TOWARDS THE THEORY OF CONTROL IN OBSERVABLE QUANTUM SYSTEMS

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Abstract. The fundamental mathematical definitions of the controlled feedback Markov dynamics of quantum-mechanical systems are introduced with regard to the dynamical reduction and filtering of quantum states in the course of quantum measurement in either discrete or continuous real time. The concept of sufficient coordinates for the description of a posteriori quantum states in a given class is introduced and it is proved that they form a classical Markov process with values in either state operators or state vector space. The general problem of optimal control of a quantum-mechanical system is discussed and the corresponding Bellman equation in the space of sufficient coordinates is derived. The results are illustrated in the example of control of the semigroup dynamics of a quantum system that is instantaneously observed at discrete times and evolves between measurement times according to the Schrödinger equation.

1. Introduction

The encouraging outlook for the application of coherent quantum optics (lasers) for communications and control has been recently stimulated by the steadily growing demands for greater accuracy of observation and monitoring, particularly under the "extreme" conditions of very faint signals at extremely great (astronomical) distances. On the other hand, instances of the successful exploitation of mathematical methods from information and control theory for the investigation of many physical phenomena in the microscopic world have also stimulated interest in the theoretical study, using general cybernetic principles, of the possibilities of dynamical systems described at the quantum-mechanical level [19] [15] [16] [1] [4]. It has been shown in [9] that it is natural to regard many physical problems as control problems for distributed systems described by standard quantum-mechanical equations. In particular, the possibility of the transition of a physical system from one microscopic state to another can be investigated [2] by the methods of the theory of controllability on Lie groups generated by the Schrödinger equation with a controlled Hamiltonian.

General problems in the theory of quantum dynamical systems with observation, control and feedback channels can be handled on the basis of the recent development [2] of an operational theory of open-loop quantum systems, for which the mathematical formalism was set down in [10] [13]. The investigation, undertaken in [3], of the dynamical observation and feedback control optimization problems for such systems has provided a means for solving these problems in the case of linear
Markov systems of the boson type, in particular for a controllable and observable quantum oscillator \(^4\). This work was based on the multistage quantum-statistical decision theory originally described in \(^4[4]\) for the problems of the optimal dynamical measurement and control of classical (i.e., commutative) Markov processes with quantum observation channels.

In the present article we describe a simplified problem of optimal feedback control of quantum dynamical systems which does not involve quantum-statistical decision theory. Here the observable subsystem at the output of the observable channel is regarded as classical and amendable to description at the macroscopic level, whereas the controlled entity remains a quantum dynamical system. In other words, we assume here, in contrast with \(^4[4][5][6]\), that the “instrument” at the output of the quantum-mechanical system is given, rather than to be optimized, and it is required only to find the optimal macroscopic feedback for a given performance criteria. The results obtained in this setting are special in relation to \(^4[4][2][3]\) as they correspond to the semiclassical case of commutativity of the algebra of output observables. They nonetheless deserve special consideration both from the methodological and from the practical point of view when the observation channels are given and cannot be optimized for the optimal feedback control purpose.

2. **CONTROLLABLE QUANTUM DYNAMICAL SYSTEMS WITH OBSERVATION**

Here we introduce the mathematical concept of controllable quantum system with observation channel on the basis of the operational theory on open-loop physical systems and quantum processes \(^2[2][10][13]\). Such systems are open by the definition, and the necessary concepts borrowed from the algebraic theory of open quantum systems are described in the Appendix.

Let \(\mathcal{H}\) be the Hilbert space of representation of a certain quantum-mechanical system regarded as an observable and controllable system and let \(\mathfrak{A}\) be the von Neumann algebra of admissible physical quantities \(Q \in \mathfrak{A}\) which is generated (see Appendix 1) by the dynamical variables of this system, acting as operators in \(\mathcal{H}\). The pair \(\{\mathcal{H}, \mathfrak{A}\}\) plays the role of a measurable space \(\{X, \mathcal{A}\}\) representing \(^1[14]\) the corresponding classical dynamical system in the phase space \(X\) of it’s point states, endowed with the Borel \(\sigma\)-algebra \(\mathcal{A}\) of admissible events \(A \in \mathcal{A}\). The simple systems normally treated in traditional texts on quantum mechanics, for example \(^1[17]\), correspond to the algebras \(\mathfrak{A} = \mathcal{B}(\mathcal{H})\) of all bounded operators in \(\mathcal{H}\), but the models that emerge from quantum field theory and statistical mechanics \(^1[12]\) are described by the more general algebras \(\mathfrak{A}\).

Normal states of the quantum-mechanical system at every time \(t \in \mathbb{R}\) are determined by the linear functionals \(\varrho_t : Q \mapsto \langle \rho_t, Q \rangle\) of the quantum-mechanical expectations \(\langle Q \rangle_t = \langle \rho_t, Q \rangle\) of all the physical quantities \(Q \in \mathfrak{A}\) for this system, and are described by the densities \(\rho_t\) as the positive elements associated with von Neumann algebra \(\mathfrak{A}\) (see Appendix 2). In the case of semifinite algebras \(^1[11]\), as in the simple case \(\mathfrak{A} = \mathcal{B}(\mathcal{H})\), the states \(\varrho_t\) are usually represented by the trace one operators \(\rho_t\) in \(\mathfrak{A}\) (or affiliated with \(\mathfrak{A}\)) as

\[
\varrho_t (Q) = \text{tr} \{\rho_t Q\} = \langle \rho_t, Q \rangle.
\]

The Master evolution \(t \mapsto \varrho_t\) of a quantum-mechanical system controlled on each time interval \([t, t+\tau]\) by a segment \(u^\tau_t = \{u(r) : r \in [t, t+\tau]\}\) of certain parameters \(u(t)\) is usually described by linear unital completely positive transformations of the
3. The Markov family

\[(2.3) \ M^\tau (u^\tau) = M^\tau_0 (u^\tau_0) \quad \forall \tau > 0.\]

They are determined by the composition of the functionals \( \rho_t : \mathfrak{A} \mapsto \mathbb{C} \) with controlled transfer operators \( M^\tau_t (u^\tau_t) \) as the maps \( \mathfrak{A} \mapsto \mathfrak{A} \) defined in the Appendix 3. The Markov family \( \{M^\tau_t\}_{t \in \mathbb{R}, \tau > 0} \) of these maps must satisfy the consistency condition

\[(2.3) \ M^\tau_0 (u^\tau_0) M^\tau_{r+r'} (u^\tau_{r+r'}) = M^\tau_{r+r'} (u^\tau_{r+r'}) \quad \forall \tau, r, r' > 0,\]

as quantum analog of the controlled Chapman-Kolmogorov equation. Here \( u^\tau = \{u(r + \tau') \}_{r < \tau} \) is a segment of an admissible control function \( u(t) \in U(t) \) of length \( \tau > 0 \) and \( u^\tau + r' \) denotes the composition \( \{u^\tau_r, u^\tau_{r+r'}\} \) of segments \( u^\tau_r \) and \( u^\tau_{r+r'} \). Note that for stationary (time shift-invariant) systems, where \( U(t) = U \), \( (2.3) \) specifies the cocycle conditions for transfer operators \( M^\tau (u^\tau) = M^\tau_0 (u^\tau_0) \) independent of \( r \) with respect to the shift \( u^\tau \mapsto u^{\tau+r} \) given by \( u_r(t) = u(t + r) \).

