Spectral gap estimates for Brownian motion on domains with sticky-reflecting boundary diffusion

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Abstract

Introducing an interpolation method we estimate the spectral gap for Brownian motion on general domains with sticky-reflecting boundary diffusion associated to the first nontrivial eigenvalue for the Laplace operator with corresponding Wentzell-type boundary condition. In the manifold case our proofs involve novel applications of the celebrated Reilly formula.

1 Introduction and statement of main results

Brownian motion on smooth domains with sticky-reflecting diffusion along the boundary has a long history, dating back at least to Wentzell [34]. As a prototype consider a diffusion on the closure \( \overline{\Omega} \) of a smooth domain \( \Omega \) with Feller generator \((\mathcal{D}(A), A)\)

\[
\mathcal{D}(A) = \{ f \in C_0(\overline{\Omega}) | Af \in C_0(\Omega) \}
\]

\[
Af = \Delta f|_{\Omega} + (\beta \Delta^\tau f - \gamma \frac{\partial f}{\partial \nu})|_{\partial \Omega}
\] (1.1)

where \( \frac{\partial}{\partial \nu} \) is the outer normal derivative, \( \Delta^\tau \) is the Laplace-Beltrami operator on the boundary \( \partial \Omega \) and \( \beta > 0, \gamma \in \mathbb{R} \). The case of pure sticky reflection but no diffusion along the boundary corresponds to the regime \( \beta = 0 \); models with \( \beta > 0 \) have appeared recently in interacting particle systems with singular boundary or zero-range pair interaction [1, 7, 13, 19, 27]. The first rigorous process constructions on special domains \( \Omega \) were given in [16, 33, 37] and were later extended to jump-diffusion processes on general domains [6] cf. [32]. An efficient construction in symmetric cases was given by Grothaus and Voßhall via Dirichlet forms in [15]. Qualitative regularity properties of the associated semigroups were studied e.g. in [14]. In this note we address the problem of estimating the spectral gap for such processes, which is a natural question also in algorithmic applications. To our knowledge this question has been considered only for \( \beta = 0 \) by Kennedy [17] and Shouman [30]. However, for \( \beta > 0 \) the properties of the process change

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Spectral gap for domains with boundary diffusion

significantly, which is indicated by the fact that the energy form of $A$ now also contains a boundary part and which also constitutes the main difference to the closely related work [18].

In the sequel we treat the case when $\gamma > 0$ which corresponds to an inward sticky reflection at $\partial \Omega$. Our ansatz to estimate the spectral gap is based on a simple interpolation idea. To this aim assume that $\Omega$ and $\partial \Omega$ have finite (Hausdorff) measure so that we may choose $\alpha \in (0, 1)$ for which

$$\frac{\alpha}{1 - \alpha} \frac{|\partial \Omega|}{|\Omega|} = \gamma.$$ 

Introducing $\lambda_\Omega$ and $\lambda_\partial$ as normalized volume and Hausdorff measures on $\Omega$ and $\partial \Omega$ and setting

$$\lambda_\alpha = \alpha \lambda_\Omega + (1 - \alpha) \lambda_\partial,$$

we find that $-A$ is $\lambda_\alpha$-symmetric with first nonzero eigenvalue/spectral gap characterized by the Rayleigh quotient

$$\sigma_{\alpha, \beta} = \inf_{f \in C^1(\Omega), \text{Var}\lambda_\alpha f > 0} \frac{\mathcal{E}_{\alpha, \beta}(f)}{\text{Var}\lambda_\alpha f},$$

where

$$\text{Var}\lambda_\alpha f = \int \Omega f^2 d\lambda_\alpha - \left( \int \Omega f d\lambda_\alpha \right)^2$$

and

$$\mathcal{E}_{\alpha, \beta}(f) = \alpha \int \Omega \|\nabla f\|^2 d\lambda_\Omega + (1 - \alpha) \int_{\partial \Omega} \beta \|\nabla^\tau f\|^2 d\lambda_\partial,$$

and $\nabla^\tau$ denotes the tangential derivative operator on $\partial \Omega$.

This representation of $\sigma_{\alpha, \beta}$ formally interpolates between the two extremal cases of the spectral gap for reflecting Brownian motion on $\Omega$ when $\alpha = 1$ and for Brownian motion on the surface $\partial \Omega$ when $\alpha = 0$. As our main result, in Proposition 2.1 we propose a simple method to estimate $\sigma_{\alpha, \beta}$ from below using only $\sigma_0$ and $\sigma_1$ and estimates for certain bulk-boundary interaction terms which are independent of $\alpha$. The method can lead to quite good results which is illustrated by the example when $\Omega = B_1 \subset \mathbb{R}^d$ is a $d$-dimensional unit ball. When $d = 2$ and $\beta = 1$, for instance, it yields the estimate

$$\sigma_{\alpha} \geq \frac{8(1 + \alpha) \sigma_\Omega}{8(1 - \alpha) \sigma_\Omega + 16\alpha + 3\alpha(1 - \alpha) \sigma_\Omega} \quad \text{with} \quad \alpha = \frac{\gamma}{2} + \frac{\gamma}{2},$$

where $\sigma_0 \approx 3.39$ is the spectral gap for the Neumann Laplacian on the 2-dimensional unit ball, c.f. Section 3.1. – In case when $\Omega$ is a $d$-dimensional manifold with Ricci curvature bounded from below by $k_R > 0$ and with boundary $\partial \Omega$ whose second fundamental form $\Pi_{\partial \Omega}$ is bounded from below by $k_2 > 0$ we obtain (again with $\beta = 1$, for simplicity) that

$$\sigma_{\alpha} \geq \min \left( \frac{dk_R}{C_{\Omega} dk_R + (1 - \alpha)(d - 1)}, \frac{dk_R}{C_{\partial \Omega} 2(1 - \alpha) dk_R + \alpha k_2 k_R C_{\partial \Omega} + \alpha(1 - \alpha)(d - 1) k_2} \right),$$

where $C_{\Omega}$ and $C_{\partial \Omega}$ are the usual (Neumann) Poincaré constants of $\Omega$ and $\partial \Omega$ respectively. To derive this result we combine Escobar’s lower bound [9] on the first Steklov eigenvalue [12, 20] of $\Omega$ with a novel estimate on the optimal zero mean trace Poincaré constant of $\Omega$ [22, 26], for which we obtain that

$$\int \Omega f^2 dx \leq \frac{d - 1}{dk_R} \int \Omega |\nabla f|^2,$$
Spectral gap for domains with boundary diffusion

for all $f \in C^1(\Omega)$ with $\int_{\partial\Omega} f dS = 0$, and which is of independent interest. The proof is based on a novel application of Reilly’s formula [28] which is also used for a complementary lower bound of $\sigma$ independent of the interpolation approach stating that

$$\sigma_\alpha \geq \min \left( \frac{d k_2}{3d - 1} \frac{\alpha |\partial\Omega|}{|\Omega|} \left( 1 - \frac{d}{d-1} \right) k_R \right),$$

but which is generally weaker for small values of $\alpha$, c.f. Section 3.2.

The interpolation approach also yields a sufficient condition for the continuity of $\sigma_\alpha$ at $\alpha \in \{0, 1\}$, which in general may fail. In Section 2.2 we present sufficient conditions for continuity and discontinuity of $\sigma_\alpha$ at $\{0, 1\}$ which hints towards a phase transition in the associated family of variational problems.

We conclude with the discussion of two applications of the method in non-standard or singular situations, c.f. Sections 3.3 and 3.4.

2 An interpolation approach

2.1 Generalized framework

It will be convenient to work with a slight generalisation of the setup above. To this aim let $\Omega$ be an open domain in $\mathbb{R}^d$ or a Riemannian manifold with a piecewise smooth boundary $\partial\Omega$. Let $\Sigma$ be a smooth compact and connected subset of $\partial\Omega$. We denote by $\partial\Sigma$ the boundary of $\Sigma$ in the space $\partial\Omega$, i.e. $\partial\Sigma = \Sigma \cap \partial\Omega \setminus \Sigma$.

We consider two probability measures $\lambda_\Omega$ and $\lambda_\Sigma$ with support $\Omega$ and $\Sigma$, which are absolutely continuous with respect to the Lebesgue and the Hausdorff measure on $\Omega$ and $\Sigma$, respectively.

