Thermal properties of a two-dimensional Duffin-Kemmer-Petiau oscillator under an external magnetic field in the presence of a minimal length

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Abstract

In this article, we investigate the solutions of a two-dimensional Duffin-Kemmer-Petiau (DKP) oscillator within an external magnetic field in a minimal length (ML) formalism of the Heisenberg generalized uncertainty principle. First, we obtain the eigensolutions in the ordinary quantum mechanics. Then, we examine the DKP oscillator in the presence of a ML for the spin-zero and spin-one sectors. We determine an energy eigenvalue equation in both cases with the corresponding eigenfunctions. We show that in the limit where the ML correction vanishes, the energy eigenvalue equations become identical with the habitual quantum mechanical ones. Finally, we employ the Euler-Mclaurin summation formula and obtain the thermodynamic functions of the DKP oscillator in the high-temperature scale. We conclude that in the literature there are several articles with the misleading thermal properties of the relativistic oscillators.

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I. INTRODUCTION

In recent years, the number of extensive research subjected the Generalized Uncertainty Principle (GUP) is seen to be increased in various fields of physics, like, in quantum mechanics [1], quantum electrodynamics [2], quantum cosmology [3], quantum gravity [4], black-hole physics [5] and its thermodynamics [6]. The general idea of the GUP depends on a modified commutation relation for position and momentum operators of the standard Heisenberg algebra. This deformed algebra is defined by $[\hat{x}_i, \hat{p}_i] = i\hbar \delta_{ij} \left(1 + \beta p^2\right)$ [7–10]. Here, $\beta$ is a small positive parameter which is called as the minimal length (ML) parameter. As a consequence of the deformation in the algebra, Heisenberg uncertainty relation is modified with $(\Delta x) (\Delta p) \geq \hbar \left(1 + \beta (\Delta p)^2\right)$. Note that when $\beta$ goes to zero, we obtain the standard uncertainty relationship as given in the OQM.

In the ordinary non-relativistic quantum mechanical approach, the Schrödinger equation and its exact solution is the subject of many theoretical investigations to describe physical processes [11–13]. In the GUP perspective, we separate a recent study of Bhat et al. among the others [14–16] because they calculated a correction term to the energy eigenvalue function of a gravitational quantum well by employing infinite extra dimensions [17].

Two well-known differential equations, namely the Klein-Gordon (KG) and the Dirac equations, are being frequently studied in the relativistic quantum mechanics with the Heisenberg Uncertainty Principle [18–23] and with the GUP. For example, in the presence of the ML, Jana et al. obtained an exact solutions to the KG equation by employing a linear and vector potential energy [24]. In 2018, Boumali et al. investigated a two dimensional KG oscillator under the GUP [25]. In another study, Elviyanti et al. employed Asymptotic Iteration Method to examine the Hultén potential energy in the KG equation in the ML [26]. Besides them, in 2013, Menculini et al. studied the relativistic Landau problem within the presence of the ML and reported an exact solution of the wave function in the momentum space [27]. In 2015, Pedram et al. studied the two dimensional Dirac equation by employing a non-varying magnetic field in a ML and commented that the solution that is given by Menculini et al. is a subset of the general solution which is correlated with the even quantum numbers [28]. In a very recent paper, Hamil et al. investigated the Dirac oscillator in two and three dimensions. They announced that their results can predict the upper bound of the GUP parameter of the relativistic Landau levels in graphene [29].
Another relativistic equation is the Duffin-Kemmer-Petiau (DKP) equation. It is a first-order relativistic equation that describes the dynamics of a spin-zero and spin-one boson together [30–33]. In the literature, we see various studies have been carried by employing the DKP equation, among them we would like to emphasize the following articles that examine: a two dimensional DKP oscillator under the effect of a magnetic field [34], a deformed DKP oscillator with Snyder-de Sitter commutation relations in a momentum space [35], exact solutions of the DKP equation with Aharonov-Bohm and Coulomb potential energies in the commutative [36] and non commutative [37] space-times which are produced by cosmic string, DKP oscillator are written in this curved space-time [38], DKP oscillator that moves in a uniform magnetic field with the Snyder-de Sitter model [39], the dynamic of the DKP particles in the space-time initiated with a spinning cosmic string [40]. In addition to these studies, we would like to mention a remarkable article by Lunardi, which discusses the equivalency of the spin-1 and spin-0 representation of the DKP equation in one dimension [41].

DKP equation is also examined with the GUP. For instance, Falek et al. investigated a three dimensional DKP oscillator [42]. Then, in 2015, they employed a step function in DKP equation in existence of a ML and obtained the modified probabilities of the transmission and reflection [43]. Wang et al. studied the spin-one sector of the DKP oscillator in non commutative space in two dimension [44]. Recently, Chargui revisited the DKP equation with linear potential energies in one dimension and reported the inaccurate discussions that were done in the literature [45].

Recently, we have observed that scientists have also begun to analyze the thermal properties of the models they address. The pioneering work in this area was given by Pacheco et al. [46] in 2003. They investigate the thermodynamic properties of a one-dimensional Dirac oscillator. Later, they extended their work to three dimensions [47]. Nouicer handled the one-dimensional Dirac oscillator problem in a ML and examined its statistical properties in [48]. Hamil et al. studied the Dirac and Klein-Gordon oscillators and reported their thermal properties on anti-de Sitter space [49]. Wu et al. investigated thermodynamic functions of a two-dimensional DKP oscillator in the GUP formalism [50].

Note that taking the existence of an external magnetic field into account in quantum physics leads us to understand significant phenomena. For instance, in the presence of an external magnetic field, the energy levels of hydrogen-like atoms split. This is known as the
Zeeman effect. Although there are two papers on the subject [34, 50], to our knowledge, a two-dimensional DKP oscillator problem has not examined in the existence of an external magnetic field in a ML. In this paper, our main motivation is to obtain the modification of the energy eigenvalue function, hence the thermal properties, as a result of the external magnetic field. We believe that this analysis is a requirement needed in this area and will, therefore, fill a gap in the research literature.

