Error-transparent evolution: the ability of multi-body interactions to bypass decoherence

Os Vy\textsuperscript{1}, Xiaoting Wang\textsuperscript{2} and Kurt Jacobs\textsuperscript{1,2,3,4}

\textsuperscript{1}Advanced Science Institute, RIKEN, Wako-shi 351-0198, Japan
\textsuperscript{2}Department of Physics, University of Massachusetts at Boston, 100 Morrissey Boulevard, Boston, MA 02125, USA
\textsuperscript{3}Hearne Institute for Theoretical Physics, Louisiana State University, Baton Rouge, LA 70803, USA
E-mail: kurt.jacobs@umb.edu

\textit{New Journal of Physics} 15 (2013) 053002 (13pp)
Received 26 October 2012
Published 7 May 2013
Online at http://www.njp.org/
doi:10.1088/1367-2630/15/5/053002

\textbf{Abstract.} We observe that multi-body interactions, unlike two-body interactions, can implement any unitary operation on an encoded system in such a way that the evolution is uninterrupted by noise that the encoding is designed to protect against. Such ‘error-transparent’ evolution is distinct from that usually considered in quantum computing, as the latter is merely correctable. We prove that the minimum body-ness required to protect (i) a qubit from a single type of Pauli error, (ii) a target qubit from a controller with such errors and (iii) a single qubit from all errors is three-body, four-body and five-body, respectively. We also discuss applications to computing, coherent feedback control and quantum metrology. Finally, we evaluate the performance of error-transparent evolution for some examples using numerical simulations.

\textsuperscript{4} Author to whom any correspondence should be addressed.

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
Contents

1. Error transparency 3
   1.1. Error-correction codes support error transparency 3
   1.2. Coherent feedback control with encoded auxiliary systems 4
   1.3. What degree of ‘body-ness’ is needed? 4
   1.4. When are perturbative interactions useful? 7

2. Simulations 8

3. Conclusion 10

Acknowledgments 11

References 11

Precision control of quantum systems is important in a number of areas. These include potential applications of quantum computing, such as simulating many-body systems [1–7], quantum metrology [8–11] and a great variety of experiments that probe quantum behavior [12–17]. The problem of controlling quantum systems can be divided into two main tasks. The first and simpler task is that of applying a unitary operator to the system to direct its motion. The second and more general task is that of modifying the von Neumann entropy of the system. This allows one to combat noise. Because the laws of physics, and thus unitary evolution, are logically reversible [18, 19], the only way to reduce the entropy of a system is to transfer this entropy to another quantum system (although sometimes this fact may be obscured by the use of measurement theory). In this paper, we concern ourselves only with coherent control (including coherent feedback control) in which no explicit measurements are made. Quantum error correction (QEC) is an example of a coherent method of combating noise, and it has the special property that it preserves quantum information stored in the system. A simpler form of control that allows noise reduction is so-called ‘coherent feedback’ in which no prior coding is used, but joint unitary operations are applied to the system and an auxiliary [20]. We show here that by using QEC methods, one can design multi-body Hamiltonians whose evolution is unaffected (to first order) by noise. This can be used, for example, to protect a system from noise in an auxiliary system that is being used to implement coherent feedback control. As we show, this can be achieved by replacing the physical auxiliary system with an encoded system. We note that ‘error-transparent’ evolution is distinct from evolution that is merely correctable—for example, the transversal evolution that enables fault-tolerant gates [21] is not error transparent. We emphasize that what we do here is not to construct new error-correcting codes, but to describe the notion of error-transparent evolution and determine the resources required to realize it using existing codes.

The next section is divided into four parts. In the first we review briefly how error-correcting codes work and define precisely what we mean by ‘error transparency’ (ET). We then show how to construct a Hamiltonian that generates error-transparent evolution for a given code. In the second part we discuss the application of ET to coherent feedback control. In the third part we determine the ‘body-ness’ required for a Hamiltonian that is transparent to various kinds of errors. In the last part we discuss briefly under what circumstances it would be advantageous to use many-body interactions that are generated perturbatively. In section 2 we
consider an application of ET to quantum metrology, and present some numerical simulations. We conclude by discussing the origin of the advantage provided by many-body interactions.

