Derivation and classical solutions for the Wave equation

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Abstract. The problem of string vibration prompted mathematicians to find a suitable equation to express this phenomenon, which led to the birth of the Fourier series and integrals. The physical law of string vibration can be expressed by wave equation and solved by Fourier series. The study on the wave equation is fraught with mathematical and physical controversy. D.Bernoulli based on results from physical experiments to make a bold conjecture that the shape of a vibrating string can be described as a combination of trigonometric series, which was inspired and validated by work from Fourier on study on heat equation. This study reviews the wave equation using Fourier as a tool. This paper not only gives the derivation of the wave equation but also explores its solution and corresponding properties.

1. Introduction
The research on partial differential equations arose in the 18th century to develop physical models for continuous media [1-5]. A vibrating string is the motion of a string fixed at its end and allowed to vibrate freely. D’Alembert in 1747 introduced the wave equation in one dimension in 1747[6-9]. D.Bernoulli based on results from physical experiments to make a bold conjecture that the shape of a vibrating string can be described as a combination of trigonometric series, which was inspired and validated by work from the study of Fourier on the heat equation [10, 11]. First of all, string vibration phenomena that can be observed are introduced to help us understand them [8, 12]. They are simple harmonic motion, standing wave and traveling wave, harmonic and tone superposition [8, 12].

2. Preliminary
Here is a general solution to the equation:

\[ ay''(t) + by'(t) + cy = 0. \]

Suppose \( y(t) = e^{nt} \). There will be \( y'(t) = me^{nt}, y''(t) = m^2e^{nt} \).

Then this research substitute \( y(t), y'(t), y''(t) \) into \( ay''(t) + by'(t) + cy = 0 \).

It can get: \( am^2e^{nt} + bme^{nt} + ce^{nt} = 0 \).
Since $e^{mt} \neq 0$, there is $am^2 + bm + c = 0$. Thus, this paper gets a quadratic equation of one variable and it just needs to solve the equation $am^2 + bm + c = 0$.

Case(I) $b^2 - 4ac > 0$: it has $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. That is,

$$
m_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, m_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.
$$

Then moving $m_1, m_2$ back to $y(t) = e^{mt}$. It will be,

$$
y(t) = e^{m_1 t}, y(t) = e^{m_2 t}.
$$

Hence, the general solution of the equation is $y(t) = C_1 e^{m_1 t} + C_2 e^{m_2 t}$, where $C_1$ and $C_2$ are two arbitrary coefficients.

Case (II) $b^2 - 4ac = 0$: it will be $m = -\frac{b}{2a}$.

The solution of the equation is $y(t) = C_1 e^{mt} + C_2 xe^{mt}$.

Case (III) $b^2 - 4ac < 0$: it has $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$, where $\alpha = -\frac{b}{2a}, \beta = \frac{\sqrt{4ac-b^2}}{2a}$.

The solution of the equation is $y_1(t) = e^{(\alpha+i\beta)t}, y_2(t) = e^{(\alpha-i\beta)t}$.

3. Main Works

3.1. Simple harmonic motion

Harmonic vibration is the most basic vibration system, which is composed of a horizontal spring fixed to the wall and an object with a mass of $m$, and assumes that there is no friction between the system and the surface. When the system is in equilibrium, the spring neither stretches nor compresses. When the force is applied to the object, the body is offset from the equilibrium position, and then the object is released. Due to the restoring force of the spring, the body will experience simple harmonic motion. To describe this phenomenon mathematically, this study assumes that the spring is ideal and satisfies Hooke’s law, that is, the restoring force $F$ where this paper uses the symbol $y''$ to represent the second derivative of $y$ to $t$, that is, the acceleration of the system. When $c=\sqrt{k/m}$, the second-order ordinary differential equation becomes

$$
y''(t) + c^2y(t) = 0
$$

Then it gets the general solution of equation (1)

$$
y(t) = acos(ct) + bsin(ct).
$$

Because $a$ and $b$ are constants and can be any real number, in order to further determine a solution of the equation, that is, to find the unknown constants $a$ and $b$, this paper imposes two initial conditions: the initial position and velocity of mass $y(0)$ and $y'(0)$. In this case, the solution of the physical problem is unique and is given by the following formula.

