CONTINUOUS COHOMOLOGY OF THE GROUP
OF VOLUME-PRESERVING AND SYMPLECTIC
DIFFEOMORPHISMS, MEASURABLE TRANSFER
AND HIGHER ASYMPTOTIC CYCLES

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Topology of a manifold is reflected in its diffeomorphism group. It is challenging therefore to understand the diffeomorphism group $Diff(M)$ both as a topological and discrete group. Twenty years ago, some work has been done, in connection with characteristic classes of foliations, in constructing continuous cohomology classes for $Diff(M)$. For $M$ closed oriented $n$-dimensional manifold, a class in $H_{cont}^{n+1}(Diff(M), \mathbb{R})$ has been explicitly written down by Bott [Bo] [Br]. This class is defined as follows. The group $Diff(M)$ acts in the multiplicative group $C^\infty_+(M)$ of positive smooth functions, and on its torsor $A_n(M)$ of volume forms. Hence one gets a cocycle in $H_{cont}^1(Diff(M), C^\infty_+(M))$, defined by
\[ \lambda(f) = f^*(v) = \text{Jac}_v(f), \]
where $v \in A_n(M)$ and $f \in Diff(M)$. The Bott class is
\[ \int_M \log \lambda \cup d \log \lambda \cup \ldots \cup d \log \lambda \]

The nontriviality of Bott class had been shown for $M = S^1$ [Br], and recently for $S^n$ [BCG], $\mathbb{C}P^n$ [Go] by restricting to finite-dimensional Lie groups in $Diff(M)$. In fact, the restriction of the Bott class on $SO(n,1) \subset Diff(S^n)$ gives the hyperbolic volume class, whereas the restriction on $PSL(n+1, \mathbb{C}) \subset Diff(\mathbb{C}P^n)$ gives the Borel class.

By its construction, the Bott class vanishes on the group $Diff_v(M)$ of volume-preserving diffeomorphisms. Moreover, since it is defined by an invariant closed $(n+1)$-form in the space $A_n(M)$ where $Diff(M)$ acts, and by a theorem of Brooks [Br] there are no more invariant forms there, one gets just one class in dimension $(n+1)$ for a fixed manifold $M$. This contrasts sharply the usual intuition coming from the study of finite-dimensional semisimple group, where there is a range of continuous cohomology classes.

In this paper we construct, for a closed manifold $M^n$ with a volume form $\nu$, a series of continuous cohomology classes in $H_{cont}^{\kappa}(Diff_v(M), \mathbb{R})$ for all $\kappa = 5, 9, \ldots$. The classes will be shown nontrivial already for a torus $T^n$. We also will construct, for a symplectic manifold $(M, w)$, a series of classes in $H^{2\kappa}(Sympl(M), \mathbb{R})$.

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for $\kappa = 1, 3, \ldots$. Again, these are nontrivial for a torus $T^n$ with standard symplectic structure.

Working harder, we will show that for the smooth moduli space of stable vector bundles over a Riemann surface $\mathcal{M}$ with its Kähler structure, our class in $H^2(Symp(\mathcal{M}_g), \mathbb{R})$ is nontrivial and restricts to a generator of $H^2(Map_g, \mathbb{R})$, where $Map_g$ is the mapping class group:

**Theorem (3.6).** $H^2(Symp(\mathcal{M}_g), \mathbb{R})$ is nontrivial. Moreover, the homomorphism $Map_g \to Symp(\mathcal{M}_g, \mathbb{R})$ induces a nontrivial map in the second real cohomology.

In both cases, our classes arise from action on a “principal homogeneous space” $X$ which in the case of $Diff_{\nu}(M)$ will be the space of Riemannian metrics with volume form $\nu$, and in the case of $Sympl(M)$ will be the twistor variety, introduced in [Re1]. In that paper we have studied the symplectic reduction of $Sympl$ to the Hamiltonian action of subgroups of $Diff$ on $SL_n(\mathbb{R})$. The transfer map [Gu] will send these classes to $H^*_{cont}(Diff_{\nu}(M))$.

We will not however prove a rigorous comparison theorem relating these two types of construction in the present paper. However we do use the transfer map to define a new source of classes in $H^*(Diff_{\nu}(M))$ coming from the fundamental group of $M$. Namely, a map

$$S : H^\kappa(\pi_1(M), \mathbb{R}) \to H^\kappa(Diff_{\nu}^\sim(M), \mathbb{R})$$

will be constructed where $Diff_{\nu}^\sim(M)$ is the connected component of $Diff_{\nu}(M)$. For $\kappa = 1$, the dual of this map, a character

$$S^\vee : Diff_{\nu}^\sim(M) \to H_1(M, \mathbb{R})$$

has been known for forty years [Sch] and called the asymptotic cycle map. One can view our map $S$ as “higher” asymptotic cycle map.

For $M$ a closed surface with an area form, the groups $Diff_{\nu}(M)$ and $Sympl(M)$ coincide. The two previously described constructions produce a class in $H^2_{cont}(Diff_{\nu}(M))$ which we will show to lie in bounded cohomology group $H^2_b(Diff(M), \mathbb{R})$. For $f, g \in Diff_{\nu}(M)$ we give an explicit formula for a cocycle $\ell(f, g)$ representing this class. For any lamination on $M$ [Th] one can exhibit quite a different formula, using the expression for Euler class from [BG].

The following application of dynamical nature will be proven. Let $F_2$ be a free group in two generators, and let, for some words $h_i, k_i$ in $F_2$, a sum $\sum_{i=1}^\infty a_i(h_i, k_i)$, $\Sigma|a_i| < \infty$ be a cycle for $\ell^1$-homology of $F_2$. This homology has dimension $2^{|\Sigma|}$, as shown in [M]. Let $M$ be a closed surface with an area form $\nu$. Given $f, g \in Diff_{\nu}(M)$ one has a homomorphism $F_2 \to Diff_{\nu}(M)$, so the words $h_i, k_i$ may be viewed as diffeomorphisms in $Diff_{\nu}(M)$.

**Theorem (4.2).** Suppose $\sum_{i=1}^\infty a_i \ell(h_i, k_i) \neq 0$. Then the group generated by $f, g$ in $Diff_{\nu}(M)$ is not amenable.
The significance of Theorem 4.2 stems from the fact that the condition \( \sum a_i \ell(h_i, k_i) \neq 0 \) is \( C^1 \)-open on \( f, g \). Therefore one gets a domain in \( Diff_\nu(M) \times Diff_\nu(M) \), such that any pair \((f, g)\) in it generate a “big” group in \( Diff_\nu(M) \). One can see this result as a step towards “Tits alternative” for the infinite-dimensional Lie group \( Diff_\nu(M) \).

