SYMMETRIC DECOMPOSITIONS, TRIANGULATIONS AND REAL-ROOTEDNESS

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Abstract. Polynomials which afford nonnegative, real-rooted symmetric decompositions have been investigated recently in algebraic, enumerative and geometric combinatorics. Brändén and Solus have given sufficient conditions under which the image of a polynomial under a certain operator associated to barycentric subdivision has such a decomposition. This paper gives a new proof of their result which generalizes to subdivision operators in the setting of uniform triangulations of simplicial complexes, introduced by the first named author. Sufficient conditions under which these decompositions are also interlacing are described. Applications yield new classes of polynomials in geometric combinatorics which afford nonnegative, real-rooted symmetric decompositions. Some interesting questions in \(f\)-vector theory arise from this work.

1. Introduction

Polynomials with nonnegative coefficients and only real roots arise frequently in mathematics, especially in algebra, combinatorics and geometry \([16, 39]\). Real-rootedness implies strong conditions on the coefficients, such as unimodality and log-concavity (for missing definitions, see Section \([2]\), and provides a powerful method to prove these important properties. Real-rooted polynomials with nonnegative and symmetric coefficients form an especially nice class of polynomials. They have a property which is stronger than unimodality, namely \(\gamma\)-positivity, and their coefficients peak at a predictable position.

Polynomials with nonnegative but not necessarily symmetric coefficients are amenable to techniques suitable for polynomials with symmetric coefficients via their symmetric decompositions. Every polynomial \(p(x) \in \mathbb{R}[x]\) of degree at most \(n\) can be written uniquely in the form \(p(x) = a(x) + xb(x)\) for some polynomials \(a(x), b(x) \in \mathbb{R}[x]\) of degrees at most \(n\) and \(n - 1\), respectively, such that \(a(x) = x^n a(1/x)\) and \(b(x) = x^{n-1} b(1/x)\). One then hopes that these two symmetric polynomials have nice properties and \(p(x)\) is said to have a nonnegative, unimodal, \(\gamma\)-positive or real-rooted symmetric decomposition with respect to \(n\) if both \(a(x)\) and \(b(x)\) have the corresponding property. To motivate this paper better, let us discuss two important examples from geometric combinatorics.

The first example comes from the theory of face enumeration of simplicial complexes. A convenient way to record the face numbers of a simplicial complex \(\Delta\) is the \(h\)-polynomial,
defined by the formula

$$h(\Delta, x) = \sum_{i=0}^{n} f_{i-1}(\Delta) x^i (1-x)^{n-i},$$

where $f_i(\Delta)$ is the number of $i$-dimensional faces of $\Delta$ and $n-1$ is its dimension. The $h$-polynomial has nonnegative coefficients if $\Delta$ is Cohen–Macaulay over some field and, in particular, if $\Delta$ triangulates a ball or sphere [41, Chapter II]. If $\Delta$ triangulates a sphere, then $h(\Delta, x)$ has symmetric coefficients and its unimodality and $\gamma$-positivity have been major topics of research in the past few decades; see [1] [2] [6, Section 3] [16, Section 7.3.2] [31, 39, 41]. Although the $\gamma$-positivity of $h(\Delta, x)$ is conjectured to hold for all flag triangulations $\Delta$ of the sphere [24], no reasonable guess for when $h(\Delta, x)$ is real-rooted exists.

It is an open problem to decide whether this is the case for barycentric subdivisions of boundary complexes of polytopes [19, Question 1], a special class of flag triangulations of the sphere. An affirmative answer to this question has been given for simplicial polytopes in [10] and only very recently for cubical polytopes in [10].

On the other hand, if $\Delta$ triangulates an $(n-1)$-dimensional ball, then $h(\Delta, x)$ will typically not have symmetric coefficients, but has the symmetric decomposition

$$h(\Delta, x) = h(\partial \Delta, x) + (h(\Delta, x) - h(\partial \Delta, x))$$

with respect to $n-1$, where $\partial \Delta$ stands for the boundary complex of $\Delta$. It seems natural to investigate under what conditions this symmetric decomposition has nice properties. As a consequence of [40, Theorem 2.1], (2) is nonnegative provided that no facet of $\Delta$ has all its vertices in $\partial \Delta$. Thus, one expects that (2) has even better behavior when $\partial \Delta$ is a vertex-induced subcomplex of $\Delta$. Indeed, under this assumption, the unimodality of (2) follows for a large family of triangulations $\Delta$ of the ball from [3, Theorem 46] and one can speculate that (2) is also $\gamma$-positive when $\Delta$ is flag. In fact, the latter statement can be shown to be equivalent to the equator conjecture, already posed in [21]. Once again, the real-rootedness of (2) has been much less studied. This can be deduced from part (b) of Theorem [11, Theorem 4.6.2] stated in the sequel, when $\Delta$ is the barycentric subdivision of a simplicial ball (see [9, Section 8]), but should be expected to hold in much more general situations. For instance, it seems natural to ask, in the spirit of [19, Question 1], whether it holds for all barycentric subdivisions of polyhedral balls.

The second example comes from Ehrhart theory. Let $P \subseteq \mathbb{R}^N$ be any $n$-dimensional convex polytope with vertices in $\mathbb{Z}^N$. The Ehrhart polynomial of $P$ [13, Part Two] [42, Section 4.6] is the unique polynomial $\iota(P; x)$ for which $\iota(P; m)$ is equal to the number of elements of $mP \cap \mathbb{Z}^N$ for every $m \in \mathbb{N}$. The function $h^*(P, x)$ defined by the formula

$$\sum_{m \geq 0} \iota(P; m)x^m = \frac{h^*(P, x)}{(1-x)^{n+1}}$$

is a well studied polynomial of degree at most $n$ with nonnegative coefficients, called the $h^*$-polynomial of $P$. Stapledon [43] showed that $h^*(P, x)$ has a nonnegative symmetric decomposition with respect to $n$ whenever $P$ contains a lattice point in its relative interior (and in fact, to the best of our knowledge, the concept of a symmetric decomposition first
appeared in [43]). While the question of unimodality of the $h^*$-polynomial has long been studied [18], its $\gamma$-positivity and real-rootedness have been investigated more recently for several special classes of lattice polytopes [12, 22, 28, 29, 35, 36, 38] and the unimodality and real-rootedness of its symmetric decomposition have been addressed too [17, 30, 37].

Other examples of $\gamma$-positive and real-rooted symmetric decompositions in enumerative combinatorics of geometric flavor can be found in [6, Section 5] [7, 8, 25, 26].

This discussion suggests that polynomials having nonnegative, real-rooted symmetric decompositions arise naturally in combinatorics and are worth of further study, and that developing more techniques to prove this property is desirable. The starting point for this paper is the following theorem. Let us denote by $\mathbb{R}_n[x]$ the space of all polynomials of degree at most $n$, with real coefficients.

**Theorem 1.1.** Let $h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$ be a polynomial with nonnegative coefficients and define $D_n(h(x)) \in \mathbb{R}_n[x]$ by the equation

$$
\sum_{m \geq 0} f(m)x^m = \frac{D_n(h(x))}{(1-x)^{n+1}},
$$

where $f(x) = \sum_{i=0}^{n} c_i x^i (1 + x)^{n-i}$.

(a) ([15, Theorem 4.2]) The polynomial $D_n(h(x))$ has only real roots.

(b) ([17, Theorem 2.13]) If the inequalities

$$
c_0 + c_1 + \cdots + c_i \leq c_n + c_{n-1} + \cdots + c_{n-i}
$$

hold for all $0 \leq i \leq \lfloor n/2 \rfloor$, then $D_n(h(x))$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$.

The map $D_n$ is closely related to the subdivision operator $\mathcal{E} : \mathbb{R}[x] \to \mathbb{R}[x]$; see [15, Section 4] and references therein, [16, Sec 7.3.3] and Example 3.1. It is the unique linear operator $D_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ which has the property that $D_n(h(\Delta, x)) = h(sd(\Delta), x)$ for every $(n-1)$-dimensional simplicial complex $\Delta$, where $sd(\Delta)$ is the barycentric subdivision of $\Delta$. Part (a) of Theorem 1.1 was applied in [19] to prove that $h(sd(\Delta), x)$ has only real roots for every Cohen–Macaulay simplicial complex $\Delta$. Part (b) was applied in [17], among other situations, to prove that $h(sd(\Delta), x)$ and $h^*(Z, x)$ have a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every doubly Cohen–Macaulay simplicial complex $\Delta$ and every $n$-dimensional lattice zonotope $Z$ having an interior lattice point, respectively.

