Spectrum of the 2-Flavor Schwinger Model from the Heisenberg Spin Chain

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We study the strong coupling limit of the 2-flavor lattice Schwinger model in the Hamiltonian formalism using staggered fermions. We show that the problem of finding the low lying states is equivalent to solving the Heisenberg antiferromagnetic spin chain. We find good agreement with the continuum theory.

Very appealing is the idea that solvable models for simple spin-$\frac{1}{2}$ quantum antiferromagnets are related to quantized gauge theories. In particular, we think it is interesting to exhibit a gauge theory with the same low-lying excitations of the antiferromagnetic Heisenberg chain, since this system admits spin-$\frac{1}{2}$ quark-like spinon excitations, but has physical states with integer spin and an even number of spinons \[ \langle \bar{\psi} \psi \rangle \neq 0 \text{ or antiferromagnetic}. \] The main purpose of this Letter is to establish the correspondence between the low-lying excitations of the antiferromagnetic Heisenberg chain and those of the 2-flavor Schwinger model.

A relation between quantum antiferromagnets and gauge theories is widely known to appear in various contexts. For example, a mechanism of confinement for spin-Peierls antiferromagnets was introduced in \[ \[ \[ \]. \] A related idea has been pursued by Laughlin \[ \[ \] who argued that analogies exist between spectral data of gauge theories and of strongly correlated electron systems. The analogy has also been used to discuss spin ladders \[ \[ \[ \]. \]

In the context of conventional lattice gauge theory, it is known that features of the strong coupling limit, particularly those involving chiral symmetry breaking, have analogs in quantum spin systems \[ \[ \]. \] In many cases, the problem of finding the ground state of the strongly coupled lattice gauge theory is equivalent to finding the ground state of a generalized quantum antiferromagnet \[ \[ \[ \]. \] This could possibly be exploited to obtain non-perturbative information about the structure of the gauge theory states.

This approach suffers from the fact that strong coupling limits of gauge theories are highly non-universal. In four or less dimensions, the continuum limit and the universal behavior occur at weak coupling. There are many choices of strong coupling theory which should produce identical continuum physics. In spite of this difficulty, there exist strong coupling computations which claim some degree of success \[ \[ \]. \] In a previous paper \[ \[ \[ \], \] we re-examined the lattice Schwinger model using staggered fermions in the Hamiltonian formulation. The staggered fermions possess a discrete chiral symmetry and we showed how chiral symmetry breaking, which arises from the axial anomaly in the continuum theory, is the result of spontaneous symmetry breaking in the strong coupling limit. The solution of the strong coupling problem was identical to solving a particular type of Ising system with long-ranged interaction. We showed that the mass of the elementary excitation and the chiral condensate could be computed reliably from an extrapolation to weak coupling using Padé approximants. Our analysis improved previous ones by taking careful account of all discrete symmetries of the continuum theory.

In this Letter, we shall test these ideas further by examining the ground state and low energy excitations of the strongly coupled 2-flavor Schwinger model. We make the point that, even with its inherent nonuniversality, the most straightforward strong coupling limit makes accurate predictions of the masses and quantum numbers of the known elementary excitations of the continuum model. It also gives an intuitive picture of the nonperturbative features of the ground state.

We show that to find the strong coupling ground state, we must solve for the ground state of the Heisenberg antiferromagnetic spin chain. This state and its excitations are known explicitly from the Bethe ansatz \[ \[ \]. \] We show that these states have the spectrum and quantum numbers which are expected for the ground state and massless excitations of the 2-flavor Schwinger model in the continuum. We stress that this is a non-trivial correspondence which would not be easy to find at weak coupling. Chiral symmetry breaking in the Schwinger model would be identical to formation in the antiferromagnet of either commensurate density waves when \[ \langle \bar{\psi} \psi \rangle \neq 0 \text{ or antiferromagnetic}. \] The Bethe ansatz solution of the antiferromagnet shares the property of the 2-flavor Schwinger model that there are no condensates of this form. The lowest lying excitations of the antiferromagnet are a singlet and a triplet \[ \[ \]. \] We shall show that they have identical parity and G-parity to the lowest lying excitations of the continuum 2-flavor Schwinger model \[ \[ \]. \]

Aside from the massless excitations, we find good agreement at strong coupling with the quantum num-
bers and mass of the Schwinger model boson, which we find is created from the Bethe ansatz ground state by charge transport. We show that essential quantities in the strong coupling expansion can be expressed in terms of spin-spin correlators of the quantum antiferromagnet.

