Superconformal Symmetry in Linear Sigma Model on Supermanifolds

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abstract

We consider a gauged linear sigma model in two dimensions with Grassmann odd chiral superfields. We investigate the Konishi anomaly of this model and find out the condition for realization of superconformal symmetry on the world-sheet. When this condition is satisfied, the theory is expected to flow into conformal theory in the infrared limit. We construct superconformal currents explicitly and study some properties of this world-sheet theory from the point of view of conformal field theories.
1 Introduction

It has recently proposed that the topological string theory on a supermanifold $\mathbb{CP}^{3|4}$ describes a twistor string \cite{1}, which is associated with super Yang-Mills theory and conformal supergravities in four dimensions. This topological B-model provides a powerful prescription for computing amplitudes of the super Yang-Mills theory.

This surprising correspondence attracts a lot of attention and various works have been done in this topic. Super Yang-Mills amplitudes are further investigated and interesting relations between the Yang-Mills theory and the topological string have been developed. On the other hand, string theories on the twistor spaces \cite{2}--\cite{9}, \cite{10, 11} or supermanifolds themselves were studied to formulate mirror symmetry for supermanifolds \cite{12}--\cite{19} and related string dualities \cite{20}--\cite{29}, \cite{30, 31}.

It is interesting to study geometry of supermanifolds \cite{32, 33} and to find profound structures beyond the bosonic manifolds. We investigated critical string theories on flat supermanifolds \cite{11} toward construction of consistent strings on these manifolds. It was a first step to understand the strings on the superspace as world-sheet theories and to obtain consistency conditions for them. Also we analyzed supermanifolds from the point of view of the gauged linear sigma models \cite{34}, \cite{12--14, 10} in order to understand properties of curved superspace backgrounds. For backgrounds for usual bosonic manifolds, Ricci-flat condition is necessary to construct consistent vacua of strings. If this condition is satisfied, conformal theories are realized on the world-sheet and physics of infrared (IR) region is controlled by conformal algebra. For models with supersymmetry, superconformal currents play important roles in describing physical properties of the IR dynamics.

In this paper, we shall consider gauged linear sigma models on weighted projective superspaces and investigate geometrical properties from viewpoints of world-sheet theories.

In section 2, we review gauged linear sigma models with a $U(1)$ gauge field and chiral superfields shortly. In section 3, we explain $U(1)$ R-symmetries and conservation of associated currents at the classical level. But one of the $U(1)$ symmetries is broken at the quantum level and induces anomaly. In section 4, we investigate this Konishi anomaly \cite{35, 36} for this $U(1)$ symmetry and construct anomaly equations explicitly. In deriving this result, we take two approaches; covariant calculation by the path integral methods and evaluation of one-loop effects in the light-cone gauge. Then we obtain the condition of anomaly cancellation for this symmetry and construct conserved currents explicitly. These supersymmetric currents flow into superconformal currents in the IR limit if there is no anomaly. In section 5, we calcu-
late operator products of these currents and study superconformal algebra of the IR theory. The lowest component of the field strength of vector superfield plays essential roles in these curved backgrounds. Together with gaugino, they behave as ghost fields. Also we discuss holomorphic forms and associated extended algebra for a few concrete cases. Section 6 is devoted to conclusions and discussions.

2 Model

In this section we review $\mathcal{N} = 2$ gauged linear sigma model shortly [34]. We want to describe supermanifolds by using two dimensional gauged linear sigma models. For simplicity we consider a $U(1)$ gauge group. The supercoordinates of the world-sheet are denoted by $(x_0, x_1, \theta^\pm, \bar{\theta}^\pm)$. We set the metric of the world-sheet to be $\eta_{ij} = \text{diag}(-1, +1)$. This model has $\mathcal{N} = 2$ supersymmetry on the world-sheet and there are four types of supercharges $Q_\pm$ and $\overline{Q}_\pm$

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\bar{\theta}^\pm (\partial_0 \pm \partial_1), \quad \overline{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - i\theta^\pm (\partial_0 \pm \partial_1).$$

(2.1)

Associated superderivatives $D_\pm$ and $\overline{D}_\pm$ are defined:

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm (\partial_0 \pm \partial_1), \quad \overline{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm (\partial_0 \pm \partial_1).$$

(2.2)

We shall take $m$ bosonic and $n$ fermionic coordinates and consider an $(m|n)$-dimensional superspace. Since these coordinates are regarded as the lowest components of chiral superfields, we introduce $m$ bosonic chiral superfields and $n$ fermionic chiral superfields,

$$\Phi_i^I(x), \quad (I = 1, 2, \cdots, m), \quad \Xi^A_0(x), \quad (A = 1, 2, \cdots, n).$$

Firstly an ordinary bosonic chiral superfield $\Phi_i^I$ is defined by $\overline{D}\Phi_i^I = 0$ and expressed in terms of component fields as

$$\Phi_0^I = \phi^I + \sqrt{2} (\theta^+ \psi^I_+ + \theta^- \psi^I_-) + 2\theta^+ \theta^- F^I$$

$$-i\theta^- \overline{\theta} (\partial_0 - \partial_1) \phi^I - i\theta^+ \overline{\theta}^+ (\partial_0 + \partial_1) \phi^I - \theta^+ \theta^- \overline{\theta}^+ (\partial_0^2 - \partial_1^2) \phi^I$$

$$-i\sqrt{2} \theta^+ \theta^- \overline{\theta}^- (\partial_0 - \partial_1) \psi^I_+ + i\sqrt{2} \theta^+ \theta^- \overline{\theta}^- (\partial_0 + \partial_1) \psi^I_-.$$  \quad (2.3)

Here $\phi^I$ and $F^I$ are bosons, while $\psi^I_+$ and $\psi^I_-$ are fermions. In the same way, we can define the fermionic chiral superfield $\Xi^A_0$ in component expansion:

$$\Xi^A_0 = \xi^A + \sqrt{2} (\theta^+ b^A_+ + \theta^- b^A_-) + 2\theta^+ \theta^- \chi^A$$

$$-i\theta^- \overline{\theta} (\partial_0 - \partial_1) \xi^A - i\theta^+ \overline{\theta}^+ (\partial_0 + \partial_1) \xi^A - \theta^+ \theta^- \overline{\theta}^+ (\partial_0^2 - \partial_1^2) \xi^A$$

$$-i\sqrt{2} \theta^+ \theta^- \overline{\theta}^- (\partial_0 - \partial_1) b^A_- + i\sqrt{2} \theta^+ \theta^- \overline{\theta}^- (\partial_0 + \partial_1) b^A_+.$$ \quad (2.4)
Fields $\xi^A$ and $\chi^A$ are Grassmann odd, while $b_+^A$ and $b_-^A$ are Grassmann even. So this chiral superfield behaves totally as a Grassmann odd superfield.

In order to construct a projective supermanifold, we consider the $U(1)$ gauge theory by introducing an abelian vector superfield $V$. It is written in the Wess-Zumino gauge,

$$V = -\sqrt{2}\theta^+\bar{\theta}^+ \sigma - \sqrt{2}\theta^+\bar{\theta}^- \sigma + \theta^+\bar{\theta}^- (v_0 - v_1) + \theta^+\bar{\theta}^+ (v_0 + v_1)$$
$$-2i\theta^+\theta^- (\bar{\theta}^+ \lambda_+ + \bar{\theta}^- \lambda_-) - 2i\bar{\theta}^+ \bar{\theta}^- (\theta^+ \lambda_+ + \theta^- \lambda_-) + 2\theta^+\theta^- \bar{\theta}^+ \bar{\theta}^- D,$$

and associated field strength of the superfield $\Sigma = (1/\sqrt{2})\bar{D}_+ D_- V$ is represented as

$$\Sigma = \sigma + i\sqrt{2}\theta^+ \lambda_+ - i\sqrt{2}\theta^- \lambda_- + \sqrt{2}\theta^+\bar{\theta}^- (D - iv_{01}) + i\theta^-\bar{\theta}^+ (\partial_0 - \partial_1)\sigma$$
$$-i\theta^+\bar{\theta}^- (\partial_0 + \partial_1)\sigma - \sqrt{2}\theta^+\theta^- (\partial_0 - \partial_1)\lambda_+ + \sqrt{2}\theta^+\bar{\theta}^- (\partial_0 + \partial_1)\lambda_-$$
$$+ \theta^+\theta^- \bar{\theta}^+ \bar{\theta}^- (\partial_0^2 - \partial_1^2)\sigma,$$

$v_{01} \equiv \partial_0 v_1 - \partial_1 v_0$.