1. Error transparency

1.1. Error-correction codes support error transparency

A traditional quantum error-correcting code (QECC) for the protection of a single qubit works in the following way [22]. We first define a set of operators that form a basis for all single-qubit operations. The basis usually used is the set of Pauli operators, denoted by $X \equiv \sigma_x$, $Y \equiv \sigma_y$, and $Z \equiv \sigma_z$. The state of a single qubit is now encoded in a two-dimensional (2D) subspace of a system consisting of $N$ physical qubits. We refer to the encoded qubit as a logical, rather than a physical qubit. This subspace, called the code space, is chosen so that the operation of $X$, $Y$ or $Z$ on just one of the physical qubits maps the code space to an orthogonal 2D space. The code space is also chosen so that each distinct error either maps the initial space to a distinct orthogonal space, or if any two errors map to the same space, then these two mappings are identical. The spaces to which the errors map the code space are called the error spaces. This construction preserves the encoded information if there is a single error on one of the physical qubits, because the 2D error space in which the information ends up tells us how the information has been mapped to this space. We can make a measurement to determine which error space the logical qubit has ended up in, without disturbing the information inside this space. Once we discover where the information is along with what transformation took it there, we then know how it is encoded in this space. This means that we still have access to the information. Note that it is not the state of the $N$-qubit system that is preserved by the error-correcting code, but merely the information that is stored in it.

During the following discussions we will focus on single operator errors for clarity; however, we now take a moment to note that any arbitrary single-qubit error can be written as a superposition of Pauli errors. So if such a superposition afflicts our encoded state, since we have defined our error spaces in terms of Pauli operators, the logical qubit will become an appropriate superposition of states in more than one error space. Thus, we can still obtain the original information by applying a transformation that maps each error space back to the code space. To do this we apply an appropriate unitary to all the physical qubits along with an auxiliary system, where this unitary correlates each of the error spaces with a different basis state for the auxiliary. In other words, this joint unitary applies a different transformation to the physical qubits for each basis state of the auxiliary system. This returns the logical qubit back to the code space with its original information intact, while the unwanted information (entropy) about the superposition that occurred is dumped into the auxiliary system. The ability to correct for a finite set of errors, that forms a basis for all possible errors, allows us to correct any arbitrary single-qubit error.

If errors happen independently to each physical qubit at a rate $\gamma$, then the probability of a single error occurring on one of the qubits in a time $t$ is $p = n \gamma t$, where $n$ is the number of physical qubits, and this is true so long as $n \gamma t \ll 1$. Usually, codes are designed to preserve the encoded information when there is only a single error on any one qubit. This provides an advantage when $p \ll 1$, because in this case the probability that two errors occur is approximately $p^2 \ll p$. Thus, a single-error QECC preserves information under first-order
effects of independent errors on physical qubits. We now show how such encoding allows logical qubits to have ‘error-transparent’ evolution to the same first-order errors.

Consider first a Hamiltonian, $H_0$, that performs a transformation only on the code space, so that it transforms the logical qubit. For each error space, we now construct a Hamiltonian that acts on this error space in a way that is equivalent to the action of $H_0$ on the code space. What we mean by equivalent is the following: the logical qubit is encoded in a specific way on the error space, and the Hamiltonian that acts on this space performs the same transformation on the logical qubit as $H_0$ performs in the code space. Let us call the code space $S_0$ and denote the error or errors that take us to the $i$th error space by $E_i$, and the Hamiltonian that acts on this error space by $H_i$. The Hamiltonian $H_i$ will apply a transformation that is equivalent to that of $H_0$ if and only if

$$H_i E_i |\psi_0\rangle = E_i H_0 |\psi_0\rangle, \quad \forall |\psi_0\rangle \in S_0. \quad (1)$$

Now consider the Hamiltonian, $H = H_0 + \sum_{i=1}^{Q} H_i$, where $Q$ is the total number of errors in our basis set (three errors for each physical qubit). If a single error occurs, then the evolution on each of the error spaces is equivalent to that on the code space. The evolution of the logical qubit is not affected in any way by the error regardless of which error space the information ends up in. This information will have been transformed in the correct way despite the error. This is what we mean by ‘error transparency’ (ET).

For a given set of error operators $E_i$ there are a set of conditions that determine whether a given coding scheme allows error correction in the manner described above [23]. This set of conditions has been shown also to be necessary for the more general formulation of error correction known as operator QEC [24]. It follows therefore that for any operator quantum error-correcting scheme, there is a corresponding Hamiltonian that generates ET evolution.

1.2. Coherent feedback control with encoded auxiliary systems

We have seen above that QECs allow, in theory, ET evolution on a logical system, but do not enable such operations on a physical system. This is interesting from the point of view of coherent feedback control, which is simply the process of coupling the system you want to control to an auxiliary system so as to perform some desired operation on the former [20, 25–27]. While we cannot use quantum codes to reduce errors in the system to be controlled, since it is necessarily a physical system, we can replace the auxiliary system with a logical (encoded) system. In this case the target could be protected against errors in the auxiliary by having it evolve under an error-transparent Hamiltonian (ETH).

1.3. What degree of ‘body-ness’ is needed?

The preceding parts discussed the existence of an ETH for logical (encoded) systems; we now focus on the structure of an ETH. If we write down an arbitrary Hamiltonian for an $N$-body system, it will, in general, have terms that simultaneously connect all $N$ of those bodies. Typically the higher the body-ness of the interaction, the harder it is to come by. So we need to know just what level of body-ness is required for an ETH.

