Some remarks on 3-partitions of multisets

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Abstract

Partitions play an important role in numerous combinatorial optimization problems. Here we introduce the number of ordered 3-partitions of a multiset \( M \) having equal sums denoted by \( S(m_1, \ldots, m_n; \alpha_1, \ldots, \alpha_n) \), for which we find the generating function and give a useful integral formula. Some recurrence formulae are then established and new integer sequences are added to OEIS, which are related to the number of solutions for the 3-signum equation.

Keywords: multiset; 3-partition of a multiset; generating function; asymptotic formula; 3-signum equation.

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1 Introduction

The \textit{signum equation} for a given sequence of integers is considered in [3], in connection with the Erdős-Surányi problem. In particular, for a given integer \( n \geq 2 \), the level \( n \) solution of this equation represents the number \( S(n) \) of ways of choosing + and − such that \( \pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0 \). This is also the number of ordered partitions of \( \{1, 2, \ldots, n\} \) in two sets with equal sums.

Andrica and Tomescu [4] conjectured an asymptotic formula for \( S(n) \):

\[
\lim_{n \to \infty} \frac{S(n)}{\sqrt{n}} = \sqrt{\frac{6}{\pi}},
\]

which was proved by analytic methods by Sullivan [11].

Starting from a problem involving derivatives, Andrica established a generating function which allowed novel approaches in the study of 2-partitions with equal sums for multisets [1]. We refer the reader to [2,3] for connections with Erdős-Suranyi representations, to [10] for general theory of multisets and to [12] for details about generating functions.

This paper is motivated by some recent results on the number of ordered 2-partitions with equal sums for multisets obtained in [5]. The study of 3-partitions of multisets differs essentially from that of 2-partitions. In Section 2 of this paper we investigate the number of ordered 3-partitions of a multiset \( M \) having equal sums, for which establish the generating function and a useful integral formula. Some particular instances related to the number of solutions for the 3-signum equation are studied in Section 3, where recurrence formulae are established and some new integer sequences are proposed.

2 3-partitions of multisets with equal sums

Partitions have direct applications to classical combinatorial optimization problems such as Bin Packing Problem (BPP), Multiprocessor Scheduling Problem (MSP) and the 0-1 Multiple Knapsack Problem (MKP) [6].

Of particular interest is the 3-partition problem, one of the famous strongly NP-complete problems [7,8]. Given a positive integer \( b \) and a set \( [n] = \{1, 2, \ldots, n\} \) of \( n = 3m \) elements, each having a positive integer size \( a_s \), such that \( \sum_{s=1}^{n} a_s = mb \). The problem has a solution if there is a partition of \( N \) into \( m \) subsets, each containing exactly three elements from \( N \), whose sum is exactly \( b \). For example, the set \( \{10, 13, 5, 15, 7, 10\} \) can be partitioned into the two sets \( \{10, 13, 7\}, \{5, 15, 10\} \), each of which sum to 30.
Here we investigate another 3-partition concept of a multiset defined for the real numbers $\alpha_1, \ldots, \alpha_n$ and the positive integers $m_1, \ldots, m_n$, denoted by

$$ M = \{\underbrace{\alpha_1, \ldots, \alpha_1}_{m_1 \text{ times}}, \ldots, \underbrace{\alpha_n, \ldots, \alpha_n}_{m_n \text{ times}}\}. $$

We call $m_s$ the \textit{multiplicity} of the element $\alpha_s$ in the multiset $M$, while the notation $\sigma(M) = \sum_{s=1}^{n} m_s \alpha_s$ represents the \textit{sum} of the elements of $M$.

\textbf{Definition 2.1} Denote by $S(m_1, \ldots, m_n; \alpha_1, \ldots, \alpha_n)$ the number of ordered 3-partitions of $M$ having equal sums, i.e., the number of triplets $(C_1, C_2, C_3)$ of pairwise disjoint subsets of $M$ such that

(i) $C_1 \cup C_2 \cup C_3 = M;$

(ii) $\sigma(C_1) = \sigma(C_2) = \sigma(C_3) = \frac{1}{3} \sigma(M).$

The number $S(m_1, \ldots, m_n; \alpha_1, \ldots, \alpha_n)$ is the constant term of the expansion of the Laurent polynomial $F(X, Y) \in \mathbb{Z}[X, Y, X^{-1}, Y^{-1}]$, defined as

$$ F(X, Y) = \left( X^{\alpha_1} + Y^{\alpha_1} + \frac{1}{(X Y)^{\alpha_1}} \right)^{m_1} \cdots \left( X^{\alpha_n} + Y^{\alpha_n} + \frac{1}{(X Y)^{\alpha_n}} \right)^{m_n}. \quad (1) $$

