On the Color-Singlet States in Many-Quark Model with the $su(4)$-Algebraic Structure. II

Determination of Ground-State Energies

Yasuhiko Tsue, Constança Providência, João da Providência and Masatoshi Yamamura

1Physics Division, Faculty of Science, Kochi University, Kochi 780-8520, Japan
2Departamento de Física, Universidade de Coimbra, 3004-516 Coimbra, Portugal
3Department of Pure and Applied Physics, Faculty of Engineering Science, Kansai University, Suita 564-8680, Japan

(Received January 20, 2013)

Ground-state energies are investigated in a many-quark model with pairing interactions, which has the $su(4)$-algebraic structure. Exact eigenstates in the boson realization method are constructed by imposing a color-singlet condition developed in the previous paper. An interaction term breaking the $su(4)$-dynamical symmetry plays an important role to determine the ground state. As a result, a quark-pairing state or a quark-triplet state as a nucleon is realized with a certain value of a variable which is regarded as an order parameter. In addition to the parameter regions in which these ground states are realized, it is shown that there are two transition regions between the quark-pairing and the quark-triplet states with different values of the order parameter.

§1. Introduction

The Bonn quark model is an interesting model, which was first introduced with a purpose of describing the nucleon and the $\Delta$-resonance as quark-triplet states. The original Bonn quark model has the following Hamiltonian

$$\tilde{H} = -\sum_m \sum_{m'} (c^*_{2m} c_{3\tilde{m'}} c_{3m'} c_{2m'} + c^*_{3m} c_{1\tilde{m'}} c_{1m'} c_{3m'} + c^*_{1m} c^*_{3m} c_{2m'} c_{1m'}) \quad (1.1)$$

where $c^*_{im}$ and $c_{im}$ are quark creation and annihilation operators with color $i$ and the angular momentum quantum number of the single quark level, $m = -j_s, -j_s + 1, \ldots, j_s$. Here, $c^*_{im} = (-1)^{j_s-m} c^*_{i-m}$. This Hamiltonian represents the quark-pairing interaction which, in general, leads to the color instability. Namely, it is possible that a color superconducting state is realized. It has been remarked that the original Bonn quark model has a dynamical $su(4)$-symmetry. Thus, a $su(4)$-symmetry breaking term is introduced in which the color $su(3)$-symmetry is retained. Namely, a $su(4)$-symmetry breaking interaction proportional to the $su(3)$-Casimir operator, $\tilde{Q}^2$, is introduced, which represents a particle-hole-type interaction in terms of the quark shell model:

$$\tilde{H}_m = \tilde{H} + \chi \tilde{Q}^2 \quad (1.2)$$

This model is called the modified Bonn quark model. In the previous paper, in Ref[2], which is hereafter referred to as (A), exact eigenstates were investigated by

\[\text{typeset using } \textit{P\TeX}.\]
the method of the boson realization. In Ref. [3], which is hereafter referred to as (B), the exact eigenstates with single-quark, quark-pair and quark-triplet structures were treated in a unified way. Further, in Ref. [4], which is referred to as (C), a phase diagram was given on the $\chi$-$N$ plane, where $N$ represents a quark number. However, in the series of previous papers (A)~(C), a color neutral quark-triplet state was only realized as a color-singlet state.

In the first paper of the present series, namely in Ref. [5], which is hereafter referred to as (I), the exact eigenstates are constructed so as to satisfy a certain condition which gives a color-singlet state in average. As a result, the color-singlet state is obtained in the color-symmetric form. In this paper, which is the second paper of the present series, the ground-state energy is reinvestigated under the condition giving a color-singlet state. In each region of the force strength $\chi$, the character of ground state is investigated, in which the quark-pairing state or quark-triplet state as a nucleon may be realized with a distinct value of a certain variable which is regarded as the order parameter of a phase transition. It is shown that there are two transition regions with different values of the order parameter.

This paper is organized as follows: In the next section, the basic scheme for searching the ground-state energies is given in the modified Bonn quark model. In §3, the condition for the ground state is investigated and in §4, the ground state is determined and the ground-state energy is derived in each area of the force strength $\chi$. In §5, numerical analysis is given. The last section is devoted to concluding remarks.

§2. Scheme for searching the ground-state energies

Our main concern in the present paper (II) is to search the minimum values of energies corresponding to the ground states. For the convenience of the discussion, we extract the relations (1-6.17) and (1-6.19):

$$E^{(m)}(N^0, n^0; 2r) = \frac{1}{2}(1 + 2\chi)F(N^0, n^0; 2r) + \frac{1}{2}n^0(2\Omega^0 - n^0) - \frac{1}{6}N^0(6\Omega^0 + 6 - N^0), \quad (0 \leq N^0 \leq 4\Omega^0) \quad (2.1)$$

$$E_l^{(m)}(N^0, n^0) = E^{(m)}(N^0, n^0; 2r = n^0) = \frac{1}{4}(1 + 6\chi)n_0^2 - \frac{1}{2}\left[(N^0 + 3 - 2\Omega^0) + 2\chi(N^0 + 3)\right]n^0 - \frac{1}{4}N^0(4\Omega^0 + 2 - N^0) + \frac{\chi}{6}N^0(N^0 + 6) \quad \text{for the area } A_l, \quad (2.2a)$$

$$E_s^{(m)}(N^0, n^0) = E^{(m)}(N^0, n^0; 2r = \frac{1}{2}(N^0 - n^0)) = \frac{1}{4}(1 + 6\chi)n_0^2 - \frac{1}{2}\left[(N^0 - 3 - 2\Omega^0) + 2\chi(N^0 - 3)\right]n^0 - \frac{1}{4}N^0(4\Omega^0 + 6 - N^0) + \frac{\chi}{6}N^0(N^0 - 6) \quad \text{for the area } A_s, \quad (2.2b)$$
Here, the areas $A_l$ and $A_s$ are shown in Fig. 1. Later, we will treat $A_l$ by decomposing it into $A_{l_1}$ and $A_{l_2}$ which indicate the areas related to $0 \leq N^0 \leq 3 \Omega^0$ and $3 \Omega^0 \leq N^0 \leq 4 \Omega^0$, respectively. The relation (2.2) shows us that $E_l^{(m)}(N^0, n^0)$ and $E_s^{(m)}(N^0, n^0)$ are quadratic in $n^0$ for a given value of $N^0$. In each area of $A_l$ and $A_s$, regarding $n^0$ as continuous variable, we first calculate the minimum values of $E_l^{(m)}(N^0, n^0)$ and $E_s^{(m)}(N^0, n^0)$, respectively, for a given value of $N^0$ and we compare with each other the two minima so obtained. The above is our scheme for searching the ground-state energies.

