Optimal spin squeezed steady state induced by the dynamics of non-hermitian Hamiltonians

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Abstract
In this work, we study the time evolution of a Coherent Spin State under the action of a Non-hermitian Hamiltonian. The Hamiltonian is modeled by a one-axis twisting term plus a Lipkin-type interaction. We show that when the Lipkin interaction is switched on, depending on the relative values of the coupling constants, the initial state evolves into a Steady Spin Squeezed State which minimizes the Uncertainty Relations, intelligent spin state. We apply this result to look for the generation of a steady intelligent spin state from an ensemble of nitrogen-vacancy colour centers in diamond coupled to a mechanical resonator.

Keywords: non-hermitian dynamics, optimal spin squeezing, hybrid physical systems, one-axes twisting, Lipkin-type interactions

1. Introduction
The one-axis-twisting (OAT) and the two-axis-twisting mechanisms have been introduced by Kitagawa and Ueda [1] to establish the concept of Spin Squeezed States and the fundamentals for their generation. From the theoretical point of view, squeezing is closely related to the analysis of Heisenberg Uncertainty Relations. It means that given a physical system, one may be interested in the minimization of the fluctuation of an observable at the expense of the increment of the fluctuation of the conjugate variable.

Since the pioneering work of Kitagawa and Ueda [1], many authors have contributed to the understanding [2–4] and to the experimental achievement of spin squeezing in atomic systems [5–7]. Recently, the interest in the study of these mechanisms has been renewed [8–19]. The characterization of spin squeezing is relevant in the analysis of potential candidates to be used in the architecture of quantum computing devices [20]. In a series of works, it has been reported the generation of Steady Spin Squeezed States in dissipative spin systems [21–31]. As an example, we can mention the analysis of phase coherence and spin squeezing in collective spin systems which are governed by OAT Hamiltonian with decay [4, 13, 15, 16, 18, 19, 21] or in systems governed by Non-hermitian Lipkin–Meshkov–Glick Hamiltonian (LMG) [10, 23, 24]. Similar results were found in the study of the behaviour of dissipative hybrid systems [25, 28, 31–38]. The reported works can be taken as an indicator that non-hermitian dynamics can be used to improve the achievement of spin squeezing in different systems.

The search for Spin Squeezed States with minimum uncertainty relations has given rise to the notion of Intelligent spin state (ISS) [39]. The first references in the literature to intelligent states is the paper of C. Argone and co-workers [39]. A considerable amount of work was devoted to the study of both the properties of ISS [40] as well as to the construction of such states [41–45]. In this work, we analyse the generation of a steady ISS in a system of spins which interact through a Non-hermitian OAT Hamiltonian plus a LMG interaction. As a physical application, we propose to search for steady ISS in diamond nanostructures [46–51].

Among other proposals, nitrogen-vacancy (NV) centers in diamond may be useful in solid quantum information processing, due to their long coherence time and to the high feasibility in their manipulation [32–34, 52–57]. The generation of entanglement among NV centers in diamond has been achieved by different mechanisms. The coupling of pairs of NV centers has been obtained directly by dipole–dipole
The coherent coupling of an ensemble of NV centers to a superconducting resonator has been reported in [60]. Also, the coupling of two separated NV electron spin ensembles in a cavity quantum electrodynamics system has recently been observed [61]. Another novel mechanism to generate long-range spin–spin interactions in NV centers in diamond has been proposed in [50]. In [50], the interaction among NV centers is mediated by their coupling via strain to the vibrational mode of a diamond mechanical nanoresonator. In the same direction, in [46, 47] it has shown that under the action of an effective phonon-induced spin–spin interaction for the ensemble of NV colour centers in diamond, the initial state evolves into a steady state that behaves as a Spin Squeezed State. In this work, we model the interaction of an ensemble of NV centers in diamond coupled to a mechanical resonator by an effective OAT plus LGM Hamiltonian for the NV centers. We investigate the possibility of the generation of a steady ISS from the time evolution of an initial prepared Coherent Spin State (CSS) under the action of this effective Hamiltonian.

The work is organized as follows. The details of the general formalism are presented in section 2. The results of the calculations are presented and discussed in section 3. In section 3.1, we present the numerical results that we have obtained from the exact diagonalization of the proposed Hamiltonian. In sections 3.2 and 3.3 we study some analytical results, in order to understand the mechanism of generation of a steady ISS. In section 3.2, the time evolution and the asymptotic behaviour of an initial CSS under the action of a Non-hermitian Hamiltonian are discussed. In section 3.3, we study the behaviour of the system, when the LMG interaction is taken into account, by performing a boson mapping and keeping terms to dominant order in the number of spins. In doing so, we explore the behaviour of the steady state on the different parameters of the model. In section 3.4 we propose a scheme to couple an ensemble of NV centers to a mechanical resonator so that the system can be modelled by an effective phonon-mediated interaction, which consists of an OAT plus a LGM interaction. We discuss the generation of a steady ISS for this effective model. Our conclusions are drawn in section 4.

2. Formalism

Let us consider a general collective system consisting of 2S elementary 1/2-pseudo-spins [32–34, 36]. The collective pseudo-spin of the system, \( S = (S_x, S_y, S_z) \), is governed by the cyclic commutation relations \( [S_i, S_j] = i \epsilon_{ijk} S_k \), where the suffixes \( i, j, k \) represent the spin components in three orthogonal directions and \( \epsilon_{ijk} \) is the Levi-Civita symbol. We shall assume that the physical properties of the system can be modeled by the Hamiltonian

\[
H = H_{\text{OAT}} + H_{\text{LMG}} + H_z, \tag{1}
\]

with

\[
H_{\text{OAT}} = \chi S_z^2, \tag{2}
\]

\[
H_{\text{LMG}} = V (S_x^2 - S_y^2), \tag{3}
\]

\[
H_z = (\epsilon - i \gamma)(S_x + S_y). \tag{4}
\]

The term \( H_{\text{OAT}} \) of equation (2) is a one-axis twisting mechanism with coupling constant \( \chi \) associated to the generation of spin squeezing [1]. The term \( H_{\text{LMG}} \) of equation (3) stands for a Lipkin-type interaction [62, 63].

In many situations, the interaction of a physical system with its environment cannot be neglected. A widely used method to deal with the so-called open quantum systems is the Feshbach projection operator formalism [64]. It can be summarized as follows. The whole Hilbert space of the system can be broken in two subspaces, the subspace of the localized discrete system and the subspace of its environment. The problem in the whole Hilbert space (localized system embedded in a well-defined environment, which is described by an hermitian Hamiltonian operator), is represented in the interior of the localized part of the system, after applying the Feshbach’s formalism, by an effective Non-hermitian Hamiltonian. Generally, the eigenstates of this effective Non-hermitian Hamiltonian are a set of states with complex eigenvalues. The real components of these eigenvalues define the energies of the different states, while the imaginary parts are inversely proportional to their lifetimes. The reader is kindly referred to [65] for further theoretical details and to [66] for an update of recent applications.

We shall assume that the particles of the proposed system, as a consequence of the interaction with its environment, have a finite lifetime which can be modeled by \( H_z \) of equation (4) [23]. The parameter \( \gamma \) is given by the inverse of the mean life of the states.

From the theoretical point of view, different physical systems can be modeled by Hamiltonians closely related to one proposed in equation (1), i.e. a system of two-component atomic condensates [8, 9, 12, 15], or an ensemble of NV centers coupling via a mechanical resonator [46–48, 50, 51].

