Extending Utility Representations of Partial Orders

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Abstract

The problem is considered as to whether a monotone function defined on a subset $P$ of Euclidean space $\mathbb{R}^k$ can be strictly monotonically extended to $\mathbb{R}^k$. It is proved that this is the case if and only if the function is separably increasing. Explicit formulas are given for a class of extensions which involves an arbitrary function. Similar results are obtained for utility functions that represent strict partial orders on abstract sets $X$. The special case where $P$ is a Pareto subset of $\mathbb{R}^k$ (or of $X$) is considered.

Key words: Extension of utility functions; Monotonicity; Utility representation of partial orders; Pareto set

1 Introduction

Suppose that a decision maker defines his or her utility function on some subset $P$ of the Euclidean space of alternatives $\mathbb{R}^k$. Utility functions are usually assumed to be strictly increasing in the coordinates which correspond to partial criteria. Therefore, it is useful to specify the conditions under which a strictly monotone function defined on $P$ can be strictly monotonically extended to $\mathbb{R}^k$.

In this paper, we demonstrate that such an extension is possible if and only if the function defined on $P$ is separably increasing. Explicit formulas for the extension are provided.

An interesting special case is where the structure of subset $P$ does not permit any violation of strict monotonicity on $P$. This is the case where $P$ is a Pareto

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set. A corollary given in Section 3 addresses this situation. The results are translated to the general case of utility functions that represent strict partial orders on arbitrary sets. We do not impose continuity requirements in these settings.

2 Notation, definitions, and main results

For any \( x, y \in \mathbb{R}^k \), \( x \geq y \) means \([x_i \geq y_i \text{ for all } i \in \{1, \ldots, k\}]; \ x \leq y \) means \([x_i \leq y_i \text{ for all } i \in \{1, \ldots, k\}]; \ x > y \) means \([x \geq y \text{ and not } x = y]; \ x < y \) means \([x \leq y \text{ and not } x = y]\). These relations \( \geq, \leq, >, \text{ and } < \) on \( \mathbb{R}^k \) will be called Paretian.

Consider an arbitrary subset \( P \) of \( \mathbb{R}^k \) and strictly increasing (with respect to the above \( > \) relation) real-valued functions \( f_P(x) \) defined on \( P \). The problem is to monotonically extend such a function \( f_P(x) \) to \( \mathbb{R}^k \), provided that it is possible, and to indicate conditions under which this is possible.

An arbitrary function \( f_P(x) \) defined on any \( P \subseteq \mathbb{R}^k \) is said to be strictly increasing on \( P \) with respect to the \( > \) relation or simply strictly increasing on \( P \) if for every \( x, y \in P \), \( x > y \) implies \( f_P(x) > f_P(y) \).

**Definition 1** A real-valued function \( f_P(x) \) defined on \( P \subset \mathbb{R}^k \) is monotonically extendible to \( \mathbb{R}^k \) if there exists a function \( f(x) : \mathbb{R}^k \to \mathbb{R} \) such that

\[ (*) \text{ the restriction of } f(x) \text{ to } P \text{ coincides with } f_P(x), \] and

\[ (**) f(x) \text{ is strictly increasing with respect to } >. \]

In this case, \( f(x) \) is a monotone extension of \( f_P(x) \) to \( \mathbb{R}^k \).

The functions \( f_P(x) \) and \( f(x) \) can be naturally considered as utility (or objective) functions. This means that \( f_P(x) \) and \( f(x) \) can be interpreted as real-valued functions that represent the preferences of a decision maker on the corresponding sets of alternatives (the alternatives are identified with \( k \)-dimensional vectors of partial criteria values or vectors of goods).

The Paretian \( > \) relation is a strict partial order on \( \mathbb{R}^k \), i.e., it is transitive and irreflexive. That is why we deal with a specific problem on the monotonic extension of functions which are monotone with respect to a partial order on their domain. We discuss some connections of this problem with the classical results of utility theory in Section 4 and turn to a more general formulation in Section 5.

For technical convenience, let us add two extreme points to \( \mathbb{R} \):

\[ \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}, \]
and extend the ordinary $>$ relation to $\sim\mathbb{R}$: for every $x \in \mathbb{R}$, set $+\infty > x > -\infty$ and $+\infty > -\infty$. This $>$ relation will determine the values of min and max functions on finite subsets of $\sim\mathbb{R}$.

The functions $\text{sup}$ and $\text{inf}$ will be considered as maps of $2^{\mathbb{R}}$ to $\sim\mathbb{R}$ defined for the empty set as follows: $\text{sup} \emptyset = -\infty$ and $\text{inf} \emptyset = +\infty$.

