Existence and Nonlinear Stability of Stationary Solutions to the Full Two-Phase Flow Model in a Half Line

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Abstract

The inflow problem for the full two-phase model in a half line is investigated in this paper. The existence and uniqueness of the stationary solution is shown by applications of center manifold theory, and its nonlinear stability of the stationary solution is established for the small perturbation.

Key words. Full two-phase flow model, stationary solution, inflow problem, nonlinear stability.

1 Introduction

Two-phase flow models play important roles in applied scientific areas, for instance, nuclear, engines, chemical engineering, medicine, oil-gas, fluidization, waste water treatment, liquid crystals, lubrication, biomedical flows [2, 4, 8], etc. In this paper, we consider the full two-phase flow model which can be formally obtained from a Vlasov-Fokker-Planck equation coupled with the compressible Navier-Stokes equations through the Chapman-Enskog expansion [6].

We consider the initial-boundary value problem (IBVP) for the two-fluid model as follows:

\[
\begin{align*}
\rho_t + (\rho u)_x &= 0, \\
(\rho u)_t + [\rho u^2 + p_1(\rho)]_x &= (\mu u)_x + n(v - u) \\
n_t + (nv)_x &= 0, \\
(nv)_t + [nv^2 + p_2(n)]_x &= (nv)_x - n(v - u), & x > 0, & t > 0,
\end{align*}
\]

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where $\rho > 0$ and $n > 0$ stand for the densities, $u$ and $v$ are the velocities of two fluids, and the constant $\mu$ is the viscosity coefficient. The pressure-densities functions take forms

$$p_1(\rho) = A_1\rho^\gamma, \quad p_2(n) = A_2n^\alpha$$

with four constants $A_1 > 0$, $A_2 > 0$, $\gamma \geq 1$ and $\alpha \geq 1$. The initial data satisfy

$$\lim_{x \to +\infty} (\rho_0, u_0, n_0, v_0)(x) = (\rho_+, u_+, n_+, v_+), \quad \rho_+ > 0, \quad n_+ > 0,$$

and inflow boundary condition imposed with

$$(\rho, u, n, v)(t, 0) = (\rho_-, u_-, n_-, u_-)$$

where $\rho_+ > 0$, $n_+ > 0$ and $u_- > 0$ are constants.

The main purpose of this paper is to prove the existence and nonlinear stability of the stationary solution in Sobolev space. The stationary solution $(\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(x)$ corresponding to the problem (1.1)-(1.5) satisfies the following system

$$
\begin{cases}
(\bar{\rho}u)_x = 0, \\
[\bar{\rho}u^2 + p_1(\bar{\rho})]_x = (\mu\bar{u}_x)_x + \bar{n}(\bar{v} - \bar{u}), \\
(\bar{n}\bar{v})_x = 0, \\
[\bar{n}\bar{v}^2 + p_2(\bar{n})]_x = (\bar{n}\bar{v}_x)_x - \bar{n}(\bar{v} - \bar{u}),
\end{cases}
$$

and the boundary condition and spatial far field condition

$$(\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(0) = (\rho_-, u_-, n_-, v_-), \quad \lim_{x \to +\infty} (\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(x) = (\rho_+, u_+, n_+, u_+),$$

$$\inf_{x \in \mathbb{R}^+} \bar{n}(x) > 0, \quad \inf_{x \in \mathbb{R}^+} \bar{\rho}(x) > 0.$$  

Integrating (1.6)$_1$, (1.6)$_3$ over $(x, +\infty)$ and $(0, x)$, we obtain

$$\bar{u} = \frac{\rho_+ u_+}{\bar{\rho}}, \quad \bar{v} = \frac{n_+ u_+}{n} = \frac{n_- u_-}{\bar{n}},$$

which implies that the relationships

$$u_+ = \frac{\rho_-}{\rho_+} = \frac{n_+ u_+}{n_- u_-} > 0, \quad u_+ = \frac{\rho_-}{\rho_+} = \frac{n_-}{n_+}$$

are necessary for the existence of stationary solutions to the boundary value problem (BVP) (1.6)-(1.8).

Define the Mach number $M_+$ and sound speed $c_+$ at the spatial far field as follows

$$M_+ := \frac{|u_+|}{c_+}, \quad c_+ := \frac{A_1\gamma \rho_+^\gamma + A_2\alpha n_+^\alpha}{\rho_+ + n_+}.$$

Then, we have the following results about the existence and uniqueness of the stationary solution.
**Theorem 1.1.** Assume that \( \delta := |u_+ - u_-| > 0, u_+ > 0, \frac{u_-}{u_+} = \frac{\rho_+}{\rho_-} = \frac{n_+}{n_-} \) hold. Then there exists a set \( \Omega_+ \subset \mathbb{R}_+ \) such that if \( u_- \in \Omega_+ \) and \( \delta \) sufficiently small, there exists a unique strong solution \((\bar{\rho}, \bar{u}, \bar{n}, \bar{v})\) to the problem (1.6)-(1.8) which satisfies either for the supersonic or subsonic case \( M_+ \neq 1 \) that

\[
|\partial_b^k(\bar{\rho} - \rho_+, \bar{u} - u_+, \bar{n} - n_+, \bar{v} - u_+)| \leq C \delta e^{-mx}, \quad k = 0, 1, 2, \quad (1.12)
\]

or for the sonic case \( M_+ = 1 \) that \( \bar{u}_x \geq 0, \bar{v}_x \geq 0 \) and

\[
|\partial_b^k(\bar{\rho} - \rho_+, \bar{u} - u_+, \bar{n} - n_+, \bar{v} - u_+)| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad k = 0, 1, 2, \quad (1.13)
\]

where \( C > 0, m > 0 \) are positive constants.

**Remark 1.1.** In contrast to isentropic Navier-Stokes equations [3, 7] and full compressible Navier-Stokes equations [9, 10], where there is no stationary solutions for the supersonic case, there exist stationary solutions for supersonic, subsonic and sonic cases to the IBVP (1.1)-(1.5).

Then, we have the nonlinear stability of the stationary solution for (1.1)-(1.5) as follows.

**Theorem 1.2.** Let the same conditions in Theorem 1.1 hold and assume that it holds

\[
|p'_{1}(\rho_+) - p'_{2}(n_+)| \leq \sqrt{2}u_+ \min\{ (1 + \frac{\rho_+}{n_+})[(\gamma - 1)p'_{1}(\rho_+)]^{\frac{1}{2}}, (1 + \frac{n_+}{\rho_+})[(\alpha - 1)p'_{2}(n_+)]^{\frac{1}{2}} \}
\]

for the sonic case \( M_+ = 1 \). Then, there exists a positive constant \( \varepsilon_0 \) such that

\[
\|(\rho_0 - \bar{\rho}, u_0 - \bar{u}, n_0 - \bar{n}, v_0 - \bar{v})\|_{H^1} + \delta \leq \varepsilon_0,
\]

the problem (1.1)-(1.5) has a unique global solution \((\rho, u, n, v)(t, x)\) satisfying

\[
\begin{align*}
(\rho - \bar{\rho}, u - \bar{u}, n - \bar{n}, v - \bar{v}) & \in C([0, +\infty); H^1), \\
(\rho - \bar{\rho}, n - \bar{n})_x & \in L^2([0, +\infty); L^2), \\
(u - \bar{u}, v - \bar{v})_x & \in L^2([0, +\infty); H^1),
\end{align*}
\]

and

\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}_+} |(\rho, u, n, v)(t, x) - (\bar{\rho}, \bar{u}, \bar{n}, \bar{v})(x)| = 0. \quad (1.14)
\]

**Notation.** We denote by \( \| \cdot \|_{L^p} \) the norm of the usual Lebesgue space \( L^p = L^p(\mathbb{R}_+), 1 \leq p \leq \infty \).