Let $D : C^1(\Omega) \mapsto \Gamma_0(\Omega)$ and $D^\tau : C^1(\partial\Omega) \mapsto \Gamma_0(\partial\Omega)$ denote given first order gradient operators mapping differentiable functions into (tangential) vector fields on $\Omega$ and on $\partial\Omega$, respectively, and for $\alpha \in [0, 1]$ let

$$\lambda_\alpha := \alpha \lambda_\Omega + (1 - \alpha) \lambda_\Sigma,$$

$$\mathcal{E}_\alpha(f) := \alpha \int_{\Omega} \|D f\|^2 d\lambda_\Omega + (1 - \alpha) \int_{\Sigma} \|D^\tau f\|^2 d\lambda_\Sigma, \quad f \in D_0,$$

where $D_0 \subset C^1(\Omega)$ is dense in $C_0(\Omega)$. We assume that for $\alpha \in [0, 1]$ the quadratic form $(\mathcal{E}_\alpha, D_0)$ is a pre-Dirichlet form on $L^2(\Omega, \lambda_\alpha)$ whose closure we shall denote by $(\mathcal{E}_\alpha, D)$, c.f. [15] for details. We wish to estimate from above $\sigma_\alpha^{-1} = C_\alpha$, where $C_\alpha$ is the optimal Poincaré constant given by

$$C_\alpha := \sup_{f \in D_0} \frac{\text{Var}_{\lambda_\alpha} f}{\mathcal{E}_\alpha(f)}. \quad (2.1)$$

In the interpolation method presented below it is assumed that $C_\alpha$ are known or can be estimated at the two extremals $\alpha \in \{0, 1\}$. For instance, when $D = \nabla$, $D^\tau = \nabla^\tau$ are the standard gradient resp. tangential gradient operators and $\lambda_\Omega$ and $\lambda_\Sigma$ are normalized Lebesgue resp. Hausdorff measures on $\Omega$ and $\Sigma \subset \partial\Omega$, $C_{1\Omega} := C_1$ is the optimal Poincaré constant associated to the Laplace operator on $\Omega$ with Neumann boundary conditions, whereas $C_{\Sigma} := C_0$ is the optimal Poincaré constant associated to the Laplace-Beltrami operator on $\Sigma$ with Neumann boundary conditions on $\partial\Sigma$. 
The following proposition establishes an estimate of \( C_\alpha \) in terms of \( C_\Omega \) and \( C_\Sigma \).

**Proposition 2.1.** Assume there exists constants \( K_{\Sigma,\Omega}, K_1, K_2 \) such that for any \( f \in \mathcal{D}_0 \)

\[
\text{Var}_\lambda \ f \leq K_{\Sigma,\Omega} \int_{\Omega} \| Df \|^2 d\lambda_{\Omega},
\]

and

\[
\left( \int_{\Omega} f d\lambda_{\Omega} - \int_{\Sigma} f d\lambda_{\Sigma} \right)^2 \leq K_1 \int_{\Omega} \| Df \|^2 d\lambda_{\Omega} + K_2 \int_{\Sigma} \| D^T f \|^2 d\lambda_{\Sigma},
\]

then it holds for any \( \alpha \in (0, 1) \).

\[
C_\alpha \leq \max \left( C_\Omega + (1 - \alpha)K_1, \alpha K_2, \frac{(1 - \alpha)K_{\Sigma,\Omega}C_\Sigma + \alpha C_\Omega C_\Sigma + \alpha(1 - \alpha)(K_{\Sigma,\Omega}K_2 + C_\Sigma K_1)}{(1 - \alpha)K_{\Sigma,\Omega} + \alpha C_\Sigma} \right).
\]

**Proof.** By definition of \( C_\Sigma \) and by (2.2), for any \( f \in \mathcal{D}_0 \)

\[
\text{Var}_\lambda \ f \leq t K_{\Sigma,\Omega} \int_{\Omega} \| Df \|^2 d\lambda_{\Omega} + (1 - t)C_\Sigma \int_{\Sigma} \| D^T f \|^2 d\lambda_{\Sigma},
\]

for any \( t \in [0, 1] \). Let \( \alpha \in (0, 1) \). For any \( f \in \mathcal{D}_0 \) and any \( t \in [0, 1] \)

\[
\text{Var}_\lambda \ f = \alpha \text{Var}_\lambda \ f + (1 - \alpha) \text{Var}_\lambda \ f + \alpha(1 - \alpha) \left( \int_{\Omega} f d\lambda_{\Omega} - \int_{\Sigma} f d\lambda_{\Sigma} \right)^2 \leq \left( C_\Omega + \frac{(1 - \alpha)t}{\alpha} K_{\Sigma,\Omega} + (1 - \alpha)K_1 \right) \alpha \int_{\Omega} \| Df \|^2 d\lambda_{\Omega} + ((1 - t)C_\Sigma + \alpha K_2) (1 - \alpha) \int_{\Sigma} \| D^T f \|^2 d\lambda_{\Sigma}.
\]

Therefore,

\[
C_\alpha \leq \inf_{t \in [0, 1]} \max \left( C_\Omega + \frac{(1 - \alpha)t}{\alpha} K_{\Sigma,\Omega} + (1 - \alpha)K_1, (1 - t)C_\Sigma + \alpha K_2 \right).
\]

For any positive constants \( a, b, c, d \), we have

\[
\inf_{t \in [0, 1]} \max (a + bt, c - dt) = \begin{cases} a & \text{if } c - a < 0, \\ c - d & \text{if } c - a > b + d, \\ \frac{b + c + d}{b + d} & \text{if } 0 \leq c - a \leq b + d. \end{cases}
\]

Therefore

\[
C_\alpha \leq \begin{cases} C_\Omega + (1 - \alpha)K_1 & \text{if } \alpha K_2 - (1 - \alpha)K_1 + C_\Sigma - C_\Omega < 0, \\ \alpha K_2 & \text{if } \alpha K_2 - (1 - \alpha)K_1 - C_\Omega > \frac{1 - \alpha}{\alpha} K_{\Sigma,\Omega}, \\ \frac{(1 - \alpha)K_{\Sigma,\Omega}C_\Sigma + \alpha C_\Omega C_\Sigma + \alpha(1 - \alpha)(K_{\Sigma,\Omega}K_2 + C_\Sigma K_1)}{(1 - \alpha)K_{\Sigma,\Omega} + \alpha C_\Sigma} & \text{if } 0 \leq \alpha K_2 - (1 - \alpha)K_1 + C_\Sigma - C_\Omega. \end{cases}
\]

The last term is equivalent to the announced result. \( \square \)
2.2 Continuity of $C_\alpha$

In general, the function $\alpha \mapsto C_\alpha$ might have discontinuities at $\alpha \in \{0, 1\}$ in which cases an upper bound for $C_\alpha$ which interpolates continuously between $C_0$ and $C_1$ cannot exist. For example, when $\Omega = (0, b) \times (0, 1) \subset \mathbb{R}^2$ and $\Sigma = [0, b] \times \{0\}$, straightforward computations yield

$$\lim_{\alpha \to 0} C_\alpha = \max \left\{ C_\Sigma, \frac{4}{\pi^2} \right\},$$

where $C_\Sigma = \frac{1}{\pi^2}$. Hence $\alpha \mapsto C_\alpha$ is discontinuous at $\alpha = 0$ if and only if $b < 2$. – To generalize this to the framework of Section 2.1 let $C^1_b(\Omega) = \{ f \in C^1(\Omega) : f = 0 \text{ on } \Sigma \}$ and

$$C_0 := \sup_{f \text{ non constant}} \frac{\int_{\Omega} f^2 \, d\lambda}{\int_{\Omega} \| Df \|^2 \, d\lambda}.$$

(If $D = \nabla$, $C_0$ is the inverse of the spectral gap for Brownian motion on $\Omega$ with killing on $\Sigma$ and normal reflection at $\partial \Omega \setminus \Sigma$.) We can then record the following statement as a partial corollary to Proposition 2.1.

**Proposition 2.2.** In the setting of proposition 2.1 it holds that

$$\lim_{\alpha \to 0} C_\alpha \geq C_0.$$

In particular, if $C_\Sigma < C_0$, then $\alpha \mapsto C_\alpha$ is discontinuous at $\alpha = 0$. Conversely, if $C_\Sigma \geq C_\Omega + K_1$ then $\alpha \mapsto C_\alpha$ is continuous at 0. If $C_\Omega \geq K_2$ continuity at 1 holds.

**Proof.** To prove the second statement, take a non constant function $g \in C^1_b(\Omega)$ and estimate

$$\lim_{\alpha \to 0} C_\alpha = \lim_{\alpha \to 0} \sup_{f \text{ non constant}} \frac{\text{Var}_{\lambda_\alpha} \, f}{\mathcal{E}_\alpha(f)} \geq \lim_{\alpha \to 0} \frac{\text{Var}_{\lambda_\alpha} \, g}{\mathcal{E}_\alpha(g)}$$

$$= \lim_{\alpha \to 0} \frac{\alpha \text{Var}_{\lambda_\alpha} \, g + (1 - \alpha) \text{Var}_{\lambda_\Sigma} \, g + \alpha(1 - \alpha) \left( \int_{\Omega} g \, d\lambda_\Omega - \int_{\Sigma} g \, d\lambda_\Sigma \right)^2}{\alpha \int_{\Omega} \| Dg \|^2 \, d\lambda + (1 - \alpha) \int_{\Sigma} \| D^2g \|^2 \, d\lambda_\Sigma}.$$

Since $g = 0$ on $\Sigma$, we obtain

$$\lim_{\alpha \to 0} C_\alpha \geq \lim_{\alpha \to 0} \frac{\alpha \text{Var}_{\lambda_\alpha} \, g + \alpha(1 - \alpha) \left( \int_{\Omega} g \, d\lambda_\Omega \right)^2}{\alpha \int_{\Omega} \| Dg \|^2 \, d\lambda} = \frac{\int_{\Omega} g^2 \, d\lambda}{\int_{\Omega} \| Dg \|^2 \, d\lambda}.$$ 

Taking the supremum over $g \in C^1_b(\Omega)$ yields the first statement.

