Doubly Special Relativity: facts and prospects

J. Kowalski–Glikman*
Institute for Theoretical Physics
University of Wroclaw
Pl. Maxa Borna 9
Pl–50-204 Wroclaw, Poland

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Abstract

In this short review of Doubly Special Relativity I describe first the relations between DSR and (quantum) gravity. Then I show how, in the case of a field theory with curved momentum space, the Hopf algebra of symmetries naturally emerges. I conclude with some remarks concerning DSR phenomenology and description of open problems.

1 Introduction: What is DSR?

The definition of Doubly Special Relativity (DSR) (Amelino-Camelia, 2001 and 2002, see Kowalski-Glikman, 2005 for review) is deceptively simple. Recall that Special Relativity is based on two postulates: Relativity Principle for inertial observers and the existence of a single observer independent scale associated with velocity of light. In this DSR replaces the second postulate by assuming existence of two observer independent scales: the old one of velocity plus the scale of mass (or of momentum, or of energy). That’s it.

Adding new postulate has consequences, however. The most immediate on is the question: what does the second observer–independent scale mean physically? Before trying to answer this question, let us recall the concept of

*e-mail address jurekk@ift.uni.wroc.pl
an observer–independent scale. It can be easily understood, when contrasted with the notion of dimensionful coupling constant, like Planck constant $\hbar$ or gravitational constant $G$. What is their status in relativity? Do they transform under Lorentz transformation? Well, naively, one would think that they should because they are given by dimensional quantities. But of course they do not. The point is that there is a special operational definition of these quantities. Namely each observer, synchronized with all the other observers, by means of the standard Einstein synchronization procedure, measures their values in an identical quasi–static experiment in her own reference frame (like Cavendish experiment). Then the relativity principle assures that the numerical value of such a constant will turn out to be the same in all experiments (the observers could check the validity of relativity principle by comparing values they obtained in their experiments). With an observer independent scale the situation is drastically different. Like the speed of light it cannot be measured in quasi-static experiments; all the observers now measure a quantity associated with a single object (in Special Relativity, all the observers could find out what the speed of light is just by looking at the same single photon.)

Now DSR postulates presence of the second observer–independent scale. What is the physical object that carries this scale, like the photon carrying the scale of velocity of light? We do not know. One can speculate that black hole remnants will do, but to understand them we need, presumably, the complete theory of quantum gravity. Fortunately, there is another way one can think of the observer–independent scale. If such a scale is present in the theory, and since, as explained above, it is operationally defined in terms of experiments, in which one physical object is observed by many distinct observers, who all measure the same value of the scale, it follows that the scale must appear as a parameter in the transformation rules, relating observers to each other. For example velocity of light is present as a parameter in Lorentz transformations. If we have a theory of spacetime with two observer independent scales, both should appear in the transformations. As an example one can contemplate the following form of infinitesimal action of Lorentz generators, rotations $M_i$ and boosts $N_i$ satisfying the standard Lorentz algebra, on momenta (so called DSR1), with the scale of mass $\kappa$

$$[M_i, P_j] = \epsilon_{ijk} P_k, \quad [M_i, P_0] = 0$$

$$[N_i, P_j] = \delta_{ij} \left( \frac{1}{2} \left( 1 - e^{-2P_0/\kappa} \right) + \frac{P^2}{2\kappa} \right) - \frac{1}{\kappa} P_i P_j,$$
This algebra is a part of $\kappa$-Poincaré quantum algebra, see Majid and Ruegg, 1994. One can also imagine a situation in which the scale $\kappa$ appears not in the rotational, but in the translational sector of the modified, deformed Poincaré group.

One may think of the second scale also in terms of synchronization of observers. Recall that the velocity of light scale is indispensable in Special Relativity because it provides the only meaningful way of synchronizing different observers. However this holds for spacetime measurements (lengths and time intervals) only. To define momenta and energy, one must relate them to velocities. On the other hand, using the momentum scale, one could, presumably, make both the spacetime and momentum space synchronization, independently, and perhaps could even describe the phase space as a single entity. Thus it seems that in DSR the primary concept would be the phase space not the configuration one.

In the limit when the second scale is very large (or very small depending on how the theory is constructed) the new theory should reduce to the old one, for example when the second scale $\kappa$ of DSR goes to infinity, DSR should reduce to Special Relativity. Putting it another way we can think of DSR as some sort of deformation of SR. Following this understanding some researchers would translate the acronym DSR to Deformed Special Relativity. But of course, deformation requires a deformation scale, so even semantically both terms are just equivalent, just stressing different aspects of DSR. Note that in addition to the modified, deformed algebra of spacetime symmetries, like the one in eq. (1), the theory is to be equipped with an additional structure(s), so as to make sure that its algebra cannot be reduced to the standard algebra of spacetime symmetries of Special Relativity, by rearrangement of generators. Only in such a case DSR will be physically different from Special Relativity.

