CONTIGUOUS RELATIONS OF $3\phi_2$-SERIES

A CHUANAN WEI AND B DIANXUAN GONG

A Department of Information Technology
Hainan Medical College, Haikou 571199, China
B College of Sciences
Hebei United University, Tangshan 063009, China

ABSTRACT. According to Abel's lemma and the method of linear combinations, we establish numerous contiguous relations of $3\phi_2$-series, which can be regarded as $q$-analogues of the contiguous relations of $3\phi_2$-series due to Krattenthaler and Rivoal [12] or Chu and Wang [3].

1. INTRODUCTION

For two complex numbers $x$ and $q$ with $|q| < 1$, define the $q$-shifted factorial by

$$(x; q)_n = \begin{cases} 
\prod_{i=0}^{n-1} (1 - xq^i), & n > 0; \\
1, & n = 0; \\
\frac{1}{1 - q^n}, & n < 0.
\end{cases}$$

The fractional form of it reads as

$$[\alpha, \beta, \cdots, \gamma \mid q]_n = \frac{(\alpha; q)_n(\beta; q)_n \cdots (\gamma; q)_n}{(A; q)_n(B; q)_n \cdots (C; q)_n}.$$

Following Gasper and Rahman [5], the basic hypergeometric series can be defined by

$$r_\phi \left[ \begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} \mid q; z \right] = \sum_{k=0}^{\infty} \frac{(a_1; q)_k(a_2; q)_k \cdots (a_r; q)_k}{(b_1; q)_k(b_2; q)_k \cdots (b_s; q)_k} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} \frac{1 + s - r}{z^k},$$

where $\{a_i\}$ and $\{b_j\}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Throughout the paper, we shall also use the shifted basic hypergeometric series

$$r_\phi^s \left[ \begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} \mid q; z \right] = \sum_{k=0}^{\infty} \frac{(a_1; q)_k(a_2; q)_k \cdots (a_r; q)_k}{(b_1; q)_k(b_2; q)_k \cdots (b_s; q)_k} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_k} \frac{1 + s - r}{(1 - q^k)z^k},$$

whose summation index begins essentially with $k = 1$, instead of $k = 0$.

For a complex sequence $\{\tau_k\}$, define respectively the forward difference operator $\Delta$ and the backward difference operator $\nabla$ by

$$\Delta \tau_k = \tau_k - \tau_{k+1} \quad \text{and} \quad \nabla \tau_k = \tau_k - \tau_{k-1}.$$ 

Then Abel's lemma (cf. [3]) can be stated as follows.

Lemma 1. For two complex sequences $\{U_k\}$ and $\{V_k\}$, there holds the relation:

$$\sum_{k=0}^{\infty} U_k \Delta V_k = \sum_{k=0}^{\infty} V_k \nabla U_k$$

provided that one of the series on both sides converges, $U_{-1}V_0 = 0$ and $U_kV_{k+1} \to 0$ as $k \to \infty$.

2010 Mathematics Subject Classification: Primary 33D15 and Secondary 11J72.

Key words and phrases. Abel's lemma; Two-term contiguous relation of $3\phi_2$-series; Three-term contiguous relation of $3\phi_2$-series.

Email addresses: weichuanan@yahoo.com.cn (C. Wei), gongdianxuan@yahoo.com.cn (D. Gong).
There are many contiguous relations in the literature. Several interesting ones can be seen in
the papers [2-4, 6-12, 14-17] and [13-20]. Implied by the work just mentioned, we shall
give numerous contiguous relations of $3_2\phi_2$-series in terms of Lemma 1 and the method of linear
combinations.

The structure of this paper is arranged as follows. In section 2, we shall use the Lemma 1 to found
combinations. Performing the substitutions $a \rightarrow q^k$, $b \rightarrow q^k$, $c \rightarrow q^k$, $d \rightarrow q^k$, $e \rightarrow q^k$ for Theorem 2 and then
letting $q \rightarrow 1$, we recover the following relation.

2. Four three-term contiguous relations of $3_2\phi_2$-series

In this section, we show four patterns A, B, C and D satisfied by three $3_2\phi_2$-series through Abel’s
lemma. Throughout this section, we assume that the parameters of all the $3_2\phi_2$-series are subject to
the condition $|bd/qace| < 1$ in order that Lemma 1 can be applied smoothly.

2.1. Pattern A.

Define two sequences by

$$U_k = \left[ \frac{qa}{q}, \frac{d/a}{d} \big| q \right]^k_k \text{ and } V_k = \left[ \frac{c}{b}, \frac{e}{d/qa} \big| q \right]^{(b)}_k,$$

Then it is not difficult to check the limiting relation

$$U_{-1}V_0 = \lim_{n \rightarrow \infty} U_nV_{n+1} = 0$$

and the finite differences

$$\nabla U_k = \left[ \frac{a}{q}, \frac{d/qa}{d} \big| q \right]^{(k)}_k, \quad \nabla V_k = \left[ \frac{c}{qb}, \frac{e}{d/qa} \big| q \right]^{(k)}_k \left( \frac{(1-b)(1-d/qa)-(1-c)(1-e)b}{(1-b)(1-d/qa)} + \frac{(1-bd/qace)}{(1-b)(1-d/qa)} 1-q^k \right).$$

In accordance with Lemma 1, we can manipulate the following $3_2\phi_2$-series:

$$3_2\phi_2 \left[ \frac{a, c, e}{b, d} \big| q; \frac{bd}{qace} \right] = \sum_{k \geq 0} V_k \nabla U_k = \sum_{k \geq 0} U_k \nabla V_k = \frac{(1-b)(1-d/qa)-(1-c)(1-e)b}{(1-b)(1-d/qa)} \sum_{k \geq 0} \left[ \frac{qa}{q}, \frac{c}{qb}, \frac{e}{d/qa} \big| q \right]^{(k)}_k \left( \frac{bd}{qace} \right)^k$$

$$+ \frac{(1-bd/qace)}{(1-b)(1-d/qa)} \sum_{k \geq 0} (1-q^k) \left[ \frac{qa}{q}, \frac{c}{qb}, \frac{e}{d/qa} \big| q \right]^{(k)}_k \left( \frac{bd}{qace} \right)^k.$$

Shifting the summation index $k \rightarrow k+1$ for the last sum, we obtain the following relation.

**Theorem 2 (Pattern A).** For five complex numbers $\{a, b, c, d, e\}$ subject to the condition $|bd/qace| < 1$, there holds the three-term contiguous relation of $3_2\phi_2$-series:

$$3_2\phi_2 \left[ \frac{a, c, e}{b, d} \big| q; \frac{bd}{qace} \right] = Aq_33_2\phi_2 \left[ \frac{qa, c, e}{qb, d} \big| q; \frac{bd}{qace} \right] + \Delta_33_2\phi_2 \left[ \frac{q^2a, qc^e}{q^2b, q^2d} \big| q; \frac{bd}{qace} \right], \quad (1a)$$

$$3_2\phi_2 \left[ \frac{a, c, e}{b, d} \big| q; \frac{bd}{qace} \right] = Aq_33_2\phi_2 \left[ \frac{qa, c, e}{qb, d} \big| q; \frac{bd}{qace} \right] + Aq_3\phi_2^* \left[ \frac{qa, c, e}{qb, d} \big| q; \frac{bd}{qace} \right], \quad (1b)$$

where the coefficients $A_q$, $\Delta_q$ and $\Delta_q$ are defined by

$$A_q := A_q(a, c, e; b, d) = \frac{(1-b)(1-d/qa)-(1-c)(1-e)b}{(1-b)(1-d/qa)},$$

$$\Delta_q := \Delta_q(a, c, e; b, d) = \frac{(1-bd/qace)(1-qac)(1-c)(1-e)b}{(1-b)(1-qbd)(1-d/qa)qace}.$$
Corollary 3 (Theorem 1). For five complex numbers \( \{a, b, c, d, e\} \) subject to the condition \( \Re(b + d - a - c - e) > 1 \), there holds the three-term contiguous relation of \( 3F_2 \)-series:

\[
3F_2\left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k.
\]

where the coefficients \( A, \beta, \gamma \) are given by

\[
A := A(a, c, e; b, d) = \frac{(1 + a - d)b + ce}{(1 + a - d)b},
\]

\[
\beta := \beta(a, c, e; b, d) = \frac{(1 + a + e - b - d)(1 + a)ce}{(1 + a - d)(1 + b)bd},
\]

\[
\gamma := \gamma(a, c, e; b, d) = \frac{1 + a + c + e - b - d}{(1 + a - d)b}.
\]

In Corollary 3 the hypergeometric series and shifted hypergeometric series have been offered by

\[
\sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k.
\]

where the shifted factorial is

\[
(x)_n = \begin{cases} \prod_{i=1}^{n} (x + i), & \text{for } n > 0; \\ 1, & \text{for } n = 0; \\ \frac{1}{\prod_{i=1}^{-n} (x+i)}, & \text{for } n < 0. 
\end{cases}
\]

2.2. Pattern B.

For two sequences defined by

\[
U_k = \left[ \begin{array}{c} c, q \\ d/q, qce/d \end{array} \right]_k \quad \text{and} \quad V_k = \left[ \begin{array}{c} a, q^2ce/b \\ q, b \end{array} \right]_k \left( \frac{bd}{qace} \right)^k,
\]

we can easily verify the limiting relation

\[
U_0V_{-1} = \lim_{n \to \infty} U_{n+1}V_n = 0
\]

and the finite differences

\[
\Delta U_k = \left[ \begin{array}{c} c, q \\ d/q, qce/d \end{array} \right]_k \left( \frac{d}{a} \right)^k \left( \frac{1 - qd}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right),
\]

\[
\nabla V_k = \left[ \begin{array}{c} a/q, qce/b \\ q, b \end{array} \right]_k \left( \frac{bd}{qace} \right)^k \left( \frac{1 - qd}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right).
\]

By means of Lemma 3 we can reformulate the following \( 3\phi_2 \)-series:

\[
\sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k \left( \frac{1 - qd}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right) = \sum_{k=0}^{\infty} V_k \Delta U_k = \sum_{k=0}^{\infty} U_k \nabla V_k
\]

\[
= \sum_{k=0}^{\infty} \left[ \begin{array}{c} a/q, q \\ q, b \end{array} \right]_k \left( \frac{bd}{qace} \right)^k + \left( \frac{1 - qd}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right),
\]

Shifting the summation index \( k \to k + 1 \) for the last sum, we get the following relation.

Theorem 4 (Pattern B). For five complex numbers \( \{a, b, c, d, e\} \) subject to the condition \( |bd/qace| < 1 \), there holds the three-term contiguous relation of \( 3\phi_2 \)-series:

\[
3\phi_2\left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k \left( \frac{1 - qd}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right),
\]

\[
3\phi_2\left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{k!} \left( \frac{d}{a} \right)^k \left( \frac{1 - qd}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right)\left( \frac{1}{1 - qe/d} \right).
\]
where the coefficients $\mathcal{B}_q$, $\mathbb{B}_q$ and $\mathcal{B}_q$ are defined by

$$
\mathcal{B}_q := \mathcal{B}_q(a, c, e; b, d) = \frac{(1 - qce/d)(1 - q/d)}{(1 - q/d)(1 - qe/d)},
$$

$$
\mathcal{B}_q := \mathcal{B}_q(a, c, e; b, d) = \frac{(1 - c)(1 - e)(1 - bd/qace)q}{(1 - b)(1 - qe/d)(1 - qe/d)d},
$$

$$
\mathbb{B}_q := \mathbb{B}_q(a, c, e; b, d) = \frac{(1 - qace/bd)(1 - q/d)}{(1 - a/q)(1 - qe/d)(1 - qe/d)}.
$$

Employing the substitutions $a \to q^a$, $b \to q^b$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Theorem 4 and then letting $q \to 1$, we recover the following relation.

**Corollary 5 (Theorem 2).** For five complex numbers $\{a, b, c, d, e\}$ subject to the condition $\text{Re}(b + d - a - c - e) > 1$, there holds the three-term contiguous relation of $_3F_2$-series:

$$
\begin{align*}
&_3F_2 \left[ \begin{array}{c}
    a, c, e \\
    b, d
\end{array} \right] \left[ \begin{array}{c}
    1
\end{array} \right] = \mathcal{B} \cdot _3F_2 \left[ \begin{array}{c}
    a - 1, c, e \\
    b, d - 1
\end{array} \right] \left[ \begin{array}{c}
    1
\end{array} \right] + \mathbb{B} \cdot _3F_2 \left[ \begin{array}{c}
    a + 1, c, e \\
    b + 1, d
\end{array} \right] \left[ \begin{array}{c}
    1
\end{array} \right],
\end{align*}
$$

$$
\begin{align*}
&_3F_2 \left[ \begin{array}{c}
    a, c, e \\
    b, d
\end{array} \right] \left[ \begin{array}{c}
    1
\end{array} \right] = \mathcal{B} \cdot _3F_2 \left[ \begin{array}{c}
    a - 1, c, e \\
    b, d - 1
\end{array} \right] \left[ \begin{array}{c}
    1
\end{array} \right] + \mathbb{B} \cdot _3F_2 \left[ \begin{array}{c}
    a + 1, c, e \\
    b + 1, d
\end{array} \right] \left[ \begin{array}{c}
    1
\end{array} \right],
\end{align*}
$$

where the coefficients $\mathcal{B}$, $\mathbb{B}$ and $\mathcal{B}$ are given by

$$
\begin{align*}
\mathcal{B} := \mathcal{B}(a, c, e; b, d) &= \frac{(1 + c + e - d)(1 - d)}{(1 + c - d)(1 + e - d)}, \\
\mathbb{B} := \mathbb{B}(a, c, e; b, d) &= \frac{(1 + a + c + e - b - d)ce}{(1 + c - d)(d - e + 1)b}, \\
\mathcal{B} := \mathcal{B}(a, c, e; b, d) &= \frac{(1 + a + c + e - b - d)(1 - d)}{(1 - a)(1 + c - d)(d - e - 1)}.
\end{align*}
$$

2.3. Pattern C.

Define two sequences by

$$
U_k = \left[ \begin{array}{c}
    q_c, q_e \\
    q_b, q_d
\end{array} \right] k \quad \text{and} \quad V_k = \left[ \begin{array}{c}
    a, qce \\
    b, d
\end{array} \right] k \left( \frac{bd}{qace} \right)^k.
$$

Then it is not hard to check the limiting relation

$$
U_{-1}V_0 = \lim_{n \to \infty} U_nV_{n+1} = 0
$$

and the finite differences

$$
\nabla U_k = \left[ \begin{array}{c}
    c, e \\
    q_c, q_e
\end{array} \right] k q^k,
$$

$$
\tilde{\Delta} V_k = \left[ \begin{array}{c}
    a, qce \\
    q_b, q_d
\end{array} \right] k \left( \frac{bd}{qace} \right)^k \left\{ \frac{(1 - b)(1 - d) - (1 - a)(1 - qe)}{1 - b(1 - d)} \frac{1 - qe}{1 - b(1 - d)} + \frac{(1 - bd)(1 - qe)}{1 - b(1 - d)} \frac{1 - qk}{1 - b(1 - d)} \right\}.
$$

According to Lemma 1, we can recombine the following $_3\phi_2$-series:

$$
\begin{align*}
&_3\phi_2 \left[ \begin{array}{c}
    a, c, e \\
    b, d
\end{array} ; q \left( \frac{bd}{ace} \right) \right]
&= \sum_{k \geq 0} V_k \nabla U_k = \sum_{k \geq 0} U_k \tilde{\Delta} V_k \\
&= \frac{(1 - b)(1 - d) - (1 - a)(1 - qe)}{(1 - b)(1 - d)} \sum_{k \geq 0} \left[ a, qc, qe \\
\right] \left[ q_b, q_d \right] k \left( \frac{bd}{qace} \right)^k \\
&\quad + \frac{(1 - bd)(1 - qace)}{(1 - b)(1 - d)} \sum_{k \geq 0} (1 - q^k) \left[ a, qc, qe \\
\right] \left[ q_b, q_d \right] k \left( \frac{bd}{qace} \right)^k.
\end{align*}
$$

Shifting the summation index $k \to k + 1$ for the last sum, we derive the following relation.

