CONJECTURES ON L-FUNCTIONS FOR PROJECTIVE BUNDLES ON DEDEKIND DOMAINS

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Abstract. Let $O_K$ be the ring of integers in an algebraic number field $K$ with $S := \text{Spec}(O_K)$. Let $T$ be a regular scheme of finite type over $S$ and let $X$ be a scheme of finite type over $T$ with a stratification of closed subschemes

$\emptyset = X_{n+1} \subseteq X_n \subseteq \cdots \subseteq X_2 \subseteq X_1 := X$

with $X_i - X_{i+1} = A^d_i$. We prove that if the Soulé conjecture holds for $T$ and the Beilinson-Soule vanishing conjecture holds for $X$, it follows the Soulé conjecture on special values of L-functions holds for $X$. As a special case we get an approach to Soulé's conjecture on special values of L-functions for flag bundles and grassmannian bundles on $S$ using induction and the geometry of flag bundles. We moreover reduce the study of the Beilinson-Soule vanishing conjecture and the Soulé conjecture on special values of L-functions to the study of affine regular schemes of finite type over $\mathbb{Z}$. Hence we get an approach to the Birch and Swinnerton-Dyer conjecture for abelian schemes using affine regular schemes of finite type over $\mathbb{Z}$.

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1. Introduction

If $O_K$ is the ring of integers in an algebraic number field, it follows the rank of the group $K'_m(\mathcal{O}_K)$ and the weight space $K'_m(\mathcal{O}_K)_{(i)}$ is well known for all integers $m \geq 0, i \geq 1$ (see [2]).

The aim of this paper is to study the Soulé’s conjecture and Beilinson-Soule vanishing conjecture (see Conjecture [1] 3.3.1 3.3.3) for arithmetic schemes. Let $T$
be a regular scheme of finite type over \( \mathcal{O}_K \) and \( X \) be a scheme of finite type over \( T \) with a stratification \( X_i \subseteq X \) of closed subschemes such that \( X_i - X_{i+1} \) is affine space over \( T \). In Theorem 4.8 we prove that if Soule’s conjecture \( 3.3.3 \) holds for \( T \) and if the vanishing conjecture \( 3.3.1 \) holds for \( X_i \) it follows \( 3.3.3 \) holds for \( X \):

\[
\text{ord}_{s=k}(L(X, s)) = \chi(X, k).
\]

Here \( \chi(X, i) \) is the K-theoretic Euler characteristic of \( X \). In particular we get an approach to the study of the Soule Conjecture 3.3.3 on special values of L-functions for flag bundles, grassmannian bundles and projective bundles over \( \mathcal{O}_K \) (see Corollary 4.9). We prove the Beilinson-Soule vanishing conjecture and conjecture on special values of L-functions for any \( P_d \)-bundle on \( \text{Spec}(\mathcal{O}_K) \). If \( A \) is an abelian scheme over \( \mathcal{O}_K \) it follows the Soule Conjecture 3.3.3 for \( A \) is one way to formulate a version of the \textit{Birch and Swinnerton-Dyer conjecture} for \( A \).

We reduce the study of the Beilinson-Soule vanishing conjecture and the Soule conjecture on special values of L-functions to the study of affine regular schemes of finite type over \( \mathbb{Z} \). Hence we get an approach to the Birch and Swinnerton-Dyer conjecture for abelian schemes using affine regular schemes of finite type over \( \mathbb{Z} \).

We use the projective bundle formula for algebraic K-theory and an elementary construction of eigenvectors for the Adams operator to calculate the weight space

\[
K_m(\mathbb{P}(E^*))_{(i)}
\]

for any pair of integers \( m \geq 0, i \geq 1 \), any finite rank projective \( \mathcal{O}_X \)-module \( E \) for any algebraic number field \( K \) (see Theorem 5.4). This illustrates the possibility to do explicit computations for the K-theory of projective bundles and more general flag bundles.

2. **Algebraic K-theory and Adams operations**

Let \( \mathcal{O}_K \) be the ring of integers in an algebraic number field \( K \) and let \( S = \text{Spec}(\mathcal{O}_K) \). Let \( X \) be a scheme of finite type over \( S \).

In this section we introduce some notation from Soule’s original paper [9]: Let \( M(X) \) denote the category of coherent \( \mathcal{O}_X \)-modules and let \( BQM(X) \) denote the simplicial classifying set of \( M(X) \). Let \( BQP(X) \) denote the simplicial classifying set of \( P(X) \), where \( P(X) \) is the category of locally trivial finite rank \( \mathcal{O}_X \)-modules.

By definition

\[
(2.0.1) \quad K_m(X) := \pi_{m+1}(BQM(X))
\]

\[
(2.0.2) \quad K_m(X) := \pi_{m+1}(BQP(X))
\]

where \( m \) is an arbitrary integer. If \( X \) is a regular scheme it follows \( K_m(X) = K_m(X) \) and \( K_m(X) = 0 \) for \( m < 0 \). Assume \( X \) is a scheme of finite type over \( \mathbb{Z} \) and assume \( u : X \to M \) is a closed immersion into a scheme \( M \) where \( M \) is a regular scheme of finite type over \( \mathbb{Z} \) of dimension \( D \). Define \( K^h_X(M) \) as the homotopy group of the fiber of the canonical map \( BQP(M) \to BQP(M - X) \).

**Definition 2.1.** If \( Y \) is a regular scheme of finite type over \( \mathbb{Z} \), there is for every positive integer \( k \geq 0 \) an action

\[
\psi^k : K_m(Y) \to K_m(Y)
\]
with the following properties: If $L$ is the class of a line bundle in $K_0(Y)$ it follows
\[ \psi^k(L) := L^k. \]
The map $\psi^k$ is the $k$'th Adams operator for $K_m(Y)$.

The map $\psi^k$ is functorial in the sense that for any map $p : Y \to Y'$ of regular schemes $Y, Y'$ of finite type over $\mathbb{Z}$ it follows
\[ \psi^k(p^*x) = p^*(\psi^k(x)) \]
for any element $x \in K_m(Y')$. The abelian group $K_*(Y) := \oplus_{m \geq 0} K_m(Y)$ is a graded commutative ring and the endomorphism
\[ \psi^k : K_*(Y) \to K_*(Y) \]
is a ring homomorphism: $\psi^k(xy) = \psi^k(x)\psi^k(y)$ for any $x \in K_m(Y), y \in K_n(Y)$. The operation $\psi^k$ induce canonically a ring homomorphism
\[ \psi^k : K_*(Y) \otimes \mathbb{Q} \to K_*(Y) \otimes \mathbb{Q} \]
(let $K_m(Y)_\mathbb{Q} := K_m(Y) \otimes \mathbb{Q}$) and we define
\[ K_m(Y)_{\mathbb{Q}}^{(i)} := \{ x \in K_m(Y)_\mathbb{Q} : \text{ such that } \psi^k(x) = k^i x. \} \]

There is a direct sum decomposition
\[ K_m(Y)_\mathbb{Q} \cong \oplus_{i \in \mathbb{Z}} K_m(Y)_{\mathbb{Q}}^{(i)} \]
and the space $K_m(Y)_{\mathbb{Q}}^{(i)}$ is independent of choice of positive integer $k$. By definition we let
(2.1.1) \[ K_m(X)_{(i)} := K_m^X(M)_{\mathbb{Q}}^{(D-i)}. \]

When $X$ is regular we may choose $M = X$. It follows
\[ K_m(X)_{(i)} = K_m(X)_{\mathbb{Q}}^{(D-i)} \]
where $D = \text{dim}(X)$. Hence when $X$ is a regular scheme of finite type over $\mathbb{Z}$ we may use the K-theory of the category $P(X)$ of finite rank algebraic vector bundles on $X$ and the Adams operations on $K_m(X)_{\mathbb{Q}}$ to calculate the group $K_m(X)_{(i)}$ introduced in Soules paper.

