Research Article

Resolvability in Subdivision of Circulant Networks $C_n[1, k]$

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Circulant networks form a very important and widely explored class of graphs due to their interesting and wide-range applications in networking, facility location problems, and their symmetric properties. A resolving set is a subset of vertices of a connected graph such that each vertex of the graph is determined uniquely by its distances to that set. A resolving set of the graph that has the minimum cardinality is called the basis of the graph, and the number of elements in the basis is called the metric dimension of the graph. In this paper, the metric dimension is computed for the graph $G_n[1, k]$ constructed from the circulant graph $C_n[1, k]$ by subdividing its edges. We have shown that, for $k \geq 2$, $G_n[1, k]$ has an unbounded metric dimension, and for $k = 3$ and $4$, $G_n[1, k]$ has a bounded metric dimension.

1. Introduction

Resolvability of graphs becomes an important parameter in graph theory due to its wide applications in different branches of mathematics, such as facility location problems, chemistry, especially molecular chemistry [1], the method of positioning robot networks [2], the optimization problem in combinatorics [3], applications in pattern recognition and image processing [4], and the problems of sonar and Coast Guard LORAN [5].

The resolvability of graphs depends on the distances in graphs. The distance between two vertices in a connected graph is the smallest distance connecting those two vertices. The representation of a vertex $u$ with respect to the set $W$ is denoted by $r(u, W)$ and is defined as a $k$-tuple $(d(u, w_1), \ldots, d(u, w_k))$, where $w_1, \ldots, w_k \in W$. The set $W$ is called the resolving set [1] or sometimes locating set [5] if each vertex of the graph has a unique representation with respect to $W$. A resolving set of the graph that has the minimum cardinality is called the basis of the graph, and the number of elements in the basis is called the metric dimension of the graph, generally denoted by $β(G)$.

Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension was first introduced by Slater in [5, 6] and studied independently by Harary and Melter in [7]. Applications of this invariant to the navigation of robots in networks are discussed in [2], and applications to chemistry are given in [1], while applications to the problem of pattern recognition and image processing, some of which involve the use of hierarchical data structures, are given in [4].

A family of connected graphs $F$ is said to have a bounded metric dimension if the metric dimension of each graph in $F$ is bounded above by a positive integer. Otherwise, $F$ has an unbounded metric dimension.

If every graph $F$ has a constant metric dimension, then $F$ is said to have constant metric dimension. A connected graph $G$ has $β(G) = 1$ if and only if $G$ is the path [1]; cycles $C_n$ have metric dimension 2 for every $n \geq 3$. Also, generalized Peterson graphs $P(n, 2)$, antiprism $A_n$, and circulant graphs $C_n[1, 2]$ are families of graphs with constant metric dimension. The families of graphs having constant metric dimension are studied in [8–23].

It is important to note that to determine the graph has a bounded metric dimension is an NP-complete problem [19]. Some bounds for this parameter, in terms of the diameters of the graph, are given in [2], and it was shown in [1, 2, 4, 20] that the metric dimension of the tree can be determined efficiently. However, it is highly unlikely to determine the
dimension of the graph unless the graph belongs to such family for which the distance between vertices can be computed systematically.

Geometrically, by subdividing an edge, we mean to insert a new vertex in the edge such that the existing edge is divided into two edges. The subdivision of the graph $G$ is a graph obtained after performing a sequence of edge subdivision. Subdivision of graphs is an important tool to determine whether the graph is planar or not. In [24], plane graphs are characterized using subdivision as follows:

A necessary and sufficient condition of a graph to be planar is that each of its subdivision is planar. In this paper, we have investigated the resolvability of subdivision of circulant graph $C_n[1,k]$ for $k \geq 2$. It is shown that, for $k = 2$, this class has an unbounded metric dimension, and for $k = 3$ and 4, it has a bounded metric dimension.

2. Metric Dimension of Subdivision of Circulant Graph $C_n[1,k]$ for $k \geq 2$

The circulant graphs are an important class of graphs that can be used in local area networks.

A circulant graph on $n$ vertices and $m$ parameters $a_1, \ldots, a_m$, where each parameter $a_i$ is at most half of $n$, is denoted by $C_n[a_1, a_2, \ldots, a_m]$. If $\{v_1, \ldots, v_m\}$ are vertices of $C_n[a_1, a_2, \ldots, a_m]$, then there is an edge between two vertices $v_i$ and $v_j$ if and only if $|i - j|$ is one of $a_i$. The parameters of $a_i$ are called generators of $C_n[a_1, a_2, \ldots, a_m]$.

The graph $G_n[a_1, \ldots, a_k]$ is a graph obtained from $C_n[a_1, a_2, \ldots, a_m]$ by subdividing all the edges of $G_n[a_1, a_2, \ldots, a_m]$ except the edges between vertices $v_i$ and $v_{i+1}$.