Indeed, assume that in the product $\left( X^{\alpha_s} + Y^{\alpha_s} + \frac{1}{(X Y)^{\alpha_s}} \right)^{m_s}$ we have selected $c_1^s$ terms equal to $X^{\alpha_s}$, $c_2^s$ terms equal to $Y^{\alpha_s}$, and $c_3^s$ terms equal to $\frac{1}{(X Y)^{\alpha_s}}$, with $s = 1, \ldots, n$, and notice that in this case we must have $c_1^s + c_2^s + c_3^s = m_s$. Such a selection contributes to the free term if and only if

$$ X^{\sum_{s=1}^{n} c_1^s \alpha_s} \cdot Y^{\sum_{s=1}^{n} c_2^s \alpha_s} \cdot \frac{1}{(X Y)^{\sum_{s=1}^{n} c_3^s \alpha_s}} = 1, $$

which is equivalent to

$$ \sum_{s=1}^{n} c_1^s \alpha_s = \sum_{s=1}^{n} c_2^s \alpha_s = \sum_{s=1}^{n} c_3^s \alpha_s. $$

This means that the sets

$$ C_j = \{\underbrace{\alpha_1, \ldots, \alpha_1}_{c_1^j \text{ times}}, \ldots, \underbrace{\alpha_n, \ldots, \alpha_n}_{c_n^j \text{ times}}\}, \quad j = 1, 2, 3, $$

represent a partition of $M$ which also satisfies property (ii) in Definition 2.1.
Ordering (1) in the increasing order of integer powers, one can write

\[ F(X, Y) = \sum_{m \in \mathbb{Z}} P_m(Y)X^m = \sum_{m \in \mathbb{Z}} Q_m(X)Y^m, \quad (2) \]

where \( P_m(Y) \) and \( Q_m(X) \) are Laurent polynomials. Also, notice that the free term of \( F(X, Y) \) is the free term of \( P_0(Y) \) and \( Q_0(X) \).

Clearly, we can write

\[ F(X, Y) = \prod_{s=1}^{n} \left( X^{\alpha_s} + Y^{\alpha_s} + \frac{1}{(XY)^{\alpha_s}} \right)^{m_s} = P_0(Y) + \sum_{m, j \neq 0} P_j(Y)X^j. \quad (3) \]

Considering \( X = \cos t + i \sin t \), in (3) and integrating with respect to \( t \) over the interval \([0, 2\pi]\), one obtains the following integral representation of the polynomial

\[ P_0(Y) = \frac{1}{2\pi} \int_0^{2\pi} \prod_{s=1}^{n} \left( X^{\alpha_s} + Y^{\alpha_s} + \frac{1}{(XY)^{\alpha_s}} \right)^{m_s} dt. \quad (4) \]

Setting \( Y = 1 \) in (3) one obtains

\[ F(X, 1) = \prod_{s=1}^{n} \left( X^{\alpha_s} + 1 + \frac{1}{X^{\alpha_s}} \right)^{m_s} = P_0(1) + \sum_{j \in \mathbb{Z}, j \neq 0} P_m(1)X^j, \quad (5) \]

which by symmetry in \( X \) and \( X^{-1} \) gives that

\[ P_m(1) = P_{-m}(1), \quad j \in \mathbb{Z}. \]

Also, from (4) we deduce that

\[ P_0(1) = \frac{1}{2\pi} \int_0^{2\pi} \prod_{s=1}^{n} \left( X^{\alpha_s} + 1 + \frac{1}{X^{\alpha_s}} \right)^{m_s} dt. \quad (6) \]

Since \( X^{\alpha_s} + 1 + \frac{1}{X^{\alpha_s}} = 1 + 2 \cos \alpha_s t \), we have

\[ P_0(1) = \frac{1}{2\pi} \int_0^{2\pi} \prod_{s=1}^{n} (1 + 2 \cos \alpha_s t)^{m_s} dt. \quad (7) \]

Note that

\[ P_0(1) = S(m_1, ..., m_n; \alpha_1, ..., \alpha_n) + R(m_1, ..., m_n; \alpha_1, ..., \alpha_n), \quad (8) \]
where \( R(m_1, \ldots, m_n; \alpha_1, \ldots, \alpha_n) \) is the sum of the coefficients different from the free term of \( P_0(Y) \). This is also equivalent to finding the number of solutions of the 3-signum equation for a multiset

\[
\sum_{s=1}^{n} \left( \sum_{j=1}^{m_s} \varepsilon_{s,j} \alpha_s \right) = 0, \tag{9}
\]

where \( \varepsilon_{s,j} \in \{-1, 0, 1\} \), and corresponds to \( P_0(1) \). Furthermore, setting \( X = 1 \) in (5) we obtain \( F(1, 1) = 3^{m_1+\cdots+m_n} = \sum_{m \in \mathbb{Z}} P_m(1) \), that is the sum of all the coefficients in all polynomials is \( 3^{m_1+\cdots+m_n} \).

