Lie II theorem for Lie algebroids via higher groupoids

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Abstract

Following Sullivan’s spacial realization of a differential algebra, we construct a universal integrating Lie 2-groupoid for every Lie algebroid. Then we show that unlike Lie algebras which one-to-one correspond to simply connected Lie groups, Lie algebroids (integrable or not) one-to-one correspond to a sort of étale Lie 2-groupoids with 2-connected source fibres. Finally, we discuss how to lift Lie algebroid morphisms to Lie 2-groupoid morphisms (Lie II Theorem).

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1 Introduction

Lie II theorem for Lie algebras is about how to lift morphisms of Lie algebras to morphisms of Lie groups. A Lie algebroid is the infinitesimal data for a Lie groupoid, as a Lie algebra is for a Lie group. More precisely, a Lie algebroid over a manifold $M$ is a vector
bundle $\pi : A \to M$ with a real Lie bracket $[,]$ on its space of sections $H^0(M, A)$ and a bundle map $\rho : A \to TM$ such that the Leibniz rule

$$[X, f \cdot Y](x) = f(x) \cdot [X, Y](x) + \rho(X)(f)(x) \cdot Y(x)$$

holds for all $X, Y \in H^0(M, A)$, $f \in C^\infty(M)$ and $x \in M$. Hence when $M$ is a point, a Lie algebroid becomes a Lie algebra. Also a tangent bundle $TM \to M$ is certainly a Lie algebroid with $[,]$ the Lie bracket of vector fields. The next example is a Poisson manifold $P$ with $A = T^*P \to P$ and $[df, dg] = d\{f, g\}$ determined by the Poisson bracket $\{,\}$ (see the book [8] for a friendly introduction). Other fundamental examples come from bundles of Lie algebras of Douady-Lazard (the case when $\rho = 0$); Lie algebra actions on manifolds by Palais [26]; abstract Atiyah sequences of Almeida, Molino [2] and Mackenzie [18].

Thus Lie II theorem for Lie algebroids is about how to lift an infinitesimal morphism from the level of Lie algebroids to a global morphism. Its version having Lie groupoids as the global objects integrating Lie algebroids is well-known [19] [23]. The subject of Lie groupoids itself was widely studied by many people—Connes, Haefliger, Hilsum, Pradines [27], Skandalis from 60’s to 80’s—for applications to foliations on manifolds. However, unlike (finite dimensional) Lie algebras which always have their associated Lie groups, Lie algebroids do not always have their associated Lie groupoids [1] [2]. The complete integrability criteria is given in a remarkable work of Crainic and Fernandes [10]. However, we claim the situation is not totally hopeless: if we are willing to enter the world of stacks, we do have the full one-to-one correspondence (Lie III) parallel to the classical one for Lie algebras [33].

Here an étale stacky Lie groupoid $\mathcal{G} \Rightarrow M$ (we call this sort of groupoid $W$-groupoid, as its existence was first conjectured by Alan Weinstein [35] [7]) is a groupoid in the category of differentiable stacks with $\mathcal{G}$ an étale stack and $M$ a manifold (see [38] for the exact definition), and we call the procedure of passing from an infinitesimal object to a global object “integration”. This problem is already very interesting, as shown by Cattaneo and Felder [9], in the case of Poisson manifolds: the object integrating $A = T^*P$ is the phase space of the Poisson sigma model and is further a symplectic $W$-groupoid [34].

Therefore my effort in this paper is to study the functoriality of this new integration procedure, for example how to integrate a morphism $A \to B$ of Lie algebroids to a global morphism from a universal stacky groupoid of $A$ to any stacky groupoid of $B$ (Lie II). The results of the paper are positive. We show

**Theorem 1.1.** The $W$-groupoid $\mathcal{G}(A)$ constructed in [32] is source-2-connected\(^1\).

\(^1\)Source-2-connectedness means that the source fibres have trivial homotopy groups $\pi_{\leq 2}$.
Thus we call $\mathcal{G}(A)$ the universal W-groupoid of $A$. Using stacky groupoids, one has one more degree of connectedness. A simple example is the algebroid $A = TS^2$. In this case, $A$ is integrable to the pair groupoid $S^2 \times S^2 \to S^2$, but $\mathcal{G}(A) = \tilde{S}^2 \times \tilde{S}^2/B\mathbb{Z}$. Here $\tilde{S}^2$ is the $B\mathbb{Z}$ gerbe on $S^2$ presented by the action groupoid $S^3 \times \mathbb{R} \to S^3$ with $\mathbb{R}$ acting via the projection $\mathbb{R} \to S^1$ and the usual Hopf $S^1$ action on $S^3$. Analogously to simply-connected coverings, $S^2$ is the $\pi_2$-trivial covering of $S^2$ (See Remark 6.3). Hence even with objects as simple as $S^2$ we expect interesting examples of $\mathcal{G}(A)$ to appear. Moreover the property that $\pi_2 = 0$ might also appeal to symplectic geometers.

But why more connectedness? This is probably best understood via the following viewpoint elaborated in Section 3 for every Lie algebroid $A$, (notice that tangent bundles are Lie algebroids), we associate to $A$ a simplicial set $S(A) = [...S_2(A) \Rightarrow S_1(A) \Rightarrow S_0(A)]$ with,

$$ S_i(A) := \{ \text{Lie algebroid morphisms } T\Delta^i \xrightarrow{f} A, \text{such that } f|_{\text{vertices of } \Delta^i} = 0. \} \quad (1) $$

Here $\Delta^i$ is the $i$-dimensional standard simplex viewed as a smooth Riemannian manifold with boundary, hence it is isomorphic to the $i$-dimensional closed ball. Then the face and degeneracy maps are induced by pullbacks of the tangent maps of natural maps $d_k : \Delta^{i-1} \to \Delta^i$ and $s_k : \Delta^i \to \Delta^{i-1}$. With this language, we also understand [9] and [10] in a new light: $S_1(A)$ and $S_2(A)$ give the space of fields and Hamiltonian symmetries respectively in the Poisson sigma model [9] in the case of Poisson manifolds; or the space of $A$-paths $P_aA$ and of $A$-homotopies [10] respectively in general (see Section 3.1).

The simplicial set $S(A)$ is not entirely unknown to us: in the case of a Lie algebra $g$, let $\Omega^1(\Delta^n, g)$ be the space of $g$-valued 1-forms on $\Delta^n$, then we have\(^2\)

$$ \text{hom}_{d.g.a.}(T\Delta^n, g) = \{ \alpha \in \Omega^1(\Delta^n, g) | d\alpha = \frac{1}{2}[\alpha, \alpha] \} $$

$$ = \{ \text{flat connections on the trivial } G\text{-bundle } G \times \Delta^n \to \Delta^n \}, $$

where $G$ is a Lie group of $g$.

In fact for a differential algebra $D$, Sullivan [31] constructed a spatial realization of $D$, which is defined as the space of differential graded maps $\text{hom}_{d.g.a.}(D, \Omega^\bullet(\Delta^n))$, and the simplicial set $S(A)$ is a parallel construction on the geometric side. Sullivan’s construction also appears in the work of Ševera [30] to integrate a non-negatively graded super-manifold with a degree 1 vector field of square 0 (NQ-manifold), which is the original motivation for our simplicial set $S(A)$. In fact a Lie algebroid can be viewed as a “degree 1” NQ-manifold, and a Courant algebroid as a “degree 2” NQ-manifold [17, 29]. Courant algebroids are now widely used in Hitchin’s and Gualtieri’s generalized complex geometry [15, 13]. This construction also appears in the works of Getzler [12] and Henriques [14] to integrate an $L_\infty$-algebra $L$, where the simplicial set is $\text{hom}_{d.g.a.}(C^\bullet(L), \Omega^\bullet(\Delta^n))$ with $C^\bullet(L)$ the Chevalley-Eilenberg cochains on $L$. This simplicial set is further proved to be a Kan simplicial manifold for all nilpotent $L_\infty$-algebras in [12] and for all $L_\infty$-algebras in [14].

In [38] we found a 1-1 correspondence between stacky Lie groupoids and Lie 2-groupoids up to a certain Morita equivalence. Under this correspondence, $\mathcal{G}(A)$ corresponds to the 2-truncation of $S(A)$, while the usual Lie groupoid of $A$ (when $A$ is integrable) corresponds to only the 1-truncation of $S(A)$. It is this higher truncation that is the hidden reason for

\(^2\)More precisely it is $-\alpha$ the connection 1-form.
the higher connectedness. Notice that in the case of a (finite dimensional) Lie algebra \( g \), \( \mathcal{G}(g) \) is the simply connected Lie group, and this Lie group is always 2-connected.

However it is not obvious that \( S(A) \) is a simplicial manifold let alone a Kan simplicial manifold. Hence its 2-truncation being a Lie 2-groupoid is not immediate but proved in,

**Theorem 1.2.** Given a Lie algebroid \( A \), the 2-truncation of the simplicial set \( S(A) \),

\[
S_2(A)/S_3(A) \Rightarrow S_1(A) \Rightarrow S_0(A),
\]

is a Lie 2-groupoid that corresponds to the W-groupoid \( \mathcal{G}(A) \) constructed in [33] under the correspondence of Theorem 1.4 of [33].

In this theorem \( S_{\geq 1}(A)'s \) are infinite dimensional spaces. As a reply to a question of Getzler and Roytenburg, it turns out that it is not necessary to take everything in the infinite dimensional space. This Lie 2-groupoid is Morita equivalent to a finite dimensional Lie 2-groupoid, \( E \Rightarrow P \Rightarrow M \), arising in the form of local Lie groupoids à la Pradines, where \( \dim P = \dim A \) and \( \dim E = 2 \dim A - \dim M \) (See (13)).

Finally, after finding this universal W-groupoid \( \mathcal{G}(A) \), we have the expected

**Theorem 1.3** (Lie II for Lie algebroids). Let \( \phi \) be a morphism of Lie algebroids \( A \to B \), \( \mathcal{G} \) a W-groupoid whose algebroid is \( B \). Then up to 2-morphisms, there exists a unique morphism \( \Phi \) of W-groupoids \( \mathcal{G}(A) \to \mathcal{G} \) such that \( \Phi \) induces the Lie algebroid morphism \( \phi : A \to B \).

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## 2 Technical tools

In this section we recall some basic knowledge on the technical tools that we need later in the paper.

### 2.1 Groupoids, stacks and higher groupoids

We first recall necessary knowledge in the theory of Lie groupoids and differentiable stacks [3, 21, 28, 22] (in the context of orbifolds). An H.S. (Hilsum-Skandalis) morphism between two Lie groupoids \( K := K_1 \Rightarrow K_0 \) and \( K' := K'_1 \Rightarrow K'_0 \) is given by a right-principal bibundle \( E \), which is called the H.S. bibundle,

\[
\begin{array}{ccc}
K_1 & \xrightarrow{E} & K'_1 \\
\downarrow J_l & & \downarrow J_r \\
K_0 & \xleftarrow{E} & K'_0.
\end{array}
\]

More precisely, a bibundle \( E \) is a manifold on which both \( K \) and \( K' \) act compatibly

\[
\Phi_l : K_1 \times_{s,K_0,J_l} E \to E, \quad \Phi_r : E \times_{J_r,K'_0,t} K'_1 \to E,
\]
via the moment maps $J_l$ and $J_r$. $E$ is furthermore said to be right-principal, if the right action of $K'$ is principal, that is, $J_l$ is a surjective submersion and $id \times \Phi_r : E \times_j K_0' \otimes K_1' \cong E \times_j K_0 \times_j K_1 E$ is an isomorphism. If $E$ is also left-principal (defined in a similar way), $E$ is called a Morita bibundle and it gives a Morita equivalence between $K$ and $K'$. Then there is a dictionary between differentiable stacks and Lie groupoids:

\[
\begin{align*}
\text{differentiable stacks} & \leftrightarrow \text{Lie groupoids} \\
\text{morphisms} & \leftrightarrow \text{H.S. morphisms} \\
\text{isomorphisms} & \leftrightarrow \text{Morita equivalences}.
\end{align*}
\]

Given a differentiable stack $\mathcal{X}$, a Lie groupoid $X := X_1 \Rightarrow X_0$ corresponding to it is called a groupoid presentation of $\mathcal{X}$, and $X_0$ is called a chart of $\mathcal{X}$.

There are also various version of higher Lie groupoids. One uniform way to describe Lie $n$-groupoids is via its nerve, by requiring the nerve to be a simplicial manifold whose homotopy groups are trivial above $\pi_n$. We refer the reader to the introduction of [38] and the references therein for a general discussion of these matters, and only recall briefly the definitions we need here.

A simplicial set (resp. manifold) $X$ is made up by sets (resp. manifolds) $X_n$ and structure maps

\[
d^i_n : X_n \to X_{n-1} \quad \text{(face maps)} \\
s^i_n : X_n \to X_{n+1} \quad \text{(degeneracy maps)} \quad i \in \{0,1,2,\ldots,n\}
\]

that satisfy suitable coherence conditions. The first two examples are the simplicial $m$-simplex $\Delta[m]$ and the horn $\Lambda[m,j]$, defined as

\[
\begin{align*}
(\Delta[m])_n &= \{ f : (0,1,\ldots,n) \to (0,1,\ldots,m) | f(i) \leq f(j), \forall i \leq j \}, \\
(\Lambda[m,j])_n &= \{ f \in (\Delta[m])_n | \{0,\ldots,j-1,j+1,\ldots,m\} \not\subseteq \{f(0),\ldots,f(n)\} \}.
\end{align*}
\]

**(Example 2.1)** Given a Lie groupoid $G := G_1 \Rightarrow G_0$, we complete it to a simplicial manifold $NG$, with $NG_n = G_1 \times_{s,G_0,t} G_1 \times_{s,G_0,t} \cdots \times_{s,G_0,t} G_1$, $n$ copies of $G_1$, for $n \geq 1$ and $NG_0 = G_0$. The face and degeneracy maps are

\[
d_k(g_1,\ldots,g_n) =
\begin{cases}
(g_2,\ldots,g_n) & k = 0 \\
(g_1,\ldots,g_kg_{k+1},\ldots,g_n) & 0 < k < n, \\
(g_1,\ldots,g_n) & k = n
\end{cases}
\]

\[
s_k(g_1,\ldots,g_n) =
\begin{cases}
(1_t(g_1),g_1,\ldots,g_n) & k = 0 \\
(g_1,\ldots,g_k,1_t(g_{k+1}),g_{k+1},\ldots,g_n) & 0 < k < n, \\
(g_1,\ldots,g_n,1_s(g_n)) & k = n
\end{cases}
\]

This construction is known as the nerve of a Lie groupoid.

