VOLUME DISTORTION IN HOMOTOPY GROUPS

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Abstract. Given a finite CW complex \( X \) and an element \( \alpha \in \pi_n(X) \), what are the properties of a geometrically optimal representative of \( \alpha \)? We study the optimal volume of \( k\alpha \) as a function of \( k \). Asymptotically, this function, whose inverse, for reasons of tradition, we call the volume distortion, turns out to be an invariant with respect to the rational homotopy of \( X \). We provide a number of examples and techniques for studying this invariant, with a special focus on spaces with few rational homotopy groups. Our main theorem characterizes those \( X \) in which all non-torsion homotopy classes are undistorted, that is, their distortion functions are linear.

The object of quantitative topology is to make more concrete various notions coming from the existential results of algebraic topology. Thus, while classical rational homotopy theory gives an exhaustive family of algebraic correlates to rational homotopy classes of simply-connected spaces, a quantitative homotopy theory seeks to give geometric examples or descriptions linked to the algebraic properties of these objects.

The term “quantitative homotopy theory” seems to have first been used by Gromov in the conference paper \([\text{Gro}99]\) and in Chapter 7 of the near-simultaneous book \([\text{Gro}98]\), although the ideas date back as far as \([\text{Gro}78]\). Construed broadly, however, this program fits into a tradition of extracting metric information from topological data which includes problems from systolic geometry, geometric group theory, and other areas. Thus, some of the important asymptotic invariants of geometric group theory are the Dehn function, growth of groups, and distortion of group elements and subgroups, all of which have higher-dimensional analogues.

Higher-dimensional isoperimetric functions of groups, that is, of their Eilenberg–MacLane spaces, have been studied in some detail, notably by Gromov \([\text{Gro}96]\), Alonso–Wang–Pride \([\text{AWP}99]\), Brady–Bridson–Forester–Shankar \([\text{BBFS}09]\), and Young \([\text{You}11]\). A common theme of this body of literature is the plurality of possible definitions, many of which are equivalent in the one-dimensional case. The subject of growth of higher homotopy groups was broached in chapters 2 and 7 of \([\text{Gro}98]\) with a number of examples and a conjecture for simply-connected spaces.

Here we analyze a higher-dimensional analogue of distortion. Heuristically, the distortion of a group element \( \alpha \) is given by

\[
\delta_\alpha(k) = \max\{m \mid \text{size}(\alpha^m) \leq k\}.
\]

If the group is the fundamental group of a space, word length is a natural measure of size. In \( \pi_n(X) \), on the other hand, there are various possible measures, leading once again to a plurality of definitions: we can choose to minimize the Lipschitz constant of a representative, its volume, or more generally the \( m \)-dilation for some \( 1 \leq m \leq n \), that is, how much the map \( f : S^n \to X \) distorts \( m \)-dimensional tangent subspaces. In all of these situations, it is natural to consider an element undistorted if the best asymptotics are attained by composing \( \alpha \) with a degree \( k \) map \( S^n \to S^n \). Thus an element is Lipschitz undistorted if its Lipschitz distortion is \( \sim Ck^n \), and volume undistorted if its volume distortion is \( \sim Ck \).

In \([\text{Gro}99]\), Gromov states a conjecture about the Lipschitz distortion of homotopy groups.

Conjecture (Gromov). A class \( \alpha \in \pi_nX \) of a simply-connected finite CW complex \( X \) is Lipschitz undistorted if and only if \( \alpha \) has nonzero image under the rational Hurewicz map. If \( \alpha \) is distorted, then its Lipschitz constant grows polynomially, and at most as \( k^{1/n+1} \).

Using different models of rational homotopy theory, it is possible to give various estimates for Lipschitz distortion from above and below. However, a full proof of the conjecture remains elusive. In this paper, we instead focus on volume distortion, which has not been previously studied.

In contrast with the Lipschitz case, volume distortion is trivial for simply connected spaces: every homotopy class is either undistorted or has a multiple with zero volume, as shown in Theorem 3.1. But for finite complexes in general, it’s an interesting invariant: we construct examples in which the volume distortion of
an element is $k^r$ for various rational $r$ and where it is $\exp(\sqrt{k})$. One feature that makes volume distortion convenient for non-simply-connected spaces is its invariance up to rational homotopy. More precisely:

**Theorem A.** Suppose two compact connected CW-complexes $X$ and $Y$ are rationally equivalent, that is, there is a space $Z$ and maps $X \to Z \leftarrow Y$ which induce isomorphisms on $\pi_1$ and $\pi_n \otimes \mathbb{Q}$ for every $n$. If $\alpha \in \pi_n(X)$ and $\beta \in \pi_n(Y)$ have the same image in $\pi_n(Z)$, then they have asymptotically equivalent volume distortion functions.

In order to show this, we develop some results in the rational homotopy theory of non-simply-connected spaces, the topic of section 1. As discussed in Example 2.6, the equivalent statement in the Lipschitz setting appears to be false.

We uncover three main causes of volume distortion. Some maps retract to a lower skeleton of the space $X$; this is the simplest possibility, since all such maps have zero volume. This case is exemplified by the Hopf map $S^3 \to S^2$. Distortion may also be caused by the action of the fundamental group on $\pi_n(X)$. Thus in the mapping torus of a degree 2 map $f : S^n \to S^n$, for every $k$, $2^k$ times the generator has a representative with volume 1. In both these cases, the volume distortion function is eventually infinite. These two effects are the topic of section 3, in which we show that they produce distortion in a wide range of spaces, excluding not only individual elements but finite-dimensional subspaces. While the definition extends easily, this is a significantly different concept: in Example 3.10(1), we see that a subspace might be distorted even if no element is. One can think of this as distortion in an irrational direction in $\mathbb{R}^3$; this is the simplest possibility, since all such maps have zero volume. This case is exemplified by the Hopf $\mathbb{Z}_n$-equivariant lift of the surface of a cube in $T^4 \cong \mathbb{R}^4$ with side length $k$. Since this lift has volume $Ck^3$, the volume distortion of $\alpha$ is $O(k^{4/3})$. Such effects are the main source of distortion in delicate spaces, which are the topic of section 6.

These three sources of distortion are sufficient to prove two general theorems about spaces in which distortion is present. Our main theorem characterizes spaces in which no homotopy groups have volume distortion. It turns out, however, that a natural definition of volume distortion in this context includes not only individual elements but finite-dimensional subspaces. While the definition extends easily, this is a significantly different concept: in Example 3.10(1), we see that a subspace might be distorted even if no element is. One can think of this as distortion in an irrational direction in $\pi_n(X) \otimes \mathbb{R}$, induced by an irrational eigenvector of the action of $\pi_1(X)$ on $\pi_n(X)$.

**Theorem B.** A finite CW complex $X$ has no volume distortion in $\pi_n(X)$ for $n \geq 2$ if and only if the following conditions hold:

1. $X$ is rationally equivalent to the total space of a tower of fibrations over $B\pi_1 X$ with fibers of the form $(S^{2n+1})^r$;
2. for each $i$, the monodromy representation $\rho : \pi_1 X \to GL(r_i, \mathbb{Q})$ is elliptic, i.e. its image is contained in a conjugate of $O(r_i, \mathbb{Q})$;
3. and for each $i$, the Euler class $\epsilon_i \in H^{2n+2} (X; M_p)$ vanishes when considered in the $L_\infty$ cohomology $H^{2n+2}_\infty (X; \mathbb{Q} \langle \pi_1 X \rangle)$ of the universal cover.

Note that while the definition of $L_\infty$ cohomology relies on a metric on $X$, it is insensitive to small-scale geometry, and so only depends on the topology of $X$. Intuitively, spaces with no distortion must have universal covers which are, in some sense, finite distance from being a product $\prod_i S^{2n_i+1} \times \overline{B\pi_1 X}$. In particular, $\pi_1 X$ must have property $F_\infty(\mathbb{Q})$, that is, $B\pi_1 X$ should be rationally equivalent to a complex with finite skeleta.

One corollary is that if $X$ is the unit tangent bundle of a closed $n$-manifold which is aspherical or has non-amenable fundamental group, then the class in $\pi_n(X)$ corresponding to the fiber is never distorted. More generally, the distortion of the total space of a fibration with fiber $S^n$ is equivalent to a certain $n$-dimensional filling function of the base space. For a given Euler class $\omega \in H^{n+1}(X; \mathbb{Q})$, we write this function as $FV^n_{X;\omega}(\omega)$.

Thus, for example, we construct a sequence of groups $D_n$, related to the Baumslag-Solitar group $BS(1,2)$, for which $FV^n_{D_n;\omega}(\omega)(k) \cong 2^{\sqrt{k}}$ for any nonzero $\omega \in H^{n+1}(D_n; \mathbb{Q})$. This implies that nontrivial fibrations $S^n \to X \to BD_n$ have volume distortion of the form $\exp(\sqrt{k})$ as well.

We say volume distortion is *infinite* if there is a finite-dimensional subspace of $\pi_n(X) \otimes \mathbb{Q}$ in which arbitrarily large vectors have representatives of bounded volume. Our second complete characterization,
however, is of when a complex has *weakly infinite* volume distortion. Here we take the minimum volume functional on $\pi_n(X)$ to be a restriction of that on $H_n(X)$; this is natural because for simply connected spaces, least volume functionals are the same for homology and homotopy, at least for $n \geq 3$. We say that (a finite-dimensional subspace of) $\pi_n(X) \otimes \mathbb{Q}$ is weakly infinitely distorted in (a finite-dimensional subspace of) $H_n(\tilde{X}; \mathbb{Q})$ if arbitrarily large vectors in the latter at a bounded distance from the former are represented by integral chains of bounded volume.

**Theorem C.** A finite CW complex $X$ has no weakly infinite volume distortion in $\pi_n(X)$ for $n \geq 2$ if and only if the following conditions hold:

1. $X$ is rationally equivalent to the total space of a fibration over $B\pi_1 X$ with fiber $\Pi_{i=1}^s (S^{2n_i+1})^{r_i}$;
2. the monodromy representation $\pi_1 X \to GL(\pi_\ast(X) \otimes \mathbb{Q})$ is elliptic;
3. and for every $n \geq 2$, the group $H_n(\tilde{X}; \mathbb{Q})$ splits as a $\mathbb{Q}\pi_1 X$-module into the image of the Hurewicz map and its complement.

Which weakly infinitely distorted classes are in fact infinitely distorted remains largely open.

Because many of our bounds rely on cellular maps, the results are inherently asymptotic—we do not aspire to minimize volume of specified maps in specified geometries. Others, however, have worked in this direction. [DCST13] and [Wen14] have found that well-known maps uniquely minimize the Lipschitz constant of maps between round spheres and from products of spheres to spheres, respectively. Larry Guth in [Gut08a] and [Gut08b] provides bounds, depending on the metric in the domain and range, on Lipschitz constants of Hopf-like maps of spheres, as well as on their $k$-dilation for various $k$, a more general measure of the size of a map that includes both volume and Lipschitz constant. In a recent paper [Gut13], Guth addresses the minimal $k$-dilation of certain torsion homotopy classes of spheres.

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1. **Algebraic preliminaries**

We start this section by proving two simple propositions that will find their uses in later sections, before moving on to a more self-contained discussion developing the rational homotopy theory of finite non-simply-connected complexes. We will tacitly assume all spaces to be connected. As a matter of notation, we write $h_n : \pi_n(X) \otimes \mathbb{Q} \to H_n(X; \mathbb{Q})$ to mean the rational Hurewicz homomorphism.

**Proposition 1.1.** Suppose $X$ is a simply connected complex and $[\alpha] \in \pi_n(X)$ is zero under the Hurewicz map. Then $[\alpha]$ has a representative $f : S^n \to X^{(n-1)}$. More generally, if $h_n([\alpha]) = 0$, then there is some $k > 0$ so that $k[\alpha]$ has a representative $f : S^n \to X^{(n-1)}$. 


Proof. The relative Hurewicz theorem and exact sequences for pairs give us the diagram

\[
\begin{array}{ccc}
\pi_n(X^{(n-1)}) & \to & \pi_n(X) \\
\downarrow & & \downarrow \\
0 & \to & H_n(X) \\
\end{array}
\]

Thus the sequence

\[
\pi_n(X^{(n-1)}) \to \pi_n(X) \to H_n(X)
\]

is exact. This gives us the integral result. The rational version follows. \(\square\)

**Proposition 1.2.** Suppose \((X,A)\) is a CW pair of finite connected \(n\)-complexes such that the inclusion \(i : A \to X\) induces isomorphisms on \(\pi_i \otimes \mathbb{Q}\) for \(i \leq n-1\). Then the induced map \(\iota_* : \pi_n(A) \otimes \mathbb{Q} \to \pi_n(X) \otimes \mathbb{Q}\) is injective.

**Proof.** By the relative Hurewicz theorem, \(\iota_* : H_i(\tilde{A}; \mathbb{Q}) \to H_i(\tilde{X}; \mathbb{Q})\) is also an isomorphism. Moreover, since \(H_{n+1}(\tilde{X}, \tilde{A}) = 0\), \(\iota_* : H_n(\tilde{A}) \to H_n(\tilde{X})\) is injective. Consider now the fibrations

\[
K(\pi_n(A), n) \to \tilde{A}(n) \to \tilde{A}(n-1)
\]

and

\[
K(\pi_n(X), n) \to \tilde{X}(n) \to \tilde{X}(n-1)
\]

within the Postnikov towers of \(\tilde{A}\) and \(\tilde{X}\) respectively. By assumption, \(H_0(\tilde{A}(n-1); M) = H_0(\tilde{X}(n-1); M) = M\) for any module \(M\). \(H_i(\tilde{A}(n-1)) = H_i(\tilde{X}(n-1)) = 0\), and by Hurewicz \(H_1(\tilde{A}(n-1); \mathbb{Q}) \cong H_1(\tilde{X}(n-1); \mathbb{Q})\) for all \(i\). Thus the Serre spectral sequence gives us a homomorphism of exact sequences

\[
\begin{array}{cccccc}
H_{n+1}(K(\pi_n A, n); \mathbb{Q}) & \to & H_{n+1}(\tilde{A}(n-1); \mathbb{Q}) & \to & \pi_n(A) \otimes \mathbb{Q} & \to & H_n(\tilde{A}; \mathbb{Q}) & \to & H_n(\tilde{A}(n-1); \mathbb{Q}) & \to & 0 \\
\downarrow & & \downarrow \psi & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_{n+1}(K(\pi_n X, n); \mathbb{Q}) & \to & H_{n+1}(\tilde{X}(n-1); \mathbb{Q}) & \to & \pi_n(X) \otimes \mathbb{Q} & \to & H_n(\tilde{X}; \mathbb{Q}) & \to & H_n(\tilde{X}(n-1); \mathbb{Q}) & \to & 0.
\end{array}
\]

\(H_{n+1}(K(\pi, n))\) is torsion for any group \(\pi\), and so a diagram chase shows that \(\ker(\iota_*) = 0\). \(\square\)

The rational homotopy of finite complexes can a priori be very complicated. At the very least, there are many such spaces with complicated fundamental groups \(\Gamma\) for which higher homotopy groups are infinitely generated as modules over \(\mathbb{Z}\Gamma\). The great achievement of [Wal65] was to show that in some sense this complexity is determined by the fundamental group: that is, if two spaces have isomorphic fundamental groups, then their homotopy groups differ by a finite amount. In this section, we develop rational results that mimic Wall’s. Moreover, we show that the rational isomorphisms produced are effective, in the sense that the torsion in the difference has bounded exponent, despite perhaps being vastly infinitely generated.

To do this, we first need to define the appropriate notions of equivalence.

**Definition.** For two CW complexes \(X\) and \(Y\), we say a map \(f : X \to Y\) is **rationally \(n\)-connected** if it induces an isomorphism on \(\pi_1\) and \(\pi_k(f) \otimes \mathbb{Q} := \pi_k(M_f, X) \otimes \mathbb{Q} = 0\), where \(M_f\) is the mapping cylinder, for \(2 \leq k \leq n\). If this is true for every \(k\), we say \(f : X \to Y\) is a **rational equivalence**.

We say that \(X\) is **rationally equivalent** to \(Y\) \((X \simeq_{\mathbb{Q}} Y)\) if there is a CW complex \(Z\) with rational equivalences \(X \to Z\) and \(Y \to Z\). \(X\) is **rationally \(n\)-equivalent** to \(Y\) if there is a CW complex \(Z\) and maps \(X \to Z\) and \(Y \to Z\) which induce isomorphisms on \(\pi_1\) and \(\pi_k \otimes \mathbb{Q}\) for \(2 \leq k \leq n\).

**Definition.** An abelian group \(A\) has **bounded exponent** if \(pA = 0\) for some \(p \in \mathbb{N}\). We call the smallest such \(p\) the **period** of the group.

**Proposition 1.3.** The class \(\mathcal{BE}\) of abelian groups of bounded exponent is an acyclic Serre ideal, that is:

1. it’s closed under subquotients and extensions;
2. if \(A \in \mathcal{BE}\) and \(B\) is any abelian group, then \(A \otimes B, \text{Tor}_1(A, B) \in \mathcal{BE}\);
3. if \(A \in \mathcal{BE}\), then \(H_n(A) \in \mathcal{BE}\) for every \(n > 0\).
Proof. Closure under subquotients, extensions, and tensor products is immediate. Suppose \( A \) has period \( p \). Since Tor_1(\mathbb{Z}/k\mathbb{Z}, B) = \{ x \in B : kx = 0 \}, it has period \( k \). Now, the Tor functor commutes with filtered colimits and direct sums; in particular, \[
\text{Tor}_1(A, B) = \bigcup_{G \subset A \text{ finite}} \text{Tor}_1(G, B) = \bigcup_{G \subset A \text{ finite}} \bigoplus_{G = \oplus_i \mathbb{Z}/k_i\mathbb{Z}} \text{Tor}_1(\mathbb{Z}/k_i\mathbb{Z}, B)
\] has period \( p \).

Since \( H_n(\mathbb{Z}/k\mathbb{Z}) = 0 \) for even \( n \) and \( \mathbb{Z}/k\mathbb{Z} \) for odd \( n \), the Künneth formula tells us that for finite \( G \), the period of \( H_n(G) \) is at most that of \( G \). According to Theorem 9.5.9 of [Spa81], for any group \( A, H_n(A) \cong \bigoplus H_n(G) : G \subset A \text{ finite} \}. \] Hence if \( A \) has period \( p \) then \( H_n(A) \) has period at most \( p \). \( \square \)

Proposition 1.4. Let \( \Gamma \) be a group and \( M \) a finitely generated torsion \( \mathbb{Z}\Gamma \)-module. Then \( M \) is of bounded exponent as a group.

Proof. Let \( M = \langle a_1, \ldots, a_m \rangle \). Then \( k_i a_i = 0 \) for some \( k_i > 0 \). Let \( r = \text{lcm}\{k_i : 1 \leq i \leq n\} \). A general element \( a \in M \) is given by \( a = \sum_j t_j g_j a_{i_j} \), where \( t_j \in \mathbb{Z} \) and \( g_j \in \Gamma \). Thus \( ra = 0 \).

From an algebraic-topological perspective, any map can be replaced by the inclusion into its mapping cylinder. Thus the next proposition may be thought of as a very weak invertibility result for rational \( n \)-equivalences. In fact, it will be our main tool for proving rational invariance.

Proposition 1.5. Let \((X, K)\) be a rationally \( n \)-connected CW pair. Then the inclusion of cellular chain complexes \( i_k : C_k(\tilde{X}; \mathbb{Q}) \to C_k(\tilde{X}; \mathbb{Q}) \) is a chain homotopy equivalence through dimension \( n \). In particular, for \( 0 \leq k \leq n \), there is a \( \pi_1 \)-equivariant homotopy inverse \( j_k : C_k(\tilde{X}; \mathbb{Q}) \to C_k(\tilde{K}; \mathbb{Q}) \) such that \( j_k \circ i_k = \text{id} \) and \( i_k \circ j_k \) is homotopic to the identity via a \( \pi_1 \)-equivariant chain homotopy \( u_k : C_k(\tilde{X}; \mathbb{Q}) \to C_{k+1}(\tilde{X}; \mathbb{Q}) \) which is zero on the image of \( i \). We call \( j_k \) a lifting homomorphism.

