Two-point Functions in Affine Current Algebra and Conjugate Weights

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Abstract

The two-point functions in affine current algebras based on simple Lie algebras are constructed for all representations, integrable or non-integrable. The weight of the conjugate field to a primary field of arbitrary weight is immediately read off.

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1 Introduction

Two-point functions are the simplest non-trivial correlators one may consider in (extended) conformal field theory. Nevertheless, results in the case of general representations of affine current algebras are still lacking, except for $SL(2)$ where invariance under (loop) projective transformations immediately produces the result. However, very recently we have provided the general solution in the case of affine $SL(N)$ current algebra using the differential operator realization of simple Lie algebras provided in [1], and explicit realizations of fundamental representations in terms of fermionic creation and annihilation operators [2].

The objective of the present work is to construct general two-point functions in affine current algebras based on any simple Lie algebra and for all representations, integrable or non-integrable. Again the construction is based on the differential operator realization of simple Lie algebras provided in [1], in addition to well-known results for general fundamental representations and their conjugate (or contragredient) representations.

Besides providing us with new insight in the general structure of conformal field theory based on affine current algebra, a motivation for studying two-point functions in affine current algebra is found in the wish to understand how to generalize to higher groups the proposal by Furlan, Ganchev, Paunov and Petkova [3] for how Hamiltonian reduction of affine $SL(2)$ current algebra works at the level of correlators. A simple proof of the proposal in that case is presented in [4] based on the work [5] on correlators for degenerate (in particular admissible) representations in affine $SL(2)$ current algebra. Explicit knowledge on two-point functions may be seen as a first step in the direction of understanding that generalization.

Furthermore, an immediate application of knowing the two-point functions is to determine the weight of the conjugate (primary) field to a primary field of an arbitrary weight. This result is valuable since conjugate representations play important roles in various respects, see e.g. [6]. For non-integrable representations such weights are in general not known.

The remaining part of this presentation is organized as follows. In Section 2 we review our differential operator realization [1] while fixing the notation. In Section 3 the construction of two-point functions is provided and the conjugate weights are derived. Section 4 contains some concluding remarks, whereas an illustration is given in Appendix A where we consider elements of the simple Lie algebra $G_2$.

2 Notation

Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$. $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The set of (positive) roots is denoted $(\Delta_+)$ $\Delta$ and the simple roots are written $\alpha_i, \ i = 1, ..., r$. $\alpha^\vee = 2\alpha/\alpha^2$ is the root dual to $\alpha$. Using the triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$$

(1)
the raising and lowering operators are denoted $e_\alpha \in g_+$ and $f_\alpha \in g_-$, respectively, with $\alpha \in \Delta_+$, and $h_i \in h$ are the Cartan operators. In the Cartan-Weyl basis we have

$$[h_i, e_\alpha] = (\alpha_i^\vee, \alpha)e_\alpha, \quad [h_i, f_\alpha] = -(\alpha_i^\vee, \alpha)f_\alpha$$

and

$$[e_\alpha, f_\alpha] = h_\alpha = G^{ij}(\alpha_i^\vee, \alpha^\vee)h_j$$

where the metric $G_{ij}$ is related to the Cartan matrix $A_{ij}$ as

$$A_{ij} = \alpha_i^\vee \cdot \alpha_j = (\alpha_i^\vee, \alpha_j) = G_{ij}\alpha_j^2 / 2$$

The Dynkin labels $\Lambda_k$ of the weight $\Lambda$ are defined by

$$\Lambda = \Lambda_k \Lambda^k, \quad \Lambda_k = (\alpha_k^\vee, \Lambda)$$

where $\{\Lambda^k\}_{k=1,...,r}$ is the set of fundamental weights satisfying

$$(\alpha_i^\vee, \Lambda^k) = \delta_i^k$$

Elements in $g_+$ may be parameterized using “triangular coordinates” denoted by $x^\alpha$, one for each positive root, thus we write general Lie algebra elements in $g_+$ as

$$g_+(x) = x^\alpha e_\alpha \in g_+$$

We will understand “properly” repeated root indices as in (7) to be summed over the positive roots. Repeated Cartan indices as in (5) are also summed over. The matrix representation $C(x)$ of $g_+(x)$ in the adjoint representation is defined by

