Higher-order properties and Bell-inequality violation for the three-mode enhanced squeezed state

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Abstract

By extending the usual two-mode squeezing operator \(S_2 = \exp \left[ i\lambda (Q_1 P_2 + Q_2 P_1) \right] \) to the three-mode squeezing operator \(S_3 = \exp \left\{ i\lambda \left[ Q_1 (P_2 + P_3) + Q_2 (P_1 + P_3) + Q_3 (P_1 + P_2) \right] \right\} \), we obtain the corresponding three-mode squeezed coherent state. The state’s higher-order properties, such as higher-order squeezing and higher-order sub-Poissonian photon statistics, are investigated. It is found that the new squeezed state not only can be squeezed to all even orders but also exhibits squeezing enhancement comparing with the usual cases. In addition, we examine the violation of Bell-inequality for the three-mode squeezed states by using the formalism of Wigner representation.

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1 Introduction

Widespread attention has been paid in recent years to a class of optical field states that are called squeezed states [1], where the fluctuation in one quadrature is less than that of the vacuum state [2]. Squeezed states have manifestly nonclassical properties, which may find application in low-noise optical communications as well as in the gravitational-wave detection [3]. The two-mode squeezed state [4], which is composed by idler mode and signal mode resulting from a parametric down conversion amplifier [5], is a typical entangled state of continuous variable [6]. Recently, many groups have devoted to the research on nonclassical properties of two-mode squeezed state [7–11]. Lee [7] and Cave’ group [8] have studied the nonclassical photon statistics of two-mode squeezed states, respectively.

Theoretically, two-mode squeezed vacuum state is constructed by the two-mode squeezing operator \(S_2 = \exp \left[ \lambda \left( a_1 a_2 - a_1^\dagger a_2^\dagger \right) \right] \) acting on the vacuum state \(|00\rangle\),

\[
S_2 |00\rangle = \sec \lambda \exp \left( -a_2^\dagger a_2^\dagger \tanh \lambda \right) |00\rangle , \tag{1}
\]

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where $\lambda$ is a squeezing parameter. Considering the coordinate and momentum operators, respectively,
\[ Q_j = \frac{1}{\sqrt{2}} \left( a_j + a_j^\dagger \right), \quad P_j = \frac{1}{i\sqrt{2}} \left( a_j - a_j^\dagger \right), \] (2)
one can recast $S_2$ into the form
\[ S_2 = \exp \left[ i\lambda \left( Q_1 P_2 + Q_2 P_1 \right) \right]. \] (3)
In the state $S_2 \ket{00}$, the variances of two-mode quadrature operators [2]
\[ X = \frac{1}{2} \left( Q_1 + Q_2 \right), \quad Y = \frac{1}{2} \left( Q_1 + Q_2 \right) \] (4)
take the standard form, i.e.,
\[ \langle 00 | S_2^{-1} X^2 S_2 | 00 \rangle = e^{-2\lambda}, \quad \langle 00 | S_2^{-1} Y^2 S_2 | 00 \rangle = e^{2\lambda}. \] (5)

In the present paper, we shall extend the two-mode squeezed states to the three-mode case and investigate their some nonclassical properties. Based on the Fan’s ideas [12], we construct the following unitary operator in the three-mode Fock space
\[ S_3 = \exp \left\{ i\lambda \left[ Q_1 P_2, Q_2 P_3, Q_3 P_1 \right] \right\}. \] (6)
Then a question naturally arises: Is the operator $S_3$ also a three-mode squeezing operator? If yes, what is its corresponding squeezed state? What is particular nonclassical properties of these squeezed states? Facing with these questions, this work is arranged as follows. In Sec. 2, to answer the above questions, we first derive the normally ordered form of $S_3$ by virtue of the technique of integration within an ordered product (IWOP) of operators and analyze if the squeezing exists. Then the corresponding squeezed states $S_3 \ket{\vec{a}}$ are obtained, where $\ket{\vec{a}} \equiv \ket{\alpha_1, \alpha_2, \alpha_3}$ is a three-mode coherent state. In Secs. 3 and 4, we concentrate our attention on studying the higher-order squeezing and higher-order sub-Poissonian photon statistics in the state $S_3 \ket{\vec{a}}$. We find that $S_3 \ket{\vec{a}}$ can be squeezed to all even orders and the variances of the three-mode quadrature operators exhibit stronger squeezing comparing with the shown case in Ref. [12]. So we call $S_3$ the three-mode enhancing squeezing operator. We devote Sec. 5 to evaluate the violation of Bell-inequality for the state $S_3 \ket{\vec{a}}$ using the formalism of the Wigner representation in phase space based on the parity operator and the displacement operation. Finally, a brief summary is given in Sec. 6.

