Gravitomagnetic Flux Quantization in Superconductors and a Method for the Experimental Detection of Gravitomagnetism in the Terrestrial Laboratory

Clovis J. de Matos
2, Nieuwsteeg 2311 SB Leiden The Netherlands
e-mail: cdematos@club-internet.fr

Robert E. Becker*
131 Old County Road, # 168 Windsor Locks, CT 06096 USA
e-mail: roberete.becker@cwix.com

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*Main contact
Abstract

It is extraordinarily difficult to detect the extremely weak gravitomagnetic (GM) field of even as large a body as the earth. To detect the GM field, the gravitational analog of an ordinary magnetic field, in a modest terrestrial laboratory should be that much more difficult. Here we show, however, that for certain superconductor configuration and topologies, it should be possible to detect a measurable GM field in the terrestrial laboratory, by using the properties of superconductors imposed by quantum mechanical requirements. In particular, we show that the GM Flux should be quantized in a superconductor with non-vanishing genus, just like the ordinary magnetic flux. And this magnetically induced, quantized GM Flux, for sufficiently high quantum number and favorable geometries, should be detectable, and distinguishable from the effects produced by an ordinary magnetic field.
Contents

1 Introduction 3

2 Weak Gravitational Fields & Gravitomagnetism 5
   2.1 Gravitational Scalar Potential 7
   2.2 Space Curvature 8
   2.3 Gravitomagnetic Vector Potential 8
   2.4 The Einstein-Maxwell-Type Gravitational Equations 9
   2.5 The Equations of Motion in the Weak Field Approximation 11
   2.6 From Electromagnetism to Gravity 13

3 Superconductivity 16
   3.1 Quantization of the Magnetic Flux 19

4 Quantization of the Gravitomagnetic Flux in Superconductors 23

5 Gravitomagnetic Flux Experiment 29
   5.1 Gravitomagnetic Flux Experiment Concept 29
   5.2 Gravitomagnetic Force Calculation 31

6 Conclusion 34

7 References 36
Chapter 1

INTRODUCTION

One of the predictions of General Theory of Relativity, unveiled eighty years ago, is the existence, in a linear approximation, of a gravitational analog of the magnetic field, known as the gravitomagnetic (GM) or prorotational field. In the form of the Lens-Thirring Effect, this aspect of the gravitational field has been studied extensively since then. Indeed, there are two large scale space missions, Gravity Probe-B and LAGEOS III, planned for launch near the turn of the millennium, designed to detect the exceedingly weak GM field produced by the rotation of the earth on its own axis.

It is the very weakness of the GM field of the entire earth that makes it so difficult, if not outright impossible, to measure the GM field in a terrestrial laboratory. Indeed, it has required extraordinary engineering ingenuity and precision to design gyroscopes sensitive enough to detect the $10^{-15} \text{Hz}$ precession induced by the earth at the orbit of Gravity Probe-B. Likewise, artificially generating a detectable GM field in the laboratory is clearly hopless, though one of us has elsewhere proposed a variant of a magnetic resonance experiment, utilizing scanning probe microscopies and nanotechnologies, to attempt this.

The present method for the experimental detection of gravitomagnetism (earlier versions are [1], [2]), proposes to overcome the problem of the weakness of the GM field by taking advantage of the unique properties of macroscopically coherent systems such as superconductors (SC). It is by now well known that the magnetic flux threading a superconductor is quantized with a quantum number $n$. We will show that the GM field enters superconducting flux relations in the same manner as the magnetic field, and that the GM flux should also be quantized. Furthermore, if the magnetic flux quantum number is fixed for a given physical configuration, that same quantum number governs the GM flux quantum as well. And this GM flux quantum, for sufficiently high $n$, should also be macroscopically detectable. This experiment is designed to detect the GM field represented by this quantized GM flux.

One obvious question that must be asked is "if the GM field of even a body as massive as earth is so tiny, how is it conceivable that a small SC can generate a field large enough to be readily measurable in the laboratory?" To address this, first note that in the presence of an external GM field there is an additional term in the usual quantized magnetic fluxoid relation. This GM contribution to the magnetic flux is correctly taken to be negligible because the external GM field is so tiny.
Similarly, there is a magnetic term in the quantized GM flux relation. For substantial magnetic fields, such as those that would result in a high magnetic flux quantum \( n \), this term is not small. Thus, a large GM flux quantum may be detected due to a magnetically-induced GM field in the SC. The exact physical mechanism responsible for this "amplification" is still nebulous. One of the objectives of this experimental method is to help elucidate such a mechanism. Initial steps in identifying possible theoretical mechanisms are outlined below in section 4.

It should be emphasized that a GM field large enough to be detectable in a terrestrial laboratory is predicted to occur only in macroscopic coherent systems such as superconductors or superfluids. (Though the effect should also occur in superfluids with a toroidal topology, the supercarriers in superfluids are much more massive than the Cooper pairs in SC, leading to a much smaller GM flux quantum than can be found in a SC).

Ultimately, the effect to be studied, as with so many unusual phenomenon, owes its postulated existence to the primacy of quantum mechanics, and the even more special properties of macroscopic, coherent systems.

Non-detection of the effect would actually raise equally important theoretical questions, such as: "If the Maxwell approximation to general relativity is correct, why is there not a GM flux quantum just as there is for the magnetic field?" After all, both vector potentials appear "symmetrically" in the quantum mechanical canonical momentum. "Could the \( U(1) \) gauge field and topological description of the GM field in the Maxwell approximation be wrong, even if the Maxwell approximation to the Einstein equations is correct? What is the fundamental topological relationship between magnetism, GM, and rotation?"
Chapter 2

WEAK GRAVITATIONAL FIELDS & GRAVITOMAGNETISM

In Einstein’s general theory of relativity there exist gravitational analogues of the electric and magnetic fields. We are already familiar with the analogy between the electric field of a charge and the gravitational field of a mass. It is well known that this analogy breaks down almost immediately, because there are two kinds of charge and one kind of mass and because two particles with the same type of charge repel, whereas particles with the same type of mass attract. Nevertheless, if we are cognizant of this distinction, we can still apply the analogy and still obtain useful results.

We will show that the Einstein field equations not only can be made to agree with this well known analogy between the electric and the gravitational field, but they can also give rise to a gravitational analog to the magnetic field. Its name is the Gravitomagnetic field. This gravitomagnetic field has the dimensions of \((\text{rad} \cdot s^{-1})\) in the MKS unit system (this unit system will be used in our discussion), and is closely related to Coriolis-type forces which arise from the principle of general relativity.

In deriving this analogy between some of the gravitational forces and the static and induction fields of electromagnetism, the following assumptions have been made:

1. The mass densities are positive.

2. All motions are much slower than the speed of light, so that special relativity can be neglected. (Often special relativistic effects will hide general relativistic effects).

3. The kinetic or potential energy of all the bodies being considered is much smaller than their mass energy.

4. The gravitational fields are always weak enough so that superposition is valid.

5. The distance between objects is not so large that we have to take retardation into account. (This can be ignored when we have a stationary problem where the fields have already been prescribed and are not changing with time).

The procedure for linearizing Einstein’s field equations is included in all texts on general relativity \[3\]. We start with Einstein field equation:

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}
\]  

(2.1)
Due to assumptions 2 and 4, the metric tensor can be approximated by

\[ g_{\alpha\beta} \simeq \eta_{\alpha\beta} + h_{\alpha\beta} \]  

(2.2)

where greek indices \( \alpha, \beta = 0, 1, 2, 3 \) and \( \eta_{\alpha\beta} = (+1, -1, -1, -1) \) is the flat spacetime metric tensor, and \( |h_{\alpha\beta}| << 1 \) is the perturbation to the flat metric. All the rationale that follows is correct up to the first order in \( |h_{\alpha\beta}| \). We assume that the \( h_{\mu\nu,\alpha\beta} \) are infinitesimally small up to first order; so \( R_{\alpha\beta} \) and \( R \) are also correct to first order. Therefore, we can consider \( g_{\alpha\beta} = \eta_{\alpha\beta} \). Using this form of the metric, the Ricci tensor can be calculated from the contraction of the Riemann tensor and the curvature scalar can be calculated from the contraction of the Ricci tensor

\[ R_{\alpha\beta} \simeq -\frac{1}{2} \Box h_{\alpha\beta} \]  

(2.3)

\[ R \simeq \eta^{\alpha\beta} R_{\alpha\beta} = \frac{1}{2} \Box h, \]  

(2.4)

where in obtaining (2.3) and (2.4) we choose our coordinate system so that we have the following "gauge" condition

\[ \left[ h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h \right]_{,\beta} = 0. \]  

(2.5)

If we substitute the Ricci tensor (2.3) and the curvature scalar (2.4) into Einstein’s equations, we obtain

\[ -\frac{1}{2} \Box h_{\alpha\beta} + \frac{1}{4} \eta_{\alpha\beta} \Box h = \frac{8\pi G}{c^4} T_{\alpha\beta} \]  

(2.6)

We now define the gravitational potential as

\[ \overline{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2} \eta_{\alpha\beta} h; \]  

(2.7)

substituting equation (2.7) into equation (2.6) and rearranging, we get

\[ \Box \overline{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}. \]  

(2.8)

If we write out the D’Alembertian operator, we have

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \overline{h}_{\alpha\beta} - \nabla^2 \overline{h}_{\alpha\beta} = -\frac{16\pi G}{c^4} T_{\alpha\beta}. \]  

(2.9)

This is the basic equation upon which the analogies between electromagnetism and gravity are based.

