Multilevel approximation of backward stochastic differential equations

D. Becherer *  
Institut für Mathematik  
Humboldt Universität zu Berlin  
Unter den Linden 6  
D-10099 Berlin, Germany

P. Turkedjiev†  
Centre de Mathématiques Appliquées  
Ecole Polytechnique and CNRS  
Route de Saclay  
F 91128 Palaiseau cedex, France

December 11, 2014

Abstract

We develop a multilevel approach to compute approximate solutions to backward differential equations (BSDEs). The fully implementable algorithm of our multilevel scheme constructs sequential martingale control variates along a sequence of refining time-grids to reduce statistical approximation errors in an adaptive and generic way. We provide an error analysis with explicit and non-asymptotic error estimates for the multilevel scheme under general conditions on the forward process and the BSDE data. It is shown that the multilevel approach can reduce the computational complexity to achieve precision $\epsilon$, ensured by error estimates, essentially by one order (in $\epsilon^{-1}$) in comparison to established methods, which is substantial. Computational examples support the validity of the theoretical analysis, demonstrating efficiency improvements in practice.

1 Introduction

The concept of Multilevel Monte Carlo has been introduced by [18] as a simulation method for the efficient computation of linear expectations $E[\Phi(X)]$ of functions $\Phi$ of diffusion processes $X$; see also [27]. Multilevel Monte Carlo (MLMC) is an active research area, evolving in many directions; For instance, [13] studies the case where $X$ is the solution to a Lévy-driven stochastic differential equation and [4] develops a multilevel approach for the problem of optimal stopping, where an expectation $E[\Phi(X_\tau)]$ is maximized over a family of stopping times $\tau$.

Our paper develops a novel multilevel approximation algorithm for solutions to backward stochastic differential equations, which can be seen as a non-linear generalization of the probabilistic Feynman-Kac representation for linear expectations of diffusions, with many applications in optimal control and

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*Authors acknowledge support from German Science Foundation DFG, Berlin Mathematical School and MATHEON
†Corresponding author: turkedjiev@cmap.polytechnique.fr Research supported by the Chair Financial Risks of the Risk Foundation, the FiME Laboratory, and the Chair Finance and Sustainable Development, under the aegis of Louis Bachelier Finance and Sustainable Growth laboratory, a joint initiative with Ecole Polytechnique.
mathematical finance, see e.g. [13]. To this end, we consider backward stochastic differential equations (BSDEs) of the form

\[ Y_i = \Phi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + N_{i,T}, \quad t \in [0, T], \]

on a filtered probability space \((\Omega, \mathcal{F}_t, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\), satisfying the usual conditions with finite horizon \(T < \infty\) and a \(q\)-dimensional Brownian motion \(W\). The terminal condition \(\Phi\) is a deterministic function satisfying some standard conditions (see Section 2), while \(X\) is an exogenously given Markov process with fixed initial value \(X_0 = x_0\), and \((N_{i,T})_{0 \leq t \leq T}\) is a martingale orthogonal to \(W\). A solution to (1) is a suitable pair \((Y,Z)\) of \(\mathbb{R} \times (\mathbb{R}^q)\)-valued processes. Typically, BSDEs cannot be solved explicitly and one uses discrete time approximation. Fixing a time-grid \(\pi := \{0 = t_0, \ldots, t_N = T\}\), let \((X_i)_{0 \leq i \leq N}\) be a suitable discrete time approximation of \((X_t)_{0 \leq t \leq T}\) on \(\pi\). We will build upon analysis in [24] on so-called multi-step forward dynamical programming (MDP) equations

\[ Y_i := \mathbb{E}[\Phi(X_N) + \sum_{j=i}^{N-1} f_j(X_j, Y_{j+1}, Z_j)(t_{j+1} - t_j)|\mathcal{F}_{t_i}], \quad \text{and} \]

\[ (t_{i+1} - t_i) \times Z_i := \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^\top(\Phi(X_N) + \sum_{j=i+1}^{N-1} f_j(X_j, Y_{j+1}, Z_j)(t_{j+1} - t_j))|\mathcal{F}_{t_i}]; \]

this process \((Y_i, Z_i)\) \((i = N, \ldots, 0)\) is called the discrete BSDE solution. Further, we make use of a known splitting technique to decompose the discrete BSDE into the sum \((Y, Z) = (\bar{y} + \tilde{y}, z + \tilde{z})\) of a the components of a system of two (discrete) BSDEs given by

\[ y_i := \mathbb{E}[\Phi(X_N)|\mathcal{F}_{t_i}] \quad \text{and} \quad (t_{i+1} - t_i) \times z_i := \mathbb{E}[(W_{t_{i+1}} - W_{t_i})^\top\Phi(X_N)|\mathcal{F}_{t_i}] \]

for \(i = N, \ldots, 0\), and, likewise,

\[ \bar{y}_i := \mathbb{E} \left[ \sum_{j=i}^{N-1} f_j(X_j, y_{j+1} + \bar{y}_{j+1}, z_j + \bar{z}_j)(t_{j+1} - t_j) \bigg| \mathcal{F}_{t_i} \right] \quad \text{and} \]

\[ (t_{i+1} - t_i) \times \bar{z}_i := \mathbb{E} \left[ (W_{t_{i+1}} - W_{t_i})^\top \sum_{j=i}^{N-1} f_j(X_j, y_{j+1} + \bar{y}_{j+1}, z_j + \bar{z}_j)(t_{j+1} - t_j) \bigg| \mathcal{F}_{t_i} \right]. \]

We call the system (3 - 4) for \((y, z)\) and \((\bar{y}, \bar{z})\) the splitting scheme. By adding together the equations (3) and (4), one recovers the original discrete BSDE (2). In order to solve the system (3 - 4), one must first solve for \((y, z)\) and then use that solution to solve \((\bar{y}, \bar{z})\). In general, one must approximate the conditional expectation operator to obtain a fully implementable algorithm, and for this we will make use of Monte Carlo least-squares regression, a method initiated in the BSDE context by [21]. Our method will be to develop a novel multi-grid algorithm, which we term the multilevel algorithm, in order to efficiently approximate \((y, z)\), then to use the so-called least-squares multi-step-forward dynamical programming algorithm (LSMDP) [24] to approximate \((\bar{y}, \bar{z})\). In this paper, we focus on the error between the solution of (3 - 4) and our fully implementable scheme; this is in the spirit of [32][24]. The present paper is not concerned with the error from using time-discretizing
schemes, like (2), to approximate (1); for analysis of this error, one can refer to extensive research [40, 9, 8, 20, 22, 29, 38, 28, 11, 16, 39, 33, 12].

Thanks to improved regularity properties, the LSMDP algorithm can be solved much more efficiently for (4) than for (2), particularly in high dimension. We show in Section 3.3 that the complexity of the multilevel algorithm is typically $O(\varepsilon^{-2d} \ln(\varepsilon^{-1} + 1)^d)$ whereas solving (2) with LSMDP incurs a complexity of $O(\varepsilon^{-4d} \ln(\varepsilon^{-1} + 1)^d)$, where $\varepsilon$ is the precision and $d$ is the dimension of $X$. The multilevel algorithm sequentially builds approximations of $(y, z)$ on a refining sequence of dyadic time-grids $\{\pi^{(k)} : k \geq 0\}$ and takes the form of an adaptive martingale control variates algorithm: assuming we have already constructed the the solution to (3) on the time-grid $\pi^{(k-1)} := \{0 = t_0^{(k-1)}, \ldots, t_{2^k-1}^{(k-1)} = T\}$, which we denote $(y^{(k-1)}, z^{(k-1)})$, we use it to construct the solution on $\pi^{(k)} := \{0 = t_0^{(k)}, \ldots, t_{2^k-1}^{(k)} = T\}$ as follows:

\[
y_i^{(k)} := \mathbb{E}[\Phi(X_{2^k}^{(k)}) - \sum_{j=\alpha(i)+1}^{2^k-1} z_j^{(k-1)}(W_{j+1}^{(k-1)} - W_j^{(k-1)})|\mathcal{F}_{t_i^{(k)}}],
\]

\[
(t_{i+1}^{(k)} - t_i^{(k)}) \times z_i^{(k)} := \mathbb{E}[(W_{t_{i+1}^{(k)}}^{(k)}) - W_{t_i^{(k)}}^{(k)})]^{-1} \left(\Phi(X_{2^k}^{(k)}) - y_i^{(k)} - \sum_{j=\alpha(i)+1}^{2^k-1} z_j^{(k-1)}(W_{j+1}^{(k-1)} - W_j^{(k-1)})\right)|\mathcal{F}_{t_i^{(k)}}],
\]

for $\alpha(i) := \max\{0 \leq j \leq 2^k - 1 : t_j^{(k-1)} \leq t_i^{(k)}\}$ and $i \in \{0, \ldots, 2^k - 1\}$. In order to solve (5), one must first solve $y_0^{(k)}$, then $z_0^{(k)}$, and proceed to the time-grid $\pi^{(k+1)}$. Observe that, under the conditional expectation operator, the multilevel scheme (5) matches the discrete BSDE (3) whenever $\pi = \pi^{(k)}$. However, when the conditional expectation is replaced by the Monte Carlo least-squares operator, the multilevel formulation suffers from substantially less variance than the LSMDP formulation. Indeed, we demonstrate in Section 3.3 that the complexity of the multilevel algorithm is typically $O(\varepsilon^{-2d} \ln(\varepsilon^{-1} + 1)^d)$ whereas the LSMDP algorithm for (3) incurs a complexity of $O(\varepsilon^{-3d})$; here, $\varepsilon$ is the precision and $d$ is the dimension of $X$. We see that we have an order one improvement (up to log terms) by using the multilevel algorithm, which is substantial. The overall complexity of using the splitting scheme with multilevel to approximate (3-4) is

\[O(\varepsilon^{-2d} \ln(\varepsilon^{-1} + 1)^d) + O(\varepsilon^{-4d/2} \ln(\varepsilon^{-1} + 1)^d)\]

which should be compared to the complexity of the LSMDP scheme $O(\varepsilon^{-4d} \ln(\varepsilon^{-1} + 1)^d)$ for (2), so we see a substantial overall gain in the complexity of the algorithm. The reduction of the complexity is largely because one needs to generate fewer simulations of the process $X$. This has the secondary effect that it reduces the memory needed to run the algorithm. Since we are typically working with high dimensional problems (e.g., $d \geq 5$), the memory usage is typically very high, therefore reducing the memory usage is extremely important for practical implementation.

To conclude the introduction, we summarise the novelty of our results and compare them to the existing literature. The majority of this paper is dedicated to the analysis of the multilevel scheme, which is, to the best of our knowledge, the first adaptive multi-grid algorithm for variance reduction in the approximation of BSDEs. We make great efforts to keep our results applicable in high generality and give examples of many situations of interest in which our assumptions are valid, see Section 2 - including some instances of discontinuous or path-dependent Markov process $X$. The
applicability of our results is not exclusive to the examples we present. We mention that the splitting scheme (3 - 4) has been studied in the literature in the past. In the continuous time setting, it was used by [22][39] in order to determine regularity properties of the BSDE (1) when the filtration $(\mathcal{F}_t)_{t \geq 0}$ is generated by the Brownian motion $W$. It has been used in the discrete time setting by [7] to design a numerical scheme based on a martingale basis technique, and in [24] for a proxy scheme for BSDEs. The common idea for both techniques is to make efficient use of some a-priori knowledge for good approximation of the solution $(y, z)$ to the discrete BSDE (3), if such is available e.g. though analytic knowledge of a suitable martingale basis or of approximate PDE solutions. In comparison, the multilevel scheme does not require such a-priori knowledge. We demonstrate that obtaining the approximation of $(y, z)$ is the most expensive part in approximating the splitting scheme without multilevel; this explains the overall efficiency gains that can be obtained by a (generic) multilevel approximation of $(y, z)$. We present explicit error estimates for our algorithm and demonstrate in a quantitative manner that we are able to obtain substantial complexity improvements; we also provide numerical examples to corroborate these claims. In the latter part of this paper, we determine explicit error estimates for the splitting scheme with multilevel used to approximate the solution of (3 - 4); $(\bar{y}, \bar{z})$ is approximated with an LSMDP scheme once $(y, z)$ is computed with the multilevel scheme.

We also use results on the improved regularity [39] of $(\bar{y}, \bar{z})$ from (4), compared to $(Y, Z)$ from (2), to demonstrate that one can obtain better complexity because one can choose a lower dimensional regression basis.

**Organization of the paper:** Section 1.1 provides some notation used within paper. In Section 2, we state the assumptions to be used throughout the paper, and give several examples to show that these assumptions permit our algorithm to apply in high generality. In Section 3, we present the multilevel scheme and compute explicit error estimates for the fully implementable scheme. These error estimates are then used to perform a complexity analysis that demonstrates a theoretical improved efficiency of the multilevel scheme compared to the LSMDP scheme. In Section 4, we perform the error analysis for the LSMDP scheme used to approximate $(\bar{y}, \bar{z})$ given that $(y, z)$ (3) has been computed using the multilevel scheme. Finally, in Section 5, we present a complexity analysis of the splitting scheme with multilevel, and compare it to the LSMDP scheme with and without splitting to demonstrate the efficiency gains. Numerical examples are included throughout to demonstrate the improved efficiency of the fully-implementable multilevel algorithm in actual computations.

### 1.1 Notation and conventions

For a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and sub-$\sigma$-algebra $\mathcal{G}$, we write $L_2(\mathcal{G})$ for the space of $\mathcal{G}$-measurable, square integrable random variables. Constants are always understood to be finite and non-negative. Filtrations in continuous time are taken to satisfy the usual condition of right continuity and completeness. Markov processes and semimartingales in continuous time are taken to have c\ad paths. Inequalities between random variables (c\ad processes) are understood to hold almost everywhere with respect to $\mathbb{P} (\mathbb{P} \otimes dt)$. For any vector or matrix $V$, we denote its transpose by $V^\top$. The usual Euclidian norm on some $\mathbb{R}^n$ (or $\mathbb{R}^{m \times n}$) is denoted by $|\cdot|$. For any functions $f : \mathbb{R}^k \to \mathbb{R}^n$, the supremum norm is denoted by $|f(\cdot)|_\infty := \sup_{x \in \mathbb{R}^k} |f(x)|$. For $L \geq 0$ and $l \in \mathbb{N}$, we define the truncation function $T_L : \mathbb{R}^l \to \mathbb{R}^l$ by $T_L(x) := (-L \vee x_1 \wedge L, \ldots, -L \vee x_l \wedge L)$. The multilevel approach is working along a refining sequence of time-grids and a sequence of approximating processes evolving on those. To this end, we introduce the following notation. For each $k \geq 0$, we denote by $\pi^{(k)} := \{t_0^{(k)}, \ldots, t_{2^k}^{(k)}\}$ a time-
grid with $2^k$ time-points, $\Delta_t^{(k)} := t_{i+1}^{(k)} - t_i^{(k)}$ the $(k+1)$-th time increment and $\Delta W_i^{(k)} := W_i^{(k+1)} - W_i^{(k)}$ the $k$-th level $(i+1)$-th Brownian increment. To deal with the the referencing of time indicies between the multiple time grids $\pi^{(k)}$, we define functions $\alpha : \{0, \ldots, 2^k\} \to \{0, \ldots, 2^{k-1}\}$ by

$$\alpha(i) := \alpha^{(k)}(i) := \max\{j \in \{0, \ldots, 2^{k-1}\} : t_j^{(k-1)} \leq t_i^{(k)}\}.$$  

To ease notation, we simply write $\alpha$ for $\alpha^{(k)}$ whenever the level $k$ is clear from the context. For the $\sigma$-algebras $\mathcal{F}_i^{(k)} := \mathcal{F}_{t_i^{(k)}}^{(i)}$ ($i = 0, \ldots, 2^k$) we denote the respective conditional expectations by $\mathbb{E}^k[.] := \mathbb{E}[.]|\mathcal{F}_i^{(k)}]$. A stochastic process $X = (X_t)_{t \in [0,T]}$ that is piecewise constant with nodes at the time points of $\pi^{(k)}$ we call discrete and write $X_t^{(k)} := X_{t_i^{(k)}}$. In straightforward way, any $(\mathcal{F}_i^{(k)})$-adapted process can be seen as a cadlag process in continuous time. We say that $X$ is $(\mathcal{F}_i^{(k)})$-adapted if $X_t$ is $\mathcal{F}_i^{(k)}$-measurable for each $i \in \{0, \ldots, 2^k\}$, and call it an $(\mathcal{F}_i^{(k)})$-martingale if it is a martingale in the filtration $(\mathcal{F}_i^{(k)})$. Finally, a $\pi^{(k)}$-Markov chain is a discrete process which is a $(\mathcal{F}_i^{(k)})$-Markov chain.

## 2 Assumptions

In this section, we state the conditions on the Markov processes/chains $X$, the time-grids $\pi^{(k)}$, the terminal function $\Phi$ and the driver $f$ for the paper. As in [24], we strive for a high level of generality under which the subsequent analysis is valid, to make results applicable to as wide a class of problems as possible. This includes but should not be restricted to the concrete examples of relevant practical problems, that are detailed in Section 2.4 to explain and illustrate our general assumptions, which might appear overly abstract at first sight. Section 2.2 derives elementary consequences for representations and a-priori estimates that will be useful in what follows. Some additional assumptions that will be required only for the analysis in Sections 4 and 5 will be detailed in Section 2.3.

### 2.1 General assumptions

The following conditions will hold throughout the entirety of this paper.

(A$_\Phi$) The function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is measurable and is uniformly bounded by $C_\Phi$.

(A$_X$) There is a family of $(\mathcal{F}_t)$-Markov processes $\{(X_s^{(t,x)})_{s \in [t,T]} : (t, x) \in [0,T] \times \mathbb{R}^d\}$ which share the same (possibly time-inhomogenous) Markov dynamics, in the sense that for the same semigroup of contraction operators $P_{t,s}$ ($t \leq s$) acting on bounded measurable functions $h : \mathbb{R}^d \to \mathbb{R}$ it holds

$$P_{t,s}h(x) = \mathbb{E}[h(X_s^{(t,x)})] \quad \text{and} \quad P_{t,s}h(X_t^{(u,x)}) = \mathbb{E}[h(X_s^{(u,x)}|\mathcal{F}_t)] \quad \text{for } 0 \leq u \leq t \leq s \leq T. \quad (7)$$

We let $X_s^{(t,x)} = x$ for $s \in [0,t]$. This family satisfies the following properties:

(i) the Markov process $X$ is in this family and satisfies $X = X_{(0,x_0)}$;

(ii) there is a constant $C_X$ such that, for all $x, x' \in \mathbb{R}^d$ and $s \in [t,T]$,

$$\mathbb{E}[|\Phi(X_s^{(t,x)}) - \Phi(X_t^{(t,x')})|^2] \leq C_X |x - x'|^2 \quad \text{and} \quad \mathbb{E}[|X_s^{(t,x)} - x|^2] \leq C_X (t - s);$$

$$\mathbb{E}[|\Phi(X_s^{(t,x)}) - \Phi(X_t^{(t,x')})|^2] \leq C_X |x - x'|^2 \quad \text{and} \quad \mathbb{E}[|X_s^{(t,x)} - x|^2] \leq C_X (t - s);$$
(iii) there exist deterministic functions \( u : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) and \( v : [0, T] \times \mathbb{R}^d \to (\mathbb{R}^q)^\top \), measurable, such that, for any \((t,x)\) in \([0, T] \times \mathbb{R}^d\), the square integrable (bounded) martingale \( y_s^{(t,x)} = E[\Phi(X^{(t,x)}_T)] \mid \mathcal{F}_s \) \((s \in [0, T])\) and the predictable integrand \( z^{(t,x)} \) from the Itô martingale representation

\[
y_s^{(t,x)} = \Phi(X^{(t,x)}_T) - \int_s^T z^{(t,x)}_r \, dW_r, \quad s \in [0, T],
\]

admit versions \( y_s^{(t,x)} = u(s, X_s^{(t,x)}) \) and \( z_s^{(t,x)} = v(s, X_s^{(t,x)}) \).

(iv) there exist constants \( \theta \in (0, 1) \) and \( C_X \geq 0 \) such that, for all \( t \in [0, T) \) and \( x \in \mathbb{R}^d \), \(|v(t, x)| \) is bounded by \( C_X/(T - t)^{(1 - \theta)/2} \). Moreover, the functions \( u(t, \cdot) \) and \( v(t, \cdot) \) are Lipschitz continuous with Lipschitz constants \( C_X/(T - t)^{(1 - \theta)/2} \) and \( C_X/(T - t)^{1 - \theta/2} \) respectively.

