ORDERABLE GROUPS AND BUNDLES

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Abstract. We define what is meant by a strict total order in a category having subobjects, products and fibre products. This allows us to define the notions of an ordered bundle \( X \) and an ordered \( G \)-set; when \( G = \pi_1(X) \) we relate these structures to orderings of \( \pi_1(X) \). We apply this to prove a theorem of Farrell \cite{farrell} relating right-orderings of \( \pi_1(X) \) to embeddings of the universal cover \( \tilde{X} \hookrightarrow \mathbb{R} \times X \), and generalize it by relating bi-orderings of \( \pi_1(X) \) to embeddings of the path space \( P(X) \hookrightarrow \mathbb{R} \times X \times X \).

1. Introduction

A group \( G \) is right-orderable if there exists a strict total ordering \( < \) of the elements of \( G \) such that \( g < h \) implies \( gf < hf \) for all \( f, g, h \in G \). A group \( G \) is bi-orderable if there exists a right-ordering of \( G \) that is also invariant under left multiplication, so \( g < h \) implies \( fg < fh \) for all \( f, g, h \in G \).

There is a well-documented connection between orderability and topology. A number of recent articles have focused on orderability of the fundamental group of a 3-manifold and its topological implications \cite{2, 4, 3}. Another strong direction of research is the investigation of the space of orderings of a group, and the algebraic consequences of its topological structure \cite{8, 11, 9, 10}.

In the body of work connecting topology and orderability, a notable outlier is the remarkable work of Farrell \cite{farrell}, which re-characterizes right-orderability of fundamental groups in terms of covering spaces. While his theorem is very appealing in its statement (see Corollary 4.6), to the best of our knowledge it has yet to be practically applied in any other topological works.

This note represents an effort to place Farrell’s theorem in a natural topological context. As a result, our new approach provides a different proof of a more general statement, and allows for the analysis of both right-orderable and bi-orderable groups.

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2. Some categories, notation and background

Though left-orderability has become popular amongst topologists, our choice of right-orderability is a consequence of several common topological conventions. In particular, the definition of concatenation of paths \( f \) and \( g \) as

\[
(f \ast g)(s) = \begin{cases} 
    f(2s) & \text{if } s \in [0, \frac{1}{2}], \\
    g(2s - 1) & \text{if } x \in [\frac{1}{2}, 1].
\end{cases}
\]

naturally produces a right action on the fibres of the standard construction of the universal covering space.

That construction is as follows. For the remainder of this paper fix a path connected, locally path connected and semilocally simply connected space \( B \) with basepoint \( b_0 \), and set \( G = \pi_1(B, b_0) \). Recall that the universal covering space of \( B \) is constructed as

\[
\widetilde{B} = \{ \alpha : [0, 1] \to B | \alpha(0) = b_0 \}/\sim
\]

where the equivalence \( \sim \) is homotopy fixing the endpoints. Thus, an element of \( \widetilde{B} \) is an equivalence class \([\alpha]\) of a path \( \alpha : [0, 1] \to B \) satisfying \( \alpha(0) = b_0 \).

The required covering map \( p : \widetilde{B} \to B \) has formula \( p([\alpha]) = \alpha(1) \), in \( \widetilde{B} \) the basepoint is the equivalence class of the constant path \( \alpha(t) = b_0 \) for all \( t \in [0, 1] \).

The space \( \widetilde{B} \) admits a left action of the group \( G \) as follows. Given \( g \in G \) represented by a loop \( \gamma : [0, 1] \to B \) and \( \alpha \in \widetilde{B} \), define a left action (by deck transformations) by \( g \cdot [\alpha] = [\gamma \ast \alpha] \). On the other hand, the fibre \( F = p^{-1}(b_0) \) admits a right action defined by \( [\alpha] \cdot g = [\alpha \ast \gamma] \), in fact the fibre (over \( b_0 \)) of any covering space \( E \to B \) admits a right action defined in a similar way using lifts of paths.

Let \( \textbf{Cov}(B) \) denote the category whose objects are covering spaces \( p : E \to B \) of \( B \). Given two objects \( p_1 : E_1 \to B \) and \( p_2 : E_2 \to B \) of \( \textbf{Cov}(B) \), a morphism between them is a continuous map \( f : E_1 \to E_2 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
E_1 & \xrightarrow{f} & E_2 \\
\downarrow{p_1} & & \downarrow{p_2} \\
B & \xrightarrow{p} & B
\end{array}
\]

A right \( G \)-set is a set \( X \) equipped with a right action of a group \( G \). Given \( x \in X \) we will write the action of \( g \in G \) on \( x \) as \( x \cdot g \). Together with \( G \)-equivariant maps of sets, \( G \)-sets form a category \( G\text{-Set} \).