We can obtain a lower bound on the body-ness required to realize an ETH for a single qubit, by examining the distance requirement of QECs. To protect against single-qubit errors, the codewords must be chosen so that a single error on one codeword does not produce the same state as any other error on a different codeword. Otherwise it would not be possible to
recover the correct initial state from the final state, and the information would be lost. This means, equivalently, that no two errors acting on one codeword can produce another codeword. It therefore requires at least three errors to transform one codeword to another; this is described by saying that the code has ‘distance three’.

To produce unitary operations on the code space, without leaving the code space, a Hamiltonian must have a matrix element that connects the codewords directly. Since errors on at least three different physical qubits are required to connect any two codewords, a matrix element that connects the codewords must simultaneously change the state of at least three qubits and is therefore a three-body interaction.

The above argument does not tell us whether three-body interactions are sufficient to protect from all single-qubit errors. But we can show that they are sufficient to protect against just one of the three Pauli errors (e.g. $X$), rather than an arbitrary error. In this case a three-qubit (classical) code is sufficient. If we only wish to protect against $X$ (bit-flip) errors, then we can use the codewords $|0_L⟩ ≡ |000⟩$ and $|1_L⟩ ≡ |111⟩$ for logical zero and logical one, respectively. The code space is thus $\{ |000⟩, |111⟩ \}$, and the three error spaces, each corresponding to a bit-flip error on each of the three physical qubits, are $\{ |100⟩, |110⟩ \}$, $\{ |010⟩, |101⟩ \}$ and $\{ |001⟩, |110⟩ \}$.

A Hamiltonian that performs a general operation on the code space is then
\[ H_0 = a |0_L⟩⟨0_L| + b |1_L⟩⟨1_L| + c |0_L⟩⟨1_L| + c^* |1_L⟩⟨0_L|. \] (2)

The corresponding Hamiltonians on each of the error spaces are obtained by applying each of the errors to $H_0$ (this is the general ETH construction). Thus the ETH is
\[ H = H_0 + \sum_{i=1}^3 X_i H_0 X_i. \] (3)

If we want to use the above three-qubit states to control a fourth qubit, the resulting Hamiltonian must now simultaneously change the states of all four systems; it must be four-body. Thus, an ETH that provides transparency to a logical system controlling a ‘target’ system must take the ‘target’ system’s body-ness into consideration.

If we wish to realize evolution that is transparent to all errors on the physical qubits, then the question of the minimal body-ness is more complex. It is clear that a five-body interaction is sufficient, since a single qubit can be protected from all errors by a five-qubit code [28, 29]. But it is no longer clear that a three-body Hamiltonian is sufficient. The code may still be distance three, and the coding states connectable by a three-body Hamiltonian (this is true of the seven-qubit Calderbank–Shor–Steane (CSS code). But connecting the two coding states is not all that an ETH must do. It must generate an evolution that is specific to each of the error spaces. Since each of the error spaces is defined by the joint states of all the physical qubits, this specificity may require that the action of the Hamiltonian is conditional on the states of other qubits. For an action to be conditional on the state of a qubit, that qubit must be involved in the interaction. The total required body-ness is therefore obtained by counting the number of qubits whose state must be changed, as well as those that this change must depend upon. This can also be understood by noting that an action that is conditional on a qubit when viewed in one basis is instead an active change in the state of that qubit when viewed in another basis.

As an example, let us try to use a three-body interaction to realize evolution that is transparent to a $Z$ error using the seven-qubit CSS code. Note that a three-body Hamiltonian

\[ H = H_0 + \sum_{i=1}^3 X_i H_0 X_i \]

If the errors were not self-inverse, then one would have $H = H_0 + \sum_{i=1}^3 X_i^\dagger H_0 X_i$ instead.
can apply any logical operation to the code space. The three-body logical $X$ operator for this code is $\bar{X} = I I I I X X X \ [22, 30, 31]$. To obtain a Hamiltonian that acts in an equivalent way on the error space for a $Z$ error, we sandwich $\bar{X}$ between two copies of the operator for this error. This gives e.g. $I I I I Z I I \bar{X} I I I I Z I I = - \bar{X}$. So if we set $H_0 = \bar{X}$, then one of the error-space Hamiltonians is $H = - \bar{X} = -H_0$. Adding this to $H_0$ cancels both of them, giving zero. Thus there is no three-body ETH for the seven-qubit CSS code.

We now prove that a five-body interaction is necessary to realize transparency to all single-qubit errors. We do this by showing that if an $n$-body Hamiltonian exists that is ET for a logical qubit for arbitrary single-qubit errors, then only $n$ qubits are required to correct all single-qubit errors on these same $n$ qubits. But since the smallest single-qubit code has five qubits, $n$ must be no less than five.

