EXCEPTIONAL SETS IN WARING’S PROBLEM: TWO SQUARES AND $s$ Biquadrates

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Abstract. Let $R_s(n)$ denote the number of representations of the positive number $n$ as the sum of two squares and $s$ biquadrates. When $s = 3$ or 4, it is established that the anticipated asymptotic formula for $R_s(n)$ holds for all $n \leq X$ with at most $O(X^{(9-2s)/8+\varepsilon})$ exceptions.

1. Introduction

Waring’s problem for sums of mixed powers involving one or two squares has been widely investigated. In 1987-1988, Brüdern [1, 2] considered the representation of $n$ in the form

$$n = x_1^2 + x_2^2 + y_1^{k_1} + \cdots + y_s^{k_s},$$

with $k_1^{-1} + \cdots + k_s^{-1} > 1$. Linnik [8] and Hooley [6] investigated sums of two squares and three cubes. In 2002, Wooley [11] investigated the exceptional set related to the asymptotic formula in Waring’s problem involving one square and five cubes. Recently, Brüdern and Kawada [3] established the asymptotic formula for the number of representations of the positive number $n$ as the sum of one square and seventeen fifth powers.

Let $R_s(n)$ denote the number of representations of the positive number $n$ as the sum of two squares and $s$ biquadrates. Very recently, subject to the truth of the Generalised Riemann Hypothesis and the Elliott-Halberstam Conjecture, Friedlander and Wooley [4] established that $R_3(n) > 0$ for all large $n$ under certain congruence conditions. They also showed that if one is prepared to permit a small exceptional set of natural numbers $n$, then the anticipated asymptotic formula for $R_s(n)$ can be obtained. To state their results precisely, we introduce some notations. We define

$$\mathcal{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-2-s} S_2(q,a)^2 S_4(q,a)^s e(-na/q),$$

(1.1)
where the Gauss sum \( S_k(q, a) \) is defined as

\[
S_k(q, a) = \sum_{r=1}^{q} e(ar^k/q).
\]

As in [4], we refer a function \( \psi(t) \) as being a \textit{sedately increasing function} when \( \psi(t) \) is a function of a positive variable \( t \), increasing monotonically to infinity, and satisfying the condition that when \( t \) is large, one has \( \psi(t) = O(t^\delta) \) for a positive number \( \delta \) sufficiently small in the ambient context. Then we introduce \( E_s(X, \psi) \) to denote the number of integers \( n \) with \( 1 \leq n \leq X \) such that

\[
|R_s(n) - c_3 \Gamma(\frac{5}{4})^4 \mathcal{G}_s(n)n^{s/4}| > n^{s/4}\psi(n)^{-1},
\]

where \( c_3 = \frac{2}{3}\sqrt{2} \) and \( c_4 = \frac{1}{4}\pi \). Friedlander and Wooley [4] established the upper bounds

\[
E_3(X, \psi) \ll X^{1/2+\varepsilon}\psi(X)^2
\]

and

\[
E_4(X, \psi) \ll X^{1/4+\varepsilon}\psi(X)^4,
\]

where \( \varepsilon > 0 \) is arbitrary small.

The main purpose of this note is to establish the following result.

\textbf{Theorem 1.1.} Suppose that \( \psi(t) \) is a \textit{sedately increasing function}. Let \( E_s(X, \psi) \) be defined as above. Then for each \( \varepsilon > 0 \), one has

\[
E_3(X, \psi) \ll X^{3/8+\varepsilon}\psi(X)^2
\]

and

\[
E_4(X, \psi) \ll X^{1/8+\varepsilon}\psi(X)^2,
\]

where the implicit constants may depend on \( \varepsilon \).

We establish Theorem 1.1 by means of the Hardy-Littlewood method. In order to estimate the corresponding exceptional sets effectively, we employ the method developed by Wooley [10, 11].

