Finite Field Theories
and
Causality

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Overview

1. Prelude
   - UV divergences
   - History

2. Preliminaries
   - Operator-valued distributions
   - S-matrix: basic properties

3. The method of Epstein and Glaser
   - Inductive construction of the perturbative S-matrix
   - Base case
   - Examples

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UV divergences

Naive example of a 'UV divergence'

Consider Heaviside-\(\Theta\)- and Dirac-\(\delta\)-distributions in 1-dim. 'configuration space':

Product \(\Theta(x)\delta(x)\) obviously ill-defined !

Fourier transforms

\[
\mathcal{F}\{\delta\}(k) = \hat{\delta}(k) = \int dx \, \delta(x) e^{-ikx} = 1,
\]

\[
\hat{\Theta}(k) = \lim_{\epsilon \searrow 0} \int dx \, \Theta(x) e^{-ikx - \epsilon x} = \lim_{\epsilon \searrow 0} \left. \frac{i e^{-ikx - \epsilon x}}{k - i \epsilon} \right|_{0}^{\infty} = -\frac{i}{k - i0}.
\]

Calculate nevertheless the Fourier transform of the ill-defined product

\[
\mathcal{F}\{\Theta\delta\}(k) = \int dx \, e^{-ikx} \Theta(x)\delta(x) = \int dx \, e^{-ikx} \int \frac{dk'}{2\pi} \hat{\Theta}(k') e^{+ik'x} \int \frac{dk''}{2\pi} \hat{\delta}(k'') e^{+ik''x}
\]

Since \(\int dx \, e^{i(k' + k'' - k)x} = 2\pi \delta(k' + k'' - k)\), we obtain the convolution

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\mathcal{F}\{\Theta\delta\}(k) = \frac{1}{2\pi} \int dk' \, \hat{\Theta}(k') \hat{\delta}(k - k') = -\frac{i}{2\pi} \int \frac{dk'}{k' - i0}
\]

Obvious problem in x-space → divergent integral in k-space !
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Obvious problem in $x$-space $\rightarrow$ divergent integral in $k$-space!
**Interpretation of UV divergences**

In **perturbative quantum field theory**, the rôle of the **Heaviside \( \Theta \)-distribution** is taken over by the **time-ordering operator**.

'**Textbook**' expression for the perturbative scattering matrix:

\[
S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \ldots \int_{-\infty}^{+\infty} dt_n \ T[H_{\text{int}}(t_1) \ldots H_{\text{int}}(t_n)]
\]

\[
= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4 x_1 \ldots \int d^4 x_n \ T[H_{\text{int}}(x_1) \ldots H_{\text{int}}(x_n)].
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\( H_{\text{int}}(t) \ \{H_{\text{int}}(x)\} \): Interaction Hamiltonian \{density\}. \( H_{\text{int}}(t) = \int d^3 x \ H_{\text{int}}(x) \).

A time-ordered expression à la

\[
T[H_{\text{int}}(x_1) \ldots H_{\text{int}}(x_n)] = \sum_{\text{Perm. } \Pi} \Theta(x_{\Pi_1}^0 - x_{\Pi_2}^0) \ldots \Theta(x_{\Pi_{(n-1)}}^0 - x_{\Pi_n}^0) H_{\text{int}}(x_{\Pi_1}) \ldots H_{\text{int}}(x_{\Pi_n})
\]

is formal (ill-defined) ! Products of \( H_{\text{int}} \) too singular to be multiplied by \( \Theta \)-distributions.
In perturbative quantum field theory, the rôle of the Heaviside Θ-distribution is taken over by the time-ordering operator.

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E. C. G. Stückelberg (∼1949)
- Construction of the S-matrix by means of causality and unitarity.

N. N. Bolgolyubov et al. (∼1959)
- Reformulation of the causality condition (see below).
- Adiabatic switching (see below).
- UV divergences persist.

H. Epstein & V. Glaser
(Annales de l’institut Henri Poincaré (A), Physique théorique, 19 (211-295) 1973)
- Inductive construction of the perturbation series by means of Poincaré invariance and causality (unitarity plays no immediate rôle).
  → no UV divergences
  → Feynman rules only hold on tree-level
  → Loop diagrams rather technical
  → Finite dispersion integrals instead of divergent Feynman integrals
  → New strategy to treat the infrared problem by adiabatic switching of interaction
  → Only scalar field theory
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Crucial observation: Free field operators are operator-valued distributions.