And if \( p = 2 \), we write \( \| \cdot \|_{L^p(\mathbb{R}_+)} = \| \cdot \|. \) \( H^s(\mathbb{R}_+) \) stands for the standard \( s \)-th Sobolev space over \( \mathbb{R}_+ \), equipped with its norm \( \| f \|_{H^s(\mathbb{R}_+)} = \| f \|_s = (\sum_{i=0}^{s} \| \partial^i f \|^2)^{\frac{1}{2}}. \)
\( C([0, T]; H^1(\mathbb{R}_+)) \) represents the space of continuous functions on the interval \([0, T]\) with values in \( H^1(\mathbb{R}_+) \).

The rest of this paper will be organized as follows. We investigate the existence and uniqueness of the stationary solution in Section 2, and gain the nonlinear stability of the solution in Section 3.
2 Existence of Stationary Solution

We prove Theorem 1.1 on the existence and uniqueness of the stationary solution to (1.6)-(1.8) with $u_+ > 0$ and $\delta$ sufficiently small as follows. The natural idea is to apply the center manifold theory [1] to the BVP (1.6)-(1.8), where it is necessary to get the bound estimates of $\overline{u}_x(0)$ or $\overline{v}_x(0)$.

Lemma 2.1. Assume that $u_+ > 0$, $\frac{u_+}{u_-} = \frac{u_+}{\overline{u}_x} = \frac{u_+}{v_x}$, $\delta := |u_+ - u_-| > 0$ hold with $\delta$ sufficiently small. Then the solution $(\rho, \overline{u}, \overline{v}, \overline{v})$ to the BVP (1.6)-(1.8) satisfies

$$|\overline{u}_x(0)| \leq C|u_- - u_+|, \quad |\overline{v}_x(0)| \leq C|u_- - u_+|,$$

where $C > 0$ is a positive constant.

Proof. Due to $\rho = \frac{u_+}{\overline{u}_x}$ and $\overline{v} = \frac{n+u_+}{v}$, we obtain

$$(\rho_+ u_+ \overline{u} + A_1 \rho_+ \overline{v} \overline{u} - \gamma \overline{u})_x = (\mu \overline{u}_x)_x + \frac{n+u_+}{\overline{v}} (\overline{v} - \overline{u}),$$

$$(n_+ u_+ \overline{v} + A_2 n_+ \overline{v} - \alpha \overline{v})_x = (n+u_+ \overline{v}_x)_x - \frac{n+u_+}{\overline{v}} (\overline{v} - \overline{u}).$$

Adding (2.2) to (2.2)2 together, then integrating the resulted equation over $(0, \infty)$, we have

$$\mu \overline{u}_x(0) + \frac{n+u_+}{u_-} \overline{v}_x(0) = \frac{1}{u_+} [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+ + A_2 \alpha n_+)] (u_- - u_+) + O(|u_- - u_+|^2).$$

Multiplying (2.2)1 by $\overline{u}$, (2.2)2 by $\overline{v}$ respectively, adding them together and integrating the resulted equation over $(0, \infty)$ lead to

$$\int_0^{\infty} (\mu \overline{u}_x^2 + n_+ u_+ \overline{v}_x^2) dx + \int_0^{\infty} \frac{n+u_+}{\overline{v}} (\overline{v} - \overline{u})^2 dx$$

$$= \int_0^{\infty} [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+ + A_2 \alpha n_+)] (u_- - u_+) + O(|u_- - u_+|^2)$$

$$= O(|u_- - u_+|^2),$$

where we have used (2.3).

Multiplying (2.2)1 by $\overline{u}^b (b < -\max(\alpha, \gamma) - 1)$, (2.2)2 by $\overline{v}^b$ respectively, adding them together and integrating the resulted equation over $(0, \infty)$ lead to

$$\int_0^{\infty} (-b \mu \overline{u}_x^2 \overline{u}^{b-1} - b n_+ u_+ \overline{v}_x^2 \overline{v}^{b-1}) dx + \int_0^{\infty} \frac{n+u_+}{\overline{v}} (\overline{v} - \overline{u})(\overline{u}^{b-1} - \overline{v}^{b-1}) dx$$

$$= u_+ \mu \overline{x}_x(0) + n_+ u_+ \overline{v}_x(0) - u_+^2 [(\rho_+ + n_+) u_+^2 - (A_1 \gamma \rho_+ + A_2 \alpha n_+)] (u_- - u_+) + O(|u_- - u_+|^2)$$

$$= O(|u_- - u_+|^2),$$

where we have used (2.3).

Multiplying (2.2)2 by $\overline{v}_x^2$ and then integrating the resulted equality over $(0, \infty)$ yield

$$\int_0^{\infty} n_+ u_+ \overline{v}_x^2(0) + \int_0^{\infty} n_+ u_+ \overline{v}_x^2 dx$$

$$= \int_0^{\infty} A_2 \alpha n_+ \overline{v}_x^{(\alpha+2)}(0) \overline{v}_x^2 dx - \int_0^{\infty} \frac{n+u_+}{\overline{v}} (\overline{v} - \overline{u}) \overline{v}_x dx := \sum_{i=1}^{2} I_i.$$
It is easy to see that the system (2.6). Using (2.3), we have
\begin{equation}
|I_1| + |I_2| \leq C \int_0^\infty \frac{\rho_+ u_+}{\bar{v}} (\bar{v} - \bar{u})^2 \, dx + C \int_0^\infty \bar{v}^{\delta - 2} \bar{c}_x^2 \, dx \leq C |u_- - u_+|^2, \tag{2.7}
\end{equation}
With the help of (2.6), (2.7) and (2.3), we get
\begin{equation}
|\bar{v}_x(0)| \leq C |u_- - u_+|, \quad |\bar{u}_x(0)| \leq C |u_- - u_+|. \tag{2.8}
\end{equation}

We use the manifold theory [1] to obtain the existence and uniqueness of the stationary solution (1.6)-(1.8) of full two-phase flow model. Firstly, it is necessary to reformulate the system (1.6) into a $3 \times 3$ system of ordinary differential equations of the first-order.

