An algorithm to generate anisotropic rotating fluids with vanishing viscosity

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Abstract

Starting with generic stationary axially symmetric spacetimes depending on two spacelike isotropic orthogonal coordinates $x^1, x^2$, we build anisotropic fluids with and without heat flow but with vanishing viscosity. In the first part of the paper, after applying the transformation $x^1 \rightarrow J(x^1)$, $x^2 \rightarrow F(x^2)$ (with $J(x^1), F(x^2)$ regular functions) to general metrics coefficients $g_{ab}(x^1, x^2) \rightarrow g_{ab}(J(x^1), F(x^2))$ with $G_{x^1 x^2} = 0$, being $G_{ab}$ the Einstein’s tensor, we obtain that $\tilde{G}_{x^1 x^2} = 0 \rightarrow G_{x^1 x^2}(J(x^1), F(x^2)) = 0$. Therefore, the transformed spacetime is endowed with an energy-momentum tensor $T_{ab}$ with expression $g_{ab}Q_i + \text{heat term}$ (where $g_{ab}$ is the metric and $\{Q_i\}, i = 1..4$ are functions depending on the physical parameters of the fluid), i.e. without viscosity and generally with a non-vanishing heat flow. We show that after introducing suitable coordinates, we can obtain interior solutions that can be matched to the Kerr one on spheroids or Cassinian ovals, providing the necessary mathematical machinery.

In the second part of the paper we study the equation involving the heat flow and thus we generate anisotropic solutions with vanishing viscosity.

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heat flow. In this frame, a class of asymptotically flat solutions with vanishing heat flow and viscosity can be obtained. Finally, some explicit solutions are presented with possible applications to a string with anisotropic source and a dark energy-like equation of state.

1 Introduction

Exact metrics describing rotating, axially symmetric, isolated bodies are of great astrophysical interest. Many methods have been developed in the literature [1]-[25] to build physically viable interior metrics. In particular, handable solutions describing a rotating body with a perfect fluid source are not at our disposal, with the exception of the Van Stockum solutions [15] representing pressureless dust spacetimes. Unfortunately, asymptotically flat solutions with a dust source are plagued by a curvature singularity at the origin of the polar coordinate system leading to a non-globally well-defined mass function. Further, the technique named “displace, cut, fill and reflect” has been applied to the van Stockum class of solutions to build a rotating disk immersed in rotating dust (see [17] and references therein). This method generates a distributional source of matter with rather unusual properties to describe ordinary galaxies. In [17], the disk is obtained starting from Bonnor [16] dust solutions. Although perfect fluids seem more appropriate to describe ordinary astrophysical objects, anisotropic fluids (see [4, 11]) are finding applications in physical situations where very compact objects come in action.

In the literature, only few solutions are present depicting global physically reasonable sources suitable for isolated rotating bodies. To this purpose, see the exception given in [1, 3] for a perfect fluid. Quite remarkably, recently [26] a procedure suitable for the static case has been proposed, and thus extended to the stationary rotating one [27, 28], in order to obtain a physically viable anisotropic interior source for the Kerr metric. This solution has interesting properties. In particular, the solution in [27] is equipped with a non-vanishing energy-momentum flux in the equatorial plane. Inspired by the results in [27], we explore, from a mathematically point of view, the presence of viscosity and of a heat flow term in the equations governing axially-symmetric rotating anisotropic sources.

To this purpose, we show that, starting from a given stationary axially-symmetric seed spacetime (also an interior non-vacuum solution) with vanishing viscosity, the map \( x^1 \to J(x^1), x^2 \to F(x^2) \), applied to the metric functions \( g_{ab} \), can be used to obtain global interior anisotropic fluids with heat flow and without viscosity, (being \( x^1, x^2 \) two spacelike, orthogonal isotropic co-
ordinates). Generally, the so generated solutions have a non-vanishing heat flow. To obtain interior solutions with vanishing heat flow, a suitable procedure is also given. In this context, a class of asymptotically flat solutions without heat flow and with mass-term is presented.

In section 2 we present our technique to generate anisotropic fluids without viscosity and with heat flow with some examples. In section 3 we show how the procedure of section 2 can be used to obtain interior solutions matching the Kerr one on spheroids or Cassianian ovals boundary surfaces. In section 4 we study the technique to obtain metrics with vanishing heat flow. In section 4 we also outline a simple procedure to obtain asymptotically flat metrics representing anisotropic fluid with vanishing heat flow and positive total mass and present a regular class of solutions with a $G_3$ group of motion and containing the van Stockum metric [15] Section 5 collects some final remarks and conclusions.

2 General case with vanishing viscosity

The general expression for a metric describing stationary axially symmetric spacetimes is given by [23]

$$ds^2 = g_{ab} dx^a dx^b, \quad g_{ab} = -V_a V_b + W_a W_b + S_a S_b + L_a L_b,$$  \hspace{1cm} (1)

where $(x^1, x^2)$ are spacelike orthogonal isotropic coordinates related to the canonical Weyl coordinates $(\rho, z)$ by $\rho_{,x^1} = z_{,x^2}, \rho_{,x^2} = -z_{,x^1}, x^3 = \phi$ is the angular coordinate, $x^4 = t$ is the time coordinate, and subindices with “comma” denote partial derivative and

$$V_a = \left(0, 0, \frac{N(x^1, x^2)}{\sqrt{f(x^1, x^2)}}, -\sqrt{f(x^1, x^2)}\right),$$

