$N = 1$ FROM $N = 2$ SUPERSTRINGS

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\textbf{Abstract}

We give a simple proof that a particular class of $N = 2$ superstrings are equivalent to the $N = 1$ superstrings. This is achieved by constructing a similarity transformation which transforms the $N = 2$ BRST operators into a direct sum of the BRST operators for the $N = 1$ string and topological sectors.

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One of the long standing problems in string theories is how to give a formalism which allows to interpolate between various string theories. Recently a very interesting discovery in this direction has been made \[1\]. It has been shown that the \(N = 0(N = 1)\) strings can be viewed as a special class of vacua for \(N = 1(N = 2)\) superstrings. The equivalence of these \(N = 1\) and \(N = 0\) strings has been discussed in the original paper \[1\] and is further confirmed in refs. \[2, 3\] (see also \[4\]).

The purpose of this paper is to give a simple and explicit proof that the above choice of the vacua in \(N = 2\) superstrings is equivalent to the \(N = 1\) superstring. That this is true has also been argued in ref. \[1\] for the scattering amplitudes. The argument, however, is rather complicated involving picture-changing operators and instanton-number-changing operators and it is not clear if the operator algebras are also isomorphic.

The most straightforward and transparent approach to this problem is the one in ref. \[3\], where it has been shown that the BRST operator for the \(N = 1\) superstrings is transformed into a direct sum of the BRST operators for the \(N = 0\) string and topological sectors by a similarity transformation. We will show that it is also possible to construct such a similarity transformation for the \(N = 2\) superstrings and demonstrate the equivalence. This may appear straightforward but, since the argument in ref. \[1\] is involved and it is not clear if that argument is applicable to the operator algebra, it is better to give a simpler proof at the operator level. We also find that the proof is not quite trivial.

Let us start by recapitulating the \(N = 2\) formulation of the \(N = 1\) superstrings in ref. \[1\]. Take the \(N = 1\) superstrings with super stress-energy matter tensors \(T_m\) and \(G_m\) with central charge \(c_m = 15\), and add to the system fermionic fields \((\eta_1, \xi_1)\) of spin \((3/2, -1/2)\) and bosonic \((\beta_1, \gamma_1)\) of spin \((1, 0)\).\[1\] This gives a system with central charge 6 which can then be used as a matter sector for an \(N = 2\) superstring. Indeed, it is possible to construct \(N = 2\) generators for this system. For this purpose, it is convenient to use

\[1\]We slightly change the notation of the fields from ref. \[1\].
the bosonization
\[ \gamma_1 = \eta_2 e^\phi, \quad \beta_1 = e^{-\phi} \partial \xi_2. \] (1)

The \( N = 2 \) generators of the system are then
\[
\begin{align*}
G^- &= \eta_1, \\
G^+ &= \gamma_1 G_m + \xi_1 \left( T_m - \frac{3}{2} \beta_1 \partial \gamma_1 - \frac{1}{2} \partial \beta_1 \gamma_1 \right) - \gamma_1^2 \eta_1 + \partial (\xi_1 \xi_2 \eta_2) + \partial^2 \xi_1 + \eta_1 \xi_1 \partial \xi_1, \\
T &= T_m - \frac{3}{2} \beta_1 \partial \gamma_1 - \frac{1}{2} \partial \beta_1 \gamma_1 - \eta_1 \partial \xi_1 - \frac{1}{2} \partial (\eta_1 \xi_1 + \eta_2 \xi_2), \\
J &= -\eta_1 \xi_1 + \eta_2 \xi_2.
\end{align*}
\] (2)

Here the generators \( G^+ \) and \( T \) are written in mixed notation, and it is more convenient to express these solely in terms of independent fields \( (\eta_1, \xi_1), (\eta_2, \xi_2) \) and \( \phi \):
\[
\begin{align*}
G^+ &= \eta_2 e^\phi G_m + \xi_1 \left( T_m + \partial \xi_2 \eta_2 - \frac{1}{2} (\partial \phi)^2 - \partial^2 \phi \right) \\
&\quad - \eta_1 \eta_2 \partial \eta_2 e^{2\phi} + \partial (\xi_1 \xi_2 \eta_2) + \partial^2 \xi_1 + \eta_1 \xi_1 \partial \xi_1, \\
T &= T_m - \frac{3}{2} \eta_1 \partial \xi_1 - \frac{1}{2} \partial \eta_1 \xi_1 - \frac{3}{2} \eta_2 \partial \xi_2 - \frac{1}{2} \partial \eta_2 \xi_2 - \frac{1}{2} (\partial \phi)^2 - \partial^2 \phi.
\end{align*}
\] (3)

