A quantum-field model of the conformally flat space is formulated using a standard field-theoretical technique, a probability interpretation and a way to establish the classical limit. The starting point is the following: after conformal transformation of the Einstein–Hilbert action, the conformal factor represents a scalar field with the negative kinetic term and the self-interaction inspired by the cosmological constant. (It has been found that quanta of such action have a negative value as a sequence of the negative energy.) The metric energy-momentum tensor of this scalar field is proportional to the Einstein tensor for the initial metric. Therefore, a vacuum state of the field is treated as a classical space. In such vacuum the zero mode is a scale factor of the flat Friedmann Universe. It is shown that conformal factor may be viewed as an inflaton field, and its small non-homogeneities represent gauge invariant scalar metric perturbations.

PACS number: 98.80.Hw Quantum Cosmology

I. INTRODUCTION

The metric of the conformally flat space-time only contains one essential variable – a conformal factor which is a scale factor of the expanding Universe. The negative energy of this variable leads to the difficulties in the quantum cosmology based on the Wheeler–DeWitt equation (for example, the wave function of the Universe is non-normalizable, a probability current density is not positive definite). Different ways are used to overcome these obstacles. Matter or scale factor sometimes are treated as non-dynamical. Neglecting an interaction between matter and scale factor allows one to factorize the wave function and to include a minus sign into the time or the energy in the stationary Schrödinger equation for scale factor. For example, Narlikar and Padmanabhan introduce stationary states of the quantum geometry as a solution of such an equation to prevent classical singularity. However, in this approach the conformal factor is treated as a physical variable while its energy is considered as an unphysical one. Hence it seems rather strange that this energy may compensate the energy of the created matter. Another interesting result of the quantization of the conformal factor via Feynman path integral derived by these authors is the growing of the quantum fluctuations near the singularity. But the Euclidian path integral for gravity is divergent because the Euclidian gravitational action is not positive-definite. Gravitational waves carry positive energy and gravitational potential energy is negative. Conformal rotation (using the complex conformal factor) makes an Euclidian path integral convergent by changing the sign of the action. But this method is applicable only if the conformal factor is an unphysical variable (as in asymptotically flat space-time).

In Friedmann–Robertson–Walker Universe, the conformal factor is a single degree of freedom. It must be a physical variable because the expansion of the Universe is an observational fact.

The talk given at the V International Conference on Gravitation and Astrophysics of Asian–Pacific Countries, October 1–7 2001, Moscow, Russia.

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1 The reduced phase space quantization allows one to ignore the negative sign if the scale factor remains a single degree of freedom. A special variable must be presented in extended phase space to become a time-like parameter after reduction. If such a variable was the scale factor, a possibility of a study on the quantum properties of the early Universe (for example, discussed in 2–3) would disappear.

2 A related problem is that the sign of the Euclidian action for scale factor must be changed with respect to the matter case to get a physically accepted probability for the birth of the Universe. 

3 This condition also constrains validity of a conclusion about positivity of the gravitational energy. Moreover, the mentioned conclusion concerns the energy of gravitational waves which do not describe the expansion of the Universe.

4 Theoretical argumentation why the conformal factor is a physical degree of freedom is done in 3–4.
To conclude the brief review of theoretical works dealing with quantization of the conformal factor, one can say that there is no completely satisfying quantum picture of the expansion of the Universe. The main questions are gravitational degrees of freedom and gravitational energy.

Experimental data point out an accelerating expansion of the Universe, and hence, the cosmological constant may carry a significant contribution into the energy density \( \Lambda \) [13]. It is a difficult question to the quantum field theory why the expected vacuum energy density is much greater than that corresponding to the cosmological constant [16]. Inflation theories exploit a state equation related to the cosmological constant but they do not explain the mentioned contradiction [17,18]. Meanwhile, the large scale structure of the Universe and the cosmic background radiation anisotropy are treated as consequences of quantum fluctuation of inflaton field [19] or quantum metric fluctuation [20]. A modern tool for studying the inflation is a non-equilibrium phase transition dynamics [21] in the field theoretical Schrödinger picture [22]

So the quantum model of the conformally flat space-time looks interesting from a theoretical viewpoint and may link quantum cosmology to the observational data.

In this paper, attempt is made to quantize the conformally flat space-time in the Schrödinger picture by a standard field theoretical technique. Two main circumstances are taken into account. First, the geometrical and field properties of the space-time are split in the Einstein – Hilbert action via a conformal transformation. The equation of motion for the gravitational field (conformal factor) is derived by variation of the action with respect to a field variable and represents a Klein – Gordon equation with additional Penrous – Chernicov – Tagirov term [23]. The metric energy-momentum tensor [24] of the field is derived by variation of the action with respect to new metric (Minkowski metric for conformally flat space-time discussed here) and is proportional to Einstein equations for the initial metric. Thus a vacuum state of the field (where the energy-momentum tensor average is zero) is considered as a classical space-time. Second, it is taken into account that a system with negative energy needs to be quantized using Planck constant with a negative sign according to the Bohr – Sommerfeld rule and to analogy between wave mechanic and geometric optics.

The paper is organized as follows. Section II represents the conformally flat space-time with the cosmological constant as a self-interacting scalar field in Minkowski space-time. In Section III quantization of the oscillating modes is fulfilled while zero mode is considered as a classical quantity. Section IV gives a relation of the conformal factor to an inflaton field and to scalar metric perturbations. The Summary reviews the construction of the model, discusses main results, and compares with some related works. Quantization of the system with the negative energy (inverted harmonic oscillator) is prepared in Appendix B. Some used formulae for Bessel functions are given in Appendix A.

II. REPRESENTATION OF THE CONFORMALLY FLAT SPACE-TIME BY A SCALAR FIELD

To describe the conformally flat Universe by a scalar field in Minkowski space-time, we use conformally flat coordinates. Let us suppose that there is a general coordinate transformation

\[
x = x(\hat{x}), \quad g_{ij}(x) = \frac{\partial \hat{x}^i}{\partial x^l} \frac{\partial \hat{x}^m}{\partial x^j} \hat{g}_{lm}(\hat{x}), \quad i, j = 1, 2, 3, 4
\]

(1)

\[
ds^2 = g_{ij}(x)dx^i dx^j = \hat{g}_{ij}(\hat{x})d\hat{x}^i d\hat{x}^j,
\]

(2)

after which the new metric has a common multiplier (conformal factor):

\[
\hat{g}_{ij}(\hat{x}) = a^2(\hat{x})g'_{ij}(\hat{x}),
\]

(3)

\[
ds^2 = a^2(\hat{x})(ds')^2, \quad (ds')^2 = g'_{ij}(\hat{x})d\hat{x}^i d\hat{x}^j.
\]

(4)

The equations for metric \( \hat{g}_{ij}(\hat{x}) \) following from Einstein – Hilbert action [4]

\[5\] Scalar metric perturbations play an important role in theoretical explanation of the CMB anisotropy due to Sachs – Wolfe effect [2]. As in quantum field theory matter is described by 4-covariant quantities, quantum fluctuations of the conformal factor (4-scalar) may be interesting as a kind of metric perturbations.

