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Semismall perturbations, semi-intrinsic ultracontractivity, and integral representations of nonnegative solutions for parabolic equations

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Abstract

We consider nonnegative solutions of a parabolic equation in a cylinder $D \times I$, where $D$ is a noncompact domain of a Riemannian manifold and $I = (0, T)$ with $0 < T \leq \infty$ or $I = (-\infty, 0)$. Under the assumption [SSP] (i.e., the constant function 1 is a semismall perturbation of the associated elliptic operator on $D$), we establish an integral representation theorem of nonnegative solutions: In the case $I = (0, T)$, any nonnegative solution is represented uniquely by an integral on $(D \times \{0\}) \cup (\partial_M D \times [0, T])$, where $\partial_M D$ is the Martin boundary of $D$ for the elliptic operator; and in the case $I = (-\infty, 0)$, any nonnegative solution is represented uniquely by the sum of an integral on $\partial_M D \times (-\infty, 0)$ and a constant multiple of a particular solution. We also show that [SSP] implies the condition [SIU] (i.e., the associated heat kernel is semi-intrinsically ultracontractive).

Key Words and Phrases: semismall perturbation, semi-intrinsic ultracontractivity, parabolic equation, nonnegative solution, integral representation, Martin boundary, elliptic equation
1 Introduction

This paper is a continuation of [34]. It is concerned with integral representations of nonnegative solutions to parabolic equations and perturbation theory for elliptic operators.

We consider nonnegative solutions of a parabolic equation

$$\left( \frac{\partial}{\partial t} + L \right) u = 0 \quad \text{in} \quad D \times I,$$  \hspace{1cm} (1.1)

where $\frac{\partial}{\partial t} = \frac{\partial}{\partial t}$, $L$ is a second order elliptic operator on a noncompact domain $D$ of a Riemannian manifold $M$, and $I$ is a time interval: $I = (0, T]$ with $0 < T \leq \infty$ or $I = (-\infty, 0)$.

During the last few decades, much attention has been paid to the structure of all nonnegative solutions to a parabolic equation, perturbation theory for elliptic operators, and their relations. (See [1], [2], [4], [5], [6], [11], [14], [17], [19], [20], [22], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [36], [37], [38], [40], [41], [42].) Among others, Murata [34] has established integral representation theorems of nonnegative solutions to the equation (1.1) under the condition $[IU]$ (i.e., intrinsic ultracontractivity) on the minimal fundamental solution $p(x, y, t)$ for (1.1). Furthermore, he has shown that $[IU]$ implies $[SP]$ (i.e., the constant function 1 is a small perturbation of $L$ on $D$). It is known ([30]) that $[SP]$ implies $[SSP]$ (i.e., 1 is a semismall perturbation of $L$ on $D$).

In this paper, we show that $[SSP]$ implies $[SIU]$ (i.e., semi-intrinsic ultracontractivity) and give integral representation theorems of nonnegative solutions to (1.1) under the condition $[SSP]$. We consider that $[SSP]$ is one of the weakest possible condition for getting "explicit" integral representation theorems.

Now, in order to state our main results, we fix notations and recall several notions and facts. Let $M$ be a connected separable $n$-dimensional smooth manifold with Riemannian metric of class $C^0$. Denote by $\nu$ the Riemannian measure on $M$. $T_xM$ and $TM$ denote the tangent space to $M$ at $x \in M$ and the tangent bundle, respectively. We denote by $\text{End}(T_xM)$ and $\text{End}(TM)$ the set of endomorphisms in $T_xM$ and the corresponding bundle, respectively. The inner product on $TM$ is denoted by $\langle X, Y \rangle$, where $X, Y \in TM$; and $|X| = \langle X, X \rangle^{1/2}$. The divergence and gradient with respect to the metric on $M$ are denoted by $\text{div}$ and $\nabla$, respectively. Let $D$ be a noncompact domain of $M$. Let $L$ be an elliptic differential operator on $D$ of the form

$$Lu = -m^{-1}\text{div}(mA\nabla u) + Vu,$$  \hspace{1cm} (1.2)
where $m$ is a positive measurable function on $D$ such that $m$ and $m^{-1}$ are bounded on any compact subset of $D$, $A$ is a symmetric measurable section on $D$ of $\text{End}(TM)$, and $V$ is a real-valued measurable function on $D$ such that

$$V \in L^p_{\text{loc}}(D, md\nu) \quad \text{for some } p > \max\left(\frac{n}{2}, 1\right).$$

Here $L^p_{\text{loc}}(D, md\nu)$ is the set of real-valued functions on $D$ locally $p$-th integrable with respect to $md\nu$. We assume that $L$ is locally uniformly elliptic on $D$, i.e., for any compact set $K$ in $D$ there exists a positive constant $\lambda$ such that

$$\lambda|\xi|^2 \leq \langle A_x \xi, \xi \rangle \leq \lambda^{-1}|\xi|^2, \quad x \in K, \ (x, \xi) \in TM.$$

We assume that the quadratic form $Q$ on $C_0^\infty(D)$ defined by

$$Q[u] = \int_D (\langle A \nabla u, \nabla u \rangle + Vu^2) md\nu$$

is bounded from below, and put

$$\lambda_0 = \inf \left\{ Q[u]; u \in C_0^\infty(D), \ \int_D u^2 md\nu = 1 \right\}.$$

Then, for any $a < \lambda_0$, $(L - a, D)$ is subcritical, i.e., there exists the (minimal positive) Green function of $L - a$ on $D$. We denote by $L_D$ the selfadjoint operator in $L^2(D, md\nu)$ associated with the closure of $Q$. The minimal fundamental solution for (1.1) is denoted by $p(x, y, t)$, which is equal to the integral kernel of the semigroup $e^{-tL_D}$ on $L^2(D, md\nu)$.

Let us recall several notions related to [SSP].

[II] $\lambda_0$ is an eigenvalue of $L_D$; and there exists, for any $t > 0$, a constant $C_t > 0$ such that

$$p(x, y, t) \leq C_t \phi_0(x)\phi_0(y), \quad x, y \in D,$$

where $\phi_0$ is the normalized positive eigenfunction for $\lambda_0$.

This notion was introduced by Davies-Simon [13], and investigated extensively because of its important consequences (see [7], [8], [9], [10], [12], [23], [24], [31], [34], [42], and references therein). It looks, on the surface, not related to perturbation theory. But it has turned out ( [34]) that [II] implies the following condition [SP] for any $a < \lambda_0$. 

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The constant function 1 is a small perturbation of $L - a$ on $D$, i.e., for any $\varepsilon > 0$ there exists a compact subset $K$ of $D$ such that

$$
\int_{D \setminus K} G(x, z)G(z, y)m(z)\nu(z) \leq \varepsilon G(x, y), \quad x, y \in D \setminus K,
$$

where $G$ is the Green function of $L - a$ on $D$.

This condition is a special case of the notion introduced by Pinchover [37]. Recall that [SP] implies the following condition [SSP] (see [30]).

The constant function 1 is a semismall perturbation of $L - a$ on $D$, i.e., with $x^0$ being a fixed reference point in $D$, for any $\varepsilon > 0$ there exists a compact subset $K$ of $D$ such that

$$
\int_{D \setminus K} G(x^0, z)G(z, y)m(z)\nu(z) \leq \varepsilon G(x^0, y), \quad y \in D \setminus K.
$$

This condition [SSP] implies that $L_D$ admits a complete orthonormal base of eigenfunctions $\{\phi_j\}_{j=0}^\infty$ with eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ repeated according to multiplicity; furthermore, for any $j = 1, 2, \ldots$, the function $\phi_j/\phi_0$ has a continuous extension $[\phi_j/\phi_0]$ up to the Martin boundary $\partial_M D$ of $D$ for $L - a$ (see Theorem 6.3 of [38]).

We show in this paper that [SSP] also implies the following condition [SIU].

$\lambda_0$ is an eigenvalue of $L_D$; and there exist, for any $t > 0$ and compact subset $K$ of $D$, positive constants $A$ and $B$ such that

$$
A \phi_0(x)\phi_0(y) \leq p(x, y, t) \leq B \phi_0(x)\phi_0(y), \quad x \in K, \quad y \in D.
$$

This notion was introduced by Bañuelos-Davis [9], where they called it one half IU. Here we should recall that [IU] implies that for any $t > 0$ there exists a constant $c_t > 0$ such that

$$
c_t \phi_0(x)\phi_0(y) \leq p(x, y, t), \quad x, y \in D.
$$

We see that the same argument as in the proof of Theorem 3.1 in [25] (or the argument in the proof of Theorem 1.2 below) shows that [SIU] implies the following condition [NUP] (i.e., non-uniqueness for the positive Cauchy problem).
The Cauchy problem

\[(\partial_t + L)u = 0 \quad \text{in} \quad D \times (0, T), \quad u(x, 0) = 0 \quad \text{on} \quad D \tag{1.3}\]

admits a nonnegative solution \(u\) which is not identically zero.

We say that [UP] holds for (1.3) when any nonnegative solution of (1.3) is identically zero. We note that [UP] implies that the constant function 1 is a "big" perturbation of \(L - a\) on \(D\) in some sense (see Theorem 2.1 of [32]).

Fix \(a < \lambda_0\), and suppose that [SSP] holds. Let \(D^* = D \cup \partial_M D\) be the Martin compactification of \(D\) for \(L - a\), which is a compact metric space. Denote by \(\partial_mD\) the minimal Martin boundary of \(D\) for \(L - a\), which is a Borel subset of the Martin boundary \(\partial_M D\) of \(D\) for \(L - a\). Here, we note that \(\partial_M D\) and \(\partial_mD\) are independent of \(a\) in the following sense: if [SSP] holds, then for any \(b < \lambda_0\) there is a homeomorphism \(\Phi\) from the Martin compactification of \(D\) for \(L - a\) onto that for \(L - b\) such that \(\Phi|D = \text{identity}\), and \(\Phi\) maps the Martin boundary and minimal Martin boundary of \(D\) for \(L - a\) onto those for \(L - b\), respectively (see Theorem 1.4 of [30]).

