Mass of Pseudoscalar Gluonium: A Higher-Loop 
QCD Sum-Rule Estimate

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Abstract: Higher-loop corrections to the pseudoscalar ($0^{-+}$) gluonium correlation function will be used to obtain the leading gluon condensate contributions to the subtraction-independent QCD sum-rules. The effect of these higher-loop corrections on the sum-rule estimates of the pseudoscalar gluonium mass will be investigated. The final results of this analysis compare favourably with $SU(3)$ lattice simulations.

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The existence of QCD bound states consisting solely of gluons is one of the conceptual predictions of QCD that has not been experimentally verified [1]. Experimental searches for gluonia states are hampered by the possibility of mixing with ordinary quark mesons of the same quantum numbers, so a dynamical signature is necessary to identify the gluonium content of observed mesons [2].

An approach to modelling mesons in QCD which has been successful in many applications is QCD sum-rules [3]. In this approach the condensates of QCD are included in the correlation functions of currents, providing a parameterization of non-perturbative vacuum effects.

Although sum-rules have been useful for quark mesons, it has been difficult to obtain a conclusive analysis of gluonium [4-7]. Part of this difficulty for scalar gluonium can be traced to a reliance on low-energy theorem subtraction constants to introduce the gluon condensate \( \langle \alpha_s G^2 \rangle \) (the most reliably determined gluonic condensate) into the sum-rule analysis. In contrast, sum-rules independent of the low-energy subtraction constants do not contain \( \langle \alpha_s G^2 \rangle \) to lowest order. Thus higher-loop contributions may be significant because they actually provide the leading \( \langle \alpha_s G^2 \rangle \) behaviour in the (subtraction-independent) sum-rules.

In previous work, the effect of these higher-loop corrections [8] was analyzed for scalar \((0^{++})\) gluonium. It was found that higher-loop effects were significant in the subtraction-independent sum-rules, leading to a mass prediction \( m_{0^{++}} = (1.7 \pm 0.2) \text{GeV} \) for pure QCD [9], a result in reasonable agreement with lattice predictions [10].

The purpose of this paper is to investigate higher-loop effects on the sum-rule for pseudoscalar \((0^{-+})\) gluonium in pure QCD. For the pseudoscalar, the \( N_f = 0 \) (pure QCD) limit is likely a better estimate of the gluonium mass when quark effects are included than in the scalar case as will be discussed below. Our sum-rule estimate of the pseudoscalar mass and the \( m_{0^{-+}}/m_{0^{++}} \) mass ratio will be compared with \( SU(3) \) lattice values [10] and with previous sum-rule estimates [6,7].
The correlation function for pseudoscalar gluonium in pure QCD is defined in terms of a renormalization group (RG) invariant current \([11]\).

\[
\Pi(q^2) = i \int d^4x \, e^{iq \cdot x} \langle \Omega | T (j(x) j(0)) | \Omega \rangle
\]

\[
j(x) = \alpha_s G^a_{\mu\nu} \tilde{G}^a_{\mu\nu} \quad \tilde{G}^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} G^a_{\alpha\beta}
\]

Perturbative contributions to \(\Pi(Q^2 = -q^2)\) are known to two-loops \([12]\), although some effort is required to relate these results to the following expression.

\[
\Pi(Q^2) = -2 \left( \frac{\alpha_s}{\pi} \right)^2 Q^4 \log \frac{Q^2}{\nu^2} \times \left( 1 + \frac{\alpha_s}{\pi} \left[ 1 + \alpha_s \left( \frac{41}{4} - \frac{29}{4} \log \frac{Q^2}{\nu^2} \right) \right] \right)
\]

\[
+ \text{QCD condensate terms}
\]

Divergent constants and polynomials in \(Q^2\) have been ignored since they are zero when the sum-rule is formed.

