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Behaviour of unimorph ring-shaped piezoelectric actuator in fluidic valve

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Abstract: This article focuses on the behaviour analysis of an unimorph ring-shaped piezoelectric plate that is used as an actuator in a fluidic valve. First, the system modelling is discussed by using the material mechanical properties of the piezoelectric actuator and the non-piezoelectric substrate. Then, two differential equations are established for the displacements along the radial and transverse directions. After that, based on the clamped edge conditions, the transverse displacement solution is obtained as an explicit function of the applied voltage. On this basis, the analytical relationship between the transverse displacement and the applied voltage is displayed. Finally, three different voltages are applied to the ring-shaped piezoelectric plate of a fluidic valve. The measured experimental results are almost consistent to the analytical results, which illustrates the validity of the established system model.

1 Introduction

Piezoelectric materials are widely utilised in engineering fields [1, 2]. Especially, the piezoelectric actuator has been deeply applied in flow control applications. A relatively new application for the piezoelectric actuator is the micro-fluidic valves. The design of these fluidic valves requires the development of analytical models to predict the behaviour of the piezoelectric actuator.

In the devoted literature, the predicting methods of the behaviour of the piezoelectric actuator can be split into three main categories. The first one is the equivalent strain model of piezoelectric plate. This method transforms the piezoelectric effect into the strain or stress of the actuator by using the constitutive relations of plane [3]. For instance, an analytical equation for the passive plate deflection is derived in [4] based on the assumption that the strain distribution is linear along the thickness direction of the passive plate of the piezoelectric actuator.

In [5], the deflected behaviour of unimorph circular piezoelectric diaphragm actuators was analysed under electrical loading. The second one is the equivalent multilayer thin plate model. Based on the Kirchhoff hypothesis, the complex multilayer piezoelectric actuator is treated as a multilayer thin plate with complex constitutive relation. Thus, the three-dimensional coupling modelling problem is transformed into a two-dimensional coupling modelling problem [6, 7]. In [8], the modal response of asymmetric piezoelectric actuators was deduced by means of energy variational method. The coupling model of the asymmetric piezoelectric actuator was established in [9] by using the circular thin plate elastic theory and the ring elasticity theory, and the maximum displacement of the drive centre position was obtained. The coupling dynamic behaviour of piezoelectric actuator was deduced in [10] by assuming that the middle plane of elastic layer is the curved neutral surface. The last one is the three-dimensional electro-solid coupling model [11]. In [12], the three-dimensional analytic solution of simply supported symmetric piezoelectric actuator was obtained by using the method of double Fourier series expansion and eigenvalue calculation. By using state space method, the three-dimensional analytic solution of simply supported symmetric piezoelectric actuator is presented in [13]. Similarly, the three-dimensional vibration coupling solution of composite rectangular piezoelectric actuator was deduced in [14].

In this paper, a ring-shaped piezoelectric actuator is applied in the flow control of a micro-fluidic valve. The behaviour analysis of the piezoelectric actuator is critical for the design of the fluidic valve and the controller to achieve the accurate flow control. To this end, the ring-shaped piezoelectric actuator is discussed here. Moreover, the experimental results are illustrated to show the efficiency of the analysis method for the ring-shaped piezoelectric plate.

The paper is organised as follows. Section 2 discusses the system modelling of the ring-shaped unimorph ring-shaped piezoelectric plate. Section 3 gives the analytical and experimental results.

2 Theoretical analysis

2.1 System modelling

The composite plate with a thin ring-shaped piezoelectric layer and a non-piezoelectric substrate is shown in Fig. 1. In Fig. 1, the notations \( r \) and \( z \) denote the radial and transverse directions, respectively, \( r_0 \) is the outer radius, \( r_1 \) and \( r_2 \) are the inner radiuses, and \( h_2 \) and \( h_3 \) are the thickness of the piezoelectric layer and the non-piezoelectric substrate, respectively.

For the deflected plate, the radial and circumferential strain–displacement relationships are (see [15, eqs. (6.22–6.23)])

\[
\begin{align*}
\epsilon_r &= \frac{du}{dr} + (z + z_c)\theta_p, \\
\epsilon_\theta &= \frac{u}{r} + (z + z_c)\theta_b,
\end{align*}
\]

where \( \epsilon \) is the strain, \( \varphi_0 = -\partial w/\partial r \) is the radial curvature, \( \psi_0 = -\frac{1}{r}\partial w/\partial \theta \) is the circumferential curvature, \( w \) is the transverse displacement \((z\) direction), \( u \) is the displacement in the radial direction \((z = -z_c)\), and \( \theta \) is the transverse slope.