Given $f_p(x)$, define two auxiliary functions for every $x \in \mathbb{R}^k$:

\begin{align}
a(x) &= \text{sup} \{ f_p(y) \mid y \leq x, y \in P \}, \\
b(x) &= \text{inf} \{ f_p(z) \mid z \geq x, z \in P \}.
\end{align}

By this definition, $a(x)$ and $b(x)$ can be infinite.

It follows from the transitivity of the Paretian $>$ relation that for every function $f_p(x)$, functions $a(x)$ and $b(x)$ are nonstrictly increasing with respect to $>$:

For all $x, x' \in \mathbb{R}^k$, $x' > x$ implies $[a(x') \geq a(x)$ and $b(x') \geq b(x)]$. (3)

Moreover,

For any $x \in P$, $a(x) \geq f_p(x) \geq b(x)$. (4)

It can be easily shown that $f_p(x)$ is nonstrictly increasing if and only if

$b(x) \geq a(x)$ for all $x \in \mathbb{R}^k$. (5)

The following definition involves a strengthening of (5).

**Definition 2** A function $f_p(x)$ defined on $P \subset \mathbb{R}^k$ is separably increasing if for any $x, x' \in \mathbb{R}^k$, $x' > x$ implies $b(x') > a(x)$.

**Definition 3** Given $P \subset \mathbb{R}^k$, let us say that $P' \subset P$ is an upper set if for some $a \in \mathbb{R}^k$, $P' = \{ x \mid x \geq a, x \in P \}$; $P'' \subset P$ is a lower set if for some $a \in \mathbb{R}^k$, $P'' = \{ x \mid x \leq a, x \in P \}$.

**Proposition 4** If $f_p(x)$ defined on $P \subset \mathbb{R}^k$ is separably increasing, then

(a) $f_p(x)$ is strictly increasing;
(b) $f_p(x)$ is upper-bounded on lower sets and lower-bounded on upper sets; in other terms, there are no $x \in \mathbb{R}^k$ such that $a(x) = +\infty$ or $b(x) = -\infty$;
(c) For every $x \in \mathbb{R}^k$, $b(x) \geq a(x)$;
(d) For every $x \in P$, $b(x) = a(x) = f_p(x)$. 


All proofs are given in Section 6. Proposition 4 and other statements are proved there in the more general case of utility functions that represent strict partial orders on arbitrary sets (cf. Section 5).

Observe that there are functions $f_P(x)$ that are strictly increasing, upper-bounded on lower sets and lower-bounded on upper sets, but are not separably increasing. An example is

$$f_P(x) = \begin{cases} x_1, & x_1 \leq 0, \\ x_1 - 1, & x_1 > 1, \end{cases} \quad (6)$$

where $P = ] - \infty, 0] \cup ]1, + \infty[ \subset \mathbb{R}^1$. This function satisfies (a) and (b) (and, as well as all nonstrictly increasing functions, it also satisfies (c) and (d)) of Proposition 4, but it is not separably increasing. Indeed, $b(1) = 0 = a(0)$.

Suppose that $f_P(x)$ defined on $P \subset \mathbb{R}^k$ is separably increasing. Below we prove that this is a necessary and sufficient condition for the existence of monotone extensions of $f_P(x)$ to $\mathbb{R}^k$ and demonstrate how such extensions can be constructed.

Let $u(x) : \mathbb{R}^k \to \mathbb{R}$ be any strictly increasing (with respect to the Paretian $>$) and bounded function defined on the whole space $\mathbb{R}^k$. Suppose that $\alpha, \beta \in \mathbb{R}$ are such that

$$\alpha < u(x) < \beta \quad \text{for all } x \in \mathbb{R}^k. \quad (7)$$

As an example of such a function, adduce

$$u_{\text{example}}(x) = \frac{\beta - \alpha}{\pi} \left( \arctan \sum_{i=1}^{k} x_i + \frac{\pi}{2} \right) + \alpha. \quad (8)$$

Consider also the special case of strictly increasing functions $u_1(x)$ such that

$$0 < u_1(x) < 1. \quad (9)$$

These functions can be obtained from the strictly increasing functions $u(x)$ that satisfy (7) as follows:

$$u_1(x) = (\beta - \alpha)^{-1}(u(x) - \alpha). \quad (10)$$

For an arbitrary strictly increasing function $u_1(x)$ that satisfies (9), every $\alpha$ and $\beta > \alpha$, and every $x \in \mathbb{R}^k$, let us define
\[ f(x) = \max \left\{ a(x), \min \{ b(x), \beta \} - \beta + \alpha \right\} (1 - u_1(x)) 
+ \min \left\{ b(x), \max \{ a(x), \alpha \} - \alpha + \beta \right\} u_1(x). \]  

(11)

Note that for every separably increasing \( f_P(x) \), function \( f(x) : \mathbb{R}^k \to \mathbb{R} \) is well defined, i.e., the two terms in the right-hand side of (11) are finite. This follows from item (b) of Proposition 4.