Define the so-called fibre functor \( F : \textbf{Cov}(B) \to G\text{-Set} \) by \( F(p : E \to B) = p^{-1}(b_0) \). Then \( F \) is the functor that chooses the fibre above \( b_0 \) for every covering space \( p : E \to B \). The action of \([\gamma] \in G \) on \( e \in p^{-1}(b_0) \) is defined as the end of the unique path lifting \( \gamma \) in \( E \) starting from \( e \). On
maps $f$ between covering spaces, the functor $F$ acts by restricting $f$ to the fibre above $b_0$. This map is always equivariant for the action of $G$.

On the other side, let $U : G\text{-}Set \to \text{Cov}(B)$ be the functor that associates to a $G$-set $X$ the covering space

$$U(X) = p : X \times_G \tilde{B} \to B$$

where $X \times_G \tilde{B} = (X \times \tilde{B})/\sim$ for $([\alpha], x) \sim (g^{-1}\cdot [\alpha], x \cdot g)$ for all $g \in G$ and the map $p$ has formula $p([\alpha], x) = \alpha(1)$. Here, the set $X$ is topologized with the discrete topology, and the topology on $U(X)$ is a quotient of the product topology. For maps, define the action of $U$ as follows: $U(f : X \to X') = h$, where

$$h : (X \times \tilde{B})/\sim \to (X' \times \tilde{B})/\sim$$

has formula $h([\alpha], x) = ([\alpha], f(x))$.

**Proposition 2.1.** The functors $F : \text{Cov}(B) \to G\text{-}Set$ and $U : G\text{-}Set \to \text{Cov}(B)$ are inverse equivalences of categories.

**Proof.** One can check that $F \circ U \cong \text{Id}_{G\text{-}Set}$ and $U \circ F \cong \text{Id}_{\text{Cov}(B)}$ [7, Chapter 3, Sections 6 and 8].

\[ \square \]

**Example 2.2.** Let $\ast$ be the trivial action of $G$ of a one element set, the corresponding covering space is the trivial cover $B \to B$. The object $\ast$ is the terminal object of the category $G\text{-}Set$, and $B \to B$ is terminal in the category of coverings.

**Example 2.3.** Let $G_r$ be the right $G$-set defined by $G$ acting on itself by multiplication on the right. By construction, the fibre of $\tilde{B} \to B$ at $b_0$ is $G$ and the right $G$-set corresponding to $\tilde{B}$ is $G_r$.

**Example 2.4.** Let $H$ be a subgroup of $G$, the right action of $G$ on itself defines a right action of $G$ on the set $H \setminus G$ of right $H$-cosets. The corresponding covering space is the quotient $H \setminus \tilde{B}$ of $\tilde{B}$ by the left action of $H$.

**Example 2.5.** Let $G_{\text{conj}}$ be the right action of $G$ on itself by conjugation: $h \cdot g = g^{-1}hg$ and let $\tilde{B}_{\text{conj}}$ be the corresponding covering space. Its fibre at $b \in B$ is naturally isomorphic to $\pi_1(B, b)$, so each fibre is equipped with a group structure. In particular, $\tilde{B}_{\text{conj}}$ has at least one global section given by the unit element in each fibre, so it is not connected.

**Example 2.6.** The product $G \times G$ is naturally the fundamental group of $B \times B$ at $(b_0, b_0)$, so there exists an equivalence between covering spaces of $B \times B$ and right $G \times G$-sets.

The path space $P(B)$ of $B$ is constructed in a similar way to the universal covering spaces. Set

$$P(B) = \{\alpha : [0, 1] \to B\}/\sim$$
where $\sim$ is homotopy fixing the endpoints of the paths. There’s a covering map $p : P(B) \to B \times B$ given by $p(\alpha) = (\alpha(0), \alpha(1))$. Observe that the fibre $p^{-1}(b_0, b_0)$ is precisely all those paths which begin and end at the point $b_0$, considered up to homotopy. Thus there is a canonical identification of the map $p$.

For a path $\alpha$ with $p(\alpha) = (b_0, b_0)$ the natural action of $G \times G$ on $f = [\alpha]$ is given by

$$f \cdot (g, h) = [\gamma^{-1} \ast \alpha \ast \beta] = g^{-1} fh,$$

where $\gamma, \beta$ are paths representing the fundamental group elements $g, h$.

The corresponding right $G \times G$-set is $rG_r$, defined as $G$ with an action by multiplication on the left and on the right: $f \cdot (g, h) = g^{-1} fh$.

Remark that the pull back of $P(B)$ along $B \to B \times B : b \mapsto (b_0, b)$ is exactly $\tilde{B}$.

