Existence and non-existence of global solutions for a heat equation with degenerate coefficients

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Abstract
In this paper, the parabolic problem $u_t - \text{div}(\omega(x) \nabla u) = h(t)f(u) + l(t)g(u)$ with non-negative initial conditions pertaining to $C_b(\mathbb{R}^N)$, will be studied, where the weight $\omega$ is an appropriate function that belongs to the Muckenhoupt class $A_{1+\frac{2}{N}}$, and the functions $f, g, h$ and $l$ are non-negative and continuous. The main goal is to establish the global and non-global existence of non-negative solutions. In addition, will be obtained both the so-called Fujita’s exponent and the second critical exponent in the sense of Lee and Ni (Trans Am Math Soc 333(1):365–378, 1992), in the particular case when $h(t) \sim t^r (r > -1), l(t) \sim t^s (s > -1), f(u) = u^p$ and $g(u) = (1 + u)\ln(1 + u)^p$.

The results of this paper extend those obtained by Fujishima et al. (Calc Var Partial Differ Equ 58:62, 2019) that worked when $h(t) = 1, l(t) = 0$ and $f(u) = u^p$.

Keywords Heat equation · Non-global solution · Global solution · Degenerate coefficients · Fujita exponent

Mathematics Subject Classification 35K05 · 35A01 · 35K58 · 35K65 · 35B33

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1 Introduction

Let $T > 0$ and $N \geq 1$. We consider the following heat equation

$$
\begin{cases}
  u_t - \text{div}(\omega(x) \nabla u) = h(t) f(u) + l(t) g(u) & (x, t) \in \mathbb{R}^N \times (0, T), \\
u(0) = u_0 \geq 0 & x \in \mathbb{R}^N,
\end{cases}
$$

where $u_0 \in C_b(\mathbb{R}^N)$; $h, l : [0, \infty) \to [0, \infty)$ are continuous functions; the functions $f, g : [0, \infty) \to [0, \infty)$ are locally Lipschitz continuous; and the spatial function $\omega : \mathbb{R}^N \to [0, \infty)$ is a weight, which can be

(A) $\omega(x) = |x_1|^a$ with $a \in [0, 1)$ if $N = 1, 2$; and $a \in [0, 2/N)$ if $N \geq 3$,

or

(B) $\omega(x) = |x|^b$ with $b \in [0, 1)$.

Here $x_1$ is the first coordinate of $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $| \cdot |$ is the Euclidean norm of $\mathbb{R}^N$. Note that the operator $\text{div}(\omega(x) \nabla u)$ under the conditions (A) or (B) on $\omega(x)$ is not self-adjoint (this particularity has been commented on in [17, p. 6]). Something important to note is that the degenerate operator $\text{div}(\omega(x) \nabla u)$ when $\omega(x) = |x_1|^a$, is related to the fractional Laplacian, through the Caffarelli–Silvestre extension (see [5, 9, 38] and [17, p. 3]). The fractional Laplacian has extensive applications in several physical and biological models with anomalous diffusion; for example, see [30] and the references therein.

Equation (1) appears in models that describe the processes of heat propagation in inhomogeneous media, see e.g. [28, 29, 40, 45], where the authors studied thermal phenomena related to the following equation

$$
v_t - |x|^\beta_1 \text{div}(|x|^\beta_2 \nabla v^m) = 0 \quad (m \geq 1).
$$

Recently in [4], Bonforte and Simonov obtained a priori Harnack inequalities and Hölder continuity for a class of nonnegative local weak solution of (2), where the weights $|x|^\beta_1$ and $|x|^\beta_2$ do not necessarily belong to the Muckenhoupt class $A_2$.

Several authors extensively studied the problem (1) when $\omega(x) = 1$. Let us mention, for instance, the seminal work of Fujita [18], who studied problem (1), and showed the existence of the following value $p^*(0) := 1 + 2/N$ known as critical Fujita exponent, which determines both the global and the non-global existence of nonnegative solutions. Specifically, he proved the following result: Assume that $\omega(x) = 1, l(t) = 0, h(t) = 1$, and $f(u) = up$ with $p > 1$.

- If $1 < p < p^*(0)$, then problem (1) does not admit any nontrivial nonnegative global solution.
- If $p > p^*(0)$, then problem (1) has global solutions, depending on the size of the initial condition.

See also [1, 2, 27, 39] for more details on these settings. Another significant result in this direction is that obtained by Lee and Ni in [33], who determined the so-called second critical exponent $\rho^*(0) = \frac{2}{p-1}$ when $p^*(0) < p$ under suitable decay conditions on the initial data. More precisely: Suppose the same previous conditions on (1) with $p^*(0) < p$.

- If $0 < \rho < \rho^*(0)$, then problem (1) has no global solutions for any $\psi \in I_\rho$.
- If $\rho^*(0) < \rho < N$, then for any $u_0 \in I^\rho$ there exists $\Lambda(0) > 0$ such that the problem (1) with initial data $\lambda u_0$ has a global solution for all $0 < \lambda < \Lambda(0)$.

Here

$I^\rho = \{ \psi \in C_b(\mathbb{R}^N), \psi \geq 0 \text{ and } \limsup_{|x| \to \infty} |x|^{\rho} \psi(x) < \infty \}$, and

$I_\rho = \{ \psi \in C_b(\mathbb{R}^N), \psi \geq 0 \text{ and } \liminf_{|x| \to \infty} |x|^{\rho} \psi(x) > 0 \}$.
for \( \rho > 0 \). For other works in this line of research, the readers can refer to [42–44].

Fujishima et al. [17] established the first results with optimal conditions for the global existence of a problem related to (1) and showed the following Fujita-type result:

**Theorem 1** [17] Assume either (A) or (B). Let \( \alpha = a \) for the case (A) or \( \alpha = b \) for the case (B). Assume \( h(t) = 1, l(t) = 0, f(t) = t^p \), and \( p^*(\alpha) = 1 + \frac{2-\alpha}{N} \).

(i) If \( 1 < p \leq p^*(\alpha) \), then problem (1) has no nontrivial global-in-time solutions.

(ii) If \( p^*(\alpha) < p \), then problem (1) has nontrivial global-in-time solutions.

