A Semi-parametric Realized Joint Value-at-Risk and Expected Shortfall Regression Framework

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Abstract

A new realized joint Value-at-Risk (VaR) and expected shortfall (ES) regression framework is proposed, through incorporating a measurement equation into the original joint VaR and ES regression model. The measurement equation models the contemporaneous dependence between the realized measures (e.g. Realized Variance and Realized Range) and the latent conditional quantile. Further, sub-sampling and scaling methods are applied to both the realized range and realized variance, to help deal with inherent micro-structure noise and inefficiency. An adaptive Bayesian Markov Chain Monte Carlo method is employed for estimation and forecasting, whose properties are assessed and compared with maximum likelihood estimator through simulation study. In a forecasting study, the proposed models are applied to 7 market indices and 2 individual assets, compared to a range of parametric, non-parametric and semi-parametric models, including GARCH, Realized-GARCH, CARE and Taylor (2017) joint VaR and ES quantile regression models, one-day-ahead Value-at-Risk and Expected Shortfall forecasting results favor the proposed models, especially when incorporating the sub-sampled Realized Variance and the sub-sampled Realized Range in the model.

Keywords: Quantile Regression, Realized Variance, Realized Range, Sub-sampling, Markov Chain Monte Carlo, Value-at-Risk, Expected Shortfall.
1 INTRODUCTION

Value-at-Risk (VaR) are widely employed by worldwide financial institutions and corporations to assist their decision making on capital allocation and risk management, since its introduction by J.P. Morgan in the RiskMetrics model at 1993. VaR is a quantitative tool to measure and control financial risk, which represents the market risk as one number and has become a standard measurement for capital allocation and risk management of financial institutions. However, VaR has been criticized because it cannot measure the expected loss for violations and is not mathematically coherent, in that it can favour non-diversification. Expected Shortfall (ES), proposed by Artzner et al. (1997, 1999), gives the expected loss, conditional on returns exceeding a VaR threshold, and is a coherent measure, thus in recent years it has become more widely employed for tail risk measurement and is chosen by the Basel Committee on Banking Supervision.

It is thus very important for institutions to have access to highly accurate VaR and ES forecasts and forecasting models, allowing accurate capital allocation, both to avoid default as well as over-allocation of funds. Both daily Value-at-Risk (VaR) and Expected Shortfall (ES) are studied in this paper, as recommended in the Basel II and III Capital Accords.

The 1-period VaR for holding an asset, and the conditional 1-period VaR, or ES, are formally defined via

\[
\alpha = Pr[r_{t+1} < \text{VaR}_{\alpha,t+1} | \Omega_t] ; \quad \text{ES}_{\alpha,t+1} = E[r_{t+1} | r_{t+1} < \text{VaR}_{\alpha}, \Omega_t]
\]

where \( r_{t+1} \) is the one-period return from time \( t \) to time \( t + 1 \), \( \alpha \) is the quantile level and \( \Omega_t \) is the information set at time \( t \).

Volatility estimation and prediction play a key role in calculating accurate VaR or ES forecasts. Since the introduction of the Auto-Regressive Conditional Heteroskedastic (ARCH) model of Engle (1982) and the generalized (G)ARCH of Bollerslev (1986), both employing squared returns as model input, many different volatility measures and models have been developed. Parkinson (1980) and Garman and Klass (1980) propose the daily high-low range as a more efficient volatility estimator compared to the daily squared
return. The availability of high frequency intra-day data has generated several popular and efficient realized measures, including Realized Variance (RV) (Andersen and Bollerslev, 1998, Andersen et al. 2003); and Realized Range (RR) (Martens and van Dijk, 2007, Christensen and Podolskij, 2007). In order to deal with the well-known, inherent microstructure noise accompanying high frequency volatility measures, Zhang, Mykland and Aıt-Sahalia (2005) and Martens and van Dijk (2007) design the sub-sampling and scaling processes, respectively, aiming to provide smoother and more efficient realized measures. In this paper the method of sub-sampling is extended to apply to the realized range measure.

Hansen, Huang and Shek (2011) extend the parametric GARCH model framework by proposing the Realized-GARCH (Re-GARCH), adding a measurement equation that contemporaneously links unobserved volatility with a realized measure. Gerlach and Wang (2016) extend the Re-GARCH model through employing RR as the realized measure and illustrate that the proposed Re-GARCH-RR framework can generate more accurate and efficient volatility, as well as VaR and ES forecasts compared to traditional GARCH and Re-GARCH models. Hansen and Huang (2016) recently extend the parametric Re-GARCH framework to include multiple realized measures.

However, the tail-risk forecast performance of parametric volatility models heavily depends on the choice of error distribution. The non-parametric conditional autoregressive VaR (CAViaR) models proposed by Engle and Manganelli (2004) can estimate quantiles (VaR) directly without a return distribution assumption. Gerlach, Chen and Chan (2011) generalize CAViaR models to a fully nonlinear family under a semi-parametric framework which incorporate asymmetric Laplace (AL) distribution for the likelihood construction. However, the CAViaR type models cannot directly estimate the ES. A joint semi-parametric model that directly estimates VaR and ES is proposed by Taylor (2017), and is referred as ES-CAViaR model. Through incorporating AL distribution with a time-varying density’s scale, the likelihood can be built to enable the joint estimation of the conditional VaR and conditional ES. Fissler and Ziegel (2016) develop a family of joint loss functions (or ”scoring rules”) of the associated VaR and ES series that are strictly consistent for the true VaR and ES series, i.e. they are uniquely minimized by the true
VaR and ES series. Under specific choices of functions in the join loss function of Fissler and Ziegel (2016), it can be shown that such loss function is exactly the same as the negative of AL log-likelihood function presented in Taylor (2017). Patton, Ziegel and Chen (2017) propose new dynamics models for VaR and ES through adopting the generalized autoregressive score (GAS) framework (Creal, Koopman and Lucas (2013) and Harvey (2013)) and utilizing the loss functions in Fissler and Ziegel (2016).

Motivated by Taylor (2017) and Hansen, Huang and Shek (2011), we extend the ES-CAViaR framework through incorporating a measurement equation which models the dependence between realized measures and quantile. The new framework is named as Realized-ES-CAViaR. In addition, the scaled and sub-sampled realized measures are also employed, to tackle the micro-structure noise and potential inefficiency. Moreover, an adaptive Bayesian MCMC algorithm is adopted to estimate the proposed models, extending that in Gerlach and Wang (2016). In the empirical study, the proposed Realized-ES-CAViaR models employing various realized measures as inputs, are assessed with their VaR and ES forecasting performance. Over the forecast period 2008-2016, the empirical results illustrate that Realized-ES-CAViaR models out-perform both the existing ES-CAViaR models and a range of competing models and methods such as the standard GARCH, realized GARCH models and conditional autoregressive Expec tile (CARE, Taylor, 2008).

The paper is organized as follows: Section 2 briefly reviews the existing ES-CAViaR model and Realized-GARCH, and proposes the Realized-ES-CAViaR class of models. The associated likelihood and the adaptive Bayesian MCMC algorithm for parameter estimation are presented in Section 3. The simulation studies are discussed in Section 4. Section 5 reviews some realized measures. Section 6 presents the forecasting study and back testing results. Section 7 concludes the paper and discusses future work.
2 MODEL PROPOSED

2.1 ES-CAViaR Models

Koenker and Machado (1999) note that the usual quantile regression estimator is equivalent to a maximum likelihood estimator when assuming that the data are conditionally Asymmetric Laplace (AL) with a mode at the quantile, i.e. if $r_t$ is the data on day $t$ and $Pr(r_t < Q_t|\Omega_{t-1}) = \alpha$ then the parameters in the model for $Q_t$ can be estimated using a likelihood based on:

$$p(r_t|\Omega_{t-1}) = \frac{\alpha(1-\alpha)}{\sigma} \exp \left( - (r_t - Q_t)(\alpha - I(r_t < Q_t)) \right),$$

for $t = 1, \ldots, n$ and where $\sigma$ is a nuisance parameter.

Taylor (2017) extends this result to incorporate the associated ES quantity into the likelihood expression. By noting a link between $ES_t$ and a dynamic $\sigma_t$, the likelihood expression thus is transformed into the conditional density function:

$$p(r_t|\Omega_{t-1}) = \frac{\alpha(1-\alpha)}{ES_t} \exp \left( - \frac{(r_t - Q_t)(\alpha - I(r_t < Q_t))}{\alpha ES_t} \right),$$

allowing a likelihood function to be built and maximised, given model expressions for $Q_t$ and $ES_t$. Taylor (2017) notes that the negative logarithm of the resulting likelihood function is strictly consistent for $Q_t$ and $ES_t$ considered jointly, i.e. it fits into the class of strictly consistent joint functions for VaR and ES developed by Fissler and Zeigel (2016).

Taylor (2017) incorporates two different ES components that describe the dynamics between VaR and ES and also avoid ES estimates crossing the corresponding VaR estimates, as presented in [1] (ES-CAViaR-AR: ES-CAViaR with an autoregressive ES to VaR difference component) and [2] (ES-CAViaR-Exp: ES-CAViaR with an exponential ES to VaR ratio component):
\[ r_t = Q_t + z_t, \]  
\[ Q_t = \beta_0 + \beta_1 |r_{t-1}| + \beta_2 Q_{t-1}, \]  
\[ \text{ES}_t = Q_t - x_t, \]  
\[ x_t = \begin{cases} 
\gamma_0 + \gamma_1 (Q_{t-1} - r_{t-1}) + \gamma_2 x_{t-1} & \text{if } r_{t-1} \leq Q_{t-1}; \\
 x_{t-1} & \text{otherwise}, 
\end{cases} \]

where \( \gamma_0 \geq 0, \gamma_1 \geq 0, \gamma_2 \geq 0 \) to ensure that the VaR and ES estimates do not cross.

\[ r_t = Q_t + z_t, \]  
\[ Q_t = \beta_0 + \beta_1 |r_{t-1}| + \beta_2 Q_{t-1}, \]  
\[ \text{ES}_t = x_t Q_t, \]  
\[ x_t = 1 + \exp(\gamma_0), \]

where \( \gamma_0 \) is unconstrained.