## 2 Normally ordered form of $S_3$

In this section, we derive the normally ordered expansion $S_3$ and then obtain the corresponding squeezed state. Because these operators $Q_1 P_2, Q_1 P_3, Q_2 P_1, Q_2 P_3, Q_3 P_1, Q_3 P_2$ in Eq. (6) do not make up a closed Lie algebra, we utilize the IWOP technique [13] to disentangle $S_3$. For this purpose, we rewrite $S_3$ using the matrix form as
\[ S_3 = \exp \left[ i\lambda \left( Q_1, Q_2, Q_3 \right) A \right] \] (7)
with
\[ A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad j, k = 1, 2, 3, \] (8)
here and henceforth the repeated indices represent the Einstein summation notation.

Appealing to the Baker-Hausdorff formula
\[ e^C B e^{-C} = B + [C, B] + \frac{1}{2} [C, [C, B]] + \frac{1}{3} [C, [C, [C, B]]] + \cdots, \] (9)
where $C$ and $B$ are operators, one can calculate out
\[
S_3^{-1}Q_kS_3 = Q_k - \lambda Q_jA_{jk} + \frac{1}{2}i\lambda^2 [Q_jA_{ik}, Q_iA_{ik}] + \cdots \\
= Q_i(e^{-\lambda A})_{ik} = (e^{-\lambda A})_{ki}Q_i,
\] (10)
and
\[
S_3^{-1}P_kS_3 = P_k + \lambda Q_iA_{ik} + \frac{1}{2}i\lambda^2 [P_iA_{ik}, Q_iA_{ik}] + \cdots \\
= (e^{\lambda A})_{ki}P_i.
\] (11)

It is implied from Eqs. (10) and (11) that the action of $S_3$ on the three-mode coordinate eigenstate $|\vec{q}\rangle$, whose expression in Fock space is [14]
\[
|\vec{q}\rangle = \pi^{-3/4} \exp[-\frac{\vec{q}^2}{2} + \sqrt{2\vec{q}}\bar{a} + \frac{1}{2}\bar{a}^\dagger a]|0\rangle,
\] (12)
leads to
\[
S_3 |\vec{q}\rangle = |\Lambda|^{1/2} |\Lambda\vec{q}\rangle, \Lambda = e^{-\lambda A}, |\Lambda| = \det \Lambda,
\] (13)
where $\bar{a} = (\bar{a}_1^1, \bar{a}_2^4, \bar{a}_3^3), \vec{q} = (q_1, q_2, q_3)$ and tilde represents transposition of matrix. According to properties of matrix $A$ in Eq. (8), by expanding the exponential term $e^{-\lambda A}$, we get
\[
\Lambda = e^{-\lambda A} = \begin{pmatrix}
u_1 & v_1 & v_1 \\
v_1 & u_1 & v_1 \\
v_1 & v_1 & u_1
\end{pmatrix},
\] (14)
where
\[
u_1 = \frac{1}{3}(e^{-2\lambda} + 2e^\lambda), v_1 = \frac{1}{3}(e^{-2\lambda} - e^\lambda).
\] (15)
Similarly,
\[
\Gamma = e^{\lambda A} = \begin{pmatrix}
u_2 & v_2 & v_2 \\
v_2 & u_2 & v_2 \\
v_2 & v_2 & u_2
\end{pmatrix},
\] (16)
where
\[
u_2 = \frac{1}{3}(e^{2\lambda} + 2e^{-\lambda}), v_2 = \frac{1}{3}(e^{2\lambda} - e^{-\lambda}),
\] (17)
which will be used in the rest of this article.