The dependence on the number of flavours \(N_f\) and colours \(N_c\) has been explicitly given in (2) to illustrate that the flavour dependence is rather weak. In particular, the transition between pure QCD and \(N_f = 3\) is negligible.

\[
\Pi(Q^2) = -2 \left( \frac{\alpha_s}{\pi} \right)^2 Q^4 \log \frac{Q^2}{\nu^2} \left[ 1 + \alpha_s \left( \frac{29}{4} - \frac{9}{4} \log \frac{Q^2}{\nu^2} \right) \right] \quad N_c = 3 = N_f
\]

\[
\Pi(Q^2) = -2 \left( \frac{\alpha_s}{\pi} \right)^2 Q^4 \log \frac{Q^2}{\nu^2} \left[ 1 + \alpha_s \left( \frac{31}{12} \log \frac{Q^2}{\nu^2} \right) \right] \quad N_c = 3 \quad N_f = 0
\]

The situation is quite different for scalar gluonium, where apart from a normalization constant \(A\), the perturbative contributions to the correlation function \(\Phi(Q^2)\) are

\[
\Phi(Q^2) = A \left( \frac{\alpha_s}{\pi} \right)^2 Q^4 \log \frac{Q^2}{\nu^2} \left[ 1 + \alpha_s \left( \frac{51}{4} - \frac{11}{4} \log \frac{Q^2}{\nu^2} \right) \right] \quad N_c = 3 \quad N_f = 0
\]

illustrating a stronger flavour dependence. The weak flavour dependence in the pseudoscalar case, combined with an estimated small mixing \([13]\) with quark mesons, suggest
that the pure QCD prediction of the pseudoscalar mass is a good approximation when quark effects are included.

The gluon condensate $\langle \alpha_s G^2 \rangle$ contributions to the correlation function of the pseudoscalar current have only been calculated to lowest order \[4\]. The general form of the $\langle \alpha_s G^2 \rangle$ portion of $\Pi(Q^2)$ to one-loop, with the dependence on momentum given explicitly, is

$$
\Pi(Q^2) = \frac{\alpha_s}{\pi} \left[ b_0 + \frac{\alpha_s}{\pi} \left( b_1 \log \frac{Q^2}{\nu^2} + b'_1 \right) \right] \langle \alpha_s G^2 \rangle + \ldots \tag{5}
$$

$b_0 = -4\pi; \quad b_1, b'_1$ unknown.

The terms proportional to $b_0, b'_1$ do not contribute to the sum-rule since they are independent of the momentum $Q^2$. Thus the leading $\langle \alpha_s G^2 \rangle$ contribution to the sum-rule comes from the one-loop logarithmic correction proportional to $b_1$.

The one-loop calculation was carried out explicitly for scalar gluonium to demonstrate that the operator-product expansion coefficient is infrared finite \[8\]. However, an explicit calculation is not necessary if the constant $b_1$ is only needed for the purpose of a sum-rule analysis. Since $\Pi(Q^2)$ is constructed from an RG invariant current, the renormalized correlation function satisfies the following RG equation

$$
0 = \left( \nu \frac{\partial}{\partial \nu} + \beta \frac{\partial}{\partial \alpha_s} \right) \Pi(Q^2) + \text{condensate anomalous dimensions} \tag{6}
$$

For the gluon condensate terms, the anomalous dimension of $\langle \alpha_s G^2 \rangle$ is zero to one-loop. Applying (6) to the $\langle \alpha_s G^2 \rangle$ dependence in (5) shows that RG invariance determines $b_1$.

$$
0 = -2b_1 \left( \frac{\alpha_s}{\pi} \right)^2 \langle \alpha_s G^2 \rangle + b_0 \beta_1 \left( \frac{\alpha_s}{\pi} \right) \langle \alpha_s G^2 \rangle + \text{terms independent of } \langle \alpha_s G^2 \rangle \tag{7}
$$

$$
b_1 = \frac{1}{2} \beta_1 b_0 = -\frac{11}{4} b_0 \quad (N_c = 3, \ N_f = 0)
$$

