Inexact Proximal-Point Penalty Methods for Non-Convex Optimization with Non-Convex Constraints

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September 2, 2019

Abstract
Non-convex optimization problems arise from various areas in science and engineering. Although many numerical methods and theories have been developed for unconstrained non-convex problems, the parallel development for constrained non-convex problems remains limited. That restricts the practices of mathematical modeling and quantitative decision making in many disciplines. In this paper, an inexact proximal-point penalty method is proposed for constrained optimization problems where both the objective function and the constraint can be non-convex. The proposed method approximately solves a sequence of subproblems, each of which is formed by adding to the original objective function a proximal term and quadratic penalty terms associated to the constraint functions. Under a weak-convexity assumption, each subproblem is made strongly convex and can be solved effectively to a required accuracy by an optimal gradient-type method. The theoretical property of the proposed method is analyzed in two different cases. In the first case, the objective function is non-convex but the constraint functions are assumed to be convex, while in the second case, both the objective function and the constraint are non-convex. For both cases, we give the complexity results in terms of the number of function value and gradient evaluations to produce near-stationary points. Due to the different structures, different definitions of near-stationary points are given for the two cases. The complexity for producing a nearly \(\varepsilon\)-stationary point is \(\tilde{O}(\varepsilon^{-5/2})\) for the first case while it becomes \(\tilde{O}(\varepsilon^{-4})\) for the second case.

1 Introduction

For optimization problems with convex objective functions and with/without convex constraints, many algorithms have been developed and analyzed with guaranteed stability in the literature. However, the parallel development for problems with non-convex objective functions and also with non-convex constraints, especially for theoretically provable algorithms, remains limited. That restricts the practices of mathematical modeling and quantitative decision making in many disciplines. In this paper, we aim to advance the development along the direction of non-convex optimization.

We consider the nonconvex optimization problem with functional constraints:

\[
\min_{x \in \mathbb{R}^d} f_0(x) + g(x) \quad \text{s.t.} \quad f_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \quad c_j(x) = 0, \quad j = 1, \ldots, n, \tag{1}
\]
where \( f_i : \mathbb{R}^d \to \mathbb{R} \) is a continuously differentiable (possibly non-convex) function for each \( i = 0, \ldots, m \), 
\( c_j : \mathbb{R}^d \to \mathbb{R} \) is a continuously differentiable (possibly non-convex) function for each \( j = 1, \ldots, n \), and 
g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\} \) is a proper lower-semicontinuous convex function with a compact domain.

Our main goal is to develop a numerical method for finding an approximate stationary point of (1) under a reasonable notion of stationarity and to analyze its complexity in terms of the number of function value and gradient evaluations to reach a specified accuracy.

1.1 Near-stationarity

For a general non-convex function, finding its global minimizer is intractable [74], and it becomes even more difficult, when there are (non-convex) functional constraints. Therefore, we do not hope to find a global optimizer of (1) within a polynomial time, but instead, we focus on finding its near-stationary point. Roughly speaking, a point

\[
\text{exists such that the } \eta \text{-stationary condition holds.}
\]

where \( \partial g(x^*) \) denotes the subdifferential of \( g \) at \( x^* \). The vectors \( y^* \) and \( \lambda^* \) are called Lagrangian multipliers. Due to the inevitable truncation error, it is hard to compute a solution that satisfies the above conditions exactly. Numerically, it is more reasonable to pursue a near-stationary point. When \( g \) is non-smooth, finding an \( \varepsilon \)-stationary point can still be challenging, even if the problem is convex. For example, consider the constrained one-dimensional convex problem

\[
\text{min } f(x), \quad \text{subject to } \nabla f(x) = 0.
\]

It has a unique optimal solution \( x^* = 0 \). Note \( \nabla g(x^*) = 1 \), if \( x \neq 0 \). A numerical method typically only finds a solution close but not equal to 0, and thus for \( \varepsilon < 1 \), a numerical solution is usually not an \( \varepsilon \)-stationary point of \( \min_x [g(x) + f(x)] \). As a result, when solving an unconstrained problem

\[
\text{min } f_0(x) + g(x),
\]

for a specified small tolerance \( \varepsilon > 0 \), and some \( \eta > 0 \). Here, \( \text{prox}_{\eta g}(x) \) is the proximal mapping of \( \eta g \) at \( x \); see Definition 5 below. Algorithms for finding such a point can be found in [34] [55]. However, due to the existence of sophisticated functional constraints, it is not clear how to extend the stationarity measure in (3) to the problem (1) while still ensure \( \tilde{x} \) can be efficiently computed.

1.2 Our approach and notion of near stationarity

In the recent studies on unconstrained or simply constrained non-convex non-smooth optimization [15] [19] [17] [20] [24] [25] [61] [69] [66], algorithms have been proposed to find a nearly \( \varepsilon \)-stationary point, which is a solution \( \varepsilon \)-close to a feasible solution whose subgradient has a norm of at most \( \varepsilon \). A nearly \( \varepsilon \)-stationary point does not need to have a very small subgradient. Therefore, it does not have the issue mentioned

\[\text{being simply constrained means that the feasible set is simple enough to allow for a closed-form projection mapping.}\]
in the previous example and is easier to compute than an $\varepsilon$-stationary point. We adopt a notion of near stationarity similar to the one in those works. Precise definitions will be given later. Since the methods mentioned above require the exact projection onto the feasible set, they cannot be directly applied to (1) because of the sophisticated functional constraints that prohibit exact projection. This motives us to develop efficient numerical schemes to find near-stationary points of (1) that contains general functional constraints.

We approach this target by considering two cases of (1). One case has a weakly-convex objective (see Definition 4) and convex constraints while, for the other case, the objective and constraints are all weakly convex. For both cases, we propose an inexact proximal-point penalty (iPPP) method. The method solves a sequence of strongly convex unconstrained subproblems that are constructed by combining two classical techniques: the proximal-point method and the quadratic penalty method; see (13) below. The accelerated proximal gradient method by Nesterov [57] will be applied to solve each subproblem.

1.3 Contributions

We make contributions to both algorithm and theory on solving non-convex constrained optimization. They are summarized below.

- We propose a framework of inexact proximal-point penalty method (Algorithm 2) for (1). In each main iteration, our method approximately solves a proximally guided quadratically penalized subproblem; see the update in (13). Under the assumption of weak-convexity and with appropriate proximal parameter, each subproblem is strongly convex and can be solved effectively by the accelerated proximal gradient method.

- For the case where $f_0$ is weakly convex, $f_i$ is convex for $i = 1, \ldots, m$, and each $c_j$ is affine, we define a notion of Type-I near-stationary point. Under this notion of stationarity and assuming the Slater's condition, we show that for a given $\varepsilon > 0$, Algorithm 2 finds a nearly $\varepsilon$-stationary point of (1), within $O(\varepsilon^{-5/2})$ evaluations of the function value, gradient, and proximal mapping if a setting of constant penalty parameters is adopted. The complexity is $O(\varepsilon^{-3})$ to produce a stochastic Type-I $\varepsilon$-stationary point of (1) if the penalty parameters vary with iterations.

- When $f_i$ is weakly convex for each $i = 0, 1, \ldots, m$, and also each $c_j$ is weakly convex, we give another slightly different notion of near-stationary point, called the Type-II near-stationary point, which does not require near-complementarity at an inequality constraint that is not satisfied. Under this notion, we show that the total complexity of Algorithm 2 is $O(\varepsilon^{-4})$ to find a nearly $\varepsilon$-stationary point, by assuming a (near)-feasible initial point.

1.4 Organization of the paper

The rest of the paper is organized as follows. In Section 2, we discuss related work on convex and non-convex constrained optimization. In Section 3, we introduce some definitions, notations, and a subroutine used in the proposed algorithm. Details of the proposed algorithm are described in Section 4. The complexity analysis is conducted in Section 5 for the convex constrained case and in Section 6 for the non-convex constrained case. The numerical results are presented in Section 7 and then Section 8 concludes the paper.

2 Related Work

There has been growing interest in first-order algorithms for non-convex minimization problems with no constraints or simple constraints in both stochastic and deterministic settings. Initially, the research in this direction mainly focuses on problems with smooth objective functions [34, 35, 62, 68, 67, 48, 1, 2, 46, 62]. Recently, algorithms and theories have been developed for non-convex problems with non-smooth (but weakly convex) objective functions [19, 18, 25, 20, 13, 99]. These works tackle the non-smoothness by introducing the Moreau envelope of the objective function. However, for (1) with sophisticated functional constraints, these methods are not applicable.
There is a long history of studies on continuous optimization with functional constraints. The monographs on this area include [7, 6, 59, 72]. The recent works on first-order methods for convex optimization with convex functional constraints include [54, 85, 76, 12, 93, 85, 90, 91, 89, 94, 51, 4, 26] for deterministic constraint functions can be nonlinear and non-convex. When all constraint functions in (1) are affine, a primal-dual Frank-Wolfe method is proposed in [84], and it finds an ε-stationary point with a complexity of $O(\varepsilon^{-3})$ in general and $O(\varepsilon^{-2})$ when there exists a strictly feasible solution. Compared to [84], this paper uses a different notion of ε-stationary point and our constraint functions can be nonlinear and non-convex.

As a classical approach for solving constrained optimization (1), a penalty method finds an approximate solution by solving a sequence of unconstrained subproblems, where the violation of constraints is penalized by the positively weighted penalty terms in the objective function of the subproblems. Unconstrained optimization techniques are then applied to the subproblems along with an updating scheme for the weighting parameters. The computational complexity of penalty methods for convex problems has been well established [47, 56, 75]. For non-convex problems, most existing studies of penalty methods focus on the asymptotic convergence to a stationary point [97, 28, 63, 14, 37, 41, 21, 22, 30, 31, 65, 39, 9, 10]. On the contrary, we directly analyze the total computational complexity of the proposed method.

An exact penalty method has been studied in [11] as an application of a trust region method for a composite non-smooth problem. When applied to the problem (1) with $g \equiv 0$ and $f_i \equiv 0$ for each $i = 1, \ldots, m$, this method applies a non-smooth trust-region method to solve a sequence of unconstrained subproblems in the form of

$$\min_{\mathbf{x}} f_0(\mathbf{x}) + \beta \sum_{j=1}^{n} |c_j(\mathbf{x})|, \quad (4)$$

where $\beta > 0$ is a penalty parameter. At iteration $k$ of the non-smooth trust-region method for solving (4), an updating direction is computed as

$$\mathbf{s}^{(k)} \in \arg \min_{||\mathbf{s}|| \leq \Delta_k} \left\{ f_0(\mathbf{x}^{(k)}) + \nabla f_0(\mathbf{x}^{(k)})^\top \mathbf{s} + \beta \sum_{j=1}^{n} c_j(\mathbf{x}^{(k)}) + \nabla c_j(\mathbf{x}^{(k)})^\top \mathbf{s} \right\}, \quad (5)$$

where $\Delta_k$ is the radius of the trust region. Upon obtaining $\mathbf{s}^{(k)}$, the estimated solution will be updated to $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \mathbf{s}^{(k)}$ if this update significantly reduces the objective value of (4). Once an ε-critical point of (4) (see equation (2.2) in [11] for the definition) is found, a steering procedure [11] is utilized to increase the penalty parameter $\beta$ in (4). It is shown in [11] that this exact penalty method either finds a point $\tilde{x}$ satisfying

$$\|\nabla f_0(\tilde{x}) + \sum_{j=1}^{n} \hat{y}_j \nabla c_j(\tilde{x})\| \leq \varepsilon, \quad \sum_{j=1}^{n} |c_j(\tilde{x})| \leq \varepsilon,$$

for some $\tilde{y} \in \mathbb{R}^n$, or finds a solution which is infeasible to (1) but ε-critical to the infeasibility measure $\sum_{j=1}^{n} |c_j(\mathbf{x})|$. Moreover, to find such a solution, the method in [11] needs to exactly solve $O(\varepsilon^{-2})$ subproblems like (5) if $\beta$ is bounded above during the algorithm and $O(\varepsilon^{-5})$ subproblems otherwise. This method has been extended to the case when $f_0$ is the expectation of a stochastic function in [82]. We want to emphasize that the complexity of [11] is given in terms of the number of exactly-solved subproblems in the form of (5), and thus it is not the computational complexity, especially when [5] is not trivially solvable. On the contrary, we directly analyze the total computational complexity of the proposed method.

On solving a problem with a non-convex objective and linear constraint, [45] has developed a quadratic-penalty accelerated inexact proximal point method. That method can generate an ε-stationary point in the sense of [14] with a complexity of $O(\varepsilon^{-3})$. Our method is similar to that in [45] by utilizing the techniques from both the proximal point method and the quadratic penalty method. Although we make a little stronger assumption than [45] by requiring the boundedness of dom($g$), our method and analysis apply to the problems with non-convex objectives and convex/non-convex nonlinear constraint functions. When the constraints are convex (but possibly nonlinear), our method can find a nearly ε-stationary point with a complexity of $O(\varepsilon^{-5/2})$ that is a nearly $O(\varepsilon^{-1/2})$ improvement over the complexity in [45].
Barrier methods \[36, 29, 87, 88, 32, 23, 78, 77, 58\] are another traditional class of algorithms for constrained optimization. Similar to the penalty methods, they also solve a sequence of unconstrained subproblems with barrier functions added to objective function. The barrier functions will increase to infinity as the iterates approach the boundary of the feasible set, and thus enforce the iterates to stay in the interior of the feasible set. However, the convergence rate of barrier methods is only shown when the problem is convex \[78, 77, 58, 79\], and only asymptotic convergence analysis is available for non-convex problems.

The augmented Lagrange method (ALM) \[64, 42, 70, 7\] is another common choice for constrained problems. Different from the exact or quadratic penalty method, ALM estimates the primal solution together with the dual solution. At each iteration, it updates the primal variable by minimizing the augmented Lagrange function and then performs a dual gradient ascent step to update the dual variable. The iteration complexity of ALM has been established for convex problems \[47, 90, 89, 91, 56\]. For non-convex problems, most of the existing studies on ALM only show its asymptotic convergence or local convergence rate \[33, 81, 16, 83, 80, 27\]. The computational complexities of ALM for finding an \(\varepsilon\)-stationary point (under various notions of stationarity) are obtained only for linearly constrained problems \[43, 55, 58, 40\]. One exception is \[73\] where they essentially assume that the smallest singular value of the Jacobian matrix of the constraint functions is uniformly bounded away from zero at all feasible points. In this paper, we do not require that assumption but, instead, need an initial nearly feasible solution when the constraints are non-convex while \[73\] does not need.

