Sasaki–Einstein 7-manifolds and Orlik’s conjecture

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Abstract

We study the homology groups of certain 2-connected 7-manifolds admitting quasi-regular Sasaki–Einstein metrics, among them, we found 52 new examples of Sasaki–Einstein rational homology 7-spheres, extending the list given by Boyer et al. (Ann Inst Fourier 52(5):1569–1584, 2002). As a consequence, we exhibit new families of positive Sasakian homotopy 9-spheres given as cyclic branched covers, determine their diffeomorphism types and find out which elements do not admit extremal Sasaki metrics. We also improve previous results given by Boyer (Note Mat 28:63–105, 2008) showing new examples of Sasaki–Einstein 2-connected 7-manifolds homeomorphic to connected sums of $S^3 \times S^4$. Actually, we show that manifolds of the form $\#^k (S^3 \times S^4)$ admit Sasaki–Einstein metrics for 22 different values of $k$. All these links arise as Thom–Sebastiani sums of chain type singularities and cycle type singularities where Orlik’s conjecture holds due to a recent result by Hertling and Mase (J Algebra Number Theory 16(4):955–1024, 2022).

Keywords  Links of weighted hypersurfaces · Orlik’s conjecture · Rational homology 7-spheres · Sasaki–Einstein metrics

Mathematics Subject Classification 53C25 · 57R60

1 Introduction

In the last 20 years, several techniques have been developed to determine the class of manifolds that admit metrics of positive Ricci curvature. For odd dimensions, Boyer, Galicki and collaborators established the abundance of positive Einstein metrics, actually Sasaki–Einstein metrics. In dimension 5, there are quite remarkable results (see [7, 16, 21]) that lead us to think it is conceivable to give a complete classification there. For dimension 7, there are several important results, in particular 7-manifolds arising as links of hypersurface singularities were studied intensively using as framework the seminal work of Milnor in [22] where
$S^3$-bundles over $S^4$ with structure group $SO(4)$ were studied. Using the Gysin sequence for these fiber bundles, one obtains three classes of 7-manifolds determined by the Euler class $e$:

a) If $e = \pm 1$ then the fiber bundle has the same homology of the 7-sphere.
b) If $|e| \geq 2$ then the fiber bundle is a rational homology 7-sphere.
c) If $e = 0$ then the fiber bundle has the homology of $S^3 \times S^4$.

Milnor showed that manifolds situated on the class described in item a) were homeomorphic to $S^7$. Furthermore, he proved that some of these bundles are not diffeomorphic to $S^7$ and as a result of that exhibited the first examples of exotic spheres.

An interesting example of manifolds described in b) is the Stiefel manifold of 2-frames in $\mathbb{R}^5$, $V_2(\mathbb{R}^5)$. It is known that this manifold can be realized as a link of quadric in $\mathbb{C}^5$, and that $V_2(\mathbb{R}^5)$ is a rational homology 7-sphere and moreover admits regular Sasaki–Einstein metric [1, 2]. This example was a source of new techniques that led to establish the existence of many Sasaki–Einstein structures. Actually in [2], a method for proving the existence of Sasaki–Einstein metrics on links of hypersurface singularities is described. In a sequel of papers [4, 7, 8, 11], Boyer, Galicki and their collaborators showed the existence of Sasaki–Einstein structures on exotic spheres, 2-connected rational homology 7-spheres and connected sums of $S^3 \times S^4$, respectively.

In general, the homeomorphism type of links is not easy to be determined. However, for certain cases, one can calculate the integral homology of the links through formulas derived by Milnor and Orlik in [23] and Orlik in [24]. In fact, Boyer exhibited 14 examples [11] of Sasaki–Einstein 7-manifolds arising from links of isolated hypersurface singularities from elements of the well-known list of 95 codimension one K3 surfaces [12, 19]. For these, the third integral homology group is completely determined. In [17], Gomez calculated the torsion for the third integral homology group explicitly for 10 examples of Sasaki–Einstein links of chain type singularities (also known as Orlik polynomials).

In this paper, we benefit from a recent result by Hertling and Mase in [18] where they show that the Orlik conjecture is valid for chain type singularities, cycle type singularities and Thom–Sebastiani sums of them, and we improve results given in [5, 11, 17]. From the list of 1936 Sasaki–Einstein 7-manifolds realized as links from the list given in [5, 20], we detect 1673 that are links of hypersurface singularities of these types. Thus, via Orlik’s algorithm we calculate the third homology group for this lot. Among them, we found 52 new examples of 2-connected rational homology spheres admitting Sasaki–Einstein metrics. We also found 124 new examples of 2-connected Sasaki–Einstein 7-manifolds of the form $#2k(S^3 \times S^4)$, improving a result of Boyer in [11]. Six of these new examples are links of quasi-smooth Fano threefold coming from Reid’s list of 95 weighted codimension 1 K3 surfaces [12, 19], the rest of them are links taken from the list given in [20].

In recent years, Sasaki–Einstein geometry has been intensely studied, under certain circumstances, metrics of this type have real Killing spinors [15] which play an important role in the context of superstring theory and in $M$-theory. Also a string theory conjecture known as the AdS/CFT correspondence relates superconformal field theories and Sasaki–Einstein metrics in dimensions 5 and 7 [27]. And certainly, rational homology spheres can be used to construct positive Sasakian structures in homotopy 9-spheres [6]. Thereby, it is important to have as many examples as possible of Sasaki–Einstein manifolds.

This paper is organized as follows: in Sect. 2 we give some preliminaries on the topology of links of hypersurface singularities. In Sect. 3, as a consequence of the explicit calculations on the topology of new rational Sasaki–Einstein homology 7-spheres given in this paper, we can apply a method of Savel’ev [25] to produce new examples of homotopy 9-spheres admitting positive Ricci curvatures and determine the diffeomorphism types of these exam-
ples. Moreover, we find out which elements do not admit extremal Sasaki metrics. Then, we present Table 1 (listing new examples of rational homology 7-spheres admitting Sasaki–Einstein metrics), Table 2 (listing new examples of 7-links homeomorphic to connected sums of $S^3 \times S^4$ admitting Sasaki–Einstein metrics produced from the Johnson and Kollár list) and Table 3 (listing new examples of Sasaki–Einstein 7-links homeomorphic to connected sums of $S^3 \times S^4$ produced from Cheltsov’s list). In these three tables, we list the weights, one quasihomogenous polynomial generating the link, the type of singularity, the degree, the Milnor number and finally the third homology group. In Section 4, we give a link to the four codes implemented in MATLAB, these codes determine whether or not the links come from the admissible type of singularities where Orlik’s conjecture is valid, and compute the homology groups of the links under discussion. We also give links to three additional tables (listing 7-links with nonzero third Betti number and with torsion).

2 Preliminaries: Sasaki–Einstein metrics on links and Orlik’s conjecture

In this section, we briefly review the Sasakian geometry of links of isolated hypersurface singularities defined by weighted homogeneous polynomials. We describe the explicit constructions of Sasaki–Einstein manifolds given by Boyer and Galicki [3]. Then, we give some known facts on the topology of links of hypersurface singularities [22, 23] and state Orlik’s conjecture. We also set up a table with the necessary conditions to obtain links where this conjecture is known to be valid.

2.1 Links and Sasaki–Einstein metrics

Consider the weighted $\mathbb{C}^*$ action on $\mathbb{C}^{n+1}$ given by

$$(z_0, \ldots, z_n) \mapsto (\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n)$$

where $w_i$ are the weights which are positive integers and $\lambda \in \mathbb{C}^*$. Let us denote the weight vector by $w = (w_0, \ldots, w_n)$. We assume $\gcd(w_0, \ldots, w_n) = 1$. Recall that a polynomial $f \in \mathbb{C}[[z_0, \ldots, z_n]]$ is said to be a weighted homogeneous polynomial of degree $d$ and weight $w = (w_0, \ldots, w_n)$ if for any $\lambda \in \mathbb{C}^* = \mathbb{C}\{0\}$, we have

$$f(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \ldots, z_n).$$

We are interested in those weighted homogeneous polynomials $f$ whose zero locus in $\mathbb{C}^{n+1}$ has only an isolated singularity at the origin. The link $L_f(w, d)$ (sometimes written $L_f$) is defined by

$$L_f(w, d) = f^{-1}(0) \cap S^{2n+1},$$

where $S^{2n+1}$ is the $(2n+1)$-sphere in $\mathbb{C}^{n+1}$. By the Milnor fibration theorem [22], $L_f(w, d)$ is a closed $(n-2)$-connected $(2n-1)$-manifold that bounds a parallelizable manifold with the homotopy type of a bouquet of $n$-spheres. Furthermore, $L_f(w, d)$ admits a quasi-regular Sasakian structure in a natural way, see, for instance [26]. Moreover, if one considers the locally free $S^1$-action induced by the weighted $\mathbb{C}^*$ action on $f^{-1}(0)$ the quotient space of the link $L_f(w, d)$ by this action is the weighted hypersurface $Z_f$, a Kähler orbifold. Actually, we
have the following commutative diagram [3]

\[ L_f(w, d) \longrightarrow S^{2n+1}_w \]
\[ \downarrow \pi \quad \downarrow \]
\[ Z_f \longrightarrow \mathbb{P}(w), \]

where \( S^{2n+1}_w \) denotes the unit sphere with a weighted Sasakian structure, \( \mathbb{P}(w) \) is a weighted projective space coming from the quotient of \( S^{2n+1}_w \) by a weighted circle action generated from the weighted Sasakian structure. The top horizontal arrow is a Sasakian embedding and the bottom arrow is a Kählerian embedding and the vertical arrows are orbifold Riemannian submersions.

It follows from the orbifold adjunction formula that the link \( L_f \) admits a positive Ricci curvature if the quotient orbifold \( Z_f \) by the natural action \( S^1 \) is Fano, which is equivalent to

\[ |w| - d_f > 0, \quad (1) \]

Here, \( |w| = \sum_{i=0}^m w_i \) denotes the norm of the weight vector \( w \) and \( d_f \) is the degree of the polynomial \( f \). Furthermore, in [2], Boyer and Galicki found a method to obtain 2-connected Sasaki–Einstein 7-manifolds from the existence of orbifold Fano Kähler–Einstein hypersurfaces \( Z_f \) in weighted projective 4-space \( \mathbb{P}(w) \). Actually, they showed a more general result:

**Theorem 2.1** The link \( L_f(w, d) \) admits a Sasaki–Einstein structure if and only if the Fano orbifold \( Z_f \) admits a Kähler–Einstein orbifold metric of scalar curvature \( 4n(n+1) \).

In [20], Johnson and Kollár give a list of 4442 quasi-smooth Fano threefold \( Z \) anti-canonically embedded in weighted projective 4-spaces \( \mathbb{P}(w) \). Moreover, they show that 1936 of these threefold admit Kähler–Einstein metrics. Thus, such Fano threefold give rise to Sasaki–Einstein metrics on smooth 7-manifolds realized as links of isolated hypersurface singularities defined by weighted homogenous polynomials. In [5], they extracted from this list 184 2-connected rational homology 7-spheres. They also determined the order of \( H_3(L_f(w, d), \mathbb{Z}) \). In [17], Gomez used Orlik’s conjecture to calculate the homology of 10 elements of the list given in [5], all the 7-manifolds found there are links of chain type singularities.

In this paper, we completely determine the third homology group for 1673 of 2-connected Sasaki–Einstein 7-manifolds from the list 1936 smooth 7-manifolds realized as links of isolated hypersurface singularities from the list given in [5]. Among them, we found 52 new examples of 2-connected rational homology spheres admitting Sasaki–Einstein metrics. We also found 124 new examples of 2-connected Sasaki–Einstein 7-manifolds of the form \( \#2k(S^3 \times S^4) \), improving a result of Boyer in [11], see Theorem 3.1. Of that lot, 118 come from the list given in [20], the other 6 examples are links of quasi-smooth Fano threefold coming from Reid’s list of 95 weighted codimension 1 K3 surfaces, where all members of the list but 4 admit Kähler–Einstein metrics, see [12].

**2.2 The topology of links and Orlik’s conjecture**

In this section, we review some classical results on the topology of links of quasi-smooth hypersurface singularities. Recall that the Alexander polynomial \( \Delta_f(t) \) in [22] associated to a link \( L_f \) of dimension \( 2n - 1 \) is the characteristic polynomial of the monodromy map

\[ h_* : H_n(F, \mathbb{Z}) \rightarrow H_n(F, \mathbb{Z}) \]
induced by the $S^1_w$-action on the Milnor fiber $F$. Therefore, $\Delta_f(t) = \det (tI - h_*)$. Now both $F$ and its closure $\overline{F}$ are homotopy equivalent to a bouquet of $n$-spheres $S^n \vee \cdots \vee S^n$, and the boundary of $\overline{F}$ is the link $L_f$, which is $(n-2)$-connected. The Betti numbers $b_{n-1}(L_f) = b_n(L_f)$ equal the number of factors of $(t - 1)$ in $\Delta_f(t)$. The following are standard facts, see [3].

