THE ERMAKOV-LEWIS INVARIANTS OF THE
GROSS-PITAEVSKII EQUATION

José Maria Filardo Bassalo

Fundação Minerva

Paulo de Tarso Santos Alencar

Professor Aposentado da UFPA

Daniel Gemaque da Silva

Professor de Ensino Médio - Amapá

Antonio Boulhosa Nassar

Extension Program-Department of Sciences, University of California, Los Angeles, California 90024

M. Cattani

Instituto de Física da USP, 05389-970, São Paulo, SP

ABSTRACT: In this work we study the Ermakov-Lewis invariants of the non-linear Gross-Pitaevskii equation.

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1. Introduction

Many years ago, in 1967[1], H. R. Lewis has shown that there is a conserved quantity, that will be indicated by I, associated with the time dependent harmonic oscillator (TDHO) with frequency ω(t), given by:

\[ I = \frac{1}{2}[(\ddot{q} + \dot{\alpha}) q + (\frac{\dot{q}}{\alpha})^2] , \quad (1.1) \]

where q and α obey, respectively the equations:

\[ \ddot{q} + \omega^2(t) q = 0 , \quad \ddot{\alpha} + \omega^2(t) \alpha = \frac{1}{\alpha^3} . \quad (1.2,3) \]
On the other hand, as the above expressions have also been obtained by V. P. Er-
makov [2] in 1880, the invariants determination of time dependent physical systems is also
known as the Ermakov-Lewis problem. So, considerable efforts have been devoted to
solve this problem and its generalizations, in the last thirty years, and in many works have
been published on these subjects [3,4].

In the present work we investigate the existence of these invariants for the one-
dimensional non-linear Gross-Pitaevskii equation with the potential \( V(x, t) \) given by:

\[
V(x, t) = \frac{1}{2} m \omega^2(t) x^2 , \quad (1.4)
\]

which is the time dependent harmonic oscillator potential.

2. Gross-Pitaevskii Equation

Em 1961[5,6], E. P. Gross and, independently, L. P. Pitaevskii proposed a non-linear
Schrödinger equation to represent time dependent physical systems, given by:

\[
i \hbar \frac{\partial \psi(x, t)}{\partial t} = - \frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{1}{2} m \omega^2(t) x^2 \psi(x, t) + g|\psi(x, t)|^3 , \quad (2.1)
\]

where \( \psi(x, t) \) is a wavefunction and \( g \) is a constant. constante.

Writing the wavefunction \( \psi(x, t) \) in the polar form, defined by the Madelung-Bohm
transformation[7,8], we get:

\[
\psi(x, t) = \phi(x, t) e^{i S(x, t)} , \quad (2.2)
\]

where \( S(x , t) \) is the classical action and \( \phi(x, t) \) will be defined in what follows.

Substituting Eq.(2.2) into Eq.(2.1) and taking the real and imaginary parts of the
resulting equation, we get[9]:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_{qu})}{\partial x} = 0 , \quad (2.3)
\]

\[
\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \omega^2(t) x = - \frac{1}{m} \frac{\partial}{\partial x} (V_{qu} + V_{GP}) , \quad (2.4)
\]

where:

\[
\rho(x, t) = \phi^2(x, t) , \quad (2.5) \quad \text{(quantum mass density)}
\]

\[
v_{qu}(x, t) = \frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} , \quad (2.6) \quad \text{(quantum velocity)}
\]

\[
V_{qu}(x, t) = - \frac{\hbar^2}{2m \sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} , \quad (2.7) \quad \text{(Bohm quantum potential)}
\]

and
\[ V_{GP} = \frac{\hbar}{m} \rho . \quad (2.8) \quad \text{(Gross-Pitaevskii potential)} \]

In order to integrate Eq.(2.4) let us assume that the expected value of quantum force is equal to zero for all times \( t \), that is,

\[ < \frac{\partial V_{qu}}{\partial x} > \rightarrow 0 \iff \frac{\partial V_{qu}}{\partial x} \bigg|_{x = q(t)} , \quad < x > = q(t) . \quad (2.9a-c) \]