### 2.2 Realized-GARCH Models

The Realized-GARCH framework is proposed in Hansen, Huang and Shek (2011). The absolute value Realized-GARCH (Abs-Re-GARCH) specification can be written as:

**Abs-Re-GARCH**

\[ r_t = \sqrt{h_t} z_t, \]  
\[ \sqrt{h_t} = \beta_0 + \beta_1 X_{t-1} + \beta_2 \sqrt{h_{t-1}}, \]  
\[ X_t = \xi + \varphi h_t + \tau_1 z_t + \tau_2 (z_t^2 - 1) + u_t, \]

where \( X_t \) is a realized measure and the third equation is called the *measurement equation*. Here \( z_t \overset{\text{i.i.d.}}{\sim} D(0, 1) \) and \( u_t \overset{\text{i.i.d.}}{\sim} D(0, \sigma_u^2) \) and \( D \) is chosen to be Normal distribution here. Hansen, Huang and Shek (2011) employed \( x_t \) as realized volatility and realized kernel.
Compared to the conventional GARCH model, the Re-GARCH employs a measurement equation, which captures the contemporaneous relation between unobserved volatility and a realized measure. The superiority of Re-GARCH compared to GARCH and GARCH-X is well demonstrated, e.g. in Hansen, Huang and Shek (2011), Watanabe (2012) and Gerlach and Wang (2016).

The main advantage of a measurement equation is that more information about the latent volatility can be incorporated into the likelihood. Furthermore, asymmetric effects of positive and negative return shocks on the volatility are incorporated, e.g. through \( \tau_1 z_t + \tau_2 (z_t^2 - 1) \).

### 2.3 Realized-ES-CAViaR Models

Motivated by the ES-CAViaR and Realized-GARCH models, the Realized-ES-CAViaR-AR and Realized-ES-CAViaR-Exp models are proposed as:

**Realized-ES-CAViaR-AR (Re-ES-CAViaR-AR):**

\[
\begin{align*}
    r_t &= Q_t + z_t, \\
    Q_t &= \beta_0 + \beta_1 X_{t-1} + \beta_2 Q_{t-1}, \\
    X_t &= \xi + \phi |Q_t| + \tau_1 \epsilon_t + \tau_2 (\epsilon_t^2 - E(\epsilon^2)) + u_t, \\
    ES_t &= Q_t - x_t, \\
    x_t &= \begin{cases} 
        \gamma_0 + \gamma_1 (Q_{t-1} - r_{t-1}) + \gamma_2 x_{t-1}, & r_{t-1} \leq Q_{t-1}; \\
        x_{t-1}, & \text{otherwise}
    \end{cases}
\end{align*}
\]

**Realized-ES-CAViaR-Exp (Re-ES-CAViaR-Exp):**

\[
\begin{align*}
    r_t &= Q_t + z_t, \\
    Q_t &= \beta_0 + \beta_1 X_t + \beta_2 Q_{t-1}, \\
    X_t &= \xi + \phi |Q_t| + \tau_1 \epsilon_t + \tau_2 (\epsilon_t^2 - E(\epsilon^2)) + u_t, \\
    ES_t &= x_t Q_t, \\
    x_t &= 1 + \exp(\gamma_0),
\end{align*}
\]
where $X_t$ represents various realized measures to be discussed in Section 5. $u_t \sim D(0, \sigma_u^2)$ and $D$ is again chosen to be Normal. Besides the ES components in the models, the three equations as in the order in (4) or (5) are: the return equation, the quantile equation and the measurement equation.

Compared to the ES-CAViaR or Realized-GARCH models which have only one return-related “error”, there are two return-related “error” series in the proposed Realized-ES-CAViaR type models: one is the additive $z_t = r_t - Q_t$, which is assumed to follow an asymmetric Laplace distribution with time varying scale, so that likelihood can be constructed based on this AL density to jointly estimate the conditional VaR and conditional ES. However, the framework does not rely on an AL or any distribution assumption for the returns. The other one is the multiplicative $\epsilon_t = \frac{r_t}{Q_t}$, that appears in the measurement equation and is employed to capture the well known leverage effect. Again, if $Q_t$ is a multiple of $\sqrt{h_t}$ then, we will have $E(\epsilon_t) = 0$, as usual, but to keep a zero mean asymmetry term $(\epsilon_t^2 - E(\epsilon^2))$, we need to know

$$E(\epsilon^2) = E\left(\frac{r_t^2}{Q_t^2}\right).$$

The Realized-ES-CAViaR model does not include this second moment information. Thus, we substitute it with an empirical estimate $E(\epsilon^2) \approx \bar{\epsilon}^2$, being the sample mean of the squared multiplicative errors. We note that $E(\epsilon_t^2 - \bar{\epsilon}^2) = 0$ is preserved if $\bar{\epsilon}^2$ is an unbiased estimate. Therefore, the term $\tau_1 \epsilon_t + \tau_2 (\epsilon_t^2 - \bar{\epsilon}^2)$ still generates an asymmetric response in volatility to return shocks. Further, the sign of $\tau_1$ is expected to be opposite to that from an Re-GARCH model, since the quantile $Q_t$ is negative for the lower quantile levels, e.g. $\alpha = 1\%$, considered in the paper.

The Realized-ES-CAViaR framework can be easily extended into other nonlinear Realized-ES-CAViaR versions, e.g. by choosing the quantile dynamics in Gerlach, Chen and Chan (2011); However, we focus solely on the Realized-ES-CAViaR-SAV (Symmetric Absolute Value defined by the quantile equation) type models in this paper.

In order to guarantee that the series $Q_t$ does not diverge, a necessary condition for both Realized-ES-CAViaR models in (4) and (5) is $\beta_2 + \beta_1 \phi < 1$, subsequently enforced during estimation. This condition can be derived through substituting the measurement
equation into the quantile equation in either (4) or (5). Further, $\gamma_0 \geq 0$, $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ are applied in Realized-ES-CAViaR-AR framework to ensure that the VaR and ES estimates do not cross.

3 LIKELIHOOD AND BAYESIAN ESTIMATION

3.1 ES-CAViaR Log Likelihood Function with AL

Gerlach, Chen and Chan (2011) use an asymmetric Laplace distribution as the error distribution in the observation equation for a CAViaR model, to make the construction of a likelihood function feasible. Taylor (2017) also incorporate the AL distribution in the ES-CAViaR type models and allowed the density’s scale to be time-varying, and showed that it can be used to estimate conditional and VaR & ES simultaneously. This enables a joint model of conditional VaR and ES to be estimated by maximizing an AL log-likelihood, as follows:

$$
\ell(r; \theta) = \sum_{t=1}^{n} \left( \log \frac{(\alpha - 1)}{\text{ES}_t} + \frac{(r_t - Q_t)(\alpha - I(r_t \leq Q_t))}{\alpha \text{ES}_t} \right).
$$

(6)

3.2 Realized-ES-CAViaR Log Likelihood

Because the Realized-ES-CAViaR framework has a measurement equation, with $u_t \overset{i.i.d.}{\sim} N(0, \sigma_u^2)$, the full log-likelihood function for Realized-ES-CAViaR (as in (4) and (5)) is the sum of the log-likelihood $\ell(r; \theta)$ for the quantile equation and the log-likelihood $\ell(x|r; \theta)$ from the measurement equation:

$$
\ell(r, x; \theta) = \ell(r; \theta) + \ell(x|r; \theta) = \sum_{t=1}^{n} \left( \log \frac{(\alpha - 1)}{\text{ES}_t} + \frac{(r_t - Q_t)(\alpha - I(r_t \leq Q_t))}{\alpha \text{ES}_t} \right)
$$

(7)

$$
-\frac{1}{2} \sum_{t=1}^{n} \left( \log(2\pi) + \log(\sigma_u^2) + u_t^2/\sigma_u^2 \right),
$$

$$
\ell(x|r; \theta)
$$
where $u_t = X_t - \xi - \phi |Q_t| - \tau_1 \epsilon_t - \tau_2 (\epsilon_t^2 - \bar{\epsilon}_t^2)$, $t = 1, \ldots, n$.

In the Re-GARCH framework, the measurement equation variable $x - t$ contributes to volatility estimation, thus the in-sample and the predictive log-likelihoods are improved compared to the classical GARCH. We expect that the measurement equation in the Realized-ES-CAViaR also facilitates an improved estimate for $Q_t$, leading to more accurate VaR and ES forecasts.

### 3.3 Maximum Likelihood Estimation

Firstly, we implement and extend the Maximum Likelihood estimation approach in Taylor (2017). More specifically, to assist the optimization of Re-ES-CAViaR model, candidate parameter vectors are generated to pick the optimal starting values. In the first step, the quantile equation parameters ($\beta_0, \beta_1, \beta_2$) are estimated separately by optimizing the likelihood of a quantile regression. Then the measurement equation parameters ($\xi, \phi, \tau_1, \tau_2, \sigma_u$) and ES component parameters ($\gamma_0, \gamma_1, \gamma_2$ for Re-ES-CAViaR-AR, or $\gamma_0$ for Re-ES-CABiaR-Exp) are randomly sampled. 10,000 random candidate parameter vectors are used for Re-ES-CABiaR-Exp, and 50,000 for Re-ES-CAViaR-AR due to the relatively larger number of parameters involved. Last but not least, the estimates for $\beta_0, \beta_1, \beta_2$ in the first step are combined with the randomly sampled candidates in the second step, the parameter set that maximizes the log-likelihood function (7) is selected as the starting values for the constrained optimization routine $fmincon$ in Matlab.

### 3.4 Bayesian Estimation

Given a likelihood function, and the specification of a prior distribution, Bayesian algorithms can be employed to estimate the parameters of the proposed Re-ES-CAViaR models.

An adaptive MCMC method, extended from that in Gerlach and Wang (2016), is employed. Three blocks are employed: $\theta_1 = (\beta_0, \beta_1, \beta_2, \phi), \theta_2 = (\xi, \tau_1, \tau_2, \sigma_u), \theta_3 = (\gamma_0, \gamma_1, \gamma_2)$ for Re-ES-CAViaR-AR and $\theta_3 = (\gamma_0)$ for Re-ES-CAViaR-Exp, via the motiva-
tion that parameters within the same block are more strongly correlated in the posterior (likelihood) than those between blocks. Priors are chosen to be uninformative over the possible stationarity and positivity regions, e.g. \( \pi(\theta) \propto I(A) \), which is a flat prior for \( \theta \) over the region \( A \).