1. $L_f$ is a rational homology sphere if and only if $\Delta_f(1) \neq 0$.
2. $L_f$ is a homotopy sphere if and only if $\Delta_f(1) = \pm 1$.
3. If $L_f$ is a rational homology sphere, then the order of $H_{n-1}(L_f, \mathbb{Z})$ equals $|\Delta_f(1)|$.

There is a remarkable theorem of Levine (see [22], p. 69) that determines the diffeomorphism type for homotopy spheres. More precisely, we have

**Theorem 2.2** Let $L_f$ be homeomorphic to the $(2n-1)$-sphere for $n$ odd. $L_f$ is diffeomorphic to the standard sphere if $\Delta_f(-1) \equiv \pm 1 (\text{mod} 8)$ and $L_f$ is diffeomorphic to the exotic Kervaire sphere if $\Delta_f(-1) \equiv \pm 3 (\text{mod} 8)$.

In the case that $f$ is a weighted homogeneous polynomial, there is an algorithm due to Milnor and Orlik [23] to calculate the free part of $H_{n-1}(L_f, \mathbb{Z})$. The authors associate to any monic polynomial $f$ with roots $\alpha_1, \ldots, \alpha_k \in \mathbb{C}^*$ its divisor

$$
\text{div } f = \langle \alpha_1 \rangle + \cdots + \langle \alpha_k \rangle
$$

as an element of the integral ring $\mathbb{Z}\left[\mathbb{C}^*\right]$. Let $\Lambda_n = \text{div } (t^n - 1)$. Then, the divisor of $\Delta_f(t)$ is given by

$$
\text{div } \Delta_f = \prod_{i=0}^{n} \left( \frac{\Lambda_{u_i}}{v_i} - \Lambda_1 \right),
$$

(2)

where the $u_i'$s and $v_i'$s are given terms of the degree $d$ of $f$ and the weight vector $w = (w_0, \ldots, w_n)$ by the equations

$$
u_i = \frac{d}{\gcd(d, w_i)}, \quad v_i = \frac{w_i}{\gcd(d, w_i)}. \tag{3}
$$

Using the relations $\Lambda_a \Lambda_b = \gcd(a, b) \Lambda_{\text{lcm}(a, b)}$, Equation (2) takes the form

$$
\text{div } \Delta_f = (-1)^{n+1} \Lambda_1 + \sum a_j \Lambda_j, \tag{4}
$$

where $a_j \in \mathbb{Z}$ and the sum is taken over the set of all least common multiples of all combinations of the $u_0, \ldots, u_n$. Then, the Alexander polynomial has an alternative expression given by

$$
\Delta_f(t) = (t - 1)(-1)^{n+1} \prod_j \left( t^j - 1 \right)^{a_j},
$$

and

$$
b_{n-1}(L_f) = (-1)^{n+1} + \sum a_j. \tag{5}
$$

Moreover, Milnor and Orlik gave an explicit formula to calculate the free part of $H_{n-1}(L_f, \mathbb{Z})$:

$$
b_{n-1}(L_f) = \sum (-1)^{n+1-s} \frac{u_{i_1} \cdots u_{i_s}}{v_{i_1} \cdots v_{i_s} \text{lcm}(u_{i_1}, \ldots, u_{i_s})}. \tag{6}
$$
where the sum is taken over all the \(2^{n+1}\) subsets \(\{i_1, \ldots, i_s\}\) of \(\{0, \ldots, n\}\). In [24], Orlik gave a conjecture which allows to determine the torsion of the homology groups of the link in terms of the weight of \(f\).

**Conjecture 2.3** (Orlik) Let \(L_f\) denote a link of an isolated hypersurface singularity defined by a weighted homogenous polynomial \(f\) with weight vector \(w = (w_0, \ldots, w_n)\) and degree \(d\). Consider \(\{i_1, \ldots, i_s\} \subset \{0, 1, \ldots, n\}\) the set of ordered set of \(s\) indices, that is, \(i_1 < i_2 < \cdots < i_s\). Let us denote by \(I\) its power set (consisting of all of the \(2^s\) subsets of the set), and by \(J\) the set of all proper subsets. Given a \((2n + 2)\)-tuple \((u, v) = (u_0, \ldots, u_n, v_0, \ldots, v_n)\) of integers given as in Equation (3), let us define inductively a set of \(2^s\) positive integers, one for each ordered element of \(I\), as follows:

\[
c_\emptyset = \gcd(u_0, \ldots, u_n),
\]

and if \(\{i_1, \ldots, i_s\}\) is ordered, then

\[
c_{i_1, \ldots, i_s} = \frac{\gcd(u_0, \ldots, \hat{u}_{i_1}, \ldots, \hat{u}_{i_s}, \ldots, u_n)}{\prod_{j \in J} c_{j_1, \ldots, j_t}}.
\]

Similarly, we also define a set of \(2^s\) real numbers by

\[
k_\emptyset = \epsilon_{n+1},
\]

and

\[
k_{i_1, \ldots, i_s} = \epsilon_{n-s+1} \sum_{l} (-1)^{s-l} \frac{u_{j_1} \cdots u_{j_t}}{v_{j_1} \cdots v_{j_t} \lcm(u_{j_1}, \ldots, u_{j_t})}
\]

where

\[
\epsilon_{n-s+1} = \begin{cases} 0 & \text{if } n - s + 1 \text{ is even} \\ 1 & \text{if } n - s + 1 \text{ is odd} \end{cases},
\]

respectively. Finally, for any \(j\) such that \(1 \leq j \leq r = \lfloor \max\{k_{i_1, \ldots, i_s}\} \rfloor\), where \(\lfloor x \rfloor\) is the greatest integer less than or equal to \(x\), we set \(d_j = \prod_{k_{i_1, \ldots, i_s} \geq j} c_{i_1, \ldots, i_s}\). Then,

\[
H_{n-1}(L_f, \mathbb{Z})_{tor} = \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_r.
\]

In a rather recent paper, Hertling and Mase [18] showed that this conjecture (open for more than 45 years old) is true for the following cases:

1. **Chain type singularity**: that is, a quasihomogeneous singularity of the form

\[
f = f(x_1, \ldots, x_n) = \sum_{i=2}^{n} x_i^{a_i} \quad \text{for some } n \in \mathbb{N} \text{ and some } a_1, \ldots, a_n \in \mathbb{N}.
\]

2. **Cycle type singularity**: a quasihomogeneous singularity of the form

\[
f = f(x_1, \ldots, x_n) = \sum_{i=1}^{n-1} x_i^{a_i} x_{i+1} + x_n^{a_n} x_1
\]

for some \(n \in \mathbb{Z}_{\geq 2}\) and some \(a_1, \ldots, a_n \in \mathbb{N}\) which satisfy for even \(n\) neither \(a_j = 1\) for all even \(j\) nor \(a_j = 1\) for all odd \(j\).
(3) Thom–Sebastiani iterated sums of singularities of chain type or cycle type. Recall, for \( f \) and \( g \) singularities, the Thom–Sebastiani sum is given by
\[
f + g = f(x_1, \ldots, x_{n_f}) + g(x_{n_f+1}, \ldots, x_{n_f+n_g}).
\]
Any iterated Thom–Sebastiani sum of chain type singularities and cycle type singularities is also called an invertible polynomial.

(4) Although Brieskorn-Pham singularities, or BP singularities
\[
f = f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^{a_i}
\]
for some \( n \in \mathbb{N} \) and some \( a_1, \ldots, a_n \in \mathbb{Z}_{\geq 2} \) are special cases of iterated sums of chain type, we prefer to label them as an independent type of singularity.

In order to use Orlik’s conjecture, we need to find from the two lists of Kähler–Einstein orbifolds aforementioned, elements with weights \( w = (w_0, w_1, w_2, w_3, w_4) \) and degree \( d = (w_0 + \cdots + w_4) - 1 \) that can be represented by BP singularities, chain type singularities, cycle type singularities or Thom–Sebastiani sums of these types of singularities. Thus, given a weight vector \( w = (w_0, w_1, w_2, w_3, w_4) \) one needs to determine if there exist exponents \( a_i \)'s that verify certain arithmetic condition. We include these conditions in the table below.

Most of the examples that we describe in this article have large weights and degrees, so in order to avoid monotonous calculations, codes were implemented in MATLAB (see (d) in Appendix) to determine for a given weight vector \( w = (w_0, w_1, w_2, w_3, w_4) \) the exponents \( a_i \)'s, such that the singularity can be written as a chain type, cycle type or Thom–Sebastiani sum of them. We also wrote a code to compute the Betti numbers and the numbers \( b_i \) which generate the torsion in \( H_3(L_f, \mathbb{Z}) \).

Next, we discuss an interesting result on the topology of the link if the degree \( d \) and the weight vector \( w \) are such that \( \gcd(d, w_i) = 1 \) for all \( i \). Several elements in the tables we present in Sect. 3, satisfy this condition. Notice that, if the type of singularities is restricted to the ten cases described on the table given above, then \( \gcd(d, w_i) = 1 \) for all \( i \) forces the singularity to be of cycle type or of type cycle+cycle. We restrict to dimension 7, but this remark can be easily generalized for any dimension.

**Lemma 2.4** Consider a 7-manifold \( M \) arising as a link of a hypersurface singularity with degree \( d \) and such that \( \gcd(d, w_i) = 1 \) for all \( i = 0, \ldots, 4 \). Then \( \mu + 1 = d(b_3 + 1) \) and \( H_3(M, \mathbb{Z})_{tor} = \mathbb{Z}_d \). In particular, for hypersurface singularities that determine a rational homology sphere with \( \gcd(d, w_i) = 1 \) for all \( i \) we obtain \( \mu = d - 1 \).

**Proof** Indeed, from Equation (6) we have
\[
b_3 = \sum (-1)^{5-s} \left[ \frac{u_{i_1} \cdots u_{i_s}}{v_{i_1} \cdots v_{i_s} \text{lcm}(u_{i_1} \cdots u_{i_s})} \right],
\]
where the sum is over all subsets \( \{i_1 \ldots i_s\} \subset \{0, 1, 2, 3, 4\} \) with \( i_1 < \cdots < i_s \) and
\[
u_i = \frac{d}{\gcd(d, w_i)}, \quad v_i = \frac{w_i}{\gcd(d, w_i)}, \quad \forall i = 0, \ldots, 4.
\]
Since \( \gcd(d, w_i) = 1 \) we have \( u_i = d \) and \( v_i = w_i \). Then, the above formula can be simplified as follows:
\[
b_3 = -1 + \sum \frac{1}{w_{i_1}} - \sum \frac{d}{w_{i_1}w_{i_2}} + \sum \frac{d^2}{w_{i_1}w_{i_2}w_{i_3}} - \sum \frac{d^3}{w_{i_1}w_{i_2}w_{i_3}w_{i_4}} + \frac{d^4}{w_0w_1w_2w_3w_4}.
\]
| Type                | Polynomial                                                                 | Condition                                                                 |
|--------------------|---------------------------------------------------------------------------|---------------------------------------------------------------------------|
| BP                 | $z_0^a + z_1^a + z_2^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = a_1w_1 = a_2w_2 = a_3w_3 = a_4w_4$                           |
| Chain              | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                              | $d = a_0w_0 = w_0 + a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$ |
| Cycle              | $z_4z_0 + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = w_4 + a_0w_0 = w_0 + a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$ |
|                   | $z_0^a + z_1^a + z_2^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = w_0 + a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$ |
| BP + Chain         | $z_0^a + z_1^a + z_2^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = w_0 + a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$ |
|                   | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = a_0w_0 = w_0 + a_1w_1 = a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$         |
| BP + Cycle         | $z_0^a + z_1^a + z_2^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$         |
|                   | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = a_0w_0 = a_1w_1 = a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$               |
| Chain + Chain      | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = a_0w_0 = w_0 + a_1w_1 = a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$         |
| Chain + Cycle      | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = a_0w_0 = a_1w_1 = a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$               |
|                   | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = a_0w_0 = a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$         |
| Cycle + Cycle      | $z_0^a + z_0z_1 + z_1z_2 + z_2z_3 + z_3z_4$                            | $d = a_0w_0 = w_0 + a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$ |
| BP + Chain + Chain | $z_0^a + z_1^a + z_2^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = a_1w_1 = a_2w_2 = a_3w_3 = w_3 + a_4w_4$                     |
| BP + Chain + Cycle | $z_0^a + z_1^a + z_2^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = a_1w_1 = a_2w_2 = a_3w_3 = w_3 + a_4w_4$                     |
| BP + Cycle + Cycle | $z_0^a + z_2^a + z_1^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$         |
|                   | $z_0^a + z_2^a + z_1^a + z_3^a + z_4^a$                                 | $d = a_0w_0 = w_2 + a_1w_1 = w_1 + a_2w_2 = w_2 + a_3w_3 = w_3 + a_4w_4$ |
Writing \( S_1 = \sum w_i, \ S_2 = \sum w_1w_i, \ S_3 = \sum w_1w_2w_i, \ S_4 = \sum w_1w_2w_3w_i \), we obtain
\[
b_3 = -1 + \frac{S_4}{w_0 \ldots w_4} - \frac{S_3d}{w_0 \ldots w_4} + \frac{S_2d^2}{w_0 \ldots w_4} - \frac{S_1d^3}{w_0 \ldots w_4} + \frac{d^4}{w_0 \ldots w_4} \quad (8)
\]

