CLOSED COHEN-MACaulAY COMPLETION OF BINOMIAL EDGE IDEALS

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ABSTRACT. Let CCM denote the class of closed graphs with Cohen-Macaulay binomial edge ideals and PIG denote the class of proper interval graphs. Then CCM ⊆ PIG. The PIG-completion problem is a classical problem in molecular biology as well as in graph theory and this problem is known to be NP-hard. In this paper, we study the CCM-completion problem. We give a method to construct all possible CCM-completion of a graph. We find the CCM-completion number and the set of all minimal CCM-completions for a large class of graphs. Moreover, for that class, we give a polynomial-time algorithm to compute the CCM-completion number and a minimum CCM-completion of a given graph. We investigate unmixed and Cohen-Macaulay properties of binomial edge ideals of induced subgraphs. Also, we discuss the accessible graphs completion and the Cohen-Macaulay property of binomial edge ideals of whisker graphs.

1. INTRODUCTION

Two interesting connections between simple graphs and polynomial ideals are through edge ideals (see [31]) and binomial edge ideals (see [10], [24]). Cohen-Macaulay graph ideals play a very important role in the field of combinatorial commutative algebra. Cohen-Macaulay edge ideals from arbitrary graphs can be constructed by adding some whiskers to the graph; see [30]. This has been generalized for d-uniform clutters through grafting; see [8]. Some constructions of Cohen-Macaulay binomial edge ideals have been studied in [27], using gluing of graphs and cone on graphs. Several classifications of Cohen-Macaulay and the unmixed property of binomial edge ideals have been investigated in [1], [2], [3], [7], [16], [20], [23], [28], [29]. In the field of binomial edge ideals, the study of complexity of a problem is an area which is mostly untouched. In this paper, we study the complexity of constructions of closed Cohen-Macaulay binomial edge ideals.

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Let $\Pi$ be a class of graphs. A $\Pi$-completion of a graph $G$ is a graph $H$ such that $V(H) = V(G)$, $E(G) \subseteq E(H)$, and $H \in \Pi$. The edges in $E(H) \setminus E(G)$ are called fill edges. We write $|E(H) \setminus E(G)| = \Pi_H(G)$. A $\Pi$-completion $[G]$ of $G$ is said to be minimal if there exists no $\Pi$-completion $H$ of $G$ such that $E(G) \subseteq E(H) \subseteq E([G])$. A minimum $\Pi$-completion of $G$ is a $\Pi$-completion $[G]$ of $G$ such that for any $\Pi$-completion $H$ of $G$, we have $\Pi_H(G) \geq \Pi_{[G]}(G)$. In this case, $\Pi_{[G]}(G)$ is called the $\Pi$-completion number of $G$, and we denote it by $\Pi(G)$, i.e., the minimum number of edges required to add in $G$ to get a $\Pi$-completion of $G$ is $\Pi(G)$. The $\Pi$-completion problem is to find $\Pi(G)$ of a given graph $G$. There are various types of completion problems in mathematics which have been tackled through graph theory, for example, the matrix completion problem ([13]) and the network completion problem ([17]). Graph completion problems like CHORDAL-completion, STRONGLY CHORDAL-completion, INTERVAL GRAPH-completion, have a rich history and motivations to study (see [14], [26]). These problems have several practical applications in the fields like graph modelling (missing edges correspond to lack of data), molecular biology and computational algebra.

The study of binomial edge ideals started through a particular class of graphs, known as closed graphs. Later it was proved that a graph is closed if and only if it is a proper interval graph or PIG (see [11]). Among various graph completion problems, the study of PIG-completion problem has become popular because of its presence in molecular biology, in particular, the structural study of DNA (see [9] for details). Some work have been done in this direction in terms of the complexity of the problem (see [6], [14], [26]). Since PIG-completion problem is known to be NP-hard ([9]), it is common to study the complexity of $\Pi$-completion problem for some subclasses $\Pi$ of PIG. Let us denote the class of closed graphs $G$ with Cohen-Macaulay $J_G$ by CCM, where $J_G$ denotes the binomial edge ideal of $G$; clearly CCM $\subseteq$ PIG. This paper is devoted to the study of complexity of the CCM-completion problem.

Our primary aim in this paper is to give a method to construct a graph $[G]$, which is a CCM-completion of a given graph $G$, and study the complexity of this construction. In this construction, we choose an arbitrary labelling of the graph and keep adding suitable edges only. For an arbitrary graph, our construction can compute all possible CCM-completion of that graph by taking all possible labelling. But, for a graph $G$ on $[n] = \{1, \ldots, n\}$, this method would take $n!$ iteration to compute CCM$(G)$. The question that emerges from our construction is whether we can find CCM$(G)$, and a minimum CCM-completion $[G]$ of a given graph $G$ in polynomial time.
or not. Finally, for a large class of graphs, we give a polynomial-time algorithm to compute a minimum CCM-completion and CCM-completion number. We also study some unmixed and Cohen-Macaulay properties of binomial edge ideals in terms of subgraphs. The paper is arranged in the following manner.

We first discuss some preliminaries in Section 2 which are required for the subsequent discussions. The rest of the paper is divided into two parts. The first part (Sections 3) is devoted to the CCM-completion problem, and the second part (Section 4) is devoted to the study of unmixed and Cohen-Macaulay properties of binomial edge ideals of induced subgraphs. To be precise, in Section 3, we give a method to construct a graph \([G]\), which is a CCM-completion of an arbitrary given graph \(G\). The first step in the construction is to make the graph \(G\) closed. Then we construct the graph \([G]\) by adding some special edges to \(G\). By this process, we can find all possible CCM-completion of a given graph by taking all possible labelling of the vertices. In Proposition 3.10 we show that the block graph of a connected closed graph is a path, and in Proposition 3.11 we prove that an indecomposable closed graph has no cut vertex. We give the explicit structure of closed graphs with Cohen-Macaulay binomial edge ideals (see Theorem 3.12), which help us identify CCM graphs without knowing the labelling of the vertices. In Remark 3.14 we discuss some properties of \(J_G\) for \(G \in \text{CCM}\). Since the block graph of a graph with unmixed binomial edge ideal is a tree (by [28, Proposition 1.3]), it is worth considering the class of graphs whose block graphs are trees, and we denote this class by \(\text{BT}\). In Theorem 3.15 for a graph \(G \in \text{BT}\), we find the CCM-completion number \(\text{CCM}(G)\), and the set of all minimal CCM-completions of \(G\). We write an algorithm (Algorithm 1) to compute the CCM-completion number \(\text{CCM}(G)\) and a minimum CCM-completion \([G]\) of a given graph \(G \in \text{BT}\). We study the time complexity of the algorithm, and our algorithm turns out to be a polynomial-time algorithm. However, in general, it remains to check whether the CCM-completion problem is NP-hard or not. So, we end this section by Question 3.18. In Section 4 we try to give some conditions for unmixed (respectively, Cohen-Macaulay) property of binomial edge ideals of subgraphs of those graphs whose binomial edge ideal is unmixed (respectively, Cohen-Macaulay). In Theorem 4.6 we show that for any closed graph \(G\) if \(J_G\) is Cohen-Macaulay, then \(J_H\) is Cohen-Macaulay for any subgraph \(H\) of \(G\). We prove that any graph with unmixed binomial edge ideal has an accessible completion (Proposition 4.7), and propose the Question 4.9. We end up by giving the sufficient and necessary conditions (Theorem 4.10) for the Cohen-Macaulay property of binomial edge ideals of whisker graphs.
2. Preliminaries

All graphs are assumed to be simple. For a graph $G$, $V(G)$ denotes the vertex set of $G$, and $E(G)$ denotes the edge set of $G$. This paper uses the term subgraph to mean an induced subgraph of a graph.

Let $V = \{x_1, \ldots, x_n\}$. A simplicial complex $\Delta$ on the vertex set $V$ is a collection of subsets of $V$, with the following properties:

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$;
(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a face of $\Delta$. A maximal face of $\Delta$ is called a facet of $\Delta$. A vertex $i$ of $\Delta$ is called a free vertex of $\Delta$ if $i$ belongs to exactly one facet.

