BIHARMONIC MAPS INTO A RIEMANNIAN MANIFOLD OF NON-POSITIVE CURVATURE

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Abstract. We study biharmonic maps between Riemannian manifolds with finite energy and finite bi-energy. We show that if the domain is complete and the target of non-positive curvature, then such a map is harmonic. We then give applications to isometric immersions and horizontally conformal submersions.

1. Introduction

Harmonic maps play a central role in geometry. They are critical points of the energy functional

\[ E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 v_g \]

for smooth maps \( \varphi \) of \((M, g)\) into \((N, h)\), and the Euler-Lagrange equation is that the tension filed \( \tau(\varphi) \) vanishes. By extending notion of harmonic map, in 1983, J. Eells and L. Lemaire [6] proposed a problem to consider the biharmonic maps which are, by definition, critical points of the bienergy functional

\[ E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g. \] (1.1)

After G.Y. Jiang [15] studied the first and second variation formulas of \( E_2 \), extensive studies in this area have been done (for instance, see [2], [4], [18], [19], [21], [24], [26], [12], [13], [14], etc.). Notice that harmonic maps are always biharmonic by definition.

For harmonic maps, it is well known that:

If a domain manifold \((M, g)\) is complete and has non-negative Ricci curvature, and the sectional curvature of a target manifold \((N, h)\) is non-positive, then every energy finite harmonic map is a constant map (cf. [27]).

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Therefore, it is a natural question to consider biharmonic maps into a Riemannian manifold of non-positive curvature. In this connection, Baird, Fardoun and Ouakkas (cf. [2]) showed that:

If a non-compact Riemannian manifold \((M, g)\) is complete and has non-negative Ricci curvature and \((N, h)\) has non-positive sectional curvature, then every bienergy finite biharmonic map of \((M, g)\) into \((N, h)\) is harmonic.

In this paper, we will show that

**Theorem 1.1.** (cf. Theorem 2.1) Under only the assumptions of completeness of \((M, g)\) and non-positivity of curvature of \((N, h)\),

1. every biharmonic map \(\varphi : (M, g) \to (N, h)\) with finite energy and finite bienergy must be harmonic.
2. In the case \(\text{Vol}(M, g) = \infty\), under the same assumption, every biharmonic map \(\varphi : (M, g) \to (N, h)\) with finite bienergy is harmonic.

We do not need any assumption on the Ricci curvature of \((M, g)\) in Theorem 1.1. Since \((M, g)\) is a non-compact complete Riemannian manifold whose Ricci curvature is non-negative, then \(\text{Vol}(M, g) = \infty\) (cf. Theorem 7, p. 667, [28]). Thus, Theorem 1.1, (2) recovers the result of Baird, Fardoun and Ouakkas. Furthermore, Theorem 1.1 is sharp because one can not weaken the assumptions because the generalized Chen’s conjecture does not hold if \((M, g)\) is not complete (cf. recall the counter examples of Ou and Tang [25]). The both assumptions of finitenesses of the energy and bienergy are also necessary. Indeed, there exists a biharmonic map \(\varphi\) which is not harmonic, but energy and bienergy are infinite. For example, \(f(x) = r(x)^2 = \sum_{i=1}^{m} (x_i)^2, x = (x_1, \ldots, x_m) \in \mathbb{R}^m\) is biharmonic, but not harmonic, and have infinite energy and bienergy.

As the first bi-product of our method, we obtain (cf. [22], [23])

**Theorem 1.2.** (cf. Theorem 3.1) Assume that \((M, g)\) is a complete Riemannian manifold, and let \(\varphi : (M, g) \to (N, h)\) is an isometric immersion, and the sectional curvature of \((N, h)\) is non-positive. If \(\varphi : (M, g) \to (N, h)\) is biharmonic and \(\int_M |\xi|^2_v g < \infty\), then it is minimal. Here, \(\xi\) is the mean curvature normal vector field of the isometric immersion \(\varphi\).

Theorem 1.2 (cf. Theorem 3.1) gives an affirmative answer to the generalized B.Y. Chen’s conjecture (cf. [4]) under natural conditions.

For the second bi-product, we can apply Theorem 1.1 to a horizontally conformal submersion (cf. [1],[3]). Then, we have
Theorem 1.3. (cf. Corollary 3.4) Let \((M^m, g)\) be a non-compact complete Riemannian manifold \((m > 2)\), and \((N^2, h)\), a Riemannian surface with non-positive curvature. Let \(\lambda\) be a positive function on \(M\) belonging to \(C^\infty(M) \cap L^2(M)\), and \(\varphi : (M, g) \to (N^2, h)\), a horizontally conformal submersion with a dilation \(\lambda\). If \(\varphi\) is biharmonic and \(\lambda|\hat{H}|_g \in L^2(M)\), then \(\varphi\) is a harmonic morphism. Here, \(\hat{H}\) is trace of the second fundamental form of each fiber of \(\varphi\).