**Theorem 6 (Pattern C).** For five complex numbers $\{a, b, c, d, e\}$ subject to the condition $|bd/qace| < 1$, there holds the three-term contiguous relation of $_3\phi_2$-series:

$$
\begin{align*}
&_3\phi_2 \left[ \begin{array}{c}
    a, c, e \\
    b, d
\end{array} ; q \left( \frac{bd}{ace} \right) \right]
&= C_{3\phi_2} \left[ a, qc, qe \\
\right] \left[ q_b, q_d \right] \left( \frac{bd}{qace} \right) + C_{3\phi_2} \left[ qa, q^2c, q^2e \\
\right] \left[ q^2b, q^2d \right] \left( \frac{bd}{qace} \right),
\end{align*}
$$

$$
\begin{align*}
&_3\phi_2 \left[ \begin{array}{c}
    a, c, e \\
    b, d
\end{array} ; q \left( \frac{bd}{ace} \right) \right]
&= C_{3\phi_2} \left[ a, qc, qe \\
\right] \left[ q_b, q_d \right] \left( \frac{bd}{qace} \right) + C_{3\phi_2}^* \left[ qa, q^2c, q^2e \\
\right] \left[ q_b, q^2d \right] \left( \frac{bd}{qace} \right).
\end{align*}
$$
where the coefficients $C_q$, $Q_q$ and $E_q$ are defined by

$$C_q := C_q(a, c, e; b, d) = \frac{(1 - b)(1 - d) - (1 - a)(1 - qce) \frac{bd}{qace}}{(1 - b)(1 - d)},$$

$$Q_q := Q_q(a, c, e; b, d) = \frac{(1 - bd/qace)(1 - a)(1 - qc)(1 - qe) bd}{(1 - b)(1 - d)(1 - qb)(1 - qd) qace},$$

$$E_q := E_q(a, c, e; b, d) = \frac{(1 - bd/qace)}{(1 - b)(1 - d)}.$$

Performing the substitutions $a \rightarrow q^a$, $b \rightarrow q^b$, $c \rightarrow q^c$, $d \rightarrow q^d$, $e \rightarrow q^e$ for Theorem 8 and then letting $q \rightarrow 1$, we recover the following relation.

Corollary 7 (Theorem 3). For five complex numbers $\{a, b, c, d, e\}$ subject to the condition $\Re(b + d - a - c - e) > 1$, there holds the three-term contiguous relation of $\phi_2$-series:

$$\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_1 = C \phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_1 + \phi_2 \left[ \begin{array}{c} a, d, e + 1 \\ b, d + 1 \end{array} \right]_1,$$

$$\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_1 = \phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_1 + \phi_2 \left[ \begin{array}{c} a, d, e + 1 \\ b, d + 1 \end{array} \right]_1,$$

where the coefficients $C$, $Q$ and $E$ are given by

$$C := C(a, c, e; b, d) = \frac{bd - a(1 + c + e)}{bd},$$

$$Q := Q(a, c, e; b, d) = \frac{(b + d - a - c - e - 1)(1 + c)(1 + e)a}{(1 + b)(1 + d)bd},$$

$$E := E(a, c, e; b, d) = \frac{b + d - a - c - e - 1}{bd}.$$

2.4. Pattern D.

For two sequences defined by

$$U_k = \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_k q^k$$

and

$$V_k = \left[ \begin{array}{c} c, e \\ q, d \end{array} \right]_k q^k,$$

we can verify without difficulty the limiting relation

$$U_0 V_{-1} = \lim_{n \rightarrow \infty} U_{n-1} V_n = 0$$

and the finite differences

$$\Delta U_k = \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_k q^k \left( \frac{1}{1 - qa/b} \right) \left( \frac{1}{1 - qd/b} \right),$$

$$\nabla V_k = \left[ \begin{array}{c} c/q, e/q \\ q, d/q \end{array} \right]_k q^k \left( \frac{bd}{qace} \right)^k \left( \frac{1}{1 - qace/bd} \right) \left( \frac{1 - q/b}{1 - q/e} \right) \left( \frac{1 - qk}{q/k} \right).$$

In terms of Lemma 3 we can re-compose the following $\phi_2$-series:

$$\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] q^{k+1} \frac{bd}{qace} = \sum_{k \geq 0} V_k \Delta U_k = \sum_{k \geq 0} U_k \nabla V_k$$

$$= \sum_{k \geq 0} \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_k q^k \frac{bd}{qace}^k + \frac{(1 - qace/bd)}{(1 - q/c)(1 - q/e)} \sum_{k \geq 0} (1 - qk)^1 \left[ \begin{array}{c} a, c/q, e/q \\ q, b/q, d/q \end{array} \right]_k q^k \frac{bd}{qace}^k.$$

Shifting the summation index $k \rightarrow k + 1$ for the last sum, we deduce the following relation.

Theorem 8 (Pattern D). For five complex numbers $\{a, b, c, d, e\}$ subject to the condition $|bd/qace| < 1$, there holds the three-term contiguous relation of $\phi_2$-series:

$$\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] q \frac{bd}{qace} = \phi_2 \left[ \begin{array}{c} a, c/q, e/q \\ b/q, d/q \end{array} \right] q \frac{bd}{qace} + \phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] q \frac{bd}{qace},$$

$$\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] q \frac{bd}{qace} = \phi_2 \left[ \begin{array}{c} a, c/q, e/q \\ b/q, d/q \end{array} \right] q \frac{bd}{qace} + \phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] q \frac{bd}{qace}.$$
where the coefficients $\mathcal{D}_q$, $\mathbb{D}_q$ and $\mathcal{D}_q$ are defined by

$$
\mathcal{D}_q := \mathcal{D}_q(a, c, e; b, d) = \frac{(1 - q/b)(1 - q/d)a}{(1 - qa/b)(1 - qa/d)},
$$

$$
\mathbb{D}_q := \mathbb{D}_q(a, c, e; b, d) = \frac{(1 - a)(1 - qace/bd)q}{(1 - qa/b)(1 - qa/d)ce},
$$

$$
\mathcal{D}_q := \mathcal{D}_q(a, c, e; b, d) = \frac{(1 - qace/bd)(1 - q/b)(1 - q/d)a}{(1 - qa/b)(1 - qa/d)(1 - c/q)(1 - e/q)}
$$

Employing the substitutions $a \to q^a$, $b \to q^b$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Theorem 8 and then letting $q \to 1$, we recover the following relation.

**Corollary 9 ([4] Theorem 4).** For five complex numbers $\{a, b, c, d, e\}$ subject to the condition $Re(b + d - a - c - e) > 1$, there holds the three-term contiguous relation of $3\,F_2$-series:

$$
3\,F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] = \mathcal{D}_q \cdot \mathcal{D}_q \left[ \begin{array}{c} a, c - 1, e - 1 \\ b - 1, d - 1 \end{array} \right] + \mathbb{D}_q \cdot \mathcal{D}_q \left[ \begin{array}{c} a, c - 1, e - 1 \\ b - 1, d - 1 \end{array} \right],
$$

where the coefficients $\mathcal{D}$, $\mathbb{D}$ and $\mathcal{D}$ are given by

$$
\mathcal{D} := \mathcal{D}(a, c, e; b, d) = \frac{(1 - b)(1 - d)}{(1 + a - b)(1 + a - d)},
$$

$$
\mathbb{D} := \mathbb{D}(a, c, e; b, d) = \frac{a(1 + a + c + e - b - d)}{(1 + a - b)(1 + a - d)},
$$

$$
\mathcal{D} := \mathcal{D}(a, c, e; b, d) = \frac{1 + a + c + e - b - d - 1}{(1 + a - b)(1 + a - d)(1 - c)(1 - e)}
$$

**Remark:** The convergent conditions for the contiguous relations that will emerge in the next two sections are easy to be confirmed. For simplifying the expressions, we shall not lay out them one by one.

3. Nine two-term contiguous relations of $3\phi_2$-series

In terms of one or two patterns of $A$, $B$, $C$ and $D$, we offer nine two-term contiguous relations of $3\phi_2$-series in this section. Each subsection will be labeled by the corresponding patterns.

3.1. $A\&A$

Let Eq. [14] stand for Eq. [15] under the parameter replacements

$$
a \to q^a, \quad c \to q^c, \quad b \to q^b, \quad d \to q^d.
$$

Then consider the linear combination of two equations

$$
\text{Eq. [14]} - \text{Eq. [15]} \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] = \frac{(1 - q/b)(1 - d/q)}{(1 - b)(1 - d/qa)} \quad \text{with} \quad d = \frac{qabe(q - c)}{qab + ce - bc - ace}.
$$

With this specific value $d$, we can check that the right member of the last equation vanishes. After some simplification, we attain the following relation.

**Theorem 10** (Two-term contiguous relation of $3\phi_2$-series).

$$
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, qab + ce - bc - ace \end{array} \right] = \frac{q^b(q - c)}{c(qab + ce - bc - ace)}
$$

$$
3\phi_2 \left[ \begin{array}{c} qa, qb, e \\ a, c - 1 + e - 1 \end{array} \right] = \frac{q^b(q - c)}{c(qab + ce - bc - ace)}
$$

$$
\times \frac{(1 - q/b)(qab + ce - bc - ace)}{(1 - b)(qab + 6ce + ce - bc - qbe - ace)}
$$

Performing the substitutions $a \to q^a$, $b \to q^b$, $c \to q^c$, $e \to q^e$ for Theorem 10 and then letting $q \to 1$, we recover the following relation.
**Corollary 11** ([4, Theorem 5]).

\[
\begin{align*}
3F2 \left[ \frac{a, c, e}{b, 1 + e - \frac{a}{1 + e}} \right] & = 3F2 \left[ \frac{a + 1, c - 1, e}{b + 1, e - \frac{a}{1 + e}} \right] \\
& \times \frac{(1 + b - e)(ab + ce - ae - e)}{b(a + ab + ce - ae - e)}.
\end{align*}
\]

Specifying the parameters, in Theorem 10, by

\[a \to \alpha, \quad c \to \beta, \quad e \to \gamma, \quad b \to qa,\]

we achieve the following relation with one free parameter less.

**Proposition 12** (Two-term contiguous relation of \(_3\phi_2\)-series).

\[
\begin{align*}
3\phi_2 & \left[ \frac{\alpha, \beta, \gamma}{qa, q^2a, \beta + \gamma - qa\beta - qa\gamma} \right] \\
& = 3\phi_2 \left[ \frac{q^3\alpha q^3(1 - \beta)}{\beta(q^2\alpha^2 + \beta\gamma - qa\beta - qa\gamma)} \right];
\end{align*}
\]

Employing the substitutions \(\alpha \to q^a, \beta \to q^b, \gamma \to q^c\) for Proposition 12 and then letting \(q \to 1\), we recover the following relation.

**Corollary 13** ([12, Theorem 2], see also [4, Proposition 6]).

\[
\begin{align*}
3F2 & \left[ \frac{a, \beta, \gamma + \alpha(\alpha - 1)}{a + 2, \gamma + \alpha(\alpha - 1) + 1} \right] \\
& = 3F2 \left[ \frac{a + 1, \beta - 1, \gamma}{a + 2, \gamma + \alpha(\alpha - 1) + 1} \right] \\
& \times \frac{(2 + \alpha - \beta)(\alpha + \alpha^2 + \beta\gamma - \alpha\gamma - \gamma)}{(1 + \alpha)(2a + \alpha^2 + \beta\gamma - \alpha\beta - \alpha\gamma - \gamma)}.
\end{align*}
\]

### 3.2. A\&B.

Let \(\text{Eq} \ (2a)\) stand for \(\text{Eq} \ (2a)\) under the parameter replacements

\[a \to q^a, \quad b \to d, \quad d \to q^b.\]

Then consider the linear combination of two equations

\[
\text{Eq}(1a) - \text{Eq} \ (2a) \quad \frac{(1 - qa)(1 - qb/c)(1 - qdb/e)}{(1 - b)(1 - q)/(qa/d - 1)} \quad \text{with} \quad d = \frac{q^3ace(1 - b)}{q^2ab + qce + ce - qace - qbc - qbe}.
\]

With this specific value \(d\), we can verify that the right member of the last equation vanishes. After some simplification, we establish the following relation.

**Theorem 14** (Two-term contiguous relation of \(_3\phi_2\)-series).

\[
\begin{align*}
3\phi_2 & \left[ \frac{a, c, e}{q^2ace(1 - b)} \right] \\
& = 3\phi_2 \left[ \frac{q^2b(1 - b)}{q^2ab + qce + ce - qace - qbc - qbe} \right];
\end{align*}
\]

Performing the substitutions \(a \to q^a, \ b \to q^b, \ c \to q^c, \ e \to q^e\) for Theorem 14 and then letting \(q \to 1\), we recover the following relation.

**Corollary 15** ([4, Theorem 7]).

\[
\begin{align*}
3F2 & \left[ \frac{a, c, e}{b, 2 + 2a + \frac{a}{b - 1}} \right] \\
& = 3F2 \left[ \frac{a + 1, c - 1, e}{b + 1, e - \frac{a}{1 - c}} \right] \\
& \times \frac{(1 + a)(1 + b - c)(1 + b - e)}{(1 + b)(1 + a + ab + ce - c - e - ac - ae)}.
\end{align*}
\]
Specifying the parameters, in Theorem 14 by
\[ a \rightarrow \alpha, \quad c \rightarrow \beta, \quad e \rightarrow \gamma, \quad b \rightarrow qa, \]
we found the following relation with one free parameter less.

**Proposition 16** (Two-term contiguous relation of 3\( \phi_2 \)-series).
\[
3\phi_2 \left[ \begin{array}{c} \alpha, \beta, \gamma \\ qa, \quad q^3 \alpha \frac{q^3(1 – qa)}{\beta – q^3 \alpha} \end{array} \right] = \frac{q^3 \alpha (1 – qa)}{q^3 \alpha^2 + q^3 \alpha \beta + q^3 \alpha \gamma – q^3 \alpha q \alpha \beta – q^2 \alpha \gamma}
\]
\[
= 3\phi_2 \left[ \begin{array}{c} \alpha, \beta, \gamma \\ qa, \quad q^3 \alpha \frac{q^3(1 – qa)}{\beta – q^3 \alpha} \end{array} \right] \times \frac{q^3 \alpha (1 – qa)}{q^3 \alpha^2 + q^3 \alpha \beta + q^3 \alpha \gamma – q^3 \alpha q \alpha \beta – q^2 \alpha \gamma}
\]

Employing the substitutions \( \alpha \rightarrow q^\alpha, \quad \beta \rightarrow q^\beta, \quad \gamma \rightarrow q^\gamma \) for Proposition 16 and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 17** (see also Prop. 8).
\[
3F_2 \left[ \begin{array}{c} \alpha, \beta, \gamma \\ \alpha + 1, 3 + 2 \alpha – \beta – \gamma + \frac{\beta \gamma}{\alpha + 1} \end{array} \right] = 3F_2 \left[ \begin{array}{c} \alpha, \beta, \gamma \\ \alpha + 1, 3 + 2 \alpha + \beta – \gamma + \frac{\beta \gamma}{\alpha + 1} \end{array} \right] \times \frac{(1 + \alpha) (\alpha – \beta + 2) (\alpha – \gamma + 2)}{(2 + \alpha) (2 + 3 \alpha + 2 \beta + 2 \gamma – \alpha \beta – \alpha \gamma – \beta – \gamma)}
\]

### 3.3. A&B.

Let Eq. (2a) stand for Eq. (2a) under the parameter replacements
\[ a \rightarrow q^2 a, \quad b \rightarrow qa, \quad d \rightarrow qd. \]

Then consider the linear combination of two equations
\[
Eq. (1a) – Eq. (2a) \frac{(1 – qa)(1 – d/e)(1 – d/e) b}{(1 – b)(1 – d)(qa – d)} \quad \text{with} \quad d = \frac{ce(qa – b)}{qab + ce – bc – be}.
\]

With this specific value \( d \), we can check that the right member of the last equation vanishes. After some simplification, we obtain the following relation.

