NONLINEAR WAVE INTERACTIONS FOR THE
BENJAMIN-ONO EQUATION

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Abstract. We study the interaction of suitable small and high frequency waves evolving by the flow of the Benjamin-Ono equation. As a consequence, we prove that the flow map of the Benjamin-Ono equation can not be uniformly continuous on bounded sets of $H^s(\mathbb{R})$ for $s > 0$.

1. Introduction

In this paper we continue our investigations around the Cauchy problem for the Benjamin-Ono equation

\begin{equation}
  u_t + Hu_{xx} + uu_x = 0,
\end{equation}

where $H$ denotes the Hilbert transform. The Benjamin-Ono equation is a model for the propagation of one dimensional internal waves (see [3]). In [17] it was shown that the Benjamin-Ono equation is globally well-posed in $H^1(\mathbb{R})$, (see also [11, 13, 15, 16]). Most likely the lower bound on $s$ is not optimal. The local well-posedness implies that the flow map is continuous on $H^s(\mathbb{R})$. The main purpose of this paper is to show that no further regularity holds. More precisely we are going to show that the flow map of the Benjamin-Ono equation can not be uniformly continuous on bounded sets of $H^s(\mathbb{R})$ for $s > 0$ and thus we significantly extend the recent work [14] of Molinet, Saut and the second author where it is shown that the flow map of the Benjamin-Ono equation can not be of class $C^2$ as map on $H^s(\mathbb{R})$.

Our method of proof is based on the description of the effect of a suitable small low frequency perturbation to a high frequency wave evolving by the flow of the Benjamin-Ono equation. We believe that these considerations are of independent interest and we refer to the next sections for more details. We now state our result concerning the lack of uniform continuity of the flow map of the Benjamin-Ono equation.

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Theorem 1. Let $s > 0$. There exist two positive constants $c$ and $C$ and two sequences $(u_n)$ and $(\tilde{u}_n)$ of solutions of (1.1) such that for every $t \in [0, 1]$,
\[
\sup_n \|u_n(t, \cdot\|_{H^s(\mathbb{R})} + \sup_n \|\tilde{u}_n(t, \cdot\|_{H^s(\mathbb{R})} \leq C,
\]
$(u_n)$ and $(\tilde{u}_n)$ satisfy initially
\[
\lim_{n \to \infty} \|u_n(0, \cdot\|_{H^s(\mathbb{R})} - \|\tilde{u}_n(0, \cdot\|_{H^s(\mathbb{R})} = 0,
\]
but, for every $t \in [0, 1]$,
\[
\liminf_{n \to \infty} \|u_n(t, \cdot\|_{H^s(\mathbb{R})} - \|\tilde{u}_n(t, \cdot\|_{H^s(\mathbb{R})} \geq c \sin t.
\]

The case $s < -1/2$ can be easily treated by using the high speed limits of solitary waves (see [2]). The construction of the solutions of Theorem 1 relies on a separation of the transport and the dispersion effects. This phenomenon becomes much less clear when $s \leq 0$. The statement of failure of uniform continuity is fairly strong: None of the maps $u_0 \to u(t)$ can be uniformly continuous in balls. We can obtain a similar statement for lower values of $s$ if consider uniform continuity of the map
\[
H^s \ni u_0 \to u \in C([0, 1]; H^s)
\]
instead. For instance, in [9] failure of the uniform continuity of the map (1.2) in the context of supercritical wave a Schrödinger equations is obtained but no result for the map $u_0 \to u(t)$, $t > 0$ is available.

We are inspired by the following observation on the Burgers equation. If $v$ solves the Burgers equation $v_t = vv_x$ then so does
\[
w(t, x) := v(t, x + \omega t) + \omega.
\]
However, the shift in the spatial variable is not norm continuous, considered as a map from $\mathbb{R}$ to the bounded operators on $H^s(\mathbb{R})$. Our strategy will be to construct solutions of 'fixed' frequency, localized in $x$, to which we add a smooth solution, which changes the speed of the wave.

