ESTIMATES ON TRANSITION DENSITIES OF SUBORDINATORS WITH JUMPING DENSITY DECAYING IN MIXED POLYNOMIAL ORDERS

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Abstract. In this paper, we discuss estimates on transition densities for subordinators, which are global in time. We establish the sharp two-sided estimates on the transition densities for subordinators whose Lévy measures are absolutely continuous and decaying in mixed polynomial orders. Under a weaker assumption on Lévy measures, we also obtain a precise asymptotic behaviors of the transition densities at infinity. Our results cover geometric stable subordinators, Gamma subordinators and much more.

Keywords and phrases: subordinator; transition density, transition density estimates; polynomially decaying Lévy measure;

1. Introduction and Main results

Since there are only a few known examples of stochastic processes for which the transition density can be computed explicitly, estimates and asymptotic behaviors of transition densities of stochastic processes are extremely important and have studied a lot. When the process is symmetric and has a strong Markov property, there are many beautiful results on this topic (see [1, 2, 3, 6, 9, 13, 14, 11, 12, 16, 24, 25, 26, 32, 39] and references therein for estimates for symmetric jump processes). But when the process is a non-symmetric jump process, estimates and asymptotic behavior of its transition density are known much less. See [10, 17, 18, 19, 28, 27, 29, 34, 35, 38, 36, 40] and references therein. In this paper, we discuss estimates on transition densities for a large class of non-decreasing Lévy processes on $\mathbb{R}$.

Let $S = (S_t)_{t \geq 0}$ be a subordinator, that is, a non-decreasing Lévy process on $\mathbb{R}$ with $S_0 = 0$. The process $S$ is characterized by its Laplace exponent $\phi$ which is given by

$$Ee^{-\lambda S_t} = e^{-t\phi(\lambda)} \quad \text{for all } t, \lambda \geq 0.$$ 

It is well known that $\phi$ is a Bernstein function with $\phi(0) = 0$ and there exists a unique constant $a \geq 0$ and a Borel measure $\nu$ on $(0, \infty)$ satisfying $\int_0^{\infty} \min\{1, s\} \nu(ds) < \infty$ such that

$$\phi(\lambda) = a\lambda + \int_0^{\infty} (1 - e^{-\lambda s}) \nu(ds). \quad (1.1)$$

The constant $a$ is called the drift and $\nu$ is called the Lévy measure of $S$ in the literature.

The main objective of this paper is to obtain two-sided estimates on the transition density for a large class of subordinators. Note that except a few special cases (see [8]), the transition probability density of subordinators can not be computed explicitly along side an expression for the Lévy measure. Through subordination and inverse subordination, the sharp estimates of the transition density of subordinators provide the sharp estimates of heat kernel of subordinate Markov process and two-sided estimates for the fundamental solution, respectively. See [27, Section 5], [15, Section 4], and [7, 45].

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Our assumptions are quite general; imposing only (mixed) polynomially decaying conditions locally at zero (or infinity) on the density of the Lévy measure. This paper is a continuation of the authors’ previous work in [20]. In [20], we studied tail probabilities of subordinators under various decaying conditions on the tail of the Lévy measure. In this paper, we concentrate on the case when the density of Lévy measure is (mixed) polynomially decaying. (cf. conditions (S.Poly.) and (L.Poly.) in [20].)

Recently in [27], estimates for transition density of subordinators have been also studied (see [27, Theorem A and Theorem 4.15]). In our context, their main assumptions on subordinators can be interpreted as our condition (S) or (G) holds with $0 < \alpha_1 \leq \alpha_2 < 1$. (See, (1.2) and (1.3) below.) In this paper, by imposing scaling conditions on the Lévy density directly, we allow the upper scaling index $\alpha_2$ at zero to be bigger than 1. (cf. [21, Remark 1.3(1)].) Moreover, we establish the large time counterpart of that result. In this situation, we even allow that the lower scaling index at zero can be negative (see (L-3) below). Hence, our results cover geometric stable subordinators. (See Example 3.3 and Section 4.1. below.)

In analysis on distributions of subordinators, by considering the subordinator $\tilde{S}_t = S_t - at$, we may assume that $a = 0$ without loss of generality. Hence, we always assume that $a = 0$ in this paper. Moreover, we always assume that the Lévy measure $\nu$ has a density function $\nu(x)$ and the following Hartman-Wintner type condition holds throughout this paper.

**(E)** There exists a constant $T_0 \in [0, \infty)$ such that

$$\liminf_{x \to 0} x \nu(x) = 1/T_0,$$

with a convention that $1/0 = \infty$.

In particular, the condition (E) implies that $\nu(0, \infty) = \infty$ and hence the subordinator $S_t$ is not a compounded Poisson process. Moreover, as a consequence of this condition, we obtain the existence and boundedness of the transition density function.

**Proposition 1.1.** For all $t > T_0$, the transition density $p(t, x)$ of the subordinator $S_t$ exists and is a continuous bounded function on $(0, \infty)$ as a function of $x$.

**Proof.** According to [31, (64) and (74)], (see also [37, (HW$_{1/4}$)]), it suffices to show that

$$\liminf_{|\xi| \to \infty} \frac{\text{Re } \phi(i\xi)}{\log(1 + |\xi|)} \geq \frac{1}{T_0}.$$

We first assume that $T_0 > 0$. Fix an arbitrary $\varepsilon > 0$. Then, by the assumption (E), there exists a constant $\delta > 0$ such that $\nu(x) \geq (1 - \varepsilon)T_0^{-1}x^{-1}$ for $x \in (0, \delta)$. On the other hand, since a Gamma subordinator, whose Laplace exponent is $\log(1 + \lambda)$, has the Lévy density $s^{-1}e^{-s}$, we get the following equalities:

$$\log(1 + \xi^2) = \text{Re } \log(1 + i\xi) = \text{Re } \int_0^{\infty} (1 - e^{-i\xi s})s^{-1}e^{-s}ds = \int_0^{\infty} (1 - \cos(\xi s))s^{-1}e^{-s}ds.$$

It follows that

$$\liminf_{|\xi| \to \infty} \frac{\text{Re } \phi(i\xi)}{\log(1 + |\xi|)} = \liminf_{\xi \to \infty} \frac{\int_0^{\infty} (1 - \cos(\xi s))\nu(s)ds}{\int_0^{\infty} (1 - \cos(\xi s))s^{-1}e^{-s}ds} \geq \frac{1 - \varepsilon}{T_0} \liminf_{\xi \to \infty} \frac{\int_{\delta}^{\infty} (1 - \cos(\xi s))s^{-1}e^{-s}ds}{\int_0^{\infty} (1 - \cos(\xi s))s^{-1}e^{-s}ds} \geq \frac{1 - \varepsilon}{T_0} \left(1 - \limsup_{\xi \to \infty} \frac{\int_{\delta}^{\infty} s^{-1}e^{-s}ds}{\log(1 + \xi)} \right) \geq \frac{1 - \varepsilon}{T_0}.$$

Hence, we get the result by letting $\varepsilon \to 0$. 
Then, we also deduce the result for the case when \( T_0 = 0 \) by letting \( T_0 \to 0 \).

Now, we enumerate our other main assumptions for the Lévy measure \( \nu \).

**(S-1)** There are constants \( c_1 > 0, R_1 \in (0, \infty] \) and \( \alpha_1 > 0 \) such that
\[
\frac{\nu(r)}{\nu(R)} \geq c_1 \left( \frac{R}{r} \right)^{1+\alpha_1} \quad \text{for all } 0 < r \leq R < R_1; \tag{1.2}
\]

**(S-2)** There are constants \( c_2 > 0, R_1 \in (0, \infty] \) and \( \alpha_2 > 0 \) such that
\[
\frac{\nu(r)}{\nu(R)} \leq c_2 \left( \frac{R}{r} \right)^{1+\alpha_2} \quad \text{for all } 0 < r \leq R < R_1; \tag{1.3}
\]

**(S-3)** There are constants \( c_3 > 0 \) and \( R_1 \in (0, \infty] \) such that
\[
\sup_{r \geq R_1} \nu(r) \leq c_3, \tag{1.4}
\]

with a convention that \( \sup \emptyset = 0 \);

**(S-3*)** There are constants \( c_4, c_5 > 0 \) and \( R_1 \in (0, \infty] \) such that
\[
c_4 \sup_{u \geq r} \nu(u) \leq \nu(r) \quad \text{and} \quad c_5 \nu(r) \leq \nu(2r) \quad \text{for all } r \geq R_1/2. \tag{1.5}
\]

**(S)** There exist a common constant \( R_1 \in (0, \infty] \) and constants \( c_1, c_2, c_3 > 0, \alpha_2 \geq \alpha_1 > 0 \) such that (S-1), (S-2) and (S-3) hold.

**(L-1)** There are constants \( c_6 > 0, R_2 > 0 \) and \( \alpha_3 > 0 \) such that
\[
\frac{\nu(r)}{\nu(R)} \geq c_6 \left( \frac{R}{r} \right)^{1+\alpha_3} \quad \text{for all } 0 < r \leq R < \infty; \tag{1.6}
\]

**(L-2)** There are constants \( c_7 > 0, R_2 > 0 \) and \( \alpha_4 > 0 \) such that
\[
\frac{\nu(r)}{\nu(R)} \leq c_7 \left( \frac{R}{r} \right)^{1+\alpha_4} \quad \text{for all } 0 < r \leq R < \infty; \tag{1.7}
\]

**(L-3)** There are constants \( c_8, c_9 > 0 \) and \( R_3 > 0 \) such that
\[
\frac{\nu(r)}{\nu(R)} \geq c_8 \left( \frac{R}{r} \right)^{-c_9} \quad \text{for all } 0 < r \leq R < R_3; \tag{1.8}
\]

**(L)** There exist a common constant \( R_2 \in (0, \infty) \) and constants \( c_6, c_7, c_8, c_9, R_3 > 0, \alpha_4 \geq \alpha_3 > 0 \) such that (L-1), (L-2) and (L-3) hold.

**(G)** The condition (S) holds with \( R_1 = \infty \).

**Remark 1.2.** (1) The condition (S-1) implies the condition (E) with \( T_0 = 0 \) and the condition (L-3) with \( R_3 = R_1 \).

(2) The constant \( \alpha_1 \) in the condition (S-1) should be less than 1. Indeed, since we have
\[
\infty > \int_0^r s \nu(s) ds \geq c_1 \nu(r) r^{-1-\alpha_1} \int_0^r s^{-\alpha_1} ds \quad \text{for all } r \in (0, R_1),
\]
it must hold that \( \alpha_1 < 1 \).

(3) A truncated \( \alpha \)-stable subordinator, whose Lévy measure \( \nu(ds) \) is given by
\[
\nu(ds) = s^{-1-\alpha} 1_{(0,1)}(s) ds \quad (0 < \alpha < 1),
\]
satisfies the condition (S) with \( R_1 = 1 \).

(4) Clearly, the condition (S-3*) implies the condition (S-3) with the same constant \( R_1 \). Indeed, we get \( \sup_{r \geq R_1} \nu(r) \leq c_4^{-1} \nu(R_1) \) under the condition (S-3*).
(5) Let $0 < \alpha_1 \leq \alpha_2 < 1$ and $m$ be a finite measure on $[\alpha_1, \alpha_2]$. Let $S$ be a subordinator without drift whose Lévy measure $\nu(dx)$ is given by

$$
\nu(dx) = \left( \int_{\alpha_1}^{\alpha_2} \frac{\beta}{\Gamma(1 - \beta)x^{\beta+1}} m(d\beta) \right) dx.
$$

Then, we can see that the subordinator $S$ satisfies the condition (G). Note that if $\alpha_1 = \alpha_2 = \alpha \in (0, 1)$ and $m$ is a Dirac measure on $\alpha$, then $S$ is a $\alpha$-stable subordinator.

(6) A geometric stable subordinator, whose Laplace exponent is $\log(1 + \nu)$ for $\nu \in (0, 1)$, has the Lévy density $\nu(x)$ such that $c_1^{-1}x^{-1} \leq \nu(x) \leq c_1 x^{-1}$ for $x \in (0, 1)$ while $c_1^{-1}x^{-1-\alpha} \leq \nu(x) \leq c_1 x^{-1-\alpha}$ for $x \in [1, \infty)$, for some constant $c_1 > 1$. Hence it satisfies the condition (L) while not satisfy (S). Note that the condition (E) is satisfied with $T_0 = 1/\alpha > 0$.

(7) The condition (L-3) is very mild. For instance, if the Lévy density is almost decreasing, then it holds trivially. Therefore, every subordinator whose Laplace exponent is a complete Bernstein function satisfies that assumption since its Lévy measure has a completely monotone density. (See [31] Chapter 16) for examples of complete Bernstein functions.)

Following [33], we let $H(\lambda) = \phi(\lambda) - \lambda \phi'(\lambda)$ and we define

$$
b(t) = (\phi' \circ H^{-1})(1/t) \quad \text{and} \quad w(r) = \nu(r, \infty).
$$

The function $H$ has an important role in estimates for the distributions of the subordinators. (see, e.g. [33, 39]) Also, the function $b$ is used in authors’ previous paper [20] (the definition of $b$-function in [20] is the same as $tb(t)$ in this paper) to describe a displacement with the highest probability of given subordinator at time $t$. We can see that $b(t)$ is strict increasing and $b(t) < \phi'(0)$.

From the definitions, we see that for every $\lambda > 0$,

$$
H(\lambda) \geq \int_0^{1/\lambda} \left( 1 - e^{-\lambda s} - \lambda s e^{-\lambda s} \right) \nu(s) ds \geq \frac{1}{2e} \lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds \tag{1.9}
$$

and

$$
H(\lambda) \geq \int_{1/\lambda}^{\infty} \left( 1 - e^{-\lambda s} - \lambda s e^{-\lambda s} \right) \nu(s) ds \geq \frac{e - 2}{e} w(1/\lambda). \tag{1.10}
$$

These inequalities follow from the facts that $1 - e^{-x} - xe^{-x} = e^{-x} (e^x - 1 - x) \geq (2e)^{-1}x^2$ for all $0 \leq x \leq 1$ and $1 - e^{-x} - xe^{-x} \geq e^{-1}(e - 2)$ for all $x \geq 1$. In particular, (1.10) implies that for all $t > 0$,

$$
w^{-1}(2e/t) \leq w^{-1}(e(\nu - 2)^{-1}) \leq H^{-1}(1/t)^{-1}. \tag{1.11}
$$

Let

$$
D(t) := \max_{s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]} sH(s^{-1}).
$$

Then we define a function $\theta : (0, \infty) \times [0, \infty) \to (0, \infty)$ by

$$
\theta(t, y) := \begin{cases} 
H^{-1}(1/t)^{-1} & \text{if } y \in [0, H^{-1}(1/t)^{-1}], \\
\min \{ s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}] : tsH(s^{-1}) = y \} & \text{if } y \in [H^{-1}(1/t)^{-1}, D(t)], \\
w^{-1}(2e/t) & \text{if } y \in (D(t), \infty). 
\end{cases} \tag{1.12}
$$

Note that $\theta(t, y) \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]$ for all $t > 0$ and $y \geq 0$. In particular, for each fixed $y \geq 0$, we have $\lim_{t \to 0} \theta(t, y) = 0$ and $\lim_{t \to \infty} \theta(t, y) = \infty$. However, neither $t \mapsto \theta(t, y)$ nor $y \mapsto \theta(t, y)$ is a monotone function in general.

Following [33], for $t > 0$ and $x \in (0, 2b'(0))$, we abbreviate

$$
\sigma = \sigma(t, x) := (\phi')^{-1}(x/t). \tag{1.13}
$$
(This function is denoted by $\lambda_t$ in [33].) Since $\phi'$ is non-increasing, $\sigma$ is a non-increasing function on $x$ for each fixed $t$ and a non-decreasing function on $t$ for each fixed $x$. From the definitions and the monotonicities of $H, \sigma$ and $b$, we have that for every $t > 0$, \[ tH(\sigma) > 1 \text{ for all } x \in (0, tb(t)) \quad \text{and} \quad tH(\sigma)|_{(t,x)=tb(t)} = 1. \]

Hereinafter, we denote $a \land b := \min\{a, b\}$ and $a \lor b := \max\{a, b\}$. The following theorems are the main results of this paper.

**Theorem 1.3.** Let $S$ be a subordinator satisfying (S). Then, for every $T > 0$, there exist constants $c_1, c_2, c_3, c_5 > 1$ and $c_4 > 0$ such that the following estimates hold for all $t \in (0, T]$.

1. It holds that for all $x \in (0, tb(t)]$, \[ \frac{c^{-1}_1}{\sqrt{t(-\phi''(\sigma))}} \exp \left( -tH(\sigma) \right) \leq p(t, x) \leq \frac{c_1}{\sqrt{t(-\phi''(\sigma))}} \exp \left( -tH(\sigma) \right), \quad (1.14) \]
where $\sigma$ is defined as $(1.13)$. In particular, it holds that for all $x \in (0, tb(t)]$, \[ c_2^{-1}H^{-1}(1/t) \exp \left( -2tH(\sigma) \right) \leq p(t, x) \leq c_2H^{-1}(1/t) \exp \left( -\frac{t}{2}H(\sigma) \right) \quad (1.15) \]

2. It holds that for all $y \in [0, R_1/2)$, \[ c_3^{-1}H^{-1}(1/t) \min \left\{ 1, \frac{tv(y)}{H^{-1}(1/t)} + \exp \left( -\frac{c_4y}{\theta(t, y/(8e^2))} \right) \right\} \leq p(t, tb(t) + y) \leq c_3H^{-1}(1/t) \min \left\{ 1, \frac{tv(y)}{H^{-1}(1/t)} + \exp \left( -\frac{y}{8\theta(t, y/(8e^2))} \right) \right\}, \quad (1.16) \]
where $\theta(t, y)$ is defined as $(1.12)$. In particular, for all $y \in (D(t), R_1/2)$, \[ c_5^{-1}tv(y) \leq p(t, tb(t) + y) \leq c_5tv(y). \quad (1.17) \]

Moreover, if $S$ also satisfies the condition (S-3*), then $(1.16)$ holds for all $y \in [0, \infty)$ and $(1.17)$ holds for all $y \in (D(t), \infty)$.

**Theorem 1.4.** Let $S$ be a subordinator satisfying (E) and (L).

1. There exist constants $T_1 > T_0$, $c_1, c_2, c_3, c_5 > 1$ and $c_4 > 0$ such that for all $t \in [T_1, \infty)$, $(1.14)$ holds for all $x \in (0, tb(t)]$, $(1.15)$ holds for all $x \in [tb(T_1), tb(t)]$, $(1.16)$ holds for all $y \in [0, \infty)$ and $(1.17)$ holds for all $y \in (D(t), \infty)$.

2. If $T_0 = 0$ in the condition (E), then for every $T > 0$, there are comparison constants such that for all $t \in [T, \infty)$, $(1.14)$ holds for all $x \in (0, tb(t)]$, $(1.15)$ holds for all $x \in [tb(T), tb(t)]$, $(1.16)$ holds for all $y \in [0, \infty)$ and $(1.17)$ holds for all $y \in (D(t), \infty)$.

**Corollary 1.5.** Let $S$ be a subordinator satisfying (G). Then, there exist constants $c_1, c_2, c_3, c_5 > 1$ and $c_4 > 0$ such that for all $t \in (0, \infty)$, $(1.14)$ and $(1.15)$ hold for all $x \in (0, tb(t)]$, $(1.16)$ holds for all $y \in [0, \infty)$ and $(1.17)$ holds for all $y \in (D(t), \infty)$.

Our main theorems also cover the cases when $\alpha_3 \leq 1$ and $\alpha_4 \geq 2$. In such cases, the exponential term in the right tail estimates may have an efficient effect on estimates at specific times while have no role in other time values. (See, Section 4.2.) Note that since the condition (E) guarantees the existence of a continuous bounded transition density function $p(t, x)$ only for $t > T_0$, we should choose the constant $T_1$ bigger than $T_0$ in Theorem 1.4.

If we impose additional conditions on decaying orders of the density of Lévy measure, then we can simplify the right tail estimates in our theorems. Consider the following further conditions:

**S.Pure** Condition (S) holds with $\alpha_2 < 2$. 

**Corollary 1.8.** Let $c$ be a constant such that the above estimates hold for all $t$. Therefore, there exists a constant $c_1 > 1$ such that

$$c_1^{-1} \left( H^{-1}(1/t) \land \nu(y) \right) \leq p(t, tb(t) + y) \leq c_1 \left( H^{-1}(1/t) \land \nu(y) \right).$$

(1.18)

Therefore, there exists a constant $c_2 > 1$ such that for all $t \in (0, T]$ and $x \in (0, R_1/2)$,

$$c_2^{-1} \min \left\{ H^{-1}(1/t) \exp \left( -2tH(\sigma) \right), \nu((x - tb(t))_+) \right\} \leq p(t, x) \leq c_2 \min \left\{ H^{-1}(1/t) \exp \left( -\frac{t}{2}H(\sigma) \right), \nu((x - tb(t))_+) \right\}.$$  

(1.19)

Moreover, if $S$ also satisfies the condition (S-3*), then (1.18) holds for all $t \in (0, T]$ and $y \in [0, \infty)$, and (1.19) holds for all $t \in (0, T]$ and $x \in (0, \infty)$.

**Corollary 1.7.** Let $S$ be a subordinator satisfying (S.Pure). Then, for every $T > 0$, there exists a constant $c_1 > 1$ such that for all $t \in (0, T]$ and $y \in [0, \infty)$, and (1.19) holds for all $t \in [T, \infty)$ and $x \in [tb(T), \infty)$.

Moreover, if $T_0 = 0$ in the condition (E), then for every $T > 0$, there are comparison constants such that (1.18) holds for all $t \in [T, \infty)$ and $y \in [0, \infty)$, and (1.19) holds for all $t \in [T, \infty)$ and $x \in [tb(T), \infty)$.

Under the condition (L.Mixed), we can find a monotone function which is easy to compute and can play the same role as the function $\theta$. Define

$$\mathcal{H}(r) := \inf_{s \geq r} \frac{1}{sH(s^{-1})} \quad \text{and} \quad \mathcal{H}^{-1}(u) := \sup\{r \in \mathbb{R} : \mathcal{H}(r) \leq u\}.$$  

Recall that under the condition (L.Mixed), $\phi'(0)$ is finite. See (3.69) and a line below.

**Corollary 1.9.** Let $S$ be a subordinator satisfying (E) and (L.Mixed). Then, there exist constants $c_1 > 1$ and $c_2, c_3 > 0$ such that for all $t \in [T_1, \infty)$ and $y \in [0, \infty)$,

$$c_1^{-1} H^{-1}(1/t) \min \left\{ 1, \frac{t\nu(y)}{H^{-1}(1/t)} + \exp \left( -\frac{c_2 y}{\mathcal{H}^{-1}(t/y)} \right) \right\} \leq p(t, tb(t) + y) \leq c_1 H^{-1}(1/t) \min \left\{ 1, \frac{t\nu(y)}{H^{-1}(1/t)} + \exp \left( -\frac{c_3 y}{\mathcal{H}^{-1}(t/y)} \right) \right\}.$$  

Moreover, if $T_0 = 0$ in the condition (E), then for every $T > 0$, there are comparison constants such that the above estimates hold for all $t \in [T, \infty)$ and $y \in [0, \infty)$. 

**Remark 1.6.** (1) Since $\alpha_1$ should be less than 1, (see Remark 1.2(2),) there is no analogous condition to (L.Mixed) concerning the condition (S).

(2) We have $\phi'(0) < \infty$ under the condition (L.Mixed). Indeed, we see that

$$\phi'(0) = \int_0^{R_3} s \nu(s) ds + \int_{R_3}^\infty s \nu(s) ds \leq c + c_0^{-1} \nu(R_3) R_3^{1+\alpha_3} \int_{R_3}^\infty s^{-\alpha_3} ds < \infty.$$  

Under either of the conditions (S.Pure) or (L.Pure), we obtain pure jump type estimates on the right tails of $p(t, x)$.

Recall that $\sigma = (\phi')^{-1}(x/t)$ for $t > 0$ and $x \in (0, t\phi'(0))$. In the following corollary, we let $\sigma = 0$ for $t > 0$ and $x \geq t\phi'(0)$ so that $x \mapsto \sigma$ is a non-increasing function on $(0, \infty)$ for each fixed $t > 0$. We use the notation $z_+ = \max\{z, 0\}$ for $z \in \mathbb{R}$. 

**Corollary 1.7.** Let $S$ be a subordinator satisfying (S.Pure). Then, for every $T > 0$, there exists a constant $c_1 > 1$ such that for all $t \in (0, T]$ and $y \in [0, \infty)$, and (1.18) holds for all $t \in [0, T]$ and $x \in (0, \infty)$. 

**Corollary 1.8.** Let $S$ be a subordinator satisfying (E) and (L.Pure). Then, there exist constants $T_1 > T_0$ and $c_1 > 1$ such that (1.18) holds for all $t \in [T_1, \infty)$ and $y \in [0, \infty)$, and (1.19) holds for all $t \in [T_1, \infty)$ and $x \in [tb(T_1), \infty)$.

Moreover, if $T_0 = 0$ in the condition (E), then for every $T > 0$, there are comparison constants such that (1.18) holds for all $t \in [T, \infty)$ and $y \in [0, \infty)$, and (1.19) holds for all $t \in [T, \infty)$ and $x \in [tb(T), \infty)$. 

Under the condition (L.Mixed), we can find a monotone function which is easy to compute and can play the same role as the function $\theta$. Define
The above corollary may be considered as a counterpart of [1, Theorem 1.5(2)] where a similar result was obtained for symmetric jump processes. (See, Section 5.)

In this paper, we also discuss the precise asymptotical properties of densities of subordinators. (cf, [21, 27].) The asymptotic expressions are given in terms of \( \pi_t \) and its derivatives. Under the condition (L-3) we show in Corollary 3.3 that the density of the subordinator is asymptotically equal to \( (2\pi t(-\phi''(\sigma)))^{-1/2} \exp \left( -tH(\sigma) \right) \) as \( t \to \infty \). If, in addition, the constant \( T_0 = 0 \) in the condition (E) is zero then the same result holds as \( x \to 0 \). In Example 3.1 we apply Corollary 3.3 to geometric stable subordinators and get the exact asymptotic behavior of the transition density of geometric stable subordinators as \( t \to \infty \). Up to authors' knowledge, since Pillai introduced the series formula (3.22) of the transition density of geometric stable subordinator in [40] in 1990, its exact asymptotic behaviors given in (3.28) and (3.31)–(3.33) have been unknown.

**Notations:** In this paper, the positive constants \( T_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, R_1, R_2 \) and \( R_3 \) will remain the same. Lower case letters \( c 's \) without subscripts denote strictly positive constants whose values are unimportant and which may change even within a line, while values of lower case letters with subscripts \( c_i, i = 0, 1, 2, \ldots \) are fixed in each statement and proof, and the labeling of these constants starts anew in each proof.