This field strength is the twisted chiral superfield and the kinetic part of the gauge field is denoted by

$$L_{gauge} = -\frac{1}{e^2} \int d^4\theta \bar{\Sigma} \Sigma$$

where $e$ is a gauge coupling constant. We then assign $U(1)$ charges $Q_I$ on $\Phi^I_0$ and $q_A$ on $\Xi^A_0$. Then ungauged $(m|n)$-dimensional target space is reduced to a projective manifold due to this $U(1)$ gauge field. Now let us write down the Lagrangian of $\mathcal{N} = (2, 2)$ gauged linear sigma model

$$L = L_{\text{kin}} + L_{\text{gauge}},$$

$$L_{\text{kin}} = \int d^4\theta \left( \sum_{i=1}^m \Phi^I_0 e^{2Q_I V} \Phi^I_0 + \sum_{A=1}^n \Xi^A_0 e^{2q_A V} \Xi^A_0 \right).$$

When one considers the IR limit, this model turns to describe physics of the sigma model with target space $\mathbb{WCP}_{(Q_1, Q_2, \ldots, Q_m|q_1, q_2, \ldots, q_n)^{m-1|n}}$, namely $(m - 1|n)$-dimensional projective space.

### 3 Current and $U(1)$ Symmetry

The physics of the model in the IR region depends on the charges $(Q_I, q_A)$. When $\sum_I Q_I - \sum_A q_A > 0$, the model is asymptotic free. If $\sum_I Q_I - \sum_A q_A = 0$ is satisfied, the theory is scale invariant in the one-loop approximation and we can expect to construct conformal
currents of the underlying IR theory. We shall discuss this conformal theory. Now we introduce a current $J$,

$$J = \frac{1}{2} \sum_I D_- (\Phi_I^l e^{2Q_I V}) e^{-2Q_I V} D_- (e^{2Q_I V} \Phi_I^l) + \frac{1}{2} \sum_A D_- (\Xi_A^0 e^{2q_A V}) e^{-2q_A V} D_- (e^{2q_A V} \Xi_A^0) + \frac{2i}{e^2} \Sigma \partial_\Sigma. \quad (3.1)$$

We can verify conservation of the current classically $D_+ J = 0$ by using equations of motion,

$$D_+ D_- (\Phi_0^l e^{2Q_I V}) = 0, \quad (3.2)$$
$$D_+ D_- (\Xi_0^0 e^{2q_A V}) = 0, \quad (3.3)$$
$$\sum_I Q_I \Phi_I^l e^{2Q_I V} \Phi_I^l + \sum_A q_A \Xi_A^0 e^{2q_A V} \Xi_A^0 = \frac{1}{2 \sqrt{2} e^2} (D_- D_+ \Sigma + D_+ D_- \Sigma). \quad (3.4)$$

This current (3.1) has expansion in terms of supercoordinates $\theta^\pm, \overline{\theta}^\pm$. In order to write down formulae of currents explicitly, we expand superfields in terms of $\theta^-$ and $\overline{\theta}^-$

$$\Phi_I^l = \phi_I^l + \sqrt{2} \theta^{-} \Lambda_I^l - 2i \theta^{-} \overline{\theta}^- \partial_- \phi_I^l,$$
$$\Xi_0^A = \xi_0^A + \sqrt{2} \theta^- \overline{\Lambda}_0 - 2i \theta^- \overline{\theta}^- \partial_- \xi_0^A,$$
$$V = \Psi + \theta^- \overline{\theta}^- (v_0 - v_1 - 2ia) - \sqrt{2} \theta^- \overline{\theta}^- \Sigma - \sqrt{2} \theta^- \overline{\theta}^- \Sigma',$$
$$a = \theta^+ \overline{\lambda}_- + \overline{\theta}^+ \lambda_- + i \theta^+ \overline{\theta}^+ D,$$
$$\Sigma = \Sigma' + \frac{i}{\sqrt{2}} \overline{\theta}^+ \Upsilon + 2i \theta^- \overline{\theta}^- \partial_- \Sigma',$$
$$\partial_\pm \equiv \frac{1}{2} (\partial_0 \pm \partial_1),$$

where component fields are defined as

$$\Phi_I^l = \phi_I^l + \sqrt{2} \theta^+ \psi_I^l - 2i \theta^+ \overline{\theta}^+ \partial_+ \phi_I^l,$$
$$\Xi_0^A = \xi_0^A + \sqrt{2} \theta^+ b_A^+ - 2i \theta^+ \overline{\theta}^+ \partial_+ \xi_0^A,$$
$$\Lambda_I^l = \psi_I^l - \sqrt{2} \theta^+ F_I^l - 2i \theta^+ \overline{\theta}^+ \partial_+ \psi_I^l,$$
$$\overline{\Lambda}_0^A = b_A^+ - \sqrt{2} \theta^+ \chi_A^l - 2i \theta^+ \overline{\theta}^+ \partial_+ b_A^+,$$
$$\Psi = \theta^+ \overline{\theta}^+ (v_0 - v_1),$$
$$\Sigma' = \sigma + i \sqrt{2} \theta^+ \overline{\chi}_+ - 2i \theta^+ \overline{\theta}^+ \partial_+ \sigma,$$
$$\Upsilon = -2 \lambda_- + 2i \theta^+ (D - iv_01) + 4i \theta^+ \overline{\theta}^+ \partial_+ \lambda_-.$$
If we introduce fields defined as follows:

\[ \Phi^I = e^{Q_1 \Psi_0} \Phi^I_0, \quad \Xi^A = e^{\eta A \Psi_0} \Xi^A_0, \]
\[ \Lambda^I = e^{Q_1 \Psi_0} (\Lambda^I_0 - 2Q_1 \vec{\theta}^+ \Sigma^I \Phi^I_0), \]
\[ \tilde{\Lambda}^A = e^{\eta A \Psi_0} (\tilde{\Lambda}^A_0 - 2q_A \vec{\theta}^+ \Sigma^A \Xi^A_0), \]
\[ (\mathcal{D}_0 - \mathcal{D}_1) \Phi^I = (\partial_0 - \partial_1 + iQ_1(v_0 - v_1 - 2ia)) \Phi^I, \]
\[ (\mathcal{D}_0 - \mathcal{D}_1) \Xi^A = (\partial_0 - \partial_1 + iq_A(v_0 - v_1 - 2ia)) \Xi^I, \]

then component expansion of \( J \) is represented by currents \((J, G, \overline{G}, T)\)