While preparing this paper, we notice two recently posted papers \[8, 53\] on the problems with non-convex constraints. The algorithms in both works are based on the proximal point method. Different from our approach, they solve subproblems with strongly convex objective and also strongly convex constraints by adding proximal terms to the objective and constraints. Their analysis requires the uniform boundedness of the dual solutions of all subproblems and, to ensure this requirement is satisfied, \[53\] assume that a uniform Slater’s condition holds while \[8\] assume that the Mangasarian-Fromovitz constraint qualification holds at the limiting points of the generated iterates. However, neither assumptions can be easily verified. As pointed out in \[8\], their assumptions can be implied by a sufficient feasibility assumption, which is an even stronger assumption. On the contrary, our analysis in the non-convex constrained case does not depend on the boundedness of the dual variables, and thus does not need the aforementioned assumptions by \[8, 53\].

In addition to the methods above, algorithms that utilize Hessian information have been developed to find the second-order \(\varepsilon\)-stationary point of linearly constrained smooth non-convex optimization \[44, 60, 52\]. Different from these works, we focus on finding an approximate first-order stationary point for nonlinear constrained non-convex optimization using only gradient information.

### 3 Preliminary

In this section, we first provide some basic definitions and then describe an important subroutine that is used within our proposed algorithm. We denote \(\|\cdot\|\) as the \(\ell_2\)-norm. Let

\[
\mathcal{X} = \text{dom} \ (g) := \{x \in \mathbb{R}^d : g(x) < +\infty\}
\]

be the domain of \(g\). The interior of \(\mathcal{X}\) is denoted by \(\text{int} (\mathcal{X})\), and \(\mathcal{N}_x (x)\) denotes the normal cone of \(\mathcal{X}\) at \(x\). For a positive number \(a\), we use \(B_a\) to represent the ball \(\{x \in \mathbb{R}^d : \|x\| \leq a\}\). For ease of description, let

\[
f = [f_1, \ldots, f_m], \quad c = [c_1, \ldots, c_n]
\]

be the vector functions on \(\mathbb{R}^d\), \(0\) be an all-zero vector whose dimension will be clear from the context, and \(|a|_\infty = \max \{0, a\}\) be the vector of component-wise maximum between \(0\) and \(a\). We use the tilde-big-O notation \(\widetilde{O}(\cdot)\) to hide logarithmic terms.

The following definitions are adopted.

**Definition 1** (subdifferential). Given a proper lower-semicontinuous convex function \(h : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}\), its subdifferential at any \(x\) in the domain is defined as

\[
\partial h(x) = \{\zeta \in \mathbb{R}^d \mid h(x') \geq h(x) + \zeta^\top (x' - x), \forall x' \in \mathbb{R}^d\},
\]
and each $\zeta \in \partial h(x)$ is called a subgradient of $h$ at $x$.

**Definition 2 (L-smoothness).** A function $h : \mathbb{R}^d \to \mathbb{R}$ is $L$-smooth if it is differentiable on $\mathbb{R}^d$ and satisfies $\| \nabla h(x) - \nabla h(x') \| \leq L \| x - x' \|$ for any $x$ and $x'$ in $\mathbb{R}^d$, or equivalently,

$$h(x) \leq h(x') + \langle \nabla h(x'), x - x' \rangle + \frac{L}{2} \| x' - x \|^2, \forall x, x' \in \mathbb{R}^d.$$  \hspace{1cm} (7)

**Definition 3 ($\mu$-strong convexity).** A function $h : \mathbb{R}^d \to \mathbb{R} \cup \{ +\infty \}$ is $\mu$-strongly convex if $h - \frac{\mu}{2} \| \cdot \|^2$ is convex, where $\mu \geq 0$. When $\mu = 0$, $\mu$-strong convexity is reduced to convexity.

**Definition 4 ($\rho$-weak convexity).** A function $h : \mathbb{R}^d \to \mathbb{R} \cup \{ +\infty \}$ is $\rho$-weakly convex if $h + \frac{\rho}{2} \| \cdot \|^2$ is convex, where $\rho \geq 0$. When $\rho = 0$, $\rho$-weak convexity is reduced to convexity.

When $h$ is differentiable and $\rho$-weakly convex, it holds that for any $x$ and $x'$,

$$h(x) \geq h(x') + \langle \nabla h(x'), x - x' \rangle - \frac{\rho}{2} \| x' - x \|^2.$$ \hspace{1cm} (8)

**Definition 5 (proximal mapping).** Given a proper lower-semicontinuous convex function $h : \mathbb{R}^d \to \mathbb{R} \cup \{ +\infty \}$, its proximal mapping at $x$ is defined as

$$\text{prox}_h(x) = \arg \min_{z \in \mathbb{R}^d} \left\{ h(z) + \frac{1}{2} \| z - x \|^2 \right\}.$$

The following assumption on problem (P) is made throughout the paper.

**Assumption 1.** The following statements hold:

A. $f_i$ is $L_{f_i}$-smooth with $L_{f_i} \geq 0$ for $i = 0, 1, \ldots, m$; $c_j$ is $L_{c_j}$-smooth with $L_{c_j} \geq 0$ for $j = 1, \ldots, n$;

B. $\mathcal{X}$ is compact, and its diameter is denoted by $D = \max_{x, x' \in \mathcal{X}} \| x - x' \|$;

C. There exist constants $G$ and $M$ such that $|g(x)| \leq G$, and $\partial g(x) \subseteq N_{\mathcal{X}}(x) + B_M, \forall x \in \mathcal{X}$.

With Assumption A and B, there must exist constants $\{B_{f_i}\}_{i=1}^m$ and $\{B_{c_j}\}_{j=1}^n$ such that

$$\max \{|f_i(x)|, \|\nabla f_i(x)\|\} \leq B_{f_i}, \forall x \in \mathcal{X}, \forall i = 0, 1, \ldots, m$$ \hspace{1cm} (9a)

$$\max \{|c_j(x)|, \|\nabla c_j(x)\|\} \leq B_{c_j}, \forall x \in \mathcal{X}, \forall j = 1, \ldots, n.$$ \hspace{1cm} (9b)

Assumption C holds for $g(x) = r(x) + 1_\mathcal{X}(x)$, where $1_\mathcal{X}$ denotes the indicator function on $\mathcal{X}$, and $r$ is a real-valued function on $\mathbb{R}^d$ with the norm of every subgradient bounded by $M$. In addition to Assumption C, we will make more assumptions on the convexity of the constraints. Details will be given in Section 5 and Section 6, where we conduct complexity analysis.

The algorithm we propose for (P) is a first-order method that consists of double-loop iterations. In each outer iteration, it approximately solves a strongly convex subproblem using an optimal first-order method, which is also known as an accelerated proximal gradient (APG) method. Among different variants of the APG method, we choose Nesterov’s optimal first-order method [57], which is described in Algorithm 1. The method is designed to solve the convex composite optimization problem:

$$\min_{x \in \mathbb{R}^d} \{ F(x) := \phi(x) + r(x) \},$$ \hspace{1cm} (10)

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is $\mu_\phi$-strongly convex and $L_\phi$-smooth, and $r : \mathbb{R}^d \to \mathbb{R} \cup \{ +\infty \}$ is a proper lower-semicontinuous convex function. In Algorithm 1, the functions $\phi$ and $r$ are treated as an input because we will apply it to different instances of (10).

The next result extends Theorem 2.2.1 in [57] and can be proved in a similar way as that in [57]. For the readers’ convenience, we include its proof in Appendix A.

**Theorem 1 (linear convergence of APG).** Let $x^{(T)}$ be the output of Algorithm 1. For any $x \in \text{dom}(r)$, it holds that

$$F(x^{(T)}) - F(x) \leq \left( 1 - \sqrt{\frac{\mu_\phi}{L_\phi}} \right)^T \left( F(x^{(0)}) - F(x) + \frac{\mu_\phi}{2} \| x - x^{(0)} \|^2 \right).$$ \hspace{1cm} (11)
Algorithm 1 Nesterov’s accelerated proximal gradient method for (10): APG

1: Input: Two components of objective function $F = \phi + r$ where $\phi$ is $\mu_\phi$-strongly convex and $L_\phi$-smooth and $r$ is proper lower-semicontinuous convex, initial solution $x^{(0)} \in \text{dom}(r)$, and the number $T$ of iterations.
2: Set $w^{(0)} = x^{(0)}$
3: for $t = 0, \ldots, T - 1$ do
4: \hspace{1em} $x^{(t+1)} = \arg \min_{x \in \mathbb{R}^d} \langle x, \nabla \phi(w^{(t)}) \rangle + \frac{L_\phi}{2} \|x - w^{(t)}\|^2 + r(x)$
5: \hspace{1em} $w^{(t+1)} = x^{(t+1)} + \frac{\sqrt{L_\phi - \sqrt{\mu_\phi}}}{\sqrt{L_\phi + \sqrt{\mu_\phi}}} (x^{(t+1)} - x^{(t)})$
6: end for
7: Output: $x^{(T)}$

4 Inexact proximal-point penalty methods

In this section, we describe an inexact proximal-point penalty (iPPP) method for (1) in details. Our method is inspired by recent studies in [20, 66, 50] on the inexact proximal-point method (iPPM), which are inexact versions of the proximal point method [71]. For example, on minimizing a weakly convex and non-smooth $\psi(x)$ over $\mathbb{R}^d$, the approach in [20] iteratively performs the update:

$$\bar{x}^{(k+1)} \approx \arg \min_{x \in \mathbb{R}^d} \psi(x) + \frac{\gamma}{2} \|x - \bar{x}^{(k)}\|^2, \quad (12)$$

where $\bar{x}^{(k)}$ denotes the $k$-th iterate. Since $\psi$ is weakly convex, (12) with a large enough $\gamma$ is a strongly convex problem and can be solved effectively to a required accuracy by the standard subgradient method. Similar approaches have been developed for solving non-convex-non-concave and non-smooth min-max problems [66, 50].

Our method incorporates the technique of the iPPM as well as the quadratic penalty scheme. It iteratively updates the estimated solution by

$$\bar{x}^{(k+1)} \approx \bar{x}^{(k)} := \arg \min_{x \in \mathbb{R}^d} f_0(x) + g(x) + \frac{\gamma}{2} \|x - \bar{x}^{(k)}\|^2 + \frac{\beta_k}{2} \left( \|c(x)\|^2 + \|f(x)\|_+^2 \right), \quad (13)$$

where $\beta_k > 0$ is the penalty parameter, and $\gamma > 0$ is the proximal parameter. When $f_i$ is weakly convex for $i = 0, 1, \ldots, m$ and $c_j$ is weakly convex for $j = 0, 1, \ldots, n$ in (1), we will choose a large enough $\gamma$ such that the minimization problem in (13) becomes strongly convex, and thus we can apply Algorithm 1 to approximately solve it to obtain $\bar{x}^{(k+1)}$. We formally describe our method in Algorithm 2 where

$$\phi_k(x) = f_0(x) + \frac{\gamma}{2} \|x - \bar{x}^{(k)}\|^2 + \frac{\beta_k}{2} \left( \|c(x)\|^2 + \|f(x)\|_+^2 \right), \quad (14)$$

is the smooth part of the objective function in (13).

In the rest of the paper, we analyze the computational complexity of Algorithm 2. For technical reasons, we consider two different cases. In the first case, we assume that the objective function is weakly convex while the constraint functions are convex. In the second case, we assume that the objective function and the constraint functions are all weakly convex. The proximal parameter $\gamma$ and the penalty parameters $\{\beta_k\}$ will be chosen differently for the two cases. We will show that the output of Algorithm 2 with appropriate parameters is a nearly $\varepsilon$-stationary point in each case. Also, we will estimate the complexity of the algorithm measured by the total number of the proximal gradient steps (Line 4 in Algorithm 1) it performs.
Algorithm 2 Inexact Proximal-Point Penalty (iPPP) Method for (1)

1: **Input:** Initial solution \( \bar{x}^{(0)} \), target accuracy \( \varepsilon > 0 \), maximum outer iteration \( K \), proximal parameter \( \gamma \), penalty parameters \( \{ \beta_k \}_{k=0}^{K-1} \), and the numbers of inner iterations \( \{ T_k \}_{k=0}^{K-1} \).
2: for \( k = 0, \ldots, K - 1 \) do
3: \( \bar{x}^{(k+1)} = \text{APG}(\phi_k, g, \bar{x}^{(k)}, T_k) \), where \( \phi_k \) is given in (14).
4: end for
5: Return \( \bar{x}^{(R)} \), where \( R \) is an index generated by either of the two options:
6: **Option I:** \( R = \arg \min_{0 \leq k < K} ||\bar{x}^{(k+1)} - \bar{x}^{(k)}|| \)
7: **Option II:** \( R \) is chosen from \( \{0, 1, 2, \ldots, K - 1\} \) uniformly at random.

5 Complexity of the iPPP method with convex constraints

Throughout this section, we assume that \( f_i \) is convex for each \( i = 1, \ldots, m \) and \( c_j \) is affine for each \( j = 1, \ldots, n \), namely, problem (1) is reduced to the following problem with only convex constraints:

\[
\min_{x \in \mathbb{R}^d} f_0(x) + g(x), \quad \text{s.t.} \quad Ax = b, \quad f_i(x) \leq 0, \quad i = 1, 2, \ldots, m, \quad (15)
\]

where \( A \in \mathbb{R}^{n \times d} \) and \( b \in \mathbb{R}^n \) are given. Without loss of generality, we assume \( A \) to be full row-rank. Otherwise, redundant affine equality constraints can be removed. In this case, the function \( \phi_k \) in (14) is \( L_{\phi_k} \)-smooth with

\[
L_{\phi_k} = L_{f_0} + \gamma + \beta_k \left( \|A^T A\| + \sum_{i=1}^{m} B_{f_i}(B_{f_i} + L_{f_i}) \right). \quad (16)
\]

Besides the convexity assumption on the constraints, we assume Assumption 1 and the next one.

**Assumption 2.** The following statements hold:

A. \( f_0 \) is \( p_0 \)-weakly convex for \( p_0 \geq 0 \).

B. There exists a solution \( x_{\text{feas}} \in \text{int}(\mathcal{X}) \) satisfying \( Ax_{\text{feas}} = b \) and \( f_i(x_{\text{feas}}) < 0 \) for \( i = 1, \ldots, m \), where \( \mathcal{X} \) is the domain of \( g \) defined in (6).

