A Study on Product-Sum of Triangular Fuzzy Numbers
Mohamed Ali A\textsuperscript{1} and Rajkumar N\textsuperscript{2}

Received: 18 October 2021 / Accepted: 07 December 2021 / Published online: 21 December 2021
©Sacred Heart Research Publications 2017

Abstract

We study the problem: if $\tilde{a}_i$, $i \in N$ are fuzzy numbers of triangular form, then what
is the membership function of the infinite (or finite) sum $\tilde{a}_1 + \tilde{a}_2 + \cdots$ (defined via the
sup-product-norm convolution)

Key words: Triangular fuzzy number, Product-sum.
AMS classification: 05c72, 47s40.

1 Introduction

A fuzzy number is a convex fuzzy subset of the real line $\mathbb{R}$ with a normalized
membership function. A triangular fuzzy number $\tilde{a}$ denoted by $(a, \alpha, \beta)$ is defined as

$$
\tilde{a}(t) = \begin{cases}
1 - \frac{a-t}{\alpha} & \text{if } a - \alpha \leq t \leq a \\
1 & \text{if } a \leq t \leq b \\
1 - \frac{t-b}{\beta} & \text{if } a \leq t \leq b + \beta \\
0 & \text{otherwise}
\end{cases}
$$

where $a \in \mathbb{R}$ is the center and $\alpha > 0$ is the left spread, $\beta > 0$ is the right spread of $\tilde{a}$. If $\alpha = \beta$, then the triangular fuzzy number is called symmetric triangular fuzzy
number and denoted by $(a, \alpha)$.

If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers, then their product-sum $\tilde{a} + \tilde{b}$ is defined as,

$$(\tilde{a} + \tilde{b})(z) = \sup_{x+y=z} \tilde{a}(x)\tilde{b}(y)$$

The support $\supp \tilde{a}$ of a fuzzy number $\tilde{a}$ is defined as

$$\supp \tilde{a} = \{ t \in \mathbb{R} | \tilde{a}(t) > 0 \}$$

\textsuperscript{1}Department of Mathematics, Islamiah College (Autonomous) Vaniyambadi 635 752, Tirupattur District, Tamil Nadu India. Email:mohamedalihashim@gmail.com
\textsuperscript{2}Department of Mathematics, Islamiah College (Autonomous) Vaniyambadi 635 752, Tirupattur District, Tamil Nadu India. Email:valvanrk2009@gmail.com
2 Preliminaries

Definition 2.1 Fuzzy set

Let us take a set \( \tilde{A} \), which is defined by \( \tilde{A} = \{ (x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0, 1] \} \).
If in the pair \( (x, \mu_{\tilde{A}}(x)) \), the first one, \( x \) belongs to the classical set \( A \) and the second one \( \mu_{\tilde{A}}(x) \) belongs to the interval [0,1], then set \( \tilde{A} \) is called a fuzzy set. Here \( \mu_{\tilde{A}}(x) \) is called a Membership function.

Definition 2.2 Interval-valued fuzzy set (IVFS)

An IVFS \( \tilde{A} \) on \( \mathbb{R} \) is defined by
\[
\tilde{A}_n = \{(x, (\mu_{\tilde{A}_U}(x), \mu_{\tilde{A}_L}(x))) : x \in \mathbb{R}\}
\]
where \( x \in \mathbb{R} \) and \( \mu_{\tilde{A}_U}(x) \), maps \( \mathbb{R} \) into \([0, \lambda] \), \( \mu_{\tilde{A}_L}(x) \), maps \( \mathbb{R} \) into \([0, \omega] \) \( \forall x \in \mathbb{R} \). \( \mu_{\tilde{A}_U}(x) \leq \mu_{\tilde{A}_L}(x) \). (\( \lambda \) and \( \omega \) are the maximum value of upper and lower membership function, respectively)

Definition 2.3 Non-linear interval-valued fuzzy number (IVFN)

An IVFN is denoted by
\[
\tilde{A}_{nLIVFN} = \{(a_1, b, c_1; \lambda), (a, b, c; \omega) ; n_1, n_2, n_3, n_4\}
\]
where \( 0 < \omega \leq \lambda \leq 1 \) and \( a_1 < a < b < c < c_1 \)