Adding (1.6)2, (1.6)4 together and substituting $\bar{\rho} = \frac{\rho_+ u_+}{\bar{v}}$, $\bar{n} = \frac{n_+ u_+}{\bar{v}}$ into the resulted equation, then integrating the resulted equality over $(x, \infty)$, we gain
\begin{equation}
\rho_+ u_+ (\bar{u} - 1) + A_1 \rho_+ \bar{u}^\gamma (\bar{u}^{-\gamma} - 1) + n_+ u_+ (\bar{v} - 1) + A_2 n_+^2 u_+^\alpha (\bar{v}^{-\alpha} - 1) = \mu \bar{u}_x + n_+ u_+ \frac{\bar{v}_x}{\bar{v}}. \tag{2.9}
\end{equation}
We consider the following system
\begin{equation}
\begin{cases}
(\rho_+ u_+ \bar{u} + A_1 \rho_+^\gamma \bar{u}^\gamma (\bar{u}^{-\gamma} - 1) + n_+ u_+ (\bar{v} - 1) + A_2 n_+ u_+^\alpha (\bar{v}^{-\alpha} - 1) = \mu \bar{u}_x + n_+ u_+ \frac{\bar{v}_x}{\bar{v}}, \tag{2.10}
\end{cases}
\end{equation}
It is easy to see that the system (2.10) contains a second-order equation and a first-order equation. Hence, the system (2.10) can be reformulated into the following system
\begin{equation}
\begin{align*}
\bar{u}_x &= \bar{w}, \\
\bar{w}_x &= \frac{1}{\mu} [\rho_+ u_+ \bar{u} - A_1 \rho_+^{\gamma+1} \bar{u}^{-\gamma} - n_+ u_+ (1 - \frac{\bar{u}}{\bar{v}})], \\
\bar{v}_x &= \frac{\bar{v}}{n_+ u_+} [\rho_+ u_+ (\bar{u} - u_+) + A_1 \rho_+ \bar{u}^\gamma (\frac{\bar{u}}{\bar{v}}) - 1) + n_+ u_+ (\bar{v} - u_+) + A_2 n_+ u_+^\alpha (\frac{\bar{u}^\gamma}{\bar{v}} - 1) - \mu \bar{w}] . \tag{2.11}
\end{align*}
\end{equation}
The boundary condition satisfies
\begin{equation}
(\bar{u}, \bar{w}, \bar{v})(0) = (u_+, \bar{u}_x(0), u_-), \quad \lim_{x \to \infty} (\bar{u}, \bar{w}, \bar{v})(x) = (u_+, 0, u_+). \tag{2.12}
\end{equation}
Define the perturbation near the far field state $(u_+, 0, u_+)$ as $(\bar{u}, \bar{w}, \bar{v}) = (\bar{u} - u_+, \bar{w}, \bar{v} - u_+)$. The system (2.11), (2.12) can be rewritten as follows:
\begin{equation}
\begin{align*}
\frac{d}{dx} \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{v} \end{pmatrix} = \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{v} \end{pmatrix} + \begin{pmatrix} g_1(\bar{u}, \bar{w}, \bar{v}) \\ g_2(\bar{u}, \bar{w}, \bar{v}) \\ g_3(\bar{u}, \bar{w}, \bar{v}) \end{pmatrix}, \tag{2.13}
\end{align*}
\end{equation}
\begin{equation}
(\bar{u}, \bar{w}, \bar{v})(0) := (\bar{u}_-, \bar{w}_-, \bar{v}_-) = (u_-- u_+, \bar{u}_x(0), u_- - u_+), \quad \lim_{x \to \infty} (\bar{u}, \bar{w}, \bar{v}) = (0, 0, 0),
\end{equation}
where the matrix $J_+$ is the defined as follows:

$$
J_+ = \begin{pmatrix}
0 & 1 & 0 \\
\frac{n_+}{\mu u_+}(\rho_+ u_+^2 - A_1 \gamma \rho_+^\gamma) & -\frac{\mu}{n_+} & \frac{1}{n_+ u_+} (n_+ u_+^2 - A_2 \alpha n_+^\alpha) \\
\frac{1}{\mu u_+} (\rho_+ u_+^2 - A_1 \gamma \rho_+^\gamma) & -\frac{\mu}{n_+} & \frac{1}{n_+ u_+} (n_+ u_+^2 - A_2 \alpha n_+^\alpha)
\end{pmatrix}
$$

(2.14)

and $\bar{g}_1, \bar{g}_2, \bar{g}_3$ are nonlinear functions defined by

$$
\begin{align*}
\bar{g}_1(\bar{u}, \bar{w}, \bar{v}) &= 0, \\
\bar{g}_2(\bar{u}, \bar{w}, \bar{v}) &= \frac{1}{2} \left( 2 \frac{n_+}{\mu} \frac{1}{u_+} \bar{w}^2 - 2 \frac{n_+}{\mu} \frac{1}{u_+} \bar{u} \bar{v} + 2 \frac{A_1 \gamma (\gamma + 1) \rho_+^\gamma}{\mu u_+} \bar{w} \bar{u} \right) + O(|\bar{v}|^3 + |\bar{u}|^3 + |\bar{w}|^3), \\
\bar{g}_3(\bar{u}, \bar{w}, \bar{v}) &= \frac{1}{2} \frac{A_1 \gamma (\gamma + 1) \rho_+^\gamma}{n_+ u_+^2} \bar{u}^2 + 2 \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^\gamma}{n_+ u_+^2} \bar{u} \bar{v} + (2 \frac{n_+ u_+^2 - A_2 \alpha n_+^\alpha}{n_+ u_+^2} + A_2 \alpha (\alpha + 1) n_+^\alpha) \bar{v}^2 \\
&\quad - 2 \frac{\mu}{n_+ u_+} \bar{u} \bar{v} + O(|\bar{v}|^3 + |\bar{u}|^3 + |\bar{w}|^3).
\end{align*}
\tag{2.15}
$$

Three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of matrix $J_+$ satisfy

$$
\begin{align*}
\lambda_1 \lambda_2 \lambda_3 &= \frac{\lambda_1 + \lambda_2 + \lambda_3}{\mu u_+}, \\
\lambda_1 + \lambda_2 + \lambda_3 &= \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^\gamma}{\mu u_+} + \frac{n_+ u_+^2 - A_2 \alpha n_+^\alpha}{n_+ u_+}, \\
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 &= \rho_+ \frac{(u_+^2 - A_1 \gamma \rho_+^\gamma) (u_+^2 - A_2 \alpha n_+^\alpha)}{\mu u_+^2} - 1 - \frac{n_+}{\mu}.
\end{align*}
\tag{2.16}
$$

If $M_+ > 1$, it is easy to check

$$
\lambda_1 \lambda_2 \lambda_3 < 0,
$$

(2.17)

and

$$
u_+^2 > \min \{ A_1 \gamma \rho_+^\gamma - 1, A_2 \alpha n_+^\alpha - 1 \}.
$$

(2.18)

Moreover, we obtain

$$
\lambda_1 + \lambda_2 + \lambda_3 > 0, \quad \text{for} \quad u_+^2 > \max \{ A_1 \gamma \rho_+^\gamma - 1, A_2 \alpha n_+^\alpha - 1 \}
$$

(2.19)

and get

$$
\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 < 0, \quad \text{for} \quad u_+^2 > \max \{ A_1 \gamma \rho_+^\gamma - 1, A_2 \alpha n_+^\alpha - 1 \}.
$$

(2.20)

Due to (2.17), (2.19), (2.20), we have

$$
\Re \lambda_1 > 0, \Re \lambda_2 > 0, \lambda_3 < 0, \quad \text{for} \quad M_+ > 1.
$$

(2.21)