**Definition 2.1.** A graph is said to be complete if there is an edge between every pair of two vertices. Complete graph on $n$ vertices is denoted by $K_n$. A clique of a given graph is a subset of vertex set on which the induced subgraph is complete.

**Definition 2.2.** The clique complex $\Delta(G)$ of a graph $G$ is the simplicial complex whose faces are the cliques of $G$. Hence, a vertex $v$ of a graph $G$ is called free vertex if it belongs to only one maximal clique of $\Delta(G)$.

**Definition 2.3.** A path graph $P$ on $n$ vertices is a graph whose vertices can be ordered as $v_1, \ldots, v_n$ such that $E(P) = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$.

**Definition 2.4.** A path in a graph $G$ is a sequence of vertices $v_0, v_1, \ldots, v_s$ such that $\{v_i, v_{i+1}\} \in E(G)$ for $i = 0, \ldots, s - 1$. If $G$ is labelled with $V(G) = [n]$, then a path $i_0, \ldots, i_s$ is said to be directed if either $i_k < i_{k+1}$ for all $k$ or $i_k > i_{k+1}$ for all $k$.

Let $G$ be a graph and $v$ be a vertex of $G$. The neighbor set of $v$ in $G$, denoted by $N_G(v)$, is $N_G(v) = \{u \in V(G) \mid \{u, v\} \in E(G)\}$. We write $N_G[v]$ to denote the set $N_G(v) \cup \{v\}$. Note that for a free vertex $v$, $N_G[v] = F$, where $F$ is the facet of $\Delta(G)$ with $v \in F$. A vertex $v$ is said to be a cut vertex or cut point of $G$, if $G \setminus \{v\}$ has more number of connected components than $G$.

**Definition 2.5.** Let $G$ be a graph on the vertex set $V(G) = [n]$ and $K$ be a field. The binomial edge ideal $J_G \subseteq R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of $G$ is the ideal generated by the binomials $f_{ij} = x_i y_j - x_j y_i$, such that $i < j$ and $\{i, j\} \in E(G)$, where $E(G)$ denotes the edge set of $G$.

Let us first recall some notations and results from [11] and [27].

**Theorem 2.6 ([11], Theorem 7.2]).** Let $G$ be a graph on $[n]$, and let $<$ be the lexicographic order on $R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$, induced by
$x_1 > x_2 > \cdots > x_n > y_1 > y_2 > \cdots > y_n$. The following conditions are equivalent:

(i) The generators $f_{ij}$ of $J_G$ form a quadratic Gröbner basis;

(ii) For all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$, one has $\{j, l\} \in E(G)$ if $i = k$, and $\{i, k\} \in E(G)$ if $j = l$.

A graph $G$ is said to be closed with respect to a given labelling of vertices if it satisfies any of the equivalent conditions of Theorem 2.6. By a closed graph $G$, we mean a graph $G$ which is closed with respect to a suitable labelling of its vertices.

**Proposition 2.7.** ([11]; Proposition 7.24) Let $G$ be a connected graph on $[n]$, which is closed with respect to the given labelling. The following conditions are equivalent:

(i) $J_G$ is unmixed;

(ii) $J_G$ is Cohen-Macaulay;

(iii) $G$ satisfies the condition that whenever $\{i, j+1\}$, with $i < j$ and $\{j, k+1\}$, with $j < k$, are edges of $G$, then $\{i, k+1\}$ is an edge of $G$.

Let $T \subseteq [n]$, and $\overline{T} = [n] \setminus T$. Let $G[T]$ denotes the induced subgraph of $G$ on $T$. By $G \setminus T$, we denote the induced subgraph $G[\overline{T}]$. Let $c_G(T)$ (sometime we write $c(T)$ if the graph is clear from the context) denotes the number of connected components of the induced subgraph on $\overline{T}$, namely $G[\overline{T}]$ or $G \setminus T$. If each $i \in T$ is a cut point of the graph $G \setminus (T \setminus \{i\})$, then we say that $T$ has cut point property for $G$ or $T$ is a cutset of $G$. We denote by $C(G)$ the set of all cutsets of $G$.

**Proposition 2.8** ([27, Proposition 2.1]). Let $G$ be a graph, $\Delta(G)$ its clique complex and $v \in V(G)$. Then the following conditions are equivalent:

(i) There exists $T \in C(G)$, such that $v \in T$.

(ii) $v$ is not a free vertex of $\Delta(G)$.

**Lemma 2.9** ([27, Lemma 2.2]). Let $G$ be a graph, $v \in V(G)$, such that $v$ is a free vertex in $\Delta(G)$. Let $F$ be the facet of $\Delta(G)$, with $v \in F$, and $T \subseteq V(G)$ with $F \setminus \{v\} \not\subseteq T$. The following conditions are equivalent:

(i) $T \in C(G)$.

(ii) $v \not\in T$ and $T \in C(G \setminus \{v\})$.

**Lemma 2.10** ([27, Lemma 2.5]). Let $G$ be a connected graph. The following conditions are equivalent:

(i) $J_G$ is unmixed.

(ii) For all $T \in C(G)$, we have $c(T) = |T| + 1$.

**Definition 2.11** ([3, Definition 2.2]). Let $G$ be a simple graph. A cutset $T \in C(G)$ is said to be accessible if there exists $v \in T$ such that $T \setminus \{v\}$
The graph $G$ is called accessible if $J_G$ is unmixed and $T$ is accessible for all $T \in \mathcal{C}(G)$.

**Definition 2.12.** Let $B_1, \ldots, B_r$ be the blocks of a graph $G$. The block graph of $G$, denoted by $B(G)$, is the graph defined as follows:

- $V(B(G)) = \{B_1, \ldots, B_r\}$.
- $E(B(G)) = \{\{B_i, B_j\} \mid V(B_i) \cap V(B_j) \neq \emptyset\}$.

By [28, Proposition 1.3], the block graph of a connected graph with unmixed binomial edge ideal is a tree.

For a graph $G$, the Cohen-Macaulay (resp. unmixed) property of $J_G$ is equivalent to the Cohen-Macaulay (resp. unmixed) property of the binomial edge ideals of its connected components. Therefore, we will assume all given graphs are connected unless otherwise stated.

### 3. CCM-completion Problem with Algorithm

In this Section, from an arbitrary graph $G$ we construct a new graph $[G]$, by adding some edges to $G$, such that $[G]$ is a CCM-completion of $G$. Also, we give an algorithm to find the CCM-completion number for a large class of graphs and discuss its complexity.

**Definition 3.1.** A graph $G$ is called an interval graph if for all $v \in V(G)$, there exists an interval $I_v = [l_v, r_v]$ of the real line such that $I_v \cap I_w \neq \emptyset$ if and only if $\{v, w\} \in E(G)$. If, in addition, the intervals can be chosen such that there is no proper containment among them, then $G$ is called a proper interval graph or simply a PI graph.

Let $G$ be a graph. A set of intervals $\{I_v\}_{v \in V(G)}$, as in Definition 3.1, is called an interval representation of $G$. Let $G$ be a graph on the vertex set $[n]$. Then $G$ satisfies the proper interval ordering with respect to the given labelling, if for all $i < j < k$, with $\{i, k\} \in E(G)$, it follows that $\{i, j\}, \{j, k\} \in E(G)$. We say that $G$ admits a proper interval ordering if $G$ satisfies the proper interval ordering for a suitable relabelling of its vertices.

**Theorem 3.2 ([21, Theorem 1]).** A graph $G$ is a proper interval graph if and only if $G$ has a proper interval ordering.

**Theorem 3.3 ([11, Theorem 7.9]).** A graph $G$ is a closed graph if and only if $G$ is a proper interval graph.

**Remark 3.4.** From Theorem 3.2 and Theorem 3.3, we can say that a graph $G$ is a closed graph if and only if $G$ has a proper interval ordering.
Corollary 3.5. Suppose $G$ is a closed graph with respect to a suitable labelling of vertices. Let $G'$ be the graph obtained by adding an edge $\{i, k\}$, with $i < k$ to $G$. Then $G'$ is closed if and only if $\{i, j\}, \{j, k\} \in E(G')$, for all $j$ with $i < j < k$.

