THE TANGENT GROUPOID OF A HEISENBERG MANIFOLD

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ABSTRACT. As a step toward proving an index theorem for hypoelliptic operators Heisenberg manifolds, including those on CR and contact manifolds, we construct an analogue for Heisenberg manifolds of Connes’ tangent groupoid $G_M$ of a manifold $M$. As it is well known for a Heisenberg manifold $(M, H)$ the relevant notion of tangent is rather that of Lie group bundle of graded 2-step nilpotent Lie groups $GM$. We then construct the tangent groupoid of $(M, H)$ as a differentiable groupoid $G_H M$ encoding the smooth deformation of $M \times M$ to $GM$. In this construction a crucial use is made of a refined notion of privileged coordinates and of a tangent approximation result for Heisenberg diffeomorphisms.

1. Introduction

This paper is part of a general project to obtain an analogue of the Atiyah-Singer index theorem ([1], [2]) for hypoelliptic operators on Heisenberg manifolds. Recall that a Heisenberg manifold $(M, H)$ consists of a manifold $M$ together with a distinguished hyperplane bundle $H \subset TM$. This includes as main examples the Heisenberg group, (codimension 1) foliations, contact manifolds, confoliations and CR manifolds. In this context the main geometric operators, although hypoelliptic, are not elliptic, so the elliptic calculus cannot be used. However, a natural substitute to the classical pseudodifferential calculus is provided by the Heisenberg calculus of Beals-Greiner [3] and Taylor [17]. Thus an analogue of the Atiyah-Singer theorem in the Heisenberg setting should yield an equality between an analytic index, defined in terms of the Fredholm indices of hypoelliptic elements of the Heisenberg calculus, and an index defined by analytic means. For instance, in the case of CR manifolds such an index theorem is motivated by Fefferman’s program of relating the hypoelliptic analysis of the Kohn-Rossi complex to the CR differential geometric data of the manifold [10].

On the other hand, Connes [7, Sect. II.5] (see also [14]) gave a simple proof of the Atiyah-Singer index theorem which is general enough to be carried out in many other settings. The crucial technical tool used by Connes is the tangent groupoid of a manifold, that is the differentiable groupoid which encodes the smooth deformation of $M \times M$ to $TM$ (see [7], [13]).

As a step towards proving an index theorem in the Heisenberg setting, we construct in this paper an analogue for Heisenberg manifolds of Connes’ tangent groupoid. The feasibility of such construction has actually been conjectured in [4, p. 74] and [15, p. 37]. Our approach is, however, different from that suggested in [4, p. 74] and can be divided in two steps.

The first step consists in suitably describing the tangent Lie group bundle $GM$ of a Heisenberg manifold $(M, H)$. The latter is a bundle of graded 2-step nilpotent Lie groups which is the relevant substitute for the Heisenberg manifold category of the classical tangent space $TM$. There are various descriptions of $GM$ in the literature ([3], [9], [11], [12], [16]). Our description here stems from the existence of a real-valued Levi form,

\begin{equation}
\mathcal{L} : H \times H \longrightarrow TM/H.
\end{equation}
Then $GM$ is the bundle $TM/H \oplus H$ equipped with the grading and Lie group law given by

\[
(1.2) \quad t.(X_0 + X') = t^2X_0 + tX', \quad t \in \mathbb{R},
\]

\[
(1.3) \quad (X_0 + X').(Y_0 + Y') = X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y',
\]

for sections $X_0, Y_0$ of $TM/H$ and sections $X', Y'$ of $H$.

It is important to relate the above description $GM$ to the tangent nilpotent approximations of previous approaches ([4], [3], [9], [11], [12], [16]). More precisely given a point $x \in M$ the tangent Lie group $G_x M$ is obtained as the Lie group associated to a Lie algebra of model vector fields in privileged coordinates centered at $x$. We point out that by using a refined notion of privileged coordinates, which we call Heisenberg coordinates (see Definition 2.18), this approach coincides with ours (Proposition 2.20).

An important consequence of the equivalence between these two descriptions of $GM$ is a tangent approximation result for Heisenberg diffeomorphisms (Proposition 2.21), which will play a crucial role in our construction of the tangent groupoid of a Heisenberg manifold (see below). This result states that in Heisenberg coordinates a Heisenberg diffeomorphism is well approximated by the Lie group isomorphism between the tangent groups at the points. Here we really need to work in Heisenberg coordinates since in general privileged coordinates we only get a Lie algebra isomorphism between the Lie algebras of the tangent group and the corresponding Lie group isomorphism does not approximate the Heisenberg diffeomorphism (compare [4, Prop. 5.20]).

The second step is the actual construction the tangent groupoid $G_H M$ of a Heisenberg manifold $(M, H)$ as a $b$-differentiable groupoid encoding the deformation of $M \times M$ to $GM$. In particular, at the set-theoretic level we have

\[
(1.4) \quad G_H M = GM \sqcup (M \times M \times (0, \infty)).
\]

While the definition of $G_H M$ as an abstract groupoid is similar to that of Connes’ tangent groupoid, the approach to endow $G_H M$ with a smooth structure differs from that of the standard proof of the smoothness of Connes’ tangent groupoid ([7], [13], [5]). In particular, at two stages we make a crucial use of the Heisenberg coordinates and of the tangent approximation of Heisenberg diffeomorphisms alluded to above. First, in order to obtain a consistent topology and a manifold structure for $G_H M$ and, second, to prove that the product of $G_H M$ is smooth (Proposition 3.5). In addition, we show that the construction of $G_H M$ is functorial with respect to Heisenberg diffeomorphisms (Proposition 3.8).

Beside potential applications towards an index theorem for hypoelliptic operators on Heisenberg manifolds, the construction of the tangent groupoid $G_H M$ is also interesting from the sole point of view of Carnot-Caratheodory geometry. Indeed, Gromov [12] and Bellaïche [4] proved that the tangent group at a point of a Carnot-Caratheodory is tangent to the manifold in a topological sense (i.e. in terms of Gromov-Hausdorff limits) but, here, in the special case of Heisenberg manifolds the construction of the tangent groupoid of a Heisenberg manifold shows that this tangence occurs in a differentiable sense.

In fact, by refining the privileged coordinates of [4] it should be possible to associate a tangent groupoid to any Carnot-Caratheodory manifold. In this case the tangent Lie group bundle $GM$ should be replaced by an orbibundle of Lie groups, which becomes an actual Lie group bundle when the Caratheodory distribution is equiregular in the sense of [12].

Let us now describe the organization of the paper. In Section 2 after recalling the main facts about Heisenberg manifolds we describe the tangent group bundle of a Heisenberg manifold in we construct in Section 3 the tangent groupoid of a Heisenberg manifold.
2. The tangent Lie group bundle of a Heisenberg manifold

In this section, after having recalled the main definitions and examples about Heisenberg manifolds, we describe the tangent Lie group bundle of a Heisenberg manifold in terms of an intrinsic Levi form. We then relate this approach to the nilpotent approximation of vector fields of previous approaches using Heisenberg coordinates, which refines the privileged coordinates of $\mathbb{R}$ and $\mathbb{H}$. As a consequence we get a tangent approximation result for Heisenberg diffeomorphism which will be crucial later on in the construction of the tangent groupoid of a Heisenberg manifold.

2.1. Heisenberg manifolds.

**Definition 2.1.** 1) A Heisenberg manifold is a smooth manifold $M$ equipped with a distinguished hyperplane bundle $H \subset TM$.

2) A Heisenberg diffeomorphism $\phi$ from a Heisenberg manifold $(M, H)$ onto another Heisenberg manifold $(M, H')$ is a diffeomorphism $\phi : M \to M'$ such that $\phi^* H = H'$.

**Definition 2.2.** Let $(M^{d+1}, H)$ be a Heisenberg manifold. Then:

1) A (local) $H$-frame for $TM$ is a (local) frame $X_0, X_1, \ldots, X_d$ so that $X_1, \ldots, X_d$ span $H$.

2) A local Heisenberg chart is a local chart with a local $H$-frame of $TM$ over its domain.

The main examples of Heisenberg manifolds are the following.

a) **Heisenberg group.** The $(2n + 1)$-dimensional Heisenberg group $H^{2n+1}$ is $\mathbb{R}^{2n+1} = \mathbb{R} \times \mathbb{R}^n$ equipped with the group law,

$$(2.1) \quad x.y = (x_0 + y_0 + \sum_{1 \leq j \leq n} (x_{n+j} y_j - x_j y_{n+j}), x_1 + y_1, \ldots, x_{2n} + y_{2n}).$$

A left-invariant basis for its Lie algebra $\mathfrak{h}^{2n+1}$ is then provided by the vector-fields,

$$(2.2) \quad X_0 = \frac{\partial}{\partial x_0}, \quad X_j = \frac{\partial}{\partial x_j} + x_{n+j} \frac{\partial}{\partial x_0}, \quad X_{n+j} = \frac{\partial}{\partial x_{n+j}} - x_j \frac{\partial}{\partial x_0}, \quad 1 \leq j \leq n,$$

which for $j, k = 1, \ldots, n$ and $k \neq j$ satisfy the relations,

$$(2.3) \quad [X_j, X_{n+k}] = -2\delta_{jk} X_0, \quad [X_0, X_j] = [X_j, X_k] = [X_{n+j}, X_{n+k}] = 0.$$

In particular, the subbundle spanned by the vector field $X_1, \ldots, X_{2n}$ yields a left-invariant Heisenberg structure on $H^{2n+1}$.

- **Foliations.** Recall that a (smooth) foliation is a manifold $M$ together with a subbundle $\mathcal{F} \subset TM$ which is integrable in the Frobenius’ sense, i.e. so that $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. Therefore, any codimension 1 foliation is a Heisenberg manifold.