Now, we are ready to state our main results. In the following theorems we assume that [SSP] holds for some fixed \(a < \lambda_0\).

**Theorem 1.1** The condition [SSP] implies [SIU].

**Theorem 1.2** Assume [SSP]. Then, for any \(\xi \in \partial_M D\) there exists the limit

\[
\lim_{D \ni y \to \xi} \frac{p(x,y,t)}{\phi_0(y)} \equiv q(x,\xi,t), \quad x \in D, \ t \in \mathbb{R}. \tag{1.4}
\]

Here, as functions of \((x,t)\), \(\{p(x,y,t)/\phi_0(y)\}_y\) converges to \(q(x,\xi,t)\) as \(y \to \xi\) uniformly on any compact subset of \(D \times \mathbb{R}\). Furthermore, \(q(x,\xi,t)\) is a continuous function on \(D \times \partial_M D \times \mathbb{R}\) such that

\[
q > 0 \quad \text{on} \quad D \times \partial_M D \times (0, \infty), \tag{1.5}
\]

\[
q = 0 \quad \text{on} \quad D \times \partial_M D \times (-\infty, 0], \tag{1.6}
\]

\[(\partial_t + L)q(\cdot,\xi,\cdot) = 0 \quad \text{on} \quad D \times \mathbb{R}. \tag{1.7}\]

**Theorem 1.3** Assume [SSP]. Consider the equation (1.1) for \(I = (0, T)\) with \(0 < T \leq \infty\). Then, for any nonnegative solution \(u\) of (1.1) there exists
a unique pair of Borel measures $\mu$ on $D$ and $\lambda$ on $\partial M \times [0, T)$ such that $\lambda$ is supported by the set $\partial M \times [0, T)$, and

$$u(x, t) = \int_D p(x, y, t) \, d\mu(y) + \int_{\partial M \times [0, t)} q(x, \xi, t-s) \, d\lambda(\xi, s)$$  \hspace{1cm} (1.8)

for any $(x, t) \in D \times I$.

Conversely, for any Borel measures $\mu$ on $D$ and $\lambda$ on $\partial M \times [0, T)$ such that $\lambda$ is supported by $\partial M \times [0, T)$ and

$$\int_D p(x_0, y, t) \, d\mu(y) < \infty, \hspace{1cm} 0 < t < T, \hspace{1cm} (1.9)$$

$$\int_{\partial M \times [0, t)} q(x_0, \xi, t-s) \, d\lambda(\xi, s) < \infty, \hspace{1cm} 0 < t < T, \hspace{1cm} (1.10)$$

where $x_0$ is a fixed point in $D$, the right hand side of (1.8) is a nonnegative solution of (1.1) for $I = (0, T)$ with $0 < T \leq \infty$.

The proof of this theorem will be given in Sections 4 and 5. It is based upon the abstract integral representation theorem established in [34], without assuming [IU], via a parabolic Martin representation theorem and Choquet's theorem (see [18], [21], [35]). Its key step is to identify the parabolic Martin boundary.

This theorem is an improvement of Theorem 1.2 of [34]; where the condition [IU], which is more stringent than [SSP], is assumed. It is also an answer to a problem raised in Remark 4.13 of [34]. Note that (1.8) gives explicit integral representations of nonnegative solutions to (1.1) provided that the Martin boundary $\partial M$ of $D$ for $L_a$ is determined explicitly. We consider that [SSP] is one of the weakest possible condition for getting such explicit integral representations.

Let us recall that when [UP] holds for (1.3), the structure of all nonnegative solutions to (1.1) for $I = (0, T)$ is extremely simple. Namely, the following theorem holds (see [5]).

Fact AT

Assume [UP]. Then, for any nonnegative solution $u$ of (1.1) with $I = (0, T)$, there exists a unique Borel measure $\mu$ on $D$ such that

$$u(x, t) = \int_D p(x, y, t) \, d\mu(y), \hspace{1cm} (x, t) \in D \times I. \hspace{1cm} (1.11)$$
Conversely, for any Borel measure $\mu$ on $D$ satisfying (1.9), the right hand side of (1.11) is a nonnegative solution of (1.1) with $I = (0, T)$.

It is quite interesting that when [UP] holds, the elliptic Martin boundary disappears in the parabolic representation theorem; while it enters in many cases of [NUP].

Finally, we state an integral representation theorem for the case $I = (-\infty, 0)$.

**Theorem 1.4** Assume [SSP]. Consider the equation (1.1) for $I = (-\infty, 0)$. Then, for any nonnegative solution $u$ of (1.1) there exists a unique pair of a nonnegative constant $\alpha$ and a Borel measure $\lambda$ on $\partial M \times (-\infty, 0)$ supported by the set $\partial m \times (-\infty, 0)$ such that

$$u(x, t) = \alpha e^{-\lambda_0 t} \phi_0(x) + \int_{\partial M \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s)$$  \hspace{1cm} (1.12)

for any $(x, t) \in D \times (-\infty, 0)$.

Conversely, for any nonnegative constant $\alpha$ and a Borel measure $\lambda$ on $\partial M \times (-\infty, 0)$ such that it is supported by $\partial m \times (-\infty, 0)$ and

$$\int_{\partial M \times (-\infty, t)} q(x^0, \xi, t - s) d\lambda(\xi, s) < \infty, \quad -\infty < t < 0,$$  \hspace{1cm} (1.13)

the right hand side of (1.12) is a nonnegative solution of (1.1).

This theorem is an improvement of Theorem 6.1 of [34], where [IU] is assumed instead of [SSP].

Here, in order to illustrate a scope of Theorems 1.3 and 1.4, we give a simple example. Further examples will be given in Section 7.

**Example 1.5** Let $D$ be a domain in $\mathbb{R}^2$ with finite area. Then, by Theorem 6.1 of [33], the constant function $1$ is a small perturbation of $L = -\Delta$ on $D$.

Thus Theorems 1.3 and 1.4 hold true for the heat equation

$$(\partial_t - \Delta) u = 0 \quad \text{in} \quad D \times I.$$

Note that there exist many bounded planar domains for which the heat semigroup is not intrinsically ultracontractive (see Example 1 of [13] and Section 4 of [9]). Thus, the last assertion of this example is new for such domains.
The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1.1, and Theorem 1.2 is proved in Section 3. Sections 4 and 5 are devoted to the proof of Theorem 1.3. In Section 4 we show it in the case of $I = (0, \infty)$. In Section 5 we show it in the case of $I = (0, T)$ with $0 < T < \infty$ by making use of results to be given in Section 4. Theorem 1.4 is proved in Section 6. Finally we shall give two more concrete examples in Section 7 with emphasis on sharpness of concrete sufficient conditions of [SSP].

2 [SSP] implies [SIU]

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 We may and shall assume that $a = 0 < \lambda_0$. Let $G$ be the Green function of $L$ on $D$. For any $t > 0$, put
\[
G_t(x, y) = \int_{t}^{\infty} p(x, y, s) ds,
\]
\[
G^t(x, y) = \int_{0}^{t} p(x, y, s) ds.
\]
Then $G = G_t + G^t$. Let us show that for any $t > 0$ and any compact subset $K$ of $D$ there exists a constant $A > 0$ such that
\[
A \phi_0(x) \phi_0(y) \leq p(x, y, t), \quad x \in K, \quad y \in D.
\]  
(2.1)

Fix a compact subset $K$. We may assume that $x^0 \in K$. Let $K_1 \subset D$ be a compact neighborhood of $K$. Then the same argument as in the proof of Theorem 1.5 of [30] shows that
\[
C^{-1} G(x^0, z) \leq \phi_0(z) \leq CG(x^0, z), \quad z \in D \setminus K_1,
\]  
(2.2)

for some constant $C > 0$. Fix $t > 0$, and put
\[
\epsilon_t = \frac{1}{2\lambda_0} \left( 1 - e^{-t\lambda_0} \right).
\]

By [SSP] and (2.2), there exists a compact subset $K_2 \supset K_1$ such that
\[
\int_{D \setminus K_2} \phi_0(z) G(z, y) d\mu(z) \leq \epsilon_t \phi_0(y), \quad y \in D \setminus K_2,
\]  
(2.3)