\[6\] Notations correspond to Landau, Lifshitz book "Classical Theory of Fields" [25]. Geometric units system \( 16\pi G = c = 1 \) (\( G \) is the Newtonian constant and \( c \) is speed of light) is used. Conformal factor \( a(x) \) has dimension of the length, and cosmological constant \( \Lambda \) has a dimension of the length in to the minus square. It is convenient to have the Planck constant in formulae apparently when quantization is discussed. In usual for quantum gravity system of units \( 16\pi G = c = h = 1 \) all the quantities, whose dimension expresses in terms of mass, length and time units, become dimensionless.
can be written as equations for conformal factor \( a(\tilde{x}) \) and metric \( g'_{ij}(\tilde{x}) \) 

\[
\tilde{G}_{ij} = \tilde{R}_{ij}(\tilde{x}) - \frac{1}{2} \tilde{R}(\tilde{x}) \tilde{g}_{ij}(\tilde{x}) - \Lambda \tilde{g}_{ij}(\tilde{x}) \\
= R'_{ij}(\tilde{x}) - \frac{1}{2} R'(\tilde{x}) g'_{ij}(\tilde{x}) \\
+ a^{-2}(\tilde{x})[-\nabla_i \nabla_j a^2(\tilde{x}) + 6 \partial_i a(\tilde{x}) \partial_j a(\tilde{x}) \\
+ g'_{ij}(\tilde{x})(\Box a^2(\tilde{x}) - 3 R^m a(\tilde{x}) \partial_m a(\tilde{x}) - \Lambda a^4(\tilde{x}))].
\]

From the other hand, the Einstein – Hilbert action (5) can be rewritten by using the conformal transformation formulae corresponding to factorization of the metric (3)

\[
\tilde{R}(\tilde{x}) = a^{-2} \left( R'(\tilde{x}) - 6 a^{-1}(\tilde{x}) \Box a(\tilde{x}) \right), \quad \Box a(\tilde{x}) = \frac{1}{\sqrt{-g'}} \partial_i \left( \sqrt{-g'} (g')^{ij} \partial_j a \right)
\]

\[
\sqrt{-g(\tilde{x})} = a^4(\tilde{x}) \sqrt{-g'(\tilde{x})}.
\]

As a result, an action for a scalar field \( a(\tilde{x}) \) with negative kinetic term in metric \( g'_{ij}(\tilde{x}) \) appears

\[
S[g'(\tilde{x}), a(\tilde{x})] = -\int d^4 \tilde{x} \sqrt{-g'} \left( \frac{1}{2} (g')^{ij} \partial_i a \partial_j a + \frac{1}{12} a^2 R' + \frac{\lambda}{4!} a^4 \right),
\]

where \( \lambda = 4 \Lambda \). Total divergence and numerical factor are neglected. Variation of this action with respect to metric \( g'_{ij} \) gives an energy-momentum tensor of the conformal scalar field \( a(\tilde{x}) \)

\[
T^{(\text{metr})}_{ij} = T^{(\text{can})}_{ij} - \frac{1}{6} \left( R'_{ij} + g'_{ij} \Box - \nabla_i \nabla_j \right) a^2,
\]

where the canonical energy-momentum tensor has the form

\[
T^{(\text{can})}_{ij} = -\partial_i a \partial_j a + g'_{ij} \left( \frac{1}{2} \partial^m a \partial_m a + \frac{1}{12} a^2 R' + \frac{\lambda}{4!} a^4 \right).
\]

The metric energy-momentum tensor (10) is proportional to the Einstein equations (6)

\[
T^{(\text{metr})}_{ij} = -\frac{a^2}{6} \tilde{G}_{ij}.
\]

This fact can be directly deduced if we mention that both quantities are derived by variation of the Einstein – Hilbert action with respect to the metrics which are proportional each other as it follows from (3) 7. The metric energy-momentum tensor trace is equal to zero on the equation of motion for \( a(\tilde{x}) \)

\[
\frac{1}{6} a^3 \tilde{G} = - a^{-1} T = \Box a - \frac{1}{6} a R' - \frac{\lambda}{6} a^3.
\]

So this equation of motion expresses a condition for conformal invariance. The interaction that arises due to the cosmological constant is a unique possible conformally invariant one. We see from (12) that the metric energy-momentum tensor (11) must be equal to zero to be consistent with Einstein equations (6). Let us suppose that there is a state in quantum field theory of the conformal factor \( a(x) \) with a vanishing averaged value for the energy-momentum tensor

7 If one considers the conformal factor as a new variable, an additional condition must be added (for example, fixing the scalar curvature or determinant of the metric) to preserve the number of independent variables.

8 The author is indebted to Professor N.A. Chernikov for this remark.
\langle 0 | T_{ij} | 0 \rangle = 0, \quad \langle 0 | 0 \rangle = 1. \tag{14}

Such a state naturally is a vacuum state and corresponds to the classical theory.

In the case of the conformally flat space-time we can consider the transformation $g \leftrightarrow \tilde{x}$ as a transformation to the conformally flat coordinates $\tilde{x}$. Hence, the metric $\tilde{g}'_{ij}(\tilde{x})$ arising due to factorization of the $\tilde{g}_{ij}(\tilde{x})$ is Minkowski one:

$$g'_{ij}(\tilde{x}) = \tilde{g}_{ij} = (+, -, -, -), \tag{15}$$

and the curvature depending terms disappear from the action (10) and from the energy-momentum tensor (10).

The energy-momentum tensor (17) differs from the canonical one by second derivative terms. But the corresponding Hamiltonian may be derived in usual form after removing a total divergence.

The solution to the Einstein equations (6) is the de Sitter space-time with the curvature scalar $R = -\lambda$. In coordinates $\tilde{x}$ it looks like the spatially flat Friedmann – Robertson – Walker Universe with a cosmological constant. Isometries of the conformally flat space-time preserve the factorization of the metric $\tilde{g}_{ij}(\tilde{x})$ and action (10). All conformally flat spaces have the same conformal group symmetry [30]. In particular, de Sitter group generators undergo to Poincaré group ones in the limit of the large curvature radius [31] (small cosmological constant). A correct description of the classical Universe will be guaranteed by a conserved energy-momentum tensor providing that the field $a(x)$ has a non-zero vacuum averaged value $\langle a(\eta) \rangle$ (it will be shown that $\eta$ corresponds to conformal time in the classical limit) playing the role of a scale factor for the classical metric

$$a(\eta, x) = a(\eta) + \delta a(\eta, x), \tag{18}$$

where

$$a(\eta) \gg |\delta a(\eta, x)|. \tag{19}$$

Vacuum fluctuations will cause increase of $a(\eta)$ due to instability of the system with the action (16).

The result of the analysis is the following. After factorization of the conformal factor, the Einstein – Hilbert action looks like a conformal scalar field action with the negative kinetic term. The Einstein equations for the initial metric (4) are proportional to the metric energy-momentum tensor of the field (10). Hence, the classical conformally flat space-time can be represented as a vacuum state of the unstable scalar field in Minkowski space-time. The energy-momentum tensor vacuum average must be zero and the vacuum field average is a classical scale factor.