We prepare the article as follows. In sect. II, we review the irreducible representations of the DKP equation for both the spin-zero and spin-one particles. In sect. III, we derive an exact solution to the two-dimensional DKP oscillator under the effect of an homogeneous magnetic field by employing the polar coordinates in the momentum space within the rules of ordinary quantum mechanics (OQM). In sect. IV, we investigate an exact solution of the problem in the generalized uncertainty principle. In sect. V, we obtain the thermodynamic functions by employing Euler-Maclaurin summation formula in the high temperature approach. Moreover, we demonstrate the correct thermal properties in the figures. We conclude the article in sect. VI.

II. FORMALISM OF THE DUFFIN KEMMER PETIAU EQUATION

In flat space-time, the DKP equation is given as [30–33]

\[(i\hbar \beta^\mu \partial_\mu - Mc)\Psi(\vec{r}, t) = 0, \quad \mu = 0, 1, 2, 3\]  \hspace{1cm} (1)

where \(M, \hbar\) and \(c\) denote the mass, reduced Planck constants, and speed of light, respectively. \(\beta^\mu\) matrices satisfy the DKP algebra

\[\beta^\kappa \beta^{\nu} \beta^{\lambda} + \beta^{\lambda} \beta^{\nu} \beta^{\kappa} = g^{\kappa\nu} \beta^{\lambda} + g^{\nu\lambda} \beta^{\kappa}.\]  \hspace{1cm} (2)

where \(\kappa, \nu, \lambda = 0, 1, 2, 3\). We examine the problem in the Minkowski space-time and we use the metric tensor, \(g^{\mu\nu}\), with the diag \((1, -1, -1, -1)\) signature. Note that there are 126 independent elements in the algebra. These elements can be reduced into three irreducible representations of dimensions one, five, and ten. Among them, one-dimensional representation is the trivial one. The others, namely the five and ten-dimensional representations, describe spin-zero and spin-one particle dynamics, respectively.
For a scalar particle the spin-zero representation is given with $5 \times 5$ matrices

$$
\beta^0 = \begin{pmatrix}
\rho_{2\times2} & \hat{\rho}_{2\times3} \\
\hat{\rho}_{3\times2}^T & \tilde{\rho}_{3\times3}
\end{pmatrix},
$$

(3)

$$
\beta^i = \begin{pmatrix}
\hat{\rho}_{2\times2} & \rho^i_{2\times3} \\
-\rho^i_{3\times2} & \tilde{\rho}_{3\times3}
\end{pmatrix},
$$

(4)

where $i = 1, 2, 3$. Here, $\hat{\rho}_{2\times2}, \tilde{\rho}_{2\times3}, \tilde{\rho}_{3\times3}$ are zero matrices. Note that the subscripts indicate the row and column numbers. The non-zero matrices are given as:

$$
\rho_{2\times2} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
$$

(5)

$$
\rho^1_{2\times3} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \rho^2_{2\times3} = \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \rho^3_{2\times3} = \begin{pmatrix}
0 & 0 & -1 \\
0 & 0 & 0
\end{pmatrix}.
$$

(6)

For a vector particle the spin-one representation is given with $10 \times 10$ matrices

$$
\beta^0 = \begin{pmatrix}
\bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & -I_{3\times3} & \tilde{\rho}_{3\times3}^T \\
\bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & \tilde{\rho}_{3\times3}^T \\
-I_{3\times3} & \bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & \tilde{\rho}_{3\times3}^T \\
\hat{\rho}_{1\times3} & \hat{\rho}_{1\times3} & \hat{\rho}_{1\times3} & 0
\end{pmatrix},
$$

(7)

$$
\beta^k = \begin{pmatrix}
\bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & iK^j \\
\bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & iK^j \\
\bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & \bar{\rho}_{3\times3} & iK^j \\
iK^j & \hat{\rho}_{1\times3} & \hat{\rho}_{1\times3} & 0
\end{pmatrix},
$$

(8)

where $j = 1, 2, 3$, and

$$
\tilde{\rho}_{3\times1}^T = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad I = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

(9)

$$
s^1 = i \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad s^2 = i \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad s^3 = i \begin{pmatrix}
0 & -1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
$$

(10)

$$
K^1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad K^2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad K^3 = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

(11)
III. DKP OSCILLATOR WITHIN OQM

In this section, we formulate the DKP oscillator and investigate its solution for spin-zero and spin-one particles within OQM. We assume that the potential energy and external magnetic field that are under the consideration are time-independent, thus, we decompose the total wave function into time and spatial parts.

A. Case of the spin-zero particle

In $(2+1)$ dimension a DKP oscillator for a spin-zero particle can be expressed with a non-minimal coupling of an external magnetic field as follows \[34\]

\[
\beta_0 E - c\beta_1 \left( p_x - \frac{eA_x}{c} - iM\omega^0 x \right) - c\beta_2 \left( p_y - \frac{eA_y}{c} - iM\omega^0 y \right) - Mc^2 \equiv 0 \tag{12}
\]

Here $E$ is the energy eigenvalues while $\omega$ is the frequency of the oscillator. We denote the momentum and position operators with bold letters. The matrix $\eta^0$ is defined by $\eta^0 \equiv 2(\beta^0)^2 - 1$. Note that the square of the $\eta^0$ is the identity matrix. We assume the components of the vector potential to be $A_x = -\frac{B}{2} y$ and $A_y = \frac{B}{2} x$, where $B$ is the strength of the magnetic field. The spatial wave function has five components and we express its transpose form with

\[
\Psi^T \equiv \left( \psi_1 \ \psi_2 \ \psi_3 \ \psi_4 \ \psi_5 \right). \tag{13}
\]