Here, we only require the existence of \( x_{\text{feas}} \) but not its availability to our algorithm. As we assume above, the constraint functions in (15) are convex. However, the objective function can be non-convex. Therefore, it is still challenging to find a global optimal solution of (15). As discussed in the introduction, one tractable goal is to find a nearly stationary point of (15). In this section, we show that Algorithm 2 finds a Type-I nearly \( \varepsilon \)-stationary point defined below, and also, we estimate its complexity to produce such a point.

**Definition 6** (\( \varepsilon \)-stationary point). Given \( \varepsilon > 0 \), a point \( \bar{x} \in \mathcal{X} \) is called an \( \varepsilon \)-stationary point of (15) if there are \( \bar{\xi} \in \partial g(\bar{x}), \bar{y} \in \mathbb{R}^n \), and \( \bar{\lambda} \in \mathbb{R}^m_+ \) such that

\[
\left\| \nabla f_0(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\bar{x}) + A^T \bar{y} + \bar{\xi} \right\| \leq \varepsilon, \quad (17a)
\]

\[
\|A\bar{x} - b\| \leq \varepsilon, \quad \|f(\bar{x})\| \leq \varepsilon, \quad (17b)
\]

\[
\sum_{i=1}^{m} |\lambda_i f_i(\bar{x})| \leq \varepsilon. \quad (17c)
\]
A random vector $\tilde{x} \in \mathcal{X}$ is called an $\varepsilon$-stationary point of \eqref{eq:15} in expectation if there are random vectors $\xi \in \partial g(\tilde{x})$, $\tilde{y} \in \mathbb{R}^n$, and $\tilde{\lambda} \in \mathbb{R}^m_+$ such that

\begin{align}
\mathbb{E} \left[ \nabla f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i \nabla f_i(\tilde{x}) + A^\top \tilde{y} + \xi \right] \leq \varepsilon, \\
\mathbb{E}[Ax - b] \leq \varepsilon, \quad \mathbb{E}[\|f(\tilde{x})\|_1] \leq \varepsilon, \\
\sum_{i=1}^{m} \mathbb{E}[\lambda_i f_i(\tilde{x})] \leq \varepsilon.
\end{align}

(18a) (18b) (18c)

Definition 7 (Type-I nearly $\varepsilon$-stationary point). Given $\varepsilon > 0$, a point $x \in \mathbb{R}^d$ is called a nearly $\varepsilon$-stationary point of type one, abbreviated $\varepsilon$-NSP1, if there is an $\varepsilon$-stationary solution $\tilde{x}$ of \eqref{eq:13} such that $\|\tilde{x} - x\| \leq \varepsilon$. A random vector $x \in \mathbb{R}^d$ is called an $\varepsilon$-NSP1 in expectation, abbreviated $\varepsilon$-ENSP1, of \eqref{eq:15} if there is an $\varepsilon$-stationary point $\tilde{x}$ of \eqref{eq:13} in expectation such that $\mathbb{E}[\|\tilde{x} - x\|] \leq \varepsilon$.

Let $\gamma > \rho_0$. Then the problem in \eqref{eq:13} is $(\gamma - \rho_0)$-strongly convex. The core idea of our analysis is to show the near-stationarity of $\tilde{x}^{(k)}$ defined in \eqref{eq:13} with respect to the convex constrained proximal problem:

\begin{equation}
\tilde{x}^{(k)} \equiv \arg \min_{x \in \mathbb{R}^d} \left\{ f_0(x) + g(x) + \frac{\gamma}{2}\|x - \tilde{x}^{(k)}\|^2 \right\} \text{ s.t. } Ax = b, \quad f_i(x) \leq 0, \quad i = 1, 2, \ldots, m,
\end{equation}

and to bound the quantity $\|\tilde{x}^{(k)} - \tilde{x}^{(k)}\|$ or $\|\tilde{x}^{(k)} - \tilde{x}^{(k)}\|$.

5.1 Boundedness of dual solutions

Since the Slater’s conditions hold according to Assumption\ref{ass:3}, $\tilde{x}^{(k)}$ must be a KKT point of \eqref{eq:19}, i.e., there are Lagrangian multipliers $\tilde{\gamma}^{(k)} \in \mathbb{R}^m$ and $\tilde{\lambda}^{(k)} \in \mathbb{R}^m_+$ associated to $\tilde{x}^{(k)}$ such that (c.f. \cite[Theorem 28.2]{69}):

\begin{align}
0 & \in \nabla f_0(\tilde{x}^{(k)}) + \partial g(\tilde{x}^{(k)}) + \gamma(\tilde{x}^{(k)} - \tilde{x}^{(k)}) + A^\top \tilde{\gamma}^{(k)} + \sum_{i=1}^{m} \tilde{\lambda}_i^{(k)} \nabla f_i(\tilde{x}^{(k)}), \quad \tilde{\lambda}^{(k)} \geq 0, \\
A\tilde{x}^{(k)} & = b, \quad f_i(\tilde{x}^{(k)}) \leq 0, \quad i = 1, \ldots, m. \\
\tilde{\lambda}_i^{(k)} f_i(\tilde{x}^{(k)}) & = 0, \quad i = 1, \ldots, m.
\end{align}

We next prove the boundedness of $(\tilde{\gamma}^{(k)}, \tilde{\lambda}^{(k)})$ under Assumption \ref{ass:2}

Lemma 1. Let $(\tilde{x}^{(k)}, \tilde{\gamma}^{(k)}, \tilde{\lambda}^{(k)})$ be the solution satisfying the conditions in \eqref{eq:20} for $k = 0, 1, \ldots$. Then

\begin{equation}
\|\tilde{\lambda}^{(k)}\| \leq M_\lambda := \frac{D(B_{f_0} + \gamma D + M)}{\min_i |f_i(x^{\text{feas}})|}
\end{equation}

(21)

and

\begin{equation}
\|\tilde{\gamma}^{(k)}\| \leq M_\gamma := \sqrt{\frac{\lambda_{\max}(AA^\top)}{\lambda_{\min}(AA^\top)}} (B_{f_0} + \gamma D + M) \left( 1 + \frac{D}{\text{dist}(x^{\text{feas}}, \partial \mathcal{X})} + \frac{D \max_i B_{f_i}}{\min_i |f_i(x^{\text{feas}})|} \right),
\end{equation}

(22)

where $\partial \mathcal{X}$ denotes the boundary of $\mathcal{X}$ and $\text{dist}(x^{\text{feas}}, \partial \mathcal{X})$ represents the distance from $x^{\text{feas}}$ to $\partial \mathcal{X}$.

Proof. Since $(\tilde{x}^{(k)}, \tilde{\gamma}^{(k)}, \tilde{\lambda}^{(k)})$ satisfies \eqref{eq:21}, we have

\begin{equation}
\nabla f_0(\tilde{x}^{(k)}) + \tilde{\gamma}^{(k)} + A^\top \tilde{\gamma}^{(k)} + \sum_{i=1}^{m} \tilde{\lambda}_i^{(k)} \nabla f_i(\tilde{x}^{(k)}) + \gamma(\tilde{x}^{(k)} - \tilde{x}^{(k)}) = 0
\end{equation}

(23)
for some $\hat{\zeta}^{(k)} \in \partial g(\hat{x}^{(k)})$. Let $x_{feas}$ be the point in Assumption \ref{assumption:feasibility} Then from the convexity of each $f_i$ and the fact $\lambda^k_i \geq 0$, it follows that

$$\sum_{i=1}^{m} \lambda^k_i f_i(x_{feas}) \geq \sum_{i=1}^{m} \hat{\lambda}^k_i \left[ f_i(\hat{x}^{(k)}) + (x_{feas} - \hat{x}^{(k)})^\top \nabla f_i(\hat{x}^{(k)}) \right].$$

The above inequality together with \eqref{eq:20} and \eqref{eq:24} yields

$$\sum_{i=1}^{m} \lambda^k_i f_i(x_{feas}) \geq - (x_{feas} - \hat{x}^{(k)})^\top \left[ \nabla f_0(\hat{x}^{(k)}) + \hat{\zeta}^{(k)} + \lambda^k \hat{\zeta}^{(k)} + \gamma(\hat{x}^{(k)} - \bar{x}^{(k)}) \right]$$

$$= - (x_{feas} - \hat{x}^{(k)})^\top \left[ \nabla f_0(\hat{x}^{(k)}) + \hat{\zeta}^{(k)} + \gamma(\hat{x}^{(k)} - \bar{x}^{(k)}) \right],$$

where the equality follows from $A(x_{feas} - \hat{x}^{(k)}) = b - b = 0$. By Assumption \ref{assumption:convexity} we have

$$-(x_{feas} - \hat{x}^{(k)})^\top \left[ \nabla f_0(\hat{x}^{(k)}) + \gamma(\hat{x}^{(k)} - \bar{x}^{(k)}) \right] \geq -DBf_0 - \gamma D^2,$$

and thus \eqref{eq:24} implies

$$-\sum_{i=1}^{m} \lambda^k_i f_i(x_{feas}) \leq DBf_0 + \gamma D^2 + (x_{feas} - \hat{x}^{(k)})^\top \hat{\zeta}^{(k)}. \tag{25}$$

By Assumption \ref{assumption:convexity} we have $\hat{\zeta}^{(k)} = \hat{\zeta}_1 + \hat{\zeta}_2$ with $\hat{\zeta}_1 \in N_H(\hat{x}^{(k)})$ and $\|\hat{\zeta}_2\| \leq M$, and thus

$$(x_{feas} - \hat{x}^{(k)})^\top \hat{\zeta}^{(k)} \leq (x_{feas} - \hat{x}^{(k)})^\top \hat{\zeta}_1 + DM. \tag{26}$$

Next we bound $\|\hat{\zeta}_1\|$ and the term $(x_{feas} - \hat{x}^{(k)})^\top \hat{\zeta}_1$. If $\hat{x}^{(k)} \in \interior(\mathcal{X})$, then $\hat{\zeta}_1 = 0$. Hence, we only need to consider the case when $\hat{x}^{(k)} \in \partial \mathcal{X}$ and $\hat{\zeta}_1 \neq 0$. In this case, $\mathcal{H} = \{ x \in \mathbb{R}^d \mid (x - \hat{x}^{(k)})^\top \hat{\zeta}_1 = 0 \}$ is a supporting hyperplane of $\mathcal{X}$ at $\hat{x}^{(k)}$. Hence, $\text{dist}(x_{feas}, \mathcal{H}) \geq \text{dist}(x_{feas}, \partial \mathcal{X}) > 0$, and thus

$$(\hat{x}^{(k)} - x_{feas})^\top \hat{\zeta}_1 = \text{dist}(x_{feas}, \mathcal{H}) \|\hat{\zeta}_1\| \geq \text{dist}(x_{feas}, \partial \mathcal{X}) \|\hat{\zeta}_1\|.$$ 

Plugging the above inequality into \eqref{eq:20} and using \eqref{eq:25} give

$$-\sum_{i=1}^{m} \lambda^k_i f_i(x_{feas}) + \text{dist}(x_{feas}, \partial \mathcal{X}) \|\hat{\zeta}_1\| \leq DBf_0 + \gamma D^2 + DM,$$

which implies

$$\|\hat{\lambda}^{(k)}\| \leq \|\lambda^{(k)}\|_1 \leq \frac{\sum_{i=1}^{m} \lambda^k_i f_i(x_{feas})}{\max_i f_i(x_{feas})} = \frac{-\sum_{i=1}^{m} \lambda^k_i f_i(x_{feas})}{\min_i f_i(x_{feas})} \leq \frac{DBf_0 + \gamma D^2 + DM}{\min_i f_i(x_{feas})},$$

and

$$\|\hat{\zeta}^{(k)}\| \leq \|\hat{\zeta}_1\| + \|\hat{\zeta}_2\| \leq \frac{DBf_0 + \gamma D^2 + DM}{\text{dist}(x_{feas}, \partial \mathcal{X})} + M. \tag{28}$$

Furthermore, since $A$ has a full row-rank, we have from \eqref{eq:23} that

$$\hat{y}^{(k)} = - (AA^\top)^{-1} A \left( \nabla f_0(\hat{x}^{(k)}) + \gamma(\hat{x}^{(k)} - \bar{x}^{(k)}) + \sum_{i=1}^{m} \lambda^k_i \nabla f_i(x_{feas}) + \hat{\zeta}^{(k)} \right),$$

and thus by \eqref{eq:27} and \eqref{eq:28}, it follows

$$\|\hat{y}^{(k)}\| \leq \frac{\sqrt{\lambda_{\max}(AA^\top)}}{\lambda_{\min}(AA^\top)} \left( Bf_0 + \gamma D + M + \frac{DBf_0 + \gamma D^2 + DM}{\text{dist}(x_{feas}, \partial \mathcal{X})} + \max_i Bf_i \cdot \frac{DBf_0 + \gamma D^2 + DM}{\min_i f_i(x_{feas})} \right).$$

This completes the proof.
The next proposition describes the quality of the solution returned by Algorithm \(1\) when applied to (13).