\[ \mathcal{J} = J + 2\sqrt{2}i\theta^+ G + 2\sqrt{2}i\bar{\theta}^+ \overline{G} + 4\theta^- \bar{\theta}^- T, \]
\[ J = \sum I \Lambda^I \tilde{\Lambda}^I - \sum A \tilde{\Lambda}^\dagger \Lambda^\dagger A + \frac{2i}{e^2} \Sigma^I \partial_\Sigma^I, \]  
\[ (3.5) \]
\[ G = -\sum I \frac{1}{2} \Lambda^I (\overline{\mathcal{D}_0 - \mathcal{D}_1}) \Phi^I + \sum A \frac{1}{2} \tilde{\Lambda}^A \overline{\mathcal{D}_0 - \mathcal{D}_1} \Xi^A \Xi^\dagger A + \frac{i}{2e^2} \Sigma^I \partial_\Sigma^I, \]
\[ (3.6) \]
\[ \overline{G} = -\sum I \frac{1}{2} (\mathcal{D}_0 - \mathcal{D}_1) \Phi^I \cdot \tilde{\Lambda}^I - \sum A \frac{1}{2} (\mathcal{D}_0 - \mathcal{D}_1) \Xi^A \cdot \tilde{\Lambda}^A + \frac{i}{2e^2} \Sigma^I \partial_\Sigma^I, \]
\[ (3.7) \]
\[ T = \sum I \frac{1}{2} (\mathcal{D}_0 - \mathcal{D}_1) \Phi^I (\overline{\mathcal{D}_0 - \mathcal{D}_1}) \Phi^I + \sum A \frac{1}{2} (\mathcal{D}_0 - \mathcal{D}_1) \Xi^A \overline{\mathcal{D}_0 - \mathcal{D}_1} \Xi^\dagger A \]
\[ - \frac{i}{4e^2} \Sigma \partial_\Sigma - \frac{1}{e^2} \Sigma^I \partial_\Sigma^I \Sigma^\dagger \Sigma^\dagger \]
\[ - \frac{i}{4} \sum I \left[ \Lambda^I (\overline{\mathcal{D}_0 - \mathcal{D}_1}) \tilde{\Lambda}^I - (\mathcal{D}_0 - \mathcal{D}_1) \Lambda^I \cdot \tilde{\Lambda}^I \right] \]
\[ + \frac{i}{4} \sum A \left[ \tilde{\Lambda}^A (\overline{\mathcal{D}_0 - \mathcal{D}_1}) \Lambda^A - (\mathcal{D}_0 - \mathcal{D}_1) \tilde{\Lambda}^A \cdot \Lambda^A \right]. \]  
\[ (3.8) \]

These are grouped into an \( \mathcal{N} = 2 \) multiplet and the current in the highest component is identified with the energy momentum tensor. If we concentrate on the left movers, then
component fields are expressed explicitly
\[ J = \sum_I \psi^I \overline{\psi}^I - \sum_A b^A \overline{b}^A + \frac{2i}{e^2} \sigma \partial_+ \varphi, \quad (3.9) \]
\[ G = -\sum_I \overline{\psi}^I \overline{D}_- \phi^I + \sum_A b^A \overline{D}_- \xi^A - \frac{i}{e^2} \sigma \partial_- \lambda, \quad (3.10) \]
\[ \overline{G} = -\sum_I D_- \overline{\phi}^I \cdot \overline{\psi}^I + \sum_A D_- \overline{\xi}^A \cdot \overline{b}^A - \frac{i}{e^2} \lambda_- \partial_+ \varphi, \quad (3.11) \]
\[ T = -\sum_I 2\overline{D}_- \overline{\phi}^I \overline{D}_- \phi^I + \frac{i}{2} \sum_I (\overline{D}_- \overline{\psi}^I \cdot \overline{\psi}^I - \overline{\psi}^I \overline{D}_- \overline{\phi}^I) \]
\[ + \sum_A 2\overline{D}_- \overline{\xi}^A \overline{D}_- \xi^A - \frac{i}{2} \sum_A (\overline{D}_- b^A \cdot \overline{b}^A - b^A \overline{D}_- \overline{b}^A) \]
\[ - \frac{i}{e^2} \lambda_- \partial_- \lambda - \frac{i}{e^2} (\partial_- \sigma \partial_+ \varphi - \sigma \partial_+ \varphi), \quad (3.12) \]
where \( \overline{D}_- \equiv (1/2)(D_0 - D_1)|_{a=0} \). They could be superconformal currents if there is no anomaly.

Our model has two \( U(1) \) R-symmetries \( U(1)_R \) and \( U(1)_L \) at the classical level. They act on the supercoordinates \( (\theta^+, \theta^-, \overline{\theta}^+, \overline{\theta}^-) \)
\[ U(1)_R; (e^{+i\alpha} \theta^+, \theta^-, e^{-i\alpha} \overline{\theta}^+, \overline{\theta}^-), \]
\[ U(1)_L; (\theta^+, e^{+i\beta} \theta^-, \overline{\theta}^+, e^{-i\beta} \overline{\theta}^-), \quad \alpha, \beta \in \mathbb{R}. \]
But they are generally anomalous at the quantum level. In the next section, we shall discuss this anomaly.

### 4 Anomaly

We shall look at the \( U(1)_L \) symmetry. It acts on coordinates \( (\theta^-, \overline{\theta}^-) \) as \( (e^{+i\beta} \theta^-, e^{-i\beta} \overline{\theta}^-) \). Then relevant superfields \( (\Lambda_-, \overline{\Lambda}_-, \Sigma') \) transform to \( e^{-i\beta}(\Lambda_-, \overline{\Lambda}_-, \Sigma') \). This \( U(1)_L \) symmetry is conserved classically, but broken at quantum level and induces the Konishi anomaly \[35\] \[36\]. Now we compute this Konishi anomaly.

First we can rewrite Lagrangian by using component expansion
\[ S = \frac{1}{2} \int d^2y \int d\theta^+ d\overline{\theta}^+ \left[ \sum_I \overline{\Lambda}'^I \Lambda^I + \sum_A \overline{\Lambda}^-_A \Lambda^A + \frac{i}{e^2} (\overline{\Sigma}' \partial_+ \Sigma' - \partial_- \Sigma' \cdot \Sigma') \right] \]
\[ + \sum_I \frac{i}{2} \overline{\Phi}^I (D_0 - D_1) \Phi^I - \sum_I \frac{i}{2} (D_0 - D_1) \overline{\Phi}^I \cdot \Phi^I \]
\[ + \sum_A \frac{i}{2} \overline{\Xi}^A (D_0 - D_1) \Xi^A - \sum_A \frac{i}{2} (D_0 - D_1) \overline{\Xi}^A \cdot \Xi^A + \frac{1}{4e^2} \overline{\Upsilon}^{AA}. \]
In order to evaluate the anomaly, we consider transformations of the relevant fields under chiral $U(1)$ symmetry\(^6\)

\[
(\Lambda_-, \tilde{\Lambda}_-, \Sigma') \rightarrow e^{iA}(\Lambda_-, \tilde{\Lambda}_-, \Sigma') \equiv (\hat{\Lambda}_-, \hat{\tilde{\Lambda}}_-, \hat{\Sigma}').
\] (4.1)

Here the chiral superfield $A$ satisfies the condition $\overline{D}_+ A = 0$. The infinitesimal transformation $\delta S$ of the action can be expressed as

\[
\delta S = \frac{1}{2} \int d^2y \int d\theta^+ A(-i\overline{D}_+) \left[ \Lambda_- \Lambda_- + \tilde{\Lambda}_- \tilde{\Lambda}_- - \frac{2i}{e^2} (\partial \cdot \Sigma') \Sigma' \right] \equiv -\frac{1}{2} \int d^2y \int d\theta^+ A(-i\overline{D}_+)J.
\] (4.2)

Then the partition function $Z$ is represented under this $U(1)$ symmetry

\[
Z = \int D[\Lambda_-, \tilde{\Lambda}_-, \Sigma'] e^{iS} = \int D[\Lambda_-, \tilde{\Lambda}_-, \Sigma'] S\det \frac{(\hat{\Lambda}_-, \hat{\tilde{\Lambda}}_-, \hat{\Sigma}' )}{(\Lambda_-, \tilde{\Lambda}_-, \Sigma')} e^{i(S + \delta S)}. 
\] (4.3)

In this formula, $J$ is the same current as the one given in Eq.(3.5). Superdeterminant $S\det(\cdot \cdot \cdot)$ is the Jacobian of this transformation

\[
S\det \frac{(\hat{\Lambda}_-, \hat{\tilde{\Lambda}}_-, \hat{\Sigma}' )}{(\Lambda_-, \tilde{\Lambda}_-, \Sigma')} = S\det (-iA\overline{D}_+) = e^{S\text{tr}(-iA\overline{D}_+)}.
\] (4.4)

