Functional quantization of Generalized Scalar Duffin-Kemmer-Petiau Electrodynamics

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Abstract

The main goal of this work is to study systematically the quantum aspects of the interaction between scalar particles in the framework of Generalized Scalar Duffin-Kemmer-Petiau Electrodynamics (GSDKP). For this purpose the theory is quantized after a constraint analysis following Dirac’s methodology by determining the Hamiltonian transition amplitude. In particular, the covariant transition amplitude is established in the generalized non-mixing Lorenz gauge. The complete Green’s functions are obtained through functional methods and the theory’s renormalizability is also detailed presented. Next, the radiative corrections for the Green’s functions at $\alpha$-order are computed; and, as it turns out, an unexpected $m_p$-dependent divergence on the DKP sector of the theory is found. Furthermore, in order to show the effectiveness of the renormalization procedure on the present theory, a diagrammatic discussion on the photon self-energy and vertex part at $\alpha^2$-order are presented, where it is possible to observe contributions from the DKP self-energy function, and then analyse whether or not this novel divergence propagates to higher-order contributions. Lastly, an energy range where the theory is well defined: $m^2 \ll k^2 < m_p^2$ was also found by evaluating the effective coupling for the GSDKP.

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1 Introduction

The Duffin-Kemmer-Petiau (DKP) equation is a first-order relativistic theory for the description of spin 0 and spin 1 bosons with a similar form as the Dirac equation\(^1\). Substantiated on Imbert’s experiments\(^3\), which suggested strong classical and quantum contradictions for longitudinal plane waves displacement, de Broglie states that a possible non-zero rest mass to photons would be the right interpretation for that phenomenon\(^4\). In fact, de Broglie also suggested that the photon should be formed by the combination of two leptons and such a combination should be responsible for assign a mass to photon. Driven by this idea and with a deep knowledge on the algebraic structure of Dirac’s equation (relativistic equation for spin 1/2 particle), de Broglie begins his search of a first-order equation in the hope of obtaining an equation for a massive particle of spin 1, his massive photon\(^5\).

Petiau was the first to obtain the matrix algebra of DKP\(^6\). Simultaneously and completely unaware to the work of Petiau, Kemmer wrote the second-order Proca equations and the Klein-Gordon-Fock (KGF) equation as a set of coupled first-order equations. Kemmer then conjectures about the existence of a matrix form describing this system of coupled equations, on which irreducible representations representing particles of spin 0 (scalar particles) and spin 1 (vectorial particles). Duffin develops the desired algebra for Kemmer’s theory\(^[8–10]\).

The DKP formalism allows an unified treatment of the scalar and vector fields\(^3\) and the wealth of couplings in the DKP formalism made the theory initially well received; in fact, due to its unique algebraic structure this formalism enjoys a plenty of couplings incapable of being expressed in the theories of KGF and Proca\(^[12–16]\). However, the equivalence of DKP and KGF in the free and minimal electromagnetic coupling cases\(^[17–20]\), both in classical and quantum pictures, led to a decreased interest in DKP theory. Although the KGF formalism is apparently simpler when compared to the algebraic treatment of the DKP theory in a classical picture, this point of view changes dramatically in a quantum picture: the similar form between the DKP and Dirac Lagrangian expressions allows a very simplified mechanism to study scalar phenomena, once the mimetism with Dirac theory can be used to understanding of the physical meaning of all the quantities obtained from the DKP theory\(^[21, 22]\).

In the last years the DKP theory has been studied on QCD at large and short distances by Gribov\(^[23]\), in the scattering K\(^+\)-nucleus\(^[24]\), covariant Hamiltonian dynamics\(^[25]\), in generalization to curved space-time\(^[26]\), in a five-dimensional Galilean covariance\(^[27]\), in the context of classical gauge invariance\(^[28]\), in the Epstein–Glaser causal method\(^[29]\) and so on.

Although the extensive research concerning the DKP theory within the framework of gauge theory, it is desirable to consider its interaction with distinct gauge fields. It is a well-known fact that Maxwell’s electrodynamics is considered as being one of the most successful physical theories; de-

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1. The historical development of this theory, among others, can be found in\(^ [1, 2]\).
2. Forsooth, Géhéniau decomposed Petiau’s sixteen-dimensional algebra in terms of irreducible representations of ten dimensions (representing particles of spin 1), five dimensions (representing particles of spin 0), and a trivial representation without physical meaning of one dimension\(^ [7]\).
3. This formalism can be extended to describe non-Abelian and gravitational fields\(^ [11]\).
spite that, the research in pursuing gauge-invariant alternatives extensions in order to supplement it is an ongoing subject of study [31,35]. Among the several features that these variants are endowed, the main difference between these theories is due to the nonlinearity of the field equation, e.g. Born-Infeld and Euler-Heisenberg Lagrangians, while the linearity of the field equation with higher-order derivatives, e.g. Bopp-Podolsky [32,33].

It is a well-known fact that higher-order derivative (HD) theories have [34,35], in the light of effective field theory [36], better renormalization properties than the conventional ones. One of the most interesting contributions to show the effectiveness of the HD terms in field theory is the Bopp-Podolsky electrodynamics, a generalization of the Maxwell electromagnetic field [37]. Moreover, the Ref. [41] showed that the Podolsky Lagrangian is the only linear generalization of Maxwell electrodynamics that preserves invariance under $U(1)$. An important feature concerning the Bopp-Podolsky electrodynamics is the gauge fixing, i.e. on how to fix the correct physical degrees of freedom, since the usual Lorentz condition is not suitable. It was shown in the Ref. [42] that the natural condition in the Bopp-Podolsky electrodynamics is the generalized Lorentz condition, $\Omega[A] = (1 + a^2 \Box) \partial^\mu A_\mu$. However, there are alternative gauge conditions that allow the same identification, in particular, and in order to preserve the order of the field equation, the so-called non-mixing gauge term [43,44], $\Omega[A] = (1 + a^2 \Box)^{1/2} \partial^\mu A_\mu$, which is a pseudo-differential operator [45].

The quantum-particle of this field is called Podolsky photon and the interaction between these particles and fermionic (scalar) fields is known as (scalar) generalized quantum electrodynamics, GQED$_4$ (GSQED$_4$) [46–48]. It was shown in these series of papers that the Podolsky photon has the quality of controlling UV divergencies, leading to an almost finite theory, since the finiteness depends on the divergence degree of the diagram. Further analysis with Bopp-Podolsky electrodynamics were realized at thermodynamical equilibrium [49,50], with boundary conditions [51], and in the presence of external sources [52]. Furthermore, in previous analysis the authors have determined a bound for the free parameter $m_p \geq 350$ GeV [44,47].

In particular, it should be noted that higher-derivative theories have a Hamiltonian which is not bounded from below [53] and that the addition of such terms leads to the existence of negative norm states (or ghosts states) jeopardizing thus the unitarity [54]. Despite the fact that many attempts to restore the unitarity by means of overcoming these ghost states, no one has been able to give a general method to deal with them [55,56]. Nonetheless, recently in Ref. [57] a procedure was suggested for including interactions in free HD systems without breaking their stability, remarkably it was shown that the dynamics of the GQED is stable at both classical and quantum level.

Therefore, based on the positive aspects and outcome of both Bopp-Podolsky electrodynamics and DKP theory, it is rather natural a systematic study of their interaction. Moreover, this work concerns itself in the scalar sector of the DKP theory, describing the electromagnetic interaction between scalar

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4This idea is successful in the case of the attempt to quantize gravity, where the (non-renormalizable) Einstein action is supplied by terms containing higher powers of curvature leading to a renormalizable [37]. Also, a new impetus in exploring appealing quantum theories such as $f(R)$-gravity [38].

5A non-Abelian version of the Bopp-Podolsky electrodynamics was studied and deeply analyzed in [39,40].
fields in a different phenomenologically way, which will certainly complement the known results in the literature [21,48]. This work is therefore devoted to the analysis of the Generalized Scalar Duffin-Kemmer-Petiau Electrodynamics. In Sec. 2 the covariant transition amplitude is derived by a constraint analysis in the non-mixing Lorentz gauge condition. In Sec. 3 the Schwinger-Dyson-Fradkin equations are calculated, and the complete expressions for the basic Green’s functions is obtained. In Sec. 4 the Ward-Takahashi-Fradkin identities are derived and subsequently, in Sec. 5, the renormalizability of the present theory is established. In Sec. 6 the radiative corrections at one loop are computed and the (finite) counter-terms are presented. In Sec. 7 the photon propagator and the vertex at two-loops order are discussed diagrammatically. In Sec. 8 the authors present their final remarks and prospects. In the whole work the metric signature $(+,-,-,-)$ for the Minkowski spacetime is used.