The main result of this section is

**Theorem 5** Suppose that \( f_P(x) \) is a real-valued function defined on some \( P \subset \mathbb{R}^k \). Then \( f_P(x) \) is monotonically extendible to \( \mathbb{R}^k \) if and only if \( f_P(x) \) is separably increasing. Moreover, for every separably increasing \( u_1(x) : \mathbb{R}^k \to \mathbb{R} \) that satisfies (9) and every \( \alpha, \beta \in \mathbb{R} \) s.t. \( \alpha < \beta \), the function \( f(x) \) defined by (11) is a strictly increasing extension of \( f_P(x) \) to \( \mathbb{R}^k \).

The same family of extensions \( f(x) \) can be generated using all strictly increasing functions \( u(x) \) satisfying (7).

**Proposition 6** For every separably increasing \( f_P(x) \) and every strictly increasing \( u(x) : \mathbb{R}^k \to \mathbb{R} \) that satisfies (7), the function

\[ f(x) = (\beta - \alpha)^{-1} \left( \max \left\{ a(x) - \alpha, \min \{ b(x) - \beta, 0 \} \right\} (\beta - u(x)) 
+ \min \left\{ b(x) - \beta, \max \{ a(x) - \alpha, 0 \} \right\} (u(x) - \alpha) \right) 
+ u(x) \]  

(12)

is a strictly increasing extension of \( f_P(x) \) to \( \mathbb{R}^k \). This function coincides with (11) where \( u_1(x) \) is defined by (10).

It is straightforward to verify that function (12) coincides with (11) where \( u_1(x) \) is defined by (10). This fact will be employed in the proofs of Theorem 5 and some propositions, after which the remaining statement of Proposition 6 will follow from Theorem 5.

In (12) we assume \( -\infty - \alpha = -\infty \) and \( +\infty - \beta = +\infty \) whenever \( a(x) = -\infty \) and \( b(x) = +\infty \), respectively, since \( \alpha \) and \( \beta \) are finite by definition.

To simplify (12), we partition \( \mathbb{R}^k \setminus P \) into four regions:

\[ A = \{ x \in \mathbb{R}^k \setminus P \mid (\exists y \in P : y < x) \& (\exists z \in P : x < z) \}, \]
\[ L = \{ x \in \mathbb{R}^k \setminus P \mid (\exists y \in P : y < x) \& (\exists z \in P : x < z) \}, \]
\[ U = \{ x \in \mathbb{R}^k \setminus P \mid (\exists y \in P : y < x) \& (\exists z \in P : x < z) \}, \]

5
\[ N = \{ x \in \mathbb{R}^k \setminus P \mid (\neg \exists y \in P : y < x) \land (\neg \exists z \in P : x < z) \}. \] (13)

Obviously, every two of these regions have empty meet and \( \mathbb{R}^k = P \cup A \cup L \cup U \cup N \).

**Proposition 7** For every separably increasing \( f_P(x) \), the function \( f(x) \) defined by (12) can be represented as follows:

\[
f(x) = \begin{cases} 
  f_P(x), & x \in P, \\
  \min \{b(x) - \beta, 0\} + u(x), & x \in L, \\
  \max \{a(x) - \alpha, 0\} + u(x), & x \in U, \\
  u(x), & x \in N, \\
  \text{not simplified expression} \ (12), & x \in A.
\end{cases}
\] (14)

Proposition 7 clarifies the role of \( u(x) \) in the definition of \( f(x) \). According to (14), \( u(x) \) determines the rate of growth of \( f(x) \) on \( L \) and \( U \), and \( f(x) = u(x) \) on the set \( N \) which consists of \( > \)-neutral points with respect to the elements of \( P \).

Now let us give one more representation for \( f(x) \), which can be used in (14) when \( x \in A \). Define four regions of other nature in \( \mathbb{R}^k \):

\[
S_1 = \{ x \in \mathbb{R}^k \mid b(x) - a(x) \leq \beta - \alpha \}, \\
S_2 = \{ x \in \mathbb{R}^k \mid b(x) - a(x) \geq \beta - \alpha \text{ and } b(x) \leq \beta \}, \\
S_3 = \{ x \in \mathbb{R}^k \mid b(x) - a(x) \geq \beta - \alpha \text{ and } a(x) \geq \alpha \}, \\
S_4 = \{ x \in \mathbb{R}^k \mid a(x) \leq \alpha \text{ and } b(x) \geq \beta \}.
\]

It is easily seen that \( \mathbb{R}^k = S_1 \cup S_2 \cup S_3 \cup S_4 \).

