1.** There exist constants $0 < c \leq C$ such that for every $U = (\phi, A)$,
\[
(1 - C|y'|^2) e_{\epsilon,\lambda}'(U) + c \left[ |D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 \right] \leq e_{\epsilon,\lambda}(U)
\]
uniformly for $y = (y^\tau, y^\nu) \in (-T_1, T_1) \times S^1 \times B_\nu(\rho_0)$, where
\[
|F_\tau|^2 = \sum_{0 \leq \alpha \leq 3, \ 0 \leq \beta \leq 1} |F_{\alpha\beta}|^2, \quad |F_\nu|^2 = |F_{23}|^2,
\]
\[
|D_\tau \phi|^2 = |D_0 \phi|^2 + |D_1 \phi|^2, \quad |D_\nu \phi|^2 = |D_2 \phi|^2 + |D_3 \phi|^2,
\]
and
\[
e_{\epsilon,\lambda}'(U) := \frac{1}{2} |D_\nu \phi|^2 + \frac{\epsilon^2}{2} |F_\nu|^2 + V_\epsilon(\phi)
\]

**Proof.** The pointwise inequalities (5.15) follow from (5.9), (1.6), (1.8) by routine computations. \(\square\)

5.3. **A differential energy inequality.** We next prove

**Lemma 5.2.** Assume that $U$ is a smooth solution of (5.5), (5.6) on $(-T_1, T_1) \times S^1 \times B_\nu(\rho_0)$. Then there exists $C > 0$ such that
\[
\partial_\alpha e_{\epsilon,\lambda}(U) \leq C \left( |D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 + |y'|^2 e_{\epsilon,\lambda}'(U) \right) - 2 \partial_\tau T_0^0(U)
\]
pointwise.

As is well-known, the tensor $T_0^0$ satisfies an exact conservation law $\partial_\alpha (T_0^0 \sqrt{-g}) = 0$ for $\alpha = 0, \ldots, 3$. However, (5.19) is more useful for our purposes. Surprisingly, it does not seem to be easy to derive (5.19) directly from the exact conservation law.
Proof. We take the inner product of (5.5) with $D_0 \phi$ to find

$$-\langle D_\alpha (g^{\alpha \beta} D_\beta \phi), D_0 \phi \rangle - \langle b \cdot D \phi, D_0 \phi \rangle + \langle V'_0(\phi), D_0 \phi \rangle = 0.$$  

Note that $\langle V'_0(\phi), D_0 \phi \rangle = \partial_0 V'_0(\phi)$. Also, using the commutation relation

$$[D_\alpha, D_\beta] = iF_{\beta \alpha},$$

we compute

$$-\langle D_\alpha (g^{\alpha \beta} D_\beta \phi), D_0 \phi \rangle = -\partial_\alpha \langle g^{\alpha \beta} D_\beta \phi, D_0 \phi \rangle + \langle g^{\alpha \beta} D_\beta \phi, [D_\alpha, D_0] \phi \rangle$$

$$= -\partial_\alpha \langle g^{\alpha \beta} D_\beta \phi, D_0 \phi \rangle + \langle g^{\alpha \beta} D_\beta \phi, D_0 D_\alpha \phi \rangle + \langle g^{\alpha \beta} D_\beta \phi, [D_\alpha, D_0] \phi \rangle$$

$$= -\partial_\alpha \langle g^{\alpha \beta} D_\beta \phi, D_0 \phi \rangle + \frac{1}{2} \partial_0 \langle g^{\alpha \beta} D_\beta \phi, D_0 \phi \rangle$$

$$- \frac{1}{2} \langle \partial_0 g^{\alpha \beta} \rangle \langle D_\beta \phi, D_\alpha \phi \rangle + F_{0\alpha} \langle g^{\alpha \beta} D_\beta \phi, i \phi \rangle.$$  

Since $-g^{0\beta} \xi_0 + \frac{1}{2} g^{\alpha \beta} \xi_0 \xi_\beta = \frac{1}{2} \alpha^{\alpha \beta} \xi_0 \xi_\beta$, by collecting terms we conclude that

$$\partial_0 \left( \frac{1}{2} a^{\alpha \beta} \langle D_\alpha \phi, D_\beta \phi \rangle + V'_0(\phi) \right) = \langle b \cdot D \phi, D_0 \phi \rangle + \partial_\alpha \langle g^{\alpha \beta} D_\beta \phi, D_0 \phi \rangle$$

$$+ \frac{1}{2} \langle \partial_0 g^{\alpha \beta} \rangle \langle D_\beta \phi, D_\alpha \phi \rangle - F_{0\alpha} \langle g^{\alpha \beta} \rangle \langle D_\beta \phi, i \phi \rangle.$$  

We now rewrite the last term on the right-hand side. First, using the equation (5.6),

$$F_{0\alpha} \langle g^{\alpha \beta} \rangle \langle D_\beta \phi, i \phi \rangle = -c^2 F_{0\alpha} \left( \partial_\beta F^{\beta \alpha} + g^{\alpha \gamma} b^\gamma F_{\gamma \gamma'} \right).$$  

We will leave the second term as it is. As for the first term, note that

$$F_{0\alpha} \partial_\beta F^{\beta \alpha} = \partial_\beta (F_{0\alpha} F^{\beta \alpha}) - \partial_\beta F_{0\alpha} F^{\beta \alpha}$$

$$= \partial_\beta (F_{0\alpha} F^{\beta \alpha}) + (\partial_\alpha F_{0\beta} + \partial_0 F_{\alpha \beta}) F^{\beta \alpha}$$

$$= \partial_\beta (F_{0\alpha} F^{\beta \alpha}) + \alpha_0 (F_{0\beta} F^{\beta \alpha}) - F_{0\beta} \partial_\alpha F^{\beta \alpha} - \partial_0 F_{\alpha \beta} F^{\beta \alpha}$$

$$= 2 \partial_\beta (F_{0\alpha} F^{\beta \alpha}) - F_{0\beta} \partial_\alpha F^{\beta \alpha} - \partial_0 F_{\alpha \beta} F^{\beta \alpha}$$

$$= 2 \partial_\beta (F_{0\alpha} F^{\beta \alpha}) - F_{0\beta} \partial_\alpha F^{\beta \alpha} - \frac{1}{2} \partial_0 (F_{\alpha \beta} F^{\beta \alpha}) + \frac{1}{2} \partial_0 (g^{\alpha \gamma} g^{\beta \gamma'}) F_{\alpha \beta} F_{\gamma \gamma'}.$$  

Hence

$$F_{0\alpha} \partial_\beta F^{\beta \alpha} = \partial_\beta (F_{0\alpha} F^{\beta \alpha}) - \frac{1}{4} \partial_0 (F_{\alpha \beta} F^{\beta \alpha}) + \frac{1}{4} \partial_0 (g^{\alpha \gamma} g^{\beta \gamma'}) F_{\alpha \beta} F_{\gamma \gamma'}.$$  

