SOME IDENTITIES FOR THE RIEMANN ZETA-FUNCTION

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Abstract. Several identities for the Riemann zeta-function $\zeta(s)$ are proved. For example, if $s = \sigma + it$ and $\sigma > 0$, then

$$\int_{-\infty}^{\infty} \left| \frac{(1 - 2^{-s})\zeta(s)}{s} \right|^2 dt = \frac{\pi}{\sigma} (1 - 2^{-1-2\sigma})\zeta(2\sigma).$$

Let as usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ ($\Re s > 1$) denote the Riemann zeta-function. The motivation for this note is the quest to evaluate explicitly integrals of $|\zeta(s)|^2$, $k \in \mathbb{N}$, weighted by suitable functions. In particular, the problem is to evaluate in closed form

$$\int_{0}^{\infty} (3 - \sqrt{8} \cos(t \log 2)|\zeta(\frac{1}{2} + it)|^{2k} \frac{dt}{(\frac{1}{4} + t^2)^k} \quad (k \in \mathbb{N}).$$

When $k = 1, 2$, this may be done, thanks to the identities which will be established below. The first identity in question is given by

**THEOREM 1.** Let $s = \sigma + it$. Then for $\sigma > 0$ we have

$$\int_{-\infty}^{\infty} \left| \frac{(1 - 2^{-s})\zeta(s)}{s} \right|^2 dt = \frac{\pi}{\sigma} (1 - 2^{-1-2\sigma})\zeta(2\sigma).$$

Since $\lim_{s \to 1}(s - 1)\zeta(s) = 1$, then setting in (1) $\sigma = \frac{1}{2}$ we obtain the following

**Corollary 1.**

$$\int_{0}^{\infty} (3 - \sqrt{8} \cos(t \log 2)|\zeta(\frac{1}{2} + it)|^{2} \frac{dt}{\frac{1}{4} + t^2} = \pi \log 2.$$
THEOREM 2. Let $\chi_A(x)$ denote the characteristic function of the set $A$, and let

$$
\varphi(x) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \int_{1}^{x} \chi_{[2m-1,2m)}(u) \chi_{[2n-1,2n)}(u) \frac{du}{u} \quad (x \geq 1).
$$

Then for $\sigma > 0$ we have

$$
\int_{1}^{\infty} \varphi(x)x^{-s-1} dx = (1 - 2^{1-s}) \zeta^2(s).
$$

From (4) we obtain the following

Corollary 2.

$$
\int_{0}^{\infty} (3 - \sqrt{8} \cos(t \log 2))^2 |\zeta(\frac{1}{2} + it)|^4 \frac{dt}{(\frac{1}{2} + t^2)^2} = \pi \int_{1}^{\infty} \varphi^2(x) \frac{dx}{x^2}.
$$

The integral on the right-hand side of (5) is elementary, but nevertheless its evaluation in closed form is complicated.

Proof of Theorem 1. We start from (see e.g., [1, Chapter 1]) the identity

$$
(1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{\infty} (-1)^{n-1} n^{-s} \quad (\sigma > 0)
$$

and

$$
\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{\sigma^2 + x^2} dx = \frac{\pi}{\sigma} e^{-|\alpha|\sigma} \quad (\alpha \in \mathbb{R}, \sigma > 0),
$$

which follows by the residue theorem on integrating $e^{in\alpha}/(\sigma^2 + z^2)$ over the contour consisting of $[-R, R]$ and semicircle $|z| = R, 3m z > 0$ and letting $R \to \infty$. By using (6) and (7) it is seen that the left-hand side of (1) becomes

$$
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \left( \frac{m}{n} \right)^{it} \frac{dt}{\sigma^2 + t^2}.
$$

$$
= \frac{\pi}{\sigma} \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} \sum_{n<m} (-1)^{m+n} (mn)^{-\sigma} \int_{-\infty}^{\infty} \frac{\cos(t \log \frac{m}{n})}{\sigma^2 + t^2} dt
$$