$$C^b_a(x) = -x^\beta f^\beta_a$$

Now, a differential operator realization $\{\bar{J}_a(x, \partial, \Lambda)\}$ of the simple Lie algebra $g$ generated by $\{j_a\}$ is found to be

$$\bar{E}_\alpha(x, \partial) = V^\beta_\alpha(x)\partial_\beta$$
$$\bar{H}_i(x, \partial, \Lambda) = V^\beta_i(x)\partial_\beta + \Lambda_i$$
$$\bar{F}_\alpha(x, \partial, \Lambda) = V^-\beta_\alpha(x)\partial_\beta + P^\beta_\alpha(x)\Lambda_j$$

where

$$V^\beta_\alpha(x) = [B(C(x))]^\beta_\alpha$$
$$V^\beta_i(x) = -[C(x)]^\beta_i$$
$$V^-\beta_\alpha(x) = \left[e^{-C(x)}\right]^-\alpha^\beta$$
$$P^\beta_\alpha(x) = \left[e^{-C(x)}\right]^\beta_-\alpha$$

$B$ is the generating function for the Bernoulli numbers

$$B(u) = \frac{u}{e^u - 1} = \sum_{n \geq 0} \frac{B_n}{n!} u^n$$
whereas \( \partial_\beta \) denotes partial differentiation wrt \( x^\beta \). Closely related to this differential operator realization is the equivalent one \( \{ J_\alpha(x, \partial, \Lambda) \} \) given by

\[
E_\alpha(x, \partial, \Lambda) = -\tilde{F}_\alpha(x, \partial, \Lambda) \\
F_\alpha(x, \partial, \Lambda) = -\tilde{E}_\alpha(x, \partial, \Lambda) \\
H_i(x, \partial, \Lambda) = -\tilde{H}_i(x, \partial, \Lambda)
\]

(12)

The matrix functions \( \{ J_\alpha \} \) are defined in terms of universal power series expansions, valid for any Lie algebra, but ones that truncate giving rise to finite polynomials of which the explicit forms depend on the Lie algebra in question. Details on the truncations and the resulting polynomials may be found in \cite{1}.

### 2.1 Affine Current Algebra

Associated to a Lie algebra is an affine Lie algebra characterized by the central extension \( k \), and associated to an affine Lie algebra is an affine current algebra whose generators are conformal spin one fields and have amongst themselves the operator product expansion

\[
J_\alpha(z)J_\beta(w) = \frac{\kappa_{\alpha\beta}}{(z - w)^2} + \frac{f_{\alpha\beta\gamma} J_\gamma(w)}{z - w}
\]

(13)

where regular terms have been omitted. \( \kappa_{\alpha\beta} \) and \( f_{\alpha\beta\gamma} \) are the Cartan-Killing form and the structure coefficients, respectively, of the underlying Lie algebra.

It is convenient to collect the traditional multiplet of primary fields in an affine current algebra (which generically is infinite for non-integrable representations) in a generating function for that \cite{3, 5, 1}, namely the primary field \( \phi_\Lambda(w, x) \) which must satisfy

\[
J_\alpha(z)\phi_\Lambda(w, x) = -J_\alpha(x, \partial, \Lambda) \phi_\Lambda(w, x) \\
T(z)\phi_\Lambda(w, x) = \frac{\Delta(\phi_\Lambda)}{(z - w)^2} \phi_\Lambda(w, x) + \frac{1}{z - w} \partial \phi_\Lambda(w, x)
\]

(14)

Here \( J_\alpha(z) \) and \( T(z) \) are the affine currents and the energy-momentum tensor, respectively, whereas \( J_\alpha(x, \partial, \Lambda) \) are the differential operator realizations. \( \Delta(\phi_\Lambda) \) denotes the conformal dimension of \( \phi_\Lambda \). The explicit construction of primary fields for general simple Lie algebra and arbitrary representation is provided in \cite{1}.

