A geometrical introduction to screw theory

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Abstract
This work introduces screw theory, a venerable but little known theory aimed at describing rigid body dynamics. This formulation of mechanics unifies in the concept of screw the translational and rotational degrees of freedom of the body. It captures a remarkable mathematical analogy between mechanical momenta and linear velocities, and between forces and angular velocities. For instance, it clarifies that angular velocities should be treated as applied vectors and that, under the composition of motions, they sum with the same rules of applied forces. This work provides a short and rigorous introduction to screw theory intended for an undergraduate and general readership.

1. Introduction

The second law of Newtonian mechanics states that if $\mathbf{F}$ is the force acting on a point particle of mass $m$ and $\mathbf{a}$ is its acceleration, then $ma = \mathbf{F}$. In a sense, the physical meaning of this expression lies in its tacit assumptions, namely that forces are vectors, that is, elements of a vector space, and as such they sum. This experimental fact embodied in the second law is what prevents us from considering the previous identity as a mere definition of force.

Coming to the study of the rigid body, one can deduce the first cardinal equation of mechanics $\mathbf{M} \ddot{\mathbf{C}} = \mathbf{F}$, where $\mathbf{C}$ is the affine point of the centre of mass, $M$ is the total mass and $\mathbf{F}^{\text{ext}} = \sum_i \mathbf{F}_{i}^{\text{ext}}$ is the resultant of the external applied forces. This equation does not fix the dynamical evolution of the body; indeed one needs to add the second cardinal equation of mechanics $\mathbf{L}(O) = \mathbf{M}(O)$, where $\mathbf{L}(O)$ and $\mathbf{M}(O)$ are respectively the total angular momentum and the total mechanical momentum with respect to an arbitrary fixed point $O$. Naively adding the applied forces might result in an incorrect calculation of $\mathbf{M}(O)$. As is well known, one must take into account the line of action of each force $\mathbf{F}_{i}^{\text{ext}}$ in order to determine the central axis, namely the locus of allowed application points of the resultant.

These considerations show that applied forces do not really form a vector space. This unfortunate circumstance can be amended considering, in place of the force, the field of mechanical momenta that it determines (the so-called dynamical screw). These types of fields
are constrained by the law which establishes the change of the mechanical momenta under change of pole

\[ M(P) - M(Q) = F \times (P - Q). \]

An analogy between momenta and velocities, and between force resultant and angular velocity, is apparent considering the so-called fundamental formula of the rigid body, namely a constraint which characterizes the velocity vector field of the rigid body

\[ v(P) - v(Q) = \omega \times (P - Q). \]

The correspondence can be pushed forward for instance by noting that the concept of instantaneous axis of rotation is analogous to that of central axis. Screw theory explores these analogies in a systematic way and relates them to the Lie group of rigid motions on the Euclidean space.

Perhaps, one of the most interesting consequences of screw theory is that it allows us to fully understand that angular velocities should be treated as vectors applied to the instantaneous axis of rotation, rather than as free vectors. This fact is not at all obvious. Let us recall that the angular velocity is defined through the Poisson theorem, which states that, given a frame \( K' \) moving with respect to an absolute frame \( K \), any normalized vector \( e' \) which is fixed with respect to \( K' \) satisfies

\[ \frac{de'}{dt} = \omega \times e', \]

in the original frame \( K \), where \( \omega \) is unique. The uniqueness allows us to unambiguously define \( \omega \) as the angular velocity of \( K' \) with respect to \( K \). As the vectors \( e' \) are free, their application point is not fixed and so, according to this traditional definition, \( \omega \) is not given an application point.

This fact seems close to intuition. Indeed, let us consider Foucault’s 1851 famous experiment performed at the Paris Observatory. By using a pendulum he was able to prove that the earth rotates with an angular velocity which coincides with that inferred from the observation of distant stars. Of course, the choice of Paris was not essential, and the measurement would have returned the same value for the angular velocity were it performed at any other place on the earth. In fact, the reason for assigning to the angular velocity an application point in the instantaneous axis of rotation becomes clear only in very special applications, and in particular when the composition of rigid motions is considered. This fact will be fully justified in section 3. Here we just wish to illustrate how, using the analogy between forces and angular velocities, it is possible to solve non-trivial problems on the composition of motions.

Consider, for instance, four frames \( K_i, i = 0, 1, 2, 3 \), where \( K_0 \) is the absolute frame and \( K_{i+1}, i = 0, 1, 2 \), moves with respect to \( K_i \) with an angular velocity \( \omega_{i,i+1} \). Let us suppose that at the given instant of interest, and for \( i = 0, 1, 2 \), the instantaneous axes of rotation of \( K_{i+1} \) as seen from \( K_i \), all lie in the same plane as shown in figure 1. We can apply the well-known rules of statics, for instance using the funicular polygon [15, 5], to obtain the angular velocity \( \omega_{0,3} \) and the instantaneous axis of rotation of \( K_3 \) with respect to \( K_0 \).

It is also interesting to observe that if a frame \( K_2 \) rotates with angular velocity \( \omega \) with respect to \( K_1 \), and \( K_1 \) rotates with angular velocity \( -\omega \) with respect to \( K_0 \), and if the two instantaneous axes of rotation are parallel and separated by an arm of length \( d \), then, at the given instant, \( K_2 \) translates with velocity \( \omega d \) in a direction perpendicular to the plane determined by the two axes. As a consequence, any act of translation can be reduced to a composition of acts of pure rotation.

This result is analogous to the usual observation that two opposite forces \( F \) and \( -F \) with arm \( d \) generate constant mechanical momenta of magnitude \( dF \) and direction perpendicular
to the plane determined by the two forces. As a consequence, any applied momentum can be seen as the effect of a couple of forces.

Of course, screw theory has other interesting consequences and advantages. We invite the reader to discover and explore them in the following sections.

The key ideas leading to screw theory included in this paper have been taught at a second year undergraduate course of ‘Rational Mechanics’ at the Faculty of Engineering of Florence University (saved for the last technical section). We shaped this text so as to be used by our students for self-study and by any other scholars who might want to introduce screw theory in an undergraduate course. Indeed, we believe that it is time to introduce this beautiful approach to mechanics already at the level of undergraduate University programmes.

1.1. Comments on previous treatments

Screw theory is venerable (for an account of the early history see [3]). It originated from the works of Euler, Mozzi and Chasles, who discovered that any rigid motion can be obtained as a rotation followed by a translation in the direction of the rotation’s axis (this is the celebrated Chasles’s theorem, which was actually first obtained by Giulio Mozzi [2]), and by Poinsot, Chasles and Möbius, who noted the analogy between forces and angular velocities for the first time [3].

It was developed and reviewed by Sir R Ball in his 1870 treatise [1], and further developed, especially in connection with its algebraic formulation, by Clifford, Kotelnikov, Zeylinger, Study and others. Unfortunately, by the end of the nineteenth century it was essentially forgotten to be then fully rediscovered only in the second half of the twentieth century. It remains largely unknown and keeps being rediscovered by various authors interested in rigid body mechanics (including this author).

Unfortunately, screw theory is usually explained following descriptive definitions rather than short axiomatic lines of reasoning. As a result, the available introductions are somewhat unsatisfactory to the modern mathematical and physical minded reader. Perhaps for this reason, some authors among the few who are aware of the existence of screw theory claim that it is too complicated to deserve to be taught. For instance, the last edition of Goldstein’s textbook

![Figure 1. Graphical determination of $K_3$ motion with respect to $K_0$ by using the funicular polygon method. This method was originally developed for finding the central axis in problems of statics.](image-url)
[7] includes a footnote which, after introducing the full version of Chasles’ theorem (section 5), which might be regarded as the starting point of screw theory, comments

[... ] there seems to be little present use for this version of Chasles’ theorem, nor for the elaborate mathematics of screw motions elaborated at the end of the nineteenth century.