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2. Preliminaries and statement of main theorem

In this section, we prepare materials for the first and second variational formulas for the bienergy functional and biharmonic maps. Let us recall the definition of a harmonic map \(\varphi : (M, g) \to (N, h)\), of a compact Riemannian manifold \((M, g)\) into another Riemannian manifold \((N, h)\), which is an extremal of the energy functional defined by

\[ E(\varphi) = \int_M e(\varphi) v_g, \]

where \(e(\varphi) := \frac{1}{2} |d\varphi|^2\) is called the energy density of \(\varphi\). That is, for any variation \(\{\varphi_t\}\) of \(\varphi\) with \(\varphi_0 = \varphi\),

\[ \frac{d}{dt} \bigg|_{t=0} E(\varphi_t) = -\int_M h(\tau(\varphi), V) v_g = 0, \quad (2.1) \]

where \(V \in \Gamma(\varphi^{-1}TN)\) is a variation vector field along \(\varphi\) which is given by \(V(x) = \frac{d}{dt}|_{t=0}\varphi_t(x) \in T_{\varphi(x)}N, (x \in M)\), and the tension field is given by \(\tau(\varphi) = \sum_{i=1}^m B(\varphi)(e_i, e_i) \in \Gamma(\varphi^{-1}TN)\), where \(\{e_i\}_{i=1}^m\) is a locally defined frame field on \((M, g)\), and \(B(\varphi)\) is the second fundamental form of \(\varphi\) defined by

\[ B(\varphi)(X, Y) = (\nabla_X d\varphi)(Y) = (\nabla_X d\varphi)(Y) = \nabla_X(d\varphi(Y)) - d\varphi(\nabla_X Y), \quad (2.2) \]

for all vector fields \(X, Y \in \mathfrak{X}(M)\). Here, \(\nabla\), and \(\nabla^N\), are connections on \(TM, TN\) of \((M, g)\), \((N, h)\), respectively, and \(\nabla\), and \(\nabla\) are the
induced ones on $\varphi^{-1}TN$, and $T^*M \otimes \varphi^{-1}TN$, respectively. By \eqref{eq:2.1}, $\varphi$ is harmonic if and only if $\tau(\varphi) = 0$.

The second variation formula is given as follows. Assume that $\varphi$ is harmonic. Then,

$$
\frac{d^2}{dt^2}\bigg|_{t=0} E(\varphi_t) = \int_M h(J(V), V) v_g,
$$

where $J$ is an elliptic differential operator, called the \textit{Jacobi operator} acting on $\Gamma(\varphi^{-1}TN)$ given by

$$
J(V) = \Delta V - R(V),
$$

where $\Delta V = \nabla^* \nabla V = -\sum_{i=1}^m \{\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V\}$ is the \textit{rough Laplacian} and $R$ is a linear operator on $\Gamma(\varphi^{-1}TN)$ given by $R(V) = \sum_{i=1}^m R^N(V, d\varphi(e_i))d\varphi(e_i)$, and $R^N$ is the curvature tensor of $(N, h)$ given by $R^N(U, V) = \nabla^N_U \nabla^N_V - \nabla^N_V \nabla^N_U - \nabla^N_{[U, V]}$ for $U, V \in \mathfrak{X}(N)$.

J. Eells and L. Lemaire \cite{6} proposed polyharmonic ($k$-harmonic) maps and Jiang \cite{15} studied the first and second variation formulas of biharmonic maps. Let us consider the \textit{bienergy functional} defined by

$$
E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,
$$

where $|V|^2 = h(V, V), V \in \Gamma(\varphi^{-1}TN)$. Then, the first variation formula of the bienergy functional is given (the first variation formula) by

$$
\frac{d}{dt}\bigg|_{t=0} E_2(\varphi_t) = -\int_M h(\tau_2(\varphi), V) v_g.
$$

Here,

$$
\tau_2(\varphi) := J(\tau(\varphi)) = \Delta(\tau(\varphi)) - R(\tau(\varphi)),
$$

which is called the \textit{bitension field} of $\varphi$, and $J$ is given in \eqref{eq:2.4}.