On the other hand, the Milnor number is given by
\[
\mu = \left( \frac{d}{w_0 - 1} \right) \ldots \left( \frac{d}{w_4 - 1} \right).
\]

Then,
\[
\mu = \frac{d^5 - S_1d^4 + S_2d^3 - S_3d^2 + S_4d - w_0 \ldots w_4}{w_0 \ldots w_4}.
\]

From (8), we obtain \( \mu + 1 = d(b_3 + 1) \). For the second claim, since \( u_i = d \) and \( v_i = w_i \), for all \( i \), we obtain \( c_{ij} = \gcd(d, \ldots, d) = d \) and \( c_{i_1 \ldots i_s} = 1 \). Clearly, \( k_{ij} = 1 \). Then, for \( j = 1 \), we get \( d_1 = d \), while in other cases, \( d_j = 1 \). Therefore, from Equation (7) one concludes that if \( \gcd(d, w_i) = 1 \) for all \( i = 0, \ldots, n \) then \( H_3(M, \mathbb{Z})_{tor} = \mathbb{Z}_d \).

\section{3 Results}

In this section, we present three tables of new examples of two classes of 7-manifolds: rational homology 7-spheres, and 7-manifolds homeomorphic to connected sums of \( S^3 \times S^4 \), all admitting Sasaki–Einstein metrics. These new examples are extracted from two lists of Kähler–Einstein orbifolds, the first one given by Johnson and Kollár in [20] (see (a) in Appendix for the link to the list) and the other one given by Cheltsov in [12]. In these three tables, we list the weights, the quasihomogenous polynomial generating the link, the type of singularity, the degree, the Milnor number and finally the third homology group. But, first, as an application of these results, we discuss the diffeomorphism type of homotopy 9-spheres admitting positive Ricci curvature, given as cyclic branched covers whose branching locus is one of the rational homology 7-spheres in Table 1, we also discuss the existence of extremal Sasakian metrics on them.

\subsection{3.1 Positive Ricci curvature on homotopy 9-spheres}

Homotopy \( n \)-spheres that bound a parallelizable manifold form the Kervaire–Milnor group, denoted by \( bP_{n+1} \). In particular, when \( n \equiv 1 \bmod 4 \) the Kervaire–Milnor group has at most two elements, the standard sphere and the Kervaire sphere. The existence of positive Sasakian metrics on elements of \( bP_{n+1} \) was studied in detail by Boyer, Galicki and collaborators, where the authors exhibited inequivalent families of homotopy 9-spheres admitting this type of metrics based on their list of 184 Sasakian-Einstein rational homology 7-spheres. Furthermore, in [7], it is shown that for \( n \geq 2 \), the \( (4n + 1) \)-dimensional standard and Kervaire spheres both admit many families of inequivalent Sasakian-Einstein metrics. We apply methods developed in [5] and [6] (which were originally given by Savel’ev in [25]) to the list of the new 52 rational homology 7-spheres presented in this paper to construct homotopy spheres that admit positive Sasakian structures. Let us briefly review some basic facts on links viewed as branched covers. The main reference here is [6].

Let \( f = f(z_1, \ldots, z_n) \) be a quasi-smooth weighted homogeneous polynomial of degree \( d_f \) in \( n \) complex variables, and let \( L_f \) denote its link. Let \( w_f = (w_1, \ldots, w_n) \) be the corresponding weight vector. We consider cyclic branched covers constructed as the link \( L_g \).
of the polynomial

\[ g = z_0^p + f(z_1, \ldots, z_n), \]

for \( p > 1 \). Then \( L_g \) is a \( p \)-fold branched cover of \( S^{2n+1} \) branched over the link \( L_f \). The degree of \( L_g \) is \( d_g = \text{lcm}(p, d_f) \), and the weight vector is \( \mathbf{w}_g = \left( \frac{d_f}{\gcd(p, d_f)}, \frac{p}{\gcd(p, d_f)} \right)^{d_f} \).

We have

\[ |\mathbf{w}_g| - d_g = \frac{d_f + p(|w_f| - 1)}{\gcd(p, d_f)}. \tag{9} \]

Thus, from Equation (1) and Theorem 2.1 it follows that \( L_g \) admits positive Ricci curvature for all \( p > 0 \) if \( L_f \) admits a Sasaki–Einstein metric.

Furthermore, in [6] the authors showed that the link \( L_g \) is a homotopy sphere if and only if \( L_f \) is a rational homology sphere. Let us recall the argument, since it will be useful for our purpose. Here, we assume that \( \gcd(p, d_f) = 1 \). From Equation (2), it is easy to see that \( u_0 = p \) and \( v_0 = 1 \) and that the \( u' \)'s and \( v' \)'s corresponding to \( g \) are equal to the ones given by \( f \) for \( i > 0 \). One then applies Equation (4) and obtains the equalities

\[
\text{div} \Delta_g = (\Lambda_p - 1) \text{div} \Delta_f = \left( \Lambda_p - 1 \right) \left( (-1)^n \Lambda_1 + \sum a_j \Lambda_j \right) \\
= \sum_j \gcd(p, j) a_j \Lambda_{\text{lcm}(p, j)} - \sum_j a_j \Lambda_j + (-1)^n \Lambda_p + (-1)^n \Lambda_1. 
\]

From the condition \( \gcd(p, d_f) = 1 \), we have \( \gcd(p, \frac{d_f}{\gcd(d_f, w_f)}) = \gcd(p, u_i) = 1 \) for all \( i \). Since the \( j \)'s run through all the least common multiples of the set \( \{u_1, \ldots, u_n\} \) that for all \( j, \gcd(p, j) = 1 \). It follows that

\[ b_{n-1} (L_g) = \sum_j a_j - \sum_j a_j + (-1)^n + (-1)^n = 0. \]

So \( L_g \) is a rational homology sphere. Now, the computation of the Alexander polynomial for \( L_g \) leads to:

\[
\Delta_g(t) = (t - 1)^{(-1)^n + 1} (t^p - 1)^n \prod_j \left( t^{p_j - 1} - 1 \right)^{a_j} \left( t^{j - 1} - 1 \right)^{-a_j} \\
= (t_1^{p-1} + \cdots + t + 1)^n \prod_j \left( t^{p_j - 1} + \cdots + t + 1 \right)^{a_j} \\
= (t_1^{p-1} + \cdots + t + 1)^n \prod_j \left( t_1^{(p-1)j} + \cdots + t_1^j + 1 \right)^{a_j}. \tag{10} 
\]

This gives

\[ \Delta_g(1) = p^{\sum_j a_j + (-1)^n}. \]

Thus from Equation (5), it follows that \( L_g \) is a homotopy sphere if and only if \( L_f \) is a rational homology sphere.

In [6], the authors made use of Equation (10) and apply Theorem 2.2 to determine the diffeomorphism type for \( L_g \) for 7-dimensional links \( L_f \) coming from their list of rational homology 7-spheres given in [5]. If \( L_f \) has odd degree they proved the following
Theorem 3.1 (Boyer et al. [6]) Let $L_g$ be the link of a $p$-branched cover of $S^{2n+1}$ branched over a link $L_f$ in their list of 184 rational homology spheres given in [5]. Suppose that degree $d_f$ of $L_f$ is odd and that $\gcd(p, d_f) = 1$. Then, for $p$ odd, $L_g$ is diffeomorphic to the standard 9-sphere $S^9$. For $p$ even, $L_g$ is diffeomorphic to $S^9$ if $|H_3(L_f, \mathbb{Z})| \equiv \pm 1(\text{mod } 8)$ and $L_g$ is diffeomorphic to the Kervaire exotic sphere $\Sigma_9$ if $|H_3(L_f, \mathbb{Z})| \equiv \pm 3(\text{mod } 8)
abla$

The next theorem is an improvement of Theorems 7.6 and Theorems 7.7 in [6] and its proof is very similar to the one given there but we include a proof for reference.

Theorem 3.2 For each member of the 52 rational homology spheres in the list given in Table 1, the link $L_g$ of a $p$-branched cover of $S^{2n+1}$ branched over a link $L_f$ is a homotopy sphere admitting positive Ricci curvature provided that the degree $d_f$ of $L_f$ satisfies $\gcd(p, d_f) = 1$. Furthermore, if $L_f$ is taken to be one of the 49 rational homology spheres with odd degree listed in Table 1, then $L_g$ admits Sasakian metrics with positive Ricci curvature and these are diffeomorphic to the standard sphere $S^9$. For the remaining three members with even degree in Table 1, we have

(1) For $L_f$ with $w = (118, 118, 185, 135, 35)$ with degree $d_f = 590$, $L_g$ is diffeomorphic to the standard $S^9$ if $p^3 \equiv \pm 1(\text{mod } 8)$ and diffeomorphic to the exotic Kervaire 9-sphere $\Sigma_9$ if $p^4 \equiv \pm 3(\text{mod } 8)$. (2) For $L_f$ with $w = (64, 512, 475, 375, 175)$ with degree $d_f = 1600$, $L_g$ is diffeomorphic to the standard $S^9$ if $p^3 \equiv \pm 1(\text{mod } 8)$ and diffeomorphic to the exotic Kervaire 9-sphere $\Sigma_9$ if $p^3 \equiv \pm 3(\text{mod } 8)$. (3) For $L_f$ with $w = (3532, 7064, 5355, 115, 1595)$ with degree $d_f = 17660$, $L_g$ is diffeomorphic to the standard $S^9$ if $p^2 \equiv \pm 1(\text{mod } 8)$ and diffeomorphic to the exotic Kervaire 9-sphere $\Sigma_9$ if $p^2 \equiv \pm 3(\text{mod } 8)$.

Proof Our list of rational homology 7-spheres admitting Sasaki–Einstein metrics contains 49 elements with odd degree and all of these have order $|H_3| \equiv 1(\text{mod } 8)$; thus, the first part of the theorem follows from Theorem 3.1. For the remaining three members with even degree in Table 1, the weight vectors can be written as $w = (w_0, w_1, w_2, w_3, w_4) = (m_2v_0, m_2v_1, m_2v_2, m_3v_3, m_3v_4)$ where the $v_i$’s are given as in Equation (3), $\gcd(m_2, m_3) = 1$ and $m_2m_3 = d$ where $m_2$ is odd and $m_3$ is even. From here, one obtains directly that $u_0 = u_1 = u_2 = m_3$ and $u_3 = u_4 = m_2$. Thus, applying the relations $\Lambda_a\Lambda_b = \gcd(a, b)\Lambda_{\text{lcm}(a, b)}$ in Equation (2), one obtains

$$\text{div} \Delta_f = \alpha(w)\beta(w)\Lambda_{d} + \beta(w)\Lambda_m - \alpha(w)\Lambda_{m_2} - 1,$$

with $\alpha(w) = \frac{m_2}{v_3v_4} - \frac{1}{v_3} - \frac{1}{v_4}$ and $\beta(w) = \left(\frac{m_3}{v_0v_1} - \frac{1}{v_1} - \frac{1}{v_0}\right)\left(\frac{m_3}{v_2} - 1\right) + \frac{1}{v_2}$ positive integers depending on the weights. Since the link is a rational homology sphere, $b_3(L_f)$ which is given by the number of factors of $t - 1$ in $\Delta_f(t)$, is zero, equivalently

$$(\alpha(w) + 1)(\beta(w) - 1) = 0,$$

which implies $\beta(w) = 1$. So we can rephrase the expression given above for $\text{div} \Delta_f$:

$$\text{div} \Delta_f = \alpha(w)\Lambda_{d} + \Lambda_m - \alpha(w)\Lambda_{m_2} - \Lambda_1. \tag{11}$$

It follows from Equation (4) that $\sum_{j \text{ even}} a_j = \alpha(w) + 1$.