$$
y(t) = y(0)cos(ct) + \frac{y'(0)}{c}sin(ct).
$$

Through the sum-difference product formula of trigonometric, the following formula could be got, where the constant $A > 0, \varphi \in \mathbb{R}$

$$
acosct + bsinct = Acos(ct - \varphi).
$$

In the above formula, $A = \sqrt{a^2 + b^2}$ is the “amplitude” of the motion, $c$ is its “natural frequency”, $\varphi$ is its “phase”, (which is not uniquely determined until an integer multiple of $2\pi$), and $\frac{2\pi}{c}$ is the “period”, of the motion.

The typical graph of the function $acos(ct - \varphi)$ shown in figure 2 shows the wavy pattern obtained by translating and stretching (or shrinking) the usual cost pattern.
By studying the simple harmonic motion, it will be seen the relationship between trigonometric functions and complex numbers, as given in Euler’s equation $e^{it} = \cos t + i\sin t$. In order to obtain specific solution of the system, it should determine two factors. One is the position and another one is the velocity.

3.2. Standing wave and traveling wave
It has been proved that vibrating strings can be observed by one-dimensional waves. Here, it will describe two kinds of motion that are suitable for simple graphical representation.

3.2.1. Standing wave
A transmission state of a plane wave on a transmission line in which the amplitude of a plane wave changes exponentially along the direction of propagation and the phase changes linearly along the transmission line. The wave that the waveform propagates forward relative to the standing wave is called the traveling wave or a traveling wave is a wave that travels outward.

3.2.2. Traveling wave
Although the waveform changes with time, it does not move in any direction. This phenomenon is called standing wave [6]. Standing waves are composed of two columns of waves with the same frequency, the same amplitude, the same vibration direction and the opposite propagation direction. It is a common wave in the vibration of strings.

Traveling wave is a kind of wave that is often observed in nature. For $u(x, t)$, when $t = 0$, there is an initial profile $F(x)$, $u(x, t)$ equals to $F(x)$ when $t$ weak 0. As $t$ evolves, the section moves $ct$ units to the right., where $c$ is a positive constant and represents the velocity of the wave at $t$ time, namely.

$$u(x, t) = F(x - ct).$$

Figure 3 graphically describes this situation.
Similarly, $u(x,t) = F(x + ct)$ is a one-dimensional traveling wave moving to the left.

3.3. Derivation of wave equation

Suppose there is a uniform string is placed on the $x$-$y$ plane. The string lies along the $x$-axis with the starting point $x = 0$ and the end point $x = L$. This article uses $u(x,t)$ to denote displacement of this string at the position $x$ at the time $t$.

This paper firstly subdivides uniformly the string into $N$ parts along the $x$-axis. The $x$-coordinate of the $n^{th}$ particle is at $x_n = nL/N$. In this sense, it can simplify this vibrating string as a system of $N$ particles, each of which vibrates only in the vertical direction; however, the vibration of each particle will be associated with its neighboring particles through the tension of the string.

Then, this research lets $y_n$ represent $u(x_n,t)$ and use $h$ to be $x_{n+1} - x_n$ ($h = L/N$). Under the condition that the string has constant density $\rho > 0$, it can make this string have equal mass $\rho h$ to each particle. By invoking the Newton's law, $\rho h y_n''(t)$ to be value of the force that acts on the $n^{th}$ particle at time $t$. Thus, it could be assumed that this force is due to the effect of the two nearby particles and it can further assume that the force from the right of the $n^{th}$ particle is proportional to $(y_{n+1} - y_n)/h$, where $h$ is the distance between $x_{n+1}$ and $x_n$; therefore, the force could be written as

$$\tau \frac{1}{h} (y_{n+1} - y_n)$$

where $\tau$ is a positive constant that represent the coefficient of tension.

