A STUDY OF PHASE TRANSITION IN BLACK HOLE THERMODYNAMICS

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This paper deals with five-dimensional black hole solutions in (a) Einstein-Maxwell-Gauss-Bonnet theory with a cosmological constant and (b) Einstein-Yang-Mills-Gauss-Bonnet theory for spherically symmetric space time. In both the cases the possibility of phase transition is examined and it is analyzed whether the phase transition is a Hawking-Page type phase transition or not.

Keywords : Black hole, Thermodynamics and phase transition

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I. INTRODUCTION

The geometry of the event horizon of a black hole (BH) and its thermodynamics are nicely interrelated. BH temperature (known as Hawking temperature in literature) is proportional to surface gravity on the horizon while entropy is related to the area of the event horizon [by Hawking, S. W.(1975) and Bekenstein, J. D.(1973)] and they follow the first law of thermodynamics [by Bekenstein, J. D.(1973)]. Although, the statistical nature of the BH thermodynamics is completely unknown, yet the thermodynamical stability of the BH is characterized by the sign of its heat capacity. In fact, a BH is said to be thermodynamically unstable [as Schwarzschild BH (SBH)] if heat capacity < 0 while if heat capacity changes sign in the parameter space such that it diverges [by Hut, P.(1977)] then ordinary thermodynamics tells us that it is a second order phase transition [by Davies, P. C. W.(1977,1977 and 1989)]. Further for extremal BH there exists a critical point and phase transition takes place from an extremal BH to its non-extremal counter part.

Gross et. al. [by Gros, D.J. et al (1982)] had shown that the SBH having negative specific heat is in an unstable equilibrium with T, the temperature of the heat reservoir. In fact due to small fluctuations the SBH will either decay (to hot flat space) or grow absorbing thermal radiation in the heat reservoir [by York, J. W.(1986)]. However it is possible to have a thermodynamically stable BH having positive specific heat through a phase transition from thermal radiation [by Myung, Y. S.(2008 and 2007)]. In this context, a mechanism of phase transition was introduced by Hawking and Page [Hawking, S. W. et al(1983)] showing the transition between thermal AdS space and Schwarzschild-AdS (SAdS) BH [Myung, Y. S.(2008 and 2008), Brown, J. D. et al (1994) and Witten, E.(1983)].

In this work, we will analyze the thermodynamical quantities namely the free energy, specific heat etc. for possibility of a phase transition for both Einstein-Maxwell-Gauss-Bonnet (EMGB)- BH and Einstein-Yang-Mills-Gauss-Bonnet (EYMG) BH.

II. PHASE TRANSITION OF EMGB BHS

The action in five dimensional space time \((M, g_{\mu\nu})\) that represents Einstein-Maxwell theory with a Gauss-Bonnet term and a cosmological constant has the expression [Boulware, D. G. et al.(1985), Willhtire, D. L.(1986 and 1988), Thibeault, M. et al(2006)]

\[
S = \frac{1}{2} \int_M d^5 x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \alpha R_{GB} \right]
\]  

(1)

where \(R_{GB} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\), is the Gauss-Bonnet term, \(\alpha\) is the GB coupling parameter having dimension \((\text{length})^2\) (\(\alpha^{-1}\) is related to string tension in heterotic super string theory), \(\Lambda\) is the...
cosmological constant and $F_{\mu\nu} = (\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})$ is the usual electromagnetic field tensor with $A_{\mu}$, the vector potential. Now variation of this action with respect to the metric tensor and $F_{\mu\nu}$ gives the modified Einstein field equations and Maxwell’s equations:

$$G_{\mu\nu} - \alpha H_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$$  \hspace{1cm} (2)

and

$$\nabla_{\mu}F^{\mu}_{\nu} = 0$$  \hspace{1cm} (3)

where $H_{\mu\nu}$ is the Lovelock tensor given by

$$H_{\mu\nu} = \frac{1}{2}g_{\mu\nu}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2) - 2RR_{\mu\nu} + 4R_{\mu}^{\lambda}R_{\lambda\nu} + 4R^\sigma\rho R_{\mu\rho\nu\sigma} - R_{\mu}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma}$$  \hspace{1cm} (4)

and

$$T_{\mu\nu} = 2F_{\mu}^{\lambda}F_{\nu\lambda} - \frac{1}{2}F_{\lambda\mu}F^{\lambda\sigma}g_{\mu\nu}$$  \hspace{1cm} (5)

is the electromagnetic stress tensor.

(Note that the modified Einstein field equations (2) do not contain any derivatives of the curvature terms and hence the field equations remain second order).

