HERMITE-HADAMARD TYPE INEQUALITIES FOR HARMONICALLY $(\alpha, m)$-CONVEX FUNCTIONS

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ABSTRACT. The author introduces the concept of harmonically $(\alpha, m)$-convex functions and establishes some Hermite-Hadamard type inequalities of these classes of functions.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality is well known in the literature as Hermite-Hadamard integral inequality

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq \frac{f(a) + f(b)}{2}. \tag{1.1}$$

The class of $(\alpha, m)$-convex functions was first introduced in [7], and it is defined as follows:

**Definition 1.** The function $f : [0, b] \rightarrow \mathbb{R}$, $b > 0$, is said to be $(\alpha, m)$-convex where $(\alpha, m) \in [0, 1]^2$, if we have

$$f \left( tx + (1 - t)y \right) \leq tf(y) + (1 - t)f(x)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

It can be easily that for $(\alpha, m) \in \{(0, 0), (\alpha, 0), (1, 0), (1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: increasing, $\alpha$-starshaped, starshaped, $m$-convex, convex, $\alpha$-convex.

Denote by $K_{\alpha m}^\circ(b)$ the set of all $(\alpha, m)$-convex functions on $[0, b]$ for which $f(0) \leq 0$. For recent results and generalizations concerning $(\alpha, m)$-convex functions (see [2, 3, 4, 5, 7, 8, 9, 10, 11]).

In [6], the author gave harmonically convex and established some Hermite-Hadamard’s inequality for harmonically convex functions as follows:

**Definition 2.** Let $I \subseteq \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex, if

$$f \left( \frac{xy}{tx + (1 - t)y} \right) \leq tf(y) + (1 - t)f(x) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0, 1]$. If the inequality in (1.2) is reversed, then $f$ is said to be harmonically concave.

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Theorem 1. Let \( f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L[a, b] \) then the following inequalities hold

\[
(1.3) \quad f \left( \frac{2ab}{a+b} \right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.
\]

The above inequalities are sharp.

In [6], the author gave the following identity for differentiable functions.

Lemma 1. Let \( f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a differentiable function on \( I^o \) and \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \) then

\[
\frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx = \frac{ab(b-a)}{2} \int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} f' \left( \frac{ab}{tb+(1-t)a} \right) dt.
\]

The main purpose of this paper is to introduce the concept of harmonically \((\alpha, m)\)-convex functions and establish some new Hermite-Hadamard type inequalities for these classes of functions.

2. Main Results

Definition 3. The function \( f : (0, b^*] \to \mathbb{R} \), \( b^* > 0 \), is said to be harmonically \((\alpha, m)\)-convex, where \( \alpha \in [0, 1] \) and \( m \in (0, 1] \), if

\[
(2.1) \quad f \left( \frac{xy}{ty + m(1-t)x} \right) = f \left( \left( \frac{t}{x} + \frac{m(1-t)}{y} \right)^{-1} \right) \leq t^\alpha f(x) + m(1-t)^\alpha f(y)
\]

for all \( x, y \in (0, b^*] \) and \( t \in [0, 1] \). If the inequality in (2.1) is reversed, then \( f \) is said to be harmonically \((\alpha, m)\)-concave.

The following proposition is obvious

Proposition 1. Let \( f : (0, b^*] \to \mathbb{R} \) be a function.

a) if \( f \) is \((\alpha, m)\)-convex and nondecreasing function then \( f \) is harmonically \((\alpha, m)\)-convex.

b) if \( f \) is harmonically \((\alpha, m)\)-convex and nonincreasing function then \( f \) is \((\alpha, m)\)-concave.

The following result of the Hermite-Hadamard type holds.

Theorem 2. Let \( f : (0, \infty) \to \mathbb{R} \) be a harmonically \((\alpha, m)\)-convex function with \( \alpha \in [0, 1] \) and \( m \in (0, 1] \). If \( 0 < a < b < \infty \) and \( f \in L[a, b] \), then one has the inequality

\[
(2.2) \quad \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \left\{ \frac{f(a) + \alpha f(mb)}{\alpha + 1}, \frac{f(b) + \alpha f(ma)}{\alpha + 1} \right\}.
\]
Proof. Since $f : (0, \infty) \to \mathbb{R}$ is a harmonically $(\alpha, m)$-convex function, we have, for all $x, y \in I$

$$f\left(\frac{xy}{ty + (1-t)x}\right) = f\left(\frac{myx}{tmy + m(1-t)x}\right) \leq t^\alpha f(x) + m(1-t^\alpha)f(my)$$

which gives:

$$f\left(\frac{ab}{tb + (1-t)a}\right) \leq t^\alpha f(a) + m(1-t^\alpha)f(mb)$$

and

$$f\left(\frac{ab}{ta + (1-t)b}\right) \leq t^\alpha f(b) + m(1-t^\alpha)f(ma)$$

for all $t \in [0, 1]$. Integrating on $[0, 1]$ we obtain

$$\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt \leq \frac{f(a) + am f(mb)}{\alpha + 1}$$

and

$$\int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) dt \leq \frac{f(b) + am f(ma)}{\alpha + 1}.$$ 

