An exact solution of the moving boundary problem for the expansion of a plasma cylinder in a magnetic field

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Abstract

An exact analytic solution has been obtained for a uniformly expanding, neutral, infinitely conducting plasma cylinder in an external uniform and constant magnetic field. The electrodynamical aspects related to the emission and transformation of energy have been considered as well. The results obtained can be used in analysing the recent experimental and simulation data.

Keywords: plasma expansion, boundary value problem, magnetic field

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1. Introduction

Many processes in physics involve boundary surfaces which requires the solution of boundary and initial value problems. The introduction of a moving boundary into the physics usually precludes the achievement of an exact analytic solution of the problem and recourse to approximation methods is required [1–3] (see also references therein). In the case of a moving plane boundary a time-dependent translation of the embedding space immobilizes the boundary at the expense of the increased complexity of the differential equation. It is the aim of this work to present an example of a soluble moving boundary and initial value problem in cylindrical geometry.

The problems with moving boundary arise in many area of physics. One important example is sudden expansion of hot plasma with a sharp boundary in an external magnetic field which is particularly of interest for many astrophysical and laboratory applications (see, e.g., [4] and references therein). Such kind of processes arise during the dynamics of solar flares and flow of the solar wind around the earth’s magnetosphere, in active experiments with plasma clouds in space, and in the course of interpreting a number of astrophysical observations [3–11].

To study the radial dynamics and evolution of the initially spherical or cylindrical plasma cloud both analytical and numerical approaches were developed (see, e.g., Refs. [4–14] and references therein). The plasma cloud is shielded from the penetration of the external magnetic field by means of surface currents circulating inside the thin layer on the plasma boundary. Ponderomotive forces resulting from interaction of these currents with the magnetic field would act on the plasma surface as if there were magnetic pressure applied from outside. After some period of accelerated motion, plasma gets decelerated as a result of this external magnetic pressure acting inward. The plasma has been considered as a highly conducting media with zero magnetic field inside. From the point of view of electrodynamics it is similar to the expansion of a superconductor in a magnetic field. An exact analytic solution for a uniformly expanding, highly conducting plasma sphere in an external uniform and constant magnetic field has been obtained in [12]. The non-relativistic limit of this theory has been used by Raizer [13] to analyse the energy balance (energy emission and transformation) during the plasma expansion. The similar problem has been considered in Ref. [8] within one-dimensional geometry for a plasma layer. In our previous paper [14] we obtained an exact analytic solution for the uniform relativistic expansion of the highly conducting plasma sphere in the presence of a dipole magnetic field. In the present paper we study the uniform expansion of the highly conducting plasma cylinder in the presence of a constant magnetic field. For this geometry we found again an exact analytical solution which can be used in analysing the recent experimental and simulation data (see, e.g., Refs. [3–11] and references therein).

2. Moving boundary problem

We consider the moving boundary problem of the highly conducting plasma cylinder expansion in the vacuum. Consider a cylindrical region of space with radius \( \rho = R(t) \) at the time \( t \) containing a neutral infinitely conducting plasma which has expanded at \( t = 0 \) (with \( R(0) = 0 \)) to its present state from a linear source located at \( \rho = 0 \). We assume that at any time \( t \) the plasma cylinder is unbounded in \( z \) direction (i.e. the cylinder is located at \( -\infty < z < \infty \)). To solve the boundary problem we introduce the cylindrical coordinate system \((\rho, \varphi, z)\) with the \( z \)-axis along the plasma cylinder symmetry axis and the azimuthal angle \( \varphi \) is counted from the plane \((x, z)\) containing the vector of the constant and homogeneous magnetic field \( \mathbf{H}_0 \). The angle \( \theta \) between the vector \( \mathbf{H}_0 \) and the \( z \)-axis is arbitrary. The unperturbed magnetic field is expressed by the vector potential, \( \mathbf{A}_0 = \frac{1}{\mu_0}[\mathbf{H}_0 \times \mathbf{r}] \) with the

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Boundary conditions should be imposed at the cylindrical surface \( \rho = R(t) \) and at infinity. Because of the finite propagation velocity of the perturbed electromagnetic field the magnetic field at infinity will remain undisturbed for all finite times. Further, no incoming wave-type solutions are permitted. Thus, for all finite times \( \mathbf{H}(\mathbf{r}, t) \to \mathbf{H}_0 \) at \( \rho \to \infty \). The boundary condition at the expanding cylindrical surface is \( H_\rho = 0 \) which is equivalent to the relation \( \chi(\beta) = 0 \) (see Eq. (3)) with \( \beta = v/c \). In addition imposing that \( \mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 \) at \( \rho \geq ct \) we obtain another boundary condition \( \chi(1) = 1 \).