- **Contact manifolds.** Opposite to foliations are contact manifolds: a contact structure on a manifold $M^{2n+1}$ is given by a non-vanishing 1-form $\theta$ on $M$ such that $d\theta$ is non-degenerate on $H = \ker \theta$. In particular, $(M, H)$ is a Heisenberg manifold. In fact, by Darboux’s theorem any contact manifold $(M^{2n+1}, \theta)$ is locally contact-diffeomorphic to the Heisenberg group $H^{2n+1}$ equipped with its standard contact form $\theta^0 = dx_0 + \sum_{j=1}^n (x_j dx_{n+j} - x_{n+j} dx_j)$.

- **Confoliations.** According to Elyashberg-Thurston a confoliation structure on an oriented manifold $M^{2n+1}$ is given by a global non-vanishing 1-form $\theta$ on $M$ such that $(d\theta)^n \wedge \theta \geq 0$. In particular, when $d\theta \wedge \theta = 0$ (resp. $(d\theta)^n \wedge \theta > 0$) we are in presence of a foliation (resp. a contact structure). In any case the hyperplane bundle $H = \ker \theta$ defines a Heisenberg structure on $M$.

- **CR manifolds.** A CR structure on an orientable manifold $M^{2n+1}$ is given by a rank $n$ complex subbundle $T_{1,0} \subset T_{\mathbb{C}} M$ which is integrable in Frobenius’ sense and such that $T_{1,0} \cap T_{0,1} = \{0\}$, where $T_{0,1} = \overline{T_{1,0}}$. Equivalently, the subbundle $H = \mathcal{R}(T_{1,0} \otimes T_{0,1})$ has the structure of a complex bundle of (real) dimension $2n$. In particular, $(M, H)$ is a Heisenberg manifold.
The main example of a CR manifold is that of the (smooth) boundary $M = \partial D$ of a complex domain $D \subset \mathbb{C}^n$. In particular, when $D$ is strongly pseudoconvex (or strongly pseudoconcave) with defining function $\rho$ then $\theta = i(\partial - \bar{\partial})\rho$ is a contact form on $M$.

2.2. The tangent Lie group bundle. A simple description of the tangent Lie group bundle of a Heisenberg manifold $(M^{d+1}, H)$ is given as follows.

**Lemma 2.3.** The Lie bracket of vector field induces on $H$ a 2-form with values in $TM/H$,

\[(2.4) \quad \mathcal{L} : H \times H \to TM/H,\]

so that for any sections $X$ and $Y$ of $H$ near a point $m \in M$ we have

\[(2.5) \quad \mathcal{L}_m(X(m), Y(m)) = [X, Y](m) \mod H_m.\]

**Proof.** We only need to check that given two sections $X$ and $Y$ of $H$ near $m \in M$ the value of $[X, Y](m)$ modulo $H_m$ depends only on those of $X(m)$ and $Y(m)$. Indeed, if $f$ and $g$ are smooth functions near $m$ then we have

\[(2.6) \quad [fX, gY](m) = f(m)g(m)[X, Y](m) - Y(f(m)X(m) + X(g(m)Y(m)) = f(m)g(m)[X, Y](m) \mod H_m.\]

This shows that if $X(m)$ or $Y(m)$ vanish then so does the class of $[X, Y](m)$ modulo $H_m$. Therefore, the latter only depends on the values of $X(m)$ and $Y(m)$. Hence the result. \(\square\)

**Definition 2.4.** The 2-form $\mathcal{L}$ is called the Levi form of $(M, H)$.

The Levi form $\mathcal{L}$ allows us to define a bundle $\mathfrak{g}M$ of graded Lie algebras by endowing $(TM/H) \oplus H$ with the smooth fields of Lie Brackets and gradings such that

\[(2.7) \quad [X_0 + X', Y_0 + Y']_m = \mathcal{L}_m(X', Y') \quad \text{and} \quad t.(X_0 + X') = t^2X_0 + tX' \quad t \in \mathbb{R},\]

for $m \in M$ and $X_0, Y_0$ in $T_mM/H_m$ and $X', Y'$ in $H_m$.

**Definition 2.5.** The bundle $\mathfrak{g}M$ is called the tangent Lie algebra bundle of $M$.

**Proposition 2.6.** The Lie algebra bundle is 2-step nilpotent and contains the normal bundle $TM/H$ in its center.

**Proof.** It follows from (2.7) that $TM/H$ is contained in the center of $\mathfrak{g}M$ and that the Lie bracket maps into $TM/H$, so that $\mathfrak{g}M$ is 2-step nilpotent. \(\square\)

Since $\mathfrak{g}M$ is nilpotent its associated graded Lie group bundle $GM$ can be described as follows. As a bundle $GM$ is $(TM/H) \oplus H$ and the exponential map is merely the identity. In particular, the grading of $GM$ is as in (2.7). Moreover, as $\mathfrak{g}M$ is actually 2-step nilpotent the Campbell-Hausdorff formula gives

\[(2.8) \quad (\exp X)(\exp Y) = \exp(X + Y + \frac{1}{2}[X, Y]) \quad \text{for sections} \quad X, Y \quad \text{of} \quad \mathfrak{g}M.\]

From this we deduce that the product on $GM$ is such that

\[(2.9) \quad (X_0 + X')(Y_0 + Y') = X_0 + Y_0 + \frac{1}{2}\mathcal{L}(X', Y') + X' + Y',\]

for sections $X_0, Y_0$ of $TM/H$ and sections $X', Y'$ of $H$.

**Definition 2.7.** The bundle $GM$ is called the tangent Lie group bundle of $M$.

In fact, the fibers of $GM$ as classified by the Levi form $\mathcal{L}$ as follows.
Proposition 2.8. 1) Let \( m \in M \). Then \( \mathcal{L}_m \) has rank \( 2n \) if, and only if, as a graded Lie group \( G_mM \) is isomorphic to \( \mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n} \).

2) The Levi form \( \mathcal{L} \) has constant rank \( 2n \) if, and only if, \( GM \) is a fiber bundle with typical fiber \( \mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n} \).

Proof. In this proof we let \( g \) be a Riemannian metric on \( H \). Moreover, since \( GM \) is already a Lie group bundle in order to show that this is a fiber bundle with typical fiber a given Lie group it is enough to prove the result locally. Therefore, without any loss of generality we may assume that the normal bundle \( TM/H \) is orientable, so that it admits a global non-vanishing section \( X_0 \). Then we let \( A \) denote the smooth section of \( \text{End} \, H \) such that

\[
\mathcal{L}(X,Y) = g(X,A^YX_0) \quad \text{for sections } X, Y \text{ of } H.
\]

1) Let \( m \in M \). Since \( \mathcal{L}_m \) is real-antisymmetric its rank has to be an even integer, say \( \text{rk} \mathcal{L}_m = 2n \). Let us first assume that \( \mathcal{L}_m \) is non-degenerate, i.e. \( A_m \) is invertible. Let \( A_m = J_m|A_m| \) be the polar decomposition of \( A_m \) and on \( H_m \) define the positive definite scalar product

\[
h_m(X,Y) = \frac{1}{2} g_m(X,|A_m|^Y) \quad X, Y \in H_m.
\]

Notice that \( J_m \) is anti-symmetric and unitary with respect to \( h_m \). Thus, \( J_m^2 = -J_m^t J_m = -1 \), i.e. \( J_m \) is a unitary complex structure on \( H_m \). Therefore, we can construct a basis \( X_1, \ldots, X_{2n} \) of \( H_m \) which is orthonormal with respect to \( h_m \) and such that \( X_{n+j} = J_m X_j \) for \( j = 1, \ldots, n \).

On the other hand, for \( X \) and \( Y \) in \( H_m \subset \mathfrak{g}_m \) we have

\[
[X,Y]_m = \mathcal{L}_m(X,Y) = g_m(X,A_mY)X_0 = h_m(X,JYX_0).
\]

Thus, for \( j = 1, \ldots, n \) and \( k = 1, \ldots, n \) we get

\[
[X_j,X_{n+j}] = 2h_m(X_j,J^2X_j)X_0 = -2h_m(X_j,X_j)X_0 = -2X_0,
\]

\[
[X_j,X_k] = h_m(X_j,JX_k)X_0 = -h_m(X_{n+j},X_k)X_0 = 0.
\]

These relations are the same as those in (2.3) for the Lie algebra of \( \mathbb{H}^{2n+1} \). Thus \( G_mM \) is isomorphic to \( \mathbb{H}^{2n+1} \) as a graded Lie group.

Now, assume that \( A_m \) has a non-trivial kernel. Then as \( A_m \) is real antisymmetric with respect to \( g_m \) we have an orthogonal direct sum \( H_m = \text{im} \, A_m \oplus \ker A_m \). In fact, it follows from (2.10) that if \( X \in \ker A_m \) and \( Y \in H_m \) then

\[
[X,Y]_m = \mathcal{L}_m(X,Y) = g_m(X,A_mY)X_0 = 0.
\]

Thus \( \ker A_m \) is contained in the center of \( \mathfrak{g}_mM \). Moreover, as \( A_m \) is invertible on \( \text{im} \, A_m \) the same reasoning as above shows that the Lie subalgebra \( (T_mM/H_m) \oplus \text{im} \, A_m \) is isomorphic to the (graded) Lie algebra \( \mathfrak{h}^{2n+1} \) of \( \mathbb{H}^{2n+1} \). Therefore, \( \mathfrak{g}_mM = (T_mM/H_m) \oplus \text{im} \, A_m \oplus \ker A_m \) is isomorphic to \( \mathfrak{h}^{2n+1} \times \mathbb{R}^{d-2n} \), and so \( G_mM \) is isomorphic to \( \mathfrak{h}^{2n+1} \times \mathbb{R}^{d-2n} \).