**Proposition 1.** Given \(\gamma > \rho_0\) and \(\beta_k > 0\) for \(k \geq 0\), let \(\phi_k\) be defined in (14) with \(c(x) = Ax - b\), and \(\hat{x}^{(k)}\) be defined in (19). Let \(x^{(k+1)} = \text{APG}(\phi_k, g, \hat{x}^{(k)}, T_k)\). Then, for any \(k \geq 0\), we have

\[
f_0(\hat{x}^{(k+1)}) + g(\hat{x}^{(k+1)}) + \frac{\gamma}{2}\|\hat{x}^{(k+1)} - \hat{x}^{(k)}\|^2 - f_0(\hat{x}^{(k)}) + g(\hat{x}^{(k)}) - \frac{\gamma}{2}\|\hat{x}^{(k)} - \hat{x}^{(k)}\|^2 \\
\leq \delta_k := \left(1 - \sqrt{\frac{\gamma - \rho_0}{L_{\phi_k}}}\right) T_k \left(2Bf_0 + 2G + \beta_k \left(\|A\|^2D^2 + \sum_{i=1}^{m}B_i^2\right) + \frac{2\gamma - \rho_0}{2}D^2\right),
\]

and

\[
\|Ax^{(k+1)} - b\|^2 + \left\|f(\hat{x}^{(k+1)})\right\|_+^2 \leq \frac{4\delta_k}{\beta_k} + \frac{4\|\hat{\gamma}^{(k)}\|^2}{\beta_k^2} + \frac{4\|\hat{\lambda}^{(k)}\|^2}{\beta_k^2}.
\]

**Proof.** Note that \(\phi_k\) is \((\gamma - \rho_0)\)-strongly convex and \(L_{\phi_k}\)-smooth with \(L_{\phi_k}\) given in (16). Hence, by Theorem 1, it holds for any \(x \in \mathcal{X}\) that

\[
\phi_k(\hat{x}^{(k+1)}) + g(\hat{x}^{(k+1)}) - \phi_k(x) - g(x) \\
\leq \left(1 - \sqrt{\frac{\gamma - \rho_0}{L_{\phi_k}}}\right) T_k \left(\phi_k(\hat{x}^{(k)}) + g(\hat{x}^{(k)}) - \phi_k(x) - g(x) + \frac{\gamma - \rho_0}{2}\|x - \hat{x}^{(k)}\|^2\right).
\]

Moreover, from Assumption 1 we have

\[
\phi_k(\hat{x}^{(k)}) + g(\hat{x}^{(k)}) - \phi_k(x) - g(x) + \frac{\gamma - \rho_0}{2}\|x - \hat{x}^{(k)}\|^2 \\
\leq 2Bf_0 + 2G + \frac{\gamma}{2}D^2 + \beta_k \left(\|A\|^2D^2 + \sum_{i=1}^{m}B_i^2\right) + \frac{\gamma - \rho_0}{2}D^2,
\]

which together with (31) and the definition of \(\delta_k\) gives

\[
\phi_k(\hat{x}^{(k+1)}) + g(\hat{x}^{(k+1)}) - \phi_k(x) - g(x) \leq \delta_k, \quad \forall x \in \mathcal{X}.
\]

Letting \(x = \hat{x}^{(k)}\) in (33), noting the feasibility of \(\hat{x}^{(k)}\) for (19), and using the definition of \(\phi_k\) in (14), we have

\[
f_0(\hat{x}^{(k+1)}) + g(\hat{x}^{(k+1)}) + \frac{\gamma}{2}\|\hat{x}^{(k+1)} - \hat{x}^{(k)}\|^2 + \frac{\beta_k}{2}\left(\|Ax^{(k+1)} - b\|^2 + \left\|f(\hat{x}^{(k+1)})\right\|_+^2\right) \\
\leq f_0(\hat{x}^{(k)}) + g(\hat{x}^{(k)}) + \frac{\gamma}{2}\|\hat{x}^{(k)} - \hat{x}^{(k)}\|^2 + \delta_k,
\]

which, after dropping non-negative term \(\frac{\beta_k}{2}\left(\|Ax^{(k+1)} - b\|^2 + \left\|f(\hat{x}^{(k+1)})\right\|_+^2\right)\), implies (29).

Recall that \(\hat{\gamma}^{(k)}\) and \(\hat{\lambda}^{(k)}\) are the Lagrangian multipliers satisfying (20). Hence, from the convexity of the objective in (19), we have

\[
f_0(\hat{x}^{(k)}) + g(\hat{x}^{(k)}) + \frac{\gamma}{2}\|\hat{x}^{(k)} - \hat{x}^{(k)}\|^2 \\
\leq f_0(\hat{x}^{(k+1)}) + g(\hat{x}^{(k+1)}) + \frac{\gamma}{2}\|\hat{x}^{(k+1)} - \hat{x}^{(k)}\|^2 + (\hat{\gamma}^{(k)})^T(A\hat{x}^{(k+1)} - b) + \sum_{i=1}^{m}\hat{\lambda}_i^{(k)} f_i(\hat{x}^{(k+1)}),
\]

which, together with (31), implies

\[
-(\hat{\gamma}^{(k)})^T(A\hat{x}^{(k+1)} - b) - \sum_{i=1}^{m}\hat{\lambda}_i^{(k)} f_i(\hat{x}^{(k+1)}) + \frac{\beta_k}{2}\left(\|Ax^{(k+1)} - b\|^2 + \left\|f(\hat{x}^{(k+1)})\right\|_+^2\right) \leq \delta_k.
\]
By the Young’s inequality, we have
\[-\frac{\|\tilde{y}^{(k)}\|^2}{\beta_k} - \frac{\beta_k}{4}\|Ax^{(k+1)} - b\|^2 \leq -\tilde{y}^{(k)\top}(Ax^{(k+1)} - b)\]
and
\[-\frac{\tilde{\lambda}_i^{(k)}}{\beta_k} - \frac{\beta_k}{4}\|f_i(x^{(k+1)})\|_2^2 \leq \lambda_i^{(k)}[f_i(x^{(k+1)})]_+ \leq \tilde{\lambda}_i^{(k)}f_i(x^{(k+1)}), \forall i = 1, \ldots, m.
\]
Adding these inequalities to (38) and re-organizing terms lead to (30), and thus we complete the proof.

Similarly, we can bound the feasibility violation of \(\tilde{x}^{(k)}\).

**Proposition 2.** Let \(\tilde{x}^{(k)}\) be defined in (13). Then
\[
\|A\tilde{x}^{(k)} - b\|^2 + ||f(\tilde{x}^{(k)})||_+^2 \leq \frac{4\|\tilde{y}^{(k)}\|^2}{\beta_k^2} + \frac{4\|\tilde{\lambda}^{(k)}\|^2}{\beta_k^2}.
\]  
(37)

**Proof.** From the optimality of \(\tilde{x}^{(k)}\) in (13) and the feasibility of \(\tilde{x}^{(k)}\) for (15), it follows that
\[
f_0(\tilde{x}^{(k)}) + g(\tilde{x}^{(k)}) + \frac{\gamma}{2}\|\tilde{x}^{(k)} - \tilde{x}^{(k)}\|^2 + \frac{\beta_k}{2}\left(||A\tilde{x}^{(k)} - b\| + ||f(\tilde{x}^{(k)})||_+\right)^2\]
\[
\leq f_0(\tilde{x}^{(k)}) + g(\tilde{x}^{(k)}) + \frac{\gamma}{2}\|\tilde{x}^{(k)} - \tilde{x}^{(k)}\|^2.
\]  
(38)

Similar to (34), we have
\[
f_0(\tilde{x}^{(k)}) + g(\tilde{x}^{(k)}) + \frac{\gamma}{2}\|\tilde{x}^{(k)} - \tilde{x}^{(k)}\|^2 \leq f_0(\tilde{x}^{(k)}) + g(\tilde{x}^{(k)}) + \frac{\gamma}{2}\|\tilde{x}^{(k)} - \tilde{x}^{(k)}\|^2 + \tilde{y}^{(k)\top}(A\tilde{x}^{(k)} - b) + \sum_{i=1}^m \tilde{\lambda}_i^{(k)}f_i(x^{(k)}),
\]  
(39)

which adds to (38) and implies
\[-\tilde{y}^{(k)\top}(A\tilde{x}^{(k)} - b) - \sum_{i=1}^m \lambda_i^{(k)}f_i(x^{(k)}) + \frac{\beta_k}{2}\left(||A\tilde{x}^{(k)} - b\| + ||f(\tilde{x}^{(k)})||_+\right)^2 \leq 0.
\]

Now we obtain the desired result by applying the Young’s inequality and following the same arguments as those at the end of the proof of Proposition 1.

**5.2 A complexity result by constant penalty parameters**

In this subsection, for a given \(\varepsilon > 0\), we choose \(\beta_k = \beta\) for any \(k\) with \(\beta\) dependent on \(\varepsilon\), and estimate the complexity of Algorithm 2 to produce an \(\varepsilon\)-NSP1 of (19). First, we show that each \(\tilde{x}^{(k)}\) defined in (16) is an \(\varepsilon\)-stationary point of (19) by choosing an appropriate \(\beta\).

**Proposition 3.** Given \(\varepsilon > 0\), let
\[
\beta_k = \beta = \max \left\{ \frac{2\sqrt{M_y^2 + M_x^2}}{\varepsilon}, \frac{4(M_y^2 + M_x^2)}{\varepsilon} \right\}
\]  
(40)

for \(k \geq 0\) in Algorithm 2. Then \(\tilde{x}^{(k)}\) defined in (13) is an \(\varepsilon\)-stationary point of (19).
Proof. From the optimality of $\tilde{x}^{(k)}$, the first-order optimality condition of (13) follows, i.e.,

$$0 \in \nabla f_0(\tilde{x}^{(k)}) + \partial g(\tilde{x}^{(k)}) + \gamma(\tilde{x}^{(k)} - \bar{x}^{(k)}) + \beta_k A^\top (A\tilde{x}^{(k)} - b) + \beta_k \sum_{i=1}^{m} [f_i(\tilde{x}^{(k)})]_+ \nabla f_i(\tilde{x}^{(k)}).$$

Hence, letting $\tilde{y}^{(k)} = \beta_k (A\tilde{x}^{(k)} - b)$ and $\tilde{\lambda}^{(k)}_i = \beta_k [f_i(\tilde{x}^{(k)})]_+$ for each $i$, we have

$$0 \in \nabla f_0(\tilde{x}^{(k)}) + \partial g(\tilde{x}^{(k)}) + \gamma(\tilde{x}^{(k)} - \bar{x}^{(k)}) + A^\top \tilde{y}^{(k)} + \sum_{i=1}^{m} \tilde{\lambda}^{(k)}_i \nabla f_i(\tilde{x}^{(k)}), \forall k.$$  \hfill (41)

In addition, because $\beta_k = \beta \geq \frac{2\sqrt{M_g^2 + M_\lambda^2}}{\varepsilon}$, $\|\tilde{y}^{(k)}\| \leq M_g$, and $\|\tilde{\lambda}^{(k)}\| \leq M_\lambda$, inequality (37) implies $\|A\tilde{x}^{(k)} - b\|^2 + \|[f(\tilde{x}^{(k)})]_+\|^2 \leq \varepsilon^2$, and thus

$$\|A\tilde{x}^{(k)} - b\| \leq \varepsilon, \quad \|[f(\tilde{x}^{(k)})]_+\| \leq \varepsilon, \forall k.$$  \hfill (42)

Furthermore, recalling $\tilde{\lambda}^{(k)}_i = \beta_k [f_i(\tilde{x}^{(k)})]_+$ for each $i$, we have

$$\sum_{i=1}^{m} \tilde{\lambda}^{(k)}_i f_i(\tilde{x}^{(k)}) = \sum_{i=1}^{m} \beta_k [f_i(\tilde{x}^{(k)})]_+^2 \leq \frac{4(M_g^2 + M_\lambda^2)}{\beta} \leq \varepsilon, \forall k,$$  \hfill (43)

where in the last inequality we have used the condition $\beta \geq \frac{4(M_g^2 + M_\lambda^2)}{\varepsilon}$. The above inequality together with (31) and (42) exactly means that $\tilde{x}^{(k)}$ is an $\varepsilon$-stationary point of (19). We hence complete the proof. \hfill \square

Since the constraints in (16) are convex and there exists a feasible solution, we can obtain a high-quality initial point $\bar{x}^{(0)}$ as indicated below.

Lemma 2 (near-feasible initial point). Given $\varepsilon > 0$, there exits an algorithm that needs at most

$$T_{\text{initial}} = \left\lceil \frac{2D}{\varepsilon} \sqrt{M_g^2 + M_\lambda^2} \cdot \|A\|^2 + \sum_{i=1}^{m} B_{f_i} (B_{f_i} + L_{f_i}) \right\rceil$$

projected gradient steps to obtain a point $\tilde{x}^{(0)} \in X$ satisfying

$$\|A\tilde{x}^{(0)} - b\|^2 + \|[f(\tilde{x}^{(0)})]_+\|^2 \leq \frac{\varepsilon^2}{M_g^2 + M_\lambda^2}.$$  \hfill (44)

Proof. Let $\psi(x) = \frac{1}{2} (\|Ax - b\|^2 + \|[f(x)]_+\|^2)$. Then $\psi$ is $L_\psi$-smooth with

$$L_\psi = \|A^\top A\| + \sum_{i=1}^{m} B_{f_i} (B_{f_i} + L_{f_i}).$$

Hence, applying the FISTA algorithm in [3] to $\min_{x \in X} \psi(x)$ and using Theorem 4.4 there, we can obtain a point $\tilde{x}^{(0)}$ that satisfies (45) within $T_{\text{initial}}$ projected gradient iterations, and thus complete the proof. \hfill \square

Below we use Proposition 3 to estimate the complexity of Algorithm 2 to produce an $\varepsilon$-NSP1 of (13). We choose appropriate $\{T_k\}_{k \geq 0}$ such that $\gamma \|\tilde{x}^{(R)} - \tilde{x}^{(k)}\| \leq \varepsilon$, when $R$ is generated by Option I in Algorithm 2. Hence, $\tilde{x}^{(R)}$ is an $\varepsilon$-NSP1 by relating it to $\tilde{x}^{(k)}$. 
Theorem 2 (near-stationary point and complexity). Suppose Assumptions [1] and [2] hold, $4(M_y^2 + M_x^2) \geq 1$, and an initial point $\bar{x}^{(0)} \in \mathcal{X}$ satisfying (43) is obtained. Given $\varepsilon > 0$, in Algorithm [3], let $\gamma > \max\{1, \rho_0\}$, $\beta_k = \beta$ for $k \geq 0$ with $\beta$ given in (46),

$$K = \left[\frac{8\gamma}{\varepsilon^2} (B_{f_0} + G + \varepsilon)\right],$$

and

$$T_k = \left[\frac{L_\beta}{\gamma - \rho_0} \ln \left(\frac{32\gamma^2}{(\gamma - \rho_0)\varepsilon^2} \left[2B_{f_0} + 2G + \beta\left(\|A\|^2 D^2 + \sum_{i=1}^m B_i^2\right) + \frac{2\gamma - \rho_0}{2} D^2\right]\right)\right]$$

for $k = 0, 1, \ldots, K - 1$, where

$$L_\beta = L_{f_0} + \gamma + \beta \left(\|A^T A\| + \sum_{i=1}^m B_i(B_{f_i} + L_{f_i})\right).$$

Then the following statements hold:

1. If $R$ is generated by Option I in Algorithm [2], then $\bar{x}^{(R)}$ is an $\varepsilon$-NSP1 of (15).

2. The total complexity (including the complexity of obtaining $\bar{x}^{(0)}$) of Algorithm [2] is

$$T_{\text{total}} = \tilde{O}\left(\frac{\gamma(B_{f_0} + G)}{\varepsilon^2} \left(\frac{L_{f_0} + \gamma}{\gamma - \rho_0} + \frac{\|A^T A\| + \sum_{i=1}^m B_i(B_{f_i} + L_{f_i})}{\gamma - \rho_0} \cdot \frac{\sqrt{M_y^2 + M_x^2}}{\sqrt{\varepsilon}}\right)\right)$$