**Theorem 1.** A five-body interaction is necessary and sufficient to realize transparency to all single-qubit errors.

**Proof.** Let us assume that we have a QECC that encodes a single logical qubit in $M$ physical qubits, and that the two logical states are connected by an $n$-body Hamiltonian, $H_0$, with $n < M$. We will refer to the $n$-qubits that $H_0$ acts on as the ‘active’ qubits, and the other $M - n$ qubits as the ‘passive’ qubits. $H_0$ can perform any operation on the logical qubit, and since it commutes with all errors on the passive qubits, it automatically performs the same operation on all error spaces for these qubits. We therefore only have to worry about the errors on the $n$ active qubits. Since there are three independent errors for each active qubit, there are $3n$ error spaces for the active qubits. We will label these error spaces by $j = 1, \ldots, 3n$. Note that for each of the logical states, there is a unique state in each error space that is the error-state equivalent. We will denote these by $|0\rangle_j$ and $|1\rangle_j$. Let us now assume that for each error space there is a Hamiltonian, $H_j$, that performs the correct operation on it (the operational equivalent to $H_0$). Note that the code may be degenerate, so that some of the $H_j$ may be the same. Note also that each of the Hamiltonians acts as the identity on all the passive qubits. Now we wish to perform error correction on the active qubits. This means that we need to perform a unitary transformation that maps each of the error spaces back to the original space. To do this we need to bring up some additional ancilla qubits, because a unitary transformation cannot map from a larger space to a smaller space. Let us denote the states of the ancilla space as $|a\rangle$. Since the $H_j$ perform arbitrary operations on each of the error spaces, we can specialize each of them to an operator $P_j$ that gives $P_j|0\rangle_j = |0\rangle_j$ and $P_j|1\rangle_j = |1\rangle_j$. We now use these to form a joint Hamiltonian that correlates each of the error spaces with an ancilla state. This Hamiltonian is given by

$$H = \hbar \lambda \sum_{j=1}^{3n} P_j \otimes (|j\rangle_a \langle 0|_a + |0\rangle_a \langle j|_a).$$

Starting the ancilla state as $|0\rangle_a$, after a time $\tau = 2\pi/\lambda$ the joint state of the active and ancilla qubits is

$$\sum_{j=0}^{3n} \rho_j \otimes |j\rangle_a \langle j|_a,$$

\[ \text{(5)} \]
where \( \rho_j \) is a state confined to the error space \( j \) (\( j = 0 \) denotes the code space). Now that the states on each of the orthogonal error spaces are correlated with orthogonal ancilla states, we can apply a unitary operation that maps each error space back to the code space. This unitary is

\[
U = U_j \otimes |j\rangle_a \langle j_l|
\]  

(6)

where \( U_j \) is the error that takes the code space to error space \( j \) (we are using the fact that the errors are all self-inverse). Since the \( E_j \) are all single-qubit errors on the active qubits, the unitary \( U \) does not perform any action on the passive qubits. Since error correction can be performed for all errors on the active qubits, by only operating on the active qubits, we do not need the passive qubits for error correction. Thus \( n \) qubits are sufficient for a full error-correcting code. Since it has already been established that \( \text{five} \) qubits is the minimum requirement for a code that corrects all errors, \( n \geq 5 \) and thus the Hamiltonian must be five-body. Further, using a five-qubit code, a five-body Hamiltonian is sufficient to realize transparency for all errors.

The above proof not only shows that five-body interactions are required for full ET, but also that if we use an \( n \)-qubit QECC to create a logical qubit, the ETH for this qubit will require \( n \)-body interactions. Furthermore, if the logical qubit is being used to control another ‘target’ system, then the ETH will require at least an \((n + 1)\)-body interaction.

1.4. When are perturbative interactions useful?

Genuine multi-body interactions are harder to come by than two-body interactions. So let us consider under what conditions one could reduce the effects of noise by generating effective multi-body interactions from two-body interactions. This can be done perturbatively \([32, 33]\) using a time-independent perturbation expansion and, in a very similar way, using rapid time-dependent control \([34–38]\). We now discuss the first of these methods. We will refer to the systems for which we want to obtain an all-body interaction collectively as the ‘primary system’. This will be the physical qubits of the error-correcting code and any other qubits that the logical qubit will be used to control. We will refer to the system employed to generate the all-body interaction as the auxiliary system. The method works by choosing the auxiliary system to have much larger gaps between its energy levels than the primary system (this means that it also has much faster dynamics than the primary system). We then couple all the subsystems in the primary system to the auxiliary system, so that this interaction constitutes a perturbation on the auxiliary Hamiltonian. One then uses time-independent perturbation theory to diagonalize the perturbed Hamiltonian to \( k \)th order in the perturbation. If the energy gaps of the auxiliary system have size \( \hbar \Delta \), and the energy of the interaction and the gaps of the primary system are of size \( \hbar \omega \), then the \( k \)th-order terms in the new effective Hamiltonian have speed (frequency) \( \Delta (\omega / \Delta)^k \) and contain all \( k \)-body interactions between the target systems.