Were it written in the fifties of the last century, this claim could have been shared, but further on screw theory has become a main tool for robotics [9] where it is ordinarily used. Furthermore, while elaborate, the mathematics of screw theory simplifies the development of mechanics. Admittedly, however, some people could be dissatisfied with available treatments and so its main advantages can be underestimated. We offer here a shorter introduction which, hopefully, could convince these readers of taking a route into screw theory.

Let us comment on some definitions of screw that can be found in the literature, so as to justify our choices.

The first approach, that this author does not find appealing, introduces the screw by means of the concept of motor. This formalism depends on the point of reduction, and one finds the added difficulty of proving the independence of the various deductions from the chosen reduction point. It hides the geometrical content of the screw and makes proofs lengthier. Nevertheless, it must be said that the motor approach could be convenient for reducing screw calculations to a matter of algebra (the so-called screw calculus [3]).

In a similar vein, some references, including Selig’s [14], introduce the screw from a matrix formulation that tacitly assumes that a choice of reference frame has been made (thus losing the invariance at sight of the definition).

Still concerning the screw definition, some literature follows the practical and traditional approach which introduces the screw from its properties (screw axis, pitch, etc) [1, 8, 6], such as in old fashioned linear algebra where one would have defined a vector from its direction, chapter and verse, instead of defining it as an element of a vector space. (To complicate matters, some authors define the screw up to a positive constant; in other words, they work with a projective space rather than a vector space.) This approach could be more intuitive but might also give a false confidence of understanding, and it is less suited for a formal development of the theory. It is clear that the vector space approach in linear algebra, while less intuitive at first, proves to be much more powerful than any descriptive approach. Of course, one has to complement it with the descriptive point of view in order to help the intuition. In my opinion, the same type of strategy should be followed in screw theory, with a maybe more formal introduction, giving a solid basis, aided by examples to help the intuition. Since descriptive intuitive approaches are not lacking in the literature, this work aims at giving a short introduction of more abstract and geometrical type.

It should be said that at places there is an excess of formality in the available presentations of screw theory. I refer to the tendency of giving separate definitions of screws, one for the kinematical twist describing the velocity field of the rigid body, and the other for the wrench describing the forces acting on the body. This type of approach, requiring definitions for screws and their dual elements (sometimes called co-screws), lengthens the presentation and forces the introduction and use of the dual space of a vector space, a choice which is not so popular especially for undergraduate teaching.

One who adopts this point of view argues that it should also be adopted for forces in mechanics, which should be treated as 1-forms instead as vectors. This suggestion, inspired by the concept of conjugate momenta of Lagrangian and Hamiltonian theory, sounds more modern, but would be geometrically well founded only if one could develop mechanics without any mention of the scalar product. The scalar product allows us to identify a vector space with
its dual and hence to work only with the former. If what really matters is the pairing between a vector space and its dual, then we could dispense with the scalar product as it is unnecessary for that pairing. It is easy to realize that in order to develop mechanics we need a vector space (and/or its dual) as well as a scalar product and an orientation (although most physical combinations of interest might be rewritten so as to get rid of it, e.g. the kinetic energy is \( T = \frac{1}{2} p |v| \)).

Analogously, in screw theory, it could seem more appealing to look at kinematical twists as screws and to dynamical wrenches as co-screws, but geometrically this choice does not seem compelling, and in fact it is questionable, given the price to be paid in terms of length and loss of unity of the presentation. Therefore, we are going to use just one mathematical entity—the screw—emphasizing the role of the screw scalar product in identifying screws and dual elements.

In this work, I took care in introducing screw theory in as coordinate independent a way as possible, but avoiding the traditional descriptive route. In this approach the relation to the Lie algebra of rigid maps becomes particularly transparent. Finally, most approaches postpone the definition of screw until after the examples of systems of applied forces from which the idea of screw can been derived. I think that it is better to introduce the screw first and then to look at the applications.

In this way, through some key choices, I have obtained a hopefully clear and straightforward introduction to screw theory, which is at the same time mathematically rigorous. My hope is that after reading these notes, the reader will share the author’s opinion that screw theory is indeed ‘the right’ way of teaching rigid body mechanics as the tight relation with the Lie group of rigid maps suggests.

2. Abstract screw theory

In this section, we define the screw and prove some fundamental properties. Specific applications will appear in the following sections.

Let us denote with \( E \) the affine Euclidean space modelled on the three-dimensional vector space \( V \). The space \( V \) is endowed with a positive definite scalar product \( : V \times V \rightarrow \mathbb{R} \), and is given an orientation. This structure is represented with a triple \( (V, \cdot, o) \), where \( o \) denotes the orientation. Note that thanks to this structure a vector product \( \times : V \times V \rightarrow V \) can be naturally defined on \( V \). Points in \( E \) are denoted as capital letters, e.g. \( P, Q, \ldots \), while points in \( V \) are denoted as \( a, b, \ldots \). We shall repeatedly use the fact that the mixed product \( a \cdot (b \times c) \) changes sign under odd permutations of its terms and remains the same under even permutations. A vector field is a map \( f : E \rightarrow V \).

An applied vector is an element of \( E \times V \), namely a pair \((Q, v)\) where \( Q \) is the application point of \((Q, v)\). A sliding vector is an equivalence class of applied vectors, where two applied vectors \((Q, v)\) and \((Q', v')\) are equivalent if \( v = v' \) and for some \( \lambda \in \mathbb{R}, Q' = \lambda v \), namely they have the same line of action. We shall preferably use the concept of applied vector even in those cases in which it could be equivalently replaced by that of sliding vector. The reason is that the concept of sliding vector is superfluous because it is more convenient to regard applied and sliding vectors as special types of screws.

Occasionally, we shall use the concept of reference frame which is defined by a choice of origin \( O \in E \), and of positive oriented orthonormal basis \( \{e_1, e_2, e_3\} \) for \((V, \cdot, o)\). Once a reference frame has been fixed, any point \( P \in E \) is univocally determined by its coordinates \( x_i, i = 1, 2, 3 \), defined through the equation \( P = O + \sum_i x_i e_i \).

Remark 2.1. In order to lighten the formalism, we shall consider different physical vector quantities, such as position, velocity, linear momenta, force and mechanical momenta, as
elements of the same vector space $V$. A more rigorous treatment would introduce a different vector space for each of these concepts. The reader might imagine to have fixed the dimension units. It is understood that, say, linear momenta cannot be summed to a force even though in our treatment they appear to belong to the same vector space.

**Definition 2.2.** A screw is a vector field $s : E \rightarrow V$ which admits some $s \in V$ in such a way that for any two points $P, Q \in E$

$$s(P) - s(Q) = s \times (P - Q).$$

(1)

For any screw $s$, the vector $s$ is unique; indeed if $s$ and $s'$ satisfy the above equation, then subtracting the corresponding equations $(s' - s) \times (P - Q) = 0$ and from the arbitrariness of $P$, $s' = s$. The vector $s$ is called the resultant of the screw. If the resultant of the screw vanishes, then $s(P)$ does not depend on $P$ and the screw is said to be constant. Equation (1) is the constitutive equation of the screw.

**Definition 2.3.** If $s$ is a screw, then the quantity $s(P) \cdot s$ does not depend on the point and is called the scalar invariant of the screw. The vector invariant of the screw is the quantity (independent of $P$) and defined by

$$f = s(P), \quad \text{if } s = 0,$$

$$f = \frac{s(P) \cdot s}{s \cdot s} s, \quad \text{if } s \neq 0.$$

Thus, if $s \neq 0$ the vector invariant of the screw is the projection of $s(P)$ on the direction given by the resultant, and it is actually independent of $P$.

