INVARIANT METRICS AND LAPLACIANS ON SIEGEL-JACOBI SPACE

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Abstract. In this paper, we compute Riemannian metrics on the Siegel-Jacobi space which are invariant under the natural action of the Jacobi group explicitly and also provide the Laplacians of these invariant metrics. These are expressed in terms of the trace form.

1. Introduction

For a given fixed positive integer $n$, we let

$$H_n = \{ Z \in \mathbb{C}^{(n,n)} \mid Z = {}^tZ, \quad \text{Im} \, Z > 0 \}$$

be the Siegel upper half plane of degree $n$ and let

$$Sp(n, \mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} \mid {}^tMJ_nM = J_n \}$$

be the symplectic group of degree $n$, where

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$ 

We see that $Sp(n, \mathbb{R})$ acts on $H_n$ transitively by

$$M \cdot Z = (AZ + B)(CZ + D)^{-1},$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $Z \in H_n$.

For two positive integers $n$ and $m$, we consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \{ (\lambda, \mu; \kappa) \mid \lambda, \mu \in \mathbb{R}^{(m,n)}, \kappa \in \mathbb{R}^{(m,m)}, \kappa + \mu^t\lambda \text{ symmetric} \}$$

endowed with the following multiplication law

$$(\lambda, \mu; \kappa) \circ (\lambda', \mu'; \kappa') = (\lambda + \lambda', \mu + \mu'; \kappa + \kappa' + \lambda^t\mu' - \mu^t\lambda').$$

We define the semidirect product of $Sp(n, \mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$G^J := Sp(n, \mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

dowed with the following multiplication law

$$(M, (\lambda, \mu; \kappa)) \cdot (M', (\lambda', \mu'; \kappa')) = (MM', (\bar{\lambda} + \lambda', \bar{\mu} + \mu'; \kappa + \kappa' + \bar{\lambda}^t\mu' - \bar{\mu}^t\lambda')).$$

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with \( M, M' \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa), (\lambda', \mu'; \kappa') \in H^m_{\mathbb{R}} \) and \((\lambda, \mu)M'\). We call this group \( G^J \) the Jacobi group of degree \( n \) and index \( m \). We have the natural action of \( G^J \) on \( \mathbb{H}_n \times \mathbb{C}^{(m,n)} \) defined by

\[
(1.2) \quad \left( M, (\lambda, \mu; \kappa) \right) \cdot (Z, W) = \left( M \cdot Z, (W + \lambda Z + \mu)(CZ + D)^{-1} \right),
\]

where \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), (\lambda, \mu; \kappa) \in H^m_{\mathbb{R}} \) and \((Z, W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)} \). The homogeneous space \( \mathbb{H}_n \times \mathbb{C}^{(m,n)} \) is called the Siegel-Jacobi space of degree \( n \) and index \( m \). We refer to [2-3], [6-7], [11], [14-21] for more details on materials related to the Siegel-Jacobi space.

For brevity, we write \( \mathbb{H}_{n,m} := \mathbb{H}_n \times \mathbb{C}^{(m,n)} \). For a coordinate \((Z, W) \in \mathbb{H}_{n,m}\) with \( Z = (z_{\mu \nu}) \in \mathbb{H}_n \) and \( W = (w_{kl}) \in \mathbb{C}^{(m,n)} \), we put

\[
\begin{align*}
Z &= X + iY, \quad X = (x_{\mu \nu}), \quad Y = (y_{\mu \nu}) \text{ real}, \\
W &= U + iV, \quad U = (u_{kl}), \quad V = (v_{kl}) \text{ real}, \\
dZ &= (dz_{\mu \nu}), \quad dZ = (d\overline{z}_{\mu \nu}), \quad dY = (dy_{\mu \nu}), \\
dW &= (dw_{kl}), \quad dW = (d\overline{w}_{kl}), \quad dV = (dv_{kl}),
\end{align*}
\]

where \( \delta_{ij} \) denotes the Kronecker delta symbol.

C. L. Siegel [12] introduced the symplectic metric \( ds_n^2 \) on \( \mathbb{H}_n \) invariant under the action (1.1) of \( Sp(n, \mathbb{R}) \) given by

\[
(1.3) \quad ds_n^2 = \sigma \left( Y^{-1} dZ Y^{-1} d\overline{Z} \right)
\]

and H. Maass [8] proved that the differential operator

\[
(1.4) \quad \Delta_n = 4 \sigma \left( Y^{-1} \left( Y \frac{\partial}{\partial Y} \right) \frac{\partial}{\partial \overline{Z}} \right)
\]

is the Laplacian of \( \mathbb{H}_n \) for the symplectic metric \( ds_n^2 \). Here \( \sigma(A) \) denotes the trace of a square matrix \( A \).

In this paper, for arbitrary positive integers \( n \) and \( m \), we express the \( G^J \)-invariant metrics on \( \mathbb{H}_n \times \mathbb{C}^{(m,n)} \) and their Laplacians explicitly.
In fact, we prove the following theorems.

**Theorem 1.1.** For any two positive real numbers $A$ and $B$, the following metric

$$
\begin{align*}
 ds^2_{n,m;A,B} &= A \sigma(Y^{-1}dZ Y^{-1}d\overline{Z}) \\
 &\quad + B \left\{ \sigma(Y^{-1}tV V Y^{-1}dZ Y^{-1}d\overline{Z}) + \sigma(Y^{-1}(dW) d\overline{W}) \\
 &\quad - \sigma(V Y^{-1} dZ Y^{-1}t(d\overline{W})) - \sigma(V Y^{-1}d\overline{Z} Y^{-1}t(dW)) \right\}
\end{align*}
$$

is a Riemannian metric on $\mathbb{H}_{n,m}$ which is invariant under the action (1.2) of the Jacobi group $G^J$.

**Theorem 1.2.** For any two positive real numbers $A$ and $B$, the Laplacian $\Delta_{n,m;A,B}$ of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$ is given by

$$
\begin{align*}
 \Delta_{n,m;A,B} &= \frac{4}{A} \left\{ \sigma \left( Y^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z} \right) + \sigma \left( V Y^{-1} V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right) \\
 &\quad + \sigma \left( V \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial W} \right) + \sigma \left( V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial Z} \right) \right\} \\
 &\quad + \frac{4}{B} \sigma \left( Y \frac{\partial}{\partial W} \left( \frac{\partial}{\partial W} \right) \right).
\end{align*}
$$

The following differential form

$$
\text{dv} = (\det Y)^{-(n+m+1)} [dX] \wedge [dY] \wedge [dU] \wedge [dV]
$$

is a $G^J$-invariant volume element on $\mathbb{H}_{n,m}$, where

$$
[dX] = \wedge_{\mu \leq \nu} dx_{\mu \nu}, \quad [dY] = \wedge_{\mu \leq \nu} dy_{\mu \nu}, \quad [dU] = \wedge_{k,l} du_{kl} \quad \text{and} \quad [dV] = \wedge_{k,l} dv_{kl}.
$$

The point is that the invariant metric $ds^2_{n,m;A,B}$ and its Laplacian $\Delta_{n,m;A,B}$ are expressed in terms of the trace form.