For the non-global existence results in Theorem 1, the authors powerfully used Lemma 3 in [17], which was obtained through an iterative method, that has been widely used in several other works, e.g. see [3, 6–8, 14]. Unfortunately, this same approach cannot be used for the case of problem (3), due to logarithmic nonlinearity \((u + 1)(\ln(u + 1))^q\).

In the current work, we are interested in establishing conditions for the global and non-global existence of solutions of (1), adapting the ideas found in [36, 37]. Meier, in [37], studied the phenomenon of blow-up for the problem (1), when \( \omega(x) = 1, l(t) = 0 \) and \( f \) is continuously differentiable with \( f' \geq 0 \) and satisfies the integral condition in \((\Phi_1)\) (this condition is given later). More specifically, he showed that if \( f \) is convex with \( f(0) = 0 \), then the solution of the problem (1) blow-up in finite time, provided

\[
\int_{\|T(t)\|_{L^\infty}}^\infty \frac{d\sigma}{f(\sigma)} < \int_0^\tau h(\sigma)d\sigma \quad \text{for some } \tau > 0,
\]

where \( \{T(t)\}_{t \geq 0} \) is the Dirichlet heat semigroup in a smooth domain \( \Omega \) (see [39, p. 440]). Later, Loayza and Da Paixão [36] improved those results and established conditions for the global and non-global existence of nonnegative solutions of problem (1), when \( l = 0 \) and \( f \) is locally Lipschitz continuous. The authors also applied their results to obtain Fujita-type results (see [36, Theorem 1.7]). Notice, though, that they did not address the critical case \( q = 1 + \frac{2(\nu+1)}{N} \).

We apply our results (see Theorem 2) to obtain both Fujita-type results and the second critical exponent, in the sense of [33], for a problem related to the following equation with logarithmic nonlinearity:

\[
\begin{aligned}
&u_t - \text{div}(\omega(x)\nabla u) = h(t)u^p + l(t)(u + 1)(\ln(u + 1))^q, \\
&u(0) = u_0 \geq 0, \\
&x \in \mathbb{R}^N, \\
&x \in \mathbb{R}^N,
\end{aligned}
\]

with suitable \( h(t) \) and \( l(t) \) (see Corollary 3 and Remark 1). As far as we are aware, results concerned to second critical exponent for problems related to (3) with \( h(t) = 0 \) have not been addressed until now, even in the case when \( \omega(x) = 1 \). Since the first work in 1979 by Galaktionov et al. [19], problems with logarithmic nonlinearities have received considerable attention in various research papers due to their multiple applications in physics and applied sciences (see [11–13, 16, 20–23, 34, 35]).

It should be mentioned that the techniques used in this paper can be applied to treat the following more general problem:

\[
u_t - \text{div}(\omega(x)\nabla u) = \sum_{i=1}^k h_i(t)f_i(u) \quad (k \in \mathbb{N}),
\]

where \( h_i \in C[0, \infty) \) and \( f_i \in C[0, \infty) \) for \( i = 1, \ldots, k \).

To prepare an accurate statement of the non-global existence result, we assume that \( f \) or \( g \) lies in

\[
\Phi = \{ \gamma : [0, \infty) \to [0, \infty) : \gamma \text{ satisfies } (\Phi_1) \text{ and } (\Phi_2) \}\]
where
\[ (\Phi_1) \int_{z}^{\infty} \frac{d\sigma}{\gamma(\sigma)} < \infty \] for all \( z > 0 \) and \( \gamma(S(t)v_0) \leq S(t)\gamma(v_0) \) for all \( 0 \leq v_0 \in C_b(\mathbb{R}^N) \) and \( t > 0 \).

\( (\Phi_2) \) The function \( \gamma \) is nondecreasing such that \( \gamma(s) > 0 \) for all \( s > 0 \).

Here \( S(t)\phi(x) := \int_{\mathbb{R}^N} \Gamma(x, y, t) \phi(y) dy \), where \( \Gamma(x, y, t) \) is the fundamental solution of
\[
v_t - \text{div}(\omega(x) \nabla v) = 0, \quad x \in \mathbb{R}^N, \quad t > 0
\]
with a pole in \( (y, 0) \). Also, \( \Gamma(x, y, t) \) is nonnegative, and continuous for all \( (x, t) \neq (y, 0) \), see [25, 26] for more details.

The integral condition in \( (\Phi_1) \) is related to the Osgood-type condition (see, e.g. [32]), while the second condition in \( (\Phi_1) \) is satisfied if, for example, \( \gamma \) is convex (see Lemma 6).

Our main result is the following:

**Theorem 2** Assume either (A) or (B), and suppose that \( f, g : [0, \infty) \to [0, \infty) \) are locally Lipschitz continuous functions such that \( f(0) = g(0) = 0 \).

(i) If \( f, g, f(s)/s \) and \( g(s)/s \) are nondecreasing in some interval \( (0, m] \), and there exists nontrivial \( 0 \leq v_0 \leq C_b(\mathbb{R}^N) \) satisfying \( \|v_0\|_{\infty} \leq m \) and the condition
\[
\int_0^{\infty} h(\sigma) \frac{f(\|S(\sigma)v_0\|_{\infty})}{\|S(\sigma)v_0\|_{\infty}} d\sigma + \int_0^{\infty} l(\sigma) \frac{g(\|S(\sigma)v_0\|_{\infty})}{\|S(\sigma)v_0\|_{\infty}} d\sigma < 1,
\]
then there exists \( \delta > 0 \), such that for \( \delta v_0 = u_0 \), the solution of (1) is a global solution.

(ii) Let \( 0 \leq u_0 \in C_b(\mathbb{R}^N) \), \( u_0 \neq 0 \) and suppose that one of the following conditions hold:

(a) \( f \in \Phi \) and there exists \( \tau > 0 \) such that
\[
\int_0^{\infty} \frac{d\sigma}{f(\sigma)} \leq \int_0^{\tau} h(\sigma)d\sigma
\]

(b) \( g \in \Phi \) and there exists \( \tau > 0 \) such that
\[
\int_0^{\infty} \frac{d\sigma}{g(\sigma)} \leq \int_0^{\tau} l(\sigma)d\sigma
\]

Then the solution of problem (1) with initial condition \( u_0 \) is non-global.