**Proposition 2.4.** The screws form a vector space $S$ and the map which sends $s$ to $s$ is linear.

**Proof.** If $s_1$ and $s_2$ are two screws, then

$$s_1(P) - s_1(Q) = s_1 \times (P - Q),$$

$$s_2(P) - s_2(Q) = s_2 \times (P - Q).$$

Multiplying by $\alpha$ the first equation and adding the second multiplied by $\beta$ we obtain

$$(\alpha s_1 + \beta s_2)(P) - (\alpha s_1 + \beta s_2)(Q) = (\alpha s_1 + \beta s_2) \times (P - Q),$$

(2)

which implies that the screws form a vector space and that the resultant of the screw $\alpha s_1 + \beta s_2$ is $\alpha s_1 + \beta s_2$, that is, the map $s \rightarrow s$ is linear.

Given two screws $s_1$ and $s_2$, let us consider the quantity

$$\langle s_1, s_2 \rangle(P) := s_1 \cdot s_2(P) + s_2 \cdot s_1(P).$$

**Proposition 2.5.** For any two points $P, Q \in E$, $\langle s_1, s_2 \rangle(P) = \langle s_1, s_2 \rangle(Q)$.

**Proof.** By definition, $s_1(P) - s_1(Q) = s_1 \times (P - Q)$ and $s_2(P) - s_2(Q) = s_2 \times (P - Q)$; thus,

$$s_1 \cdot s_2(P) + s_2 \cdot s_1(P) = s_1 \cdot (s_2(Q) + s_2 \times (P - Q)) + s_2 \cdot (s_1(Q) + s_1 \times (P - Q))$$

$$= s_1 \cdot s_2(Q) + s_2 \cdot s_1(Q) + [s_1 \cdot (s_2 \times (P - Q))] + s_2 \cdot [s_1 \times (P - Q)]$$

$$= s_1 \cdot s_2(Q) + s_2 \cdot s_1(Q).$$

According to the previous result, we can simply write $\langle s_1, s_2 \rangle$ in place of $\langle s_1, s_2 \rangle(P)$.
Proposition 2.8. The map \( \alpha : V \to S \) given by \( v \mapsto s(P) := v \) sends a (free) vector to a constant screw. The map \( \beta : E \times V \to S \) given by \( (Q, w) \mapsto s(P) := w \times (P - Q) \) sends an applied vector to a screw. The map \( \gamma : E \times V \times V \to S \) given by \( ((Q, w), v) \mapsto s(P) := v + w \times (P - Q) \) sends a pair given by an applied vector and a free vector to a screw.

The screws in the image of \( \alpha \) will be called constant or free screws. The screws in the image of \( \beta \) will be called applied screws. Clearly, by the constitutive equation of the screw, the map \( \gamma \) is surjective. In particular, each screw is the sum of a free screw and an applied screw.

Proposition 2.7. Let \( \gamma_O = \gamma(O, \cdot, \cdot) : V \oplus V \to S \); then this linear map is bijective.

Its inverse \( \gamma_O^{-1} : S \to V \oplus V \) is called motor reduction at \( O \). Once we agree on the reduction point \( O \), any pair \((s, s(O))\) as in the previous proposition is called a motor at \( O \). Sometimes we shall write \( s^O \) for \( s(O) \); thus, the motor at \( O \) reads \((s, s^O)\). Often, for reasons that will soon be clear, we will prefer to represent the ordered pair in a column form of two elements of \( V \). We can write the found bijective correspondence between \( S \) and \( V^2 \) as follows:

\[
s \in S \xrightarrow{\text{origin } O} \begin{pmatrix} s \\ s^O \end{pmatrix} \in V \oplus V.
\]

In this representation, the screw scalar product is given by \( \langle s_1, s_2 \rangle = s_1 \cdot s_2^O + s_2 \cdot s_1^O \); thus, this is mediated by the matrix \(
\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}
\), where \( I : V \to V \) is the identity map.

Let us now recall that any point \( O \in E \) can be used as origin, namely it allows us to establish a bijective correspondence between \( E \) and \( V \) given by \( P \mapsto P - O \). If we additionally introduce a positive oriented orthonormal base, then we further have the linear isomorphism \( V \xrightarrow{\text{base}} \mathbb{R}^3 \); thus, as a result, given a full reference frame the screw gets represented by an element of \( \mathbb{R}^6 \) in which the first three components are those of \( s \), while the last three components are those of \( s^O \).

Definition 2.9. Given a screw \( s \in S \), the screw axis of \( s \) is the set of points for which the screw field has minimum modulus.

Proposition 2.10. The screw axis coincides with the set \( E \) if \( s \neq 0 \) and with a line of direction \( s \) if \( s = 0 \). In both cases, if \( Q \) is any point on the screw axis, then

\[
s(P) = f + s \times (P - Q).
\]

As a consequence, the screw axis is the set of points for which the screw field coincides with \( f \). For any point \( Q \) on the axis, the motor reduction at \( Q \) is \( s \oplus f \).

Let us observe that the former term on the right-hand side of equation (3) is proportional to \( s \) and independent of the point, while the latter term is orthogonal to \( s \) and dependent on the point.
Figure 2. Reduction of a screw to the simplest system of applied screws (case $s \neq 0$).

Proof. Let us suppose $s \neq 0$, the other case being trivial. Let $A$ be any point; then it is easy to check that the axis which passes through $Q$ in direction $s$ where

$$Q = A + \frac{s \times s(A)}{s \cdot s}$$

is made of points $R$ for which $s(R) = f$. Using the constitutive equation of screws we find that equation (3) holds. If $P$ is another point for which $s(P) = f$, then that same equation gives $s \times (P - Q) = 0$, which implies that $P$ stays in the axis. Thus, the found axis is the locus of points $P$ for which $s(P) = f$. Equation (3) and the fact that $f \propto s$ imply that this axis is made of points for which the screw field is minimal. The other claims follow easily. □

Remark 2.11. Usually the vector invariant and the screw axis are defined only for $s \neq 0$. However, we observe that it is convenient to extend the definition as done here in such a way that equation (3) holds for any screw. The case $s = 0$ is admittedly special and can be called degenerate.

Remark 2.12. The composition of applied vectors is nothing but the addition of the corresponding screws in the vector space $S$. The resultant screw can then be represented with its motor in the screw axis which is given by the resultant $s$ aligned with the axis and the invariant vector $f$ having the same direction (figure 2). In this sense, the composition of applied vectors does not give an applied vector. The operation of composition is closed only if the full space of screws is considered.

Remark 2.13. Two systems of applied vectors are said to be equivalent if they determine the same screw. One often looks for the simplest way of representing a screw by applied vectors. This is accomplished as follows. The screw is the sum of the free screw given by $f$ and the applied screw determined by the resultant $s$ applied on the screw axis. The free screw is generated by two opposite applied vectors $s_2, -s_2$, placed in a plane perpendicular to $f$ and such that their magnitude times their arm gives $f$. This reduces any screw to the sum of at most three applied screws (two if $s = 0$). If $s \neq 0$, then the number can be reduced to 2 regarding $s$ as the sum $\frac{1}{2}s + \frac{1}{2}s$, and absorbing one term of type $\frac{1}{2}s$ through a redefinition of $s_2$, and analogously for the other (see figure 2). The arm can be chosen in such a way that the resultants of the two applied screws are perpendicular. In summary, any screw is generated
by two applied screws whose resultants are either opposite with screw axes belonging to the same plane (if \( s = 0 \)), or equal in magnitude and perpendicular (if \( s \neq 0 \)).

