Transverse flow in relativistic viscous hydrodynamics

Rudolf Baier, Paul Romatschke, and Urs Achim Wiedemann

1Fakultät für Physik, Universität Bielefeld, D-33501 Bielefeld, Germany
2Department of Physics and Astronomy, University of Stony Brook, NY 11794, USA
3RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973-5000, USA

(Dated: March 31, 2022)

Hydrodynamic model simulations of Au-Au collisions at RHIC have indicated recently, that with improved simulations in the coming years, it may be feasible to quantify the viscosity of the matter produced in heavy ion collisions. To this end, a consistent fluid dynamic description of viscous effects is clearly needed. In this note, we observe that a recently used, approximate form of the 2nd order Israel-Stewart viscous hydrodynamic equations of motion cannot account consistently for the transverse flow fields, which are expected to develop in heavy ion collisions. We identify the appropriate equations of motion.

I. INTRODUCTION

Ultra-relativistic nucleus-nucleus collisions exhibit large collective phenomena, such as transverse flow. In the discussion of the dynamical origin of this collectivity, relativistic hydrodynamics plays a central role. This is so, since the densities attained in nucleus-nucleus collisions imply very small mean free paths (< 1 fm), a condition, which hydrodynamics can naturally account for, while other approaches to multi-particle dynamics (such as parton cascades) have difficulties to control. Indeed, simulations based on ideal fluid dynamics successfully describe the main characteristics of soft momentum distributions in nucleus-nucleus collisions at RHIC [1, 2, 3]. However, the results of these simulations may depend significantly on the modeling of the initial conditions and freeze-out. This raises the question whether the success of ideal hydrodynamics is indicative of an almost viscosity-free matter produced at RHIC, or to what extent matter of non-negligible viscosity can be accounted for by ideal hydrodynamics due to a compensatory choice of initial conditions and freeze-out [4]. To address this point, the study of the hydrodynamical equations of motion in the presence of viscous corrections is clearly needed.

Including dissipative corrections of ideal fluid dynamics to first order in a gradient expansion is known to lead to an acausal dynamics, exhibiting significant unphysical features [5]. A consistent relativistic dissipative hydrodynamics requires a gradient expansion to second order [6, 7]. The application of this Israel-Stewart theory to relativistic heavy ion collisions is at the very beginning [4, 8, 9, 10, 11, 12]. In this note, we show that a specific approximation of the Israel-Stewart equations of motion [11, 12], used recently, does not allow for a consistent description of transverse flow, and we identify the full equations of motion, which should be used.

For simplicity, we consider hot matter with an ultra-relativistic equation of state (EoS) $\epsilon = 3p$. In the absence of conserved charges (vanishing chemical potential) and for finite shear viscosity $\eta$, the energy-momentum tensor takes the form

$$T^{\mu\nu} = (\epsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu} + \Pi^{\mu\nu}.$$  \hspace{1cm} (1)

Here $\epsilon, p$ are the energy density and pressure, $u^{\mu}$ denotes the local fluid velocity, $u^{\mu}u_{\mu} = 1$, and $\Pi^{\mu\nu}$ is the traceless shear tensor. The tensor $T^{\mu\nu}$ is conserved,

$$d^{\mu}T^{\mu\nu} = 0,$$ \hspace{1cm} (2)

where $d^{\mu}$ is the covariant derivative. In the Israel-Stewart theory, the evolution equation of the shear tensor is given by

$$\tau_D^{\alpha\beta}D^{\alpha\beta} + \Pi^{\mu\nu} = \eta(\nabla^{\mu}u^{\nu})$$  \hspace{1cm} (3)
with $\nabla^\mu \equiv \Delta^{\mu\nu} d_\nu$, $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$, $\langle \nabla_\mu u_\nu \rangle \equiv \nabla_\mu u_\nu + \nabla_\nu u_\mu - \frac{2}{3} \Delta_{\mu\nu} \nabla_\alpha u^\alpha$, and $D = u^\mu d_\mu$.

In the limit of vanishing relaxation time $\tau_D$, one finds the defining equation for $\Pi^{\mu\nu}$ in first order dissipative hydrodynamics. Here and in what follows, we neglect for simplicity other dissipative corrections (bulk viscosity, heat conductivity), as well as contributions from the vorticity to (3).