In this paper, the resolvability of $G_n[a_1, k]$ is investigated. Let $u_i$ be the added vertex in each of the edge $v_i v_{i+k}$. Thus, the graph $G_n[1,k]$ has $2n$ vertices and $3n$ edges. Let $x_i$ and $x_j$ be vertices of $G_n[1,k]$; then, the gap between vertices $x_i$ and $x_j$ is defined to be $|i - j|$, where $1 \leq i < j \leq n$.

In the following theorem, it is shown that the metric dimension of the graph $G_n[1,2]$ is unbounded.

**Theorem 1.** For $n \geq 9$, \[ \beta(G_n[1,2]) = \left\lfloor \frac{n}{3} \right\rfloor. \] (1)

**Proof.** Let $W = \{ x_i: 1 \leq i \leq q \}$ be a minimum resolving set of $G_n[1,2]$. We have two cases: either $x_i = v_i$ or $x_i = u_i$, for some $i$. \[ \square. \]

**Claim 1.** If $x_i = v_i$ for some $i$, then $v_{i+2}$ also belongs to $W$ because, otherwise, $v_{i+2}$ and $u_{i+1}$ will have the same representation.

**Claim 2.** If $x_i = u_i$ for some $i$, then $u_{i+3}$ must belong to $W$ because, otherwise, $v_{i+3}$ and $u_{i+2}$ will have the same representation.

Both these cases imply that the two consecutive vertices in $W$ can have at most distance 3. Thus, the gap between two vertices of $W$ is at most 3. Since vertices presented on the outer cycle are $n$, therefore, $q \geq \lfloor n/3 \rfloor$. Hence,

\[ \beta(G_n[1,2]) \geq \left\lfloor \frac{n}{3} \right\rfloor. \] (2)

To prove the upper bound, consider the set $W = \{ u_{i-2}: i = 1, 2, \ldots, \lfloor n/3 \rfloor \}$ of vertices of $G_n[1,2]$. The construction of $W$ shows that every vertex in $W$ determines a gap of size 3.

For $1 \leq i \leq \lfloor n/3 \rfloor$, let $S = \{ v_i, v_{i+1}, v_{i+2}, v_{i+3}, u_{i+1}, u_{i+2} \}$ be the set of vertices determined by the two consecutive vertices $u_i$ and $u_{i+3}$. It is enough to show that every vertex in $S$ is uniquely determined by some vertices in $W$.

The vertices $v_i$ and $v_{i+2}$ are the only vertices in $G_n[1,2]$ that are at distance 1 from $u_i$, but $d(v_i, u_{i+3}) = 4$ and $d(v_{i+2}, u_{i+3}) = 2$.

The vertices $v_{i-1}$, $v_{i+1}$, $v_{i+3}$, $u_{i+2}$, and $u_{i+2}$ are the only vertices in $G_n[1,2]$ that are at distance 2 from $u_i$. The vertices $v_{i-1}$ and $u_{i+2}$ also have the same distance 5 from $u_{i+3}$, but they can be resolved by the vertex $u_{i+3}$. The vertices $v_{i+1}$ and $u_{i+2}$ also have the same distance 3 from $u_{i+3}$, but they can be resolved by the vertex $u_{i+6}$. The vertex $v_{i+3}$ is the unique vertex in $G_n[1,2]$ such that $d(v_{i+3}, u_{i+3}) = 1$ and $d(v_{i+3}, u_i) = 2$.

The vertex $u_{i+3}$ is the unique vertex in $G_n[1,2]$ such that $d(u_{i+1}, u_{i+3}) = 2$ and $d(u_{i+1}, u_i) = 3$.

This shows that every vertex in the set $S$ is uniquely determined by some vertices in the set $W$. Thus, $W$ becomes a resolving set, and

\[ \beta(G_n[1,2]) \leq \left\lfloor \frac{n}{3} \right\rfloor. \] (3)

From equations (2) and (3), we have $\beta(G_n[1,2]) = \lfloor n/3 \rfloor$.

In the next results, it is shown that the graph $G_n[1,k]$ has constant metric dimension for $3 \leq k \leq 4$.

**Theorem 2.** For $n \geq 14$, if $n = 2(rm \text{mod} 3)$, then, \[ \beta(G_n[1,3]) = 3. \] (4)

**Proof.** Let $W = \{ u_1, u_s, u_{3s} \}$ be the set of vertices in $G_n[1,3]$. It is enough to show that every vertex of the graph $G_n[1,3]$ is determined uniquely by some of the vertices in $W$. For this, the representations of each vertex are calculated as follows.

The vertices $v_1, \ldots, v_7$ have representations $(1, 3, 4), (2, 2, 3), (2, 1, 3), (1, 2, 2), (2, 2, 1), (3, 1, 2), (3, 2, 2)$, respectively.

The representations of the remaining vertices $v_i$: $7 \leq i \leq n$ of $G_n[1,3]$ are calculated as follows:
The vertices $u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ of $G_n[1, 3]$ have representations \((3, 3, 2), (2, 3, 3), (4, 2, 3), (4, 3, 3), (3, 3, 4), (3, 2, 4)\), respectively.