As usual, we write \( e(z) \) for \( e^{2\pi iz} \). Whenever \( \varepsilon \) appears in a statement, either implicitly or explicitly, we assert that the statement holds for each \( \varepsilon > 0 \). Note that the "value" of \( \varepsilon \) may consequently change from statement to statement. We assume that \( X \) is a large positive number, and \( \psi(t) \) is a sedately increasing function.
2. Preparations

Throughout this section, we assume that $X/2 < n \leq X$. For $k \in \{2, 4\}$, we define the exponential sum

$$f_k(\alpha) = \sum_{1 \leq x \leq P_k} e(\alpha x^k),$$

where $P_k = X^{1/k}$. We take $s$ to be either 3 or 4. By orthogonality, we have

$$R_s(n) = \int_0^1 f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha) d\alpha. \tag{2.1}$$

When $Q$ is a positive number, we define $M(Q)$ to be the union of the intervals

$$M_Q(q, a) = \{ \alpha : |q\alpha - a| \leq Q X^{-1} \},$$

with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. Whenever $Q \leq X^{1/2}/2$, the intervals $M_Q(q, a)$ are pairwise disjoint for $1 \leq a \leq q \leq Q$ and $(a, q) = 1$. Let $\nu$ be a sufficiently small positive number, and let $R = P_4^\nu$. We take $M = M(R)$ and $m = (R/N, 1 + R/N] \setminus M$. Write

$$v_k(\beta) = \int_0^{P_k} e(\gamma \beta) d\gamma.$$

One has the estimate

$$v_k(\beta) \ll P_k (1 + X|\beta|)^{-1/k}.$$

For $\alpha \in M_{X^{1/2}/2}(q, a) \subseteq M(X^{1/2}/2)$, we define

$$f^*_k(\alpha) = q^{-1} S_k(q, a) v_k(\alpha - a/q). \tag{2.2}$$

It follows from Theorem 4.1 [9] that whenever $\alpha \in M_{X^{1/2}/2}(q, a)$, one has

$$f_k(\alpha) - f^*_k(\alpha) \ll q^{1/2} (1 + X|\alpha - a/q|)^{1/2} X^\varepsilon. \tag{2.3}$$

We define the multiplicative function $w_k(q)$ by taking

$$w_k(p^{u_k+v}) = \begin{cases} kp^{-u-1/2}, & \text{when } u \geq 0 \text{ and } v = 1, \\ p^{-u-1}, & \text{when } u \geq 0 \text{ and } 2 \leq v \leq k. \end{cases}$$

Note that $q^{-1/2} \leq w_k(q) \ll q^{-1/k}$. Whenever $(a, q) = 1$, we have

$$q^{-1} S_k(q, a) \ll w_k(q).$$

Therefore for $\alpha = a/q + \beta \in M_{X^{1/2}/2}(q, a) \subseteq M(X^{1/2}/2)$, one has

$$f^*_k(\alpha) \ll w_k(q) P_k (1 + X|\beta|)^{-1/k} \ll P_k q^{-1/k} (1 + X|\beta|)^{-1/k}. \tag{2.4}$$

The following conclusion is (4.1) in [4].
Lemma 2.1. One has
\[
\int_{\mathbb{R}} f_2(\alpha)^2 f_4(\alpha)^\varepsilon e(-n\alpha) d\alpha = c_s \Gamma\left(\frac{5}{4}\right)^4 \mathcal{G}_s(n) n^{s/4} + O(n^{s/4 - \kappa + \varepsilon}),
\]
for a suitably small positive number \(\kappa\).

The next result provides the value of the Gauss sum \(S_2(q,a)\).

Lemma 2.2. The Gauss sum \(S_2(q,a)\) satisfies the following properties.

(i) If \((2a,q) = 1\), then
\[
S_2(q,a) = \left(\frac{a}{q}\right) S_2(q,1).
\]

Here by \(\left(\frac{a}{q}\right)\) we denote the Jacobi symbol.