Since Theorem 1.1 is closely related to barycentric subdivision, it is natural to wonder whether there is a similar result which applies to more general types of triangulations. Indeed, part (a) of the theorem was generalized in [9] in the framework of uniform triangulations of simplicial complexes, of which barycentric subdivision is a prototypical example. The operator $D_n$ is replaced there by an operator $D_{\mathcal{F}, n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$ which depends on a triangular array of numbers $\mathcal{F}$ and maps $h(\Delta, x)$ to the $h$-polynomial of a triangulation of $\Delta$ for every $(n-1)$-dimensional simplicial complex $\Delta$, provided that the $f$-vector (prescribed by $\mathcal{F}$) of the restriction of this triangulation to a face of $\Delta$ depends only on the dimension of that face (the collection of $f$-vectors of these restrictions
is precisely the information encoded in $\mathcal{F}$). One of the main results of [9] describes conditions on $\mathcal{F}$ which guarantee that $D_{\mathcal{F},n}(h(x))$ has only real roots for every polynomial $h(x) \in \mathbb{R}_n[x]$ with nonnegative coefficients. This result is more general than Theorem 1.1 (a) and specializes to the latter and to a result of Jochemko [29] on the real-rootedness of the $h^*$-polynomial in the important special cases of barycentric and edgewise subdivisions, respectively. The purpose of this paper is to prove that, somewhat unexpectedly, part (b) too of Theorem 1.1 is valid when the operator $D$ is replaced by $D_{\mathcal{F},n}$ under the same assumptions on $\mathcal{F}$ as those in [9]. This provides a useful tool to address questions about the real-rootedness of symmetric decompositions in geometric combinatorics and shows that uniform triangulations provide a good framework to study this phenomenon as well.

To avoid a longer discussion in this introduction, we postpone the exact formulation of our main result until Section 4 and list some of its consequences instead, to demonstrate its applicability.

The following statement is our first application. The operator $U^n$, defined there, is the operator $D_{\mathcal{F},n}$ associated to the $r$-fold edgewise subdivision. The real-rootedness of symmetric decompositions of polynomials of the form $U^n(h(x))$ was studied in the context of Ehrhart theory in [30]. The following theorem complements the results of [30]; the conclusion of part (b) is shown in [30, Theorem 1.1] under stronger assumptions on $h(x)$ (but for a possibly larger range of values of $r$).

**Theorem 1.2.** Let $h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$ be a polynomial with nonnegative coefficients. Given a positive integer $r$, define $U^n_r(h(x)) \in \mathbb{R}_n[x]$ by the formula

$$U^n_r(h(x)) = \frac{U^n(h(x))}{(1-x)^n} = \sum_{m \geq 0} a_{rm} x^m \quad \text{if} \quad h(x) = \sum_{m \geq 0} a_m x^m.$$

(a) If the inequalities (4) hold for $0 \leq i \leq \lfloor n/2 \rfloor$, then $U^n_r(h(x))$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every $r \geq n$.

(b) If $c_n = 0$ and the inequalities

$$c_0 + c_1 + \cdots + c_i \geq c_{n-1} + c_{n-2} + \cdots + c_{n-i}$$

hold for all $1 \leq i \leq \lfloor n/2 \rfloor$, then $U^n_r(h(x))$ has a nonnegative, real-rooted symmetric decomposition with respect to $n-1$ for every $r \geq n$.

Our second application generalizes Theorem 1.1. Indeed, the operator $D_{n,r}$, defined in the following statement, reduces to $D_n$ for $r = 1$; it coincides with the operator $D_{\mathcal{F},n}$ defined by a generalization of barycentric subdivision, termed as the $r$-colored barycentric subdivision in [9]. Part (a) coincides with [9, Proposition 7.5].

**Theorem 1.3.** Let $h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$ be a polynomial with nonnegative coefficients. Given a positive integer $r$, define $D_{n,r}(h(x)) \in \mathbb{R}_n[x]$ by the equation

$$f(0) + \sum_{m \geq 1} (f(rm) - f(rm - 1)) x^m = \frac{D_{n,r}(h(x))}{(1-x)^n},$$

where $f(x) = \sum_{i=0}^{n} c_i x^i (1+x)^{n-i}$. 

The polynomial $D_{n,r}(h(x))$ has only real roots.
(b) If the inequalities (4) hold for all $0 \leq i \leq [n/2]$, then $D_{n,r}(h(x))$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$.

Applications of Theorems 1.2 and 1.3 to the $h$-polynomials of $r$-fold edgewise subdivisions and $r$-colored barycentric subdivisions of doubly Cohen–Macaulay simplicial complexes and triangulations of balls are given in Section 5. The symmetric decomposition (2), in particular, is shown there to be real-rooted for new classes of triangulations of the ball.

A lattice zonotope is defined as the Minkowski sum of finitely many line segments in $\mathbb{R}^N$, whose vertices lie in $\mathbb{Z}^N$. The following statement can be deduced from Theorem 1.3 (see Section 5). Recall that $\iota(\mathcal{Z}; x)$ stands for the Ehrhart polynomial of a lattice polytope $\mathcal{Z}$ and note that the polynomial $h^*(\mathcal{Z}, x)$, defined by (1), reduces to the $h^*$-polynomial $h^*(\mathcal{Z}, x)$ for $r = 1$. Thus, parts (a) and (b) of the following statement generalize the main results of [12] and [17, Section 4], respectively. In the notation of Theorem 1.2 we have $h^*(\mathcal{Z}, x) = \mathcal{U}^m_r(h^*(\mathcal{Z}, x))$ for every $r \geq 1$.

**Corollary 1.4.** Let $\mathcal{Z}$ be an $n$-dimensional lattice zonotope and for positive integers $r$, define $h^*_r(\mathcal{Z}, x) \in \mathbb{R}_+[x]$ by the equation

\[
1 + \sum_{m \geq 1} (\iota(\mathcal{Z}; rm) - \iota(\mathcal{Z}; rm - 1)) x^m = \frac{h^*_r(\mathcal{Z}, x)}{(1 - x)^n}.
\]

(a) The polynomial $h^*_r(\mathcal{Z}, x)$ has only real roots for every $r \geq 1$.
(b) If $\mathcal{Z}$ has a lattice point in its relative interior, then $h^*_r(\mathcal{Z}, x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every $r \geq 1$.

Our final application identifies a class of doubly Cohen–Macaulay simplicial complexes, namely that of one-coskeleta of Cohen–Macaulay simplicial complexes, whose uniform triangulations have $h$-polynomials with especially nice symmetric decompositions. This result, stated here for edgewise and $r$-colored barycentric subdivisions, is new even for barycentric subdivisions. We recall that a nonnegative, real-rooted symmetric decomposition $p(x) = a(x) + xb(x)$ is said to be interlacing if $a(x)$ is interlaced by $b(x)$ (see [17, Theorem 2.6] for a number of equivalent conditions).

**Theorem 1.5.** Let $\Gamma$ be any $n$-dimensional simplicial complex with nonnegative $h$-vector and let $\Delta$ be the $(n-1)$-dimensional skeleton of $\Gamma$.

(a) The polynomial $\mathcal{U}^m_r(h(\Delta, x))$ has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to $n$ for every $r \geq n$. Moreover, $\mathcal{U}^m_r(h(\Delta, x))$ interlaces $\mathcal{U}^{m+1}_r(h(\Gamma, x))$ for every $r \geq n + 1$.
(b) The polynomial $D_{n,r}(h(\Delta, x))$ has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to $n$ for every $r \geq 1$. Moreover, $D_{n,r}(h(\Delta, x))$ interlaces $D_{n+1,r}(h(\Gamma, x))$.

The proof of Theorem 1.1 given in [17], uses a lot of technical properties of the subdivision operators $D_n$ and $E$. The proof of our more general theorem, given in Section 4,
involves no such technicalities and essentially uses only the universal recurrence for the polynomials $D_{F,n}(x^k)$ [9, Lemma 6.3] and basic facts about real-rooted polynomials.

The structure of the remainder of this paper is as follows. Section 2 fixes notation and recalls useful definitions and facts about simplicial complexes and real-rooted polynomials. Section 3 reviews the basics of the enumerative theory of uniform triangulations [9]. The main result (Theorem 4.1) of this paper is stated and proven in Section 4. The sufficient conditions provided for the real-rooted symmetric decompositions, considered there, to be interlacing lead to new inequalities that the $h$-vector of a Cohen–Macaulay simplicial complex may or may not satisfy (see Corollary 4.3) and raise questions in $f$-vector theory which are of independent interest (see Section 7). Theorems 1.2 and 1.3 are deduced from Theorem 4.1 in Section 5 and some of their own consequences (see Corollaries 5.1 and 5.4) are discussed there. For the proof of Theorem 1.3, one needs to verify that the operator $D_{n,r}$ satisfies the crucial conditions of [9, Theorem 6.1], a problem that was left open in [9, Section 7]. This nontrivial fact requires special treatment and is proven in Section 5.2. Theorem 1.5 is stated more generally, in the setting of uniform triangulations, and proven in Section 6. Section 7 concludes with remarks and questions that are raised by this work.

