Spreading properties of a three-component reaction-diffusion model for the population of farmers and hunter-gatherers

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Abstract

In this paper, we investigate the spreading properties of solutions of farmer and hunter-gatherer model which is a three-component reaction-diffusion system. Ecologically, the model describes the geographical spreading of an initially localized population of farmers into a region occupied by hunter-gatherers. This model was proposed by Aoki, Shida and Shigesada in 1996. By numerical simulations and some formal linearization arguments, they concluded that there are four different types of spreading behaviors depending on the parameter values. Despite such intriguing observations, no mathematically rigorous studies have been made to justify their claims. The main difficulty comes from the fact that the comparison principle does not hold for the entire system. In this paper, we give theoretical justification to all of the four types of spreading behaviors observed by Aoki et al.. Furthermore, we show that a logarithmic phase drift of the front position occurs as in the scalar KPP equation. We also investigate the case where the motility of the hunter-gatherers is larger than that of the farmers, which is not discussed in the paper of Aoki et al..

Key words: Farmer and hunter-gatherer model; long time behavior; spreading speed; logarithmic correction

1 Introduction

Early in the stone age, our ancestors lived as hunter-gatherers, which means, instead of growing their food, they lived on hunting, fishing and gathering berries and eggs of birds in the forest. As humans evolved, agriculture gradually appeared independently in different parts of the globe. At least 11 separate regions of the Old and New World were identified as independent centers of the origin of agriculture. Tracing the origins of early farming and its spreading has been the major subject of interest for a long time. For instance, the study of the origins of farming in the Near East and its dispersal to Europe has been done by many archaeologists, anthropologists, linguists, and geneticists. Many archaeological evidences certified that agriculture emerged about 11,000 years ago in the Near East before reaching North Europe about 5,000 years later. Furthermore, genetic studies tended to support that farmers in the North Europe have noticeable genetic affinity to Near East populations. It suggests that agriculture did not only spread solely across Europe as a cultural process, but also in concert with a migration of people. This fact motivated ecologists to model this kind of geographical spreading of an initially localized farmers into a region occupied by hunters and gatherers as a reaction-diffusion process in an infinite habitat.

In 1996, Aoki, Shida and Shigesada [1] proposed the following three-component reaction-diffusion system to study the process of Neolithic transition from hunter-gatherers to farmers (actually, in [1], they only considered the case $N = 1$, but in the present paper we formulate the problem in a general space dimension $N \geq 1$):

\[
\begin{align*}
\partial_t F &= D \Delta F + r_f F(1 - (F + C)/K), \\
\partial_t C &= D \Delta C + r_c C(1 - (F + C)/K) + e(F + C)H, \\
\partial_t H &= D_h \Delta H + r_h H(1 - H/L) - e(F + C)H.
\end{align*}
\]
1.1 Observations by Aoki et al.

The population densities of initial farmers, converted farmers and hunter-gatherers are represented by $F$, $C$ and $H$, respectively. This model contains seven positive parameters: $D_h$ is the diffusion coefficient of hunter-gatherers, which is assumed to be greater than or equal to the diffusion coefficient $D > 0$ of farmers; $r_f$, $r_c$ and $r_h$ are the intrinsic growth rates; $K$ and $L$ are the carrying capacities of farmers and hunter-gatherers; $c$ is the conversion rate of hunter-gatherers to farmers. Note that, in [1], $D_h$ is always assumed to be equal to $D$. However, we do not see any reason to assume that the hunter-gatherers diffuse at the same speed as the farmers. It may be more natural to imagine that the hunter-gatherers would diffuse faster than the hunters. For this reason, in the present paper, we assume $D_h \geq D$ rather than $D_h = D$. As we will see later, the case $D_h > D$ is much harder to analyze than the case $D_h = D$.

As shown in [1], by a suitable change of variables, the above system is converted to:

$$
\begin{align*}
\partial_t F &= \Delta F + aF(1 - F - C), \\
\partial_t C &= \Delta C + C(1 - F - C) + sH(F + C), \quad \text{in } \mathbb{R}^N, \\
\partial_t H &= d\Delta H + bH(1 - H - g(F + C)),
\end{align*}
$$

where $d = D_h/D \geq 1$, $a = r_f/r_c$, $b = r_h/r_c$, $s = eL/r_c$ and $g = eK/r_h$. We consider the following initial condition

$$
H(0, x) \equiv 1, \ C(0, x) \equiv 0, \ F(0, x) = F_0(x) \geq 0 \ (F_0 \neq 0),
$$

where $F_0(x)$ is a compactly supported continuous function. The reason why we consider such initial data is because our goal is to understand how agriculture spread over a region that was originally occupied by hunter-gatherers. Thus our problem is to analyze the “spreading fronts” of the farmer populations that start from localized initial data.

In the special case $a = 1$, by setting $G = F + C$, the total population density of all farmers, the system (1.1) reduces to the two-component system of predator-prey type:

$$
\begin{align*}
\partial_t G &= \Delta G + G(1 + sH - G), \\
\partial_t H &= d\Delta H + bH(1 - H - gG),
\end{align*}
$$

The long time behavior of the system (1.1) (with $a = 1$) has been studied by Hilhost, Mimura and Weidenfel in [12]. Among other things they proved that the solution converges to a spatially constant steady state as $t \to \infty$. They also considered the case where the coefficients $s$, $g$ are replaced by $s/\varepsilon$, $g/\varepsilon$ and studied the limit of the solution as $\varepsilon \to 0$. However, how fast the front propagates to infinity was not discussed in [12]. The first result on the spreading speeds for predator-prey systems was obtained by Ducrot, Giletti and Matano in [7], where they treated two-species predator-prey system that are different from (1.3).

In the general case $a \neq 1$, the ODE system corresponding to (1.1) possesses the following four different types of steady states:

$$
(F, C, H) = \begin{cases} 
(0, 0, 0), \ (0, 0, 1), \\
(\hat{F}, \hat{C}, 0), \ \text{where} \ \hat{F} + \hat{C} = 1, \ \hat{F}, \hat{C} \geq 0, \\
(0, C^*, H^*), \ \text{where} \ C^* = (1 + s)/(1 + sg), \ H^* = (1 - g)/(1 + sg).
\end{cases}
$$

The first two steady states always exist and unstable. The third one is a line of neutral equilibria, which always exists and stable if $g \geq 1$. The fourth one exists and is stable if and only if $g < 1$. It implies that there exist different expanding patterns which is determined by the value of $g$. Let us remark that $g = eK/r_h$, which means the value of $g$ highly depends on the value of the conversion rate $e$. Note that, throughout this paper, we call the case $g \geq 1$ by high conversion rate case and the case $g < 1$ by low conversion rate case. Ecologically speaking, if the conversion rate is sufficiently high, hunter-gatherers will completely convert into farmers, whereas hunter-gatherers and farmers can coexist if the conversion rate is low enough.

1.1 Observations by Aoki et al.

In the above mentioned paper [1], the authors observed four different types of spreading behaviors depending on the parameter values, namely those of $a$, $s$ and $g$ in the system (1.1). Their observation of the spreading fronts
was done by numerical simulations. Strictly speaking, they did not consider truly localized initial data for the farmers, but they set $C_0 \equiv 0$ and chose $F_0$ to be a Heaviside function: $F_0(x) = 1 (x \leq 0)$, $F_0(x) = 0 (x > 0)$. In order to estimate the speed of the spreading fronts, they again relied mainly on numerical simulations, but also calculated by formal analysis of the minimal speed of the traveling wave (comprising of the advancing front of the farmers and the retreating front of the hunter-gatherers) and confirmed that the numerically observed spreading speed well agreed with the formally calculated minimal traveling wave speed. The following figures illustrate the shape of what they call the transient waveforms.

**Figure 1:** High conversion rate case, $a > 1 + s$.

**Figure 2:** High conversion rate case, $a < 1 + s$.

**Figure 3:** Low conversion rate case, $a > 1 + s$.

**Figure 4:** Low conversion rate case, $a < 1 + s$.

In order to make our motivation clearer, we give a brief explanation of what was observed by Aoki et al. before stating our main results. As mentioned above, their observation was done by numerical simulation combined with a formal analysis of the minimal speed of traveling wave.
The spreading speed of the solution of the system (1.1) is always determined by $\max\{2\sqrt{a}, 2\sqrt{1+s}\}$.

The behaviors of solutions on the leading edge where the hunter-gatherers have little contact with the farmers are almost the same between the high conversion rate case and the low conversion rate case (see Figure 1-4, Leading edge).

In the case where $a < 1 + s$, a wave of advance of initial farmers $F$ is not generated (see Figure 2 and Figure 4). If, in addition that $g < 1$, the initial farmers $F$ disappear entirely (see Figure 4). Whereas, if $g \geq 1$, behind the wavefront, farmers have almost reached carrying capacity and hunter-gatherers have just about disappeared (see Figure 2, Final zone).

In the case where $a > 1 + s$, an advancing wavefront of initial farmers $F$ is also generated (see Figure 1 and Figure 3). If, in addition that $g < 1$, the waveform is a small peak with leading edge and trailing edge that converge to 0 (see Figure 3).

The goal of the present paper is to give rigorous justification to all of the above observations, and also to discuss the case where $d > 1$ that has not been treated in [1]. The case $d > 1$, turns out to be much harder to analyze. Further, we also prove logarithmic drift of the fronts for some cases.

### 1.2 Outline of the paper

We will present our main results in Section 2. Some results are stated for the special case $d = 1$ (Theorems 2.4 and 2.8 and part of Theorems 2.10 and 2.12), which is the case that was treated in [1]. Other results are stated for $d \geq 1$. The proof of the main results will be carried out in the subsequent sections.

In Section 3, we prove Theorem 2.3 which is concerned with the behaviors at the leading edge that appears in all of Figure 1-4. Since there is little interaction between the farmers and the hunter-gatherers on the leading edge, the analysis of this zone is rather straightforward.

In Section 4, we prove Theorem 2.12 on the logarithmic phase drift of the front. The logarithmic phase drift for the scalar KPP equation was studied in detail by Bramson [3, 4] by using a probabilistic method. More recently, a much simpler PDE proof has been proposed by Hamel, Nolen, Requejoffre and Ryzhik [13]. Their method is based on a super and sub-solution argument. Our proof of Theorem 2.12 is also based on a super and sub-solution argument, but the sub-solution is quite different from that in [13] since their sub-solution does not work for systems of equations.

In Section 5, we prove Theorems 2.4 and 2.5 that are concerned with the uniform positivity of solution in the final zone. Theorem 2.4 is concerned with the case $1 + s \geq a$ (which corresponds to Figure 2 and 4 above), while Theorem 2.5 is concerned with the case $1 + s < a$ (which corresponds to Figure 1 and 3 above). Intriguingly, the proof of the two theorem are quite different: the proof of Theorem 2.4 uses a limit argument that was employed in [7] to show "pointwise spreading". The proof of Theorem 2.5 uses the result on the logarithmic drift stated in Theorem 2.12.

In Section 6, we will study final asymptotic profiles of solutions in the final zone and complete the proof of Theorems 2.8 and 2.10, thus confirming the profiles shown in Figure 1-4. The proof is based on the conclusion of the results in Theorems 2.4 and 2.5 and a certain limit argument.

Finally in the appendix, we give the proof of Proposition 4.1 which plays an important role in the proof of Theorem 2.12 in Section 4.

### 2 Main results

Front propagation for scalar reaction-diffusion equations has been studied extensively and there is vast literature on this theme. Early in 1937, Fisher [9] and Kolmogorov, Petrovsky and Piskunov [11] introduced a scalar reaction-diffusion equation with monostable nonlinearity as a model equation in population genetics, which studies the propagation of dominant gene in homogeneous environment. In particular, [11] made an important early analysis of the structure of the set of traveling waves for a special class of monostable reaction-diffusion equation, the so-called KPP equation. Application of reaction-diffusion equations to ecology was
Moreover, the quantity \( c^* \) coincides with the minimal speed of traveling wave solutions of the equation \((2.2)\) connecting \( 0 \) to \( 1 \).

**Remark 2.2** Usually the term “spreading front” regards to a front of a solution that propagates to infinity from localized initial data, typically those that are compactly supported. The term “spreading speed” refers to a quantity \( c^* \) for which the last two estimates of Proposition 2.1 hold. Thus the distance between the sphere of radius \( c^* t \) and the actual position of the front is of order \( o(t) \).

### 2.1 Uniform spreading properties

In this subsection, we present our main results on the spreading properties of solutions of the system \((1.1)\). Throughout this paper, we define the two quantities \( c^* \) and \( c^{**} \) as \( c^* := \max\{2\sqrt{a}, 2\sqrt{1+s}\} \) and \( c^{**} := \min\{2\sqrt{a}, 2\sqrt{1+s}\} \). Our first result is about the analysis of the leading edge, which provides an upper estimate on the spreading speed. As we observed from Figure 1-4, for all four cases, the behaviors of solutions are almost same on the leading edge. The first three theorems hold regardless of the size of the conversion rate.

**Theorem 2.3** For any given \( c > c^* \), the solution \((F, C, H)\) of the system \((1.1)\) with the initial data \((1.2)\) satisfies:

\[
\lim_{t \to \infty} \sup_{|x| \geq ct} \left( |1 - H(t, x)| + F(t, x) + C(t, x) \right) = 0. \tag{2.2}
\]

The most difficult part of the analysis is the behaviors of the solutions in the final zone, where original hunter-gatherers, initial farmers and converted farmers heavily interact with each other. A large part of the present thesis is devoted to the analysis in this final zone. Our third result deals with the propagation of farmers in the final zone, which provides an lower estimate on the spreading speed. As we observed from Figure 1-4, for all four cases, the spreading speed is determined by the larger value of \( 2\sqrt{1+s} \) and \( 2\sqrt{a} \).
Theorem 2.4 If \( d = 1 \) and \( 1 + s \geq a \), then for any given \( 0 \leq c < c^* \), there exists \( \varepsilon > 0 \) such that the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2) satisfies:

\[
\liminf_{t \to \infty} \inf_{\|x\| \leq ct} (F + C)(t, x) \geq \varepsilon,
\]

\[
\limsup_{t \to \infty} \sup_{\|x\| \leq ct} H(t, x) \leq 1 - \varepsilon.
\]

Theorem 2.5 If \( 1 + s < a \), then for any given \( 0 \leq c < c^* \), there exists \( \varepsilon > 0 \) such that the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2) satisfies:

\[
\liminf_{t \to \infty} \inf_{\|x\| \leq ct} (F + C)(t, x) \geq \varepsilon,
\]

\[
\limsup_{t \to \infty} \sup_{\|x\| \leq ct} H(t, x) \leq 1 - \varepsilon.
\]

Remark 2.6 In the present paper, we use two different approaches to prove the results of Theorem 2.4 and Theorem 2.5, respectively. We specially note that the result of Theorem 2.4 could be extended to the general case \( d \geq 1 \) by applying a similar argument to that in section 5, and the details will be provided in our future work.

Remark 2.7 The results of Theorem 2.3, Theorem 2.4 and Theorem 2.5 can be regarded as an analogue of the well-known "hair-trigger effect" for scalar monostable equation [2]. Moreover, one may find that the spreading speed of the system is always equal to \( c^* \), which is only determined by the parameters in the \( F \)-equation and \( C \)-equation. This is due to the initial data \( H_0(x) \equiv 1 \), which means hunter-gatherers have already spread to the whole region.

At last, we show our main results about the asymptotic profiles of solutions in the final zone. The precise results will be presented in two cases, the high conversion rate case \((g \geq 1)\) and low conversion rate case \((g < 1)\), respectively. For the high conversion rate case, our results read as:

Theorem 2.8 (High conversion rate case, \( 1 + s \geq a \)) If \( d = 1 \), \( 1 + s \geq a \) and \( g \geq 1 \), then for any given \( 0 \leq c < c^* \), the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2) satisfies:

\[
\limsup_{t \to \infty} \sup_{\|x\| \leq ct} H(t, x) = 0,
\]

\[
\limsup_{t \to \infty} \sup_{\|x\| \leq ct} |1 - (F + C)(t, x)| = 0.
\]

Moreover, for any given \( c^* < c_1 < c_2 < c^* \), it holds:

\[
\limsup_{t \to \infty} \sup_{c_1 t \leq \|x\| \leq c_2 t} \left( (F(t, x) + |1 - C(t, x)|) \right) = 0.
\]

Theorem 2.9 (High conversion rate case, \( 1 + s < a \)) If \( 1 + s < a \) and \( g \geq 1 \), then for any given \( 0 \leq c < c^* \), the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2) satisfies:

\[
\limsup_{t \to \infty} \sup_{\|x\| \leq ct} H(t, x) = 0,
\]

\[
\limsup_{t \to \infty} \sup_{\|x\| \leq ct} |1 - (F + C)(t, x)| = 0.
\]

For the high conversion rate case, Theorem 2.8 and Theorem 2.9 imply that all of the original hunter-gatherers convert to farmers at last. However, the explicit profiles of the \( F \)-component and \( C \)-component in the final zone are yet to be investigated. Nevertheless, for the low conversion rate case, we will show that the \( F \)-component always converges to 0 in the final zone. In order to do this, one need first prove that the population...
density of hunter-gatherers would stay positive uniformly. From some ecological observations, this may happen if the conversion rate is small enough or the product of the intrinsic growth rate and the diffusion speed of hunter-gatherers is large enough. Then, we can investigate how the \( C \)-component and \( H \)-component behave in the final zone by considering the dynamics of the underlying ODE system:

\[
\begin{align*}
C_t &= C(1 - C) + sCH, \\
H_t &= bH(1 - H - gC).
\end{align*}
\] (2.12)

We expect the solution of the PDE system (1.1) to converge uniformly to the equilibrium \((C^*, H^*)\) in the final zone as \( t \to +\infty \). We prove the conjecture by the established fact that a strict Lyapunov function exists. Note that, the ODE system (2.12) is a well-known Lotka-Volterra system with logistic growth rate. In population dynamics, especially prey-predator systems, ODE models always admit a strict Lyapunov function. The concrete example will be given in the later section.

**Theorem 2.10 (Low conversion rate case)** If \( g < 1 \), then for any given \( 0 \leq c < c^* \), for the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2), it holds:

1. there exists \( \varepsilon > 0 \), such that:

\[
\lim_{t \to \infty} \inf_{\|x\| \leq ct} H(t, x) \geq \varepsilon,
\] (2.13)

\[
\lim_{t \to \infty} \inf_{\|x\| \leq ct} C(t, x) \geq 1 + \varepsilon,
\] (2.14)

and

\[
\lim_{t \to \infty} \sup_{\|x\| \leq ct} F(t, x) = 0,
\] (2.15)

provided that

\[ g < \frac{\min\{1, a\}}{\min\{1, a\} + s} \text{ or } bd \geq \frac{c^*}{1 - g}. \]

2. if \( d = 1 \), one has:

\[
\lim_{t \to \infty} \sup_{\|x\| \leq ct} \left( F(t, x) + |C^* - C(t, x)| + |H^* - H(t, x)| \right) = 0,
\] (2.16)

provided that

\[ g < \frac{\min\{1, a\}}{\min\{1, a\} + s} \text{ or } b \geq \frac{c^*}{1 - g}. \]

**Remark 2.11** Note that, if we assume \( b \leq 1 \) as in [17] for the ecological motivation that the intrinsic growth rate of hunter-gatherers is supposed to be smaller than or equal to that of converted farmers, then for the first statement in Theorem 2.10, the condition \( bd \geq c^*/(1 - g) \) could hold for large enough \( d \).

### 2.2 Logarithmic correction

Recall that, in Figure 3 and Figure 4, the behaviors of the \( F \)-component are slightly different on the wavefront. In Figure 4, it is described by Corollary 3.2 that the distribution of initial farmers converges to 0 everywhere. However, in Figure 3, the distribution of initial farmers is peaked with leading and trailing edges converging to 0. The results stated in Theorem 2.4 and Theorem 2.5 can not explain why a small peak may occur on the wavefront. More precisely, the spreading speed being equal to \( c^* \) does not mean that the front propagates parallel to the traveling wave of speed \( c^* \). Even for the scalar KPP equation, this does not hold true. Therefore, we are motivated to consider the behaviors of solutions in the area enough close to \( c^* t \).

A famous result of Bramson in [3, 4] showed that for the scalar KPP equation, there is a backward phase drift of order \( O(\log t) \) from the position \( c^*(f)t \). More precisely, Bramson gave a sharp asymptotics of the location of the level sets of the solution \( u(t, x) \) of the scalar KPP equation by using some probabilistic arguments. Let
$E_m(t)$ be the set of points in $(0, +\infty)$ where $u(t, \cdot) = m$ and $m \in (0, 1)$. Then, there exists a constant $B$ and a shift $x_m$ depending on $m$ and the initial data $u_0$ such that

$$E_m(t) \subset [c^*(f) t - \frac{3}{2\lambda^*} \ln t - x_m - \frac{B}{t}, c^*(f) t - \frac{3}{2\lambda^*} \ln t - x_m + \frac{B}{t}]$$

for $t$ large enough, with $\lambda^* = c^*(f)/2$. Recently, this result has been explained in simple PDE terms by Hamel et al. [13]. They showed that, for every $m \in (0, 1)$ there exists $B > 0$ such that

$$E_m(t) \subset [c^*(f) t - \frac{3}{2\lambda^*} \ln t - B, c^*(f) t - \frac{3}{2\lambda^*} \ln t + B]$$

for $t$ large enough.