2 Canonical transition amplitude

The Lagrangian density describing the GSDKP is defined by

$$\mathcal{L} = \frac{i}{2} \bar{\psi} \beta^\mu (\partial_\mu \psi) - \frac{i}{2} \left( \partial_\mu \bar{\psi} \beta^\mu \right) \psi - m \bar{\psi} \psi + eA_\mu \bar{\psi} \beta^\mu \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha^2}{2} \partial_\mu F_{\mu \beta} \partial_\alpha F^{\alpha \beta}, \quad (2.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic field-strength tensor and $\beta^\mu$ are the DKP matrices that obey the algebra

$$\beta^\mu \beta^\nu \beta^\theta + \beta^\theta \beta^\nu \beta^\mu = \beta^\mu \eta^{\nu \theta} + \beta^\theta \eta^{\nu \mu}. \quad (2.2)$$

For further information about the representation of the matrices $\beta^\mu$ see the appendix B. Although DKP theory is formally similar to Dirac theory, there are several subtle contrasting behaviour already in classical level. For instance, the conjugated field of the fermionic theory is characterized as $\bar{\psi} = \psi^\dagger \gamma^0$, whereas the conjugate DKP field is defined such as $\bar{\psi} = \psi^\dagger \eta^0$, where $\eta^0 = 2(\beta^0)^2 - 1$. Moreover, one can sample for an arbitrary four-vector $p$ the following relation is satisfied

$$\hat{\rho}(p^2 - \hat{p}^2) = 0. \quad (2.3)$$

This shows another contrast with the fermionic theory, since $\hat{\rho}^2 \neq p^2$. Nonetheless, this relation combined with plane wave solutions for the free field equations leads to $p^2 = m^2$.

Classically this theory is invariant under local gauge transformations

$$\psi \to e^{i\alpha(x)} \psi, \quad A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \alpha(x). \quad (2.4)$$

The Euler-Lagrange equations are obtained as usual from the Hamiltonian principle

$$\left[ i \beta^\mu (\partial_\mu - ieA_\mu) - m \right] \psi = 0, \quad (2.5)$$

$$(1 + a^2 \Box) \partial_\mu F^{\lambda \mu} = e \bar{\psi} \beta^\lambda \psi. \quad (2.6)$$

6Throughout the text the following compact notation $\hat{O} = \beta^\mu O_\mu$ will be used.
The translational space-time invariance of the Lagrangian density leads to the canonical Hamiltonian

\[
H_c = \int d^4x \left[ (\partial_0 \bar{\psi}) \frac{\partial \mathcal{L}}{\partial (\partial_0 \bar{\psi})} + \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} (\partial_0 \psi) + \frac{\partial \mathcal{L}}{\partial (\partial_0 A_v)} (\partial_0 A_v) - \mathcal{L} \right].
\]  

(2.7)

Thus the canonical momenta associated with the DKP fields \( (\bar{\psi}, \psi) \) are

\[
p = \frac{\partial \mathcal{L}}{\partial (\partial_0 \bar{\psi})} = -i \beta^0 \bar{\psi},
\]

(2.8)

\[
\bar{p} = \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = i \bar{\psi} \beta^0,
\]

(2.9)

whereas the canonical momenta for gauge fields are obtained from the Ostrogradski method \([34]\). This method consists in defining the dynamics of the system in a first-order form, i.e., the dynamics takes place in a spanned phase space characterized by the independent variables \( A_\mu, \Gamma_\mu \equiv \partial_0 A_\mu, \Phi^\mu \)

\[
\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \partial_0 \Gamma^\mu} - 2 \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 \Gamma^\mu)} - \partial_0 \frac{\partial \mathcal{L}}{\partial (\partial_0 \Gamma^\mu)},
\]

\[
= F^{\mu 0} + a^2 [\eta^{\mu \nu} \partial_\nu A^{00} - \partial_0 \partial_\nu A^{0 \nu}],
\]

(2.10)

\[
\Phi^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_0 \Gamma^\mu)} = a^2 [\partial_\nu A^{\nu 0} - \eta^{\nu 0} \partial_\nu A^{\nu 0}].
\]

(2.11)

From the above momentum expressions, the constraint structure of the theory can be studied by following Dirac’s approach to singular systems \([58]\). In this approach it is possible to obtain the set of first-class constraints

\[
\varphi_1 = \Phi_0 \approx 0, \quad \varphi_2 = \Pi_0 - \partial_0 \Phi^k \approx 0, \quad \varphi_3 = e \bar{\psi} \beta^0 \psi - \partial^k \Pi_k \approx 0,
\]

(2.12)

and the set of second-class constraints

\[
\chi^{(1)} = p + i \frac{1}{2} \beta^0 \bar{\psi} \approx 0, \quad \bar{\chi}^{(1)} = \bar{p} - i \frac{1}{2} \bar{\psi} \beta^0 \approx 0,
\]

(2.13)

\[
\chi^{(2)} = [1 - (\beta^0)^2] [i \beta^i \partial_i \psi(x) - m \bar{\psi}(x) + e \bar{\psi} A_i(x) \psi(x)] \approx 0,
\]

(2.14)

\[
\bar{\chi}^{(2)} = [-i \partial_i \bar{\psi}(x) \bar{\beta}^i + m \bar{\psi}(x) - e \bar{\psi}(x) \beta^i A_i(x)] [1 - (\beta^0)^2] \approx 0.
\]

(2.15)

The weak equality \( \approx \) is understood in according to Dirac’s sense.

With the full set of first-class and second-class constraints determined, the next step is to obtain the functional generator. The transition amplitude in the Hamiltonian form is written in the following way \([59]\)

\[
Z = N \int D\mu \exp \left\{ i \int d^4x [(\partial_0 \bar{\psi}) p + \bar{p} (\partial_0 \psi) + \Pi^\mu (\partial_0 A_\mu) + \Phi^\mu (\partial_0 \Gamma^\mu) - \mathcal{H}_c] \right\}
\]

(2.16)

where the canonical Hamiltonian is given by

\[
\mathcal{H}_c = \Pi_0 \Gamma_0 + \Pi_k \Gamma^k + \Phi_k (\partial^k \Gamma_0 - \partial_k \Gamma_0) + \frac{\Phi^k}{2a^2} - i \bar{\psi} \beta^i \partial_i \psi + m \bar{\psi} \psi
\]

\[
- e \bar{\psi} A_i + \frac{1}{4} F_{kj} F^{kj} + \frac{1}{4} (\Gamma_j - \partial_j A_0)^2 - \frac{a^2}{2} (\partial_j \Gamma_j - \partial_j \partial_j A_0)^2,
\]

(2.17)
and the integration measure is defined in such a way that it transforms as an scalar at the constrained phase space

\[ D\mu = D\Phi^v D\Gamma_v D\Pi^\mu DA_\mu D\bar{\psi} D\psi D\bar{p} Dp \delta(\Theta_l) \det \left\| \{ \Theta_l, \Theta_m \} \right\|^2. \]  

(2.18)

Now, the complete set of constraints for the GSDKP is

\[ \Theta_l = \left\{ \chi^{(1)}, \bar{\chi}^{(1)}, \chi^{(2)}, \bar{\chi}^{(2)}, \varphi_1, \varphi_2, \varphi_3, \Sigma_1, \Sigma_2, \Sigma_3 \right\}, \]  

(2.19)

in which a suitable gauge conditions for the first-class constraints are chosen as the generalized radiation conditions [42]

\[ \Sigma_1 = \Gamma_0(x) \approx 0, \quad \Sigma_2 = A_0 \approx 0, \quad \Sigma_3 = (1 + a^2 \Box)(\bar{\nabla} \cdot \bar{A}) \approx 0. \]  

(2.20)

After integrating over the gauge and fermionic momenta the transition amplitude \( Z \) is explicitly written

\[ Z = N \int D\mu D\bar{\psi} D\psi \det \left| (1 + a^2 \bar{\nabla}^2) \bar{\nabla}^2 \right| \delta((1 + a^2 \Box)(\bar{\nabla} \cdot \bar{A})) \times \exp \left[ i \int d^4x \left\{ \bar{\psi} (i\beta^\mu \nabla_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial^\mu F_{\mu\beta} \partial_\alpha F^{\alpha\beta} \right\} \right]. \]  

(2.21)

Although the above expression is correct its form is not explicitly covariant; then it is not convenient for purposes of calculation. However the Faddeev-Popov-DeWitt ansatz [60] allows a covariant form for the amplitude of vacuum-vacuum transition.

Hence, using the Faddeev-Popov-DeWitt ansatz in the non-mixing gauge condition [43]

\[ \Omega(A) = (1 + a^2 \Box)^{\frac{1}{2}} \partial^\mu A_\mu \]  

(2.22)

the transition amplitude can be written as

\[ Z = N \int D\mu D\bar{\psi} D\psi \exp \left\{ i \int d^4x \left[ \bar{\psi} (i\beta^\mu \nabla_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} \partial^\mu F_{\mu\beta} \partial_\alpha F^{\alpha\beta} \right] \right\}. \]  

(2.23)

The choice of using the non-mixing gauge condition, \((1 + a^2 \Box)^{\frac{1}{2}} \partial^\mu A_\mu\), is rather justified by calculation purposes because it preserves the order of the field equation [44]; since the natural choice in the Podolsky theory, the generalized Lorenz term \((1 + a^2 \Box)(\partial_\mu A^\mu)\) complicates the theory’s quantization once it increases the order of the field equation. Then the non-mixing gauge term is related to a pseudodifferencial operator [45].