III. FIELD-THEORETICAL SCHRÖDINGER PICTURE

The most suitable approach to quantization of the model (16), (17) is the Schrödinger picture which has advantages in time-dependent problems [23]. The aim of the quantization is to find a ground state of the system and to construct creation and annihilation operators. This is done in the way proposed Guth and Pi [32] where the inflaton field with self-interaction $\lambda \phi^4$ was quantized (in the de Sitter metric) to study the “slow-rollover” phase transition in the new inflation model. Here the problem has been formulated in Minkowski metric but due to instability of the system, the field zero mode dynamics leads to an effect appearing in the non-stationary metric. The field $a(x)$ is decomposed into the Fourier series.

9 The explicit form of such transformation for Robertson – Walker metric is done in the textbook by Lightman at al. [28], problem 19.8.

10 The conformal Killing vectors of the Robertson – Walker metric in conformally flat coordinates are presented in [29].
\[
a(\eta, x) = \sqrt{2}a_o(\eta) + \sum_k [a^+_{k}(\eta) \cos \vec{k} \cdot \vec{x} + a^-_{k}(\eta) \sin \vec{k} \cdot \vec{x}],
\]

in a cub with a side \(b = 2\pi\) using real functions with the following properties
\[
a^+_{-k}(\eta) \equiv a^+_{k}(\eta), \quad a^-_{-k}(\eta) \equiv -a^-_{k}(\eta).
\]
The approximation
\[
a_o(\eta) \gg a^\pm_{k}(\eta)
\]
corresponds to the classical limit condition (19). Further simplification is to neglect the interaction between the modes \(a^\pm_{k}(\eta)\) in virtue of a small value of \(\lambda\). Then the action (16) takes the form
\[
S = \int d\eta \left\{ -\dot{a}_o^2(\eta) - \frac{\lambda}{3!} a_o^4(\eta) + \sum_k \left[ -\dot{a}_k^2(\eta) + (k^2 - \lambda a_o^2(\eta)) a_k^2(\eta) \right] \right\},
\]
where only one sign modes are presented, their notation is simplified and a space integration is prepared.

**A. Solution of a classical problem**

The action (23) leads to equations of motions
\[
\begin{aligned}
\ddot{a}_o(\eta) - \frac{4}{3!} a_o^3(\eta) - \lambda a_o(\eta) \sum_k a_k^2(\eta) &= 0, \\
\ddot{a}_k(\eta) + (k^2 - \lambda a_o^2(\eta)) a_k(\eta) &= 0,
\end{aligned}
\]
which self-consistently describe independent oscillators \(a_k(\eta)\) in external field \(a_o(\eta)\). Not self-consistent analytically solvable problem does not take into account the \(a_k(\eta)\)-modes influence on zero mode
\[
\begin{aligned}
\ddot{a}_o(\eta) - \frac{4}{3!} a_o^3(\eta) &= 0, \\
\ddot{a}_k(\eta) + (k^2 - \lambda a_o^2(\eta)) a_k(\eta) &= 0.
\end{aligned}
\]
The first integral for a zero mode equation
\[
\dot{a}_o^2(\eta) - \frac{\lambda}{6} a_o^4(\eta) = 0.
\]
must be equal to zero according to the correspondence condition (12). The second integration constant can be fixed by assumption that the origin of the time \(\eta\) (the conformal time for the Friedmann – Robertson – Walker interval (26) coincides with the origin of the proper time in synchronous coordinate system
\[
dt_{pr} = \sqrt{2}a_o(\eta)d\eta.
\]
So \(a_o(\eta)\) is a scale factor of the flat Friedmann Universe with the cosmological constant
\[
a_o = \frac{\sqrt{6/\lambda}}{1 - \eta} = \sqrt{\frac{6}{\lambda}} \exp \left\{ \sqrt{\frac{\lambda}{12}} t_{pr} \right\}
\]
considered in the intervals
\[
0 \leq \eta < 1, \quad 0 \leq t_{pr} < \infty, \quad \sqrt{\frac{6}{\lambda}} \leq a_o < \infty.
\]
The equation for oscillating modes $a_k(\eta)$ can be reduced to the equation for Bessel functions:

$$
\frac{d^2 \xi(\tau_k)}{d\tau_k^2} + \frac{1}{\tau_k} \frac{d \xi(\tau_k)}{d\tau_k} + \left(1 - \frac{\nu^2}{\tau_k^2}\right) \xi(\tau_k) = 0
$$

with the index $\nu = 5/2$ by change of variables

$$
\tau_k = k(1 - \eta), \quad a_k(\tau_k) = \xi(\tau_k) \sqrt{\tau_k}.
$$

The general solution for $a_k(\eta)$ is a linear combination

$$
a_k(\eta) = \frac{1}{\sqrt{2}} \left[ A_k \psi_k(\eta) + A_k^\dagger \psi_k^*(\eta) \right],
$$

where $A_k, A_k^\dagger$ are coefficients, and $\psi_k(\eta), \psi_k^*(\eta)$ are expressed in terms of Hankel functions

$$
\psi_k(\eta) = \frac{1}{2} \sqrt{\pi \bar{h}} (1 - \eta) H^{(1)}_{5/2}(k(1 - \eta)), \quad \psi_k^*(\eta) = \frac{1}{2} \sqrt{\pi \bar{h}} (1 - \eta) H^{(2)}_{5/2}(k(1 - \eta)).
$$

Functions $\psi_k(\eta), \psi_k^*(\eta)$ satisfy the orthogonality condition

$$
\psi_k(\eta) \frac{\partial}{\partial \eta} \psi_k^*(\eta) - \psi_k^*(\eta) \frac{\partial}{\partial \eta} \psi_k(\eta) = i\hbar.
$$

Using this relation, the coefficients $A_k, A_k^\dagger$ can be written as follows:

$$
A_k = \frac{\sqrt{2}}{i\hbar} \left( a_k(\eta) \frac{\partial}{\partial \eta} \psi_k^*(\eta) - \psi_k^*(\eta) \frac{\partial}{\partial \eta} a_k(\eta) \right), \quad A_k^\dagger = -\frac{\sqrt{2}}{i\hbar} \left( a_k(\eta) \frac{\partial}{\partial \eta} \psi_k(\eta) - \psi_k(\eta) \frac{\partial}{\partial \eta} a_k(\eta) \right).
$$

The equality

$$
A_k^* = A_k^\dagger
$$

holds as the field $a(\eta, x)$ has a real value.