We will show in the next paper that this theorem holds for \( M \) symplectic of higher dimension. For that purpose we ill use Lagrangian measurable foliations and Lyot-Vergne Maslov class to show that our class in \( H^2(Symp(\mathcal{M}_g, \mathbb{R}) \) is bounded. See also the end of [BG].

In [Re2] we defined the “symplectic Chern-Simons” classes \( K_{alg}^{2i-1}(Symp(M)) = \pi_{2i-1}(\mathbb{R}) \to \mathbb{R}/A \), where \( A \) is the group of periods of the Cartan form in \( \Omega^{2i-1}_{cl}(\text{Symp}^{\top}(M)) \), introduced in [Re2], on the Hurewitz image of \( \pi_{2i-1}(\text{Symp}^{\top}(M)) \) in \( H_{2i-1}(\text{Symp}^{\top}(M), \mathbb{R}) \). The real classes introduced in the present paper seem to be in the same relation to the symplectic Chern-Simons classes as Borel classes in \( H_{cont}^*(SL_3(K, \mathbb{R}) \) are to proper Chern-Simons classes \((K = \mathbb{R}, \mathbb{C}) \). The “symplectic Chern-Simons classes” of [Re2] have remarkable rigidity property: for a continuous family of representations of a f.g. group \( \Gamma \) into \( \text{Symp}(M) \), the pull-back of these classes are constant in \( H^*(\Gamma) \). This contrasts strikingly the famous non-rigidity of the Bott class, proved by Thurston. In fact, Thurston exhibited a family of homomorphism \( \tau_1(S) \to Diff(S^1) \), where \( S \) is a closed surface of genus two, with varying Godbillon-Vey class (which coincides with the Bott class for \( Diff(S^1) \)).

We do not know if the real classes constructed in the present paper in \( H^*(Diff_\nu(M)) \) and \( H^*(\text{Symp}(M)) \) are rigid. However, we introduce a new “Chern-Simons” class in \( H^3(Diff_\nu(S^3), \mathbb{R}/\mathbb{Z}) \) which is rigid and restricts to usual Chern-Simons class on \( H^3(SO(4), \mathbb{R}/\mathbb{Z}) \). This uses the invariant scalar product on Lie \( (Diff_\nu(S^3)) \) in much the same way we used invariant polynomials on Lie \( (\text{Symp}(M)) \) in [Re2].

6.6 Theorem (Chern-Simons class in \( Diff_\nu(S^3) \)). There exists a rigid class in \( H^3(Diff_\nu(S^3), \mathbb{R}/\mathbb{Z}) \) whose restriction on \( SO(4) \approx S^3 \times S^3/\mathbb{Z}_2 \) coincides with the sum of standard Chern-Simons classes. Moreover, for \( M = S^3/\Gamma \) there exists a class in \( H^3(Diff_\nu(M, \mathbb{R}/\mathbb{Z}) \) whose restriction on \( S^3 \) is \( |\Gamma| \) times the standard Chern-Simons class.

1. Forms on the space of metrics

We work with the manifold \( M \) with the fixed volume form \( \nu \). Define the space \( \mathcal{P} \) as the Frechet manifold of \( C^\infty \)-Riemannian metrics on \( M \), whose volume form is \( \nu \). Obviously, \( Diff_\nu(M) \) acts on \( \mathcal{P} \). We can look at \( \mathcal{P} \) as a space of sections of a fibration \( \mathcal{P} \to \mathbb{F} \to M \) with a fiber \( SL_N(\mathbb{R})/SO(N) \), where \( N = \dim M \). Clearly, \( \mathcal{M} \) is contractible. For any \( n = 5, 9, \ldots \) fix the Borel form: a \( SL_N(\mathbb{R}) \)-invariant closed \( n \)-form on \( SL_N(\mathbb{R})/SO(N) \), normalized as in [Bo]. For a vector space \( V \) of dimension \( N \) with a volume form \( \nu \) this gives a canonical choice of a closed form on the space \( \mathcal{P}^V \) of Euclidean metrics on \( V \) with determinant \( \nu \). Call this form \( \psi_n^V \). Now, we define a form on \( \mathcal{P} \) by \( \psi_n = \int_M \psi_n^{T_x M} \). That means the following: let \( g \in \mathcal{P} \) a Riemannian metric on \( M \). Let \( h_1, \ldots, h_n \in T_g \mathcal{P} \) be symmetric bilinear smooth 2-forms. Define \( \psi_n(h_1, \ldots, h_n) = \int_M \psi_n^{T_x M}(h_1(x), \ldots, h_n(x)) \, d\nu \).

Lemma (1.1). The form \( \psi \in \Omega^n(\mathcal{P}) \) is closed and \( Diff_\nu(M) \)-invariant.
Proof. The invariance is obvious from definition. To prove the closedness, observe first that a form \( \psi_n(x_1, \ldots, x_m)(h_1, \ldots, h_n) = \sum_{j=1}^m \lambda_j \psi_n^{T_z_j(M)}(h_1(x_j), \ldots, h_n(x_j)) \) is closed as a pull-back of a closed form under the map \( P \mapsto \prod_{j=1}^m P^{T_z_j(M)} \). Now one approximates \( \psi \) by \( \psi_n(x_1, \ldots, x_m) \) to show that \( \psi \) is closed.

1.2 The definition of the classes. We will now apply a general theory of regulators, as presented in [Re1], section 3. For a Frechet-Lie group \( G \), we have an inclusion \( \iota : \text{Diff}_\nu(M) \to \text{Diff}_\nu^\delta(M) \). The class \( \gamma_n \) is defined as \( \gamma_n \in H^n(\text{Diff}_\nu^\delta(M), \mathbb{R}) \) is defined as \( r(\psi_n) \).

Theorem (1.3). The class \( \gamma_n \) lies in the image of the natural map

\[
H^n_{\text{cont}}(\text{Diff}_\nu(M), \mathbb{R}) \to H^n(\text{Diff}_\nu^\delta(M), \mathbb{R}).
\]

The proof follows from Proposition 1.3 below.

