Echo Spectroscopy of Atomic Dynamics in a Gaussian Trap via Phase Imprints

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Abstract. We report on the collapse and revival of Ramsey fringe visibility when a spatially dependent phase is imprinted in the coherences of a trapped ensemble of two-level atoms. The phase is imprinted via the light shift from a Gaussian laser beam which couples the dynamics of internal and external degrees of freedom for the atoms in an echo spectroscopy sequence. The observed revivals are directly linked to the oscillatory motion of atoms in the trap. An understanding of the effect is important for quantum state engineering of trapped atoms.

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1 Introduction

A trapped gas of atoms can act as dispersive medium with a refractive index which depends on the internal state of the atoms. The state-dependent phase shift of probe laser light propagating through an ensemble of Cs atoms has recently been used to observe Rabi flopping on the clock transition non-destructively [1,2]. Such non-destructive measurements of a collective atomic quantum state component holds the promise to predict the outcome of subsequent measurements beyond the standard quantum limit [3]. This reduction in uncertainty is referred to as conditional squeezing and the resulting nonclassical atomic state may be used to increase the precision of atomic clocks [4].

In a recent paper [5], we considered the effect of inhomogeneous light shifts on the atomic quantum state evolution when using a Gaussian laser beam for dispersive probing of an ensemble of atoms confined in a dipole trap. The spatial intensity distribution of the probe beam implies that an atom experiences a position dependent differential ac Stark shift of the clock levels and the atomic cloud acquires a spatial phase imprint as illustrated schematically in Fig. 1. In the present paper we shall focus on the fact that the individual atoms are not stationary in the trap, but move about to explore regions of different probe light intensities. This is of consequence for how well their inhomogeneous phase spread can be compensated for by using Hahn echo techniques. Specifically, we shall investigate the degradation of Ramsey fringe contrast in echo spectroscopy as a result of atomic movement between two perturbing light pulses on either side of the echo pulse. The fringe contrast can be observed to revive at half-integer multiples of the radial trap period and we demonstrate how this effect can be used to measure the trap frequency in situ without actually exciting collective oscillation modes. The specific form of the collapse and revival of fringe visibility is modelled readily when the anharmonicity of the trapping potential is taken into account.

2 Ramsey Interrogation and Echo Spectroscopy

2.1 Ramsey spectroscopy of two-level atoms

In Ramsey spectroscopy, a collection of two-level atoms with quantum levels $|\downarrow\rangle$ and $|\uparrow\rangle$ separated by an energy difference $\hbar \Omega_0$ interacts with two near resonant fields of...
angular frequency \( \Omega \approx \Omega_0 \) separated by a time \( T \). With all the atoms initially in the state \( |↓⟩ \) the first field interaction — a \( \pi/2 \)-pulse — produces an equal coherent superposition state \( (|↑⟩ + |↓⟩)/\sqrt{2} \) for each atom. The atoms now evolve freely for the time \( T \) during which a phase \( \phi = (\Omega - \Omega_0)T \) is accumulated by the state: \( (|↑⟩ + e^{i\phi}|↓⟩)/\sqrt{2} \). Finally, a second \( \pi/2 \)-pulse is applied and the population difference of the two states is measured. This quantity will vary periodically with \( \phi \) giving rise to so-called Ramsey fringes when either \( T \) or \( \Omega \) are scanned.

If the two energy levels \( |↑⟩ \) and \( |↓⟩ \) are perturbed differentially in any way during the free evolution period, the energy splitting \( \hbar \Omega_0 \) and thus \( \phi \) will be affected. This may occur homogeneously such that all the atoms are perturbed by the same amount causing an over all shift of the Ramsey fringe pattern \([7]\). Essentially, this fringe shift constitutes an interferometric measurement of the perturbation strength. For nonuniform perturbations the phase \( \phi \) differs from atom to atom and in the Ramsey measurement of the whole collection of atoms the fringe shift is accompanied by a degraded visibility as a result of inhomogeneous dephasing \([8]\).

2.2 Coherence echoes

Effects of any form of inhomogeneous dephasing encountered in Ramsey spectroscopy can, to a large extent, be cancelled by introducing a so-called echo-pulse (\( \pi \)-pulse) between the \( \pi/2 \)-pulses at time \( t_1 \) in the Ramsey sequence \([8]\). The echo pulse essentially inverts the sign of the phase \( \phi \) accumulated so far by each atom and if subsequently each atom encounters the same amount of perturbation as prior to the \( \pi \)-pulse a rephasing will occur at time \( t_2 = t_1 \) after the echo pulse. In this sense the perturbation is reversible e.g. for a collection of stationary atoms in an inhomogeneous light field. Irreversible dephasing may result from fluctuating perturbations such as the variations in phase shift due to noise in the intensity of the inhomogeneous light field or movement of atoms therein \([9]\).