The minimal coupling DKP functional generator with the higher-derivative Podolsky term can be written as

\[ \mathcal{L}' [\eta, \bar{\eta}, J_\mu] = \int D\mu (\psi, \bar{\psi}, A_\mu) \exp \left[ iS_{eff} \right] \]  

(2.24)
where the effective action is defined by
\[
S_{\text{eff}} = \int d^4x \left[ \bar{\psi} (i\beta^\mu \partial_{\mu} - m + e\beta^\mu A_\mu) \psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{a^2}{2} \partial^\mu F_{\mu\beta} \partial^\alpha F^{\alpha\beta} \right.
\]
\[
- \frac{1}{2\xi} (\partial^\mu A_\mu) \left( 1 + a^2 \Box \right) (\partial^\mu A_\mu) + \bar{\psi} \eta + \bar{\eta} \psi + A_\mu J^\mu \right].
\]
(2.25)
and \( \eta, \bar{\eta} \) and \( J_\mu \) are the sources from fundamental fields involved, namely \( A_\mu, \psi, \bar{\psi} \).

3 Schwinger-Dyson-Fradkin equations

It has been known for a long time that it is possible to describe all content of a particular field theory as a set of field equations in the Heisenberg description. The most elegant way of studying such equations and extract the physical content is the functional formulation, consisting in an infinite chain of differential equations that relates different Green’s function in an exact manner [61,62]. This infinite tower of equations refers to the Schwinger-Dyson-Fradkin (SDF) equations.

The propose of this section is to determine the complete SDF equations for the basic propagators, for the gauge and DKP fields, and also for the vertex function using the functional generator defined by the equation (2.24).

3.1 The Schwinger-Dyson-Fradkin equations for the photon propagator

The complete expression for the gauge-field propagator can be determined by means of the functional generator (2.24) leading to the Schwinger variational equation for the gauge field, in which \( S \) differs from \( S_{\text{eff}} \) by source terms
\[
\left[ \frac{\delta S}{\delta A_\gamma (x)} \right] \bigg|_{\delta \bar{\eta} \over \delta \bar{\eta}, \delta \eta \over \delta \eta, \delta J_\mu \over \delta J_\mu} + J^\gamma (x) \right] \mathcal{Z} [\eta, \bar{\eta}, J_\mu] = 0.
\]
(3.1)
It should be remarked that the field limits are related to the functional Fourier transform as
\[
A_\gamma (x) \to \frac{1}{i} \frac{\delta}{\delta J^\gamma (x)}, \quad \bar{\psi} (x) \to \frac{1}{i} \frac{\delta}{\delta \bar{\eta} (x)}, \quad \psi (x) \to \frac{1}{i} \frac{\delta}{\delta \eta (x)}.
\]
(3.2)
Nonetheless, in solving the above relation the generating functional \( \mathcal{Z} [J] = \exp \{iW [J]\} \) for the connected Green’s functions must be introduced. Then the Schwinger variational equation becomes
\[
-J^\gamma (x) = -ie \frac{\delta}{\delta \eta (x)} \beta^\gamma \left( \frac{\delta W}{\delta \bar{\eta} (x)} \right) + e \frac{\delta W}{\delta \eta (x)} \beta^\gamma \frac{\delta W}{\delta \bar{\eta} (x)}
\]
\[
+ \left[ T^\gamma_\mu + \frac{1}{\xi} L^\gamma_\mu \right] \left( 1 + a^2 \Box \right) \frac{\delta W}{\delta J^\mu (x)}.
\]
(3.3)
The last equation can be interpreted as the complete Podolsky field equation subjected to a external source \( J^\gamma \). On this equation \( T^\gamma_\mu \) and \( L^\gamma_\mu \) are differential projectors
\[
T^\gamma_\mu + L^\gamma_\mu = g^\gamma_\mu, \quad L^\gamma_\mu = \frac{\partial^\gamma \partial^\mu}{\Box}.
\]
(3.4)
In order to obtain the complete gauge-field propagator it proves convenient to introduce the generating functional for the one particle irreducible (1PI) Green’s functions as well, which is related to $W$ by a functional Legendre transformation

$$ \Gamma [\psi, \bar{\psi}, A_\mu] = W [\eta, \bar{\eta}, J_\mu] - \int d^4x \left( \bar{\psi} \eta + \bar{\eta} \psi + A^\mu J_\mu \right). \tag{3.5} $$

Hence, rewriting (3.3) in terms of the 1PI $\Gamma [\psi, \bar{\psi}, A_\mu]$ and differentiating the resulting expression with respect to $A_\nu (y)$

$$ \frac{\delta^2 \Gamma}{\delta A_\nu (y) \delta A_\gamma (x)} = -ie\beta^\gamma \frac{\delta}{\delta A_\nu (y)} \left( \frac{\delta^2 W}{\delta \eta (x) \delta \bar{\eta} (x)} \right) + \left[ T^{\gamma \nu} + \frac{1}{\xi} L^{\gamma \nu} \right] \left( 1 + a^2 \Box \right) \delta^{(4)} (x, y). \tag{3.6} $$

From the above definitions, one can obtain identities relating the connected and 1PI two-point functions. For instance, it follows that for the DKP field

$$ i \int d^4z \mathcal{S} (x, z; A) \frac{\delta^2 \Gamma}{\delta \psi (y) \delta \bar{\psi} (z)} = \delta^{(4)} (x - y), \tag{3.7} $$

in which the complete DKP propagator is defined such as

$$ \mathcal{S} (x, z; A) = i \left. \frac{\delta^2 W [\eta, \bar{\eta}, J_\mu]}{\delta \eta (z) \delta \bar{\eta} (x)} \right|_{\eta = \bar{\eta} = 0}. \tag{3.8} $$

Another important quantity to be defined is the complete DKP-photon 1PI vertex function

$$ e\Gamma^\nu (x, z; y) = \left. \frac{\delta^3 \Gamma}{\delta A_\nu (y) \delta \psi (z) \delta \bar{\psi} (x)} \right|_{A_\nu = \psi = \bar{\psi} = 0}, \tag{3.9} $$

which after some algebraic manipulation makes the equation (3.6) possible to be rewritten as

$$ \frac{\delta^2 \Gamma}{\delta A_\nu (y) \delta A_\gamma (x)} = \left[ T^{\gamma \nu} + \frac{1}{\xi} L^{\gamma \nu} \right] \left( 1 + a^2 \Box \right) \delta^{(4)} (x, y) 
$$

$$ + ie^2 \int d^4ud^4wTr \left[ \mathcal{S} (x, u; A) \beta^\gamma \mathcal{S} (w, x; A) \Gamma^\nu (u, w; y) \right]. \tag{3.10} $$

The second term of (3.10) can be identified with the polarization operator, $\Pi^{\gamma \nu}$,

$$ \Pi^{\gamma \nu} (x, y) = ie^2 \int d^4ud^4wTr \left[ \mathcal{S} (x, u; A) \beta^\gamma \mathcal{S} (w, x; A) \Gamma^\nu (u, w; y) \right], \tag{3.11} $$

defined as the sum of all compact self-energy photon parts. The absence of a $(-1)$ factor comes from the fact that there is a bosonic loop related to the DKP field, not a fermionic as in the Dirac field.

Then the gauge field satisfies an identity as (3.7); therefore

$$ (\mathcal{D}^{\nu \rho})^{-1} (z, y) = \frac{\delta^2 \Gamma}{\delta A_\rho (y) \delta A_\gamma (z)}, \tag{3.12} $$

relates the inverse of the complete (and free) photon propagator to the 1PI Green’s function. Thus the expression for the photon’s inverse complete propagator in momentum representation is

$$ (\mathcal{D}^{\nu \rho})^{-1} (p) = (D^{\nu \rho})^{-1} (p) + \Pi^{\nu \rho} (p). \tag{3.13} $$
We can represent the above equation diagrammatically, as in Figure 1.

The expression (3.13) can be solved in order to find

\[
iD^\gamma\nu(p) = \left[\eta^\gamma\nu - \frac{p^\gamma p^\nu}{p^2}\right] + \frac{\xi}{p^2 (1 - a^2 p^2)} \frac{p^\gamma p^\nu}{p^2},
\]

\[(3.14)\]

in which the scalar polarization \(\Pi(p)\) is related to the scalar polarization \(\Pi^\gamma\nu(p)\) through the structure

\[
\Pi^\gamma\nu(p) = (-p^2 \eta^\gamma\nu + p^\gamma p^\nu) \Pi(p).
\]

For the free propagator, namely \(\Pi(p) = 0\) on (3.14) and \(a = m_p^{-1}\), one has the expression

\[
iD^\gamma\nu(p) = \left[\eta^\gamma\nu - (1 - \xi) \frac{p^\gamma p^\nu}{m_p^2}\right] \left[\frac{1}{p^2} - \frac{1}{p^2 - m_p^2}\right] - (1 - \xi) \frac{p^\gamma p^\nu}{(p^2)^2}.
\]

\[(3.16)\]

Note that there are no mixing between the massless and massive poles, in contrast with the usual generalized Lorenz condition, owing to the non-mixing gauge fixing. It should be remarked that in Ref. [57] a procedure was suggested for including interactions in free HD systems without breaking their stability (ghost modes) and it holds for GSDKP. In addition, previous results in the fermionic and mesonic generalized theories [46–48] also motivate an attention to the present theory, once the propagator (3.16) has a UV finite behavior (in the light of effective theories) and an interesting renormalized behavior.