First, we notice that the term $F = (1/2) \cdot (1 + 2 \chi) F(N^0, n^0; 2r)$ in the expression (2.1) is negative definite, if $\chi$ obeys

$$1 + 2 \chi < 0 , \quad \text{i.e.,} \quad \chi < -\frac{1}{2} .$$

(2.3)

Therefore, in this case, the maximum value of $F(N^0, n^0; 2r)$ determines the minimum value of energy, which appears at the point $2r = 0$, that is, $2s \neq 0$ or $2l \neq 0$. This case is contradictory to the requirement that $F(N^0, n^0; 2r)$ should be as small as possible. Next, we consider the case

$$1 + 2 \chi = 0 , \quad \text{i.e.,} \quad \chi = -\frac{1}{2} .$$

(2.4)

In this case, independently of the magnitude of $F(N^0, n^0; 2r)$, the term $F$ vanishes. This indicates that the energies in the case $2r < 2r_m$ are the same as that in the case $2r = 2r_m$, in other word, the case $2r = 2r_m$ is not toward the smaller direction in energy. For the above reason, we will be concerned only with the case

$$1 + 2 \chi > 0 , \quad \text{i.e.,} \quad \chi > -\frac{1}{2} .$$

(2.5)

Certainly, the procedure for minimizing the energy in the case (2.5) automatically leads to the condition required to $F(N^0, n^0; 2r)$. The above tells that, in the case

Fig. 1. Representation of the two areas of the $n^0, N^0$-plane considered in Eqs. (2.2a) and (2.2b).
χ ≤ −1/2, the present model loses its meaning for the many-quark model. This point was already stressed in (A) qualitatively.

In (A) and (C), we showed the minimum value of the energy as a function of \( N^0 \) and the value of \( n^0 \) which minimizes the energy. In (A) and (C), we treated the case \( n_0 = 0 \), but essentially the same is valid for \( n_0 \neq 0 \). Also, we mentioned that \( n^0 \) can be regarded as the order parameter of the phase transition between the quark-pairs \( (n^0 = 0) \) and the quark-triplets \( (n^0 = N^0/3) \). The results obtained in this investigation seem to be quite natural and acceptable. However, this investigation was not based on the “color-singlet” states minimizing \( F(N^0, n^0; 2r) \). Further, the results were provided only for the area \( A_{l1} \). Therefore, we must reexamine the results presented in (A) and (C).

Following the procedure already mentioned, we are able to obtain the result in each area. In the area \( A_{l1} \), we obtain the following:

\[
(l_{1; 1}) \quad -\frac{1}{2} < \chi < -\frac{1}{6} \cdot \frac{\Omega^0 + 6}{\Omega^0 + 2}, \quad \chi \leq -\frac{1}{2}, \quad n^0 = 0 \quad \text{for} \quad 0 \leq N^0 \leq 3\Omega^0, \quad (2.6a)
\]

\[
(l_{1; 2}) \quad -\frac{1}{6} \cdot \frac{\Omega^0 + 6}{\Omega^0 + 2} < \chi \leq -\frac{1}{6}, \quad n^0 = 0 \quad \text{for} \quad 0 \leq N^0 \leq \frac{6(2\Omega^0 - 3 - 6\chi)}{5 + 6\chi},
\]

\[
(l_{1; 3}) \quad -\frac{1}{6} < \chi \leq \frac{2\Omega^0 - 3}{6}, \quad n^0 = 0 \quad \text{for} \quad 0 \leq N^0 \leq \frac{2\Omega^0}{1 + 2\chi} - 3,
\]

\[
(l_{1; 4}) \quad \frac{2\Omega^0 - 3}{6} < \chi < +\infty, \quad n^0 = \frac{N^0}{3} \quad \text{for} \quad 3\Omega^0 - \frac{9}{2} - 9\chi \leq N^0 \leq 3\Omega^0, \quad (2.6c)
\]

\[
(l_{1; 4}) \quad \frac{2\Omega^0 - 3}{6} < \chi < +\infty, \quad n^0 = \frac{N^0}{3} \quad \text{for} \quad 0 \leq N^0 \leq 3\Omega^0. \quad (2.6d)
\]

The results (2.6) coincide with those shown in the relation (C-4.1). In the case (2.6b), we find that, at the point \( N^0 = 6(2\Omega^0 - 3 - 6\chi)/(5 + 6\chi) \), a phase transition occurs. The order parameter \( n^0 \) changes from \( n^0 = 0 \) to \( n^0 = N^0/3 \). In the area \( A_{l2} \), we obtain the following results:

\[
(l_{2; 1}) \quad -\frac{1}{2} < \chi \leq -\frac{1}{2} \cdot \frac{2\Omega^0 + 3}{4\Omega^0 + 3},
\]
\[ (l_2; 2) \quad - \frac{1}{2} \cdot \frac{2\Omega^0 + 3}{4\Omega^0 + 3} < \chi \leq -\frac{1}{6}, \]

(i) \( n^0 = 0 \) for \( 3\Omega^0 \leq N^0 \leq 4\Omega^0 \), \quad (2.7a)

\[ (l_2; 3) \quad - \frac{1}{6} < \chi < +\infty, \]

(i) \( n^0 = 4\Omega^0 - N^0 \) for \( 3\Omega^0 \leq N^0 \leq 4\Omega^0 \). \quad (2.7b)

In the case (2.7b), the phase transition occurs at the point \( N^0 = \frac{2(4\Omega^0 - 3) + 6(2\Omega^0 - 1)\chi}{3 + 10\chi} \).