The solution of Eq. (5) subject to the initial and boundary conditions may be written as

\[
\chi(\zeta) = 1 - \frac{\mathcal{U}(\zeta)}{\mathcal{F}(\zeta)} \frac{\mathcal{F}(\beta)}{\mathcal{F}(\beta)},
\]

where

\[
\left( \frac{\mathcal{U}(\zeta)}{\mathcal{F}(\zeta)} \right) = \frac{\sqrt{1 - \zeta^2}}{\zeta} \pm \frac{1}{\zeta} \ln \frac{1 + \sqrt{1 - \zeta^2}}{\zeta}.
\]

The complete solution may finally be written in the form at \( vt \leq \rho \leq ct \) (or \( \beta \leq \zeta \leq 1 \))

\[
\begin{align*}
H_\rho &= \mathcal{H}_0 \cos \varphi \left[ \frac{\mathcal{F}(\zeta)}{\mathcal{F}(\beta)} \right], \\
H_\varphi &= -\mathcal{H}_0 \sin \varphi \left[ \frac{\mathcal{F}(\zeta)}{\mathcal{F}(\beta)} \right], \\
H_z &= \mathcal{H}_0 \left[ 1 + \frac{1}{\mathcal{F}(\beta)} \ln \frac{1 + \sqrt{1 - \zeta^2}}{\zeta} \right], \\
E_\rho &= 0, \\
E_\varphi &= \frac{\mathcal{H}_0}{\mathcal{F}(\beta)} \sqrt{1 - \zeta^2} \\
E_z &= \mathcal{H}_0 \sin \varphi \left[ \frac{2}{\mathcal{F}(\beta)} \right].
\end{align*}
\]

\( \mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0, \mathbf{E}(\mathbf{r}, t) = 0 \) at \( \rho > ct \) (or \( \zeta > 1 \)) and \( \mathbf{H}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) = 0 \) at \( \rho < R(t) = vt \) (or \( \zeta < \beta \)). From Eqs. (3), (4), (9) and (10) it can be easily checked that the boundary condition on the moving surface, \( \mathcal{E}(\mathbf{r}) = -\frac{1}{\gamma} [\mathbf{v} \times \mathbf{H}(\mathbf{r})] \) (or \( \mathcal{E}_\varphi(R) = \beta H_\varphi(R), \mathcal{E}_z(R) = -\beta H_z(R) \)), is satisfied automatically.

It is also imperative to determine the surface current density induced due to the moving boundary. This can be done employing the Maxwell’s equation \( \mathbf{j} = \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{H}) - \frac{4\pi}{c} \mathbf{j}_0 \). It is clear that the surface current has only two components and \( \mathbf{j}_0 = i_0 \delta(\rho - R) \) with \( \alpha = z, \varphi \), where \( i_0 \) is the linear surface current. Using Eqs. (9) and (10) as well as the Maxwell’s equation we obtain

\[
\begin{align*}
i_\zeta &= \frac{c}{4\pi} \left[ H_\varphi(R) + \beta E_\varphi(R) \right] = \frac{c}{4\pi} H_\varphi(R) \\
&= -i_{0z} \sin \varphi \frac{2}{\gamma^2 \beta^2 \mathcal{F}(\beta)}, \\
i_\varphi &= -\frac{c}{4\pi} \left[ H_z(R) - \beta E_z(R) \right] = -\frac{c}{4\pi} H_z(R) \\
&= -i_{0\varphi} \frac{1}{\gamma^2 \beta^2 \mathcal{F}(\beta)}.
\end{align*}
\]

Here \( \gamma^{-2} = 1 - \beta^2 \) is the relativistic factor of the expanding boundary and \( i_{0\|} = (c/4\pi)H_{0\|} \). Note that the moving bound-
Consider now briefly the non-relativistic limit of Eqs. (9) and (10). This limit can be obtained using at $\zeta \to 0$ and $\beta \to 0$ the asymptotic expression $\mathcal{U}(\zeta)/F(\beta) = F'(\zeta)/F(\beta) = \beta^2/\zeta^2 = R^2/\rho^2$ which yields $H_0 = H_{0i}$, and

$$\bar{H}_p = H_{0i}\cos \varphi \left(1 + \frac{R^2}{\rho^2}\right), \quad \bar{H}_\varphi = -H_{0i}\sin \varphi \left(1 + \frac{R^2}{\rho^2}\right). \quad (13)$$

In the lowest order with respect to the factor $\beta$ the components of the electric field are given by $E_\varphi = \beta H_{00}(R/\rho)$, $E_z = 2\beta H_{00} \sin \varphi(R/\rho)$. It is seen that the parallel component $H_\varphi$ of the magnetic field remains unchanged in the case of non-relativistic expansion.