In "burn-in" period, an "epoch" method in Chen et al. (2017) is employed. For the initial "epoch" of the burn-in period, a Metropolis algorithm (Metropolis et al., 1953) employing a mixture of 3 Gaussian proposal distributions, with a random walk mean vector, is utilised for each block of parameters. The proposal variance-covariance matrix of each block in each mixture element is \( C_i \Sigma \), where \( C_1 = 1; C_2 = 100; C_3 = 0.01 \), with \( \Sigma \) initially set to \( \frac{2.38}{\sqrt{d_i}} I_{d_i} \), where \( d_i \) is the dimension of the parameter block \( (i) \), and \( I_{d_i} \) is the identity matrix of dimension \( d_i \). This covariance matrix is subsequently tuned, aiming towards a target acceptance rate of 23.4% (if \( d_i > 4 \), or 35% if \( 2 \leq d_i \leq 4 \), or 44% if \( d_i = 1 \)), as standard, via the algorithm of Roberts, Gelman and Gilks (1997).

In order to enhance the convergence of the chain, at the end of first epoch, e.g. 20,000 iterations, the covariance matrix for each parameter block is calculated, after discarding (say) the first 2,000 iterations. The covariance matrix is then used in the proposal distribution in the next epoch (of 20,000 iterations). After each epoch, the standard deviations of each parameter chain in that epoch is calculated and compared to that from the previous epoch. This process is continued until the mean absolute percentage change of the standard deviation is below a pre-specified threshold, e.g. 10%. In the empirical study, on average it takes 3-4 Epochs to observe this absolute percentage change lower than 10%; thus, the chains are run for 60,000-80,000 iterations in total as a burn-in period. A final epoch is run, for say 10,000 iterates, employing an "independent" Metropolis Hastings algorithm with a mixture of three Gaussian proposal distributions, for each block. The mean vector for each block is set as the sample mean vector of the last epoch iterates (after discarding the first 2,000 iterates) in the burn-in period. The proposal variance-covariance matrix in each element is \( C_i \Sigma \), where \( C_1 = 1; C_2 = 100; C_3 = 0.01 \) and \( \Sigma \) is the sample covariance matrix of the last epoch iterates in the burn-in period for that block (after discarding the first 2,000 iterates). Then all the IMH iterates (after discarding the first 2,000 iterates) are employed to calculate the tail risk forecasts, and
their posterior mean is used as the final forecast.

We employ the Gelman-Rubin diagnostic (Gelman et al., 2014) to diagnose the convergence of the adapted MCMC method. Further, an effective sample size testing is incorporated to evaluate the efficiency of the MCMC (Gelman et al., 2014). To save space, we do not present the MCMC efficiency results in the paper. In general, the Gelman-Rubin statistics is very close to 1 (e.g. < 1.1) for each parameter under different block settings, meaning excellent convergency for each parameter using the adapted MCMC. Through closer check of the between- and within-chain variances, we observe very small within-chain variances for each parameter, which leads to a close to 1 Gelman-Rubin statistics and suggests good convergence property. In addition, the effective sample sizes for all parameters are much larger than the benchmark suggested in Gelman et al. (2014), which as well proves the adapted MCMC algorithm is efficient.

4 SIMULATION STUDY

A simulation study is conducted to compare the Bayesian method and the maximum likelihood (ML) estimation approach for the Realized-ES-CAViaR models, with respect to parameter estimation and one-step-ahead VaR and ES forecasting accuracy. With the simulated data sets, both the mean and Root Mean Square Error (RMSE) values are calculated for the MCMC and ML methods, to illustrate their respective bias and precision.

1000 replicated datasets of size \( n = 1900 \) are simulated from the following specific absolute value Realized-GARCH model, specified as in (8). During the simulation, only positive values \( X_t \) are kept since it represents the realized measure. Sample size \( n = 1900 \) is chosen to be close to the real forecasting sample size and is employed for each simulated data set.
\[ r_t = \sqrt{h_t} \varepsilon_t^*, \quad (8) \]
\[ \sqrt{h_t} = 0.02 + 0.10X_{t-1} + 0.85\sqrt{h_{t-1}}, \]
\[ X_t = 0.1 + 0.9\sqrt{h_t} - 0.02\varepsilon_t^* + 0.02(\varepsilon_t^2 - 1) + u_t, \]
\[ \varepsilon_t^* \overset{\text{i.i.d.}}{\sim} N(0, 1), u_t \overset{\text{i.i.d.}}{\sim} N(0, 0.3^2). \]

In order to calculate the corresponding Realized-ES-CAViaR true parameter values, a mapping from the square root Realized-GARCH to the Realized-ES-CAViaR is required. With \( \text{VaR}_t = Q_t = \sqrt{h_t} \Phi^{-1}(\alpha) \), then 
\[ \sqrt{h_t} = \frac{Q_t}{\Phi^{-1}(\alpha)} = \frac{\text{VaR}_t}{\Phi^{-1}(\alpha)} \]
where \( \Phi^{-1}(\alpha) \) is the standard Gaussian inverse cdf. Further, with \( \varepsilon_t^* \overset{\text{i.i.d.}}{\sim} N(0, 1) \), we have 
\[ \epsilon_t = \frac{r_t}{Q_t} = \frac{r_t}{\sqrt{h_t}\Phi^{-1}(\alpha)} = \frac{\varepsilon_t^*}{\Phi^{-1}(\alpha)}. \]
Substituting back into the Abs-Re-GARCH and measurement equations of (8), the corresponding Re-ES-CAViaR (without ES component) specification can be written as:

\[ Q_t = 0.02\Phi^{-1}(\alpha) + 0.10\Phi^{-1}(\alpha)X_{t-1} + 0.85Q_{t-1}, \quad (9) \]
\[ X_t = 0.1 - \frac{0.9}{\Phi^{-1}(\alpha)}|Q_t| - 0.02\Phi^{-1}(\alpha)\epsilon_t + 0.02\Phi^{-1}(\alpha)^2(\epsilon_t^2 - \frac{1}{\Phi^{-1}(\alpha)^2}) + u_t, \]
allowing true parameter values to be calculated or read off. These true values are presented in Table 1 and 2.

It is more complicated to calculate the true values for the parameters in the ES equation for the ES equation, since there is not an exact one-one mapping between the ES equation in the Re-ES-CAViaR models and the Abs-Re-GARCH model. With the model in (8), the true in-sample and one-step-ahead \( \alpha \) level VaR and ES forecasts can be exactly calculated for each dataset; i.e. 
\[ \text{VaR}_t = \sqrt{h_t}\Phi^{-1}(\alpha), \text{ and } \text{ES}_t = -\sqrt{h_t}\frac{\phi(\Phi^{-1}(\alpha))}{\alpha}, \]
where \( \phi() \) is the standard Normal pdf. For the Re-ES-CAViaR-Exp model, the implied value of \( \gamma_0 \) can be solved for each \( t \); then the average of these values is treated as the true \( \gamma_0 \) in each simulated dataset. Finally, the average of all these “True” \( \gamma_0 \) values from the 1000 datasets is presented in the “True” column of Tables 2.
However, in the Re-ES-CAViaR-AR Model, the true values of $\gamma_0, \gamma_1, \gamma_2$ cannot be solved exactly, nor in a similar manner to that for the Re-ES-CAViaR-Exp model. A different approach is taken, where for each dataset, 50,000 random sets of trial values of $\gamma_0, \gamma_1, \gamma_2$ are proposed, and the set that maximizes the log-likelihood (conditional upon the true series $Q_t$) is selected as the "true" values. Finally, the average of all "true" $\gamma_0, \gamma_1, \gamma_2$ values from the 1000 datasets is presented in the "True" column Table 1 (0.1024, 0.1869 and 0.2752 respectively). The standard deviations of selected $\gamma_0, \gamma_1, \gamma_2$ across 1000 datasets are 0.0669, 0.2198 and 0.2554 respectively.

The true one-step-ahead $\alpha$ level VaR and ES forecasts can be calculated exactly via (8). For each simulated dataset, the true value of VaR$_{n+1}$ and ES$_{n+1}$ are recorded; the averages of these, over the 1000 datasets, are given in the “True” column of Table 1 and 2 respectively.

The Re-ES-CAViaR-AR and Re-ES-CAViaR-Exp models are fit to the 1000 datasets generated, the Bayesian estimator from the adaptive MCMC method in Section 3.1 and the ML estimator are both employed to reveal the properties of these two estimators.

Estimation results for Re-ES-CAViaR-AR are summarized in Table 1 where the boxes indicate the preferred model in terms of minimum bias (Mean) and maximum precision (minimum RMSE). Firstly, we can see that both MCMC and ML generate relatively accurate parameter estimates and VaR and ES forecasts, which proves the validity of both methods as discussed in Section 3. The results are clearly in favour of the MCMC estimator: for all 11 parameters as well as VaR and ES forecasts, the MCMC approach generates less bias and higher precision.

MLE provides better estimations for the Re-ES-CAViaR-Exp model than for the Re-ES-CAViaR-AR model, probably due to a simpler model specification. However, the adaptive MCMC estimator still produces more favourable results with respect to the parameter estimation (smaller bias for 6 out of 9 parameters, and higher precision for 7 out of 9 parameters), as well as the VaR (smaller bias and higher precision) and ES (higher precision) forecasts.

Last, in order to further illustrate why the MCMC produces the better estimation
Table 1: Summary statistics for the two estimators of the Realized-ES-CAViaR-AR model, with data simulated from Model 8.

| Parameter | True | MCMC | ML |
|-----------|------|------|-----|
| \( \beta_0 \) | -0.0465 | -0.0618 | -0.0935 |
| \( \beta_1 \) | -0.2326 | -0.2549 | -0.2827 |
| \( \beta_2 \) | 0.8500 | 0.8269 | 0.7887 |
| \( \xi \) | 0.1000 | 0.1837 | 0.2279 |
| \( \phi \) | 0.3869 | 0.3372 | 0.3028 |
| \( \tau_1 \) | 0.0465 | 0.0412 | 0.1253 |
| \( \tau_2 \) | 0.1082 | 0.0952 | 0.2269 |
| \( \sigma_u \) | 0.3000 | 0.2800 | 0.3284 |
| \( \gamma_0 \) | 0.1024 | 0.0840 | 0.1538 |
| \( \gamma_1 \) | 0.1869 | 0.2341 | 0.2609 |
| \( \gamma_2 \) | 0.2752 | 0.2775 | 0.3211 |
| \( \text{VaR}_{n+1} \) | -1.2494 | -1.2507 | -1.2517 |
| \( \text{ES}_{n+1} \) | -1.4311 | -1.4137 | -1.5791 |

Note: A box indicates the favored estimators, based on mean and RMSE.

