Approximate Stiffness Element Method for Complex Section Beam Modeling

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Abstract. In this paper, a finite element method of simulating complex section beam element with approximate stiffness element is presented. By establishing an approximate stiffness element, the basic mechanical properties of the element are equivalent to the original complex section beam element’s. The approximate stiffness element has several sub-section beam elements, which intersect with other beam elements in space. The problem that missing of intersection points among complex section beam and other beams, which is caused by using conventional beam element, is solved. The error between the approximate stiffness element and the original complex section beam element is analyzed by theoretical deduction, and the feasibility and accuracy of the approximate stiffness element method are verified by the physical test of a bus body.

1. Introduction
The beam element-based finite element model has the advantages of simple modeling and high efficiency in calculation. It is widely used in the analysis of frame structures in the fields of automobile, architecture, aviation, etc [1-5]. However, if a beam that intersects multi-layered beams in space (Figure 1a) is modeled using the conventional beam element method, it may lose the points of intersection with other beams (Figure 1b), because there is only one beam axis in conventional beam element. Some scholars [6-7] dealt with this kind of problem by moving the upper and lower beams in parallel, but this will change the arrangement of the beams, which will lead to an error that cannot be ignored. Some scholars modified the section characteristics of beam elements to increase the compatibility of the quantity [8-12], while others used solid element to model the intersection zone to realize the intersection of multi-beam elements [13-14], but the coupling of different types of elements will reduce the efficiency of modeling.

In order to solve the problem that missing of intersection points among complex section beam (CSB) and other beams when using conventional beam element, this paper presents a method of simulating CSB with approximate stiffness element (ASE), the basic mechanical properties of the ASE are similar to those of the original CSB element, and the space intersection between the ASE and other beams is formed.
2. Material and method

2.1. Definition of approximate stiffness element

There is an arbitrary section beam $O$, which intersects with a group of transverse beams whose number is $n$, and the distances between each transverse beam and the axis of the beam $O$ are different in the projection of the beam $O$ section. If all beams are modeled by conventional beam element, at least $n - 1$ transverse beams will lose the point of intersection with beam $O$, as shown in Figure 2a. Therefore, the original section is divided into sub-sections, each sub-section is modeled by beam elements, each sub-section beam has its own beam axis, to ensure the intersection with other beams. As shown in Figure 2b.

As shown in Figure 3, the sub-section beam elements are connected with the rigid beam elements at the longitudinal spacing to form a space composite structural element, in which a load equivalent to that of the original beam element is applied at the joint of the composite structural element, if the deformation of the composite structural element is the same as that of the original beam element, the composite structural element is considered to be equivalent to that of the original beam element, and the composite structural element is called the approximate stiffness element of the complex section beam.
2.2. Mechanical analysis of approximate stiffness element

The coordinate system for establishing the ASE model is shown in Figure 4a. In the figure, the original CSB $OO'$ is numbered 0, $L_i$ ($i = 0 \sim 6$) denotes element lengths, and the beam's local coordinate system is labeled $\bar{x}, \bar{y}$, take the length direction of the beam as the $\bar{x}$ direction. Under external moment $M_{O}$, $M_{O'}$ and forces $F_{x0}$, $F_{y0}$, $F_{xO'}$, $F_{yO'}$, displacement $u_{O}$, $v_{O}$, $u_{O'}$, $v_{O'}$ and rotation $\theta_{O}$, $\theta_{O'}$ occur at the beam joints.

The original CSB is divided into three sub-section beams ①, ② and ③, which are connected by rigid beams ④, ⑤, ⑥ and ⑦. The numbers of the elements and joints are shown in Figure 4b. The length of the CSB element $OO'$ is $L_0$, the distances between joint 3 and the original CBS point $O$, and the distance between joint 4 and point $O'$, are both $L_e$.

(a) Deformation of the original CSB element
(b) Deformation of ASE

Figure 4. Deformation of original CSB element and ASE under external load

When $i = 0 \sim 3$, $e_i$ denote the element numbers, where the CSB element number is $e_0$, $A_i$ denote the section areas of the beam, $L_i$ denote the lengths of the element, $E$ is the elastic modulus of the material, and $EI_i$ are the flexural rigidity of the beams. The following conditions can be satisfied:

\[
\begin{align*}
A_0 &= A_1 + A_2 + A_3 \\
EI_0 &= EI_1 + EI_2 + EI_3
\end{align*}
\]  

(1)