3 3-partitions with equal sums of the set \( \{1, \ldots, n\} \)

When \( \alpha_s = s \) and \( m_s = 1 \) for \( s = 1, \ldots, n \) one obtains

\[
F_n(X,Y) = \prod_{s=1}^{n} \left( X^s + Y^s + \frac{1}{(XY)^s} \right) = \sum_{m \in \mathbb{Z}} P_{n,m}(Y)X^m. \tag{10}
\]

The computation of polynomials \( P_{n,m}(Y) \) can be done recursively.

**Theorem 3.1** The following recurrence is valid for \( m \in \mathbb{Z} \) and \( n \geq 1 \).

\[
P_{n,m}(Y) = P_{n-1,m-n}(Y) + Y^n P_{n-1,m}(Y) + Y^{-n} P_{n-1,m+n}(Y). \tag{11}
\]

Also, for \( m = 0 \) we have

\[
P_{n,0}(Y) = P_{n-1,-n}(Y) + Y^n P_{n-1,0}(Y) + Y^{-n} P_{n-1,n}(Y). \tag{12}
\]

**Proof.** The following formula can be established.

\[
F_n(X,Y) = F_{n-1}(X,Y) \left( X^n + Y^n + \frac{1}{(XY)^n} \right)
\]

\[
= \left( \sum_{m \in \mathbb{Z}} P_{n-1,m}(Y)X^m \right) \left( X^n + Y^n + \frac{1}{(XY)^n} \right)
\]

\[
= \sum_{m \in \mathbb{Z}} \left( P_{n-1,m-n}(Y) + Y^n P_{n-1,m}(Y) + Y^{-n} P_{n-1,m+n}(Y) \right) X^m.
\]

\[
\square
\]

From simple computations we obtain the numbers in Table 1.
\[
P_{2,0}(Y) = Y^3
\]
\[
P_{3,0}(Y) = \frac{2}{Y^2} + Y^6
\]
\[
P_{4,0}(Y) = \frac{2}{Y^2} + \frac{2}{Y^3} + 2Y + Y^{10}
\]
\[
P_{5,0}(Y) = \frac{2}{Y^2} + \frac{2}{Y^3} + 6 + 2Y^3 + 2Y^6 + Y^{15}
\]
\[
P_{6,0}(Y) = \frac{2}{Y^2} + \frac{4}{Y^3} + \frac{1}{Y} + 6 + 8Y^3 + 6Y^6 + 2Y^9 + 2Y^{12} + Y^{21}
\]
\[
P_{7,0}(Y) = \frac{8}{Y^4} + \frac{4}{Y^3} + \frac{6}{Y^2} + \frac{10}{Y} + \frac{8}{Y^2} + 10Y + 8Y^4 + 14Y^7 + 8Y^{10} +
6Y^{13} + 2Y^{16} + 2Y^{19} + Y^{28}
\]
\[
P_{8,0}(Y) = \frac{4}{Y^4} + \frac{6}{Y^3} + \frac{10}{Y^2} + \frac{18}{Y} + \frac{22}{Y^2} + \frac{22}{Y} + 18 + 22Y^3 + 16Y^6 + 18Y^9 +
18Y^{12} + 14Y^{15} + 8Y^{18} + 6Y^{21} + 2Y^{24} + 2Y^{27} + Y^{36}
\]

Table 1

Polynomials \(P_{n,0}(Y)\) and their coefficients for \(n = 2, 3, 4, 5, 6, 7, 8.\)

Setting \(Y = 1\) in (10) one obtains

\[
F_n(X, 1) = \prod_{s=1}^{n} \left( X^s + 1 + \frac{1}{X^s} \right) = \sum_{m \in \mathbb{Z}} P_{n,m}(1)X^m.
\]

By the symmetry in \(X\) and \(X^{-1}\), we obtain \(P_{n,-m}(1) = P_{n,m}(1)\) for \(m \in \mathbb{Z}\).