E.g.: The scalar (neutral) field with mass $m$

\[
\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E}} \left[ a(k)e^{-ikx} + a^+(k)e^{ikx} \right], \quad E = \sqrt{k^2 + m^2},
\]

must be smeared out by rapidly decreasing test functions $g(x)$ in the Schwartz space $S(\mathbb{R}^4)$ ($S(\mathbb{R}^3)$) in order to get an operator in Fock space, formally

\[
\varphi(g) = \int d^4x \varphi(x)g(x). \quad \varphi(x)|0\rangle \text{ is not a Fock state!}
\]

The same arguing applies, e.g., to the interaction Hamiltonian densities used in perturbation theory constructed from normally ordered products of free fields:

- QED: Spinor field $\psi(x)$, photon field $A_\mu(x) \rightarrow \mathcal{H}_{int} = -e : \bar{\psi}(x)\gamma^\mu \psi(x) : A_\mu(x)$.
- $\varphi^3$-Theory $\rightarrow \mathcal{H}_{int} = \frac{\lambda}{3!} : \varphi(x)^3 :.$
- . . .
It is therefore *most natural* to replace the *problematic expression*

\[
S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots d^4x_n T[H_{int}(x_1) \ldots H_{int}(x_n)]
\]

by

\[
S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n), \quad g \in S(\mathbb{R}^4),
\]

where \(g(x)\) is a locally varying coupling constant and \(T_1(x) = -iH_{int}(x)\).

\(T_n(x_1, \ldots, x_n) \simeq T[T_1(x_1) \ldots T_1(x_n)]\)

is a *well-defined* (divergence free) time-ordered product with

\[
T_n(\ldots, x_i, \ldots, x_j, \ldots) = T_n(\ldots, x_j, \ldots, x_i, \ldots) \quad \forall i, j
\]

by construction.
In the causal (Epstein-Glaser-) approach, the $T_n$ are free of any UV divergences.

$$S_n(g) = \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, T_n(x_1, \ldots, x_n)g(x_1)\ldots g(x_n)$$

is well-defined at every order $n$ of the perturbation expansion (even when massless fields are present).

→ No statements about the convergence of the full series!

→ Infrared problems arise in the adiabatic limit $g(x) \to 1$, since $1 \not\in S(\mathbb{R}^4)$.

→ Performing the adiabatic limit is a delicate task (→ existence and uniqueness!):
   Limit has to be taken such that observable quantities (cross sections) remain finite.

→ Typical approach: Rescaling $g(x)$ according to

$$\lim_{\epsilon \downarrow 0} g(\epsilon x) \rightarrow g(x) = g(0) = \text{const.} \sim \text{coupling constant.}$$

→ No further regularizations necessary (→ finite photon mass).
The perturbative $S$-matrix

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \ldots d^4 x_n \, T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n) = 1 + T$$

can be formally inverted

$$S(g)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4 x_1 \ldots d^4 x_n \, \tilde{T}_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n)$$

$$= (1 + T)^{-1} = 1 + \sum_{r=1}^{\infty} (-T)^r.$$

$$\longrightarrow \tilde{T}_n(X) = \sum_{r=1}^{n} (-1)^r \sum_{\mathcal{P}_r} T_{n_1}(X_1) \ldots T_{n_r}(X_r),$$

where $X = \{x_1, \ldots, x_n\}$ is a disordered set and $\sum_{\mathcal{P}_r}$ denotes all partitions of $X$ into

$r$ disjoint subsets

$$X = X_1 \cup \ldots \cup X_r, \quad X_j \neq \emptyset, \quad X_i \cap X_j = \emptyset, \quad |X_j| = n_j.$$
We assume that \( g(x) = g_1(x) + g_2(x) \) can be decomposed such that
the supports of \( g_1(x) \) and \( g_2(x) \) are space-like separated, i.e.

\[ \exists \text{ reference frame such that } x \in \text{supp}(g_1) \Rightarrow x^0 < 0 \text{ and } y \in \text{supp}(g_2) \Rightarrow y^0 > 0. \]

Causality condition

\[ S(g_1 + g_2) = S(g_2)S(g_1) \]

\[ \forall g_1, g_2 \text{ where } \text{supp}(g_1) < \text{supp}(g_2). \]

This implies

\[ T_n(x_1, \ldots, x_n) = T_m(x_1, \ldots, x_m)T_{n-m}(x_{m+1}, \ldots, x_n) \]

if \( \{x_1, \ldots, x_m\} > \{x_{m+1}, \ldots, x_n\} \).