As an application of our Theorem, we obtain the following Corollary.

**Corollary 3** Assume either (A) or (B). Let \( a = a \) in the case (A) and \( a = b \) in the case (B). Suppose \( p > 1 \), \( q > 1 \) and \( (h, l) \in (C[0, \infty))^2 \) with \( h(t) \sim t^r \) \((r > -1)\) and \( l(t) \sim t^r \) \((s > -1)\) for \( t \) large enough.

(i) If \( 1 < p \leq 1 + \frac{(2-\omega)(r+1)}{N} \) or \( 1 < q \leq 1 + \frac{(2-\omega)(s+1)}{N} \), then the problem (3) has no nontrivial global solutions.

(ii) Let \( 1 + \frac{(2-\omega)(r+1)}{N} < p \) \( \) and \( 1 + \frac{(2-\omega)(s+1)}{N} < q \).

(a) If \( 0 < \rho < \max \left\{ \frac{(2-\omega)(r+1)}{p-1}, \frac{(2-\omega)(s+1)}{q-1} \right\} \), then the problem (3) has no global solution for all initial data \( \psi \in I_\rho \).

(b) If \( \max \left\{ \frac{(2-\omega)(r+1)}{p-1}, \frac{(2-\omega)(s+1)}{q-1} \right\} < \rho \), then for every initial data \( u_0 \in I^p \), there exists \( \Lambda(\alpha) > 0 \) such that the problem (3) with initial data \( \lambda u_0 \) has nontrivial global solutions for all \( 0 < \lambda < \Lambda(\alpha) \).
**Remark 1** Consider $\alpha = a$ in the case (A) and $\alpha = b$ in case (B).

(i) Note that the results in Corollary 3 are sharp, since putting $p = q$ in Corollary 3, we obtain the following critical Fujita exponent

$$p^*(\alpha) = \max \left\{ 1 + \frac{(2 - \alpha)(r + 1)}{N}, 1 + \frac{(2 - \alpha)(s + 1)}{N} \right\}.$$

Also, the second critical exponent is given by

$$\rho^*(\alpha) = \max \left\{ \frac{(2 - \alpha)(r + 1)}{p - 1}, \frac{(2 - \alpha)(s + 1)}{p - 1} \right\}.$$

(ii) It is possible to use arguments similar to those in the proof of Corollary 3 to demonstrate other sharp results under slight modifications in Corollary 3. For instance, if $l(t) = 0$ and $h(t) = 1$ in Corollary 3, we can recover the Fujita’s results obtained in [17] (see Theorem 1). In addition, in this same case, we obtain the following second critical exponent

$$\rho^*(\alpha) = \frac{2 - \alpha}{p - 1}.$$

On the other hand, when $h(t) = 0$, respectively can get the next Fujita exponent and the second critical exponent

$$p^*(\alpha) = 1 + \frac{(2 - \alpha)(s + 1)}{N} \quad \text{and} \quad \rho^*(\alpha) = \frac{(2 - \alpha)(s + 1)}{p - 1}. $$

### 2 Preliminaries and toolbox

For more details of what is presented in this section, articles [10, 17, 24–26] and the references therein should be examined. Let us remember some concepts:

- $C[0, \infty)$ is the set of nonnegative functions defined on the interval $[0, \infty)$.
- $C_b(\mathbb{R}^N)$ is the set of bounded continuous functions defined on $\mathbb{R}^N$.
- The Banach space $L^p(\mathbb{R}^N)$ is defined as usual, and the norm is denoted by

$$\|\psi\|_p = \left( \int_{\mathbb{R}^N} |\psi(x)|^p \, dx \right)^{1/p}$$

for $1 \leq p < \infty$. When $p = \infty$

$$\|\psi\|_\infty := \inf \left\{ K \geq 0 : |\psi(x)| \leq K, \text{ for almost every } x \in \mathbb{R}^N \right\}.$$

The set $L^\infty((0, T), C_b(\mathbb{R}^N))$ is defined as follows

$$L^\infty((0, T), C_b(\mathbb{R}^N)) = \left\{ u : (0, T) \rightarrow C_b(\mathbb{R}^N) : \|u\| = \sup_{t \in (0, T)} \|u(t)\|_\infty < \infty \right\} \quad (8)$$

is a Banach space equipped with the usual norm $\| \cdot \|$. We say that $f_1(t) \sim f_2(t)$ for $t > 0$ sufficiently large, when there exist constants $k_1 > 0$, $k_2 > 0$ and $t_0 > 0$ such that

$$k_1 \ f_2(t) \leq f_1(t) \leq k_2 \ f_2(t),$$

for all $t > t_0$.

The positive part of a real-valued function $\phi$ is defined by

$$\phi^+(t) = \max[\phi(t), 0].$$

The negative part of $\phi$ is defined analogously.
Definition 1 The Muckenhoupt class $A_p$, with $1 < p < \infty$, is the set of locally integrable nonnegative functions $w$ that satisfy
\[
\left( \int_Q w \, dx \right) \left( \int_Q w^{-\frac{1}{p-1}} \, dx \right) < K,
\]
for every cube $Q$ and some constant $K > 0$. For $p = 1$, $w$ belongs to the Muckenhoupt class $A_1$ if there exists a constant $K > 0$ such that
\[
\int_Q w \, dx \leq K \inf_Q w,
\]
for all cube $Q$.

Solutions for problem (1) are understood in the following sense.

Definition 2 Let $u_0 \in C_b(\mathbb{R}^N)$. A function $u \in C([0, T), C_b(\mathbb{R}^N))$ is called the solution of the problem (1) on $[0, T)$, for some $T > 0$, if $u$ satisfies
\[
\begin{align*}
u(x, t) &= \int_{\mathbb{R}^N} \Gamma(x, y, t)u_0(y)dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma)h(\sigma)f(u(y, \sigma))d\sigma dy \\
&\quad + \int_0^t \int_{\mathbb{R}^N} \Gamma(x, y, t - \sigma)l(\sigma)g(u(y, \sigma))d\sigma dy
\end{align*}
\]
for all $t \in (0, T)$. When $T = \infty$, we call that $u$ is a global solution of (1). Here, $\Gamma(x, y, t)$ be the fundamental solution of (4).

