GREEN’S CONJECTURE FOR THE GENERIC CANONICAL CURVE.

MONTSERRAT TEIXIDOR I BIGAS

INTRODUCTION

Let \( C \) be a non-singular curve of genus \( g \) over an algebraically closed field \( k \) of characteristic zero. Let \( K \) be the canonical sheaf on \( C \). If \( C \) is not hyperelliptic, the map associated to the complete canonical series \(|K|\)

\[ C \to \mathbb{P}^{g-1} \]

is an embedding and the image curve is projectively normal. If the curve is neither trigonal nor a plane quintic, the ideal of \( C \) is known to be generated by quadrics. Continuing in this vein, Mark Green made the conjecture that the resolution of the ideal of \( C \) in \( \mathbb{P}^{g-1} \) should depend on the linear series that \( C \) has.

To make this precise, one defines property \( N_p \). Take a minimal resolution of the ideal sheaf of \( C \) in \( \mathbb{P}^{g-1} \). Then one says that property \( N_0 \) holds if \( C \) is projectively normal, \( N_1 \) means that the ideal of the curve is generated by quadrics, \( N_2 \) means that in addition the syzygies among these quadrics are generated by linear relations...In general \( N_p \) means that \( N_{p-1} \) holds and the \( p^{th} \) syzygies are generated by linear relations.

Define the Clifford index of \( C \) by

\[
\text{Cliff}(C) = \min\{\text{deg}L - 2(h^0(C, L) - 1) | L \in \text{Pic}C, h^0(C, L) \geq 2, h^1(C, L) \geq 2\}
\]

In particular, for a curve that is generic in the sense of moduli, the Clifford index of \( C \) is given by \([ (g - 1)/2 \] while the most special curves from the point of view of Clifford index are hyperelliptic curves that have \( \text{Cliff}(C) = 0 \). Then

Conjecture 0.1. [Green](cf. [G] 5.1) The curve \( C \) has property \( N_p \) if and only if \( \text{Cliff}(C) > p \).

The only if part of \([0.1]\) was proved by Green and Lazarsfeld (cf. [G] Appendix), The conjecture has been proved for \( g \leq 8 \) (cf [S1]) and for \( p = 2 \) (i.e. \( C \) quatrignal) (cf [V, S2, H, P, R]).

In a slightly more modest vein

Conjecture 0.2. [Generic Green’s conjecture] (cf. [G] 5.6) The generic curve \( C \) of genus \( g \) satisfies \( N_{[(g-3)/2]} \).
Few effective results seem to be available with respect to 0.2. The only published work that we are aware of is [E] where L. Ein showed that the first two steps of the resolution of a generic canonical curve are as expected. The relevance of 0.2 is enhanced by the following result of Hirschowitz and Ramanan (cf. [H,R] Theorem 1.1).

**Theorem 0.3** (Hirschowitz-Ramanan). For odd \( g = 2k+1 \geq 5 \), Green’s Conjecture holds for the generic curve if and only if it holds for all curves of (maximal) Clifford index \( k \).

The purpose of this paper is to prove the generic Green’s Conjecture.

**Theorem 0.4.** Let \( C \) be a generic curve of genus \( g \). Then, Green’s conjecture on the syzygies of the canonical curve \( C \) holds.

As a consequence of Hirschowitz and Ramanan’s Theorem, one obtains a very concrete open dense subset of \( \mathcal{M}_g \) for odd \( g \) where Green’s Conjecture actually holds. In the language of Hirschowitz and Ramanan, this is the specific Green’s Conjecture

**Corollary 0.5.** If \( g = 2k + 1 \geq 5 \), then Green’s Conjecture holds for all curves of maximal Clifford index \( k \) (i.e. \( C \) satisfies \( N_{k-1} \) if and only if \( \text{Cliff}(C) = k \)).

The proof of 0.4 is as follows: Define a vector bundle \( E \) as the dual of the kernel of the evaluation map of the canonical linear series. Namely, \( E^* \) is defined by the exact sequence

**Definition 0.6.**

\[
0 \to E^* \to (H^0(C, K_C))^* \otimes \mathcal{O}_C \to K \to 0.
\]

From a result of Paranjape-Ramanan (cf. [P,R] Remark 2.8, p.507), Green’s Conjecture would follow from the surjectivity of the maps

\[
\wedge^r (H^0(C, K_C))^* \to H^0(\wedge^r E), r \leq \text{Cliff}(C).
\]

We want to prove that this is the case for \( C \) generic. To this end consider a hyperelliptic curve \( C_0 \). Notice that the map

\[
(H^0(C, K_{C_0}))^* \otimes \mathcal{O}_{C_0} \to E \to 0
\]

identifies \( (H^0(C, K_{C_0}))^* \) to a subspace \( W \) of \( H^0(C_0, E) \). As \( C_0 \) is hyperelliptic, this subspace is proper. We shall start by computing \( W \) and the image \( W^r \subset H^0(C_0, \wedge^r E) \) of its exterior powers \( \wedge W \). Every infinitesimal deformation of the curve \( C_0 \) determines a unique infinitesimal deformation of \( E \) that preserves it as the dual of the kernel of the canonical evaluation map. Consider the deformation of \( \wedge^r E \) that this induces. We then see that the only sections of \( H^0(\wedge^r E) \) that give rise to sections of the infinitesimal deformation of \( \wedge^r E \) are those in \( W^r \). This will conclude the proof.
Acknowledgments: I would like to thank the following people and institutions that contributed in different ways to the present work. My interest on Green’s Conjecture came from a conversation with S. Ramanan. Mark Green read a first version of the paper and made several suggestions to improve the presentation. I am visiting the Pure Mathematics Department at the University of Cambridge and I benefitted from conversations with Tony Wasserman and Nick Shepherd-Barrow as well as e-mails with Loring Tu. I am a member of the Europroj group Vector Bundles on Algebraic Curves.