Proof. We have $E(G') = E(G) \cup \{i, k\}$. Since $G$ satisfies proper interval ordering (as $G$ is closed), for $\{l, n\} \in E(G)$ we have $\{l, m\}, \{m, n\} \in E(G) \subseteq E(G')$, for all $l < m < n$. For $\{i, k\} \in E(G')$ it follows from the given condition that $\{i, j\}, \{j, k\} \in E(G')$, for all $j$ with $i < j < k$. So, $G'$ satisfies the proper interval ordering and hence $G'$ is closed.

Remark 3.6. (Construction of $[G]$ from $G$) Start with an arbitrary connected graph $G$. If it is not closed then take any labelling of vertices and for all edges $\{i, j\}, \{k, l\}$ in $G$ with $i < j$, $k < l$, add $\{j, l\}$ to $E(G)$ if $i = k$, and add $\{i, k\}$ to $E(G)$ if $j = l$ and repeat the process with the newly obtained graph. Then after a finite number of steps, we get a new graph which is closed. Let us therefore assume that $G$ is a connected graph on $[n]$, which is closed with respect to a given labelling. We now construct a Cohen-Macaulay binomial edge ideal by adding some edges to $G$. If $\{i, j + 1\}$, with $i < j$, and $\{j, k + 1\}$, with $j < k$ are edges of $G$, then we add the edge $\{i, k + 1\}$ to $G$ (if it is not there) and call the new graph $G'$. We repeat the process for $G'$. Since a finite graph has a finite number of edges, after finite number of steps, we would get a new graph $[G]$ on $[n]$ such that if $\{i, j + 1\}$ with $i < j$, and $\{j, k + 1\}$ with $j < k$, are edges of $[G]$, then $\{i, k + 1\} \in E([G])$. Hence, if one can show that $[G]$ is closed, then by Proposition 2.7 it follows that $J_{[G]}$ is Cohen-Macaulay and $J_G \subseteq J_{[G]} \subseteq K[x_1, \ldots, x_n, y_1, \ldots, y_n]$. Therefore, starting with an arbitrary connected graph, we can construct a new graph whose corresponding binomial edge ideal is Cohen-Macaulay by adding some special edges. Note that the graph we obtain finally may not be unique; it depends on our choice of labelling the vertices to make it closed. Therefore, one has to label the vertices in such a way that one requires the least number of edges to make it closed.

Proposition 3.7. The graph $G'$ constructed from $G$ by adding suitable edges (the procedure mentioned above) is closed. Hence, $[G]$ is closed.

Proof. We prove that at the end of each step of adding all the necessary edges, the graph remains closed. Let us denote the graph by $G'$, after the first step. To prove $G'$ is closed, by Corollary 3.5 it is enough to show that for each newly added edge $\{i, k + 1\}$ in $G'$, the edges $\{i, m\}$ and $\{m, k + 1\}$ belong to $G'$ for all $i < m < k + 1$. Suppose that for the edges $\{i, j + 1\}$, with $i < j$ and $\{j, k + 1\}$, with $j < k$, we add a new edge $\{i, k + 1\}$ to $G$. Now $\{i, j + 1\} \in E(G)$ implies that $\{i, i + 1\}, \{i, i + 2\}, \ldots, \{i, j\} \in [G]$. Therefore, $\{i, k + 1\}$ is closed, and $\{i, j + 1\}$ is closed, by Corollary 3.5, it is enough to show that $\{i, j + 1\}$ is closed, for all $i < j < k$. So, $G'$ satisfies the proper interval ordering and hence $G'$ is closed.
the graph $G$ is not closed with respect to any labeling. In our choice of labelling, it is not closed because of $\{1, 2\}, \{1, 3\} \in E(G)$ but $\{2, 3\} \not\in E(G)$. So add $\{2, 3\}$ to make it closed and call the new graph $G'$.\]
Now $J_{G'}$ is not Cohen-Macaulay as $\{1, 3\}, \{2, 4\} \in E(G)$ but $\{1, 4\} \not\in E(G)$. Add $\{1, 4\}$ and call the new graph $[G]$.

$[G]$ is closed and $J_{[G]}$ is Cohen-Macaulay by Proposition 2.7.

The construction described above depends on the labelling of the vertices, and to find the CCM-completion number $\text{CCM}(G)$ of a graph $G$, we have to check $n!$ cases, which cannot be executed in polynomial time. Next, we give the explicit structure of a closed Cohen-Macaulay binomial edge ideals in Theorem 3.12 which helps us identify the closed Cohen-Macaulay binomial edge ideals irrespective of labelling. Let $\text{BT}$ denotes the class of those graphs whose block graphs are trees, i.e., $\text{BT} = \{G \mid \mathcal{B}(G) \text{ is a tree}\}$. For any graph $G \in \text{BT}$, we give an algorithm (Algorithm 1) to find the CCM-completion number $\text{CCM}(G)$ in polynomial time.

**Proposition 3.10.** The block graph $\mathcal{B}(G)$ of a connected closed graph $G$ is a path graph.

**Proof.** Let $B_l, B_m, B_n$ be three blocks of $G$ such that $B_l \cap B_m \cap B_n = \{v\}$. There exist $u_t \in V(B_t)$, such that $\{v, u_t\} \in E(G)$ for $t \in \{l, m, n\}$. Note that $G[T]$ is an induced claw for $T = \{v, u_l, u_m, u_n\}$ and by [11, Proposition 7.4], we get a contradiction as $G$ is closed. Therefore, no three blocks of $G$ can have a common vertex. Let $B$ be a block of $G$ containing more than two cut vertices of $G$. Consider a labelling of $G$ with respect to which $G$ is closed and by Remark 3.4 $G$ admits a proper interval ordering with respect to this labelling. Let $i < j < k$ be three cut vertices of $G$

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**Diagram:**

- Two graphs $G$ and $G'$ with vertices labeled from 1 to 7.
- Graph $G$ has edges between vertices 1-4, 4-6, 1-3, 3-5, 5-6, and 2-7.
- Graph $G'$ has edges between vertices 1-4, 4-6, 1-3, 3-5, 5-6, 2-7, 1-2, and 5-7.
- $G'$ is the result of adding the edge $\{1, 4\}$ to $G$.

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chosen arbitrarily and belonging to $V(B)$. Let $B_i, B_j, B_k$ be three blocks other than $B$ containing $i, j, k$, respectively. By [5, Proposition 2.1], there exists a directed path from $i$ to $j$. Since $i < j$, there exists a vertex $j' \in N_B(j)$ such that $j' < j$. By proper interval ordering of $G$, we have $\{j - 1, j\} \in E(B)$ and $\{j, j + 1\} \in E(B_j)$. Therefore, no vertex of $N_B(j)$ can be greater than $j$, otherwise $j + 1$ will be adjacent to a vertex in $V(B) \setminus \{j\}$, which is not possible. Thus, there can not be any directed path from $j$ to $k$, as $j < k$, and this gives a contradiction due to [5, Proposition 2.1]. So, every block of $G$ contains at most two cut vertices of $G$. Moreover, $G$ being connected, there are exactly two blocks of $G$ each of which contains only one cut vertex of $G$. Hence the block graph $B(G)$ of $G$ is a path graph. □

Let $G$ be a closed graph. Then by Proposition 3.10, $G$ can be written as

$$G = B_1 \cup \cdots \cup B_r,$$

where $B_1, \ldots, B_r$ are blocks of $G$ with the following property: For $i < j$ and $i \in \{1, \ldots, r - 1\}$, we have $V(B_i) \cap V(B_j) \neq \emptyset$ if and only if $j = i + 1$.

**Proposition 3.11.** An indecomposable closed graph has no cut vertex.

**Proof.** Let $G$ be a closed graph with a cut vertex $v$. We show that $G$ is decomposable. By Proposition 3.10, $v$ belongs to exactly two blocks of $G$, say $B_1$ and $B_2$. If $N_{B_i}(v)$ is singleton for $i = 1, 2$, then we are done. Let $x, y \in N_{B_1}(v)$ and $z \in N_{B_2}(v)$ be any vertex. Suppose $\{x, y\} \notin E(G)$. Then the induced subgraph $G[\{x, y, z\}]$ is a claw and this gives a contradiction by [11, Theorem 7.10] to the fact that $G$ is closed. Thus, $N_{B_1}(v)$ is a clique and similarly, $N_{B_2}(v)$ is a clique. Hence $G$ is decomposable. □

**Theorem 3.12.** A graph $G$ is closed and $J_G$ is Cohen-Macaulay (equivalently, unmixed) if and only if every block of $G$ is complete and $B(G)$ is a path graph.

