Abstract

By introducing an appropriate parent action and considering a perturbative approach, we establish, up to fourth order terms in the field and for the full range of the coupling constant, the equivalence between the noncommutative Yang-Mills-Chern-Simons theory and the noncommutative, non-Abelian Self-Dual model. In doing this, we consider two different approaches by using both the Moyal star-product and the Seiberg-Witten map.

1 Introduction

The duality between the (2+1)-dimensional Maxwell-Chern-Simons (MCS) [1] and Self-Dual (SD) [2] Abelian models has been established long time ago in [3], where a parent action for both theories was introduced (see also [4] for an approach introducing a master Lagrangian which has a gauge invariance in all fundamental fields).

In view of this, it is natural to investigate if such equivalence could be extended to the non-Abelian case, by considering the Yang-Mills-Chern-Simons (YMCS) theory and the non-Abelian SD model. However, the non-Abelian situation is more involved, and the duality has been established only for the weak coupling regime [5] (see also [6] for an approach using Hamiltonian techniques). In addition, it has been argued in [7]...
that the use of a master action in the non-Abelian situation is ineffective since YMCS and SD happen to be dual to non-local theories each. The non-Abelian situation has also been tackled in [8] from a different point of view, by performing a duality mapping which is realized by an iterative embedding of Noether counterterms.

Recently, the parent action method has been brought back in [9] by considering a novel perturbative analysis which, for the full range of coupling constant, has established that the parent action actually interpolates YMCS with a dual theory whose action is non-Abelian Self-Dual model up to fourth order in the field, thus extending the proof in [3] to the non-Abelian case. In this formalism, the fourth order terms are expected to be non-local.

The present paper deals with the noncommutative (NC) extension of the duality between YMCS theory and non-Abelian SD model. In general, NC versions of the usual quantum field theories are obtained by replacing in all Lagrangians the usual product with the Groenewold-Moyal star-product [10][11] of the form

$$g(x) \star h(x) = \exp \left[ \frac{i}{2} \theta^{\alpha\beta} \partial^x_\alpha \partial^x_\beta \right] g(x) h(x) ,$$

where $g(x)$ and $h(x)$ are arbitrary functions and $\theta^{\alpha\beta}$ is an antisymmetric constant tensor.

In recent years, NC field theories have generated a great deal of attention, due to the fact that they arise as low-energy descriptions of string backgrounds with antisymmetric tensor fields [12] (see [13][14] for reviews and additional references). In particular, an important result in [12] is that there exists a mapping, the Seiberg-Witten Map (SWM), which interpolates between a gauge theory and its NC counterpart in such a way that gauge orbits are mapped into NC gauge orbits. The SWM is unique in the lowest non-trivial order of the NC parameter.

The purpose of this paper is to investigate if the results regarding the equivalence between YMCS theory and non-Abelian SD model can be extended to the NC case. In order to do this, we will first introduce a suitable master action in which all usual products are replaced with the star-product, and consider the parent action method under a perturbative approach, as in [9]. In doing this, we will explicitly show that, in fact, the action for the NC YMCS theory is equivalent to that of the NC non-Abelian SD model, up to fourth order in the field, and for the full range of the coupling constant.

Then, we will consider an alternative approach to the problem, involving to perform, to the first non-trivial order in the NC parameter, the SWM to the usual commutative master action. By considering again a perturbative approach, as in [9], we will show that the known equivalence between YMCS theory and non-Abelian SD model, up
to fourth order terms in the field and for the full range of the coupling constant, is maintained in the NC space. This result is not trivial also because commutative non-Abelian SD and YMCS theories generalize to NC field theories in different ways, due to the fact that the YMCS theory is gauge invariant, whereas the non-abelian SD model is not. It is known that, whereas gauge theories are lifted to their NC counterparts via the SWM, non-gauge theories are affected only in the products of the fields in the Lagrangian. In fact, our results could be considered as the extension to the non-Abelian case of the results in [15] regarding the survival of the equivalence between the SD and MCS Abelian models when the space-time becomes NC.

The paper is organized as follows. In Section 2 we deal with a parent action in which all usual products are replaced with star-products, and, by considering a perturbative approach, we establish, up to fourth order terms in the field and for the full range of the coupling constant, the equivalence between YMCS and non-Abelian SD models in which the usual products are replaced with star-products. In Section 3 we perform a SWM to the usual parent action and, by considering again a perturbative approach, we show that the duality between commutative YMCS and non-Abelian SD models survives when the space-time becomes NC.