As was shown in (A), the present model can be formulated not only from the side \( N^0 = 0 \) but also from the side \( N^0 = 6\Omega^0 \). We called this form the hole picture, which is obtained by replacing \( N^0 \) with \( (6\Omega^0 - N^0) \) in the relations appearing in the form from the side \( N^0 = 0 \). Therefore, it may be necessary to compare the results for \( A_{l_2} \) with those for \( A_{l_1}' \) obtained by replacing \( N^0 \) with \( (6\Omega^0 - N^0) \) in \( A_{l_1} \). In the area \( A_{l_1}' \), we obtain the following:

\[ (s; 1) \quad - \frac{1}{2} < \chi \leq -\frac{1}{6} \cdot \frac{4\Omega^0 + 9}{2\Omega^0 + 3}, \]

(i) \( n^0 = \frac{N^0}{3} \) for \( 0 \leq N^0 < \frac{3(2\Omega^0 + 3 + 6\chi)}{1 - 6\chi} \), \quad (2.8a)

\[ (s; 2) \quad - \frac{1}{6} \cdot \frac{4\Omega^0 + 9}{2\Omega^0 + 3} < \chi < +\infty, \]

(i) \( n^0 = \frac{N^0}{3} \) for \( 0 \leq N^0 \leq 2\Omega^0 \), \quad (2.8b)

\[ (s; 3) \quad - \frac{1}{2} < \chi < +\infty, \]

(i) \( n^0 = \frac{N^0}{3} \) for \( 2\Omega^0 \leq N^0 \leq 3\Omega^0 \). \quad (2.8c)

In the case (2.8a), a phase transition occurs at the point \( N^0 = 3(2\Omega^0 + 3 + 6\chi)/(1 - 6\chi) \).

The parameter \( n^0 \) changes from \( n^0 = N^0/3 \) to \( n^0 = N^0 \).
each area with the other areas. First, we discuss the relation between the

(3.2) indicates that, in the case (3.1),

In any other case, we have

Therefore, the comparisons of $E_i^{(m)}(N^0, n^0 = N^0)$ with $E_i^{(m)}(N^0, n^0 = N^0/3)$ should be compared with $E_i^{(m)}(N^0, n^0 = 0)$. For the comparison, we have the relation

The relation (3.2) indicates that, in the case (3.1), $\Delta E$ is positive. In any other case in $A_s$, we have $n^0 = N^0/3$ which belongs to the border of $A_s$ and $A_l$ and we notice the relation

Therefore, the comparisons of $E_s^{(m)}(N^0, n^0 = N^0)$ with $E_i^{(m)}(N^0, n^0)$ is reduced to those of $E_i^{(m)}(N^0, n^0 = N^0/3)$ with $E_i^{(m)}(N^0, n^0)$: The results are reduced to those

$\Delta E = E_s^{(m)}(N^0, n^0 = N^0) - E_i^{(m)}(N^0, n^0 = 0)$

\[
\Delta E = \frac{1}{2} (2\Omega^0 - N^0) + \frac{1}{2} (1 + 2\chi)(N^0 + 2) .
\]

\[
\frac{3(2\Omega^0 + 3 + 6\chi)}{1 - 6\chi} < N^0 \leq 2\Omega^0 . (3.1)
\]

In any other case, we have $n^0 = N^0/3$. On the other hand, we notice the case $(l; 1)$. Since $- (1/6) \cdot (4\Omega^0 + 9)/(2\Omega^0 + 3) < - (1/6) \cdot (\Omega^0 + 6)/(\Omega^0 + 2)$ and $2\Omega^0 < 3\Omega^0$, $E_s^{(m)}(N^0, n^0 = N^0)$ should be compared with $E_i^{(m)}(N^0, n^0 = 0)$. For the comparison, we have the relation

\[
\Delta E = E_s^{(m)}(N^0, n^0 = N^0) - E_i^{(m)}(N^0, n^0 = 0)
\]

\[
\frac{3(2\Omega^0 + 3 + 6\chi)}{1 - 6\chi} < N^0 \leq 2\Omega^0 . (3.1)
\]

In any other case, we have $n^0 = N^0/3$. On the other hand, we notice the case $(l; 1)$. Since $- (1/6) \cdot (4\Omega^0 + 9)/(2\Omega^0 + 3) < - (1/6) \cdot (\Omega^0 + 6)/(\Omega^0 + 2)$ and $2\Omega^0 < 3\Omega^0$, $E_s^{(m)}(N^0, n^0 = N^0)$ should be compared with $E_i^{(m)}(N^0, n^0 = 0)$. For the comparison, we have the relation

\[
\Delta E = E_s^{(m)}(N^0, n^0 = N^0) - E_i^{(m)}(N^0, n^0 = 0)
\]

\[
\frac{3(2\Omega^0 + 3 + 6\chi)}{1 - 6\chi} < N^0 \leq 2\Omega^0 . (3.1)
\]

In any other case, we have $n^0 = N^0/3$. On the other hand, we notice the case $(l; 1)$. Since $- (1/6) \cdot (4\Omega^0 + 9)/(2\Omega^0 + 3) < - (1/6) \cdot (\Omega^0 + 6)/(\Omega^0 + 2)$ and $2\Omega^0 < 3\Omega^0$, $E_s^{(m)}(N^0, n^0 = N^0)$ should be compared with $E_i^{(m)}(N^0, n^0 = 0)$. For the comparison, we have the relation

\[
\Delta E = E_s^{(m)}(N^0, n^0 = N^0) - E_i^{(m)}(N^0, n^0 = 0)
\]

\[
\frac{3(2\Omega^0 + 3 + 6\chi)}{1 - 6\chi} < N^0 \leq 2\Omega^0 . (3.1)
\]

In any other case, we have $n^0 = N^0/3$. On the other hand, we notice the case $(l; 1)$. Since $- (1/6) \cdot (4\Omega^0 + 9)/(2\Omega^0 + 3) < - (1/6) \cdot (\Omega^0 + 6)/(\Omega^0 + 2)$ and $2\Omega^0 < 3\Omega^0$, $E_s^{(m)}(N^0, n^0 = N^0)$ should be compared with $E_i^{(m)}(N^0, n^0 = 0)$. For the comparison, we have the relation

\[
\Delta E = E_s^{(m)}(N^0, n^0 = N^0) - E_i^{(m)}(N^0, n^0 = 0)
\]

\[
\frac{3(2\Omega^0 + 3 + 6\chi)}{1 - 6\chi} < N^0 \leq 2\Omega^0 . (3.1)
\]
shown in the relations (2.6a) ∼ (2.6d). There does not exist any contribution of the area $A_s$ to the ground-state energy.