Eq. (3) can be derived from kinetic theory, which is one way of determining all possible terms entering the equation of motion [4]. An alternative 'short-cut' derivation is based on the constraint that up to second order in a gradient expansion, the comoving entropy density $s^\mu$ cannot decrease [6, 13].

Motivated by this constraint, one uses in the recent literature [11, 12] the equation of motion

$$\tau_D \Pi^{\mu\nu} + \Pi^{\mu\nu} = \eta < \nabla^\mu u^\nu > .$$

However, since $u_\mu \Pi^{\mu\nu} = 0$, Eq. (4) does not specify the components parallel to $u_\mu$ in (5). Thus, Eq. (5) is based on (4) and the additional approximation that $u_\mu D^{\mu\nu} = -\Pi^{\mu\nu} D u_\mu$, which is not an additional constraint but a simple consequence of the orthogonality $u_\mu \Pi^{\mu\nu} = 0$. It follows that (5) involves the additional assumption $\Pi^{\mu\nu} D u_\mu = 0$. At least for the physically relevant case of a radially symmetric collision region, this latter constraint implies necessarily (see the appendix for details of this argument)

$$Du^\mu = 0 .$$

We emphasize that (6) is a necessary consistency condition for the approximate equation of motion (5), but it is in general not satisfied for the full Israel-Stewart equation of motion (3).

II. SOLUTION FOR IDEAL HYDRODYNAMIC EVOLUTION

Is the additional assumption (6) acceptable for the simulation of relativistic heavy ion collisions? To address this question, we study here the resulting solution of ideal hydrodynamics, which serves as the reference in comparison with viscous effects [11, 12, 14, 15]. Our starting point is Eq. (6), which in the sense of a 'proof by contradiction' is assumed to be valid in this section. We also use

$$\nabla^\mu p = 0 ,$$

$$D\epsilon = -(\epsilon + p) \nabla_\mu u^\mu \rightarrow Dp = -\frac{4}{3} p \nabla_\mu u^\mu ,$$

which follow from the conservation laws (2), together with (6). We consider a system with longitudinal boost-invariance [16] and cylindrical symmetry in the transverse direction. The independent variables are $\tau, r, \phi$ and rapidity $\eta$. Because of the symmetries, the pressure $p = p(\tau, r)$ and radial flow velocity $v = v(\tau, r)$,

$$v = \frac{u^r}{u^\tau} , \quad v \leq 1 ,$$

are functions of $\tau$ and $r$ only. Using the metric components, $g_{\tau\tau} = 1, g_{rr} = -1, g_{r\eta} = -\tau^2, g_{\phi\phi} = -r^2$, the relativistic hydrodynamic equations become

$$(\partial_\tau + v \partial_r) p(\tau, r) = 0 ,$$

$$(\partial_\tau + v \partial_r) p(\tau, r) + \frac{4}{3} p(\tau, r) [\gamma^2 (\partial_\tau + v \partial_r) v(\tau, r) + \frac{1}{\tau} + \frac{r}{v}] = 0 ,$$

$$(\partial_\tau + v \partial_r) v(\tau, r) = 0 ,$$
where \( \gamma^{-2} = 1 - v^2 \), and \( \partial_r = \frac{\partial}{\partial r}, \partial_\tau = \frac{\partial}{\partial \tau} \). This set of quasilinear partial differential equations can be solved by the method of characteristics.

From Eq. (12), one finds that along the characteristic \( \tau, r(\tau) \) that fulfills the equation \( \frac{dr}{d\tau} = v \), the value of \( v \) is constant,

\[
\frac{dv}{d\tau} = 0. \tag{13}
\]

Therefore, one can integrate \( \int dr = \int v d\tau \), obtaining

\[
v\tau - r = \text{const} = f(v), \tag{14}
\]

where we used \( \frac{dv}{d\tau} = 0 \) to rewrite the constant as an arbitrary function of \( v \). Eq. (14) is the general solution of Eq. (12). It has been discussed previously in a different context, see e.g. Biro [17, 18].

