Decoupling in Non-Perturbative Background Fields: the Thermal Sphaleron

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Abstract

Standard decoupling of heavy fermions may fail when there are non-perturbative variations in a scalar field which gives masses to the fermions. One situation of phenomenological relevance is the case of sphalerons in the presence of fermions at finite temperatures. The free energy of a simple model is determined using a non-perturbative technique to study the effect of fermions on the scalar field. The effects of quantum and thermal fermionic fluctuations on the free energy of the thermal sphaleron are calculated, including contributions from the gradients of the scalar field to all orders.

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1 Introduction

When a particle is made heavy by increasing a dimensionful parameter, its virtual effects on physical quantities are suppressed by inverse powers of the parameter. Under these conditions the particle is said to decouple\[1\]. However, decoupling fails when a particle is made heavy by increasing a dimensionless coupling. This is the case for instance when a fermion mass is made heavy by increasing a Yukawa coupling, while the Higgs vacuum expectation value is held fixed. Therefore, chiral gauge theories in particular can be sensitive to the short distance physics associated with heavy fermions, as a consequence of this *perturbative* non-decoupling\[2\].

Recently, Banks and Dabholkar have pointed out that decoupling may also fail when there are non-perturbative variations in a scalar field which gives masses to fermions\[3\]. More specifically, these authors noted that the effects of heavy fermions cannot be summarized by a local effective Lagrangian for the light degrees of freedom if the scalar field which gives the fermion its mass goes through a zero. Heuristically, this *non-perturbative* non-decoupling can be understood as stemming from the local vanishing of the fermion mass. To see this more clearly, consider the fermion determinant in a scalar background $\phi(x)$. The effective action for the the scalar field may be expressed as

$$S_{\text{eff}}[\phi] = S[\phi] - i \ln \det [i \mathcal{D} - g \phi], \quad (1)$$

where $S[\phi]$ is the classical scalar field action, $i \mathcal{D}$ is a gauge covariant derivative and $g$ is a Yukawa coupling. To evaluate the determinant in an arbitrary background one may attempt a gradient expansion. However, any such approximation is surely doomed. The leading order term in such an expansion is given by $(\partial_\mu \phi)^2 / g^2 \phi^2$. It is clear that this approach fails in the case of a vanishing scalar field. Thus, the effects of a heavy fermion can only be summarized by a *non-local* effective lagrangian.

This non-decoupling can be recognized by an apparent paradox\[3\]. Begin by considering a chiral gauge theory with heavy mirror fermions for all light fermions. The fermion number
currents for the light fermions and for the mirror fermions are assumed to be anomalous, while their difference is exactly conserved. Furthermore, assume that at energies small compared to the heavy fermion mass, there exists a local effective lagrangian which describes the physics of the light degrees of freedom. Now consider the decay of a light fermion through the fermion number anomaly. In the constrained instanton background \(^4\), the scalar will necessarily pass through zero, and furthermore, all of the flavor eigenstates will have zero mode solutions to the Dirac operator. In the absence of mixing between heavy and light fermions, a non-vanishing decay amplitude necessarily requires heavy fermions on external legs to soak up their zero modes. But, this contradicts the initial assumption that a local effective theory incorporates all of the physics of the heavy fermions.

This paper will consider the effects of heavy fermions on background scalar fields with non-perturbative variations. Specifically, the effect of heavy fermions on sphalerons at finite temperature will be addressed\(^5\).

It is by now widely accepted that fermion number violation occurs at finite temperatures due to the production and decay of sphalerons\(^6, 7\). Sphaleron processes may then play an instrumental role in determining the baryon asymmetry in the universe\(^8\). Considering the discussion given above, it is not hard to imagine that heavy fermions may influence the production and decay of sphalerons. In the process of sphaleron production and decay, the fermion number of each flavor of fermion must be violated, as a consequence of the fermion number anomaly discussed in the case of instantons above. These fermions may receive masses exceeding the mass of the classical sphaleron through Yukawa couplings to the scalar field, since the sphaleron mass is determined at tree-level from the bosonic sector of the theory only. If the heavy fermions do not mix with lighter fermions, then the sphaleron process from a state with no heavy fermions to a state with heavy fermions is prohibited strictly on the basis of kinematics\(^9\). Setting aside for the moment the obvious technical problems

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\(^1\)In realistic models with fermionic mixing, the rate of sphaleron decay will be suppressed by a Cabbibo angle which could be quite small for heavy fermions.
accompanying a large Yukawa coupling, this simple scenario illustrates that fermions with large Yukawa couplings to scalars are expected to modify the scalar background field\textsuperscript{4}. 

One can imagine at least two ways in which the scalar field could behave under the influence of heavy fermions. It may be that the sphaleron is stabilized by the effects of heavy fermions, if its negative modes are turned positive by fermion loop effects\textsuperscript{2}. It may also be that the sphaleron mass is simply boosted by effects of heavy fermions, but remains an unstable saddlepoint. This latter effect will be demonstrated in the following pages.

The sphaleron rate in thermal equilibrium, $\Gamma$, is controlled by the Boltzmann exponential\textsuperscript{6}

$$\Gamma \propto C \times \exp \left[ -\beta E_{sph} \right]$$ (2)

for temperatures $T \equiv 1/\beta$ below the mass of the sphaleron, $E_{sph}$. The sphaleron is a solution of the time-independent classical equations of motion for scalar and gauge fields determined by minimizing the energy functional. In the one-loop approximation, the prefactor $C$ is determined by calculating the gaussian and zero-mode fluctuations around the sphaleron. This factor represents the entropy associated with the sphaleron, and it combines with the energy to give the free energy of the sphaleron. However, this free energy will not in general be a minimum of the free energy functional. The self-consistent method for determining the sphaleron rate is to first calculate the free energy functional and then minimize it. This determines a new configuration, the \textit{thermal sphaleron}. The rate is then given by the equation

$$\Gamma \propto \exp \left[ -\beta F_{sph} \right].$$ (3)

This rate will include all of the effects of fermionic fluctuations.

In this paper such a self-consistent procedure is performed in a simple model, in the limits where the fermion mass is greater than or less than the temperature\textsuperscript{11, 12}. The fermionic determinant is found for an arbitrary scalar background, using a WKB approximation, so

\textsuperscript{2}A situation like this has been considered recently in an investigation of sphaleron stability in the presence of zero-modes\textsuperscript{10}. 

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that the effects of gradients to all orders are included. Thus, the problems involved in non-locality mentioned above may be addressed.

In the high $T$ limit, it is shown that the finite temperature effects serve to make the gradient expansion well-defined, as expected. This allows a comparison of the WKB result with the gradient expansion. The fermion mass is effectively controlled by the temperature, and the gradients are suppressed by inverse powers of this dimensionful quantity. However, the fermions do not decouple in the high $T$ limit, since they contribute to temperature-dependent renormalizations of the parameters in the theory which are physically observable. In the simple model considered here, the only effect of the fermions in this limit is a temperature-dependent renormalization of the scalar mass. In the low $T$ limit, it is shown that the heavy fermions also do not decouple, however their effects cannot be summarized by a temperature-dependent renormalization of the bare parameters. Furthermore, it is shown that the gradients are still suppressed in the regime where quantum fluctuations become important.