Next, we consider the overlap of the areas $A_{l_2}$ and $A'_{l_1}$. The overlap in this case consists of the regions composed of the following combinations:

\[ C_1 \; ; \; n^0 = 0 \, , \; n^{0'} = 0 \, , \]  
\[ C_2 \; ; \; n^0 = 4\Omega^0 - N^0 \, , \; n^{0'} = 0 \, , \]  
\[ C_3 \; ; \; n^0 = 0 \, , \; n^{0'} = 2\Omega^0 - \frac{N^0}{3} \, , \]  
\[ C_4 \; ; \; n^0 = 4\Omega^0 - N^0 \, , \; n^{0'} = 2\Omega^0 - \frac{N^0}{3} \, , \]  
\[ C_5 \; ; \; n^0 = 4\Omega^0 - N^0 \, , \; n^{0'} = \frac{(4\Omega^0 - N^0 + 3) + 2\chi(6\Omega^0 - N^0 + 3)}{1 + 6\chi} \, . \]  

(3.4a) \quad (3.4b) \quad (3.4c) \quad (3.4d)

For the combinations $C_1 \sim C_4$, we can prove that $\Delta E (= E_{l_2} - E'_{l_1})$ are positive:

\[ \Delta E > 0 \quad \text{for} \quad C_1 \sim C_4 \, . \]  

(3.6)

For this proof, the following forms are useful:

\[ \Delta E = (1 + 2\chi)(\Omega^0 + 1)X \quad \text{for} \quad C_1 \, , \]  
\[ \Delta E = \frac{1}{4}(3 + 10\chi)X^2 + \frac{1}{2}(1 + 2\chi)(\Omega^0 + 5)X \]  
\[ - \frac{1}{4} \left[(1 + 6\chi)\Omega^0 + 2(1 + 2\chi)\right] \Omega^0 \quad \text{for} \quad C_2 \, , \]  
\[ \Delta E = \frac{1}{36}(5 + 6\chi)X^2 + \frac{1}{2}(1 + 2\chi)(\Omega^0 + 1)X \]  
\[ + \frac{1}{4} \left[(1 + 6\chi)\Omega^0 + 6(1 + 2\chi)\right] \Omega^0 \quad \text{for} \quad C_3 \, , \]  
\[ \Delta E = \frac{8}{9}(1 + 3\chi)X^2 + 2(1 + 2\chi)X \quad \text{for} \quad C_4 \, . \]  

(3.7a) \quad (3.7b) \quad (3.7c) \quad (3.7d)

Here, $X$ denotes

\[ X = N^0 - 3\Omega^0 \, . \]  

(3.8)

For the combination $C_5$, a rather lengthy consideration may be necessary. Our task is to investigate the overlap of the cases ($l_2 \, ; \, 3(i)$) and ($l'_1 \, ; \, 3(ii)$) shown in the relations (2.7c) and (2.9c), respectively. In this case, the overlap range of $\chi$ is, at most,

\[ - \frac{1}{6} < \chi < \frac{2\Omega^0 - 3}{6} \, . \]  

(3.9)

Further, we have the relation

\[ 3\Omega^0 + \frac{9}{2}(1 + 2\chi) \leq N^0 \leq \min \left( \frac{4(1 + 3\chi)\Omega^0}{1 + 2\chi} + 3, \; 4\Omega^0 \right) \, . \]  

(3.10)
Fig. 2. The behavior of $g_B(\chi)$, $g_D(\chi)$ and $g_0(\chi)$ are illustrated. In the area (I) and (II), $\Delta E < 0$. The area (III) is divided into two areas.

Of course, the relation (3.10) gives us

$$3\Omega^0 + \frac{9}{2}(1 + 2\chi) < 4\Omega^0.$$  \hfill (3.11)

The relation (3.11) is rewritten as

$$\Omega^0 > g_B(\chi), \quad g_B(\chi) = \frac{9}{2}(1 + 2\chi), \quad \text{i.e.,} \quad -\frac{1}{6} < \chi < \frac{2\Omega^0 - 9}{18}. \hfill (3.12)$$

The relation (3.10) leads to the following relations:

$$g_B(\chi) < X < \frac{(\Omega^0 + 3) + 6(\Omega^0 + 1)\chi}{1 + 2\chi}, \hfill (3.13)$$

$$g_B(\chi) < X < \Omega^0. \hfill (3.14)$$

The relations (3.13) and (3.14) are valid, respectively in the ranges

$$-\frac{1}{6} < \chi < -\frac{3}{2(2\Omega^0 + 3)}, \quad (3.15)$$

$$-\frac{3}{2(2\Omega^0 + 3)} < \chi < \frac{2\Omega^0 - 9}{18}. \quad (3.16)$$

The energy difference $\Delta E$ is given as

$$\Delta E = \frac{4\chi(1 + 3\chi)}{1 + 6\chi}X^2 + \frac{(1 + 2\chi)(1 + 12\chi)}{1 + 6\chi}X + \frac{6\chi(1 + 2\chi)}{1 + 6\chi}\Omega^0 + \frac{9}{4} \cdot \frac{(1 + 2\chi)^2}{1 + 6\chi} = f(X). \hfill (3.17)$$

Since $X > 0$, all the terms appearing in the function $f(X)$ are positive, if $\chi \geq 0$. Therefore, we have

$\text{if} \quad \chi \geq 0, \quad \Delta E > 0 \quad \text{in the range} \quad X > 0. \hfill (3.18)$
The behavior of $g_1(\chi)$ and $g_2(\chi)$ are illustrated together with $g_B(\chi)$, $g_D(\chi)$ and $g_0(\chi)$. The area (III) is divided into two areas, (III)' and (III)". In the area (III)', $\Delta E < 0$ and in (III)" the regions of $\Delta E < 0$ and $\Delta E > 0$ coexist. In the area (IV), $\Delta E < 0$. In the area (V), $\Delta E < 0$ and in (V)', the regions of $\Delta E < 0$ and $\Delta E > 0$ coexist. Also, in the area (VI) and (VI)', the regions of $\Delta E < 0$ and $\Delta E > 0$ coexist. In the area (VII), $\Delta E > 0$. The area (V)' is too small to show the size definitely.