2. Preliminaries

This section fixes notation and explains background and terminology on real polynomials, simplicial complexes and their triangulations which will be useful in the sequel.

2.1. Polynomials. We recall that $\mathbb{R}_n[x]$ stands for the space of polynomials of degree at most $n$ with real coefficients. A polynomial $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{R}_n[x]$ is called

- **symmetric**, with center of symmetry $n/2$, if $a_i = a_{n-i}$ for all $0 \leq i \leq n$,
- **unimodal**, with a peak at position $k$, if $a_0 \leq a_1 \leq \cdots \leq a_k \geq a_{k+1} \geq \cdots \geq a_n$,
- **$\gamma$-positive**, with center of symmetry $n/2$, if $p(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$ for some nonnegative reals $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$,
- **log-concave**, if $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i < n$,
- **real-rooted**, if every root of $p(x)$ is real, or $p(x) = 0$.

Every $\gamma$-positive polynomial is symmetric and unimodal and every real-rooted and symmetric polynomial with nonnegative coefficients is $\gamma$-positive; see [6, 16, 39] for more information on the connections among these concepts.

A real-rooted polynomial $p(x)$, with roots $\alpha_1 \geq \alpha_2 \geq \cdots$, is said to **interlace** a real-rooted polynomial $q(x)$, with roots $\beta_1 \geq \beta_2 \geq \cdots$, if

$$\cdots \leq \alpha_2 \leq \beta_2 \leq \alpha_1 \leq \beta_1.$$

By convention, the zero polynomial interlaces and is interlaced by every real-rooted polynomial. A sequence $(p_0(x), p_1(x), \ldots, p_m(x))$ of real-rooted polynomials with nonnegative coefficients is called **interlacing** if $p_i(x)$ interlaces $p_j(x)$ for $0 \leq i < j \leq m$. The importance of this concept for us comes from the fact that every nonnegative linear combination $p(x)$ of $p_0(x), p_1(x), \ldots, p_m(x)$ is then real-rooted; moreover, $p(x)$ interlaces $p_m(x)$ and is interlaced by $p_0(x)$.
A standard way to produce interlacing sequences in combinatorics is the following. Suppose that \( p_0(x), p_1(x), \ldots, p_m(x) \) are real-rooted polynomials with nonnegative coefficients and set

\[
q_k(x) = x \sum_{i=0}^{k-1} p_i(x) + \sum_{i=k}^{m} p_i(x)
\]

for \( k \in \{0, 1, \ldots, m + 1\} \). Then, if the sequence \( (p_0(x), p_1(x), \ldots, p_m(x)) \) is interlacing, so is \( (q_0(x), q_1(x), \ldots, q_{m+1}(x)) \); see [16, Corollary 8.7] for a more general statement. For an extensive treatment of real-rooted polynomials and the theory of interlacing, see [23].

The \( k \)th Veronese \( r \)-section operator is defined on polynomials, or formal power series, by the formula

\[
S^r_k \left( \sum_{n \geq 0} a_n x^n \right) = \sum_{n \geq 0} a_{rn+k} x^n.
\]

We note that

\[
S^r_j(x^i f(x)) = \begin{cases} 
S^r_{j-i}(f(x)), & \text{if } i \leq j \\
x S^r_{r-i+j}(f(x)), & \text{if } i > j
\end{cases}
\]

for \( i, j \in \{0, 1, \ldots, r-1\} \).

2.2. Simplicial complexes. We assume familiarity with basic notions from algebraic, enumerative and topological combinatorics on simplicial complexes; excellent resources on these topics are [14, 27, 41]. All simplicial complexes considered here will be abstract and finite. Following [9], we denote by \( \sigma_n \) the abstract simplex \( 2^V \) on an \( n \)-element vertex set \( V \).

For the remainder of this section, \( \Delta \) will be an \( (n-1) \)-dimensional simplicial complex. The sequence \( h(\Delta) := (h_0(\Delta), h_1(\Delta), \ldots, h_n(\Delta)) \) of coefficients of the \( h \)-polynomial \( h(\Delta, x) = \sum_{i=0}^{n} h_i(\Delta) x^i \), already defined in the introduction by Equation (1), is called the \( h \)-vector of \( \Delta \). As mentioned there, \( h(\Delta) \) has nonnegative entries whenever \( \Delta \) is Cohen–Macaulay (over some field). We note that \( h_0(\Delta) = 1 \) and \( h_n(\Delta) = (-1)^{n-1} \tilde{\chi}(\Delta) \), where \( \tilde{\chi}(\Delta) \) is the reduced Euler characteristic of \( \Delta \); in particular, \( h_n(\Delta) = 0 \) if the geometric realization of \( \Delta \) is contractible. We will be interested in simplicial complexes which satisfy the inequalities

\[
h_0(\Delta) + h_1(\Delta) + \cdots + h_i(\Delta) \leq h_{n}(\Delta) + h_{n-1}(\Delta) + \cdots + h_{n-i}(\Delta)
\]

for \( 0 \leq i \leq \lfloor n/2 \rfloor \) (equivalently, for \( 0 \leq i \leq n \)). Doubly Cohen–Macaulay simplicial complexes [11, Section III.3] have this property. A larger family of simplicial complexes which satisfy inequalities [9] was introduced and studied in [33] under the name uniformly Cohen–Macaulay simplicial complexes. Here we will use the term Cohen–Macaulay* simplicial complex instead, to avoid confusion with our terminology “uniform triangulation”. Thus, \( \Delta \) is Cohen–Macaulay* if \( \Delta \) and the simplicial complexes obtained from it by removing any (single) facet of \( \Delta \) are Cohen–Macaulay of dimension \( n-1 \). Every doubly
Cohen–Macaulay simplicial complex is Cohen–Macaulay* (see [33, Proposition 2.8]) and every Cohen–Macaulay* simplicial complex satisfies [9] for all \( i \).

By the term triangulation of \( \Delta \) we will always mean a geometric triangulation. Thus, a simplicial complex \( \Delta' \) is a triangulation of \( \Delta \) if there exists a geometric realization of \( \Delta' \) which geometrically subdivides one for \( \Delta \).

Barycentric and edgewise subdivisions are important triangulations of \( \Delta \). The barycentric subdivision of \( \Delta \), denoted by \( \text{sd}(\Delta) \), is defined as the simplicial complex of all chains of nonempty faces of \( \Delta \). The edgewise subdivision depends on a positive integer \( r \) and a linear ordering of the vertex set \( V(\Delta) \) of \( \Delta \) (although its face numbers are independent of the latter). Given such an ordering \( v_1, v_2, \ldots, v_m \), we denote by \( V_r(\Delta) \) the set of maps \( f : V(\Delta) \to \mathbb{N} \) such that \( \text{supp}(f) \in \Delta \) and \( f(v_1) + f(v_2) + \cdots + f(v_m) = r \), where \( \text{supp}(f) \) is the set of all \( v \in V(\Delta) \) for which \( f(v) \neq 0 \). For \( f \in V_r(\Delta) \), we let \( \iota(f) : V(\Delta) \to \mathbb{N} \) be the map defined by setting \( \iota(f)(v_j) = f(v_1) + f(v_2) + \cdots + f(v_j) \) for \( j \in \{1, 2, \ldots, m\} \). The \( r \)-fold edgewise subdivision of \( \Delta \), denoted by \( \text{esd}_r(\Delta) \), is the simplicial complex on the vertex set \( V_r(\Delta) \) of which a set \( E \subseteq V_r(\Delta) \) is a face if the following two conditions are satisfied:

- \( E \cap F(\Delta) = \emptyset \) and
- \( \chi(f) - \chi(g) \in \{0, 1\}^{V(\Delta)} \), for all \( f, g \in E \).

The simplicial complexes \( \text{sd}(\Delta) \) and \( \text{esd}_r(\Delta) \) can be realized as triangulations of \( \Delta \). This is well known for the former; for the latter, see [8, Section 5] and references therein.

A simplicial complex \( \Delta \) is called flag if every clique in the one-skeleton of \( \Delta \) is a face of \( \Delta \); see [3, Section 3] [31, Section 5.2] [41, Section III.4] for information about this very interesting class of simplicial complexes.

3. Uniform triangulations and subdivision operators

This section summarizes the background on uniform triangulations of simplicial complexes and their associated subdivision operators [9] which are necessary in order to state and prove our main results.