**Proof.** Let $G$ be closed and $J_G$ be Cohen-Macaulay. Consider a labelling of $G$ with respect to which $G$ is closed. By Proposition 3.10, $B(G)$ is a path. Let $B$ be a block of $G$ containing two cut vertices $i$ and $j$ of $G$ with $i < j$. Let $B_i$ and $B_j$ be two blocks other than $B$ containing $i$ and $j$ respectively. Since $J_G$ is Cohen-Macaulay, $G$ is accessible by [3, Theorem 3.5]. Therefore, by [3, Proposition 4.10], we have $\{i, j\} \in E(B)$. Since $i < j$ and $\{i, j\} \in E(B)$, proper interval ordering of $G$ implies $\{i, i + 1\}, \{j - 1, j\} \in E(B)$, $\{i - 1, i\} \in E(B_i)$ and $\{j, j + 1\} \in E(B_j)$. Thus, for any vertex $v \in V(B)$ we have $i \leq v \leq j$. Again, using proper interval ordering, we can say $B$ is complete as $\{i, j\} \in E(B)$. Now consider a block $B'$ of $G$ containing only one cut vertex $k$ of $G$. By proper interval ordering of $G$, all vertices in $N_{B'}(k)$ are either less than $k$ or greater than
$k$. Since $G$ is accessible, by [3, Theorem 1.2], we have $N_{B'}[k] = V(B')$. Hence, it follows from the proper interval property of $G$ that $B'$ is complete. Conversely, if every block of $G$ is complete and $B(G)$ is a path, then $G$ is closed by [11, Theorem 7.10] and $J_G$ is Cohen-Macaulay by [7, Theorem 1.1]. □

**Remark 3.13.** Let $G$ be a closed graph such that $J_G$ is Cohen-Macaulay. Then choosing a closed labelling of $G$, we can write $G$ as

$$G = K_{m_1} \cup \cdots \cup K_{m_r},$$

where $V(K_{m_i}) = \{m_1 + \cdots + m_{i-1} - i + 1 + k \mid 1 \leq k \leq m_i\}$ for $i = 2, \ldots, r$ and $V(K_{m_1}) = \{1, \ldots, m_1\}$.

Let $G$ be a simple graph. We denote the Castelnuovo-Mumford regularity of $R/J_G$ by $\reg(R/J_G)$, the Hilbert series of $R/J_G$ by $\Hilb R/J_G(t)$, the $i$'th Betti number by $\beta_i(R/J_G)$ and the $(i, j)$-th graded Betti number by $\beta_{i,j}(R/J_G)$.

**Remark 3.14.** The following are some properties of closed graphs with Cohen-Macaulay binomial edge ideals. Let $G = K_{m_1} \cup \cdots \cup K_{m_r}$ be any closed graph such that $J_G$ is Cohen-Macaulay, as in Remark 3.13, and $R_i = K[\{x_j, y_j \mid j \in V(K_{m_i})\}]$ for $1 \leq i \leq r$.

(1) By [22, Theorem 1.1], we have $\reg(R/J_H) \geq d(H)$, where $d(H)$ is the diameter of $H$, i.e., the length of the longest induced path in $H$. By [15, Theorem 3.2], we have $\reg(R/J_H) \leq \clq(H)$ for any closed graph $H$, where $\clq(H)$ denotes the number of maximal cliques of $H$. Now, Theorem 3.12 implies $d(G) = \clq(G)$ and so, we get

$$\reg(R/J_G) = d(G) = \clq(G) = r.$$

(2) Using [32, Lemma 2.9] and [12, Proposition 3], we have

$$\beta_q(R/J_G) = \sum_{i_1 + \cdots + i_r = q} \beta_{i_1}(R_1/J_{K_{m_1}}) \cdots \beta_{i_r}(R_r/J_{K_{m_r}})$$

$$= \sum_{i_1 + \cdots + i_r = q} i_1 \binom{m_1}{i_1 + 1} \cdots i_r \binom{m_r}{i_r + 1},$$

where $1 \leq q \leq n - 1$ and $\beta_0(R/J_G) = 1$.

(3) From [4, Corollary 1], we get $\Hilb_{R_i/J_{K_{m_i}}}(t) = \frac{1 + (m_i - 1)t}{(1 - t)^{m_i + 1}}$, for $1 \leq i \leq r$. Hence by [19, Corollary 3.3], we have

$$\Hilb_{R/J_G}(t) = (1 - t)^{2r-2} \prod_{i \in [r]} \Hilb_{R_i/J_{K_{m_i}}}(t)$$
\[(1 - t)^{2r-2} \prod_{i \in [r]} \frac{1 + (m_i - 1)t}{(1 - t)^{m_i+1}} = \prod_{i \in [r]} \frac{1 + (m_i - 1)t}{(1 - t)^{n+1}}.\]

(4) By [12] Corollary 7, \(\beta_{n-1,n+2r-2}(R/J_G)\) is an extremal Betti number of \(R/J_G\) and \(\beta_{n-1,n+2r-2}(R/J_G) = \prod_{i \in [r]} (m_i - 1).\)

Let \(T\) be a tree. There exists one and only one path between any two vertices of \(T\). Consider a path \(u = u_0, u_1, \ldots, u_n = v\) in \(T\) between \(u\) and \(v\). Let \(C_{0}^{uv}\) and \(C_{n}^{uv}\) denote the connected components of \(T \setminus \{u_1\}\) containing \(u_0\) and \(T \setminus \{u_{n-1}\}\) containing \(u_n\), respectively. Let \(C_{i}^{uv}\) denote the connected component of \(T \setminus \{u_{i-1}, u_{i+1}\}\) containing the vertex \(u_i\) for \(i = \{1, \ldots, n-1\}\). Then \(V(C_{0}^{uv}), \ldots, V(C_{n}^{uv})\) creates a partition on \(V(T)\).

Let \(G\) be a connected graph such that no three blocks of \(G\) share a common vertex, i.e., \(B(G)\) is a tree. Let \(B_i\) and \(B_j\) be two blocks of \(G\). Then \(B_i\) and \(B_j\) are two vertices in \(B(G)\). Since \(B(G)\) is a tree, there exists a unique path between \(B_i\) and \(B_j\) in \(B(G)\). Let \(B_i = B_{i_0}^{ij}, B_{i_1}^{ij}, \ldots, B_{i_s}^{ij}, B_s^{ij} = B_j\) be the unique path between \(B_i\) and \(B_j\) in \(B(G)\). Then \(V(C_{0}^{ij}), \ldots, V(C_{n}^{ij})\) makes a partition on \(V(B(G))\). Now consider \(D_k^{ij} = \cup_{B \in V(C_{k}^{ij})} V(B)\), as a subset of \(V(G)\) for \(k = \{0, \ldots, s\}\). Note that \(D_0^{ij} = V(B_i)\) and \(D_s^{ij} = V(B_j)\) as \(i, j\) are pendant vertices of \(G\). Let
\[
[G]_{ij} = K_{|D_0^{ij}|} \cup \cdots \cup K_{|D_s^{ij}|},
\]
where \(V(K_{|D_k^{ij}|}) = D_k^{ij}\) for \(k = 0, \ldots, s\). Then \([G]_{ij}\) is a closed graph such that \(J_{[G]_{ij}}\) is Cohen-Macaulay and \(J_G \subseteq J_{[G]_{ij}}\). We write
\[
d_{ij} = \text{CCM}_{[G]_{ij}}(G) = |E([G]_{ij})| - |E(G)|.
\]
Note that \([G]_{ij} = [G]_{ji}\), and \(|E([G]_{ij})| = \sum_{k=0}^{s} \binom{|D_k^{ij}|}{2} \).