Also, by (12) we obtain the recurrence generating the sequence \(\{P_{n,0}(1)\}_{n \geq 1}\):

\[
P_{n,0}(1) = P_{n-1,-1}(1) + P_{n-1,0}(1) + P_{n-1,1}(1) = P_{n-1,0}(1) + 2P_{n-1,1}(1).
\]

Sequence \(P_{n,0}(1)\) has provided new context for the OEIS sequence A007576:

1, 1, 3, 7, 15, 35, 87, 217, 547, 1417, 3735, 9911, 26513, 71581, 194681, 532481, \ldots

By applying (7) to this case, one obtains the integral formula

\[
P_{n,0}(1) = S_3(n) + R_3(n) = \frac{1}{2\pi} \int_0^{2\pi} \prod_{s=1}^{n} (1 + 2 \cos st) \, dt,
\]

where \(S_3(n) = S(1, \ldots, 1; 1, \ldots, n)\) and \(R_3(n) = R(1, \ldots, 1; 1, \ldots, n)\).

The free term \(S_3(n)\) of (13) has been added by us to OEIS as A317577:

0, 0, 0, 0, 6, 6, 0, 18, 54, 0, 258, 612, 0, 3570, 8880, 0, 55764, 142368, 0, 947946,
For $n = 3k + 1$, the number $\frac{n(n+1)}{2}$ is not divisible by 3, hence $S_3(n) = 0$. The following identity holds $S_3(n) = 6 \cdot a(n)$, where $a(n)$ is sequence A112972. This is also the third row of the triangle $T(n,k)$ indexed as A275714 in OEIS.

The sequence $R_3(n)$ is new, and has the numerical values

$$1, 1, 3, 7, 9, 29, 87, 199, 493, 1417, 3477, 9299, 26513, 68011, 185801, \ldots$$

Recall that $P_{n,0}(1)$ (15) represents the free term in the expansions (10) and (13), hence corresponds to the number of solutions of the 3-signum equation

$$\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \cdots + \varepsilon_n \cdot n = 0,$$

where $\varepsilon_s \in \{-1, 0, 1\}$, $s = 1, \ldots, n$.

As the monomials in the $F_n(X,Y)$ expansion have the form $X^\alpha Y^\beta (XY)^{-\gamma}$, a term is independent of $X$ if and only if $\alpha = \gamma$. For a given $n$, we have to enumerate all the partitions $A, B, C$ of $[n]$ having the property $\sigma(A) = \sigma(C)$. The problem is equivalent to finding all triplets $(\alpha, \beta, \gamma)$ such that $\alpha, \beta, \gamma \geq 0$, $\alpha = \gamma$ and $\alpha + \beta + \gamma = \sigma([n])$.

For example, when $[n] = \{1, 2, 3, 4\}$ we have $\sigma([n]) = 10$. Table 2 presents all such partitions and the possible configurations $\varepsilon_s$, $s = 1, \ldots, 4$ with (16). This also clearly illustrates that we have $P_{4,0}(1) = 7$.

| $\alpha$ | $\beta$ | $\gamma$ | $A$ | $B$ | $C$ | $\varepsilon_1$ | $\varepsilon_2$ | $\varepsilon_3$ | $\varepsilon_4$ | Multiplicity |
|----------|---------|----------|-----|-----|-----|----------------|----------------|----------------|----------------|--------------|
| 0        | 10      | 0        | $\emptyset$ | $[n]$ | $\emptyset$ | 0 | 0 | 0 | 0 | 1 |
| 3        | 4       | 3        | $\{1,2\}$ | $\{4\}$ | $\{3\}$ | 1 | 1 | -1 | 0 | 1 |
|          |         |          | $\{3\}$ | $\{4\}$ | $\{1,2\}$ | -1 | -1 | 1 | 0 | 1 |
| 4        | 2       | 4        | $\{1,3\}$ | $\{2\}$ | $\{4\}$ | 1 | 0 | 1 | -1 | 1 |
|          |         |          | $\{4\}$ | $\{2\}$ | $\{1,3\}$ | -1 | 0 | -1 | 1 | 1 |
| 5        | 5       | 5        | $\{1,4\}$ | $\emptyset$ | $\{2,3\}$ | 1 | -1 | -1 | 1 | 1 |
|          |         |          | $\{2,3\}$ | $\emptyset$ | $\{1,4\}$ | -1 | 1 | 1 | -1 | 1 |

Table 2: Partitions of $[n]$ into 3 subsets when $n = 4$ and $k = 3$.

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