On the other hand, bearing in mind Theorem 2.2, we pay attention to $\Delta_g(-1)$: since $p$ is odd and $d$ even, from Equation (10) we have

$$\Delta_g(-1) = \prod_{j \text{ even}} p^{a_j} = p^{\sum_{j \text{ even}} a_j} = p^{\alpha(w) + 1}.$$
Since $L_f$ is a rational homology sphere $H_3(L_f, \mathbb{Z})_{tor} = \Delta_f(1)$, so in order to compute $\Delta_f(1)$ we rewrite equation (11) in terms of the Alexander polynomial:

$$\Delta_f(t) = (t^d - 1)^{a(w)}(t^{m_3} - 1)(t^{m_2} - 1)^{-a(w)}(t - 1)^{-1}$$

$$= (t^d + \ldots + 1)^{a(w)}(t^{m_3} + \ldots + 1)(t^{m_2} - 1 + \ldots + 1)^{-a(w)},$$

Thus, $\Delta_f(1) = (t^d)^{a(w)}m_{32}^{a(w)} = m_{3}^{a(w)}+1$. Now, the result follows as a direct consequence of Theorem 2.2 by checking the torsion of $H_3(L_f, \mathbb{Z})$ at Table 1.

\[ \square \]

**Remark 1** In [9] it is shown that for a link $L_f(w, d)$ of an isolated hypersurface singularity such that

$$I > n \min_i \{w_i\}, \quad (12)$$

where $I = |w| - d$, then $L_f(w, d)$ cannot admit any Sasakian-Einstein metric. Furthermore, in [9] Theorem 5.5, it is proven that if additionally $2w_i < d$ for all but at most one of the $i$’s, then the Sasaki cone is one dimensional and thus there is no extremal Sasakian metric on the link $L_f$ (see [10] for the definition of extremal Sasakian metrics). One can easily check that each of the 52 rational homology spheres $L_f$ in Table 1 satisfies this last inequality $2w_i < d$ for all $i$. Thus $2w_i < d$ for all but at most one of the $i$’s for the associated link $L_g$. So in order to determine values for $p$, such that the link $L_g$ does not admit extremal Sasakian metrics, we need to give conditions on the integer number $p$ such that the associated link $L_g$ with weight $w_g = (d_f, pw_0, pw_1, pw_2, pw_3, pw_4)$ and index $I = d_f + p$ satisfies (12). Let us consider two cases:

- First, if we consider $\min_i \{w_i\} = d_f$, then $d_f \leq pw_0$. Thus

$$I > n \min_i \{w_i\} \iff d_f + p > 5d_f \iff p > 4d_f$$

Therefore, the inequality (12) is verified when $p \in [4d_f, +\infty[$.

- On the other hand, if we consider $\min_i \{w_i\} = pw_0$, then $pw_0 \leq d_f$. Similarly, we obtain

$$I > n \min_i \{w_i\} \iff d_f + p > 5pw_0 \iff \frac{d_f}{5w_0 - 1} > p.$$

Since $\frac{d_f}{5w_0 - 1} < \frac{d_f}{w_0}$, then the inequality (12) is satisfied when $p < \frac{d_f}{5w_0 - 1}$.

Thus, the inequality (12) is verified when $p \in ]1 : \frac{d_f}{5w_0 - 1} \cup ]4d_f ; +\infty[.$

It follows that for the 49 links $L_f$ with odd degree we obtain links $L_g$ associated to the hypersurface $g = z_0^p + f(z_1, \ldots, z_5)$, diffeomorphic to the standard $S^9$, that do not admit extremal Sasaki metrics for $p$’s lying in the interval $]1 : \frac{d_f}{5w_0 - 1} \cup ]4d_f ; +\infty[.$ For the remaining three cases with even degree, one has to take into account the arithmetic conditions given in Theorem 3.2, for instance,

(1) Taking $w_f = (35, 118, 118, 135, 185)$ with $d_f = 590$. Then, $w_g = (590, pw_f)$ with $I_g = 590 + p$. Thus, if $p \in \mathbb{Z}$ and $p \in ]1 : \frac{590}{174} \cup ]2360 ; +\infty[,$ then $L_g$ cannot admit extremal Sasakian metrics. For instance, for $p = 3$ the link $L_g$ with weights $(590, 3w_f)$ and degree $d_g = 1770$, which is diffeomorphic to the standard 9-sphere $S^9$, does not admit extremal Sasakian metrics.

(2) Now we take $w_f = (64, 175, 375, 475, 512)$ with $d_f = 1600$. We obtain $w_g = (1600, pw_f)$, where $I_g = 1600 + p$. Similarly we obtain that $L_g$ cannot admit extremal
Sasakian metrics. For \( p \in \mathbb{Z} \) and \( p \in \mathbb{N} \), \( w_3 \mid 6400 \); \( +\infty \). For instance, for \( p = 3 \) the link \( L_g \) with weights \((1600, 3w_f)\) and degree \( d_g = 4800 \), which is diffeomorphic to the Kervaire 9-sphere \( \Sigma_9 \), does not admit extremal Sasakian metrics.

(3) For the vector weight \( w_f = (115, 1595, 3532, 5355, 7064) \) with \( d_f = 17660 \). We have \( w_g = (17660, pw_f) \), where \( I_g = 17660 + p \). We obtain that \( L_g \) cannot admit extremal Sasakian metrics for \( p \in \mathbb{Z} \) and \( p \in \mathbb{N} \). For instance, for \( p = 3 \) the link \( L_g \) with weights \((17660, 3w_f)\) and degree \( d_g = 52980 \), which is diffeomorphic to the standard 9-sphere \( S^9 \), does not admit extremal Sasakian metrics.

### 3.2 Rational homology 7-spheres admitting Sasaki–Einstein structures from Johnson–Kollár list

In [20], Johnson and Kollár gave a list of 4442 well-formed quasi-smooth \( \mathbb{Q} \)-Fano threefold of index one anti-canonically embedded in \( \mathbb{C}^{2d} \). This list contains 1936 threefold that admit a \( \mathbb{C}^{2d} \)-type metric. Theorem 2.1 implies that the corresponding link admits Sasaki–Einstein metrics. These links were studied in [5] and they found 184 rational homology 7-spheres. In the following table, we present 52 new examples of 2-connected rational homology spheres admitting Sasaki–Einstein metrics.

All the rational homology 7-spheres that we exhibit in Table 1 come from polynomials that are of cycle type or that have that type of singularity as part of its Thom–Sebastiani representation. The examples found in [5] also have that particular feature. Actually, in [5], Lemma 3.3 provides examples of cycle type singularities, while Lemma 3.10 provides examples of two blocks both of cycle type singularities. Our list gives seven elements of cyclic type (the last seven in Table 1) and 45 elements, each of which has a polynomial representation that contains a cyclic block of two or three terms. Moreover, all these 45 examples of rational homology spheres come from weight vectors satisfying \( w = \{w_0, w_1, w_2, w_3, w_4\} = \{m_2v_0, m_2v_1, m_2v_2, m_3v_3, m_3v_4\} \) where \( \gcd(m_2, m_3) = 1 \) and \( m_2m_3 = d \). As mentioned in the proof of Theorem 3.2 (see also Lemma 3.10 in [5]), we obtain Equation (11) with \( \alpha(w) = \frac{m_2}{v_3v_4} - \frac{1}{v_3} - \frac{1}{v_4} \). This equation gives us the equality

\[
m_2 = v_3 + v_4(1 + \alpha(w)v_3).
\]

Multiplying by \( m_3 \), we have

\[
d = w_3 + w_4a_4 \tag{13}
\]

where \( a_4 > 1 \) is an integer. Analogously, we get \( m_2 = v_4 + v_3(1 + \alpha(w)v_4) \). Multiplying by \( m_3 \), we obtain

\[
d = w_4 + w_3a_3 \tag{14}
\]

From (13) and (14), it follows that there exist a cycle block \( z_4z_3^{a_4} + z_3z_4^{a_4} \) that is a summand of some invertible polynomial associated to \( w \). Thus, for each of the 45 examples of rational homology spheres there exists a polynomial representation that contains at least one cyclic block of two terms. We conjecture that all rational homology 7-spheres arising as links of hypersurface singularities can be given by polynomials that contain a cycle singularity term in its representation as a Thom–Sebastiani sum, at least for orbifolds with index 1.

