LARGE-$N_c$ BARYONS

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ABSTRACT: A spin-flavor symmetry emerges for baryons in the large-$N_c$ limit. Large-$N_c$ baryons form irreducible representations of the spin-flavor algebra, and their static properties can be computed in a systematic expansion in $1/N_c$. Symmetry relations for static baryon matrix elements are obtained at various orders in the $1/N_c$ expansion by neglecting subleading $1/N_c$ corrections. Equivalent relations arise in the quark and Skyrme models, which satisfy the same large-$N_c$ group theory as QCD. The $1/N_c$ expansion yields useful results for QCD baryons with $N_c = 3$.

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1 INTRODUCTION

Quantum chromodynamics, the theory of the strong interactions, is an $SU(3)$ gauge theory of quarks and gluons. At low energies, the running coupling constant of the theory is large, and the colored quarks and gluons are confined into
colorless mesons and baryons. It is not known how to calculate the structure and interactions of mesons and baryons directly in terms of the underlying quark-gluon dynamics because the theory is strongly coupled at low energies. The large $N_c$ formulation of QCD [1] provides a framework for studying the nonperturbative QCD dynamics of hadrons in a systematic expansion in the parameter $1/N_c$, where $N_c$ is the number of colors. Recently, significant progress has been made in the study of baryons in $1/N_c$. A spin-flavor symmetry arises for large-$N_c$ baryons, and can be used to classify large-$N_c$ baryon states and matrix elements. The spin-flavor structure of the baryon $1/N_c$ expansion is calculable, and yields predictions for the static properties of baryons for any $N_c$. These new developments will be the focus of this review. We begin with a brief introduction of large $N_c$ QCD before turning to this work. For surveys of large $N_c$ QCD, see Refs. [2–4].

Large $N_c$ QCD is the $SU(N_c)$ gauge theory of quarks and gluons, where the number of colors $N_c$ is a parameter of the theory [1]. For large $N_c$, the number of gluons is $N_c^2 - 1 = O(N_c^2)$, whereas the number of quarks is $O(N_c)$, so gluons are much more prevalent than quarks. An analysis of the $N_c$-dependence of quark-gluon diagrams is required to define the large-$N_c$ limit of the theory. The leading $N_c$-dependence of quark-gluon diagrams can be determined by replacing the gluon gauge field $(A^A)_{ij}$ in the adjoint color representation by a tensor $(A^A)^i_j$ with one lower index and one upper index in the fundamental color representation, as depicted in Fig. 1. When Feynman diagrams are drawn using double line notation, the power of $N_c$ of a diagram is obtained by counting the number of closed quark loops since each quark loop implies an unconstrained summation over the color index of the quark, which produces a factor of $N_c$. 't Hooft analyzed the $g$- and $N_c$-counting of vacuum Feynman diagrams in double line notation, and found that the graphs are proportional to

$$\left(g^2 N_c\right)^{\frac{1}{2}V_3 + V_4} N_c^{\chi},$$

(1)

where $g$ is the coupling constant of the theory, $V_n$ are the number of $n$-point vertices of quarks and gluons in a given diagram, and the Euler character $\chi = 2 - 2H - L$ is a topological invariant depending of the number of holes $H$ and the number of quark loops $L$ of the diagram. Diagrams with large numbers of vertices grow with arbitrarily large powers of $N_c$ unless the large-$N_c$ limit $N_c \to \infty$ is taken with $g^2 N_c$ held fixed. This constraint can be implemented by performing the rescaling

$$g \to \frac{g}{\sqrt{N_c}}$$

(2)

in the large-$N_c$ QCD Lagrangian. After this rescaling, all $N_c$-dependence of the theory is manifest, and the new coupling constant $g$ is order unity. The running of the coupling constant is governed by the $\beta$ function of large-$N_c$ QCD. All $N_c$-dependence factors out of the large-$N_c$ QCD $\beta$ function equation under the rescaling Eq. (2), so the rescaled coupling constant $g$ of the large-$N_c$ theory becomes strong at the same scale $\Lambda_{\text{QCD}}$ as the $N_c = 3$ theory. For energies $E \leq O(\Lambda_{\text{QCD}})$, the large-$N_c$ theory is strongly coupled, and is presumed to exhibit confinement.

Most of the initial work on large-$N_c$ QCD concentrated on mesons [1, 2, 3].
Large-$N_c$ mesons are color singlet bound states of a quark and an antiquark,

$$\sum_{i=1}^{N_c} \bar{q}_i q_i,$$

where the color summation is given explicitly. The $N_c$-dependence of meson amplitudes can be determined by studying quark-gluon diagrams. The leading order graphs have a single quark loop and no holes, so they can be written in a plane in double line notation. Arbitrary numbers of planar gluons can be exchanged inside the single quark loop without affecting the $N_c$-counting of the diagram. Diagrams with additional quark loops or nonplanar gluons are systematically suppressed in $1/N_c$. Each additional quark loop is suppressed by $1/N_c$ and each nonplanar gluon is suppressed by $1/N_c^2$. Meson self-couplings are analyzed by studying planar diagrams with meson insertions on the quark loop (see Fig. 2). Each meson insertion is accompanied by a factor of $1/\sqrt{N_c}$, which is the normalization factor for the meson quark operator Eq. (3).

Baryons in large-$N_c$ QCD were studied by Witten [2]. Large-$N_c$ baryons are color singlet bound states of $N_c$ valence quarks. The number of quarks in a baryon grows with $N_c$, and changes as the parameter $N_c$ is varied. The color indices of the $N_c$ quarks are contracted with the $SU(N_c)$ color $\epsilon$-symbol to form a color singlet state,

$$\epsilon_{i_1i_2i_3\cdots i_{N_c}} q_1^{i_1} q_2^{i_2} q_3^{i_3} \cdots q_{N_c}^{i_{N_c}},$$

so a large-$N_c$ baryon is totally antisymmetric in the color indices of its $N_c$ valence quarks. At leading order, the large-$N_c$ baryon is described by quark-gluon diagrams consisting of $N_c$ valence quark lines with arbitrary planar gluons exchanged between the quark lines. Each sea-quark loop is suppressed by $1/N_c$ and each nonplanar gluon exchange is suppressed by $1/N_c^2$. (For a description of planar baryon diagrams, see Ref. [4].) The baryon mass grows with $N_c$,

$$M_{\text{baryon}} \sim O(N_c),$$

and baryons become infinitely heavy in the large-$N_c$ limit. For massless quarks, the only dimensional parameter of the large-$N_c$ QCD theory is $\Lambda_{\text{QCD}}$, and $M_{\text{baryon}} \sim N_c \Lambda_{\text{QCD}}$.

The number of quarks inside a large-$N_c$ baryon grows as $N_c$ in the large-$N_c$ limit, but the size of the baryon is governed by $\Lambda_{\text{QCD}}$ and remains fixed. Thus, as $N_c$ becomes large, the quarks in the baryon become denser and denser. Witten realized that this situation could be described by a Hartree approximation in which each quark in the $N_c$-quark bound state interacts with the average potential produced by the other $O(N_c)$ quarks [4]. In the large-$N_c$ limit, the Hartree picture
becomes exact, and each quark experiences the same average Hartree potential.
When $N_c$ is very large, the Hartree Hamiltonian for a large-$N_c$ baryon is given
by the sum of $N_c$ identical quark Hamiltonians

$$\mathcal{H}_{\text{baryon}} = N_c \mathcal{H}_{\text{quark}},$$

(6)

where $\mathcal{H}_{\text{quark}}$ is the Hamiltonian for any one quark in the baryon, and is $O(1)$.
The $N_c$-dependence of the Hartree Hamiltonian is trivial for large $N_c$, so in- stead of solving for the baryon many-body wavefunction, one can consider the much simpler problem of finding the wavefunction of a single quark. The baryon wavefunction is given by the product of $N_c$ identical quark wavefunctions,

$$\Psi (x_1, \ldots, x_{N_c}) = \prod_{i=1}^{N_c} \phi (x_i).$$

(7)

For the lowest-lying ground state baryons, all of the quarks are in the same rotationally invariant ground state solution of the Hartree Hamiltonian $\mathcal{H}_{\text{quark}}$, and the $N_c$ quark wavefunctions are combined in an $s$-wave, so the baryon wavefunction reduces to

$$\prod_{i=1}^{N_c} \phi (r).$$

(8)

Note that the baryon wavefunction is totally antisymmetric in the $N_c$ color indices of the quarks, so it is totally symmetric in the bosonic wavefunctions $\phi (r)$ of the quarks. The Hartree Hamiltonian can be given in closed form only for heavy quarks. However, one expects the qualitative picture and the large-$N_c$ power counting to remain valid for light quarks where the relativistic Hartree equations are not known.

A second picture of large-$N_c$ baryons is obtained by considering the weakly-coupled meson sector of large-$N_c$ QCD. It has been speculated that large-$N_c$ baryons are solitons of the large-$N_c$ meson Lagrangian

$$\mathcal{L}_{\text{meson}} = N_c \mathcal{L} \left( \phi / \sqrt{N_c} \right),$$

(9)

where $\mathcal{L}$ is an arbitrary polynomial of the meson fields $\phi$. The Skyrme model \[35–37\] is an explicit model realization of this idea using the chiral Lagrangian for pseudoscalar pions. The soliton mass is order $N_c$, but its size and shape is independent of $N_c$. The large-$N_c$ couplings of mesons to baryons can be obtained by expanding the action in the meson functional integral

$$Z = \int \mathcal{D}\phi \ e^{iN_cS/\hbar}, \quad S = \int d^4x \ \mathcal{L} \left( \phi / \sqrt{N_c} \right)$$

(10)

about the baryon soliton configuration, and solving the linear equations of motion for the semiclassical meson field in the presence of a background baryon soliton \[36\].

The $N_c$-dependence of baryon-meson scattering amplitudes can be determined by studying quark-gluon diagrams. Typical graphs which contribute to the process baryon + meson $\rightarrow$ baryon + meson are given in Fig. 3, where it is to be understood that each of the $N_c$ quarks in the baryon has a different color from any of the other quarks. In Fig. 3(a), both mesons are inserted on the same quark line of the baryon, so the color structure of the quark lines is preserved.
There are $O(N_c)$ possible choices for the quark line with the meson insertions, and each meson couples to a quark-antiquark pair with amplitude $1/\sqrt{N_c}$, so the overall diagram is $O(1)$. Alternatively, the two mesons can be inserted on two different quark lines of the baryon, as shown in Fig. 3(b). In this case, a gluon must be exchanged between the two quark lines to transfer energy between the incoming and outgoing mesons. The gluon interchanges the colors of quarks, so color conservation requires the exchange of the two quark lines. There are $N_c(N_c - 1)/2 = O(N_c^2)$ possible choices for the pair of quark lines with meson insertions; each meson insertion is accompanied by a factor of $1/\sqrt{N_c}$; and the gluon exchange is proportional to $g^2/N_c$, so the overall diagram is $O(1)$. The generalization to multimeson-baryon scattering amplitudes is straightforward. In general, the scattering amplitude for the process baryon + meson $\rightarrow$ baryon + $(n-1)$ mesons is $O(N_c^{1-n/2})$, since each additional meson in the scattering process reduces the scattering amplitude by a factor of $1/\sqrt{N_c}$.

Mesons are strongly coupled to baryons for large $N_c$. Again, quark-gluon diagrams can be used to determine the $N_c$-dependence of baryon-meson vertices. Typical graphs depicting the coupling of a baryon to a single meson are given in Fig. 4. The meson can be inserted on a single quark line in the baryon without affecting the color structure of the quark lines (Fig. 4(a)). There are $O(N_c)$ possible choices for the quark line with the meson insertion, and the meson insertion is accompanied by a factor of $1/\sqrt{N_c}$, so the overall diagram is $O(\sqrt{N_c})$. Instead, the quark line of the meson can be reinserted into the baryon at another position (Fig. 4(b)). This color rearrangement is mediated by the exchange of a gluon. There are $O(N_c^2)$ possible choices for the pair of quark lines involved; the meson insertion gives a factor of $1/\sqrt{N_c}$; and the gluon exchange yields a factor of $g^2/N_c$, so the overall diagram is $O(\sqrt{N_c})$. The large-$N_c$ power counting of vertices containing a single baryon and $n$ mesons is determined from the same quark-gluon diagrams analyzed for baryon-meson scattering amplitudes. A baryon coupling to $n$ mesons is $O(N_c^{1-n/2})$, since each additional meson in a vertex is accompanied by a factor of $1/\sqrt{N_c}$.