In the framework of DSR we want to understand if there are any modifications to the standard particle kinematics as described by Special Relativity, at very high energies, of order of Planck scale. The motivation is both phenomenological and theoretical. First there are indications from observations of cosmic rays carrying energy higher than GZK cutoff that the standard Special relativistic kinematics might be not an appropriate description of particle scatterings at energies of order of $10^{20} \text{ eV}$ (in the laboratory frame). Similar phenomenon, the violation of the corresponding cutoff predicted by
the standard Special Relativistic kinematics for ultra-high energy photons seems also to be observed. It should be noted however that in both these cases we do not really control yet all the relevant astrophysical details of the processes involved (for example in the case of cosmic rays we do not really know what are the sources, though it it is hard to believe that they are not at the cosmological distances.) The extended discussion of these issues can be found, for example, in Aloisio et. al., 2005. If violation of the GZK cutoff is confirmed, and if indeed the sources are at the cosmological distances, this will presumably indicate the deviation from Lorentz kinematics. One of the major goals of DSR is to work out the robust theoretical predictions, concerning magnitude of such effects. I will briefly discuss the “DSR phenomenology” below.

2 Gravity as the origin of DSR

The idea of DSR arose from the desire to describe possible deviations from the standard Lorentz kinematics on the one hand and, contrary to the Lorentz breaking schemes, to preserve the most sacred principle of physics – the relativity principle. Originally the view was that one may be forced by phenomenological data to replace Special Relativity by DSR, and then, on the basis of the latter one should construct its curved space extension, “Doubly General Relativity”. Then it has been realized that, in fact, the situation is likely to be quite opposite: DSR might be the correct flat space limit of gravity coupled to particles (see Amelino-Camelia et. al., 2004 and Freidel et. al., 2004)

We are thus facing the fundamental theoretical question: is Special Relativity indeed, as it is believed, the correct limit of (quantum) gravity in the case when spacetime is flat? From the perspective of gravity flat Minkowski spacetime is some particular configuration of gravitational field, and us such is to be described by theory of gravity. It corresponds to configurations of gravitational field in which this field vanishes. However equations governing gravitational field are differential equations and thus describe the solutions only locally. In the case of Minkowski space particle kinematics we have to do not only with (flat) gravitational field but also with particles themselves. The particles are, of course, the sources of gravitational field and even in flat space limit the trace of particles’ back reaction on spacetime might remain in the form of some global information, even if locally, away from the locations
of the particle, the spacetime is flat. Of course we know that in general relativity energy-momentum of matter curves spacetime, and the strength of this effect is proportional to gravitational coupling (Newton’s constant.) Thus we are interested in the situation, in which the transition from general relativity to special relativity corresponds to smooth switching off the couplings. In principle two situations are possible (in 4 dimensions):

1. weak gravity, semiclassical limit of quantum gravity:

\[ G, \hbar \to 0, \quad \sqrt{\frac{\hbar}{G}} = \kappa \text{ remains finite} \quad (2) \]

2. weak gravity, small cosmological constant limit of quantum gravity:

\[ \Lambda \to 0, \quad \kappa \text{ remains finite} \quad (3) \]

The idea is therefore to devise a controllable transition from the full (quantum) gravity coupled to point particles to the regime, in which all local degrees of freedom of gravity are switched off. Then it is expected that locally, away from particles’ worldlines gravity will take the form of Minkowski (for \( \Lambda = 0 \)) or (Anti) De Sitter space, depending on the sign of \( \Lambda \). Thus it is expected that DSR arises as a limit of general relativity coupled to point particles in the topological field theory limit. To be more explicit, consider the formulation of gravity as the constrained topological field theory, proposed in Freidel and Starodubtsev, 2005.

\[ S = \int \left( B_{IJ} \wedge F^{IJ} - \frac{\alpha}{4} B_{IJ} \wedge B_{KL} \epsilon^{IJKL} - \frac{\beta}{2} B^{IJ} \wedge B_{IJ} \right) \quad (4) \]

