Energy, Central Charge, and the BPS Bound for 1+1 Dimensional Supersymmetric Solitons

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Abstract

We consider one-loop quantum corrections to soliton energies and central charges in the supersymmetric $\phi^4$ and sine-Gordon models in 1+1 dimensions. In both models, we unambiguously calculate the correction to the energy in a simple renormalization scheme and obtain $\Delta H = -\frac{m_2}{2\pi}$, in agreement with previous results. Furthermore, we show that there is an identical correction to the central charge, so that the BPS bound remains saturated in the one-loop approximation. We extend these results to arbitrary 1+1 dimensional supersymmetric theories.

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I. INTRODUCTION

One-loop quantum corrections to the energies and central charges of solitons in 1+1 dimensional supersymmetric theories remain controversial despite many attempts to compute them. Both are formally given by the sum of zero-point contributions from small oscillations about the classical soliton field. However, these sums are divergent, so it is necessary to renormalize the theory, using an available counterterm that is consistent with supersymmetry. The calculation involves many subtleties, such as the boundary conditions placed on the oscillating fields in the presence of the soliton, the regularization of divergences, and the implementation of a definite renormalization scheme. A review of the literature shows a wide variety of conflicting results [1–5]. Recently we have developed some new tools for dealing with the regularization, renormalization and calculation of one-loop corrections to the energies of classical field configurations, which has encouraged us to take a fresh look at this problem.

First, we are able to calculate quantum corrections to classical field configurations that are not solutions to the classical field equations [6]. Our method relies on calculating phase shifts, which are determined simply by the equations of single-particle quantum mechanics and can be calculated for any background field configuration, whether or not it is a solution to the classical equations of motion. For example, in a theory with spontaneous symmetry breaking where \( \phi = \pm 1 \) are the trivial vacua, \( \phi_0(x) \) is a soliton that interpolates between \( \phi = -1 \) and \( \phi = 1 \), and \( \tilde{\phi}_0(x) \) is the corresponding antisoliton, we can study configurations of the form \( \phi_{cl}(x, x_0) = 1 + \tilde{\phi}_0(x + x_0) + \phi_0(x - x_0) \). These configurations interpolate between the trivial vacuum \( (x_0 = 0) \) and an arbitrarily far separated soliton-antisoliton pair \( (x_0 \to \infty) \). For all \( x_0 \), \( \phi_{cl} \) tends to the same trivial vacuum as \( x \to \pm \infty \). This technique enables us to study supersymmetric solitons indirectly, as the limit of a sequence of (non-supersymmetric) configurations. In practice, we use our numerical study of finite \( x_0 \) configurations to guide our analytic calculation for the isolated soliton. As a result, we can avoid the ambiguities associated with defining fermionic boundary conditions in the nontrivial topological sector. The phase shifts also give us insight into the bound state spectrum of the theory through Levinson’s theorem. Through this analysis, we will see that the bosonic and fermionic oscillations about a soliton are closely related in supersymmetric models, but that there are important differences in the phase shifts and bound state spectra.

Second, we use a cutoff-independent method to regulate potentially divergent integrals over high frequency small oscillations [6,7]. We write the one-loop correction to the energy as an integral over the density of states in the continuum plus bound state contributions. The density of states is given, in turn, by the scattering phase shifts. The divergent contributions originate in the first and second Born approximations to the phase shifts, which can be identified precisely with specific Feynman diagrams. We regulate the continuum integral by subtracting the divergent Born approximations from the density of states, and add this contribution back in using standard perturbation theory. The result is then the sum of a finite, cutoff-independent integral and a Feynman diagram computation. All cutoff dependence has been isolated in the Feynman diagrams, on which we impose renormalization conditions that are defined in the topologically trivial sector of the theory. In this paper, we will choose a simple scheme in which we require that the boson tadpole graph vanishes, with no further renormalization. We consider both the sine-Gordon and \( \phi^4 \) models and find
a one-loop correction to the soliton energy of $-m/2\pi$ in both cases, a result that is in fact general to supersymmetric solitons with reflection symmetry.

Finally, we consider the central charge, a topological term that arises in the supersymmetry algebra. It also receives one-loop corrections, which must be calculated in the same renormalization scheme. The BPS bound guarantees that to all orders in the quantum theory, the energy must be greater than or equal to the magnitude of the central charge. At the classical level this bound is saturated. We show that, independent of renormalization scheme, it remains saturated in the one-loop approximation. This result is similar to results in higher dimensions where, if the bound is saturated classically, it must remain saturated to all orders in the quantum theory because of “multiplet shortening” \cite{8}.

In §II we summarize the properties of supersymmetric solitons and antisolitons, and the small fluctuations of bosonic and fermionic fields about these backgrounds. In §III we calculate the renormalized one-loop correction to the energy of the supersymmetric soliton. In §IV we calculate the expectation value of the central charge in the soliton background. Our results are summarized in §V.