(Note that the above large-$N_c$ power counting rules apply for a single quark flavor. For more quark flavors, one must keep track of the flavor of each quark line in the quark-gluon diagrams. The flavor content of the mesons and baryons places restrictions on which quark lines can be chosen for the meson insertions. If the baryon contains $O(N_c)$ quarks of the relevant quark flavors, then the $N_c$ counting remains the same. However, if only $O(1)$ quarks in the baryon are a relevant flavor, then further suppressions occur.)

Recently, it has been realized that the spin and flavor structure of baryons simplifies for large $N_c$. Consistency of the $N_c$-power counting rules for baryon-meson scattering amplitudes and baryon-meson vertices leads to nontrivial constraints on static baryon matrix elements [8,9]. Large-$N_c$ consistency conditions for the scattering of low-energy pions with baryons imply that baryons satisfy a contracted spin-flavor algebra in the large $N_c$ limit, a result derived by Dashen & Manohar [8] and by Gervais & Sakita [40]. Dashen & Manohar and Jenkins showed that subleading $1/N_c$ corrections are constrained by additional large-$N_c$ consistency conditions [8,9], so the spin-flavor structure of large-$N_c$ baryons with finite $N_c$ can be analyzed in a systematic expansion in powers of $1/N_c$ about the large-$N_c$ limit. The spin-flavor structure and the order in $1/N_c$ of the subleading $1/N_c$ corrections are specified by the algebraic constraints. The subleading $1/N_c$
corrections break the leading order spin-flavor symmetry of large-\(N_c\) baryons, and are parametrized by operator products of the baryon spin-flavor generators \([1]\), which have known baryon matrix elements. The baryon spin-flavor generators can be written as Skyrme model operators or nonrelativistic quark model operators \([8, 11, 21]\). The formulation of the \(1/N_c\) expansion using static quark operators has a natural connection to the Hartree picture \([14]\) and to quark-gluon diagrams \([8]\), while the formulation using Skyrme model operators has a natural connection to the soliton picture and Skyrme model. Symmetry relations amongst baryon couplings and amplitudes are obtained by neglecting subleading terms in the \(1/N_c\) expansion \([8, 11, 14, 16, 21, 23]\). The large-\(N_c\) Skyrme model \([37, 39, 41, 42, 44, 47]\) and the large-\(N_c\) quark model \([6, 43]\) both share the same spin-flavor group theory for baryons as large-\(N_c\) QCD \([40, 42]\), and yield equivalent model-independent predictions for baryons in an expansion in \(1/N_c\).

The recent work on the baryon \(1/N_c\) expansion distinguishes between group-theoretic relations of the Skyrme and quark models which are model-independent and true in large-\(N_c\) QCD, and results which are model-dependent and not consequences of large-\(N_c\) QCD. Connections to other large-\(N_c\) models of baryons are found as well \([17–20]\). It has been shown that the baryon \(1/N_c\) expansion can be usefully applied to QCD baryons with \(N_c = 3\). The complete pattern of spin-flavor breaking for QCD baryons requires the inclusion of \(SU(3)\) flavor symmetry breaking, which is order 30% and comparable to the suppression factor \(1/N_c\). A combined expansion in \(1/N_c\) and \(SU(3)\) flavor symmetry breaking has been performed for many baryon quantities, including baryon axial couplings \([8, 11, 14, 16, 21, 23]\), masses \([11, 14, 21, 22, 24, 25]\), and magnetic moments \([8, 10, 15, 16, 23]\). Other phenomenological applications of the baryon \(1/N_c\) expansion include heavy quark baryons containing a single charm or bottom quark \([9, 26]\), baryon chiral perturbation theory \([8, 9, 11, 13, 15, 19, 24, 25]\), excited baryons \([27–29]\), and nuclear physics \([33, 34]\). Additional related work can be found in Refs. \([30–32]\).

2 SPIN-FLAVOR SYMMETRY FOR BARYONS

2.1 Large-\(N_c\) Consistency Conditions

Large-\(N_c\) constraints on baryon static matrix elements are derived by considering baryon-meson scattering at low energies \(E \sim O(1)\). The baryon mass \(M\) is \(O(N_c)\), so the baryon acts as a heavy static source for the scattering of mesons at low energies. The absorption of the incoming meson by the heavy baryon results in an intermediate baryon state which is offshell by a four-momentum of order unity. The momentum of an intermediate baryon can be written as

\[
P^\mu = M v^\mu + k^\mu \equiv M v^\mu, \tag{11}\]

where \(v^\mu\) is the four-velocity of the initial baryon and \(k^\mu\) is a residual offshell momentum \([18]\). The \(M v^\mu\) piece of the momentum is \(O(N_c)\) while the residual momentum is \(O(1)\), so the intermediate baryon four-velocity

\[
v^\mu = v^\mu + O\left(\frac{1}{N_c}\right) \tag{12}\]

is equal to the initial baryon four-velocity in the large-\(N_c\) limit, and there is no recoil of the baryon. In the large-\(N_c\) limit, the incident and emitted mesons have
the same energy since no energy is transferred to the infinitely heavy baryon, but the three-momentum of the meson changes in the scattering process. The propagator of the intermediate baryon reduces to
\[
i \frac{(P+M)}{P^2 - M^2} \rightarrow i \frac{1 + \gamma}{2k \cdot v},
\]
which does not involve the \(O(N_c)\) baryon mass and is manifestly \(O(1)\). In the rest frame of the baryon, the velocity projection operator projects out the two large components of the baryon Dirac spinor, and the baryon propagator reduces to the nonrelativistic propagator \(i/E\). The baryon-meson scattering amplitude is \(O(1)\) for any energy, but a baryon-meson vertex grows as \(O(\sqrt{N_c})\). Individual baryon-meson diagrams describing the absorption and then emission of a meson by a baryon contain two baryon-meson vertices, and grow as \(N_c\). The total scattering amplitude grows as \(N_c\) and violates unitarity unless there are cancellations amongst the \(O(N_c)\) baryon-meson scattering diagrams. These cancellations are described by large-\(N_c\) consistency conditions which constrain the baryon-meson couplings. Large-\(N_c\) consistency conditions can be derived for general meson couplings to baryons. However, it is easiest to derive results for low-energy pions because the leading order baryon-meson diagrams reduce to tree diagrams \([8]\). A general analysis for an arbitrary meson involves a much larger class of diagrams \([46]\).

At low energies, a pion is derivatively coupled to baryons, and the \(\pi + B \rightarrow B'\) vertex is described by
\[
\frac{\partial \mu}{f_\pi} (A^{\mu a})_{B'B},
\]
where \(f_\pi\) is the pion decay constant,
\[
(A^{\mu a})_{B'B} = \langle B' | (\bar{q} \gamma^\mu \gamma_5 \tau^a q)_{\text{QCD}} | B \rangle
\]
is the isovector baryon axial vector current matrix element, and \(a = 1, 2, 3\) is the isospin index of the pion. The baryon-pion coupling is \(O(\sqrt{N_c})\) because the isovector baryon axial vector current matrix element is \(O(N_c)\) and \(f_\pi \sim O(\sqrt{N_c})\). In the large \(N_c\) limit, the baryon is infinitely heavy compared to the pion, so the baryon-pion coupling reduces to the static baryon coupling
\[
\frac{\partial \mu}{f_\pi} (A^{ia})_{B'B},
\]
where \(i = 1, 2, 3\). Notice that the \(p\)-wave pion coupling has spin 1 and isospin 1. It is useful to keep the \(N_c\)-dependence of the baryon axial vector current manifest by defining
\[
A^{ia} \equiv gN_cX^{ia},
\]
where the baryon matrix elements \(X^{ia}\) are \(O(1)\) and have a well-defined large \(N_c\) limit. (It is to be understood in the following that \(A^{ia}\) and \(X^{ia}\) refer to baryon matrix elements.) An \(O(1)\) coupling constant \(g\) is inserted in the definition so that a convenient normalization for \(X^{ia}\) can be chosen later on.

The amplitude for pion-baryon scattering at fixed pion energy \(E\) is dominated in the large \(N_c\) limit by the diagrams displayed in Fig. 5. The total amplitude of the two diagrams is proportional to
\[
N_c [X^{ia}, X^{jb}],
\]

The matrix elements of $\sigma^J$ can produce a baryon state with intermediate states in the scattering. The propagator of the intermediate baryon the intermediate baryon in Fig. 5(a) is off-shell by an energy $E$ whereas the intermediate baryon in Fig. 5(b) is off-shell by an energy $-E$. The product of the $X$'s in Eq. (13) is a matrix product which sums over all possible baryon intermediate states. Only intermediate baryon states which are degenerate with the initial and final baryons in the large $N_c$ limit contribute in the sum. Although each individual diagram in Fig. 5 contains two baryon-pion vertices and is $O(N_c)$, the total scattering amplitude is at most $O(1)$, which leads to the constraint

$$N_c \left[ X^{ia}, X^{jb} \right] \leq O(1) . \tag{19}$$

The $1/N_c$ expansion for the operator $X^{ia}$ is given by

$$X^{ia} = X_0^{ia} + \frac{1}{N_c} X_1^{ia} + \frac{1}{N_c^2} X_2^{ia} + \ldots . \tag{20}$$

The large-$N_c$ consistency condition implies that

$$\left[ X_0^{ia}, X_0^{jb} \right] = 0, \tag{21}$$

which constrains baryon-pion couplings at leading order in the $1/N_c$ expansion. Eq. (21) was derived by Dashen and Manohar [8] and by Gervais and Sakita [40].

The solution of the large-$N_c$ consistency condition Eq. (21) can be given in closed form [8]. Assume that there is a baryon state with $J = I = \frac{1}{2}$ which can be identified with the nucleon, and consider the scattering process $N + \pi \rightarrow N + \pi$. If the only intermediate baryon involved in the scattering is the nucleon, then the consistency condition Eq. (21) must be satisfied for the nucleon matrix elements of the operator $X_0^{ia}$. The nucleon matrix elements of $X_0^{ia}$ are proportional to the nucleon matrix elements of $\sigma^J \tau^a$ by the Wigner-Eckart theorem. Since the nucleon matrix elements of $\sigma^J \tau^a$ do not commute, Eq. (21) is not satisfied and there must be additional baryon states degenerate with the nucleon which participate as intermediate states in the scattering. The $p$-wave pion coupled to the nucleon can produce a baryon state with $J = I = \frac{3}{2}$, which can be identified with the $\Delta$. Eq. (21) for $N + \pi \rightarrow N + \pi$ scattering with intermediate nucleon and $\Delta$ states yields an equation involving $g_{\pi N \Delta}$ and $g_{\pi NN}$. The large-$N_c$ consistency condition determines $g_{\pi N \Delta}$ in terms of $g_{\pi NN}$. Next, consider $N + \pi \rightarrow \Delta + \pi$ scattering. The allowed intermediate states are $N$ and $\Delta$, so Eq. (21) yields an equation which determines $g_{\pi N \Delta}$ in terms of $g_{\pi NN}$ and $g_{\pi N \Delta}$. The consistency condition for $\Delta + \pi \rightarrow \Delta + \pi$ scattering cannot be satisfied unless there is an additional baryon state with $J = I = \frac{5}{2}$. The consistency condition for this scattering fixes the pion coupling of the $\Delta$ state to the $J = I = \frac{5}{2}$ state in terms of the other couplings. The recursion continues ad infinitum. Thus, the solution of the large-$N_c$ consistency condition Eq. (21) requires an infinite tower of degenerate baryons with $I = J = \frac{1}{2}, \frac{3}{2}, \ldots$. All of the pion couplings of these baryons are determined in terms of one overall normalization $g$. The matrix elements of $X_0^{ia}$ for the degenerate baryon states $|JJ_z, II_z\rangle$ are completely specified, and are given by

$$\langle J'J_z', I'z' | X_0^{ia} | JJ_z, II_z \rangle = \sqrt{\frac{2J + 1}{2J' + 1}} \left( \begin{array}{cc} I & 1 \\ I_z & a \end{array} \right) \left( \begin{array}{cc} J & 1 \\ J_z & i \end{array} \right) \left( \begin{array}{cc} J' & 1 \\ J'_z & i' \end{array} \right) . \tag{22}$$
The $p$-wave pions have diagonal couplings to each baryon and off-diagonal couplings between baryons with spins and isospins differing by $J = I = 1$. Thus, the degeneracy requirement for the baryon states is that the mass splittings of baryons between adjacent baryons in the spin tower are at most order $1/N_c$. The solution Eq. (22) of the large-$N_c$ consistency condition implies that the ratios of all pion-baryon couplings are determined in the large-$N_c$ limit. For example, one finds that

$$\frac{g_{\pi\Delta N}}{g_{\pi NN}} = \frac{3}{2},$$

which is the model-independent result derived previously in the context of the Skyrme model [37]. The ratio $g_{\pi\Delta\Delta}/g_{\pi NN}$ also is determined. The overall normalization $g$ of the couplings, however, is not determined by the large-$N_c$ constraint.

