Improvement of three-dimensional mathematical model for the simulation of impact of high-speed metallic plates

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Abstract. The work is devoted to the development of the authors software package Turbulence Problem Solver originally created for the study of three-dimensional problems of hydrodynamic instabilities. Mathematical model is based on the three-dimensional two-component Euler equations, the numerical algorithm is based on the grid-characteristics scheme of the second approximation order in space. The functionality of the mathematical model and the numerical algorithm of the package is extended to consider the problem of high-speed metallic plates impact. The detailed description of the proposed numerical approach is presented. The problem of high-speed metallic plates impact is solved using stiffened gas equation of state. The parameters of the equation of state are calibrated on the basis of computational results obtained with the use of wide-range equations of states for the metals.

1. Introduction
The study of the mechanisms and the laws of hydrodynamic instability development and turbulent flows in different fundamental and practical problems is among the most complicated modern issues of the continuum media mechanics. The best practices in natural and numerical experiments show that the modeling of the hydrodynamic instability development should be based on three-dimensional (3D) models the program realization of which demands huge computational costs.

The authors of the paper have developed for several years an expandable software for the numerical solution of 3D Euler equations for the ideal gas using different numerical schemes including high-order methods [1,2]. The software provides the solution of some problems of the hydrodynamic instability development theory. In [3] a spectral representation of kinetic energy for a vortex cascade of instability in a compressible inviscid shear flow was considered, and the Rayleigh-Taylor instability was studied. A comparative analysis was given to the spectral decompositions of kinetic energy for both problems. The classical Kolmogorov 5/3 power law was proved to hold for developed turbulent flows under consideration. In [4] the numerical simulation was used to study the Kolmogorov flow in a shear layer of a compressible inviscid medium. It was shown that the mechanism of the onset of turbulence has an essentially 3D nature. For the turbulent flows computed, the classical Kolmogorov 5/3 power law also held in the inertial range.
2. Statement of the problem
Consider the interaction of the lead plate (parallelepiped $5 \times 5 \times 2$ mm$^3$) with the density 11 300 kg/m$^3$ with the steel plate (parallelepiped $5 \times 5 \times 3$ mm$^3$) with the density 7900 kg/m$^3$ (figure 1). The lead plate is thrown to the direction of steel one with the velocity 500 m/s. We assume that at the initial stage of the impact during first 10 $\mu$s the metals behave as pseudo-fluids [5] so the gas dynamics approach is valid. The initial pressure is $10^5$ Pa everywhere.

So the computational area is the cube with the edge length 5 mm. We set the inflow conditions at the boundaries $x = 0$ and 5 mm and periodic conditions at the others. The inflow values at $x = 0$ corresponds to the initial conditions for the steel plate and at $x = 5$ mm to the lead one. As a result of metal plates interaction two shock waves (SW) are formed. The computation lasts up to the moment of SW arrival to the boundaries $x = 0$ or 5 mm. Computational grid is uniform with the cell size $h$ and contains $100 \times 100 \times 100 = 10^6$ cells.

3. Mathematical model and the computational algorithm
Mathematical model is based on 3D non-stationary two-component Euler equations [3, 4] supplemented by stiffened gas EOS [8] (or two-terms EOS [9]). In the Cartesian frame the...
governing system of equations is written as:
\[
U_t + F(U)_x + G(U)_y + H(U)_z = 0, 
\]
\[
U = \begin{bmatrix} 
\rho u \\
\rho \nu^2 + p \\
\rho u w \\
\rho_2 
\end{bmatrix}, 
F = \begin{bmatrix} 
\rho_1 u \\
\rho \nu^2 + p \\
(e + p) u \\
\rho_2 u 
\end{bmatrix}, 
G = \begin{bmatrix} 
\rho v \\
\rho \nu^2 + p \\
(e + p) v \\
\rho_2 v 
\end{bmatrix}, 
H = \begin{bmatrix} 
\rho_1 w \\
\rho \nu w \\
\rho \nu^2 + p \\
\rho_2 w 
\end{bmatrix}, 
\]
\[
\rho = \rho_1 + \rho_2, 
\varepsilon = \rho (u^2 + v^2 + w^2)/2 + \rho \varepsilon(p, \rho), 
\varepsilon(p, \rho) = \frac{p + \gamma p_0}{\rho(\gamma - 1)}.
\]
Here \( U \) is the conservative variables vector, \( F, G \) and \( H \) are the differential flux vectors. The standard notations are used, namely \( t \) is the time, \( (x, y, z) \) are the Cartesian coordinates, \( \rho \) is the density (index 1 stands for one metal and index 2 for another), \( (e, p) \) are the components of the gas velocity vector, \( p \) is the pressure, \( e \) and \( \varepsilon \) are the total and internal specific energy. The EOS is defined by the constant parameters \( \gamma \) and \( p_0 \).