**Definition 2.2.** Let $X$ be a scheme of finite type over $\mathbb{Z}$ and let $i : X \to M$ be a closed embedding into a regular scheme $M$ of finite type over $\mathbb{Z}$ with $D := \text{dim}(M)$.

Define
\[ K'_m(X) := \oplus_{m \in \mathbb{Z}} K_m^X(M) \]
and $K'_m(X)_{\mathbb{Q}} := K'_m(X) \otimes \mathbb{Q}$. Define $K'_m(X)_{(j)} := K_m^X(M)_{\mathbb{Q}}^{(D-j)}$. The $\mathbb{Q}$-vector space $K'_m(X)_{(j)}$ is the weight space of weight $j$.

The following result calculates $K_m(\mathcal{O}_K)_{\mathbb{Q}}$ and $K_m(\mathcal{O}_K)_{\mathbb{Q}}^{(i)}$ for all $m, i$:

**Theorem 2.3.** The following holds:
(2.3.1) \[ K_m(\mathcal{O}_K)_{\mathbb{Q}} = 0 \text{ for all } m < 0 \]
(2.3.2) \[ K_0(\mathcal{O}_K)_{\mathbb{Q}} = \mathbb{Q} \]
(2.3.3) \[ K_m(\mathcal{O}_K)_{\mathbb{Q}} = 0 \text{ for } m = 2i, i \neq 0 \]
(2.3.4) \[ K_m(\mathcal{O}_K)_{\mathbb{Q}} = \mathbb{Q}^{i+1+r_2} \text{ for } m \equiv 1 \mod 4 \]
(2.3.5) \[ K_m(\mathcal{O}_K)_{\mathbb{Q}} = \mathbb{Q}^{r_2} \text{ for } m \equiv 3 \mod 4. \]
Moreover
\begin{align}
(2.3.6) \quad K_{2i-1}(\mathcal{O}_K)^{(i)}_Q &= \mathbb{Q}^{r_1+r_2} \text{ for } i \equiv 0 \mod 2 \\
(2.3.7) \quad K_{2i-1}(\mathcal{O}_K)^{(i)}_Q &= \mathbb{Q}^{r_2} \text{ for } i \equiv 1 \mod 2.
\end{align}

Theorem 2.3 follows from Borels calculation of the ranks of the algebraic K-groups of the ring of integers \( \mathcal{O}_K \) in an algebraic number field \( K \). Here \( r_1 \) and \( r_2 \) are the real and complex places of \( K \). In the case when \( K = \mathbb{Q} \) is the field of rational numbers and \( \mathcal{O}_K = \mathbb{Z} \) is the ring of integers, it follows \( r_1 = 1 \) and \( r_2 = 0 \) (see [2] and [8] for a proof of Theorem 2.3).

3. Reduction of Beilinson-Soules conjectures on vanishing and L-functions to the affine regular case

In this section we reduce the study of the Beilinson-Soule vanishing conjecture and Soule’s conjecture on special values of L-functions to the study of affine regular schemes of finite type over \( \mathbb{Z} \).

Definition 3.1. Let \( X \) be a quasi projective scheme of finite type over \( S \) and let \( i \in \mathbb{Z} \). Let
\[ \chi(X, i) := \sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim_{\mathbb{Q}}(K_m(X)^{(i)}) \]
be the Euler characteristic of \( X \) of type \( i \).

Definition 3.2. Let \( X \) be a scheme of finite type over \( S \). Let
\[ L(X, s) := \prod_{x \in X^{cl}} \frac{1}{1 - N(x)^{-s}} \]
be the L-function of \( X \). Here we view \( s \) as a complex variable and the infinite product is taken over the set of closed points \( x \) in \( X^{cl} \). By definition \( N(x) := \# \kappa(x) \) where \( \kappa(x) \) is the residue field of \( x \).

Note: Since \( X \) is of finite type over \( \mathbb{Z} \) and \( x \) is a closed point, it follows \( \kappa(x) \) is a finite field.

Example 3.3. The Dedekind L-function.

If \( K \) is an algebraic number field with ring of integers \( \mathcal{O}_K \) and \( S := \text{Spec}(\mathcal{O}_K) \), it follows \( L(S, s) \) is the Dedekind L-function of \( K \). In particular \( L(\text{Spec}(\mathbb{Z}), s) \) is the Riemann zeta function.

In Soules’ paper [9] the following conjecture is stated:

Conjecture 1. (Conjecture 2.2 in [9]) Let \( X \) be a quasi projective scheme of finite type over \( \mathbb{Z} \) and let \( i \in \mathbb{Z} \) be an integer.

\begin{align}
(3.3.1) \quad &\text{For fixed integer } i \text{ the group } K_m(X)^{(i)} \text{ is zero for almost all integers } m. \\
(3.3.2) \quad &\dim_{\mathbb{Q}}(K_m(X)^{(i)}) \text{ is finite for all } m, i. \\
(3.3.3) \quad &\text{ord}_{s=i}(L(X, s)) = \sum_{m \geq 0} (-1)^{m+1} \dim_{\mathbb{Q}}(K_m(X)^{(i)}) \text{ for all } i \geq 1.
\end{align}

Note: The Conjecture 3.3.3 is mentioned in Wiles’ CLAY Math desciption of one of the Millenium Problems: See [12] for information on the Birch and Swinnerton-Dyer conjecture for elliptic curves. In [12] Conjecture 3.3.3 is referred to as due
to Tate, Lichtenbaum, Deligne, Bloch, Beilinson and others. Conjecture 3.3.1 is sometimes referred to as the Beilinson-Soule vanishing conjecture. If \(E\) is a relative elliptic curve over \(\mathcal{O}_K\), it follows Conjecture 3.3.3 is a version of the Birch and Swinnerton-Dyer conjecture for \(E\) using K-theory. The version given in [12] is formulated for an elliptic curve \(\mathbb{F} / \mathbb{Q}\) and the group of rational points \(\mathbb{F}(\mathbb{Q})\), hence the conjecture in [12] is similar to Conjecture 3.3.3. Hence we may view the conjecture mentioned in [12] as a special case of Conjecture 3.3.3. Note moreover that if \(X_{\text{red}}\) is the reduced scheme of \(X\) it follows \(L(X, s) = L(X_{\text{red}}, s)\) and \(K_m'(X) = K_m'(X_{\text{red}})\), hence Conjecture 1 holds for \(X\) if and only if it holds for \(X_{\text{red}}\).

**Example 3.4.** Conjecture 1 for Dedekind L-functions.

If \(S := \text{Spec}(\mathcal{O}_K)\) with \(K\) an algebraic number field, it follows 3.3.1, 3.3.2 and 3.3.3 holds by the work of Borel [2].

**Example 3.5.** Conjecture 1 for finite fields.

Let \(k\) be a finite field. It follows \(K'_m(k) = 0\) hence \(K'_m(k) = 0\) for all integers \(m, j\) and it follows 3.3.1 holds for \(S := \text{Spec}(k)\). One also checks 3.3.3 holds for \(S\).

Note: In the case when \(X\) is a regular scheme of dimension \(D\) it follows there is an equality of groups

\[K_m(X)_{(i)} \cong K_m(X)^{(D-i)}_Q\]

where \(K_m(X)\) is the K-theory of the category \(P(X)\) of locally trivial finite rank \(\mathcal{O}_X\)-modules.