A simplicial set $X$ is Kan if any map from the horn $\Lambda[m,j]$ to $X$ $(m \geq 1, j = 1,\ldots,m)$, extends to a map from $\Delta[m]$ to $X$. In the language of groupoids, the Kan condition corresponds to the possibility of composing various morphisms. In an $n$-groupoid, the only well defined composition law is the one for $n$-morphisms. This motivates the following definition.

**Definition 2.2.** [14] A Lie $n$-groupoid $X$ ($n \in \mathbb{N} \cup \infty$) is a simplicial manifold that satisfies the conditions $Kan(m,j) \forall m \geq 1, 0 \leq j \leq m$ and $Kan!(m,j) \forall m > n, 0 \leq j \leq m$, where:
Kan\((m, j)\): The restriction map \(\text{hom}(\Delta[m], X) \to \text{hom}(\Lambda[m, j], X)\) is a surjective submersion.

Kan!\((m, j)\): The restriction map \(\text{hom}(\Delta[m], X) \to \text{hom}(\Lambda[m, j], X)\) is a diffeomorphism.

When \(n = \infty\), a Lie \(\infty\)-groupoid is also called a Kan simplicial manifold.

Remark 2.3. A Lie \(n\)-groupoid \(X\) is determined by its first \((n+1)\)-layers \(X_0, X_1, \ldots, X_n\) and some structure maps. For example a Lie 1-groupoid is exactly determined by a Lie groupoid structure on \(X_1 \Rightarrow X_0\). In fact a Lie 1-groupoid is the nerve of a Lie groupoid. A Lie 2-groupoid is exactly determined by \(X_2 \Rightarrow X_1 \Rightarrow X_0\) together with a sort of 3-multiplications and face and degeneracy maps satisfying certain compatibility conditions. This is made precise in [38, Section 2]. Hence in this paper, we often write only the first three layers of a Lie 2-groupoid.

Remark 2.4. It is not clear that \(\text{hom}(\Lambda[m, j], X)\) is a manifold. In general, as in [14, Section 2], we sometimes talk about the limit (for example fibre-product) of a diagram in \(\mathcal{C}\), the category of differential manifolds, before knowing its existence. For this purpose, we use the Yoneda functor

\[\text{yon} : \mathcal{C} \to \{\text{Sheaves on } \mathcal{C}\}\]

\[X \mapsto (T \mapsto \text{hom}(T, X))\]

to embed \(\mathcal{C}\) to the category of sheaves on \(\mathcal{C}\). Hence a limit of objects of \(\mathcal{C}\) can always be viewed as the limit of the corresponding representable sheaves using \(\text{yon}\). The limit sheaf is representable if and only if the original diagram has a limit in \(\mathcal{C}\). Viewed as a sheaf, \(\text{hom}(\Lambda[m, j], X)\) is representable by [14, Lemma 2.4], thus it is a manifold always.

For our convenience, (since in the end we will apply Kan replacement to local Lie groupoids), we need simplicial manifolds with certain properties, described below,

1. **Property A**: for all \(0 \leq j \leq m\), the restriction map \(\text{hom}(\Delta[m], X) \to \text{hom}(\Lambda[m, j], X)\) is a submersion;

2. **Property B**: there are isomorphisms

\[\text{hom}(\Lambda[2, 0], X) \cong \text{hom}(\Lambda[2, 1], X) \cong \text{hom}(\Lambda[2, 2], X),\]

which are compatible with the face maps \(\text{hom}(\Delta[2], X) \to \text{hom}(\Lambda[2, j], X)\).

Example 2.5. A local Lie groupoid \(G^{\text{loc}} := (U, V) \Rightarrow M\) resembles a Lie groupoid, with the difference that the multiplication is only defined on \(V \times_M V\) where \(V\) is an open tubular neighborhood of the identity section \(M \to U\). We also require here that the inverse map \(i\) maps \(V\) into \(V\). This gives us a simplicial manifold \(NG^{\text{loc}}\) (the nerve of local Lie groupoid), with \(NG^{\text{loc}}_0 = M\), \(NG^{\text{loc}}_1 = V\),

\[NG^{\text{loc}}_n = \{(g_1, \ldots, g_n) \in V \times_M V \times_M \cdots \times_M V | g_i \cdots g_j \in V, \forall 1 \leq i \leq j \leq n\},\]

and face and degeneracy maps similar to the ones in the nerve of a groupoid (Example 2.1). Then we have

**Lemma 2.6.** Constructed as above, \(NG^{\text{loc}}\) is a simplicial manifold satisfying properties A and B.
Proof. First of all, we notice that the multiplication \( m : V \times_M V \to U \) is a submersion: it is easy to argue that the groupoid multiplication is surjective, by using inverse elements. By lifting this argument to the tangent groupoid, we see that the multiplication map is a submersion. Since being a submersion is a local property, this argument applies to local Lie groupoids. Thus \( NG_2^{\text{loc}} = m^{-1}(V) \) is a manifold. The natural embedding \( NG_2^{\text{loc}} \to \text{hom}(\Delta[2, 1], NG^{\text{loc}}) = V \times_M V \) is a submersion. Moreover, with the inverse map \( i : V \to V \), it is easy to see that \( NG^{\text{loc}} \) satisfies Property B. Thus \( NG^{\text{loc}} \) satisfies Property A for \( m = 1, 2 \).

It is not hard to see that \( NG_n^{\text{loc}} = \text{hom}(\Lambda[n, j], NG^{\text{loc}}) \) for all \( 0 \leq j \leq n \); for example,

\[
\text{hom}(\Lambda[3, 1], NG^{\text{loc}}) = \{(g_1, g_2, g_3) | (g_1, g_2) \in NG_2^{\text{loc}}, (g_1, g_2g_3) \in NG_2^{\text{loc}}, (g_2, g_3) \in NG_2^{\text{loc}}\} \\
= \{(g_1, g_2, g_3) | g_i \cdot \ldots \cdot g_j \in V, \forall 1 \leq i, j \leq 3\} \\
= NG_3^{\text{loc}}.
\]

The induction in [14, Lemma 2.4] shows that if we know that \( NG^{\text{loc}} \) satisfies Property A for \( m \leq n - 1 \) (thus naturally \( NG_m^{\text{loc}} \) and \( \text{hom}(\Lambda[m, j], NG^{\text{loc}}) \) are representable for \( m \leq n - 1 \), then \( \text{hom}(\Lambda[n, j], NG^{\text{loc}}) \) is representable. Thus \( NG_n^{\text{loc}} = \text{hom}(\Lambda[n, j], NG^{\text{loc}}) \) is a manifold.

Thus \( NG^{\text{loc}} \) is actually a Lie 2-groupoid (the point is that it is a simplicial manifold) satisfying properties A and B.

There is another sort of higher groupoid, whose space of arrows is a differentiable stack \( \mathcal{G} \) and whose space of object is a manifold \( M \). \( \mathcal{G} \Rightarrow M \) satisfies the usual groupoid axioms up to some 2-morphisms, which in turn satisfy some higher coherences. We call such an object a stacky Lie (SLie for short) groupoid. When the stack is further étale, we call it a \( W \)-groupoid as in the introduction. We refer the reader to [38] for a complete definition and the following result: There is a 1-1 correspondence of Lie 2-groupoids and SLie groupoids up to a certain equivalence. Given an SLie groupoid \( \mathcal{G} \Rightarrow M \) with a groupoid presentation \( G := (G_1 \Rightarrow G_0) \) of \( \mathcal{G} \), its associated Lie 2-groupoid \( Y \) is given by

\[
Y_0 = M, \quad Y_1 = G_0, \quad Y_2 = E_m,
\]

where \( E_m \) is the bundle representing the multiplication \( m : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G} \) of the SLie groupoid. Conversely, given a Lie 2-groupoid \( Y \), let \( b(Y_2) := d_2^{-1}(s_0(Y_0)) \) be the set of bigons in \( Y_2 \). Then \( d_0, d_1 : b(Y_2) \Rightarrow Y_1 \) is a Lie groupoid, and the stack \( \mathcal{G} \) presented by \( b(Y_2) \Rightarrow Y_1 \) is an SLie groupoid over \( Y_0 \).

2.2 Hypercovers, generalized morphisms, Morita equivalences, and truncations

Definition 2.7. A strict map \( f : Z \to X \) of Lie \( n \)-groupoids is a hypercover if the natural map from \( Z_k = \text{hom}(\Delta^k, Z) \) to the pull-back

\[
\text{hom}(\partial \Delta[k] \to \Delta[k], Z \to X) = PB(\text{hom}(\partial \Delta[k], Z) \to \text{hom}(\partial \Delta[k], X) \leftarrow X_k)
\]

is a surjective submersion for \( 0 \leq k \leq n - 1 \) and an isomorphism\(^3\) for \( k = n \).

\(^3\)When \( n = \infty \), namely in the case of Kan simplicial manifolds, the requirement of isomorphism is empty.
Definition 2.8. A generalized morphism of Lie n-groupoids from X to Y consists of a zig-zag of strict maps \( X \xrightarrow{\sim} Z \rightarrow Y \), where the map \( Z \xrightarrow{\sim} X \) is an hypercover.

Definition 2.9. A 2-morphism\(^4\) between two generalized morphisms of Lie n-groupoids, \( X \xleftarrow{\sim} Z \rightarrow Y \) and \( X \xleftarrow{\sim} Z' \rightarrow Y \), is an isomorphism of Lie n-groupoids \( Z \cong Z' \) which commutes with the maps \( Z, Z' \rightarrow X \) and \( Z, Z' \rightarrow Y \).

Definition 2.10. Two Lie n-groupoids X and Y are Morita equivalent if there is a Lie n-groupoid Z such that both of the maps \( X \xleftarrow{\sim} Z \rightarrow Y \) are hypercovers.

We denote a hypercover as \( X \xleftarrow{\sim} Y \), a generalized morphism as \( X \xrightarrow{\sim} Y \), and a Morita equivalence as \( X \xleftrightarrow{\sim} Y \).

Hypercovers of Lie n-groupoids may also be understood as higher analogues of pull-backs of Lie groupoids via a surjective submersion. Let \( X \) be a 2-groupoid, let \( Z_1 \Rightarrow Z_0 \) be two manifolds with structure maps as in (2) up to the level \( n = 1 \), and let \( f_n : Z_n \rightarrow X_n \) be a map preserving the structure maps \( d^n_k \)'s and \( s^n_k \)'s for \( n \leq 1 \). Then \( \text{hom}(\partial \Delta[n], Z) \) is defined for \( n \leq 1 \). We further suppose that \( f_0 : Z_0 \rightarrow X_0 \) is a surjective submersion (hence \( Z_0 \times Z_0 \times X_0 \times X_1 \) is a manifold) and do the same for \( Z_1 \rightarrow Z_0 \times X_0 \times X_0 \times X_{1,1} \). That is to say, we suppose that the induced maps from \( Z_k \) to the pull-backs \( \text{hom}(\partial \Delta[k], Z) \times_{\text{hom}(\partial \Delta[k], X)} X_k \) are surjective submersions for \( k = 0, 1 \). Then we form

\[
Z_2 = \text{hom}(\partial \Delta[2], Z) \times_{\text{hom}(\partial \Delta[2], X)} X_2,
\]

which is a manifold (see \[38, Lemma 2.4\]).

Moreover there are maps \( d^2_0 : Z_2 \rightarrow Z_1 \) induced by the natural projections \( \text{hom}(\partial \Delta[2], Z) \rightarrow Z_1 \), and maps \( s^1_0 : Z_1 \rightarrow Z_2 \) defined by

\[
s^1_0(h) = (h, h, s^0_0(d^1_0(h))), \quad s^1_0(f_1(h))) = (s^0_0(d^1_0(h)), h, h, s^1_0(f_1(h))).
\]

Similarly we define higher \( Z_i \)'s inductively. As shown in \[38, Section 2.2\], \( Z_2 \Rightarrow Z_1 \Rightarrow Z_0 \) is a Lie 2-groupoid and we call it the pull-back Lie 2-groupoid of \( X \) by \( f \). Then \( f : Z \rightarrow X \) is a hypercover with \( f_2 \) the natural projection \( f_2 : Z_2 \rightarrow X_2 \). In particular, \( Z \) is Morita equivalent to \( X \). Sometimes \( f \) does not make the induced maps from \( Z_k \) to \( \text{hom}(\partial \Delta[k], Z) \times_{\text{hom}(\partial \Delta[k], X)} X_k \) surjective submersions for \( k = 0, 1 \), but \( Z_2 \) is still a manifold. In this case, we can still construct a pull-back Lie 2-groupoid as above. However, the pullback is not obviously Morita equivalent to \( X \) any more. Nevertheless, the following lemma, that we will use later, says that in a certain case this still holds.