We combine (5.20)-(5.22) and collect all terms involving $\partial_0$ on the left-hand side, to find that

$$\partial_0 \left( \frac{1}{2} a^{\alpha \beta} \langle D_\alpha \phi, D_\beta \phi \rangle + \frac{c^2}{4} (F_{\alpha \beta} F^{\alpha \beta} - 4 F_{0\alpha} F^{\beta \alpha}) + V'_0(\phi) \right)$$

$$= \partial_\xi \left( g^{\beta \alpha} \langle D_\beta \phi, D_0 \phi \rangle + \frac{c^2}{4} F_{0\alpha} F^{\beta \alpha} \right) + \langle b \cdot D \phi, D_0 \phi \rangle + \frac{c^2}{4} F_{0\alpha} g^{\alpha \gamma} b^\gamma F_{\gamma \gamma'}$$

$$+ \frac{1}{2} \langle \partial_0 g^{-\alpha \beta} \rangle \langle D_\beta \phi, D_\alpha \phi \rangle + \frac{c^2}{4} \partial_0 (g^{\alpha \gamma} g^{\beta \gamma'}) F_{\alpha \beta} F_{\gamma \gamma'}.$$  

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Note that the left-hand side is just ∂₀εₖ,ₗ(U), and the first term on the right-hand side is −2∂₀T₀(ϕ, A). So we just need to estimate the other terms on the right-hand side. First, by (5.10) and (5.11), which we recall is essentially the assumption that Ω is minimal,
\[ \langle b \cdot Dϕ, D₀ϕ \rangle + \frac{1}{2} (\partial₀g^αβ)(Dαϕ, Dβϕ) \leq C \left( |D_τϕ|^2 + |y'|^2 |D_vϕ|^2 \right). \]

Next,
\[ \frac{e^2}{4} \partial₀(g^{αγ}g^{β′γ′})F_αβF_γγ′ = \frac{e^2}{2} (\partial₀g^{αγ})g^{β′γ′}F_αβF_γγ′ \leq \frac{e^2}{2} \left\| g^{β′γ′} \right\|_{L∞} |(\partial₀g^{αγ})F_αβF_γγ'| \leq Ce^2 \left( |F_τ|^2 + |y'|^2 |F_v|^2 \right), \]
using (5.9)–(5.10). Finally, from (5.9) and (5.11) we similarly estimate
\[ e^2F₀αg^{αγ}b^γF_γγ′ \leq C e^2 \left( |F_τ|^2 + |y'|^2 |F_v|^2 \right). \]

\[ \square \]

5.4. weighted energy estimate. In this subsection we establish an estimate that plays a key role in the proof of Theorem 1.4. We first introduce some notation.

Given a configuration U on S¹ × B₀(R) for some R > 0, if U is a configuration on S¹ × B₀(R) for some R > 0, then we will use the notation
\[ D_m(U; R) := \int_{y^1 ∈ S¹} |D m(U(y^1); R)| dy^1 \]
for m ∈ Z, where here U(y^1) denotes the function y' → U(y^1, y'). We recall that Dₘ is defined in (3.1).

The main result of this section is

**Proposition 5.3.** Assume that U is a smooth solution of (5.5)–(5.6) on (−T₁, T₁) × S¹ × B₀(ρ₀) and that m is a nonzero integer such that (3.4) is satisfied. Then there exist positive constants Cₗ, C, κ₂ and ρ₁ ∈ (0, ρ₀/2], independent of U and ofε ∈ (0, 1], such that the following hold: if we define

\[ ζ₁(s) := \int_{S¹} \left( \int_{B_v(ρ₁−c, s)} (1 + κ₂|y'|^2)εₖ,ₗ(U)dy' − \mathcal{E}^n \right) dy^1 \Big|_{y^0 = s} \]
(5.25)
\[ ζ₂(s) := D_m(U(s, ·); \frac{1}{2} ρ₁) \]
(5.26)
\[ ζ₃(s) := \int_{S¹} \left( |D_v|^2 + \frac{1}{2} |F_τ|^2 + |y'|^2 εₖ,ₗ(U) \right) dy' dy^1 \Big|_{y^0 = s} \]
(5.27)
(where the notation |D_τϕ|, |F_τ| etc is introduced in (5.16)–(5.18)) then
\[ ζᵢ(s) ≤ C \max\{ζ₁(0), ζ₂(0), ε^2\}, \text{ for } i = 1, 2, 3, \text{ } 0 < s < s_{max} := \min\{T₁, ρ₁/2cₗ\}. \]

The proof follows that of Proposition 10 in [12].
Also, arguing as in (5.32), we deduce that
\begin{equation}
\epsilon^\nu_{\epsilon,\lambda}(U) + |D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 \leq c^{-1} \epsilon_{\epsilon,\lambda}(U)
\end{equation}
and
\begin{equation}
(1 + \kappa_2 |y_\nu|^2) \epsilon_{\epsilon,\lambda}(U) \geq c(|D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2) + (1 + |y_\nu|^2) \epsilon_{\epsilon,\lambda}(U)
\end{equation}
in \((-T_1, T_1) \times S^1 \times B_\nu(2\rho_1)\). This may be done due to Lemma 5.1. It is convenient to use
the notation \(\zeta_0 := \max\{\zeta_1(0), \zeta_2(0)\}\) and
\[
W_\nu(s) := B_\nu(\rho_1 - c_2 s), \quad W(s) := S^1 \times W_\nu(s)
\]
for \(c_2\) to be fixed below.

1. We show
\begin{equation}
\zeta_1(s) \leq \zeta_0 + C \int_0^s \zeta_3(s') ds' \quad \text{for } s \in (0, s_{\text{max}}].
\end{equation}
Since \(\zeta_1(0) \leq \zeta_0\), it suffices to prove that \(\zeta_1'(s) \leq C \zeta_3(s)\). To that end we compute
\[
\zeta_1'(s) = \int_{s \times W(s)} (1 + \kappa_2 |y_\nu|^2) \partial_0 \epsilon_{\epsilon}(U) - c_2 \int_{s \times S^1 \times \partial W_\nu(s)} (1 + \kappa_2 |y_\nu|^2) \epsilon_{\epsilon,\lambda}(U) = I + II.
\]
By Lemma 5.2
\[
I \leq C \int_{s \times W(s)} (1 + \kappa_2 |y_\nu|^2) \left(|D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 + |y_\nu|^2 \epsilon_{\epsilon,\lambda}(U)\right)
- 2 \int_{s \times W(s)} (1 + \kappa_2 |y_\nu|^2) \partial_1 T^\nu_0(U).
\]
We integrate by parts to find that
\[
2 \int_{s \times W(s)} (1 + \kappa_2 |y_\nu|^2) \partial_1 T^\nu_0(U) \leq 4\kappa_2 \int_{s \times W(s)} |y_\nu| |T^\nu_0(U)|
+ 2 \int_{s \times S^1 \times \partial W_\nu(s)} (1 + \kappa_2 |y_\nu|^2) |T^\nu_0|
\]
where \(T^\nu_0 := (T^2_0, T^3_0) = (T^1_0, T^2_0)\). We see from the definition \((5.14)\) of \(T^j_0\) and the uniform bounds \((5.3)\) on \((g_0^{0\alpha})\) that
\begin{equation}
|T^\nu_0| \leq C \left(|D_\nu \phi| \ |D_\tau \phi| + \epsilon^2 |F_\nu| \ |F_\tau| \right) \leq C \Big( \epsilon_{\epsilon,\lambda}(U) + |D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 \Big)
\end{equation}
and it follows from this and \((5.29)\) that we may choose \(c_2\) large enough that
\[
2 \int_{s \times S^1 \times \partial W_\nu(s)} (1 + \kappa_2 |y_\nu|^2) |T^\nu_0| + II \leq 0.
\]
Also, arguing as in \((5.32)\), we deduce that
\[
|y_\nu| |T^\nu_0| \leq C \left(|D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 + |y_\nu|^2 \epsilon_{\epsilon,\lambda}(U)\right),
\]
and by combining these estimates, we find that $\zeta_1' \leq C \zeta_3$, completing the proof of (5.31).

