Vanishing of Gravitational Particle Production 
in the 
Formation of Cosmic Strings

Iver Brevik
Division of Mechanics, Norwegian University of Science and Technology, N-7034 Trondheim, Norway

Bjørn Jensen
NORDITA, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Abstract
We consider the gravitationally induced particle production from the quantum vacuum which is defined by a free, massless and minimally coupled scalar field during the formation of a gauge cosmic string. Previous discussions of this topic estimate the power output per unit length along the string to be of the order of $10^{68}$ ergs/sec/cm in the s-channel. We find that this production may be completely suppressed. A similar result is also expected to hold for the number of produced photons.

PACS number(s): 11.27.+d, 98.80.Cq

\footnote{1 e-mail:iver.h.brevik@mtf.ntnu.no}
\footnote{2 e-mail:bjensen@gluon.uio.no}
\footnote{3 On leave of absence from Institute of Physics, University of Oslo, Norway.}
1 Introduction

When a long and straight gauge cosmic string is formed, it is expected that the resulting spacetime geometry outside the string core to a very high degree of approximation can be described by a flat space with a conical deficit angle. Previous studies indicate that the formation of such strings is accompanied by an intense burst of particles and radiation which are released from the quantum vacuum. The power output per unit length along a string during the formation process may be very large as to be of the order of $10^{68}$ ergs/sec/cm for a free, massless and minimally coupled scalar field [1]. The power output is strongly suppressed when one considers higher angular momentum quantum numbers. A similar estimate for the production of photons was deduced in [2]. The purpose of this communication is to show that the energy produced during the formation of a gauge cosmic string may be significantly lower than what previous estimates have indicated. It is found in particular that there is no energy production in the zero-angular momentum sector when one considers the free, massless and minimally coupled scalar-field sector to leading order in the string tension. We also expect a similar conclusion to hold for the number of produced photons.

2 A String Geometry

Let us first briefly review some properties of the geometry which is induced by a gauge cosmic string. It is in general believed that the spacetime geometry outside a long and straight gauge cosmic string can be approximated by the spacetime found outside a corresponding fundamental string, since the typical radius of a cosmic string is typically of the order of $10^{-30}$ cm. Hence, the cosmic string action can be approximated by the fundamental string action

$$S = -\frac{T}{2} \int d^2 \sigma \sqrt{-h} \partial_\mu X^A \partial_\nu X^B h^{\mu\nu} g_{AB},$$  \hspace{1cm} (1)

where Greek letters denote world-sheet indices, while capital Latin ones denote target-space coordinates. $h$ denotes the world-sheet geometry, $g$ the target space geometry, and $T (> 0)$ is the string tension. It was found in [3, 4] that such a string which is static, straight and infinitely long will, in the Weyl gauge and using cylinder coordinates, give rise to the geometry

$$ds^2 = a_0^{-1}(-dt^2 + dr^2 + dz^2) + a_0 r^2 d\phi^2$$  \hspace{1cm} (2)
where \( a_0 = 1 - 4G\mu_0 \), and \( \mu_0(=T) \) is the proper energy density in the string. In the following we will use both \( \mu_0 \) and \( T \) to denote the proper energy density in the string. It is at the outset put no restrictions on the value of \( \mu_0 \) so that \( \mu_0 \) parametrises a single and infinitely large family of geometries. The string source is assumed to be positioned at \( r = 0 \), such that the region \( r > 0 \) is a flat vacuum region. The coordinates are all assumed to be independent of each other, and of \( \mu_0 \). The world-sheet geometry is related to the target space geometry by

\[
h_{\mu\nu} = \partial_\mu X^A \partial_\nu X^B g_{AB}.
\]

By adjusting the \( t \) and \( z \) coordinates to coincide with the timelike and spacelike coordinates in the string world-sheet we have that

\[
h_{\mu\nu} = a_0^{-1} \eta_{\mu\nu}.
\]

Clearly, when \( \mu_0 > (4G)^{-1} \) we see that the world-sheet geometry changes sign, while the orientation of the induced geometry in the string world-sheet is unchanged. It is correspondingly straightforward to see that the target space geometry changes signature when \( \mu_0 \) becomes greater that \( (4G)^{-1} \). The geometry in eq.(2) can be brought in the form

\[
ds^2 = -dT^2 + dR^2 + dZ^2 + a_0^2 R^2 d\phi^2
\]

with the use of the rescalings \( T = a_0^{-1/2} t \), \( R = a_0^{-1/2} r \) and \( Z = a_0^{-1/2} z \) provided that \( \mu_0 < (4G)^{-1} \). This last expression for the string geometry can be derived directly from the Levi-Civita form of the line-element [3, 4].