Since echo spectroscopy nullifies the effect of reversible perturbations it has proven to be a powerful tool for investigating the nature and magnitude of atomic decoherence due to the combined effect of irreversible processes including dephasing and spontaneous decay of atoms \([9,10,11,12]\).

Several studies focus on the influence of the inhomogeneous dipole trapping light and it has been demonstrated that an echo pulse is efficient in compensating the effect of the trapping laser on a time scale \( \ll \hbar/\Delta E \), where \( \Delta E \) is the differential ac Stark shift between \( |↑⟩ \) and \( |↓⟩ \) \([10]\). Furthermore, it has been shown that the differential light shift for an oscillator mode of a far-off-resonant trap (FORT) is well described by its time averaged value \([9]\) such that inhomogeneous dephasing introduced by the Gaussian trapping laser beam profile can in many cases be reversed in an echo sequence.

2.3 Inhomogeneous light shifts and trap dynamics

In the present paper we shall consider the perturbations from an auxiliary inhomogeneous pulse of light applied before the echo pulse and an identical light pulse applied after the echo pulse as indicated in Fig. 2. For a stationary atomic ensemble perfect rephasing would be expected except for decoherence of the atomic state due to irreversible spontaneous scattering events and quantum fluctuations (shot noise) of the two light pulses. Hence, a measurement of the Ramsey fringe visibility would appear to be ideally suited for determining the amount of decoherence introduced by a given laser beam. The situation is somewhat complicated when atoms change positions between the application of the two light pulses. To elaborate on this issue we consider a simple 1D model of particles evolving in a harmonic trap. The distributions of atoms in momentum and position are represented as a dynamical phase space plot [see Fig. 3(a)]. The application of a light pulse with Gaussian intensity profile will imprint a phase which only depends on the position if we assume that the duration of the light pulse is much shorter than an oscillation period \( T \) in the trap. Now, during the free evolution the imprint will rotate in dynamical phase space at angular frequency \( \omega = 2\pi/T \). In Fig. 3(b), we outline the situation for echo spectroscopy in the cases (i) \( \tau_1 = \tau_2 = \pi/4\omega \) and (ii) \( \tau_1 = \tau_2 = \pi/2\omega \), where \( \tau_1 \) and \( \tau_2 \) are the durations from the light pulse before and after to the echo pulse, respectively. Complete rephasing of the atoms is encountered only if \( \tau_1 + \tau_2 = \pi \mod \pi/\omega \), i.e. the atoms oscillate in the trap for a half-integer multiple of a trap period between the two light pulses. For an echo spectroscopy experiment this implies that the Ramsey fringe contrast will depend on the time separation \( \tau_1 + \tau_2 \) between the light pulses.

3 Experimental

3.1 Setup

Details of our experimental setup and atomic sample preparation can be found in \([5]\). The starting point for the experiment presented in the present paper is an ensemble of \( \sim 10,000 - 50,000 \) Cs atoms polarized in the \( 6S_{1/2}(F = 3, m_F = 0) \equiv |↑⟩ \) clock state and confined by a \( \sim 4 \) W Yb:YAG laser beam focussed to a waist of \( \sim 40 \mu \text{m} \). This dipole trap is characterized by oscillation frequencies in
the order of a kHz radially and a few Hz axially. The clock transition $|1⟩$ $\leftrightarrow$ $6S_{1/2}(F = 4, m_F = 0) \equiv |1⟩$ is driven using microwave radiation around 9.2 GHz applied in a $\pi/2 - \pi - \pi/2$ echo spectroscopy sequence as depicted in Fig. 2. At the end of this sequence we determine the fraction of atoms residing in the $|\uparrow⟩$ state. The probing of atoms is performed with a beam of light propagating along the trap axis with a waist of 18 $\mu$m located at the center of the trap. The frequency of the probe light is blue detuned by 160 MHz from the $6S_{1/2}(F = 5) \rightarrow 6P_{3/2}(F = 6)$ transition and via the dispersive atom-light interaction the probe light experiences a phase shift proportional to the number of atoms involved in a $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atoms residing in the $|\uparrow⟩$ state. The probing of a fraction of atomic projection noise [2]. The two probes of this two-color scheme are merged in a single mode optical fiber to ensure good spatial overlap and hence enter the interferometer at the same input port.