We use the symbol "\( : = \)" to denote a definition, which is read as "is defined to be." Recall that \( \min \{ a, b \} \) and \( \max \{ a, b \} \).

The notation \( f(x) \asymp g(x) \) means that there exist comparison constants \( c_1, c_2 > 0 \) such that \( c_1 g(x) \leq f(x) \leq c_2 g(x) \) for the specified range of the variable \( x \). On the other hand, the notation \( f(x) \lessgtr g(x) \) means that there exist comparison constants \( c_3, c_4, c_5, c_6 > 0 \) such that \( c_3 g_1(x) + g_2(x) h_1(x) \leq f(x) \leq c_5 (g_1(x) + g_2(x) h_2(x)) \) for the specified range.

2. **Auxiliary functions and basic estimates**

Recall that

\[
H(\lambda) = \phi(\lambda) - \lambda \phi'(\lambda) = \int_0^\infty (1 - e^{-\lambda s} - \lambda s e^{-\lambda s}) \nu(s) ds, \\
b(t) = (\phi' \circ H^{-1})(1/t) = \int_0^\infty s e^{-H^{-1}(1/t)s} \nu(s) ds, \quad w(r) = \nu(r, \infty).
\]

Since \( \phi \) is a Bernstein function with \( \phi(0) = 0 \), we see that \( H(0) = 0 \) and \( H \) is strictly increasing on \((0, \infty)\). Also, it is easy to see that \( H(R) \leq (R/r)^2 H(r) \) for all \( R \geq r > 0 \). Recall that \( H \) satisfies (1.9) and (1.10). Moreover, it holds that

\[
e^{-1}(\lambda^2 \int_0^{1/\lambda} sw(s) ds \leq H(\lambda) \leq 5\lambda^2 \int_0^{1/\lambda} sw(s) ds \quad \text{for all } \lambda > 0. \quad (2.1)
\]

Indeed, by the Fubini’s theorem, \( \phi(\lambda)/\lambda = \int_0^\infty \int_0^x e^{-\lambda u} \nu(s) du ds = \int_0^\infty e^{-\lambda u} \int_u^\infty \nu(s) ds du = \int_0^\infty e^{-\lambda u} w(u) du \) for all \( \lambda > 0 \). It follows that

\[
\frac{H(\lambda)}{\lambda^2} = -\left( \frac{\phi(\lambda)}{\lambda} \right)' = \int_0^\infty e^{-\lambda s} sw(s) ds \quad \text{for all } \lambda > 0.
\]

Since \( \int_0^{1/\lambda} e^{-\lambda s} sw(s) ds \geq e^{-1} \int_0^{1/\lambda} sw(s) ds \geq e^{-1} w(1/\lambda) \int_0^{1/\lambda} s ds = 2^{-1} e^{-1} \lambda^{-2} w(1/\lambda) \) and \( \int_0^\infty e^{-\lambda s} sw(s) ds \leq w(1/\lambda) \int_0^\infty se^{-\lambda s} ds = 2 e^{-1} \lambda^{-2} w(1/\lambda) \), we get

\[
\int_0^{1/\lambda} e^{-\lambda s} sw(s) ds \leq \int_0^\infty e^{-\lambda s} sw(s) ds \leq 5 \int_0^{1/\lambda} e^{-\lambda s} sw(s) ds \quad \text{for all } \lambda > 0.
\]

Therefore, since \( e^{-1} \leq e^{-\lambda s} \leq 1 \) for \( s \in [0, 1/\lambda] \), we get (2.1).
Recall that the condition (L-3) is weaker than the condition (S-1). (See, Remark 12 (1).) Hence, the following lemma also hold under the condition (S-1).

We denote $\phi^{(n)}$ the $n$-th derivative of the function $\phi$.

**Lemma 2.1.** Suppose that (L-3) holds.

1. For every $\lambda > 0$, there are constants $c_n > 1$, $n = 1, 2, \ldots$ such that
   \[
   e^{-1} \int_0^{1/\lambda} s^n \nu(s) ds \leq |\phi^{(n)}(\lambda)| \leq c_n \int_0^{1/\lambda} s^n \nu(s) ds \quad \text{for all } \lambda \geq \lambda_0 \text{ and } n \geq 1.
   \]

2. For every $\lambda > 0$, there are constants $c_n' > 1$, $n = 1, 2, \ldots$ such that
   \[
   c_n^{-1} |\phi^{(n)}(2\lambda)| \leq |\phi^{(n)}(\lambda)| \leq c_n' |\phi^{(n)}(2\lambda)| \quad \text{for all } \lambda \geq \lambda_0 \text{ and } n \geq 1.
   \]

3. For every $\lambda > 0$, there are constants $c_n'' > 0$, $n = 1, 2, \ldots$ such that
   \[
   |\lambda \phi^{(n+1)}(\lambda)| \leq c_n'' |\phi^{(n)}(\lambda)| \quad \text{for all } \lambda \geq \lambda_0 \text{ and } n \geq 1.
   \]

**Proof.** (1) First, we see that for all $\lambda > 0$ and $n \geq 1$,
   \[
   |\phi^{(n)}(\lambda)| \geq \int_0^{1/\lambda} s^n \nu(s) ds \geq e^{-1} \int_0^{1/\lambda} s^n \nu(s) ds.
   \]

On the other hand, we have that for all $\lambda \geq 2R_3^{-1}$ and $n \geq 1$,
   \[
   |\phi^{(n)}(\lambda)| = \int_0^{1/\lambda} s^n \nu(s) ds + \int_{R_3}^{\infty} s^n \nu(s) ds + \lambda^n \int_0^{\infty} s^n \nu(s) ds \\
   \leq \int_0^{1/\lambda} s^n \nu(s) ds + \lambda^n \int_0^{\infty} s^n \nu(s) ds + \lambda^n \int_0^{\infty} s^n \nu(s) ds \\
   \leq \int_0^{1/\lambda} s^n \nu(s) ds + \lambda^n \int_0^{\infty} s^n \nu(s) ds,
   \]
   where $c_9 > 0$ is the constant in (L-8). We used the assumption (1.8) and the fact that for every $n \geq 1$, there exists a constant $c > 0$ such that $x^n \leq ce^{x/2}$ for all $x \geq 0$ in the first inequality and the change of the variables $u = \lambda s$ in the second inequality.

Using the assumption (L-8) (twice) and the inequality $x^n \leq ce^{x/2}$ again, it also hold that for all $\lambda \geq 2R_3^{-1}$,
   \[
   \int_0^{1/\lambda} s^n \nu(s) ds \geq \nu(1/\lambda) \int_0^{1/\lambda} s^n \nu(s) ds \geq c \nu(1/\lambda) \int_0^{1/\lambda} s^n \nu(s) ds \\
   \geq c \lambda^{-n-1} \nu(1/\lambda) \geq c \lambda^{-n-1} R_3^{-n-1} \nu(R_3) \geq c \lambda^{-n-1} R_3^{-\lambda/2} \nu(R_3),
   \]
   (2.4)

We deduce from (2.3) and (2.4) that $|\phi^{(n)}(\lambda)| \leq c \int_0^{1/\lambda} s^n \nu(s) ds$ for all $\lambda \geq 2R_3^{-1}$ and $n \geq 1$. Then, by considering the constants inf $\lambda \in [\lambda_0, 2R_3^{-1}]$ $|\phi^{(n)}(\lambda)|^{1/2}$, we get the desired result.

(2) By (1), the change of the variables and the assumption (1.8), we have that for $\lambda \geq 2R_3^{-1}$ and $n \geq 1$,
   \[
   |\phi^{(n)}(2\lambda)| \geq \int_0^{1/2\lambda} s^n \nu(s) ds = 2^{-n-1} \int_0^{1/\lambda} s^n \nu(s) ds \geq c \int_0^{1/\lambda} s^n \nu(s) ds \geq |\phi^{(n)}(2\lambda)|.
   \]

By (1), we also get
   \[
   |\phi^{(n)}(\lambda)| \geq e^{-1} \int_0^{1/\lambda} s^n \nu(s) ds \geq |\phi^{(n)}(2\lambda)|.
   \]
By considering \( \inf_{\lambda \in [\lambda_0, 2\lambda_0^{-1}]} \left| \phi^{(n)}(\lambda)/\phi^{(n)}(2\lambda) \right| \) and \( \sup_{\lambda \in [\lambda_0, 2\lambda_0^{-1}]} \left| \phi^{(n)}(\lambda)/\phi^{(n)}(2\lambda) \right| \), we get the result.

(3) By (1), we have that for all \( \lambda \geq \lambda_0 \) and \( n \geq 1 \),
\[
|\lambda \phi^{(n+1)}(\lambda)| \asymp \int_0^{1/\lambda} (\lambda s)s^n\nu(s)ds \leq \int_0^{1/\lambda} s^n\nu(s)ds \asymp |\phi^{(n)}(\lambda)|,
\]
which yields the result. \( \square \)

**Lemma 2.2.** Suppose that (L-1) holds.

(1) For every \( \lambda_0 > 0 \), there are constants \( c_n > 1 \), \( n = 1, 2, \ldots \) such that
\[
e^{-1} \int_0^{1/\lambda} s^n\nu(s)ds \leq |\phi^{(n)}(\lambda)| \leq c_n \int_0^{1/\lambda} s^n\nu(s)ds \quad \text{for all } 0 < \lambda \leq \lambda_0 \) and \( n \geq 1 \).

(2) For every \( \lambda_0 > 0 \), there are constants \( c'_n > 1 \), \( n = 1, 2, \ldots \) such that
\[
c_n^{-1}|\phi^{(n)}(2\lambda)| \leq |\phi^{(n)}(\lambda)| \leq c'_n|\phi^{(n)}(2\lambda)| \quad \text{for all } 0 < \lambda \leq \lambda_0 \) and \( n \geq 1 \).

(3) For every \( \lambda_0 > 0 \), there are constants \( c''_n > 0 \), \( n = 1, 2, \ldots \) such that
\[
|\lambda \phi^{(n+1)}(\lambda)| \leq c''_n|\phi^{(n)}(\lambda)| \quad \text{for all } 0 < \lambda \leq \lambda_0 \) and \( n \geq 1 \).

**Proof.** (1) By (2.2), it remains to prove the upper bounds. By (1.6) and the first line in (2.4), we have that for all \( 0 < \lambda \leq R_2^{-1} \) and \( n \geq 1 \),
\[
|\phi^{(n)}(\lambda)| \leq \int_0^{1/\lambda} s^n\nu(s)ds + \int_0^{\infty} s^n e^{-\lambda s}\nu(s)ds \leq \int_0^{1/\lambda} s^n\nu(s)ds + c\nu(1/\lambda) \int_1^{\infty} s^n e^{-\lambda s}ds \\
\leq \int_0^{1/\lambda} s^n\nu(s)ds + c\lambda^{-n-1}\nu(1/\lambda) \leq c\int_0^{1/\lambda} s^n\nu(s)ds.
\]
Again, by considering the constants \( \inf_{\lambda \in [R_2^{-1}, \lambda_0]} \left| (\phi^{(n)}(\lambda))^{-1} \int_0^{1/\lambda} s^n\nu(s)ds \right| \), we get the result.

(2) As in the proof of (1), it suffices to prove for \( 0 < \lambda \leq R_2^{-1} \) and \( n \geq 1 \). By (1), the change of the variables and (1.6),
\[
|\phi^{(n)}(2\lambda)| \asymp \int_0^{2R_2} s^n\nu(s)ds + \int_0^{1/(2\lambda)} s^n\nu(s)ds \\
= \int_0^{2R_2} s^n\nu(s)ds + 2^{-n-1} \int_0^{1/\lambda} s^n\nu(s/2)ds \\
\asymp \int_0^{2R_2} s^n\nu(s)ds + \int_0^{1/\lambda} s^n\nu(s)ds \asymp |\phi^{(n)}(\lambda)|.
\]
(3) We get the result by the same proof as the one for Lemma 2.1(3). \( \square \)

**Lemma 2.3.** Suppose that (S-1) holds.

(1) There are constants \( c_1, c_2 > 0 \) such that
\[
w(r) \geq c_1 r\nu(2r) \quad \text{and} \quad w(r) \leq c_2 r\nu(r) \quad \text{for all } 0 < r < R_1/2.
\]
In particular, if we further assume that (S-2) holds, then \( w(r) \asymp r\nu(r) \) for all \( r \in (0, R_1/2) \).

(2) There is a constant \( c_3 > 0 \) such that
\[
\frac{w(r)}{w(R)} \geq c_3 \left( \frac{R}{r} \right)^{\alpha_1} \quad \text{for all } 0 < r < R < R_1/2.
\]
Moreover, for all

(3) By (2.1), the change of the variables and (2), we have that for all $r_0 \leq r \leq R < \infty$.

In particular,

$$\frac{H^{-1}(t)}{H^{-1}(s)} \leq c_4^{-1/\alpha_1} \left( \frac{t}{s} \right)^{1/\alpha_1}$$

for all $H(r_0) \leq s \leq t < \infty$.

(4) For every $\lambda_0 > 0$, there are comparison constants such that

$$H(\lambda) \asymp \lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds \asymp \lambda^2 (-\partial''(\lambda))$$

for all $\lambda_0 \leq \lambda < \infty$.

Proof. (1) By (1.2), we have that for all $r \in (0, R_1/2)$,

$$w(r) \geq \int_r^{2r} \nu(s) ds = \nu(2r) \int_r^{2r} \frac{\nu(s)}{\nu(2r)} ds \geq c_1 \nu(2r) \int_r^{2r} (2r/s)^{1+\alpha_1} ds \geq c_1 \nu(2r)$$

and

$$w(r) = \int_r^{R_1} \nu(s) ds + w(R_1) \leq \int_r^{R_1/2} \frac{\nu(s)}{\nu(2r)} ds + \frac{w(R_1)}{\nu(2r)} \int_r^{R_1} \nu(s) ds$$

$$\leq c\nu(r) \int_r^{R_1/2} \frac{\nu(s)}{\nu(2r)} ds \leq c c_1^{-1} r^{1+\alpha_1} \nu(r) \int_r^{R_1/2} s^{-1-\alpha_1} ds \leq c c_1^{-1} \alpha_1^{-1} r \nu(r).$$

Moreover, if (S-2) holds, then there is a constant $c > 0$ such that $\nu(2r) \geq c \nu(r)$ for all $0 < r < R_1/2$. Hence, we obtain $w(r) \asymp r \nu(r)$ for all $r \in (0, R_1/2)$.

(2) By (1) and (1.2), for all $0 < 2r \leq R < R_1/2$,

$$\frac{w(r)}{w(R)} \geq c \frac{r \nu(2r)}{R \nu(R)} \geq c \frac{r}{R} \left( \frac{R}{2r} \right)^{1+\alpha_1} = c 2^{-1-\alpha_1} \left( \frac{R}{r} \right)^{\alpha_1}.$$ 

On the other hand, for all $0 < r \leq R \leq (R_1/2) \wedge (2r)$, by the monotonicity of $w$,

$$\frac{w(r)}{w(R)} \geq 1 \geq 2^{-\alpha_1} \left( \frac{R}{r} \right)^{\alpha_1}.$$ 

Hence, we get the result in both cases.

(3) By (2.1), the change of the variables and (2), we have that for all $2R_1^{-1} < r \leq R$,

$$\frac{H(R)}{H(r)} \geq \frac{1}{5e} \left( \frac{R}{r} \right)^2 \frac{1}{5e} \left( \frac{R}{r} \right)^{1+\alpha_1} \frac{sw(s)}{sw(s)} ds \geq c_3 \left( \frac{R}{r} \right)^{\alpha_1}.$$ 

Moreover, for all $r_0 \leq r \leq 2R_1^{-1} < R$, we see that

$$\frac{H(R)}{H(r)} \geq \frac{H(R)}{H(2R_1^{-1})} \geq c_3 \left( \frac{R}{2R_1^{-1}} \right)^{\alpha_1} \geq c_3 \frac{r_0}{2R_1^{-1}} \left( \frac{R}{r} \right)^{\alpha_1}.$$ 

Lastly, for all $r_0 \leq r \leq R \leq 2R_1^{-1}$, we get $H(R)/H(r) \geq 1 \geq (r_0 R_1/2)^{\alpha_1} (R/r)^{\alpha_1}$. Hence, the assertion holds.

(4) As in the proof of Lemma (2.1) we may and do assume that $\lambda_0 > 2R_1^{-1}$. By (1), we get

$$\lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds \geq c_2^{-1} \lambda^2 \int_0^{1/\lambda} sw(s) ds \geq c_2^{-1} \lambda^2 w(1/\lambda) \int_0^{1/\lambda} s ds = (2c_2)^{-1} w(1/\lambda).$$
Hence, from the definition of $H$, we get

$$H(\lambda) \leq \lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds + w(1/\lambda) \leq (2c_2 + 1)\lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds.$$  

Therefore, by combining with (1.9), we obtain the first comparison. Then, according to (5.6) and (5.7) in Lemma 5.1, (1.10) and the first comparison imply $H(\lambda) \asymp \lambda^2(-\phi''(\lambda))$. This completes the proof. 

Similar results to Lemma 2.3 hold under the condition (L-1).

**Lemma 2.4.** Suppose that (L-1) holds.

(1) There are constants $c_1, c_2 > 0$ such that

$$w(r) \geq c_1r \nu(2r) \quad \text{and} \quad w(r) \leq c_2r \nu(r) \quad \text{for all} \quad r \geq R_2.$$ 

In particular, if we further assume that (L-2) holds, then $w(r) \asymp r \nu(r)$ for all $r \in [R_2, \infty)$.

(2) There is a constant $c_3 > 0$ such that

$$\frac{w(r)}{w(R)} \geq c_3 \left( \frac{R}{r} \right)^{\alpha_3} \quad \text{for all} \quad R_2 \leq r \leq R < \infty.$$ 

In particular,

$$\frac{H(R)}{H(r)} \geq c_4 \left( \frac{R}{r} \right)^{\alpha_3 \wedge (3/2)} \quad \text{for all} \quad 0 < r \leq R \leq r_0.$$ 

(3) For every $r_0 > 0$, there is a constant $c_4 > 0$ such that

$$H(R) \geq c_4 \left( \frac{R}{r} \right)^{\alpha_3 \wedge (3/2)} \quad \text{for all} \quad 0 < r \leq R \leq r_0.$$ 

In particular,

$$\frac{H^{-1}(t)}{H^{-1}(s)} \leq c_4^{-1/(\alpha_3 \wedge (3/2))} \left( \frac{t}{s} \right)^{(1/\alpha_3) \nu(2/3)} \quad \text{for all} \quad 0 < s \leq t \leq H(r_0).$$ 

(4) For every $\lambda_0 > 0$, there are comparison constants such that

$$H(\lambda) \asymp \lambda^2 \int_0^{1/\lambda} s^2 \nu(s) ds \asymp \lambda^2(-\phi''(\lambda)) \quad \text{for all} \quad 0 < \lambda \leq \lambda_0.$$ 

**Proof.** (1) For all $r \geq R_2$, we see from (1.6) that $w(r) \geq \int_r^{2r} \nu(s) ds \geq cr \nu(2r)$ and $w(r) \leq cr^{1+\alpha_3 \nu(r)} \int_0^{\infty} s^{-1-\alpha_3} ds \leq cr \nu(r)$. Moreover, if (L-2) holds further, then $\nu(r) \asymp r \nu(r)$ for all $r \geq R_2$ and hence $w(r) \asymp r \nu(r)$ for all $r \geq R_2$.

(2) This follows from (1.6) and (1). (See, the proof of Lemma 2.3(2).)

(3) By the proof of Lemma 2.3(3), using the monotonicity of $H$, it suffices to show that

$$\frac{H(R)}{H(r)} \geq c_3 \left( \frac{R}{r} \right)^{\alpha_3 \wedge (3/2)} \quad \text{for all} \quad 0 < r \leq R \leq (2R_2)^{-1}. \quad (2.5)$$ 

According to (2.1) and (1), for all $0 < \lambda \leq (2R_2)^{-1}$,

$$\lambda^{-2} H(\lambda) \asymp \int_0^{1/\lambda} s^2 \nu(s) ds \leq c \int_0^{R_2} s^2 \nu(s) ds + c \int_{R_2}^{1/\lambda} s^2 \nu(s) ds \leq c \int_{R_2}^{1/\lambda} s^2 \nu(s) ds.$$ 

The holds since $\int_{R_2}^{1/\lambda} s^2 \nu(s) ds \geq \int_{R_2}^{2R_2} s^2 \nu(s) ds \geq c \int_0^{R_2} s^2 \nu(s) ds$ where we have used (1.6). Therefore, in view of (1.9), we get

$$H(\lambda) \asymp \lambda^2 \int_{R_2}^{1/\lambda} s^2 \nu(s) ds \quad \text{for all} \quad 0 < \lambda \leq (2R_2)^{-1}. \quad (2.6)$$
Let $\alpha_3' = \alpha_3 \wedge (3/2) \in (0, 2)$. Observe that by (1.6), for all $0 < r \leq R \leq (2R_2)^{-1}$,
\[
\int_{1/R}^{1/r} s^2 \nu(s) ds = \int_{1/R}^{1/r} s^{-1-\alpha_3'} s^{1+\alpha_3'} \nu(s) ds \leq cR^{-1-\alpha_3'} \nu(1/R) \int_{1/R}^{1/r} s^{1-\alpha_3'} ds
\]
\[
\leq cR^{\alpha_3'-2} R^{-1-\alpha_3'} \nu(1/R) = cR^{\alpha_3'-2} R^{-3} \nu(1/R) \leq c(R/r)^{2-\alpha_3'} \int_{1/(2R)}^{1/R} s^2 \nu(s) ds.
\]
Thus
\[
\int_{R^2}^{1/r} s^2 \nu(s) ds \leq \int_{R^2}^{1/R} s^2 \nu(s) ds + c(R/r)^{2-\alpha_3'} \int_{1/(2R)}^{1/R} s^2 \nu(s) ds \leq c(R/r)^{2-\alpha_3'} \int_{R^2}^{1/R} s^2 \nu(s) ds.
\]
It follows that by (2.6), for all $0 < r \leq R \leq (2R_2)^{-1}$,
\[
\frac{H(R)}{H(r)} \geq c \left( \frac{R}{r} \right)^{2 \int_{R^2}^{1/r} s^2 \nu(s) ds} \geq c \left( \frac{R}{r} \right)^2 \frac{\int_{R^2}^{1/R} s^2 \nu(s) ds}{(R/r)^{2-\alpha_3'} \int_{R^2}^{1/R} s^2 \nu(s) ds} \geq c \left( \frac{R}{r} \right)^{\alpha_3'}.
\]
This proves (2.5).

(4) The first comparison follows from (1.9) and (2.6). Then, using (33) (5.6) and (5.7) in Lemma 5.1, we obtain the second comparison from the first comparison and (1.10).

Now, we give some basic properties of the $b$-function. It is easy to verify that $b$ is strictly increasing, $\lim_{t \to 0} b(t) = 0$ and $\lim_{t \to \infty} b(t) = \phi'(0) \in (0, \infty]$. Also, by (20) Lemma 2.4,
\[
\frac{1}{\phi^{-1}(1/t)} \leq tb(t) \leq \frac{1}{\phi^{-1}(c_s/t)}, \quad c_s = \frac{e - 2}{e^2 - e}, \quad \text{for all } t > 0.
\]

The following lemma is useful when $\phi$ is not comparable to the function $H$.

**Lemma 2.5.** For every $a_2 \geq a_1 > 0$ and $a_3 > 0$, it holds that for all $t > 0$,
\[
\frac{1}{\phi^{-1}(1/t)} \leq tb(t) \leq \frac{1}{\phi^{-1}(c_s/t)},
\]
\[
b(t/a_1) - b(t/a_2) \leq 2e^{a_2} - 2e^{a_2} H^{-1}(a_2/t)^{-1} \leq \frac{2e^2 - 4e + 1}{e - 2} a_2 H^{-1}(a_1/t)^{-1},
\]
\[
tb(t/a_2) - tb(t/(4a_3)) \geq \frac{1}{2} \frac{tH^{-1}(4a_3/t)^2 (\phi' \circ H^{-1})(4a_3/t)}.\]

In particular, if the condition (S-1) holds, (resp. (L-1) holds) then for every $a_2 \geq a_1 > 0$, $a_3 > 0$ and $T > 0$, there exist $c_1, c_2 > 0$ such that for all $t \in (0, T)$, (resp. $t \in [T, \infty)$),
\[
tb(t/a_1) - tb(t/a_2) \leq c_1 H^{-1}(1/t)^{-1} \quad \text{and} \quad tb(t/a_3) - tb(t/(4a_3)) \geq c_2 H^{-1}(1/t)^{-1}.
\]

**Proof.** By the mean value theorem and (1.10), we have
\[
b(t/a_1) - b(t/a_2) = \int_{0}^{\infty} \left( e^{-sH^{-1}(a_1/t)} - e^{-sH^{-1}(a_2/t)} \right) s \nu(s) ds
\]
\[
\leq H^{-1}(a_2/t) \int_{0}^{H^{-1}(a_2/t)^{-1}} s^2 \nu(s) ds + \int_{H^{-1}(a_2/t)^{-1}}^{\infty} se^{-sH^{-1}(a_1/t)} \nu(s) ds
\]
\[
\leq \frac{2eH(H^{-1}(a_2/t))}{H^{-1}(a_2/t)} + \frac{e^{-1}}{H^{-1}(a_1/t)} \int_{H^{-1}(a_2/t)^{-1}}^{\infty} \nu(s) ds
\]
\[
= \frac{2e^{a_2}}{tH^{-1}(a_2/t)} + \frac{e^{-1}}{H^{-1}(a_1/t)} w(H^{-1}(a_2/t)^{-1})
\]
\[
\leq \frac{2e^{a_2}}{tH^{-1}(a_1/t)} + \frac{H(H^{-1}(a_2/t))}{(e-2)H^{-1}(a_1/t)} = \frac{2e^2 - 4e + 1}{e - 2} \frac{a_2}{tH^{-1}(a_1/t)}.
\]
In the second inequality, we used the fact that for every \( \lambda > 0 \), the map \( s \mapsto se^{-\lambda s} \) on \([0, \infty)\) has the maximum value \( \lambda^{-1}e^{-1} \).

On the other hand, we also have that by the mean value theorem,

\[
b(t/a_3) - b(t/(4a_3)) = \int_0^\infty \left( e^{-sH^{-1}(a_3/t)} - e^{-sH^{-1}(4a_3/t)} \right) s\nu(s)ds
\]

\[
\geq (H^{-1}(4a_3/t) - H^{-1}(a_3/t)) \int_0^\infty e^{-sH^{-1}(4a_3/t)} s^2\nu(s)ds \geq \frac{1}{2} H^{-1}(4a_3/t)(\phi'' \circ H^{-1})(4a_3/t).
\]

In the last inequality, we used the fact that \( H^{-1}(4a_3/t) \geq \kappa H^{-1}(\lambda) \) for all \( \kappa \geq 1 \) and \( \lambda \geq 0 \).