From Eq. (13), it is found that $S_3$ has natural representation in the coordinate basis $|\vec{q}\rangle$
\[
S_3 = \int dqS_3 |\vec{q}\rangle \langle q| = |\Lambda|^{1/2} \int dq|\Lambda\vec{q}\rangle \langle \vec{q}|.
\] (18)
It then follows that
\[
S_3^{-1}Q_kS_3 = |\Lambda| \int dq|\vec{q}\rangle \langle \Lambda\vec{q}| Q_k \int dq'||\Lambda\vec{q}'\rangle \langle \vec{q}'| = (\Lambda Q)_k
\] (19)
which is consistent with Eq. (10). Substituting Eq. (12) into Eq. (18) and using the IWOP technique as well as three-mode vacuum projector $|\vec{0}\rangle \langle \vec{0}| = \exp[-\bar{a}^\dagger a]$, we easily get the normally ordered form of $S_3$ after integrating out of $\vec{q}$
\[
S_3 = \left(\frac{\det \Lambda}{\det N}\right)^{1/2} \exp \left[\frac{1}{2}\bar{a}^\dagger \left(\Lambda N^{-1} \Lambda^\dagger - I\right) a^\dagger\right] \\
\times \exp \left[\bar{a}^\dagger (\Lambda N^{-1} - I) a\right] \exp \left[\frac{1}{2}\bar{a} (N^{-1} - I) a\right].
\] (20)
where \( N = \left( 1 + \tilde{\Lambda} \Lambda \right) / 2 \). Here we have used the mathematical formula

\[
\int d^n x \exp \left[ -\tilde{x} W x + \tilde{x} v \right] = \pi^{n/2} \left( \det W \right)^{1/2} \exp \left( \frac{1}{4} \tilde{\kappa} W^{-1} v \right) .
\]

(21)

Finally, applying the operator \( S_3 \) in Eq.(20) on the three-mode coherent state \( |\vec{\alpha}\rangle \equiv |\alpha_1, \alpha_2, \alpha_3\rangle \), we obtain the three-mode squeezed coherent state, denoted by \( S_3 |\vec{\alpha}\rangle \), as follows

\[
S_3 |\vec{\alpha}\rangle = \left( \frac{\det \Lambda}{\det N} \right)^{1/2} \exp \left[ \frac{1}{2} \hat{\alpha}^\dagger \left( \Lambda N^{-1} \tilde{\Lambda} - I \right) \alpha^\dagger + \hat{\alpha} \left( \Lambda N^{-1} - I \right) \alpha \right] |\vec{\alpha}\rangle .
\]

(22)

When \( \alpha_i = 0, i = 1, 2, 3 \), Eq.(22) reduces to the three-mode squeezed vacuum state

\[
S_3 |\vec{0}\rangle = \left( \frac{\det \Lambda}{\det N} \right)^{1/2} \exp \left[ \frac{1}{2} \hat{\alpha}^\dagger \left( \Lambda N^{-1} \tilde{\Lambda} - I \right) \alpha^\dagger \right] |\vec{0}\rangle .
\]

(23)

3 Higher-order squeezing property of \( S_3 |\vec{\alpha}\rangle \)

It is well known that the concept of higher-order squeezing, introduced by Hong and Mandel [15], is another aspect for revealing nonclassical characteristics of a quantum state [16]. When \( 2m \)-th moment in a state is less than that in the coherent state, this state is said to be squeezed to order \( 2m \). Here, we evaluate the higher-order squeezed behavior of the three-mode squeezed coherent state \( S_3 |\vec{\alpha}\rangle \).

Firstly, we derive the expression of the \( 2m \)-th moment of quadrature operators in the state \( S_3 |\vec{\alpha}\rangle \). The quadratures in the three-mode case are defined as

\[
X_3 = \frac{1}{\sqrt{6}} \sum_{j=1}^{3} Q_j = \frac{1}{2\sqrt{3}} \sum_{j=1}^{3} \left( a_j + a_j^\dagger \right) ,
\]

(24)

and

\[
Y_3 = \frac{1}{\sqrt{6}} \sum_{j=1}^{3} P_j = \frac{1}{4\sqrt{3}} \sum_{j=1}^{3} \left( a_j - a_j^\dagger \right) ,
\]

(25)

obeying \( [X_3, Y_3] = \frac{1}{2} i \).

Letting \( F = \sum_{j=1}^{n} \left( \eta_j \alpha_j^\dagger + \kappa_j \alpha_j^\dagger \right) \) and using \( [\alpha_j, \alpha_j^\dagger] = \delta_{jk} \), one can easily have

\[
\exp \left( \gamma \Delta F \right) = : \exp \left( \gamma \Delta F \right) : \exp \left[ \frac{1}{2} \gamma^2 \left( \sum_{j=1}^{n} \eta_j \kappa_j \right) \right] ,
\]

(26)

where \( \Delta F \equiv F - (F) \), \( \eta_j \), \( \kappa_j \) and \( \gamma \) are \( C \) number and : : denotes normal ordering. By expanding both sides of Eq.(26) as power series in \( \gamma \), and equating coefficients of \( (2m)! \xi^{2m} \), it is obtained that

\[
(\Delta F)^{2m} = \sum_{k=0}^{m} \frac{2m!}{(2m-2k)!k!} \left( \sum_{j=1}^{n} \eta_j \kappa_j \right)^{k} : \left( \Delta F \right)^{2m-2k} :,
\]

