Environmental bias and elastic curves on surfaces

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Abstract
The behavior of an elastic curve bound to a surface will reflect the geometry of its environment. This may occur in an obvious way: the curve may deform freely along directions tangent to the surface, but not along the surface normal. However, even if the energy itself is symmetric in the curve’s geodesic and normal curvatures, which control these modes, very distinct roles are played by the two. If the elastic curve binds preferentially on one side, or is itself assembled on the surface, not only would one expect the bending moduli associated with the two modes to differ, binding along specific directions, reflected in spontaneous values of these curvatures, may be favored. The shape equations describing the equilibrium states of a surface curve described by an elastic energy accommodating environmental factors will be identified by adapting the method of Lagrange multipliers to the Darboux frame associated with the curve. The forces transmitted to the surface along the surface normal will be determined. Features associated with a number of different energies, both of physical relevance and of mathematical interest, are described. The conservation laws associated with trajectories on surface geometries exhibiting continuous symmetries are also examined.

Keywords: elastic curves on surfaces, variational principles, symmetries

(Some figures may appear in colour only in the online journal)
1. Introduction

Linear structures are frequently encountered on surfaces. A familiar example is provided by the interface separating two phases of a fluid membrane [1], described by an energy proportional to its length [2]. Resistance to bending becomes relevant in the modeling of a semi-flexible linear polymer bound to a surface along its length or a linear protein complex that self-assembles on the surface. Even if we confine our attention to its purely geometrical degrees of freedom—typically those that turn out to be the most relevant when we zoom out to mesoscopic scales [3, 4]—the identification of the appropriate energy, nevermind attending to the dynamical behavior it predicts, is not obvious. For the environment will play a role and there is no single—physically relevant—analogue of the Euler elastic bending energy of a space curve, quadratic in the three-dimensional Frenet curvature $\kappa$ along the curve [5, 6]

$$H_B = \frac{1}{2} \int ds \kappa^2,$$

where $s$ is arc-length. The mathematically obvious generalization, quadratic in the geodesic curvature $\kappa_g$—the analogue of $\kappa$ for a surface curve

$$H_G = \frac{1}{2} \int ds \kappa_g^2,$$

while interesting in its own right, is sensitive only to the metrical properties of the surface: it quantifies the bending energy of deformations tangential to the surface but fails to respond to how the surface itself bends in space; as such it also fails to capture the physics.

The Euler elastic energy (1) of a surface bound curve does better in this respect, capturing the environmental bias associated with contact that breaks the symmetry between the two bending modes implicit in the energy. To see this, recall that the squared Frenet curvature can be decomposed into a sum of squares $\kappa^2 = \kappa_n^2 + \kappa_g^2$ [7]: the normal curvature $\kappa_n$, registers how the surface itself bends along the tangent to the curve, thus picking up the curvature deficit not captured by $\kappa_g$. Unlike $\kappa_g$, it is also fixed at any point once the tangent direction is established. In fact, its maximum and minimum are set by the surface principal curvatures; as such, and unlike geodesic curvature, it is bounded along a curve if the surface is smooth.

On the appropriate length scale, the constrained three-dimensional bending energy (1) does provide a reasonable physical description of a thin wire—of circular cross-section—bounding a soap film, or the elastic properties of a semi-flexible polymer obliged to negotiate an obstacle, a consequence of its confinement within the surface itself or its non-specific binding to it [8–12]. Intrinsic and extrinsic bending energies, characterized by $\kappa_n^2$ and $\kappa_g^2$ respectively, are weighed equally. Despite the striking difference in the natures of geodesic and normal curvatures they are treated symmetrically.

If a Euler elastic curve is often a poor caricature of a one-dimensional elastic object freely inhabiting three-dimensional space (DNA say), it fares worse as a description of a one-dimensional object inhabiting a surface. For implicit in the treatment of the two curvatures that places them on an equal footing is the assumption that the one-dimensional object possesses not only an existence but also material properties (albeit only its rigidity) that are independent of the surface constraining its movement. But in the case of protein complexes, assembling into extended one-dimensional structures on a fluid membrane or condensing along its boundary, the surface is clearly not a passive substrate. The interaction of the structure with the membrane will be reflected in the elastic properties; at a minimum, its rigidities tangential and normal the surface will differ, which establishes a distinction between how they bend intrinsically and extrinsically on the surface [13].
If the substrate is a round sphere, all directions are equivalent and the normal curvature is constant. In this case, the change amounts to a simple recalibration of the energy. In general, however, there will be physically observable consequences. Consider for example an elastic curve on a minimal surface. Locally, such surfaces minimize area; under appropriate conditions, they also minimize the surface bending energy of a symmetric fluid membrane; as such, they feature frequently in the morphology of biological membranes (see, for example, [14]). Because its mean curvature vanishes, the surface assumes a symmetric saddle shape almost everywhere. A large normal bending modulus would tend to favor low values of \( \kappa_n \) so that the equilibrium tilts to align along the ‘flat’ asymptotic directions, where \( \kappa_n = 0 \); a large geodesic bending modulus, on the other hand, would promote alignment along geodesics.

Any surface curve carries a Darboux frame. If one of its two normals is chosen to lie along the surface normal, its second co-normal is tangential. The normal and geodesic curvatures measure how fast the tangent vector rotates into the surface normal and into the co-normal respectively as the curve is followed. This frame also introduces a geometrical notion of twist: the geodesic torsion \( \tau_g \) (the rate of rotation of the surface normal into the co-normal); as such, it captures the deviation of the tangent vector from the surface principal directions. On a surface assembled curve, it may be appropriate to admit a dependence on this torsion. To justify this claim note that the normal curvature and the geodesic torsion are not independent. On a minimal surface, their squares sum to give the manifestly negative Gaussian curvature \( \kappa \) by

\[
\kappa = -\left( \kappa_n^2 + \tau_g^2 \right). 
\]

Like \( \kappa_n \), \( \tau_g \) is bounded by the surface curvature.

On an isotropic fluid membrane there is a single natural spontaneous curvature [1]; likewise elastic planar or space curves can have a preferred curvature [15–18]. However, the interaction with the substrate may also involve the introduction of effective spontaneous curvatures which are different along tangential and normal directions. For instance, rod-like proteins may assume a nematic order [19–22]. If these proteins condense—for one reason or another—into one-dimensional extended structures, they may possess not only distinct bending moduli, as in a rod [23–25], but also an anisotropic spontaneous curvature if the sub-units are curved along their length and bind preferentially along one side. A spontaneous geodesic torsion would promote deviation from the principal directions and, as a consequence, the formation of spiral structures on surfaces of negative Gaussian curvature.

A natural generalization of the bending energy (1) for a surface bound curve is

\[
H = \frac{1}{2} \int ds \left( (\kappa_n - C_n)^2 + \mu (\kappa_n - C_n)^2 + \nu (\tau_g - C_0)^2 \right),
\]

involving relative normal and torsional rigidity moduli \( \mu \) and \( \nu \), as well as constant spontaneous geodesic and normal curvatures, \( C_n \) and \( C_0 \), and torsion, \( C_0 \). This, of course, is not the most general quadratic since it does not accommodate off-diagonal elements.

The Euler–Lagrange (EL) equation describing the equilibrium states of a curve with an energy of the form (4) are presented in section 2. In this outing, the background geometry will

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4 But, even if the substrate is metrically spherical, its geodesics may be non-trivial and its normal curvature need not be constant.

5 These are the directions on the surface along which the normal curvature is a maximum or minimum.

6 The Gaussian curvature is given by the product of the two surface principal curvatures \( C_1 \) and \( C_2 \), \( \kappa_G = C_1 C_2 \), so is a surface property; it does not depend on the curve followed.

7 Unlike a space curve, along which a spontaneous Frenet curvature is ill-defined, there is no such ambiguity along surface-bound curves due to its inheritance of the Darboux frame adapted to its surface environment.
be assumed fixed, but is otherwise arbitrary. A direct approach would be to adapt the method
developed by Nickerson and Manning in the context of Euler elastic space curves
negotiating an obstacle [8, 9]. However, we will examine the problem by extending an approach,
introduced by two of the authors to examine confined Euler elastic curves. In this approach
track is kept of the chain of connections between the curve, the surface, and the Euclidean
background. This permits the breakdown of Euclidean symmetry to be correlated with the
source of tension in the curve: (minus) the normal forces transmitted to the surface [12]. For
an energy of the form (4), it would appear to be meaningless to talk about the breaking of
Euclidean symmetry, because such an energy cannot be defined without reference to the
surface environment. Yet it is still useful to think in terms of the tension along the curve. In
distinction to bound Euler elastica, the normal force transmitted to the confining surface is not
the only source of tension in the curve. In general, it will also be subjected to geometrical
tangential forces along its length. Significantly, this new tangential source of tension vanishes
only when the functional dependence of the energy can be cast in terms of the Frenet
curvature.

We will identify the boundary conditions that are physically relevant on open curves. We
follow this with an examination of several cases of special interest, among them energies that
have been been treated in the recent literature. Various possibilities of mathematical interest
are also suggested by our framework, among them generalizations of the elastic energies on
Riemannian manifolds examined by Langer and Singer [26] depending only on the surface
intrinsic geometry. One such is to take \( \mu = 0 \) and \( \nu = 0 \) in equation (4), and add a potential
depending on the local Gaussian curvature. Whereas this intrinsic dependence is reflected in
the EL equations, the forces transmitted to the surface do depend explicitly on the embedding
in space. We also examine a natural generalization of the Euler energy, with \( \mu = 1, \nu = 1 \) and
vanishing spontaneous curvatures in equation (4), so that it is symmetric not only with respect
to the interchange of the curvatures but also the interchange of curvature with torsion.

We next examine background geometries preserving some subgroup of the Euclidean
group. There will now be a conserved Noether current associated with each unbroken
symmetry, which provides a first integral of the shape equation. In particular, we examine
curves on axially-symmetric surfaces where the conserved current is a torque. We also
examine elastic curves on a helicoid, exploiting the glide rotational symmetry of this geo-
metry to identify the corresponding conserved quantity.

Various relevant identities are collected in a set of appendices. Here the less indulgent
reader will also find a Hamiltonian analysis adapted to the symmetry, which is also potentially
useful for numerical solution of the EL equations.

2. Equilibrium states

Let the surface \( \Sigma \) be described parametrically, \( \Sigma: (u^1, u^2) \mapsto X(u^1, u^2) \), and let \( e_a = \partial_a X \),
\( a = 1, 2 \) be the two tangent vectors to this surface adapted to the parametrization, and \( n \) the
unit vector normal to the surface (pointing outwards if it is closed). The surface curve \( \Gamma \) is
parametrized by arc-length, \( \Gamma: s \mapsto (U^1(s), U^2(s)) \), which can be identified as a space curve
under composition of maps, \( s \mapsto Y(s) = X(U^1(s), U^2(s)) \).