\((\mathbf{A}_x)\) The set of time-grids \( \{ \pi^k : k \geq 0 \} \) satisfies

(i) \( \pi^{(k+1)} \) is refinement of \( \pi^k \);

(ii) there exists a constant \( C_X \) such that \( \max_{0 \leq i \leq 2^k - 1} \Delta i^k \leq C_X 2^{-k} \);

(iii) there exists a constant \( c_X \) such that \( \min_{0 \leq i \leq 2^k - 1} \Delta i^k \geq c_X 2^{-k} \);

(iv) recalling the processes \( (y^{(t,x)}, z^{(t,x)}) \) solving \((\mathbf{8})\), there is a constant \( C_X \) such that for all \( k \geq 0 \), \( t^k_i \in \pi^k \) and \( x \in \mathbb{R}^d \),

\[
\sum_{j=0}^{2^k-1} \mathbb{E}\left[ \int_{t^k_i}^{t^k_{i+1}} |z_j^{(i^k,x)} - z_j^{(k,i,x)}|^2 \, dt \right] \leq C_X 2^{-k}, \quad \text{where } z_j^{(k,i,x)} := \frac{1}{\Delta j^k} \mathbb{E}_{t^k_i} \left[ \int_{t^k_i}^{t^k_{i+1}} z_t^{(i^k,x)} \, dt \right].
\]

\((\mathbf{A}_X)\) There is a family of \( \mathbb{R}^d \)-valued, \( \pi^k\)-Markov chains \( \{(X^{(k,i,x)} : x \in \mathbb{R}^d, 0 \leq i \leq 2^k, X_j^{(k,i,x)} = x \ \forall \ j \leq i\} \) satisfying the properties:

(i) recalling the parameter \( \theta \) from \((\mathbf{A}_X)\)(iv), there is a constant \( C_X \) such that, for all \( x \in \mathbb{R}^d \), \( k \geq 0 \) and \( i \in \{0, \ldots, 2^k\} \), \( \mathbb{E}[|\Phi(X^{(k,i,x)}_T) - \Phi(X^{(2^k-1,x)}_T)|^2] \leq C_X 2^{-k} \) and \( \mathbb{E}[|\Phi(X^{(2^k-1,x)}_T) - \Phi(x)|^2] \leq C_X 2^{-2\theta k} \);

(ii) for all \( k \geq 0 \) and \( j \in \{0, \ldots, 2^k\} \) and \( l \in \{j, \ldots, 2^k\} \), there exist a \( \mathcal{G}^{(k)}_j \otimes \mathcal{B}(\mathbb{R}^d) \)-measurable functions \( V^{(k)}_{j,l} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \) with a \( \sigma \)-algebra \( \mathcal{G}^{(k)}_j \subset \mathcal{F}_T \) independent of \( \mathcal{F}_i \) and containing \( \sigma(\Delta W_r : r \geq j) \), such that \( X^{(k,i,x)}_T = V^{(k)}_{j,l} (X^{(k,i,x)}_T) \);

(iii) there exists a constant \( C_X \) such that, for all \( x \in \mathbb{R}^d \), \( k \geq 1 \), \( i \in \{0, \ldots, 2^k\} \), and \( j \in \{i, \ldots, 2^k\} \)

\[
\mathbb{E}[|X_j^{(k,i,x)} - X_j^{(k-1,i,x)}|^2] \leq C_X 2^{-k}.
\]

For brevity, we denote the process \((X^{(k,0,x)}_i)_{0 \leq i \leq 2^k}\) by \((X^{(k)}_i)_{0 \leq i \leq 2^k}\).

### 2.2 Properties of the discrete BSDE derived from the assumptions

This section collects some elementary results that follow directly from the general conditions in Section 2.1 and are useful in Sections 3 – 5. Recall the family of Markov processes from \((\mathbf{A}_X)\). For every
For any time-grid $i \in \{0, \ldots, 2^k\}$ and $x \in \mathbb{R}^d$, the Kunita-Watanabe decomposition guarantees (unique) existence of a pair $(\tilde{y}^{(k)}, \tilde{z}^{(k)}) = (\tilde{y}^{(k,i,x)}, \tilde{z}^{(k,i,x)})$ of square integrable, $\pi^{(k)}$-adapted processes and a square integrable $(\mathcal{F}_{i}^{(k)})$-martingale $L^{(k)}$ such that

$$
\tilde{y}^{(k)}_l = \Phi(X^{(k,i,x)}_T) - \sum_{j=1}^{2^k-1} \tilde{z}^{(k)}_j \Delta W^{(k)}_j - \sum_{j=1}^{2^k-1} \Delta L^{(k)}_j, \quad l \geq i,
$$

(9)

where $\Delta L^{(k)}_l := L^{(k)}_{i+1} - L^{(k)}_i$, $L^{(k)}_0 = 0$, and $L^{(k)}$ is (strongly) orthogonal to $W^{(k)}$ in the sense that $(W^{(k)}_j L^{(k)}_j)_{0 \leq j \leq N}$ is an $(\mathcal{F}_{i}^{(k)})$-martingale, i.e. $\mathbb{E}^k_0[\Delta W^{(k)}_j \Delta L^{(k)}_j] = 0$ for all $j$. We will determine an explicit representation for $(\tilde{y}^{(k)}, \tilde{z}^{(k)}, L^{(k)})$ in terms the solution of the continuous time BSDE $(y^{(t^k)}, z^{(t^k), x})$ from (A$_X$)(iii) and the conditional expectation $\mathbb{E}^k_0[\cdot]$ in the next lemma. This permits one to establish important a-priori bounds on the process $\tilde{z}^{(k)}$ by Corollary 2.3.

**Lemma 2.1.** For any time-grid $\pi^{(k)}$, $k \geq 0$, $i \in \{0, \ldots, 2^k\}$, $l \geq i$, and $x \in \mathbb{R}^d$ holds

$$
\tilde{y}^{(k)}_l = \mathbb{E}^k_0[\Phi(X^{(l^k)}_T, x)] \quad \text{and} \quad \Delta L^{(k)}_l = \mathbb{E}^k_0[(\Delta W^{(k)}_l)^\top \Phi(X^{(l^k)}_T, x)],
$$

(10)

$$
\Delta L^{(k)}_l = \int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) - \tilde{z}^{(k)}_i) dW_s,
$$

(11)

$$
\Delta L^{(k)}_l \tilde{z}^{(k)}_l = \mathbb{E}^k_0[\int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) dt],
$$

(12)

**Proof.** Equalities in (10) are well known and easily obtained by taking conditional expectations of (9) itself or in product with $(\Delta W^{(k)}_l)^\top$. Equality $y^{(l^k)}_{t^{(l^k)}_i} = \mathbb{E}^k_0[\Phi(X^{(l^k)}_T, x)]$ implies

$$
y^{(k)}_i = y^{(l^k)}_{t^{(l^k)}_i} = \Phi(X^{(l^k)}_T, x) - \sum_{j=1}^{2^k-1} \tilde{z}^{(k)}_j \Delta W^{(k)}_j - \sum_{j=1}^{2^k-1} \int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) - \tilde{z}^{(k)}_i) dW_s
$$

and thereby $\Delta L^{(k)}_l = \int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) - \tilde{z}^{(k)}_i) dW_s$. Equality (12) follows easily. Indeed, multiplying with $(\Delta W^{(k)}_l)^\top$ and taking conditional expectation $\mathbb{E}^k_0[\cdot]$ yields $0 = \Delta L^{(k)}_l \tilde{z}^{(k)}_l - \mathbb{E}^k_0[\int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) dt]$. Moreover,

$$
0 = \mathbb{E}^k_0[\Phi(X^{(l^k)}_T, x) (\Delta W^{(k)}_l)^\top] - \mathbb{E}^k_0[(\Delta W^{(k)}_l)^\top \int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) dW_s] = \Delta L^{(k)}_l \tilde{z}^{(k)}_l - \mathbb{E}^k_0[\int_{t^{(l^k)}_i}^{t^{(l^k)}_{i+1}} (z^{(l^k)}_s, x) dt].
$$

which proves (12). □

Naturally, one obtains a Markov representation for the solutions of the discrete BSDEs (3 - 4).

**Lemma 2.2.** For all $k \geq 0$ and $j \in \{0, \ldots, 2^k - 1\}$, there exist deterministic functions $y^{(k)}_j : \mathbb{R}^d \to \mathbb{R}$ and $z^{(k)}_j : \mathbb{R}^d \to (\mathbb{R}^d)^\top$ such that

$$
\mathbb{E}^k_j[\Phi(X^{(k,i,x)}_0)] = y^{(k)}_j(X^{(k,i,x)}_0) \quad \text{and} \quad \frac{1}{\Delta L^{(k)}_j} \mathbb{E}^k_j[(\Delta W^{(k)}_j)^\top \Phi(X^{(k,i,x)}_0)] = z^{(k)}_j(X^{(k,i,x)}_0)
$$

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for all $i \in \{0, \ldots, 2^k\}$ and $x \in \mathbb{R}^d$. Moreover, there exist deterministic functions $\tilde{y}_i^{(k)} : \mathbb{R}^d \to \mathbb{R}$ and $\tilde{z}_i^{(k)} : \mathbb{R}^d \to (\mathbb{R}^d)^\top$, with $\tilde{y}_i^{(k)}(\cdot) = 0$, for $j \in \{0, \ldots, 2^k - 1\}$

$$
\mathbb{E}_j^k \left[ \sum_{l=j}^{N-1} f_l(X_{i}^{(k,i,x)} y_{l+1}(X_{l+1}^{(k,i,x)}), z_{l}^{(k)}(y_{l+1}(X_{l+1}^{(k,i,x)})), z_{l}^{(k)}(X_{l}^{(k,i,x)})) \Delta_{l}^{(k)} \right] = \tilde{y}_{j}^{(k)}(X_{j}^{(k,i,x)})
$$

and

$$
\mathbb{E}_j^k \left[ \frac{(\Delta W_{j}^{(k)})^\top}{\Delta_{j}^{(k)}} \left( \sum_{l=j}^{N-1} f_l(X_{l}^{(k,i,x)} y_{l+1}(X_{l+1}^{(k,i,x)}), z_{l}^{(k)}(y_{l+1}(X_{l+1}^{(k,i,x)})), z_{l}^{(k)}(X_{l}^{(k,i,x)})) \Delta_{l}^{(k)} \right) \right] = \tilde{z}_{j}^{(k)}(X_{j}^{(k,i,x)}) \quad \text{for all } i \in \{0, \ldots, 2^k\} \text{ and } x \in \mathbb{R}^d.
$$

This follows directly from $(A_X^*)$ by routine conditioning arguments from measure theory, like [24, Lemma 4.1].

Finally, we present almost sure absolute bounds, uniform in $x$, for the functions $y_i^{(k)}(x)$ and $z_i^{(k)}(x)$. Such bounds are crucial for and repeatedly used in Section 3.

**Corollary 2.3.** There exists a constant $C_X$ such that, for all $k \geq 0$, $i \in \{0, \ldots, 2^k\}$, $l \geq i$, and $x \in \mathbb{R}^d$, $|\tilde{z}_i^{(k)}| = |\tilde{z}_{l,i}^{(k)}|$ is bounded (a.s.) by

$$
C_{z,k,i} := \frac{C_X}{(T - t_i^{(k)})^{(1-\theta)/2}} \quad (13)
$$

This implies that, for all $k \geq 0$, the function $z_i^{(k)}(\cdot)$ is absolutely bounded by $C_{z,k,i}$. Further, there exists a constant $C_y$ independent of $k$ and $i$, such that $|y_i^{(k)}(\cdot)|$ is bounded by $C_y$ (a.s.).

**Proof.** Recall the version $v(t, X_t^{(k,i,x)})$ of $z_t^{(k,x)}$ from $(A_X^*)$(iii), and the absolute bound on the function $x \to v(t, x)$ from $(A_X^*)$(iv). Using the representation (12), it follows that

$$
|z_i^{(k)}(x)| \leq \frac{1}{\Delta_i^{(k)}} \left[ \mathbb{E}_i^k \left[ \int_{t_i^{(k)}}^{t_i^{(k+1)}} v(t, X_t^{(k,i,x)}) dt \right] \right] \leq \frac{C_X}{\Delta_i^{(k)}} \int_{t_i^{(k)}}^{t_i^{(k+1)}} dt \left( T - t_i^{(k)} \right)^{(1-\theta)/2} = \frac{2C_X(T - t_i^{(k)})}{\Delta_i^{(k)}(1 + \theta)(T - t_i^{(k)})^{(1-\theta)/2}} - \frac{2C_X(T - t_{i+1}^{(k)})}{\Delta_i^{(k)}(1 + \theta)(T - t_{i+1}^{(k)})^{(1-\theta)/2}} \leq \frac{2C_X \Delta_i^{(k)}}{\Delta_i^{(k)}(1 + \theta)(T - t_i^{(k)})^{(1-\theta)/2}}.
$$

In order to obtain bounds for $\tilde{z}_i^{(k)}(x)$, we use additionally the condition $(A_X^*)$(i) in order to obtain

$$
|\tilde{z}_i^{(k)}(x)| \leq |z_i^{(k)}(x) - \tilde{z}_i^{(k)}| + |\tilde{z}_i^{(k)}| \leq \frac{1}{\Delta_i^{(k)}} \mathbb{E}_i^k \left[ \Phi(X_{2x}^{(k,i,x)}) - \Phi(X_{t_i^{(k)}}^{(k,i,x)}) \right] + |\tilde{z}_i^{(k)}| \leq C_X + |\tilde{z}_i^{(k)}|
$$

We mildly abuse notation by stating $|\tilde{z}_i^{(k)}(x)| \leq C_X(T - t_i^{(k)})^{-(1-\theta)/2}$ to ease notation. The bound on $y_i^{(k)}$ is immediate from the boundedness of $\Phi(\cdot)$ in $(A_\Phi)$.

\[ \Box \]

### 2.3 Additional assumptions and properties for Section 4 and 5

The following conditions are used in Sections 4 and 5.

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Lemma 2.4. There exists a constant $C_X$ such that, for all $k \geq 0$, $i \in \{0, \ldots, 2^k - 1\}$, $x \in \mathbb{R}^d$ we have

$$|y_{i}^{(k)}(x)| \leq C_{y,i} := C_X (T - t_{i}^{(k)})(\theta_L + \theta)^2,$$

$$|z_{i}^{(k)}(x)| \leq C_{z,i} := \frac{C_X}{(T - t_{i}^{(k)})(1-(\theta_L \theta + \theta))/2}.$$
Proof. Recall the function $U(t, x)$ from Assumption $(A_t)$. The bound on $\tilde{y}_i^{(k)}(x)$ is obtained from the trivial decomposition $|\tilde{y}_i^{(k)}(x)| = |\tilde{y}_i^{(k)}(x) - U(t_i^{(k)}), x)| + |U(t_i^{(k)}, x)|$. Then, using the bounds from $(A_t)(\text{iii})$ and $(A_t^x)(\text{ii})$ (with $j = i$), the result on $|\tilde{y}_i^{(k)}(x)|$ follows. By a mild abuse of notation, we replace $C_X(T + T^\theta + \theta t)$ by $C_X$ to simplify notation.

For the bound on $|z_i^{(k)}(x)|$, we recall that $\Delta_i^{(k)}z_i^{(k)}(x) = \mathbb{E}_i^k[y_i^{(k)}(X_{i+1}^{(k)}, x) - U(t_i^{(k)}, x)]$ and treat the cases $i = 2^k - 1$ and $i < 2^k - 1$ separately. For $i = 2^k - 1$, the Cauchy-Schwarz inequality and $(A_t^x)(\text{i})$ yield

\[
(\Delta_{2^k-1}^{(k)})^2 |z_{2^k-1}^{(k)}(x)|^2 = \mathbb{E}_i^k \left[ (\Delta_{2^k-1}^{(k)})^2 |\Phi(X_{2^k-1}^{(k)}, x) - \Phi(x)| \right]^2 \leq C_X (\Delta_{2^k-1}^{(k)})^{1+2\theta},
\]

implying $|z_{2^k-1}^{(k)}(x)|^2 \leq C_X (T - t_i^{(k)})^{-1+2\theta}$, as required. For $i < 2^k - 1$, applying Cauchy-Schwarz yields

\[
(\Delta_i^{(k)})^2 |z_i^{(k)}(x)|^2 \leq 2(\Delta_i^{(k)})^2 C_X + 2(\Delta_i^{(k)})^2 C_X \frac{2(\Delta_i^{(k)})^2 C_X}{(T - t_i^{(k)})^{1-(\theta_L + \theta)}},
\]

where one exchanges $(T - t_i^{(k)})$ by $(T - t_i^{(k)})$ from $(A_t^x)$. By mild abuse of notation, we rewrite $C_X := C_X \sqrt{2C_X (1 + T^1-\theta_L + \theta)}$ to simplify the result. \(\square\)

2.4 Examples satisfying the general assumptions

This section details explicit examples of processes, time-grids and functions to illustrate and explain the conditions from Section 2.1.

Assumption $(A_t^x)$. Property (ii) is a Lipschitz continuity property of the payoff $\Phi(X_{T_i}^{(l,x)})$ with respect to the initial value $x$. It is satisfied if (a) $\Phi$ is locally Lipschitz continuous, i.e. for some constant $c$ and $l \in [0, \infty)$ holds $|\Phi(x) - \Phi(x')| \leq c|x|^l + |x|^l|x - x'|$ for all $x, x'$; or (b) Hölder continuous with Hölder exponent greater than or equal to 1/2, and, in both cases (a) and (b), $X_{T_i}^{(l,x)}$ solves an SDE (which may have jumps) with Lipschitz continuous coefficients. A lower Hölder regularity in case (b) would lower the convergence rate of the numerical scheme in Theorems 3.7, 3.9 cf. Remark 3.15.

Property (iii) is a classical property of Markovian BSDEs. It is satisfied when $X_{T_i}^{(l,x)}$ solves an SDE with deterministic (Markovian), Lipschitz continuous coefficient functions in a Brownian filtration [13, Theorem 4.1] or in a Lévy filtration [34, Proposition 4]. The property is also known to hold in the setting where $X_{T_i}^{(l,x)}$ is of the form $(S_{r_1^{(l,x)}}, \ldots, S_{r_n^{(l,x)}}, S_{T_i}^{(l,x)})$, where $S_{T_i}^{(l,x)}$ is the solution of an SDE in the Brownian filtration and $t \leq r_1 < \ldots < r_\ell \leq T$, see [34, 10].

There are two important instances where one can show Property (iv) to be valid. Firstly, suppose that $X_{T_i}^{(l,x)}$ solves an SDE with deterministic (Markovian), bounded and continuously differentiable coefficients, whose partial derivatives are bounded and Hölder continuous with diffusion coefficient

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being uniformly elliptic. Then \( v(t, x) = (\sigma(t, x)\nabla_x u(t, x))^T \) and the boundedness of \(|v(t, x)|\) follows from classical gradient bounds of parabolic PDEs \cite{15}; and \( \theta \) is equal to the Hölder exponent of \( \Phi \). Secondly, if \( \Phi \) is locally Lipschitz continuous, this result holds with \( \theta = 1 \) if \( X^{(t, x)} \) solves an SDE with deterministic (Markovian), Lipschitz continuous coefficients having linear growth; the path dependant setting \( X^{(t, x)} = (S_{t_1}^{(t, x)}, \ldots, S_{t_n}^{(t, x)}, S_{s}^{(t, x)}) \) - where \( S^{(t, x)} \) solves an SDE with deterministic (Markovian), Lipschitz continuous coefficients having linear growth - is also valid in this setting.

**Assumption** (\( A_x \)). Condition (i) is to ensure that \( t^{(k-1)}_{\alpha(i)+1} > t^{(k)}_i \) for all \( k \) and \( i \); later, when we introduce condition (\( A_X \)) in Theorem 3.7, this condition becomes crucial. It is satisfied by the time grids with points \( t^{(k)}_i := T - T(1 - i/2^k)^{1/\beta} \) for any \( \beta \in (0, 1) \), which includes the uniform time-grid.