### 3.2 The Schwinger-Dyson-Fradkin equations for the DKP propagator

This subsection is devoted to keep on deriving the SDF equations, obtaining now an integral expression for the complete DKP propagator. Starting with the Schwinger variational equation

\[
\left[\frac{\delta S}{\delta \bar{\psi}(x)} + \eta(x)\right] \mathcal{Z}[\eta, \bar{\eta}, J_\mu] = 0.
\]

\[(3.17)\]

writing it in terms of the generating functional \(W\) and then differentiating the resulting expression with respect to the source \(\eta(y)\) leads to

\[
i\delta^{(4)}(x - y) = -\left[i\beta^\mu \partial_\mu - m + e\beta^\mu \langle A_\mu \rangle - ie\beta^\mu \frac{\delta}{\delta J_\mu(x)}\right] \mathcal{Z}(x,y;A).
\]

\[(3.18)\]
Figure 2: The SDF equation for the scalar propagator.

by solving the derivative of the last term one can immediately identify
\[ \Sigma(x,z) = ie^2 \beta^\mu \int d^4u d^4v \mathcal{D}^\alpha_\mu (u,x) \mathcal{J}(x,v;A) \Gamma_\alpha (v,z;u), \] (3.19)
as the DKP self-energy function \( \Sigma \). Hence, by taking the limit of null sources
\[ i \delta^{(4)}(x-y) = -[i \beta^\mu \partial_\mu - m] \mathcal{J}(x,y;A) + \int d^4z \Sigma(x,z) \mathcal{J}(z,y;A). \] (3.20)

In momentum representation this equation becomes \( S^{-1}(p) = S^{-1}(p) + \Sigma(p) \). This equation can be viewed as in the figure 2.

The above equation can formally be written as
\[ \mathcal{J}(p) = \frac{i}{\beta^\mu p_\mu - \mathcal{M}(p)}, \] (3.21)
where the mass operator \( \mathcal{M} \) is defined by
\[ \mathcal{M}(p) = m + \Sigma(p) \] (3.22)
showing that the mass operator encompasses both the DKP self-energy \( \Sigma \) and the bare mass \( m \).

Besides, the expression for the DKP free propagator can be obtained with the help of the DKP algebra (2.2),
\[ S(p) = i \frac{1}{m} \left[ \hat{p} (\hat{p} + m) \right]^{-1}. \] (3.23)

As one can see, the self-energy function (3.19), differently from the photon function (3.11), is sensitive to the effects of the Podolsky \( m_P \)-dependent terms of (3.16) already at first order on perturbation theory.

### 3.3 The Schwinger-Dyson-Fradkin equations for the vertex part

As is well known, the SDF equations do not only depend on the fundamental Green’s functions of a given theory, but they do depend on higher-order functionals, which also satisfy their own SDF equations. This will become clear in the derivation of the vertex function. Although it should be remarked that it is possible to find a relation that connects the complete vertex function with \( \mathcal{J} \) and \( \mathcal{D} \) which contain only skeleton graphs, i.e., connected graphs [63, 64].

The starting point for the derivation of the vertex function is (3.18), this also follows from the guideline presented previously. In a similar way, on taking the derivative of the resulting expression...
with respect to the field $A_\sigma(z)$, and after some manipulations, one finds the following expression for the vertex function

$$i\Gamma^\sigma(q, p; k) = -\beta^\sigma(2\pi)^4 \delta(q - p - k) + i\Lambda^\sigma(k, p; q)$$

in which a new quantity, the vertex part, has been introduced

$$i\Lambda^\sigma(q, p; k) = ie^{2\mu} \frac{1}{(2\pi)^4} \int d^4t \mathcal{D}_{\mu\rho}(t) \mathcal{J}(t + k) \Phi^{\sigma\rho}(t + k, p; q, t)$$

$$+ e^{2\mu} \frac{1}{(2\pi)^4} \int d^4p_1 d^4p_2 \mathcal{D}_{\mu\rho}(p_1) \mathcal{J}(p_1 + k) \Gamma^\sigma((p_1 + k), p_2; q) \mathcal{J}(p_2) \Gamma^\rho(p_2, p; p_1)$$

and defined the four-point vertex function

$$e^2\Phi^{\sigma\rho}(a, w; z, s) = \frac{\delta^4\Gamma}{\delta A_\rho(s) \delta A_\sigma(z) \delta \psi(w) \delta \bar{\psi}(a)}.$$

Equation (3.25) shows explicitly that the three-point vertex function depends on the four-point one, emphasizing the tower of equations that SDF equations are. However, the present work focuses in a perturbative calculation, and then the situation here is not that complex, once the three fundamental Green’s functions of interest that can evaluate the respective radiative corrections and the effects from the HD contributions from the photon propagator had already been determined. Diagrammatically, the irreducible vertex part can be visualized in figure 3.

4 Ward-Fradkin-Takahashi identities

Although relativistic quantum systems are formulated in the framework of gauge fields, all physical observables in a field theory are gauge independent. The existence of a local gauge symmetry in a field theory generates constraint relations between the theory’s Green’s functions. These relations are known as the Ward–Fradkin–Takahashi identities. These identities, in terms of Green’s functions,
are closely related with the renormalizability of a theory. The purpose of this section is to derive such identities for GSDKP electrodynamics using a functional approach [65].

The derivation of the WTF identities is formally given in terms of the following identity upon the functional generator (2.24)

\[
\frac{\delta \mathcal{Z}[\eta, \bar{\eta}, J_\mu]}{\delta \alpha(x)} \bigg|_{\alpha=0} = 0. 
\]

(4.1)

This leads to the equation of motion satisfied by \(Z[\eta, \bar{\eta}, J_\mu]\)

\[
\left[-i \Box_\xi (1 + a^2 \Box) \frac{\partial \mu}{\partial J^\mu} - \frac{\delta}{\delta \eta} \eta + \bar{\eta} \frac{\delta}{\delta \bar{\eta}} - \frac{1}{e} \frac{\partial \mu}{\partial J^\mu} \right] \mathcal{Z} = 0. \tag{4.2}
\]

Finally, one can obtain the desired quantum equation of motion for the theory by writing (4.2) first in terms of \(W\), and then as an expression for the 1PI-generating functional \(\Gamma[\psi, \bar{\psi}, A_\mu]\) through the relation (3.5). One then obtain

\[
\left[-i \Box_\xi (1 + a^2 \Box) \frac{\partial \mu}{\partial A_\mu} - \bar{\psi} \frac{\delta \Gamma}{\delta \psi} + \frac{\delta \Gamma}{\delta \bar{\psi}} \psi + \frac{1}{e} \frac{\partial \mu}{\partial A_\mu} \right] \mathcal{Z} = 0. \tag{4.3}
\]

This is the equation that will supply all the WFT identities.

The first identity comes by applying the derivatives of (4.3) with respect to \(\psi(y)\) and \(\bar{\psi}(z)\), yielding

\[
\partial^\mu \Gamma^\mu(z, y; x) = - \delta(x - z) \Gamma(x, y) + \Gamma(x, z) \delta(x - y), \tag{4.4}
\]

where \(\Gamma(x, z) = \frac{\delta^2 \Gamma}{\delta \psi(z) \delta \psi(x)}\). Besides, writing it in momentum representation

\[
k_\mu \Gamma^\mu(p, p', k = p - p') = \mathcal{S}^{-1}(p') - \mathcal{S}^{-1}(p). \tag{4.5}
\]

Furthermore, on considering the limit of this equation as \(k \to 0\), one can find that the vertex part is related to the DKP self-energy function as

\[
\Lambda^\mu(p, p, k = 0) = - \frac{\partial}{\partial p_\mu} \Sigma(p). \tag{4.6}
\]

On the other hand, upon the differentiation of (4.3) with respect to \(A_\nu(y)\), it follows the identity

\[
\partial_\mu \Gamma^{\mu\nu}(x, y) = \Box_\xi (1 + a^2 \Box) \partial^\nu \delta^{(4)}(x - y) \tag{4.7}
\]

which, together with equation (3.11) implies that

\[
k_\mu \Pi^{\mu\nu}(k) = 0. \tag{4.8}
\]

Then, the longitudinal part does not take part of the dynamics in the sense that it is not modified by radiative corrections.

The following section will present how the renormalization program is implemented in the GS-DKP, showing that by a renormalization of the fields and physical quantities, such that the resultant, renormalized \(S\)–matrix leads to finite values for all the processes.
5 Renormalizability

This section will include the on-shell renormalization program \cite{65} for GSDKP electrodynamics. The following analysis will result in state suitable physical conditions on the Green’s functions serving as for renormalization conditions, which shall be important to determinate the renormalization constants (counterterms) in terms of (in)finite integrals as well. Besides, the resulting renormalization condition on the DKP sector will be more involving and subtle than the usual as in the Dirac theory, because $\hat{p}^2 \neq p^2$ in the DKP theory.