In the spatial case of transfer operators \( M^\tau_t (u^\tau_t) \) specified by controllable propagators \( T(u^\tau_t) : \mathcal{H} \mapsto \mathcal{H} \) satisfying the condition corresponding to \( (2.2) \),

\[T^r_{r+r'} \left( u^r_{r+r'} \right) T^r_t (u^r_t) = T^r_{r+r'} \left( u^r_{r+r'} \right),\]

the dynamics \( \rho_t \mapsto \rho_{t+r} (u^\tau_t) \) for the vector states \( \langle \rho_t, Q \rangle = \langle \psi_t | Q | \psi_t \rangle \) is described by the state-vector transformations

\[\psi_t \in \mathcal{H} \mapsto T^r_{t} (u^\tau_t) \psi_t \equiv \psi_{t+r} (u^\tau_t) \in \mathcal{H} \quad \forall t \in \mathbb{R}, \tau > 0.\]

Such transformations can be obtained, for example, as the fundamental solutions of the time-dependant Schrödinger equation with a perturbing force \( u(t) \), for which the isometric operators \( T^r_t (u^\tau_t) \) are unitary, and \( M^\tau_t (u^\tau_t) \) are the Heisenberg transformations

\[M^\tau_t (u^\tau_t) Q = T^r_t (u^\tau_t)^\dagger Q T^r_t (u^\tau_t).\]

The open-loop input-output quantum dynamical systems of the kind specified below as controllable systems with observation cannot, as a rule, be described in terms of propagators \( T^r_t (u^\tau_t) \), because the measurements induce reductions and decoherence of quantum states which are described by more general transformations. In order to define such systems as quantum dynamical objects with the classical inputs \( u(t) \) and outputs \( v(t) \in V(t) \), we shall fix a pair of two-parameter families \( \{U^r_t\}, \{V^r_t\}, t \in \mathbb{R}, \tau > 0 \) of the Cartesian product subsets \( U^r_t \subseteq \prod_{r \leq \tau} U(t + r') \)

\[V^r_t = \prod_{r \leq \tau} V(t + r'),\]

satisfying the consistency condition

\[(2.4) \ U^r_t \times U^r_{t+r} = U^r_{t+r}, \quad V^r_t \times V^r_{t+r} = V^r_{t+r}.\]

The elements \( u^\tau_t \in U^r_t, v^\tau_t \in V^r_t \) are called the admissible segments of signals, and their Cartesian compositions should also be admissible in order to make sense, for example, of \( (2.4) \). The sets \( U^r_t \) are usually endowed with Hausdorff product topologies, and the sets \( V^r_t \) with Borel product \( \sigma \)-algebras, making them consistent with \( (2.4) \). Note that the continuous signals \( u(t) \) and \( v(t) \) on \( t \in \mathbb{R} \) do not satisfy the conditions \( (2.4) \). In the time shift-invariant case \( U^r_t \) and \( V^r_t \) are given by the shift \( t \mapsto t + \tau \) of the initial sets \( U^\tau = U^\tau_0 \) and \( V^\tau = V^\tau_0 \).
Definition 1. A controllable quantum dynamical system with observation is described by a family \( \{ \Pi_t^r \}_{r \in \mathbb{R}, \tau > 0} \) of controllable transfer-operator measures \( \Pi_t^r (u_t^r, dv_t^r) : \mathfrak{A} \to \mathfrak{A} \) (see Appendix 4) defined on the spaces \( U_t^\tau \supseteq u_t^\tau, V_t^\tau \supseteq dv_t^\tau \) and satisfying the condition

\[
(2.5) \quad \Pi_t^r (u_t^r, dv_t^r) \Pi_t^r (u_t^{r+\tau}, dv_t^{r+\tau}) = \Pi_t^{r+\tau} (u_t^{r+\tau}, dv_t^{r+\tau})
\]

for any \( r \in \mathbb{R}, \tau, r' > 0 \), where \( u_t^{r+\tau} = (u_t^r, u_t^{r+\tau}) \), \( dv_t^{r+\tau} = dv_t^r \times dv_t^{r+\tau} \).

The superoperators \( \Pi_t^r (u_t^r, dv_t^r) \) are assumed to be continuous in \( u_t^r \), \( \sigma \)-additive with respect to \( dv_t^r \) (in the strong operator sense) and in the time shift-invariant case not to depend explicitly on \( t \) [the index \( t \) in the condition \( (2.5) \) specifying the cocycle dependence on \( r \) can now be omitted].

Due to positivity \( Q \in \mathfrak{A}_+ \mapsto \Pi_t^r (u_t^r, dv_t^r) Q \in \mathfrak{A}_+ \) and the normalization condition \( \Pi_t^0 (u_t^0, v_t^0) = \Pi_t^r (u_t^0, v_t^0) \), the mappings \( \Pi_t^r \) determine, for a given instantaneous state \( \rho_t \in \mathfrak{A}_+ \) and control function \( u_t^r \), the future \( (\tau > 0) \) states

\[
(2.6) \quad \rho_{t+\tau} (u_t^r, dv_t^r) = \rho_t \Pi_t^\tau (u_t^r, dv_t^r)
\]

of the quantum-mechanical process, normalized to the probabilities

\[
(2.7) \quad \pi_t^r (u_t^r, dv_t^r) = (\rho_t, \Pi_t^r (u_t^r, dv_t^r) I) \equiv \rho \Pi_t^r (u_t^r, dv_t^r) I
\]

of the events \( v_t^r \in dv_t^r \). The ratio of the state \( \rho_{t+\tau} \) to the initial state \( \rho_t \) determines conditional states, depending in general non-linearly on \( \rho = \rho_t \):

\[
(2.8) \quad \rho_t^\tau = \frac{\rho \Pi_t^\tau (u_t^r, dv_t^r)}{\rho_t \Pi_t^r (u_t^r, dv_t^r) I}.
\]

Note that the normalized conditional states are well-defined only for the measurable events \( dv_t^r \subseteq V_t^r \) of non-zero probability \( (2.7) \), with which the system transfers from the state \( \rho_t = \rho \) as a result of the control action \( u_t^r \) and the observation \( dv_t^r \) on an interval of length \( \tau \).