In this way, we can write Eq.(2.4) into two parts:

\[ \frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \omega^2 x = k(t) \left[ x - q(t) \right] , \quad (2.10) \]

\[ - \frac{1}{m} \frac{\partial}{\partial x} \left( V_{qu} + V_{GP} \right) = \frac{\partial}{\partial x} \left( \frac{\hbar^2}{2 m^2} \frac{\rho}{\sqrt{\pi m \sigma^2(0)}} - \frac{g \rho}{m} \right) = k(t) \left[ x - q(t) \right] . \quad (2.11) \]

Performing the differentiation indicated in Eq.(2.11) we get,

\[ \frac{\hbar^2}{4 m^2} \left[ \frac{1}{\rho(x, 0)} \frac{\partial^3 \rho(x, 0)}{\partial x^3} - \frac{2}{\rho^2(x, 0)} \frac{\partial \rho(x, 0)}{\partial x} \frac{\partial^2 \rho(x, 0)}{\partial x^2} + \frac{1}{\rho(x, 0)^4} \left( \frac{\partial \rho(x, 0)}{\partial x} \right)^3 \right] + \frac{g \rho}{m} \frac{\partial \rho(x, 0)}{\partial x} = k(t) \left[ x - q(t) \right] . \quad (2.12) \]

To integrate Eq.(2.12) it is necessary to known the initial condition for \( \rho(x, t) \). Let us assume that for \( t = 0 \) the physical system is represented by a normalized Gaussian wave packet, centered at \( q(0) \), that is,

\[ \rho(x, 0) = \left[ \pi \sigma(0) \right]^{-1/2} e^{-\frac{(x - q(0))^2}{\sigma(0)^2}} = \frac{1}{\sqrt{A}} e^{-\frac{B^2}{C}} , \quad (2.13) \]

\[ A = \pi \sigma(0) , \quad B = x - q(0) , \quad C = \sigma(0) . \quad (2.14-16) \]

Since Eq.(2.13) is a particular solution of Eq.(2.12), we must have:

\[ \frac{\hbar^2}{4 m^2} \left[ \frac{1}{\rho(x, 0)} \frac{\partial^3 \rho(x, 0)}{\partial x^3} - \frac{2}{\rho^2(x, 0)} \frac{\partial \rho(x, 0)}{\partial x} \frac{\partial^2 \rho(x, 0)}{\partial x^2} + \frac{1}{\rho(x, 0)^4} \left( \frac{\partial \rho(x, 0)}{\partial x} \right)^3 \right] + \frac{g \rho}{m} \frac{\partial \rho(x, 0)}{\partial x} =
\]

\[ = k(0) \left[ x - q(0) \right] . \quad (2.17) \]

Performing the differentiation indicated above, Eq.(2.17) becomes:

\[ \left[ \frac{\hbar^2}{m^2 \sigma^2(0)} - \frac{2 g}{\sqrt{\pi} \sigma(0)^2} \right] \left[ x - q(0) \right] = k(0) \left[ x - q(0) \right] \rightarrow
\]

\[ k(0) = \left[ \frac{\hbar^2}{m^2 \sigma^2(0)} - \frac{2 g}{\sqrt{\pi} \sigma(0)^2} \right] . \quad (2.18) \]

Comparing Eq.(2.18) with the Eqs.(2.12) and (2.13), by analogy we get,
\[ k(t) = \left[ \frac{h^2}{m^2 \sigma^2(t)} - \frac{2 g}{\sigma(t) m \sqrt{\pi \sigma(t)}} \right], \quad (2.19) \]

\[ \rho(x, t) = \left[ \pi \sigma(t) \right]^{-1/2} e^{-\frac{(x - q(t))^2}{\sigma(t)}} , \quad (2.20) \]

where,

\[ \sigma^2(t) = \frac{h^2}{m^2} k(t) + \frac{1}{\sigma(t) m \sqrt{\pi \sigma(t)}} . \quad (2.21) \]