The Hamiltonian, gauge constraint and non-vanishing (anti-)commutators of the continuum 2-flavor Schwinger model are

\[ H = \int dx \left( \frac{e^2}{2} E^2(x) + \sum_{a=1}^{2} \psi_{a}^{\dagger}(x) \alpha (i \partial_x + eA(x)) \psi_{a}(x) \right) \]

\[ \partial_x E(x) + \sum_{a=1}^{2} \psi_{a}^{\dagger}(x) \psi_{a}(x) - 1 \sim 0 \]

\[ [A(x), E(y)] = i \delta(x-y), \{ \psi_{a}(x), \psi_{b}^{\dagger}(y) \} = \delta_{ab} \delta(x-y) \]

A lattice Hamiltonian, constraint and (anti-)commutators that reduce to these in the naive continuum limit are

\[ H_{S} = \frac{e^2 a}{2} \sum_{i=1}^{N} E_{x} - \frac{it}{2a} \sum_{i=1}^{N} (\psi_{a,x+1}^{\dagger} e^{iA_{x}} \psi_{a,x} - h.c.) \]

\[ E_{x} - E_{x-1} + \psi_{1,x}^{\dagger} \psi_{1,x} + \psi_{2,x}^{\dagger} \psi_{2,x} - 1 \sim 0, \]

\[ [A_{x}, E_{y}] = i \delta_{xy}, \{ \psi_{a,x}, \psi_{b,y}^{\dagger} \} = \delta_{ab} \delta_{xy} \]

where the fermion fields are defined on the sites, \( x = 1, ..., N \), the gauge and the electric fields, \( A_{x} \) and \( E_{x} \), on the links \( [x; x+1] \), \( N \) is an even integer and, when \( N \) is finite, periodic boundary conditions should be used. When \( N \) is finite, the continuum limit is the two flavor Schwinger model on a circle. In Eq.(1) the constant 1 has been subtracted from the naively defined charge density operator, in order to make the gauge generator odd under the usual charge conjugation transformation. The properly defined charge density reads

\[ \rho(x) = \psi_{1,x}^{\dagger} \psi_{1,x} + \psi_{2,x}^{\dagger} \psi_{2,x} - 1 \]

and vanishes on every site occupied by only one particle.

The lattice 2-flavor Schwinger model is equivalent to a one dimensional quantum Coulomb gas on a lattice with two different kinds of particles. To see this we fix the Coulomb gauge, \( A_{x} = A \). Eliminating the non-constant electric field using the gauge constraint, one obtains the effective Hamiltonian where the constant modes of the gauge field decouple in the thermodynamic limit \( N \rightarrow \infty \). In this limit the Schwinger Hamiltonian, rescaled by the factor \( e^2 a / 2 \), becomes \( H = H_{0} + \epsilon H_{h} \), where

\[ H_{0} = \sum_{x>y} \frac{(x-y)^2}{N} - (x-y) |\rho(x) \rho(y)|, \]

\[ H_{h} = -it(R - L) \]

and \( \epsilon = t / e^2 a^2 \). In Eq.(3) the right \( R \) and left \( L \) hopping operators have been defined \( (L = R^\dagger) \)

\[ R = \sum_{x=1}^{N} R_{x} = \sum_{x=1}^{N} \sum_{a=1}^{2} \psi_{a,x+1}^{\dagger} e^{iA_{x}} \psi_{a,x} \]

On a periodic chain they commute, \( [R, L] = 0 \).

We shall consider a strong coupling perturbative expansion where \( H_{0} \) is the unperturbed Hamiltonian and \( H_{p} \) the perturbation. The \( H_{0} \) ground state is highly degenerate. Due to \( [\bar{R}, L] \), every state with one particle per site, has zero energy. There are \( 2^N \) states of this type.