For the case $n = m = 1$ and $A = B = 1$, Berndt proved in [1] (cf. [19]) that the metric $ds^2_{1,1}$ on $\mathbb{H} \times \mathbb{C}$ defined by

$$
\begin{align*}
 ds^2_{1,1} := ds_{1,1;1,1} &= \frac{y + v^2}{y^2} (dx^2 + dy^2) + \frac{1}{y} (du^2 + dv^2) \\
 &\quad - \frac{2v}{y} (dx du + dy dv)
\end{align*}
$$

is a Riemannian metric on $\mathbb{H} \times \mathbb{C}$ invariant under the action (1.2) of the Jacobi group and its Laplacian $\Delta_{1,1}$ is given by

$$
\begin{align*}
 \Delta_{1,1} := \Delta_{1,1;1,1} &= y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + (y + v^2) \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \\
 &\quad + 2y v \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right).
\end{align*}
$$

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Notations: We denote by $\mathbb{R}$ and $\mathbb{C}$ the field of real numbers, and the field of complex numbers respectively. The symbol “$:=$” means that the expression on the right is the definition of that on the left. For two positive integers $k$ and $l$, $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For a square matrix $A \in F^{(k,k)}$ of degree $k$, $\sigma(A)$ denotes the trace of $A$. For any $M \in F^{(k,l)}$, $tM$ denotes the transpose matrix of $M$. $E_n$ denotes the identity matrix of degree $n$. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = tABA$.

2. Proof of Theorem 1.1

Let $g = (M, (\lambda, \mu; \kappa))$ be an element of $G^J$ with $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R})$ and $(Z, W) \in \mathbb{H}_{n,m}$ with $Z \in \mathbb{H}_n$ and $W \in \mathbb{C}^{(m,n)}$. If we put $(Z_*, W_*) := g \cdot (Z, W)$, then we have

$$Z_* = M \cdot Z = (AZ + B)(CZ + D)^{-1},$$

$$W_* = (W + \lambda Z + \mu)(CZ + D)^{-1}.$$

Thus we obtain

$$dZ_* = dZ[(CZ + D)^{-1}] = t(CZ + D)^{-1} dZ(CZ + D)^{-1} \quad (2.1)$$

and

$$dW_* = dW(CZ + D)^{-1} + \{\lambda - (W + \lambda Z + \mu)(CZ + D)^{-1}C\}dZ(CZ + D)^{-1} \quad (2.2)$$

Here we used the following facts that

$$d(CZ + D)^{-1} = -(CZ + D)^{-1}C dZ(CZ + D)^{-1}$$

and that $(CZ + D)^{-1}C$ is symmetric.

We put

$$Z_* = X_* + iY_*, \quad W_* = U_* + iV_*, \quad X_*, Y_*, U_*, V_* \text{ real}.$$

From [9], p.33 or [13], p.128, we know that

$$Y_* = Y \{(CZ + D)^{-1}\} = t(C\overline{Z} + D)^{-1} Y(CZ + D)^{-1} \quad (2.3)$$

First of all, we recall that the following matrices

$$t(b) = \begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix}, \quad b = t b \text{ real},$$

$$g_0(h) = \begin{pmatrix} t h & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad h \in GL(n, \mathbb{R}),$$

$$J_n = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$$

generate the symplectic group $Sp(n, \mathbb{R})$ (cf. [4], [5]). Therefore the following elements
Therefore, and generate the Jacobi group \(G^f\). So it suffices to prove the invariance of the metric \(ds^2_{n,m;A,B}\) under the action of the generators \(t(b; \lambda, \mu, \kappa), g(h)\) and \(\sigma_n\). For brevity, we write

\[
\begin{align*}
(a) &= \sigma(Y^{-1}dZ Y^{-1}d\overline{Z}), \\
(b) &= \sigma(Y^{-1t}VV^{-1}dZ Y^{-1}d\overline{Z}), \\
(c) &= \sigma(Y^{-1t}(dW)d\overline{W}), \\
(d) &= -\sigma(VY^{-1}dZ Y^{-1t}(dW) + VY^{-1}d\overline{Z} Y^{-1t}(dW))
\end{align*}
\]

and

\[
\begin{align*}
(a)_* &= \sigma(Y_*^{-1}dZ_* Y_*^{-1}d\overline{Z}_*), \\
(b)_* &= \sigma(Y_*^{-1}tV_*V_*^{-1}dZ_* Y_*^{-1}d\overline{Z}_*), \\
(c)_* &= \sigma(Y_*^{-1t}(dW_*)d\overline{W}_*), \\
(d)_* &= -\sigma(V_* Y_*^{-1}dZ_* Y_*^{-1t}(dW_*) + V_* Y_*^{-1}d\overline{Z}_* Y_*^{-1t}(dW_*))
\end{align*}
\]

**Case I.** \(g = t(b; \lambda, \mu, \kappa)\) with \(b = t'b\) real and \((\lambda, \mu; \kappa) \in H^{(n,m)}_{\mathbb{R}}\).

In this case, we have

\[Z_* = Z + b, \quad Y_* = Y, \quad W_* = W + \lambda Z + \mu, \quad V_* = V + \lambda Y\]

and

\[dZ_* = dZ, \quad dW_* = dW + \lambda dZ.\]

Therefore

\[
\begin{align*}
(a)_* &= \sigma(Y_*^{-1}dZ_* Y_*^{-1}d\overline{Z}_*) = \sigma(Y^{-1}dZ Y^{-1}d\overline{Z}) = (a), \\
(b)_* &= \sigma(Y^{-1t}VV^{-1}dZ Y^{-1}d\overline{Z}) + \sigma(Y^{-1t}V \lambda dZ Y^{-1}d\overline{Z}) \\
&\quad + \sigma(t'V Y^{-1}dZ Y^{-1}d\overline{Z}) + \sigma(t' \lambda dZ Y^{-1}d\overline{Z}), \\
(c)_* &= \sigma(Y^{-1t}(dW)d\overline{W}) + \sigma(Y^{-1t}(dW) \lambda d\overline{Z}) \\
&\quad + \sigma(Y^{-1}dZ t' \lambda d\overline{W}) + \sigma(Y^{-1}dZ t' \lambda d\overline{Z})
\end{align*}
\]
and
\[(d)_* = -\sigma(V Y^{-1} dZ Y^{-1} t(dW)) - \sigma(\lambda dZ Y^{-1} t(dW))
- \sigma(V Y^{-1} dZ Y^{-1} t \lambda) - \sigma(\lambda dZ Y^{-1} t(dW))
- \sigma(V Y^{-1} dZ Y^{-1} t \lambda).
\]

Thus we see that
\[(a) = (a)_* \quad \text{and} \quad (b) + (c) + (d) = (b)_* + (c)_* + (d)_*.
\]

Hence
\[ds^2_{n,m;A,B} = A(a) + B\{(b) + (c) + (d)\}\]
is invariant under the action of \(t(B; \lambda, \mu, \kappa)\).

