Topical Review

Gromov–Witten invariants and localization*

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Abstract
We give a pedagogical review of the computation of Gromov–Witten invariants via localization in 2D gauged linear sigma models. We explain the relationship between the two-sphere partition function of the theory and the Kähler potential on the conformal manifold. We show how the Kähler potential can be assembled from classical, perturbative, and non-perturbative contributions, and explain how the non-perturbative contributions are related to the Gromov–Witten invariants of the corresponding Calabi–Yau manifold. We then explain how localization enables efficient calculation of the two-sphere partition function and, ultimately, the Gromov–Witten invariants themselves.

Keywords: Gromov–Witten, Kahler potential, Calabi–Yau manifold

(Some figures may appear in colour only in the online journal)

1. Introduction

Many of the early studies of conformal field theories in two dimensions were motivated by the connection of these theories to perturbative string theory. When the string theory is being compactified on a compact manifold $X$ (typically a Calabi–Yau manifold), the resulting conformal field theory can be described in terms of the nonlinear sigma model with target space $X$. One of the interesting features of these theories is the phenomenon of mirror symmetry [2–4]: two different Calabi–Yau manifolds $X$ and $Y$ can lead to conformal field theories which are identical save for a relabeling of the action of the superconformal algebra.

The celebrated paper of Candelas, de la Ossa, Green, and Parkes [5] exploited mirror symmetry to provide a new way to calculate instanton contributions to the sigma model (now known as ‘Gromov–Witten invariants’) [6, 7]), appealing to the fact that instanton-corrected correlation functions in one theory corresponded to correlation functions in the other theory

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which receive no quantum corrections. This powerful method, eventually formalized as a mathematical ‘mirror theorem’ [8, 9], only works when the mirror partner of a given Calabi–Yau manifold is known. Subsequent developments in mathematics (cf. [10]) suggest that it should be possible to determine the Gromov–Witten invariants without recourse to the mirror, and that has now been achieved in a physics context as well [11]. This new physical method for finding Gromov–Witten invariants is the subject of the present review.

The method is a by-product of a recent theme in the study of supersymmetric quantum field theories, which formulates a given theory on a sphere or product of spheres, and evaluates physical quantities such as the partition function by means of localization. This theme was pioneered in four dimensions by Pestun [12], and has subsequently been extended to a number of different dimensions and contexts, many of which are covered in the collection of reviews [1].

For theories in dimension two with \( \mathcal{N} = (2, 2) \) supersymmetry, the formulation on the two-sphere and the corresponding localization computations were carried out in [13, 14]. The authors of [11] then recognized that there was a connection between the partition function on the two-sphere and the Zamolodchikov metric on the conformal manifold of the theory, formulating this as a precise conjecture. They also showed how the conjecture would enable the calculation of Gromov–Witten invariants from the data of the partition function, without needing a mirror Calabi–Yau manifold.

Compelling arguments in favor of the conjecture were soon given in [15, 16]. We present here instead a more recent argument [17] which explains the result as arising from an anomaly of the conformal field theory. We review that argument in section 2.

In section 3 we then discuss two kinds of sigma models: the nonlinear sigma model with Calabi–Yau target space (including the classical Kähler potential on the conformal manifold), and the ‘gauged linear sigma model’ of [18], which is where we shall carry out our localization computations. Quantum corrections to the Kähler potential, including the non-perturbative corrections associated to Gromov–Witten invariants, are discussed in section 4.

The determination of the two-sphere partition function via localization, and the corresponding method for calculating Gromov–Witten invariants, is reviewed in section 5. Finally, in section 6 we discuss the evaluation of the partition function via residues and show how to obtain Gromov–Witten invariants explicitly. We have collected supplementary material in an appendix.

We have drawn heavily upon [11, 17, 19–22] in preparing this review.

2. Kähler potentials and 2-sphere partition functions

Let us consider the exactly marginal operators for a two-dimensional conformal field theory. These are operators \( O_I \) having the property that, if added to the action with coupling constants \( \lambda^I \)

\[
\delta S = \frac{1}{\pi} \sum_I \int d^2x \lambda^I O_I(x),
\]

(2.1)

they leave the theory conformally invariant. The coupling constants \( \lambda^I \) parameterize the conformal manifold \( \mathcal{M} \) of the theory, and the two-point functions

\[
\langle O_I(x)O_J(y) \rangle = \frac{g_{IJ}(\lambda^k)}{(x-y)^2}
\]

(2.2)
determine the Zamolodchikov metric $g_{IJ}$ on $\mathcal{M}$ [23].

In momentum space the two-point functions (2.2) take the form

$$\langle O_I(p)O_J(-p) \rangle \sim p^4 \log \left( \frac{\Lambda^2}{p^2} \right).$$  \hspace{1cm} (2.3)

Having a logarithmic behavior with cutoff $\Lambda$ does not violate scale invariance since any rescaling of $\Lambda$ can be compensated with a contact term. However, although they do not spoil conformal invariance, these logarithms lead to the non-conservation of the dilatation charge in the presence of non-vanishing background fields (the original ‘conformal anomaly’). This can be detected by promoting the couplings $\lambda^I$ to fields [24]. Then the anomaly induces a term in the energy-momentum trace of the rough form

$$T^\mu_\mu = g_{IJ} \lambda^I \Box \lambda^J + \cdots.$$ \hspace{1cm} (2.4)

In this section, we shall discuss a further conformal anomaly under variation of the 2D metric, following [17]. This anomaly was first observed in [25], and is again consistent with scale invariance due to the possibility of contact terms in the two-point function.

We assume as in [17] that the given conformal field theory can be regulated in a diffeomorphism-invariant way, including a metric $\gamma_{\mu\nu}$ on the 2D spacetime as well as spacetime-dependent couplings $\lambda^I$. The partition function of the theory on this spacetime then depends on the metric and couplings, taking the form $Z[\gamma_{\mu\nu}; \lambda^I]$.

We consider an infinitesimal Weyl transformation

$$\delta_\sigma \gamma_{\mu\nu} = 2 \gamma_{\mu\nu} \delta_\sigma$$ \hspace{1cm} (2.5)

(where the infinitesimal $\delta_\sigma$ has compact support) and ask for the corresponding variation $\delta_\sigma \log Z$ of the partition function. A precise form of the infinitesimal Weyl variation of $\log Z$ is derived in [25] and takes the form

$$\delta_\sigma \log Z = \frac{c}{24\pi} \int d^2x \, \delta_\sigma \sqrt{\gamma} \, R - \frac{1}{4\pi} \int d^2x \, \delta_\sigma \sqrt{\gamma} \, g_{IJ} \gamma_{\mu\nu} \partial_\mu \lambda^I \partial_\nu \lambda^J,$$ \hspace{1cm} (2.6)

where $R$ is the Ricci scalar. The first term is a universal contribution due to the central charge $c$ of the theory. It is argued in [17] that no anomalies other than (2.6) are possible.

The ‘conformal anomaly’ functional (2.6) describes a sigma model with target space $\mathcal{M}$, and is not the Weyl variation of any local counterterm. It is therefore cohomologically nontrivial.

There is an allowed local counterterm of the form

$$\int d^2x \, \sqrt{\gamma} \, R \, F(\lambda^I)$$ \hspace{1cm} (2.7)

whose Weyl variation is

$$\delta_\sigma \int d^2x \, \sqrt{\gamma} \, R \, F(\lambda^I) = -2 \int d^2x \, \sqrt{\gamma} \, \Box (\delta_\sigma) F(\lambda^I).$$ \hspace{1cm} (2.8)

Thus, (2.6) can be shifted by terms of the form (2.8).

In the case of an $\mathcal{N} = (2, 2)$ theory, exactly marginal operators can either be chiral or twisted chiral:

\footnote{We establish some notation and properties for these theories in an appendix.}
\[ \delta S = \frac{1}{\pi} \int d^2x \left( \sum_{I} \lambda^I \int d^2 \theta \mathcal{O}_I(x, \theta) + \sum_{A} \tilde{\lambda}^A \int d^2 \theta \tilde{\mathcal{O}}_A(x, \theta) + \text{c.c.} \right), \] 

(2.9)

where \( \mathcal{O}_I \) is chiral and \( \tilde{\mathcal{O}}_A \) is twisted chiral. The analysis in [17] assumes that the parameters \( \lambda^I \) and \( \tilde{\lambda}^A \) can be promoted to background chiral and twisted chiral superfields respectively, and we make the same assumption.

We wish to supersymmetrize the conformal anomaly (2.6) and the counterterm (2.7). In order to do so, we place the theory in curved superspace [26]. The possibilities for doing so with \( \mathcal{N} = (2, 2) \) supersymmetry were analyzed in [27, 28], and amount to a coupling to supergravity.

There are two distinct supergravities to which we could couple, known as \( U(1)_V \) and \( U(1)_A \) [29]; the label indicates whether the \( U(1) \) symmetry preserved in the Poincaré supergravity theory is vector or axial. From the point of view of the \( \mathcal{N} = (2, 2) \) SCFT, the theory has an \( R \)-symmetry of the form \( U(1)_V \times U(1)_A \), and we can couple either factor (but not both) to a background gauge field. As in [17], we assume\(^2\) that the theory can regularized so as to preserve diffeomorphism invariance, supersymmetry, and either \( U(1)_V \) or \( U(1)_A \); once this is done, the other \( R \)-symmetry cannot be preserved by the regularization scheme. In particular, our assumptions imply that there are no gravitational anomalies and that \( c_L = c_R \).

Since every two-dimensional metric is conformally flat, the conformal factor \( \sigma \) may be used to specify the metric. When we supersymmetrize, the conformal factor becomes part of a superfield. In the case of \( U(1)_A \) supergravity, it is the scalar in a twisted chiral superfield \( \Sigma \). As in [17], we make the same assumption.

\(^2\) This issue is discussed in detail in [27] building on the general framework of [26] (see also [30]).

\(^3\) This product structure holds at smooth points; at certain singular points we may need to take the quotient of the product by a finite group [31, 32].
We also need a supersymmetric version of the allowed local counterterm. In the $U(1)_V$ case this takes the form [16]

$$S_V = \frac{1}{4\pi} \int d^2x d^2\theta \mathcal{R} F(\tilde{\lambda}) + \text{c.c.} = \frac{1}{4\pi} \int d^2x d^4\theta \mathcal{S} F(\tilde{\lambda}) + \text{c.c.} \quad (2.13)$$

where $\mathcal{R} = \mathcal{D}^2 \mathcal{S}$ is the twisted chiral curvature superfield in superconformal gauge. The counterterm (2.13) depends only on the twisted chiral parameters $\tilde{\lambda}$ and the dependence is holomorphic. Under a super-Weyl transformation,

$$\delta_{\mathcal{S}} S_V = \frac{1}{4\pi} \int d^2x d^4\theta \left( \delta \mathcal{S} F(\tilde{\lambda}) + \delta \mathcal{S} F(\tilde{\lambda}) \right). \quad (2.14)$$

The effect of adding a local counterterm of the form (2.13) is to shift the twisted chiral Kähler potential

$$K_\text{tc} \rightarrow K_\text{tc} + F(\tilde{\lambda}) + F(\tilde{\lambda}). \quad (2.15)$$

The chiral Kähler potential $K_c$ is unchanged by the addition of counterterms.