Here \( F^{IJ} \) is the curvature of \( SO(4,1) \) connection \( A^{IJ} \), and \( B_{IJ} \) is a two-form valued in the algebra \( SO(4,1) \). The dimensionless parameters \( \alpha \) and \( \beta \) are related to gravitational and cosmological constants, and Immirzi parameter. The \( \alpha \) term breaks the symmetry, and for \( \alpha \neq 0 \) this theory is equivalent to general relativity. On the other hand there are various limits in which this theory becomes a topological one. For example, for \( \alpha \to 0 \) all the local degrees of freedom of gravity disappear, and only the topological ones remain. One hopes that after coupling this theory to point particles, one derives DSR in an appropriate, hopefully natural, limit. This hope is based on experience with the 2+1 dimensional case, which I will now discuss.
3 Gravity in 2+1 dimensions as DSR theory

It is well known that gravity in 2+1 does not possess local degrees of freedom and is described by a topological field theory. Even in the presence of point particles with mass and spin the 2+1 dimensional spacetime is locally flat. Thus 2+1 gravity is a perfect testing ground for DSR idea. There is also a simple argument that it is not just a toy model, but can tell us something about the full 3+1 dimensional case. It goes as follows.

As argued above what we are interested in is the flat space limit of gravity (perhaps also the semiclassical one in the quantum case.) Now consider the situation when we have 3+1 gravity coupled to planar configuration of particles. When the local degrees of freedom of gravity are switched off this configuration has translational symmetry along the direction perpendicular to the plane. But now we can make the dimensional reduction and describe the system equivalently with the help of 2+1 gravity coupled to the particles. The symmetry algebra in 2+1 dimensions must be therefore a subalgebra of the full 3+1 dimensional one. Thus if we find that the former is not the 2+1 Poincaré algebra but some modification of it, the latter must be some appropriate modification of the 3+1 dimensional Poincaré algebra. Thus if DSR is relevant in 2+1 dimensions, it is likely that it is going to be relevant in 3+1 dimensions as well.

Let us consider the analog of the situation 2, listed in the previous section. We start therefore with the 2+1 gravity with positive cosmological constant. Then it is quite well well established (see for example Noui and Roche, 2003) that the excitations of 3d quantum gravity with cosmological constant transform under representations of the quantum deformed de Sitter algebra SO_q(3,1), with \( z = \ln q \) behaving in the limit of small \( \Lambda h^2/\kappa^2 \) as \( z \approx \sqrt{\Lambda h}/\kappa \), where \( \kappa \) is equal to inverse 2+1 dimensional gravitational constant, and has dimension of mass.

I will not discuss at this point the notion of quantum deformed algebras (Hopf algebras) in much details It suffices to say that quantum algebras consist of several structures, the most important for our current purposes would be the universal enveloping algebra, which could be understand as an algebra of brackets among generators, which are equal to some analytic functions of them. Thus the quantum algebra is a generalization of a Lie algebra, and it is worth observing that the former reduces to the latter in an appropriate limit. The other structures of Hopf algebras, like co-product and antipode, are also relevant in the context of DSR, and I will introduce
them in the next section.

In the case of quantum algebra $SO_q(3,1)$ the algebraic part looks as follows (the parameter $z$ used below is related to $q$ by $z = \ln q$)

$$
[M_{2,3}, M_{1,3}] = \frac{1}{z} \sinh(zM_{1,2}) \cosh(zM_{0,3})
$$

$$
[M_{2,3}, M_{1,2}] = M_{1,3}
$$

$$
[M_{2,3}, M_{0,3}] = M_{0,2}
$$

$$
[M_{2,3}, M_{0,2}] = \frac{1}{z} \sinh(zM_{0,3}) \cosh(zM_{1,2})
$$

$$
[M_{1,3}, M_{1,2}] = -M_{2,3}
$$

$$
[M_{1,3}, M_{0,3}] = M_{0,1}
$$

$$
[M_{1,3}, M_{0,1}] = \frac{1}{z} \sinh(zM_{0,3}) \cosh(zM_{1,2})
$$

$$
[M_{1,2}, M_{0,2}] = -M_{0,1}
$$

$$
[M_{1,2}, M_{0,1}] = M_{0,2}
$$

$$
[M_{0,3}, M_{0,2}] = M_{2,3}
$$

$$
[M_{0,3}, M_{0,1}] = M_{1,3}
$$

$$
[M_{0,2}, M_{0,1}] = \frac{1}{z} \sinh(zM_{1,2}) \cosh(zM_{0,3})
$$

Observe that on the right hand sides we do not have linear functions generators, as in the Lie algebra case, but some (analytic) functions of them. However we still assume that the brackets are antisymmetric and, as it is easy to show, that Jacobi identity holds. Note that in the limit $z \to 0$ the algebra (5) becomes the standard algebra $SO(3,1)$, and this is the reason for using the term $SO_q(3,1)$.

