ON TWISTING OF FINITE-DIMENSIONAL HOPF ALGEBRAS

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Abstract. In this paper we study the properties of Drinfeld’s twisting for finite-dimensional Hopf algebras. We determine how the integral of the dual to a unimodular Hopf algebra $H$ changes under twisting of $H$. We show that the classes of cosemisimple unimodular, cosemisimple involutive, cosemisimple quasitriangular finite-dimensional Hopf algebras are stable under twisting. We also prove the cosemisimplicity of a coalgebra obtained by twisting of a cosemisimple unimodular Hopf algebra by two different twists on two sides (such twists are closely related to biGalois extensions \cite{S}), and describe the representation theory of its dual. Next, we define the notion of a non-degenerate twist for a Hopf algebra $H$, and set up a bijection between such twists for $H$ and $H^*$. This bijection is based on Miyashita-Ulbrich actions of Hopf algebras on simple algebras \cite{DT2, U}. It generalizes to the non-commutative case the procedure of inverting a non-degenerate skew-symmetric bilinear form on a vector space. Finally, we apply these results to classification of twists in group algebras and of cosemisimple triangular finite-dimensional Hopf algebras in positive characteristic, generalizing the previously known classification in characteristic zero.

1. Introduction

In this paper we study some properties of Drinfeld’s twisting operation for finite-dimensional Hopf algebras. In particular, for a finite-dimensional unimodular Hopf algebra $H$, we determine how the integral of the dual Hopf algebra $H^*$ changes under twisting. This allows us to show that the classes of cosemisimple unimodular, cosemisimple involutive, cosemisimple quasitriangular finite-dimensional Hopf algebras are invariant under twisting for any characteristic of the ground field $k$.

We also consider coalgebras obtained from $H$ by twisting it using two different twists, on the left and on the right (this reduces to Hopf algebra twisting if these two twists are the same). We show that such a coalgebra is always cosemisimple if $H$ is cosemisimple and unimodular. In particular, this applies to the situation when one of the two twists equals to 1 (twisting on one side). In this case, the unimodularity assumption can be dropped, and this result is a known theorem of Blattner and Montgomery. We also extend the result of \cite{EG2} by describing the algebra structure of the dual of the coalgebra obtained by twisting of $\mathbb{C}[G]$, the group algebra of a finite group $G$, by means of two different twists.

Twisting $H$ on one side defines not simply a coalgebra but actually an $H$-module coalgebra $C$ which is the regular $H$-module, such that the $H$-module algebra extension $k \subset C^*$ is Galois. It is easy to show that the converse is also true: any $H$-module coalgebra with such properties comes from twisting $H$ on one side by a twist, which is unique up to a gauge transformation. Thus, our results imply that if $H$ is cosemisimple, then any $H$-module coalgebra with said properties is cosemisimple.
Next, we define a non-degenerate twist for $H$ as a twist for which the corresponding $H$-coalgebra is simple (this generalizes the notion of a minimal twist for group algebras). The above results, combined with Masuoka’s generalization of the Skolem-Noether theorem [Mas] and the technique of Miyashita-Ulbrich actions [DT2, U], allow us to show that there exists a natural bijection between the set $NT(H)$ of gauge classes of non-degenerate twists for $H$, and the set $NT(H^*)$; in particular, these sets (which are finite in the semisimple cosemisimple case by a theorem of H.-J. Schneider [Sch]) have the same number of elements. For example, if $H = \mathbb{C}[V]$, where $V$ is a vector space over a finite field of odd characteristic, then $NT(H)$ is the set of symplectic structures on $V^*$, $NT(H^*)$ is the set of symplectic structures on $V$, and our bijection is the usual inversion. More generally, if $G$ is a finite group and $H = \mathbb{C}[G]$ then this bijection is the Movshev bijection [Mov] between gauge classes of minimal twists for $\mathbb{C}[G]$ and cohomology classes of non-degenerate 2-cocycles on $G$ with values in $\mathbb{C}^\times$.

Finally, we apply the results of invariance of cosemisimplicity under twisting to the problem of classification of twists in group algebras, and of finite-dimensional cosemisimple triangular Hopf algebras in any characteristic (over algebraically closed fields), which generalizes the previous results of Movshev [Mov] and the second and third authors [EG1] in characteristic zero. The answer turns out to be very simple: the classification is the same as in characteristic zero, except that the subgroup $K$ with an irreducible projective representation of dimension $|K|^{1/2}$ has to be of order coprime to the characteristic. In particular, this gives a Hopf algebraic proof of the known fact that a group of central type in characteristic $p$ has order coprime to $p$.

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We are most grateful to the referee, for the explanation how to use the Miyashita-Ulbrich actions and the results proved in the papers [DT1, DT2, Sc, U] , to strengthen several statements in Section 5; in particular to drop the semisimplicity assumption from Theorem 2.7 and to show that $D_H \circ D_H = \text{id}$.

2. Preliminaries

Throughout this paper $k$ denotes an algebraically closed field of an arbitrary characteristic. We use Sweedler’s notation for the comultiplication in a coalgebra: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, where summation is understood.

Let $H$ be a Hopf algebra (see [M, Sw] for the definition) over $k$ with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. The following notion of a twisting deformation of $H$ is due to V. Drinfeld [D].

Definition 2.1. A twist for $H$ is an invertible element $J \in H \otimes H$ that satisfies

\[(\Delta \otimes \text{id})(J)(J \otimes 1) = (\text{id} \otimes \Delta)(J)(1 \otimes J).\]

Remark 2.2. Applying $\varepsilon$ to (1) one sees that $c = (\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J)$ is a non-zero scalar for any twist $J$, cf. [DT1, p. 813]. In particular, one can always
replace \( J \) by \( e^{-1}J \) to normalize it in such a way that
\[
(\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1.
\]
We will always assume that \( J \) is normalized in this way.

We use a convenient shorthand notation for \( J \) and \( J^{-1} \), writing \( J = J^{(1)} \otimes J^{(2)} \) and \( J^{-1} = J^{-1 \text{ (1)}} \otimes J^{-1 \text{ (2)}} \), where of course a summation is understood. We also write \( J_{21} \) for \( J^{(2)} \otimes J^{(1)} \) etc.