If the event \(dv_t^r = V_t^r \) is certain, \( \pi_t^r (u_t^r, dv_t^r) = 1 \), the states \( (2.8) \) are unconditional such that \( \rho_{t+\tau} \) coincide with \textit{a priori} states \( \rho_{t+\tau} (u_t^r) \), whose controlled evolution is linear. Such evolution is described by expression \( (2.2) \), in which \( M_t^r (u_t^r) = \Pi_t^r (u_t^r, V_t^r) \) denotes controllable transforms of the open-loop quantum-mechanical system, corresponding to the absence of observation. However in general the process of precise measurement of the output signal \( v_t^r \) on an interval of length \( \tau > 0 \) takes the quantum system from \textit{a priori} state \( \rho = \rho_t \) to the \textit{a posteriori} state \( \rho_{t+\tau} = \rho M_t^r (u_t^r, v_t^r) \), where the quasi-linear mapping \( \rho \mapsto \rho M_t^r (u_t^r, v_t^r) \) is given by expression \( (2.3) \), in the limit \( dv_t^r \downarrow \{ v_t^r \} \) almost everywhere with respect to the measure \( (2.7) \). For example, let the measures \( \Pi_t^r \) have the density functions

\[
(2.9) \quad \Pi_t^r (u_t^r, dv_t^r) = \int_{dv_t^r} P_t^r (u_t^r, v_t^r) \mu_t^r (u_t^r, dv_t^r),
\]

where \( P_t^r (u_t^r, v_t^r) : \mathfrak{A} \to \mathfrak{A} \) denotes completely positive superoperators [see the Appendix, (A.4)], say, of the form \( (3.10) \), continuous with respect to \( u_t^r \) and integrable with respect to \( v_t^r \) in the strong operator sense with respect to specified positive
measures $\mu^v_t$ on $V^v_t$. Then the \textit{a posteriori} transfer operators $M^v_{\rho^v_t}(u^v_t, v^v_t)$ coincide, up to normalization, with $P^v_t(u^v_t, v^v_t)$:

\begin{equation}
M^v_{\rho^v_t}(u^v_t, v^v_t) = \frac{P^v_t(u^v_t, v^v_t)}{\langle \rho, P^v_t(u^v_t, v^v_t) I \rangle},
\end{equation}

where the ratio is defined for those $u^v_t \in U^v_t$, $v^v_t \in V^v_t$ for which the densities

\begin{equation}
p^v_t(u^v_t, v^v_t) = \langle \rho, P^v_t(u^v_t, v^v_t) I \rangle \equiv \rho P^v_t(u^v_t, v^v_t) I
\end{equation}

of the probability measure (2.10) with respect to $\mu^v_t$ are non-vanishing.

The following theorem states that the \textit{a posteriori} mapping (2.10) in fact determines the state-valued classical Markov process, which was introduced in the classical case by Stratonovich in [20] (He called this probability measure-valued process secondary, or conditional (\textit{a posteriori}) Markov process).

\textbf{Theorem 1.} The family\{M^v_{\rho^v_t}\} of a posteriori\ transfer operators\ M^v_{\rho^v_t}(u^v_t, v^v_t) satisfies, with respect to the operator composition, the consistency condition

\begin{equation}
M^v_{\rho^v_t}(u^v_t, v^v_t) M^v_{\rho^v_t}(u^v_t, v^v_t) = M^v_{\rho^v_t}(u^{\tau+t}, v^{\tau+t})
\end{equation}

almost everywhere under the measure (2.7), where $\rho'$ is the density of $\varrho' = \rho M^v_{\rho^v_t}(u^v_t, v^v_t)$.

\textbf{Proof.} It is required to verify the property (2.12) for conditional mappings (2.8), for which it follows at once from the definition and (2.5). Then it applies also in the single point limit $dv^v_t \downarrow \{v^v_t\}$. In the case (2.9) the condition (2.12) is simply verified by computing the product (2.12) of the \textit{a posteriori} transfer operators (2.10); for this purpose it is necessary to invoke the corresponding condition

\begin{equation}
P^v_t(u^v_t, v^v_t) P^v_{t+\tau}(u^{\tau+t}, v^{\tau+t}) = P^v_{t+\tau}(u^{\tau+t}, v^{\tau+t}).
\end{equation}

It is sufficient to require this composition condition for $V^{\tau+t}$ almost everywhere (mod $\mu^v_t$) and this will guarantees the satisfaction of condition (2.2) if

\begin{equation}
\mu^v_t(u^v_t, dv^v_t) \mu^v_{t+\tau}(u^{\tau+t}, dv^{\tau+t}) = \mu(u^{\tau+t}, dv^{\tau+t}).
\end{equation}

\square

\textbf{Remark 1.} If the superoperator densities $P^v_t$ of the transition measures (2.5) preserve unity: $P^v_t(u^v_t, v^v_t) I = p^v_t(u^v_t, v^v_t) I$, the ratio (2.10) determines $\rho$-independent transfer operators $M^v_t(u^v_t, v^v_t)$ describing the controllable quantum dynamics of a system with two inputs $u$ and $v$. The second is an observable stochastic process with probability measures $\pi^v_t(u^v_t, v^v_t) = p^v_t(u^v_t, v^v_t) \mu^v_t(u^v_t, v^v_t)$ independent of the state of the system. The \textit{a posteriori} mappings (2.8) in this case are linear, $\varrho^v_t = \rho M^v_t(u^v_t, v^v_t)$, almost everywhere under the measure $\pi^v_t$.

3. \textbf{Sufficient Coordinates of Quantum-Mechanical Systems}

The description of the dynamics of simple closed-loop quantum-mechanical systems for a certain class of initial states is known to be often reducible to the determination of the time evolution of certain coordinates, the role of which can be taken, for example, by vectors $\psi \in \mathcal{H}$, if only vector initial states are considered. Some aspects of the controllability of closed-loop quantum-mechanical systems described by a sufficient coordinate $\psi_t \in \mathcal{H}$, satisfying the controlled Schrödinger equation have been recently investigated in [8].
The concept of sufficient coordinates, which is introduced below for general controllable quantum dynamical systems with observation and is intimately related to the classical notion of sufficient statistics [20], plays an even greater role for quantum control theory than the analogous concept in stochastic control theory, because it permits control problems for quantum-mechanical systems to be reduced to classical control problems with localized or distributed parameters.

**Definition 2.** Let $X$ be a measurable space\(^1\), and let $\{\varrho_{x,t}\}_{x \in X, t \in \mathbb{R}}$ be a family of states given, for every $t \in \mathbb{R}$, by a measurable mapping $x \mapsto \varrho_{x,t}$ of the space $X$ into the space of states $\mathcal{H}$ of a quantum-mechanical system at time $t$ such that the controlled evolution (2.5) of the system during an observation leaves this family up to a normalization $\pi_{x,t}$ invariant:

$$\varrho_{x,t} \Pi_t^\tau (u^\tau_t, dv^\tau_t) = \pi_{x,t} (u^\tau_t, dv^\tau_t) \varrho_{f^\tau_{x,t} (u^\tau_t, v^\tau_t), t+t}$$

Then $x \in X$ is called a sufficient coordinate for $\{\varrho_{x,t}\}$, the controlled stochastic evolution of which $x \in X \mapsto f^\tau_{x,t} (u^\tau_t, v^\tau_t) \equiv x^\tau_t \in X$ is described by the mappings $f^\tau_{x,t} : U^\tau_t \times V^\tau_t \rightarrow X$, continuous with respect to $u^\tau_t \in U^\tau_t$ and measurable with respect to $v^\tau_t \in V^\tau_t$ almost everywhere under the measure

$$\pi_{x,t} (u^\tau_t, dv^\tau_t) = \langle \rho_x, \Pi_t^\tau (u^\tau_t, dv^\tau_t) \rangle.$$

Proceeding from (3.1) taken in the limit $dv^\tau_t \searrow \{v^\tau_t\}$, we note that the density operators $x = \rho$ form the sufficient coordinate space for the family of all normal states $\varrho$ on $\mathfrak{A}$. It is given by the a posteriori mapping $f^\tau_{p,t} (u^\tau_t, v^\tau_t) = M^\tau_{p,t} (u^\tau_t, v^\tau_t) \rho$, provided only [as in the case (2.3)] that there exists the derivative

$$M^\tau_{p,t} (u^\tau_t, v^\tau_t) = \frac{\Pi_t^\tau (u^\tau_t, dv^\tau_t)}{\pi_{p,t} (u^\tau_t, dv^\tau_t)}$$

as the limit $dv^\tau_t \searrow \{v^\tau_t\}$.