We now use

\[
\gamma^2 (\partial_r + v \partial_\tau) v = \partial_r v, \quad p = \tilde{p} (r\tau)^{-4/3}, \tag{15}
\]

to rewrite Eq. (11) as

\[
(\partial_\tau + v \partial_r) \tilde{p} = -\frac{4}{3} \tilde{p} \partial_r v. \tag{16}
\]

Using again the method of characteristics, we find

\[
d \ln \tilde{p}^{3/4} = -(\partial_r v) d\tau, \quad v d\tau = dr. \tag{17}
\]

Since the second equation is the same characteristic as before we know that along this characteristic \( \frac{dv}{d\tau} = 0 \). Rewriting

\[
\partial_r v = \frac{1}{\tau - f'(v)} \tag{18}
\]

from Eq. (14) with \( f'(v) = \frac{df}{dv} \), one can thus integrate Eq. (17), obtaining

\[
\tilde{p}^{-3/4} = (\tau - f'(v)) \times \text{const.} = (\tau - f'(v)) g(v), \tag{19}
\]

where we introduced another arbitrary function \( g(v) \) for the same reasons as above. In summary, we find for the general solution of Eq. (11)

\[
p^{3/4} = (r\tau)^{-1}(\partial_r v)\tilde{g}(v), \tag{20}
\]

where \( \tilde{g}(v) = g^{-1}(v) \). The remaining Eq. (10) can be used to constrain the functions \( f(v) \) and \( g(v) \). Inserting Eq. (20) into Eq. (10) and using Eqs. (12,14,18) we find

\[
- v f_0 g (f')^2 + \tau \left[ (1 - v^2) f f' g' + g (1 + v^2) f'^2 + g f (3v f' + (1 - v^2) f'') \right] - \tau^2 \left[ (1 - v^2) (f + v f') g' + g (2 v f + (2 + 3v^2) f' + v (1 - v^2) f'') \right] + \tau^3 \left[ (1 + 2v^2) g + v (1 - v^2) g' \right] = 0. \tag{21}
\]

For solutions relevant in heavy ion collisions, we require that at all spatial points \( r \) and for all times \( \tau > 0 \), the pressure and its time derivative are finite. In particular, we require \( 0 < p(\tau, r = 0) < \infty \), and \( | \partial_\tau p(\tau, r = 0) | < \infty \). According to Eq. (20), the first condition leads either to (i)
\[
\lim_{r \ll 1} \partial_r v \sim r \quad \text{or to (ii) } \lim_{r \ll 1} \bar{g}(v) \sim r.
\]
In the next paragraph, we shall show that case (i) is realized. Because of Eq. (12), case (i) implies \( \partial_r v(\tau, r = 0) = 0 \). For case (ii), one considers \( \partial_r v|_{r=0} \neq 0 \). Then, to satisfy \( \partial_\tau p < \infty \), one has to require \( \lim_{r \ll 1} \partial_\tau (\bar{g}(v) \sim r) \), and this leads again to \( \partial_\tau v(\tau, r = 0) = 0 \). As a consequence of Eq. (14) we thus find in both cases \( v(\tau, r = 0) = 0 \).

Since \( v(\tau, r = 0) = 0 \), obviously \( v \) does not depend on \( \tau \) at \( r = 0 \). Thus, for Eq. (21) to hold at all times \( \tau \), each power of \( \tau \) in (21) has to vanish separately. One can exploit this property in a neighborhood around \( v = 0 \) by doing a Taylor expansion of (21) in \( v \) around \( v = 0 \). One then finds that not only the \( v \)-dependent prefactors of the powers \( \tau^n \) in (21) must vanish at \( v = 0 \), but also all their derivatives must vanish. Thus, these prefactors must be zero and one finds

\[
f(v) = 0, \quad g(v) = \text{const} \times \frac{1}{v}(1 - v^2)^{3/2},
\]
which leads to

\[
v(\tau, r) = \frac{r}{\tau}, \quad p(\tau, r) = \text{const} \times (\tau^2 - r^2)^{-2}.
\]

This velocity field \( v \) and pressure \( p \) are physical for \( r < \tau \), only. At \( r = 0 \) the pressure behaves as \( p(\tau, r = 0) \simeq 1/\tau^4 \), which differs significantly from the Bjorken cooling law \( p(\tau) \simeq \tau^{-4/3} \) [16]. This exact analytic solution (22) of ideal relativistic hydrodynamics in \((1 + 3)\) dimensions, constrained by \( D\mu = 0 \) and restricted by boost invariance and cylindrical symmetry, has also been discussed previously in [19].