The conformal anomaly will affect the partition function whenever the theory is placed on a curved manifold with non-trivial topology. In particular, for compactification on a two-sphere, both the dependence on the radius (via the central charge) and the radius-independent part of the anomaly will be visible in the partition function. If we compactify so as to preserve the $U(1)_A$ symmetry, the partition function will detect $K_c(\lambda, \bar{\lambda})$ and be independent of $\tilde{\lambda}$; on the other hand, if we compactify so as to preserve the $U(1)_V$ symmetry (as we will do here), the partition function takes the form

$$Z_{\mathcal{S}} = \left( \frac{r}{r_0} \right)^{c/3} e^{-K_\text{tc}(\tilde{\lambda}, \bar{\lambda})} \quad (2.16)$$

(where $r_0$ is a fixed scale), as conjectured in [11]$.^6$ This quantity is independent of scale and can be calculated in the ultraviolet, for example on a gauged linear sigma model, or directly in the infrared.

3. Metrics on conformal manifolds

We now introduce two classes of $\mathcal{N} = (2, 2)$ theories which give rise to conformal theories in the infrared.

3.1. Nonlinear sigma models

A nonlinear sigma model whose target is a Calabi–Yau manifold $X$ of complex dimension $n$ is a 2D quantum field theory with $\mathcal{N} = (2, 2)$ supersymmetry which is expected to flow to a conformal theory of central charge $c = 3n$ in the infrared. In fact, the $\beta$-function of such a theory vanishes at one-loop, although there are in general higher loop corrections [33]$^7$. We choose the action of the $\mathcal{N} = (2, 2)$ algebra on the nonlinear sigma model in such a way that the chiral marginal operators correspond to the harmonic $(n-1, 1)$-forms on $X$, and the twisted chiral

$^5$ Although in this paper we allow shifts such as (2.15), the authors of [17] argue further that allowing local shifts of the Kähler potential would be incompatible with having a single Lagrangian valid throughout spacetime. Hence (they argue), such local shifts must be forbidden and $K_\text{tc}$ should actually be globally defined.

$^6$ The radial dependence was suppressed in [11].

$^7$ In spite of these perturbative corrections, one still expects a CFT in the infrared [34].
marginal operators correspond to the harmonic \((1,1)\)-forms. Thus, the chiral conformal manifold \(\mathcal{M}_c\) corresponds to the ‘moduli space of \(X\)’ studied in algebraic geometry which specifies the possible complex structures on \(X\). The twisted chiral conformal manifold \(\mathcal{M}_{tc}\), however, has no straightforward identification in mathematics. Near the ‘large radius limit’ boundary point it is parameterized by the choice of complexified Kähler form on \(X\), which is a complex combination of the Kähler form \(\omega\) and the Kalb–Ramond two-form field \(B\) (which is only well-defined up to shifts by an integral two-form). For this reason, \(\mathcal{M}_{tc}\) is sometimes referred to as the ‘complexified Kähler moduli space’, with coordinate \(t = i\omega + B\).

The Zamolodchikov metric on the chiral conformal manifold \(\mathcal{M}_c\) can be identified [35] with the Weil–Petersson metric which was described by Tian [36] and Todorov [37]. For this description, we consider the family of complex manifolds \(X \rightarrow \mathcal{M}_c\), corresponding to the variation of complex structure, and let \(\Omega\) be a nonvanishing relative holomorphic \(n\)-form on \(\pi^{-1}(U)\) over an open set \(U \subset \mathcal{M}_c\). Then the function

\[
K_c := -\log \left( \int_X \Omega \wedge \overline{\Omega} \right),
\]

which is a real-valued function on \(U\), is a Kähler potential for the Zamolodchikov metric restricted to \(U\). Any other choice of \(\Omega\) takes the form \(e^{-F} \Omega\) for a nonvanishing holomorphic function \(e^{-F}\) on \(U\). If we make such a change, then

\[
-\log \left( \int_X \Omega \wedge \overline{\Omega} \right) \mapsto -\log \left( \int_X e^{-F} \Omega \wedge \overline{\Omega} \right) = -\log \left( \int_X \Omega \wedge \overline{\Omega} \right) + F + F',
\]

as expected for a Kähler potential. Due to some powerful non-renormalization theorems [24, 38], this formula for the Kähler potential on \(\mathcal{M}_c\) is not subject to quantum corrections.

On the other hand, the twisted chiral conformal manifold \(\mathcal{M}_{tc}\) has a classical approximation in terms of the Kähler cone \(K_X\) of \(X\), complexified to \(H^2(X, \mathbb{R}) + iK_X\) by the inclusion of the Kalb–Ramond field. In the simplest case\(^8\), there are line bundles \(L_j\) whose first Chern classes \(c_1(L_j)\) form a basis for \(H^2(X, \mathbb{Z})\) and also generate the Kähler cone:

\[
K_X = \mathbb{R}_{>0} c_1(L_1) + \cdots + \mathbb{R}_{>0} c_1(L_s).
\]

If \(t_1, \ldots, t_s\) are the corresponding complex coordinates on \(H^2(X, \mathbb{R}) + iK_X\), then \(e^{2\pi i t_1}, \ldots, e^{2\pi i t_s}\) are local coordinates on the twisted chiral conformal manifold. With respect to these coordinates, the Kähler potential for the Zamolodchikov metric on \(\mathcal{M}_{tc}\) has a classical expression

\[
K_{tc} = -\log \left( \frac{1}{(2\pi)^s} \exp \left( \sum t_j F_j \right) \wedge \exp \left( \sum t_j F_j \right) \right) + \cdots,
\]

where \(F_j\) is the curvature of a connection on the bundle \(L_j\), expressed as a 2-form (with indices suppressed), and the exponential is computed as a power series in which differential forms of even degree are multiplied using the wedge product.

Using the fact that \(L_j := c_1(L_j)\) is an integral cohomology class represented by the differential form \(\frac{1}{2\pi i} F_j\), (3.4) can be rewritten in terms of integral cohomology (evaluated on the fundamental homology class \([X]\)) as

\(^8\)Other cases can be handled by expressing \(K_X\) as a union of such ‘integer basis’ cones up to automorphism; see [39].
\[ e^{-K_u} = \left( \exp \left( \sum_2 \text{Im}(t_j)L_j \right) \right) [X] + \cdots \] (3.5)

where in this formula, the multiplication in the power series expansion is represented by cup product. Only the term in the exponential of degree \( n \) contributes to this classical expression, which can be written as

\[ e^{-K_u} = \frac{1}{n!} \left( \sum_2 \text{Im}(t_j)L_j \right)^n [X] + \cdots \] (3.6)

and depends on the intersection pairings among the integral divisors \( L_j \), which are specified by the cohomology ring of \( X \).

In either form, this classical expression is subject to both perturbative and non-perturbative corrections, to be discussed in the next section.

3.2. Gauged linear sigma models

Another approach to conformal field theories on Calabi–Yau manifolds is to start with a Lagrangian theory in the UV known as a gauged linear sigma model [18].

A gauged linear sigma model (GLSM) is formulated in \( \mathcal{N} = (2, 2) \) superspace, and involves a compact gauge group \( G \) as well as \( N \) chiral matter multiplets transforming in a representation \( \Psi : G \to U(N) \). We denote the corresponding representation of the Lie algebra \( g \) of \( G \) by \( \psi : g \to \mathfrak{u}(N) \), so that

\[ \Psi(e^{2\pi i Y}) = e^{2\pi i \psi(Y)} \] (3.7)

for \( Y \in g \). To streamline our later analysis, we fix a Cartan subgroup \( H \subset G \) (i.e. a maximal connected abelian subgroup) with corresponding Cartan subalgebra \( \mathfrak{h} \subset g \), and choose coordinates \( \phi_J \) on the complex vector space \( \mathbb{C}^N \) such that each \( \phi_J \) is a simultaneous eigenvector for \( \Psi|_H \). The eigenvalues of \( \Psi|_H \) can be specified by means of the weight lattice \( \Lambda_{\text{wt}} \subset \mathfrak{h}^* \) of \( G \), which gives the eigenvalues for the corresponding representation of \( \mathfrak{h} \). That is, for each \( \phi_J \) there is a weight vector \( w_J \in \Lambda_{\text{wt}} \subset \mathfrak{h}^* \) such that for \( h = e^{2\pi i Y} \in H \),

\[ \Psi(h)(\phi_J) = e^{2\pi i w_J,Y} \phi_J, \] (3.8)

using a dot to denote the pairing between \( \mathfrak{h}^* \) and \( \mathfrak{h} \).

One of the interaction terms in the Lagrangian is specified by means of a \( G \)-invariant ‘superpotential’ polynomial \( W(\phi_1, \ldots, \phi_N) \). We will also construct a term in the Lagrangian from Lie algebra characters \( \xi : g \to \mathfrak{u}(1) \) which arise from one-dimensional representations \( \Xi : G \to U(1). \) It is convenient to choose a basis \( \xi_1, \ldots, \xi_k \) for the lattice of such characters. All of these characters are trivial on the commutator \([g,g] \) and so factor through the abelianization \( a = g/[g,g] \).

To construct the GLSM, we begin with \( N \) chiral superfields \( \Phi_J \) (i.e. satisfying \( \overline{D}_+ \Phi_J = \overline{D}_- \Phi_J = 0 \)) interacting via the holomorphic superpotential \( W(\Phi_1, \ldots, \Phi_N) \). The model is invariant under the action of \( G \) (via \( \Xi \)) on \( \mathbb{C}^N \) and we gauge this action, preserving \( \mathcal{N} = (2, 2) \) supersymmetry, by introducing a \( g \)-valued vector multiplet \( V \) with invariant field strength \( \Sigma = \frac{1}{\sqrt{2}} \overline{D}_+ \overline{D}_- V \). This last field is twisted chiral, which means that \( \overline{D}_+ \Sigma = D_- \Sigma = 0 \). We include a topological theta angle \( \vartheta \) and a Fayet–Iliopoulos D-term with coefficient \( \zeta \), each

\[ \text{We follow the usual physics convention of putting an } i \text{ in the exponential map so that the Lie algebra consists of Hermitian operators. We also put a factor of } 2\pi \text{ to clarify the integral structure.} \]
taking values in $\mathfrak{a}^* = \text{Ann}(\mathfrak{g}, \mathfrak{g}) \subset \mathfrak{g}^*$. These terms are naturally written in terms of the complex combination $\tau = i\zeta + \frac{1}{2\pi} \vartheta$ or its exponential $z = e^{2\pi i \tau} = e^{-2\pi i \vartheta + i\varphi}$. We introduce a pairing between $\mathfrak{a}^*$ and $\mathfrak{g}$ defined by

$$\langle \tau, \Sigma \rangle := \sum \tau_a \text{tr}_{\mathfrak{g}}(\Sigma),$$

which is independent of the choice of basis. The resulting Lagrangian density is

$$\mathcal{L} = \int d^4 \theta \left( \|e^{\psi(V)} \Phi \|^2 - \frac{1}{4e^c} \|\Sigma\|^2 \right) + \left( \int d\theta^+ d\bar{\theta}^- W(\Phi_1, \ldots, \Phi_N) + \text{c.c.} \right) + \left( \frac{i}{\sqrt{2}} \int d\theta^+ d\bar{\theta}^- \langle \tau, \Sigma \rangle + \text{c.c.} \right),$$

where for simplicity of notation, we have set all gauge couplings of irreducible factors of $G$ to a single value $e$. The marginal couplings of these theories are the coefficients of the superpotential (for $\mathcal{M}_d$) and the choice of D-term coefficient and $\theta$-angle (for $\mathcal{M}_c$).