Then, by Lemmas 2.3 and 2.4, we can see that the last assertion holds.

\[\square\]

3. Main results

In this section, we give proofs for our main results. Recall that we always assume that \( S = (S_t) \) is a subordinator without drift, whose Lévy measure has a density function satisfying the condition \((E)\) with the constant \( T_0 \in [0, \infty)\).

3.1. Estimates on left tail probabilities. In this subsection, we study estimates on \( p(t, x) \) when \( x \) is small. We first present a result established in [27], which holds under the condition \((S-1)\). Recall from \((L-1)\) that we use the abbreviation \( \sigma = \sigma(t, x) = (\phi')^{-1}(x/t) \) for \( t > 0 \) and \( 0 < x < t\phi'(0) \).

**Proposition 3.1.** Suppose that the condition \((S-1)\) holds. Then, for every \( T > 0 \), there exists a constant \( M_0 > 0 \) such that for all \( t \in (0, T] \) and \( x \in (0, tb(t/M_0)] \),

\[
p(t, x) \asymp \frac{1}{\sqrt{t(-\phi''(\sigma))}} \exp(-tH(\sigma)). \tag{3.1}
\]

**Proof.** According to Lemma 2.3(3) and (4), we can see that for every \( x_0 > 0 \), the condition \(-\phi'' \in \text{WLSC}(c_1 - 2, c, x_0)\) in [27, Theorem 3.3] is satisfied with some constant \( c > 0 \). Since \( x \mapsto \sigma \) decreases for each fixed \( t \), we have that for \( t \in (0, T] \) and \( x \in (0, tb(t/M_0)] \),

\[
\sigma \geq ((\phi')^{-1} \circ b)(t/M_0) = H^{-1}(M_0/t) \geq H^{-1}(M_0/T).
\]

Also, by the above inequality and Lemma 2.3(4), there exists a constant \( c_1 > 0 \) such that

\[
t\sigma^2(-\phi''(\sigma)) \geq c_1 tH(\sigma) \geq c_1 tH(H^{-1}(M_0/t)) = c_1 M_0.
\]

Hence, the result follows from [27, Theorem 3.3]. \[\square\]

Now, we establish left tail probabilities under the conditions \((L-1)\) and \((L-3)\). Since subordinators can not decrease, if \( x \) is small compare to \( t \), then left tail probabilities mainly depend on small jumps of subordinators. This is why we impose the assumption \((L-3)\) on small jumps in the condition \((L)\).

Define a function \( \mathcal{M} : (0, \infty) \times (0, \infty) \times (-\infty, \infty) \to \mathbb{C} \) by

\[
\mathcal{M}(s, z, u) := \phi(z + \frac{iu}{\sqrt{s(-\phi''(z))}}) - \phi(z) - \phi'(z) \frac{iu}{\sqrt{s(-\phi''(z))}}. \tag{3.2}
\]

In the settings of [27], the Laplace exponent \( \phi \) should satisfy a lower weak scaling condition at infinity (i.e., the lower Matuszewska index (at infinity) of the function \( \phi(\lambda)1_{\{\lambda \geq 1\}} \) should be strictly bigger than \( 0 \)). It follows that a map \( u \mapsto e^{-t\mathcal{M}(t, \sigma, u)} \) for fixed \( t > 0 \) decreases at least subexponentially. This property has an important role in the proof of [27, Theorem 3.3]. Unlike [27], in our settings, the Laplace exponent \( \phi \) can be slowly varying at infinity so...
that the map $u \mapsto e^{-t\mathcal{M}(t,\sigma,u)}$ can decays in polynomial order. Therefore, we should bound the integral $\int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)} du$ more carefully in the following proposition.

**Proposition 3.2.** Suppose that the conditions (L-1) and (L-3) holds. Then, there exist $T_1 > T_0$, $M_0 > 0$ and comparison constants such that (3.1) holds for all $t \in [T_1, \infty)$ and $x \in (0, tb(t/M_0)]$.

Moreover, if $T_0 = 0$ in the condition (E), then for every $T > 0$, there exist $M_0 > 0$ and comparison constants such that (3.1) holds for all $t \in [T, \infty)$ and $x \in (0, tb(t/M_0)]$.

**Proof.** Recall that $\mathcal{M}$ is defined in [5.2]. Since $\phi'(\sigma) = x/t$, by the Fourier-Mellin inversion formula (see, e.g. [43, (4.3)]), and the change of the variables, we have

$$p(t, x) = \frac{e^{-t\phi(\sigma)+\sigma x}}{2\pi} \int_{-\infty}^{\infty} \exp \left( -t \left( \phi(\sigma) + iu + \phi(\sigma) \right) + iux \right) du $$

$$= \frac{e^{-t\phi(\sigma)-\sigma \phi'(\sigma)}}{2\pi} \int_{-\infty}^{\infty} \exp \left( -t \left( \phi(\sigma) + iu - \phi(\sigma) - iu \phi'(\sigma) \right) \right) du$$

$$= \frac{e^{-tH(\sigma)}}{2\pi \sqrt{t(-\phi''(\sigma))}} \int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)} du, \quad (3.3)$$

whenever the integral converges. Note that if $|e^{-t\mathcal{M}(t,\sigma,u)}|$ is integrable on $\mathbb{R}$ with respect to $u$, then

$$\int_{-\infty}^{\infty} e^{-t\mathcal{M}(t,\sigma,u)} du = \int_{0}^{\infty} (e^{-t\mathcal{M}(t,\sigma,u)} + e^{-t\mathcal{M}(t,\sigma,-u)}) du.$$ 

Since the complex conjugate of $\mathcal{M}(t, \sigma, u)$ is given by

$$\overline{\mathcal{M}(t, \sigma, u)} = \int_{0}^{\infty} \left( 1 - \exp \left( - \left( \sigma + \frac{-iu}{\sqrt{t(-\phi''(\sigma))}} \right) s \right) \nu(s) ds - \phi(\sigma) - \phi'(\sigma) \right) \frac{-iu}{\sqrt{t(-\phi''(\sigma))}}$$

$$= \mathcal{M}(t, \sigma, -u),$$

we have that $e^{-t\mathcal{M}(t,\sigma,u)} + e^{-t\mathcal{M}(t,\sigma,-u)} \in \mathbb{R}$ for all $t, \sigma > 0$ and $u \in \mathbb{R}$. Hence, $p(t, x)$ is a real number whenever $|e^{-t\mathcal{M}(t,\sigma,u)}|$ is integrable on $\mathbb{R}$ with respect to $u$.

Let $T > T_0$ be a constant which will be chosen later and fix any $\delta > 0$ such that $T \geq T_0 + \delta$. We claim that the integral in (3.3) converges for all $t \geq T$ and $\sigma > 0$. Indeed, by a similar proof to that of Proposition 1.1 for $\varepsilon = \delta/(2T_0 + 2\delta)$, there are constants $\sigma_0 > 0, \xi_0 > 1$ such that

$$\nu(s) \geq \frac{(1 - \varepsilon/2)}{T_0} s^{-1} \text{ for all } s \in (0, -\sigma_0^{-1} \log(1 - \varepsilon/2)) \quad (3.4)$$

and

$$\int_{0}^{\log(1-\varepsilon/2)} (1 - \cos(\xi s)) s^{-1} ds \geq \frac{1 - \varepsilon}{1 - \varepsilon + \varepsilon^2/4} \log(1 + \xi) \text{ for all } \xi \geq \xi_0. \quad (3.5)$$
It follows that for all $|u| > \xi_0(\sigma_0 \lor \sigma) \sqrt{t(-\phi''(\sigma))}$,

$$
\text{Re} t \mathcal{M}(t, \sigma, u) = t \int_0^\infty e^{-\sigma s} \left(1 - \sqrt{\frac{us}{t(-\phi''(\sigma))}}\right) \nu(s) ds \\
\geq \frac{(1 - \varepsilon/2)^2 t}{T_0} \int_0^{-\log(1-\varepsilon/2)\delta_0 \lor \sigma} \left(1 - \sqrt{\frac{us}{t(-\phi''(\sigma))}}\right) s^{-1} ds \\
= \frac{(1 - \varepsilon/2)^2 t}{T_0} \int_0^{-\log(1-\varepsilon/2)\delta_0 \lor \sigma} \left(1 - \sqrt{\frac{us}{(\sigma_0 \lor \sigma)\sqrt{t(-\phi''(\sigma))}}\right) s^{-1} ds \\
\geq \frac{(1 - \varepsilon)t}{T_0} \log \left(1 + \frac{u}{(\sigma_0 \lor \sigma)\sqrt{t(-\phi''(\sigma))}}\right).
$$

(3.6)

Since $(1 - \varepsilon)t/T_0 \geq (1 - \varepsilon)T_0 \geq (1 - \varepsilon)(T_0 + \delta)/T_0 = 1 + \delta/(2T_0) > 1$, we see from (3.6) that $|e^{-t\mathcal{M}(t, \sigma, u)}| = e^{-\text{Re} t \mathcal{M}(t, \sigma, u)}$ is integrable on $\mathbb{R}$ with respect to $u$. This yields that (3.1) holds.

Next, we will show that there exists a constant $M_0 > 1$ such that for all $t \in [T, \infty)$ and $x \in (0, tb(t/M_0))$,

$$
\sqrt{\pi} \leq \int_{-\infty}^\infty e^{-t\mathcal{M}(t, \sigma, u)} du \leq 2\sqrt{\pi},
$$

(3.7)

which implies (3.1) in view of (3.3).

Define

$$
\mathcal{T}_0 = \mathcal{T}_0(t, \sigma) := (\sigma_0 \lor \sigma)\sqrt{t(-\phi''(\sigma))} \quad \text{and} \quad \mathcal{T} = \mathcal{T}(t, \sigma) := \sigma\sqrt{t(-\phi''(\sigma))}.
$$

Clearly, we have $\mathcal{T}_0 \geq \mathcal{T}$. For $\sigma > \sigma_0$, we see from (3.3) that $\mathcal{T}^2 \geq \sigma^2 t \int_0^{1/\sigma} s^2 e^{-\sigma s \nu(s)} ds \geq c_1 \sigma^2 t \int_0^{1/\sigma} s^2 s^{-1} ds = c_1 t/2 \geq c_1 T/2$. On the other hand, for $\sigma \leq \sigma_0$, we see from Lemma 2.4(4) and the monotonicity of the function $\sigma$ that $\mathcal{T}^2 \geq c_2 t H(\sigma) \geq c_2 t (H \circ (\phi')^{-1} \circ b)((t/M_0)) = c_2 M_0$. It follows that

$$
\mathcal{T}_0^2 \geq \mathcal{T}^2 \geq (c_1 T/2) \wedge (c_2 M_0).
$$

(3.8)

We claim that

$$
\lim_{T \to \infty} \int_{-\infty}^\infty e^{-t\mathcal{M}(t, \sigma, u)} du = \int_{-\infty}^\infty e^{-\frac{1}{2}u^2} du = \sqrt{2\pi},
$$

(3.9)

which yields the desired result. Indeed, if (3.9) is true, then there exists a constant $c_3 > 0$ such that (3.7) holds for $\mathcal{T} \geq c_3$. By choosing $T = 2c_1^{-1}c_3^2$ and $M_0 = c_2^{-1}c_3^2$, we get the result from (3.5). Now, we prove (3.9).

First, we note that according to (3.6),

$$
\left| \int_{|u| > \xi_0 \mathcal{T}_0} e^{-t\mathcal{M}(t, \sigma, u)} du \right| \leq 2 \int_{\xi_0 \mathcal{T}_0}^\infty (1 + \frac{u}{\mathcal{T}_0})^{-(2T_0 + \delta)/(2T_0)} du.
$$

(3.10)

On the other hand, by Taylor’s theorem, we have

$$
\left| t \mathcal{M}(t, \sigma, u) - \frac{1}{2} u^2 \right| = \left| t(\phi(\sigma + \frac{i \sigma}{T} u) - \phi(\sigma) - \phi'(\sigma) \frac{i \sigma}{T} u - \frac{1}{2} u^2 \right| \\
\leq \frac{1}{2} u^2 \sup_{z \in [-|u|, |u|]} \left| (\phi''(\sigma + \frac{i \sigma}{T} z) \frac{z^2}{2} - 1 \right| \\
= \frac{1}{2} \left\{ \sup_{z \in [-|u|, |u|]} \left| (\phi''(\sigma + \frac{i \sigma}{T} z) \right| - \phi''(\sigma) \frac{i \sigma}{T} z \right\}.
$$
Note that
\[
\sup_{z \in [-|u|,|u|]} \left| -\phi''(\sigma + \frac{j\sigma}{T}) + \phi''(\sigma) \right| \leq \sup_{z \in [-|u|,|u|]} \int_{0}^{\infty} s^2 e^{-zs} \left| \cos\left(\frac{\sigma z s}{T}\right) - 1 - i \sin\left(\frac{\sigma z s}{T}\right) \right| \nu(s) ds
\]
\[
= 2 \sup_{z \in [-|u|,|u|]} \int_{0}^{\infty} s^2 e^{-zs} |\sin\left(\frac{\sigma z s}{2T}\right)||\nu(s) ds \leq \frac{\sigma |u|}{T} \int_{0}^{\infty} s^3 e^{-zs} \nu(s(s)) = \frac{\sigma |u|}{T} \phi''(\sigma).
\]

We used the fact that $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$ in the second inequality. Hence, we get
\[
|t\mathcal{M}(t, \sigma, u) - \frac{1}{2} u^2| \leq \frac{\sigma \phi''(\sigma)}{2T(-\phi''(\sigma))}|u|^3.
\]
Then, combining with the fact that $|e^z - 1| \leq |z|e^{|z|}$ for $z \in \mathbb{C}$, it follows that for all $u \in \mathbb{R}$,
\[
\left| e^{-t\mathcal{M}(t, \sigma, u)} - e^{-\frac{j}{2} u^2} \right| = e^{-\frac{j}{2} u^2} \left| \exp\left(\frac{1}{2} u^2 - t\mathcal{M}(t, \sigma, u) - 1\right) \right| \leq \frac{\sigma \phi''(\sigma)}{2T(-\phi''(\sigma))}|u|^3 \exp\left( - \frac{1}{2} u^2 + \frac{\sigma \phi''(\sigma)}{2T(-\phi''(\sigma))}|u|^3 \right). \quad (3.11)
\]

Below, we consider the cases $\sigma > \sigma_0$ and $\sigma \leq \sigma_0$, separately.

(Case 1): Assume that $\sigma > \sigma_0$. By Lemma 2.1(3), there exists a constant $c_4 > 0$ such that $\sigma \phi''(\sigma) \leq c_4(-\phi''(\sigma))$. Let $\xi_1 = (2c_4)^{-1} \wedge \xi_0$. Then, according to (3.11),
\[
\left| \int_{|u| \leq \xi_1 T} \left( e^{-t\mathcal{M}(t, \sigma, u)} - e^{-\frac{j}{2} u^2} \right) du \right| \leq \frac{c_4}{T} \int_{0}^{\xi_1 T} u^3 \exp\left( - \frac{1}{2} \left( - \frac{c_4 \xi_1}{2} u^2 \right) \right) du
\]
\[
\leq \frac{c_4}{T} \int_{0}^{\xi_1 T} u^3 \exp\left( - \frac{1}{2} \left( \frac{c_4 \xi_1}{2} u^2 \right) \right) du \leq \frac{c_4}{T} \int_{0}^{\infty} u^3 \exp\left( - \frac{1}{4} u^2 \right) du \leq \frac{c_5}{T}. \quad (3.12)
\]

On the other hand, note that $\sigma |u|/T > \sigma_0 \xi_1$ for $|u| > \xi_1 T$. Hence, by Lemma 2.1(1), for all $|u| > \xi_1 T$,
\[
\operatorname{Re} t\mathcal{M}(t, \sigma, u) \geq t \int_{0}^{T/(\sigma |u|)} \frac{1}{T} \cos\left(\frac{\sigma u s}{T}\right) e^{-\sigma s} \nu(s) ds
\]
\[
\geq t \frac{\cos 1}{2} \frac{\sigma^2 u^2}{T^2} e^{-T/|u|} \int_{0}^{T/(\sigma |u|)} s^2 \nu(s) ds \geq c_6 e^{-1/\xi_1} t \frac{\sigma^2 u^2}{T^2} |\phi''(\sigma |u|/T)|. \quad (3.13)
\]

In the first inequality above, we used the fact that $1 - \cos x \geq \frac{\sin^2 x}{2}$ for all $|x| \leq 1$. It follows that
\[
\left| \int_{\xi_1 T < |u| \leq \xi_0 \xi_0} e^{-t\mathcal{M}(t, \sigma, u)} du \right| \leq 2 \xi_0 T \max_{\xi_1 T < |u| \leq \xi_0 T} \exp\left( - c_6 e^{-1/\xi_1} t \frac{\sigma^2 u^2}{T^2} |\phi''(\sigma u/T)| \right)
\]
\[
\leq 2 \xi_0 T \exp\left( - c_6 e^{-1/\xi_1} t \xi_1^2 \sigma^2 |\phi''(\sigma \xi_0)| \right)
\]
\[
\leq 2 \xi_0 T \exp\left( - c_7 t \sigma^2 |\phi''(\sigma)| \right) = 2 \xi_0 T \exp\left( - c_7 T^2 \right). \quad (3.14)
\]

We used the fact that $T_0 = T$ under the assumption $\sigma > \sigma_0$ in the first inequality and Lemma 2.1(2) in the third inequality.
Finally, by the triangle inequality and inequalities \(3.10\), \(3.12\) and \(3.14\), we obtain
\[
\left| \int_{\mathbb{R}} (e^{-tM(t,\sigma,u)} - e^{-\frac{u^2}{2}})du \right| \\
\leq \int_{|u| \leq \xi_1 T} (e^{-tM(t,\sigma,u)} - e^{-\frac{u^2}{2}})du + \int_{|u| > \xi_1 T} e^{-tM(t,\sigma,u)}du + \int_{|u| > \xi_1 T} e^{-\frac{u^2}{2}}du \\
\leq C_5 + 2\xi_0 T \exp\left( -c_7 T^2 \right) + 2 \int_{\xi_0 T}^{\infty} \left( 1 + \frac{u}{T} \right)^{-2(Tb+\delta)/(2Tb)}du + 2 \int_{\xi_1 T}^{\infty} e^{-\frac{u^2}{2}}du \\
\rightarrow 0 \quad \text{as} \quad T \rightarrow \infty.
\] (3.15)

This proves \(3.3\).

(Case 2): Assume that \(\sigma \leq \sigma_0\). We follow the proof given in (Case 1). First, using Lemma 2.2(3) instead of Lemma 2.1(3), \(3.12\) still hold with possibly different constants \(\xi_1\) and \(c_5\).

Next, note that \(\sigma|u|/T \leq \xi_0 \sigma_0\) for \(|u| \leq \xi_0 T_0\) in this case. Hence, by Lemma 2.2(1), we see that \(3.13\) holds for all \(|u| \leq \xi_0 T_0\) with a different constant \(c_6\). Also, by Lemma 2.2(4), we have that \(T^2 \approx T H(\sigma)\) and \(\sigma^2 u^2 T^{-2} |\phi''(\sigma|u|/T)| \approx H(\sigma|u|/T)\) for all \(|u| \leq \xi_0 T_0\).

Then, by \(3.13\) and Lemma 2.2(3) and (4), we have that for \(\alpha_3 := \alpha_3 \wedge (3/2)\),
\[
\left| \int_{T < |u| \leq \xi_0 T_0} e^{-tM(t,\sigma,u)}du \right| \\
\leq 2 \int_{\xi_1 T}^{\xi_0 T_0} \exp\left( -c_8 T H(\sigma u/T) \right)du + 2 \int_{\xi_1 T}^{\xi_0 T_0} \exp\left( -c_9 T H(\sigma u)H(\sigma) \right)du \\
\leq \frac{c_9}{2} T H(\sigma) \xi_1^\alpha_3 \int_{\xi_1 T}^{\infty} (T H(\sigma u)H(\sigma))^{-2/\alpha_3}du \leq c_{11} T^{-1-4/\alpha_3} \exp\left( -c_{12} T^2 \right).
\] (3.16)

We used the change of the variables in the first inequality and the fact that there exists a constant \(c > 0\) such that \(e^{-x^2/2} \leq ce^{-x^2/\alpha_3}\) for all \(x > \xi_1\) in the third inequality.

Using \(3.16\) instead of \(3.10\), we see that \(3.15\) still valid. Hence, we obtain \(3.9\).

To complete the proof, we further assume that \(T_0 = 0\). Choose any \(0 < T \leq 2c_1^{-1} c_3\). To prove the second assertion, it suffices to show that there exist constants \(c_12 > 1\) and \(M_0 \geq c_2^{-1} c_3^2\) such that for all \(t \in [T, 2c_1^{-1} c_3]\) and \(x \in (0, t B(t/M_0)]\),
\[
c_1^{-1} \leq \int_{-\infty}^{\infty} e^{-tM(t,\sigma,u)}du \leq c_12,
\] (3.17)
in view of \(3.3\). Note that \(3.6\) is still valid with possibly different constants \(\varepsilon, \sigma_0\) and \(\xi_0\). Hence, we have \(\int_{-\infty}^{\infty} e^{-tM(t,\sigma,u)}du \in \mathbb{R}\) for all \(t \in [T, 2c_1^{-1} c_3]\) and \(\sigma > 0\). Also, since inequalities \(3.10\), \(3.12\), \(3.14\) and \(3.16\) still work, by a similar argument to \(3.15\), we see that there exists a constant \(c_13 > 0\) such that if \(T = \sigma \sqrt{I(-\phi''(\sigma))} \geq c_{13}\), then \(3.17\) holds. Hence, it remains to prove that for a set \(A := \{(t, \sigma) : t \in [T, 2c_1^{-1} c_3], \sigma > 0, T < c_{13}\}\),
\[
\inf_{(t, \sigma) \in A} \int_{-\infty}^{\infty} e^{-tM(t,\sigma,u)}du \times \sup_{(t, \sigma) \in A} \int_{-\infty}^{\infty} e^{-tM(t,\sigma,u)}du \geq 1.
\] (3.18)

Recall that \(T^2 \geq c_2 M_0\) if \(\sigma \leq R_2^{-1}\). By taking \(M_0 = c_2^{-1} c_1^2\), we have \(A \subset [T, 2c_1^{-1} c_3] \times [R_2^{-1}, \infty)\). On the other hand, since \(T_0 = 0\), we have
\[
\lim_{\sigma \to \infty} \sigma^2 (-\phi''(\sigma)) \geq \lim_{\sigma \to \infty} e^{-1/\sigma} \int_{0}^{1/\sigma} s(\nu(s))ds \geq (2e)^{-1} \lim_{\sigma \to \infty} \inf_{0 < \nu < 1/\sigma} (s(\nu(s))) = \infty.
\] (3.19)
Thus, there exists a constant \( \sigma_1 > 0 \) such that \( T^2 \geq T\sigma^2(\phi'(\sigma)) \geq c_{13}^2 \) for all \( \sigma \geq \sigma_1 \) and hence \( A \subset [T,2c_1^{-1}c_3] \times [R_2^{-1},1] =: A_0 \). Clearly, \( (t,\sigma) \mapsto \int_{-\infty}^{\infty} e^{-t\Lambda(t,\sigma,0)} du \) is a continuous function on \( A_0 \). Therefore, we deduce (3.18) from the extreme value theorem. \( \square \)

As a consequence, we obtain the following corollary.

**Corollary 3.3.** Suppose that the condition (L-3) holds. Then, for every \( N > 0 \),

\[
\lim_{t \to \infty} p(t,x)\sqrt{t(-
ophi''(\sigma))} \exp{(tH(\sigma))} = (2\pi)^{-1/2} \quad \text{uniformly in } x \in (0,N]. \quad (3.20)
\]

If we further assume that the constant \( T_0 = 0 \) in the condition (E), then for every \( N > 0 \),

\[
\lim_{x \to 0} p(t,x)\sqrt{t(-
ophi''(\sigma))} \exp{(tH(\sigma))} = (2\pi)^{-1/2} \quad \text{uniformly in } t \in [N,\infty). \quad (3.21)
\]

**Proof.** Let \( T = \sigma_0 \sqrt{t(-
ophi''(\sigma))} \). Fix the constant \( \sigma_0 \) satisfying (3.4) and (3.5) with \( \delta = 1 \).

Observe that \( \sigma = (\phi')^{-1}(x/t) \to \infty \) as \( t \to \infty \). Hence, there exists a constant \( T_0 \) such that \( \sigma > \sigma_0 \) for all \( t > T_0 \) and \( x \in (0,N] \). As we observed in the the proof of Proposition 3.2, \( T^2 \geq c_1 t/2 \) if \( \sigma > \sigma_0 \). Since Lemma 2.1 holds under the condition (L-3) only, we can use it and follow (Case 1) in the proof of Proposition 3.2. Thus, (3.15) holds for all \( x \in (0,N] \) if \( t > T_0 \) (so that \( \sigma > \sigma_0 \)). Since \( \lim_{t \to \infty} T \geq \lim_{t \to \infty}(c_1 t/2)^{1/2} = \infty \), (3.20) follows from (3.3) and (3.9).

Now, we further assume that \( T_0 = 0 \). Since \( \sigma \) also go to infinity as \( x \to 0 \), there exists a constant \( x_N > 0 \) such that \( \sigma > \sigma_0 \) for all \( t \geq N \) and \( x \in (0,x_N) \). Hence, (3.15) holds for all \( t \geq N \) if \( x < x_N \). Moreover, by (3.19), we get that \( \lim_{x \to 0} T = \infty \) uniformly in \( t \in [N,\infty) \) since \( t \mapsto \sigma \) is increasing. Therefore, we also deduce (3.21) from (3.3) and (3.9). \( \square \)

A similar result to Corollary 3.3 is obtained in [27, Section 3]. Note that since the condition (L-3) is very mild and do not require any lower scaling assumptions, our result covers geometric stable subordinators and Gamma subordinators which are not covered in [27, Corollary 3.6].

**Example 3.4.** Let \( 0 < \alpha \leq 1 \) and \( S_t \) be a geometric \( \alpha \)-stable subordinator whose Laplace exponent is given by \( \log(1 + \lambda \alpha) \). When \( \alpha = 1 \), \( S_t \) is called a Gamma subordinator in the literature. It is known that the density of the Lévy measure \( \nu(x) \) and the transition density \( p(t,x) \) of \( S_t \) are equal to

\[
\nu(x) = \frac{\alpha}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^\alpha n}{\Gamma(1 + \alpha n)} \quad \text{and} \quad p(t,x) = \frac{x^{\alpha-1}}{\Gamma(t)} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(t + n) x^\alpha n}{n! \Gamma(\alpha t + \alpha n)}. \quad (3.22)
\]

where \( \Gamma(t) = \int_0^\infty y^{t-1} e^{-y} dy \) is the Gamma function. (See, [5, Example 5.11], [30, Section 9.2] and [11, Theorem 4.2].) Then, we can see that the subordinator \( S_t \) satisfies the conditions (E) with \( T_0 = 1/\alpha \), and (L-3). Thus, we can apply (3.20). (We can not expect that (3.21) holds since \( T_0 > 0 \).) Below, we get the exact asymptotic behavior of \( p(t,x) \) given in (3.22) as \( t \to \infty \), from (3.20).