1.3 Simplices in \( P \) and a Dupont-type construction. Fix two metrics \( g_1, g_2 \) in \( P \). We can join them by a segment in two different ways. First, there is a straight line segment \( I_{g_1,g_2}(t) : t \mapsto t \cdot g_1 + (1 - t)g_2 \). Second, there is a geodesic segment \( J_{g_1,g_2}(t) : t \mapsto (x \mapsto c(t,g_1(x),g_2(x))) \). Here \( t \in [0,1], x \in M, g_1(x), g_2(x) \in \mathcal{P}^{T_z}(M) \). Now, having \( n \) metrics \( g_1, \ldots, g_n \) in \( P \) we define two singular simplices \( I_{g_1\ldots g_n} : \sigma \to P \) and \( J_{g_1\ldots g_n} : \sigma \to P \) by induction as a joint of \( g_1 \) and \( J_{g_2\ldots g_n} \). Using straight line segments (resp. geodesic segments, comp [Th2]).

Proposition (1.3). Both \( \gamma_n^I \) and \( \gamma_n^J \) are continuous cocycles, representing \( \gamma_n \).

Proof. The proof mimics the finite-dimensional case, cf. [Du], and is therefore omitted.

2. Non-triviality

We will prove that the class \( \gamma_n \) in discrete group cohomology, and consequently classes of \( \gamma_n^I \) and \( \gamma_n^J \) in continuous cohomology are non-trivial in general. For that purpose, consider a torus \( T^N = \mathbb{R}^N / \mathbb{Z}^N \) with a standard volume form \( dx_1 \ldots dx_N \). We have an inclusion

\[
SL(N, \mathbb{Z}) \hookrightarrow \text{Diff}_\nu(T^N)
\]
Proposition (2.1). The class $\gamma_n$ restricts to the Borel class in $H^n(SL(N,\mathbb{Z}),\mathbb{R})$ and is therefore nontrivial for $N$ big enough.

Proof. Let $\mathcal{P}_0$ be the space of left-invariant metrics on $T^N$ with the determinant $\nu$; as a manifold, $\mathcal{P}_0 \approx SL_N(\mathbb{R})/SO(N)$. The embedding $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is $SL_N(\mathbb{Z})$-invariant, and the pull-back of the form $\psi_n$ on $\mathcal{P}_0$ is the Borel form on $\mathcal{P}_0$. Now by [Re1], section 3, $r(\psi_n)$ coincides with the Borel class.

3. Cohomology of symplectic diffeomorphisms

We will now adapt the theory for the group $\text{Sympl}(M)$ of symplectic diffeomorphisms of a compact symplectic manifold $M$. For this purpose, we will introduce a new ($\infty$-dimensional) contractible manifold $Z$, on which $\text{Sympl}(M)$ acts, preserving some differential forms of even degree.

3.1 Principal transformation space. Let $\mathfrak{F}$ be the fibration over $M^{2n}$, whose fiber over $x \in M$ consists of complex structures in $T_xM$, say $J$, such that $\omega_x$ is $J$-invariant and the symmetric form $\omega(J \cdot, \cdot)$ is positive definite. Alternatively, $\mathfrak{F}$ is a $Sp(2n,\mathbb{R})/U(n)$ fiber bundle over $M$, associated to the $Sp(2n,\mathbb{R})$-frame bundle. The principal transformation space $Z$ is defined as a space of $C^\infty$-sections of $\mathfrak{F}$. So a point in $Z$ is just an almost-complex structure on $M$, tamed by $\omega$, in the sense of Gromov [Gr]. Since the Siegel upper half-plane $Sp(2n,\mathbb{R})/U(n)$ is contractible, the space $Z$ is contractible, too.

3.2 Forms on $Z$. Fix an $Sp(2n,\mathbb{R})$-invariant form on $Sp(2n,\mathbb{R})/U(N)$. This induces a form $\varphi^{T_xM}$ on $\mathfrak{F}_x$ for each $x \in M$ and a form

$$\varphi = \int_M \varphi^{T_xM} \cdot \omega^n$$

as in 1.1. Obviously, this form $\varphi$ is $\text{Sympl}(M)$-invariant. Recall that the ring of $Sp(2n,\mathbb{R})$-invariant forms on $Sp(2n,\mathbb{R})/U(n)$ is generated by forms in dimensions $2, 6, \ldots$ [Bo].

Correspondingly, we have $\text{Sympl}(M)$-invariant closed forms, in same dimensions.

We single out the symplectic (Kähler) form on $Sp(2n,\mathbb{R})/U(n)$, which may be described as follows. For $J \in Sp(2n,\mathbb{R})/U(n)$, the tangent space $T_JSp(2n,\mathbb{R})/U(n)$ consists of operators $A : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying $AJ = -JA$ and $\langle Ax, y \rangle = \langle Ay, x \rangle$, where $\langle \cdot , \cdot \rangle$ is the symplectic structure. Alternatively, $A$ is self-adjoint in the Euclidean scalar product $\langle J \cdot , \cdot \rangle$ and skew-commutes with $J$. The Kähler form on $T_JSp(2n,\mathbb{R})/U(n)$ is given by $\langle A, B \rangle = Tr JAB$.

3.3 Simplices on $Z$. For two almost-complex structures $J_1, J_2$, tamed by $\omega$, we define a segment $\mathcal{J}(t) : t \mapsto (c(t, J_1(x), J_2(x))$ where $c(t, J_1(x), J_2(x))$ is the geodesic segment in the Hermitian symmetric space of nonpositive curvature $Sp(2n,\mathbb{R})/U(n)$, joining $J_1(x)$ and $J_2(x)$. For a collection $J_1, \ldots, J_n$ define a singular simplex $K(J_1, \ldots, J_n)$ as in 1.3.

3.4 Continuous cohomology classes in $\text{Sympl}(M)$: a definition. For any generator of the ring of $Sp(2n,\mathbb{R})$-invariant form on $Sp(2n,\mathbb{R})/U(n)$ we define a continuous cohomology class in $H_{cont}(\text{Sympl}(M),\mathbb{R})$ by the explicit formula.
\[ \delta(f_1, \ldots, f_n) = \int_{K(J_0, f_1, J_0, \ldots, f_1 f_2 \ldots f_n, J_0)} \varphi \]

where \( J_0 \) is any fixed tamed almost-complex structure, and \( \varphi \) is a form of 3.2.

### 3.5 Non-triviality

Let \( M \) be a flat torus \( \mathbb{R}^{2n} / \mathbb{Z}^{2n} \) with a standard symplectic structure \( dx_1 \wedge dx_2 + \ldots + dx_{2n-1} \wedge dx_{2n} \). As in 2.1, we have an \( Sp(2n, \mathbb{Z}) \)-invariant embedding \( Sp(2n, \mathbb{R})/U(n) \hookrightarrow \mathcal{X} \), and the classes of 3.4 on \( \text{Sympl}(M) \) restrict to Borel classes on \( Sp(2n, \mathbb{Z}) \), nontrivial for big \( n \) [B].

