Vortex Lattices in Rotating Atomic Bose Gases with Dipolar Interactions

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We show that dipolar interactions have dramatic effects on the groundstates of rotating atomic Bose gases in the weak interaction limit. With increasing dipolar interaction (relative to the net contact interaction), the mean-field, or high filling fraction, groundstate undergoes a series of transitions between vortex lattices of different symmetries: triangular, square, “stripe”, and “bubble” phases. We also study the effects of dipolar interactions on the quantum fluids at low filling fractions. We show that the incompressible Laughlin state at filling fraction $\nu = 1/2$ is replaced by compressible stripe and bubble phases.

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Ultra-cold atomic Bose gases have emerged as remarkable systems with which to study the unusual response of Bose-condensed systems to rotation. Experiments have allowed the imaging of lattices of quantised vortices\textsuperscript{1,2}, and the study of their collective dynamics\textsuperscript{2}. New vortex lattice structures in two-component condensates have been observed\textsuperscript{2}, and a novel regime of vortex density in which the vortex cores overlap\textsuperscript{1,2} has been accessed\textsuperscript{2}. Theory shows that, at very high vortex density\textsuperscript{2,4}, atomic Bose gases should undergo a transition into novel uncondensed phases closely related to the incompressible liquid of the fractional quantum Hall effect\textsuperscript{1,2,5,6,7,8}.

Although Bose condensation can occur for a non-interacting Bose gas, the formation of arrays of quantised vortices under rotation relies on non-vanishing (repulsive) interparticle interactions. In typical atomic Bose condensates, the interactions are so short-ranged that they can be viewed as local (contact) interactions. However, significant additional non-local interactions can arise if the atoms have intrinsic or induced electric or magnetic dipole moments\textsuperscript{9,10}. The recent achievement of the Bose condensation of chromium (which has a large permanent magnetic dipole moment)\textsuperscript{10} has opened the door to the experimental study of dipolar-interacting Bose gases. It is important to ask what are the effects of non-local interactions on the properties of the vortex lattices.

In this paper, we study the effects of dipolar interactions on the groundstate of a weakly interacting atomic Bose gas under rotation. We show that the additional non-local interaction leads to dramatic changes in the nature of the groundstate. Within mean-field theory, the triangular vortex lattice predicted for contact interactions is replaced by vortex lattices of different symmetries: square, “stripe” and “bubble” phases. Furthermore, we study the properties at very high vortex density where the groundstates for contact interactions are incompressible quantum fluids. We show that with increasing dipolar interactions the Laughlin state, which is the groundstate for pure contact interactions at filling fraction $\nu = 1/2$\textsuperscript{4}, is replaced by compressible states that are well described by stripe and bubble phases.

We consider a system of bosonic atoms of mass $M$ confined to a harmonic trap with cylindrical symmetry about the $z$-axis. We denote the natural frequencies of the trap by $\omega_\parallel$ and $\omega_\perp$ in the axial and transverse directions, and the associated trap lengths by $a_{\parallel,\perp} \equiv \sqrt{\hbar/(M\omega_\parallel,\perp)}$. The particles are taken to interact through both contact interactions and additional dipolar interactions. We consider the atomic dipole moments to be aligned with the $z$-axis, as in the experiments reported in Ref.\textsuperscript{11}, such that the net two-body interaction is

\begin{equation}
V(r) = \frac{4\pi\hbar^2 a_s}{M} \delta^3(r) + C_d \frac{x^2 + y^2 - 2z^2}{(x^2 + y^2 + z^2)^{\gamma/2}} \quad (1)
\end{equation}

where $a_s$ is the $s$-wave scattering length arising from contact interactions\textsuperscript{12}, and $C_d$ is a measure of the strength of the dipolar interactions.

