HOMOTOPIC HOPF-GALOIS EXTENSIONS REVISITED

ALEXANDER BERGLUND AND KATHRYN HESS

Abstract. In this article we revisit the theory of homotopic Hopf-Galois extensions introduced in [10], in light of the homotopical Morita theory of comodules established in [3]. We generalize the theory to a relative framework, which we believe is new even in the classical context and which is essential for treating the Hopf-Galois correspondence in [12]. We study in detail homotopic Hopf-Galois extensions of finite-type differential graded algebras over a field, for which we establish a descent-type characterization analogous to the one Rognes provided in the context of ring spectra [25]. An interesting feature in the differential graded setting is the close relationship between homotopic Hopf-Galois theory and Koszul duality theory.

Contents

1. Introduction 1
1.1. The classical framework 2
1.2. The homotopic framework 3
1.3. Structure of this paper 3
1.4. Conventions 4

2. Elements of homotopical Morita theory 4
2.1. Homotopy theory of modules 4
2.2. Homotopical Morita theory for comodules 6

3. Relative Hopf-Galois extensions 11
3.1. The descent and Hopf functors 11
3.2. The classical Hopf-Galois framework 16
3.3. The homotopic Hopf-Galois framework 17

4. Homotopic Hopf-Galois extensions of chain algebras 19
4.1. Homotopy theory of chain modules and comodules 20
4.2. Homotopy theory of chain comodule algebras 22
4.3. Homotopic relative Hopf-Galois extensions of chain algebras 25

References 28

1. Introduction

The theory of Hopf-Galois extensions of associative rings, introduced by Chase and Sweedler [6] and by Kreimer and Takeuchi [18], generalizes Galois theory of fields, replacing the action of a group by the coaction of a Hopf algebra. Inspired by
Rognes’ theory of Hopf-Galois extensions of ring spectra [25], the second author laid the foundations for a theory of homotopic Hopf-Galois extensions in an arbitrary monoidal model category in [10], but the necessary model category structures were not well enough understood to make it possible to compute many examples. Since then, considerable progress has been made in elaborating these model category structures (e.g., [1, 15]), so that the time is ripe to revisit this subject.

In this article we develop anew the theory of homotopic Hopf-Galois extensions, in light of the homotopical Morita theory of comodules established in [3]. Moreover we generalize the theory to a relative framework, which we believe is new even in the classical context and which is essential for treating the Hopf-Galois correspondence in [12]. We also provide a descent-type characterization of homotopic Hopf-Galois extensions of finite-type differential graded algebras over a field, analogous to [25, Proposition 12.1.8].

1.1. The classical framework. Classical Hopf-Galois extensions show up in a wide variety of mathematical contexts. For example, faithfully flat HG-extensions over the coordinate ring of an affine group scheme $G$ correspond to $G$-torsors. By analogy, if a Hopf algebra $H$ is the coordinate ring of a quantum group, then an $H$-Hopf-Galois extension can be viewed as a noncommutative torsor with the quantum group as its structure group. It can moreover be fruitful to study Hopf algebras via their associated Hopf-Galois extensions, just as algebras are studied via their associated modules.

For an excellent introduction to the classical theory of Hopf-Galois extensions, we refer the reader to the survey articles by Montgomery [22] and Schauenburg [26]. We recall here only the definition and two elementary examples, which can be found in either of these articles.

**Definition 1.1.** Let $k$ be a commutative ring, and let $H$ be a $k$-bialgebra. Let $\varphi : A \to B$ be a homomorphism of right $H$-comodule algebras, where the $H$-coaction on $A$ is trivial.

The homomorphism $\varphi$ is an $H$-Hopf-Galois extension if

1. the composite

$$B \otimes_A B \xrightarrow{\beta \otimes \text{id}_B} B \otimes_A B \otimes H \xrightarrow{\mu \otimes H} B \otimes H,$$

where $\beta$ denotes the $H$-coaction on $B$, and $\mu$ denotes the multiplication map of $B$ as an $A$-algebra, and

2. the induced map

$$A \to B^{\text{co}H} := \{ b \in B \mid \beta(b) = b \otimes 1 \}$$

are both isomorphisms.

**Notation 1.2.** The composite in (1), often denoted $\beta_\varphi : B \otimes A B \to B \otimes H$, is called the Galois map.

**Examples 1.3.**

1. **[22] Example 2.3** Let $k \subset E$ be a field extension. Let $G$ be a finite group that acts on $E$ through $k$-automorphisms, which implies that its dual $k^G = \text{Hom}(k[G], k)$ coacts on $E$. The extension $E^G \subset E$ is $G$-Galois if and only if it is a $k^G$-Hopf-Galois extension.

2. **[26] Theorem 2.2.7** Let $k$ be a commutative ring, $H$ a bialgebra over $k$ that is flat as $k$-module, and $A$ a flat $k$-algebra. The trivial extension $A \to A \otimes H : a \mapsto a \otimes 1$ is then an $H$-Hopf-Galois extension if $A \otimes H$ admits a cleaving, i.e., a convolution-invertible morphism of $H$-comodules $H \to A \otimes H$. In particular, the unit map $k \to H$ is an $H$-Hopf-Galois extension if and only if $H$ is a Hopf algebra.
1.2. The homotopic framework. In his monograph on Galois extensions of structured ring spectra [25], Rognes formulated a reasonable, natural definition of homotopic Hopf-Galois extensions of commutative ring spectra. Let \( \varphi: A \to B \) be a morphism of commutative ring spectra, and let \( H \) be a commutative ring spectrum equipped with a comultiplication \( H \to H \wedge H \) that is a map of ring spectra, where \(- \wedge -\) denotes the smash product of spectra. Suppose that \( H \) coacts on \( B \) so that \( \varphi \) is a morphism of \( H \)-comodules when \( A \) is endowed with the trivial \( H \)-coaction. If the Galois map \( \beta_{\varphi}: B \wedge A \to B \wedge H \) (defined as above) and the natural map from \( A \) to (an appropriately defined model of) the homotopy coinvariants of the \( H \)-coaction on \( B \) are both weak equivalences, then \( \varphi: A \to B \) is a homotopic \( H \)-Hopf-Galois extension in the sense of Rognes.

The unit map \( \eta \) from the sphere spectrum \( S \) to the complex cobordism spectrum \( MU \) is an \( S[BU] \)-Hopf-Galois extension in this homotopic sense. The diagonal \( \Delta: BU \to BU \times BU \) induces the comultiplication \( S[BU] \to S[BU] \wedge S[BU] \), the Thom diagonal \( MU \to MU \wedge BU \), gives rise to the coaction of \( S[BU] \) on \( MU \), and \( \beta_{\eta}: MU \wedge MU \to MU \wedge S[BU] \) is the Thom equivalence.

In [25, Proposition 12.1.8], Rognes provided a descent-type characterization of homotopic Hopf-Galois extensions. Let \( A^e_H \) denote Carlsson’s derived completion of \( A \) along \( B \) [5]. Rognes proved that if \( \varphi: A \to B \) is such that \( \beta_{\varphi} \) is a weak equivalence, then it is a homotopic \( H \)-Hopf-Galois extension if and only if the natural map \( A \to A^e_H \) is a weak equivalence, which holds if, for example, \( B \) is faithful and dualizable over \( A \) [25, Lemma 8.2.4].

1.3. Structure of this paper. We begin in Section 2 by summarizing from [3] those elements of the homotopical Morita theory of modules and comodules in a monoidal model category that are necessary in this paper. In particular we recall conditions under which a morphism of corings induces a Quillen equivalence of the associated comodule categories (Corollary 2.36).

In Section 3 we introduce a new theory of relative Hopf-Galois extensions, insisting on the global categorical picture. We first treat the classical case, then introduce the homotopic version, providing relatively simple conditions under which a morphism of comodule algebras in a monoidal model category is a relative homotopic Hopf-Galois extension (Proposition 3.29).

We furnish a concrete illustration of the theory of relative homotopic Hopf-Galois extensions in Section 4, where we consider the monoidal model category \( \text{Ch}^{\text{fin}}_k \) of finite-type, non-negatively graded chain complexes over a field \( k \). After recalling from [3] the homotopy theory of modules and comodules in this case, we elaborate the homotopy theory of comodule algebras in \( \text{Ch}^{\text{fin}}_k \), recalling the necessary existence result for model category structures from [1], then describing and studying a particularly useful fibrant replacement functor, given by the cobar construction (Theorem 4.20). Finally, we describe in detail the theory of relative homotopic Hopf-Galois extensions of differential graded algebras of finite type over a field \( k \).

In particular we establish the existence of a useful family of relative homotopic Hopf-Galois extensions analogous to the classical normal extensions (Lemma 4.24). We apply this family to proving, under reasonable hypotheses, that a morphism of comodule algebras is a relative homotopic Hopf-Galois extension if and only if it satisfies effective homotopic descent (Proposition 4.25), a result analogous to [25, Proposition 12.1.8] for commutative ring spectra. As a consequence we establish an intriguing relationship between Hopf-Galois extensions and Koszul duality, implying in particular that, under reasonable hypotheses, if \( A \to B \) is a homotopic Hopf-Galois extension with respect to some Hopf algebra \( H \), where \( B \) is weakly contractible, then \( H \) is Koszul dual to \( A \) (Proposition 4.27).
1.4. Conventions.

- All forgetful functors are denoted $\mathcal{U}$.
- Let $\mathcal{C} \xrightarrow{L} \mathcal{D} \xleftarrow{R} \mathcal{E}$ be an adjoint pair of functors. If $\mathcal{C}$ is endowed with a model category structure, and $\mathcal{D}$ admits a model category structure for which the fibrations and weak equivalences are created in $\mathcal{E}$, i.e., a morphism in $\mathcal{D}$ is a fibration (respectively, weak equivalence) if and only if its image under $R$ is a fibration (respectively, weak equivalence) in $\mathcal{C}$, then we say that it is right-induced by the functor $R$. Dually, if $\mathcal{D}$ is endowed with a model category structure, and $\mathcal{C}$ admits a model category structure for which the cofibrations and weak equivalences are created in $\mathcal{E}$, i.e., a morphism in $\mathcal{E}$ is a cofibration (respectively, weak equivalence) if and only if its image under $L$ is a cofibration (respectively, weak equivalence) in $\mathcal{C}$, then we say that it is left-induced by the functor $L$.

2. Elements of homotopical Morita theory

In this section we recall from [3] those elements of homotopical Morita theory for modules and comodules that are necessary for our study of homotopic Hopf-Galois extensions in monoidal model categories. Since the definitions and results in [3] are couched in a more general framework than we need in this article, we specialize somewhat here, for the reader’s convenience.

2.1. Homotopy theory of modules. Let $(\mathcal{V}, \otimes, k)$ be a monoidal category. Let $\text{Alg}_\mathcal{V}$ denote the category of algebras in $\mathcal{V}$, i.e., of objects $A$ in $\mathcal{V}$ together with two maps $\mu: A \otimes A \to A$ and $\eta: k \to A$ that satisfy the usual associativity and unit axioms. Dually, the category of coalgebras in $\mathcal{V}$, i.e., objects in $\mathcal{V}$ that are endowed with a coassociative comultiplication and a counit, is denoted $\text{Coalg}_\mathcal{V}$.

A right (respectively, left) module over an algebra $A$ is an object $M$ in $\mathcal{V}$ together with a map $\rho: M \otimes A \to M$ (respectively, $\lambda: A \otimes M \to M$) satisfying the usual axioms for an action. We let $\mathcal{V}_A$ (respectively, $A\mathcal{V}$) denote the category of right (respectively, left) $A$-modules in $\mathcal{V}$. We usually omit the multiplication and unit from the notation for an algebra and the action map from the notation for an $A$-module.

Schwede and Shipley established reasonable conditions, satisfied by many model categories of interest, under which module categories inherit a model category structure from the underlying category.

**Theorem 2.1.** [29, Theorem 4.1] Let $\mathcal{V}$ be a symmetric monoidal model category. If $\mathcal{V}$ is cofibrantly generated and satisfies the monoid axiom, and every object of $\mathcal{V}$ is small relative to the whole category, then the category $\mathcal{V}_A$ of right $A$-modules admits a model structure that is right induced from the adjunction $\mathcal{V} \xrightarrow{- \otimes A} \mathcal{V}_A$, and similarly for the category $A\mathcal{V}$ of left $A$-modules.

**Convention 2.2.** Henceforth, we assume always that $\mathcal{V}$ is a symmetric monoidal model category such that the adjunction $\mathcal{V} \xrightarrow{- \otimes A} \mathcal{V}_A$
right-induces a model category structure on $\mathcal{V}_A$, for every algebra $A$ in $\mathcal{V}$, and similarly for $A\mathcal{V}$. Whenever we refer to weak equivalences, fibrations, or cofibrations of $A$-modules, we mean with respect to this right-induced structure.

The tensor product of a right and a left $A$-module over $A$ is construction that appears frequently in this article.

**Definition 2.3.** Given right and left $A$-modules $M_A$ and $AN$, with structure maps $\rho: M \otimes A \rightarrow M$ and $\lambda: A \otimes N \rightarrow N$, their tensor product over $A$ is the object $M \otimes_A N$ in $\mathcal{V}$ defined by the following coequalizer diagram:

$$
\begin{array}{c}
M \otimes A \otimes N \\
\rho \otimes 1 \\
1 \otimes \lambda
\end{array}
\xrightarrow{\rho \otimes 1}
\begin{array}{c}
M \otimes N \\
M \otimes_A N
\end{array}
$$

The special classes of modules defined below, which are characterized in terms of tensoring over $A$, play an important role in this article.