The above means that if $\chi \geq 0$, the energies in $A_{L_2}$ are larger than those of $A_{L_1}$ for $N_0 > 3\Omega^0$. From the above reason, we will investigate $\chi < 0$ in the cases (3.13) and (3.14), together with the relations (3.15) and (3.16). Of course, the relation (3.16) is changed to $-3/(2(2\Omega_0^0 + 3)) < \chi < 0$. Then, the relations (3.15) and (3.16) are, respectively, rewritten as

$$\Omega^0 > g_0(\chi) \quad \text{and} \quad \Omega^0 < g_0(\chi),$$

and

$$g_0(\chi) = \frac{3}{4} \cdot \frac{1 + 2\chi}{\chi} \quad \left( -\frac{1}{6} < \chi < 0 \right) \quad \text{(3.20)}$$

First, we treat the case (3.13) combined with the relations (3.12) and (3.19). The functions related to this case are $g_B(\chi)$, $g_D(\chi)$ and $g_0(\chi)$. The functions $g_B(\chi)$ and $g_0(\chi)$ are defined in the relations (3.12) and (3.19), respectively. The function $g_D(\chi)$ is defined through the discriminant of the quadratic function $f(X)$ given in the relation (3.17), $D$:

$$D = -\frac{96\chi^2(1 + 2\chi)(1 + 3\chi)}{(1 + 6\chi)^2} \left( \Omega^0 - g_D(\chi) \right). \quad \text{(3.22)}$$

$$g_D(\chi) = \frac{(1 + 2\chi)(1 - 6\chi)^2}{96\chi^2(1 + 3\chi)}. \quad \text{(3.23)}$$

The behaviors of $g_B(\chi)$, $g_0(\chi)$ and $g_D(\chi)$ are illustrated in Fig. 2. Since $g_B(\chi) > 0$ and $g_0(\chi) > 0$, $\Delta E < 0$ and $\Delta E > 0$ are determined by $\Omega^0 > g_D(\chi)$ and $\Omega^0 < g_D(\chi)$,
Y. Tsue, C. Providência, J da Providência and M. Yamamura

respectively. In the areas (I) and (II), $\Delta E < 0$. In (III), $D > 0$, but, in the present frame, $\Delta E > 0$ or $< 0$ cannot be determined.

Next, we treat the case (3.14) combined with the relations (3.12) and (3.20). In addition to $g_B(\chi)$, $g_0(\chi)$ and $g_D(\chi)$, we introduce the functions $g_1(\chi)$ and $g_2(\chi)$ which are defined as follows:

$$f(g_B(\chi)) = \frac{6\chi(1 + 2\chi)}{1 + 6\chi} \left( \Omega^0 - g_1(\chi) \right) = f_1,$$

$$g_1(\chi) = -\frac{9}{8\chi} (1 + 2\chi)^2 (1 + 18\chi),$$

and

$$f(\Omega^0) = \frac{4\chi(1 + 2\chi)}{1 + 6\chi} \left( \Omega^0 - \frac{9(1 + 2\chi)^2}{16\chi(1 + 3\chi)} \cdot \frac{1}{g_2(\chi)} \right) \left( \Omega^0 - g_2(\chi) \right) = f_2.$$  

Here, it should be noted that $(\Omega^0 - 9(1 + 2\chi)^2/(16\chi(1 + 3\chi)) \cdot (1/g_2(\chi)))$ is positive. It may be clear from the definition of $g_i(\chi)$ for $i = 1$ and 2, that if $\Omega^0 < g_i(\chi)$, $f_i > 0$ and vice versa.

Figure 3 shows the behaviors of $g_1(\chi)$ and $g_2(\chi)$, together with $g_B(\chi)$, $g_0(\chi)$ and $g_D(\chi)$ which are shown in Fig.2. With the use of the relation between $g_i(\chi)$ and $f_i$, we obtain the following feature for $\Delta E < 0$ and $\Delta E > 0$: (i) In the area (IV), $\Delta E < 0$. (ii) In the area (V), $\Delta E < 0$ and in (V)', the regions of $\Delta E < 0$ and $\Delta E > 0$ coexist. (iii) In the areas (VI) and (VI)', the regions of $\Delta E < 0$ and $\Delta E > 0$ coexist. (iv) In the area (VII), $\Delta E > 0$. Detail discussion on the coexistence of $\Delta E < 0$ and $\Delta E > 0$ is done in next section.
Finally, we discuss the area (III) in Fig. 2. As is shown in Fig. 3, the area (III) is divided into two areas, (III)’ and (III)”. In the area (III)’, \( \Delta E < 0 \) and in (III)”, \( \Delta E > 0 \) coexist. In next section, we will discuss the meaning of the line \( \chi = -(4 - \sqrt{13})/18 \).

§4. Determination of the ground-state energies

We will continue the discussion in §3. The present model contains two parameters, \( \Omega^0 \) and \( \chi \). The parameter \( \Omega^0 \) denotes the available degeneracy of the single-particle levels and determines the framework of the model. On the other hand, \( \chi \) determines the dynamics caused by the model. Therefore, it may be natural to investigate the dynamics by changing \( \chi \) for a fixed value of \( \Omega^0 \). From Figs. 2 and 3, we can learn that if \( \chi \) changes from \( \chi = -1/6 \) to \( \chi = 0 \) for a given \( \Omega^0 \), there appear various features. The case \( 9/2 < \Omega^0 < \Omega^0_c = (7 + \sqrt{13})/3 + \sqrt{3(28921 + 4784\sqrt{13})}/54 (= 10.42 \cdots) \) may be the simplest, but physically interesting. Other cases are trivial or too complicated. In this paper, we will investigate mainly this case.