2.2 Contracted Spin-Flavor Algebra

The contracted spin-flavor algebra for large-$N_c$ baryons is given by the large-$N_c$ consistency condition

$$[X_0^{ia}, X_0^{jb}] = 0,$$  \hspace{1cm} (24)

the spin and flavor algebras

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \quad [I^a, I^b] = i\epsilon^{abc} I^c, \quad [J^i, I^a] = 0,$$  \hspace{1cm} (25)

and the commutation relations

$$[J^i, X_0^{ja}] = i\epsilon^{ijk} X_0^{ka}, \quad [I^a, X_0^{ib}] = i\epsilon^{abc} X_0^{ic}. \hspace{1cm} (26)$$

The commutators Eq. (26) follow because $X_0^{ia}$ transforms under spin and isospin as an irreducible tensor operator with spin 1 and isospin 1.

It is useful to compare the contracted $SU(4)$ spin-flavor algebra Eqs. (24)-(26) to the $SU(4)$ spin-flavor algebra. The $SU(4)$ generators $J^i$, $I^a$ and $G^{ia}$ satisfy the spin-flavor algebra Eq. (23) and

$$[J^i, G^{ia}] = i\epsilon^{ijk} G^{ka}, \quad [I^a, G^{ib}] = i\epsilon^{abc} G^{dc},$$

$$[G^{ia}, G^{jb}] = \frac{i}{4} \delta^{ab} \epsilon^{ijk} J^k + \frac{i}{4} \delta^{ij} \epsilon^{abc} I^c. \hspace{1cm} (27)$$

The spin-flavor generator $G^{ia}$ with $J = I = 1$ has matrix elements which are $O(N_c)$ for large-$N_c$ baryons, so $G^{ia}/N_c$ has a nonvanishing large-$N_c$ limit. The large-$N_c$ contracted $SU(4)$ spin-flavor algebra for baryons is obtained from the $SU(4)$ algebra by rescaling the spin-flavor generator $G^{ia}$ by a factor of $1/N_c$ and taking the large $N_c$ limit,

$$\lim_{N_c \to \infty} \frac{G^{ia}}{N_c} \to X_0^{ia}. \hspace{1cm} (28)$$

The spin ⊗ flavor algebra is unaffected by this contraction, and the commutators which are homogeneous in $G^{ia}$ yield the commutators Eq. (24). The commutation relation for $[G^{ia}, G^{jb}]$ is divided by two powers of $N_c$, and reproduces the large-$N_c$ consistency condition Eq. (23) in the limit $N_c \to \infty$, since baryon matrix elements of $J$ and $I$ are at most $O(N_c)$.

The contracted spin-flavor algebra for large-$N_c$ baryons arises because the axial vector current for large-$N_c$ baryons grows with $N_c$. The rescaled axial vector
current $X_0^a$ has a well-defined large-$N_c$ limit, and can be treated like a classical commuting object. The usual spin and flavor algebra of baryons is extended to a spin-flavor algebra in the large-$N_c$ limit when $X_0^a$ is included in the algebra. Note that a spin-flavor symmetry which mixes internal and Lorentz symmetries can emerge for large-$N_c$ baryons because the baryon field is static in the large $N_c$ limit.

2.3 Large-$N_c$ Baryon Representations

All possible irreducible representations of the contracted spin-flavor algebra have been classified using the theory of induced representations in Ref. [21]. The eigenvalues of the commuting generator $X_0^a$ can be used to label states. A standard reference state with coordinate $X_0^a$ diagonalized can be chosen for each irreducible representation. The reference state can be regarded as the analogue of the highest weight state of an irreducible representation of a Lie algebra. All states in an irreducible representation are obtained from the reference state by applying group transformations, just as all states in an irreducible representation can be obtained from the highest weight state by making group transformations. The abelian coordinate $X_0^a$ is not affected by the action of the generator $X_0$, so all states are obtained by spin and isospin transformations of the reference state. The standard reference state for irreducible representations corresponding to large $N_c$ baryons is

$$X_0^{ia} = \delta^{ia} = \text{diag}(1, 1, 1),$$  

(29)

where $i = 1, 2, 3$ and $a = 1, 2, 3$ label the rows and columns of the $3 \times 3$ matrix. This reference state is invariant under transformations of a $SU(2)$ little group generated by

$$K = I + J,$$

(30)

so a representation of the little group must be specified to completely label the states. The states $|X_0^{ia}, K, k\rangle$ of an irreducible representation are labeled by the abelian coordinate $X_0^a$ and representations $|Kk\rangle$ of the $SU(2)$ little group.

The basis states $|X_0^{ia}, K, k\rangle$ of an irreducible representation diagonalize the baryon axial vector currents, but are not states of definite spin and isospin. Baryons are states of definite spin and isospin, so it is necessary to project the basis states $|X_0^{ia}, K, k\rangle$ onto states which diagonalize spin and isospin. These new basis states $|II_z, JJ_z; K\rangle$ can be identified with baryons, but now the states do not diagonalize the baryon axial vector current. Thus, the baryon axial vector current has off-diagonal matrix elements connecting baryon states with different spin and isospin.

Large-$N_c$ baryon representations contain all spin and isospin representations $(J, I)$ which are consistent with a given value of $K$. For the physical case of $N_c$ odd, the lowest allowed baryon spin is $J = \frac{1}{2}$. The irreducible representation with $K = 0$ is given by the infinite tower of $(J, I)$ states

$$ \left( \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2} \\ \frac{5}{2}, \frac{5}{2} \end{array} \right), \ldots \right.$$

(31)

The irreducible representation with $K = \frac{1}{2}$ corresponds to the infinite tower of states

$$ \left( \begin{array}{c} \frac{1}{2}, 0 \\ \frac{1}{2}, 1 \\ \frac{3}{2}, 1 \\ \frac{3}{2}, 2 \\ \frac{5}{2}, 2 \\ \frac{5}{2}, 3 \end{array} \right), \ldots \right.$$

(32)
The irreducible representation with \( K = 1 \) is the infinite tower
\[
\left( \frac{1}{2}, \frac{1}{2} \right), \left( \frac{1}{2}, \frac{3}{2} \right), \left( \frac{3}{2}, \frac{1}{2} \right), \left( \frac{3}{2}, \frac{3}{2} \right), \left( \frac{5}{2}, \frac{1}{2} \right), \left( \frac{5}{2}, \frac{3}{2} \right), \left( \frac{5}{2}, \frac{5}{2} \right), \left( \frac{7}{2}, \frac{1}{2} \right), \left( \frac{7}{2}, \frac{3}{2} \right), \left( \frac{7}{2}, \frac{5}{2} \right), \ldots,
\]
while the irreducible representation for \( K = \frac{3}{2} \) is the infinite tower
\[
\left( \frac{1}{2}, 1 \right), \left( \frac{1}{2}, 2 \right), \left( \frac{3}{2}, 0 \right), \left( \frac{3}{2}, 1 \right), \left( \frac{3}{2}, 2 \right), \left( \frac{3}{2}, 3 \right), \ldots.
\]

The low-spin states of the different \( K \) towers have the correct spin and isospin quantum numbers to be identified with the spin-\( \frac{1}{2} \) octet and spin-\( \frac{3}{2} \) decuplet baryon states of QCD if the quantum number \( K \) is related to the number of strange quarks in a baryon, \( K = N_s/2 \). The \( K = 0 \) tower contains strangeness zero baryons such as the nucleon state \( (\frac{1}{2}, \frac{1}{2}) \) and the \( \Delta \) state \( (\frac{3}{2}, \frac{1}{2}) \). The \( K = \frac{1}{2} \) tower contains strangeness \(-1\) baryons \( \Lambda(\frac{1}{2}, 0) \), \( \Sigma(\frac{1}{2}, 1) \) and \( \Sigma^*(\frac{3}{2}, 1) \). The \( K = 1 \) tower contains the strangeness \(-2\) baryons \( \Xi(\frac{1}{2}, \frac{1}{2}) \) and \( \Xi^*(\frac{3}{2}, \frac{1}{2}) \), while the \( K = \frac{3}{2} \) tower contains the strangeness \(-3\) baryon \( \Omega(\frac{3}{2}, 0) \). The other states in the large-\( N_c \) towers correspond to baryons which exist for \( N_c \to \infty \), but not for \( N_c = 3 \). Notice that the towers with \( K > 0 \) contain extra baryon states even for \( J = \frac{1}{2} \) and \( J = \frac{3}{2} \).

For large and finite \( N_c \), one can work with the \( SU(2N_F) \) spin-flavor algebra with \( N_F \) light quark flavors rather than the contracted spin-flavor algebra. The lowest-lying baryon spin-flavor representation is given by the completely symmetric \( SU(2N_F) \) representation with \( N_c \) boxes shown in Fig. 6. Under the decomposition \( SU(2N_F) \to SU(2) \otimes SU(N_F) \), the spin-flavor representation yields the tower of spin and flavor representations shown in Fig. 7. For \( N_F = 2 \), the ground state baryon representation produces the tower of baryon states
\[
J = I = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2}.
\]

The baryon tower is now finite dimensional, unlike the baryon tower Eq. (33) of the contracted spin-flavor algebra. The baryon tower Eq. (33) reduces to the states \( N \) and \( \Delta \) for \( N_c = 3 \).

It is instructive to consider how the quark spin and flavor is arranged in the large-\( N_c \) baryon. The large-\( N_c \) nucleon with \( J = I = \frac{1}{2} \) is the tensor product of \( \nu = (N_c - 1)/2 \) up and down quarks arranged in a spin and flavor singlet
\[
\frac{1}{2} |(ud - du) \times (\uparrow\downarrow - \downarrow\uparrow)|
\]
and one additional up or down quark with spin up or down. The spin and isospin quantum numbers of the nucleon are determined by this one additional quark. For example, the large-\( N_c \) proton state is the tensor product of \( \nu \) down quarks and \( (\nu + 1) \) up quarks. It is clear that the low-spin baryon wavefunctions are highly ordered in the spin and flavor wavefunctions of the \( N_c \) quarks. This ordering can be described by a collective coordinate. In a quark model, the baryon annihilation operator can be written as
\[
q_{i_1\alpha_1} q_{i_2\alpha_2} \cdots q_{i_N\alpha_N} A_{i_1\alpha_1} A_{i_2\alpha_2} \cdots A_{i_N\alpha_N},
\]
where the collective coordinate \( A \) is a \( 2 \times 2 \) matrix with rows and columns labeled by spin and isospin. The states of the Skyrme model also are labeled by a
collective coordinate. The large-$N_c$ basis states $|X_0^{ia}, K, k\rangle$ are identical to the Skyrme model states with the coordinate $X_0^{ia}$ related to the collective coordinate $A$ by

$$X_0^{ia} = \frac{1}{2} \text{Tr} \left( A \sigma^i A^{-1} \tau^a \right).$$

(38)

Notice that the operator $X_0^{ia}$ is completely symmetric in its vector and isovector indices $i$ and $a$, and that the soliton occurs in a $2 \times 2$ subspace.

### 2.4 1/$N_c$ Corrections

Large-$N_c$ consistency conditions also can be derived for the 1/$N_c$ corrections of static baryon matrix elements [8,9,11]. The 1/$N_c$ corrections to the baryon mass and axial couplings are described here. The general form of the baryon 1/$N_c$ expansion is discussed as well.