We will assume that these theories admit both a vector-like symmetry $U(1)_V$ and an axial-like symmetry $U(1)_A$. In flat space, a given action of $U(1)_V \times U(1)_A$ may be modified by a global symmetry but on $S^2$, changing the charges of the fields (other than by adding gauge charges) produces a distinct theory [13, 14]. For this reason, we shall regard the specification of these charges as part of the data of the theory. We design our choice with the expectation that, should the theory flow to a superconformal theory in the IR, the specified $U(1)_V \times U(1)_A$ will become the $R$-symmetry which is part of the superconformal algebra. In general we allow these $R$-charges to be rational numbers, although for many purposes it is best if they are integers up to gauge transformation. We will study this theory on $S^2$ preserving the $U(1)_V$ symmetry, in order to analyze the metric on the twisted chiral conformal manifold.

Having specified an $R$-charge $q \in \mathbb{Q}$ for a given superfield, the general form of the vector-like symmetry is

$$e^{i\alpha \lambda} : \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\alpha q} \mathcal{F}(x^\mu, e^{-i\alpha} \theta^\pm, e^{i\alpha} \bar{\theta}^\pm)$$

while the general form of the axial-like symmetry is

$$e^{i\beta \lambda} : \mathcal{F}(x^\mu, \theta^\pm, \bar{\theta}^\pm) \mapsto e^{i\beta q} \mathcal{F}(x^\mu, e^{\mp i\beta} \theta^\pm, e^{\pm i\beta} \bar{\theta}^\pm).$$

Note that the superpotential, if nonzero, must have $R$-charge 2.

In flat space, such a symmetry can have an anomaly in the presence of gauge fields. A quick computation [18, 19] shows that the anomaly is given by a function on the Lie algebra proportional to $V \mapsto \text{tr}(\psi(V))$; we require that this vanish identically so that the symmetries are not anomalous. Since the action of the continuous part of $G$ on the monomial $\Phi_1 \cdots \Phi_N$ is via $\exp(\text{tr}(\psi(V)))$, this is the same as requiring that $\Phi_1 \cdots \Phi_N$ be invariant under the continuous part of the gauge group $G$. In addition, in order to ensure integral $R$-charges up to gauge

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10 Globally, as discussed in [40–42], we should identify $e^{-2\pi \zeta + i\varphi}$ with an element of $\text{Hom}(\pi_1(G), \mathbb{C}^*)^{G_0}$, where $G_0$ is the connected component of $G$.

11 Both $\tau$ and $z$ are local coordinates on the twisted chiral conformal manifold. To avoid cluttering our formulas, we suppress the ‘tilde’ on these variables which should be present for consistency with the notation of section 2.

12 More precisely, the coefficients account for the marginal chiral couplings with some redundancy; see [43] or [20] for an account of this.
transformation, we require that $\Phi_1 \cdots \Phi_N$ be invariant under the entire group $G$ \cite{44}. In other words, we must require that
\[ \Psi(G) \subset SU(N), \tag{3.13} \]
and we will impose that requirement henceforth.

Finally, using the $R$-symmetry and calculating as in \cite{45}, one finds that the central charge $c$ of the fixed-point CFT is determined by
\[ c = \frac{d}{3} - \sum_{J=1}^{N} q_J, \tag{3.14} \]
where the sum extends over all of the chiral superfields, and where $d$ is the difference between the number of chiral fields and the number of gauge fields.

### 3.3. Phases of an abelian GLSM

As explained in detail in \cite{18}, a GLSM with an abelian gauge group and an anomaly-free $R$-symmetry (i.e. $\sum w_J = 0$) can be described very explicitly at low energy and in many cases coincides with a nonlinear sigma model with target a Calabi–Yau manifold. In such cases, the correspondence only holds when the FI-parameters are in a certain range of values, and the typical GLSM has other phases with different low-energy descriptions in addition to the geometric phase(s).

The low-energy analysis begins by mapping out the space of classical vacua of the theory. The algebraic equations of motion for the auxiliary fields $D_a$ in the vector multiplets and $F_J$ in the chiral multiplets can be solved:
\[ D_a = -e^2 \left( \sum_{J=1}^{N} (w_J)_a |\phi_J|^2 - \zeta_a \right) \tag{3.15} \]
\[ F_J = -\frac{\partial W}{\partial \phi_J}. \tag{3.16} \]

The potential energy for the bosonic zero modes is then
\[ U = \frac{1}{2e^2} \sum_{a=1}^{b} D_a^2 + \sum_{J=1}^{N} |F_J|^2 + \sum_{a,b} \sigma_a \sigma_b \sum_{J=1}^{N} (w_J)_a (w_J)_b |\phi_J|^2, \tag{3.17} \]
where $\phi_J$, $\sigma_a$ are the lowest components of $\Phi_J$, $\Sigma_a$ respectively. The space of classical vacua is the quotient by $G$ of the set of zeros of $U$.

Suppressing any solutions with $\sigma \neq 0$ (which are absent for generic values of the parameters), the space of solutions is
\[ (D^{-1}(0) \cap \text{Crit}(W))/G \subset D^{-1}(0)/G. \tag{3.18} \]

Since $G$ is abelian, this quotient has a description as a toric variety of dimension $N - h$, as we now review. The group $G$ has a natural complexification $G_{\mathbb{C}}$ in which each $U(1)$ factor is promoted to the complex group $\mathbb{C}^*$; the action of $G$ on $V$ extends to an action of $G_{\mathbb{C}}$. For any choice of FI-parameters, the space $D^{-1}(0)/G$ has a description as a GIT quotient
\[ \mathbb{C}^N // G_{\mathbb{C}} = (\mathbb{C}^N - Z_\zeta)/G_{\mathbb{C}}, \tag{3.19} \]
where \( Z_\zeta \) is the union of all \( G_C \)-orbits which do not meet \( D^{-1}(0) \). (The dependence on the FI-parameters \( \zeta \) comes from their inclusion in the D-term equation (3.15)).

The nature of the quotient space changes as \( \zeta \) varies, and can be systematically described by a construction known as the ‘secondary fan’ \([46–49]\). Let \( \mathcal{J} \subset \{1, \ldots, N\} \) be a collection of \( h \) indices such that the weight vectors \( \{w_J, J \in \mathcal{J}\} \) are linearly independent; we wish to know if \( D^{-1}(0) \) contains entries in which

\[
\phi_J = 0 \text{ for all } J \notin \mathcal{J}.
\]

(If so, then the corresponding orbit \( O_{\mathcal{J}} \) lies in the quotient space \([13]\).) To answer this, note that if we impose (3.20) then the D-term equations become

\[
\zeta = \sum_{J \in \mathcal{J}} |\phi_J|^2 w_J.
\]

(3.21)

In other words, since all coefficients on the right side of (3.21) are nonnegative, \( \zeta \) must lie in the cone \( C_{\mathcal{J}} \) in FI-parameter-space \( \mathfrak{a}^* \) which is generated by the weight vectors \( \{w_J, J \in \mathcal{J}\} \).

Thus, for a given \( \zeta \in \mathfrak{a}^* \), to determine which orbits \( O_{\mathcal{J}} \) lie in \( D^{-1}(0)/G \) we simply determine which cones \( C_{\mathcal{J}} \) contain \( \zeta \). This decomposition into cones describes the secondary fan, and the regions it defines in \( \mathfrak{a}^* \) are called the phases of the GLSM.

To illustrate this construction, we work it out in a particular example \([14]\) which we will follow throughout the paper. We use an example which has been studied extensively in the literature \([19, 50–55]\).

Consider an anti-canonical hypersurface in the toric variety obtained from the weighted projective fourfold \( \mathbb{P}^{1,1,2,2,2} \) by blowing up its singular locus. This can be described by a GLSM as follows. Let \((\gamma_1, \gamma_2) \in G = U(1) \times U(1)\) act on the vector space \( \mathbb{C}^7 \) via

\[
(\phi_0, \phi_1, \ldots, \phi_6) \mapsto (\gamma_1^{-4} \phi_0, \gamma_1^{2} \phi_1, \gamma_2 \phi_2, \gamma_1 \phi_3, \gamma_1 \phi_4, \gamma_1 \phi_5, \gamma_1 \phi_6).
\]

(3.22)

We specify \( R \)-charges of these fields in terms of two arbitrary rational parameters \( q_1 \) and \( q_2 \) to be determined later, as

| field   | \( \phi_0 \) | \( \phi_1, \phi_2 \) | \( \phi_3, \phi_4, \phi_5 \) | \( \phi_6 \) |
|---------|--------------|---------------------|-----------------------------|----------|
| \( R \)-charge | \( 2 - 4q_1 \) | \( q_2 \) | \( q_1 \) | \( q_1 - 2q_2 \) |

(3.23)

For the superpotential, which is a \( G \)-invariant polynomial of \( R \)-charge two, we choose

\[
W(\phi_0, \phi_1, \ldots, \phi_6) = \phi_0 F_{(4,0)}(\phi_1, \ldots, \phi_6),
\]

(3.24)

where \( F_{(4,0)} \) is a generic homogeneous polynomial of bi-degree \( (4,0) \) with respect to the gauge group \( G = U(1) \times U(1) \).

The D-term equations are

\[
\zeta_1 = -4|\phi_0|^2 + |\phi_3|^2 + |\phi_4|^2 + |\phi_5|^2 + |\phi_6|^2,
\]

\[
\zeta_2 = |\phi_1|^2 + |\phi_2|^2 - 2|\phi_6|^2.
\]

(3.25)

(3.26)

\[13\] A bit more concretely, for each \( G_{\mathcal{J}} \) containing \( \zeta \), the complementary set of indicies \( \{1, \ldots, N\} - \mathcal{J} \) labels the coordinates for one of the toric coordinate charts of \( D^{-1}(0)/G \).

\[14\] The example we use is an abelian GLSM, which does not exhibit the full power of the localization method to compute Gromov–Witten invariants, since those invariants can also be computed by mirror symmetry for abelian GLSMs with a geometric phase. However, we avoid some of the complications of nonabelian GLSMs by working with this particular example.
We then find the secondary fan data which is illustrated in figure 1: for each pair \( \{ J, J' \} \) we have indicated the region(s) which are included in the cone generated by \( w_J \) and \( w_{J'} \). This leads to four phases, labeled by Roman numerals in the figure.