(27)

which is a basic formula in calculating \( 2m \)-th moment of two-mode fields. Considering Eqs.(10) and (11), we calculate the \( 2m \)-th moment of the three-mode quadrature \( X_3 \) in the state \( S_3 |\vec{\alpha}\rangle \), i.e.,

\[
\langle \Delta X_3 \rangle^{2m} = \langle \vec{\alpha} \rangle S_3^{-1} \left( X_3^2 - \langle X_3 \rangle^2 \right)^m S_3 |\vec{\alpha}\rangle
\]

\[
= \left( \frac{1}{12} \right)^{m} \langle \vec{\alpha} \rangle \left\{ \Delta \sum_{k=1}^{3} \left[ (e^{-\lambda A})_{jk} \left( a_j + a_j^\dagger \right) \right] \right\}^{2m} |\vec{\alpha}\rangle
\]

\[
= \left( \frac{1}{12} \right)^{m} \langle \vec{\alpha} \rangle \langle \Delta Q \rangle^{2m} |\vec{\alpha}\rangle ,
\]

(28)
where
\[ Q \equiv \sum_{k=1}^{3} \left[ (e^{-\lambda A})_{jk} (a_j + a_j^\dagger) \right]. \tag{29} \]

As a result of Eqs. (27) and (14), we obtain
\[ (\Delta Q)^{2m} = \sum_{k=0}^{m} \frac{2m!}{(2m-2k)!k!} : (\Delta Q)^{2m-2k} : \left( \sum_{i,j=1}^{3} \frac{(e^{-2\lambda A})}{2}_{ij} \right)^k. \tag{30} \]

Further, we have
\[ \langle \alpha | : (\Delta Q)^{2m-2k} : | \alpha \rangle = \left\{ \left[ \sum_{k=1}^{3} \left( e^{-\lambda A} \right)_{jk} (a_j + a_j^\dagger) \right]^2 - \langle Q \rangle^2 \right\}^{m-k}. \tag{31} \]

On the other hand,
\[ \langle Q \rangle^2 = \left\{ \langle \alpha | : \sum_{k=1}^{3} \left[ (e^{-\lambda A})_{jk} (a_j + a_j^\dagger) \right] : | \alpha \rangle \right\}^2 = \left[ \sum_{k=1}^{3} \left( e^{-\lambda A} \right)_{jk} (a_j + a_j^\dagger) \right]^2. \tag{32} \]

By putting Eq. (32) into Eq. (31), it is easily obtained that
\[ \langle \alpha | : (\Delta Q)^{2m-2k} : | \alpha \rangle = \delta_{mk}. \tag{33} \]

Thus, Considering Eqs. (30) and (33), the final expression for 2m-order squeezing of X3 in Eq. (28) is
\[ \langle \Delta X_3 \rangle^{2m} = \left( \frac{1}{4} \right)^m (2m-1)!! e^{-4m\lambda}, \tag{34} \]
where \((2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1)\) and we have used the result of Eq. (14), i.e.,
\[ \sum_{j,k=1}^{3} (e^{-2\lambda A})_{jk} = 3e^{-4\lambda}. \tag{35} \]

Similarly, the 2m-order squeezing for Y3 is expressed as
\[ \langle \Delta Y_3 \rangle^{2m} = \left( \frac{1}{4} \right)^m (2m-1)!! e^{4m\lambda}. \tag{36} \]

From Eqs. (34) and (36), it is easily seen that when \(\lambda = 0\),
\[ \langle \Delta X_3 \rangle^{2m} = \langle \Delta Y_3 \rangle^{2m} = \left( \frac{1}{4} \right)^m (2m-1)!! \tag{37} \]
which is just 2m-th moment in the coherent state. According to the definition of higher-order squeezing, \(S_3 | \alpha \rangle\) can be squeezed to all even orders.

