SHADOW OF NONCOMMUTATIVITY

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Abstract

We analyse the structure of the $\kappa = 0$ limit of a family of algebras $A_\kappa$ describing noncommutative versions of space-time, with $\kappa$ a parameter of noncommutativity. Assuming the Poincaré covariance of the $\kappa = 0$ limit, we show that, besides the algebra of functions on Minkowski space, $A_0$ must contain a nontrivial extra factor $A^I_0$ which is Lorentz covariant and which does not commute with the functions whenever it is not commutative. We give a general description of the possibilities and analyse some representative examples.

1 Introduction

In this paper we wish to give a heuristic analysis of the structure of a noncommutative space-time, as might arise from the interplay of quantum theory with gravity, in the limit where the space-time is the usual commutative Minkowski space, that is in the limit when the Planck length tends to zero (when gravity is switched off). We shall show that in this limit, one necessarily captures a non-trivial extra factor in the limit algebra besides the algebra $C(M)$ of functions on Minkowski space $M$. In view of what is known of particle interactions, it is natural to expect that this extra factor has something to do with gauge theory. We have been aware of this conclusion for some time; it was indeed one of our main motivations in studying gauge theory over the noncommutative algebra of matrix-valued functions [9]; the model for the extra factor being then described by the matrix algebra. We shall see however that a matrix algebra is too small to be the extra factor and that furthermore a noncommutative extra factor cannot commute with the algebra $C(M)$.

We recall briefly the arguments which suggest that the interplay between quantum theory and gravitation leads to a noncommutative space-time. There is first an old semi-classical argument which is recalled in [8], showing that localization loses its meaning at distances of the order of the Planck length $\lambda_p$. The argument is that, because of quantum theory, in order to localize an event to within $\Delta x^\mu \sim a$, one needs to transfer an amount of energy of order $1/a$ and that then, in view of general relativity, if $a$ is too small, say $a < \lambda_p$, the
energy would create a “black hole”. In fact this semi-classical argument can be made more precise [8] and leads to limitations of the form $\Delta x^0 \sum \Delta x^k \gtrsim \lambda_p^2$ and $\sum_{k < \ell} \Delta x^k \Delta x^\ell \gtrsim \lambda_p^2$ in order to avoid the above phenomena. This suggests that the $x^\mu$ do not commute, that the space-time is noncommutative. It is worth noticing here that this is not the only conclusion. For instance, one can argue that since in this argument the $x^\mu$ have a length scale, then the metric must enter somewhere and that the above uncertainty relations can be consequences of the quantization of the metric instead of a “quantization” of space-time itself.

There is however a second argument of a different nature. In classical general relativity, one solves locally Einstein’s equations and one extends maximally the solutions. Doing this one obtains possibly a space-time carrying a non-trivial topology. This extension is not just a mathematical artifact but is physically relevant. Consider, for instance, the analogue of the classical self-energies of point charges. More generally, this has to be taken into account in order to use the old physical idea [6] that gravitation acts as an ultraviolet “regularization” because of the fact that it is attractive and has also its own energy-momentum density source. A problem arises when one quantizes the gravitational field. The topology of space-time must then be sensitive to the states of the field. This suggests again that the functions on space-time should be replaced by elements of a noncommutative $\ast$-algebra acting on the same states as the quantized gravitation field.

In short there are several arguments suggesting that the space-time becomes noncommutative on the scale of Planck length. Although this is certainly not the only possibility, it is worth studying the consequences which follow from such an assumption when the gravitational interaction is switched off. It is the aim of next section to analyze the “shadow” of such a noncommutativity in the limit when one recovers the Poincaré-covariant physical theory in the usual Minkowski space. This analysis is sharpened in Section 3 where it is pointed out that the limit has a structure of crossed product and the Hochschild 2-cocycle associated with the deformation is identified as a group cocycle of the dual of the group.
of space-time translations. Then, in Section 4, we shall discuss in this context various recent proposals and, in particular, the generalizations of gauge theory using noncommutative differential calculi [7], [8], [9].

2 The commutative space-time limit

Let $\mathcal{A}_0$ be a one-parameter family of associative algebras. Here we think of $\kappa$ as being the gravitational constant $\lambda_0^2$ and we think of $\mathcal{A}_\kappa$ as being a noncommutative version of the functions on space-time. Technically, we use the framework of formal deformation theory of associative algebras [10], [1]. This means that we assume that, as vector spaces, all the $\mathcal{A}_\kappa$ coincide with a fixed vector space $E$ and that, for $f, g \in E$, one can expand their product $(fg)\kappa$ in $\mathcal{A}_\kappa$ as

$$(fg)\kappa = fg + \kappa c(f, g) + o(\kappa^2) \quad (1)$$

where $fg = (fg)_0$ is the product in $\mathcal{A}_0$. We also assume that there is a distinguished element, $1 \in E$, which is the unit for each algebra $\mathcal{A}_\kappa$. The bilinear map $(f, g) \mapsto c(f, g)$ is then a normalized Hochschild 2-cocycle of $\mathcal{A}_0$ with values in $\mathcal{A}_0$ [10]. In terms of commutators, (1) yields

$$[f, g]_\kappa = [f, g] - i\kappa \{f, g\} + o(\kappa^2) \quad (2)$$

where the bracket $(f, g) \mapsto \{f, g\}$ is defined by

$$\{f, g\} = i(c(f, g) - c(g, f)) \quad (3)$$

and where $[f, g]_\kappa$ and $[f, g]$ denote the commutator in $\mathcal{A}_\kappa$ and in $\mathcal{A}_0$ respectively. The first order in $\kappa$ of the identity $[h, (fg)\kappa]_\kappa = ([h, f]_\kappa g)_\kappa + (f[h, g]_\kappa)_\kappa$ yields the condition

$$i([h, c(f, g)] - c([h, f], g) - c(f, [h, g])) = f\{h, g\} - \{h, fg\} + \{h, f\}g \quad (4)$$