We substitute Eq. (13) into Eq. (12) and find five coupled equations as follows:

\[-Mc^2 \psi_1 + E \psi_2 + c(p_x - M\bar{\omega} y + iM\omega x)\psi_3 + c(p_y + M\bar{\omega} x + iM\omega y)\psi_4 = 0. \tag{14}\]

\[E\psi_1 - Mc^2 \psi_2 = 0. \tag{15}\]

\[c(p_x - M\bar{\omega} y - iM\omega x)\psi_1 + Mc^2 \psi_3 = 0. \tag{16}\]

\[c(p_y + M\bar{\omega} x - im_0\omega y)\psi_1 + Mc^2 \psi_4 = 0. \tag{17}\]

\[\psi_5 = 0. \tag{18}\]

Here $\bar{\omega} \equiv \frac{|e| B}{2Mc}$. The fifth component of the wave equation is zero. By writing the second, third and fourth components in terms of the first component, we obtain

\[
\left[ c^2(p_x - M\bar{\omega} y + iM\omega x)(p_x - M\bar{\omega} y - iM\omega x) \right. \\
+ c^2(p_y + M\bar{\omega} x + iM\omega y)(p_y + M\bar{\omega} x - iM\omega y) + \left( M^2 c^4 - E^2 \right) \left. \right] \psi_1 = 0. \tag{19}\]
We use the operators that are defined with polar coordinates in the momentum space 

\[ x \equiv i\hbar \left( \cos \theta \frac{\partial}{\partial p} - \sin \theta \frac{1}{p} \frac{\partial}{\partial \theta} \right), \]

\[ y \equiv i\hbar \left( \sin \theta \frac{\partial}{\partial p} + \cos \theta \frac{1}{p} \frac{\partial}{\partial \theta} \right), \]

\[ p_x \equiv p \cos \theta, \]

\[ p_y \equiv p \sin \theta, \]

and we find

\[ \left[ p^2 - 2M \hbar \left( \omega + i\tilde{\omega} \frac{\partial}{\partial \theta} \right) - M^2 \hbar^2 \Omega^2 \left( \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \theta^2} \right) - \varsigma \right] \Psi_1 = 0. \]  

where

\[ \Omega^2 \equiv \tilde{\omega}^2 + \omega^2, \]

\[ \varsigma \equiv \frac{E^2 - M^2 c^4}{c^2}. \]

Then, we express the wave function with the product of spatial and angular functions

\[ \Psi_1(p, \theta) \equiv f(p) e^{im\theta}, \]

where \( m = 0, \pm 1, \pm 2, \pm 3, \ldots \). After we substitute the decomposed wave function into Eq. (24), we obtain

\[ \left[ \frac{d^2}{dp^2} + \frac{1}{p} \frac{d}{dp} - \frac{m^2}{p^2} + \left( \kappa_m^2 - k^2 p^2 \right) \right] f(p) = 0. \]

Here,

\[ \kappa_m^2 \equiv \frac{2M \hbar (\omega - m\tilde{\omega}) + \varsigma}{M^2 \hbar^2 \Omega^2}, \]

\[ k^2 \equiv \frac{1}{M^2 \hbar^2 \Omega^2}. \]

We consider the following Ansatz

\[ f(p) \equiv p^m e^{-\frac{4p^2}{k^2}} F(p), \]

and replace Eq. (31) in Eq. (28). Then, and we get

\[ F''(p) + \left( \frac{2m + 1}{p} - 2kp \right) F'(p) - \left( 2k \left( m + 1 \right) - \kappa_m^2 \right) F(p) = 0. \]  

We introduce a new transformation, namely \( t = kp^2 \), and employ it in Eq. (32). We obtain

\[ t \frac{d^2 F(t)}{dt^2} + (m + 1 - t) \frac{dF(t)}{dt} - \left( \frac{1}{2} \left( m + 1 \right) - \frac{\kappa_m^2}{4k} \right) F(t) = 0, \]
which is known as the Kummer’s differential equation and its solutions are expressed in terms of confluent hypergeometric functions \[51\].

\[
F(t) = N_1 \, _1F_1 \left( \frac{1}{2}(m+1) - \frac{k_m^2}{4k}, m+1, t \right) + N_2 \, _1U_1 \left( \frac{1}{2}(m+1) - \frac{k_m^2}{4k}, m+1, t \right). \tag{34}
\]

Here, \(_1F_1 \left( \frac{1}{2}(m+1) - \frac{k_m^2}{4k}, m+1, t \right)\) and \(_1U_1 \left( \frac{1}{2}(m+1) - \frac{k_m^2}{4k}, m+1, t \right)\) are the first and second kind confluent hypergeometric function. Note that we denote the normalization constants with \(N_1\) and \(N_2\). We chose \(N_2 = 0\) since the second kind confluent hypergeometric function does not satisfy the boundary conditions. Then, we adopt the condition for obtaining a polynomial type solution from confluent hypergeometric function

\[
\frac{1}{2}(m+1) - \frac{k_m^2}{4k} = -n \tag{35}
\]

where \(n = 0, 1, 2, 3, \ldots\). We derive the energy spectrum function as

\[
E_{n,m} = \pm Mc^2 \left[ 1 + \frac{2\hbar(m\tilde{\omega} - \omega)}{Mc^2} + \frac{\hbar\Omega(4n + 2(m + 1))}{Mc^2} \right]^{1/2}. \tag{36}
\]

This result is in a good agreement with \[34\]. Finally, we express the first component of the wave function as follows

\[
\Psi_{1n,m}(p, \theta) = C_{n,m} p^m e^{-kp^2/2} \, _1F_1 \left( -n; m+1; kp^2 \right) e^{im\theta}. \tag{37}
\]

where \(C_{n,m}\) is the normalization constant. The general solution of the first component of the wave function is the linear combination of separable solutions.

\[
\Psi_1(p, \theta, t) = \sum_{n=0}^{\infty} \sum_{m=-m_c}^{m_c} \Psi_{1n,m}(p, \theta) \exp \left( -\frac{iE_{n,m}t}{\hbar} \right) \tag{38}
\]

Note that, to avoid the divergency, the quantum number \(m\) is truncated with a finite value, \(m_c\).