It will be useful to generalize the equivalence of $\text{Cov}(B)$ and $G\text{-Set}$ by considering not only actions of $G$ on sets but also on any topological space. The category $G\text{-Space}$ of $G$-spaces is defined as that of topological spaces equipped with a right action of $G$ (continuous for the discrete topology on $G$) and $G$-equivariant maps. There is an obvious fully faithful functor

$$G\text{-Set} \subset G\text{-Space}.$$

For $E \to B$ a topological space over $B$, we shall denote by $\tilde{E}$ the pull-back of $E$ over $\tilde{B}$. A locally trivial bundle (we shall say simply a bundle) $p : E \to B$ is \textit{locally constant} if $\tilde{E} \to \tilde{B}$ is a trivial bundle. (Such bundles can be characterized as bundles for which there exists a choice of trivializing charts $\{(U_i, \phi_i)\}_{i \in I}$ such that the transition functions $t_{ij} : U_i \cap U_j \to G$ are constant, hence their name.) A map $f : E_1 \to E_2$ between two locally constant bundles over $B$ is said to be \textit{locally constant} if the induced map $\tilde{f} : \tilde{E}_1 \simeq F_1 \times \tilde{B} \to \tilde{E}_2 \simeq F_2 \times \tilde{B}$ over $\tilde{B}$ is constant, i.e. $\tilde{f}(x, \tilde{b}) = (g(x), \tilde{b})$ for some continuous map $g : F_1 \to F_2$. Let $\text{LCB}(B)$ be the category of locally constant bundles over $B$ and locally constant maps.

If $\text{Top}/B$ is the category of spaces over $B$ with maps over $B$, there is an obvious inclusion functor $\text{LCB}(B) \to \text{Top}/B$ which is faithful but not full (not every map between locally constant spaces over $B$ is locally constant).

Covering spaces over $B$ are a special case of locally constant $G$-bundles over $B$. It can be proved that every map of covering spaces is locally constant. The natural inclusion functor is thus full and faithful

$$\text{Cov}(B) \subset \text{LCB}(B).$$

Trivial bundles are also examples of locally constant bundles, but not every map between trivial bundles is locally constant.

\textbf{Proposition 2.7}. The functors $F : \text{LCB}(B) \to G\text{-Space}$ sending $p : E \to B$ to $p^{-1}(b_0)$ and $U : G\text{-Space} \to \text{LCB}(B)$ sending $X$ to $X \times_G \tilde{B} \to B$ are inverse equivalences of categories.
Lemma 2.8. If the $E$ of $E$ are locally constant bundles, so is $E$ over $E$ such that the map $F \rightarrow B$ is still a locally trivial map. If $E \rightarrow B$ is locally constant, not every sub-bundle is locally constant.

Proof. Let us assume that the locally constant bundles are stable by pull-backs and composition. If now the set $\text{im}(f) = \{ y \in E_2 | \exists x \in E_1, y = f(x) \}$ is defined as the image of the map of bundles is not a bundle, but in the case of locally constant bundles the image is a locally constant bundle: if $\tilde{f} : \tilde{E}_1 \rightarrow \tilde{E}_2 : (x, \tilde{b}) \mapsto (g(x), \tilde{b})$ is the map induced by $f$ over $B$, then $\text{im}(\tilde{f}) = \text{im}(f) = \text{im}(g) \times \tilde{B}$.

The diagonal $\Delta_E$ of a bundle $p : E \rightarrow B$ is defined as the image of the map $E \rightarrow E \times_B E : x \mapsto (x, x)$, it is isomorphic to $E \rightarrow B$.

Proof. Similar to proposition 2.1.

Summarizing, we have the following diagram of categories

$$
\begin{array}{ccc}
\text{Cov}(B) & \simeq & G\text{-Set} \\
\downarrow & & \downarrow \\
\text{LCB}(B) & \simeq & G\text{-Space}
\end{array}
$$

where the horizontal arrows are equivalences of categories and the vertical ones fully faithful embeddings.

We now recall some notions that will be needed in the next section. If $p : E \rightarrow B$ is a bundle, a sub-bundle of $E$ is defined as a subspace $F \subset E$ such that $F \rightarrow B$ is still a locally trivial map. If $E \rightarrow B$ is locally constant, not every sub-bundle is locally constant.

For $i = 1, 2, 3$, let $p_i : E_i \rightarrow B$ be three bundles over $B$ and let $f_1 : E_1 \rightarrow E_3$ and $f_2 : E_2 \rightarrow E_3$ be two bundle maps. The fibre product of $f_1$ and $f_2$, noted $E_1 \times_{E_3} E_2$ can be defined as follows. Consider $f_1 \times f_2 : E_1 \times E_2 \rightarrow E_3 \times E_3$, with each product topologized with the product topology. Then $f_1 \times f_2$ has the formula $(f_1 \times f_2)(e_1, e_2) = (f_1(e_1), f_2(e_2))$. Define

$$E_1 \times_{E_3} E_2 = \{(e_1, e_2) \in E_1 \times E_2 \mid f_1(e_1) = f_2(e_2)\},$$

and topologize it using the subspace topology. There is an obvious map $E_1 \times_{E_3} E_2 \rightarrow E_3$ and by composition with $p_3$ a map $p_4 : E_1 \times_{E_3} E_2 \rightarrow B$.

In the particular case where $E_3 = B$, $E_1 \times_B E_2$ is simply called the product of $E_1$ and $E_2$.