In particular, when \(m = 1\), Eqs. (34) and (36) reduce to the usual quadrature squeezing for the state \(S_3 | \alpha \rangle\), namely,
\[ \langle \Delta X_3 \rangle^{2m} = \langle \Delta Y_3 \rangle^{2m} = \left( \frac{1}{4} \right)^m (2m-1)!! \tag{38} \]
which shows that \(S_3\) is also a three-mode squeezing operator. In addition, Eq. (38) clearly indicates that the squeezed states \(S_3 | \alpha \rangle\) exhibit stronger squeezing \(e^{-4\lambda}\) in one quadrature than that \(e^{-2\lambda}\) of the usual squeezed state in Eq. (5). This is a way of enhanced squeezing. That is the reason why we call \(S_3\) three-mode enhanced squeezing operator.
4 Higher-order sub-Poissonian photon statistics in $S_3 |\bar{\alpha}\rangle$

In this section, we focus on investigating higher-order sub-Poissonian statistics in the state $S_3 |\bar{\alpha}\rangle$. The concept of higher-order sub-Poissonian statistics was introduced in [17–19] in terms of factorial moment.

By using $\langle R^{(k)} \rangle = \langle R (R-1) \cdots (R-k+1) \rangle = \langle a^{ik} a^k \rangle$, with $R = a^0 a$, a parameter $P_k$ can be defined as

$$P_k = \frac{\langle a^{ik} a^k \rangle}{\langle a^0 a \rangle^k} - 1$$

(40)

with $k$ is a positive integer. For all $k \geq 2$, $P_k = 0$ corresponds to the Poisson distribution. The negative or positive values of parameter $P_k$ for $k > 2$ indicate higher-order sub-Poissonian or higher-order super-Poissonian statistics, respectively.

For the three-mode case of higher-order sub-Poissonian statistics in $S_3 |\bar{\alpha}\rangle$, analogue to the single-mode case, we can define the joint operator $A = \sum_{j=1}^{3} a_j$ to study the joint sub-Poissonian distribution of $S_3 |\bar{\alpha}\rangle$. Referring to Eqs. (10) and (11), the similar transformation of $A, A^\dagger$ under the operator $S_3$ is

$$S_3^{-1} A S_3 = \frac{1}{\sqrt{2}} (v A + u A^\dagger) , S_3^{-1} A S_3 = \frac{1}{\sqrt{2}} (v A^\dagger + u A) ,$$

(41)

where

$$u = e^{-2\lambda} + e^{2\lambda}, v = e^{-2\lambda} - e^{2\lambda}.$$ 

(42)

According to coherent completeness $\int \frac{d^2 \beta}{\pi^2} \langle \tilde{\beta} | \tilde{\beta} \rangle = 1$ with $|\tilde{\beta}\rangle = |\beta_1, \beta_2, \beta_3\rangle$, we acquire the following equation

$$\langle A^{ik} A^k \rangle = \langle \bar{\alpha} | (S_3^{-1} A S_3)^k (S_3^{-1} A S_3)^k | \bar{\alpha} \rangle = \int \frac{d^2 \tilde{\beta}}{\pi^2} \langle \bar{\alpha} | [v A + u A^\dagger]^k | \tilde{\beta} \rangle \langle \tilde{\beta} | [v A + u A^\dagger]^k | \bar{\alpha} \rangle .$$

(43)

It is easy to check $[A, A^\dagger] = 1$, which is fit for the following equation [20] for $A$ in normal ordering form

$$(u A + v A^\dagger)^m = (-i \sqrt{uv} / 2)^m : H_m \left(i \sqrt{u / 2v} A + i \sqrt{v / 2u} A^\dagger \right) :,$$

(44)

where $H_m$ is the $m$-order Hermite polynomial. By putting Eq (44) into (43), we have

$$\langle A^{ik} A^k \rangle = \left(-uv / 2 \right)^k \int \frac{d^2 \tilde{\beta}}{\pi^2} \langle \bar{\alpha} | H_k \left(i \sqrt{v / 2u} A + i \sqrt{u / 2v} A^\dagger \right) | \tilde{\beta} \rangle \langle \tilde{\beta} | H_k \left(i \sqrt{u / 2v} A + i \sqrt{v / 2u} A^\dagger \right) | \bar{\alpha} \rangle$$

$$= \left(-uv / 2 \right)^k \int \frac{d^2 \tilde{\beta}}{\pi^2} \xi^k H_k (\xi) \exp \left[- \sum_{j=1}^{3} \left( |\beta_j|^2 + |\alpha_j|^2 + \beta_j^* \alpha_j + \beta_j \alpha_j^* \right) \right]$$

(45)

where

$$\xi \equiv i \left( \sqrt{u / 6v} \sum_{j=1}^{3} \beta_j + \sqrt{v / 6u} \sum_{j=1}^{3} \alpha_j^* \right),$$

(46)

and

$$\zeta \equiv i \left( \sqrt{v / 6u} \sum_{j=1}^{3} \alpha_j + \sqrt{u / 6v} \sum_{j=1}^{3} \beta_j^* \right).$$