This condition is, of course, intuitively clear.
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**Explicit construction of $S_2(g)$**

Causality and translation invariance requires that the commutator $D_2(z = x_1 - x_2)$

$$D_2(x_1 - x_2) = (-i)^2 [\mathcal{H}_{\text{int}}(x_1), \mathcal{H}_{\text{int}}(x_2)] = [T_1(x_1), T_1(x_2)] = 0 \quad \text{for} \quad (x_1 - x_2)^2 < 0$$

has causal support on the closed light-cone $\mathcal{V} = \bar{\mathcal{V}}^+ \cup \bar{\mathcal{V}}^-$ in the sense of distributions.

We introduce (primed) advanced and retarded distributions $A'_2(z)$ and $R'_2(z)$:

**Splitting of $D_2$**

$$R_2 = +D_2 \bigg|_{\bar{\mathcal{V}}^+ \cup \{0\}}, \quad A_2 = -D_2 \bigg|_{\bar{\mathcal{V}}^- \setminus \{0\}},$$

$$R'_2 = -T_1(x_2)T_1(x_1), \quad A'_2 = -T_1(x_1)T_1(x_2).$$

The non-trivial (!) splitting of $D_2$ corresponds to time-ordering:

$$T[T_1(x_1)T_1(x_2)] = T_2(x_1, x_2) = R_2 - R'_2 = A_2 - A'_2.$$

Test: $R_2 - R'_2$ for $z^0 > 0$ and $z^0 < 0$.

$z^0 > 0$ : $T_1(x_1)T_1(x_2) - T_1(x_2)T_1(x_1) + T_1(x_2)T_1(x_1)$

$z^0 < 0$ : $+T_1(x_2)T_1(x_1)$
Explicit construction of $S_2(g)$

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$$D_2(x_1 - x_2) = (-i)^2[\mathcal{H}_{int}(x_1), \mathcal{H}_{int}(x_2)] = [T_1(x_1), T_1(x_2)] = 0$$

for $(x_1 - x_2)^2 < 0$ has causal support on the closed light-cone $\mathcal{V} = \bar{\mathcal{V}}^+ \cup \bar{\mathcal{V}}^-$ in the sense of distributions.

We introduce (primed) advanced and retarded distributions $A_2'(z)$ and $R_2'(z)$:

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$$T[T_1(x_1)T_1(x_2)]'' = T_2(x_1, x_2) = R_2 - R_2' = A_2 - A_2'.$$

Test: $R_2 - R_2'$ for $z^0 > 0$ and $z^0 < 0$.

$$z^0 > 0 : \quad T_1(x_1)T_1(x_2) - T_1(x_2)T_1(x_1) + T_1(x_2)T_1(x_1)$$

$$z^0 < 0 : \quad +T_1(x_2)T_1(x_1)$$

$$D_2(z) = R_2(z) - A_2(z) = R_2'(z) - A_2'(z)$$
Examples

$\varphi^3$ – Theory

With $T_1(x) = \frac{i\lambda}{3!} : \varphi(x)^3:$ and $\langle 0 | T(\varphi(x_1) \varphi(x_2)) | 0 \rangle = i \Delta_F(x_1 - x_2)$, standard Wick ordering leads to

$$T_2(x_1, x_2) = -\frac{\lambda^2}{3!} : \varphi(x_1)^3 \varphi(x_2)^3 : - \frac{9\lambda^2}{3!} : \varphi(x_1)^2 \varphi(x_2)^2 : i \Delta_F(x_1 - x_2)$$

$$- \frac{18\lambda^2}{3!} : \varphi(x_1) \varphi(x_2) : [i \Delta_F(x_1 - x_2)]^2 - \frac{\lambda^2}{3!} [i \Delta_F(x_1 - x_2)]^3.$$