A smooth map $\varphi$ of $(M, g)$ into $(N, h)$ is said to be \textit{biharmonic} if $\tau_2(\varphi) = 0$.

Then, we can state our main theorem.

\textbf{Theorem 2.1.} Assume that $(M, g)$ is complete and the sectional curvature of $(N, h)$ is non-positive.

(1) Every biharmonic map $\varphi : (M, g) \to (N, h)$ with finite energy $E(\varphi) < \infty$ and finite bienergy $E_2(\varphi) < \infty$, is harmonic.

(2) In the case $\text{Vol}(M, g) = \infty$, every biharmonic map $\varphi : (M, g) \to (N, h)$ with finite bienergy $E_2(\varphi) < \infty$, is harmonic.
3. Proof of Main Theorem and Two Applications

In this section we will give a proof of Theorem 2.1 which consists of four steps.

(The first step) For a fixed point \( x_0 \in M \), and for every \( 0 < r < \infty \), we first take a cut-off \( C^\infty \) function \( \eta \) on \( M \) (for instance, see [16]) satisfying that

\[
\begin{align*}
0 \leq & \eta(x) \leq 1 \quad (x \in M), \\
\eta(x) = & 1 \quad (x \in B_r(x_0)), \\
\eta(x) = & 0 \quad (x \notin B_{2r}(x_0)), \\
|\nabla \eta| & \leq \frac{2}{r} \quad (x \in M).
\end{align*}
\]

(3.1)

For a biharmonic map \( \varphi : (M, g) \to (N, h) \), the bitension field is given as

\[
\tau_2(\varphi) = \overline{\Delta}(\tau(\varphi)) - \sum_{i=1}^m R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i) = 0,
\]

(3.2)

so we have

\[
\int_M \langle \overline{\Delta}(\tau(\varphi)), \eta^2 \tau(\varphi) \rangle v_g = \int_M \eta^2 \sum_{i=1}^m \langle R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i), \tau(\varphi) \rangle v_g \\
\leq 0,
\]

(3.3)

since the sectional curvature of \((N, h)\) is non-positive.

(The second step) Therefore, by (3.3) and noticing that \( \overline{\Delta} = \nabla^\ast \nabla \), we obtain

\[
0 \geq \int_M \langle \overline{\Delta}(\tau(\varphi)), \eta^2 \tau(\varphi) \rangle v_g \\
= \int_M \langle \nabla \tau(\varphi), \nabla(\eta^2 \tau(\varphi)) \rangle v_g \\
= \int_M \sum_{i=1}^m \langle \nabla_{e_i}\tau(\varphi), \nabla_{e_i}(\eta^2 \tau(\varphi)) \rangle v_g \\
= \int_M \sum_{i=1}^m \left\{ \eta^2 \langle \nabla_{e_i}\tau(\varphi), \nabla_{e_i}\tau(\varphi) \rangle + e_i(\eta^2) \langle \nabla_{e_i}\tau(\varphi), \tau(\varphi) \rangle \right\} v_g \\
= \int_M \eta^2 \sum_{i=1}^m \left| \nabla_{e_i}\tau(\varphi) \right|^2 v_g + 2 \int_M \sum_{i=1}^m \langle \eta \nabla_{e_i}\tau(\varphi), e_i(\eta) \tau(\varphi) \rangle v_g.
\]

(3.4)
where we used $e_i(\eta^2) = 2\eta e_i(\eta)$ at the last equality. By moving the
second term in the last equality of (3.4) to the left hand side, we have
\[
\int_M \eta^2 \sum_{i=1}^m |\nabla e_i \tau(\varphi)|^2 v_g \leq -2 \int_M \sum_{i=1}^m \langle \eta \nabla e_i \tau(\varphi), e_i(\eta) \tau(\varphi) \rangle v_g
\]
\[
= -2 \int_M \sum_{i=1}^m \langle V_i, W_i \rangle v_g, \tag{3.5}
\]
where we put $V_i := \eta \nabla e_i \tau(\varphi)$, and $W_i := e_i(\eta) \tau(\varphi)$ ($i = 1 \cdots, m$).

Now let recall the following Cauchy-Schwartz inequality:
\[
\pm 2 \langle V_i, W_i \rangle \leq \epsilon |V_i|^2 + \frac{1}{\epsilon} |W_i|^2 \tag{3.6}
\]
for all positive $\epsilon > 0$ because of the inequality $0 \leq |\sqrt{\epsilon} V_i \pm \frac{1}{\sqrt{\epsilon}} W_i|^2$.