However,

$$\int_0^1 f\left(\frac{ab}{tb + (1-t)a}\right) dt = \int_0^1 f\left(\frac{ab}{ta + (1-t)b}\right) dt = \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx$$

and the inequality (2.2) is obtained. $\square$

**Theorem 3.** Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$, $a, b \in I^\circ$ with $a < mb$, $m \in (0, 1]$ and $f' \in L[a, b]$. If $|f|^q$ is harmonically $(\alpha, m)$-convex on $[a, b]$ for $q \geq 1$, with $\alpha \in [0, 1]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{ab(b-a)}{2^{2-1/q}} \left[ \lambda(\alpha, q; a, b) |f'(a)|^q + m\mu(\alpha, q; a, b) |f'(mb)|^q \right]^\frac{1}{q},$$

where

$$\lambda(\alpha, q; a, b) = \frac{2\beta(1, \alpha + 2)}{b^{2q}} \cdot 2F_1 \left(2q, 1; \alpha + 3; 1 - \frac{a}{b}\right) - \frac{\beta(1, \alpha + 1)}{b^{2q}} \cdot 2F_1 \left(2q, 1; \alpha + 2; 1 - \frac{a}{b}\right) + \frac{2^{2q-\alpha} \beta(2, \alpha + 1)}{(a+b)^{2q}} \cdot 2F_1 \left(2q, 2; \alpha + 3; 1 - \frac{2a}{a+b}\right),$$

$$\mu(\alpha, q; a, b) = \lambda(0, q; a, b) - \lambda(\alpha, q; a, b),$$

$\beta$ is Euler Beta function defined by

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0,$$
and \( _2F_1 \) is hypergeometric function defined by

\[
_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt, \
\]

\( c > b > 0, \, |z| < 1 \) (see [1]).

**Proof.** From Lemma [1] and using the power mean inequality, we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{a-b} \int_a^b \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b-a)}{2} \left\{ \frac{1}{2} \left[ \int_0^1 \frac{1-2t}{(tb+(1-t)a)^2} \left| f'(\frac{ab}{tb+(1-t)a})^q \right| \, dt \right] \right\}^{\frac{1}{q}} 
\leq \frac{ab(b-a)}{2^q-1/q} \left[ \lambda(\alpha, q; a, b) \left| f'(a) \right|^q + m(\lambda(0, q; a, b) - \lambda(\alpha, q; a, b)) \right]^{\frac{1}{q}}.
\]

Hence, by harmonically \((\alpha, m)\)-convexity of \( |f|^q \) on \([a, b] \), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{a-b} \int_a^b \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b-a)}{2^q-1/q} \left[ \lambda(\alpha, q; a, b) \left| f'(a) \right|^q + m(\lambda(0, q; a, b) - \lambda(\alpha, q; a, b)) \right]^{\frac{1}{q}}.
\]

It is easily check that

\[
\int_0^1 \frac{|1-2t|^{\alpha}}{(tb+(1-t)a)^{2q}} \, dt = 2 \int_0^{1/2} \frac{|1-2t|^{\alpha}}{(tb+(1-t)a)^{2q}} \, dt - \int_0^1 \frac{|1-2t|^{\alpha}}{(tb+(1-t)a)^{2q}} \, dt 
= \frac{2\beta(1, \alpha+2)}{b^{2q}} \cdot 2F_1 \left( 2q, 1; \alpha+3; 1 - \frac{a}{b} \right) - \frac{\beta(1, \alpha+1)}{b^{2q}} \cdot 2F_1 \left( 2q, 1; \alpha+2; 1 - \frac{a}{b} \right) 
+ \frac{2^{2q-\alpha} \beta(2, \alpha+1)}{(a+b)^{2q}} \cdot 2F_1 \left( 2q, 2; \alpha+3; 1 - \frac{2a}{a+b} \right) = \lambda(\alpha, q; a, b) 
\int_0^1 \frac{|1-2t|^{\alpha}}{(tb+(1-t)a)^{2q}} \, dt = \lambda(0, q; a, b) - \lambda(\alpha, q; a, b).
\]