For hydrodynamic simulations of central heavy ion collisions, the initial conditions satisfy usually \( v(\tau_0, r) = 0 \) for finite time \( \tau_0 \) and all values of \( r \) [11, 12, 13]. This initial condition is incompatible with the requirement (22) that \( f(v) = 0 \) in a finite region around \( v = 0 \), and so it is incompatible with Eq. (21). Moreover, in numerical solutions of the full ideal hydrodynamic equations of motion (for which \( D\mu \neq 0 \) in general), one finds that the velocity field for small but fixed values of \( r \) increases with \( \tau \) for some time (see e.g. Fig. 1 in [15]). In contrast, for the solution (23), the velocity \( v(\tau, r = \text{const}) \) decreases with \( \tau \). This illustrates, that \( D\mu = 0 \) is not a physically justified approximation in the context of heavy ion collisions, and so it should not be adopted for the refined formulation of viscous hydrodynamics. For reliable results, the full Israel-Stewart equation (3) rather than Eq. (5) has to be used.

Acknowledgments

PR was supported by BMBF 06BI102. UAW acknowledges the support of the Department of Physics and Astronomy, University of Stony Brook, and of RIKEN, Brookhaven National Laboratory, and the U.S. Department of Energy [DE-AC02-98CH10886] for providing the facilities essential for the completion of this work.

APPENDIX A: EQUATION (5) IMPLIES \( D\mu = 0 \)

In this appendix, we consider a longitudinally boost-invariant system with cylindrical symmetry in the transverse plane. We show that approximating the full hydrodynamic equations (3) by (5)
amounts to the assumption $Dw^\mu = 0$. We start from the observation that Eq. 3 agrees with Eq. (5), if

$$\Pi^{\mu\alpha} Du_\alpha = 0. \quad (A1)$$

Since only $u^\tau, u^r$ are non-vanishing, we find (choosing $\mu = r$)

$$\Pi^{\tau\tau} Du^\tau - \Pi^{r\tau} Du^\tau = 0. \quad (A2)$$

Using $u_\nu \Pi^{\nu\mu} = 0$ and $u^\tau = \sqrt{1 + (u^r)^2}$, this expression can be written as

$$\Pi^{r\tau} \left( \frac{u^r}{u^\tau} \right)^2 - 1) Du^\tau = 0. \quad (A3)$$

Since $u^\tau$ is in general unequal $u^r$, this means that $Dw^\mu = 0$ if $\Pi^{\mu\nu}$ is non-vanishing. In contrast, for the full viscous equation of motion (3), $Dw^\mu \neq 0$ in general.

[1] P. F. Kolb and U. Heinz, in Quark-Gluon Plasma 3, edited by R. C. Hwa and X.-N. Wang (World Scientific, Singapore, 2004), p. 634.
[2] U. Heinz, J. Phys. G 31 (2005) S717.
[3] U. W. Heinz, arXiv:nucl-th/0512049.
[4] R. Baier, P. Romatschke and U. A. Wiedemann, arXiv:hep-ph/0602249.
[5] W.A. Hiscock and L. Lindblom, Phys. Rev. D 31 (1985) 725.
[6] W. Israel, Ann. Phys. (N.Y.) 100 (1976) 310.
[7] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) 118 (1979) 349.
[8] M. Prakash, M. Prakash, R. Venugopalan and G. Welke, Phys. Rept. 227 (1993) 321.
[9] A. Muronga, Phys. Rev. Lett. 88 (2002) 062302 [Erratum-ibid. 89 (2002) 159901]; and Phys. Rev. C 69 (2004) 034903.
[10] A. Muronga and D. H. Rischke, arXiv:nucl-th/0407114 (v2).
[11] A. K. Chaudhuri and U. Heinz, arXiv:nucl-th/0504022.
[12] U. Heinz, H. Song and A. K. Chaudhuri, arXiv:nucl-th/0510014.
[13] D. H. Rischke, arXiv:nucl-th/9808004. Hadrons in Dense Matter and Hadrosynthesis, ed. by J. Cleymans, H.B. Geyer and F.G. Scholz, Springer Lecture Notes in Physics 516, 21 (1999).
[14] D. Teaney, Phys. Rev. C 68 (2003) 034913.
[15] D. A. Teaney, J. Phys. G 30 (2004) S1247.
[16] J. D. Bjorken, Phys. Rev. D 27 (1983) 140.
[17] T. S. Biro, Phys. Lett. B 487 (2000) 133.
[18] T. S. Biro, Phys. Lett. B 474 (2000) 21.
[19] T. Csorgo, F. Grassi, Y. Hama and T. Kodama, Phys. Lett. B 565 (2003) 107.