Therefore, for (3.5), we obtain
\[
-2 \int_M \sum_{i=1}^m \langle V_i, W_i \rangle v_g \leq \epsilon \int_M \sum_{i=1}^m |V_i|^2 v_g + \frac{1}{\epsilon} \int_M \sum_{i=1}^m |W_i|^2 v_g. \tag{3.7}
\]
If we put $\epsilon = \frac{1}{2}$, we obtain, by (3.5) and (3.7),
\[
\int_M \eta^2 \sum_{i=1}^m |\nabla e_i \tau(\varphi)|^2 v_g \leq \frac{1}{2} \int_M \sum_{i=1}^m \eta^2 |\nabla e_i \tau(\varphi)|^2 v_g
\]
\[
+ 2 \int_M \sum_{i=1}^m e_i(\eta)^2 |\tau(\varphi)|^2 v_g. \tag{3.8}
\]
Thus, by (3.8) and (3.1), we obtain
\[
\int_M \eta^2 \sum_{i=1}^m |\nabla e_i \tau(\varphi)|^2 v_g \leq 4 \int_M |\nabla \eta|^2 |\tau(\varphi)|^2 v_g
\]
\[
\leq \frac{16}{r^2} \int_M |\tau(\varphi)|^2 v_g. \tag{3.9}
\]

(The third step) Since $(M, g)$ is complete and non-compact, we can
tend $r$ to infinity. By the assumption $E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g < \infty$, the
right hand side goes to zero. And also, if $r \to \infty$, the left hand side
of (3.9) goes to $\int_M \sum_{i=1}^m |\nabla e_i \tau(\varphi)|^2 v_g$ since $\eta = 1$ on $B_r(x_0)$. Thus, we obtain
\[
\int_M \sum_{i=1}^m |\nabla e_i \tau(\varphi)|^2 v_g = 0. \tag{3.10}
\]
Therefore, we obtain, for every vector field $X$ in $M$,
\[
\nabla_X \tau(\varphi) = 0. \tag{3.11}
\]
Then, we have, in particular, \(|\tau(\varphi)|\) is constant, say \(c\). Because, for every vector field \(X\) on \(M\), at each point in \(M\),

\[
X |\tau(\varphi)|^2 = 2\langle \nabla_X \tau(\varphi), \tau(\varphi) \rangle = 0.
\]  

(3.12)

Therefore, if \(\text{Vol}(M, g) = \infty\) and \(c \neq 0\), then

\[
\tau_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g = \frac{c^2}{2} \text{Vol}(M, g) = \infty 
\]  

(3.13)

which yields a contradiction. Thus, we have \(|\tau(\varphi)| = c = 0\), i.e., \(\varphi\) is harmonic. We have (2).

(The fourth step) For (1), assume both \(E(\varphi) < \infty\) and \(E_2(\varphi) < \infty\). Then, let us consider a 1-form \(\alpha\) on \(M\) defined by

\[
\alpha(X) := \langle d\varphi(X), \tau(\varphi) \rangle, \quad (X \in \mathfrak{X}(M)).
\]  

(3.14)

Note here that

\[
\int_M |\alpha| v_g = \int_M \left( \sum_{i=1}^{m} |\alpha(e_i)|^2 \right)^{1/2} v_g 
\]

\[
\leq \int_M |d\varphi| |\tau(\varphi)| v_g 
\]

\[
\leq \left( \int_M |d\varphi|^2 v_g \right)^{1/2} \left( \int_M |\tau(\varphi)|^2 v_g \right)^{1/2} 
\]

\[
= 2 \sqrt{E(\varphi) E_2(\varphi)} < \infty. 
\]  

(3.15)

Moreover, the divergent \(\delta \alpha := -\sum_{i=1}^{m} \langle \nabla_{e_i} \alpha(e_i) \rangle \in C^\infty(M)\) turns out (cf. [6], p. 9) that

\[
-\delta \alpha = |\tau(\varphi)|^2 + \langle d\varphi, \nabla \tau(\varphi) \rangle = |\tau(\varphi)|^2. 
\]  

(3.16)

Indeed, we have

\[
-\delta \alpha = \sum_{i=1}^{m} e_i \langle d\varphi(e_i), \tau(\varphi) \rangle - \sum_{i=1}^{m} \langle d\varphi(\nabla_{e_i} e_i), \tau(\varphi) \rangle 
\]

\[
= \sum_{i=1}^{m} \left( \nabla_{e_i} (d\varphi(e_i)) - d\varphi(\nabla_{e_i} e_i) \right), \tau(\varphi) \rangle 
\]

\[
+ \sum_{i=1}^{m} \langle d\varphi(e_i), \nabla_{e_i} \tau(\varphi) \rangle 
\]

\[
= \langle \tau(\varphi), \tau(\varphi) \rangle + \langle d\varphi, \nabla \tau(\varphi) \rangle 
\]

which is equal to \(|\tau(\varphi)|\) since \(\nabla \tau(\varphi) = 0\).