Condition (iv) is the most complex of the requirements, and has been studied extensively in recent years. Let us first consider the case of the Brownian filtration. Then, condition (iv) is satisfied for (locally) Lipschitz continuous \( \Phi \) and uniform time-grids if \( X^{(t^{(k)}_i, x)} \) is the solution of an SDE with deterministic (Markovian), Lipschitz continuous coefficients of linear growth \cite{40, 37} local Lipschitz continuity is meant as described in (\( A_X \)). For Hölder continuous (fractionally smooth) \( \Phi \), it is satisfied by the time-grids with points \( t^{(k)}_i := T - T(1 - i/2^k)^{1/\beta} \) if \( \beta \) is less than the Hölder (fractional smoothness) exponent of \( \Phi \), and \( X^{(t^{(k)}_i, x)} \) solves a continuous SDE with deterministic (Markovian), bounded and twice continuously differentiable coefficients \( \sigma(t, x) \) for the drift and \( \sigma(t, x) \) for the volatility, whose partial derivatives are bounded and Hölder continuous, and \( \sigma \) is uniformly elliptic \cite{22}; note that the time-grid also satisfies properties (i)-(iii), cf. \cite{39} Lemma 5.3 for a proof of (ii). We remark that this rate of convergence may not be optimal, cf. \cite{20, 31}. The path dependent setting \( X^{(t^{(k)}_i, x)} = (S_{t_1}^{(t^{(k)}_i, x)}, \ldots, S_{t_n}^{(t^{(k)}_i, x)}, S_{s}^{(t^{(k)}_i, x)}) \) with \( \Phi \) fractionally smooth and \( S^{(t, x)} \) being the solution of an SDE with bounded, twice continuously differentiable coefficients, whose partial derivatives are bounded and Hölder continuous, also satisfies the condition if suitable time grids are used; cf. \cite{16}. In a filtration generated by a Lévy process, \cite{8} showed that the uniform time grid was sufficient to have this property if the terminal condition is of the form \( \Phi(X_T) \), for \( X^{(t^{(k)}_i, x)} \) solving an SDE with Lipschitz continuous coefficients of linear growth, and \( \Phi \) is Lipschitz continuous.

**Assumption** (\( A_X \)). Condition (i) is a “good-approximation” criterion for the Markov process by the Markov chain. It is satisfied if \( \Phi \) is (locally) Lipschitz continuous and \( X^{(t^{(k)}_i, x)} \) solves an SDE (with jumps) whose coefficients are deterministic (Markovian), Lipschitz continuous and have linear growth; \( X^{(k, i, x)} \) may be the Euler scheme approximation of \( X^{(t^{(k)}_i, x)} \) on the time-grid \( \pi^{(k)} \). However, if the terminal condition has a lower regularity, the Euler scheme might not satisfy this condition; for example, in the case where \( \Phi \) has only bounded variation, see \cite{1} Theorem 5.4. Higher order approximation schemes may be required for Hölder exponent \( \theta \) less than 1.

Condition (ii) is slightly stronger requirement on the Markov chain than the basic definition; it is satisfied by most approximation schemes for SDEs, including the Euler scheme.

Condition (iii) is a typical estimate required in multilevel Monte Carlo type approximation schemes for SDEs, cf. \cite{18, 19, 17} and references therein. It is a property satisfied, for instance, by the Euler scheme for an SDE with deterministic (Markovian), Lipschitz continuous coefficients of linear growth. If a convergence rate for the Markov chains were lower, a lower rate of convergence of the BSDE multilevel scheme would be obtained; see Remark 3.16.

**Assumption** (\( A_f \)). The condition (\( A_f \)) of Lipschitz continuous driver is standard in the literature \cite{32, 6, 8, 10} for \( \theta_L = 1 \), and has more recently been extended to the setting \( \theta_L < 1 \) \cite{24, 23, 39}. The case \( \theta_L < 1 \) allows to treat some cases of quadratic BSDEs \cite{24}. 

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In this section, we construct approximations the functions $y_i(x)$ and $z_i(x)$ to the novel multilevel structure of the algorithm. In Section 3.3, a comprehensive error analysis gives an upper bound for the global error

\[
\max_{0 \leq i \leq 2^k - 1} \mathbb{E}[|y_i^{(k)}(x) - y_i^{(k,M)}(x)|^2] + \sum_{i=0}^{2^k-1} \mathbb{E}[|z_i^{(k)}(x) - z_i^{(k,M)}(x)|^2] \Delta(k),
\]

and shows how it depends on numerical parameters (the number of Monte Carlo simulations, the choice of basis functions) and global error on the level $k-1$ of the algorithm. This error analysis enables us, in Section 3.3, to calibrate the numerical parameters of the multilevel algorithm and to compare its complexity to that of alternative algorithms.

### 3.1 Preliminaries

This section introduces ordinary least-squares regression (OLS) to approximate the conditional expectation operator in the multilevel scheme. We will build on a general but versatile Definition 3.1 for OLS to express our algorithms concisely. OLS admits an elementary theory (see Proposition 3.11), that enables a general (distribution-free) but tight error analysis in Section 3.3.
Algorithm 1. Initialize by setting $y_1^{(0)}(\cdot) := \Phi(\cdot), y_0^{(0)}(\cdot) := \mathbb{E}[\Phi(X_1^{(0)})]$ and $Tz_0^{(0)}(\cdot) := \mathbb{E}[W_T \Phi(X_1^{(0)})].$ Recursively for $k \geq 1,$ assume that $(y_i^{(k-1)}(\cdot), z_i^{(k-1)}(\cdot))$ have already been computed, set $y_{2k}^{(k)}(\cdot) = \Phi(\cdot),$ and, for any $i \in \{0, \ldots, 2^k - 1\},$ let $K_i$ be the space $L_2(\mathcal{B}(\mathbb{R}^d), \mathbb{P} \circ (X_i^{(k)})^{-1}; (\mathbb{R}^l)^\top)$ for $l' \in \{1, q\},$ and

$$\begin{align*}
y_i^{(k)}(\cdot) & \text{ solves } \text{OLS}(S_{Y,i}^{(k)}(\bar{x}, \bar{x}, \bar{w}), K_i, \nu_k) \text{ for } \\
y_i^{(k)}(\cdot) & \text{ solves } \text{OLS}(S_{Z,i}^{(k)}(\bar{x}, \bar{x}, \bar{w}), K_q, \nu_k) \text{ for }
\end{align*}$$

(20)

for $\bar{x} = (x_0, \ldots, x_2^k) \in \mathbb{R}^{(2^k+1) \times d}, \bar{x} = (\bar{x}_0, \ldots, \bar{x}_{2^k-1}) \in \mathbb{R}^{(2^k-1) \times d}, \bar{w} = (w_0, \ldots, w_{2^k-1}) \in \mathbb{R}^{d \times q},$ and $\nu_k$ being the law of $(X_0^0, \ldots, X_0^{k-1}, X_0^{(k-1)}, \ldots, X_0^{2^k-1}, \Delta W_0^k, \ldots, \Delta W_{2^k-1}^k)$.

Intuition for Definition 3.1. In Algorithm 1 above, we are using Definition 3.1 with respect to the (theoretical) law instead of the empirical measure (as in Algorithm 2). Here, $l = (2^k+1) \times d + 2^k \times q$ and $l' = 1$ (resp. $q$). The function $S(\cdot)$ is given by $S_{Y,i}^{(k)}(\cdot)$ (resp. $S_{Z,i}^{(k)}(\cdot)$), which is deterministic, hence there is no need for a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ here. Finally, the measure $\nu$ is the law of the trajectories of the Markov chain $X^{(k)}$ and the Brownian increments $\Delta W^{(k)},$ i.e. $\nu = \nu_k.$

Algorithm 1 in this form is not really implementable, but illustrates two computational issues that
we are going to overcome with the empirical least-squares regression algorithm in Section 3.2 below: firstly, the linear space \( K_1 \) (resp. \( K_q \)) is usually infinite dimensional, which is infeasible for actual computations; secondly, generic actual computation of the integrals \( \int \) is hindered in general by the fact that the law \( \nu \) may not be available in explicit terms.

### 3.2 Fully implementable algorithm

To avoid regression onto possibly infinite dimensional spaces \( K_1 \) and \( K_q \) as in Algorithm [1], we regress on predetermined (user defined) finite dimensional subspaces, defined as linear spans of finite sets of so-called basis functions:

**Definition 3.2 (Finite dimensional approximation spaces).** For each \( k \geq 1 \) and \( i \in \{0, \ldots, 2^k - 1\} \), define finite-dimensional functional linear spaces of dimension \( K_{Y,i}^{(k)} \) (resp. \( K_{Z,i}^{(k)} \)) by

\[
K_{Y,i}^{(k)} := \text{span} \{ p_{Y,k,i,1}, \ldots, p_{Y,k,i,K_{Y,k,i}} \} \text{ for } p_{Y,k,i,j} : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } \mathbb{E}[|p_{Y,k,i,j}(X_i^{(k)})|^2] < +\infty, \\
K_{Z,i}^{(k)} := \text{span} \{ p_{Z,k,i,1}, \ldots, p_{Z,k,i,K_{Z,k,i}} \} \text{ for } p_{Z,k,i,j} : \mathbb{R}^d \to \mathbb{R}^q \text{ s.t. } \mathbb{E}[|p_{Z,k,i,j}(X_i^{(k)})|^2] < +\infty.
\]

The minimal error afforded by these approximation spaces is denoted

\[
T_{1,i}^{(Y,k)} := \inf_{\phi \in K_{Y,i,k}^{(k)}} \mathbb{E}\left[ |\phi(X_i^{(k)}) - y_i^{(k)}(X_i^{(k)})|^2 \right], \\
T_{1,i}^{(Z,k)} := \inf_{\phi \in K_{Z,i,k}^{(k)}} \mathbb{E}\left[ |\phi(X_i) - z_i^{(k)}(X_i)|^2 \right].
\]

To avoid integration with respect to some (computationally inaccessible) law \( \nu \), as in Algorithm [1], the next Algorithm [2] will use simulation to approximate it by the empirical measure.

**Definition 3.3 (Simulations and empirical measures).** For \( k \geq 0 \), generate \( M_k \geq 1 \) independent copies (simulations) \( \mathcal{C}_k := \{ (\Delta W^{(k,m)}, X^{(k,m)}, X^{(k-1,m)} ) : m = 1, \ldots, M_k \} \) of the trajectories of the Markov chains and the Brownian increments \( (\Delta W^{(k)}, X^{(k)}, X^{(k-1)}) \). Denote by \( \nu_M^{(k)} \) the empirical probability measure of the \( \mathcal{C}_k \)-simulations, i.e.

\[
\nu_M^{(k)} = \frac{1}{M_k} \sum_{m=1}^{M_k} \delta_{(X_0^{(k,m)}, \ldots, X_0^{(k,m)}, \ldots, X_0^{(k-1,m)}, \ldots, X_0^{(k-1,m)}, \ldots, X_0^{(k-1,m)}, \Delta W_0^{(k,m)}, \ldots, \Delta W_0^{(k,m)}, \ldots, \Delta W_{2^k-1}^{(k,m)}, \ldots, \Delta W_{2^k-1}^{(k,m)}, \ldots, \Delta W_{2^k-1}^{(k,m)})}.
\]

Denote by \( X^{(m)} \) the concatenation of the trajectories of the Markov chains \( X^{(k,m)}, X^{(k-1,m)} \) and the Brownian increments \( \Delta W^{(k,m)} \), i.e.

\[
X^{(m)} := (X_0^{(k,m)}, X_0^{(k-1,m)}, X_0^{(k-1,m)}, \ldots, X_0^{(k-1,m)}, \Delta W_0^{(k,m)}, \ldots, X_0^{(k-1,m)}, \Delta W_{2^k-1}^{(k,m)}).
\]

**Notation and assumptions for the simulations.** Each \( \mathcal{C}_k \) forms a cloud of simulations. Without loss of generality, up to a generation of extra simulations, we assume \( M_k \geq \max_{0 \leq i \leq 2^k-1} K_{Y,i}^{(k)} \cup K_{Z,i}^{(k)} \). Furthermore, let the clouds of simulations (\( \mathcal{C}_k : k \geq 0 \)) be independently generated. All clouds are defined on one probability space \( (\Omega^{(M)}, \mathcal{F}^{(M)}, \mathbb{P}^{(M)}) \). To construct the probability space that supports the analysis of our algorithm, we simply extend the previous probability space supporting \( (\Delta W^{(k)}, X^{(k)})_{k \geq 0} \), which serves as a generic element for any single simulations, by passing to the usual product space \( (\Omega, \mathcal{F}, \mathbb{P}) = (\Omega, \mathcal{F}, \mathbb{P}) \otimes (\Omega^{(M)}, \mathcal{F}^{(M)}, \mathbb{P}^{(M)}) \). To simplify notation, we write \( \mathbb{P} \) (resp. \( \mathbb{E} \)) instead of \( \mathbb{P} \) (resp. \( \mathbb{E} \)).
In the sequel, we will frequently use conditioning to integrate only with respect to a specific cloud of simulations, rather than to take global expectation; the following \( \sigma \)-algebras will be used for this.

**Definition 3.4.** For every \( k \geq 0 \) and \( i \in \{0, \ldots, 2^k - 1\} \), define the \( \sigma \)-algebras

\[
\mathcal{F}_k := \sigma(C_k, \ldots, C_0), \quad \mathcal{F}_{k,i} := \mathcal{F}_k \vee \sigma(X^{(k,m)}_j, X^{(k-1,m)}_{\alpha(j)} : 1 \leq m \leq M_i, \ 1 \leq j \leq i)
\]

and let \( \mathbb{E}_k[\cdot] \) (resp. \( \mathbb{E}_{k,i}[\cdot] \)) be the conditional expectation with respect to \( \mathcal{F}_k \) (resp. \( \mathcal{F}_{k,i} \)).

Now, we are in position to formulate a fully implementable algorithm:

**Algorithm 2.** Initialize by setting \( y^{(0,M)}_1(\cdot) = \Phi(\cdot) \) and

\[
y_0^{(0,M)}(\cdot) = \frac{1}{M_0} \sum_{m=0}^{M_0} \Phi(X^{(0,m)}_1) \quad \text{and} \quad z_0^{(0,M)}(\cdot) = \frac{1}{M_0} \sum_{m=0}^{M_0} \Phi(X^{(0,m)}_1) \Delta W^{(0,m)}_0.
\]

Recursion for \( k \geq 1 \): Assume that \( \{y^{(k-1,M)}(\cdot), z^{(k-1,M)}(\cdot)\} \) have already been computed. Set \( y^{(k,M)}_i(\cdot) := \Phi(\cdot) \), and, for each \( i \in \{0, \ldots, 2^k - 1\} \), compute first \( y^{(k,M)}_i(\cdot) \) and then \( z^{(k,M)}_i(\cdot) \) as follows:

\[
y^{(k,M)}_i(\cdot) := \mathcal{T}_{C_y}(\psi^{(k,M)}_{Y,i}(\cdot)) \quad \text{and} \quad z^{(k,M)}_i(\cdot) = \mathcal{T}_{C_z,k,i}(\psi^{(k,M)}_{Z,i}(\cdot)),
\]

where the bounds \( C_y \) and \( C_{z,k,i} \) are from Corollary 2.5, the truncation functions \( \mathcal{T}_C(\cdot) \) are from Section 1.7, and

\[
\begin{align*}
\psi^{(k,M)}_{Y,i}(\cdot) & \text{ solves } \text{OLS}(S^{(k,M)}_{Y,i}(\bar{x}, \bar{x}, \bar{w}), K^{(k)}_{Y,i}, \nu_M^{(k)}), \\
S^{(k,M)}_{Y,i}(\bar{x}, \bar{x}, \bar{w}) & := \Phi(\bar{x}_{2^k}) - \sum_{j=\alpha(i)+1} z^{(k-1,M)}(\bar{x}_j)(w_{2j} + w_{2j+1}), \\
\psi^{(k,M)}_{Z,i}(\cdot) & \text{ solves } \text{OLS}(S^{(k,M)}_{Z,i}(\bar{x}, \bar{x}, \bar{w}), K^{(k)}_{Z,i}, \nu_M^{(k)}), \\
S^{(k,M)}_{Z,i}(\bar{x}, \bar{x}, \bar{w}) & := \frac{w_i}{\Delta_i^{(k)} \cdot} \left( S^{(k,M)}_{Y,i}(\bar{x}, \bar{x}, \bar{w}) - y^{(k,M)}_i(\bar{x}_i) \right),
\end{align*}
\]

for \( \bar{x} = (x_0, \ldots, x_{2^k}) \in \mathbb{R}^{(2^k+1) \times d}, \bar{x} = (\bar{x}_0, \ldots, \bar{x}_{2^k-1}) \in \mathbb{R}^{(2^k-1+1) \times d}, \bar{w} = (w_0, \ldots, w_{2^k-1}) \in \mathbb{R}^{2^k \times q} \).

**Intuition for Definition 3.1.** In Algorithm 2 above, we are clearly in the empirical measure setting of Definition 3.1. Here \( l = (2^k+1) \times (2^{k-1}+1) \times 2^k q \) and, for each \( m \in \{1, \ldots, M_k\} \), the \( \mathbb{R}^l \)-valued random variable \( X^{(m)} \) is the trajectory of the Markov chains and Brownian increments as given in Definition 3.3, \( \nu_M^{(k)} \) is the empirical measure \( \nu_M^{(k)} \). The probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) is \((\Omega, \mathcal{F}^{(M)}, \mathbb{P})\), i.e. the space generated all the sample clouds \( \{C_k : k \geq 0\} \). The random function \( S(\cdot) \) is the sample dependent function \( S^{(k,M)}_{Y,i}(\cdot) \) (resp. \( S^{(k,M)}_{Z,i}(\cdot) \)), which is clearly \( \mathcal{F}_k \)-measurable.

Actual computation of OLS in the Algorithm 2 uses numerical linear algebra [25].
3.3 Error analysis

In this section, we determine upper bounds for the global error of Algorithm 2

$$\tilde{E}(k) := \max_{0 \leq i \leq 2^k - 1} \tilde{E}(Y, k, i) + \sum_{i=0}^{2^k-1} \tilde{E}(Z, k, i)\Delta_i^{(k)}$$

(23)

on each level $k \geq 0$, for local error terms given by $\tilde{E}(Y, k, i) := \mathbb{E}[|y_i^{(k)}(X_i^{(k)}) - y_i^{(k,M)}(X_i^{(k)})|^2]$ and $\tilde{E}(Z, k, i) := \mathbb{E}[|z_i^{(k)}(X_i^{(k)}) - z_i^{(k,M)}(X_i^{(k)})|^2]$. In order to do so, it will suffice to find upper bounds for the error terms

$$\tilde{E}(Y, k, i) := \mathbb{E}\left[\frac{1}{M_k} \sum_{m=1}^{M_k} |y_i^{(k)}(X_i^{(k,m)}) - y_i^{(k,M)}(X_i^{(k,m)})|^2\right],$$

$$\tilde{E}(Z, k, i) := \mathbb{E}\left[\frac{1}{M_k} \sum_{m=1}^{M_k} |z_i^{(k)}(X_i^{(k,m)}) - z_i^{(k,M)}(X_i^{(k,m)})|^2\right]$$

(24)

thanks to the relationship in Proposition 3.5 (similar to [24 Prop.4.10]):

**Proposition 3.5.** For each $k \in \{0, \ldots, \kappa\}$ and $i \in \{0, \ldots, 2^k - 1\}$, we have

$$\tilde{E}(Y, k, i) \leq 2\tilde{E}(Y, k, i) + \frac{2028(K_{Y,i}^{(k)} + 1)C_Y^2 \log(3M_k)}{M_k},$$

$$\tilde{E}(Z, k, i) \leq 2\tilde{E}(Z, k, i) + \frac{2028(K_{Z,i}^{(k)} + 1)qC_z^2 \log(3M_k)}{M_k};$$

we recall that $C_y = C_\Phi$ and $C_{z,k,i} = C_X^2/(T - t_i^{(k)})^{(1-\theta)/2}$ from Corollary 2.3.

Since $y_i^{(k,M)}(\cdot)$ and $z_i^{(k,M)}(\cdot)$ is computed with the samples $X_i^{(k,m)}$, which are also used in the empirical norm inside the expectation of $\tilde{E}(Y, k, i)$ and $\tilde{E}(Z, k, i)$, it turns out that the error analysis of $\tilde{E}(Y, k, i)$ and $\tilde{E}(Z, k, i)$ is more tractable than that of $\tilde{E}(Y, k, i)$ and $\tilde{E}(Z, k, i)$ and an important aim for our analysis will be to find upper bounds for these terms; Proposition 3.5 then allows us to compute upper bounds the $\tilde{E}(Y, k, i)$ and $\tilde{E}(Z, k, i)$ from $\tilde{E}(Y, k, i)$ and $\tilde{E}(Z, k, i)$ and a correction in terms of the number of basis functions, the number of simulations, the time-grid, and the almost sure bounds $C_y$ and $C_{z,k,i}$. It turns out that the correction term is of the same order as one of the error terms in the estimate of $\tilde{E}(Z, k, i)$, up to the $\ln(M_k)$ term; see Theorems 3.7 and 3.9. Therefore, the impact of the correction terms on the convergence rate of the global error is essentially the same as the impact of the terms $\tilde{E}(Y, k, i)$ and $\tilde{E}(Z, k, i)$. The proof of Proposition 3.5 is analogous to the proof of [24 Proposition 4.10], as the latter involves only almost sure bounds and general concentration of measure inequalities ([24 Proposition 4.9]). Therefore, we provide no proof here but refer to that paper. The correction terms in Proposition 3.5 have an interpretation as the error due to interdependence between the cloud used to construct $(y_i^{(k,m)}(\cdot), z_i^{(k,m)}(\cdot))$ and the sample used for the empirical norm.