If the manifold $M$ is chosen to be five dimensional spherically symmetric space-time having the line element

$$ds^2 = -B(r)dt^2 + B^{-1}(r)dr^2 + r^2(d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2d\theta_3^2))$$  \hspace{1cm} (6)

with

$0 \leq \theta_1, \theta_2 \leq \pi, \quad 0 \leq \theta_3 \leq 2\pi$,

then solving the above field equations one obtains [Hawking, S. W. (1975) and Wilthire, D. L.(1986 and 1988)]

$$B(r) = 1 + \frac{r^2}{4\alpha} - \frac{r^2}{4\alpha} \sqrt{1 + \frac{16M\alpha}{\pi r^4} - \frac{8Q^2\alpha}{3r^6} + \frac{4\Lambda\alpha}{3}}$$  \hspace{1cm} (7)

Here in an orthonormal frame the non-null components of the electromagnetic tensors are $F_{\dot{t}\dot{r}} = -F_{\dot{r}\dot{t}} = \frac{Q}{2r}$. Note that in the limit $\alpha \to 0$ one may recover the Einstein-Maxwell solution with a cosmological constant. Further, in the limit with $\Lambda = 0$ we have the five-dimensional Reissner-Nordström Solution and hence the parameters $M(> 0)$ and $Q$ can be identified as the mass and charge respectively. Moreover, for the solution (7) to be well defined, the radial coordinate $r$ must have a minimum value ($r_{\text{min}}$) so that the expression within the square root is positive definite, i.e, the solution (7) is well defined for $r > r_{\text{min}}$ where $r_{\text{min}}$ satisfies

$$1 + \frac{16ma}{\pi r_{\text{min}}^4} - \frac{8Q^2\alpha}{3r_{\text{min}}^6} + \frac{4\Lambda\alpha}{3} = 0$$

The surface $r = r_{\text{min}}$ corresponds to a curvature singularity. However, depending on the values of the parameters this singular surface may be surrounded by the event horizon (having radius $r_h$ such that $B(r_h) = 0$) and the solution (7) describes a black hole solution known as EMGB BH. On the other hand, if no event horizon exists then the above solution represents a naked singularity.

A. Calculation of thermodynamic quantities

We can write the mass parameter for EMGB BH as (from $B(r_h) = 0$)

$$M = \pi \left[ \frac{Q^2}{6} r_h^{-2} + \alpha + \frac{1}{2} r_h^2 - \frac{\Lambda}{12} r_h^4 \right]$$  \hspace{1cm} (8)
The surface area of the event horizon is given by, \( A = 2\pi^2 r_h^3 \) and hence the entropy of the black hole by [Boulware, D. G. (1985)] takes the form

\[
S = \frac{K_B A}{4G\hbar} = \frac{K_B \pi^2}{2G\hbar} r_h^3
\]

Now choosing \( \hbar = 1 \) and the Boltzmann constant appropriately, we have,

\[
S = r_h^3
\]  

(9)

To find the thermodynamic quantities we follow Chakraborty et.al.(2008) and Biswas et. al.(2009) and find

- **Hawking Temperature** \((T_H) = -\frac{\pi}{9r_h^3} (Q^2 - 3r_h^4 + \Lambda r_h^6)\)  

(10)

- **Specific Heat** \((c_Q) = T \left( \frac{\partial S}{\partial T} \right)_Q = 3r_h^3 \frac{(Q^2 - 3r_h^4 + \Lambda r_h^6)}{(-5Q^2 + 3r_h^4 + \Lambda r_h^6)}\)

(11)

- **Free Energy** \((F) = M - T_H S = \frac{\pi}{36} (10Q^2r_h^2 - 36\alpha + 6r_h^4 + \Lambda r_h^6)\)

(12)

B. Graphical analysis of phase transition

The figures 1 – 3 shows the variation of the above thermodynamical quantities with the variation of the radius of the event horizon while figure 4 represents the functional dependence of \( F \) over \( T_H \).

In each figure there are three set of diagrams corresponding to \( Q = 0.5, 1, \text{ and } 2 \) respectively. In figure 1(a) we have plotted \( T_H \) for \( \Lambda = -1 \) with \( Q = 0.5, 1, \text{ and } 2 \) represented by solid line, dashed line and dotted line respectively. Similar are the figures 1(b) and 1(c) for \( \lambda = 0 \) and \( \lambda = +1 \) respectively.

In the figures 1(a) – (c) \( T_H \) has a maximum (and also a minimum for \( \lambda = -1 \)) at some finite \( r_h \) while asymptotically, \( T_H \to +\infty \) for \( \lambda = -1 \), \( T_H \to 0 \) for \( \lambda = 0 \) and \( T_H \to -\infty \) for \( \lambda = +1 \) respectively.