(47)

Noticing that Hermite polynomial can be gotten from differential of mother function, i.e.,

$$H_m (x) = \frac{\partial^m}{\partial t^m} \exp \left(2xt - t^2 \right) \big|_{t=0} ,$$

(48)
Eq. (45) can be simplified as
\[
\langle A^k A^k \rangle = \frac{(-uv)^k}{2^k} \frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} \int d^2 \beta \exp \left[ 2\xi t_1 - t_1^2 + 2\xi t_2 - t_2^2 - \sum_{j=1}^3 \left( \beta_j^2 + |\alpha_j|^2 + \beta_j^* \alpha_j + \beta_j \alpha_j^* \right) \right] \Bigg|_{t_1, t_2 = 0}.
\] (49)

Through integrating out variables \( \beta \), it is obtained that
\[
\langle A^k A^k \rangle = \frac{(-uv)^k}{2^k} \frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} \exp \left( -t_1^2 - t_2^2 + t_1 G + t_2 M - \frac{2v}{u} t_1 t_2 \right) \bigg|_{t_1, t_2 = 0},
\] (50)

where
\[
G \equiv i \sum_{j=1}^3 \left( \sqrt{\frac{2u}{3v}} \alpha_j^* - \sqrt{\frac{2v}{3u}} \alpha_j \right), \quad M \equiv i \sum_{j=1}^3 \left( \sqrt{\frac{2u}{3v}} \alpha_j - \sqrt{\frac{2v}{3u}} \alpha_j^* \right).
\] (51)

A careful observation of Eq. (50) reveals that if we get rid of \(-\frac{2v}{u} t_1 t_2\) in the exponent, the rest just are the mother function of Hermite polynomial. Fortunately we can express \( \exp \left( -\frac{2v}{u} t_1 t_2 \right) \) in the following way
\[
\exp \left( -\frac{2vt_1 t_2}{u} \right) = \sum_{n=0}^{\infty} \frac{(-2v)^n}{n! u^n} \frac{\partial^{2n}}{\partial G^n \partial M^n} \exp \left[ -t_1^2 - t_2^2 + t_1 G + t_2 M \right] \bigg|_{t_1, t_2 = 0}.
\] (52)

Then substituting Eq. (52) into Eq. (50) and considering the property of Hermite polynomial Eq. (48), i.e.,
\[
\frac{\partial}{\partial x} H_n(x) = 2n H_{n-1}(x),
\] (53)
we get the expression for the average of \( A^k A^k \) in the state \( S_3 |\bar{\alpha} \rangle \)
\[
\langle A^k A^k \rangle = \frac{(-uv)^k}{2^k} \sum_{n=0}^{\infty} \frac{(-2v)^n}{24^n u^n (k-n)!^2} \frac{\partial^{2n}}{\partial G^n \partial M^n} \exp \left[ -t_1^2 - t_2^2 + t_1 G + t_2 M \right] \bigg|_{t_1, t_2 = 0}
\] (54)

Especially, when \( k = 1 \), noticing Eqs. (42) and (54), one has
\[
\langle A^\dagger A \rangle = \left[ GM - \frac{1}{8} \tanh (-2\lambda) \right] \sinh (4\lambda),
\] (55)
where
\[
GM = \frac{2}{3} \sum_{j,k=1}^3 (\alpha_j^* \alpha_k + \alpha_k^* \alpha_j) - \frac{4}{3} \coth (-4\lambda) \sum_{j,k=1}^3 \alpha_j^* \alpha_k.
\] (56)

For \( k = 2 \),
\[
\langle A^2 A^2 \rangle = \frac{(uv)^2}{2^2} \left[ \frac{v^2}{2u u^2} - \frac{v}{2u} H_1 \left( \frac{G}{2} \right) H_1 \left( \frac{M}{2} \right) + H_2 \left( \frac{G}{2} \right) H_2 \left( \frac{M}{2} \right) \right]
\] = \sinh^2 (-4\lambda) \left[ 2^{-3} \tanh^2 (-2\lambda) + 2GM \tanh (2\lambda) + (G^2 - 2)(M^2 - 2) \right]
\] (57)

where
\[
G^2 = \frac{4}{3} \sum_{j,k=1}^3 \alpha_j^* \alpha_k - \frac{2}{3} \sum_{j,k=1}^3 [\alpha_j \alpha_k \tanh (-2\lambda) + \alpha_j^* \alpha_k^* \coth (-2\lambda)],
\] (58)
and
\[
M^2 = \frac{4}{3} \sum_{j,k=1}^3 \alpha_j^* \alpha_k - \frac{2}{3} \sum_{j,k=1}^3 [\alpha_j \alpha_k \coth (-2\lambda) + \alpha_j^* \alpha_k^* \tanh (-2\lambda)].
\] (59)
Figure 1: Negative value of $P_2$ changing with $\alpha_3$ by letting $\alpha_1 = \alpha_2 = \lambda = 1$ to reflect the sub-Poissonian distribution of $S_3 |\vec{\alpha}\rangle$