An \( f \)-triangle of size \( d \in \mathbb{N} \cup \{\infty\} \) is simply a triangular array \( \mathcal{F} = (f_{\mathcal{F}}(i, j))_{0 \leq i \leq j \leq d} \) of nonnegative integers (where \( i, j \) are finite numbers). A triangulation \( \Delta' \) of a simplicial complex \( \Delta \) of dimension less than \( d \) is called \( \mathcal{F} \)-uniform if for all \( 0 \leq i \leq j \leq d \), the restriction of \( \Delta' \) to any face of \( \Delta \) of dimension \( j - 1 \) has exactly \( f_{\mathcal{F}}(i, j) \) faces of dimension \( i - 1 \). We say that \( \mathcal{F} \) is feasible if every simplex of dimension less than \( d \) has an \( \mathcal{F} \)-uniform triangulation. The barycentric subdivision \( \text{sd}(\Delta) \) and the \( r \)-fold edgewise subdivision \( \text{esd}_r(\Delta) \) are prototypical examples of uniform triangulations of \( \Delta \).

For every \( f \)-triangle \( \mathcal{F} \) of size \( d \), there exist linear operators

\[
\mathcal{E}_{\mathcal{F}} : \mathbb{R}_d[x] \to \mathbb{R}_d[x], \quad \mathcal{D}_{\mathcal{F}, n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x], \text{ for } n \in \{0, 1, \ldots, d\} \setminus \{\infty\}
\]

such that \( f(\Delta', x) = \mathcal{E}_{\mathcal{F}}(f(\Delta, x)) \) and \( h(\Delta', x) = \mathcal{D}_{\mathcal{F}, n}(h(\Delta, x)) \) for every simplicial complex \( \Delta \) of dimension \( n - 1 \), every \( \mathcal{F} \)-uniform triangulation \( \Delta' \) of \( \Delta \) and all finite \( n \leq d \).
Thus, setting \( p_{F,n,k}(x) := D_{F,n}(x^k) \) for \( k \in \{0,1,\ldots,n\} \), we have

\[
D_{F,n}(h(x)) = \sum_{k=0}^{n} c_k p_{F,n,k}(x)
\]

for every polynomial \( h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}[x] \) and

\[
h(\Delta', x) = \sum_{k=0}^{n} h_k(\Delta)p_{F,n,k}(x)
\]

for every \( F \)-uniform triangulation \( \Delta' \) of an \((n - 1)\)-dimensional simplicial complex \( \Delta \).

Following the notation of \([9]\), we write \( h_F(\Delta, x) \) for the right-hand side of Equation (11), so that \( h(\Delta', x) = h_F(\Delta, x) \) for every \( F \)-uniform triangulation \( \Delta' \) of \( \Delta \).

**Example 3.1.** By \([9\text{ Section 5}]\) we have \( E_F(x^n) = \sum_{k=0}^{n} f^*_F(k,n)x^k \) for every \( n \leq d \), where \( f^*_F(k,n) \) is the number of interior \((k - 1)\)-dimensional faces of any \( F \)-uniform triangulation of the simplex \( \sigma_n \).

In particular, for the \( f \)-triangle associated to barycentric subdivision we have \( E_F(x^n) = \sum_{k=0}^{n} k! S(n,k)x^k \) for every \( n \in \mathbb{N} \), where \( S(n,k) \) are the Stirling numbers of the second kind. Thus, \( E_F : \mathbb{R}[x] \to \mathbb{R}[x] \) coincides with the subdivision operator \( E : \mathbb{R}[x] \to \mathbb{R}[x] \) of \([16\text{ Section 7.3.3}](16)\), mentioned in the introduction, and \( D_{F,n} : \mathbb{R}_n[x] \to \mathbb{R}_n[x] \) coincides (see, for instance, \([17\text{ Lemma 2.7}](17)\)) with the operator \( D_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x] \) of Theorem \([11]\).

The polynomial \( p_{F,n,k}(x) \) was shown to have nonnegative coefficients \([9\text{ Theorem 4.1}](9)\) for every \( k \in \{0,1,\ldots,n\} \) and every feasible \( f \)-triangle \( F \) of size at least \( n \). Following the notation of \([9\text{ [Section 5]}](9)\), we also set

\[
p_{F,n-1,n}(x) = \theta_F(\sigma_n, x) := h_F(\sigma_n, x) - h_F(\partial \sigma_n, x)
\]

and we consider the sequences

\[
P_{F,n} := (p_{F,n-1,0}(x), p_{F,n-1,1}(x), \ldots, p_{F,n-1,n}(x))
\]

\[
Q_{F,n} := (p_{F,n,0}(x), p_{F,n,1}(x), \ldots, p_{F,n,n}(x)).
\]

The polynomial \( \theta_F(\sigma_n, x) \) does not always have nonnegative coefficients. This is the case under some mild assumptions on the triangulation which defines \( F \); see \([9\text{ Remark 6.1 (b)}](9)\) or our discussion in the introduction. Let us introduce the following useful terminology.

**Definition 3.2.** We say that a feasible \( f \)-triangle \( F \) of size at least \( n \) has the interlacing property with respect to \( n \) if \( Q_{F,m} \) is an interlacing sequence for every \( m \in \{0,1,\ldots,n\} \), and that \( F \) has the strong interlacing property with respect to \( n \), if the following conditions hold:

(i) \( h_F(\sigma_m, x) \) is a real-rooted polynomial for all \( 2 \leq m < n \).

(ii) \( \theta_F(\sigma_m, x) \) is either identically zero, or a real-rooted polynomial of degree \( m - 1 \) with nonnegative coefficients which is interlaced by \( h_F(\sigma_{m-1}, x) \), for all \( 2 \leq m \leq n \).

We also say that a feasible \( f \)-triangle of infinite size has the (strong) interlacing property, if it does so with respect to every \( n \in \mathbb{N} \).
The proof of [9, Theorem 6.1] shows that if $\mathcal{F}$ has the strong interlacing property with respect to $n$, then $Q_{\mathcal{F},n}$ and $P_{\mathcal{F},m}$ for $m \leq n$ are interlacing sequences (that hopefully explains our terminology). Thus, given also that $p_{\mathcal{F},n,0}(x) = h_{\mathcal{F}}(\sigma_n, x)$ and $p_{\mathcal{F},n,n}(x) = x^n h_{\mathcal{F}}(\sigma_n, 1/x)$ (see our discussion in the sequel), the following statement is included in the results of [9].

**Theorem 3.3.** ([9]) Let $\mathcal{F}$ be any feasible $f$-triangle of size at least $n$ which has the strong interlacing property with respect to $n$. Then, $\mathcal{F}$ has the interlacing property with respect to $n$. In particular:

- $D_{\mathcal{F},n}(h(x))$ is real-rooted, is interlaced by $h_{\mathcal{F}}(\sigma_n, x)$ and it interlaces $x^n h_{\mathcal{F}}(\sigma_n, 1/x)$ for every polynomial $h(x) \in \mathbb{R}_n[x]$ with nonnegative coefficients.
- $h_{\mathcal{F}}(\Delta, x)$ is real-rooted, is interlaced by $h_{\mathcal{F}}(\sigma_n, x)$ and it interlaces $x^n h_{\mathcal{F}}(\sigma_n, 1/x)$ for every $(n - 1)$-dimensional simplicial complex $\Delta$ with nonnegative $h$-vector.

The crucial strong interlacing property is especially easy to verify for the barycentric subdivision [9, Example 7.1], since then $\theta_{\mathcal{F}}(\sigma_n, x) = 0$ for every $n \in \mathbb{N}$. It was also verified for the $r$-fold edgewise subdivision when $r \geq n$ and for certain triangulations interpolating between barycentric and edgewise subdivisions [9, Section 7]. Moreover, it was conjectured to hold for the antiprism triangulation [11, Section 5], in which case only the claim about interlacing in condition (ii) of Definition 3.2 is open. We will also verify the strong interlacing property for the $r$-colored barycentric subdivision for every positive integer $r$ in Section 5.2 and will deduce from that and Theorem 4.1 many of the results stated in the introduction.

The following proposition collects some useful properties of the polynomials $p_{\mathcal{F},n,k}(x)$.

**Proposition 3.4.** ([9]) For every feasible $f$-triangle $\mathcal{F}$ of size at least $n$ we have:

1. $x^n p_{\mathcal{F},n,k}(1/x) = p_{\mathcal{F},n,n-k}(x)$, for $k \in \{0, 1, \ldots, n\}$
2. $x^n p_{\mathcal{F},n-1,n}(1/x) = p_{\mathcal{F},n-1,n}(x)$
3. $p_{\mathcal{F},n,k}(x) = x \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x) + \sum_{i=k}^{n} p_{\mathcal{F},n-1,i}(x)$, for $k \in \{0, 1, \ldots, n\}$.