$$
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} (-1)^m m^{-\sigma} \sum_{n=1}^{m-1} (-1)^n n^{-\sigma} \cdot e^{-\sigma \log \frac{m}{n}} \right)
$$

$$
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{m=1}^{\infty} (-1)^m m^{-2\sigma} \sum_{n=1}^{m-1} (-1)^n \right)
$$

$$
= \frac{\pi}{\sigma} \left( \zeta(2\sigma) + 2 \sum_{k=1}^{\infty} (-1)^{2k} (2k)^{-2\sigma} (-1) \right) = \frac{\pi}{\sigma} (1 - 2^{1-2\sigma}) \zeta(2\sigma).
$$
Some identities for the Riemann zeta-function

This holds initially for \( \sigma > 1 \), but by analytic continuation it holds for \( \sigma > 0 \) as well.

We shall provide now a second proof of Theorem 1. As in the formulation of Theorem 2, let \( \chi_A(x) \) denote the characteristic function of the set \( A \), and let the interval \([a, b]\) denote the set of numbers \( \{x : a \leq x < b\} \). Then, for \( \sigma > 0 \), we have

\[
\int_1^\infty x^{-s-1} \sum_{n=1}^\infty \chi(2n-1,2n)x \, dx = \sum_{n=1}^\infty \int_{2n-1}^{2n} x^{-s-1} \, dx
\]

(8)

\[
= \frac{1}{s} \sum_{n=1}^\infty \left((2n-1)^{-s} - (2n)^{-s}\right) = \frac{(1 - 2^{1-s})\zeta(s)}{s}
\]

in view of (6). Now we invoke Parseval’s identity for Mellin transforms (see e.g., [1] and [3]). We need this identity for the modified Mellin transforms, defined by

\[
F^*(s) \equiv m[f(x)] := \int_1^\infty f(x)x^{-s-1} \, dx.
\]

The properties of this transform were developed by the author in [2]. In particular, we need Lemma 3 of [2] which says that

\[
\int_1^\infty f(x)g(x)x^{1-2\sigma} \, dx = \frac{1}{2\pi i} \int_{\Re s = \sigma} F^*(s)\overline{G^*(s)} \, ds
\]

(9)

if \( F^*(s) = m[f(x)], G^*(s) = m[g(x)], \) and \( f(x), g(x) \) are real-valued, continuous functions for \( x > 1 \), such that

\[
x^{\frac{1}{2} - \sigma} f(x) \in L^2(1, \infty), \quad x^{\frac{1}{2} - \sigma} g(x) \in L^2(1, \infty).
\]

From (8) and (9) we obtain, for \( \sigma > 0 \),

\[
\int_1^\infty \frac{1}{x^2} \left(\sum_{n=1}^\infty \chi(2n-1,2n)(x)\right)^2 x^{1-2\sigma} \, dx = \frac{1}{2\pi i} \int_{\Re s = \sigma} \left(\frac{(1 - 2^{1-s})\zeta(s)}{s}\right)^2 \, ds.
\]

But as \( \chi_A^2(x) = \chi_A(x) \), it is easily found that the left-hand side of the above identity equals

\[
\sum_{m=1}^\infty \sum_{n=1}^\infty \int_1^\infty \chi(2m-1,2m)(x)\chi(2n-1,2n)(x)x^{-1-2\sigma} \, dx
\]

\[
= \sum_{n=1}^\infty \int_{2n-1}^{2n} x^{-1-2\sigma} \, dx = \frac{(1 - 2^{1-2\sigma})\zeta(2\sigma)}{2\sigma}
\]

in view of (6), and (1) follows.