### 3. Analysis of the energy balance

Previously significant attention has been paid [8,11,3,14] to the question of what fraction of energy is emitted and lost in the form of electromagnetic pulse propagating outward of the expanding plasma. In this section we consider the energy balance during the plasma cylinder expansion in the presence of the homogeneous magnetic field. When the plasma cylinder of the zero initial radius is created at $t = 0$ and starts expanding, external magnetic field $H_0$ is perturbed by the electromagnetic pulse, $H'(r, t) = H(r, t) - H_0$, $E(r, t)$, propagating outward with the speed of light. The tail of this pulse coincides with the moving plasma boundary $\rho = R(t)$ while the leading edge is at $\rho = ct$. Ahead of the leading edge, the magnetic field is not perturbed and equals $H_0$ while the electric field is zero.

Our starting point is the energy balance equation (Poynting equation)

$$\nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} - \frac{\partial}{\partial t} \left(\frac{E^2 + H^2}{8\pi}\right). \quad (14)$$

where $\mathbf{S} = \frac{\varepsilon}{\varepsilon_0} \mathbf{[E \times H]}$ is the Poynting vector and $\mathbf{j} = j_x \mathbf{e}_x + j_z \mathbf{e}_z$ (with $|\mathbf{e}_x| = |\mathbf{e}_z| = 1$) is the surface current density. The energy emitted to infinity is measured as a Poynting vector integrated over the lateral surface $S_\varphi$ of the cylinder with radius $\rho_\delta$, length $l$, and the volume $\Omega_r$ (control cylinder) enclosing the plasma cylinder ($\rho_\delta > R$ or $0 \leq t < \rho_\delta/v$). Integrating over time and over the volume $\Omega_r$, Eq. (14) can be represented as

$$W_S(t) = W_J(t) + \Delta W_{EM}(t), \quad (15)$$

where

$$W_S(t) = l_0 \rho_\delta \int_0^t d\tau' \int_0^{2\pi} S_\varphi d\varphi, \quad W_J(t) = -\int_0^t d\tau' \int_{\Omega_r} \mathbf{j} \cdot \mathbf{E} d\mathbf{r}. \quad (16)$$

Here $S_\varphi = \frac{\varepsilon_0}{4\pi}(E_z H_x - E_x H_z)$ is the radial component of the Poynting vector. Note that the total flux of the energy over the bases of the control cylinder determined by the Poynting’s vector component $S_\varphi$ vanishes due to the symmetry reason. $W_{EM}(t)$ and $\Delta W_{EM}(t) = W_{EM}(0) - W_{EM}(t)$ are the total electromagnetic energy and its change (with minus sign) in a volume $\Omega_r$, respectively. $W_J(t)$ is the energy transferred from plasma cylinder to electromagnetic field and is the mechanical work with minus sign performed by the plasma on the external magnetic pressure. At $t = 0$ the electromagnetic fields are given by $\mathbf{H}(r, t) = \mathbf{H}_0$ and $\mathbf{E}(r, t) = 0$. Hence $W_{EM}(0)$ is the total energy of the magnetic field in a volume $\Omega_r$ and is given by $W_{EM}(0) = Q = \pi \rho_\delta^2 (H_0^2/8\pi)$. Then the change of the electromagnetic energy $\Delta W_{EM}(t)$ in a volume $\Omega_r$ can be evaluated as

$$\Delta W_{EM}(t) = Q - \int_{\Omega_r} \frac{E^2 + H^2}{8\pi} d\mathbf{r}. \quad (17)$$