1. Identification of the vector bundle $E$ and its space of sections

**Notations** In this section and section 3, $C = C_0$ denotes a hyperelliptic curve and $L$ the unique line bundle of degree two with two sections.

If $F$ is a vector bundle on a curve $C$, we write $H^i(F)$ for $H^i(C, F)$ if there is no danger of confusion. If several curves are involved or if we are considering sections on an open set only, we shall make this clear.

As in the previous section $E$ will denote the vector bundle defined in 0.6, $W = W^1$ will be the image of $H^0(K)^* \subset H^0(E)$ and $W^r$ the image of the exterior powers of $W$ in $H^0(\wedge^r E)$.

**Proposition 1.1.** Let $\pi$ be the map $C_0 \to \mathbb{P}^1$ associated to $L$. Let $H$ be the space of sections of $L$. Then,

$$E \cong S^{g-2} H^* \otimes \wedge^2 H^* \otimes L$$

Moreover, the map $(H^0(K))^* \to H^0(E)$ can be identified to the natural inclusion

$$\varphi_1 : S^{g-1} H^* \to S^{g-2} H^* \otimes \wedge^2 H^* \otimes H$$

**Proof.** Note that the canonical sheaf $K_C$ on $C$ is of the form

$$K = L^{g-1} = \pi^*(\mathcal{O}_{\mathbb{P}^1}(g-1))$$

and

$$H^0(C, K_C) = S^{g-1}(H) = \pi^*(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1)))$$

Therefore, the exact sequence defining $E^*$ is the pull-back of the exact sequence in $\mathbb{P}^1$

$$0 \to \text{Ker} \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1)) \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(g-1) \to 0 \quad (*)$$

Denote by $\tilde{H}$ the space of sections $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Tensoring $(*)$ with $\mathcal{O}(1)$ and taking global sections, one obtains

$$0 \to H^0(\text{Ker} \otimes \mathcal{O}(1)) \to S^{g-1} \tilde{H} \otimes \tilde{H} \to S^g \tilde{H} \to 0$$

Hence, $H^0(\text{Ker} \otimes \mathcal{O}(1)) \equiv S^{g-2} \tilde{H} \otimes \wedge^2 \tilde{H}$ (cf. for example [F, H] p.224 15.20). Using the exact sequence $(*)$, one checks that $\text{Ker}$ is a direct sum of line bundles of degree $-1$. Hence $\text{Ker} = S^{g-2} \tilde{H} \otimes \wedge^2 \tilde{H} \otimes \mathcal{O}(-1)$. As $E^* = \pi^*(\text{Ker})$, $H = \pi^* \tilde{H}$, the result follows.

$\square$
Proposition 1.2. Let the notations be as above. Choose \( r \leq g - 1 \) Then,
\[
\wedge^r E = \wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*) \otimes L^\otimes r
\]
Moreover, the map
\[
\wedge^r H^0(E) \to H^0(\wedge^r E)
\]
can be identified to
\[
\varphi_r : \wedge^r (S^{g-1} H^*) \to \wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*) \otimes S^r H
\]
and is an immersion. In particular, its image \( W^r \) has dimension \( \binom{g}{r} \).

Proof. We can give a coordinate description of the map 
\[
\varphi_1 : S^{g-1} H^* \to S^{g-2} H^* \otimes \wedge^2 H^* \otimes H
\]
as follows. For any positive integer \( k \), identify \( S^k H \) with the space of polynomials of degree at most \( k \) in a variable \( x \) and \( S^k H^* \) with the space of polynomials of degree at most \( k \) in one variable \( y = x^* \). Identify \( \wedge^2 H^* \to k \) by the isomorphism \( y \wedge 1 \to 1 \).

Then, the map \( \varphi_1 \) acts by
\[
\varphi_1(y^k) = y^k \otimes x - y^{k-1} \otimes 1
\]
with the convention that on the right hand side \( y^{g-1} = 0, y^{-1} = 0 \). By taking the wedge products of this map, one then sees that
\[
\varphi_r(y^{k_1} \wedge ... \wedge y^{k_r}) = \sum_{0 \leq \epsilon_1 \leq 1} (-1)^{\epsilon_1 + ... + \epsilon_r} y^{k_1 - \epsilon_1} \wedge ... \wedge y^{k_r - \epsilon_r} \otimes x^{r-(\epsilon_1 + ... + \epsilon_r)}
\]
again with the convention that on the right hand side \( y^{g-1} = 0, y^{-1} = 0 \).

A basis of \( \wedge^r (S^{g-1} H^*) \) consists of the elements \( y^{k_1} \wedge ... \wedge y^{k_r} \), \( 0 \leq k_1 < ... < k_r \leq g - 1 \). The images of these elements are linearly independent as their leading terms (i.e. the term of highest degrees on \( x \) and \( y \) jointly) obviously are. Therefore, the dimension of \( W^r \) is as stated.

2. General set up for infinitessimal deformations

Recall 2.1. We recall the basic set up for deformations of a curve, a vector bundle and its space of sections.