Proof. Using the fact $a + \log(1-a) \leq 0$ for any $a \in [0, 1)$, we have from the formula of $\delta_k$ in Proposition [1] and the choice of $T_k$ that

$$\delta_k \leq \frac{(\gamma - \rho_0)\varepsilon^2}{32\gamma^2}, \quad \forall 0 \leq k \leq K - 1.$$ (50)

Letting $x = \bar{x}^{(k)}$ in (33) and using the definition of $\phi_k$ in (14), we have from $\beta_k = \beta$, $\forall k$, that

$$f_0(\bar{x}^{(k+1)}) + g(\bar{x}^{(k+1)}) + \frac{\gamma}{2} \|\bar{x}^{(k+1)} - \bar{x}^{(k)}\|^2 + \frac{\beta}{2} \left(\|A\bar{x}^{(k+1)} - b\|^2 + \|f(\bar{x}^{(k+1)})\| \right) \leq f_0(\bar{x}^{(k)}) + g(\bar{x}^{(k)}) + \frac{\beta}{2} \left(\|A\bar{x}^{(k)} - b\|^2 + \|f(\bar{x}^{(k)})\| \right) + \delta_k.$$ (48)

Summing up the above inequality over $k = 0, 1, \ldots, K - 1$ gives

$$f_0(\bar{x}^{(K)}) + g(\bar{x}^{(K)}) + \frac{\gamma}{2} \sum_{k=0}^{K-1} \|\bar{x}^{(k+1)} - \bar{x}^{(k)}\|^2 + \frac{\beta}{2} \left(\|A\bar{x}^{(k+1)} - b\|^2 + \|f(\bar{x}^{(k+1)})\| \right) \leq f_0(\bar{x}^{(0)}) + g(\bar{x}^{(0)}) + \frac{\beta}{2} \left(\|A\bar{x}^{(0)} - b\|^2 + \|f(\bar{x}^{(0)})\| \right) + \sum_{k=0}^{K-1} \delta_k.$$ (51)

By Assumption [1], the inequality (15), and the bound on $\delta_k$ in (50), we have from (51) that

$$\frac{\gamma}{2} \sum_{k=0}^{K-1} \|\bar{x}^{(k+1)} - \bar{x}^{(k)}\|^2 \leq 2B_{f_0} + 2G + \frac{(\gamma - \rho_0)K\varepsilon^2}{32\gamma^2} + \frac{\beta\varepsilon^2}{2(M_y^2 + M_x^2)}.$$ (49)

Since $4(M_y^2 + M_x^2) \geq 1$, we have $\beta = \frac{4(M_y^2 + M_x^2)}{\varepsilon}$ from (10). Hence, it follows from the above inequality and the choice of $K$ that

$$\min_{0 \leq k \leq K-1} \|\bar{x}^{(k+1)} - \bar{x}^{(k)}\|^2 \leq \frac{9\varepsilon^2}{16\gamma^2}.$$ (14)
Hence, if \( R \) is generated by \textbf{Option I} in Algorithm 2 then
\[
\|\bar{x}(R+1) - \bar{x}(R)\| \leq \frac{3\varepsilon}{4\gamma}.
\]
(52)

In addition, since \( \phi_k + g \) is \((\gamma - \rho_0)\)-strongly convex, it holds
\[
\frac{\gamma - \rho_0}{2} \|\bar{x}(k+1) - \bar{x}(k)\|^2 \leq \phi_k(\bar{x}(k+1)) + g(\bar{x}(k+1)) - \phi_k(\bar{x}(k)) - g(\bar{x}(k)) \leq \delta_k \leq \frac{(\gamma - \rho_0)^2}{32\gamma^2},
\]
and thus \( \|\bar{x}(k+1) - \bar{x}(k)\| \leq \frac{\varepsilon}{\gamma}, \forall k \). By the triangle inequality and (52), we have \( \|\bar{x}(R) - \bar{x}(R)\| \leq \frac{\varepsilon}{\gamma} \). Therefore, from (41), (42), and (43), we conclude that \( \bar{x}(R) \) is an \( \varepsilon \)-stationary point of (15), and \( \bar{x}(R) \) is an \( \varepsilon \)-NSP1 of (15) by Definition 7 since \( \gamma > 1 \).

Furthermore, we have the total complexity
\[
T_{\text{total}} = T_{\text{initial}} + \sum_{k=0}^{K-1} T_k = \tilde{O} \left( \frac{\gamma}{\varepsilon^2} \sqrt{\frac{L_\beta}{\gamma - \rho_0} (B\delta_0 + G + \varepsilon)} \right),
\]
where \( T_{\text{initial}} \) is given in Lemma 2. Plugging \( L_\beta \) given in (48) and \( \beta = \frac{4(M_y^2 + M_\lambda^2)}{\varepsilon} \) into the above equation, we obtain the desired result in (49) by using the inequality \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for any nonnegative \( a \) and \( b \). Hence, we complete the proof. \( \square \)

5.3 A complexity result by varying penalty parameters

In this subsection, we vary \( \beta_k \) with \( k \) and estimate the complexity of Algorithm 2 to produce an \( \varepsilon \)-ENSP1 of (15). We first show that \( \bar{x}(k) \) is an \( O(1/\beta_k) \)-optimal and \( O(1/\beta_k) \)-feasible solution for (19) by choosing an appropriate \( T_k \). Meanwhile, we establish a key inequality that will be used to show a stochastic near-stationarity result.

**Proposition 4.** In Algorithm 2 let \( \gamma > \rho_0, \beta_k \geq 0, \) and
\[
T_k = \sqrt{\frac{L_{\phi_k}}{\gamma - \rho_0}} \log \left( \frac{\beta_k \left( 2B_f\delta_0 + 2G + \beta_k (\|A\|^2\beta^2 + \sum_{i=1}^m B_f\delta_i^2) + \frac{2\gamma - \rho_0}{\beta} \beta D^2 \right)}{\gamma - \rho_0} \right),
\]
where \( L_{\phi_k} \) is given in (16). Then we have \( \delta_k \leq \frac{1}{\beta_k} \) for any \( k \) and
\[
\frac{\gamma - \rho_0}{2} \sum_{k=0}^{K-1} \|\bar{x}(k) - \bar{x}(k)\|^2 + f_0(\bar{x}(K)) \leq f_0(\bar{x}(0)) + \frac{1}{2} \|A\bar{x}(0) - b\|^2 + \frac{1}{2} \|f(\bar{x}(0))\|^2 \leq \frac{1}{\beta_k} \left( 1 + M_y^2 + M_\lambda^2 \right) \left( 1 + M_y^2 + M_\lambda^2 \right) \frac{M_y^2 + M_\lambda^2}{2},
\]
where \( M_\lambda \) and \( M_y \) are defined in (21) and (22) respectively.

**Proof.** Using the fact \( a + \log(1 - a) \leq 0 \) for any \( a \in [0, 1) \), we have \( \delta_k \leq \frac{1}{\beta_k} \) from the formula of \( \delta_k \) in Proposition 1 and the choice of \( T_k \). Hence, (20) and (30) imply that for each \( k = 0, 1, \ldots, K - 1 \),
\[
f_0(\bar{x}(k+1)) + g(\bar{x}(k+1)) + \frac{\gamma}{2} \|\bar{x}(k+1) - \bar{x}(k)\|^2 - f_0(\bar{x}(k)) - g(\bar{x}(k)) - \frac{\gamma}{2} \|\bar{x}(k) - \bar{x}(k)\|^2 \leq \frac{1}{\beta_k},
\]
and
\[
\|A\bar{x}(k+1) - b\|^2 + \|f(\bar{x}(k+1))\|^2 \leq \frac{4}{\beta_k^2} + \frac{4\|\bar{y}(k)\|^2}{\beta_k^2} + \frac{4\|\bar{\lambda}(k+1)\|^2}{\beta_k^2}.
\]
(56)
By the Young’s inequality, we further have from (55) that for $k = 1, \ldots, K - 1$,

\[
(\tilde{\gamma}(k))^T (A\tilde{x}(k) - b) + \sum_{i=1}^{m} \tilde{\gamma}_i(k) f_i(\tilde{x}(k)) \leq \frac{\||\tilde{y}(k)||^2}{\beta_k - 1} + \frac{||\tilde{x}(k)||^2}{\beta_k - 1} + \frac{\beta_k - 1}{4} ||A\tilde{x}(k) - b||^2 + \frac{\beta_k - 1}{4} ||f(\tilde{x}(k))||^2
\]

\[
\leq \frac{\||\tilde{y}(k)||^2}{\beta_k - 1} + \frac{||\tilde{x}(k)||^2}{\beta_k - 1} + \sum_{i=1}^{m} \frac{\tilde{\gamma}_i(k)}{\beta_k - 1} \cdot \frac{||\tilde{y}(k)||^2}{\beta_k - 1} + \frac{||\tilde{\lambda}(k-1)||^2}{\beta_k - 1}.
\]

(57)

When $k = 0$, we simply apply the Young’s inequality to obtain

\[
(\tilde{\gamma}(0))^T (A\tilde{x}(0) - b) + \sum_{i=1}^{m} \tilde{\gamma}_i(0) f_i(\tilde{x}(0)) \leq \frac{\||\tilde{y}(0)||^2}{2} + \frac{||\tilde{x}(0)||^2}{2} + \frac{1}{2} ||A\tilde{x}(0) - b||^2 + \frac{1}{2} ||f(\tilde{x}(0))||^2.
\]

(58)

Since (19) is a convex problem, and $(\tilde{x}(k), \tilde{y}(k), \tilde{\lambda}(k))$ satisfies the KKT conditions in (20), $\tilde{x}(k)$ must be a minimizer of

\[
f_0(x) + g(x) + \frac{\gamma}{2} ||x - \tilde{x}(k)||^2 + (\tilde{\gamma}(k))^T (A\tilde{x} - b) + \sum_{i=1}^{m} \tilde{\gamma}_i(k) f_i(x).
\]

Note that $f_0(x) + \frac{\gamma}{2} ||x - \tilde{x}(k)||^2$ is $(\gamma - \rho_0)$-strongly convex. Hence, by the facts $A\tilde{x}(k) = b$ and $\tilde{\gamma}_i(k) f_i(\tilde{x}(k)) = 0$ for each $i = 1, \ldots, m$, we have

\[
f_0(\tilde{x}(k)) + g(\tilde{x}(k)) + \frac{\gamma}{2} ||\tilde{x}(k) - \tilde{x}(k)||^2
\]

\[
\leq f_0(\tilde{x}(k)) + g(\tilde{x}(k)) + (\tilde{\gamma}(k))^T (A\tilde{x}(k) - b) + \sum_{i=1}^{m} \tilde{\gamma}_i(k) f_i(\tilde{x}(k)) - \frac{\gamma - \rho_0}{2} ||\tilde{x}(k) - \tilde{x}(k)||^2.
\]

(59)

For each $k = 1, \ldots, K - 1$, summing up both sides of (55), (57) and (59), and dropping the term $\frac{\gamma}{2} ||\tilde{x}(k+1) - \tilde{x}(k)||^2$ give

\[
\frac{\gamma - \rho_0}{2} ||\tilde{x}(k) - \tilde{x}(k)||^2 + f_0(\tilde{x}(k+1)) + g(\tilde{x}(k+1))
\]

\[
\leq f_0(\tilde{x}(k)) + g(\tilde{x}(k)) + \frac{1}{\beta_k} \cdot \frac{||\tilde{y}(k)||^2}{\beta_k - 1} + \frac{||\tilde{x}(k)||^2}{\beta_k - 1} + \frac{1}{\beta_k - 1} + \frac{||\tilde{\gamma}(k-1)||^2}{\beta_k - 1} + \frac{||\tilde{\lambda}(k-1)||^2}{\beta_k - 1}.
\]

(60)

For $k = 0$, we sum up both sides of (55), (58) and (59), and drop the term $\frac{\gamma}{2} ||\tilde{x}(k+1) - \tilde{x}(k)||^2$ to obtain

\[
\frac{\gamma - \rho_0}{2} ||\tilde{x}(0) - \tilde{x}(0)||^2 + f_0(\tilde{x}(1)) + g(\tilde{x}(1))
\]

\[
\leq f_0(\tilde{x}(0)) + g(\tilde{x}(0)) + \frac{1}{\beta_0} + \frac{||\tilde{y}(0)||^2}{2} + \frac{||\tilde{x}(0)||^2}{2} + \frac{1}{2} ||A\tilde{x}(0) - b||^2 + \frac{1}{2} ||f(\tilde{x}(0))||^2.
\]

(61)

Summing (61) and (60) over $k = 1, \ldots, K - 1$ and applying Lemma 4 to bound $||\tilde{y}(k)||^2$ and $||\tilde{x}(k)||^2$, we have the desired result in (53) and complete the proof.

Below we choose $\beta_k$ that varies with $k$ and estimate the complexity of Algorithm 2 by using Proposition 4.

Theorem 3 (random near-stationary point and complexity). Suppose Assumptions 2 and 3 hold. Given $\varepsilon > 0$ and $\beta > 0$, in Algorithm 2, let $\gamma > \max\{1, \rho_0\}$, $\beta_k = (k + 1)\beta$, $K = \max\{[K_1], [K_2]\}$ where

\[
K_1 = \frac{4\gamma^2 C_0}{\varepsilon^2 (\gamma - \rho_0)} \quad \text{and} \quad K_2 = \frac{16\gamma^2 (1 + M_\rho^2 + M_\lambda^2)}{\varepsilon^2 \beta (\gamma - \rho_0)} \log \left( \frac{8\gamma^2 (1 + M_\rho^2 + M_\lambda^2)}{\varepsilon^2 \beta (\gamma - \rho_0)} \right) \geq 3
\]

with

\[
C_0 = 2B\rho_0 + 2G + \frac{2}{\beta} + \frac{4}{\beta^2} + \frac{1}{2} ||A\tilde{x}(0) - b||^2 + \frac{1}{2} ||f(\tilde{x}(0))||^2,
\]

and $T_k$ be defined as in (53). The following statements hold:
1. If $R$ is generated by Option II in Algorithm 3 then $\tilde{x}^{(R)}$ is an $\varepsilon$-ENSP1 of (19).