The action in eq.(1) is not invariant under target space diffeomorphisms. Using eq.(5) we find that the string action takes the form

\[
S = -\frac{T}{2} \int d^2 \sigma \partial_\mu X^A \partial_\nu X^B \eta^{\mu\nu} \eta_{AB},
\]

where \( \eta \) denotes the corresponding Minkowski geometry, while using the geometry in eq.(2) we find that

\[
S = -\frac{T}{2a_0} \int d^2 \sigma \partial_\mu X^A \partial_\nu X^B \eta^{\mu\nu} \eta_{AB}.
\]
Hence, in the geometry eq.(2) we can perceive the string to carry a renormalised tension $T_{\text{ren}}$ which is given by

$$ T_{\text{ren}} \equiv \frac{T}{1 - 4GT} \Rightarrow T = \frac{T_{\text{ren}}}{1 + 4G T_{\text{ren}}} . $$

(8)

Since $\mu_0$ (or $T$) denotes the canonical energy measure of the string relative to the geometry in eq.(5), it is more correct to formulate the geometry in eq.(2) in terms of the energy measure of the string relative to the coordinate system which is used there, i.e. in terms of $T_{\text{ren}}$. The string geometry then takes the form

$$ ds^2 = b_0 (-dt^2 + dr^2 + dz^2) + b_0^{-1} r^2 d\phi^2 , $$

(9)

where $b_0 \equiv 1 + 4GT_{\text{ren}} = (1 - 4G\mu_0)^{-1} \geq 1$.

In [1], and in a large number of consecutive studies (see [2, 7, 8], e.g.), the induced string geometry was taken in the form

$$ ds^2 = -dt^2 + dr^2 + dz^2 + a_0^2 r^2 d\phi^2 . $$

(10)

We have used the same coordinates in eq.(10) as in the other equations above. We can do this since it is assumed (implicitly) in all the references above that the coordinates used in eq.(10)

(1) cover the complete spacetime manifold ($r > 0$)

(2) do not depend on $\mu_0$.

The singularity structure carried by the manifold described by eq.(10) coincide with the singularity structure in the manifold described in eq.(2) at $r = 0$

---

4 One can also come to this conclusion without the explicit use of the string action by a very simple and straightforward calculation of the energy density in the string in the coordinate system in eq.(2), when $T$ is defined as the proper energy density in the string.

5 Note that this string tension renormalisation is of a purely classical and non-perturbative nature, and does therefore represent an additional string tension renormalisation mechanism in addition to the perturbative ones of quantum mechanical origin (see [3], and the references therein).

6 The reason for identifying the coordinates in eq.(2) and eq.(10) is to be able to relate the number of particles released in string formation when calculated in the geometry in eq.(2) and eq.(10), since quantum field theory is not generally covariant.
(a curvature singularity), but the singularity structures differ significantly at
the point $\mu = (4G)^{-1}$ in the $\mu_0$-parameter space where the geometry in
eq(2) changes signature and $T_{\text{ren}} \to \infty$. Hence, since eq.(2) is derived with
no particular restriction on the value of $\mu_0$ eq.(2) and eq.(10) describe two
different (families of) manifolds. However, at each fixed point $\mu_0$ in the $\mu_0$-
parameter space, and such that $0 \leq \mu_0 < (4G)^{-1}$ (which is the physically
interesting regime), these spaces can be brought into manifestly correspond-
ing forms via the rescalings following eq.(5). These spaces can also be so
related when $\mu_0 > (4G)^{-1}$. However, in this case we must in addition to a
set of rescalings also change the signature of either the geometry in eq.(10),
or the geometry in eq.(2). Note that $a_0$ is the same in both eq.(2) and eq.(10)
since it is a function of the proper energy density in the source. One way to
give a precise description of the relation between the geometry in eq.(2) and
the geometry in eq.(10) is to observe \[3, 4\] that eq.(2) can be written in the
form
\[ ds^2 = a_0^{-1}(-dt^2 + dr^2 + dz^2 + a_0^2r^2d\phi^2). \]  

Hence, under the assumptions in (1) and (2) above, it follows that the geo-
metry which is used in a majority of previous works on gauge cosmic string
theory is conformally related to the geometry in eq.(2), i.e. one can go from
eq(2) and to the form in eq.(10) by multiplication with an overall constant
scale factor.