Recall the following results:

**Lemma 3.6.** Let \(X\) be of finite type over \(S\) with \(X = U \cup V\) a disjoint union of two subschemes \(U, V\). It follows \(L(X, s) = L(U, s) L(V, s)\). If \(U \subseteq X\) is an open subscheme with \(Z := X - U\) it follows \(L(X, s) = L(U, s) L(Z, s)\). Assume \(X, Y\) are schemes of finite type over \(S\) such that for any closed point \(t \in S\) there is an isomorphism \(X_t \cong Y_t\) of fibers. It follows there is an equality of L-functions \(L(X, s) = L(Y, s)\).

**Proof.** Assume we may write \(X\) as a disjoint union \(X = U \cup V\). It follows \(X^{\text{cl}} = \cup U^{\text{cl}} \cup V^{\text{cl}}\). We get

\[L(X, s) = \prod_{x \in X^{\text{cl}}} \frac{1}{1 - N(x)^{-s}} = \prod_{x \in U^{\text{cl}}} \frac{1}{1 - N(x)^{-s}} \prod_{x \in V^{\text{cl}}} \frac{1}{1 - N(x)^{-s}} = L(U, s) L(V, s)\]

and the first claim follows. We moreover get

\[L(X, s) = \prod_{x \in X^{\text{cl}}} \frac{1}{1 - N(x)^{-s}} = \prod_{t \in S^{\text{cl}}} \prod_{x \in X_t} \frac{1}{1 - N(x)^{-s}} = \prod_{t \in S^{\text{cl}}} L(X_t, s) = \prod_{t \in S^{\text{cl}}} L(Y_t, s) = L(Y, s).\]

The Lemma follows.
Corollary 3.7. Let $E, F$ be locally trivial $\mathcal{O}_K$-modules of rank $d + 1$. It follows $L(\mathbb{P}(E^*), s) = L(\mathbb{P}(F^*), s)$. Let $\mathbb{A}^d$ be affine space over $S$. It follows $L(\mathbb{A}^d, s) = L(S, s - d)$. Assume $T$ is a regular scheme of finite type over $\mathcal{O}_K$ and $\mathbb{A}^d_T$ is affine $d$-space over $T$. It follows conjecture 3.3.3 holds for $T$ if and only if holds for $\mathbb{A}^d_T$. More generally: If $\pi : E \to T$ is a vector bundle of rank $l$ on $T$ is follows 3.3.3 holds for $T$ if and only if it holds for $E$. Moreover

$$L(\mathbb{P}(E^*), s) = L(S, s) L(S, s - 1) \cdots L(S, s - d).$$

Proof. Let $\dim(T) = n$. We get

$$\chi(\mathbb{A}^d_T, k) = \sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim_{\mathbb{Q}}(K'_m(\mathbb{A}^d_T)(k)) =$$

$$\sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim_{\mathbb{Q}}(K_m(\mathbb{A}^d_T)^{(d+n-k)}) =$$

$$\sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim_{\mathbb{Q}}(K_m(T)^{(n-(k-d)}) =$$

$$\sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim_{\mathbb{Q}}(K'_m(T)(k-d)) = \chi(T, k - d).$$

Hence

$$\chi(\mathbb{A}^d_T, k) = \chi(T, k - d).$$

Assume $\text{ord}_{s=k}(L(T, s)) = \chi(T, k)$. We get

$$\text{ord}_{s=k}(L(\mathbb{A}^d_T, s)) = \text{ord}_{s=k}(L(T, s - d)).$$

Let $t := s - d$, we get

$$\text{ord}_{s=k-d}(L(T, t)) = \chi(T, k - d) = \chi(\mathbb{A}^d_T, k)$$

hence the conjecture holds for $\mathbb{A}^d_T$. The converse is proved similarly. Since $L(E, s) = L(\mathbb{A}^d_T, s)$ and $\chi(E, k) = \chi(\mathbb{A}^d_T, k)$, it follows 3.3.3 holds for $E$ if and only if it holds for $\mathbb{A}^d_T$ which is if and only if 3.3.3 holds for $T$. The rest of the proof follows from Lemma 3.6 and the Corollary is proved. □

The following Lemma is by some authors referred to as the Jouanolou trick:

Lemma 3.8. Let $T := \text{Spec}(B)$ be an affine scheme of finite type over $\mathbb{Z}$ and let $X \subseteq \mathbb{P}^n_T$ be a quasi projective scheme over $T$. It follows there is an affine scheme $W := \text{Spec}(B)$ and a surjective map $\pi : W \to X$ where the fibers of $\pi$ is affine $l$-space $\mathbb{A}^l$.

Proof. This is proved in [7], Lemma 1.5. □

The affine $\mathbb{A}^l$-fibration $W$ constructed in Lemma 3.8 is an affine torsor for $U$.

Note that if $U \subseteq \mathbb{P}^n_Z$ is a quasi projective scheme and $\pi : W \to U$ is an affine torsor with fiber $\mathbb{A}^l$ constructed in Lemma 3.8 it follows $L(W, s) = L(\mathbb{A}^l_U, s)$ since $W$ and $\mathbb{A}^l_U$ have the same fibers. By construction there is an isomorphism

$$\pi^* : K'_m(U) \cong K'_m(W)$$

of abelian groups inducing an isomorphism

$$(3.8.1) \quad \pi^*_{(j-l)} : K'_m(U)_{(j-l)} \cong K'_m(W)_{(j)}$$

for all integers $j$. Since $W$ has fibers $\mathbb{A}^l$ it follows $\dim(W) = d + l$ where $d := \dim(U)$. Hence we get the following result:
Lemma 3.9. Let $U \subseteq \mathbb{P}^n_Z$ be a quasi projective scheme and let $p : W \to U$ be the torsor constructed in Lemma 3.8 with fiber $\mathbb{A}^1_U$. It follows $L(W, s) = L(\mathbb{A}^1_U, s)$ and $\chi(W, j) = \chi(U, j - l)$ for all integers $j$.

Proof. Since $W$ and $\mathbb{A}^1_U$ have the same fibers it follows $L(W, s) = L(\mathbb{A}^1_U, s)$ is an equality of L-functions. By Formula 3.8.1 we get an equality

$$\chi(U, j - l) := \sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim \mathcal{Q}(K_m(U)(j-l)) =$$

$$\sum_{m \in \mathbb{Z}} (-1)^{m+1} \dim \mathcal{Q}(K_m(W)(j)) = \chi(W, j),$$

hence $\chi(U, j - l) = \chi(W, j)$ and the Lemma follows. \qed

Lemma 3.10. Let $X$ be a scheme of finite type over $\mathbb{Z}$ and let $Z \subseteq X$ be a closed subscheme with open complement $U := X - Z$. If conjecture 3.3.1 and 3.3.3 holds for $Z$ and $U$ it follows conjecture 3.3.1 and 3.3.3 holds for $X$.

Proof. Assume $K'_m(Z)(j) = K'_m(U)(j) = 0$ for almost all $m$. There is a long exact localization sequence

$$\cdots \to K'_m(Z)(j) \to K'_m(X)(j) \to K'_m(U)(j) \to$$

$$\to K'_{m-1}(Z)(j) \to K'_{m-1}(X)(j) \to K'_{m-1}(U)(j) \to \cdots$$

hence there are integers $m_1 \leq m_2$ with the following properties: For all integers $m$ with $m \leq m_1$ or $m_2 \leq m$ it follows $K'_m(Z)(j) = K'_m(U)(j) = 0$. It follows by the long exact localization sequence that $K'_m(X)(j) = 0$ for all $m \leq m_1$ and $m_2 \leq m$, hence Conjecture 3.3.1 holds for $X$. If Conjecture 3.3.3 holds for $Z$ and $U$ we get the following: $L(X, s) = L(Z, s)L(U, s)$. We get

$$\text{ord}_{s=k}(L(X, s)) = \text{ord}_{s=k}(L(Z, s)) + \text{ord}_{s=k}(L(U, s)) =$$

$$\chi(Z, k) + \chi(U, k) = \chi(X, k)$$

since the Euler characteristic is additive with respect to $Z, U$, hence Conjecture 3.3.3 holds for $X$. The Lemma follows. \qed

We may reduce the study of Conjecture 3.3.1 and 3.3.3 to the study of affine regular schemes of finite type over $\mathbb{Z}$, with a systematic use of localization, induction on dimension and the Jouanolou trick from Lemma 3.8.