This implies that, if $h$ is an element of the center $Z(\mathcal{A}_0)$ of $\mathcal{A}_0$, then the endomorphism $\delta_h$ of $\mathcal{A}_0$ defined by $\delta_h(f) = \{h, f\}$ is a derivation of $\mathcal{A}_0$. The center $Z(\mathcal{A}_0)$ of $\mathcal{A}_0$ is stable under the derivations of $\mathcal{A}_0$ and therefore $Z(\mathcal{A}_0)$
is stable under the bracket (3). On the other hand, if $f$, $g$, and $h$ are elements of $Z(A_0)$, then the lowest non-trivial order in $\kappa$ (i.e. the second order) of $[[[f, g]_\kappa, h]_\kappa = [[[f, h]_\kappa, g]_\kappa + [f, [g, h]_\kappa]_\kappa$ yields the condition
\[
\{\{f, g\}, h\} = \{\{f, h\}, g\} + \{f, \{g, h\}\} \tag{5}
\]
So one can summarize the above discussion by the following proposition:

**PROPOSITION 1** The center $Z(A_0)$ of $A_0$ is a (commutative) Poisson algebra with Poisson bracket given by (3) and one defines a linear mapping $z \mapsto \delta_z$ of $Z(A_0)$ into the Lie algebra $\text{Der}(A_0)$ of all derivations of $A_0$ by setting $\delta_z(f) = \{z, f\}$ for $z \in Z(A_0)$ and $f \in A_0$ which satisfies $\delta_{z+z'} = z\delta_{z'} + z'\delta_z$ for $z, z' \in Z(A_0)$.

This result is known, it is the first part of Proposition 1.2 of [12] (see also [5]).

We wish to represent the noncommutative analogue of real functions by hermitian elements. This leads us to add the following reality condition to the above general structure. We assume that the $A_\kappa$ are complex $*$-algebras such that all the underlying complex involutive vector spaces coincide. Thus the vector space $E$ must be a complex vector space equipped with an antilinear involution, $f \mapsto f^*$, and a distinguished hermitian element $1_l = 1_l^*$. The parameter $\kappa$ is real (consistently) and one has $(fg)^*_\kappa = (g^*f^*)_\kappa$ and $(1_l)^*_\kappa = (1_l)^*_\kappa = 1_l$. It follows that the normalized cocycle $c$ satisfies $c(f, g)^* = c(g^*, f^*)$, which implies that the bracket (3) is real, i.e. $\{f, g\}^* = \{f^*, g^*\}$. Therefore, the set $Z_\mathbb{R}(A_0)$ of all hermitian elements of the center $Z(A_0)$ of $A_0$ is a real (commutative) Poisson algebra and the map $z \mapsto \delta_z$ induces a real linear mapping of $Z_\mathbb{R}(A_0)$ into the real Lie algebra $\text{Der}_\mathbb{R}(A_0)$ of all hermitian derivations of $A_0$.

We now return to our specific problem. The $A_\kappa$ are noncommutative versions of the algebra of functions on space-time and we wish to recover Poincaré-invariant physics on Minkowski space $M$ in the limit $\kappa = 0$. We thus assume that the Poincaré group $\mathfrak{P}$ acts by $*$-automorphisms $(\Lambda, a) \to \alpha_{(\Lambda, a)}$ on $A_0$, that $A_0$ contains as a $*$-subalgebra the (commutative) algebra $C(M)$ of (smooth) functions on Minkowski space and that the action of $\mathfrak{P}$ on $A_0$ induces its usual action
on $C(M)$ associated with the corresponding transformations of $M$. We now argue that $C(M)$ cannot be the whole $A_0$. In fact, assume that $C(M)$ is equal to $A_0$. Then, in view of Proposition 1, there is a Poisson bracket on $M$. This Poisson bracket is non trivial since we have assumed that the $A_\kappa$ are noncommutative. On the other hand there does not exist a non trivial Poincaré invariant Poisson bracket on $M$. It seems unreasonable to us that the Poincaré invariance be broken at the first order in $\kappa$. In fact, at this order, we expect a Poincaré invariant theory involving a spin-2 field linearly coupled to the other fields. Once this hypothesis is accepted, it follows that the inclusion $C(M) \subset A_0$ must be a strict one; the $\kappa = 0$ limit $A_0$ of the $A_\kappa$ must contain an extra factor. It follows also that the normalized 2-cocycle $c$ of $A_0$ defined by (1) is Poincaré invariant, so one has the condition

$$\alpha_{(\Lambda, a)}(c(f, g)) = c(\alpha_{(\Lambda, a)}(f), \alpha_{(\Lambda, a)}(g)), \forall(\Lambda, a) \in \mathfrak{p}, \forall f, g \in A_0$$

(6)

which implies the invariance of the bracket (3).