Contributions to $\Pi(Q^2)$ from dimension six and dimension eight gluonic condensates are also known to lowest order \[4\]. As will be seen below, the one-loop logarithmic correction to $\langle g G^a \rangle = \langle g f_{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \rangle$ is the leading contribution in one of the sum-rules and
is thus important. Writing the $\langle gG^3 \rangle$ terms as

$$\Pi(Q^2) = c_0 \left( \frac{\alpha_s}{\pi} \right)^2 \frac{\langle gG^3 \rangle}{Q^2} + c_1 \left( \frac{\alpha_s}{\pi} \right)^3 \log \frac{Q^2}{\nu^2} \frac{\langle gG^3 \rangle}{Q^2}$$

(8)

and applying the RG equation including the $\langle gG^3 \rangle$ anomalous dimension [14],

$$0 = \left( \nu \frac{\partial}{\partial \nu} + \beta \frac{\partial}{\partial \alpha_s} \right) \Pi(Q^2)$$

(9)

leads to the result

$$c_1 = -\frac{29}{4} c_0 \quad (N_c = 3, \, N_f = 0)$$

(10)

Collecting the above results leads to the following expression for $\Pi(Q^2)$, valid to two-loops in perturbative terms, one-loop in $\langle \alpha_s G^2 \rangle$ and $\langle gG^3 \rangle$, and to lowest order in higher dimension condensates.

$$\Pi(Q^2) = Q^4 \log \frac{Q^2}{\nu^2} \left[ a_0 + a_1 \log \frac{Q^2}{\nu^2} \right] + \frac{\alpha_s}{\pi} \left( b_0 + b_1 \frac{\alpha_s}{\pi} \right) \langle \alpha_s G^2 \rangle + \left( \frac{\alpha_s}{\pi} \right)^2 b_1 \log \frac{Q^2}{\nu^2} \langle \alpha_s G^2 \rangle + \frac{\alpha_s}{\pi} \left[ c_0 + \frac{\alpha_s}{\pi} c_1 \log \frac{Q^2}{\nu^2} \right] \frac{\langle gG^3 \rangle}{Q^2} + \frac{\alpha_s^2 G^4}{Q^4}$$

(11)

with

$$a_0 = -2 \left( \frac{\alpha_s}{\pi} \right)^2 \left[ 1 + \frac{31}{4} \frac{\alpha_s}{\pi} \right] \quad a_1 = \frac{11}{2} \left( \frac{\alpha_s}{\pi} \right)^3$$

$$b_0 = -4 \pi \quad b_1 = -\frac{11}{4} b_0 = 11 \pi$$

$$c_0 = -8 \pi^2 \quad c_1 = -\frac{29}{4} c_0 = 8 \pi^2 \frac{29}{4}$$

$$d_0 = 8 \pi^2 \frac{\alpha_s}{\pi} \langle \alpha_s^2 G^4 \rangle = \alpha_s^2 \left( \langle (f_{abc} G_{\mu
u}^b)^2 \rangle + 10 \langle (f_{abc} G_{\mu
u}^b G_{\rho\sigma}^c)^2 \rangle \right)$$

The correlation function satisfies a dispersion relation with two subtraction constants, relating the QCD prediction $\Pi(Q^2)$ to the phenomenological quantity $\text{Im} \Pi(t)$:

$$\Pi(Q^2) = \Pi(0) - \Pi'(0) Q^2 + \frac{Q^4}{\pi} \int_0^\infty dt \frac{\text{Im} \Pi(t)}{t^2 (t + Q^2)}.$$ 

(12)

The subtraction constant $\Pi'(0)$ is the slope of the $U_A(1)$ topological charge, and has been estimated using sum-rule techniques [6]. Families of sum-rules can be constructed from
by Borel-transforming the dispersion relation weighted with (positive) powers of $Q^2$

\[
\mathcal{R}_k(\tau, s_0) = \frac{1}{\tau} \hat{L} [Q^{2k}\Pi(Q^2)] - \frac{1}{\pi} \int_{s_0}^{\infty} dt \, t^k e^{-t\tau} \text{Im}\Pi_{eqn.11}(t)
\]

\[
\hat{L} \equiv \lim_{N \to \infty} \lim_{Q^2 \to \infty} \left( \frac{-Q^2}{N - 1}! \left( \frac{\partial}{\partial Q^2} \right)^N \right)
\]