As the weight function $\omega(x)$ satisfies the conditions (A) or (B), it follows that $\omega(x)$ lies in the Muckenhoupt classes $A_{1+\frac{\mu}{2}}$ and $A_2$. That is
\[
c_0 = \sup_Q \left( \int_Q \omega(x) dx \right) \left( \int_Q \omega(x)^{-\frac{\mu}{2}} dx \right)^{\frac{2}{\mu}} < \infty
\]
and
\[
C_0 = \sup_Q \left( \int_Q \omega(x) dx \right) \left( \int_Q \omega(x)^{-\frac{\mu}{2}} dx \right) < \infty
\]
respectively. Where the supremum is taken over all cubes $Q$ in $\mathbb{R}^N$. Also, the weight function $\omega(x)^{-\frac{\mu}{2}}$ satisfies a doubling and reverse doubling condition of order $\mu$ with $\mu > 1/2$. This means that there exist constants $c_1, c_2 > 0$ such that
\[
\int_{B_1(x)} \omega(y)^{-\frac{\mu}{2}} dy \leq c_1 s^{\mu N} \int_{B_R(x)} \omega(y)^{-\frac{\mu}{2}} dy
\]
and
\[
\int_{B_{1/2}(x)} \omega(y)^{-\frac{\mu}{2}} dy \geq c_2 s^{\mu N} \int_{B_R(x)} \omega(y)^{-\frac{\mu}{2}} dy.
\]
This makes it possible for us to use the lower and upper bound estimates for the fundamental solutions of (4), obtained by Gutiérrez et al. [24, 25], which were used in paper [17], respectively.

Under the conditions (A) or (B), the fundamental solution $\Gamma = \Gamma(x, y, t)$ of (4) verifies the following properties.
Lemma 5 \[17, Lemma 2.4\] Then there exists a positive constant \( C(\alpha, x) \) for convex, then
\[
\text{Lemma 4}
\]
Where the constant \( c \) can be taken so that it depends only on \( N \) and \( x \). Assume either (A) or (B). Let
\[
\text{Lemma 6}
\]
The following Lemma will be used for the proof of non-global existence for (1).
\[
\text{Lemma 5 [17, Lemma 2.4]} \quad \text{Assume either (A) or (B). Let } \phi \in L^\infty(\mathbb{R}^N), \phi \geq 0, \text{ and } \phi \neq 0. \text{ Then there exists a positive constant } C(\alpha, N), \text{ depending only on } \alpha \text{ and } N, \text{ such that}
\]
\[
S(t) \phi(x) \geq C(\alpha, N) t^{-\frac{N}{\alpha}} \int_{|y| \leq t^{-\frac{1}{\alpha}}} \phi(y) dy,
\]
for \(|x| \leq t^{-\frac{1}{\alpha}}\) and \( t > 0 \). Here \( \alpha = a \) in the case (A) and \( \alpha = b \) in the case (B).

The following Lemma will be used for the proof of non-global existence for (1).
\[
\text{Lemma 6} \quad \text{Assume either (A) or (B). If } 0 \leq u_0 \in L^\infty(\mathbb{R}^N) \text{ and } f : [0, \infty) \longrightarrow [0, \infty) \text{ is convex, then}
\]
\[
S(t) f(u_0) \geq f(S(t)u_0)
\]
Proof. By \((K_1)\) property and the fact that \(\Gamma > 0\) (see \((K_5)\)), we can use Jensen’s inequality, and so we get

\[
S(t)f(u_0(x)) = \int_{\mathbb{R}^N} \Gamma(x, y, t) f(u_0(y)) dy \geq f\left( \int_{\mathbb{R}^N} \Gamma(x, y, t) u_0(y) dy \right) = f(S(t)u_0(x)).
\]

\(\Box\)

For the second critical exponent result, we need the following version of \([33, \text{Lemma 2.12}]\)

**Proposition 7** Assume either (A) or (B). Let \(\alpha = a\) in the case (A) and \(\alpha = b\) in case (B).

(i) If \(0 \leq u_0 \in C_b(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\) and \(u_0 \neq 0\), then \(\|S(t)u_0\|_{L^\infty(\mathbb{R}^N)} \sim t^{-\frac{N}{2-\alpha}}\), for all \(t > 0\) sufficiently large.

(ii) If \(\psi \in I_\rho\), then there exists a constant \(C_1 > 0\) such that

\[
\|S(t)\psi\|_{L^\infty(\mathbb{R}^N)} \geq C_1 t^{-\frac{\rho}{2-\alpha}},
\]

for all \(t > 0\) sufficiently large.

(iii) If \(\psi \in I^\rho\) with \(0 < \rho \leq N\), then there exists a constant \(C_2(N, \alpha) > 0\) depending only on \(\alpha\) and \(N\) such that

\[
\|S(t)\psi\|_{L^\infty(\mathbb{R}^N)} \leq C_2(N, \alpha) \|\psi\|_{L^N/\rho'} t^{-\frac{\rho'}{2-\alpha}},
\]

for any \(\rho' \in (0, \rho)\) and \(t > 0\).

(iv) If \(\psi \in I^\rho\) with \(N < \rho\), then there exists a constant \(C_3(N, \alpha) > 0\) depending only on \(\alpha\) and \(N\) such that

\[
\|S(t)\psi\|_{L^\infty(\mathbb{R}^N)} \leq C_3(N, \alpha) \|\psi\|_{L^1} t^{-\frac{N}{2-\alpha}},
\]

for any \(t > 0\).

