A Smashing Subcategory of the Homotopy Category of Gorenstein Projective Modules

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Abstract Let $A$ be an artin algebra of finite CM-type. In this paper, we show that if $A$ is virtually Gorenstein, then the homotopy category of Gorenstein projective $A$-modules, denote $K(A\text{-GP})$, is always compactly generated. Based on this result, it will be proved that the homotopy category of projective $A$-modules, denote $K(A\text{-P})$, is a smashing subcategory of $K(A\text{-GP})$ and the corresponding Verdier quotient is also compactly generated. Furthermore, it turns out that the inclusion functor $i: K(A\text{-P}) \to K(A\text{-GP})$ induces a recollement of $K(A\text{-GP})$.

Keywords Gorenstein projective modules · Compactly generated homotopy categories · Smashing subcategory · Recollements

1 Introduction

Let $\mathcal{X}$ be a class of left modules over an associative ring $R$ which is closed under set-indexed coproducts and direct summands. Holm and Jørgensen [13] study the general question of when the homotopy category $K(\mathcal{X})$ of $\mathcal{X}$ is compactly generated. They give a number of sufficient conditions on $R$ and $\mathcal{X}$ which ensure that $K(\mathcal{X})$ is compactly generated.

Let $A$ be an artin algebra and $A\text{-Mod}$ the category of $A$-modules. Denote by $A\text{-P}$ the full subcategory of projective $A$-modules, $A\text{-GP}$ the full subcategory of
Gorenstein projective $A$-modules, and $A\text{-Gproj}$ the full subcategory of all finitely-generated Gorenstein projective modules. As is well known, the homotopy category $K(A\text{-P})$ is compactly generated [15, Theorem 2.4].

Gorenstein projective modules and algebras of finite Cohen–Macaulay type receive a lot of attention (See e.g. [1, 4–6, 8–10, 12, 14, 16, 17, 19]). Recall from [4, 6] that an artin algebra $A$ is of finite Cohen–Macaulay type (simply, CM-type) if there are only finitely many isomorphism classes of finitely-generated indecomposable Gorenstein projective $A$-modules. We are interested in the compact generatedness of the homotopy category $K(A\text{-GP})$ of an artin algebra $A$ of finite CM-type.

In Section 2, we first show that if $A$ is virtually Gorenstein of finite CM-type, then $K(A\text{-GP})$ is compactly generated. Next, based on this result, we show that $K(A\text{-P})$ is a smashing subcategory of $K(A\text{-GP})$ and the Verdier quotient $K(A\text{-GP})/K(A\text{-P})$ is also compactly generated.

The concept of recollement goes back to the work of Beilinson et al. [2]. In Section 3, we show the existence of recollements of the homotopy category $K(A\text{-GP})$.

2 Conditions for Compact Generatedness

Our aim in this section is to show that $K(A\text{-GP})$ is compactly generated provided $A$ is virtually Gorenstein of finite CM-type. So based on the result of Bruns and Herzog [6, Proposition 2.11], and the result of Jørgensen [16], $K(A\text{-P})$ is a smashing subcategory of $K(A\text{-GP})$ and the Verdier quotient $K(A\text{-GP})/K(A\text{-P})$ is also compactly generated.

Our strategy for the compact generatedness of $K(A\text{-GP})$ is to give sufficient conditions on $A$. We will use the following lemma.

**Lemma 2.1** [4, Theorem 4.10] Let $A$ be an artin algebra. Then $A$ is virtually Gorenstein of finite CM-type if and only if any Gorenstein projective $A$-module is a direct sum of finitely-generated modules.

Now we are ready to state and prove our first main theorem in this section.

**Theorem 2.2** Let $A$ be a virtually Gorenstein artin algebra of finite CM-type. Then $K(A\text{-GP})$ is a compactly generated triangulated category.