The $SO(3,1)$ Lie algebra is the 2+1 dimensional de Sitter algebra and it is well known how to obtain the 2+1 dimensional Poincaré algebra from it. First of all one has to single out the energy and momentum generators of right physical dimension (note that the generators $M_{\mu\nu}$ of [5] are dimensionless): one identifies three-momenta $P_\mu \equiv (E, P_i) \ (\mu = 1, 2, 3, \ i = 1, 2)$ as appropriately rescaled generators $M_{0,\mu}$ and then one takes the Inömi–Wigner contraction limit. In the quantum algebra case, the contraction is a bit more tricky, as one has to convince oneself that after the contraction the structure one obtains is still a quantum algebra. Such contractions has been first discussed in Lukierski et. al., 1991.
Let us try to contract the algebra (5). To this aim, since momenta are dimensionful, while the generators $M$ in (5) are dimensionless, we must first rescale some of the generators by an appropriate scale, provided by combination of dimensionful constants present in definition of the parameter $z$

\[
E = \sqrt{\Lambda} h M_{0,3} \\
P_i = \sqrt{\Lambda} h M_{0,i} \\
M = M_{1,2} \\
N_i = M_{1,3}
\] (6)

Taking now into account the relation $z \approx \sqrt{\Lambda} h / \kappa$, which holds for small $\Lambda$

\[
[M_{2,3}, M_{1,3}] = \frac{1}{z} \sinh(z M_{1,2}) \cosh(z M_{0,3})
\]

we find

\[
[N_2, N_1] = \frac{\kappa}{h \sqrt{\Lambda}} \sinh(h \sqrt{\Lambda} / \kappa M) \cosh(E / \kappa)
\] (7)

Similarly from

\[
[M_{0,2}, M_{0,1}] = \frac{1}{z} \sinh(z M_{1,2}) \cosh(z M_{0,3})
\]

we get

\[
[P_2, P_1] = \sqrt{\Lambda} h \kappa \sinh(\sqrt{\Lambda} h / \kappa M) \cosh(E / \kappa)
\] (8)

Similar substitutions can be made in other commutators of (5). Now going to the contraction limit $\Lambda \to 0$, while keeping $\kappa$ constant we obtain the following algebra

\[
[N_i, N_j] = -M \epsilon_{ij} \cosh(E / \kappa) \\
[M, N_i] = \epsilon_{ij} N^j \\
[N_i, E] = P_i \\
[N_i, P_j] = \delta_{ij} \kappa \sinh(E / \kappa) \\
[M, P_i] = \epsilon_{ij} P^j \\
[E, P_i] = 0 \\
[P_2, P_1] = 0
\]

This algebra is called the three dimensional $\kappa$-Poincaré algebra (in the standard basis.)
Let us pause for a moment here to make couple of comments. First of all, one easily sees that in the limit $\kappa \to \infty$ from the \(\kappa\)-Poincaré algebra one gets the standard Poincaré algebra. Second, we see that in this algebra both the Lorentz and translation sectors are deformed. However, in the case of quantum algebras one is free to change the basis of generators in an arbitrary, analytic way (contrary to the case of Lie algebras, where only linear trasformations of generators are allowed.) It turns out that there exists such a change of the basis that the Lorentz part of the algebra becomes classical (i.e., undeformed.) This basis is called bicrossproduct one, and the Doubly Special Relativity model (both in 3 and 4 dimensions) based on such an algebra is called DSR1. In this basis the 2+1 dimensional \(\kappa\)-Poincaré algebra looks as follows

\[
\begin{align*}
[N_i, N_j] &= -\epsilon_{ij} M \\
[M, N_i] &= \epsilon_{ij} N^j \\
[N_i, E] &= P_i \\
[N_i, P_j] &= \delta_{ij} \frac{\kappa}{2} \left( 1 - e^{-2E/\kappa} + \frac{\vec{P}^2}{\kappa^2} \right) - \frac{1}{\kappa} P_i P_j \\
[M, P_i] &= \epsilon_{ij} P^j \\
[E, P_i] &= 0 \\
[P_i, P_2] &= 0.
\end{align*}
\] (10)

The algebra (10) is nothing but the 2+1 dimensional analogue of the algebra (11) we started our discussion with. Thus we conclude that in the case of 2+1 dimensional quantum gravity on de Sitter space, in the flat space, i.e., vanishing cosmological constant limit the standard Poincaré algebra is replaced by (quantum) \(\kappa\)-Poincaré algebra.

