Growth and inequalities in a physicist’s view

Angelo Tartaglia

INAF and Politecnico di Torino

Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Abstract

The constraints on a continuous growth in a finite environment are formally analyzed, adding the effect of the necessary dynamics of costs.

The unavoidable global collapse is deduced. The effect of competition in a growing economic system on the evolution of unequal share of wealth is also analyzed and discussed, showing the necessity of increase of inequalities under the premises of growth and competition.

*Electronic address: angelo.tartaglia@polito.it
I. INTRODUCTION

A general paradigm of contemporary economy is of course growth and an old open question is about its global sustainability over an indefinite future. On one side ordinary common sense should immediately suggest that infinite growth in a finite environment is impossible, on the other one should clarify what kind of ”growth” we are speaking about. The issue has been debated since long time ago, at least for some peculiar aspects as, mainly, the world population. Thomas Malthus [1] focused on the long term consequences of a mismatch between the growth rates of the human kind and of the food production. Apparently no worry was presented on the possibility that primary resources could globally become exhausted, probably under the feeling that their amount was practically infinite; an attitude understandable at the end of the 18th century, but totally unjustified in today’s globalized economy. The problem of growth, still referred to the world population, was faced again fifty years later, by Pierre Verhulst [2], presenting the only possible growth law in limited availability of food conditions.

The very problem of global sustainability of growth, at least in the attention of the general public, was discussed, more than one century after the above quoted examples, in *The limits to growth* [3] authored by Donella Meadows, Dennis Meadows, Jørgen Randers and William Behrens III on behalf of a research group at MIT. The study had been commissioned by the Club of Rome and was based on the then novel approach of *System Dynamics* applied to the world economy as a complex system of inter-human relations embedded into a wider material complex system corresponding to the environment where humans live in.

Here I would like to discuss once more the dynamics of growth from a strictly physical point of view. The starting point is the remark that whatever one means by ”economical growth” it has unavoidably a material basis: in a way or another the growth implies an increase of the amount of matter manipulated and transformed, then an increase of the amount of energy used to transform and move matter around. In the rest of the paper the units, be it explicitly or implicitly, will be physical units, such as kilograms, joules and so on. These units are unaffected by inflation, deflation and the like. No trend discussed in the following will be expressed in terms of money, since the latter is de facto a human convention regulating the reciprocal right of access of humans to goods, services and resources.

Once we decide to stay with matter and energy we must take note that they are governed
and constrained by laws which are not decided by parliaments or dictators, are not affected by the ups and downs of stock exchanges nor are they sensible to schools of thought or journalistic comments. Laws constraining matter and energy are *discovered*, not decided, by science. For what matters here, the main constraint was formulated by Lavoisier [4], inspired by what he saw in chemical reactions, but somehow echoing ancient formulations going back to Lucretius ("...nullam rem e nihilo gigni...": from nothing comes nothing) [5] and earlier to Empedocles (5th century BC). Wording this constraint as "*Nothing can be created or destroyed; everything is transformed*" we directly include both matter and energy, so complying with modern relativity which states the equivalence of the two. By the way, the matter-energy conservation law cannot be overthrown by any technological progress, because technology applies science, does not trespass it.

One more precision is in order, concerning the "container" of our socio-economic system, i.e. our planet. The earth is a closed, non-isolated system. "Closed" means that the exchange of matter with the outside is negligible with respect to the total mass. Non-isolated means that the earth exchanges energy with the outside in the form of radiation: the input comes from the sun (not considering a marginal contribution from cosmic rays, not originating from our star); the output is again in the form of radiation emitted outwardly by the ground and the atmosphere. The approximate balance between the two fluxes is governed by the laws of thermodynamics treating the earth as a "gray body".