2. Next, recalling the definition (5.26), (5.24) of $\zeta_2$,

$$\zeta_2(s) \leq \zeta_0 + \zeta_2(s) - \zeta_2(0)$$

$$\leq \zeta_0 + \left| \int_{s_1} \left( D_{\nu}^m(U(s, y^1); 0, y^1) - D_{\nu}^m(U(0, y^1); 0, y^1) \right) dy^1 \right|,$$

so it follows immediately from Proposition 3.1 that

$$\zeta_2(s) \leq C \zeta_0 + C \int_0^s \zeta_3(s')ds' \quad \text{for } s \in (0, s_{\max}].$$

3. We next show that

$$\zeta_3(s) \leq C(\zeta_1(s) + \zeta_2(s) + O(\epsilon^2)) \quad \text{for } s \in (0, s_{\max}].$$

Since (5.30) implies that

$$\zeta_1(s) \geq c \zeta_3(s) + \int_{s_1} \left( \int_{W_\nu(s)} e_{\nu,\lambda}(U) - e_{\nu,\lambda}(m) \right) dy^1,$$

it suffices to show that

$$|S^1| \epsilon_{\nu,\lambda} - \int_{s \times W(s)} e_{\nu,\lambda}(U) \leq C \zeta_2(s) + O(\epsilon^2).$$

The $s$ variable plays no role in this argument, so we regard it as fixed and do not display it. We will say that $y^1 \in S^1$ is good if

$$|D_{\nu}^m(U(y^1))| \leq \kappa_1,$$

where $\kappa_1$ was fixed in Proposition 3.2 and $y^1$ is bad otherwise. As usual we estimate the size of the bad set by Chebyshev’s inequality:

$$\left| \{ y^1 \in S^1 : y^1 \text{ is bad} \} \right| \leq \frac{1}{\kappa_1} \int_{s \times S^1} |D_{\nu}^m(U)| dy^1 = C \zeta_2(s).$$

So $\left| \{ y^1 \in S^1 : y^1 \text{ is good} \} \right| \geq |S^1| - C \zeta_2(s)$, and we obtain the estimate we seek by applying the lower energy bounds from Proposition 3.2 in the normal variables for every good $y^1$. Indeed,

$$\int_{s \times W(s)} e_{\nu,\lambda}(U) \geq \int_{\{y \in S^1 \text{ is good} \}} \left( \int_{W_\nu(s)} e_{\nu,\lambda}(U) dy^1 \right) dy^1$$

$$\geq \left( |S^1| - C \zeta_2(s) \right) \left( \epsilon_{\nu,\lambda} - C \epsilon^2 \right)$$

$$\geq |S^1| \epsilon_{\nu,\lambda} - C \zeta_2(s) - C \epsilon^2.$$

Rearranging gives (5.35).

4. We gather the previous steps to conclude

$$\zeta_3(s) \leq C(\zeta_1(s) + \zeta_2(s) + O(\epsilon^2)) \leq C \zeta_0 + C \int_0^s \zeta_3(s')ds' + CO(\epsilon) \leq C \zeta_0 + C \int_0^s \zeta_3(s')ds',$$

since $\zeta_0 \geq \epsilon^2$. Then by Gronwall’s inequality,

$$\zeta_3(s) \leq C \zeta_0 \quad \text{for } s \in (0, s_{\max}],$$
and hence from Steps 1 and 2,
\[ \zeta_1(s), \zeta_2(s) \leq C \zeta_0 \]
as needed.

6. PROOF OF THEOREM 1.4

In this section we complete the proof of Theorem 1.4. This involves, among other things, combining the weighted energy estimates of Proposition 5.3, expressed in the standard coordinate system and effective near \( \Gamma \), with energy estimates in the standard coordinate system, effective away from \( \Gamma \).

In this section we will write
\[ e_{\epsilon, \lambda}(U, \eta) = \frac{1}{2} |D\phi|^2 + \frac{\epsilon^2}{2} |F|^2 + \frac{\lambda}{8\epsilon^2} (|\phi|^2 - 1)^2; \]
\[ e_{\epsilon, \lambda}(U, g) = \frac{1}{2} a^{\mu\nu} (D_{\mu}\phi, D_{\nu}\phi) + \frac{\epsilon^2}{4} \left( F_{\alpha\beta} F^{\alpha\beta} - 2 F^0 0 \right) + \frac{\lambda}{8\epsilon^2} (|\phi|^2 - 1)^2 \]
for the natural energy densities with respect to standard coordinates and normal coordinates respectively, where we raise indices with \( (\cdot, \cdot) \) for the natural energy densities with respect to standard coordinates and normal coordinates respectively, where we raise indices with \( (g^{\alpha\beta}) \) in the second expression. It is straightforward to check from the definitions and (5.29) that there exists a \( C \) independent of \( U \) and \( \epsilon \in (0, 1] \) such that
\[ \frac{1}{C} e_{\epsilon, \lambda}(U, \eta)(\psi(y)) \leq e_{\epsilon, \lambda}(U, g)(y) \leq C e_{\epsilon, \lambda}(U, \eta)(\psi(y)) \]
for \( y \in (-T_1, T_1) \times S^1 \times B_\nu(\rho_1) \).

Proposition 6.1. Assume that \( \Gamma, T_0, T_1, \rho_0, \mathcal{N} \) are as in the statement of Theorem 1.4 and assume that \( \lambda > 0 \) and \( m \in \mathbb{Z} \) satisfy the conditions in Theorem 1.4. Let \( \rho_1, \kappa_2 \) denote the constants found in Proposition 5.3.

Let \( U = (\varphi, A) \) solve the abelian Higgs model (1.2)-(1.3) on \( \mathbb{R}^{1+3} \), and let \( U = \psi^* U \), so that \( U \) solves (5.5)-(5.6) on \( (-T_1, T_1) \times S^1 \times B_\nu(\rho_0) \).

Define
\[ \tilde{\zeta}_1(s) := \int_{S^1} \left( \int_{B_\nu(\rho_1/2)} (1 + \kappa_2 |\gamma|)^2 e_{\epsilon, \lambda}(U, g) d\gamma - \zeta^m \right) d\gamma \bigg|_{\gamma^0 = s}; \]
\[ \tilde{\zeta}_2(s) := D_m(U(s, \cdot); \frac{1}{2} \rho_1); \]
\[ \tilde{\zeta}_3(s) := \int_{S^1} \int_{B_\nu(\rho_1/2)} \left( |D_\tau \phi|^2 + \epsilon^2 |F_\tau|^2 + |\gamma|^2 e_{\epsilon, \lambda}(U) \right) d\gamma \right) \bigg|_{\gamma^0 = s}; \]
and
\[ \tilde{\zeta}_4(t) := \int_{\{t \times \mathbb{R}^3 \setminus \mathcal{N}_1 \}} e_{\epsilon, \lambda}(U, \eta) d\tau \]
where
\[ \mathcal{N}_1 := \psi \left( (-T_1, T_1) \times S^1 \times B_\nu(\rho_1/2) \right) \cap \left( (-T_0, T_0) \times \mathbb{R}^3 \right). \]

Finally, let \( \zeta_0 := \max\{\tilde{\zeta}_1(0), \tilde{\zeta}_2(0), \tilde{\zeta}_3(0), \epsilon^2\} \).

