Chimera states in a population of identical oscillators under planar cross-coupling

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Abstract. We report the existence of chimera states in an assembly of identical nonlinear oscillators that are globally linked to each other in a simple planar cross-coupled form. The rotational symmetry breaking of the coupling term appears to be responsible for the emergence of these collective states that display a characteristic coexistence of coherent and incoherent behaviour. The finding, observed in both a collection of van der Pol oscillators and chaotic Rössler oscillators, further simplifies the existence criterion for chimeras, thereby broadens the range of their applicability to real-world situations.

Keywords. Synchronization; chimera; Rössler system; van der Pol oscillator.

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Chimera state, a curious emergent behaviour of a network of coupled oscillators, has attracted a great deal of attention in recent years [1–10]. A surprising and nonintuitive aspect of this collective state is its composition of coexisting synchronous and asynchronous behaviour – an asymmetric pattern arising from a purely symmetric situation where all oscillators are identical and are coupled symmetrically. This spontaneous splitting of the oscillators into two subpopulations of coherent and incoherent oscillators was first discovered by Kuramoto and Battogtokh [1] for a system of phase oscillators that were coupled in a non-local fashion and subsequently studied by a number of researchers including Abrams and Strogatz [2] who named it a chimera state. The basic interest in this state has increased over the years as a rich variety of such states have been discovered in various model systems [2,3,6,7,14] and more importantly as experimental demonstration of chimeras have also been achieved in laboratory systems [11–14]. Further interest has been sparked by the possibility of invoking these collective states to model phenomena such as unispheric sleep in certain mammals where one half of the brain sleeps (showing
high-amplitude low-frequency coherent neuronal signals) while the other half is awake and displays incoherent electrical activity [15].

A key question that has received some attention in recent times is that of the basic conditions necessary for a chimera state to exist in a system of coupled oscillators. It was long assumed (on the basis of the original findings [1,2]) that chimera states can only exist in coupled phase oscillators and only under the restrictive condition of a ‘non-local’ coupling. More recent work has shown that these conditions are not absolutely necessary and chimeras can exist in systems where both phase and amplitude variations are important and also in globally coupled systems [16,17]. Kaneko [18] observed coexisting ordered (periodic state) and disordered (chaotic state) populations in an assembly of globally coupled chaotic maps. Schmidt et al [17] demonstrated that an ensemble of Stuart–Landau oscillators with a nonlinear global coupling could give rise to chimera states. Sethia and Sen [16] further opened up the field by showing that it was not necessary for the coupling to be nonlinear and a linear complex planar coupling was capable of producing chimera states. These chimera states had variations in both amplitudes and phases. In this work we have tried to probe the question of the existence criterion even further by simplifying both the nature of the global couplings and by expanding the search for these states to ensembles of van der Pol (VDP) oscillators [19] as well as chaotic Rossler oscillators.

We define a cross-coupling in the real plane to construct a globally coupled network of identical oscillators, which is now expressed by \( \dot{X}_i = F(X_i) + K \Gamma g \), where the second term on the right side defines the coupling where \( g \) is a \( m \times 1 \) matrix involving the dynamical variables of the network, \( \Gamma \) is a \( m \times m \) real matrix and \( K \) is a coupling constant. We first consider a globally coupled network of identical VDP oscillators under such a planar-type cross-coupling with the following model equations:

\[
\begin{align*}
\dot{x}_i &= y_i + K[\bar{x} - x_i - \epsilon(\bar{y} - y_i)], \\
\dot{y}_i &= b(1 - x_i^2)y_i - x_i + K[\bar{y} - y_i + \epsilon(\bar{x} - x_i)],
\end{align*}
\]

where

\[
\bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j \quad \text{and} \quad \bar{y} = \frac{1}{N} \sum_{j=1}^{N} y_j
\]

and the coupling matrix \( \Gamma \) and \( g \) are obtained from eq. (1),

\[
\Gamma = \begin{pmatrix} 1 - \epsilon \\ \epsilon \end{pmatrix}, \quad g = \begin{pmatrix} \bar{x} - x_i \\ \bar{y} - y_i \end{pmatrix}
\]

and \( \epsilon \) is the cross-coupling strength. To the conventional global mean-field coupling (involving similar variables of the system) whose strength is controlled by \( K \), we have added a cross-coupling term, that employs the other variables, in a linear form and its strength is, particularly, tunable by the parameter \( \epsilon \). Such a coupling typically appears in a hyper-network structure [20,21]. A similar form of cross-coupling between the activator and the inhibitor variables was also used earlier [7] in a network of FitzHugh–Nagumo oscillators [22] to observe multichimera states. However, there the coupling had a non-locality feature. In contrast, we maintain a global coupling without any spatial variation. In the coupling, the matrix \( \Gamma \) operates on the vector \( g \) when the rotational symmetry [23]}
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is lost for non-zero $\epsilon$ values. The interplay of the global mean-field coupling and the linear cross-coupling is crucial for the emergence of chimera states.