First order perturbations to the vacuum energy vanish. The leading term in the vacuum energy is of order \( \epsilon^2 \)

\[ E^{(2)}_{0} = -2 < RL > \]

where the expectation values are defined on the degenerate subspace and \( \Pi \) is a projection operator which projects orthogonal to the set of states with one particle per site. Due to the vanishing of the charge density on the ground states of the \( H_{0} \), the commutator \([H_{0}, H_{h}] = H_{h}\) holds on any linear combination of the degenerate ground states. Consequently, from Eq.(3) one finds

\[ E^{(2)}_{0} = -2 < RL > \]

On the ground state the combination \( RL \) can be reexpressed in terms of the Heisenberg Hamiltonian. By introducing the Schwinger spin operators

\[ \tilde{S}_{x} = \psi_{a,x}^{\dagger} \sigma_{ab} \psi_{b,x} \]

the Heisenberg Hamiltonian \( H_{J} = \sum_{x=1}^{N} (\tilde{S}_{x} \cdot \tilde{S}_{x+1} - \frac{1}{4}) \) reads

\[ H_{J} = - \sum_{x=1}^{N} \frac{1}{2} L_{x} R_{x} + \frac{1}{4} \rho(x) \rho(x+1) \]

so that on the degenerate subspace we have

\[ < H_{J} > = < \sum_{x=1}^{N} (\frac{1}{2} L_{x} R_{x}) > \]

Taking into account that products of \( L_{x} \) and \( R_{y} \) at different points have vanishing expectation values on the ground states, and using Eq.(3), Eq.(4) reads

\[ E^{(2)}_{0} = 4 < H_{J} > \]

The problem of determining the correct ground state, on which to perform the perturbative expansion, is then reduced to the diagonalization of the Heisenberg spin Hamiltonian. As is well known, in one dimension \( H_{J} \) is exactly diagonalizable \( [\bar{S}], [\bar{R}] \).

On a given site, the presence of a flavor 1 particle can be represented, in the spin model, by the presence of a spin up, a flavor 2 particle by the presence of a spin down. The number of the \( H_{J} \) eigenstates is \( 2^N \). Among these, the spin singlet with lowest energy is the non degenerate ground state \([g.s. >] \). Consequently, \([g.s. >] \) is the ground state that must be used in the strong coupling perturbation theory of the two flavor Schwinger.
model. \( g.s. > \) is translationally invariant, namely it is invariant under the discrete chiral symmetry \( \mathbb{Z}_2 \). Therefore, at variance with the one flavor case, the chiral symmetry cannot be spontaneously broken even in the infinite coupling limit. \( g.s. > \) has vanishing charge density on each site, so that it does not support any electric flux \( \rho(x)|g.s. > = 0 \), \( E_x|g.s. > = 0 \) (\( x = 1, \ldots, N \)). \( g.s. > \) is expressed as a linear combination of all the states with \( N/2 \) spins up and \( N/2 \) spins down. The eigenvalue of the Heisenberg Hamiltonian on this state in the thermodynamic limit is known \( 8 \), \( H_{J\parallel}|g.s. > = (-N \ln 2)|g.s. > \), and it provides the second order correction Eq. \( 4 \), \( E_g^{(2)} = -4N \ln 2 \), to the vacuum energy.

We now show that it is possible to identify the low lying excitations of the Schwinger model with those of the Heisenberg model and that the mass gaps of any other excitation can be expressed as functions of e.v.'s of powers of \( H_J \) and spin-spin correlation functions.

The continuum 2-flavor Schwinger model excitations are characterized by the quantum numbers of \( P \)- and \( G \)-parity. For the massive model, in the limit where the mass of the fermions is small compared to \( e^2 \) (strong coupling), Coleman showed \( 11 \) that the lightest state is \( I^{PG} = 1^+ \) and the next state up is \( I^{PG} = 0^{++} \). The Schwinger model in this limit is equivalent to a sine-Gordon in \( d = 1 + 1 \) and the low lying states can be interpreted as soliton-antisoliton states. There are also other states in the theory, way up in mass, e.g. a pseudoscalar isosinglet \( I^{PG} = 0^{--} \), \( G \)-odd with mass \( m = \sqrt{2/\pi} e \). All these excitations can be reobtained on the lattice. In the limit of vanishing fermion mass the first two correspond to massless Heisenberg states, the massive states are instead generated by fermion transport.

