Flux enhancement and multistability induced by time delays in a feedback controlled flashing ratchet

F. J. Cao

Departamento de Física Atómica, Molecular y Nuclear, Universidad Complutense de Madrid,
Avenida Complutense s/n, 28040 Madrid, Spain. and
LERMA, Observatoire de Paris, Laboratoire Associé au CNRS UMR 8112,
61, Avenue de l'Observatoire, 75014 Paris, France.

M. Feito

Departamento de Física Atómica, Molecular y Nuclear,
Universidad Complutense de Madrid,
Avenida Complutense s/n, 28040 Madrid, Spain

Feedback controlled ratchets are thermal rectifiers that use information on the state of the system to operate. We study the effects of time delays in the feedback for a protocol that performs an instantaneous maximization of the center-of-mass velocity in the many particle case. For small delays the center-of-mass velocity decreases for increasing delays (although not as fast as in the few particle case). However, for large delays we find the surprising result that the presence of a delay can improve the flux performance of the ratchet. In fact, the maximum flux obtained with the optimal periodic protocol is attained. This implies that the delayed feedback protocol considered can perform better than its non-delayed counterpart. The improvement of the flux observed in the presence of large delays is the result of the emergence of a new dynamical regime where the presence of the delayed feedback stabilizes quasiperiodic solutions that resemble the solutions obtained in a certain closed-loop protocol with thresholds. In addition, in this new regime the system presents multistability, i.e. several quasiperiodic solutions can be stable for a fixed time delay.

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I. INTRODUCTION

Brownian motors or ratchets are mechanisms that induce transport rectifying the motion of Brownian particles through the introduction of a time-dependent perturbation that drives the system out of equilibrium [1]. This type of systems allow to get insight in non-equilibrium processes [1]. In addition, they are also important due to their applications to many fields as nanotechnology and biology [1, 2].

The two main types of ratchets are rocking ratchets [3, 4] and flashing ratchets [4, 5]. In rocking ratchets (also called tilting ratchets) the perturbation acts as a time-dependent additive driving force, which is unbiased on the average, while in flashing ratchets (also called pulsating ratchets) the time-dependent perturbation changes the potential shape without affecting its spatial periodicity. An example of a flashing ratchets is a ratchet that operates switching on and off a spatially periodic asymmetric potential. In this particular case it can be seen that a simple periodic or random switching can rectify thermal fluctuations and produce a net current of particles.

A new class of ratchets that use information on the state of the system to operate have been introduced in Ref. [6]. These feedback ratchets (or closed-loop ratchets) are able to increase the net current and the power output of collective Brownian ratchets [6, 7, 8, 9]. Feedback can be implemented monitoring the positions of the particles (see for example Refs. [10, 11]) and subsequently using the information gathered to decide whether to switch on or off the ratchet potential according to a given protocol. In addition, feedback ratchets have been recently suggested as a mechanism to explain the stepping motion of the two-headed kinesin [12].

The first feedback protocol proposed was the so-called instantaneous maximization of the center-of-mass velocity [6], which switches on the potential only if switching on would imply a positive displacement for the center-of-mass position (i.e., if the net force with the potential on would be positive). The instantaneous maximization protocol gives the maximum current in the case of one particle and performs better than any open-loop protocol for few particles. However, it has a very low performance in the many particle case given an average center-of-mass velocity smaller than that obtained with an optimal periodic protocol. (We call many particle case the case when the fluctuations of the net force are smaller than its maximum absolute value.) An improvement of the instantaneous maximization protocol is the threshold protocol [7], which consist on introducing two threshold values in order to switch the potential before the net force reaches a zero value. In this way, there is an increase of the performance for many particles up to velocity values equaling the ones of the optimal open-loop periodic protocol.
In order to check if it is experimentally feasible to obtain the increase of performance theoretically predicted for the few particle case one important question is to check the effects of time delays in the feedback that would be present in any experimental implementation. These time delays in the feedback come from the fact that the measure, transmission, processing, and action steps take a finite time interval \[13, 14\]. Time delays in the feedback also appear naturally in complex systems with self regulating mechanisms \[13, 16\]. Recently, we have investigated the effects that the delay has in the operation of feedback controlled ratchets in the few particle case \[17\]. We have found that even in the presence of time delays feedback controlled ratchets can give better performance than the corresponding optimal open-loop ratchet, although time delays decrease the performance.

In this paper we investigate the effects of time delays in the instant maximization protocol for the many particle case. We find that for small delays the asymptotic average center-of-mass velocity decreases for increasing delays (although not as fast as in the few particle case). However, if we continue increasing the time delay the average velocity starts to increase up to the value obtained for an optimal open-loop protocol. This surprising result makes that for many particles the instant maximization protocol gives greater average velocities in the presence of delay than in its absence. In Sec. II we present the evolution equations of the system. In the next section, Sec. III we briefly review the results for zero delays that will be useful, and thereafter we expose the results in the two dynamical regimes: small delays and large delays. Finally, in Sec. IV we summarize and discuss the results.