Conversely, suppose that \( G_mM \) is isomorphic to \( \mathfrak{h}^{2n+1} \times \mathbb{R}^{d-2n} \). Then \( \mathfrak{g}_mM \) is isomorphic to \( \mathfrak{h}^{2n+1} \times \mathbb{R}^{d-2n} \), so admits a basis \( X_0, \ldots, X_d \) such that

\[
[X_j,X_{n+j}] = -2X_0 \quad \text{and} \quad [X_j,X_k] = [X_l,X_k] = 0,
\]

for \( j = 1, \ldots, n \) and \( k = 1, \ldots, d \) with \( k \neq n + j \) and \( l = 2n + 1, \ldots, d \). Since \( \mathcal{L}_m(X,Y) = [X,Y] \) for \( X \) and \( Y \) in \( H_m \) it follows from this that \( \mathcal{L}_m \) has rank \( 2n \).

2) Assume that \( \mathcal{L} \) has constant rank \( 2n \). Thus everywhere we have \( \text{rk} \, A_m = 2n \), so that we get a vector bundle splitting \( H = \text{im} \, A \oplus \ker A \). Furthermore, the polar decomposition of \( A_m \) is smooth with respect to \( m \), i.e. \( J \) and \( |A| \) are smooth sections of \( \text{End} \, H \). Therefore, the above process for constructing the basis \( X_0, X_1, \ldots, X_d \) can be carried out near every point \( m \in M \) in such way to yield a smooth \( H \)-frame satisfying the relations (2.13)–(2.14). Therefore, near every point of \( M \) we get a Lie bundle trivialization of \( GM \) as a trivial fiber bundle with fiber \( \mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n} \). Consequently, \( GM \) is fiber bundle with typical fiber \( \mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n} \).
Conversely, assume that $GM$ is a fiber bundle with typical fiber $\mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n}$. Then at every point $m \in M$ the Lie group $G_mM$ is isomorphic to $\mathbb{H}^{2n+1} \times \mathbb{R}^{d-2n}$. Thus $\mathcal{L}$ has constant rank $2n$ by the first part of the proposition.

In presence of a foliation or a contact structure we have more precise results.

**Proposition 2.9.** Let $(M, H)$ be a Heisenberg manifold. Then the following are equivalent.

(i) $(M, H)$ is a foliation.

(ii) $(M, H)$ is Levi flat, i.e. $\mathcal{L}$ vanishes.

(iii) As a Lie group bundle $GM$ coincides with $(TM/H) \oplus H$.

**Proof.** It follows from the very definition of $\mathcal{L}$ that it vanishes if, and only if, for any vector field $X$ and $Y$ in $H$ the vector field $[X, Y]$ is in $H$, that is if, and only if, $H$ is a foliation.

On the other hand, in view of the definition of the group law of $GM$ the Levi form $\mathcal{L}$ vanishes if, and only if, the group law is $XY = X + Y$, i.e. $GM$ is the Abelian Lie group bundle $(TM/H) \oplus H$. Hence the result.

**Proposition 2.10.** Suppose that $(M^{2n+1}, H)$ is a Heisenberg manifold such that $TM/H$ is orientable. Then the following are equivalent:

(i) $M$ admits a contact form annihilating $H$.

(ii) The Levi form $\mathcal{L}$ is everywhere non-degenerate.

(iii) The Lie group tangent bundle $GM$ is a fiber bundle with typical fiber $\mathbb{H}^{2n+1}$.

**Proof.** Since the normal line bundle $TM/H$ is orientable it admits a global non-vanishing smooth section $X_0$. Let $\theta$ be the section of $(T^*M/H^*)$ such that $\theta(X_0) = 1$. We shall see $\theta$ as a 1-form on $M$ annihilating on $H$. Then for any sections $X$ and $Y$ of $H$ we have

$$\mathcal{L}(X, Y) = \theta([X, Y])X_0 = -d\theta(X, Y)X_0. \quad (2.17)$$

This shows that $\mathcal{L}$ and $\theta|_H$ have same rank. Thus, $\theta$ is a contact form if, and only if, $\mathcal{L}$ is everywhere non-degenerate. Combining this with Proposition 2.8 proves the proposition.

Finally, let $\phi : (M, H) \rightarrow (M', H')$ be a Heisenberg diffeomorphism from $(M, H)$ onto another Heisenberg manifold $(M', H')$. Since we have $\phi_*H = H'$ we see that $\phi'$ induces a smooth vector bundle isomorphism $\phi'$ from $TM/H$ onto $TM'/H'$.

**Definition 2.11.** We let $\phi'_H : (TM/H) \oplus H \rightarrow (TM'/H') \oplus H'$ is the vector bundle isomorphism such that

$$\phi'_H(m)(X_0 + X') = \phi'(m)X_0 + \phi'(m)X', \quad (2.18)$$

for any $m \in M$ and any $X_0 \in T_m/H_m$ and $X' \in H_m$.

**Proposition 2.12.** The vector bundle isomorphism $\phi'_H$ is an isomorphism of graded Lie group bundles from $GM$ onto $GM'$.

**Proof.** First, it follows from (2.18) that $\phi'_H$ is graded, i.e. we have $\phi'_H(t.X) = t.\phi'_H(X)$ for any $t \in \mathbb{R}$ and any section $X$ of $GM$.

Second, if $X$ and $Y$ are sections of $H$ then we have

$$\mathcal{L}(\phi'_H(X), \phi'_H(Y)) = [\phi_*X, \phi_*Y] = \phi'_H([X, Y]) = \phi'_H(\mathcal{L}_m(X, Y)) \mod H'. \quad (2.19)$$

In view of (2.9) this implies that $\phi'_H$ is a Lie group bundle isomorphism from $GM$ onto $GM'$.

**Corollary 2.13.** The Lie group bundle isomorphism class of $GM$ depends only the Heisenberg diffeomorphism class of $(M, H)$. 

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2.3. Heisenberg coordinates and nilpotent approximation of vector field. In the sequel it will be useful to combine the above intrinsic description of $GM$ with a more extrinsic description of the tangent Lie group at a point in terms of the Lie group associated to a nilpotent Lie algebra of model vector field. Incidentally, this will show that our approach is equivalent to previous ones ([3], [4], [11], [12], [16]).

First, let $m \in M$ and let us describe $\mathfrak{g}_m M$ as the graded Lie algebra of left-invariant vector field on $G_m M$ by identifying any $X \in \mathfrak{g}_m M$ with the left-invariant vector field $L_X$ on $G_m M$ given by

\begin{equation}
L_X f(x) = \frac{d}{dt} f(x(t \exp(X)))\big|_{t=0} - \frac{d}{dt} f(x(tX))\big|_{t=0}, \quad f \in C^\infty(G_m M).
\end{equation}

This allows us to associate to any vector field $X$ near $m$ a unique left-invariant vector field $X^m$ on $G_m M$ such that

\begin{equation}
X^m = \begin{cases} 
L_{X_0(m)} & \text{if } X(m) \notin H_m, \\
L_{X(m)} & \text{otherwise},
\end{cases}
\end{equation}

where $X_0(m)$ denotes the class of $X(m)$ modulo $H_m$.

**Definition 2.14.** The left-invariant vector field $X^m$ is called the model vector field of $X$ at $m$.

Let us look at the above construction in terms of a $H$-frame $X_0, \ldots, X_d$ near $m$, that is of a local trivialization of the vector bundle $(TM/H) \oplus H$. For $j, k = 1, \ldots, d$ we let

\begin{equation}
\mathcal{L}(X_j, X_k) = [X_j, X_k] = L_{jkX_0} \mod H.
\end{equation}

With respect to the coordinate system $(x_0, \ldots, x_d)$ corresponding to $X_0(m), \ldots, X_d(m)$ we can write the product law of $G_m M$ as

\begin{equation}
x.y = (x_0 + \frac{1}{2} \sum_{j,k=1}^d L_{jkX_0} x_j x_k, x_1 + y_1, \ldots, x_d + y_d).
\end{equation}

Then the vector fields $X^m_j, j = 1, \ldots, d$, in (2.21) are just the left-invariant vector field corresponding to the vectors of the canonical basis $e_j$, i.e., we have

\begin{equation}
X^m_j = \frac{\partial}{\partial x_0} \quad \text{and} \quad X^m_j = \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{k=1}^d L_{jkX_0} \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq d.
\end{equation}

In particular, for $j, k = 1, \ldots, d$ we have the relations,

\begin{equation}
[X^m_j, X^m_k] = L_{jkX_0}, \quad [X^m_j, X^m_0] = 0.
\end{equation}

Let $X$ be a vector field near $m$. Then $X$ is of the form $X = a_0(x)X_0 + \ldots + a_d(x)X_d$ near $m$ and its model vector field $X^m$ is thus given by the formula

\begin{equation}
X^m = \begin{cases} 
a_0(m)X^m_0 & \text{if } a_0(m) \neq 0, \\
a_1(m)X^m_1 + \ldots + a_dX^m_d & \text{otherwise}.
\end{cases}
\end{equation}

Now, let $\kappa : \text{dom} \kappa \to U$ be a Heisenberg chart near $m = \kappa^{-1}(u)$ and let $X_0, \ldots, X_d$ be the associated $H$-frame of $TU$. Then there exists a unique affine coordinate change $v \to \psi_u(v)$ such that $\psi_u(0) = 0$ and $\psi_uX_j(0) = \frac{\partial}{\partial x_j}$ for $j = 0, 1, \ldots, d$. Indeed, if for $j = 1, \ldots, d$ we set $X_j(x) = \sum_{k=0}^d B_{jk}(x) \frac{\partial}{\partial x_k}$ then one checks that

\begin{equation}
\psi_u(x) = A(u)(x-u), \quad A(u) = (B(u)^t)^{-1}.
\end{equation}

**Definition 2.15** ([3]). 1) The coordinates provided by $\psi_u$ are called the privileged coordinates at $u$ with respect to the $H$-frame $X_0, \ldots, X_d$.