#### 2.4.1 Masses: No 1/$N_c$ Correction in Flavor Symmetry Limit

Consistency conditions are derived for the baryon mass by considering the effect of a baryon mass splitting on the pion-baryon scattering amplitude. The intermediate baryon propagator for nondegenerate baryons is given by $i/(k \cdot v - \Delta M)$, where $\Delta M$ is a baryon mass splitting. Expanding this propagator to first order in a power series in $(\Delta M/(k \cdot v))$ yields the constraint

$$\left[X_0^{ia}, [X_0^{jb}, M]\right] \leq O\left(\frac{1}{N_c}\right).$$

(39)

The baryon mass $M$ has an expansion in 1/$N_c$ of the form

$$M = N_c M_0 + M_1 + \frac{1}{N_c} M_2 + \ldots.$$  

(40)

Eq. (39) implies that $M_0$ and $M_1$ satisfy

$$\left[X_0^{ia}, [X_0^{jb}, M_0]\right] = 0, \quad \left[X_0^{ia}, [X_0^{jb}, M_1]\right] = 0.$$  

(41)

The only solution of these large-$N_c$ consistency conditions is $M_0 = M_1 = 1$, so the baryon tower is degenerate upto 1/$N_c$ corrections as expected,

$$M = m_0 N_c 1 + O\left(\frac{1}{N_c}\right).$$

(42)

The 1/$N_c$ correction to the baryon mass is constrained by considering the scattering pion + baryon $\rightarrow$ baryon + 2 pions, which yields the constraint

$$\left[X_0^{ia}, [X_0^{jb}, [X_0^{kc}, M]]\right] \leq O\left(\frac{1}{N_c^2}\right).$$

(43)

This constraint yields a large-$N_c$ consistency condition for $M_2$,

$$\left[X_0^{ia}, [X_0^{jb}, [X_0^{kc}, M_2]]\right] = 0,$$

(44)

which has solutions $M_2 = J^2, I^2$ and $X_0^{ia} X_0^{ia}$. The operator $I^2 = J^2$ and the operator $X_0^{ia} X_0^{ia} = 3$, so the only independent solution is $M_2 = J^2$. Thus, the baryon mass is given by

$$M = m_0 N_c 1 + m_2 \frac{1}{N_c} J^2 + \ldots.$$  

(45)
The hyperfine mass splittings of the $J = I = \frac{1}{2}, \frac{3}{2}, \ldots$ baryon tower are depicted in Fig. 8. For low-spin baryons with $J \sim O(1)$, the hyperfine splitting $J^2/N_c$ is order $1/N_c$, whereas for baryons with $J \sim O(N_c)$, the hyperfine splitting is order $N_c$ with respect to the low-spin baryons. The $1/N_c$ expansion is under control for baryons with fixed and finite spin and isospin when the large-$N_c$ limit is taken, but the expansion breaks down for baryons with spin and isospin $O(N_c)$.

The mass spectrum of the baryon tower with $K = 0$ can be related to the mass spectra of baryon towers with different $K$ by considering kaon-baryon scattering [11]. The masses of the different $K$ towers are related by

$$M = m_0 N_c I + m_1 K + \ldots ,$$  \hspace{1cm} (46)

so the baryon towers with differing strangeness are not degenerate, and the leading mass splitting between different baryon towers is linear in strangeness. Higher order corrections also can be classified.

### 2.4.2 Axial Couplings: No $1/N_c$ Correction for $N_F = 2$

A consistency condition for the $1/N_c$ correction to the baryon axial vector current is obtained by considering the scattering process baryon + pion $\rightarrow$ baryon + 2 pions, shown in Fig. 9. Each baryon-pion vertex is $O(\sqrt{N_c})$, so individual diagrams are $O(N_c^{3/2})$ but the total amplitude is at most $O(1/\sqrt{N_c})$. Summing over all the tree diagrams yields the constraint [8]

$$N_c^{3/2} [X^{ia}, [X^{jb}, X^{kc}]] \leq O\left(\frac{1}{\sqrt{N_c}}\right) ,$$  \hspace{1cm} (47)

which implies that the double commutator of three $X$'s vanishes at least as fast as $O(1/N_c^2)$,

$$[X^{ia}, [X^{jb}, X^{kc}]] \leq O\left(\frac{1}{N_c^2}\right) .$$  \hspace{1cm} (48)

Expanding $X^{ia}$ in $1/N_c$ gives the large-$N_c$ consistency condition for $X_1^{ia}$

$$[X_0^{ia}, [X_0^{jb}, X_1^{kc}]] + [X_1^{ia}, [X_1^{jb}, X_0^{kc}]] = 0 .$$  \hspace{1cm} (49)

The unique solution to this consistency condition is $X_1^{ia} \propto X_0^{ia}$. There is no other candidate operator since the operators $\epsilon^{abc}I^bX_0^{ic}$ and $\epsilon^{ijk}J^jX_0^{ka}$ do not transform in the same manner as the axial current under time reversal, and the operator $\epsilon^{ijk} \epsilon^{abc}X_0^{jb}X_0^{kc}$ is equal to $2X_0^{ia}$. Eq. (44) implies that $X^{ia}$ is equal to $X_0^{ia}$ times $(1+c/N_c)$ where $c$ is an arbitrary coefficient, upto corrections of order $1/N_c^2$. The $1/N_c$ proportionality factor can be absorbed into the unknown coupling $g$, so

$$X^{ia} = X_0^{ia} + O\left(\frac{1}{N_c^2}\right) ,$$  \hspace{1cm} (50)

and all ratios of pion-baryon couplings are determined as before upto corrections of $O(1/N_c^2)$.

The above derivation is valid for any baryon spin tower with fixed $K$. The normalization of the baryon axial vector currents for different spin towers with differing strangeness can be related by considering kaon-baryon scattering [11].
The result is
\[
\langle J' J'_z, I' I'_z; K | A^a | J J_z, I I_z; K \rangle = N_c g(K) (-1)^{2J'+J-J'-K} \times 
\sqrt{(2I+1)(2J+1)} \left\{ \begin{array}{ccc} 1 & I & I' \\ K & J' & J \end{array} \right\} \left( \begin{array}{ccc} I \ & 1 \ & I' \\ I_z \ & a \ & I'_z \end{array} \right) \left( \begin{array}{ccc} J \ & 1 \ & J' \\ J_z \ & i \ & J'_z \end{array} \right)
\]
with the couplings of different \(K\) towers related by \(g(K) = g(0) + g_1 K/N_c + \ldots\), where \(g_1\) is an arbitrary coefficient. The value of \(g_1\) can be determined in the \(SU(3)\) flavor symmetry limit \([1]\). The leading correction to the coupling constants \(g(K)\) of different \(K\) towers is linear in strangeness, so there is an equal spacing rule for the pion-baryon couplings of baryons with different strangeness \([1]\).

Higher order \(1/N_c\) corrections to the baryon axial vector current can be studied by considering baryon-pion scatterings to a baryon + \(n\) pions, where \(n > 2\). The first nontrivial correction \(X^{ia}_2 = J P^a\) occurs at \(O(1/N_c^2)\), so the large-\(N_c\) values of baryon-pion coupling ratios should be violated at this level. The measured couplings are observed to satisfy the relations at the 10\% level, which is what one would naively expect for \(1/N_c = 1/3\).

### 2.4.3 General Form of \(1/N_c\) Expansion

It is possible to write down the \(1/N_c\) expansion for a baryon operator directly without deriving the large-\(N_c\) consistency conditions. The \(1/N_c\) expansion for any static baryon operator for \(N_F = 2\) light quark flavors is of the form
\[
N_c \mathcal{P} \left( X_0, \frac{J}{N_c}, \frac{I}{N_c} \right),
\]
where \(\mathcal{P}\) is a polynomial in its arguments. It is to be understood that all possible independent operator products constructed out of the generators of the contracted spin-flavor algebra appear in the \(1/N_c\) expansion with arbitrary coefficients. The generator \(X^{ia}_0\) has matrix elements of order unity for all baryons in a given irreducible representation of the spin-flavor algebra. The parameters \(J/N_c\) and \(I/N_c\) are \(O(1/N_c)\) for baryons with spin and isospin of order unity at the bottom of the baryon towers. For these baryons, operators with more powers of \(J/N_c\) and \(I/N_c\) are suppressed in \(1/N_c\) compared to operators with fewer powers. Baryons will \(J \sim O(N_c)\) at the top of the baryon tower, have \(J/N_c\) and \(I/N_c\) of order 1, so all terms in the \(1/N_c\) expansion are equally important, and the \(1/N_c\) expansion is not predictive. The \(1/N_c\) expansion can be extended to baryon towers with different \(K\) by including additional operators such as \(K/N_c\) in the polynomial \(\mathcal{P}\). This extension gives the form of the \(1/N_c\) expansion for \(N_F = 3\) light quark flavors with arbitrary \(SU(3)\) breaking because only isospin flavor symmetry is imposed.

### 3 \(1/N_c\) EXPANSION

The spin-flavor symmetry of large-\(N_c\) baryons provides the organizational framework for the baryon \(1/N_c\) expansion. As we have seen, the \(1/N_c\) expansion of any static baryon operator is given in terms of operator products of the baryon spin-flavor generators. Not all operator products are linearly independent, so redundant operators must be eliminated from the expansion using operator identities. Each independent operator product appears at a definite order in \(1/N_c\),
accompanied by an uncalculable coefficient. The $1/N_c$ expansion is predictive for baryons with finite spin and flavor quantum numbers because only a finite number of operator products occur at any given order in the $1/N_c$ expansion.

There are two natural operator realizations of the baryon spin-flavor algebra which can be used to construct the operator basis of the $1/N_c$ expansion. The first uses the spin-flavor generator $X_0^{ia}$ in operator products, whereas the second realization uses the $SU(2N_F)$ commutation relations with spin-flavor generator $G^{ia}$. The operators $X_0^{ia}$ and $G^{ia}/N_c$ differ at subleading order $1/N_c$. The ambiguity in the choice of spin-flavor generator arises because the contracted spin-flavor algebra for baryons is exact only in the large $N_c$ limit, so the matrix elements of the spin-flavor generators which complete the spin-flavor algebra are known only to leading order in $1/N_c$ up to a normalization factor. When $X_0^{ia}$ is used, the operator basis of the $1/N_c$ expansion is the same as the operator basis of the large-$N_c$ Skyrme model, whereas the operator basis constructed using $G^{ia}$ is the same as the operator basis of the large-$N_c$ nonrelativistic quark model. Both operator bases parametrize the same large-$N_c$ physics since all operator products appear in the $1/N_c$ expansion with uncalculated coefficients. In principle, these coefficients are calculable in terms of the complicated QCD dynamics, but in practice the coefficients are treated as unknowns since any such calculation would be tantamount to solving QCD exactly in the baryon sector. The coefficients of the $1/N_c$ expansion for large-$N_c$ QCD baryons are different from the coefficients of the large-$N_c$ Skyrme model and the large-$N_c$ quark model, but group theoretic predictions which do not depend on the coefficients will be the same for large-$N_c$ QCD and these models.

The general form of the $1/N_c$ expansion using $X_0^{ia}$ was explained briefly in the previous section. The solution was given for three light quark flavors with exact isospin symmetry and arbitrary $SU(3)$ flavor breaking. The analysis using $G^{ia}$ will be performed in this section. The discussion is first restricted to the case of two light quark flavors with exact isospin flavor symmetry. The generalization to $N_F = 3$ light quark flavors with $SU(3)$ flavor symmetry breaking is presented later in the section.