Lemma 2.8. If the $E_i$ are bundles over $B$, then $E_1 \times_{E_3} E_2 \rightarrow E_3$ is a bundle over $E_3$ and $E_1 \times_{E_3} E_2 \rightarrow B$ is a bundle over $B$. Moreover, if the $E_i$ are locally constant bundles so is $E_1 \times_{E_3} E_2 \rightarrow B$.

Proof. Let us assume that the $E_i$ are bundles, i.e. locally trivial maps, but such maps are stable by pull-backs and composition. If now the $E_i$ are locally constant bundles, we need to prove that $E_1 \times_{E_3} E_2 \rightarrow B$ is trivial when pulled-back on $\tilde{B}$. For this we use that $E_1 \times_{E_3} E_2 \simeq \tilde{E}_1 \times \tilde{E}_2$ and as all of $\tilde{E}_i$ are trivial bundles, so is their fibre product. □
All these notions restrict to covering spaces: any locally trivial sub-bundle of a covering space is a covering space and the fibre product of covering spaces is still a covering space, etc.

Let $X$ be a $G$-space, a subspace $Y \subset X$ is call a sub-$G$-space if it is stable by the action of $G$.

For $i = 1, 2, 3$, let $X_i$ be three $G$-spaces, and let $f_1 : X_1 \to X_3$ and $f_2 : X_2 \to X_3$ be two bundle maps. The fibre product of $f_1$ and $f_2$ is denoted $X_1 \times_{X_3} X_2$ and defined as follows. The underlying set is the fibre product of the underlying sets, i.e. the set of pairs $(x_1, x_2)$ such that $f_1(x_1) = f_2(x_2)$ and the action is defined by $(x_1, x_2) \cdot g = (x_1 \cdot g, x_2 \cdot g)$. In the case where $E_3$ is the $G$-set * (see Example 2.2), the fibre product is simply called the product and is denoted $X_1 \times X_2$.

The image of a morphism $f : X_1 \to X_2$ of $G$-spaces is defined as the set $\text{im}(f) = \{y \in X_2 | \exists x \in X_1, y = f(x)\}$ topologized with the subspace topology. It is clear that the action of $G$ on $X_2$ restricts to $\text{im}(f)$.

The diagonal $\Delta_X$ of a $G$-space $X$ is defined as the image of the map $X \to X \times X : x \mapsto (x, x)$ with the obvious action. It is isomorphic to $X$.

All these notions restricts to $G$-sets: any sub-$G$-space of a $G$-set is a $G$-set and the fibre product of $G$-sets is still a $G$-set, etc.

The notions of subobjects, products, fibre products, image and diagonal can be defined in purely categorical terms and are thus preserved under equivalences of categories [6]. As a consequence, all previous notions on locally constant bundles over $\mathcal{B}$ correspond to the same notion of $G$-spaces. For example, there is a bijection between locally trivial sub-bundles of a locally trivial bundle $E \to B$ and sub-$G$-spaces of the corresponding fibre $X$.

3. ORDERED STRUCTURES

We start by recalling some facts about relations and total orders on a given set. If $R \subset X \times X$ is a relation on a set $X$, the opposite relation $R^{op}$ is defined as $R^{op} = \{(x, y) | yRx\}$, or equivalently as the inclusion $R \to X \times X \cong X \times X$ where the map $\sigma$ is the permutation of the two factors.

For $i = 1, 2$, if $R_i \subset X \times X$ are two relations on $X$, their composition $R_2 \circ R_1$, is defined as $R_2 \circ R_1 = \{(x, z) | \exists z \in X, xR_1y \text{ and } yR_2z\}$. Equivalently, if $s_i$ and $t_i$ are respectively the first and second projections of $R_i$ on $X$, the fibre product of $t_1 : R_1 \to X$ and $s_2 : R_2 \to X$ is the set $R_1 \times_X R_2 : \{(x, y, z) | xR_1y \text{ and } yR_2z\}$. The maps $s_1$ and $t_2$ define a map $R_1 \times_X R_2 \to X \times X$ and $R_2 \circ R_1$ can be defined as the image of this map.

Recall that a strict total order on a set $X$ is a relation $<$ such that:

1. (Transitivity) $(x < y \text{ and } y < z) \Rightarrow x < z$,
2. (Irreflexivity) not $x < x$,
3. (Antisymmetry) not $(x < y \text{ and } y < x)$,
4. (Totality) $x \neq y \Rightarrow (x < y \text{ or } y < x)$. 

If \((X_1, \prec_1)\) and \((X_2, \prec_2)\) are two strictly totally ordered sets, a map \(f : X_1 \to X_2\) is order-preserving if \(x \prec_1 y \Rightarrow f(x) \prec_2 f(y)\).

Let \(R\) be the subobject of \(X \times X\) corresponding to a strict total order \(<\). Using the previous considerations, these axioms can be written as the following conditions on subobjects of \(X \times X\):

1. (Transitivity) \(R \circ R \subset R\),
2. (Irreflexivity) \(\Delta_X \cap R = \emptyset\) (where \(\Delta_X \subset X \times X\) is the diagonal),
3. (Antisymmetry) \(R \cap R^{\text{op}} = \emptyset\),
4. (Totality) \(R \cup R^{\text{op}} = X \times X \setminus \Delta_X\).