Note that the cones \( C_J \) are not necessarily phase regions in and of themselves; in the example, \( C_{16} \) is the union of phases I and II.

The geometry of the various quotients is best described by determining the sets \( Z_\zeta \) which are excluded from the quotient. If we label those sets according to phase region, then by examining which variables are allowed to vanish together, we find that

\[
Z_I = \{ \phi_1 = \phi_2 = 0 \} \cup \{ \phi_3 = \phi_4 = \phi_5 = \phi_6 = 0 \} \\
Z_{II} = \{ \phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0 \} \cup \{ \phi_6 = 0 \} \\
Z_{III} = \{ \phi_0 = 0 \} \cup \{ \phi_6 = 0 \} \\
Z_{IV} = \{ \phi_0 = 0 \} \cup \{ \phi_1 = \phi_2 = 0 \}.
\]

(3.27) (3.28) (3.29) (3.30)

Each phase has a geometric description [19]: in phase I, we get a line bundle over the blowup of \( \mathbb{P}^{(1,1,2,2,2)} \) along its singular locus, in phase II, we get a line bundle over \( \mathbb{P}^{(1,1,2,2,2)} \) itself, in phase III we get \( C^5/\mathbb{Z}_8 \), and in phase IV we get

\[
(C^3 \times \mathcal{O}_{\mathbb{P}^1(2)}) / \mathbb{Z}_4.
\]

(3.31)

The phase of relevance for comparison to the nonlinear sigma model is the geometric phase, phase I.

We still must impose the F-term equations, and in doing so, we can be more specific concerning our ‘generic’ assumptions about the superpotential (3.24). We assume that \( F_{(4,0)}(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, 1) \) is a transverse homogeneous polynomial in five variables (which means that the origin is the only common zero of the partial derivatives), and that \( F_{(4,0)}(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, 0) \), which is independent of \( \phi_1 \) and \( \phi_2 \), is a transverse homogeneous polynomial of three variables. The F-term equations are

\[
\frac{\partial W}{\partial \phi_J} = 0, \quad J = 0, \ldots, 6.
\]

(3.32)
To solve the F-term equations, we note that $\partial W/\partial \phi_0 = F$, and that $\phi_0$ divides $\partial W/\partial \phi_j$ for $J \neq 0$. Thus, one solution is

$$\phi_0 = F = 0. \tag{3.33}$$

If $\phi_0 \neq 0$ but $\phi_0 = 0$, then $(\partial F/\partial \phi_j)_{\phi_0 = 0} = 0$ for $J = 3, 4, 5$, which implies (by transversality) that

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0. \tag{3.34}$$

Moreover, there is no monomial appearing in $F$ which involves only the variables $\phi_1$ and $\phi_2$, so $F$ vanishes on the locus (3.34) and we see that (3.34) provides a second solution to the F-term equations. Finally, if $\phi_0 \neq 0$ and $\phi_6 \neq 0$, then the transversality of $F(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, 1)$ implies that

$$\phi_1 = \phi_2 = \phi_3 = \phi_4 = \phi_5 = 0 \tag{3.35}$$

is also a solution (because there is no power of $\phi_6$ is a monomial).

In phase I, we see that that only possible solution is $\{ \phi_0 = F = 0 \}$. This defines a hypersurface $F = 0$ inside the zero-section $\{ \phi_0 = 0 \}$ of the line bundle. In other words, we get a hypersurface in the blowup of $\mathbb{P}^{(1,1,2,2)}$ along its singular locus. The requirement (3.13) (guaranteeing anomaly-free $R$-symmetry and integral $R$-charges up to gauge transformation) is precisely the condition for the hypersurface to be Calabi–Yau. This is a general phenomenon for toric hypersurfaces [56].

The other solutions to the F-term equations are relevant in other phases: in phase II, we again get (3.33); in phase III, we get (3.35); and in phase IV, we get (3.34).

4. Quantum corrections to the Kähler potential

In this section, we discuss quantum effects in the $\mathcal{N} = (2, 2)$ supersymmetric two-dimensional nonlinear $\sigma$-model on a Calabi–Yau manifold $X$ of arbitrary dimension. As noted above, the Kähler potential $K_c$ on the chiral conformal manifold is not subject to corrections. However, the Kähler potential $K_e$ on the twisted chiral conformal manifold is subject to perturbative corrections which have been determined in detail in [22], as well as non-perturbative instanton corrections described in terms of Gromov–Witten invariants. We describe the perturbative corrections both in terms of expressions in the Riemannian curvature (integrated over $X$) and in terms of cohomology classes (evaluated on the fundamental homology class of $X$).

4.1. Nonlinear $\sigma$-model action and the effective action

Under the renormalization group an $\mathcal{N} = (2, 2)$ supersymmetric, two-dimensional, nonlinear $\sigma$-model with Kähler target space $X$ (of complex dimension $n$), flows in the infrared to a conformal fixed point characterized by vanishing $\beta$-functions. In this section, the $\beta$-function of the target space Kähler form is of particular interest, which vanishes at tree level but is nonzero at one-loop:

$$\frac{1}{\alpha'} \beta_{ij} = R_{ij} + \Delta \omega_{ij}(\alpha') = R_{ij} + \alpha'^3 \frac{\zeta(3)}{48} T_{ij} + O(\alpha'^5). \tag{4.1}$$

Here $\alpha'$ is the coupling constant in the nonlinear $\sigma$-model. At leading one-loop order, the Ricci tensor $R_{ij}$ appears; $\Delta \omega_{ij}$ then comprises all higher loop corrections, which are exact in cohomology, i.e. $\Delta \omega = d\rho$ with some global one form $\rho$ on $X$. The tensor $T_{ij}$ is the first non-vanishing subleading correction at four loops [58], which has been explicitly
calculated in [59]. (The five-loop correction at order $O(\alpha'^4)$ has been shown to vanish [60].) Thus, at leading order the vanishing $\beta$-function $\beta_1 = 0$ requires a Ricci-flat Kähler metric and hence a Calabi–Yau target space. However, this Ricci-flat Calabi–Yau target space metric gets further corrected at higher loops.

To analyze these corrections, it is useful to adopt an effective action point of view for the target space geometry. Namely, we interpret the condition for the vanishing $\beta$-function as the Euler–Lagrange equation for the metric $g_{\beta\gamma}$ arising from an action functional [58, 61]. The relevant effective action $S_{\text{eff}}[g]$ takes the form

$$ S_{\text{eff}}[g] = \int \sqrt{g} [R(g) + \Delta S(\alpha', g)], $$

(4.2)

with the corrections $\Delta S(\alpha', g)$. The leading correction arises at fourth loop order $\alpha'^3$ and enjoys the expansion

$$ \Delta S(\alpha', g) = \alpha'^3 S^{(4)}(g) + \alpha'^5 S^{(6)}(g) + \ldots $$

(4.3)

Here the $n$-th loop correction $S^{(n)}(g)$ is a scalar functional of the metric tensor and the Riemann tensor. A proposal for the structure of these terms was put forward in [62].

Since the effective action $S_{\text{eff}}[g]$ gets corrected beyond the leading contribution, we expect the classical metric on $\mathcal{M}_c$ to receive further corrections from higher loop orders. Using mirror symmetry, the tree level term and four-loop correction in the case of Calabi–Yau threefolds were determined to be [5]

$$ e^{-K_c} = \frac{1}{3!} \left( \sum_{j=1}^{k} 2 \text{Im} \ t_j L_j \right)^3 [X] + \alpha'^3 \left( -\frac{1}{4\pi^2} \zeta(3) c_3(X) \cup \left( \sum_{j=1}^{k} 4\pi \text{Im} \ t_j L_j \right)^0 \right) [X] + O(e^{2\pi i \alpha'}) $$

(4.4)

expressed in terms of the Chern class $c_3(X)$ and a special value of the Riemann $\zeta$-function. The appearance of the $\zeta$-value $\zeta(3)$ (of transcendental weight three) indicates its origin as a four-loop counterterm of the $\mathcal{N} = (2, 2)$ supersymmetric nonlinear $\sigma$-model [59].

In general, further corrections in $\alpha'$ appear for Calabi–Yau target spaces of higher dimension $n > 3$. They take the following form in which $\alpha'$ is indicated explicitly (although it is set equal to 1 elsewhere in this review):

$$ e^{-K_c} = \exp \left( \sum_{j=1}^{k} 2 \text{Im} \ t_j L_j \right) + \frac{1}{(2\pi)^n} \sum_{k=0}^{n} \alpha'^k \chi_k [X] + O(e^{2\pi i \alpha'}) $$

(4.5)

where the characteristic class $\chi_k$ arises from the perturbative loop corrections at loop order $k + 1$. (Thanks to [60] we expect $\chi_4$ to vanish.) Due to the appearance of higher curvature tensors in the corrections $\Delta \omega_{\beta\gamma}$ of the $\beta$-function (4.1), we can expect that integrating such curvature tensors can be expressed in terms of the Chern classes of the tangent bundle of the target space $X$. Furthermore, the loop corrections appearing in $\Delta \omega_{\beta\gamma}$ at a given loop order $k + 1$, i.e. at order $\alpha'^k$, give rise to corrections with transcendentality degree $k$, which is a general property of loop corrections of supersymmetric two-dimensional $\sigma$-models [63]. As a result, the cohomology classes $\chi_k$ are homogeneous elements of transcendental degree $k$ in the graded polynomial ring over all products of multiple $\zeta$-values up to transcendental weight $k$

$$ \chi_k \in \mathcal{H}^k(X, \mathbb{Q}) [\zeta(m_1, m_2)_{2 \leq m_1, m_2 \leq k}, \ldots, \zeta(1, \ldots, 1)]_k. $$

(4.6)
The transcendental weight of a multiple $\zeta$-value $\zeta(m_1, \ldots, m_d)$ is given by the sum $m_1 + \ldots + m_d$, and the multiple zeta functions $\zeta(m_1, \ldots, m_d)$ generalize the Riemann zeta function according to \cite{64}

$$\zeta(m_1, \ldots, m_d) = \sum_{n_1 > n_2 > \ldots > n_d} \frac{1}{n_1^{m_1} \cdots n_d^{m_d}}.$$  (4.7)

Note that there are many non-trivial relations over $\mathbb{Q}$ among such multiple $\zeta$-values; see for instance \cite{65}.

4.2. Perturbative corrections to the Kähler potential

Perturbative corrections to $e^{-K}$ were found in \cite{22}, assuming that all corrections take the universal form (4.5), by using mirror symmetry and period computations to determine the values of $\chi_k$. The answers can be expressed in terms of a characteristic class known as the ‘gamma class’ \cite{55, 66–70}, but we will take a more direct approach and present the corrections explicitly both in terms of Riemannian curvature and in terms of Chern classes.