The Hamiltonian of equation (1) can be diagonalized exactly in the basis of states \( \mathcal{A}_k = \{ |k\} \) with

\[
|k\rangle = |S, -S + k\rangle = \left[ \frac{(2S - k)!}{(S + k)!} \right]^{1/2} S_z^k |S, -S\rangle. \tag{5}
\]

In this basis

\[
S^2|k\rangle = S(S + 1) |k\rangle, \quad S_z|k\rangle = (-S + k)|k\rangle. \tag{6}
\]

2.1. Time evolution

As pointed out before, dissipative dynamics can be captured by introducing non-Hermitian descriptions. The Hamiltonian of an open quantum system consists formally of a first order interaction term describing the closed (isolated) system with discrete states and a second-order term caused by the interaction of the discrete states with the environment. By using the Feshbach [64] projection operator formalism, the solution of the problem in the whole function space (with discrete as well as scattering states, and an Hermitian Hamilton operator \( \mathcal{H} \)) can be represented in the interior of the localized part of
the system by the set of eigenfunctions of an effective Non-
hermitian Hamiltonian.

As the Hamiltonian of equation (1) is non-hermitian, we have
\[ H|\tilde{\phi}_a\rangle = E_a|\tilde{\phi}_a\rangle, \quad (7) \]
and
\[ H'|\tilde{\psi}_a\rangle = T_a|\tilde{\psi}_a\rangle. \quad (8) \]
Both sets of eigenstates, \( \mathcal{A}_H = \{ |\tilde{\phi}_a\rangle \} \) and \( \mathcal{A}_{H'} = \{ |\tilde{\psi}_a\rangle \} \), are non-orthonormal basis of the Hilbert space, \( \mathcal{H} \). It is straightforward to prove \cite{65, 66, 67} that
\[ T_a = \tilde{E}_a^*. \quad (9) \]
Also, the set \( \{ |\tilde{\psi}_a\rangle, |\tilde{\phi}_a\rangle \} \) forms a bi-orthonormal basis of \( \mathcal{H} \), with
\[ \langle \tilde{\psi}_a|\tilde{\phi}_b\rangle = \delta_{a,b}. \quad (10) \]
The spectrum of the Hamiltonian of equation (1) depends on the value of the parameters \( \chi, \nu, \epsilon \) and \( \gamma \) \cite{68}. If \( \epsilon = 0 \), the Hamiltonian \( H \) of equation (1) is a pseudo-hermitian operator, and its spectrum has complex-conjugate pair eigenvalues. It means that \( H \) is isospectral to \( H' \). Otherwise, if \( \epsilon \neq 0 \), the spectrum of \( H \) contains complex (non-complex conjugate pairs) eigenvalues, and the eigenvalues of \( H' \) are complex conjugate to the eigenvalues of \( H \), equation (9).

In the basis \( \mathcal{A}_k \), a general initial state can be written as
\[ |I\rangle = \sum_k c_k |k\rangle. \quad (11) \]
In terms of the basis formed by the eigenvectors of \( H \) the initial state is given by
\[ |I\rangle = \sum_{a} \tilde{c}_a |\tilde{\phi}_a\rangle, \quad (12) \]
\[ \tilde{c}_a = \sum_k (\tilde{T}^{-1})_{ak} c_k, \quad (13) \]
with \( \tilde{T} \) the transformation matrix from the basis \( \mathcal{A}_k \) to the basis \( \mathcal{A}_H \). We shall assume that the initial state is normalized, that is \( \langle I|I \rangle = 1 \). The initial state of equation (12) evolves in time as
\[ |I(t)\rangle = e^{-iHt}|I\rangle, \quad (14) \]
If \( H \) can be diagonalized, \( \tilde{c}_a(t) \) is given by
\[ \tilde{c}_a(t) = e^{-i\tilde{E}_a t} \tilde{c}_a. \]
In order to work with the basis formed by the eigenstates of \( H, \mathcal{A}_H \), to calculate the expectation value of a given observable, \( \tilde{\sigma} \), we have to equip the linear vector space with a scalar product. The reader is kindly refer to \cite{68} and references therein. That is, we look for a metric operator \( \mathcal{S} \), i.e. an operator which is selfadjoint and positive definite. The Hilbert space \( \mathcal{H} \) with the inner product \( \langle \Psi | \Sigma | \Phi \rangle_\mathcal{S} = \langle \Psi | S | \Phi \rangle \) is the new physical linear space \( \mathcal{H}_\mathcal{S} = (\mathcal{H}, \langle . | . \rangle_\mathcal{S}) \). In terms of the eigenvectors of the symmetry operator \( \mathcal{S} \), the initial state reads
\[ |I(t)\rangle = \sum_{\beta} \tilde{c}_\beta(t) |\tilde{\phi}_\beta\rangle, \quad \tilde{c}_\beta(t) = \sum_{a} (\tilde{T}^{-1})_{\beta a} \tilde{c}_a(t), \quad (15) \]
with \( \tilde{T} \) being the transformation matrix from the basis \( \mathcal{A}_H \) to the basis \( \mathcal{A}_S \). We are in condition of evaluating the mean value of an operator \( \tilde{\sigma} \) as a function of time as
\[ \langle \tilde{\sigma}(t) \rangle = (|I(t)\rangle \tilde{\sigma} |I(t)\rangle)_\mathcal{S} \]
\[ = \sum_{\alpha,\beta} \tilde{c}_\alpha(t) \tilde{c}_\beta^* \langle \tilde{\phi}_\beta|\tilde{\sigma}|\tilde{\phi}_\alpha\rangle_\mathcal{S}. \quad (16) \]
As it is reported in \cite{68}, the form of the metric operator depends on the spectrum of \( H \). It can be summarized as follows. If the spectrum of \( H \) contains complex conjugate pair eigenvalues, there exists a self-adjoint symmetry operator such that \( \mathcal{S} H = H' \mathcal{S} \). It reads
\[ \mathcal{S} = \sum_{j=1}^{\infty} \delta(\mathcal{E}_j - \tilde{E}_j^*) (a_j |\tilde{\psi}_j\rangle \langle \tilde{\psi}_j| + a_j^* |\tilde{\psi}_j\rangle \langle \tilde{\psi}_j|), \quad (17) \]
with \( a_j \in \mathbb{C} \) and \( \text{Im}(a_j) = 0 \). This operator is not positive so that we shall make use of the formalism of Krein Spaces. After the diagonalization of \( \mathcal{S} \), we have \( \mathcal{S}_k = \mathcal{D}_k \mathcal{D}_k^{-1} = \mathcal{D}_k \mathcal{D}_k^{-1} + \mathcal{D}_k \mathcal{D}_k^{-1} = \mathcal{S}_k + \mathcal{S}_k \). The diagonal matrix with positive elements and \( \mathcal{D} \) the diagonal matrix with negative entries. Finally, the metric operator is given by
\[ \mathcal{S} = \mathcal{S}_k + \mathcal{S}_k. \]
If the Non-hermitian Hamiltonian \( H \) has real eigenvalues or some eigenvalues are complex (non-complex conjugate pairs), the metric operator is given by
\[ \mathcal{S} = \sum_{j=1}^{\infty} |\tilde{\psi}_j\rangle \langle \tilde{\psi}_j|. \quad (18) \]

### 2.2. Spin-squeezing parameters and intelligent spin states

The concept of spin squeezing has arisen mainly from two considerations: the study of particle correlations and entanglement \cite{69}, as well as the improvement of measurement precision in experiments \cite{70}. Spin squeezing characterizes the sensitivity of a state with respect to rotations. It is related to the reduction of the fluctuation of one of the components of the total spin, beyond the uncertainty limit, at expenses of the enhancement of the fluctuation of another spin component. The definition of spin squeezing is not unique, and it depends on the context where squeezing is considered. We refer the interested reader to \cite{2} for a complete review on this subject.