(ii) If \(q\) is odd, then
\[
S_2(q,1) = \begin{cases} 
q^{1/2}, & \text{if } q \equiv 1 \pmod{4}, \\
iq^{1/2}, & \text{if } q \equiv 3 \pmod{4}.
\end{cases}
\]

(iii) If \((2,a) = 1\), then
\[
S_2(2^m,a) = \begin{cases} 
0, & \text{if } m = 1, \\
2^{m/2}(1+i^m), & \text{if } m \text{ is even}, \\
2^{(m+1)/2} e(a/8), & \text{if } m > 1 \text{ and } m \text{ is odd}.
\end{cases}
\]

(iv) If \((q_1,q_2) = 1\), then
\[
S_2(q_1q_2,a_1q_2 + a_2q_1) = S_2(q_1,a_1) S_2(q_2,a_2).
\]

Proof. These properties can be found in Lemma 2 [5]. \(
\square
\)

3. The Proof of Theorem 1.1

Let \(\tau\) be a fixed sufficiently small positive number. Set \(Y = P_4^{3/2+\tau} \psi(X)^2\).

We define \(m_1 = m \setminus M(X^{1/2}), m_2 = M(X^{1/2}) \setminus M(Y), m_3 = M(Y) \setminus M(P_4)\) and \(m_4 = M(P_4) \setminus M\). Let \(\eta(n)\) be sequence of complex numbers satisfying \(|\eta(n)| = 1\). Let \(Z\) be a subset of \(\{n \in \mathbb{N} : X/2 < n \leq X\}\). We abbreviate \(\text{card}(Z)\) to \(Z\). Then we introduce the exponential sum \(E(\alpha)\) by
\[
E(\alpha) = \sum_{n \in Z} \eta(n) e(-n\alpha).
\]

For \(1 \leq j \leq 4\), we define
\[
I_j = \int_{m_j} \left|f_2(\alpha)^2 f_4(\alpha)^\varepsilon E(\alpha)\right| d\alpha.
\]

Lemma 3.1. Let \(I_1\) be defined in (3.1). Then we have
\[
I_1 \ll P_4^{\varepsilon - 1/2 + \frac{1}{4} + \varepsilon} Z^{1/2} + P_4^{\varepsilon - 1/2 + \varepsilon} Z.
\]
Proof. For any \( \alpha \in \mathfrak{m}_1 \), there exist \( a \) and \( q \) with \( 1 \leq a \leq q \leq 2X^{1/2} \) and \((a, q) = 1\) such that \( |q\alpha - a| \leq X^{-1/2}/2 \). Since \( \alpha \in \mathfrak{m}_1 \), we conclude that \( q > X^{1/2}/2 \). It follows from Weyl's inequality (Lemma 2.4 [9]) that
\[
f_2(\alpha) \ll P_2^{1/2+\varepsilon} \quad \text{for} \quad \alpha \in \mathfrak{m}_1.
\]

Thus we have
\[
I_1 \ll P_2^{1+\varepsilon} \int_{\mathfrak{m}_1} |f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha
\]
\[
\ll P_2^{1+\varepsilon} \left( \int_0^1 |f_4(\alpha)^6| d\alpha \right)^{1/3} \left( \int_0^1 |f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha \right)^{1/3}.
\]

By Hua’s inequality (Lemma 2.5 [9]) and Schwartz’s inequality,
\[
\int_0^1 |f_4(\alpha)^6| d\alpha \ll \left( \int_0^1 |f_4(\alpha)^4| d\alpha \right)^{1/2} \left( \int_0^1 |f_4(\alpha)^8| d\alpha \right)^{1/2} \ll P_4^{7/2+\varepsilon}.
\]

When \( s = 4 \), one has the bound \( \int_0^1 |f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2| d\alpha \ll P_4 Z + P_4^{\varepsilon} Z^2 \).