As special cases of Equation (11), and applying the recurrence (14) for $k = 0$, we also have

$h_{\mathcal{F}}(\sigma_n, x) = p_{\mathcal{F},n,0}(x) = \sum_{k=0}^{n} p_{\mathcal{F},n-1,k}(x)$

$h_{\mathcal{F}}(\partial \sigma_n, x) = \sum_{k=0}^{n-1} p_{\mathcal{F},n-1,k}(x)$

The following lemma will be useful in Section 4.

**Lemma 3.5.** Let $h(x) \in \mathbb{R}_{n-1}[x]$. 

(a) $\mathcal{D}_{F,n}(x^a h(1/x)) = x^n a(x)$ for all $a(x) \in \mathbb{R}_n[x]$.

(b) The symmetric decomposition of $\mathcal{D}_{F,n}(h(x))$ with respect to $n-1$ is nonnegative and real-rooted (respectively, nonnegative, real-rooted and interlacing), if and only if so is the symmetric decomposition of $\mathcal{D}_{F,n}(x^a h(1/x))$ with respect to $n$.

Proof. Part (a) follows from Equation (10) and the symmetry property (12). It also implies that for the symmetric decompositions $\mathcal{D}_{F,n}(h(x)) = a(x) + x b(x)$ and $\mathcal{D}_{F,n}(x^a h(1/x)) = a(x) + x b(x)$ of $\mathcal{D}_{F,n}(h(x))$ and $\mathcal{D}_{F,n}(x^a h(1/x))$ with respect to $n-1$ and $n$, respectively, one has $\tilde{a}(x) = x^{n-1} b(1/x)$ and $\tilde{b}(x) = x^{n-1} a(1/x)$. Part (b) follows from these facts.

4. The main theorem

This section states and proves the main results of this paper, using only the theory of Section 3 and basic facts about real-rooted polynomials. Throughout it, $\mathcal{F}$ stands for a feasible $f$-triangle of size at least $n$.

**Theorem 4.1.** Let $\mathcal{F}$ be a feasible $f$-triangle which has the strong interlacing property with respect to $n$. Let $h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$ be a polynomial with nonnegative coefficients.

(a) If the inequalities (7) hold for $0 \leq i \leq \lfloor n/2 \rfloor$, then $\mathcal{D}_{F,n}(h(x))$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$.

If, additionally, $c_i c_{n-i} - c_{i+1} c_{n-i}$ for all $0 \leq i \leq n-1$, then this decomposition is also interlacing.

(b) If $c_n = 0$ and the inequalities (5) hold for $1 \leq i \leq \lfloor n/2 \rfloor$, then $\mathcal{D}_{F,n}(h(x))$ has a nonnegative, real-rooted symmetric decomposition with respect to $n-1$.

If, additionally, $c_i c_{n-i-1} - c_{i+1} c_{n-i}$ for all $1 \leq i < n-1$, then this decomposition is also interlacing.

The proof is based on the following lemma.

**Lemma 4.2.** For every $h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x]$, the symmetric decomposition $\mathcal{D}_{F,n}(h(x)) = a(x) + x b(x)$ of $\mathcal{D}_{F,n}(h(x))$ with respect to $n$ is given by

\begin{align}
 a(x) &= (c_0 + c_1 + \cdots + c_n) p_{\mathcal{F},n-1,n}(x) + \\
 &\quad + \sum_{i=0}^{n-1} (c_0 + c_1 + \cdots + c_i + (c_0 + c_1 + \cdots + c_{n-i-1}) x) p_{\mathcal{F},n-1,i}(x) \\
 b(x) &= \sum_{i=0}^{n-1} (c_n + c_{n-1} + \cdots + c_{n-i} - c_0 - c_1 - \cdots - c_i) p_{\mathcal{F},n-1,i}(x).
\end{align}

Proof. Let $a(x)$ and $b(x)$ be defined by (16) and (17), respectively. Properties (12) and (13) of the $p_{\mathcal{F},n-1,k}(x)$ directly imply that $x^n a(1/x) = a(x)$ and $x^{n-1} b(1/x) = b(x)$. Moreover, using recurrence (14), we get

$$
\mathcal{D}_{F,n}(h(x)) = \sum_{k=0}^{n} c_k p_{\mathcal{F},n,k}(x) = \sum_{k=0}^{n} c_k \left( x \sum_{i=0}^{k-1} p_{\mathcal{F},n-1,i}(x) + \sum_{i=k}^{n} p_{\mathcal{F},n-1,i}(x) \right).
$$
Changing the order of summation gives
\[ D_{F,n}(h(x)) = \sum_{i=0}^{n} \left( \sum_{k>i} c_k x + \sum_{k=0}^{i} c_k \right) p_{F,n-i}(x) = (c_0 + c_1 + \cdots + c_n) p_{F,n-n}(x) + \sum_{i=0}^{n-1} (c_0 + c_1 + \cdots + c_i + c_{i+1} x + \cdots + c_n x) p_{F,n-i}(x) = a(x) + x b(x) \]
and the proof follows. \( \square \)

Proof of Theorem 4.1 Because of Lemma 3.5, part (b) follows by applying part (a) to \( x^n h(1/x) \). We now prove part (a).

For the first statement, we only need to show that the polynomials \( a(x) \) and \( b(x) \), defined by Equations (16) and (17), are real-rooted. This is clear for \( b(x) \), since it is a nonnegative linear combination of the elements of the interlacing sequence \( Q_{F,n-1} \). By definition, we also have \( a(x) = \sum_{i=0}^{n} \lambda_i(x)p_{F,n-i}(x) \) for some polynomials \( \lambda_i(x) \) of degree at most one which have nonnegative coefficients and appear explicitly in (16). The nonnegativity of the \( c_i \) easily implies that \( (\lambda_n(x), \lambda_{n-1}(x), \ldots, \lambda_0(x)) \) is an interlacing sequence. This observation, the fact (pointed out before Theorem 3.3) that \( \mathcal{P}_{F,n} \) is also interlacing and [16, Lemma 7.8.3] imply that \( a(x) \) is real-rooted as well.

For the second statement, let us write \( h_F(x) = D_{F,n}(h(x)) \). By [17, Theorem 2.6], to prove that the real-rooted symmetric decomposition of part (a) is interlacing, it suffices to show that \( h_F(x) \) is interlaced by \( x^n h_F(1/x) \). For the latter, by [16, Lemma 7.8.4], it suffices to show that \( (\lambda x + \mu) x^n h_F(1/x) + h_F(x) \) is real-rooted for all positive reals \( \lambda, \mu \). Since
\[ x^n h_F(1/x) = \sum_{k=0}^{n} c_k x^n p_{F,n,k}(1/x) = \sum_{k=0}^{n} c_k p_{F,n,n-k}(x), \]
we have
\[ (\lambda x + \mu) x^n h_F(1/x) + h_F(x) = (\lambda x + \mu) \sum_{k=0}^{n} c_k p_{F,n,n-k}(x) + \sum_{k=0}^{n} c_k p_{F,n,k}(x) \]
\[ = \sum_{k=0}^{n} \mu_k(x) p_{F,n,k}(x), \]
where \( \mu_k(x) = (c_k + c_{n-k} \mu) + c_{n-k} \lambda x \) for every \( k \in \{0, 1, \ldots, n\} \). Once again, it is routine to verify that the sequence \( (\mu_n(x), \mu_{n-1}(x), \ldots, \mu_0(x)) \) is interlacing if \( c_i c_{n-i-1} \leq c_{i+1} c_{n-i} \) for every \( i \in \{0, 1, \ldots, n-1\} \). Another application of [16, Lemma 7.8.3] then shows that \( \sum_{k=0}^{n} \mu_k(x) p_{F,n,k}(x) \) is real-rooted and the proof follows. \( \square \)

The following corollary produces large families of polynomials in face enumeration which admit nonnegative, real-rooted symmetric decompositions.

Corollary 4.3. Let \( F \) be a feasible \( f \)-triangle which has the strong interlacing property with respect to \( n \).
(a) The polynomial $h_F(\Delta, x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every $(n-1)$-dimensional Cohen–Macaulay* simplicial complex $\Delta$. If, additionally, $\Delta$ satisfies the inequalities

$$\frac{h_0(\Delta)}{h_n(\Delta)} \leq \frac{h_1(\Delta)}{h_{n-1}(\Delta)} \leq \cdots \leq \frac{h_{n-1}(\Delta)}{h_1(\Delta)} \leq \frac{h_n(\Delta)}{h_0(\Delta)},$$

then this decomposition is also interlacing.