**Case II.** \(g = g(h)\) with \(h \in GL(n, \mathbb{R})\).

In this case, we have
\[Z_* = {}^t h Z h, \quad Y_* = {}^t h Y h, \quad W_* = Wh, \quad V_* = Vh\]
and
\[dZ_* = {}^t h dZ h, \quad dW_* = dW h.
\]
Therefore by an easy computation, we see that each of \((a), (b), (c)\) and \((d)\) is invariant under the action of all \(g(h)\) with \(h \in GL(n, \mathbb{R})\). Hence the metric \(ds^2_{n,m;A,B}\) is invariant under the action of all \(g(h)\) with \(h \in GL(n, \mathbb{R})\).

**Case III.** \(g = \sigma_n = \left(\begin{array}{cc} 0 & -E_n \\ E_n & 0 \end{array}\right); (0,0;0)\).

In this case, we have
\[(2.4) \quad Z_* = -Z^{-1} \quad \text{and} \quad W_* = WZ^{-1}.
\]
We set
\[\theta_1 := \text{Re} Z^{-1} \quad \text{and} \quad \theta_2 := \text{Im} Z^{-1}.
\]
Then \(\theta_1\) and \(\theta_2\) are symmetric matrices and we have
\[(2.5) \quad Y_* = -\theta_2 \quad \text{and} \quad V_* := \text{Im} W_* = V\theta_1 + U\theta_2.
\]
It is easy to see that
\[(2.6) \quad Y = -Z\theta_2 Z = -\overline{Z}\theta_2 Z,
\]
\[(2.7) \quad \theta_1 Y + \theta_2 X = 0
\]
and
\[(2.8) \quad \theta_1 X - \theta_2 Y = E_n.
\]
According to (2.6) and (2.7), we obtain
\[(2.9) \quad X = (-\theta_2)^{-1}\theta_1 Y \quad \text{and} \quad Y^{-1} = \theta_1(-\theta_2)^{-1}\theta_1 - \theta_2.
\]
From (2.1) and (2.2), we have

\[ dZ^* = Z^{-1} dZZ^{-1} \]  

(2.10)

and

\[ dW^* = dWZ^{-1} - WZ^{-1}dZZ^{-1} = (dW - WZ^{-1}dZ)Z^{-1}. \]  

(2.11)

Therefore we have, according to (2.6) and (2.10),

\[
(a)^* = \sigma \left( (-\theta_2)^{-1} Z^{-1} dZ Z^{-1} (-\theta_2)^{-1} Z^{-1} dZZ^{-1} \right) \\
= \sigma \left( Y^{-1} dZ Y^{-1} dZ \right) = (a).
\]

According to (2.5)-(2.10), we have

\[
(b)^* = \sigma \left( (-\theta_2)^{-1} (\theta_1^t V + \theta_2^t U)(V\theta_1 + U\theta_2)(-\theta_2)^{-1} Z^{-1} dZY^{-1} dZZ^{-1} \right) \\
= \sigma \left( \{^t U - (-\theta_2)^{-1} \theta_1^t V\} \{U - V\theta_1 (-\theta_2)^{-1}\} Z^{-1} dZY^{-1} dZZ^{-1} \right) \\
= \sigma \left( \{^t W + (iE_n - (-\theta_2)^{-1} \theta_1) ^t V\} \{W - V(iE_n + \theta_1 (-\theta_2)^{-1})\} \\
Z^{-1} dZY^{-1} dZZ^{-1} \right),
\]

\[
(c)^* = \sigma \left( (-\theta_2)^{-1} \left( Z^{-1} ^t (dW) - Z^{-1} dZZ^{-1} ^t W \right) \left( dWZ^{-1} - WZ^{-1} dZZ^{-1} \right) \right) \\
= \sigma \left( (-\theta_2)^{-1} Z^{-1} ^t (dW) dWZ^{-1} - (-\theta_2)^{-1} \left( Z^{-1} ^t (dW) WZ^{-1} dZZ^{-1} \right) \right) \\
- (-\theta_2)^{-1} Z^{-1} dZZ^{-1} ^t WdWZ^{-1} \right) \\
+ (-\theta_2)^{-1} Z^{-1} dZZ^{-1} ^t WdWZ^{-1} \right) \\
= \sigma \left( Y^{-1} ^t (dW) dW \right) - \sigma \left( Y^{-1} ^t (dW) WZ^{-1} dZ \right) \\
- \sigma \left( Y^{-1} dZZ^{-1} ^t WdW \right) + \sigma \left( Y^{-1} dZZ^{-1} ^t WdWZ^{-1} dZ \right)
\]
and

\[
(d)_* = -\sigma \left( (V \theta_1 + U \theta_2)(-\theta_2)^{-1} Z^{-1}dZ Z^{-1}(-\theta_2)^{-1} \left\{ Z^{-1}t(d\overline{W}) - \overline{Z}^{-1}d\overline{Z} \overline{Z}^{-1} W \right\} \right)
\]

\[
- \sigma \left( (V \theta_1 + U \theta_2)(-\theta_2)^{-1} \overline{Z}^{-1}d\overline{Z} \overline{Z}^{-1}(-\theta_2)^{-1} \left\{ Z^{-1}t(dW) - Z^{-1}dZ Z^{-1}tW \right\} \right)
\]

\[
= -\sigma \left( (V \theta_1 + U \theta_2)(-\theta_2)^{-1} Z^{-1}dZ Y^{-1}t(d\overline{W}) \right) + \sigma \left( (V \theta_1 + U \theta_2)(-\theta_2)^{-1} Z^{-1}dZ Y^{-1}t(d\overline{W}) \right)
\]

\[
- \sigma \left( (V \theta_1 + U \theta_2)(-\theta_2)^{-1} \overline{Z}^{-1}d\overline{Z} Y^{-1}t(dW) \right) + \sigma \left( (V \theta_1 + U \theta_2)(-\theta_2)^{-1} \overline{Z}^{-1}d\overline{Z} Y^{-1}dZ Z^{-1}tW \right).
\]