Then we can conclude that
\[
(3.2) \quad I_1 \ll P_4^{4-\frac{1}{4} + \frac{s-3}{2} + \varepsilon} Z^{1/2} + P_4^{s-\frac{1}{4} + \varepsilon} Z.
\]

Indeed when \( s = 3 \), the estimate (3.2) holds with \( P_4^{s-\frac{1}{4} + \varepsilon} Z \) omitted. \( \square \)

Lemma 3.2. Let \( I_2 \) be given by (3.1). Then one has
\[
J_2 \ll P_4^{4-\frac{1}{4} + \frac{s-3}{2} + \varepsilon} Z^{1/2} + P_4^{s-\tau/2+\varepsilon} \psi(X)^{-1} Z.
\]

Proof. We introduce
\[
J_1 = \int_{m_2} |(f_2(\alpha) - f_2^*(\alpha))^2 f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha
\]

and
\[
J_2 = \int_{m_2} |f_2^*(\alpha)^2 f_4(\alpha)^s \mathcal{E}(\alpha)| d\alpha.
\]

Note that \( |f_2(\alpha)|^2 \ll |f_2(\alpha) - f_2^*(\alpha)|^2 + |f_2^*(\alpha)|^2 \), where \( f_2^*(\alpha) \) is defined in (2.2). Then one has
\[
(3.3) \quad I_2 \ll J_1 + J_2.
\]

In view of (2.3), we know \( f_2(\alpha) - f_2^*(\alpha) \ll P_2^{1/2+\varepsilon} \) for \( \alpha \in \mathfrak{m}_2 \). The argument leading to (3.2) also implies
\[
(3.4) \quad J_1 \ll P_4^{4-\frac{1}{4} + \frac{s-3}{2} + \varepsilon} Z^{1/2} + P_4^{s-\frac{1}{4} + \varepsilon} Z.
\]

One has, by Schwartz’s inequality, that
\[
J_2 \leq \left( \int_{m_2} |f_4(\alpha)^6| d\alpha \right)^{1/2} J_1^{1/2} \ll P_4^{7/4+\varepsilon} J_1^{1/2},
\]

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where $\mathcal{J}$ is defined as

$$
\mathcal{J} = \int_{m_2} |f_2^*(\alpha)|^4 f_4(\alpha)^{2(s-3)} \mathcal{E}(\alpha)^2 d\alpha.
$$

In order to handle $\mathcal{J}$, we need the following estimate

$$
\int_{m_2} |f_2^*(\alpha)|^4 |e(-h\alpha)| d\alpha = \begin{cases} O(P_4^{4+\epsilon}Y^{-1}), & \text{when } 0 < |h| \leq 2X, \\ O(P_4^{4+\epsilon}), & \text{when } h = 0. \end{cases}
$$

(3.5)

Recalling the definition of $f_2^*(\alpha)$, we conclude that

$$
\int_{m_2} |f_2^*(\alpha)|^4 |e(-h\alpha)| d\alpha
$$

$$
= \sum_{q \leq X^{1/2}/2}^{*} \int_{|\beta| \leq \frac{1}{2qX^{1/2}}} q^{-4} \left( \sum_{a=1}^{q} |S_2(q, a)|^4 |e(-ha/q)| \right) |v_2(\beta)|^4 |e(-h\beta)| d\beta,
$$

where the notations $\sum^{*}$ and $f^{*}$ mean either $q > Y$ or $Xq|\beta| > Y$. Whenever $(a, q) = 1$, one has by Lemma 2.2 that

$$
|S_2(q, a)| = |S_2(q, 1)| \leq (2q)^{1/2}.
$$

We obtain

$$
\left| \sum_{a=1}^{q} |S_2(q, a)|^4 |e(-ha/q)| \right| = |S_2(q, 1)|^4 \left| \sum_{a=1}^{q} e(-ha/q) \right|
$$

$$
\leq 4q^2 \left| \sum_{a=1}^{q} e(-ha/q) \right| \leq 4q^2 (q, h),
$$

whence

$$
\int_{m_2} |f_2^*(\alpha)|^4 |e(-h\alpha)| d\alpha \ll P_2^4 \sum_{q \leq X^{1/2}/2}^{*} \int_{|\beta| \leq \frac{1}{2qX^{1/2}}} q^{-2}(q, h) \frac{q^{-2}(q, h)}{(1 + X|\beta|)^2} d\beta.
$$

