Crowd-Anticrowd Theory of Multi-Agent Market Games

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Abstract

We present a dynamical theory of a multi-agent market game, the so-called Minority Game (MG), based on crowds and anticrowds. The time-averaged version of the dynamical equations provides a quantitatively accurate, yet intuitively simple, explanation for the variation of the standard deviation (‘volatility’) in MG-like games. We demonstrate this for the basic MG, and the MG with stochastic strategies. The time-dependent equations themselves reproduce the essential dynamics of the MG.

Agent-based games have great potential application in the study of fluctuations in financial markets. Challet and Zhang’s Minority Game (MG)\cite{1,2} offers possibly the simplest example and has been the subject of much research\cite{3}. The MG comprises an odd number of agents $N$ choosing repeatedly between option 0 (e.g. buy) and option 1 (e.g. sell). The winners are those in the minority group, e.g. sellers win if there is an excess of buyers. The outcome at each timestep represents the winning decision, 0 or 1. A common bit-string of the $m$ most recent outcomes is made available to the agents at each timestep\cite{3}. The agents randomly pick $s$ strategies at the beginning of the game, with repetitions allowed - each strategy is a bit-string of length $2^m$ which predicts the next outcome for each of the $2^m$ possible histories. Agents reward successful strategies with a (virtual) point. At each turn of the basic MG, the agent uses her most successful strategy, i.e. the one with the most virtual points. Here we develop a dynamical theory for MG-like games based on the formation of crowds and anticrowds.

The number of agents holding a particular combination of strategies can be written as a $D \times D \times \ldots$ ($s$ terms) dimensional matrix $\Omega$, where $D$ is the total number of available strategies. For $s = 2$, this is simply a $D \times D$ matrix where the entry $(i, j)$ represents the number of agents who picked strategy $i$ and then $j$. The strategy labels are given by the decimal representation of the strategy plus unity, for example the strategy 0101 for $m = 2$ has strategy label $5+1=6$. $\Omega$ is fixed at the beginning of the game (‘quenched disorder’) and can represent either the full strategy space or the reduced strategy space\cite{1}, depending on the choice of $D$. $\Sigma$ is another time-independent matrix, containing all the strategies in the required space in their binary form: $\Sigma_{r,h}+1$ describes the prediction of strategy $r$ given the history $h$ (where $h$ is the decimal corresponding to the $m$-bit binary history string).

We introduce a vector $\underline{n}(t)$: this contains the number of agents using each strategy at time $t$, in order of increasing strategy label. The vector $S(t)$ contains the virtual score for each strategy at time $t$ in order of increasing strategy label. The vector $R(t)$ lists the strategy label in order of best-to-worst virtual points score at time $t$; if any strategies are tied in points then the strategy with the lower-value label is listed first. The vector $\rho(t)$ shows the rank of the strategy listed in order of increasing strategy label at time $t$. Hence $R(t)$ and $\rho(t)$ can be found from $S(t)$ using simple sort operations. The vector $\underline{n}(t)$ is the sum of

two terms
\[ \mathbf{v}(t) = \mathbf{v}^0(t) + \mathbf{v}^d(t). \] (1)

Here \( \mathbf{v}^0(t) \) gives the number of agents using each strategy; however where any strategies are tied in virtual score, \( \mathbf{v}^0(t) \) assumes that the agent will use the strategy with the lower-value label by virtue of the definition of \( \mathbf{R}(t) \). The term \( \mathbf{v}^d(t) \) accounts for tied strategies, and hence provides a correction to \( \mathbf{v}^0(t) \).