Then there exists a constant \( C \), independent of \( U \) and \( \epsilon \in (0, 1] \), such that
\[ \tilde{\zeta}_i(s) \leq C \zeta_0 \text{ for } i = 1, 2, 3 \text{ and } -T_1 \leq s \leq T_1, \quad \tilde{\zeta}_4(t) \leq C \zeta_0 \text{ for } -T_0 \leq t \leq T_0. \]
We follow the proof of [12, Theorem 22]. We start by presenting all the details, to illustrate the basic argument, and we refer to [12] for the final part of the proof.

The proof will use some standard energy estimates for $U$ and $U$ respectively, which we recall for the reader's convenience.

**Lemma 6.2.** Let $U$ be a smooth finite-energy solution of the abelian Higgs model (1.2)-(1.3) in standard coordinates on $\mathbb{R}^{1+3}$. For any $a < b$ and any bounded Lipschitz function $\chi: (a,b) \times \mathbb{R}^3 \to \mathbb{R}$

$$
(6.3) \quad \left| \int_{\{b\} \times \mathbb{R}^3} e_\epsilon(U,\eta)\chi \, dx - \int_{\{a\} \times \mathbb{R}^3} e_\epsilon(U,\eta)\chi \, dx \right| \leq \int_{(a,b) \times \mathbb{R}^3} e_\epsilon(U,\eta)|D\chi| \, dx \, dt.
$$

**Proof.** Recall the energy identity

$$
(6.4) \quad \partial_t e_\epsilon(U,\eta) - \partial_i \left( \delta^{ij}(D_j \varphi, D_0 \varphi) + \epsilon^2 F_{ij} F_{00} \right) = 0
$$

for solutions of (1.2)-(1.3). This is standard and also can be deduced from (5.23) (replacing $(g_{\alpha\beta})$ by $(\eta_{\alpha\beta})$). We integrate by parts and use the fact that $|D_i \varphi, D_0 \varphi) + \epsilon^2 F_{ij} F_{00}| \leq e_\epsilon,\lambda(U,\eta)$ and routine estimates to deduce (6.3). If $\chi$ has unbounded support, then one can approximate it by functions with compact support and use the fact that $U$ has finite energy to pass to limits and obtain (6.3). \hfill \Box

**Lemma 6.3.** Let $U$ be a smooth solution of (5.5)-(5.6) on $(-T_1,T_1) \times S^1 \times B_\nu(\rho_0)$. Then for the number $\rho_1 \in (0,\rho_0]$ from Proposition 5.3 given $-T_1 \leq a < b \leq T_1$ and $\chi \in W^{1,\infty}((-T_1,T_1) \times S^1 \times B_\nu(2\rho_1))$, if $\chi(s,\cdot)$ has compact support in $S^1 \times B_\nu(2\rho_1)$ for every $s \in [a,b]$, then

$$
(6.5) \quad \left| \int_{\{b\} \times S^1 \times B_\nu(2\rho_1)} e_\epsilon(U,g)\chi - \int_{\{a\} \times S^1 \times B_\nu(2\rho_1)} e_\epsilon(U,g)\chi \right| \leq C \int_{(a,b) \times S^1 \times B_\nu(2\rho_1)} e_\epsilon(U,g) \left(|\chi| + |D\chi|\right).
$$

**Proof.** We fix $\rho_1$ as in the proof of Proposition 5.3 so that (5.29) holds. Then the proof of (6.5) is exactly like the proof of (6.3) in Lemma 6.2 above, except that we use for example (5.23) and (5.29) in place of their counterparts in standard coordinates. \hfill \Box

We will also often use the fact that there exists some $C > 0$ such that

$$
(6.6) \quad \frac{1}{C} \leq |\det D\psi(y)| = \sqrt{-g(y)} \leq C
$$

for all $y \in (-T_1,T_1) \times S^1 \times B_\nu(\rho_0)$. This is a straightforward consequence of the definition of $\psi$. A similar estimate holds for the restriction of $\psi$ to $\{y : y^0 = 0\}$, which we will call $\psi^0$.

**Proof of Proposition 6.1.** Recall that we defined $\zeta_i(s)$ for $i = 1,2,3$ in the statement of Proposition 5.3. Comparing these with the definitions of $\tilde{\zeta}_i(s)$, we see that $\zeta_2(s) = \tilde{\zeta}_2(s)$ and $\zeta_1(0) \leq \tilde{\zeta}_1(0) + C\zeta_4(0)$, using (6.6) and a change of variables. Thus $\zeta_i(0) \leq C\zeta_0$ for $i = 1,2$, and then Proposition 5.3 immediately implies that

$$
(6.7) \quad \tilde{\zeta}_i(s) \leq \zeta_i(s) \leq C\zeta_0 \quad \text{for } i = 1,2,3 \text{ and } 0 \leq s \leq s_1 := \min\{T_1, \rho_1/2c_*\}
$$

In particular, if we write $B_\nu(\rho') \setminus \rho := B_\nu(\rho') \setminus B_\nu(\rho)$,
then the definition of $\zeta_3$ implies that
\begin{equation}
(6.8) \quad \int_{\{s\} \times S^1 \times B_\rho\left(\frac{\rho_1}{4} \setminus \frac{\rho_1}{4}\right)} e_{e,\lambda}(U, g) \leq C(\rho_1)\zeta_3(s) \leq C\zeta_0 \quad \text{for } s \in [0, s_1].
\end{equation}

Then it follows from a change of variables and (6.1), (6.6) that
\begin{equation}
(6.9) \quad \int_{\psi(\{0, s_1\} \times S^1 \times B_\rho\left(\frac{\rho_1}{4} \setminus \frac{\rho_1}{4}\right))} e_{e,\lambda}(U, \eta) \leq C\zeta_0.
\end{equation}

2. Next we consider the standard coordinate system, and we show that the energy of $U$ is small away from $\Gamma$ for all $t \in [0, t_1]$, for some $t_1 > 0$. The idea is to apply Lemma 6.2 with $a = 0$ and $b = t \in (0, t_1]$ and a suitable cutoff function $\chi$, and to use (6.9) to estimate the terms appearing on the right-hand side of (6.3).

To carry this out, let $\chi : [-T_0, T_0] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ be a smooth function such that
\begin{align*}
\chi &= 1 \text{ on } ([-T_0, T_0] \times \mathbb{R}^3) \setminus \mathcal{N}_1 \\
\chi &= 0 \text{ on } ([-T_0, T_0] \times \mathbb{R}^3) \cap \psi \left((-T_1, T_1) \times S^1 \times B_\rho\left(\frac{\rho_1}{4}\right)\right).
\end{align*}

Thus $D\chi$ is supported on $([-T_0, T_0] \times \mathbb{R}^3) \cap \psi \left((-T_1, T_1) \times S^1 \times B_\rho\left(\frac{\rho_1}{4}\right)\right)$.