Moreover, Ducrot [6] extended this proposition to the multi-dimensional case by showing that there is a logarithmic backward phase drift.

For the system (1.1), the estimate of the upper bound for the level set is rather straightforward. However, in the final zone, since three components heavily interact with each other, it is hard to estimate the level set of the system. Hence, we just provide a slightly weak result in the present paper. Our precise result reads as:

**Theorem 2.12** For any $R > 0$, for the solution $(F, C, H)$ of the system (1.1) with the initial data (1.2), it holds:

$$\liminf_{t \to +\infty} \inf_{x \in B_R, e \in SN-1} (F + C) \left( t, x + \left( c^* t - \frac{(N + 2)c^*}{\min \{1, a\}} \ln t \right) e \right) > 0 \text{ if } a > 1 + s, \quad (2.17)$$

$$\limsup_{t \to +\infty} \sup_{x \in B_R, e \in SN-1} H \left( t, x + \left( c^* t - \frac{(N + 2)c^*}{\min \{1, a\}} \ln t \right) e \right) < 1 \text{ if } a > 1 + s. \quad (2.18)$$

Moreover, if $d = 1$, it holds:

$$\liminf_{t \to +\infty} \inf_{x \in B_R, e \in SN-1} C \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) > 0 \text{ if } a < 1 + s, \quad (2.19)$$

$$\limsup_{t \to +\infty} \sup_{x \in B_R, e \in SN-1} H \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) < 1 \text{ if } a < 1 + s, \quad (2.20)$$

$$\liminf_{t \to +\infty} \inf_{x \in B_R, e \in SN-1} H \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) > 0 \text{ if } a \neq 1 + s. \quad (2.21)$$

Furthermore, for the special case $a = 1 + s$ and $d = 1$, it holds:

$$\liminf_{t \to +\infty} \inf_{x \in B_R, e \in SN-1} (F + C) \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) > 0, \quad (2.22)$$

$$\limsup_{t \to +\infty} \sup_{x \in B_R, e \in SN-1} H \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) < 1, \quad (2.23)$$

$$\liminf_{t \to +\infty} \inf_{x \in B_R, e \in SN-1} H \left( t, x + \left( c^* t - \frac{N}{c^*} \ln t \right) e \right) > 0. \quad (2.24)$$

**Proposition 2.13** If $a > 1 + s$, then for the solution $(F, C, H)$ of the system (1.1) with initial data (1.2), for any $c < c^*$, it holds:

$$\limsup_{t \to +\infty} \sup_{|x| \geq ct} F(t, x) > 0. \quad (2.25)$$

**Remark 2.14** By combining with Theorem 2.3 and Theorem 2.10, Proposition 2.13 explains the reason why a small peak of initial farmers can be observed on the wavefront just in the case $a > 1 + s$ and $g < 1$. 

8
3 Upper estimates on the spreading speeds

In this section, we deal with the spreading properties of solutions of the system (1.1) on the leading edge and complete the proof of Theorem 2.3. Since there is no interaction between farmers and hunter-gatherers on the leading edge, the analysis of this zone is rather straightforward. We first prove that the initial farmers $F$ and converted farmers $C$ cannot propagate faster than the speed $c^*$. Hence, on the leading edge, the contact between farmers and hunter-gatherers will never happen. Therefore, the population density of hunter-gatherers remain unchanged in this zone. This follows from a simple comparison argument.

We first concern with the analysis of the $F$-component and $C$-component on the leading edge and begin the proof with some simple upper estimates on the spreading speed. Note that, the nonlinear term $aF(1 - C - F)$ in the $F$-equation is nonincreasing on the value of $C$, and the $C$-component is always nonnegative. Since $c^* \geq 2\sqrt{a}$, for all $e \in S^{N-1}$, one can construct a well-known super-solution $F_e := A_1 e^{-c^*(x \cdot e - c^* t)/2}$ of the equation

$$\partial_t F = \Delta F + aF(1 - F).$$

Applying the comparison principle, one has

$$\lim_{t \to +\infty} \sup_{\|x\| \geq ct} F(t, x) \leq \lim_{t \to +\infty} \sup_{\|x\| \geq ct} \inf_{x \in S^{N-1}} F_e(t, x) = 0,$$

provided that $A_1$ is large enough such that $F_e(0, x) \geq F(0, x)$.

**Remark 3.1** Even if $1 + s > a$, one may find that the function $F_e^* = A^* e^{-\sqrt{a}x \cdot e - 2\sqrt{a}t}$ is always a super-solution of the $F$-component, provided that $A^*$ is large enough. It implies that, if $1 + s > a$, then for any $c > 2\sqrt{a}$, it holds:

$$\lim_{t \to +\infty} \sup_{\|x\| \geq ct} F(t, x) = 0.$$

Moreover, concluding from Theorem 2.10 and Remark 3.1, the following corollary is an immediate result.

**Corollary 3.2** If $g < 1$ and $a < 1 + s$, then the solution of the system (1.1) with the initial data (1.2) satisfies:

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^N} F(t, x) = 0,$$

provided that

$$g < \frac{\min\{1, a\}}{\min\{1, a\} + s} \text{ or } db \geq \frac{c^*}{1 - g}.$$

To get the estimate of the $C$-component, we need to construct another suitable super-solution. Note that the nonlinear term $C(1 - C - F) + s(C + F)H$ in the $C$-equation is neither monotonous increasing nor decreasing with respect to the value of $F$. Hence, to construct a super-solution for the $C$-component, it is necessary to control the value of $F$ suitably. However, thanks to the fact that the $F$-component is sufficient small on the leading edge, one just need to deal with the following equation

$$\partial_t C = \Delta C + C(1 - C) + s(C + F_e).$$

Then, we introduce the new function

$$\overline{C}_e := A_2 e^{-\lambda(x \cdot e - ct)} \text{ for all } c > c^*, e \in S^{N-1}.$$

One may find that, for large enough $A_2 > 0$, there exists $\lambda = c^*/2$ such that $\overline{C}_e$ satisfies

$$\partial_t \overline{C}_e - \Delta \overline{C}_e - \overline{C}_e(1 + s - \overline{C}_e) - sF_e \geq \left((c\lambda - \lambda^2 - 1 - s)A_2 - sA_1\right)e^{-c^*/2(x \cdot e - ct)} \geq 0.$$
Thus \( C_e \) is a super-solution of the equation (3.3) for any \( t \geq 0 \) and \( x \in \mathbb{R}^N \).

Moreover, \( C(0, x) = 0 \) for any \( x \in \mathbb{R}^N \). We can now apply the comparison principle again to conclude that for all \( c' > c \), \[
\lim_{t \to +\infty} \sup_{\|x\| \geq c't} C(t, x) < \lim_{t \to +\infty} \sup_{\|x\| \geq c't} \inf_{e \in S^{N-1}} C_e(t, x) = 0. \tag{3.4}
\]

For the reason that \( c \) can be chosen arbitrarily close to \( c^* \), one can conclude that the \( C \)-component does not spread faster that the spreading speed \( c^* \).

To complete this section, we deal with the upper estimate of the \( H \)-component on leading edge. One may find that the \( H \)-component is always able to stay positive outside of the farmer’s range. The main idea is to construct a suitable sub-solution for the \( H \)-component as follows:

\[
H_e := 1 - g(A_1 + A_2)e^{-c^*(x - e - ct)/(2d)} \quad \text{for all } c > c^*, e \in S^{N-1}.
\]

In previous part of this subsection, we have shown that the \( F \)-component and \( C \)-component can be controlled from above for all \( c > c^* \) and \( e \in S^{N-1} \) by

\[
A_1 e^{-c^*(x - e - ct)/2} \text{ and } A_2 e^{-c^*(x - e - ct)/2}.
\]

Since \( bH(1 - H - g(F + C)) \) is nonincreasing with respect to the value of \( F + C \), we just need to construct sub-solution for \( H \) that satisfies

\[
\begin{cases}
\partial_t H = d\Delta H + bH(1 - g(A_1 + A_2)e^{-c^*(x - e - ct)/(2d)} - H), \\
H(0, x) = 1.
\end{cases}
\]

(3.5)

Furthermore, since \( d \geq 1 \), one has

\[
(A_1 + A_2)e^{-c^*(x - e - ct)/(2d)} \geq (A_1 + A_2)e^{-c^*(x - e - ct)/2} \quad \text{for all } x \cdot e \geq ct.
\]

Hence, for each \( e \in S^{N-1} \), \( H_e \) is a sub-solution of the equation (3.5). More precisely, one has

\[
\begin{align*}
\partial_t H_e - d\Delta H_e - bH_e \left( 1 - g(A_1 + A_2)e^{-c^*(x - e - ct)/(2d)} - H_e \right) \\
\leq - \left( cc^*/2d - (c^*)^2/4d \right) \left( g(A_1 + A_2)e^{-c^*(x - e - ct)/(2d)} \right) \leq 0.
\end{align*}
\]

Moreover, on the one hand, since \( H(0, x) = 1 \), one gets that

\[
H(0, x) \geq H_e(0, x).
\]

On the other hand, one can choose \( A_1 \) and \( A_2 \) large enough such that, for any \( e \in S^{N-1} \),

\[
H_e(t, x) < 0 \leq H(t, x), \quad t \geq 0, \quad x \cdot e = ct.
\]

Now, one can now infer from the comparison principle that

\[
\lim_{t \to \infty} \inf_{\|x\| \geq c't} H(t, x) \geq \lim_{t \to \infty} \inf_{\|x\| \geq c't} \sup_{e \in S^{N-1}} H_e(t, x) = 1 \quad \text{for all } c' > c.
\]

Since we can choose \( c \) arbitrarily close to \( c^* \), the proof of Theorem 2.3 is complete.

4 Logarithmic Bramson correction

At first, we would like to introduce an important result provided in [6, 13]. Consider the linear equation with drift term as follows:

\[
\partial_t z = \partial^2 z + \left( c^* - \frac{\delta}{t + t_0} + \frac{N - 1}{\xi + s t_0(t)} \right) \partial_x^2 z + \lambda^2 z, \quad t > 0, \quad \xi > 0,
\]

(4.1)
where

\[ \lambda^* = c^*/2 \quad \text{and} \quad \xi^\delta(t) := c^*(t + t_0) - \delta \ln \frac{t + t_0}{t_0}. \]

Then the following proposition holds:

**Proposition 4.1** Let \( z^\delta_{t_0}(\xi, t) \) be the solution of the equation (4.1) with boundary condition

\[ \xi^\delta(t, 0) = 0 \quad \text{for all} \quad t > 0, \]

and the initial data

\[ \xi^\delta(t_0, \xi) = e^{-\lambda^*\xi} \zeta_0(t_0^{-1/2}\xi) \geq 0 \quad \text{for all} \quad \xi \geq 0, \]

where \( \zeta_0(\cdot) \) is a nontrivial compactly supported smooth function. Then it holds:

\[ \xi^\delta(t, \xi) = \frac{(t + t_0)^{\gamma-\frac{1}{2}}}{t_0} \xi e^{-\lambda^*\xi} \left\{ \int_0^\infty \zeta_0(\rho)d\rho + h_1(t, t_0) \right\} e^{-\frac{\rho^2}{t + t_0} + h_2(t, \xi, t_0)}, \quad \xi \geq 0, \quad t \geq 0, \quad (4.2) \]

where \( \gamma := \delta \lambda^* - \frac{N+1}{2} \), \( h_1 \) and \( h_2 \) are smooth functions satisfying

\[
\begin{align*}
|h_1(t, t_0)| &\leq B_1 t_0^{-1/2} \|\zeta_0\|_m, \\
|h_2(t, \xi, t_0)| &\leq B_2 \left\{ \int_0^1 \|\zeta_0\|_m + \left( \frac{t_0}{t + t_0} \right)^{\frac{\rho^2}{t + t_0}} \right\} e^{-\frac{\rho^2}{t_0^2 + t_0}}, \quad \xi \geq 0, \quad t \geq 0,
\end{align*}
\]

for some positive constants \( B_1 \) and \( B_2 \). Here, the norm \( \|\|_m \) is defined as

\[ \|\zeta_0\|_m^2 := \int_0^\infty \zeta_0(\rho)^2 e^\frac{\rho^2}{t_0^2}d\rho. \]

Note that, this proposition is slightly different from those of [6] [13]. Hence, we will give the proof in the appendix for the sake of completeness.

Now, let us consider the solution \( (F, C, H) \) of the system (1.1) with spherically symmetric initial data \((F_0, C_0, H_0)\). By spatially homogeneity of the system and the uniqueness of the solution, the solution \((F, C, H)\) is also spherically symmetric. By changing the variables, the solution \((F(t, r), C(t, r), H(t, r))\) where \(r = ||x||\) satisfies the following one-dimensional system

\[
\begin{align*}
\partial_t F &= \partial^2_r F + \frac{N-1}{r} \partial_r F + aF(1 - F - C), \\
\partial_t C &= \partial^2_r C + \frac{N-1}{r} \partial_r C + C(1 - F - C) + sH(F + C), \quad \xi > 0, \quad t > 0, \quad (4.3) \\
\partial_t H &= d\partial^2_r H + d\frac{N-1}{r} \partial_r H + bH(1 - H - g(F + C)),
\end{align*}
\]

We consider the moving frame as \( \xi = r - \xi_{t_0}(t) \). Then, the functions

\[(F(t, \xi), C(t, \xi), H(t, \xi)) := (F(t, \xi + \xi_{t_0}(t)), C(t, \xi + \xi_{t_0}(t)), H(t, \xi + \xi_{t_0}(t))\]

satisfy the system as follows:

\[
\begin{align*}
\partial_t F &= \partial^2_r F + \left( c^* - \frac{N + 2}{2\lambda^*(t + t_0)} + \frac{N-1}{\xi + \xi_{t_0}(t)} \right) \partial_r F + aF(1 - F - C), \\
\partial_t C &= \partial^2_r C + \left( c^* - \frac{N + 2}{2\lambda^*(t + t_0)} + \frac{N-1}{\xi + \xi_{t_0}(t)} \right) \partial_r C + C(1 - F - C) + sH(F + C), \quad \xi > -\xi_{t_0}(t), \quad t > 0. \\
\partial_t H &= d\partial^2_r H + d\left( c^* - \frac{N + 2}{2\lambda^*(t + t_0)} + \frac{N-1}{\xi + \xi_{t_0}(t)} \right) \partial_r H + bH(1 - H - g(F + C)),
\end{align*}
\]
4.1 Upper estimates on the location of the wavefront

Moreover, for any \( \mu > (a + s)/2 \), and \((\bar{u}(t, \xi), \bar{v}(t, \xi))\) be the solution of

\[
\begin{aligned}
\partial_t \bar{u} &= \partial^2_{\xi} \bar{u} + \left(2e^a - \frac{\delta^*}{t + t_0} + \frac{N - 1}{\xi + \xi_0^d(t)}\right) \partial_\xi \bar{u} + a\bar{u}, \\
\partial_t \bar{v} &= \partial^2_{\xi} \bar{v} + \left(2e^a - \frac{\delta^*}{t + t_0} + \frac{N - 1}{\xi + \xi_0^d(t)}\right) \partial_\xi \bar{v} + (1 + s)\bar{v} + s\bar{u},
\end{aligned}
\]

with the boundary condition \((\bar{u}(t, 0), \bar{v}(t, 0)) = (0, 0)\), and with compactly supported initial data

\[\bar{u}_0(\xi) = \frac{a - 1 - s}{s} \bar{v}_0(\xi) = e^{-\lambda^* \xi} \xi^0(t_0^{-1/2}) \text{ for all } \xi \geq 0.\]

Then under the same notation as in Proposition 4.1, one has

\[\left(\bar{u}(t, \xi), \bar{v}(t, \xi)\right) = \left(z_0^{\delta^*}(t, \xi), \frac{s}{a - 1 - s} \xi_0^d(t, \xi)\right) \text{ for all } \xi \geq 0, \ t \geq 0.\]

Moreover, for any \( \mu > 0 \), \( (\bar{u}^*(t, x), \bar{v}^*(t, x)) := \mu(\bar{u}(t, \|x\| - \xi_0^d(t)), \bar{v}(t, \|x\| - \xi_0^d(t))) \) satisfies

\[
\begin{aligned}
\partial_t \bar{u}^* &\geq \Delta \bar{u}^* + a\bar{u}^*(1 - F - C), \\
\partial_t \bar{v}^* &\geq \Delta \bar{v}^* + \bar{v}^*(1 - F - C) + sH(\bar{u}^* + \bar{v}^*),
\end{aligned}
\]

\[\xi > \xi_0^d(t), \ t > 0,\]

where \((F, C, H)\) is the solution of the system (1.1) with the initial data (1.2).

Next, we define the functions \(F(t, x)\) and \(C(t, x)\) as:

\[
F(t, x) = \begin{cases} 
B^*, & \|x\| \leq \xi_0^d(t) + A, \\
\min\{B^*, \mu \bar{u}(t, \|x\| - \xi_0^d(t))\}, & \|x\| > \xi_0^d(t) + A,
\end{cases}
\]

\[
C(t, x) = \begin{cases} 
1 + s, & \|x\| \leq \xi_0^d(t) + A, \\
\min\{1 + s, \mu \bar{v}(t, \|x\| - \xi_0^d(t))\}, & \|x\| > \xi_0^d(t) + A,
\end{cases}
\]

where \(B^* := \sup_{(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N} F(t, x)\). Furthermore, in the case of \(d = 1\), we define \(H(t, x)\) as

\[
H(t, x) = \begin{cases} 
0, & \|x\| \leq \xi_0^* + A, \\
\max\{0, 1 - g\mu(a(t, \|x\| - \xi_0^d(t)) + v(t, \|x\| - \xi_0^d(t)))\}, & \|x\| > \xi_0^*(t) + A.
\end{cases}
\]

Note that, we choose constant \(\mu > 0\) large enough such that, for all \(t \geq 0\),

\[
\mu \bar{u}(t, A) = \mu \xi_0^d(t), \ t > B^*, \quad \mu \bar{v}(t, A) = \frac{\mu \xi_0^d(t)}{a - 1 - s} > 1 + s,
\]

\[
\mu g(\bar{u}(t, A) + \bar{v}(t, A)) = \frac{\mu g(a - 1)}{a - 1 - s} \xi_0^* > 1.
\]

We remark that by applying Proposition 4.1 such \(\mu > 0\) always exists for sufficiently large \(t_0 > 0\). Then by applying the comparison principle, one obtain

\[F(t, x) \leq F(t, x), C(t, x) \leq C(t, x), H(t, x) \geq H(t, x) \text{ for all } x \in \mathbb{R}^N, \ t > 0. \quad (4.4)\]
Remark 4.2  Note that, for the case \( d > 1 \), the functions \( \mathcal{F} \) and \( \mathcal{C} \) are still super-solutions for the \( F \)-component and the \( C \)-component. However, the function \( \mathcal{H} \) could no longer be a suitable sub-solution for the \( H \)-component.