**Proof** Note that (i) is a direct consequence of Lemmas 4 and 5. Item (iii) follows from \(\psi \in L^{N/\rho'}(\mathbb{R}^N)\) and Lemma 4. Item (iv) follows from \(\psi \in L^1(\mathbb{R}^N)\) and Lemma 4. Now, we prove (ii). In fact, since that \(\psi \in I_\rho\), we have that there exists \(C_3 > 0\) such that \(\psi(x) \geq C_3 |x|^{-\rho}\) for \(|x|\) sufficiently large; then by Lemma 5 and \(t > 0\) sufficiently large, we have

\[
S(t)\psi(x) \geq C(\alpha, N)^{-1} t^{-\frac{N}{2-\alpha}} \int_{|y| \leq t^{1/2+\alpha}} \psi(y) dy \\
\geq C t^{-\frac{N}{2-\alpha}} \int_{t^{1/2+\alpha} \leq |y| \leq t^{1/2+\alpha}} |y|^{-\rho} dy \\
\geq C t^{-\frac{N}{2-\alpha}} t^{-\frac{\rho}{2-\alpha}} \int_{t^{1/2+\alpha} \leq |y| \leq t^{1/2+\alpha}} dy \geq Ct^{-\frac{\rho}{2-\alpha}}.
\]

Thus, the lemma follows. \(\Box\)

## 2.1 Local existence

This subsection proves the existence and uniqueness of local solutions of (1) employing the known fixed point method. There exist several pieces of literature in this frame; see by example [10].
Thus, uniqueness follows from Gronwall’s inequality.

**Lemma 9** (Comparison principle) Assume either (A) or (B). Let \( h, l \in C[0, \infty) \) and \( f, g \in C[0, \infty) \) are locally Lipschitz continuous. Then for every \( u_0 \in C_b(\mathbb{R}^N) \), there exists a unique solution \( u \) of problem (1) defined on \([0, T]\), for some \( 0 < T < \infty\).

**Proof** Let \( u_0 \in C_b(\mathbb{R}^N) \). We define \( \Psi : K \to L^\infty((0, T), C_b(\mathbb{R}^N)) \) by

\[
\Psi(u)(t) = S(t)u_0 + \int_0^t S(t - \sigma)h(\sigma)f(u(\sigma))d\sigma + \int_0^t S(t - \sigma)l(\sigma)g(u(\sigma))d\sigma, \tag{12}
\]

where \( K = \{ u \in L^\infty((0, T); C_b(\mathbb{R}^N)), \| u(t) \|_\infty \leq c_1 \| u_0 \|_\infty + 1, \forall t \in (0, T) \} \). Note that, \( K \) is a complete metric space equipped with the distance induced by the norm of \( L^\infty((0, T), C_b(\mathbb{R}^N)) \).

First, we show that \( \Psi(u) \in K \). Indeed, by (10), we have

\[
\| \Psi(u) \|_\infty \leq c_1 \| u_0 \|_\infty + c_1 \int_0^T h(\sigma)\| f(u(\sigma)) \|_\infty d\sigma + \int_0^T l(\sigma)\| g(u(\sigma)) \|_\infty d\sigma
\]

\[
\leq c_1 \| u_0 \|_\infty + \max_{s \in [0,T]}\{ f(s), g(s) \} \left( \sup_{s \in [0,T]} h(s) \right) c_1 \| u_0 \|_\infty T + \max_{s \in [0,M]}\{ f(s), g(s) \} \left( \sup_{s \in [0,T]} l(s) \right) \left( c_1 \| u_0 \|_\infty + 1 \right) T,
\]

where \( M = \| u_0 \|_\infty + 1 \). Choosing \( T > 0 \) small enough, we have \( \Psi(u) \in K \).

Now, arguing as in (13), we can show that for \( u, v \in K \),

\[
\sup_{t \in [0,T]} \| \Psi(u)(t) - \Psi(v)(t) \|_\infty \leq C \left( \sup_{s \in [0,T]} h(s) + \sup_{s \in [0,T]} l(s) \right) T \sup_{t \in [0,T]} \| u(t) - v(t) \|_\infty, \tag{14}
\]

therefore \( \Psi \) is a strict contraction for \( T \) short enough so that

\[
C \left( \sup_{s \in [0,T]} h(s) + \sup_{s \in [0,T]} l(s) \right) T < 1.
\]

Then \( \Psi \) has a unique fixed point \( u \in K \), i.e.

\[
u(t) = S(t)u_0 + \int_0^t S(t - \sigma)h(\sigma)f(u(\sigma))d\sigma + \int_0^t S(t - \sigma)l(\sigma)g(u(\sigma))d\sigma, \tag{15}\]

for \( t \in (0, T) \). Note that, by (K4) property and defining \( u(0) = u_0 \), we have that \( u \in C([0, T), C_b(\mathbb{R}^N)) \), and thus is a solution of (1).

To demonstrate uniqueness, suppose that \( u_1, u_2 \in C([0, T), C_b(\mathbb{R}^N)) \) are solutions of (1) such that \( u_1(0) = u_2(0) = u_0 \), then arguing as in (13), we have

\[
\| u_1(t) - u_2(t) \|_\infty \leq C \int_0^T \| u_1(\sigma) - u_2(\sigma) \|_\infty d\sigma, \text{ for } t \in [0, T).
\]

Thus, uniqueness follows from Gronwall’s inequality. \( \square \)

The following Lemma is a version of the comparison principle.

**Lemma 10** (Comparison principle) Assume either (A) or (B). Let \( h, l \in C[0, \infty)^2 \), \( f, g \in C[0, \infty)^2 \) are nondecreasing locally Lipschitz functions, and \( u, v \in C([0, T], C_b(\mathbb{R}^N)) \) be solutions of problem (1). If \( u(0) \leq v(0), \text{ then } u(t) \leq v(t) \text{ for all } t \in [0, T] \).
Proof Note that it is sufficient to show that \([u - v]^+ \leq 0\). Let \(u(0) = u_0, v(0) = v_0\) and \(M_0 = \max\{\|u(t)\|_\infty, \|v(t)\|_\infty : t \in [0, T]\}\). Since \(u_0 \leq v_0\), then from (\(K_5\)) we have
\[
u(t) - v(t) \leq \int_0^t S(t - \sigma) (h(\sigma)[f(u(\sigma)) - f(v(\sigma))] + l(\sigma)[g(u(\sigma)) - g(v(\sigma))]) d\sigma.
\]
Thus, since \(f\) and \(g\) are nondecreasing and Lipschitz continuous on \([0, M_0]\), it follows from Lemma 4 that
\[
\|u(t) - v(t)\|_\infty \leq C \int_0^t \|u(\sigma) - v(\sigma)\|_\infty d\sigma.
\]
The lemma is now a direct consequence of Gronwall’s inequality.

3 Global and non-global existence

In this section we proof Theorem 2.