Let $x^\mu \in C(M)$ be minkowskian coordinates. Then the algebra $C(M)$ is generated by the $x^\nu$ and the action of the Poincaré group on it is given by

$$\alpha_{(\Lambda, a)} x^\mu = \Lambda^{-1 \mu}_{\nu}(x^\nu - a^\nu \mathbb{1})$$

(7)

By choosing an origin, one can identify $C(M)$ with the Hopf algebra of functions on the group of translations. Since $C(M)$ is a subalgebra of $A_0$, the latter is (in particular) a bimodule over $C(M)$. Furthermore, by restricting attention to the action of translations, $A_0$ is in fact a bicovariant bimodule over the algebra of functions on the group of translations [13], [2]. By standard arguments [13], [2], $A_0$ is isomorphic as left $C(M)$-module to $C(M) \otimes A_0^I$ where $A_0^I$ denotes the subalgebra of translationally invariant elements of $A_0$: $A_0^I = \{ f \in A_0 | \alpha_{(1, a)} f = f, \forall a \}$. In fact $A_0$ is isomorphic to $C(M) \otimes A_0^I$ as $(C(M), A_0^I)$-bimodule. Thus in order to recover the complete structure of algebra of $A_0$ in the representation $C(M) \otimes A_0^I$, it is sufficient to describe the right multiplication by elements of
\( \mathcal{C}(M) \) of elements of \( \mathcal{A}_0^I \). For each minkowskian coordinate \( x^\mu \), it follows from (7) that \( \mathcal{A}_0^I \) is stable under the derivation \( f \mapsto \text{ad}(x^\mu)(f) = [x^\mu, f] \) and therefore one has \( fx^\mu = x^\mu f - \text{ad}(x^\mu)(f), \forall f \in \mathcal{A}_0^I \) which can be written in the representation \( \mathcal{A}_0 = \mathcal{C}(M) \otimes \mathcal{A}_0^I \)

\[
fx^\mu = x^\mu \otimes f - 1 \otimes \text{ad}(x^\mu)(f), \forall f \in \mathcal{A}_0^I
\]  
(8)

From this one deduces the right multiplication by elements of \( \mathcal{C}(M) \) i.e. the right \( \mathcal{C}(M) \)-module structure of \( \mathcal{C}(M) \otimes \mathcal{A}_0^I \), which is in fact isomorphic to \( \mathcal{A}_0^I \otimes \mathcal{C}(M) \) as right \( \mathcal{C}(M) \)-module. In the following, we shall denote by \( X^\mu \) the four commuting derivations of \( \mathcal{A}_0^I \) induced by the \( \text{ad}(x^\mu) \). The algebra \( \mathcal{A}_0^I \) is invariant under the automorphisms \( \alpha_{(\Lambda,0)} \). We shall denote by \( \alpha^I : \Lambda \mapsto \alpha^I_\Lambda \), the corresponding homomorphism of the Lorentz group into the group \( \text{Aut}(\mathcal{A}_0^I) \) of all \(*\)-automorphisms of \( \mathcal{A}_0^I \).

One can thus summarize the above discussion by the following presentation of \( \mathcal{A}_0 \). One starts with a unital \(*\)-algebra \( \mathcal{A}_0^I \) equipped with four commuting antihermitian derivations \( X^\mu \) and an action \( \Lambda \mapsto \alpha^I_\Lambda \) of the Lorentz group by automorphisms of \( \mathcal{A}_0^I \) such that

\[
\alpha^I_\Lambda \circ X^\mu = \Lambda^{-1\nu}X^\nu \circ \alpha^I_\Lambda
\]  
(9)

and \( \mathcal{A}_0 \) is “generated” as unital \(*\)-algebra by \( \mathcal{A}_0^I \) and four hermitian elements \( x^\mu \) with relations \( x^\mu x^\nu = x^\nu x^\mu \) and \( x^\mu f = f x^\mu + X^\mu(f), f \in \mathcal{A}_0^I \). We put quotes on the word generated in the above sentence because we do not pay attention here to functional analysis aspects, for instance appropriate completions, etc. The Poincaré group acts on \( \mathcal{A}_0 \) by the action \( \alpha_{(\Lambda,a)} \) on \( \mathcal{C}(M) \) defined by (7) and by \( \alpha_{(\Lambda,a)}(f) = \alpha^I_\Lambda(f) \) for \( f \in \mathcal{A}_0^I \).

We have assumed that the bracket (3) is not identically zero on \( \mathcal{C}(M) \). This implies that the \( c^{\mu\nu} = c(x^\mu, x^\nu) \) do not all vanish. On the other hand, it follows from (3) that \( c^{\mu\nu} \) are elements of \( \mathcal{A}_0^I \) and (3) implies then that one has

\[
\alpha^I_\Lambda(c^{\mu\nu}) = \Lambda^{-1\mu}_a \Lambda^{-1\nu}_b c^{ab}
\]  
(10)
Thus the homomorphism $\alpha^I : \mathfrak{L} \to \text{Aut}(A^I_0)$ of the Lorentz group $\mathfrak{L}$ into the group of $\ast$-automorphisms of $A^I_0$ is never trivial. This places severe restrictions on the structure of the extra factor $A^I_0$. For instance, $A^I_0$ cannot be a finite-dimensional matrix algebra because on such an algebra all automorphisms are inner and, on the other hand, it is known that the Lorentz group has no nontrivial finite-dimensional unitary representation. The same argument shows that, if $A^I_0$ admits an injective (eventually unbounded) $\ast$-representation in a Hilbert space, then $A^I_0$ must be infinite-dimensional.