Next, the construction of $\psi$ implies that $\partial_0\psi > 0$ everywhere and that $\psi$ maps the set $\{y : y^0 = 0\}$ into the set $\{(t, x) : t = 0\}$, and these facts imply that there exists some $t_1 > 0$ such that
\begin{equation}
(6.10) \quad \text{supp}(D\chi) \cap (\{0, t_1\} \times \mathbb{R}^3) \subset \psi(\{0, s_1\} \times S^1 \times B_\rho\left(\frac{\rho_1}{2} \setminus \frac{\rho_1}{4}\right)).
\end{equation}

Now we apply Lemma 6.2 to find that for $t \in (0, t_1]$,
\begin{equation}
(6.11) \quad \int_{\{t\} \times \mathbb{R}^3} e_{e,\lambda}(U, \eta)\chi \leq \int_{\{0\} \times \mathbb{R}^3} e_{e,\lambda}(U, \eta)\chi + \int_{[0, t_1] \times \mathbb{R}^3} e_{e,\lambda}(U, \eta)|D\chi|.
\end{equation}

Moreover, (6.10) and (6.9) imply that
\begin{equation}
(6.12) \quad \int_{[0, t_1] \times \mathbb{R}^3} e_{e,\lambda}(U, \eta)|D\chi| \leq \|D\chi\|_\infty \int_{\psi(\{0, s_1\} \times S^1 \times B_\rho\left(\frac{\rho_1}{4}\right))} e_{e,\lambda}(U, \eta) \leq C\zeta_0.
\end{equation}

And using properties of the support of $\chi$ with (6.9) and definition of $\zeta_0$,
\begin{equation}
(6.13) \quad \int_{\{0\} \times \mathbb{R}^3} e_{e,\lambda}(U, \eta)\chi = \zeta_4(0) + \int_{\psi(\{0\} \times S^1 \times B_\rho\left(\frac{\rho_1}{4}\right))} e_{e,\lambda}(U, \eta) \leq C\zeta_0.
\end{equation}

Since $\chi = 1$ on the complement of $\mathcal{N}_1$, it follows that
\begin{equation}
(6.14) \quad \zeta_4(t) = \int_{\{t\} \times \mathbb{R}^3 \setminus \mathcal{N}_1} e_{e,\lambda}(U, \eta) \leq C\zeta_0 \quad \text{for every } t \in (0, t_1].
\end{equation}

3. Now for $\sigma \geq 0$, define
\begin{align*}
\zeta_1(s; \sigma) := & \int_{S^1} \left(\int_{B_\rho\left(\rho_1 - 1, s\right)} (1 + k_2|y^\nu|^2) e_{e,\lambda}(U) dy^\nu - E^m_\lambda\right) dy^1 \bigg|_{y^0 = s + \sigma} \\
\zeta_2(s; \sigma) := & \int_{0}^{\mathbb{R}} (U(s + \sigma, \cdot; 1/2 \rho_1)) \\
\zeta_3(s; \sigma) := & \int_{S^1} \int_{B_\rho\left(\rho_1 - 1, s\right)} \left(|\partial_\tau v|^2 + c^2 |F_\tau|^2 + |y^\nu|^2 e_{e,\lambda}(U)\right) dy^\nu dy^1 \bigg|_{y^0 = s + \sigma}.
\end{align*}
We claim that there exists some \( \sigma_1 \in (0, s_1] \) and a constant \( C > 0 \) so that
\[
\zeta_i(0; \sigma_1) \leq C \zeta_0 \quad \text{for } i = 1, 2, 3. \tag{6.12}
\]
This will allow us to apply Proposition \ref{prop:weighted-energy-estimates} to extend the weighted energy estimates for \( U \) beyond time \( y^0 = s_1 \).

For \( i = 2 \), note that \( \zeta_2(s; \sigma) = \zeta_2(s + \sigma) \), so that in particular \( \zeta_2(0; \sigma) = \zeta_2(\sigma) \leq C \zeta_0 \) for every \( \sigma \in (0, s_1] \), by \eqref{eq:zeta-1-bound}. For \( i = 1, 3 \), it suffices to find some \( \sigma_1 \in (0, s_1] \) such that \( \sigma_1 < \frac{\eta_1}{4c} \) and
\[
\zeta_i(0; \sigma_1) \leq \zeta_i(\sigma_1) + C \int S^1 \int_{B_y(\rho_1 \backslash \frac{3}{4} \rho_1)} e_{\epsilon, \lambda}(U, g) \bigg|_{y^0 = \sigma_1} \leq C \zeta_0 \tag{6.13}
\]
since then the definitions, \eqref{eq:rho-1} and \eqref{eq:zeta-1} imply that
\[
\zeta_i(0; \sigma_1) \leq \zeta_i(\sigma_1) + C \int S^1 \int_{B_y(\rho_1 \backslash \frac{3}{4} \rho_1)} e_{\epsilon, \lambda}(U, g) \bigg|_{y^0 = \sigma_1} \leq C \zeta_0
\]
proving \eqref{eq:zeta-1-bound}.

We will deduce \eqref{eq:zeta-1-bound} by using Lemma \ref{lem:weighted-energy-estimates} with a suitable cutoff function \( \chi \), and using \eqref{eq:rho-1} and a change of variables to estimate the terms appearing on the right-hand side of \eqref{eq:zeta-1-bound}.

To carry this out, let \( \chi : (-T_1, T_1) \times S^1 \times B_y(\rho_0) \to [0, 1] \) be a smooth cutoff function, independent of \( y^0 \in [0, 1] \), with support in \( S^1 \times B_y(2\rho_1 \backslash \frac{3\rho_1}{2}) \), and such that \( \chi = 1 \) on \( S^1 \times B_y(\rho_1 \backslash \frac{3\rho_1}{4}) \). We also fix \( \sigma_1 > 0 \) so small that
\[
\psi([0, \sigma_1] \times S^1 \times B_y(\rho_1 \backslash \frac{3\rho_1}{2})) \subset ([0, t_1] \times \mathbb{R}^3) \setminus \mathcal{N}_1. \tag{6.14}
\]
Then
\[
\int S^1 \int_{B_y(\rho_1 \backslash \frac{3}{4} \rho_1)} e_{\epsilon, \lambda}(U, g) \bigg|_{y^0 = \sigma_1} \leq \int S^1 \int_{B_y(2\rho_1)} e_{\epsilon, \lambda}(U, g) \chi \bigg|_{y^0 = \sigma_1} \leq \int S^1 \int_{B_y(2\rho_1)} e_{\epsilon, \lambda}(U, g) \chi \bigg|_{y^0 = 0} + C(\chi) \int_{[0, \sigma_1] \times S^1 \times B_y(2\rho_1 \backslash \frac{3\rho_1}{2})} e_{\epsilon, \lambda}(U, g)
\]
The first term on the right-hand side is bounded by \( C(\zeta_0(0) + \tilde{\zeta}_4(0)) \leq C \zeta_0 \), and it follows from \eqref{eq:zeta-1-bound}, \eqref{eq:rho-1}, \eqref{eq:zeta-1} and a change of variables that the second term on the right-hand side is bounded by \( C \zeta_0 \). Thus we have proved \eqref{eq:zeta-1-bound}, and hence also \eqref{eq:zeta-1-bound}.