In the causal approach, one constructs first

$$D_2(x_1 - x_2) = -\frac{\lambda^2}{3!} [ : \varphi(x_1)^3 : ; : \varphi(x_2)^3 : ]$$

$$= \ldots - \frac{9\lambda^2}{3!} : \varphi(x_1)^2 \varphi(x_2)^2 : i \Delta(x_1 - x_2) + \ldots,$$

where $\Delta(x_1 - x_2)$ is the Pauli-Jordan distribution, which can be decomposed into the positive- and negative-frequency Pauli-Jordan distributions

$$\Delta(z) = \Delta^+(z) + \Delta^-(z),$$

$$\Delta^{\pm}(z) = \mp \frac{i}{(2\pi)^3} \int d^4k \Theta(\pm k^0) \delta(k^2 - m^2)e^{-ikx}. $$
\( \varphi^3 \) – Theory

With \( T_1(x) = \frac{i\lambda}{3!} : \varphi(x)^3 : \) and \( \langle 0 | T(\varphi(x_1)\varphi(x_2)) | 0 \rangle = i\Delta_F(x_1 - x_2) \), standard Wick ordering leads to

\[
T_2(x_1, x_2) = -\frac{\lambda^2}{3!^2} : \varphi(x_1)^3 \varphi(x_2)^3 : -\frac{9\lambda^2}{3!^2} : \varphi(x_1)^2 \varphi(x_2)^2 : i\Delta_F(x_1 - x_2) \nonumber
\]
\[
-\frac{18\lambda^2}{3!^2} : \varphi(x_1)\varphi(x_2) : [i\Delta_F(x_1 - x_2)]^2 - \frac{\lambda^2}{3!} [i\Delta_F(x_1 - x_2)]^3.
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D_2(x_1 - x_2) = -\frac{\lambda^2}{3!^2} [: \varphi(x_1)^3 :, \varphi(x_2)^3 :] = \ldots -\frac{9\lambda^2}{3!^2} : \varphi(x_1)^2 \varphi(x_2)^2 : i\Delta(x_1 - x_2) + \ldots,
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\Delta(z) = \Delta^+(z) + \Delta^-(z),
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\]
Splitting of $D_2$ (tree level)

In order to get the retarded C-number part $D_2^{\text{tree}}(z) := \Delta(z)$, we simply multiply by $\Theta(z^0)$: $R_2^{\text{tree}}(z) = \Theta(z^0)\Delta(z)$.

From $\hat{\Theta}(k) = \frac{i(2\pi)^3}{k^0 + i\epsilon} \delta^{(3)}(\vec{k})$ follows

$$\hat{R}_2^{\text{tree}}((k^0, \vec{0})) = \frac{i}{2\pi} \int dp^0 \frac{\hat{\Delta}(p^0, \vec{0})}{k^0 - p^0 + i0} = \frac{i}{2\pi} \int dt \frac{\hat{\Delta}(tk^0, \vec{0})}{1 - t + i0},$$

or, from the Lorentz covariance of $\Delta$ ($\hat{\Delta}$) we obtain a dispersion relation

$$\hat{R}_2^{\text{tree}}(k) = \frac{i}{2\pi} \int dt \frac{\hat{\Delta}(tk)}{1 - t + i0} \text{ for } k^0 \in V^+.$$

Splitting of $D_2^{\text{tree}}$

$$\hat{\Delta}(k) = -2\pi i \text{sgn}(k^0)\delta(k^2 - m^2) \rightarrow$$

$$\hat{R}_2^{\text{tree}}(k) = \int dt \frac{\text{sgn}(tk^0)\delta(t^2k^2 - m^2)}{1 - t + i0} =$$

$$\int dt \frac{[\delta(t - \frac{m}{\sqrt{k^2}}) - \delta(t + \frac{m}{\sqrt{k^2}})]}{2\sqrt{k^2}m(1 - t + i0)} = \frac{1}{k^2 - m^2}.$$

$$\hat{R}_2^{\text{tree}}(k) = \frac{1}{(2\pi)^4} \hat{D}_2(k) \ast \mathcal{F}\{\Theta(z^0)\}(k)$$
**Splitting of \( D_2 \) (tree level)**

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**Splitting of \( D_2^{\text{tree}} \)**

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\hat{\Delta}(k) = -2\pi i \text{ sgn}(k^0) \delta(k^2 - m^2) \rightarrow
\]