$$W_a = \left(0, 0, \frac{H(x^1, x^2)}{\sqrt{f(x^1, x^2)}}, 0\right),$$

$$S_a = \left(e^{\frac{N(x^1, x^2)}{2}}, 0, 0, 0\right),$$

$$L_a = \left(0, e^{-\frac{N(x^1, x^2)}{2}}, 0, 0\right),$$

an orthonormal basis. For $H(x^1, x^2) = \rho(x^1, x^2)$ the metric is expressed in the so called Papapetrou gauge. It is a simple matter to verify that the tensor $T_{ab}$ generated via $G_{ab}$ by [11] has the following non-zero components:
generally we have an anisotropic source matter. Ordinary stars are not composed of anisotropic matter: it is generally accepted that perfect fluids can depict ordinary stars. However, anisotropic fluids are raising increasing interest in the literature. Anisotropic fluids are expected to arise for very compact objects such as neutron stars, where the suitable equation of state is a matter of debate. For example, in \[29\] it is shown that anisotropies can modify the physical parameters of the fluid and also the critical mass, the stability and the redshift of the stars. Moreover, anisotropic fluids are used to study boson stars, where anisotropic stresses are naturally required \[30\]. It should also be noticed that the emission of gravitational waves from a given source is dependent on its equation of state, and thus the study of anisotropic energy-momentum tensors can be of great astrophysical interest in order to have possible hints on the frequency’s emission of gravitational waves allowed in a general context. The use of anisotropic fluids is thus important for rotating fluids. This is in part also motivated by the fact that, thanks to unsolved formidable mathematical issues principally due to the complexity of the equations involved, no physically reasonable perfect fluid sources for realistic rotating isolated objects have been found at present day. Anisotropic fluids are thus a natural arena to study the effects of rotation in general relativity, in particular when a huge rotation is expected to modify the equation of state of non-rotating or slowly rotating objects. Dissipative effects are also expected, for example, in rotating neutron stars, where convective modes come into action. Moreover, anisotropic fluids can arise from magnetic fields and collisionless relativistic particles thanks to gravitational waves emission.

Concerning the most general energy momentum tensor generated by (1) we have:

\[
T_{ab} = EV_aV_b + P_{x1}S_aS_b + P_{x2}L_aL_b + P_\phi W_aW_b + K (V_aW_b + W_aV_b) + \mathcal{H} (S_aL_b + L_aS_b),
\]

where \(E\) is the energy density, \(\{P_{x1}, P_{x2}, P_\phi\}\) the principal stresses, \(K\) the heat flow parameter and \(\mathcal{H}\) the viscosity one. Parameter \(K\) depicts heat transfer in a given space equipped with a non-uniform temperature distribution. In our frame, it is associated to the basis \(V_a\) and \(W_a\) and thus it is related, thanks to the axisymmetry of our background metric (1), to transfer heat along the rotation axis. This effect is also present in the interior Kerr solution found in [27, 28].

Concerning the viscosity parameter \(\mathcal{H}\), as well known, it is not simple to treat [31] and it is expected to depict peculiar dissipative properties of the fluid. In practice, viscosity depicts internal friction due to a relative velocity between two adjacent elements of the fluid. In particular, for the
background \([\mathcal{H}]\), the friction is expected to be on the \(x^1 - x^2\) plane. Hence, from a physical point of view, the request of a vanishing viscosity practically implies that no internal friction is present in the \(x^1 - x^2\) plane and the only dissipative effect is the one due to the transfer heat along the rotation axis. This is an assumption that is expected to hold, for example, in ordinary situations where magnetic fields play a non-relevant role. From a mathematical point of view, the vanishing of \(\mathcal{H}\) is related to the vanishing of the \(T_{x^1x^2}\) component of the energy momentum tensor. For \(G_{x^1x^2}\) we found

\[
G_{x^1x^2} = \frac{1}{2f^2H^2} A(x^1, x^2),
\]

\[
A(x^1, x^2) = f_x f_x^1 H^2 + 2f^2 HH_{x^1x^2} + f_{x^1} NN_{x^1} + f_{x^2} NN_{x^2} - f_{x^1} HH_{x^2} - v_x f^2 HH_{x^2} - v_{x^1} f^2 HH_{x^1} - f_{x^2} f_{x^1} N^2 - f f_{x^2} HH_{x^1} - f^2 N_{x^2} N_{x^1}.
\]

(4)

Our technique is the following. Suppose to start with a generic line element \(\mathcal{M}\): by applying the map \(x^1 \rightarrow J(x^1), \ x^2 \rightarrow F(x^2)\) to the metric functions \(g_{ab}\) only, we generate a new spacetime with line element given by:

\[
ds^2 = \tilde{g}_{ab} dx^a dx^b
\]

(5)

where \(\tilde{g}_{ab} = \tilde{g}_{ab}(J(x^1), F(x^2))\) with the coordinates used in \(\mathcal{M}\) left unchanged \([\mathcal{M}]\). With this map, the \(\mathcal{M}\) is transformed to \(\tilde{G}_{x^1x^2}\):

\[
G_{x^1x^2} \rightarrow \tilde{G}_{x^1x^2} = \frac{1}{2\tilde{H}^2\tilde{f}^2} F x^2 J x^1 \tilde{A}(J(x^1), J(x^2)),
\]

(6)

\[
\tilde{A}(J(x^1), F(x^2)) = A((x^1 \rightarrow J(x^1)), (x^2 \rightarrow F(x^2))),
\]

(7)

where in \(\tilde{G}\) the partial derivatives with respect to \((x^1, x^2)\) are substituted by the partial derivatives with respect to \(J, F\):

\[
\tilde{A}(J, F) = \tilde{f}_F \tilde{f}_J \tilde{H}^2 + 2\tilde{f}^2 \tilde{H} \tilde{H}_{,F} + \tilde{f} \tilde{f}_F \tilde{N}_F \tilde{N}_{,J} + \tilde{f} \tilde{f}_F \tilde{N}_F \tilde{N}_{,J} - \tilde{f} \tilde{f}_J \tilde{H}_{,F} - \tilde{v}_F \tilde{f} \tilde{H} \tilde{H}_{,F} - \tilde{v}_F \tilde{f} \tilde{H} \tilde{H}_{,J} - \tilde{f}_J \tilde{f}_J \tilde{N}^2 - \tilde{f} \tilde{f}_F \tilde{H}_{,J} - \tilde{f} \tilde{f}_F \tilde{N}_{,F} \tilde{N}_J.
\]