The rest of the paper is structured as follows. Section 2 contains a review of the 1+1 Abelian Higgs model which has many of the features of the electroweak theory necessary for a study of finite temperature fermion number violation. In Section 3, the finite temperature effective action for the 1+1 Abelian Higgs model is calculated to all orders in gradients, using a WKB approximation. The details of these calculations are left for the appendix. Section 4 contains a discussion of the contribution of heavy fermions to the free energy, and calculations of the thermal sphaleron in high and low temperature regimes. Finally, the last section contains a discussion of the results.

2 The Model

The 1+1 dimensional Abelian Higgs Model, chirally coupled to fermions, contains several features necessary for a study of finite temperature fermion number violation [13, 14, 15].
The Minkowski action of the model is given by:

\[ S = N \int d^2 x \left( -\frac{1}{4} F^2_{\mu \nu} + |D_\mu \Phi|^2 - \lambda \left( |\Phi|^2 - v^2 / 2 \right) \right) \]

\[ + \sum_a N \int d^2 x \bar{\psi}^a_L \epsilon D_L \psi^a_L + \bar{\psi}^a_R iD_L \psi^a_R + g \bar{\psi}^a_L \Phi \psi^a_L + g^* \bar{\psi}^a_R \Phi^* \psi^a_R. \]

The \( U(1) \) gauge field \( A_\mu \) is chirally coupled to \( N \) flavors of fermions through the covariant derivative, \( D_{L,R} \equiv \gamma^\mu (\partial_\mu \pm ieA_\mu) \). The complex scalar \( \Phi \) is responsible for spontaneously breaking the \( U(1) \) symmetry. The dimensionful parameters in the theory are \( \lambda \) with dimensions of mass-squared, and \( e \) and \( g \) with dimensions of mass. The scalar mass is \( M_\phi = \sqrt{2\lambda v} \) and the fermion mass is \( m_f = gv \). By a rescaling of the scalar field, it can be shown that the limit of weak scalar self-coupling corresponds to \( v >> 1 \).

The work presented in the following sections is based on a large \( N \) expansion of the free energy of this model so that the effects of a large Yukawa coupling can be considered\(^4\). An overall \( N \) has been included in the bosonic part of eq.4 to make this expansion well-defined\(^4\). The large \( N \) limit is taken with the parameters \( e, g, \lambda \) and \( v \) in eq.4 held fixed. Then, the tree-level bosonic action and the one-loop fermion determinant are the leading-order \( O(N) \) contributions to the free energy. Bosonic loops are suppressed by powers of \( 1/N \) and are neglected.

The model contains a conserved gauge current, \( J_\mu = \sum_a \bar{\psi}^a_L \gamma_\mu \psi^a_L - \bar{\psi}^a_R \gamma_\mu \psi^a_R \), and global fermion number currents associated with each fermion flavor, \( J^a_\mu = \bar{\psi}^a_L \gamma_\mu \psi^a_L + \bar{\psi}^a_R \gamma_\mu \psi^a_R \), which obey the anomalous conservation equation

\[ \partial^\mu J^a_\mu = \frac{e}{4\pi} \epsilon^{\mu \nu} F_{\mu \nu} \quad \forall a. \]

As in the electroweak theory, the anomaly appears in the divergence of a vector current due to the asymmetry in couplings between left- and right-handed fermions to the gauge field.

\(^3\)CP violating phases will be ignored.

\(^4\)An action with canonically normalized kinetic terms but rescaled parameters is obtained by rescaling the bosonic fields by \( 1/\sqrt{N} \).
The vacuum structure of this theory is non-trivial and is described by the homotopy group, \( \pi_1(S^1) = \mathbb{Z} \), as it is in the electroweak theory. Each inequivalent vacuum is labelled by the Chern-Simons number:

\[
N_{CS} \equiv \frac{e}{2\pi} \int d x \epsilon_{0\nu} A^\nu
\]

which takes values in \( \mathbb{Z} \). Imposing periodic boundary conditions in a box of size \( L \), the classical vacuum with topological number \( N_{CS} \) is given by

\[
\Phi = \frac{v}{\sqrt{2}} e^{i\alpha(x)} , \quad A_\mu = \frac{1}{e} \partial_\mu \alpha(x),
\]

where \( \alpha(0) - \alpha(L) = 2\pi N_{CS} \) and \( N_{CS} \in \mathbb{Z} \). Working in the temporal gauge \( A_0 = 0 \), the Coulomb gauge condition \( \partial_1 A_1 = 0 \) fixes the remaining time-independent gauge freedom and eq.\( 6 \) determines the vacuum gauge field to be \( \{ A_0 = 0, A_1 = 2\pi N_{CS}/eL \} \).

Fermion number violation occurs for each fermion flavor when the spacetime integral of eq.\( 5 \) is non-vanishing. This is the case for gauge fields with non-vanishing winding number, defined as

\[
\nu \equiv \frac{e}{4\pi} \int d^2 x \epsilon_{\mu\nu} F^{\mu\nu}.
\]

Configurations with non-zero winding number describe transitions between adjacent vacua since

\[
N_{CS}(t = +\infty) - N_{CS}(t = -\infty) = \frac{e}{2\pi} \int dt \int d x \frac{\partial}{\partial t} \epsilon_{0\nu} A^\nu = \nu.
\]

Transitions between adjacent vacua, whether by tunnelling or by finite temperature fluctuations, are necessarily accompanied by fermion number violation, according to eq.\( 6 \) and eq.\( 8 \).

At zero temperature, the Abrikosov-Nielsen-Olesen vortices are Euclidean configurations, or instantons, which describe such tunnelling events\[16\]. At finite temperature, sphalerons, to be discussed below, are assumed to be the relevant configurations for transitions between adjacent vacua.

In the absence of fermions, it is well-known that the 1+1 ABH model does not exist in the Higgs phase. Instantons destroy the long range order and a confining phase arises
However, the existence of fermion zero modes in the instanton background implies the vanishing of instanton contributions to fermion-number-conserving Greens functions. In particular, the effective potential which determines the ground state of the theory receives no contribution from instantons, and there is no restoration of symmetry when fermions are included in the model. Therefore, this model is expected to exhibit the symmetry-breaking implied by the tree level potential.

The 1+1 ABH theory contains a non-contractible loop in the configuration space of the bosonic sector of the theory which interpolates between vacua differing by one unit of $N_{CS}$. The field configuration of maximum energy on this loop is the sphaleron, which determines the height of the free energy barrier between adjacent vacua in the broken phase of the theory. The gauge field on the non-contractible loop has the vacuum form $\{A_0 = 0, A_1 = 2\pi N_{CS}\tau/L\}$.

The scalar field on the non-contractible loop is now simply described by a scalar field in the form $\Phi'(x) \equiv e^{-2\pi i N_{CS}(\tau)x/L} \Phi(x)$, which removes the dependence on the constant gauge field. Identifying the Chern-Simons number on the loop, $N_{CS}(\tau)$, with the loop parameter itself $\tau = [0, 1]$, and solving the time-independent equations of motion then gives the scalar field:

\[
\begin{align*}
Re \Phi'(x) &= \frac{v}{\sqrt{2}} \tanh \sqrt{\frac{\Lambda v^2}{2}} x, \\
Im \Phi'(x) &= \frac{v}{\sqrt{2}} \cos \pi \tau,
\end{align*}
\]  

where $\hat{v}^2 \equiv v^2 \sin^2 \pi \tau$. By convention, the Imaginary part of $\Phi'$ has been chosen to be spatially constant. So, the Real part of $\Phi'$ interacts in a “slice” of the Higgs potential, which is a double-well potential. The solution is then a familiar “kink” configuration of the theory of a single scalar field interacting through a double-well potential, with a mass controlled by the loop parameter $\tau$.