From the stiffness matrix of each beam, according to the assembling principle of the whole stiffness matrix, and considering the influence of the rigid body, the stiffness matrix of ASE is:

\[
K = \sum_{e} K^{e} = 
\begin{bmatrix}
K_{11}^{e_1} & K_{12}^{e_1} & K_{13}^{e_1} & 0 & K_{15}^{e_1} & 0 \\
K_{21}^{e_2} & K_{22}^{e_2} & 0 & K_{24}^{e_2} & 0 & K_{26}^{e_2} \\
K_{31}^{e_3} & 0 & K_{33}^{e_3} & K_{34}^{e_3} & K_{35}^{e_3} & 0 \\
0 & K_{42}^{e_4} & K_{43}^{e_4} & K_{44}^{e_4} & 0 & K_{46}^{e_4} \\
K_{51}^{e_5} & 0 & K_{53}^{e_5} & 0 & K_{55}^{e_5} & K_{56}^{e_5} \\
0 & K_{62}^{e_6} & 0 & K_{64}^{e_6} & K_{65}^{e_6} & K_{66}^{e_6}
\end{bmatrix}, \quad i = 0, 1, 2, 3
\]  

(2)

Note that the ASE receives an external force $F$ and a nodal displacement $\delta$, the equilibrium equation for the ASE is

\[
F = K\delta
\]  

(3)

Each component of $F$ and $\delta$ is $F_i$, $\delta_i$, $i = 0 \sim 3$, respectively.

According to the plane assumption of the bending-torsion deformation of the beam, the rigid body displacement relation of the left end node is as follows:
\[
\delta_i = \begin{bmatrix}
\frac{u_i}{\theta_i} \\
u_i \\
\frac{u_3}{\theta_i} \\
u_3 \\
\frac{u_5}{\theta_i} \\
u_5
\end{bmatrix} = \begin{bmatrix}
u_o + (L_4 + L_e) \sin \theta_o \\
\frac{u_o}{\theta_o} + (L_4 + L_e) \cos \theta_o \\
u_o - L_e \cos \theta_o \\
u_o + (L_e - L_o) \cos \theta_o \\
u_o - (L_o - L_e) \sin \theta_o \\
u_o + (L_o - L_e) \sin \theta_o
\end{bmatrix}
\]

Note \(\delta = \begin{bmatrix}
(L_4 - L_o) \sin \theta_o \\
(L_o - L_e) \cos \theta_o
\end{bmatrix} \), there is

\[(5)\]
\[\delta_i - 2\delta_i + \delta_i = 0\]

Similarly, note \(\delta = \begin{bmatrix}
(L_4 - L_o) \sin \theta_o \\
(L_o - L_e) \cos \theta_o
\end{bmatrix} \), the rigid body displacement relation of the right end is

\[(6)\]
\[\delta_i - 2\delta_i + \delta_i = 0\]

Thus, conditions (5) and (6) are rigid constraint equations of ASE.

Use \(\delta_i \) to represent the \(j\)-th component of \(\delta\), and the penalty function method is used to deal with the constraint equation. After adding the constraint equations, the potential energy expression of the ASE is as follow:

\[(7)\]
\[
\Pi = \frac{1}{2} \delta^T K' \delta + \frac{1}{2} c_1 \sum_{j=1}^{3} (\delta_{i_j} - 2\delta_{i_j} + \delta_{i_j} - \delta_{i_j})^2 + \frac{1}{2} c_2 \sum_{j=1}^{3} (\delta_{i_j} - 2\delta_{i_j} + \delta_{i_j} - \delta_{i_j})^2 - \delta^T F
\]

Where \(c_1 = c_2 = c = E \times 10^7\) are large enough coefficients. According to the principle of minimum potential energy, there are

\[
\frac{\partial \Pi}{\delta c_1} = 0, \quad \frac{\partial \Pi}{\delta c_2} = 0, \quad \frac{\partial \Pi}{\delta \delta} = 0, \quad \text{that is}
\]

\[
K\delta + \begin{bmatrix}
C & 0 & -2C & 0 & C & 0 \\
0 & C & 0 & -2C & 0 & C \\
-2C & 0 & 4C & 0 & -2C & 0 \\
0 & -2C & 0 & 4C & 0 & -2C \\
C & 0 & -2C & 0 & C & 0 \\
0 & C & 0 & -2C & 0 & C
\end{bmatrix} \delta = 0
\]

where \(C = \begin{bmatrix}
c & 0 & 0 \\
0 & c & 0 \\
0 & 0 & c
\end{bmatrix}\).

Then the global stiffness equation coupled with rigid body displacement constraint equation can be obtained as
When the boundary conditions and loads are added, the deformation of the ASE can be obtained. For the original CSB element $OOC$, the equilibrium equation is:

$$K^{e0} \delta^0 = F^{e0}$$  \hspace{0.5cm} (10)

For a given displacement constraint, note that the resultant force acting on the left nodes of the ASE is $F_l$, the resultant force acting on the right nodes is $F_r$, and the external force acting on the ASE is:

$$F = \begin{bmatrix} F_l \\ F_r \end{bmatrix} = \begin{bmatrix} F_1 + F_4 + F_5 \\ F_2 + F_4 + F_6 \end{bmatrix}$$  \hspace{0.5cm} (11)

And because in the initial state,

$$L_1 = L_2 = L_3 = L_0$$  \hspace{0.5cm} (12)