As reported in \cite{2}, different parameters have been introduced to account for the amount of squeezing achieved in a given spin state. We can mention the proposal of Wineland et al \cite{70} in the context of spectroscopy, or the extensively used squeezing parameter constructed by Sørensen et al \cite{69} to detect squeezing from an entanglement criterion. More recently, the so call nematic-spin squeezing has been introduced to deal with systems composed of particles of...
spin \( s = 1 \) \([71–73]\). The extension to systems of particles with spin \( s = J \), with \( J > 1/2 \), has been advanced in \([3]\). Among others, we shall adopt the definition given by Kitagawa and Ueda \([1]\) to characterize the squeezing properties of a system composed of particles with spin \( s = 1/2 \).

Following the work of \([1]\), we define a set of orthogonal axes \( \{ n_x, n_y, n_z \} \), such that \( n_x \) is the unitary vector pointing along the direction of the mean value of total spin, \( S \). Next, we fix the direction \( n_x \) by looking at the minimum value of variance \( \Delta^2 S_x \). The Heisenberg Uncertainty Relation for the spin components \( S_x \) and \( S_y \) reads

\[
\Delta^2 S_x \Delta^2 S_y \geq \frac{1}{4} |\langle S \rangle|^2.
\]

Consequently, we define the squeezing parameter as

\[
\zeta^2(x') = \frac{2(\Delta S_x)^2}{|\langle S \rangle|^2}.
\]

The state is a Spin Squeezed State if \( \zeta^2(x') < 1 \). So defined, the parameter of equation \((20)\) is \(SU(2) \) invariant \([74]\). For completeness, we shall define

\[
\zeta^2(y') = \frac{2(\Delta S_y)^2}{|\langle S \rangle|^2}.
\]

When the minimum value of the Heisenberg Uncertainty Relation, equation \((19)\), is achieved and \( \zeta^2 < 1 \) the state is called ISS \([39–44]\).

We shall measure the squeezing in decibels, which is the unit typically used by experimentalists. The amount of squeezing in dB is defined as

\[
q_k = 10 \log_{10}[\zeta^2_k].
\]

In terms of \( q_x \) and \( q_y \), an ISS satisfies the condition \( q_x = -q_y (\zeta^2(x') = 1) \), with \( q_x \neq 0 (\zeta^2(x') \neq 1) \) and \( q_y \neq 0 (\zeta^2(y') \neq 1) \).

3. Results and discussions

Let us first present and discuss the general results that we have obtained for the time evolution of a CSS \([75]\) through the action of the Hamiltonian of equation \((1)\). The initial CSS has the form

\[
|I(\theta_0, \phi_0)\rangle = \mathcal{N} \sum_{k=0}^{2^S} z(\theta_0, \phi_0)^k \left( \begin{array}{c} 2^S \end{array} \right)^{1/2} |k\rangle,
\]

with \( z(\theta_0, \phi_0) = e^{-i\phi_0}\tan(\theta_0/2) \). The angles \( (\theta_0, \phi_0) \) define the direction \( \vec{m}_0 = (-\sin \theta_0 \cos \phi_0, -\sin \theta_0 \sin \phi_0, \cos \theta_0) \), such that \( \vec{S} \cdot \vec{m}_0 |I\rangle = -|\langle I| \rangle \) \([75]\).

In section 3.1, we shall analyse the numerical results obtained from the exact diagonalization of the Hamiltonian of equation \((1)\). We shall complement this information with the analytical results of sections 3.2 and 3.3. Finally, in section 3.4 we shall investigate the possibility of generating a steady ISS in diamond nanostructures.
Figure 2. Polar angle of the unit vector along the direction of the mean value of the total spin, $\theta_{\|}$, as a function of time. The parameters are the same as in figure 1. In Panels (a) and (b) are displayed the results obtained when the initial CSS is prepared with $(\theta_0, \phi_0) = (\pi/4, 0)$ and $(\pi/8, 0)$, respectively.

Figure 3. Contribution of the $k$th state of the basis $A_k$ to the state $|I(t)\rangle$ of equation (14), as a function of time, equation (27). The parameters are those of figures 1 and 2. In Panels (a) and (b) are displayed the results obtained when the initial CSS is prepared with $(\theta_0, \phi_0) = (\pi/4, 0)$ and $(\pi/8, 0)$, respectively.

squeezing depends on the value of $\theta_0$. Initial states with $\theta_0$ smaller than $\pi/4$ favor the appearance of squeezing as a function of the time. However, independent of the preparation of the initial state, it evolves to an asymptotic steady state which behaves as an ISS, i.e. $q_{\epsilon^+} + q_{\epsilon} = 0$. To understand the nature of this asymptotic steady ISS, we have studied the dependence, as a function of time, of the polar angle of the unit vector along the direction of the mean value of the quasi-spin operator $\langle \hat{S} \rangle$. The corresponding results are shown in figure 2. The parameters are the same as those of figure 1. In Panels (a) and (b) we have displayed the results obtained when the initial CSS is prepared with $(\theta_0, \phi_0) = (\pi/4, 0)$, and with $(\pi/8, 0)$, respectively. The system evolves to a state with $\langle \hat{S} \rangle$ pointing in the $z$-direction, with $\langle \hat{S}_z \rangle = -S$, independent of the choice in the initial CSS.

In figure 3, we show the contribution of the $k$th state of the basis $A_k$, equation (5), to the state $|I(t)\rangle$ of equation (14), as a function of time

$$w(k) = |\langle k | I(t) \rangle|^2. \quad (27)$$

We have adopted the same parameters of are those of figures 1 and 2. In Panels (a) and (b) are displayed the results obtained when the initial CSS is prepared with $(\theta_0, \phi_0) = (\pi/4, 0)$ and $(\pi/8, 0)$, respectively. From the analysis of figure 3, it can be concluded that as the state evolves in time, the dominant contributions to the state come from the channels with low values of $k$. This fact is in correspondence with the results of figure 2.

In figure 4, we show the dependence, as a function of the relative coupling constant $\eta$, of the squeezing parameters for the steady state ($t > T_c$, $t = 120 (\mu$ sec)), in units of (dB). In Panels (a), (b) and (c) we study systems with $N = 5$, $N = 45$ and $N = 101$ spins, respectively. With solid lines we show the results which we have obtained from the exact diagonalization of the Hamiltonian of equation (1), for and initial CSS with $(\theta_0, \phi_0) = (\pi/4, 0)$, equation (23). The dotted-line is used to show the behaviour of $\zeta^2_{\epsilon^+} \zeta^2_{\epsilon}$ in units of (dB) $(q_{\epsilon^+} + q_{\epsilon})$. The results presented in figure 4 support the idea of the existence of to regions with different squeezing properties. The initial coherent state evolves into a steady ISS for $\eta < 1$, and loose the squeezing properties if $\eta > 1$. In the next sections, we shall present some analytical results to understand this property, and we shall discuss the rest of the curves of the figure.

Next, we shall study the persistence of a steady ISS as the number of spins is increased. Figure 5 shows the behaviour of the squeezing parameters of the steady state, $\zeta^2_{\epsilon^+}$ and $\zeta^2_{\epsilon}$, as a function of the number of spins of the system, in units of (dB), see equation (22). The curves have been computing at instant $t = 120$. In Panels (a)–(d) we show the results that we have obtained when the relative coupling constant $\eta$ takes the value $\eta = 0.25$, $\eta = 0.50$, $\eta = 0.75$ and $\eta = 0.95$, respectively. We have chosen an initial CSS with $(\theta_0, \phi_0) = (\pi/4, 0)$. The rest of the parameters are those of figure 1. We have plotted with circles the value of the product $\zeta^2_{\epsilon^+} \zeta^2_{\epsilon}$ in units of (dB) $(q_{\epsilon^+} + q_{\epsilon})$, equation (22)). The line at constant value 0 is just plotted as a guide. As can be observed from the figure,
except for systems with a small number of spins at large values of $\eta$, the steady state behaves as an ISS. Also, it can be observed that the amount of squeezing achieved in the steady state is increased as the value of the relative coupling constant approaches $\eta \to 1$.