**Theorem 3.15.** Let \(G \in \mathcal{B}(T)\) be a graph. The set of all minimal completion of \(G\) is \([G]_{ij} \mid B_i, B_j \text{ pendant vertices of } B(G)\}. Moreover,
\[
\text{CCM}(G) = \min \{d_{ij} \mid B_i, B_j \text{ pendant vertices of } B(G)\},
\]
and if \(\text{CCM}(G) = d_{pq}\), then \([G]_{pq}\) is a minimum \(\text{CCM}\)-completion of \(G\).

**Proof.** By Theorem 3.12, any closed graph with Cohen-Macaulay binomial edge ideal is such that every block is complete and block graph is a path as mentioned in Remark 3.13. Suppose \([G]\) is a closed graph with Cohen-Macaulay binomial edge ideal such that \(V([G]) = V(G)\) and \(E(G) \subseteq
Then by Theorem 3.12, we can write
\[ G = K_{m_1} \cup \cdots \cup K_{m_r}, \]
where for \( i < j \), \( V(K_{m_i}) \cap V(K_{m_j}) = \emptyset \) if \( j \neq i + 1 \) and \( V(K_{m_i}) \cap V(K_{m_{i+1}}) \) is a singleton set containing a cut vertex. Note that there is a block \( B_{m_i} \) of \( G \) such that \( V(B_{m_i}) \subseteq V(K_{m_i}) \) and \( B_{m_i} \) contains exactly one cut vertex of \( G \) for \( i \in \{1, r\} \). Now consider the graph \( [G]_{m_1m_r} \). Then \( E([G]_{m_1m_r}) \subseteq E([G]) \) is clear. Also, observe that \( [G]_{ij} \) is a minimal CCM-completion of \( G \) for every pair of pendant vertices \( \{B_i, B_j\} \) of \( B(G) \). Therefore, \( \{[G]_{ij} \mid B_i, B_j \text{ pendant vertices of } B(G)\} \) is the collection of all minimal CCM-completion of \( G \). Since any minimum CCM-completion is a minimal CCM-Completion, we have
\[
CCM(G) = \min \{d_{ij} \mid B_i, B_j \text{ pendant vertices of } B(G)\},
\]
and if \( CCM(G) = d_{pq} \), then \( [G]_{pq} \) is a minimum CCM-completion of \( G \).

**Corollary 3.16.** Let \( G \) be a closed graph. Then the CCM-completion number of \( G \) is given by
\[
CCM(G) = \sum_{B \text{ block of } G} \left( \frac{|V(B)|}{2} \right) - |E(G)|.
\]

**Proof.** By Proposition 3.10, \( B(G) \) is a path graph. So, \( G \) can be decomposed as in 3.3. There are only two pendant vertices of \( B(G) \), say \( B_p \) and \( B_q \). Thus, by Theorem 3.15, the only minimal CCM-completion of \( G \) will be \( [G]_{pq} \), and so, it is the only minimum CCM-completion. Hence, \( CCM(G) = d_{pq} \). Since complete graph on the vertex set \( V(B) \) of a block \( B \) of \( G \) contains \( \left( \frac{|V(B)|}{2} \right) \) edges, the result follows easily.

We give the following algorithm to find the CCM-completion number \( CCM(G) \) and a minimum CCM-completion of \( G \), for any graph \( G \in BT \).

**Algorithm 1.** **Input:** A graph \( G \) such that \( B(G) \) is a tree.

**Output:** \( CCM(G) = d \) and a minimum CCM-completion \( [G]_{pq} \) of \( G \).

**Step-1:** Find the block graph \( B(G) \) of \( G \).

**Step-2:** Collect all the pendant vertices of \( B(G) \) and let \( \mathcal{P} = \{B_1, \ldots, B_r\} \) be the set of all pendant vertices of \( B(G) \).

**Step-3:** Set \( d = n^2, p = 0, q = 0 \).

**Step-4:** Set \( T = \{(B_i, B_j) \mid B_i, B_j \in \mathcal{P} \text{ with } i < j\} \) i.e., \( T \) is the collection of all ordered pairs \( (B_i, B_j) \) such that \( B_i, B_j \) are pendant vertices of \( B(G) \) with \( i < j \).
Step-5: Take an element $(B_i, B_j) \in T$.

Step-6: Find the unique path from $B_i$ to $B_j$ in $B(G)$ and suppose the path is $B_i = B_i^0, \ldots, B_i^{s_{ij}} = B_j$.

Step-7: Set $C_0^{ij} = \text{the connected component of } B(G) \setminus \{B_i^1\}$ containing $B_i^0$ i.e., $V(C_0^{ij})$ is the collection of those vertices of $B(G)$ which are connected to $B_i^0$ via a path in $B(G) \setminus \{B_i^1\}$. Similarly, we set $C_{s_{ij}}^{ij} = \text{the connected component of } B(G) \setminus \{B_i^{s_{ij}-1}\}$ containing $B_i^{s_{ij}}$, and for $1 \leq k \leq s_{ij} - 1$, $C_k^{ij} = \text{the connected component of } B(G) \setminus \{B_k^{ij-1}, B_k^{ij+1}\}$ containing $B_k^{ij}$.

Step-8: For each $k \in \{0, \ldots, s_{ij}\}$, consider
\[ D_k^{ij} = \bigcup_{B \in V(C_k^{ij})} V(B) \]
as a subset of $V(G)$.

Step-9: Compute
\[ d_{ij} = \sum_{k=0}^{s_{ij}} \left( \frac{|D_k^{ij}|}{2} \right) - |E(G)|. \]

Step-10: If $d_{ij} < d$, then set $d = d_{ij}$, $p = i$, $q = j$.

Step-11: Update $T = T \setminus \{(B_i, B_j)\}$.

Step-12: If $T \neq \emptyset$, then go to Step-5, else go to Step-13.

Step-13: Construct the graph $[G]_{pq}$ with
\[ V([G]_{pq}) = V(G), \]
and
\[ E([G]_{pq}) = \bigcup_{k=0}^{s_{pq}} \{\{u, v\} \mid u, v \in D_k^{pq}, u \neq v\}. \]

Step-14: Output $d$ and $[G]_{pq}$.

**Correctness:** The correctness of the algorithm follows from Theorem 3.15.

**Time complexity:** Let the given graph $G$ has $V(G) = [n]$. Then by the algorithm given in [25], we can compute $B(G)$ in $O(n^2)$ time. Now, checking whether a vertex of $B(G)$ is a pendant vertex or not takes $O(|V(B(G))|^2)$ time. Therefore, all the pendant vertices of $B(G)$ can be found in Step-2 by $O(n^2)$ time (because $|V(B(G))| \leq n$). Step-3 requires constant time. Since the number of pendant vertices of $B(G)$ is $r$ and $\binom{r}{2} < n^2$, Step-4 can be executed in $O(n^2)$ time. Step-5 takes $O(1)$ time. We can find
path between two vertices in $B(G)$ using either the BFS or the DFS method (see [18]) in $O(|V(B(G))| + |E(B(G))|)$ time and thus, $O(n^2)$ time is sufficient for Step-6. Since $|V(B(G))| \leq n$, we can find any induced subgraph of $B(G)$ in $O(n^2)$ time. In any induced subgraph of $B(G)$, we can find the connected component containing one fixed vertex in $O(n^2)$ time using the DFS. Now in Step-7, we have to do this process $s_{ij} + 1$ times. Therefore, Step-7 requires $O(n^3)$ time as $s_{ij} + 1 \leq n$. Clearly, we have $|V(C_k^{ij})| \leq |V(B(G))| \leq n$, and so, we can find $D_k^{ij}$ in $O(n)$ time for each $k \in \{0, \ldots, s_{ij}\}$. Since $s_{ij} + 1 \leq n$, total time required to process Step-8 is $O(n^2)$. In Step-9, we are taking sum of $s_{ij} + 1$ terms and so it will take $O(n)$ time (as $s_{ij} + 1 \leq n$). Step-10 and Step-11 are $O(1)$ processes. Checking $T = \phi$ or $T \neq \phi$ takes constant time. We repeat Step-5 to Step-12 $\binom{n}{2}$ times. Since one iteration of Step-5 to Step-12 can be done in $O(n^3)$ time, $\binom{n}{2}$ iteration of Step-5 to Step-12 will take $O(n^5)$ time (because $\binom{n}{2} \leq n^2$).