Theorem 3.11. Assume Conjecture 3.3.1 and 3.3.3 holds for any affine regular scheme of finite type over $\mathbb{Z}$. It follows Conjecture 3.3.1 and 3.3.3 holds for any quasi projective scheme $U$ of finite type over $\mathbb{Z}$.

Proof. One first proves using induction, the long exact localization sequence and Jouanolou trick that Conjecture 3.3.1 holds for any affine scheme $S := \text{Spec}(A)$ of finite type over $\mathbb{Z}$. Then again using Jouanolou trick, one proves Conjecture 3.3.1 holds for any quasi projective scheme $U \subseteq \mathbb{P}^n_Z$ of finite type over $\mathbb{Z}$.

Assume Conjecture 3.3.3 holds for all affine regular schemes $S := \text{Spec}(A)$ of finite type over $\mathbb{Z}$. Let $\dim(S) = 1$. It follows the singular subscheme $S_s \subseteq S$ is a finite set of closed points with finite residue fields and Conjecture 3.3.3 holds for $S_s$. We use here the fact that the K-theory of a scheme $X$ is the same as the K-theory of the associated reduced scheme $X_{\text{red}}$. The singular scheme $S_s$ may be non-reduced but we can pass to the reduced scheme associated to $S_s$. Let $U := S - S_s$. It
follows $U \subseteq \mathbb{P}^2_Z$ is a quasi projective regular scheme and hence there is a affine torsor $p : W \to U$ with fibers affine $l$-space $\mathbb{A}^l$. It follows since $W$ is an $\mathbb{A}^l$-fibration that $W$ has the same fibers as relative affine space $\mathbb{A}^l_U$ over $U$. Hence by Lemma 3.7 it follows there is an equality of L-functions

\[ L(W, s) = L(\mathbb{A}^l_U, s) \]

Since $W$ is an affine regular scheme of finite type over $\mathbb{Z}$ it follows Conjecture 3.3.3 holds for $W$. We get by Lemma 3.9

\[ \text{ord}_s = \text{ord}_s = \chi(W, k) = \chi(U, k - l) = \chi(\mathbb{A}^l_U, k). \]

Hence Conjecture 3.3.3 holds for $\mathbb{A}^l_U$. By Lemma 3.7 since Conjecture 3.3.3 holds for $\mathbb{A}^l_U$, it holds for $U$. Hence Conjecture 3.3.3 holds for $S$, and hence it holds for $S$. By induction on the dimension it follows 3.3.3 holds for any affine scheme $S$ of finite type over $\mathbb{Z}$.

Assume $U \subseteq \mathbb{P}^2_Z$ is a quasi projective scheme and let $p : W \to U$ be an affine torsor with $W := \text{Spec}(B)$ where $B$ is a finitely generated $\mathbb{Z}$-algebra. It follows by assumption 3.3.3 holds for $W$. By the same argument as above it follows 3.3.3 holds for $\mathbb{A}^l_U$ and again by Lemma 3.7 it follows 3.3.3 holds for $U$. The Theorem follows.

□

Note: A result similar to Theorem 3.11 for Conjecture 3.3.2 is mentioned in Soule’s original paper [9] in Example 2.4.

Example 3.12. Conjecture 3.3.1 for Abelian schemes.

Let $A \subseteq \mathbb{P}^2_T$ is a projective abelian scheme of finite type over $T := \text{Spec}(B)$, where $K$ is an algebraic number field and $\mathcal{B}$ a finitely generated and regular $\mathcal{O}_K$-algebra. If Conjecture 3.3.1 and 3.3.3 holds for all affine regular schemes $\text{Spec}(A)$ of finite type over $\mathbb{Z}$, it follows from Theorem 3.11 Conjecture 3.3.1 and 3.3.3 holds for any abelian scheme $A \subseteq \mathbb{P}^2_T$. Hence we have reduced the study of the Birch and Swinnerton-Dyer conjecture for abelian schemes to the study of affine regular schemes $\text{Spec}(A)$ of finite type over $\mathbb{Z}$.

Example 3.13. Algebraic K-theory for an affine regular scheme of finite type over $\mathbb{Z}$.

Let $S := \text{Spec}(A)$ where $A$ is a finitely generated and regular $\mathbb{Z}$-algebra. It follows from [11], Section IV, 1.16.1 there is an embedding

\[ K_*(S) \otimes \mathbb{Q} \subseteq H_*(\text{GL}(A), \mathbb{Q}) \]

where $\text{GL}(A)$ is the infinite general linear group of $A$. The embedding in 3.13.1 realize $K_*(S) \otimes \mathbb{Q}$ as the primitive elements in the Hopf algebra $H_*(\text{GL}(A), \mathbb{Q})$. In the paper [2] Borel calculates the K-groups $K_*(\mathcal{O}_K) \otimes \mathbb{Q}$ for any algebraic number field $K$ using the embedding 3.13.1. This is Theorem 2.3.

4. Special values of L-functions for projective bundles and flag bundles on Dedekind domains.

In this section we prove the Beilinson-Soule conjecture 3.3.1 and Soule conjecture 3.3.3 for any projective bundle $\mathbb{P}(E^*)$ on $\mathcal{O}_K$ where $K$ is an algebraic number
field. We also prove a similar result for any flag bundle $F(E)$ on $\mathcal{O}_K$ assuming the existence of a stratification $X_i$ of $F(E)$ where Conjecture 3.3.1 holds for $X_i$. Hence we get an approach to the study of special values of L-functions for flag bundles using induction and the geometry of flag bundles.

**Example 4.1.** Conjecture 3.3.3 for $\mathbb{P}^d$-bundles over Spec($\mathcal{O}_K$)

**Theorem 4.2.** Let $K$ be an algebraic number field and let $S := \text{Spec}(\mathcal{O}_K)$. Let $F(E^*)$ be a $\mathbb{P}^d$-bundle on $S$. It follows Conjecture 3.3.3 holds for $F(E^*)$.

**Proof.** Let

$$\mathbb{P}^d_S := \text{Proj}(\mathcal{O}_K[x_0, ..., x_n])$$

be projective $n$-space over $\mathcal{O}_K$. Let $E$ be a rank $n+1$ projective $\mathcal{O}_K$-module and let $F(E^*)$ be the $\mathbb{P}^n$-bundle of $E$. It follows by Lemma 3.7 that $L(F(E^*), s) = L(\mathbb{P}^d_S, s)$ and $\chi(F(E^*), j) = \chi(\mathbb{P}^d_S, j)$, hence Conjecture 3.3.3 holds for $F(E^*)$ if and only if it holds for $\mathbb{P}^d_S$. Let $n = 1$ and let $\mathbb{P}^1_S := \text{Proj}(\mathcal{O}_K[x_0, x_1])$. Let $S := V(x_1) \cong \text{Spec}(\mathcal{O}_K) := S$ and let $D(x_1) = \mathbb{A}^1_S = \text{Spec}(\mathcal{O}_K[\frac{x_1}{x_1}])$. It follows

$$\chi(\mathbb{P}^1_S, k) = \chi(\mathbb{A}^1_S, k) + \chi(S, k)$$

and

$$L(\mathbb{P}^1_S, s) = L(\mathbb{A}^1_S, s) L(S, s).$$

Hence

$$\text{ord}_{s=k}(L(\mathbb{P}^1_S, s)) = \text{ord}_{s=k}(L(\mathbb{A}^1_S, s)) + \text{ord}_{s=k}(L(S, s)) = \chi(\mathbb{A}^1_S, k) + \chi(S, k) = \chi(\mathbb{P}^1_S, k),$$