(8)

In what follows, functions marked with a "tilde" will be in agreement with notation \([\mathcal{M}]\). Therefore, starting with a generic seed metric with \(G_{x^1x^2} = 0\), the transformed solution has again \(\tilde{G}_{x^1x^2} = 0\), provided that \((x^1, x^2)\)

\footnote{It is worth to be noticed that the map is applied to the metric functions \(g_{ab}\) and not to the line element \(ds^2 = g_{ab} dx^a dx^b\), i.e. it is not obviously a coordinate transformation.}
are spacelike orthogonal isotropic coordinates. As a result, the energy-momentum tensor so generated can be written as
\[
\tilde{T}_{ab} = \tilde{E} \tilde{V}_a \tilde{V}_b + \tilde{\mathcal{P}}_{x^1} \tilde{S}_a \tilde{S}_b + \tilde{\mathcal{P}}_{x^2} \tilde{L}_a \tilde{L}_b + \mathcal{P}_\phi \tilde{W}_a \tilde{W}_b + \mathcal{K}(\tilde{V}_a \tilde{W}_b + \tilde{W}_a \tilde{V}_b),
\]
(9)
where \(\tilde{E}\) denotes the transformed energy-density of the source, \(\tilde{\mathcal{P}}_{x^1}, \tilde{\mathcal{P}}_{x^2}, \mathcal{P}_\phi\) the transformed principal stresses, and \(\mathcal{K}\) the heat flow.

For the heat flow obtained from (9), we obtain
\[
\mathcal{K} = e^{-\tilde{v}} \left[ \tilde{N}_{,\alpha,\alpha} - \tilde{N}_{\alpha,\alpha} \tilde{H}_{,\alpha} - \tilde{N} \tilde{f}_{,\alpha,\alpha} + \tilde{N} \tilde{f}_{,\alpha} \tilde{H}_{,\alpha} \right],
\]
(10)
where a summation with respect to \(\alpha = (x^1, x^2)\) is implicit. Equation (10) will be discussed in the section 4.

Another equation of interest is the following
\[
\tilde{G}_{x^1 x^1} + \tilde{G}_{x^2 x^2} = -e^{-\tilde{v}} (\mathcal{P}_\rho + \mathcal{P}_z) = -\tilde{H}_{,\alpha,\alpha} \tilde{H}_{,\alpha}.
\]
(11)
Equation (11) gives us information about the equation of state of the source of the transformed solution. In fact, if \(\tilde{H}(x^1, x^2)\) is a harmonic function (\(\tilde{H}_{,\alpha,\alpha} = 0\)), and if \((\tilde{P}_{x^1}, \tilde{P}_{x^2}) \neq 0\), then the generated solution has equation of state \(\mathcal{P}_{x^1} = -\mathcal{P}_{x^2}\): this is the case of generating solutions with \(\tilde{H} = \rho(x^1, x^2)\) (isotropic coordinates). When \(\mathcal{P}_{x^1} = \mathcal{P}_{x^2} = 0\), the generated solution can be a perfect fluid with a dust source or an anisotropic fluid with string tension \(\mathcal{P}_\phi\). An example of a class of solutions containing both these equation of state will be given at the end of section 4.

If we want to consider more general situations, we have to choose a non-harmonic function for \(\tilde{H}\). Thus, if we start with a seed metric with \(H = \rho(x^1, x^2)\), i.e. a harmonic expression with \(\rho_{,\alpha,\alpha} = 0\), then by taking \(J(x^1) \neq x^1, F(x^2) \neq x^2\), we can build solutions with \(\mathcal{P}_{x^1} \neq -\mathcal{P}_{x^2}\). In summary, if we want to build perfect fluid sources, then a necessary but (obviously) not sufficient condition is that \(\tilde{H}\) is not a harmonic function.

This consideration means that our method is also compatible with the generation of perfect fluid sources with non-vanishing hydrostatic pressure.

Concerning the eigenvalues of (9), i.e. \(\|T_{ab} - \lambda g_{ab}\| = 0\), we have \(\lambda_\rho = P_\rho, \lambda_z = P_z\) and with the eigenvalues \(\lambda_{t,\phi}\) that can be complex conjugate. If \(K = 0\), then \(\lambda_t = -E, \lambda_\phi = P_\phi\).
2.1 An example: dust seed source

We use canonical Weyl coordinates. The line element appropriate for a stationary axially-symmetric dust source is

\[ ds^2 = e^v [d\rho^2 + dz^2] + \rho^2 d\phi^2 - (dt - N d\phi)^2, \]  

(12)

where \( N \) and \( v \) depend on the canonical Weyl coordinates \((\rho, z)\). For (12), Einstein’s equations \( G_{ab} = -T_{ab} \) give

\[ \tilde{\nabla}^2 N = 0, \quad E = \frac{e^{-v} N^2_{,\alpha}}{\rho^2}, \quad \tilde{\nabla}^2 = \partial_{,\alpha,\alpha} - \frac{1}{\rho} \partial_{,\rho}, \]

\[ v_{,\rho} = \frac{N^2_{,z} - N_{,\rho}^2}{2\rho}, \quad v_{,z} = -\frac{N_{,\rho} N_{,z}}{\rho}, \]  

(13)

In particular, the component \((\rho z)\) is

\[ G_{\rho z} = -\frac{1}{2\rho^2} \left[ N_{,\rho} N_{,z} + \rho v_{,z} \right] \]  