**Definition 2.14.** The pitch \( p \in \mathbb{R} \) of a screw \( s \), with \( s \neq 0 \), is that constant such that
\[
\int s = p \frac{2\pi}{2\pi - s}.
\]
If \( s = 0 \) and \( \int \neq 0 \), then we set by definition \( p = +\infty \).

Clearly, for a non-trivial screw, the pitch vanishes if and only if the screw is an applied screw, and the pitch equals \(+\infty\) if and only if the screw is a free screw. The screws with a given pitch do not form a vector subspace.

**Remark 2.15.** Using the pitch the screw can be rewritten as
\[
s(P) = a \left[ \frac{p}{2\pi} e + e \times (P - Q) \right],
\]
where \( s = ae \), with \( e \) the normalized vector and \( a \geq 0 \). The quantity \( a \) is called the amplitude of the screw. It must be said that for Sir R S Ball [1] the screw is \( s/a \). However, it is not particularly convenient to regard \( s/a \) as a fundamental object since these types of normalized screws do not form a vector space. Sir R S Ball would refer to our screws as screw motions. We prefer to use our shorter terminology (shared by [14]) because, for a dynamical screw \( d \), which we shall later introduce, no actual motion needs to take place. Note also that the normalization of the pitch is chosen in such a way that, integrating the screw vector field by a parameter \( 2\pi \), i.e. by making a full rotation, one gets a diffeomorphism which is a translation by \( p \) along the screw axis. In other words, with the chosen normalization, the pitch gives the translation of the screw for any full rotation.

### 2.1. The commutator

Every screw is a vector field; thus we can form the Lie bracket \([s_1, s_2]\) of two screws [10]. In this section, we check that this commutator is itself a screw and calculate its resultant.

**Proposition 2.16.** The Lie bracket \( s = [s_1, s_2] \) is a screw with resultant \( s = -s_1 \times s_2 \) and satisfies
\[
s(P) = s_2 \times s_1(P) - s_1 \times s_2(P).
\]

**Remark 2.17.** Some authors define the commutator of two screws as minus the Lie bracket.

**Proof.** Let \( s_1 \) and \( s_2 \) be two screws
\[
s_1(P) - s_1(Q) = s_1 \times (P - Q),
\]
\[
s_2(P) - s_2(Q) = s_2 \times (P - Q).
\]
Let us fix a Cartesian coordinate system \( \{ x^i \} \), then the Lie bracket reads
\[
s_i = s_i^j \partial_j s^i - s_i^j \partial_j s^i.
\]
Note that \( s_i^j \partial_j s^i(P) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [s_i^j(P + \epsilon s_1(P)) - s_i^j(P)] \) which, using equation (7), becomes \( s_i^j \partial_j s^i(P) = [s_2 \times s_1(P)]^i \). Inverting the roles of \( s_1 \) and \( s_2 \), we calculate the second term; thus, we obtain the interesting expression
\[
s(P) = s_2 \times s_1(P) - s_1 \times s_2(P).
\]
Let us check that it is a screw; indeed

\[ s(P) - s(Q) = s_2 \times s_1(P) - s_1 \times s_2(P) - s_2 \times s_1(Q) + s_1 \times s_2(Q) \]

\[ = s_2 \times [s_1(P) - s_1(Q)] - s_1 \times [s_2(P) - s_2(Q)] \]

\[ = s_2 \times [s_1 \times (P - Q)] - s_1 \times [s_2 \times (P - Q)] \]

\[ = [s_2 \cdot (P - Q)]s_1 - [s_1 \cdot (P - Q)]s_2 = (-s_1 \times s_2) \times (P - Q), \]

which also proves that the resultant is as claimed.

The relation between the commutator and the scalar product is clarified by the following result.

**Proposition 2.18.** Let \( s_1, s_2, s_3 \) be three screws; then

\[ \langle s_1, [s_3, s_2] \rangle + \langle [s_3, s_1], s_2 \rangle = 0. \] (8)

Furthermore, the quantity \( \langle s_1, [s_3, s_2] \rangle \) reads

\[ \langle s_1, [s_3, s_2] \rangle = s_3(P) \cdot (s_1 \times s_2) + s_2(P) \cdot (s_3 \times s_1) + s_1(P) \cdot (s_2 \times s_3), \]

is independent of \( P \) and does not change under cyclic permutations of its terms.

**Proof.** We use equation (5)

\[ \langle s_1, [s_3, s_2] \rangle = s_1 \cdot [s_2 \times s_3(P) - s_3 \times s_2(P)] + s_1(P) \cdot (-s_3 \times s_2) \]

\[ = s_3(P) \cdot (s_1 \times s_2) + s_2(P) \cdot (s_3 \times s_1) + s_1(P) \cdot (s_2 \times s_3). \]

This expression changes sign under exchange of \( s_1 \) and \( s_2 \); thus we obtain the desired conclusion.

### 2.2. The dual space and the reference frame reduction to \( \mathbb{R}^6 \)

Given a screw \( s \in S \), it is possible to construct the linear map \( \langle s, \cdot \rangle : S \to \mathbb{R} \) which is an element of the dual space \( S^* \).

**Proposition 2.19.** The linear map \( \langle s, \cdot \rangle \) sends every screw to zero (namely, it is the null map), if and only if \( s = 0 \).

**Proof.** If \( s \) is such that \( s \neq 0 \), then the scalar product with the free screw \( s'(P) := s \) shows that

\[ 0 = \langle s, s' \rangle = s^2, \]

a contradiction.

If \( s \) is a constant screw with vector invariant \( f \), then the screw scalar product with the applied screw \( s'(P) := f \times (P - Q) \), where \( Q \) is some point, gives

\[ 0 = \langle s, s' \rangle = f^2; \]

hence \( f = 0 \) and thus \( s \) is the null screw.

We have shown that the linear map \( s \to \langle s, \cdot \rangle \) is injective. We wish to show that \( s \to \langle s, \cdot \rangle \) is surjective, namely any element of the dual vector space \( S^* \) can be regarded as the scalar product with some screw. We could deduce this fact using the injectivity and the equal finite dimensionality of \( S \) and \( S^* \), but we shall proceed in a more detailed way which will allow us to introduce a useful basis for the space of screws and its dual.

Let us choose \( Q \in E \), and let \( \{e_1, e_2, e_3\} \) be a positive oriented orthonormal base for \( (V, \cdot, o) \), where \( o \) denotes the orientation. Namely, assume that we have made a choice of reference frame. The six screws, \( f_i = ((Q, e_i), 0), m_i = ((Q, 0), e_i), i = 1, 2, 3 \), generate the whole space \( S \). Indeed, if \( s \) is a screw and \( ((Q, s), s(Q)) \) is its motor at \( Q, s = a_1e_1 + a_2e_2 + a_3e_3, \)
\[ s(Q) = b_1 e_1 + b_2 e_2 + b_3 e_3, \]
then \((Q, s) = \sum_{i=1}^{3} \{a_i f_i + b_i m_i\}\). As a consequence, each reference frame establishes a bijection between the screw space \(S\) and \(\mathbb{R}^6\) as follows:

\[
 s \in S \xrightarrow{\text{reference frame}} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} \in \mathbb{R}^6,
\]
where \(\bar{a}, \bar{b} \in \mathbb{R}^3\) (vectors in \(\mathbb{R}^3\) are denoted with a bar, while the boldface notation is reserved for vectors in \(V\)).