This completes the proof. \( \square \)

**Theorem 4.** Let \( f : I \subset (0, \infty) \to \mathbb{R} \) be a differentiable function on \( I^o \), \( a, b \in I \) with \( a < mb \), \( m \in (0, 1] \) and \( f' \in \mathbb{L}[a, b] \). If \( |f'|^q \) is harmonically \((\alpha, m)\)-convex on
[a, b] for q ≥ 1, with α ∈ [0, 1], then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b - a)}{2} \lambda^{1 - \frac{q}{\gamma}} (0, q; a, b) \left[ \lambda(\alpha, 1; a, b) |f'(a)|^q + m \mu(\alpha, 1; a, b) |f'(mb)|^q \right]^{\frac{1}{q}},
\]
where λ and μ is defined as in Theorem 3.

Proof. From Lemma 1, power mean inequality and the harmonically \((\alpha, m)\)-convexity of \(|f'|^q\) on \([a, b]\), we have,
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b - a)}{2} \left( \int_{0}^{1} \left| \frac{1 - 2t}{(tb + (1 - t)a)^2} \right| f' \left( \frac{ab}{tb + (1 - t)a} \right) \, dt \right)^{1 - \frac{q}{\gamma}} 
\times \left( \int_{0}^{1} \frac{1 - 2t}{(tb + (1 - t)a)^2} \left[ |f'(a)|^q + m \mu(\alpha, 1; a, b) |f'(mb)|^q \right] \, dt \right)^{\frac{1}{q}} 
\leq \frac{ab(b - a)}{2} \lambda^{1 - \frac{q}{\gamma}} (0, q; a, b) \left[ \lambda(\alpha, 1; a, b) |f'(a)|^q + m \mu(\alpha, 1; a, b) |f'(mb)|^q \right]^{\frac{1}{q}}.
\]

\[\square\]

Theorem 5. Let \(f : I \subset (0, \infty) \to \mathbb{R}\) be a differentiable function on \(I^g\), \(a, b \in I\) with \(a < mb\), \(m \in (0, 1]\), and \(f' \in L[a, b]\). If \(|f'|^q\) is harmonically \((\alpha, m)\)-convex on \([a, b]\) for \(q > 1, \frac{1}{p} + \frac{1}{q} = 1\), with \(\alpha \in [0, 1]\), then
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2} \, dx \right| 
\leq \frac{ab(b - a)}{2} \left( \frac{1}{p + 1} \right)^{\frac{1}{p}} \left( \nu(\alpha, q; a, b) |f'(a)|^q + (\nu(0, q; a, b) - \nu(\alpha, q; a, b)) |f'(b)|^q \right)^{\frac{1}{q}}
\]
where
\[
\nu(\alpha, q; a, b) = \frac{\beta(1, \alpha + 1)}{b^{2q}} F_1 \left( 2q, 1; \alpha + 2; 1 - \frac{a}{b} \right).
\]
Proof. From Lemma 1, Hölder’s inequality and the harmonically \((\alpha, m)\)-convexity of \(|f'|^q\) on \([a, b]\), we have,
\[
\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
\leq \frac{ab(b-a)}{2} \left( \int_0^1 |1 - 2t|^p dt \right)^{\frac{1}{p}} \\
\times \left( \int_0^1 \frac{1}{(tb + (1-t)a)^{2q}} |f' \left( \frac{ab}{tb + (1-t)a} \right)|^q dt \right)^{\frac{1}{q}} \\
\leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \int_0^1 \frac{t^\alpha} {(tb + (1-t)a)^{2q}} \left[ |f'(a)|^q + (1-t^\alpha)|f'(b)|^q \right] dt \right)^{\frac{1}{q}} \\
\leq \frac{ab(b-a)}{2} \left( \frac{1}{p+1} \right)^{\frac{1}{p}} \left( \nu(\alpha, q; a, b) |f'(a)|^q + (\nu(0, q; a, b) - \nu(\alpha, q; a, b)) |f'(b)|^q \right)^{\frac{1}{q}},
\]
where an easy calculation gives
\[
\int_0^1 \frac{t^\alpha} {tb + (1-t)a)^{2q}} dt \\
= \frac{\beta (1, \alpha + 1)}{b^{2q}} {}_2F_1 \left( 2q, 1; \alpha + 2; 1 - \frac{a}{b} \right) \\
= \nu(\alpha, q; a, b)
\]
and
\[
\int_0^1 \frac{1 - t^\alpha}{tb + (1-t)a)^{2q}} dt \\
= \nu(0, q; a, b) - \nu(\alpha, q; a, b).
\]
This completes the proof. \(\square\)

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