If \((X_1, \prec_1)\) and \((X_2, \prec_2)\) are two strictly totally ordered sets, corresponding to subobjects \(R_1 \subset X_1 \times X_1\) and \(R_2 \subset X_2 \times X_2\), a map \(f : X_1 \to X_2\) is order-preserving if and only if \(f \times f\) sends \(R_1\) into \(R_2\).

**Example 3.1.** Let \((\mathbb{R}, <)\) be the set of real numbers with its canonical strict total order, the corresponding subobject \(R_\prec \subset \mathbb{R}^2\) is the set \(R_\prec = \{(x, y) | x < y\}\).

An important remark for the sequel is that these definitions make sense not only for sets but in any category \(\mathcal{C}\) where subobjects, products and fibre products exist. In particular these definitions make sense in the categories \(\text{Cov}(B), \text{LCB}(B), \text{G-Set}\) and \(\text{G-Space}\).

Let \(\mathcal{C}\) be one of the categories \(\text{Cov}(B), \text{LCB}(B), \text{G-Set}\) and \(\text{G-Space}\). A relation on an object \(X \in \mathcal{C}\) is a subobject \(R \subset X \times X\). The opposite of a relation \(R\) is the relation \(R^{\text{op}}\) defined by \(R \subset X \times X \overset{\sigma}{\cong} X \times X\), where \(\sigma\) is the permutation of the two factors. For two relations \(R_1 \subset X \times X\) and \(R_2 \subset X \times X\), if \(s_1\) and \(t_1\) are respectively the first and second projections of \(R_1\) on \(X\), the fibre product of \(t_1 : R_1 \to X\) and \(s_2 : R_2 \to X\) is the set \(R_1 \times_X R_2 : \{(x, y, z) | xR_1 y y R_2 z\}\). The composition \(R_2 \circ R_1\) is defined as the image of the map \(R_1 \times_X R_2 \to X \times X\) induced by \(s_1\) and \(t_2\).

**Definition 3.2.** A relation \(R \subset X \times X\) on an object of \(\mathcal{C}\) is said to be a strict total order if it satisfies the following properties

1. (Transitivity) \(R \circ R \subset R\),
2. (Irreflexivity) \(\Delta_X \cap R = \emptyset\),
3. (Antisymmetry) \(R \cap R^{\text{op}} = \emptyset\),
4. (Totality) \(R \cup R^{\text{op}} = X \times X \setminus \Delta_X\).

In particular, for such an order \(R\), we have \(X \times X = R \sqcup R^{\text{op}} \sqcup \Delta_X\) where \(\sqcup\) denotes the disjoint union.

**Definition 3.3.** A locally constant bundle \(p : E \to B\) is orderable if there exists a locally constant sub-bundle \(R \subset E \times_B E\) such that \(R\) satisfies the axioms of a strict total order.

Suppose that \(E_1\) and \(E_2\) are ordered bundles with order relations \(R_1 \subset E_1 \times_B E_1\) and \(R_2 \subset E_2 \times_B E_2\). We call a map \(f : E_1 \to E_2\) order-preserving if the map \(f \times f : E_1 \times_B E_1 \to E_2 \times_B E_2\) satisfies \((f \times f)(R_1) \subset R_2\).
Lemma 3.4. If \( R \subseteq E \times_B E \) is a strict total order on some bundle \( E \to B \), and if \( B' \to B \) is any continuous map, the relation \( R' \subseteq E' \times_{B'} E' \) where \( E' = E \times_B B' \) and \( R' = R \times_{B \times B} B' \times B' \) is a strict total order on \( E' \to B' \).

This lemma applies in particular in the case where \( B' \) is a point, and says that a strict total order on \( E \to B \) gives a strict total order in every fibre of \( E \to B \). In the same way a bundle can be thought of as the continuous family of its fibres, a strict total order on a bunde can be thought of as a continuous family of strict total orders.

We will not use the following but it gives a way to construct ordered bundles.

Lemma 3.5. If \( f : E_1 \to E_2 \) is an injective locally constant map of locally constant bundles and if \( R_2 \subseteq E_2 \times_B E_2 \) is a strict total order on \( E_2 \), then \( R_1 = (f \times f)^{-1}(R_2) \) is a strict total order on \( E_1 \).