Numerical algorithm is based on the spatial directions splitting technique:
\[
U_{ijk}^{n+1} = L_x L_y L_z U_{ijk}^n.
\]
Here \( i, j, k \) stand for the spatial indexes, \( n \)—for the time index, \( L_x, L_y \) and \( L_z \) are the one-dimensional hyperbolic operators in the splitting method. Let us describe the action of \( L_x \) operator. The one-dimensional subsystem (1) is written in the characteristics form:
\[
U_t + A U_x = 0, 
\]
\[
A = \left( \frac{\partial F}{\partial U} \right)^6_{i,j=1}, 
A = \left[ 
\begin{array}{cccccc}
-\rho \omega_1 & 0 & 0 & 0 & -\rho u \omega_1 \\
\rho w & 0 & 0 & \rho w & 0 & \rho w \\
0 & \rho w & 0 & 0 & \rho w & 0 \\
0 & 0 & \rho w & 0 & 0 & \rho w \\
\end{array}
\right].
\]
Here the notations from [10] are used:
\[
\omega_1 = \rho_1/\rho, 
\varepsilon = (\partial p/\partial \varepsilon)_p, 
\rho_p = (\partial p/\partial \rho)_\varepsilon, 
q^2 = u^2 + v^2 + w^2, 
b = p_c/\rho, 
\theta = q^2 - \varepsilon/p + \rho p_p/\rho_p, 
h = (e + p)/\rho.
\]
The characteristic analysis for the matrix \( A \) gives:
\[
A = \Omega_R \Lambda \Omega_L, 
\]
\[
\Omega_L = \left[ 
\begin{array}{cccc}
\frac{\theta + wc}{2} & -\frac{c - u}{2} & -\frac{v}{2} & -\frac{w}{2} \\
\frac{c^2 - \theta \omega_1}{2} & \omega_1 & \frac{\theta}{2} & -\frac{wc}{2} \\
-\frac{\theta}{2} & \omega_1 & \frac{\theta}{2} & -\frac{wc}{2} \\
\frac{\theta}{2} & \frac{c^2 - \theta}{2} & -\frac{c - u}{2} & -\frac{v}{2} \\
\end{array}
\right], 
\Omega_R = \left[ 
\begin{array}{cccc}
\omega_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\right].
\]
\[
c = \sqrt{b(h + \theta - q^2)} = \sqrt{p_b + p p_0/\rho^2}.
\]
Table 1. Reference values courtesy of professor I V Lomonosov (IPCP RAS).

|        | SW | $p$ (GPa) | $\rho$ (kg/m$^3$) | $u$ (m/s) | $D$ (km/s) |
|--------|----|-----------|-------------------|-----------|------------|
| In steel | 7.99 | 8 246 | 211 | 4.72 |
| In lead  | 7.99 | 12 830 | 211 | 1.97 |

Table 2. Parameters of two-terms EOS (4).

| Metal | $\rho_0$ (g/cm$^3$) | $c_0$ (km/s) | $\gamma$ | $p_0 = \rho_0 c_0^2 / \gamma$ (GPa) |
|-------|----------------------|--------------|----------|----------------------------------|
| Copper | 8.93 [9, 15] | 3.97 [9, 15] | 3.0–3.5 [16] | 46.9 (for $\gamma = 3.0$) |
| Iron  | 7.87 [9, 15] | 5.0 [9, 15] | 3.0–3.5 [16] | 65.6 (for $\gamma = 3.0$) |