II. THE MODEL

The feedback ratchet we consider consists of \( N \) Brownian particles at temperature \( T \) in a periodic potential \( V(x) \). The force acting on the particles is \( F(x) = -V'(x) \), where the prime denotes spatial derivative. The state of this system is described by the positions \( x_i(t) \) of the particles satisfying the overdamped Langevin equations

\[
\gamma \dot{x}_i(t) = \alpha(t) F(x_i(t)) + \xi_i(t); \quad i = 1, \ldots, N, \quad (1)
\]

where \( \gamma \) is the friction coefficient (related to the diffusion coefficient \( D \) through Einstein’s relation \( D = k_B T / \gamma \)) and \( \xi_i(t) \) are Gaussian white noises of zero mean and variance \( \langle \xi_i(t) \xi_j(t') \rangle = 2 \gamma k_B T \delta_{ij} \delta(t - t') \). The control policy uses the sign of the net force per particle,

\[
f(t) = \frac{1}{N} \sum_{i=1}^{N} F(x_i(t)), \quad (2)
\]
as follows: The controller measures the sign of the net force and, after a time \( \tau \), switches the potential on (\( \alpha = 1 \)) if the net force was positive or switches the potential off (\( \alpha = 0 \)) if the net force was negative. Therefore, the delayed control protocol considered is

\[
\alpha(t) = \Theta(f(t - \tau)), \quad (3)
\]

with \( \Theta \) the Heaviside function [\( \Theta(x) = 1 \) if \( x > 0 \), else \( \Theta(x) = 0 \)].

As ratchet potential we consider the ‘smooth’ potential of period \( L \) and height \( V_0 \) (Fig. 1a),

\[
V(x) = \frac{2V_0}{3\sqrt{3}} \left[ \sin \left( \frac{2\pi x}{L} \right) + \frac{1}{2} \sin \left( \frac{4\pi x}{L} \right) \right]. \quad (4)
\]

We have also verified that analogous results are obtained for the ‘saw-tooth’ potential of period \( L \), i.e. \( V(x) = V(x + L) \), height \( V_0 \), and asymmetry parameter \( \alpha < 1/2 \) (Fig. 1b),

\[
V(x) = \begin{cases} \frac{V_0 x}{L} & \text{if } 0 \leq \frac{x}{L} \leq \alpha, \\ V_0 - \frac{V_0}{2} \left( \frac{x}{L} - \alpha \right) & \text{if } \alpha < \frac{x}{L} \leq 1. \end{cases} \quad (5)
\]
The height \( V_0 \) of the potential is the potential difference between the potential at the minimum and at the maximum, while \( \alpha L \) is the distance between the minimum and the maximum positions. In view of this definition, the ‘smooth’ potential (4) has asymmetry parameter \( \alpha = 1/3 \).

Throughout the rest of this paper, we will use units where \( L = 1, k_B T = 1, \) and \( D = 1 \).

We consider in this paper the many particles case that is characterized by the fact that the typical fluctuations

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Panel (a): ‘Smooth’ potential (4) for \( V_0 = 5k_B T \). Panel (b): ‘Saw-tooth’ potential (5) for \( V_0 = 5k_B T \) and \( \alpha = 1/3 \). Units: \( L = 1, k_B T = 1 \).}
\end{figure}
of the net force are smaller than the maximum values of its absolute value.

III. DELAYED MANY PARTICLE FEEDBACK RATCHET

We study the effects of time delays in the previous feedback controlled Brownian ratchets in the many particles case, considering both the ‘smooth’ potential and the ‘saw-tooth’ potential for various potential heights and different initial conditions.

We find that the system presents two regimes separated by a delay \( \tau_{\text{min}} \) for which the center-of-mass velocity has a minimum; see Fig. 2. In the small delay regime (\( \tau < \tau_{\text{min}} \)) the flux decreases with increasing delays as one could expect. On the contrary, in the large delay regime (\( \tau > \tau_{\text{min}} \)) we have observed and explained a surprising effect, namely, the center-of-mass velocity increases for increasing delays and the system presents several stable solutions. We have found that this critical time delay \( \tau_{\text{min}} \) is inversely proportional to the potential height \( \tau_{\text{min}} \propto 1/V_0 \) with a proportionality constant that mildly depends on the number of particles.