Next, we deal with the construction for the case \( a = 1 + s \) and \( d = 1 \). Under the same notation as in Proposition 4.1, we consider
\[
(\hat{u}(t, \xi), \hat{v}(t, \xi)) = \left( z_{\delta_0}^*(t, \xi), (1 + st)z_{\delta_0}^*(t, \xi) \right) \quad \text{for all } \xi \geq 0, \ t \geq 0,
\]
where \( z_{\delta_0}^*(\xi, t) \) is the solution of the equation (4.1) with boundary condition \( z_{\delta_0}^*(t, 0) = 0 \) for all \( t > 0 \). Then, we define
\[
(\hat{u}^*(t, x), \hat{v}^*(t, x)) := \mu(\hat{u}(t, \|x\| - \xi_{\delta_0}^*(t)), \hat{v}(t, \|x\| - \xi_{\delta_0}^*(t))),
\]
which satisfies
\[
\begin{align*}
\partial_t \hat{u}^* &= \Delta \hat{u}^* + a \hat{v}^*, \\
\partial_t \hat{v}^* &= \Delta \hat{v}^* + (1 + s) \hat{v}^* + s \hat{u}^*,
\end{align*}
\quad \xi \geq \xi_{\delta_0}^*(t), \ t > 0.
\]
Then we choose constant \( \mu > 0 \) large enough such that, for all \( t \geq 0 \),
\[
\begin{align*}
\mu \hat{u}(t, A) &= \mu z_{\delta_0}^*(t, A) > B^*, \quad \mu \hat{v}(t, A) = \mu (1 + st)z_{\delta_0}^*(t, A) > 1 + s, \\
g \mu(\hat{u}(t, A) + \hat{v}(t, A)) &= \mu g(2 + st)z_{\delta_0}^*(t, A) > 1.
\end{align*}
\]
Then we construct super and sub-solutions as that for the case \( a > 1 + s \),
\[
\hat{F}(t, x) = \begin{cases} 
B^*, & \|x\| \leq \xi_{\delta_0}^*(t) + A, \\
\min\{B^*, \mu \hat{u}(t, \|x\| - \xi_{\delta_0}^*(t))\}, & \|x\| \geq \xi_{\delta_0}^*(t) + A,
\end{cases}
\]
\[
\hat{C}(t, x) = \begin{cases} 
1 + s, & \|x\| \leq \xi_{\delta_0}^*(t) + A, \\
\min\{1 + s, \mu \hat{u}(t, \|x\| - \xi_{\delta_0}^*(t))\}, & \|x\| \geq \xi_{\delta_0}^*(t) + A,
\end{cases}
\]
\[
\hat{H}(t, x) = \begin{cases} 
0, & \|x\| \leq \xi_{\delta_0}^*(t) + A, \\
\max\{0, 1 - g \mu(\hat{u}(t, \|x\| - \xi_{\delta_0}^*(t)) + \hat{v}(t, \|x\| - \xi_{\delta_0}^*(t)))\}, & \|x\| \geq \xi_{\delta_0}^*(t) + A.
\end{cases}
\]
where \( B^* := \sup_{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^N} F(t, x) \). By applying the comparison principle, one has
\[
F(t, x) \leq \hat{F}(t, x), \ C(t, x) \leq \hat{C}(t, x), \ H(t, x) \geq \hat{H}(t, x) \quad \text{for all } x \in \mathbb{R}^N, \ t > 0. \quad (4.5)
\]

Next, we deal with the construction for the case \( a < 1 + s \). Recall that, in Remark 4.1, we showed that \( \hat{F}^*(t, x) = \min\{A^*, A^* e^{-\sqrt{d}\|x\|-2\sqrt{d}t}\} \geq F(t, x) \) for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N \). Now, let us consider super-solutions of the \( F \)-component and \( C \)-component as
\[
\hat{F}(t, x) = \begin{cases} 
\hat{F}^*(t, x), & \|x\| \leq \xi_{\delta_0}^*(t) + A, \\
\min\{\hat{F}^*(t, x), \mu \hat{u}(t, \|x\| - \xi_{\delta_0}^*(t))\}, & \|x\| \geq \xi_{\delta_0}^*(t) + A,
\end{cases}
\]
\[
\hat{C}(t, x) = \begin{cases} 
1 + s, & \|x\| \leq \xi_{\delta_0}^*(t) + A, \\
\min\{1 + s, \mu \hat{u}(t, \|x\| - \xi_{\delta_0}^*(t))\}, & \|x\| \geq \xi_{\delta_0}^*(t) + A,
\end{cases}
\]
and in the case of \( d = 1 \), we consider a sub-solution of the \( H \)-component as
\[
\hat{H}(t, x) = \begin{cases} 
0, & \|x\| \leq \xi_{\delta_0}^*(t) + A, \\
\min\{0, 1 - g \mu(\hat{u}(t, \|x\| - \xi_{\delta_0}^*(t)) + \hat{v}(t, \|x\| - \xi_{\delta_0}^*(t)))\}, & \|x\| \geq \xi_{\delta_0}^*(t) + A,
\end{cases}
\]
where the function \((\tilde{u}(t, \xi), \tilde{v}(t, \xi))\) is the solution of

\[
\begin{align*}
\partial_t \tilde{u} &= \partial_{xx}^2 \tilde{u} + \left(2e^s - \frac{\delta^*}{t + t_0} + \frac{N - 1}{\xi + \xi^*_{t_0}(t)} \right) \partial_t \tilde{u} + (1 + s - \varepsilon)\tilde{u}, \\
\partial_t \tilde{v} &= \partial_{xx}^2 \tilde{v} + \left(2e^s - \frac{\delta^*}{t + t_0} + \frac{N - 1}{\xi + \xi^*_{t_0}(t)} \right) \partial_t \tilde{v} + (1 + s)\tilde{v} + s\tilde{u},
\end{align*}
\]

\(\xi > 0, t > 0,
\]

with Dirichlet boundary condition \((\tilde{u}(t, 0), \tilde{v}(t, 0)) = (0, 0)\) for all \(t > 0\) and compactly supported initial data \(\tilde{u}(0, \xi) = \tilde{\nu}(0, \xi) = e^{-\lambda \xi} \xi_0(t_0^{1/2} \xi)\). Let us choose \(\varepsilon = \sqrt{a}(\sqrt{1 + s} - \sqrt{a})\). Then under the same notation as that in Proposition 4.1,

\[
(\tilde{u}(t, \xi), \tilde{v}(t, \xi)) = \left( e^{-\varepsilon t \xi^*_{t_0}}(t, \xi), \frac{s(2 - e^{-\varepsilon t})}{\varepsilon} \xi^*_{t_0}(t, \xi) \right)
\]

Here, we choose \(\mu > 0\) large enough such that

\[
\mu\tilde{u}(t, A) = \mu e^{-\varepsilon t \xi^*_{t_0}}(t, A) > \tilde{F}^*(t, x)\|x\| = \xi^*_{t_0}(t) + A, \quad \mu\tilde{v}(t, A) = \frac{2s(2 - e^{-\varepsilon t})}{\varepsilon} \xi^*_{t_0}(t, A) > 1 + s,
\]

\[
\mu g(\tilde{u}(t, A) + \tilde{v}(t, A)) = \mu g \frac{2s + (1 - s)e^{-\varepsilon t}}{\varepsilon} \xi^*_{t_0}(t, A) > 1.
\]

We remark that inferring from Proposition 4.1 such \(\mu\) always exists for sufficiently large \(t_0 > 0\). Then, for the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2),

\[
(\tilde{u}^*(t, x), \tilde{v}^*(t, x)) := (\mu\tilde{u}(t, \|x\| - \xi^*_{t_0}(t)), \mu\tilde{v}(t, \|x\| - \xi^*_{t_0}(t))
\]

satisfies

\[
\left\{ \begin{array}{l}
\partial_t \tilde{u}^* \geq \Delta \tilde{u}^* + a\tilde{u}^*(1 - F - C), \\
\partial_t \tilde{v}^* \geq \Delta \tilde{u}^* + \tilde{v}^*(1 - F - C) + sH(\tilde{u}^* + \tilde{v}^*),
\end{array} \right. \quad \|x\| > \xi^*_{t_0}(t), t > 0.
\]

By applying the comparison principle, one has

\[
F(t, x) \leq \tilde{F}(t, x), \quad C(t, x) \leq \tilde{C}(t, x), \quad H(t, x) \geq \tilde{H}(t, x), \quad \text{for all} \ x \in \mathbb{R}^N, t > 0.
\]

Then, the following propositions are an immediate result from the upper estimates (4.4), (4.5) and (4.6).

**Proposition 4.3** If \(a \neq 1 + s\), for the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2), it holds:

\[
\limsup_{t \to +\infty} \sup_{c^* \frac{\beta + 3}{\beta - 1} \ln t + r \leq \|x\|} F(t, x) \to 0 \quad \text{as} \ r \to +\infty, \quad (4.7)
\]

\[
\limsup_{t \to +\infty} \sup_{c^* \frac{\beta + 3}{\beta - 1} \ln t + r \leq \|x\|} C(t, x) \to 0 \quad \text{as} \ r \to +\infty. \quad (4.8)
\]

Moreover, if \(d = 1\), one has:

\[
\liminf_{t \to +\infty} \inf_{c^* \frac{\beta + 3}{\beta - 1} \ln t + r \leq \|x\|} H(t, x) \to 1 \quad \text{as} \ r \to +\infty. \quad (4.9)
\]

**Proposition 4.4** If \(a = 1 + s\), for the solution \((F, C, H)\) of the system (1.1) with the initial data (1.2), it holds:

\[
\limsup_{t \to +\infty} \sup_{c^* \frac{\beta + 3}{\beta - 1} \ln t + r \leq \|x\|} F(t, x) \to 0 \quad \text{as} \ r \to +\infty \quad (4.10)
\]

\[
\limsup_{t \to +\infty} \sup_{c^* \frac{\beta + 3}{\beta - 1} \ln t + r \leq \|x\|} C(t, x) \to 0 \quad \text{as} \ r \to +\infty. \quad (4.11)
\]

Moreover, if \(d = 1\), one has:

\[
\liminf_{t \to +\infty} \inf_{c^* \frac{\beta + 3}{\beta - 1} \ln t + r \leq \|x\|} H(t, x) \to 1 \quad \text{as} \ r \to +\infty. \quad (4.12)
\]
Furthermore, the following proposition which is concerned with the upper estimates of the location of the wavefront holds:

**Proposition 4.5** If \( d = 1 \), then for the solution \( (F, C, H) \) of the system (1.1) with the initial data (1.2) and for any \( R > 0 \), it holds:

\[
\liminf_{t \to +\infty} \inf_{|x| \leq R, r \in S^{N-1}} H \left( t, x + \left( c^* t - \frac{N + 2}{2c^*} \ln t \right) e \right) > 0 \text{ if } a \neq 1 + s, \tag{4.13}
\]

\[
\liminf_{t \to +\infty} \inf_{|x| \leq R, r \in S^{N-1}} H \left( t, x + \left( c^* t - \frac{N}{2c^*} \ln t \right) e \right) > 0 \text{ if } a = 1 + s. \tag{4.14}
\]

**Proof of Proposition 4.5** We first deal with the proof the statement (4.13). To proceed by contradiction, we assume that there exist \( r_0 \in \mathbb{R}, e_\infty \in S^{N-1} \) and a sequence of times \( t_n \to +\infty \) such that

\[
H \left( t_n, \left( c^* t_n - \frac{N + 2}{2c^*} \ln t_n + r_0 \right) e_\infty \right) \to 0 \text{ as } n \to +\infty.
\]

Up to extraction of a subsequence, the functions \( H_n(t, x) = H(t + t_n, x + c^* t_n - \frac{N + 2}{c^*} \ln t_n + r_0) e_\infty \) converge locally uniformly to an entire solution \( H_\infty \) that satisfies

\[
\partial_t H_\infty = \Delta H_\infty + bH_\infty (1 - gF_\infty - gC_\infty - H_\infty).
\]

Since \( 0 \leq H_\infty \leq 1 \) in \( \mathbb{R} \times \mathbb{R} \) and \( H_\infty(0, 0) = 0 \). The strong maximum principle implies that \( H_\infty \equiv 0 \). However, from the estimate (4.13), \( H_\infty(0, r e_\infty) \geq 1/2 \) for \( r \) large enough. One has reached a contradiction which implies that the estimate (4.13) holds true. Since the (4.14) follows from the same argument, the proof of this proposition is thereby complete. \( \square \)

### 4.2 Lower estimates on the location of the wavefront

In this subsection, we complete the proof of Theorem 2.12 by showing lower estimates for the total population \( F + C \) of farmers, and the upper estimate for the \( H \) at the position \( c^* t - O(\ln t) \). Let us first consider the case \( a = 1 + s \), which can be proved by simple comparison argument.

**Proposition 4.6** Let \( (F, C, H) \) be the solution of the system (1.1) with the initial data (1.2) and \( G(t, x) := F(t, x) + C(t, x) \). If \( a = 1 + s \) and \( d = 1 \), then for any \( R > 0 \), it holds:

\[
\liminf_{t \to +\infty} \inf_{|x| \leq R, r \in S^{N-1}} G \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) > 0, \tag{4.15}
\]

\[
\limsup_{t \to +\infty} \sup_{|x| \leq R, r \in S^{N-1}} H \left( t, x + \left( c^* t - \frac{N + 2}{c^*} \ln t \right) e \right) < 1. \tag{4.16}
\]

**Proof of Proposition 4.6** In the case of \( d = 1 \), we know that \( 1 - \max \{1, g\} (F + C) \) is a suitable sub-solution of \( H \), such that

\[
\frac{\partial_t H - \Delta H}{(F + C)(1 - F - C) + sHC} \geq (aF + C)(1 - G) + s(1 - \max \{1, g\}) C
\]

\[
= aG - (aF + C + \max \{1, g\} sC) G \geq (a - kG) G,
\]

where \( k := \max \{a, 1 + \max \{1, g\} sC\} \). Then by applying the argument in [13] for scalar KPP equation and Proposition 4.1 one can complete the proof of Proposition 4.6. \( \square \)

Next, we deal with the case \( a > 1 + s \). Note that, in this case, our estimate is not very sharp, since the coefficient \( c^*(N + 2)/\min \{1, a\} \) in front of the \( \ln t \) is greater than \( (N + 2)/c^* \) which has been proved by Bramson and Hamel et al. for the scalar KPP equation.
Proposition 4.7 Let \((F,C,H)\) be the solution of the system \((1.1)\) with the initial data \((1.2)\) and \(G(t,x) := F(t,x) + C(t,x)\). If \(a > 1 + s\), then for any \(R > 0\), it holds:

\[
\liminf_{t \to +\infty} \inf_{|x| \leq R, e^{S^N_1}} G(t,x) + \left( c^* t - \frac{c^*(N+2)}{\min\{1,a\}} \ln t \right) < 0, \quad (4.17)
\]

\[
\limsup_{t \to +\infty} \sup_{|x| \leq R, e^{S^N_1}} H(t,x) + \left( c^* t - \frac{c^*(N+2)}{\min\{1,a\}} \ln t \right) < 1. \quad (4.18)
\]

Remark 4.8 Proposition 4.7 holds for any \(d \geq 1\). Indeed, for all \(d \geq 1\), the functions \(F^d\) and \(C^d\) introduced in subsection 4.1 are always suitable super-solutions of the \(F\)-component and the \(C\)-component. However, the function \(H^d\) is not a sub-solution of the \(H\)-component anymore if \(d \neq 1\). As a matter of fact, in the proof of Proposition 4.7 only the properties of \(F\) and \(C\) will be used.

To prove Proposition 4.7, one need to construct a suitable sub-solution of \(G\) near the wavefront. To do this, a lower estimate of \(F\) in the region which moves with a speed slightly faster that \(c^*\) is very important.

Lemma 4.9 Let \((F,C,H)\) be the solution of the system \((1.1)\) with the initial data \((1.2)\). If \(a > 1 + s\), then there exist \(A_2 > 0\) and \(t_0 > 0\) such that the following holds

\[
F(t,x) \geq A_2( t + t_0) - \frac{N+2}{2} (\|x\| - c^* (t + t_0)) e^{-\lambda^* (\|x\| - c^*(t + t_0))}, \quad (4.19)
\]

where

\[
c^* (t + t_0) \leq \|x\| \leq c^* (t + t_0) + \sqrt{t + t_0} \quad \text{and} \quad t > 1.
\]

Proof of Lemma 4.9 Let \(z_{t_0}(t,\xi) (\xi \geq \xi_{t_0}^0(t), t \geq 0)\) be the solution of

\[
\begin{cases}
\partial_t z_{t_0} = \Delta z_{t_0} + \left( c^* + \frac{N+2}{\xi + \xi_{t_0}^0(t)} \right) \partial_\xi z_{t_0} + \lambda z_{t_0}, & \xi > 0, t > 0,
z_{t_0}(t,0) = 0, & t > 0,
z_{t_0}(0,\xi) = e^{-\lambda^* \xi} \xi_0(0,\xi), & \xi \geq 0,
\end{cases}
\]

where \(\xi_0 \geq 0\) is a nontrivial compactly supported smooth function and

\[
c^* := 2\sqrt{a}, \quad \lambda^* = c^*/2, \quad \xi_{t_0}^0(t) := c^*(t + t_0).
\]

By applying Proposition 4.1, there exist \(t_0 > 0\), \(A_1 > 0\) and \(A_2 > 0\) such that

\[
\begin{align*}
z_{t_0}(t,\xi) &\leq A_1(t + t_0) - \frac{N+2}{\xi} \quad \text{for all} \quad \xi > 0, t > 0, \\
z_{t_0}(t,\xi) &\geq A_2(t + t_0) - \frac{N+2}{\xi} \quad \text{for all} \quad 0 < \xi < \sqrt{t + t_0}, t > 0.
\end{align*}
\]

From (4.4), one may obtain that, for \(t > 0\) and \(\xi_{t_0}^0(t) \leq \|x\| \leq \xi_{t_0}^0(t) + \sqrt{t + t_0},

\[
C(t,x) \leq \overline{C}(t,x) = \mu \nu(t,\|x\| - \xi_{t_0}^0(t)) \\
\leq A_3(\|x\| - \xi_{t_0}^0(t)) e^{-\lambda^* (\|x\| - \xi_{t_0}^0(t))} \\
\leq A_3 \delta^* \log \frac{t + t_0}{t_0} t^{-\frac{N+2}{2}},
\]

where \(\delta^* := \frac{N+2}{\lambda^*}\) and \(\xi_{t_0}^0(t) := c^*(t + t_0) - \delta \log \frac{t + t_0}{t_0} \).

Let us introduce a new function \(F^\varepsilon(t,x) := \varepsilon \omega(t) z_{t_0}(t,\|x\| - c^*(t + t_0))\). One may find that

\[
\partial_t F^\varepsilon - \Delta F^\varepsilon - a F^\varepsilon (1 - F^\varepsilon - C) = \left( \frac{\dot{\omega}}{\omega} + a F^\varepsilon + C \right) F^\varepsilon \\
\leq \left( \frac{\dot{\omega}}{\omega} + (\varepsilon \omega + c_3 \delta^* \log \frac{t + t_0}{t_0}) t^{-\frac{N+2}{2}} \right) F^\varepsilon.
\]
Then, we choose a suitable $\omega(t)$ as

$$
\omega(t) = e^{-f(t, x, \eta, e) - \log \frac{c}{\sqrt{t_0 + t}} t}.
$$

It is not difficult to check that $0 < \omega \leq 1$, $\inf t \geq t_0 \omega(t) = \omega(\infty) > 0$ and

$$
\partial_t F - \Delta F - aF(1 - F - C) \leq 0 \text{ for all } \|x\| \geq c^e(t + t_0) \text{ and } t > 0.
$$

For sufficiently small $\varepsilon > 0$, it holds $F(1, x) \geq F^e(0, x)$ for all $x \in \mathbb{R}^N$. Then, by applying the comparison principle, one obtains

$$
F(t, x) \geq F^e(t - 1, x), \quad x \in \mathbb{R}^N, \quad t > 1.
$$

Therefore, from the estimate (4.21), one can conclude that

$$
F(t, x) \geq \varepsilon \omega(\infty) c_2(t + t_0) - \frac{\eta^2}{2} (\|x\| - c^e(t + t_0)) e^{-\nu^e(\delta + c^e(t + t_0))},
$$

where

$$
c^e(t + t_0) \leq \|x\| \leq c^e(t + t_0) + \sqrt{t + t_0} \text{ and } t > 1.
$$

The proof is complete. \qed

Now we are ready to deal with the proof of Proposition 4.7.