**Remark 2.3.** Let \( x \in H \) be an invertible element such that \( \varepsilon(x) = 1 \). If \( J \) is a twist for \( H \) then so is \( J^x := \Delta(x)J(x^{-1} \otimes x^{-1}) \). The twists \( J \) and \( J^x \) are said to be **gauge equivalent**.

Given a twist \( J \) for \( H \) one can define a new Hopf algebra \( H^J \) with the same algebra structure and counit as \( H \), for which the comultiplication and antipode are given by
\[
\Delta^J(h) = J^{-1}\Delta(h)J,
\]
\[
S^J(h) = Q_J^{-1}S(h)Q_J, \quad \text{for all } h \in H,
\]
where \( Q_J = S(J^{(1)})J^{(2)} \) is an invertible element of \( H \) with the inverse \( Q_J^{-1} = J^{-1 \text{ (1)}}S(J^{-1 \text{ (2)}}) \).

The element \( Q_J \) satisfies the following useful identity (cf. [Ma, (2.17)]) :
\[
\Delta(Q_J) = (S \otimes S)(J_{21}^{-1})(Q_J \otimes Q_J)J^{-1}.
\]
In particular, we have,
\[
\Delta(Q_J^{-1}S(Q_J)) = J(Q_J^{-1}S(Q_J) \otimes Q_J^{-1}S(Q_J))(S^2 \otimes S^2)(J^{-1}).
\]

Twists for Hopf algebras of finite groups were studied in [Mov] and [EG] in the case when \( \text{char}(k) \) is prime to the order of the group. Such twists were classified up to a gauge equivalence and were used in [EG] to classify all semisimple cosemi-simple triangular Hopf algebras over \( k \). We refer the reader to the survey [G2] for a review of the recent developments in the classification of triangular Hopf algebras.

In subsequent sections we will use the following lemma which is a direct consequence of the twist equation (1).

**Lemma 2.4.** Let \( J \) be a twist for \( H \). Then
\[
S(J^{(1)})J^{(2)}(1) \otimes J^{(2)}(2) = (Q_J \otimes 1)J^{-1},
\]
\[
S(J^{(1)}(1)) \otimes S(J^{(1)}(2))J^{(2)} = (S \otimes S)(J^{-1})(1 \otimes Q_J)
\]
\[
J^{-1}(1) \otimes J^{-1}(2)S(J^{-2}) = J(1 \otimes Q_J^{-1}),
\]
\[
J^{-1}(1)S(J^{-2}(1)) \otimes S(J^{-2}(2)) = (Q_J^{-1} \otimes 1)(S \otimes S)(J).
\]

3. **Twisting of unimodular Hopf algebras**

Let \( H \) be a unimodular Hopf algebra and let \( J \) be a twist for \( H \). Let \( \lambda \in H^* \) (respectively, \( \rho \in H^* \)) be a non-zero left (respectively, right) integral on \( H \), i.e.,
\[
h_{(1)}(\lambda, h_{(2)}) = \langle \lambda, h \rangle 1, \quad \langle \rho, h_{(1)} \rangle h_{(2)} = \langle \rho, h \rangle 1, \quad \text{for all } h \in H.
\]

**Remark 3.1.** Note that in the case of a finite-dimensional \( H \), by [R, Theorem 3], for all \( g, h \in H \) we have
\[
\langle \lambda, gh \rangle = \langle \lambda, hS^2(g) \rangle, \quad \langle \rho, gh \rangle = \langle \rho, S^2(h)g \rangle.
\]
\textbf{Remark 3.2.} Let $H$ be an arbitrary Hopf algebra and let $\lambda$ and $\rho$ be as above. We will use the following invariance properties of integrals on a Hopf algebra, which are straightforward to check:

\begin{align}
(7) \quad g(1)\langle \lambda, hg(2) \rangle &= S(h(1))\langle \lambda, h(2)g \rangle, \\
(8) \quad \langle \rho, g(1)h \rangle g(2) &= \langle \rho, gh(1) \rangle S(h(2)),
\end{align}

for all $g, h \in H$.

\textbf{Remark 3.3.} Let $H$ be an arbitrary finite-dimensional Hopf algebra and let $\lambda$ be as above. Recall that $H$ is a Frobenius algebra with non-degenerate bilinear form $H \otimes H \rightarrow k$ given by $h \otimes g \mapsto \langle \lambda, hg \rangle$ (cf. \cite{M}, \cite{Sw}).

Our first goal is to describe the integrals on the twisted Hopf algebra $H^J$ in the case when $H$ is finite-dimensional.

Recall that $H$ acts on $H^*$ on the left via $h \mapsto \phi = \phi(1)(\phi(2), h)$ and on the right via $\phi \mapsto h = (\phi(1), \phi(2))$.

Let us denote $u_j := Q^{-1}_j S(Q_j)$.

\textbf{Theorem 3.4.} Let $H$ be a finite-dimensional unimodular Hopf algebra and let $J$ be a twist for $H$. Let $\lambda$ and $\rho$ be, respectively, non-zero left and right integrals on $H$. Then the elements $\lambda_J := u_j \mapsto \lambda$ and $\rho_J := \rho \mapsto u_j^{-1}$ are, respectively, non-zero left and right integrals on $H^J$.

\textbf{Proof.} We have

\begin{equation}
\Delta(u_j) = J(u_j \otimes u_j)(S^2 \otimes S^2)(J^{-1}).
\end{equation}

by equation (8). We need to check that $S(h(1)\langle \lambda_J, h(2) \rangle) = \langle \lambda_J, h \rangle 1$ for all $h \in H^J$, where $\Delta^J(h) = h(1) \otimes h(2)$. To this end, we compute, using Remark 3.1 and equation (9):

\begin{align*}
S(h(1)\langle \lambda_J, h(2) \rangle) &= S(J^{-1}h(1)J(1))\langle \lambda, J^{-2}h(2)J(2)u_j \rangle \\
&= S(J(1))S(h(1))S(J^{-1}(1))\langle \lambda, h(2)J(2)u_j S^2(J^{-2}) \rangle \\
&= S(J(1))J(2)u_j S^2(J^{-2}) S(J^{-1}) \\
&\times \langle \lambda, h J(2)u_j S^2(J^{-2}) \rangle.
\end{align*}

Since $\lambda$ is non-degenerate, the identity in question is equivalent to

\begin{equation}
1 \otimes u_j = (S(J(1))J(2)u_j S^2(J^{-2}) S(J^{-1}) \otimes S^2(J^{-2})),
\end{equation}

which reduces to equation (\ref{u2}) by Lemma 2.4.