3 Crossed-product structure of $A_0$

There is another useful presentation of $A_0$ which is a sort of exponentiation of the previous one and which can be used for a functional-analytic development of the framework. Let $\mathfrak{T}^*$ be the space $\mathbb{R}^4$ considered as the dual of the group $\mathfrak{T}$ of translations of the Minkowski space $M$. Instead of taking the $x^\mu$ as generators of $\mathcal{C}(M)$ we now choose the exponentials $u(k) = \exp(i k_\mu x^\mu)$ for $k \in \mathfrak{T}^*$. One has

$$u(0) = 1, \quad u(k)u(\ell) = u(k + \ell), \quad u(k)^* = u(-k) = u(k)^{-1} \quad (11)$$

and $(\ref{7})$ now yields with $(\Lambda k)_\mu = k_\nu \Lambda^{-1\nu}_\mu$

$$\alpha(\Lambda, a)u(k) = u(\Lambda k) \exp(-i(\Lambda k)_\mu a^\mu) \quad (12)$$

If $f \in A^I_0$, then $\tau_k(f) = u(k)f u(-k)$ is also in $A^I_0$ and $k \mapsto \tau_k$ is a homomorphism of the additive group $\mathfrak{T}^*$ into the group of $\ast$-automorphisms of $A^I_0$. This homomorphism is in fact the exponential version of the derivations $X^\mu$, $(\tau_k = \exp(i k_\mu X^\mu))$, and $(\ref{8})$ is replaced by

$$\alpha^I_\Lambda \circ \tau_k = \tau_{\Lambda k} \circ \alpha^I_\Lambda \quad (13)$$

The $\ast$-algebra $A_0$ is generated by the $\ast$-algebra $A^I_0$ and the $u(k), k \in \mathfrak{T}^*$ with the relations $(\ref{14})$ and the relation

$$u(k)f = \tau_k(f)u(k), \quad \forall f \in A^I_0 \text{ and } \forall k \in \mathfrak{T}^* \quad (14)$$
The action of the Poincaré group on \( \mathcal{A}_0 \) is given by (12) and by

\[
\alpha_{(\Lambda,a)}(f) = \alpha^I_\Lambda(f), \quad \forall f \in \mathcal{A}_0^I
\]

The consistency is ensured by the relation (13). We can summarize the above discussion by the following proposition:

**PROPOSITION 2** The \(*\)-subalgebra \( \mathcal{A}_0^I \) of translationally invariant elements of \( \mathcal{A}_0 \) is equipped with an action \( \tau \) of the dual group \( \mathfrak{T}^* \) of the group \( \mathfrak{T} \) of translations by \(*\)-automorphisms and \( \mathcal{A}_0 \) is isomorphic to the crossed product of \( \mathcal{A}_0^I \) with \( \mathfrak{T}^* \) for \( \tau \). Furthermore \( \mathcal{A}_0^I \) is equipped with an action \( \alpha^I \) of the Lorentz group by \(*\)-automorphisms which is connected with \( \tau \) by (13) and the action of the Poincaré group on \( \mathcal{A}_0 \) is given by (12) and (15).

In this proposition, the crossed product is taken in the category of unital \(*\)-algebras. However one can accommodate various functional analytic generalizations by working in the appropriate categories of topological \(*\)-algebras. This partly depends what one has in mind for the algebra \( \mathcal{C}(M) \) of functions on space-time (the linear decomposition of the elements of \( \mathcal{C}(M) \) over the \( \mathfrak{u}(k) \) being the Fourier transformation).

Let us now consider the description of the cocycle \( c \) in this framework. The restriction of \( c \) to \( \mathcal{C}(M) \times \mathcal{C}(M) \) is an \( \mathcal{A}_0 \)-valued normalized Hochschild 2-cocycle on \( \mathcal{C}(M) \) which is described by the \( c(u(k), u(\ell)), k, \ell \in \mathfrak{T}^* \). The invariance and the reality conditions yield

\[
\alpha_{(\Lambda,a)}c(u(k), u(\ell)) = c(u(\Lambda k), u(\Lambda \ell)) \exp(-i(\Lambda(k + \ell))_\mu a^\mu) \quad (16)
\]

\[
c(u(k), u(\ell))^* = c(u(-\ell), u(-k)) \quad (17)
\]

Define \( \gamma(k, \ell) \in \mathcal{A}_0 \) for \( k, \ell \in \mathfrak{T}^* \) by

\[
c(u(k), u(\ell)) = \gamma(k, \ell)u(k + \ell) \quad (18)
\]

It follows from (16) (with \( \Lambda = I \)) that the \( \gamma(k, \ell) \) are translationally invariant, that is \( \gamma(k, \ell) \in \mathcal{A}_0^I, \forall k, \ell \in \mathfrak{T}^* \). The reality condition (17) becomes

\[
\gamma(k, \ell)^* = \tau_{k+\ell} \gamma(-\ell, -k) \quad (19)
\]
and (16) yields the action
\[ \alpha_A^I \gamma(k, \ell) = \gamma(\Lambda k, \Lambda \ell) \] (20)

On the other hand the cocycle relation on C(M)
\[ u(k)c(u(\ell), u(m)) - c(u(k)u(\ell), u(m)) \]
\[ + c(u(k), u(\ell)u(m)) - c(u(k), u(\ell))u(m) = 0 \]
is equivalent to the relation
\[ \tau_k \gamma(\ell, m) - \gamma(k + \ell, m) + \gamma(k, \ell + m) - \gamma(k, \ell) = 0 \] (21)
and, since \( \gamma(k, 0) = \gamma(0, k) = 0 \) follows from \( c(1, x) = c(x, 1) = 0 \), one has the following result:

**LEMMA 1** The mapping \( \gamma : \mathfrak{T}^* \times \mathfrak{T}^* \rightarrow \mathcal{A}_0^I \) defined by (18) is an \( \mathcal{A}_0^I \)-valued normalized 2-cocycle of the group \( \mathfrak{T}^* \) for \( \tau \) which is real, in the sense of (19), and Lorentz-invariant, in the sense of (20).