Let \( T = Y' \) be the unit tangent vector to \( \Gamma \) and \( N = n(U(s)) \) the unit vector normal to the
surface along the curve\(^8\). The associated Darboux frame is then defined by \( \{T, L, N\} \) (see figure 1). The structure equations (analogues of the Frenet–Serret

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\(^8\) Here, and elsewhere, prime represents derivation with respect to \( s \).
The geodesic curvature, $\kappa_g$, involves acceleration and thus two derivatives along the curve, whereas the normal curvature, $\kappa_n$, and the geodesic torsion, $\tau_g$, depend only on the tangent vector, and thus involve a single derivative. The geodesic torsion is significantly different from its Frenet counterpart, which involves three derivatives, in this respect. The two are related as follows: $\tau = \tau_g - \tau^\theta$, where $\theta$ is the angle rotating one frame into the other; the missing order in derivatives is captured by a derivative of $\theta$.

More explicitly, the geodesic curvature can be cast in the intrinsic form, $\kappa_g = \nabla_t \cdot L = l^a b V_b l_a$, where $V_a$ is the covariant derivative compatible with the metric induced on the surface, $g_{ab} = e_a \cdot e_b$; $\kappa_g$ and $\tau_g$ involve the surface extrinsic curvature $K_{ab} = e_a \cdot \partial_b n$ through the identifications.

We now identify the equations describing the equilibrium states of a surface curve with an energy given by

$$H = \int ds \left( \mathcal{H}(\kappa_g, \kappa_n, \tau_g) + V \right),$$

where, for the moment, $\mathcal{H}$ is an arbitrary function of its three arguments, and $V$ is some potential that depends on the position upon the surface but not on the tangent vectors. These dependence could be through the curvatures of the surface. The important point is that the Hamiltonian (7) depends only on the geometric degrees of freedom associated with the curve on the surface. To begin with we will not make any assumptions concerning the symmetry of the surface. Because we do not need to.

9 The Frenet frame $\{T, N, B\}$ is given in terms of the Darboux frame by $N = \cos \theta L - \sin \theta N$ and $B = \sin \theta L + \cos \theta N$.

10 We can expand $T$ with respect to the basis of vectors adapted to the surface parametrization $\{e_a, a = 1, 2\}$, as follows $T = t^a e_a$, where $t^a = U^a$. Similarly, one can expand $L = l^a e_a$.

11 Our description of the variational principle is also valid if $H$ depends on derivatives of the curvatures and torsion.
2.1. The EL equation and the normal force: results

To examine the response of the energy (7) to a deformation of the curve we extend the simpler framework developed by two of the authors to study the confinement of elastic curves described by the energy (1), quadratic in the Frenet curvature [12]. The Frenet frame is, of course, a poor choice if \( \kappa_g \) and \( \kappa_n \) are treated asymmetrically, never mind contemplating an explicit dependence on \( \tau_g \); nor is it sufficient to simply tweak the framework slightly to accommodate a Darboux frame. The interested reader will find the complete details of the derivation in section 2.2.

The tension along the curve is given by the vector \( \mathbf{F} = F^T \mathbf{T} + F^L \mathbf{L} + F^N \mathbf{N} \), where the components are given by

\[
F^T = \kappa_g \mathbf{H}_g + \kappa_n \mathbf{H}_n + \tau_g \mathbf{T}_g - \mathbf{H} - V - c, \\
F^L = \mathbf{H}_g' + \tau_g \mathbf{H}_n - \kappa_n \mathbf{T}_g, \\
F^N = -\mathbf{H}_n' - \frac{\tau_g}{\kappa_n} \left( \mathbf{T}_g - \kappa_g \mathbf{H}_n \right) - \kappa_g \mathbf{T}_g + \frac{V'}{\kappa_n};
\]

the parameter \( c \) is a constant associated with the fixed length of the surface curve, and\(^12\)

\[
\mathbf{H}_g = \frac{\partial \mathbf{H}}{\partial \kappa_g}, \quad \mathbf{H}_n = \frac{\partial \mathbf{H}}{\partial \kappa_n}, \quad \mathbf{T}_g = \frac{\partial \mathbf{H}}{\partial \tau_g}. \tag{9}
\]

In equilibrium the tension is not conserved, \( \mathbf{F}' = -\lambda \), where \( \lambda \) is an external force normal to the curve due to its shaping by the surface geometry, \( \lambda = \lambda^L \mathbf{L} + \lambda^N \mathbf{N} \). Its component tangent to the surface given by

\[
\lambda^L = \frac{K_G}{\kappa_n} \left( \mathbf{T}_g' + \kappa_n \mathbf{H}_n - \kappa_g \mathbf{H}_n \right) + \frac{\tau_g V'}{\kappa_n} + \mathbf{V}_L V, \tag{10}
\]

where \( \mathbf{V}_L V = l^2 \partial_a V \) is the directional derivative of \( V \) along \( \mathbf{L} \). This tangential source of tension vanishes whenever the energy depends only on the Frenet curvature and in particular, for the Euler elastic bending energy (1). The EL equation, describing equilibrium states, is given by \( \varepsilon_L := \mathbf{L} \cdot \mathbf{F}' + \lambda^L = 0 \). Collecting terms, one can express

\[
\varepsilon_L = \mathbf{H}_g' + \left( \frac{\tau_g \mathbf{H}_n'}{\kappa_n} \right) + (K - 2\kappa_n) \left( \mathbf{T}_g' - \kappa_g \mathbf{H}_n \right) + \left( K_G + \kappa_n^2 \right) \mathbf{H}_n' - \left( \kappa_n' - 2\kappa_g \tau_g \right) \mathbf{T}_g' - \kappa_g (\mathbf{H} + V + c) + \mathbf{V}_L V = 0. \tag{11}
\]

Here \( K \) is twice the local mean curvature\(^13\). The magnitude of the normal force is given by \( \lambda^N = -\mathbf{N} \cdot \mathbf{F}' \), or explicitly

\[
\lambda^N = \mathbf{H}_n' + \left( \frac{\tau_g \mathbf{H}_n'}{\kappa_n} \right) + \kappa_g \mathbf{T}_g' - \frac{V'}{\kappa_n} - \tau_g \mathbf{H}_n' + \left( \kappa_n^2 - \tau_g^2 \right) \mathbf{H}_n + 2\kappa_n \tau_g \mathbf{T}_g - \kappa_n (\mathbf{H} + V + c). \tag{12}
\]

It also is completely determined by the geometry.

\(^12\) Partial derivatives are replaced by their functional counterparts if a dependence on derivatives with respect to arc-length is contemplated.

\(^13\) It is given by the sum of the two principal curvatures, \( K = C_1 + C_2 \).
2.2. Derivation

Three Lagrange multipliers (given as the components of a vector $\mathbf{F}$) are introduced to identify the tangent vector $\mathbf{T}$ with $\mathbf{Y}$, another six ($\lambda_{IJ}$) to identify $\mathbf{T}$ with another orthonormal frame $\mathbf{F}_{L}$, and three further multipliers $\lambda, \kappa_{g}, \kappa_{n}$ and $\tau_{g}$, to define the curvatures and torsion in terms of this frame. A vector-valued Lagrange multiplier $\mathbf{\lambda}$ identifies the space curve with a curve on the surface, $\mathbf{Y}(s) = \mathbf{X}(U^{1}(s), U^{2}(s))$, just as it did in [12] for a constrained Euler elastic curve; thus far this closely follows the derivation of the shape equation there.

What distinguishes the variational principle adapted to the Darboux frame from its Frenet counterpart is the need to introduce another vector-valued multiplier, $\Lambda$, to identify the frame with the surface normal $\mathbf{n}(U(s))$. This is the constraint that identifies the frame as the Darboux frame, and in turn identifies the curvatures associated with the frame as $\kappa_{g}, \kappa_{n}$ and $\tau_{g}$. This may appear to represent an overkill: but—as we will see—if this constraint is overlooked, one runs into mathematical inconsistencies which indicate that something is amiss.

We thus construct the modified functional $H_{C}(s, U^{a}, \kappa_{g}, \kappa_{n}, \tau_{g}, \mathbf{Y}, \mathbf{T}, \mathbf{L}, \mathbf{N}, \mathbf{F}, \mathbf{\lambda}, \Lambda, \lambda_{IJ}, \lambda_{\mathbf{g}}, \mathbf{\lambda}_{\mathbf{n}}, T_{\mathbf{g}})$:

\[
H_{C} = H(\kappa_{g}, \kappa_{n}, \tau_{g}, V) + \int ds \mathbf{F} \cdot (\mathbf{T} - \mathbf{Y}) + \int ds \mathbf{\lambda} \cdot \left[ \mathbf{Y}(s) - \mathbf{X}(U^{a}(s)) \right] + \int ds \mathbf{A} \cdot (\mathbf{N} - \mathbf{n}(U^{a}(s))) + \frac{1}{2} \int ds \lambda_{TT}(\mathbf{T} \cdot \mathbf{T} - 1) + \frac{1}{2} \int ds \lambda_{LL}(\mathbf{L} \cdot \mathbf{L} - 1) + \frac{1}{2} \int ds \lambda_{NN}(\mathbf{N} \cdot \mathbf{N} - 1)
\]

\[
+ \int ds \lambda_{TL} \mathbf{T} \cdot \mathbf{L} + \int ds \lambda_{LN} \mathbf{L} \cdot \mathbf{N} + \int ds \lambda_{NT} \mathbf{N} \cdot \mathbf{T}
\]

\[
+ \int ds \lambda_{Tg} (\mathbf{T} \cdot \mathbf{L} - \kappa_{g}) + \int ds \lambda_{Ng} (\mathbf{L} \cdot \mathbf{N} - \tau_{g}) + \int ds \lambda_{N\mathbf{g}} (\mathbf{N} \cdot \mathbf{T} - \kappa_{n}) \tag{13}
\]

The introduction of appropriate Lagrange multipliers frees all three Darboux frame vectors and the curvatures defined by their rotation to be varied independently. Note that the Lagrange multiplier $\lambda_{TT}$ ensures that the parameter $s$ is arc-length.

The variable $\mathbf{Y}$ appears only in the second and third terms in equation (13). The variation of $H_{C}$ with respect to $\mathbf{Y}$ is thus given by

\[
\delta_{\mathbf{Y}}H_{C} = \int ds (\mathbf{F}' + \mathbf{\lambda}) \cdot \delta\mathbf{Y}. \tag{14}
\]

Thus, in equilibrium, one finds that

\[
\mathbf{F}' = -\mathbf{\lambda}. \tag{15}
\]

The fate of boundary terms arising from total derivatives will be addressed in section 2.4. In addition, in section 4.1 it will be seen that $\mathbf{F}$ is the tension along the curve. It is not conserved however: external forces associated with the contact constraint acting along the curve are captured by the multiplier $\mathbf{\lambda}$ [23]. Surface curvature breaks the translational invariance of $H$.

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14 To lighten the notational burden we use the same symbol for the multiplier as we do for the solutions of the unconstrained EL equation for $\kappa_{g}, \kappa_{n}$ and $\tau_{g}$.