The proof of the statement regarding $\rho$ is completely similar. \hfill \Box

\textbf{Remark 3.5.} Suppose that $H$ is cocommutative. Then it is easy to show that $u_j$ coincides with the Drinfeld element $u$ of the triangular Hopf algebra $(H^J, J^{-1}J)$. This motivated the notation $u_j$.

The next Corollary shows that the class of cosemisimple unimodular Hopf algebras over $k$ is closed under twisting. This result is true regardless of the characteristic of $k$ (in characteristic 0 it easily follows from the result of Larson and Radford \cite{LR}).
Corollary 3.6. If $H$ is a cosemisimple unimodular Hopf algebra, then so is its twisting deformation $H^J$.

Proof. Clearly, $H^J$ is unimodular, since twisting preserves integrals in $H$. Cosemisimplicity of $H^J$ is equivalent to $\langle \lambda_J, 1 \rangle \neq 0$ by Maschke’s theorem. By Theorem 3.4, we have

$$\langle \lambda_J, 1 \rangle = \langle \lambda, Q^{-1}_J S(Q_J) \rangle = \langle \lambda, J^{-1}(J^{-2}) S(J^{2}) S^2(J^{1}) \rangle = \langle \lambda, J^{1}(J^{-1}) S(J^{-2}) S(J^{2}) \rangle = \langle \lambda, 1 \rangle \neq 0$$

where we used Remark 3.1.

Remark 3.7. In [G1, Theorem 1.3.6] it was shown that a cosemisimple quasitriangular Hopf algebra is automatically unimodular. Since the property of being quasitriangular is preserved under a twisting deformation, it follows from Corollary 3.6 that the class of cosemisimple quasitriangular Hopf algebras over a field of an arbitrary characteristic is closed under twisting.

Remark 3.8. Suppose that $H$ is a finite-dimensional cosemisimple involutive Hopf algebra, i.e., $S^2 = \text{id}$. Then $u_J = Q^{-1}_J S(Q_J)$ is a grouplike element in $H^J$ by Equation (6). Thus, we have that $\lambda(1) = \lambda_J(u_J^{-1}) \neq 0$ (where $\lambda_J$ was defined in Theorem 3.4) if and only if $u_J = 1$, i.e., if and only if $S(Q_J) = Q_J$. This means that $(S^J)^2 = \text{id}$, i.e., the class of finite-dimensional cosemisimple involutive Hopf algebras over a field of an arbitrary characteristic is closed under twisting.

Remark 3.9. We expect that the class of finite-dimensional cosemisimple Hopf algebras is closed under twisting. This would follow from (a weak form of) a conjecture of Kaplansky, saying that a cosemisimple finite-dimensional Hopf algebra is unimodular.

Below we prove a somewhat more general statement regarding the cosemisimplicity of the coalgebra obtained by deforming $H$ by means of two different twists, see Theorem 3.13 below. This result will be used in later sections.

Recall [P] that an algebra $A$ is said to be separable if there exists an element $e = e^{(1)} \otimes e^{(2)} \in A \otimes A$, where a summation is understood, such that

$$ae^{(1)} \otimes e^{(2)} = e^{(1)} \otimes e^{(2)} a \quad \text{for all } a \in A \quad \text{and} \quad e^{(1)}e^{(2)} = 1.$$  

Such an element $e$ is called a separability element for $A$. A separable algebra over a perfect field is always finite-dimensional and semisimple [P, 10.2]. For Hopf algebras the notions of separability and semisimplicity coincide; moreover, a semisimple Hopf algebra is automatically finite-dimensional.

Passing from algebras to coalgebras, one can define a notion dual to separability as follows, cf. [I].

Definition 3.10. A coalgebra $C$ is said to be coseparable if there exists a bilinear form $\psi : C \otimes C \rightarrow k$ such that

$$c(1)\psi(c(2) \otimes d) = \psi(c \otimes d(1))d(2) \quad \text{for all } c,d \in C \quad \text{and} \quad \psi(\Delta(c)) = \varepsilon(c).$$

Such a form $\psi$ is called a coseparability pairing for $C$. 
Remark 3.11. For a coalgebra $C$ over an algebraically closed field the notions of coseparability and cosemisimplicity are equivalent.

Let $J, L$ be twists for $H$, then

$$\hat{\Delta}(h) := L^{-1}\Delta(h)J, \quad h \in H$$

defines a new coassociative comultiplication in $H$ for which $\varepsilon$ is still a counit. We will denote this twisted coalgebra by $H^{(L,J)}$. The coalgebra $H^{(L,J)}$ is called ($H^{(L,J)}$)-biGalois (resp. ($H^{(L,J)}$)-biGalois) coalgebra. One sees that $H^{(L,J)} = H^{(L,1)} \otimes_H H^{(1,J)}$.

Remark 3.12. The two-sided twist $(12)$ is closely related to biGalois extensions $(7)$. The coalgebra $H^{(1,J)}$ (resp. $H^{(L,J)}$, resp. $H^{(L,J)}$) is called ($H^{(L,J)}$)-biGalois (resp. ($H^{(L,J)}$)-biGalois, resp. ($H^{(L,J)}$)-biGalois) coalgebra. One sees that $H^{(L,J)} = H^{(L,1)} \otimes_H H^{(1,J)}$.

Theorem 3.13. Let $H$ be a finite-dimensional cosemisimple unimodular Hopf algebra. Then $H^{(L,J)}$ is a cosemisimple coalgebra.

Remark 3.14. This theorem is a generalization of Corollary 3.6. Namely, it becomes Corollary 3.6 if $L = J$.

Proof. Let $V := S(Q_L)$ and $W := Q_J^{-1}$. We will show that

$$\psi(g \otimes h) = \langle \lambda, hWS(g)V \rangle, \quad h, g \in H^{(L,J)},$$

is a coseparability pairing for $H^{(L,J)}$. To this end we need to check that $\psi$ satisfies the conditions of Equation (11).