Thus starting from \( (\mathcal{A}_0^I, \tau, \alpha^I) \), one reconstructs \( \mathcal{A}_0 \) with its Poincaré automorphism by using Proposition 2 and one reconstructs the restriction to \( \mathcal{C}(M) \times \mathcal{C}(M) \) of the Hochschild cocycle \( c \) from the \( \mathcal{A}_0^I \)-valued group cocycle \( \gamma \) as in Lemma 1 by formula (18). The missing items are the values of \( c \) on \( \mathcal{A}_0^I \times \mathcal{C}(M) \) and on \( \mathcal{A}_0^I \times \mathcal{A}_0^I \). Indeed let \( f \) and \( g \) be arbitrary elements of \( \mathcal{A}_0^I \) and \( k, \ell \) be in \( \mathfrak{T}^* \). Then the cocycle relation implies
\[ c(f u(k), u(\ell) g) = c(f, \tau_{k+\ell}(g)) u(k + \ell) \]
\[ + c(f, \tau_{k+\ell}(g), u(k + \ell)) - f u(k) c(u(\ell), g) - c(f, u(k)) u(\ell) g \]
\[ + f c(u(k + \ell), g) - f c(\tau_{k+\ell}(g), u(k + \ell)) \] (22)
This relation expresses the cocycle \( c \) on \( \mathcal{A}_0 \times \mathcal{A}_0 \) in terms of its restrictions to \( \mathcal{C}(M) \times \mathcal{C}(M) \), to \( \mathcal{A}_0^I \times \mathcal{A}_0 \), to \( \mathcal{A}_0^I \times \mathcal{C}(M) \) and to \( \mathcal{C}(M) \times \mathcal{A}_0^I \). By taking into account the reality condition (17), the restriction of \( c \) to \( \mathcal{C}(M) \times \mathcal{A}_0^I \) can be expressed in terms of its restriction to \( \mathcal{A}_0^I \times \mathcal{C}(M) \). Thus \( c \) is known whenever its restrictions to \( \mathcal{C}(M) \times \mathcal{C}(M) \), to \( \mathcal{A}_0^I \times \mathcal{A}_0^I \) and to \( \mathcal{A}_0^I \times \mathcal{C}(M) \) are known. It follows from (22) that the restriction \( c' \) of \( c \) to \( \mathcal{A}_0^I \times \mathcal{A}_0^I \) is \( \mathcal{A}_0^I \)-valued and therefore, \( c' \) is a normalized \( (\mathcal{A}_0^I \)-valued) 2-cocycle of \( \mathcal{A}_0^I \) which satisfies the invariance condition
\[ \alpha_A^{f'} c'(f, g) = c'(\alpha_A^f, \alpha_A^g), \] (23)
and the reality condition
\[ c^I(f, g)^* = c^I(g^*, f^*), \]  
(24)

for any \( f, g \in A^I_0 \) and \( \Lambda \in \mathcal{L} \).

We define \( \lambda(f, k) \) for \( f \in A^I_0 \) and \( k \in \mathfrak{T}^* \) by
\[ c(f, u(k)) - c(u(k), \tau_{-k}(f)) = \lambda(f, k)u(k) \]  
(25)

It follows from (12) that \( \lambda(f, k) \) belongs to \( A^I_0 \) and therefore, \( \lambda \) is a map of \( A^I_0 \times \mathfrak{T}^* \) into \( A^I_0 \), linear in the first factor, which satisfies the Lorentz-invariance condition
\[ \alpha^I_{\Lambda}\lambda(f, k) = \lambda(\alpha^I_{\Lambda}f, \Lambda k) \]  
(26)

the normalization conditions
\[ \lambda(f, 0) = 0, \ \lambda(1, k) = 0 \]  
(27)

and the reality condition
\[ \lambda(f, k)^* = -\tau_k(\lambda(\tau_{-k}(f^*), -k)) \]  
(28)

The cocycle relation for \( c \) implies that one has
\[ f^\gamma(k, \ell) - \gamma(k, \ell)f = \tau_k(\lambda(\tau_{-k}(f), \ell)) - \lambda(f, k + \ell) + \lambda(f, k) \]  
(29)

and
\[ c^I(f, g) - \tau_k(c^I(\tau_{-k}(f), \tau_{-k}(g))) = f\lambda(g, k) - \lambda(fg, k) + \lambda(f, k)g \]  
(30)

for any \( k, \ell \in \mathfrak{T}^* \) and \( f, g \in A^I_0 \). Conversely if \( c^I, \gamma, \lambda \) as above are such that \( c^I \) is a Hochschild 2-cocycle on \( A^I_0 \) and such that (21), (29) and (30) are satisfied then \( c \) is a 2-cocycle on \( A_0 \), i.e. the cocycle relation for \( c \) is equivalent to the cocycle relation for \( c^I \) and the relations (21), (29) and (30).