and it follows 3.3.3 holds for any $\mathbb{P}^1$-bundle on $S$. Assume the conjecture holds for any $\mathbb{P}^d-1$-bundle on $S$ and let $F(E^*)$ be a $\mathbb{P}^d$-bundle on $S$. We may choose $E$ to be the trivial rank $d+1$ $\mathcal{O}_K$-module and study $\mathbb{P}^d_S := \text{Proj}(\mathcal{O}_K[x_0, ..., x_d])$. Let $Z := V(x_d)$ and let $U := D(x_d)$. It follows $Z \cong \mathbb{P}^d_S-1$ and $U \cong \mathbb{A}^d_S$. Hence Conjecture 3.3.3 holds for $Z$ and $U$ and by Lemma 3.7 it follows 3.3.3 holds for $\mathbb{P}^d_S$ and $F(E^*)$ for any $E$. The Theorem is proved. \(\square\)

Note: Theorem 4.2 is a generalization of Borel’s classical result on Spec($\mathcal{O}_K$) to higher dimensional schemes. The picard group Pic($\mathcal{O}_K$) is a finite nontrivial group in general, and given any set of elements $\mathcal{L}_i$ for $i = 0, ..., d$ we get a locally trivial $\mathcal{O}_K$-module $E := \oplus \mathcal{L}_i$ of rank $d+1$ and a $\mathbb{P}^d$-bundle $F(E^*)$.

**Theorem 4.3.** Let $S := \text{Spec}(A)$ where $A$ is a finitely generated and regular $\mathbb{Z}$-algebra. Let $X \subseteq \mathbb{P}^d_S$ be a quasi projective regular scheme of dimension $d$. If conjecture 3.3.3 holds for all affine regular schemes of finite type over $\mathbb{Z}$, it follows Conjecture 3.3.3 holds for $X$.

**Proof.** By Lemma 3.8 there is an affine torsor

$$p : W \to X$$

with $W := \text{Spec}(B)$ with $\text{dim}(W) = d + l$. The map $p$ induce an isomorphism at $K$-theory

$$p_* : K^r_m(X) \to K^r_m(W)$$

and weight spaces

$$p_* : K^r_m(X)_i \to K^r_m(W)_i$$.
The projective bundle formula and the Adams operation.

Example 4.5. Let $A$ be a finitely generated and regular $\mathbb{Z}$-algebra and let $X \subseteq \mathbb{P}^n_S$ be a quasi projective and regular scheme with $S := \text{Spec}(A)$. Assume Conjecture 3.3.2 holds for all affine regular schemes of finite type over $\mathbb{Z}$. It follows $\chi(X, i)$ is an integer for all $i \in \mathbb{Z}$.

Proof. This follows from Theorem 3.3 since in this case $\chi(X, i)$ is a finite sum of integers. \hfill \square

Example 4.5. The projective bundle formula and the Adams operation.

In the following we calculate the $K$-theory of any finite rank projective bundle on $S := \text{Spec}(\mathcal{O}_K)$ using the projective bundle formula and Borel’s calculation of $K_m(\mathcal{O}_K)$.

The projective bundle formula says the following. There is a canonical pull back morphism

$$\pi^* : K_*(S) \to K_*(\mathbb{P}(E^*))$$

inducing maps

$$\pi^* : K_m(S) \to K_m(\mathbb{P}(E^*))$$

and an isomorphism

(4.5.1) \[ K_*(\mathbb{P}(E^*)) \cong K_*(S) \otimes_{K_0(S)} K_0(\mathbb{P}(E^*)) \cong K_*(S) \otimes_{\mathbb{Z}} \mathbb{Z}/(t^{d+1}). \]

with $t := 1 - L$ and $L := [\mathcal{O}_{\mathbb{P}(E^*)}(-1)] \in K_0(\mathbb{P}(E^*))$. The Adams operation $\psi^k$ acts as follows:

$$\psi^k(t) := 1 - \psi^k(L) = 1 - L^k.$$ 

We get for any element $zt^l \in K_m(\mathbb{P}(E^*)) \cong K_m(S)\{1, t, \ldots, t^d\}$ the following formula:

$$\psi^k(zt^l) = \psi^k(z)(1 - L^k)^l \in K_m(\mathbb{P}(E^*)) .$$

The isomorphism

$$K_m(\mathbb{P}(E^*)) \cong K_m(S)\{1, t, \ldots, t^d\}$$

is an isomorphism of $K_0(S)$-modules. In Theorem 4.6 we use formula 4.5.1 and Theorem 2.3 to calculate $K_m(\mathbb{P}(E^*))$ for all integers $m$.

Theorem 4.6. Let $\mathbb{P}(E^*)$ be a $\mathbb{P}^d$-bundle on $S$. The following holds:

(4.6.1) \[ K_0(\mathbb{P}(E^*)) \cong \mathbb{Q}^{d+1} . \]

(4.6.2) \[ K_m(\mathbb{P}(E^*)) \cong 0 \quad \text{for} \quad m = 2i, i \neq 0 . \]

(4.6.3) \[ K_m(\mathbb{P}(E^*)) \cong \mathbb{Q}^{r_1+r_2} \otimes \mathbb{Q}^{d+1} \quad \text{for} \quad m \equiv 1 \text{ mod } 4 . \]

(4.6.4) \[ K_m(\mathbb{P}(E^*)) \cong \mathbb{Q}^{r_2} \otimes \mathbb{Q}^{d+1} \quad \text{for} \quad m \equiv 3 \text{ mod } 4 . \]

Proof. The Theorem follows from Theorem 2.3 and the formula 4.5.1. \hfill \square

Corollary 4.7. Let $T$ be a scheme of finite type over $\mathbb{Z}$ with the property that Conjecture 3.3.2 holds for $T$. Let $\mathbb{P}(E^*)$ be a $\mathbb{P}^d$-bundle on $T$. It follows Conjecture 3.3.2 holds for $\mathbb{P}(E^*)$. In particular it follows Conjecture 3.3.2 holds for any $\mathbb{P}^d$-bundle on $\mathcal{O}_K$. 

Since $W$ is affine and finite dimensional it follows for a fixed $i + l$ the group $K_m(W)_{i+l} = 0$ for almost all $m$ by assumption. Hence the same holds for $K'(X)_{i+i}$. The Theorem follows. \hfill \square

Corollary 4.4. Let $A$ be a finitely generated and regular $\mathbb{Z}$-algebra and let $X \subseteq \mathbb{P}^n_S$ be a quasi projective and regular scheme with $S := \text{Spec}(A)$. Assume Conjecture 3.3.2 holds for all affine regular schemes of finite type over $\mathbb{Z}$. It follows $\chi(X, i)$ is an integer for all $i \in \mathbb{Z}$.