For the remaining vertices, we have
\[
\begin{align*}
\beta(G_n[1, 3]) &\leq 3. \\
\end{align*}
\]

Now, to compute the lower bound, suppose, on contrary, that $W$ is a minimum resolving set of $G_n[1, 3]$ of cardinality 2. We have the following possibilities to choose the vertices of $W$.

2.1. If $W$ Contains Both Vertices from $v_i$. One can suppose without losing any generality that $W = \{v_i, v_j, 2 \leq j \leq n\}$. However, in this case, if $j \equiv 2 \pmod{3}$, then
\[
\begin{align*}
\beta(G_n[1, 3]) &\leq 3. \\
\end{align*}
\]

and if $j \equiv 0, 1 \pmod{3}$, then
\[
\begin{align*}
\beta(G_n[1, 3]) &\leq 3. \\
\end{align*}
\]
and if \( j \equiv 1, 2 \) (mod 3),
\[
r(u_{j+3}, W) = r(v_{j+4}, W)
\]
\[
\begin{cases}
\left\lceil \frac{2j + 3}{3} \right\rceil, & \text{for } 2 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\left\lceil \frac{2n - 2j - 3}{3} \right\rceil, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq j \leq n.
\end{cases}
\]

(11)

2.3. If One Each from \( v_i \) and \( u_j \) Belongs to \( W \). One can suppose without losing any generality that \( W = \{ v_i, u_j : 1 \leq i \leq j \leq n \} \). However, in this case, if \( j \equiv 0 \) (mod 3), then
\[
r(u_{j+3}, W) = r(v_{j+5}, W)
\]
\[
\begin{cases}
\left\lceil \frac{2j + 3}{3} \right\rceil, & \text{for } 3 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor - 2, \\
\left\lceil \frac{2n - 2j - 8}{3} \right\rceil, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq j \leq n,
\end{cases}
\]

(12)

and if \( j \equiv 1, 2 \) (mod 3),
\[
r(u_{j+3}, W) = r(v_{j+4}, W)
\]
\[
\begin{cases}
\left\lceil \frac{2j + 3}{3} \right\rceil, & \text{for } 1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor, \\
\left\lceil \frac{2n - 2j - 6}{3} \right\rceil, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq j \leq n.
\end{cases}
\]

(13)

Thus, there is no resolving set of \( G_n[1, 3] \) having two vertices. This implies that
\[
\beta(G_n[1, 3]) \geq 3.
\]

(14)

From (7) and (14), we get
\[
\beta(G_n[1, 3]) = 3.
\]

(15)

Theorem 3. For \( n \geq 15 \), if \( n \equiv 0, 1 \) (mod 3), then
\[
\beta(G_n[1, 3]) = 4.
\]

(16)

Proof. Let \( W = \{ v_1, v_2, v_3, v_4 \} \) be the set of vertices in \( G_n[1, 4] \). It is enough to show that every vertex of the graph \( G_n[1, 4] \) is uniquely determined by some vertices in \( W \). For this, the representations of each vertex are calculated as follows.

The representations of outer vertices \( v_i : 1 \leq i \leq n \) of \( G_n[1, 3] \) are calculated as follows:

If \( i = \lfloor (n + 1)/2 \rfloor + 1 \),
\[
r(v_i, W) = \begin{cases} 
\left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 8}{3} \right\rfloor, & \text{for } 5 \leq i \leq \left\lfloor \frac{n + 1}{2} \right\rfloor, \\
\left\lfloor \frac{2n - 2i + 2}{3} \right\rfloor, \left\lfloor \frac{2n - 2i + 4}{3} \right\rfloor, \left\lfloor \frac{2n - 2i + 6}{3} \right\rfloor, \left\lfloor \frac{2n - 2i + 8}{3} \right\rfloor, & \text{for } \left\lfloor \frac{n + 1}{2} \right\rfloor + 4 \leq i \leq n.
\end{cases}
\]

(17)

If \( i = \lfloor (n + 1)/2 \rfloor + 1 \),
\[
r(v_i, W) = \begin{cases} 
\left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 8}{3} \right\rfloor, & \text{for } n = 6k \text{ and } k \geq 3, \\
\left\lfloor \frac{2i - 4}{3} \right\rfloor, \left\lfloor \frac{2i - 4}{3} \right\rfloor, \left\lfloor \frac{2i - 4}{3} \right\rfloor, \left\lfloor \frac{2i - 7}{3} \right\rfloor, & \text{for } n = 6k + 1 \text{ and } k \geq 3, \\
\left\lfloor \frac{2i - 4}{3} \right\rfloor, \left\lfloor \frac{2i - 4}{3} \right\rfloor, \left\lfloor \frac{2i - 6}{3} \right\rfloor, \left\lfloor \frac{2i - 6}{3} \right\rfloor, & \text{for } n = 6k + 3 \text{ and } k \geq 2, \\
\left\lfloor \frac{2i - 5}{3} \right\rfloor, \left\lfloor \frac{2i - 3}{3} \right\rfloor, \left\lfloor \frac{2i - 5}{3} \right\rfloor, \left\lfloor \frac{2i - 8}{3} \right\rfloor, & \text{for } n = 6k + 4 \text{ and } k \geq 2.
\end{cases}
\]