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where \(d\mu(z) = m(z)\,d\nu(z)\). Since
\[
\frac{\phi_0(y)}{\lambda_0} = \int_D G(y, z) \phi_0(z)\,d\mu(z),
\]
and \(G(y, z) = G(z, y)\), (2.3) yields
\[
\frac{\phi_0(y)}{\lambda_0} \leq \int_{K_2} G_t(z, y) \phi_0(z)\,d\mu(z) + \int_{K_2} G'(z, y) \phi_0(z)\,d\mu(z) + \epsilon_t \phi_0(y)
\]
for any \(y \in D \setminus K_2\). By Fubini’s theorem,
\[
\int_D G_t(z, y) \phi_0(z)\,d\mu(z) = \int_{t}^{\infty} ds \int_D p(z, y, s) \phi_0(z)\,d\mu(z)
\]
\[
= \int_{t}^{\infty} e^{-\lambda_0s} \phi_0(y)\,ds
\]
\[
= \frac{1}{\lambda_0} e^{-\lambda_0t} \phi_0(y).
\]
Thus
\[
\int_{K_2} G_t(z, y) \phi_0(z)\,d\mu(z) \leq \frac{1}{\lambda_0} e^{-\lambda_0t} \phi_0(y).
\]
This together with (2.4) implies
\[
\epsilon_t \phi_0(y) \leq \int_{K_2} G'(z, y) \phi_0(z)\,d\mu(z).
\]
Choose a compact subset \(K_3\) whose interior includes \(K_2\). By the parabolic Harnack inequality, there exists a constant \(C_1\) depending on \(t, K_2, K_3\) such that
\[
p(z, y, s) \leq C_1 p(x, y, 2t),
\]
for any \(x, z \in K_2, y \in D \setminus K_3\), and \(0 < s \leq t\). We have
\[
G'(z, y) = \int_{0}^{t} p(z, y, s)\,ds
\]
\[
\leq C_1 t p(x^0, y, 2t), \quad z \in K_2, \quad y \in D \setminus K_3.
\]
Thus
\[
\int_{K_2} G'(z, y) \phi_0(z)\,d\mu(z) \leq \left[C_1 t \int_{K_2} \phi_0(z)\,dz\right] p(x^0, y, 2t).
\]
This together with (2.5) implies
\[ \phi_0(y) \leq C_2 p(x^0, y, 2t), \quad y \in D \setminus K_3, \tag{2.7} \]
where
\[ C_2 = \frac{1}{\epsilon_t} C_1 t \int_{K_2} \phi_0(z) d\mu(z). \]
By the parabolic Harnack inequality,
\[ p(x^0, y, 2t) \leq C p(x, y, 3t), \quad x \in K, \; y \in D, \]
for some constant \( C > 0 \). This together with (2.7) yields the desired inequality (2.1). It remains to show that for any \( t > 0 \) and a compact subset \( K \) of \( D \) there exists a constant \( B \) such that
\[ p(x, y, t) \leq B \phi_0(x) \phi_0(y), \quad x \in K, \; y \in D. \tag{2.8} \]
Fix a compact subset \( K \). We may assume that \( x^0 \in K \). Let \( K_1 \subset D \) be a compact neighborhood of \( K \). By the parabolic Harnack inequality there exists a constant \( c > 0 \) such that
\[ c p(x^0, y, t) \leq p(z, y, 2t), \quad z \in K_1, \; y \in D. \]
Thus, for any \( y \in D \),
\[ e^{-2t\lambda_0} \phi_0(y) = \int_D \phi_0(z) p(z, y, 2t) d\mu(z) \geq \int_{K_1} \phi_0(z) p(z, y, 2t) d\mu(z) \geq c \left[ \int_{K_1} \phi_0(z) d\mu(z) \right] p(x^0, y, t). \]
This implies (2.8), since
\[ C p(x^0, y, t) \geq p(x, y, t/2), \quad x \in K, \; y \in D, \]
for some constant \( C > 0 \). (We should note that in proving (2.8) we have only used the consequence of [SSP] that \( \phi_0 \) is a positive eigenfunction.) □

**Remark 2.1** It is an open problem whether [SIU] implies [SSP] or not. Furthermore, the problem whether [SSP] implies [SP] or not in the case \( n > 1 \) is still open.
3 Parabolic Martin kernels

In this section we prove Theorem 1.2. Throughout the present section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. Let $G$ be the Green function of $L$ on $D$. For any $0 < \delta < t$, put

$$G^t_\delta(x, y) = \int_0^t p(x, y, s) \, ds.$$ (3.1)

We denote by $\partial_M D$ the Martin boundary of $D$ for $L$. In order to prove Theorem 1.2, we need two lemmas.

Lemma 3.1 Let $\xi \in \partial_M D$. Suppose that a sequence $\{y_n\}_{n=1}^\infty \subset D$ converges to $\xi$, and there exists the limit

$$\lim_{n \to \infty} \frac{G^t_\delta(z, y_n)}{\phi_0(y_n)} = w(z, t), \quad z \in D.$$ (3.2)

Then

$$\lim_{n \to \infty} \int_D G(x, z) \frac{G^t_\delta(z, y_n)}{\phi_0(y_n)} \, d\mu(z) = \int_D G(x, z) w(z, t) \, d\mu(z)$$ (3.3)

for any $x \in D$, where $d\mu(z) = m(z) \, d\nu(z)$.

Proof Fix $x \in D$. Let $K_1 \subset D$ be a compact neighborhood of $x$. By [SSP], there exists a constant $C > 0$ such that

$$C^{-1} \phi_0(y) \leq G(x, y) \leq C \phi_0(y), \quad y \in D \setminus K_1.$$ (3.4)

Let $\epsilon > 0$. Then there exists a compact subset $K \supset K_1$ such that

$$\int_{D \setminus K} G(x, z) \frac{G(z, y)}{G(x, y)} \, d\mu(z) < \frac{\epsilon}{3C}, \quad y \in D \setminus K.$$ (3.5)

Thus, for $n$ sufficiently large,

$$\int_{D \setminus K} G(x, z) \left[ \frac{G^t_\delta(z, y_n)}{\phi_0(y_n)} \right] \, d\mu(z) \leq \int_{D \setminus K} G(x, z) \left[ C \frac{G(z, y_n)}{G(x, y_n)} \right] \, d\mu(z) < \frac{\epsilon}{3}.$$ (3.6)
By Fatou's lemma,
\[
\int_D G(x, z) w(z, t) \, d\mu(z) \leq \epsilon^3.
\]
By Theorem 1.1, there exist constants \(A_1\) and \(A_2\) such that
\[
A_1 \varphi_0(x) \varphi_0(y) \leq p(x, y, \delta) \leq A_2 \varphi_0(x) \varphi_0(y), \quad x \in K, \; y \in D.
\]
Then, for any \(t > \delta\), the semigroup property yields
\[
A_1 e^{-\lambda_0 (t - \delta)} \varphi_0(x) \varphi_0(y) \leq p(x, y, t) \leq A_2 e^{-\lambda_0 (t - \delta)} \varphi_0(x) \varphi_0(y),
\]
for any \(x \in K, \; y \in D\). Thus there exists a constant \(B > 0\) such that for any
\[
\lim_{n \to \infty} \int_K G(x, z) \left[ G(t \delta(z, y_n)) \varphi_0(y_n) \right] \, d\mu(z) = \int_K G(x, z) w(z, t) \, d\mu(z).
\]
Therefore, for \(n\) sufficiently large,
\[
\left| \int_D G(x, z) \left[ G(t \delta(z, y_n)) \varphi_0(y_n) \right] \, d\mu(z) - \int_D G(x, z) w(z, t) \, d\mu(z) \right| < \epsilon.
\]
This shows (3.3).

By Lemma 6.1 of [38], it follows from [SSP] that there exists the limit
\[
\lim_{D \ni y \to \xi} G(y, z) \varphi_0(y) = h(\xi, z),
\]
and \(h\) is a positive continuous function on \(\partial M D \times D\). From this we show the following lemma.

**Lemma 3.2**
Under the same assumptions as in Lemma 3.1, one has
\[
\int_D h(\xi, z) G(t \delta(z, x)) \, d\mu(z) = \lim_{n \to \infty} \int_D G(y_n, z) \varphi_0(y_n) G(t \delta(z, x)) \, d\mu(z) = \int_D G(x, z) w(z, t) \, d\mu(z),
\]
for any \(x \in D\).
Proof Fix $x \in D$. Let $K_1 \subset D$ be a compact neighborhood of $x$. By Theorem 1.1, (3.4) and (3.5), there exists a constant $C_1 > 0$ such that

$$C_1 G(z, x) \leq G^t_\delta(z, x) \leq G(z, x), \quad z \in D \setminus K_1.$$  

Let $\epsilon > 0$. By [SSP], there exists a compact subset $K \supset K_1$ such that

$$\int_{D \setminus K} \left[ \frac{G(y_n, z)}{\phi_0(y_n)} \right] G^t_\delta(z, x) \, d\mu(z) < \frac{\epsilon}{3}, \quad (3.8)$$

for $n$ sufficiently large. By Fatou’s lemma,

$$\int_{D \setminus K} h(\xi, z) G^t_\delta(z, x) \, d\mu(z) \leq \frac{\epsilon}{3}. \quad (3.9)$$

On the other hand, for any sufficiently large $n$

$$\left[ \frac{G(y_n, z)}{\phi_0(y_n)} \right] G^t_\delta(z, x) \leq C_2, \quad z \in K,$$

where $C_2$ is a positive constant. By Lebesgue’s dominated convergence theorem,

$$\lim_{n \to \infty} \int_K \frac{G(y_n, z)}{\phi_0(y_n)} G^t_\delta(z, x) \, d\mu(z) = \int_K h(\xi, z) G^t_\delta(z, x) \, d\mu(z). \quad (3.10)$$

Combining (3.8), (3.9) and (3.10), we get the first equality. It remains to show the second equality of (3.7). By Fubini’s theorem and the symmetry $p(x, y, t) = p(y, x, t)$,

we have

$$\int_D G(y_n, z) G^t_\delta(z, x) \, d\mu(z) = \int_0^\infty \int_0^t ds \, p(y_n, x, r + s)$$
$$= \int_D G(x, z) G^t_\delta(z, y_n) \, d\mu(z).$$