Lemma 2.11. Given a simplicial morphism \( f : K \rightarrow Y \) of two Lie 2-groupoids, if

1. \( K_0 \cong Y_0 \),
2. there are a manifold \( Z_1 \), and surjective submersions \( K_1 \xleftarrow{g_1} Z_1 \xrightarrow{h_1} Y_1 \),
3. there exists a morphism \( \alpha : Z_1 \rightarrow b(Y_2) \) such that \( f_1 \circ g_1 = \alpha \cdot h_1 \),
4. the natural map \( K_2 \rightarrow \text{hom}(\partial \Delta[2] \rightarrow \Delta[2], K \rightarrow Y) \) is an isomorphism,

then \( K \) is Morita equivalent to \( Y \).

\(^4\)This is not the most general notion of 2-morphism, however it is sufficient for the purposes of this paper.
Proof. Take \( Z_0 = K_0 = Y_0, g_0 = h_0 = \text{id} \). We pullback \( K \) and \( Y \) via \( g_1 \) and \( h_1 \) to \( Z_1 \). To show \( K \) and \( Y \) are Morita equivalent, we only need to show that the map induced by \( f \)

\[
\text{hom}(\partial \Delta[2] \to \Delta[2], Z \to Y) \to \text{hom}(\partial \Delta[2] \to \Delta[2], Z \to K),
\]

(6)
is an isomorphism. Then (6) follows from items 3 and 4:

\[
\text{RHS} = \text{hom}(\partial \Delta[2], Z) \times_{h_1, \text{hom}(\partial \Delta[2], Y)} \text{hom}(\Delta[2], Y)
\]

\[
= \text{hom}(\partial \Delta[2], Z) \times_{f_1 \circ g_1, \text{hom}(\partial \Delta[2], Y)} \text{hom}(\Delta[2], Y)
\]

\[
= \text{hom}(\partial \Delta[2], Z) \times_{g_1, \text{hom}(\partial \Delta[2], K)} \text{hom}(\partial \Delta[2], K) \times_{f_1, \text{hom}(\partial \Delta[2], Y)} \text{hom}(\Delta[2], Y)
\]

\[
= \text{hom}(\partial \Delta[2], Z) \times_{\text{hom}(\partial \Delta[2], K)} \text{hom}(\Delta[2], K) = \text{LHS}.
\]

Remark 2.12. Given a simplicial morphism \( f : K \to Y \) of Lie 2-groupoids, if

1. \( K_0 \cong Y_0 \) and there is a submanifold \( X_1 \) of \( K_1 \) such that \( X_1 \xrightarrow{f_1|_{X_1}} Y_1 \) is a submersion,

2. the composed map \( h_1' : \text{hom}(\Lambda[2,1], K) \times_{\text{hom}(\Lambda[2,1], Y)} Y_2 \to Y_2 \xrightarrow{d_1} Y_1 \) is a surjective submersion,

3. there exist a morphism \( \mu : \text{hom}(\Lambda[2,1], K) \to K_1 \) and a morphism \( \alpha' : \text{hom}(\Lambda[2,1], K) \times_{\text{hom}(\Lambda[2,1], Y)} Y_2 \to b(Y_2) \) such that the composed map \( g_1' : \text{hom}(\Lambda[2,1], K) \times_{\text{hom}(\Lambda[2,1], Y)} Y_2 \to \text{hom}(\Lambda[2,1], K) \xrightarrow{\mu} K_1 \) makes \( f_1 \circ g_1' = \alpha \cdot h_1' \),

4. the natural map \( g_1 := g_1' \sqcup p_{X_1} : Z_1 := \text{hom}(\Lambda[2,1], K) \times_{\text{hom}(\Lambda[2,1], Y)} Y_2 \sqcup X_1 \times Y_1, d_0 b(Y_2) \to Y_1 \) is a surjective submersion,

5. the natural map \( K_2 \to \text{hom}(\partial \Delta[2] \to \Delta[2], K \to Y) \) is an isomorphism,

then \( Z_1, g_1, h_1 := h_1' \sqcup d_1 \circ p_{b(Y_2)} \) and an extension of \( \alpha' \) to \( \alpha : Z_1 \to b(Y_2) \) obtained by addition of \( \alpha = p_{b(Y_2)} : X_1 \times Y_1, d_0 b(Y_2) \to b(Y_2) \) satisfy the conditions required by the above lemma.

Definition 2.13. Given a Kan simplicial manifold \( X \), we can perform a certain operation \( \tau_n \) called \( n \)-truncation (it is denoted by \( \tau_{\leq n} \) in [1], Section 3) for \( n \in \mathbb{Z}^\geq 1 \),

\[
\tau_n(X)_k = X_k, \forall k \leq n - 1, \quad \tau_n(X)_k = X_k/\sim_k, \forall k \geq n,
\]

where two elements \( x \sim_k y \) in \( X_k \) if they are simplicially homotopic relative to their \((n-1)\)-skeletons.\(^4\)

\(^4\)When \( k = n \), this means explicitly that \( d_i x = d_i y, 0 \leq i \leq k \), and there exists \( z \in X_{k+1} \) such that \( d_k(z) = x, d_{k+1}(z) = y \), and \( d_i z = s_{k-1} d_i x = s_{k-1} d_i y, 0 \leq i < k \).
2.3 Kan replacement

Here we recall the construction of a canonical functor $Kan$ from the category of simplicial manifolds with properties A and B (they are called invertible local Kan simplicial manifolds in [37]) to the category of weak Kan simplicial manifolds.

Let $J$ be the set of inclusions as below,

$$J := \{ \Lambda[k, j] \to \Delta[k] : 0 \leq j \leq k \geq 3, \} \cup \{ \Lambda[2, 1] \to \Delta[2] \}. \quad (7)$$

Given a simplicial manifold with property A, we construct a series of simplicial manifolds

$$X = X^0 \to X^1 \to X^2 \to \cdots \to X^\beta \to \cdots \quad (8)$$

by an inductive pushout:

$$\begin{array}{c}
\bigsqcup_{(\Lambda[k,j] \to \Delta[k]) \in J} \Lambda[k,j] \times \text{hom}(\Lambda[k,j], X^\beta) \\
\downarrow \\
\bigsqcup_{(\Lambda[k,j] \to \Delta[k]) \in J} \Delta[k] \times \text{hom}(\Lambda[k,j], X^\beta) \\
\downarrow \\
X^{\beta+1}
\end{array} \quad (9)$$

Then we let $Kan(X) = \text{colim}_{\beta \in \mathbb{N}} X^\beta$. The construction of $Kan(X)$ makes it clear that the morphism $X \to Kan(X)$ is functorial. However $Kan(X)$ is only a weak Kan simplicial manifold, whose definition is omitted here since we will not use it in this paper. But we still have

**Theorem 2.14.** [37, Thm. 3.6] If $W$ is Lie 2-groupoid, then $\tau_2(Kan(W))$ is a Lie 2-groupoid which is Morita equivalent to $W$.

Notice that given any finite simplicial set $A$ (both $\Lambda[k,j]$ and $\Delta[k]$ are such), the natural map of sets is an isomorphism

$$\text{colim}_\beta \text{hom}(A, X^\beta) \cong \text{hom}(A, Kan(X)). \quad (10)$$

We make some calculation for Kan replacement:

First of all, $X_0 = X_0^1 = X_0^2 = \cdots = Kan(X)_0$, and

$$X_1^1 = X_1 \sqcup (X_1 \times_{X_0} X_1)$$

$$X_1^2 = X_1^1 \sqcup X_1^1 \times_{X_0} X_1^1$$

$$= X_1^1 \sqcup (X_1 \times_{X_0} X_1 \sqcup X_1 \times_{X_0} X_1 \times_{X_0} (X_1 \times_{X_0} X_1)$$

$$\sqcup (X_1 \times_{X_0} X_1) \times_{X_0} X_1 \sqcup (X_1 \times_{X_0} X_1) \times_{X_0} X_1 \times_{X_0} (X_1 \times_{X_0} X_1)) \quad (11)$$

$$\vdots$$

$$Kan(X)_1 = X_1 \sqcup (X_1 \times_{X_0} X_1) \sqcup (X_1^1 \times_{X_0} X_1^1) \sqcup (X_1^2 \times_{X_0} X_1^2) \cdots,$$

which can be represented by the following picture:

$$Kan(X)_1 : \begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array} \cdots$$
A calculation shows that
\[
X_2^1 = X_2 \sqcup X_1 \times X_0 X_1 \sqcup X_1 \times X_0 X_1 \sqcup X_1 \times X_0 X_1 \\
\sqcup (\sqcup_{j=0}^{3} \text{hom}(\Lambda[3, j], X))
\]
\[
X_2^2 = X_2^1 \sqcup X_1^1 \times X_0 X_1^1 \sqcup X_1^1 \times X_0 X_1^1 \sqcup X_1^1 \times X_0 X_1^1 \\
\sqcup (\sqcup_{j=0}^{3} \text{hom}(\Lambda[3, j], X^1))
\]

Inside $X_2^1$, there are three copies of $X_1 \times X_0 X_1$. The first is an artificial filling of the horn $X_1 \times X_0 X_1$, and the second two are images of degeneracies of $X_1 \times X_0 X_1$ in $X_1^1$. The same for $X_2^2$, etc. We represent an element in $X_2^1$ as

\[
X_2 : \triangle, \quad X_1 \times X_0 X_1 : \quad \text{hom}(\Lambda(3, j), X) : \quad \triangle \ldots 4 \text{ such (12)}
\]

plus those degenerate ones in the other two copies of $X_1 \times X_0 X_1$. Thus further we represent a non-degenerate element in $X_2^2$ as

\[
X_2^2 : \text{described as above}
\]

\[
X_1^1 \times X_0 X_1^1 : \quad \text{hom}(\Lambda[3, j], X^1) : \quad \triangle \quad \triangle \quad \triangle \quad \triangle \ldots (13)
\]

3 The universal groupoid of an algebroid $A$

The first construction of universal groupoid of a Lie algebroid $A$ (even though not the one we use here) is contained in [10], where a topological groupoid (called Weinstein groupoid therein) is constructed via a certain path method which we will recall soon. This topological groupoid can be enriched with a differentiable stack structure [33] and the latter is the universal groupoid we use in this paper. We are going to show that this path method can be extended to higher levels, known as Sullivan’s construction. This gives a simplicial version of this universal stacky groupoid. Then we will also give a Lie 2-groupoid version and show the relation between all these versions.

3.1 The stacky groupoid version

We first recall [10] that an $A$-path of a Lie algebroid $A$ is a $C^1$ path $a(t) : \Delta^1 \to A$ with base map $\gamma(t) : \Delta^1 \to M$ satisfying

\[
\rho(a(t)) = \dot{\gamma}(t),
\]

where $\dot{\Box}$ denotes the derivative of $t$. If it further satisfies boundary condition $a(0) = a(1) = \dot{a}(0) = \dot{a}(1) = 0$, we call it an $A_0$-path as in [33]. This notation is useful for technical reasons
Banach manifold and $P$ (see Theorem 3.8). We call $P_aA$ and $P_0A$ the space of $A$-paths and $A_0$-paths respectively. By [10] Lemma 4.6] and a slightly modified argument (see also [32, Section 2]), $P_aA$ is a Banach manifold and $P_0A$ is a Banach submanifold of $P_aA$.

**Remark 3.1.** Since we choose $C^1$-paths, our Banach manifolds in this paper are all Hilbert manifolds, namely manifolds based on Hilbert open charts. In fact, as long as we take $C^r$-morphisms, and $r$ is finite, this holds always true [3 Chap.3.1]. However we keep on the terminology of Banach manifolds to match with the references such as [14, 16].

There is an equivalence relation in $P_aA$, called $A$-homotopy [10].

**Definition 3.2.** Let $a(t, s)$ be a $C^1$-family of $A$-paths. Assume that the base paths $\gamma(t, s) := \rho \circ a(t, s)$ have fixed end points. Fix a connection $\nabla$ on $A$. Two $A$-paths $a_0$ and $a_1$ are $A$-homotopic if there are $C^1$-morphisms $a(t, s)$, $b(t, s)$: $\Delta^1 \times \Delta^1 \to A$ satisfying

$$\partial_t b - \partial_s a = T_\nabla(a, b), \quad b(0, s) = 0, \quad b(1, s) = 0$$

Here $T_\nabla$ is the torsion of the connection defined by $T_\nabla(\alpha, \beta) = \nabla_{\rho(\beta)}\alpha - \nabla_{\rho(\alpha)}\beta + [\alpha, \beta]$. The restriction of an $A$-homotopy on $P_0A$ gives an $A$-homotopy of $A_0$-paths.

**Remark 3.3.** This definition is only apparently different from the one in [10] by Lemma 1.5 (iii) therein. We need this version since it coincides with the simplicial picture.