By (3.16) and \(E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g < \infty\), the function \(-\delta \alpha\) is also integrable over \(M\). Thus, together with (3.15), we can apply Gaffney's
theorem (see 5.1 in Appendices, below) for the 1-form $\alpha$. Then, by integrating (3.16) over $M$, and by Gaffney’s theorem, we have

\[ 0 = \int_M (-\delta \alpha) v_g = \int_M |\tau(\varphi)|^2 v_g, \]

which yields that $\tau(\varphi) = 0$.

We have Theorem 2.1. □

Our method can be applied to an isometric immersion $\varphi : (M, g) \to (N, h)$. In this case, the 1-form $\alpha$ defined by (3.14) in the proof of Theorem 2.3 vanishes automatically without using Gaffney’s theorem since $\tau(\varphi) = m \xi$ belongs to the normal component of $T_{\varphi(x)} N$ ($x \in M$), where $\xi$ is the mean curvature normal vector field and $m = \dim(M)$. Thus, (3.16) turns out that

\[ 0 = -\delta \alpha = |\tau(\varphi)|^2 + \langle d\varphi, \nabla \tau(\varphi) \rangle = |\tau(\varphi)|^2 \]

which implies that $\tau(\varphi) = m \xi = 0$, i.e., $\varphi$ is minimal. Thus, we obtain

**Theorem 3.1.** Assume that $(M, g)$ is a complete Riemannian manifold, and let $\varphi : (M, g) \to (N, h)$ is an isometric immersion, and the sectional curvature of $(N, h)$ is non-positive. If $\varphi : (M, g) \to (N, h)$ is biharmonic and $\int_M |\xi|^2 v_g < \infty$, then $\varphi$ is minimal. Here, $\xi$ is the mean curvature normal vector field of the isometric immersion of $\varphi$.

We also apply Theorem 2.1 to a horizontally conformal submersion $\varphi : (M^m, g) \to (N^n, h) \ (m > n \geq 2)$ (cf. [3], see also [11]). In the case that a Riemannian submersion from a space form of constant sectional curvature into a Riemann surface $(N^2, h)$, Wang and Ou (cf. [29], see also [20]) showed that it is biharmonic if and only if it is harmonic.

We treat with a submersion from a higher dimensional Riemannian manifold $(M, g)$ (cf. [1]). Namely, let $\varphi : M \to N$ be a submersion, and each tangent space $T_x M \ (x \in M)$ is decomposed into the orthogonal direct sum of the vertical space $\mathcal{V}_x = \text{Ker}(d\varphi_x)$ and the horizontal space $\mathcal{H}_x$:

\[ T_x M = \mathcal{V}_x \oplus \mathcal{H}_x, \]  

(3.19)

and we assume that there exists a positive $C^\infty$ function $\lambda$ on $M$, called the dilation, such that, for each $x \in M$,

\[ h(d\varphi_x(X), d\varphi_x(Y)) = \lambda^2(x) g(X, Y), \quad (X, Y \in \mathcal{H}_x). \]  

(3.20)

The map $\varphi$ is said to be horizontally homothetic if the dilation $\lambda$ is constant along horizontally curves in $M$. 


If \( \varphi : (M^m, g) \to (N^n, h) \) \((m > n \geq 2)\) is a horizontally conformal submersion . Then, the tension field \( \tau(\varphi) \) is given (cf. [1], [3]) by

\[
\tau(\varphi) = \frac{n - 2}{2} \lambda^2 d\varphi \big( \text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right) \big) - (m - n) d\varphi \left( \hat{H} \right),
\]

(3.21)

where \( \text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right) \) is the \( \mathcal{H} \)-component of the decomposition according to (3.19) of \( \text{grad} \left( \frac{1}{\lambda^2} \right) \), and \( \hat{H} \) is the trace of the second fundamental form of each fiber which is given by \( \hat{H} = \frac{1}{m - n} \sum_{k=n+1}^{m} \mathcal{H}(\nabla_{e_k} e_k) \), where a local orthonormal frame field \( \{e_i\}_{i=1}^{m} \) on \( M \) is taken in such a way that \( \{e_{ix}|i=1,\cdots,n\} \) belong to \( \mathcal{H}_x \) and \( \{e_{ix}|j=n+1,\cdots,m\} \) belong to \( V_x \) where \( x \) is in a neighborhood in \( M \). Then, due to Theorems 2.1 and (3.21), we have immediately