The corrections to the Kähler potential involve the Riemann curvature tensor, which we denote by $\mathcal{R}$ (suppressing indices) and regard as a differential-form-valued endomorphism of the tangent bundle of $M$. If we take the trace over tangent bundle indices, we obtain a 2-form $\text{tr}(\mathcal{R})$ which is just the familiar Ricci tensor $R_{ij}$ with indices suppressed. We will also consider traces of higher powers (i.e. composing the endomorphism with itself a number of times): $\text{tr}(\mathcal{R}^\ell)$ defines a $2\ell$-form.

We can now state the perturbative corrections to the classical metric (3.4) which were derived in \cite{22}, in the case of $c_1(X) = 0$:

$$K_{fc} = -\log \frac{1}{(2\pi)^n} \int_X \exp \left( 2 \text{ Re} \left( \sum \ell_j \bar{F}_j \right) - 2 \sum_{k=1}^\infty \frac{\zeta(2k + 1)}{2k + 1} \text{tr} \left( (i\mathcal{R}/2\pi)^{2k+1} \right) \right) + O(e^{2\pi i t}).$$  (4.8)

To write this in terms of integer cohomology classes, we need to use Newton’s identities which express $\sum_{j=1}^n x_j^{\ell}$ in terms of the elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_n$ of $\{x_1, x_2, \ldots, x_n\}$. If we write

$$\sum_{j=1}^n x_j^{\ell} = P_\ell(\sigma_1, \sigma_2, \ldots, \sigma_n),$$  (4.9)

then by Chern–Weil theory,

$$\int_X \text{tr} \left( (i\mathcal{R}/2\pi)^{\ell} \right) = P_\ell(c_1, c_2, \ldots, c_\ell).$$  (4.10)

Thus, we can express the perturbative corrections to (3.5) in the form

$$e^{-K} = \exp \left( \sum_{j=1}^n 2 \text{ Im}(\ell_j) L_j - \frac{2}{(2\pi)^n} \sum_{k=1}^\infty \frac{\zeta(2k + 1)}{2k + 1} P_{2k+1}(c_1, c_2, \ldots, c_{2k+1}) \right) [X] + O(e^{2\pi i t}),$$  (4.11)

where $c_1 = 0, c_2, c_3$, are the Chern classes of $X$. 


For Calabi–Yau threefolds, the perturbative correction (4.4) found in [5] gives the complete answer. However, in order to evaluate these expressions for Calabi–Yau manifolds of higher dimension, it is convenient to have the first several (odd) Newton’s identities at our disposal, which we give with $\sigma_1$ set to 0:

\[
\begin{align*}
P_1|_{\sigma_1=0} &= 0 \\
P_3|_{\sigma_1=0} &= 3\sigma_3 \\
P_5|_{\sigma_1=0} &= -5\sigma_2\sigma_3 + 5\sigma_5 \\
P_7|_{\sigma_1=0} &= 7\sigma_2^2\sigma_3 - 7\sigma_2\sigma_5 - 7\sigma_3\sigma_4 + 7\sigma_7
\end{align*}
\]  

(4.12)

Setting $c_1 = 0$ and expanding the exponential in (4.11), we find the first few perturbative corrections

\[
\begin{align*}
\chi_3 &= -2\zeta(3)c_3 \\
\chi_4 &= 0 \\
\chi_5 &= 2\zeta(5)(c_2c_3 - c_5) \\
\chi_6 &= 2\zeta(3)^2c_3^2 \\
\chi_7 &= -2\zeta(7)(c_2^2c_3 - c_3c_4 - c_2c_5 + c_7)
\end{align*}
\]

(4.13)

in terms of the Chern classes $c_k$ of the Calabi–Yau $n$-fold $X$. The contributions $\chi_3$ and $\chi_4$ are exactly what appear in [5, 71].

4.3. Nonperturbative corrections and Gromov–Witten invariants

The nonperturbative corrections to a two-dimensional nonlinear sigma model are due to instantons, i.e. action-minimizing maps from a Euclidean spacetime to the target manifold, and the relevant corrections to the metric on $M_{\tau c}$ are given by instantons of genus zero.

In order to describe the instanton corrections to the Zamolodchikov metric, we must first describe instanton corrections to certain other quantities in the theory. The twisted chiral operators in the theory have a natural ring structure determined by the two-point and three-point genus zero correlation functions\(^\text{15}\) in terms of a ‘Frobenius algebra’ structure [72]. This determination goes as follows: given a ring $R$ with a nondegenerate bilinear form

\[
(-, -) : R \times R \to \mathbb{C},
\]

(4.14)

there is a natural trilinear map

\[
\langle ABC \rangle := (A \ast B, C),
\]

(4.15)

where $\ast$ denotes the product in the ring. (When evaluated on a basis of $R$, this gives the structure constants for the ring.) Conversely, whenever we are given a bilinear form (4.14) and a trilinear form (4.15), we get a product on $R$.

These two-point and three-point correlation functions can be directly computed in the closely related ‘topological sigma model’ [7], in which the spins of the fields are modified and a suitable projection is performed\(^\text{16}\). The twisted chiral operators in the topological theory can be identified with harmonic forms (or their cohomology classes) on the target manifold $X$, and the ring structure in the classical theory is simply the cup product pairing in cohomology.

---

\(^{15}\) These are computed as certain extremal three- and four-point functions. I thank the referee for a comment on this point.

\(^{16}\) We are considering here the ‘A-model’ of [73].
However, this ring structure is deformed by instanton contributions [7, 74, 75] in the quantum theory, giving rise to the ‘quantum cohomology ring’ of $X$ (which is known to be associative [76–78]).

To describe the instanton contributions, we represent cohomology classes $A$, $B$, and $C$ by algebraic cycles on $X$; the correlation function is calculated by integrating over the space of maps $\pi$ from the genus one spacetime with three points $p$, $q$, $r$ specified such that $\pi(p)$ lies in $A$, $\pi(q)$ lies in $B$, and $\pi(r)$ lies in $C$; a standard localization procedure reduces the computation to determining the space of volume-minimizing maps. The classical contribution to the correlation function comes from constant maps, and simply counts common points of intersection of $A$, $B$, and $C$. Since the two-point function can also be expressed in terms of intersections, the Frobenius algebra construction reproduces the familiar cup product on cohomology (which counts intersections of the corresponding algebraic cycles), as mentioned above.

For nonconstant maps, it turns out that the image of $\pi$ has a deformation space whose dimension is expected to be $\dim X - 3$ (since the spacetime has genus zero). Imposing the conditions on $\pi(p)$, $\pi(q)$, and $\pi(r)$ then cuts down the dimension to zero, and the corresponding maps can be counted. If we fix the homology class $\eta$ of the image, then the Gromov–Witten invariant $GW^{X,\eta}_{0,3}(A, B, C)$, which has a precise mathematical definition, is intended to count the number of maps.

Each instanton contribution is weighted by the instanton action $e^{i\ell} = e^{2\pi i r \cdot \eta}$, which we often denote by $q^\eta$. The quantum product can be written as

$$A \star B := A \cup B + \sum_\eta (A \star B)_\eta q^\eta$$

(4.16)

where $(A \star B)_\eta$ is the unique class satisfying

$$([A \star B] \cup C) [X] = GW^{X,\eta}_{0,3}(A, B, C)$$

(4.17)

for all $C$. We can restrict the sum (4.16) to only range over those classes $\eta$ in $H_2(X, \mathbb{Z})$ whose intersection with each Kähler class is nonnegative.

Due to orbifold singularities in the deformation spaces, the mathematical definition of Gromov–Witten invariants only guarantees that they are rational numbers, not integers. The physical reason for this is understood, and stems from a ‘multiple covering’ phenomenon. If the map $\pi$ factors through a multiple covering $S^2 \to S^2$ of degree $m$, then the homology class of the image takes the form $\eta = m \mathcal{P}$ but, as argued in [82] in dimension three and [83] in arbitrary dimension (see also [84]), the count of maps is the same.

We can take this into account by defining a modified Gromov–Witten invariant $GW^{X,\eta}_{0,3}(A, B, C)$ which should only count the maps of degree one. If we collect terms according to degree one maps, we find a total instanton contribution of

$$\sum_{\mathcal{P} \in H_2(X, \mathbb{Z})} \frac{GW^{X,\eta}_{0,3}(A, B, C)}{1 - q^\mathcal{P}} = \sum_{\mathcal{P} \in H_2(X, \mathbb{Z})} \frac{GW^{X,\eta}_{0,3}(A, B, C)}{1 - q^\mathcal{P}} \sum_{m=1}^\infty q^{m\mathcal{P}}.$$  

(4.18)

Extracting the coefficient of $q^\eta$, we obtain a formula

$$GW^{X,\eta}_{0,3}(A, B, C) = \sum_{\eta = m \mathcal{P}} GW^{X,\eta}_{0,3}(A, B, C)$$

(4.19)

17 When the deformation space fails to have the expected dimension, there is a natural way to integrate over the excess deformations to still produce a ‘count’ of maps satisfying the three conditions [79–81].
which can be used to define the modified Gromov–Witten invariants. They are expected to be integers.

For Calabi–Yau threefolds, there is one further modification which can be made to the definitions. The expected dimension of the deformation spaces in that case is zero, so we expect only a finite number of possibilities (in a fixed homology class) for the image of \( \pi \). For each rational curve of class \( \varphi \), there are \((A \cdot \varphi)\) choices for the location of \( \pi(p) \), \((B \cdot \varphi)\) choices for the location of \( \pi(q) \), and \((C \cdot \varphi)\) choices for the location of \( \pi(r) \). It is then natural to define the ‘Gromov–Witten instanton number’ of genus 0 and class \( \varphi \) to be

\[
N_\varphi = \frac{\overline{GW}_{0,3}^X(A, B, C)}{(A \cdot \varphi)(B \cdot \varphi)(C \cdot \varphi)}
\]

which is expected to be an integer, independent of \( A, B, C \), that counts the number of rational curves in class \( \varphi \).

Having spelled out in detail how instantons determine the quantum cohomology ring, we can now explain the nonperturbative corrections to the Zamolodchikov metric. If we substitute (4.11) into the perturbative expression (4.11), we obtain an expression involving cup products between twisted chiral operators (labeled by \( t^\ell \)) and their complex conjugates (labeled by \( \bar{t}^\ell \)). For the former, we can make computations in the quantum cohomology ring instead, replacing \( \cup \) by \( \ast \). For the latter, we should use the complex-conjugated cohomology ring, with instanton actions \( \tilde{q}^0 \) rather than \( q^0 \). That is, we can do those computations in complex-conjugated quantum cohomology, replacing \( \cup \) by \( \bar{\ast} \).

In other words, the prescription for nonperturbative corrections to the perturbative formula (4.11) is: perform the multiplications among holomorphic terms using \( \ast \) and among antiholomorphic terms using \( \bar{\ast} \); then multiply the pieces together using cup product18.