To prove the second assertion note that $\alpha \mapsto C_\alpha$ is the pointwise supremum of a family of continuous functions and therefore lower semi continuous. Thus $C_\Sigma = C_0 \leq \lim_{\alpha \to 0} C_\alpha$. If $C_\Sigma \geq C_\Omega + K_1$, the r.h.s. of inequality (2.4) converges to $C_\Sigma$ as $\alpha$ goes to 0, which implies that $\lim_{\alpha \to 0} C_\alpha \leq C_\Sigma$. Similarly, if $C_\Omega \geq K_2$, the r.h.s. of (2.4) converges to $C_\Omega$ as $\alpha$ goes to zero.

**Remark 2.3.** For smooth enough boundary the constant $K_2$ can always be taken equal to zero, hence by proposition 2.2 continuity at $\alpha = 1$ holds. An example where a phase transition appears at $\alpha = 0$ is given in section 3.3. In section 3.4 we present an example where $C_\Omega < K_2$ but continuity of at $\alpha = 1$ can be established via Mosco-convergence [23] of the associated Dirichlet forms, see also [24].
3 Examples

3.1 Brownian motion on balls with sticky boundary diffusion

As our first example let $\Omega := B_1$ be the unit ball in $\mathbb{R}^d$, $\Sigma = \partial \Omega$ and $D = \nabla$ and $D^r = \sqrt{\beta} \nabla^r$ with $D_0 = C^1(\bar{\Omega})$.

**Proposition 3.1.** In the case when $\Omega = B_1 \subset \mathbb{R}^d$ the optimal Poincaré constant of the generator (1.1) is bounded from above by

$$C_\alpha \leq \max \left( C_\Omega + (1 - \alpha) \frac{d + 1}{4d^2}, \frac{4(1 - \alpha)d + 4\alpha d^2 C_\Omega + \alpha(1 - \alpha)(d + 1)}{4d(\alpha d + (1 - \alpha)\beta(d - 1))} \right),$$

where $\alpha = \frac{2}{d + 1}$ and $C_\Omega$ is the optimal Poincaré constant for reflecting Brownian motion on $B_1 \subset \mathbb{R}^d$.

**Proof.** In order to apply Proposition 2.1, it is sufficient to compute the constants $C_{\Sigma}, K_{\Sigma, \Omega}, K_1$ and $K_2$.

We claim that inequalities (2.2) and (2.3) holds with

$$C_{\Sigma} = \frac{1}{\beta(d - 1)}, \quad K_{\Sigma, \Omega} = \frac{1}{d}, \quad K_1 = \frac{d + 1}{4d^2}, \quad K_2 = 0.$$

First, according to [31, Theorem 21.1], the first eigenvalue of the Laplace-Beltrami operator on the unit sphere of dimension $d - 1$ is equal to $d - 1$, thus $C_{\Sigma} = \frac{1}{\beta(d - 1)}$.

Moreover, according to [3, Theorem 4], for every $f \in C^1(\partial \Omega)$ one has

$$\left( \int_{\partial \Omega} |f|^q d\lambda_{\Sigma} \right)^{\frac{2}{q}} \leq \frac{2}{d} \int_{\partial \Omega} \|\nabla u\|^2 d\lambda_{\Omega} + \int_{\partial \Omega} f^2 d\lambda_{\Sigma},$$

for $2 \leq q < \infty$ if $d = 2$ and $2 \leq q < \frac{2d - 2}{d - 2}$ if $d \geq 3$, where $u$ is the harmonic extension of $f$ to the unit ball $\Omega$. It implies the logarithmic Sobolev inequality $\text{Ent}_{\lambda_{\Sigma}}(f^2) \leq \frac{2}{d} \int_{\partial \Omega} \|\nabla u\|^2 d\lambda_{\Omega}$. Repeating the proof of Proposition 5.1.3 in [2], we get $\text{Var}_{\lambda_{\Sigma}} f \leq \frac{1}{d} \int_{\partial \Omega} \|\nabla u\|^2 d\lambda_{\Omega}$. Moreover, since the harmonic extension of $f$ is minimizing the energy functional $\mathcal{E}_1$ under any function with boundary condition $f$, the last inequality implies for any $f \in C^1(\bar{\Omega})$

$$\text{Var}_{\lambda_{\Sigma}} f \leq \frac{1}{d} \int_{\partial \Omega} \|\nabla f\|^2 d\lambda_{\Omega}, \quad (3.2)$$

which implies $K_{\Sigma, \Omega} = \frac{1}{d}$.

Furthermore, note that $\int_{\partial \Omega} f(y)\lambda_{\Sigma}(dy) = \int_{\Omega} f(\pi_x)\lambda_{\Omega}(dx)$, where $\pi_x = \frac{x}{|x|}, x \neq 0$. Hence, using Jensen’s inequality and polar coordinates

$$\left( \int_{\Omega} f d\lambda_{\Omega} - \int_{\partial \Omega} f d\lambda_{\Sigma} \right)^2 \leq \int_{\Omega} (f(x) - f(\pi_x))^2 \lambda_{\Omega}(dx)$$

$$= \frac{1}{|\Omega|} \int_{\partial \Omega} \int_0^1 (f(ry) - f(y))^2 r^{d-1} dr dy$$

$$= \frac{1}{|\Omega|} \int_{\partial \Omega} \int_0^1 \left( \int_r^1 \frac{d}{ds} f(sy) ds \right)^2 r^{d-1} dr dy.$$
Spectral gap for domains with boundary diffusion

\[ \leq \frac{1}{|\Omega|} \int_{\partial \Omega} \int_{0}^{1} (1 - r) \left( \int_{r}^{1} \left( \frac{d}{ds} f(sy) \right)^2 ds \right) r^{d-1} \, dr \, dy \]

\[ = \frac{1}{|\Omega|} \int_{\partial \Omega} \int_{0}^{1} \left[ \int_{0}^{s} (1 - r) r^{d-1} \right] \left( \frac{d}{ds} f(sy) \right)^2 ds \, dy. \]

We separately estimate

\[ \int_{0}^{s} (1 - r) r^{d-1} \, dr = \left( \frac{s}{d} - \frac{s^2}{d+1} \right) s^{d-1} \leq \frac{d+1}{4d^2} s^{d-1}. \]

for any \( s \in [0,1] \). Hence,

\[ \left( \int_{\Omega} f \, d\lambda_{\Omega} - \int_{\partial \Omega} f \, d\lambda_{\Sigma} \right)^2 \leq \frac{d+1}{4d^2|\Omega|} \int_{\partial \Omega} \int_{0}^{1} (\nabla f(sy) \cdot y)^2 s^{d-1} \, ds \]

\[ = \frac{d+1}{4d^2|\Omega|} \int_{\partial \Omega} \int_{0}^{1} \|\nabla f(sy)\|^2 s^{d-1} \, ds \, dy \]

\[ = \frac{d+1}{4d^2} \int_{\Omega} \|\nabla f(x)\|^2 \lambda_{\Omega}(dx). \]  \hfill (3.3)

which implies \( K_1 = \frac{d+1}{4d^2} \) and \( K_2 = 0 \).

For illustration, in \( d = 2 \), we compare the bound from Proposition 3.1 for \( \beta = 1, \gamma > 0 \) to the optimal constant \( C_\alpha \) which will be computed numerically. To evaluate the bound (3.1), note that in this case

\[ C_{\Omega} = \frac{1}{\sigma_{\Omega}} \approx \frac{1}{3.39}, \]  \hfill (3.4)

where \( \sigma_{\Omega} \) is the smallest positive eigenvalue of the Laplace operator with Neumann boundary condition on the circle. It is given as the minimal positive solution to the equation \( J_m'(\sqrt{\gamma}) = 0, m \in \mathbb{N}_0 \), where \( J_m \) is the Bessel function of the first kind of parameter \( m \), defined by \( J_m(x) = \frac{1}{\pi} \int_0^\pi \cos(mt - x \sin t) \, dt \), \( x \geq 0 \).