The screw scalar product between \(s, s' \in S\) in this representation takes the form

\[
\langle s, s' \rangle = \bar{a} \cdot \bar{b}' + \bar{b} \cdot \bar{a}';
\]

thus, the screw scalar product quadratic form is given by the \(6 \times 6\) matrix

\[
\begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix},
\]
where \(I\) is the identity \(3 \times 3\) matrix.

Let us now consider the six linear functionals \(\langle m_i, \cdot \rangle, \langle f_i, \cdot \rangle, i = 1, 2, 3\). From the definition of scalar product evaluated at \(Q\), it is immediate that

\[
\langle m_i, \cdot \rangle(f_j) = \langle m_i, f_j \rangle = \delta_{ij},
\]

\[
\langle f_i, \cdot \rangle(m_j) = \langle f_i, m_j \rangle = \delta_{ij},
\]

\[
\langle m_i, \cdot \rangle(m_j) = \langle m_i, m_j \rangle = 0,
\]

\[
\langle f_i, \cdot \rangle(f_j) = \langle f_i, f_j \rangle = 0,
\]

that is, \(\{\langle m_i, \cdot \rangle, \langle f_i, \cdot \rangle; i = 1, 2, 3\}\) is the dual base to \(\{f_i, m_i; i = 1, 2, 3\}\).

Every element \(z \in S^*\) is uniquely determined by the values \(c_i, d_i, i = 1, 2, 3\), that it takes on the six base screws \(f_i, m_i, i = 1, 2, 3\). By the above formulas, the linear combination

\[
\sum_{i=1}^{3} [c_i \langle m_i, \cdot \rangle + d_i \langle f_i, \cdot \rangle] = \sum_{i=1}^{3} [c_i m_i + d_i f_i],
\]
takes the same values on the screw base and thus coincides with \(z\). We can therefore establish a bijection of the dual space \(S^*\) with \(\mathbb{R}^6\) as follows:

\[
z \in S^* \xrightarrow{\text{reference frame}} \begin{pmatrix} \bar{c} \\ \bar{d} \end{pmatrix} \in \mathbb{R}^6
\]

where \(\bar{c}, \bar{d} \in \mathbb{R}^3\) (it is convenient to distinguish this copy of \(\mathbb{R}^6\) with that isomorphic with \(S^*\) introduced above).

As a consequence

**Proposition 2.20.** The linear map \(s \rightarrow \langle s, \cdot \rangle\) from \(S\) to \(S^*\) is bijective.

Thanks to this result any screw can be regarded either as an element of \(S\) or, acting with the screw scalar product, as an element of \(S^*\). It must be stressed that if \(s \in S\) is represented by \(\begin{pmatrix} c \\ d \end{pmatrix}\), then \(\langle s, \cdot \rangle \in S^*\) is represented by \(\begin{pmatrix} c \\ d \end{pmatrix}\), that is, the map from \(S\) to \(S^*\) which sends \(s\) to \(\langle s, \cdot \rangle\) is given in this representation by the matrix \(\begin{pmatrix} c & d \end{pmatrix}\). The pairing between the elements of \(S^*\) and those of \(S\) is the usual one on \(\mathbb{R}^6\). Nevertheless, it is useful to keep in mind that we are working with two copies of \(\mathbb{R}^6\) (as we consider two isomorphisms), the former isomorphic with \(S\) and the latter isomorphic with \(S^*\).

**Remark 2.21.** All this reduction to \(\mathbb{R}^6\) depends on the reference frame. As mentioned in the introduction, most references of screw theory introduce the screw starting from its reduction or using a descriptive approach (the screw has an axis, a pitch, etc). As we argued in the introduction, it is pedagogically and logically preferable to define the screw without making reference to any reference frame.
The instantaneous axis of rotation is by definition the screw axis of maps. We shall return to this non-accidental fact later on.

\[ [m_i, m_j] = 0, \]
\[ [f_i, m_j] = -[m_j, f_i] = -\sum_k \epsilon_{ijk} m_k, \]
\[ [f_i, f_j] = -\sum_k \epsilon_{ijk} f_k. \]

The reader will recognize the Lie algebra commutation relations of the group \( SE(3) \) of rigid maps. We shall return to this non-accidental fact later on.

Given a screw \( s \) we consider the map \( ad_s : S \to S \) which acts as \( s' \to ad_s s' := [s, s'] \).

Clearly, \( ad_s s' = -ad_{s'} s \) and if \( x, y, z \) are screws, then the Jacobi identity for the Lie bracket of vector fields \( [x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \) becomes

\[ ad_{ad_s, y} = ad_{ad_s, x} - ad_{ad_s, x}. \]

Let an origin \( O \) be given and let us use the isomorphism with \( V \oplus V \). Let \( s \) be represented by \((s', \bar{s})\). If we introduce a full reference frame, then it is possible to check with a little algebra that, according to the above commutations, the map \( ad_s \) is represented by the matrix

\[ ad_s \text{ origin } O \left( \begin{array}{cc} -s \times & 0 \\ -s' \times & -s \times \end{array} \right), \]

where for every \( v \in V \), \( v \times : V \to V \) is an endomorphism of \( V \) induced by the vector product. Of course, if we had kept the reference frame \( \mathbb{R}^6 \) isomorphism, then, as is customary, with \( \bar{v} \times \) we would mean the \( 3 \times 3 \) matrix \( \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \).

3. The kinematical screw and the composition of rigid motions

A rigid motion is a continuous map \( \varphi : [0, 1] \times E \to E \), which preserves the distances between points, i.e. for every \( P, Q \in E, t \in [0, 1] \), we have \( |\varphi(t, P) - \varphi(t, Q)| = |P - Q| \), and such that \( \varphi(0, \cdot) : E \to E \) is the identity map. A rigid map is the result of a rigid motion, that is, a map of type \( \varphi(1, \cdot) : E \to E \). It can be shown that every rigid map is an affine map which preserves the scalar product and is orientation preserving [11, App 6]. The rigid maps form a group usually denoted as \( SE(3) \).

In kinematics, the velocity field of bodies performing a rigid motion satisfies the fundamental formula of the rigid body

\[ \dot{v}(P) - \dot{v}(Q) = \omega \times (P - Q). \]

This formula is usually deduced from the Poisson formula for the time derivative of a normalized vector: \( \frac{\text{d}}{\text{d}t} \bar{e} = \omega \times \bar{e} \).

Equation (13) defines a screw which is called twist in the literature. Let us denote this screw with \( k \); then \( k(P) = \bar{v}(P) \) and \( k = \omega \), where \( \omega \) is the angular velocity of the rigid body. The instantaneous axis of rotation is by definition the screw axis of \( k \).

Let us recall that if a point moves with respect to a frame \( K' \) which is in motion with respect to a frame \( K \), then the velocity of the point with respect to \( K \) is obtained by summing the drag velocity of the point, as if it were rigidly connected with frame \( K' \), with the velocity relative to \( K' \). If two kinematical screws are given and summed, then the result gives a velocity field which represents (by interpreting one of the screw fields as the velocity field of the points at rest in \( K' \) with respect to \( K \)) the composition of two rigid motions. The nice fact is that
the result is independent of which screw is regarded as describing the motion of $K'$. In other words, the result has an interpretation in which the role of the screws can be interchanged.