In what follows we shall present some analytical results to clarify the generation of a steady ISS for some range of coupling constants.

### 3.2. Non-hermitian OAT model

Let us first consider the time evolution of the initial state proposed in equation (23), under the Hamiltonian

$$H_0 = \chi S_z^2 + (\epsilon - i\gamma)(S_z + S),$$  \hspace{1cm} (28)

that is in absence of the LMG interaction. The mean values of the spin components can be calculated straightforwardly, and they read

\[
\begin{align*}
\langle S_z \rangle &= -\frac{1}{2} \left( \frac{1}{1 + |\tilde{z}|^2} - |\tilde{z}|^2 \right), \\
\langle S_+ \rangle &= \frac{2S(2S^2 - 1)|\tilde{z}|^2}{(1 + |\tilde{z}|^2)^2}, \\
\langle S_- \rangle &= 2S \left( \frac{4S(2S^2 - 1)|\tilde{z}|^2}{(1 + |\tilde{z}|^2)^2} + \frac{(e^{-2i\gamma t} + |\tilde{z}|^2 e^{2i\gamma t})^{2S-1}}{(1 + |\tilde{z}|^2)^{2S}} \right),
\end{align*}
\]

being $\tilde{z} = z(\theta_0, \phi_0)e^{-\gamma t}$. For $\gamma \in \mathbb{R}$ and $\gamma > 0$, $\tilde{z} \to 0$ when $\gamma t, t = 120 (\mu \text{sec})$.

In Panels (a), (b) and (c) we plot the results obtained for ensembles with $N = 5$, $N = 45$ and $N = 101$ spins, respectively. We have fixed $\gamma = 2 \times 10^{-5}$ (GHz). Solid lines are used to show the results which we have obtained for the squeezing parameters from the exact diagonalization of the Hamiltonian of equation (1), $q_x$ and $q_y$ of equation (22), for an initial CSS with $(\theta_0, \phi_0) = (\pi/4, 0)$, equation (23). Dashed-lines correspond to the results which we have obtained by applying the boson approximation of section 3.3, $q(x, p)$ and $q(p, x)$ of equation (49). In this case, the initial state of equation (42) consists of 5 particles in mean value for Panel (a), and of 45 and 101 particles in mean value for Panels (b) and (c), respectively. With dotted-line and with dashed-dotted-line we present the results for the product of the squeezing parameters (in units of (dB)) in the exact ($q_x + q_y$) and in the approximate case ($q_{xp} + q_{p}$), respectively.

**Figure 4.** Dependence, as a function of the relative coupling constant $\eta$, of the Squeezing Parameters of the steady state ($t > T_c$, $t = 120$ (\mu sec)), in units of (dB). (approximate case respectively. We have $\mu \sec$). In this case, the initial state is increased as the value of the relative coupling constant $\eta$ is fixed to the value $\eta = 0.25$, $\eta = 0.50$, $\eta = 0.75$ and $\eta = 0.95$, respectively. The rest of the parameters are those of figure 1. We have plotted with circles the value of the product $\tilde{z}^2 S_z^2$ in (dB) ($q_x + q_y$). The line at constant value 0, is just to guide the eye.

**Figure 5.** Behaviour of the Squeezing Parameters of the steady state in units of decibels, $q_x$ and $q_y$ of equation (22), as a function of the number of spins, $N$ ($t = 120$ (\mu sec)). In Panels (a)--(d) we show the results obtained when the relative coupling constant $\eta$ is fixed to the value $\eta = 0.25$, $\eta = 0.50$, $\eta = 0.75$ and $\eta = 0.95$, respectively. The rest of the parameters are those of figure 1. We have plotted with circles the value of the product $\tilde{z}^2 S_z^2$ in (dB) ($q_x + q_y$). The line at constant value 0, is just to guide the eye.
$t \to \infty$. In this limit we find
\[
\langle S_z \rangle \to -S, \\
\langle S_i \rangle = \text{Re}(\langle S^i \rangle) \to 0, \\
\langle S^2 \rangle = \frac{1}{2} \text{Re}(\langle S^2 \rangle) + \frac{1}{4} \langle \{S^i, S_i\} \rangle - \frac{S}{2}, \\
\langle S_y \rangle = \text{Im}(\langle S^y \rangle) \to 0, \\
\langle S^x \rangle = -\frac{1}{2} \text{Re}(\langle S^x \rangle) + \frac{1}{4} \langle \{S^i, S_i\} \rangle + \frac{S}{2}.
\]
(30)

Consequently, \( \langle S \rangle \to -S \hat{\epsilon} \), with
\[
\Delta^2 S_x \to \frac{S}{2}, \quad \Delta^2 S_y \to \frac{S}{2}.
\]
(31)

This results indicates that the initial CSS, \( |\theta(t_0, \phi_0)\rangle \), evolves, asymptotically, to the state with \( |\theta(\pi, 0)\rangle = |S, -S\rangle \), independent of the orientation of the state at \( t = 0 \).

### 3.3. Non-hermitian LMG model

The purpose of this section is to provide an analytical Hamiltonian which accounts for the behaviour of the system in the stationary regime, when the Lipkin interaction is included.

We shall perform a Holstein–Primakoff boson mapping \[79–81\] of the Hamiltonian of equation (1). The generators of the \( su(2) \) algebra, in terms of the boson creation operator, \( b^\dagger \), and of the boson annihilation operator, \( b \), read
\[
S_+ = b^\dagger \sqrt{2S} - b b^\dagger \approx \sqrt{2S} b^\dagger, \\
S_- = \sqrt{2S} - b b^\dagger b \approx \sqrt{2S} b, \\
S_z = b^\dagger b - S.
\]
(32)

The nonlinearity introduced by the square-root term in equation (32) ensures that two excitations can not take place at the same spin. If we consider delocalized spin waves involving a large number of spins compared to the number of excitations, the probability that a given spin is excited is inversely proportional to the number of spins \( N \). Therefore, as long as only a few delocalized spin excitations are considered, it is reasonable to adopt the approximation given in equation (32) \[82\]. As observed from figure 3, this assumption is valid after a sufficiently large interval of time.

In this approximation, the Hamiltonian of equation (1) can be written as
\[
H_B = h_0 + 2\alpha K_0 + 2SV(K_+ + K_-),
\]
(33)

with
\[
K_+ = \frac{1}{2} b^2, \quad K_- = K^0, \\
K_0 = \frac{1}{2} b^\dagger b + \frac{1}{4},
\]
(34)

and
\[
h_0 = \chi S^2 - \frac{1}{2} \alpha, \\
\alpha = (\epsilon - 2S\chi - i\gamma).
\]
(35)

The set of operators \( \{K_+, K_-, K_0\} \) spans the algebra of \( su(1, 1) \), that is
\[
[K_+, K_-] = 2K_0, \\
[K_0, K_\pm] = \pm K_\pm.
\]
(36)

The time evolution operator of the system, \( U(t) = e^{-iH_0 t} \), can be easily computed if the exponential is written in normal order form \[83, 84\]. Making use of the faithful matrix representation of the operators \( su(1, 1) \)-algebra, it reads (see appendix)
\[
U(t) = e^{-iH_0 t} = e^{-i\theta(t) e^{i\pi/4} K_0 e^{i\theta(t)} K},
\]
(38)

with
\[
b_0 = \left( \cos(t\beta) \left( 1 + \frac{\alpha}{\beta} \tanh(\text{Im}t/\beta) \right) \right)^{-2},
\]
(39)

\[
b_\pm = \left( \frac{2|V|}{\beta} \tanh(\text{Im}t/\beta) \right)^{-1} b \pm \frac{\alpha}{\beta} \tanh(\text{Im}t/\beta),
\]
(40)

where, \( \phi_V = 0 \) if \( V > 0 \) and \( \phi_V = \pi \) if \( V < 0 \). We have defined the complex parameter \( \beta = \sqrt{\alpha^2 - (2V)^2} \).