If this Jacobian is not identity, it induces anomaly. We can evaluate this effect by the Fujikawa’s method. In doing this calculation actually, we need to regularize the superdeterminant and introduce the regulator $L$ [37];

\[
L = -\frac{i}{2} \overline{D}_+ e^{-Q\mathcal{Y}} (D_0 - D_1) e^{-Q\mathcal{Y}} \overline{D}_+ e^{2Q\mathcal{Y}}
\]

\[
= -\frac{i}{2} Q\mathcal{Y} e^{-2Q\mathcal{Y}} D_+ e^{2Q\mathcal{Y}} + e^{-Q\mathcal{Y}} (D_0 - D_1)(D_0 + D_1) e^{Q\mathcal{Y}} + \frac{i}{2} e^{-Q\mathcal{Y}} (D_0 - D_1) D_+ \overline{D}_+ e^{Q\mathcal{Y}}.
\]

In deriving the formula in the second line, we used a relation

\[
Q\mathcal{Y} = [\overline{D}_+, \overline{D}_0 - \overline{D}_1].
\] (4.5)

Then we are able to evaluate the Jacobian [43] by using this regulator

\[
e^{S\text{tr}(-iA\overline{D}_+)} = \lim_{M \rightarrow \infty} \exp[\text{Str}(-iA e^{\frac{i}{2} M^2 \overline{D}_+})],
\]

\[
\equiv \lim_{M \rightarrow \infty} \exp \left[ \int d^2y \int d\theta^+ \langle y, \theta^+, \overline{\theta}^+ \mid (-iA e^{\frac{i}{2} M^2 \overline{D}_+}) \mid y, \theta^+, \overline{\theta}^+ \rangle \right].
\] (4.6)

\(^6\)For simplicity, we abbreviate flavor indices “$I$” and “$A$” for a moment
First we compute the contribution from \( \Lambda_\_ \). We insert a complete set in the integrand and evaluate the limit

\[
\lim_{M \to \infty} \langle y, \theta^+, \bar{\theta}^+ | (-iAe^{L_+^2D_+}) | y, \theta^+, \bar{\theta}^+ \rangle
\]

\[
= \lim_{M \to \infty} \int \frac{d^2k}{(2\pi)^2} d\eta^+ d\bar{\eta}^+ \langle y, \theta^+, \bar{\theta}^+ | (-iAe^{L_+^2D_+}) | k, \eta^+, \bar{\eta}^+ \rangle \langle k, \eta^+, \bar{\eta}^+ | y, \theta^+, \bar{\theta}^+ \rangle
\]

\[
= \lim_{M \to \infty} \int \frac{d^2k}{(2\pi)^2} d\eta^+ d\bar{\eta}^+ e^{-i(ky+\eta\theta)} (-iAe^{L_+^2D_+}) e^{i(ky+\eta\theta)}
\]

\[
= \lim_{M \to \infty} \int \frac{d^2k}{(2\pi)^2} d\eta^+ d\bar{\eta}^+ (-iA)(\bar{\eta}^+ - \theta^+ k^+ \eta) \exp \left[ -\frac{Q}{2M^2} \eta^+ + \bar{L} \right]
\]

\[
= \frac{Q}{2(2\pi)^2} A\Upsilon \int d^2k e^{-k^+ \cdot k^-} = \frac{i}{8\pi} Q A\Upsilon.
\]

Here we defined the rescaled momenta \((\tilde{k}^+, \tilde{k}^-)\) and \(\tilde{L}\) as

\[
k_1 = M\tilde{k}_1, \quad k_2 = M\tilde{k}_2,
\]

\[
\tilde{L} = -\frac{i}{2} \frac{Q}{M^2} \Upsilon (D_+ + 4Q\partial_+ \Psi + \bar{\theta}^+ k^+ \eta)
\]

\[+ \frac{1}{M^2} (D_0 - D_1 + 2Q\partial_- \Psi + ik^-)(D_0 + D_1 + 2Q\partial_+ \Psi + ik^+).
\]

Finally, after summing up contributions from \(\tilde{\Lambda}_\_\) and recovering flavor indices, we obtain the Konishi anomaly equation

\[
\overline{D}_+ J = -\frac{1}{4\pi} \left( \sum_I Q_I - \sum_A q_A \right) \Upsilon.
\]

(4.7)

We can also rewrite the Konishi anomaly equation for the current \(J\) given by (3.1),

\[
\overline{D}_+ J = \frac{i\sqrt{2}}{4\pi} \left( \sum_I Q_I - \sum_A q_A \right) \overline{D}_- \Sigma.
\]

(4.8)

From this equation, we can see that the Konishi anomaly vanishes only if \(\sum_I Q_I - \sum_A q_A = 0\). In such a case, the conservation of chiral currents \((J, G, \overline{G}, T)\) in Eqs. (3.3) – (3.8) is recovered and these currents generate \(\mathcal{N} = 2\) superconformal algebra. In the IR limit \(e^2 \to \infty\), the gauge fields are decoupled. Then we can evaluate the operator product expansion of these currents by free propagators. These represent \(\mathcal{N} = 2\) superconformal algebra with central charge \(c = 3(m - n - 1)\). (See Appendix B.)

We shall also show derivation of the Konishi anomaly from another viewpoint \[38\]. We want to evaluate one-loop effects in the lowest component \(J_- \sim \sum_I \psi_I^L \psi_I^r - \sum_A b^A \bar{b}^A\) +
The limit “lim” means that \( x_1 \to x \) and \( x_2 \to x \). Under these definitions, the chiral anomaly is correctly determined from Eq. (4.11) as
\[
\langle \partial_+ \left( \sum_I \psi_I^I \psi_I^- (x) - \sum_A b^A_- b^A_+ (x) \right) \rangle \sim \left( \sum_I Q_I - \sum_A q_A \right) \langle v_- (x) \rangle. \tag{4.14}
\]

Next we shall show (4.12) and (4.13) reproduce the Konishi anomaly. Since \( \overline{D_+} J \) can be evaluated by \([\overline{Q_+}, J]\), we obtain
\[
\overline{D_+} J \sim \left( \sum_I Q_I - \sum_A q_A \right) \lambda_-, \tag{4.15}
\]

also Appendix B.)

The limit “lim” means that \( x_1 \to x \) and \( x_2 \to x \). Under these definitions, the chiral anomaly is correctly determined from Eq. (4.11) as
\[
\langle \partial_+ \left( \sum_I \psi_I^I \psi_I^- (x) - \sum_A b^A_- b^A_+ (x) \right) \rangle \sim \left( \sum_I Q_I - \sum_A q_A \right) \langle v_- (x) \rangle. \tag{4.14}
\]

Next we shall show (4.12) and (4.13) reproduce the Konishi anomaly. Since \( \overline{D_+} J \) can be evaluated by \([\overline{Q_+}, J]\), we obtain
\[
\overline{D_+} J \sim \left( \sum_I Q_I - \sum_A q_A \right) \lambda_-, \tag{4.15}
\]

\[\begin{align*}
\langle \sum_I \psi_I^I \psi_I^- (x) : \mathcal{O} \rangle &= -\frac{i}{\pi} \sum_I Q_I \int d^2 z \frac{\langle v_+ (z) \mathcal{O} \rangle}{(x_1^- - z^-)(x_2^- - z^-)}, \tag{4.9} \\
\langle \sum_A b^A_- b^A_+ (x) : \mathcal{O} \rangle &= -\frac{i}{\pi} \sum_A q_A \int d^2 z \frac{\langle v_+ (z) \mathcal{O} \rangle}{(x_1^- - z^-)(x_2^- - z^-)} \tag{4.10}.
\end{align*}\]