When spacetime is highly dynamical, for example around two colliding black holes, there is no natural, preferred way to split spacetime into space plus time. This fact has driven relativists, beginning with Einstein, to describe gravity in terms
of a unified, four dimensional spacetime with dynamical evolving four dimensional curvature.

On the other hand, astrophysicists and experimental physicists usually deal with situations where spacetime is stationary rather than dynamical, for example the spacetime around the earth, or around a quiescent black hole. In such cases stationarity dictates a preferred way to slice spacetime into three-dimensional space plus one-dimensional time [4]. Although such ”3+1 split” are rarely treated in standard textbooks on general relativity, they are used widely by professional relativists in numerical solutions of the Einstein field equations, in the quantization of general relativity, in astrophysical studies of black holes, and in analysis of laboratory experiments to test general relativity.

The 3+1 split regards three dimensional space as curved rather than Euclidean; its metric $g_{ik}$ (in an appropriate coordinate system) is just the spatial part of the spacetime metric $g_{\alpha \beta}$. In this curved three space reside two gravitational potentials: a ”gravitoelectric” scalar potential $\Phi$, which is essentially the time-time part $g_{00}$ of the space-time metric; and a ”gravitomagnetic” vector potential $\vec{A}_g$, which is essentially the time-space part $g_{0j}$ of the space-time metric. The decomposition of $g_{\alpha \beta}$ into $g_{ik}$, $\Phi$ and $\vec{A}_g$ is analogous to the decomposition of the four vector potential $A_\alpha$ into an electric scalar potential $\Psi = A_0$ and a magnetic vector potential $\vec{A} = A_j$.

2.1 Gravitational Scalar Potential

In the first approximation (zero order for the energy momentum tensor), we assume that all quantities are not varying with time. Then the time derivative of the gravitational potential is zero and all components of the energy-momentum tensor are zero except

$$T_{00} = \mu c^2.$$  \hspace{1cm} (2.10)

Equation (2.9) reduces to

$$-\Delta \overline{h}_{00} = -\frac{16\pi G}{c^2}\mu,$$  \hspace{1cm} (2.11)

which is essentially the Poisson equation, which has the solution

$$\overline{h}_{00} = -\frac{4G}{c^2}\int\int\int_V \frac{\mu}{r}dV.$$  \hspace{1cm} (2.12)

If we define the gravitational permittivity of the vacuum as

$$\gamma = \frac{1}{4\pi G},$$  \hspace{1cm} (2.13)

we get

$$\frac{c^2\overline{h}_{00}}{4} = -\frac{1}{4\pi\gamma}\int\int\int_V \frac{\mu}{r}dV.$$  \hspace{1cm} (2.14)
Comparing equation (2.14) with the scalar potential of an electric charge density

\[ \varphi = -\frac{1}{4\pi\varepsilon_0} \int \int \int_V \frac{\rho}{r} dV \]  

we see that we can construct the well known gravitational analog to the scalar potential:

\[ \Phi = \frac{c^2 \overline{h}_{00}}{4} = \frac{c^2 (g_{00} - 1)}{2}. \]  

2.2 Space Curvature

This first approximation (2.11) also determines the spatial metric. The existence of the component \( \overline{h}_{00} \) results in a relativistic interval of the form of the Schwarzschild metric (using a cartesian coordinate system \( x^1 = x, x^2 = y, x^3 = z \))

\[ ds^2 = \left( 1 + \frac{2\Phi}{c^2} \right) c^2 dt^2 - \left( 1 - \frac{2\Phi}{c^2} \right) \left( dx^2 + dy^2 + dz^2 \right). \]  

Thus, the three dimensional spatial metric will be of the form

\[ g_{ab} = \begin{bmatrix} -\left( 1 - \frac{2\Phi}{c^2} \right) & 0 & 0 \\ 0 & -\left( 1 - \frac{2\Phi}{c^2} \right) & 0 \\ 0 & 0 & -\left( 1 - \frac{2\Phi}{c^2} \right) \end{bmatrix}. \]  

In higher approximations, the additional terms in the spatial metric will be smaller than \( 2\Phi/c^2 \) by the order of \( (v/c)^2 \), and since we assume velocities much smaller than the speed of light, they will be of little experimental interest here.

2.3 Gravitomagnetic Vector Potential

In the next higher approximation (first order approximation for the energy-momentum tensor), we still assume that the potential is not varying with time, but that the masses involved have appreciable velocity or rotation. Then the energy-momentum tensor will have the components

\[ T_{00} = \mu c^2 \]  

and

\[ T_{0i} = -\mu c v_i. \]  

We then have four equations remaining: One gives us the scalar potential obtained previously, and the other three are

\[ -\Delta \overline{h}_{0i} = \frac{16\pi G}{c^3} \mu v_i. \]
This equation has the solution
\[ \bar{h}_{0i} = \frac{4G}{c^3} \iiint_{V} \frac{\mu v_i}{r} dV. \] (2.22)

If we define a gravitational permeability of space by
\[ \eta = \frac{4\pi G}{c^2} \] (2.23)
then we can substitute (2.23) into (2.22) and rearrange to get
\[ -\frac{c\bar{h}_{0i}}{4} = -\frac{\eta}{4\pi} \iiint_{V} \frac{\mu v_i}{r} dV. \] (2.24)

Thus, we can identify a mass density flow \[ \overrightarrow{p} = \mu \overrightarrow{v} \] as a gravitational equivalent to the magnetic vector potential whose components are the three components
\[ A_{gi} = -\frac{c}{4} \bar{h}_{0i} = -\frac{c}{4} g_{0i} \] (2.25)
(the factor of 4 appearing in the denominator of equation (2.25) will be explained when we will reach equation (2.59)) and thereby arrive at the isomorphism of the equations
\[ \overrightarrow{A}_{gi} = -\frac{\eta}{4\pi} \iiint_{V} \frac{\overrightarrow{p}}{r} dV \quad \text{and} \quad \overrightarrow{A} = +\frac{\mu_0}{4\pi} \iiint_{V} \frac{\overrightarrow{j}}{r} dV. \] (2.26)

2.4 The Einstein-Maxwell-Type Gravitational Equations

Let us change the linearized Einstein field equations into the form of Maxwell Equations [6]
\[ -\frac{1}{2} \left( h_{\alpha\beta,\mu} + \eta_{\alpha\beta} h_{\mu\nu} - h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha} \right) = \frac{8\pi G}{c^4} T_{\alpha\beta} \] (2.27)
\[ \frac{1}{4} \frac{\partial}{\partial x^\mu} \left( h_{\alpha\beta,\mu} - h_{\alpha\mu,\beta} + \eta_{\alpha\beta} h_{\mu\nu} - \eta_{\alpha\mu} h_{\beta\nu} \right) = -\frac{4\pi \mu_0}{c^4} T_{\alpha\beta}. \] (2.28)

For the sake of convenience we have introduced the tensor
\[ G_{\alpha\beta\mu} = \frac{1}{4} \left( h_{\alpha\beta,\mu} - h_{\alpha\mu,\beta} + \eta_{\alpha\beta} h_{\mu\nu} - \eta_{\alpha\mu} h_{\beta\nu} \right). \] (2.29)

Due to the "gauge condition" (2.3)
\[ \bar{h}_{\mu\nu} = 0 \] (2.30)
The Einstein-Maxwell-Type Gravitational Equations