**Proof** Since $A$ is virtually Gorenstein of finite CM-type, we get from Lemma 2.1 that $A\text{-GP} = \text{Add}(A\text{-Gproj})$ which means that $A\text{-GP}$ is contravariantly finite in $A\text{-Mod}$, and also each Gorenstein projective module is pure projective which means that every pure exact sequence of modules from $A\text{-GP}$ is split exact. This implies that $K(A\text{-GP})$ is a compactly generated triangulated category by [13, Theorem 3.1]. □

Recall from [11] that a complex $X^\bullet$ is $A\text{-GP}$-acyclic if the induced complex $\text{Hom}_A(G, X^\bullet)$ is acyclic for each module in $A\text{-GP}$, and the Gorenstein derived category $D_{gp}(A\text{-Mod})$ of an artin algebra $A$ is defined to be the Verdier quotient of the homotopy category $K(A\text{-mod})$ with respect to the thick subcategory $K_{gpac}(A\text{-Mod})$ which consists of all $A\text{-GP}$-acyclic complexes.
Corollary 2.3 Let $A$ be a Gorenstein artin algebra of finite CM-type. Then $D_{gp}(A\text{-Mod})$ is compactly generated.

Proof By the assumption on $A$, we see from [3, Corollary 8.3 and Corollary 8.5] that $A$ satisfies the conditions on Theorem 2.2. Hence we get that $K(A\mathcal{GP})$ is a compactly generated triangulated category. By [7, Proposition 3.5] there is a triangle-equivalence $D_{gp}(A\text{-Mod}) \cong K(A\mathcal{GP})$. This implies that $D_{gp}(A\text{-Mod})$ is compactly generated. \qed

For our second main theorem we need a definition and some lemmas.

Recall from [18] that a full subcategory $B$ of a compactly generated triangulated category $T$ is smashing if the inclusion $B \to T$ has a right adjoint which preserves coproducts.

Lemma 2.4 [18, Lemma 4.1] Let $B$ be a smashing subcategory of a compactly generated triangulated category $T$. Then $T/B$ is a compactly generated triangulated category.

Lemma 2.5 [5, Proposition 2.11] Let $T$ and $T'$ be compactly generated triangulated categories, and let $F : T \to T'$ be a fully faithful triangle functor which preserves coproducts and compact objects. Then $F$ admits a right adjoint $G : T' \to T$ which preserves coproducts.

So in view of the above lemmas, we have the following theorem.

Theorem 2.6 Let $A$ be a virtually Gorenstein artin algebra of finite CM-type. Then $K(A\mathcal{P})$ is a smashing subcategory of $K(A\mathcal{GP})$. Moreover, $K(A\mathcal{GP})/K(A\mathcal{P})$ is a compactly generated triangulated category.

Proof By the assumption on $A$, we get from Theorem 2.2 that $K(A\mathcal{GP})$ is compactly generated, and from [15, Theorem 2.4] that $K(A\mathcal{P})$ is compactly generated and each compact object $P^\bullet$ is exactly the upper bounded complex of finitely-generated projective modules. Let $i : K(A\mathcal{P}) \to K(A\mathcal{GP})$ be the inclusion functor. Note that $i$ naturally preserves coproducts. Let $\{G_i^\bullet\}_{i \in I}$ be any family objects in $K(A\mathcal{GP})$. Then we have $\text{Hom}_{K(A\mathcal{GP})}(iP^\bullet, \bigsqcup_{i \in I} G_i^\bullet) = \text{Hom}_{K(A\mathcal{GP})}(iP^\bullet, \bigsqcup_{i \in I} G_i^\bullet) \cong \bigsqcup_{i \in I} \text{Hom}_{K(A\mathcal{GP})}(iP^\bullet, G_i^\bullet) \cong \bigsqcup_{i \in I} \text{Hom}_{K(A\mathcal{GP})}(iP^\bullet, G_i^\bullet)$. This implies that $i$ preserves compact objects. Hence by Lemma 2.5 we get that $i$ admits a right adjoint $R : K(A\mathcal{GP}) \to K(A\mathcal{P})$ which preserves coproducts. This means $K(A\mathcal{P})$ is a smashing subcategory of $K(A\mathcal{GP})$. This implies by Lemma 2.4 that $K(A\mathcal{GP})/K(A\mathcal{P})$ is a compactly generated triangulated category. \qed