Originally, \( \Omega^0 \) is a positive integer and in the present case, \( \Omega^0 \) takes the values as

\[
\Omega^0 = 5, 6, 7, 8, 9 \text{ or } 10. \tag{4.1}
\]

In (A) and (C), we showed various numerical results in the case \( \Omega^0 = 6 \). Therefore, the comparison of the present results with them may be interesting. We can classify the range \(-1/6 < \chi < 0\) into the following cases:

1. \(-1/6 < \chi < \chi_0(\Omega^0)\) in (I), \tag{4.2a}
2. \(\chi_0(\Omega^0) < \chi < \chi_1(\Omega^0)\) in (IV) + (V), \tag{4.2b}
3. \(\chi_1(\Omega^0) < \chi < \chi_2(\Omega^0)\) in (VI), \tag{4.2c}
4. \(\chi_2(\Omega^0) < \chi < 0\) in (VII). \tag{4.2d}

Here, \( \chi_i(\Omega^0) (i = 0, 1, 2) \) denotes the inverse of \( \Omega^0 = g_i(\chi) \). The function \( \chi_0(\Omega^0) \) is simply expressed as

\[
\chi_0(\Omega^0) = -\frac{3}{2(2\Omega^0 + 3)}. \tag{4.3a}
\]

The function \( \chi_1(\Omega^0) \) is obtained in an extremely complicated form:

\[
\chi_1(\Omega^0) = \left[ \sqrt{3} - b + \sqrt{R} + \sqrt{3} - b - \sqrt{R} - \frac{2}{27}Z^0 \right]^{-1},
\]

\[
Z^0 = 4\Omega^0 + 99, \quad R = a^3 + b^2, \quad a = -\frac{4}{729}(Z^{02} - 4617), \quad b = \frac{4}{19683}(2Z^{03} - 13851Z^0 + 177147). \tag{4.3b}
\]

The function \( \chi_2(\Omega^0) \) is expressed in the form

\[
\chi_2(\Omega^0) = -\frac{1}{2} \cdot \frac{4\Omega^0 + 9}{(2\Omega^0 + 1)(2\Omega^0 + 9) + 2\Omega^0 \sqrt{4\Omega^0^2 + 28\Omega^0 + 55}}. \tag{4.3c}
\]
For the case $\Omega^0 = 6$, we have
\[
\chi_0(6) = -0.1, \quad \chi_1(6) = -0.041099, \quad \chi_2(6) = -0.032811. \tag{4.3d}
\]

With the aid of the properties of the quadratic function $f(X)$ (in $-\infty < X < +\infty$), we have the following features:

\(1\)' $\Delta E < 0$ for $g_B(\chi) < X < \frac{(\Omega^0 + 3) + 6(\Omega^0 + 1)\chi}{1 + 2\chi}$, \tag{4.4a}

\(2\)' $\Delta E < 0$ for $g_B(\chi) < X < \Omega^0$, \tag{4.4b}

\(3\)' $\Delta E > 0$ for $g_B(\chi) < X < X_L(\chi; \Omega^0)$, $\Delta E < 0$ for $X_L(\chi; \Omega^0) < X < \Omega^0$, \tag{4.4c}

\(4\)' $\Delta E > 0$ for $g_B(\chi) < X < \Omega^0$. \tag{4.4d}

Of course, $(1)' \sim (4)'$ correspond to $(1) \sim (4)$, respectively. Here, $X_L$ denotes the larger for two solutions of $f(X) = 0$ and it is explicitly given as
\[
X = \frac{-(1 + 2\chi)(1 + 12\chi)}{8\chi(1 + 3\chi)} + \sqrt{\frac{3}{2}} \sqrt{\frac{1 + 2\chi}{1 + 3\chi}} \sqrt{g_D(\chi) - \Omega^0} = X_L(\chi; \Omega^0). \tag{4.5a}
\]

Here, $f(X)$ is given in the relation (3.12). The equation $f(X) = 0$ is also regarded as quadratic for $\chi$ and the larger solution is obtained in the form
\[
\chi = -\frac{1}{2} \cdot \frac{4X + 9}{2(3\Omega^0 + X(2X + 5)) + (4X + 9) + 2\sqrt{(3\Omega^0 + X(2X + 5))^2 - X^2(4X + 9)}}
= \chi_L(X; \Omega^0). \tag{4.5b}
\]

The first term in the relation (4.5a) denotes the value of $X$ which makes $f(X)$ maximum and it is smaller than $g_B(\chi)$ in the case
\[
\chi < -\frac{4 - \sqrt{13}}{18}. \tag{4.6}
\]

Therefore, the smaller solution loses its meaning.

Until the present, we have investigated the effect coming from the area $A_{l_2}$ in the range $3\Omega^0 \leq N^0 \leq 4\Omega^0$. By replacing $N$ with $(6\Omega - N)$, the above effect can be included in the range $2\Omega^0 \leq N^0 \leq 3\Omega^0$. Including other areas, we can arrange the results in the ground-states as follows:

\(1\)
\[
-\frac{1}{2} < \chi < -\frac{1}{6} \cdot \frac{\Omega^0 + 6}{\Omega^0 + 2}, \quad (i) \quad 0 \leq N^0 \leq 3\Omega^0, \quad n^0 = 0, \tag{4.7a}
\]

\(2\)
\[
-\frac{1}{6} \cdot \frac{\Omega^0 + 6}{\Omega^0 + 2} < \chi < -\frac{1}{6}, \quad (i) \quad 0 \leq N^0 < \frac{6(2\Omega^0 - 3 - 6\chi)}{5 + 6\chi}, \quad n^0 = 0, \tag{4.7b}
\]
(ii) $\frac{6(2\Omega^0 - 4 - 6\chi)}{5 + 6\chi} < N^0 \leq 3\Omega^0$, \quad n^0 = \frac{N^0}{3}$, (4.7b)