Proof. This follows from Theorem 4.3, since in this case $\chi(X, i)$ is a finite sum of integers. \hfill \square
Proof. By the projective bundle formula there is an isomorphism of abelian groups
\[ K'_m(\mathbb{P}(E^*)) \cong K'_m(T)\{1, t, \ldots, t^d\}. \]
Let \( R := \mathbb{Q}[t]/(t^{d+1}) \) with \( t := 1 - L \). It follows \( \psi^k \) acts on \( R \) as follows: \( \psi^k(t) = \psi^k(1 - L) = 1 - L^k \). Let \( v \in \mathbb{Z} \) be an integer and let \( R(v) \) denote the vector space of elements \( x \in R \) with \( \psi^k(x) = k^v x \). It follows there is an inclusion of vector spaces over \( \mathbb{Q} \):
\[ K'_m(\mathbb{P}(E^*))_{(j)} \subseteq \oplus_{u + v = j} K'_m(T)_{(u)} \otimes R(v) \]
and by assumption
\[ \dim_{\mathbb{Q}}(\oplus_{u + v = j} K'_m(T)_{(u)} \otimes R(v)) < \infty \]
for all \( m, j \) it follows \( \dim_{\mathbb{Q}}(K'_m(\mathbb{P}(E^*))_{(j)}) < \infty \) for all \( m, j \) and the Corollary follows. \( \square \)

The aim of this section is to prove Conjecture 3.3.3 for all flag bundles \( \mathbb{F}(E) \) on \( S := \text{Spec} \mathcal{O}_K \) assuming Beilinson-Soule vanishing 3.3.1 for a stratification \( X_i \) of \( \mathbb{F}(E) \).

**Theorem 4.8.** Let \( T \) be a regular quasi projective scheme of finite type over \( \mathcal{O}_K \) such that Conjecture 3.3.3 holds for \( T \). Let \( X \) be a quasi projective scheme of finite type over \( T \) with a stratification of closed subschemes \( \emptyset = X_{n+1} \subseteq X_n \subseteq \cdots \subseteq X_2 \subseteq X_1 := X \) with \( X_i - X_{i+1} = \mathbb{A}^{d_i}_{T} \) such that Conjecture 3.3.1 holds for \( X_i \). It follows
\[ \text{ord}_{s=k}(L(X,s)) = \chi(X,k), \]
hence Conjecture 3.3.3 holds for \( X \).

Proof. By [9] we get for any open subscheme \( U \subseteq X \) with closed complement \( Z := X - U \) a long exact localization sequence
\[ \cdots \to K'_m(Z)_{(i)} \to K'_m(X)_{(i)} \to K'_m(U)_{(i)} \to \cdots \]
hence we get by Theorem 4.3 a well defined equality
\[ \chi(X, i) = \chi(U, i) + \chi(Z, i) \]
of Euler characteristics. It follows
\[ \chi(X, k) = \sum_{i=1}^{n} \chi(\mathbb{A}^{d_i}_{T}, k). \]
We also get
\[ L(X,s) = \prod_{i=1}^{n} L(\mathbb{A}^{d_i}_{T}, s). \]
We get
\[ \text{ord}_{s=k}(L(X,s)) = \text{ord}_{s=k}(\prod_{i=1}^{n} L(\mathbb{A}^{d_i}_{T}, s)) = \sum_{i=1}^{n} \chi(\mathbb{A}^{d_i}_{T}, k) = \chi(X, k) \]
and the conjecture holds for \( X \). \( \square \)
Corollary 4.9. Let $E$ be a finite rank projective $O_\mathcal{K}$-module and let $\mathbb{P}(E)$ a flag bundle for $E$. Assume $\mathbb{P}(E)$ has a stratification as in Theorem 4.8. Then Conjecture 3.3.3 holds for $\mathbb{P}(E)$. In particular it holds for the projective bundle $\mathbb{P}(E^\ast)$ and the grassmannian bundle $G(m, E)$ with $1 \leq m < \text{rk}(E)$. The same holds for any finite rank locally trivial $O_T$-module $F$: If it holds for $T$ it holds for $\mathbb{P}(F)$.

Proof. The proof of the Corollary is similar to the proof of Theorem 4.8. □

Example 4.10. An alternative proof of Corollary 4.9 using induction.

Given a locally trivial finite rank $O_\mathcal{K}$-module $E$ and a flag bundle $\mathbb{P}(E)$, we may ask if it is possible to give a proof of Conjecture 3.3.3 using an induction similar to Example 4.1. One wants a stratification of closed subschemes

\[ \emptyset = X_{n+1} \subseteq X_n \subseteq \cdots \subseteq X_2 \subseteq X_1 = \mathbb{P}(E) \]

with $X_i - X_{i+1} = \mathbb{A}^{d_i}$ and where the sub-schemes $X_i$ are flag schemes of dimension smaller than $\mathbb{P}(E)$, and where Conjecture 3.3.3 hold for $X_i$. This is done in Example 4.1 for $\mathbb{P}^d$-bundles on $O_\mathcal{K}$. In Example 4.5.1 the schemes $X_i$ are projective spaces over $S$ of dimension less than $d$. Such a proof would not use Beilinson-Soule vanishing [3,3.3] in general, and would therefore be more elementary.

5. Appendix A: The weight space decomposition for algebraic $K$-theory of projective bundles

In this section we calculate explicitly the weight spaces $K'_m(\mathbb{P}(E^\ast))_i$ for any $\mathbb{P}^d$-bundle on $S$ to illustrate that it is easy to make explicit calculations for projective bundles. The calculation is not necessary for the main results of the paper, but it shows how to perform such calculations using elementary methods.

Let in the following $X := \text{Proj}(\mathbb{Z}[x_0, \ldots, x_n])$ be projective $n$-space over the ring of integers $\mathbb{Z}$. By the projective bundle formula for algebraic $K$-theory we get

\[ K_m(X)_\mathbb{Q} = K_m(\mathbb{Z})_\mathbb{Q} \otimes \mathbb{Q}[t]/(t^{n+1}) = K_m(\mathbb{Z})_\mathbb{Q} \otimes \mathbb{Q}\{1, t, \ldots, t^n\}, \]

where $t = 1 - L = 1 - [O(-1)]$ with $L = [O(-1)]$ and $O(-1)$ is the tautological bundle on projective space $X := \mathbb{P}(V)$. Let $R := \mathbb{Q}[t]/(t^{n+1}) = \mathbb{Q}\{1, t, t^2, \ldots, t^n\}$. Let

\begin{equation}
(5.0.1) \quad x := \ln(1 - t) = -(t + (1/2)t^2 + (1/3)t^3 + \ldots + (1/n)t^n)
\end{equation}

in the ring $R = \mathbb{Q}\{1, t, t^2, \ldots, t^n\}$.

Lemma 5.1. Let $\psi^k$ be the $k$th Adams operator acting on $R$. The following holds for all integers $k \geq 0$:

\begin{equation}
(5.1.1) \quad \psi^k(x) = kx.
\end{equation}

\begin{equation}
(5.1.2) \quad \text{For every integer } i \geq 1 \text{ we get } \psi^k(x^i) = k^i x^i.
\end{equation}

Proof. By definition $L = [O(-1)]$ is the class in $K_0(X)$ of the tautological line bundle $O(-1)$ on projective space, hence the Adams operator $\psi^k$ acts as follows: $\psi^k(L) = L^k$. We get since $t = 1 - L$ the following calculation:

\[ \psi^k(x) = \psi^k(-(t + (1/2)t^2 + (1/3)t^3 + \ldots + (1/n)t^n) = \]
\[ \psi^k(-((1-L) + (1/2)(1-L)^2 + (1/3)(1-L)^3 + \ldots + (1/n)(1-L)^n) = \]
\[ = -((1-\psi^k(L)) + (1/2)(1-\psi^k(L))^2 + \ldots + (1/n)(1-\psi^k(L))^n) = \]
\[ \ln(\psi^k(L)) = \ln(L^k) = k \ln(L) = kx \]

by Corollary A2 in the Appendix. Claim 1 is proved. Claim 2: We get \( \psi^k(x^i) = k^i x^i \) and Claim 2 is proved. \( \square \)