We study the regime of weak interactions, when the mean interaction energy per particle is small compared to the trap energies, $\hbar\omega_{\perp,\parallel}$. The single particle states are then restricted to the groundstate of the trap in the $z$-direction (2D) and to the lowest Landau level (LLL) of the $x - y$ motion\textsuperscript{13}. While non-rotating gases are typically far from the weak interaction limit, this limit can be approached at high angular momentum owing to the reduction in particle density through the centrifugal spreading of the cloud\textsuperscript{2}. In this limit the interactions are fully specified by the Haldane pseudopotentials\textsuperscript{11}, $V_m$: the interaction energy of a pair of particles with relative angular momentum $m$. For bosons, symmetry of the wavefunction means that only even $m$ contribute. The precise form of the dipolar pseudopotentials depends on a non-trivial way on the trap asymmetry $a_{\parallel}/a_{\perp}$. For simplicity of presentation, we study the limit $a_{\parallel}/a_{\perp} \to 0$, in which the dipole forces fall off most quickly with
increasing distance. To leading order in $a/\alpha$, we find

$$V_0 = \sqrt{\frac{2}{\pi}} \frac{\hbar^2 a_s}{M a^2} + \sqrt{\frac{2}{\pi}} \frac{C_d}{a^2} - \sqrt{\frac{\pi}{2}} \frac{C_d}{a^2} \sqrt{\frac{\pi}{2}}$$

$$V_{m>0} = \sqrt{\frac{\pi}{2}} (2m-3)!! \frac{C_d}{m! \sqrt{2}} a^2$$

$V_0$ represents the net local interaction, which has contributions from both the contact and dipolar interactions; $V_{m>0}$ represent non-local interactions which arise only from the dipolar interaction. We consider the relative sizes of the non-local and local interactions to be variable, either by tuning the dipolar interaction, $C_d$, or by tuning $a_s$ close to a Feshbach resonance\cite{15} (which can allow $a_s$ to become negative). We quantify their relative size by the ratio

$$\alpha \equiv \frac{V_2}{V_0}$$

We are interested in bulk properties, so perform calculations on a periodic rectangular geometry (a torus) which accurately describes the centre of an atomic gas where the particle density is approximately uniform. In the weak interaction limit the number density of vortices is $n_\nu = 1/(\pi a^2)$, so a torus with sides $a$ and $b$ contains $N_\nu = ab/(\pi a^2)$ vortices.

First, we describe the mean-field groundstates. Within Gross-Pitaevskii (GP) mean-field theory the groundstate is assumed to be fully condensed, with condensate wavefunction, $\psi(r)$. In the weak-interaction limit, $\psi(r)$ is found by minimising the mean interaction energy for fixed average particle density, with the wavefunction constrained to states in the 2D LLL\cite{13}. The problem is mathematically equivalent to the Ginzburg-Landau model for a type-II superconductor close to $H_c2$\cite{16} generalised to a non-local interaction. A phenomenological model of this kind has been discussed in the context of superconductivity\cite{17}, but we are not aware of general solutions for the vortex lattice groundstates. We have found the mean-field groundstates for the interactions by numerical minimisation on a torus with up to $N_\nu = 24$ vortices (i.e. exploring periodic states with up to 24 vortices in the unit cell). As a function of $\alpha$, the groundstate undergoes a series of transitions between states of different translational symmetries. Representative images of the particle distributions in these states are shown in Fig.113.

The groundstates we find are: a triangular lattice of single vortices ($0 \leq \alpha \leq 0.20$); a square lattice of single vortices ($0.20 \leq \alpha \leq 0.24$); a stripe phase ($0.24 \leq \alpha \leq 0.60$)\cite{18}. The stripe phase consists of broad lines of high particle density separated by rows of closely-spaced vortices. The vortices are ordered along the rows, so the state is a "stripe crystal" with crystalline order in both directions. For $\alpha \geq 0.60$ the states consist of clusters of high particle density arranged in a triangular lattice. Owing to the similarity to crystalline states of electrons in high Landau levels\cite{20}, we refer to these as "bubble" states. We find a sequence of bubble states, which we label by the number $q$ of vortices associated to each bubble. The bubble states we find as groundstates are $q = 4$ ($0.60 \leq \alpha \leq 0.91$), $q = 5$ ($0.91 \leq \alpha \leq 1.4$), $q = 6$ ($1.4 \leq \alpha \leq 2.0$), $q = 7$ ($2.0 \leq \alpha \leq 2.7$), $q = 8$ ($2.7 \leq \alpha$). The particle distributions of the bubble states with $q \geq 4$ resemble Fig.1(d), with additional vortices confined to the honeycomb network separating the bubbles of particles. We have not looked for states with $q > 8$, but expect that states of arbitrarily large $q$ will appear.