It is worth noticing that the "instability property" of the flow of the BO equation displayed by Theorem 1 is not shared by the KdV equation which is another important model for the propagation of one directional waves. Due to the higher speed of propagation of the linearized KdV waves, the flow of the KdV equation is Lipschitz continuous on bounded sets of $H^s$, $s > -3/4$ (see [4, 5, 10]) and therefore the local flow of the Benjamin-Ono equation turns out to be quite different from that of the KdV equation. Since a Picard iteration scheme implies smooth dependence on the initial data, it is a consequence of Theorem 1 (or of the results of [14]) that the Picard iteration scheme can not be used to construct solutions to the Benjamin-Ono equation for initial data in $H^s(\mathbb{R})$. 
Our method to prove Theorem 1 is quite general and can be applied to many others PDE’s (see Remark 3 below). In particular we use only the first few integrals of the Benjamin-Ono and we do this only for the range $0 < s \leq 3/2$.

There is a number of recent papers dealing with the lack of uniform continuity for nonlinear PDE’s (see e.g. [6, 7, 8, 12]). These results are however of different nature comparing to Theorem 1. First, to our knowledge, in none of the cases considered in [6, 7, 8, 12] local well-posedness is known. Moreover, essentially all papers [6, 7, 8, 12] deal with the self interaction of a single high frequency wave which naturally lead to upper bounds on the Sobolev regularity $s$ which is consistent with the smooth dependence on the initial data for large $s$ holding in the cases considered in [6, 7, 8, 12].

The rest of the paper is organized as follows. The next section is devoted to some preliminaries. In Sections 3 and 4, we construct families of approximate solutions of the Benjamin-Ono equation. In Sections 5 and 6 we construct solutions which are close to the approximate solution. The lack of uniform continuity of the flow map (Theorem 1), proved in Section 7 is an immediate consequence.

**Notation.** In the sequel $\lambda$ will be a large real number, and $0 < \delta < 1$ will be fixed most of the time. For functions $f : \mathbb{R} \to \mathbb{R}$ we denote $f_\lambda(x) := f(x/\lambda^{1+\delta})$.

## 2. Preliminaries

In this section, we collect several preliminary results needed for the sequel. We first state a basic existence result for the Benjamin-Ono equation (see [1] [11] [17] [13] [15] [16]).

**Lemma 1.** Fix $s > 3/2$ and $\sigma \in ]3/2, s]$. Then for every $u_0 \in H^s(\mathbb{R})$ there exists a unique global solution $u \in C(\mathbb{R}; H^s(\mathbb{R}))$ of (1.1) subject to the initial data $u_0$. Moreover

$$\|u(t, \cdot)\|_{H^s(\mathbb{R})} \lesssim \|u_0\|_{H^s(\mathbb{R})}$$

if $|t| \leq \min\left(1, \frac{c\|u_0\|_{H^s(\mathbb{R})}^{-4}}{1}\right)$.

Even more is true: For initial data in $H^1(\mathbb{R})$ there exists a unique global solution $u$ with $\partial_x u \in L_{loc}^1(\mathbb{R}; L^\infty(\mathbb{R}))$ see [17]. Nevertheless we state the weaker existence result of Lemma 1 which does not depend as heavily on the specific structure of the problem.

The next lemma deals with the commutators of the Hilbert transform $H$ and scaled smooth functions with compact support.
Lemma 2. Fix $0 < \delta < 1$ and $\phi \in C^\infty_0(\mathbb{R})$. Then for any $N > 0$ there exists a positive constant $C_N$ such that for every $\alpha \in \mathbb{R}$

$$\| [H, \phi_\lambda] \cos(\lambda x + \alpha) \|_{L^2_x} \leq C_N \lambda^{-N}. \tag{2.1}$$

**Proof of Lemma 2.** Inequality (2.1) is equivalent to

$$\left\| \int_{-\infty}^{\infty} \frac{\phi_\lambda(x) - \phi_\lambda(y)}{x - y} \cos(\lambda y + \alpha) \, dy \right\|_{L^2_x} = \lambda^{\frac{1 + \delta}{2}} \left\| \int_{-\infty}^{\infty} \frac{\phi(x) - \phi(y)}{x - y} \cos(\lambda^{2+\delta} y + \alpha) \, dy \right\|_{L^2_x} \leq C_N \lambda^{-N}. \tag{2.2}$$

We fix $\eta \in C^\infty_0(\mathbb{R})$, which is identically 1 on $[-1, 1]$ and between 0 and 1 on $\mathbb{R}$. By writing

$$\frac{\phi(x) - \phi(y)}{x - y} = \eta(x - y) \int_0^1 \phi'(tx + (1 - t)y) \, dt + (1 - \eta(x - y)) \frac{\phi(x) - \phi(y)}{x - y} =: \beta(x, y) + \gamma(x, y)$$

we will deduce (2.2) after several integration by parts in the $y$ variable. Both functions $\beta$ and $\gamma$ are smooth. Integration by parts gives

$$\int_{-\infty}^{\infty} \frac{\phi(x) - \phi(y)}{x - y} \cos(\lambda^{2+\delta} y + \alpha) \, dy = \lambda^{-8N-4\delta} \int_{-\infty}^{\infty} (\partial_y^{4N} \beta(x, y) + \gamma(x, y)) \cos(\lambda^{2+\delta} y + \alpha) \, dy.$$ 