In a parameter space around $b = 3$ ($K = 0.04$), when the oscillators behave like relaxation oscillators, we observe a clear signature of chimera states in a network of size $N = 100$. We use MatLab2008 and ODE45 routine for numerical check and discard 100000 data points as transients. The initial states for $y_i$ are chosen as $(y_{i0} = A \times ((N/2) - i))$ for $i = 1$ to $N/2$ and $(y_{i0} = A \times (i - (N/2)))$ for $(i = (N/2) + 1$ to $N)$ with an added random fluctuation and $A = 0.1$. A sequence of emergent behaviours, clustering and chimera states are observed for varying $\epsilon$. We introduce a measure consisting of the time average of the synchronization error or the Euclidean distance of any arbitrarily chosen reference oscillator (say, 1) from all the other oscillators ($j = 2, \ldots, N$) as $d_{1,j} = \langle \|\vec{X}_j - \vec{X}_1\| \rangle$, where $\langle \rangle$ signifies the time average and $\vec{X}_j$ is the state vector at the $j$th node and this is plotted in figure 1. It clearly shows 1-cluster which bifurcates to 2-cluster ($\epsilon = -2.6$), 3-cluster ($\epsilon = -4.6$) and then to multicluster states and finally, to chimera states for varying $\epsilon$ keeping $K = 0.04$ constant. The multicluster states exist for intermediate $\epsilon$ values between 3-cluster and chimera states. However, the chimera states are not clearly evident from the bifurcation diagram. The chimera states are clear from the snapshots of $x_i$ in figure 2b. The sequence of behaviours (1-cluster, 2-cluster, multicluster and chimera) is also demonstrated in figure 2a that shows the temporal evolution of $x_i$ of all the nodes in each panel for various $\epsilon$. The lowest panel shows two coherent clusters for $\epsilon = -3$ followed by the upper panels with three cluster states for $\epsilon = -5$ and multicluster states for $\epsilon = -8$, respectively. Finally, the uppermost panel (for $\epsilon = -11$) clearly shows a chimera state. For further evidence of the chimera states, we present snapshots of $x_i$ ($i = 100$), in figure 2b, for different coupling strengths in four panels which correspond to the immediate left panels in figure 2a. At the bottom panel, we observe a two-cluster state, and then in the two upper panels, three clusters, multicluster and at the topmost

![Figure 1](image-url)

**Figure 1.** Network of VDP oscillators. Distance $d_{1,j}$ of oscillator-1 from all others with $\epsilon$ ($K = 0.04$). A single-cluster state bifurcates into 2-cluster, then to 3-cluster and finally to chimera states. Bifurcation points, $\epsilon = -2.6$ (2-cluster), $-4.6$ (3-cluster).
Figure 2. Transition from a 2-cluster state to chimera states in VDP network. (a) Temporal evolution of each node plotted for four different cross-coupling strengths $\epsilon = -11, -8, -5, -3$ (from top to bottom). (b) Snapshot of $x_i$ at a particular instant of time.

panel we find the chimera state. It is to be noted that, in a chimera state, we observe two subpopulations with coherence and incoherence in their amplitude dynamics. We have checked that this chimera feature persists for networks of larger sizes.

Once the chimera is confirmed in a network of the relaxation oscillator, we proceed to check if the choice of coupling can succeed in creating a chimera in a network of chaotic units. We apply the same planar-type cross-coupling to construct the globally coupled network of Rössler oscillators,

$$
\begin{align*}
\dot{x}_i &= -y_i - z_i + K(\bar{x} - x_i - \epsilon(\bar{y} - y_i)), \\
\dot{y}_i &= x_i + ay_i + K(\bar{y} - y_i + \epsilon(\bar{x} - x_i)), \\
\dot{z}_i &= bx_i + z_i(x_i - c).
\end{align*}
$$

For the 3D system,

$$
\Gamma = \begin{pmatrix}
1 & -\epsilon & 0 \\
\epsilon & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad g = \begin{pmatrix}
\bar{x} - x_i \\
\bar{y} - y_i \\
0
\end{pmatrix}.
$$