(14)

that, for dust sources, is vanishing. In [17], the map \( J(\rho) = \rho, \quad z \to |z| \) has been used to generate a disk immersed in a cloud of dust. Generally, after applying the map \( \rho \to J(\rho), \quad z \to F(z) \), for the component \((\rho z)\) we have

\[ \tilde{G}_{\rho z} = -\frac{1}{2J^2} F_{,z} J_{,\rho} \left[ \tilde{N}_{,F} \tilde{N}_{,J} + J \tilde{v}_{,F} \right] = 0. \]  

(15)

Therefore, the energy-momentum tensor of the so obtained metric can be written as [9] with

\[ \tilde{V}_a = \left( 0, 0, \tilde{N}, -1 \right), \]

\[ \tilde{W}_a = \left( 0, 0, J, 0 \right), \]

\[ \tilde{S}_a = \left( e^{\tilde{v}}, 0, 0, 0 \right), \]

\[ \tilde{L}_a = \left( 0, e^{\tilde{v}}, 0, 0 \right). \]  

(16)

Regarding the heat flow, we get

\[ \tilde{K} = \frac{e^{-\tilde{v}}}{2J} \left[ \tilde{N}_{,\alpha,\alpha} - \frac{\tilde{N}_{,\rho}}{J} J_{,\rho} \right]. \]  

(17)
We consider, as a simple example, the seed metric given by $N = c\rho^2 z, v = c^2 \rho^4 / 8 - c^2 \rho^2 z^2$. Setting $J(\rho) = \rho, F(z) = bz$, with $b$ a constant, for the generated solution we have the expression:

$$\tilde{N} = cbz\rho^2, \quad \tilde{v} = \frac{c^2 \rho^4}{8} - c^2 b^2 \rho^2 z^2,$$

$$\mathcal{P}_\rho = -\mathcal{P}_z = e^{-\tilde{v}} c^2 \rho^2 (1 - b^2), \quad \mathcal{P}_\phi = \frac{3}{4} c^2 e^{-\tilde{v}} \rho^2 (1 - b^2),$$

$$\mathcal{E} = \frac{c^2}{4} e^{-\tilde{v}} [16 b^2 z^2 + 7 b^2 \rho^2 - 3 \rho^2]. \quad (18)$$

Solution (18) is not asymptotically flat, but is regular and satisfies all energy conditions for $|b| \geq 1$. For $b^2 = 1$, dust solution is regained. Note that, for all the generated solutions, we have $\tilde{G}_{\rho\rho} + \tilde{G}_{zz} = -J(\rho)(\rho)$, that, for $J = \rho$, leads to $\mathcal{P}_\rho = -\mathcal{P}_z$. Incidentally, solution (18) has vanishing heat flow.

Another class of seed metrics are the ones asymptotically flat. For example, we can take the dipole Bonnor metric $N = c\rho^2(z^2 + \rho^2)^{-3/2}$ and perform a suitable map. Starting from an asymptotically flat solution, sufficient conditions to obtain asymptotically flat and regular metrics are:

$$\lim_{z \to \pm \infty} F(z) = +\infty, \quad F(z) \neq 0 \quad \forall z \in \mathbb{R},$$

for $|z| \to \infty, \quad F(z) = |z| + o(1)$

for $\rho \to 0, \quad J(\rho) = \rho + o(1)$, at infinity $J(\rho) = \rho + o(1)$. \quad (19)

To satisfy conditions (19), starting with the dipole Bonnor solution, the simplest map we can take is $J(\rho) = \rho, F(z) = \sqrt{z^2 + b^2}$, with $b$ a constant. Note that by setting $b = 0$ we are within the method depicted in [17].

### 3 Matching the Kerr metric on Cassinian ovals: mathematical machinery

As stated above, the map of section 2 does apply to isotropic harmonic coordinates $x^1, x^2$ with $\Delta_0(x^1, x^2) = 0, \Delta = \partial^2_{x^1} + \partial^2_{x^2}$. In this section we show how to apply the procedure depicted in section 2 to obtain interior solutions that can be matched on a suitable surface to the Kerr metric.

The Kerr metric with ADM mass $M$ and angular momentum for unit mass $a$ can be expressed in the Papapetrou gauge by using (with $a^2 < M^2$ black hole solution [32]) spheroidal prolate coordinates $\mu, \theta$ with $\rho = \sinh \mu \sin \theta$, $z =
cosh \mu \cos \theta:

\[ ds^2 = f^{-1}(\mu, \Omega) \left[ e^{2\gamma(\mu, \Omega)} (d\mu^2 + d\Omega^2) + \rho^2 d\phi^2 \right] - f(\mu, \Omega) [dt - \omega(\mu, \Omega) d\phi]^2, \]

and

\[ f = \frac{p^2 \cosh^2 \mu + q^2 \cos^2 \theta - 1}{(p \cosh \mu + 1)^2 + q^2 \cos^2 \theta}, \quad \omega = \frac{2q}{p} \frac{(p \cosh \mu + 1) \sin^2 \theta}{p^2 \cosh^2 \mu - 1 + q^2 \cos^2 \theta}, \]

\[ e^{2\gamma} = (p^2 \cosh^2 \mu - 1 + q^2 \cos^2 \theta), \]

and \( p = 1/M, \ q = a/M, \) with \( M^2 - a^2 = 1. \) We can apply, to the metric (20) the map \( \Omega \to F(\Omega) = \Omega \) and \( \mu \to J(\mu) \) where \( J(\mu = k) = \mu \) for \( k \in \mathbb{R}^+. \)

In this way we can regularize the interior solution and perform a matching on some spheroidal closed surface \( \mu = k. \) However, this surface is prolate and physical intuition suggests an oblate surface of rotation rather than a prolate one. In the mainstream present in the literature, the matching with the Kerr solution is performed on spheroids (prolate or oblate), but, for example, no perfect fluid interior solution has been found. It is thus evident the necessity to explore more general surfaces that could be suitable with an interior perfect fluid solution. In what follows we give the mathematical machinery to match the Kerr solution on general surfaces with the help of the technique of section 2.