Proof. \( R_1 \) is a locally trivial sub-bundle of \( E_1 \times_B E_1 \) by Lemma 2.8. Properties (2)-(5) of \( R \) are easy to check. Note that injectivity of \( f \) is required only to prove (5). \( \square \)

Example 3.6. Recall from Example 3.1 \( R_\prec = \{(x,y) | x < y \} \subseteq \mathbb{R}^2 \) defining the canonical order on \( \mathbb{R} \). Let \( \mathbb{R}_B = \mathbb{R} \times B \to B \) be the trivial line bundle, then \( \mathbb{R}_B \times_B \mathbb{R}_B \simeq \mathbb{R} \times \mathbb{R} \times B \to B \) is again a trivial bundle. The trivial sub-bundle \( R_\prec \times_B B \to B \) is a locally constant sub-bundle of \( \mathbb{R}_B \times_B \mathbb{R}_B \to B \) which defines a strict total order on \( \mathbb{R}_B \). Fibrewise, this order is nothing more than the canonical order on \( \mathbb{R} \).

Example 3.7. If \( \alpha : G \to \text{Homeo}_+(\mathbb{R}) \) is a group morphism, \( G \) acts on \( \mathbb{R} \) and \( R_\prec \). Then, there is a canonical strict total order \( R_\alpha \) on \( \mathbb{R} \times_G B \) given by \( R_\alpha = R_\prec \times_G B \).

We can apply Definition 3.2 to \( G \)-spaces.

Definition 3.8. A \( G \)-space \( X \) is orderable if there exists a \( G \)-invariant subspace \( R \subseteq X \times X \) satisfying the axioms of a strict total order.

Suppose that \( X_1 \) and \( X_2 \) are ordered bundles with order relations \( R_1 \subseteq X_1 \times X_1 \) and \( R_2 \subseteq X_2 \times X_2 \). A map \( f : X_1 \to X_2 \) order-preserving if the map \( f \times f : X_1 \times X_1 \to X_2 \times X_2 \) satisfies \( (f \times f)(R_1) \subseteq R_2 \).

In the case of a \( G \)-space, a relation \( R \) provides a strict total ordering \( < \) of the underlying space \( X \). Moreover, since \( R \) is \( G \)-invariant, the ordering is invariant under the right action of \( G \): for all \( x, y \in X \), \( x < y \) implies \( x \cdot g < y \cdot g \).

Example 3.9 (Right-orderable groups). Recall the right \( G \)-set \( G_r \) from Example 2.3. The group \( G \) is right-orderable if and only if \( G_r \) is orderable as a \( G \)-set. If \( G \) has ordering \( < \), there is a corresponding \( R \subseteq G_r \times G_r \) defined by \( (g,h) \in R \) if and only if \( g < h \). Conversely, given \( R \subseteq G_r \times G_r \),
since $R$ is $G$-invariant the same rule defines a right-invariant ordering $<$ of the elements of $G$.

**Example 3.10** (Bi-orderable groups). Recall the right $G \times G$-set $\ell G_r$ from Example 2.6. The group $G$ is bi-orderable if and only if $\ell G_r$ is an orderable $(G \times G)$-set. If $G$ is bi-orderable with ordering $<$, define $R \subset \ell G_r \times \ell G_r$ according to the rule $(g, h) \in R$ if and only if $g < h$. Since the ordering of $G$ is bi-invariant, $R$ is $(G \times G)$-invariant.

Conversely, if $\ell G_r$ is orderable, then $R \subset \ell G_r \times \ell G_r$ defines a bi-ordering of $G$ by the same rule: $g < h$ if and only if $(g, h) \in R$. Two-sided invariance of this ordering follows from $(G \times G)$-invariance of $R$. For if $g < h$, then $(g, h) \in R$ and for all $f \in G$ we have

$$(g, h) \cdot (f^{-1}, id) \in R \iff (fg, fh) \in R \iff fg < fh.$$ 

The calculation to show right-invariance is similar.

**Proposition 3.11.** Let $p : E \to B$ be a locally constant bundle and $X$ its fibre at $b_0$. Then $p : E \to B$ is orderable as a bundle if and only if $X$ is orderable as a $G$-space.

**Proof.** Each of the properties (1)-(5) of ordered bundles and ordered $G$-spaces is preserved under equivalence of categories, since each of properties (1)-(5) of the subobject $R$ can be defined diagrammatically. □

We also point out that order-preserving maps of bundles are sent to order-preserving maps of $G$-sets by the functor $F$, and reciprocally for the functor $U$.

4. Results

We begin with some well-known lemmas, for example see Proposition 2.1.

**Lemma 4.1.** If $(X, <)$ is a countable strictly totally ordered set, then there exists an order-preserving embedding $X \hookrightarrow \mathbb{R}$.