The BRST operator for this \( N = 2 \) superstring takes the form
\[
\begin{align*}
Q_{N=2} &= \oint \frac{dz}{2\pi i} \left[ c \left( T + \frac{1}{2} T_{gh} \right) + \frac{1}{\sqrt{2}} \gamma^+ \left( G^- + \frac{1}{2} G_{g^-} \right) \\
&\quad + \frac{1}{\sqrt{2}} \gamma^- \left( G^+ + \frac{1}{2} G_{g^+} \right) - \frac{1}{2} c_1 \left( J + \frac{1}{2} J_{gh} \right) \right],
\end{align*}
\] (4)

where \((b,c), (\beta^\pm, \gamma^\mp)\) and \((b_1, c_1)\) are the reparametrization, supersymmetry and \( U(1) \) ghosts, respectively, with the correlations
\[
\gamma^\pm(z) \beta^\mp(w) \sim c_1(z) b_1(w) \sim \frac{1}{z - w},
\] (5)

and the generators with subscript \( gh \) are those for ghosts:
\[
\begin{align*}
T_{gh} &= -2 b \partial c - \partial b c - \frac{3}{2} \beta^+ \partial \gamma^- - \frac{1}{2} \partial \beta^+ \gamma^- \\
&\quad - \frac{3}{2} \beta^- \partial \gamma^+ - \frac{1}{2} \partial \beta^- \gamma^+ - b_1 \partial c_1, \\
G_{gh}^\pm &= \frac{1}{\sqrt{2}} \left( \mp \beta^\pm c_1 \mp \partial b_1 \gamma^\pm \mp 2 b_1 \partial \gamma^\pm + 3 \beta^\pm \partial c + 2 \partial \beta^\pm c - b \gamma^\pm \right), \\
J_{gh} &= \beta^+ \gamma^- - \beta^- \gamma^+ + 2 \partial (b_1 c).
\end{align*}
\] (6)
Substituting (2), (3) and (6) into (4), we find
\[ Q_{N=2} = \oint \frac{dz}{2\pi i} \left[ \left( c + \frac{1}{\sqrt{2}} \gamma^- \xi_1 \right) T_m + \frac{1}{\sqrt{2}} \gamma^- \eta_2 e^\phi G_m + bc\partial c - \frac{1}{2} b\gamma^+ \gamma^- \right. \\
- c(\beta^+ \partial \gamma^- + \beta^- \partial \gamma^+) + \frac{1}{2} \partial c(\beta^+ \gamma^- + \beta^- \gamma^+) \\
+ c \left( -\frac{3}{2} \eta_1 \partial \xi_1 - \frac{1}{2} \partial \eta_1 \xi_1 - \frac{3}{2} \eta_2 \partial \xi_2 + \frac{1}{2} \partial \eta_2 \xi_2 - \frac{1}{2}(\partial \phi)^2 - \partial^2 \phi \right) \\
\left. \frac{1}{\sqrt{2}} \gamma^- \left[ \xi_1 \partial \xi_2 \eta_2 + \partial (\xi_1 \xi_2 \eta_2) - \xi_1 \left( \frac{1}{2}(\partial \phi)^2 + \partial^2 \phi \right) - \eta_1 \partial \eta_2 e^{2\phi} \right. \right. \\
\left. \left. + \partial^2 \xi_1 + \eta_1 \partial \xi_1 \right] + \frac{1}{\sqrt{2}} \gamma^+ \eta_1 + \frac{1}{2} e_1(\eta_1 \xi_1 - \eta_2 \xi_2 + \gamma^+ \beta^- - \gamma^- \beta^+) \\
\left. + c \partial c b_1 + \frac{1}{2} b_1(\gamma^+ \partial \gamma^- - \partial \gamma^+ \gamma^-) \right] \right) \right). \tag{7} \]

We would like to compare this with the BRST operator for the \( N = 1 \) superstring:
\[ Q_{N=1} = \oint \frac{dz}{2\pi i} \left[ cT_m - \frac{1}{2} \gamma G_m + bc\partial c - \frac{1}{4} b\gamma^2 + \frac{1}{2} \partial c \beta \gamma - c\beta \partial \gamma \right]. \tag{8} \]

Comparison between (7) and (8) suggests the identification
\[ \gamma = -\sqrt{2} \gamma^- \eta_2 e^\phi, \quad \beta = -\frac{1}{\sqrt{2}} e^{-\phi} \eta_2 \beta^+. \tag{9} \]

The \( \beta \) in this relation is chosen so as to give the correct operator product with \( \gamma \). Substituting (9) into (8), we can express the \( N = 1 \) BRST operator (8) in terms of the \( N = 2 \) fields:
\[ Q_{N=1} = \oint \frac{dz}{2\pi i} \left[ cT_m + \frac{1}{\sqrt{2}} \gamma^- \eta_2 e^\phi G_m + bc\partial c - \frac{1}{2} \partial c \beta^+ \gamma^- \\
+ c \left( -\frac{3}{2} \partial \eta_2 \xi_2 - \frac{1}{2} \partial \eta_2 \xi_2 + \frac{1}{2}(\partial \phi)^2 - \partial^2 \phi \right) \\
- \frac{1}{2} b(\gamma^-)^2 \eta_2 \partial \eta_2 e^{2\phi} - c(\beta^+ \gamma^- + \eta_2 \xi_2)(\partial \phi + \eta_2 \xi_2) \right]. \tag{10} \]