4. It follows from \eqref{eq:zeta-1-bound} and Proposition \ref{prop:weighted-energy-estimates} that
\[
\tilde{\zeta}_i(s + \sigma_1) \leq \zeta_i(s; \sigma_1) \leq C \zeta_0 \quad \text{for } i = 1, 2, 3 \text{ and } 0 < s \leq s_2 = \min\{\rho_1/2c, T_1 - \sigma_1\}
\]
Note also that \( s_2 + \sigma_1 > s_1 \) unless \( s_1 = T_1 \).

We now iterate, using Lemma \ref{lem:weighted-energy-estimates} to estimate \( e_{\epsilon, \lambda}(U, \eta) \) on \((0, t_2] \times \mathbb{R}^3) \setminus \mathcal{N}_1\) for some \( t_2 > t_1 \), with estimates of the right-hand side of \eqref{eq:zeta-1-bound} provided by \eqref{eq:zeta-1-bound}, and then combining the resulting estimate with Lemma \ref{lem:weighted-energy-estimates} and Proposition \ref{prop:weighted-energy-estimates} to extend the weighted energy estimates of \( e_{\epsilon, \lambda}(U, g) \) beyond \( y^0 = s_2 + \sigma_1 \).
To complete the proof of the theorem, then, it suffices only to show that after finitely many iterations of this argument, one can extend the bounds on \( \tilde{z}_i \) to all \( 0 \leq s \leq T_1 \) (for \( i = 1, 2, 3 \)) and \( 0 \leq t \leq T_0 \) for \( i = 4 \). (The same conclusions for \(-T_1 \leq s < 0\) and \(-T_0 \leq t < 0\) then follow by time reversal symmetry.) A proof of this may be found in [12, proof of Theorem 22] for somewhat different equations, but exactly the same proof is valid here. The point is that the proof only involves piecing together estimates in the standard and normal coordinate systems, and the algorithm for doing so applies equally to any Lorenz-invariant equation.

(In fact the argument in [12] relies on a slightly different and more complicated iteration scheme than the one suggested above, but it remains true that the arguments there can be used in this setting with essentially no change.)

We finally prove our main result.

**Proof of Theorem 1.4.** We will write \( \psi_0(y^1, y^\nu) = \psi(0, y^1, y^\nu) \). Recall that by assumption, the minimal surface \( \Gamma = \text{image}(H) \) satisfies \( \partial_0 H(0, y^1) = (1, 0, 0, 0) \) for every \( y^1 \). As a result, the normal vectors \( \bar{v}_i \), satisfy \( \bar{v}_i \equiv 0 \) for \( i = 1, 2 \), see (1.20), and hence the range of \( \psi_0 \) is an open neighborhood of \( \Gamma_0 \) in \( \{ 0 \} \times \mathbb{R}^3 \).

Also, if we define \( d_0(x) \) := \( \text{dist}(x, \Gamma_0) \), then we note that

\[(6.16) \quad d_0(\psi_0(y^1, y^\nu)) = |y^\nu|. \]

Indeed, again using the fact that \( \partial_0 H(0, y^1) = (1, 0, 0, 0) \), we deduce from (1.20) that the vectors \( \{ \partial_1 H(0, y^1), \bar{v}_1, \bar{v}_2 \} \) are orthogonal with respect to the Euclidean inner product, and it follows that \( |y^\nu| = |\psi_0(y^1, y^\nu) - \psi_0(y^1, 0)| \) and also the segment \( \psi_0(y^1, y^\nu) - \psi_0(y^1, 0) \) is orthogonal to \( T_{\psi(y^1, 0)} \Gamma_0 \). Since in addition \( |\psi_0(y^1, y^\nu) - \psi_0(y^1, 0)| \) is less than the injectivity radius of \( \Gamma_0 \) by assumption (5.1) on \( \rho_0 \), these imply (6.16).

1. We first specify initial data for (1.2)-(1.3) so that the constant \( \zeta_0 \) in Proposition 6.1 is small. The fact that the data needs to satisfy the compatibility condition (1.32) for well-posedness means that this is not completely straightforward.

First, let \( U_{1,\lambda}^m \) denote a fixed solution of the minimization problem (1.16) for \( \epsilon = 1 \), and for general \( \epsilon > 0 \), let \( U_{\epsilon,\lambda}^m \) denote the solution obtained by rescaling \( U_{1,\lambda}^m \), so that \( \phi^\epsilon(y) := \phi(\frac{y}{\epsilon}), A^\epsilon(y) := \frac{1}{\epsilon} A(\frac{y}{\epsilon}) \). Then

\[(6.17) \quad e_{\epsilon,\lambda}(y) = \frac{1}{\epsilon^2} \epsilon_{1,\lambda}^m(U_{1,\lambda}^m)(\frac{y}{\epsilon}), \quad \omega(U_{\epsilon,\lambda}^m)(y) = \frac{1}{\epsilon^2} \omega(U_{1,\lambda}^m)(\frac{y}{\epsilon}) \]

A useful property of \( U_{1,\lambda}^m \) is

\[(6.18) \quad e_{\epsilon,\lambda}(U_{1,\lambda}^m)(y^\nu) \leq C e^{-c|y^\nu|}. \]

This is proved in [9], Chapter 3, Theorem 8.1 for arbitrary finite-energy critical points of the 2d Euclidean abelian Higgs action. It follows that

\[\int_{\partial B_{\nu}(\rho_1/3)} e_{\epsilon,\lambda}(U_{1,\lambda}^m)(y^\nu) \leq C e^{-c/\epsilon}. \]

Then by the construction in the proof of Lemma 3.4 we can modify \( U_{\epsilon,\lambda}^m \) on \( \mathbb{R}^2 \setminus B_{\nu}(\rho_1/3) \) to produce a new configuration \( U_{\epsilon,\lambda}^m \) such that

\[(6.19) \quad \hat{U}_{\epsilon,\lambda}^m = U_{\epsilon,\lambda}^m \text{ in } B_{\nu}(\rho_1/3), \quad \int_{B_{\nu}(\rho_1/2) \setminus B_{\nu}(\rho_1/3)} e_{\epsilon,\lambda}(U_{\epsilon,\lambda}^m) \leq C e^{-c/\epsilon} \leq C e^2 \]
and in addition \( \tilde{U}_{e,\lambda}^m \) has the form
\[
(6.20) \quad \tilde{\phi}_{e,\lambda} = e^{i\zeta}, \quad \tilde{A}_{e,\lambda} = d\zeta \quad \text{in} \quad \mathbb{R}^2 \setminus B_\nu(\rho_1/2)
\]
for some smooth function \( \zeta \) taking values in \( \mathbb{R}/2\pi\mathbb{Z} \), which in particular implies that
\[
(6.21) \quad e_{e,\lambda}(\tilde{U}_{e,\lambda}^m) = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus B_\nu(\rho_1/2).
\]