**Theorem 18** (Two-term contiguous relation of 3\( \phi_2 \)-series).
\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, \quad qa(e + ce – bc – be) \end{array} \right] = \frac{b(qa – b)}{a(qab + ce – bc – be)}
\]
\[
= 3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, \quad qa(e + ce – bc – be) \end{array} \right] \times \frac{b(qa – b)}{a(qab + ce – bc – be)}
\]
\[
\times \frac{(1 – qa)(b – c)(b – e)}{(1 – b)(qab + ce – bc – be – qace)}.
\]

Performing the substitutions \( a \rightarrow q^a, \quad b \rightarrow q^b, \quad c \rightarrow q^c, \quad e \rightarrow q^e \) for Theorem 18 and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 19** (see Prop. 9).
\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, 1 + a – \frac{a, c, e}{1 + a – b} \end{array} \right] = 3F_2 \left[ \begin{array}{c} a, c, e \\ b + 1, 2 + a – \frac{a, c, e}{1 + a – b} \end{array} \right] \times \frac{(1 + a)(b – c)(b – e)}{b(ab + ce + b – c – e – ac – ae)}
\]

### 3.4. A&B.

Let Eq. (2a) stand for Eq. (2a) under the parameter replacements
\[ a \rightarrow qa, \quad c \rightarrow qa, \quad b \rightarrow d, \quad d \rightarrow q^2 b. \]

Then consider the linear combination of two equations
\[
Eq. (1a) – Eq. (2a) \frac{(b – a)(1 – c)(1 – qb/e)}{(1 – b)(1 – qb)(1 – qa/d)c} \quad \text{with} \quad d = \frac{qac(1 – b)}{a + c – ac – b}.
\]
With this specific value \( d \), we can verify that the right member of the last equation vanishes. After some simplification, we get the following relation.

**Theorem 20** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \begin{bmatrix} a, c, e \atop d \frac{qc+1}{a+ce-1} \end{bmatrix} 
\begin{bmatrix} q^2 \frac{bq(1-b)}{a+c-ac-1} e \end{bmatrix}
= 3\phi_2 \begin{bmatrix} qa, qc, e \atop q^2b \frac{a+ce-1}{a+ce-1} \end{bmatrix} 
\begin{bmatrix} q^b \frac{bq(1-b)}{a+c-ac-1} e \end{bmatrix} \quad q = e - \frac{b}{c}.
\]

Employing the substitutions \( a \to q^a, b \to q^b, c \to q^c, e \to q^e \) for Theorem 20 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 21** ([1] Theorem 10).  

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, 1 + a + c - \frac{ac}{e} \end{array} \right] 1 = 3F_2 \left[ \begin{array}{c} a + 1, c + 1, e \\ b + 2, 1 + a + c - \frac{ac}{e} \end{array} \right] \frac{1 + b - e}{1 + b}.
\]

Specifying the parameters, in Theorem 20 by

\[
a \to \beta, \quad c \to \gamma, \quad e \to \beta\gamma / \alpha, \quad b \to \frac{\beta + \gamma - \alpha - \beta\gamma}{1 - \alpha},
\]

we derive the following relation with one free parameter less.

**Proposition 22** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \begin{bmatrix} a, c, e \atop \frac{q\beta\gamma}{\alpha} \end{bmatrix} 
\begin{bmatrix} q^\beta \frac{\gamma}{\alpha} \beta + \gamma - \alpha - \beta\gamma \quad q^\beta \frac{\gamma}{\alpha} \beta + \gamma - \alpha - \beta\gamma \\frac{1}{1 - \alpha} \end{bmatrix}
= 3\phi_2 \begin{bmatrix} qa, q^\beta\gamma, q^\beta \gamma \atop q^\beta \frac{\gamma}{\alpha} \beta + \gamma - \alpha - \beta\gamma \quad q^\beta \frac{\gamma}{\alpha} \beta + \gamma - \alpha - \beta\gamma \\frac{1}{1 - \alpha} \end{bmatrix} \quad \alpha(qa + q^\beta\gamma - \beta\gamma - q^\beta - q^\gamma) + \beta\gamma \quad \alpha(1 + qa + q^\beta\gamma - \alpha - q^\beta - q^\gamma)
\times
\frac{\alpha(qa + q^\beta\gamma - \beta\gamma - q^\beta - q^\gamma) + \beta\gamma}{\alpha(1 + qa + q^\beta\gamma - \alpha - q^\beta - q^\gamma)}.
\]

Performing the substitutions \( a \to q^a, b \to q^b, c \to q^c, \gamma \to q^\gamma \) for Proposition 22 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 23** ([12] Theorem 9, see also [1] Proposition 11).  

\[
3F_2 \begin{bmatrix} \beta + \gamma - \alpha, \beta, \gamma \atop \beta + \gamma - \alpha + 1, \frac{\beta\gamma}{\alpha} \end{bmatrix} 1 = 3F_2 \begin{bmatrix} \beta + \gamma - \alpha, \beta + 1, \gamma + 1 \atop \beta + \gamma - \alpha + 1, \frac{\beta\gamma}{\alpha} + 2 \end{bmatrix} \frac{\alpha + \alpha^2 + \beta\gamma - \alpha\beta - \alpha\gamma}{\alpha + \beta\gamma}.
\]

### 3.5. A&C.

Let Eq. [3] stand for Eq. [6a] under the parameter replacements

\[
a \to qa, \quad c \to c/q, \quad e \to e/q, \quad d \to d/q.
\]

Then consider the linear combination of two equations

\[
\text{Eq. [1a]} - \text{Eq. [6a]} = \frac{1 - d/q}{1 - d/qa} \quad \text{with} \quad b = \frac{qce(a - 1)}{qa + ce - qe}.
\]

With this specific value \( b \), we can check that the right member of the last equation vanishes. After some simplification, we deduce the following relation.

**Theorem 24** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \begin{bmatrix} a, c, e \atop \frac{q\gamma(a - 1)}{qa + ce - qe - qe} \end{bmatrix} 
\begin{bmatrix} q^\gamma \frac{a}{\gamma^2} a \gamma - (1 - e)(1 - e) \end{bmatrix}
= 3\phi_2 \begin{bmatrix} qa, q^\gamma, q \gamma \atop d/q, \frac{a}{\gamma^2} a \gamma - (1 - e)(1 - e) \end{bmatrix} 
\begin{bmatrix} q^\gamma \frac{a}{\gamma^2} a \gamma - (1 - e)(1 - e) \end{bmatrix} \quad 1 - d/q \quad 1 - d/qa.
\]

Employing the substitutions \( a \to q^a, c \to q^c, d \to q^d, e \to q^e \) for Theorem 24 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 25** ([1] Theorem 12).  

\[
3F_2 \begin{bmatrix} a, c, e \atop d, c + e - 1 - \frac{(1 - c)(1 - e)}{a} \end{bmatrix} 1 = 3F_2 \begin{bmatrix} a + 1, c - 1, e - 1 \atop d - 1, c + e - 1 - \frac{(1 - c)(1 - e)}{a} \end{bmatrix} \frac{1 - d}{1 + a - d}.
\]
3.6. A&D.

Let Eq. (30) stand for Eq. (14) under the parameter replacements
\[ a \to c, \quad c \to q^2a, \quad e \to qe, \quad b \to q^3b, \quad d \to qd. \]

Then consider the linear combination of two equations
\[ Eq. (13) - Eq. (15) \]

Then consider the linear combination of two equations
\[ \text{with } d = \frac{qace(b-1)}{qabe + bc - qab - ce}. \]

With this specific value \( d \), we can verify that the right member of the last equation vanishes. After some simplification, we attain the following relation.

**Theorem 26** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, 1 + a + \frac{2(c-a-1)}{b}} \bigg| q^2b \right. \\
= 3\phi_2 \left[ \frac{a^2, c, qe}{q^2ace(b-1) + ce} \bigg| q \right.
\]

Performing the substitutions \( q \to q^a, b \to q^b, c \to q^c, e \to q^e \) for Theorem 28 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 27** ([1] Theorem 13).

\[
3F_2 \left[ \frac{a, c, e}{b, 1 + a + \frac{2(c-a-1)}{b}} \bigg| 1 \right. \\
= 3F_2 \left[ \frac{a + 2, c, e + 1}{b + 2, a + \frac{2(c-a-1)}{b}} \bigg| 1 \right.
\]

\[ \times \frac{1 + a(b-e)(1 + b - c)}{(1 + b)(ab + ce + b - ae - e)}. \]

3.7. B&C.

Let Eq. (31) stand for Eq. (19) under the parameter replacements
\[ a \to c, \quad c \to a/q^2, \quad e \to e/q, \quad b \to b/q, \quad d \to d/q^2. \]

Then consider the linear combination of two equations
\[ Eq. (20) - Eq. (23) \]

Then consider the linear combination of two equations
\[ \text{with } d = \frac{q^3ce(a-b)}{qab + qace - q^2bc - abe}. \]

With this specific value \( d \), we can check that the right member of the last equation vanishes. After some simplification, we achieve the following relation.

**Theorem 28** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, q^3ce(a-b)} \bigg| q \right.
\]

Employing the substitutions \( a \to q^a, b \to q^b, c \to q^c, e \to q^e \) for Theorem 28 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 29** ([1] Theorem 14).

\[
3F_2 \left[ \frac{a, c, e}{b, 2 + \frac{1}{(1-e)(1+c-a)} \bigg| 1 \right.
\]

\[ \times \frac{(1 - b)(1 + c - a)(1 + c + ae - ce - b - e)}{(1 - a)(1 + c - b)(1 + c + ae + bc - ace - ce - b - e)}. \]
3.8. C&D.

Let Eq. \( 13a \) stand for Eq. \( 13a \) under the parameter replacements

\[ c \rightarrow q^2c, \quad e \rightarrow q^2e, \quad b \rightarrow q^2b, \quad d \rightarrow q^2d. \]

Then consider the linear combination of two equations

\[ \text{Eq} \( 13a \) - Eq. \( 13a \) \]

Then consider the linear combination of two equations

\[ \text{Eq} \( 13a \) - Eq. \( 13a \) \]

With this specific value \( d \), we can verify that the right member of the last equation vanishes. After some simplification, we establish the following relation.

**Theorem 30** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, \frac{qace + qbc + qbe - qbc - q^2bce - ab}{(qace + qbc + qbe - qbc - q^2bce - ab)}}, q \right] = 3\phi_2 \left[ \frac{a, c + 2, e + 2}{b + 2, 2 + \frac{(a - 1)(c + e + 1) - ce}{(b + 1)(c + e + 1) - ce}}, q \right].
\]

Performing the substitutions \( a \rightarrow q^2a, b \rightarrow q^2b, c \rightarrow q^2c, e \rightarrow q^2e \) for Theorem 30 and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 31** ([1] Theorem 15).

\[
3F_2 \left[ \frac{a + 1, c + 1, e + 2}{b + 2, 2 + \frac{(a - 1)(c + e + 1) - ce}{(b + 1)(c + e + 1) - ce}}, \frac{1}{1} \right] = 3F_2 \left[ \frac{a + 1, c + 1, e + 2}{b + 2, 2 + \frac{(a - 1)(c + e + 1) - ce}{(b + 1)(c + e + 1) - ce}}, \frac{1}{1} \right].
\]

3.9. C&D.

Let Eq. \( 15a \) stand for Eq. \( 15a \) under the parameter replacements

\[ a \rightarrow qc, \quad c \rightarrow qa, \quad e \rightarrow q^2e, \quad b \rightarrow q^2b, \quad d \rightarrow q^2d. \]

Then consider the linear combination of two equations

\[ \text{Eq} \( 15a \) - Eq. \( 15a \) \]

With this specific value \( d \), we can check that the right member of the last equation vanishes. After some simplification, we found the following relation.

**Theorem 32** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, \frac{aq, qc, q^2e}{ab + qac - qac - ac - b}}, q \right] = 3\phi_2 \left[ \frac{a, c + 2, e + 2}{b + 2, 2 + \frac{(a - 1)(c + e + 1) - ce}{(b + 1)(c + e + 1) - ce}}, q \right].
\]

Employing the substitutions \( a \rightarrow q^2a, b \rightarrow q^2b, c \rightarrow q^2c, e \rightarrow q^2e \) for Theorem 32 and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 33** ([1] Theorem 16).

\[
3F_2 \left[ \frac{a + 1, c + 1, e + 2}{b + 2, 2 + \frac{(a - 1)(c + e + 1) - ce}{(b + 1)(c + e + 1) - ce}}, \frac{1}{1} \right] = 3F_2 \left[ \frac{a + 1, c + 1, e + 2}{b + 2, 2 + \frac{(a - 1)(c + e + 1) - ce}{(b + 1)(c + e + 1) - ce}}, \frac{1}{1} \right].
\]

4. Nineteen three-term contiguous relations of \( 3\phi_2 \)-series

By comparing two patterns of \( A, B, C \) and \( D \), we offer other nineteen three-term contiguous relations of \( 3\phi_2 \)-series. They produce several two-term contiguous relations of \( 3\phi_2 \)-series which are different from the ones before.
4.1. A & A.

Let Eq. (14) stand for Eq. (10) under the parameter replacements
\[ a \to c/q, \quad c \to qa, \quad b \to d/q, \quad d \to qb. \]

Then for an arbitrary variable \( Y_q \) the difference Eq. (11) - \( Y_q \times \text{Eq. (11)} \) results in the relation:
\[
3\phi_2 \begin{bmatrix} a, c, e \mid b, d \end{bmatrix} Y_q = 3\phi_2 \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q - 3\phi_2 \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q
\]
\[
= \left( A_q - A_q^* \right) \times Y_q
\]
\[
= \left( A_q - A_q^* \right) \times Y_q
\]
(5)

where the following notations have been used for coefficients
\[
A_q = A_q(3/a, qa, e; d/q, qb)
\]
\[
A_q^* = A_q(3/a, qa, e; d/q, qb)
\]
with \( A_q \) and \( A_q^* \) being defined in Theorem 34. Solving the equation \( 1 - \frac{A_q - A_q^*}{A_q - A_q^*} = b \) associated with the variable \( Y_q \), we obtain from equation (5) the following relation.

**Theorem 34** (Three-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \begin{bmatrix} a, c, e \mid b, d \end{bmatrix} q = Y_q \times \text{Eq. (14)} \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q - 3\phi_2 \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q + Z_q \times \text{Eq. (14)} \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by
\[
Y_q = \frac{(b-c)(b-c)(q-d)(ph-c)a}{(b-1)(qa-d)(q^2abe + acde + b^2d - bde - qabd - qabce)c},
\]
\[
Z_q = \frac{(aqace - bae)(q^2ab + acde + b^2d - bde - qabd - qabce)e}{(qa-d)(q^2abe + acde + b^2d - bde - qabd - qabce)e}
\]

Performing the substitutions \( a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e \) for Theorem 34 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 35** (14 Theorem 19),

\[
3F_2 \begin{bmatrix} a, c, e \mid b, d \end{bmatrix} 1 = Y_q \times \text{Eq. (14)} \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q - 3\phi_2 \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q + Z_q \times \text{Eq. (14)} \begin{bmatrix} qa, c/q, e \mid b, d/q \end{bmatrix} Y_q
\]

where the coefficients \( Y_q \) and \( Z_q \) are given by
\[
Y = \frac{(1 + b - c)(b-c)(b-e)(1-d)}{b(1+a-d)(1+b^2 + ae + cd + e - c - d - ab - be)}
\]
\[
Z = \frac{(1 + a + c + e - b - d)(1 + ae + cd + e - c - d - ab - ce)}{(1 + a - d)(1 + b^2 + ae + cd + e - c - d - ab - be)}
\]

Specifying the parameter \( b \to qa \) in Theorem 34 and using q-Gauss summation formula (cf. [5] p. 144):
\[
3\phi_1 \begin{bmatrix} a, b \mid c \end{bmatrix} q = \left[ c/a, c/b \mid q \right]_{\infty}
\]
(6)

we get the following relation with one free parameter less.

**Proposition 36** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \begin{bmatrix} a, c, e \mid qa, d \end{bmatrix} \frac{q}{ce} = 3\phi_2 \begin{bmatrix} qa, c/q, e \mid q^2a, d/q \end{bmatrix} \frac{q}{ce} + \frac{d/c, d/e}{q} \begin{bmatrix} q^3a^2e + qacd + acde - q^2a^2d - q^2a^2ce \mid q \end{bmatrix}
\]
\[
\frac{(q^2a^2 - q)(q^2a - qd)}{(q - c)(q - d)(q^2a - d)ce}
\]

Employing the substitutions \( a \to q^a, c \to q^c, d \to q^d, e \to q^e \) for Proposition 36 and then letting \( q \to 1 \), we recover the following relation.
Corollary 37 ([4] Proposition 1, see also [4] Proposition 20).