The function $\beta$ is compactly supported and the corresponding desired bound is obvious. If when differentiating $\gamma$ any derivative falls on $\eta$ then we are again in the compactly supported situation. By Minkowskis inequality it remains to provide the uniform bounds

$$\int_{-\infty}^{\infty} \|(1 - \eta(x - y)) \partial_y^{4N} [(\phi(x) - \phi(y))|x - y|^{-1}] \|_{L^2_x} \, dy \leq \int_{-\infty}^{\infty} (4N)! \|(1 - \eta(x - y)) \phi(x) |x - y|^{-1N-1} \|_{L^2_d} \, dy + \int_{-\infty}^{\infty} \|(1 - \eta(x - y)) \partial_y^{4N} \phi(y) |x - y|^{-1} \|_{L^2_d} \, dy \lesssim \int_{-\infty}^{\infty} (1 - \eta(z)) |z|^{-4N-1} \, dz \|\phi\|_{L^2} + \|\phi\|_{W^{4N,1}} \sup_y \|(1 - \eta(x - y)) |x - y|^{-1} \|_{L^2_d}. $$
where \( \| \phi \|_{W^{4N,1}} = \sum_{j=0}^{4N} \| \phi^{(j)} \|_{L^1} \). □

We will complete this section by evaluating the \( H^s \) norm of some high frequency localized smooth functions.

**Lemma 3.** Fix \( s \geq 0, \ 0 < \delta < 1, \ \alpha \in \mathbb{R} \) and \( \phi \in C_0^\infty(\mathbb{R}) \). Then

\[
\lim_{\lambda \to \infty} \lambda^{\frac{1-\delta}{2} - s} \| \phi_\lambda(x) \cos(\lambda x + \alpha) \|_{H^s} = \frac{1}{\sqrt{2}} \| \phi \|_{L^2}.
\]

**Proof of Lemma 3.** Write via the rescaling \( x \mapsto \lambda \frac{x + \alpha}{\lambda^2 - 1 + \delta} \).

\[
\| (1 + |D|^2)^{s/2}, \phi_\lambda \|_{L^2} \|_{L^2_x} = \lambda^{\frac{1-\delta}{2}} \| (1 + \lambda^{-2-2\delta} |D|^2)^{s/2}, \phi \|_{L^2} \|_{L^2_x}.
\]

Plancherel’s theorem then yields

\[
\| (1 + \lambda^{-2-2\delta} |D|^2)^{s/2}, \phi \|_{L^2} \|_{L^2} = \lambda^{\frac{1-\delta}{2}} \| (1 + \lambda^{-2-2\delta} |D|^2)^{s/2}, \phi \|_{L^2} \|_{L^2_x}.
\]

Since \( \phi \) is a Schwartz function, the above identity and (2.3) imply

\[
\| (1 + |D|^2)^{s/2}, \phi_\lambda \|_{L^2} \|_{L^2_x} \lesssim \lambda^{\frac{1-\delta}{2} + s - 2 - \delta}.
\]

We have

\[
(1 + |D|^2)^{s/2} \cos(\lambda x + \alpha) = (1 + |\lambda|^2)^{s/2} \cos(\lambda x + \alpha).
\]

Finally, we obtain by an integration by parts

\[
\int_{-\infty}^{\infty} (\phi_\lambda(x) \cos(\lambda x + \alpha))^2 \, dx = \int_{-\infty}^{\infty} \phi_\lambda^2(x) (\frac{1}{2} + \frac{1}{2} \cos(2\lambda x + 2\alpha)) \, dx
\]

\[
= \frac{1}{2} \lambda^{1+\delta} \| \phi \|_{L^2_x}^2 + O(\lambda^{-1} \| \phi \|_{L^2_x} \| \phi' \|_{L^2_x})
\]

which completes the proof of Lemma 3. □

### 3. First Construction of Approximate Solutions

In the rest of the paper, we shall use make of two smooth characteristic functions \( \phi \) and \( \tilde{\phi} \) in the usual manner. Namely, let \( \phi \in C_0^\infty(\mathbb{R}) \) be such that