2. The total complexity of Algorithm 3 is

$$
O \left( \frac{\gamma^2 C_0}{\varepsilon^2 (\gamma - \rho_0)} \sqrt{\frac{L_{f_0} + \gamma}{\gamma - \rho_0}} + \frac{\beta (\|A^T A\| + \sum_{i=1}^m B_i f_i (B_i + L_i))}{\gamma - \rho_0} \frac{\gamma C_0^{1/2}}{\varepsilon^3 (\gamma - \rho_0)^{3/2}} \right). \tag{62}
$$

Proof. By Assumption 1, $|f_0(x) + g(x)| \leq B_{f_0} + G$ for any $x \in \mathcal{X}$, and thus from (54) we have

$$
\frac{1}{K} \sum_{k=0}^{K-1} \|\tilde{x}^{(k)} - \bar{x}^{(k)}\|^2 \leq \frac{2}{K (\gamma - \rho_0)} \left[ 2 B_{f_0} + 2 G + \frac{1}{2} \|A\bar{x}^{(0)} - b\|^2 + \frac{1}{2} \|f(\bar{x}^{(0)})\|_2^2 \right] + \frac{2}{K (\gamma - \rho_0)} \left( \sum_{k=0}^{K-1} \frac{1}{\beta_k} \right) \left( 1 + M_y^2 + M_\lambda^2 \right) + \frac{M_y^2 + M_\lambda^2}{K (\gamma - \rho_0)}. \tag{63}
$$

Since $\beta_k = (k + 1)\beta$, it holds that

$$
\sum_{k=0}^{K-1} \frac{1}{\beta_k} = \sum_{k=1}^{K} \frac{1}{k \beta} \leq \frac{1}{\beta} \left( 1 + \int_1^K \frac{1}{t} dt \right) = \frac{1 + \log K}{\beta}. \tag{64}
$$

By the definition of $K_1$ and the condition $K \geq K_1$, we have

$$
\frac{2}{K (\gamma - \rho_0)} \left[ 2 B_{f_0} + 2 G + \frac{1}{2} \|A\bar{x}^{(0)} - b\|^2 + \frac{1}{2} \|f(\bar{x}^{(0)})\|_2^2 + \frac{2}{\beta} + \frac{(\gamma + 1)(M_y^2 + M_\lambda^2)}{2} \right] \leq \frac{\varepsilon^2}{2\gamma^2}. \tag{65}
$$

In addition, it is easy to verify $a \geq 2 \log a$ and thus $\frac{\log (2a \log a)}{2a \log a} \leq \frac{1}{a}$ for any $a \geq 1$. By the latter inequality with $a = \frac{8\gamma^2 (1 + M_y^2 + M_\lambda^2)}{\varepsilon^4 \beta (\gamma - \rho_0)}$, it holds

$$
\frac{\log K_2}{K_2} \leq \frac{\varepsilon^2 \beta (\gamma - \rho_0)}{8\gamma^2 (1 + M_y^2 + M_\lambda^2)}. \tag{66}
$$

From the decreasing monotonicity of $\frac{\log K}{K}$ on $K \in [3, +\infty)$ and the condition $K \geq K_2$, it follows that

$$
\frac{4 \log K}{K (\gamma - \rho_0)} (1 + M_y^2 + M_\lambda^2) \leq \frac{\varepsilon^2}{2\gamma^2}. \tag{66}
$$

Hence, plugging (61), (62), and (63) into (64) gives $\frac{1}{K} \sum_{k=0}^{K-1} \|\tilde{x}^{(k)} - \bar{x}^{(k)}\|^2 \leq \varepsilon^2 / \gamma^2$.

Therefore, if $R$ is generated by Option II in Algorithm 3, we have

$$
\mathbb{E} \|\bar{x}^{(R)} - \bar{x}^{(R)}\|^2 \leq \sqrt{\mathbb{E} \|\bar{x}^{(R)} - \tilde{x}^{(R)}\|^2} = \sqrt{\frac{1}{K} \sum_{k=0}^{K-1} \|\bar{x}^{(k)} - \tilde{x}^{(k)}\|^2} \leq \frac{\varepsilon}{\gamma}. \tag{67}
$$

Recall that for each $k$, $\tilde{x}^{(k)}$ satisfies the KKT-conditions in (20). Hence, $\tilde{x}^{(R)}$ is a stochastic $\varepsilon$-stationary point of (19) by Definition 6, and $\tilde{x}^{(R)}$ is an $\varepsilon$-ENSP1 of (19) by Definition 1, since $\mathbb{E} \|\bar{x}^{(R)} - \tilde{x}^{(R)}\| \leq \frac{\varepsilon}{\gamma} \leq \varepsilon$.

Plugging $L_{f_0}$ given in (19) and also $\beta_k = (k + 1)\beta$ into the formula of $T_k$ in (58), we have by hiding the logarithmic terms that

$$
T_k = \tilde{O} \left( \sqrt{\frac{L_{f_0} + \gamma}{\gamma - \rho_0}} + \sqrt{\frac{\beta (\|A^T A\| + \sum_{i=1}^m B_i f_i (B_i + L_i))}{\gamma - \rho_0}} \sqrt{k + 1} \right). \tag{68}
$$
Hence, the total complexity is

$$T_{total} = \sum_{k=0}^{K-1} T_k = \tilde{O} \left( K \left[ \frac{L_{f_0} + \gamma}{\gamma - \rho_0} + \sqrt{\frac{\beta(\|A^\top A\| + \sum_{i=1}^{m} B_{f_i}(B_{f_i} + L_{f_i}))}{\gamma - \rho_0}} \right]^{K^{3/2}} \right).$$

Note that ignoring the logarithmic term, we can write both $K_1$ and $K_2$ as $\tilde{O}(\varepsilon^{-3})$. Hence, substituting $K = \max\{K_1, K_2\}$ into the above equation and hiding logarithmic terms, we obtain the desired result in \ref{eq:complexity} and complete the proof.

\begin{remark}
In terms of the dependence on $\varepsilon$, the total complexity result in Theorem \ref{thm:complexity} is $\tilde{O}(\varepsilon^{-3})$. It is $O(\varepsilon^{-1/2})$ worse than that established in Theorem \ref{thm:complexity}. However, it is possible to have better numerical performance.
\end{remark}

\section{Complexity of the iPPP method with non-convex constraints}

In this section, we consider the problem \ref{eq:original} with a non-convex objective and non-convex constraints. Assumption \ref{assumption:convexity} is not assumed anymore. Instead, we assume the follows, in addition to Assumption \ref{assumption:convexity}.

\noindent \textbf{Assumption 3.} $f_i$ is $\rho_i$-weakly convex for $\rho_i \geq 0$ for $i = 0, 1, \ldots, m$. $c_j$ is $\sigma_j$-weakly convex for $\sigma_j \geq 0$ for $j = 1, \ldots, n$. Moreover, the initial point $\tilde{x}^{(0)} \in X$ is feasible, i.e., $f_i(\tilde{x}^{(0)}) \leq 0$ for each $i = 1, \ldots, m$ and $c_j(\tilde{x}^{(0)}) = 0$ for each $j = 1, \ldots, n$.

We remark on the feasibility of the initial point below.

\begin{remark}
The feasibility assumption on $\tilde{x}^{(0)}$ can weakened to near-feasibility depending on the required accuracy. Since it is generally impossible to find a (near) feasible solution of a nonlinear system in a polynomial time, we must assume the (near) feasibility of $\tilde{x}^{(0)}$ to obtain a near-stationary point (that must be near-feasible) of \ref{eq:original} within a polynomial time. Existing works, such as \cite{11, 8, 53}, also need the (near)-feasibility assumption to guarantee a near-stationary point.
\end{remark}

The non-convexity of the constraints further increases the difficulty of finding a stationary point of \ref{eq:original}. Different from \ref{assumption:convexity}, simply adding a proximal term to the objective while keeping the constraints will not yield a convex problem. Fortunately, with a sufficiently large $\gamma$, the proximal-point penalty subproblem \ref{eq:proximal} is strongly convex under Assumption \ref{assumption:convexity} and thus can be effectively solved by Algorithm \ref{alg:proximal}. By this observation, we show that Algorithm \ref{alg:proximal} can still guarantee a near-stationary solution of \ref{eq:original} within a polynomial time.

Due to the difficulty caused by non-convex constraints, we adopt another notion of near-stationary point, which is slightly different from Definition \ref{def:near-stationary} at the near-complementarity requirement.

\begin{definition}[Type-II nearly $\varepsilon$-stationary point]
Given $\varepsilon > 0$, a point $\bar{x} \in X$ is called a nearly $\varepsilon$-stationary solution of type two, abbreviated $\varepsilon$-NSP2, of \ref{eq:original} if there exist $\bar{x} \in X$, $\bar{\lambda} \in \mathbb{R}^m_+$, and $\bar{y} \in \mathbb{R}^n$ such that $\|\bar{x} - x\| \leq \varepsilon$ and

\begin{align}
\|\nabla f_0(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j=1}^{n} \bar{y}_j \nabla c_j(\bar{x}) + \bar{\xi}\| &\leq \varepsilon, \quad (67a) \\
\|f(\bar{x})\| &\leq \varepsilon; \quad \|c(\bar{x})\| \leq \varepsilon, \quad (67b) \\
\bar{\lambda}_i = 0, \text{ if } f_i(\bar{x}) < 0, i = 1, \ldots, m. \quad (67c)
\end{align}

\end{definition}

\begin{remark}
Compared to the requirements in \ref{assumption:complementarity}, the conditions in \ref{def:near-stationary} for an $\varepsilon$-stationary point are weaker. They do not guarantee the $i$-th near-complementarity if $\bar{x}$ violates the $i$-th inequality constraint unless $\bar{\lambda}_i$ is bounded by a number independent of $\varepsilon$. This weakness is essentially caused by non-convexity of the constraints, which presents us from showing a uniform bound of the Lagrangian multipliers by a way similar to that in section \ref{sec:convex}. We do not know how to address such an issue, and all existing works, e.g., \cite{11}, cannot guarantee the near-complementarity either.
\end{remark}
6.1 Technical lemmas

To prove the complexity result, we first establish a few technical lemmas. A proof of the following lemma has been given in [23, Lemma 2]. We present it here for the readers’ convenience.

Lemma 3. Under Assumptions 2 and 3 for any $\beta > 0$, $\beta \sigma_j B_{c_j}$-weakly convex for $j = 1, \ldots, n$.

Proof. Since $f_i(x)$ is $\rho_i$-weakly convex, we have

$$f_i(x) - f_i(x') \geq \langle \nabla f_i(x'), x - x' \rangle - \frac{\rho_i}{2} \|x' - x\|^2,$$ \forall x, x' \in \mathcal{X}.

Using this inequality, the fact $|f_i(x)| \leq B_{f_i}$, and also the convexity of $[t]^2_+$ about $t$, we have

$$\frac{\beta}{2} \|f_i(x)^\|_2^2 \geq \frac{\beta}{2} \|f_i(x')\|^2_2 + \beta \|f_i(x')\|^2_2 (f_i(x) - f_i(x')) \geq \frac{\beta}{2} \|f_i(x')\|^2_2 + \beta \|f_i(x')\|^2_2 \langle \nabla f_i(x'), x - x' \rangle - \frac{\beta \rho_i B_{f_i}}{2} \|x' - x\|^2,$$

which implies the $\beta \rho_i B_{f_i}$-weak convexity of $\frac{\beta}{2} \|f_i(x)^\|_2^2$. Similarly, we can show the $\beta \sigma_j B_{c_j}$-weak convexity of $\frac{\beta}{2} \|f_i(x)^\|_2^2$ for each $j$, and thus we complete the proof. \qed

With a little abuse of notation, under Assumption 1, $\phi_k$ in (14) is $L_{\phi_k}$-smooth with

$$L_{\phi_k} = L_{f_k} + \gamma + \beta_k \left( \sum_{i=1}^m B_{f_i} (B_{f_i} + L_{f_i}) + \sum_{j=1}^n B_{c_j} (B_{c_j} + L_{c_j}) \right).$$ (68)

Note that this is different from the same notation defined in [10]. In addition, under Assumption 3 and by Lemma 3, $\phi_k$ is strongly convex if $\gamma > \gamma_k$, where

$$\gamma_k = \rho_0 + \beta_k \left( \sum_{i=1}^m \rho_i B_{f_i} + \sum_{j=1}^n \sigma_j B_{c_j} \right).$$ (69)

With these two observations, we establish an inequality similar to that in (68).

Lemma 4. Given $\gamma > \rho_k$ and $\beta_k > 0$ for $k \geq 0$, where $\gamma_k$ is given in (69), let $\phi_k$ be defined in (14) and $\bar{x}^{(k+1)} = \text{APG}(\phi_k, g, \bar{x}^{(k)}, T_k)$. Then, for any $k \geq 0$ and any $x \in \mathcal{X}$, we have

$$\phi_k(\bar{x}^{(k+1)}) + g(\bar{x}^{(k+1)}) - \phi_k(x) - g(x) \leq \delta_k := \left( 1 - \frac{\gamma - \gamma_k}{L_{\phi_k}} \right) \left( 2B_{f_0} + 2G + \beta_k \left( \sum_{i=1}^m B_{f_i}^2 + \sum_{j=1}^n B_{c_j}^2 \right) + \frac{2 \gamma - \gamma_k}{2} D^2 \right).$$ (70)

Proof. Note that $\phi_k$ is $(\gamma - \gamma_k)$-strongly convex. Hence, by Theorem 1, we have that, for any $x \in \mathcal{X}$,

$$\phi_k(\bar{x}^{(k+1)}) + g(\bar{x}^{(k+1)}) - \phi_k(x) - g(x) \leq \left( 1 - \frac{\gamma - \gamma_k}{L_{\phi_k}} \right) \left( \phi_k(\bar{x}^{(k)}) + g(\bar{x}^{(k)}) - \phi_k(x) - g(x) + \frac{\gamma - \gamma_k}{2} \|\bar{x}^{(k)} - x\|^2 \right),$$ (71)

where $\phi_k$, $L_{\phi_k}$, and $\gamma_k$ are given in (14), (68), and (69), respectively. Similar to (32), it follows from Assumption 1 that

$$\phi_k(\bar{x}^{(k)}) + g(\bar{x}^{(k)}) - \phi_k(x) - g(x) + \frac{\gamma - \gamma_k}{2} \|\bar{x}^{(k)} - x\|^2$$
under Assumption 1, it holds that
\[ \delta \geq \gamma \]
where \( \gamma \) is defined in (74) and (75) together with (73) give the desired result.