When $h = 0$, we have

$$
\int_{m_2} |f_2^*(\alpha)|^4 |e(-h\alpha)| d\alpha \ll P_2^4 \sum_{q \leq X^{1/2}/2} q^{-1}(1 + X|\beta|)^{-2} d\beta
$$

$$
\ll P_2^4 X^{-1} \log X.
$$

When $h \neq 0$, we get

$$
\int_{m_2} |f_2^*(\alpha)|^4 |e(-h\alpha)| d\alpha \ll P_2^4 Y^{-1} \sum_{q \leq X^{1/2}/2} q^{-1} \frac{q^{-1}(q, h)}{1 + X|\beta|} d\beta
$$

$$
\ll P_2^4 Y^{-1} X^{-1} \log X \sum_{q \leq X^{1/2}/2} q^{-1}(q, h)
$$

$$
\ll P_2^4 Y^{-1} X^{-1+\epsilon}.
$$
The conclusion (3.5) is established. Now we are able to estimate $J$. When $s = 4$,

$$J = \sum_{1 \leq x_1, x_2 \leq P_4, n_1, n_2 \in \mathbb{Z}} \eta(n_1)\eta(n_2) \int_{m_2} |f_2^*(\alpha)|^4 \left| e(-x_1^4 - x_2^4 + n_1 - n_2)\alpha \right| d\alpha.$$

On applying (3.5), we can deduce that

$$J \ll \sum_{1 \leq x_1, x_2 \leq P_4, n_1, n_2 \in \mathbb{Z}} P_4^{4+\varepsilon} Y^{-1} + \sum_{x_1^4 - x_2^4 + n_1 - n_2 \neq 0} P_4^{4+\varepsilon} Z^{2Y^{-1}} + P_4^{4+\varepsilon} Z^2 + P_4^{5+\varepsilon} Z.$$

Substituting $Y = P_4^{3/2+\tau}\psi(X)^2$, we finally obtain

$$J \ll P_4^{4+1/2-\tau+\varepsilon}\psi(X)^{-2} Z^2 + P_4^{5+\varepsilon} Z,$$

whence

$$J_2 \ll P_4^{4-\tau/2+\varepsilon}\psi(X)^{-1} Z + P_4^{4+1/4+\varepsilon} Z^{1/2}.$$

Similarly, when $s = 3$, one has

$$J \ll P_4^{5/2-\tau+\varepsilon}\psi(X)^{-2} Z^2 + P_4^{4+\varepsilon} Z,$$

whence

$$J_2 \ll P_4^{3-\tau/2+\varepsilon}\psi(X)^{-1} Z + P_4^{4-1/4+\varepsilon} Z^{1/2}.$$

Therefore, we conclude that

(3.6) $$J_2 \ll P_4^{4+1/4+\varepsilon} Z^{1/2} + P_4^{4-\tau/2+\varepsilon}\psi(X)^{-1} Z.$$

Combining (3.3), (3.4) and (3.6), we conclude that

(3.7) $$I_2 \ll P_4^{4+1/4+\varepsilon} Z^{1/2} + P_4^{4-\tau/2+\varepsilon}\psi(X)^{-1} Z.$$

We complete the proof. \(\square\)

**Lemma 3.3.** Let $I_3$ be defined in (3.7). Then we have

$$I_3 \ll P_4^{4+1/4+\varepsilon} Z^{1/2} + P_4^{4-\tau/2+\varepsilon}\psi(X)^{-1} Z.$$