It is noteworthy that in the remarkable paper by Freidel and Livine \(\kappa\)-Poincaré algebra has been also found by direct quantization of 2+1 gravity without cosmological constant, coupled to point particles, in the weak gravitational constant limit. Even though the structures obtained by them and the ones one gets from contraction are very similar, their relation remains to be understood.

Let me summarize. In 2+1 gravity (in the limit of vanishing cosmological constant) the scale \(\kappa\), arises naturally. It can be also shown that instead of
the standard Poincaré symmetry we have to do with the deformed algebra, with deformation scale $\kappa$.

There is one interesting and important consequence of the emergence of $\kappa$-Poincaré algebra \[\text{(10)}\]. As in the standard case this algebra can be interpreted both as the algebra of spacetime symmetries and gauge algebra of gravity \textit{and} the algebra of charges associated with particle (energy momentum and spin.) It easy to observe that this algebra can be interpreted as an algebra of Lorentz symmetries of momenta if the momentum space is de Sitter space of curvature $\kappa$. It can be shown that one can extend this algebra to the full phase space algebra of a point particle, by adding four (non-commutative) coordinates (see Kowalski-Glikman and Nowak, 2003.) The resulting spacetime of the particle becomes the so-called $\kappa$-Minkowski spacetime with the non-commutative structure

$$[x_0, x_i] = -\frac{1}{\kappa} x_i \quad \text{(11)}$$

On $\kappa$-Minkowski spacetime one can built field theory, which in turn could be used to discuss phenomenological issues, mentioned in the Introduction. In the next section I will show how, in a framework of such a theory, one discovers the full power of quantum $\kappa$-Poincaré algebra.

## 4 Four dimensional field theory with curved momentum space

As I said above $\kappa$-Poincaré algebra can be understood as an algebra of Lorentz symmetries of momenta, for the space of momenta being the curved de Sitter space, of radius $\kappa$. Let us therefore try to built the scalar field theory on such a space (see also Daszkiewicz et al. 2005.) Usually field theory is constructed on spacetime, and then, by Fourier transform, is turned to the momentum space picture. Nothing however prevents us from constructing field theory directly on the momentum space, flat or curved. Let us see how this can be done.

Let the space of momenta be de Sitter space of radius $\kappa$

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2, \quad \text{(12)}$$

To find contact with $\kappa$-Poincaré algebra we introduce the coordinates on this
space as follows

\[
\begin{align*}
\eta_0 &= -\kappa \sinh \frac{P_0}{\kappa} - \frac{\vec{P}^2}{2\kappa} e^{\frac{\rho_0}{\kappa}}, \\
\eta_i &= -P_i e^{\frac{\rho_0}{\kappa}}, \\
\eta_4 &= \kappa \cosh \frac{P_0}{\kappa} - \frac{\vec{P}^2}{2\kappa} e^{\frac{\rho_0}{\kappa}},
\end{align*}
\]

(13)

Then one can easily check that the commutators of \( P_\mu \) with generators of Lorentz subgroup, \( SO(3,1) \) of the full symmetry group \( SO(4,1) \) of (12) form exactly the \( \kappa \)-Poincaré algebra (1).

In the standard, flat momentum space, case the action for free massive scalar field has the form

\[
S_0 = \int d^4P \mathcal{M}_0(P) \Phi(P) \Phi(-P)
\]

with \( \mathcal{M}_0(P) = P^2 - m^2 \) being the mass shell condition. In the case of de Sitter space of momenta we should replace \( \mathcal{M}_0(P) \) with some generalized mass shell condition and also modify somehow \( \Phi(-P) \), because “\(-P\)” does not make sense on curved space.

It is rather clear what should replace \( \mathcal{M}_0(P) \). It should be just the Casimir of the algebra (1). As a result of the presence of the scale \( \kappa \), contrary to the special relativistic case, there is an ambiguity here. However since the Lorentz generators can be identified with the generators of the \( SO(4,1) \) algebra of symmetries of the quadratic form (12), operating in the \( \eta_0 - \eta_3 \) sector, and leaving \( \eta_4 \) invariant it is natural to choose the mass shell condition to be just (rescaled) \( \eta_4 \), to wit

\[
m^2 = \kappa \eta_4 - \kappa^2
\]

so that

\[
\mathcal{M}_\kappa(P) = (2\kappa \sinh \frac{P_0}{2\kappa})^2 - \vec{P}^2 e^{\frac{P_0}{\kappa}} - m^2
\]

(15)

Eq. (15) is the famous dispersion relation of DSR1. Notice that it implies that the momentum is bounded from above by \( \kappa \), while the energy is unbounded.