**II. THE SUPERSYMMETRIC SOLITON**

We begin with the classical supersymmetric Lagrangian density in 1+1 dimensions

$$\mathcal{L} = \frac{m^2}{2\lambda} \left( \left( \partial_\mu \phi \right) \left( \partial^\mu \phi \right) - U(\phi)^2 + i \bar{\Psi} \slashed{D} \Psi - U'(\phi) \bar{\Psi} \Psi \right)$$

where $\phi$ is a real scalar, $\Psi$ is a Majorana fermion, our metric is diag(+, −), and $U(\phi) = \frac{m^2}{2}(\phi^2 - 1)$ for the $\phi^4$ model and $U(\phi) = 2m \sin(\phi/2)$ for the sine-Gordon model. We have rescaled the fields so that powers of $\lambda$ clearly follow powers of $\hbar$: Classical results are of order $\lambda^{-1}$, one-loop results are of order $\lambda^0$, and higher loops go as higher powers of $\lambda$. In this paper we will work to one loop and drop terms of order $\lambda$ and higher.

These models support classical soliton and antisoliton solutions, which are the solutions to

$$\phi_0'(x) = \mp U(\phi_0)$$

for the soliton and antisoliton respectively. In the $\phi^4$ model, the soliton solution is the “kink,”

$$\phi^{\text{kink}}_0(x) = \tanh \frac{mx}{2},$$

while the soliton in the sine-Gordon model is given by

$$\phi^{\text{SG}}_0(x) = 4 \tan^{-1} e^{-mx},$$

and in both cases the antisoliton is obtained from the soliton by sending $x \to -x$.

To study one-loop quantum effects we require the bosonic and fermionic normal modes in the presence of the soliton, which satisfy the eigenvalue equations
\[
\left(-\frac{d^2}{dx^2} + U'(\phi_0)^2 + U(\phi_0)U''(\phi_0)\right)\eta_k(x) = (\omega_k^B)^2\eta_k(x)
\]  
\(\gamma^0 \left(-i\gamma^1 \frac{d}{dx} + U'(\phi_0)\right)\psi_k(x) = \omega_k^F \psi_k(x).\) 

(5)

(6)

For definiteness, we choose the representation \(\gamma^0 = \sigma_2, \gamma^1 = i\sigma_3,\) so that the Majorana condition becomes simply \(\Psi^\dagger = \Psi\).

The bosonic potentials are given by

\[
U'(\phi_0)^2 + U(\phi_0)U''(\phi_0) - m^2 = -\left(\ell + \frac{1}{\ell}\right) m^2 \text{sech}^2 \frac{mx}{\ell} \equiv V_\ell(x)
\]

(7)

with \(\ell = 1\) for the sine-Gordon soliton and \(\ell = 2\) for the kink. The \(V_\ell\) are reflectionless potentials whose properties are summarized in the Appendix. We will need their phase shifts, which are defined by \(T_\ell(k) = \exp\left(i\delta_\ell(k)\right),\) where \(T\) is the transmission amplitude (the reflection amplitude \(R\) vanishes). From eq. (59), we have

\[
\delta_{B,\kink}^\ell(k) = \delta_{\ell=2}(k) = 2 \tan^{-1}\left(\frac{3mk}{2k^2 - m^2}\right),
\]

\[
\delta_{B,\SG}^\ell(k) = \delta_{\ell=1}(k) = 2 \tan^{-1}\frac{m}{k},
\]

where the branch of the arctangent is chosen so that the phase shifts are continuous and go to zero as \(k \to \infty\). The bound states of \(V_\ell\) appear as poles in \(T_\ell(k)\) at positive imaginary \(k\). The kink has bosonic bound states at \(\omega = 0, \omega = \sqrt{3}m^2\), and at threshold, \(\omega = m\). The sine-Gordon soliton has bosonic bound states at \(\omega = 0\) and at threshold. We refer to threshold states as “half-bound” because they contribute with a weight of \(\frac{1}{2}\) to Levinson’s theorem

\[
\delta(0) = \pi(n - \frac{1}{2}),
\]

(9)

as discussed in the Appendix. With this counting, Levinson’s theorem is satisfied for both the sine-Gordon soliton and the kink with the bosonic phase shifts and bound states summarized above.

As discussed in the Introduction, we compute the fermionic phase shifts by considering a widely separated soliton and antisoliton solution, with the antisoliton on the left so that \(U'(\phi) \to m\) as \(x \to \pm \infty\). We will use the second-order equation obtained by squaring eq. (6),

\[
\begin{pmatrix}
-\frac{d^2}{dx^2} + V_\ell(x) & 0 \\
0 & -\frac{d^2}{dx^2} + \tilde{V}_\ell(x)
\end{pmatrix}
\begin{pmatrix}
\psi_k(x) \\
0
\end{pmatrix}
= k^2 \begin{pmatrix}
\psi_k(x) \\
0
\end{pmatrix}
\]

(10)

for the soliton and

\[
\begin{pmatrix}
-\frac{d^2}{dx^2} + \tilde{V}_\ell(x) & 0 \\
0 & -\frac{d^2}{dx^2} + V_\ell(x)
\end{pmatrix}
\begin{pmatrix}
\psi_k(x) \\
0
\end{pmatrix}
= k^2 \begin{pmatrix}
\psi_k(x) \\
0
\end{pmatrix}
\]