3.2 Decoherence and inhomogeneous dephasing

For low optical powers the dispersive probing scheme is close to nondestructive in the sense that spontaneous scattering is very limited. However, the ratio of signal to noise (which in our case is the shot noise of light) for a phase shift measurements increases with increasing probe photon number. Hence, there is a trade off between information gained and coherence lost. Establishing an optimal balance between decoherence and measurement strength is important for quantum state engineering of a squeezed clock state population difference via a dispersive measurement. As it turns out, the optical depth of the atomic sample is the key figure of merit determining the optimal optical power and thus the amount of decoherence. Preferably, and to achieve a high degree of squeezing, the optical depth should be large such that each probe photon interacts with many atoms. In this respect the $\sim 1 : 200$ radial to axial aspect ratio of our sample provided by the dipole trap potential is favorable and gives rise to an optical depth of up to $\sim 20$ along the direction of the probe laser beam.

From these considerations, it is obviously important to have a handle on the amount of decoherence introduced by the dispersive probing scheme and echo spectroscopy would appear to be the method of choice [11]. Ideally, a light pulse derived from the probe laser could be devided into two with each part applied before and after the echo pulse, respectively, and the reduction in Ramsey fringe amplitude would then gauge the decoherence from probe light considered as a perturbation. This method relies on the complete cancellation of the reversible dephasing by the inhomogeneous light shift. However, the movement of atoms in between the two pertubing pulses may also lead to imperfect rephasing causing a Ramsey fringe reduction as outlined in [2,3]. In fact, this effect is significant and is prominently manifested in the echo spectroscopy.

3.3 Results

3.3.1 Echo Spectroscopy

As described in section 2.3 the dynamics of atoms in our dipole trap is expected to affect the ability of an echo pulse to balance the effect of two surrounding phase shifting light pulses, as can indeed be observed experimentally in echo spectroscopy. To illuminate the effect, we
present in Fig. 4 examples of Ramsey fringes as recorded in the two-color probing scheme using the echo sequence of Fig. 2 for light pulse separations $\tau_1 + \tau_2$ of 12, 250, and 500 $\mu$s, respectively. The light pulses have a duration of 2 $\mu$s which is short compared to all other relevant time scales in the experiment such as the trap oscillation period of $\sim 1000$ $\mu$s. For the traces shown, $t_1$ was kept fixed at 1500 ms while $t_2$ was scanned between 500 ms and 2500 ms. To enable the observation of Ramsey fringes in the time domain the frequency of our microwave source was detuned by 3 kHz. The detuning sets the undulation frequency of the Ramsey fringes which have their maximum amplitude at $t_2 = t_1 = 1500$ ms due to the echo rephasing pulse.

The most intriguing aspect of Fig. 4 is the collapse and revival of fringe visibility. For a time separation $\tau_1 + \tau_2 = 12$ $\mu$s between the light pulses which is short compared to the trap frequency, the echo pulse is effective in restoring the Ramsey fringe visibility. Since the light pulses are so closely spaced that the atoms hardly have time to move in between, the second pulse essentially undoes the inhomogeneous phase imprint of the first. An upper bound on the amount of decoherence can be estimated from reduction in fringe amplitude, which in this case is 76%. A completely different situation is encountered at $\tau_1 + \tau_2 = 250$ $\mu$s with a light pulse separation in the vicinity of a quarter of a radial trap period. Here the atomic ensemble develops a nonuniform phase as indicated in Fig. 3b(i) and the Ramsey fringes for individual atoms will in general not interfere constructively. Hence a certain degradation in the ensemble fringe contrast happens. Finally, for a pulse separation of $\tau_1 + \tau_2 = 500$ $\mu$s in the vicinity of half a radial trap period a Ramsey fringe revival happens. At the time of the second light pulse an atom will find itself close to the radial distance (from the trap center) it had when the first pulse was applied and hence experience the same light intensity. Due to the inverting effect of the interposed echo pulse a phase shift cancellation will occur.

**3.3.2 Revival frequency versus trap power**

The revival of Ramsey fringe visibility, resulting from interference between spatially imprinted phases in the atomic coherences, is linked to the radial trap oscillation frequency. Measurements of trap frequencies in cold atom experiments are typically performed by exciting either monopole or dipole oscillation modes, or by driving parametric losses when modulating the trap [15,16]. Using the fringe revival, we are able to extract the trap frequency in situ without actually exciting motion — an atom is tagged in its internal degrees of freedom according to its position. Towards this end we simply measure the height of the central Ramsey fringe provides measure of the fringe visibility this quantity oscillates in $\tau_1 + \tau_2$ at twice the trap frequency as discussed above. Figure 5 shows the measured revival frequency as function of the optical power of our dipole trapping beam. The observed revival frequencies are described well by a square root dependency on the optical power as expected from theory [15].

