Detection of genuinely entangled and non-separable \( n \)-partite quantum states

Ting Gao and Yan Hong
College of Mathematics and Information Science,
Hebei Normal University, Shijiazhuang 050016, China
(Dated: November 10, 2010)

Entanglement plays a fundamental role in quantum information processing and is responsible for many quantum tasks such as quantum cryptography with Bell’s theorem [1], quantum dense coding [2], quantum teleportation [3], quantum communication [1–7] and quantum computation [8, 9] etc. Thus, entanglement is not only the subject of philosophical debates, but also a new resource for tasks that cannot be performed by means of classical resources [10, 11].

Deciding whether a state is entangled or not has proven to be a very challenging problem that currently lacks a full computable solution. In the bipartite setting, there are some well-known (necessary) criteria for separability, such as the Bell inequalities [12], positive partial transposition (PPT) [13] (which is also sufficient for two-qubit or one qubit and one qutrit systems [14]), reduction [15, 16], range [17], majority [18], realignment [19–21] and generalized realignment [22] etc., which work very well in many cases, but are far from perfect [10]. For multipartite entanglement (more than two parties), the situation is even more complicated as there exist states that are inseparable under any fixed partition, but they are still not considered genuinely multipartite entangled (defined below) [23]. Likewise, there exist states that are biseparable with respect to each fixed partition, however, they are not fully separable (for some examples see Refs. [24–26]). Vast areas of multipartite state spaces are still unexplored due to the lack of suitable tools for detecting and characterizing entanglement.

Recently, Gühne and Seevinck [23] presented a method for deriving separability criteria within different classes of 3-qubit and 4-qubit entanglement using density matrix elements. Huber et al. [27] developed a general framework to identify genuinely multipartite entangled mixed quantum states in arbitrary-dimensional systems. From the framework introduced in [27], a k-separability criterion was derived in [28]. In addition, we studied the separability of \( n \)-partite quantum states and obtained practical separability criteria for different classes of \( n \)-qubit and \( n \)-qudit quantum states [29].

In this paper, we derive novel separability criteria to identify genuinely entangled and non-separable \( n \)-partite mixed quantum states. The resulting criteria are easily computable from the density matrix and no optimization or eigenvalue evaluation is needed. Below, we first describe our criteria and then provide examples in which we can detect genuine \( n \)-partite entanglement beyond all previously studied criteria. Finally, we briefly comment on the ability for our criteria to be implemented in today’s experiments without needing quantum state tomography.

An \( n \)-partite pure state \( |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \) (dim \( \mathcal{H} = d_i \geq 2 \)) is called biseparable if there is a bipartition \( j_1, j_2, \ldots, j_k | j_{k+1}, \ldots, j_n \) such that

\[
|\psi\rangle = |\psi_1\rangle_{j_1, j_2, \ldots, j_k} |\psi_2\rangle_{j_{k+1}, \ldots, j_n},
\]

where \( |\psi_1\rangle_{j_1, j_2, \ldots, j_k} \) is the state of particles \( j_1, j_2, \ldots, j_k \), \( |\psi_2\rangle_{j_{k+1}, \ldots, j_n} \) is the state of particles \( j_{k+1}, \ldots, j_n \), and \( \{ j_1, j_2, \ldots, j_n \} = \{1, 2, \ldots, n\} \). An \( n \)-partite mixed state \( \rho \) is biseparable if it can be written as a convex combination of biseparable pure states

\[
\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|,
\]

where \( |\psi_i\rangle \) might be biseparable under different partitions. If an \( n \)-partite state is not biseparable, then it is called genuine entanglement.

PACS numbers: 03.65.Ud, 03.67.-a
genuinely $n$-partite entangled. An $n$-partite pure state is fully separable if it is of the form

$$|\psi\rangle = |\psi\rangle_1|\psi\rangle_2 \cdots |\psi\rangle_n,$$

and an $n$-partite mixed state is fully separable if it is a mixture of fully separable pure states

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|,$$

where the $p_i$ forms a probability distribution, and $|\psi_i\rangle$ is fully separable. If an $n$-partite state is not fully separable, then we call it non-separable. We consider separability criteria of biseparable and fully separable $n$-qubit and $n$-qudit states.