An additional constraint concerns transformations which are so important both for matter and for energy. Transformation processes are governed by the *second principle of thermodynamics* which was initially formulated in the middle of the 19th century referred to thermal engines, but can be generalized to all processes in complex systems, relying on Boltzmann's statistical formulation of thermodynamics. For what matters here, the principle may be colloquially explained as follows. Whenever you start a physical process aiming at converting energy into something you deem useful to you (let us call it "work") you never can transform the initial amount of energy into work completely: there will always be (even in ideally perfect conditions) some "waste" you will disperse in the environment. Most often the "waste" will be residual non-retrievable heat; more generally it will be "disorder" in a form or another (technically: *entropy*). In a closed and isolated context the "waste" (which includes ordinary garbage) will accumulate; if you wish to keep your living space in order, you need to get rid of the "waste" somehow throwing it out of the window. The way nature
expels "disorder" pursuing new dynamical equilibrium states is raising the thermodynamical
temperature\(^1\).

Usually the above is consider as having little to do with economics. In what follows I
shall show it has.

II. GROWTH IN A FINITE ENVIRONMENT

Let us consider the earth as a container filled up of something I shall call "primary
resources", which include matter in any form and energy as well; the total amount of "re-
sources" be \(S\). Then let us start some process converting "primary resources" into "goods",
which means anything deemed useful or of interest to humans (including services, which
always have a material basis). I stress the fact that the mentioned process is not a peculiar
one, but rather the set of all single processes activated for specific production chains. The
picture is complete when we add a continuous push toward steadily increasing the quantity
of "goods", \(W\); leave for the moment aside any negative feed back or side effect.

The simplest growth dynamics, under these conditions, is the same as that described
by Verhulst for the world population with a finite food availability. In an elementary time
\(dt\) the increase \(dW\) is proportional to the existing stock of "goods" (every single existing
asset concurs to the global growth). To say better: the pure proportionality is corrected
by a factor feeling the proximity to the "roof" \(S\) and tending to 0 while approaching \(S\). In
symbols it is:

\[
dW = \alpha \left(1 - \frac{W}{S}\right) W dt
\]

Of course if the "primary resources" are infinite \((S \to \infty)\) the relation is a sheer propor-
tionality.

The constant \(\alpha\) is the relative initial growth rate. It is convenient to normalize the
quantity of "goods" to the total available stock of "primary resources" introducing the
variable:

\(^1\) This mechanism should not be confused with the greenhouse effect. The former produces an increase of
the global temperature; the latter leaves the global temperature unchanged but modifies the temperature
profile from the low layers to the high atmosphere.
The basic relation then becomes:

\[ dw = \alpha (1 - w) w dt \]  

As it is well known, integrating (3) one obtains a logistic curve:

\[ w = \frac{1}{1 + q e^{-\alpha t}} \]  

The constant \( q \) is related to the initial value of \( w \): \( w_0 = w(t=0) \). It is

\[ q = \frac{1}{w_0} - 1 \]  

then

\[ w = \frac{w_0}{w_0 + (1 - w_0)e^{-\alpha t}} \]  

When \( w_0 \ll 1 \) and we are close to the origin \((t \ll 1/\alpha)\) the trend is similar to an exponential: \( w \approx w_0 e^{\alpha t} \).

Fig. (1) exemplifies the growth evolution I have described.

Just to fix numbers for drawing the graph and without attaching too much relevance to the choice, I have assumed \( w_0 = 10^{-4} \) which is the same order of magnitude as the ratio between the total present energy consumption of the human kind and the incoming flux of radiative energy from the sun. The reference growth rate has been chosen to be 3% per year. The highest is \( \alpha \), the sooner the curve reaches its inflection point (at time \( t = t_i \)); it is

\[ t_i = \frac{1}{\alpha} \ln \frac{1 - w_0}{w_0} \]  

In the example of Fig. (1) it is \( t_i = 307 \) years.