**Definition 2.4.** Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.2. A left $A$-module $M$ is called

- **homotopy compact** if for every finite category $J$ and every functor $\Phi: J \rightarrow \mathcal{V}_A$, the natural map $(\lim_J \Phi) \otimes_A M \rightarrow \lim_J (\Phi \otimes_A M)$ is a weak equivalence in $\mathcal{V}$;
- **homotopy flat** if $- \otimes_A M: \mathcal{V}_A \rightarrow \mathcal{V}$ preserves weak equivalences;
- **homotopy faithful** if $- \otimes_A M: \mathcal{V}_A \rightarrow \mathcal{V}$ reflects weak equivalences; and
- **homotopy faithfully flat** if it is both homotopy faithful and homotopy flat.

Right $A$-modules of the same types are defined analogously.

It is also useful to distinguish those weak equivalences of left (respectively, right) $A$-modules that remain weak equivalences upon tensoring over $A$ with any right (respectively, left) $A$-module.

**Definition 2.5.** A morphism of left $A$-modules $f: N \rightarrow N'$ is a pure weak equivalence if the induced map $M \otimes_A f: M \otimes_A N \rightarrow M \otimes_A N'$ is a weak equivalence for all cofibrant right $A$-modules $M$. Pure weak equivalences of right $A$-modules are defined analogously.

It is easier to work in monoidal model categories in which cofibrant modules are homotopy flat, fitting our intuition of cofibrancy as a sort of projectivity.

**Definition 2.6.** We say that $\mathcal{V}$ satisfies the CHF hypothesis if for every algebra $A$ in $\mathcal{V}$, every cofibrant right $A$-module is homotopy flat.

As pointed out in [29, §4], the CHF hypothesis holds in many monoidal model categories of interest, such as the categories of simplicial sets, symmetric spectra, (bounded or unbounded) chain complexes over a commutative ring, and $S$-modules.

The following proposition highlights one of the advantages of this hypothesis.

**Proposition 2.7.** [3, Proposition 2.14] Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.3. If $\mathcal{V}$ satisfies the CHF hypothesis, then the notions of pure weak equivalence and weak equivalence coincide.

Our interest in pure weak equivalences is motivated by the next proposition, for which we need to establish a bit of terminology.
Definition 2.8. Let \( \mathcal{V} \) be a monoidal category, and let \( \varphi: A \to B \) be a morphism of algebras in \( \mathcal{V} \). The restriction/extension-of-scalars adjunction,

\[
\mathcal{V}_A \xleftarrow{\varphi_*} \mathcal{V}_B \xrightarrow{\varphi^*}
\]

is defined on objects by \( \varphi_*(M) = M \otimes_A B \), endowed with right \( B \)-action given by multiplication in \( B \), for all right \( A \)-modules \( M \), while \( \varphi^*(N) \) has the same underlying object, but with right \( A \)-action given by the composite

\[
N \otimes A \xrightarrow{N \otimes \varphi^{-1}} N \otimes B \xrightarrow{\rho} N.
\]

Remark 2.9. It is a classical result that \( \varphi^* \) is right adjoint to \( \varphi_* \). Moreover, under Convention 2.2, it is clear that the adjunction \( \varphi_* \dashv \varphi^* \) is a Quillen pair, since \( \varphi^* \) preserves fibrations and all weak equivalences.

We can now formulate a necessary and sufficient condition under which the Quillen pair \( \varphi_* \dashv \varphi^* \) is actually a Quillen equivalence.

Proposition 2.10. [3, Proposition 2.15] Let \( \mathcal{V} \) be a symmetric monoidal model category satisfying Convention 2.2, and let \( \varphi: A \to B \) be a morphism of algebras in \( \mathcal{V} \). The restriction/extension-of-scalars adjunction,

\[
\mathcal{V}_A \xleftarrow{\varphi_*} \mathcal{V}_B \xrightarrow{\varphi^*}
\]

is a Quillen equivalence if and only if \( \varphi: A \to B \) is a pure weak equivalence of right \( A \)-modules.

Remark 2.11. The proposition above is a special case of Theorem 2.23 in [3], which provides necessary and sufficient conditions for when an adjunction between \( \mathcal{V}_A \) and \( \mathcal{V}_B \) governed by an \( A \)-\( B \)-bimodule is a Quillen equivalence.

2.2. Homotopical Morita theory for comodules.

2.2.1. Review of corings and their comodules. Let \( \mathcal{V} \) be a monoidal category. For every algebra \( A \) in \( \mathcal{V} \), the tensor product \(- \otimes_A -\) endows the category of \( A \)-bimodules \( \mathcal{V}_A \) with a (not necessarily symmetric) monoidal structure, for which the unit is \( A \), viewed as an \( A \)-bimodule over itself.

Definition 2.12. An \( A \)-coring is a coalgebra in the monoidal category \((\mathcal{V}_A, \otimes_A, A)\), i.e., an \( A \)-bimodule \( C \) together with maps of \( A \)-bimodules \( \Delta: C \to C \otimes_A C \) and \( \epsilon: C \to A \), such that the diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes_A C \\
\Delta & \downarrow & \downarrow \otimes \Delta \\
C \otimes_A C & \xrightarrow{\Delta \otimes C} & C \otimes_A C \otimes_A C \\
\end{array}
\]

are commutative. A morphism of \( A \)-corings is a map of \( A \)-bimodules \( f: C \to D \) such that the diagrams

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & C \otimes_A C \\
\Delta_D & \downarrow & \downarrow f \otimes_A f \\
D & \xrightarrow{\Delta_D} & D \otimes_A D \\
\end{array}
\]

\[
\begin{array}{ccc}
C & \xrightarrow{\epsilon_C} & A \\
f & \downarrow & \downarrow \epsilon_D \\
D & \xrightarrow{\epsilon_D} & A \\
\end{array}
\]

commute.
In Section 3 we provide natural constructions of families of corings. For the moment we note only that any algebra $A$ can be seen in a trivial way as a coring over itself, where the comultiplication is the isomorphism $A \xrightarrow{=} A \otimes_A A$ and the counit is the identity.

A more general notion of morphism of corings takes into account changes of the underlying algebra as well. Note first that if $\varphi: A \to B$ is a morphism of algebras, then there is a two-sided extension/restriction-of-scalars adjunction,

$$A \xrightarrow{\varphi_*} B \xleftarrow{\varphi^*}, \quad \varphi_* \dashv \varphi^*,$$

where $\varphi_*(M) = B \otimes_A M \otimes_A B$. Moreover, $\varphi_*$ is an op-monoidal functor, i.e., there is a natural transformation

$$\varphi_*(M \otimes_A N) \to \varphi_*(M) \otimes_B \varphi_*(N),$$

which allows us to endow $\varphi_*(C)$ with the structure of a $B$-coring whenever $C$ is an $A$-coring.

**Remark 2.13.** Note that if $A$ is considered as an $A$-comodule, where $A$ is equipped with the trivial coring structure defined above, then $\varphi_*(A)$ is exactly the well known descent or canonical coring associated to the algebra morphism $\varphi$, with underlying $B$-bimodule $B \otimes_A B$.

**Definition 2.14.** A coring in $\mathcal{V}$ is a pair $(A, C)$ where $A$ is an algebra in $\mathcal{V}$, and $C$ is an $A$-coring. A morphism of corings $(A, C) \to (B, D)$ is a pair $(\varphi, f)$ where $\varphi: A \to B$ is a morphism of algebras, and $f: \varphi_*(C) \to D$ is a morphism of $B$-corings. The category of corings in $\mathcal{V}$ is denoted $\text{Coring}_\mathcal{V}$.

We will now recall the definition of a comodule over a coring.

**Definition 2.15.** Let $(A, C)$ be a coring in $\mathcal{V}$, with comultiplication $\Delta$ and counit $\epsilon$. A right $(A, C)$-comodule is a right $A$-module $M$ together with a morphism of right $A$-modules $\delta: M \to M \otimes_A C$ such that the diagrams

$$\begin{array}{ccc}
M & \xrightarrow{\delta} & M \otimes_A C \\
\downarrow{\delta} & & \downarrow{\Delta} \\
M \otimes_A C & \xrightarrow{\delta \otimes C} & M \otimes_A C \otimes_A C
\end{array}$$

are commutative. A morphism of $(A, C)$-comodules is a morphism $f: M \to N$ of right $A$-modules such that the diagram

$$\begin{array}{ccc}
M & \xrightarrow{\delta_M} & M \otimes_A C \\
\downarrow{f} & & \downarrow{f \otimes C} \\
N & \xrightarrow{\delta_N} & N \otimes_A C
\end{array}$$

commutes.

**Remark 2.16.** Every morphism of corings $(\varphi, f): (A, C) \to (B, D)$ factors in $\text{Coring}_\mathcal{V}$ as

$$(A, C) \xrightarrow{(\varphi, \text{id}_{\varphi_*(C))}} (B, \varphi_*(C)) \xrightarrow{\text{(Id}_B, f)} (B, D),$$

i.e., as a change of rings, followed by a change of corings. This easy observation is a very special case of [3 Proposition 3.30].
We let $\mathcal{C}_A^V$ denote the category of right $(A, C)$-comodules. There is an adjunction

$$\mathcal{C}_A^V \xrightarrow{U} \mathcal{A}, \quad U_A = - \otimes_A C,$$

where $U$ is the forgetful functor, and $- \otimes_A C$ is the cofree $C$-comodule functor. In particular, for any $A$-module $M$, the $C$-coaction on $M \otimes_A C$ is simply $M \otimes_A \Delta$.

The category $\mathcal{C}_A^V$ of left $(A, C)$-comodules is defined analogously.

**Remark 2.17.** Note that if $A$ is endowed with its trivial $A$-coring structure, then the adjunction above specializes to an isomorphism between $\mathcal{C}_A^V$ and $\mathcal{A}$. It follows that the theory of comodules over corings englobes that of modules over algebras.

Under reasonable conditions on $\mathcal{C}$, if $\phi, f: (A, C) \to (B, D)$ is a morphism of corings, then the restriction/extension-of-scalars adjunction on the module categories lifts to an adjunction on the corresponding comodule categories.

**Proposition 2.18.** Let $\mathcal{C}$ be a symmetric monoidal category that admits all reflexive coequalizers and coreflexive equalizers. If $(A, C)$ is a coring in $\mathcal{C}$ such that $\mathcal{C}_A^V$ admits all coreflexive equalizers, then every morphism of corings $(\phi, f): (A, C) \to (B, D)$ gives rise to an adjunction

$$\mathcal{C}_A^V \xrightarrow{\phi, f} \mathcal{C}_B^D$$

such that the following diagram of left adjoints commutes.

**Remark 2.19.** As explained in [3, Remark 3.8], if $\mathcal{C}$ is locally presentable, then $\mathcal{C}_A^V$ admits all coreflexive equalizers. On the other hand, the dual of [19, Corollary 3] implies that if $- \otimes_A C: \mathcal{C}_A \to \mathcal{C}_A$ preserves coreflexive equalizers, then $\mathcal{C}_A^V$ admits all coreflexive equalizers.

**Remark 2.20.** The commutativity of the square in the statement of Proposition 2.18 implies that for any $C$-comodule $(M, \delta)$, the $B$-module underlying $(\phi, f)_*(M, \delta)$ is $M \otimes_A B$. As shown in the proof of [3, Proposition 3.15] (in a somewhat more general context), the $D$-coaction on $M \otimes_A B$ is given by the following composite.

$$M \otimes_A B \xrightarrow{\delta \otimes B} M \otimes_A C \otimes_A B \cong M \otimes_A A \otimes_A C \otimes_A B \xrightarrow{M \otimes_C \phi \otimes B} M \otimes_A B \otimes_A C \otimes_A B$$

Since the diagram of right adjoints must also commute, we know as well that the image under $(\phi, f)^*$ of a cofree $D$-comodule $N \otimes_B D$ is the cofree $C$-comodule $\phi^*(N) \otimes_A C$.

**Notation 2.21.** When $(\phi, f) = (\text{Id}_A, f): (A, C) \to (A, D)$, we denote the induced adjunction

$$\mathcal{C}_A^V \xrightarrow{f} \mathcal{D}$$

(2.1)
and call it the \textit{coextension/corestriction-of-coefficients adjunction} or \textit{change-of-corings adjunction} associated to $f$. Note that the $D$-component of the counit of the $f_* \dashv f^*$ adjunction is $f$ itself and that for every $(A, C)$-comodule $(M, \delta)$,
\[
f_*(M, \delta) = (M, (1 \otimes f)\delta).
\]

When $(\varphi, f) = (\varphi, \text{Id}_{\varphi^*(C)})$: $(A, C) \to (B, \varphi^*(C))$, we denote the induced adjunction
\[
\mathcal{V}_A^C \xrightarrow{\text{Can}_{\varphi}} \mathcal{V}_B^{\varphi^*(C)} \xleftarrow{\text{Prim}_{\varphi}}
\]
and call it the \textit{canonical adjunction for $C$}, as a generalization of the usual canonical adjunction for descent along $\varphi$: $A \to B$, which is the case $C = A$ of the adjunction above [11], [20].

\textbf{Remark 2.22.} By Remark 2.16, the adjunction $(\varphi, f)_* \dashv (\varphi, f)^*$ can be factored as follows.
\[
\mathcal{V}_A^C \xrightarrow{\text{Can}_{\varphi}} \mathcal{V}_B^{\varphi^*(C)} \xleftarrow{\text{Prim}_{\varphi}} f_* \xrightarrow{f^*} \mathcal{V}_B^D.
\]

The right adjoint $(\varphi, f)^*$ in the adjunction governed by a morphism of corings $(\varphi, f): (A, C) \to (B, D)$ is difficult to describe in general. Under appropriate conditions on the left $A$-module underlying $C$, however, it is possible to express $(\varphi, f)^*$ as a cotensor product over $D$, dually to the expression of the left adjoint in the extension/restriction-of-scalars adjunction associated to $\varphi$ as a tensor product over $A$. The condition we need to impose on $C$ is formulated as follows.