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ a + 1, d \end{array} \right]_1 = 3F_2 \left[ \begin{array}{c} a + 1, c - 1, e \\ a + 2, d - 1 \end{array} \right]_1 + \Gamma \left[ \begin{array}{c} d, d - e + 1 \\ d - c, d - e \end{array} \right]_1
\]

where the Gamma function is given by

\[
\Gamma(s) = \int_0^\infty x^{s-1}e^{-x} \, dx \quad \text{with } \Re(s) > 0
\]

and the abbreviated expression on \( \Gamma \)-function is

\[
\Gamma \left[ \begin{array}{c} \alpha, \beta, \ldots, \gamma \\ A, B, \ldots, C \end{array} \right] = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.
\]

4.2. A&A.

Let Eq. (1d) stand for Eq. (1a) under the parameter replacements

\[
a \to c/q, \quad c \to qa.
\]

Then for an arbitrary variable \( Y_q \), the difference Eq.(1a) - \( Y_q \times \) Eq.(1a) leads us to the relation:

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] \frac{q; bd}{q; ace} = Y_q \times 3\phi_2 \left[ \begin{array}{c} qa, c/q, e \\ b, d \end{array} \right]_q \frac{q; bd}{q; ace}
\]

\[
(A_q - a_q^* Y_q) \times 3\phi_2 \left[ \begin{array}{c} qa, c/q, e \\ b, d \end{array} \right]_q \frac{q; bd}{q; ace}
\]

\[
(\alpha_q - \alpha_q^* Y_q) \times 3\phi_2 \left[ \begin{array}{c} q^2 a, q c, q e \\ q^2 b, q d \end{array} \right] \frac{q; bd}{q; ace}, \quad (7)
\]

where the following notations have been used for coefficients

\[
A_q^* = A_q(c/q, qa, e; b, d), \quad \alpha_q^* = \alpha_q(c/q, qa, e; b, d),
\]

with \( A_q \) and \( \alpha_q \) being defined in Theorem 3. Solving the equation \( \alpha_q - \alpha_q^* Y_q = 0 \) associated with the variable \( Y_q \), we derive from equation (7) the following relation.

**Theorem 38** (Three-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] \frac{q; bd}{q; ace} = Y_q \times 3\phi_2 \left[ \begin{array}{c} qa, c/q, e \\ b, d \end{array} \right]_q \frac{q; bd}{q; ace} + Z_q \times 3\phi_2 \left[ \begin{array}{c} q^2 a, q c, q e \\ q^2 b, q d \end{array} \right] \frac{q; bd}{q; ace},
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[
Y_q = \frac{(c-d)qa}{(qa-d)c}, \quad Z_q = \frac{(b-c)(qa-c)d}{(b-1)(qa-d)ce}.
\]

Performing the substitutions \( a \to q^a, \quad b \to q^b, \quad c \to q^c, \quad d \to q^d, \quad e \to q^e \) for Theorem 38 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 39** ([4] Theorem 21).

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right]_1 = Y_3 F_2 \left[ \begin{array}{c} a + 1, c - 1, e \\ b, d \end{array} \right]_1 + Z_3 F_2 \left[ \begin{array}{c} a + 1, c, e \\ b + 1, d \end{array} \right]_1,
\]

where the coefficients \( Y \) and \( Z \) are given by

\[
Y = \frac{c - d}{1 + a - d}; \quad Z = \frac{(b - c)(1 + a - c)}{b(1 + a - d)}.
\]

Specifying the parameter \( b \to qa \) in Theorem 38 and using q-Gauss summation formula 6, we deduce the following relation with one free parameter less.

**Proposition 40** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \frac{qd}{ce} = 3\phi_2 \left[ \begin{array}{c} qa, c/q, e \\ q^2 a, d \end{array} \right] \frac{qd}{ce} + \frac{d/c}{d/qd} \frac{(qa - c)(q - d)c}{(qa - 1)(qa - d)ce} \frac{qd}{ce} \frac{qa - d}{qa}.
\]

Employing the substitutions \( a \to q^a, \quad c \to q^c, \quad d \to q^d, \quad e \to q^e \) for Proposition 40 and then letting \( q \to 1 \), we recover the following relation.
Corollary 41 (Proposition 3, see also Proposition 22).

\[ 3F_2 \left[ \begin{array}{c} a, c, e \\ a + 1, d \end{array} \right] = 3F_2 \left[ \begin{array}{c} a + 1, c, e \\ a + 2, d \end{array} \right] \frac{(1 + a - c)(1 + a - e)}{(1 + a)(1 + a - d)} - \Gamma \left[ \begin{array}{c} d, d - c - e + 1 \\ d, d - e \end{array} \right] \frac{1}{1 + a - d}. \]

Instead, solving the equation \( A_q - A^*_q Y_q = 0 \) associated with the variable \( Y_q \), we attain from equation (4) the following relation.

Theorem 42 (Three-term contiguous relation of \( 3 \phi_2 \)-series).

\[ 3 \phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] q, \frac{bd}{ace} = Y_q \times 3 \phi_2 \left[ \begin{array}{c} qa, c/q, e \\ b, d \end{array} \right] q, \frac{bd}{ace} + Z_q \times 3 \phi_2 \left[ \begin{array}{c} q^2 a, qc, qe \\ b, d \end{array} \right] q, \frac{bd}{ace}, \]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[ Y_q = \frac{(c - d)(qa + bd + qd - qbd - bde - qae)qg}{(qa - d)(qace + bd + qade - qab - bde - qace)c}; \]

\[ Z_q = \frac{(1 - e)(1 - qa)(c - qa)(b - e)(qace - bde)b^2}{(1 - b)(1 - d)(1 - qb)(qa - d)(qace + bd + qade - qab - bde - qace)qae^2}. \]

Performing the substitutions \( a \rightarrow q^a, b \rightarrow q^b, c \rightarrow q^c, d \rightarrow q^d, e \rightarrow q^e \) for Theorem 42 and then letting \( q \rightarrow 1 \), we recover the following relation.

Corollary 43 (Theorem 23).

\[ 3F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] 1 = Y_3 \times 3F_2 \left[ \begin{array}{c} a + 1, c, e \\ b, d \end{array} \right] 1 + Z_3 \times 3F_2 \left[ \begin{array}{c} a + 2, c, e \\ b + 2, d + 1 \end{array} \right], \]

where the coefficients \( Y \) and \( Z \) are given by

\[ Y = \frac{(c - d)(b + a + ce - bd)}{(1 + a - d)(bc + ae + e - bd)}, \]

\[ Z = \frac{ce(1 + a)(b - e)(1 + a - c)(1 + a + e - b - d)}{bd(1 + a)(1 + a - d)(bd - ce + ae - e)}. \]

Remark: There is a tiny mistake in the original equation due to Chu and Wang [Theorem 23]. We have added a minus for the coefficient of the first \( 3F_2 \)-series on the right hand side.

Specifying the parameter \( b \rightarrow qa \) in Theorem 42 and using \( q \)-Gauss summation formula, we achieve the following relation with one free parameter less.

Proposition 44 (Two-term contiguous relation of \( 3 \phi_2 \)-series).

\[ 3 \phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] q, \frac{qd}{ce} = 3 \phi_2 \left[ \begin{array}{c} q^2 a, qc, qe \\ qa, d \end{array} \right] q, \frac{qd}{ce} \frac{(1 - c)(1 - e)(qa - d)^2}{(1 - d)(1 - qa)(1 - q^2 a)(qa - d)^2} + \frac{(d/e, d/e)}{d/d/ce} \frac{q^2 d}{q}. \]

Employing the substitutions \( a \rightarrow q^a, c \rightarrow q^c, d \rightarrow q^d, e \rightarrow q^e \) for Proposition 44 and then letting \( q \rightarrow 1 \), we recover the following relation.

Corollary 45 (Proposition 21).

\[ 3F_2 \left[ \begin{array}{c} a, c, e \\ a + 1, d \end{array} \right] 1 = 3F_2 \left[ \begin{array}{c} a + 2, c + 1, e + 1 \\ a + 3, d + 1 \end{array} \right] \frac{ce(1 + a - c)(1 + a - e)}{(1 + a + 2a + ce - d - ad)(1 + a)} + \Gamma \left[ \begin{array}{c} d, d - c - e \\ d - c, d - e \end{array} \right] \frac{(1 + a^2 + 2a + ce - d - ad)}{(1 + a)(1 + a - d)}. \]

Taking \( d = \frac{qa + ce(1 - qa)}{qa + ce - qac - qae} \) in Proposition 44 we establish the following relation.

Proposition 46 (Two-term contiguous relation of \( 3 \phi_2 \)-series).

\[ 3 \phi_2 \left[ \begin{array}{c} a, c, e \\ qa \end{array} \right] q, \frac{qa + ce(1 - qa)}{qa + ce - qac - qae} = 3 \phi_2 \left[ \begin{array}{c} q^2 a, qc, qe \\ qa + ce(1 - qa) \end{array} \right] q, \frac{qa + ce(1 - qa)}{qa + ce - qac - qae} \times \frac{(1 - qa)(qa - c)(qa - e)}{(1 - q^2 a)(qa + ce + q^2 a^2 ce - qac - qae - qae)}. \]
Performing the substitutions $a \rightarrow qa^*$, $c \rightarrow q^a$, $e \rightarrow q^c$ for Proposition 49 and then letting $q \rightarrow 1$, we recover the following relation.

**Corollary 47** ([H] Corollary 25).

\[
3F_2 \left[ \frac{a, c, e}{a + 1, 1 + a + \frac{ce}{a + a}} \right] = 3F_2 \left[ \frac{a + 2, c + 1, e + 1}{a + 3, 2 + a + \frac{ce}{a + a}} \right] \times \frac{(1 + a)(1 + a - c)(1 + a - e)}{(2 + a)(1 + 2a + a^2 + ce)}
\]

4.3. A&\&A.

Let Eq. (13) stand for Eq. (12) under the parameter replacements

\[b \rightarrow d/q, \quad d \rightarrow qb.\]

Then for an arbitrary variable $Y_q$, the difference Eq. (11) - $Y_q \times$ Eq. (12) results in the relation:

\[
3\phi_2 \left[ \frac{a, c, e}{b, d} \right] = Y_q \times 3\phi_2 \left[ \frac{a, c, e}{qb, d/q} \right] - Q_1 - \frac{bd}{ace} - A_q \times 3\phi_2 \left[ \frac{a, c, e}{qb, d/q} \right] \times \frac{qde}{ace}
\]

where the following notations have been used for coefficients

\[\begin{align*}
A_q &= A_q(a, c, e; d/q, qb), \\
A_q &= A_q(a, c, e; d/q, qb)
\end{align*}\]

with $A_q$ and $A_q$ being defined in Theorem 2. Solving the equation $1 - \frac{A_q - A_q^* Y_q}{A_q - A_q^* Y_q} = b$ associated with the variable $Y_q$, we found from equation (8) the following relation.

**Theorem 48** (Three-term contiguous relation of $3\phi_2$-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, d} \right] = Y_q \times 3\phi_2 \left[ \frac{a, c, e}{qb, d/q} \right] + Z_q \times 3\phi_2 \left[ \frac{a, c, e}{b, d} \right]
\]

where the coefficients $Y_q$ and $Z_q$ are defined by

\[\begin{align*}
Y_q &= \frac{(a - b)(b - c)(b - e)(d - q)d}{(b - a)(b - c)(b - e)(b - d)(1 - b)(qab + cde + bde - b^2d - acde - qbce)} \\
Z_q &= \frac{(a - b)(b - c)(b - e)(d - q)d}{(b - a)(b - c)(b - e)(b - d)(qab + cde + bde - b^2d - acde - qbce)}
\end{align*}\]

Employing the substitutions $a \rightarrow qa^*$, $b \rightarrow q^b$, $c \rightarrow q^c$, $d \rightarrow q^d$, $e \rightarrow q^e$ for Theorem 48 and then letting $q \rightarrow 1$, we recover the following relation.

**Corollary 49** ([H] Theorem 26).

\[
3F_2 \left[ \frac{a, c, e}{b, d} \right] = Y \times 3F_2 \left[ \frac{a, c, e}{b + 1, d - 1} \right] + Z \times 3F_2 \left[ \frac{1 + a, c, e}{b, d} \right]
\]

where the coefficients $Y$ and $Z$ are given by

\[\begin{align*}
Y &= \frac{(a - b)(b - c)(b - e)(1 - d)}{b(1 + a - d)(1 + a + b + c + e - b^2 - ce - ad)} \\
Z &= \frac{a + b - d}{(1 + a - d)(1 + a + b + c + e - b^2 - ce - ad)}
\end{align*}\]

Specifying the parameter $b \rightarrow qa$ in Theorem 49 and using $q$-Gauss summation formula 6, we obtain the following relation with one free parameter less.

**Proposition 50** (Two-term contiguous relation of $3\phi_2$-series).

\[
3\phi_2 \left[ \frac{a, c, e}{q, d} \right] = 3\phi_2 \left[ \frac{a, c, e}{q^2a, d/q} \right] + \frac{(1 - q)(q - d)(q - c)(q - e)d}{qde} \times \frac{1}{(1 - q)(q - d)(q - c)(q - e)d}
\]

Performing the substitutions $a \rightarrow qa^*$, $c \rightarrow q^c$, $d \rightarrow q^d$, $e \rightarrow q^e$ for Proposition 50 and then letting $q \rightarrow 1$, we recover the following relation.
Corollary 51 ([4, Proposition 27]).

\[
\begin{align*}
\left.\begin{array}{c}
\binom{a,c,e}{a+1,d}
\end{array}\right|1 & = \binom{a,c,e}{a+2,d-1}
\left.\begin{array}{c}
\frac{1}{1}
\end{array}\right|\frac{(1+a-c)(1+a-e)(d-1)}{(1+a)(1+a-d)(ac+ae+c+e-ce-ad-1)} \\
& + \Gamma \left[\frac{d,d-c-e}{d-c,d-e}\right] \frac{e}{(1+a-d)(ac+ae+c+e-ce-ad-1)}.
\end{align*}
\]

4.4. A&A. Let Eq* (1a) stand for Eq(1a) under the parameter replacements

\[
b \to d/q, \quad d \to qb.
\]

Then for an arbitrary variable \(Y_q\), the difference Eq(1a) - \(Y_q \times \text{Eq}^* (1a)\) leads us to the relation:

\[
\begin{align*}
\phi_2 & \binom{a,c,e}{b,d} \frac{bd}{ace} - Y_q \times \phi_2 \binom{a,c,e}{qb,d/q} \frac{bd}{ace} \\
& = (A_q - A_q^*) Y_q \times \phi_2 \binom{qa,c,e}{qb,d} \frac{bd}{ace} \\
& + (k_q - k_q^*) Y_q \times \phi_2 \binom{q^2a,c,e}{q^2b,qd} \frac{bd}{ace}.
\end{align*}
\]

Where the following notations have been used for coefficients

\[
A_q^* = A_q(a,c,e; d/q, qb),
\]

\[
k_q^* = k_q(a,c,e; d/q, qb)
\]

With \(A_q\) and \(k_q\) being defined in Theorem [2]. Solving the equation \(k_q - A_q^* Y_q = 0\) associated with the variable \(Y_q\), we get from equation (9) the following relation.

Theorem 52 (Three-term contiguous relation of \(\phi_2\)-series).

\[
\begin{align*}
\phi_2 & \binom{a,c,e}{b,d} \frac{bd}{ace} = Y_q \times \phi_2 \binom{a,c,e}{qb,d/q} \frac{bd}{ace} + Z_q \times \phi_2 \binom{qb,c,e}{qb,d} \frac{bd}{ace},
\end{align*}
\]

Where the coefficients \(Y_q\) and \(Z_q\) are defined by

\[
Y_q = \frac{(a - b)(q - d)}{(1 - b)(qa - d)}, \quad Z_q = \frac{(1 - a)(qb - d)}{(1 - b)(qa - d)}.
\]

Employing the substitutions \(a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e\) for Theorem 52 and then letting \(q \to 1\), we recover the following relation.

Corollary 53 ([4, Theorem 28]).

\[
\begin{align*}
\binom{a,c,e}{b,d} \left| 1 \right. & = \binom{a,c,e}{b+1,d-1} \left| 1 \right. + \binom{a+1,c,e}{b+1,d} \left| 1 \right. \\
Y & = \frac{(a - b)(d - 1)}{b(1 + a - d)}, \quad Z = \frac{a(1 + b - d)}{b(1 + a - d)}.
\end{align*}
\]

Specifying the parameter \(d \to qc\) in Theorem 52 and using \(q\)-Gauss summation formula [9], we derive the following relation with one free parameter less under the replacements \(a \to c, c \to a\).