The bare Lagrangian density is defined in (2.1). The standard renormalization procedure begins introducing the renormalization constants through the following replacements

\[ \psi \rightarrow Z_0^{1/2} \psi, \quad A \rightarrow Z_3 A. \]  \tag{5.1}

In this case the fully renormalized Lagrangian can be written as

\[ \mathcal{L} = \bar{\psi} (i\gamma^{\mu} - m + e\hat{A}) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2m_p^2} \partial^{\mu} F_{\mu\beta} \partial^{\alpha} F^{\alpha\beta} + \delta Z_0 \bar{\psi} i\gamma^{\mu} \psi + \delta Z_1 m \bar{\psi} \psi + \delta Z_2 e \bar{\psi} \hat{A} \psi - \delta Z_3 \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \]  \tag{5.2}

where the counterterms defined by \( \delta Z_i = Z_i - 1 \) were added, and the renormalization for the mass: \( Z_1 m = Z_0 m_0 \), and for the vertex: \( Z_2 e = Z_0 Z_3 e_0 \) were also introduced.

From the renormalized Lagrangian \eqref{5.2} one can get the general renormalized expressions for the SDF equations and WTF identities. In this scenario, on deriving the SDF equations one can conclude that the complete propagators are changed as

\[ \mathcal{D}^{\mu\nu} \rightarrow Z_3 \mathcal{D}^{\mu\nu}, \quad \mathcal{S} \rightarrow Z_0 \mathcal{S}, \quad \Gamma^{\mu} \rightarrow Z_3^{-1} \Gamma^{\mu}. \]  \tag{5.3}

Besides, from the WFT identity \eqref{4.4} it follows the equality \( Z_0 = Z_2 \), which are identically satisfied at all orders in perturbation theory. This implies that the charge renormalization is determined only by \( e = Z_3 e_0 \).

The effects from the renormalization as in \eqref{5.2} into the radiative corrections are that the self-energy functions previously derived are now added by the counterterms \( \delta Z_i \). These new self-energy functions are now denoted by the index \( \mathcal{R} \). Analysing first the photon sector, for which the renormalized self-energy function reads

\[ \Pi^{(\mathcal{R})}(p) = \Pi(p) + \delta Z_3, \]  \tag{5.4}

then \( \Pi(p) \) is the polarization scalar written in terms of the renormalized quantities.

The first renormalization condition imposes that the complete photon propagator \eqref{3.14}, with \( \tilde{\xi} = 1 \), behaves as a massless field

\[ iD^\nu(p) = \eta^\nu \frac{1}{p^2}, \quad \text{when} \quad p^2 \rightarrow 0. \]  \tag{5.5}

\footnote{The replacement \( \tilde{m}_p^2 = Z_3 m_0^2 \) is only a matter of notation, since there is not a renormalization constant associated with this parameter.}
By means of the above condition, the counterterm $\delta Z_3$ is determined by

$$\delta Z_3 = - \Pi(p)|_{p^2 \to 0}. \quad \quad (5.6)$$

The renormalization conditions in the DKP sector are easily imposed into the two-point 1PI function $\Gamma^{(R)}(p) = \hat{\rho} - m - \Sigma^{(R)}(p)$. The first on-shell condition is that the physical mass is a pole

$$\Gamma^{(R)}(p) = \hat{\rho} - m_f, \quad \text{when} \quad \hat{\rho} \to m_f, \quad \quad (5.7)$$

where

$$\Sigma^{(R)}(p) = \Sigma(p) - m\delta Z_1 I + \delta Z_0 \hat{\rho}. \quad \quad (5.8)$$

In contrast with the fermionic theory, the condition $\frac{\partial \Gamma^{(R)}(p)}{\partial \hat{\rho}}|_{\hat{\rho} \to m_f} = 1$ will not be taken, once the trilinear DKP algebra (2.2) leads to $\hat{\rho}^2 \neq p^2$, but instead $\hat{\rho}^3 = p^2 \hat{\rho}$, which complicates substantially the derivation in terms of $\hat{\rho}$ and $p^2$. Nonetheless, a convenient choice for the second renormalization condition is given by

$$\beta_\mu \frac{\partial \Gamma^{(R)}(p)}{\partial p_\mu} = \beta_\mu \beta^\mu, \quad \text{when} \quad \hat{\rho} \to m_f, \quad \quad (5.9)$$

since $\beta_\mu \beta^\mu$ has a scalar structure. These renormalization conditions, Eqs.(5.7) and (5.9), when multiplied by the l.h.s by $\beta_\nu$ and the r.h.s. by $\beta^\nu$, imply into the following expressions for the counterterm $\delta Z_0$.

$$-\delta Z_0 = \Sigma_2(p^2)|_{p^2 \to m_f^2} + \frac{m_f^2}{2} \beta_\nu \frac{\partial \Sigma_2(p^2)}{\partial p^2} \beta^\nu|_{p^2 \to m_f^2} + \frac{m_f^2}{2} \beta_\nu \frac{\partial \Sigma_1(p^2)}{\partial p^2} \beta^\nu|_{p^2 \to m_f^2}, \quad \quad (5.10)$$

and for the counterterm $m\delta Z_1$

$$m\delta Z_1 = \Sigma_1(p^2)|_{p^2 \to m_f^2} - \frac{m_f^2}{2} \beta_\nu \frac{\partial \Sigma_2(p^2)}{\partial p^2} \beta^\nu|_{p^2 \to m_f^2} - \frac{m_f^2}{2} \beta_\nu \frac{\partial \Sigma_1(p^2)}{\partial p^2} \beta^\nu|_{p^2 \to m_f^2}. \quad \quad (5.11)$$

Therefore, from Eqs.(5.10) and (5.11), the related DKP sector renormalization constants $Z_0$ and $Z_1$, respectively, can be computed in all orders of perturbation theory.

At last, in order to uncover the renormalization constants, notice that the constant $Z_0$ can be determined by considering that the renormalized vertex function (3.24), by the on-shell condition: $p^2 = q^2 = m^2$ and at a null transferred momentum limit $k^2 = (p - q)^2 \to 0$, is

$$\bar{u}(q) i\Gamma^\sigma(q, p; 0) u(p) = -(2\pi)^4 \beta^\sigma, \quad \quad (5.12)$$

or, equivalently, the vertex part (3.25) is such that

$$\bar{u}(q) i\Lambda^\sigma(q, p; 0) u(p) = 0. \quad \quad (5.13)$$

---

8 $m_f$ is defined as the zero of the DKP two-point 1PI function.
9 The following decomposition $\Sigma(p) = \rho \Sigma_2(p^2) + i \Sigma_1(p^2)$ [18] and identity $\beta_\nu \beta_\mu \beta^\mu \beta^\nu = 4I$ were used.
With this section the formal development of the theory has been concluded. Henceforth, the explicit evaluation of the radiative correction expressions for the photon polarization tensor, DKP self-energy and vertex part will be proceed. The main interest is in observing the effects from the HD terms into the UV behavior of these DKP radiative corrections. For this purpose, a detailed discussion on the divergent structure of each contributions will be done by computing their respective counterterms.

6 Radiative corrections at one loop

Once established the renormalizability of the GSDKP electrodynamics and with the Schwinger-Dyson-Fradkin equations for the main complete Green’s functions, it is time to determine the radiative corrections at the lowest order in perturbation theory. The divergences that appear in radiative corrections will be regularized by the dimensional regularization proceeding, which preserves all symmetries of the theory, in particular the gauge symmetry \[65, 66\].

6.1 The photon self-energy

Let's start the study of radiative corrections for self-energy of the photon. This quantity corresponds to the diagram shown in figure 4.

From the expression (3.11) rewritten in momentum representation

\[
\Pi^\gamma_\nu(p) = ie^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} \text{Tr} \left[ \beta^\gamma S(p-k) \beta^\nu S(-k) \right]
\]

then

\[
\Pi^\gamma_\nu(p) = -ie^2 \mu^{4-d} \frac{m^2}{m^2} \int \frac{d^d k}{(2\pi)^d} \text{Tr} \left\{ \beta^\gamma \left[ \frac{\hat{\beta} - \hat{k}}{(p-k)^2 - m^2} - 1 \right] \beta^\nu \left[ \frac{-\hat{k}(-\hat{k} + m)}{k^2 - m^2} - 1 \right] \right\}. \tag{6.2}
\]

Using the \[\beta\] matrices trace properties \[B.6\], the equation \[6.2\] can be rewritten in the following form

\[
\Pi^\gamma_\nu(p) = -ie^2 \mu^{4-d} \frac{m^2}{m^2} \int \frac{d^d k}{(2\pi)^d} \left\{ \frac{m^2(p-2k)^\gamma (p-2k)^\nu - m^2(p-k)^2 + k^2 - 2m^2}{(p-k)^2 - m^2(k^2 - m^2)} \right\} \eta^\gamma_\nu. \tag{6.3}
\]

The momentum integration of the above terms can be performed by following the well-known set of rules of the standard Feynman integrals and dimensional regularization. Thus, the above expression
reduces to

\[ \Pi_{\mu\nu}(p) = [-\eta_{\mu\nu}p^2 + p_\mu p_\nu] \Pi(p), \quad (6.4) \]

where the scalar polarization reads

\[ \Pi(p) = -\frac{e^2}{(4\pi)^2} \left[ \frac{2}{\varepsilon} - \gamma \right] - \frac{e^2}{(4\pi)^2} \int_0^1 dx (1 - 2x)^2 \ln \left( \frac{4\pi \mu^2}{m^2 - x(1 - x)p^2} \right) \quad (6.5) \]

with \( \varepsilon = 4 - d \to 0^+ \) is the ultraviolet dimensional regularization parameter. The previous result is consistent with relativistic covariance and the Ward-Fradkin-Takahashi identity, as in (4.8). The comparison between (6.5) and the known result in GSQED \[48] leads to the conclusion that they are the same.

