ONE REMARK ON CONSTRUCTION OF SEPARATED QUOTIENT-SPACE

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Abstract. We discuss elementary constructions of boundaries of symmetric spaces.

Let $M$ be a compact metric space. Let $M = \bigcup_{\alpha \in A} M_\alpha$ be a partition of $M$ ($M_\alpha \cap M_\beta = \emptyset$ if $\alpha \neq \beta$). Then the quotient-space $A$ has canonical structure of a topological space. Recall that the set $P \subset A$ is closed if and only if $\bigcup_{\alpha \in P} M_\alpha$ is a closed subset in $M$. Let $a_1, a_2, \ldots$ be a sequence in $A$. Let $a \in A$. Then $a_j \to a$ if there exist points $m_j \in M_{\alpha_j}, m \in M_\alpha$ such that $m_j \to m$ in $M$.

The space $A$ is not need to be separated in Hausdorff sense. We are interested in the following question: how to construct separated analog of the quotient-space $A$?

1. Preliminaries. Hausdorff convergence

Let $N \subset M$ be a closed subset. Denote by $M_\varepsilon$ the set of all points $m \in M$ satisfying the condition: there exist $n \in N$ such that $\rho(m, n) < \varepsilon$. Let $[M]$ be the space of all closed subsets in $M$. Hausdorff distance $d(N, N')$ in $[M]$ between $N$ and $N'$ is the infimum of $\varepsilon > 0$ such that $N \subset N_\varepsilon'$ and $N_\varepsilon' \subset N$.

Recall that the metric space $[M]$ is compact. Recall also two simple facts on Hausdorff convergence. Denote by $\bar{S}$ the closure of the set $S$. Denote by $B_\varepsilon(m)$ the ball $\rho(m, n) < \varepsilon$.

Lemma 1. Let $N_j \in [M]$. Let $K_\sigma$ $(\sigma \in \Sigma)$ be all limit points of the sequence $N_j$. Then

a) $\bigcup_{\sigma \in \Sigma} K_\sigma$ coincides with the set of all $m \in M$ such that for all $\varepsilon > 0$ the set $N_j \cap B_\varepsilon(m)$ is nonempty for infinite number of $j$.

b) $\bigcap_{\sigma \in \Sigma} K_\sigma$ coincides with the set of all $m \in M$ such that for all $\varepsilon > 0$ the set $N_j \cap B_\varepsilon(m)$ is nonempty for sufficiently large $j$.

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2. Construction of separated quotient-space

Let a partition $M = \bigcup_{\alpha \in A} M_\alpha$ satisfies the following condition

*) for each $B \subset A$ the set $\bigcup_{\alpha \in B} M_\alpha$ is the union of elements of the partition.

Fix an open subset $A \subset A$ such that quotient-topology on $A$ is separated. Denote by $A \subset [M]$ the set of subsets $M_\alpha$, $\alpha \in A$. Let our data satisfy the condition

**) the map $\alpha \mapsto M_\alpha$ is a homeomorphism of the spaces $A$ and $A$.

Definition. The separated quotient-space $[[A]]$ is the closure of $A$ in Hausdorff metrics.

Remark. Of course the construction depends on the set $A \subset A$.

3. Description of the set $[[A]]$

By lemma 1 and the condition *) the elements $N \in [[A]]$ are unions of elements $M_\alpha$ of the partition. Hence we associate to each $N \in [[A]]$ subset $S_N \subset A$ of all $\sigma \in A$ such that $M_\alpha \subset N$. Denote by $A$ the set of all subsets $S_N$. By construction we have canonical bijection $[[A]] \leftrightarrow [A]$.

The following proposition is evident.

Lemma 2. Let $S \subset A$. Then the following conditions are equivalent

a) $S \in [A]

b) There exist a sequence $a \in A$ such that each limit point of $a_j$ is an element of $S$ and each element $s \in S$ is a limit of the sequence $a_j$ in the quotient-topology on $A$.

Elements of $[A]$ we call admissible subsets.

4. Example: complete collineations

Let $M$ be the Grassmann manifold $Gr_n$ of all $n$-dimensional subspaces in $\mathbb{C}^n \oplus \mathbb{C}^n$. Let $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus 0$. Let $V \in Gr_n$. Define the subspace $\lambda V :$

$$h \oplus p \in V \iff h \oplus \lambda p \in \lambda V$$

where $h \in \mathbb{C}^n \oplus 0$, $p \in 0 \oplus \mathbb{C}^n$. Consider the partition of $Gr_n$ into $\mathbb{C}^*$-orbits. Let $Op \subset Gr_n$ be the space of graphs of invertible operators. Of course the space $Op$ coincide with the general linear group $GL_n(\mathbb{C})$. The quotient space $Op/\mathbb{C}^* = GL_n(\mathbb{C})/\mathbb{C}^*$ is the group $PGL_n(\mathbb{C})$ of invertible operators defined up to scalar multiplier.

We want to apply our construction to the space $M = Gr_n$ and $A = PGL_n(\mathbb{C})$. We have to describe all admissible subsets in $Gr_n/\mathbb{C}^*$.

Example. Let $n = 2$. Consider the sequence $Q_n = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \in PGL_2(\mathbb{C})$. Then the set of limits of $Q_n$ in $Gr_2/\mathbb{C}^*$ consists of points $V_1, \ldots, V_5$ (= subspaces in $\mathbb{C}^2 \oplus \mathbb{C}^2$) enumerated below:
\[ V_1 : (x, y; 0, 0) \]
\[ V_2 : (x, y; 0, y) \]
\[ V_3 : (x, 0; 0, y) \]
\[ V_4 : (x, 0; x, y) \]
\[ V_5 : (0, 0; x, y) \]

where \( x, y \in V \). The subspaces \( V_1, V_3, V_5 \) are stable points of the group \( \mathbb{C}^* \). The \( \mathbb{C}^* \)-orbits of \( V_2, V_4 \) are 1-dimensional complex curves.