From formulas (10), (11) and (12), the error of the external force between the ASE and the original CSB element can be obtain:

$$R_F = F - F^{e0} = \begin{bmatrix} F_l \\ F_r \end{bmatrix} - \begin{bmatrix} F_0 \\ F_0' \end{bmatrix} = \frac{E}{L_0} \begin{bmatrix} (\sin \theta_0 - \sin \theta_{O'})(A_4L_4 + A_6L_6 - A_3L_3) \\ 12(\cos \theta_0 - \cos \theta_{O'})(I_1L_3 - I_3L_1 + I_4L_2 - I_2L_4)/L_0^3 \\ 6(\cos \theta_0 - \cos \theta_{O'})(I_3L_6 - I_6L_3 - I_2L_5 + I_5L_2)/L_0 \\ (\sin \theta_0 - \sin \theta_{O'})(A_4L_4 - A_3L_3) \\ 12(\cos \theta_0 - \cos \theta_{O'})(I_1L_3 + I_3L_1 + I_4L_2 + I_2L_4)/L_0^3 \\ 6(\cos \theta_0 - \cos \theta_{O'})(I_3L_6 + I_6L_3 + I_2L_5 - I_5L_2)/L_0 \\ 0 \end{bmatrix}$$  \hspace{0.5cm} (13)

If the element length $L_0$ is small, that is, when the deformation angle is small, according to the expansion of Taylor series, we can get $\sin \theta_0 \approx \theta_0$, $\sin \theta_{O'} \approx \theta_{O'}$, $\cos \theta_0 \approx 1$, $\cos \theta_{O'} \approx 1$. Thus, formula (13) becomes

$$R_F \approx \frac{E}{L_0} \begin{bmatrix} (\theta_0 - \theta_{O'})(A_4L_4 + A_6L_6 - A_3L_3) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$  \hspace{0.5cm} (14)

Particularly, in the most ideal case, when the sub-section beam 1, 2 and 3 meet the condition $A_4L_4 + A_6L_6 - A_3L_3 = 0$ (15)

There is

$$R_F \approx \frac{E}{L_0} [0 \ 0 \ 0 \ 0 \ 0]^T = \theta$$  \hspace{0.5cm} (16)

It means the ASE is almost equivalent to the original CSB element. However, this case is seldom encountered in practical application, so when dividing the CSB section, the error caused by the division can be judged in advance according to formula (13), if the conditions permit, then the sub-sections can be divided to satisfy formula (15), and the error caused by ASE method can be minimized.

3. Results and discussion

Take the 6063-T6 aluminum alloy CSB used in a bus body as an example. Considering the condition (13), the ASE of the CSB was established, as shown in Figure 5. The elastic modulus of 6061-T6
aluminum alloy is 68GPa, its shear modulus is 14.6GPa, its yield strength is 313MPa, and its tensile strength is 400MPa. And then establish the bus body model including the ASE, as shown in Figure 6. The bus body frame was physically tested, with a total load of 4520kg. The three working conditions of horizontal bending, left ultimate torsion and right ultimate torsion were test, and analyzed by ASE model. The ASE model analysis results were compared with the test results.

Figure 5. Establishment of ASE for aluminum alloy CSB

Figure 6. Finite element model of bus body with ASE method

The analysis results of the bus body model using the ASE method are shown in Figure 7 and Table 1. The maximum displacement of bending, left ultimate torsion and right ultimate torsion conditions are 4.25 mm, 20.29 mm and 18.15 mm respectively. The maximum equivalent stress under bending, left ultimate torsion and right ultimate torsion conditions are 41.2MPa, 85.99MPa and 76.44MPa, respectively.

(a) Bending condition  (b) Left torsion condition  (c) Right torsion condition

Figure 7. Maximum equivalent stress diagram (local)

Table 1. Comparison of ASE model analysis results and actual test results

| Condition       | Analysis item         | Actual test | ASE model | Relative error |
|-----------------|-----------------------|-------------|-----------|----------------|
| Bending         | Deformation (mm)      | 4.09        | 4.25      | 3.91%          |
|                 | Maximum equivalent stress (MPa) | -         | 41.2      | -              |
| Left torsion    | Deformation (mm)      | 20.84       | 20.29     | -2.63%         |
|                 | Maximum equivalent stress (MPa) | -         | 85.99     | -              |
| Right torsion   | Deformation (mm)      | 18.5        | 18.15     | -1.90%         |
|                 | Maximum equivalent stress (MPa) | -         | 76.44     | -              |

For different test conditions of the bus body frame, the maximum error between the results of the ASE model and the actual test results is 3.91%, which can meet the engineering application requirements.
4. Conclusion
In this paper, an ASE method is proposed to simulate the CSB element. The ASE model is established, the global stiffness equation and the error of the ASE coupled with the displacement constraint equation of the rigid body are derived. The approximate simulation of the original CSB element with ASE is realized, and this method is verified by the experimental results. In the bus body analysis, the maximum error between the ASE model analysis results and the actual test results is 3.91%, which can meet the engineering application requirements.

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