The upper and lower membership function of IVFN is defined by
\[
\mu_{\tilde{A}_U}(x) = \begin{cases} 
\lambda \left( \frac{x-a_1}{b-a_1} \right)^{n_1}, & a_1 \leq x \leq b \\
\lambda, & x = b \\
\lambda \left( \frac{c_1-x}{c_1-b} \right)^{n_2}, & b \leq x \leq c_1 \\
0, & \text{otherwise}
\end{cases}
\] and
\[
\mu_{\tilde{A}_L}(x) = \begin{cases} 
\omega \left( \frac{x-a}{b-a} \right)^{n_3}, & a \leq x \leq b \\
\omega, & x = b \\
\omega \left( \frac{c-x}{c-b} \right)^{n_4}, & b \leq x \leq c \\
0, & \text{otherwise}
\end{cases}
\]

3 Product-sum of triangular fuzzy numbers

In this section we shall calculate the membership function of the product-sum \( \tilde{a}_1 + \tilde{a}_2 + \cdots + \tilde{a}_n + \cdots \) where \( \tilde{a}_i, i \in \mathbb{N} \) are fuzzy numbers of triangular form.
The following theorem can be interpreted as a central limit theorem for mutually product-related identically distributed fuzzy variables of symmetric triangular form.

Theorem 3.1 Let \( \tilde{a}_i = (a_i, \alpha), i \in \mathbb{N} \). If
\[
A := \sum_{i=1}^{\infty} a_i
\]

\[\text{Journal of Computational Mathematica} \quad \text{Page 64 of } 67\]
exists and it is finite, then with the notations
\[ \tilde{A}_n := \tilde{a}_1 + \cdots + \tilde{a}_n, \quad A_n := a_1 + \cdots + a_n, \quad n \in \mathbb{N} \]
we have
\[ \left( \lim_{n \to \infty} \tilde{A}_n \right)(z) = \exp(-|A - z|/\alpha), \quad z \in \mathbb{R} \]

**proof:** It will be sufficient to show that
\[ \tilde{A}_n(z) = \begin{cases} 
\left[ 1 - \frac{|A_n - z|}{n\alpha} \right]^n & \text{if } |A_n - z| \leq n\alpha \\
0 & \text{otherwise} 
\end{cases} \]  
(1)

for each \( n \geq 2 \), because from (1) it follows that
\[ \left( \lim_{n \to \infty} \tilde{A}_n \right)(z) = \lim_{n \to \infty} \left[ 1 - \frac{|A_n - z|}{n\alpha} \right]^n = \exp \left( - \lim_{n \to \infty} \frac{|A_n - z|}{n\alpha} / \alpha \right) = \exp(-|A - z|/\alpha), \quad z \in \mathbb{R} \]

From the definition of product-sum of fuzzy numbers it follows that
\[ \text{supp} \tilde{A}_n = \text{supp} (\tilde{a}_1 + \cdots + \tilde{a}_n) = \text{supp} \tilde{a}_1 + \cdots + \text{supp} \tilde{a}_n = \\
[a_1 - \alpha, a_1 + \alpha] + \cdots + [a_n - \alpha, a_n + \alpha] = [A_n - n\alpha, A_n + n\alpha], \quad n \in \mathbb{N} \]

We prove (1) by making an induction argument on \( n \). Let \( n = 2 \). In order to determine \( A_2(z), z \in [A_2 - 2\alpha, A_2 + 2\alpha] \) we need to solve the following mathematical programming problem:
\[ \left( 1 - \frac{|a_1 - x|}{\alpha} \right) \left( 1 - \frac{|a_2 - y|}{\alpha} \right) \to \text{max} \]
subject to \( |a_1 - x| \leq \alpha, \)
\( |a_2 - y| \leq \alpha, \quad x + y = z \)