For the Proof of Theorem 2 we need the following
LEMMA. Let $0 < a < b$. If $f(x)$ is integrable on $[a, b]$, then

$$
\left( \int_{a}^{b} f(x)x^{-s} \, dx \right)^2 = \int_{a}^{b} x^{-s} \int_{a}^{x} f(u)f(\frac{x}{u}) \, du \, dx + \int_{a}^{b} x^{-s} \int_{x/b}^{b} f(u)f(\frac{x}{u}) \, du \, dx.
$$

The identity (10) remains valid if $b = \infty$, provided the integrals in question converge, in which case the second integral on the right-hand side is to be omitted.

Proof of the Lemma. We write the left-hand side of (10) as the double integral

$$
\int_{a}^{b} \int_{a}^{b} (xy)^{-s} f(x)f(y) \, dx \, dy
$$

and make the change of variables $x = X/Y, y = Y$. The Jacobian of this transformation equals $1/Y$, hence the left-hand side of (10) becomes

$$
\int_{a}^{b} X^{-s} \left( \int_{\min(a,X/b)}^{\max(a,X/b)} f(Y) f(\frac{X}{Y}) \, dY \right) \, dX
$$

$$
= \int_{a}^{b} X^{-s} \int_{a}^{X/a} f(Y) f(\frac{X}{Y}) \, dY \, dX
$$

$$
+ \int_{a}^{b} X^{-s} \int_{X/b}^{b} f(Y) f(\frac{X}{Y}) \, dY \, dX,
$$

as asserted.

Proof of Theorem 2. We use (8) and the Lemma to obtain that (4) certainly holds with $\varphi(x)$ given by (3), since trivially $\varphi(x) \ll x$. To see that it holds for $\sigma > 0$, we note that

$$
\int_{1}^{\sqrt{x}} g(u) g\left(\frac{x}{u}\right) \frac{du}{u} = \int_{1}^{\sqrt{x}} + \int_{\sqrt{x}}^{x} = 2 \int_{\sqrt{x}}^{x} g(u) g\left(\frac{x}{u}\right) \frac{du}{u},
$$

and use (11) with

$$
g(x) = \sum_{n=1}^{\infty} \chi_{[2n-1,2n]}(x).
$$

Note then that the integrand in $\varphi(x)$ equals $1/u$ for $2m - 1 \leq u \leq 2m, 2n - 1 \leq u \leq 2n$, and otherwise it is zero. This gives the condition

$$
4mn - 2m - 2n + 1 \leq x < 4mn, \frac{1}{2} \sqrt{x} \leq n \leq \frac{1}{2}(x + 1), 1 \leq m \leq \frac{1}{2}(\sqrt{x} + 1).
$$

We also have

$$
\int_{\sqrt{x}}^{x} \chi_{[2m-1,2m]}(\frac{x}{u}) \chi_{[2n-1,2n]}(u) \frac{du}{u} \leq \int_{2n-1}^{2n} \frac{du}{u} \leq \frac{1}{2n - 1}.
$$
Therefore
\begin{equation}
\phi(x) \ll \sum_{m \leq \sqrt{x}} \frac{1}{x} \sum_{n \leq x/(4m-2)} \frac{1}{n}
\ll \sum_{m \leq \sqrt{x}} \frac{m}{x} \left(1 + \frac{x}{m^2}\right) \ll \log x.
\end{equation}

This bound shows that the integral in (4) is absolutely convergent for \( \sigma > 0 \). Thus by the principle of analytic continuation this completes the proof of Theorem 2.

Corollary 2 follows then from (4) and (9) on setting \( \sigma = \frac{1}{2} \).