To start we fix a coordinate system \( x^1, x^2, \phi, t \) with the line element in the Papapetrou form (21) together with the harmonic condition \( \Delta \rho(x^1, x^2) = 0. \) For practical purposes [22], more easy computations can be done with boundary non-null surfaces \( S \) with equation \( T(x^\alpha) = 0 \) with \( x^\alpha \) assumed to be \( x^1 \) or \( x^2 \) and with unit normal \( n_\alpha = \pm (\partial_\alpha T / \sqrt{\partial_\beta T \partial^\beta T}). \) The standard procedure [22] is to impose the continuity of the first fundamental form (the pull-back of the metric on \( S \)) and the second fundamental form \( K_{ij} \) (with \( K_{ij} = n_{ij}, \) with ";' as the covariant derivative on \( S \)). It is a simple matter to verify that a sufficient (but not necessary) way to satisfy the conditions above is to choose the metric coefficients \( g_{\alpha\beta} \) as \( C^1 \) functions on \( S \). The next step is to identify a sufficiently general and reasonable class of matching surfaces.

To this purposes, in [33] it has been shown that, within the well known Ehler method, interior rotating solutions with an energy density constant on Cassinian’s ovals quartic surfaces can be obtained. Cassinian ovals have been introduced by Giovanni Domenico Cassini in 1680 in order to substitute the ellipses with ovals to describe planets orbits. In Newtonian mechanics this task cannot be accomplished, but in general relativity, as shown in [33], Cassinian ovals can arise as a possible configuration surface suitable for
rotating bodies. Hence, we could speculate that, until now, perfect fluid Kerr interior metrics has not been obtained because non-suitable surfaces have been chosen as possible ones for rotating bodies. To this purpose, Cassinian ovals are defined as the set of points $P$ such that the product of the distances between $P$ and two fixed points $F_1 = (-\ell, 0)$ and $F_2 = (\ell, 0)$ called foci is constant. In order to obtain coordinates suitable for rotating surfaces, we write down the Cassinian equation using Weyl canonical coordinates $\rho, z$:

$$(\rho^2 + z^2 - \ell^2)^2 + 4\ell^2 z^2 = 4\ell^2 c^2,$$

(22)

with $c$ as a constant: for $c < \ell/2$ we have toroidal configurations, while for $\ell \leq c$ we have ellipsoidal-like (quartic) surfaces. We need harmonic spatial coordinates $\{m, \theta\}$ such that the boundary surface is obtained at $m = k \in \mathbb{R}$. First of all, we define $t = (\rho^2 + z^2 - \ell^2)^2 + 4\ell^2 z^2$ with $t \in [0, \infty)$. To obtain harmonic coordinates, we introduce the complex plan $\zeta$ with

$$\zeta = (\rho + iz)^2 - \ell^2 = u + iv, \quad u = \rho^2 - z^2 - \ell^2, \quad v = 2z\rho,$$

(23)

together with:

$$\zeta = \sqrt{t}(\cos \theta + i \sin \theta),$$

$$\cos \theta = \frac{\rho^2 - z^2 - \ell^2}{\sqrt{(\rho^2 + z^2 - \ell^2)^2 + 4\ell^2 z^2}}.$$

(24)

We obtain a polar representation of the coordinates $u,v$ by setting $u = e^m \cos \theta$, $v = e^m \sin \theta$ with $\sqrt{t} = e^m$ ($m \in (-\infty, \infty)$). Finally we obtain the relations between the Weyl coordinates $\rho, z$ and the harmonic Cassinian’s ones $m, \theta$:

$$\rho = \frac{Q}{\sqrt{2}}, \quad z = \frac{v}{Q\sqrt{2}},$$

$$Q = \sqrt{u + \ell^2 + \sqrt{(u + \ell^2)^2 + v^2}}.$$

(25)

With Cassinian harmonic coordinates $m, \theta$ depicting at $m = k$ Cassinian ovals, the Kerr metric can be expressed in terms of these coordinates\(^2\). However, in this paper we are only interested in presenting a further application.

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\(^2\)This can be done expressing the Kerr metric in the Weyl coordinates and thus perform the eq. (25)
of the algorithm in section 2. To this purpose, we only need to observe that in the coordinates $\mu, \theta$, the line element (20) becomes:

$$ds^2 = f^{-1}(m, \theta) \left[ e^{2\gamma(m, \theta)}(dm^2 + d\theta^2) + \rho^2(m, \theta)d\phi^2 \right] - f(m, \theta)[dt - \omega(m, \theta)d\phi]^2.$$  

To fulfill matching conditions on Cassinian ovals, the simplest map we can consider is:

$$\theta \rightarrow F(\theta) = \theta, \quad m \rightarrow J(m).$$  

(27)

With the map (27), $\theta$ is left unchanged. Starting with the metric (20) with $\Delta \rho(m, \theta) = 0$, the map (27) will transform $\rho(m, \theta) \rightarrow H(m, \theta) \neq \rho(m, \theta)$ with $H_{m,m} + H_{\theta,\theta} \neq 0$ and as a consequence $H$ is no longer a harmonic function. Thanks to equation (11), this does imply that $P_m + P_\theta \neq 0$ and more general equations of state than the ones obtained in the examples of section 2 can be obtained, leaving open the possibility to obtain perfect fluid sources. From a mathematical point of view [34], $\rho$ in (21) or (20) is nothing else but the determinant of the 2–metric $g_2$ spanned by the Killing vectors $\partial_t$ and $\partial_\phi$ and characterizes a measure of the orbits of the isometry group [34]. Hence, in order to obtain an interior source from a vacuum solution with more general equations of state than the (11), the measure of the orbits of the isometry group must be changed.