In Eq. (17) $\Omega_r$ is the volume of the control cylinder excluding the volume of the plasma cylinder (we take into account that $\mathbf{H}(r, t) = \mathbf{E}(r, t) = 0$ in a plasma cylinder). Hence the total energy flux $W_S(t)$ given by Eq. (15) is calculated as a sum of the energy loss by plasma due to the external electromagnetic pressure and the decrease of the electromagnetic energy in a control volume $\Omega_r$. For non-relativistic ($\beta \ll 1$) expansion of a one-dimensional plasma slab and for uniform external magnetic field $W_S \approx 2W_J \approx \Delta W_{EM}$, i.e., approximately the half of the outgoing energy is gained from the plasma, while the other half is gained from the magnetic energy [3]. In the case of non-relativistic expansion of highly-conducting spherical plasma with radius $R$ in the uniform magnetic field $H_0$ the outgoing energy $W_S$ is distributed between $W_J$ and $\Delta W_{EM}$ according to $W_J = 1.5Q_0$ and $\Delta W_{EM} = 0.5Q_0$ with $W_J = 2Q_0$, where $Q_0 = H_0^2 R^5/6$ is the magnetic energy escaped from the spherical plasma volume [13]. Therefore in this case the released electromagnetic energy is mainly gained from the plasma.

Consider now each energy component $W_{SJ}(t)$, $W_J(t)$ and $\Delta W_{EM}(t)$ separately. $W_S(t)$ is calculated from Eq. (15). In the first expression of Eq. (15) the $t'$-integral must be performed at $\eta \leq \tau' < t$ ($t < \eta \leq \tau'$ since at $0 \leq t < \tau'/c$ the electromagnetic pulse does not reach to the control surface yet and $S_{\delta J}(\rho_\delta) = 0$). From Eqs. (9), (10) and (16) we obtain

$$\frac{W_S(t)}{Q} = \frac{1}{\mathcal{F}^2(\beta)} \left[ 2\mathcal{F}(\beta) \mathcal{F}(\eta) - \mathcal{F}^2(\eta) + \left(1 + \frac{H_0^2}{H_0^2}ight) \left(1 - \frac{\eta^2}{\eta_0^2}\right) \right], \quad (18)$$

where $\eta = \rho_\delta/c t < 1$. In non-relativistic limit using the asymptotic expression $\mathcal{F}(\eta)/\mathcal{F}(\beta) = \beta^2/\eta^2 = (t/v)^2$ (with $v = \rho_\delta/v$ at $\beta \to 0$ and $\eta \to 0$, from Eq. (15) we obtain

$$\frac{W_S(t)}{Q} = \frac{2\eta^2}{\eta_0^2} + \frac{\eta^4}{\eta_0^4} \left(1 - \frac{H_0^2}{H_0^2}\right). \quad (19)$$

Next, we evaluate the energy loss $W_J(t)$ by the plasma which is determined by the surface current density, $\mathbf{j}$. This current has two azimuthal and axial components and is localized within thin cylindrical skin layer, $R - \delta < \rho < R + \delta$ with $\delta \to 0$, near plasma boundary. Therefore in Eq. (16) the volume $\Omega_r$ can be replaced by the volume $\Omega_r = l_0 R \delta$ which includes the space between the cylinders with $\rho = R - \delta$ and $\rho = R + \delta$. The surface current density is calculated from the Maxwell’s equation and has been determined in previous section, see Eqs. (11) and (12).
As shown below the \( r \)-integration of the term \( \mathbf{j} \cdot \mathbf{E} \) in Eq. (16) can be alternatively expressed via magnetic field. Within the skin layer we take into account that \( \mathbf{E} = -\frac{1}{c} [\mathbf{v} \times \mathbf{H}] \) and \( H_y(R) = 0 \). Then

\[
Q_J(t) = - \int_{\Omega_t} \mathbf{j} \cdot \mathbf{E} \, d\mathbf{r} = \frac{1}{4\pi} \int_{\Omega_t} \mathbf{v} \cdot [\mathbf{H} \times (\nabla \times \mathbf{H})] \, d\mathbf{r} + \frac{1}{8\pi} \int_{\Omega_t} \frac{\partial \mathbf{E}^2}{\partial t} \, d\mathbf{r} - \frac{\mathbf{v}}{c} \int_{S_{Rt}} \frac{H^2(R) - E^2(R)}{8\pi} \, dS,
\]

where \( S_R \) is the lateral surface of the expanding cylindrical plasma. As mentioned in Sec. 2 the factor \( \gamma^{-2} \) in Eq. (20) appears because of the moving boundary. In this expression the term with \( \frac{\partial E^2(r,t)}{\partial t} \) has been transformed to the surface integral using the fact that the boundary of the volume \( \Omega_t \) moves with a constant velocity \( \mathbf{v} \) and the electric field has a jump across the plasma surface. Equation (20) shows that the energy loss by the plasma per unit time is equal to the work performed by the plasma on the external electromagnetic pressure. This external pressure is formed by the difference between magnetic and electric pressures, i.e., the induced electric field tends to decrease the force acting on the expanding plasma surface. The total energy loss by the plasma cylinder is calculated as

\[
\frac{W_J(t)}{Q} = \frac{1}{Q} \int_0^t Q_J(t') \, dt' \left( 1 + \frac{H^2_0}{H^2_\perp} \right) \left[ 1 - \beta^2 \right] \left\{ 1 + \frac{H^2_0}{H^2_\perp} \right\}.
\]