### 3.6 Application to moduli spaces

Let \( S \) be a closed Riemann surface of genus \( g \geq 2 \), and let \( \mathcal{M}_g \) be a component of the representation variety \( \text{Hom}(\pi_1(S), SO(3))/SO(3) \) with Stiefel-Whitney class 1. This is known to be a smooth compact simply-connected symplectic manifold [Go2] of dimension \( 6g - 6 \). By a famous theorem of [NS], \( \mathcal{M}_g \) is identified with the moduli space of stable holomorphic vector bundles of rank 2 and odd determinant. The mapping class group \( \text{Map}_g \) acts symplectically on \( \mathcal{M}_g \), so we have an injective homomorphism \( \text{Map}_g \rightarrow \text{Sympl}(\mathcal{M}_g) \). Now we claim the following

**Theorem (3.6).** \( H^2(\text{Sympl}(\mathcal{M}_g), \mathbb{R}) \) is nontrivial. Moreover, the homomorphism \( \text{Map}_g \rightarrow \text{Sympl}(\mathcal{M}_g, \mathbb{R}) \) induces a nontrivial map in second real cohomology.

**Proof.** By the main theorem of [NS] there is a holomorphic embedding of the Teichmüller space \( T_g \) to the space of complex structures in \( \mathcal{M}_g \), tamed by Goldman’s symplectic form. In particular, we have a \( \text{Map}_g \)-invariant holomorphic embedding \( T_g \rightarrow^\alpha Z(\mathcal{M}_g) \). Let \( \Omega \) be the Kähler form of \( Z(\mathcal{M}_g) \), then \( \alpha^*(\Omega) \) is a \( \text{Map}_g \)-equivariant Kähler form on \( T_g \). We know there exist holomorphic maps \( Y \rightarrow^\pi S \), where \( S \) is a closed Riemann surface, \( Y \) is a compact complex surface and \( \pi \) is a smooth fibration by complex curves of genus \( g \), such that the corresponding holomorphic map \( \tilde{S} \rightarrow T_g \) is nontrivial. We may form a flat holomorphic fibration \( \mathcal{F} \rightarrow S \) with \( T_g \) as a fiber, associated to the homomorphism \( \pi_1(S) \rightarrow \text{Map}_g \), coming from \( \pi \). The Borel regulator of the flat fibration \( \mathcal{F} \rightarrow S \), corresponding to the form \( \alpha^*(\Omega) \) on \( T_g \), will coincide with the pullback of the class in \( H^2(\text{Sympl}(\mathcal{M}_g, \mathbb{R}) \) under the composite map \( \pi_1(S) \rightarrow \text{Map}_g \rightarrow \text{Sympl}(\mathcal{M}_g) \). The variation of complex structure \( Y \rightarrow^\pi S \) gives a holomorphic section of \( \mathcal{F} \rightarrow S \) which is not horizontal. Therefore the pullback of \( \alpha^*(\Omega) \) on \( S \) using this section will have positive integral over \( S \). By [Re1], section 3, this precisely means that the class we get in \( H^2(S, \mathbb{R}) \) is nontrivial. Therefore the map \( \text{Map}_g \rightarrow \text{Sympl}(\mathcal{M}_g) \) induces a nontrivial map in \( H^2 \).

Q.E.D.

### 4. Bounded cohomology for area-preserving diffeomorphisms

#### 4.1

Let \( M^2 \) be a compact oriented surface of any genus and let \( \nu \) be an area form on \( M \). Then \( \text{Diff}_\nu M = \text{Sympl}(M) \). The construction of 3.4 gives a class in \( H^2_{\text{cont}}(\text{Diff}_\nu M, \mathbb{R}) \).

**Theorem (4.1).** The cocyle \( \delta(h_1, h_2) \) of 3.4 is bounded. The class \([\delta]\) lives therefore in the image of the natural map

\[ H^2_0(\text{Diff}_\nu (M), \mathbb{R}) \rightarrow H^2(\text{Diff}_\nu^0 (M), \mathbb{R}) \]
Proof. Fix a tame almost-complex structure $J_0$. Then $\delta(h_1, h_2)$ is given by $\int_M \text{area}_h(\omega)$, where $\text{area}_h(x, y, z)$ is the hyperbolic area in $SL_2(\mathbb{R})/SO(2) \cong \mathcal{H}^2$ of the geodesic triangle, spanned by $x, y, z$. Therefore $|\delta(h_1, h_2)| \leq \pi \cdot \omega(M)$.

4.2 Non-amenability of two-generated subgroups of $\text{Diff}_\nu(M)$. We will apply theorem 4.1 to the following problem: given two area-preserving maps $f, g : M \rightarrow M$, when the group $\phi(f, g) \in \text{Diff}_\nu(M)$ is “big” (say, free)? When $\text{Diff}_\nu(M)$ is replaced by a finite-dimensional Lie group, this problem has been studied extensively, see e.g. [Re4], and references therein. In [Re4] we showed how the value of a (twisted) Euler class forces $2\kappa$ elements $f_1, \ldots, f_{2\kappa}$ of $SL_2(\mathbb{R})$ to generate a free group. Here we will give a criterion for $\phi(f, g)$ as above to be non-amenable. For that, denote $F(f, g)$ a free group in two generators $f, g$. Consider the $\ell^1$-homology Banach space $H_2^\ell(S, \mathbb{R}) [M]$. An element of this space has a representive $\sum_{j=1}^\infty a_j(h_j, k_j)$ with $h_j, k_j \in F, \Sigma|a_j| < \infty$ and $\sum a_j(h_j k_j - h_j - h_j) = 0$ in $\ell^1(F)$. A bounded cocycle $\ell$ induces a continuous functional

$$\sum a_j \ell(h_i, k_i) : H_2^\ell(S, \mathbb{R}) \rightarrow \mathbb{R}$$

which vanishes if $|\ell| = 0$ in $H_2^\ell(S, \mathbb{R})$.

Theorem (4.2). Let $\sum a_j(h_j, k_j)$ be any $\ell^1$-cycle in $H_2^\ell(S, \mathbb{R})$. If $\sum a_j \delta(h_j, k_j) \neq 0$, then the group $\phi(f, g)$ is non-amenable. The set of pairs $(f, g) \in \text{Diff}_\nu(M) \times \text{Diff}_\nu(M)$ satisfying this inequality, is open in $C^1$-topology.