We now describe another way to present these relations. We first extend the action \( \tau \) of \( \mathfrak{T}^* \) on \( A^I_0 \) into an action on the \( A^I_0 \)-valued Hochschild cochains on \( A^I_0 \) by setting \( \tau_k(\omega)(f_1, \ldots, f_n) = \tau_k(\omega(\tau_{-k}(f_1), \ldots, \tau_{-k}(f_n))) \) for \( k \in \mathfrak{T}^* \), \( f_i \in A^I_0 \),
where \( \omega \in C^n(\mathcal{A}_0^t, \mathcal{A}_0^t) \) is an \( \mathcal{A}_0^t \)-valued Hochschild \( n \)-cochain on \( \mathcal{A}_0^t \). Let \( C^{r,s} \) denote the space of \( C^r(\mathcal{A}_0^t, \mathcal{A}_0^t) \)-valued \( s \)-cochains on the group \( \mathfrak{T}^* \). One has two differentials on the direct sum \( C = \oplus C^{r,s} \). The first one is the composition with the Hochschild differential of \( C(\mathcal{A}_0^t, \mathcal{A}_0^t) \) which will be denoted by \( \delta^{(1,0)} \). The second one is the group differential of \( \mathfrak{T}^* \) for its action \( \tau \) on \( C(\mathcal{A}_0^t, \mathcal{A}_0^t) \) which will be denoted by \( (-1)^r \delta^{(0,1)} \). One has \( \delta^{(1,0)}(C^{r,s}) \subset C^{r+1,s} \), \( \delta^{(0,1)}(C^{r,s}) \subset C^{r,s+1} \) and \( \delta^{(1,0)} \delta^{(0,1)} = \delta^{(1,0)} \delta^{(0,1)} = 0 \). Therefore \( \delta = \delta^{(1,0)} + \delta^{(0,1)} \) is again a differential of degree one for the total degree \( r + s \). One can restrict all differentials to the space of normalized cochains (i.e. normalized group cochains with values in normalized Hochschild cochains). One has \( c^l \in C^{2,0} \), \( \lambda \in C^{1,1} \) and \( \gamma \in C^{0,2} \). Furthermore, the cocycle condition for \( c^l \) and the relations (21), (29), (30) reduce to \( (\delta^{(1,0)} + \delta^{(0,1)})(c^l + \lambda + \gamma) = 0 \). Indeed the (3,0) part of this relation \( \delta^{(1,0)} c^l = 0 \) is the cocycle relation for \( c^l \), the (2,1) part \( \delta^{(0,1)} c^l + \delta^{(1,0)} \lambda = 0 \) is relation (30), the (1,2) part \( \delta^{(0,1)} \lambda + \delta^{(1,0)} \gamma = 0 \) is relation (29) and the (0,3) part \( \delta^{(0,1)} \gamma = 0 \) is the group cocycle relation (21).

**PROPOSITION 3** The cocycle condition for \( c \) is equivalent to \( (\delta^{(1,0)} + \delta^{(0,1)})(c^l + \lambda + \gamma) = 0 \), i.e. to the cocycle condition for \( c^l + \lambda + \gamma \) in \( C \). Moreover the addition of the Hochschild coboundary of a translationally invariant 1-cochain to \( c \) is equivalent to the addition to \( c^l + \lambda + \gamma \) of a term \( (\delta^{(1,0)} + \delta^{(0,1)})(\beta^{(1,0)} + \beta^{(0,1)}) \) with \( \beta^{(1,0)} \in C^{(1,0)} \) and \( \beta^{(0,1)} \in C^{(0,1)} \).

Indeed, if one adds to \( c \) the coboundary of \( b \) with \( b \) such that \( \alpha_{(1,a)}(b(f)) = b(\alpha_{(1,a)}(f)) \) for \( a \in \mathfrak{T} \) and \( f \in \mathcal{A}_0 \), then \( \beta^{(1,0)} \) is the restriction of \( b \) to \( \mathcal{A}_0^t \) and \( \beta^{(0,1)} \) is given by \( \beta^{(0,1)}(k) = b(u(k))u(-k) \) for \( k \in \mathfrak{T}^* \); the invariance of \( b \) implies that \( \beta^{(1,0)} \) and \( \beta^{(0,1)} \) are \( \mathcal{A}_0^t \)-valued.

The above result is a particular case of results of [11] on the Hochschild cohomology of crossed products. Here the simplification comes from the fact that we are only interested on cochains of \( \mathcal{A}_0 \) which are invariant by translations (and in fact by Poincaré transformations).
It is worth noticing here that the data $c^I, \lambda$ and $\gamma$ do not determine $c$ completely. However, by using the formula (22) one can show that $c$ is determined by $c^I, \lambda$ and $\gamma$ to within the coboundary of a translationally invariant 1-cochain $b$ of $A_0$ which is such that the corresponding $\beta^{(1,0)}$ and $\beta^{(0,1)}$ as above vanish identically.

For $\kappa \neq 0$, the algebra $A_\kappa$ is generated by the noncommutative version of the functions on space-time. This implies that $A_0$ is generated by the algebra $C(M)$ of functions on space-time and the iterated applications of the cocycle $c$ on $C(M)$.

More precisely, we define an increasing filtration $F^n$ of $A_0$ by unital *-subalgebras $F^n(A_0)$ by setting $F^0(A_0) = C(M)$ and $F^{n+1}(A_0) = \{ \text{the subalgebra of } A_0 \text{ generated by } F^n(A_0) \text{ and by } c(F^r(A_0), F^s(A_0)) \}$ for $r + s = n (\in \mathbb{N})$. Then, our assumption means that one has $A_0 = \cup_n F^n(A_0) = F^\infty(A_0)$. Correspondingly one has an increasing filtration of $A^I_0$ by unital *-subalgebras $F^n(A^I_0) = F^n(A_0) \cap A^I_0$ which is such that $F^0(A^I_0) = \mathbb{C} \mathbb{I}$, $A^I_0 = \cup F^n(A^I_0) = F^\infty(A^I_0)$ and which is Lorentz-invariant, i.e. $\alpha^I_\Lambda F^n(A^I_0) \subset F^n(A^I_0)$ for $\Lambda \in \mathfrak{L}$.