First, we check that $\psi(\hat{\Delta}(h)) = \varepsilon(h)$ for all $h \in H$:

$$\psi(\hat{\Delta}(h)) = \langle \lambda, L^{-1}h(1)J^{(1)}J^{-1(1)}S(\lambda^{-2})S(L^{-2}h(2)J^{(2)})S(L^{2})S^{2}(L^{1}) \rangle$$

$$= \varepsilon(h)\langle \lambda, L^{-1}S^{2}(L^{2})S(L^{1}) \rangle$$

$$= \varepsilon(h),$$

by Remark 3.1.

Next, we compute, for all $h, g \in H$:

$$\psi(h \otimes g(1))g(2) = \langle \lambda, hWS(g(1))V \rangle g(2)$$

$$= \langle \lambda, hWS(L^{-1}g(1))J^{(1)}V \rangle L^{-2}(g(2))J^{(2)}$$

$$= \langle \lambda, hWS(J^{(1)})S(g(1))S(L^{-1})V \rangle L^{-2}(g(2))J^{(2)},$$

and also

$$h(1)\psi(h(2) \otimes g) = L^{-1}h(1)J^{(1)\langle \lambda, L^{-2}(h(2))J^{(2)}WS(g)V \rangle}$$

$$= L^{-1}h(1)J^{(1)\langle \lambda, h(2)J^{(2)}WS(g)V S^{2}(L^{-2}) \rangle}$$

$$= L^{-1}S^{-1}([J^{(2)}WS(g)V S^{2}(L^{-2})])_{(1)}J^{(1)}$$

$$\times \langle \lambda, h(J^{(2)})WS(g)V S^{2}(L^{-2}) \rangle_{(2)}$$

$$= \langle \lambda, h(J^{(2)})W(2)S(g(1))V(2)S^{2}(L^{-2}) \rangle$$

$$\times L^{-1}S(L^{-2})_{(1)}S^{-1}(V(1))g(2)S^{-1}(W(1))S^{-1}(J^{(2)})_{(1)}J^{(1)},$$

where we used Equation (12) and Remark 3.1.
Comparing the results of the above computations and using the non-degeneracy of \( \lambda \), we conclude that the following equations
\[
WS(J^{(1)}) \otimes J^{(2)} = J^{(2)}(2)W(2) \otimes S^{-1}(W(1))S^{-1}(J^{(2)}(1))J^{(1)} 
\]
(14)
\[
S(L^{-1}(1)V \otimes L^{-2}(1)) = V(2)S^{2}(L^{-2}(2)) \otimes L^{-1}(1)S(L^{-2}(1))S^{-1}(V(1))
\]
imply the identity
\[
\psi(h \otimes g(\tilde{g}))g(\tilde{g}) = h(\tilde{g})\psi(h(\tilde{g}) \otimes g).
\]
(15)
From Lemma 2.4 we see that equations (14) are equivalent to
\[
\Delta(W) = J(W \otimes W)(S \otimes S)(J_{21})
\]
(16)
\[
\Delta(S^{-1}(V^{-1})) = L^{-1}(S^{-1}(V^{-1}) \otimes S^{-1}(V^{-1}))(S \otimes S)(L_{21}^{-1}).
\]
These, in turn, are equivalent to Equation (8), therefore the proof is complete. \( \square \)

**Remark 3.15.** Suppose that \( L = 1 \). Then it is clear from the proof of Theorem 3.13 that the unimodularity assumption can be dropped. In this special case, Theorem 3.13 becomes a known result of Blattner and Montgomery [M, Theorem 7.4.2, part 2], in the case when the algebra \( A \) is one-dimensional.

**Remark 3.16.** It is clear that \( H^{(L,J)} = (H^{L,J})^{(L^{-1}J)} \). Thus, another proof of Theorem 3.13 can be obtained by combining Corollary 3.6 with [M, Theorem 7.4.2, part 2].

**Remark 3.17.** In the case when \( k = \mathbb{C} \), the field of complex numbers, and \( H = \mathbb{C}[G] \), the group algebra of a finite group \( G \), the algebraic structure of \( (H^{J})^{*} \) was described in [EG2, Theorem 3.2]. Below we give a similar description of the algebraic structure of \( (H^{(L,J)})^{*} \). Obviously, this structure depends only on gauge equivalence classes of \( L, J \).

Recall [EG1, Mox] that for any twist \( J \) of \( H = \mathbb{C}[G] \) there exists a subgroup \( K \subset G \) and a twist \( J^{*} \in \mathbb{C}[K] \otimes \mathbb{C}[K] \) which is minimal for \( \mathbb{C}[K] \) and is gauge equivalent to \( J \). Such a twist defines a projective representation \( V \) of \( K \) with \( \dim(V) = |K|^{1/2} \).

Let \( L, J \) be twists of this type and \( K_{L}, K_{J} \subset G \) be the corresponding subgroups for which they are minimal. We have \( L \in \mathbb{C}[K_{L}] \otimes \mathbb{C}[K_{L}] \) and \( J \in \mathbb{C}[K_{J}] \otimes \mathbb{C}[K_{J}] \).

Let \( V_{L} \) and \( V_{J} \) be the corresponding projective representations. Let \( Z \) be a \( (K_{L}, K_{J}) \) double coset in \( G \) and let \( g \in Z \). Let \( M_{g} := K_{L} \cap gK_{J}g^{-1} \) and define embeddings \( \theta_{L} : M_{g} \to K_{L} \) and \( \theta_{J} : M_{g} \to K_{J} \) by setting \( \theta_{L}(a) = a \) and \( \theta_{J}(a) = g^{-1}ag \) for all \( a \in M_{g} \). Denote by \( W_{L} \) (respectively, \( W_{J} \)) the pullback of the projective representation \( V_{L} \) (respectively, \( V_{J} \)) to \( M_{g} \) by means of \( \theta_{L} \) (respectively, \( \theta_{J} \)).

Let \( H_{Z}^{*} := \oplus_{g \in Z} \mathbb{C}\delta_{g} \subseteq H^{*} \), where delta-functions \( \{\delta_{g}\}_{g \in G} \) form a basis of \( H^{*} \).