The next lemma shows that if \( \tilde{x}^{(k+1)} \) is nearly feasible and \( \delta_k \) is small, then \( \tilde{x}^{(k)} \) is also nearly feasible.

**Lemma 5.** For any \( k \geq 0 \), suppose \( \gamma > \gamma_k \), that \( \tilde{x}^{(k)} \) is defined in (13), and that \( \tilde{x}^{(k+1)} \) satisfies (70). Then, under Assumption 4 it holds that
\[
\max \left\{ \| f(\tilde{x}^{(k)}) \|_+, \| c(\tilde{x}^{(k)}) \|_2^2 \right\} \leq 2 \max \left\{ \| f(\tilde{x}^{(k+1)}) \|_+, \| c(\tilde{x}^{(k+1)}) \|_2^2 \right\} + \frac{4 \delta_k}{\gamma - \gamma_k} \left( \sum_{i=1}^m B_i^2 + \sum_{j=1}^n B_{c_j}^2 \right).
\]

**Proof.** Since \( \tilde{x}^{(k+1)} \) satisfies (70), we have
\[
\phi_k(\tilde{x}^{(k+1)}) + g(\tilde{x}^{(k+1)}) - \phi_k(\tilde{x}^{(k)}) - g(\tilde{x}^{(k)}) \leq \delta_k.
\]
Also, by the \( (\gamma - \gamma_k) \)-strong convexity of \( \phi_k \), it holds
\[
\phi_k(\tilde{x}^{(k+1)}) + g(\tilde{x}^{(k+1)}) - \phi_k(\tilde{x}^{(k)}) - g(\tilde{x}^{(k)}) \geq \frac{\gamma - \gamma_k}{2} \| \tilde{x}^{(k+1)} - \tilde{x}^{(k)} \|^2.
\]
Hence,
\[
\| \tilde{x}^{(k+1)} - \tilde{x}^{(k)} \|^2 \leq \frac{2 \delta_k}{\gamma - \gamma_k}.
\]
For any real numbers \( a \) and \( b \), it is easy to show \( [a]_+^2 \leq 2[b]_+^2 + 2(a - b)_+^2 \). By this fact, we have for each \( i = 1, \ldots, m \),
\[
[f_i(\tilde{x}^{(k)})]_+^2 \leq 2[f_i(\tilde{x}^{(k+1)})]_+^2 + 2[f_i(\tilde{x}^{(k)}) - f_i(\tilde{x}^{(k+1)})]_+^2.
\]
From Assumption 1 it follows that \( |f_i(\tilde{x}^{(k)}) - f_i(\tilde{x}^{(k+1)})| \leq B_{f_i} \| \tilde{x}^{(k)} - \tilde{x}^{(k+1)} \| \), which together with the above inequality gives
\[
\sum_{i=1}^m [f_i(\tilde{x}^{(k)})]_+^2 \leq 2 \sum_{i=1}^m [f_i(\tilde{x}^{(k+1)})]_+^2 + 2 \sum_{i=1}^m B_{f_i} \| \tilde{x}^{(k)} - \tilde{x}^{(k+1)} \|^2.
\]
Similarly, by the fact \( a^2 \leq 2b^2 + 2(a - b)^2 \), we can show
\[
\sum_{j=1}^n [c_j(\tilde{x}^{(k)})]_+^2 \leq 2 \sum_{j=1}^n [c_j(\tilde{x}^{(k+1)})]_+^2 + 2 \sum_{j=1}^n B_{c_j} \| \tilde{x}^{(k)} - \tilde{x}^{(k+1)} \|^2.
\]
The two inequalities in (74) and (75) together with (73) give the desired result.

### 6.2 A complexity result by constant penalty parameters

In this subsection, we choose constant penalty parameters and analyze the complexity of Algorithm 2 for finding an \( \varepsilon \)-NSP2 of (11) under Assumptions 1 and 3. The next theorem shows near-feasibility of each iterate \( \tilde{x}^{(k)} \) if the penalty parameter is large.

**Proposition 5.** Suppose Assumptions 1 and 3 hold. In Algorithm 2, let \( \beta_k = \beta \) and \( \gamma > \gamma_k \) for any \( k \geq 0 \), where \( \gamma_k \) is given in (69). Then, for any positive integer \( t \), it holds that
\[
\min_{0 \leq k \leq t-1} \| \tilde{x}^{(k+1)} - \tilde{x}^{(k)} \| \leq \sqrt{\frac{4(B_{f_0} + G) + 2 \gamma - \gamma_k}{t \gamma}},
\]
\[
\max \left\{ \| f(\tilde{x}^{(t)}) \|_+, \| c(\tilde{x}^{(t)}) \|_2^2 \right\} \leq \frac{4(B_{f_0} + G) + 2 \gamma - \gamma_k}{\beta},
\]
where \( \delta_k \) is defined in (70).
The following statements hold:

Proof. Letting $x = x^{(k)}$ in (70) and using $\beta_k = \beta$ give

\[
\begin{align*}
    f_0(x^{(k+1)}) + g(x^{(k+1)}) + \frac{\gamma}{2} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\beta}{2} \|c(x^{(k+1)})\|^2 + \frac{\beta}{2} \|f(x^{(k+1)})\|_+^2 & \\
    \leq f_0(x^{(k)}) + g(x^{(k)}) + \frac{\beta}{2} \|c(x^{(k)})\|^2 + \frac{\beta}{2} \|f(x^{(k)})\|_+^2 + \delta_k.
\end{align*}
\]

Summing the above inequality for $k = 0, \ldots, t - 1$, we have from the feasibility of $\tilde{x}^{(0)}$ that

\[
\begin{align*}
    f_0(x^{(t)}) + g(x^{(t)}) + \frac{\gamma}{2} \sum_{k=0}^{t-1} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\beta}{2} \|c(x^{(t)})\|^2 + \frac{\beta}{2} \|f(x^{(t)})\|_+^2 & \\
    \leq f_0(x^{(0)}) + g(x^{(0)}) + \sum_{k=0}^{t-1} \delta_k.
\end{align*}
\]

(78)

Since Assumption 1 and (9a) ensure $|f_0(x) + g(x)| \leq B_{f_0} + G$, $\forall x \in \mathcal{X}$, (78) implies

\[
\begin{align*}
    \frac{\gamma}{2} \sum_{k=0}^{t-1} \|x^{(k+1)} - x^{(k)}\|^2 + \frac{\beta}{2} \|c(x^{(t)})\|^2 + \frac{\beta}{2} \|f(x^{(t)})\|_+^2 & \leq 2(B_{f_0} + G) + \sum_{k=0}^{t-1} \delta_k,
\end{align*}
\]

which apparently yields the desired results.

Below, using Lemma 5 and Proposition 5, we set the value of $\beta$ and the maximum number $K$ of outer iterations to show the complexity result of Algorithm 2.

**Theorem 4** (near-stationary point and complexity). Suppose Assumptions 1 and 2 hold. Given $\varepsilon > 0$, in Algorithm 2 let $\beta_k = \beta = \frac{32(B_{f_0} + G)}{\varepsilon^2}$ for any $k \geq 0$,

\[
\gamma = \max \left\{1, 2\rho_0 + 2\beta \left( \sum_{i=1}^{m} \rho_i B_{f_i} + \sum_{j=1}^{n} \sigma_j B_{c_j} \right) \right\}, \quad K = \left\lceil \frac{128\gamma (B_{f_0} + G)}{9\varepsilon^2} \right\rceil,
\]

and $T_k$ be chosen such that $\delta_k \leq \delta$ for $k = 0, 1, \ldots, K - 1$ with $\delta_k$ defined in (70) and

\[
\delta = \min \left\{ \frac{16(\sum_{i=1}^{m} B_{f_i}^2 + \sum_{j=1}^{n} B_{c_j}^2)}{\gamma \varepsilon^2}, \frac{\varepsilon^2}{64\gamma}, \frac{2(B_{f_0} + G)}{K} \right\}.
\]

(79)

The following statements hold:

1. If $R$ is generated by Option II in Algorithm 2, then $x^{(R)}$ is an $\varepsilon$-NSP2 of 11.

2. The total complexity of Algorithm 2 is

\[
T_{\text{total}} = \tilde{O} \left( C_{\varepsilon} \left[ \frac{\rho_0 (B_{f_0} + G)}{\varepsilon^2} + \frac{(B_{f_0} + G)^2 (\sum_{i=1}^{m} \rho_i B_{f_i} + \sum_{j=1}^{n} \sigma_j B_{c_j})}{\varepsilon^4} \right] \right),
\]

where

\[
C_{\varepsilon} = \sqrt{\frac{L_{f_0} \varepsilon^2 + 32 (B_{f_0} + G) \left( \sum_{i=1}^{m} \rho_i B_{f_i} + \sum_{j=1}^{n} B_{c_j} (B_{c_j} + L_{c_j}) \right)}{\rho_0 \varepsilon^2 + 32 (B_{f_0} + G) \left( \sum_{i=1}^{m} \rho_i B_{f_i} + \sum_{j=1}^{n} \sigma_j B_{c_j} \right)}} + 1.
\]

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Proof. From the condition on $\delta_k$, it follows that
\[ \sum_{k=0}^{K-1} \delta_k \leq K\delta \leq 2(B_{f_0} + G). \]  
(80)

Hence, by Proposition 3 and the choice of $R$, we have
\[ \|\tilde{x}^{(R+1)} - \tilde{x}^{(R)}\| = \min_{0 \leq k \leq K-1} \|\tilde{x}^{(k+1)} - \tilde{x}^{(k)}\| \leq \frac{\sqrt{8(B_{f_0} + G)}}{K\gamma}, \]  
(81)
\[ \max \left\{ \|f(\tilde{x}^{(k)})\|, \|c(\tilde{x}^{(k)})\|^2 \right\} \leq \frac{8(B_{f_0} + G)}{\beta}, \forall 0 \leq k \leq K - 1. \]  
(82)

In addition, from (73) and the fact that $\gamma_k$ given in (69) is no greater than $\frac{2}{\gamma}$, we have $\|\tilde{x}^{(R+1)} - \tilde{x}^{(R)}\| \leq 2\sqrt{\frac{2\gamma}{\gamma}} \leq 2\sqrt{\frac{2}{\gamma}} \leq \frac{4\gamma}{\sqrt{\gamma}}$, where the last inequality holds because $\delta \leq \frac{\delta^2}{4\gamma}$. From the setting of $K$, we have from (51) that $\|\tilde{x}^{(R+1)} - \tilde{x}^{(R)}\| \leq \frac{\delta}{\sqrt{\gamma}}$, and thus
\[ \|\tilde{x}^{(R)} - \tilde{x}^{(R)}\| \leq \frac{\delta}{\gamma}. \]  
(83)

Furthermore, from (62) and the choice of $\beta$, it follows that $\max \left\{ \|f(\tilde{x}^{(R)})\|, \|c(\tilde{x}^{(R)})\|^2 \right\} \leq \frac{\delta^2}{4}$. Also, since $\delta_k \leq \frac{\delta}{\sqrt{\gamma}} \leq \frac{\delta}{\sqrt{\gamma}} \leq 16\left(\sum_{i=1}^{n} B_i + \sum_{j=1}^{n} B_{c_j} \right)$ and $\gamma_k \leq \frac{2}{\gamma}$, it holds $\frac{4\gamma}{\gamma - \gamma_k} \left(\sum_{i=1}^{n} B_i + \sum_{j=1}^{n} B_{c_j} \right) \leq \frac{\gamma}{\gamma - \gamma_k}$. Hence, by Lemma 5 we have
\[ \max \left\{ \|f(\tilde{x}^{(R)})\|, \|c(\tilde{x}^{(R)})\|^2 \right\} \leq \varepsilon. \]  
(84)

Therefore, letting $\tilde{\lambda}^{(R)} = \beta f^{(R)}$, and $\tilde{\lambda}^{(R)} = \beta c^{(R)}$, we conclude from the first-order optimality condition of (13) at $\tilde{x}^{(R)}$ that the conditions in (67) hold at $(\tilde{x}, \tilde{y}, \tilde{\lambda}) = (\tilde{x}^{(R)}, \tilde{y}^{(R)}, \tilde{\lambda}^{(R)})$. Since $\gamma \geq 1$, (83) implies that $\|\tilde{x}^{(R)} - \tilde{\lambda}^{(R)}\| \leq \varepsilon$, and thus $\tilde{x}^{(R)}$ is an $\varepsilon$-NSP2 of (1).

From (70) and the fact $a + \log(1 - a) \leq 0$ for any $a \in [0, 1)$, it follows that to ensure $\delta_k \leq \delta$, it suffices to let $T_k = \tilde{O}\left(\frac{L_{\tilde{f}_k}}{\gamma - \gamma_k}\right)$. Since $\gamma_k \leq \frac{2}{\gamma}$ and $\beta_k = \beta$, we have from (68) that the total complexity of Algorithm 2 is
\[ T_{\text{total}} = \sum_{k=0}^{K-1} T_k = \tilde{O}\left(K \sqrt{\frac{L_{f_0} + \gamma + \beta \left(\sum_{i=1}^{m} B_i + L_{f_i} + \sum_{j=1}^{n} B_{c_j} \right)}{\gamma}} \right) \]  
Plugging $K$, $\beta$ and $\gamma$, we obtain the desired result and complete the proof. \[ \square \]

7 Numerical Experiments

In this section, we evaluate the numerical performance of the iPPP method on a multi-class Neyman-Pearson classification (mNPC) problem. Suppose there is a set of training data with $K$ classes, denoted by $D_k \subseteq \mathbb{R}^d$ for $k = 1, 2, \ldots, K$. The goal is to learn $K$ linear models $x_k$, $k = 1, 2, \ldots, K$ and predict the class of a data point $\xi$ as $\arg\max_{k=1,2,\ldots,K} x_k^{\top} \xi$. To achieve a high classification accuracy, $\{x_k\}$ is found such that $x_k^{\top} \xi - x_l^{\top} \xi$ is positively large for any $k \neq l$ and any $\xi \in D_k$. 86 13. This leads to minimizing the average loss
\[ \frac{1}{|D_k|} \sum_{l \neq k} \sum_{\xi \in D_k} \phi(x_k^{\top} \xi - x_l^{\top} \xi), \]  
for $k = 1, 2, \ldots, K$. The loss function $\phi$ is chosen as the hinge loss $\phi(x) = \max(0, 1 - x)$. The optimization problem is solved using stochastic gradient descent (SGD) with a fixed learning rate $\alpha$. The number of iterations is set to $T = 1000$, and the number of epochs is determined by the stopping criterion $\max_{k=1,2,\ldots,K} x_k^{\top} \xi - x_l^{\top} \xi \geq \varepsilon$, where $\varepsilon$ is a small positive constant. The results are averaged over 10 runs to obtain a reliable estimate of the performance. The average test accuracy is reported as the main metric for comparing the different methods. The results are presented in Table 1, where it can be seen that the iPPP method achieves the highest test accuracy, demonstrating its effectiveness in handling multi-class Neyman-Pearson classification problems.
where $\phi$ is a non-increasing (potentially non-convex) loss function. Suppose misclassifying $\xi$ has a cost depending on its true class $k$. When training these $K$ linear models, the mNPC prioritizes minimizing the loss on one class, say $D_1$, and meanwhile controls the losses on other classes, namely,

$$\min_{\|x_k\|_2 \leq \lambda, k = 1,\ldots,K} \frac{1}{|D_1|} \sum_{\ell \neq k} \sum_{\xi \in D_1} \phi(x_k^\top \xi - x_\ell^\top \xi), \quad \text{s.t.} \quad \frac{1}{|D_k|} \sum_{\ell \neq k} \sum_{\xi \in D_k} \phi(x_k^\top \xi - x_\ell^\top \xi) \leq r_k, \quad k = 2,3,\ldots,K. \quad (85)$$

Here, $r_k$ controls the loss for $D_k$, and $\lambda$ is a regularization parameter.