We believe that it should be possible experimentally to access these new vortex lattice groundstates. To make contact with experimental parameters, we now examine the case of a spherically symmetric trap ($a_\parallel = a_\perp$). In this case the triangular lattice is replaced by new groundstates if $a_s$ is tuned\cite{17} to values $a_s \lesssim -0.13 C_d M/\hbar^2$. Note that, despite this negative value of $a_s$, the contribution of the dipolar interaction makes $V_0$ positive, which, in the weak interaction limit, is sufficient to ensure stability to collapse. Under these circumstances, for a mean particle density $n_{3d}$, the interaction energy per particle is of order $C_d n_{3d}$, so the weak interaction limit requires

\begin{enumerate}
\item[(a)] Triangular vortex lattice ($a/b = \sqrt{3}/2$);
\item[(b)] Square vortex lattice ($a/b = 1.0$);
\item[(c)] "Stripe" phase (shown for $\alpha = 0.528$, $a/b = 0.608$);
\item[(d)] $q = 4$ "bubble" phase ($a/b = \sqrt{3}/2$).
\end{enumerate}

FIG. 1: Contour plots of the particle densities for condensed states on a torus with $N_\nu = 16$ vortices. The light (dark) shading indicates high (low) particle density (on arbitrary scales). (a) Triangular vortex lattice ($a/b = \sqrt{3}/2$); (b) Square vortex lattice ($a/b = 1.0$); (c) “Stripe” phase (shown for $\alpha = 0.528$, $a/b = 0.608$); (d) $q = 4$ “bubble” phase ($a/b = \sqrt{3}/2$).
$n_{3d} \lesssim \hbar \omega_\perp /C_d$. Choosing the value of $C_d$ appropriate for chromium\cite{11}, which has a magnetic dipole moment of 6 Bohr magnetons, and taking $\omega_\perp = 2 \pi x \times 100$ rad s$^{-1}$, the weak interaction limit should be a good approximation for densities less than about $10^{14}$ cm$^{-3}$.

We now turn to discuss the groundstates beyond the mean-field approximation. In Ref.\cite{6} it was shown that the parameter controlling the validity of mean-field theory for a rotating atomic Bose gas is the filling fraction, $\nu = n/n_V$, where $n$ and $n_V$ are the number densities (per unit area) of particles and vortices. For contact interactions, the triangular vortex lattice predicted by mean-field theory was shown to be destroyed by quantum fluctuations for $\nu < \nu_{c,tri}$, with $\nu_{c,tri} \sim 6.0$, and replaced by new groundstates which include incompressible liquids closely related to fractional quantum Hall states\cite{4,6,7,8}.

We have studied the effects of dipolar interactions on these strongly-correlated groundstates using exact diagonalisation studies. We now return to the case of $a_\parallel /a_\perp \rightarrow 0$ and focus our attention on the filling fraction $\nu = 1/2$, for which the exact groundstate for contact interactions ($\alpha = 0$) is the $\nu = 1/2$ bosonic Laughlin state\cite{4}. This is the dominant incompressible state of rotating bosons with contact interactions. We find that the exact groundstate in the presence of dipolar interactions is well described by the Laughlin state up to $\alpha \simeq 0.5$. At this point there is an abrupt transition in the groundstate and its overlap with the Laughlin state falls to a very small value [Fig.2(a)]. Our studies show that the states that replace the Laughlin state at $\alpha \gtrsim 0.5$ are compressible phases which are well described by the mean-field groundstates discussed above. Here we present evidence of a stripe phase at $\alpha = 0.528$ and of the $q = 4$ bubble phase at $\alpha = 0.758$. (For larger $\alpha$ we find evidence of bubble phases with larger $q$.) To identify these broken-symmetry states we make extensive use of the classification of energy eigenstates by a conserved momentum\cite{21}. We express the momentum as a dimensionless vector $K = (K_x, K_y)$ using units of $2\pi h / a$ and $2\pi h / b$ for the $x$ and $y$ components, and report only positive $K_x, K_y$ [states at $(\pm K_x, \pm K_y)$ are degenerate by symmetry]. We identify the broken translational symmetry of the groundstates by making use of the fact that this leads to the appearance of quasi-degeneracies in the spectrum at wavevectors equal to the reciprocal lattice vectors of the crystalline order\cite{22}.