Write \( k[t]/t^2 = k_\epsilon \). By an infinitessimal deformation of the curve \( C \) we mean a curve \( C_\epsilon \) over \( Spec k_\epsilon \) with central fiber \( C \). Similarly, by an infinitessimal deformation of a vector bundle \( F \) we mean a vector bundle over \( C \times Spec k_\epsilon \) with central fiber \( F \). By an infinitessimal deformation of the pair, we mean a curve \( C_\epsilon \) and a vector bundle \( E_\epsilon \) over \( C_\epsilon \).

Recall that the set of infinitessimal deformations of the curve \( C \) can be parametrised by \( H^1(C, T_C) \), the set of infinitessimal deformations of the vector bundle \( F \) can be parametrised by \( H^1(F^* \otimes F) \) while the set of infinitessimal deformations of a pair
consisting of a curve \( C \) and a vector bundle \( F \) on \( C \) can be parametrised by \( H^1(\Sigma_F) \)
where \( \Sigma_F \) denotes the sheaf of first order differential operators acting on \( F \).

We describe next the correspondence between these objects (cf. \[W\] proof of Prop. 1.2 and also [B.R] proof of 2.3). Assume given an element \( \nu \in H^1(T_C) \). We think of the sections of the sheaf \( T_C \) over an open set \( U \) as the set of \((k\text{-linear})\)-maps \( \mathcal{O}_U \to \mathcal{O}_U \) satisfying \( \nu(fg) = \nu(f)g + f\nu(g) \). Take an affine open cover \( C = \bigcup U_i \).
Write \( U_{ij} \) for \( U_i \cap U_j \). Represent \( \nu \) by a cocycle \( \nu = (\nu_{ij}) \), \( \nu_{ij} \in H^0(U_{ij}, T_C) \). We associate to \( \nu \) the following deformation of \( C \): Consider the trivial deformations of the \( U_i \), namely \( U_i \times \text{Spec} \, k \). Glue them along the intersections \( U_{ij} \times \text{Spec} \, k \) using the matrices

\[
\begin{pmatrix}
Id & 0 \\
\nu_{ij} & Id
\end{pmatrix}.
\]

The correspondence \( \nu_{ij} \to \mathcal{C} \), obtained in this way is a bijection.

Assume now given an element \( \varphi \in H^1(F^* \otimes F) \). Represent it by a cocycle \( \varphi_{ij} \) with \( \varphi_{ij} \in H^0(U_{ij}, \text{Hom}(F, F)) \). Consider the trivial extension of \( F \) to \( U_i \times \text{Spec} \, k \), namely \( F_{U_i} \otimes \epsilon F_{U_i} \). Take gluings on \( U_{ij} \) given by

\[
\begin{pmatrix}
Id & 0 \\
\varphi_{ij} & Id
\end{pmatrix}.
\]

This gives the correspondence between \( H^1(F^* \otimes F) \) and deformations of \( F \).

Assume now that a section \( s \) of \( F \) can be extended to a section \( s_i \) of the deformation. There exist then local sections \( s_i' \in H^0(U_i, F_{U_i}) \) such that \( (s_{U_i}, s_i) \) define a section of \( F_i \). By construction of \( F_i \) this means that

\[
\begin{pmatrix}
Id & 0 \\
\varphi_{ij} & Id
\end{pmatrix}
\begin{pmatrix}
s_{U_i} \\
s_i'
\end{pmatrix} = \begin{pmatrix}
s_{U_j} \\
s_j'
\end{pmatrix}.
\]

This can be written as \( \varphi_{ij}(s) = s_j' - s_i' \). Equivalently,

\[\varphi_{ij} \in \text{Ker}(H^1(F^* \otimes F) \to H^1(F)) \]

\[\nu_{ij} \to \nu_{ij}(s) \]

This result can be formulated using the language of Brill-Noether Theory: the set of infinitesimal deformations of the vector bundle \( F \) that have sections deforming a certain subspace \( V \subset H^0(F) \) consists of the orthogonal to the image of the Petri map

\[(2.1.1) \quad P_V : V \otimes H^0(K \otimes F^*) \to H^0(K \otimes F \otimes F^*).\]

Assume now given an element \( \sigma \in H^1(\Sigma_F) \). We think of \( \Sigma_F(U) \) as the set of additive morphisms \( \sigma : F(U) \to F(U) \) such that for a suitable element \( \nu_{\sigma} \in T_C \), \( \sigma(fs) = \nu_{\sigma}(f)s + f\sigma(s) \). Represent \( \sigma \) by a cocycle \( \sigma = (\sigma_{ij}) \), \( \sigma_{ij} \in H^1(U_{ij}, \Sigma_F) \). Consider the associated element \( (\nu_{ij}) \in H^1(T_C) \) and the corresponding deformation
$\mathcal{C}_e$ of $C$. Take then the vector bundle on $\mathcal{C}_e$ obtained by gluing the trivial extensions of $F$ on $U_i$ by means of the matrices

$$\begin{pmatrix}
Id & 0 \\
\sigma_{ij} & Id
\end{pmatrix}. $$

As in the case of deforming the line bundle alone, deformation of sections is easy to interpret: the set of infinitesimal deformations of the pair $(C, F)$ that have sections deforming a certain subspace $V \subset H^0(F)$ consists of the orthogonal to the image of the Petri map

$$\bar{P}_V : V \otimes H^0(K \otimes F^*) \rightarrow H^0(K \otimes \Sigma_F^*)$$

defined as the dual of the natural cup-product map

$$H^1(\Sigma_F) \rightarrow Hom(V, H^1(F)).$$

Consider the exact sequence

$$0 \rightarrow F^* \otimes F \rightarrow \Sigma_F \rightarrow T_C \rightarrow 0.$$