Taking the \((dZ, d\overline{W})\)-part \(\square(Z, \overline{W})\) in \((b)_* + (c)_* + (d)_*\), we have

\[
\square(Z, \overline{W}) = -\sigma \left( V Y^{-1}dZ Y^{-1}t(d\overline{W}) \right) + \sigma \left( Y^{-1}dZ (iW_* - Z^{-1}tW) d\overline{W} \right)
\]

\[
= -\sigma \left( V Y^{-1}dZ Y^{-1}t(d\overline{W}) \right) \quad \text{because} \quad W_* = W Z^{-1} \quad (\text{cf. (2.4)}).
\]

Similarly, if we take the \((d\overline{Z}, dW)\)-part \(\square(\overline{Z}, W)\) in \((b)_* + (c)_* + (d)_*\), we have

\[
\square(\overline{Z}, W) = -\sigma \left( V Y^{-1}d\overline{Z} Y^{-1}t(dW) \right) + \sigma \left( d\overline{Z} Y^{-1}t(dW)(\overline{W_*} - \overline{W} \overline{Z}^{-1}) \right)
\]

\[
= -\sigma \left( V Y^{-1}d\overline{Z} Y^{-1}t(dW) \right) \quad \text{because} \quad W_* = W Z^{-1}.
\]

If we take the \((dW, d\overline{W})\)-part \(\square(W, \overline{W})\) in \((b)_* + (c)_* + (d)_*\), we have

\[
\square(W, \overline{W}) = \sigma \left( Y^{-1}t(dW) d\overline{W} \right).
\]

Finally, if we take the \((dZ, d\overline{Z})\)-part \(\square(Z, \overline{Z})\) in \((b)_* + (c)_* + (d)_*\), we have

\[
\square(Z, \overline{Z}) = \sigma \left( \left\{ \overline{W} + (iE_n - (-\theta_2)^{-1}\theta_1)^tV \right\} \left\{ W - V(iE_n + \theta_1(-\theta_2)^{-1}) \right\} \right)
\]

\[
Z^{-1}dZ Y^{-1}d\overline{Z} \overline{Z}^{-1}
\]

\[
+ \sigma \left( Z^{-1}tW \overline{W} \overline{Z}^{-1}d\overline{Z} Y^{-1}dZ \right)
\]

\[
+ \sigma \left( \overline{W}(V \theta_1 + U \theta_2)(-\theta_2)^{-1} dZ Y^{-1}d\overline{Z} \overline{Z}^{-1} \right)
\]

\[
+ \sigma \left( \overline{W}(V \theta_1 + U \theta_2)(-\theta_2)^{-1} \overline{Z}^{-1}d\overline{Z} Y^{-1}dZ Z^{-1} \right).
\]

Since

\[
(V \theta_1 + U \theta_2)(-\theta_2)^{-1} = -U + V \theta_1(-\theta_2)^{-1}
\]

\[
= -W + V \{iE_n + \theta_1(-\theta_2)^{-1}\}
\]

\[
= -\overline{W} - V \{iE_n - \theta_1(-\theta_2)^{-1}\},
\]
we have
\[ \Box(Z, Z) = \sigma (Z^{-1} \{ iE_n - (-\theta_2)^{-1} \theta_1 \} \, tV \{ W - V(iE_n + \theta_1(-\theta_2)^{-1}) \} 
- \sigma (Z^{-1} \{ iE_n - (-\theta_2)^{-1} \theta_1 \} \, tVWZ^{-1} dZ Y^{-1} Y dZ) \]
\[ = -\sigma (Z^{-1} \{ iE_n - (-\theta_2)^{-1} \theta_1 \} \, tVV \{ iE_n + \theta_1(-\theta_2)^{-1} \} 
- \sigma (Z^{-1} \{ iE_n - (-\theta_2)^{-1} \theta_1 \} \, tV \{ W - V(iE_n + \theta_1(-\theta_2)^{-1}) \} 
- \sigma (Z^{-1} dZ Y^{-1} Y dZ). \]

By the way, according to (2.9), we obtain
\[ Z^{-1} \{ iE_n - (-\theta_2)^{-1} \theta_1 \} = (\theta_1 - i\theta_2) \{ iE_n - (-\theta_2)^{-1} \theta_1 \} \]
\[ = \theta_2 - \theta_1(-\theta_2)^{-1} \theta_1 = -Y^{-1} \]
and
\[ \{ iE_n + \theta_1(-\theta_2)^{-1} \} Z^{-1} = \theta_1(-\theta_2)^{-1} \theta_1 - \theta_2 = Y^{-1}. \]

Therefore
\[ \Box(Z, Z) = \sigma (Y^{-1} \, tV \, VY^{-1} dZ Y^{-1} dZ). \]

Hence \((a) = (a)\) and
\[ (b)_* + (c)_* + (d)_* = \Box(Z, W) + \Box(Z, W) + \Box(W, W) + \Box(Z, Z) \]
\[ = (b) + (c) + (d). \]

This implies that the metric
\[ ds^2_{n,m;A,B} = A \, (a) + B \, \{ (b) + (c) + (d) \} \]
is invariant under the action (1.2) of \(\sigma_n\).

Consequently \(ds^2_{n,m;A,B}\) is invariant under the action (1.2) of the Jacobi group \(G^J\). In particular, for \((Z, W) = (iE_n, 0)\), we have
\[ ds^2_{n,m;A,B} = A \cdot \sigma (dZ d\overline{Z}) + B \cdot \sigma (t(dW)d\overline{W}) \]
\[ = A \left\{ \sum_{\mu=1}^{n} (dx_{\mu\mu}^2 + dy_{\mu\mu}^2) + 2 \sum_{1 \leq \mu < \nu \leq n} (dx_{\mu\nu}^2 + dy_{\mu\nu}^2) \right\} \]
\[ + B \left\{ \sum_{1 \leq k \leq m, 1 \leq l \leq n} (du_{kl}^2 + dv_{kl}^2) \right\}, \]
which is clearly positive definite. Since \(G^J\) acts on \(H_{n,m}\) transitively, \(ds^2_{n,m;A,B}\) is positive definite everywhere in \(H_{n,m}\). This completes the proof of Theorem 1.1. \(\Box\)

**Remark 2.1.** The scalar curvature of the Siegel-Jacobi space \((\mathbb{H}_{n,m}, ds^2_{n,m;A,B})\) is constant because of the transitive group action of \(G^J\) on \(H_{n,m}\). In the special case \(n = m = 1\) and
A = B = 1, by a direct computation, we see that the scalar curvature of \((\mathbb{H}_{1,1}, ds_{1,1,1}^2)\) is −3.