Using the similar arguments, we have the following results:

\[
\begin{aligned}
\text{if } M_+ > 1, & \text{ then } \Re \lambda_1 > 0, \Re \lambda_2 > 0, \lambda_3 < 0, \\
\text{if } M_+ < 1, & \text{ then } \Re \lambda_1 < 0, \Re \lambda_2 < 0, \lambda_3 > 0, \\
\text{if } M_+ = 1, & \text{ then } \lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0.
\end{aligned}
\tag{2.22}
\]
In order to prove the existence of the solution \((\bar{u}, \bar{w}, \bar{v})\) to the BVP (2.13), we need to diagonalize the system (2.13). Take a linear coordinate transformation
\[
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = P^{-1} \begin{pmatrix} \bar{u} \\ \bar{w} \\ \bar{v} \end{pmatrix}, \tag{2.23}
\]
where \(()^T\) denotes the transpose of a row vector and the invertible matrix \(P\) satisfies
\[
P^{-1} J_{\mp} P = \begin{pmatrix} \lambda_1 & * & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{2.24}
\]
Thus, we have
\[
\frac{d}{dx} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & * & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} g_1(z_1, z_2, z_3) \\ g_2(z_1, z_2, z_3) \\ g_3(z_1, z_2, z_3) \end{pmatrix}, \tag{2.25}
\]
where nonlinear functions \(g_i (i = 1, 2, 3)\) and the boundary condition \((z_1, z_2, z_3)(0) = (z_{1-}, z_{2-}, z_{3-}), \lim_{x \to \infty} (z_1, z_2, z_3) = (0, 0, 0)\),

i) For the supersonic case \(M_+ \gg 1\) which satisfies \(\text{Re}\lambda_1 > 0, \text{Re}\lambda_2 > 0, \lambda_1 > 0\) and \(\text{Re}\lambda_3 < 0\), there exists a one-dimension local stable manifold
\[
W^s(0, 0, 0) = \{ (z_1, z_2, z_3) \mid z_1 = h_1^s(z_3), z_2 = h_2^s(z_3), |z_3| \text{ sufficiently small} \}, \tag{2.27}
\]
where \(h_i^s, i = 1, 2\) are smooth functions and \(h_i^s(0) = 0, \ D h_i^s(0) = 0, \ i = 1, 2\). Therefore, according to manifold theorem, if \((z_{1-}, z_{2-}, z_{3-}) \in W^s(0, 0, 0)\), then there exists a unique solution to problem (2.13) satisfying
\[
|\partial^k (z_1, z_2, z_3)| \leq C\delta e^{-\lambda x} \tag{2.28}
\]
where we used \((z_{1-}, z_{2-}, z_{3-})^T = P^{-1} (\bar{u}_-, \bar{w}_-, \bar{v}_-)^T\) and \(|(\bar{u}_-, \bar{w}_-, \bar{v}_-)| \leq C\delta\).

ii) For the subsonic case \(M_- \ll 1\) which leads to \(\lambda_1 > 0, \text{Re}\lambda_2 < 0, \text{Re}\lambda_3 < 0\), there exists a two-dimension local stable manifold
\[
W^s_2(0, 0, 0) = \{ (z_1, z_2, z_3) \mid z_3 = g^s(z_1, z_2), |(z_1, z_2)| \text{ sufficiently small} \}, \tag{2.29}
\]
where \( g^s \) is a smooth function and \( g^s(0) = 0, \ Dg^s(0) = 0 \). Thus, according to manifold theorem, if \((z_1, z_2, z_3) \in W^s_2\), then there exists a unique solution to problem (2.13) satisfying (2.28).

Finally, we prove case (iii) in Theorem 1.1.

(iii) We consider the sonic case \( M_+ = 1 \) which implies \( \lambda_1 > 0, \lambda_2 < 0, \lambda_3 = 0 \). Moreover, we have

\[
\begin{align*}
\lambda_1 + \lambda_2 &= \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma - 1}}{u_+} + \frac{\rho_+ u_+^2 - A_2 \alpha n^\alpha - 1}{u_+}, \\
\lambda_1 \lambda_2 &= \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma - 1}}{\mu u_+} (u_+^2 - A_2 \alpha n^\alpha - 1) - (1 + \frac{n}{\mu}) < 0.
\end{align*}
\]  

(2.30)

The eigenvectors of \( \lambda_1, \lambda_2, \lambda_3 \) are obtained respectively as follows

\[
\begin{align*}
 r_1 &= \begin{pmatrix} 1 \\ \frac{\lambda_1}{\mu n} (\lambda_1^2 - \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma - 1}}{\mu u_+} \lambda_1 - \frac{n}{\mu}) \end{pmatrix} \\
 r_2 &= \begin{pmatrix} 1 \\ \frac{\lambda_2}{\mu n} (\lambda_2^2 - \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma - 1}}{\mu u_+} \lambda_2 - \frac{n}{\mu}) \end{pmatrix} \\
 r_3 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{align*}
\]  

(2.31)

Hence, we have

\[
P = [r_1, r_2, r_3].
\]  

(2.32)

With the help of manifold theorem [1], there exist a local center manifold \( W^c(0, 0, 0) \) and a local stable manifold \( W^s_3(0, 0, 0) \)

\[
\begin{align*}
W^c(0, 0, 0) &= \{(z_1, z_2, z_3) \mid z_1 = f_1(z_3), z_2 = f_2(z_3), |z_3| \text{ sufficiently small}\}, \\
W^s_3(0, 0, 0) &= \{(z_1, z_2, z_3) \mid z_1 = f_1(z_2), z_3 = f_2(z_2), |z_2| \text{ sufficiently small}\},
\end{align*}
\]  

(2.33)

(2.34)

where \( f_i, f_i^s, i = 1, 2 \) are smooth functions and \( f_i^c(0) = 0, \ Df_i^c(0) = 0, \ f_i^s(0) = 0, \ Df_i^s(0) = 0, \ i = 1, 2 \).

With \( (\bar{u}, \bar{w}, \bar{v})^T = P(z_1, z_2, z_3)^T, (2.15), \) and (2.26), we gain

\[
\bar{g}_3(z^3) = az_3^2 + O(|z_1|^2 + |z_2|^2 + |z_3|^3 + |z_1 z_3| + |z_2 z_3|),
\]  

(2.35)

where

\[
a = \frac{A_1 \gamma (\gamma + 1) \rho_+^{\gamma - 1} + A_2 \alpha (\alpha + 1) n^\alpha}{2u_+^2 (\mu + n)(1 + b^2)} > 0, \quad b = \frac{\rho_+ u_+^2 - A_1 \gamma \rho_+^{\gamma - 1}}{|u_+| \sqrt{(\mu + n)n_+}}
\]  

(2.36)

The system (2.25) can be reformulated as follows

\[
\begin{align*}
 z_{1x} &= \lambda_1 z_1 + \bar{g}(z_1, z_2, z_3), \\
 z_{2x} &= \lambda_2 z_2 + \bar{g}(z_1, z_2, z_3), \\
 z_{3x} &= az_3^2 + O(|z_1|^2 + |z_2|^2 + |z_3|^3 + |z_1 z_3| + |z_2 z_3|).
\end{align*}
\]  

(2.37)

Let \( \sigma_1(x) \) be a solution to (2.37) restricted on the local center manifold satisfying the equation

\[
\sigma_{1x} = a \sigma_1^2 + O(\sigma_1^3), \quad \sigma_1(x) \to 0 \text{ as } x \to +\infty.
\]  