Observe that \( \phi'(r) = \alpha r^{\alpha-1}/(1 + r^{\alpha}) \). Hence, for every \( \lambda < \phi'(0) \), we can see that

\[
(\phi')^{-1}(\lambda) + (\phi')^{-1}(\lambda)^{1-\alpha} = \frac{\alpha}{\lambda} \quad (3.23)
\]

Fix \( N > 0 \). By (3.23), for every \( t > N/\phi'(1) \) and \( x \in (0,N] \), \( \sigma \in (1,\infty) \) is determined by

\[
\sigma + \sigma^{1-\alpha} = \frac{\alpha t}{x} \quad (3.24)
\]

We claim that

\[
\lim_{t \to \infty} \frac{\sigma t}{t} = \alpha, \quad \text{uniformly in } x \in (0,N]. \quad (3.25)
\]
Indeed, \((3.24)\) implies that \(\sigma \leq \alpha t/x\) and hence \(\sigma^{1-\alpha}x/t \leq \alpha^{1-\alpha}x^\alpha t^{-\alpha} \leq \alpha^{1-\alpha}N^\alpha t^{-\alpha}\). It follows that \(|\sigma x/t - \alpha| = |\sigma^{1-\alpha}x/t| \to 0\) as \(t \to \infty\) uniformly in \(x \in (0, N]\). Then, since

\[
-\phi''(\sigma) = \frac{\alpha(1-\alpha)\sigma^{a-2}}{1+\sigma^a} + \frac{\alpha^2\sigma^{2a-2}}{(1+\sigma^a)^2} = \frac{(1-\alpha)}{\sigma} \phi'(\sigma) + \phi(\sigma)^2 = \frac{x^2}{\xi^2} \left( \frac{(1-\alpha)t}{x} + 1 \right),
\]

we get from \((3.25)\) that

\[
\lim_{t \to \infty} \frac{x^2}{\xi^2} = \frac{1/\alpha}{1-\alpha} \lim_{t \to \infty} \frac{1/\alpha}{(1-\alpha)/\alpha+1} = 1, \quad \text{uniformly in } x \in (0, N].
\]

(3.26)

On the other hand, we get from \((3.24)\) that

\[
\exp(\alpha \phi(x)) = \exp(\alpha \phi(x) - t \sigma \phi'(\sigma)) = (1 + \sigma^\alpha)^t \exp(-\sigma x) = (\sigma^{1-\alpha} + \sigma^\alpha)^t \exp(-\sigma x) = (\alpha t/x)^{\sigma^{1-\alpha}} \exp(-\sigma x).
\]

Therefore, according to \((3.20)\), \((3.26)\) and \((3.27)\), it holds that

\[
\lim_{t \to \infty} \frac{p(t, x)x^{1-\alpha}(\alpha t)^{\alpha-1/2}(1 + \sigma^{-\alpha})^t \exp(-\sigma x)}{(2\pi)^{-1/2}}, \quad \text{uniformly in } x \in (0, N],
\]

which is equivalent to (by Stirling's formula)

\[
\lim_{t \to \infty} \frac{p(t, x)x^{1-\alpha}(\alpha t)^{\alpha-1/2}(1 + \sigma^{-\alpha})^t \exp(-\sigma x)}{(2\pi)^{-1/2}} = 1, \quad \text{uniformly in } x \in (0, N],
\]

(3.28)

where \(\sigma = (\phi')^{-1}(x/t) \in (1, \infty)\) is determined by \((3.24)\). In other words, according to \((3.22)\),

\[
\lim_{t \to \infty} (1 + \sigma^{-\alpha})^t \exp(-\sigma x) \sum_{n=0}^\infty (-1)^n \frac{\Gamma(t+n)(\alpha t)^\alpha x^n}{n! \Gamma(t+n\lambda) \Gamma(\alpha+n\lambda)} = 1, \quad \text{uniformly in } x \in (0, N].
\]

Now, we express \((3.28)\) in terms of \(t\) and \(x\) only. Let \(\zeta = (x/(\alpha t))^{\alpha}\) and define

\[
\eta_1(\lambda) = 1 - \lambda^{1/\alpha}(\phi')^{-1}(\alpha \lambda^{1/\alpha}), \quad \eta_2(\lambda) = 1 + (\phi')^{-1}(\alpha \lambda^{1/\alpha})^{-\alpha} \quad \text{for } \alpha \lambda^{1/\alpha} \leq \phi'(1).
\]

Observe that for \((t, x) \in (N/\phi'(1), \infty) \times (0, N],\) we have \(\zeta \leq (\phi'(1)/\alpha)^\alpha = 2^{-\alpha}, \ \eta_1(\zeta) = 1 - \sigma x/(\alpha t)\) and \(\eta_2(\zeta) = 1 + \sigma^{-\alpha}.\) Since \(\phi\) is a Bernstein function, \(\eta_1\) and \(\eta_2\) are infinitely differentiable. Moreover, according to \((3.23)\), it holds that for all \(\alpha \lambda^{1/\alpha} \leq \phi'(1),
\]

\[
\lambda^{-\alpha}(1 - \eta_1(\lambda)) + [\lambda^{-\alpha}(1 - \eta_1(\lambda))]^{1-\alpha} = \alpha / (\alpha \lambda^{1/\alpha}),
\]

which is equivalent to

\[
(1 - \eta_1(\lambda)) + \lambda(1 - \eta_1(\lambda))^{1-\alpha} = 1.
\]

Thus, \(\eta_1(0) = 0\) and \(\eta_1'(\lambda) = [1 - \eta_1(\lambda)]/[\lambda(1 - \eta_1(\lambda))^{\alpha} + (1 - \alpha)\lambda]\) by the implicit function theorem. Moreover, one can show the following by induction: for every \(j \geq 1,\) there exists an infinitely differentiable function \(F_j\) such that \(F_j(0), F_j'(0) < \infty\) and for all \(\alpha \lambda^{1/\alpha} \leq \phi'(1),
\]

\[
\eta_1^{(j)}(\lambda) = \frac{F_j(\lambda)}{((1 - \eta_1(\lambda))^{\alpha} + (1 - \alpha)\lambda)^{2^j}},
\]

where \(\eta_1^{(j)}\) denotes the \(j\)-th derivative of \(\eta_1.\) In particular, \(\eta_1^{(j)}(0) < \infty\) for all \(j \geq 0.\)

Besides, we also get from \((3.24)\) that \(\eta_2(\lambda)(\phi')^{-1}(\alpha \lambda^{1/\alpha}) = \alpha / (\alpha \lambda^{1/\alpha})\) and hence

\[
\eta_2(\lambda) = \frac{1}{\lambda^{1/\alpha}(\phi')^{-1}(\alpha \lambda^{1/\alpha})} = \frac{1}{1 - \eta_1(\lambda)}.
\]

In particular, we can see that \(\eta_2^{(j)}(0) < \infty\) for all \(j \geq 0.\)
Therefore, from (3.29) and (3.30), we get that
\[ \alpha t - \sigma x = \alpha t \sum_{j=1}^{k} \frac{\eta_j^{(j)}(0)}{j!} \left( \frac{x}{\alpha t} \right)^{ja} + \alpha t \varepsilon_{k+1,1}(\zeta) \left( \frac{x}{\alpha t} \right)^{(k+1)\alpha}, \] (3.29)

Moreover, from the uniqueness of the Taylor series, there exist a unique sequence \( \{\varepsilon_j\}_{j \geq 1}\) and a sequence of bounded functions \( \{\delta_j\}_{j \geq 1}\) such that for every \( k \geq 1 \) and \( (t,x) \in (N/\phi'(1), \infty) \times (0,N)\),
\[ 1 + \sigma^{-\alpha} = 1 + \sum_{j=1}^{k} \frac{\eta_j^{(j)}(0)}{j!} \left( \frac{x}{\alpha t} \right)^{ja} + \varepsilon_{k+1,2}(\zeta) \left( \frac{x}{\alpha t} \right)^{(k+1)\alpha}. \] (3.30)

For example, we can calculate that \( \eta_2'(0) = 1 \) and \( \eta_2''(0) = 2\alpha \). It follows that
\[ 1 + \sigma^{-\alpha} = 1 + \left( \frac{x}{\alpha t} \right)^{\alpha} + \alpha \left( \frac{x}{\alpha t} \right)^{2\alpha} + \varepsilon_{3,2}(\zeta) \left( \frac{x}{\alpha t} \right)^{3\alpha} \]
\[ = 1 + \delta_1 \left( \frac{x}{\alpha t} \right)^{\alpha} + (\delta_1^2/2 + \delta_2) \left( \frac{x}{\alpha t} \right)^{2\alpha} + O \left( \left( \frac{x}{\alpha t} \right)^{3\alpha} \right) \]
\[ = \left( 1 + \varepsilon_{3,3}(\zeta) \left( \frac{x}{\alpha t} \right)^{3\alpha} \right) \exp \left( \delta_1 \left( \frac{x}{\alpha t} \right)^{\alpha} + \delta_2 \left( \frac{x}{\alpha t} \right)^{2\alpha} \right), \]
and hence we see that \( \delta_1 = 1 \) and \( \delta_2 = \alpha - 1/2 \).

Note that for every \( k \geq 1 \) and \( (t,x) \in (N/\phi'(1), \infty) \times (0,N)\),
\[ |\alpha t \varepsilon_{k+1,1}(\zeta) \zeta^{k+1}| \leq \alpha^{1-(k+1)\alpha} N^{(k+1)\alpha} \|\varepsilon_{k+1,1}\| \infty t^{1-(k+1)\alpha} \]
and
\[ \left| (1 + \varepsilon_{k+1,3}(\zeta) \zeta^{k+1})^{(1-\alpha) t} - 1 \right| \leq c(1 - \alpha) \alpha^{-(k+1)\alpha} N^{(k+1)\alpha} \|\varepsilon_{k+1,3}\| \infty t^{1-(k+1)\alpha}. \]

Therefore, from (3.29) and (3.30), we get that
\[ \lim_{t \to \infty} (1 + \sigma^{-\alpha})^{(1-\alpha)t} e^{\alpha t - \sigma x} \exp \left( -t \sum_{j=1}^{[1/\alpha]} \left( \alpha \eta_j^{(j)}(0) \frac{1}{j!} + (1 - \alpha) \delta_j \right) \left( \frac{x}{\alpha t} \right)^{ja} \right) = 1, \]
uniformly in \( x \in (0,N)\).

Finally, according to (3.28), we conclude that
\[ \lim_{t \to \infty} p(t,x)x^{1-\alpha t} \Gamma(\alpha t) \exp \left( t \sum_{j=1}^{[1/\alpha]} \left( \alpha \eta_j^{(j)}(0) \frac{1}{j!} + (1 - \alpha) \delta_j \right) \left( \frac{x}{\alpha t} \right)^{ja} \right) \]
\[ = \lim_{t \to \infty} \exp \left( t \sum_{j=1}^{[1/\alpha]} \left( \alpha \eta_j^{(j)}(0) \frac{1}{j!} + (1 - \alpha) \delta_j \right) \left( \frac{x}{\alpha t} \right)^{ja} \right) \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(t+n)\Gamma(\alpha t)x^{an}}{n!\Gamma(t)\Gamma(\alpha t+a\alpha)} = 1, \]
(3.31)
uniformly in \( x \in (0,N)\).
In particular, we can check that \( \eta'(0) = 1, \eta''(0) = -2(1 - \alpha) \), \( \delta_1 = 1 \) and \( \delta_2 = \alpha - 1/2 \). From these calculations, we obtain the following two special results: if \( \alpha \in (1/2, 1] \), then
\[
\lim_{t \to \infty} p(t, x)x^{1-\alpha}t \Gamma(\alpha t) \exp \left( t \left( \frac{x}{\alpha t} \right)^\alpha \right) = \lim_{t \to \infty} \exp \left( t \left( \frac{x}{\alpha t} \right)^\alpha \right) \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(t + n) \Gamma(\alpha t) x^{\alpha n}}{n! \Gamma(t) \Gamma(\alpha + \alpha n)} = 1, \text{ uniformly in } x \in (0, N], \tag{3.32}
\]
and if \( \alpha \in (1/3, 1/2] \), then
\[
\lim_{t \to \infty} p(t, x)x^{1-\alpha}t \Gamma(\alpha t) \exp \left( t \left( \frac{x}{\alpha t} \right)^\alpha - \frac{1 - \alpha}{2} t \left( \frac{x}{\alpha t} \right)^{2\alpha} \right) = \lim_{t \to \infty} \exp \left( t \left( \frac{x}{\alpha t} \right)^\alpha - \frac{1 - \alpha}{2} t \left( \frac{x}{\alpha t} \right)^{2\alpha} \right) \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(t + n) \Gamma(\alpha t) x^{\alpha n}}{n! \Gamma(t) \Gamma(\alpha + \alpha n)} = 1, \tag{3.33}
\]
uniformly in \( x \in (0, N] \).

Since for each fixed \( n \geq 0 \), \( \lim_{t \to \infty} (\alpha t)^n \Gamma(\alpha t)/\Gamma(\alpha + \alpha n) = \lim_{t \to \infty} t^n \Gamma(t)/\Gamma(t + n) = 1 \), one may expect that for all sufficiently large \( t \),
\[
\sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(t + n) \Gamma(\alpha t) x^{\alpha n}}{n! \Gamma(t) \Gamma(\alpha + \alpha n)} \sim \sum_{n=0}^{\infty} (-1)^n \frac{t^n}{(\alpha t)^\alpha} \frac{x^{\alpha n}}{n!} = \exp \left( - t \left( \frac{x}{\alpha t} \right)^\alpha \right).
\]
However, (3.32) says that this heuristic only works when \( \alpha > 1/2 \). \( \square \)

3.2. Estimates on the transition density near the maximum value. In this subsection, we obtain maximum estimates on \( p(t, x) \). Then, we extend the left tail estimates obtained in Section 3.1 as a corollary.

**Lemma 3.5.** Let \( a \in [0, \infty), \beta, c_1 > 0 \) be constants and \( f \) be a non-negative, non-decreasing function. Assume that
\[
f(R) \geq c_1 \left( \frac{R}{r} \right)^\beta \quad \text{for all } a \leq r \leq R \text{ (resp. } 0 \leq r \leq R \leq a). \tag{3.34}
\]
Then, for every \( c_2 > 0 \), there exists a constant \( c_3 > 0 \) such that
\[
\int_a^\infty \exp(-c_2 t f(\xi)) d\xi \leq c_3 f^{-1}(1/t) \quad \text{for all } t \in (0, 1/f(a)),
\]
(resp. \( \int_0^a \exp(-c_2 t f(\xi)) d\xi \leq c_3 f^{-1}(1/t) \quad \text{for all } t \in [1/f(a), \infty) \)),
where \( f^{-1}(s) := \inf\{ r \geq 0 : f(r) > s \} \) with a convention that \( \inf \emptyset = \infty \).

**Proof.** We first assume that (3.34) holds for \( a \leq r \leq R \). Note that \( f^{-1}(1/t) \geq a \) for all \( t \in (0, 1/f(a)) \). By the assumption, we get that for all \( t \in (0, 1/f(a)) \),
\[
\int_a^\infty \exp(-c_2 t f(\xi)) d\xi \leq \int_a^{2f^{-1}(1/t)} \exp(-c_2 t f(\xi)) d\xi + \int_{2f^{-1}(1/t)}^\infty \exp \left( - c_2 \frac{f(\xi)}{f(2f^{-1}(1/t))} \right) d\xi \\
\leq 2f^{-1}(1/t) + \int_{2f^{-1}(1/t)}^{\infty} \exp \left( - c_1 c_2 \left( \frac{\xi}{2f^{-1}(1/t)} \right)^\beta \right) d\xi \\
= 2f^{-1}(1/t) \left( 1 + \int_1^\infty \exp \left( - c_1 c_2 u^\beta \right) du \right) = c_3 f^{-1}(1/t).
\]
On the other hand, assume that \( (3.34) \) holds for \( 0 \leq r \leq R \leq a \). If \( a \leq 2f^{-1}(1/t) \), then there is nothing to prove. Hence, assume that \( a > 2f^{-1}(1/t) \). Then, for all \( t \in [1/f(a), \infty) \),

\[
\int_0^a \exp(-c_2 tf(\xi))d\xi \leq \int_0^{2f^{-1}(1/t)} \exp(-c_2 tf(\xi))d\xi + \int_0^a \exp\left(-c_2 \frac{f(\xi)}{f(2f^{-1}(1/t))}\right)d\xi \\
\leq 2f^{-1}(1/t) + \int_0^a \exp\left(-c_1c_2\left(\frac{\xi}{2f^{-1}(1/t)}\right)^{\beta}\right)d\xi \\
\leq 2f^{-1}(1/t) \left(1 + \int_1^{\infty} \exp\left(-c_1c_2u^\beta\right)du\right) = c_3f^{-1}(1/t).
\]

\[ \square \]

**Proposition 3.6.** (cf. [29] Theorems 3.1 and 3.10.)

(1) Suppose that the condition (S-1) holds. Then, for every \( T > 0 \), there exists a constant \( c > 0 \) such that for all \( t \in (0, T] \),

\[
\sup_{x \in \mathbb{R}} p(t, x) \leq cH^{-1}(1/t). \tag{3.35}
\]

(2) Suppose that the condition (L-1) holds. Then, for every \( T > T_0 \), where \( T_0 \) is the constant in (E), there exists a constant \( c > 0 \) such that \( (3.35) \) holds for all \( t \in [T, \infty) \).

**Proof.** (1) By Lemma 2.3(4) (and Lemma 2.4(4) as well if \( R_1 = \infty \)) and the Fourier inversion theorem, for every \( t > 0 \) and \( x \in \mathbb{R} \),

\[
p(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi x} e^{-t\phi(-i\xi)}d\xi \leq \frac{1}{2\pi} \int_{\mathbb{R}} |e^{-i\xi x} e^{-t\phi(-i\xi)}|d\xi \\
\leq \frac{1}{\pi} \int_0^{\infty} \exp\left(-t \int_0^{\infty} (1 - \cos(\xi s)) \nu(s)ds\right) d\xi \\
\leq \frac{2R_1^{-1}}{\pi} + \frac{1}{\pi} \int_{2R_1^{-1}}^{\infty} \exp\left(-\frac{\cos 1}{2} t\xi^2 \int_0^{1/\xi} s^2 \nu(s)ds\right) d\xi \\
\leq \frac{2R_1^{-1}}{\pi} + \frac{1}{\pi} \int_{2R_1^{-1}}^{\infty} \exp\left(-c_2tH(\xi)\right) d\xi.
\]

We used the fact that \( 1 - \cos x \geq 2^{-1}(\cos 1)x^2 \) for all \( |x| \leq 1 \) in the third inequality.

By Lemmas 2.3(3) and 3.5 (and Lemma 2.4(3) as well if \( R_1 = \infty \)), there is a constant \( c_3 > 0 \) such that

\[
\int_{2R_1^{-1}}^{\infty} \exp\left(-c_2tH(\xi)\right) d\xi \leq \int_{2R_1^{-1}+H^{-1}(1/T)}^{\infty} \exp\left(-c_2tH(\xi)\right) d\xi \leq c_3H^{-1}(1/t) \text{ for all } t \in (0, T].
\]

Since \( H^{-1}(1/t) \geq H^{-1}(1/T) \) for \( t \in (0, T] \), we see that \( (3.35) \) holds.

(2) Fix any \( T' \in (T_0, T) \). By the proof of Proposition 1.1 \( \int_{\mathbb{R}} |e^{-T'\phi(-i\xi)}|d\xi \leq c_4 < \infty \). On the other hand, by the condition (E), there exists a constant \( s_0 > 0 \) such that \( \nu(s) \geq 1/(2T_0s) \) for all \( s \in (0, s_0) \). Then, by the similar arguments as the ones given in the proof of (1) and using Lemma 2.4 instead of Lemma 2.3, we get

\[
p(t, x) \leq \frac{1}{2\pi} \int_{|\xi| \leq 1/s_0} |e^{-t\phi(-i\xi)}|d\xi + \frac{1}{2\pi} \int_{|\xi| > 1/s_0} |e^{-(t-T')\phi(-i\xi)}||e^{-T'\phi(-i\xi)}|d\xi \\
\leq \frac{1}{\pi} \int_0^{1/s_0} \exp\left(-c_3tH(\xi)\right) d\xi + \frac{c_4}{2\pi} \sup_{|\xi| > 1/s_0} \left|\exp\left(-(t-T')\phi(-i\xi)\right)\right| =: I_1 + I_2.
\]
By Lemmas \[2.4] (3) and \[3.5],
\[
I_1 \leq \frac{1}{\pi} \int_0^{1/s_0 \vee H^{-1}(1/T)} \exp \left(-c_5 t H(\xi)\right) d\xi \leq c_6 H^{-1}(1/t) \quad \text{for all } t \geq T.
\]
On the other hand, we also have
\[
I_2 \leq \frac{c_4}{2\pi} \sup_{|\xi| > 1/s_0} \exp \left(-\frac{(T - T') \cos 1}{2T} t \xi^2 \int_0^{1/\xi} s^2 \nu(s) ds\right)
\leq \frac{c_4}{2\pi} \sup_{|\xi| > 1/s_0} \exp \left(-c_7 t \xi^2 \int_0^{1/\xi} s^2 s^{-1} ds\right) = \frac{c_4}{2\pi} \exp(-c_7 t).
\]

Note that the lower bound in Lemma \[2.4] (3) implies that there exists \(c_8 > 0\) such that \(H^{-1}(1/t) \geq c_8 t^{-1/(\alpha_3 \wedge (3/2))}\) for all \(t \geq T\). Since it also holds that \(\exp(-c_7 t/2) \leq c_9 t^{-1/(\alpha_3 \wedge (3/2))}\) for all \(t \geq T\), for some constant \(c_9 > 0\), we finish the proof. \(\square\)

Now, we find a range of \(x\) which achieves the maximum value of \(p(t, x)\). One of the important points in the following proposition is that \(N\) can be arbitrarily big number. This point allows us to remove the constant \(M_0\) in estimates in Corollary \[3.8\].

A similar result to the following proposition was established in \[29\] Theorem 5.3 which considers a class of Lévy processes whose Lévy measure dominates some symmetric measure. Note that since the support of the Lévy measure of a subordinator is one-sided, that is always contained in \((0, \infty)\), we can only push the \(y\)-variable to the positive direction in the following unlike \[29\] Theorem 5.3.

### Proposition 3.7

1. **Suppose that the condition \((S-1)\) holds.** Then, for every \(T > 0\) and \(N > 0\), there exists a constant \(c_1 > 1\) such that for all \(t \in (0, T]\),
\[
c_1^{-1} H^{-1}(1/t) \leq p(t, tb/(2M_0)) + y \leq c_1 H^{-1}(1/t) \quad \text{for all } 0 \leq y \leq NH^{-1}(1/t)^{-1}, \quad (3.36)
\]
where \(M_0\) is the constant in Proposition \[2.4\].

2. **Suppose that the conditions \((L-1)\) and \((L-3)\) hold.** Then, for every \(N > 0\), there is a comparison constant such that \(3.36\) holds for all \(t \in [2T_1, \infty)\) and \(y \in \{0, NH^{-1}(1/t)^{-1}\}\) with the constants \(T_1, M_0\) in Proposition \[3.5\].

Moreover, if \(T_0 = 0\) in the condition \((E)\), then for every \(T > 0\) and \(N > 0\), there is a comparison constant such that \(3.36\) holds for all \(t \in [T, \infty)\) and \(y \in \{0, NH^{-1}(1/t)^{-1}\}\).

**Proof.** By Proposition \[3.6\] it remains to prove the lower bound. Below, we give the full proof for (1) and explain main differences in the proof of (2) in the end.

For \(p \in [1, 4]\), we observe that
\[
b(t/(pM_0)) \leq b(t/M_0) \quad \text{and} \quad ((\phi')^{-1} \circ b)(t/(pM_0)) = H^{-1}(pM_0/t).
\]
Hence, by Proposition \[3.1\] and Lemma \[2.4\] (3) and (4), for all \(p \in [1, 4]\),
\[
p(t, tb/(pM_0)) \geq \frac{ce^{-pM_0}}{\sqrt{t(\phi'' \circ H^{-1}(pM_0/t))}} \geq \frac{ce^{-4M_0} H^{-1}(pM_0/t)}{\sqrt{pM_0}} \geq c_2 H^{-1}(1/t). \quad (3.37)
\]
According to Lemma \[2.5\] there is a constant \(c_3 > 0\) such that
\[
tb(t/M_0) - tb(t/(4M_0)) \geq c_3 H^{-1}(1/t)^{-1} \quad \text{for all } t \in (0, T]. \quad (3.38)
\]
Then, by the intermediate value theorem, for all \(t \in (0, T] \) and \(u \in [0, c_3 H^{-1}(1/t)^{-1}]\), there exists \(p \in [1, 4]\) such that \(tb(t/M_0) - u = tb(t/(pM_0))\). Hence, by \[3.57\], we get
\[
p(t, tb(t/M_0) - u) \geq c_2 H^{-1}(1/t) \quad \text{for all } t \in (0, T], \quad u \in [0, c_3 H^{-1}(1/t)^{-1}]. \quad (3.39)
\]
By the semigroup property and (3.39), we have that for any \( t \in (0,T) \) and \( y \geq 0 \),
\[
p(2t, 2tb(t/M_0) + y) = \int_{\mathbb{R}} p(t, tb(t/M_0) - u)p(t, tb(t/M_0) + y + u)\,du \\
\geq c_2 H^{-1}(1/t)\mathbb{P}(y \leq S_t - tb(t/M_0) \leq y + c_3 H^{-1}(1/t)^{-1}).
\]
Thus, since \( H^{-1}(1/t) \asymp H^{-1}(1/(2t)) \) for all \( t \in (0,T) \) by Lemma 2.3(3), it suffices to show that for every fixed \( N > 0 \), it holds that
\[
\inf_{t \in (0,T)} \inf_{y \in [0, N H^{-1}(1/t)^{-1}]} \mathbb{P}(y \leq S_t - tb(t/M_0) \leq y + c_3 H^{-1}(1/t)^{-1}) > 0. \tag{3.40}
\]

Let \( (t_n : n \geq 1) \) be a sequence of time variables realizing the infimimum in (3.40). Since \((0,T)\) is a bounded interval, after taking a subsequence, we can assume that \( t_n \) converges to \( t_* \in [0,T] \). If \( t_* \in (0,T) \), then since the support of the distribution of \( S_{t_*} \) is \((0,\infty)\), we obtain (3.40). Hence, we assume that \( t_* = 0 \) and all \( t_n \) are sufficiently small.