This together with Lemma 3.1 implies the second equality. □

Proof of Theorem 1.2 Let $\{y_j\}_{j=1}^\infty \subset D$ be any sequence converging to $\xi \in \partial_M D$. Put

$$u_j(x, t) = \frac{p(x, y_j, t)}{\phi_0(y_j)} \quad \text{for } t > 0, \quad u_j(x, t) = 0 \quad \text{for } t \leq 0. \quad (3.11)$$
Since [SIU] holds, it follows from the parabolic Harnack inequality and local a priori estimates for nonnegative solutions to parabolic equations (see [6] and [16]) that there exists a subsequence \( \{u_{jk}\}_{k=1}^{\infty} \) such that \( u_{jk} \) converges, as \( k \to \infty \), uniformly on any compact subset of \( D \times \mathbb{R} \) to a solution \( u \) of the equation

\[
(\partial_t + L) u = 0 \quad \text{in} \quad D \times \mathbb{R}
\]
satisfying \( u > 0 \) on \( D \times (0, \infty) \) and \( u = 0 \) on \( D \times (-\infty, 0] \). Thus, in order to prove Theorem 1.2, it suffices to show that the limit function \( u \) is independent of \( \{y_j\}_{j=1}^{\infty} \) and uniquely determined by \( \xi \). Let \( \{y_j\}_{n=1}^{\infty} \) and \( \{y'_j\}_{n=1}^{\infty} \) be two sequences in \( D \) converging to \( \xi \). Define \( u_j \) by (3.11), and \( u'_j \) by (3.11) with \( y_j \) replaced by \( y'_j \). Suppose that \( \{u_j\}_{j=1}^{\infty} \) and \( \{u'_j\}_{j=1}^{\infty} \) converge to \( u \) and \( u' \), respectively. For any \( t > \delta > 0 \), put

\[
w(z, t) = \int_{\delta}^{t} u(z, s) \, ds, \quad w'(z, t) = \int_{\delta}^{t} u'(z, s) \, ds.
\]

Then we have

\[
\lim_{n \to \infty} \frac{G_{\delta}^{t}(z, y_n)}{\phi_0(y_n)} = w(z, t), \quad \lim_{n \to \infty} \frac{G_{\delta}^{t}(z, y'_n)}{\phi_0(y'_n)} = w'(z, t).
\]

By Lemma 3.2,

\[
\int_D G(x, z) \, w(z, t) \, d\mu(z) = \int_D h(\xi, z) \, G_{\delta}^{t}(z, x) \, d\mu(z) = \int_D G(x, z) \, w'(z, t) \, d\mu(z).
\]

Thus \( w(x, t) = w'(x, t) \), which implies \( u(x, t) = u'(x, t) \). This completes the proof of Theorem 1.2.

4 Integral representations; the case \( I = (0, \infty) \)

In this section we prove Theorem 1.3 in the case \( T = \infty \).

We first state an abstract integral representation theorem which holds without [SSP]. For \( x \in D \) and \( r > 0 \), we denote by \( B(x, r) \) the geodesic ball in the Riemannian manifold \( M \) with center \( x \) and radius \( r \). Let \( x^0 \) be a reference point in \( D \). Choose a nonnegative continuous function \( a \) on \( D \) such
that \( a(x) = 1 \) on \( B(x^0, r^0) \) and \( a(x) = 0 \) outside \( B(x^0, 2r^0) \) for some \( r^0 > 0 \) with \( B(x^0, 3r^0) \subset D \). Choose a nonnegative continuous function \( b \) on \( \mathbb{R} \) such that \( 0 < b(t) < e^{\gamma t} \) on \( (1, \infty) \) for some \( \gamma < \lambda_0 \), and \( b(t) = 0 \) on \( (-\infty, 1] \). Denote by \( \beta \) the measure defined by \( d\beta(x, t) = a(x)b(t)m(x)\,dv(x)dt \). For any nonnegative measurable function \( u \) on \( Q = D \times (0, \infty) \), we write

\[
\beta(u) = \iint_Q u(x, t)\,d\beta(x, t).
\]

Denote by \( P(Q) \) the set of all nonnegative solutions of (1.1) with \( I = (0, \infty) \), and put

\[
P_{\beta}(Q) = \{ u \in P(Q) ; \beta(u) < \infty \}.
\]

Note that for any \( u \in P(Q) \) there exists a function \( b \) as above such that \( \beta(u) < \infty \); thus \( P(Q) = \bigcup_{\beta} P_{\beta}(Q) \). Furthermore, the parabolic Harnack inequality shows that if \( \beta(u) = 0 \), then \( u = 0 \). Now, let us define the \( \beta \)-Martin boundary \( \partial_M^\beta Q \) of \( Q \) with respect to \( \partial_t + L \) along the line given in [21] and [18]. Put

\[
p(x, t; y, s) = p(x, y, t - s), \quad t > s, \quad x, y \in D,
\]

\[
p(x, t; y, s) = 0, \quad \quad t \leq s, \quad x, y \in D.
\]

Define the \( \beta \)-Martin kernel \( K_\beta \) by

\[
K_\beta(x, t; y, s) = \frac{p(x, t; y, s)}{\beta(p(\cdot; y, s))}, \quad (x, t), \,(y, s) \in Q,
\]

where \( \beta(p(\cdot; y, s)) = \int_Q p(z, r; y, s)\,d\beta(z, r) \). Note that \( \beta(p(\cdot; y, s)) < \infty \) for any \((y, s) \in Q \), since \( 0 < b(t) < e^{\gamma t} \) on \( (1, \infty) \) for some \( \gamma < \lambda_0 \). Let \( \{D_j\}_{j=1}^\infty \) be an exhaustion of \( D \) such that each \( D_j \) is a domain with smooth boundary, \( D_j \Subset D_{j+1} \Subset D \), \( \bigcup_{j=1}^\infty D_j = D \), and \( B(x^0, 3r^0) \subset D_1 \). Put \( Q_j = D_j \times (1/j, j) \). For \( Y = (y, s), \, Z = (z, r) \in Q \), let

\[
\delta_\beta(Y, Z) = \sum_{j=1}^\infty 2^{-j} \sup_{X \in Q_j} \frac{|K_\beta(X; Y) - K_\beta(X; Z)|}{1 + |K_\beta(X; Y) - K_\beta(X; Z)|}.
\]

Then we see that \( \delta_\beta \) is a metric on \( Q \), and the topology on \( Q \) induced by \( \delta_\beta \) is equivalent to the original topology of \( Q \). Denote by \( Q^{\beta*} \) the completion of \( Q \) with respect to the metric \( \delta_\beta \). Put \( \partial_M^\beta Q = Q^{\beta*} \setminus Q \). A sequence \( \{Y^k\}_{k=1}^\infty \) in \( Q \)
Thus the metric $\delta_\beta$ is canonically extended to $Q^{\beta^*}$. Furthermore, $Q^{\beta^*}$ becomes a compact metric space, since by the parabolic Harnack inequality, any sequence $\{Y^k\}_{k=1}^\infty$ with no point of accumulation in $Q$ has a fundamental subsequence. We call $K_\beta(\cdot ; \Xi)$, $Q^\beta$ and $Q^{\beta^*}$ the $\beta$-Martin kernel, $\beta$-Martin boundary and $\beta$-Martin compactification for $(Q, \partial_t + L)$, respectively. Note that $\beta(K_\beta(\cdot ; \Xi)) \leq 1$ by Fatou’s lemma; and so $K_\beta(\cdot ; \Xi) \in P_\beta(Q)$. A nonnegative solution $u \in P_\beta(Q)$ is said to be minimal if for any nonnegative solution $v \leq u$ there exists a nonnegative constant $C$ such that $v = Cu$. Put

$$\partial_m^\beta Q = \left\{ \Xi \in \partial_M^\beta Q; K_\beta(\cdot ; \Xi) \text{ is minimal and } \beta(K_\beta(\cdot ; \Xi)) = 1 \right\},$$

which we call the minimal $\beta$-Martin boundary for $(Q, \partial_t + L)$.

Observe that $D \times [0, \infty)$ is embedded into $Q^{\beta^*}$, and $D \times \{0\} \subset \partial_M^\beta Q$. Indeed, with $y \in D$ fixed, for any sequence $\{Y^k\}_{k=1}^\infty$ in $Q$ with $\lim_{k \to \infty} Y^k = (y, 0)$ we have $\lim_{k \to \infty} K_\beta(x, t; Y^k) = p(x, t; y, 0) / \beta(p(\cdot ; y, 0))$; furthermore, $K_\beta(\cdot ; y, 0) \neq K_\beta(\cdot ; z, 0)$ if $y \neq z$. We also note that any sequence $\{Y^k = (y^k, s^k)\}_{k=1}^\infty$ in $Q$ with $\lim_{k \to \infty} s^k = \infty$ is a fundamental sequence, since $\lim_{k \to \infty} K_\beta(\cdot ; Y^k) = 0$. We denote by $\varpi$ the point in $\partial_M^\beta Q$ corresponding to the Martin kernel which is identically zero : $K_\beta(\cdot ; \varpi) = 0$. Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus (D \times \{0\} \cup \{\varpi\}).$$

We obtain the following abstract integral representation theorem in the same way as in the proof of Theorem 2.1 and Lemma 2.2 of [34].

**Theorem 4.1** For any $u \in P_\beta(Q)$, there exists a unique pair of finite Borel measures $\kappa$ on $D$ and $\lambda$ on $\partial_M^\beta Q \setminus (D \times \{0\})$ such that $\lambda$ is supported by the set $\mathcal{L}_m^\beta Q$,

$$u(x, t) = \int_D \frac{p(x, t; y, 0)}{\beta(p(\cdot ; y, 0))} \, d\kappa(y) + \int_{\mathcal{L}_m^\beta Q} K_\beta(x, t; \Xi) \, d\lambda(\Xi) \quad (4.1)$$
for any \((x, t) \in \mathbb{Q}\), and
\[
\beta(u) = \kappa(D) + \lambda(L \beta m \mathbb{Q}) \tag{4.2}
\]
Furthermore, the function
\[
v(x, t) = u(x, t) - \int_D p(x, t; y, 0) \beta(p(\cdot; y, 0)) d\kappa(y)
\]
is a nonnegative solution of the equation
\[
(\partial_t + L)v = 0 \text{ in } D \times \mathbb{R}
\]
such that \(v = 0\) on \(D \times (\mathbb{R} \setminus \{0\})\).