### 6.1.1 Effective charge

Lastly, the computation of the photon self-energy counterterm \( \delta Z_3 \), vide (5.6), which can be written directly as

\[ \delta Z_3 = \frac{e^2}{3(4\pi)^2} \left\{ \left[ \frac{2}{\varepsilon} - \gamma \right] + \ln \left( \frac{4\pi \mu^2}{m^2} \right) \right\} \quad (6.6) \]

showing explicitly that the ultraviolet divergence of the photon propagator is absorbed by its counterterm.

It is possible now to draw some physical conclusions associated with the running of coupling constant using as a guide the Coulomb scattering in the Born approximation \[22, 63\]. After the renormalization procedure, the expression for the complete propagator (3.14) can be rewritten in terms of the respective counterterm such as (\( \xi = 1 \))

\[ i\mathcal{D}^{\mu\nu}(p) = \eta^{\mu\nu} \left[ \frac{1}{p^2 - p_\mu^2 - m^2} \right] \left[ 1 + \left[ \frac{1}{p^2} - \frac{1}{p_\mu^2 - m^2} \right] \left[ \delta Z_3 - \Pi^{(R)}(p) \right] \right]. \quad (6.7) \]

The previous relation allows a definition of the effective charge in the regime where \( k^2 \gg m^2 \)

\[ \alpha_{(R)}(k^2) = \alpha(m^2) \left[ 1 + \left[ \frac{1}{p^2} - \frac{1}{p_\mu^2 - m^2} \right] \left[ Z_3 - 1 + \frac{\alpha}{12\pi} \ln \left( \frac{k^2}{m^2} \right) \right] \right]. \quad (6.8) \]

in which \( \alpha(m^2) = Z_3 \alpha \), and \( \alpha \) is the fine-structure constant. Besides, one can see that

\[ \alpha_{(R)}(k^2) = \alpha(m^2) \left[ 1 + \frac{\alpha(m^2)}{12\pi} \frac{1}{1 - \frac{k^2}{m^2}} \ln \left( \frac{k^2}{m^2} \right) \right]. \quad (6.9) \]

Therefore the running coupling constant expression, in the leading logarithmic approximation, is written as follows

\[ \frac{1}{\alpha_{(R)}(k^2)} = \frac{1}{\alpha(m^2)} - \frac{1}{12\pi} \frac{1}{1 - \frac{k^2}{m^2}} \ln \left( \frac{k^2}{m^2} \right). \quad (6.10) \]
The expression for the running coupling constant (6.10) displays a pole at $k^2 = m_p^2$; this expression then provides a validity regime for the theory: $m^2 \ll k^2 < m_p^2$, where the generalized DKP theory is in fact well-defined. Moreover, this behavior is in agreement with analysis for the fermionic and scalar theories [47, 48].

6.2 The DKP self-energy

In the same way as in the previous case, the radiative corrections for self-energy of the DKP particle corresponds to one diagram, as directly seen in figure 5.

From the expression (3.19) rewritten in momentum representation,

$$\Sigma(p) = \frac{ie^2 \beta^\mu}{(2\pi)^4} \int d^4k \left[ \mathcal{D}_{\mu\nu}(k) \mathcal{J}^\nu(p-k) \Gamma^\nu(p-k,p;k) \right]. \quad (6.11)$$

At the lowest order in perturbation theory with the gauge choice $\xi = 1$ and the Podolsky’s free parameter as $a^2 = m_p^{-2}$,

$$\Sigma(p) = -\frac{ie^2 m_p^2}{m} \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} B^\mu \left[ (\hat{\rho} - m) - (p-k)^2 + m^2 \right] \beta^\mu \frac{1}{[(p-k)^2 - m^2]} \frac{1}{k^2(k^2 - m_p^2)}. \quad (6.12)$$

The momentum integration is again performed using dimensional regularization. The calculation is rather direct, and the resulting expression is

$$\Sigma(p) = \frac{e^2 m_p^2}{3m(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{B^\mu \left[ (1-x)^2 \hat{\rho}^2 + (1-x) \hat{\rho} m - (1-x)^2 p^2 + m^2 \right] \beta^\mu}{m^2 x + m_p^2 y - x(1-x) p^2}$$

$$- \frac{4e^2 m_p^2}{3m(4\pi)^2} (1 - \beta^\mu \beta^\mu) \int_0^1 dx \int_0^{1-x} dy \left[ \frac{2}{\varepsilon - 1 - \gamma - \ln \left( \frac{m^2 x + m_p^2 y - x(1-x) p^2}{4\pi \mu^2} \right)} \right]. \quad (6.13)$$

Furthermore, with help of DKP algebra (and making use of an explicit representation, appendix B) [10], it is possible to show that $\beta^\mu \hat{\rho}^2 \beta^\mu = p^2$ and $\beta^\mu \hat{\rho} \beta^\mu = \hat{\rho}$. Therefore, with these identities, $\Sigma(p)$ can be conveniently separated as

$$\Sigma(p) = \hat{\rho} \Sigma_2(p^2) + \Sigma_1(p^2), \quad (6.14)$$

in which

$$\Sigma_2(p^2) = \frac{e^2 m_p^2}{3(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{1}{m^2 x + m_p^2 y - x(1-x) p^2} \quad (6.15)$$

[10] Actually, the r.h.s. of these identities is invariant under changes of representation, showing that this result is general.
and
\[
\Sigma_1(p^2) = - \left( \frac{1}{\epsilon} \right) \frac{8e^2 m_p^2}{3m(4\pi)^2} (1 - \beta^\mu \beta_\mu) + \frac{e^2 m_p^2}{3m(4\pi)^2} \int_0^1 dx \int_0^{1-x} dy \frac{m^2}{m^2 x + m_p^2 y - x (1-x) p^2} \beta^\mu \beta_\mu \\
+ \frac{4e^2 m_p^2}{3m(4\pi)^2} (1 - \beta^\mu \beta_\mu) \int_0^1 dx \int_0^{1-x} dy \left[ 1 + \gamma + \ln \left( \frac{m^2 x + m_p^2 y - x (1-x) p^2}{4\pi \mu^2} \right) \right].
\]
\[(6.16)\]

The expressions for the counter-terms of the DKP sector, \(\delta Z_0\) and \(\delta Z_1\), evaluated at \(\alpha\)-order can be presented. Then, it is time to calculate the counterterms. At first, on considering the counterterm \(\delta Z_0\), through the relation (5.10) and equations (6.15) and (6.16), it follows that for \(\zeta = \frac{m^2}{m^2} > 4\), it leads to
\[
-4\delta Z_0 = \frac{\alpha}{6\pi} \left( 4 + \beta^2 \right) \frac{1}{\epsilon_{IR}} - \frac{\alpha}{6\pi} \left( 4 + 3\beta^2 + \frac{\Xi \beta^2}{24} \right) - \frac{\alpha}{18\pi} \Xi^2 \left[ 36\beta^2 - 180 - \zeta (25\beta^2 - 109) - 4\zeta^3 (4 - \beta^2) \right] \\
- \frac{\alpha}{12\pi} \Xi \left[ (\zeta (2 + 3\beta^2) - \beta^2 + 4) \log [\Xi - 1] + 2 (7 + 2\beta^2) \log [\Xi + 1] \right] \\
+ \frac{\alpha}{36\pi} \Xi \left[ 148\zeta - 11\zeta^2 \beta^2 + 4 (4 - \beta^2) \zeta^3 - 101\zeta^2 - 24 - 12\beta^2 \right] \log \left[ \frac{\Xi - 1}{\Xi + 1} \right] \\
- \frac{\alpha}{144\pi} \Xi \left[ -122\zeta \beta^2 + 16 (4 - \beta^2) \zeta^3 - 84 - 49\beta^2 \right] \\
+ 542\zeta - \frac{101}{3} \zeta^2 + \frac{23}{3} \zeta^2 \beta^2 \right] \log \left[ \frac{\zeta - \sqrt{(\zeta - 4)\zeta}}{\zeta + \sqrt{(\zeta - 4)\zeta}} \right],
\]
\[(6.17)\]