Proof. We produce \( j_k \) and \( u_k \) by induction on \( k \). For 0-cells \( e \) of \( \tilde{X} \) outside \( \tilde{K} \), we can set \( j_0(e) \) to be any 0-cell of \( \tilde{K} \) in an equivariant way, and \( u_0(e) \) to be a path between \( e \) and \( j_0(e) \). Now suppose that we have constructed \( j_{k-1} \) and \( u_{k-1} \). Let \( e \) be a cell of \( \tilde{X} \). Then \( e + u_{k-1}(\partial e) \) represents an element of \( H_k(\tilde{X}, \tilde{K}; \mathbb{Q}) \) and so there’s a chain \( j_e \in C_k(\tilde{K}; \mathbb{Q}) \) such that \( i(j_e) - e - u_{k-1}(\partial e) = \partial u_e \) for some \( u_e \in C_{k+1}(\tilde{X}; \mathbb{Q}) \). We set \( j_k(e) = j_e \) and \( u_k(e) = u_e \). We then extend equivariantly over the equivalence class of \( e \). If \( e \) is a cell of \( \tilde{K} \), we can take \( j_e = e \) and \( u_e = 0 \) by induction. \( \square \)

Corollary 1.6. Suppose \( n \geq 2 \) and \((X, K)\) is a rationally \( n \)-connected finite CW pair. Then for \( i \leq n \), \( H_i(\tilde{X}, \tilde{K}) \) and \( \pi_i(\tilde{X}, \tilde{K}) \cong \pi_i(\tilde{X}, \tilde{K}) \) have bounded exponent. Moreover, \( H_{n+1}(\tilde{X}, \tilde{K}) \cong \pi_{n+1}(\tilde{X}, \tilde{K}) \) mod \( BE \).

Proof. Let \( j_k \) and \( u_k \) be a lifting homomorphism and the corresponding chain homotopy. There are a finite number of equivalence classes of \( k \)-cells \( e \) in \( \tilde{X} \), and each \[
j_k(e) = \sum_{k\text{-cells } e'} \frac{p_{e,e'}}{q_{e,e'}} e', u_k(e) = \sum_{(k+1)\text{-cells } e'} \frac{p_{e,e'}}{q_{e,e'}} e'.
\]
Taking the least common denominator \( q_k \) of all these summands, we get that for any \( k \)-chain \( c \) with boundary in \( \tilde{K} \), \( q_k c \) is integrally homologous to a \( k \)-chain in \( \tilde{K} \). This proves that \( H_i(\tilde{X}, \tilde{K}) \) have bounded exponent. The rest of the conclusion follows by the relative Hurewicz theorem mod \( BE \). Theorem 9.6.21 in [Spa81]. \( \square \)

Another consequence of Prop. 1.5 is a rational version of [Wal65]’s Theorem A, characterizing CW complexes with finite \( n \)-skeleton.

Theorem 1.7. Let \( X \) be a CW complex and \( n \geq 2 \). Then the following are equivalent:

1. \( X \) is rationally equivalent to a CW complex \( Y \) with finite \( n \)-skeleton.
2. \( Y \) is a CW complex \( Y \) with finite \( n \)-skeleton and a rational equivalence \( Y \to X \).
3. The group \( \Gamma := \pi_1(X) \) is finitely presented, and for every \( k \leq n \), the condition \( F_k(\mathbb{Q}) \) holds: for every finite complex \( K^{k-1} \) and rationally \((k-1)\)-connected map \( \varphi : K \to X \), \( \pi_k(\varphi) \otimes \mathbb{Q} \) is a finitely generated \( \mathbb{Q}\Gamma \)-module.
Proof. (2) clearly implies (1).

Suppose (1) is true. That is, there is a CW complex $Z$ such that $f : X \to Z$ and $g : Y \to Z$ are rational equivalences. Since $\Gamma = \pi_1(Y)$, it must be finitely presented. Now suppose that $K$ is a $(k - 1$)-complex and $\varphi : K \to X$ is a rationally $(k - 1)$-connected map. Then $\psi = f \circ \varphi$ is as well, so that, by Hurewicz,

$$\pi_k(\psi) \otimes Q = \pi_k(Z, K) \otimes Q \cong \pi_k(\tilde{Z}, \tilde{K}; Q) \cong H_k(\tilde{Z}, \tilde{K}; Q).$$

Moreover, we can assume by taking mapping cylinders that $K$ and $Y$ are subcomplexes of $Z$. Let $u_{k-1} : C_{k-1}(\tilde{Z}) \to C_k(\tilde{Z})$ be the chain homotopy from Prop. [13] given by the inclusion of $K$, and $j'_k$ and $u'_k, u'_{k-1}$ be chain maps given by the inclusion of $Y$. Then there is a homomorphism

$$F : C_k(\tilde{Y}; Q) \oplus C_{k-1}(\tilde{K}; Q) \to H_k(\tilde{Z}, \tilde{K}; Q),$$

given on cells by

$$F(e, e') = [c + u_{k-1}(\partial e) + u'_{k-1}(e') - u_{k-1}(\partial u'_{k-1}(e'))].$$

Then if $c$ is a chain in $\tilde{Z}$ with boundary in $\tilde{K}$, then $F(j'_k(c), \partial c) = [c] \in H_k(\tilde{Z}, \tilde{K}; Q)$, since the two chains are homologous via $u'_k(c)$. In other words, $F$ is onto. But the domain of $F$ is a finitely generated free $Q\Gamma$-module, and so $H_k(\tilde{Z}, \tilde{K}; Q)$ is finitely generated, proving (3).

On the other hand, suppose (3) is true. We will inductively construct a $Y$ with finite $n$-skeleton with a rational equivalence $Y \to X$. Since $\Gamma$ is finitely presented, we can construct a finite 2-complex $A$ with $\pi_1(A) = \Gamma$. This gives a 1-connected map $\varphi : A \to X$, so $\pi_2(\varphi) \otimes Q$ is finitely generated. We can add a finite number of 2-cells to kill it, building $Y^{(2)}$ and extending $\varphi$. By induction, we can build a finite complex $Y^{(n)}$ and extend $\varphi$ to it. Finally, we can add cells to build the rest of $Y$. \qed

**Corollary 1.8.** Let $n \geq 2$, and suppose $f : X \to Y$ is a $\pi_1$-isomorphic map of CW complexes such that $\pi_k(f) \otimes Q$ is finitely generated for $k \leq n$. Then $Y$ is rationally equivalent to a complex with finite $n$-skeleton if and only if $X$ is.

**Proof.** Suppose first that $Y$ has property $F_k(Q)$ for $k \leq n$. Let $k \leq n$, and suppose $K$ is a finite $(k-1)$-complex and $\varphi : K \to Y$ is a map. We would like to show that $\pi_k(\varphi) \otimes Q$ is finitely generated. Since $\pi_k(f \circ \varphi) \otimes Q$ is finitely generated, it’s enough to show that the kernel $T$ of $f_* : \pi_k(\varphi) \otimes Q \to \pi_k(f \circ \varphi) \otimes Q$ is also finitely generated. We have a commutative diagram

$$\begin{array}{cccc}
\pi_k(f) \otimes Q & & & T \\
\pi_{k+1}(K) \otimes Q & \to & \pi_k(X) \otimes Q & \to & \pi_k(\varphi) \otimes Q & \to & \pi_k(K) \otimes Q \\
\pi_{k+1}(K) \otimes Q & \to & \pi_k(Y) \otimes Q & \to & \pi_k(f \circ \varphi) \otimes Q & \to & \pi_k(K) \otimes Q
\end{array}$$

with exact rows and columns. Since $\pi_k(f) \otimes Q$ is finitely generated, a diagram chase confirms that $T$ is as well. Thus $X$ also has $F_k(Q)$.

Now suppose that $X$ is rationally equivalent to a complex with finite $n$-skeleton, and suppose that $Y$ is not. Let $3 \leq k \leq n$ be the first dimension in which a complex rationally equivalent to $Y$ necessarily has infinitely many cells, and so $\pi_k(Y, Y^{(k-1)}) \otimes Q$ is infinitely generated. Consider the homotopy pullback $g : Z \to Y^{(k-1)}$ of the homotopy fibration $f : X \to Y$; in particular, $\pi_i(g) = \pi_i(f)$ for every $i$ and the map $Z \to X$ is rationally $(k-1)$-connected. By the $Y \Rightarrow X$ direction, $Z$ is rationally equivalent to a complex $K$ with finite skeleta and indeed there is a rational equivalence $K \to Z$. Let $\varphi : K^{(k-1)} \to X$ and $\psi : K^{(k-1)} \to Y^{(k-1)}$ be the maps which factor through $Z$, so that $\pi_k(\varphi) \otimes Q$ is finitely generated by Theorem [17], whereas $\pi_k(\psi) \otimes Q$ is finitely generated by the exact sequence of triples

$$\pi_k(K, K^{(k-1)}) \otimes Q \to \pi_k(\psi) \otimes Q \to \pi_k(f) \otimes Q \to 0.$$

The exact sequence of triples

$$\pi_k(\varphi) \otimes Q \to \pi_k(Y, K^{(k-1)}) \otimes Q \to \pi_k(f) \otimes Q \to 0$$
shows that $\pi_k(Y, K^{(k-1)}) \otimes \mathbb{Q}$ is finitely generated, while the exact sequence of triples
\[ \pi_k(\psi) \otimes \mathbb{Q} \rightarrow \pi_k(Y, K^{(k-1)}) \otimes \mathbb{Q} \rightarrow \pi_k(Y, Y^{(k-1)}) \otimes \mathbb{Q} \rightarrow \pi_{k-1}(\psi) \otimes \mathbb{Q} \]
shows that $\pi_k(Y, K^{(k-1)}) \otimes \mathbb{Q}$ is infinitely generated, since $\pi_{k-1}(\psi) \otimes \mathbb{Q} \cong \pi_{k-1}(f) \otimes \mathbb{Q}$. Thus we have a contradiction. \hfill \Box

Another finiteness result we will use is that rational equivalences can be kept in the category of complexes with finite skeleta.

**Lemma 1.9.** Suppose $K$ and $L$ are CW-complexes with finite $n$-skeleta and $K \xrightarrow{f} Y \xleftarrow{g} L$ are rational equivalences. Then indeed there are rational equivalences $K \xrightarrow{h} Z \xleftarrow{g} L$ where $Z$ has finite $n$-skeleton.

**Proof.** By taking a mapping cylinder, homotoping, then taking another mapping cylinder, we can assume that $K$ and $L$ are subcomplexes of $Y$ with the same 1-skeleton. By inducting on dimension, we will build a complex $Z$ with maps

\[
\begin{array}{ccc}
K & \xrightarrow{f} & Z \\
& h \swarrow & \downarrow \searrow g \\
Y & \downarrow & L
\end{array}
\]

where $h$ is a rational equivalence, and hence so are the inclusions into $Z$. We start with $Z^{(2)} = K^{(2)} \cup L^{(2)}$; then the inclusion $Z^{(2)} \hookrightarrow Y$ is rationally 2-connected, since it induces an isomorphism on $\pi_1$ and $\pi_2(Z) \otimes \mathbb{Q}$ surjects onto $\pi_2(Y) \otimes \mathbb{Q}$.

Now suppose we have constructed $Z^{(k)}$ with rationally $k$-connected $h_k : Z^{(k)} \rightarrow Y$. Then the rational Wall theorem tells us that $\pi_{k+1}(h_k) \otimes \mathbb{Q}$ is a finitely generated $\mathbb{Q}\Gamma$-module, and so we can add in a finite number of $(k+1)$-cells to kill it, in such a way that the map $h_k$ extends to an $h_{k+1}$ which is then rationally $(k+1)$-connected. To ensure that $K$ and $L$ are included in our final $Z$, we make sure that all their $(k+1)$-cells are on the list.

By induction, $h$ is a rational equivalence, and hence so are the inclusions $K \xhookrightarrow{h} Z \xleftarrow{g} L$. \hfill \Box

### 2. Basic properties

In this section, we define distortion functions and discuss relationships between different definitions. When comparing functions $\mathbb{N} \rightarrow \mathbb{R} \cup \{\infty\}$, we will use the asymptotic relations
\[
\begin{align*}
f \lesssim g & \iff \exists A, B, C, D, f(n) \leq A g(Bn + C) + D \\
f \sim g & \iff f \lesssim g \text{ and } f \gtrsim g.
\end{align*}
\]

We start by describing what we mean by distortion in an abstract setting which encompasses any notion of distortion in both higher homotopy and homology groups.

**Definition.** Let $W$ be a vector space over $\mathbb{Q}$, $G$ an abelian group with an identification $\varphi : G \otimes \mathbb{Q} \xrightarrow{\cong} W$, and $F : G \rightarrow \mathbb{R}^+$ a subadditive functional. Define the $F$-distortion function of $\alpha \in G$ to be the function $\delta_{\alpha,F} : \mathbb{N} \rightarrow \mathbb{R} \cup \{\infty\}$ given by
\[
\delta_{\alpha,F}(k) = \sup\{m \mid F(m\alpha) \leq k\}.
\]

We say $\alpha$ is distorted if $F(k\alpha) / k \rightarrow 0$ as $k \rightarrow \infty$, and that $\alpha$ is infinitely distorted if there is a $k$ such that $\delta_{\alpha,F}(k) = \infty$.

Similarly, given a norm $\|\cdot\|$ on a finite-dimensional subspace $V \subseteq W$, we can define the $F$-distortion function of $V$ to be
\[
\delta_{V,F}(k) = \sup\{\|\varphi(\alpha)\| \mid \alpha \in G \text{ with } \varphi(\alpha) \in V \text{ and } F(\alpha) \leq k\}.
\]

Note that the asymptotics of $\delta_V$ do not depend on the norm. We say that $V$ is distorted if $k / \delta_{V,F}(k) \rightarrow 0$ as $k \rightarrow \infty$, and infinitely distorted if there is a $k$ such that $\delta_{V,F}(k) = \infty$. 

Finally, we will say that $V$ is weakly distorted in another finite-dimensional subspace $U$ if for some constant $C$,

$$\sup\{|\phi(\alpha)| : \text{dist}(\phi(\alpha), V) < C \text{ and } F(\alpha) \leq k\} \geq k.$$ 

Similarly, $V$ is weakly infinitely distorted in $U$ if there are constants $C$ and $k$ such that

$$\sup\{|\phi(\alpha)| : \text{dist}(\phi(\alpha), V) < C \text{ and } F(\alpha) \leq k\} = \infty.$$ 

These conditions also clearly do not depend on $\|\cdot\|$.

Note that the subadditivity of $F$ means that weak distortion in fact implies distortion. It will later be shown, however, that weak infinite distortion does not imply infinite distortion.

In these definitions, $G$ may be rather complicated. For example, $G = \mathbb{Z} \left[ \frac{1}{2} + \frac{1}{2} \right] \subset \mathbb{C} \cong \mathbb{Q}^2$ is infinitely generated and each generator has the same 2-norm in $\mathbb{Q}^2$.

We now specify these ideas to the study of homotopy groups. Let $X$ be a compact Riemannian manifold with boundary or a finite CW-complex with a piecewise Riemannian metric. Rademacher’s theorem, applied to some piecewise smooth local embedding of $X$ into $\mathbb{R}^N$, tells us that a Lipschitz map $f : S^n \to X$ is almost everywhere differentiable. In particular, one can define its volume as the integral over $S^n$ of the magnitude of the Jacobian.

**Definition.** Given $\alpha \in \pi_n(X)$, write

$$|\alpha|_{\text{Lip}} := \sup\{\text{Lip } f : f : S^n \to X \text{ is a Lipschitz representative of } \alpha\}.$$ 

The Lipschitz distortion of $\alpha$, written $L\delta_\alpha(k)$, is the distortion of $\alpha$ with respect to the functional $F(\alpha) = |\alpha|_{\text{Lip}}$. Similarly, write

$$|\alpha|_{\text{vol}} := \min\{\text{vol } f : f : S^n \to X \text{ is a Lipschitz representative of } \alpha\}.$$ 

The volume distortion of $\alpha$, written $V\delta_\alpha(k)$, is the distortion of $\alpha$ with respect to the volume functional.

Given a finite-dimensional vector subspace $V \leq \pi_n(X) \otimes \mathbb{Q}$ together with a norm $\|\cdot\|$, we define the distortion functions

$$L\delta_V(k) = \sup\{\|\vec{v}\| : \alpha \in \pi_n(X) \text{ such that } \alpha \otimes_{\mathbb{Q}} \vec{v} \in V \text{ with } |\alpha|_{\text{Lip}} \leq k\},$$

$$V\delta_V(k) = \sup\{\|\vec{v}\| : \alpha \in \pi_n(X) \text{ such that } \alpha \otimes_{\mathbb{Q}} \vec{v} \in V \text{ with } |\alpha|_{\text{vol}} \leq k\}.$$ 

If $\alpha$ is torsion, then $|k\alpha|_{\text{Lip}}$ and $|k\alpha|_{\text{vol}}$ are bounded, and so the corresponding distortion functions are eventually infinite. In all cases, $|k\alpha|_{\text{Lip}} \lesssim k^{1/n}\alpha$ because we can get a representative of $k\alpha$ by precomposing a map $f \in \alpha$ with a map $S^n \to S^n$ of degree $k$ and Lipschitz constant $k^{1/n}$. Thus for any $\alpha$, $L\delta_\alpha(k) \sim k^n$. Moreover, a $k$-Lipschitz map has volume at most $k^n$, so for any $\alpha$, $V\delta_\alpha(k) \leq L\delta_\alpha(k)^{1/n}$. In particular, $V\delta_\alpha(k)$ is at least linear. This motivates another definition:

**Definition.** A class $\alpha$ or subspace $V$ in $\pi_n(X) \otimes \mathbb{Q}$ is Lipschitz undistorted if $L\delta_\alpha(k) \sim k^n$, and undistorted if in addition its volume distortion is linear.

Clearly, for any subspace $V$ and any $0 \neq \alpha \in V$, $L\delta_V(k) \gtrsim L\delta_\alpha(k)$ and $V\delta_V(k) \gtrsim V\delta_\alpha(k)$. On the other hand, we will later see an example of an $X$ such that $\pi_n(X) \otimes \mathbb{Q}$ is distorted although none of its elements are. This provides extra motivation for the definitions of $L\delta_V$ and $V\delta_V$.

A Lipschitz homotopy equivalence $X \xrightarrow{f} Y$ induces an asymptotic equivalence on distortion functions, since $|\alpha|_{\text{Lip}} \leq \text{Lip } g |f_*|_{\text{Lip}}$ and $|\alpha|_{\text{vol}} \leq (\text{Lip } g)^n |f_*|_{\text{vol}} \leq (\text{Lip } f)^n (\text{Lip } g)^n |\alpha|_{\text{vol}}$. In particular, we can speak of distortion functions of finite CW complexes and compact manifolds without specifying a metric. Another way of simplifying our object of study is to restrict to a more combinatorial class of functions $S^n \to X$.

**Definition (BBFS09).** Given a CW-complex $X$ and a manifold $M^n$, call a map $f : M \to X$ admissible if $f(M) \subset X^{(n)}$ and for every interior $U$ of an $n$-cell of $X$, $f^{-1}(U)$ is a disjoint union of balls which map homeomorphically to $U$. If $M$ is compact, define the cellular volume $\text{vol}_C(f)$ to be the total number of these balls.
First, however, we need to make sure that the asymptotics of distortion functions remain the same if we only consider admissible maps. For this, we use a theorem originally from geometric measure theory.

**Theorem 2.1** (Federer–Fleming deformation theorem \cite{EPC+92}). For any finite-dimensional simplicial complex \( Y \), for example for a triangulated manifold, there is a \( c > 0 \), with the following property. Any Lipschitz \( k \)-chain \( T \) can be decomposed as \( T = Q + \partial R \), where \( R \) is a Lipschitz \((k + 1)\)-chain and \( Q \) is a smooth \( K \)-cycle whose simplices are cellular, such that \( \text{mass}_{k+1} R \leq c \text{mass}_k Q \leq c^2 \text{mass}_k T \) and \( Q \) and \( R \) are contained in the smallest subcomplex of \( Y \) containing \( T \).

Indeed, if \( N \) is a triangulated manifold, then a Lipschitz map \( f : N \to Y \) Lipschitz homotopic (via a homotopy of mass bounded by \( c^2 \text{vol} f \)) to an admissible map \( g \) with \( \text{vol}_C(g) \leq c \text{vol} f \).

The second statement, though not stated as such by \cite{EPC+92}, falls out of the proof they give. Every CW complex \( X \) has a simplicial approximation \( Y \) with a homotopy equivalence \( Y \to X \) that sends each \( k \)-simplex either homeomorphically to a cell or to \( X^{(k-1)} \); therefore, the same statement holds for CW complexes. Thus we have the following consequences.

**Corollary 2.2.** The minimal volume of a representative of \( \alpha \in \pi_n(X) \) are approximated to within a multiplicative constant, depending on \( n \) and \( X \), by admissible representatives. To find the asymptotic behavior of the distortion function of an element of \( \pi_n(X) \), it is enough to consider admissible representatives. In particular, the asymptotic behavior of the volume distortion functions of \( \pi_n(X) \) depends only on the \((n+1)\)-skeleton of \( X \).

These homotopy invariance results are also true for Lipschitz constants, using a Lipschitz version of the deformation theorem which is beyond the scope of this paper. But the following lemma gives a stronger independence result which only seems to work for volume distortion.