Conversely, for any finite Borel measures \(\kappa\) on \(D\) and \(\lambda\) on \(\partial \beta M \mathbb{Q} \setminus (D \times \{0\})\) such that \(\lambda\) is supported by the set \(L \beta m \mathbb{Q}\), the right hand side of (4.1) belongs to \(P_\beta(\mathbb{Q})\).

We put
\[
P_0(\mathbb{Q}) = \left\{ v \in P(\mathbb{Q}); \lim_{t \to 0} v(x, t) = 0 \text{ on } D \right\}.
\]
We show Theorem 1.3 on the basis of Theorem 4.1. To this end it suffices to show (1.8) for \(u \in P_0(\mathbb{Q})\). The key step in the proof is to identify \(L \beta m \mathbb{Q}\).

Under the condition \([\text{SSP}]\), we shall show that
\[
L \beta m \mathbb{Q} = \partial m D \times [0, \infty).
\]
In the remainder of this section we assume \([\text{SSP}]\). We may and shall assume that \(a = 0\) < \(\lambda_0\).

Lemma 4.2
For any domains \(U\) and \(W\) with \(U \subset W \subset D\), there exist positive constants \(C\) and \(\alpha\) such that
\[
p(x, y, t) \leq Cf(t) \phi_0(x) \phi_0(y), \ x \in U, y \in D \setminus W, t > 0, \tag{4.3}
\]
where \(f(t) = e^{-\alpha/t}\) for \(0 < t < 1\), and \(f(t) = e^{-\lambda_0 t}\) for \(t \geq 1\). Furthermore,
\[
q(x, \xi, t) \leq Cf(t) \phi_0(x), \ x \in U, \xi \in \partial M D, t > 0, \tag{4.4}
\]
\[
G(x, y) \leq C\phi_0(x) \phi_0(y), \ x \in U, y \in D \setminus W, \tag{4.5}
\]
where \(G\) is the Green function of \(L\) on \(D\).

This lemma is shown in the same way as Lemmas 4.2 and 4.4 of [34].

Let \(K(x, \xi)\) be the Martin kernel for \(L\) on \(D\) with reference point \(x_0 \in D\), i.e., \(K(x_0, \xi) = 1\), \(\xi \in \partial M D\). The following lemma gives a relation between \(K\) and \(q\).
Lemma 4.3 For any $\xi \in \partial M D$,

$$\lim_{D \ni y \to \xi} \frac{G(x, y)}{\phi_0(y)} = \int_0^\infty q(x, \xi, t) \, dt, \quad x \in D, \quad (4.6)$$

$$K(x, \xi) = \int_0^\infty \frac{q(x, \xi, t)}{\int_0^\infty q(x^0, \xi, t) \, dt} \, dt, \quad x \in D. \quad (4.7)$$

This lemma is shown in the same way as Lemma 4.5 of [34].

Lemma 4.4 Let $\xi, \eta \in \partial M D$, $0 \leq s, r < \infty$ and $C > 0$. If

$$q(x, \xi, t - s) = C q(x, \eta, t - r), \quad (x, t) \in Q,$$

then $\xi = \eta$, $s = r$ and $C = 1$.

Proof Since $q(x, \xi, t) > 0$ for $t > 0$ and $q(x, \xi, t) = 0$ for $t \leq 0$, we obtain that $s = r$. Thus $q(x, \xi, t) = q(x, \eta, t)$. This together with (4.7) implies that $K(\cdot, \xi) = K(\cdot, \eta)$ on $D$. Hence $\xi = \eta$, and so $C = 1$.

Now, let $\beta$ be a measure on $Q = D \times (0, \infty)$ as described in the beginning of this section: $d\beta(x, t) = a(x)b(t)m(x) \nu(x) \, dt$. The following proposition determines the $\beta$-Martin boundary $\partial^\beta_M Q$, $\beta$-Martin compactification $Q^\beta_*$, and $\beta$-Martin kernel $K_\beta$ for $(\partial_t + L, Q)$. Recall that $p(x, t; y, s) = p(x, y, t - s)$ and $K_\beta(\cdot; y, s) = p(\cdot; y, s)/\beta(p(\cdot; y, s))$. We write

$$q(x, t; \xi, s) = q(x, \xi, t - s)$$

for $\xi \in \partial M D$ and $0 \leq s < \infty$.

Proposition 4.5 (i) The $\beta$-Martin boundary $\partial^\beta_M Q$ of $Q$ for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial M D \times [0, \infty)$ and the one point set $\{\varpi\}$:

$$\partial^\beta_M Q = D \times \{0\} \cup \partial M D \times [0, \infty) \cup \{\varpi\}. \quad (4.8)$$

In particular, $\partial^\beta_M Q$ does not depend on $\beta$.

(ii) The $\beta$-Martin compactification $Q^\beta_*$ of $Q$ for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0, \infty)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of $\varpi$ is given by the family $\{\varpi\} \cup D^* \times (N, \infty)$, $N > 1$. In particular, $Q^\beta_*$ does not depend on $\beta$. 

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(iii) The $\beta$-Martin kernel $K_\beta$ is given as follows: For $(x, t) \in Q,$

$$K_\beta(x, t; y, 0) = \frac{p(x, t; y, 0)}{\beta(p(\cdot; y, 0))}, \quad (y, 0) \in D \times \{0\},$$

$$K_\beta(x, t; \xi, s) = \frac{q(x, t; \xi, s)}{\beta(q(\cdot; \xi, s))}, \quad (\xi, s) \in \partial_M D \times [0, \infty),$$

and $K_\beta(x, t; \varpi) = 0.$

This proposition is shown in the same way as Proposition 4.8 of [34].

**Lemma 4.6** Let $(\xi, s) \in (\partial_M D \setminus \partial_mD) \times [0, \infty).$ Then there exists a finite Borel measure $\gamma$ on $\partial_M D$ supported by $\partial_mD$ such that

$$q(\cdot; \xi, s) = \int_{\partial_mD} q(\cdot; \eta, s) d\gamma(\eta).$$

**Proof** For reader’s convenience, we give a sketch of the proof for the case $s = 0.$ (For details, see the proof of Lemma 4.10 of [34].) By the elliptic Martin representation theorem, there exists a unique finite Borel measure $\mu$ on $\partial_M D$ supported by $\partial_mD$ such that

$$K(x, \xi) = \int_{\partial_mD} K(x, \eta) d\mu(\eta).$$

This together with (4.7) implies

$$\int_0^\infty q(x, \xi, t) dt = \int_{\partial_mD} \left( \int_0^\infty q(x, \eta, t) dt \right) d\gamma(\eta),$$

where $d\gamma(\eta) = [H(x^0, \xi)/H(x^0, \eta)] d\mu(\eta)$ with

$$H(x, \eta) = \int_0^\infty q(x, \eta, t) dt.$$

For $\alpha > 0,$ denote by $G_\alpha$ the Green function of $L + \alpha$ on $D.$ By the resolvent equation and [SSP], we then have

$$\int_0^\infty e^{-\alpha t} q(x, \eta, t) dt$$

$$\quad = \int_0^\infty q(x, \eta, t) dt - \alpha \int_D G_\alpha(x, z) \left( \int_0^\infty q(z, \eta, t) dt \right) m(z) d\nu(z),$$

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for any $\eta \in \partial M D$. By combining (4.12) and (4.13), we get
\[
\int_0^\infty e^{-at} \left( \int_{\partial_m D} q(x, \eta, t) d\gamma(\eta) \right) dt = \int_0^\infty e^{-at} q(x, \xi, t) dt.
\]
Thus the Laplace transforms of $q(x, \xi, t)$ and $\int_{\partial_m D} q(x, \eta, t) d\gamma(\eta)$ coincide; and so (4.11) holds. $\square$

**Lemma 4.7** Let $(\xi, s) \in (\partial_M D \setminus \partial_m D) \times [0, \infty)$. Then $q(\cdot; \xi, s)$ is not minimal.

**Proof** For reader’s convenience, we give a proof. We have (4.11). Suppose that $q(\cdot; \xi, s)$ is minimal. Then, along the line given in the proof of Lemma 12.12 of [15], we obtain from (4.11) that the support of $\gamma$ consists of a single point. Thus, for some $\eta \in \partial_m D$ and constant $C$
\[
q(\cdot; \xi, s) = C q(\cdot; \eta, s).
\]
Hence, by Lemma 4.4, $\xi = \eta$; which is a contradiction. $\square$

**Lemma 4.8** Let $(\xi, s) \in \partial_m D \times (0, \infty)$. Then $q(\cdot; \xi, s)$ is minimal if and only if $q(\cdot; \xi, 0)$ is minimal.