**Proof.** Similarly to (3.3) and (3.4), we can derive that

(3.8) $$I_3 \ll P_4^{4+1/4+\varepsilon} Z^{1/2} + P_4^{4-\tau/2+\varepsilon} Z + K,$$

where

$$K = \int_{m_3} \left| f_2^*(\alpha)^2 f_4(\alpha)^s E(\alpha) \right| d\alpha.$$
One has
\[
K \leq \sup_{\alpha \in m_3} |f_4(\alpha)| \left( \int_{m_3} |f_2^*(\alpha)^2 f_4(\alpha)^4| d\alpha \right)^{1/2}
\times \left( \int_{m_3} |f_2^*(\alpha)^2 f_4(\alpha)^{2(s-3)} E(\alpha)^2| d\alpha \right)^{1/2}.
\]

In view of (2.3) and (2.4), we have for \(\alpha \in m_3\) that
\[
f_4(\alpha) \ll P_4 q^{-1/4} (1 + X|\alpha - a/q|)^{-1/4} + Y^{1/2} X^\varepsilon \ll P_4^{3/4 + \tau/2 + \varepsilon} \psi(X).
\]

Since \(f_2^*(\alpha) - f_2(\alpha) \ll P_2^{1/2}\) for \(\alpha \in m_3\), we easily deduce that
\[
\int_{m_3} |f_2^*(\alpha)^2 f_4(\alpha)^4| d\alpha \ll P_2^{1/2} \int_0^1 |f_2(\alpha) f_4(\alpha)^4| d\alpha + \int_0^1 |f_2(\alpha)^2 f_4(\alpha)^4| d\alpha
\ll P_4^{4 + \varepsilon}.
\]

Therefore we arrive at
\[
K \ll P_4^{11/4 + \tau/2 + \varepsilon} \psi(X) \left( \int_{m_3} |f_2^*(\alpha)^2 f_4(\alpha)^{2(s-3)} E(\alpha)^2| d\alpha \right)^{1/2}.
\]

Similarly to (3.5), we have the following estimate
\[
(3.9) \quad \int_{2\mathbb{R}(Y)} |f_2^*(\alpha)|^2 e(-h\alpha) d\alpha = \begin{cases} O(P_4^s), & \text{when } 0 < |h| \leq 2X, \\
O(P_4^s Y), & \text{when } h = 0. \end{cases}
\]