This flow of $A_0$-paths $a(\epsilon, t)$ generates a foliation $\mathcal{F}$ of codimension $\dim A$ on $P_0A$. The $A_0$-path space is a Banach submanifold of the $A$-path space and $\mathcal{F}$ is the restricted foliation of the foliation defined in [10 Section 4]. Sometimes to avoid dealing with infinite dimensional issues, we take an open cover $U_i$ of $P_0A$, and take $G_0(A)$ the disjoint union of slices $P_i \subset U_i$ that are transversal to the foliation $\mathcal{F}$. As the classic foliation theory tells us (see for example [21, Example 5.9]), $G_0(A)$ is a smooth immersion submanifold of $P_0A$, and $\mathcal{F}$ induces an étale foliation $\mathcal{F}_\epsilon$ on $G_0(A)$. Also recall that the monodromy groupoid assigned to a foliation is the groupoid whose space of objects is the manifold itself and whose space of arrows is given by leafwise paths modding out leafwise homotopies. We take the monodromy groupoid $G(A) := G_1(A) \to G_0(A)$ of $\mathcal{F}_\epsilon$. It is a finite dimensional étale Lie groupoid and it is the pull-back groupoid of the monodromy groupoid $\text{Mon}(P_0A) := \text{Mon}(P_0A) \to P_0A$ via $G_0(A) \to P_0A$. Moreover $\text{Mon}(P_0A)_1 \times_{P_0A} G_0(A) \to P_0A$ is surjective and étale, thus $G(A)$ and $\text{Mon}(P_0A)$ are Morita equivalent.

There are two maps $G_0(A) \to M$ assigning to a path its two end points. The quotient stack $\mathcal{G}(A)_1 := [G_0(A)/G_1(A)]$, which is presented by $G(A)$, is an étale differentiable stack. The two maps from $G_0(A)$ to $M$ descend to the quotient, giving two maps $s, t : \mathcal{G}(A) \to M$. We define a multiplication $m : \mathcal{G}(A)_1 \times_M \mathcal{G}(A)_1 \to \mathcal{G}(A)_1$ by concatenation $\odot$ of paths (here we need to use the Banach chart $P_0A$), namely,

$$ (a \odot b)(t) = \begin{cases} a(2t) & \text{when } t \in [0, 1/2], \\ b(2t - 1) & \text{when } t \in [1/2, 1]; \end{cases} $$

We define an inverse $i : \mathcal{G}(A)_1 \to \mathcal{G}(A)_1$, by reversing the orientation of a path; we define an identity section $e : M \to \mathcal{G}(A)_1$ by considering constant paths. These maps are defined in detail in [33]. There, we prove that these structure maps make $\mathcal{G}(A) := \mathcal{G}(A)_1 \to M$ into a $\mathcal{W}$-groupoid.

\footnote{Two $A$-paths being $A$-homotopic is in the end independent of the choice of connections.}
3.2 The simplicial version

Now to a Lie algebroid $A$ over a manifold $M$, we associate the simplicial set $S(A)$ as in \(1\) in the introduction. The first three layers of $S(A)$ are actually familiar to us:

- it is easy to check that $S_0 = M$;
- $S_1$ is exactly the $A_0$-path space $P_0A$ since a map $T\Delta^1 \to A$ can be written as $a(t)dt$ with base map $\gamma(t) : \Delta^1 \to M$. Its being a Lie algebroid map is equivalent to $\rho(a(t)) = \frac{d}{dt} \gamma(t)$ since the Lie bracket of $T\Delta^1$ is trivial and its anchor is the identity;
- bigons in $S_2$ are exactly the $A$-homotopies in $P_0A$ since a bigon $f : T(d_2^1)^{-1}(Ts_0^1(T\Delta^0)) \to A$ can be written as $a(t,s)dt + b(t,s)ds$ over the base map $\gamma(t,s)$ after a suitable choice of parametrization\(\footnote{We need the one that $\gamma(0,s) = x$ and $\gamma(1,s) = y$ being constant for all $s \in [0,1]$.}$\) of the disk $(d_2^1)^{-1}(s_0^1(\Delta^0))$. Then we naturally have $b(0,s) = f(0,s)(\frac{\partial}{\partial s}) = 0$ and $b(1,s) = f(1,s)(\frac{\partial}{\partial s}) = 0$. Moreover the morphism $f$ is a Lie algebroid morphism if and only if $a(t,s)$ and $b(t,s)$ satisfy equation \(\footnote{See Definition \(1.3\)}\) which defines the $A$-homotopy (see for example \(4\) Cor. 4.3).

Remark 3.4. We do not know whether $S(A)$ is a simplicial manifold or further a Kan simplicial manifold, namely a Lie $\infty$-groupoid as in Definition \(2.2\), though it is so for a Lie algebra \(14\). Unlike that $S_1(A)$ being a Banach manifold involves solving an ODE, it is not clear how to solve directly the corresponding PDE for $S_2(A)$ to be a Banach manifold. This is one of the open questions left at the end of \(14\).

However, although $S(A)$ has a clear geometric meaning, it involves infinite dimensional manifold, and although $G(A)$ is an étale SLie groupoid, the 2-truncation $\tau_2(S(A))$ will not be 2-étale\(\footnote{See Definition \(1.3\)}\). To achieve the étale version we need to use a sub-simplicial set based on the good étale covering $G_0(A)$ of $G(A)$ in Lemma \(3.7\) namely

$$
S_0^\text{e}(A) = M, \; S_1^\text{e}(A) = G_0(A), \; S_i^\text{e}(A) = S_i(A)|_{G_0(A)}, \; \text{for } i \geq 2,
$$

(17)

where

$$
S_i^\text{e}(A)|_{G_0(A)} := \{x \in S_i(A) : 1\text{-skeleton of } x \text{ is made up by elements in } G_0(A)\}.
$$

3.3 Lie 2-groupoid version

Now we look for the Lie 2-groupoid corresponding to $G(A) \Rightarrow M$. For this we need the notion of a good chart.

Definition 3.5. If a map $\tilde{f}$ from a manifold $N$ to an étale differentiable stack $\mathcal{X}$ with an étale chart $X_0$ comes from an embedding $f : N \to X_0$, we call $X_0$ a good étale chart and its corresponding groupoid presentation $G := G_1 \Rightarrow G_0$ a good étale groupoid presentation.

Then we have the following technical lemma (proved in \(38\) Section 3),

Lemma 3.6. Given an immersion $\tilde{e} : M \to \mathcal{G}$ from a manifold $M$ to an étale stack $\mathcal{G}$, there is a good étale chart $G_0$ of $\mathcal{G}$, i.e. $\tilde{e}$ can be lifted to an embedding $e : M \to G_0$. 
The étale chart $G_0(A) = \sqcup_i P_i$ of $G(A)$, made up by local transversals $P_i$ of the foliation $\mathcal{F}$ on $P_0A$, is usually not a good étale chart itself. But we have the following lemma:

**Lemma 3.7.** The étale chart $G_0(A) = \sqcup_i P_i$ of $G(A)$ can be completed to a good étale chart of $G(A)$.

**Proof.** We recall the construction of the local groupoid $G^{loc}(A)$ of $A$ in [10]. Take a small tubular neighborhood $\mathcal{O}$ of $M$ in the $A_0$-path space $P_0A$ so that the foliation $\mathcal{F}$ restricted to $\mathcal{O}$, which we denote by $\mathcal{F}|_\mathcal{O}$, has good transversal sections, that is, all the leaves in $\mathcal{F}|_\mathcal{O}$ intersect every transversal section only once. Take $\mathcal{O}_1 \subset \mathcal{O}$ a smaller tubular neighborhood of $M$ such that $\mathcal{O}_1 \times_M \mathcal{O}_1 \overset{\sim}{\rightarrow} \mathcal{O}$, and for all $a(t) \in \mathcal{O}_1$, the inverse path $a(1-t)$ is also in $\mathcal{O}_1$. Then the quotients $U := \mathcal{O}/(\mathcal{F}|_\mathcal{O})$ and $V := \mathcal{O}_1/(\mathcal{F}|_\mathcal{O})$ yield a local Lie groupoid over $M$.

The open set $\mathcal{O}/(\mathcal{F}|_\mathcal{O}) =: U$ can be also visualized by gluing $P_i|_\mathcal{O}$ together via the equivalence induced by $\mathcal{F}|_\mathcal{O}$ on $P_i|_\mathcal{O}$. Then, by definition, there is a natural embedding $U \rightarrow \sqcup_i P_i|_\mathcal{O} \rightarrow \sqcup_i P_i$. Composing with the representable submersion $\sqcup_i P_i \rightarrow G$, we obtain a representable submersion $U \rightarrow G$.

Although $G_0(A)$ is not necessarily a good chart, we join $U$ to $G_0(A)$, that is, we redefine $G_0(A) := U \sqcup (\sqcup_i P_i)$. Then $G_0(A)$ is still a chart of $G(A)$ since $U \rightarrow G$ is a representable submersion. Thus $G_0(A)$ becomes good and the étale groupoid $G_1(A) := G_0(A) \times_G(A)$ $G_0(A) \Rightarrow G_0(A)$ via $G_0(A) \rightarrow G(A)$ is a good étale groupoid presentation. □

To avoid redundant notation, from now on $G_0(A)$ denotes this completed good étale chart of $G(A)$ and $G(A)$ this good étale groupoid presentation. Then the multiplication bimodule $E_m^{G(A)}$ of $G(A)$ is

$$E_m^{G(A)} = (G_0(A) \times_M G_0(A)) \times_{\sqcup_i P_0A} Mon(P_0A)_1 \times_{s,P_0A,i} G_0(A),$$

with $G_0(A) \times_M G_0(A) \overset{\circ}{\rightarrow} P_0A$ given by $(a, b) \mapsto a \circ b$ as in [10] and the immersion $G_0(A) \overset{\rightarrow}{\rightarrow} P_0A$. Since $Mon(P_0A)_1 \times_{P_0A} G_0(A) \rightarrow P_0A$ is surjective and étale (see [23], Example 5.9), we verify directly that $E_m^{G(A)}$ is a manifold. Thus by the 1-1 correspondence in [10],

$$\Gamma(A) : E_m^{G(A)} \xrightarrow{\sim} G_0(A) \rightarrow M$$

is a Lie 2-groupoid. We call $\Gamma(A)$ the universal Lie 2-groupoid integrating $A$. Notice that different choices of $G_0(A)$ give the same Lie 2-groupoid up to 1-Morita equivalence (see [38], Prop.-Def.2.13).

There is also a natural infinite dimensional Lie 2-groupoid corresponding to $G(A) \Rightarrow M$ obtained by using the chart $P_0A$,

$$Y(A) : E_m \xrightarrow{\sim} P_0A \rightarrow M,$$

where $E_m = (P_0A \times_M P_0A) \times_{\sqcup_i P_0A,i} Mon(\mathcal{F})(P_0A)$ is the multiplication bimodule of $G(A)$ with respect to the chart $P_0A$. In fact [10] is simply the pullback 2-groupoid of [19] via

---

9By [16] Chap.VII.4, a tubular neighborhood can be taken isomorphic to a vector bundle over the submanifold if the submanifold is paracompact and the ambient manifold is a Hilbert manifold, which is our case (see Remark 3.1).
the map $G_0(A) \xrightarrow{i} P_0A$. Even though $G_0(A) \xrightarrow{i} P_0A$ is not a surjective submersion, these two 2-groupoids are Morita equivalent because both of them correspond to the same stacky groupoid $\mathcal{G}(A) \Rightarrow M$.

Then Theorem 1.2 is equivalent to the following:

**Theorem 3.8.** The 2-truncation $\tau_2(S^2(A))$ is exactly the 2-étale Lie 2-groupoid $\Gamma(A)$ defined in [18], and the 2-truncation $\tau_2(S(A))$ is exactly the Lie 2-groupoid $Y(A)$ defined in [19]. Hence both $\tau_2(S^2(A))$ and $\tau_2(S(A))$ corresponds to $\mathcal{G}(A)$ with the correspondence in [38].

**Proof.** To prove $\tau_2(S^2(A)) \cong \Gamma(A)$, we only need to show that $\tau_2(S^2(A))_2 \cong E^G_m(A)$ and the rest is easy to check. Thanks to the boundary condition in the definition of $P_0A$, $Mon(P_0A)_1$ is a quotient of the space of bigons in $S_2(A)$. Thus we have,

$$E^G_m(A) = S^2_2(A)/\sim,$$

where $a \sim b$ in $S^2_2(A)$ if and only if there is a path $a(\epsilon)$ in $S^2_2(A)$ with fixed boundary, i.e. $\partial(a(\epsilon)) = \partial a = \partial b$, such that $a(0) = a$ and $a(1) = b$. By [4] Lemma 4.5], a path of $A$-homotopies with fixed boundary can be connected to make a Lie algebroid morphism $T\Delta^3 \rightarrow A$ in a unique way. Thus what we mod out in (20) is exactly an element in $S_2(A)$. Hence $\tau_2(S^2(A)) \cong E^G_m(A)$. The proof of the statement $\tau_2(S(A)) \cong Y(A)$ is similar. 

4 Lie II theorem

In this section we prove Theorem 1.3. The major tool is the adaptation of Quillen’s small object argument for simplicial manifolds. It is necessary to demonstrate this since simplicial manifolds do not form a model category.

4.1 Universal Lie 2-groupoids via local groupoids

Although there is no one-to-one correspondence of Lie algebroids and Lie groupoids, Lie algebroids and local Lie groupoids do have a one-to-one correspondence. Given a Lie algebroid $A$, its corresponding local Lie groupoid $G^{loc}(A)$ is constructed explicitly as in Lemma 3.7. Then we naturally have a 2-groupoid $\tau_2(Kan(NG^{loc}(A)))$, and we prove:

**Proposition 4.1.** Given a Lie algebroid $A$, $\tau_2(Kan(NG^{loc}(A)))$ is a Lie 2-groupoid and is Morita equivalent to the universal Lie 2-groupoid $\Gamma(A)$ associated to $A$.

We prove this proposition via infinite dimensional spaces because there we can to a certain extent avoid the problem of the “non-existing strict morphism” (see [37 Section 3.2]) and simplify considerably the proof.