3. Proof of Theorem 1.2

If \((Z_*, W_*) = g \cdot (Z, W)\) with \(g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, (\lambda, \mu; \kappa) \in G^j\), we can see easily that

\[
\frac{\partial}{\partial Z_*} = (CZ + D)^t \left\{ (CZ + D) \frac{\partial}{\partial Z} \right\}
+ (CZ + D)^t \left\{ (C^tW + C^t \mu - D^t \lambda)^t \left( \frac{\partial}{\partial W} \right) \right\}
\]

and

\[
\frac{\partial}{\partial W_*} = (CZ + D) \frac{\partial}{\partial W}.
\]

For brevity, we put

\[(\alpha) := 4\sigma \left( Y^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial Z} \right), \]
\[(\beta) := 4\sigma \left( Y \frac{\partial}{\partial W} \right)^t \left( \frac{\partial}{\partial W} \right), \]
\[(\gamma) := 4\sigma \left( VY^{-1}V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right), \]
\[(\delta) := 4\sigma \left( V^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial W} \right), \]

and

\[(\epsilon) := 4\sigma \left( V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial Z} \right).
\]

We also set

\[(\alpha)_* := 4\sigma \left( Y_*^t \left( Y_* \frac{\partial}{\partial Z_*} \right) \frac{\partial}{\partial Z_*} \right), \]
\[(\beta)_* := 4\sigma \left( Y_* \frac{\partial}{\partial W_*} \right)^t \left( \frac{\partial}{\partial W_*} \right), \]
\[(\gamma)_* := 4\sigma \left( V_* Y_*^{-1}V_*^t \left( Y_* \frac{\partial}{\partial W_*} \right) \frac{\partial}{\partial W_*} \right), \]
\[(\delta)_* := 4\sigma \left( V_*^t \left( Y_* \frac{\partial}{\partial Z_*} \right) \frac{\partial}{\partial W_*} \right), \]

and

\[(\epsilon)_* := 4\sigma \left( V_*^t \left( Y_* \frac{\partial}{\partial W_*} \right) \frac{\partial}{\partial Z_*} \right).
\]

We need the following lemma for the proof of Theorem 1.2. H. Maass [8] observed the following useful fact.
Lemma 3.1. (a) Let $A$ be an $n \times k$ matrix and let $B$ be a $k \times n$ matrix. Assume that the entries of $A$ commute with the entries of $B$. Then $\sigma(AB) = \sigma(BA)$.

(b) Let $A$ be an $m \times n$ matrix and $B$ an $n \times l$ matrix. Assume that the entries of $A$ commute with the entries of $B$. Then $^t(AB) = ^tB^tA$.

(c) Let $A$, $B$ and $C$ be a $k \times l$, an $n \times m$ and an $m \times l$ matrix respectively. Assume that the entries of $A$ commute with the entries of $B$. Then $^t(A^t(BC)) = B^t(A^tC)$.

Proof. The proof follows immediately from the direct computation. □

Now we are ready to prove Theorem 1.2. First of all, we shall prove that $\Delta_{n,m;A,B}$ is invariant under the action of the generators $t(b; \lambda, \mu, \kappa)$, $g(h)$ and $\sigma_n$.

Case I. $g = t(b; \lambda, \mu, \kappa) = \left( \begin{pmatrix} E_n & b \\ 0 & E_n \end{pmatrix}, (\lambda, \mu; \kappa) \right)$ with $b = ^t b$ real.

In this case, we have

\begin{align*}
Y_\ast &= Y, \quad V_\ast = V + \lambda Y
\end{align*}

and

\begin{align*}
\frac{\partial}{\partial Z_\ast} &= \frac{\partial}{\partial Z} - \frac{\partial}{\partial Z}^t t(\lambda^t \left( \frac{\partial}{\partial W} \right)) \quad \text{and} \quad \frac{\partial}{\partial W_\ast} = \frac{\partial}{\partial W}.
\end{align*}

Using Lemma 3.1, we obtain

\begin{align*}
(\alpha)_\ast &= (\alpha) - \sigma \left( \lambda Y^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial W} \right) \\
&\quad - \sigma \left( Y^t \lambda^t Y \frac{\partial}{\partial W} \right) + \sigma \left( \lambda Y^t \lambda^t Y \frac{\partial}{\partial W} \right) \\
(\beta)_\ast &= \sigma \left( Y_\ast \frac{\partial}{\partial W_\ast} \left( - \frac{\partial}{\partial W_\ast} \right) \right) = (\beta),
\end{align*}

\begin{align*}
(\gamma)_\ast &= (\gamma) + \sigma \left( \lambda^t V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right) \\
&\quad + \sigma \left( V^t \lambda^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right) + \sigma \left( \lambda Y^t \lambda^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right),
\end{align*}

\begin{align*}
(\delta)_\ast &= (\delta) + \sigma \left( \lambda Y^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial W} \right) \\
&\quad - \sigma \left( V^t \lambda^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right) + \sigma \left( \lambda Y^t \lambda^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right)
\end{align*}

and

\begin{align*}
(\epsilon)_\ast &= (\epsilon) + \sigma \left( \lambda^t V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right) \quad - \sigma \left( \lambda^t V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right) - \sigma \left( \lambda Y^t V^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial W} \right).
\end{align*}

Thus $(\beta) = (\beta)_\ast$ and

$$(\alpha) + (\gamma) + (\delta) + (\epsilon) = (\alpha)_\ast + (\gamma)_\ast + (\delta)_\ast + (\epsilon)_\ast.$$
Hence
\[ \Delta_{n,m;A,B} = \frac{4}{B} (\beta) + \frac{4}{A} \left\{ (\alpha) + (\gamma) + (\delta) + (\epsilon) \right\} \]
is invariant under the action of all \( t(b; \lambda, \mu, \kappa) \).

**Case II.** \( g = g(h) = \left( \begin{pmatrix} t & 0 \\ 0 & h^{-1} \end{pmatrix}, (0,0;0) \right) \) with \( h \in GL(n, \mathbb{R}) \).

In this case, we have
\[ Y_\ast = t^h Y h \quad V_\ast = V h \]
and
\[ \frac{\partial}{\partial Z_\ast} = h^{-1} t \left( h^{-1} \frac{\partial}{\partial Z} \right), \quad \frac{\partial}{\partial W_\ast} = h^{-1} \frac{\partial}{\partial W}. \]

According to Lemma 3.1, we see that each of \((\alpha), (\beta), (\gamma), (\delta)\) and \((\epsilon)\) is invariant under the action of all \( g(h) \) with \( h \in GL(n, \mathbb{R}) \). Therefore \( \Delta_{n,m;A,B} \) is invariant under the action of all \( g(h) \) with \( h \in GL(n, \mathbb{R}) \).