In a non-relativistic case Eq. (21) yields:

\[
\frac{W_J(t)}{Q} = \frac{\beta^2}{\eta^2} \left( 1 + \frac{H^2_0}{H^2_\perp} \right).
\]

The change of the electromagnetic energy in a control cylinder is calculated from Eq. (17). At \( R < \rho_c < ct \) (the electromagnetic pulse fills the whole control cylinder) we obtain

\[
\frac{\Delta W_{\text{EM}}(t)}{Q} = \frac{1}{\mathcal{F}^2(\beta)} \left[ 2\mathcal{F}(\beta) \mathcal{F}(\eta) - \mathcal{F}^2(\eta) \right] - \left( 1 + \frac{H^2_0}{H^2_\perp} \right) \eta^2 \left( \frac{\beta^2}{\eta^2} - 1 \right).
\]

Comparing Eqs. (18), (21) and (23) we conclude that \( \Delta W_{\text{EM}}(t) + W_J(t) = \Delta W_S(t) \) as predicted by the energy balance equation (15).

The non-relativistic limit of Eq. (23) can be evaluated from Eqs. (19) and (22) using the relation \( \Delta W_{\text{EM}}(t) = W_S(t) - W_J(t) \). As an example in Fig. 1 we show the results of model calculations for the ratios \( \Gamma_S(t) = W_S(t)/Q_0(t) \) and \( \Gamma_J(t) = W_J(t)/Q_0(t) \) as a function of time \( \rho_c/c < t < \rho_c/v \). Here \( Q_0(t) = \pi l \eta^2 R^2 H^4_\perp/8\pi \) is the magnetic energy escaped from the plasma cylinder at time \( t \). For the relativistic factor \( \beta \) we have chosen a wide range of values. We recall that at \( 0 < t < \rho_c/c \), i.e., the electromagnetic pulse does not yet reach to the surface of the control cylinder, \( W_S(t) = 0 \). Unlike the case with uniform magnetic field discussed above (see also [8, 13]) there are no simple relations between the energy components \( W_S(t) \) and \( W_J(t) \) and
From Eq. (21) it is seen that the ratio \( \Gamma_J(t) \) is constant and is given by
\[
\Gamma_J(t) = \left(1 + \frac{\gamma^2 \beta^2 F(\beta)}{\gamma_0^2} \right)^{-2}
\]
is some kinematic factor. For non-relativistic expansion with \( \beta \ll 1 \), this factor is \( C \approx 1 \) while at \( \beta \sim 1 \) this factor is very large and behaves as \( C \approx \left( \frac{9}{8} \right) \left( 1 - \beta \right)^{-1} \gg 1 \). As expected the total energy flux, \( W_S(t) \), increases monotonically with \( t \). At the final stage (\( t = \rho/v \)) of relativistic expansion (with \( \beta \sim 1 \)) \( W_S \approx W_J \). Hence in this case the emitted energy \( W_S \) is mainly gained from the plasma cylinder.

4. Conclusion

An exact solution of the uniform radial expansion of a neutral, infinitely conducting plasma cylinder in the presence of a homogeneous magnetic field has been obtained. The electromagnetic fields are derived by using the appropriate boundary and initial conditions, Eq. (6). It has been shown that the electromagnetic fields are perturbed only within the domain extending from the surface of the expanding plasma cylinder \( \rho = R = vt \) to the surface of the expanding information cylinder \( \rho = ct \). External to the cylinder \( \rho = ct \) the magnetic field is not perturbed. In the course of this study we have also considered the energy balance during the plasma cylinder expansion. The model calculations show that the emitted energy is mainly gained from the plasma cylinder. For relativistic expansion \( W_S \approx W_J \) and the emitted energy is practically gained only from plasma cylinder.

We expect our theoretical findings to be useful in experimental investigations as well as in numerical simulations of the plasma expansion into ambient magnetic field. One of the improvements of our model will be to include the effect of the deceleration of the plasma cylinder as well as the derivation of the dynamical equation for the surface deformation. A study of this and other aspects will be reported elsewhere.

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