4 Discussion

The simplest cases correspond to the situation where $C(M)$ is in the center of $A_0$, i.e. where $\tau$ is trivial. In this case, one can assume that, up to a coboundary, $c$ is antisymmetric on $C(M)$ and in fact on $Z(A_0)$, (this corresponds to a weak regularity assumption on the commutative algebra $Z(A_0)$). It then follows from the discussion in the end of last Section that one has $A_0 = Z(A_0)$ i.e. that $A_0$ is a commutative Poisson algebra and that the $A_\kappa$ are obtained by “quantization” of $A_0$. In this case $A_0$ is, as algebra, the tensor product $C(M) \otimes A^I_0$. Since the $\{x^\mu, x^\nu\}$ are elements of $A^I_0$ and since the Lorentz group acts there by automorphisms, $A^I_0$ must contain as subalgebra an algebra of functions on a union of Lorentz orbits of antisymmetric 2-tensors. The $\{x^\mu, x^\nu\}$ are then coordinates functions on this manifold of antisymmetric tensors and the orbits occurring are labeled by the pairs $(\alpha, \beta)$ of real numbers such that $\alpha$ is in the spectrum of $g_{\mu\nu}g_{\rho\sigma}\{x^\mu, x^\nu\}\{x^\rho, x^\sigma\}$ and $\beta$ is in the spectrum of $\varepsilon_{\alpha\beta\gamma\delta}\{x^\alpha, x^\beta\}\{x^\gamma, x^\delta\}$. It is
natural to require time reversal and parity be defined and therefore to assume that whenever one has the orbit \((\alpha, \beta)\), one also has the orbit \((\alpha, -\beta)\). When furthermore one has \(\{x^\lambda, \{x^\mu, x^\nu\}\} = 0 \) \((\forall \lambda, \mu, \nu)\), then \(\mathcal{A}_0^I\) is just such an algebra of functions on a union of orbits of antisymmetric tensors. This is precisely the case for the algebra \(\mathcal{A}_0\) which is the \(\kappa = 0\) limit of the model of Doplicher, Fredenhagen and Robert \([8]\) where the orbits there are (0,1) and (0,-1). It is not very difficult to construct examples with \(\{x^\lambda, \{x^\mu, x^\nu\}\} \neq 0\).

As pointed out above, in order to have a noncommutative algebra \(\mathcal{A}_0\), it is necessary (and obviously sufficient) that the algebra \(\mathcal{C}(M)\) of functions on space-time be not included in the center of \(\mathcal{A}_0\), which means that \(\tau\) is non trivial. Since by assumption the cocycle \(\gamma\) defined by \([18]\) is non trivial, the simplest cases with \(\tau\) non trivial are the cases where \(\lambda\) and \(c^I\) vanish. In such a case, it follows from \([24]\) that the image of \(\gamma\) is in the center of \(\mathcal{A}_0^I\) and that \(F^{n+1}(\mathcal{A}_0) = F^1(\mathcal{A}_0), \forall n \in \mathbb{N}\). Therefore, in view of the discussion in the end of last section, \(\mathcal{A}_0 = F^1(\mathcal{A}_0)\) and \(\mathcal{A}_0^I\) is the commutative algebra generated by the \(\gamma(k, \ell)\), (or, equivalently, by the \(\{x^\mu, x^\nu\}\)). Thus in such a case \(\mathcal{A}_0\) is the crossed product of the commutative algebra \(\mathcal{A}_0^I\) with \(\mathfrak{S}^*\) for \(\tau\). An example of this situation is provided by the \(\kappa = 0\) limit of an example elaborated by Doplicher and Fredenhagen described in Section 2 of \([7]\). It is worth noticing here that in the \(\kappa = 0\) limits of examples of \([8]\) and \([4]\) the orbits of antisymmetric 2-tensors occuring are (0,1) and (0,-1), (recall that in these examples \(\mathcal{A}_0^I\) is commutative). This is connected with the fact that these authors construct \(\mathcal{A}_\kappa\) in such a way that physically-motivated spacetime uncertainty relations are implemented.

Generically \(\tau\) is non-trivial and \(\lambda\) and \(c^I\) do not vanish. A simple example of this kind can be easily found. For \(\kappa \in \mathbb{R}\), let \(\mathcal{A}_\kappa\) be the unital \(*\)-algebra generated by hermitian elements \(x^\mu, L^\mu, I^{\mu\nu}\) \((\mu, \nu = 0, 1, 2, 3)\) satisfying the relations

\[
\begin{align*}
[x^\mu, x^\nu] &= i\kappa I^{\mu\nu} \\
[x^\lambda, I^{\mu\nu}] &= i(g^{\lambda\nu} L^\mu - g^{\lambda\mu} L^\nu) \\
[x^\mu, L^\nu] &= i\kappa I^{\mu\nu}
\end{align*}
\]

(31)
\[ [I^{\rho}, I^{\mu\nu}] = i(g^{\lambda\nu}I^{\rho\mu} - g^{\rho\nu}I^{\lambda\mu} - g^{\lambda\mu}I^{\rho\nu}) \]
\[ [L^{\lambda}, I^{\mu\nu}] = i(g^{\lambda\nu}L^{\mu} - g^{\rho\mu}L^{\lambda\nu}) \]
\[ [L^{\mu}, L^{\nu}] = i\kappa I^{\mu\nu} \]  

where \( g^{\mu\nu} \) denotes the minkowskian metric with \( g^{00} = -1, g^{11} = g^{22} = g^{33} = 1 \).

It follows from these relations that for \( \kappa \neq 0 \), \( \mathcal{A}_\kappa \) is generated by the \( x^\mu \). This implies that \( \mathcal{A}_0 \) satisfies the property \( \mathcal{A}_0 = \bigcup_n F^n(\mathcal{A}_0) \) of the end of last section. In fact here one has again \( F^{n+1}(\mathcal{A}_0) = F^1(\mathcal{A}_0), \forall n \in \mathbb{N} \).