**B. Quantization of oscillatory modes**

As zero mode $a_o(\eta)$ is a classical quantity in the accepted approximation, only oscillating modes $a_k(\eta)$ must be quantized as independent variables. Due to the negative kinetic energy, the conjugated momentum has an opposite sign with respect to time derivative of the $a_k(\eta)$

$$
\pi_k(\eta) = \frac{\partial L}{\partial \dot{a}_k(\eta)} = -2\dot{a}_k(\eta).
$$

The momentum operator is as follows (see Appendix A)

$$
\pi_k \rightarrow i\hbar \frac{\delta}{\delta a_k}.
$$

The commutator also differs from that in a usual case by sign

$$
[a_k, \pi_{k'}] = -i\hbar \delta_{kk'}.
$$

The wave functional of the system is a product of the mode functionals in the case of the absence of interactions

$$
\Psi = \prod_k \Psi_k(\eta), \quad \Psi_k(t) \equiv \Psi_k[a_k, \eta].
$$

11 Some properties of the Bessel functions are presented in Appendix B [35].
The Schrödinger equation for $\Psi_k(\eta)$

$$-i\hbar \frac{\partial}{\partial \eta} \Psi_k(\eta) = H_k \Psi_k(\eta)$$  (41)

with the Hamiltonian

$$H_k = -\frac{\pi^2}{4} - (k^2 - \lambda a_0^2(\eta))a_k^2,$$  (42)

where $a_0(\eta)$ is a known function (28), can be solved by usual ansatz (43)

$$\Psi_k(\eta) = C_k(\eta) \exp \left\{ -2\Omega_k(\eta) a_k^2 \right\},$$  (43)

where a covariance $\Omega_k(\eta)$ is introduced

$$\Omega_k(\eta) = -\frac{i}{2\hbar} \dot{\varphi}_k^*(\eta) \varphi_k(\eta).$$  (44)

The equations for $C_k(\eta)$ and $\varphi_k(\eta)$ follow from the Schrödinger equation (41)

$$\begin{cases}
\dot{C}_k(\eta) - \frac{1}{2} \varphi_k^2(\eta) 
\end{cases}$$

The solution for $\varphi_k(\eta)$ is analogous to this for $\psi_k(\eta)$ (33)

$$\varphi_k(\eta) = \frac{1}{2} \sqrt{\pi \hbar (1 - \eta)} \left( c_1 H_\frac{1}{2}(k(1 - \eta)) + c_2 H_\frac{3}{2}(k(1 - \eta)) \right),$$  (46)

with the same normalization (34) suggesting a condition

$$|c_1|^2 - |c_2|^2 = 1.$$  (47)

The coefficient $C_k(\eta)$ in the wave functional (33) can be expressed via function $\varphi_k(\eta)$

$$\left| C_k(\eta) \right| = \left( \pi |\varphi_k|^2 \right)^{-1/4}$$  (48)

according to (45) and to normalization condition

$$\int \mathcal{D}\Psi_k(t) \Psi_k(t) = 1.$$  (49)

The real and imaginary parts of the covariance $\Omega_k(\eta)$ are as follows (49)

$$\text{Re} \Omega_k(\eta) = \frac{1}{4|\varphi_k|^2}, \quad \text{Im} \Omega_k(\eta) = -\frac{1}{4\hbar |\varphi_k|^2} \frac{\partial}{\partial \eta} |\varphi_k|^2.$$  (50)

Hence, the functional $\Psi_k(\eta)$ (33) is determined. It corresponds to the Bunch–Davies vacuum (33) of the scalar field in the de Sitter space which is a unique, completely de Sitter invariant vacuum state (32). As it was stressed in (34), owing to conformal invariance, the situation is as in flat space-time, where a unique Poincaré invariant vacuum exists.

We demand $A_k$ (33) to be an annihilation operator

$$A_k \Psi_k(\eta) = 0.$$  (51)

---

12 It must be stressed that the real part would be negative in case of usual Schrödinger equation instead of (41) and, hence, the solution (43) would be non-normalizable.

13 The author thanks Professor R. Jackiw for this reference.
This equality is satisfied by choosing \( c_1 = 1 \) in solution \( \varphi_k(\eta) \) \(^{[14]}\), hence we have \( \varphi_k(\eta) \equiv \psi_k(\eta) \) \((c_2 = 0 \text{ from } (47))\).

The annihilation \( A_k \) and creation \( A_k^\dagger \) operators \(^{[35]}\) are the following:

\[
A_k = \frac{\psi_k^*(\eta)}{\sqrt{2}} \left( 4\Omega_k(\eta)a_k - \frac{i}{\hbar}\pi_k \right), \quad A_k^\dagger = \frac{\psi_k(\eta)}{\sqrt{2}} \left( 4\Omega_k^*(\eta)a_k + \frac{i}{\hbar}\pi_k \right).
\]

They satisfy the usual commutation relation

\[
[A_k, A_{k'}^\dagger] = \delta_{kk'}
\]

which preserves a positive definitized norm of the state functional.

Using expressions for operators \( a_k, \pi_k \) via operators \( A_k, A_k^\dagger \) \(^{[52]}\)

\[
a_k = \frac{1}{\sqrt{2}}(\psi_k(\eta)A_k + \psi_k^*(\eta)A_k^\dagger),
\]

\[
\pi_k = i\hbar 2\sqrt{2}(\psi_k(\eta)\Omega_k(\eta)A_k - \psi_k^*(\eta)\Omega_k(\eta)A_k^\dagger) = -\sqrt{2}\dot{\psi}_k(\eta)A_k + \dot{\psi}_k^*(\eta)A_k^\dagger,
\]

we can rewrite the Hamiltonian \(^{[12]}\) in terms of creation and annihilation operators

\[
H = \frac{1}{2} \left\{ A_k A_k \left( \dot{\psi}_k^2(\eta) + (k^2 - \frac{6}{(1-\eta)^2})\psi_k^2(\eta) \right) \right.
+ (A_k^\dagger A_k + A_k A_k^\dagger) \left( \dot{\psi}_k^2(\eta) + (k^2 - \frac{6}{(1-\eta)^2})\psi_k^2(\eta) \right)
+ \left. A_k^\dagger A_k \left( (\dot{\psi}_k(\eta))^2 + (k^2 - \frac{6}{(1-\eta)^2})\psi_k^2(\eta) \right) \right\}.
\]

As functions \( \psi_k(\eta) \) tend to oscillator functions when \( t \to 0 \), this Hamiltonian tends to the oscillator Hamiltonian for large \( k \).