Throughout this paper, let $\rho$ be a density matrix describing an $n$-particle system whose state space is Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\dim \mathcal{H}_l = d_l$, $l = 1, 2, \cdots, n$. We denote its entries by $\rho_{ij}$, where $1 \leq i, j \leq d_1d_2 \cdots d_n$. We introduce the further notation of $|\Phi_{ij}\rangle = |\phi_i\rangle|\phi_j\rangle$ with $|\phi_i\rangle = |x_1xy_2xyz\cdots x_l\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where the local state of $\mathcal{H}_k$ is $|x\rangle$ for $k \neq i$ and $|y\rangle$ for $k = i$. Furthermore, let $P$ denote the operator that performs a simultaneous local permutation on all subsystems in $(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n)^{\otimes 2}$, while $P_i$ just performs a permutation on $\mathcal{H}_k^{\otimes 2}$ and leaves all other subsystems unchanged.

**Theorem 1** Let $\rho$ be a biseparable $n$-partite density matrix acting on Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, where $\dim \mathcal{H}_l = d_l$, $l = 1, 2, \cdots, n$. Then

$$\sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho^{\otimes 2} P_i | \Phi_{ij}\rangle} \leq \sum_{i \neq j} \langle \Phi_{ij} | P_i^{+} \rho^{\otimes 2} P_i | \Phi_{ij}\rangle + (n - 2) \sum_i \sqrt{\langle \Phi_i | P_i^{+} \rho^{\otimes 2} P_i | \Phi_i\rangle},$$

(5)

If an $n$-partite state $\rho$ does not satisfy the inequality above, then $\rho$ is genuine $n$-partite entangled.

**Proof.** To prove that inequality (5) is indeed satisfied by all biseparable states $\rho$, let us first verify that this holds for any pure state $\rho$ which is biseparable under some partition.

Suppose that $\rho = |\psi\rangle\langle \psi|$ is a biseparable pure state under the partition of $\{1, 2, \cdots, n\}$ into two disjoint subsets: $\{1, 2, \cdots, n\} = A \cup B$ with $A = \{j_1, j_2, \cdots, j_k\}$ and $B = \{j_{k+1}, \cdots, j_n\}$, and

$$|\psi\rangle = |\psi_1\rangle_{j_1j_2\cdots j_k}|\psi_2\rangle_{j_{k+1}\cdots j_n} = \sum_{i_1, i_2, \cdots, i_k} a_{i_1 i_2 \cdots i_k} |i_1i_2 \cdots i_k\rangle_{j_1j_2 \cdots j_k} \sum_{i_{k+1}, \cdots, i_n} b_{i_{k+1} \cdots i_n} |i_{k+1} \cdots i_n\rangle_{j_{k+1} \cdots j_n}$$

(6)

then

$$\rho = \sum_{i, j} \sum_{d_1, d_2, \cdots, d_n = 1} a_{i_1 i_2 \cdots i_k} b_{i_{k+1} \cdots i_n} a_{i_1 i_2 \cdots i_k}^* b_{i_{k+1} \cdots i_n}^* \delta_{i_1 j_1} \delta_{i_2 j_2} \cdots \delta_{i_n j_n}$$

(7)

Here the sum is over all possible values of $i_1, i_2, \cdots, i_n$, i.e., $\sum_{i_1, i_2, \cdots, i_n} = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_n} - 1 \cdot 1 \cdots 1 = 1$.