An affine transformation of a primary field is given by

\[
\delta_\epsilon \phi_\Lambda(w, x) = \oint_w \frac{dz}{2\pi i} \epsilon^a(z) J_a(z) \phi_\Lambda(w, x) \\
= \left\{ \epsilon^{-\alpha}(w) V^\beta_\alpha(x) \partial_\beta + \epsilon^i(w) \left( V^\beta_\gamma(x) \partial_\beta + \Lambda_i \right) + \epsilon^a(w) \left( V^\beta_\alpha(x) \partial_\beta + P^a_{-\alpha}(x) \Lambda_i \right) \right\} \phi_\Lambda(w, x)
\]

(15)

and is parameterized by the \( d \) (\( d \) is the dimension of the underlying Lie algebra) independent infinitesimal functions \( \epsilon^a(z) \).
3 Two-point Functions

Let \( W_2(z, w; x, y; \Lambda, \Lambda') \) denote a general two-point function of two primary fields \( \phi_\Lambda(z, x) \) and \( \phi_{\Lambda'}(w, y) \). From the conformal Ward identities or projective invariance the well-known conformal property of the two-point function is found to be

\[
W_2(z, w; x, y; \Lambda, \Lambda') = \frac{\delta\Delta(\phi_\Lambda)\Delta(\phi_{\Lambda'})}{(z-w)^{\Delta(\phi_\Lambda)+\Delta(\phi_{\Lambda'})}} W_2(x, y; \Lambda, \Lambda')
\]

(16)

The affine Ward identity

\[
\delta_\epsilon W_2(z, w; x, y; \Lambda, \Lambda') = \langle \delta_\epsilon \phi_\Lambda(z, x) \phi_{\Lambda'}(w, y) \rangle + \langle \phi_\Lambda(z, x) \delta_\epsilon \phi_{\Lambda'}(w, y) \rangle = 0
\]

(17)

may be recast (using (15)) into the following set of \( d \) partial differential equations

\[
\left( \tilde{J}_a(x, \partial, \Lambda) + \tilde{J}_a(y, \partial, \Lambda') \right) W_2(x, y; \Lambda, \Lambda') = 0
\]

(18)

It is easily verified that only the \( 2r \) equations for \( a = \pm \alpha_i \) are independent. By induction, this simply follows from the fact that \( \{ \tilde{J}_a \} \) is a differential operator realization of a Lie algebra. It is the general solution to the equations (18) that we shall provide below.

First we review a few basic properties of fundamental representations and their conjugate representations.

In every highest weight representation of highest weight \( \Lambda \) the weights are given by \( \lambda = \Lambda - \sum \beta \) where \( \sum \beta \) is a sum of positive roots or zero. The depth of \( \lambda \) is then defined as the height of \( \sum \beta \). In a finite dimensional irreducible highest weight module there exists a unique vector (up to trivial renormalizations) of lowest weight characterized by having maximal depth. The conjugate representation of such a representation is a highest weight representation with highest weight \( \Lambda^+ \) given by minus the lowest weight of the original one, while in general all weights in the conjugate representation are given by minus the ones in the original representation. The conjugate representation of a fundamental representation (which is a finite dimensional irreducible highest weight representation of highest weight a fundamental weight) is again a fundamental representation. Due to the uniqueness of the conjugate weight we shall write \( \Lambda^i = (\Lambda^i)^+ \). Many fundamental representations are self-conjugate, see e.g. [6].

A key property of the Kronecker product of two finite dimensional irreducible highest weight representations that we shall use, is the result that the singlet occurs in the decomposition of the product if and only if the two highest weights are conjugate, and in that case its multiplicity is one, see e.g. [6]. In particular, this statement is valid for the Kronecker product of two fundamental representations. A simple consequence of that is that in the Kronecker product \( \Lambda^i \times \Lambda^{i+} \) there exists a unique linear combination

\[
R^i = \sum C_{\mu\nu} |\lambda^{(i)}\rangle^\mu \otimes |\lambda^{(i+)}\rangle^\nu
\]

(19)

which is annihilated by the co-product

\[
\Delta(j_a) = j_a \otimes 1 + 1 \otimes j_a
\]

(20)

for all generators \( j_a \in g \). In (19) \( \{ |\lambda^{(i)}\rangle^\mu \} \) is meant to be a basis for the highest weight module with highest weight \( \Lambda^i \) whereas \( \{ |\lambda^{(i+)}\rangle^\nu \} \) is a basis for the conjugate module.
characterized by $\Lambda^{i+}$. Let us emphasize that $C_{\mu\nu}$ are uniquely given coefficients (up to an overall and in this respect irrelevant scaling factor) as soon as the two bases have been chosen. Note that $\mu$ and $\nu$ are multiple indices also counting multiplicities of the weights. This is illustrated in Appendix A where the self-conjugate fundamental representation $\Lambda^2$ of $G_2$ is considered.