It will be convenient to use the following notation of random norms in subsequent analysis; the norms are random because their values depend on the samples of Definition 3.3 and no global expectation is taken.

**Definition 3.6.** Let $\varphi : \Omega^{(M)} \times \mathbb{R}^d \to \mathbb{R}$ or $\mathbb{R}^q$ be $\mathcal{F}^{(M)} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable. For each $k \geq 0$ and
\[ i \in \{0, \ldots, 2^k - 1\}, \text{ define the random norms} \]
\[
\| \varphi \|_{k,i,\infty}^2 := \int_{\mathbb{R}^d} \| \varphi(x) \|^2 \mathbb{P} \circ (X_{i}^{(k)})^{-1}(dx) \quad \text{and} \quad \| \varphi \|_{k,i,M}^2 := \frac{1}{M_i} \sum_{m=1}^{M_i} \| \varphi(X_{i}^{(k,m)}) \|^2.
\]

The norm \( \| \cdot \|_{k,i,\infty} \) makes use of the law of \( X_i^{(k)} \), whereas \( \| \cdot \|_{k,i,M} \) makes use of the empirical measure of the samples \( \{X_{i}^{(k,m)} : m = 1, \ldots, M_i\} \). Indeed, the error terms \([24]\) can be written \( \mathcal{E}(Y, k, i) = \mathbb{E}[\| y_i^{(k)}(\cdot) - y_i^{(k,M)}(\cdot) \|_{k,i,M}^2] \) and \( \mathcal{E}(Z, k, i) = \mathbb{E}[\| z_i^{(k)}(\cdot) - z_i^{(k,M)}(\cdot) \|_{k,i,M}^2] \). Moreover, it follows from the tower law that
\[
\mathcal{E}(Y, k, i) = \mathbb{E}[\| y_i^{(k)}(\cdot) - y_i^{(k,M)}(\cdot) \|_{k,i,\infty}^2] \quad \text{and} \quad \mathcal{E}(Z, k, i) = \mathbb{E}[\| z_i^{(k)}(\cdot) - z_i^{(k,M)}(\cdot) \|_{k,i,\infty}^2].
\]

We come to the main results of this paper, the error propagation of Algorithm \([2]\). Two theorems are presented based on different assumptions. The proofs of the two theorems are very similar in that they are based on a common error decomposition technique. For this reason, we prove them simultaneously and explain where the proofs differ; the proofs are lengthy and deferred to Section \([3.4]\).

**Theorem 3.7.** In addition to the general assumptions, assume also \( (A''_k) \). For any time point \( t \in \tau^{(k_1)} \cap \tau^{(k_2)} \) that belongs to two time-grids \( \tau^{(k_1)} \) and \( \tau^{(k_2)} \) for some \( k_1, k_2 \), it holds that \( X_i^{(k_1)} = X_j^{(k_2)} \).

Then, for every \( k \geq 0 \), \( i \in \{0, \ldots, 2^k - 1\} \), the error term \( \mathcal{E}(Y, k, i) \) is bounded above by
\[
\frac{4 \times 2^{-k} K_{Y,i}^{(k)} \delta}{M_k} \{3C_X^2 + (2 + q)\} + \frac{2K_{Y,i}^{(k)}}{M_k} \sum_{j=\alpha(i)+1}^{2^{k-1}-1} \left| z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot) \right|_{\infty}^2 \Delta_j^{(k-1)} + T_{i,Y}^{(k)} \quad (25)
\]
and the error term \( \mathcal{E}(Z, k, i) \) is bounded by
\[
\frac{12\delta K_{Z,i}^{(k)} (2 + 5T^{-1-\theta}) C_X^2}{c_X M_k (T - t_i^{(k)})^{1-\theta}} + 4\delta (2 + q) K_{Z,i}^{(k)} C_X \frac{1}{c_X M_k} \\
+ \frac{2K_{Z,i}^{(k)}}{\Delta_i^{(k)} M_k} \left\{ \left| y_i^{(k)}(\cdot) - y_i^{(k,M)}(\cdot) \right|_{\infty}^2 + \sum_{j=\alpha(i)+1}^{2^{k-1}-1} \left| z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot) \right|_{\infty}^2 \Delta_j^{(k-1)} \right\} + T_{i,Z}^{(k)} \quad (26)
\]

**Remark.** The Assumption \( (A''_k) \) is trivially valid if \( X^{(k)} \) can be taken as the finite dimensional marginals of \( X \), provided those are available in closed form, like for instance for (geometric) Brownian motion. If one is computing \( X^{(k)} \) with an Euler scheme, for example, one would fix a maximal level, \( \kappa \) say, and could obtain \( X^{(k)} \) by running the Euler scheme once on the finest time-grid \( \tau^{(\kappa)} \) and selecting only the values associated \( \tau^{(k)} \) for every \( k \leq \kappa \). We remark that the assumption is not necessary for Theorem \([3.9]\).

**Remark 3.8.** The error bounds in Theorem \([3.7]\) above are not easy to apply because it appears difficult to quantify the terms in the norms \( | \cdot |_{\infty} \) more explicitly; these norms are stronger than the norms used to quantify the error \( \mathcal{E}(\cdot, k, i) \), and we have no precise estimates for them. It seems to be difficult to replace the use of this strong norm in general, cf. \([5]\) who obtain estimates using the same norm when

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using general basis functions. However, we can plug the absolute $y$- and $z$-bounds Algorithm 2 into \cite{25,26} to obtain a rough upper bound on the error
\[
\mathcal{E}(Y, k, i) \leq T^{(Y, k)}_{1, i} + \frac{16 \times 2^{-k} C^{(k)}_{Y,i}}{M_k} + \frac{8(C^2_X \vee C^2_Y)T^\theta C^{(k)}_{Y,i}}{\theta M_k},
\]
\[
\mathcal{E}(Z, k, i) \leq T^{(Z, k)}_{1, i} + \frac{16(2 + T^{1-\theta}) C^{(k)}_{Z,i}}{c_X(T - t^{(k)})1-\theta M_k} + \frac{8(C^2_X \vee C^2_Y)(1 + T^\theta \theta^{-1}) C^{(k)}_{Z,i}}{\Delta_i^{(k)} M_k}.
\]

This is the “worst-case” error estimate in the sense that we assume the terms $\left| y_j^{(k-1)}(\cdot) - y_j^{(k-1,M)}(\cdot) \right|_\infty$ and $\left| z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot) \right|_\infty$ are maximal.

We use this estimate for a crude comparison to the usual least-squares multistep forward dynamical programming (LSMDP) scheme \cite{24}, i.e. Algorithm 2 with the correction terms from level $k-1$ removed. For every $k \geq 0$ and $i \in \{0, \ldots, 2^k - 1\}$, the corresponding error estimates for the LSMDP algorithm (when the same time-grid and the same basis functions are used) are
\[
\mathcal{E}_{\text{MDP}}(Y, k, i) \leq T^{(Y, k)}_{1, i} + \frac{C^2_X C^{(k)}_{Y,i}}{M_k} \quad \text{and} \quad \mathcal{E}_{\text{MDP}}(Z, k, i) \leq T^{(Z, k)}_{1, i} + \frac{C^2_X C^{(k)}_{Z,i}}{\Delta_i^{(k)} M_k}. \tag{27}
\]

We see that the dependence on the number of basis functions $K^{(k)}_{i,j}$, the time increment $\Delta_i^{(k)}$, and the number of simulations $M_k$ is the same for both the multilevel and the MDP scheme, although the constants may differ. In this setting, the behavior of the both algorithms with respect to each of these parameters might be the same. We emphasize, however, that this is a rather rough “worst case scenario”, in which the approximations of $z^{(k-1)}$ are as bad as absolute a-priori bounds would permit. We now turn to the opposite extreme, a “best case” scenario, in which the $| \cdot |_\infty$-terms are negligible; studying \cite{25,26}, we see these terms are counter-balancing the negative impact of the time-increment $\Delta_i^{(k)}$. This leads to “best-case” error estimates
\[
\mathcal{E}_{\text{best}}(Y, k, i) \leq T^{(Y, k)}_{1, i} + \frac{16 \times 2^{-k} C^{(k)}_{Y,i}}{M_k} \quad \text{and} \quad \mathcal{E}_{\text{best}}(Z, k, i) \leq T^{(Z, k)}_{1, i} + \frac{2K^{(k)}_{Z,i}}{c_X M_k},
\]
motivating, in particular, an improvement in the dependence on $\Delta_i^{(k)}$, which no longer appears in the denominator.

Remark 3.8, despite its crude quantitative nature, is encouraging as a first comparison between the multilevel algorithm and the LSMDP algorithm. A main obstacle for more precise statements was error bounds were given in terms of very strong norms in the setting with a general basis. We next provide a more precise comparison to the LSMDP for a specific choice of basis.

Theorem 3.9. In addition to the general assumptions, assume that

(Ax) the basis functions are indicator functions, i.e. $p_{i,k,i,j} := 1_{A_{i,k,i,j}}$ on disjoint sets $A_{i,k,i,j}$; moreover, there exists $\delta \geq 1$ such that either $\mathbb{P}(X_i^{(k)} \in A_{i,k,i,j}) \geq 1/(\delta K)$.
For every $k \in \{1, \ldots, \kappa\}$, $i \in \{0, \ldots, 2^k - 1\}$, the error term $\mathcal{E}(Y, k, i)$ is bounded above by

$$
4 \times 2^{-k} \frac{K_{Y}^{(k)} \delta}{M_k} \{3C_X^2 + (2 + q)\} + \frac{2K_{Y}^{(k)} \delta}{M_k} \sum_{j=\alpha(i)+1}^{2^{k-1} - 1} \mathcal{E}(Z, k - 1, j) \Delta_j^{(k-1)} + T_{1,i}^{(Y,k)}
$$

and the error term $\mathcal{E}(Z, k, i)$ is bounded by

$$
\frac{12\delta K_{Z,i}^{(k)} (2 + 5T^{1-\theta}) C_X^2}{c_X M_k (T - t_i^{(k)})^{1-\theta}} + 4\delta (2 + q) K_{Z,i}^{(k)} C_X T^{(Z,k)} + \sum_{j=\alpha(i)+1}^{2^{k-1} - 1} \mathcal{E}(Z, k - 1, j) \Delta_j^{(k-1)}
$$

where $\delta = 1$ under the measure $\mathbb{P} \circ (X_i^{(k)})^{-1}$. Moreover, if $X_i^{(k)}$ has a density $\phi_X(x)$ that is bounded from below away from zero on a compact $A \subset \mathbb{R}^d$, it follows that $\mathbb{P}(X_i^{(k)} \in H_j) \geq \min_{x \in A} \phi_X(x) \int_{H_j} 1 \, dx$ for all sets $H_j \subset A$, so a partition of $A$ into sets $A_{.,k,i,j}$, satisfying $\int_{A_{.,k,i,j}} 1 \, dx = \text{const}/K_{i,j}^{(k)}$ for all $j = 1, \ldots, K_{i,j}^{(k)}$ would form a basis satisfying $(A_{K})$.

Remark. Let us note that Theorem 3.7 differs from Theorem 3.9 in that the latter does not require assumption $(A_{\kappa})$ and uses a weaker norm than $\| \cdot \|_{\infty}$. We believe that estimates in the weaker norm and moreover relaxing the assumption $(A_{\kappa})$ may hold true for a class of basis functions beyond $(A_{K})$. Computational examples later indeed indicate that multilevel benefits prevail beyond the assumptions under which complexity gains are proven subsequently.

Comparing the error bounds from Theorem 3.7 and Theorem 3.9 one sees that the terms given in the norm $\| \cdot \|_{\infty}$ in the former theorem have been replaced by the equivalent terms in the weaker norm $\mathcal{E}(\cdot)$ in the latter. From theorem 3.9 the upper bound for the global error $\mathcal{E}(\cdot)$ of the multilevel scheme on the level $k$ is expressed in terms of the number of time-steps, the time-increments, the number of basis functions, the bias of the basis functions, the number of simulations, and the global error on the previous level $k - 1$ (i.e. the terms $\mathcal{E}(Z, k - 1, j)$). For the remainder of this section, the order notation $O(\cdot)$ will be used: we write $g(y)$ is in $O(y)$ if there exists a constant $C$, not depending on the level $k$, such that $\limsup_{y \to 0} g(y)/y \leq C$. We set the numerical parameters – the basis functions and the number of simulations – of the multilevel algorithm so that the global error $\mathcal{E}(\cdot)$ attains a precision level $O(\varepsilon)$ for $\varepsilon > 0$. We use this calibration to compute complexity and to compare the multilevel algorithm to the LSMDP scheme more precisely than in Remark 3.8.

Remark 3.10. Our theoretical complexity analysis below applies error estimates from Theorem 3.9 and hence requires its assumption to hold; in particular, the basis is to satisfy $(A_{\kappa})$. Furthermore, it is required that the basis is such that the approximations errors $T_{1,i}^{(Y,k)}$ and $T_{1,i}^{(Z,k)}$ are of order $O(\varepsilon)$ for basis dimension $K_{Y,i}^{(k)}$ resp. $K_{Z,i}^{(k)}$ as stated in Choice of basis below. In combination, these assumptions appear restrictive, but computational examples later will indicate empirical multilevel benefits beyond these assumptions. On the other hand, there is a class of examples in which the required assumptions are satisfied, and we exhibit this class for the remainder of this remark. Suppose that the solutions $x \mapsto (y_i^{(k)}(x), \hat{z}_i^{(k)}(x))$ are periodic, that is, there exists $\lambda := (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ such...
that \(y_t^{(k)}(x), z_t^{(k)}(x)\) = \(y_t^{(k)}(x + n\lambda), z_t^{(k)}(x + n\lambda)\) for all \(n \in \mathbb{Z}\) and \(x \in \mathbb{R}^d\). As an example one can think of \(X_t^{(k)} = W_t^{i(k)}\) and \(\Phi(x) = \sin(\beta \cdot x)\), whence \(\lambda_i = 2\pi / \beta_i\). More generally, one can consider any periodic terminal condition \(\Phi\) and \(X_t^{(k)}\) to be the marginals of the solution to a stochastic differential equation whose coefficient functions have the same periodicity as \(\Phi\). For every \(t_0 \in (0, T]\) and \(\kappa \leq k\) we assume that the marginals \(X_t^{(\kappa)}\) have a density bounded from below by \(c(t_0) > 0\) (independent of \(\kappa\)) in the domain \(D := \otimes_{i=1}^d [ - \frac{N}{2}, \frac{N}{2} ]\) for all \(j\) such that \(t_j^{(\kappa)} > t\). This property is satisfied if \(X_t^{(k)}\) were the marginal of the solution to a stochastic differential equation whose generator is uniformly elliptic; hence the marginal density is bounded from below by a Gaussian density [30]. Let \(B_k, A_k, K^{(k)}\) be a hypercube partition of \(D\), and define the basis functions \(p_{\eta, k, i, j}(x)\) \((j = 1, \ldots, K^{(k)}, \eta = Y, Z)\) to be the indicator functions \(1_{A_{k, i, j}}(x)\) on the sets \(A_{k, i, j} := \bigcup_{n \in \mathbb{Z}} \{x + n \lambda : x \in B_{i, j}\}\). Then, \(\mathbb{P}(A_{k, i, j}) \geq \mathbb{P}(B_{k, i, j}) \geq c(t_0) \mu(D)/K^{(k)}\), where \(\mu\) is the Lebesgue measure; hence, the condition \((A_\kappa)\) is satisfied with \(\delta = 1/(c(t_0) \mu(D))\). \(\delta\) may therefore be considered a constant with respect to the level \(k\) and the precision level so long as one considers the global error on the interval \([t_0, T]\).

**Remark.** As an alternative case to the periodic one outlined in Remark 3.10 one may also think about a forward process \(X\) which is a diffusion reflected within some compact domain, such as to ensure its density being bounded away from zero, so that one could argue similarly as in the periodic case above. To make this ansatz rigorous, however, would require \(L_2\)-regularity properties like \((A_\pi)(ii)\) to hold for reflected diffusions \(X\). We are not aware of such results being yet available.

The error bounds [28–29] of Theorem 3.9 show that a sufficient criterion to achieve an global error \(O(\varepsilon)\) is to ensure that each of the terms in the sums in [28–29] is bounded by \(O(\varepsilon)\), and we use this criterion to develop a calibration procedure. Furthermore, assume that the assumptions of Remark 3.10 hold, namely that the basis functions satisfy \((A_k)\) and periodicity.

**Choice of basis.** We first choose a basis satisfying \((A_k)\) so that \(T^{(Y,k)}_{1,i}\) and \(T^{(Z,k)}_{1,i}\) are bounded above by \(O(\varepsilon)\) for all \(i\). Let \(k^{(k)}_{Y,i} = k^{(k)}_{Z,i}\) and the set \(\{B_{k, i, j} : j = 1, \ldots, K^{(k)}\}\) be the uniform hypercubes on the set \(D\). Thanks to \((A_\pi)(iv), (A_\pi')\), it is sufficient for the boundedness of \(T^{(Y,k)}_{1,i}\), \(T^{(Z,k)}_{1,i}\) to ensure that

\[
\max_{0 \leq i \leq 2k} \min_{\varphi \in K_{Y,i}} \mathbb{E}[(u(t_{i,k}^{(k)}, X_{i,k}^{(k)}) - \phi(X_{i,k}^{(k)}))^2] + \sum_{i=0}^{2k-1} \min_{\varphi \in K_{Z,2,i}} \mathbb{E}[(v(t_{i,k}^{(k)}, X_{i,k}^{(k)}) - \phi(X_{i,k}^{(k)}))^2] \leq O(\varepsilon).
\]

We assume (for simplicity) that \(\theta = 1\). Thanks to \((A_\pi')(iv)\), the Lipschitz constant of \(v(t, \cdot)\) is equal to \(O((T-t)^{-1/2})\), so it suffices to set the hypercube diameter at time \(t^{(k)}_i\) equal to \(\sqrt{T - t^{(k)}_i} O(\sqrt{\varepsilon})\), whence the dimension of the basis is \(K^{(k)}_{Z,i} = (T - t^{(k)}_i)^{-d/2} O(\varepsilon^{-d/2})\). Since the Lipschitz constant of \(u(t^{(k)}_i, \cdot)\) is \(O(1)\), it follows that \(T^{(Y,k)}_{1,i} \leq O(\varepsilon)\) with the same basis.

**Number of simulations.** The choice of basis fixes the number of basis functions \(K^{(k)}_{Z,i} = K^{(k)}_{Y,i}\). We choose \(M_k = \max_i O(\varepsilon^{-1} K^{(k)}_{Z,i}(\varepsilon)) = O(k^{-1-d/2} \max_i (T - t^{(k)}_i)^{-d/2}) \leq O(k^d \varepsilon^{-1-d/2})\) to ensure that all terms in [28–29] except those depending on \(\mathbb{E}(Z, k, -1, \cdot)\) and also in the correction terms in Proposition 3.5 are bounded by \(O(\varepsilon)\); observe that we do not need to worry about the terms \(1/(T - t^{(k)}_i)^{1-\theta}\), because the sum \(\sum_{i=0}^{2k-1} \Delta_i^{(k)}/(T - t^{(k)}_i)^{1-\theta}\) is bounded uniformly in \(k\).
Iteration to levels $j < k$. In the calculations above, it only remains to set parameters such that

$$
\sum_{l=(i-1)+1}^{2^{k-1}-1} \mathcal{E}(Z, k-1, l) \Delta_i^{(k-1)} \leq O(\Delta_i^{(k)})
$$

for all $i$. This is satisfied by setting the precision for the global error $\mathcal{E}(k-1)$ on level $k-1$ to be less than or equal to $O(\min_i \Delta_i^{(k)}) \leq O(2^{-k})$ in the place of $O(\varepsilon)$. Subsequently, on every level $j \leq k-1$ thereafter, we set the precision for the global error $\mathcal{E}(j)$ less than or equal to $O(\min_i \Delta_i^{(j+1)}) \leq O(2^{-j})$ and repeat the first two steps of the procedure above. For simplicity, we choose the same basis for every level, although this is possibly not optimal. The basis dimension at time $t^{(j)}_i$ is $K(j, i, \varepsilon) := (T - t^{(j)}_i)^{-d/2}O(\varepsilon^{-d/2})$ and the number of simulations on level $j < k$ is $M_j = \max_i O(j 2^{d/2} K(j, 2^j - 1, \varepsilon)) = O(j 2^{j + j d/2} \varepsilon^{-d/2})$.