\textbf{Definition 2.23.} A coring $(A, C)$ is \textit{flat} if $- \otimes_A C: \mathcal{V}_A \to \mathcal{V}_A$ preserves coreflexive equalizers.

Flatness of a coring gives us control of coreflexive equalizers in the associated comodule category.

\textbf{Proposition 2.24.} [3, Proposition 3.27] If $(A, C)$ is a flat coring, then the forgetful functor $U: \mathcal{V}_A^C \to \mathcal{V}_A$ creates coreflexive equalizers.

The following definition is dual to Definition 2.23.

\textbf{Definition 2.25.} Suppose that the monoidal category $\mathcal{V}$ admits coreflexive equalizers. Let $(A, C)$ be a coring in $\mathcal{V}$, let $M$ be a right and $N$ a left $(A, C)$-comodule. The \textit{cotensor product} $M \Box C N$ is defined as the coreflexive equalizer in $\mathcal{V}$:
\[
M \Box C N \xrightarrow{\delta_M \otimes N} M \otimes_A N \xrightarrow{M \otimes \delta_N} M \otimes_A C \otimes_A N.
\]

We can now formulate the desired explicit description of the right adjoint in the adjunction governed by a morphism of corings.

\textbf{Proposition 2.26.} [3, Proposition 3.29] Let $\mathcal{V}$ be a monoidal category admitting all reflexive coequalizers and coreflexive equalizers. Let $(A, C)$ be a flat coring in $\mathcal{V}$. If $(\varphi, f): (A, C) \to (B, D)$ is a coring morphism, then $B \otimes_A C$ admits the structure of a left $(B, D)$-comodule in $\mathcal{V}_A^C$ such that the functor $(\varphi, f)^*$ is isomorphic to the cotensor product functor $- \Box_D (B \otimes_A C)$, i.e., there is an adjunction
\[
\mathcal{V}_A^C \xrightarrow{(\varphi, f)_*} \mathcal{V}_B^D \xleftarrow{- \Box_D (B \otimes_A C)}.
\]
Remark 2.27. The left $D$-coaction on $B \otimes A C$ is given by the following composite.

$$
\begin{array}{c}
B \otimes A C \xrightarrow{B \otimes \Delta} B \otimes A C \otimes A C \\
\cong B \otimes A C \otimes A \otimes A C \xrightarrow{B \otimes C \otimes \phi \otimes C} B \otimes A C \otimes A B \otimes A C \\
\approx B \otimes A C \otimes A A \otimes A C \\
\approx \downarrow \downarrow D \otimes A C \\
\approx \downarrow \downarrow D \otimes_B B \otimes A C
\end{array}
$$

2.2.2. Homotopy theory of comodules. We now introduce homotopy theory into our discussion of comodule categories. Let $\mathcal{V}$ be a symmetric monoidal model category.

Convention 2.28. For every coring $(A, C)$ in $\mathcal{V}$ that we consider here, we suppose that $\mathcal{V}_A$ admits the model category structure right-induced from $\mathcal{V}$ and that $\mathcal{V}_{A \otimes_B}$ admits the model category structure left-induced from $\mathcal{V}_A$, via the adjunction $\mathcal{V}_{A \otimes_B} \xrightarrow{U_A} \mathcal{V}_A$.

Remark 2.29. Conditions on $\mathcal{V}$ under which the convention above holds can be found in [1], [13], and [15], where a number of concrete examples are also elaborated. In Section 4 we recall in detail the example of non-negatively graded, finite-type chain complexes over a field.

Remark 2.30. Since we assume henceforth that $\mathcal{V}$, $\mathcal{V}_A$, and $\mathcal{V}_{A \otimes_B}$ are model categories, they are in particular complete and cocomplete and thus admit all reflexive coequalizers and coreflexive equalizers.

We now recall from [3, Section 4] the conditions under which a morphism of corings $(\varphi, f): (A, C) \to (B, D)$ induces a Quillen equivalence of the associated comodule categories. We begin by breaking the problem into two pieces, according to the factorization in Remark 2.22.

Definition 2.31. Let $(A, C)$ be a coring in $\mathcal{V}$ and $B$ an algebra in $\mathcal{V}$. An algebra morphism $\varphi: A \to B$ satisfies effective homotopic descent with respect to $C$ if the adjunction

$$(2.4) \quad \mathcal{V}_C \xrightarrow{\text{Can}_{\varphi}} \mathcal{V}_B^{\varphi^{-1}(C)},$$

is a Quillen equivalence.

Sufficient conditions for effective homotopic descent were established in [3].

Proposition 2.32. [3, Corollary 4.12] Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.28. Let $\varphi: A \to B$ be a morphism of algebras in $\mathcal{V}$. If $B$ is homotopy compact and homotopy faithfully flat as a left $A$-module, then $\varphi$ satisfies effective homotopic descent.

For the other piece of the factorization, we need to introduce a notion dual to that of pure weak equivalence.

Definition 2.33. Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.28. We say that a map $f: C \to D$ of $A$-corings is a copure weak equivalence if

$$\epsilon_M: f_*f^*(M) \to M$$

is a weak equivalence for all fibrant right $D$-comodules $M$. 
Just as pure weak equivalences induce Quillen equivalences of module categories, copure weak equivalences do the same for comodule categories.

**Proposition 2.34.** [3, Proposition 4.6] Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.28. Let $A$ be an algebra in $\mathcal{V}$. The change-of-corings adjunction,

\[
\mathcal{V}_A^C \xrightarrow{f} \mathcal{V}_A^D,
\]

is a Quillen equivalence if and only if $f: C \to D$ is a copure weak equivalence of $A$-corings.

**Remark 2.35.** As pointed out in [3, Proposition 4.6], if $A$ is fibrant as an object of $\mathcal{V}$, then every copure weak equivalence of $A$-corings is a weak equivalence. Conversely, if the coring $(A, C)$ is flat, then $f^*(M) = M \square_D C$ by Proposition 2.26. In this case, if $f$ is a weak equivalence, and the functor $M \square_D - : \mathcal{D}_A \mathcal{V} \to \mathcal{V}$ preserves weak equivalences for all fibrant right $D$-comodules $M$, then the adjunction above is a Quillen equivalence. It follows that if every fibrant $D$-module is “homotopy coflat”, then every weak equivalence of corings with flat domain is copure; compare with Proposition 2.7.

As a consequence of Propositions 2.32 and 2.34, we obtain the following sufficient condition for the adjunction induced by a coring morphism to be a Quillen equivalence.

**Corollary 2.36.** Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 2.28. Let $(\phi, f): (A, C) \to (B, D)$ be a morphism of corings in $\mathcal{V}$.

If $B$ is homotopy compact and homotopy faithfully flat as a left $A$-module, and $f$ is a copure weak equivalence, then

\[
\mathcal{V}_A^C \xrightarrow{(\phi, f)_*} \mathcal{V}_B^D
\]

is a Quillen equivalence.

### 3. Relative Hopf-Galois extensions

Here we apply the results of [3] recalled in the previous section to elaborating interesting and natural generalizations first of the classical framework, then of the homotopic framework, for Hopf-Galois extensions.

#### 3.1. The descent and Hopf functors.

Let $(\mathcal{Y}, \otimes, k)$ be a symmetric monoidal category that is both complete and cocomplete. Generalizing somewhat constructions in [4], we begin by describing two important, natural ways to create corings in $\mathcal{Y}$ and the relation between these constructions.

**Definition 3.1.** Let $\text{Alg}_{\mathcal{Y}}^+$ denote the category of morphisms of algebras in $\mathcal{Y}$. The descent functor

\[
\text{Desc}: \text{Alg}_{\mathcal{Y}}^+ \to \text{Coring}_{\mathcal{Y}}
\]

sends an object $\varphi: A \to B$ to its associated canonical descent coring (also called the Sweedler coring)

\[
\text{Desc}(\varphi) = (B, (B \otimes_A B, \Delta_\varphi, \varepsilon_\varphi)),
\]

where $\Delta_\varphi$ is equal to the composite

\[
B \otimes_A B \cong B \otimes_A A \otimes_A B \xrightarrow{B \otimes_A \varphi \otimes_A B} B \otimes_A B \otimes_A B \cong (B \otimes_A B) \otimes_B (B \otimes_A B),
\]
\[\varepsilon_{\phi} = \overline{\mu}_B: B \otimes_A B \to B,\]
the morphism induced by the multiplication \(\mu_B: B \otimes B \to B\). A morphism 
\((\alpha, \beta): \phi \to \phi'\) in \(\text{Alg}^\gamma\), i.e., a commuting diagram of algebra morphisms

\[
\begin{array}{c}
A \\
\alpha
\end{array} \xrightarrow{\phi} \begin{array}{c}
A' \\
\phi'
\end{array} \xleftarrow{\beta} \begin{array}{c}
B \\
B'
\end{array} \xrightarrow{\beta} \begin{array}{c}
B' \\
\phi'
\end{array}
\]

induces a morphism of \(B\)-corings
\[
\text{Desc}(\alpha, \beta) = (\beta, \beta \otimes_A \alpha): \text{Desc}(\phi) \to \text{Desc}(\phi').
\]

**Remark 3.2.** The coring \(\text{Desc}(\phi)\) is the same as the coring \(\phi_*(A)\) of Remark 2.13. We change the notation here to emphasize the functoriality of the construction in the morphism \(\phi\).

It is not hard to check that \((B, (B \otimes_A B, \Delta_{\phi}, \varepsilon_{\phi}))\) is indeed a \(B\)-coring and that 
\((\beta, \beta \otimes_A \alpha)\) is a morphism of corings for any \((\alpha, \beta): \phi \to \phi'\). Moreover, \(\text{Desc}(\phi)\) admits two natural coaugmentations, given by the composites

\[
B \cong B \otimes_A A \xrightarrow{B \otimes_A \Delta_{\phi}} B \otimes_A B \quad \text{and} \quad B \cong A \otimes_A B \xrightarrow{\varepsilon_{\phi} \otimes_A A} B \otimes_A B.
\]

The other functor into \(\text{Coring}_\gamma\) that we consider here takes as input algebras, respectively coalgebras, endowed with extra structure given by a bialgebra \(H\).

**Remark 3.3.** If \(H\) is a bialgebra in \(\mathcal{V}\), then it is an algebra in \(\text{Coalg}_\gamma\) and a coalgebra in \(\text{Alg}_\gamma\). In particular, \((\mathbb{k}, H)\) is a coring in \(\mathcal{V}\).

**Definition 3.4.** Let \(H\) be a bialgebra in \(\mathcal{V}\). An object of the category \((\text{Coalg}_{\mathcal{V}})_H\) of \(H\)-module coalgebras in \(\mathcal{V}\) is an \(H\)-module in \(\text{Coalg}_{\mathcal{V}}\), i.e., a coalgebra \(C\) in \(\mathcal{V}\), equipped with an associative, unital morphism of coalgebras \(\kappa: C \otimes H \to C\). Morphisms in \((\text{Coalg}_{\mathcal{V}})_H\) are morphisms in \(\mathcal{V}\) that respect the comultiplication and counit and the \(H\)-action.

**Definition 3.5.** Let \(H\) be a bialgebra in \(\mathcal{V}\). An object of the category \((\text{Alg}_{\mathcal{V}})_H\) of \(H\)-comodule algebras in \(\mathcal{V}\) is an \(H\)-comodule in \(\text{Alg}_{\mathcal{V}}\), i.e., an algebra \(A\) in \(\mathcal{V}\), equipped with a coassociative, counital morphism of algebras \(\rho: A \to A \otimes H\). Morphisms in \((\text{Alg}_{\mathcal{V}})_H\) are morphisms in \(\mathcal{V}\) that respect the multiplication and unit and the \(H\)-coaction.

**Notation 3.6.** Let \(\gamma: H \to H'\) be a morphism of bialgebras. There is an induced extension/restriction-of-scalars adjunction

\[
(\text{Coalg}_{\mathcal{V}})_H \xrightarrow{\gamma^*} (\text{Coalg}_{\mathcal{V}})_{H'}.
\]

Moreover, by Proposition 2.13 there is also a change-of-corings adjunction

\[
\text{Alg}_{\mathcal{V}}^H \xrightarrow{\gamma^*} \text{Alg}_{\mathcal{V}}^{H'}.
\]

As we are using the same notation for the functors in these two different cases, we will be very careful to specify context any time we refer to a functor \(\gamma_*\) or \(\gamma^*\).

The category below of matched pairs of comodule algebras and module coalgebras is the natural domain for an interesting functor to the global category \(\text{Coring}_\gamma\) of all corings in \(\mathcal{V}\), generalizing the well known construction of a coring from any comodule algebra [10 Example 4.3(2)].
Definition 3.7. The category Pair$_F$ has as objects triples $(H, (A, \rho), (C, \kappa))$, where $H$ is a bialgebra in $\mathcal{V}$, $(A, \rho)$ is an $H$-comodule algebra, and $(C, \kappa)$ is an $H$-module coalgebra. A morphism from $(H, (A, \rho_A), (C, \kappa_C))$ to $(K, (B, \rho_B), (D, \kappa_D))$ consists of a triple $(\gamma, \varphi, \theta)$, where $\gamma: H \to K$ is a morphism of bialgebras, $\varphi: \gamma_*(A) \to B$ is a morphism of $K$-comodule algebras, and $\theta: C \to \gamma^*(D)$ is a morphism of $H$-module coalgebras.