As a final step, we must specify $J(m)$ in (27), with the conditions: $\forall m \in (-\infty, k]$ $J(m) \in C^1$ and $J(m = k) = m$. Obviously, these conditions can be easily fulfilled. As an example, we can take

$$J(m) = m + w(m - k)^b, \quad w \in \mathbb{R}^+, \quad b > 1,$$

(28)

$$J(m) = m + \sum_{i=1}^{n} w_i(m - k)^{i+\epsilon}, \quad w_i \in \mathbb{R}^+, \quad \epsilon > 0.$$  

(29)

The interior Kerr solutions generated with the maps (28) and (29) are regular and smoothly match the vacuum Kerr solution at the Cassinian surfaces $m = k$. This certainly will be matter for futures calculations. We stress that, with our procedure, the finding of physically and mathematically reasonable (perfect fluid) interior rotating solutions for real astrophysical objects must require a reasonable matching surface: Cassinian ovals can offer a possible realistic solution for this issue.

$^3$Since $\sqrt{t} = e^{m}$ this does imply that the matching surface $S$ is defined by $t = e^{2k}$. 

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4 Vanishing Heat Flow

In this second part of the paper, we study the equation governing heat flow term. To the solutions generated with the technique of this section, we can apply the transformation \( x^1 \rightarrow J(x^1), x^2 \rightarrow F(x^2) \) and thus we can generate new solutions with vanishing viscosity and generally with a non-zero heat flow term.

The equation \( K = 0 \) is nothing else but

\[
N_{,\alpha,\alpha} - \frac{1}{H} N_{,\alpha} H_{,\alpha} - \frac{N}{f} \left( f_{,\alpha,\alpha} - \frac{1}{H} f_{,\alpha} H_{,\alpha} \right) = 0. \tag{30}
\]

Equation (30) can be satisfied by taking, for example \( N = f \). Generally, if we take a solution \( N(x^1, x^2), f(x^1, x^2) \) of (30), then also \( N(G, T), f(G, T) \) is, being \( G(x^1, x^2) \) an arbitrary harmonic function and \( T(x^1, x^2) \) its harmonic conjugate (see [24]).

By posing \( \omega = \frac{N}{f} \) and introducing the Ernst-like potential \( \Phi \) such that

\[
\Phi_{,x^1} = \frac{f^2}{H} \omega_{,x^2}, \quad \Phi_{,x^2} = -\frac{f^2}{H} \omega_{,x^1}, \tag{31}
\]

we see that the integrability condition \( \Phi_{,x^1} = \Phi_{,x^2} = \Phi_{,x^2,x^1} \) for (31) leads exactly to equation (30). Further, by taking the integrability condition for (31) in terms of \( \Phi \) (i.e. \( \omega_{,x^1} = \omega_{,x^2,x^1} \)), we obtain:

\[
f \left[ \Phi_{,\alpha,\alpha} + \frac{H_{,\alpha}}{H} \Phi_{,\alpha} \right] - 2 \Phi_{,\alpha} f_{,\alpha} = 0. \tag{32}
\]

By setting \( H = \rho(x^1, x^2) \), equation (32) is one of the two equations of the Ernst method for the vacuum expressed in the Papapetrou gauge, the other being

\[
f \left[ f_{,\alpha,\alpha} + \frac{H_{,\alpha}}{H} f_{,\alpha} \right] + \Phi_{,\alpha}^2 - f_{,\alpha}^2 = 0. \tag{33}
\]

For \( H = \rho \), (32) and (33) are the Ernst equations expressed in the Papapetrou gauge. Therefore, considering both (32) and (33), we have the vacuum Ernst equations, while the equation (32) without (33) is compatible with spacetimes with anisotropic pressure and vanishing heat flow and viscosity.

In order to obtain explicit solutions, we can for example separate the (30) by posing

\[
N_{,\alpha,\alpha} - \frac{1}{H} N_{,\alpha} H_{,\alpha} = 0, \quad f_{,\alpha,\alpha} - \frac{1}{H} f_{,\alpha} H_{,\alpha} = 0. \tag{34}
\]
that, after setting $H(x^1, x^2) = \rho$, becomes $\nabla^2 N = 0$, $\nabla^2 f = 0$, that for $f = 1$ reduces to the dust case. The next step is to consider the equation $G_{x^1 x^2} = 0$, with $G_{x^1 x^2}$ given by (1). This equation involves both $v_{,x^1}$ and $v_{,x^2}$ and therefore can be integrated directly by imposing the integrability condition $v_{,x^1} x^2 = v_{,x^2} x^1$ (see [24]). However, if we take, for example, $H(x^1, x^2) = H(x^2)$, the term proportional to $v_{,x^1}$ disappears in (1), obtaining

$$
v_{,x^2} = \frac{1}{f^2 H_{,x^1}} B(x^1, x^2),
$$

$$
B(x^1, x^2) = f_{,x^2} f_{,x^1} (H^2 - N^2) + f f_{,x^2} N N_{,x^1} + f f_{,x^1} N N_{,x^2} - f f_{,x^2} H H_{,x^1} - f^2 N_{,x^2} N_{,x^1}.
$$

If we set $H(x^1, x^2) = H(x^2)$, then the term involving $v_{,x^2}$ disappears, and therefore we can easily calculate $v_{,x^1}$. Obviously, to the equation (35) we can apply the map of section 2.

### 4.1 A class of asymptotically flat solutions

In what follows, without loss of generality, we adopt cylindrical coordinates with $x^1 = \rho$, $x^2 = z$.

By inspection of equations (34), we see that given a solution $N$ for the first equation, any combination $f = x + y N$ with $\{x, y\} \in \mathbb{R}$ is a solution for the second of (34). After imposing that $N \to 0$ at spatial infinity, we can obtain asymptotically flat solutions.

Hence, looking for asymptotically flat solutions without heat flow we can take, for example, for $N$ an asymptotically flat solution of the first of (34) with $N \simeq \frac{1}{\sqrt{\rho^2 + z^2}} + o(1)$ at spatial infinity. For the metric function $f$, we could take a generic linear combination of solutions $N_a$ of the first of (34): $f = 1 - k_a N_a$ (with $k_a$ constant coefficients).