III. "COSTS"

The dynamics described in the previous section illustrates a principle situation evidencing a basic mechanism, but is, strictly speaking, unrealistic, because, as declared, it does not take
FIG. 1: Logistic evolution of the amount of "goods" produced in a finite resources scenario. The dashed line represents the exponential trend approached in the initial phase of the process. The reference growth rate has been $\alpha = 0.03$ per year; $w_0 = 10^{-4}$.

into account any back-reaction or side effect. Any physical growth mechanism requires that part of the primary resources, as well as part of the goods globally produced, be destined to insure and preserve the efficiency of the conversion process. You may include in this need maintenance, safety and the like. I shall call this quota of resources/goods "costs", $C$, reminding once more that these costs are not measured in terms of money but using physical units.

The importance of "costs" and their specific nature varies according to different peculiar processes, but here we are considering the global conversion system from resources to goods and we are interested in general features governing all processes.

Whenever we wish to increase some physical entity we need to perform some work which is proportional to the change we desire to produce and to the size of the variable we plan to increase. Using the symbols introduced so far and expressing also $C$ in terms of the total amount of resources, the elementary balance between "costs" and growth is

$$dC = \beta w dw$$

where $\beta$ is an appropriate constant.

Integrating, we obtain that
\[ C = \frac{\beta}{2} w^2 \quad (9) \]

which means that "costs" grow faster than the interesting variable.

This rule is general and the examples in the physical world are numerous. The simplest may be found in mechanics, where kinetic energy \( T \) depends quadratically on the speed \( v \) and \( \beta \) is the mass \( m \) of the object which moves:

\[ T = \frac{m}{2} v^2 \quad (10) \]

Doubling the speed of a mass quadruples its kinetic energy: the cost of a higher speed, in terms of energy, grows faster than the speed does.

Another simple example is electric current \( I \). When a current flows in a wire part of its energy is converted into waste heat \( Q \) in the wire, so that you have to lose part of the initial energy. The heat is proportional to the square of the current (Joule’s law):

\[ Q = \frac{R}{2} I^2 \quad (11) \]

If the current doubles, the energy you have to pour in to compensate for the waste heat quadruples. Now \( R \) is Ohm’s resistance of the wire.

Most often people concentrate on \( \beta \) trying and reduce it as much as possible, but disregarding the square law which is the real problem whenever the system pretends to grow.

So far we have seen that Eq. (9) is a general rule for any single physical process, however there is more when we have to deal with complex systems.

Global economy is undoubtedly a very complex system that we can schematize by a great number of knots, i.e. places (factories, workshops, agencies ...) where primary resources are converted into "goods", and by a big number of links among the knots along which matter and energy (raw materials, ware, people...) flow.

For the processes in the knots and the "current" along the links the general cost law (9) holds, but now another aspect related to complexity enters the scene.

A simple way to measure the complexity of a network is to count the relations or links among the knots. Including all possible connections \( r \) we see once more that their number depends quadratically on the number of knots, \( n \):
The number of actual links does not necessarily coincide with all possible links \( r = \frac{1}{2} n (n - 1) \) (12), however in a system which is pushed to grow the trend is towards saturation of the number of links, then saturation of the flow across each link and of the production capacity of each knot, finally toward increasing the number of knots. Growth implies also a growth of complexity, so, summing up and combining (9), holding for each element, with (12), holding for the whole network, we infer that the cost to keep the system working grows more than quadratically with respect to the output of the system:

\[
C \geq \frac{\beta}{2} w^2
\]  

(13)

Optimistically staying with the lower limit and recalling Eq. (6) we explicitly write the time evolution of the ”costs” of a growing system in a finite environment:

\[
C = \frac{\beta}{2} \frac{w_0^2}{[w_0 + (1 - w_0) e^{-\alpha t}]^2}
\]  

(14)

Constant \( \beta \) can be expressed in terms of the initial conditions. Suppose that at time 0 the initial cost \( C_0 \) be a given fraction \( \varepsilon < 1 \) of the initially available ”goods”:

\[
C_0 = \varepsilon w_0
\]  

(15)

Using (14) and (15) we get:

\[
\beta = 2 \frac{\varepsilon}{w_0}
\]  

(16)

then

\[
C = \frac{\varepsilon w_0}{[w_0 + (1 - w_0) e^{-\alpha t}]^2}
\]  

(17)

What happens with growth while time goes on is shown in Fig. (2), drawn using the same numerical values as for Fig. (1) and assuming that the initial ”cost” be 1% of the initial stock of ”goods”.