### 3.1 Baryon Operator Expansion

The static baryon matrix elements of a QCD 1-body operator, such as $(\bar{q}\gamma^\mu\gamma_5\tau^aq)_{\text{QCD}}$, have a $1/N_c$ expansion of the form

$$O^{1-\text{body}}_{\text{QCD}} = N_c \sum_n c_n \frac{1}{N_c^n} O_n ,$$

(52)

where the sum is over all independent operator polynomials $O_n$ of degree $n$ in the baryon spin-flavor generators which transform according to the same spin $\times$ flavor representation as $O_{\text{QCD}}$. Here spin refers to the $SU(2)$ algebra generated by the baryon spin $J^i$ in the rest frame of the baryon. For finite $N_c$, the sum on $n$ is over $0 \leq n \leq N_c$. Every operator coefficient $c_n(1/N_c)$ has an expansion in $1/N_c$ beginning at order unity. The factor $1/N_c^n$ is present since each spin-flavor generator in an $n^{th}$ order operator product $O_n$ is accompanied by a factor of $1/N_c$. The overall factor of $N_c$ arises because a QCD 1-body operator has matrix elements which are at most $O(N_c)$ when inserted on all quark lines of the baryon. The generalization of the $1/N_c$ expansion to a QCD $m$-body operator acting on $m$ quarks is obtained by replacing $N_c = N_c^1$ by $N_c^m$ in Eq. (52).
The baryon $1/N_c$ expansion depends on $N_c$ through the explicit $1/N_c$ prefactors and through the implicit $N_c$-dependence of the $O_n$ matrix elements. The matrix elements of the operators $O_n$ have a nontrivial dependence on $N_c$ which varies for different states of the baryon spin-flavor multiplet. This dependence on $N_c$ can be obtained by analyzing the $N_c$-dependence of the baryon spin-flavor generators. The baryon spin tower $J = I = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2}$ contains baryons at the bottom of the spin tower with spin $J$ and isospin $I$ of order unity, as well as baryons at the top of the spin tower with spin and isospin of $O(N_c)$. All of the baryon states in the multiplet have matrix elements of $G^{\alpha\beta}$ that are $O(N_c)$. Since the matrix elements of $J, I$ and $G$ are $\leq O(N_c)$ on the baryon states, $n^{th}$ order operator polynomials $O_n$ constructed from the generators have matrix elements which are $\leq O(N_n^c)$. Baryon states with $J \sim O(N_c)$ have $O_n$ matrix elements which are $O(N_n^c)$, so the explicit factor of $1/N_n^c$ is completely cancelled by the implicit $N_n^c$ dependence of the operator matrix elements. In this case, all terms in the $1/N_c$ expansion are equally important and must be retained at leading order, so the $1/N_c$ expansion is not predictive. For baryon states with $J \sim O(1)$, operator polynomials containing more factors of $J$ are systematically suppressed in $1/N_c$, and the operator expansion can be truncated at various orders in the $1/N_c$ expansion. The $1/N_c$ expansion for the low-spin baryons is predictive because not all spin-flavor structures appear at a given order in $1/N_c$.

The group theoretic structure of the $1/N_c$ expansion is determined by the operators $O_n$ and the explicit $1/N_c$ prefactors. The complicated large-$N_c$ QCD dynamics is parametrized by the unknown coefficients $c_n(1/N_c)$. The quark-gluon diagrams of large-$N_c$ QCD contribute to the $c_n(1/N_c)$ at different orders in $1/N_c$ depending on their topological structure. Diagrams of $N_c$ valence quark lines with arbitrary planar gluon exchange contribute to the coefficients at leading order in $1/N_c$. Diagrams containing quark loops and nonplanar gluons contribute to the coefficients at subleading orders. Each quark loop in a baryon diagram is suppressed by one factor of $1/N_c$, and each nonplanar gluon exchange is suppressed by $1/N_c^2$.

The operators $O_n$ which form the operator basis of the $1/N_c$ expansion have a natural interpretation as $n$-body quark operators acting on the spin and flavor indices of $n$ static quarks. This identification is possible because the spatial dependence of the quark wavefunctions is irrelevant for the computation of static baryon matrix elements of the ground state baryons, so it is only necessary to keep track of the spin, flavor and color quantum numbers of the quarks. (This point will be discussed in greater detail later in the context of the Hartree picture of large-$N_c$ baryons.) A large-$N_c$ baryon state is totally antisymmetric in the color indices of its $N_c$ constituent valence quarks, so the color structure of the large-$N_c$ baryon is trivial, and color indices of the quarks also can be omitted. When the color indices of the quarks are omitted, the quarks are treated as bosonic objects, since it is the the antisymmetry of the $SU(N_c)$ color $\epsilon$-symbol combined with the Fermi statistics of the quarks which implies that the ground state baryons contain $N_c$ quarks in the completely symmetric representation of spin $\otimes$ flavor. Define a set of quark creation and annihilation operators, $q_i^\dagger$ and $q_i^\alpha$, where $i = 1, 2$ and $\alpha = 1, \ldots, N_F$ represent the spin and flavor quantum numbers of the quarks. The creation and annihilation operators satisfy the bosonic commutation relation

$$[q_i^\alpha, q_j^\dagger_{\beta}] = \delta_{ij}^\alpha \delta_\beta, \quad (53)$$

An $n$-body quark operator is constructed in terms of these static quark creation
and annihilation operators.

A systematic classification of $n$-body quark operators is possible. There is a unique 0-body operator $\mathbb{1}$, which is the identity operator acting on baryons. This operator does not act on the quarks in the baryon. The 1-body operators that act on a single quark consist of the quark number operator $q\dagger q$ and

$$J^i = q\dagger \left( \frac{\sigma^i}{2} \otimes \mathbb{1} \right) q,$$

$$I^a = q\dagger \left( \mathbb{1} \otimes \frac{\tau^a}{2} \right) q,$$  \hspace{1cm} (54)

$$G^{ia} = q\dagger \left( \frac{\sigma^i}{2} \otimes \frac{\tau^a}{2} \right) q,$$

where $\sigma^i$ and $\tau^a$ are the $SU(2)$ spin and isospin generators acting on the spin-$\frac{1}{2}$ and isospin-$\frac{1}{2}$ indices of the quark creation and annihilation operators. The last three 1-body quark operators can be identified with the baryon spin-flavor generators $J^i$, $I^a$ and $G^{ia}$ when acting on baryon states, since the baryon matrix elements of a 1-body quark operator are obtained by summing over all possible insertions of the operator on the $N_c$ quark lines of the baryon. Thus, $J^i$, the $SU(2)$ baryon spin generator, is equal to the sum of the spins of the $N_c$ quarks in the baryon,

$$J^i = \sum_{\ell=1}^{N_c} q\dagger_{\ell} \left( \frac{\sigma^i}{2} \otimes \mathbb{1} \right) q_{\ell},$$  \hspace{1cm} (55)

where $\ell$ denotes the quark line number. Similarly, the baryon isospin generator $I^a$ is given by the sum of the isospins of the $N_c$ quarks, while the spin-flavor generator $G^{ia}$ is obtained by inserting both $\sigma^i/2$ and $\tau^a/2$ on each quark line,

$$G^{ia} = \sum_{\ell=1}^{N_c} q\dagger_{\ell} \left( \frac{\sigma^i}{2} \otimes \frac{\tau^a}{2} \right) q_{\ell}.$$  \hspace{1cm} (56)

Inserting the quark number operator on each quark line of the baryon counts the number of quarks in the baryon,

$$q\dagger q = \sum_{\ell=1}^{N_c} q\dagger_{\ell} q_{\ell} = N_c \mathbb{1}.$$  \hspace{1cm} (57)

Thus, baryon matrix elements of the quark number operator can be rewritten in terms of the 0-body baryon identity operator $\mathbb{1}$, and the only independent 1-body operators are $J^i$, $I^a$ and $G^{ia}$. The $n$-body operators for $n \geq 2$ can be written as completely symmetric operator products of degree $n$ in the baryon spin-flavor generators. An example of a 2-body operator is

$$J^i T^a = \sum_{\ell, \ell'} \left( q\dagger_{\ell} \frac{\sigma^i}{2} q_{\ell} \right) \left( q\dagger_{\ell'} \frac{\tau^a}{2} q_{\ell'} \right),$$  \hspace{1cm} (58)

which differs from the 1-body operator $G^{ia}$ because $\sigma^i/2$ and $\tau^a/2$ are inserted independently on the quark lines. Only symmetric products (anticommutators) of baryon spin-flavor generators occur because antisymmetric products (commutators) reduce to linear combinations of lower-body operators by the $SU(2N_F)$ Lie
algebra commutation relations. Not all of the \(n\)th order operator products correspond to independent operators. The classification of the independent \(n\)-body operator polynomials for \(n \geq 2\) is nontrivial, and requires additional discussion.

It is convenient to consider normal ordered \(n\)-body operators

\[ q_{j_1\beta_1}^{\dagger} \cdots q_{j_n\beta_n}^{\dagger} \mathcal{T}^{(j_1\beta_1)\cdots(j_n\beta_n)}_{(i_1\alpha_1)\cdots(i_n\alpha_n)} q_{i_1\alpha_1}^{\dagger} \cdots q_{i_n\alpha_n}^{\dagger}, \tag{59} \]

where \(\mathcal{T}\) denotes a spin and flavor tensor which saturates the spin and flavor indices of the \(n\) quark creation and annihilation operators. The order of the \(n\) creation operators or the \(n\) annihilation operators in the normal ordered operator is unimportant, since the quark creation and annihilation operators are bosonic operators. Normal ordered operators always act on \(n\) distinct quarks because \(n\) quarks are first annihilated and then created. The independent \(n\)-body operators are all normal-ordered \(n\)-body operators with traceless and completely symmetrized tensors \(\mathcal{T}\). The tensor \(\mathcal{T}\) must be completely symmetrized in its upper and lower pairs of spin and flavor indices because the quarks in the ground state baryons are in the completely symmetric representation of the spin-flavor group, so tensor operators with mixed symmetry vanish identically when acting on the ground state baryons. The tensor \(\mathcal{T}\) is also traceless, since all nonvanishing traces of the tensor \(\mathcal{T}\) reduce to lower-body operators.

It is easy to identify the independent \(n\)-body operators when the operators are written in normal ordered form, but normal ordered operators do not have simple baryon matrix elements. For example, the normal ordered 2-body operator

\[ q_{j_1\alpha_1}^{\dagger} q_{j_2\beta_2}^{\dagger} \left( \sigma^1 \right)_{i_1}^{j_1} \left( \tau^a \right)_{i_2}^{\beta_2} q_{i_1\alpha_1}^{\dagger} q_{i_2\alpha_2}^{\dagger} = \sum_{\ell \neq \ell'} \left( q_{\ell}^{\dagger} \sigma^1 q_{\ell} \right) \left( q_{\ell'}^{\dagger} \tau^a q_{\ell'} \right), \tag{60} \]

does not equal \(J^i T^a\) since the double sum over quark lines is restricted to pairs of distinct quark lines \(\ell \neq \ell'\).

In contrast to the situation for normal ordered operators, the baryon matrix elements of operator products of the baryon spin-flavor generators are simple to evaluate, but the classification of independent operator products is nontrivial. When operator products are rewritten in normal ordered form, there are tensors \(\mathcal{T}\) which are not traceless or completely symmetrized, so some linear combinations of the operator products are equivalent to lower body operators or vanish identically on the completely symmetric ground state baryon representation. Operator identities describe the operator product combinations which are redundant operators and must be eliminated from the operator expansion. The complete set of operator identities for the ground state baryons have been derived and are given in the next section. Construction of the independent operator basis \(\mathcal{O}_n\) of the baryon \(1/N_c\) expansion for any spin \(\times\) flavor representation is straightforward using the operator identities.

The formulation of the baryon \(1/N_c\) expansion using static quark operators is described in the work of Dashen \etal. Carone \etal. and Luty \& March-Russell. Dashen \etal. showed that the \(1/N_c\) expansion is given in terms of operator products of the baryon spin-flavor generators, which can be interpreted as soliton operators or quark operators. Carone \etal. analyzed the \(1/N_c\) expansion in terms of normal-ordered static quark operators in the Hartree picture, while Luty \& March-Russell studied the \(1/N_c\) expansion in terms of static quark operators using a many-body approach.
3.2 Redundant Operators (Operator Identities)

The complete set of operator identities for the ground state baryons have been derived by Dashen et al. [21]. The operator identities have an elegant group theoretic structure, and are given in terms of the Casimir invariants of the SU(2N_F) baryon spin-flavor representation. An important result of Ref. [21] is that the only nontrivial operator identities which are required are those which reduce 2-body operators to linear combinations of 1-body or 0-body operators. All identities for n-body operators with n > 2 can be obtained by recursively applying the 2-body identities on all pairs of 1-body operators appearing in a completely symmetric n\textsuperscript{th} order operator product. The operator identities have been derived for arbitrary numbers of light quark flavors N_F.