**Proof** Assume that $q(\cdot; \xi, 0)$ is minimal. Suppose that a nonnegative solution $u$ of (1.1) satisfies $u(\cdot) \leq q(\cdot; \xi, s)$ on $Q$. Put $v(x, t) = u(x, t + s)$. Then $v(\cdot) \leq q(\cdot; \xi, 0)$. Thus $v(\cdot) = C q(\cdot; \xi, 0)$ for some constant $C$. Hence
\[
u(x, t) = C q(x, t; \xi, s)
\]
for $t > s$, and $u(x, t) = 0 = C q(x, t; \xi, s)$ for $t \leq s$. This shows that $q(\cdot; \xi, s)$ is minimal. Next, assume that $q(\cdot; \xi, s)$ is minimal. Suppose that a nonnegative solution $u$ of (1.1) satisfies $u(\cdot) \leq q(\cdot; \xi, 0)$ on $Q$. Put $v(x, t) = u(x, t - s)$ for $t > s$, and $v(x, t) = 0$ for $0 < t \leq s$. Then $v(\cdot) \leq q(\cdot; \xi, s)$. Thus $v(\cdot) = C q(\cdot; \xi, s)$ for some constant $C$. Hence $u(x, t) = C q(x, t; \xi, 0)$. This shows that $q(\cdot; \xi, 0)$ is minimal. $\square$

By Theorem 4.1 and Lemmas 4.7 and 4.8, we have the following proposition.

**Proposition 4.9** There exists a Borel subset $R$ of $\partial_M D$ such that
\[
R \subset \partial_m D, \quad L_0^\beta Q = R \times [0, \infty),
\]
for any \( u \in P^0_\beta(Q) \) there exists a unique Borel measure \( \lambda \) on \( \partial_m D \times [0, \infty) \) which is supported by \( R \times [0, \infty) \) and satisfies
\[
    u(x, t) = \int_{R \times [0, \infty)} q(x, \xi, t - s) \, d\lambda(\xi, s) \quad (x, t) \in Q.
\]
(4.14)

**Lemma 4.10** Let \((\xi, s) \in \partial_m D \times [0, \infty)\). Then \(q(\cdot; \xi, s)\) is minimal.

**Proof** Suppose that \( q(\cdot; \xi, 0) \) is not minimal. Then \( \xi \notin R \) and
\[
    q(x, \xi, t) = \int_{R \times [0, \infty)} q(x, \eta, t - s) \, d\lambda(\eta, s)
\]
for some Borel measure \( \lambda \). We have
\[
    K(x, \xi) \int_0^\infty q(x^0, \xi, t) \, dt = \int_0^\infty q(x, \xi, t) \, dt = \int_{R \times [0, \infty)} d\lambda(\eta) K(x, \eta) \int_0^\infty q(x^0, \eta, t) \, dt.
\]
Thus
\[
    K(x, \xi) = \int_R K(x, \eta) \, d\Lambda(\eta)
\]
for some Borel measure \( \Lambda \). But \( \xi \in \partial_m D \setminus R \) and \( R \subset \partial_m D \). This contradicts the uniqueness of a representing measure in the elliptic Martin representation theorem. Hence \( q(\cdot; \xi, 0) \) is minimal; which together with Lemma 4.8 shows Lemma 4.10. \( \square \)

**Completion of the proof of Theorem 1.3 in the case \( I = (0, \infty) \)** By Lemma 4.10, \( R = \partial_m D \) and
\[
    \mathcal{L}^\beta_m Q = \partial_m D \times [0, \infty).
\]
Thus Proposition 4.9 shows Theorem 1.3. \( \square \)

## 5 Proof of Theorem 1.3; the case \( 0 < T < \infty \)

In this section we prove Theorem 1.3 in the case \( 0 < T < \infty \) by making use of the results in Section 4. To this end, the following proposition plays a crucial role.
Proposition 5.1 Let $\xi \in \partial_M D$ and $0 \leq s < r < \infty$. Then
\[
\int_D p(x, y, t - r)q(y, r; \xi, s)d\mu(y) = q(x, t; \xi, s), \quad x \in D, \ t > r,
\] (5.1)
where $d\mu(y) = m(y)d\nu(y)$

Proof We first show (5.1) for $\xi \in \partial_m D$. Define $u(x, t)$ by
\[
\begin{align*}
u(x, t) &= q(x, t; \xi, s), \quad 0 < t \leq r, \\
u(x, t) &= \int_D p(x, y, t - r)q(y, r; \xi, s)d\mu(y), \quad r < t < \infty.
\end{align*}
\] (5.2)
(We call $u$ the minimal extension of $q$ from $t = r$.) Then we see that $u$ is a nonnegative solution of $(\partial_t + L)u = 0$ in $D \times (0, \infty)$ such that $u(\cdot) \leq q(\cdot; \xi, s)$ on $D \times (0, \infty)$. By Lemma 4.10, $u(\cdot) = Cq(\cdot; \xi, s)$ for some constant $C$. But $u(x, t) = q(x, t; \xi, s)$ for $0 < t \leq r$. Thus $C = 1$, and so $u(\cdot) = q(\cdot; \xi, s)$.

Next, let $\xi \notin \partial_m D$. By Lemma 4.6, there exists a finite Borel measure $\gamma$ on $\partial_M D$ supported by $\partial_m D$ such that
\[
q(\cdot; \xi, s) = \int_{\partial_m D} q(\cdot; \eta, s) d\gamma(\eta).
\] (5.3)
Thus
\[
\begin{align*}
u(x, t) &= \int_{\partial_M D} q(x, t; \eta, s) d\gamma(\eta) \\
&= \int_{\partial_m D} q(x, t; \eta, s) d\gamma(\eta) \\
&= q(x, t; \xi, s).
\end{align*}
\] This proves (5.1).

Lemma 5.2 Let $\xi, \eta \in \partial_M D$, $0 \leq s, r < T$ and $C > 0$. If
\[
q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad x \in D, \ 0 < t < T,
\] (5.4)
then $\xi = \eta$, $s = r$ and $C = 1$. 

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Choose $u$ such that $\max(r, s) < u < T$, and construct minimal extensions of both sides of (5.4) from $t = u$. Then, by (5.1) we have

$$q(x, \xi, t - s) = Cq(x, \eta, t - r), \quad x \in D, \quad 0 < t < \infty.$$ 

By Lemma 4.4, this implies that $\xi = \eta$, $s = r$ and $C = 1$.

Now, let $\beta$ be a measure on $Q = D \times (0, T)$ defined by

$$d\beta(x, t) = a(x) b(t) m(x) d\nu(x) dt.$$ 

Here $a(x)$ is a nonnegative continuous function on $D$ as described in the beginning of Section 4, and $b(t)$ is a nonnegative continuous function on $R$ such that $b(t) > 0$ on $(T/2, T)$ and $b(t) = 0$ on $R \setminus (T/2, T)$. Let $K_\beta(\cdot; \Xi)$, $\partial_\beta M_Q$, $\partial_\beta m_Q$, and $Q_\beta^*$ be the $\beta$-Martin kernel, $\beta$-Martin boundary, minimal $\beta$-Martin boundary, and $\beta$-Martin compactification for $(Q, \partial_t + L)$ with $Q = D \times (0, T)$, respectively. The following proposition is an analogue of Proposition 4.5, and is shown in the same way.

**Proposition 5.3**

(i) The $\beta$-Martin boundary $\partial_\beta M_Q$ of $Q$ for $\partial_t + L$ is equal to the disjoint union of $D \times \{0\}$, $\partial M_D \times [0, T)$ and the one point set $\{\varpi\}$:

$$\partial_\beta M_Q = D \times \{0\} \cup \partial M_D \times [0, T) \cup \{\varpi\}.$$ 

In particular, $\partial_\beta M_Q$ does not depend on $\beta$.

(ii) The $\beta$-Martin compactification $Q_\beta^*$ of $Q$ for $\partial_t + L$ is homeomorphic to the disjoint union of the topological product $D^* \times [0, T)$ and the one point set $\{\varpi\}$, where a fundamental neighborhood system of $\varpi$ is given by the family $\{\varpi\} \cup D^* \times (T - \varepsilon, T)$, $0 < \varepsilon < T/2$. In particular, $Q_\beta^*$ does not depend on $\beta$.

(iii) The $\beta$-Martin kernel $K_\beta$ is given as follows: For $(x, t) \in Q$,

$$K_\beta(x, t; y, 0) = p(x, t; y, 0) \beta(p(\cdot; y, 0)),$$ 

$$(y, 0) \in D \times \{0\},$$

(5.6)

$$K_\beta(x, t; \xi, s) = q(x, t; \xi, s) \beta(q(\cdot; \xi, s)),$$

$$(\xi, s) \in \partial M_D \times [0, T),$$

(5.7)

and $K_\beta(x, t; \varpi) = 0$. 24
Lemma 5.4  Let $(\xi, s) \in (\partial_M D \setminus \partial_mD) \times [0, T)$. Then $q(\cdot; \xi, s)$ is not minimal.

Proof  Suppose that $q(\cdot; \xi, s)$ is minimal. Then we obtain from (5.3) that

$$q(x, \xi, t - s) = C q(x, \eta, t - s), \quad x \in D, \ 0 < t < T,$$

for some $\eta \in \partial_mD$ and $C > 0$. By Lemma 5.2, this is a contradiction.  \[\square\]

Lemma 5.5  Let $(\xi, s) \in \partial_mD \times [0, T)$. Then $q(\cdot; \xi, s)$ is minimal.