More generally, one may have a certain number of frames $K^{(i)}$, $i = 0, 1, \ldots, n$, of which we know the screw $k_{i+1}$ which describes the rigid motion of $K^{(i+1)}$ with respect to $K^{(i)}$. The motion of $K^{(n)}$ with respect to $K^{(0)}$ is then described by the screw $\sum_{i=1}^{n} k_i$. In particular, since the map which sends a screw to its resultant is linear, the angular velocity of $K^{(n)}$ with respect to $K^{(0)}$ is the sum of the angular velocities: $\sum_{i=1}^{n} \omega_i$. As illustrated in the introduction, the screw approach tells us something more. Indeed, one can establish the direction of the instantaneous axis of rotation of $K^{(n)}$ with respect to $K^{(0)}$ by using the same methods used to determine the central axis in a problem of applied forces. Indeed, we shall see in a moment that there is a parallelism between forces and angular velocities as they are both resultants of some screw.

4. Dynamical examples of screws

In dynamics, the most important screw is that given by the moment field, and is called wrench. Let us recall that the momentum $M(Q)$ of a set of applied forces $(P_i, F_i)$ with respect to a point $Q$ is given by

$$M(Q) = \sum_i (P_i - Q) \times F_i. \quad (14)$$

If we consider $P$ in place of $Q$, then we obtain

$$M(P) = \sum_i (P_i - P) \times F_i = \sum_i (P_i - Q + Q - P) \times F_i = M(Q) + F \times (P - Q), \quad (15)$$

where $F = \sum_i F_i$ is the force resultant. This equation shows that we are in presence of a screw $d$ such that $d(P) = M(P)$, $d = F$. The central axis of a system of forces is nothing but the screw axis.

Another example of screw is given by the angular momentum field. The angular momentum $L(Q)$ of a system of point particles located at $R_i$ with momentum $p_i$ with respect to a point $Q$ is given by

$$L(Q) = \sum_i (R_i - Q) \times p_i. \quad (16)$$

If we consider $B$ in place of $Q$, then we obtain

$$L(B) = \sum_i (R_i - B) \times p_i = \sum_i (R_i - Q + Q - B) \times p_i = L(Q) + P \times (B - Q),$$

where $P = \sum_i p_i$ is the total linear momentum. This equation shows that we are in presence of a screw $l$ such that $l(Q) = L(Q)$, $l = P$.

4.1. The cardinal equations of mechanics

Let us consider the constitutive equation of the screw of angular momentum

$$L(B) - L(Q) = P \times (B - Q).$$

The vector $L(B)$ changes in time as the distribution of velocity and mass changes. Actually, we can consider here another source of time change if we allow the point $B$ to change in time. Let us first consider the case in which the angular momentum is considered with respect to a fixed point.
By differentiating the previous equation with respect to time we get equation (15). In other words, the dynamic screw \( d \) is the time derivative of the dynamic screw \( l \)

\[
\frac{\partial l}{\partial t} = d.
\]  

(17)

We use here a partial derivative to remind us that the poles are fixed.

This equation replaces the first and second cardinal equations of mechanics. Indeed, as the map \( l \rightarrow l \) is linear it follows

\[
\frac{\partial l}{\partial t} = d,
\]  

(18)

which is the first cardinal equation \( \partial P/\partial t = F \) in disguise. (Alternatively, write \( l_t(P) = l_t(Q) + l_t \times (P - Q) \) and differentiate.) Here, the partial derivative coincides with the total derivative because the resultant is a free vector; it does not depend on the point. The second cardinal equation with respect to a point \( O \)

\[
\frac{\partial L(O)}{\partial t} = M(O),
\]

is obtained by evaluating equation (17) at the point \( O \).

4.1.1. The cardinal equation in a rigidly moving non-inertial frame. In equation (17), we have differentiated with respect to time assuming that the point with respect to which we evaluate the angular momentum does not change in time. In other words, we have adopted a Eulerian point of view. Suppose now that in space we have a vector field \( v(P) \) which describes the motion of a continuum (not necessarily a rigid body). In this case, we have to distinguish the Eulerian derivative with respect to time, which we have denoted by \( \partial/\partial t \), from the Lagrangian or total derivative with respect to time \( d/\partial t \). According to the latter, the second cardinal equation of mechanics reads

\[
\frac{dL(O)}{dt} = -v(O) \times P + M(O),
\]  

(19)

where \( O(t) \) is the moving pole. Let us differentiate

\[
L(B) - L(Q) = P \times (B - Q),
\]

with respect to time using the Lagrangian description, that is, assuming that \( B \) and \( Q \) move respectively with velocities \( v(B) \), \( v(Q) \), and considering the angular momenta with respect to the moving points. We obtain

\[
\frac{dL(B)}{dt} - \frac{dL(Q)}{dt} = F \times (B - Q) + P \times (v(B) - v(Q)).
\]  

(20)

Using the second cardinal equation (19) we find that this is the constitutive equation of the momentum screw. Nevertheless, the total derivative of the angular momentum is not a screw.

The relation between the partial and total derivative is as follows:

\[
\frac{dL(B)}{dt} = \frac{\partial L(B)}{\partial t} + \nabla_{v(B)} L,
\]

where

\[
\nabla_{v(B)} L = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [L(B + v(B)\epsilon) - L(B)] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} [P \times v(B)\epsilon] = P \times v(B);
\]

thus

\[
\frac{dL(B)}{dt} = \frac{\partial L(B)}{\partial t} + P \times v(B).
\]  

(21)
However, suppose that the velocity field is itself a screw,
\[ \mathbf{v}(P) - \mathbf{v}(R) = \omega \times (P - R), \]
so that the continuum moves rigidly; then from equation (20), using the previous results for commutators,
\[
\frac{d}{dt} \mathbf{L}(B) - \omega \times \mathbf{L}(B) = \frac{d}{dt} \mathbf{L}(Q) - \omega \times \mathbf{L}(Q)
\]
\[ = \mathbf{F} \times (B - Q) + [P \times \mathbf{v}(B) - \omega \times \mathbf{L}(B)] - [P \times \mathbf{v}(Q) - \omega \times \mathbf{L}(Q)]
\[ = \mathbf{F} \times (B - Q) + [k, l](B) - [k, l](Q) = \mathbf{F} \times (B - Q) + (-\omega \times P) \times (B - Q)
\[ = [\mathbf{F} - \omega \times P] \times (B - Q).
\]
The time derivative \( \frac{d}{dt} \) with respect to the moving frame reads by the Poisson formula
\[
\frac{d}{dt} \mathbf{R} = \frac{d}{dt} - \omega \times \mathbf{R};
\]
thus the previous result can be summarized as follows.

**Theorem 4.1.** Let us denote with \( \frac{d}{dt} \) the total derivative with respect to points that move rigidly according to a kinematical screw \( k \) of vector field \( \mathbf{v}(Q) \), and with \( \frac{d}{dt} \) the time derivative relative to the corresponding rigidly moving frame. The quantity
\[
\frac{d}{dt} \mathbf{L}(Q) \big|_R = \frac{d}{dt} \mathbf{L}(Q) - \omega \times \mathbf{L}(Q),
\]
defines a screw with resultant \( \frac{d}{dt} \mathbf{P} \big|_R = \mathbf{F} - \omega \times \mathbf{P} \). This screw coincides with the screw \( \mathbf{M}(Q) + [k, l](Q) \):
\[
\frac{d}{dt} \mathbf{l} = \frac{\partial}{\partial t} + [k, l].
\]

**Proof.** We have only to prove the last statement, which follows easily from equation (21) and the definition of commutator.

It must be remarked that in the previous result the angular momentum \( \mathbf{L} \) is calculated as in the original inertial frame, and not using the point particle velocities as given in the moving frame of twist \( k \). Nevertheless, the previous result is quite interesting as it gives a dynamical application of the commutator.

### 4.2. The inertia map

Given a rigid body, the kinematical screw \( k \) fixes the velocity of each point of the rigid body and hence determines the angular momentum screw \( l \). The map \( k \to l \) is linear and is an extension of the momentum of inertia map which includes the translational inertia provided by the mass.