(11)

for the antisoliton, where \(\tilde{V}_\ell(x) = \frac{1}{\ell} V_{\ell-1}(\frac{x}{\ell})\). An incident wave far to the left is given by

\[
\psi_k(x) = e^{ikx} \begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

(12)
with $\theta = \tan^{-1} \frac{k}{m}$. It scatters without reflection through the antisoliton becoming

$$\psi_k(x) = e^{ikx} \left( e^{i\delta_k} \right)$$

(13)

where $\delta_k(k)$ is the phase shift for the bosonic potential $V_k(x)$ and $\tilde{\delta}_k(k)$ is the phase shift for the bosonic potential $\tilde{V}_k(x)$. It then scatters without reflection through the soliton giving

$$\psi_k(x) = e^{ikx} \left( e^{i(\tilde{\delta}_k+\delta_k)} \right).$$

(14)

By rescaling the results from the Appendix, we easily obtain

$$\tilde{\delta}_k(k) = \delta_k(k) - 2 \tan^{-1} \frac{m}{k}$$

(15)

so that for the soliton/antisoliton pair,

$$\delta_F(k) = \delta_B(k) - 2 \tan^{-1} \frac{m}{k}$$

(16)

and thus for a single soliton

$$\delta_F(k) = \delta_B(k) - \tan^{-1} \frac{m}{k}$$

(17)

in both models. This result has been also been obtained in [2] and [3]. Through this analysis, we see that the deficit between the boson and fermion phase shifts is necessary so that eq. (13) correctly solves the Dirac equation in the region where $V_F(x) = -m$.

As a final check on eq. (16), we have calculated the bosonic and fermionic phase shifts numerically for a sequence of field configurations that interpolates between the trivial vacuum and a widely separated soliton/antisoliton pair. The intermediate configurations are not reflectionless (or supersymmetric), so we must solve for the phase shifts in the positive and negative parity channels separately, using the techniques of [6] generalized to include fermions.

Since $\delta_F(0) \neq \delta_B(0)$, Levinson’s theorem requires that the spectrum of fermionic and boson bound states differ. The difference is that, although there is a fermionic bound state for every bosonic bound state, the mode at $\omega = 0$ only counts as $\frac{1}{2}$ for the fermions. (The fermionic states at threshold also count as $\frac{1}{2}$, the same as in the boson case.) We can see this result analytically by observing that the residue of the pole at $k = im$ in $T_F$ is half the residue of the pole at $k = im$ in $T_B$ because of eq. (17). To see it in a more physical way, we imagine doubling the spectrum by turning $\phi$ into a complex scalar and $\Psi$ into a Dirac fermion. Then in a soliton background $\phi$ would have two zero-energy bound states, one involving its real part and one involving its imaginary part. However, $\Psi$ would have only a single zero-energy bound state, with wavefunction given by

$$\psi(x) = \left( e^{-\int_0^x V_F(y) dy} \right)$$

with $V_F(x) = U'(\phi_0)$. The corresponding solution with only an lower component is not normalizable; for an antisoliton, we would find the same situation with upper and lower
components reversed. Thus when we reduce to a Majorana fermion, we count this state as a half.

We note that the fermionic phase shift and bound state spectrum are simply given by the average of the results we would obtain for the two bosonic potentials \( V(x) \) and \( \tilde{V}(x) \). We also note that just as the bosonic zero mode arises because the soliton breaks translation invariance, the fermionic zero mode arises as a consequence of broken supersymmetry invariance (which we can think of as breaking translation invariance in a fermionic direction in superspace). For a soliton solution, only the supersymmetry generator \( Q_- \) is broken, while \( Q_+ \) is left unbroken (again the situation is reversed for an antisoliton). Thus since the supersymmetry is only half broken, it is not surprising that the corresponding zero mode counts only as a half. In both cases, acting with the broken generator on the soliton solution gives the corresponding zero mode.

These results generalize to any supersymmetric potential \( U(\phi) \) that supports a soliton \( \phi_0 \) with \( \phi_0(x) = -\phi_0(-x) \). We can still consider eq. (10), with \( V(x) \) and \( \tilde{V}(x) \) replaced by

\[
V(x) = U'(\phi_0)^2 + U(\phi_0)U''(\phi_0) - m^2
\]

and

\[
\tilde{V}(x) = U'(\phi_0)^2 - U(\phi_0)U''(\phi_0) - m^2.
\]

These are symmetric, though now not necessarily reflectionless, bosonic potentials. We decompose their solutions into symmetric and antisymmetric channels. For \( x \to \pm \infty \), these solutions are given in terms of phase shifts as

\[
\begin{align*}
\eta^S_k(x) &= \cos(kx \pm \delta^S(k)) & \eta^A_k(x) &= \sin(kx \pm \delta^A(k)) \\
\tilde{\eta}^S_k(x) &= i \cos(kx \pm \tilde{\delta}^S(k)) & \tilde{\eta}^A_k(x) &= -i \sin(kx \pm \tilde{\delta}^A(k))
\end{align*}
\]