\[
\phi(x) = \begin{cases} 
0, & |x| > 2, \\
1, & |x| < 1
\end{cases}
\]
and let $\tilde{\phi} \in C_0^\infty(\mathbb{R})$ be equal to one on the support of $\phi$.
For $\lambda \geq 1$, $0 < \delta < 1$ and $\omega \in \mathbb{R}$, we set
\[
U_{\lambda,\omega}(x) := -\omega \lambda^{-1/2} \tilde{\phi}_\lambda(x)
\]
which corresponds to the low frequency part of the approximate solution.
Next, we define the phase
\[
\Phi := -\lambda^2 t + \lambda x + \omega t
\]
which describes the phase shift compared to linear Benjamin-Ono waves.

Further, we define the high frequency part of the approximate solution
\[
u_h(t, x) := -\lambda^{-1 + \delta/2 - s} \phi_\lambda(x) \cos \Phi.
\]
The next lemma states that $u_{ap}(t, x)$ defined by
\[
(3.1) \quad u_{ap}(t, x) := U_{\lambda,\omega}(x) + u_h(t, x)
\]
almost solves the Benjamin-Ono equation when $s$ is not too large and $\lambda \gg 1$.

**Lemma 4.** Set
\[
F := (\partial_t + H\partial_x^2)u_{ap} + u_{ap} \partial_x u_{ap}.
\]
Then there exists a positive constant $C$ such that for $t \in \mathbb{R}$, $0 < \delta < 1$ one has
\[
\|F(t, \cdot)\|_{L^2(\mathbb{R})} \leq C(\lambda^{-\delta-s} + \lambda^{\frac{1+\delta}{2} - 2s} + \omega^2 \lambda^{-\frac{5+\delta}{2}} + |\omega| \lambda^{-\frac{5+3\delta}{2}} + |\omega| \lambda^{-2-\delta-s}).
\]

**Proof of Lemma 4.** Let $L := \partial_t + H\partial_x^2$. We compute
\[
Lu_{ap} + u_{ap} \partial_x u_{ap} = LU_{\lambda,\omega} + U_{\lambda,\omega} \partial_x U_{\lambda,\omega} + \partial_t(U_{\lambda,\omega} u_h) + u_h \partial_x u_h + Lu_h
\]
\[
= \lambda^{-1 + \delta/2 - s} \phi_\lambda \cos \Phi
\]
\[
-\lambda^{-1 + \delta/2 - s} \phi_\lambda (L + U_{\lambda,\omega} \partial_x) \cos \Phi
\]
\[
=: F_1 + F_2 + F_3 + F_4 + F_5,
\]
where we put the part of $\partial_\xi(U_{\lambda,\omega} u_h)$ with the derivative on $\cos \Phi$ into $F_5$. Using that $\phi_\lambda \tilde{\phi}_\lambda = \phi_\lambda$, we readily obtain that $F_5$ vanishes which is the crucial cancelation. It is worth noticing that both
\[
(3.3) \quad -\lambda^{-1 + \delta/2 - s} \phi_\lambda L(\cos \Phi)
\]
and

\begin{equation}
- \lambda^{-1+\delta} \phi_{\lambda} U_{\lambda, \omega} (\partial_x \cos \Phi)
\end{equation}

are “big” in $L^2$ compared to $u_{ap}$. Notice also that the term (3.3) comes from the linear part while (3.4) is a contribution coming from the nonlinear term.

Next we expand $F_4$ as

$$F_4 = \lambda^{3-s} [H, \phi_{\lambda}] \cos \Phi + 2\lambda^{1+\delta-\delta-s} H \{(\phi')_{\lambda} \sin \Phi\} - \lambda^{-\frac{5}{4} - \frac{5s}{2} - s} H \{(\phi'')_{\lambda} \cos \Phi\}.$$  

The first term is controlled in $L^2$ by Lemma 2. The $L^2$ norm of the other terms are readily estimated by $c \lambda^{-\delta-s}$. The $L^2$ norm of $F_3$ is easily controlled by $c \lambda^{1-\delta-s}$. It remains to estimate $F_1$, $F_2$ and $F_3$. The bound for the latter is obvious. The term $F_1$ can be handled as follows

$$\|F_1\|_{L^2} \lesssim \|U_{\lambda, \omega} \partial_x U_{\lambda, \omega}\|_{L^2} + \|HU'_{\lambda, \omega}\|_{L^2} \lesssim \omega^2 \lambda^{-\frac{5s}{2}} + |\omega| \lambda^{-\frac{2+3s}{2}}.$$  