It is interesting to find a time dependence for the vacuum fluctuations and for the one-particle excitation energy. Fluctuations are determined by real part of the covariance \( \Omega_k(\eta) \) \(^{[50]}\)

\[
\langle a_k^2 \rangle = \int D\eta \, \psi_k^*(\eta)a_k^2\psi_k(\eta) = \frac{1}{2}|\varphi_k|^2 = \frac{1}{8\Re \Omega_k(\eta)}.
\]

The one-particle state

\[
\Psi_{1k}(\eta) = A_k^\dagger \Psi_k(\eta) = \sqrt{\frac{\psi_k(\eta)}{|\psi_k|^2}} a_k \Psi_k(\eta), \quad \langle \Psi_{1k}^*(\eta)|\Psi_{1k}(\eta) \rangle = 1
\]

has the energy

\[
E_{1k} = \langle \Psi_{1k}^*(\eta)|H_k|\Psi_{1k}(\eta) \rangle - \langle \Psi_k^*(\eta)|H_k|\Psi_k(\eta) \rangle = -\dot{\psi}_k(\eta)\dot{\psi}_k^*(\eta) - \dot{\psi}_k(\eta)\dot{\psi}_k^*(\eta) \left( k^2 - \frac{6}{(1-\eta)^2} \right),
\]

where the vacuum energy is subtracted. The potential part of the energy shows that the oscillations of the mode with the wave number \( k > \sqrt{6} \) break down at the moment \( \eta = T_k \) \(^{[14]}\)

\[
T_k = 1 - \frac{\sqrt{6}}{k}.
\]

At this moment the potential becomes non-oscillatory, the amplitude and fluctuations of the mode begin to grow. Explicit time dependence of the energy is the following

\(^{14}\) This coincides with the equation of motion for \( a_k(\eta) \) \(^{[23]}\) if the solution for the \( a_o(\eta) \) \(^{[28]}\) is taken into account. The presence of the non-oscillating modes with \( k < \sqrt{6} \) at the time \( \eta = 0 \) is a sequence of the initial value of the zero mode \( a_o(\eta = 0) = \sqrt{6}/\lambda \).
\[ E_{1k} = -\frac{\hbar k}{2} \zeta^{-6}(2\zeta^6 - 6\zeta^4 - 9\zeta^2 - 18), \quad \zeta = k(1 - \eta), \]  

where an analytic expression for the spherical Bessel function is used (see Appendix B). For small \( \eta \ll 1 \) and large \( k \gg \sqrt{6} \) (that means \( \zeta \gg 1 \)) the excitation energy is like a negative oscillator energy \( E = -\hbar k \). As the time grows \( (\eta \to 1) \), the lower bound for oscillating modes wave number \( k \) rises. After time \( \eta = T_k \) \((60)\) a moment comes when the excitation energy changes the sign. It can be seen from expression for \( E_{1k} \) \((61)\) in the limit \( \zeta \ll 1 \) \((\eta \approx 1)\).

IV. RELATION OF THE CONFORMAL FACTOR TO AN INFLATON FIELD AND TO COSMOLOGICAL PERTURBATIONS

In this section two issues are discussed. The former is to show how to present conformal factor as an inflaton field. The latter shows how fluctuations of the conformal factor are included into the theory of the cosmological perturbations.

As it follows from \((51), (48)\), vacuum fluctuations of the field \( a(x) \) are similar to those of the scalar field in the de Sitter metric \((32)\). To show an equivalence of their dynamical properties explicitly, let us rewrite the equation for \( a_k \) \((25)\) in proper time \((27)\), simultaneously preparing a conformal transformation \((24)\) of the mode

\[ a_k = a_o \phi_k. \tag{62} \]

The resulting equation is the same as one for a scalar field \((32)\)

\[ \frac{d^2 \phi_k}{dt^2_{pr}} + 3H \frac{d \phi_k}{dt_{pr}} + H^2 \left(k^2 e^{-2Ht_{pr}} - 4 \right) \phi_k = 0, \tag{63} \]

where \( H \) is the Hubble parameter following from solution for \( a_o \) \((28)\)

\[ H = \frac{1}{a_o} \frac{da_o}{dt_{pr}} = \sqrt{\frac{\Lambda}{12}}. \tag{64} \]

Induced by zero mode mass term \( \mu^2 = 4H^2 \) fixes the Bessel functions index \( \nu = 5/2 \) in accordance with the result \((32)\) \((\text{formula } (3.18))\). Moreover, the solution for the zero mode of the scalar field \((\text{formula } (7.6) \text{ in } [32])\) coincides with the solution for \( a_o \) \((28)\).

Let us show that small non-homogeneities of the conformal factor correspond to those of the two independent gauge invariant functions, which represent cosmological scalar metric perturbations \((36)\). For spatially flat Robertson – Walker metric, scalar perturbations are represented by two 3-scalar functions \( \Phi(\eta, x) \) and \( \Psi(\eta, x) \) \((\Phi, \Psi \ll 1)\)

\[ ds^2 = a^2(\eta) \left[(1 + 2\Phi(\eta, x))d\eta^2 - (1 - 2\Psi(\eta, x))\delta_{\alpha\beta}dx^\alpha dx^\beta \right]. \tag{65} \]

In terms of new variables

\[ C(\eta, x) = \frac{1}{2} (\Phi(\eta, x) - \Psi(\eta, x)), \quad N(\eta, x) = \frac{1}{2} (\Phi(\eta, x) + \Psi(\eta, x)) \tag{66} \]

the perturbed interval looks like

\[ ds^2 = a^2(\eta)(1 + 2C(\eta, x)) \left[ (1 + 2N(\eta, x))d\eta^2 - (1 - 2N(\eta, x))\delta_{\alpha\beta}dx^\alpha dx^\beta \right]. \tag{67} \]

Function \( C(\eta, x) \) represents conformal factor fluctuations, and function \( N(\eta, x) \) is a generalization of the Newtonian potential. The trace of the equations for the cosmological perturbations without matter is as follows \((N(\eta, x) \equiv 0)\)

\[ \ddot{C} + 2H \dot{C} - \nabla^2 C - 2(\mathcal{H}^2 + \dot{\mathcal{H}})C = 0 \tag{68} \]

where \( \dot{C} \equiv dC/d\eta, \quad \mathcal{H} = \frac{1}{a} \frac{da}{d\eta}, \nabla^2 \) is Laplacian in the space metric \( \delta_{\alpha\beta}. \) A relation between function \( C(\eta, x) \) and conformal factor \( a(\eta, x) \) follows from comparison of the form of the intervals \((67)\) and \((2)\). The latter is following in the present paper

\[ ds^2 = a^2(\eta, x)[d\eta^2 - \delta_{\alpha\beta}dx^\alpha dx^\beta]. \tag{69} \]

Taking into account decomposition \((24)\) of the conformal factor \( a(\eta, x) \), one can deduce
where \( a_\eta(\eta) \) is considered as a known function \( \langle \delta \rangle \). Then eq. \( (58) \) for the function \( C(\eta, x) \) completely coincides with the eq. \( (24) \) for modes \( a_k(\eta) \)

\[
\ddot{a}_k + (k^2 - \lambda a_\eta^2)a_k = 0.
\]

This correspondence follows from the fact that the equation of motion for the conformal factor is proportional to the trace of the Einstein equations according to \( (44) \).

So we have showed that the conformal factor (4-scalar) represents a kind of the inflaton field and how it is connected to gauge invariant scalar metric perturbations.

V. SUMMARY

The main aim of this work is to get a consistent way to present gravitation as a field living on the flat background and to apply quantum field technique with the well known interpretation (especially, a link between quantum and classical theories) to cosmology. It seems preferable to describe the expansion of the Universe in the functional Schrödinger picture which is suitable for non-stationary processes like a phase transition. Here the problem is solved in the not self-consistent case.