**Case III.** \( g = \sigma_n = \left( \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}, (0,0;0) \right) \).

In this case, we have
\[ Z_\ast = -Z^{-1} \quad \text{and} \quad W_\ast = WZ^{-1}. \]

We set
\[ \theta_1 := \text{Re} Z^{-1} \quad \text{and} \quad \theta_2 := \text{Im} Z^{-1}. \]

Then we obtain the relations (2.5)-(2.9). From (2.6), we have the relation
\[ (3.3) \quad \theta_2 Z = -Z^{-1} Y. \]

It follows from the relation (2.3) that
\[ (3.4) \quad Y_\ast = Z^{-1} Y Z^{-1} = Z^{-1} Y Z^{-1} = -\theta_2. \]

From (2.9), we obtain
\[ (3.5) \quad \theta_1 \theta_2^{-1} \theta_1 = -Y^{-1} - \theta_2. \]

According to (3.1) and (3.2), we have
\[ (3.6) \quad \frac{\partial}{\partial Z_\ast} = Z^t \left( Z \frac{\partial}{\partial Z} \right) + Z^t \left( W^t \left( \frac{\partial}{\partial W} \right) \right) \]
and
\[ (3.7) \quad \frac{\partial}{\partial W_\ast} = Z \frac{\partial}{\partial W}. \]

From (2.6), (3.3) and Lemma 3.1, we obtain
\[ (3.8) \quad (\alpha)_\ast = (\alpha) - \sigma \left( U \theta_2 Z^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial W} \right) - \sigma \left( V \theta_2 Z^t \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial W} \right) - \sigma \left( W \theta_2^t U^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial Z} \right) + \sigma \left( W \theta_2^t U^t \left( Y \frac{\partial}{\partial W} \right) \frac{\partial}{\partial Z} \right). \]
From the relation (3.4), we see $(\beta)_* = (\beta)$. According to (3.3), (3.5) and Lemma 3.1, we obtain

$$(\gamma)_* = (\gamma) + \sigma \left( (V \theta_2^t V - V \theta_1^t U - U \theta_1^t V - U \theta_2^t U) \left( \begin{array}{c} Y \\ \frac{\partial}{\partial W} \end{array} \right) \right).$$

Using the relation (3.3) and Lemma 3.1, we finally obtain

$$(\delta)_* = \sigma \left( V \theta_1^t Z \left( \begin{array}{c} Y \\ \frac{\partial}{\partial W} \end{array} \right) \right) + \sigma \left( U \theta_2^t Z \left( \begin{array}{c} Y \\ \frac{\partial}{\partial W} \end{array} \right) \right) + \sigma \left( V \theta_1^t W \left( \begin{array}{c} Y \\ \frac{\partial}{\partial W} \end{array} \right) \right) + \sigma \left( U \theta_2^t W \left( \begin{array}{c} Y \\ \frac{\partial}{\partial W} \end{array} \right) \right).$$

Using the fact $Z^{-1} = \theta_1 + i \theta_2$, we can show that

$$(\alpha) + (\gamma) + (\delta) + (\epsilon) = (\alpha)_* + (\gamma)_* + (\delta)_* + (\epsilon)_*.$$

Hence

$$\Delta_{n,m;A,B} = \frac{4}{B} (\beta) + \frac{4}{A} \left\{ (\alpha) + (\gamma) + (\delta) + (\epsilon) \right\}$$

is invariant under the action of $\sigma_n$.

Consequently $\Delta_{n,m;A,B}$ is invariant under the action (1.2) of $G^J$. In particular, for $(Z, W) = (iE_n, 0)$, the differential operator $\Delta_{n,m;A,B}$ coincides with the Laplacian for the metric $ds^2_{n,m;A,B}$. It follows from the invariance of $\Delta_{n,m;A,B}$ under the action (1.2) and the transitivity of the action of $G^J$ on $\mathbb{H}_{n,m}$ that $\Delta_{n,m;A,B}$ is the Laplacian of $(\mathbb{H}_{n,m}, ds^2_{n,m;A,B})$.

The invariance of the differential form $dv$ follows from the fact that the following differential form

$$(\det Y)^{-(n+1)} [dX] \wedge [dY]$$

is invariant under the action (1.1) of $Sp(n, \mathbb{R})$ (cf. [13], p. 130).

4. Remark on Spectral Theory of $\Delta_{n,m;A,B}$ on Siegel-Jacobi Space

Before we describe a fundamental domain for the Siegel-Jacobi space, we review the Siegel’s fundamental domain for the Siegel upper half plane.

We let

$$\mathcal{P}_n = \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = Y^t > 0 \right\}$$

be an open cone in $\mathbb{R}^{n(n+1)/2}$. The general linear group $GL(n, \mathbb{R})$ acts on $\mathcal{P}_n$ transitively by

$$h \circ Y := hY^t h, \quad h \in GL(n, \mathbb{R}), \quad Y \in \mathcal{P}_n.$$ 

Thus $\mathcal{P}_n$ is a symmetric space diffeomorphic to $GL(n, \mathbb{R})/O(n)$. We let

$$GL(n, \mathbb{Z}) = \left\{ h \in GL(n, \mathbb{R}) \mid h \text{ is integral} \right\}.$$
be the discrete subgroup of $GL(n, \mathbb{R})$.

The fundamental domain $R_n$ for $GL(n, \mathbb{Z}) \backslash \mathcal{P}_n$ which was found by H. Minkowski [10] is defined as a subset of $\mathcal{P}_n$ consisting of $Y = (y_{ij}) \in \mathcal{P}_n$ satisfying the following conditions (M.1)-(M.2) (cf. [9] p. 123):

(M.1) $aY^t a \geq y_{kk}$ for every $a = (a_i) \in \mathbb{Z}^n$ in which $a_k, \ldots, a_n$ are relatively prime for $k = 1, 2, \ldots, n$.

(M.2) $y_{k,k+1} \geq 0$ for $k = 1, \ldots, n - 1$.

We say that a point of $R_n$ is Minkowski reduced or simply $M$-reduced.