**Proposition 8** For every separably increasing \( f_P(x) \), the function \( f(x) \) defined by (12) or (11), with \( u_1(x) \) and \( u(x) \) related by (10), can be represented as follows:

\[
f(x) = \begin{cases} 
  a(x)(1 - u_1(x)) + b(x) u_1(x), & x \in S_1, \\
  b(x) + u(x) - \beta, & x \in S_2, \\
  a(x) + u(x) - \alpha, & x \in S_3, \\
  u(x), & x \in S_4.
\end{cases}
\] (15)

The regions \( S_1, S_2, S_3, \) and \( S_4 \) meet on some parts of the border sets \( b(x) - a(x) = \beta - \alpha \), \( a(x) = \alpha \), and \( b(x) = \beta \). Accordingly, the expressions of \( f(x) \) given by Proposition 8 are concordant on these intersections.

6
3 Extendibility of arbitrary functions defined on Pareto sets

Consider the case where \( P \) is a Pareto set.

**Definition 9** A set \( P \subset \mathbb{R}^k \) is a Pareto set in \( \mathbb{R}^k \) if there are no \( x, x' \in P \) such that \( x' > x \).

Observe that for every function \( f_P(x) \) defined on a Pareto set \( P \), the set \( A \) introduced in (13) is empty. It turns out that such a function \( f_P(x) \) is separably increasing if and only if it is upper-bounded on lower sets and lower-bounded on upper sets (see Definition 3). Based on this, the following corollary from Theorem 5 and Proposition 7 is true.

**Corollary 10** Suppose that \( f_P(x) : P \to \mathbb{R} \) is a function defined on a Pareto set \( P \subset \mathbb{R}^k \). Then \( f_P(x) \) can be strictly monotonically extended to \( \mathbb{R}^k \) if and only if \( f_P(x) \) is upper-bounded on lower sets and lower-bounded on upper sets. Moreover, for such a function \( f_P(x) \) and every strictly increasing function \( u(x) : \mathbb{R}^k \to \mathbb{R} \) that satisfies (7), the function

\[
f(x) = \begin{cases} 
  f_P(x), & x \in P, \\
  \min\{b(x), \beta\} - (\beta - u(x)), & x \in L, \\
  \max\{a(x), \alpha\} + u(x) - \alpha, & x \in U, \\
  u(x), & x \in N, 
\end{cases}
\]

provides a monotone extension of \( f_P(x) \) to \( \mathbb{R}^k \).

On regions \( L \) and \( U \), \( f(x) \) is expressed through the “relative” functions \( \beta - u(x) \) and \( u(x) - \alpha \); this is equivalent to the form given in (14). A result closely related to Corollary 10 was used in Chebotarev and Shamis (1998) to construct an implicit form of monotonic scoring procedures for preference aggregation.

4 The extension problem in the context of utility theory

Extensions and utility representations of partial orders were studied since Zorn’s lemma and the Szpiro phase theorem according to which every strict partial order extends to a strict linear order.

In general, neither strict partial orders nor their linear extensions must have utility representations. Let us discuss connections between these extensions and utility representations in more detail. Recall that
Definition 11  A utility representation of a strict partial order $\succ$ on a set $X$ is a function $u : X \to \mathbb{R}$ such that for every $^2 x, y \in X$,

$$
\begin{align*}
  x \succ y & \Rightarrow u(x) > u(y), \\
  x \approx y & \Rightarrow u(x) = u(y),
\end{align*}
$$

where, by definition,

$$
\begin{align*}
  & x \approx y \iff [\forall z \in X, x \sim z \iff y \sim z], \\
  & x \sim y \iff [x \not\succ y \text{ and } y \not\succ x].
\end{align*}
$$

A sufficient condition for a strict partial order $\succ$ to have a utility representation is the existence of a countable and dense (w.r.t. the induced strict partial order) subset in the factor set $X/\approx$ (see Debreu, 1964; Fishburn, 1970). Generally, this sufficient condition is not necessary, but if $\succ$ is a strict weak order, then it is necessary.

The Paretian $>$ relation on $\mathbb{R}^k$ is a special strict partial order. Any lexicographic order on $\mathbb{R}^k$ is its strict linear extension. Such extensions have no utility representations, whereas the $>$ relation has a wide class of utility representations. $^3$ These are all functions strictly increasing in all coordinates.

Every such a strictly increasing function induces a strict weak order on $\mathbb{R}^k$ that extends $>$. Naturally, not all strict weak orders that extend $>$ can be obtained in this manner. A sufficient condition for such representability is the Archimedean property which ensures the existence of a countable and dense (w.r.t. the strict weak order) subset in $\mathbb{R}^k$.