(2.38)
which implies that there exists the monotonically increasing solution $\sigma_1(x) < 0$ to (2.38) for $\sigma_1(0) < 0$ and $|\sigma_1(0)|$ sufficiently small. Therefore, if the initial data $(z_{1-}, z_{2-}, z_{3-})$ belongs to the region $\mathcal{M} \subset \mathbb{R}^3$ associated to the local stable manifold and the local center manifold, then we have

$$
\begin{align*}
  z_i &= O(\sigma_1^2) + O(\delta e^{-cx}), \quad i = 1, 2 \\
  z_3 &= \sigma_1 + O(\delta e^{-cx}),
\end{align*}
$$

(2.39)

with $z_{3-} < 0$, the smallness of $|(z_{1-}, z_{2-}, z_{3-})|$ and

$$
\frac{\delta}{1 + \delta x} \leq |\sigma_1| \leq C \frac{\delta}{1 + \delta x}, \quad |\partial^k \sigma_1| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad C > 0, \quad k = 0, 1, 2, 3, \ldots
$$

(2.40)

It is easy to get

$$
|\partial_x^k (\tilde{\rho} - \rho_+, \tilde{u} - u_+, \tilde{n} - n_+, \tilde{v} - u_+)| \leq C \frac{\delta^{k+1}}{(1 + \delta x)^{k+1}}, \quad C > 0, \quad k = 0, 1, 2, 3, \ldots
$$

(2.41)

$$(\tilde{u} - u_+, \tilde{v} - u_+)_x = (a, a) \sigma_1^2 + O(|\sigma_1|^3)
$$

(2.42)

with the help of (2.23), (2.31) and (2.32).

### 3 Asymptotic stability of stationary solutions

The function space $Y(0, T)$ for $T > 0$ is denoted by

$$
Y(0, T) = \{ (\varphi, \eta, \psi) \mid (\varphi, \eta, \psi) \in C(0, T]; H^1(\mathbb{R}_+) \}
$$

(3.1)

$$(\varphi_x, \eta_x) \in L^2([0, T]; L^2(\mathbb{R}_+)), \quad \psi_x \in L^2([0, T]; H^1(\mathbb{R}_+)) \}
$$

Let

$$
\phi = \rho - \tilde{\rho}, \quad \psi = u - \tilde{u}, \quad \tilde{\phi} = n - \tilde{n}, \quad \tilde{\psi} = v - \tilde{v}.
$$

(3.2)

Then the perturbation $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$ satisfies the following system

$$
\begin{align*}
  \phi_t + u \phi_x + \rho \psi_x &= - (\tilde{u} \phi_x + \phi \tilde{u}_x), \\
  \psi_t + u \psi_x + p'(\rho) \rho \phi_x &= \frac{\mu \psi_{xx}}{\rho} - n(\tilde{\psi} - \psi) = F_1, \\
  \tilde{\phi}_t + v \tilde{\phi}_x + n \tilde{\psi}_x &= - (\tilde{v} \tilde{\phi}_x + \tilde{\phi} \tilde{v}_x), \\
  \tilde{\psi}_t + v \tilde{\psi}_x + p'(n) \frac{\rho}{n} \tilde{\phi}_x &= \frac{(n \tilde{\psi}_x)_x}{n} + (\tilde{\psi} - \psi) = F_2,
\end{align*}
$$

(3.3)

where

$$
\begin{align*}
  F_1 &= - \left[ \mu \left( \frac{1}{\rho} - \frac{1}{\tilde{\rho}} \right) \tilde{u}_{xx} + \tilde{u} \tilde{u}_x + \left( \frac{p'(\rho)}{\rho} - \frac{p'(\tilde{\rho})}{\rho} \right) \tilde{\rho}_x - \left( \frac{n}{\rho} - \frac{\tilde{n}}{\rho} \right) (\tilde{v} - \tilde{u}) \right], \\
  F_2 &= - \left[ \frac{1}{n} \frac{\tilde{n}}{n} \tilde{\psi}_{\tilde{x}} + \tilde{\psi} \tilde{\psi}_x + \left( \frac{p'(n)}{n} - \frac{p'(\tilde{n})}{n} \right) \tilde{n}_x \right].
\end{align*}
$$

(3.4)
The initial and boundary conditions to the system (3.3) satisfy

\[
(\phi, \psi, \tilde{\phi}, \tilde{\psi})(0, x) := (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) = (\rho_0 - \tilde{\rho}, u_0 - \tilde{u}, n_0 - \tilde{n}, v_0 - \tilde{v}),
\]

\[
\lim_{x \to \infty} (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) = (0, 0, 0, 0), \quad (\psi, \tilde{\psi})(t, 0) = (0, 0).
\]

**Proposition 3.1.** Assume that the same assumptions in Theorem 1.2 hold. Let \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\) be the solution to the problem (3.3)-(3.7) satisfying \((\phi, \psi, \tilde{\phi}, \tilde{\psi}) \in Y(0, T)\) for a certain positive constant \(T\). Then there exist positive constants \(\varepsilon\) and \(C\) independent of \(T\) such that if

\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_1 + \delta \leq \varepsilon
\]

is satisfied, it holds for an arbitrary \(t \in [0, T]\) that

\[
\| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_1 + \int_0^t \| (\phi_x, \psi_x, \tilde{\phi}_x, \tilde{\psi}_x) \|_2^2 \, d\tau + \int_0^t \| (\psi - \psi_{xx}, \tilde{\psi}_{xx}) \|_2^2 \, d\tau \leq C \| (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) \|_{1}^2.
\]

With (3.8), it is easy to verify the following Sobolev inequality

\[
|h(x)| \leq \sqrt{2} \|h\|_1^2 \|h_x\|_1^2 \quad \text{for} \quad h(x) \in H^1(\mathbb{R}_+).
\]

**Lemma 3.2** ([5]). For any function \(\psi(\cdot, t) \in H^1(\mathbb{R}_+),\) it holds

\[
\delta \int_0^\infty e^{-c_0 x} |\psi|^2 \, dx \leq C \delta (|\psi(0, t)|^2 + \|\psi_x(t)\|^2),
\]

\[
\int_0^\infty \frac{\delta^j}{(1 + \delta x)^j} |\psi|^2 \, dx \leq C \delta^{-j} (|\psi(0, t)|^2 + \|\psi_x(t)\|^2), \quad \text{for} \ j > 2,
\]

where \(\delta > 0, \ c_0 > 0, \ C > 0\) are positive constants.

With the above Lemma, we can gain the \(L^2\) estimates of \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\).