Let us recall that given some continuum and fixed a point \( Q \), the momentum of inertia map \( I : V \to V, \eta \to I_Q(\eta) \), is the linear map defined by the expression
\[
I_Q(\eta) = \sum_i m_i (R_i - Q) \times [\eta \times (R_i - Q)]
\]
where we have discretized the continuum into point masses \( m_i \) located, respectively, at positions \( R_i \). Let \( C \) be the centre of mass, namely the point defined by \( \sum_i m_i (R_i - Q) = M(C - Q) \) where \( M = \sum_i m_i \). It is easy to prove the Huygens–Steiner formula:
\[
I_Q(\eta) = IC(\eta) + M(C - Q) \times [\eta \times (C - Q)].
\]
Let us consider a rigid motion described by a kinematical screw. From equation (16),
\[ L(Q) = \sum_i m_i (R_i - Q) \times v(R_i), \tag{22} \]
where \( v(R) \) is the kinematical screw. It is clear that this map sends a screw into what has been proved to be another screw, and that this map depends on the location of the masses.

Let us differentiate with respect to time the equation \( \sum_i m_i (R_i - Q) = M(C - Q) \) and use \( v(R_i) = v(O) + \omega \times (R_i - O) \) to obtain
\[ \dot{C} = \frac{1}{M} \sum_i m_i v_i = \frac{1}{M} \sum_i m_i [v(O) + \omega \times (R_i - O)] = v(O) + \omega \times (C - O) = v(C). \]

From this equation, we also obtain \( P = M v(C) \). We have by writing \( v(R_i) = v(C) + \omega \times (R_i - C) \)
\[ L(Q) = \sum_i m_i (R_i - Q) \times v(R_i) = (C - Q) \times (M v(C)) + \sum_i m_i (R_i - C) \times [\omega \times (R_i - C)] \]
\[ = (M v(C)) \times (Q - C) + I C(\omega). \]
This equation shows how the kinematical screw determines the dynamical screw \( l \). Plugging \( Q = C \), we find \( L(C) = I C(\omega) \), and we reobtain, as we already know, \( l = P = M v(C) \).

According to equation (4), the screw axis of the angular momentum screw passes through the point
\[ Q = C + \frac{v(C) \times I C(\omega)}{M v(C)^2} \]
with direction \( v(C) \), while the instantaneous axis of rotation passes through the point
\[ O = C + \frac{\omega \times v(C)}{\omega^2} \]
and has direction \( \omega \).

### 4.3. Screw scalar product examples: kinetic energy, power and reciprocal screws

Let us consider a rigid body and let us decompose it into point particles of mass \( m_i \) located at \( R_i \) with mass velocity \( v_i \) and momentum \( p_i = m_i v_i \).

Let \( Q \) be any point,
\[ T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} \sum_i p_i (v(Q) + \omega \times (R_i - Q)) = \frac{1}{2} P \cdot v(Q) + \frac{1}{2} \sum_i p_i \cdot (\omega \times (R_i - Q)) \]
\[ = \frac{1}{2} P \cdot v(Q) + \frac{1}{2} \sum_i \omega \cdot ((R_i - Q) \times p_i) = \frac{1}{2} [v(Q) \cdot P + \omega \cdot L(Q)] = \frac{1}{2} [k, l]. \]

Thus, the kinetic energy is one half the screw scalar product between the kinetic screw and the angular momentum screw.

Let us now suppose that on each point particle of mass \( m_i \) located at \( R_i \) acts a force \( F_i \) possibly null. The power of the applied forces is the sum of the powers of the single forces. Denoting with \( L \) the work done by the forces
\[ \frac{dL}{dt} = \sum_i F_i \cdot v_i = \sum_i F_i \cdot [v(O) + \omega \times (R_i - O)] \]
\[ = F \cdot v(O) + \sum_i F_i \cdot [\omega \times (R_i - O)] = F \cdot v(O) + \sum_i \omega \cdot [(R_i - O) \times F_i] \]
\[ = F \cdot v(O) + \omega \cdot M(O) = \langle k, d \rangle , \]
that is, the total power is the screw scalar product between the kinematical screw and the dynamical screw. Since the scalar product is independent of \( O \), let us choose \( O = C \)

\[
\frac{dL}{dt} = F \cdot P/M + \omega \cdot \frac{dI_c}{dt}(\omega) + \omega \cdot (v(C) \times P) = \frac{d}{dt}\left(\frac{P^2}{2M}\right) + \omega \cdot \frac{dI_c}{dt}(\omega),
\]

where we used König’s decomposition theorem. From the kinetic energy theorem, we know that the variation of kinetic energy equals the work done by the forces on the rigid body; thus we expect that the last term vanishes. This is indeed so thanks to the following lemma.

**Lemma 4.2.** For a rigid body, \( \omega \cdot \frac{dI_c}{dt}(\omega) = 0. \)

**Proof.** Let us fix a base for the vector space \( V \) so that \( I_c \) becomes represented by a time-dependent matrix \( O(t)DO(t) \), where \( D \) is the diagonal (time-independent) matrix of the principal moments of inertia and \( O(t) \) is a time-dependent matrix giving the rotation of the principal directions of inertia with respect to a chosen fixed base of \( V \). Differentiating with respect to time we obtain \( \frac{dI_c}{dt} = -AIC + ICA \), where \( A = O(t)\frac{dO}{dt} \) is an antisymmetric matrix. However, \( \omega \) belongs to the kernel of this matrix from which the desired result follows. \( \square \)

**Remark 4.3.** The screw is particularly useful when modelling workless constraints between rigid bodies (think for example of a robotic arm and at its constituent rigid parts). Indeed, suppose that the body is made of \( N \) rigid parts and let us focus on part \( i \). The constraints will reduce the possible motions of part \( i \) for a given position of the other parts. In particular, the possible kinematical status of part \( i \) for any given relative configuration of all parts will be described by a screw subspace \( W \subset S \) of all the possible twists of part \( i \). Let \( k \in W \) and let \( d \) be the wrench acting on rigid body \( i \) as a result of the interaction with the neighboring bodies. Since, by assumption, the constraints are workless we must have \( \langle d, k \rangle = 0 \). We conclude that the vector subspace \( Z \subset S \) made of the screws which are screw-orthogonal to the allowed movements (i.e. screw-orthogonal to \( W \)) is made by all the possible wrenches acting on body \( i \) so as to make no work. Two screws with vanishing screw scalar product (i.e. screw-orthogonal) are said to be reciprocal and the subspace \( Z \) is said to be reciprocal to subspace \( W \).

## 5. Lie algebra interpretation and Chasles’ theorem

Let \( g : E \to E \) be a rigid map, namely the result of a rigid motion as it has been defined in section 3. We shall omit the proof that \( g \) is an affine map such that \( g(P + a) = g(P) + l(a) \), where \( l : V \to V \) is a linear map which preserves the scalar product and the orientation. The rigid maps form a group denoted \( SE(3) \) (which sometimes we shall simply denote \( G \)). In the coordinates induced by a reference frame, \( P' = g(P) \) has coordinates \( x' \) which are related to those of \( P \) by

\[
x' = \sum \ O'_j x'^j + b',
\]

where \( O \) is a special orthogonal \( 3 \times 3 \) matrix. This expression clarifies that \( SE(3) \) is a Lie group. The three Euler angles determining \( O \) and the three translation coefficients \( \{b'\} \) can provide a coordinate system on the six-dimensional group manifold. We stress that we regard \( SE(3) \) as an abstract Lie group, and we do not make any privileged choice of coordinates on it (we do not want to make considerations that depend on the choice of reference coordinates).
The Lie algebra \( \mathfrak{se}(3) \) (which sometimes we shall simply denote as \( \mathcal{F} \)) of \( SE(3) \) is the family of left invariant vector fields on \( SE(3) \). The Lie commutator of two Lie algebra elements is still an element of the Lie algebra. This structure can be identified with the tangent space \( TG_e \), with \( e \) being the identity element on \( G \), endowed with the Lie bracket \([\cdot,\cdot] : TG_e \times TG_e \to TG_e \).