2) The map $\psi_u$ is called the privileged-coordinate map with respect to the $H$-frame $X_0, \ldots, X_d$. 

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Lemma 2.17. Consider a diffeomorphism \( \phi \) isomorphic. In fact, an explicit isomorphism can be obtained as follows.

\[
\phi(0, \ldots, d) = (0, \ldots, d),
\]

where the \( a_{jk} \)'s are smooth functions such that \( a_{jk}(0) = 0 \).

Next, on \( \mathbb{R}^{d+1} \) we consider the dilations

\[
\delta_t(x) = t.x = (t^2x_0, tx_1, \ldots, tx_d), \quad t \in \mathbb{R},
\]

with respect to which \( \frac{\partial}{\partial x_0} \) is homogeneous of degree \(-2\) and \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \) are homogeneous of degree \(-1\). Therefore, we may let

\[
X_0^{(u)}(u) = \lim_{t \to 0} t^2 \delta_t^* X_0 = \frac{\partial}{\partial x_0},
\]

\[
X_j^{(u)}(u) = \lim_{t \to 0} t^{-1} \delta_t^* X_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^d b_{jk} x_k \frac{\partial}{\partial x_0}, \quad j = 1, \ldots, d,
\]

where for \( j, k = 1, \ldots, d \) we have let \( b_{jk} = \partial x_k a_{j0}(0) \). In fact, for any vector field \( X = a_0(x) X_0 + \ldots + a_d(x) X_d \) we have

\[
\lim_{t \to 0} t^2 \delta_t^* X = a_0(0) X_0^{(u)},
\]

\[
\lim_{t \to 0} t^{-1} \delta_t^* X = a_1(0) X_1^{(u)} + \ldots + a_d(0) X_d^{(u)} \quad \text{when } a_0(0) = 0.
\]

Observe that \( X_0^{(u)} \) is homogeneous of degree \(-2\) and \( X_1^{(u)}, \ldots, X_d^{(u)} \) are homogeneous of degree \(-1\). Moreover, for \( j, k = 1, \ldots, d \) we have

\[
[X_j^{(u)}, X_0^{(u)}] = 0 \quad \text{and} \quad [X_j^{(u)}, X_k^{(u)}] = (b_{kj} - b_{jk}) X_0^{(u)},
\]

Thus, the linear space spanned by \( X_0^{(u)}, X_1^{(u)}, \ldots, X_d^{(u)} \) is a graded 2-step nilpotent Lie algebra \( g^{(u)} \).

In particular, \( g^{(u)} \) is the Lie algebra of left-invariant vector field over the graded Lie group \( G^{(u)} \) consisting of \( \mathbb{R}^{d+1} \) equipped with the grading \( \{0\} \) and the group law,

\[
x.y = (x_0 + \sum_{j,k=1}^d b_{kj} x_j y_k, x_1 + y_1, \ldots, x_d + y_d).
\]

Now, if near \( m \) we set \( L(X_j, X_k) = [X_j, X_k] = L_{jk} X_0 \) mod \( H \) then we have

\[
[X_j^{(u)}, X_k^{(u)}] = \lim_{t \to 0} t \delta_t^* X_j, t \delta_t^* X_k = \lim_{t \to 0} t^2 \delta_t^* (L_{jk} X_0) = L_{jk}(m) X_0^{(u)}.
\]

Comparing this with \( 2.23 \) and \( 2.31 \) shows that \( g^{(u)} \) has the same constant structures as those of \( g_m M \) and is therefore isomorphic to it. Consequently, the Lie groups \( G^{(u)} \) and \( G_m M \) are isomorphic. In fact, an explicit isomorphism can be obtained as follows.

Lemma 2.17. Consider a diffeomorphism \( \phi : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1} \) of the form

\[
\phi(x_0, \ldots, x_d) = (x_0 + \frac{1}{2} c_{jk} x_j x_k, x_1, \ldots, x_d),
\]
where \( c = (c_{jk}) \), \( c^t = c \), is a symmetric matrix in \( M_d(\mathbb{R}) \). Then \( \phi \) is a graded isomorphism from \( G^{(u)} \) onto the Lie group \( G \) consisting of \( \mathbb{R}^{d+1} \) equipped with the group law,

\[
x \cdot y = (x_0 + y_0 + \sum_{j,k=1}^{d} (b_{kj} + c_{kj})x_jy_k, x_1 + y_1, \ldots, x_d + y_d).
\]

Moreover, under \( \phi \) the vector field \( X_0^{(u)}, \ldots, X_d^{(u)} \) transform into

\[
\phi_* X_0^{(u)} = \frac{\partial}{\partial x_0} \quad \text{and} \quad \phi_* X_j^{(u)} = \frac{\partial}{\partial x_j} + \sum_{k=1}^{d} (b_{kj} + c_{kj})x_k \frac{\partial}{\partial x_0}, \quad j = 1, \ldots, d.
\]

Proof. First, since \( \phi(t,x) = t \cdot \phi(x) \) for any \( t \in \mathbb{R} \), we see that \( \phi \) is graded. Second, for \( x \) and \( y \) in \( \mathbb{R}^{d+1} \) the product \( \phi(x) \cdot \phi(y) \) is equal to

\[
\phi(x_0 + y_0 + \sum_{j,k=1}^{d} b_{kj}x_jy_k, x_1 + y_1, \ldots, x_d + y_d)
\]

\[
= (x_0 + y_0 + \sum_{j,k=1}^{d} b_{kj}x_jy_k + \frac{1}{2} \sum_{j,k=1}^{d} c_{kj}(x_j + y_j)(x_k + y_k), x_1 + y_1, \ldots, x_d + y_d),
\]

\[
= (x_0 + \frac{1}{2} \sum_{j,k=1}^{d} c_{kj}x_jx_k + y_0 + \frac{1}{2} \sum_{j,k=1}^{d} c_{kj}y_jy_k + (b_{kj} + c_{kj})x_jy_k, x_1 + y_1, \ldots, x_d + y_d).
\]

Thus in view of the law group of \( G \) we have \( \phi(x \cdot y) = \phi(x) \cdot \phi(y) \), so that \( \phi \) is a Lie group isomorphism. Consequently, for \( j = 0, \ldots, d \) the vector field \( \phi_* X_j^{(u)} = \phi' \phi^{-1}(x)[X_j(\phi^{-1}(x))] \) on \( G \) is left-invariant. In fact, as \( \phi'(0) = \text{id} \) and \( X_j^{(u)}(0) = \frac{\partial}{\partial x_j} \) we see that \( \phi_* X_j^{(u)} \) is the left-invariant vector fields on \( G \) that coincides with \( \frac{\partial}{\partial x_j} \) at \( x = 0 \). Therefore, a formula for \( \phi_* X_j^{(u)} \) can be deduced from (2.31) by replacing \( b_{kj} \) by \( b_{kj} + c_{kj} \), so we get the formulas (2.39).

Now, since by (2.34) and (2.36) we have \( L_{jk} = b_{kj} - b_{kj} \) for \( j, k = 1, \ldots, d \), we deduce from Lemma 2.17 that an isomorphism of graded Lie groups from \( G^{(u)} \) onto \( G \) is given by

\[
\phi_u(x_0, \ldots, x_d) = (x_0 - \frac{1}{4} \sum_{j,k=1}^{d} (b_{kj} + b_{kj})x_jx_k, x_1, \ldots, x_d).
\]

**Definition 2.18.** Let \( \varepsilon_u = \phi_u \circ \psi_u \). Then:

1) The new coordinates provided by \( \varepsilon_u \) are called Heisenberg coordinates at \( u \) with respect to the \( H \)-frame \( X_0, \ldots, X_d \).

2) The map \( \varepsilon_u \) is called the \( u \)-Heisenberg coordinate map.

**Remark 2.19.** The Heisenberg coordinates were first introduced in [3] where they were called ”antisymmetric \( u \)-coordinates” and used as a technical tool for inverting the principal symbol of a hypoelliptic sublaplacian.

Next, Lemma 2.17 also tells us that

\[
\phi_* X_0^{(u)} = \frac{\partial}{\partial x_0} = X_0^{m} \quad \text{and} \quad \phi_* X_j^{(u)} = \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{k=1}^{d} L_{jk}x_k \frac{\partial}{\partial x_0} = X_j^{m}, \quad j = 1, \ldots, d.
\]

Since \( \phi_u \) commutes with the Heisenberg dilations (2.29) using (2.30)–(2.31) we get

\[
\lim_{t \to 0} t^2 \delta_t^* \phi_u X_0^{(u)} = X_0^{m} \quad \text{and} \quad \lim_{t \to 0} t \hat{\delta}_t^* \phi_u X_j^{(u)} = X_j^{m}, \quad j = 1, \ldots, d.
\]
Combining with $m$ and $n$, this shows that, for any vector field $X$ near $m$, as $t \to 0$ and in Heisenberg coordinates at $m$ we have
\begin{equation}
\delta^i_j X = \begin{cases} \ t^{-2}X^m + O(t^{-1}) & \text{if } X(m) \in H_m, \\ t^{-1}X^m + O(1) & \text{otherwise.} \end{cases}
\end{equation}
Therefore, we obtain:

**Proposition 2.20.** In the Heisenberg coordinates centered at $m = \kappa^{-1}(u)$ the tangent Lie group $G_m M$ coincides with $G(u)$.