**Proof of Proposition 4.7.** Let us denote the total population density of the farmers as $G(t, x) := F(t, x) + C(t, x)$. Then, one may find that

$$
\partial_t G - \Delta G \geq \min \{1, a\} G(1 - G) \quad \text{for} \quad \{(t, x) \mid G(t, x) \leq 1\}.
$$

We consider a new function as

$$
\phi(t, x) := \phi_{t_1/2, \eta, e}(t, x) := \eta e^{\min \{1, a\} (t - t^*)} \varphi_{t_1/2} \left( x - \left( c^e(t + t_0) + \frac{t_1 + t_0}{2} \right) e \right),
$$

where $\eta > 0$, $e \in S^{N^2 - 1}$, $t^* := \frac{N^2 + 2}{\min \{1, a\}} \log t$ and $\varphi_R(x) > 0$ is the eigenfunction satisfies

$$
\begin{cases}
-\Delta \varphi_R = \mu_R \varphi_R, & \|x\| < R, \\
\varphi_R(x) = 0, & \|x\| = R, \\
\varphi_R(0) = 1.
\end{cases}
$$

Since the eigenvalue satisfies $\mu_R = \mu_1 R^{-2}$, for sufficiently small $\eta > 0$ and sufficiently large $t_1 > 0$, one has

$$
\partial_t \phi - \Delta \phi - \min \{1, a\} \phi(1 - \phi) = \left( \mu_{t_1/2} - \frac{\min \{1, a\}}{2} + \frac{\eta}{2} \right) \phi \\
\leq \left( 4 \mu_1 t_1^{-1} - \frac{\min \{1, a\}}{2} + \eta \right) \phi \leq 0.
$$

By applying the estimate (4.19), if we choose

$$
\eta = A_2 \frac{\sqrt{t_1 + t_0} + \sqrt{t_1}}{2} e^{-\nu^e(\delta + c^e(t + t_0))} (1 + t_0/t_1)^{-\frac{N^2 + 2}{2}},
$$

then for $\|x - (c^e(t + t_0) + \sqrt{t_1 + t_0}) e\| < \frac{\sqrt{t_1}}{2}$, $t > t_1$ and $e \in S^{N^2 - 1}$, one may obtain

$$
F(t, x) \geq A_2 \frac{\sqrt{t_1 + t_0} + \sqrt{t_1}}{2} e^{-\nu^e(\delta + c^e(t + t_0))} (t + t_0)^{-\frac{N^2 + 2}{2}}
$$

$$
\geq \eta e^{-\nu^e(\delta + c^e(t + t_0))} \varphi_{t_1/2} (x - \left( c^e(t + t_0) + \frac{t_1 + t_0}{2} \right) e) = \phi_{t_1/2, \eta, e}(0, x, t).
$$
On the other hand, for \( \|x - (c^*(t + t_0) + \sqrt{t_1 + t_0})e\| \geq \sqrt{t_1}, t > t_1 \) and \( e \in S^{N-1} \), it holds obviously that 
\[
F(t, x) > \phi_{\sqrt{t_1/2}, \eta, c}(0, x, t) = 0.
\]

Hence, one has 
\[
G(t, x) \geq F(t, x) \geq \phi_{\sqrt{t_1/2}, \eta, c}(0, x, t) \quad \text{for all} \quad x \in \mathbb{R}^N, \ t > t_1 \quad \text{and} \quad e \in S^{N-1}.
\]

Thus, by applying the comparison principle, one can conclude that 
\[
G(\tau + t, x) \geq \phi_{\sqrt{t_1/2}, \eta, c}(\tau, x, t) \quad \text{for all} \quad x \in \mathbb{R}^N, \ 0 \leq \tau \leq \tau^*(t), \ t > t_1 \quad \text{and} \quad e \in S^{N-1}.
\]

Therefore, for any \( t_* > t_1 + \tau^*(t_1) \), it holds
\[
G(\tau^*(t_*) + t_*, x) \geq \phi_{\sqrt{t_1/2}, \eta, c}(\tau^*(t_*), x, t_*) = \eta \varphi_{\sqrt{t_1/2}}(x - (c^*(t_* + t_0) + \sqrt{t_1 + t_0})e).
\]

If we denote \( t = t_* + \tau^*(t_*) \), then one can find
\[
t_* = t - \tau^*(t - \tau^*(t_*)) = t - \frac{N + 2}{\min\{1, a\}} \log(t - \tau^*(t_*))
\]
\[
= t - \frac{N + 2}{\min\{1, a\}} \log t - \frac{N + 2}{\min\{1, a\}} \log \left( 1 - \frac{\tau^*(t_*)}{t} \right)
\]
\[
= t - \frac{N + 2}{\min\{1, a\}} \log t - \frac{N + 2}{\min\{1, a\}} \log \left( 1 - \frac{\tau^*(t_*)}{t_* + \tau^*(t_*)} \right) = t - \frac{N + 2}{\min\{1, a\}} \log t + \epsilon(t),
\]

where \( \lim_{t \to \infty} \epsilon(t) = 0 \). Hence, there exist a bounded function \( m(t) \) such that, for any large \( t \), it holds
\[
G(t, x) \geq \phi_{\sqrt{t_1/2}, \eta, c}(\tau^*(t_*), x, t_*)
\]
\[
= \eta \varphi_{\sqrt{t_1/2}}(x - (c^*(t - \frac{N + 2}{\min\{1, a\}} \log(t + m(t)) + \sqrt{t_1 + t_0})e).
\]

This estimate implies that the statement (4.17) holds true.

By applying (4.17), the statement (4.18) follows from a simple limit argument. Thus the proof of Proposition 4.7 is complete.

Next, we complete the proof of Proposition 4.13 which explain the reason why \( F \) would not uniformly converge to 0 on the wavefront in the case of \( a > 1 + s \).

Proof of Proposition 4.13

We prove by contradiction and assume that
\[
\limsup_{t \to \infty} \sup_{c_0 t \leq \|x\|} F(t, x) = 0 \quad \text{for some} \quad c_0 < c^* = 2\sqrt{a}.
\]

Then for any \( \varepsilon > 0 \) there exists \( T > 0 \) such that
\[
F(t, x) < \varepsilon \quad \text{for} \quad \|x\| \geq c_0 t, \ t \geq T.
\]

Hence, if we denote \( C := C - \alpha F \) where \( \alpha > 0 \), then for \( \|x\| \geq c_0 t \) and \( t \geq T, C \) satisfies
\[
\partial_t C - \Delta C \leq (1 + s)C + sF - \alpha c_0 F + \alpha FC
\]
\[
\leq (1 + s + \varepsilon \alpha)C + \alpha c_0 F
\]
\[
\leq (1 + s + \varepsilon \alpha)C + ((1 + s + \varepsilon \alpha) + \alpha c_0)F.
\]
Thus by choosing \( \alpha := 2s/(a - \varepsilon a - 1 - s + \sqrt{(a - \varepsilon a - 1 - s)^2 - 4\varepsilon as}) > s/(a - s - 1) \), one can obtain
\[
\partial_t C - \Delta C \leq (1 + s + \varepsilon a\alpha)C.
\]
On the other hand, for sufficiently large \( A > 0 \), for all \( e \in S^{N-1} \),
\[
\mathcal{F}_e(t, x) := Ae^{-\sqrt{\pi}(x - 2\sqrt{\pi}t)}, \quad \mathcal{C}_e(t, x) := \frac{sA}{a - s - 1} e^{-\sqrt{\pi}(x - 2\sqrt{\pi}t)}
\]
are super-solutions of \( F \) and \( C \), respectively, and hence
\[
C(t, x) \leq \inf_{e \in S^{N-1}} \mathcal{C}_e(t, x) = Ae^{-\sqrt{\pi}(\|x\|^2 - 2\sqrt{\pi}t)} \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \geq 0.
\]
Hence, for \( \lambda_1 < \sqrt{\alpha} \), there exists sufficiently large constant \( B > 0 \) such that
\[
\mathcal{C}(T, x) \leq C(T, x) \leq Ae^{-\sqrt{\pi}(\|x\|^2 - 2\sqrt{\pi}T)} \leq Be^{-\lambda_1 e^{-\sqrt{\alpha}}x}\|x\| \quad \text{for all } x \in \mathbb{R}^N \text{ and } e \in S^{N-1}.
\]
Therefore, if we take \( \lambda_1 \) such that \( \max\{c_0/2, \sqrt{1 + s + \varepsilon a\alpha}\} < \lambda_1 < \sqrt{\alpha} \), then
\[
C_e(t, x) := A_1 e^{-\lambda_1(e^{-\sqrt{\alpha}} - 2\lambda_1 t)}
\]
satisfies
\[
\partial_t C_e - \Delta C_e \geq (1 + s + \varepsilon a\alpha)C_e \quad \text{for all } x \in \mathbb{R}^N, \ t \geq 0, \ e \in S^{N-1},
\]
and
\[
C(t, x) \leq C(t, x) \leq 1 + s \leq A_1 e^{-\lambda_1(e^{-\sqrt{\alpha}} - 2\lambda_1 t)} = C_e(t, x) \quad \text{for all } \|x\| \geq c_0 t, \ t \geq T, \ e \in S^{N-1}.
\]
Hence, by applying the comparison principle,
\[
\mathcal{C}(t, x) \leq \inf_{e \in S^{N-1}} C_e(t, x) = A_1 e^{-\lambda_1(\|x\|^2 - 2\sqrt{\pi}t)} \quad \text{for all } \|x\| \geq c_0 t, \ t \geq T.
\]
Since \( C(t, x) = \mathcal{C}(t, x) + \alpha F(t, x) \), for any \( c \in (2\lambda_1, 2\sqrt{\alpha}) \), one has
\[
\limsup_{t \to \infty} \sup_{ct \leq \|x\|} C(t, x) \leq \lim_{t \to \infty} A_1 e^{-\lambda_1(c - 2\sqrt{\alpha})t} + \alpha \limsup_{t \to \infty} \sup_{ct \leq \|x\|} F(t, x) \leq \alpha \varepsilon.
\]
Since \( \varepsilon \) could be chosen arbitrarily small, one obtains
\[
\limsup_{t \to \infty} \sup_{ct \leq \|x\|} C(t, x) = 0 \quad \text{for } c \in (2\lambda_1, 2\sqrt{\alpha}).
\]
Hence, for \( G(t, x) = F(t, x) + C(t, x) \), it holds
\[
\limsup_{t \to \infty} \sup_{ct \leq \|x\|} G(t, x) = 0 \quad \text{for } c \in (2\lambda_1, 2\sqrt{\alpha}).
\]
This contradicts the statement \textbf{(4.17)} and complete the proof of Proposition \textbf{2.13}. \hfill \Box

At the end of this section, we complete the proof of Theorem \textbf{2.12} by showing a lower estimate for the \( C \)-component at the position \( c^* t - \frac{N+2}{2\lambda^*} \ln t \) in the case of \( a < 1 + s \).

**Proposition 4.10** If \( a < 1 + s \) and \( d = 1 \), then for any \( R > 0 \), it holds:
\[
\liminf_{t \to +\infty} \inf_{|x| \leq R, e \in S^{N-1}} C \left( t, x + \left( c^* t - \frac{N+2}{2\lambda^*} \ln t \right) e \right) > 0, \quad \text{(4.22)}
\]
\[
\limsup_{t \to +\infty} \sup_{|x| \leq R, e \in S^{N-1}} H \left( t, x + \left( c^* t - \frac{N+2}{2\lambda^*} \ln t \right) e \right) < 1. \quad \text{(4.23)}
\]
Proof of Proposition 4.10. Let us denote \( G(t, x) := F(t, x) + C(t, x) \) and \( H(t, x) := 1 - \max\{1, g\} G(t, x) \). Then, for \((t, x) \in \{(t, x) \mid G(t, x) \leq 1\}\), one has
\[
\partial_t H - \Delta H - bH(1 - H - gG) = -\max\{1, g\}(\partial_t G - \Delta G) - b(\max\{1, g\} - g)HG \\
\leq -\max\{1, g\}(aF + C)(1 - G) \leq 0.
\]

On the other hand, for \((t, x) \in \{(t, x) \mid G(t, x) > 1\}\), one has
\[
H(t, x) \geq 0 > H(t, x).
\]

Hence by applying the comparison principle, one can obtain that
\[
H(t, x) \geq H(t, x) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \geq 0.
\]

Thus, it follows immediately that
\[
\partial_t C - \Delta C \geq C(1 - (F + C)) + sHC \geq C(1 - (F + C)) + sHC \\
= C(1 + s - k(F + C)) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0,
\]

where \( k := 1 + s \max\{1, g\} \).

By applying the same argument as that in the proof of Proposition 2.13, one may find
\[
F(t, x) \leq Ae^{\sqrt{a}(\|x\| - \sqrt{\pi t})} \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \geq 0.
\]

Let us denote \( c_0 := \sqrt{a + \sqrt{1 + s}} \in (2\sqrt{a}, 2\sqrt{1 + s}) \) and \( \varepsilon_0 := \sqrt{a}(\sqrt{1 + s} - \sqrt{a}) \), then one can obtain
\[
F(t, x) \leq Ae^{c_0 t} \quad \text{for all } \|x\| \geq c_0 t \text{ and } t \geq 0.
\]

Therefore, for all \( \|x\| \geq c_0 t \) and \( t > 0 \), one has
\[
\partial_t C - \Delta C \geq C(1 + s - kC) - Ak e^{-c_0 t} C.
\]

By similar arguments to those in the proof of statements (4.19) and (4.17), one find that
\[
C(t, x) \geq c_2 (t + t_0)^{-\frac{N + 2}{2s}}(\|x\| - c^*(t + t_0))e^{-\lambda t}(\|x\| - c^*(t + t_0)) \\
\quad \text{for } c^*(t + t_0) \leq \|x\| \leq c^*(t + t_0) + \sqrt{t + t_0} \text{ and } t > 1, 
(4.24)
\]

\[
\liminf_{t \to \infty} \inf_{\varepsilon \in \mathbb{R}^N} C \left( t, \left( c^* t - \frac{(N + 2)c^*}{1 + s} \log t \right) e \right) (=: c_0) > 0. 
(4.25)
\]

Let \( U(z) \) be a solution of
\[
\begin{cases}
  c^* U' + U'' + U(1 + s - LU) = 0, \quad z \in \mathbb{R}, \\
  U(-\infty) = \frac{1 + s}{L}, \quad U(+\infty) = 0,
\end{cases}
\]

where \( L := \max\{k, \frac{2(1 + s)}{c_0}\} \). Then \( U'(z) < 0 \) for all \( z \in \mathbb{R} \) and
\[
\frac{U(z)}{ze^{-\lambda z}} \to B \quad \text{as } z \to \infty \quad \text{for some constant } B > 0, 
(4.26)
\]

which have been proved in [2]. Next, let us introduce two functions
\[
u_r(0)(t, x) := \omega(t)U(\|x\| - c^* t + \frac{N + 2}{2\lambda} \log t + \tau_0),
\]
\[ \omega(t) := e^{-Ak \int_0^t e^{-r_0^*} \, d\tau}. \]

There exists \( T_0 > 0 \) such that for \( c^*t - \frac{(N+2)c^*}{1+s} \log t \leq \|x\|, \quad t \geq T_0 \) and \( r_0 \geq 0 \), it holds

\[
\partial_t u_{r_0} - \Delta u_{r_0} - u_{r_0}(1 + s - ku_{r_0}) + Ake^{-c_0t}u_{r_0} = \left( \frac{N + 2}{2\lambda t} - \frac{N - 1}{\|x\|} \right) \omega U' + \left( \frac{\omega}{\omega} + Ake^{-c_0t} - (L - K\omega) \right) u_{r_0} \leq 0.
\]

From (4.24) and (4.25), there exists \( T_1 > 0 \) such that for \( \|x\| = c^*t - \frac{(N+2)c^*}{1+s} \log t, \quad t \geq T_1 \) and \( r_0 \geq 0 \), it holds

\[ C(t, x) > \frac{c_0}{2} \geq \frac{1 + s}{L} > u_{r_0}(t, x). \]

From the estimates (4.26) and (4.24), there exists \( T_2 > 0 \) such that for \( \|x\| = c^*t + \sqrt{t}, \quad t \geq T_2 \) and \( r_0 \geq 0 \),

\[
u_{r_0}(t, x) = \omega(t)U(\sqrt{t} + \frac{N + 2}{2\lambda^t} \log t + r_0) \leq 2B(\sqrt{t} + \frac{N + 2}{2\lambda^t} \log t + r_0)e^{-\lambda^t(\sqrt{t} + \frac{N + 2}{2\lambda^t} \log t + r_0)} \leq c_2(\sqrt{t} - c^*t)e^{-\lambda^t(\sqrt{t} + \frac{N + 2}{2\lambda^t} \log (t + t_0) - c^*t_0)} \leq C(t, x).
\]

Moreover, for sufficiently large \( r_0 > 0 \), one has

\[ C(T_s, x) \geq u_{r_0}(T_s, x), \quad c^*T_s - \frac{(N+2)c^*}{1+s} \log T_s \leq \|x\| \leq c^*T_s + \sqrt{T_s}, \quad \text{where } T_s := \max\{T_0, T_1, T_2\}.\]

Therefore, by applying the comparison principle, for \( c^*t - \frac{(N+2)c^*}{1+s} \log t \leq \|x\| \leq c^*t + \sqrt{T} \) and \( t \geq T_s \), one can conclude that

\[ C(t, x) \geq u_{r_0}(t, x) \geq \omega(\infty)U(\|x\| - c^*t + \frac{N + 2}{2\lambda^t} \log t + r_0).\]

This implies that the statement (4.22) holds true. Further, the statement (4.23) follows immediately from a simple limit argument. Thus, the proof of Proposition 4.10 is complete. \( \Box \)

In conclusion, Theorem 2.12 is an immediate result following from Proposition 4.3, Proposition 4.7 and Proposition 4.10.

5 Lower estimates on the spreading speed

In this section, we deal with the lower estimate on the spreading speeds of solutions of the system (1.1). The proof of Theorem 2.4 and Theorem 2.5 will be performed through several subsections. Since original hunter-gatherers, initial farmers and converted farmers heavily interact with each other in the final zone, one need very delicate analysis to get uniform upper estimate of the \( H \)-component and uniform lower estimate of the \( F + C \)-component. Note that, our result only implies that the total population density of initial farmers and converted farmers is uniformly greater than 0 in the final zone as \( t \rightarrow +\infty \). However, we still do not how to investigate the population density of each populations of farmers separately.

Before stating our arguments, we would like to introduce some basic properties at first. Denote \( X \) as a Banach space of \( \mathbb{R}^3 \)-valued boundedly and uniformly continuous functions on \( \mathbb{R}^N \) endowed with the usual sup-norm. Let us define \( \Psi(r) \subset X \) as

\[ \Psi(r) = \{ (\psi_1, \psi_2, \psi_3) \in X : 0 \leq \psi_1, 0 \leq \psi_2, \psi_1 + \psi_2 \leq r \text{ and } 0 \leq \psi_3 \leq 1 \}. \]

Although the comparison principle does not hold for the full system, one can apply it on each equation separately. Let us denote the nonlinear semiflow generated by the system (1.1) by \( Z(t) \) and add the both sides of the \( F \)-equation and \( C \)-equation of the system (1.1), then one gets

\[ \partial_t(F + C) - \Delta(F + C) = (aF + C)(1 - F - C) + sH(F + C). \] (5.1)
Since $0 \leq H \leq 1$, the right hand side of the equation (5.1) is not greater than
\[
\min\{a, 1\}\left(\frac{\max\{a, 1\} + s}{\min\{a, 1\}} - F - C\right)(F + C).
\]
By applying the comparison principle, one can conclude immediately that
\[
\sup_{x \in \mathbb{R}^N} \left(F(t, x) + C(t, x)\right) \leq m(t),
\]
where $m(t)$ is a function satisfying
\[
\frac{dm}{dt} = \min\{a, 1\}\left(\frac{\max\{a, 1\} + s}{\min\{a, 1\}} - m\right)m \quad \text{and} \quad m(0) = \sup_{x \in \mathbb{R}^N} (F(0, x) + C(0, x)).
\]
Therefore, if we introduce a new function as
\[
M(r) := \max\{r, \max\{a, 1\} + s\min\{a, 1\}\},
\]
then one may obtain that, for each $r > 0$, it holds
\[
Z(t)[\Psi(r)] \subset \Psi(M(r)) \quad \text{for all} \quad t > 0.
\]

Remark 5.1 If $H_0(x) \equiv 0$, then by applying the comparison principle, one can rewrite the system (1.1) to a two-component competition system as
\[
\begin{align*}
\partial_t F &= \Delta F + aF(1 - F - C), \\
\partial_t C &= \Delta C + C(1 - F - C),
\end{align*}
\]
of which the spreading properties are partly studied by Girardin and Lam in [10].

Furthermore, for the special case when the diffusion coefficient $d$ of the $H$-equation is equal to 1, we have a direct observation as follows:

Proposition 5.2 If $d = 1$, then there exists $\varepsilon^* > 0$ such that the solution $(F, C, H)$ of the system (1.1) satisfies:
\[
\inf_{t \geq 0, x \in \mathbb{R}^N} (F + C + H)(t, x) \geq \varepsilon^*,
\]
provided that the initial data $(F_0, C_0, H_0) \in \Psi(r)$ satisfies $F_0 + C_0 + H_0 \geq \varepsilon^*$.

Proof of Proposition 5.2. Let us first add the both sides of the $F$-equation, $C$-equation and $H$-equation. One may find that, it holds
\[
\partial_t(F + C + H) - \Delta(F + C + H) = aF + C + H - (aF + C)(F + C)
+ sH(F + C) - gbH(F + C) - bH^2 
\geq \varepsilon_2(F + C + H) - \varepsilon_3(F + C + H)^2,
\]
where $\varepsilon_2 = \min\{1, a, b\}$, $\varepsilon_3 = \max\{1, \varepsilon_1, \varepsilon_1 b\}$ and $\varepsilon_1 = \max\{1, a, s, g\}$. Therefore, by applying the comparison principle, there exists $\varepsilon^* > 0$ such that
\[
\inf_{t \geq 0, x \in \mathbb{R}^N} (F + C + H)(t, x) \geq \varepsilon^*,
\]
provided that $F_0(x) + C_0(x) + H_0(x) \geq \varepsilon^*$.

\[\square\]
5.1 Uniform spreading in the final zone \((1 + s \geq a)\)

Throughout this paper, we denote \(c^0\) as an arbitrarily chosen constant speed in \([0, c^\ast]\). The proof of Theorem 2.4 is split into three steps. We first deal with a weak spreading property which states that, in final zone, for any fixed speed \(c \in [0, c^0]\) and direction \(e\), \(H(t, cte + x)\) does not uniformly converge to 1 and \((F + C)(t, cte + x)\) does not uniformly converge to 0. Then, in the second step, we prove that, \(H(t, cte + x)\) is uniformly smaller than 1 and \((F + C)(t, cte + x)\) is uniformly greater than 0 with respect to \(t\). At last, we conclude the proof by showing that these properties hold with respect to \(\|x\| \leq ct\).