**3.3.3 Modeling the revivals**

In a perfect harmonic trap the radial oscillation period is independent of oscillation amplitude and complete revival of the Ramsey fringe would hence be expected when $\tau_1 + \tau_2$ equals half a trap period. This is clearly not the case for the revival presented in the inset of Fig. 5. A possible explanation could be sought in the anharmonicity of the Gaussian trapping potential. To investigate this
in more detail we perform numerical simulations of the phase evolution of \( N = 50,000 \) particles in a Gaussian trap when subjected to the echo spectroscopy sequence shown in Fig. 2 including a non-uniform light shift from a Gaussian laser beam. During half a radial trapping period the axial movement of even the most energetic, trapped particles will only amount to a small fraction of the probe beam Rayleigh length which characterizes the length scale for the light-atom interaction volume. Furthermore, collisions between particles are expected to be negligible for the low densities and short time scales involved \[ \text{[17].} \] In addition, we shall assume that the mean kinetic energy for trapped particles is sufficiently large for the radial frequency variation along the trap axis to be negligible compared with the local frequency spread caused by the trap potential anharmonicity. Hence, as an approximation, we shall restrict our treatment to the 2D case of radial dynamics. The details of the implementation of our numerical simulations is given in appendix \[ \text{[A].} \] An expected fringe amplitude (relative to the unperturbed case) as function of the time separation \( \tau \) between the two perturbing light pulses, \( t \equiv \tau_1 + \tau_2 \), is given by

\[
F(t) = \frac{\sum_{j=0}^{N} e^{-\rho_j^2(t/2+\tau_2)/\gamma^2 w_0^2} \cos \phi_j(t)}{\sum_{j=0}^{N} e^{-\rho_j^2(t/2+\tau_2)/\gamma^2 w_0^2}},
\]

where \( \rho_j \) and \( \phi_j \) are the radial position and echo phase [cf. Eq. \[ \text{[4].} \] ], respectively, of particle \( j \) in the ensemble. Since the atomic ensemble is probed with a Gaussian light beam, each particle contributes to the total signal by a weight according to the intensity at its radial position at the time of probing. Figure 6 shows the fringe amplitude versus light pulse separation time as recorded in echo spectroscopy experiments for various perturbing light powers \( P_{\text{pert}} \). We model our data using \( \tilde{F}(t) \) as provided by our numerical simulations. The functional behavior of \( \tilde{F}(t) \) depends on the three parameters \( T \), \( \gamma \), and \( \phi_0 \), i.e., the cloud temperature prior to the separatrix truncation (see appendix \[ \text{[A].} \] ), the probe to trap beam ratio, and the peak phase shift. These parameters are adjusted to minimize the mean square difference to our experimental data. Least squares are obtained for \( kT/U_0 = 1 \), \( \gamma = 0.6 \), and \( \phi_0/N_{\text{pert}} = 5 \times 10^{-7} \) which corresponds well to what would be expected in the experiment. Fixing the free parameters of our simulation code to these values, we plot the normalized echo amplitude in Fig. 6. We note that our simple 2D model gives an excellent over-all agreement between experiment and simulation.

The model does not include decoherence with the result that all the simulated curves in Fig. 6 indicate perfect rephasing at \( t = 0 \). In the experimental data it is evident that rephasing is not perfect especially for the large probe powers. However, the effect of trap dynamics on the fringe visibility clearly dominates over the decoherence for most pulse separations and it is unclear how much the, in principle, reversible dephasing contributes to the fringe reduction at the smallest pulse separation of 12 \( \mu \)s. Thus, measurements provide an upper bound on the (irreversible) decoherence.

4 Discussion

In summary, we have reported on the manifestation of motional dynamics of trapped atoms in echo spectroscopy on the Cs clock transition. The movement of particles implies that the ac Stark shifts from two identical light pulses generally do not cancel completely on the application of an in-between Hahn echo. Rather, the echo fringe amplitude is observed to decrease with increasing separation of the two light pulses. For obtaining squeezing on the clock transition via dispersive measurements this is unfortunate for two reasons. First, the effect limits the applicability of echo techniques to restore the adverse inhomogeneous light shift effects of a strong off-resonant probe pulse. Second, using echo spectroscopy to gauge the atomic decoherence resulting from spontaneously scattered probe photons when probing an ensemble is not straightforward. Obviously, an understanding of the effect of motional dynamics is important when dispersive measurements are used for quantum state engineering.