Then \( H_{Z}^{*} \) is a subalgebra of \( (H^{(L,J)})^{*} \) and \( (H^{(L,J)})^{*} = \oplus_{Z} H_{Z}^{*} \) as algebras.

Thus, to find the algebraic structure of \( (H^{(L,J)})^{*} \) it suffices to find the algebra structure of each \( H_{Z}^{*} \). The following result is a straightforward generalization of [EG2, Theorem 3.2].

**Theorem 3.18.** Let \( W_{L}, W_{J} \) be as above and let \( \hat{M}_{g}, \hat{W} \) be any linearization of the projective representation \( W := W_{L} \otimes W_{J} \) of \( M_{g} \). Let \( \zeta \) be the kernel of the projection \( \hat{M}_{g} \to M_{g} \), and let \( \chi : \zeta \to \mathbb{C}^{\times} \) be the character by which \( \zeta \) acts in \( W \). Then
there exists a one-to-one correspondence between isomorphism classes of irreducible representations of $H_2^*$ and isomorphism classes of irreducible representations of $\tilde{M}_g$ with $\zeta$ acting by $\chi$. If a representation $Y$ of $H_2^*$ corresponds to a representation $X$ of $\tilde{M}_g$ then

$$\dim(Y) = \sqrt{\frac{|K_L||K_J|}{|M_g|}} \dim(X).$$

Proof. The proof is analogous to [EG2, Section 4], where we refer the reader for details. Note that $|K_L|$ and $|K_J|$ are full squares, because of the minimality of the twists $L$ and $J$, so the right hand side of Equation (17) is an integer.

4. The $H$-module coalgebra associated to a twist

Let $H$ be a Hopf algebra. Recall, that a left $H$-module coalgebra $(C, \Delta_C, \varepsilon_C)$ is a coalgebra which is also a left $H$-module via $h \otimes c \mapsto h \cdot c$, such that

$$\Delta_C(h \cdot c) = \Delta(h) \cdot \Delta_C(c), \quad \text{and} \quad \varepsilon_C(h \cdot c) = \varepsilon(h) \varepsilon_C(c),$$

for all $h \in H$ and $c \in C$.

Clearly, if $H$ is finite-dimensional, then $C$ is a left $H$-module coalgebra if and only if $A := C^*$ is a left $H^*$-comodule algebra [M, 4.1.2], i.e., if and only if the algebra $A$ is a left $H^*$-comodule via $a \mapsto \gamma(a)$, such that

$$\gamma(ab) = \gamma(a)\gamma(b) \quad \text{and} \quad \gamma(1_A) = 1 \otimes 1_A$$

for all $a, b \in A$.

It is well known in the theory of Hopf algebras that cleft $H$-extensions [M, 7.2] of algebras are precisely cocycle crossed products with $H$ [M, Theorem 7.2.2]. Since a twist for a finite-dimensional $H$ is naturally a 2-cocycle for $H^*$, it is quite clear that twists for $H$ can be characterized in terms of extensions.

We will need the following definition of a Galois extension, cf. [M, 8.1.1].

**Definition 4.1.** Let $A$ be a left $H$-comodule algebra with the structure map $\gamma : A \to H \otimes A$. Then the $H$-extension of algebras $A^{co-H} \subset A$ is left $H$-Galois if the map $A \otimes A^{co-H} \to H \otimes_k A$ given by $a \otimes b \mapsto (1 \otimes a)\gamma(b)$ is bijective.

Let $H$ be a finite-dimensional Hopf algebra and $J \in H \otimes H$ be a twist for $H$. Then the left regular $H$-module coalgebra $C := H^{(J)} = H$ with comultiplication and counit

$$\Delta_{H^{(J)}}(c) = \Delta(c)J, \quad \varepsilon_{H^{(J)}}(c) = \varepsilon(c), \quad c \in C,$$

is a left $H$-module coalgebra. Moreover, the corresponding $H^*$-extension $k \subset (H^{(J)})^*$ is Galois, since $J$ is invertible.

Let $J$ and $J^x = \Delta(x)J(x^{-1} \otimes x^{-1})$ be gauge equivalent twists, where $x \in H$ is invertible and is such that $\varepsilon(x) = 1$. Then $H^{(J^x)} = H$ with the comultiplication $\Delta_{H^{(J^x)}}(c) = \Delta(c)J^x$ is another left $H$-module coalgebra such that the map $c \mapsto cx$ is an $H$-module coalgebra isomorphism between $H^{(J)}$ and $H^{(J^x)}$.

Conversely, let $C$ be an $H$-module coalgebra and let $i : C \to H$ be an isomorphism of left $H$-modules (where $H$ is viewed as a left regular module over itself), such that the corresponding $H^*$-extension $k \subset C^*$ is Galois. Then one can check by a direct computation that

$$J := (i \otimes i)\Delta_C(i^{-1}(1)) \in H \otimes H$$

is a twist for $H$ and $i : C \cong H^{(J)}$ is an $H$-module coalgebra isomorphism. Here the Galois property above is equivalent to $J$ being invertible.
Proposition 4.2. The above two assignments define a one-to-one correspondence between:
1. gauge equivalence classes of twists for \( H \), and
2. isomorphism classes of \( H \)-module coalgebras \( C \) isomorphic to the regular \( H \)-
module and such that the corresponding \( H^* \)-extension \( k \subset C^* \) is Galois.

Corollary 4.3. If \( H \) is a finite-dimensional cosemisimple Hopf algebra, then any \( H \)-module coalgebra \( C \) isomorphic to the regular \( H \)-module and such that the corresponding \( H^* \)-extension \( k \subset C^* \) is Galois, is cosemisimple.

Proof. This follows from Theorem 3.13 and Remark 3.15.

5. Non-degenerate twists and non-commutative “lowering of indices”

Let \( H \) be a finite-dimensional Hopf algebra over \( k \).

Definition 5.1. We will say that a twist \( J \) for \( H \) is non-degenerate if the corre-
sponding coalgebra \( H(J) \) is simple.

Remark 5.2. If \( H \) is a group algebra over a field of characteristic prime to the or-
der of the group then the notion of a non-degenerate twist coincides with the notion
of a minimal twist from [EG1] (i.e., this property is equivalent to the triangular
Hopf algebra \((H,J,J^{-1})\) being minimal).