bias and precision results, Figure 1 demonstrates the histogram of 1% VaR and ES forecasts from Re-ES-CAViaR-AR model estimated with MCMC and ML respectively, for 1000 simulated data sets. With respect to the VaR forecasts, MCMC and ML generate quite close bias and precision results, with both average VaR forecasts overlap the true VaR value in the plot. However, due to the challenge of estimating \( \gamma_0, \gamma_1, \gamma_2 \) in the ES component of Realized-ES-CAViaR-AR, ML frequently and largely overestimates (as high as three times) the 1% VaR, thus generates more dispersed VaR forecasts, as well as an average VaR forecast more diverged from the true VaR. Clearly, it is demonstrates the advantageous to employ MCMC estimator, especially for the Realized-ES-CAViaR-AR
Figure 1: Histogram of 1000 1% VaR and ES forecasts with 1000 simulated data sets generated from Realized-ES-CAViaR-AR estimated with MCMC and ML. “True” vertical line repeters the true VaR forecast value in Table. “Mean” vertical line repeters the average of 1000 VaR forecasts.
Table 2: Summary statistics for the two estimators of the Realized-ES-CAViaR-Exp model, with data simulated from Model 8.

| Parameter | True | Mean | RMSE | True | Mean | RMSE |
|-----------|------|------|------|------|------|------|
| $\beta_0$ | -0.0465 | -0.0647 | 0.1027 | -0.0836 | 0.2024 |      |
| $\beta_1$ | -0.2326 | -0.2500 | 0.0931 | -0.2511 | 0.1040 |      |
| $\beta_2$ | 0.8500 | 0.8270 | 0.1026 | 0.8119 | 0.1739 |      |
| $\xi$ | 0.1000 | 0.1840 | 0.1513 | 0.2273 | 0.4494 |      |
| $\phi$ | 0.3869 | 0.3367 | 0.1145 | 0.3026 | 0.3509 |      |
| $\tau_1$ | 0.0465 | 0.0404 | 0.0160 | 0.0405 | 0.0159 |      |
| $\tau_2$ | 0.1082 | 0.0969 | 0.0287 | 0.0973 | 0.0285 |      |
| $\sigma_u$ | 0.3000 | 0.2802 | 0.0203 | 0.2797 | 0.0207 |      |
| $\gamma_0$ | -1.9264 | -2.1046 | 0.3380 | -2.0886 | 0.6856 |      |
| VaR$_{n+1}$ | -1.2427 | -1.2410 | 0.0718 | -1.2405 | 0.0818 |      |
| ES$_{n+1}$ | -1.4237 | -1.4028 | 0.0887 | -1.4066 | 0.0977 |      |

Note: A box indicates the favored estimators, based on mean and RMSE.

framework.

5 REALIZED MEASURES

This section reviews popular realized measures and also the sub-sampled Realized Range.

For day $t$, denote the daily high, low and closing prices as $H_t$, $L_t$ and $C_t$, the most commonly used daily log return is:

$$r_t = \log(C_t) - \log(C_{t-1})$$

where $r_t^2$ is the corresponding volatility estimator.

Given each day $t$ is divided into $N$ equally sized intervals of length $\Delta$, subscripted by $\Theta = 0, 1, 2, ..., N$, then several high frequency volatility measures can be calculated. For day $t$, denote the $i$-th interval closing price as $P_{t-1+i\Delta}$, and the interval high and low prices as $H_{t,i} = \sup_{(i-1)\Delta < j < i\Delta} P_{t-1+j}$ and $L_{t,i} = \inf_{(i-1)\Delta < j < i\Delta} P_{t-1+j}$. Then RV is
proposed by Andersen and Bollerslev (1998) as:

$$RV_t^\Delta = \sum_{i=1}^{N} [log(P_{t-1+i\Delta}) - log(P_{t-1+(i-1)\Delta})]^2$$  \hspace{1cm} (10)

Martens and van Dijk (2007) and Christensen and Podolskij (2007) develop the Realized Range, which sums the squared intra-period ranges:

$$RR_t^\Delta = \frac{\sum_{i=1}^{N} (logH_{t,i} - logL_{t,i})^2}{4\log 2}$$  \hspace{1cm} (11)

Through theoretical derivation and simulation, Martens and van Dijk (2007) show that RR is a competitive, and sometimes more efficient, volatility estimator than RV under some micro-structure conditions and levels. Gerlach and Wang (2016) confirm that RR can provide extra efficiency in empirical tail risk forecasting, when employed as the measurement equation variable in an Re-GARCH model. To further reduce the effect of microstructure noise, Martens and van Dijk (2007) present a scaling process, as in follows:

$$ScRV_t^\Delta = \frac{\sum_{l=1}^{q} RV_{t-l}^\Delta}{\sum_{l=1}^{q} RV_{t-l}^\Delta} \cdot RV_t^\Delta,$$  \hspace{1cm} (12)

$$ScRR_t^\Delta = \frac{\sum_{l=1}^{q} RR_{t-l}^\Delta}{\sum_{l=1}^{q} RR_{t-l}^\Delta} \cdot RR_t^\Delta,$$  \hspace{1cm} (13)

where $RV_t$ and $RR_t$ represent the daily squared return and squared range on day $t$, respectively. This scaling process is inspired by the fact that the daily squared return and range are less affected by micro-structure noise than their high frequency counterparts, thus can be used to scale and smooth RV and RR, creating less micro-structure sensitive measures.

Further, Zhang, Mykland and Aǹt-Sahalia (2005) proposed a sub-sampling process, also to deal with micro-structure effects. For day $t$, $N$ equally sized samples are grouped into $M$ non-overlapping subsets $\Theta^{(m)}$ with size $N/M = n_k$, which means:

$$\Theta = \bigcup_{m=1}^{M} \Theta^{(m)}$$, where $\Theta^{(k)} \cap \Theta^{(l)} = \emptyset$, when $k \neq l$.

Then sub-sampling will be implemented on the subsets $\Theta^i$ with $n_k$ interval:

$$\Theta^i = i, i + n_k, ..., i + n_k(M - 2), i + n_k(M - 1)$$, where $i = 0, 1, 2..., n_k - 1$.  \hspace{1cm} (14)
Representing the log closing price at the $i$-th interval of day $t$ as $C_{t,i} = P_{t-1+i\Delta}$, the RV with the subsets $\Theta^i$ is:

$$RV_i = \sum_{m=1}^{M} (C_{t,i+n_k m} - C_{t,i+n_k (m-1)})^2; \text{ where } i = 0, 1, 2..., n_k - 1.$$ 

We have the $T/M$ RV with $T/N$ sub-sampling as (supposing there are $T$ minutes per trading day):

$$RV_{T/M,T/N} = \frac{\sum_{i=0}^{n_k-1} RV_i}{n_k}.$$ (14)

then, denoting the high and low prices during the interval $i + n_k (m - 1)$ and $i + n_k m$ as $H_{t,i} = \sup_{(i+n_k (m-1)) \Delta < j < (i+n_k m)} P_{t-1+j}$ and $L_{t,i} = \inf_{(i+n_k (m-1)) \Delta < j < (i+n_k m)} P_{t-1+j}$ respectively, we propose the $T/M$ RR with $T/N$ sub-sampling as:

$$RR_i = \sum_{m=1}^{M} (H_{t,i} - L_{t,i})^2; \text{ where } i = 0, 1, 2..., n_k - 1,$$ (15)

$$RR_{T/M,T/N} = \frac{\sum_{i=0}^{n_k-1} RR_i}{4\log(2)n_k},$$ (16)

For example, the 5 mins RV and RR with 1 min subsampling can be calculated as below respectively:

$$RV_{5,1,0} = (\log C_{t5} - \log C_{t0})^2 + (\log C_{t10} - \log C_{t5})^2 + ...$$

$$RV_{5,1,1} = (\log C_{t6} - \log C_{t1})^2 + (\log C_{t11} - \log C_{t6})^2 + ...$$

$$RV_{5,1} = \frac{\sum_{i=0}^{4} RV_{5,1,i}}{5},$$

$$RR_{5,1,0} = (\log H_{t0 \leq t \leq t5} - \log L_{t0 \leq t \leq t5})^2 + (\log H_{t5 \leq t \leq t10} - \log L_{t5 \leq t \leq t10})^2 + ...$$

$$RR_{5,1,1} = (\log H_{t1 \leq t \leq t16} - \log L_{t1 \leq t \leq t16})^2 + (\log H_{t11 \leq t \leq t16} - \log L_{t11 \leq t \leq t16})^2 + ...$$

$$RR_{5,1} = \frac{\sum_{i=0}^{4} RR_{5,1,i}}{4\log(2)5}.$$ 

Only intra-day return and range on the 5 minute frequency, additionally with 1 minute sub-sampling when employed, are considered in this paper. The properties of the sub-
sampled RR, compared to those of other realized measures, are assessed via simulation under three scenarios in Gerlach and Wang (2018).

6 DATA and EMPIRICAL STUDY

6.1 Data Description

Daily open, closing, high and low prices, as well as 1-minute and 5-minute price data, are downloaded from Thomson Reuters Tick History. Data are collected for seven market indices and two individual assets: S&P500 and NASDAQ (U.S.), Hang Seng (Hong Kong), FTSE 100 (UK), DAX (Germany), SMI (Swiss) and ASX200 (Australia), IBM and GE. The time range for the market indices and IBM is from January 2000 to June 2016, while for GE it starts from May 2000, after its 3:1 stock split in April 2000.

The price data are used to calculate the daily return, daily range and daily range plus overnight price jump. Further, the 5-minute prices are employed to calculate the daily RV and RR measures, while both 5- and 1-minute data are employed to produce daily scaled and sub-sampled daily RV and RR measures, as in Section 5; $q = 66$ is employed for the scaling process, approximately 3 months' daily data. Thus, additional data are collected since 3 month before the above-mentioned starting points of the sample range. Figure 2 display the time series plots of the absolute value of daily returns, $\sqrt{RV}$ and $\sqrt{RR}$ of S&P 500. Graphically, the absolute return is a noisy volatility estimator, while both $\sqrt{RV}$ and $\sqrt{RR}$ are less noisy and more efficient estimators.

6.2 Tail Risk Forecasting

For each series, one-step ahead forecast of both daily Value-at-Risk (VaR) and Expected Shortfall (ES) are estimated, as recommended in the Basel II and III Capital Accords, for the 9 return series.