These equations lead us to
\[
\partial_+ \langle \left( \sum_I \psi_I^I \psi_I^- (x) - \sum_A b^A_- b^A_+ (x) \right) : \mathcal{O} \rangle \\
\sim \left( \sum_I Q_I - \sum_A q_A \right) \langle \left( \partial_- v_+ (x) + \lim_{x_1 \to x_2} \frac{x_1^+ - x_2^+}{x_1^- - x_2^-} \partial_+ v_+ (x) \right) \mathcal{O} \rangle. \tag{4.11}
\]

Since there is ambiguity remained in Eq. (4.11), we should define the gauge invariant currents,
\[
\psi_I^I \psi_I^- (x) \equiv \lim_{x_1 \to x_2} \left[ \psi_I^I (x_1) \exp \left( i Q_I / 2 \int_{x_2}^{x_1} dx^\mu v_\mu \right) \psi_I^- (x_2) - \frac{i}{x_1^- - x_2^-} \right]
\sim : \psi_I^I \psi_I^- (x) : + \frac{Q_I}{2} \lim_{x_1 \to x_2} \frac{x_1^+ - x_2^+}{x_1^- - x_2^-} v_+ (x) + \frac{Q_I}{2} v_- (x), \tag{4.12}
\]

\[
b^A_- b^A_+ (x) \equiv \lim_{x_1 \to x_2} \left[ b^A_- (x_1) \exp \left( i q_A / 2 \int_{x_2}^{x_1} dx^\mu v_\mu \right) b^A_+ (x_2) - \frac{i}{x_1^- - x_2^-} \right]
\sim : b^A_- b^A_+ (x) : + \frac{q_A}{2} \lim_{x_1 \to x_2} \frac{x_1^+ - x_2^+}{x_1^- - x_2^-} v_+ (x) + \frac{q_A}{2} v_- (x). \tag{4.13}
\]

The same method is applicable to the computation of \( \langle \partial_+ \sum_I \psi_I^I \psi_I^- (x) + \sum_A b^A_- b^A_+ (x) : \mathcal{O} \rangle \). (See also Appendix B.)
where we used the supersymmetric transformations $[\mathcal{Q}_+, v_+] = 0$ and $[\mathcal{Q}_+, v_-] = 2i\lambda_-$. The right hand side of Eq.(4.15) reminds us of the first component of $\mathcal{D}_-\Sigma$, which is $\sqrt{2}i\lambda_-$. In fact, the supersymmetric completion of Eq.(4.15) becomes

$$\mathcal{D}_+ J \sim \left( \sum_I Q_I - \sum_A q_A \right) \mathcal{D}_- \Sigma. \quad \text{(4.16)}$$

That reproduces the results in the previous discussion and we can read that the vanishing condition of the Konishi anomaly is $\sum_I Q_I - \sum_A q_A = 0$. In this case we have conformal currents in the IR region, which are identified with $J$. We shall study such case in the next section.

5 \textbf{\textit{N}} = 2 \textbf{\textit{Superconformal Algebra}}

We consider \textit{N} = 2 currents of our model in the Euclidean case. In the IR limit $\epsilon^2 \to \infty$, the gauge fields are decoupled and we use free field representations. After appropriate redefinitions and Wick-rotations, we can write down a set of \textit{N} = 2 superconformal currents $(J, G, \overline{G}, T)$ in the Euclidean case

\begin{align*}
J &= -\sum_I \psi^I \psi^I - \sum_A \overline{b}^A b^A + \overline{\sigma} \partial \sigma, \\
G &= -i\sqrt{2} \sum_I \psi^I \partial \phi^I + i\sqrt{2} \sum_A b^A \partial \xi^A - \sqrt{2} \overline{\lambda} \partial \sigma, \\
\overline{G} &= -i\sqrt{2} \sum_I \overline{\psi}^I \partial \phi^I + i\sqrt{2} \sum_A \overline{b}^A \partial \xi^A - \sqrt{2} \lambda \partial \overline{\sigma}, \\
T &= -\sum_I \partial \overline{\phi}^I \partial \phi^I - \sum_A \partial \overline{\xi}^A \partial \xi^A - \overline{\lambda} \partial \lambda - \frac{1}{2} (\partial \overline{\sigma} \partial \sigma - \overline{\sigma} \partial^2 \sigma) \\
&\quad - \frac{1}{2} \sum_I (\overline{\psi}^I \partial \psi^I - \partial \overline{\psi}^I \cdot \psi^I) - \frac{1}{2} \sum_A (\overline{b}^A \partial b^A - \partial \overline{b}^A \cdot b^A). 
\end{align*}

Here each field has operator product expansion

\begin{align*}
\overline{\phi}^I (z) \phi^J (w) &\sim -\delta^{IJ} \log(z - w), \quad \overline{\psi}^I (z) \psi^I (w) \sim \frac{\delta^{IJ}}{z - w}, \\
\partial \overline{\xi}^A (z) \xi^B (w) &\sim \frac{\delta^{AB}}{z - w}, \quad \overline{b}^A (z) b^B (w) \sim \frac{-\delta^{AB}}{z - w}, \\
\overline{\lambda} (z) \lambda (w) &\sim \frac{1}{z - w}, \quad \overline{\sigma} (z) \partial \sigma (w) \sim \frac{1}{z - w}.
\end{align*}
Then we obtain \( \mathcal{N} = 2 \) superconformal algebra with central charge \( c = 3(m - n - 1) \)

\[
T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w),
\]

\[
T(z)J(w) \sim \frac{1}{(z-w)^2}J(w) + \frac{1}{z-w}\partial J(w),
\]

\[
T(z)G(w) \sim \frac{3/2}{(z-w)^2}G(w) + \frac{1}{z-w}\partial G(w),
\]

\[
T(z)\overline{G}(w) \sim \frac{3/2}{(z-w)^2}\overline{G}(w) + \frac{1}{z-w}\partial \overline{G}(w),
\]

\[
J(z)G(w) \sim \frac{+1}{z-w}G(w), \quad J(z)\overline{G}(w) \sim \frac{-1}{z-w}\overline{G}(w),
\]

\[
G(z)\overline{G}(w) \sim \frac{2c/3}{(z-w)^3} + \frac{2}{(z-w)^2}J(w) + \frac{1}{z-w}(2T + \partial J)(w),
\]

\[
J(z)J(w) \sim \frac{c/3}{(z-w)^2}, \quad G(z)G(w) \sim 0, \quad G(z)\overline{G}(w) \sim 0.
\]

Conformal weights and \( U(1) \) charges of fields are measured in terms of \( T \) and \( J \). Field \( \sigma \) is the lowest component of the field strength of the vector superfield \( \Sigma \). But this \( \sigma \) has conformal weight \( 1/2 \) with respect to \( T \). Similarly the gaugino \( \lambda \) in the multiplet \( \Sigma \) has weight \( 1 \). We summarize these data in Table 1.

| fields            | weights       | \( U(1) \) charges |
|-------------------|---------------|--------------------|
| \((\psi^I, \psi)\) | \((1/2, 1/2)\) | \((+1, -1)\)      |
| \((b^A, \overline{b}^A)\) | \((1/2, 1/2)\) | \((+1, -1)\)      |
| \((\partial \sigma, \overline{\sigma})\) | \((1/2, 1/2)\) | \((+1, -1)\)      |
| \((\lambda, \overline{\lambda})\) | \((0, 1)\)    | \((0, 0)\)        |
| \((\xi^A, \partial \xi^A)\) | \((0, 1)\)    | \((0, 0)\)        |