**Lemma 2.3.** Let \( (X,K) \) be a CW pair, and \( n \geq 2 \), such that the inclusion \( K \to X \) is rationally \( n \)-connected. Then there is a \( p_n(X,K) > 0 \) and \( C_n(X,K) \) such that for any map \( f : S^n \to X \) of volume \( k \) there is an admissible map \( g : S^n \to K \) for which \( p_n f \simeq g \) and \( g \) has volume \( C_n k + C_n \). If \( f \) is admissible, then as cellular chains, \( \#(\{S^n\}) = p_n j_n (f(\{S^n\})) \), where \( j_n : C_n(X) \to C_n(K) \) is a lifting homomorphism.

**Proof.** By passing to a homotopy equivalent situation, we assume that \( X \) and \( K \) have the same 1-skeleton and that all boundary maps are admissible. At the cost of increasing \( C_n \), we assume \( f \) is admissible.

First, fix some notation. For every \( i \) and \( r \), fix a degree \( r \) map \( d_r : (D^i, S^{i-1}) \to (D^i, S^{i-1}) \). Given maps \( a : I \times S^{i-1} \to X \) and \( b : D^i \to X \) such that \( a|_{\{0\} \times S^{i-1}} = b|_{\partial D^i} \), define the map \( a \vee b : D^i \to X \) to be \( "a on the outside and \( b \) on the inside." \) Finally, let \( q_n \) be as the bounded exponent from Corollary 1.6.

Suppose first that \( n = 2 \). Let \( b : S^2 \to X^{(2)} \) be an admissible map. Since \( \pi_1(X) \cong \pi_1(K) \), given a 2-cell \( e \) with attaching map \( \gamma_e : S^1 \to X^{(1)} \), we can extend \( \gamma_e \) to a map \( h_e : (D^2, S^1) \to (K, K^{(1)}) \). Define a map \( b' : S^2 \to K^{(2)} \) which agrees with \( b \) on \( b^{-1}(X^{(1)}) \) and for every cell \( e \) we replace every homeomorphic copy of \( e \) with \( h_e \). Then \( b' \) differs from \( b \) by a torsion element of \( \pi_2(X) \). Thus we can homotope \( b' \circ d_{q_n} \) to \( b'' \circ d_{q_2} \) by a homotopy that takes the preimage of each 2-cell \( e \) to a map which is \( h_e \) on half the preimage and a representative of a torsion element of \( \pi_2(X) \) on the other half, followed by a homotopy which cancels out all the torsion. This shows the lemma for \( n = 2 \), with \( p_2 = q_2 \).

Now suppose \( n \geq 3 \) and let \( \alpha \) be the cellular cycle \( f_\#(\{S^n\}) \). Let \( j_n \) be a lifting homomorphism and \( u_n \) be the corresponding chain homotopy. Then let \( h : [0,\frac{1}{2}] \times S^n \to \tilde{X} \) be a homotopy with \( h_0 = f \circ d_{q_n} ; \) as \( t \) increases, homotope through each cell \( e \) of \( q_n u_n(\alpha) \) so that \( h_\#(\{(t_r) \times S^n\}) = h_\#(\{(t_r-1) \times S^n\}) + \partial e_r \). Thus \( h_\#(\{(\frac{1}{2}) \times S^n\}) = j_n(\alpha) \). We can also ensure that \( h \) is admissible by leaving it constant for a time \( \varepsilon \) between homotoping through cells of \( A \). Moreover, we can assume that in \( h_{1/3} \), the closures of preimages of open cells in \( \tilde{X} \) are closed disks.

On the interval \( [\frac{1}{4}, \frac{1}{2}] \), we homotope, using a method inspired by \cite{Win84}, to ensure that \( h_{1/3} \) does not hit the interior of any \( n \)-cell outside \( \partial \), or map to the same cell with opposite orientations. Choose two preimages of cells with opposite orientations in \( h_{1/3}^{-1}(\tilde{X}) \). Call these \( a_1, a_2 : B^n \to S^n \). Since \( h_{1/3}^{-1}(\tilde{X}^{(n-1)}) \) is path-connected, there is a path \( \gamma \) from a point \( b_1 \) on the boundary of \( a_1(B^n) \) to a point \( b_2 \) on the boundary of \( a_2(D^n) \) such that \( h_{1/3} \circ \gamma \) maps to a loop in \( X^{(n-1)} \). Moreover, \( X^{(n-1)} \) is simply connected, so \( h_{2/3} \circ \gamma \) is nullhomotopic. We homotope a tubular neighborhood \( N \) of \( \gamma \) so that \( h_{1/3} \circ \gamma \) is the identity and \( N \) maps to \( X^{(n-1)} \). This gives us a disk \( D = a_1(B^n) \cup N \cup a_2(B^n) \) which maps to our cell with degree 0. Thus we can
homotope $h_{1/3+\varepsilon}|_{B}$ so that it maps to $\hat{X}^{(n-1)}$. Eventually, we get rid of all cells with opposite orientations, and so $h_{2/3}$ is admissible and the multiplicities of cells are the same as those of $\beta$.

However, at this point, it’s not necessarily true that $h_{2/3}$ maps to $\tilde{K}$. We have ensured that the image of $h_{2/3}$ is in $\tilde{K} \cup \tilde{X}^{(n-1)}$; moreover, by homotoping neighborhoods of paths through $\tilde{X}^{(n-1)}$ into $\tilde{K}^{(1)}$, we can ensure that $h_{2/3}^{-1}(\tilde{K})$ is connected, and $h_{2/3}^{-1}(\tilde{X} \setminus \tilde{K})$ is contained in a ball $B$ with $h_{2/3}(B) \subset \tilde{X}^{(n-1)}$. We can assume that $B$ is the upper hemisphere and $d_r$ preserves it.

**Proposition 2.4.** There is an $r = r(X,K,n) > 0$ such that $h_{2/3}|_{B} \circ d_r$ deforms rel boundary to $K^{(n-1)}$ via a homotopy $\Phi: [\frac{1}{2}, 1] \times B \rightarrow X$.

**Proof.** By Corollary [1.6] $\pi_n(\tilde{X}, \tilde{K})$ also has bounded exponent, while $\pi_{n+1}(\tilde{X}, \tilde{K}) \cong H_{n+1}(\tilde{X}, \tilde{K}) \mod \mathcal{BE}$.

Since $h_{2/3}|_{B}$ defines an element of $\pi_n(X, \tilde{K})$, there’s some $r_1(X,K,n)$ such that $h_{2/3}|_{B} \circ d_{r_1}$ deforms rel boundary to a map $\varphi: B \rightarrow K$.

By Proposition [1.2] the inclusion $\tilde{K}^{(n-1)} \hookrightarrow \tilde{X}^{(n-1)}$ induces a map on $\pi_{n-1}$ with torsion kernel. Call $\Gamma := \pi_1(X)$ and consider the diagram

$$
\begin{array}{ccccccc}
\mathbb{Z}\Gamma\#(n\text{-cells of } K) & \xrightarrow{\theta} & \pi_{n-1}(\tilde{K}^{(n-1)}) & \xrightarrow{\iota_*} & \pi_{n-1}(\tilde{K}) & \xrightarrow{\bar{\iota}_*} & 0 \\
\mathbb{Z}\Gamma\#(n\text{-cells of } X) & \xrightarrow{\iota} & \pi_{n-1}(\tilde{X}^{(n-1)}) & \xrightarrow{\bar{\iota}_*} & \pi_{n-1}(\tilde{X}) & \xrightarrow{\bar{\iota}_*} & 0 \\
\end{array}
$$

induced by the inclusions $K^{(n-1)} \hookrightarrow K$ and $X^{(n-1)} \hookrightarrow X$. From a diagram chase, we get that $\ker \iota_*$ is an extension of a bounded exponent group by $\mathbb{Z}\Gamma\#(n\text{-cells of } K)/\ker \theta$; since $\ker \iota_*$ is known to be torsion, its subgroup has bounded exponent by Lemma [1.3]. Therefore $\ker \iota_*$ itself has bounded exponent, and there is an $r_2(X,K,n)$ such that $h_{2/3}|_{B} \circ d_{r_2}$ is nullhomotopic in $\tilde{K}^{(n-1)}$ via a nullhomotopy $\psi$.

Together, $\psi \circ d_{r_1}$ and $\varphi \circ d_{r_2}$ give us a map $\chi$ with $[\chi] \in \pi_n(\tilde{K})$ which deforms through $\tilde{X}$ into $\tilde{X}^{(n-1)}$. Now, we have the diagram (with decorations mod $\mathcal{BE}$)

$$
\begin{array}{ccccccc}
\pi_{n+1}(\tilde{X}, \tilde{K}) & \xrightarrow{\psi} & \pi_{n}(\tilde{K}) & \xrightarrow{\iota_*} & \pi_{1}(\tilde{X}) & \xrightarrow{\bar{\iota}_*} & 0 \\
H_{n+1}(\tilde{X}, \tilde{K}) & \xrightarrow{\bar{\iota}_*} & H_{n}(\tilde{K}) & \xrightarrow{\bar{\iota}_*} & H_{n}(\tilde{X}), & & \\
\end{array}
$$

where the rows are exact. Since $h_{X}(\chi)$ is torsion, for some $r_3(X,K,n)$, $r_3[\chi]$ has a preimage whose Hurewicz image is 0. By Prop. [1.1] $r_3[\chi]$ has a representative which maps into $\tilde{K}^{(n-1)}$.

Set $r = r_1r_2r_3$ and let $\Phi$ be a homotopy that takes $h_{2/3}|_{B} \circ d_r$ to $\psi \circ d_{r_1} \circ d_{r_2} \vee \chi \circ d_{r_3}$ and then deforms $\chi \circ d_{r_3}$ into $\tilde{K}^{(n-1)}$. □

**Figure 1.** The stages of the homotopy $\Phi$. 
By precomposing with $d_r$ on the interval $[0, \frac{3}{4}]$, we then get a homotopy $H : [0, 1] \times S^{n-1} \to \tilde{X}$ with

$$H(t, x) = \begin{cases} h_1(d_r(x)) & \text{if } t \in [0, \frac{3}{4}] \\ h_2/d_3(d_r(x)) & \text{if } t \in \left[\frac{3}{4}, 1\right) \text{ and } x \in S^{n-1} \setminus B \\ \Phi(x) & \text{if } t \in \left[\frac{3}{4}, 1\right] \text{ and } x \in B \end{cases}$$

and $H_1$ lands in $\tilde{K}$ with $H_1([S^n]) = rq_n j_n(\alpha)$, is admissible and has no cells of opposite orientations. In particular, if we let $t_n = \max\{|j_n(e)| : e \text{ is an } n\text{-cell of } X\}$, $\|g\| \leq rq_n t_n \|f\|$. Thus we can set $p_n = rq_n$, and in our homotopy equivalent setup $C_n = rq_n t_n$ and $g = H_1$. The homotopy equivalence may impose a penalty on the constant. 

More generally, if a map $\varphi : Y \to X$ obeys the same conditions on homotopy, then using the mapping cylinder, we can prove that for any $f : S^n \to X$, $rf$ lifts to a map of volume $Ck$ for constants $C(\varphi, n)$ and $r(\varphi, n)$. As a corollary, making use of Lemma [1.9] we have Theorem A.

**Theorem 2.5** (Rational invariance of volume distortion). Let $X$ and $Y$ be rationally $n$-equivalent, with $n$-equivalences $X \to Z \leftarrow Y$. Then for any $\alpha \in \pi_n(X)$ and $\beta \in \pi_n(Y)$ which map to the same element of $\pi_n(Z)$, $V\delta_\alpha \sim V\delta_\beta$. Similarly, for any finite-dimensional $V \subseteq \pi_n(X) \otimes \mathbb{Q}$ and $W \subseteq \pi_n(Y) \otimes \mathbb{Q}$ which are sent to the same subspace of $\pi_n(Z) \otimes \mathbb{Q}$, $V\delta_V \sim V\delta_W$.

This fact allows us to ignore torsion information when studying volume distortion functions. Indeed, in our subsequent discussion, we will often speak of spaces and maps “up to rational equivalence”. However, we conjecture that such an approach is not sufficient for studying lipschitz distortion.

**Example 2.6.** Let $X$ be the total space of a fibration $S^3 \to X \to T^4$ with Euler class equal to the fundamental class $[T^4]$. Define a map $f_k : S^3 \to \tilde{T}^4$ which sends $S^3$ to the sides of a 4-cube with side length $k$. This map is $Ck$-lipschitz, for $C$ independent of $k$, and as will be discussed it lifts to a $Ck$-lipschitz map $g_k : S^3 \to \tilde{X}$ which is a representative of $k^4\alpha$, where $\alpha$ is a generator of $\pi_3(X)$. Thus $\alpha$ is lipschitz distorted in $X$ with $\delta_\alpha(k) \gtrsim k^4$.

On the other hand, let $Z$ be $T^3$ with a disk cut out and a 3-cell glued in via a degree 2 map. Since $S^2 \cup_{\deg 2} D^3$ is rationally equivalent to a point, so is $\tilde{Z} \simeq \bigvee_{\mathbb{Z}^2} (S^2 \cup_{\deg 2} D^3)$. Thus the map $i : S^3 \times \mathbb{Z} \to T^4$ sending the glued-in disk to a point is a rational equivalence. Thus we can define a pullback fibration with total space $\tilde{Y} = i^* X$, and $Y \to X$ is also a rational equivalence. On the other hand, suppose that $\delta_\alpha(k) \gtrsim k^4$. It is possible to show that this is equivalent to the existence of admissible maps $f_k : (D^4, S^3) \to \tilde{Z}$ of volume $k^4$ with admissible boundary such that $\operatorname{Lip}(f_k|_{S^3}) \lesssim k$. By composing $f_k|_{S^3}$ with a projection onto $\tilde{Z}$, we get a sequence of $Ck$-Lipschitz admissible maps $D^3 \to \tilde{Z}$ with area $C'k^3$. Although we don’t know a proof, we suspect that such maps do not exist.

This would mean that $Y$ and $X$ have asymptotically different Lipschitz distortion functions. In other words, torsion matters for Lipschitz distortion, at least for spaces with a nontrivial fundamental group.

### 3. Sources of Volume Distortion

In this section, we discuss two ways volume distortion is induced in finite complexes: homological triviality and actions by the fundamental group. Indeed, both of these induce a particularly coarse kind of distortion. Throughout the section, let $X$ be a finite CW complex with universal cover $\tilde{X}$ and fundamental group $\Gamma$. Restating the previous section’s definition, a class $\alpha$ or finite-dimensional subspace $V$ in $\pi_n(X) \otimes \mathbb{Q}$ is infinitely distorted if there is some constant $C$ such that there are arbitrarily large $k$ (respectively, arbitrarily large vectors $v \in V$) such that $|k\alpha|_{\text{vol}} \leq C$ (respectively, $|v|_{\text{vol}} \leq C$). Note that nothing we have said prevents certain multiples of an infinitely distorted $\alpha$ from having unbounded minimal volume, and indeed there are plenty of examples in which this happens. In particular, by defining distortion the way we do rather than simply studying the function $f(k) = |k\alpha|_{\text{vol}}$, we are obscuring a certain amount of complexity.

Lipschitz distortion in simply connected spaces is fairly subtle. It has been previously studied by Gromov in [Gro88, Gro99], and [Gro98]. On the other hand, volume distortion is trivial in the simply connected case.

**Theorem 3.1.** Suppose $X$ is a simply connected complex and $\alpha \in \pi_n(X) \otimes \mathbb{Q}$. Then $k\alpha$ has a representative of zero volume for some $k$ if and only if $h_n(\alpha) = 0$; otherwise $\alpha$ is undistorted.
Proof. If $h_n(\alpha) \neq 0$, then there is a cohomology class $\omega \in H^n(X; \mathbb{Q})$ which pairs nontrivially with any representative of $k\alpha$, giving some $kC$. Thus $k\alpha$ cannot have zero volume and in fact the cellular volume

$$\text{vol}(k\alpha) \geq kC/\min_{n\text{-cells } c \text{ of } X} \omega(c).$$

In particular, $\alpha$ is undistorted.

On the other hand, if $h_n(\alpha) = 0$ then some $k\alpha$ has a representative of zero volume by Proposition [L1]. 

We can also couch the observations about volume and pairing with cohomology classes in the language of metrics and differential forms.

**Definition.** Given a $k$-form $\omega$ on a piecewise Riemannian space $(X, g)$, let $\|\omega\|_\infty = \sup \langle\omega, (v_1, \ldots, v_k)\rangle$, where the supremum ranges over all orthonormal frames $(v_1, \ldots, v_k)$. We say $\omega$ is 

“homologically nontrivial in any finite cover.”

Indeed, suppose $\alpha$ is a finite cover $\omega \in \Omega^n(\tilde{X})$ such that $\langle\omega, \alpha\rangle = c \neq 0$, then $\alpha$ is volume-undistorted.

**Lemma 3.2.** If $\alpha \in \pi_n(X)$ and there is a bounded cocycle $\omega \in \Omega^n(\tilde{X})$ such that $\langle\omega, \alpha\rangle = c \neq 0$, then $\alpha$ is volume-undistorted.

**Proof.** If $f : S^n \to X$ is a map representing $k\alpha$, then $\langle\omega, f\rangle = ck$, and so $\text{vol} f \geq ck/\|\omega\|_\infty \sim k$. 

Indeed, one might hypothesize, falsely, that the following sufficient condition for a class to be undistorted is also necessary:

**Corollary 3.3.** Let $\alpha \in \pi_n(X)$. If for some finite cover $\varphi : Y \to X$, $0 \neq h_n(\varphi^{-1}_*\alpha) \in H_n(\tilde{X})$, then $\alpha$ is volume-undistorted.

**Proof.** Let $\pi : \tilde{X} \to X$ be the universal cover. It is enough to show that there is a bounded form $\omega$ on $\tilde{X}$ such that $\langle\omega, h_n(\pi^{-1}_*\alpha)\rangle \neq 0$. One such form is the pullback of the dual of $h_n(\varphi^{-1}_*\alpha)$ to $\tilde{X}$.

We have an example in which this applies nontrivially:

**Example 3.4.** Let $X = (S^2 \vee S^1 \vee S^1) \cup_f D^2$, where $f \simeq x + ax + bx$, where $a$ and $b$ generate $\pi_1 X$ and $x$ generates $\pi_2 S^2$. Then in the threefold cover $Y$ of $X$ corresponding to $\langle\langle a^3, ab\rangle\rangle$, each lift of $f$ covers each of the three lifts of $S^2$ with degree $1$; therefore $H_2(Y)$ is generated as a group by $h_2(x)$ and $h_2(ax)$, and $x$ is undistorted by Corollary 3.3.

But also an example demonstrating that it is not a necessary condition:

**Example 3.5.** According to [Mes72], the Baumslag-Solitar group $BS(2, 3) = \langle a, b \mid ab^2a^{-1}b^{-3} \rangle$ is not residually finite; in particular every surjection onto a finite group sends $g := [a^b, a] \to 1$. Moreover, $g$ and $b$ generate a free subgroup of $BS(2, 3)$. Let $X$ be a 2-complex with $\pi_1 X = BS(2, 3)$, and let $Y = (X \vee S^2) \cup_f D^2$, for some $f$ such that $[f] = y + g \cdot y + b \cdot y$, where $y$ is the generator of $\pi_2 S^2$. Then $y$ is undistorted in $Y$, since its image in $H_2(Y; \mathbb{Q})$ is nonzero and we can find a form which agrees with the form in the previous example on a coset of $\langle y, b \rangle$ and is 0 elsewhere. However, such a form cannot descend to a finite cover of $Y$.

Indeed, suppose $\varphi : Z \to Y$ is a finite cover. Then in $Z$, the disk attached via $f$ induces the relation $(2 + b)h_2(\varphi^{-1}_* y) = 0$. Moreover, there’s an $r$ for which $b^r$ acts trivially on $H_2(Z)$, so that

$$0 = (2^r - (-b)^r)h_2(\varphi^{-1}_* y) = (2^r + (1 - 1)^{-1})h_2(\varphi^{-1}_* y).$$

Thus $\varphi^{-1}_* y$ is sent via the Hurewicz map to a torsion element of $H_n(Z)$. In other words, $y$ does not become homologically nontrivial in any finite cover.

**The ubiquity of Whitehead products.** As an application of Theorem 3.1, we can show that all spaces which have a homotopy group which is infinitely generated over $\mathbb{Q}$ have infinitely distorted homotopy classes.

To do this, we must define generalized Whitehead products.

**Definition.** Let $n_1, \ldots, n_r \geq 2$, and $N = n_1 + \cdots + n_r$. Following [AA78], we refer to the $(N - 1)$-skeleton of $\prod_{i=1}^r S^{n_i}$ as the fat wedge $T = V^r_{i=1} S^{n_i}$. Let $\psi : S^{N-1} \to T$ take the $(N - 1)$-skeleton of $\prod_{i=1}^r D^{n_i} = D^N$ to $T$ by sending the boundary of each disk to a point. Equivalently, $\psi$ is the attaching map of the top cell of $\prod_{i=1}^r S^{n_i}$ in the cell structure which has one cell corresponding to each subset of $\{1, \ldots, r\}$. Then the
between 0 and 1, as a subgroup of \( \text{Homeo}(\mathbb{R}) \). Since this gives a filling for each highest-dimensional cell of \( T \).