Define \( \nu_n(s) := \nu(s)1_{(0,H^{-1}(1/t_n)^{-1})}(s) \) and let \( \tilde{S}_u \) be a subordinator without drift, whose Lévy measure is given by \( \nu_n(s)\,ds \). Then, for all \( u > 0 \), \( S_u = \tilde{S}_u + P_u \), \( \mathbb{P}\)-a.s. where \( P \) is a compounded Poisson process whose Lévy measure is given by \( \nu(s)1_{(H^{-1}(1/t_n)^{-1},\infty)}(s)\,ds \). Thus, by (1.10),
\[
\mathbb{P}(\tilde{S}_{t_n} = S_{t_n}) = \mathbb{P}(P_{t_n} = 0) = \exp(-t_n w(1/H^{-1}(1/t_n))) \\
\geq \exp(-e^{-t_n(H \circ H^{-1})(1/t_n)}) = \exp(-e^{-2}).
\]

Hence, to prove (3.40), it is enough to show that
\[
\lim_{n \to \infty} \inf_{y \in [0, N H^{-1}(1/t_n)^{-1}]} \mathbb{P}(y \leq \tilde{S}_{t_n} - t_n b(t_n/M_0) \leq y + c_3 H^{-1}(1/t_n)^{-1}) > 0. \tag{3.41}
\]

Define \( Z_n = H^{-1}(1/t_n)(\tilde{S}_{t_n} - t_n b(t_n/M_0)) \). Then, we have that for \( \xi \in \mathbb{R} \),
\[
\mathbb{E}[\exp(i\xi Z_n)] = \exp(-i\xi t_n H^{-1}(1/t_n)b(t_n/M_0)) \mathbb{E}[\exp(i\xi H^{-1}(1/t_n)\tilde{S}_{t_n})] \\
= \exp(-i\xi t_n H^{-1}(1/t_n)b(t_n/M_0) + t_n \int_0^\infty (\exp(i\xi H^{-1}(1/t_n)s) - 1) \nu_n(s)\,ds).
\]

Therefore, we get \( \mathbb{E}[\exp(i\xi Z_n)] = \exp(\Psi_n(\xi)) \) for all \( \xi \in \mathbb{R} \) and \( n \geq 1 \) where
\[
\Psi_n(\xi) = \int_0^\infty \left(e^{i\xi s} - 1 - \frac{i\xi s}{1 + s^2}\right) \lambda_n(s)\,ds - i\xi \gamma_n, \\
\lambda_n(s) = t_n H^{-1}(1/t_n)^{-1} \nu_n(H^{-1}(1/t_n)^{-1}s), \\
\gamma_n = t_n H^{-1}(1/t_n)b(t_n/M_0) - \int_0^\infty \frac{s}{1 + s^2} \lambda_n(s)\,ds,
\]
by the change of the variables. We claim that the family of random variables \( \{Z_n : n \geq 1\} \) is tight. Indeed, according to (3.3) (3.2), it holds that for all \( n \geq 1 \) and \( R > 1 \),
\[
\mathbb{P}(Z_n \geq R) \leq c_4 \left( \int_0^\infty \left( \frac{s^2}{R^2} \wedge 1 \right) \lambda_n(s)\,ds + \frac{1}{R} \left| \gamma_n + \int_0^\infty \frac{s}{1 + s^2} \lambda_n(s)\,ds - \int_0^R \frac{s^3}{1 + s^2} \lambda_n(s)\,ds \right| \right) \\
=: c_4(I_1 + I_2).
\]
First, by the change of variables and (1.9), we have
\[
I_1 = t_n \int_0^\infty \left( \frac{H^{-1}(1/t_n)^2 u^2}{R^2} \land 1 \right) \nu_n(u)du = R^{-2} t_n H^{-1}(1/t_n)^2 \int_0^{H^{-1}(1/t_n)^{-1}} u^2 \nu_n(u)du \\
\leq 2eR^{-2} t_n H(H^{-1}(1/t_n)) = 2eR^{-2}.
\]

On the other hand, by the change of variables, Lemma 2.3 (3) and (4) and (1.10),
\[
RI_2 = \left| t_n H^{-1}(1/t_n)b(t_n/M_0) - \int_0^R s \lambda_n(s)ds \right|
\]
\[
= t_n H^{-1}(1/t_n) \left( \int_0^\infty s \exp \left( -H^{-1}(M_0/t_n)s \right) \nu(s)ds - \int_0^{R H^{-1}(1/t_n)^{-1}} s \nu(s)ds \right)
\]
\[
\leq t_n H^{-1}(1/t_n) \left( \int_0^{H^{-1}(1/t_n)^{-1}} s \left( 1 - \exp \left( -H^{-1}(M_0/t_n)s \right) \right) \nu(s)ds + \int_{H^{-1}(1/t_n)^{-1}}^\infty \nu(s)ds \right)
\]
\[
\leq t_n H^{-1}(1/t_n) \left( H^{-1}(M_0/t_n) \int_0^{H^{-1}(1/t_n)^{-1}} s^2 \nu(s)ds + H^{-1}(M_0/t_n)^{-1} w(H^{-1}(1/t_n)^{-1}) \right)
\]
\[
\leq c_5 t_n H^{-1}(1/t_n) \left( H^{-1}(1/t_n)^{-1}(H \circ H^{-1})(1/t_n) \right) = c_5.
\]

We used the fact that the support of \( \nu_n \) is contained in \((0, H^{-1}(1/t_n)^{-1}]\) in the first inequality, and the mean value theorem and the fact that for every \( a > 0 \), \( \sup_{x \in (0, \infty)} x e^{-ax} = e^{-1} a^{-1} \) in the second inequality.

Therefore, we deduce that \( \mathbb{P}(Z_n \geq R) \leq c_4(2e + c_5) R^{-1} \) for all \( n \geq 1 \) and \( R > 1 \), which yields that the family \( \{Z_n : n \geq 1\} \) is tight. Then by the Prokhorov’s theorem, by taking a subsequence, we can assume that \( Z_n \) is weakly convergent to the random variable \( Z_\star \).

Now, from the weak convergence, we can prove (3.41) by showing the following:
\[
\inf_{z \in [0, N]} \mathbb{P}(z \leq Z_\star \leq z + c_3) > 0. \tag{3.42}
\]

According to [42] Theorem 8.7, \( Z_\star \) is a infinitely divisible random variable with the characteristic function
\[
\Psi_\star(\xi) = \frac{1}{2} A_\star \xi^2 - i \xi \gamma_\star + \int_0^\infty \left( e^{i \xi s} - 1 - \frac{i \xi s}{1 + s^2} \lambda_\star(s)ds \right),
\]
where the triplet \((A_\star, \gamma_\star, \lambda_\star)\) is characterized by
(i) \( \lim_{s \to 0} \sup_{n \to \infty} \left| \int_0^s s^2 \lambda_n(s)ds - A_\star \right| = 0; \)
(ii) \( \gamma_\star = \lim_{n \to \infty} \gamma_n; \)
(iii) \( \int_0^\infty f(s) \lambda_n(s)ds = \lim_{n \to \infty} \int_0^\infty f(s) \lambda_n(s)ds \) for any bounded continuous function \( f \) vanishing in a neighborhood of 0.

If \( A_\star > 0 \), then it is evident that the support of \( Z_\star \) is \( \mathbb{R} \) and hence (3.42) holds. Hence, we assume that \( A_\star = 0 \). Then, by (i) and (iii) in the above characterization, for every \( \eta \in (0, 1), \)
\[
\int_0^\eta s^2 \lambda_n(s)ds = \lim_{\varepsilon \to 0^+} \int_0^\eta s^2 \lambda_n(s)ds = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \left( \int_0^\eta s^2 \lambda_n(s)ds - \int_0^\varepsilon s^2 \lambda_n(s)ds \right)
\]
\[
= \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \int_0^\eta s^2 \lambda_n(s)ds = \lim_{n \to \infty} t_n H^{-1}(1/t_n)^2 \int_0^{\eta H^{-1}(1/t_n)^{-1}} u^2 \nu(u)du
\]
\[
\geq \lim_{n \to \infty} c_7 \eta^2 t_n H(\eta^{-1} H^{-1}(1/t_n)) \geq c_7 \eta^2 > 0.
\]
We used Lemma 2.3(4) in the first inequality and the monotonicity of $H$ in the second inequality. It follows that according to [40] Lemma 2.5, if $\int_0^1 s\lambda_s(s)ds = \infty$, then the support of $Z_s$ is $\mathbb{R}$ so that (3.42) holds. Assume that $\int_0^1 s\lambda_s(s)ds < \infty$. Then we see from (iii) in the characterization that $\limsup_{n \to \infty} \int_0^1 s\lambda_n(s)ds < \infty$. Hence, again by [40] Lemma 2.5, the support of $Z_s$ is $[-\chi, \infty)$ where $\chi = \lim_{n \to \infty} t_n H^{-1}(1/t_n)b(t_n/M_0) \geq 0$.

Since the support of $Z_s$ includes (0, $\infty$) in any cases, we see that (3.42) holds. This finishes the proof of the proposition under the condition (S-1).

Hereafter, we assume that the conditions (L-1) and (L-3) hold instead of (S-1). By Proposition 3.2 and Lemmas 2.4(3&4) and 2.5 we can follow (3.37), (3.38), and (3.39), and hence it suffice to show that for every $N > 0$, there exists a constant $c_N > 0$ such that

$$\inf_{t \in [T, \infty]} \inf_{y \in [0, NH^{-1}(1/t)^{-1}]} \mathbb{P}(y \leq S_t - tb(t/M_0) \leq y + c_N H^{-1}(1/t)^{-1}) \geq c_N. \quad (3.43)$$

(The constant $c_N$ may differ.) For convenience, we still denote a sequence of time variables realizing the infimum in (3.43) by $(t_n : n \geq 1)$. Then, after taking a subsequence, we can assume that either $t_n$ converges to $t^* \in [T, \infty)$ or $\lim_{n \to \infty} t_n = \infty$. If $t_n$ converges, then we are done. Hence, assume that $\lim_{n \to \infty} t_n = \infty$ and all $t_n$ are sufficiently large. Then, by using Lemma 2.3 instead of Lemma 2.3, we can follow the proof under the condition (S-1) and deduce the desired result.

Furthermore, note that the restriction that $t > 2T_1$ is only required to obtain (3.37). Hence, in view of the second statement of Proposition 3.2, we can see that the later assertion holds. This completes the proof. $$\square$$

Recall that $\sigma = \sigma(t, x) = (\phi')^{-1}(x/t)$. As a corollary to the above proposition, we can erase the constant $M_0$ in Propositions 3.1 and 3.2.

**Corollary 3.8.** (1) Suppose that the condition (S-1) holds. Then, for every fixed $T > 0$ and $N > 0$, it holds that for all $t \in (0, T]$,

$$p(t, x) \asymp \frac{1}{\sqrt{t(-\phi''(\sigma))}} \exp\left(-tH(\sigma)\right), \quad \text{for} \quad x \in (0, tb(t)],$$

$$p(t, tb(t) + y) \asymp H^{-1}(1/t), \quad \text{for} \quad y \in [0, NH^{-1}(1/t)^{-1}]. \quad (3.44)$$

(2) Suppose that the conditions (L-1) and (L-3) hold. Then, for every $N > 0$, there are comparison constants such that for all $t \in [2T_1, \infty), (3.44)$ holds for $x \in (0, tb(t)]$ and $y \in [0, NH^{-1}(1/t)^{-1}]$ where $T_1$ is the constant in Proposition 3.3.

Moreover, if $T_0 = 0$ in the condition (E), then for every $T > 0$ and $N > 0$, there is a comparison constant such that (3.44) holds for all $t \in [T, \infty)$, $x \in (0, tb(t)]$ and $y \in [0, NH^{-1}(1/t)^{-1}]$.

**Proof.** (1) The second comparison follows from Proposition 3.7 and Lemma 2.5. To prove the first one, in view of Proposition 3.1, it suffices to consider for $x \in [tb(t/M_0), tb(t)]$. For those $x$, we see from Lemma 2.5 and Proposition 3.7 that $p(t, x) \asymp H^{-1}(1/t)$. On the other hand, observe that for $x \in [tb(t/M_0), tb(t)], \sigma = (\phi')^{-1}(x/t) \geq H^{-1}(1/t) \geq H^{-1}(1/T)$, and hence by Lemma 2.3(3) and (4),

$$\frac{\exp\left(-tH(\sigma)\right)}{\sqrt{t(-\phi''(\sigma))}} \asymp \frac{\sigma}{\sqrt{tH(\sigma)}} \asymp H^{-1}(1/t) \asymp p(t, x). \quad (3.45)$$

This proves the corollary under the condition (S-1).

(2) Similarly, the second comparison follows from Proposition 3.7 and Lemma 2.5. Moreover, in this case, for $x \in [tb(t/M_0), tb(t)], \sigma \leq H^{-1}(M_0/t) \leq H^{-1}(M_0/(2T_1))$. Therefore, by Lemmas 2.4(3&4) and 2.5 and Proposition 3.2 we can deduce that (3.45) holds for all $t \in$...
Moreover, if the condition \((\text{S-3})\) further hold, then (3.46) holds true for all \(t \in (0, T]\) and \(y \in [0, \infty)\).

(2) Suppose that the condition \((\text{L})\) holds. Then, there exist constants \(c_1 > 1, c_2 > 0\) such that (3.46) holds for all \(t \in [2T_1, \infty)\) and \(y \in [0, \infty)\) where \(T_1\) is the constant in Proposition 3.2. Moreover, if \(T_0 = 0\) in the condition \((\text{E})\), then for every \(T > 0\), there are comparison constants such that (3.46) holds for all \(t \in [T, \infty)\) and \(y \in [0, \infty)\).

Proof. The result follows from Propositions 3.11 and 3.14. \(\square\)

**Proposition 3.11.** Under the settings of Theorem 3.10, the upper bound in (3.46) holds.
Indeed, by \([16, \text{Lemma 7.2}]\), Proposition 3.6 and Lemma 2.3(3), we see that \(Lévy measure \nu H \) from Proposition 3.6. Therefore, for the remainder part of the proof of (1), we assume that \(\delta y > H^{-1}(1/t)^{-1}. \)

Define

\[
\nu_1(s) := 1_{(0,\theta(t,\delta y)]}(s)\nu(s) \quad \text{and} \quad \nu_2(s) := \nu(s) - \nu_1(s) = 1_{(\theta(t,\delta y),\infty)}(s)\nu(s).
\]

Denote by \(S^i\) and \(H_i\) the corresponding subordinator and \(H\)-function with respect to the Lévy measure \(\nu_i\) for \(i = 1, 2\), respectively. Since \(\lim \inf_{s \to 0} sv_1(s) = \lim \inf_{s \to 0} sv(s) = 1/T_0,\) by Proposition 3.6, \(S^1_i\) has a transition density function \(p\) for every \(u > T_0.\) Recall that \(T_0 = 0\) under the condition \((S).\) Since \(S_i = S^1_i + S^2_i,\) we see that for \(t > 0,\)

\[
p(t, tb(t) + y) = \int_{\mathbb{R}} p^1(t, tb(t) + y - z)\mathbb{P}(S^2_t \in dz)
\]

\[
= \int_{\{z \leq y/4\}} p^1(t, tb(t) + y - z)\mathbb{P}(S^2_t \in dz) + \int_{\{z > y/4\}} p^1(t, tb(t) + y - z)\mathbb{P}(S^2_t \in dz)
\]

\[
\leq \sup_{z \geq 3y/4} p^1(t, tb(t) + z) + \sup_{z > y/4} \frac{\mathbb{P}(S^2_t \in dz)}{dz} := A_1 + A_2. \tag{3.47}
\]

**Step1.** First, we estimate \(A_1.\) By the semigroup property, for every \(z \geq 3y/4,\)

\[
p^1(t, tb(t) + z) = \left(\int_{u < z/2} + \int_{u \geq z/2}\right) p^1(t/2, tb(t)/2 + u)p^1(t/2, tb(t)/2 + z - u)du
\]

\[
\leq 2\mathbb{P}(S^1_{t/2} \geq \frac{t}{2}b(t) + \frac{3y}{8}) \sup_{u \in \mathbb{R}} p^1(t/2, u). \tag{3.48}
\]

We claim that there exists a constant \(c_2 > 0\) such that for all \(t \in (0, T],\)

\[
\sup_{u \in \mathbb{R}} p^1(t/2, u) \leq c_2H^{-1}(1/t). \tag{3.49}
\]

Indeed, by \([16, \text{Lemma 7.2}], \text{Proposition 3.6 and Lemma 2.3}\), we see that

\[
\sup_{u \in \mathbb{R}} p^1(t/2, u) \leq \exp \left(2^{-1}tw(\theta(t, \delta y))\right) \sup_{u \in \mathbb{R}} p(t/2, u) \leq c \exp(e)H^{-1}(2/t) \leq cH^{-1}(1/t).
\]

On the other hand, by the Chebychev's inequality and \([20, \text{Lemma 2.5}],\) for every \(\lambda > 0,\)

\[
\mathbb{P}(S^1_{t/2} \geq \frac{t}{2}b(t) + \frac{3y}{8}) \leq \mathbb{E} \left[\exp(\lambda S^1_{t/2} - \frac{\lambda t}{2}b(t) - \frac{3\lambda y}{8})\right]
\]

\[
= e^{-3\lambda y/8} \exp \left(\frac{t}{2} \int_0^{\theta(t, \delta y)} (e^{\lambda s} - 1)\nu(s)ds - \frac{t}{2} \int_0^{\infty} \lambda s e^{-H^{-1}(1/t) s} \nu(s)ds\right)
\]

\[
\leq e^{-3\lambda y/8} \exp \left(\frac{t}{2} \int_0^{\theta(t, \delta y)} (e^{\lambda s} - e^{-H^{-1}(1/t) s})\lambda s\nu(s)ds\right)
\]

\[
\leq e^{-3\lambda y/8} \exp \left(\frac{t}{2} \int_0^{\theta(t, \delta y)} (\lambda + H^{-1}(1/t))\lambda e^{\lambda s} s^2\nu(s)ds\right).
\]
We used the mean value theorem in the second and third inequalities. Thus, by letting 
\[ \lambda = \theta(t, \delta y)^{-1} \geq H^{-1} (1/t), \]
we see from (1.9) and Lemma 3.9 that
\[
\mathbb{P}( S_{t/2}^1 \geq \frac{t}{2} b(t) + \frac{3y}{8} ) \leq \exp \left( - \frac{3y}{8 \theta(t, \delta y)} + \frac{e t}{\theta(t, \delta y)^2} \int_0^{\delta y} s^2 \nu(s) ds \right) 
\leq \exp \left( - \frac{1}{8 \theta(t, \delta y)} (3y - 16e^2 t \theta(t, \delta y) H(\theta(t, \delta y)^{-1})) \right) 
\leq \exp \left( - \frac{1}{8 \theta(t, \delta y)} (3y - 16e^2 \delta y) \right) = \exp \left( - \frac{y}{8 \theta(t, \delta y)} \right). \tag{3.50}
\]

Consequently, from (3.48), (3.49) and (3.50), we deduce that
\[ A_1 \leq 2c_2 H^{-1} (1/t) \exp \left( - \frac{y}{8 \theta(t, \delta y)} \right). \tag{3.51} \]

Note that (3.47) and (3.51) hold for all \( y > \delta^{-1} H^{-1} (1/t)^{-1} \) and we have not assumed \( y < R_1/2 \) yet.

Step 2. Next, we assume \( y \in [0, R_1/2) \) and estimate \( A_2 \). Since \( S^2 \) is a compound Poisson process, for every \( z > 0 \) and \( \rho > 0 \), we have that
\[
\mathbb{P}( S_{t/2}^2 \in (z, z + \rho)) = \sum_{n=1}^\infty e^{-\rho u} u^{n} \nu_2^n(z, z + \rho) \leq \sum_{n=1}^\infty \frac{t^n \nu_2^n(z, z + \rho)}{n!}, \tag{3.52}
\]
where \( \nu_2^n \) is the \( n \)-fold convolution of the measure \( \nu_2 \). Define
\[
f(r) := \begin{cases} 
sup_{u \geq r} \nu(u) & \text{if } r < R_1/2, \\ 
sup_{u \geq R_1/2} \nu(u) & \text{if } r \geq R_1/2. 
\end{cases}
\]

Then \( f \) is a non-increasing function on \( (0, \infty) \) and \( \nu(r) \leq f(r) \) for all \( r > 0 \). Moreover, we see from the conditions (S-1) and (S-3) that \( \nu(r) \asymp f(r) \) for \( r \in (0, R_1/2) \). Also, we see that there exists a constant \( c_5 > 1 \) such that
\[ f(r) \leq c_5 f(2r) \quad \text{for all } r > 0. \]

Indeed, if \( r < R_1/4 \), then \( f(r) \leq c f(2r) \) for some constant \( c > 0 \) by (S-2). Else if \( R_1/4 \leq r < R_1/2 \), then by (S-2) and (S-3), \( f(r) \leq c \nu(R_1/4) \leq c \sup_{u \geq R_1} \nu(u) \nu^{-1}(f(2r) \leq c f(2r) \). Otherwise, if \( r \geq R_1/2 \), then \( f(r) = f(2r) \).

Now, we prove by induction that for every \( n \geq 1, z > 0 \) and \( \rho > 0 \), it holds that (cf. [34, Lemma 9 and Corollary 10])
\[ \nu_2^n(z, z + \rho) \leq (4e c_5)^n t^{1-n} f(z) \rho. \tag{3.53} \]

First, we see that \( \nu_2(z, z + \rho) \leq f(z) \rho \). Assume that (3.53) is true for \( n \). Then we have
\[
\nu_2^{(n+1)}(z, z + \rho) = \left( \int_{u < z/2} + \int_{u \geq z/2} \right) \nu_2^n(z - u, z - u + \rho) \nu_2(du) 
\leq \int_{u < z/2} (4e c_5)^n t^{1-n} f(z - u) \rho \nu_2(du) + \int_{z/2}^{z/2 + \rho} \nu_2(du) \nu_2^n(z - v, z - v + \rho) 
\leq (4e c_5)^n t^{1-n} f(z/2) \rho \nu_2(\mathbb{R}) + f(z/2) \rho \nu_2^n(\mathbb{R}) 
\leq (2e (4e c_5)^n) c_5 t^{-n} f(z) \rho \leq (4e c_5)^{n+1} t^{-n} f(z) \rho.
\]

We used the Fubini’s theorem in the first inequality and the fact that \( \nu_2(\mathbb{R}) = w(\theta(t, \delta y)) \leq w(w^{-1}(2e/t)) = 2e/t \) in the third inequality. Hence, we conclude that (3.53) holds.
It follows from (3.52) and (3.53) that since \( y \in [0, R_1/2) \),
\[
A_2 = \sup_{z>y/4} \lim_{\rho \to 0} \rho^{-1} \mathbb{P}(S_{t/2}^2 \in (z, z + \rho)) = \sup_{z>y/4} tf(z) \sum_{n=1}^{\infty} \frac{(4e\rho)^n}{n!} \leq e^{4e\rho} tf(y/4) \leq c_6 tv(y).
\]

(3.54)

Finally, we get the desired result from (3.47), (3.51) and (3.54).

Now, we further assume that (S-3*) holds and assume that \( y > (R_1/2) \lor (\delta^{-1} H^{-1}(1/t))^{-1} \). Recall that (3.47) and (3.51) still hold for those values of \( y \). Define \( f_*(r) := \sup_{u > r} \nu(u) \). Then, we see from the conditions (S) and (S-3*) that \( \nu(r) \asymp f(r) \) for all \( r \in (0, \infty) \) and there exists a constant \( c_7 > 1 \) such that \( f_*(r) \leq c_7 f_*(2r) \) for all \( r > 0 \). By following the above proof given in Step2, we get \( A_2 \leq e^{4e\rho} tf_*(y/4) \leq c_7 \nu(y) \). Thus, (3.54) still hold for those values of \( y \) and this completes the proof.

Now, we begin to prove the lower bound in Theorem 3.10. We first establish a preliminary jump type estimates for \( p(t,x) \).

**Proposition 3.12.** (1) Suppose that the conditions (S-1) and (S-3) hold. Then, for every \( T > 0 \), there exist constants \( c_1, c_2 > 0 \) such that for all \( t \in (0,T] \) and \( y \in [0, R_1/2) \),
\[
p(t, tb(t) + y) \geq c_1 H^{-1}(1/t) \min \left\{ 1, \frac{tv(y)}{H^{-1}(1/t)} \right\}.
\]

Moreover, if the condition (S-3*) further hold, then (3.55) holds true for all \( t \in (0,T] \) and \( y \in [0, \infty) \).

(2) Suppose that the conditions (L-1) and (L-3) hold. Then, there exist constants \( c_1, c_2 > 0 \) such that (3.55) holds for all \( t \in [2T_1, \infty) \) and \( y \in [0, \infty) \) where \( T_1 \) is the constant in Proposition 3.2.

Moreover, if \( T_0 = 0 \) in the condition (E), then for every \( T > 0 \), there exist \( c_1, c_2 > 0 \) such that (3.55) holds for all \( t \in (T, \infty) \) and \( y \in [0, \infty) \).

**Proof.** (1) According to Corollary 3.8 it suffices to prove (3.55) for \( y > 2H^{-1}(1/t) \). Hence, we assume \( y > 2H^{-1}(1/t) \).

Let \( \varepsilon \in (0,1/2) \) be a small constant which will be chosen later and define
\[
\begin{align*}
\mu_1(s) &:= (1 - \varepsilon) \mathbb{1}_{[H^{-1}(1/t), \infty)}(s) \nu(s), \\
\mu_2(s) &:= \varepsilon \mathbb{1}_{[1, H^{-1}(1/t), \infty)}(s) \nu(s).
\end{align*}
\]

We denote by \( T_i^s \) the corresponding subordinator with respect to the Lévy measure \( \mu_s \) for \( i = 1, 2 \), respectively. Since \( \lim \inf_{s \to 0} s \mu_1(s) = \lim \inf_{s \to 0} s \nu(s) = 1/T_0 \), by Proposition 3.1, \( T_i^s \) has a transition density function \( q^i(u, \cdot) \) for every \( u > T_0 \).

We claim that there exists a constant \( c_3 > 0 \) such that for all \( t \in (0, T] \),
\[
q^1(t, tb(t) + z) \geq c_3 H^{-1}(1/t) \quad \text{for all } z \in [0, H^{-1}(1/t)^{-1}].
\]

(3.56)
Indeed, we see from the conditions (S-1) and (S-3), Lemma 2.3 (1) and (1.10) that
\[ \sup_{s>0} |\mu_2(s)| \leq \varepsilon c (\nu(H^{-1}(1/t)^{-1}) + 1) \leq \varepsilon c (H^{-1}(1/t)w(H^{-1}(1/t)^{-1} / 2) + 1) \leq \varepsilon c (H^{-1}(1/t)H(2H^{-1}(1/t)) + 1) \leq \varepsilon c t^{-1} H^{-1}(1/t). \]

On the other hand, by Corollary 3.8 there exists \( c_5 > 0 \) such that
\[ p(t, t^b(t) + z) \geq c_5 H^{-1}(1/t) \quad \text{for all } z \in [0, H^{-1}(1/t)^{-1}]. \]

Hence, by [3, Lemma 3.1(c)], we get that for all \( z \in [0, H^{-1}(1/t)^{-1}], \)
\[ q^1(t, t^b(t) + z) \geq p(t, t^b(t) + z) - t \sup_{s>0} |\mu_2(s)| \geq (c_5 - \varepsilon c_4) H^{-1}(1/t). \]

Therefore, by taking \( \varepsilon = c_5/(2c_4) \), we obtain (3.56).