We will distinguish between the cases in which both indices $i$ and $j$ corresponding to different, or the same parts $A$ and $B$ in the bipartition with respect to $|\psi\rangle$. By calculation, one has

$$\sqrt{\langle \Phi_{ij} | \rho^{\otimes 2} P_i | \Phi_{ij}\rangle} = \langle \phi_i | \rho | \phi_j\rangle$$

$$= \langle \phi_i | \rho | \phi_i\rangle \langle \phi_j | \rho | \phi_j\rangle$$

$$\leq \frac{1}{2} \sqrt{\langle \Phi_{ij} | P_i^{+} \rho^{\otimes 2} P_i | \Phi_{ij}\rangle + \langle \Phi_{ij} | P_j^{+} \rho^{\otimes 2} P_j | \Phi_{ij}\rangle}$$

(8)

in case of either $i, j \in A$ or $i, j \in B$, and

$$\sqrt{\langle \Phi_{ij} | \rho^{\otimes 2} P_i | \Phi_{ij}\rangle} = \langle \phi_i | \rho | \phi_j\rangle$$

$$= \sqrt{\langle \phi_0 | \rho | \phi_0\rangle \langle \phi_j | \rho | \phi_j\rangle}$$

$$= \langle \Phi_{ij} | P_i^{+} \rho^{\otimes 2} P_i | \Phi_{ij}\rangle$$

(9)
in case of either \( i \in A, j \in B \) or \( i \in B, j \in A \). Here \(|\phi_0\rangle = |xx \cdots \rangle\), and \(|\phi_{ij}\rangle = |x \cdots xy \cdots x \cdots y \cdots x \rangle\) such that all particles are in the state \(|x\rangle\) except the \( i \)th and \( j \)th particles are in the state \(|y\rangle\). Combining (8) and (11) gives that
\[
\sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} \leq \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} + (n - 2) \sum_{i} \sqrt{\langle \Phi_{ii} | \rho_{ii} \rangle} \sqrt{\langle \Phi_{ii} | \rho_{ii} \rangle}.
\]

Hence, Ineq.(5) is satisfied by all biseparable \( n \)-partite pure states.

Next we show that Ineq.(5) is also true for all biseparable \( n \)-partite mixed states. Indeed, the generalization of Ineq.(5) to mixed states is a direct consequence of the convexity of its left hand side and the concavity of its right hand side, which we can see in the following.

Suppose that
\[
\rho = \sum_{m} m_{m} \rho_{m} = \sum_{m} m_{m} |\psi_{m}\rangle \langle \psi_{m}|
\]
is biseparable \( n \)-partite mixed state, where \( m_{m} = |\psi_{m}\rangle \langle \psi_{m}| \) is biseparable. Then, by Cauchy-Schwarz inequality
\[
(\sum_{k=1}^{m} x_{k} y_{k})^{2} \leq (\sum_{k=1}^{m} x_{k}^{2})(\sum_{k=1}^{m} y_{k}^{2}),
\]
one has
\[
\sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} \leq \sum_{i \neq j} \sum_{m} m_{m} \sqrt{\langle \Phi_{ij} | \rho_{m} \rangle} \sqrt{\langle \Phi_{ij} | \rho_{m} \rangle} + (n - 2) \sum_{i} \sqrt{\langle \Phi_{ii} | \rho_{ii} \rangle} \sqrt{\langle \Phi_{ii} | \rho_{ii} \rangle}.
\]

which finishes the proof of Ineq.(5).

It is worth pointing out that inequality (III) in Ref.[27], which can be rewritten as
\[
\sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} \leq (n - 2) \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle},
\]
is the corollary of this theorem. The reason is as follows: Note that the second summation in inequality (III) of Ref.[27], the right side of inequality above, can be re-expressed as
\[
(n - 2) \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} = \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} + (n - 2) \sum_{i} \sqrt{\langle \Phi_{ii} | \rho_{ii} \rangle} + (n - 3) \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle}.
\]
in case of \( n \geq 3 \) and all terms in the third summation term of the right side of above equality are expectation values of positive operators, which implies that
\[
\sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle} + (n - 2) \sum_{i} \sqrt{\langle \Phi_{ii} | \rho_{ii} \rangle} \leq (n - 2) \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho_{ij} \rangle}.
\]
FIG. 1: (Color online) Detection quality for the state $\rho^{(G-W_n)} = \frac{1-a-b}{2^n}I + a|GHZ_n\rangle\langle GHZ_n| + \beta|W_n\rangle\langle W_n|$, $n = 10$. Here the (red) line $a$ represents the threshold given by inequality (5) in Theorem 1 such that the region above it identifies genuine 10-partite entanglement. The regions above lines $b$ (blue) and $c$ (green) correspond to the genuine entanglement detected by inequalities (II) in [27] (also [23]) and (III) in [27] respectively. The area enclosed by the red curve $a$, the blue curve $b$, the green curve $c$, and the $\beta$ axis contains the genuine 10-partite entanglement detected only by inequality (5) in Theorem 1.