2.4. **Tangent approximation of Heisenberg diffeomorphisms.** Recall that if $\phi : M \to M'$ is a smooth map between (standard) smooth manifolds then, for any $m \in M$, the derivative $\phi'(m)$ yields a tangent linear approximation for $\phi$ in local coordinates around $m$. We shall now prove analogous result in the Heisenberg setting. To this end it will be useful to endow $\mathbb{R}^{d+1}$ with the pseudo-norm,
\begin{equation}
\|x\| = (x_0^2 + (x_1^2 + \ldots + x_d^2)^2)^{1/4}, \quad x \in \mathbb{R}^{d+1},
\end{equation}
so that for any $x \in \mathbb{R}^{d+1}$ and any $t \in \mathbb{R}$ we have
\begin{equation}
\|tx\| = |t| \|x\|.
\end{equation}

From now on we let $\phi : (M,H) \to (M',H')$ be a Heisenberg diffeomorphism from $(M,H)$ to another Heisenberg manifold $(M',H')$.

**Proposition 2.21.** Let $m \in M$ and set $m' = \phi(m)$. Then, in Heisenberg coordinates at $m$ and at $m'$ the diffeomorphism $\phi$ has a behavior near $x = 0$ of the form
\begin{equation}
\phi(x) = \phi_H(0)x + (O(\|x\|^2), O(\|x\|^2), \ldots, O(\|x\|^2)),
\end{equation}
where $\phi_H$ is as defined in Definition 2.11. In particular, there is no term of the form $x_j x_k$, $1 \leq j, k \leq d$, in the Taylor expansion of $\phi_0(x)$ at $x = 0$.

**Proof.** Let $X_0, \ldots, X_d$ be a $H$-frame of $TM$ over a Heisenberg chart $\kappa$ near $m$ and let $Y_0, \ldots, Y_d$ be a $H'$-frame of $TM'$ over a Heisenberg chart $\kappa_1$ near $m'$. Also, set $u = \kappa(m)$, so that in the privileged coordinates at $u$ we have $X_j(0) = \frac{\partial}{\partial x_j}$ for $j = 0, \ldots, d$. As the change of variables $\phi_u$ from the privileged coordinates to the Heisenberg coordinates at $u$ is such that $\phi_u(0) = 0$ and $\phi_u'(0) = \text{id}$ we see that in the Heisenberg coordinates at $m$ too we have $X_j(0) = \frac{\partial}{\partial x_j}$ for $j = 0, \ldots, d$. Similarly, in the Heisenberg coordinates at $m'$ we have $Y_j(0) = \frac{\partial}{\partial x_j}$ for $j = 0, \ldots, d$. As $\phi'(0)$ maps $H_0$ to $H'_0$ it then follows that with respect to the basis $\frac{\partial}{\partial x_0}, \ldots, \frac{\partial}{\partial x_d}$ the matrices of $\phi'(0)$ and $\phi_H'(0)$ take the forms,
\begin{equation}
\phi'(0) = \begin{pmatrix} a_{00} & 0 \\ B & A_\parallel \end{pmatrix} \quad \text{and} \quad \phi_H'(0) = \begin{pmatrix} a_{00} & 0 \\ 0 & A_\parallel \end{pmatrix},
\end{equation}
for some scalar $a_{00} \neq 0$ and some matrices $b \in M_d(\mathbb{R})$ and $A_\parallel \in GL_d(\mathbb{R})$. In particular, we have $\phi'(0)x = \phi_H'(0)x + x_0(0, b_{10}, \ldots, b_{d0})$. Thus, the Taylor expansion of $\phi(x)$ at $x = 0$ takes the form
\begin{equation}
\phi(x) = \hat{\phi}(x) + \theta(x), \quad \hat{\phi}(x) = (x_0 + \frac{1}{2} \sum_{j,k=1}^d c_{j,k} x_j x_k, x_1, \ldots, x_d),
\end{equation}
where $c_{j,k} = \frac{\partial^2 \phi_0}{\partial x_j \partial x_k}(0)$, $j, k = 1, \ldots, d$, and $\theta(x) = (\theta_0(x), \ldots, \delta_d(x))$ is such that
\begin{align}
\theta_0(x) &= O(|x_0| |x| + |x|^3) = O(\|x\|^4), \\
\theta_j(x) &= O(|x_0| + |x|^2) = O(\|x\|^2), \quad j = 1, \ldots, d.
\end{align}
Therefore, for completing the proof we only need to show that \( c_{jk} = 0 \) for \( j, k = 1, \ldots, d \). In fact, to reach this goal, possibly by replacing \( \phi \) by \( \phi_H'(0)^{-1} \circ \phi \), we may assume that \( \phi_H'(0) = \text{id} \). Since \( \phi_H'(0) \) is by Proposition 2.22 a Lie group isomorphism from \( G = G_0M \) onto \( G' = G_0'M' \) this implies that \( G \) and \( G' \) have same group law, i.e.

\[
(2.52) \quad x.y = (x_0 + y_0 + \frac{1}{2} \sum_{j,k=1}^{d} L_{jk}x_jx_k, x_1 + y_1, \ldots, x_d + y_d),
\]

where the structure constants are such that \( L(X_j, X_k)(0) = L(Y_j, Y_k)(0) = L_{jk}X_0(0) \). Therefore, using (2.24) we deduce that, at the level of the model vector fields (2.21), we have

\[
(2.53) \quad X_0^m = Y_0^{m'} = \frac{\partial}{\partial x_0} \quad \text{and} \quad X_j^m = Y_j^{m'} = \frac{\partial}{\partial x_j} - \frac{1}{2} \sum_{k=1}^{d} L_{jk}x_k \frac{\partial}{\partial x_0}, \quad j = 1, \ldots, d.
\]

Now, as \( \phi_H'(0) \) is the diagonal part of \( \phi'(0) \) in (2.48) we have \( \phi_*X_0(0) = Y_j(0) \mod H'_0 \) and \( \phi_*X_0(0) = Y_j(0) \) for \( j = 1, \ldots, d \). Therefore, using (2.21) we obtain

\[
(2.54) \quad (\phi_*X_j)^{m'} = Y_j^{m'} = X_j^m \quad \text{for} \quad j = 0, \ldots, d.
\]

On the other hand, as we are using Heisenberg coordinates at \( m \) and Heisenberg coordinates at \( m' \) from (2.41) we get

\[
(2.55) \quad X_j^m = \lim_{t \to 0} t\delta^*_t X_j \quad \text{and} \quad (\phi_*X_j)^{m'} = \lim_{t \to 0} t\delta^*_t \phi_*X_j = \lim_{t \to 0} (\delta^{-1}_t \circ \phi \circ \delta_t)_*(t\delta^*_t X_j).
\]

Since \( (2.49) = (2.51) \) imply that \( \lim_{t \to 0} \delta^{-1}_t \circ \phi \circ \delta_t = \hat{\phi} \) we see that

\[
(2.56) \quad (\phi_*X_j)^{m'} = \lim_{t \to 0} (\delta^{-1}_t \circ \phi \circ \delta_t)_*(t\delta^*_t X_j) = \hat{\phi}_*X_j^m.
\]

Combining this with (2.54) we then obtain

\[
(2.57) \quad \hat{\phi}_*X_j^m = (\phi_*X_j)^{m'} = X_j^m \quad \text{for} \quad j = 1, \ldots, d.
\]

Now, the form of \( \hat{\phi} \) in (2.49) allows us to apply Lemma 2.17 to get

\[
(2.58) \quad \hat{\phi}_*X_j^m = \frac{\partial}{\partial x_j} + \sum_{k=1}^{d} (\frac{1}{2} L_{jk} + c_{jk})x_k \frac{\partial}{\partial x_0}.
\]

Combining this with (2.53) and (2.57) then gives \( L_{jk} = L_{jk} - 2c_{jk} \), from which we get \( c_{jk} = 0 \) for \( j, k = 1, \ldots, d \). The proof is now complete. \( \square \)

Remark 2.22. An asymptotics similar to (2.47) is given in [4, Prop. 5.20] in privileged coordinates at \( u \) and \( u' = \kappa_1(m') \), but the leading term there is only a Lie algebra isomorphism from \( g^{(u)} \) onto \( g^{(u')} \). This is only in Heisenberg coordinates that we recover the Lie group isomorphism \( \phi_H'(m) \) as the leading term of the asymptotics.

Finally, for future purpose we mention the following version of Proposition 2.21

Proposition 2.23. In local coordinates and as \( t \to 0 \) we have

\[
(2.59) \quad t^{-1} \cdot \varepsilon_{\phi(u)} \circ \phi \circ \varepsilon_{u}^{-1}(t,x) = (\varepsilon_{\phi(u)} \circ \phi \circ \varepsilon_{u}^{-1})'_H(0)x + O(t),
\]

locally uniformly with respect to \( u \) and \( x \).

Proof. First, combining Proposition 2.21 with (2.46) we get

\[
(2.60) \quad t^{-1} \cdot \varepsilon_{\phi(u)} \circ \phi \circ \varepsilon_{u}^{-1}(t,x) = (\varepsilon_{\phi(u)} \circ \phi \circ \varepsilon_{u}^{-1})'_H(0)x + O(t).
\]

A priori this holds only pointwise with respect to \( u \) and \( x \). However, the bound of the above asymptotics comes from remainder terms in Taylor formulas at \( t = 0 \) for components of the function
\(\Psi(u, x, t) := \varepsilon_{\phi(u)} \circ \phi \circ \varepsilon_{u}^{-1}(t, x)\). Since \(\Psi\) is smooth with respect to \(u\) and \(x\) it follows that the bounds in (2.60) are locally uniform with respect to \(u\) and \(x\).