Definition 3.8. Let $H$ be a bialgebra in $\mathcal{V}$. The Hopf functor

\[ \text{Hopf}: \text{Pair}_F \to \text{Coring}_F \]

sends an object $(H, (A, \rho), (C, \kappa))$ to its associated Hopf coring,

\[ \text{Hopf}(\rho, \kappa) = (A, (A \otimes C, \Delta_{\rho, \kappa}, \varepsilon_{\rho, \kappa})) \]

where the left $A$-action is equal to

\[ A \otimes A \otimes C \xrightarrow{\mu \otimes C} A \otimes C, \]

where $\mu$ is the multiplication on $A$, and the right $A$-action is given by the composite

\[ A \otimes C \otimes A \xrightarrow{A \otimes C \otimes \rho} A \otimes C \otimes H \cong A \otimes A \otimes C \otimes H \xrightarrow{\mu \otimes K} A \otimes C. \]

The comultiplication $\Delta_{\rho, \kappa}$ is equal to the composite

\[ A \otimes C \xrightarrow{A \otimes A} A \otimes C \otimes C \cong (A \otimes C) \otimes_A (A \otimes C), \]

where $\Delta$ is the comultiplication on $C$, and $\varepsilon_{\rho, \kappa}$ is given by

\[ A \otimes C \xrightarrow{A \otimes \varepsilon} A \otimes \mathbb{k} \cong A, \]

where $\varepsilon$ is the counit of $C$.

If $(\gamma, \varphi, \theta)$ is a morphism from $(H, (A, \rho_A), (C, \kappa_C))$ to $(K, (B, \rho_B), (D, \kappa_D))$, then the morphism of $A$-bimodules underlying $\text{Hopf}(\gamma, \alpha, \theta)$ is

\[ \varphi \otimes \theta: A \otimes C \to B \otimes D. \]

The proof that $\text{Hopf}(\rho, \kappa)$ is actually an $A$-coring is somewhat fastidious, but straightforward.

Notation 3.9. An important special case of the construction above comes from taking $(C, \kappa) = (H, \mu)$, where $\mu$ is the multiplication on $H$. We simplify notation a bit and write

\[ \text{Hopf}(\rho) = \text{Hopf}(\rho, \mu). \]

The relation between the functors $\text{Desc}$ and $\text{Hopf}$ can be expressed in terms of a natural transformation, as explained below. Observe first that the proof of [21] Proposition 4.3] can easily be generalized to an arbitrary monoidal category, implying that for any morphism $\gamma: H \to K$ of bialgebras, the coequalizer $\mathbb{k} \otimes_H K$ inherits a coalgebra structure from $K$, with compatible right $K$-module structure, induced by the multiplication in $K$.

Notation 3.10. If $\gamma: H \to K$ is a morphism of bialgebras in $\mathcal{V}$, let $\text{Cof}(\gamma)$ denote the $K$-module coalgebra $\mathbb{k} \otimes_H K$, let $\mu_K$ denote its induced right $K$-action, and let $\pi_\gamma: K \to \text{Cof}(\gamma)$ denote the quotient map.

Definition 3.11. The category $\text{ComodAlg}_F$ of all comodule algebras in $\mathcal{V}$ has as objects pairs $(H, (A, \rho))$, where $H$ is a bialgebra in $\mathcal{V}$, and $(A, \rho)$ is an $H$-comodule algebra. A morphism in $\text{ComodAlg}_F$ from $(H, (A, \rho_A))$ to $(K, (B, \rho_B))$ consists of a pair $(\gamma, \varphi)$, where $\gamma: H \to K$ is a morphism of bialgebras, and $\varphi: \gamma_*(A) \to B$ is a morphism of $K$-comodule algebras.
Definition 3.12. Let $\text{ComodAlg}^\rightarrow$ denote the category of morphisms in the category $\text{ComodAlg}$. Let

$$U^\rightarrow: \text{ComodAlg}^\rightarrow \to \text{Alg}^\rightarrow: \left( (H, (A, \rho_A)) \to (K, (B, \rho_B)) \right) \mapsto (A \xrightarrow{\varphi} B)$$

be the obvious forgetful functor, and $C: \text{ComodAlg}^\rightarrow \to \text{Pair}^\rightarrow: \left( (H, (A, \rho_A)) \to (K, (B, \rho_B)), (\text{Cof}((\gamma), \bar{\mu}_K)) \right)$ the “cofiber” functor.

The Galois transformation is the natural transformation

$$\text{Gal}: \text{Desc} \circ U^\rightarrow \to \text{Hopf} \circ C$$

deﬁned on an object $(H, (A, \rho_A)) \to (K, (B, \rho_B))$ so that

$$\text{Gal}_{(\gamma, \varphi)}: \text{Desc}(\varphi) \to \text{Hopf}(\rho_B, \bar{\mu}_K)$$

is the morphism of $B$-corings given by the identity on $B$ in the algebra component and by the composite

$$B \otimes_A B \xrightarrow{B \otimes_A \rho_B} B \otimes K \xrightarrow{\bar{\mu}_B \otimes \pi_\gamma} B \otimes \text{Cof}(\gamma)$$

in the coring component, where $\bar{\mu}_B$ is induced by the multiplication in $B$; compare with the Galois map of Definition 1.1.

Remark 3.13. An object in $\text{ComodAlg}^\rightarrow$ of the form $(k, (A, \rho_A)) \to (K, (B, \rho_B))$, where $\eta: k \to K$ is the unit of $K$, is Hopf-Galois data, in the sense of [10], since one can also view a morphism of this type as a morphism of $K$-comodule algebras, where the coaction of $K$ on $A$ is trivial.

Remark 3.14. The naturality of all of the constructions seen thus far implies that a commuting diagram of comodule algebra morphisms

$$\begin{array}{ccc}
(H, A) & \to & (K, B) \\
\downarrow (\zeta, \alpha) & & \downarrow (\xi, \beta) \\
(H', A') & \to & (K', B')
\end{array}$$

The diagram below summarizes the definitions seen thus far in this section.

![Diagram]

Remark 3.13. An object in $\text{ComodAlg}^\rightarrow$ of the form

$$(k, (A, \rho_A)) \xrightarrow{[\eta \varphi]} (K, (B, \rho_B)),$$

where $\eta: k \to K$ is the unit of $K$, is Hopf-Galois data, in the sense of [10], since one can also view a morphism of this type as a morphism of $K$-comodule algebras, where the coaction of $K$ on $A$ is trivial.

Remark 3.14. The naturality of all of the constructions seen thus far implies that a commuting diagram of comodule algebra morphisms

$$\begin{array}{ccc}
(H, A) & \xrightarrow{(\gamma, \varphi)} & (K, B) \\
\downarrow (\zeta, \alpha) & & \downarrow (\xi, \beta) \\
(H', A') & \xrightarrow{(\gamma', \varphi')} & (K', B')
\end{array}$$
gives rise to a commuting diagram of functors

\[
\begin{array}{ccc}
\mathcal{Y}_A & \xrightarrow{\alpha_*} & \mathcal{Y}_A' \\
\downarrow \text{Can}_\varphi & & \downarrow \text{Can}_\varphi' \\
\mathcal{Y}_{\text{Desc}(\varphi)} & \xrightarrow{\left(\beta, \text{Desc}(\alpha, \beta)\right)_*} & \mathcal{Y}_{\text{Desc}(\varphi')} \\
\downarrow \text{Gal}(\gamma, \varphi)_* & & \downarrow \text{Gal}(\gamma', \varphi')_* \\
\mathcal{Y}_{\text{Hopf}(\rho, \bar{\mu}_K)} & \xrightarrow{\left(\beta, \text{Hopf}(\rho, \bar{\mu}_K)\right)_*} & \mathcal{Y}_{\text{Hopf}(\rho', \bar{\mu}'_K)}
\end{array}
\]

where the \(B'\)-bimodule map underlying \(H_{\xi, \beta}\) is

\[
B' \otimes \text{Cof}(\gamma) \otimes_B B' \xrightarrow{\left(\beta, \text{Hopf}(\rho, \bar{\mu}_K)\right)_*} B' \otimes \text{Cof}(\gamma') \otimes_B B' \rightarrow B' \otimes \text{Cof}(\gamma'),
\]

with the second map given by the right \(B'\)-action on \(B' \otimes \text{Cof}(\gamma')\) (cf. Definition 3.8).

We need to introduce one more functor defined on \(\text{ComodAlg}_\mathcal{Y}\), in order to set the stage for Hopf-Galois extensions and their generalizations.

**Remark 3.15.** Proposition 2.18 implies that if \(\text{Alg}_\mathcal{Y}\) admits all reflexive coequalizers and coreflexive equalizers and \(\text{Alg}_H\) admits all coreflexive equalizers, then every morphism of bialgebras \(\gamma: H \rightarrow K\) gives rise to an adjunction

\[
\text{Alg}_H \xleftarrow{\gamma_*} \text{Alg}_K.
\]

See Remark 2.19 for conditions under which these hypotheses hold.

**Definition 3.16.** Let \(H\) be a bialgebra in \(\mathcal{Y}\) with unit \(\eta: k \rightarrow H\). If the extension-of-corings functor \(\eta_*: \text{Alg}_\mathcal{Y} \rightarrow \text{Alg}_H\), which endows any algebra with a trivial \(H\)-coaction, admits a right adjoint, then we call this right adjoint the \(H\)-coinvariants functor and denote it

\[
(-)^{\text{co}H}: \text{Alg}_H \rightarrow \text{Alg}_\mathcal{Y}.
\]

**Remark 3.17.** Suppose that \(\text{Alg}_\mathcal{Y}\) admits all reflexive coequalizers and coreflexive equalizers and \(\text{Alg}_H\) admits all coreflexive equalizers. For any morphism \(\gamma: H \rightarrow K\) of bialgebras, there is a commuting diagram of adjunctions, with right adjoints on the inner triangle and left adjoints on the outer triangle,

\[
\begin{array}{ccc}
\text{Alg}_H & \xrightarrow{\gamma_*} & \text{Alg}_K \\
\downarrow \left(\gamma \circ \eta_H\right)_* & & \downarrow \left(\gamma \circ \eta_K\right)_* \\
\text{Alg}_\mathcal{Y} & \xrightarrow{\left(\gamma \circ \eta_H\right)_*} & \text{Alg}_\mathcal{Y}
\end{array}
\]

\[
\begin{array}{ccc}
\text{Alg}_K & \xleftarrow{\gamma^*} & \text{Alg}_H \\
\downarrow \left(\gamma \circ \eta_K\right)_* & & \downarrow \left(\gamma \circ \eta_K\right)_* \\
\text{Alg}_\mathcal{Y} & \xleftarrow{\left(\gamma \circ \eta_K\right)_*} & \text{Alg}_\mathcal{Y}
\end{array}
\]

since \(\gamma \circ \eta_H = \eta_K\).

Let \((\gamma, \varphi): (H, (A, \rho_A)) \rightarrow (K, (B, \rho_B))\) be a morphism in in \(\text{ComodAlg}_\mathcal{Y}\). Recall that if \(\eta^\gamma\) is the unit of the \(\gamma_* \dashv \gamma^*\)-adjunction in diagram (3.1), then the transpose of \(\varphi: \gamma_* A \rightarrow B\) is the composite

\[
A \xrightarrow{\eta_A^\gamma} \gamma^* \gamma_* A \xrightarrow{\gamma^* \varphi} \gamma^* B.
\]
Applying $(-)^{co\mathcal{H}}$, we obtain a morphism of algebras
\[
A^{co\mathcal{H}} \xrightarrow{(\eta^\gamma, \gamma^* A)^{co\mathcal{H}}} (\gamma^* B)^{co\mathcal{H}},
\]
where the last isomorphism follows from the commutativity of the diagram above.
We denote this composite morphism
\[
\varphi^{co\gamma}: A^{co\mathcal{H}} \to B^{co\mathcal{K}},
\]
which becomes simply $\varphi^{co\mathcal{H}}$ when $\gamma$ is the identity morphism on $\mathcal{H}$.

As the constructions above are clearly natural in both the bialgebra and the algebra components of a comodule algebra, we can summarize the discussion above as follows.

**Proposition 3.18.** There is a functor $\text{Coinv} : \text{ComodAlg}_{\mathcal{V}} \to \text{Alg}_{\mathcal{V}}$ that to a morphism $(\gamma, \varphi): (H, (A, \rho_A)) \to (K, (B, \rho_B))$ in $\text{ComodAlg}_{\mathcal{V}}$ associates the algebra morphism $\varphi^{co\gamma}: A^{co\mathcal{H}} \to B^{co\mathcal{K}}$.

### 3.2. The classical Hopf-Galois framework.

We have now set up the complete framework enabling us to formulate a relative version of the classical notion of Hopf-Galois extensions of rings and algebras. To simplify notation, we drop henceforth the coactions from the notation for comodule algebras.

**Definition 3.19.** A morphism $(H, A) \xrightarrow{(\gamma, \varphi)} (K, B)$ in $\text{ComodAlg}_{\mathcal{V}}$ is a relative Hopf-Galois extension if
\[
\varphi^{co\gamma}: A^{co\mathcal{H}} \to B^{co\mathcal{K}}
\]
is an isomorphism of algebras, and
\[
\text{Gal}(\gamma, \varphi): \text{Desc}(\varphi) \to \text{Hopf}(\rho_B, \mu_K)
\]
is an isomorphism of $B$-corings.

When $\mathcal{V}$ is the category of $k$-modules for some commutative ring $k$, a relative Hopf-Galois extension for $H = k$ is exactly a classical Hopf-Galois extension, as defined by Chase and Sweedler [6]. Related notions of relative Hopf-Galois extensions have been considered in [27] and [28], in the context of quotient theory of noncommutative Hopf algebras.