(b) The polynomial $h_F(\Delta, x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n-1$ for every triangulation $\Delta$ of the $(n-1)$-dimensional ball. If, additionally, $\Delta$ satisfies the inequalities

$$\frac{h_1(\Delta)}{h_{n-1}(\Delta)} \geq \frac{h_2(\Delta)}{h_{n-2}(\Delta)} \geq \cdots \geq \frac{h_{n-2}(\Delta)}{h_2(\Delta)} \geq \frac{h_{n-1}(\Delta)}{h_1(\Delta)}$$

(where terms involving an entry $h_i(\Delta) = 0$ may be ignored), then this decomposition is also interlacing.

Proof. This follows directly from Theorem 4.1, the fact that Cohen–Macaulay* simplicial complexes satisfy (9) and the fact (a consequence of [40, Lemma 2.3]) that triangulations $\Delta$ of the $(n-1)$-dimensional ball satisfy the inequalities

$$h_0(\Delta) + h_1(\Delta) + \cdots + h_i(\Delta) \geq h_{n-1}(\Delta) + h_{n-2}(\Delta) + \cdots + h_{n-i}(\Delta)$$

for all $1 \leq i \leq \lfloor n/2 \rfloor$. □

Let us record one situation in which the assumptions of Theorem 4.1 on $h(x)$ are valid trivially.

Corollary 4.4. Let $F$ be a feasible $f$-triangle which has the strong interlacing property with respect to $n$. For every $h(x) \in \mathbb{R}[x]$ of degree at most $n/2$ with nonnegative coefficients, $D_{F,n}(h(x))$ has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to $n - 1$.

In particular, $h_F(\Delta, x)$ has such a decomposition for every $(n-1)$-dimensional simplicial complex $\Delta$ with nonnegative $h$-vector which satisfies $h_i(\Delta) = 0$ for $i \geq (n+1)/2$.

Proof. This follows from part (b) of Theorem 4.1 since, under our assumptions, (5) holds trivially for $1 \leq i \leq \lfloor n/2 \rfloor$ and $c_{i+1}c_{n-i} = 0$ for every $i$. □

5. Applications

This section applies Theorem 4.1 to the $r$-fold edgewise and $r$-colored barycentric subdivision and proves Theorems 1.2 and 1.3 and Corollary 1.4. We denote by $F$ and $F_{resd}$ the $f$-triangles (of infinite size) defined by the barycentric and the $r$-fold edgewise subdivision, respectively.
5.1. The $r$-fold edgewise subdivision operator. Recall from Section 2 that $\mathcal{S}_k$ stands for the $k$th Veronese $r$-section operator. For the $r$-fold edgewise subdivision one has that $\mathcal{D}_{r\text{esd}_r,n} = \mathcal{U}_r^n : \mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$, where

$$\mathcal{U}_r^n(h(x)) = \mathcal{S}_r^n \left((1 + x + x^2 + \cdots + x^{r-1})^n h(x)\right)$$

for $h(x) \in \mathbb{R}_n[x]$; see, for instance, [20, Section 4] [5, Section 4] [9, Section 3]. Equivalently,

$$\frac{\mathcal{U}_r^n(h(x))}{(1 - x)^n} = \frac{1}{(1 - x)^n} \cdot \mathcal{S}_0^n \left((1 + x + x^2 + \cdots + x^{r-1})^n h(x)\right) = \mathcal{S}_r^n \left(\frac{h(x)}{(1 - x)^n}\right),$$

which shows that $\mathcal{U}_r^n$ coincides with the operator which appears in the statement of Theorem 1.2 under the same name.

**Proof of Theorem 1.2.** This follows directly from Theorem 4.1 and the fact [9, Example 7.2] that the $f$-triangle of the $r$-fold edgewise subdivision has the strong interlacing property with respect to $n$ for every $r \geq n$. □

The conclusion of part (b) of Theorem 1.2 was proven in [30] under stronger assumptions (see [30, Theorem 1.1]) which, for example, do not cover part (b) of the following corollary.

**Corollary 5.1.** (a) The polynomial $h(\text{esd}_r(\Delta), x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every $r \geq n$ and every $(n-1)$-dimensional Cohen–Macaulay* simplicial complex $\Delta$.

(b) The polynomial $h(\text{esd}_r(\Delta), x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n-1$ for every $r \geq n$ and every triangulation $\Delta$ of the ball of dimension $n-1$.

**Proof.** Apply Theorem 1.2 to $h(\Delta, x)$ or, alternatively, Corollary 4.3 to the $r$-fold edgewise subdivision. □

The following statement improves [30, Proposition 5.2].

**Corollary 5.2.** The polynomial $\mathcal{U}_r^n(h(x))$ has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to $n-1$ for every polynomial $h(x) \in \mathbb{R}[x]$ of degree at most $n/2$ with nonnegative coefficients and every $r \geq n$.

**Proof.** This follows from Corollary 4.3 and the fact that the $f$-triangle of the $r$-fold edgewise subdivision has the strong interlacing property with respect to $n$ for every $r \geq n$. □

5.2. The $r$-colored barycentric subdivision operator. Consider the composition of linear operators $\mathcal{D}_{n,r} := \mathcal{D}_{r\text{esd}_r,n} \circ \mathcal{D}_{r,n} = \mathcal{U}_r^n \circ \mathcal{D}_n : \mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$. Thus,

$$\mathcal{D}_{n,r}(h(x)) = \mathcal{U}_r^n(\mathcal{D}_n(h(x))) = \mathcal{S}_r^n \left((1 + x + x^2 + \cdots + x^{r-1})^n \mathcal{D}_n(h(x))\right)$$

for every $h(x) \in \mathbb{R}_n[x]$. We confirm in the proof of Theorem 1.3 given in the sequel, that $\mathcal{D}_{n,r}$ coincides with the operator which appears in the statement of this theorem, under the same name.
Clearly, we have \( D_{n,r} = D_{\mathcal{F}_r,n} \), where \( \mathcal{F}_r \) is the \( f \)-triangle of the uniform triangulation obtained by first taking the barycentric subdivision of a simplicial complex \( \Delta \) and then the \( r \)-fold edgewise subdivision of that. This triangulation (see Figure 1 for an example), termed as the \( r \)-colored barycentric subdivision in [9], was introduced in [5] in order to partially interpret geometrically the derangement polynomial for the colored permutation group \( \mathbb{Z}_r \wr \mathfrak{S}_n \); it was further studied enumeratively in [8]. As mentioned in [9, Section 3], \( \mathcal{F}_2 \) coincides with the \( f \)-triangle defined by the interval triangulation [34, Section 3.3], so all our results here apply to that as well.

The main result of this section answers in the affirmative a question from [9, Section 7].

**Theorem 5.3.** The \( f \)-triangle \( \mathcal{F}_r \) has the strong interlacing property.

We postpone the proof until the end of the section.

**Proof of Theorem 5.3.** This follows from Theorems [3.3] [4.1] and [5.3] provided \( D_{n,r} \) coincides with the operator in the statement of Theorem 1.3. Indeed, considering \( h(x) \in \mathbb{R}_n[x] \) and setting \( f(x) = \sum_{i=0}^{n} c_i x^i (1 + x)^{n-i} \), from Equation (20) we get

\[
\frac{D_{n,r}(h(x))}{(1-x)^n} = \frac{1}{(1-x)^n} \cdot S_0^r \left( (1+x+x^2+\cdots+x^{r-1})^n D_n(h(x)) \right)
\]

\[
= S_0^r \left( \frac{(1+x+x^2+\cdots+x^{r-1})^n}{(1-x)^n} D_n(h(x)) \right) = S_0^r \left( \frac{D_n(h(x))}{(1-x)^n} \right)
\]

\[
= S_0^r \left( (1-x) \sum_{m \geq 0} f(m)x^m \right) = f(0) + \sum_{m \geq 1} (f(rm) - f(rm - 1)) x^m
\]

and the proof follows. Note that we have used Equation (3) for the next to last step. □

Part (a) of Theorem 1.3 was deduced in [9, Proposition 7.5] from Theorem 1.1 and the fact that \( \mathcal{U}_r^\mu \) preserves real-rootedness for polynomials with nonnegative coefficients. The
present proof shows additionally that $D_{n,r}(h(x))$ is interlaced by $h_{\mathcal{F}_r}(\sigma_n, x)$ and interlaces $x^n h_{\mathcal{F}_r}(\sigma_n, 1/x)$; see [8] Proposition 5.1] for combinatorial interpretations of $h_{\mathcal{F}_r}(\sigma_n, x)$.

Part (b) of the following corollary can be deduced from [30] Theorem 1.1] for $r \geq n - 1$ but, to the best of our knowledge, not for other values of $r$.