A rolling window with fixed size is employed for in-sample estimation; the window size $n$ is given in Table 3 for each series, which differs due to non-trading days in each market. In order to assess the model performance during the GFC period, the forecast
period starts from the beginning of 2008. One-day ahead VaR and ES forecasts for 2111 days on average are generated for each return series from a range of models. These models include the proposed Realized-ES-CAViaR models with different input measures of volatility–RV & RR, scaled RV & RR and sub-sampled RV & RR; the ES-CAViaR models of Taylor (2017); the conventional GARCH, EGARCH and GJR-GARCH with Student-t distribution, CARE-SAV and Re-GARCH with Gaussian and Student-t observation equation error distributions. Further, a filtered GARCH (GARCH-HS) approach is also included, where a GARCH-t is fit to the in-sample data, then a standardised VaR and ES are estimated via historical simulation (using all the in-sample data) from the sample of returns (e.g. $r_1, \ldots, r_n$ divided by their GARCH-estimated conditional standard deviation (i.e. $r_t/\sqrt{h_t}$). Then the VaR and ES forecasts are obtained by multiplying the standardised VaR, ES estimates by the forecast $\sqrt{h_{n+1}}$ from the GARCH-t model. The Re-ES-CAViaR and ES-CAViaR models are estimated with adaptive MCMC, and the rest of models are estimated by MLE, either using the Econometrics toolbox in Matlab (GARCH-t, EGARCH-t, GJR-t and GARCH-HS) or code developed by the authors.
(CARE-SAV and Re-GARCH). The actual forecast sample sizes $m$, in each series, are given in Table 3.

The VaR violation rate (VRate) is employed to initially assess the VaR forecasting accuracy. VRate is simply the proportion of returns that exceed the forecasted VaR in the forecasting period, as in (17). Models with VRate closest to the nominal quantile level $\alpha = 1\%$ are preferred.

$$\text{VRate} = \frac{1}{m} \sum_{t=n+1}^{n+m} I(r_t < \text{VaR}_t),$$

where $n$ is the in-sample size and $m$ is the forecast sample size.

However, having a VRate close to the expected level is a necessary but not sufficient condition to guarantee an accurate forecasting model. Thus several standard quantile accuracy and independence tests are also employed: the unconditional coverage (UC) and conditional coverage (CC) tests of Kupiec (1995) and Christoffersen (1998) respectively, as well as the dynamic quantile (DQ) test of Engle and Manganelli (2004) and the VQR test of Gaglianone et al. (2011).

### 6.2.1 Value-at-Risk

Table 3 presents the VRates for 1% quantile for each model and each series, while Table 4 summarizes the results for each model across all series. A box indicates the model that has observed VRate closest to 1% in each series, while bold indicates the violation rate is significantly different to 1% by the UC test.

Clearly from Tables 3 and 4 Realized-ES-CAViaR models as a group have most optimal VaR forecast for most series, in terms of being closest to the nominal VRate of 1%, over the 9 return series. From Table 4 Re-ES-CAV-Exp-SSRR has VRates closest to 1% on average (1.153%), and the Re-ES-CAViaR-AR-SSRR ranks as the second best. Considering the median VRates, the Gt-HS, ES-CAViaR and Re-ES-CAViaR models actually have quite equivalently good performance. Later, we will compare Re-ES-CAViaR and ES-CAViaR in details, and provide evidence on why proposed Realized-ES-CAViaR models are preferred in VaR forecasting. All the models were anti-conservative, having
VRates on average (and median) above 1%: Re-GARCH-GG was most anti-conservative, generating 80-90% too many violations, which is not surprising since it is the only model employing the Gaussian distribution.

Table 3: 1% VaR Forecasting VRates with different models on 7 indices and 2 assets.

| Model         | S&P 500 | NASDAQ | HK  | FTSE | DAX  | SMI  | ASX200 | IBM  | GE  |
|---------------|---------|--------|-----|------|------|------|--------|------|-----|
| G-t           | 1.467%  | 1.895% | 1.652% | 1.731% | 1.362% | 1.617% | 1.702% | 1.183% | 0.945% |
| EG-t          | 1.514%  | 1.611% | 1.215% | 1.777% | 1.408% | 1.712% | 1.466% | 1.183% | 1.040% |
| GJR-t         | 1.467%  | 1.563% | 1.263% | 1.777% | 1.408% | 1.759% | 1.513% | 1.088% | 1.040% |
| Gt-HS         | 1.230%  | 1.563% | 1.263% | 1.123% | 1.127% | 1.284% | 0.898% | 1.041% | 1.181% |
| CAVt          | 1.278%  | 1.563% | 1.120% | 1.310% | 1.221% | 1.284% | 1.229% | 1.183% | 1.371% |
| Gt-HS         | 2.130%  | 1.942% | 2.818% | 1.777% | 2.800% | 1.807% | 1.560% | 1.419% | 1.323% |
| RG-RV-GG      | 1.467%  | 1.326% | 1.992% | 1.310% | 1.596% | 1.712% | 1.466% | 1.183% | 1.040% |
| RG-RV-GG      | 1.467%  | 1.326% | 1.992% | 1.310% | 1.596% | 1.712% | 1.466% | 1.183% | 1.040% |
| ES-CAV-AR     | 1.467%  | 1.516% | 1.215% | 1.216% | 1.268% | 1.236% | 0.946% | 1.230% | 0.992% |
| ES-CAV-Exp    | 1.278%  | 1.421% | 1.166% | 1.216% | 1.315% | 1.236% | 0.946% | 1.277% | 0.945% |
| ReES-CAV-AR-RV| 1.278%  | 1.705% | 2.284% | 1.123% | 1.315% | 1.427% | 0.808% | 1.041% | 1.181% |
| ReES-CAV-AR-RR| 1.088%  | 1.326% | 1.215% | 0.889% | 1.221% | 1.522% | 0.709% | 1.230% | 1.465% |
| ReES-CAV-AR-ScRV| 1.278%  | 1.705% | 1.166% | 1.123% | 1.174% | 1.236% | 0.946% | 0.993% | 1.323% |
| ReES-CAV-AR-ScRR| 1.562%  | 1.658% | 1.267% | 1.221% | 1.331% | 0.757% | 1.041% | 1.418% |
| ReES-CAV-AR-SSRV| 1.420%  | 1.516% | 1.215% | 1.076% | 1.268% | 1.379% | 0.898% | 1.088% | 1.270% |
| ReES-CAV-AR-SSRR| 1.136%  | 1.326% | 1.060% | 0.935% | 0.935% | 1.174% | 0.427% | 0.709% | 1.230% | 1.418% |
| ReES-CAV-Exp-RV| 1.278%  | 1.468% | 2.187% | 1.169% | 1.315% | 1.284% | 0.804% | 1.041% | 1.181% |
| ReES-CAV-Exp-RR| 1.088%  | 1.374% | 1.263% | 0.889% | 1.221% | 1.379% | 0.709% | 1.183% | 1.560% |
| ReES-CAV-Exp-ScRV| 1.325%  | 1.705% | 1.166% | 1.216% | 1.268% | 1.189% | 0.898% | 1.041% | 1.465% |
| ReES-CAV-Exp-ScRR| 1.278%  | 1.468% | 1.020% | 1.169% | 1.221% | 1.236% | 0.709% | 1.041% | 1.465% |
| ReES-CAV-Exp-SSRV| 1.372%  | 1.468% | 1.215% | 1.076% | 1.315% | 1.284% | 0.709% | 1.135% | 1.371% |
| ReES-CAV-Exp-SSRR| 1.278%  | 1.374% | 1.166% | 0.935% | 0.935% | 1.174% | 0.331% | 0.662% | 1.183% | 1.270% |

Note: Box indicates the most favored models based on VRates, blue shading indicates the 2nd ranked model, in each series, whilst bold indicates the violation rate is significantly different to 1% by the UC test. $m$ is the out-of-sample size, and $n$ is in-sample size. RG stands for the Realized-GARCH type models, and ReES-CAV represents the Realized-ES-CAViaR type models.

Since the standard quantile loss function is strictly consistent, e.g. the expected loss is a minimum at the true quantile series, the most accurate VaR forecasting model should minimise the quantile loss function, given as:

$$\sum_{t=n+1}^{n+m} (\alpha - I(r_t < Q_t))(r_t - Q_t), \quad (18)$$

where $Q_{n+1}, \ldots, Q_{n+m}$ is a series of quantile forecasts at level $\alpha$ for the observations $r_{n+1}, \ldots, r_{n+m}$.

As we discussed before, although it seems that both ES-CAViaR models and proposed
Table 4: Summary of 1% VaR Forecast VRates, for different models on 7 indices and 2 assets.

| Model          | Mean   | Median  |
|----------------|--------|---------|
| G-t            | 1.505% | 1.608%  |
| EG-t           | 1.437% | 1.466%  |
| GJR-t          | 1.432% | 1.466%  |
| Gt-HS          | 1.190% | 1.183%  |
| CARE           | 1.274% | 1.277%  |
| RG-RV-GG       | 1.895% | 1.798%  |
| RG-RV-tG       | 1.300% | 1.325%  |
| ES-CAV-AR      | 1.232% | 1.230%  |
| ES-CAV-Exp     | 1.200% | 1.230%  |
| ReES-CAV-AR-RV| 1.358% | 1.277%  |
| ReES-CAV-AR-RR | 1.184% | 1.230%  |
| ReES-CAV-AR-ScRV | 1.216% | 1.183% |
| ReES-CAV-AR-ScRR| 1.237% | 1.230% |
| ReES-CAV-AR-SSRV| 1.237% | 1.277% |
| ReES-CAV-AR-SSRR| 1.158% | 1.183% |
| ReES-CAV-Exp-RV| 1.300% | 1.277%  |
| ReES-CAV-Exp-RR| 1.184% | 1.230%  |
| ReES-CAV-Exp-ScRV| 1.253% | 1.230% |
| ReES-CAV-Exp-ScRR| 1.179% | 1.230% |
| ReES-CAV-Exp-SSRV| 1.216% | 1.277% |
| ReES-CAV-Exp-SSRR| 1.153% | 1.183% |

Note: Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, in each column.

Realized-ES-CAViaR models generate close VRates, Realized-ES-CAViaR models in VaR forecasting. Now the quantile loss results as presented in Table 5 and 6 provide more
evidence.

Table 5 presents the quantile loss values for each model on each return series, and shows that the proposed Realized ES-CAViaR in general rank higher than other models. Table 6 shows that on average and by median over the 9 return series, the Realized-ES-CAViaR using SubRR and SubRV perform best overall and produce quantile loss values that are clearly lower than the ES-CAViaR models. Actually the quantile loss from the ES-CAViaR models are close to those of the G-t and Gt-HS models which in general produce high quantile loss values. CARE model performs the worst in this study.