3. The tangent groupoid of a Heisenberg Manifold

In this section we construct the tangent groupoid of a Heisenberg manifold \((M, H)\) as a group encoding the smooth deformation of \(M \times M\) to \(GM\). In this construction a crucial use is made of the Heisenberg coordinates and of the tangent approximation of Heisenberg diffeomorphisms provided by Proposition 2.21.

3.1. Differentiable groupoids. Here we briefly recall the main definitions about groupoids and illustrate them by the example of Connes’ tangent groupoid.

**Definition 3.1.** A groupoid consists of a set \(G\) together with a distinguished subset \(G^{(0)} \subset G\), two maps \(r\) and \(s\) from \(G\) to \(G^{(0)}\) called the range and source maps, and a composition map,

\[
\circ : G^{(2)} = \{(\gamma_1, \gamma_2) \in G \times G; s(\gamma_1) = r(\gamma_2)\} \rightarrow G,
\]

such that the following properties are satisfied:

(i) \(s(\gamma_1 \circ \gamma_2) = s(\gamma_2)\) and \(r(\gamma_1 \circ \gamma_2) = r(\gamma_1)\) for any \((\gamma_1, \gamma_2) \in G^{(2)}\);

(ii) \(s(x) = r(x) = x\) for any \(x \in G^{(0)}\);

(iii) \(\gamma \circ s(\gamma) = r(\gamma) \circ \gamma = \gamma\) for any \(\gamma \in G\);

(iv) \((\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)\);

(v) Each element \(\gamma \in G\) has a two-sided inverse \(\gamma^{-1}\) so that \(\gamma \circ \gamma^{-1} = r(\gamma)\) and \(\gamma^{-1} \circ \gamma = s(\gamma)\).

The idea about groupoids is that they interpolate between spaces and groups. This especially pertains in the construction by Connes [11, Sect. II.5] (see also [13]) of the tangent groupoid \(G = GM\) of a smooth manifold \(M^{d}\).

At the set theoretic level we let

\[
G = TM \cup (M \times M \times (0, \infty)) \quad \text{and} \quad G^{(0)} = M \times [0, \infty),
\]

where \(TM\) denotes the (total space) of the tangent bundle of \(M\). Here the inclusion \(\iota\) of \(G^{(0)}\) into \(G\) is given by

\[
\iota(m, t) = \begin{cases} 
(m, m, t) & \text{for } t > 0 \text{ and } m \in M, \\
(m, 0) \in TM & \text{for } t = 0 \text{ and } m \in M.
\end{cases}
\]

The range and source maps of \(G\) are such that

\[
\begin{align*}
(3.4) & \quad r(p, q, t) = (p, t) \quad \text{and} \quad s(p, q, t) = (q, t) \quad \text{for } t > 0 \text{ and } p, q \in M, \\
(3.5) & \quad r(p, X) = s(p, X) = (p, 0) \quad \text{for } t = 0 \text{ and } (p, X) \in TM,
\end{align*}
\]

while the composition law is given by

\[
\begin{align*}
(3.6) & \quad (p, m, t) \circ (m, q, t) = (p, q, t) \quad \text{for } t > 0 \text{ and } m, p, q \in M, \\
(3.7) & \quad (p, X) \circ (p, Y) = (p, X + Y) \quad \text{for } t = 0 \text{ and } (p, X) \text{ and } (p, Y) \in TM.
\end{align*}
\]

In fact, the groupoid \(GM\) is a \(b\)-differentiable groupoid in the sense of the definition below.

**Definition 3.2.** A \(b\)-differentiable groupoid is a groupoid \(G\) so that \(G\) and \(G^{(0)}\) are smooth manifolds with boundary and the following properties hold:

(i) The inclusion of \(G^{(0)}\) into \(G\) is smooth;

(ii) The source and range maps are smooth submersions, so that \(G^{(2)}\) is a submanifold with boundary of \(G \times G\);

(iii) The composition map \(\circ : G^{(2)} \rightarrow G\) is smooth.
In the case of the tangent groupoid $\mathcal{G} = \mathcal{G}M$ the topology such that:

- The inclusions of $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)} := M \times M \times (0, \infty)$ into $\mathcal{G}$ are continuous and in such way that $\mathcal{G}^{(1)}$ is an open subset of $\mathcal{G}$;

- A sequence $(p_n, q_n, t_n) \in \mathcal{G}^{(1)}$ converges to $(p, X) \in TM$ if, and only if, $\lim(p_n, q_n, t_n) = (p, p, 0)$ and for any local chart $\kappa$ near $p$ we have
  \[
  \lim_{n \to \infty} t_n^{-1}(\kappa(q_n) - \kappa(p_n)) = \kappa'(p)X.
  \]

One can check that the above condition does not depend on the choice of a particular chart near $p$.

Second, the differentiable structure is obtained by combining that of $TM$ and $\mathcal{G}^{(1)} = M \times M \times (0, \infty)$ with the following chart, from an open subset of $TM \times [0, \infty]$ onto a neighborhood of the boundary $TM \subset \mathcal{G}$,

\[
\gamma(p, X, t) = \begin{cases} (p, \exp_{p}(-tX), t) & \text{if } t > 0 \text{ and } (p, tX) \in \text{dom exp}, \\
(p, X) & \text{if } t = 0 \text{ and } (p, X) \in \text{dom exp},
\end{cases}
\]

where $\exp : TM \subset \text{dom exp} \to M \times M$ is the exponential map associated to an (arbitrary) Riemannian metric on $M$ (see [7], [13], [5]).

### 3.2. The tangent groupoid of a Heisenberg manifold.

Let us now construct the tangent groupoid $\mathcal{G} = \mathcal{G}_{H}M$ of a Heisenberg manifold $(M^{d+1}, H)$. Let

\[
\mathcal{G} = G M \sqcup (M \times M \times (0, \infty)) \quad \text{and} \quad \mathcal{G}^{(0)} = M \times [0, \infty),
\]

where $G M$ denotes the (total space) of the Lie group tangent bundle of $M$. We have an inclusion $\iota : \mathcal{G}^{(0)} \to \mathcal{G}$ as in (3.3), that is

\[
\iota(m, t) = \begin{cases} (m, m, t) & \text{for } t > 0 \text{ and } m \in M, \\
(m, 0) \in GM & \text{for } t = 0 \text{ and } m \in M.
\end{cases}
\]

The range and source maps are defined in a similar way as in (3.6)–(3.7) by letting

\[
\begin{align*}
\tau(p, q, t) &= (p, t) \quad \text{and} \quad s(p, q, t) = (q, t) \quad \text{for } t > 0 \text{ and } p, q \text{ in } M, \\
\tau(p, X) &= s(p, X) = (p, 0) \quad \text{for } t = 0 \text{ and } (p, X) \in GM,
\end{align*}
\]

In addition we endow $\mathcal{G}$ with the composition law,

\[
\begin{align*}
(p, m, t) \circ (m, q, t) &= (p, q, t) \quad \text{for } t > 0 \text{ and } m, p, q \text{ in } M, \\
(p, X) \circ (p, Y) &= (p, X, Y) \quad \text{for } t = 0 \text{ and } (p, X) \text{ and } (p, Y) \text{ in } GM.
\end{align*}
\]

It is immediate to check the properties (i)–(v) of Definition 3.1, noticing that the inverse map here is given by

\[
\begin{align*}
(p, q, t)^{-1} &= (q, p, t) \quad \text{for } t > 0 \text{ and } p, q \text{ in } M, \\
(p, X)^{-1} &= (p, X^{-1}) = (p, -X) \quad \text{for } t = 0 \text{ and } (p, X) \in GM.
\end{align*}
\]

Therefore $\mathcal{G} = \mathcal{G}_{H}M$ is a groupoid.

**Definition 3.3.** The groupoid $\mathcal{G}_{H}M$ is called the tangent groupoid of $(M, H)$.

Let us now turn the groupoid $\mathcal{G} = \mathcal{G}_{H}M$ into a b-differentiable groupoid. First, we endow $\mathcal{G}$ with the topology such that:

- The inclusions of $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(1)} := M \times M \times (0, \infty)$ into $\mathcal{G}$ are continuous and in such way that $\mathcal{G}^{(1)}$ is an open subset of $\mathcal{G}$;

- A sequence $(p_n, q_n, t_n) \in \mathcal{G}^{(1)}$ converges to $(p, X) \in GM$ if, and only if, $\lim(p_n, q_n, t_n) = (p, p, 0)$ and, for any local Heisenberg chart $\kappa : \text{dom } \kappa \to U$ near $p$, we have

\[
\lim_{n \to \infty} t_n^{-1}(\kappa(q_n) - \kappa(p_n)) = (\kappa'(p) \circ \kappa)'_{H}(p)X,
\]
where \( t.x \) is the Heisenberg dilation and \( \varepsilon_u \) denotes the coordinate change to the Heisenberg coordinates at \( u \in U \) with respect to the \( H \)-frame of the Heisenberg chart \( \kappa \) (cf. Definition 2.18).