The scale $\sqrt{s_0}$ is the continuum threshold representing duality between resonance physics and QCD and $\hat{L}$ is the Borel transform operator. The continuum threshold is constrained by the finite-energy sum-rule [15].

\[
\mathcal{F}_0(s_0) = \frac{1}{\pi} \int_0^{s_0} dt \, \text{Im}\Pi(t)
\]

After some calculation, the sum-rules of interest are obtained from the QCD correlation function $\Pi(Q^2)$ in (11).

\[
\mathcal{R}_0(\tau, s_0) = -\frac{2a_0}{\tau^3} [1 - \rho_2(s_0\tau)]
\]

\[
- \frac{4a_1}{\tau^3} \left[ \frac{3}{2} - \gamma_E - \rho_2(s_0\tau) \log s_0 \tau - \frac{3}{2} e^{-s_0 \tau} \left( 1 + \frac{1}{3} s_0 \tau \right) - E_1(s_0 \tau) \right] + \frac{b_1}{\tau} \left[ 1 - e^{-s_0 \tau} \right] \langle \alpha_s G^2 \rangle + c_0 \left( \frac{\alpha_s}{\pi} \right)^2 \langle gG^3 \rangle + d_0 \langle \alpha_s^2 G^4 \rangle
\]

\[
\mathcal{R}_1(\tau, s_0) = -\frac{6a_0}{\tau^4} [1 - \rho_3(s_0\tau)]
\]

\[
- \frac{12a_1}{\tau^4} \left[ \frac{11}{6} - \gamma_E - \rho_3(s_0\tau) \log s_0 \tau - \frac{3}{2} e^{-s_0 \tau} \left( 1 + \frac{1}{3} s_0 \tau \right) - \frac{1}{3} \rho_2(s_0 \tau) - E_1(s_0 \tau) \right]
\]

\[
- \frac{b_1}{\tau^2} \left( \frac{\alpha_s}{\pi} \right)^2 [1 - \rho_1(s_0\tau)] \langle \alpha_s G^2 \rangle
\]

\[
+ \frac{c_1}{\tau} \left( \frac{\alpha_s}{\pi} \right)^3 [1 - e^{-s_0 \tau}] \langle gG^3 \rangle - d_0 \langle \alpha_s^2 G^4 \rangle
\]
\[ F_0(s_0) = \frac{1}{3} a_0 s_0^3 + \frac{2}{9} a_1 s_0^3 - b_1 s_0 \left( \frac{\alpha_s}{\pi} \right)^2 \langle \alpha_s G^2 \rangle + c_0 \langle gG^3 \rangle \quad (16) \]

\[ \rho_k(x) \equiv e^{-x} \sum_{j=0}^{k} \frac{x^j}{j!} \quad \gamma_E \equiv \text{Euler's Constant} \approx 0.5772 \]

\[ E_1(x) \equiv \int_x^\infty dy \frac{e^{-y}}{y} \quad \text{(Exponential Integral)} \]

Renormalization group improvement of the sum-rules implies that the running coupling constants \( \alpha_s(1/\tau), \alpha_s(s_0) \) respectively appear in (15) and (16). In principle, the condensates other than \( \langle \alpha_s G^2 \rangle \) should also be RG improved (only \( \langle \alpha_s G^2 \rangle \) is an RG invariant to one-loop). These effects can be explicitly considered for \( \langle gG^3 \rangle \), and are found to have a small effect, while for \( \langle \alpha_s^2 G^4 \rangle \) the vacuum saturation hypothesis used to estimate its numerical value [4,16] leads to the (RG invariant) \( \langle \alpha_s G^2 \rangle \) condensate.