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\hat{R}_2^{\text{tree}}(k) = \int dt \frac{\text{ sgn}(tk^0) \delta(t^2 k^2 - m^2)}{1 - t + i0} = \int \frac{[\delta(t - \frac{m}{\sqrt{k^2}}) - \delta(t + \frac{m}{\sqrt{k^2}})]}{2\sqrt{k^2}m(1 - t + i0)} = \frac{1}{k^2 - m^2}.
\]

\[
\hat{R}_2^{\text{tree}}(k) = \frac{1}{(2\pi)^4} \hat{D}_2(k) * \mathcal{F}\{\Theta(z^0)\}(k)
\]
Splitting of $D_2$ (loop level)

The self-energy (one-loop) part in $T_2$ is logarithmically divergent:

$$T_2^{\text{loop}}(x_1 - x_2) \sim [i \Delta_F(x_1 - x_2)]^2 \overset{\mathcal{F}}{\longrightarrow} \int \frac{d^4p}{[p^2 - m^2 + i0][(k - p)^2 - m^2 + i0]}.$$

In the causal approach, one calculates first $D_2(x_1 - x_2) = [T_1(x_1), T_1(x_2)]$, leading to

$$D_2^{\text{loop}}(x_1 - x_2) \sim [\Delta^- (x_1 - x_2)]^2 - [\Delta^- (x_2 - x_1)]^2 \overset{\mathcal{F}}{\longrightarrow} \text{sgn}(k^0) \Theta(k^2 - 4m^2).$$

Naive splitting also leads to a divergent dispersion integral:

$$\Theta(z_0)D_2^{\text{loop}}(z) \overset{\mathcal{F}}{\longrightarrow} \int dt \frac{\hat{D}_2^{\text{loop}}(tk)}{1 - t + i0} \quad (k \in V^+).$$

However, it can be shown that the retarded part can be obtained from a subtracted dispersion integral ($m \neq 0$)

$$\hat{R}_2^{\text{loop}}(k) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{D}_2^{\text{loop}}(tk)}{(t - i0)\omega^+ (1 - t + i0)} + \text{const.} \quad (k \in V^+) \quad \text{with} \quad \omega = 0.$$

$\omega$ depends on the scaling properties ($\longrightarrow$ power counting) of the causal distribution $D_2^{\text{loop}}$. 
Splitting of $D_2$ (loop level)

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$\omega$ depends on the scaling properties ($\rightarrow$ power counting) of the causal distribution $D_2^{\text{loop}}$. 
With causality as the fundamental input, it is possible to construct causal distributions

\[ A_n, R_n \text{ and } D_n(x_1, \ldots, x_n) = R_n(x_1, \ldots, x_n) - A_n(x_1, \ldots, x_n) \]

at higher orders:

\[
\text{supp } R_n(x_1, \ldots, x_n) \subseteq \Gamma^+(x_n), \\
\text{supp } A_n(x_1, \ldots, x_n) \subseteq \Gamma^-(x_n),
\]

where

\[
\Gamma^\pm(x_n) = \{(x_1, \ldots, x_n) \mid x_j \in \bar{V}^\pm(x_n) \forall j = 1, \ldots, n - 1\}, \\
\bar{V}^+(x) = \{y \mid (y - x)^2 \geq 0, y^0 \geq x^0\}, \quad \bar{V}^-(x) = \{y \mid (y - x)^2 \geq 0, y^0 \leq x^0\},
\]

and

\[
\text{supp } D_n(x_1, \ldots, x_n) \subseteq \Gamma^+(x_n) \cup \Gamma^-(x_n).
\]
Higher orders

Recipe

\( T_m \) known for \( 1 \leq m \leq n - 1 \\
\Downarrow \\
construct advanced/retarded distributions

\[
A'_n(x_1, \ldots, x_n) = \sum_{P_2} \tilde{T}_{n_1}(X)T_{n-n_1}(Y, x_n) \quad \text{and} \quad R'_n(x_1, \ldots, x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n)\tilde{T}_{n_1}
\]

with \( P_2 : \{x_1, \ldots, x_{n-1}\} = X \cup Y, X \neq \emptyset, n_1 = |X| \geq 1 \)
\Downarrow \\
allow \( X = \emptyset \)