Table 1: Conformal weights and \( U(1) \) charges of fields

By taking account of the conformal weights, we can redefine fields \( \sigma, \overline{\sigma}, \lambda, \overline{\lambda} \) by \( b^\sigma, \overline{b}^\sigma, \xi^\sigma, \overline{\xi}^\sigma \)

\[
b^\sigma = i\partial \sigma, \quad \overline{b}^\sigma = i\sigma, \quad \xi^\sigma = \lambda, \quad \partial \overline{\xi}^\sigma = \overline{\lambda},
\]

\[
\Phi^P = (\phi^I, \xi^A, \xi^\sigma, \overline{\phi}^I, \overline{\xi}^A, \overline{\xi}^\sigma) = (\Phi^P; \overline{\Phi}^P),
\]

\[
\Psi^P = (\psi^I, b^A, \overline{b}^A, \overline{\psi}^I, \overline{b}^\sigma, \overline{\xi}^A) = (\Psi^P; \overline{\Psi}^P).
\]
Also we introduce symbol $|P|$ $(P = I, A, \sigma)$ as $|I| = 0$ and $|A| = |\sigma| = 1$. Then $\Phi^P$ and $\Psi^P$ have commutation or anti-commutation relations as follows:

$$
\Phi^P \cdot \Phi^Q = (-1)^{|P||Q|} \Phi^Q \cdot \Phi^P,
$$
$$
\Psi^P \cdot \Psi^Q = (-1)^{|P||Q|} \Psi^Q \cdot \Psi^P,
$$
$$
\Phi^P \cdot \Psi^Q = (-1)^{|P||Q|+1} \Psi^Q \cdot \Phi^P.
$$

Under this setup, $N = 2$ superconformal currents are expressed by $\Phi^P$ and $\Psi^P$

$$
J = -\frac{1}{2} N_{PQ} \Psi^P \Psi^Q,
$$
$$
G = -i\sqrt{2} \cdot \frac{1}{2}(M - N)_{PQ} \Psi^P \partial \Phi^Q,
$$
$$
\overline{G} = -i\sqrt{2} \Psi^P \partial \Phi^Q \cdot \frac{1}{2}(M - N)_{QP},
$$
$$
T = -\frac{1}{2} M_{PQ} \Psi^P \partial \Phi^Q - \frac{1}{2} M_{PQ} \Psi^P \partial \Psi^Q,
$$

where $M_{PQ}$ and $N_{PQ}$ have non-zero components

$$
M_I^J = M^I_J = \delta_{IJ}, \quad M^A_B = -M_B^A = -\delta_{AB}, \quad M_{\sigma\tau} = -M_{\sigma\tau} = -1,
$$
$$
N_I^J = -N^I_J = -\delta_{IJ}, \quad N^A_B = N_B^A = \delta_{AB}, \quad N_{\sigma\tau} = N_{\sigma\tau} = 1.
$$

Also we can introduce their inverse matrices $M^{PQ}$ and $N^{PQ}$ with $M_{PR}M^{RQ} = \delta_P^Q$ and $N_{PR}N^{RQ} = \delta_P^Q$. Propagators of fields $\Phi^P$ and $\Psi^P$ are expressed as

$$
\Phi^P(z)\Phi^Q(w) \sim -M^{PQ} \log(z - w), \quad \Psi^P(z)\Psi^Q(w) \sim \frac{M^{PQ}}{z - w}.
$$

In order to analyze geometric picture of manifolds, we consider twisted conformal field theories by introducing modified energy momentum tensors $\tilde{T}^{(\pm)} = T \pm \frac{1}{2} \partial J$

$$
\tilde{T}^{(\pm)} = -\frac{1}{2} M_{PQ} \partial \Phi^P \partial \Phi^Q - \frac{1}{2}(M \pm N)_{PQ} \Psi^P \partial \Psi^Q.
$$

When one measures conformal weights $\tilde{h}$ of fields by these new energy momentum tensors, $\Psi^p$ ($p = I, A, \sigma$) and $\Psi^\overline{p}$ ($\overline{p} = \overline{I}, \overline{A}, \overline{\sigma}$) have $\tilde{h} = 0, 1$ for $\tilde{T}^{(+)}$, $\tilde{h} = 1, 0$ for $\tilde{T}^{(-)}$. Also super stress tensors $G$ ($\overline{G}$) have respectively conformal weights 1 under $\tilde{T}^{(+)}$ ($\tilde{T}^{(-)}$). So we can define BRST operators for these twisted models

$$
Q = -\frac{i}{\sqrt{2}} \int G \quad \text{for} \quad \tilde{T}^{(+)},
$$
$$
\overline{Q} = -\frac{i}{\sqrt{2}} \int \overline{G} \quad \text{for} \quad \tilde{T}^{(-)}.
$$
These BRST charges act on fields with parameters $\alpha$, $\tilde{\alpha}$ in the following way and they can be interpreted as some kinds of “differential operators” on the supermanifold

$$[\alpha Q, \Phi^p] = \alpha \Psi^p, \quad [\tilde{\alpha} Q, \Phi^p] = \tilde{\alpha} \Psi^p S_{\Phi^p},$$

$$[\alpha Q, \Psi^p] = -\alpha \partial \Phi^p S_{\Psi^p}, \quad [\tilde{\alpha} Q, \Psi^p] = -\tilde{\alpha} \partial \Phi^p,$$

$$S = \begin{pmatrix} +I_m & 0 \\ 0 & -I_{n+1} \end{pmatrix}.$$

In this setting, $\Psi^p$ ($p = I, A, \sigma$) and $\Psi^p$ ($\overline{p} = \overline{I}, \overline{A}, \overline{\sigma}$) are naively set to differential forms on the manifold

$$(d\phi^I d\xi^A d\xi^\sigma) \leftrightarrow (\psi^I b^A b^\sigma),$$

$$(d\overline{\phi}^I d\overline{\xi}^A d\overline{\xi}^\sigma) \leftrightarrow (\overline{\psi}^I \overline{b}^A \overline{b}^\sigma).$$

For the purpose of further investigation, let us bosonize fields in $\Psi^p$

$$\psi^I = e^{\varphi^I}, \quad \overline{\psi}^I = e^{-\varphi^I},$$

$$b^A = e^{\varphi^A} \eta^A, \quad \overline{b}^A = e^{-\varphi^A} \partial \zeta^A,$$

$$b^\sigma = e^{\varphi^\sigma} \eta^\sigma, \quad \overline{b}^\sigma = e^{-\varphi^\sigma} \partial \zeta^\sigma,$$

$$\varphi^I(z)\varphi^J(w) \sim \delta^{IJ} \log(z - w),$$

$$\varphi^A(z)\varphi^B(w) \sim -\delta^{AB} \log(z - w), \quad \zeta^A(z)\eta^B(w) \sim \frac{\delta^{AB}}{z - w},$$

$$\varphi^\sigma(z)\varphi^\sigma(w) \sim -\log(z - w), \quad \zeta^\sigma(z)\eta^\sigma(w) \sim \frac{1}{z - w}.$$

Then $(\psi^I, \overline{\psi}^I)$ can be identified with $(b, c)$ systems with spins $(1, 0)$. On the other hand, $(b^A, \overline{b}^A)$ and $(b^\sigma, \overline{b}^\sigma)$ correspond to “bosonic ghost” systems $(\beta, \gamma)$ with spins $(1, 0)$. Our $\mathcal{N} = 2$ algebra contains $U(1)$ current $J$ that is the number current of these ghost and bosonic ghost fields in the context of twisted models. Differential forms on the manifold can be represented as ghost fields in the twisted model.