**Complexity analysis.** We fix $\varepsilon = O(2^{-k})$, as this is usually the discretization error between $(y_k, z_k)$ and the continuous time solution (see (A.5)(iv)). There are two contributions to the computational cost: the cost of simulation of the Markov chain $X^{(k)}$ and Brownian increments $\Delta W^{(k)}$, and the cost of the regressions. The cost of computing the simulations is $O(2^l M_j)$ on level $l$ of the algorithm, therefore the overall simulation cost is $\sum_{j=0}^{k} O(2^l M_j)$. To compute the cost of the regression, one must first of all remark that there is a closed form formula for regression on indicators (see the partitioning estimate in [26]): for responses $(\psi_m)_{1 \leq m \leq M}$ corresponding to observations $(\phi_m)_{1 \leq m \leq M}$, the precise coefficient of the indicator function denoted by $H$ is given by

$$
\alpha_H = \frac{\sum_{m=1}^{M} \psi_m 1_H(\phi_m)}{\sum_{m=1}^{M} 1_H(\phi_m)};
$$

therefore, the cost of the regression on each time point is proportional to the cost of sorting the simulations into the indicators, which is proportional to the dimension $d$ times the number of simulations. This implies that the cost of the regressions on level $l$ is equal also equal to $O(2^l M_l)$. Therefore, recalling that $\varepsilon = O(2^{-k})$, the overall cost of the algorithm is

$$
\sum_{j=0}^{k} O(2^j M_j) \leq O(k) \sum_{j=0}^{k} O(2^{j(1+d)}) = O(\ln(\varepsilon^{-1} + 1)\varepsilon^{-2-d}).
$$

For comparison, we calibrate the basis functions and number of simulations for the LSMDP algorithm described in Remark 3.8 using (27) in the place of (28)–(29). We choose the same basis functions, and $M_k = O(\varepsilon^{-1} 2^k K(k, 2^k - 1, \varepsilon))$. Then, setting $\varepsilon = O(2^{-k})$, the overall complexity is $2^k \times M_k = O(\varepsilon^{-3-d})$. We observe that, in comparison to the complexity of the multilevel scheme, one factor in $\ln(\varepsilon^{-1} + 1)$ have been replaced by a factor $\varepsilon^{-1}$, which is much larger. This implies that, in comparison to MDP, the multilevel scheme has a possible efficiency gain of factor $\varepsilon$ (ignoring the log terms). In our setting, is equal to the number of time steps, which is substantial.

### 3.4 Proof of Theorems 3.7 and 3.9

We state the elementary properties of OLS (Definition 3.1) in Proposition 3.11 below. This proposition is in fact the same as [24 Proposition 4.12], and we refer the reader interested in the proof to that.
ψ

for functions

We introduce the “fictitious” functions

With the notation of Definition 3.1, suppose that

σ

provide some explicit

document. We are aware that parts (iii) and (iv) of this proposition are given in high generality, so we

paper. We are aware that parts (iii) and (iv) of this proposition are given in high generality, so we

Proposition 3.11. With the notation of Definition 3.1, suppose that

K

finite dimensional and

spanned by the functions \( \{p_1(\cdot), \ldots, p_K(\cdot)\} \). Let \( S^* \) solve \( \text{OLS}(S, K, \nu) \) (resp. \( \text{OLS}(S, K, \nu_M) \)), according to [18] (resp. [19]). The following properties are satisfied:

(i) linearity: the mapping \( S \mapsto S^* \) is linear.

(ii) contraction property: \( \|S^*\|_{L_2(B(\mathbb{R}^l), \mu)} \leq \|S\|_{L_2(B(\mathbb{R}^l), \mu)} \), where \( \mu = \nu \) (resp. \( \mu = \nu_M \)).

(iii) conditional expectation: in the case of the discrete probability measure \( \nu_M \), assume additionally that the sub-\( \sigma \)-algebra \( \mathcal{Q} \subset \bar{\mathcal{F}} \) is such that \( \{p_j(X^{(1)}), \ldots, p_j(X^{(M)})\} \) is \( \mathcal{Q} \)-measurable for every \( j \in \{1, \ldots, K\} \). Let \( S_\mathcal{Q}(\cdot) \) be any \( \bar{\mathcal{F}} \otimes B(\mathbb{R}^l) \)-measurable, \( \mathbb{R}^l \)-valued function such that such that \( S_\mathcal{Q}(X^{(m)}) := \mathbb{E}[S(X^{(m)})]\mathcal{Q} \) for each \( m \in \{1, \ldots, M\} \) \( \mathbb{P} \)-almost surely. Then \( \mathbb{E}[S^*\mathcal{Q}(\omega, x)] \) solves \( \text{OLS}(S, \mathcal{K}, \nu_M) \).

(iv) bounded conditional variance: in the case of the discrete probability measure \( \nu_M \), suppose that

\( S(\omega, x) \) is \( \mathcal{G} \otimes B(\mathbb{R}^l) \)-measurable, for \( \mathcal{G} \subset \bar{\mathcal{F}} \) independent of \( \sigma(X^{(1):M}) \), there exists a Borel measurable function \( h : \mathbb{R}^l \to \mathcal{E} \), for some Euclidean space \( \mathcal{E} \), such that the random variables

\( \{p_j(X^{(m)}) : m = 1, \ldots, M, j = 1, \ldots, K\} \) are \( \mathcal{H} := \sigma(h(X^{(m)})) : m = 1, \ldots, M \)-measurable, and there is a finite constant \( \sigma^2 \geq 0 \) that uniformly bounds the conditional variances \( \mathbb{E}[|S(X^{(m)}) - \mathbb{E}[S(X^{(m)})|\mathcal{G} \vee \mathcal{H}]|^2 | \mathcal{G} \vee \mathcal{H}] \leq \sigma^2 \) \( \mathbb{P} \)-a.s. and for all \( m \in \{1, \ldots, M\} \). Then

\[ \mathbb{E}\left[\|S^*(\cdot) - \mathbb{E}[S^*(\cdot)|\mathcal{G} \vee \mathcal{H}]\|_{L_2(B(\mathbb{R}^l), \nu_M)}^2|\mathcal{G} \vee \mathcal{H}\right] \leq \sigma^2 K/M. \]

Intuition for Proposition 3.11. The observation \( X^{(m)} \) and response \( S \) above will be \( X^{(k,m)} \) and

\( S_i^{(k)}(x, w) \), respectively, whereas the linear space \( \mathcal{K} \) will be \( \mathcal{K}_i^{(k)} \) and the measure \( \nu \) (respectively \( \nu_M \)) will be \( \nu_k \) (respectively \( \nu_M^{(k)} \)). For part (iii), we will take \( \mathcal{Q} \) to be the \( \sigma \)-algebras \( \mathcal{F}_i^{(k)} \) in Definition 3.4

the function \( \mathbb{E}[S(X^{(m)})]\mathcal{Q}(\cdot) \) will then be equal to \( y_i^{(k)} \) (respectively \( z_i^{(k)} \)), see below. For part (iv), we take \( \mathcal{E} = \mathbb{R}^l \) and the Borel function \( h : \mathbb{R}^l \to \mathcal{E} \) to be \( h(X^{(m)}) = X_i^{(k,m)} \), whence the \( \sigma \)-algebra \( \mathcal{H} \) is

\( \sigma(X^{(k,m)} : m = 1, \ldots, M_k) \). We take \( \mathcal{F}_i^{(k)} \) for \( \mathcal{G} \), whence \( \mathcal{G} \vee \mathcal{H} = \mathcal{F}_i^{(k)} \).

We now begin the proof of the two theorems. Recall the \( \sigma \)-algebras from Definition 3.4 and the soft truncation function \( \mathcal{T}_r(\cdot) \) in Section 1.1. The Lipschitz continuity (for all \( r \)) of the function \( \mathcal{T}_r(\cdot) \) implies that

\[ \mathbb{E}[\|y_i^{(k)} - y_i^{(k,M)}\|^2_{k,i,M}] = \mathbb{E}[\|\mathcal{T}_{C_i}(y_i^{(k)}) - \mathcal{T}_{C_i}(\psi_i^{(k,M)})\|^2_{k,i,M}] \leq \mathbb{E}[\|y_i^{(k)} - \psi_i^{(k,M)}\|^2_{k,i,M}] \] \( (30) \)

\[ \mathbb{E}[\|z_i^{(k)} - z_i^{(k,M)}\|^2_{k,i,M}] = \mathbb{E}[\|\mathcal{T}_{C_{i,k}}(z_i^{(k)}) - \mathcal{T}_{C_{i,k}}(\psi_i^{(k,M)})\|^2_{k,i,M}] \leq \mathbb{E}[\|z_i^{(k)} - \psi_i^{(k,M)}\|^2_{k,i,M}] \] \( (31) \)

We introduce the “fictitious” functions \( \psi_i^{(k,M)} : \mathbb{R}^d \to \mathbb{R} \) and \( \psi_i^{(k,M)} : \mathbb{R}^d \to (\mathbb{R}^q)^\top \) defined by

\( \psi_i^{(k,M)}(\cdot) \) solves \( \text{OLS}(S_i^{(k)}(x, w), \mathcal{K}_i^{(k)}, \nu_i^{(k)}) \), \( \psi_i^{(k,M)}(\cdot) \) solves \( \text{OLS}(S_i^{(k,M)}(x, w), \mathcal{K}_i^{(k)}, \nu_M^{(k)}) \),

for functions \( S_i^{(k)}(\cdot) \) and \( S_i^{(k,M)}(\cdot) \) given in [20] from Algorithm 1 the fictitious nature of \( \psi_i^{(k)}(\cdot) \) and \( \psi_i^{(k,M)}(\cdot) \) comes from the functions \( S_i^{(k)}(\cdot) \) and \( S_i^{(k,M)}(\cdot) \), which are constructed using the unknown functions

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The term $y^{(k)}(\cdot)$ and $z^{(k)}(\cdot)$, so cannot be computed explicitly. We will decompose (30) and (31) using the (random) functions $E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}(\cdot)]$ and $E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}(\cdot)]$, respectively, but first we make use of Proposition 3.11 (iii) to determine that $E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}(\cdot)]$ and $E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}(\cdot)]$ solve OLS’s. Set $Q$ to be the $\sigma$-algebra $F_{k,i}^{(M)}$. $p(X_i^{(k,m)})$ is $Q$-measurable for any $p \in K_{Y,k,i} \cup K_{Z,k,i}$. Now, since $z_j^{(k-1)}(X_j^{(k-1,m)})$ is $F_{k,j}^{(M)}$-measurable for all $j > \alpha(i)$, applying the tower property and the Markov property ($A_{X,i}^{(k)}$) yields that

$$E_{k,i}^{(M)}[S_{X,i}^{(k)}(X_m)] = E_{k,i}^{(M)}[\Phi(X_{2k}^{(k,m)})] = y^{(k)}_{i}(X_i^{(k,m)}),$$

$$E_{k,i}^{(M)}[S_{Z,i}^{(k)}(X_m)] = E_{k,i}^{(M)}[\frac{\Phi(X_{2k}^{(k,m)})\Delta W_i^{(k,m)}}{\Delta_i^{(k)}}] = z^{(k)}_{i}(X_i^{(k,m)}).$$

for all $m \in \{1, \ldots, M_k\}$, whence we finally obtain the expression

$$x \in \mathbb{R}^d \rightarrow E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}(\cdot)](x) \text{ solves } OLS(y^{(k)}_{i}(x_i), K_{Y,i}^{(k)}, \nu^{(k)}_{M,M}),$$

$$x \in \mathbb{R}^d \rightarrow E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}(\cdot)](x) \text{ solves } OLS(z^{(k)}_{i}(x_i), K_{Z,i}^{(k)}, \nu^{(k)}_{M,M}).$$

Therefore, introducing the random functions $E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}(\cdot)]$ and $E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}(\cdot)]$ on the right hand side of (30) and (31), respectively, and applying Pythagoras’ theorem, it follows that

$$E[\|y_i^{(k)} - E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}]\|_{k,i,M}^2] \leq E[\|y_i^{(k)} - E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}]\|_{k,i,M}^2] + E[\|E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}\|_{k,i,M}^2],$$

$$E[\|z_i^{(k)} - E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}]\|_{k,i,M}^2] \leq E[\|z_i^{(k)} - E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}]\|_{k,i,M}^2] + E[\|E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}\|_{k,i,M}^2].$$

Moreover, $E[\|y_i^{(k)} - E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}]\|_{k,i,M}^2] \leq T_{Y,i}^{(k)}$ and $E[\|z_i^{(k)} - E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}]\|_{k,i,M}^2] \leq T_{Z,i}^{(k)}$, and injecting this into inequalities (34) and (35) yields

$$E[\|y_i^{(k)} - E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}]\|_{k,i,M}^2] \leq T_{Y,i}^{(k)} + E[\|E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}\|_{k,i,M}^2],$$

$$E[\|z_i^{(k)} - E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}]\|_{k,i,M}^2] \leq T_{Z,i}^{(k)} + E[\|E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}\|_{k,i,M}^2].$$

To treat the second term on the right-hand side of (36) (resp. (37)), we decompose

$$E[\|E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}\|_{k,i,M}^2] \leq 2E[\|E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}\|_{k,i,M}^2] + 2E[\|E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}\|_{k,i,M}^2],$$

$$E[\|E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}\|_{k,i,M}^2] \leq 2E[\|E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}\|_{k,i,M}^2] + 2E[\|E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}\|_{k,i,M}^2].$$

We first treat the terms $E[\|\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k)}\|_{k,i,M}^2]$ and $E[\|\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k)}\|_{k,i,M}^2]$; the approach for both terms is identical, so we focus on the upper bound for the latter and only state the result for the former. We adopt an approach similar to the proof of Proposition 3.11 (iv); see [24] Appendix A to compare. First, observe using Proposition 3.11 (i) that

$$\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k)}(\cdot) \text{ solves } OLS(S_{Z,i}^{(k)}(x,w) - S_{Z,i}^{(k)}(x,w), K_{Z,i}^{(k)}, \nu_{M,M}).$$

Then, since $K_{Z,k,i}$ is finite dimensional, it has an orthonormal (with respect to the norm $\|\cdot\|_{k,i,M}^2$) basis $\{\hat{\nu}_1, \ldots, \hat{\nu}_K\}$ with $\hat{K} \leq K_{Z,k,i}$. Using the orthogonality property of $\hat{\nu}$, setting $\alpha := \int \hat{\nu}(x)\, d\sigma(x) - \hat{\nu}(x) \hat{\nu}(x)$. 

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\( S_{Z,i}^{(k,M)}(x) \) and expanding \(|\alpha|^2\) as a summation over the samples yields

\[
\|\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k,M)}\|^2_{k,i,M} = |\alpha|^2 \\
= \frac{1}{M_k^2} \sum_{m_1,m_2=1}^{M_k} \text{Tr}(\tilde{p}(\mathcal{X}(m_1))\tilde{p}^\top(\mathcal{X}(m_2))) \\
(\mathcal{S}_{Z,i}^{(k)}(\mathcal{X}(m_1)) - \mathcal{S}_{Z,i}^{(k,M)}(\mathcal{X}(m_1)))(\mathcal{S}_{Z,i}^{(k)}(\mathcal{X}(m_2)) - \mathcal{S}_{Z,i}^{(k,M)}(\mathcal{X}(m_2)))^\top.
\]

The random variables \( \{\mathcal{X}(1), \ldots, \mathcal{X}(M_k)\} \) are independent, which implies that \( \{\mathcal{S}_{Z,i}(\mathcal{X}(m_1)) - \mathcal{S}_{Z,i}(\mathcal{X}(m_1)) : m = 1, \ldots, M_k\} \) are independent conditionally on \( \mathcal{F}_{k,i}^{(M)} \). Thus, taking the conditional expectation \( \mathbb{E}_{k,i}^{(M)} \) implies that the \( (m_1, m_2) \)-terms go to 0 for \( m_1 \neq m_2 \). With matrix \( \Sigma^{(m)} := \mathbb{E}_{k,i}^{(M)}[(\mathcal{S}_{Z,i}(\mathcal{X}(m)) - \mathcal{S}_{Z,i}(\mathcal{X}(m)))(\mathcal{S}_{Z,i}(\mathcal{X}(m)) - \mathcal{S}_{Z,i}(\mathcal{X}(m)))^\top] \), it follows that

\[
\mathbb{E}_{k,i}^{(M)}[\|\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k,M)}\|^2_{k,i,M}] = \frac{1}{M_k^2} \sum_{m=1}^{M_k} \text{Tr}(\tilde{p}\tilde{p}^\top(\mathcal{X}(m))\Sigma^{(m)}) \leq \frac{1}{M_k^2} \sum_{m=1}^{M_k} \text{Tr}(\tilde{p}\tilde{p}^\top(\mathcal{X}(m))\Sigma^{(m)}),
\]

(40)

where we have used that \( \text{Tr}(AB) \leq \text{Tr}(A)\text{Tr}(B) \) for any symmetric non-negative definite matrices \( A \) and \( B \). To continue, we require a bound from above on \( \mathbb{E}[\text{Tr}(\tilde{p}\tilde{p}^\top(\mathcal{X}(m))\Sigma^{(m)})] \). Two approaches are available depending on the choice of basis: for general basis (as for Theorem 3.7), we find almost sure upper bounds for \( \text{Tr}(\Sigma^{(m)}) \) that are uniform in \( m \); on the other hand, for the special selection of basis in Theorem 3.9, the intrinsic properties of the basis are used to obtain refined bounds.

**Lemma 3.12.** For any \( k \geq 0, i \in \{0, \ldots, 2^k - 1\} \), and basis functions chosen as in Definition 3.3,

\[
\|\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k,M)}\|^2_{k,i,M} \leq \frac{K_{Z,i}^{(k)}}{\Delta_j^{(k)}} \frac{2^{k-1}-1}{M_k} \sum_{j=\alpha(i)+1}^{\infty} |z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot)|^2_{\infty} \Delta_j^{(k-1)}
\]

\[
\|\psi_{Y,i}^{(k)} - \psi_{Y,i}^{(k,M)}\|^2_{k,i,M} \leq \frac{K_{Z,i}^{(k)}}{\Delta_j^{(k)}} \frac{2^{k-1}-1}{M_k} \sum_{j=\alpha(i)+1}^{\infty} |z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot)|^2_{\infty} \Delta_j^{(k-1)}.
\]

**Proof.** We treat the terms \( \|\psi_{Z,i}^{(k)} - \psi_{Z,i}^{(k,M)}\|^2_{k,i,M} \); the proof for the terms \( \|\psi_{Y,i}^{(k)} - \psi_{Y,i}^{(k,M)}\|^2_{k,i,M} \) is the same and we exclude it. Recall the estimate (40). Thanks to the independence of the Brownian increments, one obtains the equality

\[
\text{Tr}(\Sigma^{(m)}) = \mathbb{E}_{k,i}^{(M)}[\|\mathcal{S}_{Z,i}^{(k)}(\mathcal{X}(m)) - \mathcal{S}_{Z,i}^{(k,M)}(\mathcal{X}(m))\|^2]
\]

\[
= \mathbb{E}_{k,i}^{(M)}[(y_{i}^{(k)}(X_i^{(k,m)}) - y_{i}^{(k,M)}(X_i^{(k,m)}))^2] \frac{\mathbb{E}[|\Delta W_i^{(k,m)}|^2]}{(\Delta_i^{(k)})^2} \\
+ \sum_{j=\alpha(i)+1}^{2^{k-1}-1} \mathbb{E}_{k,i}^{(M)}[|\Delta W_i^{(k,m)}|^2|z_j^{(k-1)}(X_j^{(k-1,m)}) - z_j^{(k-1,M)}(X_j^{(k-1,m)})|^2] \frac{\mathbb{E}[|\Delta W_j^{(k-1,m)}|^2]}{(\Delta_j^{(k)})^2}.
\]
Now, using \( \frac{1}{M} \sum_{m=1}^{M} [\tilde{p}\tilde{p}^T](\mathcal{X}^{(m)}) = \text{Id}_{k}\) and \( \tilde{K} \leq K_{Z,k,i} \), one substitutes the bounds of \( (40) \) into \( (41) \) in order to obtain the result. \( \square \)

In fact, one can improve on Lemma 3.12 if one assumes additional structure on the basis functions.