The graphs of \( c_Q \) in figures 2(a) - 2(c) show some distinct features. In figure 2(a) for \( \lambda = -1 \) and \( Q = 0.5 \) (the solid line) \( c_Q \) changes sign at two finite value of \( r_h \), i.e., \( c_Q \) changes form positive value to negative value and then again \( c_Q \) becomes positive. So the black hole is a stable one for small \( r_h \) and then it becomes unstable at intermediate \( r_h \) and finally it again becomes stable. So it is expected that there are two phase transitions for the evolution of the black hole. But for \( Q = 1 \) and \( Q = 2 \) (described by dash and dot lines respectively) \( c_Q \) always remains positive and hence there is no question of any phase transition. For \( \Lambda = 0 \) in figure 2(b) all three graphs have the similar behavior – the BH changes from stable phase to an unstable one by changing sign of \( c_Q \) at some finite \( r_h \). In figure 2(c) for \( \Lambda = +1 \), the graphs of \( c_Q \) for \( Q = 0.5 \) and \( Q = 1 \) cross the \( r_h \)-axis twice and there is a transition from stable BH to an unstable one and then again the BH will be stable. But for \( Q = 2 \) the BH was initially unstable and finally becomes a stable one through a phase transition.

The variation of \( F \) over \( r_h \) has been presented in figures 3(a) – (c). In all the three sets of figures the behavior of \( F \) does not change significantly for different values of \( Q \). For \( \Lambda = -1 \) (in fig. 3(a)) initially \( F \) is positive for small \( r_h \) and then decreases to negative values and falls sharply to \( -\infty \). But for other two values of \( \Lambda \) (i.e, \( \Lambda = 0 \) and \( \Lambda = +1 \)) \( F \) remains positive through with a minima at finite \( r_h \).

The graphical presentation of the free energy \( F \) over the Hawking temperature \( T_H \) are shown in figures 4(a) – (c) for \( Q = 0.5 \). In all the figures there are double points (cusp-type) (three in fig 4(a)) which possibly indicate a Hawking-Page type phase transition.

III. PHASE TRANSITION OF EYMGB BHS

Mazharimousavi and Halilsoy (2007) have recently obtained a 5-D spherically symmetric solution in EYMGB theory. The metric ansatz for 5-D spherically symmetric space-time is chosen as

\[
ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2d\Omega_3^2
\]

(13)
Fig. 1(a), 1(b) and 1(c) show the variation of $T_H$ with $r_h$ for $\Lambda (\text{mentioned as } L \text{ in the figure}) = -1$, 0 and 1 respectively for $\alpha = 0.1$ in the case of EMGB black hole.

Fig. 2(a), 2(b) and 2(c) show the variation of $c_Q$ with $r_h$ for $\Lambda = -1$, 0 and 1 respectively for $\alpha = 0.1$ in the case of EMGB black hole.

where they have expressed the metric on unit three sphere $d\Omega_3^2$ in terms of Euler angles as [Mazharimousavi and Halilsoy (2007)]

$$d\Omega_3^2 = \frac{1}{4}(d\theta^2 + d\phi^2 + d\psi^2 - 2 \cos \theta d\phi d\psi)$$ (14)

with $\theta \in [0, \pi]$, $(\phi, \psi) \in [0, 2\pi]$.

For the Yang-Mills field the energy momentum tensor is given by,

$$T_{\mu\nu} = 2F^{i\alpha}_{\mu}F_{\nu\alpha} - \frac{1}{2}g_{\mu\nu}F^{i\alpha\beta}F_{i\alpha\beta}$$ (15)

where $F^{i\alpha\beta}$ is the Yang-Mills field 2-forms such that $F^{i}_{\alpha\beta}F^{i\alpha\beta} = \frac{Q^2}{r^4}$, $Q$ the only non-zero gauge charge. The modified Einstein equations in EYMGB theory are

$$G_{\mu\nu} - \alpha H_{\mu\nu} = T_{\mu\nu}$$ (16)

Where $G_{\mu\nu}$ is the usual Einstein tensor in 5-D, $T_{\mu\nu}$ is the energy-momentum tensor given by equation (6), $\alpha$, the GB coupling parameter is chosen to be positive in the heterotic string theory and the Lovelock tensor has the expression given by equation (4).

Now solving the non vanishing components of the field equations we have [Mazharimousavi and Halilsoy (2007)]

$$U(r) = 1 + \frac{r^2}{4\alpha} \pm \sqrt{\left(\frac{r^2}{4\alpha}\right)^2 + \left(1 + \frac{m}{2\alpha}\right) + \frac{Q^2}{\alpha}}$$ (17)
Fig. 3(a), 3(b) and 3(c) show the variation of $F$ with $r_h$ for $\Lambda = -1, 0$ and 1 respectively for $\alpha = 0.1$ in the case of EMGB black hole.