### 2 The Star-Product

We begin by proposing a parent action in which all usual products are replaced with star-products. It is given by

$$I_P = \int d^3 x \text{Tr} \left[ -\mu f^\mu \star f_\mu + m \epsilon^{\alpha\mu\nu} \left( f_\alpha \star F_{\mu\nu} - A_\alpha \star \left( F_{\mu\nu} - \frac{2}{3} A_\mu \star A_\nu \right) \right) \right], \quad (2)$$

where $\mu$ and $m$ are, in principle, arbitrary coefficients, our metric convention is $\eta_{\mu\nu} = \text{diag}(-,+,+,+)$, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_M,$$

$$[A_\mu, A_\nu]_M = A_\mu \star A_\nu - A_\nu \star A_\mu. \quad (4)$$

Here $A_\mu = A_\mu^a \tau^a$ and $f_\mu = f_\mu^a \tau^a$ are fields in the adjoint representation of a non-Abelian gauge group $G$ (with $a = 1 \cdots \dim G$), and $\tau^a$ are the matrices representing $G$, such as $[\tau^a, \tau^b] = \tau^{abc} \tau^c$, where $\tau^{abc}$ are the structure constants of the group.

We shall verify that solving $I_P$, first for $f_\mu$ (in terms of $A_\mu$) and further for $A_\mu$ (in terms of $f_\mu$) we recover both the NC YMCS theory and the NC non-Abelian Self-Dual model, respectively.
Integration by parts in Eq.(1) yields

\[ g(x) \ast h(x) = g(x)h(x) + \cdots , \]
\[ g(x) \ast h(x) \ast p(x) = g(x) (h(x) \ast p(x)) + \cdots , \] (5)

where the dots stand for total derivatives. Throughout this paper, we consider boundary conditions such as the surface terms in the action vanish. Then, using Eqs.(5) we can write \( I_P \) as

\[ I_P = \int d^3 x \, Tr \left[ -\mu f^\mu f_\mu + m \epsilon^{\alpha\mu\nu} \left( f_\alpha F_{\mu\nu} - A_\alpha \left( F_{\mu\nu} - \frac{2}{3} A_\mu \ast A_\nu \right) \right) \right] . \] (6)

From the above expression we get the following equation of motion for \( f_\mu \)

\[ f_\mu = \frac{m}{2\mu} \epsilon^{\alpha\mu\nu} F_{\mu\nu} . \] (7)

Introducing this result into Eq.(6) and using again Eqs.(5) we get

\[ I_P = \int d^3 x \, Tr \left[ -\mu f^\mu f_\mu + 2m \epsilon^{\alpha\mu\nu} \left( (f_\alpha - A_\alpha) \ast \partial_\mu A_\nu + \left( f_\alpha - \frac{2}{3} A_\alpha \right) \ast A_\mu \ast A_\nu \right) \right] , \] (8)

which is the action for the NC YMCS theory. Notice that the single parameter which effectively appears in the theory is the boson mass, \( M \equiv \frac{-2\mu}{m} \).

Now we look for the Self-Dual model. Note that we can write \( I_P \) as follows

\[ I_P = \int d^3 x \, Tr \left[ -\mu f^\mu f_\mu + 2m \epsilon^{\alpha\mu\nu} \left( (f_\alpha - A_\alpha) \ast \partial_\mu A_\nu + \left( f_\alpha - \frac{2}{3} A_\alpha \right) \ast A_\mu \ast A_\nu \right) \right] , \] (9)

and using Eq.(5) we get

\[ I_P = \int d^3 x \, Tr \left[ -\mu f^\mu f_\mu + 2m \epsilon^{\alpha\mu\nu} \left( (f_\alpha - A_\alpha) \ast \partial_\mu A_\nu + \left( f_\alpha - \frac{2}{3} A_\alpha \right) \ast A_\mu \ast A_\nu \right) \right] . \] (10)

From the above expression, we find the following equation of motion for \( A_\mu \)

\[ 0 = \epsilon^{\alpha\mu\nu} \left( \partial_\mu f_\nu - 2 \partial_\mu A_\nu + [A_\mu, f_\nu]_M - [A_\mu, A_\nu]_M \right) , \] (11)
which after contracting with a Levi-Civita tensor gives rise to

$$2F_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu + [A_\mu, f_\nu]_M - [A_\nu, f_\mu]_M .$$

(12)

Now we must find a solution $A_\mu = A_\mu[f_\nu]$ which should be replaced into Eq.(9), thus getting an action which is a functional of $f_\mu$. However, as in the commutative case, the problem regarding the equation above is that it cannot be inverted.