Proof. Consider the following maps:

$$H_2^\ell(\text{Diff}_\nu(M), \mathbb{R}) \rightarrow H_2^\ell(\phi(f, g), \mathbb{R}) \rightarrow H_2^\ell(F(f, g), \mathbb{R}) \rightarrow (H_2^\ell(F(f, g), \mathbb{R}))^*$$

If $\phi(f, g)$ is amenable, then $H_2^\ell(\phi(f, g), \mathbb{R}) = 0$ [Gr2], so the image of $\delta$ in $(H_2^\ell(F(f, g), \mathbb{R}))^*$ is zero and $(\delta, \sum a_i(h_j, k_j)) = 0$, a contradiction. The last statement of the theorem is checked directly from the definition of $\delta$.

4.3 Constructing $\ell^1$-cycles. The cardinality of $\dim\mathbb{R} H_2^\ell(F(f, g), \mathbb{R})$ is $2^{\aleph_0}$ by [M]. To apply the theorem 4.2 it is useful to have explicit formulas for $\ell^1$-cycles. One way is described in [M].

5. Lie algebra cohomology

We will give the Lie algebraic analogues of the above constructed classes in $\text{Diff}_\nu(M)$ and $\text{Sympl}(M)$. Observe that some odd-dimensional classes in the Lie algebra of $\text{Sympl}(M)$ were constructed in [Re2] they induce, in general, nontrivial classes in cohomology of $\text{Sympl}(M)$ as a topological space. The even-dimensional classes constructed here always induce trivial classes in $H^*(\text{Sympl}^{\text{top}}(M), \mathbb{R})$.

5.1 Formulas for $\text{Diff}_\nu(M)$. Let $X_1, \ldots, X_{2\kappa+1} \in \text{Lie}(\text{Diff}_\nu(M))$. Fix a Riemannian metric $g$ with volume form $\nu$. Let

$$\psi(X_1, \ldots, X_{2\kappa+1}) = \int_M \text{AltTr} \prod_{j=1}^{2\kappa+1} (\nabla X_j + (\nabla X_j)^*) \cdot \nu$$
Theorem (5.1). $\psi$ defines a cocycle for $H^{2\kappa+1}(\text{Lie}(\text{Diff}_{\nu}(M)))$.

Proof. Consider a $\text{Diff}_{\nu}(M)$-equivariant evaluation map $\text{Diff}_{\nu}(M) \to M : f \mapsto (f^*)^{-1}(g)$. Then the $\text{Diff}_{\nu}(M)$-invariant forms on $M$, constructed in 1.1 induce left-invariant closed forms on $\text{Diff}_{\nu}(M)$, whose restriction on $T_{\nu} \text{Diff}_{\nu}(M)$ will be a Lie algebra cocycle. The derivative of the evaluation map $\text{Lie}(\text{Diff}_{\nu}(M)) \to T_{g} M$ is given by $X \mapsto \mathcal{L}_X g = g(\nabla X + (\nabla X)^*, \cdot)$. Accounting the formula for Borel classes (see e.g. [Re3]), one arrives above-written formula for $\psi$.

5.2 Formulas for $\text{Sympl}(M)$. Let $X_1, \ldots, X_{2\kappa} \in \text{Lie}(\text{Sympl}(M))$. Fix a tame almost-complex structure $J$. Let

$$\varphi_{2\kappa}(X_1, \ldots, X_{2\kappa}) = \int_M \text{Alt} \text{Tr} J \cdot \prod_{j=1}^{2\kappa} \mathcal{L}_{X_j} J \cdot \omega^n$$

Theorem (5.2). $\varphi$ defines a cocycle for $H^{2\kappa}(\text{Lie}(\text{Sympl}(M)))$.

Proof. Same as for 5.1.

5.3 Vanishing for $\varphi_2$ for flat torus.

Proposition (5.3). Let $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be a torus with standard symplectic structure. Then for any choice of a tame almost-complex structure, the cohomology class of $\varphi_2$ in $H^2(\text{Lie}(\text{Sympl}(M)), \mathbb{R})$ is zero.

Proof. The cohomology class of $\varphi_2$ does not depend on the choice of $J$, since $X$ is connected. Choose $J$ to be the standard complex structure. We need to work on the formula for $\varphi_2$. Let $g$ be a metric, defined by $g(J \cdot, \cdot) = \omega$ (flat in our case). We then have $\mathcal{L}_X J = [\nabla X, J]$ since $g$ is Kähler and $\nabla X J = 0$. So

$$\varphi_2(X, Y) = \int_M \text{Tr} J([\nabla X, J][\nabla Y, J] - [\nabla Y, J][\nabla X, J]) \cdot \omega^n$$

Let $X$ be Hamiltonian, so that $X = J \text{grad} f$. Then $\nabla X = J H_f$, where $H_f$ is the Hessian of $f$. If $Y$ is also Hamiltonian, say $Y = J \text{grad} h$, we have

$$\varphi_2(X, Y) = -\int_M \text{Tr} J[H_f, J][H_h, J] \cdot \omega^n$$

Direct computation shows that the last expression is zero for flat torus. Now, Lie(Sympl(M)) is a semidirect product of the ideal of Hamiltonian vector fields and an abelian subalgebra of constant vector fields, generated by (multivalued) linear Hamiltonians. Clearly, $\varphi_2(X, Y)$ is zero for all choices for $X$ and $Y$.

5.4 Vanishing of $\varphi_2$ for a symplectic surface.

Proposition (5.4). Let $(M, \omega)$ be a compact surface with a symplectic form. Then for any choice of a tame almost-complex structure, the cohomology class of $\varphi_2$ in $H^2(\text{Lie}(\text{Ham}(M)), \mathbb{R})$ is zero.

Proof. Let $g$ be as above. Again we have

$$\varphi_2(X, Y) = -\int_M \text{Tr} J[H_f, J][H_h, J] \cdot \omega$$

The proposition follows now from the following remarkable identity.
Theorem (5.4). On a compact Riemannian surface \((M, g)\) the following identity holds:

\[
\int_M Tr J[H_f, J][H_h, J] \cdot d\text{ area} = - \int K(g) \{f, h\} \cdot d\text{ area},
\]

(*)

where \(K(g)\) is the curvature of \(g\).