\[
A_n(x_1, \ldots, x_n) = A'_n(x_1, \ldots, x_n) + T_n(x_1, \ldots, x_n), \quad R_n(x_1, \ldots, x_n) = R'_n(x_1, \ldots, x_n) + T_n(x_1, \ldots, x_n)
\]
\Downarrow \\
\( T_n \) unknown, but difference distribution

\[
D_n = R'_n - A'_n = R_n - A_n
\]
can be shown to be causal: splitting \( D_n \) generates \( R_n \)
\Downarrow \\
\[
T_n = R_n - R'_n
\]
The C-number parts of \( R_n \) and \( D_n \)

\[
\hat{r}^{\text{tree,loop},\ldots}(x_1 - x_n, \ldots, x_{n-1} - x_n), \quad \hat{d}^{\text{tree,loop},\ldots}(x_1 - x_n, \ldots, x_{n-1} - x_n)
\]

go over into

\[
\hat{\rho}^{\text{tree,loop},\ldots}(p_1, \ldots, p_{n-1}) \quad \text{and} \quad \hat{\rho}^{\text{tree,loop},\ldots}(p_1, \ldots, p_{n-1})
\]

via Fourier transformation.

If at least one field in the theory is massive, it can be shown that for \( p = (p_1, p_2, \ldots) \in \Gamma^+ \)

\[
\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{d}(tp)}{(t - i0)^{\omega+1}} dt + \sum_{|\alpha|=0} c_\alpha p^\alpha \quad (\alpha : \text{multi-index}),
\]

where \( \omega \) is a rigorously defined power-counting degree of divergence.

The terms \( \sum c_\alpha p^\alpha \) correspond to the divergent parts of Feynman integrals in the standard regularization methods. They are due to the fact that the splitting of \( D_n \) into \( R_n \) and \( A_n \) is not uniquely defined at the "tip" \( x_1 = x_2 = \ldots = x_n \) of the generalized forward/backward light-cones \( \Gamma^\pm \).

In configuration space, they correspond to local terms \( \sim \sum \hat{c}_\alpha D^\alpha \delta(x_1 - x_n, \ldots, x_{n-1} - x_n) \).

Perturbation theory does not specify them, and they have to be restricted, e.g., by symmetry considerations.
Considering QCD within the causal approach, it is most natural to start from a first order gluon field coupling (matter fields neglected)

\[ T_1(x) = i\frac{g}{2} f_{abc} : A_\mu^a(x) A_\nu^b(x) F_{\nu\mu}^c(x) : , \quad F_{\nu\mu}^c(x) = \partial_\mu A_\nu^c(x) - \partial_\nu A_\mu^c(x). \]

First order gauge invariance requires additional fields (ghosts)

\[ T_1(x) = igf_{abc} : A_\mu^a(x) A_\nu^b(x) \partial_\nu A_\mu^c(x) : - igf_{abc} : A_\mu^a(x) u_b(x) \partial_\mu \tilde{u}_c(x) : . \]

At second order, tree diagrams containing C-number distributions

\[ \sim \partial_\mu \partial_\nu \Delta(x_1 - x_2) \rightarrow - k^\mu k^\nu \hat{\Delta}(k) \]

appear; the splitting is fixed up to a local term \( \sim g^{\mu\nu} \delta^{(4)}(x_1 - x_2) \)

\[ k^\mu k^\nu \hat{\Delta}(k) \overset{\text{splitting}}{\rightarrow} \frac{k^\mu k^\nu}{k^2 + i0} + C g^{\mu\nu}. \]

Second order gauge invariance determines the constant \( C \) such that the usual four-gluon coupling term is generated.
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What has been done so far

- The finite/causal/Epstein-Glaser approach has been rediscovered by Michael Dütsch and Günter Scharf (U. Zürich) in 1985.
- Complete discussion of QED and introduction to the causal approach: "Finite Quantum Electrodynamics" by G. Scharf (1989, 2nd edition 1995).
- Zürich group, 1989-1992: Interacting fields, axial anomalies, full discussion of the renormalizability of scalar QED, ...
- 1992-1997: Complete discussion of perturbative QCD, gauge theories like the standard model (including spontaneous symmetry breaking) studied.
- since 1997: Quantum gravity and supersymmetric theories considered.
- Other groups: Klaus Fredenhagen et al. (causal approach on curved space-times)

→ Talk by Ernst Werner!