**Lemma 3.4.** The condition (3.18) is independent of the choice of the Heisenberg chart \( \kappa \).

**Proof.** Assume that (3.18) holds for \( \kappa \). Let \( \kappa_1 \) be another local Heisenberg chart near \( p \) and let \( \phi = \kappa_1 \circ \kappa^{-1} \). Then, setting \( x_n = \kappa(p_n) \) and \( y_n = \kappa(q_n) \), we have

\[
(3.19) \quad t_n^{-1} \cdot \varepsilon_{\kappa_1(p_n)}(\kappa_1(q_n)) = t_n^{-1} \cdot \varepsilon_{\phi(x_n)}(\phi(y_n)) = \delta_{t_n^{-1} \circ \varepsilon_{\phi(x_n)} \circ \phi \circ \varepsilon_{x_n}^{-1} \circ \delta_{t_n}(t_n, \varepsilon_{x_n}(y_n)).
\]

On the other hand, since \( \phi \) is a Heisenberg diffeomorphism it follows from Proposition 2.23 that as \( t \) goes to zero, locally uniformly with respect to \( x \) and \( y \), we have

\[
(3.20) \quad \delta_{t^{-1} \circ \varepsilon_{\phi(x)} \circ \phi \circ \varepsilon_{x}^{-1} \circ \delta_{t}(\varepsilon_{\phi(x)} \circ \phi \circ \varepsilon_{x}^{-1})_H(0)y \rightarrow 0.
\]

Since \( (x_n, y_n, t_n) \rightarrow (\kappa(p), \kappa(p), 0) \) and \( t_n^{-1} \cdot \varepsilon_{\kappa(p_n)}(\kappa(q_n)) \rightarrow (\varepsilon_{\kappa(p)}(\kappa) \circ \kappa)'_H(p)X \) combining this with (3.19) we see that

\[
(3.21) \quad \lim_{n \rightarrow \infty} t_n^{-1} \cdot \varepsilon_{\kappa_1(p_n)}(\kappa_1(q_n)) = (\varepsilon_{\phi(\kappa(p))} \circ \phi \circ \varepsilon_{\kappa(p)}^{-1})'_H(0)[(\varepsilon_{\kappa(p)} \circ \kappa)'_H(p)X] = (\varepsilon_{\kappa_1(p)} \circ \kappa)'_H(p)X.
\]

Hence the lemma. \( \Box \)

Next, to endow \( G_H M \) with a manifold structure we cannot make use of an exponential chart as in (3.3), because unless \( GM \) is a fiber bundle the Lie algebraic structures of its fibers vary from point to point. Instead we make use of local charts as follows.

Let \( \kappa : \text{dom} \kappa \rightarrow U \) be a local Heisenberg chart near \( m \in M \). Then we get a local coordinate system near \( G_m M \subset G \) by letting

\[
(3.22) \quad \gamma_{\kappa}(x, X, t) = \begin{cases} 
(\kappa^{-1}(x), \kappa^{-1} \circ \varepsilon_x^{-1}(t.X), t) & \text{if } t > 0 \text{ and } x \text{ and } \varepsilon_x^{-1}(t.X) \text{ are in } U, \\
(\kappa^{-1}(x), \kappa^{-1} \circ \varepsilon_x^{-1}(t)_H(0)X) & \text{if } t = 0 \text{ and } (x, X) \text{ is in } U \times \mathbb{R}^{d+1}.
\end{cases}
\]

This yields a continuous embedding into \( G \) because \( \gamma_{\kappa} \) is continuous off the boundary \( t = 0 \) and if a sequence \( (x_n, X_n, t_n) \in \text{dom} \gamma_{\kappa} \) with \( t_n > 0 \) converges to \( (x, X, 0) \) then \( (p_n, q_n, t_n) = \gamma_{\kappa}(x_n, X_n, t_n) \) has limit \( (\kappa^{-1}(x), \kappa^{-1} X_H(x)) = \gamma_{\kappa}(x, X, 0), \) since we have

\[
(3.23) \quad t_n^{-1} \cdot \varepsilon_{\kappa(p_n)}(\kappa(q_n)) = X_n \longrightarrow X = \kappa_H'(\kappa(x))(\kappa^{-1})'_H(x)X.
\]

Moreover, the inverse \( \gamma_{\kappa}^{-1} \) here is given by

\[
(3.24) \quad \gamma_{\kappa}(p, q, t)^{-1} = (\kappa(p), t^{-1} \cdot \varepsilon_{\kappa(p)} \circ \kappa(q), t) \quad \text{for } t > 0,
\]

\[
(3.25) \quad \gamma_{\kappa_1}^{-1}(p, X) = (\kappa(p), \kappa_H'(p)X) \quad \text{for } (p, X) \in GM \text{ in the range of } \gamma_{\kappa_1}.
\]

Therefore, if \( \kappa_1 \) is another local Heisenberg chart near \( m \) then, in term of \( \phi = \kappa_1^{-1} \circ \kappa \), the transition map \( \gamma_{\kappa}^{-1} \circ \gamma_{\kappa_1} \) is such that

\[
(3.26) \quad \gamma_{\kappa}^{-1} \circ \gamma_{\kappa_1}(x, X, t) = \begin{cases} 
(\phi(x), t \cdot \varepsilon_{\phi(x)} \circ \phi \circ \varepsilon_{x}^{-1}(t.X), t) & \text{for } t > 0, \\
(\phi(x), \phi_H'(x)X, 0) & \text{for } t = 0.
\end{cases}
\]

This shows that \( \gamma_{\kappa}^{-1} \circ \gamma_{\kappa_1}(x, X, t) \) is smooth with respect to \( x \) and \( X \) and is meromorphic with respect to \( t \) with a possible singularity at \( t = 0 \) only. However, by Proposition 2.23 we have

\[
(3.27) \quad \lim_{t \rightarrow 0} t^{-1} \cdot \varepsilon_{\phi(x)} \circ \phi \circ \varepsilon_{x}^{-1}(t.X) = \phi_H'(x)X.
\]

Thus there is no singularity at \( t = 0 \), so that \( \gamma_{\kappa}^{-1} \circ \gamma_{\kappa_1} \) is a smooth diffeomorphism between open subsets of \( \mathbb{R}^{d+1} \times [0, \infty) \). Therefore, together with the differentiable structure of \( G^{(1)} = M \times \mathbb{R} \times (0, \infty) \) the coordinate systems \( \gamma_{\kappa} \) turn \( G \) into a smooth manifold with boundary.

Next, \( G^{(0)} = M \times [0, \infty) \) is a manifold with coordinate and, as before, the inclusion \( t : G^{(0)} \rightarrow G \) is smooth. Also, the range and source maps again are submersions off the boundary and in a coordinate system \( \gamma_{\kappa} \) near the boundary of \( G \) they are given by

\[
(3.28) \quad r(x, X, t) = (x, t) \quad \text{and} \quad s(x, X, t) = (\varepsilon_x^{-1}(t.X), t),
\]
Since $\partial_{x,t}r$ and $\partial_{x,s}r$ are always invertible it follows that $r$ and $s$ are submersions everywhere.

Now, let us look at the smoothness of the composition map.

**Proposition 3.5.** The composition map $\circ : G^2 \to G$ is smooth.

**Proof.** Since $\circ$ is clearly smooth off the boundary, we only need to understand what happens near the boundary. Using (3.35) we see that in a local coordinate system $\gamma_t$ near the boundary two elements $(x, X, t)$ and $(y, Y, t)$ can be composed iff $y = \varepsilon_x(t, X)$. Then, for $t > 0$ using (3.34) and (3.33) we see that $(x, X, t) \circ (\varepsilon_x^{-1}(t, X), Y, t)$ is equal to

\[
(3.29) \quad \gamma_t^{-1}[(\kappa^{-1}(x), \kappa^{-1}\varepsilon_x^{-1}(t, X), t) \circ (\kappa^{-1}\varepsilon_x^{-1}(t, X), \kappa^{-1} \circ \varepsilon_x^{-1}(t, X)(t, Y), t)]
\]

\[
= \gamma_t^{-1}[(\kappa^{-1}(x), \kappa^{-1} \circ \varepsilon_x^{-1}(t, X)(t, Y), t)] = (x, t^{-1}\varepsilon_x \circ \varepsilon_x^{-1}(t, X)(t, Y), t).
\]

On the other hand, for $t = 0$ from (3.34) and (3.33) we see that $(x, X, 0) \circ (x, Y, 0)$ is equal to

\[
(3.30) \quad \gamma_t^{-1}[(\kappa^{-1}, (\kappa^{-1} \circ \varepsilon_x^{-1} S_H^{-1}(0) X) \circ (\kappa^{-1} \circ \varepsilon_x^{-1} S_H^{-1}(0) Y)]
\]

\[
= \gamma_t^{-1}[(\kappa^{-1}(x), (\kappa^{-1} \circ \varepsilon_x^{-1} S_H^{-1}(0) X) \circ (\kappa^{-1} \circ \varepsilon_x^{-1} S_H^{-1}(0) Y)] = (x, X, 0).
\]

Therefore, we get

\[
(3.31) \quad (x, X, t) \circ (\varepsilon_x^{-1}(t, X), Y, t) = \begin{cases} (x, t^{-1}\varepsilon_x \circ \varepsilon_x^{-1}(t, X)(t, Y), t) & \text{for } t > 0, \\
(x, X, 0) & \text{for } t = 0. 
\end{cases}
\]

This shows that $\circ$ is smooth with respect to $x$, $X$ and $Y$ and is meromorphic with respect to $t$ with at worst a singularity at $t = 0$. Therefore, in order to prove the smoothness of $\circ$ at $t = 0$ it is enough to prove that

\[
(3.32) \quad \lim_{t \to 0^+} t^{-1}\varepsilon_x \circ \varepsilon_x^{-1}(t, X)(t, Y) = X, Y.
\]

**Lemma 3.6.** Let $\psi_u$ denote the affine change to the privileged coordinates at $u$ as in Definition 2.12. Then with respect to the law group of the $u$-group $G(u)$ we have

\[
(3.33) \quad \lim_{t \to 0} t^{-1}\psi_u \circ \psi_u^{-1}(t, w) = v, w,
\]

locally uniformly with respect to $w$.