15 An alternative constraint, equivalent to the latter, is provided by the term $\int ds \lambda_{a} \mathbf{N} \cdot \mathbf{e}_{a}(U^{a}) ds$, where the tangent vectors $\mathbf{e}_{a}$ are treated as functionals of $U^{a}$: $\mathbf{e}_{a}(U^{a}) = \frac{\partial \mathbf{s}(U^{a})}{\partial U^{a}(\mathbf{s})}. \tag{29}$

16 This may appear, at first sight, to be an overly roundabout approach to the problem. We will see, however, that there is a direct payoff: the shape equation gets cast directly in terms of the evolution of the Euclidean vector $\mathbf{F}$ along the curve. If there is a conservation law lurking, it will be sniffed out; boundary conditions become transparent. The first point was clear already in [27], the latter in [28], both in the context of membranes.
The variables $U^a$ appear in the third and fourth terms. The corresponding variation of $H_C$ with respect to $U^a(s)$ is given by
\[
\delta U^a H_C = - \int ds \left( \lambda \cdot e_a(U^a) + K_{ab} A^b - \partial_a V \right) \delta U^a.
\] (16)
Here we have defined $A^a = A \cdot e^a$, and made use of the Weingarten surface structure equations, capturing the definition of the extrinsic curvature, $\partial_a n = K_{a b} e_b$. Thus, in equilibrium
\[
\lambda \cdot e_a = - K_{a b} A^b + \partial_a V;
\] (17)
the force on the equilibrium curve generally does not act orthogonally to the surface: $\lambda \cdot e_a \neq 0$. Notice that the potential depends on the trajectory through its dependence on the functions $U^a$.

Since $H$ is a scalar under reparametrization, the tangential projection of $F'$ must vanish
\[
F' \cdot T = 0,
\] (18)
whether the curve is in equilibrium or not [29] 17. A consequence is that the projection along $T$ of the external force must vanish: $T \cdot \lambda = 0$, so that from equation (17) one gets $t^a K_{a b} A^b = V$, which can be alternatively cast in the index-free form
\[
\kappa_a A^T - \tau_a A^L = V',
\] (19)
where $A^T = l_a A^a$ and $A^L = l_a A^a$ are the projections of the surface vector $A^a$ along $T$ and $L$ respectively. One can thus express the tangential component of $\lambda$ along $L$ in the form ($V_L V = l^a \partial_a V$)
\[
\lambda^L = \lambda \cdot L = \frac{1}{\kappa_n} \left( - \kappa_G A^L + \tau_e V' \right) + V_L V.
\] (20)
Here we have used the identities for the normal curvature and geodesic torsion in terms of the relevant projections of the extrinsic curvature tensor, given by equation (6); we have also used the completeness of the tangent vectors $T$ and $L$ on the surface: $t^i l^j = g^{i j}$, to express the normal curvature along the orthogonal direction in the form: $t^i l^j K_{a b} = K - \kappa_n$; the definition of the Gaussian curvature as a two-dimensional determinant then identifies
\[
\kappa_G = \frac{1}{2} \left( K^2 - K_{a b} K_{a b} \right) = \kappa_n \left( K - \kappa_n \right) - \tau_e;
\] (21)
of which, equation (3) is a special case.

The two Lagrange multipliers $A^L$ and $\lambda^N$ remain undetermined. To anticipate, $A_L$ will appear in the homogeneous EL equation for $N$; this equation will then fix this multiplier, and with it the tangential force—through equation (20)—completely in terms of the local geometry.

From the variation of $H_C$ with respect to $\kappa_n$, $\kappa_n$ and $\tau_e$ we readily determine the Lagrange multipliers expressing their definitions in terms of the Darboux basis vectors by equations (9).

Now $H_C$ is stationary under variations with respect to $T$, $N$ and $L$ respectively when
\[
F + \lambda_{TT} T + \lambda_{TL} L + \lambda_{NT} N + H_n N' - \left( H_L L \right)' = 0,
\] (22a)

17 That it is not identically satisfied here is because we have broken the manifest reparametrization invariance of the problem by the explicit introduction of parametrization by arc-length.
\[ \lambda_{TL}T + \lambda_{LL}L + \lambda_{LN}N + H_g T' - (T_g N)' = 0, \quad (22b) \]
\[ \Lambda + \lambda_{NT}T + \lambda_{LN}L + \lambda_{NN}N + T_g L' - (H_g T)' = 0. \quad (22c) \]

Substituting the structure equations (5) into equations (22a)–(22b), one obtains

\[ F + (\lambda_{TT} + k_g H_g + k_n H_n)T + (\lambda_{TL} - H_g - \tau_g H_n)N = 0, \quad (23a) \]

\[ (\lambda_{TL} - k_n T_g)T + (\lambda_{LL} + k_g H_g + \tau_g T_g)N + (\lambda_{LN} - T_g' - k_n H_n)N = 0, \quad (23b) \]

The first equation expresses the tension \( F \) at each point along the curve as a linear combination of the Darboux vectors; the linear independence of these vectors implies that each of the six coefficients appearing in equations (23b) and (23c) must vanish, thus:

\[ \lambda_{TL} - k_n T_g = 0, \quad (24a) \]
\[ \lambda_{LL} + k_g H_g + \tau_g T_g = 0, \quad (24b) \]
\[ \lambda_{LN} - T_g' - k_n H_n = 0, \quad (24c) \]
\[ \Lambda^T + \lambda_{NT} - H_n - k_\tau T_g = 0, \quad (24d) \]
\[ \Lambda^L + \lambda_{LN} - k_\tau H_n = 0, \quad (24e) \]
\[ \Lambda^N + \lambda_{NN} + k_\tau H_n + \tau_g T_g = 0. \quad (24f) \]

Equations (24a), (24b) and (24c) determine three of the multipliers directly in terms of \( k_g, k_n \) and \( \tau_g \):

\[ \lambda_{TL} = k_n T_g, \quad (25a) \]
\[ \lambda_{LL} = -k_g H_g - \tau_g T_g, \quad (25b) \]
\[ \lambda_{LN} = T_g' + k_n H_n. \quad (25c) \]

Equation (25c) together with equation (24e) completely determines \( \Lambda^L \)

\[ \Lambda^L = -T_g' + k_g H_n - k_n H_g. \quad (26) \]

This vanishes for the Frenet energy, given by equation (1), and only for this energy.

In turn equation (26), determines \( \Lambda^T \) through equation (19)

\[ \Lambda^T = \frac{1}{k_n} (\tau_g (-T_g + k_g H_n - k_n H_g) + V'). \quad (27) \]

Together equations (20) and (26), also completely determine the tangential force \( \lambda^L \) in terms of the geometry:
\[
\lambda^L = \frac{K_G}{\kappa_n} \left( T_g + \kappa_n H_g - \kappa_g H_n \right) + \frac{\tau_g V'}{\kappa_n} + V_L V. \tag{28}
\]

Using equation (27) for \( \Lambda^T \) in equation (24d) gives

\[
\lambda_{NT} = H_n + \tau_g H_g + \kappa_g T_g + \frac{1}{\kappa_n} \left( \tau_g \left( T_g - \kappa_g H_n \right) - V' \right). \tag{29}
\]

We have now determined all of the relevant Lagrange multipliers\(^{18}\), save one: \( \lambda_{TT} \). The tension along the curve \( \mathbf{F} \) given by equation (22a) is now determined modulo this multiplier. To complete the description, one appeals again to the statement of reparametrization invariance of \( H \), equation (18), which along with equation (23a) implies that

\[
\left( \lambda_{TT} + \kappa_g H_g + \kappa_n H_n \right) + \kappa_g H' + \kappa_n H' + \tau_g T_g' - V' = 0. \tag{30}
\]

Equation (30) can be integrated to express \( \lambda_{TT} \) as a linear functional of \( H \):

\[
\lambda_{TT} = H - 2 \left( \kappa_g H_g + \kappa_n H_n \right) - \tau_g T_g + V + c,
\]

where \( c \) is a constant. Thus, modulo this constant, the vector \( \mathbf{F} \)—conserved or not—is now completely determined by the curve itself:

\[
\mathbf{F} = F^T \mathbf{T} + F^L \mathbf{L} + F^N \mathbf{N}, \tag{32}
\]

where \( F^T, F^L \) and \( F^N \) are given by 8. Using equation (15), \( \mathbf{F} \) satisfies

\[
\mathbf{F}' = -\lambda^L \mathbf{L} - \lambda^N \mathbf{N}, \tag{33}
\]

where \( \lambda^L \) is given by equation (28), and \( \lambda^N = \lambda \cdot \mathbf{N} \). There is no projection onto \( \mathbf{T} \) on account of equation (18). The EL equation is then given by \( \epsilon_L = \mathbf{F}' \cdot \mathbf{L} + \lambda^L = 0 \), where \( \mathbf{F}' \cdot \mathbf{L} = F^L + \kappa_g F^T - \tau_g F^N \), which reproduces equation (11). The magnitude of the normal force is given by \( \lambda^N = -\mathbf{F}' \cdot \mathbf{N} = - F^N + \kappa_n F^T - \tau_g F^L \), which reproduces equation (12).

Before examining the general structure, it is useful to first confirm that this framework reproduces known results for bound Euler elastic curves.

### 2.3. Bound Euler elastic curves

The bending energy defined by equation (1), can be decomposed as

\[
H_B = \frac{1}{2} \int ds \left( \kappa_g^2 + \kappa_n^2 \right), \tag{34}
\]

thus, \( H_g = \kappa_g, H_n = \kappa_n \), with \( T_g = 0 \) and \( V = 0 \). In addition, there are no tangential forces: \( \lambda^L \) given by equation (10) vanishes due to the quadratic dependence on curvature as well as the symmetry between \( \kappa_g \) and \( \kappa_n \). Note that the symmetry, itself, is not enough.\(^{19}\) The EL equation (11) reduces to

\(^{18}\) Using equation (24f) one determines \( \lambda_{NN} = -\lambda^N - \kappa_g H_g - \tau_g T_g \). One can set \( \lambda^N = 0 \) without any loss of generality. While \( \lambda_{NN} \) itself does not feature in the physical description of the curve, had the corresponding constraint been overlooked one would have run into an inconsistency.

\(^{19}\) More generally, if the dependence of \( H \) on \( \kappa_g \) and \( \kappa_n \) can be collected into a dependence on \( \kappa, \lambda^L \) will vanish.
This agrees with the equation first derived in reference [9] and, using an approach adapted to the Frenet basis, in [12]. In general, geodesics with $\kappa = 0$ are not solutions.

For this energy, the magnitude of the normal force $\lambda_N$, given by (12), transmitted to the surface assumes the form

$$\lambda_N = \kappa_n^2 - \left(\frac{\kappa_n^2 \tau_g + \kappa_g}{\kappa_g}\right)^2 + \kappa_g \left(\frac{\kappa_g^2 + \kappa_n^2}{2} - \tau_g^2 - c\right),$$

which again reproduces the result first obtained in [12]. The normal force depends non-trivially on the location on the surface.

### 2.4. Boundary conditions on an open curve

Four terms were consigned to the end points of the curve in the variational principle. Modulo the EL equations, the variation of the Hamiltonian (13) is given by $\delta H_C = -\int s \delta Q^2$, where the boundary term $\delta Q$ is identified as

$$\delta Q = F \cdot \delta Y - H_s L \cdot \delta T - T_s N \cdot \delta L - H_n T \cdot \delta N. \quad (37)$$

Several different boundary possibilities are physically relevant.

#### 2.4.1. Fixed ends

If the end points are fixed then the first and last terms in (37) vanish: $\delta Y = 0$ and $\delta N = 0$. If the tangent is also fixed, then both middle terms vanish as well, for $\delta T = 0$ and consequently $\delta L = 0$. If $T$ is not fixed one requires $H_s = 0$.

The boundary term also vanishes for a periodic curve such as a spiral on a helicoid.