The representation of the conformally flat space-time with the cosmological constant as a self-interacting scalar field in Minkowski world is based on the fact that only one essential metric variable (a conformal factor) describes the investigated model due to the conformal symmetry \( (44) \). After extraction of the conformal factor from the metric, the Einstein – Hilbert action looks like an action for the conformal scalar field with a negative kinetic term and a self-interaction inspired by a cosmological constant. The metric energy-momentum tensor for such a field is proportional to the initial Einstein equations \( (44) \). Hence, a vacuum state (in which energy-momentum tensor average vanishes) corresponds to classical space-time, and the field average is a scale factor of the Universe. From other side, mentioned condition provides a link to the Wheeler – DeWitt approach.

A role of the negative energy in quantum theory is investigated here. In this case the momentum and the velocity of the particle are contrarily directed. If an action is considered as a wave phase, then the constant phase wave will move against a momentum direction. Hence, a coefficient between the momentum and the wave vector must be negative. Further, the negative energy leads to the negative “action”-variable in the Hamilton – Jacobi theory. And the quantum of this “action” must be negative, too. Meanwhile, the minus sign before the Planck constant does not disturb the Dirac’s prescription for the quantization of a classical system. The changing in sign with respect to usual case occurs, for example, in commutator, in momentum operator, in quasi-classical wave function \( (44) \) and in the left hand side of the Schrödinger equation. Quantization of the system with the negative energy includes standard steps and a probability interpretation if one follows the procedure suggested here (see appendix \( A \).

In frames of the suggested model, the classical Universe arises at a stage of the non-stable quantum system evolution when zero mode of the conformal factor becomes much greater then oscillating modes. So the classical space is the vacuum state of this system. The large field vacuum average (zero mode) plays the role of the metric

\[ C(\eta, x) \approx \frac{1}{\sqrt{2a_\eta(\eta)}} \sum_k \left[ a_k^+(\eta) \cos k \cdot \vec{x} + a_k^-(\eta) \sin k \cdot \vec{x} \right], \quad a(\eta) = \sqrt{2a_\eta(\eta)} \]

15 There is an interesting relation: 15 (conformal group Killing vectors) plus one (conformal factor) is equal to 16 (variables in an arbitrary tensor \( \phi_{\mu\nu} \) in four dimension space-time).

16 Lorentz \( (13) \) and Levi-Civita \( (13) \) proposed to regard the Einstein tensor \( G_{\mu\nu} \) as a gravitational energy momentum tensor. It seems that such approach may change the number of the degrees of freedom. Instead of the Einstein equations without second time derivatives (constraints \( (13) \)), we now have a quantum average of the components of the conserved energy-momentum tensor.

17 The sign becomes the same as it was suggested by Linde \( (13) \).

18 An indefinite metric does not appear here in contrast with the Gupta – Bleuler quantization \( (13) \). The latter is based on the assumption that a state with the negative energy is not observable. This is fixed by weaker Lorentz condition. Without it, one can not exclude states with a negative norm, and hence the probability interpretation is away.

When fields with positive and negative energy are included into a theory, one has a difficulties with spontaneous creation of infinite number of particles of both types in the finite space-time volume \( (13) \). This remark will not be directly applicable here, at least because conformal factor describes physical space-time.

In any case, an investigation of possibility of negative gravitational energy is necessary. It is well known that Hartle – Hawking \( (13) \) and Vilenkin tunneling \( (13) \) wave functions suffers difficulties following Wheeler – DeWitt approach.
scale factor. Integral of motion for zero mode is the energy vacuum average and must be put to zero to coincide with the corresponding Einstein equation. In result we derive the correct description of the classical Universe. The existence of the field vacuum average is a sequence of the instability of the system (the kinetic energy is negative and self-interaction energy depending of the cosmological constant is positive).

The “phantom” field Lagrangian \[44,45\] has negative kinetic and positive potential parts, too \[19\]. In contrast with the conformal factor, this field is introduced only for getting a negative pressure to explain the accelerating Universe expansion.

The mentioned above peculiarities distinguish the suggested model from Thirring model \[47\] describing gravity by a scalar field in the field theoretical formulation of the General Relativity in a flat space \[20\]. In this kind of theories the Universe expansion can not be described because the positive definite Hamiltonian is postulated and small deviations from Minkowski space are supposed \[47\].

In the model introduced here, the quantum fluctuations are similar to those in the inflation theories \[32\] because the equations of motion for oscillating modes are the same in both theories. The index of Bessel functions in the solution for the conformal factor is fixed, because the “mass” parameter is connected with the cosmological constant which simultaneously determines the Hubble parameter.

Further it seems interesting to do the following. In the self-consistent case \[24\], renormalization group running of the interaction constant \(\lambda\) can be studied. It must be shown at non-zero temperature that there is a phase transition causing the emergence of the vacuum field average due to quantum fluctuations. Next, the quantum properties of the open, flat and closed Universes can be compared using the fact that the background metric curvature scalar is presented in the conformal field action \[11\]. A possible generalization for an arbitrary metric can be investigated. The most intriguing problem is to incorporate a matter into the model.

ACKNOWLEDGMENTS

The author is grateful to Prof. V.V.Papoyan and Dr. S.Gogilidze for encouragement at the beginning of this work. It is a pleasure to thank Dr. V.Smirichinskii for helpful and critical discussions. This work was supported by the Russian Foundation of Basic Research, Grant No 01–01–00708.

APPENDIX A: SYSTEMS WITH NEGATIVE ENERGY

In this Appendix the quantization of a system with the negative energy is discussed on the example of an inverted harmonic oscillator. It is shown that the negative energy causes the minus in front of Planck constant. However we do not need to change the known quantization algorithm and interpretation of a resulting quantum theory.

First of all, let us discuss a difference between a classical motion of a positive particle (with the positive kinetic energy) and of a negative particle (with the negative kinetic energy). We describe them by \(L_+\) and \(L_-\) Lagrangians respectively with the same linear potential

\[
L_+ = \frac{m\dot{x}_+^2}{2} - kx_+, \quad L_- = \frac{m\dot{x}_-^2}{2} - kx_-, \quad \dot{x}_\pm = \frac{dx_\pm}{dt},
\]

where \(m > 0, \ k > 0\). Their accelerations and velocities are contradictory to each other

\[
\ddot{x}_- = -\ddot{x}_+ = \frac{k}{m}, \quad \dot{x}_- = -\dot{x}_+ = \frac{k}{m}t.
\]

The positive particle moves towards decreasing the potential and the negative particle moves towards increasing the potential

\[
x_- = -x_+ = \frac{k}{2m}t^2.
\]

\[19\]Narlikar and Padmanabhan \[46\] pointed out the similarity between inflation and the steady state model and gave references where negative energy momentum tensor was proposed to drive the expansion and matter creation.

\[20\]Gravitation theories as field theories in Minkowski space are based on the bimetrical formalism introduced by Rosen \[48\] and Papapetrou \[49\]. An arbitrary second rank tensor field \(\phi^{\mu\nu}\) can be decomposed into direct sum of the irreducible representations of the non-homogeneous Lorentz group of the spin 2,1 and 0 \[50,51\].
From the energy expressions

\[ E_+ = \frac{m\dot{x}_+^2}{2} + kx_+ , \quad E_- = \frac{m\dot{x}_-^2}{2} + kx_- \]  

(A.4)

it follows that the classically allowed regions are above the potential barrier for a positive particle and under the barrier for a negative particle, respectively (Fig. 1).