Thus, Ineq. (13), inequality (III) in Ref. [27], follows from Theorem 1 and Ineq. (15).

Theorem 1 deserves comments. It is better than inequality (III) of Ref. [27] in the case of genuine multipartite entanglement detection for $n$-partite quantum states. This criterion detects genuine $n$-partite entanglement (for $n$-qubit states such as $W$ state mixed with white noise, and the mixture of the GHZ state and the $W$ state, dampened by isotropic noise) that had not been identified so far.

Example 1 Consider the family of $n$-qubit states

$$\rho^{(G-W_n)} = \frac{1-a-b}{2^n}I + a|GHZ_n\rangle\langle GHZ_n| + \beta|W_n\rangle\langle W_n|,$$  \hspace{1cm} (16)

the mixture of the GHZ state and the $W$ state, dampened by isotropic noise. Here

$$|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|00\cdots0\rangle + |11\cdots1\rangle)$$ \hspace{1cm} (17)

and

$$|W_n\rangle = \frac{1}{\sqrt{n}}(|00\cdots001\rangle + |00\cdots010\rangle + \cdots + |10\cdots00\rangle)$$ \hspace{1cm} (18)

are $n$-qubit GHZ state and $W$ state, respectively. For this family, our criteria can detect genuine $n$-partite ($n \geq 4$) entanglement that had not been identified so far. The detection parameter spaces of the inequality (5) in Theorem 1, inequality (III) in [27], and inequality in [23] and inequality (II) in [27] for $n = 10$, are illustrated in Fig. 1.

Example 2 Let us consider the $n$-qubit state, $W$ states mixed with white noise,

$$\rho^{(W_n)}(p) = \frac{p}{2^n}I + (1-p)|W_n\rangle\langle W_n|.$$ \hspace{1cm} (19)

By Theorem 1 above and Theorem 3 of Ref. [29], we derive that if $0 \leq p < \frac{2^n}{n(2n-3)+2^n}$, then $\rho^{(W_n)}(p)$ is genuine $n$-partite entangled, while from inequality (III) of Ref. [27], one can obtain that if $0 \leq p < \frac{2^n}{n(2n-2)+2^n}$, then $\rho^{(W_n)}(p)$ is genuine $n$-partite entangled. That is, our criteria detect $W$ state mixed with white noise, $\rho^{(W_n)}(p)$, for $0 \leq p < \frac{2^n}{n(2n-3)+2^n}$, as genuinely $n$-partite entangled, whereas inequality (III) of Ref. [27] detects it only for $0 \leq p < \frac{2^n}{n(2n-2)+2^n}$. For the special case $n = 3$ our criteria coincide. When $n = 3$, in Ref. [30] $\rho^{(W_n)}(p)$ was found to be genuinely multipartite entangled by means of the best known entanglement witness up to a threshold of $p < \frac{8}{17}$. This bound was then improved to $p < \frac{8}{17}$ [23, 27], which is also our result. When $n = 4$, both Theorem 1 and the previous results [23, 29]...
Therefore, inequality (20) holds with equality if \( \rho \) is the absolute value of matrix element \( p_{16} \) under permutation of each element \( n \) as genuine multipartite entangled, are for the first time detected by inequality (5) in Theorem 1 and the inequality (III) in [27], respectively. \( \rho(W_n)(p) \), for \( \frac{n^2}{n(n-2)+2^n} \leq p < \frac{n^2}{n(n-3)+2^n} \), as genuine \( n \)-partite \((n \geq 5)\) entangled, are for the first time detected by inequality (5) in Theorem 1.