The tensor $G_{\alpha\beta\mu}$ can be simplified to

$$G_{\alpha\beta\mu} = \frac{1}{4} (\overline{r}_{\alpha\beta,\mu} - \overline{r}_{\alpha\mu,\beta}) .$$

(2.31)

This tensor has the following properties

$$G^{\alpha\beta\mu} = -G^{\alpha\mu\beta}$$

(2.32)

$$G^{\alpha\beta\mu} + G^{\mu\alpha\beta} + G^{\beta\mu\alpha} = 0$$

(2.33)

$$G^{\alpha\beta\mu,\lambda} + G^{\alpha\lambda\beta,\mu} + G^{\alpha\mu\lambda,\beta} = 0.$$  

(2.34)

With the help of the tensor $G_{\alpha\beta\mu}$ the linearized Einstein field equations (2.8) become

$$\frac{\partial G^{\alpha\beta\mu}}{\partial x^{\mu}} = -\frac{4\pi G}{c^4} T^{\alpha\beta} .$$

(2.35)

Introducing the gravitational scalar potential and the GM vector potential we obtain equations (2.16) and (2.25), respectively

$$\Phi = \frac{c^2 h^{00}}{4}$$

(2.36)

$$A_g^i = -\frac{c}{4} h^{0i} \quad \overrightarrow{A}_g = (A_g^1, A_g^2, A_g^3) .$$

(2.37)

Introduce new signs, and substitute equation (2.36) into equation (2.31), to get the gravitoelectric field

$$G^{00i} = \frac{1}{4} \left( \overline{h}^{00,i} - \overline{h}^{0i,0} \right)$$

(2.38)

$$g^i = -c^2 G^{00i} = \frac{\partial \Phi}{\partial x^i} - \frac{\partial A_{g_i}}{\partial t}$$

(2.39)

$$\overrightarrow{g} = -\nabla \Phi - \frac{\partial \overrightarrow{A}_g}{\partial t} .$$

(2.40)

Substitute equation (2.37) into equation (2.31) to get the gravitomagnetic field

$$c G^{0ij} = A_g^i - A_g^j$$

(2.41)

or

$$\overrightarrow{B}_g = \nabla \wedge \overrightarrow{A}_g .$$

(2.42)

Using the expressions for the gravitoelectric and GM fields given as functions of tensor $\overline{r}^{\alpha\beta}$ derivatives, we can obtain from equation (2.35) the "Gauss law for the gravitoelectric field" and the "gravitational Ampere law"; and from equation (2.34) we get the "Gauss law for the gravitomagnetic field" and the "Faraday induction law"
for the gravitoelectric field. These four laws constitute the four Einstein-Maxwell-type gravitational equations, i.e.,

\[ \nabla \vec{g} = -4\pi G\mu \quad (2.43) \]

\[ \nabla \vec{B} = 0 \quad (2.44) \]

\[ \nabla \wedge \vec{g} = -\frac{\partial \vec{B}}{\partial t} \quad (2.45) \]

\[ \nabla \wedge \vec{B} = -\frac{4\pi G}{c^2} \mu \vec{v} + \frac{1}{c^2} \frac{\partial \vec{g}}{\partial t}. \quad (2.46) \]

### 2.5 The Equations of Motion in the Weak Field Approximation

In the four-dimensional equation of motion

\[ \frac{d\mu^\lambda}{d\tau} + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu = 0, \quad (2.47) \]

the gravitational effects are entirely in the metric tensor. The only forces explicitly stated are nongravitational forces. This four-dimensional equation of motion can be broken down and arranged so that it is a three-dimensional curvilinear spatial equation of motion. The gravitational effects resulting from the temporal components of the metric tensor are represented as forces due to a gravitational scalar and a gravitational vector potential. The spatial components of the metric tensor are used as the three-dimensional metric tensor.

The general equation of motion for a particle with only gravitational forces acting is given by Moller [7] as

\[ \frac{dP_\alpha}{d\tau} - \frac{1}{2} \frac{\partial g_{\mu\nu}}{\partial x^\alpha} U^\mu P^\nu = 0, \quad (2.48) \]

where \( d\tau = ds/c \),

\[ P_\alpha = mU_\alpha = m_0 g_{\alpha\lambda} U^\lambda = m_0 g_{\alpha\lambda} \frac{dx^\lambda}{dt}, \quad (2.49) \]

and

\[ \Gamma = \frac{dt}{d\tau} = \left[ 1 + \frac{2\Phi}{c^2} - \left( \frac{v}{c} \right)^2 - \frac{8}{c^2} A_g v^i \right]^{-1/2}. \quad (2.50) \]

Note that if \( \Phi = \vec{A}_g = 0 \) then we recover the usual relativistic correction:

\[ \Gamma = \frac{1}{\sqrt{1 - \left( \frac{v}{c} \right)^2}}. \quad (2.51) \]
Rearranging equation (2.48) we have

\[ \frac{1}{\Gamma} \frac{d}{dt} \left( g_{\alpha \lambda} \Gamma \frac{dx^\lambda}{dt} \right) = \frac{1}{2} \frac{\partial g_{\mu \nu}}{\partial x^\alpha} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} \]  

(2.52)

The \( \alpha = 0 \) equation gives us the conservation of mass-energy:

\[ \frac{1}{\Gamma} \frac{d}{dt} \left[ \Gamma \left( g_{00} \frac{dx_0}{dt} + g_{0i} \frac{dx^i}{dt} \right) \right] = \frac{1}{2} \frac{\partial g_{ii}}{\partial x^0} \left( \frac{dx^i}{dt} \right)^2 + \frac{1}{2} \frac{\partial g_{00}}{\partial x^0} \left( \frac{dx^0}{dt} \right)^2 + \frac{\partial g_{0i}}{\partial x^0} \left( \frac{dx^0}{dt} \right) \frac{dx^i}{dt} \]  

(2.53)

Letting \( dx^0/dt = c \) and considering \( \Gamma \simeq 1 \) and \( g_{ii} = (-1, -1, -1) \), we get:

\[ \frac{d}{dt} \left( cg_{00} + g_{0i} \frac{dx^i}{dt} \right) = \frac{c}{2} \frac{\partial g_{00}}{\partial t} + \partial g_{0i} \frac{dx^i}{dt} \]  

(2.54)

We then use our definition of the gravitational and vector potential

\[ g_{00} = 1 + \frac{2\Phi}{c^2} \quad \text{and} \quad g_{0i} = -\frac{4}{c} \mathbf{A}_g \]  

(2.55)

to get

\[ \frac{d}{dt} \left( \Phi - 4A_{g_i} v^i \right) = g_{i} v^i \]  

(2.56)

\[ - \frac{d}{dt} \left( m\mathbf{v} \right) + \frac{d}{dt} \left( 4m \mathbf{v} \cdot \mathbf{A}_g \right) = m \mathbf{g} \cdot \mathbf{v} \]  

(2.57)

From equation (2.57) we conclude that the gravitational potential energy and the gravitomagnetic potential energy are respectively given by:

\[ U_{grav} = m\Phi \]  

(2.58)

and

\[ U_{gm} = -\frac{1}{4} m \mathbf{v} \cdot \mathbf{A}_g. \]  

(2.59)

The factor 4 appearing in equations (2.59), (2.23), (2.64) and (2.63), is presumably due to gravity being associated with a spin-2 (graviton) field rather than a spin-1 field (photon).