As an example, we can take (Bonnor)

$$
H = \rho, \quad N = c \rho^2 (\rho^2 + z^2)^{-3/2}, \quad N \sim \frac{1}{\sqrt{\rho^2 + z^2}}, \quad f = 1 - k N
$$

With the solutions (35), we can integrate equation (35) to calculate the metric coefficient $v$. Generally, we obtain a very complicated expression for $v$, but with the correct asymptotic behaviour ($v \to 0$ at spatial infinity) suitable for asymptotically flat metrics, provided that the integration constant is chosen to be zero.

Obviously, to the solutions and the technique presented in this section, we
can apply the map of section 2. To this purpose, note that equation (30) (and (34)) is not invariant in form under the map considered in section 2 and as a result starting from a seed solution with vanishing heat flow, the map $x^1 \rightarrow J(x^1)$, $x^2 \rightarrow F(x^2)$ generally does not generate a solution with vanishing heat flow, but rather with vanishing viscosity. Hence, we can apply the transformation $x^1 \rightarrow J(x^1), x^2 \rightarrow F(x^2)$ of section 2 to spacetimes generated in this section to obtain new solutions with vanishing viscosity and non-zero heat flow. We can thus start with the Bonnor solution $N = \frac{1}{\sqrt{\rho^2 + z^2}}$ with the map (as a simple example)

$$J(x^1) = \rho, F(x^2) = \sqrt{z^2 + a^2}, a \in \mathbb{R},$$

(37)
to obtain regular solutions with anisotropic pressures and vanishing viscosity.

### 4.2 Solutions with a $G_3$ group of motion

As a physically interesting subcase, we consider solutions with a $G_3$ group of motion with cylindrical symmetry. In this case, retaining the condition $f = 1 - kN$, equation (30) becomes

$$\frac{H}{\rho} = \frac{N_{,\rho}}{N_{,\rho}}.$$  

(38)

Equation (38) is integrable. Moreover, equation $G_{\rho z} = 0$ is identically satisfied, and therefore solutions depend on the arbitrary metric function $v(\rho)$. As a title of example we consider the solution $H = \rho, N = \frac{\xi}{2} \rho^2$. We get

$$P_\rho = -P_z = \frac{(4v_{,\rho} - 4kcv_{,\rho} - 4\rho^2 kcv_{,\rho} + \rho^4 k^2 c^2 v_{,\rho} + 2c^2 \rho)}{2\rho e v(-2 + kcp^2)^2},$$

$$E = \frac{-e^{-v}}{2(-2 + kcp^2)^2}[-6c^2 + 12kc + (\rho^4 k^2 c^2 + 4 - 4\rho^2 kc)v_{,\rho}],$$

$$E + P_\phi = \frac{4ce^{-v}(c - 2k)}{(-2 + kcp^2)^2}.$$  

(39)

Note that, also in this case, thanks to (11), by taking $H \neq \rho$, we can obtain a more general equation of state than the one in (39). The regularity of (39) is fulfilled if $kc \leq 0$, and if, for $\rho \rightarrow 0$, $v(\rho)$ looks as follows: $v = -a^2 \rho^n + o(1)$, being $a, n$ numbers with $n \geq 2$. Further, energy conditions are satisfied by setting $v_{,\rho,\rho} \leq 0$: this is a sufficient but not necessary condition. In the limiting case $k = 0, v = -\frac{c^2 \rho^2}{4}$ and we regain
the van Stockum solution [15].

As a further example, we can look for solutions with string tension and vanishing radial pressure, i.e. with \( P_\rho = P_z = 0, P_\phi \geq 0 \). We have:

\[
v = \frac{2k - c}{k(2 - k\rho^2)} , \quad P_\rho = P_z = 0,
\]

\[
E = e^{-v}8c \frac{2k - c}{(-2 + k\rho^2)^3} , \quad P_\phi = e^{-v}4c^2k \frac{-2k + c}{(-2 + k\rho^2)^3}. \quad (40)
\]

Solution (40) is regular for \( c \geq 0, k \leq 0 \) and satisfies the weak and the strong energy conditions, but not the dominant energy condition. Note that solution (40), in the limit \( \rho \to \infty \), has vanishing \( E, P_\phi \) and \( v \).

As a final study, we can compare the features of the internal solutions generated in this paper with the Kerr interior one in terms of Cassinian ovals of section 3. Summarizing in a list we have:

- Kerr interior solution (26)-(29): Vanishing viscosity; non-vanishing heat flow; no curvature singularities; equation of state \( P_m + P_\theta \neq 0 \); matching smoothly to Kerr on Cassinian ovals.
- Solution (18): Vanishing viscosity; vanishing heat flow; no curvature singularities; string-like equation of state \( P_{x1} + P_{x2} = 0 \); non-asymptotically flat; no matching with Kerr exterior solution.
- Solutions (34) with (36): Vanishing viscosity; vanishing heat flow; curvature singularity at \((\rho, z) = (0, 0)\); equation of state \( P_\rho + P_z = 0 \); asymptotically flat; no matching with the Kerr metric.
- Solution (34) with the map (37): Vanishing viscosity; non-vanishing heat flow; no curvature singularities; equation of state \( P_\rho + P_z = 0 \); asymptotically flat; no matching with the Kerr metric.
- Solution (38)-(39): Vanishing viscosity; vanishing heat flow; no curvature singularities; equation of state \( P_\rho + P_z = 0 \); non-asymptotically flat; no matching with the Kerr metric.