**Lemma 3.3.** Under the same conditions in Proposition 3.1, then the solution \((\phi, \psi, \tilde{\phi}, \tilde{\psi})\) to the problem (3.3)-(3.7) satisfies for \(t \in [0, T]\)

\[
\| (\phi, \psi, \tilde{\phi}, \tilde{\psi}) \|_2^2 + \int_0^t \| (\psi_x, \tilde{\psi}_x, \tilde{\psi} - \psi) \|_2^2 \, d\tau \leq C \| (\phi_0, \psi_0, \tilde{\phi}_0, \tilde{\psi}_0) \|_2^2 + C (\delta + \varepsilon) \int_0^t \| (\phi_x, \tilde{\phi}_x) \|^2 \, d\tau.
\]

**Proof.** Define

\[
\Phi_1 = \rho \int_{\tilde{\rho}}^{\rho} \frac{p_1(s) - p_1(\tilde{\rho})}{s^2} \, ds, \quad \mathcal{E}_1 = \rho \left( \frac{\psi^2}{2} + \Phi_1 \right),
\]

\[
\Phi_2 = n \int_{\tilde{n}}^{n} \frac{p_2(s) - p_2(\tilde{n})}{s^2} \, ds, \quad \mathcal{E}_2 = n \left( \frac{\tilde{\psi}^2}{2} + \Phi_2 \right).
\]

Then, by (1.1) and (1.6), we gain

\[
(\mathcal{E}_1 + \mathcal{E}_2)_t + G_x + n(\psi - \tilde{\psi})^2 + \mu \psi_x^2 + n \tilde{\psi}_x^2 + n(\tilde{\psi} - \psi)^2 + R_1 + R_2 = -R_3,
\]

10
where
\[ G := [uE_1 + vE_2 + (p'_1(\rho) - p'_1(\bar{\rho}))\psi + (p'_2(n) - p'_2(\bar{n}))\bar{\psi}]_x - [\mu\psi_x + n\bar{\psi}x + \bar{\phi}\bar{\psi}x], \]
\[ R_1 := [\rho\psi^2 + p_1(\rho) - p'_1(\bar{\rho})\phi]\bar{u}_x + [n\bar{\psi}^2 + p_2(n) - p'_2(\bar{n})\bar{\phi}]\bar{v}_x, \]
\[ R_2 := \phi\psi\bar{\mu}\bar{u}_x + \frac{(p_1(\bar{\rho}))_x}{\bar{\rho}} + \phi\bar{\psi}\bar{n}\bar{u}_x + \frac{(p_2(\bar{n}))_x}{\bar{n}}, \]
\[ R_3 := \bar{\phi}(\bar{\psi} - \psi)(\bar{\psi} - \bar{\psi}) + \bar{\phi}\psi_x\bar{u}_x. \]

Integrating (3.14) over \((0, \infty)\), we get
\[ \frac{d}{dt}\int E_1 + E_2 dx + \int n(\psi - \bar{\psi})^2 + \mu\psi_x^2 + n\bar{\psi}^2 dx + \int R_1 dx + \int R_2 dx = - \int R_3 dx, \quad (3.15) \]
where we have used (3.7). With the help of (1.12), (3.8) and Sobolev embedding inequality, we obtain
\[ \int n(\psi - \bar{\psi})^2 + \mu\psi_x^2 + n\bar{\psi}^2 dx \geq C\|(\psi - \bar{\psi}, \psi, \bar{\psi})\|^2 - C(\varepsilon + \delta)\|(\psi - \bar{\psi}, \bar{\psi})\|^2, \quad (3.16) \]
For the case \(M_+ \neq 1\) in Theorem 1.2, with (1.12) and Sobolev embedding inequality, we have
\[ \int_0^\infty |R_1 + R_2 + R_3| dx \leq C\delta\|(\phi_x, \bar{\phi}_x, \psi_x, \bar{\psi}_x, \bar{\psi} - \psi)\|^2. \quad (3.17) \]
For the case \(M_+ = 1\) in Theorem 1.2, with (1.13) and Sobolev embedding inequality, we obtain
\[ \int R_1 + R_2 + R_3 dx \geq \int (\rho_+ \psi^2 + \frac{p''_1(\rho_+)}{2})\phi^2 \frac{u_x^2 - p'_1(\rho_+)}{\rho_+} + \bar{\psi}^2 dx - C(\varepsilon + \delta)\|(\phi_x, \psi_x, \bar{\phi}_x, \psi_x, \bar{\psi} - \psi)\|^2 \]
\[ \geq - C(\varepsilon + \delta)\|(\phi_x, \psi_x, \bar{\phi}_x, \psi_x, \bar{\psi} - \psi)\|^2, \quad (3.18) \]
where we use \(|p'_1(\rho_+) - p'_2(\rho_+)\| \leq \sqrt{2}\rho_+| \min\{(1 + \frac{n_+}{\rho_+})[(\gamma - 1)p'_1(\rho_+)]^{\frac{2}{\gamma}} + (1 + \frac{n_+}{\rho_+})[(\alpha - 1)p'_2(\rho_+)]^{\frac{2}{\gamma}} \)} and take \(\delta\) and \(\varepsilon\) small enough.

Integrating (3.14) over \(\mathbb{R}_+ \times [0, t]\), substituting (3.17) or (3.18) into the resulted equation, we have
\[ \|\phi, \psi, \bar{\phi}, \bar{\psi}\| + \int_0^t \|\psi_x, \bar{\psi}_x, \bar{\psi} - \psi\|^2 d\tau \]
\[ \leq C\|(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2 + C(\varepsilon + \delta)\int_0^\tau \|(\phi_x, \bar{\phi}_x)\|^2 d\tau, \quad (3.19) \]
where we take \(\varepsilon\) and \(\delta\) sufficiently small. Hence, we complete the proof of Lemma 3.3. \(\blacksquare\)

In order to complete the proof of Proposition 3.1, we need to obtain estimates of \((\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x)\).

**Lemma 3.4.** Under the same conditions in Proposition 3.1, then the solution \((\phi, \psi, \bar{\phi}, \bar{\psi})\) to the problem (3.3)-(3.7) satisfies for \(t \in [0, T]\)
\[ \|\phi_x, \bar{\phi}_x\|^2 + \int_0^t \|\phi_x, \bar{\phi}_x\|^2 \leq C\|(\phi_0, \psi_0, \bar{\phi}_0, \bar{\psi}_0)\|^2 + C\int_0^\tau (\varepsilon + \delta)\|(\psi_x, \bar{\psi}_x)\|^2. \quad (3.20) \]
Proof. Differentiating (3.3)1 in \( x \), then multiplying the resulted equation by \( \mu \phi_x \), (3.3)2 by \( \rho^2 \phi_x \) respectively, we gain

\[
(\mu \frac{\phi_x^2}{2})_t + (\mu u \frac{\phi_x^2}{2})_x + \mu \rho \rho_x \psi_x \psi_{xx} = -\mu \left(\frac{3}{2} \phi_x \phi_{x}^2 + (\phi_x \psi_x + \frac{1}{2} \phi_{xx} u_x + \psi_x \rho_x) \phi_x + (\phi u_x + \psi \rho_x) \phi_x \right),
\]

(3.21)

\[
(\rho^2 \phi_x \psi)_t - (\rho^2 \phi_x \psi)_x + 2 \rho \rho_x \phi_t \psi + \rho^2 \phi_t \phi_x + \rho^2 u \phi \psi_x + \rho^2 p'_{1}(\rho) \phi_x^2 - \mu \rho \phi_x \psi_{xx} - \rho^2 n \rho \phi_x \psi_x = \rho \phi_x \psi_x - F_1 \rho^2 \phi_x.
\]

(3.22)