### 3 Gravitational Particle Creation

Even though a cosmic string may have the impressing proper mass per unit
length $\mu_0 \sim 10^{22} \text{ g/cm} \sim 10^{43} \text{ erg/cm}$, the deviation from the Minkowski
space which the string induces in $dt = dz = 0$ hyperplanes is only of the
(less impressing) order $G\mu_0 \sim 10^{-6}$. However, it has been argued in earlier
studies that even though a static string barely distorts spacetime, the energy
production $W$ during the formation of the string may be huge, and at least of
the order of $W \sim 10^{30} \text{ erg/cm}$ during a formation time which was taken to be
of the order $\Delta t \sim 10^{-35} \text{ sec.}$. In these studies a variety of different scalar fields
as well as Maxwell fields were studied. Hence, physical particles released
from the quantum vacuum due to the creation of a string may represent a
potentially significant source of entropy in the early universe \[1\]. We will
now turn to a reassessment of these production estimates.
In the computation of the energy released in the creation of a string we will use the instantaneous approximation, which was pioneered in this context in [1]. In this approach one assumes that the creation of the string happens instantaneously. We will follow [1] in that we will assume that the initial spacetime is Minkowski space, but we will describe the final spacetime by eq.(9) (or equivalently eq.(11)) and not by eq.(10). The actual time-dependent problem can thus be captured by writing the relevant spacetime geometry as in eq.(9), but with $T_{\text{ren}}$ replaced by a time-dependent function. In our case this time-dependent function should be the Heaviside step-function $\Theta$, i.e.

$$T_{\text{ren}} \rightarrow T_{\text{ren}}(t) = T_{\text{ren}}\Theta(t - t_0) \Rightarrow b_0 \rightarrow b(t).$$ (12)

The moment of creation of the string is at $t = t_0$. In explicit calculations we will set $t_0 = 0$ without loss of any generality.

We will direct our attention to a massless and minimally coupled scalar field $\Phi(x)$ configuration with the density

$$\mathcal{L} = \sqrt{-g} \partial_A \Phi(x) \partial^A \Phi(x).$$ (13)

Since we always have that $\sqrt{-g}g^{AA} = 1$; $A \neq \phi$, it follows that the corresponding field equation reduces to the form

$$\left(\Box_3 + \frac{b^2(t)}{r} \partial_\phi^2\right)\Phi(x) = 0,$$ (14)

where $\Box_3$ is the d’Alembertian in a 2+1-dimensional Minkowski geometry for all times $t$. The reader should be aware that the form of eq.(14) does not depend on the particular substitution in eq.(12), but is valid for any general function $b(t)$. From this equation of motion it follows in particular that the continuity condition across $t = t_0$ in the time direction simply reduces to

$$\left(\partial_t \Phi(x)\right)|_{t^-_0} = \left(\partial_t \Phi(x)\right)|_{t^+_0}.$$ (15)

This fact will be of crucial importance, and will represent one main reason, for why our findings differ from those stemming from similar previous excursions into this subject.

In the following we will confine the quantum field to the interior of a straight cylinder centered at the origin $r = 0$, with the constant coordinate
radius \( r = R \). The top and bottom of the cylinder are assumed to be at the fixed coordinate positions \( z = 0 \) and \( z = L \), respectively. The cylinder thus defines a co-moving volume, since the proper volume changes in the transition from Minkowski spacetime and to the situation when a string is present. Note that the proper area of a cross section of the cylinder defined by \( dt = dz = 0 \) always equals \( \pi R^2 \). The change in the proper volume of the cylinder is thus solely induced by a “stretching” of the cylinder in the \( z \)-direction.

In the region \( t < t_0 \) we decompose the scalar field operator according to

\[
\Phi(\vec{x}, t) = \sum_j (a_j f_j(\vec{x}, t) + a_j^\dagger f_j^*(\vec{x}, t)),
\]

(16)

where \( j = (n, m, s); n, m = 0, \pm 1, \pm 2, \ldots, s = 1, 2, 3, \ldots, \) and the annihilation and the creation operators \( a_j \) and \( a_j^\dagger \) satisfy the usual canonical commutation relations. The mode functions \( f_j \), which constitute a complete set with respect to the canonical symplectic form, are given by

\[
f_{n,m,s}(\vec{x}, t) = N_1 e^{-i\omega s t} e^{ikz} e^{im\phi} J_m(\omega_s^2 - k^2)^{1/2} r,
\]