Table 6: 1% VaR Forecasting quantile loss on 7 indices and 2 assets.

| Model       | S&P 500 | NASDAQ | HK    | FTSE  | DAX  | SMI  | ASX200 | IBM  | GE  |
|-------------|---------|--------|-------|-------|------|------|--------|------|-----|
| G-t         | 81.8    | 92.1   | 98.4  | 81.5  | 93.4 | 88.0 | 69.7   | 114.8| 128.8|
| EG-t        | 80.3    | 92.2   | 90.1  | 76.9  | 92.2 | 83.1 | 67.3   | 115.0| 127.5|
| GJR-t       | 77.6    | 89.8   | 92.2  | 77.9  | 93.9 | 85.7 | 67.9   | 116.0| 126.6|
| Gt-HS       | 81.8    | 91.5   | 96.9  | 80.3  | 93.9 | 86.3 | 69.5   | 116.3| 130.6|
| CARE        | **84.2**| 95.5   | 93.0  | **82.7**| 93.3 | **89.8**| 77.3   | 112.0| **142.7**|
| RG-RV-GG    | 80.0    | 87.3   | 119   | 78.1  | **95.2**| 83.4 | 66.1   | 109.2| 112.5|
| RG-RV+G     | 77.1    | 85.3   | **108.6**| 77.0  | 91.7 | 82.0 | 65.4   | 108.0| 112.5|
| ES-CAViaR   | **84.2**| 93.5   | 94.9  | 81.1  | 92.8 | 86.3 | **71.9**| 111.3| **133.5**|
| ES-CAVia-Exp| 83.5    | 93.3   | 95.3  | 81.7  | 93.2 | 85.7 | 71.7   | 110.2| 132.1|
| ReES-CAVia-RV| 75.8    | 91.1   | **106.3**| 76.6  | 91.5 | 81.0 | 67.0   | 108.0| 111.4|
| ReES-CAVia-RR| 75.3    | **86.0**| 101.2| 76.5  | 90.0 | 78.7 | 67.8   | 111.6| **110.3**|
| ReES-CAVia-ScRR| 77.2    | 89.5   | 96.6  | 76.3  | 93.7 | 82.6 | 66.5   | **105.9**| 114.1|
| ReES-CAVia-ScRV| 76.0    | 89.1   | 91.0  | 76.3  | 91.0 | 80.2 | 68.1   | 108.5| 112.9|
| ReES-CAVia-SSRV| 74.3    | 87.5   | 90.9  | 75.8  | 90.1 | 78.5 | 67.5   | 109.2| 111.4|
| ReES-CAVia-SSRR| **72.8**| 85.9   | 96.2  | 73.8  | **88.7**| 78.4 | 67.0   | 109.8| 111.2|
| ReES-CAVia-Exp-RV| 76.0    | **97.9**| 106.2| 76.4  | 91.6 | 81.4 | **65.2**| 107.3| 112.4|
| ReES-CAVia-Exp-RR| 73.3    | 86.4   | 101.2| 76.3  | 90.0 | 79.1 | 67.4   | **110.6**| **110.2**|
| ReES-CAVia-Exp-ScRV| 77.2    | 89.8   | 96.5  | 76.3  | 93.9 | 82.9 | 66.3   | **105.6**| 114.4|
| ReES-CAVia-Exp-ScRR| 76.0    | 89.3   | 91.7  | 76.2  | 91.0 | 80.6 | 67.8   | 108.3| 113.3|
| ReES-CAVia-Exp-SSRV| 74.9    | 87.3   | 91.0  | 75.8  | 90.2 | 78.6 | 66.2   | 109.1| 112.2|
| ReES-CAVia-Exp-SSRR| **72.8**| 86.2   | 96.7  | **75.6**| **83.7**| 78.6 | 66.7   | 109.8| 111.1|

**Note:** Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, in each column.

Figure 8 and 4 provide further evidence on why the proposed Realized-ES-CAViaR type models generate clearly lower quantile loss than those models producing similarly accurate level of VRates. Specifically, the VaR violation rates of the Gt-HS, ES-CAViaR-Exp and Realized-ES-CAViaR-Exp-RR models are 1.230%, 1.278% and 1.088% respectively, for the S&P500 returns. Only looking at these violation rates, it seems that the 3 competing models generate quite close VaR forecasts. However, their quantile loss values
Table 6: Quantile loss function summary for different models on 7 indices and 2 assets.

| Model                | Mean | Mean rank |
|----------------------|------|-----------|
| G-t                  | 94.27| 17.44     |
| EG-t                 | 91.64| 13.11     |
| GJR-t                | 91.95| 14.78     |
| Gt-HS                | 94.11| 17.22     |
| CARE                 | 96.72| 18.22     |
| RG-RV-GG             | 92.33| 12.89     |
| RG-RV-tG             | 89.75| 9.33      |
| ES-CAV-AR            | 94.38| 16.89     |
| ES-CAV-Exp           | 94.07| 16.44     |
| ReES-CAV-AR-RV       | 89.85| 9.78      |
| ReES-CAV-AR-RR       | 88.60| 8.44      |
| ReES-CAV-AR-ScRV     | 89.14| 10.11     |
| ReES-CAV-AR-ScRR     | 88.12| 8.56      |
| ReES-CAV-AR-SSRV     | 87.24| 5.56      |
| ReES-CAV-AR-SSRR     | 87.43| 4.89      |
| ReES-CAV-Exp-RV      | 90.50| 10.11     |
| ReES-CAV-Exp-RR      | 88.27| 7.44      |
| ReES-CAV-Exp-ScRV    | 89.23| 10.89     |
| ReES-CAV-Exp-ScRR    | 88.27| 8.44      |
| ReES-CAV-Exp-SSRV    | 87.26| 5.44      |
| ReES-CAV-Exp-SSRR    | 87.46| 5.00      |

Note: Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, in each column. "Mean rank" is the average rank across the 7 markets and 2 assets for the quantile loss function, over the 21 models: lower is better.

are 81.8, 83.5 and 73.3 respectively, meaning the Re-ES-CAViaR-Exp-RR model is the most efficient model which produces the lowest quantile loss. Through close inspection of Figure 4, the ES-CAViaR-Exp and Gt-HS have close VaR forecasts as proved by their
quantile loss values. However, both models generate obviously more extreme (in the negative direction) level of VaR forecasts on most days, than Re-ES-CAViaR-Exp-RR does. This means the capital set aside by financial institutions to cover extreme losses, based on such VaR forecasts, is at a higher level for the Gt-HS or ES-CAViaR-Exp than for the Re-ES-CAViaR-RR.

In other words, the Re-ES-CAViaR-Exp-RR model produces VaR forecasts which suggest lower amounts of capital that are required to protect against market risk, while simultaneously produce a relatively adequate violation rate within the Basel Accords framework. For 2113 forecasting steps for S&P 500, the forecasts from Re-ES-CAViaR-Exp-RR are less extreme than those from ES-CAViaR-Exp on 1397 days (66.2%). This suggests a higher level of information (and cost) efficiency regarding risk levels for the Realized-ES-CAViaR model, likely coming from the increased statistical efficiency of the realized range series over squared returns, compared to the ES-CAViaR-Exp and Gt-HS models. Since the economic capital is determined by financial institutions’ own model and should be directly proportional to the VaR forecast, the Re-ES-CAViaR-RR model is able to decrease the cost capital allocation and increase the profitability of these institutions, by freeing up part of the regulatory capital from risk coverage into investment, while still providing sufficient and more than adequate protection against violations. The more accurate and often less extreme VaR forecasts produced by Re-ES-CAViaR-Exp-RR are particularly strategically important to the decision makers in the financial sector. This extra efficiency is also often observed for other Realized-ES-CAViaR-AR and Re-ES-CAViaR-Exp models in the other markets/assets in this study.

Further, during the periods with high volatility including the GFC when there is a persistence of extreme returns, the Re-ES-CAViaR-Exp-RR VaR forecasts ”recover” the fastest among the 3 models, presented through close inspection in Figure 4, in terms of being marginally the fastest to produce forecasts that again rejoin and follow the tail of the return series. Traditional GARCH models tend to over-react to extreme events and to be subsequently very slow to recover, due to their oft-estimated very high level of persistence, as discussed in Harvey and Chakravarty (2009). Realized-ES-CAViaR models clearly improve the performance on this aspect. Generally, the Realized-ES-CAViaR models
better describe the dynamics in the volatility, compared to the traditional GARCH model and the original ES-CAViaR type models, thus largely improving the responsiveness and accuracy of the risk level forecasts, especially after high volatility periods.

![Graph showing S&P 500 VaR Forecasts with Gt-HS, ES-CAViaR-Exp and Realized-ES-CAViaR-Exp-RR. VRates: 1.230%, 1.278% and 1.088%. Quantile loss: 81.8, 83.5 and 73.3.]

Figure 3: S&P 500 VaR Forecasts with Gt-HS, ES-CAViaR-Exp and Realized-ES-CAViaR-Exp-RR. VRates: 1.230%, 1.278% and 1.088%. Quantile loss: 81.8, 83.5 and 73.3

Several tests are employed to statistically assess the forecast accuracy and independence of violations from each VaR forecast model. Table [7] shows the number of return series (out of 9) in which each 1% VaR forecast model is rejected by each test, conducted at a 5% significance level. The Realized-ES-CAViaR models are generally less likely to be rejected by the back tests than other models. Re-ES-CAViaR-AR-ScRV and Re-ES-CAViaR-Exp-RV achieved the least number of rejections (in 2 series), followed by RG-RV-tG, Re-ES-CAViaR-AR-SSRR, Re-ES-CAViaR-Exp-RR and Re-ES-CAViaR-Exp-ScRv. The G-t is rejected in all 9 series, and the EG-t and Re-GARCH-GG models are rejected in 8 series, respectively.

6.2.2 Expected Shortfall

The same set of 21 models are employed to generate 1-step-ahead forecasts of 1% ES for all 9 series in the forecast periods.
First, to demonstrate the extra forecasting efficiency can be gained by employing the proposed Realized-ES-CAViaR models, Figure 5 visualize the ES forecasts (zoomed in) from CARE, ES-CAViaR-AR and Re-ES-CAViaR-AR-SSRR. Specifically, the ES violation rate of CARE, ES-CAViaR-AR and Re-ES-CAViaR-AR-SSRR models are 0.284%, 0.237% and 0.331% respectively, for S&P500. Gerlach and Chen (2016) illustrate that the quantile level that the 1% ES is estimated to fall at is $\approx 0.36\%$ for non-parametric models. Therefore, CARE, ES-CAViaR-AR and Re-ES-CAViaR-AR-SSRR all generate conservative and relatively accurate ES violation rate, with Re-ES-CAViaR-AR-SSRR being the one closest to the nominal level 0.36%.