Thus, the utility representations of $>$ induce a special class of strict weak orders that extend $>$. Such a strict weak order determines its utility representation up to arbitrary monotone transformations (some related specific results are given in Morkeliunas, 1986b).

In the previous sections, we considered a (utility) function $f_P$ that is defined on a subset $P$ of $\mathbb{R}^k$ and represents the restriction of $>$ to $P$. If $P$ is a Pareto subset, then this imposes no constraints on $f_P$. The problem was to find conditions under which there exist functions $f$ that $(\ast)$ reduce to $f_P$ on $P$ and $(\ast\ast)$ represent $>$ on $\mathbb{R}^k$, and to provide an explicit form of such functions.

Observe that every strictly increasing function $f_P$ induces some strict weak

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$^2$ For uniformity, we designate the elements of $X$ by boldface letters.

$^3$ They do exist, say, because $\mathbb{R}^k$ contains countable and dense (w.r.t. $>$) subsets, one of which being the set of vectors with rational coordinates.
order $\succ$ on $P$ that contains the restriction of $>$ to $P$. For $\succ$, there exists a countable and dense (w.r.t. $\succ$) subset in the factor set $P/\sim$, where $\sim$ corresponds to $\succ$. Combining this subset with the set of vectors in $\mathbb{R}^k$ that have rational coordinates, we obtain a countable and dense subset in the factor set $\mathbb{R}^k/\sim$ (the $\approx$ relation that corresponds to $>$ is the identify relation). Consequently, there exist utility functions $g$ that represent $T(> \cup \succ)$ on $\mathbb{R}^k$, where $T(\cdot)$ designates transitive closure.

The restriction of such a function $g$ to $P$ need not coincide with $f_P$, but it is related with $f_P$ by a strictly increasing transformation, since they represent the same weak order $\succ$. Thus, we obtain

**Proposition 12** For every strictly increasing function $f_P$ on $P \subset \mathbb{R}^k$, there exists a strictly increasing map $\varphi$ of the range of $f_P$ to $\mathbb{R}$ such that $\varphi(f_P)$ is monotonically extendible to $\mathbb{R}^k$. If $f_P$ is not strictly increasing, then there are no such maps.

Proposition 12 elucidates a difference between separably increasing and strictly increasing functions $f_P$ with respect to the extendibility. The former functions are monotonically extendible to $\mathbb{R}^k$ (Theorem 5), whereas for the latter functions, only some their strictly increasing transformations are extendible in the general case.

### 5 Extending utility functions on arbitrary sets

A shortcoming of the definition (11) is that it provides discontinuous extensions even for continuous functions $f_P$. On the other hand, the extension technique does not strongly rely on the structure of $\mathbb{R}^k$. Indeed, the connection of $f(x)$ with $\mathbb{R}^k$, its domain, reduces to the dependence on $a(x), b(x)$, and $u(x)$, which have a rather general nature. This enables one to translate the above results to an abstract set $X$ substituted for $\mathbb{R}^k$ and any strict partial order $>$ on $X$ substituted for the Pareto $>$ relation on $\mathbb{R}^k$. The author thanks Andrey Vladimirov for his suggestion to consider the problem in this general setting. To formulate a counterpart of Theorem 5, we will use the following notation:

$X$ is a nonempty set;
$>$ is a fixed strict partial order on $X$;
$P \subset X$; $>_P$ is the restriction of $>$ to $P$;
$z \in X$ is a **maximal element** of $>$ iff $x > z$ for no $x \in X$;
$y \in X$ is a **minimal element** of $>$ iff $x < y$ for no $x \in X$.

We will also explore all the above definitions and formulas (except for (8) and
Definition 2), where the substitution of $X$ for $\mathbb{R}^k$ is implied.

For arbitrary strict partial orders—which may have maximal and minimal elements—a somewhat stronger condition should be used in the definition of separably increasing functions. Let

$$\tilde{X} = X \cup \{-\infty, +\infty\}.$$ 

Extend $>$ to $\tilde{X}$ in the usual way to get a strict partial order on $\tilde{X}$: $+\infty > -\infty$ and $+\infty > x > -\infty$ for every $x \in X$. We use the same symbol $>$ for a strict partial order on $X$, for its extension to $\tilde{X}$, and for the ordinary “greater” relation on $\mathbb{R}$, which should not lead to confusion.

**Definition 13** A function $f_P(x)$ defined on $P \subset X$ is separably increasing if for any $x, x' \in \tilde{X}$, $x' > x$ implies $b(x') > a(x)$.

Since for every $f_P$, $b(+\infty) = +\infty$ and $a(-\infty) = -\infty$, Definition 13 implies that for a separably increasing function, $a(x) < +\infty$ and $b(x) > -\infty$ whenever $x \in X$ is a maximal and a minimal element of $>$, respectively. If $>$ has neither maximal nor minimal elements (as for the Paretian $>$ relation on $\mathbb{R}^k$), then the replacement of $\tilde{X}$ with $X$ in Definition 13 does not alter the class of separably increasing functions.