Let us spell this out explicitly for Calabi–Yau threefolds. The perturbative expression can be written as

\[
\frac{i}{6} \left( \sum t^\ell j^\ell \right)^3 - \frac{i}{2} \left( \sum t^\ell j^\ell \right) \left( \sum t^\ell j^\ell \right) + \frac{i}{2} \left( \sum t^\ell j^\ell \right) \left( \sum \overline{t^\ell j^\ell} \right)^2 - \frac{i}{6} \left( \sum \overline{t^\ell j^\ell} \right)^3 - \frac{\zeta(3)}{4\pi^3} c_3(X).
\]

When nonperturbative corrections are included, this becomes

\[
\frac{i}{6} \left( \sum t^\ell t^\ell t^\ell \left( \left( L^\ell \cup L^\ell \cup L^\ell \right) [X] + \sum_n GW_{0,3}^{X,n} (L^\ell, L^\ell, L^\ell) q^n \right) \right) \\
- \frac{i}{2} \left( \sum t^\ell t^\ell t^\ell \left( \left( L^\ell \cup L^\ell \cup L^\ell \right) [X] + \sum_n GW_{0,3}^{X,n} (L^\ell, L^\ell, L^\ell) q^n \right) \right) \\
+ \frac{i}{2} \left( \sum t^\ell t^\ell t^\ell \left( \left( L^\ell \cup L^\ell \cup L^\ell \right) [X] + \sum_n GW_{0,3}^{X,n} (L^\ell, L^\ell, L^\ell) q^n \right) \right) \\
- \frac{i}{6} \left( \sum t^\ell t^\ell t^\ell \left( \left( L^\ell \cup L^\ell \cup L^\ell \right) [X] + \sum_n GW_{0,3}^{X,n} (L^\ell, L^\ell, L^\ell) q^n \right) \right) - \frac{\zeta(3)}{4\pi^3} c_3(X).
\]

18 This prescription matches the formulas in [11] in dimension three and [71] in dimension four, as well as considerations from mirror symmetry.
5. The two-sphere partition function and Gromov–Witten invariants

5.1. The $S^2$ partition function for a GLSM

Consider an $\mathcal{N} = (2, 2)$ GLSM with gauge group $G$, chiral fields $\Psi_j$ of $R$-charge $q_j$ on which $G$ acts by the representation $\Psi : S \rightarrow SU(\mathcal{N})$, superpotential $W$, and complexified FI-parameters $\frac{q_j}{2\pi} + i \zeta \in \text{Ann}([g, g])_C \subset \mathfrak{g}^*_C$. As in section 3.2, we fix a Cartan subgroup $H$ of $G$ and weights $w_j \in \Lambda_\text{wet} \subset \mathfrak{h}^*$ describing the eigenvalues of $\Psi|_H$.

The possible fluxes of the gauge theory through the 2-sphere are GNO quantized [85], which means that they are integer-valued functions on the weight lattice, i.e. elements $m$ of the coweight lattice $\Lambda_\text{cow} \subset \mathfrak{h}$. The combinations $\sigma \pm i \frac{\pi}{2}$ are what appear in the formulas.

In [13, 14], a computation of $Z_{S^2}$ for a 2-sphere of radius $r$ is made by expanding on the Coulomb branch and using localization, after modifying the Lagrangian appropriately to put the theory on $S^3$. The original papers include twisted masses related to flavor symmetries of the theory but we shall not include those as they are not relevant for our application. Since we are only studying the theories which are conformal in the infrared, the only dependence on the radius is through a multiplicative factor. Some additional notation: $|W|$ denotes the order of the Weyl group of $G$, $\Delta^+$ denotes the set of positive roots of $G$ (a subset of the weight lattice), and $\rho = \frac{1}{2} (\sum_{\alpha \in \Delta^+} \alpha)$ is the Weyl vector. Here is the final formula, which assumes that all $R$-charges have been chosen in the range $0 < q < 2$:

$$
\frac{Z_{S^2}(z, \bar{z})}{(r/r_0)^{2/3}} = \frac{1}{|W|} \sum_{m \in \Lambda_\text{cow}} \int_h \left( \prod_{\mu=1}^{\text{rank}(G)} \frac{d\sigma_\mu}{2\pi} \right) \left( \text{rank}(G) \right) Z_{\text{class}}(\sigma, m) Z_{\text{gauge}}(\sigma, m) Z_{\text{matter}}(\sigma, m),
$$

(5.1)

where

$$
Z_{\text{class}} = \exp(\log z, i\sigma + \frac{m}{2}) \exp(\log \bar{z}, i\sigma - \frac{m}{2}) = e^{-4\pi \langle \zeta, i\sigma \rangle + \langle 0, m \rangle}
$$

$$
Z_{\text{gauge}} = (-1)^{2\rho \cdot m} \prod_{\alpha \in \Delta^+} \left( (\alpha \cdot \sigma)^2 + \frac{1}{4} (\alpha \cdot m)^2 \right)
$$

$$
Z_{\text{matter}} = \prod_j \frac{\Gamma \left( \frac{q_j}{2} - w_j \cdot i\sigma - \frac{1}{2} w_j \cdot m \right)}{\Gamma \left( 1 - \frac{q_j}{2} + w_j \cdot i\sigma - \frac{1}{2} w_j \cdot m \right)}.
$$

(5.2)

We have included an overall sign in $Z_{\text{gauge}}$, missing in [13, 14], whose necessity was pointed out in [55, 86].

5.2. The hemisphere partition function and the $tt^*$ equations

As discussed above, the partition function of a GLSM has been evaluated on a 2-sphere [13, 14]. The partition function has also been evaluated on a hemisphere, that is, a half-sphere $D^2$ equipped with the spherical metric [55, 86, 87], as well as on real projective 2-space [88]. A full discussion of these results is beyond the scope of this review, but we will briefly present the result for the hemisphere, following [55].

Note that we are organizing things slightly differently from the way the matter representation is described in [13, 14]. With our conventions, each $\phi_j$ spans a one-dimensional space which is preserved by the action of $H$ and thus is identified with a weight of the representation $\Psi$. In [13, 14], the representation was decomposed into irreducible representations of $G$ (labeled by $\phi_j$, each of which had additional indices in those papers), and then each of those irreducible representations was cast into weight spaces.
We need to specify BPS boundary conditions for the GLSM along the boundary of the hemisphere, and the natural type of boundary conditions to use are ‘B-branes’ [89, 90]. The spectrum of B-branes is locally constant over $M^{tc}$, and the data needed to specify B-branes in a GLSM was thoroughly analyzed in [54].

Having specified some B-brane data $B$, the partition function $Z_{D^2, B}$ on a hemisphere of radius $r$ is evaluated explicitly in [55]. The dependence on the radius is via an overall factor (which is the square root of the corresponding factor in $Z_{S^2}$). The dependence of the hemisphere partition function on the choice $B$ of B-brane comes through a factor $f_B(\sigma)$ in the integrand described in [55], to which we refer for further details (see also [42]).

In order to evaluate the partition function, an appropriate integration contour $\gamma \subset h_C$ must be chosen. With this understood, the result of [55] is:

$$Z_{D^2, B}(z, \bar{z}) = \left(\frac{r}{r_0}\right)^{c/6} |W| \int_{\gamma} \prod_{\mu \in \Delta^+} e^{-4\pi i (\zeta, \sigma)} \prod_{\alpha \in \Delta^+} \alpha \cdot \sigma \sinh(\pi \alpha \cdot \sigma) \prod_{j} \Gamma \left(\frac{q_j}{2} - w_j \cdot i\zeta\right) f_B(\sigma),$$

(5.3)

where $\zeta = -\frac{1}{2\pi} \text{Re} \log z$. The authors of [55] interpret this formula as specifying a BPS charge for each choice of boundary condition, and they verify that it agrees with the BPS charge in circumstances under which both can be computed.

It is natural to expect that the hemisphere partition function will play an important role in some yet-to-be-established holomorphic factorization property for two-sphere partition functions. Indeed, for the analogous three-dimensional gauge theories with $\mathcal{N} = 2$ supersymmetry, the partition function computed in [91, 92] displays such a factorization. (This fact was noted in [93], and further explored in [94].) The authors of [55] have taken an important step in this direction in the 2D case by analyzing the GLSM partition function on an annulus and studying how it can be used to glue together the results on the two hemispheres to reproduce the result on the sphere. Their calculation is not completely general, but for theories with a geometric phases they verify that such a factorization result does indeed hold.

An alternate approach to such a factorization result might proceed by means of the ‘$tt^*$ equations’ of [95], as suggested in [11, 15]. The $tt^*$ equations describe the Zamolodchikov metric in terms of topological twists of the GLSM: it is equal to the overlap of ground states in the $A$-twisted GLSM on one hemisphere, and an $\bar{A}$-twisted GLSM on the other hemisphere. The authors of [15] carry out a computation of such an overlap by means of a calculation on the squashed two-sphere, which has different limiting interpretations as the squashing parameter is varied. We refer to [15] for further details.

5.3. Extracting Gromov–Witten invariants from the partition function

We now explain how, with the Euler characteristic $\chi(X) = c_3(X)$ as additional input, the relationship between the Kähler potential and the two-sphere partition function can be used to extract the Gromov–Witten invariants from the partition function $Z_{S^2}$ in the case of a Calabi–Yau threefold $X$. The form of the partition function which we have determined for a nonlinear sigma model depends on a choice of coordinates, so our task is to use the asymptotic behavior of $Z_{S^2}$ to determine the appropriate coordinates. For ease of exposition, in this section we will assume that we have chosen FI coordinates so that a neighborhood of $\{z_a = 0 \text{ for all } a\}$ describes a geometric phase of the GLSM.
To bring the partition function $Z_{S^2}(z, \bar{z})$ into an appropriate normal form and to extract the Gromov–Witten invariants, we use the following algorithm:

1. Evaluate $Z_{S^2}(z, \bar{z}) = e^{-K}tc$ by contour integration as an expansion around large volume. (We will discuss this step in more detail in the next section.) The result can be expressed in terms of logarithmic coordinates $\tau_j = \frac{1}{2\pi i} \log z_j$, and the goal is to find a change of coordinates from $\tau_j$ to $t_j$.

2. Isolate the perturbative $\zeta(3)$ term and perform a Kähler transformation $K = K' + X^0(z) + \bar{X}^0(\bar{z})$ in order to reproduce the constant term $-\frac{\zeta(3)}{4\pi^3} \chi(X)$ in (4.4) and (4.23); (The initial $\zeta(3)$ term might have a non-constant coefficient, which gives rise to a nontrivial Kähler transformation.)

3. Read off the holomorphic part of the coefficient of $\bar{\tau}_j \bar{\tau}_k = \frac{1}{(2\pi i)^2} \log \bar{z}_j \log \bar{z}_k$, which should then be identified with $-i\frac{1}{2} \sum_{\ell} (L_j \cup L_k \cup L_\ell) [X] t_\ell$.

Use this to extract the NLSM coordinates $t_\ell$, which must have the form

$$t_\ell = \frac{1}{2\pi i} \log z_\ell + f_\ell(z),$$

(5.5)

where $f_\ell(z)$ is a holomorphic function. This determines the NLSM coordinates up to the undetermined constants $f_\ell(0)$.

4. Invert the GLSM/NLSM map (5.5) to obtain the $z_\ell$ as a function of $t_\ell$,

$$z_\ell = e^{-2\pi i f_\ell(0)} \left( q_\ell + O(q_\ell^2) \right),$$

(5.6)

where $q_\ell := e^{2\pi i t_\ell}$.