We can choose two vertices out of $n$ in $\binom{n}{2}$ ways. Now for each pair of vertices, we have to check whether it belongs to the same $D_k^{pq}$ or not for some $k \in \{0, \ldots, s_{pq}\}$. There are $s_{pq} + 1$ choices of $k$. Since $\binom{n}{2} < n^2$ and $s_{pq} + 1 \leq n$, total time required in Step-13 is $O(n^3)$. At last, Step-14 takes $O(n^2)$ time as $|V([G]_{pq})| = |V(G)| = n$ and $|E([G]_{pq})| \leq n^2$. Since the entire algorithm can be performed in $O(n^5)$ time, it is a polynomial time algorithm.

**Example 3.17.** Consider the graph $G$ in the Figure 1. It is clear from the graph that $B(G)$ is a tree. Now $\mathcal{P} = \{B_1, B_4, B_7, B_8\}$ is the set of blocks of $G$ containing only one cut vertex of $G$, i.e., $\mathcal{P}$ is the set of all pendant

![Figure 1. A graph $G$ such that $B(G)$ is a tree](image)
vertices of $B(G)$. Consider $T$ as in Step-4 of the algorithm. Then

$$T = \{(B_1, B_4), (B_1, B_7), (B_1, B_8), (B_4, B_7), (B_4, B_8), (B_7, B_8)\}.$$ 

We have $|E(G)| = 27$ and can observe that

- $[G]_{14} = K_4 \cup K_4 \cup K_{13} \cup K_2$ and $d_{14} = \left(\frac{4}{2}\right) + \left(\frac{4}{2}\right) + \left(\frac{13}{2}\right) + \left(\frac{2}{2}\right) - 27 = 64$;
- $[G]_{17} = K_4 \cup K_4 \cup K_4 \cup K_8 \cup K_2 \cup K_3$ and $d_{17} = \left(\frac{4}{2}\right) + \left(\frac{4}{2}\right) + \left(\frac{4}{2}\right) + \left(\frac{8}{2}\right) + \left(\frac{2}{2}\right) + \left(\frac{3}{2}\right) - 27 = 23$;
- $[G]_{18} = K_4 \cup K_4 \cup K_4 \cup K_7 \cup K_5$ and $d_{18} = \left(\frac{4}{2}\right) + \left(\frac{4}{2}\right) + \left(\frac{7}{2}\right) + \left(\frac{5}{2}\right) - 27 = 22$;
- $[G]_{47} = K_2 \cup K_9 \cup K_8 \cup K_2 \cup K_3$ and $d_{47} = \left(\frac{2}{2}\right) + \left(\frac{9}{2}\right) + \left(\frac{8}{2}\right) + \left(\frac{2}{2}\right) + \left(\frac{3}{2}\right) - 27 = 42$;
- $[G]_{48} = K_2 \cup K_9 \cup K_7 \cup K_5$ and $d_{48} = \left(\frac{2}{2}\right) + \left(\frac{9}{2}\right) + \left(\frac{7}{2}\right) + \left(\frac{5}{2}\right) - 27 = 48$;
- $[G]_{78} = K_3 \cup K_2 \cup K_{13} \cup K_5$ and $d_{78} = \left(\frac{3}{2}\right) + \left(\frac{2}{2}\right) + \left(\frac{13}{2}\right) + \left(\frac{5}{2}\right) - 27 = 65$.

Therefore, $CCM(G) = d_{18} = 22$, i.e., the minimum number of edges needed to add to $G$ to make it a closed graph with Cohen-Macaulay binomial edge ideal is 22, and $[G]_{18}$ is the only minimum CCM-completion of $G$.

PIG-completion problem is NP-hard, but for $G \in \text{BT}$, the CCM-completion problem can be performed in polynomial time. Since $CCM \subseteq \text{PIG}$, the following question arise naturally.

**Question 3.18.** For an arbitrary graph $G$, can we give a polynomial-time algorithm to compute $CCM(G)$ and a minimum CCM-completion of $G$?

## 4. Unmixed and Cohen-Macaulay Property of Binomial Edge Ideals of Subgraphs

In this section, we discuss when unmixed and Cohen-Macaulay properties of binomial edge ideals are hereditary for subgraphs. Also, we study the accessible completion of graphs and Cohen-Macaulay criterion of binomial edge ideals of whisker graphs.

**Proposition 4.1.** Let $G$ be a graph such that $J_G$ is unmixed. Let $v \in V(G)$ be a free vertex in $\Delta(G)$, and $F$ the facet of $\Delta(G)$ with $v \in F$. If $F \setminus \{v\} \not\subset T$ for each $T \in C(G)$, with $|T| \neq 1$, then $J_{G \setminus \{v\}}$ is unmixed.

**Proof.** We first prove that $C(G \setminus \{v\}) \subseteq C(G)$. Let $T \in C(G \setminus \{v\})$ and $u \in T$ be any vertex. We will show that $u$ is a cut point of $H = G \setminus (T \setminus \{u\})$. Suppose that $u$ is not a cut point of $H$. Since $u$ is a cut point of $H' = G' \setminus (T \setminus \{u\})$, where $G' = G \setminus \{v\}$, there exist two vertices $u_1, u_2 \in$
$V(H')$ such that all the paths from $u_1$ to $u_2$ pass through $u$. Moreover, there exists a path $\pi = u_1, \ldots, v', v, v'', \ldots, u_2$ in $H$, that does not pass through $u$. Since $F$ is a clique and $\{v', v\}, \{v, v''\} \in E(H)$, we have $\{v', v''\} \in E(H)$ and so $\{v', v''\} \in E(H')$. Hence we can obtain a new path $\pi' = u_1, \ldots, v', v'', \ldots, u_2$ that is contained in $H'$ and does not pass through $u$, a contradiction. Therefore $u$ is a cut point of $H$ and hence $T \in C(G)$.

**Case I.** Let $T \in C(G \setminus \{v\}) \subseteq C(G)$, such that $|T| \neq 1$. Let $c(T)$ be the number of connected components when we remove $T$ from $G$. Since $v$ is a free vertex, one of these components contains $v$ and let that component be $G''$. So, in $(G \setminus \{v\}) \setminus T$, that component is $G'' \setminus \{v\} \neq \phi$ (since $F \setminus \{v\} \not\subseteq T$) and other components remain the same as in $G \setminus T$. Therefore, the number of connected components in $(G \setminus \{v\}) \setminus T$ is also $c(T)$. Since $J_G$ is unmixed, it follows from Lemma [2.10] that $c_{G \setminus \{v\}}(T) = |T| + 1$.

**Case II.** Let $T \in C(G \setminus \{v\}) \subseteq C(G)$, such that $|T| = 1$. Let $c(T)$ be the number of connected components in $G \setminus T$. Then from Lemma [2.10] $c(T) = 2$ as $J_G$ is unmixed. Let $T = \{i\}$. Then $i$ is a cut point of $(G \setminus \{v\}) \setminus T \setminus \{i\} = G \setminus \{v\}$. So the number of connected components, after removing $T$ from $G \setminus \{v\}$, is at least 2. Moreover, corresponding to each connected component of $(G \setminus \{v\}) \setminus T$, we get a connected component in $G \setminus T$. Since $c(T) = 2$, $c_{G \setminus \{v\}}(T) = 2$.

Now $G \setminus \{v\}$ is connected as $v$ is a free vertex. Therefore, for all $T \in C(G \setminus \{v\})$, we have $c_{G \setminus \{v\}}(T) = |T| + 1$. Hence, it follows from Lemma [2.10] that $J_{G \setminus \{v\}}$ is unmixed.

**Proposition 4.2.** Let $G$ be a graph such that $J_G$ is unmixed and $v \in V(G)$. Let $G \setminus \{v\}$ be connected and $J_{G \setminus \{v\}}$ be unmixed. Then the following hold good:

(i) $v$ is a free vertex in $\Delta(G)$

and

(ii) $F \setminus \{v\} \not\subseteq T$ for each $T \in C(G \setminus \{v\})$, where $F$ is the facet of $\Delta(G)$ with $v \in F$.