Let us return to the untwisted model. This $U(1)$ current is rewritten under the bosonization

$$J = \partial(\varphi^I - \varphi^A - \varphi^\sigma) = i\sqrt{c} \partial H, \quad \hat{c} = m - n - 1,$$

$$H(z)H(w) \sim -\log(z - w).$$

This formula means that the $U(1)$ current $J$ is the sum of fermion number current $J_F = -\overline{\psi}^I \psi^I = \partial \varphi^I$ and ghost number current $J_P = -\partial(\varphi^A + \varphi^\sigma)$. The set of currents $(T, G, \overline{G}, J)$
generate $\mathcal{N} = 2$ superconformal algebra with central charge $c = 3(m - n - 1)$. This algebra has algebra automorphism and NS sector and R sector are related by a spectral flow operator. Under this flow, the $(h, q) = (0, 0)$ state is transformed into states associated with primary fields $\Omega$ and $\overline{\Omega}$

$$\Omega = e^{i\sqrt{\hat{c}}H} = e^{\varphi^j - \varphi^A - \varphi^a} = e^{\varphi^j} \delta(b^A) \delta(b^a),$$
$$\overline{\Omega} = e^{-i\sqrt{\hat{c}}H} = e^{-\varphi^j + \varphi^A + \varphi^a} = e^{-\varphi^j} \delta(b^A) \delta(b^a).$$

These operators satisfy chiral primary condition $h = q/2$ with $q = \hat{c}$ for $\Omega$, $q = -\hat{c}$ for $\overline{\Omega}$. By looking at this formula, we could identify $d\xi^A, d\xi^\sigma (= d\lambda)$ with $\delta(b^A), \delta(b^\sigma)$ respectively. It means that the degree of differential forms on the supermanifold is measured by $U(1)$ charges associated with the current $J$, and differential forms are described by $\psi^I$’s, $\delta(b^A)$’s, $\delta(b^\sigma)$. In the geometric picture of the sigma model, $\Omega$ and $\overline{\Omega}$ are respectively associated with the holomorphic $\hat{c}$ form and the antiholomorphic $\hat{c}$ form.

Next we shall consider superpartners $U, \overline{U}$ of $\Omega, \overline{\Omega}$. They are defined by considering operator products

$$\overline{G}(z)\Omega(w) \sim \frac{i\sqrt{2}}{z-w} U(w), \quad G(z)\overline{\Omega}(w) \sim \frac{i\sqrt{2}}{z-w} \overline{U}(w),$$
$$U = - \sum_I \exp \left( \sum_{J \neq I} \varphi^J - \sum_A \varphi^A - \varphi^a \right) \partial \phi^I$$
$$+ \sum_A \exp \left( -2\varphi^A + \sum_I \varphi^J - \sum_{B \neq A} \varphi^B - \varphi^a \right) \partial \xi^A \partial \xi^A$$
$$+ \exp \left( -2\varphi^a + \sum_I \varphi^J - \sum_A \varphi^A \right) \partial \xi^\sigma \partial \xi^\sigma,$$
$$\overline{U} = - \sum_I \exp \left( - \sum_{J \neq I} \varphi^J + \sum_A \varphi^A + \varphi^a \right) \partial \phi^I$$
$$+ \sum_A \exp \left( 2\varphi^A - \sum_I \varphi^J + \sum_{B \neq A} \varphi^B + \varphi^a \right) \eta^A \partial \overline{\xi}^A$$
$$+ \exp \left( 2\varphi^a - \sum_I \varphi^J + \sum_A \varphi^A \right) \eta^\sigma \partial \overline{\xi}^\sigma,$$

where we omit cocycle factors for each term. These superpartners are primary fields with $h = (\hat{c} + 1)/2$ and $U, \overline{U}$ have $U(1)$ charges $q = \hat{c} - 1, -(\hat{c} - 1)$ respectively (see Table 2). Together with these $U, \overline{U}, \Omega$ and $\overline{\Omega}$, the $\mathcal{N} = 2$ superconformal algebra is enlarged to extended algebra. For $\hat{c} = 1$ case, $\Omega$ and $\overline{\Omega}$ have conformal weight $1/2$ and $q = \pm 1$. They
Table 2: Conformal weights and $U(1)$ charges of fields $\Omega, \overline{\Omega}, U$ and $\overline{U}$.

| fields | $\Omega$ | $\overline{\Omega}$ | $U$ | $\overline{U}$ |
|--------|---------|-----------------|-----|-------------|
| charge $q$ | $\hat{c}$ | $-\hat{c}$ | $\hat{c} - 1$ | $-(\hat{c} - 1)$ |
| weight $h$ | $\hat{c}/2$ | $\hat{c}/2$ | $(\hat{c} + 1)/2$ | $(\hat{c} + 1)/2$ |

turn out to be a pair of complex fermions. On the other hand, neutral fields $U$ and $\overline{U}$ have $h = 1$ and they are interpreted as derivatives of scalars, namely these fields correspond to a pair of complex bosons. Resulting algebra is direct product of $\mathcal{N} = 2$ superconformal algebra with $\hat{c} = 1$ and a pair of complex bosons and fermions. Let us see cases of $\hat{c} = 2, 3$ concretely.

5.1 $\hat{c} = 3$ case

This case corresponds to a Calabi-Yau threefold and resulting world-sheet theory is described by $c = 9$ algebra. When we put $M = \sqrt{2}\Omega$, $\overline{M} = \sqrt{2}\overline{\Omega}$, $K = iU$, $\overline{K} = i\overline{U}$, the extended algebra is expanded by eight currents $(T, G, \overline{G}, J, M, \overline{M}, K, \overline{K})$. They are shown in Fig. 1 $K$ and $\overline{K}$ are respectively superpartners of $\Omega$ and $\overline{\Omega}$. They contain subalgebra with $\mathcal{N} = 2$ superconformal symmetry with $\hat{c} = 1/3$ ($c = 1$). It is generated by currents $(\hat{T}, \hat{G}, \overline{G}, \hat{J})$

$$J = \partial \left( \sum \varphi^I - \sum A \varphi^A - \varphi^\sigma \right),$$

$$\hat{T} = \frac{1}{6} J^2, \quad \hat{J} = \frac{1}{3} J, \quad \hat{G} = \sqrt{\frac{2}{3}} \Omega, \quad \overline{\hat{G}} = \sqrt{\frac{2}{3}} \overline{\Omega}.$$  

This conformal subalgebra belongs to the $\mathcal{N} = 2$ minimal model and its spectrum is classified by $(\hat{h}, \hat{q}) = (0, 0), (1/6, \pm 1/3)$. In other words, these states are labeled by the original $U(1)$ charge $q = 0, \pm 1$. The character of this subalgebra is expressed by the Dedekind’s eta-function $\eta(\tau)$ and classical $SU(2)$ theta function $\Theta_{m,k}(\tau, \theta)$ as $\chi_q = \eta^{-1}(\tau) \Theta_{2q,3}(\tau/2, \theta)$.

5.2 $\hat{c} = 2$ case

This case is an analogue of four-dimensional hyperkähler case and resulting world-sheet theory is described by $\mathcal{N} = 4$ superconformal algebra. When we put

$$J^+ = i\Omega, \quad J^- = -i\overline{\Omega}, \quad J^3 = \frac{1}{2} J, \quad$$

$$G^+ = G, \quad G^- = \overline{G}, \quad G'^+ = \sqrt{2} U, \quad G'^- = \sqrt{2} \overline{U},$$
Fig. 1: Currents of $c = 9$ extended algebra. In this figure, $h$ is the conformal weight and $q$ is the $U(1)$ charge measured by $J$.

Eight currents $(T, G^\pm, G'^\pm, J^\pm, J)$ generate $\mathcal{N} = 4$ superconformal algebra with $c = 6$. It includes affine $\tilde{su}(2)_1$ algebra which is constructed by $J^\pm$ and $J^3 = J/2$. This symmetry seems to reflect an analogue of the hyperkähler structure of the manifold. Choice of the complex structure corresponds to pick up the Kähler class associated to the $U(1)$ current $J$. The other almost complex structures are related to $(2, 0)$-form $\Omega$ and $(0, 2)$-form $\overline{\Omega}$. Also $(G^+, G'^-)$ and $(G'^+, G^-)$ turn out to be doublets under this $su(2)$. Two components in each doublet are changed one another under the action of $J^\pm$. We summarize conformal weight $h$ and $U(1)$ charge $q$ in Fig. 2. Here we show charge $q$ measured by $J$. The $su(2)$ current $J^3$ is related to this $J$ through $J^3 = J/2$.