As mentioned previously, notice that the leading \( \langle \alpha_s G^2 \rangle \) behaviour in both sum rules, and the leading \( \langle gG^3 \rangle \) correction in the \( R_1 \) sum-rule comes from the one-loop logarithmic terms in \( \Pi(Q^2) \). These contributions have not been considered in previous studies of pseudoscalar gluonium, motivating our analysis of the sum-rule mass estimates.

The sum-rules \( R_k(\tau, s_0) \) relate a QCD prediction to a phenomenological model for \( \text{Im} \Pi(t) \). The simplest model is the narrow-width approximation for the lightest resonance,

\[ \text{Im} \Pi(t) = \pi f^2 \delta(t - M^2) \quad (17) \]

where \( M \) is the mass of the \( 0^{-+} \) state and \( f \) is its coupling to the pseudoscalar current. This model leads to the following family of sum-rules relating QCD expressions to resonance properties of the state.

\[ R_0(\tau, s_0) = f^2 M^4 e^{-M^2 \tau} \]

\[ R_1(\tau, s_0) = f^2 M^6 e^{-M^2 \tau} \quad (18) \]

\[ F_0(s_0) = f^2 M^4 \]

Taking the ratio of \( R_1/R_0 \) it is easily observed that

\[ M^2 = \frac{R_1(\tau, s_0)}{R_0(\tau, s_0)} \quad (19) \]
For small $\tau$ (high-energy) the dominance of the lightest resonance in the phenomenological model fails, while for large $\tau$ (low-energy) the OPE is no longer a good approximation. Thus at intermediate values of $\tau$ there should be a slowly varying region of the ratio (19).

To obtain a QCD prediction of the resonance properties, the conventional ($\tau$ stability) procedure [3] is to fix $s_0$ and then identify the $\tau$ stationary point of $R_1/R_0$ as the ($s_0$-dependent) value of $M^2$. This value is then used to extract $f^2$ by locating the stationary point of $e^{M^2\tau R_0/M^4}$. This procedure then leads to $M$ and $f$ as functions of the continuum threshold $s_0$. To determine the final QCD prediction, the finite-energy sum-rule (FESR) is used to constrain $s_0$ by demanding maximum agreement with the lowest FESR [15].

$$f^2(s_0)M^4(s_0) = F_0(s_0)$$

(20)

The outcome of this procedure will obviously depend on the QCD parameters used as input into the sum-rules. In an exhaustive analysis of several channels it has been concluded that an acceptable range of the gluon condensate is $\langle \alpha_s G^2 \rangle = 3(0.05 \pm 0.015)/\pi$ GeV$^4$ [17]. The dilute instanton gas approximation $\langle gG^3 \rangle \approx (0.27$ GeV$^2)\langle \alpha_s G^2 \rangle$ [18] and vacuum saturization $\langle \alpha_s^2 G^4 \rangle = \frac{15}{16}(\langle \alpha_s G^2 \rangle)^2$ [4,16] will be used for the higher-dimension gluonic condensates. These values for the higher-dimension condensates are in agreement with the conclusions for the scalar glueball sum-rule [9]. Finally, $\Lambda_{\overline{\text{MS}}}$ will be allowed to vary over a generous range. To summarize, the following QCD parameter space will be considered.

$$\langle \alpha_s G^2 \rangle = \frac{3}{\pi}(0.050 \pm 0.015) \text{ GeV}^4$$

$$\langle gG^3 \rangle = (0.27 \text{ GeV}^2)\langle \alpha_s G^2 \rangle \quad \langle \alpha_s^2 G^4 \rangle = \frac{15}{16}(\langle \alpha_s G^2 \rangle)^2$$

(21)

$$\bar{\alpha}_s(1/\tau) = -\frac{4\pi}{11 \log \tau \Lambda_{\overline{\text{MS}}}^2} \quad \Lambda_{\overline{\text{MS}}} = (0.15 \pm 0.05)\text{GeV}$$

Two-loop corrections to the running coupling constant will be applied in the (two-loop) perturbative terms.