We use the same notation as in Lemma 3.7 and we take $X(A)$ to be the pull-back simplicial manifold of $NG^{loc}(A)$ along the surjective submersion $O_1 \rightarrow V$. Then by construction, $X(A)$ satisfies properties A and B. Since $X(A)_1$ contains small paths and $X(A)_2$ contains small triangles, $X(A)_n$ is a neighborhood of the identities $Y_0(A) = M$ in $Y(A)_n$. Thus $X(A)$ can be viewed as a local part of $Y(A)$.

\[^{10}\text{see Definition 4.3}\]
Lemma 4.2. The 2-truncation $\tau_2(Kan(X(A)))$ is a Lie 2-groupoid and is Morita equivalent to $Y(A)$.

Proof. To simplify the notation, we omit "$(A)$" in the proof of this Lemma.

We first construct a morphism $f' : Kan(Y) \to Y$ inductively thanks to [10]. The construction is similar to the construction in [37, Theorem 3.6]. There we could not construct a global map. However here we are able to do so because we can choose a global section of $\text{hom}(\Lambda[2,1],Y) \to Y_2$, for example the one given by the "flat" triangle of concatenation [10],

$$P_0A \times_M P_0A \to \text{hom}(T\Delta_2,A), \quad (a,b) \mapsto \text{the flat triangle whose three sides are } a, b, a \odot b. \quad (21)$$

Clearly there is an embedding $X \hookrightarrow Y$, hence a map $Kan(X) \to Kan(Y)$. Thus we have a morphism $f : Kan(X) \to Y$.

We do not have a hypercover directly from $\tau_2(Kan(X))$ to $Y$ because $Kan(X)_1 \to Y_1$ is not surjective. Thus we will use Lemma [2,11] and verify that the conditions in Remark [2,12] are satisfied.

Given a triangle, we add a point and divide it into three triangles. Among these three triangles, one can be a weird one in $X_1 \times X_0 X_1$ as [12] shows. A triangulation is good if it is made of a sequence of such divisions. Thus, more geometrically, an element in $Kan(X)_2$ is a set of small triangles of $X_2$ matching together in a good triangulation situation. Then $f_1 : Kan(X)_1 \to Y_1$ operates by concatenating small paths into a long path and $f_2$ by composing small triangles into a big one in $Y_2$ with the flat filling of $X_1 \times X_0 X_1$ as in [21]. The embedding $f_1|_{X_1} : X_1 \to Y_1$ is a submersion in particular.

Recall that we need an intermediate $Z$ with $Z_0 = M$ and

$$Z_1 = \text{hom}(\Lambda[2,1], Kan(X)) \times_{\text{hom}(\Lambda[2,1],Y)} Y_2 \sqcup X_1 \times f_1 Y_1, d_0 b(Y_2)$$

The map $h'_1 : \text{hom}(\Lambda[2,1], Kan(X)) \times_{\text{hom}(\Lambda[2,1],Y)} Y_2 \to Y_1$ is surjective because any $A_0$-path $a_1 \in Y_1$ is $A$-homotopic to a concatenation of small paths in $X_1$. More precisely, there are elements $a_2, a_0 \in Kan(X)_1$ such that $f_1(a_0) \odot f_1(a_2) = f_1(a_1)$. Thus $A$-homotopy gives an element $\eta \in Y_2$ with three sides $f_1(a_0), a_1, f_1(a_2)$ respectively. Thus $h'_1(a_0, a_2, \eta) = a_1$. Notice that given a small path $a^\delta_1$ around $a_1$, the choice of $(a^\delta_0, a^\delta_2, \eta^\delta)$ can be made smoothly depending on $s$. Thus $h'_1$ is also a submersion.

The morphism $\mu : \text{hom}(\Lambda[2,1], Kan(X)) \to Kan(X)_1$ is obtained by taking the limit of the natural embedding $X_1^\beta \times X_0 X_1^\beta \to X_1^{\beta+1}$. It is clear that $\mu$ is a submersion, thus the composed map $g'_1$ is a submersion since the Kan projection $Y_2 \to \text{hom}(\Lambda[2,1],Y)$ is a submersion. Then joining with $pr_{X_1}$, the map $g_1 = g'_1 \sqcup pr_{X_1} : Z_1 \to Kan(X)_1$ is a surjective submersion.

Now we need to verify that the natural map obtained by composing small triangles via the Kan condition

$$Kan(X)_2 \to \text{hom}(\partial\Delta[2], Kan(X)) \times_{f, \text{hom}(\partial\Delta[2],Y)} Y_2,$$

induces an isomorphism when passing to the quotient of the 2-truncation. Then $\tau_2(Kan(X))$ is a Lie 2-groupoid and Morita equivalent to $Y$.

Thanks to the weird triangle in [12], we have a sort of division as in the second group in $\text{hom}(\Lambda[3,j],X^1)$ in [13]. After several times, we can always end up with small triangles.
with at most $\frac{1}{2}$ perimeter of the original one. Hence any $y \in Y_2$ can be divided into small triangles that lie in $X_2$. Thus the map $\phi$ is surjective up to the homotopy $\sim_2$ given by elements in $Y_3$.

On the other hand, suppose two relatively homotopic elements $(\partial z, y), (\partial z', y') \in \text{hom}(\partial \Delta[2], \text{Kan}(X)) \times \text{hom}(\partial \Delta[2], \gamma)$ $Y_2$ have preimages $x, x'$, i.e. two triangulations $x, x'$ of two homotopic triangles $\eta, \eta' \in \text{hom}(T \Delta_2, A)$ sharing the same division of the border. Since $\eta$ and $\eta'$ can be connected by a path $\eta_t \in \text{hom}(T \Delta_2, A)$, we might assume that this path is a small path and we can deform $x'$ to make it also a triangulation of $\eta$. Then $x$ and $x'$ have a common subtriangulation $\tilde{x}$ and $\tilde{x}'$ (though with a possibly different order of composition) which has the same division of the border. However different order of composition differs by an element in $\text{Kan}(X)_3$. A triangulation and its subtriangulation sharing the same border also differ by an element in $\text{Kan}(X)_3$. Hence $\phi$ is injective up to the homotopy $\sim_2$ given by elements in $\text{Kan}(X)_3$. 

\begin{proof} [Proof of Prop. 4.1] We take the Lie 2-groupoid $Y(A)$ and $X(A)$ as above. Since $X(A)$ is the pull-back of $\text{NG}^{\text{loc}}(A)$ by the surjective submersion $O_1 \to V$, the map $X(A) \to \text{NG}^{\text{loc}}(A)$ is a hypercover. Hence $\tau_2(\text{Kan}(X(A))) \to \tau_2(\text{Kan}(\text{NG}^{\text{loc}}(A)))$ is a hypercover by [37, Thm. 3.11]. Thus we have a composed Morita equivalence,

$$\Gamma(A) \xrightarrow{\sim} Y(A) \xrightarrow{\sim} \tau_2(\text{Kan}(X(A))) \xrightarrow{\sim} \tau_2(\text{Kan}(\text{NG}^{\text{loc}}(A))),$$

by Remark [2, 3] and Lemma [4, 2]
\end{proof}

4.2 Differentiating back to Lie algebroids

**Definition 4.3.** As in [38], we call a Lie 2-groupoid $X$ 2-étale if the Kan projections $X_2 \to \text{hom}(\Lambda[2, j], X)$ is étale for $j = 0, 1, 2$.

In a 2-étale Lie 2-groupoid, $X_0$ is an embedding submanifold of $X_1$ and $X_2$ via compositions of degeneracy maps. Moreover $\text{Kan}(2, 1)$ tells us that $X_2 \to X_1 \times_{d_0, X_0, d_2} X_1$ is étale. Hence there is a tubular neighborhood $V \subset X_1$ of $s_0 : X_0 \to X_1$ such that there is a unique section $\sigma_m : V \times_{d_0, X_0, d_2} V \to X_2$ extending the section given by the degeneracy map $X_0 \times_{d_0, X_0, d_2} X_0 \to X_2$. The image $d_1 \circ \sigma_m(V \times_{d_0, X_0, d_2} X_0)$ equals $V$, hence if we take $V$ small enough, we have another open neighborhood $U$ of $X_0$ in $X_1$ satisfying $V \subset d_1 \circ \sigma_m(V \times_{d_0, X_0, d_2} V) \subset U$. We define

$$m = d_1 \circ \sigma_m : V \times_{d_0, X_0, d_2} V \to U.$$

The identity embedding $X_0 \to U$ is given by $s_0$.

Finally, $\text{Kan}(2, 0)$ tells us that $X_2 \to X_1 \times_{d_2, X_0, d_1} X_1$ is étale. Hence there is a unique section $\sigma_i : U \times_{d_2, X_0, d_1} X_0 \to X_2$ extending $X_0 \times_{d_2, X_0, d_1} X_0 \to X_2$. We take $U$ small enough (for example $U \cap d_1 \circ \sigma_i(U \times_{X_0} X_0)$), and define the composed map

$$i : U \xrightarrow{id \times d_1} U \times_{d_2, X_0, d_1} X_0 \to X_2 \xrightarrow{\sigma_i} U$$

to be the inverse map. Then $m(i(i(g), g) = 1$ and $m(g, i(g)) = 1$ by construction.

Then the uniqueness of the sections $\sigma_m$ and $\sigma_i$ gives us strict Kan condition $\text{Kan}(2, j)!$ locally. Hence $V \subset U \Rightarrow X_0$ with the above structure maps is a local Lie groupoid. The
obtain a Lie 2-groupoid morphism \( \Phi : \Gamma \rightarrow X \). By construction \((NX^{loc})_n\) is a tubular neighborhood of the identity \(X_0\) in \(X_n\). Thus we naturally have an inclusion \(NX^{loc} \subset X\) of simplicial manifolds.

Similarly given a W-groupoid \(\mathcal{G} \Rightarrow M\) where \(\mathcal{G}\) is presented by \(G_1 \Rightarrow G_0\), there are neighborhoods \(V \subset U \subset G_0\) of \(M\) such that all the groupoid structure maps descend to \((U, V) \Rightarrow M\) and make \((U, V) \Rightarrow M\) a local Lie groupoid [33 Section 5]. We call it the local Lie groupoid associated to \(\mathcal{G} \Rightarrow M\) and denote it by \(G^{loc}\).

The Lie algebroid of \(X^{loc}\) (respectively \(G^{loc}\)) is defined to be the Lie algebroid of the 2-étale Lie 2-groupoid \(X\) (respectively W-groupoid \(\mathcal{G}\)).

**Example 4.4.** If we take the universal 2-étale Lie 2-groupoid \(\Gamma(A)\), the local Lie groupoid associated to it is exactly \(G^{loc}(A)\).

A W-groupoid morphism \(\Phi : (\mathcal{G} \Rightarrow M) \rightarrow (\mathcal{H} \Rightarrow N)\) is made up by a map \(\Phi_1 : \mathcal{G} \rightarrow \mathcal{H}\) between stacks and by a map \(\Phi_0 : M \rightarrow N\) such that they preserve the W-groupoid structure maps up to 2-morphisms, which should satisfy again higher coherence conditions linking the 2-commutative diagrams of \(\mathcal{G}\) and \(\mathcal{H}\) (also see [33 Section 4]). Such a morphism, in the world of 2-étale Lie 2-groupoids, corresponds to a generalized morphism \(\Psi : X \rightarrow Y\) via \(Z\) such that the morphism \(Z \rightarrow X\) makes \(Z_0 \cong X_0\). We call such generalized morphisms special generalized morphisms. Now we differentiate this generalized morphism to a Lie algebroid morphism.

First we notice that \(Z_1 \rightarrow X_1\) is a surjective submersion. Thus we choose a local section \(\sigma_1\) of \(Z_1 \rightarrow X_1\) from a small neighborhood of \(X_0\) in \(X_1\) extending the isomorphism \(X_0 \rightarrow Z_0\). Then we obtain a local morphism \(\sigma_2 : X_2^{loc} \rightarrow Z_2\) by \(x_2 \mapsto (x_2, \sigma_1(d_0(x_2)), \sigma_1(d_1(x_2)), \sigma_1(d_2(x_2)))\) since \(Z_2 \cong X_2 \times_{\text{hom}(\partial \Delta [2], X)} \text{hom}(\partial \Delta [2], Z)\). Similarly we obtain higher \(\sigma_i : X_i^{loc} \rightarrow Z_i\). It is easy to see that with \(\sigma_0 : X_0 \rightarrow Z_0\) as the isomorphism, \(\sigma : NX^{loc} \rightarrow NZ^{loc}\) is a simplicial morphism. Thus the composed simplicial morphism \(NX^{loc} \rightarrow NZ^{loc} \rightarrow NY^{loc}\) gives a local group morphism \(\phi : X^{loc} \rightarrow Y^{loc}\) since the local groupoid structure is determined by the simplicial manifold structure of the nerve. Then differentiating \(\phi\) gives us a Lie algebroid morphism \(A \rightarrow B\) where \(A\) and \(B\) are the Lie algebroid of \(X\) and \(Y\) respectively. We call \(\Psi\), or the corresponding W-groupoid morphism \(\Phi\), the integration of \(\phi\).

**Lemma 4.5.** If \(\phi\) is a Lie algebroid morphism \(A \rightarrow B\), then it induces a Lie 2-groupoid morphism \(\Phi : \Gamma(A) \rightarrow \Gamma(B)\) integrating the Lie algebroid morphism \(\phi\).