With the data in the example we see that the conflict between production and costs is reached well before the inflection point of the logistic: here around year 154. When the
Fig. 2: The black line reproduces the "goods" \( w \) and is the same as in Fig. (1). The red line represents the "costs" \( C \) with an initial value equal to 1% of the "goods".

Initial stock of "goods" is much smaller than the total amount of primary resources, the most stringent constraint comes from the costs dynamics rather than from the residual availability of resources.

IV. BENEFITS

The discussion in the previous section has started from the fact that part of the "goods" globally produced must be destined to keep the production process going. This can also be interpreted saying that the actual "advantage" or "profit" or "gain" \( G \) of the process is the difference between the gross production \( w \) and the necessary cost \( C \).

Recalling previous formulae we have:

\[
G = w - C = \frac{w_0}{w_0 + (1-w_0)e^{-\alpha t}} \left( 1 - \frac{\varepsilon}{w_0 + (1-w_0)e^{-\alpha t}} \right)
\]  

(18)

The curve is shown in Fig. (3) and uses the same numerical values as in the previous graphs.

A curve like the one in Fig. (3) is empirically known since a long time, based on observation of social or personal dynamics or on studies on human civilizations. So much so that it has been nicknamed "Seneca’s curve" [6] from a sentence written by the Roman philoso-
pher Lucius Annaeus Seneca to his pupil Lucilius: "incrementa lente exeunt, festinatur in damnun" (increases are of sluggish growth, but the way to ruin is rapid) [7]. Examples of rise and fall of historical civilizations, where the decline is much shorter than the rise, may be found for instance in Ref. [8].

Here I have not considered special cases, but the global economy and I have highlighted a basic mechanism necessarily driving growing physical systems to collapse.

V. INEQUALITIES

A recurring worry often recalled and discussed is about income inequalities. World statistics tell us that inequalities have been increasing everywhere in the last forty years or so (see Fig. 4). This trend is present in countries of different continents, with different kinds of government or regimes, and different governance of the economy, such as USA, on one side, and China, on the other.

The curves we see in Fig. 4 are irregular and noisy depending on local economical dynamics and expedients and episodical policies aimed at redistributing income, but the underlying trend looks similar for all.

The question I would like to address here is: is there any common mechanism at the base of the generation and increase of differences?

We have already discussed the dynamics of growth which is a central requirement of
FIG. 4: Share of national income in various areas of the world since 1980.

globalized economy. The other essential ingredient is *competition*, seen as being the main engine of "progress". Let me then try and analyze, always from a physical viewpoint, the dynamics of competition in economies striving for growth.

The situation is extremely complicated, but let me reduce the problem to the essence and consider just two contenders, labelled 1 and 2. Two forms of competition will be discussed. The first one is in a sense a form of passive competition: each competitor works to transform primary resources, whose stock is unique, into "goods"; each competitor wants to grow; each one’s product is of its exclusive pertinence, but the contenders do not directly interact with one another.

Adapting the approach in (3) to the new situation, I write

\[
\begin{align*}
   \frac{dw_1}{dt} &= \alpha_1 w_1 (1 - w) \\
   \frac{dw_2}{dt} &= \alpha_2 w_2 (1 - w)
\end{align*}
\]

\tag{19}

where at any moment it is \( w_1 + w_2 = w \).