Taking into account Eqs.(2.19-21), let us perform the following differentiations, remembering that \( t \) and \( x \) are independent variables:

\[ \frac{\partial \rho}{\partial t} = -\frac{1}{2} \frac{\dot{\sigma}}{\sigma} \rho + \rho \left[ \frac{(x - q)^2}{\sigma^2} \dot{\sigma} + \frac{2 (x - q)}{\sigma} \dot{q} \right] , \quad (2.22) \]

\[ \frac{\partial \rho}{\partial x} = -\frac{2 (x - q)}{\sigma} \rho , \quad (2.23) \]

From Eq.(2.3) and Eqs.(2.22-23), results

\[ \frac{\partial v_{qu}}{\partial x} - \frac{2 (x - q)}{\sigma} v_{qu} = \frac{\dot{\sigma}}{2 \sigma} - \dot{\sigma} \frac{2}{\sigma^2} (x - q)^2 - \frac{2 (x - q)}{\sigma} \dot{q} . \quad (2.24) \]

Defining

\[ p(x, t) = -2 \frac{(x - q)}{\sigma} , \quad (2.25) \]

\[ r(x, t) = \frac{\dot{\sigma}}{2 \sigma} - \dot{\sigma} \frac{2}{\sigma^2} (x - q)^2 - \frac{2 (x - q)}{\sigma} \dot{q} , \quad (2.26) \]

Eq.(2.24) becomes,

\[ \frac{\partial v_{qu}}{\partial x} + p(x, t) v_{qu} = r(x, t) , \quad (2.27) \]

which can be integrated, giving:

\[ v_{qu} = \frac{1}{u} \left[ \int r(x, t) \, dx + c(t) \right] , \quad (2.28) \]

where,

\[ u = \exp \left( \int p(x, t) \, dx \right) . \quad (2.29) \]

Using Eqs.(2.20,28,29), the function \( u \) given by Eq.(2.29) is written as:

\[ u = \exp \left( \int \left[ -\frac{2 (x - q)}{\sigma} \right] \, dx \right) = \left( \pi \sigma \right)^{1/2} \rho . \quad (2.30) \]
In this way, defining \( I = r u x \), and using Eqs.(2.26,30) we obtain:

\[
I = \int r u \, \partial x = \\
= \int \left[ \frac{\dot{\sigma}}{2 \sigma} - \frac{\dot{\sigma}}{\sigma^2} (x - q)^2 - \frac{2 (x - q)}{\sigma} \dot{q} \right] (\pi \sigma)^{1/2} \rho \, \partial x \\
\rightarrow I = I_1 - I_2 , \quad (2.31)
\]

where,

\[
I_1 = \int \left[ \frac{\dot{\sigma}}{2 \sigma} - \frac{\dot{\sigma}}{\sigma^2} (x - q)^2 \right] (\pi \sigma)^{1/2} \rho \, \partial x , \quad (2.32)
\]

and

\[
I_2 = \int \left[ \frac{2 (x - q)}{\sigma} \dot{q} \right] (\pi \sigma)^{1/2} \rho \, \partial x . \quad (2.33)
\]

To integrate Eq.(2.32) it is necessary, first to perform the differentiation\(^9\) shown bellow, where Eq.(2.23) is used:

\[
\frac{\partial}{\partial x} \left[ \frac{\dot{\sigma}}{2 \sigma} (x - q) \rho \right] = \left[ \frac{\dot{\sigma}}{2 \sigma} - \frac{\dot{\sigma}}{\sigma^2} (x - q)^2 \right] \rho . \quad (2.34)
\]

Inserting Eq.(2.34) into Eq.(2.32), results

\[
I_1 = (\pi \sigma)^{1/2} \rho \left( \frac{\dot{\sigma}}{2 \sigma} \right) (x - q) . \quad (2.35)
\]