As a consequence, inequality (3.1) becomes

\[ C_\alpha \leq \frac{8(1 - \alpha)\sigma_{\Omega} + 16\alpha + 3\alpha(1 - \alpha)\sigma_{\Omega}}{8(1 + \alpha)\sigma_{\Omega}}. \]  \hfill (3.5)

For the numerical computation of \( C_\alpha \) one notes that the generator \( A_\alpha \) associated with \( E_\alpha \) is defined on \( D(A_\alpha) \subset C^2(\Omega) \) as

\[ A_\alpha f = 1_{\Omega} \Delta f + 1_{\partial \Omega} \left( \Delta^\gamma f - \frac{2\alpha}{1 - \alpha} \frac{\partial f}{\partial \nu} \right), \]

where \( \Delta^\gamma \) and \( \frac{\partial}{\partial \nu} \) denote the Laplace-Beltrami operator and the outer normal derivative on the circle \( \partial \Omega \). Hence, an eigenvector of \( -A_\alpha \) for eigenvalue \( \lambda \geq 0 \) is a function \( f \in D(A_\alpha) \) such that

\[ A_\alpha f = -\lambda f \quad \text{in} \quad \Omega. \]

This equation is equivalent to the system of partial differential equations

\[
\begin{align*}
\Delta f = -\lambda f & \quad \text{in} \quad \Omega, \\
\Delta^\gamma f - \frac{2\alpha}{1 - \alpha} \frac{\partial f}{\partial \nu} & = -\lambda f \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
which by the continuity of $f$ can be rewritten as
\[
\begin{cases}
\Delta f = -\lambda f & \text{in } \Omega, \\
\Delta f = \Delta^* f - \frac{2\alpha}{1-\alpha} \frac{\partial f}{\partial \nu} & \text{on } \partial \Omega.
\end{cases}
\]

Passing to polar coordinates $(x_1, x_2) = (r \cos \theta, r \sin \theta) \in \Omega$ in $d = 2$ and separating variables, we obtain the set of eigenfunctions $\{f_{m,l}^c, f_{m,l}^s\}_{m,l \in \mathbb{N}_0}$,
\[
f_{m,l}^c(x_1, x_2) = J_m(\sqrt{\lambda_{m,l}} r) \cos(m\theta), \quad m, l \in \mathbb{N}_0, \\
f_{m,l}^s(x_1, x_2) = J_m(\sqrt{\lambda_{m,l}} r) \sin(m\theta), \quad m \in \mathbb{N}, \ l \in \mathbb{N}_0,
\]
where $\lambda_{m,l}, l \in \mathbb{N}_0$, are countable family of positive solutions to the equation
\[
\sqrt{\lambda} J''_m(\sqrt{\lambda}) + \frac{1 + \alpha}{1 - \alpha} J'_m(\sqrt{\lambda}) = 0 
\]
for every $m \in \mathbb{N}_0$. Since the family $\{f_{m,l}^c, m, l \in \mathbb{N}_0\} \cup \{f_{m,l}^s, m \in \mathbb{N}_0, l \in \mathbb{N}_0\}$ is dense in $L^2(\Omega, \lambda_\alpha)$ and the operator $A_\alpha$ is symmetric, the standard argument implies
\[
C_\alpha = \frac{1}{\lambda_{\alpha, \ast}},
\]
where $\lambda_{\alpha, \ast} = \min_{m,l \in \mathbb{N}_0} \lambda_{m,l}$. The resulting curves are plotted in Figure 1.

![Figure 1: The blue curve represents $\alpha \mapsto C_\alpha$ the optimal Poincaré constant when $\Omega$ is the unit ball of $\mathbb{R}^2$ with full boundary diffusion. The red curve is the upper estimate given by (3.5).](image)

**3.2 Smooth manifold with boundary**

Let $\Omega$ be a smooth compact Riemannian manifold of dimension $d$ with piecewise smooth boundary $\partial \Omega$. We denote by $\text{Ric}$ the Ricci curvature of $\Omega$ and by $\Pi$ the second fundamental form on the boundary $\partial \Omega$. 
Assume in this section that:

**Assumption (M):**  
\[ \exists k_r > 0, k_2 > 0, \quad \text{Ric} |\gamma| \geq k_R \text{id} \quad \text{and} \quad \Pi|_{\partial \Omega} \geq k_2 \text{id}. \]

As before we consider \( \Sigma = \partial \Omega, \ D = \nabla \) and \( D^\tau = \nabla^\tau \) with \( D_0 = C^1(\Pi) \).

**Proposition 3.2.** Under assumption (M), it holds that

\[
C_\alpha \leq \max \left( C_\Omega + \frac{(1 - \alpha)(d - 1)}{dk_R}, \frac{C_\Sigma}{dk_R} + \frac{2(1 - \alpha)dk_R + \alpha dk_2 k_R C_\Omega + \alpha(1 - \alpha)(d - 1)k_2}{2(1 - \alpha) + \alpha dk_2 C_\Sigma} \right) =: M_1. \tag{3.8}
\]

This statement is obtained via Proposition 2.1 and the two statements below.

**Proposition 3.3.** Under assumption (M), inequality (2.3) is satisfied with \( K_2 = 0 \) and \( K_1 = \frac{d - 1}{dk_R} \).

**Proof.** Our goal is to obtain a lower bound of

\[
\inf_{f \in C^1(\Omega)} \frac{\int_{\Omega} \| \nabla f \|^2 d\lambda_\Omega}{\left( \int_{\Omega} f d\lambda_\Omega - \int_{\Sigma} f d\lambda_\Sigma \right)^2},
\]

where we recall that \( \Sigma = \partial \Omega \). We note that

\[
\inf_{f \in C^1(\Omega)} \frac{\int_{\Omega} \| \nabla f \|^2 d\lambda_\Omega}{\left( \int_{\Omega} f d\lambda_\Omega - \int_{\Sigma} f d\lambda_\Sigma \right)^2} \geq \inf_{f \in C^1(\Omega), \int_{\Sigma} f d\lambda_\Sigma = 0} \frac{\int_{\Omega} \| \nabla f \|^2 d\lambda_\Omega}{\int_{\Omega} f^2 d\lambda_\Omega} =: \sigma.
\]

Let \( f \in C^1(\Pi) \) be a minimizer for \( \sigma \). Then \( \int_{\Sigma} f d\lambda_\Sigma = 0 \) and

\[
\int_{\Omega} \nabla f \cdot \nabla \xi d\lambda_\Omega = \sigma \int_{\Omega} f \xi d\lambda_\Omega
\]

for each \( \xi \in C^1(\Pi) \) with \( \int_{\Sigma} \xi d\lambda_\Sigma = 0 \). By integration by parts, the latter equality is equivalent to

\[
- \int_{\Omega} \Delta f \xi d\lambda_\Omega + \frac{\xi \| \nabla f \|^2}{\| \nabla \xi \|^2} \int_{\Sigma} \frac{\partial f}{\partial \nu} |\nabla f| d\lambda_\Sigma = \sigma \int_{\Omega} f \xi d\lambda_\Omega
\]

for each \( \xi \in C^1(\Pi) \) satisfying \( \int_{\Sigma} \xi d\lambda_\Sigma = 0 \). In particular, choosing \( \xi \in C^1_{c0}(\Omega) \) (which obviously satisfies \( \int_{\Sigma} \xi d\lambda_\Sigma = 0 \)), we get that \( f \) should satisfy \( -\Delta f = \sigma f \) in \( \Omega \). Hence \( \int_{\Sigma} \frac{\partial f}{\partial \nu} \xi d\lambda_\Sigma = 0 \) for every \( \xi \in C^1(\Pi) \), which is equivalent to

\[
\int_{\Sigma} \left( \frac{\partial f}{\partial \nu} - \int_{\Sigma} \frac{\partial f}{\partial \nu} d\lambda_\Sigma \right) \xi d\lambda_\Sigma = 0
\]

for every \( \xi \in C^1(\Pi) \). It follows that \( \frac{\partial f}{\partial \nu} \) is constant on \( \Sigma \). Therefore, \( f \) satisfies

\[
\begin{cases}
\Delta f = -\sigma f & \text{in } \Omega, \\
\frac{\partial f}{\partial \nu} \equiv c & \text{on } \partial \Omega, \\
\int_{\Sigma} f d\lambda_\Sigma = 0,
\end{cases}
\tag{3.9}
\]
for some constant $c$.

Moreover, recall Reilly’s formula (see [28])

$$
\int_{\Omega} (\Delta f)^2 - \|\nabla^2 f\|^2 \, dx = \int_{\Omega} \text{Ric}(\nabla f, \nabla f) \, dx \\
+ \int_{\Sigma} \left( H \left( \frac{\partial f}{\partial \nu} \right)^2 + \Pi(\nabla^\tau f, \nabla^\tau f) + 2 \Delta^\tau f \frac{\partial f}{\partial \nu} \right) \, dS
$$

(3.10)

where $dx$ and $dS$ denote the Riemannian volume resp. surface measure on $\Omega$ and $\partial \Omega$, $\nabla^2 f$ is the Hessian of $f$ and $H$ is the mean curvature of $\Sigma$ (i.e. the trace of $\Pi$). Since $f$ satisfies (3.9),

$$
\int_{\Omega} (\Delta f)^2 \, dx = -\sigma \int_{\Omega} f \Delta f \, dx = \sigma \int_{\Omega} \|\nabla f\|^2 \, dx - \sigma \int_{\Sigma} \frac{\partial f}{\partial \nu} f \, dS
$$

because $\int_{\Sigma} f dS = |\Sigma| \int_{\Sigma} f \, d\lambda_\Sigma = 0$. Furthermore, note that $\|\nabla^2 f\|^2 = \sum_{i,j} (\partial^2_{ij} f)^2 \geq \sum_{i=1}^d (\partial^2_{ii} f)^2 \geq \frac{1}{d} (\Delta f)^2$. Therefore, the l.h.s. of (3.10) is bounded by

$$
\int_{\Omega} ((\Delta f)^2 - \|\nabla^2 f\|^2) \, dx \leq \frac{d-1}{d} \int_{\Omega} (\Delta f)^2 \, dx \leq \frac{d-1}{d} \sigma \int_{\Omega} \|\nabla f\|^2 \, dx.
$$

On the other hand, by assumption (M), $H \geq 0$, $\Pi(\nabla^\tau f, \nabla^\tau f) \geq 0$ and

$$
\int_{\Omega} \text{Ric}(\nabla f, \nabla f) \, dx \geq k_R \int_{\Omega} \|\nabla f\|^2 \, dx.
$$

Since

$$
\int_{\Sigma} \Delta^\tau f \frac{\partial f}{\partial \nu} \, dS = c \int_{\Sigma} \Delta^\tau f \, dS = 0
$$

the r.h.s. of (3.10) is bounded from below by $k_R \int_{\Omega} \|\nabla f\|^2 \, dx$. It turns out that

$$
\frac{d-1}{d} \sigma \int_{\Omega} \|\nabla f\|^2 \, dx \geq k_R \int_{\Omega} \|\nabla f\|^2 \, dx,
$$

which implies that $\sigma \geq \frac{d}{d-1} k_R$. It follows that inequality (2.3) holds with $K_1 = \frac{d-1}{d \pi n}$.