Proof  Let $u$ be a nonnegative solution of $(\partial_t + L)u = 0$ in $Q$ such that $u(\cdot) \leq q(\cdot; \xi, s)$ in $Q$. For $r \in (s, T)$, let $u_r$ be the minimal extension of $u$ from $t = r$. By Proposition 5.1,

$$u_r(x, t) \leq q(x, t; \xi, s), \quad x \in D, \ t > 0.$$

By Lemma 4.10, there exists a constant $C_r$ such that $u_r(x, t) = C_r q(x, t; \xi, s)$ for $t > 0$. But $u_r(x, t) = u(x, t)$ for $0 < t < r$. Thus $C_r$ is independent of $r$; and so $u(\cdot) = C q(\cdot; \xi, s)$ in $Q$ for some constant $C$.  \[\square\]

Completion of the proof of Theorem 1.3 in the case $0 < T < \infty$

Put

$$\mathcal{L}_m^\beta Q = \partial_m^\beta Q \setminus (D \times \{0\} \cup \{\infty\}).$$

By Proposition 5.3, Lemmas 5.4 and 5.5, we get

$$\mathcal{L}_m^\beta Q = \partial_mD \times [0, T).$$

Thus, Theorem 2.1 of [34] which is an analogue of Theorem 4.1 completes the proof.  \[\square\]

6  Integral representations; the case $I = (-\infty, 0)$

In this section we prove Theorem 1.4. We begin with the following proposition, which can be shown in the same way as in the proof of Theorem 1 of [9] (see also [39]).
Proposition 6.1 Assume [SIU]. Then
\[
\lim_{t \to \infty} e^{\lambda_0 t} \frac{p(x, y, t)}{\phi_0(x)\phi_0(y)} = 1 \quad \text{uniformly in } (x, y) \in K \times D
\] (6.1)
for any compact subset $K$ of $D$.

In the rest of this section we assume [SSP]. We may and shall assume that $a = 0 < \lambda_0$. By Theorem 1.1, we have the following corollary of Proposition 6.1.

Corollary 6.2 Assume [SSP]. Then, for any compact subset $K$ of $D$ and $N > 1$,
\[
\lim_{s \to -\infty} \frac{p(x, y, t - s)}{e^{\lambda_0 s} \phi_0(y)} = e^{-\lambda_0 t} \phi_0(x) \quad \text{uniformly in } (x, y, t) \in K \times D \times (-N, 0).
\]

Lemma 6.3 The solution $e^{-\lambda_0 t} \phi_0(x)$ is minimal.

Proof Suppose that $e^{-\lambda_0 t} \phi_0(x)$ is not minimal. Then, in view of Corollary 6.2, the same argument as in the proof of Theorem 1.3 shows that for any nonnegative solution $u$ of the equation
\[
(\partial_t + L)u = 0 \quad \text{in } Q = D \times (-\infty, 0)
\]
there exists a unique Borel measure $\lambda$ on $\partial_M D \times (-\infty, 0)$ supported by the set $\partial_M D \times (-\infty, 0)$ such that
\[
u(x, t) = \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s), \quad (x, t) \in Q.
\]
Thus
\[
e^{-\lambda_0 t} \phi_0(x) = \int_{\partial_M D \times (-\infty, t)} q(x, \xi, t - s) d\lambda(\xi, s), \quad (x, t) \in Q,
\] (6.2)
for such a measure $\lambda$. Now, fix $x$. It follows from Theorems 1.1 and 1.2 that for any $\delta > 0$ there exists a positive constant $C_\delta$ such that
\[
C_\delta^{-1} \leq \frac{q(x, \xi, \tau)}{e^{-\lambda_0 \tau} \phi_0(x)} \leq C_\delta, \quad \tau \geq \delta, \ \xi \in \partial_M D.
\] (6.3)
By (4.4),
\[ q(x, \xi, \tau) \leq C e^{-\alpha/\tau} \phi_0(x), \quad \xi \in \partial_M D, \ 0 < \tau < 1, \quad (6.4) \]
for some positive constants $\alpha$ and $C$. By (6.2) and (6.3),
\[ e^{\lambda_0} \phi_0(x) \geq \int_{\partial_M D \times (-\infty, -2)} C_1^{-1} e^{-\lambda_0(1-s)} d\lambda(\xi, s). \]
Thus
\[ \int_{\partial_M D \times (-\infty, -2)} e^{\lambda_0} d\lambda(\xi, s) \leq C_1 \phi_0(x). \quad (6.5) \]
For $t < -2$ and $0 < \delta < 1$, we have
\[ \phi_0(x) = \int_{\partial_M D \times \{(t, s) \mid t \in (-\infty, t-\delta], s \in (t-\delta, t)\}} e^{\lambda_0(t-s)} q(x, \xi, s) e^{\lambda_0 s} d\lambda(\xi, s). \quad (6.6) \]
In view of (6.4) and (6.5), we choose $\delta$ so small that the integral on $\partial_M D \times (t-\delta, t)$ of the right hand side of (6.6) is smaller than $\phi_0(x)/3$. Then, in view of (6.3) and (6.5), we choose $t < -2$ with $|t|$ being so large that the integral on $\partial_M D \times (-\infty, t-\delta]$ of the right hand side of (6.6) is smaller than $\phi_0(x)/3$. This is a contradiction. 

**Completion of the proof of Theorem 1.4** By virtue of Corollary 6.2 and Lemma 6.3, the same argument as in the proof of Theorem 1.3 shows Theorem 1.4.

**7 Examples** In this section we give two examples in order to illustrate a scope of Theorem 1.3. Throughout this section $L_0$ is a uniformly elliptic operator on $\mathbb{R}^n$ of the form
\[ L_0 u = -\sum_{i,j=1}^n \partial_i \left( a_{ij}(x) \partial_j u \right), \]
where $a(x) = [a_{ij}(x)]_{i,j=1}^n$ is a symmetric matrix-valued measurable function on $\mathbb{R}^n$ satisfying, for some $\Lambda > 0$,
\[ \Lambda^{-1} \|\xi\|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq \Lambda \|\xi\|^2, \quad x, \xi \in \mathbb{R}^n. \]
7.1. Let $V(x)$ be a measurable function in $L^\infty_{\text{loc}}(\mathbb{R}^n)$, and $L = L_0 + V(x)$ on $D = \mathbb{R}^n$.

**Theorem 7.1** Suppose that there exist a positive constant $c < 1$ and a positive continuous increasing function $\rho$ on $[0, \infty)$ such that

$$c \, [\rho(|x|)]^2 \leq V(x) \leq [\rho(|x|)]^2, \quad x \in \mathbb{R}^n,$$

(7.1)

$$c \rho \left( r + \frac{c}{\rho(r)} \right) \leq \rho(r), \quad r \geq 0.$$  

(7.2)

Assume that

$$\int_{1}^{\infty} \frac{dr}{\rho(r)} < \infty.$$  

(7.3)

Then 1 is a small perturbation of $L$ on $\mathbb{R}^n$. Thus Theorem 1.3 holds true.

**Remark.** Compare this theorem with a non-uniqueness theorem of [26].

**Proof**  We first note that (7.2) yields

$$c \rho(r) \leq c \rho \left( r - \frac{c}{\rho(r)} + \frac{c}{\rho \left( r - \frac{c}{\rho(r)} \right)} \right) \leq \rho \left( r - \frac{c}{\rho(0)} \right), \quad r \geq \frac{c}{\rho(0)};$$

since $\rho$ is increasing. We show the theorem by using the same approach as in the proof of Theorem 5.1 of [31]. Put $b = c^{-2}$ and

$$\ell = \inf \{ j \in \mathbb{Z}; \rho(0) < b^j \}.$$  

For $k \geq \ell$, put $r_k = \sup \{ r \geq 0; \rho(r) \leq b^k \}$. By the continuity of $\rho$ and (7.3), $\rho(r_k) = b^k$ and $\lim_{k \to \infty} r_k = \infty$. By (7.2),

$$\rho(r_k + cb^{-k}) \leq c^{-1} \rho(r_k) = b^{1/2}b^k < b^{k+1} = \rho(r_{k+1}).$$

Thus $r_k + cb^{-k} < r_{k+1}$ for $k \geq \ell$. Define a positive continuously differentiable increasing function $\tilde{\rho}$ on $[0, \infty)$ as follows: Put $\tilde{\rho}(r) = b^\ell$ for $r \leq r_\ell$,

$$\tilde{\rho}(r) = b^{k+1} \quad \text{for} \quad r_k + cb^{-k} \leq r \leq r_{k+1} \quad (k \geq \ell);$$

and $\tilde{\rho}(r) = \rho_k(r)$ for $r_k \leq r \leq r_k + cb^{-k}$ $(k \geq \ell)$ by choosing a continuously differentiable function $\rho_k$ on $[r_k, r_k + cb^{-k}]$ such that