\( \mathbf{v}^0(t) \) is given by
\[ \mathbf{v}^0(t)_r = \sum_{i=\rho(t)}^{2^{m+1}} [\hat{f}(\Omega)]_{r,R(t)}, \] (2)

where \([\hat{f}(\Omega)]_{\alpha,\beta} = \Omega_{\alpha,\beta} + \Omega_{\beta,\alpha} - \delta_{\alpha,\beta} \Omega_{\alpha,\beta} \). The vector \( \mathbf{v}^d(t)_r \) is given by
\[ \mathbf{v}^d(t)_r = \sum_{r' \neq r} \delta_{s(t),r',s(t),r'} Sgn(r' - r) Bin_{r',r} \] (3)

where: \( Bin_{r',r} \sim B([\hat{f}(\Omega)]_{r',r}, \frac{1}{2}) \) and \( Bin_{r',r} = Bin_{r,r'} \). The standard notation \( Bin \) represents the binary distribution. Note the condition \( Bin_{r',r} = Bin_{r,r'} \) which guarantees conservation of agents, as in the basic MG. The outcome parameter \( \Upsilon(t) \) denotes which choice, 0 or 1, is the minority (and hence winning) decision at time \( t \):
\[ \Upsilon(t) = \mathcal{H}[-[\mathbf{v}(t)^T \Sigma']_{h(t)+1}] \] (4)

where \( \Sigma' = 2\Sigma - 1 \). The history, i.e. bit-string of the \( m \) most recent outcomes, and the virtual scores of the strategies are updated as follows:
\[ h(t+1) = 2[h(t) - 2^{m-1} \mathcal{H}[h(t) - 2^{m-1}]] + \Upsilon(t) \] (5)

where \( \mathcal{H} \) is the Heaviside function, and
\[ \mathbf{S}(t+1) = \mathbf{S}(t) + \Sigma'_{h(t)+1} [2\Upsilon(t) - 1]. \] (6)

Equations (1-6) are a set of time-dependent equations which reproduce the essential dynamics of the basic MG, and can be easily extended to describe MG generalizations. Iterating these equations is equivalent to running a numerical simulation, but is far easier and can even be done analytically. A slight difference may arise as a result of the method chosen for tie-breaking between strategies with equal virtual points: a numerical program will typically break this tie using a separate coin-toss for each agent, whereas the dynamical equations group together those agents using the same pair of strategies and then assign a proportion of that group to a particular strategy using a coin-toss. This difference is typically unimportant.

As an example of the implementation of these equations, consider a time \( t_e \) during the following game: \( m = 2 \), \( s = 2 \) and \( N = 101 \) in the reduced strategy space, with a strategy configuration \( \Omega \) and strategy score given as follows:
\[ \Omega = \left( \begin{array}{cccccc} 2 & 3 & 2 & 3 & 5 & 3 & 1 & 1 \\ 1 & 3 & 2 & 2 & 1 & 2 & 1 & 2 \\ 1 & 0 & 2 & 0 & 1 & 3 & 1 & 3 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 3 \\ 4 & 5 & 1 & 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 & 2 & 0 & 4 \\ 1 & 2 & 1 & 2 & 0 & 0 & 2 & 4 \\ 1 & 2 & 2 & 1 & 1 & 1 & 1 & 2 \end{array} \right) \]

\[ \mathbf{S}(t_e) = \left( \begin{array}{c} 3 \\ -1 \\ -3 \\ 1 \\ 1 \\ 1 \\ -3 \end{array} \right), \text{ with } \Sigma = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right). \]

Using these values for \( \Omega \) and \( \mathbf{S}(t_e) \) we can obtain values for \( \mathbf{v}(t) \) and ultimately \( \mathbf{S}(t_e+1) \). \( \Omega \) and \( \mathbf{S}(t_e) \) imply that
\[ \mathbf{v}^0(t_e) = \left( \begin{array}{c} 31 \\ 15 \\ 7 \\ 13 \\ 5 \\ 15 \\ 13 \\ 2 \end{array} \right), \text{ and } \mathbf{v}^d(t_e) = \left( \begin{array}{c} -3 \\ -2 \\ -5 \\ 0 \\ 2 \\ 3 \\ 0 \\ 5 \end{array} \right) \]

with probability \( \frac{105}{65536} \), yielding \( \mathbf{v}(t_e) \) when summed. (When two strategies are tied, agents holding these strategies each flip a coin to decide which strategy to
use. The separate probabilities for all tied strategies, when multiplied together, yield the probability of the current \( \mathbf{u}^{(t)} \) being chosen.