**Remark 3.4.** Instead of using $K_1$ from Proposition 3.3 another admissible choice is

$$
K_1' = \frac{|\Omega|}{|\partial \Omega|} B^2 (1 + C_{\Omega}) < \infty,
$$

where $B$ is the optimal Sobolev trace constant of $\Omega$, i.e. the norm of the embedding $H^{1,2}(\Omega) \hookrightarrow L^2(\partial \Omega)$. $B^{-2}$ is the first nontrivial eigenvalue of a Steklov-type eigenvalue problem

$$
\begin{cases}
-\Delta f + f = 0 & \text{in } \Omega \\
\frac{\partial f}{\partial \nu} = \sigma f & \text{on } \partial \Omega,
\end{cases}
$$

for which however explicit lower bounds in terms of the geometry of $\Omega$ seem yet unknown [4, 5, 11, 21, 29].

**Proposition 3.5.** Under assumption (M), inequality (2.2) holds with $K_{\Sigma, \Omega} = \frac{2}{k_2}$. 

Proof. The optimal choice for $K_{\Sigma, \Omega}$ is $\sigma^{-1}$, where $\sigma$ given by

$$\sigma = \inf_{f \in C^1(\Omega)} \frac{\int_\Omega \|\nabla f\|^2 d\lambda_\Omega}{\left(\int_\Sigma f^2 d\lambda_\Sigma\right)^2}$$

is the first nontrivial eigenvalue of the Steklov-problem c.f. [12]

$$\begin{cases}
\Delta f = 0 & \text{in } \Omega, \\
\frac{\partial f}{\partial \nu} = \sigma f & \text{on } \partial \Omega.
\end{cases}$$

Escobar [9] showed $\sigma \geq \frac{k_2}{d}$ in this case. \qed

Alternatively, we obtain another upper bound for $C_\alpha$ by a direct application of Reilly’s formula.

**Proposition 3.6.** Under assumption (M) it holds that

$$C_\alpha \leq \max \left( \frac{(3d-1)(1-\alpha)}{d\alpha k_2} \frac{|\Omega|}{|\partial \Omega|} \frac{d-1}{dk_R} \right) =: M_2. \quad (3.11)$$

**Proof.** We estimate equivalently from below the first nontrivial eigenvalue $\sigma = C_\alpha^{-1}$ for the problem

$$\begin{cases}
\Delta f + \sigma f = 0 & \text{in } \Omega, \\
\Delta^\tau f - \gamma \frac{\partial f}{\partial \nu} + \sigma f = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\gamma = \frac{\alpha}{1-\alpha} \frac{|\partial \Omega|}{|\Omega|}$. As in the proof of Proposition 3.3 we apply Reilly’s formula (3.10) to the corresponding eigenfunction $f$. In this case, for the l.h.s. we estimate

$$\int_\Omega ((\Delta f)^2 - \|\nabla^2 f\|^2) \, dx \leq \frac{d-1}{d} \int_\Omega (\Delta f)^2 \, dx = -\frac{d-1}{d} \sigma \int_\Omega f \Delta f \, dx$$

$$= \frac{d-1}{d} \sigma \int_\Omega \|\nabla f\|^2 \, dx - \frac{d-1}{d} \sigma \int_\Sigma \frac{\partial f}{\partial \nu} \, f dS$$

$$= \frac{d-1}{d} \sigma \int_\Omega \|\nabla f\|^2 \, dx - \frac{d-1}{d} \sigma \int_\Sigma (\Delta^\tau f + \sigma f) \, f dS$$

$$= \frac{d-1}{d} \sigma \int_\Omega \|\nabla f\|^2 \, dx - \frac{d-1}{d} \sigma \int_\Sigma \|\nabla^\tau f\|^2 \, dS - \frac{d-1}{d} \frac{\sigma^2}{\gamma} \int_\Sigma f^2 \, dS$$

$$\leq \frac{d-1}{d} \sigma \int_\Omega \|\nabla f\|^2 \, dx - \frac{d-1}{d} \sigma \int_\Sigma \|\nabla^\tau f\|^2 \, dS.$$
Combining the two bounds for (3.10) yields
\[ \left( \frac{d-1}{d} \sigma - k_R \right) \int_{\Omega} \| \nabla f \|^2 dx \geq \left( k_2 - \frac{3d-1}{d} \sigma \right) \int_{\Sigma} \| \nabla^\gamma f \|^2 dS, \]
which implies that either
\[ k_2 - \frac{3d-1}{d} \sigma \leq 0, \quad \text{i.e.} \quad \sigma \geq \frac{d k_2 \gamma}{3d-1}, \]
or
\[ \frac{d-1}{d} \sigma - k_R \geq 0, \quad \text{i.e.} \quad \sigma \geq \frac{d}{d-1} k_R. \]
Consequently,
\[ \sigma \geq \min \left( \frac{d k_2 \gamma}{3d-1}, \frac{d}{d-1} k_R \right). \]

\textbf{Corollary 3.7.} Under assumption (M), it holds that
\[ C_\alpha \leq \min(M_1, M_2), \]
where $M_1 = M_1(\alpha)$ and $M_2 = M_2(\alpha)$ are defined by (3.8) and (3.11), respectively.

When $\alpha$ goes to 0, $M_1$ tends to $\max(C_\Omega, \frac{d-1}{d} k_R, C_\Sigma)$ and $M_2$ tends to $+\infty$, so the estimation via the interpolation method is always stronger. When $\alpha$ goes to 1, $M_1$ tends to $C_\Omega$ and $M_2$ tends to $\frac{d-1}{d} k_R$, so the relative strength of each method depends on the values of $C_\Omega$, $d$ and $k_R$.

3.3 Brownian motion on balls with partial sticky reflecting boundary diffusion

As in Section 3.1, let $\Omega := B_1$ be the unit ball of $\mathbb{R}^2$. Now, define for a fixed $\delta \in (0, 1)$
\[ \Sigma = \{ (\cos \theta, \sin \theta) \in \partial \Omega : -\delta \pi \leq \theta \leq \delta \pi \}, \quad \Sigma_N := \partial \Omega \setminus \Sigma. \]

\textbf{Proposition 3.8.} It holds that
\[ C_\alpha \leq \max \left( C_\Omega + (1 - \alpha)K_1(\delta), \frac{4(1 - \alpha)\delta^2 + 8\alpha \delta^3 C_\Omega + 8\alpha (1 - \alpha)\delta^3 K_1(\delta)}{(1 - \alpha) + 8\alpha \delta^3} \right), \] (3.12)
where $C_\Omega = \frac{1}{\sigma_0} \approx \frac{1}{4.39}$ and $K_1(\delta) = \left( \sqrt{1 - \delta^2} + \frac{1}{4} \sqrt{\frac{3}{2}} \right)^2$.

As previously, we will start by computing the needed constants $C_\Omega$, $C_\Sigma$, $K_{\Sigma, \Omega}$, $K_1$ and $K_2$. The first constant, $C_\Omega = \frac{1}{\sigma_0} \approx \frac{1}{4.39}$, remains unchanged.

\textbf{Lemma 3.9.} The following inequalities hold true
\[ \text{Var}_{\lambda_\Sigma} f \leq C_\Sigma \int_{\Sigma} \| \nabla^\gamma f \|^2 d\lambda_\Sigma, \] (3.13)
\[ \text{Var}_{\lambda_\Omega} f \leq K_{\Sigma, \Omega} \int_{\Omega} \| \nabla f \|^2 d\lambda_\Omega, \] (3.14)
where $C_\Sigma = 4\delta^2$ and $K_{\Sigma, \Omega} = \frac{1}{28}$.
Proof. Inequality (3.13) corresponds to the Poincaré inequality of the Laplacian on the one-dimensional interval \([-\delta \pi, \delta \pi]\) with Neumann boundary conditions. It is well known (see [2, Prop. 4.5.5]) that the optimal Poincaré constant is given by \(C_\Sigma = 4\delta^2\).