The other three equations for \( \alpha = i = 1, 2, 3 \) equation (2.52) are the equations of motion:

\[ \frac{1}{\Gamma} \frac{d}{dt} \left[ \Gamma \left( g_{ii} \frac{dx^i}{dt} + g_{ij} \frac{dx^j}{dt} \right) \right] = \frac{1}{2} \frac{\partial g_{ii}}{\partial x^i} \left( \frac{dx^i}{dt} \right)^2 + \frac{1}{2} \frac{\partial g_{00}}{\partial x^0} \left( \frac{dx^0}{dt} \right)^2 + \frac{\partial g_{0i}}{\partial x^0} \left( \frac{dx^0}{dt} \right) \frac{dx^i}{dt} \]  

(2.60)

Putting \( dx^0/dt = c \) and considering \( \Gamma \simeq 1 \) and \( g_{ii} = (-1, -1, -1) \), we get:

\[ \frac{d}{dt} \left( g_{ij} \frac{dx^j}{dt} \right) = \frac{c^2}{2} \frac{\partial g_{00}}{\partial x^i} - \frac{c}{2} \frac{\partial g_{0i}}{\partial t} + c \left( \frac{\partial g_{0j}}{\partial x^i} - \frac{\partial g_{0i}}{\partial x^j} \right) \frac{dx^j}{dt}. \]  

(2.61)
We then use our definition of the gravitational and vector potential

\[ g_{00} = 1 + \frac{2\Phi}{c^2} \quad \text{and} \quad g_{0i} = -\frac{4}{c^2} A_{gi}, \quad (2.62) \]

to get:

\[ \frac{d}{dt} \left( g_{ij} \frac{dx^j}{dt} \right) = \frac{\partial \Phi}{\partial x^i} + 4 \frac{\partial \vec{A}_g}{\partial t} - 4 \left( \vec{v} \wedge \vec{B}_g \right)_i. \quad (2.63) \]

Rearranging we have:

\[ \frac{d^2 x^i}{dt^2} = \left( -\frac{\partial \Phi}{\partial x^i} - 4 \frac{\partial \vec{A}_g}{\partial t} \right) + \frac{4}{c^2} \left( \vec{v} \wedge \vec{B}_g \right)_i. \quad (2.64) \]

Using equation (2.40), we obtain

\[ m \vec{a} = m \left( \vec{g} - 3 \frac{\partial \vec{A}_g}{\partial t} \right) + 4m \vec{v} \wedge \vec{B}_g. \quad (2.65) \]

The left hand side of equation (2.64) is the total acceleration of the particle. The first term on the right hand side is one component of \( \nabla \Phi \), the gravitational static attraction; the second term is one component of \( \partial \vec{A}_g / \partial t \), the gravitational induction effect; and the third term is one component of \( \vec{v} \wedge (\nabla \wedge \vec{A}_g) \) the gravitational equivalent of the Lorentz force.

### 2.6 From Electromagnetism to Gravity

When the analogy between electrodynamics and gravity is carried out and all the constants are evaluated, we obtain an isomorphism between the gravitational and the electromagnetic quantities.
| EM Symbol | Gravitational Symbol | Value or definition |
|-----------|----------------------|---------------------|
| $\vec{E}$ | $\vec{g}$ | $= -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$ |
| $\vec{B}$ | $\vec{B}_g$ | $= \nabla \wedge \vec{A}_g$ |
| $\varphi$ | $\Phi$ | $\simeq -\frac{1}{4\pi\gamma} \iiint_V \frac{\mu}{r} dV$ |
| $\vec{A}$ | $\vec{A}_g$ | $\simeq -\frac{\eta}{4\pi} \iiint_V \frac{\mu v}{r} dV$ |
| $\rho$ | $\mu$ | $= \frac{dM}{dV}$ |
| $Q$ | $M$ | $= \iiint_V \mu dV$ |
| $\vec{j}$ | $\vec{p}$ | $= \mu \vec{v}$ |
| $I$ | $\vec{M}$ | $= \frac{dM}{dt} = \iiint_S \vec{p} \cdot \hat{n} dS$ |
| $\vec{M}$ | $\vec{J} = \frac{1}{2} \vec{L}$ | $= \frac{1}{2} I \omega$ |
| $\varepsilon$ | $\gamma$ | $= \frac{1}{4\pi G} = 1.19 \times 10^9 [kg.s^2/m^3]$ |
| $\mu$ | $\eta$ | $= \frac{4\pi \varepsilon_0}{\gamma} = 9.33 \times 10^{-27} [m/kg]$ |

Table 1: From electromagnetism to gravity

The ordinary electromagnetic (EM) magnetic flux density $\vec{B}$ can be defined by its torque on a magnetic moment

$$\vec{N} = \vec{M} \wedge \vec{B}$$  \hspace{1cm} (2.66)

where $\vec{N}$ is the torque and $\vec{M}$ the magnetic moment. Since torque is defined to be the time rate of change of angular momentum $\vec{L}$, and since $\vec{M}$ is related to $\vec{L}$ in many classical systems by

$$\vec{M} = \frac{q}{2m} \vec{L},$$  \hspace{1cm} (2.67)

where $q$ is the charge and $m$ the mass of the particle, we can rewrite (2.66) as

$$\frac{d\vec{L}}{dt} = \frac{q}{2m} \vec{L} \wedge \vec{B}.$$  \hspace{1cm} (2.68)

It is well known that (2.68) represents the precession of the angular momentum $\vec{L}$ about the direction of $\vec{B}$ at a Larmor frequency of

$$\vec{\omega} = \frac{q}{2m} \vec{B}.$$  \hspace{1cm} (2.69)
Just as a spinning magnetic moment generates a magnetic field, solutions of the Einstein Equations show that a spinning mass generates a dipolar GM field $\vec{B}_g$. In this case, the GM equivalent of the magnetic moment is the angular momentum $\vec{L}$, so that the GM torque on a spinning body can be written as

$$\vec{N} = 2 \vec{L} \wedge \vec{B}_g. \quad (2.70)$$

Comparing (2.70) with (2.68) we conclude

$$\frac{d \vec{L}}{dt} = 2 \vec{L} \wedge \vec{B}_g. \quad (2.71)$$

Thus, there is a GM precessional frequency of

$$\vec{\omega}_g = 2 \vec{B}_g. \quad (2.72)$$

It is $\vec{\omega}_g$ that we will attempt to detect in our experiment.

It should be emphasized that the previous discussion is approximate and is presented merely to provide a simple tool with which to make estimates.
Superconductors (SC) have many curious characteristics [9], [10]. They are perfect conductors, so that once a current is started for any reason, it will continue. But they also obey the Meissner effect by which a magnetic field is excluded from the interior of a bulk SC, even in an applied $\vec{B}$ field. Physically, this comes about because the $\vec{B}$ field induces surface (not bulk) currents, which set up a field opposing the applied field in the interior; the net field is $\vec{B} = 0$. These currents only flow in a layer, the London penetration depth, in which $\vec{B}$ and the current $\vec{j}$ fall-off exponentially with distance into the SC.

All the strange properties of SC ultimately derive from the quantum mechanical (QM) state of the system. Below a certain critical temperature, it becomes energetically favorable for conduction electrons in the material to form pairs such that their center of mass momentum is zero; these are Cooper pairs. At absolute zero, all the electrons in the SC form Cooper pairs, and so condense into the QM state of zero momentum.

Since electrons have a QM spin of 1/2, and since opposite spin electrons are paired, a Cooper pair is a QM object of integral spin, called a boson. Unlike fermions, which obey the Pauli Exclusion Principle, any number of bosons can occupy the same QM state. So below the critical temperature, $T_c$, the SC can be said to be in a macroscopically occupied QM state. In order to change the state, a physical phenomena would have to break all the Cooper pairs and remove the electrons from this state, known as a condensate. This requires much more energy than just breaking a single Cooper pair, and so changing QM states becomes exceedingly unlikely.

This explains properties like persistent currents. They persist, because once the QM state is set-up, it can not easily be changed. There are complications to the simple picture just described. For example, persistant London screening currents must be generated by an applied field, and then the Cooper pairs do not have zero center of mass momentum. But they do have zero QM canonical momentum, which is the correct, vanishing quantity in this case.

At $T > 0$, not all the electrons form Cooper pairs, but the ground state is still macroscopically occupied. For our purposes, this will suffice. Another complication is the existence of several types of SC. The first to be discovered was Type I SC, which obey the Meissner effect for all $B$ and all $T < T_c$. Next came Type II SC,
which obey Meissner effect only for \( B < B_{c1} \), the lower Critical Field. Between \( B_{c1} \) and \( B_{c2} \) (Upper Critical Field), an applied magnetic field can penetrate the SC and form current vortices around the field lines.

Finally, there are the famous high-\( T_c \), SC. High-\( T_c \) SC do not obey the same microscopic QM theory as do standard Type I and Type II SC (s-wave BCS theory), but for our purposes can be regarded as Type II SC: they have a \( B_{c1} \). In the non-Meissner regime of the Type II SC \([9]\), the critical property is that the magnetic field can penetrate in a vortex. The vortex core is not SC, and is composed of normal material; that is, electrons not paired into Cooper pairs. Topologically, this has enormous importance, because it means the SC material is no longer simply-connected; it now has a hole in it. The same thing can happen to any SC if one drills a hole into an otherwise bulk SC, creating a ring or doughnut shaped object.