Example 3.6. The rational homotopy of \( CP^n \) is generated by an element \( \alpha \in \pi_2(CP^n) \) and an element \( \beta \in \pi_{2n+1}(CP^n) \). With the right choice of orientation, the \((n+1)\)-fold Whitehead product \([\alpha, \cdots, \alpha] \) is \( \{ \beta \} \).

Since the fat wedge \( \vee_{i=1}^{n-1} S^m \) is actually at most \((N-2)\)-dimensional, any Whitehead product is infinitely distorted. Thus we can find infinitely distorted homotopy classes by showing certain Whitehead products to be nonzero. The example that sets the pattern is as follows. Let \( X = S^1 \vee S^2 \); then \( \pi_2(X) \cong \mathbb{Z} \).\( \mathbb{Z} \) is generated by a generator \( \alpha \in \pi_2(S^2) \), and \( \pi_3(X) \) contains Whitehead products \([i \cdot \alpha, j \cdot \alpha] \) which are linearly independent over \( \mathbb{Q} \) for each unordered pair \( i, j \). Thus only a finite number of them can be cancelled out by adding \( 2n \)-cells, and almost all of them are nonzero in rational homotopy. Indeed, when \( \pi_1(X) \cong \mathbb{Z} \), it is fairly easy to find infinitely generated submodules using Whitehead products.

Lemma 3.7. Suppose \( X \) is a complex with finite skeleta and virtually abelian fundamental group \( \Gamma \). Suppose also that \( h_{n}(\pi_{n}(X) \otimes \mathbb{Q}) \subseteq H_{n}(\tilde{X}; \mathbb{Q}) \) is infinite-dimensional as a \( \mathbb{Q} \)-vector space. Then distorted elements of \( \pi_{2n-1}(X) \otimes \mathbb{Q} \) are infinitely generated as a \( \mathbb{Q} \Gamma \)-submodule of \( \pi_{2n-1}(X) \).

Proof. By taking a finite cover, we can assume that \( \Gamma \) is abelian.

It is enough to show that the group

\[
K = \ker \left( \pi_{2n-1}(X^{2n-1}) \otimes \mathbb{Q} \xrightarrow{h_{n-1}} H_{2n-1}(\tilde{X}^{2n-1}; \mathbb{Q}) \right)
\]

is infinitely generated as a \( \mathbb{Q} \Gamma \)-module, since every element of \( K \) is infinitely distorted, and because only a finitely generated submodule is killed by \( 2n \)-cells. For this, and repeatedly, we use the fact that \( \mathbb{Q} \Gamma \) is noetherian.

Let \( M = h_{n}(\pi_{n}(X) \otimes \mathbb{Q}) \). For \( \alpha \in K \), choose a representative \( f : S^{2n-1} \to \tilde{X}^{(2n-2)} \) of some \( k \alpha \). Then there is, independent of the choice of \( k \) and \( f \), a rational homotopy invariant \( \mathbb{Q} \Gamma \)-module homomorphism \( T_{j} : \text{Hom}(M, \mathbb{Q}) \wedge \mathbb{Q} \text{Hom}(M; \mathbb{Q}) \to \mathbb{Q} \) defined by restricting the cup product in the space \( Y = X_{\dagger} D^{2n} \) to \( M \) and evaluating it on \( \frac{1}{2}|D^{2n}| \). Here, we write \( \text{Hom}(M, \mathbb{Q}) \wedge \mathbb{Q} \text{Hom}(M; \mathbb{Q}) \) to mean the graded exterior product equipped with the \( \Gamma \)-structure \( \gamma \cdot (\omega \wedge \eta) = \gamma \omega \wedge \gamma \eta \). This gives us a diagram of \( \mathbb{Q} \Gamma \)-module homomorphisms

\[
\begin{array}{ccc}
\text{Hom}(M, \mathbb{Q}) \wedge \mathbb{Q} \text{Hom}(M; \mathbb{Q}) & \xrightarrow{\varphi} & K \\
\downarrow h_{n} \wedge h_{n} & & \downarrow \varphi \\
M \wedge \mathbb{Q} M & \xrightarrow{\gamma} & \text{Hom}(\text{Hom}(M, \mathbb{Q}) \wedge \mathbb{Q} \text{Hom}(M; \mathbb{Q}))
\end{array}
\]

which commutes by the definition of the Whitehead product; moreover, since the cup products that define \( \varphi \) only depend on a compact subspace of \( \tilde{X} \), \( \text{im} \varphi \subseteq \text{im} \gamma \). In particular, this gives a surjection \( K \to M \wedge \mathbb{Q} M \), and so it’s enough to show that this latter module is infinitely generated.

Now, \( M \) is finitely generated as a module, since it’s a subquotient of the chain group \( C_{n}(\tilde{X}; \mathbb{Q}) \). Thus there is some \( \alpha \in M \) for which there is a sequence \( \gamma_{1}, \gamma_{2}, \ldots \in \Gamma \) such that all the \( \gamma_{i} \alpha \) are linearly independent. Now, \( \gamma \cdot (\alpha \wedge \gamma \alpha) = \gamma \alpha \wedge \gamma \gamma \alpha \), and so if \( \gamma \alpha = \alpha \) then this determines the other term. Thus \( \Gamma \) being abelian means that a finite set of generators can only generate a finite-dimensional set of elements of the form \( \alpha \wedge b \alpha \) for \( b \in \mathbb{Q} \Gamma \). Thus elements of the form \( \alpha \wedge \gamma \alpha \) generate an infinitely generated submodule of \( M \wedge \mathbb{Q} M \). \( \square \)

In general, though, we have no hope of finding nonzero Whitehead products of a particular order, essentially because most group rings are very complicated. In the following example, we demonstrate a finite complex \( X \) such that \( \pi_{2}(X) \) is infinite-dimensional but \( \pi_{3}(X) \) is zero.

Example 3.8. Consider Thompson’s group \( F \). Brown and Geoghegan [BG84] give a \( (F, 1) \) with two cells in each dimension; call this space \( X \). The group \( F \) acts transitively on the set \( A \) of dyadic rationals strictly between 0 and 1, as a subgroup of \( \text{Homeo}([0, 1], \{0, 1\}) \); the stabilizer of a point \( a \in A \) under this action is
isomorphic to $F \times F$, corresponding to homeomorphisms that fix $[a,1]$ and those that fix $[0,a]$. Thus $\mathbb{Q}[A]$ is a quotient module of $\mathbb{Q}F$ by four generators of the form $g_{ij} - 1$, where $g_{ij}$ generate the two copies of $F$. So we can attach a 2-cell and four 3-cells to $X^{(4)}$, as well four 4-cells corresponding to the relators of the stabilizer subgroup, to get a space $Y$ with $\pi_2(Y) \cong \mathbb{Q}[A]$ and $\pi_3(Y) \cong \mathbb{Q}[A] \otimes \mathbb{Q}[A]$. Moreover, $\mathbb{Q}[A] \otimes \mathbb{Q}[A]$ is finitely generated by $b \wedge b$ and $a \wedge b$ for any $a < b \in A$. Thus we can add two cells to get a finite complex $Z$ with $\pi_2(Z) \cong \mathbb{Q}[A]$ and $\pi_3(Z) \cong 0$.

Since in fact $F$ acts transitively on unordered $n$-element subsets of $A$, one can continue a similar construction to build a complex with finite skeleton (with $O(n^2)$ cells in dimension $n$) whose universal cover is rationally equivalent to $(S^3)^\infty$. In particular, Theorem 13 does not extend to complexes with finite skeleton.

It would be interesting to expand the class of groups for which the conclusion of Lemma 5.7 holds. One candidate is the class of virtually polycyclic groups, which are also known to have noetherian group rings [BLSS1].

In any case, when $X$ is a finite complex, it becomes much easier to produce a nonzero higher-order Whitehead product.

**Lemma 3.9.** Let $X$ be a finite CW complex. If $\pi_n(X) \otimes \mathbb{Q}$ is infinite-dimensional, then there is an infinitely distorted non-torsion element of $\pi_n(X)$ or of $\pi_{n-1}(X)$ for some $r$.

**Proof.** Suppose that no non-torsion element of $\pi_n(X)$ is infinitely distorted. In particular, the image of $\pi_n(X) \otimes \mathbb{Q}$ in $H_n(\hat{X}; \mathbb{Q})$ under the Hurewicz map is infinite-dimensional. Pick an $r$ such that $nr > \dim X$ and let $\alpha_1, \ldots, \alpha_r \in \pi_n(X)$ be such that the $h_n(\alpha_i)$ are linearly independent in $H_n(\hat{X}; \mathbb{Q})$.

Suppose first that for some $s < r$ there are $1 \leq i_1 < \cdots < i_s \leq r$ such that $[\alpha_{i_1}, \ldots, \alpha_{i_s}]$ has a non-torsion element. This gives us an infinitely distorted element of $\pi_{n-1}(X)$.

Otherwise, let $\omega_1, \ldots, \omega_r \in H^n(\hat{X})$ be cohomology classes such that $\langle \omega_i, h_n(\alpha_j) \rangle = \delta_{ij}$. By the above remark, there are numbers $k_1, \ldots, k_r > 0$ such that $[k_1 \alpha_1, \ldots, k_r \alpha_r]$ has an element with a representative $f : S^{nr-1} \to \hat{X}$. Moreover, let $Y = \hat{X} \cup_f D^{nr}$. Then the inclusion $F : D^{nr} \to Y$, which has $F|_{\partial D^{nr}} = f$, factors through a map $(S^n)^r \to Y$. This map restricts to $\alpha_i$ on the $i$th factor, and so $F^*(\omega_1 \smile \cdots \smile \omega_r)$ is a nonzero multiple of $[(S^n)^r]$ in particular, $\omega_1 \smile \cdots \smile \omega_r$ evaluates to a nonzero value on the $nr$-cell of $Y$. So $X \cup_k D^{nr} \not\cong X \cup D^{nr}$ for any $k$, and thus $f$ is a representative of a nonzero rational homotopy class, which is necessarily infinitely distorted. □

On the other hand, if all the rational homotopy groups of $X$ are finite-dimensional vector spaces, then Sullivan’s theory of minimal models says that the rational Hurewicz map is an isomorphism if and only if $X$ is rationally equivalent to a product of odd-dimensional spheres. In fact, using [AA78] it’s possible to show that otherwise there is always a nonzero higher-order Whitehead product. Thus whenever $\pi_n(X)$ is undistorted for all $n$, $\hat{X}$ must be rationally equivalent to a finite product of odd-dimensional spheres.

**Distortion via monodromy.** Another source of infinite distortion is the action by the fundamental group on $\pi_n(X)$. The most basic example is the mapping torus of a degree 2 map on $S^2$. Let $\alpha \in \pi_1(X)$ be the identity map on the sphere; then $2k\alpha$ has a cellular representative of volume 1 for every integer $k$, and any integer multiple $k\alpha$ has a representative of volume $\leq \log k$.

More generally, let $\alpha \in \pi_n(X)$, $\gamma \in \Gamma$, and suppose that the $\mathbb{Q}[Z]$-module generated by $\gamma^t \cdot \alpha$ is a $t$-dimensional $\mathbb{Q}$-vector space $V$ for some finite $t$. Then $\gamma$ acts on $V$ via a linear transformation $T$. Then either $T$ is conjugate to an element of $O(t, \mathbb{Q})$, or there is some vector $\vec{v}$ such that $T^k(\vec{v})$ increases without bound and thus $V$ is infinitely distorted.

**Definition.** We say a transformation $T \in GL(t, \mathbb{Q})$, or more generally a representation $\rho : \Gamma \to GL(t, \mathbb{Q})$, is *elliptic* if it preserves a norm on $\mathbb{Q}^t$, or equivalently if it conjugates into $O(t, \mathbb{Q})$.

One finds more interesting phenomena by looking at the distortion of individual elements of $V$. Consider $V$ as a $\mathbb{Q}[\gamma]$-module. Then $V$ decomposes as $V = \oplus_1^{t=1} \mathbb{Q}[\gamma]/p_i(\gamma)^{k_i} =: \oplus_1^{t=1} V_i$, where the $p_i$ are irreducible factors of the characteristic polynomial of $T$. Since the distortion of elements of $V$ is determined by their decomposition into elements of the $V_i$, we may assume that $V$ is irreducible.

First, we see that the distortion of a subspace need not correspond to that of an element.
Examples 3.10.  

(1) Consider the mapping torus $X$ of a map $f : \mathbb{V}_4 S^n \rightarrow \mathbb{V}_4 S^n$ whose action on $\pi_n(\mathbb{V}_4 S^n)$ (or equivalently on the homology) is given by the companion matrix of the polynomial $x^4 - 2x^3 - 2x + 1$. This polynomial is irreducible over $\mathbb{Q}$ and has four complex roots, two of which are on the unit circle and two of which are real. Call these roots $\xi, \bar{\xi}, \eta, \bar{\eta}$, and let $\bar{u}_\xi \in \mathbb{C}^4$, etc., be the corresponding eigenbasis. Then for any $0 \neq \bar{v} \in \pi_n(X) \otimes \mathbb{Q}$, the coordinates in each of these basis vectors are nonzero. On the other hand, for any admissible map $f : S^n \rightarrow X$ of volume 1, the element $[f] = T^k e_1 \in \pi_n(X) \otimes \mathbb{Q}$ has $\bar{u}_\xi$- and $\bar{u}_\eta$-coordinates at most 1. Therefore, any admissible map representing $k\bar{v}$ has volume proportional to $k$. The deformation theorem then implies that $\bar{v}$ admits neither volume nor Lipschitz distortion in $X$.

(2) Construct a space $S$ by gluing $S^3 \times I$ on both sides to a copy of $S^3$, on one side with degree 3 and on the other with degree 5. One can think of this as the “mapping torus” of multiplication by 5/3. Note that $\pi_n(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ is infinitely distorted, since $(5/3)^k$ can be represented with volume 1. However, any given element in $\pi_n(X)$ is not infinitely distorted, since there’s a finite number of integers that can be expressed as the sum of a bounded number of powers of 5/3. So a one-dimensional subspace need not have the same distortion function as one of its elements!

(3) This example demonstrates that weak infinite distortion does not necessarily imply infinite distortion. Consider the action on $\pi_n(\mathbb{V}_4 S^n) = \langle e_1, \ldots, e_4 \rangle$ given by $B = \begin{pmatrix} A & 0 \\ I & A \end{pmatrix}$, where $A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$. As in the previous example, there’s no literal mapping torus, but we can glue $(n + 1)$-cells to $\mathbb{V}_4 S^n \vee S^3$ that homotope $5\bar{e}_i$ to the relevant integer vector on the other side. Then for $\bar{v}, \bar{w} \in \mathbb{R}^2$ the vector corresponding to $(\bar{v}, \bar{w})$ shifted by $k$ is

$B^k \begin{pmatrix} \bar{v} \\ \bar{w} \end{pmatrix} = \begin{pmatrix} A^k \bar{v} \\ kA^{-k} \bar{v} + A^k \bar{w} \end{pmatrix}.$

In particular, $(B^k - B^{-k})\bar{e}_1 = 2\cos(k\alpha)\bar{e}_1 + 2k\sin(2\alpha)A^{-1}\bar{e}_3$ where $\alpha$ is the irrational angle of rotation. In this way, $\pi_n(X)$ is infinitely distorted, and indeed any rational vector in $(\bar{e}_3, \bar{e}_4)$ is weakly infinitely distorted. On the other hand,

Lemma 3.11. For any $C$, elements of $(\bar{e}_3, \bar{e}_4)$ with volume at most $C$ have length bounded by some $L(C)$.

Proof. To see this, we induct on $C$. For $C = 1$, The options are $B^k \bar{e}_j$ for $j = 3, 4$, which all have the same length.

Now fix a $C > 1$, and take a particular linear combination of integer vectors

$\bar{u} = B^{t_0} \bar{u}_0 + B^{t_1} \bar{u}_1 + \cdots + B^{t_k} \bar{u}_k$

with $\sum_i |\bar{u}_i| \leq C$ and $u_i = \begin{pmatrix} \bar{v}_i \\ \bar{w}_i \end{pmatrix}$. If $\bar{u} \in (\bar{e}_3, \bar{e}_4)$, then $\sum_i A^{t_i} \bar{v}_i = \bar{0}$. We can assume that the $t_k$ are increasing and $t_0 = 0$ since multiplying by $B^n$ doesn’t change the length of a vector in $(\bar{e}_3, \bar{e}_4)$. Moreover, we can assume that all the $\bar{w}_i$ are zero, since they contribute at most a linear amount of total length.

Note that $\bar{v}$ and $A\bar{v}$ are integer lattice points if and only if $\bar{v}$ is in the lattice generated by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, so the length of such a vector is $\sqrt{5z}$ for some integer $z$.

Now, either (1) $\sum_{i=0}^\ell A^{t_i} \bar{v}_i \neq 0$ for $\ell < k$, or (2) we know that $\|u\| \leq \max\{L(C_1) + L(C_2)\}$ for some $C_1 + C_2 = C$. In case (1), for each $\ell$, $\bar{u}_\ell = \sum_{i=\ell}^k A^{t_i-t_\ell} \bar{v}_i$ must have $\|\bar{u}_\ell\|^2 = 5^{t_\ell-t_\ell-1}z$ for some integer $z$. Since $A$ doesn’t change lengths of vectors, this dictates that $5^{t_\ell-t_\ell-1} \leq C^2$. Together with the restriction that $k < C$, this puts a bound on $t_k$ and therefore there is a finite number of choices of $\bar{u}$ satisfying (1), whose lengths are bounded by some number $L'(C)$. We can thus set

$L(C) = \max\{L'(C), L(C-1) + L(1), L(C-2) + L(2), \ldots\},$

completing the proof. \qed
Indeed, with a bit more accounting, we see that the best we can do with a given C to get
\[ \sum_{i=1}^{k} 5^{t_i-t_{i-1}} \leq C \] to have \( t_i = i \), with a total span of \( t_k = C \). Thus a map of volume C represents a map of volume \( O(C^2) \).

Since, e.g., \( \vec{e}_i \) is weakly infinitely distorted, it is in particular distorted. To estimate its actual distortion function, let \( \vec{v}_0 = \vec{v} \in \mathbb{Z}^2 \) be any vector of length k such that \( \vec{v} \) and \( A\vec{v} \) are both \( \mathbb{Z} \)-points, and for \( 1 \leq i \leq k \) let \( \vec{v}_{i+1} \) be the nearest point to \( A\vec{v}_i \) such that \( \vec{v}_{i+1} \) and \( A\vec{v}_{i+1} \) are both lattice points. Let \( \vec{v}_i = \left( \frac{v_i}{d} \right) \in \mathbb{Z}^4 \). Then \( \sum_{i=0}^{k-1} (\vec{v}_i - B\vec{u}_i) \) is represented by a map \( S^n \rightarrow X \) of volume \( O(k) \), since \( B\vec{u}_i - \vec{u}_{i+1} \) has bounded length, but represents an element in \( \pi_n(X) \) of length \( \sim k^2 \). Thus \( \forall \vec{v} \delta_{\vec{v}}(k) \sim k^2 \).

However, this depends on \( p_i \) having roots on the unit circle. If all the roots of \( p_i \) are off the unit circle, then in fact all elements of \( V \) are distorted.

**Lemma 3.12.** Suppose that all complex roots \( \lambda_j \) of \( p_i \) have \( |\lambda_j| \neq 1 \). Then any \( \alpha \in V_i \) has distortion function \( \delta_\alpha(k) \gtrsim e^{|\text{log}_L(M(p_i))|} \).

**Proof.** Let \( \lambda_1, \ldots, \lambda_r \) be the eigenvalues with multiplicity, with \( |\lambda_1|, \ldots, |\lambda_s| < 1 \) and \( |\lambda_1+\ldots+\lambda_r| > 1 \). Let \( v_1, \ldots, v_r \) be a real Jordan basis for \( V \otimes \mathbb{R} \) with respect to \( T \), and \( V = V_- \oplus V_+ \), where \( V_- \) is spanned by \( v_1, \ldots, v_s \) and \( V_+ \) by \( v_{s+1}, \ldots, v_r \). Next, let \( Q \) be such that \( T \) and \( T^{-1} \) are both in \( \mathbb{Z}_r(M_r(\mathbb{Z})) \), where \( L = \min \{ |\lambda_1|, \ldots, |\lambda_r| \} \), and \( U = \max \{ |\vec{v}| : \vec{v} \in \mathbb{Z}_r \} \).