5.1.1 First step: pointwise weak spreading property

The first step is to prove the following lemma, from which one can find that the \(H\)-component does not uniformly converge to 1, and the \(F+C\)-component does not uniformly converge to 0 in the final zone. Moreover, this property is in some sense uniform with respect to the initial data.

**Lemma 5.3** If \(d = 1\) and \(1 + s \geq a\), there exists \(\varepsilon_1 > 0\) such that, for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \(F_0 + C_0 + H_0 \geq \varepsilon^*\) and \(F_0 + C_0 \neq 0\), for all \(c \in [0, c^0]\), \(e \in S^{N-1}\) and \(x \in \mathbb{R}^N\), the solution \((F, C, H)\) of the system (1.1) satisfies:

\[
\liminf_{t \to +\infty} H(t, x + cte) \leq 1 - \varepsilon_1, \quad (5.2)
\]

\[
\limsup_{t \to +\infty} (F + C)(t, x + cte) \geq \varepsilon_1. \quad (5.3)
\]

**Remark 5.4** Note that, from the statement of Lemma 5.3, it is immediately that \(\varepsilon_1(c^0)\) can be chosen to be nonincreasing with respect to \(c^0\). Hence, we slightly change our notation and denote it as \(\varepsilon_1\).

**Proof of Lemma 5.3** For \(H_0(x) \equiv 0\), since the \(F\)-component and \(C\)-component are nonnegative, the estimate (5.2) holds immediately with \(\varepsilon_1 = 1\). Moreover, from Proposition 5.2, the statement (5.3) also holds true. Hence, without loss of generality, we assume \(H_0(x) \neq 0\). We argue by contradiction once again and assume there exist sequences

\[
\{(F_{0,n}, C_{0,n}, H_{0,n})\}_{n \geq 0}, \quad \{c_n\}_{n \geq 0} \subset [0, c^0], \quad \{x_n\}_{n \geq 0} \subset \mathbb{R}^N,
\]

\[
\{e_n\}_{n \geq 0} \subset S^{N-1} \quad \text{and} \quad \{t_n\}_{n \geq 0} \subset [0, \infty) \quad \text{such that} \quad t_n \to +\infty,
\]

such that one of the following statements hold true:

for all \(t \geq t_n\), \((F_{n} + C_{n})(t, x_n + c_n t e_n) \leq \frac{1}{n}\), \((5.4)\)

for all \(t \geq t_n\), \(H_n(t, x_n + c_n t e_n) \geq 1 - \frac{1}{n}\), \((5.5)\)

wherein \((F_n, C_n, H_n)\) denotes the solution with the initial data \((F_{0,n}, C_{0,n}, H_{0,n})\). Note without loss of generality that

\[e_n \to e_\infty \in [0, c^0]\] and \(e_n \to e_\infty \in S^{N-1}\).

Then, we first claim that

**Claim 5.5** Either (5.4) or (5.5) holds true, there exists a sequence \(\{t'_n\}_{n \geq 0}\) satisfying \(t'_n \geq t_n\) such that, for any \(R > 0\), it holds:

\[
\lim_{n \to \infty} \sup_{t \geq 0, x \in B_R} (F_{n} + C_{n})(t'_n + t, x_n + c_n (t'_n + t) e_n + x) = 0, \quad (5.6)
\]

\[
\lim_{n \to \infty} \sup_{t \geq 0, x \in B_R} |1 - H_n(t'_n + t, x_n + c_n (t'_n + t) e_n + x)| = 0. \quad (5.7)
\]
Proof of Claim 5.5. We first prove that the statement (5.5) implies the statements (5.6) and (5.7) hold true. To proceed by contradiction, we assume that for any $R > 0$, there exist $\delta > 0$, $s_n > t_n$ and $x_n' \in B_R$ such that

$$|1 - H_n(s_n, x_n + c_n s_n e_n + x_n')| \geq \delta.$$  

Due to standard parabolic estimates, possibly along a subsequence, one may assume that

$$\lim_{n \to \infty} F_n(s_n + t, x_n + c_n(s_n + t)e_n + x) = F_\infty(t, x),$$

$$\lim_{n \to \infty} C_n(s_n + t, x_n + c_n(s_n + t)e_n + x) = C_\infty(t, x),$$

$$\lim_{n \to \infty} H_n(s_n + t, x_n + c_n(s_n + t)e_n + x) = H_\infty(t, x).$$

The above convergences hold locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $(F_\infty, C_\infty, H_\infty)$ is an entire solution of the following system

$$\begin{align*}
\partial_t F_\infty &= \Delta F_\infty + c_\infty \nabla F_\infty \cdot e_\infty + a F_\infty(1 - C_\infty - F_\infty), \\
\partial_t C_\infty &= \Delta C_\infty + c_\infty \nabla C_\infty \cdot e_\infty + C_\infty(1 - C_\infty - F_\infty) + s(F_\infty + C_\infty)H_\infty, \\
\partial_t H_\infty &= \Delta H_\infty + c_\infty \nabla H_\infty \cdot e_\infty + b H_\infty(1 - g F_\infty - g C_\infty - H_\infty). 
\end{align*}$$  

(5.8)

From the strong maximum principle and the construction (5.5), one has $H_\infty \equiv 1$, and hence $F_\infty + C_\infty \equiv 0$ by considering the $H_\infty$-equation in the system (5.8). However, since the sequence $\{x_n'\} \subset B_R$ is relatively compact, $H_\infty \equiv 1$ contradicts the fact that $|1 - H_\infty(0, x'_n)| \geq \delta$. It proves that the statement (5.7) holds true. The statement (5.6) follows from the same approach.

Next, we prove that the statement (5.4) implies the statements (5.6) and (5.7) hold true. To proceed by contradiction, we assume that for any $R > 0$, there exist $\delta > 0$, $s_n > t_n$ and $x_n' \in B_R$ such that

$$(F_n + C_n)(s_n, x_n + c_n s_n e_n + x_n') \geq \delta.$$  

Due to standard parabolic estimates, possibly along a subsequence, one may assume that

$$\lim_{n \to \infty} F_n(s_n + t, x_n + c_n(s_n + t)e_n + x) = F_\infty(t, x),$$

$$\lim_{n \to \infty} C_n(s_n + t, x_n + c_n(s_n + t)e_n + x) = C_\infty(t, x),$$

$$\lim_{n \to \infty} H_n(s_n + t, x_n + c_n(s_n + t)e_n + x) = H_\infty(t, x).$$

The above convergences hold locally uniformly in $(t, x) \in \mathbb{R} \times \mathbb{R}^N$ and $(F_\infty, C_\infty, H_\infty)$ is an entire solution of the system (5.8).

From the strong maximum principle and the construction (5.4), one has $H_\infty \geq \epsilon^* > 0$, and hence $H_\infty \equiv 1$. However, since the sequence $\{x'_n\} \subset B_R$ is relatively compact, $F_\infty + C_\infty \equiv 0$ contradicts the fact that $(F_\infty + C_\infty)(0, x'_n) \geq \delta$. It proves that the statement (5.6) holds true. The statement (5.7) follows from the same approach.

Now, we can go back to the proof of Lemma 5.3. From the statements (5.6) and (5.7), for any $R > 0$ and small enough $\delta > 0$, for any $n$ large enough, one has for all $t > 0$ and $x \in \mathbb{R}^N$,

$$F_n(t_n + t, x_n + c_n(t_n + t)e_n + x) \leq \chi_{\mathbb{R}^N \setminus B_R} + \delta \chi_{B_R}(x) := \bar{F}(x),$$

and

$$H_n(t_n + t, x_n + c_n(t_n + t)e_n + x) \geq (1 - \delta) \chi_{B_R}(x) := \bar{H}(x).$$

Then one infers from the comparison principle that

$$C_n(t_n + t, x_n + c_n(t_n + t)e_n + x) \geq \bar{C}_n(t, x) \quad \text{for all} \quad t \geq 0, \quad x \in \mathbb{R}^N,$$

$$C_n(t_n + t, x_n + c_n(t_n + t)e_n + x) \leq \bar{C}_n(t, x) \quad \text{for all} \quad t \leq 0, \quad x \in \mathbb{R}^N.$$
wherein \( C_n \) is the solution of
\[
\begin{align*}
\partial_t C_n &= \Delta C_n + c_n \nabla C_n \cdot e_n + C_n (1 + sH - \bar{F} - C_n), \\
C_n(0, x) &= C_n(t, x_n + c_n t e_n + x).
\end{align*}
\tag{5.9}
\]

For each \( R > 0 \), let \( \phi_R \) be the principal eigenfunction as
\[
\begin{align*}
\Delta \phi_R &= \mu_R \phi_R \text{ in } B_R, \\
\phi_R &= 0 \text{ on } \partial B_R,
\end{align*}
\tag{5.10}
\]
that is normalized so that \( \| \phi_R \| = 1 \), and extend it by 0 outside of the ball \( B_R \). Then, we construct a stationary sub-solution \( \psi(t, x; \eta) \), for each \( \eta > 0 \), as
\[
\psi_n(t, x; \eta) = \eta e^{-c_n x e_n/2} \phi_R(x).
\]
Since \( \epsilon^0 < c^* = 2 \sqrt{1 + s} \), one can check that there exist \( \eta_0 \) depending only on \( \epsilon^0 \) such that for any \( \delta \) small enough, \( 0 < \eta \leq \eta_0 \) and \( R \) large enough, for each \( n > 0 \), \( \psi_n \) satisfies
\[
\begin{align*}
\partial_t \psi_n - \Delta \psi_n - c_n \nabla \psi_n \cdot e_n - \psi_n (1 - \psi_n + s(1 - \delta) - \delta) &= \left( c_n^2 / 4 - \mu_R - (1 + s(1 - \delta) - \delta) \right) \psi_n(t, x; \eta) + \psi_n(t, x; \eta)^2 \\
&\leq 0.
\end{align*}
\]
Moreover, since \( \text{supp} \psi_n \subset B_R \), the function \( \psi_n \) is a stationary sub-solution of the equation \( (5.9) \). Therefore, the solution of the equation \( (5.9) \) associated with initial data \( \psi_n(t, x; \eta) \) is increasing in time, and converges to a positive stationary solution that is denoted by \( p_{n, R, \delta}(x) \).

Moreover, we claim that:

**Claim 5.6** \( p_{n, R, \delta}(x) \) does not depend on the choice of \( \eta \in (0, \eta_0) \).

**Proof of Claim 5.6** To check this, let us change our notation for simplicity and denote the stationary sub-solution and this stationary solution as \( \psi \) and \( p_\eta \). We first note that the comparison principle implies that \( p_\eta \leq p_{\eta'} \) for any \( \eta < \eta' \). Next, let us assume by contradiction that there exists \( \eta_1 < \eta_0 \) with \( p_{\eta_1} \neq p_{\eta_0} \). Hence, infer from the strong maximum principle, one has \( p_{\eta_1} < p_{\eta_0} \). Moreover, there exists a point \( x_0 \in B_R \) such that \( \psi(0, x_0; \eta_0) > p_{\eta_1}(x_0) \). Indeed, if not, then \( \psi(0, x; \eta_0) \leq p_{\eta_1}(x) \) for all \( x \in \mathbb{R}^N \), which yields \( p_{\eta_1} \geq p_{\eta_0} \), and reaches a contradiction.

Then, we consider
\[
\eta^* = \sup \{ \eta \geq \eta_1 : \psi(0, x; \eta) \leq p_{\eta_1}(x) \} \quad \text{for all } x \in \mathbb{R}^N.
\]
One can deduce from the comparison principle and the strong maximum principle that
\[
\psi(0, x; \eta^*) < \psi(t, x; \eta^*) < p_{\eta_1}(x) \quad \text{for all } t > 0, x \in \mathbb{R}^N.
\]
On the other hand, from the definition of \( \eta^* \) and recalling that the function \( \psi \) has compact support \( B_R \), there exists \( x_0 \in B_R \) such that \( \psi(0, x_0; \eta^*) = p_{\eta_1}(x_0) \), which reaches a contradiction.

Now, since the initial data satisfies \( F_0(x) + C_0(x) \neq 0 \) and \( H_0(x) \neq 0 \), the strong maximum principle implies that \( C_n \) is not trivial. Hence, we can choose \( \eta \) sufficiently small such that \( C_n(0, x) \geq \psi_n(0, x; \eta) \) for all \( x \in \mathbb{R}^N \). Then, it follows from the comparison principle that for any \( R > 0 \) large enough and \( \delta > 0 \) small enough and \( n \) large enough, it holds
\[
\lim_{t \to \infty} \inf_{t \geq t} C_n(t_n + t, x_n + c_n(t_n + t)e_n + x) \geq \lim_{t \to \infty} \inf_{t \geq t} C_n(t, x) \geq p_{n, R, \delta}(x) \quad \text{for all } x \in \mathbb{R}^N. \tag{5.11}
\]

To complete the proof of this lemma, it remains to check that \( p_{n, R, \delta} \) is far away from 0 as \( n \) and \( R \) are large enough and \( \delta \) is small. Since \( p_{n, R, \delta} \) is bounded from above by 1 + \( s \), one can use standard elliptic estimates to
Moreover, since the map \( t \to \psi(t, x; \eta_0) \) is nondecreasing, one has \( p_{n, R, \delta}(0) \geq \psi(0, 0; \eta_0) \geq \eta_0 \psi_R(0) \). Note that \( \varphi_R \to 1 \) locally uniformly as \( R \to +\infty \), hence \( p_{\infty}(0) \geq \eta_0 \) and \( p_{\infty}(x) > 0 \) for all \( x \in \mathbb{R}^N \). Therefore, from the statements (5.6) and (5.11), one can reach a contradiction and completes the proof of Lemma 5.3. \( \square \)

5.1.2 Second step: pointwise strong spreading property

Next, we deal with the following improved result of Lemma 5.3.

**Lemma 5.7** If \( d = 1 \) and \( 1 + s \geq a \), there exists \( \varepsilon_2 > 0 \) such that, for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \( F_0 + C_0 + H_0 \geq \varepsilon^* \) and \( F_0 + C_0 \neq 0 \), for all \( c \in \{0, c^0\} \), \( e \in S^{N-1} \) and \( x \in \mathbb{R}^N \), the solution \((F, C, H)\) of the system (1.1) satisfies:

\[
\lim_{t \to +\infty} \sup_{x} H(t, x + cte) \leq 1 - \varepsilon_2, \tag{5.12}
\]

\[
\lim_{t \to +\infty} \inf_{x} (F + C)(t, x + cte) \geq \varepsilon_2. \tag{5.13}
\]

**Proof of Lemma 5.7** We first deal with the proof of the statement (5.12). Without loss of generality, we assume \( H_0(x) \neq 0 \). We proceed by contradiction and assume that there exists a sequences \( \{x_n\}_{n>0} \subset \mathbb{R}^N \), \( \{c_n\}_{n>0} \subset [0, c^0] \) and \( \{\epsilon_n\}_{n>0} \subset S^{N-1} \) such that

\[
\lim_{t \to +\infty} \sup_{x} H_n(t, x_n + c_n t e_n) \geq 1 - \frac{1}{n}. \tag{5.14}
\]

From Lemma 5.3 there exist two sequences \( \{t_n\}_{n>0} \) with \( t_n \to \infty \) and \( \{s_n\}_{n>0} \subset \mathbb{R}_+ \) such that for each \( n > 0 \),

\[
H_n(t_n + s_n, x_n + c_n (t_n + s_n) e_n) = 1 - \frac{1}{n},
\]

\[
H_n(t, x_n + c_n t e_n) \geq 1 - \frac{\epsilon_1}{2} \quad \text{for all } t \in [t_n, t_n + s_n],
\]

\[
H_n(t_n, x_n + c_n t n e_n) = 1 - \frac{\epsilon_1}{2}.
\]

We assume as before, possibly along a subsequence, the functions

\[
(F_n, C_n, H_n)(t_n + t, x_n + c_n (t_n + s_n) e_n + x)
\]

converge locally uniformly to \((F_{\infty}, C_{\infty}, H_{\infty})\), which is an entire solution of

\[
\begin{cases}
\partial_t F_{\infty} = \Delta F_{\infty} + a F_{\infty} (1 - C_{\infty} - F_{\infty}), \\
\partial_t C_{\infty} = \Delta C_{\infty} + C_{\infty} (1 - C_{\infty} - F_{\infty}) + s (C_{\infty} + F_{\infty}) H_{\infty}, \\
\partial_t H_{\infty} = \Delta H_{\infty} + b H_{\infty} (1 - g F_{\infty} - g C_{\infty} - H_{\infty}).
\end{cases} \tag{5.15}
\]

From the choices of sequences \( \{t_n\}_{n>0} \) and \( \{s_n\}_{n>0} \), one has \( H_{\infty}(0, 0) = 1 \), and hence \( H_{\infty} \equiv 1 \). In particular, the sequence \( \{s_n\}_{n>0} \) is unbounded since it contradicts the fact that

\[
\lim_{n \to \infty} H_n(t_n, x_n + s_n t n e_n) = 1 - \frac{\epsilon_1}{2} < 1.
\]

Therefore, we assume that \( s_n \to +\infty \) as \( n \to \infty \).
Now let us consider the limit functions as follows:

\[ \tilde{F}(t, x) = \lim_{n \to \infty} F_n(t_n + t, x_n + c_nt_e_n + x), \]

\[ \tilde{C}(t, x) = \lim_{n \to \infty} C_n(t_n + t, x_n + c_nt_e_n + x), \]

\[ \tilde{H}(t, x) = \lim_{n \to \infty} H_n(t_n + t, x_n + c_nt_e_n + x), \]

which are well defined thanks to parabolic estimates. The pair \((\tilde{F}, \tilde{C}, \tilde{H})\) is an entire solution of the system (1.1). Then we look on \((\tilde{F}, \tilde{C}, \tilde{H})\) as a solution of the system (1.1) with initial data

\[ (\tilde{F}_0, \tilde{C}_0, \tilde{H}_0) := \lim_{n \to \infty} (F_n(t_n, x_n + c_nt_e_n + x), C_n(t_n, x_n + c_nt_e_n + x), H_n(t_n, x_n + c_nt_e_n + x)). \]

Note that, it follows from Proposition 5.2 that \(\tilde{F}_0(x) + \tilde{C}_0(x) + \tilde{H}_0(x) \geq \varepsilon^*\).

Since \(\tilde{H}_0(0) = 1 - \varepsilon_1/2\), by applying Proposition 5.2, one may find \(\tilde{F}_0(x) + \tilde{C}_0(x) \neq 0\). Thus, by applying Lemma 5.3 one has

for all \(x \in \mathbb{R}^N\), \(\liminf_{t \to \infty} \tilde{H}(t, x + c_\infty te) \leq 1 - \varepsilon_1.\) \hfill (5.15)

One the other hand, for all \(t \in [0, s_n]\), it holds

\[ H_n(t_n + t, x_n + c_nt_e_n + c_nt_e_n) \geq 1 - \frac{\varepsilon_1}{2}. \]

Since \(s_n \to +\infty\), we get by the locally uniform convergence that

\[ \tilde{H}(t, c_\infty te_\infty) \geq 1 - \frac{\varepsilon_1}{2} \quad \text{for all} \quad t \geq 0, \]

which contradicts the result (5.15) concluded from Lemma 5.7. Thus, the proof of the statement (5.12) is complete. The statement (5.13) follows immediately from the same approach. \(\square\)

### 5.1.3 Third step: uniform spreading property

In this subsection, we complete the proof of Theorem 2.4 by showing that results of Lemma 5.7 holds uniform on \(|x| \leq ct\) for all \(0 \leq c < c^*\).