Let us now turn to the “\(-P\)” issue. To see what is to replace it in the theory with curved momentum space let us trace the origin of it. In Special Relativity the space of momenta is flat, and equipped with the standard group of motions. The space of momenta has the distinguished point, corresponding
to zero momentum. An element of translation group \( g(P) \) moves this point to a point of coordinates \( P \). This defines coordinates on the energy momentum space. Now we define the point with coordinates \( S(P) \) to be the one obtained from the origin by the action of the element \( g^{-1}(P) \). Since the group of translations on flat space is an abelian group with addition, \( S(P) = -P \).

Now, since in the case of interest the space of momenta is de Sitter space, which is a maximally symmetric space, we can repeat exactly the same procedure. The result, however is not trivial now, to wit

\[
S(P_0) = -P_0, \quad S(P_i) = -e^{P_b/\kappa} P_i
\]  

(16)

Actually one can check that the \( S \) operator is in this case nothing but the antipode of \( \kappa \)-Poincaré quantum algebra. Thus we can write down the action for the scalar field on curved momentum space as

\[
S_\kappa = \int d^4P \, M_{\kappa}(P) \Phi(P) \Phi(S(P))
\]

(17)

De Sitter space of momenta has the ten-dimensional group of symmetries, which can be decomposed to six “rotations” and four remaining symmetries, forming the deformed \( \kappa \)-Poincaré symmetry [1]. We expect therefore that the action [17] should, if properly constructed, be invariant under action of this group. We will find that this is indeed a case, however the story will take an unexpected turn here: the action will turn out to be invariant under action of the quantum group.

Let us consider the four-parameter subgroup of symmetries, that in the standard case would correspond to spacetime translation. It is easy to see that in the standard case the translation in spacetime fields is in the one-to-one correspondence with the phase transformations of the momentum space ones. This suggests that the ten parameter group of Poincaré symmetries in space-time translates into six parameter Lorentz group plus four independent phase transformations in the momentum space, being representations of the same algebra.

Using this insight let us turn to the case at hands. Consider first the infinitesimal phase transformation in energy direction\(^1\) (to simplify the notation I put \( \kappa = 1 \))

\[
\delta_0 \Phi(P_0, P) = i \epsilon P_0 \Phi(P_0, P),
\]

(18)

\(^1\)Note that since the function \( M \) is real, \( \delta_0 M_\kappa = \delta_i M_\kappa = 0 \).
where $\epsilon$ is an infinitesimal parameter. It follows that
\[
\delta_0 \Phi(S(P_0), S(P)) = i\epsilon S(P_0) \Phi(S(P_0), S(P)) = -i\epsilon P_0 \Phi(S(P_0), S(P)) \tag{19}
\]
and using Leibniz rule we easily see that the action is indeed invariant. Let us now consider the phase transformation in the momentum direction. Assume that in this case
\[
\delta_i \Phi(P_0, P) = i\epsilon P_i \Phi(P_0, P). \tag{20}
\]
But then
\[
\delta_i \Phi(S(P_0), S(P)) = i\epsilon S(P_i) \Phi(S(P_0), S(P)) = -i\epsilon e^{P_0} P_i \Phi(S(P_0), S(P)) \tag{21}
\]
and the action is not invariant, if we apply Leibniz rule.

The way out of this problem is to replace the Leibniz rule by the co-product one. To this end we take
\[
\delta_i \{\Phi(P_0, P) \Phi(S(P_0), S(P))\} \equiv \\
\delta_i \{\Phi(P_0, P)\} \Phi(S(P_0), S(P)) + \{e^{-P_0} \Phi(P_0, P)\} \delta_i \{\Phi(S(P_0), S(P))\} = 0
\]
i.e., we generalize Leibniz rule by multiplying $\Phi(P_0, P)$ in the second term by $e^{-P_0}$. Note that such definition is consistent with the fact that the fields are commuting, because
\[
\delta_i (\Phi(S(P_0), S(P)) \Phi(P_0, P)) = \\
(\epsilon S(P_i) + i\epsilon e^{-S(P_0)} P_i) \Phi(S(P_0), S(P)) \Phi(P_0, P) = 0.
\]
We see therefore that in order to make the action invariant with respect to infinitesimal phase transformations one must generalize the standard Leibniz rule to the non-symmetric co-product one.