where \(\beta^2 \equiv \beta_\mu \beta^\mu\) and \(\Xi \equiv \sqrt{\frac{\zeta}{\zeta - 4}}\) have been defined and also the infrared dimensional parameter as \(\epsilon_{IR} = d - 4, \epsilon_{IR} \to 0^-\). Similarly, the mass counterterm \(\delta Z_1\) through the relation (5.11) and equations...
Thus, under the condition $\zeta = \frac{m^2_p}{m^2} > 4$ one can find that

$$4\delta Z_1 = -\left(\frac{1}{\varepsilon}\right) \frac{8\alpha}{3\pi} (1 - \beta^2) \zeta + \frac{2\alpha}{3\pi} (1 + \beta^2) \frac{1}{\varepsilon_{IR}} + \frac{2\alpha}{3\pi} (1 - \beta^2) \zeta \left[1 + \gamma - \ln \left(\frac{4\pi\mu^2}{m^2}\right)\right]$$

$$+ \frac{\alpha}{72\pi} \left[4\zeta (48 - 16\zeta - 9\beta^2 + 4\zeta \beta^2) - \frac{2}{3} \zeta (4 - \beta^2) (8 - 18\zeta) - \frac{8}{3} (\beta^2 - 1) (6\zeta^2 - 36\zeta - 8)\right] + \frac{\alpha}{36\pi} \left[6\zeta \beta^2 + 4 (6 - 9\zeta + 2\zeta^2) \zeta - 3\zeta \beta^2 + 15\zeta^2 \beta^2 - 4\zeta^3 \beta^2 + 12 (\zeta + 1) + 3\beta^2 + 4 (\beta^2 - 1) (\zeta - 6) \zeta^2\right] \log[\zeta]$$

$$+ \frac{\alpha}{36\pi} \left[\frac{3}{\zeta} (4 + \beta^2) \zeta (5 - \zeta) + 4 (\beta^2 - 1) (4\zeta + \zeta^2 - 16) \zeta^2 + 4 (4 - \beta^2) (-2\zeta^2 + 13\zeta - 20) \zeta^2 + 12 (2 - \zeta) \zeta \beta^2 + 3\zeta^3 \beta^2 - 12\zeta^2 \beta^2 + 6\zeta \beta^2\right] \log\left[\frac{\zeta - 1}{\zeta + 1}\right]$$

$$+ \frac{\alpha}{72\pi} \zeta \left[\zeta (\zeta - 4) (4 (4 - \beta^2) (2\zeta - 5) \zeta + 12) - 6\zeta (\zeta^2 - 4\zeta + 2) \beta^2 + 6\zeta (4 + \beta^2) (5 - \zeta) - 4 (\beta^2 - 1) (2 + 8\zeta^3 - 32\zeta^2)\right] \log\left[\frac{\zeta - \sqrt{(\zeta - 4)\zeta - 2}}{\zeta + \sqrt{(\zeta - 4)\zeta - 2}}\right].$$

A pertinent comment is in place. Equation (6.14) has an UV divergence, proportional to the $m^2_p$-parameter. A naive thought about this divergence would present some problem with respect and spoil the WFT identity, that yielded $Z_0 = Z_2$; this is a subtle issue once the vertex part is in fact UV finite (to be treated carefully in the following). However, remarkably, this divergence is absorbed by the mass counterterm $\delta Z_1$, clearly at equation (6.18), showing therefore that the WFT identity (4.4) is satisfied at this order. A similar situation was also found in the GSQED$_4$ [48].

### 6.3 The vertex part

Finally the computation of the first radiative correction associated with the vertex function, which corresponds to the diagram depicted in figure 6.

The resulting outcome will be important to verify the validity of the WFT identity as discussed in the previous subsection.
From the expression (3.25), the vertex part at the lowest order correction is

$$\Lambda^\mu (p', p) = e^2 \mu^{4-d} \int \frac{d^d k}{(2\pi)^d} B^\sigma S (p' - k) \beta^\mu S (p - k) \beta^\nu D_{\sigma \nu} (k),$$  \hspace{2cm} (6.19)

substituting the expressions for their respective propagators

$$\Lambda^\mu (p', p) = \frac{-i e^2 \mu^{4-d} m_p^2}{m^2} \int \frac{d^d k}{(2\pi)^d} \beta^\sigma [(\hat{p}' - \hat{k})(\hat{p}' - \hat{k} + m) - (p' - k)^2 - m^2]$$

$$\times \beta^\mu [(\hat{p}' - \hat{k})(\hat{p}' - \hat{k} + m) - (p' - k)^2 - m^2] \beta^\sigma.$$  \hspace{2cm} (6.20)

This expression may be simplified by making use of the Feynman parametrization, and then be cast into a suitable form

$$\Lambda^\mu (p', p) = \frac{-i e^2 \mu^{4-d} m_p^2}{m^2} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz$$

$$\times \int \frac{d^d k}{(2\pi)^d} \frac{A^\mu + B^\mu_{\alpha \nu} k^\alpha k^\nu + C^\mu_{\alpha \nu \lambda \theta} k^\alpha k^\nu k^\lambda k^\theta}{(k^2 - b^2)^4},$$  \hspace{2cm} (6.21)

in which $b^2 = (p'x + py)^2 + p'^2 x + p'^2 y - m^2 (x + y) - m_p^2 z$ and the following tensor quantities

$$A^\mu = \beta^\sigma \left\{ [(1 - x) \hat{p}' - y \hat{p}] [(1 - x) \hat{p}' - y \hat{p} + m] - [(1 - x) p' - yp]^2 - m^2 \right\} \beta^\mu$$

$$\times \left\{ [(1 - y) \hat{p} - x \hat{p}'] [(1 - y) \hat{p} - x \hat{p}' + m] - [(1 - y) p - xp']^2 - m^2 \right\} \beta^\sigma,$$  \hspace{2cm} (6.22)

and

$$B^\mu_{\alpha \nu} = \beta^\sigma [\beta_\alpha \beta_\nu - \eta_{\alpha \nu}] \beta^\mu \left\{ [(1 - y) \hat{p} - x \hat{p}' + m] - [(1 - y) p - xp']^2 - m^2 \right\} \beta^\sigma$$

$$+ \beta^\sigma \left\{ [(1 - x) \hat{p}' - y \hat{p}] [(1 - x) \hat{p}' - y \hat{p} + m] - [(1 - x) p' - yp]^2 - m^2 \right\} \beta^\mu \beta_\alpha \beta_\nu - \eta_{\alpha \nu}$$

$$- \beta^\sigma \left\{ \beta_\alpha [(1 - x) \hat{p}' - y \hat{p} + m] + [(1 - x) \hat{p}' - y \hat{p}] \beta_\alpha + 2 [(1 - x) p' - yp_\alpha] \right\} \beta^\mu$$

$$\times \left\{ \beta_\nu [(1 - y) \hat{p} - x \hat{p}'] + m + [(1 - y) \hat{p} - x \hat{p}'] \beta_\nu + 2 [(1 - y) p - xp'] \right\},$$  \hspace{2cm} (6.23)

and

$$C^\mu_{\alpha \nu \lambda \theta} = \beta^\sigma \beta_\alpha \beta_\nu \beta_\lambda \beta_\theta - \eta_{\alpha \nu \lambda \theta} \beta^\sigma.$$  \hspace{2cm} (6.24)

The momentum integration in expression (6.21) can be evaluated and results into

$$\Lambda^\mu (p', p) = \frac{e^2 m_p^2}{(4\pi)^2 m^2} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz$$

$$\times \left[ \frac{A^\mu}{b^4} - \frac{g^\alpha g^\mu B^\mu_{\alpha \nu}}{2b^2} - \frac{\Gamma(\xi)}{4} \left[ \frac{4\pi m^2}{b^2} \right]^2 \left( g^\alpha g^\lambda \theta + g^\nu \theta g^\alpha \lambda + g^\theta g^\mu \lambda \nu \right) C^\mu_{\alpha \nu \lambda \theta} \right].$$  \hspace{2cm} (6.25)

The term $C^\mu_{\alpha \nu \lambda \theta}$ presents a logarithmic divergence. However, by means of using the DKP algebra (2.2), one can show that this term is actually vanishing its identities

$$(g^\alpha g^\lambda \theta + g^\nu \theta g^\alpha \lambda + g^\theta g^\mu \lambda \nu) C^\mu_{\alpha \nu \lambda \theta} = 0.$$  \hspace{2cm} (6.26)
Figure 7: Two-loops renormalized photon propagator.

Showing therefore that there is no divergences on the vertex part. The finite contribution is then given by

$$A^\mu(p', p) = \frac{e^2 m_p^2}{(4\pi)^2 m^2} \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz \frac{1}{((p'x + py)^2 + p'^2x + p^2y - m^2(x + y) - m_p^2z)^2}$$

$$\times \left[ \frac{A^\mu}{((p'x + py)^2 + p'^2x + p^2y - m^2(x + y) - m_p^2z)^2} - \frac{g^{\alpha\nu} B^\mu_{\alpha\nu}}{2} \right].$$

(6.27)

This result confirms the information contained in the WTF identity (4.6), $Z_0 = Z_2$, assuring that the divergence of $\Sigma$ in the term proportional to the mass $m_p$ (6.14) does not spoil the WTF identity. For sake of completeness one could calculate $\delta Z_0$ in view of the equations (5.13) and (6.25).