Obviously, if $>$ has a utility representation, then it has bounded utility representations (which can be constructed, say, by transformations like (8)).

**Theorem 14** Suppose that a strict partial order $>$ defined on $X$ has a utility representation, $P \subset X$, and $f_P: P \to \mathbb{R}$ is a utility representation of $>_P$. Then $f_P$ is monotonically extendible to $X$ if and only if $f_P$ is separably increasing. Moreover, if $u_1(x)$ is a utility representation of $>$ that satisfies (9) and $\alpha < \beta$, then (11) provides a monotone extension of $f_P$ to $X$.

If a function $f_P$ is not separably increasing according to Definition 13 but it satisfies this definition with $\tilde{X}$ replaced by $X$, then (11) can be used to obtain extensions of $f_P$ in terms of “quasiutility” functions $f: X \to \tilde{\mathbb{R}}$.

Propositions 4, 6, 7, and 8 are preserved in the case of arbitrary $X$. Proposition 12, as well as Theorem 14, is valid for the strict partial orders $>$ on $X$ that have a utility representation. The proofs given in Section 6 are conducted for this general case.

Interesting further problems are describing the complete class of extensions of $f_P$ to $\mathbb{R}^k$ (and to $X$), specifying necessary and sufficient conditions for the existence of continuous extensions and constructing them, and considering the extension problem with an ordered extension of the field $\mathbb{R}$ as the range
of \( f_p, u, \) and \( f \). An interesting result on the existence of continuous utility representations for weak orders on \( \mathbb{R}^k \) was obtained in Morkeliūnas (1986a).

6 Proofs

The proofs of all propositions and Corollary 10 are given in the general case of a strict partial order on an abstract set \( X \) (see Section 5, especially, Definition 13). Theorem 14 which generalizes Theorem 5 is proved separately.

Proof of Proposition 4. (a) Assume that \( f_p(x) \) is not strictly increasing. Then there are \( x, x' \in P \) such that \( x' > x \) and \( f_p(x') \leq f_p(x) \). Then, by (4), \( b(x') \leq f_p(x') \leq f_p(x) \leq a(x) \) holds, i.e., \( f_p(x) \) is not separably increasing.

(b) Let \( P' \) be a lower set. Then there exists \( a \in X \) such that \( P' = \{ x \mid x \leq a, x \in P \} \). Consider any \( a' \in \tilde{X} \) such that \( a' > a \). Since \( f_p(x) \) is separably increasing, \( b(a') > a(a) \). Therefore, \( a(a) < +\infty \). Since \( a(a) = \sup \{ f_p(y) \mid y \in P' \} \), \( f_p(x) \) is upper-bounded on \( P' \). Similarly, \( f_p(x) \) is lower-bounded on upper sets.

(c) Let \( x \in X \). By (a), if \( y, z \in P, z \geq x, \) and \( y \leq x \), then \( f_p(z) \geq f_p(y) \). Having in mind that \( \sup \emptyset = -\infty \) and \( \inf \emptyset = +\infty \), we obtain \( b(x) \geq a(x) \).

(d) By (a), for every \( x \in P, b(x) = f_p(x) \) and \( a(x) = f_p(x) \) hold. This completes the proof.

Now we prove Proposition 8; thereafter Proposition 8 will be used to prove Proposition 7 and Theorem 5.

Proof of Proposition 8. Let \( x \in S_1 \). Since \( b(x) - a(x) \leq \beta - \alpha \), we have

\[
\begin{align*}
\min \{ b(x), \beta \} - a(x) & \leq \beta - \alpha, \\
\min \{ b(x), \beta \} - a(x) & \leq \beta - \alpha,
\end{align*}
\]

whence

\[
\begin{align*}
a(x) & \geq \min \{ b(x), \beta \} - \beta + \alpha, \\
b(x) & \leq \min \{ a(x), \alpha \} - \alpha + \beta.
\end{align*}
\]

Therefore, (12) reduces to \( f(x) = a(x)(1 - u_1(x)) + b(x)u_1(x) \).

Let \( x \in S_2 \). Inequalities \( b(x) - a(x) \geq \beta - \alpha \) and \( b(x) \leq \beta \) imply \( a(x) \leq \alpha \), therefore, (11) reduces to \( f(x) = b(x) + u(x) - \beta \).
Let $x \in S_3$. Inequalities $b(x) - a(x) \geq \beta - \alpha$ and $a(x) \geq \alpha$ imply $b(x) \geq \beta$, therefore, (11) reduces to $f(x) = a(x) + u(x) - \alpha$.