Suppose \( h(t) = 2 \), i.e. the last two minority groups were '1' then '0'. Hence \( \Upsilon(t) = 0 \), \( h(t+1) = 0 \) and consequently

\[
\mathbf{s}(t+1) = \begin{pmatrix} 4 \\ -2 \\ -2 \\ 0 \\ 0 \\ 2 \\ 2 \\ -4 \end{pmatrix}.
\]

An expression for the time-averaged quantity called the 'volatility' (standard deviation of the number of agents choosing one particular group) can be easily found using the above formalism:

\[
\sigma_{MG} = \frac{\left[ \sum_{t=t_1}^{t_2} \left( \varepsilon(t) - \overline{\varepsilon} \right)^2 \right]^{\frac{1}{2}}}{t_2 - t_1}\]

(7)

where \( \varepsilon(t) = \left[ \mathbf{u}(t) \right]_{h(t)+1} \) and \( \overline{\varepsilon} \) is the time-average of \( \varepsilon(t) \) from time \( t_1 \) to \( t_2 \). Here \( t_1 \) and \( t_2 \) denote the time window over which the volatility is calculated. In the reduced strategy space \( \Omega \) a similar quantity to this standard deviation can also be written down using our previously introduced (time-averaged) crowd-anticrowd framework [3]:

\[
\sigma_{CA} = \frac{\sum_{t=t_1}^{t_2} \left[ \frac{1}{4} \sum_{r=1}^{2^m} \left[ \mathbf{u}(t)_r - \mathbf{u}(t)_{2^m+1-r} \right]^2 \right]^{\frac{1}{2}}}{t_2 - t_1}.
\]

(8)

For a given run of the game \( \sigma_{MG} \neq \sigma_{CA} \), however these quantities become quantitatively the same (within the limits of sample size) when averaged over initial configurations of strategies [3]. \( \sigma_{CA} \) mirrors the semi-analytic approach introduced to motivate the time-independent crowd-anticrowd theory of Ref. [3] (see Fig. 1 of Ref. [3]). Indeed, the dynamical equations can be linked more formally with our previous time-averaged approach [3]. Consider, for example, the situation where no two strategies are tied in virtual points and there are an equal number of agents having each possible pairing of strategies (low \( m \) limit and reduced strategy space), i.e. all elements in \( \Omega \) are equal and non-zero. It is then easy to show that \( n^0_\rho(t) \), reduces to \( n^0_\rho = \frac{N}{t^{2m+1}} \left[ 1 + 2(2^m+1 - \rho(t)) \right] \); this is precisely the vector of the quantity \( n_\rho \) introduced in Ref. [3] now written in order of increasing strategy label. If we allow for tied strategies, \( \mathbf{u}^0(t) \) will be non-zero thus reducing the size of large crowds and increasing the size of the smaller crowds (and hence anticrowds), thereby leading to a smaller standard deviation.

We now turn to a comparison between the standard deviation or ‘volatility’ \( \sigma \) obtained from numerical simulations and our (time-averaged) crowd-anticrowd theory. We start with the basic MG. Figure 1 shows the spread of numerical values for different numerical runs (open circles), the full crowd-anticrowd theoretical calculation (large solid circles) and various limiting analytic curves (solid lines) for which closed-form expressions were given in Ref. [3]. Fuller details are provided in Ref. [3]. The time-averaged dynamics can be described using a quantity \( P(r' = \bar{r}) \) which represents the probability that any strategy \( r' \) is the anti-correlated partner of strategy \( r \) [3]. To produce the limiting analytic curves in Fig. 1, \( P(r' = \bar{r}) \) is taken to be either a delta-function or a flat distribution. The full theory takes the relevant form of \( P(r' = \bar{r}) \) from the game. The agreement is very good, confirming that our theory captures the essential physics.