In order to deal with this problem, we assume that a solution exists at least perturbatively, and look for a solution of the form

$$A_\mu = A_\mu^{(0)} + A_\mu^{(1)} [f_\nu] + A_\mu^{(2)} [f_\nu] + \cdots .$$

(13)

Then, the lowest order in Eq.(12) yields

$$F^{(0)}_{\mu\nu} = 0 ,$$

(14)

so that $A_\mu^{(0)}$ is just a pure-gauge, and we will not consider it anymore. The following order in Eq.(12) gives

$$0 = \partial_\mu \left( 2A_\nu^{(1)} - f_\nu \right) - \partial_\nu \left( 2A_\mu^{(1)} - f_\mu \right) + \left[ A_\mu^{(0)} , 2A_\nu^{(1)} - f_\nu \right]_M - \left[ A_\nu^{(0)} , 2A_\mu^{(1)} - f_\mu \right]_M ,$$

(15)

which has solution

$$A_\mu^{(1)} = \frac{1}{2} f_\mu .$$

(16)

Now we will show that in fact the explicit expression of $A_\mu^{(2)}$ is not needed for our present purposes. Introducing Eq.(13) into Eq.(9), using Eq.(5) and discarding the pure-gauge term $A_\mu^{(0)}$ we find

$$I_P = I_P \left[ A_\mu^{(1)} \right] + \int d^3x \; Tr \left[ m\epsilon^{\alpha\mu\nu} \left( f_\nu - 2A_\nu^{(1)} \right) \left( \partial_\alpha A_\mu^{(2)} - \partial_\mu A_\alpha^{(2)} \right) \right] + \mathcal{O}(f^4) .$$

(17)

Then, using Eq.(16) we get

$$I_P = I_P \left[ A_\mu^{(1)} \right] + \mathcal{O}(f^4) .$$

(18)

From Eqs.(9, 16, 18) we finally get

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\[ I_P = \int d^3 x \ Tr \left[ -\mu f^\mu * f_\mu + \frac{m}{2} \epsilon^{\alpha \mu \nu} f_\alpha * \left( \partial_\mu f_\nu + \frac{2}{3} f_\mu * f_\nu \right) \right] + O(f^4) , \]  

(19)

where we have reproduced the action for the NC non-Abelian SD model. In this theory we also find \( M \equiv \frac{2\mu}{m} \) as the single arbitrary constant which characterizes the model. From Eqs.\((8, 19)\) we see that, as anticipated, we have shown that NC YMCS is dual to a theory which coincides with the NC non-Abelian SD model, up to fourth order in \( f_\mu \), and for the full range of the coupling constant.

In the following section, we will tackle this problem under a different point of view, namely, that of the SWM, obtaining analogous results.

### 3 Seiberg-Witten Map and Duality

Our starting point is the following parent action in the usual commutative space-time

\[ S_P = \int d^3 x \ Tr \left[ -\mu f^\mu f_\mu + 2m \epsilon^{\alpha \mu \nu} \left( f_\alpha (\partial_\mu A_\nu + A_\mu A_\nu) - A_\alpha \left( \partial_\mu A_\nu + \frac{2}{3} A_\mu A_\nu \right) \right) \right] , \]

(20)

where \( A_\mu \) and \( f_\mu \) are fields in the adjoint representation of a non-Abelian gauge group.

In this section, we will be concerned with the inverse SWM at the first non-trivial order in the NC parameter. It is given by

\[ A_\mu = A'_\mu + a'_\mu(A'_\nu, \theta) , \]

(21)

where

\[ a'_\mu = \frac{1}{4} \theta^{\rho \sigma} \left[ A'_\rho , 2\partial_\sigma A'_\mu - \partial_\mu A'_\sigma + A'_\sigma A'_\mu - A'_\mu A'_\sigma \right]_+ . \]

(22)

We emphasize that all calculations in this section will be valid up to \( O(\theta^2) \) terms. The above mapping together with \( f_\mu = f'_\mu \) lifts \( S_P \) to the following NC parent action

\[ S'_P = \int d^3 x \ Tr \left[ -\mu f'^\mu f'_\mu + 2m \epsilon^{\alpha \mu \nu} (f'_\alpha - A'_\alpha - a'_\alpha) \partial_\mu (A'_\nu + a'_\nu) \right. \]

\[ + 2m \epsilon^{\alpha \mu \nu} \left( f'_\alpha - \frac{2}{3} A'_\alpha - \frac{2}{3} a'_\alpha \right) (A'_\mu + a'_\mu) (A'_\nu + a'_\nu) \]  