The proof for the case $x \in S_4$ is straightforward.

**Proof of Proposition 7.** Let $x \in P$. Then, by item (d) of Proposition 4, $b(x) = a(x) = f_P(x)$, therefore, by (7), $b(x) - a(x) \leq \beta - a$, hence $x \in S_1$. Using Proposition 8, we have $f(x) = f_P(x)(1 - u_1(x)) + f_P(x) u_1(x) = f_P(x)$.

Let $x \in U$. Then $b(x) = +\infty$, hence (12) reduces to $f(x) = \max \{a(x) - \alpha, 0\} + u(x)$. Similarly, if $x \in L$ then $a(x) = -\infty$ and (12) reduces to $f(x) = \min \{b(x) - \beta, 0\} + u(x)$.

Finally, if $x \in N$, then $a(x) = -\infty$ and $b(x) = +\infty$, whence $a(x) < \alpha$ and $b(x) > \beta$, and Proposition 8 provides $f(x) = u(x)$.

**Proof of Theorem 5.** Let $f_P(x)$ be separably increasing. By Proposition 7, the restriction of $f(x)$ to $P$ coincides with $f_P(x)$.

Prove that $f(x)$ is strictly increasing on $\mathbb{R}^k$. This can be demonstrated directly by analyzing equation (11). Here, we give another proof, which does not require any additional calculations with min and max.

By Proposition 8, function (11) coincides with (15), where $u(x)$ is related with $u_1(x)$ by (10).

Suppose that $x, x' \in \mathbb{R}^k$ and $x' > x$. Then, by (3) and the strict monotonicity of $u(x)$ and $u_1(x)$, we have

\[
\begin{align*}
  u(x') &> u(x), \\
  u_1(x') &> u_1(x), \\
  a(x') &\geq a(x), \\
  b(x') &\geq b(x).
\end{align*}
\]

(19)

Suppose first that $x$ and $x'$ belong to the same region: $S_2, S_3,$ or $S_4$. Then (19) yields

\[
\begin{align*}
  b(x') + u(x') - \beta &> b(x) + u(x) - \beta, \\
  a(x') + u(x') - \alpha &> a(x) + u(x) - \alpha, \\
  u(x') &> u(x),
\end{align*}
\]

(20)

hence, by (15), $f(x)$ is strictly increasing on each of these regions.
If \( x, x' \in S_1 \), then by (15), (19), (9), and (c) of Proposition 4,

\[
\begin{align*}
f(x') - f(x) &\geq a(x)(1 - u_1(x')) + b(x)u_1(x') - a(x)(1 - u_1(x)) - b(x)u_1(x) \\
&= (b(x) - a(x))(u_1(x') - u_1(x)) \geq 0.
\end{align*}
\]

This implies that \( f(x') = f(x) \) is possible only if \( b(x') = b(x) \) and \( b(x) = a(x) \), i.e., only if \( b(x') = a(x) \). The last equality is impossible, since \( f_P(x) \) is separably increasing by assumption. Therefore, \( f(x') > f(x) \), and \( f(x) \) is strictly increasing on \( S_1 \).

Let now \( x \) and \( x' \) belong to different regions \( S_i \) and \( S_j \). Consider the points that correspond to \( x \) and \( x' \) in the 3-dimensional space with coordinate axes \( a(\cdot), b(\cdot), \) and \( u(\cdot) \) and connect these two points, \( (a(x), b(x), u(x)) \) and \( (a(x'), b(x'), u(x')) \), by a line segment. The projection of the line segment and the borders of the regions \( S_1, S_2, S_3, \) and \( S_4 \) to the plane \( u = 0 \) are illustrated in Fig. 1.

![Fig. 1. An example of line segment \([(a(x), b(x), u(x)), (a(x'), b(x'), u(x'))]\) in the space with coordinate axes \( a(\cdot), b(\cdot), \) and \( u(\cdot) \) projected to the plane \( u = 0 \).](image)

Suppose that \( (a_1, b_1, u_1), \ldots, (a_p, b_p, u_p), p \leq 3, \) are the points where the line
segment between \((a(x), b(x), u(x))\) and \((a(x'), b(x'), u(x'))\) crosses the borders of the regions. Then
\[
\begin{align*}
  a(x) &\leq a_1 \leq \cdots \leq a_p \leq a(x'), \\
  b(x) &\leq b_1 \leq \cdots \leq b_p \leq b(x'), \\
  u(x) &< u_1 < \cdots < u_p < u(x')
\end{align*}
\] (22) (23)
(24)

with strict inequalities in (22) or in (23) (or in the both).