**Theorem 2.** The mappings $f^\tau_{x,t}$ in (3.1) define a sufficient statistics $x^\tau_t = f^\tau_{x,t} (u^\tau_t, v^\tau_t)$ for $\varrho_t \in \{\varrho_{x,t}\}_{x \in X}$ in the sense that the a posteriori states $\varrho^\tau_t = \rho M^\tau_{x,t} (u^\tau_t, v^\tau_t)$ are determined for such $\varrho = \varrho_{x,t}$ as $\varrho^\tau_t = \varrho_{x^\tau_t, t+\tau}$ for $\varrho > 0$. Moreover, the transition probabilities

$$\pi_{x,t} (u^\tau_t, dx') = \langle \rho_x, \Pi_{x,t}^\tau (u^\tau_t, dx') \rangle,$$

from $x_t = x$ into $dx' \ni x^\tau_t$, defined by

$$\Pi_{x,t}^\tau (u^\tau_t, dx') = \Pi_t^\tau (u^\tau_t, f^{-1}_{x,t} (u^\tau_t, dx'))$$

satisfy the Chapman-Kolmogorov equation

$$f^{-1}_{x,t} (u^\tau_t, dx') = \{ v^\tau_t : f^\tau_{x,t} (u^\tau_t, v^\tau_t) \in dx' \}$$

for all $r \in \mathbb{R}$, $\tau, \tau' > 0$ and $u^\tau_r, u^\tau_{r+\tau}$, so that the sufficient statistics form a controllable Markov process.

\(^1\)For all practical purposes it is always sufficient to assume that $X$ is a standard Borel space, i.e. a complete separable metric space, also known as a Polish space (for example, $\mathbb{R}^n$, $C^0$, or any countable set).
This completes the proof.

\[ f \] \text{evolution} \
\[ \hat{\mu} \] 

The normalized vectors \[ \{\varrho_{x,t}\} \text{ in (3.3) and (3.4). For the states in the class } \{\varrho_{x,t}\}, \text{ is equivalent to the Chapman-Kolmogorov equation (3.3), as} \]
\[ \varrho_{x,t} \] \text{in accordance with (3.1). Thus equation (3.3) determines a Markov stochastic evolution } \hat{x}(t) \text{ of the sufficient coordinates } x(t) \in X, \text{ which is described, according to the main Kolmogorov theorem (see, e.g. [10], p. 48 for a standard space } X, \text{ a Markov probability measure in the functional Borel space of the trajectories } (x(t)). \]

This completes the proof. \( \square \)

We now discuss in more detail the special case, in which the transition measures \( \varrho_{x,t} \) have the superoperator densities
\[ \mathbf{P}_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t) = F_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t) \]
with respect to a given consistent family of measures \( \mu^\tau_t(u^\tau_t, dv^\tau_t) \). Here the operators \( F_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t) \) are assumed to satisfy the normalization condition
\[ \int F_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t) = I, \]
in the Hilbert space \( \mathcal{H} \), as well as the composition condition
\[ F_{\tau+\tau'}^{\hat{x}}(u^\tau_{\tau+\tau'}, v^\tau_{\tau+\tau'}) = F_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t) F_{\tau'}^{\hat{x}}(u^\tau_{\tau'}, v^\tau_{\tau'}), \]
which guarantees the fulfillment of (2.13).

One can easily see that the \textit{a posteriori} transfer operators (2.10) preserve the vectorial property of the states \( \varrho_\psi(Q) = \langle \psi | Q | \psi \rangle \) such that
\[ \varrho_{\psi,\tau}(Q) = | \psi \rangle (Q | Q \rangle \varrho_\psi(Q), \varrho_\psi \]
where \( \psi = T_{\psi,\tau}(u^\tau_t, v^\tau_t) \psi \) for any unit vector \( \psi \in \mathcal{H} \), \( \tau > 0 \) and
\[ T_{\psi,\tau}(u^\tau_t, v^\tau_t) = \frac{F_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t)}{\| F_{\tau}^{\hat{x}}(u^\tau_t, v^\tau_t) \|} \psi. \]

\textbf{Corollary 1.} The normalized vectors \( \psi \in \mathcal{H}, \| \psi \| = 1 \) in the case (3.10) form the space \( X \) of sufficient coordinates \( x = \psi \) specified by the \textit{a posteriori} mappings \( f_{\psi,\tau}(u^\tau_t, v^\tau_t) = T_{\psi,\tau}(u^\tau_t, v^\tau_t) \psi \) of the quasilinear form (3.13) for the family of all vectorial states \( \varrho_\psi \).
We note that the \textit{a priori} propagators $T_{\psi,\tau}^\ast (u_i^t, v_i^t)$ satisfy the semigroup property \cite{24, 31}:

\begin{equation}
T_{\psi,\tau+\tau}^\ast (u_i^t, v_i^t) T_{\psi,\tau}^\ast (u_i^t, v_i^t) = T_{\psi,\tau+\tau}^\ast (u_i^{t+\tau}, v_i^{t+\tau}),
\end{equation}

where $\psi = T_{\psi,\tau}^\ast (u_i^t, v_i^t) \psi$. They are nonlinear (quasilinear), and in contrast with the linear operators $F_i^\ast (u_i^t, v_i^t)$ they preserve the norm in $\mathcal{H}$. Only in the case discussed at the end of Section \ref{sec:3} where $F_i^\ast (u_i^t, v_i^t)$ $F_i^\ast (u_i^t, v_i^t) = p_i^t (u_i^t, v_i^t) I$, the operators \cite{34} are $\psi$-independent isometries $T_i^\ast (u_i^t, v_i^t) = F_i^\ast (u_i^t, v_i^t) / \sqrt{p_i^t (u_i^t, v_i^t)}$.

We note, however, that the \textit{a priori} transfer operators

\begin{equation}
M_i^\ast (u_i^t) Q = \int F_i^\ast (u_i^t, v_i^t) Q F_i^\ast (u_i^t, \hat{v}_i^t) \mu_i^\ast (u_i^t, \hat{v}_i^t)
\end{equation}

determining the controllable Markov dynamics of the quantum system \cite{34, 31} in the absence of observations, are not described by the propagators $T_i^\ast (u_i^t)$, with the exception of the degenerate case in which the \textit{a posteriori} states coincide \textit{a priori} with \textit{a priori} states, i.e., actually do not depend on the results of the observations $v_i^t$.