**Corollary 5.4.** Let $\text{sd}_r(\Delta)$ denote the $r$-colored barycentric subdivision of $\Delta$.

(a) The polynomial $h(\text{sd}_r(\Delta), x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every $(n-1)$-dimensional Cohen–Macaulay* simplicial complex $\Delta$.

(b) The polynomial $h(\text{sd}_r(\Delta), x)$ has a nonnegative, real-rooted symmetric decomposition with respect to $n - 1$ for every triangulation $\Delta$ of the $(n-1)$-dimensional ball.

**Proof.** Apply Theorem 1.3 to $h(\Delta, x)$ or, alternatively, Corollary 4.3 to the $r$-colored barycentric subdivision.

**Proof of Corollary 5.4.** It was observed in [17, Section 4] that $i(\mathcal{Z}; x) = \sum_{i=0}^{n} c_i x^i (1+x)^{n-i}$ for some nonnegative integers $c_i$ (these are the coefficients of the $h^\ast$-polynomial of the Lawrence polytope associated to $\mathcal{Z}$) which satisfy inequalities (4) for all $0 \leq i \leq \lfloor n/2 \rfloor$. Hence, the result follows by applying Theorem 1.3 to $f(x) = i(\mathcal{Z}; x)$.

We now turn our attention to the proof of Theorem 5.3. Recall that $\mathcal{F} = \mathcal{F}_1$ is the $f$-triangle defined by barycentric subdivision. We write $p_{n,k}(x)$ in place of $p_{\mathcal{F},n,k}(x)$ and recall that

$$p_{n,k}(x) = x \sum_{i=0}^{k-1} p_{n-1,i}(x) + \sum_{i=k}^{n-1} p_{n-1,i}(x)$$

for all $n \geq 1$ and $k \in \{0, 1, \ldots, n\}$ since, as mentioned in Section 5, $p_{\mathcal{F},n-1,n}(x) = 0$ in this case. A combinatorial interpretation of the polynomials $p_{\mathcal{F},n,k}(x)$ for $k \in \{0, 1, \ldots, n\}$ was given in [9] Proposition 4.7]. We will now express these polynomials and $\theta_{\mathcal{F}_r}(\sigma_n, x)$ in terms of the $p_{n,k}(x)$. For this reason, we introduce the polynomials

$$p_{n,k}^{(r,j)}(x) = S_j^r \left( (1 + x + x^2 + \cdots + x^{r-1})^n p_{n,k}(x) \right)$$

for $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, r-1\}$. For $r = 2$ they have been considered before in [4] (see, for instance, Corollary 4.7 there).

| $j$ | $k = 0$          | $k = 1$          | $k = 2$          | $k = 3$          |
|-----|------------------|------------------|------------------|------------------|
| 0   | $1 + 34x + 19x^2$| $30x + 24x^2$   | $24x + 30x^2$   | $19x + 34x^2 + x^3$|
| 1   | $7 + 40x + 7x^2$ | $4 + 40x + 10x^2$| $2 + 38x + 14x^2$| $1 + 34x + 19x^2$|
| 2   | $19 + 34x + x^2$ | $14 + 38x + 2x^2$| $10 + 40x + 4x^2$| $7 + 40x + 7x^2$|

**Table 1.** The polynomials $p_{n,k}^{(r,j)}(x)$ for $n = r = 3$. 
Proposition 5.5. For the $f$-triangle $\mathcal{F}_r$ we have

\begin{equation}
(22) \quad p_{\mathcal{F}_r,n,k}(x) = P_{n,k}^{(r,0)}(x)
\end{equation}

for all $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n\}$ and

\begin{equation}
(23) \quad \theta_{\mathcal{F}_r}(\sigma_n, x) = x \sum_{j=1}^{r-1} \sum_{k=0}^{n-1} P_{n-1,k}^{(r,j)}(x)
\end{equation}

for every $n \geq 1$.

Proposition 5.6. The polynomials $p_{n,k}^{(r,j)}(x)$ satisfy the recurrence

\begin{equation}
(24) \quad p_{n,k}^{(r,j)}(x) = x \sum_{\ell=j+1}^{r-1} \sum_{i=0}^{n-1} p_{n-1,i}^{(r,\ell)}(x) + x \sum_{i=0}^{k-1} p_{n-1,i}^{(r,j)}(x) + \sum_{i=k}^{n-1} p_{n-1,i}^{(r,j)}(x) + \sum_{\ell=0}^{j-1} \sum_{i=0}^{n-1} p_{n-1,i}^{(r,\ell)}(x)
\end{equation}

for every $n \geq 1$ and all $k \in \{0, 1, \ldots, n\}$ and $j \in \{0, 1, \ldots, r-1\}$.

Proof. From the definition of $p_{n,k}^{(r,j)}(x)$ we get

\begin{align*}
p_{n,k}^{(r,j)}(x) &= S_j^r \left( (1 + x + x^2 + \cdots + x^{r-1})^n p_{n,k}(x) \right) \\
&= S_j^r \left( \sum_{\ell=0}^{r-1} x^\ell \right) (1 + x + x^2 + \cdots + x^{r-1})^{n-1} p_{n,k}(x) \\
&= \sum_{\ell=0}^{r-1} S_j^r \left( x^\ell (1 + x + x^2 + \cdots + x^{r-1})^{n-1} p_{n,k}(x) \right).
\end{align*}

Replacing $p_{n,k}(x)$ by the right-hand side of (21) and changing the order of summation, we get

\begin{align*}
p_{n,k}^{(r,j)}(x) &= \sum_{i=0}^{k-1} \sum_{\ell=0}^{r-1} S_j^r \left( x^{\ell+1} (1 + x + x^2 + \cdots + x^{r-1})^{n-1} p_{n-1,i}(x) \right) + \\
&\quad \sum_{i=k}^{n-1} \sum_{\ell=0}^{r-1} S_j^r \left( x^\ell (1 + x + x^2 + \cdots + x^{r-1})^{n-1} p_{n-1,i}(x) \right)
\end{align*}

and applying (22) yields the desired expression for $p_{n,k}^{(r,j)}(x)$; the details are omitted. \qed

Proof of Proposition 5.5. For every $h(x) = c_0 + c_1 x + \cdots + c_n x^n \in \mathbb{R}_n[x], \ldots
\[
\mathcal{P}_{F_r,n}(h(x)) = \mathcal{U}_r^n(\mathcal{D}_n(h(x))) = \mathcal{U}_r^n \left( \sum_{k=0}^{n} c_k p_{n,k}(x) \right) = \sum_{k=0}^{n} c_k \mathcal{U}_r^n(p_{n,k}(x))
\]
\[
= \sum_{k=0}^{n} c_k \mathcal{S}_0^r \left( (1 + x + x^2 + \cdots + x^{r-1})^n p_{n,k}(x) \right) = \sum_{k=0}^{n} c_k p_{n,k}^{(r,0)}(x).
\]

This proves (22). Since \(h_{F_r}(\sigma_n, x) = p_{F_r,n,0}(x) = p_{n,0}^{(r,0)}(x)\), from the recurrence of Proposition 5.6 we get
\[
h_{F_r}(\sigma_n, x) = x \sum_{j=1}^{r-1} \sum_{k=0}^{n-1} p_{n-1,k}^{(r,j-1)}(x) + \sum_{k=0}^{n-1} p_{n-1,k}^{(r,0)}(x).
\]

Since \(\theta_{F_r}(\sigma_n, x) = h_{F_r}(\sigma_n, x) - h_{F_r}(\partial \sigma_n, x)\) and
\[
h_{F_r}(\partial \sigma_n, x) = \sum_{k=0}^{n-1} p_{F_r,n-1,k}(x) = \sum_{k=0}^{n-1} p_{n-1,k}^{(r,0)}(x)
\]
by (15) and part (a), the proof of (23) follows. \(\square\)

**Proof of Theorem 5.3** For \(j \in \{0, 1, \ldots, r-1\}\) we consider the sequence
\[
\mathcal{P}_n^{(r,j)} := (p_{n,0}^{(r,j)}(x))_{0 \leq k \leq n} = (p_{n,0}^{(r,j)}(x), p_{n,1}^{(r,j)}(x), \ldots, p_{n,n}^{(r,j)}(x))
\]
and let
\[
\mathcal{P}_{n,r} := (\mathcal{P}_n^{(r,r-1)}, \ldots, \mathcal{P}_n^{(r,1)}, \mathcal{P}_n^{(r,0)})
\]
be their concatenation, in the specified order; see see Table 1 for an example.