In a variant of the basic MG, agents pick which strategy to use stochastically at each timestep. Focussing on \( s = 2 \), numerical simulations [3] found that the larger-than-random \( \sigma \) in the ‘crowded’ regime (i.e. small \( m \)) becomes smaller-than-random when the strategy-picking rule is made increasingly stochastic. Our crowd-anticrowd theory provides a quantitative explanation of this effect. Let \( \theta \) be the probability that the agent uses the worst of her \( s = 2 \) strategies. Figure 2 shows a comparison between numerical simulation (open circles) and analytic expressions (monotonically-decreasing solid lines) obtained using our crowd-anticrowd theory (full details are given in Ref. [3]). These analytic expressions vary in their choice of \( P(r' = \bar{r}) \): the upper line \( \sigma_{\text{delta}} \) in
Fig. 2 assumes a delta function while the lower line $\sigma_{\text{flat}}$ assumes a flat distribution. The theory agrees well in the range $\theta = 0 \rightarrow 0.35$ and provides a quantitative, yet physically intuitive, explanation for the previously unexplained transition in $\sigma$ from larger-than-random to smaller-than-random as $\theta$ increases.

Above $\theta = 0.35$, the numerical data tend to flatten off while the analytic expressions predict a decrease in $\sigma$ as $\theta \rightarrow 0.5$. This is because the analytic theory averages out the fluctuations in strategy-use at each time-step. In Ref. [6] we showed how to correct this shortcoming of the analytic theory. Consider $\theta = 0.5$; Fig. 2 inset (a) shows the measured numerical distribution in $\sigma$ for $\theta = 0.5$, while inset (b) shows the result from the semi-analytic procedure introduced in Ref. [6]. The two distributions are in good agreement. Note that the non-zero average (4.7 for $N = 101$, $m = 2$ and $s = 2$) for each distribution lies below the random coin-toss limit $\sqrt{N}/2$. It is also possible to perform a fully analytic calculation of the average $\sigma_{\theta}$ in the $\theta \rightarrow 0.5$ limit [3]; this value (which is also 4.7 for $N = 101$, $m = 2$ and $s = 2$) is shown in Fig. 2.

In summary, we have demonstrated that the crowd-anticrowd approach can be applied to explain many aspects of MG games, yielding both time-averaged and time-dependent theories (see also Ref. [7]). Our efforts to develop such simplified market games in order to describe real-world financial markets are described elsewhere [8].

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FIG. 1. Theoretical crowd-anticrowd calculation (solid circles) and numerical simulations (open circles) for the standard deviation $\sigma$ in basic MG with $s = 2$ and $N = 101$. 16 numerical runs are shown for each $m$. Solid lines correspond to analytic expressions representing special cases within the time-averaged crowd-anticrowd theory of Ref. \cite{4}.

FIG. 2. Theoretical crowd-anticrowd calculation and numerical simulations (circles) for $\sigma$ vs. the probability parameter $\theta$ in the stochastic MG. Here $N = 101, m = 2$ and $s = 2$. Monotonically decreasing solid lines correspond to analytic expressions $\sigma_{\text{delta}}$ and $\sigma_{\text{flat}}$ (see text). Dashed line shows random coin-toss value. Solid arrow indicates theoretical value $\sigma_{\theta \rightarrow 0.5} = 4.7$ for $\theta \rightarrow 0.5$. Inset shows distribution of $\sigma$ values at $\theta = 0.5$ for several thousand randomly-chosen initial strategy configurations: (a) numerical simulation, (b) semi-analytic theory of Ref. \cite{6}. 

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Fig. 2