**Lemma 3.13.** In addition to the general assumptions, assume \((A_K)\) from Theorem 3.9. For any \( k \geq 0, i \in \{0, \ldots, 2^k - 1\} \),

\[
E[\|\psi^{(k)}_{Z,i} - \psi^{(k,M)}_{Z,i}\|_{k,i,M}^2] \leq \frac{K^{(k)}_{Z,i}}{\Delta^{(k)}_i M_k} \left\{ E[\|y_i^{(k)}(\cdot) - y_i^{(k,M)}(\cdot)\|_{k,i}^2] + 8qC_\chi 2^{-kT\theta \theta^{-1}} \right. \\
+ 2kq \ln(2) \sum_{j=\alpha(i)+1}^{2^k-1} E[\|z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot)\|_{k-1,j,\infty}^2] \Delta_j^{(k-1)} \bigg\},
\]

(42)

\[
E[\|\psi^{(k)}_{Y,i} - \psi^{(k,M)}_{Y,i}\|_{k,i,M}^2] \leq \frac{K^{(k)}_{Z,i}}{\Delta^{(k)}_i M_k} \sum_{j=\alpha(i)+1}^{2^k-1} E[\|z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot)\|_{k-1,j,\infty}^2] \Delta_j^{(k-1)}.
\]

(43)

**Proof.** We give the proof for \( (42) \); the proof for \( (43) \) is analogous (and simpler). Starting from \( (40) \), we apply the method of \( [5] \). For the convenience of the reader, we translate the notation of \( [5] \) to our setting: the functions \( f_j \) are equivalent to our \( \tilde{p}_j \), the \( j \)-th component of the vector \( \tilde{p} \), whence \( \text{Tr}(\tilde{p}\tilde{p}^T)(\mathcal{X}^{(m)}) = \sum_{j=1}^{K^{(k)}_{Z,i}} (f_j)^2 \); \( H^* \) is equivalent to our \( S^{(k)}_{Z,i}(\mathcal{X}^{(m)}) \); \( X \) is equivalent to our \( \mathcal{X} \) and \( X_m \) is equivalent to our \( \mathcal{X}^{(m)} \). Assume that \( P_{\mathcal{X}^{(k)}}(A_{Z,k,i,j}) \geq \delta / K^{(k)}_{Z,i} \) for all \( j \). Using the conditioning argument of \( [5] \) case (b) on page 14, it follows that

\[
E[\|\psi^{(k)}_{Z,i} - \psi^{(k,M)}_{Z,i}\|_{k,i,M}^2] \leq E \left[ \frac{1}{M_k} \sum_{m=1}^{M_k} \text{Tr}(\tilde{p}\tilde{p}^T)(\mathcal{X}^{(m)})E_{k,i}^{(M)}[\text{Tr}(\Sigma^{(m)})] \right]
\]

\[
\leq \frac{1}{M_k} E \text{Var}(H^*(X)|X) \sum_{j=1}^{K^{(k)}_{Z,i}} (f_j)^2 \quad \text{(in the equivalent notation of [5])}
\]

\[
\leq \sum_{j=1}^{K^{(k)}_{Z,i}} E[|S^{(k)}_{Z,i}(X) - S^{(k,M)}_{Z,i}(X)|^2 1_{X^{(k)}_i \in A_{Z,k,i,j}}] \leq \frac{K^{(k)}_{Z,i}}{M_k} \sum_{j=1}^{K^{(k)}_{Z,i}} E[|S^{(k)}_{Z,i}(X) - S^{(k,M)}_{Z,i}(X)|^2].
\]

To complete the proof, we obtain upper bounds on \( E[|S^{(k)}_{Z,i}(X) - S^{(k,M)}_{Z,i}(X)|^2] \):

\[
E[|S^{(k)}_{Z,i}(X) - S^{(k,M)}_{Z,i}(X)|^2] = E[\left(\left|y_i^{(k)}(X^{(k)}_i) - y_i^{(k,M)}(X^{(k)}_i)\right|^2\right)] \frac{E[|\Delta W^{(k)}_i|^2]}{\Delta^{(k)}_i^2}
\]

\[
+ \sum_{j=\alpha(i)+1}^{2^k-1} E[\left|\Delta W^{(k)}_j|^2 \right| z_j^{(k-1)}(X^{(k-1)}_j) - z_j^{(k-1,M)}(X^{(k-1)}_j)|^2] \frac{E[|\Delta W^{(k-1)}_j|^2]}{\Delta^{(k)}_j^2}.
\]
There is an interdependency issue between $\Delta W_i^{(k)}$ and $|z_j^{(k-1)}(X_j^{(k-1)} - z_j^{(k-1)}X_j^{(k-1)})|$ that we now treat; note that this interdependency does not arise when dealing with $\mathbb{E}[|z_j^{(k)}(X_j^{(M)}) - S_j^{(M)}(X_j^{(M)})|^2]$. Since $\Delta W_i^{(k)}$ has $q$ independent components, each with Gaussian distribution with mean 0 and variance $\Delta_i^{(k)}$, these components are each equal in law to $\sqrt{\Delta_i^{(k)}}N$, where $N$ has a Gaussian distribution with mean 0 and variance 1. Calculating the expectation by integration-by-parts then using Mill's inequality implies, for any $R > 0$, that

$$
\mathbb{E}[|N|^2 1_{|N| > \sqrt{R}}] = 2(\mathbb{P}(N > \sqrt{R})(R + 1) - \sqrt{Re^{-R/2}}) \leq 2\mathbb{P}(N > \sqrt{R})(R + 1 - R) \leq 2e^{-R/2}.
$$

Now, using the decomposition $\Delta W_i^{(k)} = \frac{T}{\sqrt{\Delta_i^{(k)}}R}(\Delta W_i^{(k)}) + (\Delta W_i^{(k)} - \frac{T}{\sqrt{\Delta_i^{(k)}}R}(\Delta W_i^{(k)}))$ and the almost sure bounds on the $z$ terms from Corollary 2.3 and Algorithm 2 it follows that

$$
\mathbb{E}[|\Delta W_i^{(k)}|^2 | z_j^{(k-1)}(X_j^{(k-1)}) - z_j^{(k-1,M)}(X_j^{(k-1)})|^2]
\leq qR\Delta_i^{(k)}\mathbb{E}[|z_j^{(k-1)}(X_j^{(k-1)}) - z_j^{(k-1,M)}(X_j^{(k-1)})|^2] + \frac{4qC^2_\Delta \Delta_i^{(k)}}{(T - \epsilon_j^{(k-1)})^{1-\theta}}\mathbb{E}[|N|^2 1_{|N| > R}]
\leq qR\Delta_i^{(k)}\mathbb{E}[|z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot)|^2_{k-1,j,\infty}] + \frac{8qe^{-R/2}C^2_\Delta \Delta_i^{(k)}}{(T - \epsilon_j^{(k-1)})^{1-\theta}}
\leq qR\Delta_i^{(k)}\mathbb{E}[|z_j^{(k-1)}(\cdot) - z_j^{(k-1,M)}(\cdot)|^2_{k-1,j,\infty}] + \frac{8qe^{-R/2}C^2_\Delta \Delta_i^{(k)}}{(T - \epsilon_j^{(k-1)})^{1-\theta}}.
$$

The proof is completed by selecting $R = \ln(2^{2k})$. \qed

To complete the estimate of $[38]$ and $[39]$, it remains only to bound $\mathbb{E}[||E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}||_{k,i,M}^2]$ and $\mathbb{E}[||E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}||_{k,i,M}^2]$.  

Proposition 3.14. In addition to the general assumptions, suppose that either $(A''_X)$ (from Theorem 3.7) or $(A_K)$ (from Theorem 3.8) is in force. Then, for all $k \geq 0$ and $i \in \{0, \ldots, 2^k - 1\}$,

$$
\mathbb{E}[||E_{k,i}^{(M)}[\psi_{Y,i}^{(k)}] - \psi_{Y,i}^{(k)}||_{k,i,M}^2] \leq C_1 \frac{2 \times 2^{-k}K^{(k)}_{x_i}}{M_k} \{3C^2_X + (2 + q)\} \quad \text{and} \\
\mathbb{E}[||E_{k,i}^{(M)}[\psi_{Z,i}^{(k)}] - \psi_{Z,i}^{(k)}||_{k,i,M}^2] \leq C_1 \frac{6K^{(k)}_{x_i}(2 + 5T^{1-\theta})C^2_X}{c_X M_k(T - \epsilon_i^{(k)})^{1-\theta} + 2C_1(2 + q)K^{(k)}_{x_i}C_X}{c_X M_k}.
$$

where $C_1 = \delta$ if $(A_K)$ holds and $C_1 = 1$ if $(A''_X)$ holds.

Proof. We will use Proposition 3.11(iv). For $x := (x_0, \ldots, x_{2^k}) \in \mathbb{R}^{(2^{k+1}+1) \times d}$, $\overline{x} := (\overline{x}_0, \ldots, \overline{x}_{2^k-1}) \in \mathbb{R}^{2^{k+1} \times q}$, define $h(x, \overline{x}, w) := x_i; h$ is a Borel measurable function, and $h(\chi^{(m)}) = X_i^{(k,m)}$. Denote by $\mathcal{H}$ the $\sigma$-algebra $\sigma(h^{(i)}(X_m) : m = 1, \ldots, M_k)$, which is equal to $\sigma(X_i^{(k,m)} : m = 1, \ldots, M_k)$, and by $\mathcal{G}$ the $\sigma$-algebra $\mathcal{F}_{k-1}^{(i)} \vee \sigma(X_j^{(k,m)} : j < i, m = 1, \ldots, M_k)$; then $S^{(k)}_{Y,i}(\cdot)$ and $S^{(k)}_{Z,i}(\cdot)$ are $\mathcal{G} \otimes \mathcal{B}(\mathbb{R})$-measurable, and $\mathcal{G} \vee \mathcal{H}$ is equal to $\mathcal{F}_{k,i}^{(M)}$. Since $\psi_{Y,i}^{(k,M)}$ (resp. $\psi_{Z,i}^{(k,M)}$) solves $\text{OLS}(S^{(k)}_{Y,i}(\cdot), K^{(k)}_{Y,i}, \nu^{(k)}_{M})$ (resp. $\text{OLS}(S^{(k)}_{Z,i}(\cdot), K^{(k)}_{Z,i}, \nu^{(k)}_{M})$) it only remains to find suitable (deterministic) upper bounds for expectation of $\Psi_{Y,k,i} := E_{k,i}^{(M)}[|S^{(k)}_{Y,i}(\chi^{(m)}) - E^{(M)}_{k,i}S^{(k)}_{Y,i}(\chi^{(m)})|^2]$. 

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Ψ \(Z,k,i\) := \(\mathbb{E}^{(M)}_{k,i} [S^{(k)}_{Z,i}(\mathcal{X}^{(m)}) - \mathbb{E}^{(M)}_{k,i} [S^{(k)}_{Z,i}(\mathcal{X}^{(m)})]^{2}]\) to allow us to apply Proposition 3.1 iv. The technique is similar for both \(Ψ_{Y,k,i}\) and \(Ψ_{Z,k,i}\); so we include the proof for the latter only. The strategy will be the following: first, we assume that the Markov chains \(X^{(k,m)}\) and \(X^{(k-1,m)}\) are have deterministic values \(x\) at \(t^{(k)}_{1}\) and \(\bar{x}\) at \(t^{(k-1)}_{\alpha(i)}\), respectively; then we decompose the upper bound on \(Ψ_{Z,k,i}\) by introducing the diffusion processes \(X^{(t^{(k)}_{1},x,m)}\) and \(X^{(t^{(k-1)}_{\alpha(i)},\bar{x},m)}\); eventually, we fix \(x = X^{(k,m)}_{t^{(k)}_{1}}\) and \(\bar{x} = X^{(k-1,m)}_{t^{(k-1)}_{\alpha(i)}}\) (which does not pose difficulties due to the use of the conditional expectation \(\mathbb{E}^{(M)}_{k,i}\) throughout) to obtain the final bounds.

- **Step 1 (fixing the initial value of the Markov chain at \(t^{(k)}_{1}\) and \(t^{(k-1)}_{\alpha(i)}\)):** Observe that the random variable \(\mathcal{X}^{(m)}\) depends on the sample path \(\mathcal{X}^{(m)}\) only through the values

\[
(x^{(k,m)}, \ldots, x^{(k,m,i)}_{\alpha(i)+1}, \ldots, x^{(k,m,\bar{x})}_{\alpha(i)})_{k,m,i} W^{(k,m)}_{i}, \ldots, W^{(k-1,m)}_{i}, \ldots, W^{(k,m)}_{2^{k-1}}, \ldots, W^{(k,m)}_{2^{k-1}}.
\]

i.e., it does not depend on the path \(X^{(k,m)}, X^{(k,m-1)}\) and \(\Delta W^{(k,m)}\) before the time \(t^{(k)}_{i}\). Letting \(x, \bar{x} \in \mathbb{R}^{d}\), we define

\[
\mathcal{X}^{(m,i)}(x, \bar{x}) := (x^{(k,m,i,x)}_{\alpha(i)+1}, \ldots, x^{(k,m,\bar{x})}_{\alpha(i)})_{k,m,i} W^{(k,m)}_{i}, \ldots, W^{(k-1,m)}_{i}, \ldots, W^{(k,m)}_{2^{k-1}}, \ldots, W^{(k,m)}_{2^{k-1}}.
\]

One can then write \(Ψ_{Z,k,i} = Ψ_{Z,k,i}(X^{(k,m)}, X^{(k-1,m)})\), where

\[
Ψ_{Z,k,i}(x, \bar{x}) := \mathbb{E}^{(M)}_{k,i} [S^{(k)}_{Z,i}(\mathcal{X}^{(m,i)}(x, \bar{x})) - \mathbb{E}^{(M)}_{k,i} [S^{(k)}_{Z,i}(\mathcal{X}^{(m,i)}(x, \bar{x}))]^{2}].
\]

- **Step 2 (decomposition with intermediate discrete BSDE):** Let \(\mathcal{X}^{(t,m,x)}(m \in \{1, \ldots, M_{k}\})\) be the simulation of the diffusion started at time \(t\) with value \(x\) generated with the same path of the Brownian motion as the increments \(\Delta W^{(k,m)}_{i}\). Recall the discrete BSDE \((\tilde{y}^{(k)}, \tilde{z}^{(k)})\) from section 2.2 and define

\[
\tilde{y}^{(k,m)}_{j} := \mathbb{E}^{\bar{x}}_{j} [\Phi(\mathcal{X}^{(t^{(k)}_{1},x,m)}_{T})] \quad \text{and} \quad \tilde{\Delta}^{(k)}_{j} \tilde{z}^{(k,m)}_{j} := \mathbb{E}^{\bar{x}}_{j} [(\Delta W^{(k,m)}_{j})^\top \Phi(\mathcal{X}^{(t^{(k)}_{1},x,m)}_{T})]
\]

for \(j \in \{0, \ldots, 2^{k-1} - 1\}\). For the coarse grid \(\pi^{(k-1)}\), define

\[
\tilde{y}^{(k-1,m)}_{j} := \mathbb{E}^{\bar{x}}_{j} [\Phi(\mathcal{X}^{(t^{(k-1)}_{\alpha(i)},\bar{x},m)}_{T})] \quad \text{and} \quad \tilde{\Delta}^{(k-1)}_{j} \tilde{z}^{(k-1,m)}_{j} := \mathbb{E}^{\bar{x}}_{j} [(\Delta W^{(k-1,m)}_{j})^\top \Phi(\mathcal{X}^{(t^{(k-1)}_{\alpha(i)},\bar{x},m)}_{T})]
\]

for \(j \in \{0, \ldots, 2^{k-1} - 1\}\). We use these processes to decompose \(S^{(k)}_{Z,i}(\mathcal{X}^{(m,i)}(x, \bar{x}))\) into two expressions:

\[
S^{(k)}_{Z,i}(\mathcal{X}^{(m,i)}(x, \bar{x})) = \left\{ \begin{array}{c}
\frac{\Delta W^{(k,m)}_{i}}{\Delta^{(k)}_{i}} \left\{ \Phi(\mathcal{X}^{(k,m,i,x)}_{N}) - \Phi(\mathcal{X}^{(t^{(k)}_{1},x,m)}_{T}) - \tilde{y}^{(k,m)}_{i} (\mathcal{X}^{(k,m,i,x)}_{T}) - \tilde{z}^{(k,m)}_{i} \right\} \\
- \sum_{j=\alpha(i)+1}^{2^{k-1} - 1} \left( \tilde{\Delta}^{(k)}_{j} \tilde{z}^{(k-1,m)}_{j} \Delta W^{(k-1,m)}_{j} \right)
\end{array} \right\} + \left\{ \begin{array}{c}
\frac{\Delta W^{(k,m)}_{i}}{\Delta^{(k)}_{i}} \left\{ \Phi(\mathcal{X}^{(t^{(k)}_{1},x,m)}_{T}) - \tilde{y}^{(k,m)}_{i} - \sum_{j=\alpha(i)+1}^{2^{k-1} - 1} \tilde{z}^{(k-1,m)}_{j} \Delta W^{(k-1,m)}_{j} \right\} \\
=: A_{1}(x, \bar{x}) + A_{2}(x, \bar{x}).
\end{array} \right\}
\]
The trivial inequality \((x + y)^2 \leq 2x^2 + 2y^2\) for all real \(x\) and \(y\) then yields
\[
\Psi_{Z,k,i}(x, \bar{x}) \leq 2E^{(M)}_{k,i}[A_1(x, \bar{x})^2] + 2E^{(M)}_{k,i}[A_2(x, \bar{x})^2].
\]

\[\textbf{Step 3 (bound on } E^{(M)}_{k,i}[A_1(x, \bar{x})^2])
\]
Using the Cauchy-Schwarz inequality, we have
\[
E^{(M)}_{k,i}[(z_j^{(k-1)}(X_j^{(k-1,m,\alpha(i),\bar{x})} - \bar{y}_j^{(k-1,m)})^2]
\]
\[
= \frac{1}{(\Delta_j^{(k-1)})^2}E^{(M)}_{k,i}[(\Delta W_j^{(k-1,m)})^2]
\]
\[
\leq \frac{q}{\Delta_j^{(k-1)}}\left\{E^{(M)}_{k,i}[(y_j^{(k-1)}(X_j^{(k-1,m,\alpha(i),\bar{x})} - \bar{y}_j^{(k-1,m)})^2]
\right\}.
\]

Observe that
\[
E^{(M)}_{k,i-1}[y_j^{(k-1)}(X_j^{(k-1,m,\alpha(i),\bar{x})}) - \bar{y}_j^{(k-1,m)}] = y_j^{(k-1)}(X_j^{(k-1,m,\alpha(i),\bar{x})}) - \bar{y}_j^{(k-1,m)}.
\]

Then, a shift of summation indices gives
\[
\sum_{j=\alpha(i)+1}^{2^{k-1}-1}E^{(M)}_{k,i}[(z_j^{(k-1)}(X_j^{(k-1,m,\alpha(i),\bar{x})}) - \bar{y}_j^{(k-1,m)})^2] \Delta_j^{(k-1)}
\]
\[
\leq \sum_{j=\alpha(i)+1}^{2^{k-1}-1} q\left\{E^{(M)}_{k,i}[(y_j^{(k-1)}(X_j^{(k-1,m,i,\bar{x})}) - \bar{y}_j^{(k-1,m)})^2]
\right\}
\]
\[
\leq \sum_{j=\alpha(i)+1}^{2^{k-1}-1} q\left\{E^{(M)}_{k,i}[(y_j^{(k-1)}(X_j^{(k-1,m,i,\bar{x})}) - \bar{y}_j^{(k-1,m)})^2]
\right\}
\]
\[
+ qE^{(M)}_{k,i}[(\Phi(X_N^{(k-1,m,\alpha(i),\bar{x})})) - \Phi(X_T^{(k-1,m,\alpha(i),\bar{x})})^2]
\]

Therefore, using the independence of the Brownian increments, one obtains the upper bound
\[
E^{(M)}_{k,i}[A_1(x, \bar{x})^2] = \frac{1}{\Delta_i^{(k)}}\left\{E^{(M)}_{k,i}[(\Phi(X_N^{(k,m,i,\bar{x})})) - \Phi(X_T^{(k,m,i,\bar{x})})^2]
\right\}
\]
\[
+ \sum_{j=\alpha(i)+1}^{2^{k-1}-1} q\left\{E^{(M)}_{k,i}[(y_j^{(k-1)}(X_j^{(k-1,m,\alpha(i),\bar{x})}) - \bar{y}_j^{(k-1,m)})^2]
\right\}
\]
\[
\leq \frac{1}{\Delta_i^{(k)}}\left\{2E^{(M)}_{k,i}[(\Phi(X_N^{(k,m,i,\bar{x})})) - \Phi(X_T^{(k,m,i,\bar{x})})^2]
\right\}
\]
\[
\leq \frac{(2 + q)C_X 2^{-k}}{\Delta_i^{(k)}} \leq \frac{(2 + q)C_X}{c_X}.
\]

It follows from assumption on the Markov chains \((A^X_i)(i)\) and the assumption on the time-grids \((A_{\pi})_{(iii)}\) that the terms in parenthesis in (44) can be bounded by \(C_X \max\{2^{-k} + q2^{-(k-1)}\}\). Hence,
\[
E^{(M)}_{k,i}[A_1(x, \bar{x})^2] \leq \frac{(2 + q)C_X 2^{-k}}{\Delta_i^{(k)}} \leq \frac{(2 + q)C_X}{c_X}.
\]
Step 4 (bound on $\mathbb{E}_{k,i}^{(M)}[A_2(x, \bar{x})^2]$). Using equality [9] and Lemma 2.4, we have

$$
\sum_{j=\alpha(i)+1}^{2^{k-1}-1} z_j^{(k,m)} \Delta W_j^{(k-1,m)} = \Phi(X_T^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}) - \bar{y}_{\alpha(i)+1}^{(k-1,m)} - \sum_{j=\alpha(i)+1}^{2^{k-1}-1} \Delta L_j^{(k-1,m)}
$$

where $\Delta L_j^{(k-1,m)} := \int_{t_j^{(k-1)}}^{t_{j+1}^{(k-1)}} (z_j^{(k-1,m)} - z_j^{(k-1,m)})^T dW_j(t)$ and $z_j^{(k-1,m)}$ is the process given in (A_X)(iii) with $X^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}$ in the place of $X^{(l_{\alpha(i)}^{(k-1)}, \bar{x})}$. Substituting this into the definition of $A_2(x, \bar{x})$, it follows that

$$
A_2(x, \bar{x}) = \frac{(\Delta W_i^{(k)})^T}{\Delta_i} \left\{ \Phi(X_T^{(l_{x,i}, \bar{x}, m)}) - \Phi(X_T^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}) - (y_i^{(k,m)} - \bar{y}_{\alpha(i)+1}^{(k-1,m)}) + \sum_{j=\alpha(i)+1}^{2^{k-1}-1} \Delta L_j^{(k-1,m)} \right\}.
$$

Now we square and take expectations, and treat the terms in $\Delta L, \Phi$ and $\bar{y}$ individually.