Let \( G \times E \to E \) be the above (left) group action on \( E \) so that \( g_2(g_1(P)) = (g_2g_1)(P) \). Each point \( P \in E \) induces an orbit map \( u_P : G \to E \) given by \( u_P(g) = g(P) \); thus \( u_P : TG_e \to TE_{g(P)}. \) We are interested in \( u_P \) at \( g = e \), so that \( g(P) = P \). If \( v \in TG_e \), then \( s(P) := u_P(v) \) gives, for every \( P \in E \), a vector field on \( E \) which is the image of the Lie algebra element \( v \). Such vector fields on \( E \) are called fundamental vector fields.

Let us consider the exponential \( g(t) = \exp(tv) \) which is obtained by the integration of the vector field \( v \) from \( e \). The orbit \( g(t)(P) \) passing through \( P \) is obtained from the integration of the vector field \( s \) starting from \( P \). In other words, the 1-parameter group of rigid maps \( g(t) \) coincides with the 1-parameter group of diffeomorphisms (which are rigid maps) generated by the vector field \( s \). Conversely, every such 1-parameter group of rigid maps determines a Lie algebra element. Since every screw element, once integrated, gives a non-trivial 1-parameter group of rigid maps, every screw is the fundamental vector field of some Lie algebra element. That is, the map \( u_P|_e : \mathcal{F} \to S \) is surjective. But \( \mathcal{F} \) and \( S \) are two vector spaces of the same dimensionality; thus this map is also injective. In summary, the screws are the representation on \( E \) of the elements of \( \mathfrak{se}(3) \).

It is particularly convenient to study the Lie algebra of \( SE(3) \) through their representative vector fields on \( E \); indeed many features, such as the existence of a screw axis for each Lie algebra element, become very clear.

We can now use several results from the study of Lie groups and their actions on manifolds [10, 4]. A central result is that the bijective map \( v \to s \) is linear and sends the Lie bracket to the commutator of vector fields on \( E \). In some cases, the exponential map from the Lie algebra to the Lie group is surjective (which is not always true as the example of \( SL(2, \mathbb{R}) \) shows [4, 13]). This is the case of the group \( SE(3) \) (see [12, proposition 2.9]); thus, since every element of the Lie algebra \( \mathfrak{se}(3) \) corresponds to a screw \( s \), and the exponential map corresponds to the rigid map obtained from the integration of the screw vector field by a parameter \( t \), the surjectivity of the exponential map implies that every rigid map can be accomplished as the result of the integration along a screw, or, which is the same, by a suitable rotation along an axis combined with the translation along the same axis. This is the celebrated Chasles’ theorem [14] reformulated and reobtained in the Lie algebra language.

5.1. Invariant bilinear forms and screw scalar product

Let us recall that if \( x, y \in \mathcal{F} \), then the expression \( \text{ad}_v : \mathcal{F} \to \mathcal{F} \) defined as \( \text{ad}_v(x) := [x, y] \) defines a linear map \( \text{ad}_v : \mathcal{F} \to \mathcal{F} \) called adjoint endomorphism. The trace of the composition of two such endomorphisms defines a symmetric bilinear form

\[
K(x, y) = \text{trace}(\text{ad}_x \text{ad}_y),
\]

called Killing form on \( \mathcal{F} \). The Killing form is a special invariant bilinear form \( Q \) on \( \mathcal{F} \), namely it is bilinear and satisfies (use equation (11))

\[
Q(\text{ad}_x, y) + Q(x, \text{ad}_y) = 0.
\]

Proposition 2.18 shows that the scalar product of screws provides an invariant symmetric bilinear form on the Lie algebra. We wish to establish if there is any connection with the Killing form.
As done in section 2.2, let us introduce an origin $O$ and use the isomorphism of $S$ with $V ⊕ V$. If $x, y ∈ S$ are represented by $(\hat{x}_O)$ and $(\hat{y}_O)$, then the screw scalar product is

$$\langle x, y \rangle = x · y_O + x_O · y.$$

According to the result of section 2.2, we have

$$K(x, y) = \text{trace} \left( \begin{pmatrix} -x × & 0 & -y × & 0 \\ -xO × & -x × & -yO × & -y × \\ x × (y ×) & 0 & yO × & x × (y ×) \\ xO × (y × + x × (yO ×) & x × (y ×) & - (x · y)I + y(x ·) & 0 \end{pmatrix} \right) = -4x · y.$$

We conclude that the Killing form is an invariant bilinear form which is distinct from the screw scalar product. It coincides with the Killing form of the Lie group of rotations alone and thus, it does not involve the translational information inside the $O$-terms. Therefore, the screw scalar product provides a new interesting invariant bilinear form, which is sometimes referred to as the Klein form of $\mathfrak{se}(3)$.

From equation (12), we find that $ad_x$ is represented by

$$ad_x \overset{\text{origin}}{\leftarrow} \begin{pmatrix} x × z \\ xO × z + x × zO \end{pmatrix}$$

from which, using the symmetry properties of the mixed product, we can check again that the screw scalar product is invariant

$$\langle ad_x, y \rangle + \langle x, ad_y \rangle = [(x × z) · yO + (xO × z) · yO + (x × zO) · y]$$

$$+ [(y × z) · xO + (yO × z) · x + (y × zO) · x] = 0.$$

6. Conclusions

Screw theory, although venerable, has found some difficulties in affirming itself in the curricula of the physicist and the mechanical engineer. This has changed in the last decades when screw theory has finally found application in robotics where its ability to deal with the composition of rigid motions has proved to be much superior with respect to treatments based on Euler coordinates.

We have given here a short introduction to screw theory which can provide a good starting point to a full self-study of the subject. We started from a coordinate independent definition of screw and we went to introduce the concepts of screw axis, screw scalar product and screw commutator. We introduced the dual space and showed that any frame induces an isomorphism on $\mathbb{R}^6$ which might be used to perform calculations. We then went on to consider kinematical and dynamical examples of screws, reformulating the cardinal equations of mechanics in this language.

Particularly important was the application of the screw scalar product in the expressions for the kinetic energy and power, in fact, the virtual work (power) is crucial in the formulation of Lagrangian mechanics. In this connection, we mentioned the importance of reciprocal screws. Finally, we showed that the space of screws is nothing but the Lie algebra $\mathfrak{se}(3)$, and that the screw scalar product is the Klein form.

Philosophically speaking, screw theory clarifies that the most natural basic dynamical action is not the force, but rather the force aligned with a mechanical momentum (remark 2.12).
In teaching we might illustrate the former action with a pushing finger and the latter action with a kind of pushing hand. Analogously, the basic kinematical action is not given by the act of pure rotation (or translation), but by that of rotation aligned with translation. Again, for illustration purposes this type of motion can be represented by that of a (real) screw.

Clearly, in our introduction we had to omit some arguments. For instance, we present neither the cylindroid nor the calculus of screws. Nevertheless, the arguments that we touched were covered in full generality, emphasizing the geometrical foundations of screw theory and its connection with the Lie group of rigid maps. We hope that this work will promote screw theory providing an easily accessible presentation to its key ideas.

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