The energy functional reduces to $E(\tau) = \frac{2\sqrt{2}}{3} \sqrt{\Lambda n} v^3 | \sin^3 \pi \tau |$ for configurations on this loop. The energy reaches a maximum when $\tau = 1/2$, and this defines the unstable sphaleron.
configuration. In terms of the original variables, it takes the form:

$$\Phi = \frac{v}{\sqrt{2}} \tanh \left( \sqrt{\frac{\lambda v^2}{2}} x \right) \exp \left( i \frac{\pi x}{L} \right), \quad A_1 = \frac{\pi}{e L}, \quad A_0 = 0. \quad (11)$$

This is an unstable static solution to the source-free equations of motion for the scalar field. The $\tau$-direction of the energy functional along the loop represents the only unstable direction in configuration space away from the sphaleron. The sphaleron energy, $E_{sph} = \frac{2\sqrt{2}}{3} \sqrt{\lambda} v^3 = \frac{2}{3} v^2 M_{\phi}$, has the familiar form of a non-perturbative result since the weak coupling regime is achieved for $v \gg 1$ \cite{1}. In the following section, the contribution of fermions to the free energy of this system will be calculated to determine thermal and quantum corrections to this classical sphaleron configuration. The constant gauge field in eq.11 does not contribute to the fermion determinant to be calculated, and so the gauge field will be neglected below \cite{4}. In this sense, the approach in this paper is complimentary to \cite{4} where the Yukawa coupling is neglected but bosonic fluctuations around the sphaleron are calculated.

The sphaleron plays a key role in determining the rate of fermion number violation in the theory. As determined above, the sphaleron energy is the height of the free energy barrier between adjacent vacuua. If the sphaleron is also assumed to be the unique saddlepoint of lowest free energy, then its energy controls the rate of finite temperature fluctuations between adjacent vacuua. As mentioned in the Introduction, this rate is accompanied by a Boltzmann suppression factor

$$\Gamma \propto \exp \left[ -\beta E_{sph} \right] \quad (12)$$

for temperatures much lower than the barrier height but much larger than the scalar mass, $M_{\phi} << T << v^2 M_{\phi}$. Similarly, the rate of fermion number violation which accompanies vacuum transitions is also determined by the Boltzmann exponential in this temperature range \cite{8}. The strong dependence of the rate on the energy of the sphaleron requires an accurate determination of this quantity and provides further motivation for the work in this paper.

\footnote{Analogously, the electroweak sphaleron has energy, $E_{sph} \propto M_w/\alpha_w$.}
3 The Free Energy in the WKB Approximation

The partition function for the system of fermions in a background scalar field, $\Phi$, can be represented as a Euclidean path integral,

$$Z[\Phi] = \exp \left[ -N \ln \text{det} \left( i \frac{\partial}{\partial E} - gv \right) \right] \int iD\psi D\psi^\dagger \exp \left[ -\int_0^\beta dt \int d^3x \mathcal{L}_E + \delta \mathcal{L} \right]. \tag{13}$$

The fermion fields $\psi$ and $\psi^\dagger$ are required to be anti-periodic with period $\beta \equiv 1/T$. $\mathcal{L}_E$ is the Wick rotation of the Lagrangian given in eq.4, and $\delta \mathcal{L}$ is the counterterm Lagrangian. The determinant factor is the finite temperature determinant in the constant background which serves to normalize the free energy. The counterterm Lagrangian $\delta \mathcal{L}$ is given by

$$\delta \mathcal{L} = AN (|\Phi|^2 - v^2/2), \tag{14}$$

and is sufficient to renormalize both the one point and the two point function. $A$ is chosen so that the tadpole graph vanishes when $\Phi = v$. This prescription gives

$$A = -\frac{g^2}{2\pi} \int_0^\infty \frac{dp}{\sqrt{p^2 + g^2v^2}}. \tag{15}$$

Using this value for $A$, the physical mass will be

$$m_{\text{phys}}^2 = N \left( -2\lambda v^2 - \frac{g^2}{\pi} \right). \tag{16}$$

This counterterm is sufficient to renormalize the theory at finite temperatures.

When the gaussian integral over the fermionic fields is performed, the partition function is expressed as the determinant of the operator

$$\ln Z = \frac{N}{4} \sum_\epsilon \sum_n \ln \left( \omega_n^2 + \epsilon^2 \right), \tag{17}$$

where $\omega_n$ are the Matsubara frequencies given by $\omega_n = 2\pi(2n+1)/\beta$ and $\epsilon$ are the eigenvalues of the spatial operator $O$ given by

$$O = \gamma_0^E \left[ i \partial - \lambda \phi P_L - \lambda^* \phi^* P_R \right]. \tag{18}$$
$P_L$ and $P_R$ are the projection operators on the left and right moving states, respectively.

The free energy of this system, defined by the usual thermodynamic relation, $F = -\frac{1}{\beta} \ln Z$, is an effective action for the scalar field, $\phi$. The remainder of the paper will focus on a calculation of this quantity using the WKB approximation.

Performing the sum over the Matsubara frequencies in eq.[17] leaves

$$F = -\frac{N}{2} \sum_{\sigma} \sum_{\epsilon} \left[ f(\epsilon) - f(\epsilon_0) \right]. \quad (19)$$

where $f(\epsilon)$ is the contribution to the free energy from states with energy $\epsilon$

$$f(\epsilon) \equiv \epsilon + \frac{2}{\beta} \ln \left(1 + e^{-\beta\epsilon}\right). \quad (20)$$

The sum over $\sigma$ corresponds to the sum over two helicity states, and $\epsilon_0$ are the eigenvalues of the operator $O$ with $\phi$ taking on the constant value $v$. Eq.[19] reduces to the shift in the Dirac sea in the zero temperature limit[12].

The calculation of the free energy has been recast as a calculation of the spectrum of the operator $O$. In terms of the up (+) and down (-) components of the wave function, the eigenvalue problem may be expressed as a Schrödinger equation,

$$\frac{d^2\psi_{\pm}}{dx^2} - \left( R^2 + I^2 \mp \frac{dR}{dx} + \frac{dI}{dx} \sigma_1 \right) \psi_{\pm} = -E_{\pm}^2 \psi_{\pm}. \quad (21)$$

A convenient basis for the two dimensional Dirac matrices has been chosen in which $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_3$, and $\gamma_5 = \sigma_2$. The scalar field has been expressed here in terms of $R \equiv R e(g \Phi)$ and $I \equiv I m(g \Phi)$.

The imaginary part of the scalar field will be assumed to play no role in the energetics of the sphaleron. This is clearly seen in the temporal gauge where the imaginary piece is utilized only to enforce the boundary conditions at $x = \pm \infty$. The gradient energy of the phase in eq.[11] will vanish in the infinite volume limit. Thus, the eigenvalue equations eq.[21] can be decoupled in the temporal gauge for the general sphaleron-like background.