#### 2.4.2. One or two free ends

The constraint that the curve $Y$ lie on the surface implies that, at the boundary, its variation $\delta Y$ must be tangential: we decompose it with respect to the tangent basis $\{T, L\}$

$$\delta Y = \Psi_L^T + \Psi_T^L. \quad (38)$$

The requirement that $s$ remain arc-length under variation implies that variation and differentiation with respect to arc-length commute: $\delta T = (\delta Y)'$, or equivalently $T \cdot \delta T = 0$. This implies the constraint

$$\Psi_L^T - \kappa_L \Psi_T^L = 0 \quad (39)$$

on the two component of the variation field. Thus

$$\delta T = \left(\Psi_L^T + \kappa_T \Psi_T^T\right) + \left(\tau_g \Psi_L^T - \kappa_T \Psi_T^T\right)N, \quad (40a)$$

$$\delta N = -\left(\tau_T \Psi_L^T - \kappa_T \Psi_T^T\right) + \left(\left(K - \kappa_s\right) \Psi_L^T - \tau_g \Psi_T^T\right)L. \quad (40b)$$

---

20 Note that $\delta Y = 0$ implies that both $\Psi_L^T$ and $\Psi_T^T$ defined by equation (38) vanish.

21 Note that whereas a tangential deformation at interior points can always be identified with a reparametrization, at a boundary it cannot. On the boundary, it changes the length of the curve.

22 Equivalently, $\delta U'' = \Psi_L^T + \Psi_T^T$. 
and $\delta L$ follows from the orthonormality of the Darboux frame. Notice that $\delta N$ does not involve derivatives of the deformation scalars, consistent with its vanishing when $Y$ is fixed. The boundary term (37) now reads

$$\delta Q = -H_g \Psi'_\perp + \left( H_g + 2\tau_g H_n + (K - 2\kappa_n) T_g \right) \Psi'_\perp - (H + V + c) \Psi'. \quad (42)$$

Vanishing $\delta Q$ at the end point for arbitrary $\Psi'_\perp$, $\Psi'_\parallel$ and $\Psi'$ implies the boundary conditions

$$H_g = 0, \quad H_g = -2\tau_g H_n - (K - 2\kappa_n) T_g, \quad H = -V - c, \quad (43)$$

at free end-points. For a bound elastic curve, it implies that the curve must be geodesic at a free end, so these boundary conditions read

$$\kappa_g = 0, \quad \kappa_g = -2\tau_g \kappa_n, \quad H + c = 0. \quad (44)$$

The last condition, in turn, implies $\kappa_g^2 = -2c$ so that $c < 0$. On a sphere, the only equilibrium states consistent with equation (44) are geodesic arcs.

3. Examples of surface biased bending energies

We now examine various energies of special interest.

3.1. Energies dependent only on surface intrinsic geometry

Consider an elastic curve on a surface with the intrinsically defined energy

$$H = \frac{1}{2} \int ds \left( \kappa_g^2 - \gamma K_G \right), \quad (45)$$

involving a potential proportional to the local surface Gaussian curvature, $K_G$; $\gamma$ is a constant. Higher dimensional relativistic brane world analogues of (45), with energy replaced by the action, have been considered in some detail by Armas [30].

In this case one has $H_g = \kappa_g$, $H_n = T_g = 0$ and $V = -\gamma K_G/2$; thus equation (11) can be cast completely in terms of the intrinsic geometry, independent of the extrinsic curvature.

23 The corresponding changes in the Darboux curvatures follow

$$\delta \kappa_g = \Psi'_\parallel + \kappa_n \Psi'_\perp + \left( \kappa_p \Psi' \right). \quad (41a)$$

$$\delta \kappa_n = -2\tau_g \Psi'_\perp - (\tau_g + \kappa_g (K - \kappa_n)) \Psi'_\parallel + \left( k_n \Psi' \right). \quad (41b)$$

$$\delta \tau_g = -\left( K \Psi' \right) + 2\kappa_n \Psi'_\parallel + (\kappa_n + \kappa_g \tau_g) \Psi'_\perp + \left( \tau_g \Psi' \right). \quad (41c)$$

The constraint (39) permits the $\Psi'$ dependence to be absorbed into a total derivative. It is simple to confirm that these expressions provide an alternative derivation of the single EL equation (11), as well as the boundary conditions (43).

What is missing is the underlying structure of the EL equation in terms of a tension vector, provided by the constrained variational approach provided earlier as well as the identification of the normal forces.

24 An interesting limit in its own right is one in which $H$ depends only on the potential, so that $H = \int ds \kappa_g$. One may be interested in identifying curves that avoid regions of large Gaussian curvature, so that one minimizes (or maximizes) the total Gaussian curvature along its length.

25 Recall that both the geodesic curvature and the Gaussian curvature are invariant under isometry.
If, in addition $\mathcal{K}_G$ is constant, geodesics occur as solutions. Despite the intrinsic nature of the EL equation, (46), the magnitude of the normal force binding the curve to the surface, given by equation (12), is generally non-vanishing and does depend explicitly on the extrinsic curvature:

$$\lambda^N = \frac{\mathcal{K}_G}{\mathcal{K}_n} - \tau\kappa' + \kappa_n\left(\frac{\kappa^2}{2} + \frac{\mathcal{K}_G}{2} - c\right).$$

(47)

This is true even when $\mathcal{K}_G$ is constant and along geodesics. In this case, the force is proportional to $\kappa_n$.

If the surface is a round sphere of radius $R_0$, with $\mathcal{K}_G = 1/R_0^2$, $\kappa_n = 1/R_0$ and $\tau = 0$, the two energies (34) and (45) coincide if $\gamma = -1$. Consequently equation (46) coincides with equation (35), and equation (47) with equation (36). The force depends only on the local value of the geodesic curvature.

On a round sphere, the extrinsic geometry like the metric is homogeneous and isotropic. There are, however, geometries locally isometric to a sphere, which are non-trivially embedded. Examples are provided by the axially symmetric surfaces of constant positive Gaussian curvature obtained by removing or adding a wedge spanned by two meridians from a sphere; both geometries display conical singularities [7, 31]. Whereas geodesics on a round sphere are sections of closed great circles, they generally do not close in these isometric geometries; and while $\lambda^N$ is constant along the great circles on a round sphere, it will not be along geodesics in one of these geometries.

3.1.1. Energy symmetric in curvatures and torsion. Let us now examine the energy symmetric in the curvatures and the torsion

$$H = \frac{1}{2} \int ds \left( \kappa^2_g + \kappa^2_n + \tau^2 \right).$$

(48)

On first inspection, this energy appears very different from the intrinsically defined equation (45) and, in general, it is. However, using the identity equation (21), it is evident that the two-energies (48) and (45) coincide on minimal surfaces when $\gamma = 1$. Using the identities $H_g = \kappa_g$, $H_n = \kappa_n$, $T_g = \tau_g$ and $V = 0$, the EL equation (11) reduces to

$$\kappa^2_g + \kappa_n\tau_g + (K - \kappa_n)\tau^2_g + \kappa_g(H - c) = 0,$$

(49)

where we have used equation (21) to replace $\mathcal{K}_G$ in favor of the Darboux curvatures and $K$. The magnitude of the normal force (12) is given by

$$\lambda^N = \kappa_n' + \left(\frac{\tau_g\tau^g}{\kappa_n}\right)' - \tau_g\kappa^2_g + \kappa_n(H - c).$$

(50)

On a minimal surface, the EL equation (49), unlike its counterpart (46), appears to depend explicitly on the extrinsically defined $\kappa_g$ and $\tau_g$. As we will show, however, on such a surface equation (49) can be cast in a form that is manifestly independent of the extrinsic geometry. First note that equation (49) can be cast, using equation (21), as
\[ \kappa^+_g + \kappa^-_g \left( \frac{1}{2} \kappa^2_g - \frac{1}{2} \kappa^+_G - c \right) = \kappa^+_u \tau^+_u - \kappa^-_u \tau^-_u, \]  

(51)

whereas equation (46) with \( \gamma = 1 \) can be written as

\[ \kappa^+_g + \kappa^-_g \left( \frac{1}{2} \kappa^2_g - \frac{1}{2} \kappa^+_G - c \right) = \frac{1}{2} V_L K^+_G - 2 \kappa^+_g K^-_G. \]  

(52)

On a minimal surface, however, it is shown in appendix A that

\[ V_L K^+_G/2 = \kappa^+_u \tau^+_u - \tau^+_g \kappa^+_u + 2 \kappa^+_g K^-_G. \]  

(53)

Using this identity, equations (49) and (46) with \( \gamma = 1 \) indeed coincide on such surfaces.

3.1.2. Energy quadratic in curvature and linear in geodesic torsion. The energy \( H = \kappa^2/2 + \nu \tau^+_g \) was proposed in [32] in order to model the energy of boundaries of axially symmetric cranellated disks, where the geodesic torsion term accounts for their microscopic chirality. For this energy one has that \( \tau^+_g = \lambda \), so the EL derivative in (35) has the additional contribution \( -\nu (\kappa^+_u - \kappa^-_u \tau^-_g) \). Likewise, the magnitude of normal force given by (36) has the additional term \( \nu (\kappa^+_u + \kappa^-_u \tau^-_g) \).

3.1.3. Barros Garay energy. Recently Barros and Garay considered the stationary states of an energy depending only on the normal curvature \( H = \kappa^N_n / N \), and in particular, for \( N = 2 \), for curves on space forms [33]. It is simple to show that equations (11) and (12) reduce to

\[ \kappa^+_N = 2 \tau^+_g (\kappa^N_n - 1) + \kappa^-_N \left( \frac{2}{N} \kappa^N_n - \kappa^N_n - 1 - c \right) = 0, \]  

(54)

and

\[ \lambda^N = \left( \kappa^N_n - 1 \right) - ( \kappa^+_g \tau^+_g \kappa^-_g - 1 ) + \kappa^-_n \left( \frac{2}{N} \kappa^N_n - \tau^+_g \kappa^N_n - 2 - c \right). \]  

(55)

Except for a minus sign arising from the different convention used for \( \kappa_n \) and the constant \( c \) fixing total length, the EL equation (54) coincides with expression (2.19) presented in proposition 1 of [33].

4. Residual Euclidean invariance

4.1. Translations

In general, the change in the energy along any section of a bound curve under a deformation is given by

\[ \delta H_C = \int ds (F' + \lambda) \cdot \delta Y - \int ds \delta Q', \]  

(56)

where \( \delta Q \) is given by equation (37). In equilibrium, \( F' = -\lambda \).

Under an infinitesimal translation of the surface, \( \delta Y = \delta e \), the Darboux vectors along the bound curve do not change, so that \( \delta T = 0, \delta L = 0, \) and \( \delta N = 0 \). The boundary term is then given by \( \delta Q = \delta e \cdot F \). In equilibrium
\[
\delta H_C = -\delta c \cdot \int ds F' = \delta c \cdot \int ds \lambda.
\]
Thus the vector \( F \) is identified as the tension along the curve and \( \int ds \lambda \) represents the total force on the curve.

Let the surface geometry be a generalized cylinder (not necessarily circular) so that \( H \) is invariant under translation along the \( k \) direction (say). Then the component

\[
F^Z = F \cdot \hat{k},
\]
is constant.