Proof. We were only able to prove this identity by a direct (very) long computation ([Re1]), which we will sketch here. Let \(g = e^{A(x, y)}(dx^2 + dy^2)\) in local conformal coordinates. Then \(\Gamma_{xx} = \frac{1}{2}A_x, \Gamma_{yy} = \frac{1}{2}A_y, \Gamma_{xy} = \frac{1}{2}A_x, \Gamma_{yx} = \frac{1}{2}A_y,\)

\(\Gamma_{yy} = -\frac{1}{2}A_x\). Next, \(H_f = \nabla(Grad f)\) and to the matrix of \(H_f\) is

\[
\begin{pmatrix}
-ae_{xx} + \frac{1}{2}e^{-A}(Ayf_y - Ax f_x) & e^{-A}f_y - \frac{1}{2}e^{-A}(Ayf_x + Ax f_y) \\
e^{-A}f_{xy} - \frac{1}{2}e^{-A}(Ayf_x + Ax f_y) & e^{-A}f_{yy} + \frac{1}{2}e^{-A}(Ax f_x - Ay f_y)
\end{pmatrix}
\]

and the same for \(h\). Substituting to the left side of (*) one gets

\[
-2 \left[ \int (e^{-A}f_{xy} - \frac{1}{2}e^{-A}(Ayf_x + Ax f_y)) \cdot (h_{xx} - h_{yy} + Ay h_y - Ax h_x) - \\
\int (e^{-A}h_{xy} - \frac{1}{2}(Ay h_x + Ax h_y))(f_{xx} - f_{yy} + Ay f_y - Ax f_x) \right] dxdy
\]

Twice integrating by parts, one finds this equal to

\[
\int e^{-A}[-A_{xy}f_x + AyAx f_x - A_{yy}f_x + \\
+AyAy f_x + Ayx f_y - Ax Ay f_y + AxAx f_y - Ax Ax f_y]dxdy
\]

On the other hand, the right hand side is

\[
\int_M \{f_x h_y - f_y h_x\} \cdot (Ax + Ay) e^{-A}dxdy.
\]

Again integrating by parts, one gets the same expression as above. q.e.d.

6. Chern-Simons-type class in \(H^3(Diff f_\nu(M^3), \mathbb{R}(\mathbb{Z}))\)

This section is best read in conjunction with [Re2]. In that paper, we constructed secondary classes in \(Hom(\pi_{2i-1}(B\text{Sympl}^0(M)^+, \mathbb{R}/A)\) where \(M^{2n}\) is a compact simply-connected symplectic manifold and \(A\) is a group of periods of a biinvariant \((2i-1)\)-form on \(\text{Sympl}(M)\), whose restriction on the Lie algebra is \(f_1, \ldots, f_{2i-1} \rightarrow \text{Alt} \int_M \{f_1, f_2\} f_3 \ldots f_{2i-1} \cdot \omega^n\). In particular, it implied the following results.

6.1 Theorem ([Re2]) (Chern-Simons class extends to \(\text{Sympl}(S^2)\)). There exists a rigid class in \(H^3(\text{Sympl}(S^2, \text{can}), \mathbb{R}/\mathbb{Z})\) whose restriction on \(\text{SO}(3)\) is the standard Chern-Simons class.
6.2 Theorem ([Re2]) (Chern-Simons class extends to $\text{Symp}(\mathbb{C}P^2)$). There exists a rigid class in $H^3(\text{Symp}(\mathbb{C}P^2, \text{can}), \mathbb{R}/\mathbb{Z})$ whose restriction on $SU(3)$ is the standard Chern-Simons class.

6.3 Theorem ([Re2]). There exists a rigid class in $H^3(\text{Symp}(S^2, a_1 \cdot \text{can}) \times S^2(a_2 \times \text{can})), \mathbb{R}/\mathbb{Z}), a_1 \neq a_2$, whose restriction on $SO(3) \times SO(3)$ is the sum of standard Chern-Simons classes.

Let $M^3$ be a rational homology sphere, say $f \cdot H_1(M, \mathbb{Z}) = 0, f \in \mathbb{Z}$.

6.4 The definition of the $\text{ChS}$ class. Fix a point $p \in M$ and consider the evaluation (at $p$) map

$$\text{Diff}_\nu(M) \to M.$$ 

The pull-back of $\nu$ under this map is a closed left-invariant form $\nu_p$ on $\text{Diff}_\nu(M)$, having integral periods. The general theory of [Re3] and [Re2] produces a regulator

$$\pi_3 (B \text{Diff}^\delta_\nu(M)^{+}) \to \mathbb{R}/\mathbb{Z} \quad (*)$$

A different choice of a point $p' \in M$ will give another left-invariant form $\nu_{p'}$ such that $\nu_p - \nu_{p'} = d\mu$ for a left-invariant form $\mu$. It follows from [Re3] that the regulator $(*)$ does not depend on $p$. In fact, one has a biinvariant 3-form $\omega$ on $\text{Diff}_\nu(M)$, whose values on the Lie algebra are given by $\omega(X, Y, Z) = \int_M \nu(X(p), Y(p), Z(p))d\nu(p)$. The form $\omega$ gives the same regulator as above.

To extend the regulator to $H^3(B \text{Diff}^\delta_\nu(M), \mathbb{R}/\mathbb{Z})$, we need to alter the scheme of [Re3] as follows. Since $MSO_3(B \text{Diff}^\delta_\nu(M)) \approx H_3(B \text{Diff}^\delta_\nu(M), \mathbb{Z})$ any class in $H_3(B \text{Diff}^\delta_\nu(M), \mathbb{Z})$ is represented by a map $X \xrightarrow{\nu} B \text{Diff}(M)$, or equivalently, by a representation $\pi_1(X) \xrightarrow{\nu^*} B \text{Diff}_\nu(M)$. Now, for $M$ a flat bundle $M \to E \to X$, associating to $\rho$. The form $\omega$ extends to the closed form on $E$ whose periods on fibers are 1. That gives an element $\lambda$ in $H^3(E, \mathbb{R}/\mathbb{Z})$. The spectral sequence of $E$ with $\mathbb{R}/\mathbb{Z}$-coefficients looks like

$$
\begin{array}{cccc}
\mathbb{R}/\mathbb{Z} & H^1(X, \mathbb{R}/\mathbb{Z}) & H^2(X, \mathbb{R}/\mathbb{Z}) & H^3(X, \mathbb{R}/\mathbb{Z}) \\
0 & 0 & 0 & 0 \\
H^0(X, W) & H^1(X, W) & H^2(X, W) & H^3(X, W) \\
\mathbb{R}/\mathbb{Z} & H^1(X, \mathbb{R}/\mathbb{Z}) & H^2(X, \mathbb{R}/\mathbb{Z}) & H^3(X, \mathbb{R}/\mathbb{Z})
\end{array}
$$

where $W$ is the local system whose stalk at $p$ is $H^1(M, \mathbb{R}/\mathbb{Z}) \approx H_1(M, \mathbb{Z})$. The element $\lambda$ lies in the kernel of the wedge map $H^3(E, \mathbb{R}/\mathbb{Z}) \to H^3(M, \mathbb{R}/\mathbb{Z})$. Now, the group $H^2(X, W)$ has exponent a divisor of $f$, and the image of the transgression $\delta^2 : H^1(X, W) \to H^3(X, \mathbb{R}/\mathbb{Z})$ has the same property. Therefore, $f \cdot \lambda$ induces a well-defined class in $H^3(X, \mathbb{R}/\mathbb{Z} \cdot \frac{1}{f})$. If $M$ is a $\mathbb{Z}$-homology sphere, we get a class in $H^3(X, \mathbb{R}/\mathbb{Z})$.