**Theorem 3.2.** Let \( (M^m, g) \) be a complete non-compact Riemannian manifold, and \( (N^n, h) \), a Riemannian manifold with the non-positive sectional curvature \((m > n \geq 2)\). Let \( \varphi : (M, g) \to (N, h) \) be a horizontally conformal submersion with the dilation \( \lambda \) satisfying that

\[
\int_{M} \lambda^2 \left| \frac{n - 2}{2} \lambda^2 \text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right) - (m - n) \hat{H} \right|_g^2 v_g < \infty.
\]

(3.22)

Assume that, either \( \int_{M} \lambda^2 v_g < \infty \) or \( \text{Vol}(M, g) = \int_{M} v_g = \infty \). Then, if \( \varphi : (M, g) \to (N, h) \) is biharmonic, then it is a harmonic morphism.

Due to Theorem 3.2, we have:

**Corollary 3.3.** Let \( (M^m, g) \) be a complete non-compact Riemannian manifold, and \( (N^n, h) \), a Riemannian surface with the non-positive sectional curvature \((m > n = 2)\). Let \( \varphi : (M, g) \to (N, h) \) be a horizontally conformal submersion with the dilation \( \lambda \) satisfying that

\[
\int_{M} \lambda^2 \left| \hat{H} \right|_g^2 v_g < \infty.
\]

(3.23)

Assume that, either \( \int_{M} \lambda^2 v_g < \infty \) or \( \text{Vol}(M, g) = \int_{M} v_g = \infty \). Then, if \( \varphi : (M, g) \to (N, h) \) is biharmonic, then it is a harmonic morphism.

Corollary 3.3 implies

**Corollary 3.4.** Let \( (M^m, g) \) be a non-compact complete Riemannian manifold \((m > 2)\), and \( (N^n, h) \), a Riemannian surface with non-positive curvature. Let \( \lambda \) be a positive function in \( C^\infty(M) \cap L^2(M) \), where \( L^2(M) \) is the space of square integrable functions on \( (M, g) \). Then,
every biharmonic horizontally conformal submersion $\varphi : (M^m, g) \rightarrow (N^2, h)$ with a dilation $\lambda$ and a bounded $|\hat{H}|_g$, exactly $\lambda |\hat{H}|_g \in L^2(M)$, must be a harmonic morphism.

Remark 3.5. Notice that in Corollary 3.4, (1), there is no restriction to the dilation $\lambda$ because of $\dim N = 2$. This implies that for every positive $C^\infty$ function $\lambda$ in $C^\infty(M) \cap L^2(M)$ satisfying (3.2), we have a harmonic morphism $\varphi : (M^m, g) \rightarrow (N^2, h)$.

4. Appendices

4.1. Gaffney’s theorem. In this appendix, we recall Gaffney’s theorem ([10]):

Theorem 4.1. (Gaffney) Let $(M, g)$ be a complete Riemannian manifold. If a $C^1$ 1-form $\alpha$ satisfies that $\int_M |\alpha| v_g < \infty$ and $\int_M (\delta \alpha) v_g < \infty$, or equivalently, a $C^1$ vector field $X$ defined by $\alpha(Y) = \langle X, Y \rangle$ ($\forall Y \in \mathfrak{X}(M)$) satisfies that $\int_M |X| v_g < \infty$ and $\int_M \text{div}(X) v_g < \infty$, then

$$\int_M (-\delta \alpha) v_g = \int_M \text{div}(X) v_g = 0. \quad (4.1)$$

Proof. For completeness, we give a proof. By integrating over $M$, the both hand sides of

$$\text{div}(\eta^2 X) = \eta^2 \text{div}(X) + 2\eta \langle \nabla \eta, X \rangle, \quad (4.2)$$

we have

$$\int_M \text{div}(\eta^2 X) v_g = \int_M \eta^2 \text{div}(X) v_g + 2\int_M \eta \langle \nabla \eta, X \rangle v_g. \quad (4.3)$$