Note: Formal properties of exponential power series and logarithm power series valid in the formal power series ring \( \mathbb{Q}[[t]] \) implies similar properties for exponentials and logarithms in the quotient ring \( R := \mathbb{Q}[[t]]/(t^{n+1}) \). If

\[ (5.1.3) \quad \ln(L) := \ln(1-t) = -(t + (1/2)t^2 + (1/3)t^3 + \ldots + (1/i)t^i + \ldots), \]

where \( \ln(L) \) lives in the formal power series ring \( \mathbb{Q}[[t]] \), one proves there is an equality of formal power series \( \ln(L^k) = k \ln(L) \) for all integers \( k \geq 0 \) in \( \mathbb{Q}[[t]] \). See the Appendix for general properties of formal power series. It follows the vector \( x^i \) is an eigen vector for \( \psi^k \) with eigen value \( k^i \). It follows the inclusion of vector spaces

\[ (5.1.4) \quad \mathbb{Q}\{1, x, x^2, \ldots, x^n\} \subseteq \mathbb{Q}\{1, t, t^2, \ldots, t^n\} \]

is an isomorphism of vector spaces: The vectors \( \{1, x, x^2, \ldots, x^n\} \) are linearly independent over \( \mathbb{Q} \) since they have different eigenvalues with respect to \( \psi^k \) - the kth Adams operator. Hence [5.1.3] gives a decomposition of \( R := \mathbb{Q}[t]/(t^{n+1}) \) into eigen spaces for the Adams operations \( \psi^k \) for \( k \geq 0 \). We get an isomorphism of abelian groups

\[ (5.1.5) \quad K_*(X)_{\mathbb{Q}} \cong K_*(\mathbb{Z})_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}\{1, x, x^2, \ldots, x^n\}. \]

We get the following formula for \( K_m(X)_{\mathbb{Q}} \):

\[ (5.1.6) \quad K_m(X)_{\mathbb{Q}} = 0 \text{ if } m < 0. \]
\[ (5.1.7) \quad K_m(X)_{\mathbb{Q}} = \mathbb{Q}\{1, x, x^2, \ldots, x^n\} \text{ if } m = 0. \]
\[ (5.1.8) \quad K_m(X)_{\mathbb{Q}} = 0 \text{ if } m = 1 \text{ or } m = 2k, k \geq 1. \]
\[ (5.1.9) \quad K_m(X)_{\mathbb{Q}} = \mathbb{Q}\{1, x, x^2, \ldots, x^n\} \text{ if } m = 4k + 1, k > 0. \]
\[ (5.1.10) \quad K_m(X)_{\mathbb{Q}} = 0 \text{ if } m = 4k + 3, k \geq 0. \]

For the field of rational numbers \( \mathbb{Q} \) we have \( r_1 = 1 \) and \( r_2 = 0. \)

**Lemma 5.2.** The following holds for \( K_m(X)_{\mathbb{Q}}^{(i)} \) and \( i = 0, \ldots, n:\)

\[ (5.2.1) \quad K_m(X)_{\mathbb{Q}}^{(i)} = 0 \text{ if } m < 0. \]
\[ (5.2.2) \quad K_m(X)_{\mathbb{Q}}^{(i)} = \mathbb{Q} \text{ if } m = 0. \]
\[ (5.2.3) \quad K_m(X)_{\mathbb{Q}}^{(i)} = 0 \text{ if } m = 1 \text{ or } m = 2k \text{ with } k \geq 1. \]
\[ (5.2.4) \quad K_m(X)_{\mathbb{Q}}^{(i)} = \mathbb{Q} \text{ if } m = 4k + 1 \text{ with } k > 0. \]
\[ (5.2.5) \quad K_m(X)_{\mathbb{Q}}^{(i)} = 0 \text{ if } m = 4k + 3 \text{ with } k \geq 0. \]
Proof. The Lemma follows from the discussion above: The basis \{1, x, x^2, \ldots, x^n\} gives a decomposition of \( R := Q[t]/(t^{n+1}) \) into eigen spaces for the Adams operation \( \psi^k \) and the Lemma follows from the projective bundle formula and the calculation of \( K_m(\mathbb{Z})_Q \) given above. \( \square \)

**Corollary 5.3.** For all \( m = 4k + 1 \) with \( k > 0 \) and all \( i = 0, \ldots, n \) it follows \( K_m(X_i)_Q = Q \neq 0 \).

**Proof.** This follows from Lemma 1 above. \( \square \)

Algebraic K-theory \( K_m(O_K)_Q \) is well known, the Adams eigen space \( K_m(O_K)_Q^{(i)} \) is well known by [4], Volume 1, Theorem 47 and the projective bundle formula holds for \( \mathbb{P}(E^*) \):

\[
K_n(\mathbb{P}(E^*))_Q \cong K_n(O_K)_Q \otimes Q[t]/(t^{n+1}).
\]

Hence the study of the eigen space \( K_m(\mathbb{P}(E^*))_Q^{(i)} \) is by the above calculation reduced to the study of \( K_m(O_K)_Q^{(i)} \) which is well known by Theorem 2.3. We get the following Theorem:

**Theorem 5.4.** Let \( Q \subseteq K \) be an algebraic number field with ring of integers \( O_K \). Let \( r_1, r_2 \) be the real and complex places of \( K \). Let \( \mathbb{P}(E^*) \) be a rank \( e \) projective bundle on \( S := \text{Spec}(O_K) \) and let \( K_m(\mathbb{P}(E^*))_Q \) denote the \( m \)th algebraic K-theory of the category of algebraic vector bundles on \( \mathbb{P}(E^*) \) with rational coefficients. The following holds: Let \( j \geq 0 \) be an integer.

\[
\begin{align*}
(5.4.1) & \quad K_m(\mathbb{P}(E^*))_Q^{(j)} = 0 \text{ for all } m < 0 \text{ and } m = 2k \text{ with } k \geq 1 \text{ an integer.} \\
(5.4.2) & \quad K_0(\mathbb{P}(E^*))_Q^{(j)} = Q \text{ if } j = 0, 1, 2, \ldots, e. \\
(5.4.3) & \quad K_0(\mathbb{P}(E^*))_Q^{(j)} = 0 \text{ if } j > e. \\
(5.4.4) & \quad K_{4a+1}(\mathbb{P}(E^*))_Q^{(j)} = Q^{r_1+1} \text{ if } j \text{ is in } I := 2a + 1, 2a + 2, \ldots, 2a + 1 + e. \\
(5.4.5) & \quad K_{4a+1}(\mathbb{P}(E^*))_Q^{(j)} = 0 \text{ if } j \text{ is not in } I. \\
(5.4.6) & \quad K_{4a+3}(\mathbb{P}(E^*))_Q^{(j)} = Q^{r_2} \text{ if } j \text{ is in } J := 2a + 2, 2a + 3, \ldots, 2a + 2 + e. \\
(5.4.7) & \quad K_{4a+3}(\mathbb{P}(E^*))_Q^{(j)} = 0 \text{ if } j \text{ is not in } J.
\end{align*}
\]

Here \( a \geq 0 \) is an integer.

**Proof.** This follows from the calculation of \( K_m(O_K)_Q^{(j)} \), the projective bundle formula and the eigen space decomposition \( R := Q[t]/(t^{n+1}) = Q\{1, x, x^2, \ldots, x^n\} \) of the ring \( R \), with \( x := \ln(L) := \ln(1 - t) \in R \), as described above. \( \square \)

**Example 5.5.** Example of Theorem 5.4 for terms \( m = 0, 1, 2, 3 \).

\( m = 0: \)

\[
K_0(\mathbb{P}(E^*))_Q^{(l)} = Q \text{ if } l = 0, 1, 2, \ldots, e.
\]

\[
K_0(\mathbb{P}(E^*))_Q^{(l)} = Q \text{ if } l > e.
\]

\( m = 1: \)

\[
K_1(\mathbb{P}(E^*))_Q^{(l)} = Q^{r_1+r_2-1} \text{ if } l = 1, 2, 3, \ldots, e + 1.
\]

\[
K_0(\mathbb{P}(E^*))_Q^{(l)} = 0 \text{ if } l = 0 \text{ or } l > e + 1.
\]
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$m = 2$: \( K_2(\mathbb{P}(E^*))^{(l)}_Q = 0 \).