Proposition 54 (Two-term contiguous relation of \(\phi_2\)-series).

\[
\begin{align*}
\phi_2 & \binom{a,c,e}{qa,b} \frac{qb}{ce} = \phi_2 \binom{aq,c,e}{qa,qb} \frac{qb}{ce} \left(\frac{a - b)(1 - c)}{(a - c)(1 - b)} + \frac{b/c, qb/ce}{b, qb/ce} \right) \frac{(a - 1)c}{a - c}.
\end{align*}
\]

Performing the substitutions \(a \to q^a, b \to q^b, c \to q^c, e \to q^e\) for Proposition 54 and then letting \(q \to 1\), we recover the following relation.

Corollary 55 ([4, Proposition 29]).

\[
\begin{align*}
\binom{a,c,e}{a+1,b} \left| 1 \right. & = \binom{a,c+1,e}{a+1,b+1} \left| 1 \right. + \binom{(a - b)c}{b - c, b - e + 1} \left| \frac{a}{a - e} \right.
\end{align*}
\]

Instead, solving the equation \(A_q - A_q^* Y_q = 0\) associated with the variable \(Y_q\), we deduce from equation (9) the following relation.
Theorem 56 (Three-term contiguous relation of $3\phi_2$-series).

$$3\phi_2\left[\frac{a, c, e}{b, d} \bigg| q; \frac{b d}{a c e}\right] = Y_q \times 3\phi_2\left[\frac{a, c, e}{b, d} \bigg| q; \frac{b d}{a c e}\right] + Z_q \times 3\phi_2\left[\frac{q^2 a, q c, q e}{q b, q d} \bigg| q; \frac{b d}{a c e}\right],$$

where the coefficients $Y_q$ and $Z_q$ are defined by

$$Y_q = \frac{(a - b)(q - d)(qac + cde + bd - bce - qae)}{(1 - b)(qac - d)(qac + ace + bd - bce - qae)},$$

$$Z_q = \frac{(1 - a)(1 - c)(1 - e)(1 - qa)(qb - d)(qae - bd)bd}{(1 - b)(1 - d)(1 - gb)(qac - d)(qac + ace + bd - bce - qae)qae}.$$

Employing the substitutions $a \to q^a$, $b \to q^b$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Theorem 56 and then letting $q \to 1$, we recover the following relation.

Corollary 57 (2 Theorem 30)).

$$3F_2\left[\frac{a, c, e}{b, d} \bigg| 1 \right] = Y Z F_2\left[\frac{a, c, e}{b, d} \bigg| 1 \right] + Z F_2\left[\frac{a + 2, c + 1, e + 1 + 1}{b + 2, d + 1} \bigg| 1 \right],$$

where the coefficients $Y$ and $Z$ are given by

$$Y = \frac{(a - b)(d - 1)(bd - ab - ce - b)}{(b + a + d)(a + bd - ad - ce - b)},$$

$$Z = \frac{ace(1 + a)(1 + b - d)(1 + a + c + e - b - d)}{bd(1 + b)(1 + a - d)(a + bd - ad - ce - b)}.$$

Specifying the parameter $b \to c$ in Theorem 59 and using $q$-Gauss summation formula 6, we attain the following relation with one free parameter less under the replacements $a \to c$, $c \to a$, $d \to qd$.

Proposition 58 (Two-term contiguous relation of $3\phi_2$-series).

$$3\phi_2\left[\frac{a, c, e}{qa, d} \bigg| q; \frac{qd}{ce}\right] = 3\phi_2\left[\frac{qa, q^2 c, q e}{qa, q^2 d} \bigg| q; \frac{d}{ce}\right] + \frac{d/c, q d/e}{d, d/c} \left\{ q^{\infty} \frac{(ae + cde + d - ad - de - ce)}{(a - c)e} \right\}.$$

Performing the substitutions $a \to q^a$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Proposition 58 and then letting $q \to 1$, we recover the following relation.

Corollary 59 (2 Proposition 31)).

$$3F_2\left[\frac{a, c, e}{a + 1, d} \bigg| 1 \right] = 3F_2\left[\frac{a + 1, c + 2, e + 1 + 1}{a + 2, d + 2} \bigg| 1 \right] \frac{ce(1 + c)(a - d)}{d(1 + a)(1 + d)(c - a)} + \frac{d, d - c - e}{d - c, d - e + 1} \frac{(ad - cd - ae)}{(a - c)}.$$

Taking $d = \frac{e(c - a)}{1 + ce - a - e}$ in Proposition 58, we achieve the following relation.

Proposition 60 (Two-term contiguous relation of $3\phi_2$-series).

$$3\phi_2\left[\frac{a, c, e}{qa, e^{(c - a)}} \bigg| q; \frac{q(c - a)}{e(1 + ce - a - e)}\right] = 3\phi_2\left[\frac{qa, q^2 c, q e}{qa, q^2 d \frac{e(c - a)}}{1 + ce - a - e} \bigg| q; \frac{c - a}{e(1 + ce - a - e)}\right] \times \frac{(1 - c)(1 - qae)(a - ce)}{(1 - qa)(1 + ce + qae - qae - a - e)c}.$$

Employing the substitutions $a \to q^a$, $c \to q^c$, $e \to q^e$ for Proposition 60 and then letting $q \to 1$, we recover the following relation.

Corollary 61 (2 Corollary 32)).

$$3F_2\left[\frac{a, c, e}{a + 1, \frac{ae}{a - c}} \bigg| 1 \right] = 3F_2\left[\frac{a + 1, c + 2, e + 1 + 1}{a + 2, 2 + \frac{ae}{a - c}} \bigg| 1 \right] \frac{c(1 + c)(c + e - a)}{(a + ac - c)}.$$
4.5. A&B.

Let Eq. (26) stand for Eq. (25) under the parameter replacements

\[ a \rightarrow qc, \quad c \rightarrow qa, \quad b \rightarrow d, \quad d \rightarrow q^2b. \]

Then for an arbitrary variable \( Y_q \), the difference Eq. (13) \( Y_q \times Eq^* (25) \) results in the relation:

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \mid \frac{bd}{ace} \right] - Y_q \times 3\phi_2 \left[ \begin{array}{c} qa, qc, e \\ q^2b, d \mid \frac{bd}{ace} \end{array} \right] = (A_q - B_q Y_q) \times 4\phi_3 \left[ \begin{array}{c} qa, c, e, q \left( 1 - \frac{A_q - B_q Y_q}{A_q - B_q^* Y_q} \right) \\ qb, d, 1 - \frac{A_q - B_q Y_q}{A_q - B_q^* Y_q} \mid q \frac{bd}{qace} \end{array} \right] ,
\]

where the following notations have been used for coefficients

\[ B_q = B_q(qc, qa, c, e; d, q^2b), \]

\[ B_q^* = B_q(qc, qa, c, e; d, q^2b) \]

with \( B_q \) and \( B_q^* \) being defined in Theorem 4. Solving the equation \( 1 - \frac{A_q - B_q^* Y_q}{A_q - B_q^* Y_q} = e \) associated with the variable \( Y_q \), we establish from equation (19) the following relation.

**Theorem 62** (Three-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \mid \frac{bd}{ace} \right] = Y_q \times 3\phi_2 \left[ \begin{array}{c} qa, qc, e \\ q^2b, d \mid \frac{bd}{ace} \end{array} \right] + Z_q \times 3\phi_2 \left[ \begin{array}{c} qa, c, e \\ qb, d \mid \frac{bd}{qace} \end{array} \right] ,
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[
Y_q = \frac{(1 - c)(b - a)(b - e)(qab - e)(qae - d)d}{(1 - b)(1 - qb)(qac + aed + bde - ade - bcd - qace^2)c},
\]

\[
Z_q = \frac{(1 - c)(qace - bd)(qac + aed + bde - ade - bcd - qace^2)c}{(1 - b)(d - qa)(qac + aed + bde - ade - bcd - qace^2)c}.
\]

Performing the substitutions \( a \rightarrow q^a \), \( b \rightarrow q^b \), \( c \rightarrow q^c \), \( d \rightarrow q^d \), \( e \rightarrow q^e \) for Theorem 62 and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 63** ([4] Theorem 33).

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \mid 1 \right] = Y \times 3F_2 \left[ \begin{array}{c} a + 1, c + 1, e \\ b + 2, d \mid 1 \right] + Z \times 3F_2 \left[ \begin{array}{c} a + 1, c, e \\ b + 1, d \mid 1 \right] ,
\]

where the coefficients \( Y \) and \( Z \) are given by

\[
Y = \frac{e(a - b)(b - e)(1 + b - e)(1 + a + e - d)}{b(1 + b)(1 + a - d)(ac + be + de - ac - e^2 - bc)},
\]

\[
Z = \frac{e(b + d - a - c - e - 1)(1 + b + ab + be - bd - ac)}{b(1 + a - d)(ac + be + de - ac - e^2 - bc)}.
\]

Specifying the parameter \( d \rightarrow qe \) in Theorem 63 and using \( q \)-Gauss summation formula 69, we found the following relation with one free parameter less.

**Proposition 64** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, qe \end{array} \mid \frac{qb}{ac} \right] = 3\phi_2 \left[ \begin{array}{c} qa, qc, e \\ q^2b, qe \mid \frac{qb}{ac} \end{array} \right] \left( 1 - \frac{a - b}{b(1 - b)(1 + b - e)(1 + a + e - d)} \right) + \left( q - e \right)(abc + ae + ac - be - ace - acde). \]

Employing the substitutions \( a \rightarrow q^a \), \( b \rightarrow q^b \), \( c \rightarrow q^c \), \( e \rightarrow q^e \) for Proposition 64 and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 65** ([12] Proposition 2], see also [4] Proposition 34).

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, e + 1 \mid 1 \right] = \Gamma \left[ \begin{array}{c} b, b - a - c + 1 \\ b - a + 1, b - c + 1 \end{array} \mid \frac{e(ac + ab - ab - be)}{(a - e)(c - e)} \right].
\]

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, e + 1 \mid 1 \right] = 3F_2 \left[ \begin{array}{c} a + 1, c + 1, e \\ b + 2, e + 1 \mid 1 \right] \frac{ac(b - e)(1 + b - e)}{b(1 + b)(a - e)(e - c)} + \Gamma \left[ \begin{array}{c} b, b - a - c + 1 \\ b - a + 1, b - c + 1 \end{array} \mid \frac{e(ac + ab - ab - be)}{(a - e)(c - e)} \right].
\]
Instead, solving the equation $1 - \frac{A_q - F^*_q \nu_q}{\nu_q} = b$ associated with the variable $Y_q$, we obtain from equation \[10\] the following relation.

**Theorem 66** (Three-term contiguous relation of $3\phi_2$-series).

$$3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \left( q^d \bigg| \frac{b d}{ac} \right) = Y_q \times 3\phi_2 \left[ \begin{array}{c} qa, qc, e \\ q^d b, d \end{array} \right] + Z_q \times 3\phi_2 \left[ \begin{array}{c} qa, c, e \\ d, d \end{array} \right] \left( \frac{b d}{q e} \right)$$

where the coefficients $Y_q$ and $Z_q$ are defined by

$$Y_q = \frac{(a - b)(b - c)(1 - c)(b - e)(gb - e)d^2}{(1 - b)(1 - gb)(qa - d)(qab + ad + cd - bd - ad - qac)}$$

$$Z_q = \frac{(qace - bd)(qace + adc + bd - b^2d - acde - qac)c}{(qa - d)(qace + adc + bd - b^2d - acde - qac)c}.$$

Performing the substitutions $a \to q^a$, $b \to q^b$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Theorem 66 and then letting $q \to 1$, we recover the following relation.

**Corollary 67** ([ Proposition 35]).

$$3F_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \left( q^d \bigg| \frac{b d}{ac} \right) = Y \times 3F_2 \left[ \begin{array}{c} a + 1, c + 1, e \\ b + 2, d \end{array} \right] + Z \times 3F_2 \left[ \begin{array}{c} a + 1, c, e \\ b, d \end{array} \right] \left( \frac{b d}{q e} \right)$$

where the coefficients $Y$ and $Z$ are given by

$$Y = \frac{(a - b)(b - c)(1 - b - e)c}{b(1 + b)(1 + a - d)(ac + ce + bd + b^2 - b - ab - be - 2bc)}$$

$$Z = \frac{(b + d - a - c - e - 1)(ab + be + b - bd - ac)}{(1 + a - d)(ac + ce + bd + b^2 - b - ab - be - 2bc)}.$$

**Remark:** There is a tiny mistake in the original equation due to Chu and Wang [ Proposition 35]. We have changed the parameter $e + 1$ into $e$ for the last $3F_2$-series. The similar changes should also be used in Chu and Wang [ Proposition 36 and Corollary 37].

Specifying the parameter $b \to qa$ in Theorem 66 and using $q$-Gauss summation formula 6, we get the following relation with one free parameter less.

**Proposition 68** (Two-term contiguous relation of $3\phi_2$-series).

$$3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \left( q^d \bigg| \frac{b d}{ac} \right) = 3\phi_2 \left[ \begin{array}{c} qa, qc, e \\ q^d a, d \end{array} \right] \left( q^d \bigg| \frac{b d}{ac} \right) + \frac{q d}{ce} \left[ \begin{array}{c} d, c, e \\ q d, d \end{array} \right] + \frac{q d}{ce} \left[ \begin{array}{c} q d \left( qa^2 + ad + cd - acd - qac - qad \right) \\ qa - d \end{array} \right].$$

Employing the substitutions $a \to q^a$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Proposition 68 and then letting $q \to 1$, we recover the following relation.

**Corollary 69** ([ Proposition 36]).

$$3F_2 \left[ \begin{array}{c} a, c, e \\ a + 1, d \end{array} \right] = 3F_2 \left[ \begin{array}{c} a + 1, c + 1, e \\ a + 3, d \end{array} \right] + \frac{c(1 + a + c)(1 + a + e)(1 + a + c + 2a + 2c + e + d - ad - ce)}{(a + 1)(a + 2)(1 + e)(ac + ce + 2c + e + d - ad - ce)}.$$

Taking $d = \frac{q ac(q a - 1)}{q a + ac - a - c}$ in Proposition 68 we derive the following relation.

**Proposition 70** (Two-term contiguous relation of $3\phi_2$-series).

$$3\phi_2 \left[ \frac{qa - a - e}{e qa + ac - a - c} \right] = 3\phi_2 \left[ \frac{qa, qc, e}{q ac(q a - 1)} \right] + \frac{q a^2(q a - 1)}{e(2 + a - e)} \frac{q a^2(q a - 1)}{q a + ac - a - c}.$$

Performing the substitutions $a \to q^a$, $c \to q^c$, $e \to q^e$ for Proposition 70 and then letting $q \to 1$, we recover the following relation.

**Corollary 71** ([ Corollary 37]).

$$3F_2 \left[ \begin{array}{c} a, c, e \\ a + 1, a + 1, 1 + a \end{array} \right] = 3F_2 \left[ \begin{array}{c} a + 1, c + 1, e \\ a + 3, a + 1, 1 + a \end{array} \right] \left( \frac{2 + a - e}{2 + a} \right).$$
4.6. A&B.