Evidence of stripe ordering at $\alpha = 0.528$ is presented in Fig.2(b). This shows the excitation spectrum on a torus with an aspect ratio chosen to be consistent with the mean-field groundstate. The quasi-degeneracy of the groundstate at $K = (0,0)$ with a state at $(0,3)$ indicates a strong tendency to translational symmetry breaking in the stripe pattern found in mean-field theory: three stripes lying parallel to the short axis of the torus. However, unlike the mean-field state, we find no evidence of crystalline order parallel to the stripes: the groundstate appears to be a “smectic”\cite{22} in which translational symmetry is broken in only one direction.

Evidence for the formation of the $q = 4$ bubble phase is shown in Fig.2(c). At filling fraction $\nu = 1/2$, each bubble contains $\nu q = 2$ particles, so this state can also be described as a triangular lattice of pairs of particles. The low energy states shown in Fig.2(c) as filled symbols correspond to wavevectors at the reciprocal lattice vectors of the $q = 4$ bubble state [shown in Fig.4(d)]. Although these states do not seem to be cleanly separated from the rest of the spectrum, the other low energy states (shaded symbols) can be understood as arising from a nearby (in energy) crystalline configuration in which the crystal is rotated through 90°.

The appearance of stripe and bubble states at $\nu =
1/2 indicates that these states are much more stable to quantum fluctuations than is the triangular vortex lattice (which is unstable for $\nu < \nu_{tri} \sim 6/4$). This enhanced stability can be understood within a simple Lindemann analysis, in which one asserts that quantum melting occurs when the quantum fluctuations of a lattice site exceed a multiple $c_L$ of the lattice spacing. Treating the fluctuations of each vortex independently, one expects the triangular vortex lattice to melt for $\nu < \nu_{tri} = \sqrt{3}/(2\pi c_L^2)$. For the square lattice, we find $\nu_{tri} = 1/(\pi c_L^2) = (2/\sqrt{3})\nu_{tri}$, close to that for the triangular lattice. The enhanced stabilities of the stripe and bubble phases arise from the existence of larger length scales and larger numbers of particles per unit cell (as compared to the triangular or square vortex lattices).

For the $q$ bubble phase, the bubbles form a triangular lattice with lattice constant $\sqrt{\frac{q}{\nu_0}} c_L$. Applying the Lindemann analysis to the quantum fluctuations of the centre-of-mass of a bubble, we find $\nu_{q}^2 = \pi c_L^2 - \nu_{tri}^2/q^2$. Even for the smallest ($q = 4$) bubble state, this critical filling fraction is very much smaller than that for the triangular lattice. For the stripe phases, there are two length scales: the inter-vortex separation in the directions parallel, $R_\parallel$, and perpendicular, $R_\perp$, to the stripes. One therefore expects two transitions: when fluctuations of the vortices along the stripes exceed $c_L R_\parallel$ there is a loss of order in that direction, leading to a smectic phase; when fluctuations perpendicular to the stripes exceed $c_L R_\perp$ the stripe ordering will finally be lost. Assuming that, for this final transition, of order $R_\perp / R_\parallel$ vortices must fluctuate together, we find that this loss of stripe order should occur at $\nu_{stripe} = \frac{1}{\pi c_L^2} \left( \frac{R_\perp}{R_\parallel} \right)^2 = (2/\sqrt{3})\nu_{tri}^2$. Over the range $0.24 \leq \alpha \leq 0.60$ for which the stripe is the mean-field groundstate the ratio $R_\perp / R_\parallel = 1 - 2.59$, so the critical filling fraction for the stripe can be as small as about $(1/6)\nu_{tri}^2$. While it would be desirable to have a more complete Lindemann analysis in which the collective modes of these new lattices are quantised, we believe that the simple analyses presented here capture the essential physics of the relative stability of the states to quantum fluctuations.

At filling fractions above $\nu = 1/2$ quantum fluctuations of the stripes and bubbles are strongly suppressed. For the cases in Fig. 2(b) and (c), increasing the number of particles to $N = 12$ (so that the respective filling fractions are $\nu = 2/3$ and $3/4$) leads to much improved groundstate quasi-degeneracies. Quantum fluctuations are enhanced for $\nu < 1/2$. For small non-zero $\alpha$, we find incompressible liquids at filling fractions, $\nu = p/(3p \pm 1)$, expected for composite fermions formed from bosons bound to three vortices. The state at $\nu = 1/4$ is well described by the $\nu = 1/4$ Laughlin state. For $\alpha \gtrsim 1.7$ this state is replaced by bubble phases, consistent with the expectation from the Lindemann analysis that for large $q$ these can have critical filling fractions less than $1/4$.

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