\section{Optimal Quantum Feedback Control}

Let us now discuss the optimal control of a quantum dynamical system with observation $\{E_i^t\}$. We assume that the performance of the system is measured at each time $t$ by the mathematical expectation $\langle p_i, Q_t (u_t, dv_t) \rangle$ of a certain physical quantity $Q_t (u_t, dv_t) \in \mathbb{A}$ which continuously depends in strong operator topology on the input state $u_t = \{u (t + \tau)\}_{\tau \geq 0}$ and on the output event $dv_t = d \{v (t + \tau)\}_{\tau > 0}$.

We define

\begin{equation}
Q_t (u_t, dv_t) = \Pi_i^\ast (u_i^t, dv_i^t) Q_{t+\tau} (u_{t+\tau}, dv_{t+\tau}) + S_i^\ast (u_i^t, dv_i^t).
\end{equation}

Here $S_i^\ast (u_i^t, dv_i^t) \in \mathbb{A}$ are Hermitian operators having the integral form

\begin{equation}
S_i^\ast (u_i^t, dv_i^t) = \int_0^\tau \Pi_i^\ast \left( u_i^{t+\tau}, dv_i^{t+\tau} \right) S (u (t + \tau'), t + \tau') d\tau'
\end{equation}

for a Hermitian operator-function $S (u, t) = S (u, t)^\dagger$ completely determining $Q_t$ for a certain boundary condition $Q_T (u_T, dv_T) = Q$ at the final time $T > t$. The conditions for the existence of the integral $Q_t$ its continuous dependence on $u_i^t$, and its $\sigma$-additivity with respect to $dv_i^t$, requiring the continuity in $u \in U$ and measurability in $t \in \mathbb{R}$ for the operator function $(u, t) \mapsto S (t, u) \in \mathbb{A}$ under strong operator topology, are presumed to be fulfilled. The operator $Q$, specifying the terminal risk $\langle p_T, Q \rangle$, is assumed to be Hermitian-positive.

\begin{definition}
A measurable mapping $v_t \mapsto u_t (v_t) \in U_t$ is called a \textit{non-anticipating} control strategy if its components $u (t + \tau, \cdot) : v_t \mapsto u (t + \tau)$ are determined by functions independent of $v_{t+\tau}$. It is called a \textit{retarded} control strategy if all $u (t + \tau, \cdot)$ are determined by functions $v_t' \mapsto u (t + \tau, v_t')$ for some measurable $\tau' = \tau' (t + \tau) < \tau$. A \textit{non-anticipating} strategy $u_t (\cdot)$ is called \textit{admissible} if the integral

\begin{equation}
Q_t [u (\cdot)] = \int Q_t (u_t (v_t), dv_t)
\end{equation}

is a priori measurable.
exists in strong operator topology, and it is called optimal for an initial state \( \varrho_t = \varrho \) if it realizes the extremum

\[
q(\rho, t) = \inf_{u_t(\cdot) \in U_t(\cdot)} \langle \rho, Q_t [u_t(\cdot)] \rangle,
\]

where \( U_t(\cdot) \) is a certain set of admissible strategies \( u_t(\cdot) \) [\( \varepsilon \)-optimal if \( \langle \rho, Q_t [u_t(\cdot)] \rangle \) exceeds \( \text{(4.3)} \) at most by \( \varepsilon \)].

We note that in accordance with \( \text{(4.1)} \), a strategy \( u_t(\cdot) \) is admissible with respect to \( Q_t(\cdot, \cdot) \) if and only if its segments \( u_{t+\tau}(\cdot) \) are admissible strategies with respect to \( Q_{t+\tau}(\cdot, \cdot) \) for each fixed \( v_t(\cdot) \), and if there exists measure

\[
\Pi_t^{u,t}(dv_t) = \int_{dv_t} \Pi_t^{u,t}(v_t) dv_t,
\]

specifying the operator-valued integral

\[
S_t^{u_t}(\cdot) = \int_0^\tau \Pi_t^{u,t}(dv_t) S \left( t + \tau', u \left( t + \tau', v_t(\cdot) \right) \right)
\]

for each strategy segment \( u_t(\cdot) \). The latter holds for any delayed strategy that is admissible for a given boundary condition \( Q_T(\cdot, \cdot) = Q \).

**Theorem 3.** Let the sets \( U_t^T(\cdot) \) of admissible strategy segments satisfy the condition

\[
U_t^T(\cdot) \times U_{t+\tau'}(\cdot) \subseteq U_{t+\tau'}(\cdot) \quad \forall t \in \mathbb{R}, \quad \tau, \tau' > 0.
\]

Then the minimal risk \( \text{(4.3)} \) as a function of the density operator \( \rho \) and the time \( t \) satisfies the functional equation

\[
q(\rho, t) = \inf_{u_t(\cdot) \in U_t(\cdot)} \left[ \langle \rho, S_t^{u_t}(\cdot) \rangle + \int \pi_t^{u,t}(dv_t) q(\rho(v_t), t + \tau) \right],
\]

where \( \pi_t^{u,t}(\cdot) = \langle \rho, \Pi_t^{u,t}(\cdot) I, \rho(v_t) \rangle = M_t^{u_t}(v_t, v_t) \rho \) denotes the probability measures \( \text{(2.7)} \) and a posteriori states \( \text{(2.8)} \) corresponding to an admissible strategy \( u = u_t(\cdot) \) and an initial state \( \varrho = \varrho_t \).

**Proof.** The proof of \( \text{(4.7)} \) generalizes the proof of the Bellman equation \( \text{(4.7)} \). By substitution of \( \text{(4.1)} \) into \( \text{(4.8)} \), it reduces the minimization over \( u_t(\cdot) \) by the successive minimization of \( \text{(4.1)} \), first on \( u_{t+\tau}(\cdot) \) and then on \( u_t(\cdot) \), which by condition \( \text{(4.6)} \) yields the same result as \( \text{(4.3)} \). Since the integral \( \text{(4.5)} \) does not depend on \( u_{t+\tau}(\cdot) \) and by definition,

\[
\varrho \Pi_t^{u,t}(dv_t) = \pi_t^{u,t}(dv_t) \varrho_{\rho,t},
\]

the first minimization entails finding the second term of the minimized sum \( \text{(4.7)} \):

\[
\inf_{u_{t+\tau}(\cdot) \in U_{t+\tau}(\cdot)} \int \pi_t^{u,t}(dv_t) \left[ \rho(v_t), \int Q_t(u_t(v_t), dv_t) \right] = \int \pi_t^{u,t}(dv_t) q(\rho(v_t), t + \tau).
\]

In the case of a given boundary condition \( q(\rho, t) = \langle \rho, Q \rangle \) the theorem proved above provides a constructive method of synthesizing an optimal or \( \varepsilon \)-optimal strategy \( u_{\rho,t}^{T-t}(v_t^{T-t}) \) by the successive minimization of \( \text{(4.7)} \) in reverse time. In this
case it is sufficient to restrict the discussion to Markov admissible strategies described by segments $u_{r,t}^\tau (v_t^\gamma)$, $\tau = T - t$, depending on the a priori history $v_t^\gamma$ only through the agency of their dependence on the a posteriori state $p = g_t^{t'-t}$ for any $t' > t$. Accordingly, the determination of the a posteriori quantum states $g_t^\tau$, which generate the a posteriori Markov process, enables us to reduce the optimal quantum control problem to the classical problem of stochastic control theory with usual transition probabilities and final risk functions
determined by the operators of the corresponding quantum variables $S(t,u)$ and $Q$.