(23)
which can be written as

\[ S'_P = \int d^3 x \, Tr[-\mu \, f'^\mu f'_\mu + 2m \, \epsilon^{\alpha\mu\nu} \left( f'_\alpha - A'_\alpha \right) \partial_\mu \left( A'_\nu + a'_\nu \right) + 2m \, \epsilon^{\alpha\mu\nu} \left( f'_\alpha - \frac{2}{3} A'_\alpha \right) \left( A'_\mu A'_\nu + a'_\mu A'_\nu + A'_\mu a'_\nu \right) - 2m \, \epsilon^{\alpha\mu\nu} a'_\alpha \left( \partial_\mu A'_\nu + \frac{2}{3} A'_\mu A'_\nu \right) + O(\theta^2). \]  

(24)

We will show that solving \( S'_P \), first for \( f'^\mu \) and further for \( A'^\mu \), we recover the SWM-lifted actions for the YMCS theory and the non-Abelian SD model respectively, even when both theories generalize to NC field theories in different ways.

First, from Eq.(24) we get the following equation of motion for \( f'^\mu \)

\[ f'^\mu = \frac{m}{\mu} \epsilon^{\mu\nu\lambda} \left[ \partial_\mu \left( A'_\nu + a'_\nu \right) + A'_\mu A'_\nu + a'_\mu A'_\nu + A'_\mu a'_\nu \right] + O(\theta^2), \]  

(25)

and introducing this back into Eq.(24) we find

\[ S'_P = \int d^3 x \, Tr[-\frac{m^2}{\mu} \left( \partial_\mu A'_\nu - \partial_\nu A'_\mu + A'_\mu A'_\nu - A'_\mu A'_\nu \right) \] 
\[ \times \left( \partial_\mu \left( A'^\nu + a'^\nu \right) + A'^\mu A'^\nu + a'^\mu A'^\nu + A'^\mu a'^\nu \right) - \frac{m^2}{\mu} \left( \partial_\mu a'_\nu - \partial_\nu a'_\mu + a'_\mu A'_\nu - a'_\mu A'_\nu + a'_\mu a'_\nu - a'_\nu A'_\mu \right) \] 
\[ \times \left( \partial_\mu A'^\nu + A'^\mu A'^\nu \right) - 2m \, \epsilon^{\alpha\mu\nu} A'_\alpha \left( \partial_\mu \left( A'_\nu + a'_\nu \right) + \frac{2}{3} \left( A'_\mu A'_\nu + a'_\mu A'_\nu + A'_\mu a'_\nu \right) \right) \] 
\[ - 2m \, \epsilon^{\alpha\mu\nu} a'_\alpha \left( \partial_\mu A'_\nu + \frac{2}{3} A'_\mu A'_\nu \right) + O(\theta^2), \]  

(26)

which to \( O(\theta) \) is the SWM-lifted action of the YMCS theory

\[ S'_{YMCS} = \int d^3 x \, Tr[-\frac{m^2}{\mu} \left( \partial_\mu \left( A'_\nu + a'_\nu \right) - \partial_\nu \left( A'_\mu + a'_\mu \right) \right) \] 
\[ \times \left( \partial_\mu \left( A'^\nu + a'^\nu \right) + \left( A'^\mu + a'^\mu \right) \left( A'^\nu + a'^\nu \right) \right) - \frac{m^2}{\mu} \left( \left( A'_\mu + a'_\mu \right) \left( A'_\nu + a'_\nu \right) - \left( A'_\nu + a'_\nu \right) \left( A'_\mu + a'_\mu \right) \right) \] 

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However, before doing so, we note that in fact Eq. (29) differs from the following equation of motion for $A'_\mu$

$$0 = \epsilon^{\mu\nu}(\partial_\mu f'_{\nu} - 2\partial_\nu (A'_\mu + a'_\mu) + [A'_\mu + a'_\mu, f'_{\nu}] - [A'_\mu, A'_\nu] - [a'_\mu, A'_\nu] - [A'_\mu, a'_\nu]) + O(\theta^2) ,$$

and contracting with a Levi-Civita tensor we get

$$2 \left( \partial_\mu (A'_\nu + a'_\nu) - \partial_\nu (A'_\mu + a'_\mu) + [A'_\mu + a'_\mu, A'_\nu] + [a'_\mu, A'_\nu] + [A'_\mu, a'_\nu] \right)$$

$$= \partial_\mu f'_{\nu} - \partial_\nu f'_{\mu} + [A'_\mu + a'_\mu, f'_{\nu}] - [A'_\nu + a'_\nu, f'_{\mu}] + O(\theta^2) .$$