The term $F_2$ is estimated as

$$\|F_2(t, \cdot)\|_{L^2} \lesssim |\omega| \lambda^{-2-\delta-s}$$

This completes the proof of Lemma 4. □

**Remark 1.** Notice that if $0 < s < 2$ the bound of Lemma 4 on $F$ can be simply written as

\begin{equation}
\|F(t, \cdot)\|_{L^2(\mathbb{R})} \lesssim \lambda \frac{\min(\delta, 1-\delta)}{2} - s
\end{equation}

if $|\omega| \leq 1$. The bound (3.5) implies that at least in $L^2(\mathbb{R})$, $u_{ap}(t, x)$ is a good approximate solution of the Benjamin-Ono equation.

## 4. Refined approximate solutions

When $s \geq 2$ one has to modify slightly the construction of approximate solutions, presented in the previous section. To avoid small frequency residual terms, we will chose the small frequency part of $u_{ap}$ to be a solution of the Benjamin-Ono equation with small frequency initial data. Let $u_{low}(t, x)$ be the solution of (1.1) with initial data

\begin{equation}
\begin{aligned}
&u_{low}(0, x) = -\omega \lambda^{-1} \tilde{\phi}_{\lambda}(x), & 0 < \delta < 1, \ \omega \in \mathbb{R}.
\end{aligned}
\end{equation}

In the next lemma, we collect several bounds for $u_{low}(t, x)$.

**Lemma 5.** Let $k \geq 0$. Then the following estimates hold, if $|t| \leq 1$, $\lambda \gg 1$ and $|\omega| \ll \lambda^{\frac{1-s}{2}},$

\begin{equation}
\|\partial_x^k u_{low}(t, \cdot)\|_{L^2(\mathbb{R})} \lesssim |\omega| \lambda^{-\frac{1+\delta}{2} - k(1+\delta)},
\end{equation}

\begin{equation}
\|\partial_x^k u_{low}(t, \cdot)\|_{L^2(\mathbb{R})} \lesssim |\omega| \lambda^{-\frac{1+\delta}{2} - k(1+\delta)},
\end{equation}
(4.3) \[ \| \partial_x u_{low}(t, \cdot) \|_{L^\infty(\mathbb{R})} \lesssim |\omega| \lambda^{-2-\delta}, \]

(4.4) \[ \| u_{low}(t, \cdot) - u_{low}(0, \cdot) \|_{L^2(\mathbb{R})} \lesssim |\omega| \lambda^{-2-\delta}. \]

**Proof of Lemma 5** Rescale by setting

(4.5) \[ v(t, x) := \lambda^{1+\delta} u_{low}(\lambda^{2+2\delta} t, \lambda^{1+\delta} x). \]

Then \( v \) is again a solution of the Benjamin-Ono equation. Since \( v(0, x) = -\omega \lambda^\delta \tilde{\phi}(x) \) we readily obtain for any \( s \geq 0 \) the bound

\[ \| v(0, \cdot) \|_{H^s} \lesssim |\omega| \lambda^\delta \]

and therefore by Lemma 1

(4.6) \[ \| v(t, \cdot) \|_{H^s} \lesssim |\omega| \lambda^\delta, \]

if \( |t| \leq \min(1, |\omega|^{-4} \lambda^{-4\delta}) \) and \( s > 3/2 \). But since the right hand-side of (4.6) contains a constant which is uniformly bounded for bounded \( s \), we conclude that (4.6) is valid for any real \( s \). The Sobolev embedding and (4.6) now give

(4.7) \[ \| v_x(t, \cdot) \|_{L^\infty} \lesssim |\omega| \lambda^\delta, \]

if \( |t| \leq \min(1, |\omega|^{-4} \lambda^{-4\delta}) \).

Using (4.5), we deduce from (4.7) by scaling back that

(4.8) \[ \| \partial_x u_{low}(t, \cdot) \|_{L^\infty} \lesssim |\omega| \lambda^{-2-\delta}, \]

if \( |t| \leq 1 \) which proves (4.3).