5. Fix the constant terms $f_\ell(0)$ by demanding the lowest order terms in the instanton expansion be positive; and, finally,

6. Read off the (rational) Gromov–Witten invariants $GW^{X,0}_{0,3}(A, B, C)$ from the coefficients in the $q$-expansion. The (integral) Gromov–Witten instanton numbers $N_\eta$ of genus zero (roughly, the ‘number of rational curves’) can then be obtained from the multi-covering formulas (4.19) and (4.20).

6. Evaluating the partition function

The low-energy effective theory describing the dynamics of the GLSM depends on the value of the FI-parameters [18]. The space of FI-parameters can be divided into phase regions depending on the character of the low-energy dynamics as explained in section 3.3. In this section we show how this phase structure is closely related to structure of the integrand of the two-sphere partition function, and how this observation can be used to determine Gromov–Witten invariants explicitly.

The idea is stated rather simply: when $Z_{S^2}$ is evaluated by the method of residues, the contour prescription depends on the value of the FI-parameters, which in turn affects the set of

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20 Here, guided by properties of the NLSM, we are choosing a particular Kähler potential in a geometric phase, the choice of which eliminates the possibility of further shifts of the form (2.15). Somewhat surprisingly, this choice is in general different from the globally defined Kähler potential predicted in [17]. I thank Ronen Plesser and Eric Sharpe for discussions on this point.

21 This is the analogue of the much-studied ‘mirror map’ relating chiral and twisted chiral conformal manifolds of a mirror pair [43].
poles that contribute to the integral. At certain codimension-one walls in FI-parameter space the structure of poles contributing to the \( Z_{\mathcal{C}} \) integral can change, signaling the presence of a GLSM phase transition along that wall. In particular, for abelian GLSMs we show that this phase structure is precisely the same secondary fan which governs the low-energy physics. Furthermore, we also describe how phases of non-abelian GLSMs can be understood in terms of phases of an associated ‘Cartan’ theory.

6.1. Analytic structure of the partition function

When the integrand in equation (5.1) is analytically continued to complex values of \( \sigma \in \mathfrak{h}_{\mathbb{C}} \), it becomes a meromorphic function of the integration variables. In order to evaluate this integral by means of residues, we need to identify the location of all poles in that integrand. We observe that the gauge factor \( Z_{\text{gauge}} \) never contributes poles and the integrand has the same analytic structure if this term is omitted. In fact, if we retain the same matter content but restrict the gauge group to a Cartan subgroup \( H \), then up to a constant, we simply omit the previous \( Z_{\text{gauge}} \) factor while retaining the \( Z_{\text{classical}} \) and \( Z_{\text{matter}} \) factors. For the purposes of analyzing the analytic structure of the integrand, we may thus restrict ourselves to abelian gauge theories without loss of generality. See [96] for a further discussion of this point.

The partition function for an abelian GLSM with \( G = H = U(1) \) is

\[
Z_{\mathcal{C}}(z, \bar{z}) \left( \frac{r}{r_0} \right)^{c/3} = \sum_{m \in \mathbb{Z}^\mathfrak{h}} \int_{\mathfrak{h}} \left( \prod_{\mu = 1}^{\mathfrak{h}} \frac{d\sigma\mu}{2\pi} \right) e^{-4\pi(\zeta, i\sigma) + (i\theta, m)} \prod_{j} \frac{\Gamma \left( \frac{q_j}{2} - w_j \cdot i\sigma - \frac{1}{2} w_j \cdot m \right)}{\Gamma \left( 1 - \frac{q_j}{2} + w_j \cdot i\sigma - \frac{1}{2} w_j \cdot m \right)},
\]

(6.1)

assuming as in (5.1) that \( 0 < q_j < 2 \). Recall that \( \Gamma(z) \) is a meromorphic function in the complex plane with simple poles at \( z = -n, n \in \mathbb{Z}_{\geq 0} \), with residue \( \text{Res}_{z=-n} \Gamma(z) = (-1)^n n! \) and an essential singularity at \( z = \infty \). Taking into account cancellations between zeros and poles, the \( J \)-th factor in the final product of the integrand in (6.1) has poles along the hyperplanes

\[
H_j^{(k)} : \quad \frac{q_j}{2} - w_j \cdot i\sigma - \frac{1}{2} w_j \cdot m = -k, \quad k \in \mathbb{Z}_{\geq 0}, \quad k \geq w_j \cdot m.
\]

(6.2)

The partition function integrand can be regarded as a meromorphic function on the space \( \mathbb{C}^{\text{rank}(G)} \) with poles along the hyperplanes \( H_j^{(k)} \), which we refer to as polar divisors. Note that the collection of hyperplanes \( H_j^{(k)} \) is contained in the half space \( \text{Re}(w_j \cdot i\sigma) \geq 0 \), and also satisfies \( \text{Im}(w_j \cdot i\sigma) = 0 \). The analytic structure of the integrand will be relevant in what follows, as we will evaluate the integral in (6.1) through the method of residues after choosing an appropriate way to close the integration contour in \( \mathfrak{h}_{\mathbb{C}} \).

6.2. Residues and phases

We use the multi-dimensional residue method [97, 98] to evaluate the \( h \)-dimensional integral in (6.1) for an abelian GLSM in terms of ‘Grothendieck residues’. The integration contour \( \mathfrak{h} = \mathbb{R}^{\mathfrak{h}} \) must be replaced by a closed contour \( \gamma \) which meets none of the polar divisors. This is done using a multi-dimensional analogue of the familiar Jordan lemma which replaces an improper integral over the real axis by a contour integral enclosing poles in the lower half-plane; the latter can be evaluated using residues. This can be done provided that the integrand in the lower half-plane dies off at infinity in a suitable way.
Here, we do the same thing for each complex variable in \( h_C \). The resulting integral is of a meromorphic \( h \)-form over an \( h \)-dimensional compact cycle \( \gamma \) which does not intersect the poles of the integrand. By Stokes’ theorem, if we vary the integration cycle without crossing any of the polar divisors, the value of the integral does not change. In particular, since a change of basis of \( h \) can be described by a path between the two bases leading to a one-parameter family of contours \( \gamma_i \); as long as none of the intermediate contours \( \gamma_i \) intersect any of the polar divisors, the integral does not change. For our primary integral (6.1), the growth rate is controlled by the exponential factor and so it is the sign of \( \text{Re}(\zeta, i\nu) \) which matters; if that sign remains positive then the contours \( \gamma_i \) will not encounter the polar divisors.

Each possible transverse intersection of \( h \) polar divisors is associated to an \( h \)-element subset \( J \subseteq \{1, \ldots, N\} \) such that the \( h \) vectors \( \{w_J, J \in J\} \) are linearly independent; as in section 3.3, we let \( \mathcal{J} \) denote the set of all such subsets \( J \). The corresponding polar divisors \( H_J^{(h)} \) for all \( J \in \mathcal{J} \) and \( h \in \mathbb{Z}_{\geq 0} \) (defined in (6.2)), intersect in an infinite discrete point set that we denote by \( P_{\mathcal{J}} \). The residue of the integrand at any point \( p \in P_{\mathcal{J}} \) is well defined and can be evaluated.

To determine whether the residues at the points of \( P_{\mathcal{J}} \) contribute to the integrand, we consider the basis \( \{w_J, J \in J\} \) of \( h^* \), and express the FI-parameters as \( \zeta = \sum_{J \in \mathcal{J}} \zeta w_J \). (which is possible since \( \{w_J | J \in \mathcal{J}\} \) is linearly independent). If \( \zeta > 0 \) for all \( J \in \mathcal{J} \), then with respect to the dual coordinates \( \sigma_{\mu} \) of \( h \), the contours all close in the lower half-plane and the residues at the points of \( P_{\mathcal{J}} \) are to be included in the integrand. If one of more \( \zeta \) is negative, then at least one of the contours closes in the upper half-plane rather than the lower half-plane, and the residue is excluded.

Thus, the cones \( C_{\mathcal{J}} \) which determine the phase structure of the theory (as explained in section 3.3) also determine which residues to include in an evaluation of the two-sphere partition function (6.1).

Let \( \mathcal{C} \) denote the (non-empty) intersection of all cones \( C_{\mathcal{J}} \) that contain \( \zeta \). The partition function, for \( \zeta = -\frac{1}{2\pi i} \zeta \in \mathcal{C} \), can be evaluated as

\[
Z_{\mathcal{C}}(z, \bar{z}) = \sum_{J \in \mathcal{J}} \sum_{p \in P_{\mathcal{J}}} \text{Res}_{\sigma_p = \sigma_p} \left( \sum_{\nu \in \mathbb{C}^{2r}} e^{-4\pi(\zeta, i\nu) + (i\sigma, m)} \prod_{J=1}^{N} \frac{\Gamma(\frac{\nu}{2} - w_J \cdot (i\sigma + \nu))}{\Gamma(1 - \frac{\nu}{2} + w_J \cdot (i\sigma - \frac{\nu}{2}))} \right),
\]

(6.3)

where \( \sigma_p \) denotes the coordinates of the point \( p \in P_{\mathcal{J}} \). The above expression is an infinite series, whose convergence is controlled by the exponential factor \( e^{-4\pi(\zeta, i\nu) + (i\sigma, m)} \), up to a finite shift of \( \zeta + \frac{1}{2\pi i} \theta \) due to the exponential asymptotics of the remainder of the integrand. When \( \zeta \) is sufficiently deep inside the cone \( \mathcal{C} \) all summations are convergent.

We thus see that the expression (6.1) for \( Z_{\mathcal{C}} \) is an integral of Mellin–Barnes type, which allows comparison of the behavior in different regions of the FI-parameter space. The Mellin–Barnes technique had been used earlier to study GLSMs [99–101] but here it arises as a property of the two-sphere partition function integrand, rather than being introduced as a mathematical tool to aid in understanding the theory.