**Example 3.20.** Let $H$ be a bialgebra in $\mathcal{V}$ with unit $\eta$, comultiplication $\Delta$, and multiplication $\mu$. The morphism $(\eta, \eta): (k, k) \to (H, H)$ in $\text{ComodAlg}_{\mathcal{V}}$ is a relative Hopf-Galois extension if and only if
\[
H \otimes H \xrightarrow{H \otimes \Delta} H \otimes H \otimes H \xrightarrow{\mu \otimes H} H \otimes H
\]
is an isomorphism. If $\mathcal{V}$ is the category of $k$-modules for some commutative ring $k$, then this condition is equivalent to requiring that $H$ admit an antipode, i.e., that $H$ be a Hopf algebra, in the classical sense of the word [26, Example 2.1.2]. If $\mathcal{V}$ is the category of (differential) graded $k$-modules, then, as is well known, every connected bialgebra $H$ satisfies the condition above [9, Proposition 3.8.8].

Inspired by the classical case, we make the following definition.

**Definition 3.21.** We say that a bialgebra $H$ in $\mathcal{V}$ is a Hopf algebra if the map
\[
(\eta, \eta): (k, k) \to (H, H)
\]
is a relative Hopf-Galois extension in the sense of Definition 3.19. More generally, we say that a morphism of bialgebras $\gamma: H \to K$ is a relative Hopf algebra if
\[
(\gamma, \gamma): (H, H) \to (K, K)
\]
is a morphism of fibrant replacements. Since

\[ \gamma: H \to K \] of fibrant replacements in \( \text{Alg}_H \).

By \cite[Theorem 4.4]{21}, \( K \) admits such a description if \( H \) and \( K \) are connected, while \( \gamma: H \to K \) is split injective and \( \pi_\gamma: K \to \text{Cof}(\gamma) \) is split surjective, as morphisms of graded k-modules. In particular, if \( k \) is a field, then \( \gamma \) is a relative Hopf algebra if it is injective.

For any algebra \( E \) in \( \mathcal{Y} \), and any relative Hopf algebra \( \gamma: H \to K \), let

\[ (A, \rho_A) = (E \otimes H, E \otimes \Delta_H) \quad \text{and} \quad (B, \rho_B) = (E \otimes K, E \otimes \Delta_K). \]

The morphism \( (H, (A, \rho_A)) \) \( \xrightarrow{(\gamma, E \otimes \gamma)} \) \( (K, (B, \rho_B)) \) in \( \text{ComodAlg}_{\mathcal{Y}} \) is then a generalized Hopf-Galois extension, as \( \varphi^{\text{co}\gamma} \) is simply the identity on \( E \), while \( \text{Gal}(\gamma, E \otimes \gamma) \) is given by applying the functor \( E \otimes - \) to the composite (3.2). Following classical terminology, we call this morphism a normal relative Hopf-Galois extension with normal basis \( E \).

### 3.3. The homotopic Hopf-Galois framework.

**Convention 3.22.** Henceforth \( \mathcal{Y} \) denotes a symmetric monoidal model category satisfying Convention \cite[2.2.8]{22} and the CHF hypothesis (Definition \cite[2.9]{20}).

We also require that the category \( \text{Alg}_H \) of \( H \)-comodule algebras admit the model category structure left-induced from that of \( \text{Alg}_{\mathcal{Y}} \) via the adjunction \( U \dashv (- \otimes H) \) for any bialgebra \( H \) that we consider. It follows that

\[ \text{Alg}_H \xrightarrow{\gamma^*} \text{Alg}_{\mathcal{Y}} \xrightarrow{\gamma^*} \text{Alg}_H \]

is a Quillen pair for every morphism \( \gamma: H \to K \) of bialgebras. Explicit examples of such model category structures can be found in \cite[11]{1} and \cite[10]{16}.

**Definition 3.23.** Let \( A \) be an \( H \)-comodule algebra. For any fibrant replacement \( A^f \) of \( A \) in \( \text{Alg}_{\mathcal{Y}} \), the algebra \( (A^f)^{\text{co}H} \) is a model of the homotopy coinvariants of the \( H \)-coaction on \( A \), denoted (somewhat abusively) \( A^{\text{hco}H} \).

Given an object \( (\gamma, \varphi): (H, A) \to (K, B) \) in \( \text{ComodAlg}_{\mathcal{Y}} \), we can construct an associated morphism of algebras \( \varphi^{\text{hco}}: A^{\text{hco}H} \to B^{\text{hco}K} \) as follows, inspired by Remark \cite[3.17]{3.17}.

Let

\[ i_B: B \xrightarrow{\sim} B^f \quad \text{and} \quad i_A: \gamma_* A \xrightarrow{\sim} (\gamma_* A)^f \]

be fibrant replacements in \( \text{Alg}_K \), and let \( \varphi^f: (\gamma_* A)^f \to B^f \) be an extension of \( \varphi \) to the fibrant replacements. Since \( \gamma^*: \text{Alg}_K \to \text{Alg}_{\mathcal{Y}} \) is a right Quillen functor,

\[ \gamma^*: (\varphi^f): \gamma^*: (\gamma_* A)^f \to \gamma^*: (B^f) \]

is a morphism of fibrant \( H \)-comodule algebras.

Let \( j: A \xrightarrow{\eta^A} A^f \) be any fibrant replacement in \( \text{Alg}_{\mathcal{Y}} \). The composite morphism of \( H \)-comodule algebras

\[ A \xrightarrow{\eta^A} \gamma^*: (\gamma_* A) \xrightarrow{\gamma^*: (i_A)} \gamma^*: ((\gamma_* A)^f) \]
extends to a morphism of $H$-comodule algebras
\[ i: A^f \to \gamma^*((\gamma_* A)^f), \]
since $j$ is an acyclic cofibration, and $\gamma^*((\gamma_* A)^f)$ is fibrant. A model for
\[ \varphi_{\text{hco}}: A_{\text{hco}} H \to B_{\text{hco}} K \]
is then given by the composite
\begin{equation}
(A^f)_{\text{co} H} \to (\gamma^* (\gamma_* A)^f)_{\text{co} H} \to (\gamma^* (B^f))_{\text{co} H} \cong (B^f)_{\text{co} K}.
\end{equation}

To define homotopic relative Hopf-Galois extensions, we now modify somewhat the approach of [10, Definition 3.2], categorifying both conditions instead of just one. As we see below, under reasonable hypotheses a homotopic Hopf-Galois extension in the sense of [10, Definition 3.2] also satisfies the conditions of the modified definition below.

**Definition 3.24.** A morphism $(\gamma, \varphi): (H, A) \to (K, B)$ in $\text{ComodAlg}_\mathcal{V}$ is a relative homotopic Hopf-Galois extension if both of the adjunctions
\[ \mathcal{V}_{A_{\text{hco}} H} \xrightarrow{(\varphi_{\text{hco}}^*)_*} \mathcal{V}_{B_{\text{hco}} K} \]
and
\[ \mathcal{V}_{B_{\text{co} B}} \xrightarrow{\text{Gal}(\gamma, \varphi)_*} \mathcal{V}_{\text{Hopf}(\rho_B, \beta_K)} \]
are Quillen equivalences.

A morphism $\gamma: H \to K$ of bialgebras in $\mathcal{V}$ is a relative homotopic Hopf algebra if $(\gamma, \gamma): (H, H) \to (K, K)$ is a relative homotopic Hopf-Galois extension.

**Remark 3.25.** The definition of homotopic Hopf-Galois extension is independent of the choice of fibrant replacements for $A$ and $B$ underlying the definition of $A_{\text{hco} H}$ and $B_{\text{hco} K}$, since $\mathcal{V}$ satisfies the CHF hypothesis, whence all weak equivalences of algebras are pure and therefore induce Quillen equivalences on module categories (Propositions 2.7 and 2.10).

**Remark 3.26.** In the special case of a morphism of the form $(k, A) \xrightarrow{(\eta_H, \varphi)} (H, B)$ in $\text{ComodAlg}_\mathcal{V}$, we recover a slightly modified version of the definition of a homotopic $H$-Hopf-Galois extension from [10].

**Remark 3.27.** In [25] Rognes defined homotopic Hopf-Galois extensions of commutative ring spectra in a convenient symmetric monoidal model category $\mathcal{J}$ of spectra, such as symmetric spectra and $S$-modules. According to his conventions, a morphism $(S, A) \xrightarrow{\text{(\eta_S, \varphi)}} (H, B)$ in $\text{ComodAlg}_\mathcal{J}$, where $S$ is the sphere spectrum, and $A$ and $B$ are commutative $S$-algebras, is a homotopic Hopf-Galois extension if the composite
\[ A = A_{\text{co} H} \xrightarrow{j_{\text{co} H}} A_{\text{hco} H} \xrightarrow{\varphi_{\text{hco}}^*} B_{\text{hco} H}, \]
where $j: A \xrightarrow{\sim} A^f$ is a fibrant replacement in $\text{Alg}_\mathcal{J}$, and
\[ \beta_\varphi = \text{Gal}(\eta_H, \varphi)_*: \text{Desc}(\varphi) \to \text{Hopf}(\rho_B) \]
are weak equivalences, where $B_{\text{hco} H}$ is modelled explicitly as the totalization of a certain cosimplicial “cobar”-type construction.
As it is still work in progress to show that all of conditions of Convention 3.22 hold in various incarnations of $\mathfrak{S}$ (cf. [16, Corollary 5.6]), we cannot yet apply the results below characterizing homotopic Hopf-Galois extensions to conclude that Rognes’s definition fits precisely into our framework, but we strongly suspect that it is the case.

**Remark 3.28.** The generalization of homotopic Hopf-Galois extensions to a relative framework is not merely an idle exercise. Indeed, as shown in [12], the formulation of one direction of a Hopf-Galois correspondence for Hopf-Galois extensions of differential graded algebras requires such relative extensions.

As an immediate consequence of Proposition 2.10 and Corollary 2.36, we obtain conditions under which a morphism of comodule algebras is a relative homotopic Hopf-Galois extension.

**Proposition 3.29.** Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 3.22. Let $(\gamma, \varphi): (H, A) \to (K, B)$ be a morphism in $\text{ComodAlg}_\mathcal{V}$.

If $\varphi_{\text{hco}}: A_{\text{hco}}H \to B_{\text{hco}}K$ is a weak equivalence and $\text{Gal}(\gamma, \varphi): \text{Desc}(\varphi) \to \text{Hopf}(\rho_B, \mu_K)$ is a copure weak equivalence, then $(\gamma, \varphi)$ is a relative homotopic Hopf-Galois extension.

**Corollary 3.30.** Let $\mathcal{V}$ be a symmetric monoidal model category satisfying Convention 3.22. If the unit $k$ is fibrant, then a morphism $\gamma: H \to K$ of bialgebras in $\mathcal{V}$ is a relative homotopic Hopf algebra if $\text{Gal}(\gamma, \gamma): \text{Desc}(\gamma) \to \text{Hopf}(\Delta_K, \mu_K)$ is a copure weak equivalence.

**Proof.** Since $k$ is fibrant in $\mathcal{V}$, it is fibrant in $\text{Alg}_\mathcal{V}$, whence both $H$ and $K$ are fibrant in their respective categories of comodule algebras. It follows that the identity on $k$ is a model of $\gamma_{\text{hco}}: H_{\text{hco}}H \to K_{\text{hco}}K$. $\square$

4. Homotopic Hopf-Galois extensions of chain algebras

In this section we illustrate the theory of the previous section when the underlying monoidal model category is that of chain complexes of finite-dimensional vector spaces. In particular we provide a large class of examples of homotopic Hopf-Galois extensions and prove a theorem analogous to the descent-type description of homotopic Hopf-Galois extensions in [25, Proposition 12.1.8]. The work in this section builds on [3, Section 5], the key results of which we recall below.

Let $k$ be a field, and let $\text{Ch}_k$ denote the category of non-negatively graded chain complexes of $k$-vector spaces. The category $\text{Ch}_k$ admits a model structure where the weak equivalences are the quasi-isomorphisms, the cofibrations are the degreewise injections, and the fibrations are the maps that are surjective in positive degrees [7]. This model category structure is closed monoidal with respect to the usual graded tensor product of chain complexes, where the internal hom is a truncated version of the unbounded hom-complex.

Let $\text{Ch}_k^{\text{fin}}$ denote the full monoidal subcategory of $\text{Ch}_k$, the objects of which are chain complexes that are of finite type, i.e., degreewise finite dimensional. Note that $\text{Ch}_k^{\text{fin}}$ is neither complete nor cocomplete, but does admit all degreewise-finite limits and colimits. The monoidal model category structure of $\text{Ch}_k$ therefore restricts to a monoidal model category structure on $\text{Ch}_k^{\text{fin}}$, with the same distinguished classes of morphisms, but where, as in Quillen’s original definition [24], one requires only finite completeness and cocompleteness, which suffices to define and study the associated homotopy category.

**Remark 4.1.** In a forthcoming article [13] it will be shown that the results in this section can be extended to chain complexes of over an arbitrary commutative ring.
4.1. Homotopy theory of chain modules and comodules. Let $A$ be an algebra in $\mathrm{Ch}_k$. As shown in [29, Section 4], the category $(\mathrm{Ch}_k)_A$ also admits a model category structure right-induced by the adjunction

$$
\mathrm{Ch}_k \xrightarrow{- \otimes A} (\mathrm{Ch}_k)_A.
$$

If $A$ is itself of finite type, then this structure restricts to $(\mathrm{Ch}^\text{fin})_A$, so that the adjunction

$$
\mathrm{Ch}^\text{fin}_k \xrightarrow{- \otimes A} (\mathrm{Ch}^\text{fin}_k)_A.
$$

right-induces a model category structure on $(\mathrm{Ch}^\text{fin}_k)_A$. Observe that the functor $\mathcal{U}$ preserves cofibrant objects, since all objects in $\mathrm{Ch}^\text{fin}_k$ are cofibrant. The analogous result obviously holds for left modules as well.