The operator identities for N_F = 2 light quark flavors are given in Table 1. The spin \otimes\text{ flavor representation (J, I) of each operator identity is given explicitly. The identities are grouped according to their group theoretic origin. The first identity is the SU(4) quadratic Casimir relation

$$\Lambda^A \Lambda^A = C(R) \mathbb{1},$$

(61)

where \Lambda^A, A = 1, \ldots, 15, refer to the 15 SU(4) generators \textit{J}\textsuperscript{i}, \textit{I}\textsuperscript{a} and \textit{G}\textsuperscript{ia}, and C(R) is the quadratic Casimir for the baryon spin-flavor representation Fig. 6. The second block of identities results from the SU(4) cubic Casimir relation

$$d^{ABC} \Lambda^B \Lambda^C = D(R) \Lambda^A$$

(62)

which involves the SU(4) d-symbol. The third block of identities correspond to operators which vanish identically when acting on the completely symmetric baryon spin-flavor representation.

The 1/N_c expansion for any static baryon operator can be constructed using the operator identities to reduce the set of operators to a complete and independent set. It is possible to systematically construct an operator basis which maximizes the number of \textit{J}'s in operator products by eliminating redundant operators which contain more powers of \textit{I} and \textit{G}. This reduction of operator products is summarized by an operator reduction rule: All operators in which two spin or isospin indices are contracted with a \delta or \epsilon symbol can be eliminated, with the exception of \textit{J}\textsuperscript{2}. (Note \textit{J}\textsuperscript{2} = \textit{I}\textsuperscript{2} for the baryon states, so symmetry under spin and isospin remains.) The application of the operator identities and the operator reduction rule is straightforward, and will be demonstrated in the following examples.

3.2.1 Masses

The baryon mass transforms as a spin-isospin singlet (0, 0). The 0-body operator \mathbb{1} transforms as a singlet, and is one of the operators in the 1/N_c expansion. None of the baryon spin-flavor generators transforms as a singlet, so there are no 1-body operators in the expansion. The candidate 2-body operators are the singlets \textit{J}\textsuperscript{2}, \textit{I}\textsuperscript{2} and \textit{G}\textsuperscript{2}. The operator identities give two nontrivial relations amongst these three 2-body operators, so only one of the three operators is independent. Without loss of generality, the 2-body operator appearing in the 1/N_c expansion can be chosen to be \textit{J}\textsuperscript{2}. This choice is in accord with the operator reduction rule. It is also possible to classify the higher body operators using the 2-body identities: the only higher body operators are (\textit{J}\textsuperscript{2})\textsuperscript{n}. Thus, the baryon mass has
the expansion \[ \frac{1}{N_c} M = m_0 N_c \mathbb{1} + m_2 \frac{1}{N_c} J^2 + m_4 \frac{1}{N_c^2} (J^2)^2 + \ldots, \] (63)

where each term is multiplied by an arbitrary coefficient, and the ellipsis represents higher body operators. The mass expansion has the form

\[ N_c \mathcal{P} \left( \frac{1}{N_c}, \frac{J^2}{N_c^2} \right), \] (64)

where \( \mathcal{P} \) is a general polynomial in its arguments. For \( N_c = 3 \), the \( 1/N_c \) expansion only extends to 3-body operators, and there are only two terms in the mass expansion Eq. (63). The two mass coefficients \( m_0 \) and \( m_2 \) parametrize the two masses \( N \) and \( \Delta \) in terms of the spin-independent average mass and the hyperfine mass splitting \( (\Delta - N) \). For baryons with \( J \sim O(1) \), the hyperfine mass splitting is order \( 1/N_c \), and is suppressed by a factor of \( 1/N_c^2 \) relative to the leading \( O(N_c) \) contribution to the baryon mass. The mass splitting \( (\Delta - N) \) also is proportional to \( \langle J^2 \rangle_{3/2} - \langle J^2 \rangle_{1/2} = \frac{15}{4} - \frac{3}{4} = 3 \), so one expects \( (\Delta - N) \sim 300 \text{ MeV} \) given that the average \( O(N_c) \) baryon mass is \( \sim 1 \text{ GeV} \).

### 3.2.2 Axial Couplings

The isovector baryon axial vector current transforms as a \((1,1)\) under spin \( \otimes \) isospin. There is no 0-body operator transforming as a \((1,1)\). There is one 1-body operator which has spin-1 and isospin-1, namely \( G^{ia} \). Two operator identities in Table 1 transform as a \((1,1)\). The 2-body operators containing a single spin or flavor \( \epsilon \)-symbol in the second operator identity, namely \( \epsilon^{ijk} \{ J^i, G^{jc} \} \) and \( \epsilon^{abc} \{ I^a, G^{kb} \} \), do not transform under time-reversal in the same way as the isovector axial vector current, and therefore are not allowed operators. The 2-body operators in the first operator identity are both allowed operators. The two operators are not independent, so there is one independent 2-body operator, which can be chosen to be \( J^i I^a \). Again, this choice is prescribed by the operator reduction rule. Higher-body operators are obtained by taking anticommutators of the above operators with \( J^2 \). The single allowed 3-body operator is \( \{ J^2, G^{ia} \} \), and the single allowed 4-body operator is \( \{ J^2, J^i I^a \} \). In general, the operators \( \mathcal{O}_{ia}^1 = G^{ia}, \mathcal{O}_{ia}^2 = J^i T^a \) and

\[ \mathcal{O}_{ia}^{n+2} = \{ J^2, \mathcal{O}_{ia}^{n} \} \] (65)

for \( n \geq 1 \) form a complete and independent operator basis for the \( 1/N_c \) expansion of a \((1,1)\) operator. Thus, the \( 1/N_c \) expansion for the isovector axial vector current is given by

\[ A^{ia} = a G^{ia} + b \frac{1}{N_c} J^i T^a + c \frac{1}{N_c^2} \{ J^2, G^{ia} \} + \ldots, \] (66)

where the ellipsis represents higher-body operators. For \( N_c = 3 \), there are only three terms in expansion. The three operator coefficients \( a, b \) and \( c \) of the \( 1/N_c \) expansion parametrize the three isovector couplings \( g_{\pi NN}, g_{\pi N\Delta} \) and \( g_{\pi \Delta\Delta} \). The matrix elements of \( G^{ia} \) are \( O(N_c) \), and the matrix elements of \( J \) and \( I \) are \( O(1) \) for low-spin baryons, so the \( 1/N_c \) expansion can be truncated after the first term.
upto corrections of relative order $1/N_c^2$ from the neglected $b$ and $c$ terms. Thus, one recovers the result from Sect. (2.4.2) that ratios of the isovector baryon axial vector couplings are given by $G^{a}$ up to corrections of order $1/N_c^2$ relative to the leading order.

3.3 Hartree Picture

The spin-flavor structure of large-$N_c$ baryons has been studied in the Hartree picture by Carone et al. [12]. A number of interesting results can be derived by making the zeroth order ansatz that the Hartree potential is spin and flavor independent.

If the Hartree potential is spin and flavor independent, then the $N_c$ quarks in the large-$N_c$ baryon are all in the same ground state wavefunction $\phi(r)$, and the large-$N_c$ baryon wavefunction is given by the product Eq. (8) of $N_c$ identical quark wavefunctions. The baryon wavefunction must be completely symmetric in space, spin and flavor, so the complete symmetry of the baryon wavefunction with respect to the spatial coordinates of the $N_c$ valence quarks implies that the baryon wavefunction is completely symmetric in the spin-flavor indices of the $N_c$ quarks. Thus, the spin-flavor wavefunction of the ground state baryons is given by the completely symmetric spin-flavor representation shown in Fig. 6.

A generic $n$-body operator $O_n$ appearing in the $1/N_c$ expansion for baryons can be written as

$$
\int d^3x_1 \ldots d^3x_n \phi_{j_1 \beta_1}^\dagger (x_1) \ldots \phi_{j_n \beta_n}^\dagger (x_n) \times O^{(j_1 \beta_1) \ldots (j_n \beta_n)}_{(i_1 \alpha_1) \ldots (i_n \alpha_n)} (x_1, \ldots, x_n) \phi_{i_1 \alpha_1} (x_1) \ldots \phi_{i_n \alpha_n} (x_n),
$$

where $\phi_{i \alpha}^\dagger (x)$ denotes a quark annihilation operator which annihilates a quark with spin $i$ and flavor $\alpha$ in a given wavefunction. The ansatz for the ground state baryon wavefunction implies that the quark wavefunction annihilation operator can be replaced by

$$
\phi_{i \alpha}^\dagger (x) \rightarrow \phi(r) q^{\alpha},
$$

where the quark annihilation operator $q^{\alpha}$ only acts on the spin and flavor indices of the quark. The spatial symmetry of the baryon wavefunction implies that the tensor $O_n$ in Eq. (67) does not carry any angular momentum, because terms transforming nontrivially under orbital angular momentum on any quark line will vanish when integrated over space. Thus, the operator tensor may be replaced by

$$
O^{(j_1 \beta_1) \ldots (j_n \beta_n)}_{(i_1 \alpha_1) \ldots (i_n \alpha_n)} (x_1, \ldots, x_n) \rightarrow O^{(j_1 \beta_1) \ldots (j_n \beta_n)}_{(i_1 \alpha_1) \ldots (i_n \alpha_n)} (r_1, \ldots, r_n)
$$

when acting on the ground state baryons. After the spatial integrations in Eq. (67) are performed, the Hartree operator $O_n$ reduces to a normal-ordered quark operator acting on the spin and flavor indices of $n$ static quarks, Eq. (52).

The above deductions are predicated on the assumption that the Hartree potential is spin-independent at leading order in $1/N_c$. The Hartree Hamiltonian is a spin and flavor singlet, and has an expansion of the form

$$
\mathcal{H} = h_0 N_c \mathbb{I} + h_2 \frac{1}{N_c} J^2 + \ldots,
$$

where $h_n$ are arbitrary unknown coefficients. For baryons with spins $J \sim O(1)$, the spin-dependent part of the Hartree potential is suppressed in $1/N_c$ relative to
the spin-independent part, and the initial assumption that the Hartree potential is
spin and flavor independent is justified in the large-\(N_c\) limit. The spin-dependent
part of the Hartree potential first appears at relative order \(1/N_c^2\) compared to
the leading term, so the wavefunction of low-spin baryons is not significantly
deformed by spin-dependent interactions. Thus, the Hartree picture leads to a
self-consistent picture of the spin-flavor structure of large-\(N_c\) baryons with spins of
order unity. For baryons with spins \(J \sim O(N_c)\), the spin-dependent contributions
to the energy are not corrections, but are \(O(N_c)\) and of leading order, so the
ansatz that the Hartree potential is spin and flavor independent is not valid, and
the argument breaks down. Significant deformation of the baryon wavefunction
due to spin-dependent interactions is possible for baryons with spins \(J \sim O(N_c)\).

3.4 \(SU(3)\) Flavor Symmetry

The baryon \(1/N_c\) expansion for \(N_F = 3\) light quark flavors is considerably more
complicated than for two light flavors, because baryon states and operator matrix
elements have a nontrivial dependence on \(N_c\).

The ground state baryon representation of the \(SU(2N_F)\) spin-flavor algebra is
given by the completely symmetric Young tableau with \(N_c\) boxes. For odd \(N_c\),
the decomposition of this spin-flavor representation under \(SU(2N_F) \rightarrow SU(2) \otimes
SU(N_F)\) yields the baryon spin tower of Fig. 7. For any \(N_F \geq 3\), the flavor
representations of the baryon spin tower have dimensions which depend on \(N_c\)
even for low-spin baryons. The \(SU(3)\) flavor weight diagrams for the spin \(1/2\)
and spin \(3/2\) baryons are shown explicitly in Figs. 10 and 11, respectively. The
weight diagrams reduce to octet and decuplet multiplets for \(N_c = 3\), but there
are many additional states for \(N_c > 3\). Because the baryon flavor representations
vary with \(N_c\), the identification of QCD baryons with large-\(N_c\) baryons is not
unique. The \(1/N_c\) expansion yields the most predictions for baryons with finite
spin and flavor quantum numbers, so it is best to identify the QCD baryons
with states at the top of the weight diagrams with fixed finite strangeness. Note
that the large-\(N_c\) baryon states in the \(SU(3)\) flavor weight diagrams with a given
fixed strangeness are in one-to-one correspondence with the \((J, I)\) towers of Sect.
(2.3). Although the identification of QCD baryons with large-\(N_c\) baryons is not
unique, the extrapolation of the \(1/N_c\) expansion of baryon operators to \(N_c = 3\)
is unambiguous, and all the \(1/N_c\) expansions in the literature reduce to the same
operator expansion for \(N_c = 3\).