Let Eq\(^{(2b)}\) stand for Eq\(^{(2b)}\) under the parameter replacements

\[
a \rightarrow qa, \quad c \rightarrow qa, \quad b \rightarrow qh, \quad d \rightarrow qd.
\]

Then for an arbitrary variable \(Y_q\), the difference Eq\(^{(11)}\) – Eq\(^{(2b)}\) leads us to the relation:

\[
3\phi_2 \left[ \begin{array}{c|c} a, c, e & b, d \\ \hline \phi & bd \\ \hline ace & \end{array} \right] - Y_q \times 3\phi_2 \left[ \begin{array}{c|c} qa, qc, e & qhd \\ \hline q & \phi \\ \hline ace & \end{array} \right] = (A_q - B^*_q Y_q) \times 4\phi_3 \left[ \begin{array}{c|c} qa, c, e, q \left( 1 - \frac{A_q - B^*_q Y_q}{3Y_q - B^*_q Y_q} \right) & qbd \\ \hline qh, d & \phi \\ \hline ace & \end{array} \right],
\]

where the following notations have been used for coefficients

\[
B^*_q = B_q(qc, qa, e; qb, qd),
\]

\[
B^*_q = B_q(qc, qa, e; qb, qd)
\]

with \(B_q\) and \(B^*_q\) being defined in Theorem \[4\]. Solving the equation \(1 - \frac{A_q - B^*_q Y_q}{3Y_q - B^*_q Y_q} = b\) associated with the variable \(Y_q\), we deduce from equation \[11\] the following relation.

\[
\text{Theorem 72 (Three-term contiguous relation of } 3\phi_2\text{-series).}
\]

\[
3\phi_2 \left[ \begin{array}{c|c} a, c, e & b, d \\ \hline \phi & bd \\ \hline ace & \end{array} \right] = Y_q \times 3\phi_2 \left[ \begin{array}{c|c} qa, qc, e & qhd \\ \hline q & \phi \\ \hline ace & \end{array} \right] + Z_q \times 3\phi_2 \left[ \begin{array}{c|c} qa, c, e & qbd \\ \hline q & \phi \\ \hline ace & \end{array} \right],
\]

where the coefficients \(Y_q\) and \(Z_q\) are defined by

\[
Y_q = \frac{(1 - c)(b - c)(d - c)bd}{(1 - b)(1 - d)(qae - bd)c}, \quad Z_q = \frac{(qae - bd)}{(qae - bd)c}.
\]

Employing the substitutions \(a \rightarrow q^a, b \rightarrow q^b, c \rightarrow q^c, d \rightarrow q^d, e \rightarrow q^e\) for Theorem \[72\] and then letting \(q \rightarrow 1\), we recover the following relation.

\[
\text{Corollary 73 (4 Theorem 38).}
\]

\[
3F_2 \left[ \begin{array}{c|c} a, c, e & b, d \\ \hline \phi & \end{array} \right] = Y_q 3F_2 \left[ \begin{array}{c|c} a + 1, c + 1, e & b + 1, d + 1 \\ \hline \phi & \end{array} \right] + Z_q 3F_2 \left[ \begin{array}{c|c} a + 1, c, e & b, d \\ \hline \phi & \end{array} \right],
\]

where the coefficients \(Y\) and \(Z\) are given by

\[
Y = \frac{c(b - e)(e - d)}{bd(1 + a + e - b - d)}, \quad Z = \frac{(1 + a + c + e - b - d)}{(1 + a + e - b - d)}.
\]

Specifying the parameter \(b \rightarrow qa\) in Theorem \[72\] and using \(q\)-Gauss summation formula \[6\], we attain the following relation with one free parameter less.

\[
\text{Proposition 74 (Two-term contiguous relation of } 3\phi_2\text{-series).}
\]

\[
3\phi_2 \left[ \begin{array}{c|c} a, c, e & qa, d \\ \hline \phi & \end{array} \right] = 3\phi_2 \left[ \begin{array}{c|c} qa, qc, e & qd \phi \\ \hline q & \phi \\ \hline ace & \end{array} \right] \frac{(qa - e)(1 - c)d}{(qa - 1)(1 - d)c} + \frac{d/c, qd/e}{d, qd/ce} \left( q \right)_\infty.
\]

Performing the substitutions \(a \rightarrow q^a, b \rightarrow q^b, c \rightarrow q^c, d \rightarrow q^d, e \rightarrow q^e\) for Proposition \[74\] and then letting \(q \rightarrow 1\), we recover the following relation.

\[
\text{Corollary 75 (4 Proposition 39).}
\]

\[
3F_2 \left[ \begin{array}{c|c} a, c, e & a + 1, d \\ \hline \phi & \end{array} \right] = 3F_2 \left[ \begin{array}{c|c} a + 1, c + 1, e & a + 2, d + 1 \\ \hline \phi & \end{array} \right] \frac{(1 + a - e)c}{(1 + a)d} + \Gamma \left( d, d - c - e + 1 \right),
\]

4.7. A&D.

Let Eq\(^{(2b)}\) stand for Eq\(^{(2b)}\) under the parameter replacements

\[
a \rightarrow qa, \quad c \rightarrow qc, \quad e \rightarrow qe, \quad b \rightarrow q^2b, \quad d \rightarrow qd.
\]
Then for an arbitrary variable $Y_q$, the difference Eq\[13\] – $Y_q \times \text{Eq}^*\[13\]$ results in relation:

$$3\phi_2 \left[ \frac{a, c, e}{b, d} \mid q; \frac{bd}{ace} \right] - Y_q \times 3\phi_2 \left[ \frac{qa, qc, qe}{q^2b, qd} \mid q; \frac{bd}{ace} \right] = (A_q - D_q^* Y_q) \times 3\phi_2 \left[ \frac{qa, c, e}{qb, d} \mid q; \frac{bd}{ace} \right] + (B_q - B_q^* Y_q) \times 3\phi_2 \left[ \frac{q^2a, qc, qe}{q^2b, qd} \mid q; \frac{bd}{ace} \right],$$

where the following notations have been used for coefficients:

$$D_q^* = D_q(qa, qc, qe; q^2b, qd),$$

$$B_q^* = B_q(qa, qc, qe; q^2b, qd)$$

with $D_q$ and $B_q$ being defined in Theorem \[3\]. Solving the equation $A_q - D_q^* Y_q = 0$ associated with the variable $Y_q$, we achieve from equation \[12\] the following relation.

**Theorem 76** (Three-term contiguous relation of $3\phi_2$-series).

$$3\phi_2 \left[ \frac{a, c, e}{b, d} \mid q; \frac{bd}{ace} \right] = 3\phi_2 \left[ \frac{qa, qc, qe}{q^2b, qd} \mid q; \frac{bd}{ace} \right] (a - b)(1 - c)(1 - e)bd \rightarrow (1 - b)(1 - qb)(d - 1)ace + 3\phi_2 \left[ \frac{qa, c, e}{qb, d} \mid q; \frac{bd}{ace} \right].$$

Employing the substitutions $a \rightarrow q^a$, $b \rightarrow q^b$, $c \rightarrow q^c$, $d \rightarrow q^d$, $e \rightarrow q^e$ for Theorem \[76\] and then letting $q \rightarrow 1$, we recover the following relation.

**Corollary 77** (\[4\] Theorem 40)).

$$3F_2 \left[ \frac{a, c, e}{b, d} \mid 1 \right] = 3F_2 \left[ \frac{a + 1, c + 1, e + 1}{b + 2, d + 1} \mid 1 \right] (a - b)ce \rightarrow (1 + b)bd + 3F_2 \left[ \frac{a + 1, c, e}{b + 1, d} \mid 1 \right].$$

Specifying the parameter $d$ to $qa$ in Theorem \[76\] and using $q$-Gauss summation formula \[6\], we establish the following relation with one free parameter less.

**Proposition 78** (Two-term contiguous relation of $3\phi_2$-series).

$$3\phi_2 \left[ \frac{a, c, e}{qa, b} \mid q; \frac{qb}{ce} \right] = 3\phi_2 \left[ \frac{qa, qc, qe}{qa^2, q^2b} \mid q; \frac{qb}{ce} \right] (a - b)(1 - c)(1 - e)qb \rightarrow (1 - b)(1 - qb)(qa - 1)ce + \left[ \frac{qb/c, qh/e}{qb, qh/ce} \mid q \right] \infty.$$  

Performing the substitutions $a \rightarrow q^a$, $b \rightarrow q^b$, $c \rightarrow q^c$, $e \rightarrow q^e$ for Proposition \[78\] and then letting $q \rightarrow 1$, we recover the following relation.

**Corollary 79** (\[4\] Proposition 41)).

$$3F_2 \left[ \frac{a, c, e}{a + 1, b} \mid 1 \right] = 3F_2 \left[ \frac{a + 1, c + 1, e + 1}{a + 2, b + 2} \mid 1 \right] (a - b)ce \rightarrow (1 + a)(1 + b)bd \times \Gamma \left[ \frac{b + 1, b - c - e + 1}{b - c + 1, b - e + 1} \right].$$

4.8. B&B.

Let Eq\[15\] stand for Eq\[15\] under the parameter replacements

$$a \rightarrow qc, \quad c \rightarrow a/q.$$  

Then for an arbitrary variable $Y_q$, the difference Eq\[15\] – $Y_q \times \text{Eq}^*\[15\]$ leads us to the relation:

$$3\phi_2 \left[ \frac{a, c, e}{b, d} \mid q; \frac{bd}{ace} \right] - Y_q \times 3\phi_2 \left[ \frac{a/q, qc, e}{b, d} \mid q; \frac{bd}{ace} \right] = (B_q - B_q^* Y_q) \times 3\phi_2 \left[ \frac{a/q, c, e}{b, d/q} \mid q; \frac{bd}{ace} \right] + (B_q - B_q^* Y_q) \times 3\phi_2 \left[ \frac{a/qc, qe}{qb, d} \mid q; \frac{bd}{qace} \right],$$

where the following notations have been used for coefficients

$$B_q^* = B_q(qc, a/q, e; b, d),$$

$$B_q^* = B_q(qc, a/q, e; b, d)$$

with $B_q$ and $B_q$ being defined in Theorem \[4\]. Solving the equation $B_q - B_q^* Y_q = 0$ associated with the variable $Y_q$, we found from equation \[15\] the following relation.
Theorem 80 (Three-term contiguous relation of $3\phi_2$-series).

\[
3\phi_2\left[\begin{array}{c|c} a, c, e & b, d, q \end{array}\right]_{q; \text{ace}} = \frac{Y_\phi \times 3\phi_2\left[\begin{array}{c|c} a/q, q, c, e & b, d, q \end{array}\right]_{q; \text{ace}} + Z_\phi \times 3\phi_2\left[\begin{array}{c|c} a, q, c, e & q, b, d \end{array}\right]_{q; \text{ace}}}{Y \times 3\phi_2\left[\begin{array}{c|c} a/q, q, c, e & b, d, q \end{array}\right]_{q; \text{ace}}},
\]

where the coefficients $Y_\phi$ and $Z_\phi$ are defined by

\[
Y_\phi = \frac{(a - d)(qc - d)}{(ae - d)(qc - d)}, \quad Z_\phi = \frac{(1 - e)(a - qe)(qd - bd)}{(1 - b)(ae - d)(qc - d)ace}.
\]

Employing the substitutions $a \to q^a$, $b \to q^b$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Theorem 80 and then letting $q \to 1$, we recover the following relation.

Corollary 81 (H Theorem 42]).

\[
3F_2\left[\begin{array}{c|c} a, c, e & b, d, q \end{array}\right]_{q, \text{ace}} = Y 3F_2\left[\begin{array}{c|c} a - 1, 1, a + 1, c + 1 & b, d, q, \phi \end{array}\right]_{q, \text{ace}} + Z 3F_2\left[\begin{array}{c|c} a, c + 1, a + 1 & b + 1, d, \phi \end{array}\right]_{q, \text{ace}},
\]

where the coefficients $Y$ and $Z$ are given by

\[
Y = \frac{(a - d)(1 + c + e - d)}{(a + e - d)(1 + c - d)}, \quad Z = \frac{e(1 + c - a)(b + d - a - c - e - 1)}{b(1 + c - d)(a + e - d)}.
\]

Specifying the parameter $b \to a$ in Theorem 80 and using $q$-Gauss summation formula (6), we obtain the following relation with one free parameter less under replacements $a \to qa$, $c \to c/q$.

Proposition 82 (Two-term contiguous relation of $3\phi_2$-series).

\[
3\phi_2\left[\begin{array}{c|c} a, c, e & qa, d, q \end{array}\right]_{\text{ace}} = 3\phi_2\left[\begin{array}{c|c} qa, c, qe & q^2a, d, q \end{array}\right]_{\text{ace}} \frac{(qa - c)(1 - e)d}{(1 - qa)(d - qa)ce} + \frac{d/c, d/e}{\Gamma(d/c, d/e)_{\infty}} (q a - d)^d (q - d)ce.
\]

Performing the substitutions $a \to q^a$, $c \to q^c$, $d \to q^d$, $e \to q^e$ for Proposition 82 and then letting $q \to 1$, we recover the following relation.

Corollary 83 (H Proposition 43]).

\[
3F_2\left[\begin{array}{c|c} a, c, e & a + 1, d, q \end{array}\right]_{q, \text{ace}} = 3F_2\left[\begin{array}{c|c} a + 1, 1, a + 1 & a + 2, d, q \end{array}\right]_{q, \text{ace}} \frac{e(c - a - 1)}{(1 + a)(1 + a - d)} + \Gamma(d/c, d/e)_{\frac{1 + a + e - d}{1 + a - d}} (1 + a + e - d) (1 + a - d).
\]

Taking $d = qae$ in Proposition 82 we get the following relation.

Proposition 84 (Two-term contiguous relation of $3\phi_2$-series).

\[
3\phi_2\left[\begin{array}{c|c} a, c, e & qa, qae \end{array}\right]_{q, \text{ace}} = 3\phi_2\left[\begin{array}{c|c} qa, c, qe & q^2a, qa, qae \end{array}\right]_{q, \text{ace}} \frac{(qa - c)}{(qa - 1)c}.
\]

Employing the substitutions $a \to q^a$, $c \to q^c$, $e \to q^e$ for Proposition 84 and then letting $q \to 1$, we recover the following relation.

Corollary 85 (H Corollary 44]).

\[
3F_2\left[\begin{array}{c|c} a, c, e & a + 1, a + e + 1 & q \end{array}\right]_{q, \text{ace}} = 3F_2\left[\begin{array}{c|c} a + 1, 1, a + e + 1 & a + 2, a + e + 1 & q \end{array}\right]_{q, \text{ace}} \frac{(1 + a + e)}{(1 + a)}.
\]

4.9. B&B.

Let Eq (28) stand for Eq (28) under the parameter replacements

\[
b \to d/q, \quad d \to qb.
\]

Then for an arbitrary variable $Y_\phi$, the difference Eq (28) - $Y_\phi \times$ Eq (28) results in the relation:

\[
3\phi_2\left[\begin{array}{c|c} a, c, e & b, d, q \end{array}\right]_{q, \text{ace}} - Y_\phi \times 3\phi_2\left[\begin{array}{c|c} a/q, c, e & b, d/q, q \end{array}\right]_{q, \text{ace}} = (B_\phi - B_\phi^* Y_\phi) \times 3\phi_2\left[\begin{array}{c|c} a/q, c, e & b, d, q \end{array}\right]_{q, \text{ace}} \frac{b/d}{\text{ace}} + (B_\phi - B_\phi^* Y_\phi) \times 3\phi_2\left[\begin{array}{c|c} a/q, c, e & q, b, d \end{array}\right]_{q, \text{ace}} \frac{b/d}{\text{ace}}.
\]
where the following notations have been used for coefficients

\[ E_q^* = B_q(a, c, e; d/q, qb), \]

\[ \mathcal{B}_q = B_q(a, c, e; d/q, qb) \]

with \( B_q \) and \( \mathcal{B}_q \) being defined in Theorem 88. Solving the equation \( B_q - E_q^* Y_q = 0 \) associated with the variable \( Y_q \), we derive from equation (13) the following relation.

**Theorem 86** (Three-term contiguous relation of \( 3\phi_2 \)-series).

\[ 3\phi_2 \left[ \frac{a, c, e}{b, d} \middle| q; \frac{bd}{ace} \right] = Y_q \times 3\phi_2 \left[ \frac{a, c, e}{qb, d/q} \middle| q; \frac{bd}{ace} \right] + Z_q \times 3\phi_2 \left[ \frac{a, c, e}{qb, d} \middle| q; \frac{bd}{ace} \right], \]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[ Y_q = \frac{(b-c)(b-e)(q-d)(qce-d)}{(b-1)(b-ce)(q-c)(q-ce-d)}, \quad Z_q = \frac{(1-c)(1-e)(d-qe)(qce-bd)}{(b-1)(b-ce)(q-c)(q-ce-d)}. \]

Performing the substitutions \( a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e \) for Theorem 88 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 87** (**Theorem 45**).

\[ 3F_2 \left[ \frac{a, c, e}{qa, b} \middle| b+1, d-1 \right] = Y_q 3F_2 \left[ \frac{a, c, e}{b+1, d} \middle| 1 \right] + Z_q 3F_2 \left[ \frac{a, c + 1, e + 1}{b+1, d} \middle| 1 \right], \]

where the coefficients \( Y_q \) and \( Z_q \) are given by

\[ Y = \frac{(b-c)(b-e)(d-1)(1+c-e-d)}{b(1+c-d)(1+e-d)(c+e-b)}, \quad Z = \frac{ce(1+b-d)(1+a+c+e-b-d)}{b(1+c-d)(1+e-d)(c+e-b)}. \]

Specifying the parameter \( d \to qa \) in Theorem 88 and using \( q \)-Gauss summation formula (6), we deduce the following relation with one free parameter less.