Siegel [12] determined a fundamental domain $\mathcal{F}_n$ for $\Gamma_n \backslash \mathbb{H}_n$, where $\Gamma_n = Sp(n, \mathbb{Z})$ is the Siegel modular group of degree $n$. We say that $\Omega = X + iY \in \mathbb{H}_n$ with $X, Y$ real is Siegel reduced or $S$-reduced if it has the following three properties:

(S.1) $\det(\text{Im } (\gamma \cdot \Omega)) \leq \det(\text{Im } (\Omega))$ for all $\gamma \in \Gamma_n$;

(S.2) $Y = \text{Im } \Omega$ is $M$-reduced, that is, $Y \in R_n$;

(S.3) $|x_{ij}| \leq \frac{1}{2}$ for $1 \leq i, j \leq n$, where $X = (x_{ij})$.

$\mathcal{F}_n$ is defined as the set of all Siegel reduced points in $\mathbb{H}_n$. Using the highest point method, Siegel [12] proved the following (F1)-(F3) (cf. [9], p. 169):

(F1) $\Gamma_n \cdot \mathcal{F}_n = \mathbb{H}_n$, i.e., $\mathbb{H}_n = \bigcup_{\gamma \in \Gamma_n} \gamma \cdot \mathcal{F}_n$.

(F2) $\mathcal{F}_n$ is closed in $\mathbb{H}_n$.

(F3) $\mathcal{F}_n$ is connected and the boundary of $\mathcal{F}_n$ consists of a finite number of hyperplanes.

The metric $ds_n^2$ given by (1.3) induces a metric $ds_{\mathcal{F}_n}^2$ on $\mathcal{F}_n$. Siegel [12] computed the volume of $\mathcal{F}_n$

$$\text{vol } (\mathcal{F}_n) = 2 \prod_{k=1}^{n} \pi^{-k} \Gamma(k) \zeta(2k),$$

where $\Gamma(s)$ denotes the Gamma function and $\zeta(s)$ denotes the Riemann's zeta function. For instance,

$$\text{vol } (\mathcal{F}_1) = \frac{\pi}{3}, \quad \text{vol } (\mathcal{F}_2) = \frac{\pi^3}{270}, \quad \text{vol } (\mathcal{F}_3) = \frac{\pi^6}{127575}, \quad \text{vol } (\mathcal{F}_4) = \frac{\pi^{10}}{200930625}.$$

Let $f_{kl}$ ($1 \leq k \leq m$, $1 \leq l \leq n$) be the $m \times n$ matrix with entry 1 where the $k$-th row and the $l$-th column meet, and all other entries 0. For an element $\Omega \in \mathbb{H}_n$, we set for brevity

$$h_{kl}(\Omega) := f_{kl} \Omega, \quad 1 \leq k \leq m, \ 1 \leq l \leq n.$$

For each $\Omega \in \mathcal{F}_n$, we define a subset $P_\Omega$ of $\mathbb{C}^{(m,n)}$ by

$$P_\Omega = \left\{ \sum_{k=1}^{m} \sum_{j=1}^{n} \lambda_{kl} f_{kl} + \sum_{k=1}^{m} \sum_{j=1}^{n} \mu_{kl} h_{kl}(\Omega) \mid 0 \leq \lambda_{kl}, \mu_{kl} \leq 1 \right\}.$$  

For each $\Omega \in \mathcal{F}_n$, we define the subset $D_\Omega$ of $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ by

$$D_\Omega := \left\{ (\Omega, Z) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)} \mid Z \in P_\Omega \right\}.$$

We define

$$\mathcal{F}_{n,m} := \cup_{\Omega \in \mathcal{F}_n} D_\Omega.$$
Theorem 4.1. Let
\[ \Gamma_{n,m} := \text{Sp}(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)} \]
be the discrete subgroup of \( G^J \), where
\[ H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}. \]
Then \( F_{n,m} \) is a fundamental domain for \( \Gamma_{n,m} \backslash \mathbb{H}_{n,m} \).

Proof. The proof can be found in [20]. \( \square \)

In the case \( n = m = 1 \), R. Berndt [2] introduced the notion of Maass-Jacobi forms. Now we generalize this notion to the general case.

Definition 4.1. For brevity, we set \( \Delta_{n,m} := \Delta_{n,m;1,1} \) (cf. Theorem 1.2). Let
\[ \Gamma_{n,m} := \text{Sp}(n, \mathbb{Z}) \ltimes H_{\mathbb{Z}}^{(n,m)} \]
be the discrete subgroup of \( G^J \), where
\[ H_{\mathbb{Z}}^{(n,m)} = \left\{ (\lambda, \mu; \kappa) \in H_{\mathbb{R}}^{(n,m)} \mid \lambda, \mu, \kappa \text{ are integral} \right\}. \]
A smooth function \( f : \mathbb{H}_{n,m} \rightarrow \mathbb{C} \) is called a Maass-Jacobi form on \( \mathbb{H}_{n,m} \) if \( f \) satisfies the following conditions (MJ1)-(MJ3):

(MJ1) \( f \) is invariant under \( \Gamma_{n,m} \).

(MJ2) \( f \) is an eigenfunction of the Laplacian \( \Delta_{n,m} \).

(MJ3) \( f \) has a polynomial growth, that is, there exist a constant \( C > 0 \) and a positive integer \( N \) such that
\[ |f(X + iY, Z)| \leq C |p(Y)|^N \quad \text{as} \quad \det Y \rightarrow \infty, \]
where \( p(Y) \) is a polynomial in \( Y = (y_{ij}) \).

It is natural to propose the following problems.

Problem A: Construct Maass-Jacobi forms.

Problem B: Find all the eigenfunctions of \( \Delta_{n,m} \).

We consider the simple case \( n = m = 1 \). A metric \( ds_{1,1}^2 \) on \( \mathbb{H}_1 \times \mathbb{C} \) given by
\[ ds_{1,1}^2 = \frac{y + v^2}{y^3} \left( dx^2 + dy^2 \right) + \frac{1}{y} \left( du^2 + dv^2 \right) \]
\[ - \frac{2v}{y^2} \left( dx du + dy dv \right) \]
is a \( G^J \)-invariant Kähler metric on \( \mathbb{H}_1 \times \mathbb{C} \). Its Laplacian \( \Delta_{1,1} \) is given by
\[ \Delta_{1,1} = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \]
\[ + \left( y + v^2 \right) \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) \]
\[ + 2yv \left( \frac{\partial^2}{\partial x \partial u} + \frac{\partial^2}{\partial y \partial v} \right). \]
We provide some examples of eigenfunctions of $\Delta_{1,1}$.

1. $h(x, y) = y^{\frac{s}{2}} K_{s-\frac{1}{2}}(2\pi|a|y) e^{2\pi i ax}$ ($s \in \mathbb{C}$, $a \neq 0$) with eigenvalue $s(s-1)$. Here

$$K_s(z) := \frac{1}{2} \int_0^\infty \exp \left\{ -\frac{t}{2}(t+1) \right\} t^{s-1} dt,$$

where $\text{Re } z > 0$.