Remark 5.3. A Hopf algebra possessing a non-degenerate twist need not be se-
misimple or cosemisimple. For example, in [AEG] it was explained that Swee dler’s
4-dimensional Hopf algebra \( H(t) \) := \( 1 \otimes 1 - (t/2)g x \otimes x \) (where \( g \) is the
group-like element and \( x \) the skew primitive element). It is easy to show that \( J(t) \)
is non-degenerate for \( t \neq 0 \). Indeed, it is easy to prove by a direct calculation that
the coalgebra \( H(J(t)) \) does not have group-like elements. But any 4-dimensional
coalgebra without group-like elements is necessarily simple.

Now consider a twist \( J \in H \otimes H \). Observe that \( H^*(J) := (H(J))^* \) is a right
\( H \)-module algebra in a natural way:
\[
\langle a \cdot h, c \rangle = \langle a, hc \rangle, \quad a \in H^*(J), \ c \in H(J), \ h \in H.
\]
The algebra \( H^*(J) \) is simple if and only if \( J \) is non-degenerate. In this case the
Skolem-Noether theorem for Hopf algebras [Mas] (see also [M, 6.2.4]) says that the
action of \( H \) is inner, i.e., there is a convolution invertible map \( \pi \in \text{Hom}_k(H,H^*(J)) \)
such that
\[
a \cdot h = \pi(h(1))a\pi(h(2)), \quad \text{for all } a \in H^*(J), \ h \in H,
\]
where \( \pi \) is the convolution inverse for \( \pi \). Furthermore, we may (and will) assume
that \( \pi(1) = \pi(1) = 1 \). Such a map \( \pi \) will be called a Skolem-Noether map.

It is easy to check that a Skolem-Noether map is unique up to a gauge trans-
formation, i.e., that two Skolem-Noether maps \( \pi, \pi' \) are linked by the relation
\( \pi'(h) = \eta(h(1))\pi(h(2)), \) where \( \eta \in H^* \) is invertible and \( \eta(1) = 1 \).

Remark 5.4. If \( H \) is the group algebra of a group \( G \) then a Skolem-Noether map
is the same thing as an irreducible projective representation of \( G \). Thus, the theory of
Skolem-Noether maps is a generalization of the theory of projective representations
to Hopf algebras.
Comparing the values of \((a \cdot g) \cdot h\) and \(a \cdot (gh)\), it is easy to deduce from Equation (22) that \(c(h \otimes g) := \pi(h(1)g(1))\pi(g(2))\pi(h(2))\) is a scalar for all \(g, h \in H\). Furthermore, the form \(c \in H^* \otimes H^*\) is an invertible element (by convolution invertibility of \(\pi\)), and it is a 2-cocycle for \(H\), i.e.,

\[
\begin{align*}
(c(f_1)g(1) \otimes h)c(f_2)g(2)) &= c(f \otimes (g(1)h(1))c(g(2) \otimes h(2)), \quad \text{for all } f, g, h \in H.
\end{align*}
\]

Indeed, we have

\[
\begin{align*}
(c(f_1)g(1) \otimes h)c(f_2)g(2)) &= \\
&= \pi(f_1)g(1)h(1))\pi(g(2)h(2))\pi(f_2)g(2))\pi(g(4))\pi(f_4)) \\
&= \pi(f_1)g(1)h(1))\pi(g(2)h(2))\pi(g(2))\pi(f_2), \\
&= \pi(f \otimes (g(1)h(1))c(g(2) \otimes h(2)) = \\
&= \pi(f_1)g(1)h(1))\pi(g(2)h(2))c(g(3) \otimes h(3))\pi(f_3) \\
&= \pi(f_1)g(1)h(1))\pi(g(2)h(2))\pi(g(3)h(3))\pi(g(4))\pi(f_2)) \\
&= \pi(f_1)g(1)h(1))\pi(g(2)h(2))\pi(g(2))\pi(f_2).
\end{align*}
\]

Thus, \(c\) defines a twist for \(H^*\). We will denote this twist by \(D_\pi(J)\) (the dual twist to \(J\), constructed via \(\pi\)).

It is easy to check that while the twist \(D_\pi(J)\) does depend on the choice of \(\pi\), its gauge equivalence class does not. Namely, for a fixed \(J\), the map \(\pi \rightarrow D_\pi(J)\) commutes with gauge transformations in \(H^*\).

Let \(NT(H), NT(H^*)\) be the sets of gauge equivalence classes of non-degenerate twists for \(H, H^*\), respectively. Below we explain that the map \(D_H : J \rightarrow D_\pi(J)\) establishes a duality between \(NT(H)\) and \(NT(H^*)\).

**Remark 5.5.** Of course, both \(NT(H)\) and \(NT(H^*)\) may well be empty. It is clearly so, for example, if the dimension of \(H\) is not a square.

We adopt the viewpoint of Miyashita-Ulbrich actions [DT2], which we recall next. Let \(H\) be a finite-dimensional Hopf algebra and \(R\) be a simple algebra. Suppose there is a right \(H\)-Galois coaction on \(R\), \(y \mapsto y(0) \otimes y(1)\), by which we mean an \(H\)-coaction making \(R\) right \(H\)-Galois over \(k\) (i.e. \(R\) is an \(H\)-comodule algebra and \(R_{coH} = k\)). Then we have the right \(H\)-action (the Miyashita-Ulbrich action) \(x \cdot h, x \in R, h \in H\) characterized by

\[
xy = y(0)(x \cdot y(1)), \quad x, y \in R,
\]

see [DT2, Theorem 3.4(i)]. The construction of the left Miyashita-Ulbrich action from a left \(H\)-Galois coaction on \(R\) is completely similar.

**Remark 5.6.** The Miyashita-Ulbrich actions can be defined in a more general situation, when \(k\) is an arbitrary (not necessarily algebraically closed) field and \(R\) is an Azumaya (or finite-dimensional central simple) algebra [DT2].