### 4.2. Rotations

Under an infinitesimal rotation of the surface, \( \delta Y = \delta \omega \times Y \) the Darboux vectors change accordingly: \( \delta T = \delta \omega \times T \), and similarly for \( L \) and \( N \). The corresponding boundary term (37) is given by

\[
\delta Q = \delta \omega \cdot M,
\]
where the torque of the curve about the chosen origin is given by

\[
M = Y \times F + S,
\]
with

\[
S = -\nabla T - H_s L - H_t N.
\]
The first term appearing in \( M \) is the torque with respect to the origin due to the force \( F \) acting on a segment of the curve; \( S \) is the bending moment originating in the curvature dependence of the bending energy, and is translationally invariant. If \( H = \kappa^2/2 \), one reproduces the expression

\[
S = -\kappa_s L - \kappa_n N = -\kappa B,
\]
where \( \kappa_s = \kappa \cos \theta \) and \( \kappa_n = \kappa \sin \theta \) have been used. On a sphere, centered on the origin, as described in [12], \( M \) is a constant vector. However, \( M \) generally will not be conserved in a bound equilibrium. One has instead

\[
M' = -Y \times \lambda - N \times A,
\]
where \( \lambda \) is the external force, normal to the curve, defined earlier, equations (10) and (12), and \( A \) is a vector tangent to the surface, with components defined by (26) and (27) in 2.2. Thus the source of the torque is given as a sum of two terms: one the moment of the external forces, both normal and tangential to the surface, associated with the constraint; the second an additional source for the bending moment.

In the case of bound Euler elastic curves, equation (63) simplifies. For now \( A^L = 0 \) by equation (26) so that \( A = 0 \) by equation (19) and \( \lambda^L = 0 \) by equation (20). Thus the source is given by the moments of the normal force \( \lambda^N \) alone. This also lends a physical interpretation for \( A \).

#### 4.2.1. Axially symmetric geometries.

If the surface geometry is axially symmetric, \( H \) will be invariant under rotations about the axis of symmetry. As a consequence, the projection of \( M \) along this axis, i.e. \( M^Z = M \cdot \hat{k} \) is conserved. Let \( R \) and \( Z \) represent the cylindrical polar coordinate and the height along the curve, adapted to the symmetry, and \( \alpha \) be the angle that

\[\theta\] is the angle between the Frenet normal and the Darboux conormal, and \( B \) is the Frenet binormal.
the tangent vector to the curve makes with the azimuthal direction (see appendix B). One identifies
\[ M^Z = R \left( \cos \alpha T^T - \sin \alpha E^L \right) + \cos a \left( R'H_g - Z' \cos \alpha H_n + \sin \alpha T_g \right). \] (64)
This provides a first integral of the EL equation (11), one which does not exactly leap off the page on first inspection. Differentiating it with respect to arc-length one finds, as shown in appendix C, that \( M^{Z'} = -R \sin \alpha \epsilon_6 \), so that in equilibrium \( M^Z \) is indeed conserved. Let the surface be parametrized by the length along the meridian and the polar angle \( \phi \). Their values along the curve are determined by the equations (see appendix B)
\[ \alpha' = \frac{1}{\sin \alpha}, \] (65a)
\[ R \phi' = \cos \alpha. \] (65b)

The Darboux curvatures and torsion are given by equations (B.7), (B.9) and (B.10) in appendix B, or
\[ \kappa_g = -\frac{(R \cos \alpha)}{R \sin \alpha}; \quad \kappa_n = \sin^2 \alpha C_\perp + \cos^2 \alpha C_\parallel; \quad \tau_g = \sin \alpha \cos \alpha \left( C_\parallel - C_\perp \right), \] (66)
where \( C_\perp \) and \( C_\parallel \) are the meridional and parallel curvatures respectively. Substituting the expression (8) for the components of \( F \), as well as the identities \( Z' = R \sin \alpha \epsilon_6 \) (use equation (B.4) in appendix B) and 2 cot \( 2 \tau_g = 2 \kappa_n - K \) (which follows from the definitions (B.9) and (B.10)) yields the more transparent expression
\[ M^Z = -R \sin \alpha \left( H_g + 2 \tau_g H_n + (K - 2 \kappa_n) T_g \right) \]
\[ + (R \sin \alpha) H_g - R \cos \alpha (H + V + c). \] (67)
For a bound Euler elastic curve, this expression reduces to
\[ M^Z = -R \sin \alpha \left( \kappa_g' + 2 \tau_g \kappa_n \right) + R \cos \alpha \left( \frac{\kappa_g^2 - \kappa_n^2}{2} - c \right) + R' \csc \alpha \kappa_g. \] (68)

The polar radius \( R \), as well as the curvatures \( C_\parallel \) and \( C_\perp \) appearing in equation (67) are given functions of the arc-length along the meridian \( l \). In general one can write equations (65a) and equation (67) as a pair of coupled ODEs for \( \alpha \) and \( l \) as functions of \( s \): if \( H \) is quadratic in \( \kappa_g \), equation (67) will be linear in second derivatives of \( \alpha \), or of the form \( \alpha'' = f(\alpha, \alpha', l) \), where \( \alpha \) satisfies equation (65a). These two equations provide \( R \) and \( Z \) as functions of \( s \). To complete the construction of the trajectory one uses equation (65b) to determine \( \phi \) as a function of \( s \).

Comment on boundary conditions: suppose that the two ends of a curve of fixed length \( L \) are fixed so that the elevations \( Z_0 \) and \( Z_1 \), the angle turned \( \Delta \phi = \phi_1 - \phi_0 \), as well as the tangent angles at the two points are given\(^{27}\). The non-local constraints on the length and the azimuthal angle turned completely determine the two free parameters appearing in equation (67). Solutions on a catenoid will be studied in detail elsewhere.

If free-boundary conditions are relevant, the solution is very different. Comparing equations (43) and (67), one finds that the right hand side of equation (67) vanishes at the boundary. As a consequence, \( M^Z \) also vanishes, leaving a single free parameter, \( c \).

4.2.2. Cylinders. For a circular cylinder of radius \( R_0 \) one has \( R' = 0 \) so that \( Z = l \) and \( Z' = \sin \alpha \). Now \( C_\perp = 0, C_\parallel = 1/R_0 \) and as a consequence \( \kappa_g = \alpha', \kappa_n = \cos^2 \alpha/R_0 \) and

\(^{27}\) We will suppose that \( R \) is a monotonic function of \( Z \), so that \( R_0 \) and \( R_1 \) are fixed once \( Z_0 \) and \( Z_1 \) are.
\( \tau_g = \sin 2\alpha/(2R_0) \). In addition to the rotational symmetry about the axis, the geometry also possesses translational symmetry along this axis, so that the corresponding projection of \( F, F^Z = F \cdot k \) is conserved. In general this projection reads
\[
F^Z = R C_0 \left( \sin \alpha F^T + \cos \alpha F^L \right) - R' \cos \alpha F^N,
\]
which for the cylinder reduces to
\[
F^Z = \sin \alpha F^T + \cos \alpha F^L. \tag{70}
\]
The cylinder is flat so that \( \kappa_g = 0 \); thus, in the absence of a potential, \( \lambda^L = 0 \) according to equation (10). Thus there is no tangential force along the curve. Taking into account these results, it is straightforward to confirm that \( F^{Z'} = \cos \alpha e_L \). \( F^Z \) is conserved when the EL equation is satisfied.

It is now possible to exploit the two-dimensional subgroup of the Euclidean group unbroken by motion along the cylinder to identify a quadrature. In this case \( M^Z \) given by expression (64) simplifies to give
\[
M^Z/R_0 = \cos \alpha \left( F^T - H_n/R_0 \right) - \sin \alpha \left( F^L + T_g/R_0 \right). \tag{71}
\]
Both equations (70) and (71) are of second order in derivatives of \( \alpha \) (this dependence enters through \( H_g \) which involves \( \kappa'_g \)). By taking an appropriate linear combination of \( F^Z \) and \( M^Z \), it is possible to eliminate the term involving this second derivative, \( F^L \). Specifically
\[
F^Z \sin \alpha + M^Z/R_0 \cos \alpha = F^T - \kappa_n H_n - \tau_g T_g = \kappa_g H_g - H - c, \tag{72}
\]
where the expression for \( \kappa_n \) and \( \tau_g \) on a cylinder have been used. For an energy density quadratic in the geodesic curvature of the form \( \kappa \cos \tau \), equation (72) provides a quadrature for \( \alpha \):
\[
\frac{1}{2} (\alpha')^2 + U(\alpha) = \frac{1}{2} C_g^2 + c, \tag{73}
\]
where the potential \( U \) is given by
\[
U(\alpha) = -h - F^Z \sin \alpha - M^Z/R_0 \cos \alpha, \tag{74}
\]
involving the two constants \( F^Z \) and \( M^Z \). The spontaneous curvature \( C_g \) plays no role. For bound Euler elastic curves, \( h = \kappa_g^2/2 = \cos^2 \alpha/(2R_0^2) \).

The other natural linear combination of the conservation laws provides a useful expression for \( H_g \), and with it a remarkably simple expression for the transmitted force, \( \lambda^N \). One has
\[
F^Z \cos \alpha - M^Z/R_0 \sin \alpha = F^L + \tau_g H_n
+ \left( \frac{1}{R_0} - \kappa_n \right) T_g = H_g + 2\tau_g H_n + \left( \frac{1}{R_0} - 2\kappa_n \right) T_g, \tag{75}
\]
Likewise, for bound Euler elastic curves, equation (75) reads
\[
\kappa'_g + 2\kappa_n \tau_g - F^Z \cos \alpha + \frac{M^Z}{R_0} \sin \alpha = 0. \tag{76}
\]
The magnitude of the force on the cylinder is given by

\[
\lambda^N = \mathcal{H}_n + \left[ \sec \alpha \left( \frac{\sin \alpha \mathcal{T}_s}{2} \right) + (\ln \cos \alpha) \mathcal{H}_n - \frac{R_0}{2} \sec^2 \alpha \mathcal{V} \right] \lambda^{N'} + \frac{\cos^2 \alpha}{R_0} \left( - \tan \alpha \mathcal{H}_s + \alpha' \mathcal{H}_s + \frac{\cos 2\alpha}{R_0} \mathcal{H}_n + \sin \frac{2\alpha}{R_0} \mathcal{T}_s - \mathcal{H} - \mathcal{V} \right) \right].
\] (77)

For Euler elastica, using the quadrature (73) and the second order DE (76) to eliminate the derivatives of \( \alpha \) in (77), \( \lambda^N \) reduces to the form

\[
\lambda^N = 2\kappa_n^2 \left( \frac{5}{R_0} - 6\kappa_n \right) + 3F^Z \sin \alpha \left( \frac{2}{R_0} - 5\kappa_n \right)
+ 5 \frac{M^Z}{R_0} \cos \alpha \left( \frac{2}{R_0} - 3\kappa_n \right) - 6 \frac{c}{R_0} \cos 2\alpha.
\] (78)

It depends only on \( \alpha \). Confinement within cylinders will be treated in detail elsewhere.