3 Recollements for the Homotopy Category $K(A\mathcal{GP})$

In this section, let $A$ be an artin algebra. Based on the compact generatedness of the full subcategory $K(A\mathcal{P})$ of $K(A\mathcal{GP})$, we will apply the arguments of Neeman to prove the existence of a recollement of $K(A\mathcal{GP})$. 

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Lemma 3.1 [21, Theorem 4.1], [22, Theorem 8.6.1] Let $F : T \rightarrow T'$ be a triangle functor between triangulated categories $T$ and $T'$, where $T$ is compactly generated.

1. $F$ admits a right adjoint if and only if it preserves all coproducts.
2. $F$ admits a left adjoint if and only if it preserves all products.

Theorem 3.2 Let $A$ be an artin algebra. Then the inclusion functor $i : K(A\cdot P) \rightarrow K(A\cdot GP)$ induces a recollement of the form

$$
K(A\cdot P) \xleftarrow{i} K(A\cdot GP) \xrightarrow{R} \text{Ker} R
$$

such that $\text{Ker} R \cong K(A\cdot GP)/K(A\cdot P)$ as triangulated categories.

Proof Since $A$ is an artin algebra, it follows from [15, Theorem 2.4] that $K(A\cdot P)$ is compactly generated. Note that the inclusion functor $i$ naturally preserves all coproducts and products. Then $i$ admits a right adjoint $R$, also a left adjoint. Hence by [20, Theorem 2.2] we have a recollement of the form

$$
K(A\cdot P) \xleftarrow{i} K(A\cdot GP) \xrightarrow{R} \text{Ker} R
$$

such that $\text{Ker} R \cong K(A\cdot GP)/K(A\cdot P)$ as triangulated categories. □

So in view of the above theorem, we have the following result. Let us begin by recalling some definitions.

Let $T$ be a triangulated category with the suspension functor $\Sigma$. Recall from [5, Section 2] that a torsion pair in $T$ is a pair of strict full subcategories $(\mathcal{X}, \mathcal{Y})$ of $T$ satisfying the following conditions: (1) $T(\mathcal{X}, \mathcal{Y}) = 0$; (2) $\Sigma(\mathcal{X}) \subseteq \mathcal{X}$ and $\Sigma^{-1}(\mathcal{Y}) \subseteq \mathcal{Y}$; (3) For any $T \in T$ there exists a triangle $X_T \xrightarrow{fr} T \xrightarrow{g} Y_T \xrightarrow{h} \Sigma(X_T)$. Then $\mathcal{X}$ is called a torsion class and $\mathcal{Y}$ is called a torsion-free class. A torsion, torsion-free triple, TTF-triple for short, in $T$ is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories of $T$ such that the pairs $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs.

Now we give a TTF-triple in $K(A\cdot GP)$.

Corollary 3.3 Let $A$ be an artin algebra. Then there exists a TTF-triple $(K(A\cdot P), \text{Ker} R, (\text{Ker} R)\perp)$ in $K(A\cdot GP)$.

Proof By Theorem 3.2 we have the recollement of the form

$$
K(A\cdot P) \xleftarrow{i} K(A\cdot GP) \xrightarrow{R} \text{Ker} R.
$$

Hence by [20, Theorem 2.2] we get that $(K(A\cdot P), \text{Ker} R)$ and $(\text{Ker} R, (\text{Ker} R)\perp)$ are two torsion pairs in $K(A\cdot GP)$. This means $K(A\cdot GP)$ has a TTF-triple $(K(A\cdot P), \text{Ker} R, (\text{Ker} R)\perp)$. □

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