**Proof.** A Lie algebroid morphism \(\phi : A \rightarrow B\) induces a simplicial morphism \(\phi_i : S(A)_i \rightarrow S(B)_i\). Thus it gives a Lie 2-groupoid morphism on the 2-truncations. By Theorem 3.8 we obtain a Lie 2-groupoid morphism \(\Phi : \Gamma(A) \rightarrow \Gamma(B)\). Then by what we discuss above, \(\Phi\) gives a local groupoid morphism \(\Phi^{loc} : G^{loc}(A) \rightarrow G^{loc}(B)\) which maps equivalence classes of \(A\)-paths in \(A\) to those in \(B\) via \(\phi\). Note that the local groupoids can be understood as equivalence classes of \(A\)-paths as explained in Lemma 4.2. Therefore the corresponding Lie algebroid map is exactly \(\phi\). \(\square\)

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11 Thanks to Lemma 3.6 we can construct the local groupoid at once and it is not necessary to divide \(M\) into pieces as it is done in the cited reference.
By the one-to-one correspondence between the $W$-groupoid and 2-étale Lie 2-groupoid, Theorem 1.3 is equivalent to the following theorem:

**Theorem 4.6.** Given a Lie algebroid morphism $A \xrightarrow{\phi} B$, let $G(A)$ be the universal Lie 2-groupoid of $A$, and $W$ any 2-étale Lie 2-groupoid corresponding to $B$. Then there is a special generalized morphism $\Psi : G(A) \rightarrow W$ integrating $\phi$. This generalized morphism is unique up to 2-morphisms.

**Proof.** To build the map $\Psi$, by Lemma 4.3 we only have to treat the situation when $\phi = id : A \rightarrow A$, that is, if a Lie 2-groupoid $W$ has algebroid $A$, then there is a generalized morphism $\Psi : \Gamma(A) \rightarrow W$ integrating $id : A \rightarrow A$. Then $W^{loc}$ is the local Lie groupoid of $A$ since the Lie algebroid corresponding to $W$ is $A$. Thus we have an embedding of simplicial manifolds $NG^{loc}(A) = NW^{loc} \subset W$. By Prop. 1.1 and Thm. 2.14 we obtain a composed generalized morphism
\[
\Psi : \Gamma(A) \xrightarrow{\sim} \tau_2(Kan(NG^{loc}(A))) \rightarrow \tau_2(Kan(W)) \xrightarrow{\sim} \tau_2(W) = W.
\]
Locally $\Psi$ is the identification map $G^{loc}(A) \rightarrow W^{loc}$, thus it integrates $id : A \rightarrow A$. Different choices of $G^{loc}(A)$ give us the same generalized morphism up to a 2-morphism.

On the other hand, if we have two special generalized morphisms $\Psi, \Psi' : \Gamma(A) \rightarrow H$ integrating the Lie algebroid morphism $\phi : A \rightarrow B$, via $Z$ and $Z'$, then locally we can invert the projections $Z \rightarrow \Gamma(A)$ and $Z' \rightarrow \Gamma(A)$ and get two strict morphisms $\Psi^{loc}, \Psi'^{loc} : NG^{loc}(A) \rightarrow H$. By choosing $G^{loc}(A)$ close enough to the identity section, $\Psi^{loc} = \Psi'^{loc}$ because they both integrate the Lie algebroid morphism $\phi : A \rightarrow B$. Then since the local morphism determines the global generalized morphism,
\[
\Gamma(A) \xrightarrow{\sim} \tau_2(Kan(G^{loc}(A))) \xrightarrow{\tau_2(Kan(\Psi^{loc})) = \tau_2(Kan(\Psi'^{loc}))} \tau_2(Kan(H)) \xrightarrow{\sim} \tau_2(H) = H,
\]
so we must have $\Psi = \Psi'$ up to a 2-morphism. \qed

## 5 Connectedness

In this section, we will show that $Y(A)$ and $G(A)$ are source-2-connected, that is, their source fibres have trivial homotopy groups $\pi_0, \pi_1$ and $\pi_2$. The idea is that the $n$-truncation of $S(A)$ is the universal object integrating $A$ in the world of differentiable $(n - 1)$-stacks or equivalently Lie $(n - 1)$-groupoids. On the other hand, if we take a $k$-connected cover of this $(n - 1)$-stack for $k = 1, \ldots, n$, the cover is also a universal object. Hence we must have:

*The universal object in differentiable $(n - 1)$-stacks has to be $n$-connected, by the same argument according to which the universal group for a Lie algebra is 1-connected.*

But to make this into a proper proof, to construct this $k$-connected cover, we must be equipped with homotopy theory for Lie $n$-groupoids or differentiable $n$-stacks. Notice that even for a manifold $M$, its $k$-connected cover might be a higher stack (see Section 6). Since in this article we are aiming at 2-groupoids, we give a direct proof of the 2-connectedness using the above idea. We also give some examples of 2-connected covers in the example section. In this method, we view the universal integrating object as a $(1)$-stack with an additional groupoid structure. Thus it is more natural to use the stacky Lie groupoid model for the universal object.

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Homotopy groups of simplicial schemes \[11\], simplicial sets \[20\], orbifolds \[5\], and topological stacks \[25\] are well-known or have been recently studied. They have properties that parallel those of the usual homotopy groups of topological spaces; for example, \(\pi_n\) is a group. Moreover, we believe that these theories are all the same under suitable equivalences, but sometimes not the obvious one (see \[11\]). For example, in the sense of \[20\], the simplicial homotopy groups \(\pi^n_0\) of the 2-truncation of \(S(A)\) is automatically trivial by definition when \(n \geq 2\). But we will show in Remark \[6.3\] \(\pi_2(\mathcal{G}(TS^2)) \neq 0\) with \(\pi_2\) defined in Definition \[5.1\]. Indeed they are two different though related types of homotopy groups if applied to simplicial manifolds. Here we give a version of homotopy groups of differentiable stacks which generalizes that of orbifolds in \[5\].

As before we denote 2-morphisms as \(\sim\).

**Definition 5.1.** At a point \(x : pt \to \mathcal{X}\), the \(n\)-th homotopy group is defined as,

\[
\pi_n(\mathcal{X}, x) = \{f : S^n \to \mathcal{X}, \text{ such that } f|_N \sim i_x\}/\text{homotopies},
\]

here \(i_x\) is the morphism from the north pole \(N = pt \xrightarrow{s} \mathcal{X}\), and \(f_0\) is homotopic to \(f_1\) if and only if there is \(F : S^n \times [0, 1] \to \mathcal{X}\) such that

\[
F(N, s) \sim i_x, F(\cdot, 0) \sim f_0, F(\cdot, 1) \sim f_1.
\]

Since in the main results of this paper, we are using nothing more than the definition of the homotopy groups, we omit a detailed discussion of homotopy groups.

Source connectedness for \(\mathcal{G}(A)\), i.e. \(\pi_0(s^{-1}(x), x) = 0\) is easy to see. It is implied by the fact that every \(A_0\) path \(a(t)\) can be connected to a constant path \(0_x\) via \(sa(st)\) with \(s \in [0, 1]\). Here \(x = a(0)\). Now we set off to prove the less trivial part about \(\pi_1\) and \(\pi_2\).

We recall how we prove that the universal Lie group \(G(\mathfrak{g}) = P_0\mathfrak{g}/\sim\) of a Lie algebra \(\mathfrak{g}\), constructed by \(A\)-paths modding out \(A\)-homotopies in \(\mathfrak{g}\), is simply connected. We use only Lie II theorem but without a covering space.

Given a loop \(S^1 \xrightarrow{\varphi} G\), we have a groupoid morphism \(P(S^1) \xrightarrow{\Phi_{\text{gpd}}} G\) given by \(\Phi_{\text{gpd}}(c, d) = \varphi(c)\varphi(d)^{-1}\). Here \(P(\square)\) denotes the pair groupoid \(\square \times \square \Rightarrow \square\). We use this notation from now on. The corresponding Lie algebroid morphism \(TS^1 \xrightarrow{\phi_{\text{algd}}} \mathfrak{g}\) is in fact a \(\mathfrak{g}\)-loop \(\phi_{\text{algd}} : \varphi'(t)\varphi(t)^{-1}\). By re-parametrization, we might as well assume that \(\varphi([t]) = 0\), when \(t = 0, 1\). Here we view \(S^1 = \mathbb{R}/\mathbb{Z}\) and \([\cdot]\) denotes an equivalence class. Then \(\phi_{\text{algd}}\) is in \(P_0\mathfrak{g}\), and \(\varphi\) is contractible if and only if \(\phi_{\text{algd}}\) is contractible, i.e. \(\phi_{\text{algd}} \sim 0\) in the quotient \(G(\mathfrak{g}) = P_0\mathfrak{g}/\sim\). In fact, if \(\phi_{\text{algd}} \sim 0\) via a \(\mathfrak{g}\)-homotopy \(F_\Phi : TD^2 \to \mathfrak{g}\), we integrate it to a homotopy \(F_\Phi : P(D^2) \to G\). Then \(F_\Phi\) restricted to one copy of \(D^2\) gives the homotopy which contracts \(\varphi\).

To prove \(\phi_{\text{algd}} \sim 0\), we use the fact that \(\phi_{\text{algd}}\) is a \(\mathfrak{g}\)-path coming from the groupoid morphism from the pair groupoid \(P(S^1) \xrightarrow{\Phi_{\text{gpd}}} G\), with \(\Phi_{\text{gpd}}(c, d) = \varphi(c)\varphi(d)^{-1}\). More precisely, the integration morphism \(\tilde{\Phi}_{\text{gpd}}\) of \(\phi_{\text{algd}}\) factors through \(P(S^1)\),

\[
\tilde{\Phi}_{\text{gpd}} : G(TS^1) = (\mathbb{R} \times \mathbb{R}/\mathbb{Z} \Rightarrow S^1) \xrightarrow{\text{pr}} P(S^1) \xrightarrow{\Phi_{\text{gpd}}} G,
\]

with every arrow a strict groupoid morphism. This is because the infinitesimal morphism of both \(\tilde{\Phi}_{\text{gpd}}\) and \(\Phi_{\text{gpd}}\) is \(\phi_{\text{algd}}\). (Here we use Lie II theorem).
Taking \([0, 1]\) and \([0, 0]\) in \(\mathbb{R} \times \mathbb{R}/\mathbb{Z}\), we have
\[
\tilde{\Phi}_{gpd}([0, 1]) = \tilde{\Phi}_{gpd}([0, 0]),
\]
(22)
because \(pr([0, 0]) = pr([0, 1]) = ([0], [0]) \in S^1 \times S^1 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\). Here \([\cdot]\) denotes equivalence classes in various quotients. We define maps \(\gamma_{0,1,2} : S^1 \to S^1\) by \(\gamma_0([t]) = [0]\), \(\gamma_1([t]) = [\tau(t)]\), and \(\gamma_2([t]) = [t]\), with \(t \in [0, 1]\) and
\[
\tau : [0, 1] \to [0, 1], \quad \text{which satisfies } \tau'(0) = \tau'(1) = 0, \tau'(t) \geq 0.
\]
(23)
As \(A\)-paths, \(T\gamma_1, T\gamma_2 : TS^1 \to TS^1\) differ by a re-parametrization \(\tau\). In particular, \(T\gamma_1\) and \(T\gamma_2\) are \(A\)-homotopic. Thus \(T\gamma_1\) represents \([0, 1]\) and \(T\gamma_0\) represents \([0, 0]\) in \(\mathcal{G}(TS^1)\) respectively. Then \(\Phi_{gpd}\) tells us that, on the algebroid level, \(\phi_{algd} \circ T\gamma_1 \sim \phi_{algd} \circ T\gamma_0\) are two equivalent \(g\)-paths.

But it is clear that \(\phi_{algd} \circ T\gamma_1 = \phi_{algd}\) and \(\phi \circ T\gamma_0 = 0\) the 0-path. Hence \(\phi_{algd} \sim 0\).

Now we use the same idea to prove our theorem.

\textbf{Proof of Theorem 1.1.} It is almost the same proof to show that \(\mathcal{G}(A)\) has simply connected source fibre \(s^{-1}(x)\) for all \(x \in M\). We only have to replace \(g\) by \(A\) and insert the base point \(x\). We omit the proof here.

We focus on adapting the proof for the 2-connectedness. Given \(x \in M\), let \(\varphi\) be a smooth map \((S^2, N) \to (s^{-1}(x), x)\). Here \(x : pt \to s^{-1}(x)\) denotes also the base point. Then similarly, we form a \(W\)-groupoid morphism \(\Phi_{gpd} : P(S^2) \to \mathcal{G}(A)\) by \(\Phi_{gpd}(c, d) = \varphi(c)\varphi^{-1}(d)\) with the base map \(\Phi_{gpd}^0(c) = t(\varphi(c))\). Then \(\Phi_{gpd}\) induces a Lie algebroid morphism \(\phi_{algd} : TS^2 \to A\).

For any Lie algebroid \(A\), there is another stacky Lie groupoid \(\mathcal{H}(A) := \mathcal{H}(A)_1 \Rightarrow M\) with arrow space \(\mathcal{H}(A)_1\) presented by the holonomy groupoid \(Hol(P_0A) \Rightarrow P_0A\) of the same foliation. By \[33\] Theorem 1.3, when a Lie algebroid \(A\) is integrable to a source simply-connected Lie groupoid \(G\), \(\mathcal{H}(A)\) is simply \(G\). Since the source simply-connected Lie groupoid of \(TS^2\) is \(P(S^2)\), the stacky Lie groupoid \(\mathcal{H}(TS^2)\) is this Lie groupoid \(P(S^2)\).