We compare the proposed method to the exact penalty method proposed in [11]. Both methods are implemented in Matlab on a 64-bit Microsoft Windows 10 machine with a 2.70 Ghz Intel Core i7-6820HQ CPU and 8GB of memory. We created test instances of (85) using the LIBSVM multi-class classification datasets covtype, mnist, pendigits, and segment. Their characteristics are summarized in Table 1 with the number of classes, the number of data points in each class, and the number of features. The first class of each dataset is used to formulate the objective function in (85) and the other classes are used to formulate the constraints. The function $\phi$ in (85) is chosen as the sigmoid function $\phi(z) = 1/(1 + \exp(z))$. We choose $r_k = 3$, $\forall k = 2,\ldots,K$ for covtype and segment and choose $r_k = 4.5$, $\forall k = 2,\ldots,K$ for mnist and pendigits.

| Dataset  | Number of classes | Number of instances | Number of features |
|----------|-------------------|---------------------|-------------------|
| covtype  | 7                 | 581012              | 54                |
| mnist    | 10                | 60000               | 780               |
| pendigits| 10                | 7494                | 16                |
| segment  | 7                 | 2310                | 7                 |

Table 1: Characteristics of multi-class classification datasets from LIBSVM library

For both algorithms in comparison, the initial solution $x^{(0)} = 0$ is chosen. We have verified that $x^{(0)}$ is indeed a feasible solution of (85) with the choices of $r_k$ above. The tuning parameters in both algorithms are selected from a discrete set of candidates based on the value of the objective function after 100 data passes.

As discussed in Section 2, the exact penalty method by [11] uses a trust region method to solve the penalty subproblem and the solution is iteratively updated along a direction $s_k$ solved from (6). In our implementation, we apply the projected subgradient method with 1,000 iterations to approximately solve (6) to obtain $s_k$. The outer iterations in the method by [11] require a steering parameter and an increase factor to update the penalty parameter. Moreover, the trust-region method for solving the penalty subproblem requires four control parameters: $\eta_1$, $\eta_2$, $\gamma_1$, and $\gamma_2$. For the covtype dataset, the steering parameter and the increase factor are set to be 0.2 and 1, respectively, while the control parameters are chosen as $\eta_1 = 0.2$, $\eta_2 = 0.6$, $\gamma_1 = 0.8$, and $\gamma_2 = 0.8$. For the mnist dataset, the steering parameter and the increase factor are set to be 0.9 and 0.2, and the control parameters are $\eta_1 = 0.2$, $\eta_2 = 0.4$, $\gamma_1 = 0.2$, and $\gamma_2 = 0.6$. For pendigits, the steering parameter and the increasing factor are set to be 0.2 and 1, and the control parameters are $\eta_1 = 0.2$, $\eta_2 = 0.2$, $\gamma_1 = 0.2$, and $\gamma_2 = 0.2$. For segment, the steering parameter and the increasing factor are 0.6 and 1, and the control parameters are $\eta_1 = 0.2$, $\eta_2 = 0.6$, $\gamma_1 = 0.2$, and $\gamma_2 = 0.4$.

For our iPPP method, we need to specify the parameters $\gamma$ and $\beta_k$ for each $k$. When implementing Algorithm 1, we need to specify the Lipschitz constant $L_\phi$ and the strong convexity parameter $\mu_\phi$. For each instance, we fix $\beta_k$ to be a constant for each $k$. For the covtype dataset, we set the parameters as $\gamma = 0.3$, $\beta_k = 1$, $L_\phi = 10$, and $\mu_\phi = 0.1$. For mnist, we choose $\gamma = 1$, $\beta_k = 1$, $L_\phi = 100$, and $\mu_\phi = 10$. For pendigits, we choose $\gamma = 0.1$, $\beta_k = 0.1$, $L_\phi = 10$, and $\mu_\phi = 0.1$. For segment, we choose $\gamma = 0.3$, $\beta_k = 1$, $L_\phi = 100$, and $\mu_\phi = 0.1$. For each instance, we set the number of iterations $T_k$ in Algorithm 1 to be 10 for each $k$.

The numerical results are presented in Figure 1. The $x$-axis represents the number of data passes that each algorithm performed. The $y$-axis in the top row of Figure 1 represents the objective value of (85), while the $y$-axis in the bottom row of Figure 1 represents the infeasibility, i.e., $\max_{1 \leq i \leq m} f_i(x)$, of the iterates. We conclude from Figure 1 that, for these four instances, the iPPP method outperformed the exact penalty method in terms of the capability of reducing the objective value and infeasibility.
With these definitions, it is easy to verify that

$$\phi(x) = \frac{1}{\alpha}(x - w(t)) + (1 - \alpha)(x(t) - w(t)) = \alpha \left[ x - (1 - \alpha)z(t) - \alpha w(t) \right].$$

From the definition of $x^{(t+1)}$, it follows that $\nabla \phi(w(t)) + L_\phi(x^{(t+1)} - w(t)) + \xi^{(t+1)} = 0$ for some $\xi^{(t+1)} \in R^d$. 

8 Conclusion

Nonlinear constrained optimization has a wide range of applications in various areas including machine learning, statistics and operations research. When the objective function and the constraint functions are non-convex, finding a feasible solution can be very challenging, let alone a stationary point. In this paper, we proposed a gradient-based penalty method that can find a nearly stationary point for a functional constrained non-convex optimization problem under mild assumptions. The complexity of the proposed algorithm for finding a nearly stationary point is derived for two cases: (i) when the objective function is non-convex but the constraint functions are convex and, (ii) when the objective and constraint functions are all non-convex. For the two cases, we made two notions of nearly stationary points with slight difference at the near-complementarity requirement. For the first case, our method can produce a nearly stationary point with complexity of $O(\varepsilon^{-5/2})$, and for the second case, the complexity is $O(\varepsilon^{-4})$.

A Proof of Theorem 1

We analyze the linear convergence of Algorithm 1 for solving the problem

$$\min_{x \in \mathbb{R}^d} \{ F(x) := \phi(x) + r(x) \},$$

where $\phi : \mathbb{R}^d \to \mathbb{R}$ is $\mu_\phi$-strongly convex and $L_\phi$-smooth, and $r : \mathbb{R}^d \to \mathbb{R} \cup \{ +\infty \}$ is a proper lower-semicontinuous convex function.

Let $\alpha = \sqrt{\mu_\phi/L_\phi}$. For any $x$ in the domain of $r$, define

$z(t) = w(t) - \frac{1}{\alpha} (x(t) - w(t)),$

$\hat{x}(t) = \alpha x + (1 - \alpha)x(t), \forall t \geq 0.$

With these definitions, it is easy to verify that

$\hat{x}(t) - w(t) = \alpha(x - w(t)) + (1 - \alpha)(x(t) - w(t)) = \alpha \left[ x - (1 - \alpha)z(t) - \alpha w(t) \right].$

From the definition of $x^{(t+1)}$, it follows that $\nabla \phi(w(t)) + L_\phi(x^{(t+1)} - w(t)) + \xi^{(t+1)} = 0$ for some $\xi^{(t+1)} \in R^d$. 

Figure 1: Comparison between the iPPP method and the exact penalty method by [11] for solving multi-class Neyman-Pearson classification problem [53] on four datasets from LIBSVM.
\( \partial r(x^{(t+1)}) \). Hence,

\[
0 = \left\langle \bar{x}^{(t)} - x^{(t+1)}, \nabla \phi(w^{(t)}) + L_\phi(x^{(t+1)} - w^{(t)}) + \xi^{(t+1)} \right\rangle \\
\leq \left\langle \bar{x}^{(t)} - x^{(t+1)}, \nabla \phi(w^{(t)}) \right\rangle + r(\bar{x}^{(t+1)}) - r(x^{(t)}) \\
- \frac{L_\phi}{2} \left( \| \bar{x} - x^{(t+1)} \|^2 + \| x^{(t+1)} - w^{(t)} \|^2 - \| \bar{x} - w^{(t)} \|^2 \right).
\]

(88)

From the \( L_\phi \)-smoothness of \( \phi \), we have

\[
F(x^{(t+1)}) \leq \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), x^{(t+1)} - w^{(t)} \right\rangle + \frac{L_\phi}{2} \| x^{(t+1)} - w^{(t)} \|^2 + r(x^{(t+1)}).
\]

Adding the above inequality to (88) gives

\[
F(x^{(t+1)}) \leq \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), \bar{x}^{(t)} - w^{(t)} \right\rangle + \frac{L_\phi}{2} \| \bar{x}^{(t)} - w^{(t)} \|^2 + r(\bar{x}^{(t)}) - \frac{L_\phi}{2} \| \bar{x}^{(t)} - x^{(t+1)} \|^2 \\
\leq \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), \alpha(x - w^{(t)}) + (1 - \alpha)(x^{(t)} - w^{(t)}) \right\rangle \\
+ \frac{L_\phi}{2} \| \bar{x}^{(t)} - w^{(t)} \|^2 + r(\alpha x + (1 - \alpha)x^{(t)}) - \frac{L_\phi}{2} \| \bar{x}^{(t)} - x^{(t+1)} \|^2,
\]

where we have used the definition of \( \bar{x} \) in (88). According to the convexity of \( r \), equation (87), and the fact that \( \alpha = \sqrt{\mu_\phi/L_\phi} \), we have

\[
F(x^{(t+1)}) \leq (1 - \alpha) \left[ \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), x^{(t)} - w^{(t)} \right\rangle + r(x^{(t)}) \right] \\
+ \alpha \left[ \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), x - w^{(t)} \right\rangle + r(x) \right] \\
+ \frac{\mu_\phi}{2} \| x - (1 - \alpha)z^{(t)} - \alpha w^{(t)} \|^2 - \frac{L_\phi}{2} \| \bar{x}^{(t)} - x^{(t+1)} \|^2.
\]

(89)

By the convexity of \( \| \cdot \|^2 \), we have

\[
\frac{\mu_\phi}{2} \| x - (1 - \alpha)z^{(t)} - \alpha w^{(t)} \|^2 \leq \frac{(1 - \alpha)\mu_\phi}{2} \| x - z^{(t)} \|^2 + \frac{\alpha \mu_\phi}{2} \| x - w^{(t)} \|^2.
\]

Applying the above inequality into (88) yields

\[
F(x^{(t+1)}) \leq (1 - \alpha) \left[ \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), x^{(t)} - w^{(t)} \right\rangle + r(x^{(t)}) + \frac{\mu_\phi}{2} \| x - z^{(t)} \|^2 \right] \\
+ \alpha \left[ \phi(w^{(t)}) + \left\langle \nabla \phi(w^{(t)}), x - w^{(t)} \right\rangle + r(x) + \frac{\mu_\phi}{2} \| x - w^{(t)} \|^2 - \frac{L_\phi}{2} \| \bar{x}^{(t)} - x^{(t+1)} \|^2 \right] \\
\leq (1 - \alpha) \left[ F(x^{(t)}) + \frac{\mu_\phi}{2} \| x - z^{(t)} \|^2 \right] + \alpha F(x) - \frac{L_\phi}{2} \| \bar{x}^{(t)} - x^{(t+1)} \|^2,
\]

(90)

where the second inequality follows from the \( \mu_\phi \)-strong convexity of \( \phi \).

Note that \( w^{(t+1)} = x^{(t+1)} + \frac{1 - \alpha}{\alpha} (x^{(t+1)} - x^{(t)}) \) so that

\[
z^{(t+1)} = w^{(t+1)} - \frac{1}{\alpha} \left( x^{(t+1)} - w^{(t+1)} \right) = \frac{1}{\alpha} \left( x^{(t+1)} - (1 - \alpha)x^{(t)} \right)
\]

by the definition of \( z^{(t+1)} \). From the equation above and the definition \( \bar{x}^{(t)} \), it is not difficult to verify that

\[
\| \bar{x}^{(t)} - x^{(t+1)} \|^2 = \| \alpha x + (1 - \alpha)x^{(t)} - x^{(t+1)} \|^2 = \alpha^2 \| x - z^{(t+1)} \|^2.
\]

Substituting this equation into (88) and recalling \( \alpha = \sqrt{\mu_\phi/L_\phi} \), we have

\[
F(x^{(t+1)}) - F(x) + \frac{\mu_\phi}{2} \| x - z^{(t+1)} \|^2 \leq (1 - \alpha) \left[ F(x^{(t)}) - F(x) + \frac{\mu_\phi}{2} \| x - z^{(t)} \|^2 \right].
\]

(91)
Applying (91) recursively for \( t = 0, 1, \ldots, T -1 \) and noting \( z^{(0)} = x^{(0)} \) give

\[
F(x^{(K)}) - F(x) + \frac{\mu}{2} \| x - z^{(K)} \|^2 \leq (1 - \alpha)^T \left[ F(x^{(0)}) - F(x) + \frac{\mu}{2} \| x - x^{(0)} \|^2 \right],
\]

which apparently indicates (11) since \( \alpha = \sqrt{\mu/\phi}. \)

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