(21)

where the arbitrary factors of \( \pm i \) are inserted for convenience later. We define total phase shifts \( \delta(k) = \delta^S(k) + \delta^A(k) \) for \( V(x) \) and likewise for \( \tilde{V}(x) \). For all \( x \) these wavefunctions are related by

\[
\begin{align*}
\omega_k \eta^S_k(x) &= i \left( \frac{d}{dx} + U'(\phi_0) \right) \eta^A_k(x) & \omega_k \tilde{\eta}^S_k(x) &= i \left( \frac{d}{dx} - U'(\phi_0) \right) \tilde{\eta}^A_k(x) \\
\omega_k \tilde{\eta}^A_k(x) &= i \left( \frac{d}{dx} + U'(\phi_0) \right) \tilde{\eta}^S_k(x) & \omega_k \hat{\eta}^S_k(x) &= i \left( \frac{d}{dx} - U'(\phi_0) \right) \hat{\eta}^A_k(x)
\end{align*}
\]

(22)

so that the solutions to the Dirac equation are

\[
\begin{align*}
\psi^+_k(x) &= \left( \begin{array}{c}
\eta^S_k(x) \\
\tilde{\eta}^A_k(x)
\end{array} \right) & \text{and} & \\
\psi^-_k(x) &= \left( \begin{array}{c}
\tilde{\eta}^S_k(x) \\
\eta^A_k(x)
\end{array} \right)
\end{align*}
\]

(23)

with positive and negative parity respectively.

The total fermion phase shift \( \delta_F(k) \) is is given by the sum of the phase shifts \( \delta^\pm(k) \) for the parity eigenstates. Combining eq. (21) and eq. (22), we find that the bosonic phase shifts in the symmetric and antisymmetric channels are related by

\[
\delta^{(S,A)}(k) = \delta^{(A,S)}(k) + \tan^{-1} \frac{m}{k}
\]

(24)
and the fermionic phase shifts in the positive and negative parity channels are related to the bosonic phase shifts by
\[ \delta^\pm(k) = \frac{1}{2} \left( \delta^{(S,A)}(k) + \delta^{(A,S)}(k) \right). \]

Combining eq. (24) and eq. (25) we obtain the same result, eq. (17) as we derived in the reflectionless case.

**III. ONE-LOOP CORRECTION TO THE SOLITON ENERGY**

To compute the one-loop correction to the energy, we follow the method of [6,7] and sum the quantity \( \frac{1}{2} \omega \) over bosonic and fermionic states, with the fermions entering with a minus sign as usual. We will discuss the case of an isolated soliton, but as in [6], we have checked that we obtain the same result by considering a continuous sequence of soliton-antisoliton pairs as a function of separation, \( x_0 \), in the limit \( x_0 \to \infty \). This process gives another check on the weighting of states described in the previous section.

We must sum over the bound states and over the continuum. For the latter we use an integral over \( k \) weighted by the density of states, which for both bosons and fermions is obtained using
\[ \rho(k) = \rho_0(k) + \frac{1}{\pi} \frac{d\delta}{dk} \]

where \( \rho_0 \) is the free density of states.

Thus our formal expression for the energy correction is
\[ \Delta H = \frac{1}{2} \sum_j \omega^B_j - \frac{1}{2} \sum_j \omega^F_j + \int_0^\infty \frac{dk}{2\pi \omega} \left( \frac{d\delta_B}{dk} - \frac{d\delta_F}{dk} \right) \]

where the states at threshold and the fermion bound state at \( \omega = 0 \) are weighted by \( \frac{1}{2} \) as discussed above. The free density of states has cancelled between bosons and fermions, as required by supersymmetry. To avoid infrared problems later, we use Levinson’s theorem to rewrite eq. (27) as
\[ \Delta H = \frac{1}{2} \sum_j (\omega^B_j - m) - \frac{1}{2} \sum_j (\omega^F_j - m) + \int_0^\infty \frac{dk}{2\pi (\omega - m)} \left( \frac{d\delta_B}{dk} - \frac{d\delta_F}{dk} \right) \]

where the \( \frac{1}{2} \) in eq. (27) has cancelled between bosons and fermions.

The continuum integral in eq. (28) is still logarithmically divergent at large \( k \), as we should expect since we have not yet included the contribution from the counterterm. As discussed in the Introduction, we can isolate this divergence in the contributions from the low order Born approximations to the phase shifts \( \delta_B \) and \( \delta_F \). We then identify these contributions with specific Feynman graphs, subtract the Born approximations, and add back in the associated graphs. For the boson, the divergence comes from the first Born approximation, which corresponds exactly to the tadpole graphs with a bosonic loop. For the fermion, the source of the divergence is more complicated: we subtract the first Born approximation
to the fermionic phase shift and the piece of the second Born approximation that is related to it by the spontaneous symmetry breaking of $\phi$. This subtraction corresponds exactly to subtracting the tadpole graph with a fermionic loop and the part of the graph with two external bosons and a fermionic loop that is related to the tadpole graph by spontaneous symmetry breaking (the rest of the two-point function is then finite). For both boson and fermion this subtraction amounts to simply subtracting the term proportional to $\frac{1}{k}$ that cancels the leading $\frac{1}{k}$ behavior of the phase shift at large $k$. We can identify the coefficient of these $\frac{1}{k}$ terms with the coefficients of the logarithmic divergences in the corresponding diagrams. As a result, by computing the divergences in the bosonic and fermionic diagrams, we obtain a check on eq. (17), to leading order in $\frac{1}{k}$ for $k$ large.