Similarly, to calculate \( I_2 \), seen in Eq.(2.33), we need to use Eq.(2.23) obtaining,

\[
I_2 = - (\pi \sigma)^{1/2} \dot{q} \rho . \quad (2.36)
\]

Substituting Eqs.(2.35-36) into Eq.(2.31) we see that, teremos:

\[
I = (\pi \sigma)^{1/2} \rho \left[ \frac{\dot{\sigma}}{2 \sigma} (x - q) + \dot{q} \right] . \quad (2.37)
\]

Remembering that the quantum velocity \( v_{qu} \) is defined by Eq.(2.28) and using Eqs.(2.30,37) we verify that \( v_{qu} \) can be written as:

\[
v_{qu} = \frac{\dot{\sigma}}{2 \sigma} (x - q) + \dot{q} + \frac{c(t)}{(\pi \sigma)^{1/2} \rho} . \quad (2.38)
\]

Assuming that the mass density \( \rho \to 0 \) when \( |x| \to \infty \) we verify that the parameter \( c(t) \) must be equal to zero. Consequently, \( v_{qu} \) becomes,

\[
v_{qu}(x, t) = \frac{\sigma(t)}{2 \sigma(t)} [x - q(t)] + \dot{q}(t) . \quad (2.39)
\]
Using the above Eq.(2.39) we calculate the following differentiations, remembering that $t$ and $x$ as independent variables:

$$\frac{\partial v_{uu}}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{\dot{\sigma}}{2\sigma} (x - q) + \dot{q} \right] =$$

$$= \frac{\dot{\sigma}}{2\sigma} (x - q) - \frac{(\dot{\sigma})^2}{2\sigma^2} (x - q) - \frac{\dot{\sigma}}{2\sigma} \dot{q} + \ddot{q}. \quad (2.40)$$

$$\frac{\partial v_{uu}}{\partial x} = \frac{\partial}{\partial x} \left[ \frac{\dot{\sigma}}{2\sigma} (x - q) + \dot{q} \right] = \frac{\dot{x}}{2\sigma}. \quad (2.41)$$

Now, adding the factor $\omega^2 q$ to Eqs.(2.39-41), we see that Eq.(2.10) becomes,

$$\frac{\dot{\sigma}}{2\sigma} (x - q) - \frac{(\dot{\sigma})^2}{2\sigma^2} (x - q) - \frac{\dot{\sigma}}{2\sigma} \dot{q} + \ddot{q} + \frac{\dot{q}}{2\sigma} +$$

$$+ \omega^2 x + \omega^2 q - \omega^2 q = \left( \frac{\hbar^2}{m^2 \sigma^2} - \frac{2g}{m \sqrt{\pi} \sigma} \right) (x - q) \rightarrow$$

$$\left[ \frac{\dot{\sigma}}{2\sigma} - \frac{(\dot{\sigma})^2}{4\sigma^2} + \omega^2 - \frac{\hbar^2}{m^2 \sigma^2} + \frac{2g}{m \sigma \sqrt{\pi} \sigma} \right] (x - q) + \ddot{q} + \omega^2 q = 0. \quad (2.42)$$

To satisfy Eq.(2.42), the following conditions must be obeyed:

$$\frac{\dot{\sigma}}{2} - \frac{(\dot{\sigma})^2}{4\sigma^2} + \omega^2 - \frac{\hbar^2}{m^2 \sigma^2} + \frac{2g}{m \sigma \sqrt{\pi} \sigma} = 0, \quad (2.43)$$

$$\ddot{q} + \omega^2 q = 0. \quad (2.44)$$

Putting

$$\sigma = \frac{\hbar}{m} \alpha^2, \quad (2.45)$$

we obtain,

$$\dot{\sigma} = \frac{2\hbar}{m} \alpha \dot{\alpha}, \quad \ddot{\sigma} = \frac{2\hbar}{m} \left[ (\dot{\alpha})^2 + \alpha \ddot{\alpha} \right]. \quad (2.46-47)$$