A particularly nice class of cofibrant $A$-modules can be described as follows.

**Definition 4.2.** An object $N$ in $A(\mathrm{Ch}_k)$ is $A$-semifree [8, Section 6] if it admits an increasing filtration,

$$
0 = F_{-1}N \subseteq F_0N \subseteq \cdots \subseteq F_pN \subseteq \cdots \subseteq N, \quad N = \bigcup N_p,
$$

such that for all $p \geq 0$, there is a graded $k$-vector space $V(p)$ such that

$$
F_pN/F_{p-1}N \cong A \otimes V(p)
$$

as differential graded $A$-modules. Semifree right $A$-modules are defined analogously.

If $B$ is another algebra in $\mathrm{Ch}_k$, then an $A$-$B$-bimodule is $A$-semifree as a right $B$-module if (4.1) is a filtration of right $B$-modules.

As summarized in the lemma below, semifree modules have many good properties. Recall Definitions 2.4 and 2.23.

**Lemma 4.3.** [3, Lemmas 5.3 and 5.4] Let $A$ be an algebra in $\mathrm{Ch}^\text{fin}_k$.

1. An $A$-module is cofibrant if and only if it is a retract of a semi-free $A$-module.
2. If $N$ is a cofibrant object in $A(\mathrm{Ch}^\text{fin}_k)$, then it is homotopy projective and homotopy flat. In particular, the category $A(\mathrm{Ch}^\text{fin}_k)$ satisfies the CHF hypothesis (Definition 2.6). If $N$ is $A$-semifree, then it is homotopy faithful.
3. If $N$ is a semifree left $A$-module of finite type, then $- \otimes A N$ preserves all finite limits. In particular, every $A$-semifree module in $A(\mathrm{Ch}^\text{fin}_k)$ is flat and homotopy compact.
4. Every algebra quasi-isomorphism $A \sim B$ is a pure weak equivalence.

Proposition 2.10 and Lemma 4.3(4) together imply the following characterization of those algebra morphisms in $\mathrm{Ch}_k$ that induce Quillen equivalences.

**Proposition 4.4.** Let $\varphi: A \rightarrow B$ be a morphism of algebras in $\mathrm{Ch}_k$. The induced restriction/extension-of-scalars adjunction

$$
\mathcal{V}_A \xrightarrow{\varphi^*} \mathcal{V}_B,
$$

is a Quillen equivalence if and only if $\varphi: A \rightarrow B$ is a quasi-isomorphism.

The existence of model category structures for categories of comodules over corings is somewhat delicate to establish. The next result is a special case of [15, Theorem 6.2].
Theorem 4.5. Let \( A \to k \) be an augmented algebra in \( \text{Ch}^\text{fin}_k \) such that \( H_1 A = 0 \). If \( C \) is a finite-type \( A \)-coring that is semifree as left \( A \)-module, \( H_0(k \otimes_A C) = k \), and \( H_1(k \otimes_A C) = 0 \), then the category \( (\text{Ch}^\text{fin}_k)_A^C \) of \((A,C)\)-comodules admits a model category structure left-induced by the adjunction

\[
(\text{Ch}^\text{fin}_k)_A^C \xrightarrow{U} - \otimes_A C \xleftarrow{\varphi} \text{Ch}^\text{fin}_k^A
\]

from the model structure on \( (\text{Ch}^\text{fin}_k)_A \) defined above.

Remark 4.6. As established in the course of the proof of Theorem 4.5, all limits in \( (\text{Ch}^\text{fin}_k)_A \) are in fact created in \( \text{Ch}^\text{fin}_k \) and thus in \( \text{Ch}^\text{fin}_k \) \cite[Lemma 6.8]{15}.

It follows immediately from \cite[Theorem 5.8]{15} and its proof that we can characterize the fibrations in the left-induced model structure on \( (\text{Ch}^\text{fin}_k)_A \) in a computationally useful way. Recall from \cite{10} the following definition, which dualizes the definition of a relative cell complex in a model category.

Definition 4.7. Let \( X \) be a class of morphisms in a complete category \( C \). Let \( Y : N \to C \) be a functor. If for all \( n \geq 0 \), there is a pullback

\[
\begin{array}{ccc}
Y^{n+1} & \xrightarrow{\partial} & X^{n+1} \\
\downarrow & & \downarrow \text{X}^{n+1} \\
Y^n & \xrightarrow{\text{X}^{n+1} \in \text{X}} & X^{n+1},
\end{array}
\]

then the composition of the tower

\[
\lim_n Y^n \to Y^0,
\]

is an \( X \)-Postnikov tower of countable height. The class of all \( X \)-Postnikov towers of countable height is denoted \( \text{Post}^X \).

Proposition 4.8. \cite[Theorem 5.8]{15} Let \( A \) and \( C \) satisfy the hypotheses of Theorem 4.5. Let \( \text{Fib} \) denote the class of fibrations in the right-induced model category structure on \( (\text{Ch}^\text{fin}_k)_A^C \). Every fibration in the left-induced model category structure on \( (\text{Ch}^\text{fin}_k)_A^C \) is a retract of an element of \( \text{Post}^X \text{Fib} \otimes_A C \).

The proof of the theorem below relies heavily on this characterization of the fibrations in \( (\text{Ch}^\text{fin}_k)_A^C \).

Theorem 4.9. \cite[Theorem 5.12]{3} Let \( A \) and \( C \) satisfy the hypotheses of Theorem 4.5. If \( f : C \to D \) is a morphism of \( A \)-corings that is a retract of a quasi-isomorphism of semifree left \( A \)-modules, then it is copure.

Semifreeness turns out to provide useful conditions for effective homotopic descent as well.

Theorem 4.10. \cite[Theorem 5.14]{3} Let \( A \) and \( B \) be augmented algebras in \( \text{Ch}^\text{fin}_k \) such that \( H_1 A = 0 \). If a morphism of algebras \( \phi : A \to B \) is such that \( B \) is semifree as a left \( A \)-module, then it satisfies effective homotopic descent.

Finally, putting all of the pieces together, we can describe a class of morphisms of corings that induce Quillen equivalences between the corresponding comodule categories.

Theorem 4.11. \cite[Corollary 5.15]{3} Let \( A \) and \( B \) be augmented algebras in \( \text{Ch}^\text{fin}_k \) such that \( H_1 A = 0 \). Let \( \phi : A \to B \) be a morphism of algebras in \( \text{Ch}^\text{fin}_k \) such that \( B \) is semifree (or, more generally, homotopy compact and homotopy faithfully
flat) as a left $A$-module. Let $(\varphi, f): (A, C) \to (B, D)$ be a morphism of corings of finite type such that $C$ is left $A$-semifree and $D$ is left $B$-semifree.

The adjunction

$$(\text{Ch}_{k}^{\text{fin}})_A \xrightarrow{(\varphi, f)} (\text{Ch}_{k}^{\text{fin}})_B$$

is a Quillen equivalence if and only if $f: B^*(C) \to D$ is a quasi-isomorphism.

**Remark 4.12.** The hypothesis on $\varphi: A \to B$ in the theorems above, requiring that $B$ be left $A$-semifree, is only mildly restrictive. For example, the KS-extensions (also known as relative Sullivan algebras) of rational homotopy theory [8] are classical examples of such algebra morphisms. More generally, any algebra morphism in $\text{Ch}_{k}^{\text{fin}}$ can be replaced up to homotopy by an algebra morphism such that the codomain is semifree over the domain. Indeed, every morphism $\varphi: A \to B$ admits a factorization

$$(A, d) \xrightarrow{\varphi} (B, d),$$

where $j$ is the inclusion into a free extension, and $p$ is a quasi-isomorphism, and [17] Proposition 4.3.11, Remark 4.3.12 implies that $(A \coprod TV, D)$ is left $A$-semifree.

**4.2. Homotopy theory of chain comodule algebras.** For technical reasons related to existence of model category structures on categories of comodule algebras, throughout this section all of the algebras with which we work will be augmented, i.e., equipped with an algebra map to the unit object. The category $\text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}}$ will therefore be taken to mean the category of augmented algebras and of morphisms preserving the augmentation.

Recall that for every bialgebra $H$ in $\text{Ch}_{k}^{\text{fin}}$, there is an adjunction

$$\text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}} \xleftarrow{\Omega(-; H; H)} \text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}},$$

where $\Omega$ is the forgetful functor.

**Theorem 4.13.** [1] Theorem 3.8] If $H$ is a differential graded $k$-Hopf algebra that is finite dimensional in each degree, then the adjunction $\Omega(-; H) \xrightarrow{- \otimes H} \text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}}$ left-induces a combinatorial model category structure on the category $\text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}}$ of augmented right $H$-comodule algebras.

It is important for our study of homotopic Hopf-Galois extensions to know that for any 1-connected chain bialgebra $H$, the two-sided cobar construction provides a canonical fibrant replacement functor

$$\Omega(-; H; H): \text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}} \to \text{Alg}_{H}^{H} \text{Ch}_{k}^{\text{fin}}, \quad A \mapsto \Omega(A; H; H),$$

and a natural weak equivalence of $H$-comodule algebras $\iota_{A}: A \to \Omega(A; H; H)$. We establish this result as follows.

We recall first the well known definition of the cobar construction for comodules over a conaugmented, differential graded coalgebra.

**Notation 4.14.** Let $T$ denote the free tensor algebra functor, which to any graded $k$-vector space $V$ associates the graded $k$-algebra $TV = k \oplus \bigoplus_{n \geq 1} V^{\otimes n}$, the homogeneous elements of which are denoted $v_1 | \cdots | v_n$. 
For any graded vector space $V$, we let $s^{-1}V$ denote the graded vector space with $s^{-1}V_n \cong V_{n+1}$ for all $n$, where the element of $s^{-1}V_n$ corresponding to $v \in V_{n+1}$ is denoted $s^{-1}v$.

Let $(C, \Delta, \varepsilon, \eta)$ be a coaugmented coalgebra in $\text{Ch}_k$, with coaugmentation coideal $\overline{C} = \text{coker}(\eta; \kappa \rightarrow C)$. We use the Einstein summation convention and write $\Delta(c) = c_i \otimes c^i$ for all $c \in C$ and similarly for the map induced by $\Delta$ on $\overline{C}$. If $(M, \rho)$ is a right $C$-comodule, then we apply the same convention again and write $\rho(x) = x_i \otimes c^i$ for all $x \in M$, and similarly for a left $C$-comodule.

**Definition 4.15.** Let $(C, \Delta, \varepsilon, \eta)$ be a coaugmented coalgebra in $\text{Ch}_k$, with coaugmentation coideal $\overline{C} = \text{coker}(\eta; \kappa \rightarrow C)$. For any right $C$-comodule $(M, \rho)$ and left $C$-comodule $(N, \lambda)$, let $\Omega(M; C; N)$ denote the object in $\text{Ch}_k$

$$(M \otimes T(s^{-1}\overline{C})) \otimes C, d_{\Omega},$$

where

$$d_{\Omega}(x \otimes s^{-1}c_1| \cdots | s^{-1}c_n \otimes y) = dx \otimes s^{-1}c_1| \cdots | s^{-1}c_n \otimes y$$

$$+ x \otimes \sum_{j=1}^{n} \pm s^{-1}c_1| \cdots | s^{-1}dc_j| \cdots | s^{-1}c_n \otimes y$$

$$\pm x \otimes s^{-1}c_1| \cdots | s^{-1}c_n \otimes dy$$

$$\pm x_1 \otimes s^{-1}c^1| s^{-1}c_1| \cdots | s^{-1}c_n \otimes y$$

$$+ x \otimes \sum_{j=1}^{n} \pm s^{-1}c_1| \cdots | s^{-1}c_{j,i}| \cdots | s^{-1}c_1 \otimes y$$

$$\pm x \otimes s^{-1}c_1| \cdots | s^{-1}c_n| s^{-1}c_1 \otimes y^i$$

where all signs are determined by the Koszul rule, the differentials of $M, N$, and $C$ are all denoted $d$, and $s^{-1}1 = 0$ by convention.

If $N = C$, then $\Omega(M; C; C)$ admits a right $C$-comodule structure induced from the rightmost copy of $C$.

**Remark 4.16.** The cobar construction $\Omega(M; C; C)$ is a “cofree resolution” of $M$, in the sense that the coaction map $\rho: M \rightarrow M \otimes C$ factors in $\text{Ch}_k^C$ as

$$M \xrightarrow{\tilde{\rho}} M \otimes C,$$

$$\Omega(M; C; C),$$

where $\tilde{\rho}(x) = x_i \otimes 1 \otimes c^i$ and $q(x \otimes 1 \otimes c) = x \otimes c$, while $q(x \otimes s^{-1}c_1| \cdots | s^{-1}c_n \otimes c) = 0$ for all $n \geq 1$. It is well known that the composite

$$\Omega(M; C; C) \xrightarrow{\Omega} M \otimes C \xrightarrow{M \otimes \varepsilon} M$$

is a quasi-isomorphism (cf. [23 Proposition 10.6.3] or any “extra degeneracy” argument). It follows that $\tilde{\rho}: M \rightarrow \Omega(M; C; C)$ is always a quasi-isomorphism of right $C$-comodules. Moreover, $\Omega(M; C; C)^{coC} \cong \Omega(M; C; \kappa)$.