Note that
\[
\int_{2\mathbb{R}(Y)} |f_2^*(\alpha)|^2 e(-h\alpha) d\alpha
= \sum_{q \leq Y} \int_{|\beta| \leq \frac{Y}{q^2}} q^{-2} \left( \sum_{a=1}^q |S_2(q, a)|^2 e(-ha/q) \right) |v_2(\beta)|^2 e(-h\beta) d\beta
\ll P_2^2 \sum_{q \leq Y} \int_{|\beta| \leq \frac{Y}{q^2}} q^{-1}(q, h)(1 + X|\beta|)^{-1} d\beta
\ll (\log X) \sum_{q \leq Y} q^{-1}(q, h).
\]
The desired estimate (3.9) follows easily from above. For \( s = 4 \), we derive that
\[
\int_{m_3} |f_2^* + f_4^*|^2 d\alpha \\
\leq \int_{\mathfrak{M}(Y)} |f_2^* + f_4^*|^2 d\alpha \\
= \sum_{n_1, n_2 \in \mathbb{Z}} \eta(n_1) \eta(n_2) \int_{\mathfrak{M}(Y)} |f_2^* + f_4^*| \epsilon \left( -(n_1 - n_2 + x_1^4 - x_2^4) \alpha \right) d\alpha \\
\ll P_4^{2+\varepsilon} Z^2 + P_4^{5/2+\varepsilon} \psi(X)^2 Z + P_4^{3/2+\varepsilon} \psi(X)^2 Z^{1/2},
\]
whence
\[
\mathcal{K} \ll (P_4^{15/4+\varepsilon} \psi(X) + P_4^{7/2+\varepsilon} \psi(X)^2) Z + P_4^{4+\varepsilon} \psi(X)^2 Z^{1/2}.
\]
In particular, we have
\[
\mathcal{K} \ll P_4^{4+\varepsilon} Z^{1/2} + P_4^{4-\varepsilon} \psi(X)^{-1} Z
\]
promised that \( \psi(X) \ll X^{1/64-\varepsilon} \). For \( s = 4 \), by (3.9) we have
\[
\int_{m_3} |f_2^* + f_4^*|^2 d\alpha \ll P_4^{\varepsilon} Z^2 + P_4^{3/2+\varepsilon} \psi(X)^2 Z,
\]
whence
\[
\mathcal{K} \ll P_4^{11/4+\varepsilon} \psi(X) Z + P_4^{7/2+\varepsilon} \psi(X)^2 Z^{1/2}.
\]
When \( \psi(X) \ll X^{1/64-\varepsilon} \), one has
\[
\mathcal{K} \ll P_4^{\varepsilon} Z^{1/2} + P_4^{3-\varepsilon} \psi(X)^{-1} Z.
\]
We conclude from above that
\[
(3.10) \quad \mathcal{K} \ll P_4^{4-\varepsilon} Z^{1/2} + P_4^{4-\varepsilon} \psi(X)^{-1} Z.
\]
By (3.8) and (3.10), we obtain
\[
(3.11) \quad \mathcal{I}_4 \ll P_4^{4-\varepsilon} Z^{1/2} + P_4^{4-\varepsilon} \psi(X)^{-1} Z.
\]
We complete the proof. \( \square \)

**Lemma 3.4.** Let \( \mathcal{I}_4 \) be defined in (3.7). Then we have
\[
\mathcal{I}_4 \ll Z P_4^{s-(s-2)/4+\varepsilon}.
\]

**Proof.** In view of (2.3) and (2.4), for \( \alpha \in \mathfrak{M}(q, a) \), one has
\[
f_4(\alpha) \ll P_4 w_4(q)(1 + X|\alpha - a/q|)^{-1/4} + P_4^{1/2+\varepsilon}
\ll P_4^{1+\varepsilon} w_4(q)(1 + X|\alpha - a/q|)^{-1/4}.
\]
and
\[ f_2(\alpha) \ll P_2q^{-1/2}(1 + X|\alpha - a/q|)^{-1/2}. \]

Therefore we obtain
\[
\mathcal{I}_4 \ll Z \sup_{\alpha \in m_4} |f_4(\alpha)|^{s-2} \int_{3N(P_4)} |f_4(\alpha)f_2(\alpha)|^2d\alpha \\
\ll ZP_4^{2(s-2)(1-\nu/4)+\varepsilon} P_4^2 P_2^2 \sum_{q \in P_4} w_4(q)^2 \int_{|\beta| \leq \frac{P_4}{Z}} (1 + X|\beta|)^{-3/2} d\beta \\
\ll ZP_4^{2+(s-2)(1-\nu/4)+\varepsilon} \sum_{q \in P_4} w_4(q)^2.
\]

In light of Lemma 2.4 by Kawada and Wooley [7], one can conclude that
\[
(3.12) \quad \mathcal{I}_4 \ll ZP_4^{2(s-2)(1-\nu/4)+\varepsilon} \ll ZP_4^{s-(s-2)\nu/4+\varepsilon}.
\]