When we regard the whole SC as being in a condensate state, we can describe it by a single, condensate wave function and the Schrödinger equation for a Cooper pair will be similar to the following equation \([15]\):

\[
-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \hat{H} \psi = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q \vec{A} \right)^2 \psi + q \varphi \psi.
\] (3.1)

This is the Schrödinger equation for a particle with electric charge \( q \) and mass \( m \) moving in an electromagnetic field. In equation (3.1), \( \varphi \) is the electrostatic potential and \( \vec{A} \) is the magnetic vector potential. Note that \( \varphi \) and \( \vec{A} \) are applied potentials on, not the field generated by, the moving charge, \( q \).

Notice that in the case of the Cooper pairs the charge \( q \) and the mass \( m \) that appear in equation (3.1) are respectively, twice the electron’s charge and twice the electron’s mass.

As we said above, the Cooper pairs are bosons, therefore the amplitude to find almost all the Cooper pairs present in a SC in the same QM state is very large. In an ideal, non-interacting system, the pairs will be in the state of lowest energy, the ground state.

Let \( \psi \) be the wave function of a Cooper pair in the state of lowest energy. The probability to find the pair in this state will be proportional to the electric charge density \( \rho \)

\[
\psi \psi^* \propto \rho.
\] (3.2)

Therefore, we can write the wave function in the following form:

\[
\psi (\vec{r}) = \sqrt{\rho (\vec{r})} e^{i \theta (\vec{r})},
\] (3.3)

where \( \rho \) and \( \theta \) are real functions of \( \vec{r} \). What is the physical meaning of the phase \( \theta \) in the wave function?
To answer this question we have to write down the current density for a charged particle moving in an EM field

\[ \vec{j} = \frac{1}{2} \left\{ \left[ \frac{\hat{P} - q \vec{A}}{m} \right] \psi^* \psi + \psi^* \left[ \frac{\hat{P} - q \vec{A}}{m} \psi \right] \right\} \]  

(3.4)

where \( \hat{P} = \frac{\hbar}{i} \nabla \) is the linear momentum operator. Putting equation (3.3) into (3.4), we have the following equation for the current:

\[ \vec{j} = \frac{\hbar}{m} \left( \nabla \theta - \frac{q}{\hbar} \vec{A} \right) \rho. \]  

(3.5)

Inserting \( \vec{j} = \rho \vec{v} \) into equation (3.5) we obtain:

\[ \hbar \nabla \theta = m \vec{v} + q \vec{A}. \]  

(3.6)

Therefore, the canonical momentum \( \vec{P} \) for a Cooper pair is given by:

\[ \vec{P} = \hbar \nabla \theta. \]  

(3.7)

As \( \vec{j} \) and \( \rho \) have a direct physical meaning, \( \rho \) and \( \theta \) are real quantities.
3.1 Quantization of the Magnetic Flux

Let us consider a conducting ring in a magnetic field.

![Image](image_url)

Figure 1: a ring in an external magnetic field, (a) in the normal state, (b) in the superconductive state, (c) when the external magnetic field has been removed.

In the normal state there is a magnetic field inside the ring (figure 1-(a)). When the ring becomes superconducting (at the critical temperature $T_c$), the magnetic field is expelled outside the material. At this moment there will be a given magnetic flux through the ring’s hole (figure 1-(b)). If we now remove the external applied magnetic field, the lines of the magnetic field that cross the ring’s hole will be trapped (figure 1-(c)). The magnetic flux $\Phi_m$ inside the central hole can not decrease, because $\frac{\partial \Phi_m}{\partial t}$ must be equal to the line integral of $\vec{E}$ along a closed path inside the ring. This integral is zero inside a superconductor. When we remove the external field, a circular supercurrent is established through the surface of the ring to maintain the magnetic flux constant through the ring’s hole.

Whereas screening, diamagnetic, currents are always set-up around the outer periphery of a SC to cancel an applied field in the interior (see figure 1-(b)), if a hole is present, another current, called paramagnetic current, is set-up around the inner circumference of the hole (see figure 1-(c)). The paramagnetic current ensures that once a SC is cooled below $T_c$ in an applied field, the magnetic flux penetrating the hole remains constant even if the applied field is removed.
Inside the ring the current density is zero. Therefore from the current equation (3.5) we have:

\[ \hbar \nabla \theta = q \vec{A}. \]  

(3.8)

Now we take the integral of \( \vec{A} \) over a closed path \( \Gamma \) (see figure 2) that goes around the ring hole and inside the central region of the SC bulk, in a way that it will never get close to the surface of the ring.

![Figure 2: the path \( \Gamma \) inside the superconductive ring](image)

We write this integral in the following way:

\[ q \oint _{\Gamma} \vec{A} \cdot d\vec{s} = \int _{\Sigma} \text{curl} \vec{A} \cdot d\vec{\sigma}. \]  

(3.9)

Using the Stokes Theorem, equation (3.9) becomes

\[ q \oint _{\Gamma} \vec{A} \cdot d\vec{s} = \iint _{\Sigma} \vec{B} \cdot d\vec{\sigma}. \]  

(3.10)

From the laws of electromagnetism we have:

\[ \vec{B} = \nabla \wedge \vec{A}. \]  

(3.11)

Therefore, from equations (3.10) and (3.11) we find,

\[ q \oint _{\Gamma} \vec{A} \cdot d\vec{s} = \iint _{\Sigma} \vec{B} \cdot d\vec{\sigma} = \Phi _m \]  

(3.12)
\( \Phi_m \) is the magnetic flux through the superconducting ring’s hole.

Placing equation (3.12) into equation (3.9) we get

\[
\hbar \oint_{\Gamma} \nabla \cdot d\vec{s} = q \oint_{\Sigma} \vec{B} \cdot d\vec{\sigma} = q\Phi_m.
\] (3.13)

As the integral of the gradient of a function between two points, is equal to the difference between the values that the function will assume at those points we can write

\[
\oint_{\Gamma} \nabla \cdot d\vec{s} = \theta_2 - \theta_1.
\] (3.14)

What happens if the point (1) \( \equiv \) point (2), i.e., when we go around a closed path inside the ring? The value of \( \theta \) after a complete roundtrip must give the same value when inserted into the wave function,

\[
\psi(\vec{r}) = \sqrt{\rho(\vec{r})} e^{i\theta(\vec{r})}.
\] (3.15)

This occurs if \( \theta \) changes by \( 2\pi n \) with \( n \in \mathbb{N} \). Therefore,

\[
\oint_{\Gamma} \nabla \cdot d\vec{s} = 2\pi n.
\] (3.16)

Inserting equation (3.16) into equation (3.13) we see immediately that the magnetic flux trapped inside the ring’s hole will be quantized.

\[
\frac{2\pi \hbar}{q} = \Phi_m
\] (3.17)

The magnetic flux quantum trapped inside the ring has the value:

\[
\Phi_m = \frac{\hbar}{2e}.
\] (3.18)

\( q = 2e \) because in a superconductor we have Cooper pairs as mentioned above.

Recognizing that in a SC, the supercurrent \( \vec{j} \) is proportional to the (ordinary) momentum (3.6) of Cooper pairs of charge \( 2e \) and mass \( 2m \), we have

\[
n \frac{\hbar}{2e} = n\Phi = \frac{m}{de^2} \oint_{\Gamma} \vec{j} \cdot d\vec{s} + \oint_{\Sigma} \vec{B} \cdot d\vec{\sigma}
\] (3.19)

where \( d \) in the denominator is the density of superconducting electrons, while \( d\vec{s} \) denotes a line integral around a closed contour and \( d\vec{\sigma} \) denotes a surface integral over a closed contour. The contour \( \Gamma \) is now (unlike in equation (3.13)) a general contour,
and is not necessarily in the bulk of the SC where $\vec{j} = 0$ always. Equation (3.19) defines the Magnetic Fluxoid quantum $\Phi$ on the left hand side. The left hand side is called the fluxoid rather than the flux because it is the sum of a flux term and another term. Since the first term on the right hand side is a line integral, and since the current $\vec{j}$ vanishes deep inside the SC, for any contour in the bulk SC, this term can be neglected, and we can say that the magnetic flux, defined by the second term right hand side, is quantized.

Notice that it is the flux that is quantized, not the field.