When $C$ is replaced by a bialgebra and $M$ and $N$ by comodule algebras, then the cobar construction admits a compatible multiplicative structure. The case of the trivial comodule algebra $N = \kappa$ was treated in [14].

**Lemma 4.17.** [14 Corollary 3.6] If $H$ is a bialgebra in $\text{Ch}_k^{fin}$, the cobar construction lifts to a functor

$$\Omega(-; H; \kappa): \text{Alg}_{\text{Ch}_k^{fin}}^H \rightarrow \text{Alg}_{\text{Ch}_k^{fin}}^C.$$
where for every $H$-comodule $A$

\[
(a \otimes 1)(a' \otimes 1) = aa' \otimes 1, \quad \forall a, a' \in A;
\]
\[
(a \otimes w)(1 \otimes w') = a \otimes uw', \quad \forall a \in A, w, w' \in \Omega H;
\]
\[
(1 \otimes s^{-1}h)(a \otimes 1) = (-1)^{\deg h + 1} a_i \otimes s^{-1}(ha_i), \quad \forall a \in A, h \in H.
\]

As Karpova showed in [17], the lemma above implies the existence of a compatible multiplicative structure when $N = H$ as well.

**Lemma 4.18.** [17] Section 2.1.3 If $H$ is a bialgebra in $\text{Ch}\text{fin}_k$, the cobar construction lifts to a functor

\[
\Omega(-; H; H) : \text{Alg}^H_{\text{Ch}\text{fin}_k} \to \text{Alg}^H_{\text{Ch}\text{fin}_k}
\]

such that for any $H$-comodule algebra $(A, \rho)$, the coaction map $\rho : A \to A \otimes H$ factors in $\text{Alg}^H_{\text{Ch}\text{fin}_k}$ as

\[
\xymatrix{ A \ar[rr]^{\rho} \ar[dr]_{\tilde{\rho}} & & A \otimes H, \ar[dl]^{q} \\
& \Omega(A; H; H) & }
\]

where $\tilde{\rho}(a) = a_0 \otimes 1 \otimes h^i$ and $q(a \otimes 1 \otimes h) = a \otimes h$, while $q(a \otimes s^{-1}h_1, \ldots, s^{-1}h_n \otimes h) = 0$ for all $n \geq 1$.

The two-sided cobar construction $\Omega(A; H; H)$ has particularly nice properties as a left $A$-module.

**Lemma 4.19.** If $H$ is a bialgebra in $\text{Ch}\text{fin}_k$, and $A$ is an augmented $H$-comodule algebra in $\text{Ch}\text{fin}_k$, then $\Omega(A; H; H)$ is homotopy compact and homotopy faithfully flat as a left $A$-module, with respect to the structure induced by the algebra map $\tilde{\rho} : A \to \Omega(A; H; H)$.

**Proof.** Note first that the nondifferential graded $A$-module underlying $\Omega(A; H; H)$ is $A$-free, whence $- \otimes_A \Omega(A; H; H)$ preserves kernels. The functor $- \otimes_A \Omega(A; H; H)$ also preserves all finite products, since they are isomorphic to finite sums. It follows that $- \otimes_A \Omega(A; H; H)$ preserves all finite limits and thus that $\Omega(A; H; H)$ is homotopy compact.

To see that $\Omega(A; H; H)$ is homotopy flat, observe that for any right $A$-module $M$, the graded $k$-vector space underlying $M \otimes_A \Omega(A; H; H)$ is isomorphic to $M \otimes T(s^{-1}T) \otimes H$ and admits a bounded, increasing differential filtration with

\[
\mathcal{F}^p(M \otimes_A \Omega(A; H; H)) = M_{\leq p} \otimes T(s^{-1}T) \otimes H.
\]

The $E^2_{p,q}$-term of the associated spectral sequence, which converges to $H_A(M \otimes_A \Omega(A; H; H))$, is isomorphic to $H_pM \otimes H_A \Omega(k; H; H)$, i.e., $H_pM$ when $q = 0$ and 0 when $q > 0$. An easy argument by the Zeeman comparison theorem implies that if $f : M \to N$ is a weak equivalence, then so is $f \otimes_A \Omega(A; H; H) : M \otimes_A \Omega(A; H; H) \to N \otimes_A \Omega(A; H; H)$.

Finally, suppose that $f : M \to N$ is a morphism of right $A$-modules such that

\[
f \otimes_A \Omega(A; H; H) : M \otimes_A \Omega(A; H; H) \to N \otimes_A \Omega(A; H; H)
\]

is a weak equivalence. Since $A$ is a retract of $\Omega(A; H; H)$ as an algebra, $f$ is a retract of $f \otimes_A \Omega(A; H; H)$ and is therefore also a weak equivalence. \qed

A proof essentially identical to that of [15] Theorem 7.8 enables us to establish the following result based on Lemma 4.18.
Theorem 4.20. Let \( H \) be a bialgebra in \( \text{Ch}_{k}^{\text{fin}} \). For every augmented \( H \)-comodule algebra \( (A, \rho) \), the maps \( \tilde{\rho} \) and \( q \) are a trivial cofibration and a fibration, respectively, in \( \text{Alg}_{H}^{\text{Ch}_{k}^{\text{fin}}} \). Moreover, both the source and the target of \( q \) are fibrant in \( \text{Alg}_{H}^{\text{Ch}_{k}^{\text{fin}}} \), whence \( \Omega(A; H; H) \) is a fibrant replacement of \( A \) in \( \text{Alg}_{H}^{\text{Ch}_{k}^{\text{fin}}} \).

The second part of Remark 4.16 implies that the next result is an immediate consequence of Theorem 4.20.

Corollary 4.21. Let \( H \) be a bialgebra in \( \text{Ch}_{k}^{\text{fin}} \). For every augmented \( H \)-comodule algebra \( (A, \rho) \), the algebra \( \Omega(A; H; k) \) is a model of \( A^{\text{co}H} \).

The proposition below further illustrates the utility of Theorem 4.20.

Proposition 4.22. For any 1-connected chain bialgebra \( H \) in \( \text{Ch}_{k}^{\text{fin}} \), the functor \((-)^{\text{co}H}: \text{Alg}_{H}^{\text{Ch}_{k}^{\text{fin}}} \rightarrow \text{Alg}_{\text{Ch}_{k}^{\text{fin}}} \) reflects weak equivalences between fibrant objects.

Proof. Let \( \varphi: (A, \rho_{A}) \rightarrow (B, \rho_{B}) \) be a morphism of fibrant \( H \)-comodule algebras such that \( \varphi^{\text{co}H} \) is weak equivalence. To show that \( \varphi \) is necessarily also a weak equivalence, first consider the commutative diagram of fibrant \( H \)-comodule algebras

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\tilde{\rho}_{A} \downarrow \sim & & \downarrow \sim \\
\Omega(A; H; H) & \xrightarrow{\Omega(\varphi; H; H)} & \Omega(B; H; H).
\end{array}
\]

Applying \((-)^{\text{co}H} \), we obtain a commutative diagram of algebras

\[
\begin{array}{ccc}
A^{\text{co}H} & \xrightarrow{\varphi^{\text{co}H}} & B^{\text{co}H} \\
\tilde{\rho}_{A}^{\text{co}H} \downarrow \sim & & \downarrow \sim \\
\Omega(A; H; k) & \xrightarrow{\Omega(\varphi; H; k)} & \Omega(B; H; k),
\end{array}
\]

where the vertical arrows are still weak equivalences, since \((-)^{\text{co}H} \) is a right Quillen functor, and the top horizontal arrow is a weak equivalence by hypothesis. By two-out-of-three, \( \Omega(\varphi; H; k) \) is also weak equivalence, whence, by Zeeman’s comparison theorem \([30]\), \( \Omega(\varphi; H; H) \) is also a weak equivalence. Two-out-of-three applied to the first diagram then implies that \( \varphi \) is a weak equivalence as well. \( \square \)

4.3. Homotopic relative Hopf-Galois extensions of chain algebras. We can now provide concrete examples of homotopic relative Hopf-Galois extensions in \( \text{Ch}_{k}^{\text{fin}} \), as well as conditions under which being a homotopic Hopf-Galois extension is equivalent to satisfying homotopic descent, which enables us moreover to include a generalized notion of Koszul duality in our global picture.

We begin by establishing the existence of a useful class of homotopic relative Hopf algebras.

Lemma 4.23. A morphism of 1-connected bialgebras \( \gamma: H \rightarrow K \) in \( \text{Ch}_{k}^{\text{fin}} \) is a homotopic relative Hopf algebra if \( K \) is semifree as a left \( H \)-module.

Proof. By Corollary 3.30 \( \gamma \) is a homotopic relative Hopf algebra if

\[
\text{Gal}(\gamma): \text{Desc}(\gamma) \rightarrow \text{Hopf}(\Delta K, \bar{\mu}_{K})
\]

is a copure weak equivalence, since \( k \) is fibrant in \( \text{Ch}_{k}^{\text{fin}} \). On the other hand, since \( K \) is \( H \)-semifree, the left \( H \)-modules underlying \( \text{Desc}(\gamma) \) and \( \text{Hopf}(\Delta K, \bar{\mu}_{K}) \), which are \( K \otimes_{H} K \) and \( K \otimes \text{Cof}(\gamma) \), are also left \( H \)-semifree. Theorem 4.9 implies that
it suffices therefore to prove that $\text{Gal}(\gamma, \gamma)$ is a quasi-isomorphism. As seen in Example \[3.20\] however, since $K$ is $H$-semifree and therefore $\gamma$ is injective, $\text{Gal}(\gamma, \gamma)$ is actually an isomorphism by \[21\] Theorem 4.4. □

Given a homotopic relative Hopf algebra, one can construct a homotopy-theoretic analogue of the “normal basis” extension in Example \[3.20\].

**Lemma 4.24.** Let $\gamma: H \to K$ be a homotopic relative Hopf algebra in $\text{Ch}_{k}^{\text{fin}}$, where $H$ and $K$ are 1-connected, and $K$ is $H$-semifree. Let $E$ be an augmented $K$-comodule algebra in $\text{Ch}_{k}^{\text{fin}}$.

If $A = \Omega(E; K; H)$, $B = \Omega(E; K; K)$, and $\varphi = \Omega(E; K; \gamma)$, then

$$(\gamma, \varphi): (H, A) \to (K, B)$$

is a homotopic relative Hopf-Galois extension, and $\varphi$ satisfies effective homotopic descent.

**Proof.** It is clear that $(\gamma, \varphi)$ is a morphism in $\text{ComodAlg}_{\text{Ch}_{k}^{\text{fin}}}$. Since $B$ is a fibrant object in $\text{Alg}_{k}^{K}$, and $\gamma^{*}$ is a right Quillen functor, $A$ is a fibrant object in $\text{Alg}_{\text{Ch}_{k}^{\text{fin}}}^{H}$, as $A = \gamma^{*}B$. It follows that a model of

$$\varphi^{\text{hco}}: A^{\text{hco}} \to B^{\text{hco}}K$$

is the identity on $\Omega(E; K; k)$, whence the first adjunction in Definition \[3.24\] is an actual equivalence of categories. Moreover, since

$$B \otimes_{A} B \cong \Omega(E; K; K \otimes_{H} K) \quad \text{and} \quad B \otimes \text{Cof}(\gamma) \cong \Omega(E; K; K \otimes (k \otimes_{H} K)),$$

it follows by Theorem \[4.11\] that if $\gamma: H \to K$ is a homotopic relative Hopf algebra, then $\text{Gal}(\gamma, \gamma)$ is a quasi-isomorphism, whence $\text{Gal}(\gamma, \varphi) = \Omega(E; K; \text{Gal}(\gamma, \gamma))$ is a quasi-isomorphism and therefore that the second adjunction in Definition \[3.24\] is a Quillen equivalence. We can thus conclude that $(\gamma, \varphi)$ is indeed a homotopic relative Hopf-Galois extension.

Since $K$ is left $H$-semifree, $B$ is left $A$-semifree, whence $\varphi$ satisfies effective homotopic descent by Theorem \[4.10\]. □

Applying the homotopic normal basis construction, we establish a relative analogue of \[2.20\] Proposition 12.1.8] in the differential graded context.

**Proposition 4.25.** Let $\gamma: H \to K$ be a homotopic relative Hopf algebra in $\text{Ch}_{k}^{\text{fin}}$ such that $H$ and $K$ are 1-connected, and $K$ is $H$-semifree. Let $(\gamma, \varphi): (H, A) \to (K, B)$ be a morphism in $\text{ComodAlg}_{\text{Ch}_{k}^{\text{fin}}}$ such that $A$ and $B$ are 1-connected.

If $\varphi^{\text{hco}}: A^{\text{hco}} \to B^{\text{hco}}K$ is a weak equivalence, then $(\gamma, \varphi)$ is a homotopic relative Hopf-Galois extension if and only if $\varphi$ satisfies effective homotopic descent.

**Proof.** Our strategy in this proof is to exploit a comparison of $(\gamma, \varphi)$ with a homotopic normal basis extension. Note first that since $k$ is a field, the coring $(k, K)$ is flat and thus by Proposition \[2.20\] the functor $\gamma^{*}: \text{Alg}_{k}^{K} \to \text{Alg}_{k}^{H}$ is isomorphic to the cotensor product functor $- \otimes_{K} H$.