As we have shown, the Schwinger model vacuum, is the ground state of the antiferromagnet, \( g.s. > \), thus any excitation should be created from this state. There are two different types of excitations that can be created from \( g.s. > \). Those that involve only spin flipping and those that involve fermion transport besides spin flipping. The excitations of the first type have lower energy since no electric flux is created, those of the second type have a higher energy and the lowest energy ones occur when the fermion is transported a minimal distance, since the energy is proportional to the coupling times the length of the electric flux that is created. Only the first type of excitations can be described in terms of the Heisenberg model excited states. In \( 8 \) a complete classification of the Heisenberg model excitations has been given. It was shown that any excitation can be interpreted as the scattering of quasiparticles of spin 1/2, and that in physical states there is only an even number of these kink-spin waves, so that there are only states with integer spin. In the thermodynamic limit the two lowest excitations are a triplet and a singlet, they have a dispersion relation depending on 2 parameters, (the momenta of the two kinks) and for vanishing total momentum (relative to the ground state momentum \( P_{g.s.} = 0 \) for \( N/2 \) even, \( P_{g.s.} = \pi \) for \( N/2 \) odd) they are degenerate with the ground state. This interpretation is consistent with the one given by Coleman in terms of solitons in the Schwinger model. In fact these excitations also have the correct Schwinger model quantum numbers (the spin becomes the isospin) and energies, so that they can be identified with the lowest lying excitation for this model in strong coupling. There is a whole set of excitation of this type (in the notation of Ref. \( 8 \), all those belonging to the class \( M_{AF} \) all of them, for zero momentum in the thermodynamic limit, would be gapless. They are eigenstate of the total momentum. A G-parity transformation corresponds to a translation by one site, therefore at zero momentum they have a positive G-parity (with respect to the G-parity of the ground state) as expected for the low lying Schwinger model states. For finite systems these excitations are gapped so that the P-parity is a well defined quantum number. The P-parity and spin of the lowest lying excitations has been considered in \( 11 \) and the two lowest-lying states were shown to be \( s^P = 1^- \) and \( s^P = 0^+ \), i.e. with the parity of the lowest-lying states found by Coleman in the Schwinger model.

Let us discuss their energies. The excitation masses are given by the difference between the energy of the excitation and the energy of the ground state at zero momentum. The strong coupling perturbative expansion for the energies of any of the state belonging to \( M_{AF} \), which we denote by the generic symbol \( |ex > \), have the same expression as those of the ground state. Consequently if the states \( |ex > \) are taken at zero momentum, up to the second order in the strong coupling expansion, they have the same energy of the ground state \( |g.s. > \), \( E_{g.s.}^{(2)} = -4N \ln 2 \). To this order the mass gap is zero. At higher orders a mass gap might arise, due to the fact that correlators of spin products \( S_x \cdot S_{x+d} \) with \( d > 1 \) arise and these might be different on the ground state and on the excited states. The values of these correlators is not known, only on the ground state they have been considered in \( 11 \) and their asymptotic behavior has been computed in \( 9 \). The lowest lying among the state \( |ex > \), have the correct quantum numbers, are gapless, at least up to second order in the strong coupling expansion, and can then be identified with the lowest lying excitations of the Schwinger model.

Let us now consider the states obtained by fermion transport on one site on the Heisenberg model ground state. Using the spatial component of the Schwinger model currents two states can be created, a pseudoscalar isosinglet \( I^{PG} = 0^{--} \), \( G \)-odd, and a pseudoscalar isotriplet \( I^{PG} = 1^+ \), \( G \)-even (the quantum numbers are relative to those of the ground state \( I_{g.s.}^{PG} = 0^{++} \) for \( N/2 \) even \( I_{g.s.}^{PG} = 0^{--} \) for \( N/2 \) odd). The lattice operators with the correct quantum numbers that create these states at zero momentum, acting on \( |g.s. > \), and the corresponding states, read

\[
S = R + L \quad , \quad |S > = |0^{--} > = S|g.s. > \quad (11)
\]

\[
T_+ = (T_-)^\dagger = R^{(12)} + L^{(12)} \quad (12)
\]
Due to the mapping on the Heisenberg model the norm of these states can be easily computed

\[
\langle S | S \rangle = \langle g.s. | S^1 S | g.s. \rangle = -4 \langle g.s. | H_J | g.s. \rangle
\]

\[
T_+ | T_+ > = \frac{2}{3} (N + \langle g.s. | H_J | g.s. \rangle)
\]

\[
T_0 | T_0 > = \langle T_- | T_+ > = \langle T_+ | T_+ > , \langle g.s. > \text{ is normalized to the unity.}
\]