By using Lagrange’s multipliers method and decomposition rule of fuzzy numbers into two separate parts, it is easy to see that \( A_2(z), z \in [A_2 - 2\alpha, A_2 + 2\alpha] \) is equal to the optimal value of the following mathematical programming problem:
\[ \left( 1 - \frac{a_1 - x}{\alpha} \right) \left( 1 - \frac{a_2 - y}{\alpha} \right) \to \text{max} \]
subject to \( a_1 - \alpha \leq x \leq a_1 \)
\( a_2 - \alpha \leq z - x \leq a_2, \quad x + y = z \)  
(2)
Using Lagrange’s multipliers method for the solution of (2) we get that its optimal value is
\[ \left[ 1 - \frac{|A_2 - z|}{2\alpha} \right]^2 \]
and its unique solution is
\[ X = 1/2 (a_1 - a_2 + z) \]
(where the derivative vanishes).
Indeed, it can be easily checked that the inequality
\[ \left[ 1 - \frac{|A_2 - z|}{2\alpha} \right]^2 \geq 1 - \frac{A_2 - z}{\alpha} \]
holds for each \( z \in [A_2 - 2\alpha, A_2] \)
In order to determine \( \tilde{A}_2(z), z \in [A_2, A_2 + 2\alpha] \) we need to solve the following mathematical programming problem:
\[
\begin{align*}
(1 + \frac{a_1-x}{\alpha}) \left( 1 + \frac{a_2-z+x}{\alpha} \right) & \rightarrow \max \\
\text{subject to } & a_1 \leq x \leq a_1 + \alpha, a_2 \leq z - x \leq a_2 + \alpha
\end{align*}
\]
(3)
In a similar manner we get that the optimal value of (3) is
\[ \left[ 1 - \frac{|z - A_2|}{2\alpha} \right]^2 \]
Let us assume that (1) holds for some \( n \in N \). By similar arguments we obtain
\[ \tilde{A}_{n+1}(z) = \left( \tilde{A}_n + \tilde{a}_{n+1} \right) (z) = \]
\[ \sup_{x+y=z} \tilde{A}_n(x) \cdot \tilde{a}_{n+1}(y) = \sup_{x+y=z} \left( 1 - \frac{|A_n - x|}{n\alpha} \right) \left( 1 - \frac{|a_{n+1} - y|}{\alpha} \right) = \]
\[ \left[ 1 - \frac{|A_{n+1} - z|}{(n+1)\alpha} \right]^{n+1}, z \in [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha] \]
and
\[ \tilde{A}_{n+1}(z) = 0, z \notin [A_{n+1} - (n+1)\alpha, A_{n+1} + (n+1)\alpha] \]
This ends the proof.

**Theorem 3.2** Let \( \tilde{a}_i = (a_i, \alpha, \beta), i \in N \) be fuzzy numbers of triangular form. If
A := \sum_{i=1}^{\infty} a_i \text{ exists and it is finite, then with the notations of Theorem 3.2 we have}

\left( \lim_{n \to \infty} \tilde{A}_n \right) (z) = \begin{cases} 
\exp \left( -\frac{|A-z|}{\alpha} \right) & \text{if } z \leq A \\
\exp \left( -\frac{|A-z|}{\beta} \right) & \text{if } z \geq A 
\end{cases}

4 Conclusion

In this paper, we studied about the Product-sum of triangular fuzzy numbers and also calculated the membership function of the product-sum \( \tilde{a}_1 + \tilde{a}_2 + \cdots \tilde{a}_n + \cdots \) where \( \tilde{a}_i, i \in \mathbb{N} \) are fuzzy numbers of triangular form.

References
[1] Atanassov KT, Intuitionistic Fuzzy Sets ;VII ITKR×92s Session: Sofia, Bulgarian, 1983.
[2] Dubois D, Prade H, Operations on fuzzy numbers. Int. J. Syst. Sci., Volume 9, 613-626 (1978).
[3] Dubois D and Prade H, Additions of Interactive Fuzzy Numbers, IEEE Transactions on Automatic Control, Volume 26, 926-936(1981).
[4] Dubois D, Prade H, Fundamental of Fuzzy Sets, The Handbooks of Fuzzy Sets, Springer, Volume 7 (2000).
[5] Ebrahimnejad A, A method for solving linear programming with interval-valued fuzzy variables, RAIRO-Oper. Res, Volume 52, 955-979(2018).
[6] Guanrong Chen, Trung Tat Pham, Introduction to fuzzy sets, fuzzy logic, and fuzzy control systems, (Printed by LCC press, in the United States of America, 2001).
[7] Pathinathan T, Ponnivalavan K, Reverse order Triangular, Trapezoidal and Pentagonal Fuzzy Numbers, Ann. Pure Appl. Math., Volume 9, 107-117(2015).