Mathematically, the vanishing of the first term of (3.19) makes perfect sense. However, it is a little more difficult to interpret in term of physical measurements. To do that, we will rewrite the first term by the Stokes theorem to obtain

$$n \frac{\hbar}{2e} = n\Phi = \frac{m}{d\varepsilon^2} \int \int_{\Sigma} (\nabla \times \vec{j}) \, d\sigma + \int \int_{\Sigma} \vec{B} \cdot d\sigma.$$  \hspace{1cm} (3.20)

The screening layer with the current is contained in the surface integral when the contour is in the bulk SC, so we interpret (3.20) to mean that an experiment measures the TOTAL magnetic field due to the flux through the hole (second term) PLUS the magnetic flux generated by the London current layer around the hole (first term). (In a SC, $\nabla \times \vec{j}$ is proportional to the magnetic field by the London Equations). Since that screening current is precisely the current necessary to establish and maintain the flux through the hole at a quantized value according to theory, experiments confirmed the theory when they detected a quantum of flux given by the left hand side. Indeed, if the applied field had a flux of, say, $(n + \frac{1}{4})\Phi$, a current is set up to move the trapped flux in the hole to an even quantized value of $n\Phi$. There is also a very small magnetic flux contribution from the penetration of the external field into the London layer around the hole (from which it is not screened, unlike the rest of the SC). This would show up as a slight deviation from the quantized value.

If the SC is very thin, such that its thickness is no greater than the penetration depth, effectively, there is no bulk SC, and the contour goes right through the London layer around the hole. The interpretation of this in light of (3.20) is that the London current layer is no longer inside the integration surface area and cannot contribute. Therefore, the only term left is the second, with the result that a measurement now sees a deficit in the flux, since the first term has effectively been excluded. This deficit was also observed in experiments.
Chapter 4
QUANTIZATION OF THE GRAVITOMAGNETIC FLUX IN SUPERCONDUCTORS

Not long after the detection of magnetic flux quantization in SC [16], physicists began to examine the role of GM in SC, recognizing the analogy between the magnetic and GM vector potentials [17], [18]. Ross [20] concluded that in the presence of a GM field, there is not a pure Meissner effect inside a bulk SC. Li and Torr first extended this result [21] and then proposed [19] a novel GM amplification mechanism arising from lattice ion contributions and modification of the usual EM constitutive relations.

To include the contribution of the gravitomagnetic flux we need to write down the Hamiltonian of an electrically charged particle with mass, moving simultaneously in an electromagnetic, a gravitomagnetic, and gravitational fields [20].

\[
\hat{H} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q \vec{A} - 4m \vec{A}_g \right)^2 + q\varphi - m\Phi. \tag{4.1}
\]

Inserting equation (4.1) into the Schrödinger equation \(-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \hat{H} \psi\) we get:

\[
-\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla - q \vec{A} - 4m \vec{A}_g \right)^2 \psi + q\varphi \psi - m\Phi \psi. \tag{4.2}
\]

We see clearly from this equation that the Cooper pair canonical momentum is:

\[
\vec{P} = \hbar \nabla \theta = m \vec{v} - q \vec{A} - 4m \vec{A}_g. \tag{4.3}
\]

The solution of equation (4.2) is the wave equation

\[
\psi (\vec{r}) = \sqrt{\rho(\vec{r})} e^{i\theta(\vec{r})}. \tag{4.4}
\]

Notice that this equation can also be expressed as a function of the Cooper pair mass density

\[
\psi (\vec{r}) = \frac{e}{m} \sqrt{\mu(\vec{r})} e^{i\theta(\vec{r})}. \tag{4.5}
\]

Placing (4.4) into (4.2) we will obtain the equations that govern the dynamical behavior of \(\rho\) and \(\theta\). We have to keep in mind that \(\rho (x, y, z)\) and \(\theta (x, y, z)\) are functions of \(x, y\) and \(z\). Separating the real part from the imaginary part we obtain the following couple of equations:
1. The continuity equations for electric charge and mass govern the behavior of \( \rho \) and \( \mu \)

\[
\frac{\partial \rho}{\partial t} = \nabla \cdot \rho \vec{v} = \nabla \cdot \vec{j} \quad \text{and} \quad \frac{\partial \mu}{\partial t} = \nabla \cdot \mu \vec{v} = \nabla \cdot \vec{p}.
\] (4.6)

2. The second equation governs the behavior of \( \theta \).

\[
\hbar \frac{\partial \theta}{\partial t} = -\frac{1}{2}mv^2 + q\varphi - m\Phi - \hbar^2 \frac{1}{2m} \left\{ \frac{1}{\sqrt{\rho}} \Delta (\sqrt{\rho}) \right\}
\] (4.7)

The right hand side of this equation is composed of the kinetic energy, the electrical potential energy, the gravitational potential energy and an additional term corresponding to an energy which has a "quantum nature" (we will take the density \( \rho \) to be constant, so this fourth term will vanish).

In order to make the physics behind equation (4.7) more apparent we can take the gradient of equation (4.7) and consider the canonical momentum (4.3) that we obtained above.

\[
\frac{\partial \vec{v}}{\partial t} = \frac{q}{m} \left( -\nabla \varphi - \frac{\partial \vec{A}}{\partial t} \right) + \left( -\nabla \Phi - 4 \frac{\partial \vec{A}_g}{\partial t} \right) - \vec{v} \wedge (\nabla \wedge \vec{v}) - (\vec{v} \cdot \nabla) \vec{v}
\] (4.8)

The first term on the right hand side of equation (4.8) corresponds to the electrical force, because:

\[
\vec{E} = -\nabla \varphi - \frac{\partial \vec{A}}{\partial t}.
\] (4.9)

The second term on the right hand side of equation (4.8) corresponds to the gravitational force, because as we saw above with equation (2.65), we have:

\[
\vec{g} - 3 \frac{\partial \vec{A}_g}{\partial t} = -\nabla \Phi - 4 \frac{\partial \vec{A}_g}{\partial t}
\] (4.10)

The third term on the right hand side of equation (4.8) corresponds to the Lorentz force and to the gravitational Lorentz force, because, taking the curl of \( \vec{v} \) that we extracted from equation (4.3), we obtain:

\[
\nabla \wedge \vec{v} = -\frac{q}{m} \nabla \wedge \vec{A} - 4 \nabla \wedge \vec{A}_g = -\frac{q}{m} \vec{B} - 4 \vec{B}_g.
\] (4.11)

Moving the fourth term on the right hand side of equation (4.8) to left hand side we obtain the comoving acceleration:

\[
m \left[ \frac{d \vec{v}}{dt} \right]_{\text{comoving}} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v}
\] (4.12)
Therefore, the equations of motion for an ideal electrically charged fluid with mass which is located simultaneously in an electromagnetic, a gravitational, and a gravitomagnetic field are:

\[
m \left[ \frac{d \vec{v}}{dt} \right]_{\text{Comoving}} = q \left( \vec{E} + \vec{v} \wedge \vec{B} \right) + m \left( \vec{g} - 3 \frac{\partial \vec{A}_g}{\partial t} + 4 \vec{v} \wedge \vec{B}_g \right)
\]  

(4.13)

and

\[
\nabla \wedge \vec{v} = - \frac{q}{m} \vec{B} - 4 \vec{B}_g.
\]  

(4.14)

From equation (4.14) we know that the sum of the magnetic flux and the GM flux is proportional to the circulation of the Cooper pair’s velocity around a closed path. We have indeed:

\[
\int_{\Sigma} (\nabla \wedge \vec{v}) \, d\vec{\sigma} = - \frac{q}{m} \int_{\Sigma} \vec{B} \cdot d\vec{\sigma} - 4 \int_{\Sigma} \vec{B}_g \cdot d\vec{\sigma}.
\]  

(4.15)

Moreover, we see from the middle of equation (4.11) that

\[
- \vec{v} = \frac{q}{m} \vec{A} + 4 \vec{A}_g.
\]  

(4.16)

The first term on the right hand side of equation (4.16), we interpret to be the GM vector potential induced by the applied magnetic field \([21]\). Therefore, the Cooper pair velocity is proportional to the total gravitomagnetic vector potential.

To include the contribution of GM in the magnetic fluxoid equation (3.20), one need simply add an appropriate term of \(4(2m) \vec{B}_g\) in the right hand side of this equation, which becomes

\[
n \frac{\hbar}{2e} = n \Phi = \frac{m}{de^2} \int_{\Sigma} (\nabla \wedge \vec{j}) \, d\vec{\sigma} + \int_{\Sigma} \vec{B} \cdot d\vec{\sigma} + 4 \left( \frac{2m}{2e} \right) \int_{\Sigma} \vec{B}_g \cdot d\vec{\sigma}.
\]  

(4.17)

Rearranging, we get

\[
n \frac{\hbar}{2m} = n \Phi = \frac{1}{de^2} \int_{\Sigma} (\nabla \wedge \vec{j}) \, d\vec{\sigma} + \frac{e}{m} \int_{\Sigma} \vec{B} \cdot d\vec{\sigma} + 4 \int_{\Sigma} \vec{B}_g \cdot d\vec{\sigma}
\]  

(4.18)

We have transformed the quantization relation into a form for which the left hand side has dimensions of GM-flux \((m^2/s)\). A single GM flux quantum trapped inside the ring has the value:

\[
\Phi_{g_0} = \frac{\hbar}{2m}.
\]  

(4.19)

We interpret the right hand side as being the total GM flux in the system, which is quantized. The first term is that GM flux from the moving currents in the London
layer; the third term is the ”pure” GM flux from the external field; the second term we interpret to be the GM flux induced by the applied magnetic field. Note that the GM quantum number is the same as the magnetic one for the same physical situation. Quantization of one establishes the other.