**Proof.** Let $T \in C(G)$ with $v \in T$. Since $J_G$ is unmixed, from Lemma [2.10] we have $c_G(T) = |T| + 1$. As $v \in T$, we have $G[T] = (G \setminus \{v\})[T]$. So for each $i(\neq v) \in T$, we have $G[T \cup \{i\}] = (G \setminus \{v\})[T \cup \{i\}]$. Therefore $T \setminus \{v\}$ satisfies the cut point property for $G \setminus \{v\}$. Since $J_{G \setminus \{v\}}$ is unmixed and $G \setminus \{v\}$ is connected, the number of connected components in $(G \setminus \{v\}) \setminus (T \setminus \{v\}) = G \setminus T$ is $|T \setminus \{v\}| + 1 \neq |T| + 1$, which is a contradiction. Therefore, $v \not\in T$ for each $T \in C(G)$. Hence, by Proposition [2.8] $v$ is a free vertex in $\Delta(G)$.

Let $F$ be the facet of $\Delta(G)$ with $v \in F$. Suppose $F \setminus \{v\} \subseteq T$ for some $T \in C(G \setminus \{v\})$. We know $C(G \setminus \{v\}) \subseteq C(G)$. Since $J_G$ is unmixed, the number of connected components in $G \setminus T$ is $c_G(T) = |T| + 1$. One of these
components is only \( \{v\} \) as \( v \) is a free vertex and \( F \setminus \{v\} \subseteq T \). Therefore, the number of connected components in \( (G \setminus \{v\}) \setminus T \) is \(|T|\), which is a contradiction by Lemma 2.10 to the fact that \( J_{G\setminus\{v\}} \) is unmixed. □

**Corollary 4.3.** Let \( G \) be a graph with \( v \in V(G) \), such that \( v \) is a free vertex in \( \Delta(G) \) and \( F \) is the facet of \( \Delta(G) \) containing \( v \). Suppose that for all \( T \in \mathcal{C}(G) \) we have \( F \setminus \{v\} \not\subseteq T \), then the following conditions are equivalent:

(a) \( J_G \) is unmixed;

(b) \( J_{G\setminus\{v\}} \) is unmixed.

**Proof.** From Lemma 2.9, we have \( \mathcal{C}(G) = \mathcal{C}(G \setminus \{v\}) \). From the proof of Proposition 4.1, we have that the number of connected components in \( G[T] \) is equal to the number of connected components in \( (G \setminus \{v\})[T] \), for all \( T \in \mathcal{C}(G) = \mathcal{C}(G \setminus \{v\}) \). Since \( v \) is a free vertex, we have \( G \setminus \{v\} \) is connected and hence the proof follows from Lemma 2.10. □

Suppose \( G \) is a graph, such that \( J_G \) is unmixed. From the above results, especially Corollary 4.3, we have seen that removing some special vertices from \( G \), we get some induced subgraphs of \( G \), whose corresponding binomial edge ideals are also unmixed. Therefore, a natural question arise: whether one can classify all such subgraphs of a graph \( G \). The same question can be asked if we replace unmixed by Cohen-Macaulay. The rest of this section is devoted to the following classification problem: Given a graph \( G \) that is closed with Cohen-Macaulay (respectively unmixed) \( J_G \), classify all closed induced subgraphs of \( G \) which have Cohen-Macaulay (respectively unmixed) binomial edge ideals.

**Corollary 4.4.** Let \( G \) be a graph such that \( J_G \) is Cohen-Macaulay. Suppose \( G \setminus \{v\} \) is connected and \( J_{G\setminus\{v\}} \) is Cohen-Macaulay. Then,

(i) \( v \) is a free vertex in \( \Delta(G) \)

and

(ii) \( F \setminus \{v\} \not\subseteq T \), for each \( T \in \mathcal{C}(G \setminus \{v\}) \), where \( F \) is the facet of \( \Delta(G) \) containing \( v \).

**Proof.** \( J_G \) is Cohen-Macaulay, hence unmixed. The proof follows from Proposition 4.2. □

Next, we show that any induced subgraph of a closed graph (respectively CCM graph) is closed (respectively CCM). We prove the results using the labelling of vertices with respect to which the graph is closed, and this helps us find a labelling of the subgraph with respect to which it is closed. However, it is possible to write a shorter proof if one uses the structure of closed graphs and CCM graphs discussed in Propositions 3.10, 3.11 and Theorem 3.12.
**Lemma 4.5.** Subgraph of a closed graph is closed.

*Proof.* Let $G$ be a closed graph with respect to a given labelling on $[n]$. Let us remove the vertex $m \in V(G)$. Then the vertices of $G \setminus \{m\}$ are $1, 2, \ldots, m - 1, m + 1, \ldots, n$. We rename these as $1, 2, \ldots, m, m + 1, \ldots, n - 1$, i.e., for a vertex $k \in V(G)$, if $k < m$, then label it as $k$, and if $k > m$, then label it as $k - 1$, in $G \setminus \{m\}$.

**Claim:** $G \setminus \{m\}$ is closed with respect to this labelling.

*Proof of the claim.* Let $\{i, k\} \in E(G \setminus \{m\})$, with $i < k$.

**Case I.** Let $k < m$. Then $\{i, k\} \in E(G)$ implies that $\{i, j\}, \{j, k\} \in E(G)$, for all $i < j < k < m$, since $G$ is closed. Therefore, $\{i, j\}, \{j, k\} \in E(G \setminus \{m\})$, for all $i < j < k$.

**Case II.** Let $i < m$ and $k \geq m$. Then the edge $\{i, k\} \in E(G \setminus \{m\})$ is the same edge as $\{i, k + 1\} \in E(G)$. Since $G$ is closed, $\{i, k + 1\} \in E(G)$ implies $\{i, j\}, \{j, k + 1\} \in E(G)$, for all $i < j < k + 1$. Let $i < p < k$. If $p < m$ then $\{i, p\}, \{p, k + 1\} \in E(G)$ and hence $\{i, p\}, \{p, k\} \in E(G \setminus \{m\})$. Again, if $p \geq m$, then $i < p + 1 < k + 1$ in $V(G)$ implies $\{i, p + 1\}, \{p + 1, k + 1\} \in E(G)$ and hence $\{i, p\}, \{p, k\} \in (E(G \setminus \{m\})$ since $p + 1 > m$.

**Case III.** Let $i \geq m$. Then $\{i, k\} \in E(G \setminus \{m\})$ is the same edge as $\{i + 1, k + 1\} \in E(G)$. Since $G$ is closed, $\{i + 1, k + 1\} \in E(G)$ implies that $\{i + 1, j + 1\}, \{j + 1, k + 1\} \in E(G)$, for all $m < i + 1 < j + 1 < k + 1$ and hence $\{i, j\}, \{j, k\} \in (E(G \setminus \{m\})$, for all $i < j < k$.

Combining all the cases we get that if $\{i, k\} \in E(G \setminus \{m\})$, with $i < k$, then $\{i, j\}, \{j, k\} \in (E(G \setminus \{m\})$ for all $i < j < k$. Therefore, from Theorems 3.2 and 3.3 it follows that $G \setminus \{m\}$ is closed. Since $m$ is an arbitrary vertex of $G$, any subgraph of $G$ is closed. \hfill \Box

**Theorem 4.6.** Let $G$ be a connected graph on $[n]$, which is closed with respect to the given labelling. If $J_G$ is Cohen-Macaulay, then $J_H$ is Cohen-Macaulay for any subgraph $H$ of $G$.

*Proof.* Let $m \in V(G)$ be any vertex. Remove $m$ and consider the graph $H = G \setminus \{m\}$, with the labeling mentioned in the Lemma 4.5, for which $H$ is closed. Suppose $\{i, j + 1\}$ with $i < j$, and $\{j, k + 1\}$ with $j < k$, are edges of $H$. Then, to prove $J_H$ is Cohen-Macaulay we have to show that $\{i, k + 1\} \in E(H)$.

**Case I ($i > m - 1$).** In this case $m - 1 < i < j < k$. Therefore,

$\{i, j + 1\}, \{j, k + 1\} \in E(H) \Rightarrow \{i + 1, j + 2\}, \{j + 1, k + 2\} \in E(G)$.