Similarly, differentiating (3.3)3 in \( x \), then multiplying the resultant equation by \( \tilde{\rho}_x \), (3.3)4 by \( \tilde{n} \tilde{\phi}_x \) respectively lead to

\[
(\frac{\tilde{\phi}_x^2}{2})_t + (\frac{\tilde{\phi}_x^2}{2})_x + \tilde{n} \tilde{\phi}_x \tilde{\phi}_{xx} = -\frac{3}{2} \tilde{\phi}_x \tilde{\phi}_x^2 + (\tilde{\phi}_x \psi_x + \frac{1}{2} \tilde{\phi}_x \phi_x + \tilde{\psi}_x \tilde{n}_x) \tilde{\phi}_x - (\tilde{\phi}_x \tilde{\psi}_x + \tilde{\psi}_n \tilde{n}_x) \tilde{\phi}_x,
\]

(3.23)

\[
(\tilde{\phi}_x \psi)_t - (\tilde{\phi}_x \psi)_x + \tilde{n} \tilde{\phi}_x \tilde{\phi}_x \tilde{\psi}_x + \tilde{n} \tilde{\phi}_x \tilde{\psi}_x + \tilde{n} \tilde{\phi}_x \tilde{\phi}_x + \tilde{n} (\tilde{\psi} - \psi) \tilde{\phi}_x = \tilde{n} \tilde{\phi}_x \tilde{\phi}_x \tilde{\psi}_x + \tilde{n} (\tilde{\psi} - \psi) \tilde{\phi}_x
\]

(3.24)

Adding (3.21)-(3.24) together and integrating the resulted equation in \( x \) over \([0, \infty)\) lead to

\[
\frac{d}{dt} \int (\mu \frac{\phi_x^2}{2} + v \frac{\phi_x^2}{2} + \rho^2 \phi_t \psi + \tilde{n} \tilde{\phi}_x) dx + \int (\mu u \frac{\phi_x^2}{2} + v \frac{\phi_x^2}{2} - \rho^2 \phi_t \psi - \tilde{n} \tilde{\phi}_x) dx
\]

\[
+ \int \left( \rho^2 p'_{1}(\rho) \phi_x^2 + \tilde{n} \frac{\tilde{\phi}_x^2}{n} \phi_x \right) dx = \sum_{i=1}^{4} I_i,
\]

(3.25)

where

\[
I_1 = - \int \left[ \rho^2 (\phi_t + u \phi_x) \phi_x + \tilde{n} \tilde{\phi}_x + \tilde{v} \tilde{\phi}_x \right] dx,
\]

\[
I_2 = \int -\mu \phi_x \psi_x \phi_{xx} - \tilde{n} \tilde{\phi}_x \tilde{\phi}_{xx} + \rho^2 \left( \frac{1}{\rho} - \frac{1}{\mu} \right) \phi_x \psi_{xx} + \tilde{n} \frac{\tilde{\phi}_x}{n} \tilde{n} \tilde{\phi}_x \psi_{xx} + \tilde{n} \tilde{\phi}_x \tilde{\phi}_x + \tilde{n} \tilde{\psi}_x \tilde{\phi}_x - \tilde{n} (\tilde{\psi} - \psi) \tilde{\phi}_x dx,
\]

\[
I_3 = \int \frac{3}{2} \frac{\phi_x^2}{2} + \frac{3}{2} \frac{\tilde{\phi}_x^2}{2} dx + \int \frac{\tilde{n} \tilde{\phi}_x}{n} \tilde{\phi}_x dx,
\]

\[
I_4 = \int F_1 \rho^2 \phi_x + F_2 \tilde{n} \tilde{\phi}_x dx,
\]

\[
I_5 = \int \left( \frac{1}{2} \frac{\phi_x^2}{2} + \frac{1}{2} \frac{\tilde{\phi}_x^2}{2} \right) + \frac{n}{n} \tilde{n} \tilde{\phi}_x \psi_x \phi_x + \mu (\phi_x \psi_x + \psi_x \rho_x) \phi_x
\]

\[
- \tilde{n} \tilde{\phi}_x \psi_x \phi_x + \frac{1}{2} \tilde{\phi}_x \tilde{\phi}_x \tilde{\psi}_x + \tilde{n} \tilde{\phi}_x \tilde{\phi}_x \tilde{\psi}_x - (\tilde{\phi}_x \tilde{\psi}_x + \tilde{\psi}_n \tilde{n}_x) \tilde{\phi}_x dx.
\]

We estimate terms in the left side of (3.25). The second term in the left side is estimated as follows

\[
\int (\mu u \frac{\phi_x^2}{2} + v \frac{\phi_x^2}{2} - \rho^2 \phi_t \psi - \tilde{n} \tilde{\phi}_x) dx = -u_- \mu \phi_x^2 (0, t) + \tilde{\phi}_x^2 (0, t)
\]

\[
\leq C ||(\psi, \tilde{\psi})||^2 + \eta ||(\psi_x, \tilde{\psi}_x)||^2,
\]

(3.26)

where we have used \( u_- \phi_x (0, t) + \rho \psi_x (0, t) = 0 \), \( u_- \tilde{\phi}_x (0, t) + n_- \tilde{\psi}_x (0, t) = 0 \) and (3.7). The third term is estimated as follows

\[
\int \rho^2 p'_{1}(\rho) \phi_x^2 + \tilde{n} \frac{\tilde{\phi}_x^2}{n} \phi_x dx
\]

\[
\geq \rho + p'_{1}(\rho) ||\phi_x||^2 + p'_{2}(n_+) ||\tilde{\phi}_x||^2 - C(||\phi_x||_L^\infty + \delta) ||(\phi_x, \tilde{\phi}_x)||^2
\]

\[
\geq \rho + p'_{1}(\rho) ||\phi_x||^2 + p'_{2}(n_+) ||\tilde{\phi}_x||^2 - C(\varepsilon + \delta) ||(\phi_x, \tilde{\phi}_x)||^2.
\]
Then, we turn to estimate terms in the right hand side of (3.25). With Cauchy-Schwartz inequality, Young inequality, (3.3)1, (3.3)3, we obtain

\[ |I_1| \leq C \| (\phi_x, \bar{\psi}_x) \|^2 + C \delta \| (\phi_x, \bar{\psi}_x) \|^2, \] (3.28)
\[ |I_2| \leq C \| (\phi, \bar{\phi}) \|_{L^\infty} \| (\phi_x, \bar{\psi}_x) \|^2 + C \eta \| \bar{\psi} - \psi \|^2 + C \| (\phi_x, \bar{\psi}_x) \|^2 + C \delta \| (\phi_x, \bar{\psi}_x) \|^2 \leq C \| (\phi, \bar{\phi}) \|_{L^\infty} \| (\phi_x, \bar{\psi}_x) \|^2 + C \eta \| \bar{\psi} - \psi \|^2 + C \delta \| \bar{\psi}_x \|^2, \] (3.29)
\[ |I_3| \leq C \| (\psi_x, \bar{\psi}_x) \|_{L^\infty} \| (\phi_x, \bar{\phi}_x) \|^2 + C \| \bar{\phi}_x \|_{L^\infty} \| (\phi_x, \bar{\psi}_x) \|^2 \leq C \| (\psi_x, \bar{\psi}_x) \|_{L^\infty} \| (\phi_x, \bar{\phi}_x) \|^2 + \varepsilon \| (\phi_x, \bar{\psi}_x) \|^2 \] (3.30)
\[ |I_4 + I_5| \leq C \delta \| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x) \|^2. \] (3.31)