An alternative approach to eliminate inhomogeneous differential light shift of the clock states induced by a probe pulse is to apply a simultaneous light pulse from a second laser \[ \text{[18].} \] The frequency of the second probe laser should be chosen so that the differential ac Stark shifts introduced by each probe laser exactly cancel. In the present configuration, where the two probe beams enter the Mach Zehnder interferometer via the same input port, the detunings are pegged such that simultaneous application of the two probes on a coherent superposition state yields a zero mean phase shift. At these detunings the ac Stark shift imposed on each clock state add in sign
and instead of cancelling the differential light shift, the two-color probing scheme essentially doubles it. However, injecting the two beams via different ports of the interferometer input beam splitter establishes a scheme where the detunings can be chosen to eliminate the differential light shift and achieve a state sensitive, balanced interferometer signal at the same time [19]. We are presently in the course of reconfiguring our experimental setup to this favourable scheme. Echo spectroscopy experiments along the lines the present paper should then easily indicate if light shift cancelation has been accomplished.

We recall that the primary motivation for our experimental efforts is to demonstrate squeezing on the clock transition. Here a balanced interferometer measurement of an ensemble in an equal clock state superposition will be used to predict the outcome of a subsequent measurement beyond the standard quantum limit [23,41]. The present paper and ref. [5] have investigated some adverse effects on the collective ensemble state which may accompany these “nondestructive” measurements and discussed routes to minimize them. Even when applying these strategies to eliminate inhomogeneous light shifts, the atomic trap dynamics will lead to retardation effects when comparing two consecutive measurements on an ensemble. Particle motion and a Gaussian probe beam profile implies that an atom in general contributes to the interferometer signal by different weights in the two measurements. A fuller treatment of this is beyond the scope of this paper, but it is expected that the degree of squeezing (i.e. the amount of correlation between the two pulses) is going the diminishing and increase analogous to the collapse and revival of echo fringe amplitude as reported on here.

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A Methods for numerical simulations

We obtain the initial conditions for our numerical simulations by considering a canonical ensemble of particles with mass \( m \) at a temperature \( T \). Using polar coordinates \((\rho, \theta)\), the phase space density is

\[
\sigma(\rho, \theta, p_\rho, p_\theta) \propto e^{-[(p_\rho^2 + p_\theta^2)/2m + V(\rho)]/kT},
\]

where \( k \) is the Boltzmann constant and \( p_\rho \) and \( p_\theta \) are the conjugate momenta to \( \rho \) and \( \theta \), respectively. For the Gaussian trap the confinement potential is \( V(\rho) = -U_0 e^{-(2\rho^2/\rho_s^2)} \), which has the (finite) trap depth \( U_0 \) and depends only on the radial distance \( \rho \) to the trap center at the origin. We want to restrict ourselves to particles inside the separatrix of stable motion, i.e., we disregard unbound particles for which \((p_\rho^2 + p_\theta^2)/2m + V(\rho) > 0\) (see Fig. 7). Integrating over particles inside the separatrix yields a spatial density distribution

\[
n(r) \propto [e^{-V(\rho)/kT} - 1] \propto \left\{ \begin{array}{ll} 0, & U_0 \gg kT \\ \exp\left(-\frac{U_0}{kT}, \frac{2\gamma^2}{w_\rho^2}\right), & U_0 \ll kT \end{array} \right.,
\]

which for all practical purposes can be assumed to vanish for \( \rho \gtrapprox w_\rho \). Applying a finite cut-off radius \( \rho_0 \gtrapprox w_\rho \), we assign random radii to the particles of our ensemble according to a probability density \( n(\rho) \propto \rho e^{-V(\rho)/kT} \) using the rejection method [21] on the interval \([0...\rho_0]\). Next we assign the canonical momenta \( p_\rho \) and \( p_\theta \) for each particle in correspondence with normal distributions of widths \( \sqrt{U/T} \) and \( \rho \sqrt{U/T} \), respectively. We discard untrapped particles as described above and tag each trapped particle \( j \) with a phase \( \phi_0 (\rho) = \phi_0 e^{-(2\rho^2/\rho_s^2)}, \) where \( \gamma \) is the ratio between the waists of the trapping laser and the perturbing laser and \( \phi_0 \) is the peak phase shift. We finally integrate the equations of motion using the Verlet method to obtain \( \phi_j(t) \) from which the echo phase

\[
\phi_j^{(2)}(t) = \phi_j^{(1)} - \phi_0 \exp\left(-2\rho_j^2(t)/\gamma^2 w_\rho^2\right)
\]

can be calculated.

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