We claim that \(\mathcal{P}_{n,r}\) is interlacing for every \(n \in \mathbb{N}\). This is clear for \(n = 0\), since \(\mathcal{P}_{0,r} = (0, \ldots, 0, 1)\), so we assume that \(n \geq 1\). Proposition 5.6 implies that \(p_{n,n}^{(r,j)}(x) = p_{n,0}^{(r,j-1)}(x)\) for every \(j \in \{1, 2, \ldots, r-1\}\), so \(\mathcal{P}_{n,r}\) has \(r-1\) pairs of equal consecutive elements. The same proposition shows that, when one of these elements is removed from each of these pairs, the resulting sequence is obtained from \(\mathcal{P}_{n-1,r}\) by the recipe of (7). As a result, and since doubling some elements of an interlacing sequence clearly preserves the interlacing property, the interlacing of \(\mathcal{P}_{n-1,r}\) implies that of \(\mathcal{P}_{n,r}\) and our claim follows by induction on \(n\).

We may now prove the theorem. Clearly, the polynomials \(p_{n,k}^{(r,j)}(x)\) have nonnegative coefficients. As can be inferred from their definition or Proposition 5.6, they have degree \(n - 1\) except for \(p_{n,0}^{(r,0)}(x)\), which has degree \(n\). Thus, Proposition 5.5 shows that \(\theta_{F_r}(\sigma_n, x)\) has nonnegative coefficients and degree \(n-1\). Given that \(\mathcal{P}_{n-1,r}\) is an interlacing sequence, it also shows that \(\theta_{F_r}(\sigma_n, x) / x\) is a sum of polynomials each of which interlaces \(p_{n-1,0}^{(r,0)}(x) = h_{F_r}(\sigma_{n-1}, x)\). This implies that \(\theta_{F_r}(\sigma_n, x)\) is real-rooted and interlaced by \(h_{F_r}(\sigma_{n-1}, x)\) and the proof follows. \(\square\)
6. Skeleta of simplicial complexes

This section proves and generalizes Theorem 1.5 in the setting of uniform triangulations as follows.

**Theorem 6.1.** Let \( \Gamma \) be an \( n \)-dimensional simplicial complex with nonnegative \( h \)-vector and let \( \Delta \) be the \((n - 1)\)-dimensional skeleton of \( \Gamma \).

(a) The polynomial \( h_\mathcal{F}(\Delta, x) \) has a nonnegative, real-rooted and interlacing symmetric decomposition with respect to \( n \) for every feasible \( f \)-triangle \( \mathcal{F} \) which has the strong interlacing property with respect to \( n \).

(b) The polynomial \( h_\mathcal{F}(\Delta, x) \) interlaces \( h_\mathcal{F}(\Gamma, x) \) for every feasible \( f \)-triangle \( \mathcal{F} \) which has the strong interlacing property with respect to \( n + 1 \).

**Proof.** As a direct consequence of the defining equation (1) of the \( h \)-polynomial, the entries of the \( h \)-vector of \( \Delta \) can be expressed in terms of those of the \( h \)-vector of \( \Gamma \) as

\[
h_k(\Delta) = h_0(\Gamma) + h_1(\Gamma) + \cdots + h_k(\Gamma)
\]

for \( 0 \leq k \leq n \). In particular, \( h_0(\Delta) \leq h_1(\Delta) \leq \cdots \leq h_n(\Delta) \) and this makes it obvious that the \( h_i(\Delta) \) satisfy the inequalities (11) and that \( h_i(\Delta)h_{n-i}(\Delta) \leq h_{i+1}(\Delta)h_{n-i-1}(\Delta) \) for \( 0 \leq i \leq n - 1 \). Thus, part (a) follows from part (a) of Theorem 4.1.

Since \( \mathcal{F} \) has the strong interlacing property with respect to \( n + 1 \), \( h_\mathcal{F}(\Delta, x) \) and \( h_\mathcal{F}(\Gamma, x) \) have nonnegative coefficients and only real roots by Theorem 3.3. Moreover, by Equation (11) and our previous remarks,

\[
h_\mathcal{F}(\Delta, x) = \sum_{k=0}^{n} (h_0(\Gamma) + h_1(\Gamma) + \cdots + h_k(\Gamma)) p_{\mathcal{F},n,k}(x)
\]

\[
h_\mathcal{F}(\Gamma, x) = \sum_{k=0}^{n+1} h_k(\Gamma) p_{\mathcal{F},n+1,k}(x).
\]

Expressing the \( p_{\mathcal{F},n+1,k}(x) \) in terms of the \( p_{\mathcal{F},n,k}(x) \) by (14), we compute that for any positive reals \( \lambda, \mu \),

\[
(\lambda x + \mu)h_\mathcal{F}(\Delta, x) + h_\mathcal{F}(\Gamma, x) = \sum_{k=0}^{n+1} \nu_k(x) p_{\mathcal{F},n,k}(x),
\]

where

\[
\nu_k(x) = \begin{cases} 
(\mu + 1) \sum_{i=0}^{k} h_i(\Gamma) + \left(\lambda \sum_{i=0}^{k} h_i(\Gamma) + \sum_{i=k+1}^{n+1} h_i(\Gamma)\right) x, & \text{if } 0 \leq k \leq n \\
\sum_{i=0}^{n+1} h_i(\Gamma), & \text{if } k = n + 1.
\end{cases}
\]

As in the proof of Theorem 4.1, it is routine to show that \((\nu_{n+1}(x), \ldots, \nu_1(x), \nu_0(x))\) is an interlacing sequence. Since \( \mathcal{P}_{\mathcal{F},n+1} \) is also interlacing by the proof of [9, Theorem 6.1], the result of part (b) follows by applying Lemmas 7.8.3 and 7.8.4 of [16]. \( \square \)
Proof of Theorem 1.5. Apply Theorem 6.1 to the $f$-triangles of the $r$-fold edgewise and $r$-colored barycentric subdivisions.

7. Concluding remarks and open problems

Given the crucial role played by the strong interlacing property in Theorems 3.3 and 4.1, the following question arises naturally.

Question 7.1. Which uniform triangulations have the strong interlacing property?

The inequalities (18) imply that $h_i(\Delta) \leq h_{n-i}(\Delta)$ for all $0 \leq i \leq \lfloor n/2 \rfloor$. The validity of the latter inequalities for doubly Cohen–Macaulay complexes follows from [2, Corollary 6.2] (and was earlier shown for the more restrictive class of simplicial complexes with a convex ear decomposition in [45, Corollary 3.10]). Similar remarks apply to inequalities (19). We thus ask the following questions.

Question 7.2. Which Cohen–Macaulay* simplicial complexes satisfy (18)? Do these inequalities hold for all doubly Cohen–Macaulay simplicial complexes?

Question 7.3. Which triangulations of the ball satisfy (19)?

We expect that Question 7.2 has an affirmative answer at least for interesting classes of doubly Cohen–Macaulay complexes. The inequalities $h_i(\Delta) \leq h_{n-i}(\Delta)$ we mentioned earlier, the fact that doubly Cohen–Macaulay simplicial complexes are level [41, p. 94] and [41, Proposition III.3.3 (a)] imply an affirmative answer for doubly Cohen–Macaulay complexes of dimension at most 3. Question 7.3 has an affirmative answer in three dimensions as well, since $h_1(\Delta) \geq h_3(\Delta)$ for every triangulation $\Delta$ of the 3-dimensional ball (see, for instance, [32, Section 3]) but not for every triangulated ball $\Delta$ in higher dimensions. Indeed, according to [32, Theorem 14], $(1, a, b, 1)$ is the $h$-vector of a triangulation of the 4-dimensional ball for all positive integers $a, b$ with $b \leq 1 + a(a - 1)/2$.

Some of the problems about simplicial complexes we have studied make sense for polyhedral (or even more general cell) complexes. We record two of them here, one of which has already been mentioned in the introduction. Let $sd(\mathcal{L})$ denote the barycentric subdivision of a (finite) polyhedral complex $\mathcal{L}$.

Question 7.4. (a) Does $h(sd(\mathcal{L}), x)$ have a nonnegative, real-rooted symmetric decomposition with respect to $n$ for every $(n-1)$-dimensional Cohen–Macaulay* polyhedral complex $\mathcal{L}$?

(b) Does $h(sd(\mathcal{L}), x)$ have a nonnegative, real-rooted symmetric decomposition with respect to $n-1$ for every $(n-1)$-dimensional polyhedral ball $\mathcal{L}$?

If so, are these decompositions interlacing?

Finally, we noticed in Section 6 that the $(n-1)$-skeleton of any $n$-dimensional Cohen–Macaulay simplicial complex has an increasing $h$-vector.

Question 7.5. Which Cohen–Macaulay simplicial complexes have increasing $h$-vector?
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