**B. Case of the spin-one particle**

In the spin-one case, unlike the spin-zero case, another non-trivial irreducible representation of the DKP algebra is being used. Within this algebra \(\beta\) matrices are given with \(10 \times 10\) matrices. Therefore the total wave function is expressed with ten components. Here, we define the stationary wave function with

\[
\Psi \equiv \left( \psi_1 \, \psi_2 \, \psi_3 \, \psi_4 \, \psi_5 \, \psi_6 \, \psi_7 \, \psi_8 \, \psi_9 \, \psi_{10} \right)^T. \tag{39}
\]
We substitute the wave function in Eq. (12) and obtain the following coupled equations.

\[ Mc^2 \Psi_1 + EP_1 + 2ic(p_x - M\bar{\omega}y + iM\omega x)\Psi_{10} = 0, \quad (40) \]
\[ Mc^2 \Psi_2 + EP_8 + 2ic(p_y + M\bar{\omega}x + iM\omega y)\Psi_{10} = 0, \quad (41) \]
\[ Mc^2 \Psi_3 + EP_9 = 0, \quad (42) \]
\[ Mc^2 \Psi_4 + 2ic(p_y + M\bar{\omega}x - iM\omega y)\Psi_{9} = 0, \quad (43) \]
\[ Mc^2 \Psi_5 - 2ic(p_x - M\bar{\omega}y - iM\omega x)\Psi_{9} = 0, \quad (44) \]
\[ ic(p_y + M\bar{\omega}x - iM\omega y)\Psi_7 - ic(p_x - M\bar{\omega}y - iM\omega x)\Psi_8 - Mc^2 \Psi_6 = 0, \quad (45) \]
\[ -Mc^2 \Psi_7 - EP_1 + ic(p_y + M\bar{\omega}x + iM\omega y)\Psi_6 = 0, \quad (46) \]
\[ Mc^2 \Psi_8 + EP_2 + ic(p_x - M\bar{\omega}y + iM\omega x)\Psi_6 = 0, \quad (47) \]
\[ -ic(p_y + M\bar{\omega}x + iM\omega y)\Psi_4 + ic(p_x - M\bar{\omega}y + iM\omega x)\Psi_5 - EP_3 - Mc^2 \Psi_9 = 0, \quad (48) \]
\[ ic(p_x - M\bar{\omega}y - iM\omega x)\Psi_1 + ic(p_y + M\bar{\omega}x - iM\omega y)\Psi_2 + Mc^2 \Psi_{10} = 0. \quad (49) \]

Eqs. (12), (13), (41) and (48) give the relationship between the third, fourth, fifth and ninth components. Eqs. (40), (41), (45), (46), (47) and (49) present the coupling between the rest six components. Boumali et al. showed that the relationships that give the four components are relatively easier to decompose than the relationships that give the six components [34]. They assumed that \( \Psi_1 = \Psi_2 = 0 \), thus, \( \Psi_6 = \Psi_7 = \Psi_8 = \Psi_{10} = 0 \). Here, we follow this ansatz. Then, we use Eqs. (12), (13), and (41) and write the third, fourth, fifth components in terms of the ninth component. After that we use them in Eq. (48) and obtain

\[
\left[ c^2\left(p_x - M\bar{\omega}y + iM\omega x\right)\left(p_x - M\bar{\omega}y - iM\omega x\right) \right. \\
\left. + c^2\left(p_y + M\bar{\omega}x + iM\omega y\right)\left(p_y + M\bar{\omega}x - iM\omega y\right) + \left(M^2c^4 - E^2\right) \right] \Psi_9 = 0. \quad (50)
\]

Since Eq. (50) is equivalent to Eq. (19), its solution should be the same. Therefore, we write the ninth component of the wave function as follows

\[
\Psi_{9n,m}(p, \theta) = C_{n,m}p^m e^{-\frac{k}{2}p^2} F_1 \left(-n; m + 1; kp^2\right) e^{im\theta}. \quad (51)
\]

where \( C_{n,m} \) is the normalization constant. Moreover, we obtain the energy eigenvalue equation as given

\[
E_{n,m} = \pm Mc^2 \left[ 1 + \frac{2\hbar (m\bar{\omega} - \omega)}{Mc^2} + \frac{\hbar\Omega(4n + 2(m + 1))}{Mc^2} \right]^{1/2}. \quad (52)
\]
IV. DKP OSCILLATOR IN THE MINIMAL LENGTH QUANTUM MECHANICS

In the presence of minimal length formalism, we consider the associative Heisenberg algebra generated by the coordinate, $x_i$, and the momentum, $p_j$, operators which obey the following commutation rule \[ [x_i, p_j] = i\hbar \delta_{ij} (1 + \beta p^2). \] (53)

Here, $\beta$ is the minimal length parameter and carries the inverse square of momentum unit. We employ the Heisenberg algebra representation on the momentum space where the momentum and position operators act on a momentum space wave function as given

\[ p_i \Psi(p) = p_i \Psi(p), \] (54)
\[ x_i \Psi(p) = i\hbar \left[ 1 + \beta p^2 \right] \frac{\partial}{\partial p_i} \Psi(p). \] (55)

Note that in formalism of minimal length, the scalar product of the two momentum space wave functions is defined as follows.

\[ \int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} \Psi^*(p)\Phi(p). \] (56)