**Remark 2** Following the terminology given in [5], a rational homology sphere with the same degree \( d \), Milnor number \( \mu \) and order of \( H_3 \) is a twin. We detect 10 twins in Table 1; with the exception of the pair given by the weights \((2323, 1611, 562, 151, 899)\) and \((2387, 1579, 661, 148, 771)\) with \( d = |H_3| = 5545 \) and \( \mu = 5544 \), the rest of twins
| w = (w₀, w₁, w₂, w₃, w₄) | Polynomial | Type | d   | μ    | H₃(M, ℤ) |
|-----------------|-----------|------|-----|------|----------|
| (13,143,775,620,465) | z₁⁰ + z₀z₁⁴ + z₄z₂ + z₂z₃ + z₃z₄ | Chain + Cycle | 2015 | 24192 | (ℤ₁₃)⁴ |
| (7,77,333,180,27) | z₁⁰ + z₁⁹ + z₄z₂ + z₂z₃ + z₃z₄ | BP + Cycle | 693  | 4864  | (ℤ₇₇)⁵  |
| (67,67,161,28,147) | z₀⁹ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | BP + Cycle | 469  | 2376  | (ℤ₆₇)⁵  |
| (29,667,1807,1112,417) | z₁⁰ + z₀⁶ + z₄z₂ + z₂z₃ + z₃z₄ | Chain + Cycle | 4031 | 19488 | (ℤ₂₉)⁵  |
| (493,34,1841,1315,789) | z₁⁹ + z₀z₁⁷ | Cycle + Cycle | 4471 | 16848 | (ℤ₁₇)⁵  |
| (67,67,217,84,35) | z₀⁷ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | BP + Cycle | 469  | 2376  | (ℤ₆₇)⁵  |
| (118,118,185,135,35) | z₀⁶ + z₁⁷ | BP + Cycle | 900  | 1872  | (ℤ₁₈)⁵  |
| (373,373,780,35,305) | z₀⁶ + z₁⁷ | BP + Cycle | 1865 | 5952  | (ℤ₃₇₃)⁴  |
| (113,226,715,377,39) | z₁⁰ + z₀⁶ + z₄z₂ + z₂z₃ + z₃z₄ | Chain + Cycle | 1469 | 7392  | (ℤ₁₃)⁶  |
| (253,253,545,40,175) | z₁⁰ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | BP + Cycle | 1265 | 4032  | (ℤ₂₅₃)⁴  |
| (43,1333,1875,500,1625) | z₀⁶ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | Chain + Cycle | 5375 | 15792 | (ℤ₄₃)⁴  |
| (43,1333,2375,1000,625) | z₀⁶ + z₁⁷ | Chain + Cycle | 5375 | 15792 | (ℤ₄₃)⁴  |
| (73,73,95,45,80) | z₀⁷ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | BP + Cycle | 365  | 1152  | (ℤ₇₃)⁴  |
| (185,740,1911,987,63) | z₀⁷ + z₁⁷ | BP + Cycle | 3885 | 15640 | (ℤ₁₈₅)⁵  |
| (929,1858,2849,63,805) | z₀⁷ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | Chain + Cycle | 6503 | 13920 | (ℤ₉₂₉)³  |
| (64,512,475,375,175) | z₀⁷ + z₁⁷ | Chain + Cycle | 1600 | 3213  | (ℤ₆₄)³  |
| (253,253,600,95,65) | z₀⁷ + z₁⁷ | BP + Cycle | 1265 | 4032  | (ℤ₂₅₃)⁴  |
| (127,381,793,286,65) | z₀⁷ + z₁⁷ + z₄z₂ + z₂z₃ + z₃z₄ | Chain + Cycle | 1651 | 5040  | (ℤ₁₂₇)⁴  |
| (65,650,1581,867,153) | z₀⁷ + z₁⁷ | Chain + Cycle | 3315 | 13120 | (ℤ₆₅)⁵  |
| (231,664,481,185,259) | z₁⁰ + z₀z₁⁷ | Cycle + Cycle | 1221 | 2400  | (ℤ₃₃)³  |
| (1003,683,745,2675,105) | z₁⁰ + z₀z₁⁷ | Cycle + Cycle | 9095 | 17136 | (ℤ₁₇)³  |
| Table 1 | continued |
|---------|-----------|
| $w = (w_0, w_1, w_2, w_3, w_4)$ | Polynomial | Type | $d$ | $\mu$ | $H_3(M, \mathbb{Z})$ |
| (73,584,1435,779,123) | $z_0^4 + z_0 z_1^5 + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{18}$ | Chain + Cycle | 2993 | 11880 | $(\mathbb{Z}_{73})^5$ |
| (481,962,1519,77,329) | $z_0^7 + z_0 z_1^4 + z_4 z_2^2 + z_2 z_3^4 + z_3 z_4^{10}$ | Chain + Cycle | 3367 | 7200 | $(\mathbb{Z}_{481})^3$ |
| (657,394,24693,95,3097) | $z_0^{19} + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^{82} + z_3 z_4^4$ | Chain + Cycle | 12483 | 25584 | $(\mathbb{Z}_{657})^3$ |
| (2628,1971,4693,95,3097) | $z_1^4 + z_0 z_1^5 + z_4 z_2^2 + z_2 z_3^{82} + z_3 z_4^4$ | Cycle + Cycle | 12483 | 13120 | $(\mathbb{Z}_{657})^2$ |
| (3773,98,8901,5031,1161) | $z_1^5 + z_0 z_1^{155} + z_4 z_2^2 + z_2 z_3^2 + z_3 z_4^{12}$ | Cycle + Cycle | 18963 | 37200 | $(\mathbb{Z}_{49})^3$ |
| (2069,2069,1413,102,555) | $z_3 + z_1^3 + z_4 z_2^3 + z_2 z_3^{41} + z_3 z_4^{11}$ | BP + Cycle | 6207 | 8272 | $(\mathbb{Z}_{2069})^2$ |
| (929,1858,3199,413,105) | $z_0^7 + z_0 z_1^4 + z_4 z_2^2 + z_2 z_3^8 + z_3 z_4^{58}$ | Chain + Cycle | 6503 | 13920 | $(\mathbb{Z}_{929})^3$ |
| (3532,7064,5355,115,1595) | $z_0^5 + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^{107} + z_3 z_4^{11}$ | Chain + Cycle | 17660 | 21186 | $(\mathbb{Z}_{3532})^2$ |
| (1505,6020,3357,2547,117) | $z_0^6 + z_0 z_1^4 + z_4 z_2^4 + z_2 z_3^4 + z_3 z_4^{94}$ | Chain + Cycle | 13545 | 15040 | $(\mathbb{Z}_{1505})^2$ |
| (136,119,889,635,381) | $z_1^5 z_2 + z_0 z_1^7 + z_4 z_2^2 + z_2 z_3^3 + z_3 z_4^4$ | Cycle + Cycle | 2159 | 4080 | $(\mathbb{Z}_{136})^3$ |
| (1297,3891,2653,119,1120) | $z_0^7 + z_0 z_1^2 + z_4 z_2^3 + z_2 z_3^{54} + z_3 z_4^8$ | Chain + Cycle | 9079 | 10368 | $(\mathbb{Z}_{1297})^2$ |
| (6485,2594,9197,119,3655) | $z_1^6 + z_0 z_1^6 + z_4 z_2^2 + z_2 z_3^{106} + z_3 z_4^6$ | Cycle + Cycle | 22049 | 23328 | $(\mathbb{Z}_{1297})^2$ |
| (1457,1457,1011,120,327) | $z_0^3 + z_1^3 + z_4 z_2^4 + z_2 z_3^{28} + z_3 z_4^{13}$ | BP + Cycle | 4371 | 5824 | $(\mathbb{Z}_{1457})^2$ |
| (701,701,381,123,198) | $z_0^3 + z_1^3 + z_4 z_2^2 + z_2 z_3^{13} + z_3 z_4^{10}$ | BP + Cycle | 2103 | 2800 | $(\mathbb{Z}_{701})^2$ |
| (2069,2069,1521,426,123) | $z_0^3 + z_1^3 + z_4 z_2^4 + z_2 z_3^{11} + z_3 z_4^{47}$ | BP + Cycle | 6207 | 8272 | $(\mathbb{Z}_{2069})^2$ |
| (2149,921,3193,124,3131) | $z_1^4 + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^3 + z_3 z_4^4$ | Cycle + Cycle | 9517 | 9792 | $(\mathbb{Z}_{307})^2$ |
| (289,2312,2725,125,1775) | $z_0^5 + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^6 + z_3 z_4^4$ | Chain + Cycle | 7225 | 14688 | $(\mathbb{Z}_{289})^3$ |
Table 1 continued

| \( w = (w_0, w_1, w_2, w_3, w_4) \) | Polynomial | Type | \( d \) | \( \mu \) | \( H_3(M, \mathbb{Z}) \) |
|--------------------------------|-----------|------|-----|-----|----------------|
| \((129,3612,24165,425,2635)\) | \( z_0^{85} + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^{16} + z_3 z_4^4 \) | Chain + Cycle | 10965 | 21888 | \((\mathbb{Z}_{129})^3\) |
| \((129,3612,5185,1445,595)\) | \( z_0^{85} + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^{16} \) | Chain + Cycle | 10965 | 21888 | \((\mathbb{Z}_{129})^3\) |
| \((4085,129,5745,1532,4979)\) | \( z_1 z_0^4 + z_0 z_1^6 + z_4 z_2^2 + z_2 z_3^7 + z_3 z_4^3 \) | Cycle + Cycle | 16469 | 16128 | \((\mathbb{Z}_{43})^2\) |
| \((4085,129,7277,3064,1915)\) | \( z_1 z_0^4 + z_0 z_1^6 + z_4 z_2^2 + z_2 z_3^3 + z_3 z_4^7 \) | Cycle + Cycle | 16469 | 16128 | \((\mathbb{Z}_{43})^2\) |
| \((481,962,1617,175,133)\) | \( z_1^4 z_0 + z_0 z_1^3 + z_2 z_3^{10} + z_3 z_4^{24} \) | Chain + Cycle | 3367 | 7200 | \((\mathbb{Z}_{481})^3\) |
| \((657,3942,6175,1577,133)\) | \( z_0^4 + z_0 z_1^3 + z_4 z_2^2 + z_2 z_3^{4} + z_3 z_4^{2}\) | Chain + Cycle | 12483 | 25584 | \((\mathbb{Z}_{657})^3\) |
| \((2628,1971,6175,1577,133)\) | \( z_1 z_0^4 + z_0 z_1^5 + z_4 z_2^2 + z_2 z_3^{4} + z_3 z_4^{8}\) | Cycle + Cycle | 12483 | 13120 | \((\mathbb{Z}_{657})^2\) |
| \((1945,477,1321,148,1871)\) | \( z_4 z_0^4 + z_0 z_1^3 + z_1 z_2^4 + z_2 z_3^{10} + z_3 z_4^3 \) | Cycle | 5761 | 5760 | \(\mathbb{Z}_{5761}\) |
| \((2387,1579,661,148,771)\) | \( z_4 z_0^2 + z_0 z_1^2 + z_1 z_2^6 + z_2 z_3^{33} + z_3 z_4^7 \) | Cycle | 5545 | 5544 | \(\mathbb{Z}_{5545}\) |
| \((9142,3097,1917,4129,149)\) | \( z_4 z_0^2 + z_0 z_1^3 + z_1 z_2^8 + z_2 z_3^{4} + z_3 z_4^{66} \) | Cycle | 18433 | 18432 | \(\mathbb{Z}_{18433}\) |
| \((2323,1611,562,151,899)\) | \( z_4 z_0^2 + z_0 z_1^2 + z_1 z_2^7 + z_2 z_3^{33} + z_3 z_4^6 \) | Cycle | 5545 | 5544 | \(\mathbb{Z}_{5545}\) |
| \((3073,712,2211,151,1199)\) | \( z_4 z_0^2 + z_0 z_1^2 + z_1 z_2^3 + z_2 z_3^{34} + z_3 z_4^6 \) | Cycle | 7345 | 7344 | \(\mathbb{Z}_{7345}\) |
| \((1585,189,1105,292,1439)\) | \( z_4 z_0^2 + z_0 z_1^6 + z_1 z_2^4 + z_2 z_3^{12} + z_3 z_4^3 \) | Cycle | 4609 | 4608 | \(\mathbb{Z}_{4609}\) |
| \((18277,6172,10207,1899,239)\) | \( z_4 z_0^2 + z_0 z_1^3 + z_1 z_2^3 + z_2 z_3^{14} + z_3 z_4^{146} \) | Cycle | 36793 | 36792 | \(\mathbb{Z}_{36793}\) |
have weight vectors with identical first two components \( w_0, w_1 \). As mentioned in [5], it is tempting to conjecture that twins are homeomorphic or even diffeomorphic links; however, we are not able to determine this.

### 3.3 Sasaki–Einstein 7-manifolds of the form \( #k(S^3 \times S^4) \) from the Johnson–Kollár’s list and Cheltsov’s list

In [2], it is proven that a 2-connected oriented 7-manifold \( M \) that bounds a parallelizable 8-manifold with \( H_3(M, \mathbb{Z}) \) torsion free is completely determined up to diffeomorphism by the rank of \( H_3(M, \mathbb{Z}) \). Moreover, \( M \) is diffeomorphic to \( #k(S^3 \times S^4) \# \Sigma^7 \) for some homotopy sphere \( \Sigma^7 \in bP_8 \) that bounds a parallelizable 8-manifold (one of the 28 possible smooth structures on the oriented 7-sphere). Thus, if the link has no torsion it is homeomorphic to \( k\#(S^3 \times S^4) \) where \( k \) is the rank of the third homology group. In [3], it is shown that, being of Sasaki type, the link has \( k \) even.

In the following lemma, we give certain conditions on the weights that restrict the topology of a link of an invertible polynomial. Actually, every element in Table 2 satisfies one of the conditions of this lemma. The proof of this result is a direct application of Orlik’s conjecture.

**Lemma 3.3** Given the weight vector \( w = (w_0, w_1, w_2, w_3, w_4) \) of a hypersurface singularity of an invertible polynomial \( f \). Consider the numbers \( u_i = \frac{d}{\gcd(d, w_1)} \). If the vector \( u = (u_0, u_1, u_2, u_3, u_4) \) verifies one of the following cases

1. \( u = (a, b, br_1, br_2, abr_2) \), where \( \gcd(a, b) = 1 \), \( \gcd(a, r_2) = 1 \) and \( r_2 \) is divided by \( r_1 \),
2. \( u = (a, b, abr_1, abr_2, abr_2) \), where \( \gcd(a, b) = 1 \), \( r_2 \) is divided by \( r_1 \),
3. \( u = (a, b, a^\alpha c, a^\beta c, ab) \), where \( a, b \) and \( c \) are pairwise relatively prime and \( 0 \leq \alpha \leq \beta \leq 1 \),
4. \( u = (a, b, c, abr, abcr) \), where \( a, b \) and \( c \) are pairwise relatively prime and \( \gcd(c, r) = 1 \),
5. \( u = (a^2 r, b, ca^\alpha, ca^\beta, a^2 rb) \), where \( a, b \) and \( c \) are pairwise relatively prime, \( \gcd(c, r) = 1 \) and \( 0 \leq \alpha \leq \beta \leq 2 \),
6. \( u = (a, b, a^2 b, a^\alpha c, a^\beta c) \), where \( a, b \) and \( c \) are pairwise relatively prime and \( 2 \leq \alpha \leq 3 \),

then the link associated has the form \( #k(S^3 \times S^4) \), that is, \( H_3(L_f, \mathbb{Z}) \) has no torsion.

**Proof** We only prove cases 1 and 4, the remaining cases are proven in a similar fashion. Recall that by Orlik’s algorithm, given as Conjecture 2.3, we have \( H_3(L_f, \mathbb{Z})_{tor} = \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r} \), where \( d_j = \prod_{k_{i_1, \ldots, i_s}} c_{i_1, \ldots, i_s} \), \( k_{i_1, i_2} = k_{i_1 i_2 i_3} = 0 \), it is sufficient to calculate \( k_{i_1 i_2} \) and \( k_{i_1 i_2 i_3 i_4} \), where the associated values \( c_{i_1, \ldots, i_s} \) are greater than 1.