Let $\Omega(A; H; H)$ and $\Omega(B; K; K)$ be the fibrant replacements of $A$ in $\text{Alg}_{\text{Ch}_{k}^{\text{fin}}}^{H}$ and of $B$ in $\text{Alg}_{\text{Ch}_{k}^{\text{fin}}}^{K}$ given by Corollary \[1.24\]. Recall formula \[33\] for $\varphi^{\text{hco}}$. Since $\varphi^{\text{hco}}$ is a weak equivalence by hypothesis, Proposition \[1.22\] implies that the composite

$$\Omega(A; H; H) \to \Omega(\gamma; A; K; K) \otimes_{K} H \to \Omega(B; K; K) \otimes_{K} H \cong \Omega(B; K; H)$$

is also a weak equivalence. Precomposing with $\tilde{\rho}_{A}: A \xrightarrow{\sim} \Omega(A; H; H)$, we obtain a weak equivalence of $H$-comodule algebras

$$\alpha: A \xrightarrow{\sim} \Omega(B; K; H).$$
Set $A' = \Omega(B; K; H)$, $B' = \Omega(B; K; K)$, and
$$\varphi' = \Omega(B; K; \gamma): \gamma_* A' \to B'.$$

Lemma 4.24 implies that $(\gamma, \varphi'): (H, A') \to (K, B')$ is itself a homotopic relative Hopf-Galois extension and that $\varphi'$ satisfies effective homotopic descent.

By Remark 3.13, the commuting diagram of comodule algebra morphisms

$$
\begin{array}{c}
(H, A) \\
\downarrow \alpha \quad \downarrow \tilde{\alpha} \\
(H, A') \\
\end{array}
\xymatrix{\quad (K, B) \quad \\
\quad (K, B') \quad }
$$

gives rise to a commuting diagram of functors

$$
\begin{array}{c}
\text{Can}_\varphi \\
\downarrow \text{Prim}_\varphi \\
\text{Can}_\varphi' \\
\downarrow \text{Prim}_\varphi' \\
\end{array}
\xymatrix{\quad (\text{Ch}_{k}^\text{fin})_A \quad \\
\quad (\text{Ch}_{k}^\text{fin})_{A'} \quad }
\begin{array}{c}
\rho_B \cdot \text{Desc}(\varphi) \\
\downarrow \rho_{B'} \cdot \text{Desc}(\varphi') \\
\rho_B \cdot \text{Desc}(\varphi) \\
\downarrow \rho_{B'} \cdot \text{Desc}(\varphi') \\
\end{array}
\xymatrix{\quad \text{Gal}(\gamma, \varphi) \quad \\
\quad \text{Gal}(\gamma, \varphi') \quad }
\begin{array}{c}
\rho_B \cdot H_{1k, \delta_B} \\
\downarrow \rho_{B'} \cdot H_{1k, \delta_B} \\
\rho_B \cdot H_{1k, \delta_B} \\
\downarrow \rho_{B'} \cdot H_{1k, \delta_B} \\
\end{array}
\xymatrix{\quad (\text{Ch}_{k}^\text{fin})_B \quad \\
\quad (\text{Ch}_{k}^\text{fin})_{B'} \quad }
$$

where the $B'$-bimodule map underlying $H_{1k, \delta_B}$,

$$B' \otimes \text{Cof}(\gamma) \otimes_B B' \to B' \otimes \text{Cof}(\gamma),$$

is given by the right $B'$-action on $B' \otimes \text{Cof}(\gamma)$ (cf. Definition 3.3). This map is a quasi-isomorphism, since the canonical isomorphism

$$B' \otimes \text{Cof}(\gamma) \otimes_B B \xrightarrow{\cong} B' \otimes \text{Cof}(\gamma)$$

factors as

$$B' \otimes \text{Cof}(\gamma) \otimes_B B \xrightarrow{\text{Cof}(\gamma) \otimes \tilde{\rho}_B} B' \otimes \text{Cof}(\gamma) \otimes_B B' \xrightarrow{H_{1k, \delta_B}} B' \otimes \text{Cof}(\gamma),$$

where the first map is a quasi-isomorphism by Remark 4.10 because it is equal to

$$\tilde{\rho}_{B' \otimes \text{Cof}(\gamma)}: B' \otimes \text{Cof}(\gamma) \to \Omega(B' \otimes \text{Cof}(\gamma); K; K).$$

Proposition 4.4 implies that the top adjunction is a Quillen equivalence, since $\alpha$ is a quasi-isomorphism. Because $H_{1k, \delta_B}$ is also a quasi-isomorphism, and $\text{Hopf}(\rho_B, \mu_K)$ and $\text{Hopf}(\rho_{B'}, \mu_{K'})$ are free and therefore semifree over their base algebras, Theorem 4.11 implies that the bottom adjunction is also a Quillen equivalence, since $\Omega(B; K; K)$ is homotopy compact and homotopy faithfully flat by Lemma 4.18. Moreover the two vertical adjunctions on the right side of the diagram are Quillen equivalences, by Lemma 4.21.

A “two-out-of-three” argument enables us to conclude that $(\gamma, \varphi): (H, A) \to (K, B)$ is a relative homotopic Hopf-Galois extension if and only if $\varphi: A \to B$ satisfies effective homotopic descent.
Remark 4.26. We believe that it should be possible to generalize the strategy in the proof above to many other monoidal model categories, establishing an equivalence between homotopic Hopf-Galois extensions and morphisms satisfying effective homotopic descent when the induced map on the coinvariants is a weak equivalence. The key to the proof is the existence of a well-behaved construction, replacing any (nice enough) morphism of comodule algebras by a weakly equivalent morphism of comodule algebras that is a homotopic Hopf-Galois extension and that satisfies effective homotopic descent. A “homotopic normal extension” of the sort employed in the proof above should do the trick in monoidal model categories with compatible simplicial structure.

The close relationship between homotopic Hopf-Galois extensions and morphisms satisfying effective homotopic descent enables us to include the notion of Koszul duality in our general picture as well.

Proposition 4.27. Let \((\gamma, \varphi): (H, A) \to (K, B)\) be a relative homotopic Hopf-Galois extension in \(\text{ComodAlg}_{\text{Ch}^\text{fin}_k}\) such that \(\varphi: A \to B\) satisfies effective homotopic descent.

If the unit map \(\eta: k \to B\) is a weak equivalence, then \(\text{Cof}(\gamma)\) is a generalized Koszul dual of \(A\), in the sense that homotopy category of right \(A\)-modules is equivalent to the homotopy category of right \(\text{Cof}(\gamma)\)-comodules.

Proof. Since \((\gamma, \varphi): (H, A) \to (K, B)\) is a relative homotopic Hopf-Galois extension, and \(\varphi: A \to B\) satisfies effective homotopic descent, there is a chain of Quillen equivalences

\[
\text{Ch}_{\text{Ch}^\text{fin}_k} \xrightarrow{\text{Can}_\varphi} \text{Ch}_{\text{Ch}^\text{fin}_k} \xrightarrow{\text{Desc}(\varphi)} \text{Ch}_{\text{Ch}^\text{fin}_k} \xrightarrow{\text{Gal}(\gamma, \varphi)_*} \text{Ch}_{\text{Ch}^\text{fin}_B} \xrightarrow{\text{Hopf}(\rho_B, \mu_K)} .
\]

The unit map \(\eta: k \to B\) induces a morphism in \(\text{Pair}\)

\[
(Id_K, \eta, Id_{\text{Cof}(\gamma)}): (K, k, \text{Cof}(\gamma)) \to (K, B, \text{Cof}(\gamma))
\]

and therefore a morphism of corings with underlying morphism of chain complexes

\[
\text{Id}_B \otimes \mu_K: B \otimes \text{Cof}(\gamma) \otimes B \to B \otimes \text{Cof}(\gamma),
\]

which is a weak equivalence, since \(\eta\) is a weak equivalence, and \(\mu_K \circ (\text{Id}_{\text{Cof}(\gamma)} \otimes \eta) = \text{Id}_{\text{Cof}(\gamma)}\). Because \(B\) and \(\text{Cof}(\gamma)\) are left \(k\)-semifree, and \(B \otimes \text{Cof}(\gamma)\) is left \(B\)-semifree, the induced adjunction

\[
\text{Ch}_{\text{Ch}_k} \xrightarrow{\text{Cof}(\gamma)_*} \text{Ch}_{\text{Ch}_B} \xrightarrow{\text{Hopf}(\rho_B, \mu_K)}
\]

is a Quillen equivalence as well. \(\square\)

The connection between Koszul duality and homotopic Hopf-Galois extensions hinted at here will be explored further in a forthcoming paper [2].

References

1. Marzieh Bayeh, Kathryn Hess, Varvara Karpova, Magdalena Kedziorek, Emily Riehl, and Brooke Shipley, Left-induced model category structures on diagram categories, available on the arXiv, to appear in Contemporary Mathematics, 2014.
2. Alexander Berglund and Kathryn Hess, Homotopic descent, Koszul duality, and Hopf-Galois extensions, in preparation.
3. , Homotopical Morita theory for corings, available on the arXiv.
4. Tomasz Brzeziński and Robert Wisbauer, Corings and comodules, London Mathematical Society Lecture Note Series, vol. 309, Cambridge University Press, Cambridge, 2003. MR 2012570 (2004k:16093)
5. Gunnar Carlsson, Derived completions in stable homotopy theory, J. Pure Appl. Algebra 212 (2008), no. 3, 550–577. MR MR2365333 (2008m:55018)
6. Stephen U. Chase and Moss E. Sweedler, Hopf algebras and Galois theory, Lecture Notes in Mathematics, Vol. 97, Springer-Verlag, Berlin-New York, 1969. MR 0260724 (41 #5348)
7. William G. Dwyer and Jan Spalinski, Homotopy theories and model categories, Handbook of Algebraic Topology (I.M. James, ed.), 1995, pp. 73–126.
8. Yves Félix, Stephen Halperin, and Jean-Claude Thomas, Rational homotopy theory, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, New York, 2001. MR 1802847 (2002d:55014)
9. Michiel Hazewinkel, Nadiya Gubareni, and V. V. Kirichenko, Algebras, rings and modules, Mathematical Surveys and Monographs, vol. 168, American Mathematical Society, Providence, RI, 2010. Lie algebras and Hopf algebras. MR 2724822 (2011j:16066)
10. Kathryn Hess, Homotopic Hopf-Galois extensions: foundations and examples, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 79–132. MR 2544387 (2010j:55010)
11. Kathryn Hess, A general framework for homotopic descent and codescent, available on the arXiv, 2010.
12. Kathryn Hess and Varvara Karpova, Homotopic Hopf-Galois extensions of commutative differential graded algebras, In preparation, 2014.
13. Kathryn Hess, Magdalena Kedziorek, Emily Riehl, and Brooke Shipley, Left-induced model category structures and enriched cofibrant generation, In preparation, 2014.
14. Kathryn Hess and Ran Levi, An algebraic model for the loop space homology of a homotopy fiber, Alg. Geom. Top. 7 (2007), 1699–1765.
15. Kathryn Hess and Brooke Shipley, The homotopy theory of coalgebras over a comonad, Proc. London Math. Soc. 108 (2014), 484–516.
16. Waldhausen K-theory of spaces via comodules, submitted, available on the arXiv, 2014.
17. Varvara Karpova, Homotopic Hopf-Galois extensions of commutative differential graded algebras, Ph.D. thesis, EPFL, 2014.
18. H. F. Kreimer and M. Takeuchi, Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J. 30 (1981), no. 5, 675–692. MR 625957 (83h:16015)
19. F. E. J. Linton, Coequalizers in categories of algebras, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 75–90. MR 0244341 (39 #5656)
20. Bachuki Mesablishvili, Monads of effective descent type and comonadicity, Theory Appl. Categ. 16 (2006), No. 1, 1–45 (electronic). MR MR2210664 (2006m:18002)
21. John W. Milnor and John C. Moore, On the structure of Hopf algebras, Ann. of Math. (2) 81 (1965), 211–264. MR 0174052 (36 #6480)
22. Susan Montgomery, Hopf Galois theory: a survey, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 367–400. MR 2544394 (2010g:16060)
23. Joseph Neisendorfer, Algebraic methods in unstable homotopy theory, New Mathematical Monographs, vol. 12, Cambridge University Press, Cambridge, 2010. MR 2604913 (2011j:55026)
24. Daniel G. Quillen, Homotopical algebra, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967. MR 0223432 (36 #6480)
25. John Rognes, Galois extensions of structured ring spectra. Stably dualizable groups, Mem. Amer. Math. Soc. 192 (2008), no. 898, viii+137. MR 2387923 (2009c:55007)
26. Peter Schauenburg, Hopf-Galois and bi-Galois extensions, Galois theory, Hopf algebras, and semialternian categories, Fields Inst. Commun., vol. 43, Amer. Math. Soc., Providence, RI, 2004, pp. 469–515. MR 2075600 (2005f:16065)
27. Peter Schauenburg and Hans-Jürgen Schneider, On generalized Hopf Galois extensions, J. Pure Appl. Algebra 202 (2005), no. 1–3, 188–194. MR 2163407 (2006c:16069)
28. Hans-Jürgen Schneider, Principal homogeneous spaces for arbitrary Hopf algebras, Israel J. Math. 72 (1990), no. 1-2, 167–195, Hopf algebras. MR 1098988 (92a:16047)
29. Stefan Schwede and Brooke E. Shipley, Algebras and modules in monoidal model categories, Proc. London Math. Soc. (3) 80 (2000), no. 2, 491–511. MR 1734325 (2001c:18006)
30. E. C. Zeeman, A proof of the comparison theorem for spectral sequences, Proc. Cambridge Philos. Soc. 53 (1957), 57–62. MR 0084769 (18,191f)

DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, SE-106 91 STOCKHOLM, SWEDEN
E-mail address: alexb@math.su.se
MATHGEOM, ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE, CH-1015 LAUSANNE, SWITZERLAND
E-mail address: kathryn.hess@epfl.ch