Of course we must add back all that we have subtracted, together with the contribution from the counterterm. To do so we must consider renormalization. We will use a simple renormalization scheme that is consistent with supersymmetry, in which we introduce only the subtraction

$$L \to L - CU''(\phi)U(\phi) - CU''(\phi)\bar{\Psi}\Psi$$

(29)

which is equivalent to

$$U(\phi) \to U(\phi) + \frac{\lambda}{m^2}CU''(\phi)$$

(30)

and thus preserves supersymmetry. We fix the coefficient $C$ by requiring that the boson tadpole (which includes contributions from both boson and fermion loops as we have described above) vanish. In this scheme, the counterterm completely cancels the terms we have subtracted from eq. (28), so there is nothing to add back in. In the sine-Gordon theory, this scheme also makes the physical mass of the boson equal to $m$, while in the $\phi^4$ theory, there is a one-loop correction to the physical mass of the boson from the diagram with two three-boson vertices, giving a physical mass of $m - \frac{\lambda}{4m\sqrt{3}}$. For us it is more important to guarantee that the tadpole graphs vanish, assuring us that we have chosen the correct vacuum for the theory, than to have the physical mass equal to the Lagrangian parameter $m$; for the sine-Gordon case we happen to be able to do both at once. Furthermore, such renormalization conditions can be applied uniformly to arbitrary $U(\phi)$.

Thus the effect of regularization and renormalization in our renormalization scheme is to subtract

$$\delta^{(1)}(k) = \delta_B^{(1)}(k) - \delta_F^{(1)}(k) = \frac{m}{k}$$

(31)

from the difference of the boson and fermion phase shifts, giving

$$\Delta H = \frac{1}{2} \sum_j (\omega_j^B - m) - \frac{1}{2} \sum_j (\omega_j^F - m) + \int_0^\infty \frac{dk}{2\pi} (\omega - m) \left( \frac{d\delta_B}{dk} - \frac{d\delta_F}{dk} - \frac{d\delta^{(1)}}{dk} \right)$$

$$= -\frac{m}{4} + \int_0^\infty \frac{dk}{2\pi} (\omega - m) \frac{d}{dk} \left( \tan^{-1} \frac{m}{k} - \frac{m}{k} \right) = -\frac{m}{2\pi}$$

(32)

for both the kink and sine-Gordon soliton. This result agrees with [3] and [4], and disagrees with [1], [2], and [5]. As pointed out in [3], in the case of the sine-Gordon soliton, it also
agrees with the result obtained from the Yang-Baxter equation assuming the factorization of the S-matrix [9].

We note that in the end this result depended only on eq. (17) and its implications for Levinson’s theorem. Thus, since eq. (17) holds in general for antisymmetric soliton solutions, eq. (32) gives the one-loop correction to the energy in our renormalization scheme of any supersymmetric soliton that is antisymmetric under reflection.

**IV. SUPERSYMMETRY ALGEBRA AND THE CENTRAL CHARGE**

Our second application of the apparatus developed in §II is to compute the one-loop quantum correction to the central charge in the presence of the kink or sine-Gordon soliton.

First we summarize the supersymmetry algebra. We define

\[
Q_\pm = \frac{1 \mp i \gamma^1}{2} \frac{m^2}{\lambda} \int \left( \psi(x) \right) \gamma^0 \psi dx = \frac{m^2}{\lambda} \int (\Pi \psi_\pm + (\phi' \pm U) \psi_\pm) \ dx
\]  

(33)

where \( \psi_\pm = \frac{1 \mp i \gamma^1}{2} \psi \) and \( Q_\pm = \frac{1 \mp i \gamma^1}{2} Q \). Using the canonical equal-time (anti)commutation relations, we have

\[
\frac{m^2}{\lambda} \{i \psi_\pm(x), \psi_\pm(y)\} = i \delta(x - y)
\]

\[
\frac{m^2}{\lambda} [\phi(x), \Pi(y)] = i \delta(x - y)
\]

(34)

where \( \Pi = \dot{\phi} \) is the momentum conjugate to \( \phi \) and all other (anti)commutators vanish. The supersymmetry algebra is

\[
\{Q_+, Q_-\} = 2H \pm 2Z \quad \{Q_+, Q_-\} = 2P,
\]

(35)

where \( H, P, \) and \( Z \) are given classically by

\[
H = \frac{m^2}{\lambda} \int \left( \frac{1}{2} \Pi^2 + \frac{1}{2} (\phi')^2 + \frac{1}{2} U^2 + \frac{i}{2} (\psi_+ \psi' + \psi_- \psi'_-) + i U' \psi_- \psi_+ \right) \ dx
\]

\[
P = \frac{m^2}{\lambda} \int \left( \Pi \phi' + \frac{i}{2} (\psi_+ \psi' + \psi_- \psi'_-) \right) \ dx
\]

\[
Z = \frac{m^2}{\lambda} \int \phi' U \ dx.
\]

(36)

It is easy to check that \( H \) is the same Hamiltonian as would be determined canonically from eq. (1).