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6 For notational convenience we do not include here the ideal gas contribution
7 We will assume that there are no CP violating phases.
It is convenient to follow the notation used in ref. [12] and introduce the parameters
\[ x_{cl} = \sqrt{\frac{2}{\lambda v^2}}, \quad y = \frac{g}{\sqrt{\lambda}}, \quad z = x/x_{cl}. \]  
(22)
The Compton wavelength of the scalar, \( x_{cl} \), and the inverse temperature, \( \beta \), are now the only dimensionful parameters. The size of the classical sphaleron is roughly of order \( x_{cl} \). The parameter, \( y \), is the ratio of the scalar and fermion Compton wavelengths. Now the eigenvalue equation may be written in the simple form:
\[ \left[ \frac{d^2}{dz^2} - y^2 V_\pm(z) - y^2 + x_{cl}^2 \epsilon_\pm^2 \right] \psi_\pm = 0, \]  
(23)
with a “potential” given by
\[ V_\pm(z) = \frac{\phi^2}{v^2} - 1 \mp \frac{1}{yv} \left( \frac{d\phi}{dz} \right). \]  
(24)
in terms of a canonically-normalized Real scalar field, \( \phi \equiv \sqrt{2} Re(\Phi) \).

The eigenvalue problem will now be solved using a WKB approximation [11, 12]. This approximation is expected to be valid when the scalar field varies on scales larger than the fermion Compton wavelength, or roughly for large \( y \). Since the goal is to study the effects of fermions on the sphaleron, only background scalar fields with the same topology as the sphaleron will be considered. These consist of fields which interpolate between two minima of the potential as \( x \) varies from \(-\infty\) to \(+\infty\), as in eq.[11]. For these fields, the potential \( V \) will be parity even, and the eigenstates will be classified by their parity quantum number. Furthermore, these eigenstates naturally fall into two classes: continuum states and discrete (bound) states.

### 3.1 The Continuum States

The asymptotic form of the WKB wavefunctions will be given by plane waves
\[ \psi_{\sigma}^{even}(k, z) \rightarrow \cos \left( k z \pm \frac{1}{2} \delta_{\sigma}^{even} \right), \quad \psi_{\sigma}^{odd}(k, z) \rightarrow \sin \left( k z \pm \frac{1}{2} \delta_{\sigma}^{odd} \right), \quad z \rightarrow \infty. \]  
(25)
The momenta here are given by the free dispersion relation:

\[ k = \sqrt{x_{cl}^2 \epsilon^2 - y^2}. \]  

(26)

Periodic boundary conditions in a finite box of spatial size \( L \) require that the WKB phase shifts \( \delta_{\sigma} \) be related to the momenta by: \( k_{\sigma n}^0 L + \delta_{\sigma}(k) = 2\pi n \). As the size of the box is eventually to be taken infinite, eq.\[19\] can be expanded in powers of \( \delta/L \), leaving

\[ F = -\frac{N}{2} \sum_{\sigma} \sum_{k} \left[ -\left( \frac{\partial f}{\partial \epsilon} \frac{\partial \epsilon}{\partial k} \right)_{k_0} \left( \frac{\delta_{\sigma}}{L} \right) + O\left( \frac{\delta_{\sigma}^2}{L^2} \right) \right], \]  

(27)

where the phase shifts for even and odd parity wavefunctions have been combined,

\[ \delta_{\sigma} \equiv \frac{1}{2} \left( \delta_{\sigma}^{\text{even}} + \delta_{\sigma}^{\text{odd}} \right). \]  

(28)

The phase shifts contain all of the information about the potential and are given by the WKB approximation.

\[ \delta_{\sigma}(k) = \int_{-\infty}^{\infty} dz \left[ k_{\sigma}(z) - k \right], \quad k_{\sigma}(z) = \sqrt{x_{cl}^2 \epsilon^2 - y^2 - y^2 V_{\sigma}(z)}. \]  

(29)

In the limit of large \( L \), the sum in eq.\[27\] becomes an integral and the contribution of continuum states to the free energy is:

\[ F_{\text{cont}} = -\frac{y^2 N}{4\pi x_{cl}} \sum_{\sigma} \int_{-\infty}^{\infty} dz \left[ \frac{1}{2} \left( 1 - \frac{\phi^2}{\epsilon^2} \right) - \sqrt{-V_{\sigma}} \right] \delta_{\sigma}(k), \]  

(30)

with \( \epsilon \) given in terms of \( k \) by the free particle dispersion relation, eq.\[26\].

The first term in the square brackets in eq.\[31\] is the zero temperature contribution. It is UV divergent as it stands and must be regulated, after which the momentum integral can be carried out explicitly. The counterterm discussed previously, eq.\[15\] renormalizes this term and leaves\[12\]

\[ F_{\text{cont}}^0 = -\frac{y^2 N}{4\pi x_{cl}} \sum_{\sigma} \int_{-\infty}^{\infty} dz \left[ \frac{1}{2} \left( 1 - \frac{\phi^2}{\epsilon^2} \right) - \sqrt{-V_{\sigma}} \right. \]

\[ + \left. (1 + V_{\sigma}) \ln \left( 1 + \sqrt{-V_{\sigma}} \right) \right]. \]  

(31)
The second term in the square brackets in eq.30 is the finite temperature contribution and may be written in terms of the ratio of temperature to fermion mass, \( a \equiv \beta y/x_{cl} \).

\[
F(a^2)_{cont} = -\frac{N x_{cl}}{\pi \beta^2} \sum_{\sigma} \int_{-\infty}^{\infty} dz \int_{0}^{\infty} dx x \frac{1}{\sqrt{x^2 + a^2}} \left\{ \sqrt{x^2 - a^2 V(z)} - \sqrt{x^2} \right\} \tag{32}
\]

This integral is UV finite, but cannot be evaluated analytically for arbitrary values of the temperature-to-mass ratio, \( a \). However, its limiting forms can be. The finite temperature contributions will be evaluated in the limit of large and small \( a^2 \) below. But first, the contribution of discrete states to the free energy must be included.

### 3.2 The Discrete States

The spectrum of discrete states is determined in the WKB approximation by the Bohr-Sommerfeld condition. The Schrödinger equation, eq(21), will have a discrete bound state whenever

\[
w_{\sigma}(\epsilon) \equiv \frac{1}{\pi} \int_{-\infty}^{\infty} dz k_{\sigma}(z) \Theta \left(k_{\sigma}^2(z)\right). \tag{33}\]

equals half of an odd integer. The total number of discrete states is then given by the integer closest to \( w_{\sigma}(y/x_{cl}) \). Therefore, the contribution to the free energy, eq(39), from the discrete states will be given by

\[
F_{disc} = -\frac{N}{2} \sum_{\sigma} \sum_{w_{\sigma} > 0} w_{\sigma}(y/x_{cl}) \left[ f'(\epsilon) - f(y/x_{cl}) \right]. \tag{34}\]

This sum may always be expressed in terms of an integral

\[
F_{disc} = -\frac{N}{2} \sum_{\sigma} \int_{0}^{w(y/x_{cl})} dw \left[ f'(\epsilon) - f(y/x_{cl}) \right] + Rem. \tag{35}\]

with a remainder given by the Euler-MacLaurin formula. The remainder represents the error in the free energy from an integral approximation. As the “potential” becomes deeper, the number of bounds states \( w(y/x_{cl}) \) increases, and the integral approximation becomes better.

The depth of the potential is controlled by the dimensionless parameter \( y \), the ratio of the
fermion and scalar Compton wavelengths. Thus, the integral approximation is obtained in the WKB regime, where the scalar field is slowly varying on the scales of the fermion Compton wavelength. This approximation will be discussed further below.