Fig. 4(a), 4(b) and 4(c) show the variation of $F$ with $T_H$ for $\Lambda = -1, 0$ and 1 respectively for $\alpha = 0.1$ in the case of EMGB black hole.

with $m$ as the constant of integration. Now as $\alpha \to 0$

$$U(r) \to 1 - \frac{m}{r^2} - \frac{2Q^2 \ln r}{r^2} \tag{18}$$

provided negative branch is considered. The metric coefficient $U(r)$ in (18) is identical to that of the Einstein-Yang-Mills solution and hence 'm' is interpreted as the mass of the system. In the equation (17) for $U$ the expression within the square root is positive definite for $\alpha > 0$ while the geometry has a curvature singularity at the surface $r = r_{min}$ for $\alpha < 0$. Here $r_{min}$ is the minimum value of the radial coordinate such that the function under the square root is positive. Moreover, depending on the values of the parameters $(m, Q, \alpha)$, the singular surface can be surrounded by an event horizon with radius $r_h$ so that the space-time given by equation (1) represents a black hole. However if no event horizon exists, then there will be naked singularity.

Now the metric described by equation (1) and (17) has a singularity at the greatest real and positive solution $(r_s)$ of the equation

$$r^4 + \frac{m^2}{\alpha^2} + \frac{Q^2 \ln r}{\alpha} = 0 \tag{19}$$

Note that if equation (19) has no real positive solution then the metric diverges at $r = 0$. However, the singularity is surrounded by the event horizon $r_h$, which is the positive root of (the larger one if there are two positive real roots)

$$r^2 - m - 2Q^2 \ln r = 0 \tag{20}$$
Thus if \( r_s < r_h \) then the singularity will be covered by the event horizon. While the singularity will be naked for \( r_s \geq r_h \). In this connection one may note that the event horizon is independent of the coupling parameter \( \alpha \).

A. Calculation of the thermodynamical quantities

We will discuss the thermodynamics [Aman, J. et. al.(2003)] of the black hole described above. As the event horizon \( r_h \) satisfies (20) so we have,

\[
\bullet \quad m = r_h^2 - 2Q^2 \ln r_h \tag{21}
\]

As before, with proper choice of units \( S = r_h^3 \) and we find,

\[
\bullet \quad T_H = \left( \frac{\partial m}{\partial S} \right)_Q = \frac{2}{3} r_h^{-3} \left[ r_h^2 - Q^2 \right] \tag{22}
\]

\[
\bullet \quad c_Q = \frac{3r_h^3 (r_h^2 - Q^2)}{(3Q^2 - r_h^2)} \tag{23}
\]

\[
\bullet \quad F = \frac{1}{3} r_h^2 + \frac{2}{3} Q^2 \left[ 1 - 3 \ln r_h \right] \tag{24}
\]

B. Graphical interpretation of phase transition

The figures 5(a) – (d) show the graphical representation of the thermodynamical quantities \( T_H, c_Q, F \) with \( r_h \) and also of \( F \) with \( T_H \) for the coupling parameter \( \alpha = 0.1 \) and \( Q = 1 \). From the graphs \( T_H \) increase sharply for small \( r_h \) reaches a maximum value \( T_{\text{max}} \) (at \( r_h = \sqrt{3}Q \)) and then gradually decreases for large \( r_h \). The specific heat is initially positive (for \( r_h > Q \)), gradually increases and then blows up at \( r_h = \sqrt{3}Q \) and subsequently changes sign and remains negative. The free energy is positive throughout with a minimum at \( r_h = \sqrt{3}Q \). The variation of \( F \) w.r.t. \( T_H \) shows a double point at \( T_H = T_{\text{max}} \) which is of cusp type. This corresponds to a phase transition in BH thermodynamics. As specific heat changes from positive value to negative value due to this phase transition so a stable BH for small \( r_h \) becomes an unstable one due to this phase transition.

IV. CONCLUSION

The thermodynamics of the BHs presented above show some possible phase transition where the BH changes changes from some stable configuration to a unstable one and vice versa. At these critical points the heat capacity changes sign and correspond to singularities of \( c_Q \). In EMGB BH for \( \lambda = -1 \) and \( Q = 0.5 \) there are two critical points at which \( c_Q \) changes sign and blows up, indicating a possible phase transition. Further, for the second critical point there is a transition of \( (c_Q, F) \) from \((-,-)\) sign to \((+,-)\) sign, i.e., intermediate unstable BH becomes a stable one and this phase transition is of Hawking-Page type. However, for the EYMGB BH as \( F \) is positive through so possible no second order phase transition (of Hawking-Page type) is possible. For further work, it will be interesting to examine whether Bekenstein area-entropy relation that we have used, should be modified or not.

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Fig. 4(a), 4(b) and 4(c) show the variation of $T_H$, $c_Q$ and $F$ respectively with $r_h$. 4(d) shows the variation of $F$ with $T_H$ for $\alpha = 0.1$ in the case of EYMGB black hole.

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