Take an element \( \alpha \in V_i \subseteq \pi_n(X) \otimes \mathbb{Q} \). It has a multiple which is a lattice point \( p = p_- + p_+ \), with \( p_- \in V_- \) and \( p_+ \in V_+ \).

If \( V_- = 0 \), i.e. if \( T \) only has large eigenvalues, then for any \( M > Q \sqrt{r} \), \( M = p_0 + Tp' \), where \( p_0 \) and \( p' \) are lattice points, \( |p_0| \leq Q \sqrt{r} \) and \( |p'| \leq M|p|/L \). Continuing this construction inductively gives us

\[
M p = \sum_{i=0}^{\ell} T^i p_i,
\]

with \( |p_i| \leq Q \sqrt{r} \) for every \( i \) and \( \ell \leq \log_L(M|p|) \). Thus

\[
|\alpha|_{\text{Lip}} \leq \ell Q \sqrt{r} \in O(\log |p|).
\]

If \( V_+ = 0 \), the same computation holds substituting \( T^{-1} \) for \( T \).

Now suppose \( T \) has both small and large eigenvalues. In other words, multiplying by a power of \( T \) shrinks a vector in certain irrational directions and stretches it in others. If these directions were rational, then by the above we could distort a vector going in just one of them. Our strategy will be to express our chosen vector as a sum of shrinking and expanding components as precisely as possible.

Thus we can write \( p = a + b \), where \( a \) and \( b \) are in \( Q \mathbb{Z}^r \) and \( d(a, p_-) < Q \sqrt{r} \) and \( d(b, p_+) < Q \sqrt{r} \). This gives us the following lemma.

**Lemma 3.13.** For any lattice point \( q \in V_+ \oplus V_- \), we can write \( q = a + Tb \), with \( a \) and \( b \) lattice points, \( |a_+| < Q \sqrt{r} \) and \( |b_-| < UQ \sqrt{r} \), \( |a_-| \leq |q_-| + Q \sqrt{r} \) and \( |b_+| \leq \frac{|q_+|}{2} \). The same thing holds switching \( V_+ \) and \( V_- \) components if we substitute \( T^{-1} \) for \( T \).

Applying Lemma 3.13 to \( Mp \), we get \( Mp = a + Tp_1 \) with the appropriate bounds. Applying the lemma inductively to \( p_i \) gives

\[
M p = a + \sum_{i=1}^{\ell} T^i p_i,
\]

where \( |p_i| \leq (U + 1)Q \sqrt{r} \) and \( \ell \leq \log_L(M|p|) \). We can now also apply the \( T^{-1} \) case of Lemma 3.13 to \( a \) to get

\[
M p = \sum_{i=-\ell}^{\ell} T^i p_i,
\]

and the same bounds on \( p_i \) and \( \ell \) hold. (Although \( p_0 \) is a special case, we get \( |p_0| \leq |(p_0)_+| + |(p_0)_-| \leq 2Q \sqrt{r} \leq (U + 1)Q \sqrt{r} \). Hence

\[
|\alpha|_{\text{Lip}} \leq 2U(Q \sqrt{r}) \in O(\log |p|).
\]
This analysis holds in a slightly modified form for Lipschitz distortion, which is also exponential in this situation.

The situation is more complex when $T$ is non-diagonalizable with eigenvalues on the unit circle. If it is quasiunipotent, i.e. the eigenvalues are roots of unity, then we can take a power of it which is in fact unipotent. For a unipotent $T$, there is an eigenvector $\vec{v}$ and a $\vec{u}$ for which $T\vec{u} = \vec{u} + \vec{v}$. Then $k\vec{v} = T^k\vec{u} - \vec{u}$ and so $\vec{v}$ is infinitely distorted. With a little more work one can show that other vectors in the image of $T-I$ are also distorted, and that the Lipschitz distortion is a polynomial with degree depending on sizes of Jordan blocks. If $T$ has eigenvalues with irrational angles, then individual vectors are distorted because they are weakly distorted, by the argument given in Example 3.10(3). The precise distortion function is harder to ascertain, though the argument in Example 3.10(3) generalizes to the span of the eigenspaces of such a $T$.

Summarizing this section, we see that if $X$ is a finite complex and no subspace of $\pi_*(X)$ is infinitely distorted, then:

- $\pi_*(X) \otimes \mathbb{Q}$ is finite dimensional;
- the universal cover $\tilde{X}$ is rationally equivalent to a product of odd-dimensional spheres;
- and the action of $\Gamma$ on $\pi_*(X) \otimes \mathbb{Q}$ is elliptic.

We will refer to spaces that satisfy these conditions as delicate spaces.

4. Filling functions

As a tool to help us understand distortion, we will define certain higher-dimensional filling functions of spaces. The Dehn function of a group $\Gamma$ describes the difficulty of solving the word problem in that group; specifically, $\delta(k)$ is the minimal number of conjugates of relations required to trivialize a trivial word of length $k$. This has a geometric interpretation as the cellular volume of fillings of cellular loops in the Cayley 2-complex of $\Gamma$.

There are several different ways to generalize this notion to higher dimensions. Higher-dimensional Dehn functions were first defined by [AWP99], and filling volume functions by Gromov in [Gro96]; later, other equivalent and non-equivalent definitions of filling functions have been given by [BBFS09], [You11], and [Gro09].

For our purposes, we will only need what [You11] calls homological filling functions, which concern fillings of $n$-chains with $(n+1)$-cycles. In addition, we will define directed isoperimetric functions, which concern pairings of such fillings with cohomology classes.

**Definition.** Let $X$ be a compact space with fundamental group $\Gamma$ and $n$-connected fundamental cover $\tilde{X}$. Given a Lipschitz boundary $\beta \in C_n(\tilde{X})$, define the filling volume of $\beta$ to be

$$FVol^n_X(\beta) = \inf\{\text{vol}(\alpha) \mid \alpha \in C_{n+1}(\tilde{X}) \text{ s.t. } \partial \alpha = \beta\}$$

and the filling volume function

$$FVol^n_X(k) = \sup\{FVol^n_X(\beta) \mid \beta \in C_n(\tilde{X}) \text{ s.t. } \text{vol} \beta \leq k\}.$$

One can also restrict to boundaries that look like a specific $n$-manifold $N$ to get

$$FVol^n_X(k) = \sup\{\text{FVol}^n_X(\beta) \mid \beta \in C_n(\tilde{X}) \text{ s.t. } \text{vol} \beta \leq k \text{ and } \beta = f_*[N] \text{ for some } f : N \to \tilde{X}\}.$$

By restricting chains to be cellular and maps to be admissible, one can get similar cellular definitions. Note, however, that cellular and Lipschitz filling functions of $X$ might not be asymptotically equivalent as defined earlier; e.g. one may be linear and the other sublinear. Instead, we need a slightly weaker notion of coarse equivalence,

$$f \preceq_C g \iff f(k) \leq Ag(Bk + C) + Dk + E$$

for arbitrary constants $A$, $B$, $C$, $D$, and $E$. It is easy to apply the deformation theorem to see that these functions are indeed coarsely equivalent. It’s also easy to see that the cellular versions depend only on the $(n+1)$-skeleton of $X$, and that a homotopy equivalence up to dimension $n$ between $X$ and $Y$ induces a
coarse equivalence of filling functions. This gives a well-defined notion of $FV^n_\Gamma$ and $FV^N_\Gamma$ for any group $\Gamma$ of type $\mathcal{F}_{n+1}$, that is, with a $K(\Gamma, 1)$ with finite $(n+1)$-skeleton.

We will now refine the notion of filling volume to define filling homology classes.

**Definition.** Assume that $X$ is constructed via Lipschitz attaching maps and that $\tilde{X}$ is $(n+1)$-connected. Let $p : \tilde{X} \to X$ be the universal covering map. Given an $n$-manifold $M$ and an admissible map $f : M \to \tilde{X}$, define the *chain evaluation* $p$ on $M$ to be the cellular chain whose value on a cell $c$ is the degree of the map $p \circ f$ in $H_c(X, X \setminus c)$. Suppose $[f] = 0$. Then the *filling class* $\text{Fill}(f) \in H_{n+1}(X; \mathbb{Q})$ is the homology class of $pG$ for any homological filling $G$ of $f$. This is well-defined since two fillings differ by an $(n+1)$-boundary in $X$. Given a seminorm $\|\cdot\|$ on $H_{n+1}(X; \mathbb{Q}) = H_{n+1}(\Gamma; \mathbb{Q})$, define the *directed isoperimetric function* of $\Gamma$ with respect to $\|\cdot\|$ to be

$$FV^M_{\Gamma; \|\cdot\|}(k) = \sup \{ \text{Fill}(f) \mid f : M \to \tilde{X} \text{ admissible s.t. } [f] = 0 \text{ and } \text{vol } f \leq k \}.$$ 

If $b \in C_n(\tilde{X})$ is a cellular boundary with $p_b = 0$, then we can similarly define a filling class $\text{Fill}(b) \in H_{n+1}(X)$, and filling functions

$$FV^n_{\Gamma; \|\cdot\|}(k) = \sup \{ \text{Fill}(b) \mid b \in C_n(\tilde{X}) \text{ cellular s.t. } p_b = 0 \text{ and } \text{vol } b \leq k^{1/n} \}.$$ 

More generally, suppose $X$ is any finite complex with Lipschitz attaching maps, $p : \tilde{X} \to X$ is the universal covering map, and $\|\cdot\|$ is a seminorm on $H_{n+1}(X; \mathbb{Q})/p_*H_{n+1}(\tilde{X}; \mathbb{Q})$. Then a map $f : M \to \tilde{X}$ with $[f] = 0$, or a boundary $b \in C_n(\tilde{X})$ with $p_b = 0$ has a filling class $\text{Fill} f \in H_{n+1}(X; \mathbb{Q})/p_*H_{n+1}(\tilde{X}; \mathbb{Q})$, and we can define the filling functions $FV^M_{\Gamma; \|\cdot\|}$ and $FV^n_{\Gamma; \|\cdot\|}$ as above.

**Example 4.1.**

1. Define the *cellular norm* of $h \in H_{n+1}(X)$ by

$$\|h\|_{\text{cell}} = \min \left\{ \sum a_i \mid a_i c_i \text{ is a cellular representative of } h \right\}.$$ 

Then $FV^M_{\tilde{X}; \|\cdot\|_{\text{cell}}}$ is defined whenever $H_{n+1}(\tilde{X}) = 0$. Moreover, for any other seminorm $\|\cdot\|$ on $H_{n+1}(X)$, $\|\cdot\| \leq C\|\cdot\|_{\text{cell}}$ for some $C$.

2. Suppose that $e \in H^{n+1}(X; \mathbb{Q})$ is a cohomology class such that $p^*e = 0$. Then $\langle e, \cdot \rangle$ defines a seminorm on $H_{n+1}(X; \mathbb{Q})/p_*H_{n+1}(\tilde{X}; \mathbb{Q})$.

In order to compare usual and directed isoperimetric functions, we need to show that our definition maximizes over a large enough class of maps. Once this is done, it’s clear that for any $X$, $n$, and norm $\|\cdot\|$, $FV^n_{X; \|\cdot\|}(k) \leq FV^n_{\tilde{X}; \|\cdot\|}(k)$.

**Lemma 4.2.** Let $X$ be a finite complex with admissible boundary maps.

1. Let $f : S^n \to \tilde{X}$ be an admissible map of volume $k$ with $[f]$ a boundary. Then $f$ can be deformed via a homotopy with $(n+1)$-volume $Ck$ to an admissible map $g$ of volume $Ck$ with $[g] = 0$.

2. Let $b \in C_n(\tilde{X})$ be a boundary of volume $k$. Then $b$ is homologous via an $(n+1)$-chain of volume $Ck$ to a boundary $c$ of volume $Ck$ with $p_c = 0$.

**Proof.** We only prove (1); the proof of (2) is similar.

We work in $X$, since all the maps we are considering lift to $\tilde{X}$. We know $\|f\|_{\text{cell}} \leq \text{vol } f \leq Ck$. Since $X$ is finite, there’s an $(n+1)$-chain $c$ with $\partial c = [f]$ and $\|c\|_{\text{cell}} \leq Ck$.

Let $a$ be the maximal volume of an attaching map $f_i$ of an $(n+1)$-cell $c_i$. Then a map $D^n \to X$ which takes the disk to a balloon starting at a basepoint and with head $f_i$ is has volume at most $a$. By mapping the upper hemisphere of $S^n$ to $X$ via $Ck$ balloons corresponding to the cells of $-c$ and the lower hemisphere via $f$, we create a map $g$ of volume $Ck$ with $[g] = 0$. Since each $f_i$ can be nullhomotoped through $c_i$, this map is homotopic to $f$ via a homotopy with volume given by $\|c\|_{\text{cell}}$. \qed

This also allows us to show that directed isoperimetric inequalities satisfy the same invariance results as the usual kind.

**Proposition 4.3.** Given a norm $\|\cdot\|$ on $H_{n+1}(\Gamma; \mathbb{Q})$, $FV^n_{\Gamma; \|\cdot\|}$ depends up to coarse equivalence only on $\Gamma$, justifying the notation.
Proof. Suppose \( X \) and \( Y \) are two complexes with fundamental group \( \Gamma \) and \((n+1)\)-connected fundamental cover. In particular, we can find a cellular map \( h : X^{(n+2)} \to Y \) which induces an isomorphism on \( H_{n+1} \).

Now suppose \( f : S^n \to X \) is an admissible map of volume \( k \) with \( |\tilde{f}| = 0 \). Then \( h \circ f \) has volume \( Ck \) and is homotopic to a map \( g \) of volume \( Ck \) which is admissible with the same degrees on \( n \)-cells, so that \( |\tilde{g}| = 0 \) also. Moreover, an admissible filling of \( f \) gives a corresponding cellular filling of \( g \), and since \( h \) is induced by the corresponding map of chain complexes, which lifts to that on \( X^{(n+2)} \), \( h_* \text{Fill}(g) = \text{Fill}(f) \). \( \square \)

It remains to cite some general results about filling volumes. First, we show that in dimensions other than 2, every cellular boundary is induced by a map \( f : S^n \to X \); we say that every boundary is spherical. This means that isoperimetric functions are equivalent whether or not we require boundaries to be spherical.

Lemma 4.4. Let \( X \) be a finite complex with universal cover \( \tilde{X} \) and \( n \geq 3 \). Then for every integral boundary \( c \in C_n(\tilde{X}) \), there is an admissible \( f : S^n \to X \) with \( f_\#(\langle S^n \rangle) = c \) and no cells of opposite orientations. In particular, \( \text{FV}^c_{S^n} \sim \text{FV}^c_X \) and for every norm \( \|\cdot\| \) on \( H^{n+1}(X) \), \( \text{FV}^c_{S^n,\|\cdot\|} \sim \text{FV}^c_X,\|\cdot\| \).

Proof. This proof generalizes Remark 2.6(4) in [BBFP09]. Let \( c \) be a boundary in \( C_n(\tilde{X}) \), and take an admissible map \( g : (D^n, S^{n-1}) \to (\tilde{X}, \tilde{X}^{(n-1)}) \) such that \( g([D^n]) = c \) and with no cells of opposite orientations, for example, by mapping the boundary to a sum of attaching maps. Then \( g_\#([D^n]) = 0 \in H_n(\tilde{X}, \tilde{X}^{(n-1)}) \), and thus, by the relative Hurewicz theorem, \( [g] = 0 \in \pi_n(\tilde{X}, \tilde{X}^{(n-1)}) \). This means \( \partial g = 0 \in \pi_{n-1}(\tilde{X}^{(n-1)}) \), and so, taking the disk to be the upper hemisphere of \( S^n \), \( g \) extends to a map \( f : S^n \to X \) with the desired properties. \( \square \)

Finally, it’s worth remarking that for groups, though not for all spaces, homological filling functions are always finite.

Lemma 4.5. Let \( \Gamma \) be a group of type \( \mathcal{F}_n \). Then for every \( 1 \leq t \leq n \), \( \text{FV}^t_t(k) < \infty \).

Proof. A consequence of Theorem 1 of [AWP99] is that \( \delta^t_t(k) < \infty \) for every \( 1 \leq t \leq n \). By Lemma 1.3 for \( t \geq 3 \), \( \text{FV}^t_t(k) < \delta^t_t(k) \), which completes the proof. The same is true for \( t = 1 \), since every 1-cycle is a union of circles. For \( t = 2 \), it may be harder to fill a surface of positive genus than a sphere. However, consider an admissible map \( f : \Sigma_g \to B\Gamma \). We can give \( \Sigma_g \) a Riemannian metric with volume form \( d\text{vol} \) so that \( f \) has Lipschitz constant 1 and \( \int_{\Sigma_g} d\text{vol} \leq 2 \text{vol} f \). By Gromov’s systolic inequality for surfaces (Theorem 11.3.1 in [Kat07]) there is a nontrivial loop \( \gamma : S^1 \to \Sigma_g \) of length at most \( C \log g \sqrt{g^{-1} \text{vol} f} \). Let \( g : \Sigma_g \to B\Gamma \) be a map which coincides with \( f \) on \( \Sigma_g \setminus \gamma(S^1) \) and with a filling of \( f \circ \gamma \) on the disks surgering \( \gamma \). Then a filling of \( g \) is also a filling of \( f \), and hence

\[
\text{FV}^{\Sigma_g}_{\Gamma^t}(k) \leq \text{FV}^{\Sigma_g}_{\Gamma^t}(k + 2\delta^t_t(C \log g \sqrt{g^{-1}k})).
\]

For a given \( k \), there is a maximum \( G \) for which \( C \log G \sqrt{G^{-1}k} \geq 1 \). For any \( g > G \), then, and for any admissible \( f : \Sigma_g \to B\Gamma \), there is a nontrivial loop \( \gamma : S^1 \to \Sigma_g \) whose image is constant, allowing us to surger with no penalty. Hence \( \text{FV}^2_{\Gamma^t}(k) \leq \text{FV}^{\Sigma_G}_{\Gamma^t}(k) \), which in turn is bounded, by induction on \( g \). \( \square \)

To conclude the section, we give some examples of directed isoperimetric functions.

Examples 4.6. (1) Suppose \( \Gamma \) is a group of type \( \mathcal{F}_{k+1} \) with Dehn function \( f(k) \). By a theorem of [You11], for \( n \geq 2 \), \( \text{FV}^n_p(k) \geq f(k) \) because if \( \gamma \) is hard to fill in \( \Gamma \), then \( \gamma \times \cdots \times \gamma \) is hard to fill in \( \Gamma^n \). However this doesn’t tell us anything about directed isoperimetric functions, because if \( D \) is a filling of \( \gamma \), then \( D \times \gamma \times \cdots \times \gamma \) is a filling of \( \gamma \times \cdots \times \gamma \) which is zero in \( H_{n+1}(\Gamma^n) \).

(2) Suppose \( M^{n+1} \) is a closed oriented smooth manifold with fundamental group \( \Gamma \). By dualizing a handle decomposition, we see that filling an \( n \)-boundary in \( \tilde{M} \) is equivalent to finding a 0-cochain cobounding a compactly supported 1-cochain in the Cayley graph of \( \Gamma \). Thus any \( n \)-boundary \( b \) has a unique filling; moreover, \( b = \partial a + b_- \), where \( \text{vol} b = \text{vol} \partial a + \text{vol} b_- \) and \( b_- \) has a filling by positively oriented copies of the top cell while \( b_- \) has a filling by negatively oriented copies. Thus \( \text{FV}^n_m(k) \), \( \text{FV}^n_{\mathcal{M}(|[M],\cdot|)}(k) \), the isoperimetric problem for domains in \( \tilde{M} \), and the boundary problem for sets in \( \Gamma \) are all equivalent. In particular, \( \text{FV}^n_m(k) \sim \text{FV}^n_{\mathcal{M}(|[M],\cdot|)}(k) \) is linear if and only if \( \Gamma \) is non-amenable.
Let the \( n \)th diamond group be
\[
D_n = \langle b_1(1), b_1(2), \ldots, b_n(1), b_n(2), a \mid b_j(i)^{-1}ab_j(i) = a^2, [b_j(i), b_j(k)] = 0 \text{ for } j \neq \ell \rangle.
\]
We can think of this as \( F_2^n \) with an extra generator and some relations, or we can define \( D_n \) inductively by setting \( D_0 = \mathbb{Z} \) and \( D_n \) to be a multiple HNN extension of \( D_{n-1} \), specifically, the fundamental group of the graph of groups with a single vertex \( D_{n-1} \) and two edges each labeled by the automorphism \( a \mapsto a^2, b_j(i) \mapsto b_j(i) \). This last definition gives a construction for a \( (n+1) \)-dimensional classifying complex \( X_n \) for \( D_n \), starting with an \( S^1 \) with one 1-cell and setting \( X_n \) to be the appropriate quotient space of \( X_{n-1}(0,1) \times [0,1] \) with the product cell structure.