Then, since \( S_t = T_1^1 + T_2^1 \) and \( T_2^1 \) is a compounded Poisson process, by (3.56) and (1.10), for all \( t \in (0, T] \) and \( y \in [0, \infty), \)
\[ p(t, t^b(t) + y) = \int_\mathbb{R} q^1(t, t^b(t) + y - z) \mathbb{P}(T_2^1 \in dz) \]
\[ \geq c_3 H^{-1}(1/t) \mathbb{P}(T_2^1 \in [y - H^{-1}(1/t)^{-1}, y]) \]
\[ \geq c_3 H^{-1}(1/t) c t w((y - H^{-1}(1/t)^{-1}, y]) \exp \left( - c t w(H^{-1}(1/t)^{-1}) \right) \]
\[ \geq c_3 H^{-1}(1/t) c t H^{-1}(1/t)^{-1} \exp \left( - 4c t w(H^{-1}(1/t)^{-1}) \right) \inf_{u \in [y - H^{-1}(1/t)^{-1}, y]} \nu(u) \]
\[ \geq c_6 t \inf_{u \in [y/2, y]} \nu(u). \]

We see from the condition (S-1) that \( \inf_{u \in [y/2, y]} \nu(u) \propto \nu(y) \) for all \( y \in (2H^{-1}(1/t)^{-1}, R_1/2) \). Moreover, if the condition (S-3*) further hold, then \( \inf_{u \in [y/2, y]} \nu(u) \propto \nu(y) \) for all \( y \in (2H^{-1}(1/t)^{-1}, \infty) \). Hence, we get the results.

(2) Fix any \( N > 2 \) such that \( NH^{-1}(1/T)^{-1} \geq R_2 \). In view of Corollary 3.8 we can assume that \( y > NH^{-1}(1/t)^{-1} \geq R_2 \). Then, by repeating the proof for (1), we get the desired result. The proof for the second assertion is exactly the same.

\[ \Box \]

**Lemma 3.13.** (1) **Suppose that the condition (S-1) holds.** Then, for every \( a > 0 \) and \( T > 0 \), there exists \( c_1 > 0 \) such that for all \( t \in (0, T] \) and \( y \in [H^{-1}(1/t)^{-1}, R_1/2] \),
\[ \exp \left( - \frac{ay}{w^{-1}(2e/t)} \right) \leq c_1 \frac{tv(y)}{H^{-1}(1/t)}. \]

Moreover, if the condition (S-3*) further hold, then (3.57) holds true for all \( t \in (0, T] \) and \( y \in [H^{-1}(1/t)^{-1}, \infty) \).

(2) **Suppose that the condition (L-1) holds.** Then, for every \( a > 0 \) and \( T > 0 \), there exists \( c_1 > 0 \) such that (3.57) holds for all \( t \in [T, \infty) \) and \( y \in [H^{-1}(1/t)^{-1} \lor R_2, \infty) \).

**Proof.** Since the proofs are similar, we only give the proof for (1). By (1.11) and Lemma 2.3 (1) and (2), we have that for all \( y \in [H^{-1}(1/t)^{-1}, R_1/2] \),
\[ \exp \left( - \frac{ay}{w^{-1}(2e/t)} \right) \leq c_0 \left( \frac{w^{-1}(2e/t)}{y} \right)^{\alpha_1+1} \leq c \frac{w(y)}{w(w^{-1}(2e/t))} \frac{w^{-1}(2e/t)}{y} \leq c \frac{tv(y)}{H^{-1}(1/t)}. \]

In the first inequality above, we used the fact that for every \( p > 0 \), there exists a constant \( c(p) > 0 \) such that \( c^x \geq c(p)x^p \) for all \( x > 0 \).
Next, we further assume that the condition (S-3*\) holds. Since both conditions (S-1) and (S-3*) hold, there exist constants c_2, c_3 \in (0, 1) such that
\begin{equation}
\sup_{u \geq r} c_2 \nu(u) \leq \nu(r) \quad \text{and} \quad c_3 \nu(r) \leq \nu(2r) \quad \text{for all} \quad r > 0.
\end{equation}
(3.59)

Let \( r_0 = (\log(c_2^{-1}c_3^{-1}) + 1)(a^{-1} + 1)H^{-1}(1/T)^{-1} > 0. \) Then, \( r_0 > H^{-1}(1/T)^{-1} \geq H^{-1}(1/t)^{-1} \) for all \( t \in (0, T) \) and by (3.10),
\begin{equation}
\exp \left( - \frac{ar_0}{w^{-1}(2e/t)} \right) \leq \exp \left( - \frac{ar_0H^{-1}(1/T)}{2c_2c_3} \right) \leq c_2c_3.
\end{equation}
(3.60)

We first note that, using the condition (3.58), we can see that the condition (S-1) holds with \( R_1 = 9r_0 \) (after changing the constant \( c_1 \) therein). Therefore, we see that (3.58) holds for all \( t \in (0, T] \) and \( y \in [H^{-1}(1/t)^{-1}, 4r_0] \).

Now, assume that \( y \in (4r_0, \infty) \). Choose \( n \in \mathbb{N} \) such that \( 2^{n-1}r_0 < y \leq 2^n r_0. \) Then, by (3.59) and (3.60), and using (3.58) for \( y = r_0 \), it holds that for all \( t \in (0, T] \),
\begin{align*}
\frac{tv(r_0)}{H^{-1}(1/t)} \geq c_2 \frac{tv(2^n r_0)}{H^{-1}(1/t)} \geq c_2c_3 \frac{tv(r_0)}{H^{-1}(1/t)} \geq c_0^{-1} c_2 c_3 \exp \left( - \frac{ar_0}{w^{-1}(2e/t)} \right) \\
\geq c_0^{-1} \exp \left( - \frac{(n+1)ar_0}{w^{-1}(2e/t)} \right) \geq c_0^{-1} \exp \left( - \frac{2^{-n-1}ar_0}{w^{-1}(2e/t)} \right) \geq c_0^{-1} \exp \left( - \frac{ar_0}{w^{-1}(2e/t)} \right).
\end{align*}

The fifth inequality above holds since \( n \geq 3. \) This completes the proof.

\begin{proposition}
\begin{enumerate}
\item Suppose that the conditions (S-1) and (S-3) hold. Then, for every \( T > 0 \), there exist constants \( c_1, c_2 > 0 \) such that for all \( t \in (0, T] \) and \( y \in [0, R_1/2] \),
\begin{equation}
p(t, tb(t) + y) \geq c_1 H^{-1}(1/t) \min \left\{ 1, \frac{tv(y)}{H^{-1}(1/t)} + \exp \left( - \frac{c_2 y}{\theta(t, y/(8e^2))} \right) \right\}
\end{equation}
(3.61)
Moreover, if the condition (S-3*) further hold, then (3.61) holds true for all \( t \in (0, T] \) and \( y \in [0, \infty) \).
\item Suppose that the conditions (L-1) and (L-3) hold. Then, there exist constants \( c_1, c_2 > 0 \) such that (3.61) holds for all \( t \in [2T_1, \infty) \) and \( y \in [0, R_1] \) where \( T_1 \) is the constant in Proposition 4.2.
Moreover, if \( T_0 = 0 \) in the condition (E), then for every \( T > 0 \), there exist \( c_1, c_2 > 0 \) such that (3.61) holds for all \( t \in [T, \infty) \) and \( y \in [0, \infty) \).
\end{enumerate}
\end{proposition}

\textbf{Proof.} We first give the proof for (2). Suppose that the conditions (L-1) and (L-3) hold. Since the proof for the case when \( T_0 = 0 \) is easier, we only give the proof for the case when \( T_0 > 0 \).

Let \( \rho = (16e^2 T_1 H(w^{-1}(e/T_1)^{-1})^{-1} \wedge (4e^2)^{-1} \). Then, by the monotonicities of \( H \) and \( w \),
\begin{equation}
\frac{1}{8e^2 \rho H(w^{-1}(2e/t)^{-1})^{-1}} \geq 2T_1 \quad \text{for all} \quad t \geq 2T_1.
\end{equation}
(3.62)

By Corollary 3.8, Lemma 3.13 and Proposition 3.12, it remains to prove that there are constants \( c_1, c_2 > 0 \) such that for all \( t \in [2T_1, \infty) \) and \( y \in [2\rho^{-1} H^{-1}(1/t)^{-1}, 8e^2 D(t)] \),
\begin{equation}
p(t, tb(t) + y) \geq c_1 H^{-1}(1/t) \exp \left( - \frac{c_2 y}{\theta(t, y/(8e^2))} \right).
\end{equation}

Fix \( t \in [2T_1, \infty) \), \( y \in [2\rho^{-1} H^{-1}(1/t)^{-1}, 8e^2 D(t)] \) and we simply denote \( \theta := \theta(t, y/(8e^2)) \). Then, since \( 2\rho^{-1} \geq 8e^2 \), by Lemma 3.9, we have
\begin{equation}
8e^2 t \theta H(\theta^{-1}) = y.
\end{equation}
(3.63)

Let \( n = \lfloor \rho y / \theta \rfloor := \max \{ m \in \mathbb{Z} : m \leq \rho y / \theta \} \). Then, since \( \theta \leq H^{-1}(1/t)^{-1} \), we have
\begin{equation}
n \geq \rho y H^{-1}(1/t) - 1 \geq 1.
\end{equation}
We claim that there exist constants $\kappa_1 \in (0, 1)$ and $\kappa_2 \in (1, \infty)$ independent of $t$ and $y$ such that
\[
\kappa_1 H^{-1}(n/t)^{-1} \leq y/n \leq \kappa_2 H^{-1}(n/t)^{-1}.
\] (3.64) Indeed, first note that (3.64) is equivalent to $H(\kappa_1 n/y) \leq n/t \leq H(\kappa_2 n/y)$.

Since $\rho/\theta \leq \rho w^{-1}(e/T_1)^{-1}$, by Lemma 2.4(3) and (3.63), there exists a constant $c_3 \in (0, 1)$ independent of $t$ and $y$ such that for every $\kappa \in [1, y/n)$,
\[
H(\kappa n/y) \geq c_3 \kappa^\delta \rho^\rho \theta^\theta = c_3 \kappa^\delta \theta \rho^\theta / (8e^2 t^3) \geq c_3 \rho \kappa^\delta n / (8e^2 t^3)
\]
where $\alpha_3 = \alpha_3 \wedge (3/2)$. Hence, if $y/n \geq (8e^2 c_3^{-1} \rho^{-1})^{1/\alpha_3}$, then by choosing $\kappa_2$ bigger than $(8e^2 c_3^{-1} \rho^{-1})^{1/\alpha_3}$, we get the upper bound in (3.64). Otherwise, if $y/n < (8e^2 c_3^{-1} \rho^{-1})^{1/\alpha_3}$, then we have
\[
(8e^2 c_3^{-1} \rho^{-1})^{1/\alpha_3} \geq y/n \geq \rho^{-1} \theta \geq \rho^{-1} w^{-1}(2e/t)
\]
which implies that $t \approx 1$. It follows that $y \times \theta \approx n \approx 1$. Therefore, by choosing $\kappa_2 > (8e^2 c_3^{-1} \rho^{-1})^{1/\alpha_3}$ sufficiently large, we obtain the upper bound in (3.64).

On the other hand, we also have that by Lemma 2.4(3) and (3.63),
\[
H(\kappa_1 n/y) \leq c_4 \kappa_1^\delta \rho^\rho \theta^\theta = c_4 \kappa_1^\delta \theta \rho^\theta / (8e^2 t^3)
\]
Therefore, by choosing $\kappa_1 = (4e^3 c_3^{-1} \rho^{-1} \alpha_3)\wedge (1/2)$, we get $H(\kappa_1 n/y) \leq \rho y / (2t) \leq n/t$, which proves the lower bound in (3.64).

Define $z = y + tb(t) - tb(t/n)$ and $z_j = jz/n$ for $j = 1, ..., n - 1$. Then, according to Lemma 2.5 (3.64) and Lemma 2.4(2),
\[
y \leq z \leq y + 2enH^{-1}(n/t)^{-1} + e^{-1} t^{-1}H^{-1}(1/t)^{-1} w(H^{-1}(n/t)^{-1})
\]
\[
\leq (2e \kappa_1^{-1} + 1)y + e^{-1} t^{-1}H^{-1}(1/t)^{-1} w(\theta / (\rho c_2))
\]
\[
\leq (2e \kappa_1^{-1} + 1)y + c_5 \kappa_1^\delta tH^{-1}(1/t)^{-1} w(\theta)
\]
\[
\leq (2e \kappa_1^{-1} + 1)y + 2e c_5 \kappa_1^\delta H^{-1}(1/t)^{-1} \leq c_6 y.
\]

We used the definition that $\theta \geq w^{-1}(2e/t)$ in the fourth inequality and the assumption that $y \geq 8e^2 H^{-1}(1/t)^{-1}$ in the last inequality. Then, by (3.64), for any $u \in (z_j - z/(2n), z_j + z/(2n))$ and $v \in (z_{j+1} - z/(2n), z_{j+1} + z/(2n))$ for some $j = 1, ..., n - 2$, we have
\[
|u - v| \leq 2z/n \leq 2c_6 y/n \leq 2c_6 \kappa_2 H^{-1}(n/t)^{-1}.
\]

Moreover, we see from (3.62) and (3.63) that
\[
\frac{t}{n} \geq \frac{t \theta}{\rho y} = \frac{1}{8e^2 \theta H(\theta^{-1})} \geq \frac{1}{8e^2 \rho H(w^{-1}(2e/t)^{-1})} \geq 2T_1.
\]

Thus, by Corollary 3.3 there exists a constant $c_7 \in (0, 1)$ independent of $t$ and $y$ such that
\[
p(t/n, (t/n)b(t/n) + v - u) \geq c_7 H^{-1}(n/t),
\]
for any $u \in (z_j - z/(2n), z_j + z/(2n))$ and $v \in (z_{j+1} - z/(2n), z_{j+1} + z/(2n))$ for some $j = 1, ..., n - 2$. Then, by the semigroup property and (3.64), we get
\[
p(t, tb(t) + y)
\]
\[
\geq \int_{z_{j+1} - z/(2n)}^{z_{j+1} + z/(2n)} ... \int_{z_{j-1} - z/(2n)}^{z_{j-1} + z/(2n)} \left( p(t/n, (t/n)b(t/n) + u_1) p(t/n, (t/n)b(t/n) + u_2 - u_1) ... \times p(t/n, (t/n)b(t/n) + u_{n-1} - u_{n-2}) p(t/n, (t/n)b(t/n) + z - u_{n-1}) du_1 ... du_{n-1} \right.
\]
\[
\geq (c_7 H^{-1}(n/t))^n (z/n)^{n-1} \geq c_7^n H^{-1}(n/t) \geq H^{-1}(1/t) \exp \left( -n \log c_7^{-1} \right).
\]
Since $n \approx y/\theta$, we obtain the desired result.
Now, we assume that the conditions (S-1) and (S-3) hold and follow the above proof. In this case, we simply let \( \rho = (4e^2)^{-1} \). Since \( \theta^{-1} \geq H^{-1}(1/t)^{-1} \geq H^{-1}(1/T)^{-1} \) in this case, by using Lemma 2.3 instead of Lemma 2.4, we obtain (3.64). Then, we get the result by exactly the same proof as the one given in the above. We note that there is no difference in the proof for the second assertion in (1).

3.4. Proofs of Theorems 1.3 and 1.4 and Corollaries 1.5, 1.7, 1.8 and 1.9.

Lemma 3.15. (1) Suppose that the condition (S-1) holds. Then, for every fixed \( T > 0 \), there exist constants \( c_1, c_2 > 0 \) such that for all \( t \in (0, T] \) and \( x \in (0, tb(t)] \),

\[
 c_1 H^{-1}(1/t) \exp \left( -2t H(\sigma) \right) \leq \exp \left( \frac{-t H(\sigma)}{\sqrt{t(1-\phi''(\sigma))}} \right) \leq c_2 H^{-1}(1/t) \exp \left( -\frac{t}{2} H(\sigma) \right). \tag{3.65}
\]

In particular, if the condition (S-1) holds with \( R_1 = \infty \), then (3.65) holds for all \( t \in (0, \infty) \) and \( x \in (0, tb(t)] \).

(2) Suppose that the condition (L-1) holds. Then, for every fixed \( T > 0 \), there exist constants \( c_1, c_2 > 0 \) such that (3.65) holds for all \( t \in [T, \infty) \) and \( x \in [tb(T), tb(t)] \).

Proof. (1) Observe that for all \( t \in (0, T] \) and \( x \in (0, tb(t)] \), we have \( \sigma \geq H^{-1}(1/t) \geq H^{-1}(1/T) \) and \( t H(\sigma) \geq 1 \). Hence, by Lemma 2.3(3) and (4), we get

\[
 c_3 \sigma \exp(-2t H(\sigma)) \leq \frac{\exp(-t H(\sigma))}{\sqrt{t(1-\phi''(\sigma))}} \leq \frac{\exp(-t H(\sigma))}{\sigma \sqrt{t H(\sigma)}} \leq \sigma \exp(-t H(\sigma)). \tag{3.66}
\]

Moreover, by applying Lemma 2.3(4) again,

\[
 \sigma \exp(-t H(\sigma)) = H^{-1}(1/t) \frac{\sigma}{H^{-1}(1/t)} \exp(-t H(\sigma)) \leq c_4 H^{-1}(1/t) \left( \frac{H(\sigma)}{1/t} \right)^{1/\alpha_1} \exp(-t H(\sigma)) \leq c_5 H^{-1}(1/t) \exp \left( -\frac{t}{2} H(\sigma) \right) \tag{3.67}
\]

and

\[
 \sigma \exp(-2t H(\sigma)) = H^{-1}(1/t) \frac{\sigma}{H^{-1}(1/t)} \exp(-2t H(\sigma)) \geq H^{-1}(1/t) \exp(-2t H(\sigma)). \tag{3.68}
\]

This proves the first assertion. If we further assume that \( R_1 = \infty \), then by combining Lemmas 2.3 and 2.4, we can see that (3.66), (3.67) and (3.68) holds for all \( t \in (0, \infty) \) and \( x \in (0, tb(t)] \) since \( t H(\sigma) \geq 1 \) on those values of \( t \) and \( x \).

(2) Note that for all \( t \in [T, \infty) \) and \( x \in [tb(T), tb(t)] \), we have \( \sigma \leq H^{-1}(1/T) \) and \( t H(\sigma) \geq 1 \). Hence, by using Lemma 2.3 instead of Lemma 2.3, we can follow the proof for (1) and conclude that (2) also holds.

**Proof of Theorems 1.3 and 1.4.** The results follow from Corollary 3.8, Theorem 3.10 and Lemmas 3.13 and 3.15.

**Proof of Corollary 1.5.** The results follow from Theorems 1.3 and 1.4.

**Proof of Corollaries 1.7 and 1.8.** Since the proofs are similar, we only give the proof for Corollary 1.7. Under the condition (S.Pure), by [20, Lemma 2.1(iii)], we have that \( w^{-1}(2e/t) \asymp H^{-1}(1/t)^{-1} \) for all \( t \in (0, T] \) and hence \( D(t) \asymp H^{-1}(1/t)^{-1} \) for all \( t \in (0, T] \). Then, by Theorem 1.3(2), 1.17 and Corollary 3.8, we get that (1.18) holds.

On the other hand, note that by the condition (E), \( \nu((x - tb(t))^+) = \nu(0) = \infty \) for all \( x \leq tb(t) \). Thus, by joining (1.18) and (1.15) together, we also deduce (1.19).
Proof of Corollary 1.4 Since the proofs for the case $T_0 = 0$ and the case $T_0 > 0$ are similar, we give the proof for the case $T_0 > 0$ only. Let $T_1 > 0$ is the constant in Theorem 1.4(1). Let $\alpha' = \alpha_3 \wedge (3/2)$. Since $\alpha_3 > 1$, we also have that $\alpha' > 1$. Observe that for every $t \geq T_1$, by Lemma 2.3(4) and (L.Mixed),

$$
0 \leq t\phi'(0) - tb(t) = t \int_0^{H^{-1}(1/t)} (-\phi''(\lambda))d\lambda \leq c \int_0^{H^{-1}(1/t)} \lambda^{-2} \frac{H(\lambda)}{H(1/t)} d\lambda
$$

$$
\leq cH^{-1}(1/t)^{-\alpha'} \int_0^{H^{-1}(1/t)} \lambda^{-2+\alpha'}d\lambda \leq c_1 H^{-1}(1/t)^{-1},
$$

(3.69)

for some constant $c_1 > 1$. Moreover, by Lemma 2.4(3), there exists $c_2 \in (0, 1)$ such that

$$
H(\kappa \lambda) \geq c_2 \kappa^{\alpha'} H(\lambda) \quad \text{for all } \kappa \geq 1, \quad 0 < \lambda \leq \kappa^{-1}.
$$

(3.70)

Let $y_t = y + t\phi'(0) - tb(t)$. Note that $y \leq y_t \leq y + c_1 H^{-1}(1/t)^{-1}$. By Theorem 1.4(1), we have that for every $t \geq T_1$,

$$
p(t, t\phi'(0) + y) = p(t, tb(t) + y_t) \simeq H^{-1}(1/t) \min \left\{ 1, \frac{t\nu(y_t)}{H^{-1}(1/t)} + \exp \left( - \frac{cy_t}{\theta(t, y_t/(8e^2))} \right) \right\}.
$$

(3.71)

Define

$$
F(t, y) = \min \left\{ 1, \frac{t\nu(y)}{H^{-1}(1/t)} + \exp \left( - \frac{y}{\mathcal{H}^{-1}(t/y)} \right) \right\}.
$$

Then, in view of (3.71), it remains to prove that for all $t \geq T_1$ and $y \geq 0$,

$$
F(t, y) \simeq \min \left\{ 1, \frac{t\nu(y_t)}{H^{-1}(1/t)} + \exp \left( - \frac{cy_t}{\theta(t, y_t/(8e^2))} \right) \right\} =: G(t, y, c).
$$

(3.72)

We prove (3.72) by considering several cases. We use the following notations below.

$$
\varepsilon_1 := (c_2/(8e^2))^{1/(\alpha' - 1)} \in (0, 1), \quad \kappa_1 := c_2^{-1/(\alpha' - 1)} > 1, \quad \theta := \theta(t, y_t/(8e^2)),
$$

(i) Suppose that $0 \leq y_t < 8e^2 H^{-1}(1/t)^{-1}$. Then, we have $\theta = H^{-1}(1/t)^{-1} \geq y_t/(8e^2)$ and hence $G(t, y, 1) \asymp 1$.

We claim that it also holds that $F(t, y) \asymp 1$ which yields the desired result in this case. To prove this claim, we consider the following two cases:

(a) If $t \geq 1/H(\varepsilon_1)$, then we see from (3.70) that

$$
\mathcal{H}(\varepsilon_1 H^{-1}(1/t)^{-1}) \leq \frac{H^{-1}(1/t)}{\varepsilon_1 H(\varepsilon_1 H^{-1}(1/t)^{-1})} \leq \frac{tH^{-1}(1/t)}{c_2 \varepsilon_1^{-1}} \leq \frac{tH^{-1}(1/t)}{8e^2} \leq \frac{t}{y_t} \leq \frac{t}{y}.
$$

Thus, $\mathcal{H}^{-1}(t/y) \geq \varepsilon_1 H^{-1}(1/t)^{-1} \geq \varepsilon_1 y/(8e^2)$ and hence $F(t, y) \asymp 1$.

(b) If $T_1 \leq t \leq 1/H(\varepsilon_1)$, then $y \leq y_t < 8e^2/\varepsilon_1$ and hence from the monotonicity, we get

$$
\mathcal{H}^{-1}(t/y) \geq \mathcal{H}^{-1}(\varepsilon_1 H(1/t)/8e^2) \geq \varepsilon_1 \mathcal{H}^{-1}(\varepsilon_1 H(1/t)/8e^2) \geq \varepsilon_1 H^{-1}(1/t)^{-1} \leq y/(8e^2)\text{ and hence } F(t, y) \asymp 1.
$$

(ii) Suppose that $8e^2 H^{-1}(1/t)^{-1} \leq y_t < 8e^2D(t)$. Then, by Lemma 3.9 we have $y_t = 8e^2 t \theta H(\theta^{-1})$. Denote by $\varepsilon_2 := \varepsilon_2(t, y) = \theta^{-1}(1/t) \in (0, 1)$ so that $\theta = \varepsilon_2 H^{-1}(1/t)^{-1}$.

(a) Assume that $y \leq c_1 H^{-1}(1/t)^{-1}$. Then we see from (3.69) and (3.70) that

$$
2c_1 H^{-1}(1/t)^{-1} > y_t = 8e^2 t \theta H(\theta^{-1}) \geq 8e^2 c_2 \varepsilon_2^{1-\alpha'} H^{-1}(1/t)^{-1},
$$

provided that $\theta \geq 1$. Hence, if $\theta \geq 1$, then $\varepsilon_2 \geq (4c_2 c_1^{1/(\alpha' - 1)}) > 0$ and hence $y_t \asymp \theta \asymp H^{-1}(1/t)^{-1}$. Therefore, combining with the results in (i)(a) and (i)(b), we can deduce that $F(t, y) \asymp G(t, y, 1) \asymp 1$ in this case. Otherwise, if $\theta < 1$, then $w^{-1}(2e/t) \leq \theta < 1$ and hence $t < 2ew(1)$. By the similar proof to the one given in (i)(b), we can also deduce that $F(t, y) \asymp G(t, y, 1) \asymp 1$. 


(b) Assume that \( y \geq c_1 H^{-1}(1/t)^{-1} \). By the proof given in (i)(b), we may assume that \( H^{-1}(1/t)^{-1} \geq R_3 \) and \( w^{-1}(2e/t) \geq \varepsilon_1 \). Since (3.69) implies that \( y \leq y_t \leq 2y \) in this case, by the condition \((L)\), we get \( \nu(y) \asymp \nu(y_t) \). Hence, it remains to prove that \( \mathcal{H}^{-1}(t/y) \asymp \theta \) in this case. First, we see that by (3.70),
\[
\mathcal{H}(\varepsilon_1) \leq \frac{1}{\varepsilon_1 \theta H(\varepsilon_1^{-1} \theta^{-1})} \leq \frac{1}{c_2 \varepsilon_1^{1-a \gamma} \theta H(\theta^{-1})} = \frac{8e^2 \theta H(\theta^{-1})}{y_t} > \frac{t}{y}
\]
and hence \( \mathcal{H}^{-1}(t/y) \geq \varepsilon_1 \theta \). On the other hand, since \( \kappa_1 = c_2^{-1/(a \gamma)} \), by (3.70),
\[
\mathcal{H}(\kappa_1 \theta) = \inf_{\kappa \geq \kappa_1} \frac{1}{\kappa \theta H(\kappa^{-1} \theta^{-1})} \geq \frac{c_2 \kappa_1^{a \gamma - 1}}{\theta H(\theta^{-1})} = \frac{8e^2 t}{y_t} > \frac{t}{y}
\]
and hence \( \mathcal{H}^{-1}(t/y) \leq \kappa_1 \theta \). Therefore, we obtain \( \mathcal{H}^{-1}(t/y) \asymp \theta \).