The constitutive equations for the piezoelectric layer are

\[
\begin{align*}
\sigma_r &= \Delta_{11}(\sigma_{1p} - \nu\sigma_{2p}) - d_{31}E_z, \\
\sigma_\theta &= \Delta_{12}(\sigma_{1p} - \nu\sigma_{2p}) - d_{31}E_z, \\
E_z &= -d_{31}(\sigma_{1p} + \sigma_{2p}) + \epsilon^p_{31}E_z,
\end{align*}
\]
where $\nu = -S_m^E/S_p^E$ is Poisson’s ratio, $\sigma_{p}^{i}$ is the stress of the piezoelectric layer, $\varepsilon_{33}^{p}$ is the permittivity of the piezoelectric, $d_{31}$ is the piezoelectric constant, $D_{i}$ is the charge density, $E_{i}$ is the electric field strength, and $S_{11}^{m}$ is the elastic compliance constant for the non-piezoelectric substrate.

The constitutive equations for the non-piezoelectric substrate are

\[
\begin{align*}
\sigma_{m} &= S_m(\sigma_m - \nu \sigma_{hm}), \\
\epsilon_{m} &= S_m^{-1}(\epsilon_m + \nu \epsilon_{hm}), \\
\sigma_{hm} &= S_m^{-1}(\sigma_m - \nu \sigma_{hm}), \\
\epsilon_{hm} &= S_m^{-1}(\epsilon_m + \nu \epsilon_{hm}).
\end{align*}
\]

(ii) Non-piezoelectric substrate:

\[
\begin{align*}
\sigma_{m} &= S_m(\sigma_m - \nu \sigma_{hm}), \\
\epsilon_{m} &= S_m^{-1}(\epsilon_m + \nu \epsilon_{hm}), \\
\sigma_{hm} &= S_m^{-1}(\sigma_m - \nu \sigma_{hm}), \\
\epsilon_{hm} &= S_m^{-1}(\epsilon_m + \nu \epsilon_{hm}).
\end{align*}
\]

Let

\[
\begin{align*}
K_p &= S_p^{E/(1 - \nu^2)}, \\
K_m &= S_m^{E/(1 - \nu^2)}, \\
E_m &= (1 + \nu)d_{31}E_{i}.
\end{align*}
\]

Then, one obtains

\[
\begin{align*}
\frac{\sigma_{p}^{i}}{\sigma_{hp}^{i}} &= K_p \left[ \frac{1}{\nu} \frac{\partial u}{\partial r} \right] + [z + z_c] \left[ \frac{\rho_{i}}{\rho_{h}} \right] + K_m \left[ \frac{E_m}{E_{i}} \right],
\end{align*}
\]

and

\[
\begin{align*}
N_r &= \int_{0}^{h_p} \sigma_{rp}^{i} dz + \int_{0}^{h_m} \sigma_{rm}^{i} dz, \\
N_{\theta} &= \int_{0}^{h_p} \sigma_{r \theta}^{i} dz + \int_{0}^{h_m} \sigma_{r \theta m}^{i} dz, \\
M_r &= \int_{0}^{h_p} \sigma_{rp}^{i}(z + z_c) dz + \int_{0}^{h_m} \sigma_{rm}^{i}(z + z_c) dz, \\
M_{\theta} &= \int_{0}^{h_p} \sigma_{r \theta}^{i}(z + z_c) dz + \int_{0}^{h_m} \sigma_{r \theta m}^{i}(z + z_c) dz.
\end{align*}
\]

In the absence of external forces, the equilibrium equations for any radial section of the axisymmetric plate are (see [15, eqs. (6.15–6.19)]):

\[
\begin{align*}
\frac{dN_r}{dr} + \frac{N_r - N_{\theta}}{r} &= 0, \\
\frac{dM_r}{dr} + \frac{M_r - M_{\theta}}{r} &= Q_r,
\end{align*}
\]

where $Q_r \approx 0$ is the shear resultant neglected here. It is noted that there are no initial or external forces applied to the plate.
Next, let us analyse the second equation

\[ \frac{dM}{dr} = K_p h_p \left( \frac{h_p}{2} + z_0 \right) \left( \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} \right) \]

Furthermore,

\[ \frac{dM}{dr} = K_p h_p \left[ \frac{-h_p}{2} + z_0 \right] \left( \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} \right) \]

It follows that

\[ \frac{dM}{dr} + \frac{M_f - M_0}{r} = G_u \left( \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} \right) \]

where

\[ G_u = K_p h_p \left( \frac{h_p}{2} + z_0 \right) + K_w h_w \left( \frac{h_w}{3} + z_0 \right) \]

Therefore, from (11) and (12), the equations in (10) yield

\[ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0, \]

\[ \frac{d^2 w}{dr^2} + \frac{1}{r} \frac{dw}{dr} - \frac{w}{r^2} = 0. \]

It is noted that the equations in (13) are independent of the electric field \( E \), which is given by \( V/h_p \), where \( V \) is an applied voltage. Notice that the piezoelectric field term in the right-hand side in (6) is not a function of \( r \). Then, the differentiations of the radial force resultant, \( N_r \), and the moment per unit length, \( M_r \), with respect to \( r \) make \( E \) terms vanish. Moreover, the subtractions of force resultant \( N_r - N_0 \) and moments per unit length \( M_r - M_0 \) in the equilibrium equations of (10) also cause the \( E \) terms to disappear. As a result, the piezoelectric field term does not explicitly appear in the governing displacement equations. However, under specific edge conditions and different radius of the piezoelectric layer, the displacement of a circular piezoelectric plate could be affected by the piezoelectric field term.