### 4.3. Glide rotations: helicoids

A helicoid consists of an infinite double spiral staircase winding about a fixed axis [31]; it can be represented very simply in terms of a height function over the orthogonal plane; one half is described by \( (r, \theta) \rightarrow h(r, \theta) = p\theta \), where \( p \) is its pitch. Its other half is described by \( h(r, \theta) = p(\pi + \theta) \). The surface is smooth along \( r = 0 \), which forms the boundary of each staircase. It is straightforward to confirm that the helicoid is a minimal surface satisfying \( \nabla^2 h = 0 \), where \( \nabla^2 \) is the Laplacian on the plane. Curves of constant \( r \) form helices.

The symmetry of a helicoid is a glide rotation, i.e. the composition of an infinitesimal rotation by an angle \( \delta \theta \) about the rotation axis, with a translation by a height \( \delta h = p \delta \theta \) along it, given by \( \delta \mathbf{Y} = \delta \mathbf{k} \times \mathbf{Y} + p \delta \mathbf{k} \). The relevant conserved quantity is then

\[
G^Z = M^Z + p F^Z = (\mathbf{M} + p \mathbf{F}) \cdot \mathbf{k}.
\] (79)

One has

\[
F^Z = \frac{1}{\sqrt{r^2 + p^2}} \left( p \left( \cos \beta \mathbf{T} + \sin \beta \mathbf{F} \right) + r F^N \right),
\] (80 a)

\[
M^Z = \frac{1}{\sqrt{r^2 + p^2}} \left( \cos \beta \left( r^2 F^T - p \mathcal{T}_s \right) + \sin \beta \left( r^2 F^L - p \mathcal{H}_n \right) - r \left( p F^N + \mathcal{H}_s \right) \right),
\] (80 b)

where \( \beta \) is the angle that the tangent vector \( \mathbf{T} \) makes with the helical direction \( \hat{e}_\theta \) (see appendix D). Both projections involve all three components of \( \mathbf{F} \). Therefore \( G^Z \) is given by
\[ G^Z = \frac{1}{\sqrt{r^2 + p^2}} \left( \cos \beta \left( (r^2 + p^2)F_T - pT_g \right) + \sin \beta \left( (r^2 + p^2)F_L - pH_a \right) - rH_g \right) \]
\[ = \sqrt{r^2 + p^2} \left( \sin \beta \left( H_g + 2rH_a - 2\kappa_a T_g \right) \right) \]
\[ + \left( \cos \beta \kappa_g - \frac{r}{r^2 + p^2} H_g - \cos \beta (H + V + c) \right). \tag{81} \]

The sum does not involve \( F_N \). As shown in appendix E, its derivative is proportional to the EL derivative, or \( G^{\text{EL}} = \sqrt{r^2 + p^2} \sin \beta \varepsilon_L \), so that it is conserved in equilibrium and provides a first integral of the EL equation.

An alternative derivation for \( M^Z \) and \( G^Z \) within a Hamiltonian formulation is presented in appendix F.

5. Discussion

The physics of linear polymers or protein complexes bound to fluid membranes is, in general, enormously complicated and modeling their behavior at a microscopic level involves a significant computational effort. However, just as essential aspects of the physics of fluid membranes on mesoscopic scales are captured by the geometrical degrees of freedom of the membrane surface, elements of the interaction between the surface-bound protein and its substrate can be understood on such scales in terms of the geometrical environment: the linear structure can be modeled as a curve on a surface with a potential energy that reflects its interaction with this environment. The accommodation of such an environmental bias has never been examined in any systematic way, either by mathematicians or physicists, even though it is a natural question either from the point of view of dynamical systems or of differential geometry, and an obvious generalization of the study of geodesics. Along the way, one is forced to think how curves best negotiate energetically costly obstacles on a surface consistent with its environmental biases. To address this problem, it is necessary to extend existing geometrical methods in the calculus of variations to accommodate elastic energies capturing these biases.

We have shown how the conservation laws associated with any residual continuous Euclidean symmetry can be identified, focusing not only on the obvious axially symmetric surfaces but also on surfaces—exemplified by a helicoid—with glide rotational symmetry. Teasing out the physics has involved mathematics of interest in its own right which warrants closer inspection.

It is beyond the scope of this paper to catalog the equilibrium states on even the simplest geometries. In future publications we will examine the ground states of physically interesting variants of the elastic energy (4) on cylinders, catenoids and helicoids as well as some more exotic geometries. For spheres there is not a lot to say that was not already reported in [12].28 In our work thus far, we do find time and time again that it is invaluable to first understand how geodesics and curves of constant geodesic curvature behave on the surface. Geodesics are not without interest themselves as the configurations of a taut string negotiating the surface. Curves of constant geodesic curvature, on the other hand, may be interpreted as domain boundaries of a two-phase system living on the surface. For helicoids and catenoids, there is a one to one mapping between such curves on one and those on the other: this is

28 This is not the case for higher dimensional analogs of this problem, as demonstrated in [34] where the confinement of spheres within hyperspheres was examined.
because each turn of a helicoid is isometric to a catenoid so that the trajectories of constant geodesic curvature in one imply those in the other [7, 31], even if the two are very different from a physical point of view. This distinction, as emphasized throughout this paper, is captured by the normal forces. Of course, from a mathematical point of view, the intrinsic aspect of this subject is classical; what is surprising is that it appears not to have been treated from a physically relevant point of view, that addresses the three-dimensionality of the problem. Likewise, it is useful to examine curves along which \( \kappa_n \) or \( \tau_g \) vanish or are constant. Of course, in an elastic system, none of these curves tend to appear as equilibrium states. In general, the constancy of any one of \( \kappa_g \), \( \kappa_n \) or \( \tau_g \) will be incompatible with that of the other two. This frustration of access to the ground state gives rise to interesting, and on occasion counterintuitive, behavior. In particular, geodesics tend to be incompatible with \( \kappa_n = 0 \).

Indeed, the later identity is only possible along asymptotic directions, which occur only if the local Gaussian curvature is non-positive.

An obvious omission in this paper is the response of the membrane to the elastic curve. The surface generally will not play a passive role. This is clearly true if the elastic curve forms the boundary. Whereas a line tension induced on the boundary of a fluid membrane will tend to heal the disrupted edge [36–38], a boundary providing resistance to bending—or selecting some preferred curvature—can provide stable ends to an otherwise unstable membrane preventing its closure. The inner boundary of the recently discovered Terasaki ramps in the rough endoplasmic reticulum are believed to be stabilized by a mechanism of this kind due to the condensation, along it, of reticulons [39]. Unfortunately, as anyone who has taken a moment to ponder these boundary conditions will have noted—even in the apparently simple scenario where the boundary energy is dominated by line tension—getting them right is a challenge. The interaction between a fluid membrane and an elastic boundary will be taken up in a subsequent paper.

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Appendix A. Identity for \( \mathbf{F}_L \mathcal{K}_G \) on a minimal surface

The identity (53) between the normal derivative of \( \mathcal{K}_G \) and the behavior of the Darboux curvatures and torsion follows as a consequence of the Gauss–Codazzi–Mainardi equations. On a minimal surface the Gauss–Codazzi equation reads

\[
\mathcal{K}_G = -1/2K_{ab}K^{ab} = -\left( \kappa_n^2 + \tau_g^2 \right).
\]

Taking the directional derivative along the normal to the curve \( \mathbf{L} \) and using the Codazzi–Mainardi equations \( V_a K_{bc} = V_b K_{ac} \), one has

\[29\] Animated nicely in its wikipedia entry [35].
\[ V_L K_G = -K^{ab} l^c V_{l} K_{ab} = -K^{ab} l^c V_{l} K_{bc} \]
\[ = -K^{ab} \left[ V_a \left( l^c K_{bc} \right) - K_{bc} V_a l^c \right]. \] (A.2)

On a minimal surface the extrinsic curvature tensor can be expanded with respect to the tangential Darboux basis vectors \{T, L\} as
\[ K_{ab} = (t_a l_b - l_a t_b) \kappa_a - (t_a l_b + l_a t_b) \tau_\gamma, \]
where \( t^l \) and \( l^l \) are their projections onto the basis adapted to the parametrization of the surface. It follows that
\[ \kappa_\tau = -\kappa_b l^b + \kappa_a t^a. \]
Using these expressions along with the definition of the geodesic curvature \( \kappa \)
\[ \kappa = \left\langle \nabla_l T, T \right\rangle, \]
one finds for each of the two term on the second line in equation (A.2)
\[ -K^{ab} V_a \left( l^c K_{bc} \right) = \kappa_a \tau_\gamma - \kappa_b \tau_\gamma + \kappa_\tau K_G + 1/2 V_L K_G; \] (A.3a)
\[ K^{ab} K_{bc} V_a l^c = \kappa_\tau K_G. \] (A.3b)

Summing terms and simplifying, equation (53) is identified.

**Appendix B. Curves on axially symmetric surfaces**

An axially-symmetric surface is described by its embedding
\[ \Sigma: (l, \varphi) \rightarrow X(l, \varphi) = (R(l) \cos \varphi, R(l) \sin \varphi, Z(l)), \] (B.1)
into three-dimensional space, where \( l \) is arc length along the meridian and \( \varphi \) is the polar angle along the parallel. \( R(l) \) is the polar radius, and \( Z(l) \) the corresponding height, see figure 2. The two tangent vectors adapted to this parametrization will be denoted \( e_l \) and \( e_\varphi \). They also form the principal directions on the surface with corresponding curvatures \( \kappa_\perp \) along the meridian, and \( \kappa_\parallel \) along the parallel. A curve parametrized by arc-length \( s \) on such a surface is described by the embedding \( s \rightarrow (l(s), \varphi(s)) \), or \( \Gamma: s \rightarrow Y(s) = R(s) \hat{\rho}(s) + Z(s) \hat{k}, \)
where \( \hat{\rho}(s) = (\cos \varphi(s), \sin \varphi(s), 0) \) is the unit vector along the polar radial direction. The corresponding unit vector in the azimuthal direction is \( \hat{\varphi}(s) = (\sin \varphi(s), \cos \varphi(s), 0) \). On our surface \( R \) and \( Z \) will depend on \( s \) through \( l \). It is straightforward to expand the tangent vector along the curve \( T = Y' \) (where the prime denotes differentiation with respect to arc-length, \( \alpha \)) as well as its Darboux counterpart \( L = N \times T \), with respect to the the adapted basis vectors:
\[ T = \cos \alpha \hat{\rho} + \sin \alpha e_l, \quad L = -\sin \alpha \hat{\rho} + \cos \alpha e_l. \] (B.2)

The unit vector tangent to the curve is now characterized by the angle \( \alpha(s) \) that it makes with the parallel direction \( \hat{\rho}; \) parametrization by arc-length implies \( (l'^2 = R'^2 + Z'^2) \)
\[ R'^2 + Z'^2 + R^2 \varphi'^2 = 1, \] (B.3)
so that equations (65a) and (65b) follow. The principal curvatures on the curve are given by
\[ C_\parallel = \csc \alpha \frac{Z'}{R}, \quad C_\perp = \csc^3 \alpha (R'Z'' - Z'R''). \] (B.4)

The two surface unit tangent vectors change along the curve as
\[ \hat{\rho}' = -\cot \alpha R'/R e_l - \cos \alpha C_\parallel N, \quad e_l' = \cot \alpha R'/R \hat{\rho} - \sin \alpha C_\perp N. \] (B.5)
Using these expressions one can decompose the acceleration with respect to the Darboux basis as:

30 The shorthand \( R(s) \) for \( R(l(s)) \), and similarly for \( Z \), is understood.
This decomposition identifies the geodesic and normal curvatures. Firstly, one has that the geodesic curvature of $\Gamma$: $\kappa_g = \frac{\alpha'}{R} \cot \alpha$ is given by

$$\kappa_g = -\frac{(R \cos \alpha)'}{R \sin \alpha}.$$  

(B.7)

If $\alpha \neq 0$, geodesic curves satisfy the remarkably simple Clairaut’s relationship [7, 40]

$$\cos \alpha = \frac{C}{R},$$

(B.8)

where $C$ is constant with dimensions of distance. In particular, meridians with constant $\alpha = \pi/2$ (and extremal parallels with $dR/dl = 0$, when they exist) are also geodesic. Under the change of chirality, $\alpha \to \pi - \alpha$, $\kappa_g$ changes sign: $\kappa_g \to -\kappa_g$.