Since the support of $\eta^2 X$ is compact, the left hand side must vanish. So, we have

$$\int_M \eta^2 \text{div}(X) v_g = -2\int_M \eta \langle \nabla \eta, X \rangle v_g. \quad (4.4)$$
Therefore, we have

\[
\left| \int_{B_r(x_0)} \text{div}(X) v_g \right| \leq \left| \int_M \eta^2 \text{div}(X) v_g \right|
\]

\[
= 2 \left| \int_M \eta \langle \nabla \eta, X \rangle v_g \right|
\]

\[
\leq 2 \int_M \eta |\nabla \eta| |X| v_g
\]

\[
\leq \frac{4}{r} \int_M |X| v_g.
\] (4.5)

By the assumption that \( \int_M |X| v_g < \infty \), the right hand side goes to 0 if \( r \) tends to infinity. Since \( B_r(x_0) \) goes to \( M \) as \( r \rightarrow \infty \), due to completeness of \((M, g)\), and the assumption that \( \int_M \text{div}(X) v_g < \infty \), we have \( \int_M \text{div}(X) v_g = \lim_{r \rightarrow \infty} \int_{B_r(x_0)} \text{div}(X) v_g = 0. \)  

4.2. The Bochner-type estimations and alternative proof of main theorem. We first show the Bochner-type estimations for the tension fields of biharmonic maps into a Riemannian manifold \((N, h)\) of non-positive curvature. We write down them for clarity since at least we do not find them in the literature. Then we can give an alternative proof of our main theorem (cf. Theorem 2.1) by using them.

**Lemma 4.2.** Assume that the sectional curvature of \((N, h)\) is non-positive, and \( \varphi : (M, g) \rightarrow (N, h) \) is a biharmonic mapping. Then, it holds that

\[
\Delta |\tau(\varphi)|^2 \geq 2 |\nabla \tau(\varphi)|^2
\] (4.6)

in \( M \). Here, \( \Delta = \sum_{i=1}^m (e_i^2 - \nabla_{e_i} e_i) \) is the Laplace-Beltrami operator of \((M, g)\).

**Proof.** Let us take a local orthonormal frame field \( \{e_i\}_{i=1}^m \) on \( M \), and \( \varphi : (M, g) \rightarrow (N, h) \), a biharmonic map. Then, for \( V := \tau(\varphi) \in \).
we have
\[
\frac{1}{2} \Delta |V|^2 = \frac{1}{2} \sum_{i=1}^{m} \left\{ e_i^2 |V|^2 - \nabla e_i e_i |V|^2 \right\}
\]
\[
= \sum_{i=1}^{m} \left\{ e_i h(\nabla e_i V, V) - h(\nabla e_i V, e_i V) \right\}
\]
\[
= \sum_{i=1}^{m} \left\{ h(\nabla e_i \nabla e_i V, V) - h(\nabla e_i \nabla e_i e_i V, V) \right\}
\]
\[
+ \sum_{i=1}^{m} h(\nabla e_i V, \nabla e_i V)
\]
\[
= h(-\Delta V, V) + |\nabla V|^2
\]
\[
= h(-\mathcal{R}(V), V) + |\nabla V|^2
\]
\[
\geq |\nabla V|^2,
\] (4.7)
because for the second last equality, we used $\Delta V - \mathcal{R}(V) = J(V) = 0$
for $V = \tau(\varphi)$, due to the biharmonicity of $\varphi : (M, g) \to (N, h)$, and
for the last inequality of (4.7), we used

\[
h(\mathcal{R}(V), V) = \sum_{i=1}^{m} h(R^N(V, \varphi_* e_i)\varphi_* e_i, V) \leq 0
\] (4.8)
since the sectional curvature of $(N, h)$ is non-positive. 

By Lemma 4.2, we have

Lemma 4.3. Under the same assumptions as Lemma 4.2, we have

\[
|\tau(\varphi)| |\Delta |\tau(\varphi)| \geq 0.
\] (4.9)

Proof. Due to Lemma 4.2, we have

\[
2 |\nabla \tau(\varphi)|^2 \leq \Delta |\tau(\varphi)|^2
\]
\[
= 2 |\tau(\varphi)| |\Delta |\tau(\varphi)| + 2 |\nabla |\tau(\varphi)| |
\] (4.10)

Thus, we have

\[
|\tau(\varphi)| |\Delta |(\tau(\varphi)| \geq |\nabla \tau(\varphi)|^2 - |\nabla |\tau(\varphi)| |
\]
\[
\geq 0
\] (4.11)
on the set of points in $M$ where $|\tau(\varphi)| > 0$. Here, to see the last
inequality of (4.11), it suffices to notice that for all $V \in \Gamma(\varphi^{-1}TN)$,

\[
|\nabla V| \geq |\nabla |V||
\] (4.12)
on the set of points in $M$ where $|V| > 0$, which follows from that

$$
|V| |\nabla|V| | = \frac{1}{2} |\nabla|V| |^2 |
$$

$$
= \frac{1}{2} |\nabla h(V, V)|
$$

$$
= |h(\nabla V, V)|
$$

$$
\leq |\nabla V| |V|.
$$

(4.13)

This proves Lemma 4.3. $\square$

Now, we are in position to give an alternative proof of Theorem 2.1.