Proof of Theorem 2 (i) We apply arguments used in [41]. Consider \(\delta > 0\) such that
\[
\delta < \frac{1}{\beta + 1} \min \left\{1, \frac{m}{\|v_0\|_\infty}\right\},
\]
where \(\beta > 0\) satisfies
\[
\int_0^\infty h(\sigma) f\left(\frac{\|S(\sigma)v_0\|_\infty}{\|S(\sigma)v_0\|_\infty}\right) d\sigma + \int_0^\infty l(\sigma) g\left(\frac{\|S(\sigma)v_0\|_\infty}{\|S(\sigma)v_0\|_\infty}\right) d\sigma < \frac{\beta}{\beta + 1}.
\]
Set \(u_0 = \delta v_0 \in C_b(\mathbb{R}^N)\). Now we define the sequence \(\{u^k\}_{k \geq 0}\) by \(u^0 = S(t)u_0\) and
\[
u^k(t) = S(t)u_0 + \int_0^t S(t - \sigma) h(\sigma) f(\mu^k-1(\sigma)) d\sigma + \int_0^t S(t - \sigma) l(\sigma) g(\mu^k-1(\sigma)) d\sigma
\]
for \(k \in \mathbb{N}\) and \(t > 0\).

We claim
\[
u^k(t) \leq (1 + \beta)S(t)u_0
\]
for \(k \geq 0\) and \(t > 0\). We show our claim by induction on \(k\). If \(k = 0\), then the claim (16) is trivial. Consider now that (16) holds, for some \(k \in \mathbb{N}\). Since \(\|S(t)v_0\|_\infty \leq m\) for \(t > 0\) and \(f, g, f(\cdot), g(\cdot)\) are nondecreasing on \((0, m]\), by \((K_1), (K_2), (K_3)\) and Fubini’s theorem we have
\[
u^{k+1}(t) \leq S(t)u_0 + \int_0^t S(t - \sigma) \left[h(\sigma) f((1 + \beta)S(\sigma)u_0]\right] d\sigma
\]
\[
\quad + \int_0^t S(t - \sigma) \left[l(\sigma) g((1 + \beta)S(\sigma)u_0)\right] d\sigma
\]
\[
= S(t)u_0 + \int_0^t h(\sigma)S(t - \sigma) f((1 + \beta)S(\sigma)u_0) \left(\frac{1}{1 + \beta}S(\sigma)u_0\right) d\sigma
\]
\[
\quad + \int_0^t l(\sigma)S(t - \sigma) g((1 + \beta)S(\sigma)u_0) \left(\frac{1}{1 + \beta}S(\sigma)u_0\right) d\sigma
\]
\[
\leq S(t)u_0 + [(1 + \beta)S(t)u_0] \int_0^t h(\sigma) \frac{f \left(\|S(\sigma)v_0\|_\infty\right)}{\|S(\sigma)v_0\|_\infty} d\sigma
\]
Thus, the claim follows. Also, using the fact $f$ and $g$ are nondecreasing, and by induction on $k$, we have $u^k \leq u^{k+1}$ for all $k \in \mathbb{N}$. Since the sequence $\{u^k\}_{k \geq 0}$ is bounded [by (16)], then by Monotone Convergence Theorem, we conclude that $u = \lim_{k \to \infty} u^k$ is a global solution of (1).

Proof of Theorem 2 (ii) First, suppose that $f \in \Phi$ and satisfies (6). In order to show the non-global existence, we argue by contradiction. Suppose that there exists a global solution $u \in C([0, \infty), C_b(\mathbb{R}^N))$ of (1) with nonnegative initial condition $u_0 \neq 0$; this is $u(t) = S(t)u_0 + \int_0^t S(t-s)h(s)f(u(s)) \, ds + \int_0^t S(t-s)l(s)g(u(s)) \, ds,$

for $t > 0$.

Let $s > 0$ be fixed and let $t > 0$ such that $0 < t < s$. Then, from $(K_1), (K_2), (K_5), (\Phi_1)$ and Lemma 6, we have

$$S(s-t)u(t) \geq \Theta(t), \quad (17)$$

where $\Theta(t) := S(s)u_0 + \int_0^t h(s)f(S(s-s)u(s)) \, ds$. Note that, $\Theta(t) := \Theta(\cdot, t)$ is absolutely continuous on $[0, s]$, consequently is differentiable almost everywhere on $[0, s]$ and

$$\Theta'(t) = h(t)f(S(s-t)u(t)). \quad (18)$$

Since $f$ is nondecreasing, then from (17) and (18), we have

$$\Theta'(t) \geq h(t)f(\Theta(t)). \quad (19)$$

From $(\Phi_1)$ and $(K_5)$, this implies that

$$\int_{\|S(s)u_0\|_{\infty}}^{\infty} \frac{d\sigma}{f(\sigma)} \geq \int_{\Theta(t)}^{\Theta(s)} \frac{d\sigma}{f(\sigma)} > \int_{\Theta(0)}^{\Theta(s)} \frac{d\sigma}{f(\sigma)} = \frac{\Theta(s)}{f(\Theta(s))} \geq \int_{\Theta(0)}^{\Theta(s)} \frac{d\sigma}{f(\sigma)} = \int_0^{\infty} \frac{d\sigma}{f(\sigma)}$$

$$= - \int_0^s \left( \int_{\Theta(t)}^{\infty} \frac{d\sigma}{f(\sigma)} \right) \, dt = \int_0^s \frac{\Theta'(t)}{f(\Theta(t))} \, dt \geq \int_0^s h(t) \, dt.$$

However, this contradicts (6) (the same contradiction is obtained when $g \in \Phi$ and satisfies (7)). So that the second part of the Theorem 2 is proved. \qed

4 An application to a problem with logarithmic nonlinearities

In this section we proof Corollary 3.