**Theorem 2** Every fully separable \( n \)-partite state \( \rho \) satisfies

\[
\sqrt{\langle \Phi | \rho \otimes^2 P | \Phi \rangle} \leq \left( \prod_{A \in S} \langle \Phi | P_A^+ \rho \otimes^2 P_A | \Phi \rangle \right)^{\frac{2^n+1}{2}}, \tag{20}
\]

for fully separable states \( |\Phi\rangle \), where \( S \) is the set of all nonempty proper subsets of \( \{1, 2, \ldots, n\} \), the permutation operators \( P_A \) are the operators permuting the two copies of all subsystems contained in the set \( A \), and \( P \) is the total permutation operator, permuting the two copies.

This inequality is equality for fully separable \( n \)-partite pure states.

**Proof.** We start by showing that the Ineq. (20) holds for pure states. So, let us suppose that \( \rho \) is \( n \)-partite fully separable pure state and \( |\Phi\rangle = |\Phi_1\rangle |\Phi_2\rangle \) with fully separable \( n \)-partite states \( |\Phi_1\rangle \) and \( |\Phi_2\rangle \). The left side of Ineq. (20) is the absolute value of matrix element \( \langle \Phi_1 | \rho | \Phi_2 \rangle \):

\[
\sqrt{\langle \Phi | \rho \otimes^2 P | \Phi \rangle} = |\langle \Phi_1 | \rho | \Phi_2 \rangle|, \tag{21}
\]

since \( P \) simply permutes \( |\Phi_1\rangle \) and \( |\Phi_2\rangle \), i.e., \( P|\Phi_1\rangle \otimes |\Phi_2\rangle = |\Phi_2\rangle \otimes |\Phi_1\rangle \). Due to its fully separability, \( \rho \otimes^2 \) is invariant under permutation of each element \( A \) of \( S \):

\[
P_A^+ \rho \otimes^2 P_A = \rho \otimes^2. \tag{22}
\]

Thus,

\[
\sqrt{\langle \Phi | \rho \otimes^2 P | \Phi \rangle} = |\langle \Phi_1 | \rho | \Phi_2 \rangle| \leq \sqrt{\langle \Phi_1 | \rho | \Phi_1 \rangle \langle \Phi_2 | \rho | \Phi_2 \rangle} = \sqrt{\langle \Phi | \rho \otimes^2 | \Phi \rangle} = \left( \prod_{A \in S} \sqrt{\langle \Phi | P_A^+ \rho \otimes^2 P_A | \Phi \rangle} \right)^{\frac{2^n+1}{2}}, \tag{23}
\]

as claimed. Here we have used the positivity of density matrix in the first inequality and the cardinality \( |S| \) of \( S \) being \( 2^n - 2 \) (\( S \) has exactly \( 2^n - 2 \) elements) in the third equality. In fact, for any fully separable pure state \( \rho \), straightforward computation yields

\[
|\langle \Phi_1 | \rho | \Phi_2 \rangle| = \sqrt{\langle \Phi_1 | \rho | \Phi_1 \rangle \langle \Phi_2 | \rho | \Phi_2 \rangle}. \tag{24}
\]

Therefore, inequality (20) holds with equality if \( \rho \) is fully separable pure state.
It remains to show that Ineq. (20) holds if $\rho$ is mixed state. Now we suppose that $\rho = \sum p_i \rho_i$ is a fully separable $n$-partite mixed state, where $\rho_i$ are fully separable pure states. As the absolute value is convex, i.e., $|a + b| \leq |a| + |b|$ for arbitrary complex number $a$ and $b$, and Ineq. (20) is satisfied by fully separable pure state $\rho_i$, one gets
\[
\sqrt{\langle \Phi | \rho^{\otimes 2} P | \Phi \rangle} = \left| \langle \Phi_1 | \rho_2 \rangle \right| \leq \sum_i p_i | \langle \Phi_1 | \rho_i_2 \rangle | = \sum_i p_i \sqrt{\langle \Phi | \rho_i_2 \rangle}.
\]
By continuously using the Hölder inequality
\[
\sum_{k=1}^m |x_k y_k| \leq \left( \sum_{k=1}^m |x_k|^p \right)^{1/p} \left( \sum_{k=1}^m |y_k|^q \right)^{1/q} (p, q > 1, 1/p + 1/q = 1),
\]
we obtain that
\[
\sum_i p_i \left( \Pi_{A \in S} \langle \Phi | P_A^+ \rho_i^{\otimes 2} P_A | \Phi \rangle \right)^{1/2} \leq \left( \Pi_{A \in S} \langle \Phi | P_A^+ \rho^{\otimes 2} P_A | \Phi \rangle \right)^{1/2},
\]
where in the second inequality we have used $\rho^{\otimes 2} - \sum_i p_i^{2} \rho_i^{\otimes 2} = \sum_i p_i \rho_i \otimes \rho_j \geq 0$, since density matrices $\rho_i$ are positive semi-definite, i.e., $\rho_i \geq 0$. Combining Ineqs. (25) and (27) gives Ineq. (20), as required. This completes the proof.