Let us consider the case in which the quantum states $p$ are considered in a certain class $\{q_{x,t}\}$ for which sufficient coordinates exist.

**Corollary 2.** Let $f_{x,t}^\tau : U_t^\times V_t^\tau \rightarrow X$ denote mappings satisfying the conditions of Theorem 2. Then in problem (4.3) for $p \in \{q_{x,t}\}$ it is sufficient to restrict the discussion to Markov strategies described by measurable mappings $u_t^\tau : X \times V_t^\tau \rightarrow U_t^\tau$ satisfying the consistency condition

\[
\left( u_{x,r}^\tau (v_t^\gamma), u_{r',r+t}^\tau (v_{r+t}^\gamma) \right) = u_{x,r}^{\tau+t'} (v_{r+t}^{\tau+t'},\tau),
\]

where $x' = f_{x,t}^{u_t^\tau} (v_t^\gamma) = f_{x,t}^\tau (u_t^\tau (v_t^\gamma), v_t^\gamma)$. In particular, the instantaneous control functions $u_x (\tau)$ for any $\tau \in [t,T]$ are determined by functions $u(\tau,x)$ of the point state $x$ in accordance with the equation

\[
u_t^\tau (x,t,u,v_t^\gamma) = u_t^\tau (t + \tau, f_{x,t}^{u_t^\tau} (v_t^\gamma)).
\]

The foregoing assertion, which follows directly for the “maximum” sufficient coordinate $x(t) = \rho(t)$ from the optimality equation (4.4), is readily proved on the basis of the properties formulated for sufficient coordinates in Theorem 2.

The further simplification of problem (4.3) entails utilizing the specific properties of the Markov process $x(t) = f_{x,0}^{u_t^\tau} (v_0^\gamma)$, the role of which is logically assigned to sufficient coordinates of the fewest possible dimensions.

For example, in the case where the generator for the transition probabilities $\pi_{x,t}^\tau (dx')$, defined as the $t$-continuous strong limit

\[
L(t) q_t (x) = \lim_{\tau \rightarrow 0} \int_X \pi_{x,t}^\tau (u_t^\tau, dx') (q(x', t + \tau) - q(x, t + \tau)),
\]

exists on some set $D(X)$ of bounded and measurable functions $x \mapsto q(x,t)$, depending continuously on $t$, the optimality equation (4.3) is written in the infinitesimal form

\[
-\frac{\partial}{\partial t} q_t(x,t) = \inf_{u \in U} \{ s(x,t,u) + L(t) q_t(x) \},
\]

where $s(x,t,u) = \langle \rho_{x,t}, S(t,u) \rangle$. Equation (4.11), which represents the standard Bellman equation for controlled Markov processes in continuous time, can be used, together with a boundary condition $q(x,T) = \langle \rho_{x,T}, Q \rangle \in D(X)$, to seek optimal or $\varepsilon$-optimal Markov control functions $u(t)$ directly as functions $u(t,x)$ of the instantaneous state $x$. 

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5. Quantum Control with Discrete Observation

As an example here we consider the controlled dynamics of a simple quantum system described between discrete measurement times \( T = \{ t_k \} \) by the Schrödinger equation
\[
ih \frac{\partial}{\partial t} \psi(t) = H(t, u(t)) \psi(t), \quad t \in T.
\]
Here \( H(t, u) \) is the controlled Hamiltonian, i.e., a self-adjoint operator in \( \mathcal{H} \) with a dense domain of definition \( \mathcal{D} \subseteq \mathcal{H} \), written in the usual form
\[
H(t, u) = H_0(t) + \sum_{i=1}^{m} u_i(t) H_i(t),
\]
where \( u_i(t) \in \mathbb{R}; \ H_i(t) \) are simple functions of \( t \). Under the stated assumption there exists a unique consistent family \( \{ T^\tau_i \} \) of unitary propagators \( T^\tau_i(u_i^\tau) \), representing for any \( \psi(t - \tau) = \varphi \in \mathcal{D} \), a solution of (5.1) between adjacent measurement times \( t_k < t_{k+1} \) in the form \( \psi(t) = T^\tau_{t-t_k} (u^\tau_{t-t_k}) \psi, \ t \in [t_k, t_{k+1}] \), where
\[
\lim_{\tau \rightarrow 0^+} \psi(t) = \psi.
\]

Let \( E_{v,k} \) denote Hermitian projectors, which determine orthogonal decompositions \( \mathcal{I} = \sum_{v \in V_k} E_{v,k} \) of the unit operator in \( \mathcal{H} \) and specify measurements at times \( t_k \) of quantum physical quantities described by self-adjoint operators
\[
A_k = \sum_{v \in V_k} v E_{v,k}
\]
with discrete spectra \( V_k \subseteq \mathbb{R} \).

As a result of measurement of the quantity \( A_k \) there occurs a reduction 18 of the quantum state, \( \varrho \rightarrow g \Pi_{v,k}, v \in V_k \), described by the superoperators \( \Pi_{v,k} Q = E_{v,k} Q E_{v,k} \), which determines a priori transfer operators
\[
(5.3) \quad \Pi_k Q = \sum_{v \in V_k} E_{v,k} Q E_{v,k}.
\]
The states \( \varrho_{v,k} = g \Pi_{v,k} \) to which the system transfers instantaneously depending on the result of this measurement \( v \in V_k \) are normalized to the probabilities \( \pi_{v,k} = \langle \rho, E_{v,k} \rangle \) of these transitions, where if \( \varrho = \varrho_{v} \) is a vector state \( \varrho_{v} (Q) = \langle \psi | Q | \psi \rangle \), the states \( \varrho_{v,k} \) are also vectorial, determined by the projections \( \psi_{v,k} = E_{v,k} \psi \). The product \( E_{v,k} T^\tau_i (u^\tau_i) = F_{v,k}^\tau (u^\tau_i) \) for \( \tau = t_k - t \) determines a transformation \( \psi(t) \mapsto E_{v,k} \psi(t_k) \) corresponding to the evolution (5.1) on the interval \([t, t_k]\) with subsequent measurement of the quantity \( A_k \).

We introduce the notation \( F_{v,k} (u_k) = F_{v,t_k}^\tau (u^\tau_k) \), where \( \tau_k = t_{k+1} - t_k \), and we set \( V^\tau_k = \prod_{k \in K^\tau} V_k \), where \( K^\tau = \{ k : t_k \in [t, t+\tau) \} \) is the set of all indices of times in the interval \([t, t+\tau]\) (in the case of an empty set \( K^\tau = \emptyset \) we assume that \( V^\tau_k \) consists of some single point \( \{ w \} \)).