A. Case of the spin-zero particle

In this subsection, we investigate the DKP oscillator that is examined in Sec. III A in the ML formalism. Since we assume that the oscillator is under the effect of an equivalent external magnetic field, we obtain the same five coupled equations that are given in Eqs. (14), (15), (16), (17), and (18), thus, Eq. (19). We employ the momentum and position operators, which are defined by the ML formalism with Eq. (54) and Eq. (55), in the Eq. (19). In the polar coordinates, we find a differential equation for the first component of the wave function as follows

\[ \left\{ p^2 - 2M\hbar \left( 1 + \beta p^2 \right) \left[ \omega + i\tilde{\omega} \frac{\partial}{\partial \theta} \right] + \beta M\hbar \left( \Omega^2 \frac{\partial}{\partial p} - 2i\tilde{\omega} \frac{\partial}{\partial \theta} \right) \right\} \psi_1 = 0. \] (57)

Note that, when the ML parameter is taken to be zero, Eq. (57) turns to Eq. (21). Next, we separate the wave function to spatial and angular functions as given in Eq. (27), then
we use in Eq. (57). We find
\[
\left\{ p^2 - 2M\hbar (1 + \beta p^2) \left[ (\omega - m\tilde{\omega}) + \beta M\hbar \left( \Omega^2 p \frac{d}{dp} + 2m\tilde{\omega} \right) \right] \\
- M^2 \hbar^2 \Omega^2 (1 + \beta p^2)^2 \left( \frac{d^2}{dp^2} + \frac{1}{p} \frac{d}{dp} - \frac{m^2}{p^2} \right) - \varsigma \right\} f(p) = 0. \tag{58}
\]

Then, we follow the paper of Jana et al. There, they showed that a second order differential equation in the form
\[
\left[ - a(p) \frac{d^2}{dp^2} + b(p) \frac{d}{dp} + c(p) \right] \phi(p) = \varsigma \phi(p), \tag{59}
\]
is transformed to a Schrödinger-type differential equation by employing
\[
\zeta(p) \equiv \frac{\frac{da(p)}{dp} + 2b(p)}{4a(p)}, \tag{60}
\]
\[
\rho(p) \equiv \exp \int \zeta(p) dp, \tag{61}
\]
\[
\phi(p) \equiv \rho(p) \varphi(p). \tag{62}
\]

We match Eq. (59) with Eq. (58) to determine \(a(p), b(p),\) and \(c(p)\) functions. We find
\[
a(p) = M^2 \hbar^2 \Omega^2 (1 + \beta p^2)^2, \tag{63}
\]
\[
b(p) = -2\beta M^2 \hbar^2 \Omega^2 p(1 + \beta p^2) - \frac{M^2 \hbar^2 \Omega^2 (1 + \beta p^2)^2}{p}, \tag{64}
\]
\[
c(p) = p^2 - 2M\hbar(1 + \beta p^2) \left( (\omega - m\tilde{\omega}) + 2\beta M\hbar m\tilde{\omega} \right) + \frac{M^2 \hbar^2 \Omega^2 (1 + \beta p^2)^2 m^2}{p^2}. \tag{65}
\]

We calculate \(\zeta(p),\) then, \(\rho(p),\) and obtain
\[
\phi(p) = \frac{1}{\sqrt{p}} \varphi(p) \tag{66}
\]
which converts Eq. (59) to
\[
\left[ - a(p) \frac{d^2}{dp^2} + \left( \frac{a(p)}{p} + b(p) \right) \frac{d}{dp} + \left( c(p) - \frac{3a(p)}{4p^2} - \frac{b(p)}{2p} \right) \right] \varphi(p) = \varsigma \varphi(p). \tag{67}
\]

Next, we use a variable change
\[
q \equiv \int \frac{dp}{\sqrt{a(p)}}, \tag{68}
\]
that transforms the \(p\) coordinate to \(q\) coordinate via
\[
p = \frac{1}{\sqrt{\beta}} \tan \left( \frac{\hbar \Omega \sqrt{\beta} q}{\hbar} \right) \tag{69}
\]
in Eq. (67). After a straightforward calculation we obtain

$$-\frac{d^2\chi(q)}{dq^2} + \beta M^2 h^2 \Omega^2 \left( \frac{\xi_1(\xi_1 - 1)}{\sin^2(Mh\Omega\sqrt{\beta}q)} + \frac{\xi_2(\xi_2 - 1)}{\cos^2(Mh\Omega\sqrt{\beta}q)} \right) \chi(q) = \sigma \chi(q) \quad (70)$$

where

$$\sigma \equiv \varsigma + \frac{1}{\beta}, \quad (71)$$
$$\xi_1(\xi_1 - 1) \equiv \left( m^2 - \frac{1}{4} \right), \quad (72)$$
$$\xi_2(\xi_2 - 1) \equiv \frac{1}{\beta M^2 h^2 \Omega^2} - \frac{2(\omega - m\tilde{\omega})}{\beta M h^2 \Omega^2} - \frac{4\tilde{\omega}m}{\Omega^2} + \left( m^2 + \frac{3}{4} \right). \quad (73)$$

We introduce a new coordinate transformation of the form

$$z \equiv \sin^2 \left( Mh\Omega\sqrt{\beta}q \right) \quad (74)$$

and we get

$$z(1 - z) \frac{d^2u(z)}{dz^2} + \left( \frac{1}{2} - z \right) \frac{du(z)}{dz} + \frac{1}{4} \left[ \sigma \frac{1}{\beta M^2 h^2 \Omega^2} - \xi_1(\xi_1 - 1) \frac{1}{z} - \xi_2(\xi_2 - 1) \frac{1}{1 - z} \right] u(z) = 0. \quad (75)$$

We propose the general solution as an Ansatz

$$u(z) \equiv z^{\xi_1/2} (1 - z)^{\xi_2/2} v(z) \quad (76)$$

and we obtain the hypergeometric equation in the following form of

$$z(1 - z) \frac{d^2v(z)}{dz^2} + \left[ \xi_1 + \frac{1}{2} - z(1 + \xi_1 + \xi_2) \right] \frac{dv(z)}{dz}$$
$$- \frac{1}{4} \left[ \xi_1 + \xi_2 \right] v(z) = 0. \quad (77)$$