There is another force originated from the left side of the string, which is:

$$\tau \frac{1}{h} (y_{n-1} - y_n).$$
Conclusively, it can get the relationship of oscillators $y_n(t)$, which is:

$$\rho h y''_n(t) = \frac{1}{h} \{y_{n+1}(t) + y_{n-1}(t) - 2y_n(t)\} \quad (2)$$

Notice that for any second differentiable functions $F(X)$, there is:

$$\frac{F(x + h) + F(x - h) - 2F(x)}{h^2} \rightarrow F''(x) \quad \text{as} \quad h \rightarrow 0.$$ 

Thus, from the equation (2), there is:

$$\rho \frac{\partial^2 u}{\partial t^2} = \tau \frac{\partial^2 u}{\partial x^2}$$

or

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad \text{with} \quad c = \sqrt{\tau/\rho}.$$

This equation is known as the one-dimension wave equation, or simpler the wave equation.

3.4. Solution to the wave equation

To obtain a closed form for the solution to the partial differential equation $\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, for the simplicity, the technique of changing variables is employed. In details, the variable $x$ is substituted by $a \cdot X$, where the coefficient $a$ is a positive number for further scaling purpose; the variable $t$ is substituted by $b \cdot T$, where the coefficient $b$ is a real number for further scaling purpose. Thus, the solution $u(x,t)$ can be rewritten as $u(X,T)$. Notice that

$$\frac{\partial u}{\partial X} = a \frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial X^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

Thus, by choosing a suitable coefficient $a$, with similar techniques for choosing a suitable coefficient $b$, the wave equation can be rewritten as

$$\frac{\partial^2 u}{\partial T^2} = \frac{\partial^2 u}{\partial X^2} \quad \text{on} \quad 0 \leq x \leq \pi.$$

Next, we will explain in detail how to obtain closed form of solution to the wave equation by invoking the method of traveling waves and Superposition of standing waves.

3.4.1 Traveling waves

The vital conclusion is: if $F$ is any twice differentiable function, then $u(x, t) = F(x + t)$ and $u(x, t) = F(x - t)$ solve the wave equation. Note that the graph of $u(x, t) = F(x - t)$ is the graph of $F$ at time $t=0$. It became a graph of $F$ moving to the right by 1 at time $t=1$. Therefore, this study recognizes $F(x - t)$ as the wave traveling to the right with the speed of 1. Similarly, $u(x, t) = F(x + t)$ is the wave traveling to the left by velocity of 1 as shown in Figure 6.

![Traveling wave](image-url)
This discussion for tone and its combinations enables us to observe the linearity property for the solutions to the wave equation. This means if \( u(x, t) \) and \( v(x, t) \) is solutions, then \( \alpha u(x, t) + \beta v(x, t) \) are also particular solutions, where \( \alpha \) and \( \beta \) is any constants. Therefore, this paper could superpose two waves traveling in opposite directions to find that whenever \( F \) and \( G \) are twice differentiable functions, then:

\[
 u(x, t) = F(x + t) + G(x - t)
\]

is the solution of wave equation. In fact, it has been used in all the solutions this study could have. This study now temporarily drops the presumption of \( 0 \leq x \leq \pi \), assume that \( u \) is a twice differentiable function which solves the wave equation for all real \( x \) and \( t \). Consider the new variable set \( \xi = x + t \), \( \eta = x - t \), define \( v(\xi, \eta) = u(x, t) \). The change of variable formula satisfied.

\[
 \frac{\partial^2 v}{\partial \xi \partial \eta} = 0.
\]

By integrating twice, it gets that \( v(\xi, \eta) = F(\xi) + G(\eta) \). It implies that \( u(x, t) \) can be written as

\[
 u(x, t) = F(x + t) + G(x - t),
\]

for two functions \( F \) and \( G \).