Case (1): from Orlik’s algorithm we obtain \( c_0 = b, c_01 = r_1, c_{012} = r_2, c_{123} = a \) and \( c_{i_1, \ldots, i_s} = 1 \) in other cases. Thus, we just need to find \( k_{01} \):

\[
k_{01} = 1 - \frac{1}{v_0} - \frac{1}{v_1} + \frac{ab}{v_0 v_1 \text{ lcm}(a, b)} = \frac{(v_0 - 1)(v_1 - 1)}{v_0 v_1} < 1.
\]

It follows that \( d_j = 1 \), for all \( j \). Thus, there is no torsion.

Case (4): we obtain \( c_{02} = b, c_{12} = a, c_{012} = r, c_{013} = c \) and \( c_{i_1, \ldots, i_s} = 1 \) in other cases. This time we need to find \( k_{02} \) and \( k_{12} \):

\[
k_{02} = 1 - \frac{1}{v_0} - \frac{1}{v_2} + \frac{ab}{v_0 v_2 \text{ lcm}(a, b)} = \frac{(v_0 - 1)(v_2 - 1)}{v_0 v_2} < 1
\]

\[
k_{12} = 1 - \frac{1}{v_1} - \frac{1}{v_2} + \frac{ac}{v_1 v_2 \text{ lcm}(a, c)} = \frac{(v_1 - 1)(v_2 - 1)}{v_1 v_2} < 1
\]
Calculating $ui$, Table 3, we present these six elements and include the two elements found by Boyer in [11], examples of Sasaki–Einstein 7-manifolds with no torsion on the third homology group. We compute the third homology group for this lot and found six new chain type, cycle type or an iterated Thom–Sebastiani sum of chain type singularities and list given in [19]. From this list, we found that 88 of these links are links of singularities of $k$ manifolds of the form $\sum_i w_i = d$ can be used to generate $\mathbb{Q}$-Fano threefold [19]. These threefolds are hypersurfaces $X_d$ of degree $d$ in weighted projective spaces of the form $\mathbb{C}P(1, w_1, w_2, w_3, w_4)$ with $\sum_{i=1}^4 w_i = d$. In [12], Cheltsov studied these sort of $\mathbb{Q}$-Fano threefold and proved that 91 elements admit Kähler–Einstein orbifold metrics and thereby the links associated to them admit Sasaki–Einstein metrics. The four that fail the test are numbers 1, 2, 4, and 5 in the list given in [19]. From this list, we found that 88 of these links are links of singularities of chain type, cycle type or an iterated Thom–Sebastiani sum of chain type singularities and cycle type singularities. We compute the third homology group for this lot and found six new examples of Sasaki–Einstein 7-manifolds with no torsion on the third homology group. In Table 3, we present these six elements and include the two elements found by Boyer in [11], the ones with weights $(1, 1, 4, 6)$ and $(1, 1, 6, 14, 21)$.

This list is not necessarily exhaustive for Cheltsov’s list, since neither numbers 52, 81 nor 86 in [19] admit descriptions in terms of the type of singularities where Orlik’s conjecture is known to be valid. However, our computer program suggests that these members do have torsion.

Also, it is interesting to notice that the $\mathbb{Q}$-Fano threefold considered by Cheltsov are $d$-fold branch covers for certain weighted projective spaces branched over orbifold $K3$ surfaces of degree $d$. It follows from a well-known result on links of branched covers (see Proposition 2.1 in [14]) that the 91 Sasaki–Einstein links associated to the threefold of Cheltsov’s list can be realized as $d$-fold branched covers of $S^7$ branched along the submanifolds $\#k(S^2 \times S^3)$. Actually, we extend Theorem 4.6 in [11].

**Theorem 3.6** There exist Sasaki–Einstein metrics on the 7-manifolds $M^7$ which can be realized as $d$-fold branched covers of $S^7$ branched along the submanifolds $\#k(S^2 \times S^3)$ where $k$ ranges from 3 to 21, except $k = 17$. The $\mathbb{Q}$-Fano threefold, the homology of $M^7$ and $k$ are given in Table 1 in Appendix, (b).
| w = (w₀, w₁, w₂, w₃, w₄) | Polynomial | Type | d   | μ   | H₃(M, ℤ) |
|---------------|------------|------|-----|-----|----------|
| (9,15,5,10,7) | z₀⁵ + z₁³ + z₁z₂⁶ + z₂z₃⁴ + z₃z₄⁵ | BP + Chain | 45  | 1216| ℤ⁳²     |
| (16,56,7,21,13) | z₀⁵ + z₁⁷ + z₁z₂⁸ + z₂z₃⁵ + z₃z₄⁷ | BP + Chain | 112 | 2970| ℤ⁳⁰     |
| (35,15,15,9,32) | z₀³ + z₁⁵ + z₁z₂⁶ + z₂z₃¹⁰ + z₃z₄³ | BP + Chain | 105 | 1752| ℤ²⁴     |
| (50,30,15,9,47) | z₀³ + z₁⁵ + z₁z₂⁸ + z₂z₃¹⁵ + z₃z₄³ | BP + Chain | 150 | 2472| ℤ²⁴     |
| (35,56,63,9,153) | z₀⁹ + z₀z₁⁵ + z₁z₂⁸ + z₂z₃³ + z₃z₄⁴ | Chain + Chain | 315 | 5328| ℤ²⁴     |
| (141,9,138,95,41) | z₀³ + z₁⁷ + z₁z₂⁴ + z₂z₃⁴ + z₃z₄⁸ | BP + Chain | 423 | 6112| ℤ¹⁶     |
| (48,10,115,25,43) | z₀⁵ + z₁⁴ + z₁z₂⁵ + z₂z₃⁵ + z₃z₄⁵ | BP + Chain | 240 | 3940| ℤ²⁰     |
| (49,35,105,10,47) | z₀³ + z₁⁴ + z₁z₂⁵ + z₂z₃¹⁴ + z₃z₄⁵ | BP + Chain | 245 | 3168| ℤ¹⁶     |
| (714,476,119,11,109) | z₀² + z₁³ + z₁z₂⁸ + z₂z₃¹¹ + z₃z₄³ | BP + Chain | 1428| 34294| ℤ²⁶    |
| (65,65,39,15,12) | z₀³ + z₁⁵ + z₁z₂⁹ + z₂z₃¹³ + z₃z₄⁵ | BP + Chain | 195 | 2928| ℤ²⁴     |
| (119,51,153,12,23) | z₀³ + z₁⁷ + z₁z₂⁹ + z₂z₃¹⁷ + z₃z₄¹⁵ | BP + Chain | 357 | 6680| ℤ²⁰     |
| (57,38,95,12,27) | z₀⁵ + z₁⁶ + z₁z₂⁵ + z₂z₃⁴ + z₃z₄⁸ | BP + Chain | 228 | 2814| ℤ¹⁸     |
| (81,45,12,131,137) | z₀⁵ + z₁⁹ + z₁z₂³⁰ + z₂z₃³ + z₃z₄² | BP + Chain | 405 | 4288| ℤ¹⁶     |
| (236,12,87,207,167) | z₀⁵ + z₁⁹ + z₁z₂³⁰ + z₂z₃³ + z₃z₄² | BP + Chain | 708 | 6492| ℤ¹²     |
| (16,14,49,21,13) | z₀⁷ + z₁⁸ + z₁z₂³ + z₂z₃³ + z₃z₄⁷ | BP + Chain | 112 | 1782| ℤ¹⁸     |
| (35,21,14,13,23) | z₀⁶ + z₁⁴ + z₁z₂⁵ + z₂z₃⁷ + z₃z₄⁴ | BP + Chain | 105 | 1312| ℤ¹⁶     |
| (495,135,736,107,13) | z₀³ + z₁¹¹ + z₁z₂³ + z₂z₃⁷ + z₃z₄¹⁰⁶ | BP + Cycle | 1485| 29680| ℤ²⁰    |
| (17,14,105,75,45) | z₁⁵ + z₀z₂⁷ + z₂z₃³ + z₃z₄⁴ | Chain + Cycle | 255 | 3856| ℤ¹⁶     |
| (133,38,19,14,63) | z₀⁷ + z₁⁷ + z₁z₂⁴ + z₂z₃⁹ + z₃z₄⁴ | BP + Chain | 266 | 4524| ℤ²⁴     |
| \(w = (w_0, w_1, w_2, w_3, w_4)\) | Polynomial | Type | \(d\) | \(\mu\) | \(H_3(M, \mathbb{Z})\) |
|---|---|---|---|---|---|
| (119,21,14,153,51) | \(z_0^3 + z_1^{17} + z_1 z_2^{24} + z_4 z_3^2 + z_3 z_4^4\) | BP + Chain + Cycle | 357 | 6272 | \(\mathbb{Z}^{32}\) |
| (33,77,77,14,31) | \(z_0^3 + z_1^{11} + z_1 z_2^{11} + z_3 z_4^4\) | BP + Chain | 231 | 2400 | \(\mathbb{Z}^{12}\) |
| (315,90,135,14,77) | \(z_0^2 + z_1^7 + z_1 z_2^2 + z_4^{15} + z_3 z_4^8\) | BP + Chain + Chain | 630 | 6952 | \(\mathbb{Z}^{16}\) |
| (15,20,85,17,119) | \(z_0^17 + z_0 z_1^{12} + z_2^3 + z_2 z_3^{10} + z_3 z_4^2\) | Chain + Chain | 255 | 6016 | \(\mathbb{Z}^{32}\) |
| (24,30,15,35,17) | \(z_0^2 + z_1^4 + z_1 z_2^6 + z_2 z_3^3 + z_3 z_4^5\) | BP + Chain | 120 | 1236 | \(\mathbb{Z}^{12}\) |
| (65,39,15,18,59) | \(z_0^2 + z_1^4 + z_2^3 + z_2 z_3^{10} + z_3 z_4^3\) | BP + Chain | 195 | 2176 | \(\mathbb{Z}^{16}\) |
| (51,15,24,77,89) | \(z_0^2 + z_1^7 + z_1 z_2^{10} + z_2 z_3^3 + z_3 z_4^2\) | BP + Chain | 255 | 2656 | \(\mathbb{Z}^{16}\) |
| (143,26,39,15,207) | \(z_0^3 + z_0 z_1^{11} + z_1 z_2^6 + z_2 z_3^6 + z_3 z_4^2\) | Chain + Chain | 429 | 9176 | \(\mathbb{Z}^{32}\) |
| (215,15,315,33,68) | \(z_0^3 + z_1^{13} + z_1 z_2^5 + z_2 z_3^{10} + z_3 z_4^9\) | BP + Chain | 645 | 13848 | \(\mathbb{Z}^{24}\) |
| (176,48,15,171,119) | \(z_0^2 + z_1^{11} + z_2 z_3^2 + z_2 z_3^3 + z_3 z_4^5\) | BP + Chain | 528 | 4908 | \(\mathbb{Z}^{12}\) |
| (115,69,92,15,55) | \(z_0^3 + z_1^5 + z_1 z_2^3 + z_2 z_3^3 + z_3 z_4^6\) | BP + Chain + Chain | 345 | 2552 | \(\mathbb{Z}^{12}\) |
| (117,65,104,285,15) | \(z_0^3 + z_0 z_1^{15} + z_1 z_2^5 + z_2 z_3^{14} + z_3 z_4^{20}\) | BP + Chain + Cycle | 585 | 5920 | \(\mathbb{Z}^{16}\) |
| (395,15,65,474,237) | \(z_0^3 + z_1^{19} + z_1 z_2^{18} + z_2 z_3^2 + z_3 z_4^3\) | BP + Chain + Cycle | 1185 | 16128 | \(\mathbb{Z}^{24}\) |
| (132,110,275,15,129) | \(z_0^2 + z_1^9 + z_1 z_2^4 + z_3 z_4^3\) | BP + Chain + Chain | 660 | 4956 | \(\mathbb{Z}^{12}\) |
| (228,570,15,125,203) | \(z_0^5 + z_1^7 + z_1 z_2^3 + z_2 z_3^3 + z_3 z_4^2\) | BP + Chain | 1140 | 11244 | \(\mathbb{Z}^{12}\) |
| (655,15,130,367,799) | \(z_0^3 + z_0 z_1^{16} + z_1 z_2^5 + z_2 z_3^5 + z_3 z_4^2\) | BP + Chain | 1965 | 23320 | \(\mathbb{Z}^{20}\) |
| (740,222,999,15,245) | \(z_0^3 + z_1^{10} + z_1 z_2^7 + z_2 z_3^5 + z_3 z_4^9\) | BP + Chain + Chain | 2220 | 26070 | \(\mathbb{Z}^{18}\) |
| (200,16,127,39,19) | \(z_0^2 + z_1^{15} + z_2 z_3^3 + z_2 z_3^2 + z_3 z_4^9\) | BP + Cycle | 400 | 9576 | \(\mathbb{Z}^{24}\) |
| (135,240,765,17,1139) | \(z_0^7 + z_0 z_1^8 + z_2 z_3 + z_2 z_3^9 + z_3 z_4^2\) | Chain + Chain | 2295 | 37264 | \(\mathbb{Z}^{20}\) |
| (77,33,3,18,71) | \(z_0^3 + z_1^7 + z_1 z_2^5 + z_2 z_3^1 + z_3 z_4^3\) | BP + Chain | 231 | 1920 | \(\mathbb{Z}^{12}\) |
| (4928,896,2921,19,1093) | \(z_0^3 + z_1^{11} + z_2 z_3^3 + z_2 z_3^3 + z_3 z_4^9\) | BP + Cycle | 9856 | 98550 | \(\mathbb{Z}^{10}\) |
| w = (w_0, w_1, w_2, w_3, w_4) | Polynomial | Type | d | \( \mu \) |
|---|---|---|---|---|
| (55,22,99,20,25) | \( z_0^4 + z_1^{10} + z_1 z_2^2 + z_1 z_3 + z_3 z_4^8 \) | BP + Chain + Chain | 220 | 2574 |
| (2990,20,745,349,1877) | \( z_0^2 + z_1^{299} + z_1 z_2^5 + z_2 z_3^{15} + z_3 z_4^3 \) | BP + Chain | 5980 | 73854 |
| (77,21,35,66,33) | \( z_0^3 + z_1^{11} + z_1 z_2^6 + z_4 z_3^3 + z_3 z_4^5 \) | BP + Chain + Cycle | 231 | 1680 |
| (203,21,294,45,47) | \( z_0^3 + z_1 z_2^2 + z_2 z_3^7 + z_3 z_4^{12} \) | BP + Chain | 609 | 8992 |
| (119,119,51,21,48) | \( z_0^3 + z_1 z_2^7 + z_3 z_4^7 \) | BP + Chain | 357 | 2472 |
| (27,66,207,23,299) | \( z_0^2 + z_0 z_1 + z_2 + z_2 z_3 + z_3 z_4^2 \) | Chain + Chain | 621 | 10360 |
| (172,473,1978,23,1311) | \( z_0^2 + z_0 z_1 z_2^8 + z_2 z_3^6 + z_3 z_4^3 \) | Chain + Chain | 3956 | 55890 |
| (6615,2835,9472,23,901) | \( z_0^3 + z_1 + z_4 z_3^5 + z_2 z_3 + z_3 z_4^2 \) | BP + Cycle | 19845 | 238128 |
| (28,182,91,39,25) | \( z_0^3 + z_1 z_2^2 + z_2 z_3^7 + z_3 z_4^3 \) | BP + Chain | 364 | 4068 |
| (195,25,190,157,409) | \( z_0^3 + z_1 z_2^3 + z_2 z_3 + z_3 z_4^2 \) | BP + Chain | 975 | 4528 |
| (1045,418,190,25,413) | \( z_0^2 + z_1 z_2^7 + z_2 z_3^6 + z_3 z_4^5 \) | BP + Chain | 2090 | 13416 |
| (9610,620,25,3839,5127) | \( z_0 + z_1 z_2 + z_2 z_3 + z_3 z_4^4 \) | BP + Chain | 19220 | 253674 |
| (4928,896,3277,731,25) | \( z_0^2 + z_1 + z_4 z_3^2 + z_2 z_3^9 + z_3 z_4^{365} \) | BP + Cycle | 9856 | 98550 |
| (170,102,51,27,161) | \( z_0^4 + z_1 z_2^5 + z_2 z_3 + z_3 z_4^2 \) | BP + Chain | 510 | 2792 |
| (92,414,207,27,89) | \( z_0^9 + z_1 z_2^7 + z_2 z_3^7 + z_3 z_4^2 \) | BP + Chain | 828 | 5912 |
| (1251,27,414,371,1691) | \( z_0^1 + z_1 z_2^3 + z_3 + z_4^9 \) | BP + Chain | 3753 | 24744 |
| (28,161,658,47,423) | \( z_0^4 + z_1 z_2^7 + z_2 z_3 + z_3 z_4^2 \) | BP + Chain | 1316 | 18810 |
| (455,105,28,191,587) | \( z_0^4 + z_1 z_2^7 + z_2 z_3 + z_3 z_4^2 \) | BP + Chain | 1365 | 9336 |
| (854,28,105,229,493) | \( z_0^2 + z_1 z_2 + z_2 z_3 + z_3 z_4^3 \) | BP + Chain | 1708 | 14580 |
| (16646,4756,29,1147,10715) | \( z_0^2 + z_1 z_2 + z_2 z_3 + z_3 z_4^2 \) | BP + Chain | 33292 | 406386 |
| (84,35,77,30,195) | \( z_0^5 + z_1 z_2^5 + z_3 + z_4^2 \) | BP + Chain + Chain | 420 | 2940 |
$$w = (w_0, w_1, w_2, w_3, w_4)$$