(3) $-\frac{1}{6} < \chi < \chi_0(\Omega^0)$,

(i) $0 \leq N^0 < \frac{2\Omega^0}{1 + 2\chi} - 3$, \quad n^0 = 0,

(ii) $\frac{2\Omega^0}{1 + 2\chi} - 3 < N^0 < 3\Omega^0 - \frac{9}{2}(1 + 2\chi)$, \quad n^0 = n^{0\dagger},

(iii) $3\Omega^0 - \frac{9}{2}(1 + 2\chi) < N^0 \leq 3\Omega^0$, \quad n^0 = \frac{N^0}{3}$, (4.7c)

(4) $\chi_0(\Omega^0) < \chi < \chi_1(\Omega^0)$,

(i) $0 \leq N^0 < \frac{2\Omega^0}{1 + 2\chi} - 3$, \quad n^0 = 0,

(ii) $\frac{2\Omega^0}{1 + 2\chi} - 3 < N^0 < 2\Omega^0$, \quad n^0 = n^{0\ast},

(iii) $2\Omega^0 < N^0 < 3\Omega^0 - \frac{9}{2}(1 + 2\chi)$, \quad n^0 = n^{0\dagger},

(iv) $3\Omega^0 - \frac{9}{2}(1 + 2\chi) < N^0 \leq 3\Omega^0$, \quad n^0 = \frac{N^0}{3}$, (4.7d)

(5) $\chi_1(\Omega^0) < \chi < \chi_2(\Omega^0)$,

(i) $0 \leq N^0 < \frac{2\Omega^0}{1 + 2\chi} - 3$, \quad n^0 = 0,

(ii) $\frac{2\Omega^0}{1 + 2\chi} - 3 < N^0 < 2\Omega^0$, \quad n^0 = n^{0\ast},

(iii) $2\Omega^0 < N^0 < 3\Omega^0 - X_L(\chi; \Omega^0)$, \quad n^0 = n^{0\dagger},

(iv) $3\Omega^0 - X_L(\chi; \Omega^0) < N^0 < 3\Omega^0 - \frac{9}{2}(1 + 2\chi)$, \quad n^0 = n^{0\ast},

(v) $3\Omega^0 - \frac{9}{2}(1 + 2\chi) < N^0 \leq 3\Omega^0$, \quad n^0 = \frac{N^0}{3}$, (4.7e)

(6) $\chi_2(\Omega^0) < \chi < \frac{2\Omega^0 - 3}{6}$,

(i) $0 \leq N^0 < \frac{2\Omega^0}{1 + 2\chi} - 3$, \quad n^0 = 0,

(ii) $\frac{2\Omega^0}{1 + 2\chi} - 3 < N^0 < 3\Omega^0 - \frac{9}{2}(1 + 2\chi)$, \quad n^0 = n^{0\ast},

(iii) $3\Omega^0 - \frac{9}{2}(1 + 2\chi) < N^0 \leq 3\Omega^0$, \quad n^0 = \frac{N^0}{3}$, (4.7f)

(7) $\frac{2\Omega^0 - 3}{6} < \chi < +\infty$,

(i) $0 \leq N^0 \leq 3\Omega^0$, \quad n^0 = \frac{N^0}{3}$, (4.7g)
Here, \( n^0\) and \( n^0*\) denote

\[
n^0 = N^0 - 2\Omega^0, \quad n^0* = \frac{(N^0 - 2\Omega^0 + 3) + 2\chi(N^0 + 3)}{1 + 6\chi}.
\]  

(4.8)

If \( n^0\) is replaced with \( n^0*\), the result \( (4.7) \) reduces to the result \( (C.4.1) \). With the use of the expression \((2.2a)\), the ground-state energies are calculated except the case \( n^0 = n^0\), in which the energies are calculated by the relation \((2.2a)\) under the replacement of \( N^0 \) with \( (6\Omega^0 - N^0) \).

§5. Numerical analysis

In this section, we give numerical results in the region \( A_{11} \). The quantity \( n^0 \), which is regarded as an order parameter of a phase transition between the quark-triplets and the quark-pairs, is given in Eq. \((4.7)\) in the various areas. The ground-state energies are also calculated numerically by using Eq. \((2.2a)\). In this section, we fix the parameter \( \Omega^0 = 6 \). Thus, we show the behaviors of the ground-state energy and the order parameter in the region \( 0 \leq N^0 \leq 3\Omega^0 \) (= 18) with various force

![Fig. 5](image1.png)  
**Fig. 5.** The behaviors of energy (left panel) and the order parameter \( n^0 \) (right panel) are shown as functions of \( N^0 \) in the case (1) in Eq.\((4.7a)\) with \( \chi = -1/3 \). The model parameter \( \Omega^0 \) is taken as 6.

![Fig. 6](image2.png)  
**Fig. 6.** The behaviors of energy (left panel) and the order parameter \( n^0 \) (right panel) are shown as functions of \( N^0 \) in the case (2) in Eq.\((4.7b)\) with \( \chi = -1/5 \). The model parameter \( \Omega^0 \) is taken as 6.
Fig. 7. The behaviors of energy (left panel) and the order parameter \( n^0 \) (right panel) are shown as functions of \( N^0 \) in the case (3) in Eq. (4.7c) with \( \chi = -1/8 \). The model parameter \( \Omega^0 \) is taken as 6.

Fig. 8. The behaviors of energy (left panel) and the order parameter \( n^0 \) (right panel) are shown as functions of \( N^0 \) in the case (4) in Eq. (4.7d) with \( \chi = -1/16 \). The model parameter \( \Omega^0 \) is taken as 6.

strength of the particle-hole interaction, \( \chi \).

In Fig. 7, the ground-state energy on the left panel and the order parameter on the right panel are, respectively, shown as a function of \( N^0 \) with \( \chi = -1/3 \), namely the results are based on Eq. (4.7a). In this case, the order parameter \( n^0 \) is identical to 0. Thus, the quark-pair state is realized. Almost the same behavior is seen in Fig. 2 in (A).