Then the projection \(Mon(P_0TS^2) \to Hol(P_0TS^2)\) induces a stacky groupoid morphism \(pr : \mathcal{G}(TS^2) \to P(S^2)\) whose corresponding infinitesimal morphism is \(id : TS^2 \to TS^2\). The composed stacky groupoid morphism
\[
\tilde{\Phi}_{gpd} : \mathcal{G}(TS^2) \xrightarrow{pr} P(S^2) \xrightarrow{\Phi_{gpd}} \mathcal{G}(A),
\]
still corresponds infinitesimally to \(\phi_{algd}\). Thus \(\phi_{algd}\) integrates to a \(W\)-groupoid morphism \(\Phi_{gpd} : \mathcal{G}(TS^2) \to \mathcal{G}(A)\) factoring through the pair groupoid \(P(S^2)\).

Then we have a 2-commutative diagram
\[
\begin{array}{ccc}
P_0TS^2 & \xrightarrow{\phi_1} & P_0A, \\
\downarrow & & \downarrow \\
\mathcal{G}(TS^2) & \xrightarrow{pr} & P(S^2) \xrightarrow{\Phi_{gpd}} \mathcal{G}(A)
\end{array}
\]

where \(\phi_1\) is the natural map induced by \(\phi_{algd}\) between the path spaces. The triangle 2-commutes by the definition of \(pr\). The big square 2-commutes since \(\Phi_{gpd}\) has its infinitesimal morphism \(\phi_{algd}\). Thus the right tetrahedron 2-commutes.
By Lemma 5.2, $\Phi_{gpd}$ is presented by a strict groupoid morphism $\phi_{gpd} : Hol(P_0T^2S) = P_0T^2S \times_{S^2 \times S^2} P_0T^2 = \Phi_{gpd}\phi_1 \to P_0A \times_\mathcal{G}(A) P_0A = Mon(P_0A)$. Thus on the level of Lie groupoids presenting the differentiable stacks of the arrow space, we have strict maps of Lie groupoids,

$$\tilde{\phi}_{gpd} : Mon(P_0T^2S) \xrightarrow{pr} Hol(P_0T^2S) \xrightarrow{\phi_{gpd}} Mon(P_0A).$$

A Lie algebroid morphism $T^2S \to A$ is an $A$-homotopy such that it gives a loop in $P_0A$ along the foliation $\mathcal{F}$ (see Section 3.1). Thus a re-parametrization$^{12}$ of the map $id : T^2S \to T^2S$ gives a loop in $P_0A$ based at the constant path $0_N$, thus representing an element $\epsilon \in Mon(P_0T^2S)$. However $pr(\epsilon) = 1_{0_N} \in Hol(P_0T^2S)$ because any groupoid Morita equivalent to a manifold must have trivial isotropy groups. Therefore

$$\tilde{\phi}_{gpd}(\epsilon) = \phi_{gpd}(1_{0_N}) = 1_{\phi_1(0_N)} = 1_{0_s}.$$

Notice that $\tilde{\phi}_{gpd}(\epsilon) \in Mon(P_0A)$ is represented by a re-parametrization of the $A$-homotopy $\phi_{algd} : T^2S \to A$, and $1_{0_s}$ is represented by the constant $A$-homotopy $T^2S \overset{0_s}{\to} A$. A re-parametrization of $\phi_{algd}$ is connected to $\phi_{algd}$ by a path of $A$-homotopies because a representation of any element in $S_n(A)$ is homotopic to itself via an element in $S_{n+1}(A)$ [4, Remark 3.6]. What we mod out to form the monodromy groupoid $Mon(P_0A)$ is paths of $A$-homotopies. By [4, Lemma 4.5], a path of $A$-homotopies between $\phi_{algd}$ and the constant $A$-homotopy $0_s$ gives a Lie algebroid morphism $TD^3 \to A$. We then arrive at a higher homotopy $F_{\phi} : TD^3 \to A$ such that its restriction to the tangent bundle of the boundary of $D^3$ is $F_{\phi}|_{TS^2} = \phi_{algd}$.

Thus the rest is clear: $F_{\phi}$ integrates to a $W$-groupoid morphism $F_\Phi : P(D^3) \to \mathcal{G}(A)$. Its restriction $F_{\Phi}|_{P(S^2)}$ corresponds infinitesimally to the Lie algebroid morphism $\phi_{algd}$ since $F_{\phi}|_{TS^2} = \phi_{algd}$. By Theorem 1.3, we have $F_{\Phi}|_{P(S^2)} \sim \Phi_{gpd}$. Moreover $F_{\Phi}|_{D^3 \times N}$ fixes the base point $N \times N$. Therefore $F_{\Phi}|_{D^3 \times N}$ gives a homotopy between $\phi(\sim F_{\Phi}|_{S^2 \times N})$ and the constant map $i_x = F_{\Phi}|_{N \times N}$. Hence the source fibre $s^{-1}(x)$ is $2$-connected.

$\Box$

**Lemma 5.2.** Suppose that a morphism of differentiable stacks $X \xrightarrow{\Phi} Y$ and a morphism of their charts $X_0 \xrightarrow{\phi} Y_0$ fit into the following $2$-commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow{p_x} & \alpha & \downarrow{p_y} \\
X_0 & \xrightarrow{\phi} & Y_0
\end{array}
$$

where $p_x : X_0 \to X$ and $p_y : Y_0 \to Y$ are the chart projection maps. Then the H.S. morphism presenting $\Phi$ between the groupoid $X$ and $Y$ is a strict Lie groupoid morphism.

$^{12}$We can always reparametrize an $A$-path so that it becomes an $A_0$-path. See also [23].
Proof. We form the following 2-commutative diagram

There are two composed maps $X_1 \rightarrow X_0 \rightarrow Y_0 \rightarrow Y$, one goes through the upper $Y_0$ and the other goes through the lower $Y_0$. These two maps are the same up to a 2-morphism since the front, back, right, and bottom faces of the diagram are 2-commutative. Thus by the property of pull-backs, there exists a morphism $\phi_1 : X_1 \rightarrow Y_1$ making all the faces of the diagram 2-commutative. This map can be chosen canonically as in (26) after we describe fibre products of stacks in more detail: Given $U \in \mathcal{C}$ the category of differential manifolds, $X_1 = X_0 \times_{p_x, \mathcal{X}, p_x} X_0$ as a stack is the category fibred over $\mathcal{C}$ with

- **objects over** $U$: $(a, \gamma, a')$, where $a : U \rightarrow X_0$, $a' : U \rightarrow X_0$ are objects in $X_0$ viewed as a stack, $\gamma : p_x(a') \rightarrow p_x(a)$ is a morphism in $\mathcal{X}$;

- **morphisms over** $U \xrightarrow{\phi} V$: $(a, \gamma, a') \rightarrow (b, \kappa, b')$ which is made up by morphisms $f : a \rightarrow b$ and $f' : a' \rightarrow b'$ in $X_0$ over $U \xrightarrow{\phi} V$ such that

  \[
  \begin{array}{ccc}
  p_x(a) & \xrightarrow{\gamma} & p_x(a') \\
  p_x(f) & \downarrow & p_x(f') \\
  p_x(b) & \xleftarrow{\kappa} & p_x(b')
  \end{array}
  \]

  commutes.

Then the map $\phi_1$ can be explicitly described as

\[
\begin{align*}
(a, \gamma, a') & \mapsto (\phi(a), \alpha_a^{-1} \circ \Phi(\gamma) \circ \alpha_{a'}, \phi(a')) \\
(f, f') & \mapsto (\phi(f), \phi(f'))
\end{align*}
\]

(26) on the level of objects and morphisms respectively. Let us now explain how this formula makes sense. On the object level, $\phi_1$ is well-defined because the 2-morphism $\alpha : p_y \circ \phi \Rightarrow \Phi \circ p_x$ fits in the diagram

\[
\begin{array}{ccc}
p_y \circ \phi(a) & \xrightarrow{\alpha_a} & p_y \circ \phi(a') \\
\Phi \circ p_x(a) & \xrightarrow{\Phi(\gamma)} & \Phi \circ p_x(a')
\end{array}
\]
On the morphism level, $\phi_1$ is well-defined because the following diagram commutes by the naturality of $\alpha$:

\[
\begin{array}{ccc}
p_y\phi(a) & \xrightarrow{\alpha_a} & \Phi p_x(a) \\
p_y\phi(f) & \xrightarrow{\Phi p_x(f)} & \Phi p_x(a') \\
p_y\phi(b) & \xrightarrow{\alpha_b} & \Phi p_x(b) \\
\end{array}
\]

Now we claim that $\phi_1$ and $\phi$ together form a groupoid morphism. First of all, we recall the groupoid structure [3] on $X_1 \Rightarrow X_0$. The source and target are simply the right and left projections to $X_0$ respectively; the multiplication on $X_1$ is defined as

\[(a, \gamma, a') : (a', \gamma', a'') = (a, \gamma \circ \gamma', a''), \quad (f, f') : (f', f'') = (f, f'')\]

where $\gamma \circ \gamma'$ is simply the composition of morphisms in $\mathcal X$; the identity element is $(a, id_{p_x(a)}, a)$ for each object $a$ in $X_0$.

Then $\phi_1$ obviously preserves the source and target and multiplication. To verify that $\phi_1$ also preserves the identity element, one uses the fact that functors preserve identity morphisms. Then the inverse is automatically preserved. Thus $(\phi_1, \phi)$ is a groupoid morphism. Then $(\phi_1, \phi)$ induces a stack morphism $\Phi'$ (unique up to a unique 2-morphism) which makes the diagram [24] 2-commute. In fact the morphism $\Phi$ in diagram [24] is totally determined by $\phi$ on the image of $p_x$, which is a sub-prestack of $\mathcal X$. But since $p_x$ is an epimorphism, the stackification of the prestack $\text{Im}(p_x)$ is isomorphic to $\mathcal X$ by [21] 52.]. Thus any possible $\Phi$ making diagram [24] commute is unique up to a unique 2-morphism by [21] 51.]. Hence the groupoid morphism corresponds to the stack morphism $\Phi$ up to a 2-morphism.

If one prefers to use the language of H.S. morphism, here is a hint at how the H.S. morphism presenting $\Phi$ between $X$ and $Y$ comes from the strict morphism $(\phi_1, \phi)$. The H.S. morphism is

\[X_0 \times_{p_x \phi, Y, p_y} Y_0 \cong X_0 \times_{p_y \phi, Y, p_y} Y_0 \cong X_0 \times_{\phi, Y_0, t} Y_0 \times_{p_y \phi, Y, p_y} Y_0 \cong X_0 \times_{\phi, Y_0, t} Y_1.\]

**Remark 5.3.** One might expect that Theorem [1.1] would give a new proof of the fact that the simply connected Lie group $G$ has $\pi_2(G) = 0$. By Theorem [1.1] we only have $\pi_2(\mathcal G(g)_1) = 0$. Now $\mathcal G(g)_1$ is presented by the groupoid $\text{Mon}(P_0g)$ and the topological quotient $P_0g/\text{Mon}(P_0g)_1$ equals $G$. Thus $\mathcal G(g)_1 = G$ if and only if the monodromy groupoid $\text{Mon}(P_0g)$ has trivial isotropy groups. This means that for any $a \in P_0g$, there is a unique $g$-homotopy up to higher homotopies between $a$ and itself. Now $g$-paths correspond to $G$-paths and $g$-homotopies correspond to usual homotopies on $G$. The above statement is equivalent to the statement that for any $G$-path $g(t)$ there is a unique homotopy up to higher homotopy between $g(t)$ and itself. This exactly requires that $\pi_2(G) = 0$. In summary, $\mathcal G(g)_1 = G$ is equivalent to the fact that $\pi_2(G) = 0$. Thus Theorem [1.1] can not give a new proof for $\pi_2(G) = 0$. Rather $\pi_2(G) = 0$ implies that $G$ is the universal object integrating $g$ even in the world of 2-groups. This is not true any more for Lie 2-algebras or higher Lie algebras (see [14] Theorem 7.5)).
6 Examples

Now we give some examples of stacky groupoids of the form $G(A)$. The purpose of this section is to show that, for a manifold $M$ with $\pi_2(M)$ nontrivial, $G(TM)$ is not the traditional homotopy groupoid $\tilde{M} \times \tilde{M}/\pi_1(M)$ where $\tilde{M}$ is the simply connected cover of $M$. The second homotopy group $\pi_2(M)$ will play a role too.

Example 6.1 ($G(TM)$). The set of Lie algebroid $C^1$-morphisms $TN \to TM$ is equal to the set of $C^1$-maps $\text{Mor}(N,M)$. With a re-parametrization\footnote{Here to simplify calculations we will make later, we stop using the 0-boundary condition as in $P_0A$. We are able to do so because $\text{Mon}(P_0A)$ is Morita equivalent to $\text{Mon}(P_A)$ and both of them present $G(A)$ (see Section 4).} the stack $G(TM)_1$ is presented by the groupoid $G_1 \Rightarrow G_0$ with $G_1 = \text{Mor}(D^2, M)/\text{Mor}(D^3, M)$ the space of bigons (the quotient is similarly given by $\beta_1 \sim \beta_2$ if $\beta_1, \beta_2$ have the same boundary and bound an element in $\text{Mor}(D^3, M)$), and $G_0 = \text{Mor}(I, M)$. Here a $k$-dimensional disk $D^k$ is viewed as a $k$-dimensional simplex $\Delta^k$ with many degenerated faces. If $\pi_2(M) = 0$, then two disks $\beta_1, \beta_2 \in \text{Mor}(D^2, M)$ sharing the same boundary bound an element in $\text{Mor}(D^3, M)$. If $\pi_1(M) = 0$, we see that $G_1 = \text{Mor}(S^1, M)$. Then two paths $\alpha_1, \alpha_2 \in G_0$ sharing the same end points bound exactly one element in $G_1$. Thus $G(TM)_1 = M \times M$ and $G(TM)$ is the pair groupoid $M \times M \Rightarrow M$.