The system (3) is numerically integrated as follows:

$$
\frac{U_{n+1}^i - U_{n}^i}{\Delta t_n} + \frac{\mathbf{F}_{n+1/2}^i - \mathbf{F}_{n-1/2}^i}{h} = 0,
$$

$$
\mathbf{F}_{n+1/2}^i = \Omega_R[\alpha_k\lambda_k]\Omega_L U_{n-1}^i + \Omega_R[\beta_k\lambda_k]\Omega_L U_{n+1}^i + \Omega_R[\gamma_k\lambda_k]\Omega_L U_{n}^i + \Omega_R[\delta_k\lambda_k]\Omega_L U_{n+2}^i,
$$

where the coefficients $\alpha_k$, $\beta_k$, $\delta_k$ and $\gamma_k$ are chosen to provide second approximation order of the scheme in space and monotonicity property [11]. For all matrices coefficients computations the Roe averaging procedure is used taking into account the stiffened gas EOS (2) [10, 12].

Note that the described numerical approach is valid for the general EOS if the partial derivatives $(\partial p / \partial \varepsilon)_\rho$ and $(\partial p / \partial \rho)_\varepsilon$ are known.

4. Results of the numerical experiments

To reproduce quantitatively reasonable characteristics of metal plates impact the parameters of stiffened gas EOS should be fitted using either experimental data or the results of simulations taking into account the wide-range EOS for metals. We use the results of simulations courtesy of professor I V Lomonosov (IPCP RAS) obtained using wide-range EOS [13] and the numerical approach [14]. The results of I V Lomonosov will be referred as reference values further (table 1).

Apparently the stiffened gas EOS is the simplest one that provides the hydrodynamic effects during plates collision and the quantitative agreement within tens of percents. Stiffened gas EOS (2) is the simplification of more general two-terms EOS [9]:

$$
\varepsilon(p, \rho) = \frac{p + \gamma \rho_0}{\rho(\gamma - 1)} - \frac{c_0^2}{\gamma - 1}.
$$

The known parameters of two-terms EOS for some metals are summarized in table 2. The density of the cold substance and the speed of sound in the cold substance are denoted as $\rho_0$ and $c_0$ correspondingly.

Figures 2–4 illustrate the predicted fields of density, velocity and pressure at the time moment 0.5 $\mu$s after the impact. Density distribution in figure 2 shows both shock waves and contact surface which moves in the direction of the steel plate. The figures correspond to the parameters of stiffened gas EOS $\gamma = 3.0$ and $p_0 = 25$ GPa. The parameters were found as a result of parametric calculations in which we compared the calculated characteristics of both shock waves with the reference values. Table 3 demonstrates the finally achieved agreement with the relative error.
Table 3. Predicted characteristics of shock waves.

| SW   | $p$ (GPa) | $\rho$ (kg/m$^3$) | $u$ (m/s) | $D$ (km/s) |
|------|-----------|-------------------|-----------|------------|
| In steel | 7.24 (–7%) | 8 595 (+4%) | 272 (+29%) | 3.40 (–28%) |
| In lead  | 7.24 (–7%) | 12 294 (–5%) | 272 (+29%) | 2.38 (+21%) |

Figure 3. Predicted distribution of velocity (m/s) after 0.5 $\mu$s.

Figure 4. Predicted distribution of pressure (GPa) after 0.5 $\mu$s.

5. Conclusion
The three-dimensional mathematical model based on the multicomponent Euler equations supplemented with the stiffened gas equation of state for the numerical investigation of the metal plates impact is described. The details of the computational algorithm based on the grid-characteristics approach including all the matrices are presented. The parameters of the stiffened gas equation of state are calibrated on the basis of the computations using wide-range equation of state for the metals. For the characteristics of the shock waves which are formed after the impact of two metal plates the maximum error is in the range of 30%. The main reason for relatively big error is the usage of Euler equations with single equation of state for the whole mixture. The development of the multi-fluid model based on the approaches from [8] is the subject of the following studies. Nevertheless the proposed numerical technique is suitable for the numerical simulation of the three-dimensional instability development on the contact surface of the high-speed colliding metallic plates.

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