Applying for \( w \) the logistic growth of Eq. 6 any of the equations in (19) (index \( i \) is either 1 or 2) becomes

\[
\frac{dw_i}{dt} = \alpha_i w_i \left( 1 - \frac{w_0}{w_0 + (1 - w_0) e^{-\alpha t}} \right) dt
\]

\tag{20}
Integrating we get

$$w_i = \frac{w_{i0}}{(w_0 + (1 - w_0) e^{-\alpha t})^{\frac{\alpha_i}{\alpha}}}$$  \hspace{1cm} (21)

What happens in time depends on the values of $\alpha_1$ and $\alpha_2$. If they both equal $\alpha$, the initial difference between the competitors grows in its absolute value following a logistic curve, but stays fixed as a fraction of the total amount of ”goods”: relative differences are frozen. If the basic growth rates of the competitors are different, under the condition of a fixed total amount of primary resources, they cannot be independent from one another, but in any case the one who has the higher value of $\alpha_i$ prevails and continuous to grow following a logistic trend; the difference between the two also grows logistically.

Considering costs and supposing their initial value to be the same fraction $\varepsilon$ of the initial stock of ”goods” for both contenders, it is

$$C_i = \varepsilon \frac{w_i^2}{w_{i0}}$$  \hspace{1cm} (22)

and the typical ”gain” is

$$G_i = w_i \left(1 - \varepsilon \frac{w_i}{w_{i0}}\right)$$  \hspace{1cm} (23)

Each competitor follows its own Seneca’s curve; the one who has a higher basic growth rate meets its collapse earlier than the other.

So far however the situation is not really competition. In a more realistic situation, when two competitors meet one wins and the other loses; part of the ”goods” of the loser are transferred to the winner. In statistical terms the highest probability is that the winner coincides with the stronger one, i.e. the one who has more ”goods”; in the long run this is what happens. A simple way to formally describe the dynamics of growth in these conditions is to write

$$\begin{cases}
    dw_1 = \alpha w_1 (1 - w) \, dt + \gamma (w_1 - w_2) \, w_2 \, dt \\
    dw_2 = \alpha w_2 (1 - w) \, dt - \gamma (w_1 - w_2) \, w_2 \, dt
\end{cases}$$  \hspace{1cm} (24)

Now $\gamma$ is an empirical (positive) parameter measuring the efficacy of the difference in wealth of the competitors in producing the transfer of ”goods” from the loser to the winner. System (24) is written conventionally assuming that the upper hand is with competitor 1,
who gains a contribution proportional to the assets of 2 (which is correspondingly lost by 2). For simplicity we also assume that the basic growth rate $\alpha$ is the same for both.

Summing and subtracting the equations in (24) from each other, recalling that $w_1 + w_2 = w$ and introducing $\Delta = w_1 - w_2$ so that $w_2 = \frac{w - \Delta}{2}$ we get:

$$\begin{cases}
\frac{dw}{dt} = \alpha w (1 - w) \\
\frac{d\Delta}{dt} = \alpha \Delta (1 - w) + \gamma \Delta (w - \Delta)
\end{cases}$$

(25)

The first equation is once more Eq. (3) whose solution is (6). Let us introduce this result into the second equation; we are left with

$$\frac{d\Delta}{dt} = \alpha \Delta (1 - w_0) + \left(1 - \frac{w_0}{w_0} + e^{-\alpha t}ight)$$

(26)

The solution is:

$$\Delta = \left(1 - \frac{w_0}{w_0}\right)^{1 - \frac{\gamma}{\alpha}} e^{-\alpha t} \left(\frac{1}{\Delta_0 - \frac{1}{w_0}} + \left(1 - \frac{w_0}{w_0} + e^{-\alpha t}\right)^{\frac{\gamma}{\alpha} - 1}\right)$$

(27)