**Definition.** Let \( V \in Gr_n \). Then

a) *Kernel* \( \text{Ker} \ V = V \cap (\mathbb{C}^n \oplus 0) \)

b) *Image* \( \text{Im} \ V \) is the projection of \( V \) to \( 0 \oplus \mathbb{C}^n \).

c) *Domain* \( \text{Dom} \ V \) is the projection of \( V \) to \( \mathbb{C}^n \oplus 0 \).

d) *Indefiniteness* \( \text{Indef} \ V = V \cap (0 \oplus \mathbb{C}^n) \).

**Remark.** Let \( V \in Gr_n \) Then the subspace \( V \) induces by the obvious way the invertible operator

\[ \text{Dom} V / \text{Ker} V \to \text{Im} V / \text{Indef} V \]

We denote this operator by \( < V > \).

**Definition.** *Hinge* in \( \mathbb{C}^n \) is a collection

\[ \mathcal{P} = (Q_0, P_1, Q_1, P_2, Q_2, \ldots, P_k, Q_k) \]

where \( Q_j, P_j \) are elements of \( Gr_n \) defined up to multiplier and 0.

\[ Q_j = \text{Ker} Q_j \oplus \text{Indef} Q_j \]
\[ P_j \neq \text{Ker} P_j \oplus \text{Indef} P_j \]

1. For each \( j = 1, 2, \ldots, k \)

\[ \text{Ker} P_j = \text{Ker} Q_j = \text{Dom} P_{j+1} \]
\[ \text{Im} P_j = \text{Im} Q_j = \text{Indef} P_{j+1} \]

2.

\[ Q_0 = \mathbb{C}^n \oplus 0 ; \text{Dom} P_1 = \mathbb{C}^n \]
\[ Q_k = 0 \oplus \mathbb{C}^n ; \text{Im} P_k = \mathbb{C}^n. \]

**Remark.** Let \( P \) be the graph of an invertible operator \( \mathbb{C}^n \to \mathbb{C}^n \). Then

\[ (\mathbb{C}^n \oplus 0, P, 0 \oplus \mathbb{C}^n) \]

is a hinge.

**Remark.** The elements \( Q_0, \ldots, Q_{k+1} \) of a hinge are completely defined by the elements \( P_1, \ldots, P_k \). The subspaces \( Q_j \) are fixed points of the group \( \mathbb{C}^* \). The \( \mathbb{C}^* \)-orbits of \( P_j \) are 1-dimensional complex curves.
Theorem. The space $[PGL_n]$ of all admissible subsets in $Gr_n/C^*$ coincides with the space of all hinges.

The space $[PGL_n]$ coincide with the complete collineation space constructed by Semple (see [2]). It is a smooth algebraic variety and the group $PGL_n$ is an open dense subset in $[PGL_n]$. On equivalence of these two constructions see see [8]. Complete collineations is a partial case of complete symmetric varieties, see De Concini, Procesi [3].

5. Example. Furstenberg-Satake compactification of riemannian symmetric space

We will only discuss the case $PGL_n(\mathbb{R})/SO(n)$. Consider the space $\mathbb{R}^n \oplus \mathbb{R}^n$ provided by a skew-symmetric bilinear form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $\mathcal{L}$ be the grassmannian of all Lagrangian subspaces in $\mathbb{R}^n \oplus \mathbb{R}^n$. Denote by $\mathbb{R}^*$ the multiplicative group of real positive numbers. This group acts on $\mathcal{L}$ by multiplications of linear relations on scalars.

Denote by $R$ the open subset in $\mathcal{L}$ consisting of graphs of operators $S : \mathbb{R}^n \to \mathbb{R}^n$. It is easy to see that

$$\{\text{matrix } S \text{ is symmetric}\} \iff \{\text{the graph of } S \text{ is an element of } \mathcal{L}\}$$

The group $GL_n(\mathbb{R})$ acts on $R$ by the formula $g : S \mapsto g^t S g$. The stabilizer of the point $S = E$ is the orthogonal group $O(n)$. Hence $GL_n(\mathbb{R})$-orbit $X$ of $E$ is a homogeneous space $GL_n(\mathbb{R})/O(n)$. Points of $X$ correspon to positive definite matrices $S$.

Now we apply the construction of the sections 2-3 to the space $\mathcal{L}$ and to the open subset $X = GL_n(\mathbb{R})/O(n)$. Then the completion consists of hinges

$$P = (Q_0, P_1, Q_1, \ldots, P_k, Q_k)$$

such that $P_j \in \mathcal{L}, Q_j \in \mathcal{L}$ and the operators $< P_j >$ (see section 4) are positive definite.

6. Example. Boundary of Bruhat-Tits building

Let $Q_p$ be a $p$-adic field. Let $M$ be the space of all $\mathbb{Z}_p$-submodules in $\mathbb{Q}_p$. Let $B \subset M$ be the space of all lattices. The group $\mathbb{Q}_p^*$ act on $M$ in a natural way. Then the corresponding separated quotient-space consists of collections

$$(R_0, T_1, R_1, \ldots, T_k, R_k)$$

where $0 = R_0 \subset T_1 \subset R_1 \subset T_2 \ldots \subset R_k = \mathbb{Q}_p^n$ are elements of $M$ defined up to multiplier, $R_j$ are subspaces and images of $T_j$ in $R_j/R_{j-1}$ are lattices.

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