The general solution of the hypergeometric function is given as

$$v(z) = C_{12} F_1 \left( \frac{1}{2} \left( \xi_1 + \xi_2 + \frac{1}{M h \Omega \sqrt{\beta} \beta} \right), \frac{1}{2} \left( \xi_1 + \xi_2 - \frac{1}{M h \Omega \sqrt{\beta} \beta} \right), \xi_1 + \frac{1}{2}, z \right) \quad (78)$$
$$+ C_{22} z^{\xi_1/2} \xi_1 F_1 \left( \frac{1}{2} \left( 1 - \xi_1 + \xi_2 + \frac{1}{M h \Omega \sqrt{\beta} \beta} \right), \frac{1}{2} \left( 1 - \xi_1 + \xi_2 - \frac{1}{M h \Omega \sqrt{\beta} \beta} \right), \xi_1 + \frac{3}{2}, \xi_1 + \xi_2 \right)$$

where $C_1$ and $C_2$ are normalization constants. Note that, Eqs. (72) and (73) are quadratic equations, thus, they have two roots. In both, we take the higher root from the quadratic formula. Therefore, the boundary condition requires that the normalization constant $C_2$ should be set to zero. Then, we employ the quantization condition

$$\frac{1}{2} \left( \xi_1 + \xi_2 + \frac{1}{M h \Omega \sqrt{\beta} \beta} \right) = -n \quad (79)$$
where \( n \) is an integer. After simple calculus, we obtain the energy function

\[
E_{n,m} = \pm Mc^2 \left[ 1 + \frac{2\hbar (m\tilde{\omega} - \omega)}{Mc^2} + \frac{\hbar \Omega (4n + 2(m + 1))}{Mc^2} \sqrt{\alpha_m + \frac{\Lambda_{n,m}}{M^2 c^4}} \right]^{\frac{1}{2}}.
\]

where

\[
\alpha_m \equiv \left[ 1 + 2\beta M\hbar (m\tilde{\omega} - \omega) + \beta^2 M^2 \hbar^2 (\Omega^2 (m^2 + 1) - 4m\tilde{\omega} \omega) \right]
\]

\[
\Lambda_{n,m} \equiv \beta (M\hbar \Omega)^2 \left[ 2m^2 + 4n^2 + 4nm + 4n + 2(m + 1) - \frac{\tilde{\omega} \omega}{\Omega^2 m} \right]
\]

We would like to emphasize that in the limit of \( \beta = 0 \), \( \alpha \) goes to one while \( \Lambda \) disappears. Therefore, Eq. (80) turns to Eq. (36), which means we end with the habitual quantum mechanical result. Moreover, in the other limit, where the external magnetic field does not exist, \( \tilde{\omega} \) becomes zero and the energy eigenvalue function becomes identical to the one given in Eq. (34) in [50].

**B. Case of the spin-one particle**

In this subsection, we examine the DKP oscillator for a spin-one particle in the ML formalism. In Sec. IIIB we took the six components out of ten components of the wave function as zero. Here, we follow that Ansatz and start by expressing the differential equation of the ninth component of the total wave function in the polar coordinates in the momentum space

\[
\left\{ p^2 - 2M\hbar (1 + \beta p^2) \left[ (\omega + i\tilde{\omega} \frac{\partial}{\partial \theta}) + \beta M\hbar \left( \Omega^2 p \frac{\partial}{\partial p} - 2i\tilde{\omega} \omega \frac{\partial}{\partial \theta} \right) \right] \\
- M^2 \hbar^2 \Omega^2 (1 + \beta p^2)^2 \left( \frac{\partial^2}{\partial p^2} + \frac{1}{p} \frac{\partial}{\partial p} + \frac{1}{p^2} \frac{\partial^2}{\partial \theta^2} \right) - \varsigma \right\} \Psi_9 = 0.
\]

Since Eq. (83) is identical to Eq. (57), we go on the same procedures. Therefore we obtain the energy eigenvalue equation that is given in (80). The main difference occurred in the spin-zero and spin-one sectors are seen in the components of the total wave function.
V. THERMAL PROPERTIES

In this section, we investigate the statistical properties of the two-dimensional DKP oscillator by examining the thermodynamic functions such as Helmholtz free energy, internal energy, entropy and specific heat function in the presence of a ML. Since the energy eigenvalue functions are the same in the spin-zero and spin-one sectors, we can derive thermal properties by adopting Eq. (80) for both sectors. Note that, in all figs. we assume $M=\hbar=c=k_B=\omega=1$.

We assume that the DKP oscillator is in an equilibrium state at a finite temperature, $T$, in a canonical ensemble. We use definition of the partition function

$$Z \equiv \sum_{n=0}^{\infty} \sum_{m=-m_c}^{m_c} \left[ \exp \left( -\frac{|E_{n,m}|}{k_BT} \right) + \exp \left( \frac{|E_{n,m}|}{k_BT} \right) \right]. \quad (84)$$

where $k_B$ denotes the Boltzmann constant. Then, we express the partition function

$$Z = \sum_{n=0}^{\infty} \sum_{m=-m_c}^{m_c} \left[ \exp \left( -\frac{M^2c^2}{k_BT} \sqrt{1 + 2\hbar(\tilde{\omega} - \omega) + \frac{\hbar\Omega(4n + 2(m + 1))}{Mc^2} \sqrt{\frac{\alpha_m + \Lambda_{n,m}}{M^2c^4}}} \right) 
+ \exp \left( \frac{M^2c^2}{k_BT} \sqrt{1 + 2\hbar(\tilde{\omega} - \omega) + \frac{\hbar\Omega(4n + 2(m + 1))}{Mc^2} \sqrt{\frac{\alpha_m + \Lambda_{n,m}}{M^2c^4}}} \right) \right]. \quad (85)$$