The price one pays for creating a multi-body interaction is a reduction in the rate of the effective dynamics. That is, if one has a two-body interaction at rate \( \omega \), there are two choices. One can use this interaction to perform the task at speed \( \omega \) without using an ETH, or one can use it to generate an effective multi-body interaction performing the task error transparently yet slower. In particular, if one obtains a \( k \)-body effective interaction by using the two-body interaction to perturb a faster single-body dynamics that has rate \( \Delta \) (or by using quantum control at rate \( \Delta \)), then the rate of the effective dynamics is \( \omega_k = \omega (\omega / \Delta)^{(k-1)} \). Making the ratio \( \omega / \Delta \) smaller makes it simpler to obtain the effective interaction with high accuracy. Now, the effects of decoherence on a transformation are proportional to the decoherence rate, \( \gamma \), divided by the
speed of the transformation. For a $k$-body interaction generated in the manner above, this speed is $\omega_k$. The probability of an error, $p$, is of the order of $\gamma/\omega_k$. So to see if the error probability is reduced by using an effective multi-body interaction, we need to compare the $p$ that we get when using the two-body interaction by itself (no error correction) to that which we get by using an effective $k$-body interaction to implement an error-transparent operation. The former is $p = \gamma/\omega$. When we use an ETH, and thus a QECC, we only have an error in the logical qubit if there are errors in two of the physical qubits. So the probability of an error is approximately the square of the probability for a single error to occur, multiplied by the number of physical qubits, $n$. This is $p' = n(\gamma/\omega)^2 = np^2(\Delta/\omega)^{(2k-2)}$. The use of an effective $k$-body interaction could thus reduce the effects of decoherence if $n(\gamma/\omega) < (\omega/\Delta)^{(2k-2)}$. Since $\omega/\Delta$ is necessarily less than unity, effective multi-body interactions will only be useful when $p = \gamma/\omega$ is already rather small.

2. Simulations

Error-transparent evolution could be useful in any situation in which one is not concerned with the evolution of a particular system but merely with the evolution itself. Examples of this are information processing (computation), coherent feedback control (where the evolution of the controller is merely a means to control a second system) and quantum metrology.

In quantum metrology one wishes to determine the value of a constant (or parameter) that appears in the Hamiltonian via the evolution that the Hamiltonian induces in a quantum system [39]. Such a parameter’s value is determined by first preparing some initial state, allowing it to evolve under the Hamiltonian, and inferring the parameter from subsequent measurements. Such a procedure is clearly protected from noise if the sought-after parameter appears in an ETH. Since the physical qubits contain all the information regarding the logical qubit, the parameter can be inferred equally well from a logical system as from a real system, and this is independent of whether the logical qubit ends in the code space or one of the error spaces. We note that Caves and collaborators have shown that multi-body interactions can be used to enhance quantum metrology in a completely different manner, by changing the way in which the accuracy scales with the number of subsystems [40, 41].

We now present some simulations, largely to check that our analysis above is not flawed. Let us say that we have a single qubit that evolves under the Hamiltonian $H = \hbar \omega Z = \hbar \omega (|0\rangle\langle 0| - |1\rangle\langle 1|)$, and we want to protect this from $X$ errors. To do this we use our previously discussed three-qubit encoding into logical states: $|000\rangle$ and $|111\rangle$. The equivalent evolution for this logical code space is given by $H_0 = \hbar \omega (|000\rangle\langle 000| - |111\rangle\langle 111|)$. Now to turn this into an ETH, we add a new term with each error appropriately applied to $H_0$:

$$H = H_0 + IIXH_0IIX + IXIH_0IXI + XIIH_0XII. \quad (7)$$

Since the logical qubit is transparent only to a single error, its effective error rate will be lower only for times short compared to the inverse error rate. Measuring time in units of the oscillation period $\tau = \pi/\omega$, we choose the single-qubit error rate to be $\gamma = \alpha(1/\tau)$ where $\alpha \ll 1$. The error probability for a physical qubit during this time is $p = \gamma \tau = \alpha$. The total probability of an error occurring on any of the three physical qubits is therefore approximately $p_{\text{tot}} = 3p$. Since, in our error-transparent framework, the logical qubit only experiences an error when there are at least two errors on the physical qubits, the effective error rate for the logical qubit is lower. The probability that there are two errors in time $\tau$ on any of the three qubits is $p_{\text{tot}}^2$, but we note that
two errors on a single qubit cancel each other out. Thus there are just three ways to obtain two errors and so the approximate error probability for the logical qubit in time $\tau$ is $p_L = 3p^2$. The effective error rate for the logical qubit is then $\gamma_L = 3p^2/\tau = 3\alpha^2/\tau = 3\alpha\gamma$. So the code, along with the ETH, reduces the effective error rate by a factor of approximately $3\alpha$; this will guide us in comparing the simulations.