At the classical level, using eq. (2),

\[
H_{cl} = \frac{m^2}{2\lambda} \int \left( \phi_0'(x)^2 + U(\phi_0)^2 \right) \ dx = \mp \frac{m^2}{\lambda} \int U(\phi_0(x)) \phi_0'(x) \ dx = \mp Z_{cl},
\]

(37)

for the soliton and antisoliton respectively.

The hermiticity of \( Q_\pm \) gives the BPS bound on the expectation values of \( H \) and \( Z \) in any quantum state:
\[ \langle H \rangle \geq |\langle Z \rangle|. \quad (38) \]

Classically, the values of \( H \) and \(|Z|\) are equal so this bound is saturated. We have found a negative correction to \( H \) at one-loop, so if there is no correction to \( Z \), eq. (38) will be violated.

To unambiguously compute the corrections to the central charge for a soliton, it is easier to consider corrections to \( Q_+^2 = H + Z \), which is zero classically (for the antisoliton we should consider \( Q_-^2 \)). One reason to consider \( Q_+^2 \) rather than \( H \) and \( Z \) separately is that this quantity is finite and independent of the renormalization scheme. Using eq. (29) and eq. (30) we see explicitly that the contribution from the counterterm cancels:

\[ \Delta H_{ct} = C \int U''(\phi_0) U(\phi_0) \, dx = -C \int U''(\phi_0) \phi'_0 \, dx = -\Delta Z_{ct} \quad (39) \]

(and we only need consider the tree-level contribution since the counterterm coefficient \( C \) is already order \( \lambda^0 \)).

Next we expand \( \phi(x) = \phi_0(x) + \eta(x) \), where the soliton solution \( \phi_0 \) is an ordinary real function of \( x \). Neglecting terms of order \( \eta^3 \) and higher (which give higher-loop corrections), we obtain

\[ \langle H + Z \rangle_\phi = \frac{m^2}{2\lambda} \int \left\langle \Pi^2 + \left[ \left( \frac{d}{dx} + U'(\phi_0) \right) \eta \right]^2 + i\Psi_+ \left( \frac{d}{dx} - U'(\phi_0) \right) \Psi - i\Psi_- \left( \frac{d}{dx} + U'(\phi_0) \right) \Psi_+ \right\rangle_\phi \, dx, \quad (40) \]

where \( \langle \rangle_\phi \) denotes expectation value in the classical soliton background.

To evaluate this expression, we decompose the fields \( \eta \) and \( \Psi \) using creation and annihilation operators for the small oscillations around \( \phi_0 \). The small oscillation modes will be given in terms of the eigenmodes of the bosonic potentials \( V_t(x) \) and \( \tilde{V}_t(x) = \frac{1}{2} V_\ell^{-1}(\tilde{\phi}) \). For any mode \( \eta_k(x) \) of \( V_t(x) \) with nonzero energy \( \omega_k = \sqrt{k^2 + m^2} \), there is a mode \( \tilde{\eta}_k(x) \) of \( \tilde{V}_t(x) \) with the same energy, related by

\[ \omega_k \eta_k(x) = i \left( \frac{d}{dx} + U'(\phi_0) \right) \eta_k \]
\[ \omega_k \tilde{\eta}_k(x) = i \left( \frac{d}{dx} - U'(\phi_0) \right) \tilde{\eta}_k. \quad (41) \]

We use these wavefunctions to obtain

\[ \eta(x) = \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi \omega_k}} \left( a_k \eta_k(x) e^{-i\omega_k t} + a_k^\dagger \eta_k^*(x) e^{i\omega_k t} \right) + \eta_{\omega=0}(x) a_{\omega=0} \]
\[ \Psi(x) = \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi \omega_k}} \left( b_k \psi_k(x) e^{-i\omega_k t} + b_k^\dagger \psi_k^*(x) e^{i\omega_k t} \right) + \psi_{\omega=0}(0) b_{\omega=0} \quad (42) \]

where \( \eta_{-k}(x) = \eta_k^*(x) \), the creation and annihilation operators obey

\[ [a_k, a_{k'}^\dagger] = [b_k, b_{k'}^\dagger] = \delta(k - k') \quad (43) \]
with all other (anti)commutators vanishing, and

$$\psi_k(x) = \sqrt{\omega_k} \left( \eta_k(x) \right).$$  \hfill (44)

We note that the integral over \(k\) also includes discrete contributions from the bound states (which correspond to imaginary values of \(k\)). These are understood to give discrete contributions to the results that follow (with Dirac delta functions replaced by Kronecker delta functions appropriately). However, we have explicitly indicated the contribution from the bound states at \(\omega = 0\) following [10].