The analysis of the \(N_c\)-dependence of operator products appearing in the \(1/N_c\)
expansion is subtle for \(N_F = 3\) light quark flavors because the operator matrix
elements have a different \(N_c\)-dependence in different parts of the flavor weight
diagram. The \(N_c\)-dependence of operator products is determined by the \(N_c\)-
dependence of the baryon spin-flavor generators. The \(SU(6)\) spin-flavor genera-
tors are given by

\[
J^i = q^\dagger \left( \frac{\sigma^i}{2} \otimes 1 \right) q, \\
T^a = q^\dagger \left( 1 \otimes \frac{\lambda^a}{2} \right) q, \\
G^{\alpha \alpha} = q^\dagger \left( \frac{\sigma^i}{2} \otimes \frac{\lambda^a}{2} \right) q,
\]  
(71)
where $\lambda^a$ are Gell-Mann $SU(3)$ matrices. The matrix elements of the baryon spin $J$ are $O(1)$ for baryons at the bottom of the spin tower and $O(N_c)$ for baryons at the top of the spin tower, as before. The matrix elements of the other spin-flavor generators are much more subtle. Unlike the case for two light flavors, matrix elements of the spin-flavor generators $T^a$ and the matrix elements of the spin-flavor generators $G^{ia}$ do not have the same $N_c$-dependence everywhere in the flavor weight diagram. Consider baryons at the top of the spin weight diagrams with finite strangeness. These baryons have matrix elements of the spin-flavor generators $T^a$, which are $O(1)$, $O(\sqrt{N_c})$, and $O(N_c)$ for $a = 1, 2, 3, a = 4, 5, 6, 7$, and $a = 8$, respectively, and matrix elements of the spin-flavor generators $G^{ia}$ which are $O(N_c)$, $O(\sqrt{N_c})$ and $O(1)$, for $a = 1, 2, 3, a = 4, 5, 6, 7$, and $a = 8$, respectively.

For baryon states in other regions of the weight diagrams, different combinations of the $T$'s and $G$'s will be order $N_c$, $\sqrt{N_c}$ and 1. Thus, the matrix elements of $T^a$ are not suppressed relative to matrix elements of $G^{ia}$ everywhere in the flavor weight diagram, and it is not possible to neglect the matrix elements of $T^a$ or $G^{ia}$ for any $a$ unless one restricts one's attention to baryons in a certain region of the flavor weight diagram, such as baryons with finite flavor quantum numbers. Irregardless of this choice, the baryon $1/N_c$ expansion is predictive for baryons with $J \sim O(1)$ because the matrix elements of the operator products with more powers of $J/N_c$ are suppressed.

The operator identities for $N_F = 3$ light quark flavors are given in Table 2, which is taken from Ref. [21]. Many new operator structures appear at $N_F = 3$ because there is a $d$-symbol for the $SU(3)$ flavor group. The elimination of redundant operators is not as simple for three light quark flavors as for two light quark flavors, so the operator reduction rule is modified. The operator reduction rule for $N_F = 3$ light quark flavors is: All operator products in which two flavor indices are contracted using $\delta^{ab}$, $d^{abc}$, or $f^{abc}$ or two spin indices on $G$'s are contracted using $\delta^{ij}$ or $\epsilon^{ijk}$ can be eliminated.

### 3.5 Flavor Symmetry Breaking

An additional complication for $N_F = 3$ light quark flavors is that effects of $SU(3)$ flavor symmetry breaking are comparable to $1/N_c$ corrections, so flavor symmetry breaking cannot be neglected. $SU(3)$ flavor symmetry breaking can be included perturbatively in the $1/N_c$ expansion. The combined expansion in $1/N_c$ and $SU(3)$ flavor symmetry breaking produces a rich pattern of spin-flavor symmetry breaking which is not dominated either by $1/N_c$ or flavor symmetry breaking corrections.

The baryon $1/N_c$ expansion for finite $N_c$ only extends to operator products of $N_c^{1\text{th}}$ order in the baryon spin-flavor generators, so the perturbative expansion in $SU(3)$ flavor breaking extends only to a given finite order in flavor symmetry breaking. For a flavor and spin singlet baryon operator with a $1/N_c$ expansion beginning with the 0-body operator $1$, the expansion in flavor symmetry breaking goes to $N_c^{1\text{th}}$ order. For baryon operators with $1/N_c$ expansions beginning with a 1-body operator, such as a flavor octet or a spin-1 baryon operator, the flavor symmetry breaking expansion goes to order $(N_c - 1)$. In general, the flavor symmetry breaking expansion extends to order $(N_c - n)$ for a baryon operator with a $1/N_c$ expansion beginning with a $n$-body operator.

The incorporation of flavor symmetry breaking into the baryon $1/N_c$ expansion is best illustrated by an example.
3.5.1 SU(3) Flavor Analysis of Baryon Masses

The masses of the baryon octet and decuplet can be analyzed in a combined expansion in $1/N_c$ and SU(3) flavor symmetry breaking \[22\]. The combined expansion is given by

$$M = M^1 + M^8 + M^{27} + M^{64},$$ \hspace{1cm} (72)

where the $1/N_c$ expansions of the flavor singlet, octet, 27 and 64 contributions to the baryon masses are

\begin{align*}
M^1 &= N_c \mathbb{I} + \frac{1}{N_c} J^2, \\
M^8 &= T^8 + \frac{1}{N_c} \left\{ J^i, G^{i8} \right\} + \frac{1}{N_c^2} \left\{ J^2, T^8 \right\}, \\
M^{27} &= \frac{1}{N_c} \left\{ T^8, T^8 \right\} + \frac{1}{N_c^2} \left\{ T^8, \left\{ J^i, G^{i8} \right\} \right\}, \\
M^{64} &= \frac{1}{N_c^2} \left\{ T^8, \left\{ T^8, T^8 \right\} \right\},
\end{align*} \hspace{1cm} (73)

respectively, and it is to be understood that each operator is multiplied by an unknown coefficient. The $1/N_c$ expansion extends only up to 3-body operators for QCD baryons with $N_c = 3$. The coefficients of the flavor singlet operators in $M^1$ are $O(1)$ in SU(3) flavor symmetry breaking, while the coefficients of the flavor-octet operators in $M^8$ are proportional to one power of SU(3) flavor symmetry breaking $\epsilon$. The SU(3) flavor breaking parameter $\epsilon$ is proportional to $m_s/\Lambda_\chi$, and is order 30%. The flavor-27 operators are second order in SU(3) flavor symmetry breaking, and the one flavor-64 operator is third order in SU(3) flavor symmetry breaking. Each operator also occurs at a definite order in $1/N_c$, which is given by the explicit factor of $1/N_c$ in front of the operator times the leading $N_c$-dependence of the operator matrix element.

The eight mass operators in the $1/N_c$ expansion Eq. (73) parametrize the eight baryon masses $N$, $\Delta$, $\Lambda$, $\Sigma$, $\Sigma^*$, $\Xi$, $\Xi^*$, and $\Omega$ of the spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet baryons. Each operator of the combined $1/N_c$ and flavor symmetry breaking expansion corresponds to a unique linear combination of octet and decuplet baryon masses. Mass relations between octet and decuplet baryon masses are obtained by neglecting subleading operators in the combined $1/N_c$ and flavor symmetry breaking expansion. For example, the mass combination corresponding to the $J^2/N_c$ operator is

$$\frac{1}{8} \left( 2N + \Lambda + 3\Sigma + 2\Xi \right) - \frac{1}{10} \left( 4\Delta + 3\Sigma^* + 2\Xi^* + \Omega \right).$$ \hspace{1cm} (74)

Neglect of the $J^2/N_c$ operator relative to the leading singlet operator implies that the mass combination Eq. (74) vanishes. Thus, the average mass of the baryon octet is equal to the average mass of the average mass of the baryon decuplet when the hyperfine operator $J^2/N_c$ is neglected. The eight baryon mass combinations are graphed in Fig. 12. The dimensionless quantity $\sum B_i/(\sum |B_i|/2)$ is graphed for each mass combination. The hierarchy of baryon masses is in excellent agreement with the $1/N_c$ and flavor symmetry breaking factors of the combined analysis.
4 SUMMARY OF MAIN RESULTS

Many results have been derived in the literature using the baryon $1/N_c$ expansion. A brief description of some of the main applications is given here.

4.1 Axial Couplings

The baryon axial vector couplings have been studied extensively. The analysis for $N_F = 2$ light quark flavors is given in the original paper of Dashen & Manohar \[8\]. The extension to $N_F = 3$ light quark flavors with arbitrary $SU(3)$ flavor symmetry breaking is given by Dashen et al. \[11\] using the operator $X_0^{ia}$. The analysis with exact $SU(3)$ flavor symmetry imposed at leading order in $1/N_c$ also is given. In Luty \[14\], the $SU(3)$ flavor symmetry analysis at leading order in $1/N_c$ is given using quark operators. The complete $1/N_c$ expansion with exact $SU(3)$ flavor symmetry and perturbative $SU(3)$ breaking is derived in Dashen et al. \[21\]. A detailed fit to the hyperon decay data and measured decuplet $\rightarrow$ octet + pion axial couplings is performed in Ref. \[23\].

The two-flavor analysis was presented in Sect. 2 of this review. For three light quark flavors, a number of additional results are found. In the $SU(3)$ flavor symmetry limit, the pion couplings of the spin-$\frac{1}{2}$ octet and spin-$\frac{3}{2}$ decuplet baryons are described by the $1/N_c$ expansion \[1, 14, 21\]

$$A^{ia} = a_1 G^{ia} + b_2 \frac{1}{N_c} J^i T^{ia} + b_3 \frac{1}{N_c^2} \left\{ J^i, \left\{ J^j, G^{ja} \right\} \right\} + c_3 \frac{1}{N_c^2} \left( \left\{ J^2, G^{ia} \right\} - \frac{1}{2} \left\{ J^i, \left\{ J^j, G^{ja} \right\} \right\} \right).$$

(75)

The four coefficients $a_1$, $b_2$, $b_3$ and $c_3$ parametrize the four pion couplings $D$ and $F$ of the baryon octet, $C$ of decuplet-to-octet transitions, and $H$ of the decuplet \[18\]. At leading order in $1/N_c$, the expansion can be truncated after two operators \[11, 14, 21\], and one obtains the two relations

$$C = -2D, \quad H = 3D - 9F,$$

(76)

which are valid up to corrections of relative order $1/N_c^2$. The first relation is an $SU(6)$ relation, which explains why an $SU(6)$ coupling is found in chiral perturbation theory calculations \[18\].

The $F/D$ ratio can be extracted by analyzing the baryon-pion couplings for arbitrary large $N_c$ \[11, 21\]. If QCD baryons are identified with the large-$N_c$ baryons at the top of the $SU(3)$ flavor weight diagrams in Figs. 10 and 11 with strangeness of order unity, then the $b_2$ term in Eq. (75) is of relative order $1/N_c^2$ compared to the $a_1$ term for the pion couplings $a = 1, 2, 3$; of relative order $1/N_c$ for the kaon couplings $a = 4, 5, 6, 7$; and of relative order 1 for the $\eta$ couplings $a = 8$. Using the pion couplings of baryons with finite strangeness, one obtains $F/D = 2/3 + O(1/N_c^2)$. The determinations from the kaon and $\eta$ couplings are $F/D = 2/3 + O(1/N_c)$ and $F/D = 2/3 + O(1)$, respectively.