**Lemma 5.8** If \(d = 1\) and \(1 + s \geq a\), there exists \(\varepsilon_3 > 0\) such that, for any \(c \in [0, c_0]\), for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \(F_0 + C_0 + H_0 \geq \varepsilon^*\) and \(F_0 + C_0 \neq 0\), the solution \((F, C, H)\) of the system (1.1) satisfies:

\[ \liminf_{t \to +\infty} \inf_{|x| \leq ct} (F + C)(t, x) \geq \varepsilon_3, \]

\[ \limsup_{t \to +\infty} \sup_{|x| \leq ct} H(t, x) \leq 1 - \varepsilon_3. \]

**Proof of Lemma 5.8** We proceed by contradiction and assume that there exist sequences \(\{t_n\}_{n \geq 0}\) with \(t_n \to +\infty\), \(\{e_n\}_{n \geq 0} \subset [0, c_0]\) and \(\{e_n\}_{n \geq 0} \subset S^N-1\) such that

\[ \lim_{n \to +\infty} H(t_n, c_ne_ne_n) = 1. \] \hfill (5.16)

Without loss of generality, possibly along a subsequence, we assume that \(e_n \to e_\infty\) and \(e_n \to e_\infty\) as \(n \to +\infty\). Choose some small \(\delta > 0\) such that \(e_\infty + \delta < c^*\), and define the sequence

\[ t'_n := \frac{c_ne_n}{e_\infty + \delta} \in [0, t_n) \quad \text{for all} \quad n \geq 0. \]
Let us first consider the case when the sequence \( \{c_n t_n\}_{n \geq 0} \) is bounded, which may happen if \( c_\infty = 0 \). Then one can infer from the strong maximum principle that as \( n \to +\infty \) that \( c_n t_n e_n \to x_\infty \in \mathbb{R}^N \) and
\[
H(t_n + t, c_n t_n e_n + x) \to 1
\]
locally uniformly. Thus, one obtains that \( H(t_n, 0) \to 1 \), which already contradicts the result of Lemma 5.7 with \( c = 0 \). Therefore, one can assume that \( t_n' \to +\infty \). Then, by applying Lemma 5.7 again, one has
\[
H(t_n', (c_\infty + \delta)t_n' e_\infty) \leq 1 - \varepsilon
\]
for each \( n \) large enough.

Next, let us consider the functions as follows:
\[
\begin{align*}
\tilde{F}_n(t, x) &= F(t_n' + t, c_n t_n e_\infty + x), \\
\tilde{C}_n(t, x) &= C(t_n' + t, c_n t_n e_\infty + x), \\
\tilde{H}_n(t, x) &= H(t_n' + t, c_n t_n e_\infty + x),
\end{align*}
\]
and define the sequences
\[
\tilde{c}_n := \frac{c_n t_n \|c_n e_\infty\|}{t_n - t_n'} \to 0 \quad \text{and} \quad \tilde{e}_n := \frac{e_n - e_\infty}{\|e_n - e_\infty\|}.
\]
Using the above notations, one can rewrite statements (5.16) and (5.17) as
\[
\tilde{H}_n(0, 0) \leq 1 - \varepsilon \quad \text{and} \quad \tilde{H}_n(t_n - t_n', \tilde{c}_n(t_n - t_n') \tilde{e}_n) \to 1.
\]
By introducing two time sequences
\[
\tilde{t}_n := \sup \left\{ 0 \leq t \leq t_n - t_n' \mid \tilde{H}_n(t, \tilde{c}_n \tilde{e}_n) > 1 - \frac{\varepsilon}{2} \right\} \in (0, t_n - t_n'), \\
\tilde{s}_n := t_n - t_n' - \tilde{t}_n,
\]
one may find that the following properties hold true:
\[
\tilde{H}_n(\tilde{t}_n, \tilde{c}_n \tilde{t}_n \tilde{e}_n) = 1 - \frac{\varepsilon}{2},
\]
\[
\tilde{H}_n(t, \tilde{c}_n \tilde{t} \tilde{e}_n) \leq 1 - \frac{\varepsilon}{2} \quad \text{for all} \quad t \in [\tilde{t}_n, \tilde{t}_n + \tilde{s}_n],
\]
\[
\tilde{H}_n(\tilde{t}_n + \tilde{s}_n, \tilde{c}_n(\tilde{t}_n + \tilde{s}_n) \tilde{e}_n) \to 1 \quad \text{as} \quad n \to +\infty.
\]
Proceeding the argument as in the proof of Lemma 5.7, one can reach a contradiction. The other statement follows from the same approach. The proof of Lemma 5.8 and Theorem 2.4 are complete. \( \square \)

### 5.2 Uniform spreading in the final zone \((1 + s < a)\)

Before dealing with the proof of Theorem 2.5, we note that, by proceeding the same argument in subsection 5.1, one can obtain a lemma as follows:

**Lemma 5.9** If \( d = 1 \) and \( 1 + s < a \), for any \( c \in [0, c^*] \), there exists \( \varepsilon > 0 \) such that, for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \( F_0 + C_0 + H_0 \geq \varepsilon^* \) and \( F_0 + C_0 \neq 0 \), the solution \((F, C, H)\) of the system (1.1) satisfies:
\[
\liminf_{t \to +\infty} \inf_{\|x\| \leq ct} (F + C)(t, x) \geq \varepsilon,
\]
\[
\limsup_{t \to +\infty} \sup_{\|x\| \leq ct} H(t, x) \leq 1 - \varepsilon.
\]
Indeed, for all $c \in [0, c^*)$, one could always conclude a similar weak pointwise spreading result as Lemma 5.3. However, to obtain the strong pointwise spreading property for $c^* < c < c^*$, the argument in Lemma 5.7 is not workable anymore. More precisely, one could not ensure $\tilde{F}(0, x) \neq 0$, which is a necessary condition to apply the weak pointwise spreading property to reach the contradiction. In this section, we complete the proof of Theorem 2.5 by applying a totally different approach.

**Proof of Theorem 2.5** We complete the proof by showing that for any $0 < c_1 < c_2 < c^*$, there exists $\varepsilon > 0$ such that

$$\liminf_{t \to +\infty} \inf_{c_1 t \leq \|x\| \leq c_2 t} (F + C)(t, x) \geq \varepsilon.$$  

To do this, we start by assuming there exist sequences $\{t_n\}_{n \geq 0} \subset \mathbb{R}_+$ with $t_n \to +\infty$ and $\{x_n\}_{n \geq 0} \subset \mathbb{R}^N$ with $c_1 t_n \leq \|x_n\| \leq c_2 t_n$, such that $(F + C)(t_n, x_n) \leq 1/n$.

We first add the both sides of the $F$-equation and $C$-equation, and find $G = F + C$ satisfies

$$\partial_t G \geq \Delta G + \min\{1, a\} G(1 - G) \quad \text{for all} \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^N.$$  

Then, let us consider a stationary sub-solution of the $G$-equation as $\varphi_R(x) := \eta \phi_R(x)$, where $\phi_R(x)$ is the principal eigenfunction defined as (5.10). One can check that

$$\eta \Delta \phi_R(x) + \min\{1, a\} \eta \phi_R(x)(1 - \eta \phi_R(x)) \geq 0,$$

provided that $\eta$ is small enough and $R$ is large enough. Recall that, in Theorem 2.12, we proved that, for all $R' > R > 0$, there exists $\varepsilon' > 0$ such that

$$\liminf_{t \to +\infty} \inf_{x \in B_{R'}, e \in S^{N-1}} G\left(t, x + \left(c^* t - \frac{c^* (N + 2)}{\min\{1, a\}} \ln t\right) e\right) > \varepsilon' \quad \text{if} \quad 1 + s < a.$$

Hence, for any $t' > T_0$, one can choose $\eta$ small enough such that

$$G(t', x) \geq \varphi_R\left(x - \left(c^* t' - \frac{c^* (N + 2)}{\min\{1, a\}} \ln t'\right) e\right) \quad \text{for all} \quad x \in \mathbb{R}^N, \ e \in S^{N-1}.$$  

Since $\varphi_R(x)$ is a stationary sub-solution of the $G$-equation, by applying the comparison principle, one obtains

$$G(t, x) \geq \varphi_R\left(x - \left(c^* t' - \frac{c^* (N + 2)}{\min\{1, a\}} \ln t'\right) e\right) \quad \text{for all} \quad t > t', \ x \in \mathbb{R}^N, \ e \in S^{N-1}.$$  

This implies that, for any $t' \geq T_0$ and $t \geq t'$, it holds

$$G\left(t, \left(c^* t' - \frac{c^* (N + 2)}{\min\{1, a\}} \ln t'\right) e\right) \geq \eta \phi_R(0). \quad (5.18)$$

Moreover, since $0 < c_1 < c_2 < 2\sqrt{a}$, for each large enough $n$, one can find $t_n > t'_n \geq T_0$ such that

$$x_n = \left(c^* t'_n - \frac{c^* (N + 2)}{\min\{1, a\}} \ln t'_n\right) e.$$  

Thus, from the estimate (5.18), one gets $G(t_n, x_n) \geq \eta \phi_R(0)$, which contradicts that

$$G(t_n, x_n) = F(t_n, x_n) + C(t_n, x_n) \leq \frac{1}{n} \to 0 \quad \text{as} \quad n \to +\infty.$$  

Therefore, the proof of Theorem 2.5 is complete. \hfill $\square$

**Remark 5.10** Note that, one also can prove Theorem 2.4 for the case $d = 1$ by applying the same argument as above.
6 Asymptotic profiles in the final zone

In section 5, we have already shown that the propagation of farmers occurs with the speed $c^*$. However, whether the profiles of solutions converges to the steady states $(\hat{F}, \hat{C}, 0)$ or $(0, C^*, H^*)$ are still unknown. In this section, we mainly deal with the asymptotic profiles of solutions in the final zone. From the numerical work of Aoki et al., we expect that the profiles of solutions in the final zone are different between the high conversion rate case and the low conversion rate case. Therefore, we split the justification of the numerical results into two parts by dealing with the cases $g \geq 1$ and $g < 1$, respectively.

6.1 Asymptotic profiles in the high conversion rate case ($g \geq 1$)

Our first result comes from a direct observation on the $F$-equation of the system (1.1).

Proposition 6.1 If $g \geq 1$, then for any $0 \leq c < \min\{2, 2\sqrt{a}\}$, the solution $(F, C, H)$ of the system (1.1) with the initial data (1.2) satisfies:

$$\lim_{t \to \infty} \sup_{\|x\| \leq ct} H(t, x) = 0,$$

$$\lim_{t \to \infty} \sup_{\|x\| \leq ct} |1 - (F + C)|(t, x) = 0.$$  

Proof of Proposition 6.1 Let us consider the solution $G_2(t, x)$ of the following equation

$$\begin{cases}
\partial_t G_2 = DG_2 + \min\{1, a\} G_2(1 - G_2), \\
G_2(0, x) = G_{2,0}(x).
\end{cases}$$

Note that, $G_2(t, x) \leq 1$ for all $t \geq 0$ and $x \in \mathbb{R}^N$, provided that the initial data $G_{2,0}(x) \leq 1$. Moreover, the function $G = F + C$ satisfies

$$\partial_t (F + C) = D(F + C) + (aF + C)(1 - F - C) + sH(F + C)$$

$$\geq DG + \min\{1, a\} G(1 - G).$$

The last inequality holds for all $(t, x) \in \{ (t, x) | G(t, x) \leq 1 \}$. Thus, by applying the comparison principle and the spreading properties of KPP equation, one can claim that:

Claim 6.2 For both cases $g \geq 1$ and $g < 1$, it holds:

$$\lim_{t \to +\infty} \inf_{\|x\| \leq ct} (F + C)(t, x) \geq 1,$$

for all $c \in [0, \min\{2, 2\sqrt{a}\})$.

Now, let us choose sequences $\{c_n\}_{n \geq 0} \subset [0, c_0]$ where $0 \leq c_0 < \min\{2, 2\sqrt{a}\}$, $\{t_n\}_{n \geq 0} \subset \mathbb{R}_+$ with $t_n \to +\infty$ and $\{x_n\}_{n \geq 0} \subset \mathbb{R}^N$ with $\|x_n\| \leq c_n t_n$. Then, we consider the limit functions

$$\lim_{n \to +\infty} F(t_n + t, x_n + x) = F_\infty(t, x),$$

$$\lim_{n \to +\infty} C(t_n + t, x_n + x) = C_\infty(t, x),$$

$$\lim_{n \to +\infty} H(t_n + t, x_n + x) = H_\infty(t, x),$$

which converge locally uniformly to $(F_\infty, C_\infty, H_\infty)$, an entire solution of the system

$$\begin{cases}
\partial_t F_\infty = DF_\infty + aF_\infty(1 - C_\infty - F_\infty), \\
\partial_t C_\infty = DC_\infty + C_\infty(1 - C_\infty - F_\infty) + s(F_\infty + C_\infty)H_\infty, \\
\partial_t H_\infty = d\Delta H_\infty + bH_\infty(1 - gF_\infty - gC_\infty - H_\infty).
\end{cases}$$
Note that, the lower estimate (6.3) implies that \((F_\infty + C_\infty)(t, x) \geq 1\) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\).

Next, we show that \(H_\infty(t, x) \equiv 0\). Indeed, since \((F_\infty + C_\infty)(t, x) \geq 1\) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\), one may find \(H_\infty\) is a sub-solution of \(\tilde{H}_\infty\) which satisfies

\[
\begin{cases}
\partial_t \tilde{H}_\infty = \Delta \tilde{H}_\infty + b\tilde{H}_\infty(1 - g - \tilde{H}_\infty), \\
\tilde{H}_\infty(0, x) = H_\infty(0, x).
\end{cases}
\]

It is clear that, if \(g > 1\), the function \((t, x) \rightarrow e^{-(g-1)(t+t_0)}\) is a super-solution of the above equation for any \(t > -t_0\). Since \(H_\infty(-t_0, x) \leq 1\) for any \(t_0 \in \mathbb{R}_+\), it follows from the comparison principle that

\(H_\infty(0, x) \leq e^{-(g-1)t_0}\).

By passing the limit as \(t_0 \to +\infty\), one gets that \(H_\infty \equiv 0\). Therefore, for any arbitrarily chosen sequences \(\{t_n\}_{n \geq 0} \subset \mathbb{R}_+\) with \(t_n \to +\infty\) and \(\{x_n\}_{n \geq 0} \subset \mathbb{R}^N\) with \(||x_n|| \leq ct_n\), one can obtain

\[
\lim_{n \to +\infty} H(t_n, x_n) = 0,
\]

which implies (6.1) holds true. If \(g = 1\), one can consider the super-solution as \(1/(bt + bt_0 + b)\), for all \(-t_0 < t < +\infty\) where \(t_0 \in \mathbb{R}_+\).

To complete the proof of this proposition, we consider the limit functions \((F_\infty, C_\infty, H_\infty)\) again, for which sequences \(\{t_n\}_{n \geq 0} \subset \mathbb{R}_+\) with \(t_n \to +\infty\) and \(\{x_n\}_{n \geq 0} \subset \mathbb{R}^N\) with \(||x_n|| \leq ct_n\) are chosen arbitrarily. The statement (6.1) and the strong maximum principle implies that \(H_\infty \equiv 0\). Then, by applying Claim 6.2 again, one can conclude that \((F_\infty + C_\infty) \equiv 1\), which completes the proof of the statement (6.2).

However, to investigate the profiles of solutions in the region of \(e^{*}t \leq ||x|| \leq c^0t\), one need to apply the uniform lower estimate of the \(F + C\)-component in Theorem 2.4 and Theorem 2.5. Since the arguments for proving Theorem 2.8 and Theorem 2.9 are almost same, here we just show the proof of Theorem 2.8.

**Proof of Theorem 2.8** Let us choose sequences \(\{c_n\}_{n \geq 0} \subset [0, c^0]\), \(\{t_n\}_{n \geq 0} \subset \mathbb{R}_+\) with \(t_n \to +\infty\) as \(n \to +\infty\), and \(\{x_n\}_{n \geq 0} \subset \mathbb{R}^N\) with \(||x_n|| \leq c_n t_n\). Next, we prove

\[
\lim_{n \to +\infty} H(t_n, x_n) = 0 \quad \text{and} \quad \lim_{n \to +\infty} (F + C)(t_n, x_n) = 1.
\]

We consider the limit functions

\[
\lim_{n \to +\infty} F(t_n + t, x_n + x) = F_\infty(t, x),
\]

\[
\lim_{n \to +\infty} C(t_n + t, x_n + x) = C_\infty(t, x),
\]

\[
\lim_{n \to +\infty} H(t_n + t, x_n + x) = H_\infty(t, x),
\]

which converge locally uniformly to \((F_\infty, C_\infty, H_\infty)\), an entire solution of the system

\[
\begin{cases}
\partial_t F_\infty = \Delta F_\infty + aF_\infty(1 - C_\infty - F_\infty), \\
\partial_t C_\infty = \Delta C_\infty + C_\infty(1 - C_\infty - F_\infty) + s(F_\infty + C_\infty)H_\infty, \\
\partial_t H_\infty = \Delta H_\infty + bH_\infty(1 - gF_\infty - gC_\infty - H_\infty).
\end{cases}
\]

The result of Theorem 2.4 implies that \(H_\infty(x) \leq 1 - \varepsilon\) and \((F_\infty + C_\infty) \geq \varepsilon\) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\). Adding the both sides of the \(F_\infty\)-equation and \(C_\infty\)-equation, one may find that \(1 - (1-\varepsilon)e^{-(1-\varepsilon)(t+t_0)}\) is a sub-solution of \((F_\infty + C_\infty)(t, x)\) for all \(x \in \mathbb{R}^N\) and \(t > -t_0\) where \(t_0 \in \mathbb{R}_+\). By passing \(t_0 \to +\infty\), one obtains that \((F_\infty + C_\infty)(0, x) \geq 1\). Since the sequences \(\{t_n\}_{n \geq 0}, \{c_n\}_{n \geq 0}\) and \(\{x_n\}_{n \geq 0}\) are chosen arbitrarily, one can conclude that

\[
\liminf_{t \to +\infty} \inf_{||x|| \leq ct} (F + C)(t, x) \geq 1 \quad \text{for all} \quad c \in [0, c^0].
\]
By applying the same argument as Proposition \[6.1\], the above statement implies that, if \( g \geq 1 \),

\[
\limsup_{t \to \infty} \sup_{||x|| \leq ct} H(t, x) = 0 \text{ for all } c \in [0, c^0].
\]

Then, one can conclude that

\[
\limsup_{t \to \infty} \sup_{||x|| \leq ct} |1 - (F + C)(t, x)| = 0 \text{ for all } c \in [0, c^0].
\]

Since \( c^0 \) can be chosen arbitrary close to \( c^* \), the proof of the statement \[2.8\] is complete. Furthermore, by applying Remark \[3.1\] the statement \[2.9\] follows immediately.

\[ \square \]

6.2 Asymptotic profiles in the low conversion rate case \((g < 1)\)

The key point of studying the asymptotic profiles in the low conversion rate case is to provide a uniform lower estimate of the \( H \)-component in the final zone. However, for the general case, it is hard to give the necessary and sufficient condition under which the \( H \)-component is uniform positive from below. In this subsection, we first show two sufficient conditions for obtaining the uniform lower estimate of the \( H \)-component. Then, we show that the \( C \)-component and \( H \)-component would converge to \((C^*, H^*)\) as \( t \to +\infty \) in the final zone.

The first sufficient condition means if conversion rate \( g \) is small enough, then the population density of hunter-gatherers alway stay uniform positive.

**Lemma 6.3** If \( g < \min\{1, a\}/(\min\{1, a\} + s) \), there exists \( \varepsilon > 0 \) such that, for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \( H_0 \equiv 1 \), the solution \((F, C, H)\) of the system \((1.1)\) satisfies:

\[
\liminf_{t \to +\infty} \inf_{x \in \mathbb{R}^N} H(t, x) \geq \varepsilon.
\]

**Proof of Lemma 6.3** The proof of this lemma is rather straightforward. Adding the both sides of the \( F \)-equation and \( C \)-equation of the system \((1.1)\), the function \( G = F + C \) satisfies

\[
\partial_t (F + C) - \Delta (F + C) = (aF + C)(1 - F - C) + sH(F + C).
\]

The right hand of the above equation can be rewritten as

\[
(F + C)(1 + sH - F - C) + (a - 1)F(1 - F - C) \leq G(1 + s - G) \text{ if } a \geq 1, \ G \geq 1,
\]

\[
(F + C)(a + sH - aF - aC) + (1 - a)C(1 - F - C) \leq G(a + s - aG) \text{ if } a \leq 1, \ G \geq 1.
\]

Since \( H(t, x) \leq 1 \), \( F(t, x) \geq 0 \) and \( C(t, x) \geq 0 \) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\), then by applying the comparison principle, one obtains

\[
\limsup_{t \to +\infty} \sup_{x \in \mathbb{R}^N} (F + C)(t, x) \leq \langle \min\{1, a\} + s \rangle / \min\{1, a\}.
\]

Thus, for any sufficiently small \( \varepsilon > 0 \), there exist \( T > 0 \), such that

\[
\sup_{x \in \mathbb{R}^N} G(t, x) \leq \frac{\min\{1, a\} + s}{\min\{1, a\}} + \varepsilon \text{ for all } t \geq T.
\]

Then, by applying the comparison principle again, one has \( H(t, x) \geq H(t, x) \) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\), which satisfies the equation as follows:

\[
\begin{aligned}
\partial_t H &= \Delta H + bH(1 - g\varepsilon - g(\min\{1, a\} + s) / \min\{1, a\} - H), \\
H(0, x) &= 1.
\end{aligned}
\]
Therefore, one can conclude that
\[
\liminf_{t \to +\infty} \inf_{x \in \mathbb{R}^N} H(t, x) \geq 1 - g(\min \{1, a\} + s) / \min \{1, a\} - \varepsilon g,
\]
which completes the proof. \(\square\)

The second sufficient condition states that if the diffusion speed \(d\) and intrinsic growth rate \(b\) of hunter-gatherers are large enough, then the population density of hunter-gatherers always stay uniform positive.