Proof of Corollary 3 (i) Note that the functions $f(t) = t^p \, (p > 1)$ and $g(t) = (1 + t)[\ln(1 + t)]^q \, (q > 1)$ are convex functions, then the second property in $(\Phi_1)$ follows from Lemma 6.
Also, $f$ and $g$ satisfies the integral condition in ($\Phi_1$), since
\[
\int_{\mathbb{Z}}^{\infty} \frac{d\sigma}{f(\sigma)} = \int_{\mathbb{Z}}^{\infty} \frac{d\sigma}{\sigma^p} = (p-1)^{-1}z^{1-p} \quad (z > 0),
\]
\[
\int_{\mathbb{Z}}^{\infty} \frac{d\sigma}{g(\sigma)} = \int_{\mathbb{Z}}^{\infty} \frac{d\sigma}{(1+\sigma)[\ln(1+\sigma)]^q} = (q-1)^{-1}[\ln(1+z)]^{1-q} \quad (z > 0).
\]

Similarly, when arguing by contradiction, we suppose that there exists a nontrivial global solution $u(t) > 0$ for all $t$ large enough. Since $\lim_{t \to \infty} (1+t^{-\frac{N}{\alpha}}) = c$ and $1 < q < 1 + \frac{(2-\alpha)(s+1)}{N}$ for all $t$ large enough and some positive constant $c$; thus, from (20), we get
\[
\left[ \int_{\|S(t)u_0\|_{\infty}}^{\infty} \frac{d\sigma}{f(\sigma)} \right]^{-1} \int_{0}^{t} l(\sigma) d\sigma \geq (q-1)[\ln(1+\|S(t)u_0\|_{\infty})]^q \cdot \int_{0}^{t} l(\sigma) d\sigma \\
\geq C \cdot \left[ \ln(1 + c \cdot t^{-\frac{N}{\alpha}}) \cdot t^{\frac{s+1}{\alpha}} \right]^{q-1} \\
\geq C \cdot \left[ c t^{-\frac{N}{\alpha}} \cdot t^{\frac{s+1}{\alpha}} \right]^{q-1} > 1,
\]
for $t > 0$ sufficiently large; since $\lim_{t \to \infty} \frac{\ln(1+ct^{-\frac{N}{\alpha}})}{t^{-\frac{N}{\alpha}}} = c$ and $1 < q < 1 + \frac{(2-\alpha)(s+1)}{N}$.
Similarly, when $p < 1 + \frac{(2-\alpha)(r+1)}{N}$ we have
\[
\left[ \int_{\|S(t)u_0\|_{\infty}}^{\infty} \frac{d\sigma}{g(\sigma)} \right]^{-1} \int_{0}^{t} h(\sigma) d\sigma > 1,
\]
for $t$ large enough. Thus, from Theorem 2-(ii), we obtain that the solution of problem (3), with initial condition $0 \leq u_0 \neq 0$, is nonglobal.

Critical cases $p = 1 + \frac{(2-\alpha)(r+1)}{N}$ or $q = 1 + \frac{(2-\alpha)(s+1)}{N}$. In this part, we adapt the previous arguments joint with the ideas in [17, p. 17]. First, consider $q = 1 + \frac{(2-\alpha)(s+1)}{N}$; arguing by contradiction, we suppose that there exists a nontrivial global solution $u \in C([0, \infty), C_b(\mathbb{R}^N))$ of (3), since $u(0) = u_0 \in C_b(\mathbb{R}^N)$ is nontrivial and $l(t) \sim t^s$ ($s > -1$) for $t$ large enough, there exists $r_0 > 1$ such that $M = \int_{B_{r_0}(0)} u_0(y) dy \neq 0$ and
\[
\int_{0}^{r_0} l(\sigma) d\sigma \neq 0. \text{ Thus, from Lemma 5, we have}
\]
\[
\int_{0}^{t} l(\sigma) d\sigma \geq \frac{CM}{t^{\frac{N}{\alpha}}} \quad (22)
\]
for all $|x| \leq \frac{1}{t^{\frac{1}{\alpha}}}$ and $t \geq r_0$. We can also choose $r_0$ in such a way that
\[
\ln(1 + CM t^{-\frac{N}{\alpha}}) \geq C M t^{-\frac{N}{\alpha}} \quad (23)
\]
for all $t \geq r_0$; since $\lim_{t \to \infty} \ln(1 + CM t^{-\frac{N}{\alpha}}) = CM$. From estimates (2.11) and (2.12) in [17, p. 10], we have
\[
\begin{align*}
\int_{|x| \leq \frac{1}{t^{\frac{1}{\alpha}}}} \Gamma(x, y, t) dx & \geq C_0 > 0,
\end{align*}
\]
\[
\text{for all } x \in \mathbb{R}^N, y \in B_{r_0}(0), t \geq r_0.
\]
\[
\text{Springer}
\]
for all \( |y| \leq t^{1/z_0} \). Since \( t + r_0 - \sigma \leq t \) and \( \sigma \leq t + r_0 - \sigma \) for \( 1 < r_0 \leq \sigma \leq t/2 \), by (22), (23) and (24) we have
\[
\int_{|x| \leq t^{1/z_0}} u(x, t + r_0) \, dx \\
\geq \int_{|x| \leq t^{1/z_0}} \int_{r_0}^{t^{1/2}} \sigma^s \int_{|y| \leq (t + r_0 - \sigma)^{1/z_0}} \Gamma(x, y, t + r_0 - \sigma) \\
\times (1 + u(y, \sigma)) \left[ \ln(1 + u(y, \sigma)) \right]^q \, dy \, d\sigma \, dx \\
\geq \int_{|x| \leq t^{1/z_0}} \int_{r_0}^{t^{1/2}} \sigma^s \int_{|y| \leq (t + r_0 - \sigma)^{1/z_0}} \Gamma(x, y, t + r_0 - \sigma) \\
\times \left[ \ln(1 + u(y, \sigma)) \right]^q \, dy \, d\sigma \\
\geq C_0 \int_{r_0}^{t^{1/2}} \sigma^s \int_{|y| \leq t^{1/z_0}} \left[ \ln(1 + u(y, \sigma)) \right]^q \, dy \, d\sigma \\
\geq C_0 \int_{r_0}^{t^{1/2}} \sigma^s \int_{|y| \leq t^{1/z_0}} \left[ \ln(1 + C M \sigma^{-N/z_0}) \right]^q \, dy \, d\sigma \\
\geq C^q C_0 M^q \int_{r_0}^{t^{1/2}} \sigma^{-N(z_0-1)/z_0 + s} \left( \int_{|y| \leq t^{1/z_0}} \sigma^{-N/z_0} \, dy \right) \, d\sigma \\
= C^q C_0 M^q \int_{r_0}^{t^{1/2}} \sigma^{-1} \, d\sigma = C^q C_0 M^q \ln \left( \frac{t}{2r_0} \right), \quad \text{for } t > 2r_0. \tag{25}
\]