Hence, in the framework of our differential operator realization there exists a unique polynomial $R^i(x, y)$ (again up to scaling) for all $i = 1, ..., r$ satisfying

$$\left( J_a(x, \partial, \Lambda^i) + J_a(y, \partial, \Lambda^{i+}) \right) R^i(x, y) = 0$$

(21)

This polynomial may then be decomposed as in (19)

$$R^i(x, y) = \sum C_{\mu\nu} b(x, \Lambda^i, \{j_l\}_{\mu}) b(y, \Lambda^{i+}, \{j'_l\}_{\nu})$$

(22)

with the same coefficients $C_{\mu\nu}$. The relation between the realizations is

$$|\lambda^{(i)}\rangle_{\mu} = f_{\alpha j_1} ... f_{\alpha j_{n(\mu)}} |\Lambda^i\rangle$$

$$b(x, \Lambda^i, \{\tilde{j}_l\}_{\mu}) = \tilde{F}_{\alpha j_1}(x, \partial, \Lambda^i) ... \tilde{F}_{\alpha j_{n(\mu)}}(x, \partial, \Lambda^i) 1$$

$$= \left( V^\beta_{-\alpha j_1}(x) \partial_\beta + P^i_{-\alpha j_1}(x) \right) ...$$

$$\cdot \left( V^\beta_{-\alpha j_{n(\mu)-1}}(x) \partial_\beta + P^i_{-\alpha j_{n(\mu)-1}}(x) \right) P^i_{-\alpha j_{n(\mu)}}(x)$$

(23)

By construction we have $\alpha_{j_{n(\mu)}} = \alpha_i$ for $n(\mu) > 0$.

We are now in a position to state our main result:

**Proposition**

The two-point function of the primary fields $\phi_{\Lambda}(z, x)$ and $\phi_{\Lambda'}(w, y)$ in an affine current algebra is (up to an irrelevant normalization constant) given by

$$W_2(z, w; x, y; \Lambda, \Lambda') = \frac{\delta_{\Delta(\phi_{\Lambda}), \Delta(\phi_{\Lambda'})}}{(z - w)^{\Delta(\phi_{\Lambda}) + \Delta(\phi_{\Lambda'})}} \prod_{i=1}^r \left( R^i(x, y) \right)^{p_i(\Lambda, \Lambda')}$$

(24)

where

$$p_i(\Lambda, \Lambda') = \Lambda_i$$

$$= \Lambda_{i+}$$

(25)

and $R^i(x, y)$ is given by (22).

**Proof**

As remarked above, we only need to consider the actions of $\tilde{E}_{\alpha j}(x, \partial) + \tilde{E}_{\alpha j}(y, \partial)$ and $\tilde{F}_{\alpha j}(x, \partial, \Lambda) + \tilde{F}_{\alpha j}(y, \partial, \Lambda')$ for $j = 1, ..., r$. That the $r$ former operators respect (18) follows directly from (21). From (21) we also have

$$\left( V^\beta_{-\alpha}(x) \partial_\beta + V^\beta_{-\alpha}(y) \partial_\beta \right) R^i(x, y) = - \left( P^i_{-\alpha}(x) + P^i_{-\alpha}(y) \right) R^i(x, y)$$

(26)
This implies that

\[
\left( \tilde{F}_{\alpha_j}(x, \partial, \Lambda) + \tilde{F}_{\alpha_j}(y, \partial, \Lambda') \right) W_2(x, y; \Lambda, \Lambda') = \left( \sum_{i=1}^{r} (\Lambda_i - p_i(\Lambda, \Lambda')) P^{\Lambda}_{-\alpha_j}(x) + \sum_{i=1}^{r} (\Lambda'_i + p_i(\Lambda, \Lambda')) P^{\Lambda'}_{-\alpha_j}(y) \right) W_2(x, y; \Lambda, \Lambda') \quad (27)
\]

from which we obtain (25).