As \( |b_\pm| < 1 \) (see appendix), we can introduce the squeezing parameter \( \zeta = re^{i(\beta + \phi_V + \pi/2)} \), such that
\[
b_\pm = (\zeta/|\zeta|) \tanh(|\zeta|).
\]
(41)

In what follows, we shall study the evolution of the state
\[
|\psi\rangle = N \sum_n \frac{(\sqrt{2S})^n}{\sqrt{n!}} |n\rangle = D(\sqrt{2S}|0\rangle),
\]
(42)

where \( D(\eta) = e^{i\varphi_\eta} \) is the displacement operator. The proposed initial state of equation (42) is the limit to dominant order in the number of spins of the CSS of equation (23). This state evolves in time as (see appendix)
\[
U|\psi\rangle = e^{-iH_0 t} e^{i\theta(t) e^{i\pi/4} K_0 e^{i\theta(t)} K} |\psi\rangle = e^{i\varphi_\eta} D(\sqrt{2S}\text{Re}^t)|0\rangle.
\]
(43)

The parameters \( R_0 \) and \( R_- \) are given by
\[
R_0 = \frac{b_0}{1 - |b_\pm|^2},
\]
(44)

\[
R_- = \text{Re}^t, \quad R_0 - b_\pm,
\]
(45)

and \( S_\lambda(\zeta) \) stands for the squeezing operator, \( S_\lambda(\zeta) = e^{i\lambda\zeta K} \).

We are now, in condition to compute the uncertainty relations of the operators
\[
x = \frac{1}{\sqrt{2}} (b^\dagger + b), \\
p = \frac{1}{\sqrt{2}} (b^\dagger - b),
\]
(46)

on the state of equation (43). After some cumbersome algebra
(see appendix) it can be proved that
\[ \Delta^2 x = \frac{1}{2} \left( \frac{-\cos(\phi + \phi_\gamma) + 2\rho}{1 - \rho^2} + \frac{1 + \rho^2}{1 - \rho^2} \right), \]
\[ \Delta^2 p = \frac{1}{2} \left( \frac{\cos(\phi + \phi_\gamma) + 2\rho}{1 - \rho^2} + \frac{1 + \rho^2}{1 - \rho^2} \right) \tag{47} \]
with \( \rho = |b_\gamma| = \tanh|\zeta| \). Consequently we can define the associated squeezing parameters \( Q(x, p) \) and \( Q(p, x) \) as
\[ Q(x, p) = 2\Delta^2 x, \quad Q(p, x) = 2\Delta^2 p. \tag{48} \]
The system is squeezed in \( x(p) \) when \( Q(x, p) < 1 \) \( (Q(p, x) < 1) \). As we have stated before, we can measure the squeezing factor in decibels, that is
\[ q(x, p) = 10 \log_{10}[Q(x, p)], \quad q(p, x) = 10 \log_{10}[Q(p, x)]. \tag{49} \]

Our objective is to study the behaviour of the system after a long interval of time \( (t \to \infty) \).

Due to decoherence, it is straightforward to show that
\[ \lim_{t \to \infty} b_\gamma = e^{k(\phi_\gamma + \phi)} \rho_L. \tag{50} \]

In the previous expression, \( \sigma \) stands for the sign function of \( (\epsilon - 2\Sigma \chi) \), and
\[ \phi = -\arctan \left( \frac{\beta_\gamma - \Gamma}{\beta_\gamma - \sigma} \right), \quad \rho_L = \frac{\eta}{\sqrt{\eta^2 + (\beta_\gamma - \Gamma)^2}}. \tag{51} \]

being
\[ \beta_\gamma^2 = \frac{1}{2} \left( (1 - \eta^2 - \Gamma^2)^2 + 4 \Gamma^2 \pm (1 - \eta^2 - \Gamma^2) \right). \tag{52} \]

To leading order in \( \Gamma \), the phase factor \( \phi \) can be written as
\[ \phi \approx \begin{cases} \arctan \left( \frac{\Gamma}{\sqrt{1 - \eta^2 - \sigma}} \right) & 0 < \Xi < 1 \\ \arctan \left( \sigma \left( \sqrt{\eta^2 - 1 - \Gamma} \right) \right) & \Xi > 1, \end{cases} \tag{53} \]

where \( \Xi \) is given in equation (26). We can identify two regions, in the space of coupling constants \( \eta \) and \( \Gamma \), with different squeezing properties for the steady state of the system. Region I corresponds to values of \( \eta \) and \( \Gamma \) that satisfy the condition \( \Xi^2 < 1 \), and Region II for values of \( \eta \) and \( \Gamma \) that satisfy \( \Xi^2 > 1 \).

In Region I, for small values of \( \Gamma \), the phase \( \phi \) of \( b_\gamma \) becomes approximately null, \( \phi < < 1 \), so that \( b_\gamma \approx e^{k(\phi_\gamma + \phi)} \rho_L \). Then, the squeezing parameters \( Q(x, p) \) and \( Q(p, x) \) of equation (48) for \( \phi_\gamma = 0 \), take the form
\[ Q(x, p) \to \frac{1 + \rho_L}{1 - \rho_L} = e^{2|\zeta|}, \tag{54} \]
\[ Q(p, x) \to \frac{1 - \rho_L}{1 + \rho_L} = e^{-2|\zeta|}. \tag{55} \]

and
\[ Q(x, p)Q(p, x) \to 1. \tag{56} \]

Thus, the steady state of the system behaves as an ISS. Similar expressions hold for \( \phi_\gamma = \pi \), but with the exchange of the roles of \( Q(x, p) \) and \( Q(p, x) \).

In Region II, the behaviour of the system is entirely different. The phase \( \phi \) of \( b_\gamma \) is no longer null, \( \phi \neq 0 \), moreover for values of \( \eta \) sufficiently large \( \phi \to \pm \pi/2 \), depending on \( \sigma \). In this case the squeezing parameters \( Q(x, p) \) and \( Q(p, x) \), equation (48), take the form
\[ Q(x, p) \to \frac{1 + \rho_L}{1 - \rho_L}, \quad Q(p, x) \to \frac{1 + \rho_L}{1 - \rho_L}. \tag{57} \]

Thus, in Region II, the asymptotic steady state is not a squeezed state.

Let us compare these analytical results with the ones discussed in section 3.1.

In view of equation (32) and of equation (46), to leading order in the number of spins
\[ \frac{S_\gamma}{S} = \frac{1}{2\sqrt{S}} (S_+ + S_-) \to x, \quad \frac{S_\gamma}{S} = -\frac{1}{2\sqrt{S}} (S_+ - S_-) \to -p. \tag{58} \]

Under the action of the Hamiltonian of equation (1), the initial state of equation (23) evolves to a steady state which points in the \( z \)-direction, the squeezing parameters \( \{\zeta_x^2, \zeta_y^2\} \) should give the same information as \( \{Q(x, p), Q(p, x)\} \).