Finally, eq.\ref{eq:35} may be cast in the same form as the contribution from the continuum states, eq.\ref{eq:32}, by changing variables from $w$ to $\epsilon$.

$$F_{\text{disc}} = \frac{N}{2\pi} \sum_{\sigma} \int_{-\infty}^{\infty} dz \int_{y \sqrt{1 + V_{\sigma} / x_{cl}}}^{y / x_{cl}} d\epsilon \left[ 1 - \frac{2}{1 + e^{\beta \epsilon}} \right] \sqrt{x_{cl}^2 \epsilon^2 - y^2 (1 + V_{\sigma})} \quad (36)$$

The first term in the square brackets is the zero temperature contribution. The integral over $\epsilon$ can be done explicitly to give:

$$F_{\text{disc}}^0 = -\frac{y^2 N}{4\pi x_{cl}} \sum_{\sigma} \int_{-\infty}^{\infty} dz \left[ \sqrt{-V_{\sigma}} + (1 + V_{\sigma}) \ln \left( \frac{\sqrt{1 + V_{\sigma}}}{1 + \sqrt{-V_{\sigma}}} \right) \right] \quad (37)$$

The second term in the square brackets is the finite temperature contribution and may be written in terms $a$, defined previously. Changing variables to $u = \beta \epsilon$ gives:

$$F(a^2)_{\text{disc}} = \frac{N}{\pi} \sum_{\sigma} \int_{-\infty}^{\infty} dz \int_{a\sqrt{1 + V_{\sigma}}}^{a} \frac{du}{1 + e^u} \sqrt{u^2 - a^2 (1 + V_{\sigma})} \quad (38)$$

Finally, the sum of the two zero temperature pieces, eq.\ref{eq:31} and eq.\ref{eq:37}, and the two finite temperature pieces, eq.\ref{eq:32} and eq.\ref{eq:38}, yields the total free energy, eq.\ref{eq:19}.

### 3.3 The High Temperature Expansion

The high temperature limit will provided a check on the previous expressions. One expects the leading finite temperature effects to be summarized in a temperature-dependent mass term for the scalar field. The high temperature expansion is defined as $a^2 \ll 1$. The free energy is not an analytic function of the temperature in this limit, and special care must be taken to obtain the expansion for small $a$ (see Appendix). Including only the first non-trivial dependence on $a^2$, eq.\ref{eq:32} is:

$$F(a^2)_{\text{cont}} = \frac{y^2 N}{2\pi x_{cl}} \sum_{\sigma} \int_{-\infty}^{\infty} dz \left[ \frac{1}{2} \sqrt{-V_{\sigma}(z)} - \frac{1}{4} (1 + V_{\sigma}(z)) \left( 2 \gamma - \ln \left( \frac{a^2}{\pi^2} \right) \right) \right. \left. - \frac{1}{2} (1 + V_{\sigma}(z)) \ln \left( 1 + \sqrt{-V_{\sigma}(z)} \right) \right] + O(a^2) \quad (39)$$
When this is combined with the zero temperature contribution, eq.\ref{eq:31}, the square root and logarithmic terms cancel precisely. Then, the contribution of the continuum states in the high temperature limit is:

\[
F(a^2)_{\text{cont}} = \frac{y^2 N}{4\pi x_{cl}} \sum_{\sigma} \int_{-\infty}^{\infty} dz \left[ \frac{1}{2} \left( 1 - \frac{\phi^2}{v^2} \right) - \frac{1}{2} V_{\sigma}(z) \left( 2 \gamma - 1 + \ln \left( \frac{a^2}{\pi^2} \right) \right) \right].
\]  
(40)

The field gradients contained in \( V \) will cancel in the sum over \( \sigma \), leaving only a temperature-dependent mass term for the scalar field. Thus, it is clear that the finite temperature effects have made the theory local, in the sense that there is a well-defined gradient expansion.

The contribution of discrete states follows from eq.\ref{eq:36}. It is not hard to demonstrate that the Euler-MacLaurin remainder is suppressed by at least \( O(a) \). Furthermore, since the integral in eq.\ref{eq:36} is cut off at the scale \( y/x_{cl} \), the Boltzmann exponential may be expanded in the high temperature approximation, and the zero temperature contribution cancels out, leaving

\[
F_{\text{disc}} = O(a).
\]
(41)

As expected, low-lying bound states do not contribute to the free energy for high temperatures.

Combining eq.\ref{eq:40} and eq.\ref{eq:41} the total free energy in the high temperature approximation is given by

\[
F = \frac{y^2 N}{4\pi x_{cl}} \left( 2\gamma + \ln \left( \frac{a^2}{\pi^2} \right) \right) \int_{-\infty}^{\infty} dz \left( 1 - \frac{\phi^2}{v^2} \right).
\]
(42)

We may now calculate the critical temperature in the high temperature approximation by combining this result with the tree-level scalar action from eq.\ref{eq:4}. We will define critical temperature as the temperature for which the curvature at the origin vanishes. This is not necessarily the temperature at which the phase transition occurs. If the transition is first order, then bubble nucleation will induce the phase transition at a temperature above the critical temperature as defined here. Therefore, the critical temperature may overestimate the temperature at which the transition occurs, but this will be sufficient for this paper.
Requiring $\frac{\partial}{\partial \phi^2} F_{\text{total}} = 0$ at the origin gives

$$T_c = \pi e^{-\gamma/4} \frac{y}{x_{cl}} \exp \left[ \frac{2\pi v^2}{y^2} \right].$$

Interestingly enough, there is a range of temperature available in which a high temperature expansion, $T >> y/x_{cl}$, is consistent with a broken phase, $T < T_c$, and the sphaleron effects are of interest. So, for at least some range of heavy fermion mass, the leading temperature-dependent effect on the sphaleron is summarized by introducing a temperature-dependent mass for the scalar field, which in turn implies a temperature-dependent sphaleron mass. Thus, the sphaleron mass decreases with the scalar mass as the critical temperature is approached from below. However, notice that in the region of validity of the high $T$ expansion ($v >> y$), it is not expected that fermions will have a large effect on the sphaleron mass. Contrary to the high $T$ case, in the low $T$ limit the broken phase can be maintained for a wider range of heavy fermion masses. Therefore, we would expect that in this limit fermions will play a more important role in determining the sphaleron free energy.

### 3.4 The Low Temperature Expansion

The low temperature expansion is defined as $a^2 >> 1$. It can be developed from eq.32 by changing variables from $x$ to $u \equiv \beta \epsilon = \sqrt{x^2 + a^2}$. The finite temperature contribution to the free energy from the continuous states is then:

$$F(a^2)_{\text{cont}} = -\frac{N x_{cl}}{\pi \beta^2} \sum_{\sigma} \int_{-\infty}^{\infty} dz \int_{a}^{\infty} \frac{du}{1 + e^u} \left\{ \sqrt{u^2 - a^2(1 + V_\sigma)} - \sqrt{u^2 - a^2} \right\}. \quad (44)$$

In the limit that the temperature is much less than the fermion mass, the Boltzmann distribution is dominated by low energy states and the exponential in the integrand dominates.

$$F(a^2)_{\text{cont}} = -\frac{N x_{cl}}{\pi \beta^2} \sum_{\sigma} \int_{-\infty}^{\infty} dz \int_{a}^{\infty} du e^{-u} \left\{ \sqrt{u^2 - a^2(1 + V_\sigma)} - \sqrt{u^2 - a^2} \right\}. \quad (45)$$
The second term in the curly brackets is independent of \(V\) and is simply expressed in terms of the MacDonald function, \(K_1(a)\).

\[
\int_a^\infty du e^{-u} \sqrt{u^2 - a^2} = a K_1(a) = \sqrt{\frac{u\pi}{2}} \left(1 + O(1/a)\right) e^{-a}, \quad a \gg 1. \tag{46}
\]

The first term in the curly brackets cannot be evaluated in terms of simple functions. However, an integration by parts gives the leading temperature dependence immediately.

\[
\int_a^\infty du e^{-u} \sqrt{u^2 - a^2(1+V)} = a\sqrt{-V} e^{-a} + \int_a^\infty du e^{-u} \frac{u}{\sqrt{u^2 - a^2(1+V)}}
\]

\[
= a\sqrt{-V} (1 + O(1/a)) e^{-a}, \quad a \gg 1. \tag{47}
\]

So, all contributions of continuum states to the free energy are exponentially suppressed, \(O(e^{-a})\), when the fermion mass greatly exceeds the temperature.