Then introducing Eqs. (55) and (57) into (40) for $k = 2$, we easily obtain the result of $P_2$. In order to clearly observe the sub-Poissonian statistics in the state $S_3 |\vec{\alpha}\rangle$, we plot Fig. 1 to visualize the change of $P_2$ with $\alpha_3$ by simply letting $\alpha_1 = \alpha_2 = \lambda = 1$. It clearly shows that the parameter $P_2 < 0$ when $-0.5 < \text{Re} (\alpha_3) < 0.5$ but unchanged along with $\text{Im} (\alpha_3)$. The negative value of $P_2$ displays the sub-Poissonian distribution of $S_3 |\vec{\alpha}\rangle$. According to Ref. [21], the higher-order sub-Poissonian statistics may become useful for their use in detection of higher-order squeezing.

5 Bell-inequality Violation for $S_3 |\vec{\alpha}\rangle$

The quantum nonlocality for continuous variable states has attracted much attention. Wigner function representation of the Bell-inequality has been developed using a parity operator as a quantum observable [22–24]. In this section, we turn our attention to evaluate the violation of the Bell-inequality for squeezed coherent states $S_3 |\vec{\alpha}\rangle$ using the formalism of the Wigner representation in phase space based on the parity operator and the displacement operation.

For the three-mode system, the correlation function is the expectation of the operator

$$\Pi (\vec{\beta}) = \prod_{j=1}^{3} D_j (\beta_j) (-1)^{a_j^\dagger a_j} D_j^\dagger (\beta_j)$$

which is an equivalent definition of the Wigner function, namely

$$W_3 (\vec{q}, \vec{p}) = W_3 (\vec{\beta}) = \frac{1}{\pi^3} \left\langle \Pi (\vec{\beta}) \right\rangle$$

where $\vec{\beta} \equiv (\beta_1, \beta_2, \beta_3) = \frac{1}{\sqrt{2}} (\vec{q} + i \vec{p})$ and $D_j (\beta_j) = \exp[\beta_j a_j^\dagger - \beta_j^* a_j]$ is phase-space displacement operators acting on mode $j$. Thus $\Pi_j (\beta_j)$ is a product of displaced parity operators given as

$$\Pi_j (\beta_j) = D_j (\beta_j) \sum_{n=0}^{\infty} \left( |2k\rangle \langle 2k| - |2k+1\rangle \langle 2k+1| \right) D_j^\dagger (\beta_j)$$

corresponding to the measurement of an even (parity +1) or an odd (parity −1) number of photons in mode $j$. Within the framework of local realistic theories, the Wigner representation of Bell
Further, considering Eqs. (14), (16) and (66), it is obtained that

\[ \vec{Q} \]

invariance under similar transformations as well as Eqs. (10), (11), (68), we have

\[ \Delta_{1}(q_1, p_1) = \delta(q_1 - Q_1) \delta(p_1 - P_1) \]

and its normal ordering form is

\[ \Delta_{1}(q_1, p_1) = \frac{1}{\pi} : \exp[-(q_1 - Q_1)^2 - (p_1 - P_1)^2] : \]

where symbols \( : : \) denotes Weyl ordering. Note that the order of Bose operators \( a_i, a_i^\dagger \) within a normally ordered product and a weyl ordered product can be permuted. The Weyl ordering has a remarkable property, i.e., the order-invariance of Weyl ordered operators under similar transformations which means

\[ U^{-1} : F(a, a^\dagger) : U = : U^{-1}F(a, a^\dagger)U : \]

as if the fence \( : : \) does not exist.