$SU(3)$ symmetry breaking leads to a rich hierarchy of couplings \[21, 23\]. One particularly interesting new result, mentioned in Sect. (2.4.2), is that there is an equal spacing rule for the pion axial vector couplings to baryons with differing strangeness \[11\]. This feature of the symmetry breaking is seen in the experimental data.
4.2 Masses

The baryon masses have been analyzed in many contexts. The first derivation of the subleading $J^2/N_c$ hyperfine operator is given by Jenkins [9]. The three-flavor analysis with $SU(3)$ flavor breaking is given by Dashen *et al.* [11]. The analysis was performed using the operator $K$. The same mass relations subsequently were obtained by Luty [14] using quark operators with manifest $SU(3)$ flavor symmetry. The large-$N_c$ result Eq. (64) for exact flavor symmetry is given by Carone *et al.* [12] and Luty & March-Russell [13]. Jenkins & Lebed [22] analyzed the isospin $I = 0, 1, 2, 3$ mass splittings of the baryon octet and decuplet to all orders in a combined expansion in $1/N_c$ and flavor symmetry breaking. The analysis from Ref. [22] of the $I = 0$ isospin-averaged masses was presented in Sect. (3.5.1). Results also were obtained for the isospin splittings of the baryon masses. A sample result of this analysis is that the Coleman-Glashow mass relation,

\[ (p - n) - (\Sigma^+ - \Sigma^-) + (\Xi^0 - \Xi^-) = 0, \]  

(77)

is $O(1/N_c)$ in the $1/N_c$ expansion, so the mass relation should be more accurate than is predicted by a flavor symmetry breaking analysis alone. Many additional predictions for the isospin splittings are found as well.

Nonanalytic corrections to the baryon masses have been calculated in chiral perturbation theory in conjunction with the $1/N_c$ expansion in Refs. [9,11,13,24,25]. The leading $m_s^{3/2}$ correction to the flavor $27$ baryon masses of the octet and decuplet is calculated in Jenkins [24]. Subleading chiral corrections are considered in Bedaque & Luty [25].

4.3 Magnetic Moments

The baryon magnetic moments have been studied by many authors. The first result by Dashen & Manohar [8] shows that the isovector baryon magnetic moments are proportional to $X_0^a$ in the large-$N_c$ limit up to a correction of relative order $1/N_c^2$, so the ratios of the isovector magnetic moments are determined for $N_F = 2$ flavors up to a correction of relative order $1/N_c^2$. The leading $m_s^{1/2}$ chiral correction to the baryon magnetic moments is observed to be $O(N_c)$ by Luty *et al.* [15]. A flavor $SU(3)$ analysis of the isoscalar and isovector magnetic moments including $SU(3)$ flavor symmetry breaking is given by Jenkins & Manohar [16]. An $SU(3)$ group theory decomposition of the leading chiral correction is given in Ref. [23], and a detailed comparison is made to the experimental data.

The magnetic moments are found to satisfy the $1/N_c$ hierarchy predicted by the $1/N_c$ expansion. The accuracy of magnetic moment relations is in good agreement with the theoretical predictions.

4.4 Other Applications

Many additional applications of the $1/N_c$ expansion have been considered in the literature. A few of the more important applications are briefly remarked upon below.
4.4.1 Heavy Quark Baryons

The couplings of baryons containing a single heavy quark to pions and mesons containing a single heavy quark are considered in Ref. [9]. The pion couplings of the heavy quark baryons are related to the pion couplings of the baryon octet and decuplet.

The masses of heavy baryons containing a single charm or bottom quark are studied in Ref. [26]. The heavy baryons are shown to satisfy a heavy quark spin-flavor symmetry as well as a large-$N_c$ light quark spin-flavor symmetry. The masses are studied to all orders in a combined expansion $1/N_c$, $SU(3)$ flavor symmetry breaking $\epsilon$, and the heavy quark symmetry breaking parameter $\sim \Lambda_{QCD}/m_Q$. Mass relations between charm baryon splittings and bottom baryon splittings are obtained, as are relations between mass splittings of heavy quark baryons and the baryon octet and decuplet. The most accurate mass relations can be used to predict the masses of unmeasured charm and bottom baryons.

4.4.2 Excited Baryons

Excited baryons also can be studied in the $1/N_c$ expansion. For finite large $N_c$, the first excited baryons are identified with the $SU(2N_F)$ representation shown in Fig. 13. The pion couplings of the excited baryons to the ground state baryons are studied in the large-$N_c$ limit in Carone et al. [27] using quark operators. Pirjol & Yan [28] derived large-$N_c$ consistency conditions for the pion couplings of the excited baryons and analyzed the masses and mixings of the baryons. Many of the large-$N_c$ predictions for excited baryons are seen in the observational data [29].

4.4.3 Nuclear Physics

The nuclear potential has been studied in the $1/N_c$ expansion [33,34]. The $1/N_c$ suppressions of various phenomenological terms can be predicted, and are found to accurately reproduce the known magnitudes of these terms.

4.4.4 Chiral Perturbation Theory

Large-$N_c$ consistency conditions can be derived by studying chiral loop corrections to baryon amplitudes. The constraint that the chiral loop corrections do not grow with more powers of $N_c$ than the leading order tree-level terms leads to consistency conditions equivalent to the ones presented in this review [8,9]. Cancellations between loop corrections with different intermediate baryons are required at leading order in $1/N_c$ for consistency of the $N_c$ counting of the chiral expansion [3,9,11].

In Ref. [24], the baryon chiral Lagrangian is formulated in an expansion in $1/N_c$ for finite $N_c$ to all orders in $1/N_c$, and it is shown how to perform chiral loop calculations at finite $N_c$.

5 CONCLUSIONS

The $1/N_c$ expansion has many important implications for the spin and flavor properties of baryons. Large-$N_c$ baryons satisfy a contracted spin-flavor algebra, which is used to organize the group theory of the baryon $1/N_c$ expansion. The baryon $1/N_c$ expansion is formulated in terms of operator products of the baryon
spin-flavor generators, whose baryon matrix elements are known. The spin-flavor structure and $1/N_c$ suppression factors of the baryon operator expansion are determined. A wide variety of symmetry predictions and relations for baryon static matrix elements can be obtained by neglecting subleading operators in the baryon $1/N_c$ expansion. The accuracy of various symmetry relations is predicted by the $1/N_c$ expansion. The spin-flavor algebra of large-$N_c$ baryons connects the quark and soliton pictures of large-$N_c$ baryons. Many successes of the nonrelativistic quark model and the Skyrme model can be understood in terms of the spin-flavor structure of the baryon $1/N_c$ expansion in large-$N_c$ QCD.

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Table 1: $SU(4)$ Operator Identities for $N_F = 2$.

| Expression | Condition | \( N = 2 \) |
|------------|-----------|--------------|
| \( \{J^i, J^j\} + \{I^a, I^a\} + 4 \{G^{ia}, G^{ia}\} = \frac{3}{2} N (N + 4) \) | \( (0, 0) \) |
| \( 2 \{J^i, G^{ia}\} = (N + 2) I^a \) | \( (0, 1) \) |
| \( 2 \{I^a, G^{ia}\} = (N + 2) J^i \) | \( (1, 0) \) |
| \( \frac{1}{2} \{J^k, I^c\} - \epsilon^{ijk} \epsilon^{abc} \{G^{ia}, G^{jb}\} = (N + 2) G^{kc} \) | \( (1, 1) \) |
| \( \{I^a, I^a\} - \{J^i, J^i\} = 0 \) | \( (0, 0) \) |
| \( 4 \{G^{ia}, G^{ib}\} = \{I^a, I^b\} \) | \( (I = 2) \) |
| \( \epsilon^{ijk} \{J^i, G^{jc}\} = \epsilon^{abc} \{I^a, G^{kb}\} \) | \( (1, 1) \) |
| \( 4 \{G^{ia}, G^{ja}\} = \{J^i, J^j\} \) | \( (J = 2) \) |

\( (2, 0) \)
Table 2: SU(6) Operator Identities for $N_F = 3$.

| Equation | \[d^{abc} \{G^{ia}, G^{ib}\} + \left(\frac{2}{3}\right) \{J^i, G^{ic}\} + \frac{1}{4} d^{abc} \{T^a, T^b\} = \frac{2}{3} (N + 3) \ T^c\] | (0, 0) |
|----------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------|
|          | $\{T^a, G^{ia}\} = \frac{2}{3} (N + 3) \ J^i$                                                                                                                                             | (1, 0)  |
|          | $\frac{1}{3} \ \{J^k, T^c\} + d^{abc} \ {G^{ia}, G^{kb}} - \epsilon^{ijk} f^{abc} \ {G^{ia}, G^{jb}} = \frac{4}{3} (N + 3) \ G^{kc}$             | (1, 8)  |
|          | $-12 \ \{G^{ia}, G^{ia}\} + 27 \ \{T^a, T^a\} - 32 \ \{J^i, J^i\} = 0$                                                                                                                    | (0, 0)  |
|          | $d^{abc} \ {G^{ia}, G^{ib}} + \frac{9}{4} d^{abc} \ {T^a, T^b} - \frac{10}{9} \ \{J^i, G^{ic}\} = 0$                                                                                   | (0, 8)  |
|          | $4 \ \{G^{ia}, G^{ib}\} = \{T^a, T^b\}$                                                                                                                                                    | (0, 27) |
|          | $\epsilon^{ijk} \ \{J^i, G^{jc}\} = f^{abc} \ \{T^a, G^{kb}\}$                                                                                                                         | (1, 8)  |
|          | $3 \ d^{abc} \ \{T^a, G^{kb}\} = \{J^k, T^c\} - \epsilon^{ijk} f^{abc} \ \{G^{ia}, G^{ib}\}$                                                                                   | (1, 8)  |
|          | $\epsilon^{ijk} \ \{G^{ia}, G^{jb}\} = f^{acg} d^{bhc} \ \{T^g, G^{kh}\}  \ \{J^i, J^j\}$                                                                                             | (10 + \overline{10}) |
|          | $3 \ \{G^{ia}, G^{ia}\} = \{J^i, J^j\}$                                                                                                                                                    | (2, 0)  |
|          | $3 \ \{G^{ia}, G^{jb}\} = \{J^i, G^{jc}\}$                                                                                                                                             | (2, 8)  |

Figure 1: Double line notation for a gluon.

Figure 2: A typical planar diagram contributing to a meson three-point vertex at leading order.
Figure 3: Quark-gluon diagrams for the scattering process baryon + meson → baryon + meson.

Figure 4: Quark-gluon diagrams for baryon coupling to a meson.

Figure 5: Leading order diagrams for the scattering $B + \pi \rightarrow B' + \pi$.

Figure 6: SU($2N_F$) spin-flavor representation for the ground state baryons. The Young tableau has $N_c$ boxes.

Figure 7: SU(2) $\otimes$ SU($N_F$) representations for the ground state baryons. The SU($2N_F$) representation decomposes into a tower of baryon states with $J = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2}$. Each Young tableau has $N_c$ boxes.
Figure 8: Hyperfine mass splittings for the tower of large-$N_c$ baryon states with $J = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N_c}{2}$. The $J^2/N_c$ operator leads to a mass splitting of $O(1/N_c)$ between baryons with spins $J \sim O(1)$, and to a mass splitting of $O(1)$ between baryons with spins $J \sim O(N_c)$. The mass splitting between the baryon states with $J = \frac{1}{2}$ and $J = \frac{N_c}{2}$ is $O(N_c)$.

Figure 9: A leading order diagram for the scattering $B + \pi \rightarrow B' + \pi + \pi$. Additional tree graphs with all permutations of the pion vertices contribute at leading order.

Figure 10: $SU(3)$ flavor weight diagram for the spin-$\frac{1}{2}$ baryons. The numbers denote the multiplicity of the weights. The long side of the weight diagram contains $\frac{1}{2} (N_c + 1)$ weights.
Figure 11: $SU(3)$ flavor weight diagram for the spin-$\frac{3}{2}$ baryons. The numbers denote the multiplicity of the weights. The long side of the weight diagram contains $\frac{1}{2} (N_c - 1)$ weights.

Figure 12: Isospin-averaged baryon mass combinations of Jenkins & Lebed [22]. The error bars are experimental. A hierarchy of baryon masses in $1/N_c$ and $SU(3)$ flavor symmetry breaking $\epsilon \sim 0.3$ is predicted by the theoretical analysis. The open circle is a mass combination of order $1/N_c^2$; the three solid triangles are mass relations of order $\epsilon/N_c$, $\epsilon/N_c^2$, and $\epsilon/N_c^3$; the open squares are relations of order $\epsilon^2/N_c^2$, and $\epsilon^2/N_c^3$; and the $\times$ is a relation of order $\epsilon^3/N_c^3$. 
Figure 13: $SU(2N_F)$ spin-flavor representation for the first excited baryons. The Young tableau has $N_c$ boxes.