In particular, if $\rho$ is fully separable $n$-qubit state, then this theorem for $|\Phi \rangle = 00 \cdots 0 |11 \cdots 1)$ implies
\[
|\rho_{1,2^n} | \leq (\rho_{2,2} 3,3 \rho_{4,4} \cdots \rho_{2^n-1,2^n-1})^{1/2},
\]
the first inequality of Theorem 4 in Ref. [29], which is necessary and sufficient condition [29] for GHZ state mixed with white noise, $\rho(p) = (1 - p)|\text{GHZ}_n\rangle \langle \text{GHZ}_n| + \frac{p}{2^n} I$ as fully separable, where $|\text{GHZ}_n\rangle = \frac{1}{\sqrt{2^n}} (|00 \cdots 0\rangle + |11 \cdots 1\rangle)$.

For detection of non-separable quantum states, Theorem 2 is as strong as the PPT criterion and criterion (*) in Ref. [28]. Consider the most general maximally entangled state (general GHZ states) for $n$-qudits mixed with white noise
\[
\rho = p |\Psi \rangle \langle \Psi | + \frac{1 - p}{d^n} 1_{d^n},
\]
where
\[
|\Psi \rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle^\otimes n.
\]
Direct calculation of inequality (20) yields that these states are non-separable (not fully separable) if
\[
p > \frac{1}{1 + d^n - 1},
\]
which is exactly the threshold detected by PPT criterion and criterion (*) in Ref. [28].

**Theorem 3** Suppose that $\rho$ is a fully separable $n$-partite state. Then the following inequality
\[
\sum_{i \neq j} \sqrt{\langle \Phi_{ij} | \rho^{\otimes 2} P | \Phi_{ij} \rangle} \leq \sum_{i \neq j} \sqrt{\langle \Phi_{ij} | P_i^+ \rho^{\otimes 2} P_i | \Phi_{ij} \rangle}
\]
holds with equality if $\rho$ is a pure state.

**Proof.** Note that the left side of the inequality (32) minus the right side of (32) is a convex function of the matrix $\rho$ entries (since the left side is the summation of absolute values of density matrix elements and the right hand
is the summation of the square root of a product of two diagonal matrix elements). Consequently, it suffices to prove the validity for fully separable pure states and validity for mixed states is guaranteed.

Similar to the proof of Theorem 1 we need only to prove that inequality (32) holds for fully separable pure states. Suppose that $\rho$ is a pure state. Since $\rho$ is a fully separable pure state, this gives

$$|\langle \phi_i | \rho | \phi_j \rangle| = \sqrt{\langle \phi_i | \rho | \phi_i \rangle \langle \rho | \phi_j \rangle} = \sqrt{\langle \phi_0 | \rho | \phi_0 \rangle \langle \rho | \phi_j \rangle}, \quad (33)$$

$$P_i^+ \rho \otimes^2 P_i = \rho \otimes^2, \quad (34)$$

where $|\phi_0\rangle$ and $|\phi_{ij}\rangle$ are the same as that in Theorem 1. Applying these two equalities, we have