The discrete states will give the dominant contribution to the free energy in the low temperature regime. This contribution is obtained from eq.37 and eq.38. The finite temperature part is:

\[
F(a^2)_{\text{disc}} = -2 \frac{N x_{cd}}{\beta \pi} \sum_\sigma \int_{-\infty}^\infty dz \int_a^a \frac{du}{1 + e^u} \sqrt{u^2 - a^2(1 + V_\sigma)}. \tag{48}
\]

For some point in the \(z\)-integration, the lower bound on the \(u\)-integral vanishes. Therefore, a simple expansion of the exponential as in eq.45 will not be valid at this point. This is precisely the case where the background scalar goes through a zero. However, the large factor \(a \gg 1\) allows \(-V\) to be quite close to unity and still allow the expansion. One requires \(a(1 + V) \gg 1\) which is \(-V << 1 - 1/a\). If \(-V\) is not within \(1/a << 1\) of unity, then the low temperature expansion of the exponential is valid.

If \(-V\) goes to unity at some point \(z_0\), the \(z\)-integral can be partitioned into three regions: \(z < z_0 - b\), \(z_0 - b < z < z_0 + b\), and \(z > z_0 + b\). If \(b\) is greater than \(O(1/a)\), then the contribution of the first and third regions is exponentially damped since \(-V\) is not near
unity. Then, the finite-T contribution to the free energy in the low temperature regime is approximately:

\[
F(a^2)_{\text{disc}} = -2 \frac{N_{xcl}}{\beta^2 \pi} \sum_\sigma \int_{z_0-b}^{z_0+b} dz \int_{a\sqrt{1+V}}^a \frac{du}{1+e^u} \sqrt{u^2 - a^2(1 + V_\sigma)} + O(e^{-a}). \tag{49}
\]

The total free energy is obtained by adding the \( T \)-independent contributions, eq. 31 and eq. 37. The remaining integral contains a region of size \( 2b \) in which \(-V\) approaches unity. A numerical computation of the free energy in this regime will be discussed in the next section. The qualitative behavior of eq. 49 is however clear: the finite-T contribution to the free energy decreases as the fermion mass increases in the low temperature regime.

4 The Thermal Sphaleron

As discussed in the previous sections, the thermal sphaleron is determined by minimizing the free energy while imposing the appropriate boundary conditions. In this section, this calculation will be performed in the high and low \( T \) limits.

In the high \( T \) regime, the boundary conditions are temperature dependent due to the temperature dependence of the scalar expectation value. The scalar part of the free energy can be expressed in the form

\[
F = \frac{N}{x_{cl}} \int_{-\infty}^{\infty} dz \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial z} \right)^2 + \frac{1}{2v^2} \left( \phi^2 - v^2(T) \right)^2 \right] \tag{50}
\]

with a temperature-dependent vev defined as

\[
v^2(T) = v^2 - \frac{y^2}{4\pi} \left( \gamma + \ln \frac{a}{\pi} \right) \tag{51}
\]

Fermionic fluctuations are not expected to effect the form of the classical sphaleron which is given by

\[
\phi(x) = v \tanh \left( \frac{x}{x_{cl}} \right). \tag{52}
\]

---

\(^8\)This was demonstrated in the case of the soliton at zero temperature in ref. [12]
So, only the effect on the size of the sphaleron relative to the size of the classical sphaleron \( x_{cl} \) will be considered here. The phase in eq.11 may be ignored for this purpose. It is easily seen from eq.50 that the thermal sphaleron, in the high \( T \) case, is given by a simple modification of the classical solution. One replaces the vev \( v \) in eq.52 by \( v(T) \) determined by eq.51, including its appearance in the scalar Compton wavelength \( x_{cl} \). The free energy of the ansatz in the high \( T \) approximation is then given by

\[
F_{sph} = \frac{4N v^2(T)}{3 x(T)} .
\]  

(53)

where a temperature-dependent scalar Compton wavelength has been defined for convenience

\[
x(T) = \sqrt{\frac{2}{\lambda v^2(T)}} .
\]  

(54)

This is just the classical sphaleron energy with the parameters appropriately changed to include the finite temperature effects. The free energy of the sphaleron is an increasing function of the fermion mass. Furthermore, the free energy increases only logarithmically with the temperature in two dimensions. As expected, when \( v >> y \), the thermal sphaleron reduces to the classical sphaleron because in this regime the classical contribution from the scalar sector overwhelms the fermionic fluctuations. Unfortunately, as was shown in the previous section, the high \( T \) approximation will break down in the regime where quantum fluctuations become important (i.e. \( y \gtrsim v \)).

It is interesting at this point to compare the free energy of the thermal sphaleron to the free energy of the classical sphaleron including the one-loop fermion contribution, which is given by

\[
F_{cl} = \frac{4Nv^2}{3x_{cl}} \left[ 1 + \frac{3y^2}{4\pi v^2} \left( \gamma + \ln \frac{a}{\pi} \right) \right] .
\]  

(55)

This free energy is indeed slightly higher than the thermal sphaleron free energy, eq.53, as expected. Of course, this comparison should only be made at temperatures low enough that the temperature-dependent vev of the scalar field is still near \( v \).
In general, it may or may not be simple to solve for the thermal sphaleron in the high $T$ limit. This depends upon whether or not the corrections to the gradient terms are suppressed by finite temperature effects. In the Standard Model at one loop order, the gradient corrections are suppressed because there is no renormalization of the kinetic term [21]. This is also the case in the 1+1 Abelian Higgs model. However, in this model, the suppression persists to all orders in the loop expansion because only the mass and the vacuum energy require renormalization in two dimensions. The suppression of the corrections to the gradient terms allows one to determine the free energy functional by calculating the fermionic fluctuations in a constant background. The effect of the fermions is to give the couplings in the scalar sector temperature dependence. The thermal sphaleron is given by the classical sphaleron with its parameters appropriately replaced with temperature-dependent parameters. This is the standard procedure when working in the high $T$ limit.[22]