We normalize the wavefunctions \(\eta_k\) such that

$$\int \frac{dk}{2\pi} \eta_k(x)^* \eta_k(y) = \delta(x - y)$$  \hfill (45)

which implies

$$\int \frac{dk}{2\pi} \tilde{\eta}_k(x)^* \tilde{\eta}_k(y) = \delta(x - y).$$  \hfill (46)

With this normalization, the fields \(\eta\) and \(\Psi\) obey canonical commutation relations. Elementary algebra yields

$$\left( \frac{d}{dx} + U'(\phi_0) \right) \eta = -i \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi}} \sqrt{\omega_k} \left( a_k \tilde{\eta}_k(x) e^{-i\omega_k t} - a_k^* \eta_k(x) e^{i\omega_k t} \right)$$

$$\Pi(x) = -i \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi}} \sqrt{\omega_k} \left( a_k \eta_k(x) e^{-i\omega_k t} - a_k^* \eta_k(x) e^{i\omega_k t} \right)$$

$$\Psi_+ = \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi}} \left( b_k \eta_k(x) e^{-i\omega_k t} + b_k^* \eta_k(x) e^{i\omega_k t} \right)$$

$$\Psi_- = \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi}} \left( b_k \eta_k(x) e^{-i\omega_k t} + b_k^* \eta_k(x) e^{i\omega_k t} \right)$$

$$i \left( \frac{d}{dx} + U'(\phi_0) \right) \Psi_+ = \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi}} \omega_k \left( b_k \tilde{\eta}_k(x) e^{-i\omega_k t} - b_k^* \tilde{\eta}_k(x) e^{i\omega_k t} \right)$$

$$i \left( \frac{d}{dx} - U'(\phi_0) \right) \Psi_- = \sqrt{\frac{\lambda}{m^2}} \int \frac{dk}{\sqrt{4\pi}} \omega_k \left( b_k \eta_k(x) e^{-i\omega_k t} - b_k^* \eta_k(x) e^{i\omega_k t} \right).$$  \hfill (47)

Thus we find

$$\langle H + Z \rangle_\phi = \int dx \int \frac{dk}{8\pi} \omega_k |\eta_k(x)|^2$$

$$+ \int dx \int \frac{dk}{8\pi} \omega_k |\tilde{\eta}_k(x)|^2$$

$$- \int dx \int \frac{dk}{8\pi} \omega_k |\eta_k(x)|^2 - \int dx \int \frac{dk}{8\pi} \omega_k |\tilde{\eta}_k(x)|^2 = 0$$  \hfill (48)

and the BPS bound remains saturated. (If we instead considered an antisoliton, we would find the same result for \(\langle Q^2 \rangle_\phi = \langle H - Z \rangle_\phi\), with the roles of \(\Psi_+\) and \(\Psi_-\) reversed.) Our result disagrees with [3] and [4], which claim that there is no correction to the central charge at one loop in this renormalization scheme. We note that this result did not depend on any specific properties of \(U\), so it holds for any supersymmetric soliton satisfying eq. (2).
The second line of eq. (48) is simply the unregulated fermionic contribution to the energy, and is explicitly equal to minus the average of the contributions from the bosonic potentials \( V_\ell \) and \( \tilde{V}_\ell \), in agreement with what we found in §II. As a final consistency check, we recalculate the full one-loop correction to the energy and central charge using our expansion in terms of quantum fields. For \( \Delta H \) we obtain, again neglecting \( \eta^3 \) terms,

\[
\Delta H = \langle H \rangle_\phi - H_{cl} = \frac{m^2}{2\lambda} \int \left\langle \Pi^2 + \eta \left( -\frac{d^2}{dx^2} + U'(\phi_0)^2 + U(\phi_0)U''(\phi_0) \right) \eta \right. \\
\left. + i\Psi_+ \left( \frac{d}{dx} - U'(\phi_0) \right) \Psi_- + i\Psi_- \left( \frac{d}{dx} + U'(\phi_0) \right) \Psi_+ \right\rangle dx \\
= \Delta H_{ct} + \int dx \int \frac{dk}{4\pi} \omega_k |\eta_k(x)|^2 - \int dx \int \frac{dk}{8\pi} \omega_k \left( |\tilde{\eta}_k(x)|^2 + |\eta_k(x)|^2 \right). \tag{49}
\]

To relate this expression to the formalism of §III, we consider the Green’s function for the bosonic field

\[
G(x, y, t) = i T \left\langle \eta(x, t) \eta(y, 0) \right\rangle \\
= i \int \frac{dk}{4\pi \omega_k} \left( e^{i\omega_k t} \eta^*_k(x) \eta_k(y) \Theta(t) + e^{-i\omega_k t} \eta_k(x) \eta^*_k(y) \Theta(-t) \right) \tag{50}
\]

and its Fourier transform

\[
G(x, y, \omega) = \int G(x, y, t) e^{i\omega t} dt = \int \frac{dk}{2\pi} \left( \frac{\eta_k(x) \eta^*_k(y)}{\omega^2 - \omega_k^2 - i\epsilon} \right) \tag{51}
\]

whose trace gives the density of states according to

\[
\rho_B(\omega) = \frac{2\omega}{\pi} \int G(x, x, \omega) dx \tag{52}
\]

giving as a result

\[
\rho_B(k) = \frac{1}{\pi} \int dx |\eta_k(x)|^2. \tag{53}
\]

Similarly for the fermions we find

\[
\rho_F(k) = \frac{1}{2\pi} \int dx \left( |\eta_k(x)|^2 + |\tilde{\eta}_k(x)|^2 \right). \tag{54}
\]

These results enable us to verify that eq. (19) is in agreement with eq. (32).