6.3. Gromov–Witten invariants of an example

We consider again the example arising from the resolution of the degree eight hypersurface in the weighted projective space \( \mathbb{P}^{(1,1,2,2)} \), using the GLSM description introduced in [102,103].

\[\text{Page 22}\]

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\[\text{Page 22}\]

22 I am grateful to my collaborators Jim Halverson, Hans Jockers, Vijay Kumar, Joshua Lapan, and Mauricio Romo for their assistance with this example.
The residues which contribute to the integral correspond to the cones $\sigma$ are positive, we close the $\sigma$-integration in the lower half-planes of the complexified $\sigma$-planes. The residues which contribute to the integral correspond to the cones $C$, illustrated in the first quadrant of figure 1. Computing the residues and bearing in mind the charge condition (6.4), a straightforward but somewhat tedious algebraic manipulation yields the partition function

$$Z_F(z, \bar{z}) = \left| z_1 \right|^{q_1} \left| z_2 \right|^{q_2} \text{Res}_{\epsilon=0} \left[ \frac{\pi^5 \sin \pi \epsilon_1}{\sin^3 \pi \epsilon_1 \cdot \sin^2 \pi \epsilon_2 \cdot \sin \pi (\epsilon_1 - 2 \epsilon_2)} \times \left( \sum_{k_1, k_2=0}^{+\infty} \frac{\Gamma(4(k_1 - \epsilon_1) + 1)}{\Gamma((k_1 - \epsilon_1) + 1)^3} \frac{\zeta_1^{k_1-\epsilon_1} \zeta_2^{k_2-\epsilon_2}}{\Gamma(k_2 - \epsilon_2 + 1)^2} \right) \times \left( \sum_{k_1, k_2=0}^{+\infty} \frac{\Gamma(4(k_1 - \epsilon_1) + 1)}{\Gamma((k_1 - \epsilon_1) + 1)^3} \frac{\zeta_1^{k_1-\epsilon_1} \zeta_2^{k_2-\epsilon_2}}{\Gamma(k_2 - \epsilon_2 + 1)^2} \right) \right] .$$

(6.5)

We identify the partition function with the exponentiated Kähler potential, and we follow the algorithm in section 5.3 to arrive at the Kähler potential in flat coordinates. Using the Euler characteristic $\chi = -168$ as an overall normalization for the $\zeta^{(3)}$ term, we obtain the transformed Kähler potential $K'(z, \bar{z})$

$$e^{-K'(z, \bar{z})} = -\frac{1}{8\pi^3 \left| z_1 \right|^{q_1} \left| z_2 \right|^{q_2} \left| X^0(z) \right|^2} \times \sum_{k_1, k_2=0}^{\infty} \frac{(4k_1)!}{(k_1+1)(k_1+k_2)^2(k_1-2k_2)!} \Gamma^{k_1} \zeta_1^{k_1} \zeta_2^{k_2}.$$

(6.6)

We observe that the relevant Kähler transformation involves the ‘fundamental period’ $X^0(z)$ familiar from the toric mirror symmetry program [56, 102]. (This is a common feature of all of the examples which have been computed explicitly [11].) From the $(\log \zeta \log \zeta)$ terms, $k, \ell = 1, 2$, we extract the NLSM coordinates, which have the expansions
\begin{align}
2\pi i t_1 &= \log z_1 + 104 z_1 - z_2 + 9780 z_1^2 + 48 z_1 z_2 - \frac{3}{2} z_2^2 + \ldots,
2\pi i t_2 &= \log z_2 + 48 z_1 + 2 z_2 + 6408 z_1^2 - 96 z_1 z_2 + 3 z_2^2 + \ldots.
\end{align}

Inverting these maps, from the Kähler potential we find the triple intersection numbers of the Calabi–Yau hypersurface
\begin{align}
L_1^3 = 8, \quad L_1^2 L_2 = 0, \quad L_1 L_2^2 = 4, \quad L_2^3 = 0,
\end{align}
where \( L_1 \) and \( L_2 \) are the divisors associated to the Kähler coordinates \( t_1 \) and \( t_2 \), and the genus zero Gromov–Witten instanton numbers \( N_{\delta, d} \), of degree \((d_1, d_2)\) listed in table 1. These same numbers had earlier been calculated by means of mirror symmetry [50] but the present calculation is direct.

Our example Calabi–Yau threefold \( X \) exhibits an extremal transition to a different Calabi–Yau threefold \( \hat{X} \) [50]. Such extremal transitions have been studied in detail in [103–107], and a transition arises for the given example as follows. As the curves in the homology class dual to the divisor \( L_2 \) in \( X \) are blown down, \( X \) develops (generically) four nodal singularities. The resulting singular threefold \( X_{\text{sing}} \) is a degree eight hypersurface in \( \mathbb{P}^{1,1,2,2,2} \) with the nodal points induced from the singularities of the weighted projective space. Alternatively, we may embed \( X_{\text{sing}} \) into \( \mathbb{P}^5 \) as the complete intersection of the degree two polynomial
\[ \tilde{F}_2(\eta_1, \ldots, \eta_6) = \eta_1 \eta_2 - \eta_3^2 \] and a degree four polynomial \( \tilde{F}_4(\eta_1, \ldots, \eta_6) \) with the homogeneous coordinates \( \eta_1, \ldots, \eta_6 \) of \( \mathbb{P}^5 \). \( X_{\text{sing}} \) embedded in \( \mathbb{P}^{1,1,2,2,2} \) is associated to the degree eight polynomial
\[ F_8(\phi_1, \ldots, \phi_5) = \tilde{F}_4(\phi_1^2, \phi_2^2, \phi_1 \phi_2, \phi_3, \phi_4, \phi_5) \] in terms of the weighted homogeneous coordinates \( \phi_1, \ldots, \phi_5 \) of \( \mathbb{P}^{1,1,2,2,2} \). Perturbing the polynomial \( \tilde{F}_2 \), we obtain the smooth and deformed Calabi–Yau threefold \( \hat{X} \) as the complete intersection of degree \((2,4)\) in \( \mathbb{P}^5 \).

Due to this extremal transition, the genus zero Gromov–Witten instanton numbers of \( \hat{X} \) appear as the sum [108]
\begin{equation}
N_{\delta}(\hat{X}) = \sum_{d=0}^{+\infty} N_{\delta,d}(X).
\end{equation}

From table 1, we extract the invariants
\begin{equation}
N_{\delta}(\hat{X}) = 1280, \quad 92288, \quad 15655168, \quad 3883902528, \quad 1190923282176, \ldots,
\end{equation}
which are in agreement with the genus zero Gromov–Witten instanton numbers of \( \mathbb{P}^5[2,4] \) calculated (again using mirror symmetry) in [109].

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Appendix. $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions

We study two-dimensional theories with $\mathcal{N} = (2, 2)$ supersymmetry. There are two basic types of supermultiplet, the notation for which is obtained by dimensional reduction from that of [110]. A chiral supermultiplet has components $(\phi, \psi, F)$ while a vector supermultiplet has components $(v, \sigma, \lambda, D)$.

In Euclidean signature, we use a complex coordinate $z$ on spacetime, and consider $\sqrt{dz}$ and $\sqrt{d\bar{z}}$ as bases for the two spinor bundles $S_+$ and $S_-$ of opposite chirality. In components, we can write
\[
\psi = \psi_- \sqrt{dz} + \psi_+ \sqrt{d\bar{z}}.
\] (A.1)

In Minkowski signature, with time coordinate $x^0$ and spatial coordinate $x^1$, the metric has components $\eta_{00} = -1, \eta_{11} = 1, \eta_{01} = 0$. The fermionic coordinates $\theta^\pm$ and $\bar{\theta}^\pm$ are complex, related by complex conjugation (i.e. $(\theta^\pm)^\ast = \bar{\theta}^\pm$). The $\pm$ indices denote chirality. In particular,
\[
\begin{bmatrix}
\cosh \gamma & \sinh \gamma \\
\sinh \gamma & \cosh \gamma
\end{bmatrix} \in SO(1, 1)
\] (A.2)
acts by $e^{\pm \gamma/2}$ on $\theta^\pm$ and also by $e^{\pm \gamma/2}$ on $\bar{\theta}^\pm$.

A superfield is a function $\Phi$ of these variables, and can be expanded into the thetas. $\Phi$ is bosonic if $[\theta^\alpha, \Phi] = 0$ and fermionic if $\{\theta^\alpha, \Phi\} = 0$.

We introduce the combinations $x^\pm := x^0 \pm x^1$ and the corresponding differential operators
\[
\partial^\pm = \frac{\partial}{\partial x^\pm} := \frac{1}{2} \left( \frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1} \right).
\] (A.3)

There are two natural sets of differential operators on superspace:
\[
\begin{align*}
Q^\pm &= \frac{\partial}{\partial \theta^\pm} + i \bar{\theta}^\pm \partial^\pm \\
\overline{Q}^\pm &= -\frac{\partial}{\partial \bar{\theta}^\pm} - i \theta^\pm \partial^\pm
\end{align*}
\] (A.4, A.5)
which satisfy $\{Q_\pm, \overline{Q}_\pm\} = -2i \partial^\pm$, and
\[
D^\pm = \frac{\partial}{\partial \theta^\pm} - i \bar{\theta}^\pm \partial^\pm
\] (A.6)
\[
\overline{D}^\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i \theta^\pm \partial^\pm
\] (A.7)
which anti-commute with the first set, and satisfy $\{D_\pm, \overline{D}_\pm\} = 2i \partial^\pm$.

A chiral superfield $\Phi$ satisfies $\overline{D}_\pm \Phi = 0$. Its component description is:
\[
\Phi(x^\mu, \theta^\pm, \bar{\theta}^\pm) = \phi(y^\pm) + \theta^\alpha \psi^\alpha(y^\pm) + \theta^+ \bar{\theta}^- F(y^\pm)
\] (A.8)
where $y^\pm = x^\pm - i \theta^\pm \bar{\theta}^\pm$ and $F(y^\pm)$ is a non-propagating ‘auxiliary’ field in the multiplet. A twisted chiral superfield $U$ satisfies $\overline{D}_\pm U = D_\pm U = 0$.

A vector multiplet is a real superfield $V = V(x^\mu, \theta^\pm, \bar{\theta}^\pm)$ which under a gauge transformation $\Phi \mapsto e^{iA} \Phi$ transforms as $V \mapsto V + i(A - \bar{A})$, so that the Lagrangian

\[
J. Phys. A: Math. Theor. 50 (2017) 443004
\]
\[
\int d^4 \theta \Phi e^\Phi
\]
is invariant under gauge transformations.

There is a gauge transformation putting the gauge field into Wess–Zumino gauge, where it takes the form

\[
V = \theta^- \theta^+ (v_0 - v_1) + \theta^+ \theta^- (v_0 + v_1) - \theta^- \theta^+ \sigma - \theta^+ \theta^- \bar{\sigma} + i \theta^- \theta^+ (\theta^- \lambda_- + \theta^+ \lambda_+) \\
+ i \partial^- \bar{\theta} - (\theta^- \lambda_- + \theta^+ \lambda_+) + \theta^- \theta^+ \bar{\theta} \bar{\theta} D
\]
(A.9)

where \(\sigma\) is a complex scalar field, \(\lambda_\pm\) and \(\bar{\lambda}_\pm\) define a Dirac fermion, and \(D\) is an auxiliary real scalar field. The components \(v_\mu\) define the covariant derivatives \(D_\mu = \partial_\mu + i v_\mu\).

The field strength of \(V\) is

\[
\Sigma := D_\tau D_\tau V
\]
(A.10)

and it has a component expansion

\[
\Sigma = \sigma (\tilde{y}) + i \theta^+ \bar{\lambda}_+ (\tilde{y}) - i \theta^- \lambda_- (\tilde{y}) + \theta^+ \bar{\theta} \left[ D(\tilde{y}) - iv_0 (\tilde{y}) \right],
\]
(A.11)

where \(\tilde{x}^\pm := x^\pm \mp i \theta^\pm \theta^\pm\) and \(v_0 = \partial_0 v_1 - \partial_1 v_0\) is the curvature of the connection.

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