If $Y \to B \text{Diff}^\delta_\nu(M)$ is a map, bordant to $\varphi$, then the same argument as in [Re2] proves that the value of the corresponding class in $H^3(Y, \mathbb{R}/\mathbb{Z} \cdot \frac{1}{f})$ on $[Y]$ is the same as for $X$. So we constructed a well-defined map

$$H_3(\text{Diff}^\delta_\nu(M), \mathbb{Z}) \to \mathbb{R}/\mathbb{Z} \cdot \frac{1}{f}.$$
6.5 Invariant scalar product on $\text{Lie}(\text{Diff}_\nu(M))$, the Cartan form and rigidity of ChS class. Here we will prove that the ChS class

$$H_3(\text{Diff}_\nu^\delta(M), \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

of the previous section is rigid for $M \approx S^3$. For that purpose we need to work with principal flat bundles rather than with flat associated bundles. The clue is that the form $\omega$ constructed above on $\text{Diff}_\nu(M)$ can be viewed as a Cartan form, associated with an invariant scalar product on $\text{Lie}(\text{Diff}_\nu(M))$.

We are going to prove similar results for the group $\text{Diff}_\nu(M)$ of volume-preserving diffeomorphisms of a compact oriented three-manifold. Throughout this section, $M$ is assumed to be a rational homology sphere, that is, $H_1(M, \mathbb{Z})$ is torsion.

Let $X \in \text{Lie}(\text{Diff}_\nu(M))$ a vector field with $\text{div} X = 0$. The form $\langle X, X \rangle$ is closed, whence exact: $d\mu = X \cdot (X \nu)$. An immediate computation shows that $\langle X, X \rangle$ does not depend on the choice of $\mu$. Moreover $\langle X, X \rangle$ is a quadratic form, invariant under the adjoint action of $\text{Diff}_\nu(M)$. By Arnold [A], $\langle X, X \rangle$ is the asymptotic self-linking number of trajetories of $X$. We need the following elementary lemma (the proof of left to the reader)

**Lemma (6.5).** For any $X, Y, Z \in \text{Lie}(\text{Diff}_\nu(M))$,

$$\Omega(X, Y, Z) = \omega(X, Y, Z)$$

that is, the forms $\Omega$ and $\omega$ coincide.

Now, as in [Re2] we define a biinvariant form $\Omega$ on $\text{Diff}_\nu(M)$ by $\Omega(X, Y, Z) = \langle [X, Y], Z \rangle$ on the Lie algebra.

**Lemma (6.6).** Let $M = S^3/\Gamma$ where $S^3$ is considered as a compact Lie group and the finite subgroup $\Gamma$ acts from the right. Then the pullback of $\Omega$ by the natural map $S^3 \to \text{Diff}_\nu(M)$ is $\frac{1}{|\Gamma|}$ (volume form of $S^3$).

**Proof.** It is clearly enough to check this for $\Gamma = \{1\}$. Let $v \in \text{Lie}(S^3)$ and $X$ is the corresponding right-invariant vector field. Let $\mu$ be a right-invariant 1-form, defined by $(v, \cdot)$ on $\text{Lie}(S^3)$. Then $d\mu = X \cdot \nu$ and $\mu \wedge (X \cdot \nu) = \nu$. q.e.d.

**6.6 Theorem (Chern-Simons class in $\text{Diff}_\nu(S^3)$).** There exists a rigid class in $H^3(\text{Diff}_\nu(S^3), \mathbb{R}/\mathbb{Z})$ whose restriction on $SO(4) \approx S^3 \times S^3/\mathbb{Z}_2$ coincides with the sum of standard Chern-Simons classes. Moreover, for $M = S^3/\Gamma$ there exists a class in $H^3(\text{Diff}_\nu(M), \mathbb{R}/\mathbb{Z})$ whose restriction on $S^3$ is $|\Gamma|$ times the standard Chern-Simons class.

**Proof.** By the general theory of regulators, developed in [Re3], section 3, and [Re2], the invariant form $\Omega$ gives rise to a map

$$\pi_3(B \text{Diff}_\nu^\delta(M^+)) \to \mathbb{R}/A$$

where $A$ is the group of periods of $\Omega$ on the Hurewitz image of $\pi_3(\text{Diff}_\nu(M))$ in $H_3(\text{Diff}_\nu(M), \mathbb{Z})$. Moreover, if $\text{Diff}_\nu(M)$ is homotopically equivalent to $S^3$ or $SO(4)$ this extends to a map

$$H_3(B \text{Diff}_\nu^\delta(M)) \to \mathbb{R}/A$$
By Hatcher [H] and Ivanov [I] this is exactly the case for \( M = S^3/\Gamma \). Moreover, periods of \( \Omega \) are \( 2\pi^2 \cdot \mathbb{Z} \) and \( 2\pi^2 \cdot \frac{1}{\kappa} \mathbb{Z} \), respectively. Since \( \Omega \) is a Cartan form, associated to an invariant polynomial in \( \text{Lie}(Diff_\nu(M)) \), it is rigid by Cheeger-Simons [Che-S].

### 6.6 Case of Seifert manifolds.
Let \( \Gamma \) be a uniform lattice in \( SL_2(\mathbb{R}) \), then \( M = SL_2(\mathbb{R})/\Gamma \) is a Seifert manifold. There is a cohomology class \( \beta \in H^3(SL_2(\mathbb{R}), \mathbb{R}) \), called the Seifert volume class [BGo], such that for any \( \Gamma \subset SL_2(\mathbb{R}) \), the restriction of \( \beta \) on \( \Gamma \) is \( \text{vol}(SL_2(\mathbb{R})/\Gamma) \) times the fundamental class. Then the computation of 6.4 gives the class in \( H^3(Diff_\nu(M), \mathbb{R}) \), whose restriction on \( \hat{SL}_2(\mathbb{R}) \) is \( \beta \), subject to the condition that \( Diff_\nu(M) \) is contractible. It is not known to the author if this is true for all such \( M \), comp. [FJ].