We perform simulations to compare this evolution among four scenarios: (i) a single physical qubit evolving under $H$ with $X$ errors at rate $\gamma$; (ii) the same single qubit with $X$ errors but at rate $0.03\gamma$; (iii) the three-qubit logical state evolving under $X$ noise on each of its three physical qubits; and (iv) the same logical state with $X$ errors but this time evolving under the ETH. If we denote the density matrix of a physical qubit by $\rho$, then the evolution due to the $X$-errors at rate $\gamma$ is given by the master equation $\dot{\rho} = -\gamma [X, \rho]$ for each physical qubit, since the Lindblad operator is equal to $X$ [43]. For each of the four scenarios we start the qubit in state $|1\rangle + |0\rangle$ (and the equivalent logical state for the last two scenarios) and evolve them for one cycle; then we check the probability that they have returned to their initial states. In the case of the logical qubit, this ‘success’ probability is the sum of the probabilities that the qubit is in the correct state in the coding space and any of the error spaces.

In figure 1(a) we plot the probabilities as a function of the (dimensionless) error rate $\gamma/\omega$. We see from these plots that the ETH, the best performing out of the four, indeed suppresses the errors in its evolution and transforms them, as expected, from first- to second-order errors. Further, we see that for small $\gamma\tau$ the logical qubit evolves in the same way as a physical qubit with the reduced error rate given by $3\alpha\gamma$. We also show that the logical qubit evolves merely under $H_0$ rather than the full ETH, and its error rate is faster than each physical qubit, as we would expect.

As our second example, we consider coherent feedback control of a single ‘target’ qubit by an auxiliary qubit encoded using the five-qubit stabilizer code [28, 29] and the seven-qubit CSS code [22, 30, 31]; with the addition of the target qubit this gives us six- and eight-body dynamics, respectively. Recall from section 1.2 that coherent feedback control is simply the use of an interaction with an auxiliary system to implement some operation on the system of interest (the ‘target’). Here our goal is to swap the state of the target with that of the auxiliary, and to do this as well as possible in the presence of a thermalizing environment for the target, and damping for the auxiliary. We will start the target in the ground state, and the auxiliary (logical) qubit in the logical 1 code state, as per the definitions of the codes given in [22]. We choose the ground state of the physical qubits to correspond to the physical state $|0\rangle$ in the definitions of the codes.

The thermalizing environment is modeled for each qubit using the standard thermal Markovian master equation for a two-level system, namely [42]

$$\dot{\rho} = -\gamma(\hat{n} + 1)(\sigma^+\sigma + \rho\sigma^+\sigma - 2\sigma\rho\sigma^+) - \gamma\hat{n}(\sigma^+\rho + \rho\sigma\sigma^+ - 2\sigma\rho\sigma^+). \quad (8)$$

Here $\sigma$ is the lowering operator for the two-level system, and $\hat{n}$ sets the temperature, with $\hat{n} = 0$ corresponding to zero temperature. The Hamiltonian that performs the swap is $H = \omega(\sigma_{\text{targ}}\sigma_{\text{aux}}^+ + \sigma_{\text{targ}}^+\sigma_{\text{aux}})$. For the target we choose $\gamma_{\text{targ}} = 10^{-4}\omega$ and $\hat{n} = 1$, and for the physical qubits that encode the auxiliary we choose $\hat{n} = 0$ and explore a range of values of $\gamma_{\text{aux}}$ between 0 and 0.1$\omega$. In this example, the physical qubit could represent the first two states of a nanomechanical oscillator that we wish to prepare in the one-phonon Fock state.

The success of the ETH is measured by the probability that the target qubit finishes in the excited state, and we plot this as a function of $\gamma_{\text{aux}}/\omega$. We do this for five different scenarios: (i) using a single physical auxiliary; (ii) using a logical auxiliary with the five-qubit stabilizer
Figure 1. Here we show the increased fidelity that can be achieved using ETHs. The probability of finishing in the desired state is shown versus the ratio of the error rate to the swap rate ($\gamma/\omega$), for two scenarios (a) Four scenarios in which a physical qubit or a logical qubit evolve under $Z = \sigma_z$ with frequency $\omega$ subjected to $X = \sigma_x$ noise. Dark dashed line: single qubit with no error correction or transparency; light solid line: single qubit with an effective second-order error rate realized by the ETH; light dashed line: three-qubit logical state with no transparency; dark solid line: three-qubit logical state with a three-body ETH. (b) A single qubit being controlled by various logical qubit controllers. Light solid lines: control via a logical qubit encoded with five-qubit and seven-qubit codes (the latter uses a Monte-Carlo simulation and so the squares give approximate error bars); dark solid lines: the equivalent logical controllers with ETHs; light dashed line: control via a single, unprotected, controller qubit.

code, but without an ETH; (iii) the same as (ii) but using the seven-qubit CSS code; (iv) the same as (ii) but now with a full ETH; and (v) the same as (iii) but with a full ETH.