(18)
If \( i = \lceil (n + 1)/2 \rceil + 2 \),
\[
\begin{align*}
\mathbf{r}(v_i, W) &= \begin{cases} 
\left[ \frac{2i-10}{3}, \frac{2i-8}{3}, \frac{2i-8}{3}, \frac{2i-8}{3} \right], & \text{for } n = 6k \text{ and } k \geq 3, \\
\left[ \frac{2i-6}{3}, \frac{2i-6}{3}, \frac{2i-6}{3}, \frac{2i-6}{3} \right], & \text{for } n = 6k + 1 \text{ and } k \geq 3, \\
\left[ \frac{2i-8}{3}, \frac{2i-6}{3}, \frac{2i-6}{3}, \frac{2i-8}{3} \right], & \text{for } n = 6k + 3 \text{ and } k \geq 2, \\
\left[ \frac{2i-10}{3}, \frac{2i-7}{3}, \frac{2i-4}{3}, \frac{2i-7}{3} \right], & \text{for } n = 6k + 4 \text{ and } k \geq 2.
\end{cases}
\end{align*}
\]

If \( i = \lceil (n + 1)/2 \rceil + 3 \),
\[
\begin{align*}
\mathbf{r}(v_i, W) &= \begin{cases} 
\left[ \frac{2i-14}{3}, \frac{2i-12}{3}, \frac{2i-10}{3}, \frac{2i-10}{3} \right], & \text{for } n = 6k \text{ and } k \geq 3, \\
\left[ \frac{2i-11}{3}, \frac{2i-8}{3}, \frac{2i-8}{3}, \frac{2i-8}{3} \right], & \text{for } n = 6k + 1 \text{ and } k \geq 3, \\
\left[ \frac{2i-10}{3}, \frac{2i-10}{3}, \frac{2i-8}{3}, \frac{2i-8}{3} \right], & \text{for } n = 6k + 3 \text{ and } k \geq 2, \\
\left[ \frac{2i-12}{3}, \frac{2i-12}{3}, \frac{2i-9}{3}, \frac{2i-6}{3} \right], & \text{for } n = 6k + 4 \text{ and } k \geq 2.
\end{cases}
\end{align*}
\]

The vertices \( u_1, u_2, u_3, u_4, u_5, u_6, u_{n-1}, u_n \) have representations \((1, 2, 2, 1), (2, 1, 2, 2), (3, 2, 1, 2), (3, 3, 2, 1), (2, 1, 2, 3), (2, 2, 1, 2)\), respectively.

The representations of the remaining vertices \( u_i \) of \( G_n[1, 3] \) are calculated as follows:

\begin{align*}
\mathbf{r}(u_i, W) &= \begin{cases} 
\left[ \frac{2i+1}{3}, \frac{2i-1}{3}, \frac{2i-3}{3}, \frac{2i-5}{3} \right], & \text{for } 5 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \\
\left[ \frac{2n-2i-1}{3}, \frac{2n-2i+1}{3}, \frac{2n-2i+3}{3}, \frac{2n-2i+5}{3} \right], & \text{for } \left\lceil \frac{n}{2} \right\rceil + 3 \leq i \leq n - 2.
\end{cases}
\end{align*}

If \( i = \lfloor n/2 \rfloor \),
\[
\begin{align*}
\mathbf{r}(u_i, W) &= \begin{cases} 
\left[ \frac{2i-1}{3}, \frac{2i-3}{3}, \frac{2i-3}{3}, \frac{2i-3}{3} \right], & \text{for } n = 6k \text{ and } k \geq 3, \\
\left[ \frac{2i-2}{3}, \frac{2i-1}{3}, \frac{2i-2}{3}, \frac{2i-5}{3} \right], & \text{for } n = 6k + 1 \text{ and } k \geq 3, \\
\left[ \frac{2i-1}{3}, \frac{2i-1}{3}, \frac{2i-1}{3}, \frac{2i-4}{3} \right], & \text{for } n = 6k + 3 \text{ and } k \geq 2, \\
\left[ \frac{2i-1}{3}, \frac{2i-1}{3}, \frac{2i-1}{3}, \frac{2i-4}{3} \right], & \text{for } n = 6k + 4 \text{ and } k \geq 2.
\end{cases}
\end{align*}

If \( i = \lfloor n/2 \rfloor + 1 \),

\[
\begin{align*}
\begin{cases}
\lfloor \frac{2i-6}{3} \rfloor, & \text{for } n = 6k \text{ and } k \geq 3, \\
\lfloor \frac{2i-7}{3} \rfloor, & \text{for } n = 6k+1 \text{ and } k \geq 3, \\
\lfloor \frac{2i-9}{3} \rfloor, & \text{for } n = 6k+3 \text{ and } k \geq 2, \\
\lfloor \frac{2i-8}{3} \rfloor, & \text{for } n = 6k+4 \text{ and } k \geq 2.
\end{cases}
\end{align*}
\]