Since the first term is relatively small, the important question is whether the interpretation of the second term is valid. This second term will not be small in general, if for no other reason than the factor of $1/m$ is so large. Traditional interpretations of the flux quantization relations emphasize that what is physically measured in an experiment is only one term of those relations, say, the magnetic flux integral. Each term is measured separately. The left hand side, that which is actually quantized, is not measurable and just represents a mathematical agglomeration of the individual terms on the right hand side. Support for this interpretation comes from experiments [9], [10] on very thin SC with thickness less than the London penetration depth. Unlike bulk SC, thin SC do not exhibit magnetic flux quantization. Explanation for this is offered by observing that the current line integral in equation (3.19) is non-vanishing for thin SC because the contour can not be placed in the bulk where the Meissener Effect prevails. Since this term does not vanish, and the left hand side of equation (3.19) is fixed, this must mean the magnetic flux integral is less than the flux quantum, which accounts for the measurements. This is certainly mathematically sensible and persuasive.

However, as we saw in equation (3.20), we can write this equation in a form where the current integral is also written as a flux integral, so that all terms have the same form. Now, this term will not automatically vanish in all cases, since $\nabla \wedge \vec{J}$ is proportional to the magnetic field in a SC by one of the London Equations. And the surface of integration includes that region containing the paramagnetic current, where a magnetic field will exist. Then, a more physically natural and persuasive explanation for this term is that it represents the magnetic flux contribution of the paramagnetic current, which generates the magnetic field sufficient to maintain a quantized flux. It is the left hand side of the equation which is actually measured, representing the combined contributions of the external magnetic field, a contribution from any induced magnetic fields (e.g. the negligible external GM-induced magnetic field), and the paramagnetic field. In a bulk SC, these contributions sum to ensure the measured magnetic flux is quantized.

In a very thin SC, the integration surface is so small, it chops off part of the paramagnetic region. There is simply not enough SC left to maintain the entire quantized flux; it is not a complete SC system. This explains why the total flux from all the sources on the right hand side is insufficient to impose the quantized value.

A corresponding interpretation of the GM flux quantization relation equation (4.18) implies that what would be measured in a GM experiment is the quantized GM flux value on the left hand side of equation (4.18), which is entirely of GM nature. That is, the GM flux Quantum represents a physically observable quantity, just as the magnetic flux quantum does. Quantization of the one imposes quantization on
the other. As in the purely magnetic case, a sufficiently large external magnetic field must be present to boost the SC into a \( n > 0 \) quantum state. Though the Earth’s GM field pervades the terrestrial environment, it is not strong enough to do this. For this reason, quantized GM flux due to an external GM field alone will not be observable. But because the magnetic and GM fluxes are tied together by the same quantum number in the same quantum state, if a sufficiently large external magnetic field is present, it should be possible to observe its GM counterpart. Indeed, the proposed experiment may offer the first direct test of the thin SC explanation by searching for different fields representing the multiple terms on the right hand side of the flux equations.

An objection, both cogent and obvious, can be made to this interpretation on the grounds that there is no known mechanism for producing a macroscopically measurable GM field. Exotic mechanisms can be imagined. Coherent zero point motion was suggested for a similar purpose in the context of superfluid HeII [1]. It turns out this was not the first time Coherent zero point motion was proposed as an underlying mechanism operative in superfluids. Twenty years ago, Shenoy and Biswas [11] proved by detailed statistical mechanical arguments using a coherent state formalism in coordinate space that coherent, macroscopic zero point motion characterizes and defines an interacting Bose gas (that is, a non-ideal Bose gas) superfluid system. It is only ideal Bose gases that display Off-Diagonal Long-Range Order (ODLRO) and form a true condensate; real superfluids do not possess ODLRO. Instead, any interaction effectively mediates a form of phase coherence which is reflected in a momentum space dispersion. Coherent zero point motion is at first glance counterintuitive, if not downright oxymoronic: Zero point motion is, after all, supposed to be inherently random. A very schematic way of thinking about this physically is terms of random phase space orbits of the Bose gas constituents, which are nonetheless correlated in position along each such orbit, has been suggested [1].

Outlandish as some of these ideas may appear, there are some precedents for the role of “background” contributions in condensed-matter systems. Certain thin film superfluid helium mixture systems [12] have excited states, each of which separates out into completely independent quantum subsystems. Extensive quantities are determined by simply summing the contribution of each individual state. The Fractional Quantum Hall Effect can be understood in terms of a system of anyonic, fractionally charged quasi-particle excitations coupled to magnetic flux quanta [13]. These collective excitations are “hierarchically” correlated and themselves form a condensate [13], [1]. Thus, it is not entirely inconceivable that an amplification effect from repeated “reuse” of hierarchies of underlying collective coherent constituents might begin to explain the predicted macroscopic GM flux quantum. Pushing such speculation to the limits, one may even posit that such interactions may explain the extraordinary coherence distances of recent entangled quantum state teleportation experiments [14].
if it does not physically contribute to a GM flux, but just represents the magnetic torque that a magnetized body would feel if such a body were present in the SC hole, then the other "pure" GM term is far too small to be measurable. On the other hand, if the two terms of equation (4.18) and the fluxoid itself are GM in origin, the effect may be large and definitely noticeable given the right circumstances.
Chapter 5

GRAVITOMAGNETIC FLUX EXPERIMENT

5.1 Gravitomagnetic Flux Experiment Concept

The actual experiment concept consists of suspending a SC from a balancing-torsion mechanism akin to that used in the Cavendish experiment (see figure 3). A hole is scooped out of the bulk SC, such that a small, rotating cylinder (C) of non-magnetic material can be placed inside. The cylinder has its long axis oriented in the horizontal plane, and spins along that axis.

The cylinder is supported by a mechanism which allows rotation around the long axis, but resists rotation around any other. The SC itself is free to rotate in the horizontal plane. There is a small mirror (M1) or other such rotation sensor on the SC suspension assembly.

If a strong enough total GM fluxoid quantum is present in the hole, due to a large applied magnetic field, the cylinder ought to precess about an axis perpendicular to the horizontal plane. However, the cylinder is constrained from precessing. Since the SC, to which the cylinder should be coupled by virtue of the GM field, is free to rotate in the precession plane, the cylinder’s precessional angular momentum is transferred to the SC. The SC then twist about its suspension, permitting a measurement (R1) of the degree of turn from the sensor on the suspension. Alternatively, a direct measurement of the cylinder’s precession can be attempted.

To calibrate the apparatus and ensure that there are no lingering magnetic effects in the cylinder, the cylinder set-up experiment should first be run with no SC present, with just the applied field. Next, the experiment should be repeated with a SC, but at $T > T_c$, where there should be no quantization effects. Finally, the full SC plus cylinder tests can be run, varying the applied magnetic field as desired, as well as other parameters. A magnetized cylinder should be substituted for the non-magnetic test body, not only to verify the basic experimental set-up in case no GM effect is seen, but to also verify that an experimental effect is present even in the well-known case of the SC magnetic field.

Flexibility in the experiment design should be sufficient to permit variations in the following experimental parameters:

1. The axis of the SC can be moved smoothly and without induced vibrations in, or causing collisions between, the rotating cylinder and the SC. This can be
Figure 3: The Gravitomagnetic Cavendish balance
done under cryogenic and vacuum conditions.

2. The speed and direction of the cylinder rotation can be changed by varying the flow rates of the helium or hydrogen jets directed at the top and bottom of the cylinder, under cryogenic and vacuum conditions.

3. The magnitude of the trapped magnetic flux can be adjusted after each run of the experiment sequence by changing the applied magnetic field.

4. The composition of the cylinder and its mass properties (e.g. moment of inertia) are adjustable by switching out cylinder samples.