The canonical map $\pi : \Sigma_F \rightarrow T_C$ is defined by $\pi(\sigma) = \nu_\sigma$. The map $i : F^* \otimes F \rightarrow \Sigma_F$ sends an element of $F^* \otimes F$ (considered as an endomorphism of $F$) to itself. One obtains a commutative diagram

$$\begin{array}{cccc}
0 & \rightarrow & H^1(F^* \otimes F) & \rightarrow & H^1(\Sigma_F) & \rightarrow & H^1(T_C) & \rightarrow & 0 \\
\downarrow & & \downarrow \bar{P}_V & & \downarrow \bar{P}_V & & \downarrow \bar{P}_V^* & & \\
0 & \rightarrow & ImP^* & \rightarrow & Hom(V, H^1(F)) & \rightarrow & (KerP)^* & \rightarrow & 0
\end{array}$$

and its dual (cf. [A,C]p.18)

$$\begin{array}{cccc}
0 & \rightarrow & H^0(K \otimes F \otimes F^*) & \leftarrow & H^0(K \otimes (\Sigma_F)^*) & \leftarrow & H^0(2K) & \leftarrow & 0 \\
\uparrow & & \uparrow \bar{P}_V & & \uparrow \bar{P}_V & & \uparrow \bar{P}_V^* & & \\
0 & \leftarrow & ImP & \leftarrow & V \otimes H^0(K \otimes F^*) & \leftarrow & KerP & \leftarrow & 0
\end{array}$$

When $V = H^0(F)$, we shall write $P_F$ instead of $P_V$. When $V$ and $F$ are clear, we shall suppress them from the notations.

We shall later use the following result. Its proof appears in [I] Lemma 2.12.

**Lemma 2.2.** Let $M$ be a line bundle on a curve $C$ with two independent sections $s_0, s_1$ and such that $|K \otimes M^{-2}|$ has a section $t$. Denote by $D_1$ the fixed part of the series determined by $s_0, s_1$, denote by $D_2$ the divisor corresponding to the section $t$. Denote by $R$ the ramification divisor of the map $C \rightarrow \mathbf{P}^1$ associated to the series $<s_0, s_1>$. Then, $P_M(s_0 \otimes ts_1 - s_1 \otimes ts_0)$ corresponds to the divisor $2D_1 + D_2 + R$. In particular it is non-zero.
Lemma 2.3. Assume that \( F = \bigoplus_{i=1}^{n} F_i \) is a direct sum of vector bundles. Then
\[
\Sigma F = \bigoplus_{T_C} \Sigma F_i \oplus \bigoplus_{i \neq j} F_i^* \otimes F_j^*
\]
Here \( \bigoplus_{T_C} \Sigma F_i \) denotes the fibered product over \( T_C \) of the \( \Sigma F_i \).

Proof. Consider an open set \( U \). Let \( \sigma : F(U) \to F(U) \) be a first order differential operator acting on \( F \). Using the decomposition of \( F \) as a direct sum, \( \sigma \) admits a representation as a matrix \( (\sigma_{ij}) \) where \( \sigma_{ij} : F_i \to F_j \). Take a local section \( s_k \in F_k(U) \).

One then checks that
\[
\sigma((0...0,f s_k,0...0)) = (\sigma_{1k}(f s_k),...,\sigma_{nk}(f s_k))
\]
Using that \( \sigma((0...0,f s_k,0...0)) = f\sigma((0...0,s_k,0...0)) + \nu(f)(0...0,s_k,0...0) \), we find that \( \sigma_{kk} \) is a first order differential operator corresponding to the same \( \nu \) as \( \sigma \) while \( \sigma_{ij} \) is \( \mathcal{O}_C \)-linear if \( i \neq j \).

Lemma 2.4. Assume that \( F = \bigoplus_{i=1}^{n} F_i \) is a direct sum of vector bundles. Denote by \( P_i, \bar{P}_i, P'_i \) the Petri maps corresponding to the \( F_i \). Then \( P'_F \) can be obtained as the composition
\[
\text{Ker} P \to \bigoplus_{i=1}^{n} \text{Ker} P_i \to H^0(2K)
\]
where the first map is the projection and the second map is \( \frac{1}{n} \bigoplus P_i' \)

Proof. Consider the right hand square in 2.1.3 for each one of the \( F_i \). As in 2.3, consider the fibered product of the \( \Sigma F_i \) over \( T_C \). One then has a commutative square
\[
\bigoplus_{i=1}^{n} H^1(T_C) \to H^1(\Sigma F_i) \downarrow \bigoplus_{i=1}^{n} P_i^* \downarrow \bigoplus_{i=1}^{n} \text{Hom}(H^0(F_i),H^1(F_i)) \to \bigoplus_{i=1}^{n} \text{Hom}(H^0(F_i),H^1(F_i)) \to \bigoplus_{i=1}^{n} \text{coker} P_i^*
\]
Take also the corresponding square for \( F \)
\[
H^1(\Sigma F) \downarrow \bar{P}_* \downarrow H^1(T_C) \to H^0(F), H^1(F) \to \text{coker} P^*
\]
Consider the cube that has these diagrams as back and front faces respectively. We define four maps in the side edges. The maps
\[
\bigoplus_{i=1}^{n} H^1(T_C) \to H^1(\Sigma F_i) \to H^1(\Sigma F)
\]
and
\[
\bigoplus_{i=1}^{n} \text{Hom}(H^0(F_i),H^1(F_i)) \to \text{Hom}(H^0(F),H^1(F))
\]
are natural diagonal injections (with zeroes on the terms corresponding to a pair \( F_i, F_j, i \neq j \)). Notice that from 2.3, the first map is well defined. With these definitions, the left hand square commutes.