(17)

where the normalisation factor \( N_1 \) is given by

\[
N_1 = (2\omega_s V_1)^{-1/2}((\partial_r J_m)(\omega_s^2 - k^2)^{1/2} r)|_{r=R}^{-1},
\]

(18)

\[
V_1 \equiv \pi LR^2, \quad k = \frac{2\pi n}{L}.
\]

(19)

The \( \omega_s \)'s are determined from the equation \( J_m(\omega_s^2 - k^2)^{1/2} R) = 0 \). \( s \) does therefore denote a radial quantum number. The canonical vacuum state is defined as \( a_j|0\rangle_{\text{in}} = 0 \). When \( t > t_0 \) we similarly expand the quantum field as

\[
\Phi(\vec{x}, t) = \sum_j (b_j g_j(\vec{x}, t) + b_j^\dagger g_j^*(\vec{x}, t)).
\]

(20)

The mode-functions \( g_j \) are given by

\[
g_j(\vec{x}, t) = N_2 e^{-iW_s t} e^{ikz} e^{im\phi} J_{\nu_m}(\nu_m^2 - k^2)^{1/2} r)
\]

(21)

with \( \nu_m \equiv b_0|m| \). The \( W_s \)'s are determined from \( J_{\nu_m}(\nu_m^2 - k^2)^{1/2} R) = 0 \).

The canonical symplectic form is defined by

\[
(\psi_n, \psi_m) = i \int \Sigma d^3 x \sqrt{g_3} N^A \psi_n^\dagger \partial_A \psi_m^* ,
\]

(22)

7
where $\vec{N}$ is defined as a future pointing unit vector, $\vec{N}^2 = -1$, which is everywhere orthogonal to the spacelike hypersurface $\Sigma$. $d^3x \equiv drdzd\phi$ and $g_3$ is the determinant of the induced 3-geometry in $\Sigma$. We choose the normal vector field to be the canonical one, i.e. we choose

$$\vec{N}_{|t_0^-} = \partial_t, \quad \vec{N}_{|t_0^+} = b_0^{-1/2} \partial_t . \quad (23)$$

With this normalisation we find that

$$(g_j, g_j) = i \int_{\Sigma} d^3x \left( g_j \tilde{\partial}_t g_j^* \right)_{|t_0^+} . \quad (24)$$

It follows that $N_2$ has the same form as $N_1$ except that $\omega \to W$ and $|m| \to \nu_m$, of course.

On the spacelike hypersurface $\Sigma$ of instantaneous creation of the string we will have the following general relation between $f$-modes and $g$-modes

$$f_j(\vec{x}, t_0^-) = \sum_{j'} (\alpha_{j'j} g_{j'}(\vec{x}, t_0^+) + \beta_{j'j} g_{j'}^*(\vec{x}, t_0^+)) . \quad (25)$$

The total number of produced particles $|\beta|^2$ as they are defined in the $t > t_0$ region relative to the incoming vacuum $|0\rangle_{\text{in}}$ is formally given by

$$|\beta|^2 \equiv \sum_{j'} |\beta_{j'j}|^2 \equiv \sum_{j'j} \langle 0 | N_{j'j} | 0 \rangle_{\text{in}} , \quad (26)$$

where $N_{j'j} \equiv b_{j'}^\dagger b_j$. From eq.(25) we then easily deduce that

$$\beta_{j'j} = i \int_{\Sigma} d^3x (f_j |_{t_0^-} (\sqrt{g_3} \vec{N} g_{j'}^*)_{|t_0^+} - g_{j'}^* |_{t_0^-} (\sqrt{g_3} \vec{N} f_j)_{|t_0^+} ) . \quad (27)$$

$$= i \int_{\Sigma} d^3x (f_j |_{t_0^-} (\partial_t g_{j'}^*)_{|t_0^+} - g_{j'}^* |_{t_0^-} (\partial_t f_j)_{|t_0^-} ) . \quad (28)$$

Since we have

$$f_{n,0,s} |_{t_0^-} = g_{n,0,s} |_{t_0^+} , \quad \partial_t f_{n,0,s} |_{t_0^-} = \partial_t g_{n,0,s} |_{t_0^+} , \quad (29)$$

we can immediately conclude that

$$\beta_{n,0,s;n,0,s} = 0 . \quad (30)$$
However, even though the diagonal elements in the scattering matrix $\beta_{s',s}$ vanish, the off-diagonal elements may not. The explicit form for $\beta_{j',j}$ is

$$
\beta_{j',j} = \frac{b_0^{1/2} (2(s' - s) - |m| + |m'| b_0)}{(2s' + \frac{3}{2}) + (|m'| - |m| - 2s - \frac{3}{2}) b_0},
$$