7 Photon propagator and vertex at two loops

In spite of the DKP self-energy and the photon self-energy were both successfully renormalized, in comparison with the usual theory and also with GQED$_4$ [46,47], the GSDKP$_4$ presents an unexpected novel divergent structure. This novel divergence is closely related to the one as in the self-energy in GSQED$_4$ [48]. In this way, it is important to complete the discussion of this new of divergence ($m_p$-dependent one), present in the DKP self-energy equation (6.13), with further information and details. In particular, the diagrammatic analysis of the $\alpha^2$-order photon polarization tensor and the vertex function will be done in order to conclude whether this divergence propagates and if the original counter-terms are sufficient to control it correctly. The interest in these functions is driven mainly by the divergence structure embedded on it from the DKP self-function. Furthermore, discussing the photon polarization tensor in the light of the higher-order terms, once the $\alpha$-order calculation is not sensitive to these effects.

The diagrams presented in this order are depicted in the figures 7 and 8 for the two-loop photon polarization tensor and vertex part, respectively.

The diagrams are separated in such a way that the divergent diagram and its respective counter-term diagram were written together. This allows to highlight the action of the counter-terms as well as a better reading concerning the cancellation of the $m_p$-dependent divergent parts.

It is worth stressing nonetheless that this work does not have the intention of presenting here a formal proof concerning the complete renormalizability of the theory. But the belief that a qualitative
(diagrammatic) discussion does provide all the necessary information to the renormalizability of the theory make sense, especially regarding those $m_p$-dependent divergent diagrams. At last, this section can be concluded stating that the original counter-terms, in particular $\delta Z_1$, are sufficient to absorb all the primitive divergences of all Green’s function.

8 Concluding remarks

The phenomenological interaction between scalar fields and generalized photons, from the point of view of GSDKP electrodynamics was systematically studied. The first point to note is the implementation of the non-mixing gauge within the Faddeev-Popov-De Witt ansatz to obtain a covariant expression for the functional generator. Also due to the presence of a novel divergence, the theory’s renormalizability was carefully analyzed in full detail.

The quantization of the GSDKP took place by the canonical path-integral formalism. Based on this approach, the Schwinger-Dyson-Fradkin equations have been derived for the basic Green’s functions. In particular, these equations provide non-perturbative information in nature of the complete Green’s functions. Besides, the SDF equations were determined in the generalized non-mixing gauge, which, in contrast with the generalized Lorenz condition, gave an expression for the photon’s propagator in which the transversal and the longitudinal sectors are not mixed, as one can see in equation (3.16). Along the formal development, the WFT identities have also been determined in which the gauge symmetry is proved to hold at quantum level as well. Also, the on-shell renormalization program was applied by including the respective counter-terms. It should be noticed that due to the particular structure of the DKP algebra one of the renormalization conditions in the DKP sector had to be changed in comparison with the Dirac theory, basically because of the relation $\hat{p} (\hat{p}^2 - p^2) = 0$.

In particular, it is important to emphasize that the DKP field is often employed in nuclear physics to describe mesons [21], when it is possible to say that it has a mesonic algebraic structure [67]. But the DKP fields are described by the DKP algebra, while the fermionic field obeys a Clifford
algebra. Although the complete quantum structure of the scalar field, seen by means of SDF diagrams, is exactly as those from GQED$_4$\cite{46}. In the present case, scalar DKP theory, the same diagram phenomenology for the electromagnetic interaction between scalar or fermionic fields happens.

After concluding the formal development of the GSDKP, the evaluation of the respective one-loop radiative corrections for the photon and DKP propagator, and for the vertex part was done. For the DKP field self-energy and vertex part, these radiative correction expressions have a very interesting behavior. First, it was found that the DKP field self-energy had an UV divergence, displayed in a term proportional to $m_p$, as in the equation (6.16). At first this seemed to be a problematic situation, since after evaluating the one-loop correction to the vertex part a finite result was found, what could naively be interpreted as a violation of the the WFT identity, $Z_0 = Z_2$. Nonetheless, after evaluating explicitly the DKP sector counterterms, this $m_p$-dependent divergence was in fact absorbed by the mass counterterm $Z_1$. In order to verify if the renormalizability still holds in higher-orders, i.e., whether the $m_p$-dependent divergence do not propagate to higher-loops, the diagrammatic analysis of the photon self-energy and vertex part at two-loop was performed, showing that the respective original counter-terms are sufficient to absorb all the primitive divergences.

The information gathered from the study of the radiative corrections of GSDKP$_4$ can be compared with the results for GSQED$_4$\cite{48}. Then, from the self-energy of the photon one has the same running of the effective charge (as well the validity regime for the theory: $m^2 \ll k^2 < m_p^2$), from the self-energy of the scalar particle the divergence appears proportional to $m_p$ and the vertex has no ultraviolet divergence. What is interesting to note is that GSQED$_4$ has two vertices and GSDKP has just one, this fact makes the analysis of higher loops apparently much easier in the framework of DKP theory rather than in the scalar QED. Hence, based on the present outcome, it is possible to extend the present analysis to study GSDKP at thermodynamical equilibrium within the Matsubara-Fradkin formalism. This matter will be further elaborated and requires deeper investigations.

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A Dimensional regularization identities

The momentum integrals were evaluated throughout the paper by means of the useful dimensional regularization results [66]

\[
\int \frac{d^d k}{(2\pi)^d [k^2 - b^2]^{\alpha}} \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \right) = i \left( -\frac{1}{4\pi^2} \right)^{\frac{d}{2}} \Gamma(\alpha - 1 - \frac{d}{2}), \tag{A.1}
\]

\[
\int \frac{d^d k}{(2\pi)^d [k^2 - b^2]^{\alpha}} k^\lambda k^\nu \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{\Gamma(\alpha - 1 - \frac{d}{2})}{\Gamma(\alpha)} \right) = i \left( -\frac{1}{4\pi^2} \right)^{\frac{d}{2}} \eta^\lambda \eta^\nu \Gamma(\alpha - 1 - \frac{d}{2}), \tag{A.2}
\]

\[
\int \frac{d^d k}{(2\pi)^d [k^2 - b^2]^{\alpha}} k^\mu k^\nu k^\lambda k^\theta \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{\Gamma(\alpha - 2 - \frac{d}{2})}{\Gamma(\alpha)} \right) = i \left( -\frac{1}{4\pi^2} \right)^{\frac{d}{2}} \eta^\mu \eta^\nu \eta^\lambda \eta^\theta \Gamma(\alpha - 2 - \frac{d}{2}), \tag{A.3}
\]

in which \( \eta^\lambda \eta^\lambda = d \). And the gamma’s function properties

\[
\Gamma(-n + \varepsilon) = \frac{(-1)^n}{n!} \left[ \frac{1}{\varepsilon} + \Psi_1(n + 1) + O(\varepsilon) \right] \tag{A.4}
\]

in which

\[
\Psi_1(n + 1) = 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma. \tag{A.5}
\]

and \( \gamma \) as the Euler-Mascheroni constant. Also useful

\[
z \Gamma(z) = \Gamma(z + 1), \quad \chi^{-\frac{e^\varepsilon}{\chi}} \sim 1 - \frac{\varepsilon}{2} \ln \chi. \tag{A.6}
\]

B \( \beta \)-matrices properties

The \( \beta \) matrices satisfy the following algebra

\[
\beta^\mu \beta^\nu \beta^\theta + \beta^\theta \beta^\nu \beta^\mu = \beta^\mu \eta^\nu \theta + \beta^\theta \eta^\nu \mu. \tag{B.1}
\]

A particular representation of this algebra can be given by [28]

\[
\beta^0 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
\beta^1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0
\end{pmatrix}, \tag{B.2}
\]

\[
\beta^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0
\end{pmatrix}, \quad
\beta^3 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0
\end{pmatrix}. \tag{B.3}
\]
The algebra (B.1) can be used to show that

\[ \beta^\mu \beta^\nu \beta_\mu = \beta^\nu, \quad \beta^\mu \hat{p}^2 \beta_\mu = p^2, \quad \beta^\mu \hat{p} \beta_\mu = \hat{p}. \]  

(B.4)

In such a way that the extension of the DKP algebra over a d-dimensional spacetime leads to the following algebraic identities

\[ \beta^\mu \beta^\nu \beta_\mu = d, \quad \beta^\mu \beta^\nu \beta_\mu + \beta^\nu \beta_\nu \beta^\mu = (1 + d) \beta^\mu. \]  

(B.5)

The useful property of the trace

\[ Tr (\beta_{\mu_1} \beta_{\mu_2} \cdots \beta_{\mu_{2n-1}}) = 0, \]
\[ Tr (\beta_{\mu_1} \beta_{\mu_2} \beta_{\mu_3} \beta_{\mu_2n}) = \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} \cdots \eta_{\mu_{2n-1} \mu_{2n}} + \eta_{\mu_2 \mu_3} \eta_{\mu_4 \mu_5} \cdots \eta_{\mu_{2n-1} \mu_1}. \]  

(B.6)

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