The space \( X_n \) has \( 2^n (n+1) \)-cells \( e_I \) corresponding to elements \( I \in \{1,2\}^n \). It’s easy to see that the only \( (n+1) \)-cycles are multiples of \( \sigma = \sum_{I \in \{1,2\}^n} (-1)^{|I|} e_I \), and so \( H_{n+1}(X_n) \cong \mathbb{Z} \).

**Theorem 4.7.** Let \( h \neq 0 \in H^{n+1}(D_n) \). Then \( FV_{D_n}^h(k)(k) \sim FV_{D_n}^h(k) \sim 2 \sqrt{\pi} \).

**Proof.** Let \( \rho_n : D_n \to D_n \) be the monodromy homomorphism \( a \mapsto a^2, b_j(i) \mapsto b_j(i) \) used in the construction of \( D_{n+1} \). We write \( I_1 \) and \( I_2 \) for the two intervals used to construct \( X_n \) from \( X_{n-1} \). Also let \( Z \) be the union of the two copies of \( X_{n-1}(1/2) \) in \( X_n \) and \( \tilde{Z} \) its preimage in the universal cover \( \tilde{X} \).

Finally, notice that \( \tilde{X}_n \) consists of glued-together copies of \( X_{n-1}(0,1) \), which we call layers \( Y_e \), indexed by edges of the Bass-Serre tree corresponding to the graph of groups. For each layer, let \( Z_e \) be the corresponding copy of \( X_{n-1} \times \{1/2\} \).

To show that \( 2 \sqrt{\pi} \leq FV_{D_n}^h(k)(k) \), we construct, by induction on \( n \), a chain \( \tau_n(k) \) in \( \tilde{X}_n \) which descends to \( K \), where \( 2^k < K < 2^{n+k} \), and whose boundary has volume \( O(k^n) \). In \( D_1 \), this is similar to the usual demonstration that \( BS(1,2) \) has exponential Dehn function; namely, we take \( \tau_1(k-1) \) to be the disk bounded by \( b_1(1)^{-k}ab_1(1)^k b_1(2)^{-k}a^{-1}b_1(2)^k \). Notice that \( \rho_1(\tau_1(k-1)) \) gives a disk that differs from \( \tau_1(k) \) only by two cells.

Now suppose we have constructed \( \tau_{n-1}(k) \), and that \( \rho_{n-1}(\tau_{n-1}(k-1)) \) gives us a chain that differs from \( \tau_{n-1}(k) \) by \( O(k^{n-2}) \) cells. We construct \( \tau_n(k) \) from lifts of \( (-1)^j \tau_{n-1}(j) \times I_1 \) for \( j = 1, \ldots, k \) and \( \ell = 1, 2 \), such that the two copies of \( \tau_{n-1}(k) \times \{1\} \) cancel out and \( \tau_{n-1}(j-1) \times \{0\} \) cancels out with \( \tau_{n-1}(j-1) \times \{1\} \), except for the aforementioned \( O(k^{n-2}) \) cells. Thus in total
\[
\text{vol}(\partial \tau_n(k)) \leq 2kO(k^{n-1}) + 2(k-1)O(k^{n-2}) + 2^{1+2n},
\]
where the first term comes from each \( \partial \tau_{n-1}(j) \times I_1 \) and the last term comes from the two copies of \( \tau_{n-1}(1) \times \{0\} \). Moreover, for each layer, \( \rho_n(\tau_{n-1}(j) \times I_1) \) differs from \( \tau_{n-1}(j+1) \times I_1 \) by \( O(k^{n-2}) \) cells. Thus \( \rho_n(\tau_n(k)) \) differs from \( \tau_n(k+1) \) by \( 2kO(k^{n-2}) + 2^{n+1} = O(k^{n-1}) \) cells. This completes the inductive step.

To show that \( FV_{D_n}^h(k) < 2 \sqrt{\pi} \), we do another induction. It’s clear that \( FV_{D_n}^h(k) \sim 2^k \). For the inductive step, we adapt and strengthen the argument of [BBFS09]’s Theorem 7.2. This theorem states that for any \( n \)-manifold \( M \) and group \( H \) which is a multiple HNN extension, the filling volume of a map \( f : M \to BH \) is the sum of the filling volumes of layers. Specifically, in the case of \( H = D_n \), given \( f : M \to \tilde{X}_n \) admissible and transverse to \( \tilde{Z} \), and setting \( N_e := f^{-1} \tilde{Z}_e \) and \( g_e = f|N_e \), we have
\[
FV(f_*[M]) = \sum e FV(g_e_*[N_e]).
\]
This formulation is obtained by taking the minimum of homotopical Dehn functions over all relevant geometries. On the other hand, \( \text{vol} f = \sum \text{vol } N_e + H \), where \( H \) stands for any “horizontal” volume which is the difference between fillings of the various \( N_e \) meeting at a given vertex of the Bass-Serre tree.

Now suppose that \( \text{vol } f = 2^k \) and \( \text{vol } f_k \leq k^n \). In particular, for some \( e_0 \), the \( \text{vol } g_{e_0} \geq 2^k / CK^n \sim 2^k \). By inductive assumption, \( \text{vol } g_{e_0} \geq k^{n-1} \). Moreover, suppose \( e_1 \) is the edge adjacent to \( e_0 \) for which \( \text{vol } g_{e_0} \) is greatest. Then \( \text{vol } g_{e_0} \leq 10 \text{Vol } g_{e_1} \), because \( \rho \) multiplies areas by at most 2 and vertices in the tree have degree 6.

Now assume that \( k \) is large enough that \( 2^{k/2} > (\text{vol } f)^2 \), and so if we pick \( e_{j+1} \) adjacent to \( e_j \) on the opposite vertex from \( e_{j-1} \), again such that \( \text{vol } g_{e_{j-1}} \) is greatest, then for \( j \leq k/8 \),...
FVol $N_{ij} \geq 2^{k-4j}/Ck^n$ and thus $\text{vol} g|_{N_{ij}} \geq (k/2)^{n-1}$. Thus $\text{vol} f \geq \sum_{j=1}^{k/8} \text{vol} g|_{N_{ij}} \geq k^n$. This completes the induction and the proof. \hfill \Box

5. \textit{L}_\infty \textit{ COHOMOLOGY AND FILLINGS}

Correspondences between isoperimetry and the cohomology theories that turn up in coarse geometric settings have been noted a number of times in the literature, notably by Block and Weinberger [BW92], Attie, Block and Weinberger [ABW92], Gersten [Ger94], and Nowak and Spakula [NS10]. The main technical theorem of this section generalizes most of these results using a technique from the theory of algorithms, the duality theorem for linear programming problems. In effect, a linear programming problem seeks to optimize a linear function subject to a number of linear constraints. In the dual linear program, the role of the constraints and variables is switched. The theorem of linear programming duality states that the optimum solution to the original and dual programs is the same. Our proof proceeds by translating our two conditions into this formal setting and demonstrating that they generate dual linear programs and are therefore equivalent. For a more detailed discussion of these ideas, see a textbook on algorithms, such as [CLR01].

One can think of this theorem as a greatly expanded generalization of the classical max flow–min cut theorem from graph theory. For a very differently flavored application of linear programming duality to isoperimetric problems, see [KKI13].

\textbf{Theorem 5.1 (Isoperimetric duality).} Suppose $Y$ is a metric CW complex in which balls intersect a finite number of cells, and write $E_n(Y)$ for the set of $n$-cells of $Y$. For $F = \mathbb{Q}$ or $\mathbb{R}$, let $\omega \in C^{n+1}(Y; F^r)$ be any cocycle. For each $e \in E_n(Y)$, fix a (perhaps asymmetric) polyhedral norm $N_e$ on $F^r$, and let $N'_e$ be the dual norm on $(F^r)^*$. Then the following are equivalent:

1. for all chains $\sigma \in C_{n+1}(Y; (F^r)^*)$, $\langle \omega, \sigma \rangle \leq \sum_{e \in E_n(Y)} N'_e(\partial \sigma(e))$;
2. $\omega = d\alpha$ for a cochain $\alpha \in C^n(Y; F^r)$ with $N_e(\langle \alpha, e \rangle) \leq 1$ for every $e \in E_n(Y)$.

\textbf{Proof.} Fix $R$, let $*$ be a basepoint in $Y$, and let $\{e_i : i \in I\}$ be an enumeration of the $(n+1)$-cells that intersect the open ball $B_R(\ast)$, and $\{f_j : j \in J\}$ be an enumeration of the $n$-cells which either intersect $B_R(\ast)$ or are incident to $e_i$ for some $i$. We denote the coefficient of $f_j$ in $\partial e_i$ by $\partial_j e_i$. For each $j$, we can write the norm $N_{f_j}$ as

$$N_{f_j}(v_1, \ldots, v_r) = \max \left\{ \sum_{k=1}^r c(j, \ell)k v_k : \ell = 1, \ldots, L_j \right\}$$

for constants $L_j$ and constant vectors $c(j, \ell)$. With these notations in place, condition (2) holds restricted to $B_R(\ast)$ if and only if the linear programming problem

$\text{(5.2) maximize } \sum_{i \in I} \sum_{k=1}^r x_{i,k} \text{ subject to }$

$$\begin{cases} \text{for } i \in I, 1 \leq k \leq r, & 0 \leq x_{i,k} \leq |\langle \omega, e_i \rangle| & (A_{i,k}) \\ \text{for } i \in I, 1 \leq k \leq r, & x_{i,k} = \text{sign}(\langle \omega, e_i \rangle) \sum_{j \in J} \partial_j e_i \alpha_{j,k} & (C_{i,k}) \\ \text{for } j \in J, 1 \leq \ell \leq L_j, & \sum_{k=1}^r c(j, \ell) k \alpha_{j,k} \leq 1, & (B_{j,\ell}) \end{cases}$$

has the maximal possible solution, $x = \sum_{i \in I} \|\langle \omega, e_i \rangle\|_1$. By linear programming duality, this is equivalent to $x$ being the

$$\text{minimal value of } \sum_{i \in I} \sum_{k=1}^r |\langle \omega, e_i \rangle| A_{i,k} + \sum_{j \in J} \sum_{\ell=1}^{L_j} B_{j,\ell} \text{ subject to }$$

$$\begin{cases} \text{for } i \in I, 1 \leq k \leq r, & A_{i,k} + C_{i,k} \geq 1 & (x_{i,k}) \\ \text{for } j \in J, 1 \leq k \leq r, & \sum_{\ell=1}^{L_j} c(j, \ell) k B_{j,\ell} = \sum_{i \in I} \text{sign}(\langle \omega, e_i \rangle) \partial_j e_i C_{i,k} & (a_{j,k}) \\ \text{for } i \in I, 1 \leq k \leq r, & A_{i,k} \geq 0 & \text{for } j \in J, 1 \leq \ell \leq L_j, & B_{j,\ell} \geq 0. \end{cases}$$

$$\text{(5.3)}$$
Then $\Gamma = B$ group, by Corollary 1.8 but not of an aspherical complex with finite skeleta \cite{BB97}.

in the dual basis to the standard basis. Then the right side of $(\alpha_{j,k})$ adds up to $\partial \sigma(f_j)_k$. So picking the $B_{j,\ell}$ amounts to adding together nonnegative multiples of the vectors $\tilde{c}(j, \ell)$ for each $\ell$, and $\sum_{\ell} B_{j,\ell}$ is the sum of these coefficients, i.e. the dual norm $N^*_{\ell}(\partial \sigma(f_j))$. Thus, keeping in mind that $A_{i,k} \geq 1 - C_{i,k}$, we can bound the quantity $M$ being minimized in \cite{53} as

$$M \geq \sum_{i \in I} \sum_{k=1}^r |(\omega, e_i)_k| (1 - C_{i,k}) + \sum_{j \in J} \sum_{j=1}^{l_j} B_{j,\ell} = x - \langle \omega, \sigma \rangle + \sum_f N'_{f}(\partial \sigma(f)).$$

If (1) is true, then $M \geq x$ for every $R$ and choice of $\sigma$. On the other hand, if (1) is not true, we can scale any counterexample $\sigma$ so that $|C_{i,k}| \leq -1$ and so we can set $A = 1 - C_{i,k}$ and $M \leq x$. This demonstrates that (2) implies (1), and that (1) implies that for every $R$ there is an $\alpha_R$ which satisfies (2) when restricted to $B_R(\ast)$. To get an $\alpha$ as desired on all of $Y$, we take a weak-* accumulation point of the $\alpha_R$. \hfill$\Box$

Mutatis mutandis, the proof gives a similar theorem for chains with the same kind of bound. There are a number of corollaries obtained through specific choices of norms.

**Corollary 5.2.** The following are equivalent for a cochain $\omega \in C_{n+1}(Y; \mathbb{F}^r)$:

1. there is a constant $K$ such that for all chains $\sigma \in C_{n+1}(Y; \mathbb{F})$, $\|\omega, \sigma\|_\infty \leq K \text{vol}(\partial \sigma)$;
2. $\omega$ is an $L_\infty$ coboundary.

**Proof.** The corollary, as well as the next one, is actually obtained by applying the theorem separately to each coordinate of $\omega$. \hfill$\Box$

This generalizes the theorem for 0-chains proved in \cite{BW92} and its Poincaré dual mentioned in \cite{ABW92}. Similarly, if we take $r = 1$, choose a basepoint $\ast \in Y$ and define $N_{\ast}(x) = |x/f(d(\ast, e))|$ for some non-decreasing function $f : [0, \infty) \to [0, \infty)$, we recover an analogous theorem for the groups $H^1_n$ introduced by \cite{NS10}, generalizing their Theorem 4.2:

**Corollary 5.3.** The following are equivalent for a chain $\sigma \in C_{n+1}(Y, \ast; \mathbb{F}^r)$:

1. there is a constant $K$ such that for all cochains $\omega \in C_{n+1}(Y; \mathbb{F})$, $\|\omega, \sigma\|_\infty \leq K \sum e f(d(\ast, e))|d\omega(e)|$;
2. $\sigma$ is a boundary in $H^1_n(Y; \mathbb{F})$.

6. Finite approximations of Postnikov towers

Suppose now that $X$ is a delicate space, and once again let $\Gamma = \pi_1 X$. By general nonsense, the inclusion $X \to B\Gamma$ is homotopy equivalent to a fibration with fiber rationally homotopy equivalent to $\prod_{i=1}^r S^{2n_i+1}$. Indeed, up to rational homotopy $B\Gamma$ also has finite skeleta.

**Lemma 6.1.** Suppose $X$ is a finite complex and $\tilde{X}$ is rationally equivalent to a complex $F$ with finite skeleta. Then $\Gamma = \pi_1 X$ is of type $F_{\infty}(\mathbb{Q})$, i.e. $B\Gamma$ is rationally equivalent to a complex with finite skeleta.

**Proof.** Let $f : X \to B\Gamma$ be the canonical map. Since for every $n$, $\pi_n(f) \cong \pi_{n-1}(F)$ is finitely generated as a group, by Corollary \cite{68} $B\Gamma$ is rationally equivalent to a complex with finite skeleta. \hfill$\Box$

Note that $B\Gamma$ need not itself have finite skeleta. Thus for example, if $L$ is a flag triangulation of $\Sigma \mathbb{R}P^2$, then the Bestvina-Brady group $H_L$ is finitely presented and the fundamental group of a $\mathbb{Q}$-aspherical 3-complex, but not of an aspherical complex with finite skeleta \cite{BB97}.

In our case, for any $N$, the lemma gives us a sequence of compact spaces

$$\prod_{i=1}^r S^{2n_i+1} \to X \to B\Gamma^{(N)} ,$$

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which maps to a fibration via rational $N$-equivalences. We can pick $N$ sufficiently large that in a desired range, these maps rationally obey the homotopy exact sequence of a fibration and the Serre spectral sequence. Moreover, the fibration splits into a tower of fibrations by products of same-dimensional spheres which is rationally equivalent to a Postnikov tower.

We study the total space $X$ by studying each step of this tower. Thus in the rest of this section, a \textit{quasi-Postnikov $n$-pair}, for $n$ odd, will denote finite complexes $(X, B)$ and a map $\pi : X \to B$ such that $(S^n)^* \to X \to B$ is a rational homotopy fibration up to a large dimension $N$, and such that the universal covering fibration $\bar{X} \to \bar{B}$ is trivial up to rational homotopy. We start by defining invariants for such pairs.

**Definition.** Let $K(Q, n)^* = F \to X \to B$ be a fiber bundle with monodromy representation $\rho : \pi_1 B \to GL(H_n(F; Q))$, with a corresponding $Q\pi_1B$-module $M_\rho$. This defines a bundle of groups $\mathcal{H}(\rho)$ over $B$ with fiber $H_n(F; Q)$. Define a class $eu \in H^{n+1}(B; \mathcal{H}(\rho)) = H^{n+1}(B; M_\rho)$ as the obstruction to lifting a singular $(n+1)$-simplex in $B$ to $X$, given a lifting of the singular chain complex through dimension $n$. By analogy, we refer to this as the \textit{Euler class} of the bundle.

We need to show that the Euler class is well-defined, that is,

1. the cochain is a cocycle;
2. the choice of lifts of $n$-simplices determines it up to a coboundary.

We can assume fixed lifts for simplices in $C_*(B)$ for $* \leq n - 1$. Notice then that two ways to lift $n$-simplices giving representatives $c_1$, $c_2$ of the Euler class give us a cocycle $b \in C^n(B; M_\rho)$ with $\partial b = c_1 - c_2$. This proves (2). For (1), let $f : (\Delta^{n+2}) \to B$ be a simplex, and choose a lifting of the singular chain complex through dimension $n$ which gives $g : \Delta^n \to B$ to $\bar{g}$, which induces an obstruction cocycle $c$. Finally, let $f_i$ be the restriction of $f$ to the $(n+1)$-simplex opposite vertex $v_i$, and let $f_{ij}$ be the restriction to the $n$-simplex which does not include $i$ or $j$. To show that $eu(\partial f) = 0$, we lift the $n$-simplices containing $v_0$, and then extend by homotopy lifting to a map $F$ on the star of $v_0$. This gives a way of lifting $\partial f_0$ which is nullhomotopic again via the homotopy lifting property. Moreover, $c(f_0) = \sum_{i=1}^{n+2} [F_{ij} - \bar{f}_{ij}] = -\sum_{i=1}^{n+2} [c(f_i)]$, completing the proof.

Now suppose that $\pi : X \to B$ is a quasi-Postnikov pair of rational homotopy type as above. In order to work within the finite context, we would like to come up with a cellular representative of the Euler class in $C^n(X; M_\rho)$. So let $Z_\pi$ be the mapping cylinder of $\pi$, and let $j_\pi : C_n(Z_\pi; Q) \to C_n(X; Q)$ be a lifting homomorphism. By Lemma 23, some multiple of the attaching map $f$ of each $(n+1)$-cell of $B$, say $p_n f$ lifts to a $g : S^n \to X$ with $g[S^n] = p_n j_\pi(f[S^n])$. This gives us an element $[g]/p_n \in \pi_n(X, j_\pi(*) \otimes Q)$ associated to the cell, where $*$ is the basepoint of the attaching map. After pulling back to the fiber, we have our cellular representative.

As in the untwisted case, this Euler class relates to the Serre spectral sequence of the bundle.

**Definition.** Let $K(Q, n)^* = F \to X \to B$ be a fiber bundle with monodromy representation $\rho : \pi_1 B \to GL(H_n(F; Q))$. The $E_{n+1}$ page of the rational cohomology spectral sequence for $X$ includes a differential,

$$d_{n+1} : H^0(B; H^n(F; M_\rho)) \to H^n(F; Q)^\rho \to H^{n+1}(B; Q)$$

which determines the rest of the differential and multiplicative structure on the $E_{n+1}^{s,t}$. Define a class $\varepsilon \bar{\pi} \in H^{n+1}(B; H_n(F; Q)/\rho)$ by taking $\bar{\pi} (\alpha) = \bar{\psi}$ such that $d_{n+1} \omega(\alpha) = \omega(\bar{\psi})$ for every $\omega \in H^n(F; Q)^\rho$. Note also that this map is the dual differential in the homology spectral sequence.

Notice that in the range $0 \leq k < n$, the cohomology of such a bundle obeys a Gysin sequence

$$\cdots \to H^{n+k}(B; Q) \to H^{n+k}(E; Q) \to H^k(B; H^n(F; M_\rho)) \xrightarrow{\varepsilon \bar{\pi}} H^{n+k+1}(B; Q) \to \cdots,$$

where $\alpha \leadsto \varepsilon \bar{\pi}$ is defined via the natural fiberwise pairing between $H^n(F; M_\rho)$ and $H_n(F; M_\rho)$. Dually, there is also a long exact sequence in homology in this range, which ends with

$$\cdots \to H_{n+1}(B; Q) \xrightarrow{\varepsilon \bar{\pi}} H_n(F; Q)/\rho \to H_n(E; Q) \to H_n(B; Q) \to 0.$$

After this series of remarks, we would do well to justify our notation.