**Proposition 88** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[ 3\phi_2 \left[ \frac{a, c, e}{qa, b} \middle| q, \frac{qb}{ce} \right] = 3\phi_2 \left[ \frac{a, c, e}{qa, q} \middle| q; \frac{b}{ce}, \frac{(1-c)(1-e)(a-b)}{(1-b)(a-c)(a-e)} \right] \]

\[ + \frac{b/c, b/e}{b, b/ce} \left[ q; \frac{(1-a)(ce-a)}{(a-c)(a-e)} \right] \]

Employing the substitutions \( a \to q^a, b \to q^b, c \to q^c, e \to q^e \) for Proposition 88 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 89** (**Proposition 46**).

\[ 3F_2 \left[ \frac{a, c, e}{a+1, b} \middle| 1 \right] = 3F_2 \left[ \frac{a, c + 1, e + 1}{a+1, b+1} \middle| 1 \right] \frac{ce(b-a)}{b(a-c)(a-e)} \]

\[ + \Gamma \left[ \frac{b, b-c-e}{b-c, b-e} \right] \frac{a(a-e-c)}{(a-c)(a-e)} \]

Taking \( a = ce \) in Proposition 88 we attain the following relation.

**Proposition 90** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[ 3\phi_2 \left[ \frac{ce, c, e}{qce, b} \middle| q, \frac{qb}{ce} \right] = 3\phi_2 \left[ \frac{ce, c, e}{qce, q} \middle| q; \frac{b}{ce}, \frac{(b-cc)}{(b-1)ce} \right] \]

Performing the substitutions \( b \to q^b, c \to q^c, e \to q^e \) for Proposition 90 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 91** (**Corollary 47**).

\[ 3F_2 \left[ \frac{c + e, c, e}{c + e + 1, b} \middle| 1 \right] = 3F_2 \left[ \frac{c + e, c + 1, e + 1}{c + e + 1, b+1} \middle| 1 \right] \frac{(b-c-e)}{b} \]
Let Eq. \([\text{(15)}] \) stand for Eq. \([\text{(10)}] \) under the parameter replacements

\[ a \rightarrow c, \quad c \rightarrow a/q^2, \quad e \rightarrow e/q, \quad b \rightarrow b/q, \quad d \rightarrow d/q^2. \]

Then for an arbitrary variable \( Y_q \), the difference Eq. \([\text{(13)}] - Y_q \times \text{Eq. \([\text{(15)}] \)) leads us to the relation:

\[
3\phi_2 \left[ \begin{array}{ccc} a, c, e \\ b, d \end{array} \bigg| q; \frac{bd}{ace} \right] - Y_q \times 3\phi_2 \left[ \begin{array}{ccc} a/q^2, c, e/b \\ b/q, d/q^2 \end{array} \bigg| q; \frac{bd}{ace} \right] = \left( B_q - C_q^* Y_q \right) \times 4\phi_3 \left[ \begin{array}{ccc} a/q, c, e \left(1 - \frac{B_q - C_q^* Y_q}{3\phi_3 - C_q^* Y_q} \right) \\ b, d, q \left(1 - \frac{B_q - C_q^* Y_q}{3\phi_3 - C_q^* Y_q} \right) \end{array} \bigg| q; \frac{bd}{qace} \right],
\]

where the following notations have been used for coefficients

\[ C_q^* = C_q(c, a/q^2, e/q; b/q, d/q^2), \]

\[ \mathcal{C}_q = \mathcal{C}_q(c, a/q^2, e/q; b/q, d/q^2) \]

with \( C_q \) and \( \mathcal{C}_q \) being defined in Theorem \([\text{4} \text{]} \). Solving the equation \( 1 - \frac{B_q - C_q^* Y_q}{3\phi_3 - C_q^* Y_q} = d/q^2 \) associated with the variable \( Y_q \), we achieve from equation \([\text{15} \text{)} \) the following relation.

**Theorem 92** (Three-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \begin{array}{ccc} a, c, e \\ b, d \end{array} \bigg| q; \frac{bd}{ace} \right] = Y_q \times 3\phi_2 \left[ \begin{array}{ccc} a/q^2, c, e/q \\ b/q, d/q^2 \end{array} \bigg| q; \frac{bd}{ace} \right] + Z_q \times 3\phi_2 \left[ \begin{array}{ccc} a/q, c, e/b \\ d/q^2 \end{array} \bigg| q; \frac{bd}{qace} \right],
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[
Y_q = \frac{(q - b)(q - d)(q^2 - d)(bd^2 + q^2abc + q^3ace - q^3bce - qabd - qade)ace}{(q - a)(q^2c - d)(q^2c - d)(q^2c - d)(ae - dq)^2},
\]

\[
Z_q = \frac{(q - d)(q^2 - d)(bd - qade)(abde + q^2bcd + q^3ace - q^3bce - qabd - qade)ace}{(q - a)(q^2c - d)(q^2c - d)(ae - dq)^2}.
\]

Employing the substitutions \( a \rightarrow q^a, b \rightarrow q^b, c \rightarrow q^c, d \rightarrow q^d, e \rightarrow q^e \) for Theorem \([\text{92} \text{)} \) and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 93** (\([\text{4} \text{)} \) Theorem \([\text{48} \text{)} \)

\[
3F2 \left[ \begin{array}{ccc} a, c, e \\ b, d \end{array} \bigg| q; \frac{bd}{ace} \right] = Y_q \times 3F2 \left[ \begin{array}{ccc} a - 2, c, e - 1 \\ b - 1, d - 2 \end{array} \bigg| q; \frac{bd}{ace} \right] + Z_q \times 3F2 \left[ \begin{array}{ccc} a - 1, c, e \\ b, d - 2 \end{array} \bigg| q; \frac{bd}{qace} \right],
\]

where the coefficients \( Y \) and \( Z \) are given by

\[
Y = \frac{(b - 1)(d - 1)(d - 2)(2b + 2d + cd + de + 2ad - ad - ae - bd - 3a - c - e - d^2 - 1)}{(a - 1)(1 + c - d)(2 + c - d)(1 + e - d)(d - a - e)},
\]

\[
Z = \frac{(d - 1)(d - 2)(b + d - a - c - e - 1)(1 + a + c + ae + bd - ad - ce - 2b - e)}{(a - 1)(1 + c - d)(2 + c - d)(1 + e - d)(d - a - e)}.
\]

Specifying the parameter \( d \rightarrow qa \) in Theorem \([\text{92} \text{)} \) and using \( q \)-Gauss summation formula \([\text{6} \text{)} \), we establish the following relation with one free parameter less.

**Proposition 94** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \begin{array}{ccc} a, c, e \\ qa, b \end{array} \bigg| q; \frac{gb}{ce} \right] = \frac{(a - 1)(a - b)(q - b)(q - b)c^2e^2}{(a - c)(a - qc)(a - e)(q - b)eb^2},
\]

\[
+ \left[ \begin{array}{ccc} b/c, e/b \\ b, ce/b \end{array} \bigg| q^\infty \right] \frac{(a - 1)(ce - b)(a^2be + q^2abc + q^3ace - q^3bce - qa^2b)}{(a - c)(a - qc)(a - e)(q - b)b^2}.
\]

Performing the substitutions \( a \rightarrow q^a, b \rightarrow q^b, c \rightarrow q^c, e \rightarrow q^e \) for Proposition \([\text{94} \text{)} \) and then letting \( q \rightarrow 1 \), we recover the following relation.

**Corollary 95** (\([\text{4} \text{)} \) Proposition \([\text{49} \text{)} \)

\[
3F2 \left[ \begin{array}{ccc} a, c, e \\ a + 1, b \end{array} \bigg| q; \frac{gb}{ce} \right] = \frac{a(a - 1)(a - b)(b - 1)}{(a - c)(a - c - 1)(a - e)(e - 1)} + \Gamma \left[ \begin{array}{ccc} b, b - c - e \\ b - c, e \end{array} \bigg| q; \frac{gb}{ce} \right] \frac{a(c + e - b)(a^2 + ce + e + b - c - ae - ab - 1)}{(a - c)(a - c - 1)(a - e)(e - 1)}.
\]
Taking \( b = \frac{q ace(a - q)}{q ac + q^2 ace - q^2 ce - q ac} \) in Proposition 94, we found the following relation under the replacements \( a \to q^2 a, \ e \to q^e \).

**Proposition 96** (Two-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \frac{a, e}{qa, q a c e(qa - 1)} \right] = \left( \frac{q^2 a(q a - 1)}{q^2 a e + q ac - ce - q^2 a^2} \right) \times \left( \frac{q^2 a(q a - 1)}{q^2 a e + q ac - ce - q^2 a^2} \right)
\]

Employing the substitutions \( a \to q^a, \ b \to q^b, \ c \to q^c, \ e \to q^e \) for Proposition 96 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 97** (Corollary 50).

\[
3 F_2 \left[ \frac{a, e}{a + 1, a + 1} \right] = 3 F_2 \left[ \frac{a + 2, c, e + 1}{a + 3, 2 + a - e + \frac{ce}{a + 1}} \right] \times \\
(1 + a)(2 + a - e)(1 + a - e)
\]

### 4.11. B&D.

Let Eq. 120 stand for Eq. 120 under the parameter replacements

\[
a \to a/q, \ c \to q c, \ e \to q e, \ b \to q b.
\]

Then for an arbitrary variable \( Y_q \), the difference Eq. 210 - \( Y_q \times \) Eq. 120 results in the relation:

\[
3\phi_2 \left[ \frac{a, c, e}{b, d} \right] = 3\phi_2 \left[ \frac{a/q, c, e}{q b, d} \right] - Y_q \times 3\phi_2 \left[ \frac{a/q, c, e}{q b, d} \right] \times \\
\left( B_q - D_q Y_q \right)
\]

where the following notations have been used for coefficients

\[
D_q = D_q(a/q, q c, q e; q b, d),
\]

\[
D_q = D_q(a/q, q c, q e; q b, d)
\]

with \( D_q \) and \( D_q \) being defined in Theorem 5. Solving the equation \( 1 = \frac{B_q - D_q Y_q}{Y_q} = d/q^2 \) associated with the variable \( Y_q \), we obtain from equation 16 the following relation.

**Theorem 98** (Three-term contiguous relation of \( 3\phi_2 \)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, d} \right] + Y_q \times 3\phi_2 \left[ \frac{a/q, c, e}{q b, d} \right] + 3\phi_2 \left[ \frac{a/q, c, e}{q b, d} \right] = Y_q \times 3\phi_2 \left[ \frac{a/q, c, e}{q b, d} \right] + 3\phi_2 \left[ \frac{a/q, c, e}{q b, d} \right],
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[
Y_q = \frac{(a - q b)(a - d)(1 - c)(1 - e)(q^3 ace + q^2 abce + bcd - q abd - q^3 bce - q acde)q d}{(a - q)(1 - b)(q c - d)(q e - d)(q^3 ace + q^2 bcd e + bcd - q^2 bcd - q^2 bcd - q acde)},
\]

\[
Z_q = \frac{(q - a)(q^2 - d)(q acde - bcd)(q^2 ce + q acde + ad - q acde - q acde)}{(q^3 ace + q^2 bcd e + bcd - q^2 bcd - q^2 bcd - q acde)}.
\]

Performing the substitutions \( a \to q^a, \ b \to q^b, \ c \to q^c, \ d \to q^d, \ e \to q^e \) for Theorem 98 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 99** (Theorem 51).

\[
3 F_2 \left[ \frac{a, c, e}{b, d} \right] = Y_3 F_2 \left[ \frac{a - 1, c + 1, e + 1}{b + 1, d} \right] + Z_3 F_2 \left[ \frac{a - 1, c + 1, e + 1}{b + 1, d} \right],
\]

where the coefficients \( Y \) and \( Z \) are given by

\[
Y = \frac{ce(a - b - 1)(a - d)(1 + c + e - d + (d - 2)(b + d - a - c - e - 1))}{b(1 - a)(1 + c - d)(1 + e - d)(1 + c - d - a - c - e - 1)},
\]

\[
Z = \frac{(d - 1)(d - 2)(b + a + c + e - b - d)(1 + c + e + ce + ad - ac - ae - a - d)}{(1 - a)(1 + c - d)(1 + e - d)(1 + c - d - a - c - e - 1)}.
\]
Performing the substitutions $a \rightarrow qa$, $b \rightarrow q^b$, $c \rightarrow q^c$, $e \rightarrow q^e$ for Proposition 100 and then letting $q \rightarrow 1$, we recover the following relation.

**Corollary 101** [\[52\) Proposition 52]).

Theorem 102 (Three-term contiguous relation of $3\phi_2$-series).

$$3\phi_2 \left[ a, c, e \bigg| a, b \bigg| q \frac{qb}{ce} \right] = \frac{a/b}{c/e} \left[ a/q, c/q, e \bigg| q \frac{qb}{ce} \right] \frac{(a-b)(a-b)(1-e)(1-e)ce}{(a-c)(a-c)(1-e)ce - qbe - qce - qae - a^2} + \frac{b/c}{b/e} \left[ a/q, b/q, c \bigg| q \frac{qb}{ce} \right] \frac{ce(1-a)(a-b)(1-e)ce - qbe - qce - qae - a^2}{(a-c)(a-c)(1-e)ce - qbe - qce - a^2}.$$
Proposition 104 (Two-term contiguous relation of \(3\phi_2\)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \mid q: \frac{qd}{ce} = 3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \mid q: \frac{qc}{ce} = \frac{(a - d)(1 - q)(1 - c)(1 - e)d}{(1 - qa)(d - 1)(ace + cd - cde - ce - ad)}
\]

Employing the substitutions \(a \to q^a, c \to q^c, d \to q^d, e \to q^e\) for Proposition 104 and then letting \(q \to 1\), we recover the following relation.

Corollary 105 (Proposition 54).

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ a + 1, d \end{array} \right] = 3F_2 \left[ \begin{array}{c} a, c + 1, e + 1 \\ a + 2, d + 1 \end{array} \right] \mid \frac{ce(d - a)}{d(1 + a)(ad + ce - ac - ae)} \]

\[
+ \Gamma \left[ \begin{array}{c} d, d - c - e + 1 \\ d - c, d - e \end{array} \right] \mid \frac{a}{(ad + ce - ac - ae)}.
\]

4.13. B&D.

Let Eq. (13) stand for Eq. (14) under the parameter replacements

\(a \to c, \quad c \to a, \quad e \to qe, \quad b \to qb\).

Then for an arbitrary variable \(Y_q\), the difference Eq. (14) - \(Y_q \times Eq. (14)\) results in the relation:

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] = \frac{bd}{ace} - Y_q \times 3\phi_2 \left[ \begin{array}{c} a, c, e \\ qb, d \end{array} \right] \mid q: \frac{bd}{ace},
\]

where the following notations have been used for coefficients

\(\mathcal{D}_q^* = \mathcal{D}_q(c, a, qe; qb, d), \quad \mathcal{D}_q = \mathcal{D}_q(c, a, qe; qb, d)\)

with \(\mathcal{D}_q\) and \(\mathcal{D}_q^*\) being defined in Theorem 55. Solving the equation \(\mathcal{B}_q - \mathcal{D}_q^* Y_q = 0\) associated with the variable \(Y_q\), we attain from equation (15) the following relation.

Theorem 106 (Three-term contiguous relation of \(3\phi_2\)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] \mid q: \frac{bd}{ace} = Y_q \times 3\phi_2 \left[ \begin{array}{c} a, c, e \\ qb, d \end{array} \right] \mid q: \frac{bd}{ace} + Z_q \times 3\phi_2 \left[ \begin{array}{c} a, c, e \\ qb, d \end{array} \right] \mid q: \frac{bd}{ace},
\]

where the coefficients \(Y_q\) and \(Z_q\) are defined by

\[
Y_q = \frac{(b - c)(qce - d)}{(b - 1)(qce - dc)}, \quad Z_q = \frac{(1 - c)(qace - bd)}{(1 - b)(qce - dc)}.
\]

Performing the substitutions \(a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e\) for Theorem 106 and then letting \(q \to 1\), we recover the following relation.