To treat the terms in $\Delta L$, we apply property (A_x)(iv) to obtain that $\sum_{j=\alpha(i)+1}^{2^{k-1}-1} \mathbb{E}_{k,i}^{(M)}[|\Delta L_j^{(k-1,m)}|^2]$ is bounded by $2C\chi 2^{-k}$; this upper bound is independent of the starting value $\bar{x}$ of $X^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}$. In order to treat the terms involving $\Phi$, we apply assumption (A_X)(ii) to obtain

$$
\mathbb{E}_{k,i}^{(M)}[|\Phi(X_T^{(l_{x,i}, \bar{x}, m)}) - \Phi(X_T^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)})|^2] \leq C_X |x - X_{t_i^{(k)}}^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}|^2.
$$

Remark 3.15. In (45), one can see the impact of condition (A_X)(ii): it is needed to obtain the upper bound $O(2^{-k})$ of the terms in $\Phi$, once we put $x = X_{t_i^{(k)}}^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}$, and take expectations.

Finally, to treat the terms in $\bar{y}$, we use that $\bar{y}_i^{(k,m)} - \bar{y}_{\alpha(i)+1}^{(k-1,m)}$ is equal to

$$
\bar{y}_i^{(k,m)} - \bar{y}_{\alpha(i)+1}^{(k-1,m)} = \bar{y}_2^{(k,m)} - \bar{y}_{\alpha(i)+1}^{(k-1,m)} + \mathbb{E}_{k,2(\alpha(i)+1)}^{(M)} [\Phi(X_T^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}) - \Phi(X_T^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)})].
$$

The terms in $\Phi$ are treated as in (45). We further expand $(\bar{y}_i^{(k)} - \bar{y}_2^{(k)})$ using [9]

$$
\bar{y}_i^{(k,m)} - \bar{y}_{2(\alpha(i)+1)}^{(k,m)} = \sum_{j=\alpha(i)+1}^{2(\alpha(i)+1)} \left\{ z_j^{(k)} \Delta W_j^{(k,m)} + \Delta L_j^{(k,m)} \right\}.
$$

Squaring and taking conditional expectations, we obtain from (A_x)(iv) and Corollary 2.3 that

$$
\mathbb{E}_{k,i}^{(M)}[|\bar{y}_i^{(k,m)} - \bar{y}_{2(\alpha(i)+1)}^{(k,m)}|^2] = \sum_{j=\alpha(i)+1}^{2(\alpha(i)+1)} \mathbb{E}_{k,i}^{(M)}[z_j^{(k)}^2 |\Delta W_j^{(k,m)}|^2 + |\Delta L_j^{(k,m)}|^2] \leq \frac{2C_X^2 2^{-k}}{(T - t_i^{(k)})^{1-\theta}} + C_X 2^{-k}.
$$

To conclude Step 4, we combine the above upper bounds to obtain

$$
\Delta_i^{(k)} \mathbb{E}_{k,i}^{(M)}[A_2(x, \bar{x})^2] \leq 3 \times (3C_X |x - X_{t_i^{(k)}}^{(l_{\alpha(i)}^{(k-1)}, \bar{x}, m)}|^2 + \frac{2C_X^2 2^{-k}}{(T - t_i^{(k)})^{1-\theta}} + 2C_X 2^{-k}).
$$

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Therefore, plugging $x = X_i^{(k,m)}$ and $\bar{x} = X_i^{(k-1,m)}$, we obtain
\[
\Delta_i^{(k)} E_{k,i}^{(M)}[A_2(X_i^{(k,m)}, X_i^{(k-1,m)})^2] \leq 9C_X |X_i^{(k,m)} - X_i^{(k-1,m)}|^2 + \frac{6C_X^2 2^{-k}}{(T - t_i^{(k)})^{1-\theta}} + 6C_X 2^{-k} \tag{46}
\]

**Concluding the proof.** The proof of the proposition under $\mathbf{A}_X^{(m)}$ is now completed by observing that $|X_i^{(k,m)} - X_i^{(k-1,m)}|^2 = 0$ in (46), piecing together the estimates obtained in **Steps 1-4** on $E_{k,i}^{(M)}[S^k_i(X^{(m)}) - E_{k,i}^{(M)}[S^k_i(X^{(m)})]^2]$ to find that there is a deterministic bound, and applying Proposition 3.14(iv). On the other hand, if $\mathbf{A}_K$ were in force, we again (as in proof of Lemma 3.13) use the conditioning arguments of [5, case (b) on page 14] in order to obtain
\[
E[\|E_{k,i}^{(M)}[\psi_i^{(k)}(x) - \psi_i^{(k)}]\|_2^2 \leq \frac{\delta K_{Z,i}^k [E_{k,i}^{(M)}[|S^k_i(X^{(1)}) - E_{k,i}^{(M)}[S^k_i(X^{(1)})]^2]]}{M_k},
\]

By combining the estimates obtained in **Steps 1-4** on $E_{k,i}^{(M)}[|S^k_i(X^{(1)}) - E_{k,i}^{(M)}[S^k_i(X^{(1)})]^2]$, substituting them into the above inequality, one sees that the expectation $E[\|E_{k,i}^{(M)}[\psi_i^{(k)}(x) - \psi_i^{(k)}]\|_2^2]_{k,i,M}$ is bounded by
\[
6\delta K_{Z,i}^k \left(3C_X \frac{E[|X_i^{(k)} - X_i^{(k-1,m)}|^2]}{\Delta_i^{(k)}} + \frac{(2 + 2T^{1-\theta})C_X^2}{c_X (T - t_i^{(k)})^{1-\theta}} + 2\delta (2 + q) K_{Z,i}^k C_X \right) c_X M_k 
\leq \frac{6\delta K_{Z,i}^k (2 + 5T^{1-\theta})C_X^2}{c_X M_k (T - t_i^{(k)})^{1-\theta}} + 2\delta (2 + q) K_{Z,i}^k C_X \frac{c_X M_k}{c_X M_k} \tag{47}
\]

where we have used $\mathbf{A}_X^{(m)}(\text{iii})$ in the last inequality, and $\mathbf{A}_X^{(m)}(\text{iii})$ for the bound $\Delta_i^{(k)} \geq c_X 2^{-k}$.

**Remark 3.16.** We see in equation (47) the impact of assumption $\mathbf{A}_X^{(m)}(\text{iii})$: were a lower rate of convergence assumed, the overall rate of convergence with respect to $k$ of the upper bound in Theorem 3.9 would be lower.

The proof is completed by estimating the terms in (38) using Lemma 3.12 (resp. Lemma 3.13) and Proposition 3.14 and substituting the estimates into (36) – (37).

### 3.5 Computational examples

The computational examples in this section illustrate and compare the actual errors and efficiency of different simulation schemes to the BSDE with zero generator [3] to support the results of the theoretical analysis based on error estimates. We consider cases of BSDE with analytically known solutions in order to investigate the actual global mean squared errors (MSE) of the approximate solutions for the respective approximation schemes. The MSEs are computed by Monte Carlo on a fine time grid in the same way as the global error in [23]. The overall MSE is the sum of the MSEs with respect to the $Y$ and the $Z$ components, corresponding to the first (with squared maximum over time) respectively second (time-weighted) summand.
3.5.1 Sine payoff

At first let us consider a $q = 1$ dimensional example with $T = 1$, $X = W$, and terminal condition $\Phi(x) = \sin(x)$. For a regression basis, we took Hermite polynomials up to degree 7, adjusted for time so that \{1, p_1(t, W_t), \ldots, p_7(t, W_t)\} is orthonormal for all time $t > 0$, i.e. $K = 8$. We run the multilevel (ML) scheme with $M_k = 40 \times K \times 2^k$ simulations at final level $k$, while at any lower level $j < j+1 \leq k$ the number of simulations $M_j = 2M_{j+1}$ doubles. The overall complexity for ML up to level $k$ is therefore $C_{ML} = O(k \times 2^{2k})$. For comparison, we run two instances of the MDP scheme: At first (MDP1) with $M_k = 40 \times K \times 2^k$ simulations, and then (MDP2) with a much higher number $M_k = 40 \times K \times 2^{2k}$ of simulations. The complexity in the first case is $C_{MDP,1} = O(2^{2k})$ and $C_{MDP,2} = O(2^{3k})$ in the second. Figure 1 shows the log of the global mean squared errors (MSE) of the MDP1, MDP2 and the ML scheme vs. the log of the number of time steps $k = \log_2 N$. The respective regression lines are (ML) $-0.88x - 5.0$, (MDP1) $-0.05x - 5.7$, respectively (MDP2) $-0.97 - 6.5$. This example supports the results of Theorems 3.7 and 3.9 and the subsequent complexity analysis in Section 3.3; indeed, one sees that one needs to have $2^k$-times as many simulations for MDP as for multilevel to achieve a convergence rate of about $-1$. Moreover, one sees that results the computational results indicate, that efficiency gains as stated in Theorem 3.9 may be obtained, beyond the assumptions of the theorem, for a wider class of basis functions than allowed by $(A_X)$, as the basis functions used in this example do not satisfy this condition.

3.5.2 A multi-dimensional example

Let the forward process be a Brownian motion $X = W$ in dimension $q = 3$ and consider the terminal condition $\Phi(X_T) = \prod_{i=1}^q X_T^i$ with $T = 1$. This is beyond the assumptions used in the complexity analysis of Section 3.3, the boundedness assumption $(A_{\Phi})$, and the Lipschitz assumption $(A_X)$(iv). This example is to compare the the MSE separately in the contributions of the $Y$- and the $Z$-part of the
multilevel (ML) and the MDP scheme. Moreover, we compare two different sets of regression bases. The first regression basis (‘indicator’) consists of indicator functions on equiprobable hypercubes of a partition of \( R^3 \) into \( K = 8^3 = 512 \) sets. The second regression basis (‘linear’) consists of functions, each being affine within one hypercube of a partition of \( R^3 \) into \( K = 5^3 = 125 \) sets and vanishing outside. The linear basis contains \( 4 \times 5^3 = 500 \) of regression functions, so that both bases have about the same size. Number simulations is \( M = 2 \times 10^6 \) for both schemes. For ML, the same number of simulations is used at each level.

| \( \log_2(N) \) | 2   | 3   | 4   | 5   | 6   | 7   |
|-----------------|-----|-----|-----|-----|-----|-----|
| ML Y (linear)   | 0.1528 | 0.1266 | 0.1215 | 0.1194 | 0.1183 | 0.1190 |
| ML Z (linear)   | 0.0334 | 0.0184 | 0.0160 | 0.0157 | 0.0166 | 0.0185 |
| MDP Y (linear)  | 0.1578 | 0.1316 | 0.1253 | 0.1236 | 0.1231 | 0.1222 |
| MDP Z (linear)  | 0.0358 | 0.0245 | 0.0301 | 0.0462 | 0.0786 | 0.1441 |
| ML Y (indicator) | 0.5815 | 0.5454 | 0.5356 | 0.5331 | 0.5310 | 0.5306 |
| ML Z (indicator) | 0.1509 | 0.1219 | 0.1148 | 0.1135 | 0.1154 | 0.1210 |
| MDP Y (indicator) | 0.5865 | 0.5465 | 0.5351 | 0.5318 | 0.5297 | 0.5297 |
| MDP Z (indicator) | 0.1514 | 0.1253 | 0.1230 | 0.1330 | 0.1550 | 0.2044 |

The table of global mean squared errors, Table 1, shows that the multilevel scheme achieves lower errors for the \( Z \)-part for finer time grids, whereas errors for the \( Y \)-part are similar. The multilevel scheme shows a higher reduction of error in \( Z \) for the linear basis. Error reduction by factors beyond \( 1/2 \) for \( k = \log_2 N \geq 4 \) are significant, even when noting that the computational cost for MDP with \( N = 2^{k+1} \) time steps are basically equal to that for ML with \( N = 2^k \) steps at final level \( k \). Compared to MDP, errors (in \( Z \)) for the multilevel scheme begin to increase at a later stage \( k = \log_2 N \) and increase at a much milder rate, regardless of the choice of the basis; this is best seen by comparing Figure 2 with Figure 3. This effect fits with the results of Theorems 3.7 and 3.9, which state that the error of multilevel scheme is less affected by the number of time points than the MDP scheme \(^{(27)} \); the error of the multilevel scheme should increase only logarithmically with \( N \). and indeed Figure 2 shows an error curve increasing only mildly at large \( k \). Note that for a fixed number of simulations, as here, it is inevitable that statistical errors increase and take over at some stage; such simply means that more simulations would be required for larger \( k = \log_2 N \). The advantages of the linear over the indicator basis can understood in the sense that in this example the bias from \( L^2 \) projection on the function space spanned by this basis is smaller, whereas the indicator basis would require a finer partition to achieve the same. This example shows that efficiency gains from multilevel can be realized in actual computations; and moreover indicates that efficiency gains may be expected in a more general context beyond the specific assumptions required in Section 3.3.

4 Completing the splitting algorithm

Fix \( k > 0 \). In this section, approximate the second part of the split system \(^{(4)} \), namely the functions \( \bar{y}_i^{(k)} : \mathbb{R}^d \to \mathbb{R} \) and \( \bar{z}_i^{(k)} : \mathbb{R}^d \to (\mathbb{R}^q)^\top \) in Lemma 2.2. Omitting the superscript \((k)\) to ease notation in
what follows, we recall that these functions satisfy

$$\bar{y}_i(X_i) := \mathbb{E}_i \left[ \sum_{j=i+1}^{2^k-1} f_j(X_j, y_{j+1}^{(k)}(X_{j+1}^{(k)}), \bar{y}_{j+1}(X_{j+1}^{(k)}), z_{j+1}(X_{j+1}^{(k)}) + \bar{z}_{j+1}(X_{j+1}^{(k)})) \Delta_j \right],$$

$$\Delta_i \times \bar{z}_i(X_i) := \mathbb{E}_i \left[ \Delta W_i^{(k)} \left( \sum_{j=i+1}^{2^k-1} f_j(X_j^{(k)}, y_{j+1}(X_{j+1}^{(k)}), \bar{y}_{j+1}(X_{j+1}^{(k)}), z_{j+1}(X_{j+1}^{(k)}) + \bar{z}_{j+1}(X_{j+1}^{(k)})) \Delta_j \right) \right].$$

Let the functions \((y_j(\cdot), z_j(\cdot))_{0 \leq j \leq N-1}\) be approximated using the multilevel algorithm, and \(\pi\) denote the time-grid \(\pi^{(k)}\) of the highest level of the multilevel algorithm. We use least-squares multistep dynamical programming (LSMDP) from[24] for the discrete BSDE with zero terminal condition and random driver

$$f_j^{(M)}(y, z) := f_j(X_j^{(k)}, y_{j+1}^{(k,M)}(X_{j+1}^{(k)}), y_{j+1}^{(k,M)}(X_{j+1}^{(k)}), z_{j+1}^{(k,M)}(X_{j+1}^{(k)})) \quad (48)$$

to approximate \(\bar{y}_i(\cdot)\) and \(\bar{z}_i(\cdot)\). We maintain the superscript \(k\) in notation \(X^{(k)}, \Delta W^{(k)}, \) and \(f^{(k)}\) to remind that the time-grid in use is \(\pi^{(k)}\), although LSMDP does not make use of earlier (coarser) time-grids. The driver has two sources of randomness: the random functions \((y^{(k,M)}, z^{(k,M)})\), which depend on the samples used in the multilevel algorithm, and the Markov chain \(X^{(k)}\). The notation

$$f_j^{(M)}(x_1, x_2, y, z) := f_j(x_1, y_{j+1}^{(k,M)}(x_2), y_{j+1}^{(k,M)}(x_2), z_{j+1}^{(k,M)}(x_1) + z),$$

will be helpful in the sequel. We briefly recall the LSMDP algorithm for the convenience of the reader. Like the algorithm in Section[3] LSMDP is a least-squares Monte Carlo algorithm; the difference in
the choice of basis functions and the generation of simulations compared to the multilevel algorithm of Section 3 is threefold: firstly, since there is no use of multiple levels, only simulations of the Markov chain $X^{(k)}$ are generated; secondly, independent clouds of simulations are generated for every time-point, which means that the empirical measure for each time-point is independent of the empirical measure used at any other time-point; thirdly, the choice of basis functions is different to that used for the multilevel scheme. We formalize this in the following definitions.

**Definition 4.1** (Finite dimensional approximation spaces). For $i \in \{0, \ldots, 2^k - 1\}$, we finite functional linear spaces $K_{Y,i}$ and $K_{Z,i}$ of dimensions $K_{Y,i}$ and $K_{Z,i}$, given by

$$
\begin{align*}
K_{Y,i} &:= \text{span}\{p_{Y,i}^{(1)}, \ldots, p_{Y,i}^{(K_{Y,i})}\}, \text{ for } p_{Y,i}^{(l)} : \mathbb{R}^d \to \mathbb{R} \text{ s.t. } E[|p_{Y,i}^{(l)}(X_i)|^2] < +\infty, \\
K_{Z,i} &:= \text{span}\{p_{Z,i}^{(1)}, \ldots, p_{Z,i}^{(K_{Z,i})}\}, \text{ for } p_{Z,i}^{(l)} : \mathbb{R}^d \to (\mathbb{R}^q)^\top \text{ s.t. } E[|p_{Z,i}^{(l)}(X_i)|^2] < +\infty.
\end{align*}
$$

We suppress the subscript $k$ in the notation of the basis functions and linear spaces to distinguish them from those in Definition 3.2. The functions $\bar{y}_i(\cdot)$ and $\bar{z}_i(\cdot)$ will be approximated in the linear spaces $K_{Y,i}$ and $K_{Z,i}$, respectively. We define

$$
T_{1,i}^Y := \inf_{\phi \in K_{Y,i}} E\left[|\phi(X_i) - \bar{y}_i(X_i)|^2\right] \quad \text{and} \quad T_{1,i}^Z := \inf_{\phi \in K_{Z,i}} E\left[|\phi(X_i) - \bar{z}_i(X_i)|^2\right];
$$

as in Definition 3.2, these are the best approximation errors possible with the chosen basis functions.

**Definition 4.2** (Simulations and empirical measures). For $i \in \{0, \ldots, 2^k - 1\}$, generate $M_i \geq 1$ independent copies $C_{k,i} := \{(\Delta W_{i}^{(k,i,m)}, X^{(k,i,m)}_{i}) : m = 1, \ldots, M_i\}$ of $(\Delta W_{i}^{(k)}, X^{(k)})$: $C_{k,i}$ forms the cloud of simulations used for the regressions at time $i$. We assume that the clouds of simulations $(C_{k,i} : 0 \leq i < N)$ are independently generated, and are also independently generated from the clouds $\{C_k : 0 \leq k \leq k\}$ of Definition 4.2 used for the multilevel algorithm. Let $\nu_{i,M}^{(k)}$ denote the empirical
measure of the $C_{k,i}$-simulations, i.e.

$$
\nu_{k,M}^{(k)} = \frac{1}{M} \sum_{m=1}^{M} \delta_{\Delta W_{i}^{(k,i,m)},X_{i}^{(k,i,m)},...,X_{2k}^{(k,i,m)}}.
$$

We use the additional subscript $i$ in the notation for the clouds of simulations $C_{k,i}$ and the empirical measure $\nu_{k,M}^{(k)}$ for the LSMDP algorithm to distinguish them from those used for the multilevel algorithm and to specify the time-point. As in Section 3, we enlarge the probability space, while continuing to denote it for simplicity by $(\Omega,F,P)$, to contain also the simulations used for the LSMDP and the multilevel algorithms; recall that the prototype processes $W^{(k)}$ and $X^{(k)}$ are independent of all simulation clouds.