This implies that for all \( m > 0 \), there exist \( T_m > 0 \), such that
\[
\int_{|x| \leq T_m^{1/z_0}} u(x, T_m) \, dx \geq m. \tag{26}
\]

Also, from (26) and Lemma 5, we obtain
\[
\| S(t) u(T_m) \|_\infty \geq S(t) u(x, T_m) \geq C(\alpha, N)^{-1} m t^{-N/z_0} \tag{27}
\]
for \( |x| < t^{1/z_0} \) and for all \( t > T_m \). Note that, \( v(t) := u(t + T_m) \) is also a global solution of (3) with initial condition \( u(T_m) := u(\cdot, T_m) \), since \( u \) is a global solution. Then, from (26), (27), arguing similarly to (21), and recalling that \( q = 1 + \frac{(2-\alpha)(s+1)}{N} \), we deduce that
\[
\int_{\| S(t) u(T_m) \|_\infty}^\infty \left[ \frac{d\sigma}{g(\sigma)} \right]^{-1} \int_0^t l(\sigma) \, d\sigma \geq C \left[ C(\alpha, N)^{-1} m t^{-N/z_0} \cdot t^{s-1} \right] q^{-1} \\
= C C(\alpha, N)^{(q-1)m^{q-1}} > 1,
\]
for \( m > 0 \) large enough. Thus, from Theorem 2-(ii), we have that \( v \) is a non-global solution which is a contradiction; thus, the solution \( u \) is non-global. The proof is similar when we suppose that \( p = 1 + \frac{(2-\alpha)(s+1)}{N} \) and omit it here. Thus, the proof is completed. \( \square \)
Proof of Corollary 3 (ii) (a) The proof is similar to the given above by using Proposition 7-(ii) instead of Lemma 5.

Proof of Corollary 3 (ii) (b) First note that
\[ \max \left\{ \frac{(2 - \alpha)(r + 1)}{p - 1}, \frac{(2 - \alpha)(s + 1)}{q - 1} \right\} < N. \]

We have two cases:

\[ (\text{CASE} \max \left\{ \frac{(2 - \alpha)(r + 1)}{p - 1}, \frac{(2 - \alpha)(s + 1)}{q - 1} \right\} < \rho \leq N). \]

Let \( \max \left\{ \frac{(2 - \alpha)(r + 1)}{p - 1}, \frac{(2 - \alpha)(s + 1)}{q - 1} \right\} < \rho' < \rho, u_0 \in I^p, f(t) = t^p, g(t) = (1 + t)[\ln(1 + t)]^q \) and consider \( \lambda > 0 \), which will be chosen later. Note that \( f(t), g(t), \frac{f(t)}{t}, \frac{g(t)}{t} \) are non-decreasing functions. Thus by Lemma 4 and Proposition 7-(iii) we have

\[
\int_{\sigma}^{\infty} h(\sigma) \left( \frac{\|S(\sigma)(\lambda u_0)\|_\infty}{\|S(\sigma)(\lambda u_0)\|_\infty} \right) d\sigma + \int_{\sigma}^{\infty} l(\sigma) \left( \frac{\|S(\sigma)(\lambda u_0)\|_\infty}{\|S(\sigma)(\lambda u_0)\|_\infty} \right) d\sigma 
\]

\[
\leq \left( \sup_{s \in [0, t_0]} h(s) \right) \int_{0}^{t_0} [\lambda c_1 \|u_0\|_\infty]^{p-1} d\sigma + \int_{0}^{\infty} \sigma^{r-\left(\frac{p-1}{2-\alpha}\right)} d\sigma 
\]

\[
+ \left( \sup_{s \in [0, t_0]} l(s) \right) \int_{0}^{t_0} (1 + \lambda c_1 \|u_0\|_\infty)[\lambda c_1 \|u_0\|_\infty]^{q-1} d\sigma 
\]

\[
+ \int_{0}^{\infty} \sigma^{s} \left( 1 + \frac{\|S(\sigma)(\lambda u_0)\|_\infty}{\|S(\sigma)(\lambda u_0)\|_\infty} \right)[\ln(1 + \|S(\sigma)(\lambda u_0)\|_\infty)]^{q} d\sigma 
\]

\[
\leq \left( \sup_{s \in [0, t_0]} h(s) \right) \int_{0}^{t_0} [\lambda c_1 \|u_0\|_\infty]^{p-1} d\sigma 
\]

\[
+ [C_2(N, \alpha)\lambda \|u_0\|_{L^{N/\alpha'}}]^{p-1} \int_{0}^{\infty} \sigma^{r-\left(\frac{p-1}{2-\alpha}\right)} d\sigma 
\]

\[
+ \left( \sup_{s \in [0, t_0]} l(s) \right) \int_{0}^{t_0} (1 + \lambda c_1 \|u_0\|_\infty)[\lambda c_1 \|u_0\|_\infty]^{q-1} d\sigma 
\]

\[
+ (1 + \lambda c_1 \|u_0\|_\infty)[C_2(N, \alpha)\lambda \|u_0\|_{L^{N/\alpha'}}]^{q-1} \int_{0}^{\infty} \sigma^{s-\left(\frac{q-1}{2-\alpha}\right)} d\sigma 
\]

where \( t_0 > 1 \) was choose such that \( h(t) = t^r \) and \( l(t) = t^s \) for all \( t \geq t_0 \). Note that the hypothesis implies that the above integrals are bounded. Thus, the assertion hold by Theorem 2 and Lemma 9, since \( l < 1 \) for \( \lambda \) short enough.

\[ (\text{CASE} \max \left\{ \frac{(2 - \alpha)(r + 1)}{p - 1}, \frac{(2 - \alpha)(s + 1)}{q - 1} \right\} < N < \rho). \]

In this case, we can use the item (iv) of Proposition 7 instead of the item (iii); thus the proof follows step to step the same ideas of the above proof. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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