If \( i = \lfloor n/2 \rfloor + 2 \),

\[
\begin{align*}
\begin{cases}
\lfloor \frac{2i-7}{3} \rfloor, & \text{for } n = 6k \text{ and } k \geq 3, \\
\lfloor \frac{2i-9}{3} \rfloor, & \text{for } n = 6k+1 \text{ and } k \geq 3, \\
\lfloor \frac{2i-11}{3} \rfloor, & \text{for } n = 6k+3 \text{ and } k \geq 2, \\
\lfloor \frac{2i-8}{3} \rfloor, & \text{for } n = 6k+4 \text{ and } k \geq 2.
\end{cases}
\end{align*}
\]

This shows that every vertex of the graph \( G_n[1,4] \) is uniquely determined by some of the vertices in \( W \). Hence, \( W \) become a resolving set, and

\[ \beta(G_n[1,4]) \leq 4. \]  \hfill (25)

Let \( n = 6k+l \) and \( l \in \{0, 1, 3, 4\} \). We show that there is no resolving set of \( G_n[1,4] \) with three elements. Suppose, on contrary, that \( W \) is a minimum resolving set of \( G_n[1,4] \) of cardinality 3. We have the following possibilities to choose the vertices of \( W \). \hfill \Box

2.4. If Only \( v_i \in W \). One can suppose without losing any generality that \( W = \{v_i, v_j, v_k\} \), where \( 2 \leq i \leq 3k \) and \( i+1 \leq j \leq 6k \). In this case, the vertices of \( G_n[1,4] \) that have the same representation for all choices of the resolving set are given as follows:

\[
\begin{align*}
\begin{cases}
r(u_{n-i-1}) = r(v_{n-i-1}) = \left(2i, \left\lfloor \frac{2i+3}{3} \right\rfloor, \left\lfloor \frac{2j+4}{3} \right\rfloor\right), & \text{for } i \equiv 0 \pmod{3}, j \equiv 0, 1 \pmod{3} \text{ and for every value of } l, \\
r(u_{n+3l/2}) = r(v_{n+3l/2}) = \left(\frac{n-3}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+5}{3} \right\rfloor\right), & \text{for } l = 1, 3 \text{ and } i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3}, \\
r(u_{n+2l}) = r(v_{n+2l}) = \left(\frac{n}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+3}{3} \right\rfloor\right), & \text{for } 1l = 0, 4 \text{ and } i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3}, \\
r(u_{n-i}) = r(v_{n-i}) = \left(1, \left\lfloor \frac{2i}{3} \right\rfloor, \left\lfloor \frac{2j+1}{3} \right\rfloor\right), & \text{otherwise and for all } l.
\end{cases}
\end{align*}
\]
2.5. If Two Vertices from \( v_i \) and One Vertex from \( u_j \) Belong to \( W \). One can suppose without losing any generality that \( W = \{v_i, v_j, u_j\} \), where \( 2 \leq i \leq 3k \) and \( i \leq j \leq 6k \).

\[
\begin{align*}
    r(u_{n-3}) &= r(v_{n-1}) = \left(2, \left\lfloor \frac{2i + 3}{3} \right\rfloor, \left\lfloor \frac{2j + 4}{3} \right\rfloor\right), \\
    r(u_{(n+3)/2}) &= r(v_{(n+5)/2}) = \left(\frac{n-3}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+5}{3} \right\rfloor\right), \\
    r(u_{n/2}) &= r(v_{(n+2)/2}) = \left(\frac{n}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+3}{3} \right\rfloor\right), \\
    r(u_{n-2}) &= r(v_n) = \left(1, \left\lfloor \frac{2i}{3} \right\rfloor, \left\lfloor \frac{2j+1}{3} \right\rfloor\right),
\end{align*}
\]

for \( i \equiv 0 \pmod{3}, j \equiv 0, 1 \pmod{3} \) and for all \( l \), for \( l = 1, 3 \) and \( i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3} \), for \( l = 0, 4 \) and \( i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3} \), otherwise and for all \( l \).