**Proof.** Given an enumeration of $X = \{x_0, x_1, \ldots\}$, construct an order-preserving map $t : X \to \mathbb{R}$ as follows. Set $t(x_0) = 0$, and if $t(x_0), \ldots, t(x_i)$ have already been defined, there are three cases to consider when defining $t(x_{i+1})$. If $x_{i+1}$ is either larger or smaller than all of $\{x_0, \ldots, x_i\}$, set

$$t(x_{i+1}) = \begin{cases} \max\{t(x_0), \ldots, t(x_i)\} + 1 & \text{if } x_{i+1} > \max\{x_0, \ldots, x_i\} \\ \min\{t(x_0), \ldots, t(x_i)\} - 1 & \text{if } x_{i+1} < \min\{x_0, \ldots, x_i\} \end{cases}$$

If neither of these cases hold, then there exist $j, k \in \{0, \ldots, i\}$ such that $x_j < x_{i+1} < x_k$ and there is no $i_0 \in \{0, \ldots, i\}$ such that $g_j < g_{i_0} < g_k$. In this case set

$$t(x_{i+1}) = \frac{t(x_j) + t(x_k)}{2}$$

This map preserves the order by construction and is injective. □
Lemma 4.2. [4] cf. Lemma 1.10] If $X$ is an orderable $G$-set and $t : X \to \mathbb{R}$ an order-preserving embedding, then there exists a right action of $G$ on $\mathbb{R}$ by order-preserving homeomorphisms making $t$ a $G$-equivariant map of $G$-sets.

Proof. Outside of the interval

$$I = (\inf(t(X)), \sup(t(X)))$$

define the action of $G$ to be the trivial action.

Inside the interval $I$ we proceed as follows. Evidently $G$ acts in an order-preserving way on $t(X)$ according to the rule $(t(e)) \cdot g = t(e \cdot g)$, we extend this action to the closure $\overline{t(X)}$ in such a way that every $g \in G$ acts continuously on $t(X)$. Specifically, for $x \in \overline{t(X)} \setminus X$ choose a monotone sequence of points $t(x_i)$ converging to $x$ from above (or below). Assume that the sequence is monotone increasing (the case of monotone decreasing is similar), since $x \in I$ there exists $x' \in X$ with $t(x') > x$. For every element $g \in G$, define the action of $g$ on $x$ as follows:

$$x \cdot g = \lim_{i \to \infty} t(x_i \cdot g)$$

This limit exists because the sequence $\{t(x_i \cdot g)\}$ is monotone increasing and bounded above by $t(x' \cdot g)$. It is not hard to see that this definition is independent of our choice of sequence $\{t(x_i)\}$.

Now $\overline{t(X)}$ is closed, and the complement $\mathbb{R} \setminus \overline{t(X)}$ is a union of open intervals with the $G$-action defined at their endpoints. Extend the $G$-action affinely across all intervals, this defines the required action of $G$ on $\mathbb{R}$. □

The next lemma allows us to understand those bundles with structure group $\text{Homeo}_+(\mathbb{R})$.

Lemma 4.3. [5] Theorem 1.1.1] Equipped with the compact open topology, the group $\text{Homeo}_+(\mathbb{R})$ is a contractible space.

Proof. For every $f \in \text{Homeo}_+(\mathbb{R})$ and $t \in [0, 1]$ define

$$H(f, t) = (1 - t)f(x) + tx.$$  

If $f$ is order-preserving, then $H(f, t)$ is order-preserving for all $t \in [0, 1]$, and $H(f, 0) = f$ while $H(f, 1) = \text{id}$. Therefore $\text{Homeo}_+(\mathbb{R})$ is contractible. □

In the following proposition and theorem, we will use the notation $p : E \to B$ to denote a bundle that is isomorphic to

$$U(X) = X \times G \tilde{B} \to B$$

for some $G$-set $X$. In particular, the fibre of $E$ over $b_0$ is isomorphic to $X$.

Proposition 4.4. For an ordered $G$-set $X$, the following are equivalent:

1. There exists an order-preserving injective map $t : X \hookrightarrow \mathbb{R}$.
2. There exists an order-preserving injective map of bundles $f : E \hookrightarrow \mathbb{R} \times B$. Here, $\mathbb{R} \times B$ is equipped with the natural order of $\mathbb{R}$ on the fibres.
Proof. By Lemma 4.2, \( \mathbb{R} \) can be equipped with a \( G \)-action so that \( t \) is an equivariant map. Thus the map \( t : X \to \mathbb{R} \) can be considered as a map of \( G \)-spaces.

Correspondingly there is an order-preserving locally constant map \( f' \) of locally constant bundles

\[
\begin{array}{ccc}
E & \xrightarrow{f'} & \mathbb{R} \times G \widetilde{B} \\
\downarrow & & \downarrow \\
B & \xleftarrow{\pi_2} & \mathbb{R} \times B
\end{array}
\]

The structure group of \( \mathbb{R} \times G \widetilde{B} \) is \( \text{Homeo}_+(\mathbb{R}) \), which is contractible by Lemma 4.3. So the bundle is trivial [12, Corollary 12.3]. There exists an isomorphism \( \mathbb{R} \times G \widetilde{B} \simeq \mathbb{R} \times B \) (which is not locally constant) and this isomorphism is given by a function with values in the structure group \( \text{Homeo}_+(\mathbb{R}) \), hence it preserves the order. By composition, we deduced an injective map \( f : E \to \mathbb{R} \times B \), which is still order preserving (but no longer locally constant).