Finally, the substitution of (3.26)-(3.31) into (3.25) leads to

\[ \frac{d}{dt} \int (\phi_x^2 + \bar{\phi}_x^2 + \bar{n} \phi_x \bar{n} \bar{\psi}_x) \, dx + \| (\phi_x, \bar{\phi}_x) \|^2 \leq C \| (\psi_x, \bar{\psi}_x) \|^2 + C \varepsilon \| (\psi_x, \bar{\psi}_x) \|^2 \] (3.32)

where we let \( \delta, \varepsilon \) and \( \eta \) small enough. Integrating (3.32) over \([0, t]\), we obtain

\[ \| (\phi_x, \bar{\phi}_x) \|^2 + \int_0^t \| (\phi_x, \bar{\phi}_x) \|^2 \, dt \leq C \| (\phi_0, \bar{\phi}_0, \psi_0, \bar{\psi}_0, \phi_0x, \bar{\phi}_0x) \|^2 + C \varepsilon \int_0^t \| (\psi_x, \bar{\psi}_x) \|^2, \] (3.33)

where we have used (3.13) and Cauchy-Schwartz inequality. Hence, we complete the proof of (3.4). \( \square \)

**Lemma 3.5.** Assume that the same conditions in Proposition 3.1 hold, then the solution \( (\phi, \psi, \bar{\phi}, \bar{\psi}) \) to the problem (3.3)-(3.7) satisfies for \( t \in [0, T] \)

\[ \| (\psi_x, \bar{\psi}_x) \|^2 + \int_0^t \| (\psi_x, \bar{\psi}_x) \|^2 \, dt \leq C \| (\phi_0, \psi_0, \phi_0x, \psi_0x, \bar{\phi}_0, \bar{\psi}_0, \phi_0x, \bar{\phi}_0x, \bar{\psi}_0x) \|^2. \] (3.34)

**Proof.** Multiplying (3.3)2 by \(-\psi_{xx}\), (3.3)4 by \(-\bar{\psi}_{xx}\) respectively, then adding them together and integrating the resulted equation in \( x \) over \( \mathbb{R}_+ \) to gain

\[ \frac{d}{dt} \int \frac{\psi_x^2}{2} + \frac{\bar{\psi}_x^2}{2} \, dx + \int \mu \frac{1}{\rho} \psi_{xx}^2 + \bar{\psi}_{xx}^2 \, dx = - \sum_{i=1}^3 K_i, \] (3.35)

where

\[ K_1 = - \int \left[ \psi_x \psi_{xx} + \bar{n} \bar{\psi}_x \bar{n} \bar{\psi}_{xx} + \frac{n}{\rho} (\bar{\psi} - \psi) \psi_{xx} - \frac{p'_1(\rho)}{\rho} \phi_x \psi_{xx} - \frac{n}{\rho} (\bar{\psi} - \psi) \bar{\psi}_{xx} - \frac{p'_2(n)}{n} \bar{\phi}_x \bar{\psi}_{xx} \right] \, dx, \]
\[ K_2 = \int \left[ -\bar{n} \psi \psi_{xx} + \mu \bar{n} \bar{\psi}_{xx} - \frac{1}{\rho} \psi_{xx} - \frac{p'_1(\rho)}{\rho} \phi_x \psi_{xx} \bar{n} \bar{\psi}_x + \frac{n}{\rho} (\bar{\psi} - \psi) \bar{\psi}_{xx} \bar{n} \bar{\psi}_x \right] \, dx, \]
\[ + \left( \bar{n} \bar{\psi}_x \right) \frac{1}{n} \bar{n} \bar{\psi}_{xx} - \frac{p'_2(n)}{\rho} \psi_x \bar{n} \bar{\psi}_x + \frac{n}{\rho} (\bar{\psi} - \psi) \bar{\psi}_{xx} \bar{n} \bar{\psi}_x + \frac{n}{\rho} \bar{n} \bar{\psi}_x \bar{n} \bar{\psi}_{xx} + \bar{n} \bar{\psi}_x \bar{n} \psi \bar{n} \bar{\psi}_x \right] \, dx, \]
\[ K_3 = - \int \left[ \frac{(\bar{\phi}_x \psi)}{n} \bar{n} \bar{\psi}_{xx} + \frac{1}{n} \bar{n} \bar{\psi}_x \bar{n} \psi \bar{n} \bar{\psi}_{xx} \right] \, dx. \]
First, we estimate terms in the left side of (3.35). With \( \frac{1}{\rho} = \left( \frac{1}{\rho_x} - \frac{1}{\rho_y} \right) + \frac{1}{\rho_0} \), the second term is estimated as follows:

\[
\int \frac{\mu}{\rho} \psi_{xx}^2 + \psi_{xx}^2 dx \geq \frac{\mu}{\rho^2} \| \psi_{xx} \|^2 + \frac{1}{16} \| \bar{\psi}_{xx} \|^2 + C\| \phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi \|^2 \tag{3.36}
\]

Then, we turn to estimate each term in the right side of (3.35). Using (1.12), Sobolev inequality and Cauchy-Schwarz inequality, we have

\[
|K_1| \leq \frac{\mu}{16\rho^+} \| \psi_{xx} \|^2 + \frac{1}{16} \| \bar{\psi}_{xx} \|^2 + C\| \phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi \|^2, \tag{3.37}
\]

\[
|K_2| \leq C\| \phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x \|^2 + C\| \psi_{xx} \bar{\psi}_{xx} \|^2, \tag{3.38}
\]

\[
|K_3| \leq C\| \phi_x \| L^\infty \| (\bar{\psi}_x, \bar{\psi}_{xx}) \|^2 + C\| \bar{\psi}_x \| L^\infty \| \bar{\psi}_{xx} \|^2 \| \bar{\phi}_x \|
\]

\[
\leq C\| (\bar{\psi}_x, \bar{\psi}_{xx}) \|^2 + C\| \bar{\psi}_x \| + \| \bar{\psi}_{xx} \| \| \bar{\phi}_x \|
\]

\[
\leq C\| (\bar{\psi}_x, \bar{\psi}_{xx}) \|^2. \tag{3.39}
\]

With \( \delta \) and \( \varepsilon \) small enough and the substitution of (3.36)-(3.39) into (3.35), we obtain

\[
\frac{d}{dt} \int \psi_x^2 + \psi_{xx}^2 dx + \frac{\mu}{2\rho^+} \| \psi_{xx} \|^2 + \frac{1}{2} \| \bar{\psi}_{xx} \|^2 \leq C|| (\phi_x, \psi_x, \bar{\phi}_x, \bar{\psi}_x, \bar{\psi} - \psi) ||^2 \tag{3.40}
\]

Integrating (3.40) in \( \tau \) over \([0, t]\), we gain

\[
\| (\psi_x, \bar{\psi}_x) \|^2 + \int_0^t \| (\psi_{xx}, \bar{\psi}_{xx}) \|^2 d\tau \leq C\| (\phi_0, \psi_0, \phi_{0x}, \psi_{0x}, \bar{\phi}_0, \bar{\psi}_0, \bar{\psi}_{0x}, \bar{\psi}_{0x}) \|^2, \tag{3.41}
\]

where we use (3.13), (3.20) and the smallness of \( \varepsilon \). Then we obtain the desired estimate (3.40) and complete the proof of Lemma 3.5. \( \square \)

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