The rule of how an algebra acts on (tensor) product of objects is called the co-product, and denoted by $\Delta$. If Leibniz rule holds the coproduct is trivial $\Delta \delta = \delta \otimes 1 + 1 \otimes \delta$. Quantum groups can be characterized by the fact that Leibniz rule is generalized to a non-trivial coproduct rule. We discovered that in the case of $\kappa$-Poincaré algebra it takes the form
\[
\Delta \delta_0 = \delta_0 \otimes 1 + 1 \otimes \delta_0, \quad \Delta \delta_i = \delta_i \otimes 1 + e^{-P_0} \otimes \delta_i \tag{22}
\]
One can check that, similarly, the co-product for rotational part of the symmetry algebra is also non-trivial. The presence of non-trivial co-product in the algebraic structure of DSR theory has, presumably, far reaching consequences for particle kinematics. I will return to this point below.
5 DSR phenomenology

DSR has emerged initially from the quantum gravity phenomenology investigations, as a phenomenological theory, capable of describing possible future observations disagreeing with predictions of Special Relativity. Two of these effects, the possible energy dependence of the speed of light, which could be observed by GLAST satellite, and the mentioned already, possible violation of the GZK cutoff, which could be confirmed by Pierre Auger Observatory were quite extensively discussed in the literature. Let me now briefly describe what would be the status of this (possible) effects vis a vis the approach of DSR I have been analyzed above.

The prediction of energy dependence of the speed of light is based on the rather naive observation that since in (some formulations of) DSR the dispersion relation is being deformed, the formula for velocity $v = \partial E/\partial p$ gives, as a rule, the result which differs from this of Special Relativity. It turns out however that this conclusion may not stand if the effects of non-commutative spacetime are taken into account.

In the classical theory the non-commutativity is replaced by the nontrivial structure of the phase space of the particle, and, as in the standard case, one calculates the three velocity of the particle as the ratio of $\dot{x} = \{x,H\}$ and $\dot{t} = \{t,H\}$: $v = \dot{x}/\dot{t}$. Then it can be generally proved that the effect of this nontrivial phase space structure cancels neatly the effect of the modified dispersion relation (see Daszkiewicz et. al. 2004 for details.) Thus, in the framework of this formulation of DSR, the speed of massless particles is always 1, though there are deviations from the standard Special Relativistic formulas in the case of massive particles. However the leading order corrections are here of order of $m/\kappa$, presumably beyond the reach of any feasible experiment.

Similarly one can argue that deviations from the GZK cutoff should be negligibly small in any natural DSR theory. The reasoning goes as follows (similar argument can be found in Amelino-Camelia 2003.) Consider experimental measurement of the threshold energy for reaction $p + \gamma = p + \pi^0$, which is one of the relevant ones in the ultra high energy cosmic rays case, but details are not relevant here. To measure this energy we take the proton

\footnote{It should be stressed that DSR has been originally proposed as an idea, not a formally formulated theory, and therefore it may well happen that the particular realization of this idea described above could be replaced by another one in the future.}
initially at rest and bombard it by more and more energetic photons. At some point, when the photon energy is of order of $E_{th}^0 = 145$ MeV, the pion is being produced. Note that the threshold energy is just $E_{th}^0$, exactly as predicted by Special Relativity, and the corrections of DSR (if any) are much smaller than the experimental error bars $\Delta E_{th}^0$. Thus whichever kinematics is the real one we have the robust result for the value of the threshold energy.

Now there comes the major point. Since DSR respects the Relativity Principle by definition, we are allowed to boost the photon energy down to the CMB energy (this cannot be done in the Lorentz breaking schemes, where the velocity of the observer with respect to the ether matters), and to calculate the value of the corresponding rapidity parameter. Now we boost the proton with the same value of rapidity, using the DSR transformation rules, and check what is the modified threshold. Unfortunately, the leading order correction to the standard Special Relativistic transformation rule would be of the form $\sim \alpha E_{proton}/\kappa$, where $E_{proton}$ is the energy of the proton after boost, and $\alpha$ is the numerical parameter, fixed in any particular formulation of DSR. It is natural to expect that $\alpha$ should be of order 1, so that in order to have sizeable effect we need $\kappa$ of order of $10^{19}$ eV, quite far from the expected Planck scale\(^3\). One may contemplate the idea that since the proton is presumably, from the perspective of the Planck scale physics, a very complex composite system, we do not have to do here with “fundamental” $\kappa$, but with some effective one instead, but then this particular value should be explained (it is curious to note in this context that, as observed in Amelino-Camelia 2003, $10^{19}$ eV is of order of the geometric mean of the Planck energy and the proton rest mass.) However the conclusion for now seems inevitably be that with the present formulation of DSR, the explanation of possible violation of GZK cutoff offered by this theory is, at least, rather unnatural.