**Lemma 6.4** If \(bd \geq c^* / (1 - g)\), for any \(c \in [0, c^*)\), there exists \(\varepsilon > 0\) such that, for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \(F_0 \neq 0\) and \(H_0 \equiv 1\), the solution \((F, C, H)\) of the system \((1.1)\) satisfies:
\[
\liminf_{t \to +\infty} \inf_{||x|| \leq ct} H(t, x) \geq \varepsilon.
\]

The proof of this lemma is similar to that for Theorem 2.4. The first step is to show a weak pointwise property as follows:

**Lemma 6.5** If \(bd \geq c^* / (1 - g)\), there exists \(\varepsilon_1 > 0\) such that, for any given initial data \((F_0, C_0, H_0) \in \Psi(r)\) satisfying \(H_0 \neq 0\) for all \(c \in [0, c^0]\), \(e \in S^{N-1}\) and \(x \in \mathbb{R}^N\), the solution \((F, C, H)\) of the system \((1.1)\) satisfies:
\[
\limsup_{t \to +\infty} H(t, cte + x) \geq \varepsilon_1.
\]

**Proof of Lemma 6.5** For \(F_0(x) + C_0(x) \equiv 0\), the lemma holds immediately with \(\varepsilon_1 = 1\). Hence, without loss of generality, we assume \(F_0(x) + C_0(x) \neq 0\). We argue by contradiction once again and assume there exist sequences
\[
\{(F_{0,n}, C_{0,n}, H_{0,n})\}_{n \geq 0}, \quad \{c_n\}_{n \geq 0} \subset [0, c^0], \quad \{x_n\}_{n \geq 0} \subset \mathbb{R}^N,
\]
\[
\{e_n\}_{n \geq 0} \subset S^{N-1} \quad \text{and} \quad \{t_n\}_{n \geq 0} \subset [0, \infty) \quad \text{such that} \quad t_n \to +\infty,
\]
such that the following statement holds true
\[
\text{for all} \quad t \geq t_n, \quad H_n(t, x_n + c_n t e_n) \leq \frac{1}{n}, \quad (6.4)
\]
wherein \((F_n, C_n, H_n)\) denotes the solution with the initial data \((F_{0,n}, C_{0,n}, H_{0,n})\). Note without loss of generality that
\[
c_n \to c_\infty \in [0, c^0] \quad \text{and} \quad e_n \to e_\infty \in S^{N-1}.
\]

Then, by applying a similar argument to that for Claim 5.5 one can claim that

**Claim 6.6** If \((6.4)\) holds true, then there exists a sequence \(\{t'_n\}_{n \geq 0}\) satisfying \(t'_n \geq t_n\) such that, for any \(R > 0\), it holds:
\[
\lim_{n \to \infty} \sup_{t \geq 0, x \in B_R} (F_n + C_n)(t'_n + t, x_n + c_n(t'_n + t)e_n + x) \leq 1, \quad (6.5)
\]
\[
\lim_{n \to \infty} \sup_{t \geq 0, x \in B_R} H_n(t'_n + t, x_n + c_n(t'_n + t)e_n + x) = 0. \quad (6.6)
\]

Now, we can go back to the proof of Lemma 6.5 From the statement \((6.5)\), for any \(R > 0\) and small enough \(\delta > 0\), for any \(n\) large enough, one has for all \(t > 0\) and \(x \in \mathbb{R}^N\),
\[
(F_n + C_n)(t_n + t, x_n + c_n(t_n + t)e_n + x) \leq \min \{1, a\} \chi_{\mathbb{R}^N \setminus B_R} + (1 + \delta) \chi_{B_R}(x) := \mathcal{G}(x).
\]
Then one infers from the comparison principle that
\[
H_n(t_n + t, x_n + c_n(t_n + t)e_n + x) \geq H_n(t_n) \text{ for all } t \geq 0, \quad x \in \mathbb{R}^N,
\]

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wherein \( H_n \) is the solution of the equation

\[
\begin{align*}
\partial_t H_n &= d \Delta H_n + c_n \nabla H_n \cdot e_n + b H_n (1 - H_n - g G), \\
H_n(0, x) &= H_n(t_n, x_n + c_n t_n e_n + x).
\end{align*}
\] (6.7)

Then, we consider a stationary sub-solution \( \psi(x; \eta) \), for each \( \eta > 0 \),

\[
\psi(t, x; \eta) = \eta e^{-c_n x \cdot e_n / 2 \phi_R(x)}.
\]

Since \( \epsilon^0 < 2 \sqrt{db(1 - g)} \), one can check that there exist \( \eta_0 \) depending only on \( \epsilon^0 \) such that for any \( \delta \) small enough, \( 0 < \eta \leq \eta_0 \) and \( R \) large enough, the function \( \psi(x) \) is a stationary sub-solution of the equation (6.7). Therefore, the solution of the equation (6.7) associated with initial data \( \psi(x; \eta) \) is increasing in time, and converges to some positive stationary solution that denote by \( p_{n,R,\delta}(x) \). Moreover, the stationary state \( p_{n,R,\delta}(x) \) does not depend on the choice of \( \eta \in (0, \eta_0] \). Then, we can choose \( \eta \) sufficiently small such that \( H_n(t_n, x_n + c_n t_n e_n + x) \) is far away from \( \eta > \delta > 0 \) small enough and \( n \) large enough

\[
\liminf_{t \to \infty} H_n(t_n + t, x_n + c_n(t_n + t) e_n + x) \geq \liminf_{t \to \infty} H_n(t_n, x) \geq p_{n,R,\delta}(x) \text{ for all } x \in \mathbb{R}^N.
\] (6.8)

To complete the proof of lemma, it remains to check that \( p_{n,R,\delta} \) is far from 0 as \( n \) and \( R \) are large enough and \( \delta \) is small. Since \( p_{n,R,\delta}(0) \) is bounded from above by 1, one can use standard elliptic estimates to get that, as \( n \to +\infty, R \to +\infty \) and \( \delta \to 0 \), the function \( p_{n,R,\delta}(x) \) converges locally uniformly to a stationary solution \( p_\infty(x) \) of the equation

\[
d\Delta p_\infty + c_n \nabla p_\infty \cdot e_n + b p_\infty (1 - g - p_\infty) = 0.
\]

Moreover, since the map \( t \to \psi(t, x; \eta_0) \) is nondecreasing, one has \( p_{n,R,\delta}(0) \geq \psi(0, 0; \eta_0) \geq \eta_0 \psi_R(0) \). Note that \( \varphi_R \to 1 \) locally uniformly as \( R \to +\infty \), hence \( p_\infty(0) \geq \eta_0 \) and \( p_\infty(x) > 0 \) for all \( x \in \mathbb{R}^N \). Therefore, from the statements (6.6) and (6.8), we reached a contradiction and proved the Lemma 6.5.

By applying the similar argument to that for Lemma 6.7 and Lemma 6.8, one can complete the proof of Lemma 6.4. Then, by proceeding the proof of Theorem 2.4, one can immediately conclude a lemma as follows:

**Lemma 6.7** If \( g < 1 \), then for any given \( 0 \leq c < c^* \), there exists \( \epsilon > 0 \), such that, for the solution \( (F, C, H) \) of the system (1.1) with the initial data (1.2), it holds:

\[
\liminf_{t \to \infty} \inf_{\|x\| \leq ct} (F + C)(t, x) \geq \epsilon,
\] (6.9)

provided that

\[
g < \frac{\min\{1, a\}}{\min\{1, a\} + s} \text{ or } bd \geq \frac{c^*}{1 - g}.
\]

With the uniform lower estimate of the \( H \)-component, we can first prove the \( F \)-component converges to 0 in the final zone.

**Lemma 6.8** If \( g < 1 \), then for any given \( 0 \leq c < c^* \), the solution \( (F, C, H) \) of the system (1.1) with the initial data (1.2) satisfies:

\[
\lim_{t \to \infty} \sup_{\|x\| \leq ct} F(t, x) = 0,
\] (6.10)

provided that

\[
g < \frac{\min\{1, a\}}{\min\{1, a\} + s} \text{ or } bd \geq \frac{c^*}{1 - g}.
\]
As a matter of fact, we can consider the strictly convex functional as
\[ F(t, x + x) = F_\infty(t, x), \]
\[ C(t, x + x) = C_\infty(t, x), \]
\[ H(t, x + x) = H_\infty(t, x), \]
which converge locally uniformly to \((F_\infty, C_\infty, H_\infty)\), an entire solution of the system
\[
\begin{align*}
\partial_t F_\infty &= \Delta F_\infty + aF_\infty(1 - C_\infty - F_\infty), \\
\partial_t C_\infty &= \Delta C_\infty + C_\infty(1 - C_\infty - F_\infty) + s(F_\infty + C_\infty)H_\infty, \\
\partial_t H_\infty &= d\Delta H_\infty + bH_\infty(1 - gF_\infty - gC_\infty - H_\infty).
\end{align*}
\]

The result of Lemma 6.4 and Lemma 6.7 imply that \((F_\infty + C_\infty) \geq \varepsilon\) and \(H_\infty \geq \varepsilon\) for all \((t, x) \in \mathbb{R} \times \mathbb{R}^N\). Adding the both sides of the \(F_\infty\)-equation and \(C_\infty\)-equation, one may find that \(G_\infty = F_\infty + C_\infty\) satisfies
\[
\partial_t G_\infty - \Delta G_\infty \geq \min\{1, a\}G_\infty(1 + s\varepsilon - G_\infty) \text{ for all } (t, x) \in \{(t, x) \mid G_\infty(t, x) \leq 1 + s\varepsilon\},
\]
and \(G_\infty(t, x) \geq \varepsilon\) when \(t = -t_0\). Then, by passing \(t_0 \to +\infty\), one has \(G_\infty(t, x) \geq 1 + s\varepsilon\) for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^N\). Then, it implies that
\[
\partial_t F_\infty - \Delta F_\infty \leq -a s\varepsilon F_\infty,
\]
and hence \(F_\infty \equiv 0\). Since the sequences \(\{t_n\}_{n \geq 0}\) and \(\{x_n\}_{n \geq 0}\) are chosen arbitrarily, the proof of this lemma is complete.

Then, for the special case \(d = 1\), we can investigate the profiles of the \(C\)-component and \(H\)-component in the final zone by considering the dynamics of the underlying ODE system:
\[
\begin{align*}
C_t &= C(1 - C) + sCH, \\
H_t &= bH(1 - H - gC).
\end{align*}
\]
We expect the solution of the PDE system (1.1) to converge uniformly to the equilibrium \((C^*, H^*)\) as \(t \to +\infty\).

Let us introduce the set \(\Sigma = \{(C, H) \in \mathbb{R}^2 : 0 < C < 1 + s, 0 < H < 1\}\), in which \((C^*, H^*)\) is the unique singular point. There exists a strictly convex function \(\Phi : \Sigma \to \mathbb{R}\) of class \(C^2\) that attains its minimum point at \((C^*, H^*)\) and satisfies
\[
(C(1 - C + sH), bH(1 - H - gC)) \cdot \nabla \Phi(C, H) \leq 0 \text{ for all } (C, H) \in \Sigma.
\]
The function \(\Phi\) is a strict Lyapunov function in the sense that: if \((C, H)\) denotes the solution of corresponding Ode system (2.12) with the initial date \((C_0, H_0)\), then
\[
\Phi(C(t), H(t)) = \Phi(C_0, H_0) \text{ for all } t > 0 \implies (C_0, H_0) = (C^*, H^*).
\]
As a matter of fact, we can consider the strictly convex functional as
\[
\Phi(C, H) := bg \int_{C^*}^C \frac{\eta - C^*}{\eta} + s \int_{H^*}^H \frac{\xi - H^*}{\xi}.
\]
It is not difficult to check that, for all \((C, H) \in \Sigma\), it holds
\[
(C(1 - C + sH), bH(1 - H - gH)) \cdot \nabla \Phi(C, H) = -bg(C - C^*)^2 - bs(H - H^*)^2 \leq 0.
\]
Furthermore, for any solution \((C, H)\) of the ODE system, one has
\[
\Phi(C, H)_t = bg(C - C^*)(1 - C + sH) + bs(H - H^*)(1 - H - gH)
\]
\[
= -bg(C - C^*)^2 - bs(H - H^*)^2,
\]
and it is a strict Lyapunov function. Since \(\Phi\) is bounded from below, we assume without loss of generality that \(\Phi \geq 0\), and the equality only holds at the unique minimizer \((C^*, H^*)\).

Let us argue by contradiction and assume that there exist \(c \in [0, c^*]\) and a sequence \(\{(t_k, x_k)\}_{k \geq 0} \subset (0, \infty) \times \mathbb{R}^N\) such that \(t_k \to +\infty\) and \(\epsilon > 0\) such that for all \(k > 0\),
\[
\|x_k\| \leq ct_k \quad \text{and} \quad |C(t_k, x_k) - C^*| + |H(t_k, x_k) - H^*| \geq \delta.
\]
Consider the sequence of functions \((F_k, C_k, H_k)(t, x) = (F, C, H)(t + t_k, x + x_k)\). Now, let us fix \(c' > 0\) such that \(c < c' < c^*\). Therefore, there exist \(N > 0\) large enough and \(\epsilon > 0\) small enough such that, for \(k \geq 0\) and \(t \in \mathbb{R}\), one has
\[
t + t_k \geq A \quad \text{and} \quad \|x\| \leq c't + (c' - c)t_k \Rightarrow \begin{cases}
F_k(t, x) \leq \frac{1}{k}, \\
\epsilon \leq C_k(t, x) \leq 1 + s - \epsilon, \\
\epsilon \leq H_k(t, x) \leq 1 - \epsilon.
\end{cases}
\]
Then, by parabolic estimates, possibly along a subsequence, one may assume that
\[
(F_k, C_k, H_k)(t, x) \to (F_\infty, C_\infty, H_\infty)(t, x) \quad \text{locally uniformly for } (t, x) \in \mathbb{R} \times \mathbb{R}^N,
\]
where \((F_\infty, C_\infty, H_\infty)\) is a bounded entire solution and satisfies
\[
\sup_{(t, x) \in \mathbb{R} \times \mathbb{R}^N} F_\infty(t, x) = 0,
\]
\[
\inf_{(t, x) \in \mathbb{R} \times \mathbb{R}^N} C_\infty(t, x) > 0 \quad \text{and} \quad \inf_{(t, x) \in \mathbb{R} \times \mathbb{R}^N} H_\infty(t, x) > 0,
\]
\[
\sup_{(t, x) \in \mathbb{R} \times \mathbb{R}^N} C_\infty(t, x) < 1 + s \quad \text{and} \quad \sup_{(t, x) \in \mathbb{R} \times \mathbb{R}^N} H_\infty(t, x) < 1.
\]
Moreover, it holds
\[
|C_\infty(0, 0) - C^*| + |H_\infty(0, 0) - H^*| > 0.
\]
In order to reach a contradiction, we claim that
\[\textbf{Claim 6.9} \quad \text{Let } (U, V) \text{ be a bounded entire solution satisfying the above estimates, then } (U, V)(t, x) \equiv (C^*, H^*).\]

**Proof of Claim 6.9** To prove this claim, we consider
\[
W(t, x) := \Phi(U(t, x), V(t, x)).
\]
Then one has
\[
W_t - \Delta W = - (\Phi_{UU}|\nabla U|^2 + 2\Phi_{UV} \nabla U \cdot \nabla V + \Phi_{VV} |\nabla V|^2)
\]
\[
+ \Phi_{UU}(1 - U + sV) + \Phi_{VV} bV(1 - V - gU)
\]
\[
\leq 0.
\]
Choose sequences \(\{t_n\}_{n \geq 0}\) and \(\{x_n\}_{n \geq 0}\) such that
\[
\lim_{n \to \infty} W(t_n, x_n) = \sup_{(t, x) \in \mathbb{R} \times \mathbb{R}^N} W(t, x).
\]

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By considering the sequence
\[ W_n(t, x) = W(t + t_n, x + x_n) = \Phi(U_n(t, x), V_n(t, x)), \]
where \( U_n(t, x) = U(t + t_n, x + x_n) \) and \( V_n(t, x) = V(t + t_n, x + x_n) \), then one obtains, possibly along a subsequence, \( (U_n, V_n) \to (U_\infty, V_\infty) \) locally uniformly and \( W_n \to W_\infty := \Phi(U_\infty, V_\infty) \) locally uniformly where \( (U_\infty, V_\infty) \) is an entire solution of the system (2.12). Note that \( W_\infty \) satisfies
\[ W_\infty(0, 0) = \sup_{(t, x)} W(t, x) = \sup_{(t, x)} W_\infty(t, x), \]
and \( W_\infty \) is a sub-solution of the heat equation, hence \( W_\infty(t, x) \equiv W_\infty(0, 0) \) is a constant function. The strict convexity of \( \Phi \) provide that
\[ U_\infty(t, x) \equiv U_\infty(t), \quad V_\infty(t, x) \equiv V_\infty(t), \]
\[ \left( U_\infty(1 - U_\infty + sV_\infty), bV_\infty(1 - V_\infty - gU_\infty) \right) \cdot \nabla \Phi(U_\infty, V_\infty) \equiv 0. \]
Also by using the fact that \( W_\infty \) is a constant, we have \( \Phi(U_\infty(t), V_\infty(t)) = \Phi(U_\infty(0), V_\infty(0)) \) for all \( t \in \mathbb{R} \). Since the Lyapunov function is strict, one obtains that \( U_\infty(t) = C^*, \quad V_\infty(t) = H^* \). Hence, one can conclude that
\[ 0 \leq W[U, V](t, x) \leq \Phi(C^*, H^*) = 0, \]
which completes the proof. \( \square \)

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**Appendix: Proof of Proposition 4.1**

We consider the equation
\[ \partial_t z = \partial_x^2 z + \left( c^* - \frac{\delta}{t + t_0} + \frac{N - 1}{\xi + \xi_\delta(t)} \right) \partial_x z + \lambda^* z, \quad t > 0, \xi > 0, \tag{4.1} \]
where \( \lambda^* = c^*/2 \) and \( \xi_\delta(t) := c^*(t + t_0) - \delta \ln \frac{t + t_0}{t_0} \).

We prove the following proposition.