The map \(D_H\) can be clearly explained in terms of the Miyashita-Ulbrich actions as follows. Let \(\cdot\) be the corresponding right Miyashita-Ulbrich action of \(H\) \([24]\). The left \(H^*\)-coaction \(x \mapsto x(-1) \otimes x(0)\) \(x \in R\) adjoint to this action is a left \(H^*\)-Galois coaction on \(R\) by [1], Satz 2.8, [DT2, Theorem 5.1]. The left Miyashita-Ulbrich action \(t \triangleright x (t \in H^*, x \in R\) arising from the last \(H^*\)-coaction coincide with the one that is adjoint to the original \(H\)-coaction, since we have

\[
y(0)x(-1), y(1)x(0) = y(0)(x \cdot y(1)) = xy = (x(-1) \triangleright y)x(0),
\]
and hence, $y_0(t, y(1)) = t \triangleright y$ for all $t \in H^*$ and $y \in R$, since $x \mapsto x_{(-1)} \otimes x(0)$ is Galois, see \cite{DT2} (3.1)]]. Thus we have a natural 1-1 correspondence between the right $H$-Galois coactions on $R$ and the left $H^*$-Galois coactions on $R$. This can be regarded as a very special case of \cite{DT2} Theorem 6.20]. As is easily seen, this induces a 1-1 correspondence between the sets of respective equivalence classes of coactions, where two right coactions are equivalent if the corresponding comodule algebras are isomorphic. By Proposition 4.2, this gives a bijective map between $NT(H)$ and $NT(H^*)$.

**Theorem 5.7.** The map $D_H : NT(H) \to NT(H^*)$ is a bijection and $D_H^{-1} = D_{H^*}$.

**Proof.** Write $R = H^*_R(j)$. The map $\pi : H \to R$ is left $H^*$-linear, where $H^*$ acts on $R$ by the Miyashita-Ullrich action, since we have

\[
(x_{(-1)} \triangleright \pi(h))x(0) = x\pi(h) = \pi(h_{(1)})(x \cdot h_{(2)}) = \pi(h_{(1)})(x_{(-1)} \otimes h_{(2)})x(0) = \pi(x_{(-1)} \triangleright h)x(0), \quad h \in H, x \in R.
\]

It follows from [DT1, Theorems 9,11] that $R$ is cleft right $H$-Galois with coinvariants $k$ (by counting dimension), and

\[
\pi : H_{(D_H(j))} \to R
\]

is an isomorphism of right $H^*$-comodule algebras. From formula (20) we see that the bijective map between $NT(H)$ and $NT(H^*)$ mentioned above coincides with $D_H$. It also follows from the construction of this map that $D_H^{-1} = D_{H^*}$. \hfill \square

**Remark 5.8.** When $H$ is semisimple and cosemisimple, the set $NT(H)$ is finite. This follows from the result of H.-J. Schneider \cite{Sel}, stating that the number of isomorphism classes of semisimple $H$-comodule algebras of fixed dimension is finite.

**Example 5.9.** Let $F$ be a finite field of characteristic $p \neq 2$, and $\psi : F \to \mathbb{C}^\times$ a nontrivial additive character (for instance, $F = \mathbb{Z}/p\mathbb{Z}$, $\psi(n) = e^{2\pi i n/p}$). Let $V$ be a finite-dimensional vector space over $F$. Suppose that $H = \mathbb{C}[V]$. In this case $NT(H)$ can be identified with the set of non-degenerate alternating bilinear forms $\omega$ on $V^*$, by the assignment $\omega \mapsto J_\omega$, where $J_\omega$ is defined by

\[
((\psi \circ \chi_1) \otimes (\psi \circ \chi_2))(J_\omega) = \psi \left( \frac{1}{2} \omega(\chi_1, \chi_2) \right), \quad \chi_i \in V^*.
\]

On the other hand, $H^*$ is isomorphic to $\mathbb{C}[V^*]$, via $\chi \mapsto \psi \circ \chi \in H^*$. Therefore, $NT(H^*)$ is identified with the set of non-degenerate alternating bilinear forms on $V$. Thus, it is easy to guess the canonical map $D_H : NT(H) \to NT(H^*)$ in this case. Namely, it is not hard to check that this map is the usual inversion of a non-degenerate form, which in classical tensor calculus is often called the “lowering of indices” \cite{DFN}. This motivated the title of this subsection.

**Example 5.10.** This example is a generalization of the previous one. Now assume that $H = k[G]$, where $G$ is a finite group of order coprime to $\text{char}(k)$. In this case, $NT(H^*)$ is the set of cohomology classes of 2-cocycles on $G$ with values in $\mathbb{C}^\times$, which are non-degenerate in the sense of \cite{Mov}. Furthermore, the bijection $D_H$ in this case is nothing but the bijection from \cite{Mov} between such cohomology classes and classes of non-degenerate twists for $k[G]$, which plays a central role in the group theoretic description of twists in $k[G]$. Thus, the theory of this subsection is a generalization of the theory of \cite{Mov} and \cite{EG1} to the non-cocommutative case.
Example 5.11. Let us give an example of a non-degenerate twist for a Hopf algebra which does not come from group theory. Let $A$ be a finite-dimensional Hopf algebra and $\{a_i\}$ and $\{\phi_i\}$ be dual bases in $A$ and $A^*$ respectively. Then it is straightforward to check that

$$J := \sum_i (\phi_i \otimes 1) \otimes (\varepsilon \otimes a_i)$$

is a twist for the Hopf algebra $H := A^{*op} \otimes A$. Then the twisted Hopf algebra is $H^J \cong D(A)^{*op}$, the (opposite) dual of the Drinfeld double of $A$. Furthermore, it was proved in [Lu, Theorem 6.1] that $(H^J)^* \cong A^*A$, where $A$ acts on its dual via $\rightarrow$. This algebra $A^*A$ is called the Heisenberg double in the literature, and is known to be simple [M, Corollary 9.4.3]. Thus, $J$ is a non-degenerate twist for $H$.

Remark 5.12. Hopf algebras with a non-empty set of non-degenerate twists may be called "quantum groups of central type" by analogy with [EG1]. It would be interesting to find more examples of such Hopf algebras.

6. The classification of finite-dimensional triangular cosemisimple Hopf algebras in positive characteristic

The classification of isomorphism classes of finite-dimensional triangular cosemisimple Hopf algebras over $\mathbb{C}$ was obtained in [EG1] (since by [LR], such Hopf algebras are also semisimple). Namely, in [EG1] they were put in bijection with certain quadruples of group-theoretical data. Our next goal is to classify finite-dimensional triangular cosemisimple Hopf algebras over a field $k$ of characteristic $p > 0$ in similar terms.