Next we analyse an example where $\pi_2(M) \neq 0$.

Example 6.2 ($G(TS^2)$). We take $M = S^2$ in Example 6.1 and we use the same notation as there. We claim that in this case, $G_1 \Rightarrow G_0$ is Morita equivalent to the action groupoid $S^3 \times S^3 \times (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \Rightarrow S^3 \times S^3$. Here $(\mathbb{R} \times \mathbb{R})/\mathbb{Z}$ is a quotient group by the diagonal $\mathbb{Z}$ action $(r_1, r_2) \cdot n = (r_1, r_2, r_1 + n)$, and the action of this quotient group is given by the projection $(\mathbb{R} \times \mathbb{R})/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} = S^1 \times S^1$ and the product of the $S^1$ action on $S^3$. Our convention of target and source maps are $t(p, q, [r_1, r_2]) = (p, q)$ and $s(p, q, [r_1, r_2]) = (p \cdot [r_1], q \cdot [r_2])$.

To show this, we give the associated complex line bundle $L \to S^2$ of the $S^1$-principal bundle $S^3 \to S^2$ a Hermitian metric and a compatible connection. We denote $\xi//\gamma$ as the result of the parallel transport of a vector $\xi \in L_{\gamma(0)}$ along a path $\gamma$ in $S^2$ to $L_{\gamma(1)}$. Since parallel transport is isometric: $L_{\gamma(0)} \to L_{\gamma(1)}$, it preserves the $S^1$ bundle $S^3 \subset L$ and the angle $\text{ang}(\xi_1, \xi_2)$ between $\xi_1$ and $\xi_2$. Here the angle $\text{ang}(\cdot, \cdot) L \oplus L \to S^1$ is point-wise the usual angular map (or argument map) $\mathbb{C} \to S^1$.

It satisfies

\[
\text{ang}(\xi_1, \xi_2) + \text{ang}(\xi_2, \xi_3) = \text{ang}(\xi_1, \xi_3) \quad \text{and} \quad \text{ang}(\xi_1, \xi_2) = -\text{ang}(\xi_2, \xi_1).
\]

Therefore for two paths $\gamma_1$ and $\gamma_2$ sharing the same end points, we can define the angle $\text{ang}(\gamma_1, \gamma_2)$ between them to be

\[
\text{ang}(\gamma_1, \gamma_2) := \text{ang}(\xi//\gamma_1, \xi//\gamma_2), \quad \text{for } \xi \in T_{\gamma(0)}S^2. \tag{27}
\]

Since parallel transport preserves the angle, this definition does not depend on the choice of $\xi$ as the following calculation shows,

\[
\text{ang}(\xi_1//\gamma_1, \xi_1//\gamma_2) = \text{ang}(\xi_1//\gamma_1, \xi_2//\gamma_1) + \text{ang}(\xi_2//\gamma_1, \xi_2//\gamma_2) + \text{ang}(\xi_2//\gamma_2, \xi_1//\gamma_2) = \text{ang}(\xi_1, \xi_2) + \text{ang}(\xi_2, \xi_1) + \text{ang}(\xi_2//\gamma_1, \xi_2//\gamma_2) = \text{ang}(\xi_2//\gamma_1, \xi_2//\gamma_2).
\]
In fact $\text{ang}(\gamma_1, \gamma_2) = \int_D \omega_{\text{area}}$ where $\partial D = -\gamma_1 + \gamma_2$ and $\omega_{\text{area}}$ is the standard symplectic (area) form on $S^2$.

As shown in the above picture, because of $\pi_2(S^2)$, $G_1$ is not simply the pair groupoid $G_0 \times_{S^2 \times S^2} G_0$—the set of paths with matched ends. We use $\text{ang} : G_0 \times_{S^2 \times S^2} G_0 \to S^1$ as in (27), then we have $G_1 = (G_0 \times_{S^2 \times S^2} G_0) \times_{\text{ang}, S_1, pr} \mathbb{R}$ where $pr : \mathbb{R} \to S^1$ is the projection. For example (28) corresponds to $(\gamma_1, \gamma_2, r)$ with $|r| = 2$. The pull-back groupoid of $G_1 \Rightarrow G_0$ along the projection (a surjective submersion) $S^3 \times_{\pi,S^2,t} G_0 \times_{s,S^2,s} S^3 \to G_0$ with $s(\gamma) = \gamma(0)$, $t(\gamma) = \gamma(1)$, and $\pi : S^3 \to S^2$, is

$$(S^3 \times_{S^2} G_0 \times_{S^2} S^3 \times S^2 G_0 \times_{S^2} S^3) \times_{\text{ang}, S_1, pr} \mathbb{R} \Rightarrow S^3 \times_{S^2} G_0 \times_{S^2} S^3.$$

Now the pull-back groupoid of $S^3 \times S^3 \times (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \Rightarrow S^3 \times S^3$ along the projection (also a surjective submersion) $S^3 \times_{S^2} G_0 \times_{S^2} S^3 \to S^3 \times S^3$ defined by $(\xi, \gamma, \xi') \mapsto (\xi'//\gamma, \xi/\gamma^{-1})$ is

$$(S^3 \times_{S^2} G_0 \times_{S^2} S^3 \times S^2 G_0 \times_{S^2} S^3) \times_{(S^3 \times S^3) \times (S^3 \times S^3)} S^3 \times S^3 \times (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \Rightarrow S^3 \times_{S^2} G_0 \times_{S^2} S^3.$$  

They are isomorphic as Lie groupoids with the morphism from the second to the first given by

$$(\xi, \gamma, \xi) \mapsto (\xi, \gamma', \xi')$$

on the base of the groupoid, where $\gamma'$ is a path sharing the same end points of $\gamma$ and such that $\text{ang}(\gamma, \gamma') = \text{ang}(\xi, \xi'/\gamma)$. On the level of morphisms, we define

$$(\xi_1, \gamma_1, \xi_1', \xi_2, \gamma_2, \xi_2', [(r_1, r_2)]) \mapsto (\xi_1, \gamma_1', \xi_1', \xi_2, \gamma_2, \xi_2', r_2 - r_1).$$  

(30)

We need to verify that this map does land on the correct manifold. By (29) we have $(\xi_1/\gamma_1^{-1}) \cdot [-r_2] = \xi_2/\gamma_2^{-1}$, thus

$$-[r_2] = \text{ang}(\xi_1/\gamma_1^{-1}, \xi_2/\gamma_2^{-1})$$

$$= \text{ang}(\xi_1/\gamma_1^{-1}, \xi_2/\gamma_1^{-1}) + \text{ang}(\xi_2/\gamma_1^{-1}, \xi_2/\gamma_2^{-1})$$

$$= \text{ang}(\xi_1, \xi_2) + \text{ang}(\gamma_1^{-1}, \gamma_2^{-1}),$$

hence $\text{ang}(\gamma_1, \gamma_2) = [r_2] + \text{ang}(\xi_1, \xi_2)$ and similarly $\text{ang}(\gamma_1, \gamma_2) = [-r_1] - \text{ang}(\xi_1', \xi_2')$. Therefore

$$[r_2 - r_1] = 2\text{ang}(\gamma_1, \gamma_2) - \text{ang}(\xi_1, \xi_2) + \text{ang}(\xi_1', \xi_2').$$

\footnote{Such $\gamma'$ can be chosen canonically in the following way. Take the usual Riemannian metric on $S^2$. Let $\phi_t(s)$ be the trajectory of $\dot{\gamma}(t)$. Then when we increase the angle $\text{ang}(\dot{\gamma}(t), \dot{\gamma}(t)/\phi_t(s))$ starting from 0, there is a unique $\gamma'$ such that $\text{ang}(\gamma, \gamma') = \alpha$ for any $\alpha \in S^1$.}
Since
\[ \text{ang}(\xi_1, \xi_1'//\gamma_1) + \text{ang}(\xi_1', \gamma_1, \xi_2'//\gamma_1) + \text{ang}(\xi_2', \gamma_1, \xi_2//\gamma_2) + \text{ang}(\xi_2', \gamma_2, \xi_2) = \text{ang}(\xi_1, \xi_2), \]
we have
\[ -\text{ang}(\xi_1, \xi_2) + \text{ang}(\xi_1', \xi_2') = -\text{ang}(\xi_1, \xi_1'//\gamma_1) + \text{ang}(\xi_2', \xi_2//\gamma_2) - \text{ang}(\gamma_1, \gamma_2). \]

Hence
\[ [r_2 - r_1] = \text{ang}(\gamma_1, \gamma_2) + (-\text{ang}(\xi_1', \xi_1'//\gamma_1) + \text{ang}(\xi_2', \xi_2//\gamma_2)) \]
\[ = \text{ang}(\gamma_1, \gamma_2) + \text{ang}(\gamma_1', \gamma_1) + \text{ang}(\gamma_2', \gamma_2) \]
\[ = \text{ang}(\gamma_1, \gamma_2'). \]

Therefore (30) is well-defined. It is not hard to see that it is indeed a groupoid isomorphism.

Therefore \( G_1 \Rightarrow G_0 \) and \( S^3 \times S^3 \times (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \Rightarrow S^3 \times S^3 \) are Morita equivalent via this third groupoid (29). Therefore \( \mathcal{G}(TS^2) \) is presented by \( S^3 \times S^3 \times (\mathbb{R} \times \mathbb{R})/\mathbb{Z} \Rightarrow S^3 \times S^3 \).

**Remark 6.3** (on 2-connected covers). It is well-known that the cohomology group \( H^2(M, \mathbb{Z}) \) corresponds to \( S^1 \)-bundles via Chern classes. Let \( B\mathbb{Z} \) denote the differentiable stack corresponding to the Lie group \( \mathbb{Z} \). Then \( B\mathbb{Z} \) is a stacky Lie group with the multiplication given by the group(oid) morphism \( \mathbb{Z} \times \mathbb{Z} \xrightarrow{+} \mathbb{Z} \) on the level of presenting groupoids. The morphism + is a groupoid morphism because \( \mathbb{Z} \) is abelian. Then from a stacky viewpoint, \( H^2(M, \mathbb{Z}) \) also corresponds to \( B\mathbb{Z} \)-gerbes: Given a covering \( U_i \) of \( M \), and \( g_{ijk} \) representing a Čech class in \( H^2(M, \mathbb{Z}) \), there is a stack \( \mathcal{G} \) presented by groupoid \( \sqcup U_{ij} \times \mathbb{Z} \Rightarrow \sqcup U_i \) with the groupoid multiplication \( (x_{ij}, n) \cdot (x_{jk}, m) = (x_{ik}, n + m + g_{ijk}) \) and the source and target maps inherited from the groupoid \( \sqcup U_{ij} \Rightarrow \sqcup U_i \) which presents \( M \). Here \( x_{ij} \) denotes a point in \( U_{ij} \). An \( A \)-gerbe corresponds to an \( A \)-groupoid central extension for an abelian group \( A \) [3]. It is clear that \( \mathcal{G} \) is a \( \mathbb{Z} \)-gerbe (or equivalently a \( B\mathbb{Z} \)-principal bundle [36]) over \( M \) from the following diagram of groupoid central extension:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} \times \sqcup U_i & \longrightarrow & \sqcup U_{ij} \times \mathbb{Z} & \longrightarrow & \sqcup U_i & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\sqcup U_i & & \sqcup U_i & & \sqcup U_i & & \sqcup U_i &
\end{array}
\]

For example, corresponding to the Hopf fibration \( S^3 \rightarrow S^2 \), there is a \( \mathbb{Z} \) gerbe or \( B\mathbb{Z} \) principal bundle denoted as \( \tilde{S}^2 \). Now, apply the long exact sequence of homotopy groups to the \( B\mathbb{Z} \)-fibration \( \tilde{S}^2 \rightarrow S^2 \). Since \( \pi_{n\neq 1}(B\mathbb{Z}) = 0 \) and \( \pi_1(B\mathbb{Z}) = \mathbb{Z} \), we have \( \pi_1(\tilde{S}^2) = \pi_2(S^2) = 0 \) and \( \pi_{2\geq 3}(\tilde{S}^2) = \pi_{2\geq 3}(S^2) \). Hence we can view \( \tilde{S}^2 \) as a 2-connected “covering” of \( S^2 \). Although \( S^3 \) also has the same property of homotopy groups, \( \tilde{S}^2 \) seems to be a more suitable candidate than \( S^3 \) since \( S^2 \) has the same dimension as \( S^2 \) given \( \dim B\mathbb{Z} = 0 \).

In this example, \( S^3 \times \mathbb{R} \Rightarrow S^3 \) presents the stack \( \tilde{S}^2 \). What we prove above shows that the stacky groupoid \( \mathcal{G}(TS^2) \) is not the usual one, \( S^2 \times S^2 \), but rather \( S^2 \times \tilde{S}^2 / B\mathbb{Z} \). This is to ensure that it has 2-connected source-fibre, in agreement with the general result we proved in Theorem 1.1. This resembles the construction of the groupoid integrating \( TM \) when \( M \) is a non-simply-connected manifold. The source simply connected groupoid of \( TM \) is \( \tilde{M} \times \tilde{M} / \pi_1(M) \). Here, for \( \mathcal{G}(TS^2) \), the construction is comparable to this, but on a higher level—the aim is to kill \( \pi_2 \) of the source fibre. Further, using the long exact sequence of homotopy groups, we have \( \pi_2(\tilde{S}^2 \times \tilde{S}^2 / B\mathbb{Z}) = \pi_1(B\mathbb{Z}) = \mathbb{Z} \) since \( \pi_2(\tilde{S}^2 \times \tilde{S}^2) = 0 \).
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