Calling in the initial value of the difference between the competitors, $\Delta_0$, we may get rid of the integration constant $K$:

$$K = \left(1 - \frac{1}{w_0}\right) (1 - w_0)^{1 - \frac{\gamma}{\alpha}}$$

(28)

so that

$$\Delta = e^{-\alpha t} \left(\frac{1}{\Delta_0 - \frac{1}{w_0}} + \left(1 - \frac{w_0}{w_0} + e^{-\alpha t}\right)^{\frac{\gamma}{\alpha} - 1}\right)$$

(29)

Now the behaviour is maybe clearer returning to:

$$\begin{cases}
w_1 = \frac{w + \Delta}{2} = \frac{w_0}{2w_0 + 2(1 - w_0)e^{-\alpha t}} + \frac{e^{-\alpha t}}{2} \left(\frac{1}{\Delta_0 - \frac{1}{w_0}} + \left(1 - \frac{w_0}{w_0} + e^{-\alpha t}\right)^{\frac{\gamma}{\alpha} - 1}\right)\\
w_2 = \frac{w - \Delta}{2} = \frac{w_0}{2w_0 + 2(1 - w_0)e^{-\alpha t}} - \frac{e^{-\alpha t}}{2} \left(\frac{1}{\Delta_0 - \frac{1}{w_0}} + \left(1 - \frac{w_0}{w_0} + e^{-\alpha t}\right)^{\frac{\gamma}{\alpha} - 1}\right)
\end{cases}$$

(30)

The time evolution is visible in Fig. 5 where $\Delta_0$ has been assumed to be $1/3$ of $w_0$ so that $w_{10} = 2w_{20}$, and $\gamma = 0.01$. 

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FIG. 5: The black solid line is the time evolution of $w_1$; the red line is $w_2$. The dotted line, sum of the other two, is the logistic growth of the global production of "goods".

Including costs and looking at "gains", under the assumption that the initial "costs" are the same fraction $\varepsilon$ of $w_{i0}$, the relevant equations are:

\[
\begin{align*}
G_1 &= w_1 - 2\frac{\varepsilon}{w_0+\Delta_0} w^2_1 \\
G_2 &= w_2 - 2\frac{\varepsilon}{w_0-\Delta_0} w^2_2
\end{align*}
\]  \hspace{1cm} (31)

Reproducing the curves in a graph I obtain Fig. [6] were the numerical values are the same as in previous figures, including $\varepsilon = 0.01$.

VI. CONCLUSION

The real world with which humans interact is indeed a quite complicated system: cause/effect relationships are in general non-linear and the evolution of the whole system is basically chaotic (in the technical meaning of the word). I have considered extremely simple situations, with the aim of highlighting the fundamental mechanisms of the machine’s operation. Numerical values used in the examples are arbitrary but not entirely unrealistic, so that also the time scale of the figures is plausible. Those mechanisms are embedded in the real world and all superposed noises and non-linearities can camouflage them in various ways, but never subvert their essence and implications. Certain initial conditions invariably produce certain consequences.
FIG. 6: The black solid line is the time evolution of $G_1$; the red line is $G_2$.

The initial axioms of our globalized economy are two: growth and competition. The arena in which the global game is played is finite. As we have seen, the dynamics of production and costs leads the system to collapse and the details of collapse are irrelevant. Adding competition, we have seen that the situation does not change as regards the final outcome but in addition income inequalities grow up to the final collapse phase. This result has been exposed in terms of equations and graphs, but it is also perfectly and intuitively exemplified by the Monopoly game: at the beginning all players are approximately at the same level; in the end the winner has everything and the others are left with nothing.

If we do not like the ending, we have to change the initial conditions, intended as the rules of the game. The debate on these issues is normally encumbered by mountains of political, social, emotional, rhetorical factors, including some sort of faith in magics and the irrational refusal to look further beyond the immediate and local context. Unfortunately no irrational emotionality is able to influence those parts of the rules of the game that are not under our jurisdiction. This is physics.

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[8] Joseph A. Tainter, *The collapse of complex societies*, Cambridge University Press (1988)