We now define
\[
(6.22) \quad (\varphi^0, A^0) = (\psi_0^{-1})^*(\tilde{\phi}_{e,\lambda}^m, \tilde{A}_{e,\lambda}^m) \quad \text{on} \quad \psi_0(S^1 \times B_\nu(\rho_1)) \subset \{0\} \times \mathbb{R}^3,
\]
(where here we view \( \tilde{\phi}_{e,\lambda}^m, \tilde{A}_{e,\lambda}^m \) as functions on \( S^1 \times \mathbb{R}^2 \) that are independent of \( y^1 \).) In view of (6.20), in \( \psi_0(S^1 \times (B_\nu(\rho_1) \setminus B_\nu(\rho_1/2)) \), \( U^0 \) has the form
\[
\varphi^0 = e^{i\zeta}, \quad A^0 = d\zeta \quad \text{for} \quad \zeta := \zeta \circ \psi_0^{-1}.
\]

Now fix a smooth \( q : \mathbb{R}^3 \setminus [\psi_0(S^1 \times B_\nu(\rho_1/2))] \to \mathbb{R}/2\pi\mathbb{Z} \) such that \( q = \zeta \) on the intersection of the domains of \( q \) and \( \zeta \), and define
\[
(6.23) \quad \varphi^0 = e^{iq}, \quad A^0 = dq \quad \text{in} \quad \mathbb{R}^3 \setminus \psi_0(S^1 \times B_\nu(\rho_1/2)).
\]

The choice of \( q \) implies that (6.22) and (6.23) are consistent.

Finally, we let \( \mathcal{U} \) denote the solution of (1.2)-(1.3) in the temporal gauge \( A_0 = 0 \), with initial data
\[
(6.24) \quad (\varphi, A_1, \ldots, A_3)|_{t=0} = (\varphi^0, A_1^0, \ldots, A_3^0) := \mathcal{U}^0, \quad \partial_t(\varphi, A_1, \ldots, A_3)|_{t=0} = 0.
\]

The existence of the solution \( \mathcal{U} \) follows from the discussion in Sections 1.6, 1.7

2. We claim that in this gauge, and for the initial data (6.22), (6.23) and (6.24) above,
\[
(6.25) \quad \tilde{\zeta}_i(0) \leq Ce^2 \quad \text{for} \quad i = 1, 2, 4
\]

We remind the reader that these quantities are defined in the statement of Proposition 6.1.

First, it is routine to check from (6.22), (6.23) that \( e_{e,\lambda}(\mathcal{U}, \eta) = 0 \) in \( \mathbb{R}^3 \setminus \psi_0(S^1 \times B_\nu(\rho_1/2)) \), and hence that \( \tilde{\zeta}_4(0) = 0 \).

The quantities \( \tilde{\zeta}_1, \tilde{\zeta}_2 \) are expressed in terms of the \( U = (\phi, A) = \psi^* \mathcal{U} \). So we must translate our assumptions about \( \mathcal{U} \) at \( t = 0 \) into information about \( U \) for \( y^0 = 0 \).

Let us write \( A^0(y_1, y^\nu) := \sum_{i=1}^3 A_i(0, y^1, y^\nu)dy^i \) to denote the spatial part of \( A \) at time \( y^0 = 0 \), and \( U^0 = (\phi^0, A^0) \), where \( \phi^0(y_1, y^\nu) := \phi(0, y^1, y^\nu) \). This says that \( U^0 = i^*U^0 \), for \( i(y_1, y^\nu) = (0, y^1, y^\nu) \). As a result,
\[
U^0 = i^*\psi^* \mathcal{U} = (\psi \circ i)^* \mathcal{U} = \psi_0^* \mathcal{U} = \psi_0^* U^0
\]
using the fact for the final equality that \( \text{Image}(\psi_0) \subset \{0\} \times \mathbb{R}^3 \). Hence (6.22) implies that \( U^0 = (\tilde{\phi}_{e,\lambda}^m, \tilde{A}_{e,\lambda}^m) \), or in other words that
\[
\phi(0, y^1, y^\nu) = \tilde{\phi}_{e,\lambda}^m(y^\nu),
\]
\[
\sum_{i=1}^3 A_i(0, y^1, y^\nu)dy^i = \tilde{A}_{e,\lambda}^{m,1}(y^\nu)dy^2 + \tilde{A}_{e,\lambda}^{m,2}(y^\nu)dy^3.
\]

As a result,
\[
(6.26) \quad D_1\phi = 0, \quad F_{ij} = F_{ji} = 0 \quad \text{for} \quad j = 2, 3
\]
when \( y^0 = 0 \), everywhere in \( S^1 \times B_\nu(\rho_1) \). Next, we can write the identity \( U = \psi^* U \) explicitly as

\[
\phi = \varphi \circ \psi, \quad A_\alpha = \frac{\partial \psi^\mu}{\partial y^\alpha} A_\mu \circ \psi
\]

From these we check that

\[
D_y \phi = \frac{\partial \psi^\mu}{\partial y^\alpha} (D_{x^\nu} \varphi) \circ \psi = \sum_{k=1}^{3} \frac{\partial \psi^k}{\partial y^\alpha} (D_{x^\nu} \varphi) \circ \psi
\]

due to the temporal gauge and the initial condition \( \partial_t \varphi = 0 \). (Here for example \( D_y \phi = \frac{\partial}{\partial y^\alpha} - iA_0 \).) Recall that we have assumed that the initial velocity of \( \Gamma \) vanishes. This states that \( \partial_y h(0, y^1) = 0 \), and then it follows from the explicit form (1.21) of \( \psi \) that

\[
\frac{\partial \psi^k}{\partial y^\alpha} (0, y^1, y^\nu) = O(|y^\nu|) \text{ for } k = 1, 2, 3.
\]

Thus for \( y^0 = 0 \),

\[
|D_y \phi|^2 \leq C|y^\nu|^2 \sum_{k=1}^{3} |(D_{x^\nu} \varphi) \circ \psi|^2 \leq C|y^\nu|^2 \sum_{k=1}^{3} D_y \phi^2 \leq C|y^\nu|^2 |D_y \phi|^2.
\]

Similarly, (6.27), the temporal gauge, and the initial conditions imply that for \( y^0 = 0 \),

\[
F_{\alpha\beta} = \sum_{i,j=1}^{3} \frac{\partial \psi^i}{\partial y^\alpha} \frac{\partial \psi^j}{\partial y^\beta} (\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}) \circ \psi,
\]

so, again using the fact that \( \frac{\partial \psi^k}{\partial y^\alpha} (0, y^1, y^\nu) = O(|y^\nu|) \), we see that

\[
F_{0k}^2 \leq C|y^\nu|^2 \sum_{i,j=1}^{3} F_{ij}^2 \leq C|y^\nu|^2 \sum_{i,j=1}^{3} F_{ij}^2 \leq C|y^\nu|^2 |F_\nu|^2, \quad k = 1, 2, 3.
\]

Combining (6.26), (6.28) and (6.29), and recalling (5.15), we find that

\[
e_{\epsilon,\lambda}(U, U^*) U(0, y^1, y^\nu) \leq (1 + C|y^\nu|^2) e_{\epsilon,\lambda}(U^m_{1,\lambda})(y^\nu)
\]

for all \( y^1 \in S^1 \). We have chosen \( U^m_{1,\lambda} \) exactly so that it satisfies \( \int_{\mathbb{R}^2} e_{\epsilon,\lambda}(U^m_{1,\lambda}) = E^m_{\lambda} \), so we can use (6.19), (6.21), together with (6.17) and a change of variables, to find that

\[
\tilde{\zeta}_1(0) \leq C\epsilon^2 + C \int_{S^1 \times B_\nu(\rho_1/2)} |y^\nu|^2 e_{\epsilon,\lambda}(U^m_{1,\lambda})(y^\nu) dy^1 dy^\nu
\]

\[
\leq C\epsilon^2 + C \int_{\mathbb{R}^2} |y^\nu|^2 e_{1,\lambda}(U^m_{1,\lambda}) dy^\nu
\]

\[
\leq C\epsilon^2.
\]

The finiteness of the second moment in the last inequality follows from the exponential decay estimate (6.18).