### 7. Measurable transfer and higher asymptotic cycles
We will first outline here an alternative approach in defining the classes of 1.2 in \( Diff_\nu(M) \). For \( M \) a locally symmetric space of nonpositive curvature, this approach also leads to new classes in \( H^*_{cont}(Diff_\nu(M), \mathbb{R}) \), different from those of 1.2.

Let \( \mathfrak{G} = Diff_\nu(M) \) and \( \mathfrak{G}_0 \subset \mathfrak{G} \) is a closed group, stabilizing a fixed point \( p \in M \). Let \( \mathfrak{G}^\sim \) be the connected component of \( \mathfrak{G} \) and let \( \mathfrak{G}_0^\sim = \mathfrak{G} \cap \mathfrak{G}_0 \). Fix a measurable section \( s : M \to \mathfrak{G} \) such that \( s(p) = p \). We will always assume that \( s(M) \) is compact.

#### 7.1 Ergodic cocycle in non-abelian cohomology [Gu]
Define a map \( \psi : \mathfrak{G} \times M \to \mathfrak{G}_0 \) by \( g\cdot s(q) = s(g\cdot q) = s(g)\cdot s(q) \). We will view it as a map \( \mathfrak{G} \to \mathcal{F}(M, \mathfrak{G}_0) \). Here \( \mathcal{F}(M, \mathfrak{G}_0) \) is the group of measurable functions from \( M \) to \( \mathfrak{G}_0 \) with compact closure of the image. \( \mathfrak{G} \) acts on \( \mathcal{F}(M, \mathfrak{G}_0) \) by the argument change and \( \psi \) is a cocycle for the non-abelian cohomology \( H^1(\mathfrak{G}, \mathcal{F}(M, \mathfrak{G}_0)) \).

#### 7.2 Measurable transfer [Gu]
Now let \( f : \mathfrak{G}_0 \to \mathcal{F}(M, \mathfrak{G}_0) \) be a locally bounded (say, continuous) cocycle. Define \( F : \mathfrak{G} \times \mathcal{F}(M, \mathfrak{G}_0) \to \mathbb{R} \) as \( F = \int_M \varphi(g_1, m) \cdot \varphi(g_2, m) \cdots \varphi(g_n, m) \cdot dv(m) \). This defines a cohomology class in \( H^n(\mathfrak{G}, \mathbb{R}) \), independent of the choices of \( s \) and \( f \) [Gu].

Now, we have the tangential representation \( \mathfrak{G}_0 \to SL(T_p(M)) \). Pulling back the usual Borel classes on \( \mathfrak{G}_0 \), we construct cohomology classes in \( H^i(\mathfrak{G}_0, \mathbb{R}) \) for \( i = 5, 9, \ldots \). The transfer will map these to classes in \( H^i(\mathfrak{G}, \mathbb{R}) \), which we have constructed in 1.2. We do not prove the comparison theorem here, however.

#### 7.3 Supertransfer
We will now define a map
\[
H^\kappa(\pi_1(M), \mathbb{R}) \overset{S}{\to} H^\kappa(Diff_\nu(M), \mathbb{R})
\]
in the following way. We know that \( \pi_0(\mathfrak{G}_0^\sim) \approx \pi_1(M)/\pi_1(\mathfrak{G}^\sim) \). This defines a homomorphism \( \mathfrak{G}_0^\sim \to \pi_0(\mathfrak{G}_0^\sim) \to \pi_1(M)/\pi_1(\mathfrak{G}^\sim) \), and a map \( H^\kappa(\pi_1(M)/\pi_1(\mathfrak{G}^\sim), \mathbb{R}) \to H^\kappa(\mathfrak{G}_0^\sim, \mathbb{R}) \).

In many interesting cases one knows that \( \pi_1(\mathfrak{G}^\sim) = 1 \). If \( M \) is a surface of genus \( g \geq 2 \), a result of Earle and Eells says that \( \mathfrak{G}^\sim \) is contractible. For \( M \) locally symmetric of rank \( \geq 2 \) [FJ]. For any \( M \) such that \( \pi_1(\mathfrak{G}^\sim) = 1 \), we get \( \pi_0(\mathfrak{G}_0) \approx \pi_1(M) \) so that there is a map.
Now, composing with the measurable transfer \( H^\kappa(\mathfrak{G}^\sim) \to H^\kappa(\mathfrak{G}^\sim) \) we arrive to a desired map

\[
S : H^\kappa(\pi_1(M), \mathbb{R}) \to H^\kappa(\mathfrak{G}^\sim, \mathbb{R})
\]

### 7.4 Higher asymptotic cycles

The dual to the above-constructed map \( S \) is \( S^\vee \):

\[
S^\vee : H_\kappa(\mathfrak{G}^\sim, \mathbb{R}) \to H_\kappa(\pi_1(M), \mathbb{R})
\]

As we will see now, this is higher version of the classical asymptotic cycle character

\[
\mathfrak{G}^\sim \xrightarrow{\tau} H_1(M, \mathbb{R})
\]

Indeed, for \( \kappa = 1 \) the map \( S^\vee \) will act as follows: let \( g \in \mathfrak{G}^\sim \) be a volume-preserving map, isotopic to identity. Fix an isotopy \( g(t, x) \) such that \( g(0, \cdot) = \text{id} \) and \( g(1, \cdot) = g \). For \( x \in M, g(t, x) \) is a path from \( x \) to \( g(x) \) and may be considered as a 1- current. Now, the integral

\[
\int_M [g(t, x)]d\nu(x)
\]

is a closed current, defining an element in \( H_1(M, \mathbb{R}) \). This will be \( S^\vee(g) \).

Now, the definition of the asymptotic cycle map [Sch] gives the following recepy: for an element \( z \in H^1(M, \mathbb{Z}) \) let \( f : M \to S^1 \) be a representing map. The map \( f \circ g - f : M \to S^1 \) is zero-homotopic, so it comes from the map \( F : M \to \mathbb{R} \).

Now, \( \int_M F(\text{mod } \mathbb{Z}) \) is the image of \( \tau(f) \) on \( z \). If \( f \) is isotopic to identity, \( \tau(f) \) lifts to \( H_1(M, \mathbb{R}) \). It is easy to check that \( (df, \int_M [g(t, x)]d\nu) = (\tau(f), z) \), which proves \( S^\vee = \tau \) in dimension 1.

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