Moreover, let us decompose the normalized Hausdorff measure \(\lambda_\delta\) on the sphere \(\partial \Omega\) into the normalized Hausdorff measure \(\lambda_\Sigma\) on \(\Sigma\) and the normalized Hausdorff measure \(\lambda_N\) on \(\Sigma_N\): \(\lambda_\delta = \delta \lambda_\Sigma + (1 - \delta) \lambda_N\). Therefore

\[
\text{Var}_{\lambda_\delta} f = \delta \text{Var}_{\lambda_\Sigma} f + (1 - \delta) \text{Var}_{\lambda_N} f + \delta(1 - \delta) \left( \int_\Sigma f d\lambda_\Sigma - \int_{\Sigma_N} f d\lambda_N \right)^2 \geq \delta \text{Var}_{\lambda_\Sigma} f.
\]

Furthermore, recall that by inequality (3.2), for any \(f \in C^1(\Omega)\), \(\text{Var}_{\lambda_\delta} f \leq \frac{1}{2} \int_\Omega \|\nabla f\|^2 d\lambda_\Omega\). It implies (3.14).

\[\Box\]

**Lemma 3.10.** It holds that

\[
\left( \int_\Omega f d\lambda_\Omega - \int_{\Sigma} f d\lambda_\Sigma \right)^2 \leq K_1(\delta) \int_\Omega \|\nabla f\|^2 d\lambda_\Omega
\]

with \(K_1(\delta) = \left( \sqrt{1 - \delta^2} + \frac{\sqrt{3}}{\pi} \right)^2\).

**Proof.** For every \(x \in \Omega \setminus \{0\}\) with polar coordinates \((r, \theta)\), \(r \in (0, 1), \theta \in (-\pi, \pi]\), denote by \(p_x\) the point of coordinates \((1, \delta \theta)\) on \(\Sigma\). Obviously, \(\int_\Sigma f(y) \lambda_\Sigma(dy) = \int_\Omega f(p_x) \lambda_\Omega(dx)\) and by Jensen’s inequality

\[
I := \left( \int_\Omega f d\lambda_\Omega - \int_{\Sigma} f d\lambda_\Sigma \right)^2 \leq \int_\Omega (f(x) - f(p_x))^2 \lambda_\Omega(dx).
\]

Define \(g(r, \theta) := f(r \cos(\theta), r \sin(\theta))\). Then

\[
I \leq \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi (g(r, \theta) - g(1, \delta \theta))^2 r dr d\theta \leq \left( \sqrt{J_1} + \sqrt{J_2} \right)^2,
\]

where \(J_1 = \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi (g(r, \theta) - g(1, \delta \theta))^2 r dr d\theta\) and \(J_2 = \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi (g(r, \delta \theta) - g(1, \delta \theta))^2 r dr d\theta\). On the one hand

\[
J_1 = \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi \left( \int_{\partial \theta}^\theta \frac{\partial g}{\partial \theta} (r, u) du \right)^2 r dr d\theta \leq \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi (r, u) du \int_{\partial \theta}^\theta \left( \frac{\partial g}{\partial \theta} \right)^2 (r, u) dr d\theta
\]

\[
\leq (1 - \delta)^2 \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi \left( \frac{\partial g}{\partial \theta} \right)^2 (r, u) du \int_{\partial \theta}^\theta r dr d\theta \leq (1 - \delta)^2 \frac{1}{\pi} \int_\Omega \|\nabla f\|^2 d\lambda_\Omega.
\]

On the other hand

\[
J_2 \leq \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi (1 - r) \int_r^1 \left( \frac{\partial g}{\partial r} \right)^2 (s, \delta \theta) ds dr d\theta \leq \frac{1}{\pi} \int_0^1 \int_{-\pi}^\pi \left( \frac{\partial g}{\partial r} \right)^2 (s, \delta \theta) \int_r^1 (1 - r) ds dr d\theta.
\]

For every \(s \in [0, 1]\), \(\int_0^1 (1 - r) dr = \frac{s^2}{2} - \frac{s^3}{3} \leq \frac{3s}{16}\), thus

\[
J_2 \leq \frac{3}{16\pi} \int_0^1 \int_{-\pi}^\pi \left( \frac{\partial g}{\partial r} \right)^2 (s, u) du ds dr d\theta \leq \frac{3}{16\pi} \int_\Omega \|\nabla f\|^2 d\lambda_\Omega.
\]

The proof of the lemma is completed by putting together (3.15), (3.16) and (3.17). \[\Box\]
Spectral gap for domains with boundary diffusion

0.2 0.4 0.6 0.8 1

\( \alpha \)

\( C \)

\( \Omega \)

\( \Sigma \)

\( \delta = 0.5 \)

\( \delta = 0.9 \)

Figure 2: The above two figures show the upper estimate given by the r.h.s of (3.12). In the case \( \delta = 0.9 \) (Figure 2b), the curve interpolates between the extremal constants \( C_\Sigma \) and \( C_\Omega \), as opposed to the half-sphere case (Figure 2a).

**Proof of Proposition 3.8.** We apply Proposition 2.1 with \( C_\Omega = \frac{1}{\sigma_\Omega} \), \( C_\Sigma = 4\delta^2 \), \( K_{\Sigma\Omega} = \frac{1}{2\pi} \), \( K_1(\delta) = \left( \sqrt{1 - \delta\pi} + \frac{1}{2}\sqrt{\frac{3}{\delta}} \right)^2 \) and \( K_2 = 0 \).

For \( \delta \) sufficiently large, the map \( \alpha \mapsto C_\alpha \) is continuous at \( \alpha = 0 \). Indeed, by Proposition 2.2, a sufficient condition is \( C_\Sigma(\delta) > C_\Omega + K_1(\delta) \), that is

\[ 4\delta^2 > \frac{1}{\sigma_\Omega} + \left( \sqrt{1 - \delta\pi} + \frac{1}{4}\sqrt{\frac{3}{\delta}} \right)^2, \]

which is satisfied for any \( \delta \geq 0.862 \).

### 3.4 Ball with a needle

Our final example is the unit ball \( \Omega = B_1 \) of \( \mathbb{R}^2 \) with a needle \( L \) of length \( L \) attached to one point of the boundary, i.e. \( L := \{ (x, 0) : 1 \leq x \leq L + 1 \} \), see Figure 3. The attachment point and the endpoint of the needle are denoted by \( x_0 := (1, 0) \) and \( x_L = (L + 1, 0) \), respectively.

In that setting, we define \( \tilde{\Omega} = B_1 \cup L, \Sigma = \partial B_1 \cup L \) and

\[ \lambda_\alpha = \alpha \lambda_\Omega + (1 - \alpha) \lambda_\Sigma, \]

where \( \lambda_\Omega \) is as previously the normalized Lebesgue measure on \( \Omega \) and \( \lambda_\Sigma = \frac{2\pi}{2\pi + \tau} \lambda_\theta + \frac{\tau}{2\pi + \tau} \lambda_\mathcal{L} \), with \( \lambda_\theta \) and \( \lambda_\mathcal{L} \) being the normalized Hausdorff measures on \( \partial \Omega \) and \( L \), respectively. We choose

\[ \mathcal{D}_0 = \left\{ f \in C_0(\tilde{\Omega}) \cap C^1(\tilde{\Omega} \setminus \{x_0\}) \mid \frac{\partial f}{\partial e_1} + \frac{\partial f}{\partial e_2} + \frac{\partial f}{\partial e_3} = 0 \text{ at } x_0 \right\}, \]

where \( e_1 = (0, 1) \), \( e_2 = (0, -1) \) and \( e_3 = (1, 0) \) are the three "tangent" vectors to \( \Sigma \) at point \( x_0 \), and \( D := \nabla, D^\tau := \sqrt{\mathcal{N}} \nabla^\tau \), which is well defined in \( \Sigma \setminus \{x_0\} \). With this choice, for \( \alpha \in [0, 1] \) (\( \mathcal{E}_\alpha, \mathcal{D}_0 \) is
a pre-Dirichlet form on $L^2(\Omega, \lambda_\alpha)$, whose closure generates Brownian motion on $\Omega$ with sticky boundary diffusion on $\Sigma$, i.e. whose generator is given by

$$A_\alpha(f) = \Delta f_{\Omega} + \beta \Delta_{\Sigma} f_{\Sigma} - \frac{\alpha}{1 - \alpha} \frac{2\pi}{\pi} \frac{L}{\partial f/\partial \nu}_{\partial \Omega},$$

with $\Delta_{\Sigma}$ being the generator of the canonical diffusion on $\Sigma$ with reflecting boundary condition at $x_L$. As before, the optimal Poincaré constant $C_\alpha$ for $A_\alpha$ is given by

$$C_\alpha := \sup_{f \in D_0} \frac{\text{Var}_{\lambda_\alpha} f}{\mathbb{E}_\alpha(f)},$$

and let $C_\Omega := C_1$ and $C_\Sigma := C_0$. In this case the following estimate is obtained.