This research next connects the result with the physical motion of a string. Thus, it is necessary to add a restriction on the space variable, \( 0 \leq x \leq \pi \). This study also adds an initial condition \( u(x, 0) = f(x) \). Notice that the string has fixed end points, which means that for all \( t \), \( u(0, t) = u(\pi, t) = 0 \). Then it makes an odd and periodic extension on domain of the function \( f \) from \([0, \pi]\). After that, \( u(x, t) \) needs to be modified to solve wave equation on all \( R \), and \( u(x, 0) = f(x) \) for all \( x \in R \). Therefore, \( u(x, t) = F(x + t) + G(x - t) \), and set \( t = 0 \) it finds that

\[
 F(x) + G(x) = f(x).
\]

for the reason that many choices of \( F \) and \( G \) would be satisfied this equation, this paper suggests that an initial condition be imposed on \( u \), which means \( g(X) \) will be used to represent the initial velocity of the string:

\[
 \frac{\partial u}{\partial t}(x, 0) = g(x),
\]

As well as for \( g(0) = g(\pi) = 0 \). Similarly, it first expands \( g \) to all \( R \) by making it odd on \([-\pi, \pi]\) and periodic of period \( 2\pi \). Thus, the position and velocity now transfer to the following:

\[
 \begin{align*}
 F(x) + G(x) &= f(x), \\
 F'(x) - G'(x) &= g(x).
\end{align*}
\]

Differentiating the first equation and plus it to the second one, it can get:

\[
 2F'(x) = f'(x) + g(x)
\]

Thus, there is

\[
 F(X) = \frac{1}{2} [f(x) + \int_0^x g(y)dy] + C_1.
\]

and

\[
 G(X) = \frac{1}{2} [f(x) - \int_0^x g(y)dy] + C_2.
\]

Since \( F(X) + G(X) = f(X) \), it will be got that \( C_1 + C_2 = 0 \), therefore, the final solution for the wave equation with the given initial conditions is:

\[
 u(x, t) = \frac{1}{2} [f(x + t) + f(x - t)] + \frac{1}{2} \int_{x-t}^{x+t} g(y)dy
\]

Noticing that the expansion chosen for \( f \) and \( g \) assure that the string always has a fixed end, as for all \( t \), \( u(0, t) = u(\pi, t) = 0 \).

The above procedure from \( t \geq 0 \) to \( t \in R \) and the return to \( t \geq 0 \) showed the time reversal property of the wave equation. In other words, the solution \( u \) of the wave equation on \( t \geq 0 \) can only need to be set \( u^-(x, t) = u(x, -t) \), then this paper can get the solution \( u- \) defined for negative time \( t < 0 \). This is a fact because of the invariance of the wave equation under the transformation \( t \rightarrow -t \).
3.4.2 Superposition of standing waves

Assume \( u(x, t) \) is in the form of \( \phi(X)\psi(T) \). By computation, this article gets that second derivative, with regard to time variable, of \( u(x, t) \) is \( \phi(X)\psi''(T) \) and the second derivative of, with regard to space variable, of \( u(x, t) \) is \( \phi''(X)\psi(T) \).

Thus, there is:

\[
\phi''(x) \phi(x) = \psi''(t) \psi(t)
\]

Divided by the \( \phi(x)\psi(x) \) on both sides, it can get

\[
\frac{\phi''(x)}{\phi(x)} = \frac{\psi''(t)}{\psi(t)}
\]

The purpose of this form is to make both sides of the whole equation only involve variable \( x \) or \( t \). So, if the left and right sides of the equation are to be equal, then there will be no part about \( x \) or \( t \) after the calculation. This means that the left and right sides of the equation will get two equal constants. Usually, it is assumed that the constant as \( \lambda \), and it will simplify the wave equation to the following system of equations:

\[
\begin{align*}
\psi''(t) - \lambda \psi(t) &= 0 \\
\phi''(x) - \lambda \phi(x) &= 0
\end{align*}
\]

(3)

In this system of equations, it only needs to solve any one of them to know the solution of the other equation. At the same time, it is known that when \( \lambda \) is greater than or equal to 0, \( \psi(t) \) will not oscillate with the change of \( t \), so this paper only needs to discuss the situation when \( \lambda \) is less than 0. Now when assuming \( \lambda = -m^2 \), it can get:

\[
\psi(t) = A \cos(mt) + B \sin(mt)
\]

By the same way, it can get the second solution of the equation (3) is:

\[
\phi(x) = \tilde{A} \cos(mx) + \tilde{B} \sin(mx).
\]