For three-mode case, the Weyl ordering form of the Wigner operator is

\[ \Delta_3(\vec{q}, \vec{p}) = : \delta(\vec{q} - \vec{Q}) \delta(\vec{p} - \vec{P}) : \]

where \( \vec{Q} = (Q_1, Q_2, Q_3) \) and \( \vec{P} = (P_1, P_2, P_3) \). Thus, according to the principle of Weyl ordering invariance under similar transformations as well as Eqs. (10), (11), (68), we have

\[ W_{\vec{a}} = \langle \vec{a} | S_3^{-1} : \delta(\vec{q} - \vec{Q}) \delta(\vec{p} - \vec{P}) : S_3 | \vec{a} \rangle \]

Further, considering Eqs. (14), (16) and (66), it is obtained that

\[ W_{\vec{a}} = \frac{1}{\pi^3} \langle \vec{a} | : \exp\left[ \left( -e^{-\lambda A} \vec{q} - \vec{Q} \right)^2 - \left( e^{\lambda A} \vec{p} - \vec{P} \right)^2 \right] : | \vec{a} \rangle \]

\[ = \frac{1}{\pi^3} \exp \left[ e^{-\lambda A} \vec{q} - \frac{1}{\sqrt{2}} (\vec{a} + \vec{a}^*) \right] \left( e^{\lambda A} \vec{p} - \frac{1}{i\sqrt{2}} (\vec{a}^* - \vec{a}) \right)^2 \]

\[ = \frac{1}{\pi^3} \exp \left[ -A_{ji} \Lambda_{jk} q_j k_k - \sum_{j=1}^{3} \sigma_j^2 + 2q_k \Lambda_{kj} \sigma_j - \sum_{j=1}^{3} \chi_j^2 + 2p_k \Gamma_{kj} \chi_j \right] \]

\[ = \frac{1}{\pi^3} \exp \left[ -\sum_{j=1}^{3} (u_j^2 + 2u_j^2) q_j^2 - \sigma_j^2 + 2u_j q_j \sigma_j \right] - \sum_{j>k=1}^{3} \left[ (2u_1 v_1^2 + v_1^2) q_j k_k + 2v_1 q_j \sigma_k \right] \]

\[ - \sum_{j=k=1}^{3} \left[ (2u_2 v_2 + v_2^2) p_j k_k + 2v_2 p_j \chi_k \right] \]
Figure 2: Violation of the Bell-inequality $B(3)$ for the state $S_3 |\vec{\alpha}\rangle$ by using parity measurements with $\alpha_1 = 0.4$, $\alpha_2 = 0.5$, $\alpha_3 = 0.6$, $\beta_1 = \beta_2 = \beta'_3 = 0$, $\beta_3 = -b$, and $\beta'_1 = \beta'_2 = b$.

where $\sigma_j = \frac{1}{\sqrt{2}} (\alpha_j + \alpha_j^*)$ and $\chi_j = \frac{1}{\sqrt{2}} (\alpha_j - \alpha_j^*)$. Especially, when $\alpha_j = 0$, Eq. (70) reduces to the expression of Wigner function for the three-mode squeezed vacuum state $S_3 |\vec{0}\rangle$ in Eq. (23).

Now we put Eq. (70) with corresponding variables into three-mode Bell-inequality Eq. (64). This Bell-inequality has 13 variables and it is highly nontrivial to find the global maximum values of $B(3)$ for all 13 variables. To deal with this problem, in Fig.2 we plot the maximal Bell violation of the state $S_3 |\vec{\alpha}\rangle$ as a function of $\lambda$ by defining $\alpha_1 = 0.4$, $\alpha_2 = 0.5$, $\alpha_3 = 0.6$, $\beta_1 = \beta_2 = \beta'_3 = 0$, $\beta_3 = -b$, and $\beta'_1 = \beta'_2 = b$ ($b$ is a positive constant associated with the displacement magnitude). From Fig.2, we see that when the $\lambda \in (0, 1)$, $B(3) > 2$, the violation of the Bell-inequality exists for positive squeezing parameter. Another interesting phenomenon reflected by Fig.2 is that $B(3)$ will converge to a constant more than 2 independent of change of $\lambda$.

6 Conclusion

In summary, we generalize the usual two-mode squeezed state to three-mode operator $S_3$ and present the normal ordered form by the IWOP technique. Then, we apply the operator $S_3$ on three-mode coherent state $|\vec{\alpha}\rangle$ to get a new squeezed state $S_3 |\vec{\alpha}\rangle$. It is found that the state $S_3 |\vec{\alpha}\rangle$ displays enhanced higher-order squeezed property, which proves that $S_3$ is an enhanced squeezing operator. We also formulate the explicit expression of $P_k$ in order to describe higher-order sub-Poissonian statistics by taking $k = 2$ as an example. Finally, we discover that the state $S_3 |\vec{\alpha}\rangle$ violates the Bell-inequality by using the formalism of the Wigner representation.

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