We now turn to the proof of (4.2) and (4.4). Differentiating (4.5) and using (4.6) (with \( s = k \)) yields

(4.9) \[ \| \partial_x^k u_{low}(t, \cdot) \|_{L^2} \lesssim |\omega| \lambda^{-\frac{k+\delta}{2}-k(1+\delta)}, \quad k = 0, 1, 2, \ldots \]

if \( |t| \leq 1 \). Estimate (4.9) proves (4.2). Next, using (4.8), (4.9) and the equation satisfied by \( u_{low} \) gives

\[ \| \partial_t u_{low}(t, \cdot) \|_{L^2} \lesssim \| \partial_x^2 u_{low}(t, \cdot) \|_{L^2} + \| \partial_x u_{low}(t, \cdot) \|_{L^\infty} \| u_{low}(t, \cdot) \|_{L^2} \lesssim |\omega| \lambda^{-2-\delta}, \]

if \( |t| \leq 1 \). We now observe that (4.4) can be deduced from the above bound via the fundamental theorem of calculus, applied to \( u_{low} \) in the time variable. This completes the proof of Lemma 5.

We now set for \( \lambda \geq 1, 0 < \delta < 1 \) and \( |\omega| \ll \lambda \frac{1-\delta}{2} \),

(4.10) \[ u_{ap}(t, x) := u_{low}(t, x) - \lambda^{-\frac{1}{2} - \frac{\delta}{2} - s} \phi_\lambda(x) \cos(-\lambda^2 t + \lambda x - \lambda t u_{low}(0, x)). \]

The above function is an approximate solution for \( \lambda \gg 1 \) and \( s > 0 \).
Lemma 6. Let $s > 0$, $0 < \delta < 1$, $|\omega| \ll \lambda^{-\frac{1-s}{2}}$ and $|t| \leq 1$. Set
\[
F := (\partial_t + H \partial_x^2) u_{ap} + u_{ap} \partial_x u_{ap}.
\]
Then there exist positive constants $C$ and $\lambda_0$ such that for $\lambda \geq \lambda_0$ one has
\[
\|F(t, \cdot)\|_{L^2(\mathbb{R})} \leq C \left( \lambda^{-\delta-s} + \lambda^{\frac{1-\delta}{2}-2s} \right).
\]

Proof of Lemma 6. Set $\Phi := -\lambda^2 t + \lambda x + \omega t$. We observe that
\[
u_{ap}(t, x) = u_{low}(t, x) - \lambda^{-\frac{1}{2} - \delta - \delta} \phi_\lambda(x) \cos \Phi.
\]
Then, as in the proof of Lemma 4, we can write again
\[
(\partial_t + H \partial_x^2) u_{ap} + u_{ap} \partial_x u_{ap} = F_1 + F_2 + F_3 + F_4 + F_5,
\]
where $U_{\lambda, \omega}$ is simply replaced by $u_{low}$. Since $u_{low}$ is a solution of the Benjamin-Ono equation, we deduce that $F_1 = 0$ which eliminates the problem with small frequency residual terms. A difficulty however appears since now $F_5$ is not vanishing anymore. We will however be able to control $F_5$ by the aid of Lemma 5. Using that $\tilde{\phi} = \phi_\lambda$, we readily obtain that
\[
F_5 = \lambda^{\frac{1-\delta}{2}-s}(u_{low}(t, x) - u_{low}(0, x)) \phi_\lambda(x) \sin \Phi.
\]
Using Lemma 5, we get
\[
\|F_5(t, \cdot)\|_{L^2} \lesssim \lambda^{\frac{1-\delta}{2}-s} |\omega| \lambda^{-2-\delta} \lesssim \lambda^{-1-2\delta-s}.
\]
It remains to bound $F_2, F_3$ and $F_4$. Observe that $F_3$ and $F_4$ are exactly as in Lemma 4 and therefore
\[
\|F_3(t, \cdot)\|_{L^2} + \|F_4(t, \cdot)\|_{L^2} \lesssim \lambda^{-\delta-s} + \lambda^{\frac{1-\delta}{2}-2s}.
\]
The term $F_2$ reads
\[
F_2 = -\cos \Phi \frac{\partial_x}{\partial_x} \left\{ u_{low}(t, x) \lambda^{\frac{1-\delta}{2}-s} \phi_\lambda(x) \right\}.
\]
Using Lemma 5 and the assumption on $|\omega|$, we obtain
\[
\|F_2(t, \cdot)\|_{L^2} \lesssim \lambda^{-s} \|\partial_x u_{low}(t, \cdot)\|_{L^\infty} + \lambda^{\frac{1-\delta}{2}-s} \|u_{low}(t, \cdot)\|_{L^2} \lesssim \lambda^{-\frac{1+\delta}{2}-s}.
\]
Collecting (4.11), (4.12) and (4.13) completes the proof of Lemma 6. \qed
In order to prove Theorem 1, we need to show that the family of approximate solutions constructed in Sections 3 and 4 are indeed close to the “real” solutions of the Benjamin-Ono equation at least up to $t = 1$.