We display the results in figure 1(b). The results show us that indeed the control achieved on the target qubit is greatly improved using the ETHs. Further, as expected, the five-qubit code performs better than the seven-qubit code, since in the latter the effective physical error rate is greater by a factor of $7/5$. We also see, as expected, that the logical qubits perform much worse than a single-qubit controller when they do not have an ETH. Yet, on the other hand, we thankfully see that both of the ETH cases out-perform using just a single, noisy, controller qubit.

3. Conclusion

In this paper we have shown that multi-body interactions can generate evolution that is uninterrupted by single-qubit errors. It is worth noting that this ability can be viewed as resulting
from the fact that the body-ness of an ETH is greater than that of the Hamiltonians generating the errors. The error-correcting codes we have considered here work precisely because the errors on different subsystems are assumed to be independent. If the subsystems that make up the bath were to couple to the physical qubits via \(k\)-body interactions, then they would likely induce errors that were correlated across \(k-1\) physical subsystems. In this case, \(k\)-body interactions would not be able to reduce the effects of the errors. One can therefore summarize the power of multi-body interactions to implement error-transparent operations in the following way: error-transparent operations can be realized in a multi-body system if the bath interacts with the system via \(k\)-body interactions, and the bodies of the system interact with each other via \(m\)-body interactions, with \(m > 2k\). Of course, this characterization is blurred somewhat by the fact that, given appropriate timescale separations, effective multi-body interactions can be used to obtain a degree of ET.

Acknowledgments

KJ and XW are partially supported by the NSF under project number PHY-0902906, and by the Intelligence Advanced Research Projects Activity (IARPA) via Department of Interior National Business Center contract number D11PC20168. KJ and OV are partially supported by the NSF under project number PHY-1005571, and by the ARO MURI grant number W911NF-11-1-0268. The US Government is authorized to reproduce and distribute reprints for governmental purposes notwithstanding any copyright annotation thereon.

Disclaimer. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of IARPA, DoI/NBC or the US Government.

References

[1] Abrams D S and Lloyd S 1997 Simulation of many-body Fermi systems on a universal quantum computer Phys. Rev. Lett. 79 2586–9
[2] Wu L-A, Byrd M S and Lidar D A 2002 Polynomial-time simulation of pairing models on a quantum computer Phys. Rev. Lett. 89 057904
[3] Brown K R, Clark R J and Chuang I L 2006 Limitations of quantum simulation examined by simulating a pairing Hamiltonian using nuclear magnetic resonance Phys. Rev. Lett. 97 050504
[4] Simon J, Bakr W S, Ma R, Tai M E, Preiss P M and Greiner M 2011 Quantum simulation of antiferromagnetic spin chains in an optical lattice Nature 472 307–12
[5] Barreiro J T, Müller M, Schindler P, Nigg D, Monz T, Chwalla M, Hennrich M, Roos C F, Zoller P and Blatt R 2011 An open-system quantum simulator with trapped ions Nature 470 486–91
[6] Porras D, Ivanov P A and Schmidt-Kaler F 2012 Quantum simulation of the cooperative Jahn–Teller transition in 1D ion crystals Phys. Rev. Lett. 108 235701
[7] Casanova J, Mezzacapo A, Lamata L and Solano E 2012 Quantum simulation of interacting fermion lattice models in trapped ions Phys. Rev. Lett. 108 190502
[8] Wineland D J, Bollinger J J, Itano W M, Moore F L and Heinzen D J 1992 Spin squeezing and reduced quantum noise in spectroscopy Phys. Rev. A 46 R6797–800
[9] Wineland D J, Bollinger J J, Itano W M and Heinzen D J 1994 Squeezed atomic states and projection noise in spectroscopy Phys. Rev. A 50 67–88
[10] Huelga S F, Macchiavello C, Pelizzari T, Ekert A K, Plenio M B and Cirac J I 1997 Improvement of frequency standards with quantum entanglement Phys. Rev. Lett. 79 3865–8