Consider \(f(x)\) represented by (15) as a function \(\hat{f}(a, b, u)\) of \(a(x), b(x),\) and \(u(x)\). Then, using the fact that \(\hat{f}(a, b, u)\) is nondecreasing in \(a\) and \(b\) on each region, strictly increasing in \(u\) on \(S_2, S_3,\) and \(S_4,\) and strictly increasing in \(u\) on \(S_1\) unless \(a(x) = b(x')\) (which is not the case, since \(f_P(x)\) is separably increasing), we obtain
\[
\begin{align*}
  f(x) &= \hat{f}(a(x), b(x), u(x)) < \hat{f}(a_1, b_1, u_1) < \cdots < \hat{f}(a_p, b_p, u_p) \\
  &= \hat{f}(a(x'), b(x'), u(x')) = f(x').
\end{align*}
\] (25)

This completes the proof that \(f(x)\) is strictly increasing.

It remains to prove that if \(f_P(x)\) is not separably increasing, then it cannot be strictly monotonically extended to \(\mathbb{R}^k.\) Indeed, if there are \(x, x' \in \mathbb{R}^k\) such that \(x' > x\) and \(b(x') \leq a(x),\) then strict monotonicity of \(f(x)\) requires \(f(x') \leq b(x')\) and \(f(x) \geq a(x)\) to hold, whence \(f(x) \geq f(x'),\) and strict monotonicity is violated. Theorem 5 is proved.

**Proof of Corollary 10.** Suppose that a strict partial order \(>\) on \(X\) has a utility representation. Let \(f_P\) be a utility representation of \(>_P,\) where \(>_P\) is the restriction of \(>\) to a Pareto set \(P \subset X.\) Then, since the \(>\) relation is transitive, the set \(A\) is empty. It remains to prove that \(f_P\) is separably increasing if and only if it is upper-bounded on lower sets and lower-bounded on upper sets. If \(f_P\) is separably increasing, then these boundedness conditions are satisfied by Proposition 4. Suppose now that \(f_P\) is upper-bounded on lower sets and lower-bounded on upper sets and assume that \(f_P\) is not separably increasing. Then there exist \(x, x' \in X\) such that \(x' > x\) and \(b(x') \leq a(x).\) This is possible only if \((a)\ b(x') = +\infty\) or \((b)\ a(x) = -\infty\) or \((c)\) there are \(y, z \in P\) such that \(y \leq x\) and \(z \geq x'.\) However, in \((a)\ a(x) = +\infty\) and \(x \in X,\) hence \(f_P\) is not upper-bounded on a lower set; in \((b)\ b(x') = -\infty\) and \(x' \in X,\) hence \(f_P\) is not lower-bounded on an upper set; in \((c)\) by transitivity, \(z > y,\) which contradicts the definition of Pareto set. Therefore, \(f_P\) is separably increasing, and the corollary is proved.
Proof of Theorem 14. If \( f_P \) is not separably increasing, then it is not monotonically extendible to \( X \). Indeed, if the implication \( \mathbf{x}' > \mathbf{x} \Rightarrow b(\mathbf{x}') > a(\mathbf{x}) \) is violated for some \( \mathbf{x}, \mathbf{x}' \in X \), then the argument is the same as for Theorem 5. Otherwise, if this implication is violated for \( \mathbf{x} \in \bar{X} \setminus X \), then \( \mathbf{x}' > \mathbf{x} \) implies \( \mathbf{x} = -\infty \), and since \( b(\mathbf{x}') \leq a(\mathbf{x}) = -\infty \), \( \mathbf{x}' \in X \) holds, hence \( f(\mathbf{x}') \) cannot be evaluated without violation of (17); the case of \( \mathbf{x}' \in \bar{X} \setminus X \) is considered similarly.

The argument in the proof of Theorem 5 works here with no change to demonstrate that if \( > \) has a utility representation and \( f_P \) is separably increasing, then every \( f \) defined by (11) satisfies condition (17) of utility representability. It remains to show that \( f \) satisfies (18). Let \( \mathbf{x}, \mathbf{y} \in X \) and \( \mathbf{x} \approx \mathbf{y} \). Then for every \( \mathbf{z} \in X \), \( [\mathbf{x} \succ \mathbf{z} \iff \mathbf{y} \succ \mathbf{z} \text{ and } \mathbf{z} \succ \mathbf{x} \iff \mathbf{z} \succ \mathbf{y}] \) (see, e.g., Fishburn, 1970). Consequently, \( a(\mathbf{x}) = a(\mathbf{y}) \) and \( b(\mathbf{x}) = b(\mathbf{y}) \). Since \( u \) represents \( > \), \( u(\mathbf{x}) = u(\mathbf{y}) \) holds. Therefore, \( f(\mathbf{x}) = f(\mathbf{y}) \). This completes the proof.

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