(iii) Suppose that \( y_t > 8e^2 D(t) \). If \( y < c_1 H^{-1}(1/t)^{-1} \), then by the proof given in (ii)(a), we get the result. Hence, suppose that \( y \geq c_1 H^{-1}(1/t)^{-1} \) and hence \( y_t \leq 2y \). By the proof given in (ii)(b), we may assume that \( H^{-1}(1/t)^{-1} \geq R_2 \) and \( \nu(y) \asymp \nu(y_t) \). Moreover, by Lemma 2.4 and the condition \((L)\), we see that \( tH^{-1}(1/t)^{-1} \nu(y_t) \leq ct \nu(y_t) \leq ct \nu(y) \leq ct \nu(y_t) \leq c \). Moreover, by Lemma 3.13 and the condition \((L)\), for any fixed \( a > 0 \),
\[
\exp \left( -\frac{ay_t}{\theta(t,y_t/(8e^2))} \right) \leq \exp \left( -ac \frac{c_1 \nu(y_t) \nu(y)}{w^{-1}(2e/t)} \right) \leq \exp \frac{ct \nu(y_t) \nu(y)}{H^{-1}(1/t)^{-1}} \leq \frac{ct \nu(y_t) \nu(y)}{H^{-1}(1/t)^{-1}}
\]
Thus, \( G(t,y,1) \asymp tH^{-1}(1/t)^{-1} \nu(y) \) in this case. Therefore, it remains to prove that there exists \( c_3 > 0 \) such that
\[
\exp \left( -\frac{y}{\mathcal{H}^{-1}(t/y)} \right) \leq c_3 \frac{\nu(y)}{H^{-1}(1/t)^{-1}}.
\]
(3.73)
As before, we may assume that \( w^{-1}(2e/t) \geq 1 \). Then, since \( \theta = w^{-1}(2e/t) \) in this case, by (3.70) and Lemma 3.9
\[
\mathcal{H}(\kappa_1 w^{-1}(2e/t)) = \inf_{\kappa \geq \kappa_1} \frac{1}{\kappa \theta H(\kappa^{-1} \theta^{-1})} \geq \frac{c_2 \kappa_1^{a \gamma - 1}}{\theta H(\theta^{-1})} = \frac{8e^2 t}{y_t} > \frac{t}{y},
\]
which implies that \( \mathcal{H}^{-1}(t/y) \leq \kappa_1 w^{-1}(2e/t) \). Hence, we get (3.73) from Lemma 3.13. This completes the proof.

4. Examples

In this section, we provide non-trivial and concrete examples of subordinators which our main results can be applied to.

Recall that \( b(t) = (\phi' \circ H^{-1})(1/t) \) and \( \sigma = (\phi')^{-1}(x/t) \).

4.1. Polynomially decaying Lévy measure perturbed by a logarithmic function. In this subsection, we use the notation \( \log^p x := (\log x)^p \) for \( p \in \mathbb{R} \) and \( x > 1 \). Suppose that \( \gamma_1 \in [0, 1] \) and \( p \in \mathbb{R} \). If \( \gamma_1 = 0 \) we further assume that \( p > 0 \) and, if \( \gamma_1 = 1 \) then we further assume that \( p < -1 \).

Let \( f : (1, \infty) \to (0, \infty) \) be a measurable function satisfying
\[
\int_1^\infty f(r)dr < \infty, \quad c_1 \sup_{u \geq r} f(u) \leq f(r) \quad \text{and} \quad c_2 f(r) \leq f(2r) \quad \text{for all} \quad r \geq 1. \quad (4.1)
\]
for some constants \( c_1, c_2 > 0 \) and
\[
\nu(s) := 1_{(0 < s \leq 1)} s^{-1-\gamma_1} \log^p (1 + 1/s) + 1_{\{s > 1\}} f(s).
\]
Let $S$ be a subordinator with the Laplace exponent $\phi(\lambda) = \int_0^\infty (1 - e^{-\lambda t}) \nu(s)ds$. We see that $S$ satisfies the conditions (E) with $T_0 = 0$. Thus, by Proposition [1.1] for every $t > 0$, the transition density $p(t, x)$ of $S_t$ exists and is a continuous bounded function. In the following, we show that, using our main results, we can get the precise two-sided estimates on $p(t, x)$.

4.1.1. Small time estimates. Below, we assume that $\gamma_1 > 0$ and obtain estimates on $p(t, x)$ for $t \in (0, 2]$. Note that the conditions (S,Pure) and (S-3*) hold. Using Lemmas [2.3] [1&4] and [2.7] and (2.7), for every fixed $\lambda > 0$, we get that for all $\lambda \geq \lambda_0$,

$$H(\lambda) \asymp \lambda^{\gamma_1} \log^p(1 + \lambda), \quad H^{-1}(\lambda) \asymp \lambda^{1/\gamma_1} \log^{-p/\gamma_1}(1 + \lambda),$$

$$\phi'(\lambda) \asymp \lambda^{-1} \phi(\lambda) \asymp 1_{\{\gamma_1 \in (0,1)\}} \lambda^{\gamma_1 - 1} \log^p(1 + \lambda) + 1_{\{\gamma_1 = 1\}} \log^{p+1}(1 + \lambda),$$

$$(\phi')^{-1}(1/\lambda) \asymp 1_{\{\gamma_1 \in (0,1)\}} \lambda^{1/(1-\gamma_1)} \log^{p/(1-\gamma_1)}(1 + \lambda) + 1_{\{\gamma_1 = 1\}} \exp \left( c \lambda^{-1/(p+1)} \right),$$

$$\lambda^{-1} b(1/\lambda) \asymp 1_{\{\gamma_1 \in (0,1)\}} \lambda^{-1} \log^p(1 + \lambda) + 1_{\{\gamma_1 = 1\}} \lambda^{-1} \log^{p+1}(1 + \lambda). \quad (4.2)$$

In particular, $tb(t) \asymp H^{-1}(1/t)^{-1}$ for $t \in (0, 2]$ unless $\gamma_1 = 1$.

(i) Suppose that $\gamma_1 \in (0, 1)$. Then, for all $t \in (0, 2]$ and $x \in (0, tb(t)]$, by (4.2),

$$tb(t) \asymp H^{-1}(1/t)^{-1} \asymp t^{1/\gamma_1} \log^{p/\gamma_1}(1 + 1/t),$$

$$\sigma \asymp (t/x)^{1/(1-\gamma_1)} \log^{p/(1-\gamma_1)}(1 + t/x),$$

$$H(\sigma) \asymp \sigma^{\gamma_1} \log^p(1 + \sigma) \asymp (t/x)^{\gamma_1/(1-\gamma_1)} \log^{p/(1-\gamma_1)}(1 + t/x).$$

Hence, by Theorem [1.3] and Corollary [1.7] we have that for all $t \in (0, 2]$,

$$p(t, x) \begin{cases} \asymp t^{-1/\gamma_1} \log^{-p/\gamma_1}(1 + 1/t) \exp \left( -ct \left( \frac{t}{x} \right)^{\frac{1}{\gamma_1}} \log^{\frac{x}{\gamma_1}}(1 + \frac{t}{x}) \right), & \text{if } x \in (0, 2t^{1/\gamma_1} \log^{p/\gamma_1}(1 + 1/t)], \\
\asymp tx^{-\gamma_1} \log^p(1 + 1/x), & \text{if } x \in (2t^{1/\gamma_1} \log^{p/\gamma_1}(1 + 1/t), 1], \\
\asymp tf(x), & \text{if } x \in (1, \infty). \end{cases} \quad (4.3)$$

In the first and second comparison in (4.3), we used the following observation: In this case, for every fixed $a > 0$, by (4.2), it holds that for all $t \in (0, 2]$, $tb(t) \asymp H^{-1}(1/t)^{-1}$ and

$$tH \circ \sigma|_{x=ab(t)} = t(H \circ (\phi')^{-1}(ab(t))) \asymp t(H \circ (\phi')^{-1}(b(t))) = 1.$$ 

Hence, according to Corollary [3.8] and Lemma [3.15] for every fixed $a > 0$, we get $p(t, atb(t)) \asymp t^{-1/\gamma_1} \log^{-p/\gamma_1}(1 + 1/t) \asymp H^{-1}(1/t)$ for all $t \in (0, 2]$. Therefore, we can use $(0, 2t^{1/\gamma_1} \log^{p/\gamma_1}(1 + 1/t)]$ instead of $(0, tb(t) + H^{-1}(1/t)^{-1}]$, and use $(2t^{1/\gamma_1} \log^{p/\gamma_1}(1 + 1/t), 1]$ instead of $(tb(t) + H^{-1}(1/t)^{-1}, 1]$ in (4.3).

(ii) Suppose that $\gamma_1 = 1$ and $p < -1$. Then, for all $t \in (0, 2]$ and $x \in (0, tb(t)]$, by (4.2),

$$tb(t) \asymp t \log^{p+1}(1 + 1/t), \quad H^{-1}(1/t)^{-1} \asymp t \log^p(1 + 1/t), \quad \sigma \asymp \exp \left( c(t/x)^{-1/(p+1)} \right). \quad (4.4)$$

Moreover, for all $t \in (0, 2]$ and $x \in (0, tb(t)]$, we get

$$\exp(-ctH(\sigma)) \asymp \exp(-ct\sigma^c) \asymp \exp \left( -ct \exp \left( c(t/x)^{-1/(p+1)} \right) \right).$$
Thus, by Theorem 1.3 and Corollaries 1.7 and 3.8 we have that for all $t \in (0, 2]$,
\[
 p(t, x) \begin{cases} 
 \simeq t^{-1} \log^p(1 + 1/t) \exp \left(-ct \exp \left(c \left( \frac{t}{x} \right)^{\frac{1}{p+1}} \right) \right), & \text{if } x \in (0, tb(t) + t \log^p(1 + 1/t)], \\
 \asymp t(x - tb(t))^{-2} \log^p(1 + 1/(x - tb(t))), & \text{if } x \in (tb(t) + t \log^p(1 + 1/t), 1], \\
 \asymp tf(x), & \text{if } x \in (1, \infty). 
\end{cases}
\]

In particular, for $t \in (0, 2]$ and $x \in (tb(t), tb(t) + t \log^p(1 + 1/t)]$, by (4.4) and Corollary 5.8
\[
1 \asymp \exp \left(-ct \exp \left(c (t/x)^{-1/(p+1)} \right) \right) \quad \text{and} \quad p(t, x) \asymp t^{-1} \log^p(1 + 1/t) \asymp H^{-1}(1/t)^{-1}.
\]

We note that for $t \in (0, 2]$,
\[
p(t, tb(t)) \asymp t^{-1} \log^p(1 + 1/t) \asymp t^{-1} \log^{p-2}(1 + 1/t) \asymp p(t, 2tb(t)).
\]

4.1.2. Large time estimates. Next, we further assume that $f(s) = s^{-1-\gamma_2} \log^q(1 + s)$ so that
\[
\nu(s) = 1_{\{0 < s \leq 1\}} s^{-1-\gamma_2} \log^p(1 + 1/s) + 1_{\{s > 1\}} s^{-1-\gamma_2} \log^q(1 + s),
\]

for some $\gamma_2 \in (1, \infty)$ and $q \in \mathbb{R}$, and obtain estimates on $p(t, x)$ for $t \in (2, \infty)$. Clearly, (4.1) is satisfied. Since the condition \textbf{(L.Mixed)} holds, by Remark 1.6(2) and Lemma 2.1(4), for every fixed $\lambda_0 > 0$, we get that for all $\lambda \in (0, \lambda_0]$,
\[
\phi(\lambda) \asymp \lambda, \quad \phi'(\lambda) \asymp \phi'(0) \asymp 1,
\]

\[
H(\lambda) \asymp \begin{cases} 
\lambda^{\gamma_2} \log^q(1 + 1/\lambda), & \text{if } \gamma_2 < 2, \\
\lambda^2 \log^{q+1}(1 + 1/\lambda), & \text{if } \gamma_2 = 2, q > -1, \\
\lambda^2 \log(1 + 1/\lambda), & \text{if } \gamma_2 = 2, q = -1, \\
\lambda^2, & \text{if } \gamma_2 = 2 \text{ and } q < -1, \text{ or } \gamma_2 > 2,
\end{cases}
\]

\[
H^{-1}(\lambda) \asymp \begin{cases} 
\lambda^{1/\gamma_2} \log^{-q/\gamma_2}(1 + 1/\lambda), & \text{if } \gamma_2 < 2, \\
\lambda^{1/2} \log^{-(q+1)/2}(1 + 1/\lambda), & \text{if } \gamma_2 = 2, q > -1, \\
\lambda^{1/2} \log^{-1}(1 + 1/\lambda), & \text{if } \gamma_2 = 2, q = -1, \\
\lambda^{1/2}, & \text{if } \gamma_2 = 2 \text{ and } q < -1, \text{ or } \gamma_2 > 2.
\end{cases}
\]

We also have that for all $s \geq \lambda_0$,
\[
\mathcal{H}^{-1}(s) \asymp \begin{cases} 
s \log^{q+1}(1 + s), & \text{if } \gamma_2 = 2, q > -1, \\
s \log(1 + s), & \text{if } \gamma_2 = 2, q = -1, \\
s, & \text{if } \gamma_2 = 2 \text{ and } q < -1, \text{ or } \gamma_2 > 2,
\end{cases}
\]

Note that even though $t$ is large, $\sigma$ can be arbitrary big. Hence, to obtain large time estimates on $p(t, x)$, we still need estimates for $\phi''(\lambda)$ and $H(\lambda)$ for large $\lambda$ to calculate the function in (1.14). To cover the case $\gamma_1 = 0$, we observe that by Lemma 2.1 and 2.1, if $\gamma_1 = 0$ and $\lambda_0 > 0$, we get that for all $\lambda \in [\lambda_0, \infty)$,
\[
\phi'(\lambda) \asymp \lambda^{-1} \log^p(1 + \lambda), \quad (\phi')^{-1}(1/\lambda) \asymp \lambda \log^p(1 + \lambda), \quad |\phi''(\lambda)| \asymp \lambda^{-2} \log^p(1 + \lambda),
\]

\[
w(1/\lambda) \asymp \int_{1/\lambda}^{2/\lambda_0} s^{-1} \log^p(1 + 1/s) ds + 1 \asymp \log^{p+1}(1 + \lambda),
\]

\[
H(\lambda) \asymp \lambda^2 \int_0^{1/\lambda} sw(s) ds \asymp \log^{p+1}(1 + \lambda).
\]
Thus, if \( \gamma_1 = 0 \) and \( p > 0 \), then for all \( t \in [2, \infty) \) and \( x \in (0, t(\phi' \circ H^{-1})(1)] \), since \( \sigma \geq H^{-1}(1) \),

\[
\sigma \asymp \frac{t}{x} \log^p(1 + \frac{t}{x}), \quad |\phi''(\sigma)| \asymp \frac{x^2}{t^2} \log^{-p}(1 + \frac{t}{x}), \quad H(\sigma) \asymp \log^{p+1}(1 + \frac{t}{x}),
\]

and hence

\[
\exp\left( -tH(\sigma) \right) \asymp t^{-1/2} \left( \frac{t}{x} \right) \log^p \left( 1 + \frac{t}{x} \right) \exp\left( -(ct + 1) \log^{p+1} \left( 1 + \frac{t}{x} \right) \right)
\]

\[
\asymp t^{1/(2p+2)} x^{-1} \exp\left( -ct \log^{p+1} \left( 1 + \frac{t}{x} \right) \right) \exp\left( -\log \left( 1 + \frac{t}{x} \right) \right)
\]

\[
\asymp t^{-(2p+1)/(2p+2)} \exp\left( -ct \log^{p+1} \left( 1 + \frac{t}{x} \right) \right).
\]

(4.7)

Define

\[
p_S(t, x, c) :=
\begin{cases}
  t^{-(2p+1)/(2p+2)} \exp\left( -ct \log^{p+1} \left( 1 + \frac{t}{x} \right) \right), & \text{if } \gamma_1 = 0, \\
  t^{-1} \log^{-p} \left( 1 + \frac{t}{x} \right) \exp\left( -ct \exp\left( \left( \frac{t}{x} \right)^\frac{1}{p} \right) \right), & \text{if } \gamma_1 = 1,
\end{cases}
\]

\[
\asymp t^{-1/\gamma_1} \log^{-\gamma_1/\gamma_2} (1 + t) \exp\left( -ct \sigma^{\gamma_2} \log^q \left( 1 + \frac{t}{x} \right) \right), & \text{if } x \in (t(tb(1)), t\phi'(0)),
\]

\[
y^{-1/\gamma_2} \log^q (1 + y), & \text{if } x = t\phi'(0) + y, \ y \in [0, t^{1/\gamma_2} \log^{q/\gamma_2} (1 + t)],
\end{cases}
\]

The function \( p_S(t, x, c) \) with \( \gamma_1 > 0 \) appears in small time left tail estimates on \( p(t, x) \) in Section 4.1.1. We will see that the function \( p_S(t, x, c) \) also appears in large time left tail estimates on \( p(t, x) \).

(i) Suppose that \( \gamma_2 < 2 \). Since both conditions (L.Pure) and (L.Mixed) hold in this case, by (4.5), (4.7), Theorem 1.3 and Corollaries 1.8 and 1.9 it holds that for all \( t \in [2, \infty) \),

\[
p(t, x) \begin{cases}
  \asymp p_S(t, x, c), & \text{if } x \in (0, tb(1)), \\
  \asymp t^{-1/\gamma_2} \log^{-\gamma_2/\gamma_2} (1 + t) \exp\left( -ct \sigma^{\gamma_2} \log^q \left( 1 + \frac{t}{x} \right) \right), & \text{if } x \in (tb(1), t\phi'(0)),
\end{cases}
\]

\[
\times t^{-1/\gamma_2} \log^{-\gamma_2/\gamma_2} (1 + t), & \text{if } x = t\phi'(0) + y, \ y \in [0, t^{1/\gamma_2} \log^{q/\gamma_2} (1 + t)],
\]

\[
\times ty^{-1/\gamma_2} \log^q (1 + y), & \text{if } x = t\phi'(0) + y, \ y \in [t^{1/\gamma_2} \log^{q/\gamma_2} (1 + t), \infty).
\end{cases}
\]

In the second comparison, we used the following observation: by Corollary 3.8, (3.69) and (3.5), we get that for all \( t \in [2, \infty) \) and \( x \in (tb(t), t\phi'(0)) \),

\[
p(t, x) \asymp H^{-1}(1/t) \asymp t^{-1/\gamma_2} \log^{-\gamma_2/\gamma_2} (1 + t) \quad \text{and} \quad t\sigma^{\gamma_2} \log^q \left( 1 + \frac{t}{x} \right) \asymp tH(\sigma) \leq 1. \quad (4.8)
\]

Note that by Lemma 2.4(4) and (4.5), for \( t \in [2, \infty) \) and \( x \in (tb(t), t\phi'(0)) \),

\[
\phi'(0) - \frac{x}{t} = \int_0^\sigma (-\phi''(u))du \asymp \int_0^\sigma u^{\gamma_2-2} \log^q (1 + 1/u)du \asymp \sigma^{\gamma_2-1} \log^q (1 + 1/\sigma)
\]

(4.9)

and hence

\[
\sigma \asymp \left( \phi'(0) - \frac{x}{t} \right)^{1/(\gamma_2-1)} \log^{-\gamma_2/(\gamma_2-1)} \left( 1 + \frac{1}{\phi'(0) - x/t} \right).
\]

(ii) Suppose that \( \gamma_2 = 2, q > -1 \). By (4.5), (4.7), Theorem 1.3 and Corollary 1.9 it holds
that for all $t \in [2, \infty)$,

$$p(t, x) \begin{cases} 
\simeq p_S(t, x, c), & \text{if } x \in (0, t b(1)], \\
\simeq t^{-1/2} \log^{-q(1)/2}(1 + t) \exp \left( - c t \sigma^2 \log^{q+1} (1 + 1/\sigma) \right), & \text{if } x \in (t b(1), t \phi'(0)), \\
\simeq t^{-1/2} \log^{-q(1)/2}(1 + t), & \text{if } x = t \phi'(0) + y, \ y \in [0, t^{1/2} \log^{q(1)/2}(1 + t)), \\
\simeq t^{-1/2} \log^{-q(1)/2}(1 + t) \exp \left( - c y^2 \frac{y}{t \log^{q+1}(1 + t)} \right) + t y^{-3} \log^{q}(1 + y), & \text{if } x = t \phi'(0) + y, \ y \in [t^{1/2} \log^{q(1)/2}(1 + t), \infty). 
\end{cases}$$

We used a similar argument to (4.8) in the second comparison. In the last comparison, we used the facts that the exponential term is dominated by $t \nu$ and hence by (4.6), for all $\gamma > 0$, that

$$\gamma \frac{t x}{c} \exp \left( t \phi(0) + y \right) \leq \gamma t x \log^{q(1)+1}(1 + y),$$

and hence by (4.6), for all $t \in [2, \infty)$ and $y \in [0, D(t))$, (cf. 1 Corollary 6.3),

$$\exp \left( - c \frac{y}{\mathcal{H}^{-1}(t/y)} \right) \simeq \exp \left( - c \frac{y^2}{t \log^{q+1}(1 + t/y)} \right) \simeq \exp \left( - c \frac{y^2}{t \log^{q+1}(1 + t)} \right). \quad (4.10)$$

In particular, we can see that for every fixed $\epsilon > 0$, there are comparison constants such that $p(t, t \phi'(0) + y) \asymp t y^{-3} \log^{q}(1 + y)$ for all $y \geq t^{1/2} \log^{q(1)+1+\epsilon/2}(1 + t)$.

We also note that by a similar calculation to (4.9), for $t \in [2, \infty)$ and $x \in (t b(1), t b(t))$,

$$\sigma \asymp \left( \phi'(0) - \frac{x}{t} \right) \log^{-(q+1)} \left( 1 + \frac{1}{\phi'(0) - x/t} \right).$$

(iii) Suppose that $\gamma_2 = 2$, $q = -1$. By (1.5), (1.7), Theorem 1.4 and Corollary 1.9, it holds that for all $t \in [2, \infty)$,

$$p(t, x) \begin{cases} 
\simeq p_S(t, x, c), & \text{if } x \in (0, t b(1)], \\
\simeq t^{-1/2} \log^{-1}(1 + t) \exp \left( - c t \sigma^2 \log (1 + 1/\sigma) \right), & \text{if } x \in (t b(1), t \phi'(0)), \\
\simeq t^{-1/2} \log^{-1}(1 + t), & \text{if } x = t \phi'(0) + y, \ y \in [0, t^{1/2} \log \log(1 + t)), \\
\simeq t^{-1/2} \log^{-1}(1 + t) \exp \left( - c t \frac{y^2}{\log \log(1 + t)} \right) + t y^{-3} \log^{-1}(1 + y), & \text{if } x = t \phi'(0) + y, \ y \in [t^{1/2} \log \log(1 + t), \infty). 
\end{cases}$$

We used a similar argument to (4.8) in the second comparison. Also, the last comparison holds by a similar argument to the one which is used to obtain (4.10).

In particular, we can see that for every fixed $\epsilon > 0$, there are comparison constants such that $p(t, t \phi'(0) + y) \asymp t y^{-3} \log^{-1}(1 + y)$ for all $y \geq t^{1/2} \log^{\epsilon}(1 + t)$.

We also note that by a similar calculation to (4.9), for $t \in [2, \infty)$ and $x \in (t b(1), t b(t))$,

$$\sigma \asymp \left( \phi'(0) - \frac{x}{t} \right) \log^{-1} \log \left( 1 + \frac{1}{\phi'(0) - x/t} \right).$$
(iv) Suppose that either $\gamma_2 = 2$, $q < -1$ or $\gamma_2 > 2$. By [1.3], [1.7], Theorem [1.4] and Corollary [1.9] it holds that for all $t \in [2, \infty)$,

$$
p(t, x) \begin{cases} 
\asymp p_S(t, x, c), & \text{if } x \in (0, tb(1)], \\
\asymp t^{-1/2} \exp \left(-c t \sigma^2\right), & \text{if } x \in (tb(1), t\sigma'(0)), \\
\asymp t^{-1/2}, & \text{if } x = t\sigma'(0) + y, \ y \in [0, t^{1/2}), \\
\asymp t^{-1/2} \exp \left(-\frac{c y^2}{t}\right) + ty^{-\gamma_2} \log^q(1 + y), & \text{if } x = t\sigma'(0) + y, \ y \in [t^{1/2}, \infty).
\end{cases}
$$

We used a similar argument to (1.8) in the second comparison.

In particular, we can see that for every fixed $\varepsilon > 0$, there are comparison constants such that $p(t, t\sigma'(0) + y) \asymp ty^{-\gamma_2} \log^q(1 + y)$ for all $y \geq t^{1/2} \log^{1/2+\varepsilon}(1 + t)$. Indeed, for all $t \in [2, \infty)$ and $y \geq t^{1/2} \log^{1/2+\varepsilon}(1 + t)$,

$$
t^{-1/2} \exp \left(-\frac{c_1 y^2}{t}\right) \leq t^{-1/2} \exp \left(-\frac{c_1}{2} \log^{1+2\varepsilon}(1 + t)\right) \exp \left(-\frac{c_1 y^2}{2t}\right) \\
\leq c_2 t^{-1/2-\gamma_2} \left(\frac{t}{y^2}\right)^{1+\gamma_2} = c_2 \frac{t}{y^{1/2}} \frac{\log(1 + y)}{y^{1+\gamma_2}} \leq c_2 \frac{t \log(1 + y)}{y^{1+\gamma_2}}.
$$

Note that $\phi''(0) < \infty$ in this case. Hence, we get that for $t \in [2, \infty)$ and $x \in (tb(1), t\sigma(t))$, by the mean value theorem, $\phi''(H^{-1}(1/2)) \leq \sigma^{-1}(\phi'(0) - x/t) \leq \phi''(0)$ and hence

$$
\frac{\phi'(0) - x/t}{\phi''(0)} \leq \sigma \leq \frac{\phi'(0) - x/t}{\phi''(H^{-1}(1/2))}.
$$

4.2. An example of varying transition density estimates. In this subsection, we give an example of subordinator whose transition density has the estimates given in Theorem 1.4 and the exponential term in estimates only appears at specific time ranges.