New Journal of Physics 15 (2013) 053002 (http://www.njp.org/)
[11] Shaji A and Caves C M 2007 Qubit metrology and decoherence Phys. Rev. A 76 032111
[12] Hofheinz M et al 2009 Synthesizing arbitrary quantum states in a superconducting resonator Nature 459 546–9
[13] Kubanek A, Koch M, Sames C, Ourjoumtsev A, Pinkse P W H, Murr K and Rempe G 2009 Photon-by-photon feedback control of a single-atom trajectory Nature 462 898
[14] Barthel C, Medford J, Marcus C M, Hanson M P and Gossard A C 2010 Interlaced dynamical decoupling and coherent operation of a singlet–triplet qubit Phys. Rev. Lett. 105 266808
[15] Leroux I D, Schleier-Smith M H and Vuletić 2010 Implementation of cavity squeezing of a collective atomic spin Phys. Rev. Lett. 104 073602
[16] Teufel J D, Donner T, Li D, Harlow J W, Allman M S, Cicak K, Sirois A J, Whittaker J D, Lehnert K W and Simmonds R W 2011 Sideband cooling micromechanical motion to the quantum ground state arXiv:1103.2144
[17] Marshall W, Simon C, Penrose R and Bouwmeester D 2003 Towards quantum superpositions of a mirror Phys. Rev. Lett. 91 130401
[18] Plenio M B and Vitelli V 2001 The physics of forgetting: Landauer’s erasure principle and information theory Contemp. Phys. 42 25–60
[19] Maruyama K, Nori F and Vedral V 2009 Colloquium: the physics of Maxwell’s demon and information Rev. Mod. Phys. 81 1–23
[20] Lloyd S 2000 Coherent quantum feedback Phys. Rev. A 62 022108
[21] Gottesman D 1998 Theory of fault-tolerant quantum computation Phys. Rev. A 57 127–37
[22] Gottesman D 2009 An introduction to quantum error correction and fault-tolerant quantum computation Phys. Rev. Lett. 94 180501
[23] Knill E and Laflamme R 1997 Theory of quantum error-correcting codes Phys. Rev. A 55 900
[24] Kribs D, Laflamme R and Poulin D 2005 A unified and generalized approach to quantum error correction Phys. Rev. Lett. 94 180501
[25] Nurdin H I, James M R and Petersen I R 2009 Coherent quantum LQG control Automatica 45 1837
[26] Dong D and Petersen I R 2010 Quantum control theory and applications: a survey IET Control Theory Appl. 4 2651
[27] Jacobs K and Wang X 2012 Coherent feedback that beats all measurement-based feedback protocols arXiv:1211.1724
[28] Bennett C H, DiVincenzo D P, Smolin J A and Wootters W K 1996 Mixed-state entanglement and quantum error correction Phys. Rev. A 54 3824–49
[29] Knill E and Laflamme R 1997 Theory of quantum error-correcting codes Phys. Rev. A 55 900
[30] Calderbank A R and Shor P W 1996 Good quantum error-correcting codes exist Phys. Rev. A 54 1098–105
[31] Steane A M 1996 Error correcting codes in quantum theory Phys. Rev. Lett. 77 793–7
[32] Kempe J, Kitaev A and Regev O 2006 The complexity of the local Hamiltonian problem SIAM J. Comput. 35 1070
[33] Jacobs K and Landahl A J 2009 Engineering giant nonlinearities in quantum nanosystems Phys. Rev. Lett. 103 067201
[34] Jacobs K 2007 Engineering quantum states of a nano-resonator via a simple auxiliary system Phys. Rev. Lett. 99 117203
[35] Jacobs K, Tian L and Finn J 2009 Engineering superposition states and tailored probes for nanoresonators via open-loop control Phys. Rev. Lett. 102 057208
[36] Dinerman J and Santos L F 2010 Manipulation of the dynamics of many-body systems via quantum control methods New J. Phys. 12 055025
[37] Kitagawa T, Oka T, Brataas A, Fu L and Demler E 2011 Transport properties of nonequilibrium systems under the application of light: photoinduced quantum Hall insulators without landau levels Phys. Rev. B 84 235108
[38] Tanamoto T, Becker D, Stojanović V M and Bruder C 2012 Preserving universal resources for one-way quantum computing Phys. Rev. A 86 032327
[39] Kok P and Lovett B W 2010 Introduction to Optical Quantum Information Processing (Cambridge: Cambridge University Press)
[40] Boixo S, Flammia S T, Caves C M and Geremia J M 2007 Generalized limits for single-parameter quantum estimation Phys. Rev. Lett. 98 090401
[41] Boixo S, Datta A, Davis M J, Flammia S T, Shaji A and Caves C M 2008 Quantum metrology: dynamics versus entanglement Phys. Rev. Lett. 101 040403
[42] Breuer H-P and Petruccione F 2007 The Theory of Open Quantum Systems (Oxford: Oxford University Press)
[43] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)