5 Conclusions and final remarks

In this paper, we have presented methods to generate anisotropic fluids. Anisotropic sources are of great astrophysical interest, for example in the context of very compact objects at high densities. Many generating methods are present in the literature to obtain rotating fluid solutions. The most
famous one is the Ehlers method \cite{35} where new stationary exterior solutions and interior ones with a one-parameter family are obtained starting from static vacuum solutions. Within this beautiful method, generally one generates solutions with curvature singularities. In \cite{6} Geroch showed that it is possible to obtain an infinite-parameter family of solutions. In \cite{36} a technique has been presented for generating a two parameter family starting from vacuum solutions. With these techniques generally it is not easy to have a sound physical control of the so generated energy-momentum tensor when rotating sources are investigated \cite{13, 19, 37, 38, 39, 40, 41, 42}, or the generating methods have a particular specific equation of state \cite{39}. Moreover, in order to depict astrophysical objects, we need solutions representing an isolated body or a solution that can be matched to the exterior Kerr one on a suitable surface of rotation \cite{13, 19, 37, 38, 39, 40, 41}.

With the algorithms presented in this paper, we can build global solutions with and without heat flow and vanishing viscosity. In particular, under a suitable choice of the seed metric, we can also obtain global solutions that are also asymptotically flat.

In the first part of the paper, we have presented the transformation properties of a generic stationary axially symmetric line element, representing a vacuum or fluid filled solution, under the map $x^1 \rightarrow J(x^1), x^2 \rightarrow F(x^2)$. We have shown that, starting with a seed metric with vanishing viscosity, $G_{x^1x^2} = 0$, the generated solution has again $\tilde{G}_{x^1x^2} = 0$. Generally, the source of the so obtained metric is composed of anisotropic fluid with heat flow. The limit appropriate for thin disk is given by setting $F(z) = |z|$ in the Weyl coordinates. In the literature \cite{17}, the disk has been matched with the exterior dust dipole Bonnor solution. As a consequence, due to the fact that asymptotically flat dust solutions have not mass term, in the solution so obtained unavoidably will appear exotic matter with negative mass and total zero mass. Therefore, we argue that, for the reasonings made in this paper, we can obtain more realistic solutions with positive mass term and asymptotically flat, with a progress with respect to the method in \cite{17}. It is also interesting to note that for the starting seed solution, we can also take a vacuum solution of the Ernst equations, where any fluid is absent and obviously the viscosity is zero. In particular, after applying a suitable map to the Kerr metric, we can obtain regular asymptotically flat solutions representing fluids with vanishing viscosity, non-zero heat flow and, after setting $J(x^1) = \rho$, equation of state with $P_\rho = -P_z$.

Moreover, after taking $F(x^2) = \text{const.}$ (or $H(x^1) = \text{const.}$), we can build solutions with at least a $G_3$ group of motion.

In section 4 we have shown how, within the technique of section 3, we can
easily build interior solutions that can be matched to the Kerr one on suitable boundary surfaces. In particular, we present the general framework to match the Kerr solution on Cassinian ovals that are very interesting surface suitable to describe a rotating star.

Summarizing, the following possible advantages with the procedures outlined in this paper with respect to the ones usually present in the literature are:

1. We can obtain anisotropic fluids from known solutions with a $T_{ab}$ with the specific feature to have a vanishing viscosity, i.e. the friction for the fluid is absent in particular in the $x^1 - x^2$ plane by the properties of the component $G_{x^1 x^2}$ analyzed in section 2.

2. By means of the function $H$, we can obtain, thanks to equation (11), anisotropic fluids with equation of state more general than the string ones with $P_{x^1} = -P_{x^2}$.

3. As shown at the end of section 2, we can generate asymptotically flat solutions, starting from the Bonnor ones, that are regular on the whole axis of rotation and potentially solving the zero-mass problem of this class of solutions.

4. In section 4 we showed that it is possible to generate asymptotically flat solutions with also vanishing heat flow and with generally some curvature singularity. Starting from these solutions and after applying the map of section 2, we can build asymptotically flat solutions with non vanishing heat flow, but with vanishing viscosity and regular everywhere.

5. In section 3 it is argued that, thanks to relativistic effects due to the rotation, the shape of the source can be more general than the one provided by spheroids. To this regard, more general coordinates, the Cassinian ones, are introduced that permit us to explore more general revolution surfaces than the spheroidal ones expected in a Newtonian context. To these new coordinates, the algorithm of section 2 can be applied to obtain interior Kerr solutions that are regular everywhere. To the best of my knowledge, this line of research is absolutely new and is certainly matter for future investigations.

As a final consideration, we discuss the algorithm outlined in this paper with similar approaches recently appeared in the literature. Our algorithm to find interior rotating solutions and in particular the one that is capable
to be smoothly matched to the Kerr one can be related to the interesting recent works present in [26, 27, 28, 43]. The algorithm in the aforementioned papers is based on an ansatz applied to the Kerr exterior solution, and calculated at the boundary surface representing a spheroid. Also, in these papers the starting point is the line element in the Weyl-Papapetrou form [1], and the interior solution is obtained by guessing in a suitable way some component of the metric coefficients in such a way that the static solution represents a perfect fluid. In this way regular metrics, satisfying the energy conditions and representing anisotropic fluids with heat flow and viscosity are obtained. In fact, the so generated solutions have the components \( t \phi \) and \( (x^1 x^2) \) of \( T_{ab} \) non-vanishing. Hence, the generated solutions in [27, 28] can be seen in light of our results as solutions generated from guessing components of the Kerr metric that do not preserve the condition \( G_{x^1 x^2} = 0 \) assuring a vanishing viscosity. However, our technique and the one in [27, 28] show that exists a systematic way to guess a stationary vacuum solution in order to obtain a generic anisotropic source with and without viscosity and heat flow. We stress that, in order to obtain interior Kerr solutions, more general boundary surfaces can be considered. In practice, general relativistic effects can lead to a modification of the usual spheroids as boundary surfaces: Cassinian ovals, for their properties as quartic curves, are the natural candidates.

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