**Proof of the lemma.** Let $\lambda_v(w) = v, w$ and $\mu_u(w) = t^{-1}\psi_u \circ \psi_u^{-1}(t, v)$. For $w = 0$ we have

\[
(3.34) \quad \mu_0(0) = t^{-1}\psi_u \circ \psi_u^{-1}(0) = t^{-1}\psi_u(0, 0) = v = \lambda_v(0).
\]

Remark also that $\mu_t$ and $\lambda_v$ both are affine maps and we have

\[
(3.35) \quad \mu_t' = \delta^{-1}_t \circ \psi_t' \circ (\psi^{-1}_{\psi^{-1}_{\psi^{-1}_u}(t, v)})' \circ \delta_t.
\]

Next, let $X_0, \ldots, X_d$ be the $H$-frame associated to the Heisenberg chart $\kappa$, seen as a $H$-frame on $U = \text{ran } \kappa$, and set $w_0 = 2$ and $w_1 = \ldots = w_d = 1$. Recall that by (2.30) and (2.31) for $j = 0, \ldots, d$ we have $X_j(u) = (\psi^{-1}_u)'(\delta x_j)$. Therefore, we get

\[
(3.36) \quad (\delta^*_\psi u X_j)(v) = \delta^{-1}_t \circ \psi_t[u X_j(\psi^{-1}_u \circ \delta_t(v))] = \delta^{-1}_t \circ \psi_t' \circ (\psi^{-1}_{\psi^{-1}_u}(t, v))' \circ \delta_t[\delta x_j].
\]

Combining this with (3.35) we thus obtain

\[
(3.37) \quad t^{w_j}(\delta^*_\psi u X_j)(v) = \delta^{-1}_t \circ \psi_t' \circ (\psi^{-1}_{\psi^{-1}_u}(t, v))'[t^{w_j}[\delta x_j]] = \delta^{-1}_t \circ \psi_t' \circ (\psi^{-1}_{\psi^{-1}_u}(t, v))' \circ \delta_t[\delta x_j] = \mu'_t[\delta x_j].
\]
Now, for \( j = 1, \ldots, d \) let \( X_j^{(u)} \) be the left-invariant vector field on \( G^{(u)} \) such that \( X_j^{(u)} = \partial_{x_j} \). Recall that by the very definition of \( G^{(u)} \) we have \( X_j^{(u)} = \lim_{t \to 0} t^u(\delta_j^* \psi_u X_j) \). Thus,

\[
X_j^{(u)}(v) = \lim_{t \to 0} \mu'_t[\partial_{x_j}].
\]

In fact, as \( X_j^{(u)} \) is left-invariant we have

\[
X_j^{(u)}(v) = (\lambda_v X_j^{(u)})(v) = \lambda'_v[X_j^{(u)}(0)] = \lambda'_v[\partial_{x_j}].
\]

Therefore, we have \( \lim_{t \to 0} \mu'_t[\partial_{x_j}] = \lambda'_v[\partial_{x_j}] \) for \( j = 0, \ldots, d \), which yields

\[
\lim_{t \to 0} \mu'_t = \lambda'_v.
\]

Since by (3.34) we have \( \mu_t(0) = \lambda_v(0) \) and since \( \mu_t \) and \( \lambda_v \) both are affine maps it follows that as \( t \) goes to zero \( \mu_t(w) = t^{-1} \psi_u \circ \psi_u^{-1}(t.w) \) converges to \( \lambda_v(w) = v.w \) locally uniformly with respect to \( w \). Hence the claim.

Next, let \( \phi_x \) be the \( x \)-coordinate-to-Heisenberg-coordinate map given by (2.11). Recall that \( \phi_x \) is an isomorphism of graded Lie groups from \( G^{(x)} \) to the tangent group \( G_x = (\kappa_s GM)_x \). Therefore, as \( \varepsilon_x = \phi_x \circ \psi_x \), we get

\[
t^{-1} \varepsilon_x \circ \varepsilon_x^{-1}(t.X) (t.Y) = \delta_x^{-1} \circ \phi_x \circ \psi_x \circ \psi_x^{-1} \circ \phi_x \circ \delta_x^{-1}(t.X) \circ \delta_t(Y)
\]

where we have let \( v = \phi_x^{-1}(X) \) and \( w_t = \phi_x^{-1}(t.X) \). Combining this with (3.39) we then get

\[
\lim_{t \to 0} t^{-1} \varepsilon_x \circ \varepsilon_x^{-1}(t.X) (t.Y) = \phi_x \circ \psi_x \circ \psi_x^{-1}(Y) = \phi_x(\phi_x^{-1}(X), \phi_x^{-1}(Y)) = X.Y.
\]

This proves (3.38) and so completes the proof of the smoothness of the composition map.

Summarizing all this we have proven:

**Theorem 3.7.** The groupoid \( G_H M \) is a b-differentiable groupoid.

Let us now look at the effect of a Heisenberg diffeomorphism \( \phi : (M, H) \to (M', H') \) on the groupoid \( G_H M \). To this end consider the map \( \Phi_H : G_H M \to G_H M' \) given by

\[
\Phi_H(p, q, t) = (\phi(p), \phi(q), t) \quad \text{for } t > 0 \text{ and } p, q \in M,
\]

\[
\Phi_H(p, X) = (\phi(p), \phi'_H(p) X) \quad \text{for } (p, X) \in GM.
\]

Then for \( t > 0 \) and \( p, q \in M \) we have

\[
r_{M'} \circ \Phi_H(p, q, t) = (\phi(q), t) = \Phi_H \circ r_M(p, q, t),
\]

\[
s_{M'} \circ \Phi_H(p, q, t) = (\phi(p), t) = \Phi_H \circ s_M(p, q, t),
\]

while for \( (p, X) \in GM \) we have

\[
r_{M'} \circ \Phi_H(p, X) = r_{M'} \circ \Phi_H(p, X) = (\phi(p), 0) = \Phi_H \circ r_M(p, X) = \Phi_H \circ s_M(p, X).
\]

Thus \( r_{M'} \circ \Phi_H = \Phi_H \circ r_M \) and \( s_{M'} \circ \Phi_H = \Phi_H \circ s_M \). Incidentally, we have \( \Phi_H(G_H^{(2)} M) = G_H^{(2)} M' \). Furthermore, for \( t > 0 \) and \( m, p, q \in M \) we get

\[
\Phi_H(m, p, t) \circ_M \Phi_H(p, q, t) = (\phi(m), \phi(q), t) = \Phi_H[(m, p, t) \circ_M (p, q, t)],
\]

and for \( p \) in \( M \) and \( X, Y \) in \( G_p M \) we obtain

\[
\Phi_H(p, X) \circ_M \Phi_H(p, Y) = (\phi(p), \phi'_H(p)(X,Y)) = \Phi_H[(p, X) \circ_M \Phi_H(p, Y)].
\]
All this shows that $\Phi_H$ is a morphism of groupoids. In fact, the map defined as in (3.43) and (3.44) by replacing $\phi$ by $\phi^{-1}$ is an inverse for $\Phi_H$, so $\Phi_H$ is in fact a groupoid isomorphism from $G_H M$ onto $G_{H'} M'$.

Next, it follows from (3.43) that $\Phi_H$ is continuous off the boundary. To see what happens at the boundary consider a sequence $(p_n, q_n, t_n)$ converging to $(p, X) \in GM$ and let $\kappa$ be a local Heisenberg chart for $M'$ near $p' = \phi(p)$. Then pulling back the $H'$-frame of $\kappa$ by $\phi$ turns $\kappa \circ \phi$ into a Heisenberg chart, so that setting $(p'_n, q'_n, t_n) = \Phi_H(p_n, q_n, t_n)$ we get

\begin{equation}
(3.50) \quad t_n^{-1} \varepsilon_{\kappa(q'_n)}(\kappa(q'_n)) = t_n \varepsilon_{\kappa \circ \phi(p_n)}(\kappa \circ \phi(q_n)) \longrightarrow (\kappa \circ \phi)'_H(p)X = \kappa'_H(p)(\phi'_H(p)X).
\end{equation}

Thus $\Phi_H$ is continuous from $G_H M$ to $G_{H'} M'$.

In fact, it also follows from (3.43) that $\Phi_H$ is smooth off the boundary. Moreover, if $\kappa$ is a local Heisenberg chart for $M'$ then $\Phi_H \circ \gamma_{\kappa \circ \phi}(p, X, t)$ coincides for $t > 0$ with

\begin{equation}
(3.51) \quad (\phi(\phi^{-1} \circ \kappa^{-1}(x), \phi^{-1} \circ \kappa^{-1} \circ \varepsilon_x^{-1}(t, X)), t) = (\kappa^{-1}(x), \kappa^{-1} \circ \varepsilon_x^{-1}(t, X), t) = \gamma_{\kappa}(x, X, t),
\end{equation}

while for $t = 0$ it is equal to

\begin{equation}
(3.52) \quad (\phi(\phi^{-1} \circ \kappa^{-1}(x), \phi'_H(\phi^{-1} \circ \kappa^{-1}(x))(\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0, X)), 0)
= (\kappa^{-1}(x), (\kappa^{-1} \circ \varepsilon_x^{-1})'_H(0, X), 0) = \gamma_{\kappa}(x, X, 0).
\end{equation}

Hence $\gamma_{\kappa} \circ \phi \circ \gamma_{\kappa \circ \phi} = \text{id}$, which shows that $\Phi_H$ is smooth map. Since similar arguments show that $\Phi_H^{-1}$ is smooth, it follows that $\Phi_H$ is a diffeomorphism. We have thus proved:

**Proposition 3.8.** The map $\Phi_H : G_H M \to G_{H'} M'$ given by (3.43)-(3.44) is an isomorphism of $b$-differentiable groupoids. Hence the isomorphism class of $b$-groupoids of $G_H M$ depends only on the Heisenberg-diffeomorphism class of $(M, H)$.

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