The map
\[
\bigoplus_{i=1}^{n} (\text{Ker} P_i)^* \to (\text{Ker} P)^*
\]
is defined as the dual of the natural projection. By dualisation, one can then check that the bottom face commutes.

If we take as the fourth map the homotethy
\[ \times n : H^1(T_C) \to H^1(T_C) \]
then, the top face commutes too. As \( H^1(\Sigma_F) \to H^1(T_C) \) is onto, this shows that the right hand square commutes. Dualising this square, one obtains the result in the Lemma.

3. Deformations of \( E \)

We apply the set up of the previous section to the hyperelliptic curve \( C_0 \) and the vector bundles \( \wedge^r E \). As in section 1, \( L \) denotes the hyperelliptic line bundle (i.e. the line bundle on \( C \) of degree two with two sections). We shall assume in all that follows that \( r \leq g - 1 - r \)

**Proposition 3.1.** The Petri map \( P_{H^0(\wedge^r E)} \) (cf. (2.1.2)) associated to the vector bundle \( \wedge^r (E) \) gives by restriction an isomorphism
\[
P_{W^r} : W^r \otimes H^0(K \otimes (\wedge^r E)^*) \to H^0(K \otimes \wedge^r E \otimes (\wedge^r E)^*)
\]

**Proof.** From \( \ref{L2} \)
\[
\wedge^r E = \wedge^r (S^{g-2}H^* \otimes \wedge^2 H^* ) \otimes L^r
\]
so one has
\[
\wedge^r E^* = \wedge^r (S^{g-2}H \otimes \wedge^2 H) \otimes L^{-r}
\]
Hence,
\[
H^0(\wedge^r E) = \wedge^r (S^{g-2}H^* \otimes \wedge^2 H^*) \otimes S^r H.
\]
and
\[
H^0(K \otimes \wedge^r E^*) = \wedge^r (S^{g-2}H \otimes \wedge^2 H) \otimes S^{g-1-r} H.
\]
Then, the Petri map \( P_{H^0(E)} \) and its restriction \( P_{W^r} \) to \( W^r \otimes H^0(K \otimes (\wedge^r E)^*) \) can be written as the tensor product with the vector space \( \wedge^r (S^{g-2}H \otimes \wedge^2 H) \) of the diagram
\[
\begin{array}{c}
\wedge^r (S^{g-2}H^* \otimes \wedge^2 H^*) \otimes S^r H \otimes S^{g-1-r} H \\
\uparrow \\
W^r \otimes S^{g-1-r} H \\
\uparrow \\
\wedge^r (S^{g-2}H^* \otimes \wedge^2 H^*) \otimes S^{g-1} H
\end{array}
\]
Therefore, it is enough to show that the map \( p_{W^r} \) in the lower row of this diagram is an isomorphism. Recall that \( \varphi_r \) identifies \( \wedge^r S^{g-1} H^* \) to its image \( W^r \) (where \( \varphi_r \) is the natural immersion \( \wedge^r S^{g-1} H^* \to \wedge^r (S^{g-2}H^* \otimes \wedge^2 H^*) \otimes S^r \) defined in \( \ref{L2} \)). The natural product map \( S^r H \otimes S^{g-1-r} H \to S^{g-1} H \) can be identified to \( P_{L^r} \). Then, \( p_{W^r} = (Id \otimes P_{L^r})o(\varphi_r \otimes Id) \). We shall show that \( p_{W^r} \) is an isomorphism by exhibiting
its inverse \( q_{W^r} \). Using the notations in [1,2], and the identification \( \wedge^2 H^* \cong k \), one obtains

\[
p_{W^r}(y^{k_1} \wedge ... \wedge y^{k_r} \otimes x^a) = (Id \otimes P_{L^r})o(\varphi_r \otimes Id)(y^{k_1} \wedge ... \wedge y^{k_r} \otimes x^a) =
\]

\[
= Id \otimes P_{L^r} \left( \sum_{0 \leq \epsilon_i \leq 1} (-1)^{\epsilon_1 + ... + \epsilon_r} y^{k_1 - \epsilon_1} \wedge ... \wedge y^{k_r - \epsilon_r} \otimes x^{r - (\epsilon_1 + ... + \epsilon_r)} \otimes x^a \right) =
\]

\[
= \sum_{0 \leq \epsilon_i \leq 1} (-1)^{\epsilon_1 + ... + \epsilon_r} y^{k_1 - \epsilon_1} \wedge ... \wedge y^{k_r - \epsilon_r} \otimes x^{a + r - (\epsilon_1 + ... + \epsilon_r)}
\]

with the convention that on the right hand side \( y^{-1} = 0 \), \( y^{g-1} = 0 \). We describe \( q_{W^r} \) as follows. Assume given integers \( 0 \leq j_1 < ... < j_r \leq g - 2, 0 \leq b \leq g - 1 \). There is then a value \( l \) with \( 0 \leq l \leq r \) such that \( j_1 + 1 \leq b \leq j_{i+1} \). Define then