$$\sum_{i\neq j} \sqrt{\langle \phi_{ij} | \rho \otimes^2 P | \phi_{ij} \rangle} = \sum_{i\neq j} |\langle \phi_i | \rho | \phi_j \rangle| = \sum_{i\neq j} \sqrt{\langle \phi_i | \rho | \phi_i \rangle \langle \rho | \phi_j \rangle} = \sum_{i\neq j} \sqrt{\langle \phi_0 | \rho | \phi_0 \rangle \langle \rho | \phi_j \rangle} = \sum_{i\neq j} \sqrt{\langle \phi_{ij} | P_i^+ \rho \otimes^2 P_i | \phi_{ij} \rangle}, \quad (35)$$

as desired. This completes the proof.

For the $n$-qubit W state mixed with white noise, $\rho^{(W_n)}(p)$, equation (32) detects entanglement for

$$p < \frac{2^n}{2^n + n}, \quad (36)$$

that is, $\rho^{W_n}(p)$ is entangled (not fully separable) if $p < \frac{2^n}{2^n + n}$.

Our criteria are experimentally accessible without quantum state tomography. Each term in the left hand side of our criteria can be determined by measuring two observables, while each term in the right hand side can be determined by one observable. For any fixed $|\phi_{ij}\rangle$, Eq. (3) and Eq. (32) can be determined by $n^2 + 1$ and $n^2 + n + 1$ density matrix elements, respectively. For any fixed $|\Phi\rangle$, Eq. (20) can be determined by $2^n - 1$ density matrix elements. Compared to the $(d_i^2 - 1)(d_j^2 - 1)\cdots(d_n^2 - 1)$ measurements needed for quantum state tomography, which requires an exponentially increasing since $(d_i^2 - 1)(d_j^2 - 1)\cdots(d_n^2 - 1) = (d^2 - 1)^n$ in case of all subsystems with same dimension $d$, the numbers of density matrix elements in our criteria not only grows significantly slower with $n$, but have great advantage of being independent of the dimension $d$ of the subsystem $l$, $l = 1, 2, \ldots, n$.

The observables associated with each term (diagonal matrix elements) of the right hand side in Eq. (3) and Eq. (32) can be implemented by means of local observables, which can be seen from the following expressions $|\phi_0\rangle\langle \phi_0 | = P \otimes^n$, $|\phi_{ij}\rangle\langle \phi_{ij} | = P^{\otimes(i-1)} \otimes Q \otimes P^{\otimes(j-i-1)} \otimes Q \otimes P^{\otimes(n-j)}$, and $|\phi_i\rangle\langle \phi_i | = P^{\otimes(i-1)} \otimes Q \otimes P^{\otimes(n-i)}$, where $P = |x\rangle\langle x |$ and $Q = |y\rangle\langle y |$. Similarly, each term of the right hand side in Eq. (20) can also be determined by local measurement. Thus, determining one diagonal matrix element requires only a single local observable.

From $\sqrt{\langle \phi_{ij} | \rho \otimes^2 P | \phi_{ij} \rangle} = |\langle \phi_i | \rho | \phi_j \rangle|$ and $\sqrt{\langle \phi_{ij} | \rho \otimes^2 P | \phi_{ij} \rangle} = |\langle \Phi_1 | \rho | \Phi_2 \rangle|$, next, we should determine modulus of the off diagonal elements $|\langle \phi_i | \rho | \phi_j \rangle|$ by measuring two observables $O_{ij}$ and $\tilde{O}_{ij}$, and $|\langle \Phi_1 | \rho | \Phi_2 \rangle|$ by measuring $O$ and $\tilde{O}$, since $(O_{ij}) = 2\text{Re}(\langle \phi_i | \rho | \phi_j \rangle)$, $(\tilde{O}_{ij}) = -2\text{Im}(\langle \phi_i | \rho | \phi_j \rangle)$, $(O) = 2\text{Re}(\langle \Phi_1 | \rho | \Phi_2 \rangle)$, and $(\tilde{O}) = -2\text{Im}(\langle \Phi_1 | \rho | \Phi_2 \rangle)$. Here $O_{ij} = |\phi_i\rangle\langle \phi_j | + |\phi_j\rangle\langle \phi_i |$, $\tilde{O}_{ij} = -i|\phi_i\rangle\langle \phi_j | + i|\phi_j\rangle\langle \phi_i |$, $O = |\Phi_1\rangle\langle \Phi_2 | + |\Phi_2\rangle\langle \Phi_1 |$, and $\tilde{O} = -i|\Phi_1\rangle\langle \Phi_2 | + i|\Phi_2\rangle\langle \Phi_1 |$.