![Diagram](image1)

**FIG. 1.** The positive particle \(+m\) moves towards decreasing the potential above the potential barrier and the negative particle \(-m\) moves towards increasing the potential under the barrier.

The inverted oscillator has the following Lagrangian:

\[ L = -\frac{m\dot{x}^2}{2} + \frac{m\omega^2x^2}{2} \]  

(A.5)

leading to the standard equation of motion

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = -m\ddot{x} - m\omega^2x = 0. \]  

(A.6)

With the momentum

\[ p = \frac{\partial L}{\partial \dot{x}} = -m\dot{x} \]  

(A.7)

one derives the Hamiltonian (coinciding with energy)

\[ H = p\dot{x} - L = -\frac{1}{2} \left( \frac{p^2}{m} + m\omega^2x^2 \right). \]  

(A.8)

Hence, the energy is negative or zero

\[ E = H \leq 0, \]  

(A.9)

and particle moves under the potential (Fig. 2).

![Diagram](image2)

**FIG. 2.** A particle with negative energy \(E\) oscillates under the potential.
Using usual Poisson Bracket definition

\[
\{A, B\} = \frac{\partial A}{\partial x} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial x}, \quad \{x, p\} = 1,
\]  

(A.10)

for arbitrary function \(A = A(x, p), B = B(x, p)\) one gets the Hamiltonian equations

\[
\begin{align*}
\dot{x} &= \{x, H\} = -\frac{p}{m}, \\
\dot{p} &= \{p, H\} = m\omega^2 x,
\end{align*}
\]  

(A.11)

which correspond to the momentum definition (A.7) and to the Lagrangian equation (A.6) but differ from the usual case by its sign.

The straight corollary of the negative energy is the negative “action”-variable in the Hamilton – Jacoby theory where a momentum is a partial derivative of the principal Hamilton function \[52\]

\[p = \frac{\partial S}{\partial x}.
\]  

(A.12)

The Hamilton – Jacoby equation

\[
H \left( x, \frac{\partial S}{\partial x} \right) + \frac{\partial S}{\partial t} = 0
\]  

(A.13)

with the negative oscillator Hamiltonian \[52\] can be solved by substitution

\[S(x, E, t) = W(x, E) - Et,
\]  

(A.14)

where \(W(x, E)\) is the characteristic Hamiltonian function. It follows from this solution that the “action”-variable is negative

\[J = \oint p dx = \int \frac{\partial W(x, E)}{\partial x} dx = \frac{2\pi}{\omega} E \leq 0.
\]  

(A.15)

So we have seen that two circumstances must be taken into account at the quantization of a negative energy system.

First, an “action”-quantum would be negative if it is considered as the least possible portion of the “action” \[52\] in the Bohr – Sommerfeld quantization. Let us suggest the following:

\[J = -nh,
\]  

(A.16)

where a quantum number \(n\) is positive or zero, \(h\) is the Planck constant.

Second, a surface \[52\]

\[S(\vec{x}, E, t) = W(\vec{x}, E) - Et = W_o = \text{const}
\]  

(A.17)

moves along momentum \(\vec{p} = \nabla W(\vec{x}, E)\) if \(E > 0\)

\[S(t) = W_o = W_o - Et, \quad W_o < W_+,
\]  

(A.18)

and against it if \(E < 0\) (Fig. 3)

\[S(t) = W_o = W_o - Et, \quad W_o > W_-.
\]  

(A.19)

![FIG. 3. The wave \(S(\vec{x}, E, t) = W_o\) moves along momentum \(\vec{p}\) if \(E > 0\) and against it if \(E < 0\).]
Hence, the direction of a wave vector
\[ \vec{k} = \nabla \Phi(\vec{x}, \omega, t), \]  
(A.20)
where \( \Phi(\vec{x}, \omega, t) \) is a wave phase, coincides with the direction of the momentum if \( E > 0 \) and contradicts it if \( E < 0 \)
\[
\begin{cases}
\vec{k} \uparrow \uparrow \vec{p} & E > 0, \\
\vec{k} \uparrow \downarrow \vec{p} & E < 0.
\end{cases}
\]  
(A.21)
Putting
\[ \vec{p} = -\hbar \vec{k}, \]  
(A.22)
for the case \( E < 0 \), one gets the following relation:
\[ \Phi = -\frac{S}{\hbar}, \]  
(A.23)
Note that the wave vector coincides with the velocity as \( \vec{p} = -m\vec{v} \), and the circular frequency of the wave process is positive
\[ \omega = \frac{\partial \Phi}{\partial t} = -\frac{1}{\hbar} \frac{\partial S}{\partial t} = -\frac{E}{\hbar} > 0, \quad \text{where } E < 0. \]  
(A.24)
Does the quantum mechanics allow a negative “action”-quantum? According to Dirac the main point in quantization of a classical system is the replacement of a classical Poisson bracket by a quantum one. The latter must be Hermitian as the former is real. The minus sign in front of the Planck constant
\[ \{A, B\} = \frac{1}{i\hbar} \frac{1}{\hbar} (AB - BA) = \frac{1}{i\hbar} [A, B] \]  
(A.25)
does not disturb this condition. Using formulae (A.10), (A.25), one gets a commutation relation
\[ [x, p] = -i\hbar, \]  
(A.26)
where the momentum operator has the form
\[ \hat{p} \equiv i\hbar \frac{\partial}{\partial x}, \]  
(A.27)
The quasi-classical wave function reads as following according to (A.23)
\[ \Psi(x, t) = \exp \left\{ -\frac{i}{\hbar} S(x, t) \right\} \]  
(A.28)
and must satisfy the Schrödinger equation
\[ -i\hbar \frac{d}{dt} \Psi(x, t) = H(x, \hat{p})\Psi(x, t). \]  
(A.29)
The energy operator changes the sign with respect to a usual case, too
\[ E \rightarrow -i\hbar \frac{d}{dt}. \]  
(A.30)
The Schrödinger equation (A.29) undergoes into the Hamilton–Jacoby equation (A.13) in the classical limit \( \hbar \to 0 \). The probability conservation law holds
\[ \frac{d\rho}{dt} + \nabla \cdot \vec{j} = 0, \]  
(A.31)
where
\[ \rho = \Psi^*(x,t)\Psi(x,t), \quad \vec{j} = \frac{i\hbar}{2m} (\Psi(x,t)\nabla\Psi^*(x,t) - \Psi^*(x,t)\nabla\Psi(x,t)). \tag{A.32} \]

The quantity \( \rho \) is positive and has a meaning of the probability density under the normalizing condition

\[ \int \rho dx = \int \Psi^*(x,t)\Psi(x,t) dx = 1. \tag{A.33} \]

The current \( \vec{j} \) coincides with the wave vector

\[ \vec{j} \uparrow \vec{k}. \tag{A.34} \]

The Heisenberg equations are the following:

\[ \frac{d\hat{A}}{dt} = \frac{\partial \hat{A}}{\partial t} - \frac{1}{i\hbar}[\hat{A},H]. \tag{A.35} \]

Then, according to the definition of the average of the time derivative

\[ \langle \frac{d\hat{A}}{dt} \rangle = \frac{d}{dt} \langle \hat{A} \rangle, \tag{A.36} \]

one gets the Ehrenfest theorems

\[ \begin{cases} \frac{d}{dt} \langle \hat{x} \rangle = \langle \frac{-1}{\hbar}[\hat{x},H] \rangle = \langle \hat{p} \rangle / m, \\ \frac{d}{dt} \langle \hat{p} \rangle = \langle \frac{-1}{\hbar}[\hat{p},H] \rangle = m\omega^2 \langle x \rangle, \end{cases} \tag{A.37} \]

which correspond to classical Hamiltonian equations (A.11).