On the other hand, suppose we an order-preserving injective map \( f : E \to \mathbb{R} \times B \), say \( f = (f_1, f_2) \). Thus we have a map \( f_1 : E \to \mathbb{R} \) which respects the order on each fibre, and by restriction to the fibre \( X \) we get an order-preserving map \( t : X \hookrightarrow \mathbb{R} \).

We arrive now at our main theorem.

**Theorem 4.5.** For a countable \( G \)-set \( X \), the following are equivalent.

1. \( X \) is orderable.
2. \( p : E \to B \) is orderable as a locally constant bundle.
3. There exists an embedding \( E \to \mathbb{R} \times B \) such that

\[
\begin{array}{ccc}
E & \xrightarrow{f} & \mathbb{R} \times B \\
\downarrow & & \downarrow \\
B & \xleftarrow{\pi_2} & \mathbb{R} \times B
\end{array}
\]

commutes.

**Proof.** The equivalence of statements (1) and (2) was already observed in Proposition 3.11. By Lemma 4.1 if \( X \) is ordered and countable then there exists an injective, order-preserving map \( X \to \mathbb{R} \). By Proposition 4.4 there exists an embedding \( E \to \mathbb{R} \times B \). Thus (1) implies (3).

We will finish the proof by proving that (3) implies (2). Let \( R' \subset E \times_B E \) be the pull back of the canonical order relation \( R \times B \subset \mathbb{R} \times B \) by the map \( f : E \to \mathbb{R} \times B \). Then \( R' \) satisfies all the axioms of a strict total order, but because \( f \) is not a locally constant map, we cannot use Lemma 2.8 to deduce that \( R' \) is a sub-covering space of \( E \times_B E \).

To prove that \( R' \) is indeed a sub-covering, it is sufficient to prove that it is a union of path connected components of \( E \times_B E \). Remark that a bundle
map \( f : E \to \mathbb{R}_B = \mathbb{R} \times B \) over \( B \) is the same thing as a map \( g : E \to \mathbb{R} \).

From such a map \( g \) the partition \( R^\prec \sqcup R'_{op} \sqcup \Delta \) of \( \mathbb{R}^2 \) is pulled back by \( g \) to a partition \( R' \sqcup (R')_{op} \sqcup D \) of \( E \times_B E \) where \( D = (g \times g)^{-1}(\Delta) \), but because \( f \) is injective, \( D = \emptyset \).

So \( R' \) is a sub-covering if we prove that no path starting in \( R' \) can arrive in \( (R')_{op} \). Let us assume that we have such a path; by composing with \( g \), we obtain a path in \( \mathbb{R}^2 \) starting in \( R^\prec \) and ending in \( R^\prec_{op} \), i.e. a family of numbers \( x(t) \) and \( y(t) \) such that \( x(0) < y(0) \) and \( x(1) > y(1) \). Then, the topology of \( \mathbb{R} \) is such that we must have \( x(t) = y(t) \) for some \( t \), but this would give two elements \( e_1 \) and \( e_2 \) such that \( f(e_1) = f(e_2) \) and contradict the injectivity of \( f \).

\[ \square \]

**Corollary 4.6.** [4, cf. Theorem 2.3] Suppose \( G \) is a countable group and \( p : E \to B \) a covering space.

The right cosets \( p_* (\pi_1(E, e_0)) \backslash G \) form an orderable right \( G \)-set if and only if there exists a bundle embedding \( E \hookrightarrow \mathbb{R} \times B \) such that

\[
\begin{array}{ccc}
E & \xrightarrow{f} & \mathbb{R} \times B \\
\downarrow{p} & & \downarrow{\pi_2} \\
B & & 
\end{array}
\]

commutes. In particular, \( G \) is right orderable if and only if there is a bundle embedding \( \tilde{B} \hookrightarrow \mathbb{R} \times B \).

**Proof.** The fibre of the covering \( p : E \to B \) can be naturally identified with the countable \( G \)-set \( p_* (\pi_1(E, e_0)) \backslash G \), which is isomorphic to \( G \) in the case that \( E = \tilde{B} \). The result now follows from the equivalence of (1) and (3) in Theorem 4.5. \[ \square \]

The generality of Theorem 4.5 allows for an analysis of other ordered structures. Recall the path space \( P(B) \to B \) from Example 2.6 whose fibre is the \( G \times G \)-set \( \ell G_r \).

**Corollary 4.7.** A countable group \( G \) is bi-orderable if and only if there exists an embedding \( P(B) \to \mathbb{R} \times B \times B \) such that

\[
\begin{array}{ccc}
P(B) & \xrightarrow{f} & \mathbb{R} \times (B \times B) \\
\downarrow{p} & & \downarrow{\pi_2} \\
B \times B & & 
\end{array}
\]

commutes.

**Proof.** The group \( G \) is bi-orderable if and only if the fibre of \( p : P(B) \to B \times B \) is an ordered \( G \times G \)-set. By Theorem 4.3 this happens if and only if there is an embedding \( P(B) \to \mathbb{R} \times B \times B \) making the above diagram commute. \[ \square \]
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