\( m = 3: \)

\( K_3(\mathbb{P}(E^*))^{(l)}_Q = \mathbb{Q}^2 \) if \( l = 2, 3, 4, \ldots, e + 2 \).

\( K_3(\mathbb{P}(E^*))^{(l)}_Q = 0 \) if \( I \neq 0, 1 \) or \( l > e + 2 \).

**Example 5.6.** Schubert calculus for algebraic K-theory.

In a future paper a similar theory and calculation will be developed for the algebraic K-theory \( K_*(\mathbb{G}(m, E)) \) of the grassmannian \( \mathbb{G}(m, E) \) of \( E \). The aim of this study is to introduce and study Schubert calculus for the K-theory of the grassmannian and flag schemes \( \mathbb{F}(E) \) of a bundle \( E \) over \( S := \text{Spec}(\mathcal{O}_K) \), and to relate this study to Bloch’s higher Chow groups. In [6], Proposition 3.1 (Berthelot’s talk) the following formula is proved: Let \( S \) be a noetherian scheme, \( E \) a locally trivial \( \mathcal{O}_S \)-module of rank \( n \) and \( P := \{ p_1, \ldots, p_k \} \) a set of positive integers with \( \sum_i p_i = n \) and \( \mathbb{F}_P(E) := \mathbb{F}(P, E) \) the flag bundle of \( E \) of type \( P \), it follows the canonical morphism

\[ K_*(S) \otimes_{K_*(S)} K_*(\mathbb{F}_P(E)) \cong K_*(\mathbb{F}_P(E)) \]

is an isomorphism. Hence a formula similar to the projective bundle formula is known for flag bundles. One wants to calculate weight space decomposition

\[ K_m(\mathbb{F}_P(E))_Q \cong \oplus_{i \in \mathbb{Z}} K_m(\mathbb{F}_P(E))^{(i)} \]

for all integers \( m \).

**Corollary 5.7.** Let \( X \) be a scheme of finite type over \( \text{Spec}(\mathcal{O}_K) \). There are no integers \( M, L \gg 0 \) with the property that \( K_m(X)^{(l)}_Q = 0 \) for \( m \geq M \) and \( l \geq L \).

**Proof.** Choose an integer \( a \) such that \( 2a + 1 \geq \max \{ M, L \} \). It follows from Theorem [A4] that \( K_{4a+1}(\mathbb{P}(E^*))^{(2a+1)}_Q = \mathbb{Q}^{r_1 + r_2} \neq 0 \). By choice \( 4a + 1 \geq M \) and \( 2a + 1 \geq L \).

\[ \square \]

6. APPENDIX B: SOME GENERAL PROPERTIES OF FORMAL POWER SERIES

In this section we recall some well known elementary facts on formal power series, logarithm power series and maps of abelian groups.

Recall the following results from [Bour], page A.IV.39 on formal power series:

Let

\[ l(g(x)) := \sum_{n \geq 1} (-1)^{n-1}(1/n)(g(x))^n \in \mathbb{Q}[x]. \]

For any \( g(x) \in \mathbb{Q}[x] \). Define the following formal power series:

\[ \text{Log}(g(x)) := l(g(x) - 1) \]

For any power series \( g(x) \in \mathbb{Q}[x] \). It follows

\[ \text{Log}(1 - x) = l(-x) = -(x + (1/2)x^2 + (1/3)x^3 + (1/4)x^4 + \cdots) \in \mathbb{Q}[x]. \]

Let \( A \) be a commutative unital ring containing the field \( \mathbb{Q} \) of rational numbers. Let \( \text{nil}(A) \) be the nilradical of \( A \). Let \( 1 - \text{nil}(A) \) denote the set of elements on the form \( 1 - u \) with \( u \in \text{nil}(A) \). It follows \( 1 - u \) is a multiplicative unit in \( A \). The set \( 1 - \text{nil}(A) \) has a multiplication: \( (1 - u)(1 - v) = 1 - u - v + uv = 1 - z \) with
\[ z = -u - v + uv, \] and the element \( z \) is again in \( \text{nil}(A) \). Hence \( (1 - u)(1 - v) = 1 - z \) is in \( 1 - \text{nil}(A) \). It follows \( 1 - \text{nil}(A) \) is a subgroup of the multiplicative group of units in \( A \).

**Lemma 6.1.** (A1) Let \( u \in \text{nil}(A) \) be an element with \( u^{k+1} = 0 \). Define the following map:

\[ \ln : 1 - \text{nil}(A) \to \text{nil}(A) \]

by

\[ \ln(1 - u) := -(u + (1/2)u^2 + (1/3)u^3 + \cdots + (1/k)u^k) \in \text{nil}(A). \]

It follows \( \ln \) is a morphism of groups: For any two elements \( 1 - u, 1 - v \in 1 - \text{nil}(A) \) it follows

\[ \ln((1 - u)(1 - v)) = \ln(1 - u) + \ln(1 - v). \]

**Proof.** From \([3]\), page A.IV.40 we get

\[ \log(1 - x) = l(1 - x - 1) = l(-x) \in \mathbb{Q}[[x]]. \]

The following holds for the powerseries \( l(x) \):

\[ l(x + y + xy) = l(x) + l(y) \]

in the ring \( \mathbb{Q}[[x, y]] \). We may for any two elements \( u, v \) in \( \text{nil}(A) \) define a map

\[ f : \mathbb{Q}[[x, y]] \to A \]

by \( f(x) = u, f(y) = v \). It follows \( f \) induce a well defined map of rings

\[ f' : \mathbb{Q}[[x, y]]/I \to A \]

where \( I = \ker(f) \). In the ring \( \mathbb{Q}[[x, y]] \) we get the following formula:

\[ \log((1 - x)(1 - y)) = \log(1 - x - y + xy) = l(-x - y + xy) = l(-x) - l(-y) = \log(1 - x) + \log(1 - y). \]

It follows the same formula holds in the quotient ring \( \mathbb{Q}[[x, y]]/I \). Hence we get the following formula for the map \( \ln \) (viewing \( u \) and \( v \) as elements in the quotient \( \mathbb{Q}[[x, y]]/I \)):

\[ \ln((1 - u)(1 - v)) = \log((1 - x)(1 - y)) = \log(1 - x) + \log(1 - y) = \ln(1 - u) + \ln(1 - v). \]

Hence the map \( \ln \) is a map of groups. \( \square \)

Note: Lemma A1 may also be proved using Bell polynomials.

**Corollary 6.2.** (A2) Use the notation from Lemma A1. If \( 1 - u \in 1 - \text{nil}(A) \) the following holds for any integer \( k \geq 1 \):

\[ \ln((1 - u)^k) = k \ln(1 - u). \]

**Proof.** This follows from Lemma A1 and an induction. \( \square \)

Example: Let \( A := \mathbb{Q}[t]/(t^{e+1}) \) with \( \text{nil}(A) = (t) \) define the following logarithm map \( (u \in \text{nil}(A)) \):

\[ \ln(1 - u) := -(u + (1/2)u^2 + (1/3)u^3 + \cdots + (1/e)u^e) \in A. \]

It follows

\[ \ln((1 - u)^k) = k \ln(1 - u) \] (6.2.1)
for any integer $k \geq 1$. The property \ref{6.2.1} is well known when we consider the logarithm function defined for real numbers, and the above section proves it holds for formal power series.

Note: Formal properties of exponentials and logarithms in $\mathbb{Q}[[t]]$ can also be proved using Bell polynomials.

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