The $\Gamma$ matrix operates here on the $g$ vector to break the rotational symmetry for non-zero $\epsilon$ values. The Rössler system parameters are considered in the chaotic regime ($a = 0.36, b = 0.4, c = 4.5$). We apply three different cross-coupling strengths ($\epsilon = -4.0, -3.7, -2.8$) and $K = 0.08$ when we identify two-cluster, multicluster and single chimera regimes. This sequence of emergent behaviours, particularly the multichimera state under global coupling is a new feature. To characterize the clustering and chimera, we use the same Euclidean distance measure $d_{1,j}$ from all the other oscillators ($j = 2, \ldots, N$) as described earlier. The $d_{1,j}$ of each node of the network ($N = 100$) is plotted in figure 3 as a function of the cross-coupling strength $\epsilon$. For $\epsilon = -4.75$ and below, the $d_{1,j}$ plot of all the nodes confirms a single cluster: all oscillators are completely synchronized. With a slight increase in $\epsilon$ the whole population splits into two synchronized clusters. In contrast to the common notion of attaining symmetry, an inhomogeneity
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Figure 3. Network of Rössler oscillators. $d_{1,j}$ is the distance of oscillator-1 from all others with $\epsilon$. Bifurcation from a single cluster to two cluster, chimera and multichimera states.

is created with increasing $\epsilon$. A similar inhomogeneity with increasing coupling strength was reported [24] in a globally coupled CGLE system. However, they did not notice any chimera state. The coupling plays an important role in the case where it is not a simple all-to-all global coupling but a mixed cross-coupling. As a result, we record the splitting of a single cluster to two-cluster states with increasing inhomogeneity and finally observe chimera and multichimera states for larger coupling strengths. The first arrow (figure 3) from left at $\epsilon = -4.0$ confirms one such two-cluster states. Next, at $\epsilon = -3.7$, as indicated by the middle arrow, $d_{1,j}$ is measured where one set of nodes shows zero value while other nodes are scattered with finite values. However, this is not so clear in figure 3, which is actually a multichimera state but clarified subsequently by a snapshot plot of $x_i$

Figure 4. Multicluster and multichimera states in a globally coupled Rössler oscillator. (a) Time series $x_i$ for all nodes $i = 1–100$, (b) snapshots of $x_i$ (left) for all nodes for $K = 0.08$ and $\epsilon = -4.0, -3.7, -2.8$, respectively from bottom to top panels, similar snapshots of phase $\theta_i$ at right panels.

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in figure 4. Another example (right arrow) has been taken at $\epsilon = -2.8$ which confirms the existence of chimera state. Numerical distance ($d_{1,j}$) is determined by taking 250000 data points after discarding 200000 data points.

To further clarify the collective behaviour in the chaotic Rössler network, we plot $x_i$ of all the nodes in figure 4a. Colour coding of the top left panel shows two clustered states for $\epsilon = -4.0$, while the middle panel exhibits multichimera states ($\epsilon = -3.7$) and the lower panel shows single chimera (middle region is coherent). Furthermore, we take snapshots of $x_i$ and phase $\theta_i$ at an instant of time in figure 4b for all the 100 oscillators. They reconfirm the existence of two-cluster, multichimera and single chimera states from top to bottom panels corresponding to $\epsilon = -4.0, -3.7, -2.8$, respectively. Similar two-cluster, multiclerusted and single clustered chimera are also reflected in the snapshots of instant phase $\theta_i$ for all the oscillators at right panels. The last two panels (middle and lower) at right in figure 4b exhibit coexistence of randomly distributed phases with coherent phases leading to multichimera and chimera states, respectively.

To summarize, we have observed chimera states in networks of VDP oscillators as well as chaotic Rössler oscillators using a planar-type global coupling. For van der Pol oscillators, the evolutionary path to the chimera state follows a sequence of single-cluster state to a two-cluster state and then to a chimera state as the cross-coupling strength $\epsilon$ is varied. For the globally coupled Rössler system in the chaotic regime, we observe both chimera and multichimera states. The chimera states for both the oscillator systems show amplitude and phase fluctuations in the incoherent part of the subpopulation while they are coherent in the other subpopulations of the network. It is evident that non-isochronicity [24–27] is an important factor for the emergence of chimera in limit cycle systems such as VDP system under global coupling while the rotational symmetry breaking [23] plays an additional role. Non-isochronicity in limit cycle systems allows amplitude fluctuations in the non-coherent population in the chimera state. For the Rössler system operating in the chaotic regime, amplitude fluctuations are intrinsic to the dynamics where the presence of non-isochronicity is redundant. It would be worth experimenting this coupling in a host of other systems to verify the conceptual basis of this mechanism. As the linear cross-coupling form is easy to implement, it can also be tried out experimentally and provide a useful paradigm for a broad range of applications of the chimera state.

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