Define an increasing sequence $(a_n)_{n \geq 0}$ as follows:

$$
a_0 := 0, \quad a_1 := 3, \quad a_{n+1} := \exp(a_{n}^{3/2}) \quad \text{for } n \geq 1.
$$

Using this $(a_n)_{n \geq 0}$, we define a non-decreasing function $\psi : (0, \infty) \to (0, \infty)$ as follows:

$$
\psi(r) = \begin{cases} 
(4/3)r^{1/2} & \text{for } r \in (0, a_1], \\
r^4 + \psi(a_{2n-1}) - a_{2n-1}^4 & \text{for } r \in (a_{2n-1}, a_{2n}], \\
(4/3)r^{1/2} + \psi(a_{2n}) - (4/3)a_{2n}^{1/2} & \text{for } r \in (a_{2n}, a_{2n+1}].
\end{cases}
$$

One can easily check that there exist $c_2 \geq c_1 > 0$ such that

$$
\frac{c_1}{4} \leq \frac{\psi(r)}{\psi(r)} \leq c_2 R^{4/3} \quad \text{for all } 0 < r \leq R.
$$

Let

$$
\Phi(r) := \frac{r^2}{2 \int_0^r s \psi(s)^{-1} ds}.
$$

Then by [1. Lemma 2.4] and [1.12], there exists a constant $c_3 > 0$ such that

$$
\frac{c_3}{4} \leq \frac{\Phi(R)}{\Phi(r)} \leq \left(\frac{R}{r}\right)^2 \quad \text{for all } 0 < r \leq R.
$$
4.2.1. Preliminary calculations.

**Lemma 4.1.** For every \( \varepsilon \in (0, 1) \), there exists \( N \in \mathbb{N} \) such that for every \( n \geq N \), the following estimates hold:

1. For every \( r \in [a_{2n+1}^{1-\varepsilon}, a_{2n+1}] \),
   \[
   \frac{4}{3} r^{1/2} \leq \psi(r) \leq 2 r^{1/2} \quad \text{and} \quad r^{1/2} \leq \Phi(r) \leq 2 r^{1/2}. \tag{4.14}
   \]

2. For every \( r \in [a_{2n}^{1-\varepsilon}, a_{2n}] \),
   \[
   \frac{1}{2} r^{4} \leq \psi(r) \leq r^{4} \quad \text{and} \quad \frac{2(1 - \varepsilon)r^2}{3 \log r} \leq \Phi(r) \leq \frac{2r^2}{\log r}. \tag{4.15}
   \]

**Proof.** From the definition (4.11) of the sequence \((a_n)\), by choosing \( N \) sufficiently large, we can assume that \([a_{2n+1}^{1-\varepsilon}, a_{2n+1}] \subset (a_{2n}, a_{2n+1}] \) and \([a_{2n}^{1-\varepsilon}, a_{2n+1}] \subset (a_{2n}, a_{2n+1}] \) for all \( n \geq N \).

First, we prove the assertions for the function \( \Phi \). Fix \( a \) and \((4.16)\), we can see that for all sufficiently large \( n \),

\[
\epsilon \rightarrow \infty \text{ we can assume that } a \leq a_{2n_{\epsilon}} \text{ for all sufficiently large } n, \]

Moreover, for all sufficiently large \( n \) and \( r \in [a_{2n+1}^{1-\varepsilon}, a_{2n+1}] \), by (4.11),

\[
\psi(r) \leq \left(1 + \frac{a_{2n}^{1-\varepsilon}}{a_{2n+1}^{1-\varepsilon}/2}\right) \frac{4}{3} r^{1/2} \leq \left(1 + a_{2n}^{4 \exp (-2^{-1}(1-\varepsilon)a_{2n}^{3/2})}\right) \frac{4}{3} r^{1/2}.
\]

Similarly, for all sufficiently large \( n \) and \( r \in [a_{2n}^{1-\varepsilon}, a_{2n}] \),

\[
\psi(r) \geq \left(1 - \frac{a_{2n-1}^{1-\varepsilon}}{a_{2n}^{1-\varepsilon}/2}\right) r^{4} \geq \left(1 - a_{2n-1}^{4 \exp (-4(1-\varepsilon)a_{2n-1}^{3/2})}\right) r^{4}.
\]

Since \( \lim_{x \to \infty} x^4 e^{-4(1-\varepsilon)x^{3/2}} = \lim_{x \to \infty} x^4 e^{-2^{-1}(1-\varepsilon)x^{3/2}} = 0 \), we deduce the results for \( \psi \).

Now, we prove the assertions for the function \( \Phi \). Fix \( \varepsilon' \in (0, 1 - \varepsilon) \). By using the results for \( \psi \) and (4.16), we can see that for all sufficiently large \( n \), it holds that for \( r \in [a_{2n+1}^{1-\varepsilon}, a_{2n+1}] \),

\[
\frac{2}{3} r^{3/2} (1 - a_{2n+1}^{3e/2}) \leq \frac{2}{3} r^{3/2} (1 - (a_{2n+1}^{1-\varepsilon}/r)^{3/2}) = \frac{2}{3} (r^{3/2} - a_{2n+1}^{3(1-\varepsilon)/2})
\]

Next, by (4.17), for all sufficiently large \( n \) and \( r \in [a_{2n}^{1-\varepsilon}, a_{2n}] \), we get

\[
\frac{1}{2} a_{2n}^{3/2} \leq 2 \int_0^r \psi(s)^{-1} ds \leq r^{3/2} \quad \text{and hence} \quad r^{1/2} \leq \Phi(r) \leq 2 r^{1/2}. \tag{4.17}
\]

Note that for all sufficiently large \( n \), by (4.16),

\[
\int_{a_{2n-1}}^r \frac{s}{s^4 + \psi(a_{2n-1}) - a_{2n-1}^3} ds \leq \int_{a_{2n-1}}^r \frac{s}{(s - a_{2n-1})a_{2n-1}^3 + \psi(a_{2n-1})} ds
\]

\[
\leq \psi(a_{2n-1})^{-1} \int_{a_{2n-1}}^{a_{2n-1}+1} s ds + \int_{a_{2n-1}}^r s^{-2} ds \leq \frac{3}{4} a_{2n-1}^{-1/2}(a_{2n-1} + 1) + 1 \leq a_{2n-1}^{-1/2}.
\]

\[
\frac{4}{3} r^{1/2} \leq \psi(r) \leq 4 r^{1/2} \quad \text{for all } r \geq 1. \tag{4.16}
\]
Thus, by combining the above inequality with (4.17) and (4.18), we have that for all sufficiently large \( n \) and \( r \in [a_{2n}^{-1-\varepsilon}, a_{2n}], \)

\[
\frac{1}{2} \log r \leq \frac{1}{2} a_{2n-1}^{3/2} \leq 2 \int_0^r s^\psi(s)^{-1} ds \leq (1 + a_{2n-1}^{-1}) a_{n-1}^{3/2} \leq \frac{3}{2} a_{2n-1}^{3/2} \leq \frac{3}{2(1-\varepsilon)} \log r,
\]

and hence

\[
\frac{2(1-\varepsilon)r^2}{3 \log r} \leq \Phi(r) \leq \frac{2r^2}{\log r}.
\]

This completes the proof. \( \square \)

Let \( t_n = a_n^2/(\log a_n) \) and \( t_{n+1} = a_{n+1}^{1/2} \) for \( n \geq 1 \). Since \( \exp(x^{3/2}) \geq 4x^4 \) for \( x \geq 10 \), we can check that \( t_n \geq 4t_{n+1} \) for all \( n \geq 2 \). As a corollary to Lemma 4.1, we obtain the following estimates for the inverse functions of \( \Phi \) and \( \psi \), respectively.

**Lemma 4.2.** There exists \( N \in \mathbb{N} \) such that for every \( n \geq N \), the following estimates hold:

1. For every \( t \in [t_{2n+1}/2, t_{2n+1}] \), it holds that

\[
\Phi^{-1}(t) \asymp \psi^{-1}(t) \asymp t^2.
\]

2. For every \( t \in [t_{2n}/2, t_{2n}] \), it holds that

\[
\Phi^{-1}(t) \asymp t^{1/2}(\log t)^{1/2} \quad \text{and} \quad \psi^{-1}(t) \asymp t^{1/4}.
\]

**Proof.** (1) For all sufficiently large \( n \) and \( t \in [t_{2n+1}/2, t_{2n+1}] \), by (4.12), (4.13) and Lemma 4.1, we have \( \Phi(t^2) \asymp \Phi(a_{2n+1}) \asymp \psi(t^2) \asymp \psi(a_{n+1}) \asymp t_{2n+1} \asymp t \). Then, we get the result from (4.12) and (4.13).

(2) For all sufficiently large \( n \) and \( t \in [t_{2n}/2, t_{2n}] \), we have \( t^{1/2}(\log t)^{1/2} \asymp a_{2n} \) and \( t^{1/4} \asymp a_{2n}^{1/2}(\log a_n)^{-1/4} \). Since for all sufficiently large \( n \), \( \Phi(a_{2n}) \asymp \psi(a_{2n}^{1/2}(\log a_n)^{-1/4}) \asymp t_{2n} \asymp t \) by Lemma 4.1 (2) with \( \varepsilon = 2/3 \), we obtain the results. \( \square \)

4.2.2. **Construction of subordinator and its transition density estimates.** Let \( S \) be a subordinator without drift whose Lévy measure \( \nu(dr) \) is given by

\[
\nu(dr) = \frac{1}{r \psi(r)} dr,
\]

i.e., the Laplace exponent is given by \( \phi(\lambda) = \int_0^\infty (1 - e^{-\lambda r}) \nu(ds) \). Since \( \nu \) satisfies the condition (E) with \( T_0 = 0 \), we see that the subordinator \( S_t \) has a transition density function \( p(t, x) \) for all \( t > 0 \). The following theorem is the main result in this example.

Recall that \( b(t) = (\phi' \circ H^{-1})(1/t) \) for \( t > 0 \).

**Theorem 4.3.** (1) For every \( n \geq 2 \) and \( t \in [(1/2)t_{2n+1}, t_{2n+1}] \), it holds that

\[
p(t, tb(t) + y) \asymp t^{-2} \land \frac{t}{y^2} \psi(y) \quad \text{for all} \quad y \geq 0.
\]

(2) For every \( n \geq 2 \) and \( t \in [(1/2)t_{2n}, t_{2n}] \), it holds that

\[
p(t, tb(t) + y) \asymp t^{-1/2}(\log t)^{-1/2} \land \left( \frac{t}{y^2 \psi(y)} + t^{-1/2}(\log t)^{-1/2} \exp\left(-c \frac{y^2}{t \log t}\right) \right) \quad \text{for all} \quad y \geq 0.
\]
Remark 4.4. Note that for every \( n \geq 2 \), \( t \in [(1/2)t_{2n}, t_{2n}] \) and \( y \in [a_{2n}, a_{2n}(\log a_{2n})^{1/3}] \), since \( \lim_{n \to \infty} a_{2n} = \infty \), we have

\[
t^{-1/2}(\log t)^{-1/2} \exp \left( - c_1 \frac{y^2}{t \log t} \right) \geq c_2 a_{2n}^{-1} \exp \left( - c_3 \frac{y^2}{a_{2n}^2} \right) \geq c_2 a_{2n}^{-1} \exp \left( - c_3 (\log a_{2n})^{-1/3} \log a_{2n} \right) = c_2 a_{2n}^{-1-c_3(\log a_{2n})^{-1/3}} \geq c_4 a_{2n}^{-2},
\]

while

\[
\frac{t}{y^2} \leq c_5 a_{2n}^{-2} \log a_{2n} = c_5 a_{2n}^{-3}(\log a_{2n})^{-1}.
\]

Hence, we see that the exponential term is the dominating factor in heat kernel estimates at those intervals. Therefore, we deduce that although the lower index \( \alpha_1 < 1 \), the exponential term in (1.10) is indispensable in heat kernel estimates. (cf. [1] Theorem 1.5.)

Proof of Theorem 4.3. Since \( S \) satisfies the condition (G), by Lemmas 2.3 and 2.4,

\[
H^{-1} = 2r^{-2} \int_0^r \frac{s}{\psi(s)} ds = \Phi(r)^{-1}, \quad w(r) = rw(r) = \psi(r)^{-1}
\]

for all \( r > 0 \). (4.19)

Hence, by the scaling property of the function \( w \), we get

\[
H^{-1}(1/t) \sim \Phi^{-1}(1) t^{-1} \quad \text{and} \quad w^{-1}(2e/t) \sim \psi^{-1}(t) \quad \text{for all} \quad t > 0.
\]

We simply denote by \( \theta = \theta(t, y/(8e^2)) \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}] \).

(1) By Lemma 4.2(1) and (4.20), there is \( N \in \mathbb{N} \) such that for all \( n \geq N \),

\[
H^{-1}(1/t) \sim w^{-1}(2e/t) \quad \text{for all} \quad t \in [(1/2)t_{2n+1}, t_{2n+1}].
\]

Since \( H^{-1}(1/t)^{-1} \sim w^{-1}(2e/t)^{-1} \times 1 \) for \( 2 \leq n < N \) and \( t \in [(1/2)t_{2n+1}, t_{2n+1}] \), after taking comparison constants larger, we can see that (4.21) holds for all \( n \geq 2 \). It follows that \( \theta(t, y) \sim t^2 \) for all \( t \in [(1/2)t_{2n+1}, t_{2n+1}] \) and \( y \geq 0 \). Hence, by Lemma 5.13 for fixed \( a > 0 \) and all \( t \in [(1/2)t_{2n+1}, t_{2n+1}] \),

\[
\exp \left( - \frac{ay}{\theta} \right) \leq 1 \quad \text{for} \quad y \in [0, H^{-1}(1/t)^{-1}]
\]

and

\[
\exp \left( - \frac{ay}{w^{-1}(2e/t)} \right) \leq c_1 \frac{t v(y)}{H^{-1}(1/t)} \quad \text{for} \quad y \in (H^{-1}(1/t)^{-1}, \infty).
\]

Therefore, we get the result from Corollary 1.5.

(2) By Lemma 4.2(2), (4.20) and using the same argument as the one given in the proof of (1), we get that for all \( n \geq 2 \),

\[
H^{-1}(1/t)^{-1} \sim t^{1/2}(\log t)^{1/2} \quad \text{and} \quad w^{-1}(2e/t) \sim t^{1/4} \quad \text{for all} \quad t \in [(1/2)t_{2n}, t_{2n}].
\]

Then, by (4.19), (4.22) and Lemma 4.1(2) with \( \varepsilon = 4/5 \), we have that for all \( t \in [(1/2)t_{2n}, t_{2n}] \),

\[
D(t) = \max_{s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]} \frac{\Phi(s)}{s} \geq t \max_{s \in [w^{-1}(2e/t), H^{-1}(1/t)^{-1}]} \frac{\log s}{s} \sim t^{3/4} \log t.
\]

From (4.10), we see that for all \( t \in [(1/2)t_{2n}, t_{2n}] \) and \( y \geq D(t) \geq ct^{3/4} \log t \),

\[
t^{-1/2}(\log t)^{-1/2} \exp \left( - c_1 \frac{y^2}{t \log t} \right) \leq ct^{-1/2}(\log t)^{-1/2} \left( \frac{t \log t}{y^2} \right)^{11/2} = c t \frac{t^4}{y^2} \frac{t^4}{y^6} \leq c \frac{t}{y^4} \phi(y).
\]

Hence, by (4.22), Corollaries 1.5 and 3.3 and Lemma 5.13 it suffices to show that

\[
\frac{y}{\theta} \leq \frac{y^2}{t \log t} \quad \text{for all} \quad t \in [(1/2)t_{2n}, t_{2n}], \quad y \in [H^{-1}(1/t)^{-1}, D(t)].
\]
Since $\theta \in [w^{-1}(2\epsilon/t), H^{-1}(1/t^{-1})]$, by (4.22), there are constants $c_1, c_2 > 0$ such that $c_1 t^{1/4} \leq \theta \leq c_2 t^{1/2}(\log t)^{1/2}$. Since $t\theta H(\theta^{-1}) = y$, by (4.19) and Lemma 4.1 with $\epsilon = 4/5$, (as before, it suffices to consider large $t$ only,)

$$y\theta = t\theta^2 H(\theta^{-1}) \asymp t^2 \Phi(\theta) \asymp t \log \theta \asymp t \log t.$$ 

This proves (4.23).

5. Relationship between subordinator and symmetric jump processes

In this short section, we discuss a resemblance between transition density estimates on subordinators and heat kernel estimates on symmetric jump processes.

Let $S$ be a subordinator without drift whose Lévy density is $\nu(r) = 1/(r\psi(r))$. We assume that $\psi$ is non-decreasing and that $S$ satisfies the condition (G). Define

$$\Phi(r) = \frac{r^2}{2 \int_0^1 s\psi(s)^{-1} ds}.$$ 

Then, by Lemmas 2.3 and 2.4, we have

$$H(r) \asymp \Phi(r^{-1})^{-1} \quad \text{and} \quad w(r) \asymp \psi(r)^{-1} \quad \text{for all } r > 0. \quad (5.1)$$

Following [1], (1.16)], we define

$$\mathcal{K}_\infty(r) := \begin{cases} \sup_{1 \leq s \leq r} s^{-1} \Phi(s) & \text{if } r \geq 1, \\ \Phi(r) & \text{if } 0 < r < 1. \end{cases}$$

If we further assume that (L:Mixed) holds, that is $\alpha_3 > 1$, then we can see from (5.1) that

$$\mathcal{K}_\infty(r) \asymp \mathcal{K}(r) = \inf_{s \geq r} \frac{1}{sH(s^{-1})} \quad \text{for all } r \geq 1. \quad (5.2)$$

Let $X = (X_t, x \in \mathbb{R}, t \geq 0)$ be a pure jump symmetric Markov process on $\mathbb{R}$ whose jumping kernel $J(x,y)$ satisfies

$$J(x,y) \asymp \frac{1}{|x-y|\psi(|x-y|)} , \quad x,y \in \mathbb{R},$$

that is, its associated Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(\mathbb{R})$ is given by

$$\mathcal{E}(f,g) = \int_{\mathbb{R} \times \mathbb{R} \setminus \text{diag}} (f(x) - f(y))(g(x) - g(y))J(x,y)dxdy, \quad f,g \in \mathcal{F},$$

$$\mathcal{F} = \{ f \in L^2(\mathbb{R}) : \mathcal{E}(f,f) < \infty \}.$$ 

According to [1], Theorem 1.5(2)], under the conditions (G) and (L:Mixed), the process $X$ admits a transition density $p^X(t,x,y)$ enjoying the following estimates: for all $(t,x,y) \in [1,\infty) \times \mathbb{R} \times \mathbb{R},$

$$p^X(t,x,y) \simeq \Phi^{-1}(t)^{-1} \wedge \left( \frac{t}{|x-y|\psi(|x-y|)} + \Phi^{-1}(t)^{-1} \exp \left( -c \frac{|x-y|}{\mathcal{K}_\infty^{-1}(t/|x-y|)} \right) \right).$$

Hence, in view of (5.1), (5.2) and Corollary 1.3, we see that if the Lévy density for a subordinator and a symmetric jump process are decaying in the same order and the conditions (G) and (L:Mixed) hold, then right tail estimates on the transition density for the subordinator and off-diagonal estimates on the one for the symmetric jump process on $\mathbb{R}$ are the same.
References

[1] J. Bae, J. Kang, P. Kim and J. Lee, Heat kernel estimates for symmetric jump processes with mixed polynomial growths. *Ann. Probab.* 47 (2019), no. 5, 2830–2868.

[2] M. T. Barlow, R. F. Bass, Z.-Q. Chen and M. Kassmann, Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.* 361 (2009), no. 4, 1963–1999.

[3] M. T. Barlow, A. Grigor’yan and T. Kumagai, Heat kernel upper bounds for jump processes and the first exit time. *J. Reine Angew. Math.* 626 (2009), 135–157.

[4] N. H. Bingham, C. M. Goldie and J. L. Teugels, Regular variation. Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, 1987.

[5] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song and Z. Vondraček, Potential analysis of stable processes and its extensions. Edited by Piotr Graczyk and Andrzej Stos. Lecture Notes in Mathematics, 2009. Springer-Verlag, Berlin, 2009.

[6] K. Bogdan, T. Grzywny and M. Ryznar, Density and tails of unimodal convolution semigroups. *J. Funct. Anal.* 266 (2014), no. 6, 3543–3571.

[7] K. Bogdan, A. Stós and P. Sztonyk, Harnack inequality for stable processes on *d*-sets, *Studia Math.* 158 (2003), no. 2, 163–198.

[8] J. Burridge, A. Kuznetsov, M. Kwaśnicki and A. E. Kyprianou, New families of subordinators with explicit transition probability semigroup. *Stochastic Process. Appl.* 124 (2014), no. 10, 3480–3495.

[9] Z.-Q. Chen, Symmetric jump processes and their heat kernel estimates. *Science in China Series A: Mathematics* 52 (2009), no. 7, 1423–1445.

[10] Z.-Q. Chen, E. Hu, L. Xie and X. Zhang, Heat kernels for non-symmetric diffusion operators with jump. *J. Differential Equations* 263 (2017), 6576–6634.

[11] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on *d*-sets. *Stochastics Process. Appl.* 108 (2003), no. 1, 27–62.

[12] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for jump processes of mixed types on metric measure spaces. *Probab. Theory Related Fields* 140 (2008), no. 1, 277–317.

[13] Z.-Q. Chen, P. Kim and T. Kumagai, On heat kernel estimates and parabolic Harnack inequality for jump processes on metric measure spaces. *Acta Mathematica Sinica, English Series* 25 (2009), no. 7, 1067–1086.

[14] Z.-Q. Chen, P. Kim and T. Kumagai, Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.* 363 (2011), no. 9, 5021–5055.

[15] Z.-Q. Chen, P. Kim, T. Kumagai and J. Wang, Time Fractional Poisson Equations: Representations and Estimates *J. Funct. Anal.* 278 (2020), Article 108311.

[16] Z.-Q. Chen, T. Kumagai and J. Wang, Stability of heat kernel estimates for symmetric non-local Dirichlet forms. To appear in *Mem. Amer. Math. Soc.*

[17] Z.-Q. Chen and X. Zhang, Heat kernels and analyticity of non-symmetric jump diffusion semigroups. *Probab. Theory Related Fields* 165 (2016), no. 1-2, 267–312.

[18] Z.-Q. Chen and X. Zhang, Heat kernels for time-dependent non-symmetric stable-like operators. *J. Math. Anal. Appl.* 465 (2018), no. 1, 1–21.

[19] Z.-Q. Chen and X. Zhang, Heat kernels for non-symmetric non-local operators. In *Recent developments in nonlocal theory*, De Gruyter, Berlin, (2018), 24–51.

[20] S. Cho and P. Kim, Estimates on the tail probabilities of subordinators and applications to general time fractional equations. To appear in *Stochastic Process. Appl.*, arXiv:1905.00341.

[21] R. A. Doney and V. M. Rivero, Asymptotic behaviour of first passage time distributions for subordinators. *Electron. J. Probab.* 20 (2015), no. 91, 28.

[22] M. A. Fahrenwaldt, Heat kernel asymptotics of the subordinator and subordinate Brownian motion. *J. Evol. Equ.* 19 (2019), no. 1, 33–70.

[23] M. Fukushima, Y. Oshima and M. Takeda, Dirichlet forms and symmetric Markov processes(2nd ed.). de Gruyter, Berlin, 2010.

[24] A. Grigor’yan, E. Hu and J. Hu, Lower estimates of heat kernels for non-local Dirichlet forms on metric measure spaces. *J. Funct. Anal.* 272 (2017), no. 8, 3311–3346.

[25] A. Grigor’yan, E. Hu and J. Hu, Two-sided estimates of heat kernels of jump type Dirichlet forms. *Advances in Math.* 330 (2018), 433-515.

[26] A. Grigor’yan, J. Hu and K.-S. Lau, Estimates of heat kernels for non-local regular Dirichlet forms. *Trans. Amer. Math. Soc.* 366 (2014), no. 12, 6397–6441.

[27] T. Grzywny, L. Łeżyj and B. Trojan, Transition densities of subordinators. arXiv:1812.06793.

[28] T. Grzywny and K. Szczypkowski, Estimates of heat kernels of non-symmetric Lévy processes. arxiv:1710.07793.
[29] T. Grzywny and K. Szczypkowski, Lévy processes: concentration function and heat kernel bounds. arXiv:1907.00778.

[30] R. Gorenflo, A. A. Kilbas, F. Mainardi and S. V. Rogosin, Mittag-Leffler Functions, Related Topics and Applications. Springer Monographs in Mathematics. Springer, Heidelberg, 2014.

[31] P. Hartman and A. Wintner, On the infinitesimal generators of integral convolutions. *Amer. J. Math.* 64 (1942), 273–298.

[32] J. Hu and X. Li. The Davies method revisited for heat kernel upper bounds of regular Dirichlet forms on metric measure spaces. *Forum Math.* 30 (2018), no. 5, 1129–1155.

[33] N. C. Jain and W. E. Pruitt, Lower tail probability estimates for subordinators and nondecreasing random walks. *Ann. Probab.* 15 (1987), no. 1, 75–101.

[34] K. Kaleta and P. Sztonyk, Estimates of transition densities and their derivatives for jump Lévy processes. *J. Math. Anal. Appl.* 431 (2015), no. 1, 260–282.

[35] P. Kim and J. Lee, Heat kernels of non-symmetric jump processes with exponentially decaying jumping kernel. *Stochastic Process. Appl.* 129 (2019), no. 6, 2130–2173.

[36] P. Kim, R. Song and Z. Vondraček, Heat kernels of non-symmetric jump processes: beyond the stable case. *Potential Anal.* 49 (2018), no. 1, 37–90.

[37] V. Knopova and R. L. Schilling, A note on the existence of transition probability densities of Lévy processes. *Forum Math.* 25 (2013) no. 1, 125–149.

[38] T. Kulczycki and M. Ryznar, Transition density estimates for diagonal systems of SDEs driven by cylindrical α-stable processes. *ALEA Lat. Am. J. Probab. Math. Stat.* 15 (2018), no. 2, 1335–1375.

[39] A. Mimica, Heat kernel estimates for subordinate Brownian motions. *Proc. Lond. Math. Soc.* (3) 113 (2016), no. 5, 627–648.

[40] J. Picard, Density in small time for Lévy processes. *ESAIM Probab. Statist.* 1 (1997) 357–389.

[41] R. N. Pillai, On Mittag-Leffler functions and related distributions. *Ann. Inst. Statist. Math.* 42 (1990), no. 1, 157–161.

[42] K.-I. Sato, Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, 1999.

[43] J. L. Schiff, The Laplace Transform: Theory and Applications. Springer Science & Business Media, New York, 2013.

[44] R. L. Schilling, R. Song, and Z. Vondraček, Bernstein functions, second ed., de Gruyter Studies in Mathematics, vol. 37, Walter de Gruyter & Co., Berlin, 2012.

[45] R. Song, Sharp bounds on the density, Green function and jumping function of subordinate killed BM. *Probab. Theory Relat. Fields* 128 (2004) 606–628.

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