We employ the following abbreviations

$$y_{n,m}^2 \equiv 1 + 2\hbar(\tilde{\omega} - \omega) + \frac{\hbar\Omega(4n + 2(m + 1))}{Mc^2} \sqrt{\frac{\alpha_m + \Lambda_{n,m}}{M^2c^4}}, \quad (86)$$

$$\gamma \equiv \frac{M^2c^2}{k_BT}. \quad (87)$$

and we rewrite the partition function

$$Z = \sum_{n=0}^{\infty} \sum_{m=-m_c}^{m_c} \left[ \exp \left( -\gamma y_{n,m}^2 \right) + \exp \left( \gamma y_{n,m}^2 \right) \right]. \quad (88)$$

Since the first and second term has the $\gamma \to -\gamma$ symmetry, we decide to take the half of the terms into account and use the Euler-Maclaurin summation formula

$$\sum_{n=0}^{\infty} F_m(n) = \frac{1}{2} F_m(0) + \int_0^{\infty} dn F_m(n) - \sum_{\rho=1}^{\infty} \frac{B_{2\rho}}{(2\rho)!} F_m^{(2\rho-1)}(0), \quad (89)$$

to compute the partition function as Noucier has done in [48]. Note that, $B_{2\rho}$ represents the Bernoulli numbers while $F^{(2\rho-1)}$ denotes the order of the derivative. First, we perform the
integral term
\[ \int_0^\infty dn F_m(n) = \frac{Mc^2}{2h\Omega} \int_{y_{0,m}}^{+\infty} y_{n,m} y_{n,m} \left[ \frac{1}{1 - \beta (Mc)^2 (1 - y_{n,m}^2)} \right] \frac{1}{2} \exp \left( -\gamma y_{n,m} \right), \quad (90) \]

where
\[ y_{0,m} = \sqrt{1 + \frac{2h (m\bar{\omega} - \omega)}{Mc^2} + \frac{2h\Omega (m + 1)}{Mc^2} \sqrt{\alpha_m} + \frac{\Lambda_{0,m}}{M^2 c^4}}. \quad (91) \]

We expand the square root to its Taylor series and evaluate the integral in high temperature region. We obtain
\[ \int_0^\infty dn F_m(n) = \frac{Mc^2}{2h\Omega \sqrt{1 - \beta (Mc)^2}} \sum_n \frac{(2n)!}{(2^n n!)^2} \left( \frac{\beta (Mc)^2}{\beta (Mc)^2 - 1} \right)^n \frac{\Gamma (2n + 2)}{\gamma^{2n+2}}. \quad (92) \]

On the other hand, the first and the third terms in the summation formula are constituted from \( F_m(n) \) \( \left| \frac{dF_m(n)}{dn} \right|_{n=0} \), \( \frac{d^2F_m(n)}{dn^2} \) \( \left| n=0 \right. \), \( \cdots \) functions and depend on the reciprocal temperature parameter. Therefore at higher temperatures, we ignore the contributions from these terms, since they are negligible [35]. We expand the summation and keep the only terms that are up to the first order of the ML parameter. We obtain the half of the partition function as given
\[ Z = \sum_{m=-m_c}^{m_c} \left\{ \frac{2Mc^2}{\hbar\Omega \sqrt{1 - \beta (Mc)^2}} \left[ \frac{k_B T}{Mc^2} \right]^2 + 6 \left( \frac{\beta (Mc)^2}{\beta (Mc)^2 - 1} \right) \left( \frac{k_B T}{Mc^2} \right)^4 \right\} \]
\[ - \left( \frac{1}{\beta M\hbar\Omega} \right) \left( \sqrt{1 - \beta (Mc)^2 (1 - y_{0,m}^2)} - 1 \right). \quad (93) \]

Note that, in the OQM limit at high temperature the half of the partition function becomes
\[ Z = \sum_{m=-m_c}^{m_c} \frac{k_B^2 T^2}{Mc^2 \hbar\Omega}. \quad (94) \]

We use the definition of the Helmholtz free energy
\[ F \equiv -k_B T \ln (Z) \quad (95) \]
and express the half of the Helmholtz free energy function in the form of
\[ F = -k_B T \ln \left\{ \sum_{m=-m_c}^{m_c} \left[ \frac{Mc^2}{\hbar\Omega \sqrt{1 - \beta (Mc)^2}} \left( \frac{k_B T}{Mc^2} \right)^2 + 6 \left( \frac{\beta (Mc)^2}{\beta (Mc)^2 - 1} \right) \left( \frac{k_B T}{Mc^2} \right)^4 \right] \right. \]
\[ - \left. \left( \frac{1}{2\beta M\hbar\Omega} \right) \left( \sqrt{1 - \beta (Mc)^2 (1 - y_{0,m}^2)} - 1 \right) \right\}. \quad (96) \]
There, in the OQM we get the half of the Helmholtz free function as follows

\[ F = -k_B T \ln \left( \sum_{m=-m_c}^{m_c} \frac{k_B^2 T^2}{M c^2 \hbar \Omega} \right). \]  

(97)

We demonstrate the Helmholtz free energy function in fig. 1. We observe that Helmholtz free energy functions are separated from each other in strong magnetic fields.

The entropy function is derived from the Helmholtz free energy function via

\[ S \equiv -\frac{\partial F}{\partial T}. \]  

(98)

Therefore we find the half of the entropy function as follows

\[ S = k_B \ln \left\{ \sum_{m=-m_c}^{m_c} \left[ \frac{M c^2}{\hbar \Omega \sqrt{1 - \beta (M c)^2}} \left\{ \left( \frac{k_B T}{M c^2} \right)^2 + 6 \left( \frac{\beta (M c)^2}{\beta (M c)^2 - 1} \right) \left( \frac{k_B T}{M c^2} \right)^4 \right\} \right. \right. \]

\[ \left. \left. - \left( \frac{1}{2 \beta M c \hbar \Omega} \right) \left( \sqrt{1 - \beta (M c)^2 (1 - y_0^2)} - 1 \right) \right\} \right. \]

\[ + \sum_{m=-m_c}^{m_c} \frac{k_B}{Z} \frac{M c^2}{\hbar \Omega \sqrt{1 - \beta (M c)^2}} \left[ 2 \left( \frac{k_B T}{M c^2} \right)^2 + 24 \left( \beta (M c)^2 \right) \left( \frac{k_B T}{M c^2} \right)^4 \right] \left( \frac{k_B T}{M c^2} \right)^4 \]. \]  

(99)

In the OQM limit the half of the entropy function reduces to

\[ S = k_B \left[ \ln \left( \sum_{m=-m_c}^{m_c} \frac{k_B^2 T^2}{M c^2 \hbar \Omega} \right) + 2 \right]. \]  

(100)

We illustrate the entropy functions in fig. 2. In strong magnetic fields, we observe that entropy functions differ from each other at relatively high temperatures.