In Fig. 8, the ground-state energy and the order parameter are shown in the case \( \chi = -1/5 \), namely the results are based on Eq. (4.7b). In this case, the order parameter \( n^0 \) changes from 0 to \( N^0/3 \). Thus, as the particle number \( N^0 \) increases, the quark-pair state changes to the quark-triplet state directly. This behavior is seen in Fig. 3 in (A).

However, in Fig. 7, another behavior of the order parameter \( n^0 \) is seen. The ground-state energy and the order parameter are shown in the case \( \chi = -1/8 \) based on Eq. (4.7c). In this case, the order parameter \( n^0 \) changes from 0 to \( N^0/3 \). However, in the region \( 13 < N^0 < 14.625 \) (if \( N^0 \) is integer, \( 13 < N^0 \leq 14 \)), the transition region from quark-pair state to color-singlet quark-triplet state is open with \( n^0 = n^0_{\Omega} \). The change of order parameter is not continuous, while the order parameter is changed.
Fig. 9. The behaviors of energy (left panel) and the order parameter $n^0$ (right panel) are shown as functions of $N^0$ in the case (5) in Eq. (4.7e) with $\chi = -1/27$. The model parameter $\Omega^0$ is taken as 6.

Fig. 10. The behaviors of energy (left panel) and the order parameter $n^0$ (right panel) are shown as functions of $N^0$ in the case (5) in Eq. (4.7f) with $\chi = 1$. The model parameter $\Omega^0$ is taken as 6.

continuously shown in Fig.4 in (A). For the point between $n^0 = 0$ or $n^0 = N^0/3$ and $n^0 = n^0\dagger$, the ground-state energy is not continuously changed with respect to the change of $N^0$. The dotted curve in the left panel represents the ground state energy with $n^0 = n^0\dagger$. However, as the particle number $N^0$ increases, the quark-pair state changes to the quark-triplet state through the transition region.

In Fig.8 the ground-state energy and the order parameter are shown in the case $\chi = -1/16$ based on Eq. (4.7d). In this case, the order parameter $n^0$ changes from 0 to $N^0/3$ through $n^0 = n^0\ast$ and $n^0 = n^0\dagger$. For the point between $n^0 = n^0\ast$ or $n^0 = N^0/3$ and $n^0 = n^0\dagger$, the ground-state energy is not continuously changed with respect to the change of $N^0$. The dotted curve in the left panel represents the ground state energy with $n^0 = n^0\dagger$. In this case, also, as the particle number $N^0$ increases, the quark-pair state changes to the quark-triplet state through two transition regions characterized by $n^0\ast$ and $n^0\dagger$.

Also, in Fig.9 in the case $\chi = -1/27$ based on Eq. (4.7e), the order parameter $n^0$ changes from 0 to $N^0/3$ through $n^0 = n^0\ast$, $n^0 = n^0\dagger$ and $n^0 = n^0\dagger$. For the point between $n^0 = n^0\ast$ and $n^0 = n^0\dagger$, the ground-state energy is not continuously changed with respect to the change of $N^0$ similar to the case $\chi = -1/8$. Here, the
§6. Concluding remarks

In this paper, the ground-state energies were calculated in the modified Bonn quark model. The color-singlet condition was imposed, which was developed in the previous paper. It was indicated that the modified Bonn quark model has a meaning in the regions $\chi > -1/2$, where $\chi$ represents the strength of the particle-hole-type interaction. The ground-state energies were determined in each area of $\chi$ and the ground states were also determined. Namely, in a certain parameter region, the color-singlet quark-triplet state is the ground state, and in another region, the quark-color-pairing state is the ground state. Further, it was shown that there were two transition regions between the quark-pairing and the quark-triplet state, which were distinguished by the value of the order parameter.

Finally, we give a comment on the phase transition induced by the present model. For this aim, we show the phase diagram obtained by the relation (4.7). It is drawn for the case $\Omega^0 = 6$ in Fig.12. Since $n^0$ plays a role of the order parameter, the present model induces four phases and the areas specified by $n^0 = N^0/3$ and $n^0 = 0$ are in the
The phase diagram is shown in the $\chi-N^0$ plane (left panel). The order parameter is regarded as $n^0$. In the right panel, the details are shown in the region of $-0.17 < \chi < 0$. Here, $\chi_i (i = 0, 1, 2)$ represent $\chi_i (\Omega^0)$, respectively.

quark-triplet and the quark-pair phase, respectively. In the original Bonn model, the case $\chi = 0$ was adopted. The region $N^0 \sim 0$ or $6\Omega^0$ and the region $N^0 \sim 3\Omega^0$ are in the quark-pair and the quark-triplet phase, respectively. Further, we must note that in the former and the latter regions the quark system has a low and a high density, respectively. We observe, however, that the situation predicted by the present model is in contradiction with the common understanding, according to which color-superconductivity is expected to provide a reliable description of quark matter at high densities. In the case where $\chi$ is different from $\chi = 0$, the situation does not change, if we treat $\chi$ as a constant in the whole region $0 \leq N^0 \leq 6\Omega^0$. Although the original and the modified Bonn quark model present us various features of many-quark system, it may be not permitted for these models to avoid the investigation of the above-mentioned problem. In (III), we will discuss this problem.

References

1) K. Bleuler, H. Hofestädt, S. Merk and H. R. Petry, Z. Naturforsch. 38a (1983) 705.
2) Y. Tsue, C. Providência, J. da Providência and M. Yamamura, Prog. Theor. Phys. 121 (2009), 1237.
3) Y. Tsue, C. Providência, J. da Providência and M. Yamamura, Prog. Theor. Phys. 122 (2009), 693: Errata, ibid 122 (2009), 1065.
4) Y. Tsue, C. Providência, J. da Providência and M. Yamamura, Prog. Theor. Phys. 122 (2009), 911.
5) Y. Tsue, C. Providência, J. da Providência and M. Yamamura, to appear in Prog. Theor. Phys. 126 (2011).
6) M. G. Alford, A. Schmitt, K. Rajagopal and T. Schäfer, Rev. Mod. Phys. 80 (2008), 1455, and references cited therein.

7) Y. Tsue, C. Providência, J. da Providência and M. Yamamura, submitted to Prog. Theor. Phys.