Furthermore there is (for any \( \kappa \)) an action of the Poincaré group \( \mathfrak{P} \) by \(*\)-automorphisms \((\Lambda, a) \mapsto \alpha(\Lambda, a)\) on \( \mathcal{A}_\kappa \) given by

\[ \alpha(\Lambda, a)x^\mu = \Lambda^{-1\mu}(x^\nu - a^\nu \mathbb{1}), \quad \alpha(\Lambda, a)L^\mu = \Lambda^{-1\mu}L^\nu, \quad \alpha(\Lambda, a)I^{\mu\nu} = \Lambda^{-1\mu}_\alpha \Lambda^{-1\nu}_\beta I^{\alpha\beta} \]

The commutation relations (32) between the \( I^{\mu\nu} \) and the \( L^\lambda \) are the relations of the Lie algebra of \( SO(4,1) \) if \( \kappa > 0 \), of \( SO(3,2) \) if \( \kappa < 0 \) and of the Poincaré group if \( \kappa = 0 \). It follows that the \( I^{\mu\nu} \) and the \( L^\lambda \) generate the corresponding enveloping algebra. The \( x^\mu - L^\mu \) are in the center \( Z(\mathcal{A}_\kappa) \) of \( \mathcal{A}_\kappa \). Therefore the algebra \( \mathcal{A}_\kappa \) is the tensor product of the commutative algebra generated by the \( x^\mu - L^\mu \) with the above enveloping algebra. In fact the center \( Z(\mathcal{A}_\kappa) \) of \( \mathcal{A}_\kappa \) is generated by the \( x^\mu - L^\mu \) and the two casimirs \( c_2 \) and \( c_4 \) given by

\[ c_2 = \kappa g_{\lambda\mu} g_{\rho\nu} I^{\lambda\rho} I^{\mu\nu} + 2g_{\alpha\beta} L^{\alpha \beta} \]

\[ c_4 = g^{\rho\nu} \varepsilon_{\rho\lambda\mu\nu} L^\lambda I^{\mu\nu} \varepsilon_{\rho'\lambda'\mu'\nu'} L^\lambda I^{\mu'\nu'} \]

where \( \varepsilon_{\rho\lambda\mu\nu} \) is the completely antisymmetric tensor with \( \varepsilon_{0123} = 1 \). Therefore \( \mathcal{A}_0 \) is the tensor product of the commutative algebra generated by the \( x^\mu - L^\mu \) with the enveloping algebra of the Poincaré Lie algebra generated by the \( I^{\mu\nu} \) and the \( L^\lambda \). Notice however that the \( I^{\mu\nu} \) and the \( L^\lambda \) have nothing to do with the physical Poincaré group \( \mathfrak{P} \) acting on \( \mathcal{A}_0 \) by the automorphisms \( \alpha(\Lambda, a) \); in particular the \( L^\lambda \) have the dimension of a length. The algebra \( \mathcal{A}_0^I \) is simply the subalgebra generated by the \( I^{\mu\nu} \) and the \( L^\lambda \) (\( \simeq \) enveloping algebra of a Poincaré
The $\tau_k$ are given by $\tau_k(I^{\mu\nu}) = I^{\mu\nu} + k^\mu L^\nu - k^\nu L^\mu$ and $\tau_k(L^\lambda) = L^\lambda$.

One can compute the cocycle $c$, e.g. one has $c(x^\mu, x^\nu) = -\frac{1}{2} I^{\mu\nu}$. Finally, allowing exponentials, the cocycle $\gamma$ is given by

$$\gamma(k, \ell) = -\frac{i}{2} \left( k_\mu \ell_\nu I^{\mu\nu} - \left( \frac{2}{3} k_\rho \ell^\rho + \frac{1}{3} \ell_\rho \ell^\rho \right) k_\lambda L^\lambda + \left( \frac{1}{3} k_\rho \ell^\rho + \frac{2}{3} k_\rho k^\rho \right) \ell_\lambda L^\lambda \right)$$

and one has $\lambda(L^\mu, k) = -k_\nu \left( I^{\mu\nu} + \frac{1}{2}(k^\mu L^\nu - k^\nu L^\mu) \right)$ and $c'(L^\mu, L^\nu) = -\frac{1}{2} I^{\mu\nu}$ and so forth. Since $c_2$ and $c_4$ are in the center of $A_\kappa$ and since they are translationally invariant it is natural to fix them and to add the corresponding relations in the definition of the $A_\kappa$. Since the element $c_2$ has the dimension of a length squared, there are three natural ways to fix it: (i) $c_2 = \kappa$, (ii) $c_2 = -\kappa$, (iii) $c_2 = 0$. Any of these choices leads to $g_{\mu\nu}L^\mu L^\nu = 0$ in $A_0$ (in fact in $A^{I}_0$). Remembering that $A^{I}_0$ has the structure of the enveloping algebra of a Poincaré Lie algebra, the latter condition is the analogue of a zero mass condition (although here it has a different meaning as we pointed out). Thus if one assumes furthermore that $c_4$ is fixed in such a way that the representations occurring are “zero mass” representations with strictly positive “spin”, one finds again a characteristic two sheeted structure for $A^{I}_0$ (by allowing the two helicities). The origin of the frequent occurrence of this two sheeted structure in this context is obviously due to the fact that the full Lorentz group is not connected.

In all the above examples the $A_\kappa$ are Poincaré-covariant. This is not needed, only $A_0$ must have this property. In fact one can easily produce an example where only $A_0$ is Poincaré covariant by deforming the universal enveloping algebras occurring in the previous example for $\kappa \neq 0$ (in the sense used in the theory of quantum groups).

If one wishes to establish a connection between the extra factor $A^{I}_0$ here and the one occurring in recent noncommutative versions of gauge theory ([9], [3], [4]), one should remember that, according to our analysis, $A^{I}_0$ must be infinite dimensional and that it can be noncommutative only if some of its elements do not commute with the functions on space-time. Concerning the first point,
one could expect that, by some contractibility argument, a finite dimensional approximation can be found. However the second point remains. This suggests that it is worth trying to enlarge the setting of the noncommutative models of gauge theory.

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