Now we must look for a solution $A'_\mu = A'_\mu[f'_{\mu}]$. However, the equation above cannot be inverted. To deal with this problem, we proceed as in the case of the previous section, and assume that a solution exists at least perturbatively, namely

$$A'_\mu = A'_\mu^{(0)} + A'_\mu^{(1)} [f'_{\mu}] + A'_\mu^{(2)} [f'_{\mu}] + \cdots.$$  

All there is to do now is to solve Eq. (29) order by order, as in the previous section. However, before doing so, we note that in fact Eq. (29) differs from the following equation

$$2 \left( \partial_\mu (A'_\nu + a'_\nu) - \partial_\nu (A'_\mu + a'_\mu) + [A'_\mu + a'_\mu, A'_\nu] + [a'_\mu, A'_\nu] + [A'_\mu, a'_\nu] \right)$$

$$= \partial_\mu f'_{\nu} - \partial_\nu f'_{\mu} + [A'_\mu + a'_\mu, f'_{\nu}] - [A'_\nu + a'_\nu, f'_{\mu}] ,$$

only by $O(\theta^2)$ terms. The interesting advantage regarding Eq. (31) is that it will easily allow us to find the explicit expressions of $A'_\mu^{(0)} + a'_\mu^{(0)}$ and $A'_\mu^{(1)} + a'_\mu^{(1)}$ (up to $O(\theta^2)$ terms), if not of $A'_\mu^{(0)}$ and $A'_\mu^{(1)}$. For our present purposes, it will suffice. Note that, in
fact, the above equation has the same formal structure as Eq.(12), and thus we easily find the solution

\[ A_\mu' + a_\mu' = \frac{1}{2} f_\mu' + O(\theta^2) , \]

(32)

where \( A_\mu'(0) + a_\mu'(0) \) remains as a pure-gauge (also up to \( O(\theta^2) \) terms). As in the case of the previous section, it can be shown that \( A_\mu'' + a_\mu'' \) only contributes to \( O(f'^4) \) and so we do not need to compute it. In fact it can be shown that, up to \( O(\theta^2) \) terms, the following identity stands

\[ S'_P = S'_P \left[ A_\mu' + a_\mu' \right] + O(f'^4) . \]

(33)

From Eqs.(23, 32, 33) we find (up to \( O(\theta^2) \) terms)

\[ S'_P = S'_{SD} + O(f'^4) , \]

(34)

where \( S'_{SD} \) is the action of the SWM-lifted non-Abelian SD model

\[ S'_{SD} = \int d^3x \ Tr \left[ -\mu f'^\mu f'_\mu + \frac{m}{2} \varepsilon^{\rho\mu\nu} f'_\rho \left( \partial_\rho f'_\nu + \frac{2}{3} f'_\rho f'_\nu \right) \right] . \]

(35)

In this way, we have shown that the duality between commutative YMCS theory and non-Abelian SD model, up to fourth order terms in the field, survives when the space-time becomes NC, even when both theories generalize to NC field theories in different ways. We point out that it would be interesting to extend our results to higher orders in \( \theta \).

One last comment concerns the relation between our approach and the results included in the previous literature regarding NC Chern-Simons theories. It is already known [16][17][18] that the SWM connects the commutative and NC versions of two related models, namely, the Chern-Simons theory and the Wess-Zumino-Witten model. We may wonder if similar results could be found in the present case. In this respect, it should be mentioned that, in fact, the SWM lifts the NC Yang-Mills action (defined through the usual Moyal product) into a non-polynomial commutative action which differs from the usual one. So, in principle, we should not expect the SWM to map the NC YMCS action Eq.(8) into its commutative version, due to the presence of the Yang-Mills term. This is to be contrasted with our result Eq.(27) which, as pointed out before, corresponds to the SWM-lifted action of the YMCS theory.
We hope that the novel perturbative method considered in this article could be helpful to establish other dual equivalences between models, and also to simplify the treatment of NC non-Abelian mathematical structures.

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