\[
q_{W^r}(y_{j_1} \wedge ... \wedge y_{j_r} \otimes x^b) = \sum_{0 \leq t_i \leq j_i - j_{i-1}, 1 \leq s_i \leq j_{i+1} - j_i} (-1)^{r - l} y_{j_1 - t_1} \wedge ... \wedge y_{j_i - t_i} \wedge y_{j_{i+1} + s_i} \wedge ... \wedge y_{j_r + s_r} \otimes x^{b - r - t_1 - ... - t_i + s_i + ... + s_r}
\]

with the conventions \( j_{r+1} = g - 1, j_{-1} = -1 \). Notice that the map \( q_{W^r} \) is well defined as \( 0 \leq b - r - t_1 - ... - t_i + s_{i+1} + ... + s_r \leq g - 1 - r \) and \( 0 \leq j_1 - t_1 < ... < j_i - t_i < j_{i+1} + s_{i+1} \leq ... \leq j_r + s_r \leq g - 1 \). It is a slightly tedious but straightforward computation to show that the composition \( p_{W^r}oq_{W^r} = Id \). As the two vector spaces involved have the same dimension, this suffices in order to prove the isomorphism.

**Corollary 3.2.** For any given infinitesimal deformation \( \nu \) of the curve \( C_0 \), there is a unique infinitesimal deformation \( \sigma \) of the pair consisting of \( C_0 \) and the vector bundle \( \wedge^r E \) such that \( W^r \) can be extended to a space of sections of the deformation.

**Proof.** Consider the Petri map

\[
P_{W^r} : W^r \otimes H^0(K \otimes F^*) \rightarrow H^0(K \otimes F \otimes F^*)
\]

Consider the commutative diagram (2.1,3) in case \( F = \wedge^r E, V = W^r \)

\[
\begin{array}{ccc}
0 & \rightarrow & H^1((\wedge^r E)^* \otimes \wedge^r E) \rightarrow H^1(\Sigma_{\wedge^r E}) \rightarrow H^1(T_C) \rightarrow 0 \\
\downarrow P^* & & \downarrow \bar{P}^* \downarrow P^* \downarrow P^* \downarrow P^* \\
0 & \rightarrow & ImP^* \rightarrow Hom(W^r, H^1(\wedge^r E)) \rightarrow (ker P)^* \rightarrow 0
\end{array}
\]

From [3,1], \( P^* \) is an isomorphism, \( Ker P = 0 \). Hence, every element in \( H^1(T_C) \) has a unique inverse image in \( H^1(\Sigma_F) \) that belongs to the kernel of \( \bar{P}^* \). This proves the result.

**Corollary 3.3.** For any given infinitesimal deformation \( \nu \) of the curve \( C_0 \), the unique infinitesimal deformation \( \sigma \) of the vector bundle \( \wedge^r E \) that preserves it as the exterior power of the dual of the evaluation map is the \( \sigma \) above.
Proof. An infinitessimal deformation of $E$ as the dual of the kernel of the evaluation map preserves the subspace $W = W^1$ as space of sections. Hence, an infinitessimal deformation of $\wedge^r E$ as the exterior product of this dual preserves $W^r$ as space of sections. By the unicity of such deformation, the result follows.

Proposition 3.4. Take $\nu$ a generic infinitessimal deformation of $C_0$. Let $\sigma$ be the deformation of $\wedge^r E$ associated to $\nu$ as in $\S 3.3$. If $\tilde{W}$ is a subspace of $H^0(\wedge^r(E))$ that strictly contains $W^r$, then $\tilde{W}$ does not extend to a space of sections of the infinitessimal deformation of $\wedge^r E$ corresponding to $\sigma$.

Proof. It is enough to prove the result when $\tilde{W}$ has dimension $a = \dim W^r + 1$. Denote by $S$ the Schubert cycle of subspaces of dimension $a$ of $H^0(\wedge^r E)$ that contain $W^r$ ($S \subset \Gr(a, H^0(\wedge^r E))$).

Consider the diagram (2.1.4) for the case $F = \wedge^r E, V = \tilde{W}$. We obtain

\[
\begin{array}{c}
0 & \leftarrow & H^0(K \otimes (\wedge^r E) \otimes (\wedge^r E)^*) & \leftarrow & H^0(K \otimes (\Sigma_{\wedge^r E})^*) & \leftarrow & H^0(2K) & \leftarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \leftarrow & Im P & \leftarrow & \tilde{W} \otimes H^0(K \otimes (\wedge^r E)^*) & \leftarrow & Ker P_{\tilde{W}} & \leftarrow & 0 \\
\end{array}
\]

Define $P' = P'_{H^0(\wedge^r(E))}$.

We first show that the result would follow from the following statement:

(*) There is an element $\alpha \in H^0(2K)$ such that $\alpha \in P'(\tilde{W})$ for all $\tilde{W}$ in $S$.

If (*) holds, consider a direction of deformation $\nu$ in $H^1(T_C)$ not orthogonal to $\alpha$ and its corresponding deformation $\sigma_\nu$. Then, $P'_{\tilde{W}}(\sigma_\nu) \neq 0$ for all $\tilde{W}$. Therefore, no $\tilde{W}$ extends to a space of sections of the deformation corresponding to $\sigma_\nu$.