This can be seen from figure 4, where we present, by using dashed-lines, the results obtained for \( Q(x, p) \) and \( Q(p, x) \) in units of \( (\text{dB}) \), equation (49), for the CSS of equation (42) with \( N \) particles in mean value. With dashed-dotted-line we present the results for the product of the squeezing parameters in units of \( (\text{dB}) \) \( (q(x, p) + q(p, x)) \). Clearly, for systems with more than 9 spins, the initial CSS evolves into a steady ISS for \( \Xi^2 < 1 \), and it loses the squeezing properties if \( \Xi^2 > 1 \).

We complete our analytical results by analysing the behaviour of the phase \( \phi \) of equation (51). The results are presented in figure 6. We have considered a system with \( N = 45 \) spins and with \( \gamma = 2 \times 10^{-5} \) (GHz). The numerical results are in agreement with the analytical estimations of Subsection 3.3. That is, in Region I the phase \( \phi \) is null, \( \phi = 0 \) and consequently, the steady state is an ISS, while in Region II \( \phi \to -\pi/2 \) for increasing values of the coupling constant \( \eta \), and the steady state is no longer an ISS.

From the results we have presented, it can be inferred that dissipative mechanisms can be used to improve the achievement of squeezing in different spin system [23, 25].
3.4. Application to phonon-induced spin–spin interactions in diamond nanostructures

Let us consider the spin–spin interaction, among NV centers in diamond, mediated through the coupling of the spins to a magnetic nano-resonator [46, 47, 50].

An NV center has a ground state with spin 1 and a zero-field splitting $D = 2.88$ GHz between the states $|1, 0\rangle$ and $|1, \pm 1\rangle$ [53]. If an external magnetic field, $B_0$, along the crystalline axis of the NV center, is applied an additional Zeeman splitting between $|1, \pm 1\rangle$ sub-levels occurs. Then, it is possible to isolate the subsystem $|1, 0\rangle$ and $|1, -1\rangle$ [25, 34, 46, 47].

The mechanical resonator can be described by the Hamiltonian $H_{\text{r}} = \omega_r b^\dagger b$, where $\omega_r$ stands for the frequency of the fundamental vibrational mode of the resonator, and $b$ ($b^\dagger$) is the corresponding annihilation (creation) operator. We shall choose $\omega_r$ almost in resonance with the splitting of the states $|1, 0\rangle$ and $|1, -1\rangle$ so that the NV center can be modeled by a two-level system. The motion of the magnetic mechanical resonator produces a magnetic gradient field on the NV centers so that within this two-level subspace the Hamiltonian of the system can be modeled as

$$H_{\text{NV}} = \omega_r b^\dagger b + \delta \sigma_z + g_1 (\sigma_z b^\dagger + b^\sigma_z) + g_2 (\sigma_z b^\dagger + b^\sigma_z),$$

where $\delta = D - \gamma_e B_0$ is the energy gap between the ground state $|1, 0\rangle$ and the state $|1, -1\rangle$, being $\gamma_e$ the gyromagnetic ratio of an electron. We have assumed an asymmetric interaction between the NV centers and the single mode mechanical resonator, which is model by the effective coupling constants parameter $g_1$ and $g_2$. The operators $\sigma_x$, $\sigma_y$, $\sigma_z$ are collective spin operators for the ensemble of NV centers in diamond, $\sigma_n = \sum_i \sigma_n i$. They fulfill the usual angular momentum commutation relations. We shall consider that the intensity of the external magnetic field is fixed in order to have a detuning $\delta \approx 0$.

We can apply an unitary transformation of the form

$$U = e^{-i(\delta/\omega_r)(\sigma_z b^\dagger - b^\sigma_z) - i(\gamma/\omega_r)(\sigma_y b - b^\sigma_y)}$$

to the Hamiltonian of equation (59). To leading order in $g_1/\omega_r$ and $g_2/\omega_r$, together with the assumption that $\delta \approx 0$, the effective Hamiltonian, $H_{\text{eff}} = U H U^{-1}$, takes the form

$$H_{\text{eff}} \approx H_0 + H_{\text{sp}} + H_{\text{int}},$$

$$H_0 = -2 \frac{g_1^2 + g_2^2}{\omega_r} S(S + 1),$$

$$H_{\text{sp}} = \omega_r b^\dagger b + 2 \frac{g_1^2 - g_2^2}{\omega_r} (1 + 2 b^\dagger b) \sigma_z,$$

$$H_{\text{int}} = 2 \frac{g_1^2 + g_2^2}{\omega_r} \sigma_z^2 - 4 \frac{g_1 g_2}{\omega_r} (\sigma_z^2 - \sigma_y^2).$$

The coherence of the ensemble of NV centers highly depends on contents of paramagnetic impurities in diamond. The main magnetic impurities are neutral nitrogen atoms, P1 centers. NV centers are created from nitrogen doped in diamond in two steps: (i) irradiation with protons or electrons of a diamond crystal to produce vacancies, (ii) annealing at 800°C–1000°C to allow the vacancies to migrate and form the NV defect. This method leaves, unavoidably, a significant residual concentration of P1 centers. The contribution of P1 centers to the decoherence of the NV ensemble has been thoroughly studied [85, 86]. In particular, it was shown that the coherence time of the NV ensemble depends on the concentration of P1 centers. We shall consider the ensemble of P1 centers as a spin-bath for the NV centers, and we shall model its environment effect through a line-width related to the inverse of the characteristic coherence time of the order of $T_C = 100$ (μsec) [32, 76–78, 85, 86]. That is

$$H_\gamma = -\frac{\gamma_e}{2} (\sigma_z + S).$$

Thus, the Hamiltonian of the NV ensemble reads

$$H_{\text{NV,eff,ph}} = H_{\text{eff}} + H_\gamma.\tag{65}$$

In order to generate a steady ISS, we initialize the ensemble of NV centers as CSS, $|I(\theta_0, \phi_0)\rangle$, along the direction $\hat{n}_0 = (-\sin(\theta_0) \cos(\phi_0), -\sin(\theta_0) \sin(\phi_0), \cos(\theta_0))$ of the collective Bloch sphere. As it is well known, the CSS satisfies the condition $\sigma_\alpha \hat{n}_0(\text{CSS}) |I(\theta_0, \phi_0)\rangle = -S I(\theta_0, \phi_0)$, and it has equal transverse variances, $S/2$. This state can be prepared by using optical pumping and microwave spin manipulation applied to the the NV ensemble [50].