**Proposition 4.1 (6)** Let \( z^\delta_{t_0}(\xi, t) \) be the solution of the equation (4.1) with boundary condition
\[ z^\delta_{t_0}(t, 0) = 0 \text{ for all } t > 0 \]
and the initial data
\[ z^\delta_{t_0}(0, \xi) = e^{-\lambda^* \xi} \zeta_0(t_0^{-1/2} \xi) \geq 0 \text{ for all } \xi \geq 0, \]
where \( \zeta_0(\cdot) \) is a nontrivial compactly supported smooth function. Then it holds:
\[ z^\delta_{t_0}(t, \xi) = \left( \frac{t + t_0}{t_0} \right)^{\gamma + \frac{1}{2}} \xi e^{-\lambda^* \xi} \left\{ \int_0^\infty \zeta_0(\rho) \rho d\rho + \frac{h_1(t, t_0)}{\sqrt{\pi}} e^{-\frac{\rho^2}{4(t + t_0)}} + h_2(t, \xi, t_0) \right\}, \quad \xi \geq 0, \ t \geq 0, \tag{4.2} \]
where \( \gamma := \delta \lambda^* - \frac{N + 1}{2} \), \( h_1 \) and \( h_2 \) are smooth functions satisfying
\[
\begin{align*}
|h_1(t, t_0)| &\leq B_1 \left( \frac{t_0}{t_0 + t_0} \right)^{\gamma + \frac{1}{2}} \| \zeta_0 \|_m, \\
|h_2(t, \xi, t_0)| &\leq B_2 \left( \frac{t_0^{1/4} \| \zeta_0 \|_m}{(t + t_0)^{1/4}} + \left( \frac{t_0}{t + t_0} \right)^{1/4} \right)^{\frac{1}{2}} \left( \frac{t_0}{(t + t_0)^{1/2}} \right)^{1/4} e^{-\frac{\xi^2}{8(t + t_0)}}, \\
&\quad \xi \geq 0, \ t \geq 0,
\end{align*}
\]
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for some positive constants $B_1$ and $B_2$. Here, the norm $\|\cdot\|_m$ is defined as

$$\|\zeta\|_m^2 := \int_0^\infty \zeta(\rho)^2 \rho^k e^{-\frac{\rho^2}{4\tau}} d\rho.$$ 

**Proof**: Let $z(t, \xi) = z_{t_0}^\delta(t, \xi)$ be the solution of the equation (4.1) with boundary condition

$$z_{t_0}^\delta(t, 0) = 0 \quad \text{for all } t > 0$$

and the initial data

$$z_{t_0}^\delta(0, \xi) = e^{-\lambda^* \xi} \zeta(0, 1/2 \xi) \geq 0 \quad \text{for all } \xi \geq 0,$$

where $\zeta(\cdot)$ is a nontrivial compactly supported smooth function. By using new coordinates $\rho = \frac{\xi}{\sqrt{\tau + t_0}}$, $\tau = \log \frac{t + t_0}{t_0}$ and a new unknown function $e^{\lambda^* \xi} z(t, \xi) = \tilde{\zeta}(t, \rho)$, the equation (4.1) could be rewritten as follows:

$$
\begin{aligned}
\partial_t \zeta &= \partial_{\rho}^2 \zeta + \frac{\rho}{2} \partial_{\rho} \zeta + \tilde{\zeta} + e^{-\tau} \sqrt{t_0} \{ \alpha_{1, t_0} \partial_{\rho} \zeta + (\rho + 1) \alpha_{2, t_0} \zeta \} + \gamma \zeta, \quad \rho > 0, \quad \tau > 0, \\
\zeta(\tau, 0) &= 0, \quad \tau > 0, \\
\tilde{\zeta}(0, \xi) &= \zeta(0, \xi), \quad \xi \geq 0,
\end{aligned}
$$

(6.11)

where

$$\gamma := \delta \lambda^* - \frac{N - 1}{2} - 1,$$

$$\alpha_{1, t_0}(\tau, \rho) := \frac{N - 1}{2 \lambda^* + \frac{\tau}{\sqrt{t_0}} \rho - \frac{\delta \tau}{\sqrt{t_0}} e^{-\tau}} - \delta,$$

$$\alpha_{2, t_0}(\tau, \rho) := \frac{N - 1}{\rho + 1} \frac{-\delta \tau}{2 \lambda^* + \frac{\tau}{\sqrt{t_0}} \rho - \frac{\delta \tau}{\sqrt{t_0}} e^{-\tau}}.$$

Define $\zeta(\tau, \rho) := e^{-\gamma \tau} \tilde{\zeta}(\tau, \rho)$ ($\rho \geq 0, \quad \tau \geq 0$), $\mathcal{L} \varphi := \frac{d^2}{d\rho^2} \varphi + \frac{\rho}{2} \frac{d}{d\rho} \varphi + \varphi,$

$$m(\rho) := e^{\frac{\rho^2}{4}}, \quad \mathcal{L}(\varphi) := \{ \phi \in L^2_m(0, \infty) \mid \phi', \phi'' \in L^2_m(0, \infty), \phi(0) = 0 \},$$

$$L^2_m(0, \infty) := \{ \phi \in L^2(0, \infty) \mid \| \phi \|_m^2 := (\phi, \phi)_m := \int_0^\infty \phi(\rho)^2 m(\rho) d\rho < \infty \}.$$ 

Then, it holds

$$\frac{d}{d\tau} \zeta = \mathcal{L} \zeta + e^{-\tau} \sqrt{t_0} \{ \alpha_{1, t_0} \partial_{\rho} \zeta + (\rho + 1) \alpha_{2, t_0} \zeta \}, \quad \tau > 0.
$$

(6.12)

Here we remark that $\mathcal{L}$ is a self-adjoint operator whose resolvent is compact and eigenvalues and corresponding eigenfunctions are as follows:

$$\lambda_k := -(k - 1), \quad \varphi_k := \left\| \frac{d^{2k-1}}{d\rho^{2k-1}} \frac{1}{m} \right\|_m^{-1} \frac{d^{2k-1}}{d\rho^{2k-1}} \frac{1}{m} \left( \varphi_1(\rho) = \pi^{-\frac{1}{2}} e^{-\frac{\rho^2}{4}} \right).$$

We also remark some useful inequalities:

$$\|\zeta\|_m^2 \leq \|\partial_{\rho} \zeta\|_m^2,$$

(6.13)

$$\| (\rho + 1) \zeta \|_m \leq 3 \sqrt{2} \| \partial_{\rho} \zeta \|_m + \sqrt{6} \| \zeta \|_m,$$

(6.14)

$$2 \| Q \zeta \|_m^2 \leq \| \partial_{\rho} Q \zeta \|_m^2,$$

(6.15)

$$\frac{3}{4} \| \tilde{L} \zeta \|_m^2 \leq \| \partial_{\rho} \zeta \|_m^2 \leq \frac{5}{4} \| \tilde{L} \zeta \|_m^2,$$

(6.16)
\[\mathcal{L}\varphi := \mathcal{L}\varphi - \varphi = \frac{d^2}{dt^2}\varphi + \theta \frac{d}{dt}\varphi\] and \(Q\zeta := \zeta - \langle \zeta, \varphi_1 \rangle_m \varphi_1\) which is the component of \(\zeta\) which orthogonal to \(\varphi_1\).

Then there exist \(c > 0, T_0 > 0\) depending only on
\[
\sup_{t_0 \geq \frac{1}{c} T_0} \|\alpha_{i,t_0}\|_{L^\infty([0,\infty)^2)} < \infty, \quad \sup_{t_0 \geq \frac{1}{c} T_0} \|\partial_\rho \alpha_{i,t_0}\|_{L^\infty([0,\infty)^2)} < \infty \quad (i = 1, 2)
\] such that the following holds for any \(t_0 \geq T_0:\)
\[
\begin{align*}
\|\zeta(\tau)\|_m &\leq c\|\zeta_0\|_m, \quad (6.17) \\
\|\zeta(\tau), \varphi_1\|_m - \langle \zeta_0, \varphi_1 \rangle_m &\leq \frac{e}{\sqrt{T_0}} \|\zeta_0\|_m, \quad (6.18) \\
\|Q\zeta(\tau)\|_m^2 &\leq \frac{c e^{-\tau}}{\sqrt{T_0}} \|\zeta_0\|_m^2 + \frac{(2 - \sqrt{m})}{\sqrt{T_0}} \|Q\zeta_0\|_m^2, \quad (6.19) \\
\|\tilde{Q}\zeta(\tau)\|_m^2 &\leq \frac{c e^{-\tau}}{\sqrt{T_0}} \|\zeta_0\|_m^2 + \frac{(2 - \sqrt{m})}{\sqrt{T_0}} \|\tilde{Q}\zeta_0\|_m^2. \quad (6.20)
\end{align*}
\]
As a matter of fact,
\[
\frac{d}{d\tau}\|\zeta\|_m^2 = 2\langle \zeta, \dot{\zeta} \rangle_m = I_1 + I_2,
\]
where we denote \(\dot{\zeta} := \frac{d}{d\tau}\zeta\) and
\[
I_1 := 2\langle \zeta, \mathcal{L}\zeta \rangle_m, \quad I_2 := \frac{2e^{-\frac{\tau}{2}}}{\sqrt{T_0}} \langle \zeta, \alpha_{1,t_0}\partial_\rho \zeta + (\rho + 1)\alpha_{2,t_0}\zeta \rangle_m.
\]
Integrating by part, one has
\[I_1 = -2(\|\partial_\rho \zeta\|_m - \|\zeta\|_m).\]
By applying Schwarz inequality and (6.14),
\[
|\langle \zeta, \alpha_{1,t_0}\partial_\rho \zeta + (\rho + 1)\alpha_{2,t_0}\zeta \rangle_m| \leq \|\zeta\|_m(\|\alpha_{1,t_0}\|_{L^\infty} \|\partial_\rho \zeta\|_m + \|\alpha_{2,t_0}\|_{L^\infty} (\rho + 1)\|\zeta\|_m)
\leq (\|\alpha_{1,t_0}\|_{L^\infty} + 3\sqrt{2}\|\alpha_{2,t_0}\|_{L^\infty})\|\zeta\|_m\|\partial_\rho \zeta\|_m + \sqrt{6}\|\alpha_{2,t_0}\|_{L^\infty} \|\zeta\|_m^2.
\]
Hence, if we denote \(C := \|\alpha_{1,t_0}\|_{L^\infty} + 3\sqrt{2}\|\alpha_{2,t_0}\|_{L^\infty}\), then it holds
\[I_2 \leq \frac{e^{-\frac{\tau}{2}}}{\sqrt{T_0}}(\|\partial_\rho \zeta\|_m^2 - \|\zeta\|_m^2) + \frac{3Ce^{-\frac{\tau}{2}}}{\sqrt{T_0}} \|\zeta\|_m^2.
\]
Thus by (6.13), for \(t_0 \geq C^2/4\), one has
\[
\frac{d}{d\tau}\|\zeta\|_m^2 \leq -\left(2 - \frac{Ce^{-\frac{\tau}{2}}}{\sqrt{T_0}}\right)(\|\partial_\rho \zeta\|_m^2 - \|\zeta\|_m^2) + \frac{3Ce^{-\frac{\tau}{2}}}{\sqrt{T_0}} \|\zeta\|_m^2 \leq \frac{3Ce^{-\frac{\tau}{2}}}{\sqrt{T_0}} \|\zeta\|_m^2.
\]
Then, one can obtain that
\[
\|\zeta\|_m^2 \leq \frac{e^{\frac{\tau}{2}}}{\sqrt{T_0}} \|\zeta_0\|_m^2 \leq e^{\frac{\tau}{2}} \|\zeta_0\|_m^2.
\]
Therefore the inequality (6.17) holds true. By (6.17), \(\mathcal{L}\varphi_1 = 0, \|\varphi_1\|_m = \|\partial_\rho \varphi_1\|_m = 1\) and integrating by part, one has
\[
\frac{d}{d\tau}\langle \zeta, \varphi_1 \rangle_m = \frac{2e^{-\frac{\tau}{2}}}{\sqrt{T_0}} \langle \alpha_{1,t_0}\partial_\rho \zeta + (1 + \rho)\alpha_{2,t_0}\zeta, \varphi_1 \rangle_m
\]
\[
= -\frac{2e^{-\frac{\tau}{2}}}{\sqrt{T_0}} \langle \zeta, \alpha_{1,t_0}\partial_\rho \varphi_1 + \partial_\rho \alpha_{1,t_0}\varphi_1 + \alpha_{1,t_0}\rho \varphi_1 - (\rho + 1)\alpha_{2,t_0}\varphi_1 \rangle_m
\]
\[
\leq \frac{2Ce^{\frac{\tau}{2}}e^{-\frac{\tau}{2}}}{\sqrt{T_0}} \langle \alpha_{1,t_0}\partial_\rho \varphi_1 + \partial_\rho \alpha_{1,t_0}\varphi_1 + \alpha_{1,t_0}\rho \varphi_1 - (\rho + 1)\alpha_{2,t_0}\varphi_1 \rangle_m \|\zeta_0\|_m.
\]
Thus, if we denote $C_1 := 4e^{\sqrt{m}}(\alpha_{1,t_0} \partial_{\rho} \varphi_1 + \partial_{\rho} \alpha_{1,t_0} \varphi_1 + \alpha_{1,t_0} \partial^{2}_{\rho} \varphi_1 - (\rho + 1) \alpha_{2,t_0} \varphi_1) / m$, it holds

$$||\langle \zeta, \varphi_1 \rangle_m - \langle \zeta_0, \varphi_1 \rangle_m|| \leq \frac{C_1}{\sqrt{t_0}} (1 - e^{-\frac{\tau}{2}}) \| \zeta_0 \|_m.$$  

Therefore the inequality (6.18) holds. Inferring from the facts that $\langle Q\psi, Q\varphi \rangle_m = \langle Q\psi, \varphi \rangle_m, Q\tilde{L} = \tilde{L} Q$, one has

$$\frac{\partial}{\partial \tau} \| \tilde{L} Q \|_m^2 = 2 \langle \tilde{L}^2 Q \zeta, \tilde{L} Q \zeta \rangle_m = (\tilde{L}^2 Q \zeta, \zeta)_m = I_1 + I_2 + I_3,$$

where

$$I_1 := 2(\tilde{L}^2 Q \zeta, L \zeta)_m = 2(\tilde{L}^2 Q \zeta, \tilde{L} Q \zeta + Q \zeta)_m = -2(\| \partial_{\rho} \tilde{L} Q \zeta \|_m^2 - \| \tilde{L} Q \zeta \|_m^2),$$

$$I_2 := \frac{2e^{-\frac{\tau}{2}}}{\sqrt{t_0}}(\tilde{L}^2 Q \zeta, \alpha_{1,t_0} \partial_{\rho} Q \zeta + (\rho + 1) \alpha_{2,t_0} Q \zeta)_m,$$

$$I_3 := \frac{2e^{-\frac{\tau}{2}}}{\sqrt{t_0}}(\tilde{L}^2 Q \zeta, \alpha_{1,t_0} \partial_{\rho} P \zeta + (\rho + 1) \alpha_{2,t_0} P \zeta)_m \ (P \varphi := \langle \varphi, \varphi \rangle_m \varphi_1 = \varphi - Q \varphi).$$

By (6.14), (6.15), (6.16) and $\| \partial_{\rho} \zeta \|_m = - \langle \tilde{L} \zeta, \zeta \rangle_m \leq \| \tilde{L} \zeta \|_m \| \zeta \|_m$, one may find

$$\langle \tilde{L}^2 Q \zeta, Q \zeta \rangle_m = - \langle \partial_{\rho} \tilde{L} Q \zeta, \partial_{\rho} Q \zeta \rangle_m + \partial_{\rho} \alpha_{1,t_0} \partial_{\rho} P \zeta + \alpha_{2,t_0} \partial_{\rho} P \zeta \rangle_m$$

$$\leq C_2 \| \partial_{\rho} \tilde{L} Q \zeta \|_m \| \partial_{\rho} \tilde{L} Q \zeta \|_m \| Q \zeta \|_m \| \partial_{\rho} \tilde{L} Q \zeta \|_m + \| Q \zeta \|_m.$$ 

Thus, one conclude that

$$I_2 \leq \frac{C_2}{2\sqrt{t_0}} (3e^{-\frac{\tau}{2}} + 4e^{-\frac{\tau}{2}} + 2) \| \partial_{\rho} \tilde{L} Q \zeta \|_m^2 + \frac{3C_2}{2\sqrt{t_0}} e^{-\tau} \| Q \zeta \|_m^2.$$ 

By similar computation,

$$I_3 \leq \frac{C_2}{\sqrt{t_0}} \| \partial_{\rho} \tilde{L} Q \zeta \|_m^2 + \frac{C_2 e^{-\tau}}{\sqrt{t_0}} \| \zeta \|_m^2,$$

where

$$C_2 := \| \partial_{\rho} \alpha_{1,t_0} \partial_{\rho} \varphi_1 + \alpha_{1,t_0} \partial_{\rho} \varphi_1 - (\rho + 1) \partial_{\rho} \alpha_{2,t_0} \varphi_1 + (\rho + 1) \partial_{\rho} \alpha_{2,t_0} \varphi_1 + \alpha_{2,t_0} \varphi_1 \|_m.$$ 

Thus by applying the fact $2 \| \tilde{L} Q \zeta \|_m^2 \leq \| \partial_{\rho} \tilde{L} Q \zeta \|_m^2$ and (6.17), one has

$$\frac{\partial}{\partial \tau} \| \tilde{L} Q \zeta \|_m^2 \leq - \left( 2 - \frac{C_2 (3e^{-\frac{\tau}{2}} + 4e^{-\frac{\tau}{2}} + 2) + 2C_2'}{\sqrt{t_0}} \right) \| \tilde{L} Q \zeta \|_m^2 + \frac{(3C_2 + 2C_2')e^{-\frac{\tau}{2}}}{\sqrt{t_0}} e^{-\tau} \| \zeta_0 \|_m^2.$$ 

Therefore the inequality (6.20) holds. Since the proof of (6.19) is similar to that of (6.20), we omit it. Next, one can obtain that

$$\langle Q \zeta (\tau, \rho) \rangle^2 m(\rho) \rangle \leq \langle \partial_{\rho} ((Q \zeta)^2) m(\rho) \rangle \leq \int_{0}^{\infty} |\partial_{\rho} ((Q \zeta)^2) m(\rho) | d\rho$$

$$\leq 2 \int_{0}^{\infty} |Q \zeta \partial_{\rho} Q \zeta| m d\rho + \int_{0}^{\infty} (Q \zeta)^2 \frac{\rho}{2} m d\rho$$

$$= 4 \int_{0}^{\infty} |Q \zeta \partial_{\rho} Q \zeta| m d\rho \leq 4 \| Q \zeta \|_m \| \partial_{\rho} Q \zeta \|_m \leq 4 \| Q \zeta \|_m \| \tilde{L} Q \zeta \|_m^2$$

$$\leq \| \tilde{L} Q \zeta \|_m^2 + 3\| Q \zeta \|_m^2.$$ 

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Similarly, it also holds
\[ |(\partial_\rho Q\zeta(\tau, \rho))^2| \leq 4\|\partial_\rho Q\zeta\|_m\|\partial^2_\rho Q\zeta\|_m \leq 2\sqrt{5}\|Q\zeta\|_m^2\|\tilde{L}Q\zeta\|_m^2 \leq \frac{\sqrt{5}}{2}\|Q\zeta\|_m^2 + \frac{3\sqrt{5}}{2}\|\tilde{L}Q\zeta\|_m^2. \]

By (6.19) and (6.20), one has
\[
\frac{\sqrt{5}}{2}\|\tilde{Q}\zeta\|_m^2 + 3\sqrt{5} - \frac{\sqrt{5}}{2}\|\tilde{L}Q\zeta\|_m^2 \leq 2\sqrt{5}\|Q\zeta\|_m^2 + \frac{\sqrt{5}}{2}e^{-(2-\frac{\sqrt{5}}{2})\tau} (\|Q\zeta\|_m^2 + 3\|\tilde{L}Q\zeta\|_m^2).
\]

Hence by \(\|Q\zeta\|_m \leq \|\zeta_0\|_m\) and \(\|\tilde{L}Q\zeta\|_m \leq \|\tilde{L}\zeta_0\|_m\), one obtains
\[
|Q(\tau, \rho)| \leq \sqrt{\frac{4\sqrt{5}e^{-\frac{\tau}{2}}}{t_0^{1/4}}} \|\zeta\|_m + \sqrt{3e^{-(1-\frac{\sqrt{5}}{2})\tau}} (\|\zeta_0\|_m + \|\tilde{L}\zeta_0\|_m) e^{-\frac{\tau}{2}},
\]
\[
|\partial_\rho Q\zeta(\tau, \rho)| \leq \frac{\sqrt{2\sqrt{5}e^{-\frac{\tau}{2}}}}{t_0^{1/4}} \|\zeta\|_m + \sqrt{3e^{-(1-\frac{\sqrt{5}}{2})\tau}} (\|\zeta_0\|_m + \|\tilde{L}\zeta_0\|_m) e^{-\frac{\tau}{2}}.
\]

Thus if we denote \(\tilde{h}_1(\tau) := (4\pi)^{\frac{1}{2}}|<\zeta(\tau), \varphi_1>_m - <\zeta_0, \varphi_1>_m|\) and \(\tilde{h}_2(\tau, \rho) := \frac{|Q(\zeta_\rho)|}{\rho}\), by (6.18) and (6.16), it holds
\[
|\tilde{h}_1(\tau)| \leq \frac{(4\pi)^{\frac{1}{2}}c}{\sqrt{t_0}} \|\zeta_0\|_m,
\]
\[
|\tilde{h}_2(\tau, \rho)| = \left|\frac{Q(\tau, \rho)}{\rho}\right| \leq 2\left(\frac{\sqrt{5}e^{-\frac{\tau}{2}}}{t_0^{1/4}} \|\zeta\|_m + e^{-(1-\frac{\sqrt{5}}{2})\tau} (\|\zeta_0\|_m + \|\tilde{L}\zeta_0\|_m) e^{-\frac{\tau}{2}}\right).
\]

On the other hand, one has
\[
Q(\tau, \rho) = P(\tau, \rho) + Q(\tau, \rho) = <\zeta(\tau), \varphi_1>_m \varphi_1(\rho) + Q(\zeta(\tau), \rho)
\]
\[
= \rho \{ (\zeta_0, \varphi_1)_m + (4\pi)^{\frac{1}{2}}\tilde{h}_1(\tau) \} \frac{\varphi(\rho)}{\rho} + \tilde{h}_2(\tau, \rho).
\]

Since
\[
z(t, \xi) = \left(\log \frac{t + t_0}{t_0}\right)^{1/4}e^{-\frac{\pi^2}{4}(\log \frac{t + t_0}{t_0})^2},
\]
if we denote \(B_1 = (4\pi)^{1/4}, B_2 = \max\{4\sqrt{5}, \frac{\pi^2}{4}\}\) and \(h_1(t, t_0) := \tilde{h}_1\left(\frac{\log (t + t_0)}{t_0}\right), h_2(t, \xi, t_0) := \tilde{h}_2\left(\frac{\log (t + t_0)}{t_0}, \frac{\xi}{\sqrt{t + t_0}}\right)\), then we obtain the conclusion (4.2) of Proposition 4.1 and complete the proof. \(\square\)

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