| Polynomial                                      | Type           | $d$ | $\mu$  | $H_3(M, \mathbb{Z})$ |
|------------------------------------------------|----------------|-----|--------|----------------------|
| $(405,162,30,65,149)$                           | $z_0^2 + z_1^5 + z_2^{27} + z_2 z_3^{12} + z_3 z_4^5$ | BP + Chain   | 810     | 5288                | $\mathbb{Z}^8$ |
| $(1245,830,30,123,263)$                         | $z_0^2 + z_1^3 + z_2^{83} + z_2 z_3^{20} + z_3 z_4^9$ | BP + Chain   | 2490    | 26724               | $\mathbb{Z}^{12}$ |
| $(319,87,58,31,463)$                            | $z_0^3 + z_1^{11} + z_1 z_2^{15} + z_2 z_3^{29} + z_3 z_4^2$ | BP + Chain   | 957     | 9880                | $\mathbb{Z}^{20}$ |
| $(1323,567,1921,32,127)$                        | $z_0^3 + z_1^7 + z_4 z_3^{2} + z_2 z_3^{64} + z_3 z_4^{31}$ | BP + Cycle   | 3969    | 47616               | $\mathbb{Z}^{12}$ |
| $(341,93,62,495,33)$                            | $z_0^3 + z_1^{11} + z_1 z_2^{15} + z_4 z_3^{2} + z_3 z_4^{16}$ | BP + Chain + Cycle | 1023   | 9920                | $\mathbb{Z}^{20}$ |
| $(1173,782,102,33,257)$                         | $z_0^2 + z_1^3 + z_2^{23} + z_2 z_3^{68} + z_3 z_4^9$ | BP + Chain   | 2346    | 25068               | $\mathbb{Z}^{12}$ |
| $(440,330,33,117,401)$                         | $z_0^3 + z_1^4 + z_1 z_2^{30} + z_2 z_3^{11} + z_3 z_4^{3}$ | BP + Chain   | 1320    | 5514                | $\mathbb{Z}^{6}$  |
| $(935,330,308,1275,255)$                        | $z_0^3 + z_1^{85} + z_1 z_2^{9} + z_4 z_3^{2} + z_3 z_4^{5}$ | BP + Chain + Cycle | 2805   | 16344               | $\mathbb{Z}^{12}$ |
| $(1700,510,2295,33,563)$                        | $z_0^3 + z_1^{10} + z_1 z_2^{7} + z_2 z_3^{85} + z_3 z_4^{9}$ | BP + Chain   | 5100    | 27222               | $\mathbb{Z}^{6}$  |
| $(55,77,46,175,35)$                            | $z_0^7 + z_1^5 + z_1 z_2^{5} + z_2 z_3^{6} + z_3 z_4^{6}$ | BP + Chain + Cycle | 385    | 2232                | $\mathbb{Z}^{12}$ |
| $(189,35,182,405,135)$                         | $z_0^5 + z_1^{27} + z_1 z_2^{5} + z_4 z_3^{2} + z_3 z_4^{4}$ | BP + Chain + Cycle | 945    | 3488                | $\mathbb{Z}^{8}$  |
| $(324,270,675,35,317)$                         | $z_0^3 + z_1^6 + z_1 z_2^{3} + z_2 z_3^{7} + z_3 z_4^{3}$ | BP + Chain   | 1620    | 5212                | $\mathbb{Z}^{4}$  |
| $(6102,36,507,3899,1661)$                      | $z_0^2 + z_1^{339} + z_1 z_2^{24} + z_2 z_3^{3} + z_3 z_4^{5}$ | BP + Chain   | 12204   | 105430              | $\mathbb{Z}^{10}$ |
| $(345,45,110,37,499)$                          | $z_0^3 + z_1^{3} + z_1 z_2^{2} + z_2 z_3^{5} + z_3 z_4^{2}$ | BP + Chain   | 1035    | 10720               | $\mathbb{Z}^{12}$ |
| $(315,175,280,37,769)$                         | $z_0^5 + z_1^9 + z_1 z_2^{5} + z_2 z_3^{35} + z_3 z_4^{2}$ | BP + Chain   | 1575    | 6448                | $\mathbb{Z}^{8}$  |
| $(27306,18204,37,1475,7591)$                   | $z_0^5 + z_1^3 + z_1 z_2^{984} + z_2 z_3^{13} + z_3 z_4^{15}$ | BP + Chain   | 54612   | 658294              | $\mathbb{Z}^{14}$ |
| $(416,624,39,93,77)$                          | $z_0^3 + z_1^2 + z_1 z_2^{16} + z_2 z_3^{13} + z_3 z_4^{15}$ | BP + Chain   | 1248    | 11710               | $\mathbb{Z}^{10}$ |
| $(6615,2835,9901,1452,43)$                     | $z_0^2 + z_1^7 + z_4 z_2^{3} + z_2 z_3^{5} + z_3 z_4^{51}$ | BP + Cycle   | 19845   | 238128              | $\mathbb{Z}^{12}$ |
| $(255,45,60,47,359)$                          | $z_0^3 + z_1^{17} + z_1 z_2^{12} + z_2 z_3^{15} + z_3 z_4^{2}$ | BP + Chain   | 765     | 6496                | $\mathbb{Z}^{16}$ |
| Polynomial                                | Type                | $d$ | $\mu$ | $H_3(M, \mathbb{Z})$ |
|-------------------------------------------|---------------------|-----|-------|----------------------|
| $z_0^3 + z_1^{11} + z_1 z_0^2 + z_2 z_0^3 + z_3 z_0^4$ | BP + Chain          | 891 | 7504  | $\mathbb{Z}^{12}$   |
| $z_0^3 + z_1^4 + z_1 z_0^2 + z_2 z_0^3 + z_3 z_0^4$ | BP + Chain          | 1392| 5658  | $\mathbb{Z}^{6}$    |
| $z_0^5 + z_1^4 z_0^3 + z_4 z_0^2 + z_3 z_0^4$        | BP + Chain + Cycle  | 1935| 6512  | $\mathbb{Z}^{6}$    |
| $z_0^2 + z_1^3 z_0^2 + z_2 z_0^3 + z_3 z_0^4$          | BP + Chain          | 13200| 158494| $\mathbb{Z}^{14}$   |
| $z_0^2 + z_1^3 z_0^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 988 | 8100  | $\mathbb{Z}^{12}$   |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 1824 | 9762  | $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 5992 | 72234 | $\mathbb{Z}^{18}$   |
| $z_0^2 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 68992| 413946| $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 3969 | 47616 | $\mathbb{Z}^{12}$   |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 1560 | 5084  | $\mathbb{Z}^{4}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 17920| 107514| $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 8088 | 80878 | $\mathbb{Z}^{10}$   |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 6993 | 43080 | $\mathbb{Z}^{12}$   |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 12544| 75258 | $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 6004 | 24570 | $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 11583| 25456 | $\mathbb{Z}^{4}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 68992| 413946| $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 10192| 61146 | $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Cycle          | 17920| 107514| $\mathbb{Z}^{6}$    |
| $z_0^2 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 8300 | 36438 | $\mathbb{Z}^{6}$    |
| $z_0^3 + z_1^2 + z_2 z_0^3 + z_3 z_0^4$        | BP + Chain          | 10605| 64248 | $\mathbb{Z}^{12}$   |
### Table 2 continued