For the mechanical resonator we shall consider an initial coherent phonon state with mean number of phonons $n_{\text{ph}}$.

$$|n_{\text{ph}}\rangle = e^{-|\varphi_{\text{ph}}|^2/2} \sum_{n=0}^{\infty} \frac{e^{-n_{\text{ph}}}}{\sqrt{n!}} |n\rangle,$$

where $|n\rangle$ represents the state with $n$ phonons, and $|\varphi_{\text{ph}}|^2 = n_{\text{ph}}$. The Hamiltonian of equation (60) includes a term which couples the phonon number $\hat{n} = a^\dagger a$ to $\sigma_z$, equation (62). Thus, an initial product state of the form

$$|\psi\rangle = \langle n_{\text{ph}} |I(\theta_0, \phi_0)\rangle$$

is initialized.
\[
|I\rangle = |n_{ph}\rangle |I(\theta, \phi)\rangle,
\]
will evolve as
\[
|I(t)\rangle = e^{-i\omega_{ph}t/2} \sum_n \frac{z_{ph}^n}{\sqrt{n!}} |I_{NVE}(t, n)\rangle,
\]
with
\[
|I_{NVE}(t, n)\rangle = e^{-iH_{NVE(n)}t} |I(\theta, \phi)\rangle,
\]
and
\[
H_{NVE(n)} = \epsilon \sigma_z^2 + \chi \sigma_z^2 + V (\sigma_z^2 - \sigma_z^2) + H_{\gamma},
\]
where
\[
\epsilon = 2 \frac{g_1^2}{\omega_r} - \frac{g_2^2}{\omega_r} (1 + 2n) \sigma_z,
\]
\[
\chi = 2 \frac{g_1^2 + g_2^2}{\omega_r} \sigma_z^2,
\]
\[
V = -4 \frac{g_1g_2}{\omega_r} \sigma_z.
\]

Following the formalism presented in section 2.1, the mean value of physical operator associated to the NV centers, \(\hat{\sigma}_{NVE}\), will be computed as
\[
\langle \hat{\sigma}_{NVE}(t) \rangle = e^{-i\omega_{ph}t} \sum_{n=0}^{\infty} \frac{1}{n!} \langle I_{NVE}(t, n) | \hat{\sigma}_{NVE} | I_{NVE}(t, n) \rangle S,
\]
where the \(S\) is the corresponding metric operator [68].

In the previous section we have concluded that, for a large number of NV centers, the values of \(\Xi = n_f^2 + 1^2\) (equation (26)) can be used to characterize the appearance of a steady ISS. That is, for \(\Xi < 1\), the initial state evolves into a steady ISS. In terms of \(g_1, g_2, \omega_r, \gamma\), the number of NV centers, \(N = 2S\) and the number of phonons, \(n\), the quantity \(\Xi\) reads
\[
\Xi = \left[ \frac{2g_1^2 + 4\gamma (g_2^2/\omega)}{g_2} \right] \left( 1 - \frac{1+2n}{2S} \right) + \left( 1 + \frac{1+2n}{2S} \right).
\]

For \(\gamma/(4S) \ll 2g_1g_2/\omega_r\), the quantity \(\Xi\) depends on the relative coupling constant \(g_1/g_2\) and on the ratio of phonon numbers to the number of spins, \((1 + 2n)/(2S)\).

In figure 7, we present a contour plot of \(\Xi^2\) as a function of the ratios \(g_1/g_2\) and \((1 + 2n)/(2S)\). We have considered a system of \(N = 1001\) NV-centers. We have chosen values of \(g_1, g_2, \omega_r, \gamma\), for \(g_1/g_2 < 1\), or for \((1 + 2n)/(2S) < 0.5\) if \(g_1/g_2 > 1\). Similar results are obtained for systems with different values of the number of the NV centers, \(N = 2S\), and of the number of phonons, provided that \((1 + 2n)/(2S)\) remains the same.

In figure 8, we show the results obtained for the squeezing parameter of the steady state, as a function of the ratio \(g_1/g_2\). We have computed the mean values of the physical operators following equation (70). We have chosen an initial CSS for the NV centers with \(\theta_0 = \pi/4\) and \(\phi_0 = 0\). We have considered a system with \(N = 2S = 1001\) NV colour centers in diamond. The values of \(g_2, \omega_r\) and \(\gamma\) are those of figure 7. We have evaluated the Squeezing parameter at \(t = 120 \text{ (µs)} > T_C\) in Panels (a)–(c) we show the results obtained when the mean value of phonons is \(n_{ph} = 6, 100\) and \(250\), respectively.
the mean value of phonons is increased, the contribution from states with a large number of $n$ becomes important, so that, at fix number of NV-centers, the parameter $\Xi^2$ can be $>1$ depending on the ratio $g_1/g_2$. We have verified that the values of the squeezing parameter in the steady state are independent of the initial state adopted [47].

4. Conclusions

In this work, we have studied the behaviour of a system of spins interacting through a Non-hermitian one-axis twisting Hamiltonian plus a Lipkin-type interaction. We have analysed the time evolution of an initial CSS. We have shown, by performing the exact numerical diagonalization of the Hamiltonian, that under the action of the one-axes twisting Hamiltonian plus a Lipkin-type interaction, the initial state evolves into steady CSS pointing in the $z$-direction. This fact has been proved analytically in section 3.2. In addition, in section 3.1 we have shown that, by performing an exact diagonalization of the interaction of Hamiltonian in equation (1), when the Lipkin interaction is turned on, an initial CSS evolves into a steady ISS for a specific range of values of the relative coupling constants $\eta$ and $\Gamma$. To understand this result, we have performed a boson mapping of the $su(2)$ Hamiltonian of equation (1). To leading order in the number of spins, the Hamiltonian was written in terms of the operators of the $su(1, 1)$ algebra, and the time evolution of the system was obtained analytically. In the asymptotic limit, that is after long intervals of time compared to the characteristic coherence time of the system, the numerical results that we have presented support the idea that the behaviour of the steady state governed by the $su(2)$-Hamiltonian of equation (1) can be understood in terms of the behaviour of the steady state governed by the $su(1, 1)$-Hamiltonian of equation (33). Both from analytical and numerical results, it is observed that two well-defined regions can be identified, depending on the relative value of the coupling constants ($\eta$, $\Gamma$), with different behaviour of the steady state. For systems with more than $N \approx 9$ spins, the initial state evolves in a Steady ISS when $\Xi < 1$, equation (26). Otherwise, the asymptotic state does not behave as a squeezed state. The previously reported results indicate that the generation of a Steady ISS, for a certain range of values of ($\eta$, $\Gamma$), is the consequence of the dissipative character of the interaction. Similar results have been advanced in [25]. As a potential physical application, we have investigated the possibility of generating a Steady ISS in diamond nanostructures. We have presented an effective spin–spin interaction among NV colour centers in diamond, mediated through the interaction of the NV centers with a magnetic nano-resonator. We have studied the regimen of coupling constants so that, under the action of this effective interaction, an initial CSS evolves in time into a Steady ISS.

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Appendix

Let us consider the Lie algebra $su(1, 1)$ [84], which is spanned by the operators $[K_1, K_2, K_3]$. They satisfy the well known commutation relations

$$[K_1, K_2] = -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_1, K_3] = iK_2. \tag{A1}$$

The complex linear combinations of these operators span the algebra $su^\ast(1, 1)$, which is isomorphic to $sl(2, C)$. The time evolution operator, $U(t) = e^{-iAt}$, is an exponential form of the elements of the $su^\ast(1, 1)$ Lie algebra (equations (34) and (36)). Thus, $U(t)$ belongs to the $SU(1, 1)$ Lie group. Consequently, $U(t)$ can be represented by a matrix $G$, which depends on two complex parameters $w_1$ and $w_2$ and can be written as

$$G = \begin{pmatrix} w_1 & w_2 \\ w_2^* & w_1^* \end{pmatrix}, \tag{A2}$$

moreover, the parameters $w_1$ and $w_2$ fulfill the condition $|w_1|^2 - |w_2|^2 = 1$.

Let us determine $w_1$ and $w_2$. In doing so, we shall write $U(t)$ in normal order as

$$U(t) = e^{-iK_3t}e^{-i(2\eta K_1 + 2SV(K_+ K_-))},$$

$$= e^{-iK_3t}e^{i\eta t K_1}e^{i\mu t K_2}e^{i\xi t K_3}e^{-i\eta t K_1}, \tag{A3}$$

with $K_\pm = K_1 \pm iK_2$ and $K_0 = K_3$.