In the low $T$ limit, one is no longer bound to the regime where $v >> y$ and the effect of fermion fluctuations may become more important. In this regime, it is not possible to get a closed form expression for the size of the sphaleron. The thermal sphaleron will be determined by using the following ansatz for its shape

$$\phi_{z_0}(x) = v \tanh\left(\frac{z}{z_0}\right), \quad z = \frac{x}{x_{cl}},$$

and minimizing the free energy with respect to the parameter, $z_0$. The use of this ansatz assumes that the temperature is small enough that the scalar expectation value is still near $v$. The free energy of this ansatz in the low $T$ limit is given by

$$F = \frac{y^2 N}{x_{cl}} \left[ \frac{2v^2}{3y^2} \left( z_0 + \frac{1}{z_0} \right) + z_0 \left( \frac{3}{2\pi} - \frac{\pi}{8} \right) 
+ \frac{2}{\pi a^2} \int_{-\infty}^{\infty} dz \int_{a[\tanh(z/z_0)]}^{a} \frac{du}{e^u + 1} \sqrt{u^2 - a^2 \tanh^2(z/z_0)} \right].$$

The first term in this equation is the classical contribution, while the second term is the zero temperature contribution from quantum corrections. In the $T = 0$ limit, the size of the
sphaleron which minimizes eq. 57 is

\[ z_0 = \left[ \frac{3y^2}{2v^2} \left( \frac{3}{2 \pi} - \frac{\pi}{8} + \frac{2v^2}{3y^2} \right) \right]^{-1/2}. \]  

(58)

Note that at zero T, for the choice \( v/y = 0.1 \), the sphaleron size changes by a factor of 1/4 relative to its classical value. Thus the effect of quantum fluctuations alone will tend to increase the sphaleron mass as the fermion is made heavier. Furthermore, the temperature-dependent effects induced by the fermions are expected to become less important as the fermion mass increases, because thermal fluctuations will be suppressed.

As in the high T case, we expect the classical contribution to dominate the contribution from fermionic fluctuations in the limit that \( v \gg y \) and the size of the sphaleron will reduce to the classical value \( x_{cl} \), or \( z_0 = 1 \). However, allowing \( y \) to become large while holding the temperature fixed, fermionic fluctuations become increasingly important, and there are non-negligible temperature-dependent corrections. The free energy was calculated numerically for several values of \( z_0 \) and \( a \). The thermal sphaleron size was determined by minimizing the free energy with respect to \( z_0 \), for a given value of \( a \). Figure 1 shows the thermal sphaleron size in units of the \( T = 0 \) sphaleron mass, as a function of the parameter \( a \) for \( v/y = 0.1 \). This figure shows that in addition to the quantum fluctuations thermal fluctuations are non-negligible (yet smaller than the quantum effects) as the temperature is increased. Thus, fermionic fluctuations will tend to increase the free energy of the sphaleron relative to its classical value. Figure 2 shows the free energy of the thermal sphaleron in units of the classical sphaleron energy for several values of the parameter \( a \), with the choice \( v/y = 0.1 \). Figure 2 indicates that quantum corrections may change the sphaleron free energy by a factor of 5. Furthermore, within the validity of the low T expansion, the temperature-dependent effects are at the level of 20 percent.
5 Discussion

The 1 + 1 Abelian Higgs model has illustrated the effects of heavy fermions on a non-perturbative scalar background. In both the low and high temperature regimes, a self-consistent calculation of the free energy of the sphaleron has shown that heavy fermions will increase the mass of the thermal sphaleron. It has been shown that the thermal sphaleron has a smaller free energy than the sum of the classical sphaleron energy and its fermionic corrections. That is to say, the classical sphaleron is not a true saddlepoint of the free energy functional.

Beyond the implications this result might have for phenomenology, the results obtained here also address the interesting theoretical questions mentioned in the Introduction. In the low temperature regime, the free energy was found to be a non-local functional of the scalar field. The free energy is correctly expressed in this regime only by the summation of gradients to all orders, a summation provided by the WKB approximation. However, the net effect of the fluctuation-induced gradients is suppressed exactly when one would expect quantum fluctuations become important ($y \gtrsim v$). An heuristic explanation is that the sphaleron size acts as a cut-off for infra-red divergences. Since the sphaleron gets smaller when the fermionic mass ($y$) increases, the infrared cut-off increases, and fluctuation-induced gradients will be further suppressed. Of course, as the size of the sphaleron decreases, the kinetic term in the classical Lagrangian will become more important. Though the effects of gradients are small, thermal effects can still change the sphaleron free energy substantially. This is a consequence of the occupation of a large number of low-lying fermionic states which become available when the Yukawa coupling is increased. The $T = 0$ quantum effects can also change the sphaleron free energy substantially as a consequence of the standard perturbative non-decoupling when the fermion mass is taken to infinity while holding the scalar vacuum expectation value constant.

In the high temperature regime on the other hand, it has been shown here that the
gradient expansion is well-behaved. So, the net effect of thermal fluctuations at high $T$ effects is the localization of the theory. As expected, the fermions do not decouple in this regime, even though their masses are determined by the dimensionful parameter, $T$, since they contribute to temperature-dependent renormalizations of the parameters in the theory which are physically observable.

Thus, one may conclude that, in the simple model studied in this paper, the fermions do not decouple in the perturbative sense and can enhance the sphaleron mass. Since the gradients are suppressed, one may also conclude that the effects of the non-perturbative variation of the field discussed in the Introduction are small. One would expect similar behaviour also in four dimensions for the physical reasons discussed above. The fact that there is only perturbative non-decoupling also encourages one to expect that non-perturbative methods may not be necessary in order to calculate the effects of heavy fermions on non-perturbative scalar backgrounds in four dimensions.

6 Appendix

The integral, eq.32, is not an analytic function of $a$ in a neighborhood of $a = 0$. So, a naive Taylor expansion in the region, $a \approx 0$, is bound to fail. This is a familiar problem which occurs in the calculation of the free energy of a gas of free particles. The work in this Appendix will extract the leading non-analytic behavior near $a \approx 0$.

Consider the integral:

$$I(a^2; V) = \int_0^\infty dx^2 \sqrt{\frac{x^2 - a^2V}{x^2 + a^2}} \frac{1}{e^{\sqrt{x^2 + a^2}} + 1}$$

(59)

where $x \equiv k\beta/x_{cl}$ and $a^2 \equiv \beta^2 m^2 = \beta^2 y^2 v^2/x_{cl}^2$. The integral is finite at $a = 0$, reproducing the free energy of a gas of free fermions at high temperatures:

$$I(0; V) = 2 \frac{\pi^2}{12} T^2.$$  

(60)

This high temperature result is independent of the background field, as expected.
The first derivative of eq.59 is:

$$
\frac{\partial I}{\partial a^2} (a^2; V) = - \frac{\sqrt{-V}}{e^a + 1} - \frac{1}{2} (1 + V) \int_0^\infty \frac{d x^2}{\sqrt{(x^2 - a^2 V)(x^2 + a^2)}} \frac{1}{e^{x^2 + a^2} + 1} \tag{61}
$$

after an integration by parts. The surface term is $O(a^0)$ but the integral is logarithmically divergent as $a^2 \to 0$, so one can expect:

$$
\frac{\partial I}{\partial a^2} (a^2; V) \sim - \frac{1}{2} \sqrt{-V} + \text{const. log } a \tag{62}
$$

near $a^2 = 0$.