**Proposition 1.** Let the set \( K^{\tau - t} \) be finite for any \( t < s \). Then the chronological product
\[
(5.4) \quad F_{v,t}^{s-t} (u^{s-t}) = T_{t_0}^{s-t} (u^{s-t}_0) F_{v,1,t_1} (u^{t_1-t_1}_1) \cdots F_{v,k,t_k} (u^{t_k-t}_k) F_{v,t}^{t_k-t} (u^{t_k-t}_k),
\]
2In other words, having one-sided limits. For unbounded self-adjoint operators \( H_i(t), i = 0, \ldots, m \), this means that \( H(t, u(t)) \psi \) is a simple function for any \( \psi \in \mathcal{D} \).
where $k = \min K^{s-t}_l$, $l = \max K^{s-t}_l$, and $v = (v_k, \ldots, v_l) = v^{s-t}$, determines controllable quantum dynamical system described by superoperators $\{\Pi^T_v\}$ of the form (2.9), (3.10):

$$\Pi^T_{v,t} (u^T_v) Q = F^T_{v,t} (u^T_v) \dagger Q F^T_{v,t} (u^T_v),$$

under the counting measure $\mu^T_v = 1$ on $V^T_v \ni v$.

The proof is the verification of conditions (5.11) and (3.12), which take the form

$$\sum_{v \in V^T_v} F^r_{v,v} (u^T_v) \dagger F^r_{v,v} (u^T_v) = I \quad \forall u^T_v \in U^T_v,$$

$$F^r_{v',r+\tau} (u^T_{r+\tau}) F^r_{v,r} (u^T_r) = F^r_{v',r+\tau} (u^T_{r+\tau}),$$

where $v' \in V^T_{r+\tau}, v \in V^T_v$. They are easily verified by induction, owing to the finiteness of the product (4.4).

Because of the spatial form (5.3) of the consistent family $\{\Pi^T_{v,t}\}$, on the basis of Corollary 1 we infer that the space $X$ of normalized vectors $\psi \in \mathcal{H}$, $\|\psi\| = 1$ forms a space of sufficient coordinates, the a posteriori evolution $\psi \mapsto T^T_{v,t} (u^T_v, \psi) \psi$ of which is described by the nonlinear propagators (3.13):

$$T^T_{v,t} (u^T_v) / \|T^T_{v,t} (u^T_v) \psi\|,$$

and the a priori evolution by transfer operators of the form (5.10):

$$\Pi^T_v (u^T_v) Q = \sum_{v \in V^T_v} F^T_{v,t} (u^T_v) \dagger Q F^T_{v,t} (u^T_v) .$$

We give special consideration to the case of complete measurements described by the operators $A_k$ with a non-degenerate spectrum.

**Proposition 2.** Let $\{\psi_{v,k}\}_{v \in V_k}$ denote the complete orthonormal systems of eigenvectors of the operators $A_k$, and let $E^T_{v,k}$ be the corresponding one-dimensional projectors onto $\psi_{v,k}$. Then the a posteriori states at the times $\{t_k\}$ are vector states, which are completely determined by the last result of measurement $v_k \in V_k$:

$$\langle \rho^T_k, Q \rangle = \langle Q \psi_{v_k,k}, \psi_{v_k,k} \rangle \quad \forall t \leq t_k, \quad q = \vartheta_k,$$

and the measurement process $\{v_k\}$ is a Markov process, which is described by the controllable transition probabilities

$$\tau_{v,k} (u_k, v_k) = | \langle \psi_{v_k,k+1} | T_k (u_k) \psi_{v_k,k} \rangle |^2,$$

where $T_k (u_k) = T^T_{k \tau} (u^T_k), \tau = t_k+1 - t_k$.

This proposition follows from the property

$$E^T_{v,k} Q E^T_{v,k} = \langle \psi_{v,k} | Q \psi_{v,k} \rangle E^T_{v,k}$$

of the one-dimensional orthogonal projection operators $E^T_{v,k}$ corresponding to the eigenvectors $\psi_{v,k}$, so that the application of any state $\vartheta$ to (5.6) at $t = t_k - \tau$ leads to (5.8), up to normalization. Since the a posteriori state (5.8) does not depend on the previous measurements, the conditional probability given by expression (5.9) for the event $v_{k+1} = v$ and fixed preceding results is Markovian.

In the proposition proved above, the controllable sufficient coordinate $x_k = v_k$ can be used, provided only that the quantum system is analyzed at discrete measurement times $\{t_k\}$.
We now consider the optimal control problem for a discretely observed quantum system. Let the control performance, as a function of the initial state, be described by an operator \( H \), which is determined by the integral \( I_2 \) of some operator-valued function \( S(t, u) : \mathcal{H} \to \mathcal{H} \).

**Proposition 3.** Under the conditions of Theorem 3 for a vector initial state \( \varphi_0 \), the minimal risk

\[
q(\psi, t) = \inf_{u(t) \in U(t)} \int_{v(t)} \langle \psi | Q(t (v_t), dv_t) \psi \rangle
\]

in the intervals \((t_k, t_{k+1})\) between measurements satisfies the functional equation in variational derivatives

\[
-\frac{\partial}{\partial t} q(\psi, t) = \inf_{u(t) \in U(t)} \left( \| \psi \|_{S(t, u)}^2 + 2\hbar^{-1} \text{Im} (\delta q(\psi, t) / \delta \psi | H(t, u) \psi) \right),
\]

where \( \| \psi \|_S^2 = \langle \psi | S \psi \rangle \). At the measurement time instances \( \{t_k\} \) it satisfies the recursive equation

\[
q_k(\psi) = \inf_{u_k \in U_k} \left( \| \psi \|_{S_k(u_k)}^2 + \sum_{v \in V_k+1} \pi_{v,k}^u(\psi) q_{k+1}(\psi(v)) \right),
\]

which determines the boundary values \( q(t_k, 0, \psi) = q_k(\psi) \) for \( k \). Here \( \pi_{v,k}^u(\psi) = \| T_{v,k}(u_k) \psi \|^2 \), \( \psi(v) = T_{v,k}(u_k) \psi / \sqrt{\pi_{v,k}^u(\psi)} \), and

\[
S_k(u_k) = \int_0^{t_{k+1}} T_{t_k}(u_{t_k}) S(t, u(t)) T_{t_k}^\dagger(u_{t_k}) \, dt.
\]

Equation (5.12) is readily proved on the assumption of analyticity of the function \( \psi \mapsto q(\psi, t) \) which is natural for a quadratic boundary condition \( q(\psi, t) = \| \psi \|_Q^2 \) at some final time \( T \). Here (5.12) represents a functional version of the Bellman equation corresponding to the Schrödinger equation (5.1) and a quadratic transition cost function \( S(t, u, \psi) = \| \psi \|_{S(t, u)}^2 \). Equation (5.13) follows directly from (4.12) for \( t = t_k, \tau = t_{k+1} - t_k \) and \( q = q(x) \) if it is taken into account that the integral (4.2) now has the form (5.14).