**Algorithm 3.** Recall the the linear spaces $K_{Y,i}$ and $K_{Z,i}$ from Definition 4.2, the empirical measures $\nu_{i,M}^{(k)} : i = 0, \ldots, 2k - 1$ from Definition 4.2, the almost sure bounds from $(A_{f})^{(iii)}$, the definition of the truncation functions $T_{i}^{(k)}$ from Section 1.1, and OLS from Definition 3.1.

Set $\bar{y}_{2i}^{(M)}(\cdot) := 0$. For each $i = 2k - 1, 2k - 2, \ldots, 0$, set the random functions $\bar{y}_{i}^{(M)}(\cdot)$ and $\bar{z}_{i}^{(M)}(\cdot)$ recursively as follows: Define $\bar{y}_{i}^{(M)}(\cdot) := T_{C_{y,i}}(\psi_{Y,i}^{(M)}(\cdot))$ and $\bar{z}_{i}^{(M)}(\cdot) = T_{C_{z,i}}(\psi_{Z,i}^{(M)}(\cdot))$, where $C_{y,i} := C_{X}(T - \bar{t}_{i}^{(k)})^{(\theta_{f} + \theta)/2}$, $C_{z,i} := C_{X}(T - \bar{t}_{i}^{(k)})^{(\theta_{f} + \theta)/2}/\Delta_{i}^{(k)}$ and

$$
\begin{align}
\psi_{Y,i}^{(M)}(\cdot) & \text{ solves OLS} \left( S_{Y,i}^{(M)}(\bar{x}) , K_{Y,i} , \nu_{i,M}^{(k)} \right) \\
& \text{ for } S_{Y,i}^{(M)}(\bar{x}) := \sum_{j=1}^{2k-1} f_{k}(x_{j},x_{j+1},\bar{y}_{j}^{(M)}(x_{j}),\bar{z}_{j}^{(M)}(x_{j}))\Delta_{j}^{(k)}, \text{ and}
\\
\psi_{Z,i}^{(M)}(\cdot) & \text{ solves OLS} \left( S_{Z,i}^{(M)}(w,\bar{x}) , K_{Z,i} , \nu_{i,M}^{(k)} \right) \\
& \text{ for } S_{Z,i}^{(M)}(w,\bar{x}) := \frac{1}{\Delta_{i}^{(k)}} S_{Y,i+1}^{(M)}(\bar{x}) w^{	op}, \text{ for } w \in \mathbb{R}^{d}, \bar{x} = (x_{0}, \ldots, x_{2k}) \in (\mathbb{R}^{d})^{2k+1}.
\end{align}
$$

We now come to the main result of this section, which is the error analysis of the LSMDP algorithm.

**Theorem 4.3** (Error for the LSMDP scheme). Recall the constants $C_{y,i}$ and $C_{z,i}$ from Algorithm 3.

For each $j \in \{0, \ldots, 2k - 1\}$, define

$$
\mathcal{E}(j) := T_{1,j}^{(k)} + T_{1,j}^{(k)} + C_{S}^{2} \left( \frac{3K_{Y,j}}{M_{j}} + 2q \frac{K_{Z,j}}{\Delta_{j}^{(k)}M_{j}} \right) + 800 \left( C_{y,j}^{2}(K_{Y,j} + 1) + C_{z,j}^{2}(K_{Z,j} + 1)q \right) \frac{\log(3M_{j})}{M_{j}}.
$$

where

$$
C_{S} := \sum_{i=0}^{2k-1} \left\{ C_{f} \frac{L_{f}(C_{y,i} + C_{y} + C_{z,i} + C_{z,k,i})}{(T - \bar{t}_{i}^{(k)})(1-\theta_{f})/2} \right\} \Delta_{i}^{(k)}.
$$

Recall $C_{\pi^{(k)}}$ from $(A_{\pi})^{(k)}$ and assume that $k$ is sufficiently large so that $C_{\pi^{(k)}}L_{f}^{2}(R_{x} \vee 1) \leq (384(2q + (1 + T)e^{T/2})(1 + T))^{-1}$, and that the parameters of the multilevel algorithm are such that the global error is estimated by $\bar{E}(k) \leq \varepsilon$ for some $\varepsilon > 0$ for definition of $\bar{E}(k)$, see equation (23) and subsequent
adaptations to the analysis, which we now detail for the convenience of the reader. Firstly, one must
least-squares regression are universal, therefore these arguments require no alterations for our setting.
priori estimates [24, Proposition 3.2] admit randomness in the driver, and the properties of ordinary
[24, Theorem 4.11] relies only on conditioning arguments, a-priori estimates, concentration of measure
the application of the a-priori estimates, one must estimate
The proof of the above theorem is analogous to the proof of [24, Theorem 4.11]. Indeed, the proof of
5 Conclusion: Comparison of the schemes with and without splitting and multilevel
Using the results of Sections 3 and 4, we are now in a position to compare our algorithm (splitting
Finally, one replaces the constant $C$ exactly as the explicit value of
is Lipschitz continuous (but not necessarily differentiable) and that the driver is uniformly Lipschitz
}{24, Lemma 4.7}, only using the almost absolute bounds of
where $C_T = 8 \exp \left( 384 (R_n \vee 1) (2q + (1 + T) e^{T/2}) (1 + T) T^{\theta_L} L_2^2 / \theta_L \right)$.

The proof of the above theorem is analogous to the proof of [24, Theorem 4.11]. Indeed, the proof of
the simulations used in the multilevel algorithm for the first part of the split system. Secondly, after
the application of the a-priori estimates, one must estimate
whereas, in the respective computation in [24, equation (33)], one only needed to estimate $\mathbb{E}[|f_j(\tilde{y}_{j+1}(X_{j+1}^{(k)}), z_j(X_{j}^{(k)})) - f_j(y_{j+1}(X_{j+1}^{(k)}), z_j(X_{j}^{(k)}))|^2]$
Recalling that $f_j(M, y, z) := f_j(X_j^{(k)}, y_{j+1}(X_{j+1}^{(k)}), z_j(X_{j}^{(k)}) + y, z_j(X_{j}^{(k)})) + z$, one uses the Lipschitz continuity of $f_j(y, z)$ and the hypothesis that the approximation of $(y^{(k)}, z^{(k)})$ by $(\tilde{y}^{(k,M)}, \tilde{z}^{(k,M)})$ produces a global error less than or equal to $\varepsilon$ to estimate the error due to multilevel.
Finally, one replaces the constant $C_{(4.7)}$ in [24, Theorem 4.11] by $C_S$; the explicit value of $C_S$ is obtain
exactly as the explicit value of $C_{(4.7)}$ in [24, Lemma 4.7], only using the almost absolute bounds of
$\tilde{y}^{(M)}(\cdot)$ and $\tilde{z}^{(M)}(\cdot)$.

5 Conclusion: Comparison of the schemes with and without splitting and multilevel
Using the results of Sections 3 and 4 we are now in a position to compare our algorithm (splitting combined with multilevel) to the least-squares multistep dynamical programming (LSMDP) scheme with neither. For simplicity, we will assume that $\theta = \theta_L = 1$, meaning that the terminal condition $\Phi(\cdot)$ is Lipschitz continuous (but not necessarily differentiable) and that the driver is uniformly Lipschitz
continuous in $(y, z)$. For the remainder of this section, we write $g(y) = O(y)$ if there exists a constant
$C$ independent of $k$ and $y$ such that $g(y)/y \to C$ as $y \to 0$. For given precision level $\varepsilon > 0$, it is our
Therefore, using the assumptions and computations of Section 3.3, it follows that the overall complexity

\[ \mathcal{E}(M) := \max_{0 \leq i \leq 2^k - 1} \mathbb{E}[\|\hat{y}_i^{(M)}(X_i^{(k)}) - \hat{y}_i(X_i^{(k)})\|^2] + \sum_{i=0}^{2^k} \mathbb{E}[\|\hat{z}_i^{(M)}(X_i^{(k)}) - \hat{z}_i(X_i^{(k)})\|^2] \Delta_k^i \leq O(\varepsilon). \]  

(53)

To apply Theorem 4.3, we first provide a concentration of measure result.

**Proposition 5.1.** For each \( k \in \{0, \ldots, \kappa\} \) and \( i \in \{0, \ldots, 2^k - 1\} \), we have

\[
\mathbb{E}[\|\hat{y}_i^{(M)}(X_i^{(k)}) - \hat{y}_i(X_i^{(k)})\|^2] \leq 2\mathbb{E} \left[ \frac{1}{M_i} \sum_{m=1}^{M_i} \|\hat{y}_i^{(M)}(X_i^{(k,i,m)}) - \hat{y}_i(X_i^{(k,i,m)})\|^2 \right] + \frac{2028(Ky,i + 1)C_{y,i}^2 \log(3M_i)}{M_i},
\]

and

\[
\mathbb{E}[\|\hat{z}_i^{(M)}(X_i^{(k)}) - \hat{z}_i(X_i^{(k)})\|^2] \leq 2\mathbb{E} \left[ \frac{1}{M_i} \sum_{m=1}^{M_i} \|\hat{z}_i^{(M)}(X_i^{(k,i,m)}) - \hat{z}_i(X_i^{(k,i,m)})\|^2 \right] + \frac{2028(Kz,i + 1)C_{z,i}^2 \log(3M_i)}{M_i};
\]

we recall that \( C_{y,i} = C_X^2(T - t_i^{(k)})^{1+\theta/2} \) and \( C_{z,i} = C_X^2(T - t_i^{(k)})^{\theta/2} \).

Just as Proposition 5.5, Proposition 6.1 is analogous to Proposition 4.10. The second terms on the right hand side of the inequalities are correction terms which can be interpreted as interdependency errors due to the change of the inner measure. We see the interdependency errors have the same dependence on \( K_{-,i} \) and \( M_i \) as the last term in \( \mathcal{E}(i) \) in Theorem 4.3. Hence, to ensure (53), it is sufficient to set the numerical parameters so that the local error terms satisfy \( \mathcal{E}(i) \leq O(\varepsilon) \) for every \( i \in \{0, \ldots, 2^k - 1\} \). Using (A_X^{\mathcal{I}})(ii), we can replace \( T^Y_{2,i} \) and \( T^Z_{2,i} \) by

\[
T^Y_{2,i} := \inf_{\phi \in \mathcal{K}_{Y,i}} \mathbb{E}[\|\phi(X_i^{(k)}) - U(t_i^{(k)}, X_i^{(k)})\|^2] \quad \text{and} \quad T^Z_{2,i} := \inf_{\phi \in \mathcal{K}_{Z,i}} \mathbb{E}[\|\phi(X_i^{(k)}) - V(t_i^{(k)}, X_i^{(k)})\|^2],
\]

respectively, in the local error term \( \mathcal{E}(i) \), and choose basis functions such that \( T^Y_{2,i} \) and \( T^Z_{2,i} \) are dominated by \( O(\varepsilon) \) for every \( i \). Thanks to the Lipschitz continuity in (A_F)(iii), it is sufficient to use (for every time-point and both for \( Y \) and \( Z \)) a basis of functions on a partition of disjoint hypercubes with diameter \( O(\sqrt{\varepsilon}) \). This basis is infinite dimensional, but one can make a simple truncation to get around this problem. We assume additionally (as in Section 4.4) that, for each \( i \in \{0, \ldots, 2^k - 1\} \), \( X_i^{(k)} \) has exponential moments, so that we may set the basis in the region outside \([0, R]^d\) to zero for \( R = \ln(\varepsilon^{-1} + 1) \); this truncation induces an error \( O(\varepsilon) \), which is admissible. The dimension of the hypercube basis \( K_{l,i} \) is therefore, uniformly in \( l \in \{Y, Z\} \) and \( i \in \{0, \ldots, 2^k - 1\} \), equal to \( O(\varepsilon^{-d/2} \ln(\varepsilon^{-1} + 1)^d) \). It follows that we must choose the number of simulations \( M_i \) to be equal, uniformly in \( i \), to \( O(2^{2k} \varepsilon^{-1-d/2} \ln(\varepsilon^{-1} + 1)^d) \).

It remains only to compute the complexity of the scheme. There are two contributions to the computational cost: the cost of simulation of the Markov chain \( X^{(k)} \) and Brownian increments \( \Delta W^{(k)} \), and the cost of the regressions. The cost of simulation is equal to \( O(2^{2k} \varepsilon^{-1-d/2} \ln(\varepsilon^{-1} + 1)^d) \); the additional factor \( 2^k \) comes from re-simulation the paths of the Markov chain \( X^{(k)} \) at every time-step. Since we are using the partitioning estimate to compute the regression coefficient, the regression cost is equal to \( \sum_{j=0}^{2^k-1} O(M_j) = O(2^{2k} \varepsilon^{-1-d/2} \ln(\varepsilon^{-1} + 1)^d) \); see Section 3.3 for details on the partitioning estimate. Therefore, recalling that \( 2^k = \varepsilon^{-1} \), the overall complexity is equal to \( O(\varepsilon^{-4-d/2} \ln(\varepsilon^{-1} + 1)^d) \). Therefore, using the assumptions and computations of Section 3.3, it follows that the overall complexity
of the splitting scheme with multilevel, i.e. Algorithms 3 and 2 together, is

\[ O(\varepsilon^{-2-d} \ln(\varepsilon^{-1} + 1)) + O(\varepsilon^{-4-d/2} \ln(\varepsilon^{-1} + 1)^d). \]  

(54)

We now calibrate the LSMDP algorithm with no splitting and no multilevel, which we recall below for completeness in Algorithm 4. We then compute the complexity of this algorithm in order to provide a suitable comparison to an established algorithm and determine the possible gains of the splitting algorithm with multilevel.

**Algorithm 4.** Recall the the linear spaces \( K_{Y,i} \) and \( K_{Z,i} \) from Definition 4.1, the empirical measures \( \nu_{i,M} : i = 0, \ldots, 2^k - 1 \) from Definition 4.2, the bounds from (A\( _{T} \))(iii), and the truncation function \( T_{\ell_i}() \) from Section 1.1. Set \( y^{(M)}_{2i}() := \Phi() \). For each \( i = 2^k - 1, 2^k - 2, \ldots, 0 \), set the random functions \( y^{(M)}_{i}() \) and \( z^{(M)}_{i}() \) recursively as follows: Define \( y^{(M)}_{i}() := T_{C_y}(\psi^{(M)}_{Y,i}()) \) and \( z^{(M)}_{i}() = T_{C_z,i}(\psi^{(M)}_{Z,i}()) \), where \( C_y := C_X, C_{z,i} := C_{z,k,i} \) and

\[
\left\{ \begin{array}{ll}
\psi^{(M)}_{Y,i}() & \text{solves } \text{OLS}( S^{(M)}_{Y,i}(x), K_{Y,i}, \nu^{(k)}_{i,M} ) \\
& \text{for } S^{(M)}_{Y,i}(x) := \Phi(x_N) + \sum_{j=i}^{2^k-1} f_{k}(x_j, y^{(M)}_{j+1}(x_{j+1}), z^{(M)}_{j}(x_j)) \Delta^{(k)}_j, \text{ and} \\
\psi^{(M)}_{Z,i}() & \text{solves } \text{OLS}( S^{(M)}_{Z,i}(w,x), K_{Z,i}, \nu^{(k)}_{i,M} ) \\
& \text{for } S^{(M)}_{Z,i}(w,x) := \frac{1}{\Delta^{(k)}_i} S^{(M)}_{Y,i+1}(x) w^T, \text{ for } w \in \mathbb{R}^q, x = (x_0, \ldots, x_{2^k}) \in (\mathbb{R}^d)^{2^k+1}.
\end{array} \right.
\]

The error of this algorithm is studied in [24, Theorem 4.11]. With this algorithm, we are directly approximating the continuous time function \( v(t, \cdot) + V(t, \cdot) \). The complexity analysis for Algorithm 4 is the same as that for Algorithm 3 above, however we must take into account the additional weight due to the time-dependency of the Lipschitz coefficient: the Lipschitz constant for \( v(t, \cdot) + V(t, \cdot) \) is equal to \( O((T - t)^{-1/2}) \) for all \( t \in [0,T] \) - see assumptions (A\( _{X} \))(iv) and (A\( _{T} \))(iii). Therefore, we choose a hypercube basis for each time-point \( i \in \{0, \ldots, 2^k - 1\} \) whose cubes have diameter \( \sqrt{T - t^{(k)}_i} O(\sqrt{\varepsilon}) \). Therefore, the overall complexity of Algorithm 4 is

\[ O(\varepsilon^{-3-d/2} \ln(\varepsilon^{-1} + 1)^d) \sum_{i=0}^{2^k-1} (T - t^{(k)}_i)^{d/2} \leq O(\varepsilon^{-4-d} \ln(\varepsilon^{-1} + 1)^d). \]

Compared with the two terms in (54), the complexity of Algorithm 4 dominates: if \( d < 4 \), (54) is dominated by \( O(\varepsilon^{-4-d/2} \ln(\varepsilon^{-1} + 1)^d) \), whereas for \( d \geq 4 \), (54) is dominated by \( O(\varepsilon^{-2-d} \ln(\varepsilon^{-1} + 1)) \). Therefore, one gains two orders in \( \varepsilon \) in high dimension \( d \geq 4 \) thanks to the use of splitting algorithm with multilevel. If we were to use a splitting method but no multilevel (i.e., LSMDP as in Remark 3.8), in the zero driver part, the gain compared to pure LSMDP would still be substantial, being of order one in \( \varepsilon \), however an additional order is to be gained by multilevel (for \( d \geq 4 \)), cf. Section 3.3.

Finally, we support the theory for the non-linear generator by computational results for an example from finance. To this end, consider a \( d = 2 \)-dimensional forward process \( X = (S, H) \) for correlated geometric Brownian motions \( dS = S\sigma Sdw^1 \) with \( S_0 = 1 \), and

\[ dH = H \left( \gamma dt + \sigma^H (\rho dw^1 + \sqrt{1 - \rho^2} dw^2) \right), \quad H_0 = 1, \]

(55)
with parameters $\sigma_S = \sigma_H = 0.5, \rho = 0.6$ and $\gamma = 0.1$. Considering $S$ and $H$ as the (discounted) price processes of a liquidly tradable risky asset and of a non-tradable asset, the so called no-good-deal valuation bound $Y$ for an option $\Phi(X_T) = (H_T - S_T)^+$ to exchange at maturity $T$ one traded asset $S_T$ into one non-traded asset $H_T$ is described by the non-linear BSDE

$$dY_t = -h|Z^{(2)}_t|dt + Z^{(2)}_tdW = -|Z^{(2)}_t|dt + Z^{(1)}_tdW^{(1)}_t + Z^{(2)}_tdW^{(2)}_t, \quad Y_T = \Phi(X_T),$$  

(56)

where $Z = (Z^{(1)}, Z^{(2)})$; for a good-deal constraint that we take as $h = 0.2$, see [2, 3]. The BSDE has an explicit solution in terms of a Margrabe-type formula, see [3], and a corresponding good-deal hedging strategy can be obtained from $Z$.

The regression basis for ML and MDP is given by indicator functions on the hypercubes of a partition of $\mathbb{R}^2$ into $K = 50^2$ sets. Number of simulations is $M = 2 \times 10^6$ for both schemes; for multilevel (ML), the same number of simulations is used at every level. Results on mean squared errors are reported in Table 2. They show substantial error reduction by multilevel (ML) in combination with the splitting scheme, in particular for the MSE in $Z$ for finer time grids (larger $k = \log_2 N$), confirming insights as before also for the present example with non-zero generator.

| $\log_2(N)$ | 1   | 2   | 3   | 4   | 5   |
|-------------|-----|-----|-----|-----|-----|
| MDP $Y$     | 0.1372 | 0.0795 | 0.0515 | 0.0379 | 0.0322 |
| MDP $Z$     | 0.0161 | 0.0089 | 0.0092 | 0.0143 | 0.0253 |
| ML $Y$      | 0.1371 | 0.0791 | 0.0510 | 0.0373 | 0.0314 |
| ML $Z$      | 0.0156 | 0.0068 | 0.0039 | 0.0032 | 0.0031 |

**Acknowledgements:** We like to thank Emmanuel Gobet for advice on this paper and the thesis [38] where a first multilevel scheme has been introduced, and for pointing out the special basis used in Theorem 3.9 in particular. We thank Axel Mosch and Klebert Kenita for help with the examples.

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