The isosinglet energy up to the second order is

\[
E_{S}^{(0)} = \frac{\langle S | H_0 | S \rangle}{\langle S | S \rangle} = 1 \quad (16)
\]

\[
E_{S}^{(2)} = \frac{\langle S | H_0 | S \rangle}{\langle S | S \rangle} = 1 \quad (17)
\]

with \( \Lambda_S = \frac{N \Pi_S}{E^{(0)}_{S} - H_0} \) and \( 1 - \Pi_S \) a projection operator onto \( |S> \). Using the commutators \( \left[H_0, (\Pi_S H_0)^n S \right] = (n + 1)(\Pi_S H_0)^n S = (n = 0, 1, \ldots) \), that hold when acting on \( |g.s.> \), Eq.\((17)\) can be written in terms of spin correlators as

\[
E_{S}^{(2)} = \frac{N \sum_{x=1}^{N} \langle g.s. | \vec{S}_x \cdot \vec{S}_{x+2} - \frac{1}{4} \langle g.s. > }{\langle g.s. | H_J | g.s. \rangle}
\]

At the zeroth perturbative order the pseudoscalar triplet is completely degenerate with the isosinglet \( E_{T}^{(0)} = E_{S}^{(0)} = 1 \). The second order energy of the states \((14)\) and \((13)\) can also be computed with an analogous procedure

\[
E_{T}^{(2)} = E_{T}^{(0)} - \Delta_{DS}(T) -
\]

\[
4 \langle g.s. | H_J | g.s. \rangle + 5 \sum_{x=1}^{N} \langle g.s. | \vec{S}_x \cdot \vec{S}_{x+2} - \frac{1}{4} \langle g.s. > \]

\[
\sum_{x=1}^{N} \langle g.s. | H_J | g.s. \rangle
\]

where the introducing the operator \( \vec{V} = \sum_{x=1}^{N} \vec{S}_x \wedge \vec{S}_{x+1} \)

\[
\Delta_{DS}(T_\pm) = 12 \langle g.s. | (V_1)^2 | g.s. \rangle \]

\[
\Delta_{DS}(T_0) = 12 \langle g.s. | (V_2)^2 | g.s. \rangle \]

\[
\Delta_{DS}(T_\pm) = \Delta_{DS}(T_0), \text{ so that the triplet states (as in the continuum) have a degenerate mass gap. We verified this by direct computation on finite size systems. We used also finite size systems computations to demonstrate that } \Delta_{DS} \text{ is of zeroth order in } N \text{. The excitation masses are given by } m_T = E_T - E_{g.s.} \text{, and } m_T = E_T - E_{g.s.} \text{. Consequently, the } (N\text{-dependent}) \text{ ground state energy terms appearing in } E_T^{(2)} \text{ and } E_T^{(2)} \text{ cancel and what is left are only } N \text{ independent terms. This is a good check of our computation, being the mass an intensive quantity.}
\]

The novel idea put forward in this letter is that in the strong coupling, all the low-lying excitations of the Schwinger model can be identified with those of the Heisenberg model and the mass spectrum of the 2-flavor Schwinger model is computable in terms of spin-spin correlators of the Heisenberg model. This shows that in strong coupling there is an exact mapping between the gauge theory and the quantum antiferromagnetic Heisenberg chain. Such a mapping was conjectured in \( [6,4] \). As evidenced in \( [10,14] \), the explicit evaluation of the pertinent spin-spin correlators is far from being trivial. It is an interesting problem in its own right to derive expressions for spin correlators usable in the evaluation of the mass spectrum of the 2-flavor Schwinger model. We may devote our attention to this problem in a future publication. This work is supported in part by the Natural Sciences and Engineering Research Council of Canada, the Istituto Nazionale di Fisica Nucleare and M.U.R.S.T.

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