However, through closer inspection in Figure 5, the cost efficiency gain from Realized-ES-CAViaR models is again in a similar pattern to that from the VaR forecasts. The CARE model is slightly more conservative than Realized-ES-CAViaR here, but achieves this by sacrificing efficiency: its ES forecasts are more extreme than the Re-ES-CAViaR-AR-SSRR model’s on 1687 days (79.8%) in the forecast period. In addition, the ES-CAViaR-AR model is more extreme than Re-ES-CAViaR-AR-SSRR on 1613 days (76.3%).

Therefore, the Re-ES-CAViaR-AR employing the sub-sampled realized range clearly
Table 7: Counts of 1% VaR rejections with UC, CC, DQ and VQR tests for different models on 7 indices and 2 assets.

| Model              | UC  | CC  | DQ1 | DQ4 | VQR | Total |
|--------------------|-----|-----|-----|-----|-----|-------|
| G-t                | 6   | 6   | 7   | 7   | 5   | 9     |
| EG-t               | 5   | 3   | 4   | 7   | 2   | 8     |
| GJR-t              | 5   | 3   | 6   | 5   | 3   | 7     |
| Gt-HS              | 1   | 1   | 1   | 3   | 1   | 4     |
| CARE               | 1   | 1   | 0   | 5   | 0   | 5     |
| RG-RV-GG           | 7   | 7   | 7   | 7   | 5   | 8     |
| RG-RV-tG           | 3   | 2   | 2   | 1   | 3   | 3     |
| ES-CAV-AR          | 2   | 0   | 0   | 4   | 1   | 6     |
| ES-CAV-Exp         | 0   | 0   | 0   | 4   | 0   | 4     |
| ReES-CAV-AR-RV     | 2   | 2   | 2   | 3   | 3   | 4     |
| ReES-CAV-AR-RR     | 2   | 1   | 1   | 2   | 3   | 5     |
| ReES-CAV-AR-ScRV   | 1   | 1   | 1   | 2   | 1   | 2     |
| ReES-CAV-AR-ScRR   | 2   | 3   | 2   | 3   | 0   | 4     |
| ReES-CAV-AR-SSRV   | 1   | 1   | 4   | 1   | 6   |       |
| ReES-CAV-AR-SSRR   | 0   | 0   | 0   | 2   | 1   | 3     |
| ReES-CAV-Exp-RV    | 2   | 2   | 2   | 2   | 2   | 2     |
| ReES-CAV-Exp-RR    | 1   | 2   | 1   | 2   | 0   | 3     |
| ReES-CAV-Exp-ScRV  | 2   | 1   | 1   | 2   | 1   | 3     |
| ReES-CAV-Exp-ScRR  | 2   | 1   | 2   | 3   | 0   | 4     |
| ReES-CAV-Exp-SSRV  | 1   | 1   | 2   | 5   | 4   | 6     |
| ReES-CAV-Exp-SSRR  | 0   | 1   | 1   | 2   | 3   | 4     |

*Note:* Box indicates the model with least number of rejections, blue shading indicates the model with 2nd least number of rejections, bold indicates the model with the highest number of rejections, red shading indicates the model 2nd highest number of rejections. All tests are conducted at 5% significance level.

Improves the forecasting efficiency compared with the CARE and ES-CAViaR-AR. Again, such extra efficiency is also frequently observed for the Realized-ES-CAViaR models with other asset return series. In the next section, we will further quantify this extra efficiency gained by employing the Realized-ES-CAViaR models.
6.2.3 VaR&ES Joint Loss Function

Now, a joint VaR and ES loss function (Fissler and Ziegel, 2016) is employed to compare the models’ VaR and ES forecasts jointly, and to help clarify and quantify any extra efficiency gains from the Realized-ES-CAViaR ES forecasts compared to its competitors.

Fissler and Ziegel (2016) develop a family of loss functions that is a joint function of the associated VaR and ES series. This loss function family is strictly consistent for the true VaR and ES series, i.e. they are uniquely minimized by the true VaR and ES series. The general form of the loss function family is:

\[
S_t(r_t, VaR_t, ES_t) = (I_t - \alpha)G_1(VaR_t) - I_tG_1(r_t) + G_2(ES_t) \left( ES_t - VaR_t + \frac{I_t}{\alpha}(VaR_t - r_t) \right) - H(ES_t) + a(r_t),
\]

where \( I_t = 1 \) if \( r_t < VaR_t \) and 0 otherwise for \( t = 1, \ldots, T \), \( G_1(\cdot) \) is increasing, \( G_2(\cdot) \) is strictly increasing and strictly convex, \( G_2 = H' \) and \( \lim_{x \to -\infty} G_2(x) = 0 \) and \( a(\cdot) \) is a real-valued integrable function.

As discussed in Taylor (2017), making the choices: \( G_1(x) = 0 \), \( G_2(x) = -1/x \), \( H(x) = -\log(-x) \) and \( a = 1 - \log(1 - \alpha) \), which satisfy the required criteria, returns the
scoring function (defined $r_t$ to have zero mean):

$$S_t(r_t, VaR_t, ES_t) = -\log \left( \frac{\alpha - 1}{ES_t} \right) - \frac{(r_t - Q_t)(\alpha - I(r_t \leq Q_t))}{\alpha ES_t},$$

where the loss function is $S = \sum_{t=1}^{T} S_t$. Taylor (2017) refers \(^{19}\) as AL log score. Compared with the likelihood function in (6), function \(^{19}\) is exactly the negative of the AL log-likelihood, and is a strictly consistent scoring rule that is jointly minimized by the true VaR and ES series. We use this to informally and jointly assess and compare the VaR and ES forecasts from all models.

Tables 8 and 9 present the loss function values $S$, calculated using Equation \(^{19}\), which jointly assesses the accuracy of each model’s VaR and ES forecasts, during the forecast period for each market.

By this measure, the Re-ES-CAViaR-Exp models employing sub-sampled RV and sub-sampled RR perform best overall, with lower loss than most other models in most series and being consistently ranked better. Another observation here is that the VaR and ES joint loss from Re-ES-CAViaR-Exp specification is in general slightly lower than that of Re-ES-CAViaR-AR specification. This can be partially explained by the fact that the Re-ES-CAViaR-Exp has a simpler ES component compared to ReES-CAViaR-AR. The G-t and CARE models rank lowest among all models. Generally the Realized-ES-CAViaR models are better ranked with higher cost efficiency in ES forecasts than other models in most markets.

### 6.2.4 Model Confidence Set

The model confidence set (MCS) was introduced by Hansen, Lunde and Nason (2011), as a method to statistically compare a group of forecast models via a loss function. We apply MCS to further compare among the 21 forecasting models and the forecasts combinations. A MCS is a set of models that is constructed such that it will contain the best model with a given level of confidence, which was selected as 90% in our paper. The Matlab code for MCS testing was downloaded from "www.kevinheppard.com/MFE_Toolbox". We adapted code to incorporate the VaR and ES joint loss function \(^{19}\) in the MCS calculation. Two methods, R and SQ, are employed in the MCS to calculate the test
Table 8: VaR and ES joint loss function values, using equation (19), across the markets; $\alpha = 0.01$.

| Model         | S&P 500 | NASDAQ | HK   | FTSE | DAX  | SMI  | ASX200 | IBM  | GE   |
|---------------|---------|--------|------|------|------|------|--------|------|------|
| G-t           | 4755.0  | 5067.2 | 5144.4 | 4872.1 | 5285.2 | 4987.1 | 4531.9 | 5818.5 | 5687.7 |
| EG-t          | 4800.9  | 5068.7 | 4985.2 | 4837.6 | 5277.3 | 4905.2 | 4503.7 | 5844.2 | 5681.0 |
| GJR-t         | 4665.5  | 4967.7 | 5009.9 | 4793.8 | 5315.8 | 4993.5 | 4475.2 | 5875.6 | 5668.7 |
| Gt-HS         | 4768.8  | 5031.3 | 5100.9 | 4811.8 | 5274.5 | 4884.3 | 4510.4 | 5807.3 | 5693.5 |
| CARE          | 4836.7  | 5201.5 | 5018.5 | 4890.9 | 5231.8 | 4973.2 | 4793.4 | 5650.4 | 6128.3 |
| RG-RV-GG      | 4706.9  | 4948.3 |      | 5673.1 | 4768.5 | 5275.2 | 4878.0 | 4432.7 | 5840.0 | 5436.7 |
| RG-RV-tG      | 4590.8  | 4875.4 | 5288.3 | 4706.4 | 5146.0 | 4778.0 | 4386.4 | 5583.0 | 5440.9 |
| ES-CAV-AR     | 4844.0  | 5099.7 | 5069.0 | 4859.6 | 5237.1 | 4941.2 | 4596.9 | 5620.0 | 5752.5 |
| ES-CAV-Exp    | 4833.6  | 5068.2 | 5071.5 | 4875.6 | 5234.8 | 4909.0 | 4589.5 | 5622.7 | 5707.6 |
| ReES-CAV-AR-RV| 4551.6  | 4996.1 | 5295.6 | 4690.0 | 5161.7 | 4776.0 | 4424.5 | 5585.7 | 5416.7 |
| ReES-CAV-AR-RR| 4509.6  | 4881.5 | 5158.2 | 4675.8 | 5098.7 | 4721.8 | 4463.6 | 5637.6 | 5409.8 |
| ReES-CAV-AR-ScRV| 4605.0 | 4974.5 | 5096.5 | 4680.5 | 5217.7 | 4772.1 | 4410.4 | 5560.7 | 5405.8 |
| ReES-CAV-AR-ScRR| 4552.5 | 4949.1 | 4961.5 | 4684.2 | 5129.3 | 4715.3 | 4479.7 | 5582.0 | 5493.4 |
| ReES-CAV-AR-SSRV| 4511.6 | 4915.0 | 4976.2 | 4670.1 | 5114.1 | 4710.9 | 4443.4 | 5623.2 | 5438.5 |
| ReES-CAV-AR-SSRR| 4462.5 | 4876.5 | 5071.7 | 4661.9 | 5096.7 | 4721.8 | 4435.3 | 5626.4 | 5447.3 |
| ReES-CAV-Exp-RV| 4551.5 | 5195.2 | 5296.3 | 4679.6 | 5154.0 | 4755.9 | 4380.2 | 5583.1 | 5401.2 |
| ReES-CAV-Exp-RR| 4477.8 | 4876.2 | 5147.1 | 4667.5 | 5093.2 | 4721.2 | 4459.4 | 5629.1 | 5362.7 |
| ReES-CAV-Exp-ScRV| 4598.6 | 4966.1 | 5090.8 | 4676.1 | 5212.1 | 4759.3 | 4411.1 | 5564.1 | 5396.2 |
| ReES-CAV-Exp-ScRR| 4547.4 | 4932.5 | 4668.2 | 4674.5 | 5120.7 | 4707.6 | 4479.9 | 5592.2 | 5438.3 |
| ReES-CAV-Exp-SSRV| 4506.8 | 4895.7 | 4974.1 | 4673.1 | 5105.0 | 4702.8 | 4414.4 | 5612.9 | 5396.7 |
| ReES-CAV-Exp-SSRR| 4456.6 | 4874.3 | 5067.8 | 4656.1 | 5090.7 | 4717.7 | 4437.4 | 5613.0 | 5388.0 |

Note: Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, in each column.

statistics. Specifically, R method uses the summed absolute values in the calculation, while SQ uses summed squares, more details can be found in page 465 of Hansen, Lunde and Nason (2011).