Inserting Eqs.(2.45-47) into Eq.(2.43):

$$\frac{2\hbar}{m} \frac{[(\dot{\alpha})^2 + \alpha \ddot{\alpha}]}{\alpha^2} - \frac{4\hbar^2}{m} \frac{\alpha^2 (\dot{\alpha})^2}{\alpha^4} + \frac{2g}{m \sqrt{\pi} \alpha} - \frac{\hbar^2}{m^2 \alpha^4} + \omega^2 = 0 \rightarrow$$

$$\ddot{\alpha} + \omega^2 \alpha + \frac{2g}{\hbar \alpha^2 \sqrt{\pi} m} = \frac{1}{\alpha^4}. \quad (2.48)$$
Note that, although the above Eq.(2.48) is formally identical to that obtained by Ermakov [see Eq.(1.3)], they are different due to the Planck’s constant $\hbar$ which appears in Eq.(2.48). This is the same kind of difference found between the classical wave equation (d’Alembertian) and the quantum Schrödinger wave equation.

Finally, eliminating the factor $\omega^2$ into Eqs.(2.44) and (2.48) we get,

$$\ddot{\alpha} - \frac{\ddot{q} \alpha}{q} + \frac{2\,g\,q}{h\,\alpha^2\sqrt{\frac{2\,\bar{m}}{m}}} = \frac{1}{\alpha^2} \rightarrow \ddot{\alpha} q - \ddot{q} \alpha + \frac{2\,g\,q}{h\,\alpha^2\sqrt{\frac{2\,\bar{m}}{m}}} = \frac{q}{\alpha^2} \rightarrow$$

$$\frac{d}{dt} (\dot{\alpha} q - \dot{q} \alpha) = \frac{q}{\alpha^3} - \frac{2\,g\,q}{h\,\alpha^2\sqrt{\frac{2\,\bar{m}}{m}}} \rightarrow$$

$$(\dot{\alpha} q - \dot{q} \alpha) \frac{d}{dt} (\dot{\alpha} q - \dot{q} \alpha) = \left(\frac{q}{\alpha^3} - \frac{2\,g\,q}{h\,\alpha^2\sqrt{\frac{2\,\bar{m}}{m}}}\right) (\dot{\alpha} q - \dot{q} \alpha) \rightarrow$$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{\alpha} q - \dot{q} \alpha)^2\right] = -\frac{q}{\alpha} \frac{d}{dt} \left(\frac{q}{\alpha}\right) + \frac{2\,g\,q}{h\,\sqrt{\frac{2\,\bar{m}}{m}}} \frac{(\dot{\alpha} q - \dot{q} \alpha)}{\alpha^2} \rightarrow$$

$$\frac{d}{dt} \left[\frac{1}{2} (\dot{\alpha} q - \dot{q} \alpha)^2 + \frac{1}{2} \left(\frac{q}{\alpha}\right)^2\right] = \frac{2\,g\,q}{h\,\sqrt{\frac{2\,\bar{m}}{m}}} \frac{d}{dt} \left(\frac{q}{\alpha}\right) \rightarrow$$

$$\frac{dI}{dt} = \frac{2\,g\,q}{h\,\sqrt{\frac{2\,\bar{m}}{m}}} \frac{d}{dt} \left(\frac{q}{\alpha}\right) \, (2.49)$$

where,

$$I = \frac{1}{2} \left[\left(\dot{\alpha} q - \dot{q} \alpha\right)^2 + \left(\frac{q}{\alpha}\right)^2\right], \, (2.50)$$

which represents the **Ermakov-Lewis-Schrödinger invariant** of the time dependent harmonic oscillator (TDHO)[9]. In conclusion, we have shown that the **Gross-Pitaevskii equation** has not an **Ermakov-Lewis invariant** for the TDHO.

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