6 Conclusions

We have studied the gauged linear sigma model on the supermanifold with Grassmann odd chiral superfields, which provide Grassmann odd coordinates in the target supermanifold. In this paper we have considered the $U(1)$ gauge theory and target space of this model is reduced to the weighted projective space $\mathbb{WCP}^{m-1|n}$. This model has supersymmetry on the
Fig. 2: Currents of $\hat{c} = 2$ theory. They generate $\mathcal{N} = 4$ superconformal algebra with affine $\hat{su}(2)_1$ algebra. In this figure, $h$ is the conformal weight and $q$ is the $U(1)$ charge measured by $J$. The $su(2)$ current $J^3$ is related with $J$ through a relation $J^3 = J/2$. 
world-sheet and there are classically conserved $U(1)$ R-symmetries. If these R-symmetries are not anomalous, the resulting theory has superconformal symmetry in the IR region. But conservation of supercurrents could be broken generally due to anomalies for these $U(1)$ symmetries.

In order to study quantum properties of the theory, we have calculated the Konishi anomaly associated with the R-symmetries. We have taken two approaches in this calculation; covariant calculation by the path integral methods and evaluation of one-loop effects in the light-cone gauge. By these analyses, the condition of anomaly cancellation
\[ \sum_i Q_i - \sum_A q_A = 0 \]
is obtained. The result is consistent with the Ricci-flat condition of the supermanifold. If this condition is satisfied, then the theory flows into a superconformal theory in the IR limit. We also constructed superconformal currents explicitly in this region, and found that they generate $\mathcal{N} = 2$ superconformal algebra with central charge $c = 3(m - n - 1)$.

In this model, there are spin $1/2$ fields, $(\psi^I, \bar{\psi}^I)$ and $(b^A, \bar{b}^A)$ that contribute $+m$ and $-n$ to the central charge of this algebra. The lowest components $\sigma$ and $\bar{\sigma}$ of field strengths $\Sigma$ and $\bar{\Sigma}$ of vector superfield play essential roles in this model. The set $(\partial \sigma, \bar{\sigma})$ has conformal weights $(1/2, 1/2)$ respectively and behaves as an extra set of $(b^A, \bar{b}^A)$’s. Especially it induces extra contribution “$-1$” in the formula of $c$.

These fields are collected into the $U(1)$ current of $\mathcal{N} = 2$ superconformal algebra. By bosonizing the bosonic fields, we find that the $U(1)$ current $J$ is the sum of the fermion number current $J_F = \psi^I \bar{\psi}^I$ and ghost number current $J_P = -\partial(\varphi^A + \varphi^\sigma)$. This $J$ measures “charges” of fields, that are also identified with degrees of differential forms on the manifolds. Usually fermions $\psi^I$ and $\bar{\psi}^I$ are interpreted as differential forms on the manifold and charges associated with $J_F$ are identified with degrees of these forms. In our supermanifolds, bosonic fields $b^A, \bar{b}^A, b^\sigma, \bar{b}^\sigma$ could be identified with differential forms of Grassmann odd coordinates. From the correspondence between $U(1)$ charges and degrees of forms, we suppose that ghost number is identified with degree of differential forms of Grassmann odd coordinates.

In order to put forwards this interpretation, we discussed the holomorphic form $\Omega$ and its conjugate. Their expression has terms $\delta(b^A)\delta(\varphi^\sigma)$ and their conjugates that confirm the above supposition.

Finally we investigate structure of extended algebra generated by these $\Omega, \bar{\Omega}$ and their superpartners. The $\hat{c} = 3$ case is an analogue of the Calabi-Yau threefold and resulting theory is described by $c = 9$ algebra on the world-sheet. For $\hat{c} = 2$ case, the algebra is enlarged into $\mathcal{N} = 4$ superconformal symmetry with affine $\hat{su}(2)_1$. This model is an analogue of
four-dimensional hyperkähler manifolds. It implies that the affine \(su(2)\) reflects some kind of hyperkähler structure of the manifold. We are looking forward to further investigation towards this direction.

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A Conventions

\[
x^\pm \equiv x^0 \pm x^1, \quad \partial_\pm \equiv \frac{1}{2}(\partial_0 \pm \partial_1), \quad \partial_+ \partial_- \ln |x^+ x^-| = i\pi \delta(x^0) \delta(x^1) \equiv i\pi \delta^2(x), \quad \partial_- \frac{1}{x^+} = i\pi \delta^2(x), \quad \partial_+ \frac{1}{x^-} = i\pi \delta^2(x),
\]

\[
\int d^2\theta \equiv \frac{1}{2} \int d\theta^- d\theta^+, \quad \int d^2\bar{\theta} \equiv \frac{1}{2} \int d\bar{\theta}^+ d\bar{\theta}^-, \quad \int d^4\theta \equiv \int d^2\theta d^2\bar{\theta} = \frac{1}{4} \frac{\partial}{\partial \theta^+} \frac{\partial}{\partial \theta^-} \frac{\partial}{\partial \bar{\theta}^+} \frac{\partial}{\partial \bar{\theta}^-}.
\]

B Superconformal Algebra (Lorentzian case)

B.1 Two point functions

\[
\bar{\psi}^I(x)\phi^J(y) \sim -\frac{\delta^{IJ}}{2} \ln |x-y|^2, \quad \bar{\psi}^I_\pm(x)\psi^J_\pm(y) \sim -\frac{i\delta^{IJ}}{x^\pm - y^\pm},
\]

\[
\bar{\xi}^A(x)\xi^B(y) \sim \frac{\delta^{AB}}{2} \ln |x-y|^2, \quad \bar{\nu}^A_\pm(x)\nu^B_\pm(y) \sim \frac{i\delta^{AB}}{x^\pm - y^\pm},
\]

\[
\bar{\lambda}_\pm(x)\lambda_\pm(y) \sim -\frac{ie^2}{x^\pm - y^\pm}, \quad \bar{\sigma}(x)\sigma(y) \sim -\frac{e^2}{2} \ln |x-y|^2,
\]

\[
v_\pm(x)v_\pm(y) \sim \left(-\frac{e^2}{8} + \frac{\alpha}{2}\right)\frac{x^\pm - y^\pm}{x^\pm - y^\pm}, \quad v_\pm(x)v_\mp(y) \sim \left(\frac{e^2}{8} + \frac{\alpha}{2}\right) \ln |x-y|^2.
\]

We added \(-\frac{1}{2\pi} \int d^2y \frac{1}{8\alpha} (\partial^\mu v_\mu)^2\) to the action for gauge fixing.
B.2 $\mathcal{N} = 2$ superconformal algebra

Superconformal currents are expressed by Eqs. (3.9)–(3.12). When one considers the IR limit $e^2 \to \infty$, the gauge fields are decoupled. Then we can evaluate the operator product expansion of these currents by the free propagators (B.1)–(B.3),

$$J(x)J(y) \sim -\frac{m-n-1}{(x-y)^2},$$
$$J(x)G(y) \sim \frac{i}{x-y}G(y),$$
$$J(x)\overline{G}(y) \sim \frac{-i}{x-y}\overline{G}(y),$$
$$T(x)J(y) \sim \frac{J(y)}{(x-y)^2} + \frac{\partial_- J(y)}{x-y},$$
$$T(x)G(y) \sim \frac{3G(y)}{2(x-y)^2} + \frac{\partial_- G(y)}{x-y},$$
$$T(x)\overline{G}(y) \sim \frac{3\overline{G}(y)}{2(x-y)^2} + \frac{\partial_- \overline{G}(y)}{x-y},$$
$$T(x)T(y) \sim \frac{3(m-n-1)}{2(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial_- T(y)}{x-y},$$
$$G(x)\overline{G}(y) \sim \frac{2i(m-n-1)}{(x-y)^3} + \frac{2J(y)}{(x-y)^2} + \frac{2iT(y) + \partial_- J(y)}{x-y}. $$

These relations represent $\mathcal{N} = 2$ superconformal algebra with central charge $c = 3(m-n-1)$.

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