The corresponding normal curvature $\kappa_n = -\mathbf{T}' \cdot \mathbf{N}$ is given in terms of the angle $\alpha$ and the principal curvatures, $C_\perp$ and $C_\parallel$, by Euler’s formula

$$\kappa_n = \sin^2 \alpha C_\perp + \cos^2 \alpha C_\parallel.$$  

(B.9)

If the Gaussian curvature at a point is negative, so that $K_G = C_\perp C_\parallel < 0$, $\kappa_n$ will vanish along the tangential directions given by $\tan \alpha = \frac{C_\parallel}{C_\perp}$. On a minimal surface, $\alpha = \pi/4$ along such curves. The geodesic torsion $\tau_g = \mathbf{L}' \cdot \mathbf{N}$ assumes the form

$$\tau_g = \sin \alpha \cos \alpha \left(C_\parallel - C_\perp \right).$$  

(B.10)

It vanishes when the tangent is aligned along a principal direction. Under the change of chirality $\alpha \to \pi - \alpha$: $\tau_g \to -\tau_g$, whereas $\kappa_n$ is unchanged.
Appendix C. Calculation of $M^Z$

By either projecting equation (63) onto $\hat{k}$, or differentiating $M^Z$ given by equation (64), one obtains

$$M^Z\cdot = R \sin \alpha \left( -\epsilon_L + \lambda^L + C_1 (A^L - \cot \alpha A^T) \right). \quad (C.1)$$

We need to show that the second two terms sum to zero. We do this by showing that

$$I := \lambda^L + C_1 \left( A^L - \cot \alpha A^T \right) = -\cot \alpha V' + V_L V, \quad (C.2)$$

vanishes along curves on axisymmetric surfaces because $V_L V = \cot \alpha V'$ if $V$ depends only on the meridian arc length $l$. Thus one has that $M^Z = -R \sin \alpha \epsilon_L$.

To establish equation (C.2), we use the identity equation (20) for $\lambda^L$ along with equations (B.9) and (B.10) defining $\kappa_\theta$ and $\tau_\theta$ along curves on axisymmetric surfaces, to obtain

$$I = -\cot \alpha \frac{C_1}{\kappa_\theta} \left( \kappa_\theta A^T - \tau_\theta A^L \right) + \frac{\tau_\theta}{\kappa_\theta} V'' + V_L V. \quad (C.3)$$

Using the identity equation (19) for $\kappa_\theta A^T - \tau_\theta A^L$, as well as the identity $\tau_\theta = \cot \alpha (C_1 - \kappa_\theta)$, equation (C.2) follows.

Appendix D. Helicoids

The embedding functions describing a helicoid parametrized by a polar chart on the plane $(r, \theta)$ centered on the rotation axis is given by $X_H(r, \theta) = r \hat{r} + p \theta \hat{k}$, where $\hat{r} = (\cos \theta, \sin \theta, 0); -\infty < r < \infty$. One revolution is described by $-\pi \leq \theta \leq \pi$; the constant $p$ characterizes the pitch; if positive the helicoid is right handed; if negative, it is left handed. As $p \rightarrow 0$ the helicoid degenerates into the plane.

The tangent vectors adapted to the helicoid in this parametrization are given by $e_r = \hat{r}$ and $e_\theta = \hat{\theta} + p \hat{k}$, $\hat{\theta} = (-\sin \theta, \cos \theta, 0)$. Whereas $e_r$ is a unit vector, $e_\theta$ is not; $\hat{e}_\theta = e_\theta / \left( r^2 + p^2 \right)^{1/2}$ is normalized. The line element is $ds^2 = dr^2 + (r^2 + p^2)d\theta^2$.

The unit normal vector is $n = (-p \hat{\theta} + r \hat{k}) / \left( r^2 + p^2 \right)^{1/2}$. The corresponding curvatures are $C_x = \pm p / (r^2 + p^2)$, so that the Gaussian curvature is given by

$$K_G = -p^2 / \left( r^2 + p^2 \right)^2, \quad (D.1)$$

decaying rapidly with distance $r$ along the rulings.

D.1. Curves on the helicoid

Consider a curve on the Helicoid, $s \mapsto = r(s)\hat{r}(s) + p \theta(s)\hat{k}$, parametrized by arc-length. Its unit tangent vector can be expanded with respect to the adapted basis vectors, $T = r'e_r + \theta' e_\theta$, which can be written as $T = t^n e_n$, with components $t^n = (r', \theta')$. Arc-length parametrization implies the normalization $r'^2 + (r^2 + p^2)\theta'^2 = 1$. The surface normal to the curve is $L = -\theta'(r^2 + p^2)^{-1/2} e_r + r' \hat{e}_\theta$. Thus $t^0 = (-\theta'(r^2 + p^2)^{-1/2}, r'(r^2 + p^2)^{-1/2})$. The curve can be characterized by $b'(s)$, the angle that $T$ makes with the direction $e_\theta$, see figure 3.

31 Extending the range of $r$ permits one to treat the two spiral staircases as a single double spiral staircase.
In this parametrization the Darboux tangent basis, $T$ and $L$, is expressed as
\[
\beta = \hat{\theta} + \theta T e + e \cos \sin, \quad \sin \cos, (D.2)
\]
from which one identifies the relations $\sin \theta = r'$ and $\cos \theta = (r^2 + p^2)^{1/2} \theta'$. The geodesic curvature of the curve, $\kappa_g = T' \cdot L$, on the helicoid is
\[
\kappa_g = -\beta' + \frac{r \cos \beta}{r^2 + p^2}. (D.3)
\]
It can also be deduced from its counterpart on the catenoid, defined by equation (B.7), using the isometry between the two geometries, with the replacements $R = (r^2 + p^2)^{1/2}$ and $\alpha = -\beta$. Geodesics, as before, satisfy a Clairaut type relation
\[
\left( r^2 + p^2 \right) \cos^2 \beta = C^2. (D.4)
\]
The normal curvature $\kappa_n = K_{ab} t^{ab}$ is given by
\[
\kappa_n = \frac{p \sin 2\beta}{r^2 + p^2}. (D.5)
\]
Thus the asymptotic directions coincide with the parameter curves with $\beta = 0, \pi/2$. In particular, helices are asymptotic. The geodesic torsion $\tau_g = -K_{ab} t^{ab}$ is given by
\[
\tau_g = \frac{p \cos 2\beta}{r^2 + p^2}. (D.6)
\]
$\alpha$ is measured with respect to $\hat{\phi}$ in an anticlockwise sense, whereas $\beta$ is measured clockwise with respect to $\hat{\theta}$.
Appendix E. Calculation of $G^Z$.

Differentiating $G^Z$ given by equation (79), one gets

$$G^{Z'} = \sqrt{r^2 + p^2} \sin \beta \left( \epsilon_L - \lambda^L - \frac{p}{r^2 + p^2} (A^T - \cot \beta \Lambda^L) \right).$$  \hfill (E.1)

Using equation (19) along with expression (20) for $\lambda^L$ one gets

$$\lambda^L + \frac{p}{r^2 + p^2} (A^T - \cot \beta \Lambda^L) = \left( \frac{p}{r^2 + p^2} \left( \frac{\tau_{g}}{\kappa_n} - \cot \beta \right) - \frac{\kappa_G}{\kappa_n} \right) \Lambda^L$$
$$+ \left( \frac{p}{r^2 + p^2} + \tau_{g} \right) V' + V_L V.$$  \hfill (E.2)

From equations (D.1), (D.5) and (D.6) for $\kappa_G$, $\kappa_n$ and $\tau_{g}$ of the helicoid, one finds that

$$\frac{p}{r^2 + p^2} \left( \frac{\tau_{g}}{\kappa_n} - \cot \beta \right) - \frac{\kappa_G}{\kappa_n} = \frac{p}{r^2 + p^2} (\cot 2\beta - \cot \beta + \csc 2\beta) = 0,$$  \hfill (E.3)

$$\frac{1}{\kappa_n} \left( \frac{p}{r^2 + p^2} + \tau_{g} \right) = \cot \beta.$$  \hfill (E.4)

Thus

$$\lambda^L + \frac{p}{r^2 + p^2} (A^T - \cot \beta \Lambda^L) = \cot \beta V' + V_L V,$$  \hfill (E.5)

which vanishes on account that for a curve on the helicoid $V$ depends only on the radial coordinate $r$, so that $V_L V = - \cot \beta V'$. Thus one has that $G^{Z'} = \sqrt{r^2 + p^2} \sin \beta \epsilon_L$.

Appendix F. Hamiltonian framework adapted to symmetry

F.1. Elastic curves on surfaces with axial symmetry

Consider an energy of the general form (7), with $V = 0$ to avoid clutter. The Lagrangian density along a curve on an axially symmetric surface is given by

$$\mathcal{H}_C = \mathcal{H} \left[ \alpha, \alpha', l, \phi', \lambda_i, \lambda_{\phi} \right] + \lambda_i (l' - \sin \alpha) + \lambda_{\phi} (\phi' - \cos \alpha / R(l)), \hfill (F.1)$$

where $l$ is arc-length along a meridian, $\phi$ is the azimuthal angle, $R$ the polar radius, and $\alpha$ the angle the tangent makes with the polar radial direction. For details see appendix B. The two constraints capture the parametrization by arc-length. The dependence of $\mathcal{H}$ on the generalized coordinates and their derivatives is not arbitrary, but enters through the curvatures and the torsion, given for an axisymmetric surface by (dot is $\partial / \partial l$)

$$\kappa_{\phi}(\alpha', l') = \alpha' - \cos \alpha (\ln R(l)), \hfill (F.2a)$$
$$\kappa_\alpha(\alpha, l) = \sin^2 \alpha C_{\perp}(l) + \cos^2 \alpha C_y(l), \hfill (F.2b)$$
$$\tau_{\phi}(\alpha, l) = 1/2 \sin 2\alpha \left( C_y(l) - C_{\perp}(l), \hfill (F.2c)$$

where $C_{\perp}(l) = ZR - RZ$ and $C_y(l) = Z/R$. The momentum densities conjugate to the three generalized coordinates, $(\alpha, l, \phi)$, are now given by
\( P_\alpha = H_\alpha, \quad P_\lambda = \lambda_\alpha, \quad P_\varphi = \lambda_\varphi. \) \tag{F.3}