We start by defining the class of quadruples of group-theoretical data which is suitable in the positive characteristic case.

Definition 6.1. Let $k$ be a field of characteristic $p > 0$. A quadruple $(G, K, V, u)$ is called a triangular quadruple (over $k$) if $G$ is a finite group, $K$ is a $p'$-subgroup of $G$ (i.e., a subgroup of order coprime to $p$), $V$ is an irreducible projective representation of $K$ of dimension $\sqrt{|K|}$ over $k$, and $u \in G$ is a central element of order less or equal than $\min(2, p-1)$ (so if $p = 2$ then $u = 1$).

The notion of an isomorphism between triangular quadruples is clear. Given a triangular quadruple $(G, K, V, u)$, one can construct a triangular cosemisimple Hopf algebra $H(G, K, V, u)$ in the following way. As in [EG1], first construct the twist $J(V)$ of $k[K]$ corresponding to $V$ (well defined up to gauge transformations), and then define the triangular Hopf algebra

$$H(G, K, V, u) := (k[G]^{J(V)}, J(V)^{-1}J(V)R_u),$$

where $R_u = 1$ if $u = 1$ and

$$R_u := \frac{1}{2}(1 \otimes 1 + 1 \otimes u + u \otimes 1 - u \otimes u)$$

if $u$ is of order 2 (note that the 1/2 makes sense, since by the definition, if $u$ has order 2 then $p$ is odd). As a Hopf algebra, $H(G, K, V, u)$ is isomorphic to $k[G]^{J(V)}$, and therefore it is cosemisimple by Corollary 3.4 (as $k[G]$ is obviously unimodular).

Conversely, let $(H, R)$ be a finite-dimensional triangular cosemisimple Hopf algebra over $k$. Let $(H_{\min}, R)$ be the minimal triangular sub Hopf algebra of $(H, R)$.
Since $H_{\text{min}} \cong H_{\text{min}}^{\text{top}}$ as Hopf algebras, $H_{\text{min}}$ is cosemisimple and semisimple. Hence by [EG1], there exists a finite $p'$-group $K$ and a twist $J$ for $K$ such that

$$(H_{\text{min}}, RR_u) \cong (k[K]^J, J_{21}^{-1} J),$$

where $u \in H_{\text{min}}$ is a central group-like element satisfying $u^2 = 1$, and $R_u$ is as above. Note that $u$ is the Drinfeld element of $H_{\text{min}}$, hence of $H$ as well.

Consider now the triangular Hopf algebra $(H, R)^J = (H^J, R^J)$, and set $B := H^J$. By [G1, Theorem 1.3.6], $H$ is unimodular and hence by Corollary 3.6, $B$ is cosemisimple.

Our next goal is to show that $u$ is central in $H$. If $u = 1$, there is nothing to prove. Thus, we may assume that $u$ has order 2, which in particular implies that $p$ is odd (since the order of $u$ divides the dimension of $H_{\text{min}}$, while $p$ does not divide this dimension, as $H_{\text{min}}$ is semisimple and cosemisimple).

Let $B$ be the cocommutative Hopf superalgebra corresponding to $(B,u)$ by [AEG, Corollary 3.3.3] (this corollary was proved for characteristic zero, but applies verbatim in odd characteristic). In particular, $u \in B$ acts by parity.

**Lemma 6.2.** The Hopf superalgebra $B$ is cosemisimple.

**Proof.** Form the biproduct $\overline{B} := k[\mathbb{Z}_2] \ltimes B$ where the generator $g$ of $\mathbb{Z}_2$ acts on $B$ by parity; it is an ordinary Hopf algebra (see [AEG]). Consider the algebra $\overline{B}^*$. On one hand, $\overline{B}^* = k[\mathbb{Z}_2] \ltimes B^*$ as an algebra, and on the other hand, $\overline{B}^* = k[\mathbb{Z}_2] \otimes B^*$ as an algebra. Since $B$ is cosemisimple, the result follows. \[\square\]

Since $B$ is cocommutative and cosemisimple, it is a group algebra $k[G]$ of some finite group $G$. In particular, $B$ is purely even, and hence $u$ is central in $B$ and hence central in $H$ as well.

We conclude that $(H, R)^J = (H^J, R_u) = (k[G], R_u)$. But since $k[K]$ is embedded in $k[G]$ as a Hopf algebra, $K$ is a subgroup of $G$. We thus assigned to $(H, R)$ a triangular quadruple $(G, K, V, u)$.

In fact we have the following theorem.

**Theorem 6.3.** Let $k$ be a field of characteristic $p > 0$. The above two assignments define a one to one correspondence between:

1. isomorphism classes of finite-dimensional triangular cosemisimple Hopf algebras over $k$, and
2. isomorphism classes of triangular quadruples $(G, K, V, u)$ over $k$.

**Proof.** It remains to show that the two assignments are inverse to each other. Indeed, this follows from the results of [EG1, Section 5]. \[\square\]

**Remark 6.4.** This theorem and Remark 3.8 imply that in a finite-dimensional cosemisimple triangular Hopf algebra, one has $S^2 = \text{id}$. So Kaplansky’s conjecture that in a semisimple Hopf algebra, one has $S^2 = \text{id}$, is valid in the triangular case.

As a corollary we are now able to classify twists for finite groups in positive characteristic.

**Theorem 6.5.** Let $k$ be a field of characteristic $p > 0$. Let $G$ be a finite group. There is a one to one correspondence between:

1. gauge equivalence classes of twists for $k[G]$, and
2. pairs \((K, V)\) where \(K\) is a \(p'\) subgroup of \(G\) and \(V\) is an irreducible projective representation of \(K\) of dimension \(\sqrt{|K|}\) over \(k\), modulo inner automorphisms of \(G\).

Proof. Follows from Theorem 6.3 since for any twist \(J\) for \(k[G]\), \((k[G]^J, J_2^{-1} J)\) is cosemisimple triangular by Corollary 3.6.

Remark 6.6. Theorem 6.5 implies that the representation theory of \((k[G]^J)^*\) for twists \(J \in k[G] \otimes k[G]\) is described exactly as in [EG2].

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