Table 10 and 11 present the 90% MCS using the R and SQ methods, respectively. Column ”Total” counts the number of times that a model is included in the 90% MCS across the 9 return series.

In general, the proposed Realized-ES-CAViaR models are more likely to be included in the MCS compared to the other models. Via the test statistics calculated by R method, Realized-ES-CAViaR models employing SSRV and SSRR are included in the MCS for all assets. The ES-CAV-AR and ES-CAV-Exp are included by the MCS for only 4 and 6 times respectively. Via the SQ method, Re-ES-CAViaR-AR employing RR, SSRV and SSRR and Re-ES-CAViaR-Exp employing RR, ScRR, SSRV and SSRR are included in the MCS for all series.
| Model            | Mean loss | Mean rank |
|------------------|-----------|-----------|
| G-t              | 5132.11   | 17.78     |
| EG-t             | 5100.43   | 16.11     |
| GJR-t            | 5085.09   | 15.33     |
| Gt-HS            | 5098.29   | 16.11     |
| CARE             | 5191.62   | 17.78     |
| RG-RV-GG         | 5106.52   | 14.00     |
| RG-RV-tG         | 4977.25   | 9.33      |
| ES-CAV-AR        | 5113.35   | 16.78     |
| ES-CAV-Exp       | 5101.39   | 16.33     |
| ReES-CAV-AR-RV   | 4988.64   | 10.78     |
| ReES-CAV-AR-RR   | 4950.72   | 8.89      |
| ReES-CAV-AR-ScRV | 4969.24   | 9.22      |
| ReES-CAV-AR-ScRR | 4949.90   | 8.33      |
| ReES-CAV-AR-SSRV | 4933.66   | 7.00      |
| ReES-CAV-AR-SSRR | 4933.32   | 7.00      |
| ReES-CAV-Exp-RV  | 4999.68   | 9.67      |
| ReES-CAV-Exp-RR  | 4937.13   | 6.56      |
| ReES-CAV-Exp-ScRV| 4963.84   | 8.22      |
| ReES-CAV-Exp-ScRR| 4940.16   | 7.11      |
| ReES-CAV-Exp-SSRV| 4920.18   | 4.56      |
| ReES-CAV-Exp-SSRR| 4922.39   | 4.11      |

*Note:* Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, in each column. "Mean rank" is the average rank across the 7 markets and 2 assets for the loss function, over the 21 models: lower is better.

Overall, across several measures and tests for 1% VaR and ES forecasts in 9 financial return series, the Realized-ES-CAViaR type models generally perform in a highly favourable manner when compared to a range of competing models. Considering VRates,
Table 10: 90% model confidence set with R method across the markets and assets.

| Model               | S&P 500 | NASDAQ | HK  | FTSE | DAX  | SMI  | ASX200 | IBM  | GE  | Total |
|---------------------|---------|--------|-----|------|------|------|--------|------|-----|-------|
| G-t                 | 0       | 1      | 0   | 1    | 1    | 0    | 0      | 0    | 0   | 3     |
| EG-t                | 0       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 0   | 7     |
| GJR-t               | 0       | 1      | 1   | 1    | 1    | 1    | 0      | 0    | 0   | 6     |
| G1-ES               | 0       | 1      | 1   | 1    | 1    | 1    | 0      | 0    | 1   | 6     |
| CARE                | 0       | 1      | 1   | 1    | 1    | 1    | 0      | 1    | 0   | 6     |
| RG-RV-GG            | 0       | 1      | 0   | 1    | 1    | 1    | 1      | 1    | 1   | 7     |
| RG-RV-Exp           | 0       | 1      | 0   | 1    | 1    | 1    | 1      | 1    | 1   | 7     |
| ES-CAV-AR           | 0       | 0      | 1   | 1    | 1    | 0    | 0      | 1    | 0   | 4     |
| ES-CAV-Exp          | 0       | 1      | 1   | 1    | 1    | 1    | 0      | 1    | 0   | 6     |
| ReES-CAV-AR-RV      | 0       | 1      | 0   | 1    | 1    | 1    | 1      | 1    | 1   | 7     |
| ReES-CAV-AR-RR      | 1       | 1      | 0   | 1    | 1    | 1    | 0      | 1    | 1   | 7     |
| ReES-CAV-AR-ScRV    | 0       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 8     |
| ReES-CAV-AR-ScRR    | 0       | 1      | 1   | 1    | 1    | 1    | 0      | 1    | 1   | 7     |
| ReES-CAV-AR-SSRV    | 1       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 8     |
| ReES-CAV-AR-SSRR    | 1       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 8     |
| ReES-CAV-Exp-RV     | 0       | 0      | 0   | 1    | 1    | 1    | 1      | 1    | 1   | 6     |
| ReES-CAV-Exp-RR     | 1       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 6     |
| ReES-CAV-Exp-ScRV   | 0       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 8     |
| ReES-CAV-Exp-ScRR   | 1       | 1      | 1   | 1    | 1    | 1    | 0      | 1    | 1   | 8     |
| ReES-CAV-Exp-SSRV   | 1       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 8     |
| ReES-CAV-Exp-SSRR   | 1       | 1      | 1   | 1    | 1    | 1    | 1      | 1    | 1   | 8     |

Note: Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, based on total number of included in the MCS across the 7 markets and 2 assets, higher is better.

rejections by standard quantile tests, quantile loss and VaR and ES joint loss, the Re-ES-CAViaR-AR and Re-ES-CAViaR-Exp employing sub-sampled RV and sub-sampled RR are the most favourable models overall.

7 CONCLUSION

In this paper, a new semi-parametric framework named Realized-ES-CAViaR is proposed for estimating and forecasting financial tail risks. Through incorporating intra-day and high frequency volatility measures with a measurement equation, improvements in the out-of-sample forecasting of tail risk measures are observed, over a range of competing models including the traditional GARCH, Realized-GARCH employing realized volatility, CARE models, as well as the original ES-CAViaR models. Specifically, Realized-ES-CAViaR-AR and Realized-ES-CAViaR-Exp models employing sub-sampled RV and sub-sampled RR generate the best VaR and ES forecasts in the empirical study of 9 financial return series. With respect to the backtesting of VaR and ES forecasts, the Realized-ES-CAViaR

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### Table 11: 90% model confidence set with SQ method across the markets and assets.

| Model          | S&P 500 | NASDAQ | HK  | FTSE | DAX | SMI | ASX200 | IBM | GE | Total |
|----------------|---------|--------|-----|------|-----|-----|--------|-----|----|-------|
| G-t            | 0       | 1      | 1   | 1    | 1   | 0   | 1      | 1   | 1 | 7     |
| EG-t           | 0       | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 0 | 7     |
| GJR-t          | 0       | 1      | 1   | 0    | 1   | 1   | 0      | 0   | 0 | 5     |
| Gt-RS          | 0       | 1      | 1   | 1    | 1   | 1   | 1      | 0   | 1 | 7     |
| CARE           | 0       | 1      | 1   | 1    | 1   | 0   | 0      | 1   | 0 | 5     |
| RG-RV-GG       | 0       | 1      | 0   | 1    | 1   | 1   | 1      | 1   | 1 | 7     |
| RG-RV-tG       | 0       | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 8     |
| ES-CAV-AR      | 0       | 1      | 1   | 1    | 1   | 1   | 1      | 0   | 1 | 6     |
| ES-CAV-Exp     | 0       | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 8     |
| ReES-CAV-AR-RV | 0       | 1      | 0   | 1    | 1   | 1   | 1      | 1   | 1 | 7     |
| ReES-CAV-AR-RR | 1       | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 9     |
| ReES-CAV-AR-ScRV | 0   | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 8     |
| ReES-CAV-AR-ScRR | 0  | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 8     |
| ReES-CAV-Exp-RV | 0   | 1      | 0   | 1    | 1   | 1   | 1      | 1   | 1 | 7     |
| ReES-CAV-Exp-RR | 1   | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 9     |
| ReES-CAV-Exp-ScRV | 0 | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 8     |
| ReES-CAV-Exp-ScRR | 1  | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 9     |
| ReES-CAV-Exp-SSRV | 1 | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 9     |
| ReES-CAV-Exp-SSRR | 1 | 1      | 1   | 1    | 1   | 1   | 1      | 1   | 1 | 9     |

**Note:** Box indicates the most favoured model, blue shading indicates the 2nd ranked model, bold indicates the least favoured model, red shading indicates the 2nd lowest ranked model, based on total number of included in the MCS across the 7 markets and 2 assets, higher is better.

models are also generally less likely to be rejected than their counterparts. Further, the model confidence set results also apparently favour the proposed Realized-ES-CAViaR framework. In addition to being more accurate, the Realized-ES-CAViaR models regularly generate less extreme tail risk forecasts, allowing smaller amounts of regulatory capital for financial institutions.

To conclude, the proposed Realized-ES-CAViaR type models, especially when combining sub-sampled RV and sub-sampled RR, should be considered for financial applications in tail risk forecasting, as they allow financial institutions to more accurately allocate capital under the Basel Capital Accords (Basel III will be fully effective in 2019), to protect their investments from extreme market movements. This work could be extended by developing asymmetric and non-linear quantile regression specifications; by improving ES component of the model; by using alternative frequencies of observations for the realized measures and by extending the framework to allow multiple realized measures to appear simultaneously in the model (Hansen and Huang, 2016).
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