\( H \) does not depend explicitly on arc-length \( s \), thus the Hamiltonian density \( \mathcal{H} = \alpha' P_\alpha + \lambda' P_\lambda + \varphi' P_\varphi - H_C \) is constant. Using the relations \( l' = \sin \alpha \) and \( \varphi' = \cos \alpha/R, \) one finds that it is given explicitly by

\[ \mathcal{H} = \left( \kappa_\varphi + \cot (\ln \alpha)^' \right) P_\varphi + \sin \alpha P_\lambda + \cos \alpha \frac{P_\varphi}{R} - H. \] \tag{F.4}

The Hamilton equations for the conjugate momenta are

\[ R \left( \frac{P_\alpha}{R} \right)' = -2\tau_\alpha H_\alpha - \left( K - 2\kappa_\alpha \right) T_\varphi - \cos \alpha P_\varphi + \sin \alpha \frac{P_\varphi}{R}, \] \tag{F.5a}

\[ \sin \alpha P_\lambda = \left( \kappa_\varphi - \frac{(R \alpha')^'}{R} \right) P_\alpha + \left( \kappa_\alpha + 2\alpha' \tau_\varphi \right) H_\alpha \]

\[ + \left( \tau_\varphi + \alpha' \left( K - 2\kappa_\alpha \right) \right) T_\varphi + R' \cos \alpha \frac{P_\varphi}{R^2}, \] \tag{F.5b}

\[ P_\varphi = 0. \] \tag{F.5c}

\( P_\varphi \) is constant on account of the axial symmetry. \( P_\lambda \) is not in general constant and can be eliminated by taking an appropriate combination of equations (F.4) and (F.5a). Specifically, taking the combination \( \sin \alpha P_\lambda + \cos \alpha H \) it is possible to express the constant \( P_\varphi \) in terms of the remaining canonical variables:

\[ P_\varphi = R \sin \alpha \left( P_\alpha + 2\tau_\varphi H_\alpha + \left( K - 2\kappa_\alpha \right) T_\varphi \right) - \left( R \sin \alpha \right) P_\alpha + R \cos \alpha \left( H + \mathcal{H} \right). \] \tag{F.6}

Using the definitions of the remaining canonical momenta in (F.3), along with the identifications \( P_\kappa = -M Z \) and \( H = c \), the ‘first’ integral for a curve, given by equation (67), is reproduced. Thus, the ‘Hamiltonian’ is identified as the constant of integration associated with fixed arc-length and the component of the torque along the axis of symmetry is minus the momentum conjugate to the coordinate along that axis.

For the energy, quadratic and symmetric in the Darboux curvatures, equation (F.6) reads

\[ P_\varphi = R \sin \alpha \left( \kappa_\varphi + \frac{R}{2} \cos \alpha \left( \kappa_\varphi - C_\varphi \right) - \left( R \sin \alpha \right) \right) \left( \kappa_\varphi - C_\varphi \right) \]

\[ + R \left( \kappa_\alpha - C_\alpha \right) \left( 2 \sin \alpha \tau_\varphi + \cos \alpha/2 \left( \kappa_\alpha - C_\alpha \right) \right) \]

\[ + \nu R \left( \tau_\varphi - C_\alpha \right) \left( \sin \alpha \left( K - 2\kappa_\alpha \right) + \cos \alpha/2 \left( \tau_\varphi - C_\alpha \right) \right) + R \cos \alpha. \] \tag{F.7}

To reconstruct the curve, one needs to solve the differential equation for \( P_\varphi \)

\[ P_\alpha = (\ln \left( R \sin \alpha \right)) P_\alpha - 2\tau_\varphi H_\alpha - \left( K - 2\kappa_\alpha \right) T_\varphi + \csc \alpha P_\varphi \]

\[ \frac{P_\varphi}{R} - \cot \alpha \left( H + \mathcal{H} \right), \] \tag{F.8}

subject to the relation (F.3) which provides a relation between \( \alpha' \) and \( P_\alpha \). The conjugate moment \( P_\alpha \) can be determined from the combination \( \sin \alpha H - \cos \alpha P_\varphi \), obtaining

\[ P_\alpha = -\cos \alpha \left( P_\alpha + 2\tau_\varphi H_\alpha + \left( K - 2\kappa_\alpha \right) T_\varphi \right) - \sin \alpha \left( \kappa_\varphi P_\varphi - H - \mathcal{H} \right). \] \tag{F.9}
Using expression (8) this can be recast as

$$-P_l = \sin \alpha F_T + \cos \alpha F_L + \left( \cos \alpha (K - \kappa_n) - \sin \alpha \tau_g \right) T_g - \sin \alpha C_L,$$

so the component of $F$ along the symmetry axis, given by (69), can be written as

$$F^Z = -RC_i \theta \left( P_l + \left( \cos \alpha (K - \kappa_n) - \sin \alpha \tau_g \right) T_g - \sin \alpha C_L \right) - R' \csc \alpha F_i^N.$$

(F.10)

Note the shortcoming of this approach: it does not provide the force.

### F.1.1. Cylindrical constraint

Here $R = R_0$, a constant, $C_L = 0$, and $C_\theta = 1/R_0$. From equation (F.5b) follows that $P_l$ is constant (equation (F.11) identifies it as minus the $F^Z$). For an energy quadratic in the geodesic curvature of the form $H = 1/2 \left( \kappa_g - C_g \right)^2 + h(\kappa_n, \tau_g)$, equation (F.4) reduces to a genuine quadrature,

$$\mathcal{H} = \frac{1}{2} \alpha'^2 + V(\alpha),$$

(F.12)

where

$$V(\alpha) = -h + P_l \sin \alpha + \frac{P_\theta}{R_0} \cos \alpha - \frac{1}{2} C_g^2.$$

(F.13)

Taking into account the identifications $H = c, P_l = -F^Z$ and $P_\theta = -M^Z$, this reproduces the ‘second’ integral equation (73).

### F.1.2. Catenoidal constraint

Here $R(Z) = R_0 \cosh (Z/R_0)$ or $R(l) = (l^2 + R_0^2)^{1/2}$ and $Z(l) = R_0 \arcsinh (l/R_0)$, therefore $C_\theta = R_0/R^2 = -C_L$ and $R' = \sin \alpha \tanh Z/R_0$, thus $\kappa_g = \alpha' - 1/R_0 \cos \alpha \sech Z/R_0$, $\kappa_n = R_0/R^2 \cos 2\alpha$ and $\tau_g = R_0/R^2 \sin 2\alpha$. For the quadratic and symmetric energy in the Darboux curvatures, equation (4), the ‘first integral’, equation (F.6) reads

$$P_\theta = R \sin \alpha \alpha' - R/2 \cos \alpha \alpha'^2 + \sin^2 \alpha \cos \alpha/R \left( \tanh^2 Z/R_0 - \sech^2 Z/R_0 \right)$$

$$+ \left( \cos \alpha/R \tanh Z/R_0 + C_g \right) \left( R/2 \cos \alpha \left( \cos \alpha/R \tanh Z/R_0 + C_g \right) + R' \sin \alpha \right)$$

$$+ \mu R \left( \kappa_n - C_n \right) \left( 2 \sin \alpha \tau_g + \cos \alpha/2 \left( \kappa_n - C_n \right) \right)$$

$$+ \nu R \left( \tau_g - C_0 \right) \left( -2 \sin \alpha \kappa_n + \cos \alpha/2 \left( \tau_g - C_0 \right) \right) + Rc \cos \alpha.$$

(F.14)

### F.2. Helicoidal constraint

The effective energy is

$$\mathcal{H}_C = \mathcal{H} \left[ \beta, \beta', r, r', \theta', \lambda_j, \lambda_\theta \right] + \lambda_r (r' - \sin \beta) + \lambda_\theta \left( \theta' - \cos \beta \sqrt{r'^2 + p^2} \right).$$

(F.15)

The momentum densities conjugate to the three generalized coordinates, $(\alpha, l, \varphi)$, are given by

$$P_\theta = -\mathcal{H}_C, \quad P_l = \lambda_r, \quad P_\varphi = \lambda_\theta.$$

(F.16)

$\mathcal{H}$ does not depend explicitly on arc-length $s$, thus the Hamiltonian density $\mathcal{H} = \beta' P_\beta + r' P_r + \theta' P_\theta - \mathcal{H}_C$ is constant. Using the relations $r' = \sin \beta$ and $\theta' = \cos \beta \sqrt{r'^2 + p^2}$, one finds it is given explicitly by
The Hamilton equations for the conjugate momenta are
\[
\begin{align*}
\sqrt{r^2 + p^2} \left( \frac{P_\beta}{\sqrt{r^2 + p^2}} \right) &= 2 \left( \tau_\beta H_n - \kappa_n T_\beta \right) - P_\gamma \cos \beta + P_\theta \sin \beta \sqrt{r^2 + p^2}, \\
\left( r^2 + p^2 \right) P_\gamma &= P_\beta \cos \beta \frac{r^2 - p^2}{r^2 + p^2} + P_\theta \frac{r \cos \beta}{\sqrt{r^2 + p^2}} - 2 \frac{r}{p} \left( \kappa_n H_n + \tau_\beta T_\beta \right), \\
P_\theta &= 0.
\end{align*}
\] (F.18)

\(P_\theta\) is constant on account of the axial symmetry. \(P_\gamma\) is not in general constant but can be eliminated by taking an appropriate combination of equations (F.17) and (F.18a). Specifically from the combination \(\sin \beta P_\beta + \cos \beta \mathcal{H}\) one can solve for the constant \(P_\theta\) obtaining
\[
-P_\theta = \sqrt{r^2 + p^2} \left[ \sin \beta \left( -P_\beta + 2 \tau_\beta H_n - 2 \kappa_n T_\beta \right) \right] - \cos \beta \left( \mathcal{H} + \mathcal{H} \right) + P_\beta \left( \sqrt{r^2 + p^2} \sin \beta \right).
\] (F.19)

Taking into account the relation (F.16), along with the identifications \(P_\theta = -M^2\) and \(\mathcal{H} = c\), the ‘first’ integral for an elastic curve on a helicoid, given by equation (81), is reproduced. Thus, the ‘Hamiltonian’ is again identified as the constant of integration associated with fixed arc-length and the component of the torque along the axis of symmetry is minus the momentum conjugate to the coordinate along that axis.

To reconstruct the curve, one has to solve the ODE for \(P_\beta\)
\[
P_\beta = P_\beta \left( \ln \left( \sqrt{r^2 + p^2} \sin \beta \right) \right) + 2 \left( \tau_\beta H_n - \kappa_n T_\beta \right) - \frac{P_\theta}{\sqrt{r^2 + p^2}} - \cos \beta \left( \mathcal{H} + \mathcal{H} \right); \\
+ \csc \beta \frac{P_\theta}{\sqrt{r^2 + p^2}} - \cot \beta \left( \mathcal{H} + \mathcal{H} \right); \\
\] (F.20)

the identification in equation (F.16) provides a relation between \(\beta'\) and \(P_\beta\). The conjugate moment \(P_\gamma\) can be determined from the combination \(\sin \beta \mathcal{H} - \cos \beta P_\beta\), obtaining
\[
P_\gamma = \cos \beta \left( -P_\beta + 2 \left( \tau_\beta H_n - \kappa_n T_\beta \right) \right) + \sin \beta \left( \kappa_n P_\theta + \mathcal{H} + \mathcal{H} \right).
\] (F.21)

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