The desired estimate is established. \hfill \Box

Proof of Theorem 1.1. We denote by \( Z_s(X) \) the set of integers \( n \) with \( X/2 < n \leq X \) for which the lower bound
\[
|R_s(n) - c_s \Gamma(\frac{5}{4})^4 \mathcal{G}(n)n^{s/4}| > n^{s/4}\psi(n)^{-1}
\]
holds, and we abbreviate \( \text{card}(Z_s(X)) \) to \( Z_s \). It follows from (2.1) and Lemma 2.1 that for \( n \in Z_s(X) \),
\[
\left| \int_m f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)d\alpha \right| \gg X^{s/4}\psi(X)^{-1},
\]
whence
\[
\sum_{n \in Z_s(X)} \left| \int_m f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)d\alpha \right| \gg Z_s X^{s/4}\psi(X)^{-1}.
\]

We choose complex numbers \( \eta(n) \), with \( |\eta(n)| = 1 \), satisfying
\[
\left| \int_m f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)d\alpha \right| = \eta(n) \int_m f_2(\alpha)^2 f_4(\alpha)^s e(-n\alpha)d\alpha.
\]

Then we define the exponential sum \( \mathcal{E}_s(\alpha) \) by
\[
\mathcal{E}_s(\alpha) = \sum_{n \in Z_s(X)} \eta(n)e(-n\alpha).
\]

One finds that
\[
(3.13) \quad Z_s X^{s/4}\psi(X)^{-1} \ll \int_m \left| f_2(\alpha)^2 f_4(\alpha)^s \mathcal{E}_s(\alpha) \right| d\alpha.
\]

Note that \( m = m_1 \cup m_2 \cup m_3 \cup m_4 \). Now we conclude from Lemmata 3.1,3.1,3.1 and (3.13) that
\[
Z_s X^{s/4}\psi(X)^{-1} \ll P_4^{A-\frac{1}{2} + \frac{s-3}{2} + \varepsilon} Z_s^{1/2} + P_4^{s-\delta}\psi(X)^{-1} Z_s
\]
for some sufficiently small positive number $\delta$. Therefore we have

\[(3.14) \quad Z_sX^{s/4}\psi(X)^{-1} \ll X^{1-\frac{s}{16}+\frac{s}{8s}+\epsilon}Z_s^{1/2}.\]

The estimate (3.14) implies $Z_3 \ll X^{3/8+\epsilon}\psi(X)^3$ and $Z_4 \ll X^{7/8+\epsilon}\psi(X)^2$.

The proof of Theorem 1.1 is completed by summing over dyadic intervals. □

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References

[1] J. Brüdern, *Sums of squares and higher powers*, J. Lond. Math. Soc. (2) **35** (1987), 233–243.

[2] J. Brüdern, *A problem in additive number theory*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 27–33.

[3] J. Brüdern and K. Kawada, *The asymptotic formula in Waring’s problem for one square and seventeen fifth powers*, Monatshefte für Mathematik **162** (2011), 385–407.

[4] J. B. Friedlander and T. D. Wooley, *On Waring’s problem: two squares and three biquadrates*, Mathematika (in press), arXiv: 1211.1823.

[5] D. R. Heath-Brown and D. I. Tolev, *Lagrange’s four squares theorem with one prime and three almost prime variables*, J. Reine Angew. Math. **558** (2003), 159-224.

[6] C. Hooley, *On Waring’s problem for two squares and three cubes*, J. Reine Angew. Math. **328** (1981), 161-207.

[7] K. Kawada and T. D. Wooley, *On the Waring-Goldbach problem for fourth and fifth powers*, Proc. Lond. Math. Soc. (3) **83** (2001), 1–50.

[8] Yu. V. Linnik, *Additive problems involving squares, cubes and almost primes*, Acta Arith. **21** (1972), 413-422.

[9] R. C. Vaughan, *The Hardy-Littlewood method*, 2nd ed. Cambridge University Press, Cambridge 1997.

[10] T. D. Wooley, *Slim exceptional sets for sums of four squares*, Proc. Lond. Math. Soc. (3) **85** (2002), 1-21.

[11] T. D. Wooley, *Slim exceptional sets in Waring’s problem: one square and five cubes*, Quart. J. Math. **53** (2002), 111-118.

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