In conclusion we consider the optimal control problem described above in the complete measurement case. Making use of the fact that the process of complete measurement at discrete times \( \{t_k\} \) induces a Markov sufficient coordinate \( x_k \), we deduce the customary equation

\[
q_k(u_k) = \inf_{u_k \in U_k} \left( s_k(u_k, v_k) + \sum_{v \in V_k+1} \pi_{v,k}(u_k, v_k) q_{k+1}(v) \right),
\]

which describes the optimum risk for the control of a discrete Markov process \( \{v_k\} \) with the transition probabilities \( s_k(u_k, v_k) = \| \psi_{v_k} \|^2_{S_k(u_k)} \), a cost function \( s_k(u_k, v_k) = \| \psi_{v_k} \|^2_{Q} \), and a boundary condition of the form \( q_k(v) = \| \psi_{v_k} \|^2_{Q} \). The solution of derived Bellman equation (5.15) can be easily modelled on a computer by standard dynamic programming methods for the piecewise-constant admissible strategies, for which \( U_k = U(t_k) \subseteq \mathbb{R}^m \).
Appendix A. Notations, Definitions and Facts

(1) Let \( \{Q_i\}_{i=1} \) be a family of self-adjoint operators acting in a complex Hilbert space \( \mathcal{H} \). The von Neumann algebra generated by the family \( \{Q_i\} \) is defined as the minimal weakly closed self-adjoint sub-algebra \( \mathfrak{A} \) of bounded operators in \( \mathcal{H} \) containing the spectral projectors of this operators, along with the unit operator \( I \). It consists of all bounded operators that commute with the commutant \( \{Q_i\}' = \{Q : QQ_i = Q_iQ \ \forall i \in \mathbb{N}\} \), i.e., it is the second commutant \( \mathfrak{A}'' \) of the family \( \{Q_i\} \). The latter can be taken as the definition of the von Neumann algebra generated by the family \( \{Q_i\} \) in the case of unbounded self-adjoint operators \( Q_i \) densely defined on a domain \( D \subseteq \mathcal{H} \). The simplest example of von Neumann algebra is the algebra \( B(\mathcal{H}) \) of all bounded operators acting in \( \mathcal{H} \).

(2) A state on a von Neumann algebra \( \mathfrak{A} \) is defined as a linear ultraweakly continuous functional \( \varrho : \mathfrak{A} \to \mathbb{C} \) (which will be denoted as \( \varrho(Q) = \langle \varrho, Q \rangle \)) satisfying the positivity and normalization conditions

\[
\langle \rho, Q \rangle \geq 0, \quad \forall Q \geq 0, \quad \langle \rho, I \rangle = 1
\]

\[Q \geq 0 \text{ signifies the nonnegative definiteness } \langle \psi | Q | \psi \rangle \geq 0 \ \forall \psi \in \mathcal{H} \text{ called Hermitian positivity of } Q.\]

It is described by the density operators \( \rho \) as the elements of the algebra \( \mathfrak{A} \) with respect to a standard pairing \( \langle \rho, Q \rangle \). The linear span of all states on \( \mathfrak{A} \) is a Banach subspace \( \mathfrak{A}^* \) of the dual space \( \mathfrak{A}^* \), called predual to \( \mathfrak{A} \) as \( \mathfrak{A}^* = \mathfrak{A} \). A state \( \varrho \) is called vector state if \( \langle \rho, Q \rangle = \langle \psi | Q | \psi \rangle \) \( (\varrho = \varrho_{\psi}) \) for some \( \psi \in \mathcal{H} \). Any state is a closed convex hull of vector states \( \varrho_{\psi}, \|\psi\| = 1 \). If on an algebra \( \mathfrak{A} \) there exists a normal semi-finite trace \( Q \mapsto \text{tr} \{Q\} \), then the states on \( \mathfrak{A} \) can be described by the density operators \( \rho \in \mathfrak{A} \) (or affiliated to \( \mathfrak{A} \), if they are unbounded), determining \( \rho \) by means of the bilinear form \( \langle \rho, Q \rangle = \text{tr} \{\rho Q\} \). For the case \( \mathfrak{A} = B(\mathcal{H}) \) the density operator \( \rho \) is any nuclear positive operator with unit trace [11].

(3) Let \( \mathfrak{A}_1, \mathfrak{A}_2 \) be von Neumann algebras in respective Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), and let \( M : \mathfrak{A}_2 \to \mathfrak{A}_1 \) be a linear map that transforms the operators \( Q_2 \in \mathfrak{A}_2 \) into operators \( Q_1 \in \mathfrak{A}_1 \) (superoperator, in the terminology of [13]). We shall call the operator \( M \) a transfer operator if it is ultraweakly continuous, completely positive in the sense

\[
\sum_{i,k=1} \langle \psi_i | M \{Q_i^* Q_k\} | \psi_k \rangle \geq 0, \quad \forall Q_i \in \mathfrak{A}_2, \quad \psi_i \in \mathcal{H}_1,
\]

\((i = 1, \ldots, n < \infty)\), and unity-preserving: \( MI_2 = I_1 \). In this case the composition \( gM \) with a state \( g_1 : \mathfrak{A}_1 \to \mathbb{C} \) is a (normal) state \( \mathfrak{A}_2 \to \mathbb{C} \) described by the predual action of the superoperator \( M \) on \( g_1 \):

\[
\langle g_1, MQ_2 \rangle = \langle M^*_g g_1, Q_2 \rangle, \quad \forall Q_2 \in \mathfrak{A}_2, g_1 \in \mathfrak{A}_1^*.
\]

A transfer operator \( M \) is called spatial if

\[
MQ_2 = T^\dagger Q_2 T \quad \text{or} \quad M^*_g g_1 = T^\dagger g_1 T^\dagger,
\]

where \( T : \mathcal{H}_1 \to \mathcal{H}_2 \) is a linear isometric operator, \( T^\dagger T = I \), called the propagator, and \( T^\dagger \) is the adjoint operator. Every transfer operator on \( \mathfrak{A}_2 = B(\mathcal{H}_2) \) has a decomposition as a closed convex hull of spatial transfer operators.
(4) Let $V$ be a measurable space, and $\mathcal{B}$ its Borel $\sigma$-algebra. A mapping $\Pi : dv \in \mathcal{B} \mapsto \Pi (dv)$ with values $\Pi (dv)$ in ultraweakly continuous, completely positive superoperators $\mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ is called a transfer-operator measure if for any $\rho_1 \in \mathfrak{A}_1$, $Q_2 \in \mathfrak{A}_2$ the numerical function

$$\langle \Pi (dv), \rho_1, Q_2 \rangle = \langle \rho_1, \Pi (dv) Q_2 \rangle$$

of the set $dv \subseteq V$ is a countably additive measure normalized to unity for $Q_2 = I$. In other words, $\Pi (dv)$ is an operator-valued measure that is $\sigma$-additive in the weak (strong) operator sense and for $dv = V$ is equal to some transfer operator $M$. The quantum-state transformations $\rho_1 \mapsto \rho_2$ corresponding to ideal measurements are described by transfer-operator measures of the form

$$\Pi (dv) Q = F (v) \dagger F (v) \mu (dv),$$

where $F (v)$ denotes linear operators $\mathcal{H}_1 \rightarrow \mathcal{H}_2$, the integral under a positive numerical measure $\mu$ on $V$ is interpreted in strong operator topology, and $\int F \dagger (v) F (v) \mu (dv) = I_1$. Every transfer-operator $M : \mathfrak{A}_2 \rightarrow \mathfrak{A}_1$ for $\mathfrak{A}_2 = \mathcal{B} (\mathcal{H})$ can be represented by the integral \((A.4)\) with respect to $dv \subseteq V$ of some ideal measure $\Pi (dv)$.

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