2. $y^s$, $y^s x$, $y^s u$ ($s \in \mathbb{C}$) with eigenvalue $s(s-1)$.

3. $y^s v$, $y^s uv$, $y^s xv$ with eigenvalue $s(s+1)$.

4. $x$, $y$, $u$, $v$, $xv$, $uv$ with eigenvalue 0.

5. All Maass wave forms.

We fix two positive integers $m$ and $n$ throughout this section.

For an element $\Omega \in \mathbb{H}_n$, we set

$$L_\Omega := \mathbb{Z}^{(m,n)} + \mathbb{Z}^{(m,n)} \Omega$$

It follows from the positivity of $\text{Im } \Omega$ that the elements $f_{kl}$, $h_{kl}(\Omega)$ $(1 \leq k \leq m, 1 \leq l \leq n)$ of $L_\Omega$ are linearly independent over $\mathbb{R}$. Therefore $L_\Omega$ is a lattice in $\mathbb{C}^{(m,n)}$ and the set $\{ f_{kl}, h_{kl}(\Omega) \mid 1 \leq k \leq m, 1 \leq l \leq n \}$ forms an integral basis of $L_\Omega$. We see easily that if $\Omega$ is an element of $\mathbb{H}_n$, the period matrix $\Omega := (I_n, \Omega)$ satisfies the Riemann conditions (RC.1) and (RC.2):

**(RC.1)** $\Omega^t J_n \Omega = 0$;

**(RC.2)** $-\Omega^t J_n \Omega > 0$.

Thus the complex torus $A_\Omega := \mathbb{C}^{(m,n)}/L_\Omega$ is an abelian variety.

It might be interesting to investigate the spectral theory of the Laplacian $\Delta_{n,m}$ on a fundamental domain $\mathcal{F}_{n,m}$. But this work is very complicated and difficult at this moment. It may be that the first step is to develop the spectral theory of the Laplacian $\Delta_\Omega$ on the abelian variety $A_\Omega$. The second step will be to study the spectral theory of the Laplacian $\Delta_n$ (see (1.4)) on the moduli space $\Gamma_n \backslash \mathbb{H}_n$ of principally polarized abelian varieties of dimension $n$. The final step would be to combine the above steps and more works to develop the spectral theory of the Laplacian $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$. **Maass-Jacobi forms** play an important role in the spectral theory of $\Delta_{n,m}$ on $\mathcal{F}_{n,m}$. Here we deal only with the spectral theory $\Delta_\Omega$ on $L^2(A_\Omega)$.

We fix an element $\Omega = X + iY$ of $\mathbb{H}_n$ with $X = \text{Re } \Omega$ and $Y = \text{Im } \Omega$. For a pair $(A, B)$ with $A, B \in \mathbb{Z}^{(m,n)}$, we define the function $E_{\Omega,A,B} : \mathbb{C}^{(m,n)} \rightarrow \mathbb{C}$ by

$$E_{\Omega,A,B}(Z) = e^{2\pi i (\sigma(t^t A) + \sigma((B-AX)Y^{-1}t)V)},$$

where $Z = U + iV$ is a variable in $\mathbb{C}^{(m,n)}$ with real $U, V$.

**Lemma 4.1.** For any $A, B \in \mathbb{Z}^{(m,n)}$, the function $E_{\Omega,A,B}$ satisfies the following functional equation

$$E_{\Omega,A,B}(Z + \lambda \Omega + \mu) = E_{\Omega,A,B}(Z), \quad Z \in \mathbb{C}^{(m,n)}$$

for all $\lambda, \mu \in \mathbb{Z}^{(m,n)}$. Thus $E_{\Omega,A,B}$ can be regarded as a function on $A_\Omega$. 
Proof. We write $\Omega = X + iY$ with real $X, Y$. For any $\lambda, \mu \in \mathbb{Z}^{(m,n)}$, we have
\[
E_{\Omega; A,B}(Z + \lambda \Omega + \mu) = E_{\Omega; A,B}((U + \lambda X + \mu) + (V + \lambda Y))
\]
\[
= e^{2\pi i \{ \sigma((U + \lambda X + \mu)(B - AX)^{-1} (V + \lambda Y)) \}}
\]
\[
= e^{2\pi i \{ \sigma((U + \lambda X + \mu) + \sigma((B - AX)(Y^{-1})(V + B\lambda - AX\lambda)) \}}
\]
\[
= e^{2\pi i \{ \sigma(U) + \sigma((B - AX)(Y^{-1})(V)) \}}
\]
\[
= E_{\Omega; A,B}(Z).
\]
Here we used the fact that $tA\mu$ and $Bt\lambda$ are integral.

Lemma 4.2. The metric
\[
ds^2_{\Omega} = \sigma((\text{Im } \Omega)^{-1} t(dZ) d\overline{Z})
\]
is a Kähler metric on $A_{\Omega}$ invariant under the action (5.15) of $\Gamma^J = \text{Sp}(n, \mathbb{Z}) \ltimes H^{(m,n)}_{\mathbb{Z}}$ on $(\Omega, Z)$ with $\Omega$ fixed. Its Laplacian $\Delta_{\Omega}$ of $ds^2_{\Omega}$ is given by
\[
\Delta_{\Omega} = \sigma\left((\text{Im } \Omega) \frac{\partial}{\partial Z} t \left( \frac{\partial}{\partial Z} \right) \right).
\]
Proof. The proof can be found [20].

We let $L^2(A_{\Omega})$ be the space of all functions $f : A_{\Omega} \rightarrow \mathbb{C}$ such that
\[
||f||_{\Omega} := \int_{A_{\Omega}} |f(Z)|^2 dv_{\Omega},
\]
where $dv_{\Omega}$ is the volume element on $A_{\Omega}$ normalized so that $\int_{A_{\Omega}} dv_{\Omega} = 1$. The inner product $(\cdot, \cdot)_{\Omega}$ on the Hilbert space $L^2(A_{\Omega})$ is given by
\[
(f, g)_{\Omega} := \int_{A_{\Omega}} f(Z) g(Z) dv_{\Omega}, \quad f, g \in L^2(A_{\Omega}).
\]

Theorem 4.2. The set $\{ E_{\Omega; A,B} \mid A, B \in \mathbb{Z}^{(m,n)} \}$ is a complete orthonormal basis for $L^2(A_{\Omega})$. Moreover we have the following spectral decomposition of $\Delta_{\Omega}$:
\[
L^2(A_{\Omega}) = \oplus_{A,B \in \mathbb{Z}^{(m,n)}} \mathbb{C} \cdot E_{\Omega; A,B}.
\]
Proof. The complete proof can be found in [20].

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