Now our claim is that the BRST operator (7) is mapped into a direct sum of those for the \( N = 1 \) superstrings (10) and topological sectors by a similarity transformation
\[ e^R Q_{N=2} e^{-R} = Q_{N=1} + Q_{\text{top}} + Q_{U(1)}, \tag{11} \]

where
\[ R = \oint \frac{dz}{2\pi i} \left[ \frac{1}{\sqrt{2}} \xi_1 \left( -\gamma^- b + 3 \partial c \beta^- + 2 c \partial \beta^- + \beta^- c_1 + 2 b_1 \partial \gamma^- + \partial b_1 \gamma^- \right) \right. \\
- \left( 2 b_1 c + \frac{1}{\sqrt{2}} \beta^- c_1 + \frac{1}{\sqrt{2}} \gamma^- b_1 \xi_1 \right) \left( \partial \phi + \eta_2 \xi_2 \right) - \beta^- \gamma^- \eta_2 \partial \eta_2 e^{2\phi} \right], \tag{12} \]

3
and

\[ Q_{\text{top}} = \oint \frac{dz}{2\pi i \sqrt{2}} \gamma^+ \eta_1, \]

\[ Q_{U(1)} = -\oint \frac{dz}{2\pi i \sqrt{2}} \xi_2 (\eta_2 + \beta^+ \gamma^-). \]  

(13)

are the BRST operators for the topological sectors. It is easy to confirm that the BRST operators on the right hand side of eq. (11) all anticommute with one another and are nilpotent. With this form of the BRST operator, it is obvious that the cohomology of the \( Q_{N=2} \) is a direct product of those of \( Q_{N=1}, Q_{\text{top}} \) and \( Q_{U(1)} \). The BRST operator \( Q_{\text{top}} \) imposes the condition that \( \beta^-, \gamma^+, \eta_1 \) and \( \xi_1 \) all decouple and its cohomology consists of their vacuum alone. The modes \( \beta^+, \gamma^-, \eta_2, \xi_2 \) and \( \phi \) can be represented in terms of \( \beta, \gamma \) in eq. (9) as well as \( j \equiv (\eta_2 \xi_2 + \partial \phi) \) and \( \tilde{j} \equiv (\eta_2 \xi_2 + \beta^+ \gamma^-) \). \(^2\) The latter two together with \( b_1 \) and \( c_1 \) decouple from the physical subspace due to the condition imposed by the BRST operator \( Q_{U(1)} \) in (13). Thus we are left with only the particular combinations of the fields in (9) as well as the fields in the \( N = 1 \) superstrings and obtain one-to-one correspondence between the cohomologies of \( Q_{N=2} \) and \( Q_{N=1} \). It is also clear that the transformation manifestly keeps the operator algebra. This establishes the equivalence of the \( N = 1 \) superstrings and the \( N = 2 \) superstrings.

It is instructive to rewrite the total energy-momentum tensor for the system. First, it may be shown that

\[ T_{\text{tot}} \equiv T + T_{gh} \]

(where \( T \) and \( T_{gh} \) are given by (3) and (6)) in fact remains invariant under the similarity transformation. Second, it is interesting that it may be written as

\[ T_{\text{tot}} = T_{N=1} + \{ Q_{\text{top}}, b_{\text{top}} \} + \{ Q_{U(1)}, b_{U(1)} \} \]  

(14)

where

\[ T_{N=1} = T_m - 2b \partial c - \partial bc - c \frac{3}{2} \beta \partial \gamma - \frac{1}{2} \partial \beta \gamma \]

\(^2\)The two fermionic degrees of freedom represented by \( \eta_2, \xi_2 \) count like a single bosonic degree of freedom. Also note that the currents, \( j \) and \( \tilde{j} \), commute with \( \beta \) and \( \gamma \) in (9) and represent modes independent of \( \beta \) and \( \gamma \).
\[
T_m - 2b\partial c - \partial bc - \frac{3}{2}\beta^+\partial\gamma^- - \frac{1}{2}\partial\beta^+\gamma^-
\]
\[
- \frac{1}{2}\eta_2\partial\xi_2 - \frac{3}{2}\partial\eta_2\xi_2 - \frac{1}{2}(\partial\phi)^2 - \partial^2\phi - \beta^+\gamma^-\partial\phi
\]
\[
- \eta_2\xi_2(\partial\phi + \beta^+\gamma^-)
\]

Here one may work out that

\[
\tilde{j}j = \partial\eta_2\xi_2 - \eta_2\partial\xi_2 + \beta^+\gamma^-\partial\phi + \eta_2\xi_2(\partial\phi + \beta^+\gamma^-)
\]
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