(31)

where $\beta_{j',j}^{(10)}$ represents the resulting production estimate if we had used the geometry in eq.(10) in order to compute the particle production. We identify $\beta_{j',j}^{(10)}$ with the corresponding expression in [1]. In the derivation of this expression (which is straightforward and will therefore not be reproduced here) we assumed that $R$ is very large in order to utilise the asymptotic properties of the Bessel functions. We also put $n = n' = 0$, since the inclusion of these quantum numbers does not provide us with any additional insights. In the physically interesting regime we can effectively set $m = m' = 0$ [1], so that

$$
\frac{\beta_{s';s}}{\beta_{s',s}^{(10)}} = \frac{2(s' - s)\sqrt{1 - 4G\mu_0}}{(2s' + \frac{3}{2})(1 - 4G\mu_0 - (2s + \frac{3}{2}))},
$$

(32)

Clearly, $\beta_{s;s} = 0$, while $\beta_{s';s \neq s'} \neq 0$. However, this off-diagonal production is sub-leading. Indeed, from [1] one finds that to leading order in $4G\mu_0(<< 1)$

$$
\beta_{s';s}^{(10)} \sim 2G\mu_0 \delta_{s',s}.
$$

(33)

Hence, the potentially physically significant production of scalar particles from the quantum vacuum, due to the creation of a physically realistic cosmic string, is completely suppressed in the zero angular momentum sector.

4 Conclusion

In previous studies [1, 2, 4, 5] (e.g.) one made the replacement $a_0 \to a(t)$ in eq.(10) in order to approximate the description of cosmic string creation. Clearly, the resulting geometric structure is not simply related to the corresponding geometry in eq.(2). From this point of view it is perhaps not surprising that our results differ significantly from the results of these previous studies. The conformal form of the string geometry in eq.(11) were explored and partially utilised in [3, 4] in order to understand the properties of the usual form of the string geometry in eq.(10). In [4] this form of the
string geometry was also used in order to extract the qualitative properties of the amount of particles produced during string creation compared to the estimate one computes directly from eq.(10). When the conformal scalar field sector is considered along with the associated conformal vacuum structure \[10\], it was shown in \[9\] that the total number of particles produced in the appropriately generalised version of eq.(11) \(|\beta^{(11)}|^2\) is less than the corresponding amount produced in eq.(10) \(|\beta^{(10)}|^2\), in the instantaneous approximation. These quantities were found to be related by \[9\]

\[|\beta^{(11)}|^2 = (1 - 4G\mu_0)|\beta^{(10)}|^2. \tag{34}\]

Clearly, the conformal vacuum is a very special configuration, and is probably of greater theoretical interest than of physical significance. However, the findings in \[9\] are indeed along the lines implied by the findings in this paper, since \(|\beta^{(11)}|^2 < |\beta^{(10)}|^2\) to leading order in \(4G\mu_0\). Indeed, the \((1 - 4G\mu_0)\) factor in eq.(34) is also manifestly present in the squared version of eq.(32). However, the differences between eq.(34) and eq.(32) do also illustrate that the question of whether any physically significant particle production occur in string creation or not, is very sensitive to the nature of the coupling of the scalar field to gravity.

5 Acknowledgements

BJ thanks NORDITA for a travelling grant, and NORDITA, the Norwegian University of Science and Technology and the Niels Bohr Institute for hospitality during the time when this work was carried out.

References

[1] L. Parker, Phys.Rev.Lett.59 1369 (1987).
[2] I. Brevik and T. Toverud, Phys.Rev.D51 691 (1995).
[3] B. Jensen, Nucl.Phys.B453 413 (1995).
[4] B. Jensen, J.Math.Phys.38 1329 (1997).
[5] W.A. Hiscock, Phys.Rev.D31 3288 (1985).
[6] A. Buonanno and T. Damour, hep-th/9803025.

[7] V. Hussain, J. Pullin and E. Verdager, Phys.Lett.B232 299 (1989).

[8] G. Mendel and W.A. Hiscock, Phys.Rev.D40 282 (1989).

[9] B. Jensen, Phys.Lett.B391 53 (1997).

[10] N.D. Birrell and P.C.W. Davies, *Quantum Fields in Curved Space* (Cambridge UP, Cambridge, England, 1982).