| \( w = (w_0, w_1, w_2, w_3, w_4) \) | Polynomial | Type       | \( d \) | \( \mu \) | \( H_3(M, \mathbb{Z}) \) |
|-----------------|-----------|------------|--------|--------|-----------------|
| (2666,172,645,109,1741) | \( z_0^2 + z_1^{31} + z_2 z_3^{43} + z_3 z_4^{41} + z_4 z_2^{53} \) | BP + Chain  | 5332   | 21546  | \( \mathbb{Z}^6 \) |
| (6272,1792,2145,227,109) | \( z_0^2 + z_1^7 + z_2 z_3^{37} + z_3 z_4^{113} \) | BP + Cycle   | 12544  | 75258  | \( \mathbb{Z}^6 \) |
| (5096,1456,3355,159,127) | \( z_0^2 + z_1^7 + z_2 z_3^{43} + z_3 z_4^{29} \) | BP + Cycle   | 10192  | 61146  | \( \mathbb{Z}^6 \) |
| (10368,6912,1741,131,1585) | \( z_0^2 + z_1^3 + z_2 z_3^{11} + z_3 z_4^{13} \) | BP + Cycle   | 20736  | 41470  | \( \mathbb{Z}^2 \) |
| (10368,6912,1873,1451,133) | \( z_0^2 + z_1^3 + z_2 z_3^{11} + z_3 z_4^{145} \) | BP + Cycle   | 20736  | 41470  | \( \mathbb{Z}^2 \) |
| (18970,5420,813,137,12601) | \( z_0^2 + z_1^7 + z_2 z_3^{40} + z_3 z_4^{24} \) | BP + Chain   | 37940  | 152034 | \( \mathbb{Z}^6 \) |
| (6885,1215,2160,137,10259) | \( z_0^3 + z_1^7 + z_2 z_3^{43} + z_3 z_4^{13} \) | BP + Chain   | 20655  | 41584  | \( \mathbb{Z}^4 \) |
| (3240,2160,559,191,331) | \( z_0^3 + z_1^3 + z_2 z_3^{11} + z_3 z_4^{19} \) | BP + Cycle   | 6480   | 12958  | \( \mathbb{Z}^2 \) |
| (3240,2160,571,311,199) | \( z_0^3 + z_1^3 + z_2 z_3^{11} + z_3 z_4^{21} \) | BP + Cycle   | 6480   | 12958  | \( \mathbb{Z}^2 \) |
| (168480,112320,46837,223,9101) | \( z_0^2 + z_1^3 + z_2 z_3^{37} + z_3 z_4^{137} \) | BP + Chain   | 336960 | 673918 | \( \mathbb{Z}^2 \) |
| (41472,27648,11561,247,2017) | \( z_0^2 + z_1^3 + z_2 z_3^{28} + z_3 z_4^{81} \) | BP + Cycle   | 82944  | 165886 | \( \mathbb{Z}^2 \) |
| (168480,112320,48101,7807,253) | \( z_0^2 + z_1^3 + z_2 z_3^{37} + z_3 z_4^{1301} \) | BP + Cycle   | 336960 | 673918 | \( \mathbb{Z}^2 \) |
| (41472,27648,11809,1735,281) | \( z_0^2 + z_1^3 + z_2 z_3^{41} + z_3 z_4^{289} \) | BP + Cycle   | 82944  | 165886 | \( \mathbb{Z}^2 \) |
| (24840,16560,6947,283,1051) | \( z_0^2 + z_1^3 + z_2 z_3^{151} + z_3 z_4^{7} \) | BP + Cycle   | 49680  | 99356  | \( \mathbb{Z}^2 \) |
| (24840,16560,7051,907,323) | \( z_0^2 + z_1^3 + z_2 z_3^{47} + z_3 z_4^{151} \) | BP + Cycle   | 49680  | 99356  | \( \mathbb{Z}^2 \) |
| (18792,12528,5279,355,631) | \( z_0^2 + z_1^3 + z_2 z_3^{7} + z_3 z_4^{59} \) | BP + Cycle   | 37584  | 75166  | \( \mathbb{Z}^2 \) |
| (18792,12528,5311,547,407) | \( z_0^2 + z_1^3 + z_2 z_3^{9} + z_3 z_4^{91} \) | BP + Cycle   | 37584  | 75166  | \( \mathbb{Z}^2 \) |
| $w = (w_0, w_1, w_2, w_3, w_4)$ | Polynomial | Type | $d$ | $\mu$ | $H_3(M, \mathbb{Z})$ |
|--------------------------------|------------|------|-----|-------|-----------------|
| (1,1,4,6)                     | $z_0^{12} + z_1^{12} + z_2^{12} + z_3^2 + z_4^2$ | BP   | 12  | 2662  | $\mathbb{Z}^{222}$ |
| (7,3,10,1)                    | $z_0^3 + z_1^7 + z_2^{18} + z_3 z_4^2 + z_3 z_4^{11}$ | BP + Chain | 21  | 5280  | $\mathbb{Z}^{252}$ |
| (14,4,1,9,1)                  | $z_0^2 + z_1^7 + z_2^{24} + z_2 z_3^3 + z_3 z_4^{19}$ | BP + Chain | 28  | 9234  | $\mathbb{Z}^{330}$ |
| (11,3,5,14,1)                 | $z_0^3 + z_1^{11} + z_1 z_2^2 + z_2 z_3^2 + z_3 z_4^{19}$ | BP + Chain | 33  | 4864  | $\mathbb{Z}^{148}$ |
| (18,12,1,5,1)                 | $z_0^2 + z_1^3 + z_1 z_2^{24} + z_2 z_3^7 + z_3 z_4^{31}$ | BP + Chain | 36  | 15,190 | $\mathbb{Z}^{422}$ |
| (1,1,6,14,21)                 | $z_0^{12} + z_1^{42} + z_2^7 + z_3^3 + z_4^2$ | BP   | 42  | 20,172 | $\mathbb{Z}^{480}$ |
| (22,4,13,1)                   | $z_0^3 + z_1^{11} + z_1 z_2^8 + z_2 z_3^3 + z_3 z_4^{31}$ | BP + Chain | 44  | 7998  | $\mathbb{Z}^{182}$ |
| (33,22,6,5,1)                 | $z_0^2 + z_1^{11} + z_2^{12} + z_2 z_3^{12} + z_3 z_4^{51}$ | BP + Chain | 66  | 15,860 | $\mathbb{Z}^{240}$ |
Proof This is just a consequence of Theorem A in [13] (also see Appendix B in [3]), and the fact that for each \( k \neq 17 \) ranging from 3 to 21 there is an element in Cheltsov’s list that can be given as an hypersurface singularity that is of one of the types admissible for Orlik’s conjecture.

In Appendix (c), we also include Tables II and III which list the third homology group of links of hypersurface singularities which are neither rational homology 7-spheres nor homeomorphic to connected sums of \( S^3 \times S^4 \). Table II consists of 79 links coming from Cheltsov’s list and Table III consists of 1503 links extracted from the Johnson and Kollár list.

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Declarations

Conflict of interest We declare that the authors have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

Ethics approval and consent to participate Informed consent was obtained from all individual participants included in the study.

Consent for publication The authors have read and understood the publishing policy, and submit this manuscript in accordance with this policy.

Appendix

(a) The Johnson–Kollár list of hypersurfaces in weighted projective 4-space admitting Kähler–Einstein orbifold metrics is available at https://web.math.princeton.edu/~jmjohnso/delpezzo/KEandTiger.txt. This list includes the weight vectors followed by data on whether or not it is known if the orbifold is Kähler–Einstein.

(b) Table I, cited in Theorem 3.6 is given here https://github.com/Jcuadrosvalle/TABLES

(c) Table II and Table III, where we exhibit the third homology group of links of hypersurface singularities that are neither rational homology 7-spheres nor homeomorphic to connected sums of \( S^3 \times S^4 \), are available at https://github.com/Jcuadrosvalle/TABLES

(d) Four codes in MATLAB are available at https://github.com/Jcuadrosvalle/Codes-in-Matlab. Codes1, 2 and 3 determine whether or not the singularities are of chain type, cycle type or an iterated Thom–Sebastiani sum of chain type and cycle type singularities, Code4 computes the third homology groups of the corresponding links.
References

1. Baum, H., Friedrich, T., Grunewald, R., Kath, I.: Twistors and Killing Spinors on Riemannian Manifolds. Teubner-Texte für Mathematik, vol. 124. Teubner, Stuttgart, Leipzig (1991)
2. Boyer, C.P., Galicki, K.: On Sasakian-Einstein Geometry. Internat. J. Math. 11(7), 873–909 (2000)
3. Boyer, C.P., Galicki, K.: Sasakian Geometry. Oxford University Press, Oxford (2008)
4. Boyer, C.P., Galicki, K., Nakamaye, M.: On the Geometry of Sasakian-Einstein 5-Manifold. Math. Ann. 325, 485–524 (2003)
5. Boyer, C.P., Galicki, K., Nakamaye, M.: Einstein Metrics on Rational Homology 7-Spheres. Ann. Inst. Fourier 52(5), 1569–1584 (2002)
6. Boyer, C.P., Galicki, K., Nakamaye, M.: Sasakian geometry, homotopy spheres and positive Ricci curvature. Topology 42, 981–1002 (2003)
7. Boyer, C.P., Galicki, K., Kollár, J.: Einstein metrics on spheres. Ann. Math. 162, 557–580 (2005)
8. Boyer, C.P., Galicki, K., Kollár, J., Thomas, E.: Einstein metrics on exotic spheres in dimensions 7, 11, and 15. Exp. Math. 14(1), 59–64 (2005)
9. Boyer, C.P., Galicki, K., Simanca S.: The Sasaki cone and Extremal Sasakian Metrics Riemannian Topology and Geometric Structures on Manifolds Proceedings of the Conference on Riemannian Topology and Geometric Structures on Manifolds (2006)
10. Boyer, C.P., Galicki, K., Simanca, S.: Canonical Sasakian metrics. Comm. Math. Phys. 279(3), 705–733 (2008)
11. Boyer, C.P.: Sasakian geometry: the recent work of Krzysztof Galicki. Note Mat. 28, 63–105 (2008)
12. Cheltsov, I.: Fano varieties with many self-maps. Adv. Math. 217, 97–124 (2008)
13. Cuadros, J.: Null Sasaki $\eta$-Einstein structures in 5-manifolds. Geom. Dedicata 169, 343–359 (2014)
14. Durfee, A.H., Kaufmann, L.: Periodicity of branched cyclic covers. Math. Ann. 218, 157–174 (1975)
15. Friedrich, Th., Kath, I.: 7-dimensional compact Riemannian manifolds with Killing spinors. Comm. Math. Phys. 133, 543–561 (1990)
16. Gauntlett, J., Martelli, D., Sparks, J.: Obstructions to the existence of Sasaki-Einstein metrics. Comm. Math. Phys. 273, 803–827 (2006)
17. Gomez, R.R.: Sasaki-Einstein 7-manifolds, Orlik polynomials and homology. Symmetry 11(7), 947 (2019). https://doi.org/10.3390/sym11070947
18. Hertling, C., Mase, M.: The integral monodromy of isolated quasihomogeneous singularities. Algebra Number Theory 16(4), 955–1024 (2022)
19. Iano-Fletcher, A.R.: Working with Weighted Complete Intersections, Explicit Birational Geometry of 3-folds, London Math. Soc. Lecture Notes Ser., vol. 281, Cambridge Univ. Press, Cambridge, pp. 101–173 (2000)
20. Johnson, J.M., Kollár, J.: Fano hypersurfaces in weighted projective 4-space. Exp. Math. 10(1), 151–158 (2004)
21. Kollár, J.: Einstein metrics of five dimensional Seifert bundles. J. Geom. Anal. 15, 445–476 (2005)
22. Milnor, J.: Singular Points of Complex Hypersurfaces. Annals of Mathematical Studies, vol. 61. Princeton University Press, Princeton (1968)
23. Milnor, J., Orlik, P.: Isolated singularities defined by weighted homogeneous polynomials. Topology 9, 385–393 (1970)
24. Orlik, P.: On the homology of weighted homogenous manifolds. In: Proceedings of the Second Conference on Compact Transformation Groups (Univ. Mass, Amherst, Mass 1971) Part I (Berlin), Spring, pp. 260–269 (1972)
25. Savel’ev, I.V.: Structure of singularities of a class of complex hypersurfaces. Mat. Zametki 25(4), 497–503 (1979)
26. Takahashi, T.: Deformations of Sasakian structures and its application to the Brieskorn manifolds. Tohoku Math. J. (2) 30(1), 37–43 (1978)
27. Xie, D., Yau, S.-T.: Singularity, Sasaki-Einstein manifold, Log del Pezzo surface and $N = 1AdS/CFT$ correspondence: part I, arXiv:1903.00150 [INSPIRE] (2019)

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