Now this paper takes into account that the string is attached at \( x=0 \) and \( x=\pi \). This translates into \( \psi(0)=\psi(\pi) \), which in turn gives \( A=0 \), and if \( B \) is not 0, then \( m \) must be an integer. If \( m=0 \), the solution vanishes identically, and if \( m<=-1 \), it may rename the solutions and reduce to the case \( m>=1 \) since the function \( \sin y \) is odd and \( \cos y \) is even. Finally, it arrives at the guess that for each \( m>=1 \), the function

\[
\psi(x, t) = (A_m \cos(mt) + B_m \sin(mt)) \sin(mx)
\]

This is the solution of the wave equation, also known as a standing wave. It should be noted that when divided by \( \phi \) or \( \psi \), these two values may disappear, so the verification of \( u \) is very important.

In order to discuss the wave equation further, this research first has a more detailed understanding of standing waves. This term comes from looking at the um \( (x, t) \) graph for each fixed \( t \). First, suppose \( m=1 \) and take \( u(x, t) = \cos(t)\sin(x) \). Then, Figure 7(a) shows the graph of \( u \) with different \( t \) values.

When \( m=1 \), the fundamental tone or first harmonic of the vibrating string. When \( m=2 \), it can get

\[
u(x, t) = \cos(2t)\sin(2x),
\]

which corresponds to the first overtone or second harmonic. Note that in the equation \( u \) for \( x \) and \( t \), when \( x=\pi/2 \), no matter what the value of \( t \) is, \( u(x, t) =0 \). The points in the image that are constant no matter how \( x \) changes are called nodes. At the same time, the points with the largest changes are called anti-nodes. In all the function \( u \), the larger the value of \( m \), the more overtones or higher harmonics can be observed in the image. This means that as \( m \) increases, the frequency increases and the period decreases.

Through the above discussion, it is known that the wave equation is linear, and if \( u \) and \( v \) solve the function. Then in \( au+bv \), \( a \) and \( b \) are also linear as constants, so more solutions will be constructed through the linear combination of standing waves, which is called the superposition method. This also led to the guesses about the solution of the wave equation,

\[
u(x, t) = \sum_{m=1}^{\infty} (A_m \cos(mt) + B_m \sin(mt)) \sin(mx)
\]

(4).
Next it discusses convergence for the summation (4). If the expression obtained above gives all the solutions to the wave equation, since the wave equation is a periodic equation, it can first assume that when the time $t$ is 0, this article can study the variable $x$. When $t=0$, the shape of the graph of equation $f$ on $[0, \pi]$ is given, of course $f(0)=f(\pi)=0$, Then it can get when $t=0$, $u(x, 0) = f(x)$, hence

$$
\sum_{m=1}^{\infty} A_m \sin(mx) = f(x).
$$

Since the initial shape of the chord can be any reasonable function $f$, it is necessary to study whether it can find the value of $A_m$, and the function $f$ on a given $[0, \pi]$ (where $f(0)=f(\pi)=0$)

$$
f(x) = \sum_{m=1}^{\infty} A_m \sin(mx).
$$

Notice that

$$
\int_{0}^{\pi} \sin(mx) \sin(nx) \, dx = \begin{cases} 
0 & \text{if } m \neq n \\
\frac{\pi}{2} & \text{if } m = n
\end{cases}.
$$

By observing the whole equation, this study multiplies both sides by $\sin(nx)$ and there is:

$$
\int_{0}^{\pi} f(x) \sin(nx) \, dx = \int_{0}^{\pi} \left( \sum_{m=1}^{\infty} A_m \sin(mx) \right) \sin(nx) \, dx = \sum_{m=1}^{\infty} A_m \int_{0}^{\pi} \sin(mx) \sin(nx) \, dx = A_n \frac{\pi}{2}
$$

Then this research rearranges the formula:

$$
A_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin(nx) \, dx.
$$

4. Conclusion

This paper mainly discusses basic but fundamental ideas of how to solve wave equation in one dimension. The reader should have noticed that the solution to the wave equation is well-defined in the sense that for all time $t$, the solution makes sense. This reflects a fact that the wave equation is reversible in time, which provides an important technique in the field of signal processing. In the future, We will report more on the weak solution and the distribution theory.

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