(27)

2.6. If Two Vertices from \( u_i \) and One Vertex from \( v_j \) Belong to \( W \). One can suppose without losing any generality that \( W = \{v_i, u_i, u_j\} \), where \( 1 \leq i \leq 3k \) and \( i \leq j \leq 6k \).

\[
\begin{align*}
    r(u_{n-3}) &= r(v_{n-1}) = \left(2, \left\lfloor \frac{2i + 3}{3} \right\rfloor, \left\lfloor \frac{2j + 4}{3} \right\rfloor\right), \\
    r(u_{(n+3)/2}) &= r(v_{(n+5)/2}) = \left(\frac{n-3}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+5}{3} \right\rfloor\right), \\
    r(u_{n/2}) &= r(v_{(n+2)/2}) = \left(\frac{n}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+3}{3} \right\rfloor\right), \\
    r(u_{n-2}) &= r(v_n) = \left(1, \left\lfloor \frac{2i}{3} \right\rfloor, \left\lfloor \frac{2j+1}{3} \right\rfloor\right),
\end{align*}
\]

for \( i \equiv 0 \pmod{3}, j \equiv 0, 1 \pmod{3} \) and for all \( l \), for \( l = 1, 3 \) and \( i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3} \), for \( l = 0, 4 \) and \( i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3} \) and for \( l = \), otherwise and for all \( l \).

(28)

2.7. If \( W \) Contains All Three Vertices from \( u_i \). One can suppose without losing any generality that \( W = \{u_i, u_i, u_j\} \) is a resolving set, where \( 2 \leq i \leq 3k \) and \( i + 1 \leq j \leq 6k \).

\[
\begin{align*}
    r(u_{n-3}) &= r(v_{n-1}) = \left(2, \left\lfloor \frac{2i + 3}{3} \right\rfloor, \left\lfloor \frac{2j + 4}{3} \right\rfloor\right), \\
    r(u_{(n+3)/2}) &= r(v_{(n+5)/2}) = \left(\frac{n-3}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+5}{3} \right\rfloor\right), \\
    r(u_{n/2}) &= r(v_{(n+2)/2}) = \left(\frac{n}{3}, \left\lfloor \frac{n-i+1}{3} \right\rfloor, \left\lfloor \frac{n-2j+3}{3} \right\rfloor\right), \\
    r(u_{n-2}) &= r(v_n) = \left(1, \left\lfloor \frac{2i}{3} \right\rfloor, \left\lfloor \frac{2j+1}{3} \right\rfloor\right),
\end{align*}
\]

for \( i \equiv 0 \pmod{3}, j \equiv 0, 1 \pmod{3} \) and for all \( l \), for \( l = 1, 3 \) and \( i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3} \), for \( l = 0, 4 \) and \( i \equiv 0 \pmod{3}, j \equiv 2 \pmod{3} \), otherwise and for all \( l \).

(29)

Thus, there is no resolving set of \( G_n[1, 4] \) having three elements. Hence,

\[
\beta(G_n[1, 4]) \geq 4. \quad (30)
\]

From equations (25) and (30), we get

\[
\beta(G_n[1, 4]) = 4. \quad (31)
\]
Theorem 4. For \( n \geq 20 \), if \( n \equiv 3 \) (mod 4), then
\[
\beta(G_n[1, 4]) = 4.
\]

**Proof.** Let \( W = \{u_1, u_4, u_5, u_{10}\} \) be the set of vertices in \( G_n[1, 4] \). It is enough to show that every vertex of the graph \( G_n[1, 4] \) is uniquely determined by some vertices in \( W \). For this, the vertices \( v_1, \ldots, v_{11} \) have representations \( (1, 4, 5, 6), (2, 3, 4, 5), (3, 2, 3, 6), (2, 1, 4, 5), (1, 2, 3, 4), (2, 3, 2, 3), (3, 2, 1, 4), (4, 1, 2, 3), (3, 2, 3, 2), (4, 3, 2, 1), (5, 4, 1, 2) \), respectively.

Let \( W' = \{u_1, u_4, u_{10}\} \). Then, the representations of the vertices \( v_i \); \( 12 \leq i \leq n \) of \( G_n[1, 4] \) with respect to \( W' \) and the vertex \( u_{10} \) are calculated as follows:

\[
r(v_i | W') = \begin{cases} 
\left( \left\lfloor \frac{i}{2} \right\rfloor, \left\lfloor \frac{i - 6}{2} \right\rfloor, \left\lfloor \frac{i - 8}{2} \right\rfloor \right), & \text{if } i \text{ is divisible by } 4 \text{ and } 12 \leq i \leq \left\lceil \frac{n+9}{2} \right\rceil, \\
\left( \left\lfloor \frac{n - i + 3}{2} \right\rfloor, \left\lfloor \frac{n - i + 9}{2} \right\rfloor, \left\lfloor \frac{n - i + 11}{2} \right\rfloor \right), & \text{if } i \text{ is divisible by } 4 \text{ and } \left\lceil \frac{n+9}{2} \right\rceil + 1 \leq i \leq n, \\
\left( \left\lfloor \frac{i - 3}{2} \right\rfloor, \left\lfloor \frac{i - 5}{2} \right\rfloor, \left\lfloor \frac{n - i + 11}{2} \right\rfloor \right), & \text{if } i - 1 \text{ is divisible by } 4 \text{ and } 13 \leq i \leq \left\lceil \frac{n+9}{2} \right\rceil, \\
\left( \left\lfloor \frac{n - i + 6}{2} \right\rfloor, \left\lfloor \frac{n - i + 8}{2} \right\rfloor, \left\lfloor \frac{n - i + 10}{2} \right\rfloor \right), & \text{if } i - 1 \text{ is divisible by } 4 \text{ and } \left\lceil \frac{n+9}{2} \right\rceil + 1 \leq i \leq n, \\
\left( \left\lfloor \frac{i - 3}{2} \right\rfloor, \left\lfloor \frac{i - 4}{2} \right\rfloor, \left\lfloor \frac{i - 6}{2} \right\rfloor \right), & \text{if } i - 2 \text{ is divisible by } 4 \text{ and } 14 \leq i \leq \left\lceil \frac{n+9}{2} \right\rceil, \\
\left( \left\lfloor \frac{n - i + 5}{2} \right\rfloor, \left\lfloor \frac{n - i + 7}{2} \right\rfloor, \left\lfloor \frac{n - i + 9}{2} \right\rfloor \right), & \text{if } i - 2 \text{ is divisible by } 4 \text{ and } \left\lceil \frac{n+9}{2} \right\rceil + 1 \leq i \leq n, \\
\left( \left\lfloor \frac{i - 1}{2} \right\rfloor, \left\lfloor \frac{i - 3}{2} \right\rfloor, \left\lfloor \frac{i - 9}{2} \right\rfloor \right), & \text{if } i - 3 \text{ is divisible by } 4 \text{ and } 15 \leq i \leq \left\lceil \frac{n+1}{2} \right\rceil, \\
\left( \left\lfloor \frac{n - i + 4}{2} \right\rfloor, \left\lfloor \frac{n - i + 6}{2} \right\rfloor, \left\lfloor \frac{n - i + 12}{2} \right\rfloor \right), & \text{if } i - 3 \text{ is divisible by } 4 \text{ and } \left\lceil \frac{n+1}{2} \right\rceil + 1 \leq i \leq n.
\end{cases}
\]

The representations of the vertices \( u_i; 1 \leq i \leq 11 \) of \( G_n[1, 4] \) are calculated as follows: \( r(u_1, W) = (3, 4, 3, 4) \), \( r(u_2, W) = (4, 3, 2, 5) \), \( r(u_3, W) = (2, 3, 4, 3) \), \( r(u_4, W) = (3, 4, 3, 2) \), \( r(u_5, W) = (5, 2, 3, 4) \), \( r(u_{10}, W) = (4, 3, 4, 3) \), \( r(u_{11}, W) = (6, 5, 2, 3) \), \( r(u_{12}, W) = (3, 4, 5, 6) \), \( r(u_{13}, W) = (4, 3, 4, 7) \), \( r(u_{14}, W) = (3, 2, 5, 6) \).
The remaining vertices \( u_i \): \( 12 \leq i \leq n \) of \( G_n[1, 4] \) have the following representation with respect to \( W' \), and the vertex \( u_{10} \) is calculated as follows:

\[
\begin{align*}
\text{if } i \text{ is divisible by } 4 \text{ and } 12 \leq i \leq \left\lfloor \frac{n + 7}{2} \right\rfloor, \\
\left( \frac{i - 7}{2}, \frac{i - 4}{2}, \frac{i - 6}{2} \right), \\
\left( \frac{n - i + 11}{2}, \frac{n - i + 7}{2}, \frac{n - i + 9}{2} \right), \text{ if } i \text{ is divisible by } 4 \text{ and } \left\lfloor \frac{n + 7}{2} \right\rfloor + 1 \leq i \leq n - 3, \\
\left( \frac{i - 1}{2}, \frac{i - 3}{2}, \frac{i - 5}{2} \right), \text{ if } i - 1 \text{ is divisible by } 4 \text{ and } 13 \leq i \leq \left\lfloor \frac{n + 7}{2} \right\rfloor, \\
\left( \frac{n - i + 3}{2}, \frac{n - i + 6}{2}, \frac{n - i + 7}{2} \right), \text{ if } i - 1 \text{ is divisible by } 4 \text{ and } \left\lfloor \frac{n + 7}{2} \right\rfloor + 1 \leq i \leq n - 3, \\
\left( \frac{i + 6}{2}, \frac{i - 2}{2}, \frac{i - 4}{2} \right), \text{ if } i - 2 \text{ is divisible by } 4 \text{ and } 14 \leq i \leq \left\lfloor \frac{n + 7}{2} \right\rfloor, \\
\left( \frac{n - i + 5}{2}, \frac{n - i + 5}{2}, \frac{n - i + 7}{2} \right), \text{ if } i - 2 \text{ is divisible by } 4 \text{ and } \left\lfloor \frac{n + 7}{2} \right\rfloor + 1 \leq i \leq n - 3, \\
\left( \frac{n - i}{2}, \frac{i + 5}{2}, \frac{i - 7}{2} \right), \text{ if } i - 3 \text{ is divisible by } 4 \text{ and } 15 \leq i \leq \left\lfloor \frac{n + 7}{2} \right\rfloor, \\
\left( \frac{n - i + 4}{2}, \frac{n - i + 4}{2}, \frac{n - i + 10}{2} \right), \text{ if } i - 3 \text{ is divisible by } 4 \text{ and } \left\lfloor \frac{n + 7}{2} \right\rfloor + 1 \leq i \leq n - 3,
\end{align*}
\]