Next we use the definition of the internal energy function

\[ U \equiv k_B T^2 \frac{\partial}{\partial T} \ln Z \]  

(101)

and we find the half of the mean energy function

\[ U = \frac{k_B T}{Z} \sum_{m=-m_c}^{m_c} \frac{M c^2}{\hbar \Omega \sqrt{1 - \beta (M c)^2}} \left[ 2 \left( \frac{k_B T}{M c^2} \right)^2 + 24 \left( \frac{\beta (M c)^2}{\beta (M c)^2 - 1} \right) \left( \frac{k_B T}{M c^2} \right)^4 \right] \left( \frac{k_B T}{M c^2} \right)^4 \]  

(102)

Note that in the OQM at high-temperature scale the half of the internal energy function reduces to

\[ U = 2k_B T \]  

(103)
and the total internal energy goes to zero because of the symmetry. We plot the internal energy functions in fig. 3. In strong magnetic fields, we observe that entropy functions differ from each other at relatively high temperatures. Furthermore, in fig. 4a we demonstrate the higher temperature behaviour of the internal energy. The internal energy value diverges to zero, since the mean energy of the system is zero. At this point, we would like to emphasize that many authors in their articles \[35, 48–50\] did not consider the full partition function. Therefore, they found that the internal energy is proportional to temperature. Due to their prediction, in very high temperature the mean energy of the system should increase to infinity. This is not correct, since the mean energy saturates at zero.

Finally, we use the definition of the specific heat function in the form of

\[ C_v \equiv \frac{\partial U}{\partial T} \tag{104} \]

and evaluate the half of the specific heat function. We get

\[
C_v = \frac{1}{TZ} \sum_{m=-m_e}^{m_e} \frac{M c^2}{\hbar \Omega \sqrt{1 - \beta (M c)^2}} \left\{ 6 \left( \frac{k_B T}{M c^2} \right)^3 + 120 \left( \frac{\beta (M c)^2}{(M c^2)^2} - 1 \right) \left( \frac{k_B T}{M c^2} \right)^5 \right\} \\
- \frac{1}{TZ} \frac{M c^2}{\hbar \Omega \sqrt{1 - \beta (M c)^2}} \left[ 2 \left( \frac{k_B T}{M c^2} \right)^2 + 24 \left( \frac{\beta (M c)^2}{(M c^2)^2} - 1 \right) \left( \frac{k_B T}{M c^2} \right)^4 \right] \\
\times \left[ 2 \left( \frac{k_B T}{M c^2} \right)^3 + 24 \left( \frac{\beta (M c)^2}{(M c^2)^2} - 1 \right) \left( \frac{k_B T}{M c^2} \right)^5 \right] \right\} \tag{105}
\]

In the OQM limit, it reduces zero although the half of the specific heat function goes to a constant

\[ C_v = 2k_B. \tag{106} \]

We demonstrate the specific heat functions in fig. 5. We observe that the energy that is needed to change the temperature of the DKP oscillator changes when the oscillator is in a strong external magnetic in the ML formalism. Moreover, we plot the very high-temperature behaviour of the specific heat functions in fig. 4b. As we emphasized above, our results differ from the others because at very high temperature the specific heat functions go to zero. Note that the authors in their articles \[35, 48–50\] predicted that a non zero value.

Before we conclude the article, we plot the thermodynamic function in a comparative way where they have different magnitudes of the external magnetic field with the ML parameters. To be more precisely we plot, in fig. 6 Helmholtz free energy, in fig. 7 entropy, in fig. 8
internal energy, and in fig. specific heat functions versus the temperature at $\beta = 0$, $\beta = 0.005$, and $\beta = 0.01$ ML parameter comparison.

VI. CONCLUSION

In this article, we solved the two dimensional DKP oscillator under an external homogeneous magnetic field in the generalized uncertainty principle scenario. Initially, we reviewed the exact solutions in the momentum representation in the OQM limit for both the spin-zero and spin-one particles. After that, we obtained the exact solution of one component of the total wave functions, hence total wave functions, in the ML formalism in momentum space. We derived an analytic expression for the energy eigenvalue function depends on the ML parameter. We showed that in the absence of the ML parameter, the results become identical with the ordinary quantum mechanical ones. Finally, we investigated the thermal properties by obtaining the Helmholtz free energy, entropy, mean energy and specific heat functions in the high-temperature limit. We found that some of the existing articles in the literature have wrong results because of the misleading approach in the definition of the partition function.

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FIG. 1: Helmholtz free energy functions versus the temperature.
FIG. 2: Entropy functions versus the temperature.
FIG. 3: Internal energy functions versus the temperature.
FIG. 4: Asymptotic behaviour of the internal energy and specific heat functions.
FIG. 5: Specific heat functions versus the temperature.

(a) $\tilde{\omega} = 0$

(b) $\tilde{\omega} = 0.5$

(c) $\tilde{\omega} = 1.5$
FIG. 6: Helmholtz free energy functions versus the temperature.
FIG. 7: Entropy functions versus the temperature.
FIG. 8: Internal energy functions versus the temperature.
FIG. 9: Specific heat functions versus the temperature.