$$\rho_k(r_k) = b^k, \quad \rho_k'(r_k) = 0, \quad \rho_k(r_k + cb^{-k}) = b^{k+1}, \quad \rho_k'(r_k + cb^{-k}) = 0,$$
and
\[ 0 \leq \rho_k'(r) \leq B b^{2k}, \quad r_k \leq r \leq r_k + cb^{-k}, \]
for some constant \( B > 0 \) independent of \( k \). Then we have
\[ C^{-1} \leq \frac{\tilde{\rho}(r)}{\rho(r)} \leq C, \quad 0 \leq \tilde{\rho}'(r) \leq C \rho(r)^2, \quad r \geq 0, \tag{7.4} \]
for some positive constant \( C \). Introduce a Riemannian metric \( g = (g_{ij})_{i,j=1}^n \) by \( g_{ij} = \tilde{\rho}(|x|)^2 \delta_{ij} \). Then \( M = \mathbb{R}^n \) with this metric \( g \) becomes a complete Riemannian manifold. Furthermore, by (7.2) and (7.4), \( M \) has the bounded geometry property (1.1) of [4]. The associated gradient \( \nabla \) and divergence \( \text{div} \) are written as
\[
\nabla = \tilde{\rho}(|x|)^{-2} \nabla^0, \quad \text{div} = \tilde{\rho}(|x|)^{-n} \circ \text{div}^0 \circ \tilde{\rho}(|x|)^n,
\]
where \( \nabla^0 \) and \( \text{div}^0 \) are the standard gradient and divergence on \( \mathbb{R}^n \). Put
\[
\mathcal{L} = \tilde{\rho}(|x|)^{-2} L, \quad m(x) = \tilde{\rho}(|x|)^{2-n}, \quad A(x) = [a_{ij}(x)]_{i,j=1}^n, \quad \gamma(x) = \tilde{\rho}(|x|)^{-2} V(x).
\]
Then
\[
\mathcal{L}u = -\frac{1}{m} \text{div} (m A \nabla u) + \gamma
= -\text{div} (A \nabla u) - \left( \frac{1}{m} A \nabla^0 m, \nabla u \right)^0 + \gamma,
\]
where \( \langle \cdot, \cdot \rangle^0 \) is the standard inner product on \( \mathbb{R}^n \). Since the inner product \( \langle \cdot, \cdot \rangle \) associated with the metric \( g \) is written as
\[
\langle X, Y \rangle = \langle \tilde{\rho}^2 X, Y \rangle^0,
\]
we have
\[
\mathcal{L}u = -\text{div} (A \nabla u) - \langle \tilde{\rho}^{-2} A \nabla^0 m, \nabla u \rangle + \gamma. \tag{7.5}
\]
By (7.4),
\[
|\nabla^0 m(x)| \leq C^3 |n - 2| \tilde{\rho}(|x|) m(x).
\]
From this we have
\[
\langle \tilde{\rho}^{-2} A \nabla^0 m, \tilde{\rho}^{-2} A \nabla^0 m \rangle \leq \tilde{\rho}^{-2} \Lambda^2 (C^3 |n - 2| \tilde{\rho})^2
\leq \{ \Lambda (C^3 |n - 2|) \}^2.
\]
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By (7.1) and (7.4),
\[ cC^{-2} \leq \gamma(x) \leq C^2. \]
Thus the operator \( \mathcal{L} - cC^{-2}/2 \) has the Green function; and \( \mathcal{L} \) belongs to the class \( \mathcal{D}_M(\theta, \infty, \epsilon) \) introduced by Ancona [4], where
\[ \theta = \max \left( \Lambda, \Lambda(C^3|n-2|), C^2 \right), \quad \epsilon = cC^{-2}/2. \]
Put
\[ \mathcal{L}_2 = \tilde{\rho}(|x|)^{-2} (L + 1) = \mathcal{L} + \tilde{\rho}(|x|)^{-2}. \]
In order to apply the results of [4], we proceed to estimate \( \tilde{\rho}(|x|)^{-2} \). Let \( d(x) \) be the Riemannian distance \( \text{dist}(0,x) \) from the origin 0 to \( x \), and put
\[ \psi(r) = \int_0^r \tilde{\rho}(s) \, ds. \]
Then we see that \( d(x) = \psi(|x|) \). Denote by \( \psi^{-1} \) the inverse function of \( \psi \), and put
\[ \Phi(s) = \left[ \tilde{\rho}(\psi^{-1}(s)) \right]^{-2}, \quad s \geq 0. \]
Then
\[ 0 < \tilde{\rho}(|x|)^{-2} = \Phi(d(x)), \quad x \in M. \]
Furthermore,
\[ \int_0^\infty \Phi(s) \, ds = \int_0^\infty \Phi(\psi(r)) \tilde{\rho}(r) \, dr \]
\[ = \int_0^\infty \frac{dr}{\tilde{\rho}(r)} \leq C \int_0^\infty \frac{dr}{\rho(r)} \, dr < \infty. \]
Hence, by virtue of Corollary 6.1, Theorems 1 and 2 of [4], \( \tilde{\rho}(|x|)^{-2} \) is a small perturbation of \( \mathcal{L} \) on the manifold \( M \). That is, for any \( \varepsilon > 0 \) there exists a compact subset \( K \) of \( D = M \) such that
\[ \int_{D \setminus K} H(x,z)\tilde{\rho}(|z|)^{-2} H(z,y)\tilde{\rho}(|z|)^n \, dz \leq \varepsilon H(x,y), \quad x, y \in D \setminus K, \]
where \( dz \) is the Lebesgue measure on \( \mathbb{R}^n \), and \( H(x,z) \) is the Green function of \( \mathcal{L} \) on \( D \) with respect to the measure \( \tilde{\rho}(|z|)^n \, dz \). Denote by \( G(x,z) \) the Green function of \( L \) on \( D \) with respect to the measure \( dz \). Since \( \mathcal{L} = \tilde{\rho}(|x|)^{-2} L \), we have
\[ H(x,z) = G(x,z) \tilde{\rho}(|z|)^{2-n}. \]
Thus
\[
\int_{D \setminus K} G(x, z) \tilde{\rho}(|z|)(2-n)^{-2} G(z, y) \tilde{\rho}(|y|)^{2-n} \tilde{\rho}(|z|)^n dz \leq \varepsilon G(x, y) \tilde{\rho}(|y|)^{2-n}
\]
for any \(x, y \in D \setminus K\). Hence 1 is a small perturbation of \(L\) on \(\mathbb{R}^n\). \(\square\)

**Remark.** A sufficient condition for (7.2) is the following: \(\rho\) is a positive differentiable function on \([0, \infty)\) satisfying
\[
0 \leq \rho'(r)\rho(r)^{-2} \leq C, \quad r \geq 0,
\]
for some positive constant \(C\). Indeed, from (7.6) we have
\[
X(\delta) \equiv \rho \left( r + \frac{\delta}{\rho(r)} \right) \rho(r)^{-1} \leq \exp[C\delta X(\delta)], \quad r \geq 0, \quad \delta > 0.
\]
Put \(\delta = (2Ce)^{-1}\), and let \(\gamma \in (1, e)\) be the solution of the equation
\[
\exp[X/2e] = X.
\]
Then we get \(1 \leq X(\delta) \leq \gamma\). Thus (7.2) holds with \(c = \min(\delta, 1/\gamma)\).

The condition (7.3) is sharp, since Theorem 6.2 of [17] yields the following uniqueness theorem.

**Theorem 7.2** Suppose that there exists a positive continuous increasing function \(\rho\) on \([0, \infty)\) such that
\[
|V(x)| \leq \rho(|x|)^2, \quad x \in \mathbb{R}^n.
\]
Assume that
\[
\int_1^\infty \frac{dr}{\rho(r)} = \infty.
\]
Then [UP] holds. Thus Fact AT holds true.

**7.2.** Throughout this subsection we assume that \(D\) is a bounded domain of \(\mathbb{R}^n\). Let \(L\) be an elliptic operator on \(D\) of the form
\[
L = \frac{1}{w(x)} L_0,
\]
where \(w\) is a positive measurable function on \(D\) such that \(w, w^{-1} \in L_{loc}^\infty(D)\).
Theorem 7.3

Let $D$ be a Lipschitz domain. Suppose that there exists a positive function $\psi$ on $(0, \infty)$ such that $s^2 \psi(s)$ is increasing and $w(x) \leq \psi(\delta_D(x))$, $x \in D$, \(7.9\)

where $\delta_D(x) = \text{dist}(x, \partial D)$.

Assume that
\[
\int_0^1 s \psi(s) \, ds < \infty.
\]

(7.10)

Then 1 is a small perturbation of $L$ on $D$. Thus Theorem 1.3 holds true.

Remark.

(i) The first assertion of this theorem is implicitly shown in [17] (see Theorem 7.11 and Remark 7.12 (ii) there).

(ii) The Lipschitz regularity of the domain $D$ is assumed only for the Hardy inequality to hold for any function in $C^\infty_0(D)$. Thus, for this theorem to hold, it suffices to assume (for example) that $D$ is uniformly $\Delta$-regular John domain or a simply connected domain of $\mathbb{R}^2$ (see [3], [4]).

Proof of Theorem 7.3

For $x \in D$, put
\[
D_x = \{ y \in D ; |x - y| < \delta_D(x) \}.
\]

Then
\[
\frac{1}{2} \delta_D(x) \leq \delta_D(y) \leq 3 \left( \frac{2}{3} \delta_D(x) \right),
\]

$y \in D_x$.

Thus
\[
\delta_D(x)^2 w(y) \leq 4 \delta_D(y)^2 \psi(3 \delta_D(x)), y \in D_x.
\]

Put $\Psi(s) = 9s^2 \psi((3/2)s)$.

Then $\Psi(s)$ is increasing, and satisfies
\[
\delta_D(x)^2 \left( \sup_{y \in D_x} w(y) \right) \leq \Psi(\delta_D(x)),
\]

\[
\int_0^1 \Psi(s) s \, ds < \infty.
\]

Hence, by virtue of Proposition 9.2, Theorem 9.1' and Corollary 6.1 of [4], $w$ is a small perturbation of $L_0$ on $D$. This implies that 1 is a small perturbation of $L$ on $D$.

\[\Box\]

The condition (7.10) is sharp, since Theorem 7.8 and Lemma 7.6 of [17] yield the following uniqueness theorem.
**Theorem 7.4** Suppose that there exists a positive continuous increasing function $\psi$ on $(0, \infty)$ such that
\[
 c\psi(\delta_D(x)) \leq w(x) \leq \psi(\delta_D(x)), \quad x \in D
\]  
(7.11)
for some positive constant $c$, and
\[
 \nu \leq \frac{\psi(\eta s)}{\psi(s)} \leq \nu^{-1}, \quad s > 0, \quad \frac{1}{2} \leq \eta \leq 2,
\]  
(7.12)
for some positive constant $\nu$. Assume
\[
 \int_0^1 \left[ \psi(s) \left( \inf_{s \leq r \leq 1} r^2 \psi(r) \right) \right]^{\frac{1}{2}} ds = \infty.
\]  
(7.13)
Then [UP] holds. Thus Fact AT holds true.

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