The results of the previously described algorithm for analyzing the pseudoscalar sum-rule are more conclusive than in the scalar channel. It is possible to find a value of $s_0$
leading to precise agreement† with the FESR constraint, and also providing wide flat regions in the $\tau$ plots. These plots are shown for the optimum $s_0$ and various choices of parameter space in Figures 1-3, and summarized in Table 1. As is necessary in a sum-rule analysis, the $\tau$ stationary points occur at intermediate values, so that convergence of the QCD sum rule (small $\tau$) is balanced against dominance of the lightest resonance (large $\tau$).

Table 1 exhibits the complete range of mass predictions resulting from the QCD parameter space (21). Thus over the range of parameters considered, the $\tau$ stability analysis of the sum-rules satisfies the FESR constraint, leading to the predictions $f = (0.30 \pm 0.05)$GeV, $m_{0-+} = (2.3 \pm 0.2)$GeV, with the errors reflecting only the parameter space uncertainties.

Our analysis leads to a heavier $0^{-+}$ mass prediction than in previous sum rule estimates [6,7], indicating that the higher-loop corrections considered here are indeed important. However, our results are in reasonable agreement with $SU(3)$ lattice simulations which find $m_{0-+} = (6.3\pm0.7)\sqrt{K} \approx (2.8\pm0.3)$GeV [10]. For mass ratios, using the (higher-loop) sum rule estimate for the scalar glueball [9], we find $m_{0-+}/m_{0++} = (1.4 \pm 0.3)$. In good agreement with the $SU(3)$ lattice ratio $m_{0-+}/m_{0++} = (1.8 \pm 0.3)$.

Since our results disagree with the other sum-rule estimates, we feel that it is incumbent upon us to check our calculations using the same corrections as in [6]. A value of the continuum threshold can be found satisfying the sum-rule constraint, and the corresponding plots are shown in Figure 4. As is evident a mass scale of about 1.7GeV results, in agreement with [6]. It can thus be concluded that the neglected contribution of the gluon condensate, providing the leading $\langle \alpha_s G^2 \rangle$ behaviour in the sum-rule, has a significant effect on the sum-rule analysis.

In conclusion, the effect of higher-loop corrections on the sum-rule predictions of the pseudoscalar gluonium mass have been investigated in pure QCD. These higher-loop corrections...
corrections provide the leading \(\alpha s G^2\) behaviour in the (subtraction-independent) sum-rules, and thus their effects are significant. In the sum-rule analysis, a continuum threshold can be found in precise agreement with the lowest FESR, leading to optimum predictions
\[
m_{0-+} = (2.3 \pm 0.2)\text{GeV}
\]
over a wide range of QCD parameter space. These results compare well with lattice values, both for the pseudoscalar mass, and the pseudoscalar-scalar mass ratio.

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**Figure Captions**

Figures 1-3 illustrate the sum-rule estimates of $f$ and $M$ for the optimum values of $s_0$.

**Fig. 1:** $\langle \alpha_s G^2 \rangle = 0.033$ GeV$^4$, $\Lambda_{\overline{MS}} = 0.2$ GeV $\sqrt{s_0} = 2.7$ GeV

**Fig. 2:** $\langle \alpha_s G^2 \rangle = 0.048$ GeV$^4$, $\Lambda_{\overline{MS}} = 0.15$ GeV $\sqrt{s_0} = 2.9$ GeV

**Fig. 3:** $\langle \alpha_s G^2 \rangle = 0.062$ GeV$^4$, $\Lambda_{\overline{MS}} = 0.1$ GeV $\sqrt{s_0} = 3.1$ GeV

**Fig. 4:** Illustration of the sum-rule analysis neglecting the higher-loop effects. The parameters are: $\langle \alpha_s G^2 \rangle = 0.04$ GeV$^4$, $\Lambda_{\overline{MS}} = 0.15$ GeV $\sqrt{s_0} = 2.2$ GeV and lead to an agreement with the FESR of $(\mathcal{F}_0 - f^2 M^4)/\mathcal{F}_0 = 1 \times 10^{-2}$. 

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