A. Zero delay

The many particle ratchet in absence of delay (i.e., \( \tau = 0 \) in the model of Sec. II) have been studied in Ref. [6]. It has been shown that the net force per particle exhibits a quasideterministic behavior that alternates large periods of time \( t_{\text{on}} \) with \( f(t) > 0 \) (on dynamics) and large periods of time \( t_{\text{off}} \) with \( f(t) < 0 \) (off dynamics). The center-of-

mass velocity can be computed as

\[
\langle \dot{x}_{\text{cm}} \rangle = \frac{\Delta x(t_{\text{on}})}{t_{\text{on}} + t_{\text{off}}},
\]

with

\[
\Delta x(t_{\text{on}}) = \Delta x_{\text{on}}[1 - e^{-t_{\text{on}}/(2\Delta t_{\text{on}})}],
\]

where \( \Delta x_{\text{on}} \) and \( \Delta t_{\text{on}} \) are obtained fitting the displacement during the ‘on’ evolution for an infinite number of particles (see Ref. [8] for details).

On the other hand, for many particles the fluctuations of the net force are smaller than the maximum value of the net force. This allows the decomposition of the dynamics as the dynamics for an infinite number of particles plus the effects of the fluctuations due to the finite value of \( N \). The late time behavior of the net force \( f(t) \) for an infinite number of particles is given for the on and off dynamics by [6],

\[
f_{\infty}(t) = C_\nu e^{-\lambda_\nu(t-\tau_0)} \text{ with } \nu = \text{on, off}
\]

The coefficients \( C_\nu, \lambda_\nu, \) and \( \tau_0 \) can be obtained fitting this expression with the results obtained integrating a mean field Fokker-Planck equation obtained in the limit \( N \rightarrow \infty \) and without delay; see Refs. [6, 8] for details.

For a finite number of particles the fluctuations in the force induce switches of the potential and the times on and off are computed equating \( f_{\text{on}}(t) \) to the amplitude of the force fluctuations, resulting [6]

\[
t_{\text{on}} + t_{\text{off}} = b + d \ln N,
\]

with \( b = C_{\text{on}} + C_{\text{off}} \) and \( d = (\lambda_{\text{on}} + \lambda_{\text{off}})/(2\lambda_{\text{on}}\lambda_{\text{off}}) \).

B. Small delays

For small delays, \( \tau < \tau_{\text{min}} \), we observe that the flux decreases with the delay. See Fig. 2. We have seen that this decrease is slower than that found for the few particle case [17], and that the expressions derived to describe this decrease in the few particles case does not hold here. However, the decrease observed here can be understood by the fact that the delay implies an increase of the time interval between switches, which makes the tails of the force distribution less smooth than for no delay. See Fig. 8. The main effect of the delay is to stretch the ‘on’ and ‘off’ times of the dynamics; then, using the many particle approximation [6] we can write

\[
\langle \dot{x}_{\text{cm}} \rangle = \frac{\Delta x_{\text{on}}}{t_{\text{on}} + t_{\text{off}} + \Delta \tau} = \frac{\Delta x_{\text{on}}}{b + d \ln N + \Delta \tau},
\]

where we have found that the increase of the length of the on-off cycle \( \Delta \tau \) is proportional to the delay \( \Delta \tau \propto \tau \).
This increase is due to a change in the dynamical regime: for \( \tau > \tau_{\text{min}} \) the present net force starts to be nearly synchronized with the net force a time \( \tau \) ago. This self-synchronization gives rise to a quasiperiodic solution of period \( T = \tau \). Note that there is not a strict periodicity due to stochastic fluctuations in the ‘on’ and ‘off’ times. Looking at the \( f(t) \) dependence, Fig. 2, we see that the solutions stabilized by the selfsynchronization are similar to those obtained with the threshold protocol \( 2, 8 \). In Fig. 5 we show that the threshold protocol that has the same period gives similar center-of-mass velocity values, confirming the picture. (Differences are due to the fact that we have considered for the threshold protocol simulations with on and off thresholds of the same magnitude, while Fig. 4 shows that the effective thresholds are different.)

This picture allows to understand the increase of velocity for increasing delay, and the presence of a maximum. This maximum is related with the optimal values of the thresholds that have been shown in \( 5 \) to give a quasiperiodic solution of period \( T_{\text{on}} + T_{\text{off}} \), with \( T_{\text{on}} \) and \( T_{\text{off}} \) the optimal ‘on’ and ‘off’ times of the periodic protocol. Therefore, if we know the values of \( T_{\text{on}} \) and \( T_{\text{off}} \) for the optimal periodic protocol \( [T_{\text{on}} \sim (1 - a)^2/V_0 \text{ and } T_{\text{off}} \sim a^2/2] \) we can predict that the maximum of the center-of-mass velocity is reached for a delay

\[
\tau_{\text{max}} = T_{\text{on}} + T_{\text{off}},
\]

and has a value

\[
\langle \dot{x}_{\text{cm}} \rangle_{\text{closed}}(\tau_{\text{max}}) = \langle \dot{x}_{\text{cm}} \rangle_{\text{open}}^{\text{max}},
\]

with \( \langle \dot{x}_{\text{cm}} \rangle_{\text{open}}^{\text{max}} \) the center-of-mass velocity for the optimal open-loop protocol. Thus, this expression gives the position and height of the maximum of the delayed feedback control protocol in terms of the characteristic values of the optimal open-loop control. In particular, it implies that the position and height of the maximum for the flux is independent of the number of particles.