Let us see that in fact it suffices to prove (*) for the generic $\tilde{W} \in S$. Note that from $\S 3.1$, $Ker P_{\tilde{W}}$ is a subspace of $H^0(\wedge^r E) \otimes H^0(K \otimes (\wedge^r E)^*)$ of fixed dimension $b = h^0(K \otimes (\wedge^r E)^*)$. We have a well-defined map

$$\varphi : S \rightarrow \Gr(b, H^0(\wedge^r E) \otimes H^0(K \otimes (\wedge^r E)^*))$$

$$\tilde{W} \rightarrow Ker P_{\tilde{W}}$$

Define $\tilde{P} = \tilde{P}_{H^0(\wedge^r(E))}$. Assume that for the generic $\tilde{W} \in S$, $\varphi(\tilde{W})$ intersects $(P')^{-1}(\alpha)$ for a fixed element $\alpha \in H^0(2K)$. Denote by $S'$ the Schubert cycle of $\Gr(b, H^0(\wedge^r E) \otimes H^0(K \otimes (\wedge^r E)^*)$ of subspaces that intersect $(P')^{-1}(\alpha)$. Our assumption is that for $\tilde{W} \in S$ generic, $\varphi(\tilde{W}) \in S'$. From the irreducibility of $S$ and the closedness of $S'$, the same is true for every element in $S$.

Hence, it remains to prove (*) for generic $\tilde{W} \in S$. This will follow from the Lemma below and $\S 2.2$.

Lemma 3.5. Let $\tilde{W}$ be generic in $S$, then $P'_{\tilde{W}}(Ker P_{\tilde{W}}) = P'_{L'}(Ker P_{L'})$.

Proof. [of 3.5] As noticed before, $\dim Ker P_{\tilde{W}}$ is constant for $\tilde{W} \in S$. Therefore, the map $P'_{\tilde{W}}$ attains its maximum rank on an open set of $S$. From $\S 2.4$, the image of
$P'_W$ is contained in the image of $P'_{W'}$. Hence, the maximum possible rank of $P'_W$ is $\dim Im P'_{W'}$. It then suffices to exhibit one $\hat{W}$ satisfying the condition. We use the basis of $\wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*)$ introduced in [2.2] and the corresponding trivialisation of $\wedge^2 H^*$. Define

$$\hat{w} = \sum_{j=0}^{r-1} (-1)^j 1 \wedge y \wedge ... \wedge y^{j-1} \wedge y^{j+1} \wedge ... \wedge y^r \otimes x^{r-j}$$

Let $\hat{W}$ be the space generated by $\hat{w}$ and $W^r$. We recall from the proof of [3.1] that the map $P_W$ can be obtained by taking tensor product with $\wedge^r (S^{g-2} H \otimes \wedge^2 H)$ of the map

$$p_{W'} : \hat{W} \otimes S^{g-1-r} H \to \wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*) \otimes S^{g-1} H$$

Consider the following element of $\hat{W} \otimes S^{g-1-r} H$

$$z = \hat{w} \otimes 1 + \sum_{i=1}^{g-1} \phi_r (y \wedge y^2 \wedge ... \wedge y^{r-1} \wedge y^i) \otimes x^{i-r}$$

We want to see that $p_{W'} (z) = 0$.

We can write an explicit expression for $z$ using the convention that $y^{-1} = 0$, $y^{g-1} = 0$

$$z = \sum_{j=0}^{r-1} (-1)^j 1 \wedge y \wedge ... \wedge y^{j-1} \wedge y^{j+1} \wedge ... \wedge y^r \otimes x^{r-j} \otimes 1$$

$$+ \sum_{i=r+1}^{g-1} \sum_{0 \leq \epsilon_i \leq 1} (-1)^{\epsilon_1+...+\epsilon_r} y^{1-\epsilon_1} \wedge ... \wedge y^{r-1-\epsilon_{r-1}} \wedge y^{1-\epsilon_r} \otimes x^{r-(\epsilon_1+...+\epsilon_r)} \otimes x^{i-r}$$

$$= \sum_{j=0}^{r-1} (-1)^j 1 \wedge y \wedge ... \wedge y^{j-1} \wedge y^{j+1} \wedge ... \wedge y^r \otimes [x^{r-j} \otimes 1 - x^{r-(j+1)} \otimes x]$$

$$+ \sum_{i=r+1}^{g-2} \sum_{0 \leq \epsilon_i \leq 1} (-1)^{\epsilon_1+...+\epsilon_{r-1}} y^{1-\epsilon_1} \wedge ... \wedge y^{r-1-\epsilon_{r-1}} \wedge y^i \otimes$$

$$[x^{r-(\epsilon_1+...+\epsilon_{r-1})} \otimes x^{i-r} - x^{r-(\epsilon_1+...+\epsilon_{r-1}+1)} \otimes x^{i+1-r}]$$

It is clear from the above expression for $z$ that $z \in \ker p_{W'}$. In particular, for any element $e \in \wedge^r (S^{g-2} H \otimes \wedge^2 H)$, $z \otimes e \in \ker P_{W'}$.