**Theorem 2.** Let $1 - s < \delta < 1$ and $|\omega| \ll \lambda^{-\frac{1-\delta}{2}}$. Let $u_{\omega,\lambda}$ be the unique global solution of the Benjamin-Ono equation subject to initial data

$$u_{\omega,\lambda}(0, x) = -\omega \lambda^{-1} \tilde{\phi}_\lambda(x) - \lambda^{-\frac{1}{2} - \frac{\delta}{2}} \phi_\lambda(x) \cos \lambda x.$$

Then the identity

$$u_{\omega,\lambda}(t, x) = -\lambda^{-\frac{1}{2} - \frac{\delta}{2}} \phi_\lambda(x) \cos(-\lambda^2 t + \lambda x + \omega t) + O\left(\lambda^{-\min\{\delta, 1-\delta\} - \frac{s}{4}} + |\omega| \lambda^{-\frac{1-\delta}{2}}\right)$$

holds in $H^s_x(\mathbb{R})$, uniformly in $t \in [0, 1]$ and $\lambda \gg 1$.

**Remark 2.** The theorem describes the short time nonlinear interaction between some low and high frequency waves. If $\omega = 0$, the approximate solution propagates as a high frequency linear Benjamin-Ono wave. When $\omega \neq 0$, the approximate solution propagates as a high frequency linear dispersive wave with modified propagation speed which is the crucial nonlinear effect.

**Proof of Theorem 2.** The first step is to bound $u_{\omega,\lambda}$ in high Sobolev norms. Let $s > \frac{3}{2}$. Observe that for $\frac{3}{2} < \sigma < s$

$$\|u_{\omega,\lambda}(0, \cdot)\|_{H^s} \lesssim \lambda^{\sigma-s} + |\omega| \lambda^{-\frac{1-\delta}{2}}.$$

Therefore for $k \geq s$, it follows from Lemma 1 that

$$\|u_{\omega,\lambda}(t, \cdot)\|_{H^k} \lesssim \|u_{\omega,\lambda}(0, \cdot)\|_{H^k} \lesssim \lambda^{k-s}, \quad |t| \leq 1, \quad \lambda \gg 1.$$  

Let $0 < s \leq \frac{3}{2}$. Using the conservation laws associated to the Benjamin-Ono equation (see Lemma 3.3.2 of [1]), we get the following bound uniformly in $t \in \mathbb{R}$

$$\|u_{\omega,\lambda}(t, \cdot)\|_{H^2} \lesssim \|u_{\omega,\lambda}(0, \cdot)\|_{H^2} + \|u_{\omega,\lambda}(0, \cdot)\|_{L^2}^5 \lesssim 1 + \lambda^{2-s},$$

and therefore we obtain

$$\|u_{\omega,\lambda}(t, \cdot)\|_{H^2} \lesssim \lambda^{2-s}, \quad t \in \mathbb{R}.$$  

Let $u_{ap}$ as in (4.10) and $v_{\omega,\lambda} := u_{\omega,\lambda} - u_{ap}$. The aim is to show that $v_{\omega,\lambda}$ is small comparing to $u_{ap}$ in the $H^s$ norm.

Due to Lemma 5 we get

$$\|u_{\text{true}}(t, \cdot)\|_{H^s} \lesssim |\omega| \lambda^{-\frac{1-\delta}{2}},$$

where $u_{\text{true}}$ is the true solution of the Benjamin-Ono equation.
if $|t| \leq 1$. Next, using Lemma 3, we obtain the bound
$$
\|u_{ap}(t, \cdot)\|_{H^k(\mathbb{R})} \lesssim \lambda^{k-s},
$$
if $|t| \leq 1$ and $k \geq s$.

Therefore using (5.1) and (5.3), we get the bounds for the high Sobolev norms
$$
\|v_{\omega,\lambda}(t, \cdot)\|_{H^k} \lesssim \lambda^{k-s},
$$
if $|t| \leq 1$ and $3/2 < s < k$, and
$$
\|v_{\omega,\lambda}(t, \cdot)\|_{H^2} \lesssim \lambda^{2-s},
$$
if $t \in \mathbb{R}$ and $0 < s < 3/2$.