**Proposition 6.2.** The projection $H^{n+1}(B; M_\rho) \to H^{n+1}(B; H_n(F; Q)/\rho)$ sends $\varepsilon \bar{\pi}$. 

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Proof. We will first show that for a fiber bundle \( F \xrightarrow{i} X \xrightarrow{\pi} B \) with rationally \((n - 1)\)-connected fiber and path-connected base, and mapping cylinder \( Z_n = (X \times [0, 1]) \cup_f B \simeq B \), the diagram

\[
\begin{array}{ccc}
H_{n+1}(B; \mathbb{Q}) & \xrightarrow{\varphi} & H_{n+1}(Z_n, X; \mathbb{Q}) \\
\downarrow{\partial_{n+1}} & & \downarrow{\theta} \\
H_n(F; \mathbb{Q})/\rho & & H_n(X; \mathbb{Q})
\end{array}
\]

where \( \varphi \) is induced by the inclusion \( B \hookrightarrow Z_n \), commits and the map \( \theta \), which takes a chain \( c \mapsto i(c) \times [0, 1] \), is an isomorphism.

The triangle on the right is commutative by definition. Now we look at the differential \( \partial_{n+1} \) on the chain level. Let \( c \in C_{n+1}(B; \mathbb{Q}) \) be a singular cycle; then any simplexwise lift \( \tilde{c} \) of \( c \) to \( X \) is a representative of the corresponding class in \( F_{n+1}(X; \mathbb{Q}) \). Since \( \partial_2, \ldots, \partial_n \) are zero, \( [\tilde{c}] \) contains a chain whose boundary \( a \) is in \( F_0C_n(X; \mathbb{Q}) \), that is each simplex of \( a \) projects to a single point in \( B \); in other words there is a chain \( b \in C_{n+1}(X; \mathbb{Q}) \) such that \( \pi_\#(b) \) is degenerate and \( \partial(\tilde{c} + b) = a \). This \( a \) is a representative of \( \partial_{n+1}(c) \). Indeed, since \( E \) is path-connected, we can assume that all simplices of \( a \) map to the same point in \( B \), so that \( a \) is a chain in \( c \). Thus in \( C_*(Z_n; \mathbb{Q}) \),

\[
\partial((\tilde{c} + b) \times [0, 1]) = (\tilde{c} + b) \times \{0\} - \tilde{c} + \theta(a),
\]

so that \( c \) and \( \theta(a) \) are in the same class in \( H_{n+1}(Z_n, X; \mathbb{Q}) \). Thus \( \theta \circ \partial_{n-1} \) is the natural map \( H_{n+1}(B; \mathbb{Q}) \to H_{n+1}(Z_n, X; \mathbb{Q}) \).

On the other hand, given a way of lifting singular \( n \)-simplices in \( B \) to \( X \), we can take \((n+1)\)-simplices in \( B \) to simplices in \( Z_n \) with boundaries in \( X \times \{0\} \). As a lift of \( C_n(B; \mathbb{Q}) \) for \( n \leq n+1 \), this clearly chain homotopic to the inclusion map which gives us \( \varphi \). Thus the projection \( H^{n+1}(B; M_n) \to H^{n+1}(B; H_{n+1}(Z_n, X; \mathbb{Q})) \) sends \( eu \mapsto \varphi \).

Using the Euler class, we can characterize distortion in the total space in terms of fillings of spheres in the base. Indeed, the Euler class of a rational fibration is a third source of distortion, in addition to the two discussed in the previous section.

**Theorem 6.3.** Let \( F = (S^n)^r \xrightarrow{i} X \xrightarrow{\pi} B \) be a quasi-Postnikov pair with \( \pi \) an injective cellular map. Let \( j_* : C_*(\tilde{B}; \mathbb{Q}) \to C_*(\tilde{X}; \mathbb{Q}) \) be a lifting homomorphism, and \( \omega_i \) be the representative of the Euler class induced by \( j_* \). Then there are constants \( 0 < c \leq 1, p_n(j_*) \geq 1 \) such that for every \( \alpha \in \pi_\#(F; \mathbb{Q}) \cong H_n(F; \mathbb{Q}) \),

\[
(6.1) \quad \frac{c}{p_n} |p_n j_* \alpha|_{vol} \leq \min\{vol(f) \mid f : S^n \to B \text{ admissible and } \omega_j(Fill(f)) = \alpha\} \leq C |\alpha|_{vol} + C.
\]

**Proof.** The right inequality is true since any map \( f : S^n \to B \) deforms to an admissible one. Conversely, for any \( f : S^n \to B \) we can apply Lemma 2.3 to concoct a map \( g : S^n \to X \) representing \( po \) with \( g_\#[S^n] = p_n j_n(f_\#[S^n]) \) and \( \text{vol} g \leq C_n \text{vol} f \), where \( p_n \) and \( C_n \) depend only on \( j_n \). \( \square \)

The spirit of this result is that, for a general quasi-Postnikov pair \( X \to B \), an Euler class together with a way of lifting boundaries of \((n + 1)\)-cells from \( B \) to \( X \) gives us a way of converting small maps \( S^n \to B \) with big homological fillings into small representatives of big classes in \( \pi_* \) (\( X \)); thus the left inequality holds even if \( \pi \) is not injective, but it’s not a priori clear in what sense it’s sharp. We hope to clarify this principle with corollaries and examples.

First of all, homological isoperimetric functions give us a bound on how distorted classes may be.

**Corollary 6.4.** If \( X \) is built as a fibration \((S^n)^r \to X \to B \), then for any \( \alpha \in \pi_n X, V\delta_{\alpha}(k) \lesssim FV_B^\alpha(k) \).

When the monodromy is trivial, it’s possible to eliminate the ambiguity of the lift by only considering maps \( f : S^n \to B \) with chain evaluation \([f] = 0 \). In particular, when \( F = S^n \), we have the following neater characterization:

**Theorem 6.5.** Suppose that \( B \) is a finite CW-complex, \( n \) odd, and let \( S^n \xrightarrow{i} X \to B \) be a quasi-Postnikov pair with Euler class \( eu \neq 0 \). Let \( \alpha \) generate \( \pi_n(S^n) \). Then \( i_* \alpha \) has distortion function \( V\delta_{\alpha}(k) \sim C FV_B^{S^n}_{(eu, \alpha)}(k) \). In particular, this is true if \( B = BG \) for some group \( \Gamma \).
Proof. By taking a double-cover we can assume that the monodromy is trivial. Lemma \[4.2\] makes the Euler class unambiguous at the cost of an extra multiplicative constant, giving
\[
\frac{c}{p} |\pi_* m\alpha|_{\text{vol}} \leq \min\{\text{vol}(f) \mid f : S^n \to B \text{ admissible with } [\tilde{f}] = 0 \text{ and } \text{eu}(\text{Fill}(f)) = m\alpha\} \leq C |\pi_* m\alpha|_{\text{vol}} + C.
\]
Maximizing over functions \( f \) with volume at most \( k \) then gives us the desired asymptotic equivalence. \( \square \)

In concrete examples, the main principle is as follows: to show a class \( \alpha \in \pi_n(X) \) to be distorted, one needs to find a sequence of boundaries in \( \tilde{B} \) which lift to multiples of \( \alpha \). The exact lift depends on the exact representative of the Euler class chosen, but any pair of representatives gives lifts which diverge from each other linearly. Thus in principle, if \( \alpha \) is distorted, then we can use any representative of the Euler class to demonstrate this.

Examples 6.6. 
(1) Mineyev [Min00] shows that hyperbolic groups satisfy linear isoperimetric inequalities for filling of boundaries with cycles. In particular, if \( \Gamma \) is hyperbolic, \( \text{FV}_n^{\Gamma}(k) \) is linear for all \( n \geq 1 \), and hence so is \( \text{FV}_n^{\Gamma,(e,c)}(k) \) for any \( e \in H^{n+1}(\Gamma) \). Therefore, if \( X \) is the total space of a fibration \( S^n \to X \to B \Gamma \), then \( \pi_n(X) \) is undistorted.

(2) Conversely, if \( \Gamma = \mathbb{Z}^d \) for \( d > n \), then the isoperimetric inequality \( k^{(n+1)/n} \) is attained by fillings of round spheres in \( (n+1) \)-dimensional coordinate hyperplanes, which correspond to the generators of \( H^{n+1}(\Gamma) \). Suppose that \( X \) is a bundle over \( T^n \) with finite monodromy. Given \( 0 \neq \omega \in H^{n+1}(\Gamma) \), \( \langle \omega, x \rangle \neq 0 \) for one of these generators \( x \), so \( \text{FV}_n^{\Gamma,(e,c)}(k) \sim k^{(n+1)/n} \), and if \( X \) has Euler class \( \omega \), then \( \pi_n(X) \) is distorted with \( \text{VD}_n(k) \sim k^{(n+1)/n} \).

(3) A somewhat more subtle situation occurs when \( \Gamma = \mathbb{Z}^d \) with infinite-order monodromy. As a concrete example, we take \( \Gamma = \mathbb{Z}^5 = \langle a, b_1, \ldots, b_4 \rangle \) and fiber \( (S^3)^3 \). Let \( \hat{a} \) and \( \hat{b}_i \) be the 4-cells in \( \mathbb{R}^5 \) orthogonal to the generators of the fundamental group. Suppose that the monodromy representation \( \rho \) takes
\[
a \mapsto A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3/5 & -4/5 \\ 0 & 4/5 & 3/5 \end{pmatrix} \quad \text{and} \quad b_i \mapsto I_3.
\]

Then \( \text{H}^*(\Gamma; M_\rho) \cong \text{H}^*(\langle a \rangle; \mathbb{Z}^3/\rho(a)) \otimes \text{H}^*(\langle b_i \rangle; \mathbb{Z}^3), \) and in particular \( \text{H}^4(\Gamma; M_\rho) \cong \mathbb{Q}^7 \) generated by cochains sending the flats to any vector and disregarding the last two coordinates of the image of each \( \hat{b}_i \).

If the Euler class of the fibration takes \( \hat{a} \) to a vector \( \vec{v} \), then the situation restricted to a flat perpendicular to \( a \) is the same as in the previous example, so \( \vec{v} \) is distorted polynomially, with \( \text{VD}^k(\rho) \sim k^{4/3} \). Moreover, so is \( \rho(m)\vec{v} \), for any \( m \), as we can see by taking the relevant parallel flat. Indeed, so is any rational linear combination of such vectors, that is, any \( \rho(m)\vec{v} \) for \( m \in \mathbb{Q}\Gamma \).

On the other hand, suppose the Euler class takes \( \hat{b}_1 \) to a vector \( \vec{v} \). Of the 3-cells bounding \( \hat{b}_1 \), the ones perpendicular to \( \hat{a} \) lift to vectors differing by multiplication by \( A \). Thus when choosing a cellular representative of this Euler class we can choose to lift \( \hat{b}_1 \) to any vector with the same first coordinate. If we choose the lift to be \( (v_1, 0, 0) \), then flats perpendicular to \( \hat{b}_1 \) behave as in the previous example, so again this vector is distorted.

Thus vectors with nonzero second and third coordinate can only be distorted from \( \text{eu}(\hat{a}) \), but any flat may demonstrate distortion of vectors of the form \( (v_1, 0, 0) \).

(4) Now take \( \Gamma = D_n \), the \( n \)th diamond group defined earlier, and suppose \( Y \) is the total space of a rational homotopy fibration \( (S^n)^r \to Y \to X_n \). From the earlier discussion we see that if the monodromy is trivial, then the volume distortion function with any nontrivial Euler class will be \( \exp(k^{1/n}) \). Indeed, this is also true with other monodromy representations. Thus suppose that \( \rho : D_n \to GL_r(\mathbb{Q}) \) takes \( a \mapsto 1 \) and the \( b_j(i) \) to any elliptic image of \( F^n_q \). As a lower bound on the distortion function of \( \pi_n(Y) \), we have the inverse of \( \text{FV}^{D_n}_n \) as computed earlier. To show that this bound is sharp, we construct a sequence of representatives. Note that \( H^{n+1}(X_n; M_\rho) \cong \mathbb{Q}^r \), because coboundaries in \( C^{n+1}(X_n; M_\rho) \) are those \( \omega \) that satisfy the vector equation
\[
A(\omega) := \sum_{i_1, \ldots, i_n \in \{1, 2\}} (-1)^{\sum_{j=1}^n i_j} \prod_{j=1}^n (\rho(b_j(i_j))^{-1} - 2I)^{-1} \langle \omega, e_{i_1, \ldots, i_n} \rangle = 0,
\]
where the lifts $\tilde{e}_j$ are chosen so that they all coincide on the edge $a^{2^n}$. Now choose a representative $\omega$ of $\eu \in H^{n+1}(X_n; M_\rho)$. Then the embedding $\Sigma_k$ of $S^n$ described before, which has surface area $O(k^n)$, lifts to a representative in $\pi_n(Y) \otimes \mathbb{Q} \cong \mathbb{Q}^r$ of

$$
\theta(k) = \sum_{i_1, \ldots, i_n \in \{1, 2\}} (-1)^{\sum_{i=1}^n i_j} \sum_{j_1 + \cdots + j_n \leq k} 2^{k-\sum_{i=1}^n j_i} \rho(b_1(i_1))^{j_1} \cdots \rho(b_n(i_n))^{j_n} \langle \omega, e_{i_1}, \ldots, e_{i_n} \rangle.
$$

By induction on $n$, one finds that this vector can be rewritten as

$$
\theta(k) = 2^{k+1} A(\omega) + O \left( \max_{t \in \{0, 1\}^n} |\langle \omega, e_t \rangle| \right).
$$

In particular, the second term stays bounded as $k$ varies. But to show that $A(\omega)$ is indeed distorted, we need a more precise statement which also bounds the denominator of the remainder term.

**Lemma 6.7.** The remainder term of $\theta(k-1)$ is given as a sum of terms of the form

$$
\pm \rho(b_j(i))^k \prod_{t=1}^n (A_t - B_t)^{-1} \langle \omega, e_t \rangle,
$$

for some $i$, $j$, and $I$, and where each $A_t$ and $B_t$ is either $2I$ or $\rho(b_j'(i'))$ for some $j'$ and $i'$.

**Proof.** For $I = (i_1, \ldots, i_n) \in \{0, 1\}^n$, set $\tilde{u}_0(k, I) = 2^k \langle \omega, e_t \rangle$, and

$$
\tilde{u}_t(k, I) = \sum_{m=0}^{k-1} \rho(b_t(i_t))^m \tilde{u}_{t-1}(k-1-m).
$$

In particular, $\theta(k-1) = \sum_{I \in \{0, 1\}^n} \tilde{u}_n(k, I)$. Thus it is enough to show by induction that $\tilde{u}_t(k, I)$ is given as a sum of terms of the form

$$
\pm \rho(b_j(i))^k \prod_{t=1}^n (A_t - B_t)^{-1} \langle \omega, e_t \rangle.
$$

Clearly this is true for $t = 0$, and the general formula $\sum_{i=0}^{k-1} a^k b^{k-1-i} = (a - b)^{-1} (a^k - b^k)$ gives us the inductive step.

Thus, let $q$ be the common denominator of all possible length $n$ products of terms of the form $(A_t - B_t)^{-1}$; this quantity is independent of $k$. Then the vector $q \cdot 2^k A(\omega) \in \pi_n(X) \otimes \mathbb{Q}$ has a representative of volume $q \left( k^n + O \left( \max_{t \in \{0, 1\}^n} |\langle \omega, e_t \rangle| \right) \right)$. Thus whenever $A(\omega) \neq 0$, it is distorted with distortion function $\exp(\sqrt{k})$. Moreover, so is $\rho(m)A(\omega)$ for any $m \in \mathbb{Q}D_n$, and we can use other embeddings of $S^n$, depending on choices of branches in the Bass-Serre tree, to obtain other, perhaps distinct distorted vectors. Thus the one-to-one relationship between homology classes downstairs and homotopy classes upstairs that we see in the case of trivial monodromy, as exemplified by Theorem [6.3], does not have an equivalent in the general case.

Although the situation with volume distortion in general can be quite complicated, there is a criterion that can be used to identify pairs for which all classes are volume-undistorted.

**Definition ( [Gro91]).** Let $X$ be a compact piecewise Riemannian space. A form $\omega \in \Omega^n(X)$ is called $\tilde{d}$(bounded) if its lift $\tilde{\omega}$ to the universal cover $\tilde{X}$ is the differential of a bounded form.

Being $\tilde{d}$(bounded) is a homotopy invariant and (in the given setting) exact forms are $\tilde{d}$(bounded). Indeed, a form on a finite complex $X$ is $\tilde{d}$(bounded) if its pullback to the universal cover is zero in $L_\infty$ cohomology.

**Theorem 6.8.** Let $B$ be a finite CW complex with universal covering $\pi : \tilde{B} \to B$. Let $F := (S^n)^r \to X \xrightarrow{\rho} B$, $n \geq 3$, be a quasi-Postnikov pair with elliptic monodromy representation $\rho : \pi_1 B \to GL(H_n(F; \mathbb{Q}))$ with Euler class $\eu \in H^{n+1}(B; M_\rho)$, such that $\rho^* \eu = 0$ as a class in $H^{n+1}(B; \mathbb{Q})$. Then the following are equivalent:

1. The subspace of $\pi_n(X) \otimes \mathbb{Q}$ induced by the inclusion of the fiber is volume-undistorted.
2. For some (any) representative $\omega$ of $\eu$, there is a constant $C$ such that for all spherical boundaries $\sigma \in B_n(B)$, $\|\langle \omega, \text{Fill}(\sigma) \rangle\| \leq C \text{vol} \sigma$.  

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Theorem B then follows by first restricting to delicate spaces and then inducting on the rational Postnikov decomposition.

What does this theorem really mean? If $F \to X \to B$ is an honest fiber bundle with $F \to \tilde{X} \to \tilde{B}$ homotopically trivial, one can think of condition (4) as specifying that there is a section $\sigma : \tilde{B} \to \tilde{X}$ with a bounded amount of “twisting” around the fiber for every $n$-cell. What is bounded a priori is the number of twists, but this is equivalent to being able to find a section with a bounded Lipschitz constant, or, say, $n$-dilation. This in turn induces a trivializing map $\tilde{X} \to F \times \tilde{B}$ which is also bounded in all the same senses, that is, the bundle $\tilde{X} \to \tilde{B}$ is “coarsely trivial” in a natural sense. It is tempting to assert that when $X$ has undistorted homotopy groups, its universal cover must necessarily be coarsely trivial as a rational homotopy fibration, whatever that may mean. However, outside the important special case of actual fiber bundles, for example when $\rho$ is outside $GL(r, \mathbb{Z})$, it’s much harder to make such an assertion precise.

We continue by specifying this result to certain classes of groups and spaces.

Examples 6.9. If $\Gamma$ is an amenable group, then only the zero class in $H^{n+1}(B; M_\rho)$ is $d$-(bounded), i.e. the lifting map $H^{n+1}(B; M_\rho) \to H^{n+1}_\infty(B; \mathbb{Q}^r)$ is injective. A special case of this fact was noted in [ABW92]. In general, an invariant mean $\mu$ on $\Gamma$ provides a map $\int : C^*(\tilde{B}; \mathbb{Q}^r) \to C^*(B; M_\rho)$, where for any cell $e$,

$$\langle f(\omega, \epsilon), e \rangle \equiv \int_{\gamma \in \Gamma} \rho(\gamma)^{-1}(\langle \omega, \gamma \cdot e \rangle) d\mu.$$ 

This map commutes with coboundaries and thus is a well-defined projection inverting the lifting map. Thus in the case of an amenable fundamental group $\Gamma$ only spaces with zero Euler class have undistorted homotopy groups.

Thus, suppose that $B\Gamma$ is an $n$-dimensional manifold and let $X$ be its unit tangent bundle. Then if $\Gamma$ is amenable, the Euler characteristic of $B\Gamma$, and thus the Euler class of $X \to B\Gamma$, is zero by a result of Cheeger and Gromov [CG98]. Otherwise, the fundamental class of $B\Gamma$ lifts to zero in $H^n_\infty(X; \mathbb{Q})$, as discussed in [ABW92]. Thus in any case, $\pi_n(X)$ is undistorted.

On the other end of the spectrum, suppose $\Gamma$ is a hyperbolic group. As discussed before, $B\Gamma$ satisfies a linear isoperimetric inequality for fillings of $n$-boundaries with chains for $n > 0$, so that $H^n_\infty(B\Gamma; \mathbb{Q}^r) = 0$ for $n \geq 2$. On the other hand, Lemma 6.10. For any finite complex $B$ with universal covering map $\pi : \tilde{B} \to B$ and any elliptic monodromy representation $\rho : \pi_1 B \to GL(H_n(S^n; \mathbb{Q}))$, the lifting map $\pi^* : H^1(\tilde{X}; M_\rho) \to H^1_\infty(\tilde{X}; \mathbb{Q})$ is injective.