6 DSR – facts and prospects

Let me summarize. Above I stressed two facts that seem to be essential features of DSR theory.

First (quantum) gravity in 2+1 dimensions coupled to point particles is just a DSR theory. Since the former is rather well understood, it is a perfect

\(^3\)Note that in this reasoning we do not have to refer to any particular DSR kinematics, the form of energy-momentum conservation etc. The only input here is the Relativity Principle.
playground for trying to understand better the physics of the latter. In 3+1 dimensions the situation is much less clear. Presumably, DSR emerges in an appropriate limit of (quantum) gravity, coupled to point particles, when the dynamical degrees of freedom of gravitational field are switched off, and only the topological ones remain. However, it is not known, what exactly this limit would be, and how to perform the limiting procedure in the full dynamical theory. There is an important insight, coming from algebraic consideration, though. In 3+1 dimension one can do almost exactly the same procedure as the one, I presented for the 2+1 case above. It suffices to replace the $SO_q(3,1)$ group with the $SO_q(4,1)$. It happens however that in the course of the limiting procedure one has to further rescale the generators corresponding to energy and momentum. The possible rescalings are parametrized by the real, positive parameter $r$: for $r > 1$ the contraction does not exist, for $0 < r < 1$ as the result of contraction one gets the standard Poincaré algebra, and only for one particular value $r = 1$ one finds $\kappa$-Poincaré algebra. This result is not understood yet, and, if DSR is indeed a limit of gravity, gravity must tell us why one has to choose this particular contraction.

Second, as I explained above there is a direct interplay between non-trivial co-product and the fact that momentum space is curved. In addition, curved momentum space naturally implies non-commutative space-time. While the relation between these three properties of DSR theory has been well established, it still requires further investigations.

The presence of the non-trivial co-product in DSR theory has its direct consequences for particle kinematics. Namely the co-product can be understood as a rule of momentum composition. This fact has been again well established in the 2+1 dimensional case. However the 3+1 situation requires still further investigations. The main problem is that the co-product composition rule is not symmetric: the total momentum of the system ($\text{particle}_1 + \text{particle}_2$) is not equal, in general, to the of the total momentum of ($\text{particle}_2 + \text{particle}_1$) one. This can be easily understood in 2+1 dimension if one thinks of particles in terms of their worldlines, and where the theory takes care of the worldlines' braiding. In 3+1 dimensions the situation is far from being clear, though. Perhaps a solution could be replacing holonomies that characterize particles in 2+1 one dimensions by surfaces surrounding particles in 3+1 dimensions. If this is true, presumably the theory of gerbes will play a role in DSR (and gravity coupled with particles, for that matter.)

Related to this is the problem of “spectators”. If the co-product rule is indeed correct, any particle would feel non-local influence of other particles of
the universe. This means in particular, that LSZ theorem of quantum field
theory, which requires the existence of free asymptotic states, presumably
does not hold in DSR, and thus the whole of basic properties of QFT will
have to be reconsidered.

Arguably one of the most urgent problems of DSR is the question “what
is the momentum?” Indeed, as I mentioned above, in the $\kappa$-Poincaré case one
has a freedom do redefine momentum and energy by any function of them
and the $\kappa$ scale, restricted only by the condition that in the limit $\kappa \to \infty$
they all reduce to the standard momenta of special relativity. In particular
some of them might be bounded from above, and some not. For example
in DSR1 momentum is bounded from above and energy is not, in another
model, called DSR2 both energy and momentum are bounded, and there are
models in which neither is. Thus the question arises as to which one of them
is physical? Which momentum and energy we measure in our detectors?

There is a natural answer to this question. Namely, the physical mo men-
tum is the charge that couples to gravity. Indeed if DSR is an emerge
t theory, being the limit of gravity, the starting point should be, presumably,
gravity coupled to particles’ Poincaré charges in the canonical way.

To conclude. There seem to be an important and deep interrelations be-
tween developments in quantum gravity and understanding of DSR. Proper
control over semiclassical quantum gravity would provide an insight into the
physical meaning and relevance of DSR. And vice versa, DSR, being a possi-
ble description of ultra high energetic particle behavior will perhaps become
a workable model of quantum gravity phenomenology, to be confronted with
future experiments.

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