By Proposition 2.7, we have $\{i + 1, k + 2\} \in E(G)$, since $J_G$ is Cohen-Macaulay. Hence $\{i, k + 1\} \in E(H)$, since $i + 1 > m$. \hfill \Box
Case II \((i \leq m - 1, \ j > m - 1)\). In this case \(m - 1 < j < k\). Therefore
\[
\{i, j + 1\}, \{j, k + 1\} \in E(H) \Rightarrow \{i, j + 2\}, \{j + 1, k + 2\} \in E(G).
\]

By Proposition 2.7, we have \(\{i, k + 2\} \in E(G)\), since \(J_G\) is Cohen-Macaulay. Therefore, \(\{i, k + 1\} \in E(H)\), since \(i \leq m - 1\) and \(k + 2 > m\).

Case III \((i < m - 1, \ j = m - 1)\). In this case \(m - 1 = j < k\). Therefore
\[
\{i, j + 1\}, \{j, k + 1\} \in E(H) \Rightarrow \{i, j + 2\}, \{j + 1, k + 2\} \in E(G).
\]

\(G\) is closed, and therefore, \(\{i, j + 2\} \in E(G)\) implies \(\{i, j + 1\} \in E(G)\). By Proposition 2.7, we have \(\{i, k + 2\} \in E(G)\), since \(J_G\) is Cohen-Macaulay. Hence, \(\{i, k + 1\} \in E(H)\), since \(k + 2 > m\).

Case IV \((i < j < m - 1, \ k \geq m - 1)\). In this case,
\[
\{i, j + 1\}, \{j, k + 1\} \in E(H) \Rightarrow \{i, j + 1\}, \{j, k + 2\} \in E(G).
\]

By Proposition 2.7, we have \(\{i, k + 2\} \in E(G)\), since \(J_G\) is Cohen-Macaulay. Therefore, \(\{i, k + 1\} \in E(H)\), since \(k + 2 > m\).

Case V \((i < j < k < m - 1)\). In this case,
\[
\{i, j + 1\}, \{j, k + 1\} \in E(H) \Rightarrow \{i, j + 1\}, \{j, k + 1\} \in E(G).
\]

By Proposition 2.7, we have \(\{i, k + 1\} \in E(G)\), since \(J_G\) is Cohen-Macaulay. Therefore, \(\{i, k + 1\} \in E(H)\), since \(k + 1 < m\).

Combining all the cases, we get that if \(\{i, j + 1\}\) with \(i < j\), and \(\{j, k + 1\}\) with \(j < k\), are edges of \(H\) then \(\{i, k + 1\}\) is an edge of \(H\). Hence, from Proposition 2.7 it follows that \(J_H\) is Cohen-Macaulay. Therefore, \(m\) being an arbitrary vertex, \(J_H\) is Cohen-Macaulay for every subgraph \(H\) of \(G\). \(\square\)

Let \(G\) be a graph and \(v\) be a vertex of \(G\). We define the graph \(G_v\) as follows:

- \(V(G_v) = V(G)\),
- \(E(G_v) = E(G) \cup \{\{i, j\} \mid i, j \in N_G(v) \text{ with } i \neq j\}\).

In [3], Bolognini et al. have introduced the concept of and have proved that for a graph \(G\), Cohen-Macaulay property of \(J_G\) implies the accessible property of \(G\). They have conjectured about the converse in [3 Conjecture 1.1]. The following Proposition 4.7 gives a technique to make an unmixed binomial edge ideal accessible by adding some special edges. Let us denote the class of accessible graphs by \(\text{ACC}\),

**Proposition 4.7.** Let \(G\) be a graph such that \(J_G\) is unmixed. We get an accessible graph \(H\) with \(V(H) = V(G)\), by adding some special edges to \(G\).

**Proof.** If \(G\) is accessible, then consider \(H = G\). Suppose \(G\) is not accessible, then there exists \(T_1 \in \mathcal{C}(G)\) such that \(T\) is not accessible. Take any \(v_1 \in T_1\) and consider the graph \(G_{v_1}\). By [3 Lemma 4.5], \(\mathcal{C}(G_{v_1}) = \{T \in\)
\( \mathcal{C}(G) \mid v_1 \not\in T \) and \( J_{G_{v_1}} \) is unmixed. If \( G_{v_1} \) is accessible, then consider \( H = G_{v_1} \) and if not accessible, then repeat the process. Due to [3, Lemma 4.5], after finite steps, we get a graph \( H = (\ldots((G_{v_1})_{v_2})\ldots)_{v_r} \), such that \( H \) is accessible.

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{A non-accessible graph \( G \) with unmixed \( J_G \) and the accessible graph \( G_3 \) with Cohen-Macaulay \( J_{G_3} \).}
\end{figure}

**Example 4.8.** Consider the graph \( G \) in Figure (2a). It was shown in [3, Example 2.3] that \( J_G \) is unmixed but \( G \) is not accessible. In fact, \( \mathcal{C}(G) = \{\phi, \{2\}, \{6\}, \{2, 6\}, \{3, 5\}, \{2, 4, 6\}\} \). As in the proof of Proposition 4.7, we choose the non-accessible cutset \( \{3, 5\} \) of \( G \) and take \( 3 \in \{3, 5\} \). Now consider the graph \( G_3 \) (see Figure 2(b)). Then \( J_{G_3} \) is unmixed and \( \mathcal{C}(G_3) = \{\phi, \{2\}, \{6\}, \{2, 6\}, \{2, 4, 6\}\} \). Thus, \( G_3 \) is accessible and using Macaulay2, we get \( J_{G_3} \) is Cohen-Macaulay.

In Proposition 4.7, we see that for a graph \( G \) with unmixed \( J_G \), there exists an ACC-completion of \( G \). Thus, we propose the following question:

**Question 4.9.** For a graph \( G \) with unmixed \( J_G \), is the method discussed in Proposition 4.7 the only way to get a ACC-completion of \( G \)? Also, what will be \( \text{ACC}(G) \)? Can we find a polynomial-time algorithm to compute \( \text{ACC}(G) \)?

Let \( G \) be a graph with \( V(G) = \{x_1, \ldots, x_n\} \). The whisker graph or suspension of \( G \), denoted by \( W_G \), is the graph attaching \( n \) new vertices \( \{y_1, \ldots, y_n\} \) to \( G \) as follows:

\begin{itemize}
  \item \( V(W_G) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \),
  \item \( E(W_G) = E(G) \cup \{x_i, y_i \mid i = 1, \ldots, n\} \).
\end{itemize}

In the following Theorem 4.10, we give sufficient and necessary conditions for a whisker graph to be Cohen-Macaulay.

**Theorem 4.10.** Let \( G \) be a graph. Then the following are equivalent.

\begin{enumerate}
  \item \( J_{W_G} \) is Cohen-Macaulay.
  \item \( J_{W_G} \) is unmixed.
\end{enumerate}
(iii) $G$ is complete.

Proof. (i) $\Rightarrow$ (ii) is well known.

(ii) $\Rightarrow$ (iii): Let $V(G) = \{x_1, \ldots, x_n\}$ and $V(W_G) = V(G) \cup \{y_1, \ldots, y_n\}$ be such that $\{x_i, y_i\} \in E(W_G)$ for $1 \leq i \leq n$. Suppose $G$ is not complete. Then there exist $x_j, x_k \in V(G)$ such that $\{x_j, x_k\} \notin E(G)$. Now consider $T = N_G(x_j) = \{x_j, \ldots, x_{j_s}\}$. Then each $x \in T$ is a cut vertex of $W_G \setminus (T \setminus \{x\})$ and hence, $T \in \mathcal{C}(W_G)$. Note that $W_G \setminus T$ has $\{y_{j_1}, \ldots, y_{j_s}\}$ as connected components, with one containing $x_j$ and one containing $x_k$. Therefore, $c_{W_G}(T) > |T| + 1$, a contradiction to the fact that $J_{W_G}$ is unmixed. Hence $G$ is complete.

(iii) $\Rightarrow$ (i): Let $G$ be complete. Then $B(W_G)$ is a tree and every block of $W_G$ is complete. Thus, $J_{W_G}$ is Cohen-Macaulay by [7, Theorem 1.1].

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