Proof. Suppose $\omega \in C^1(B\Gamma; M_\rho)$ is a representative of a nonzero cohomology class; in other words, there is a loop $\gamma$ such that $\bar{v} = \langle \omega, \gamma \rangle \not\in \text{im}(\rho(\gamma) - I)$. Then $\rho(\gamma)$ is conjugate to a rotation and $\bar{v}$ has a nonzero
projection onto its invariant subspace in \(\mathbb{R}^r\). Thus \(\sum_{k=0}^{k} \rho^k(v_i)\) has unbounded length, so if \(\sigma_k\) is a lift of \(k\gamma\) to \(\tilde{B}\), then \((\pi_1(\omega, \sigma_k))\) grows linearly in \(k\) but \(\operatorname{vol}(\partial \sigma_k) = 2\), and \(\omega\) lifts to a nonzero class in \(H^1(\tilde{B}; \mathbb{Q}^r)\). \(\square\)

Now suppose that \(B\) is built up to rational homotopy as a fibration \(F = \prod_i S^{2n_i+1} \to B = B\Gamma\). Then \(H^1_\infty(\tilde{B}; \mathbb{Q}^r)\) can be computed via a Serre spectral sequence whose columns are zero except for \(p = 0, 1\). Thus for \(\Gamma\) hyperbolic, distortion in delicate spaces occurs if and only if the relevant Euler class is represented by a nonzero element of \(H^1(\tilde{B}; \mathbb{Q}^r)\). As will be proved in Theorem 6.11, such distortion is always weakly infinite, and so for \(\Gamma\) hyperbolic all subspaces of \(\pi_n(X)\) are either undistorted or weakly infinitely distorted.

Indeed, according to a preprint of Gersten [Ger92], hyperbolic groups are characterized by \(H^2_\infty(\Gamma) = 0\). Thus the equivalence above is true if and only if \(\Gamma\) is hyperbolic.

It is well known, but worth remarking, that bounded cohomology classes of groups are zero in \(L_\infty\) cohomology: for example, this is Lemma 10.3 in [Ger92]. Thus bounded Euler classes always give rise to volume proportional to their height in the \(\mathbb{R}\)-direction, rather than bounded volume. On the other hand, given a map \(f : S^n \to \mathbb{H}^n \times \mathbb{R}\), we can deform \(f\) slightly so that it is an immersion whose projection to the \(\mathbb{R}\) factor is Morse. Then both the area of \(f\) and the volume of a filling are determined by integrating hyperbolic areas and volumes over \(\mathbb{R}\), and hence one is linear in the other. Thus for any bundle \(S^n \to X \to M^n \times S^1\), \(\pi_n(X)\) is undistorted.

We conclude with a criterion for the presence of weakly infinite distortion in delicate spaces.

**Theorem 6.11.** Suppose \((S^n)^r \to X \xrightarrow{\tilde{F}} B\) is a quasi-Postnikov pair with elliptic monodromy representation \(\rho : \Gamma := \pi_1(B) \to \text{GL}(H_\infty((S^n)^r); \mathbb{Q})\) and Euler class \(e\in H^{n+1}(B; M_\rho)\). Assume also that \(B\) is itself the total space of a fibration \(F \to B \xrightarrow{\tilde{F}} A\), where \(F\) is simply connected and the universal cover \(\tilde{A}\) is rationally \((n+1)-\)connected. Then the following are equivalent:

1. There is no weakly infinite volume distortion in \(\iota_*\pi_n((S^n)^r) \otimes \mathbb{Q}\) as a subset of \(H_n(\tilde{X}; \mathbb{Q})\).
2. For every \(\gamma \in \Gamma\), \(0 = \gamma^* e\in H^{n+1}(U; M_{\rho|_{\gamma}})\), where \(U\) is the total space of \(\gamma^* p\).
3. \(H_n(\tilde{X}; \mathbb{Q})\) splits as a direct sum of \(\mathbb{Q}\Gamma\)-modules \(h_n(\pi_n(\tilde{X}) \otimes \mathbb{Q}) \oplus P\).
4. There is a subspace \(P \subset H_n(\tilde{F}; \mathbb{Q})\) which is invariant under the homological monodromy representation \(\tilde{p} : \Gamma \to \text{GL}(H_n(\tilde{F}; \mathbb{Q}))\) of \(\tilde{p}\), such that \(P \oplus h_n(\pi_n(\tilde{F}) \otimes \mathbb{Q}) = H_n(\tilde{F}; \mathbb{Q})\).

**Example 6.12.** Let \(B = S^1 \times (S^3)^3\) and \(X\) be an \(S^9\)-bundle over \(B\) with Euler class \(e = [B]\). Then the generator of \(\pi_9(X)\) is homotopic to a lift of the boundary of the 10-cell in the product cell structure of \(B\); in \(\tilde{B}\) and hence also \(\tilde{X}\), this map induces the chain \((i+1) \cdot a - i \cdot a\), where \(a\) is the top cell of \((S^3)^3\) and \(i \in \mathbb{Z} \cong \pi_1(X)\). By Lemma 1.4, we can find a representative \(f : S^9 \to X\) of \(k \in \mathbb{Z} \cong \pi_9(\tilde{X})\) with \(f_\#([S^9]) = k \cdot ([S^3]^3) - ([S^3]^3)\) and thus volume 2.

Note that \(H_9((S^3)^3 \times S^9; \mathbb{Q}) = \mathbb{Q}^2\) and the generator \(1 \in \pi_1(X)\) acts on it via the matrix \(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\). Thus as a \(\mathbb{Q}[\mathbb{Z}]/\mathbb{Z}\)-module, \(H_9((S^3)^3 \times S^9; \mathbb{Q}) \cong \mathbb{Q}[\mathbb{Z}]/([1] - 1)^2\), which does not decompose as a direct sum.

**Proof.** Note that since, by definition, the fibration \(\tilde{X} \to \tilde{B}\) is rationally trivial, the total rational homotopy fiber of \(X \xrightarrow{\tilde{F}} A\) is \(\tilde{F} := (S^n)^r \times F\). In other words, \(X\) is rationally \((n+1)\)-equivalent to the total space of a fibration with fiber \(\tilde{F}\) and base \(BT\).

We first show that (1) implies (2), by demonstrating that volume distortion over \(S^3\) is always infinite. Specifically, consider a rational homotopy fibration \(F \to U \to S^3\) with \(F\) simply connected, and suppose that \((S^n)^r \to V \to U\) is a rational homotopy fibration with monodromy \(\rho : \mathbb{Z} \to \text{GL}(\mathbb{Q}, r)\) and Euler class \(0 \neq e\in H^{n+1}(U; M_\rho)\), such that the total rational fiber of \(V \to S^1\) is \(F \times (S^n)^r\).
Without loss of generality, by taking a factor, we may assume $M_\rho \cong \mathbb{Q}[\mathbb{Z}]/I$ with $I$ a primary ideal. Indeed, since $\rho$ is elliptic, we can assume that $I = \langle q \rangle$ is a nonzero prime ideal, and thus $M_\rho$ is a simple module. We write $\omega(x) = \sum q_i x^i$.

Since $\mathbb{Q}[\mathbb{Z}]$ is a PID, the universal coefficient theorem gives a short exact sequence

$$0 \to \operatorname{Ext}_{\mathbb{Q}[\mathbb{Z}]}(H_n(\tilde{U}; \mathbb{Q}), M_\rho) \to H^{n+1}(U; M_\rho) \to \operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}]}(H_{n+1}(\tilde{U}; \mathbb{Q}), M_\rho) \to 0.$$  

By assumption, the Euler class lifts to $0 \in H^{n+1}(U; \mathbb{Q})$, and thus it pulls back to $\operatorname{Ext}_{\mathbb{Q}[\mathbb{Z}]}(H_n(\tilde{U}; \mathbb{Q}), M_\rho)$. Note that $H_n(\tilde{U}; \mathbb{Q}) = Z_n(\tilde{U}; \mathbb{Q})/B_n(\tilde{U}; \mathbb{Q})$, both of which are free modules; thus one can find bases $z_1, \ldots, z_m$ for $Z_n$ and $b_1, \ldots, b_m$ for $B_n$ such that $H_n(\tilde{U}; \mathbb{Q}) = \bigoplus_{i=1}^m \mathbb{Q}[\mathbb{Z}] z_i/\mathbb{Q}[\mathbb{Z}] b_i$ and each of the summands is semisimple. Then $\operatorname{Ext}_{\mathbb{Q}[\mathbb{Z}]}(H_n(\tilde{U}; \mathbb{Q}), M_\rho)$ takes the form $\bigoplus_{i=1}^m M_\rho/M_i$, where $M_i \subseteq M_\rho$ is the set of possible images of $b_i$ under a homomorphism $\mathbb{Q}[\mathbb{Z}] z_i \to M_\rho$. Since $M_\rho$ is simple, each $M_i$ is either $0$ or $M_\rho$. Thus if $\epsilon \neq 0$, then there is some $i$ and an element $\mu \in M_\rho$ such that any representative of $\epsilon$ takes any chain $c \in C_{n+1}(\tilde{U}; \mathbb{Q})$ bounding $b_i$ to $\mu$, and thus $b_i \in I \subseteq \mathbb{Q}[\mathbb{Z}] z_i$.

Let $t$ be the least positive number such that $q((1)^t) \cdot z_i$ is a boundary, which we may assume is $b_i$. Let $z = q((1)^t) z_i$. Then $z, z \cdot [1], \ldots, z \cdot [r - 1]$ descends to a basis for $\mathbb{Q}[\mathbb{Z}] z/\mathbb{Q}[\mathbb{Z}] b_i$. Let $S$ be the vector subspace of $Z_n(\tilde{U}; \mathbb{Q})$ generated by $\tilde{e}_1 = z, \tilde{e}_2 = z \cdot [1], \ldots, \tilde{e}_r = z \cdot [r - 1]$, and let $A_q : S \to S$ act on this basis as the companion matrix of $q$. Thus $\operatorname{mod} \mathbb{Q}[\mathbb{Z}] b_i$, the action of $A_q$ is the same as that of multiplication by $[1]$, and $A_q$ is conjugate to $\rho([1])$. Let $T : S[\mathbb{Z}] \to \mathbb{Q}[\mathbb{Z}]$ send $\tilde{e}_i [j'] \mapsto [j + j']$.

Now, let $(\xi_s)_{s \in \mathbb{N}}$ be a sequence of chains in $C_{n+1}(\tilde{U}; \mathbb{Q})$ given by

$$c_{s+1} = \sum_{j=0}^s T(A_q^j(\xi_1))(s - j)c.$$  

Then for any $s \geq r$, and for some $\bar{u}_j, \bar{v}_j \in \mathbb{Q}^r$ not depending on $s$,

$$\partial c_{s+1} = \sum_{j=0}^s T(A_q^j(\xi_1))(s - j)b_i = \sum_{j=0}^s \sum_{\ell=0}^{\deg q} q_T(A_q^j(\xi_1))(s - j + \ell)z = \sum_{j=0}^s T(q(A_q j) j)z + \sum_{j=0}^{r-1} T(A_q^j(\bar{u}_j))(j)z + \sum_{j=0}^{r-1} T(\bar{v}_j)(s + j + 1)z =: E(s + 1) + F(s + 1) + G(s + 1).$$  

For each $s > r$, $E(s) = 0$ since $q(A_q) = 0$, and $G(s)$ has constant volume. Moreover, since $A_q$ is elliptic, $F(s)$ is a free $\mathbb{Q}$-module, whereas $G(s)$ is a vector of bounded norm in a finite-dimensional vector subspace of $Z_n(\tilde{U}; \mathbb{Q})$. Thus for any $\omega \in C^m(\tilde{U}; \mathbb{Q}^r)$ such that $\omega d\omega$ is a representative of $\epsilon$, $(\omega, F(s))$ is bounded as a function of $s$, and $(\omega, F(s) + G(s)) = (\epsilon, c_s)$ is a free module of bounded norm. Thus, there’s some constant $K$ such that every $(\omega, F(s))$ is a vector of bounded norm. Thus, $c_{n+1}(S^n)''$ is weakly infinitely distorted.

$(2')$ is clearly equivalent to $(2)$. The equivalence of $(2'')$ and $(2)$ follows from $(6.10)$.

$(2') \implies (3)$. It’s enough to show the equivalent of $(3)$ for $\tilde{\rho} \circ \iota_{1*} : F_m \to GL(H_n(\tilde{F}; \mathbb{Q}))$, where $F_m$ is a free group whose quotient is $\Gamma$; tensoring with $\mathbb{Q}^\Gamma$ as an $\mathbb{Q}F_m$-module then gives $(3)$. Writing $\iota^*_1 X$ and $\iota^*_1 B$ for $\iota^{-1}(A(1))$ and $(\rho')^{-1}(A(1))$ respectively, it suffices to show that the short exact sequence of $\mathbb{Q}F_m$-modules

$$0 \to \pi_n(\iota^*_1 X) \xrightarrow{\rho_n} H_n(\iota^*_1 X) \to H_n(\iota^*_1 B) \to 0$$  

splits, i.e. that there is a module homomorphism $i : H_n(\iota^*_1 B) \to H_n(\iota^*_1 X)$ such that $\rho \circ i = \iota_{1*}$. Suppose $\iota^*_1 B$ is an $\mathbb{Q}$-module such that $\rho \circ i = \iota_{1*}$.

As before, the restriction of $j$ to $B_n(\iota^*_1 B; \mathbb{Q})$ induces a representative of the Euler class, which we may call $\omega : C_{n+1}(\iota^*_1 B; \mathbb{Q}) \to \pi_n(\iota^*_1 X) \otimes \mathbb{Q}$. We know that $\omega = \delta \eta$ for some $n$-cocycle $\eta$; in particular, $\rho_n\eta_n = 0$. Then $j = h_n\eta$ is a cocycle which induces a homomorphism $i : H_n(\iota^*_1 B) \to H_n(\iota^*_1 X)$ as desired.

Since $\pi_n(\tilde{X}) \otimes \mathbb{Q}$ is clearly $\tilde{\rho}$-invariant, $(3')$ is simply a restatement of $(3)$.  

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(3')⇒(1). Equip $H_n(\tilde{X}; \mathbb{Q}) \cong H_n(\tilde{F}; \mathbb{Q})$ with the seminorm which sends $P$ to 0 and restricts to a norm preserved by $\rho$ on $\pi_n(\tilde{X}; \mathbb{Q})$. Note that $\hat{\rho}$ preserves this seminorm. By analogy with the proof in [AWP99] that higher-order Dehn functions of groups are finite, we will show that when (3') holds, the homology classes of integral cycles of volume $k$ in $\tilde{X}$ have seminorm bounded by some $C(k)$. This is enough to show (1).

The proof is by induction on $k$. Clearly there is a $C(1)$, since there’s a finite number of $\Gamma$-equivalence classes of cycles of volume 1. Now suppose we have determined $C(i)$ for $1 \leq i < k$. Given an $n$-cycle $c$ in $\tilde{X}$ of volume $k$, assume that it contains a cell from a fundamental domain $D$. Since the action of $\Gamma$ preserves the seminorm, we can do this without loss of generality. Then either $c$ consists entirely of cells within distance $k$ of $D$ (in the graph defined by adjacency of cells, in the sense of having common $(n-1)$-cells in their boundaries) or it consists of two disjoint cycles. There are a finite number of cycles of the first kind, since there are a finite number of such cells, and so the seminorm of their homology classes is bounded by some $B(k)$. Thus we can set $C(k) = \max\{B(k)\} \cup \{C(i) + C(k-i) : 0 < i < k\}$.

Restricting to delicate spaces and induction on $n$ then gives us Theorem C.

Note, however, that weakly infinite distortion inside the homology group does not imply that $\pi_n$ is infinitely distorted. For example, we could have a homology module as in Example 3.10(3). One can construct such a space by letting $B$ be the “mapping torus” of $\prod_{i=1}^4 S^3$ corresponding to the matrix $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$, where $A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$. By Poincaré duality, this is also the monodromy of $H_0(\tilde{B}; \mathbb{Q})$ with respect to the action.

So let $X$ be the total space of the rational $(S^3)^2$-bundle over $B$ with monodromy $\rho : \mathbb{Z} \to GL(2, \mathbb{Q})$ taking $1 \mapsto A$ and nonzero Euler class in $H^n(B; M_\rho) \cong \text{Ext}_{\mathbb{Z}[\mathbb{Q}]}(M_\rho \times \mathbb{Q} \times \mathbb{Q}, M_\rho) \cong M_\rho$. By the argument of Example 3.10(3), $\pi_9(X)$ is not infinitely distorted.

On the other hand, suppose that for some $\gamma \in \Gamma$, $\gamma^* e\mu$ is nontrivial when restricted to a direct summand $\mathbb{Q} \cong L \subset M_\gamma^r e\mu$ (a module over $\mathbb{Z}[\mathbb{Q}]$) on which the action of $\gamma$ is trivial. Then it is in fact possible to find true infinite distortion—for example, if in the proof that (1) implies (2) in Theorem 6.11 we assume $M_\rho = L$, then the chain $K\partial_c s$ constructed in that proof is an integral chain of constant volume which lifts to an element of $\pi_n(U)$ proportional to $s$.

In the case when $\rho$ is trivial, this is of course a sufficient condition. We conjecture that it is sufficient in general. This is certainly true in the case $\pi_1(X) = \mathbb{Z}$.

**Theorem 6.13.** Suppose $(S^n)^r \to V \to U$ is a quasi-Postnikov pair such that $U$ is the total space of a rational homotopy fibration $F \to U \to S^1$. Let $\rho : \mathbb{Z} \to GL(r, \mathbb{Q})$ be its elliptic monodromy and $e\mu \in H^{n+1}(U; M_\rho)$ be the Euler class. Then the following are equivalent:

1. There is infinite volume distortion in $\pi_n((S^n)^r) \otimes \mathbb{Q}$ as a subset of $H_n(\tilde{V}; \mathbb{Q})$.
2. There is a finite-sheeted cover $\varphi : \tilde{U} \to U$ such that $\varphi^* M_\rho$ has a trivial submodule $L \cong \mathbb{Q}$ for which the projection of $\varphi^* e\mu \in H^{n+1}(\tilde{U}; \varphi^* M_\rho)$ onto $L$ is nonzero.

**Proof.** (2)⇒(1) is proved above. Now suppose that (2) is not true. In particular, since an elliptic element of $GL(m, \mathbb{Z})$ is unipotent, every simple factor of $M_\rho$ on which $e\mu$ is nontrivial must have the form $\mathbb{Q}[\mathbb{Z}]/(q)$ for some $q \in \mathbb{Q}[\mathbb{Z}] \setminus \mathbb{Z}[\mathbb{Z}]$. Choose such one such factor $M$, and let $\pi_M : M_\rho \to M$ be the projection map. Consider a fundamental domain $B \subset \tilde{U}$ large enough that cycles contained in $B$ generate $Z_n(\tilde{U}; \mathbb{Q})$ as a $\mathbb{Z}[\mathbb{Q}]$-module, and let $\Lambda$ be the lattice in $\mathbb{Q}^{\text{deg} q}$ generated by applying $e\mu |_M$ to cycles in $B$. Let $\sigma(\Lambda) = \min\{\|\vec{v}\| : \vec{v} \in \Lambda\}$, and $R = \sigma(\Lambda \cap r(1)\Lambda)/\sigma(\Lambda)$. We can then proceed as in Lemma 5.11, substituting $R$ for 5, to show that for any $C$, $\pi_M \circ e\mu$ is bounded on cycles of volume at most $C$. Since this is true for every factor $M$, we get (1). □

7. Open problems

A topic which seems to allow boundless possibilities for further investigation is the effect of the fundamental group on distortion. Thus, it is possible for infinite distortion to arise in a “non-local” way, that is, in a way other than the one outlined in Theorem 6.13. For what class of groups does the conclusion of Lemma 5.11 hold, such that we can expand our conclusions to complexes with finite skeleta? Finally, one can try
to expand the class of functions known to be distortion functions by finding new directed isoperimetric functions.

One can also ask whether a result similar to Theorem 13 holds for Lipschitz distortion. Indeed, the reduction to delicate spaces works just as well for Lipschitz distortion, as does a result similar to Theorem 6.5 relating distortion in a sphere bundle to a certain filling function in the base space, though as far as we know this only holds for actual sphere bundles, not up to rational homotopy. Example 2.6 gives pause to attempts to extend this to delicate spaces in general, and so the problem seems less tractable.

The Lipschitz constant and volume of a map $S^n \to X$ can be thought of as the 1-dilation and $n$-dilation, respectively. Thus the two kinds of distortion studied can be generalized further to $m$-distortion, measuring the growth of $m$-dilation as $k$ increases. The optimal $m$-dilation of homotopy classes has been studied by Guth in [Gut13] in the case of certain torsion homotopy groups of spheres. It would be interesting to extend the results of section 3 to this setting.

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