Corollary 107 (Theorem 55).

\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] \mid 1 = Y_3F_2 \left[ \begin{array}{c} a, c, e + 1 \\ b + 1, d \end{array} \right] \mid 1 + Z_3F_2 \left[ \begin{array}{c} a, c + 1, e + 1 \\ b + 1, d \end{array} \right] \mid 1,
\]

where the coefficients \(Y\) and \(Z\) are given by

\[
Y = \frac{(b - c)(1 + c + e - d)}{b(1 + e - d)}, \quad Z = \frac{c(1 + a + c + e - b - d)}{b(1 + e - d)}.
\]

Specifying the parameter \(b \to a\) in Theorem 106 and using \(q\)-Gauss summation formula (46), we achieve the following relation with one free parameter less under the replacement \(e \to e/q\).

Proposition 108 (Two-term contiguous relation of \(3\phi_2\)-series).

\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \mid q: \frac{qd}{ce} = 3\phi_2 \left[ \begin{array}{c} a, c, e \\ qa, d \end{array} \right] \mid q: \frac{d}{ce} \mid \frac{1 - c}{a - c} + \frac{d/c, d/e}{d, d/ce} \mid q: \frac{(a - 1)}{(a - c)}.
\]
Employing the substitutions \( a \to q^a, c \to q^c, d \to q^d, e \to q^e \) for Proposition 112 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 109** (Proposition 56).

\[
3 F_2 \left[ \begin{array}{c|c|c} a, c, e \\ \hline a + 1, d \\ \hline b, d \\ \end{array} \right] = 3 F_2 \left[ \begin{array}{c|c|c} a, c + 1, e \\ \hline a + 1, d \\ \hline b, d \\ \end{array} \right] + \frac{c}{(c - a)} + (d, d - c - e) \left( \frac{a}{a - c} \right).
\]

4.14. C&cC.

Let Eq. 9(a) stand for Eq. 8(a) under the parameter replacements

\( a \to q, c \to a/q \).

Then for an arbitrary variable \( Y_q \), the difference Eq. 3(a) \(- Y_q \times \) Eq. 9(a) leads us to the relation:

\[
3 \phi_2 \left[ \begin{array}{c|c|c} a, c, e \\ \hline b, d \\ \hline q; \frac{bd}{ace} \end{array} \right] - Y_q \times 3 \phi_2 \left[ \begin{array}{c|c|c} a/q, q, c \\ \hline b, d \\ \hline q; \frac{bd}{ace} \end{array} \right] = \left( C_q - C_q^* Y_q \right) \times 3 \phi_2 \left[ \begin{array}{c|c|c} a, c, e \\ \hline b, d \\ \hline q; \frac{bd}{ace} \end{array} \right] \]

\[
+ \left( C_q - C_q^* Y_q \right) \times 3 \phi_2 \left[ \begin{array}{c|c|c} qa, q^2c, q^2e \\ \hline q^2b, q^2d \\ \hline q; \frac{bd}{ace} \end{array} \right],
\]

(19)

where the following notations have been used for coefficients

\[ C_q^* = C_q(q, a/q, c/e, b/d), \]

\[ C_q^* = C_q(qc, a/q, c/e, b/d), \]

with \( C_q \) and \( C_q^* \) being defined in Theorem 3. Solving the equation \( C_q - C_q^* Y_q = 0 \) associated with the variable \( Y_q \), we establish from equation (19) the following relation.

**Theorem 110** (Three-term contiguous relation of \( 3 \phi_2 \)-series).

\[
3 \phi_2 \left[ \begin{array}{c|c|c} a, c, e \\ \hline b, d \\ \hline q; \frac{bd}{ace} \end{array} \right] = Y_q \times 3 \phi_2 \left[ \begin{array}{c|c|c} a/q, q, c \\ \hline b, d \\ \hline q; \frac{bd}{ace} \end{array} \right] + Z_q \times 3 \phi_2 \left[ \begin{array}{c|c|c} qa, q^2c, q^2e \\ \hline q^2b, q^2d \\ \hline q; \frac{bd}{ace} \end{array} \right],
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[
Y_q = \frac{(qace + qacde + bd - abd - qace - qbed)}{(qace + qacde + bd + abd - qace - qbed)}
\]

\[
Z_q = \frac{(1 - a)(a - qc)(1 - q)(1 - qe)(1 - e)(qace - bd)(bd)^2}{(1 - b)(1 - d)(1 - qb)(1 - qd)(qace + qacde + bd + abd - qace - qbed)(qace)^2}.
\]

Performing the substitutions \( a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e \) for Theorem 110 and then letting \( q \to 1 \), we recover the following relation.

**Corollary 111** (Proposition 57). \[ 3 F_2 \left[ \begin{array}{c|c|c} a, c, e \\ \hline b, d \\ \hline 1 \end{array} \right] = Y \times 3 F_2 \left[ \begin{array}{c|c|c} a - 1, c + 1, e \\ \hline b, d \\ \hline 1 \end{array} \right] + Z \times 3 F_2 \left[ \begin{array}{c|c|c} a + 1, c + 2, e + 2 \\ \hline b + 2, d + 2 \\ \hline 1 \end{array} \right], \]

where the coefficients \( Y \) and \( Z \) are given by

\[
Y = \frac{bd - a(1 + c + e)}{bd - (1 + c)(a + e)}, \quad Z = \frac{ae(1 + c)(1 + e)(1 + c - a)(b + d - a - c - e - 1)}{bd(1 + b)(1 + d)(a + e + ac + ce - bd)}.
\]

Specifying the parameter \( d \to qc \) in Theorem 111 and using \( q \)-Gauss summation formula 8, we found the following relation with one free parameter less.

**Proposition 112** (Two-term contiguous relation of \( 3 \phi_2 \)-series).

\[
3 \phi_2 \left[ \begin{array}{c|c|c} a, c, e \\ \hline q, b \\ \hline q; \frac{qe}{ae} \end{array} \right] = 3 \phi_2 \left[ \begin{array}{c|c|c} qa, q^2c, q^2e \\ \hline q^2b, q^2d \\ \hline q; \frac{qe}{ae} \end{array} \right] - \frac{b/a}{b/a} \left( \frac{ae(1 - qe) - b(1 + ae - a - qce)}{e(a - b)(1 - qe)} \right).
\]

Employing the substitutions \( a \to q^a, b \to q^b, c \to q^c, e \to q^e \) for Proposition 112 and then letting \( q \to 1 \), we recover the following relation.
Corollary 113 ([I Proposition 58]).

\[
3F2 \left[ \frac{a, c, e}{c + 1, b} \bigg| 1 \right] = 3F2 \left[ \frac{a + 1, c + 2, e + 2}{c + 3, b + 2} \bigg| 1 \right] \frac{ae(1 + e)(a - c - 1)}{b(1 + b)(1 + c)(2 + c)}
+ \Gamma \left[ \frac{b, b - a - e}{b - a, b - e} \right] \frac{(a - b)(1 + c) + ae}{(a - b)(1 + c)}
\]

Taking \( b = \frac{ae(1 - qc)}{1 + ae - a - qce} \) in Proposition 112 we obtain the following relation.

Proposition 114 (Two-term contiguous relation of \(3\phi_2\)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{qc, \frac{ae(1 - qc)}{1 + ae - a - qce}} \bigg| q; \frac{q(1 - qc)}{1 + ae - a - qce} \right] = 3\phi_2 \left[ \frac{qa, q^2c, q^2e}{q^3c, \frac{q^2ae(1 - qc)}{1 + ae - a - qce}} \bigg| q; \frac{1 - qc}{1 + ae - a - qce} \right]
\times \frac{(1 - e)(1 - qe)(1 - qc)(qc - a)}{(1 - q^2e)(1 - qce)(1 + ae + q^2ace - qae - qce - a)}
\]

Performing the substitutions \( a \to q^a, c \to q^c, e \to q^e \) for Proposition 114 and then letting \( q \to 1 \), we recover the following relation.

Corollary 115 ([I Proposition 59]).

\[
3F2 \left[ \frac{a, c, e}{c + 1, a + c + e}{c + 1}_{c+1} \bigg| 1 \right] = 3F2 \left[ \frac{a + 1, c + 2, e + 2}{c + 3, a + c + 2}{c + 3}_{c+3} \bigg| 1 \right]
\times \frac{(a - c - 1)(1 + c)(1 + e)e}{(2 + c)(1 + c + e)(1 + c + a + ac + ae)}
\]

4.15. D&D.

Let Eq [13] stand for Eq [13] under the parameter replacements

\[ a \to c/q, \quad c \to qa. \]

Then for an arbitrary variable \( Y_q \), the difference Eq [13] \( - Y_q \times Eq^{*} [13] \) results in the relation:

\[
3\phi_2 \left[ \frac{a, c, e}{b, d} ; \frac{bd}{ace} \bigg| q; \frac{bd}{ace} \right] - Y_q \times 3\phi_2 \left[ \frac{a, c/q, e}{b, d} ; \frac{bd}{ace} \bigg| q; \frac{bd}{ace} \right]
\]

\[
= (D_q - D^*_q Y_q) \times 3\phi_2 \left[ \frac{a, c/q, e/q}{b/q, d/q} ; q; \frac{bd}{ace} \bigg| q; \frac{bd}{ace} \right]

+ (D_q - D^*_q Y_q) \times 3\phi_2 \left[ \frac{qa, c}{b, d} ; q; \frac{bd}{qace} \bigg| q; \frac{bd}{qace} \right],
\]

where the following notations have been used for coefficients

\[ D^*_q = D_q(c/q, qa, e; q, d), \]

\[ D^*_q = D_q(c/q, qa, e; b, d) \]

with \( D_q \) and \( D_q \) being defined in Theorem 13. Solving the equation \( D_q - D^*_q Y_q = 0 \) associated with the variable \( Y_q \), we get from equation (20) the following relation.

Theorem 116 (Three-term contiguous relation of \(3\phi_2\)-series).

\[
3\phi_2 \left[ \frac{a, c, e}{b, d} ; \frac{bd}{ace} \bigg| q; \frac{bd}{ace} \right] = Y_q \times 3\phi_2 \left[ \frac{qa, c/q, e}{b, d} ; \frac{bd}{ace} \bigg| q; \frac{bd}{ace} \right] + Z_q \times 3\phi_2 \left[ \frac{a, c/q, e/q}{b/q, d/q} ; q; \frac{bd}{ace} \bigg| q; \frac{bd}{ace} \right],
\]

where the coefficients \( Y_q \) and \( Z_q \) are defined by

\[
Y_q = \frac{(a - 1)(c - b)(c - d)aq^2}{(c - q)(qa - b)(qa - d)c}, \quad Z_q = \frac{(q - b)(q - d)(qa - c)a}{(q - c)(qa - b)(qa - d)c}.
\]

Employing the substitutions \( a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e \) for Theorem 110 and then letting \( q \to 1 \), we recover the following relation.
Corollary 117 [1 Theorem 60]).
\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] | 1 \\
= Y \ 3F_2 \left[ \begin{array}{c} a + 1, c - 1, e \\ b, d \end{array} \right] | 1 \\
+ Z \ 3F_2 \left[ \begin{array}{c} a, c - 1, e \\ b - 1, d - 1 \end{array} \right] | 1 ,
\]
where the coefficients Y and Z are given by
\[
Y = \frac{(b - c)(d - e)}{(1 - c)(1 + a - b)(1 + a - d)}, \quad Z = \frac{(1 - b)(1 - d)(1 + a - c)}{(1 - c)(1 + a - b)(1 + a - d)}.
\]

Instead, solving the equation \(D_q \phi^2 - D_q^2 Y_q = 0\) associated with the variable \(Y_q\), we derive from equation (20) the following relation.

Theorem 118 (Three-term contiguous relation of \(3 \phi_2\)-series).
\[
3\phi_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] | q; \frac{bd}{ace} \\
= Y_q \ 3\phi_2 \left[ \begin{array}{c} qa/b, c/q, e \\ b, d \end{array} \right] | q; \frac{bd}{ace} \\
+ Z_q \ 3\phi_2 \left[ \begin{array}{c} qa, c, e \\ b, d \end{array} \right] | q; \frac{bd}{ace} ,
\]
where the coefficients \(Y_q\) and \(Z_q\) are defined by
\[
Y_q = \frac{(c - b)(d - e)aq}{(qa - b)(qa - d)c}, \quad Z_q = \frac{(qa - c)(qace - bd)}{(qa - b)(qa - d)ce}.
\]

Performing the substitutions \(a \to q^a, b \to q^b, c \to q^c, d \to q^d, e \to q^e\) for Theorem 118 and then letting \(q \to 1\), we recover the following relation.

Corollary 119 [2 Theorem 61]).
\[
3F_2 \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right] | 1 \\
= Y \ 3F_2 \left[ \begin{array}{c} a + 1, c - 1, e \\ b, d \end{array} \right] | 1 \\
+ Z \ 3F_2 \left[ \begin{array}{c} a, c - 1, e \\ b - 1, d - 1 \end{array} \right] | 1 ,
\]
where the coefficients Y and Z are given by
\[
Y = \frac{(c - b)(d - e)}{(1 + a - b)(1 + a - d)}, \quad Z = \frac{(1 + a - c)(1 + a + c - b - d)}{(1 + a - b)(1 + a - d)}.
\]

References

[1] W.N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, Cambridge, 1935.
[2] R.G. Buschman, Less simple contiguous function relations for hypergeometric functions, J. Comput. Appl. Math. 107 (1999), 127-131.
[3] J. Choi, A.K. Rathie, H.M. Srivastava, A generalization of a formula due to Kummer, Integral Transforms Spec. Funct. 22 (2011), 851-859.
[4] W. Chu, X. Wang, Abel’s Method on summation by parts and hypergeometric contiguous relations, Integral Transforms Spec. Funct. 18 (2007) 771-807.
[5] G. Gasper, M. Rahman, Basic Hypergeometric Series (2nd edition), Cambridge University Press, Cambridge, 2004.
[6] D.P. Gupta, M.E.H. Ismail, D.R. Masson, Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. II. Associated big \(q\)-Jacobi polynomials, J. Math. Anal. Appl. 171 (1992), 477-497.
[7] D.P. Gupta, D.R. Masson, Contiguous relations, continued fractions and orthogonality: An \(8\phi_7\) model, J. Comput. Appl. Math. 65 (1995), 157-164.
[8] D.P. Gupta, M.E.H. Ismail, D.R. Masson, Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual \(q\)-Hahn polynomials, J. Comput. Appl. Math. 68 (1996), 115-149.
[9] A.K. Ibrahim, M.A. Rakha, Contiguous relations and their computations for \(2F_1\) hypergeometric series, Comput. Math. Appl. 56 (2008), 1918-1926.
[10] M.E.H. Ismail, C.A. Libis, Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. II. J. Math. Anal. Appl. 141 (1989), 349-372.
[11] Y.S. Kim, A.K. Rathie, J. Choi, Three-term contiguous functional relations for basic hypergeometric series \(2\phi_1\), Commun. Korean Math. Soc. 20 (2005), 395-403.
[12] C. Krattenthaler, T. Rivoal, How can we escape Thomae’s relations, J. Math. Soc. Japan, 58 (2006) 183-210.
[13] M. Petkovšek, H. Wilf, D. Zeilberger, \(A=B\), A. K. Peters, Wellesley, 1996.
[14] S.D. Purohit, Some recurrence relations for the generalized basic hypergeometric functions, Bull. Math. Anal. Appl. 1 (2009), 22-29.
[15] M.A. Rakha, A.K. Ibrahim, On the contiguous relations of hypergeometric series, J. Comput. Appl. Math. 192 (2006), 396-410.
M.A. Rakha, A.K. Ibrahim, A.K. Rathie, *On the computations of contiguous relations for $2\!F_1$ hypergeometric series*, Commun. Korean Math. Soc. 24 (2009), 291-302.

M.A. Rakha, A.K. Rathie, P. Chopra, *On some new contiguous relations for the Gauss hypergeometric function with applications*, Comput. Math. Appl. 61 (2011), 620-629.

R.F. Swarttouw, *The contiguous function relations for the basic hypergeometric series*, J. Math. Anal. Appl. 149 (1990), 151-159.

R. Viduñas, *A generalization of Kummer’s identity*, Rocky Mountain J. Math. 32 (2002), 919-936.

R. Viduñas, *Contiguous relations of hypergeometric series*, J. Comput. Appl. Math. 153 (2003), 507-519.