From the condition on the pair $(\Lambda, \Lambda')$ in (25) it follows immediately that the conjugate weight $\Lambda^+$ to an arbitrary weight $\Lambda = \sum_{k=1}^{r} \Lambda_k \Lambda^k$, integrable or non-integrable, is given by

\[
\Lambda^+ = \sum_{k=1}^{r} \Lambda^+_k \Lambda^k = \sum_{k=1}^{r} \Lambda_{k+} \Lambda^k \quad (28)
\]

4 Conclusion

By constructing solutions to the affine Ward identities, we have provided a general expression for two-point functions in conformal field theory based on affine current algebra for all simple Lie groups and all representations, integrable or non-integrable. The construction relies on the unique existence of a singlet in the Kronecker product of a fundamental representation and its conjugate (fundamental) representation. An immediate application is the derivation of the conjugate weight to an arbitrary weight.

We hope to come back elsewhere with a discussion on two-point functions (and conjugate weights) in affine current superalgebra. In that case one may employ the recently obtained differential operator realizations of the underlying Lie superalgebras \[7\].

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A The Singlet in $\Lambda^2 \times \Lambda^2$ in the Case of $G_2$

Our labeling of the $r = 2$ simple roots in $G_2$ results in the Cartan matrix

\[
A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}
\]

This means that in the basis of fundamental weights the simple roots are given by

\[
\alpha_1 = (2, -1) \quad , \quad \alpha_2 = (-3, 2) \quad (30)
\]

The weights in the 14 dimensional and self-conjugate fundamental representation $\Lambda^2$ are easily obtained and all have multiplicity 1 except the weight $(0, 0)$ which has multiplicity
2. A particular basis in the highest weight module $\Lambda^2 = (0, 1)$ is

\[
\begin{align*}
|0, 1\rangle & = f_2 |0, 1\rangle \\
|3, -1\rangle & = f_2 |0, 1\rangle \\
|1, 0\rangle & = f_1 |3, -1\rangle \\
|-1, 1\rangle & = f_1 |1, 0\rangle \\
|-3, 2\rangle & = f_1 |-1, 1\rangle \\
|2, -1\rangle & = f_2 |-1, 1\rangle \\
|0, 0\rangle_1 & = f_1 |2, -1\rangle \\
|0, 0\rangle_2 & = f_2 |-3, 2\rangle \\
|3, -2\rangle & = f_2 |0, 0\rangle_2 \\
|-2, 1\rangle & = f_1 |0, 0\rangle_1 \\
|1, -1\rangle & = f_1 |3, -2\rangle \\
|-1, 0\rangle & = f_1 |-1, 0\rangle \\
|-3, 1\rangle & = f_1 |-1, 0\rangle \\
|0, -1\rangle & = f_2 |-3, 1\rangle
\end{align*}
\]

Here we have introduced the abbreviation $f_i = f_{\alpha_i}$. Note that $|0, 0\rangle_1$ and $|0, 0\rangle_2$ are linearly independent. The singlet linear combination $R^2$ is worked out to be

\[
R^2 = |0, 1\rangle \otimes |0, -1\rangle - |3, -1\rangle \otimes |-3, 1\rangle + |1, 0\rangle \otimes |-1, 0\rangle - |-1, 1\rangle \otimes |1, -1\rangle + (|3, -2\rangle \otimes |3, -2\rangle + 3|2, -1\rangle \otimes |-2, 1\rangle) - (12|0, 0\rangle_1 \otimes |0, 0\rangle_1 - 6|0, 0\rangle_1 \otimes |0, 0\rangle_2 - 6|0, 0\rangle_2 \otimes |0, 0\rangle_1 + 4|0, 0\rangle_2 \otimes |0, 0\rangle_2) + (|3, -2\rangle \otimes |-3, 2\rangle + 3|-2, 1\rangle \otimes |2, -1\rangle) - |1, -1\rangle \otimes |-1, 1\rangle + |-1, 0\rangle \otimes |1, 0\rangle - |-3, 1\rangle \otimes |3, -1\rangle + |0, -1\rangle \otimes |0, 1\rangle
\]

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