Now the Schrödinger picture and a filling number representation for a negative harmonic oscillator can be considered. The Schrödinger equation

\[ (H - E)\Psi(x,t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \frac{1}{2}m\omega^2 x^2 - E \right) \Psi(x,t) = 0, \tag{A.38} \]

or in coordinate representation

\[ \left( \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \Psi(x,t) = 0 \tag{A.39} \]

differs in sign from the usual case but here the energy \( E \) is less than zero. Introducing dimensionless parameters

\[ \xi = \frac{x}{x_d}, \quad x_d = \sqrt{\frac{\hbar}{m\omega}}, \quad \varepsilon = \frac{-2E}{\hbar\omega} > 0 \tag{A.40} \]

we get a usual equation

\[ \left( \frac{\partial^2}{\partial \xi^2} - \xi^2 + \varepsilon \right) \Psi(x,t) = 0. \tag{A.41} \]

Hence, the stationary states are characterized by negative energy levels

\[ E_n = -\hbar\omega \left( n + \frac{1}{2} \right) \tag{A.42} \]

and by wave functions

\[ \Psi_n(x,t) = \frac{1}{(\sqrt{\pi}n!2^n x_d)^{1/2}} \exp \left\{ -\frac{1}{2} \left( \frac{x}{x_d} \right)^2 \right\} H_n \left( \frac{x}{x_d} \right), \tag{A.43} \]

where Hermit polynomials
\[ H_n(\xi) = (-)^n e^{\xi^2} \frac{d^n}{d\xi^n} e^{-\xi^2} \]  

satisfy the orthogonality condition

\[ \int_{-\infty}^{+\infty} \Psi^*_m(x,t) \Psi_n(x,t) dx = \delta_{mn}. \]  

To undergo to the filling number representation, let us introduce operators

\[ a = \frac{1}{\sqrt{2}} \left( \xi + \frac{\partial}{\partial \xi} \right) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m\omega}{\hbar}} x - i \frac{\hat{p}}{\sqrt{\hbar m\omega}} \right), \]  

\[ a^\dagger = \frac{1}{\sqrt{2}} \left( \xi - \frac{\partial}{\partial \xi} \right) = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{m\omega}{\hbar}} x + i \frac{\hat{p}}{\sqrt{\hbar m\omega}} \right), \]

satisfying the commutation relation

\[ [a, a^\dagger] = 1 \]

according to (A.26). The operator \( a \) annuls the vacuum state

\[ |0\rangle = \Psi_0(x,t) = \frac{1}{\pi^{1/4}} e^{-\xi^2/2}, \quad \langle 0|0 \rangle = 1, \]  

\[ a|0 \rangle = 0. \]  

The operator \( a^\dagger \) acting on the vacuum gives exited states with the quantum number \( n \)

\[ |n\rangle = \Psi_n(x,t) = \frac{1}{\sqrt{n!}} (a^\dagger)^n \Psi_0(x,t), \quad \langle n|n \rangle = 1. \]

The energy operator looks as follows:

\[ H = -\hbar \omega \left( \hat{n} + \frac{1}{2} \right), \]

where a particle number operator is introduced

\[ \hat{n} = a^\dagger a, \quad \hat{n}|n \rangle = n|n \rangle. \]  

To give the interpretation of the operators \( a, a^\dagger \) let us find the value of the operators \( H \) and \( \hat{n} \) in the stations \( a^\dagger |n \rangle, \ a|n \rangle \)

\[ Ha^\dagger |n \rangle = (E_n - \hbar \omega) a^\dagger |n \rangle, \quad \hat{n}a^\dagger |n \rangle = (n + 1)a^\dagger |n \rangle, \]  

\[ Ha|n \rangle = (E_n + \hbar \omega) a|n \rangle, \quad \hat{n}a|n \rangle = (n - 1)a|n \rangle. \]  

The operator \( a^\dagger \) decreases the energy onto the quantity \( \hbar \omega \) and increases the particle number onto a unit. This operator can be called the creation operator for the particle with the negative energy \( -\hbar \omega \). The operator \( a \) increases the energy onto the quantity \( \hbar \omega \) and decreases the particle number onto a unit. This operator can be called the annihilation operator for the particle with the negative energy \( -\hbar \omega \).

So the quantum theory of the negative energy system needs a negative “action”-quantum value by preserving the usual apparatus and probability interpretation.
APPENDIX B: BESSEL FUNCTIONS

The first- and second-order Hankel functions are linear combinations of the Bessel functions

\[ H^{(1)}_\nu(\zeta) = J_\nu(\zeta) + iY_\nu(\zeta) \]

\[ H^{(2)}_\nu(\zeta) = J_\nu(\zeta) - iY_\nu(\zeta) \]

and satisfy the following orthogonality relation:

\[ H^{(1)}_\nu(\zeta) \frac{d}{d\zeta} H^{(2)}_\nu(\zeta) - H^{(2)}_\nu(\zeta) \frac{d}{d\zeta} H^{(1)}_\nu(\zeta) = -\frac{4i}{\pi \zeta}. \]

They are complex conjugated by real argument \( \zeta \in \mathbb{R} \)

\[ \left( H^{(1)}_\nu(\zeta) \right)^* = H^{(2)}_\nu(\zeta). \]

The Bessel functions \( J_{\frac{3}{2}}(\zeta) \) \( Y_{\frac{3}{2}}(\zeta) \) can be expressed via spherical Bessel functions

\[ j_{\frac{3}{2}}(\zeta) = \sqrt{\frac{3}{\pi \zeta}} J_{\frac{3}{2}}(\zeta) = \left( \frac{3}{\zeta^2} - \frac{1}{2} \right) \sin(\zeta) - \frac{3}{2} \zeta \cos(\zeta), \]

\[ y_{\frac{3}{2}}(\zeta) = \sqrt{\frac{3}{\pi \zeta}} Y_{\frac{3}{2}}(\zeta) = \left( -\frac{3}{\zeta^2} + \frac{1}{2} \right) \cos(\zeta) - \frac{3}{2} \zeta \sin(\zeta). \]

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