It is interesting to note that the bound in (12) is actually of the correct order of magnitude. Namely we have

**THEOREM 3.** For any given \( \varepsilon > 0 \) we have
\begin{equation}
\phi(x) = \frac{1}{4} \log x + \frac{1}{2} \log \left(\frac{\pi}{2}\right) + O\left(x^{\varepsilon-\frac{1}{4}}\right).
\end{equation}

**Proof of Theorem 3.** By (8) and the inversion formula for the Mellin transform \( m[f(x)] \) (see [2, Lemma 1]) we have, for any \( c > 0 \),
\begin{equation}
\phi(x) = \frac{1}{2\pi i} \int_{\text{Re } s = c} \frac{(1 - 2^{1-s})^2 \zeta^2(s)x^s}{s^2} \, ds.
\end{equation}

We shift the line of integration in (14) to \( c = \varepsilon - 1/4 \) with \( 0 < \varepsilon < 1/8 \), which clearly may be assumed. Since \( \zeta(0) = -\frac{1}{2} \) and \( \zeta'(0) = -\frac{1}{2} \log(2\pi) \), the residue at the double pole \( s = 0 \) is found to be
\begin{equation}
\frac{1}{4} \log x + A, \quad A = -\zeta'(0) - \log 2 = \frac{1}{2} \log \left(\frac{\pi}{2}\right).
\end{equation}

We use the functional equation (see e.g., [1, Chapter 1]) for \( \zeta(s) \), namely
\[ \zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \]
with
\[ \chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} e^{it(\frac{1}{2}+\frac{\pi}{2})} \left(1 + O\left(\frac{1}{t}\right)\right) \quad (t \geq 2). \]

Let \( s = \varepsilon - \frac{1}{4} + it \). Then by absolute convergence we have
\[
\int_T^{2T} \frac{(1 - 2^{1-s})^2 \zeta^2(s)x^s}{s^2} \, dt = \sum_{n=1}^{\infty} d(n)n^{\varepsilon-5/4} \int_T^{2T} \frac{(1 - 2^{1-s})^2 x^{\varepsilon-\frac{1}{4}+it}}{s^2} \left(\frac{t}{2\pi}\right)^{\frac{3}{2}-2\varepsilon} e^{iF(t,n)} \, dt + O(T^{\frac{1}{2}-2\varepsilon}),
\]
where \( d(n) \) is the number of divisors of \( n \) and
\[
F(t, n) := 2t + t \log n - 2t \log(t/2\pi), \quad \frac{d^2}{dt^2}(t \log x + F(t, n)) = -\frac{2}{t}.
\]
Hence by the second derivative test (see [1, Lemma 2.2]) the above series is
\[
\ll \sum_{n=1}^{\infty} d(n) n^{\varepsilon - 5/4} T^{-2\varepsilon} = \zeta^{2}(\frac{5}{4} - 2\varepsilon) T^{-2\varepsilon} \ll T^{-2\varepsilon}.
\]
This shows that
\[
\int_{\Re s = \varepsilon - 1/4} \frac{(1 - 2^{1-s})^{2} \zeta^{2}(s) x^{s}}{s^{2}} \, ds \ll x^{\varepsilon - 1/4},
\]
hence (13) follows from (14), (15) and the residue theorem.

In concluding, note that if we write
\[
\varphi(x) = \frac{1}{4} \log x + A + \varphi_{1}(x),
\]
where \( A \) is given by (15) then, for \( \Re s = \sigma > 0 \), (4) yields
\[
s^{2} \left( \frac{A}{s} + \frac{1}{4s^{2}} + \int_{1}^{\infty} \varphi_{1}(x)x^{-s-1} \, dx \right) = (1 - 2^{1-s})^{2} \zeta^{2}(s),
\]
and the above integral converges absolutely, for \( \sigma > -1/4 \), in view of (13). Thus by analytic continuation it follows that, for \( \sigma > -1/4 \),
\[
As + \frac{1}{4} + s^{2} \int_{1}^{\infty} \varphi_{1}(x)x^{-s-1} \, dx = (1 - 2^{1-s})^{2} \zeta^{2}(s).
\]

References

[1] A. Ivić, The Riemann zeta-function, John Wiley & Sons, New York, 1985.
[2] A. Ivić, On some conjectures and results for the Riemann zeta-function and Hecke series, Acta Arith. 109(2001), 115-145.
[3] E.C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford University Press, Oxford, 1948.