**Proposition 3.11.**

$$C_\alpha \leq \max \left( \frac{1}{\sigma_\Omega} + \frac{3}{8} (1 - \alpha), \frac{1}{\beta \gamma_L} + \alpha \frac{L^2(\pi + L)}{\beta (2\pi + L)} \right),$$

where $\gamma_L > 0$ is the smallest positive solution to

$$2 \cos(\sqrt{\gamma_L}) \cos(\sqrt{2\pi}) + \sin(\sqrt{\gamma_L}) \sin(\sqrt{2\pi}) = 0. \quad (3.18)$$

Note that $\gamma_L \leq 1$ for any $L > 0$ and if $L = 2\pi$, $\gamma_{2\pi} = \left( \frac{\arccos(-1/3)}{2\pi} \right)^2 \approx 0.0925$.

Let us compute the constants needed to apply Proposition 2.1. As we do not expect an inequality of type (2.2) to hold in that case, we set $K_{\Sigma, \Omega} := +\infty$. Moreover, $C_\Sigma$ can be computed exactly as follows.

**Lemma 3.12.** In this case, $C_\Sigma = \frac{1}{\beta \gamma_L}$.

**Proof.** The constant $\frac{1}{C_\Sigma}$ is the smallest non-zero eigenvalue $\gamma$ of the following problem:

$$\begin{cases}
\beta \Delta f = -\gamma f & \text{on } \Sigma \setminus \{ x_0 \}, \\
\frac{\partial f}{\partial \nu} = 0 & \text{at point } x_L, \\
\frac{\partial f}{\partial e_1} + \frac{\partial f}{\partial e_2} + \frac{\partial f}{\partial e_3} = 0 & \text{at point } x_0,
\end{cases}$$

Figure 3: The ball (in green) is denoted by $\Omega$, the boundary of the ball is denoted by $\partial \Omega$ and the needle (in blue) is denoted by $\mathcal{L}$. 
where $\Delta^\tau$ is the Laplace-Beltrami operator on $\partial\Omega$ and $\mathcal{L}$. A general solution to that boundary value problem is given by

$$f(x) = \begin{cases} A \cos(\sqrt{\beta} y) + B \sin(\sqrt{\beta} y) & \text{if } x = (y,0) \in \mathcal{L}, \\ C \cos(\sqrt{\beta} \theta) + D \sin(\sqrt{\beta} \theta) & \text{if } x = (\cos \theta, \sin \theta) \in \partial\Omega, \end{cases}$$

where $A$, $B$, $C$ and $D$ have to satisfy the continuity assumption of $f$ at point $x_0$ and both boundary conditions, that is:

$$\begin{cases} A = C = C \cos(\sqrt{\beta} 2\pi) + D \sin(\sqrt{\beta} 2\pi), \\ 0 = -A \sin(\sqrt{\beta} L) + B \cos(\sqrt{\beta} L), \\ 0 = B + D + C \sin(\sqrt{\beta} 2\pi) - D \cos(\sqrt{\beta} 2\pi). \end{cases}$$

A short computation shows that this system has a non-trivial solution if and only if $\gamma_{\beta}$ solves (3.18). Therefore, $\frac{1}{\Sigma} = \beta \gamma_{L}$. Obviously, $\gamma = 1$ is a solution to (3.18), thus $\gamma_{L} \leq 1$. \qed

Next, we look for the constants $K_{1}$ and $K_{2}$.

**Lemma 3.13.** Inequality (2.3) holds with $K_{1} = \frac{3}{8}$ and $K_{2} = \frac{L^{2}(\pi + L)}{\pi(2\pi + L)}$.

**Proof.** Recall that $\Sigma = \partial\Omega \cup \mathcal{L}$. Let us insert the average of $f$ over $\partial\Omega$ as follows:

$$\left( \int_{\Omega} f d\lambda_{\Omega} - \int_{\Sigma} f d\lambda_{\Sigma} \right)^{2} \leq 2 \left( \int_{\Omega} f d\lambda_{\Omega} - \int_{\partial\Omega} f d\lambda_{\partial\Omega} \right)^{2} + 2 \left( \int_{\partial\Omega} f d\lambda_{\partial\Omega} - \int_{\Sigma} f d\lambda_{\Sigma} \right)^{2} \leq \frac{3}{8} \int_{\Omega} \| \nabla f \|^{2} d\lambda_{\Omega} + 2 \left( \int_{\partial\Omega} f d\lambda_{\partial\Omega} - \int_{\Sigma} f d\lambda_{\Sigma} \right)^{2},$$

where the second inequality follows directly from (3.3). Moreover, recalling that $\lambda_{\Sigma} = \frac{2\pi}{2\pi + L} \lambda_{\Omega} + \frac{L}{2\pi + L} \lambda_{\mathcal{L}}$

$$\left( \int_{\partial\Omega} f d\lambda_{\partial\Omega} - \int_{\mathcal{L}} f d\lambda_{\mathcal{L}} \right)^{2} = \left( \frac{L^{2}}{(2\pi + L)^{2}} \right) \left( \int_{\partial\Omega} f d\lambda_{\partial\Omega} - \int_{\mathcal{L}} f d\lambda_{\mathcal{L}} \right)^{2}.$$

For every $x = (\cos \theta, \sin \theta) \in \partial\Omega$, with $\theta \in (-\pi, \pi]$, we denote by $p_{x}$ the point of $\mathcal{L}$ with coordinates $(1 + L - \frac{\theta L}{\pi}, 0)$. It follows that

$$\left( \int_{\partial\Omega} f d\lambda_{\partial\Omega} - \int_{\mathcal{L}} f d\lambda_{\mathcal{L}} \right)^{2} = \left( \int_{\partial\Omega} (f(x) - f(p_{x})) d\lambda_{\partial\Omega} \right)^{2} \leq \int_{\partial\Omega} (f(x) - f(p_{x}))^{2} d\lambda_{\partial\Omega}.$$

Denoting by $\lambda_{\beta}^{\Omega}$ and $\lambda_{\beta}^{\mathcal{L}}$ the normalized Hausdorff measures on $\partial\Omega^{+} := \{(x, y) \in \partial\Omega : y > 0\}$ and $\partial\Omega^{-} := \{(x, y) \in \partial\Omega : y < 0\}$, respectively,

$$\int_{\partial\Omega} (f(x) - f(p_{x}))^{2} d\lambda_{\partial\Omega} = \frac{1}{2} \int_{\partial\Omega^{+}} (f(x) - f(p_{x}))^{2} d\lambda_{\beta}^{\Omega} + \frac{1}{2} \int_{\partial\Omega^{-}} (f(x) - f(p_{x}))^{2} d\lambda_{\beta}^{\mathcal{L}}.$$

Moreover, for any $C^{1}$-function $g : [-\pi, L] \to \mathbb{R}$,

$$\int_{\partial\Omega} \left| g(\theta) - g(L - \frac{\theta L}{\pi}) \right|^{2} d\theta \leq \frac{\pi + L}{2} \int_{-\pi}^{L} |g'(t)|^{2} dt,$$
so we deduce, identifying $\partial \Omega^+$ with $[-\pi, 0]$ and $\mathcal{L}$ with $[0, L]$, that
\[
\int_{\partial \Omega^+} (f(x) - f(p_x))^2 d\lambda_\partial^+ \leq \frac{\pi + L}{2} \left( \pi \int_{\partial \Omega^+} \|\nabla f\|^2 d\lambda_\partial^+ + L \int_{\mathcal{L}} \|\nabla f\|^2 d\lambda_{\mathcal{L}} \right)
\]
and using symmetry to deal with $\partial \Omega^-$, we obtain
\[
\int_{\partial \Omega} (f(x) - f(p_x))^2 d\lambda_\partial \leq \frac{\pi + L}{4} \left( \pi \int_{\partial \Omega^+} \|\nabla f\|^2 d\lambda_\partial^+ + \pi \int_{\partial \Omega^-} \|\nabla f\|^2 d\lambda_\partial^- + 2L \int_{\mathcal{L}} \|\nabla f\|^2 d\lambda_{\mathcal{L}} \right)
\]
\[
\leq \frac{(\pi + L)(2\pi + L)}{2} \int_{\Sigma} \|\nabla f\|^2 d\lambda_{\Sigma}.
\]
Putting together the above inequalities, we get
\[
\left( \int_{\Omega} f \, d\lambda_{\Omega} - \int_{\Sigma} f \, d\lambda_{\Sigma} \right)^2 \leq \frac{3}{8} \int_{\Omega} \|\nabla f\|^2 d\lambda_{\Omega} + \frac{L^2}{(2\pi + L)^2} \frac{(\pi + L)(2\pi + L)}{2\beta} \int_{\Sigma} \beta \|\nabla f\|^2 d\lambda_{\Sigma}
\]
which leads to inequality (2.3) with $K_1 = \frac{3}{8}$ and $K_2 = \frac{L^2(\pi + L)}{\beta(2\pi + L)}$.