### 2.2 Edge condition

The ring-shaped piezoelectric actuator in Fig. 1 is a clamped edge plate. It implies that the boundary conditions are

\[ \begin{align*}
& u(r_s) < \infty, \\
& \frac{dw}{dr} |_{r = r_s} < \infty, \\
& w(r_s) = 0, \\
& \frac{dw}{dr} |_{r = r_s} = 0, \\
& \frac{d^2 w}{dr^2} |_{r = r_s} = 0.
\end{align*} \]
Notice that in the presence of the applied static voltage, the displacement solution is expressed as an explicit function of the applied voltage. As a result, the displacement of the ring-shaped piezoelectric actuator is controllable by changing the size of the applied voltage.

### Table 1 Physical parameters of the experimental system

| Symbols | Characteristics | Values |
|---------|-----------------|--------|
| \( r_a \) | outer radius | 25 mm |
| \( r_b \) | inner radius of the piezoelectric | 9 mm |
| \( h_p \) | thickness of the piezoelectric | 0.5 mm |
| \( h_m \) | thickness of the substrate | 0.3 mm |
| \( \nu \) | Poisson's ratio | 0.31 |
| \( d_{33} \) | piezoelectric constant | \( 3.2030 \times 10^{-9} \) mm V\(^{-1} \) |
| \( s_{31} \) | elastic compliance constant for the piezoelectric | \( 1.6233 \times 10^{-11} \) m\(^2\)N\(^{-1} \) |
| \( S_m \) | elastic compliance constant for the substrate | \( 1.4293 \times 10^{-9} \) m\(^2\)N\(^{-1} \) |

Applying the boundary conditions and the general solutions of the equations in (13), the plate deflections are given by

\[
u(r) = 0
\]

and (see (16)) where

\[
C_1 = 3d_{33}h_mS_m(h_m + h_p),
\]

\[C_2 = 4S_m^2(2r_1h_m^2S_m + 6S_m^2h_m^2S_m + S_m^2h_m^2S_m + 6S_m^2h_m^2S_m + 6S_m^2h_m^2S_m + 6S_m^2h_m^2S_m),
\]

\[
+ h_mS_m(1 + \nu),
\]

\[
C_3 = 4S_m^2(2r_1h_m^2S_m + 6S_m^2h_m^2S_m + 6S_m^2h_m^2S_m + 6S_m^2h_m^2S_m + 6S_m^2h_m^2S_m + 6S_m^2h_m^2S_m),
\]

\[
+ h_mS_m(1 + \nu).
\]

Notice that in the presence of the applied static voltage, the displacement solution is expressed as an explicit function of the voltage. As a result, the displacement of the ring-shaped piezoelectric actuator is controllable by changing the size of the applied voltage.

### 3 Experimental validation

#### 3.1 Experimental set-up

A linear positioning experimental set-up is established to demonstrate the validity of the analysis. It is shown in Fig. 2. The lead zirconate titanate amplifier is used to supply the power signal for the ring-shaped piezoelectric plate. The linear positioning equipment is utilised to measure the displacement of the ring-shaped piezoelectric plate. The ring-shaped piezoelectric plate is the key actuator of the fluidic valve to control the flow rate.

#### 3.2 Analytical result

To calculate the analytical deflections of the piezoelectric plate, the corresponding parameters are given in Table 1.

Let \( V = 100 \) V. Then, the analytical result of displacement of the ring-shaped piezoelectric plate is displayed in Fig. 3. In addition, if the DC voltage \( V \) is increased from 0 to 100 V, then the displacement of the ring-shaped piezoelectric plate is illustrated in Fig. 4.

For a given voltage, Fig. 3 shows that the displacement of the piezoelectric plate decreases significantly along the \( r \) direction. In addition, the displacement along the radius under different voltages is illustrated in Fig. 4. The relationship between applied voltage and the deflection is very important for the accurate flow control.
3.3 Experimental results

The experimental results of the deflection of the ring-shaped piezoelectric plate are shown in Fig. 5. Notice that each test is done ten trials.

It is shown in Fig. 5 that the analytical results are almost consistent with the experimental results. This demonstrates the efficiency of the employed analysis method. In addition, from the error bars displayed in Fig. 5, we can see that the designed fluidic valve has a very good consistency under the same test conditions.

4 Conclusion

The behaviour analysis of the ring-shaped piezoelectric actuator plays an important role in the design of the fluidic valve and the flow controller. To this end, the equivalent strain modelling method is employed to present the differential equations of the transverse and radial displacements. In addition, the experimental set-up is established to measure the transverse displacement along the radial direction under different applied voltages. It is verified that the established system model is almost consistent with the experimental ring-shaped piezoelectric actuator.

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6 References

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