From [2.4] the map $P'_W$ is up to a constant $c'$ ($c' = \binom{1}{r}$) the composition of the following maps: natural inclusion

$$\hat{W} \otimes \wedge^r (S^{g-2} H \otimes \wedge^2 H) \otimes S^{g-1-r} H \to \wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*) \otimes \wedge^r (S^{g-2} H \otimes \wedge^2 H) \otimes S^{g-1} H$$

the cup-product with the identity element in $\wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*) \otimes \wedge^r (S^{g-2} H \otimes \wedge^2 H)$

$$\wedge^r (S^{g-2} H^* \otimes \wedge^2 H^*) \otimes \wedge^r (S^{g-2} H \otimes \wedge^2 H) \otimes S^{g-1} H \to S^{r} H \otimes S^{g-1-r} H$$
followed by the natural product map

\[ P'_{Lr} : S^r H \otimes S^{g-1-r} H \to S^{g-1} H \]

Hence, for \( j = 0, \ldots, r-1 \) writing \( c = (-1)^j c' \),

\[ P'_{W}(z \otimes 1 \wedge x \wedge \ldots \wedge x^{j-1} \wedge x^{j+1} \wedge \ldots \wedge x^r) = c P'_{Lr}[x^{r-j} \otimes 1 - x^{r-(j+1)} \otimes x] \]

and for \( i = r+1 \ldots g - 2, 0 \leq \epsilon_i \leq 1 \) writing \( c = (-1)^{\epsilon_1+\ldots+\epsilon_{r-1}} c' \)

\[ P'_{W}(z \otimes x^{1-\epsilon_1} \wedge \ldots \wedge x^{r-1-\epsilon_{r-1}} \wedge x^i) = c P'_{Lr}[x^{r-(\epsilon_1+\ldots+\epsilon_{r-1})} \otimes x^{i-r} - x^{r-(\epsilon_1+\ldots+\epsilon_{r-1})} \otimes x^{i+1-r}] \]

As \( \ker P'_{Lr} \) is generated by elements of the form

\[ x^i \otimes x^j - x^{i+1} \otimes x^{j-1}, \quad i = 0 \ldots r - 1, \quad j = 1 \ldots g - 1 - r \]

the result is proved. \( \square \)

4. Extending the results to the generic curve.

**Proposition 4.1.** Let \( C \) be a generic curve of genus \( g \). Let \( E \) be defined as in 0.6. Denote by \( W \) the image of \((H^0(C, K_C))^* \) in \( H^0(C, E) \). Then, the natural map

\[ \psi_{C,r} : \wedge^r W \to H^0(C, \wedge^r E) \]

is injective.

Note that for \( C \) non-hyperelliptic, \( W = H^0(E) \). This follows from the projective normality of \( C \) (case \( p = 0 \) of the conjecture).

**Proof.** If for a given curve \( C \), \( \psi_{C,r} \) is injective, the same holds for every curve in a neighborhood of \( C \) in \( \mathcal{M}_g \). As \( \dim \wedge^r W = \binom{g}{r} \), [1.2] shows that \( \psi_{C_0,r} \) is injective for \( C_0 \) hyperelliptic. Hence the result follows. \( \square \)

The following proposition now concludes the proof of 0.4:

**Proposition 4.2.** Let \( C \) be a generic curve of genus \( g \). Then, \( h^0(C, \wedge^r E) = \binom{g}{r} \) and \( \wedge^r W \to H^0(C, \wedge^r E) \) is an isomorphism.

**Proof.** From [1.1], \( h^0(C, \wedge^r E) \geq \dim \text{Im}(\wedge^r W \to H^0(C, \wedge^r E)) = \dim \wedge^r W = \binom{g}{r} \).

From [3.4], \( h^0(C, \wedge^r E) \leq \dim W_{C_0} = \binom{g}{r} \). This concludes the proof. \( \square \)

**References**

[A,C] E.Arbarello, M.Cornalba, *Su una congettura di Petri*, Comm.Math. Helv. *56* (1981), 1-38.

[B,R] I.Biswas, S.Ramanan, *An infinitessimal study of the moduli space of Hitchin pairs*, J.London Math.Soc. (2) *49* (1994), 219-231.

[E] L.Ein, *A remark on the syzygies of the generic canonical curve*, J.Diff.Geom. *26* (1987), 361-365.

[F,H] W.Fulton, J.Harris. *Representation Theory, a first course*. GTM 129, Springer Verlag 1991.
[G] M. Green, Koszul cohomology and Geometry, J. Diff. Geom. 19 (1984), 125-171.

[H,P,R] K. Hulek, K. Paranjape, S. Ramanan, On a conjecture on canonical curves, J. Alg. Geom. 1 (1992), 335-359.

[H,R] A. Hirschowitz, S. Ramanan, New evidence for Green’s Conjecture on syzygies of canonical curves. Preprint alg/geom 9707017

[P,R] R. Paranjape, S. Ramanan, On the canonical ring of a curve, Algebraic Geometry and Commutative Algebra in honour of M. Nagata (1987) 503-516.

[S1] F. Schreyer, Syzygies of canonical curves and special linear series, Math. Ann. 275 (1986), 105-137.

[S2] F. Schreyer, A standard basis approach to the syzygies of canonical curves, J. Reine Angew. Math. 421 (1991), 83-123.

[T] M. Teixidor, Half canonical series on algebraic curves, Trans. AMS 302 N1 (1987), 99-115.

[V] C. Voisin, Courbes tetragonales et cohomologie de Koszul, J. Reine Angew. Math. 421 (1991), 111-121.

[W] G. Welters, Polarised abelian varieties and the heat equations, Comp. Math. 49 (1983), 173-194

Mathematics Department, Tufts University, Medford MA 02155, U.S.A.
E-mail address: teixidor@dpmms.cam.ac.uk, mteixido@tufts.edu