**Proof of Proposition 3.11.** Since $K_{\Sigma, \Omega} = \infty$, we immediately get from Proposition 2.1 that
\[
C_\alpha \leq \max \left( C_{\Omega} + (1 - \alpha)K_1, \alpha K_2, C_{\Sigma} + \alpha K_2 \right) = \max \left( C_{\Omega} + (1 - \alpha)K_1, C_{\Sigma} + \alpha K_2 \right)
\]
Therefore,
\[
C_\alpha \leq \max \left( \frac{1}{\sigma_{\Omega}} + \frac{3}{8}(1 - \alpha), \frac{1}{\beta \gamma_L} + \alpha \frac{L^2(\pi + L)}{\beta(2\pi + L)} \right),
\]
where $\sigma_{\Omega} \approx 3.39$.

**Remark 3.14.** If $\beta$ is large enough, that is if the diffusion velocity is larger on $\Sigma$ than on $\Omega$, then the first term in (3.19) dominates. Precisely, if $\beta \geq \sigma_{\Omega} \left( \frac{1}{\gamma_L} + \frac{L^2(\pi + L)}{2\pi + L} \right)$, then (3.19) rewrites for any $\alpha$
\[
C_\alpha \leq \frac{1}{\sigma_{\Omega}} + \frac{3}{8}(1 - \alpha).
\]
Conversely, if $\beta \leq \frac{1}{\gamma_L} \left( \frac{1}{\sigma_{\Omega}} + \frac{3}{8} \right)^{-1}$, then (3.19) rewrites for any $\alpha$
\[
C_\alpha \leq \frac{1}{\beta \gamma_L} + \alpha \frac{L^2(\pi + L)}{\beta(2\pi + L)}.
\]

**References**

[1] Alexander Aurell and Boualem Djehiche, *Behavior near walls in the mean-field approach to crowd dynamics*, SIAM J. Appl. Math. 80 (2020), no. 3, 1153–1174. MR 4096131

[2] Dominique Bakry, Ivan Gentil, and Michel Ledoux, *Analysis and geometry of Markov diffusion operators*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 348, Springer, Cham, 2014. MR 3155209
[3] William Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2) **138** (1993), no. 1, 213–242. MR 1230930

[4] Rodney Josué Biezuner, *Best constants in Sobolev trace inequalities*, Nonlinear Anal. **54** (2003), no. 3, 575–589. MR 1978428

[5] Julián Fernández Bonder, Julio D. Rossi, and Raúl Ferreira, *Uniform bounds for the best Sobolev trace constant*, Adv. Nonlinear Stud. **3** (2003), no. 2, 181–192. MR 1971310

[6] Jean-Michel Bony, Philippe Courrège, and Pierre Priouret, *Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum*, Ann. Inst. Fourier (Grenoble) **18** (1968), no. fasc. 2, 369–521 (1969). MR 245085

[7] Jean-Dominique Deuschel, Giambattista Giacomin, and Lorenzo Zambotti, *Scaling limits of equilibrium wetting models in (1 + 1)-dimension*, Probab. Theory Related Fields **132** (2005), no. 4, 471–500. MR 2198199

[8] Klaus-Jochen Engel, *The Laplacian on $C(\Omega)$ with generalized Wentzell boundary conditions*, Arch. Math. (Basel) **81** (2003), no. 5, 548–558. MR 2029716

[9] José F. Escobar, *The geometry of the first non-zero Stekloff eigenvalue*, J. Funct. Anal. **150** (1997), no. 2, 544–556. MR 1479552

[10] Torben Fattler, Martin Grothaus, and Robert Voßhall, *Construction and analysis of a sticky reflected distorted Brownian motion*, Ann. Inst. Henri Poincaré Probab. Stat. **52** (2016), no. 2, 735–762. MR 3498008

[11] Vincenzo Ferone, Carlo Nitsch, and Cristina Trombetti, *On a conjectured reverse Faber-Krahn inequality for a Steklov-type Laplacian eigenvalue*, Commun. Pure Appl. Anal. **14** (2015), no. 1, 63–82. MR 3299025

[12] Alexandre Girouard and Iosif Polterovich, *Spectral geometry of the Steklov problem (survey article)*, J. Spectr. Theory **7** (2017), no. 2, 321–359. MR 3662010

[13] Gisèle Ruiz Goldstein, *Derivation and physical interpretation of general boundary conditions*, Adv. Differential Equations **11** (2006), no. 4, 457–480. MR 2215623

[14] Gisèle Ruiz Goldstein, Jerome A. Goldstein, Davide Guidetti, and Silvia Romanelli, *Maximal regularity, analytic semigroups, and dynamic and general Wentzell boundary conditions with a diffusion term on the boundary*, Ann. Mat. Pura Appl. (4) **199** (2020), no. 1, 127–146. MR 4065110

[15] Martin Grothaus and Robert Voßhall, *Stochastic differential equations with sticky reflection and boundary diffusion*, Electron. J. Probab. **22** (2017), Paper No. 7, 37. MR 3613700
[16] Nobuyuki Ikeda, *On the construction of two-dimensional diffusion processes satisfying Wentzell’s boundary conditions and its application to boundary value problems*, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. **33** (1960/61), 367–427. MR 126883

[17] James Kennedy, *An isoperimetric inequality for the second eigenvalue of the Laplacian with Robin boundary conditions*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 627–633. MR 2448584

[18] Alexander V. Kolesnikov and Emanuel Milman, *Brascamp-Lieb-type inequalities on weighted Riemannian manifolds with boundary*, J. Geom. Anal. **27** (2017), no. 2, 1680–1702. MR 3625169

[19] Vitalii Konarovskyi and Max von Renesse, *Reversible coalescing-fragmentating Wasserstein dynamics on the real line*, 2017.

[20] Nikolay Kuznetsov and Alexander Nazarov, *Sharp constants in the Poincaré, Steklov and related inequalities (a survey)*, Mathematika **61** (2015), no. 2, 328–344. MR 3343056

[21] Yanyan Li and Meijun Zhu, *Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries*, Comm. Pure Appl. Math. **50** (1997), no. 5, 449–487. MR 1443055

[22] Svetlana Matculevich and Sergey Repin, *Explicit constants in Poincaré-type inequalities for simplicial domains and application to a posteriori estimates*, Comput. Methods Appl. Math. **16** (2016), no. 2, 277–298. MR 3483617

[23] Umberto Mosco, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal. **123** (1994), no. 2, 368–421. MR 1283033

[24] Delio Mugnolo, Robin Nittka, and Olaf Post, *Norm convergence of sectorial operators on varying Hilbert spaces*, Oper. Matrices **7** (2013), no. 4, 955–995. MR 3154581

[25] Delio Mugnolo and Silvia Romanelli, *Dirichlet forms for general Wentzell boundary conditions, analytic semigroups, and cosine operator functions*, Electron. J. Differential Equations (2006), No. 118, 20. MR 2255233

[26] A. I. Nazarov and S. I. Repin, *Exact constants in Poincaré type inequalities for functions with zero mean boundary traces*, Math. Methods Appl. Sci. **38** (2015), no. 15, 3195–3207. MR 3400329

[27] Andreas Nonnenmacher and Martin Grothaus, *Overdamped limit of generalized stochastic hamiltonian systems for singular interaction potentials*, 2018.

[28] Robert C. Reilly, *Applications of the Hessian operator in a Riemannian manifold*, Indiana Univ. Math. J. **26** (1977), no. 3, 459–472. MR 474149

[29] Julio D. Rossi, *First variations of the best Sobolev trace constant with respect to the domain*, Canad. Math. Bull. **51** (2008), no. 1, 140–145. MR 2384747
[30] Abdolhakim Shouman, *Generalization of Philippin’s results for the first Robin eigenvalue and estimates for eigenvalues of the bi-drifting Laplacian*, Ann. Global Anal. Geom. 55 (2019), no. 4, 805–817. MR 3951758

[31] M. A. Shubin, *Pseudodifferential operators and spectral theory*, second ed., Springer-Verlag, Berlin, 2001, Translated from the 1978 Russian original by Stig I. Andersson. MR 1852334

[32] Kazuaki Taira, *Boundary value problems and Markov processes*, third ed., Lecture Notes in Mathematics, vol. 1499, Springer, Cham, [2020] ©2020, Functional analysis methods for Markov processes. MR 4176673

[33] Satoshi Takanobu and Shinzo Watanabe, *On the existence and uniqueness of diffusion processes with Wentzell’s boundary conditions*, J. Math. Kyoto Univ. 28 (1988), no. 1, 71–80. MR 929208

[34] A. D. Ventcel, *On boundary conditions for multi-dimensional diffusion processes*, Theor. Probability Appl. 4 (1959), 164–177. MR 121855

[35] Hendrik Vogt and Jürgen Voigt, *Wentzell boundary conditions in the context of Dirichlet forms*, Adv. Differential Equations 8 (2003), no. 7, 821–842. MR 1988680

[36] Mahamadi Warma, *The Robin and Wentzell-Robin Laplacians on Lipschitz domains*, Semigroup Forum 73 (2006), no. 1, 10–30. MR 2277314

[37] Shinzo Watanabe, *On stochastic differential equations for multi-dimensional diffusion processes with boundary conditions. II*, J. Math. Kyoto Univ. 11 (1971), 545–551. MR 287612