Next, since \( |\omega(U)| \leq C(\lambda) e_{\epsilon,\lambda}(U) \), and since \( \int_{\mathbb{R}^2} \omega(U^m_{1,\lambda}) = \pi m \), one can check, again using (6.17), (6.19), (6.21), (recall also the definitions (5.24), (3.1), (3.2)) that

\[
\tilde{\zeta}_2(0) \leq C\epsilon^3 \leq C\epsilon^2,
\]

where the scaling \( \epsilon^3 \) comes ultimately from (3.2).

3. Proposition 6.1 now implies

\[
\tilde{\zeta}_s(s) \leq C\epsilon^2 \text{ for } s = 1, 2, 3 \text{ and } -T_1 \leq s \leq T_1,
\]

where \( T_1 = C\epsilon^2 \).
In particular, (6.31) implies (1.28). Next, by (6.1), (6.6), and the definition of \( d \), proving (1.27). (In fact the estimate of \( \tilde{N} \)

proving (1.27). (In fact the estimate of \( \tilde{C} \) in (6.30) is substantially stronger than (1.27).)

4. To establish (1.26), we carry out a gauge transform to arrange that

\[
A_0 = 0 \quad \text{in} \quad \psi^{-1}(\mathcal{N}_1) \subset (-T_1, T_1) \times S^1 \times B_\nu(\rho_1/2).
\]

Toward this end, let \( \chi \) be a smooth function with support in \( \text{Image}(\psi) \) such that \( \chi = 1 \) in \( \mathcal{N}_1 \), and define

\[
v = (1 - \chi)e_0 + \chi \hat{v}, \quad e_0 := (1, 0, 0, 0), \quad \hat{v} := \frac{\partial \psi^\mu}{\partial y^0} \circ \psi^{-1}.
\]

Then from (6.27) we see that (6.32) holds if and only if \( v^\mu A_\mu = 0 \) in \( \mathcal{N}_1 \). To arrange this, let \( f \) satisfy the linear transport equation

\[
v^\mu (\partial_\mu f + A_\mu) = 0 \quad \text{in} \quad (-T_0, T_0) \times \mathbb{R}^3, \quad f(0, x) = 0 \quad \text{for} \quad x \in \mathbb{R}^3.
\]

(The condition \( g_{00} < 0 \) implies that \( \hat{v} \) and hence \( v \) are timelike; thus \( v^0 \) never vanishes, so the above initial value problem is solvable.) Then after the gauge transform \( (\varphi, \mathcal{A}) \rightarrow (e^{iF} \varphi, \mathcal{A} + df) \), the equation that defines \( f \) states exactly that the new connection 1-form satisfies \( v^\mu A_\mu = 0 \). Thus we have achieved (6.32). Also, since \( \tilde{C}_i, i = 1, \ldots, 4 \) are gauge invariant, (6.30)-(6.31) still hold.

Recall the form of \( U^{\mathcal{A}} = (\phi^{\mathcal{A}}, A^{\mathcal{A}}) \):

\[
\phi^{\mathcal{A}}(y^\tau, y^\nu) = \phi^m(y^\nu), \quad A^{\mathcal{A}}(y^\tau, y^\nu) := A_1^m(y^\nu)dy^\nu + A_2^m(y^\nu)dy^\nu,
\]

where \( U^m = (\phi^m, A^m) = U^{\mathcal{A}} \) is a minimizer. We stipulate that \( U_\mathcal{A}^{\mathcal{A}} \) is exactly the same minimizer out of which \( \tilde{\phi}_m^\mathcal{A}, \tilde{A}_m^\mathcal{A} \) is constructed in Step 1. Then by a change of variables and a Poincaré inequality we have

\[
\int_{\mathcal{N}_1} |\varphi - \phi^{\mathcal{A}}|^2 + \epsilon^2 \sum_{\alpha=0}^3 |A_\alpha - A_\alpha^{\mathcal{A}}|^2
\]

\[
\leq C \int_{(-T_1, T_1) \times S^1 \times B_\nu(\rho_1/2)} |\phi - \phi^{\mathcal{A}}|^2 + \epsilon^2 \sum_{\alpha=0}^3 |A_\alpha - A_\alpha^{\mathcal{A}}|^2
\]

\[
\leq C \int_{(-T_1, T_1) \times S^1 \times B_\nu(\rho_1/2)} |\partial y^\nu (\phi - \phi^{\mathcal{A}})|^2 + \epsilon^2 \sum_{\alpha=0}^3 |\partial y^\nu (A_\alpha - A_\alpha^{\mathcal{A}})|^2
\]

\[
+ C \left( \int_{S^1 \times B_\nu(\rho_1/2)} |\phi - \phi^{\mathcal{A}}|^2 + \epsilon^2 \sum_{\alpha=0}^3 |A_\alpha - A_\alpha^{\mathcal{A}}|^2 \right)_{|y^\nu = T_1}.
\]

and

\[
(6.31) \quad \tilde{\zeta}_i(t) \leq C \epsilon^2 \quad \text{for} \quad -T_0 \leq t \leq T_0.
\]
In the first integral on the right-hand side we use the explicit form of $U^{\text{NO}}$ and the gauge $A_0 = 0$ to write
\[|\partial^y \phi (\phi - \phi^{\text{NO}})|^2 + \epsilon^2 \sum_{\alpha=0}^3 |\partial^y (A_\alpha - A^{\text{NO}}_\alpha)|^2 = |D_0 \phi|^2 + \epsilon^2 |F_\tau|^2.\]

Inserting this into (6.33) and using (6.30), we conclude the first integral is bounded by $C\epsilon^2$.

To bound the second integral, using fundamental theorem of calculus we observe
\[\left(\int_{S^1 \times B_\nu (\rho_1/2)} |\phi - \phi^{\text{NO}}|^2 + \epsilon^2 \sum_{\alpha=0}^3 |A_\alpha - A^{\text{NO}}_\alpha|^2\right)|_{y^0=0} \leq C \left(\int_{S^1 \times B_\nu (\rho_1/2)} |\phi - \phi^{\text{NO}}|^2 + \epsilon^2 \sum_{\alpha=0}^3 |A_\alpha - A^{\text{NO}}_\alpha|^2\right)|_{y^0=0}
+ C \int_0^T_1 \int_{S^1 \times B_\nu (\rho_1/2)} |\partial^y \phi|^2 + \epsilon^2 \sum_{\alpha=0}^3 (\partial^y A_\alpha)^2.\]

The second integral is bounded again using $A_0 = 0$ and (6.30), whereas $C\epsilon^2$ bounds for the first integral follow from the construction of the data. The boundary term $y^0 = -T_1$ is treated exactly the same. This gives (1.26). □

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