In the exact same way, we can calculate the correction to \( Z \) directly. We start from the classical expression for \( Z \) in eq. (36) and expand about the classical solution \( \phi = \phi_0 \), giving

\[
\Delta Z = \langle Z \rangle_\phi - Z_{cl} \\
= \Delta Z_{ct} + \frac{m^2}{2\lambda} \int \left\langle \left( \frac{d}{dx} + U' \right) \eta \right. \\
\left. \right\rangle dx \\
= \Delta Z_{ct} + \frac{m^2}{2\lambda} \int \left\langle \left( \frac{d}{dx} + U' \right) \eta \right. \\
\left. \right\rangle dx. \tag{55}
\]
After substituting the expansions of eq. (47) we obtain
\[
\Delta Z = \Delta Z_{ct} + \int dx \int \frac{dk}{8\pi} \omega_k |\tilde{\eta}_k(x)|^2 - \int dx \int \frac{dk}{8\pi} \omega_k |\eta_k(x)|^2 \\
= \frac{1}{4} \sum_j (\bar{\omega}_j - m) - \frac{1}{4} \sum_j (\omega_j - m) + \int dk (\omega_k - m) \frac{d}{dk} \left( \tilde{\delta}_1(k) - \delta_1(k) + 2\delta^{(1)}(k) \right) \\
= \frac{m}{4} - \int \frac{dk}{2\pi} (\omega - m) \frac{d}{dk} \left( \tan^{-1} \frac{m}{k} - \frac{m}{k} \right) = \frac{m}{2\pi} = -\Delta H. \tag{56}
\]

V. CONCLUSIONS

We have shown that, in a simple renormalization scheme, the the one-loop correction to the energy of any antisymmetric soliton solution to a 1+1 dimensional supersymmetric theory of the form of eq. (1) is given by $\Delta H = -m/2\pi$. Furthermore, we have shown that independent of scheme, the BPS bound is saturated at one loop.

We have seen that the zero-point energies of bosonic and fermionic oscillations around supersymmetric solitons are closely related, but do not cancel completely. In particular, their scattering phase shifts, and thus their densities of states and bound state spectra, are forced to differ by the effect of the soliton’s topology on the Dirac equation. The nontrivial topology of the soliton also gives rise to a nontrivial central charge, which receives corresponding corrections at one loop. The key technical ingredients in our calculation — the ability to calculate corrections to configurations that are not solutions to the classical field equations, the use of Levinson’s theorem to guide our treatment of the bound states at zero energy and at threshold, and the subtraction of the Born approximation to regulate integrals in a cutoff-independent fashion — enable us to resolve subtleties regarding cutoffs, boundary conditions, and the counting of states that have plagued earlier calculations.

APPENDIX: PROPERTIES OF $V_\ell$

In this section we review the properties of solutions of
\[
\left( -\frac{d^2}{dx^2} + V_\ell(x) \right) \eta(x) = k^2 \eta(x) \tag{57}
\]
with
\[
V_\ell(x) = - \left( \frac{\ell + 1}{\ell} \right) m^2 \text{sech}^2 \frac{m x}{\ell}. \tag{58}
\]
for an integer $\ell$. For more details, see [6] and [11]. The scattering from these potentials is reflectionless, with phase shift
\[
\delta_\ell(k) = 2 \sum_{j=1}^\ell \tan^{-1} \left( \frac{jm}{\ell k} \right) \tag{59}
\]
and bound states at
\[ k^2 = -\left(\frac{mj}{\ell}\right)^2 \] (60)

for \( j = 0, \ldots, \ell \). The bound state at \( k^2 = 0 \) is right at threshold, and corresponds to a state that goes to a constant as \( x \to \pm \infty \) (generically, a state at \( k = 0 \) goes to a straight line, but not necessarily one with zero slope). This state counts as a half [12] in Levinson’s theorem, so we refer to it as a “half-bound” state.

Since \( V_\ell \) is symmetric, we can separate the spectrum of wavefunctions into symmetric and antisymmetric channels. The symmetry of the bound states will alternate, with the lowest energy bound state being symmetric. The phase shift can also be decomposed into contributions from the two channels, with

\[ \delta_\ell(k) = \delta_\ell^S(k) + \delta_\ell^A(k). \] (61)

That the scattering is reflectionless is equivalent to

\[ \delta_\ell^S(k) = \delta_\ell^A(k). \] (62)

Levinson’s theorem for the two channels gives [13]

\[ \delta_\ell^A(0) = \pi n^A \]
\[ \delta_\ell^S(0) = \pi (n^S - \frac{1}{2}) \] (63)

where \( n^A \) and \( n^S \) are the numbers of antisymmetric and symmetric bound states, with threshold states counted as one half. Thus we see that the threshold states are essential in reconciling eq. (62) with eq. (63).

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