The following tables illustrate schematically the way parameter space can be scanned to test whether there is or is not a macroscopic quantized GM flux separate and distinguishable from magnetic flux. In Table 2, it is assumed there is an applied magnetic field equivalent to some integer multiple of the magnetic flux quantum. In Table 3, it is assumed that some non-integer (say, \(n + \frac{1}{4}\)) quantum of magnetic flux is applied. Each entry in the tables contains an “equivalent” magnetic (for a magnetic test body) or GM (for a non-magnetic test body) flux strength that would be left by the test body as a torque, in the different temperature regimes. If there is no macroscopic GM flux in the superconducting regime, then all entries in the second column would be zero in both tables.

|        | Magnetic Test Body | Non-magnetic Test Body |
|--------|--------------------|------------------------|
| \(T < T_c\) | \(n\)             | \(n\)                  |
| \(T > T_c\) | \(n\)             | 0                      |

Table 2: Integer Quantized Applied Magnetic Flux

|        | Magnetic Test Body | Non-magnetic Test Body |
|--------|--------------------|------------------------|
| \(T < T_c\) | \(n\)             | \(n\)                  |
| \(T > T_c\) | \(n + \frac{1}{4}\) | 0                      |

Table 3: Non-integer Quantized Applied Magnetic Flux

5.2 Gravitomagnetic Force Calculation

In this section is outlined the calculations needed to compute the forces that are expected to arise from the GM-induced precession of the cylinder based on the theory of section 4.

For a given applied magnetic field, the fluxoid quantum number is obtained from

\[
n = \pi r^2 \frac{H 2e}{\hbar}
\]

(5.1)

where \(H\) is the applied magnetic field, \(r\) is the radius of the hole. Since \(n\) is an integral quantum number, it should be rounded to the nearest integer.
Since it is believed that the GM field can penetrate the SC, unlike the magnetic field, the area that enters the GM fluxoid is that of the whole SC, not the hole. From the left hand side and the third term on the right hand side of equation (4.18) we estimate the GM field as an average over its penetration area. This will just be an estimate because, at least in the case of the magnetic field, the field is not uniform in a flat cylindrical SC as will be used for this experiment. Conversely, if the GM field is not expelled from the bulk SC, its lines should not deviate from linearity to any degree, making the average field estimate from the GM flux a better estimate than one for $\vec{B}$ itself,

$$B_g = n \frac{h}{8m \pi R^2} = \frac{1}{4m} e H \left( \frac{r}{R} \right)^2$$

where $R$ is the radius of the SC.

If $B_g$ is given by (5.2), the precession velocity of the cylinder is therefore

$$\Omega = 2B_g = \frac{1}{2m} e H \left( \frac{r}{R} \right)^2.$$  

The torque the GM field will attempt to apply to the cylinder so as to result in the precession velocity of the cylinder is therefore

$$\vec{N} = 2 \vec{L} \land \vec{B}_g = \frac{e}{2m} \left( \frac{r}{R} \right)^2 \vec{L} \land \vec{H}.$$  

Rigid body spin angular momentum is given by

$$\vec{L} = I \vec{\omega}$$

where $\vec{\omega}$ is the spin angular velocity. A cylinder rotating around its long axis has moment of inertia $I$ of

$$I = \frac{Ma^2}{2}$$

where $M$ is the mass of the cylinder and $\alpha$ is its radius. Putting both (5.3) and (5.6) into (5.4), we obtain (assuming right angles)

$$N = \frac{Ma^2e}{4m} \left( \frac{r}{R} \right)^2 \omega H.$$  

Since the cylinder will be fixed in the precessional rotation direction, but the SC will be free, the back coupling of the cylinder to SC should transfer the applied torque to the SC, resulting in a torque, and therefore force, on the SC suspension mechanism. The torque is given by

$$\vec{N} = \vec{R} \land \vec{F}.$$  

So, assuming right angles relative to the apparatus,

$$F = \frac{N}{R} = \frac{Ma^2e}{4m} \frac{r^2}{R^3} \omega H.$$  

For the following suggested parameter values
Gravitomagnetic Force Calculation

- $M$, Mass of test body: $2 \times 10^{-3}$ Kg
- $\omega$, Angular velocity of test body: 10 Rad/s
- $R$, Radius of SC: $1.1 \times 10^{-2}$ m
- $r$, Radius of SC hole: $5 \times 10^{-3}$ m
- $\alpha$, Radius of test body: $3 \times 10^{-4}$ m
- $m$, electron mass: $9.1091 \times 10^{-31}$ Kg
- $e$, electron charge: $1.6021 \times 10^{-19}$ C
- $h$, Planck constant: $6.6256 \times 10^{-34}$ J.s

the value of $n$ at the lower critical field ($2 \times 10^{-2}$ T) for the material proposed for this experiment, with $r$ as above, is from (5.1)

$$n = 7.59 \times 10^{8}.$$  

This would make the force

$$F = 29.7 \text{ Newtons!}$$

For a more reasonable magnetic field, like the earth’s ($3 \times 10^{-5}$ T) we get

$$n = 1.139 \times 10^{6}$$

and the force is

$$F = 4.46 \times 10^{-2} \text{ Newton.}$$

It is essential to use small-area holes in the SC. While it is the flux which is quantized, it is the field which induces the precession. Too large a hole means that the flux can be dominated by the hole area, rather than the field, with the result that precessional effects could be quite small, despite a large applied magnetic field. Conversely, it is not quite customary to suspend small test bodies in small holes in SC, which may explain why these effects have not been noticed before.
Chapter 6
CONCLUSION

For suitable superconductors geometries, in a sufficiently strong external magnetic field below the superconducting transitions temperature, measurable GM fields should be detected. Such GM fields obey a GM version of the familiar magnetic fluxoid relation, defining a GM Flux quantum. This GM Flux quantum, consistent with our reinterpretation of the magnetic fluxoid relation, has real physical significance. It is not the outcome of purely arbitrary algebraic manipulations of the magnetic fluxoid relation; nor is it an unmeasurable quantity which is merely the arithmetic sum of physical quantities which can only be measured separately. Rather, the flux quanta represent the total contribution of the physical sources of the fields encompassed by each quanta, each source ultimately traceable to the Hamiltonian obeyed by the condensate system. Both the magnetic and GM fields are attributed to the same carriers (in the present conception of the theory), namely, the Cooper pairs, and so must be governed by the same quantum number. Since terrestrial GM fields themselves are too weak to promote a superconductor out of the flux quantum ground state, reliance is placed on magnetic fields to boost the superconductor out of its ground state, creating the same number of non-vanishing flux quanta for both the magnetic and GM fields.

Our experiment concept takes advantage of this magnetic boost to detect GM fields "trapped" in a superconductor. While it may be preferable to perform this experiment with low-temperature superconductor systems because of their relative simplicity, high-Tc superconductors should also be amenable to this investigation. Indeed, while this concept is not without its own complexities, a proof of principle demonstration, using larger magnetic fields, should be possible with standard superconductor laboratory equipment supplemented by the Cavendish-type apparatus. If counting quanta is perhaps the most fundamental way of assessing the relative magnitude of a field, then such a detection also brings us into the "mesoscopic" regime of gravitational fields, featuring $10^2 - 10^4$ GM flux quanta.

Such demonstration would hold profound implications for the nature of the gravitational field, its quantization, its relation to other classical fields, its relation to coherence, the transition from linearized to nonlinear regimes of field equations, and the relationship between gauge fields, their field equations, and their topologies. Other systems, such as superfluids and the exotic coupled superfluid-superconductor
systems thought to exist in neutron stars, should also manifest quantized GM effects to varying degrees. Practical applications, if the effects are large enough, may also be feasible (generation of gravitational fields through time varying GM flux).

Even if no macroscopic GM field is found, this experiment still represents an important verification of the traditional fluxoid interpretation. There may be little doubt harbored in that interpretation after several decades of experimentation with thin superconductors, interferometric phase and Josephson Effect measurements. Yet, this experiment, perhaps for the first time, tests that interpretation by seeking direct, physical effects, distinguishable by the presence of the different fields that produce them, on objects of distinctive composition. A phase shift is a phase shift no matter how produced. But a GM field and a magnetic field should have quite different effects on objects composed of different materials.

A deeper, completely explanatory theory for this effect trails far behind the basic concept at present. Halting steps towards a deeper theory, invoking other speculative ideas such as coherent zero point motion, have been taken. But a complete description may have to await discovery of the effect and fuller exploration of its intricacies and ramifications.
Chapter 7
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