To determine this non-analytic term, the integral is broken into a series by use of the identity:

$$
\frac{1}{e^y + 1} = \frac{1}{2} - \sum_{-\infty}^{+\infty} \frac{y}{y^2 + \pi^2 (2n + 1)^2} \tag{63}
$$

which follows from a contour integration of the left hand side[22]. Now, the integrals over each term in this expansion are divergent, so they will be regulated by introducing:

$$
J_\epsilon (a^2; V) \equiv \int_0^\infty \frac{d x^2}{\sqrt{(x^2 - a^2 V)(x^2 + a^2)}} \frac{x^{-\epsilon}}{e^{x^2 + a^2} + 1} \tag{64}
$$

and

$$
J_\epsilon (a^2; V) \equiv J^{(1)}_\epsilon (a^2; V) + J^{(2)}_\epsilon (a^2; V). \tag{65}
$$

where

$$
J^{(1)}_\epsilon (a^2; V) \equiv - \sum_{-\infty}^{+\infty} \int_0^\infty d x^2 \frac{x^{-\epsilon}}{\sqrt{x^2 - a^2 V}} \frac{1}{x^2 + a^2 + \pi^2 (2n + 1)^2} \tag{66}
$$

and

$$
J^{(2)}_\epsilon (a^2; V) \equiv \frac{1}{2} \int_0^\infty d x^2 \frac{x^{-\epsilon}}{\sqrt{(x^2 - a^2 V)(x^2 + a^2)}}. \tag{67}
$$

A little analysis shows that $J^{(1)}$ and $J^{(2)}$ converge if $0 < \epsilon < 2$. The integrals will be estimated for small $a^2$ in this region, and then analytically continued to $\epsilon = 0$, which is a regular point of the original integral, eq.64. The result is the unique high-T expansion of $I$. 25
First, eq.66 can be rewritten as:

\[ J^{(1)}_{\epsilon}(a^2; V) = -\sum_{-\infty}^{+\infty} \frac{1}{[\pi^2(2n+1)^2]^{1/4}} \frac{1}{[1 + a^2/\pi^2(2n+1)^2]^{1/4}} \times \int_{0}^{\infty} du^2 \frac{u^{-\epsilon}}{(u^2 + 1)\sqrt{u^2 + c^2(n)}} \]  \hspace{1cm} (68)

where \( c^2(n) \) is an \( n \)-dependent constant of order \( a^2 \). To leading order in \( a^2 \) then, eq.68 is:

\[ J^{(1)}_{\epsilon}(a^2; V) = -2 \left(1 - 1/2^{1+\epsilon}\right) \zeta [1 + \epsilon] \frac{\pi^{-\epsilon}}{\cos(\pi \epsilon)} + O(a^2) . \]  \hspace{1cm} (69)

The remaining integral is elementary, and the sum can be expressed in terms of the Riemann-Zeta function.

\[ J^{(1)}_{\epsilon}(a^2; V) = -2 \left(1 - 1/2^{1+\epsilon}\right) \zeta [1 + \epsilon] \frac{\pi^{-\epsilon}}{\cos(\pi \epsilon)} + O(a^2) . \]  \hspace{1cm} (70)

Finally, this result is continued to \( \epsilon = 0 \) by means of the formula

\[ \zeta [1 + \epsilon] = -\frac{2^\epsilon \pi^{1+\epsilon} \zeta [-\epsilon]}{\sin(\pi \epsilon) \Gamma [1 + \epsilon]} \]  \hspace{1cm} (71)

which gives

\[ J^{(1)}_{\epsilon}(a^2; V) = -1/\epsilon - \gamma + \log(\pi/2) + O(\epsilon) + O(a^2) , \]  \hspace{1cm} (72)

where \( \gamma \approx 0.53 \) is the Euler Number.

Next, eq.67 can be rewritten as:

\[ J^{(2)}_{\epsilon}(a^2; V) = a^{-\epsilon} J^{(2)}_{\epsilon}(1; V) = \left(1 - \epsilon \log a + O(\epsilon^2)\right) J^{(2)}_{\epsilon}(1; V) \]  \hspace{1cm} (73)

This is the source of the non-analyticity near \( a = 0 \).

Now, \( J^{(2)}_{\epsilon}(1; V) \) is easily analyzed by expressing it in terms of two special functions\[20\].

\[ J^{(2)}_{\epsilon}(1; V) = \frac{1}{2\sqrt{-V}} B(1 - \epsilon/2, \epsilon/2) 2F_1 [1/2, 1 - \epsilon/2; 1; 1 + 1/V] \]  \hspace{1cm} (74)

26
The pole at $\epsilon = 0$ is contained in the Beta function, $B(1 - \epsilon/2, \epsilon/2)$. The hypergeometric function, $\, _2F_1$, is analytic in its second argument near 1. Expanding in a Taylor series there gives

$$J^{(2)}_\epsilon (1; V) = \frac{1}{2} B(1 - \epsilon/2, \epsilon/2) \left[ 1 - \frac{\epsilon}{2\sqrt{-V}} \frac{\partial}{\partial \epsilon} \, _2F_1 \left[ 1/2, 1; 1 + 1/V \right] + O(\epsilon^2) \right]$$

(75)

where the prime denotes differentiation with respect to the second argument of $\, _2F_1$. This function can be determined by using the various recursion relations which relate hypergeometric functions of different arguments. The result is:

$$\, _2F_1' \left[ 1/2, 1; 1; 1 + 1/V \right] = 2 \sqrt{-V} \log \left[ \frac{1}{2} (1 + \sqrt{-V}) \right].$$

(76)

Now, expanding the Beta function for small $\epsilon$ in eq.74 yields:

$$J^{(2)}_\epsilon (1; V) = 1/\epsilon - \log \left[ \frac{1}{2} (1 + \sqrt{-V}) \right] + O(\epsilon).$$

(77)

Finally, eq.73 becomes

$$J^{(2)}_\epsilon (a^2; V) = 1/\epsilon - \log(a) - \log \left[ \frac{1}{2} (1 + \sqrt{-V}) \right] + O(\epsilon).$$

(78)

Combining eq.72, eq.78 and eq.85, the poles in $\epsilon$ cancel and give

$$J_\epsilon (a^2; V) = - \log(2a/\pi) - \gamma_E - \log \left[ \frac{1}{2} (1 + \sqrt{-V}) \right] + O(a) + O(\epsilon).$$

(79)

$$\frac{\partial I}{\partial a^2} (a^2; V) = - \frac{1}{2} \sqrt{-V} + \frac{1}{2} (1 + V) \left\{ \log(2a/\pi) + \gamma_E \right. \right.$$

$$+ \log \left[ \frac{1}{2} (1 + \sqrt{-V}) \right] \right. \left. \} + O(a) + O(\epsilon).$$

(80)

Now, $I(a^2)$ can be recovered by expanding the surface term in eq.61 to lowest order in $a$, combining with eq.80, and integrating with respect to $a^2$.

$$I (a^2; V) = I (0; V) - \frac{1}{2} a^2 \sqrt{-V} + \frac{1}{4} (1 + V) a^2 \left\{ \log(a^2/\pi^2) + 2\gamma_E - 1 \right\}$$

$$\left. + \frac{1}{2} (1 + V) a^2 \log \left[ 1 + \sqrt{-V} \right] + O(a^4) \right.$$

(81)

where the integration constant is determined by eq.60. This result establishes the free energy in eq.39.
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Figure Captions

Figure 1: The ratio of the size of the thermal sphaleron to the zero temperature sphaleron as a function of the parameter $a = \beta y / x_{cl}$.

Figure 2: The free enrgy of the thermal sphaleron as a function of the parameter $a = \beta y / x_{cl}$. 

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