(The first step) For a fixed point $x_0 \in M$, and for every $0 < r < \infty$, we first take the same cutoff $C^\infty$ function $\eta$ on $M$ in Section Three.

For $0 < r < \infty$, multiply $\lambda^2$ to both hand sides of the inequality (4.9) in Lemma 4.3, and integrate over $M$, we have

$$
0 \leq \int_M \eta^2 |\tau(\varphi)| \Delta (|\tau(\varphi)|) v_g
$$

$$
= -\int_M \langle \nabla (\eta^2 |\tau(\varphi)|), \nabla |\tau(\varphi)| \rangle v_g
$$

$$
= -\int_M \eta^2 |\nabla (|\tau(\varphi)|)|^2 v_g
$$

$$
- 2 \int_M |\tau(\varphi)| \eta \langle \nabla (|\tau(\varphi)|), \nabla \eta \rangle v_g
$$

$$
= -\int_M |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g
$$

$$
- 2 \int_M \langle \eta \nabla (|\tau(\varphi)|), |\tau(\varphi)| \nabla \eta \rangle v_g.
$$

(4.14)

Therefore, by applying $A := \eta \nabla (|\tau(\varphi)|)$ and $B := |\tau(\varphi)| \nabla \eta$ to Young’s inequality: $-2AB \leq \epsilon^2 A^2 + \frac{1}{\epsilon^2} B^2$, for every positive real number $\epsilon > 0$, the right hand side of (4.14) is smaller than or equal to

$$
-\int_M |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g
$$

$$
+ \epsilon^2 \int_M |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g + \frac{1}{\epsilon^2} \int_M |\tau(\varphi)|^2 |\nabla \eta|^2 v_g
$$

$$
= -(1 - \epsilon^2) \int_M |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g + \frac{1}{\epsilon^2} \int_M |\tau(\varphi)|^2 |\nabla \eta|^2 v_g.
$$

(4.15)
Thus, we obtain
\[(1 - \epsilon^2) \int_M |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g \leq \frac{1}{\epsilon^2} \int_M |\tau(\varphi)|^2 |\nabla \eta|^2 v_g. \tag{4.16}\]

(The second step) In the inequality (4.16), the left hand side is bigger than or equal to \((1 - \epsilon^2) \int_{B_r(x_0)} |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g\) since \(\eta = 1\) on \(B_r(x_0)\), and the right hand side is smaller than or equal to \(\frac{1}{\epsilon} \int_M |\tau(\varphi)|^2\frac{1}{r^2} v_g = \frac{4}{\epsilon^2 r^2} \int_M |\tau(\varphi)|^2 v_g\), we obtain
\[(1 - \epsilon^2) \int_{B_r(x_0)} |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g \leq \frac{4}{\epsilon^2 r^2} \int_M |\tau(\varphi)|^2 v_g. \tag{4.17}\]

By putting \(\epsilon = \frac{1}{2}\), we have
\[\int_{B_r(x_0)} |\nabla (|\tau(\varphi)|)|^2 \eta^2 v_g \leq \frac{64}{3 r^2} \int_M |\tau(\varphi)|^2 v_g. \tag{4.18}\]

(The third step) Now we may take \(r \to \infty\) in (4.18), due to the assumptions that \(\int_M |\tau(\varphi)|^2 v_g < \infty\), and \((M, g)\) is complete, we obtain
\[\int_M |\nabla (|\tau(\varphi)|)|^2 v_g = 0, \tag{4.19}\]
which implies \(\nabla (|\tau(\varphi)|) = 0\), that is,
\[|\tau(\varphi)| = c \quad \text{(a constant)} \tag{4.20}\]
on \(M\). Then, the inequality (4.6) turns out that
\[0 = \Delta (|\tau(\varphi)|) \geq 2 |\nabla \tau(\varphi)|^2 \geq 0, \tag{4.21}\]
which means that
\[\nabla \tau(\varphi) = 0 \quad \text{(on } M). \tag{4.22}\]

By passing through the same procedure of the fourth step of the proof of Theorem 2.1 in Section Three, we have done. \(\square\)
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