Following the prescriptions of [83], it is possible to carry out all calculations, in either the algebra or the group, by using the faithful matrix representation of the operator algebra. It reads

$$K_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$K_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix},$$

$$K_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A4}$$

Writing (A3) in terms of the faithful matrix representation, we obtain

$$\begin{pmatrix} c + \frac{a_0}{2d} & \frac{a_1}{2d} \\ -\frac{a_1}{2d} & c - \frac{a_0}{2d} \end{pmatrix} = \begin{pmatrix} \sqrt{b_0} & -\frac{b_1}{\sqrt{2b_0}} \\ \frac{b_1}{\sqrt{2b_0}} & i\frac{1}{\sqrt{2b_0}} \end{pmatrix}, \tag{A5}$$

where

$$c = \cosh(\gamma t/\beta),$$

$$s = \sinh(\gamma t/\beta),$$

$$a_0 = -2i\alpha,$$

$$a = -i\gamma. \tag{A6}$$
Then, it results
\[
b_0 = \left( \cos(\beta) \left( 1 + \frac{\alpha}{\beta} \tanh(\beta) \right) \right)^{-2},
\]
\[
b_+ = e^{i(\phi + \pi)} \frac{2\mathcal{V}}{\beta} \frac{\tanh(\beta)}{1 + \frac{\alpha}{\beta} \tanh(\beta)},
\]
\[
b_- = b_+,
\]
where, \( \phi = 0 \) if \( V > 0 \) and \( \phi = \pi \) if \( V < 0 \). We have defined \( \beta = \sqrt{\alpha^2 - (2\mathcal{V})^2} \). We can identify
\[
w_1 = \sqrt{b_0} - b_0^2, \quad \bar{w}_1 = \frac{1}{\sqrt{b_0}},
\]
\[
w_2 = b_+ - \sqrt{b_0}, \quad \bar{w}_2 = -\frac{b_+}{\sqrt{b_0}},
\]
(\text{A10})

As \( |w_1|^2 - |w_2|^2 = 1 \), there exist \( \zeta \in \mathbb{C} \) and \( \{\theta_1, \theta_2\} \in \mathbb{R} \) so that
\[
w_1 = \cosh(\zeta)|e^{i\theta_1}|, \quad w_2 = \sinh(\zeta)|e^{i\theta_2}|.
\]
(\text{A11})

Consequently
\[
\frac{w_2}{\bar{w}_1} = b_+ = e^{i(\theta_1 - \theta_2) \tanh(\zeta)},
\]
(\text{A12})

verifying that \( |b_+| < 1 \).

It is convenient to introduce the operator of squeezing \( S_q(\zeta) = e^{\frac{i}{2} \kappa_{\zeta} - \kappa_{\zeta}^*} \), with \( \zeta = re^{i\pi} \) and \( \tau = \phi + \phi + \pi \). In terms of the complex parameter \( \zeta \), \( b_+ \) is written as \( b_+ = \frac{\zeta}{|\zeta|} \tanh(\zeta) \). It is straightforward to show, by using the faithful matrix representation, that
\[
S_q(\zeta) = e^{\frac{i}{2} \kappa_{\zeta} - \kappa_{\zeta}^*} \quad \Rightarrow \quad e^{h_{\kappa_{\zeta}} e^{\kappa_{\zeta}^*}} |\kappa_{\zeta}| e^{-\kappa_{\zeta}^*} K_{\zeta}^+ K_{\zeta}^{-},
\]
(\text{A13})

and
\[
U(t) = e^{-i\theta_0 e^{h_{\kappa_{\zeta}}} e^{h_{\kappa_{\zeta}^*}} |\kappa_{\zeta}| e^{-\kappa_{\zeta}^*} K_{\zeta}^+ K_{\zeta}^{-}} \quad \Rightarrow \quad e^{-i\theta_0 S_q(\zeta)} e^{h_{\kappa_{\zeta}}} e^{h_{\kappa_{\zeta}^*}} |\kappa_{\zeta}| e^{-\kappa_{\zeta}^*} K_{\zeta}^+ K_{\zeta}^{-},
\]
(\text{A14})

where we have defined \( R_0 = \frac{b_0}{1 - \tanh^2(\zeta)} \).

and of
\[
|\psi\rangle = e^{-\i \kappa_{\zeta}} e^{h_{\kappa_{\zeta}}} e^{h_{\kappa_{\zeta}^*}} |\kappa_{\zeta}| e^{-\kappa_{\zeta}^*} K_{\zeta}^+ K_{\zeta}^{-} |\xi\rangle,
\]
(\text{A15})

with \( D(\sqrt{2\mathcal{S}}) = \exp(\sqrt{2\mathcal{S}} a^\dagger - \sqrt{2\mathcal{S}} a) \). It is easy to prove that \( K_{\zeta} |\xi\rangle = S_{\zeta} |\xi\rangle \).

Then
\[
U |\psi\rangle = N_0 e^{-i\theta_0 S_q(\zeta)} e^{h_{\kappa_{\zeta}}} e^{h_{\kappa_{\zeta}^*}} |\kappa_{\zeta}| D(\sqrt{2\mathcal{S}}) |\xi\rangle,
\]
\[
= N_0 e^{-i\theta_0 S_q(\zeta)} e^{h_{\kappa_{\zeta}^*}} e^{h_{\kappa_{\zeta}}} |\kappa_{\zeta}| D(\sqrt{2\mathcal{S}}) |\xi\rangle,
\]
\[
= N_0 e^{-i\theta_0 R_0^{1/4}} e^{h_{\kappa_{\zeta}^*} - |\xi|} S_q(\zeta) D(\sqrt{2\mathcal{S}}) |\xi\rangle,
\]
(\text{A16})

and the normalization factor results
\[
N^{-2} = e^{i\mathcal{S}(\mathcal{R}_0 + \mathcal{R}_0 K_{\zeta}^+)} \sqrt{\mathcal{R}_0} \times \langle 0 | D(\sqrt{2\mathcal{S}}) S^+ \rangle S_q(\zeta) D(\sqrt{2\mathcal{S}}) |0\rangle
\]
\[
= e^{i\mathcal{S}(\mathcal{R}_0 + \mathcal{R}_0 K_{\zeta}^+)} \sqrt{\mathcal{R}_0}.
\]
(\text{A17})

Let us evaluate the fluctuation of the operators \( x \) and \( p \). In doing so, we shall make use of well known relations for the squeezing operator \( S_q(\zeta) \):
\[
S_q(\zeta)xS_q(\zeta)^{-1} = \langle x \rangle \langle \zeta \rangle (\cos(\tau) - \sin(\tau) \sinh(\zeta)) + \langle \zeta \rangle \langle x \rangle \sin(\tau) \sinh(\zeta),
\]
\[
S_q(\zeta)pS_q(\zeta)^{-1} = \langle p \rangle \langle \zeta \rangle (\cos(\tau) - \sin(\tau) \sinh(\zeta)) + \langle \zeta \rangle \langle p \rangle \sin(\tau) \sinh(\zeta),
\]
(\text{A18})

Finally, we can proceed to calculate the variances of \( p \) and \( x \). That is
\[
\Delta^2 p = \langle \psi | U^p | \psi \rangle - \langle \psi | U | \psi \rangle^2,
\]
\[
= \frac{1}{2} \left[ \cos(\tau) \sinh(2\tau^2) + \sinh^2(\tau) + \cosh^2(\tau) \right],
\]
(\text{A20})

and
\[
\Delta^2 x = \langle \psi | U^x | \psi \rangle - \langle \psi | U | \psi \rangle^2,
\]
\[
= \frac{1}{2} \left[ -\cos(\tau) \sinh(2\tau^2) + \sinh^2(\tau) + \cosh^2(\tau) \right].
\]
(\text{A21})

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