Without loss of generality, let $i < j$. From

$$O_{ij} = \frac{1}{2} P^{\otimes(i-1)} \otimes M \otimes P^{\otimes(j-i-1)} \otimes M \otimes P^{\otimes(n-j)} + \frac{1}{2} P^{\otimes(i-1)} \otimes \tilde{M} \otimes P^{\otimes(j-i-1)} \otimes \tilde{M} \otimes P^{\otimes(n-j)}, \quad (37)$$

$$\tilde{O}_{ij} = \frac{1}{2} P^{\otimes(i-1)} \otimes M \otimes P^{\otimes(j-i-1)} \otimes M \otimes P^{\otimes(n-j)} - \frac{1}{2} P^{\otimes(i-1)} \otimes \tilde{M} \otimes P^{\otimes(j-i-1)} \otimes \tilde{M} \otimes P^{\otimes(n-j)}, \quad (38)$$

where $M = |y\rangle\langle x | + |x\rangle\langle y |$, $\tilde{M} = i|y\rangle\langle x | - i|x\rangle\langle y |$, one can determine the left hand side in Eq.(5) by $(n^2 - n)$ local observables.

Suppose $|\Phi_1\rangle = |x_1 x_2 \ldots x_n \rangle$, $|\Phi_2\rangle = |y_1 y_2 \ldots y_n \rangle$. Let $R_l = |y_l\rangle\langle x_l | + |x_l\rangle\langle y_l |$ and $\tilde{R}_l = i|y_l\rangle\langle x_l | - i|x_l\rangle\langle y_l |$, $l = 1, 2, \ldots, n$. Following the method of [31, 32], element $\sqrt{\langle \Phi | \rho \otimes^2 P | \Phi \rangle}$ can be obtained from two local measurement settings $R_l$ and $\tilde{R}_l$, given by

$$\mathcal{M}_l = \left[ \cos \left( \frac{l\pi}{n} \right) R_l + \sin \left( \frac{l\pi}{n} \right) \tilde{R}_l \right]^\otimes n, \quad l = 1, 2, \ldots, n, \quad (39)$$

$$\tilde{\mathcal{M}}_l = \left[ \cos \left( \frac{l\pi + \pi/2}{n} \right) R_l + \sin \left( \frac{l\pi + \pi/2}{n} \right) \tilde{R}_l \right]^\otimes n, \quad l = 1, 2, \ldots, n. \quad (40)$$
These operators obey
\[
\sum_{l=1}^{n} (-1)^l \mathcal{M}_l = nO, \quad (41)
\]
\[
\sum_{l=1}^{n} (-1)^l \tilde{\mathcal{M}}_l = n\tilde{O}, \quad (42)
\]
which can be proved in the same way as \cite{31,32}.

Therefore in total at most \(5(n^2-n) + n + 1\), \(5(n^2-n) + 1\), and \(2^n + 2n - 2\) local observables are needed to test our separability criteria Eq. (45), Eq. (20), and Eq. (32), respectively.

In conclusion, we investigate \(n\)-partite quantum states from elements of density matrices and derive practical separability criteria to identify genuinely entangled and non-separable \(n\)-partite mixed quantum state. We show cases in which our criteria is stronger than all known separability criteria. In fact, our criteria detect genuine \(n\)-partite entanglement that had not been identified so far. It has the added appeal of being relatively easy to compute and requiring far fewer measurements to implement experimentally compared to full quantum tomography.

This work was supported by the National Natural Science Foundation of China under Grant No: 10971247, Hebei Natural Science Foundation of China under Grant Nos: F2009000311, A2010000344.

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