The second step provides a good bound of the $L^2$ norm of $v_{\omega,\lambda}$. Clearly

$$
\|v_{\omega,\lambda}(t, \cdot)\|_{L^2} \lesssim \lambda^{-\min(\delta, 1-\delta)-s}
$$

by Lemma 9 and the assumption $1-s < \delta < 1$.

The second endpoint is the $L^2$ estimate
$$
\|v_{\omega,\lambda}(t, \cdot)\|_{L^2} \lesssim \lambda^{-\min(\delta, 1-\delta)-s}, \quad |t| \leq 1.
$$
To prove (5.7), we multiply (5.6) by $v_{\omega,\lambda}$ and integrate over the real line,

$$
\frac{d}{dt}\|v_{\omega,\lambda}(t, \cdot)\|_{L^2}^2 \lesssim \|\partial_x u_{ap}(t, \cdot)\|_{L^\infty}\|v_{\omega,\lambda}(t, \cdot)\|_{L^2}^2 + \|v_{\omega,\lambda}(t, \cdot)\|_{L^2}\|F(t, \cdot)\|_{L^2}
$$

hence, since we have for $1-s < \delta < 1$

$$
\|\partial_x u_{ap}(t, \cdot)\|_{L^\infty} \lesssim \|\partial_x u_{low}(t, \cdot)\|_{L^\infty} + \lambda^{\frac{1-s}{2}} \lesssim |\omega|\lambda^{2-\delta} + \lambda^{\frac{1-s}{2}} \ll 1,
$$
we readily get the bound (5.7).

We now complete the proof by an interpolation argument. Let first $s > \frac{3}{2}$. Choose $k \in [s + \frac{1}{2}, s + 2]$ and interpolate between (5.4) and (5.7) as follows

$$
\|v_{\omega,\lambda}(t, \cdot)\|_{H^k} \leq \|v_{\omega,\lambda}(t, \cdot)\|_{L^2}^{\frac{k-1}{k}} \|v_{\omega,\lambda}(t, \cdot)\|_{H^2}^{\frac{1}{k}} \lesssim \lambda^{-\min(\delta, 1-\delta)}. \text{ (5.10)}
$$

If $s \leq \frac{3}{2}$ we obtain the same estimate by using $k = 2$ in the interpolation and (5.5) instead of (5.4). This completes the proof of Theorem 2. \qed
Remark 3. Notice that to derive estimate (5.2) one needs to exploit the higher conservation laws for the Benjamin-Ono equation. This fact permits us to have an ansatz for the solution up to time one as claimed in Theorem 2. If \( s > \frac{3}{2} \) we use Lemma 1 instead.

The method of proof can be generalized to many other equations. For example the corresponding to Theorem 2 result in the context of the KdV equation provides a family of essentially linear KdV waves \( (\omega \to 0) \) with the same initial data as approximate solutions and thus no instability property of the flow is displayed.

6. Lack of uniform continuity

In this section, we complete the proof of Theorem 1. We apply Theorem 2 with \( \omega = \pm 1 \) and \( \lambda = 2^n \) to obtain two families \( (u_{1,2^n}) \) and \( (u_{-1,2^n}) \) of solutions to the Benjamin-Ono equation. Notice that

\[
\|u_{1,2^n}(0, \cdot) - u_{-1,2^n}(0, \cdot)\|_{H^s} \lesssim 2^{\left(\frac{\delta - 1}{2}\right)n}
\]

and moreover due to Theorem 2, setting \( \kappa = -2^{2n}t + 2^n x \), we arrive at

\[
\|u_{1,2^n}(t, \cdot) - u_{-1,2^n}(t, \cdot)\|_{H^s} = \|2^{-\left(\frac{1 + \delta + s}{2}\right)n}\phi_{2^n}(x)(\cos(\kappa + t) - \cos(\kappa - t))\|_{H^s} + o(1),
\]

if \( |t| \leq 1 \) and where \( o(1) \to 0 \) as \( n \to \infty \). Then using Lemma 3 we get

\[
\|2^{-\left(\frac{1 + \delta + s}{2}\right)n}\phi_{2^n}(x)(\cos(\kappa + t) - \cos(\kappa - t))\|_{H^s} = \sqrt{2} |\sin t| \|\phi\|_{L^2} + o(1).
\]

The proof of Theorem 1 is completed. \( \Box \)

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