As an example we can apply these expressions to the ‘smooth’ potential with \( V_0 = 5 \) that for the optimal periodic protocol gives \( \langle \dot{x}_{\text{cm}} \rangle_{\text{open}} = 0.44 \) for \( T_{\text{on}} = 0.06 \) and \( T_{\text{off}} = 0.05 \), so we obtain \( \tau_{\text{max}} = 0.06 + 0.05 = 0.11 \) in agreement with Fig. 2.

For values of the delay of the order of or larger than \( \tau_{\text{max}} \) quasiperiodic solutions of other periods start to be stable; see Fig. 5. The periods for the net force \( f(t) \) that are found are those that fit an integer number of periods inside a time interval \( \tau \), verifying that the present net force is synchronized with the net force a time \( \tau \) ago,
that is, the quasiperiodic solutions have periods $T = \tau /2$, $T = \tau /3$, ... In addition, it can be seen that the center-of-mass velocity of the $n$ branch $\langle \dot{x}_{cm}\rangle_{\tau /n}$ whose $f(t)$ has period $T = \tau /n$ is related with that of the $T = \tau$ branch through

$$\langle \dot{x}_{cm}\rangle_{\tau /n}(\tau) = \langle \dot{x}_{cm}\rangle_{\tau}(\tau /n).$$

We highlight that several branches can be stable for the same time delay $\tau$. Whether the system finally goes to one or another stable solution depends on the initial conditions and on the particular realization of the noise. See Fig. [5]. For these branches we have found initial conditions that goes to these solutions and that remain in them during several thousands of periods, indicating that they are stable solutions or at least metastable solutions with a large lifetime.

The analogy with the threshold protocol allows to use the analytic results of [8] to get further insight in the numerical results. The behavior for large delays for the $T = \tau$ branch can be obtained using the relation

$$\langle \dot{x}_{cm}\rangle = \frac{\Delta x(\tau)}{\tau},$$

with $\Delta x(\tau)$ given by Eq. [4]. This equation gives a good prediction for the largest delays of the first branch (see Fig. [2]).

On the other hand, for very large values of the delays of the first branch the solutions in a given branch start to become unstable, what can be understood noting that this happens when the fluctuations of the net force become of the order of the absolute value of the net force. Thus, the maximum delay that gives a stable solution in the first branch is

$$\tau_{\text{inst}} = t_{\text{on}} + t_{\text{off}} = b + d \ln N,$$

where $b$ and $d$ are determined as in Eq. [9]. For example, for the ‘smooth’ potential with $V_0 = 5$, which has $b = -0.070$ and $d = 0.031$, we obtain for $N = 10^5$ particles the value $\tau_{\text{inst}} = 0.29$ in accordance with the numerical results shown in Figs. [2] and [5].

The previous results for the first branch, Eqs. [13] and [15], can be extended to other branches by direct application of the relation [13].

**IV. CONCLUSIONS**

We have studied the effects of time delays in the many particle case, where surprising and interesting results arise. Although in the many particle case without delay the instantaneous maximization protocol performs worst than the optimal open-loop protocol, the introduction of a delay can increase the center-of-mass velocity up to the values given by the optimal open-loop control protocol.

For small delays the asymptotic average velocity decreases for increasing delays, until it reaches a minimum. After this minimum, a change of regime happens and the system enters a selfsynchronized dynamics with the net force at present highly correlated with the delayed value of the net force used by the controller. This selfsynchronization stabilizes several of the quasiperiodic solutions that can fit an integer number of periods in a time interval of the length of the time delay. The stable quasiperiodic solutions have an structure similar to those solutions appearing in the threshold protocol. This analogy has allowed us to make numerical and analytical predictions using the previous results for the threshold protocol [8]. In particular, we have established the location and value of the maximum, and also the value of the time delay beyond which a quasiperiodic solution becomes unstable. The results obtained shown that for most time delays several solutions are stable and therefore the systems presents multistability; which stable solution is reached depends on the past history of the system.

The possibility to choose the quasiperiod of the solution we want to stabilize just tuning the time delay can have potential applications to easily control the particle flux. Note that we can even leave some branch just going to time delays where the branch is already unstable, and force the system to change to another branch of solutions.

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