(34)

One can easily verify that each vertex of \( G_n[1, 4] \) has unique representation with respect to \( W \). Hence, \( W \) is a resolving set, and

\[
\beta(G_n[1, 3]) \leq 4. \tag{35}
\]

Now, to prove the lower bound, it is sufficient to show that there is no resolving set of \( G_n[1, 4] \) with three elements. Suppose, on contrary, that \( W \) is a minimum resolving set of \( G_n[1, 4] \) of cardinality 3. Define
\[ A = \left\{ u_{4i-3}, v_{4i+1}, 1 \leq i \leq \frac{n-1}{2} \right\}, \]
\[ B = \left\{ u_{4i-2}, v_{4i+2}, 1 \leq i \leq \frac{n-1}{2} \right\}, \]
\[ C = \left\{ u_{4i-1}, v_{4i+3}, 1 \leq i \leq \frac{n-1}{2} \right\}, \]
\[ D = \left\{ u_{4i}, v_{4i+4}, 1 \leq i \leq \frac{n-1}{2} \right\}. \]  

(36)

It is easy to see that these are disjoint subsets of the vertex set of \( G_n \). We make the following claims.

**Claim 3.** Let \( x \) be an arbitrary vertex of the resolving set \( W \) that does not belong to \( A \). Then,
\[
d(v_{i-1}, x) = d(u_{i-4}, x) = \left\lfloor \frac{i + 2}{2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{i + 4}{2} \right\rfloor.
\]  

(37)

Thus, to resolve \( u_{i-4} \) and \( v_{i-1} \), either \( u_{i-4} \) or \( v_{i-1} \) must belong to \( W \).

**Claim 4.** Let \( x \) be an arbitrary vertex of the resolving set \( W \) that does not belong to \( B \). Then,
\[
d(v_{i}, x) = d(u_{i-3}, x) = \left\lfloor \frac{i}{2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{i + 2}{2} \right\rfloor.
\]  

(38)

Thus, to resolve \( u_{i-3} \) and \( v_{i} \), either \( u_{i-3} \) or \( v_{i} \) must belong to \( W \).

**Claim 5.** Let \( x \) be an arbitrary vertex of the resolving set \( W \) that does not belong to \( C \). Then,
\[
d(v_{i+1}, x) = d(u_{i+2}, x) = \left\lfloor \frac{i - 1}{2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{i + 1}{2} \right\rfloor.
\]  

(39)

Thus, to resolve \( u_{i+2} \) and \( v_{i+1} \), either \( u_{i+2} \) or \( v_{i+1} \) must belong to \( W \).

**Claim 6.** Let \( x \) be an arbitrary vertex of the resolving set \( W \) that does not belong to \( D \). Then,
\[
d(v_{i+2}, x) = d(u_{i-1}, x) = \left\lfloor \frac{i - 2}{2} \right\rfloor \quad \text{or} \quad \left\lfloor \frac{i}{2} \right\rfloor.
\]  

(40)

Thus, to resolve \( u_{i-1} \) and \( v_{i+2} \), either \( u_{i-1} \) or \( v_{i+2} \) must belong to \( W \).

The above claims imply that \( W \) contains at least four elements. Thus,
\[
\beta(G_n[1,4]) \geq 4.
\]  

(41)

From equations (35) and (41), we get
\[
\beta(G_n[1,4]) = 4.
\]  

(42)

3. Conclusion

The resolvability of the circulant graphs \( C_n[1,2,\ldots,k] \) has been investigated by different authors [15, 17, 18, 25]. The resolvability of barycentric subdivision of circulant graphs was investigated by Imran et al. in [8, 25], where they showed that some of these families had constant metric dimension. In this paper, we have studied the metric dimension of subdivision of circulant graphs \( G_n[1,k] \) denoted by \( G_n \) for \( 2 \leq k \leq 4 \). It is proved that the metric dimension of this family of graphs \( G_n \) has unbounded metric dimension when \( n \geq 9 \) and \( k = 2 \). It is also shown that for \( k = 3 \) and \( 4 \), the graph \( G_n[k] \) has constant metric dimension. This family of graphs \( G_n[k] \) which is obtained from subdivision of circulant graphs has interesting metric properties. The family has an unbounded metric dimension for \( k = 2 \) and bounded metric dimension for \( k = 3, 4 \). We also believe that the metric dimension increases as the value of \( k \) increases. In this context, we arise the following open question.

Open problem: compute the exact value of metric dimension or determine some good bounds in terms of other graphical parameters for the subdivision of circulant graphs \( G_n \) when \( k \geq 5 \). Also, characterize the classes of \( G_n \) that has bounded or unbounded metric dimension.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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