GLOBAL AND REGIONAL CONSTRAINED CONTROLLABILITY FOR DISTRIBUTED PARABOLIC LINEAR SYSTEMS: RHUM APPROACH

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Abstract. The aim of this paper is to study the problem of constrained controllability for distributed parabolic linear system evolving in spatial domain \( \Omega \) using the Reverse Hilbert Uniqueness Method (RHUM approach) introduced by Lions in 1988. It consists in finding the control \( u \) that steers the system from an initial state \( y_0 \) to a state between two prescribed functions. We give some definitions and properties concerning this concept and then we resolve the problem that relays on computing a control with minimum cost in the case of \( \omega = \Omega \) and in the regional case where \( \omega \) is a part of \( \Omega \).

1. Introduction. The concept of controllability is one of the most important concepts in the analysis of distributed systems. This notion can be done in an abstract way by considering various types of functional spaces and operators to introduce some definitions and establish various characterization and properties. The problem of controllability has been widely developed in several articles [12, 21, 9, 1]. The definition of this concept is equivalent to being able to drive a system exactly from an initial state to a desired one in the state space. This type of controllability (exact controllability) is a very strong property in infinite dimensions, and most systems can achieve this only in an approximate way (approximate controllability). Controllability problems for integer order systems have been widely studied and many techniques have been developed for solving such problems [21, 9, 22, 5].

The mathematical model of a real problem is obtained from measurements or approximation techniques and is often affected by perturbations [26], and the solution of such a system is approximately known. It is for these reasons that we

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are interested here in the introduction of the concept of controllability with constraints. Problems with hard constraints on the state or control have attracted several authors in the last three decades, mostly for their importance in various applications in optimal control. Indeed, it is well known that by proving the existence of a Lagrange multiplier associated with the constraints in the state, we can derive optimality conditions \[6, 19\]. For instance, Barbu and Precupanu \[2\] and Laseicka \[20\], derived the existence of a Lagrange multiplier for some optimal control problems with integral state constraints. Bergounioux \[5\] used a penalization method to prove existence of a multiplier and to derive optimality conditions for elliptic equations with state constraints. Mordukhovich \[27, 28, 29\] also was concerned about the parabolic control system under hard/pointwise constraints on control and state variables; he designed a feedback control regulator that ensures the required state performance and robust stability under any feasible perturbations and minimize an energy type functional under the worst perturbations from the given area.

Here, we solve the problem of enlarged controllability, also so-called controllability with constraints on the state, using the Reverse Hilbert Uniqueness Method. The idea was developed in 1988 by Lions for the case of the linear wave equation using a penalty method \[23\], E. Zerrik et al. extended this notion to the case of linear parabolic systems \[30, 31\] using the sub-differential analysis and Lagrangian multipliers approach. Karite et al. studied the controllability under constraints for parabolic semilinear systems \[13, 14, 15, 16\] using the same techniques but in a subregion \( \omega \) of the evolution domain \( \Omega \).

This method is closely related to the duality between controllability and observability. This duality is classical in finite dimension \[7\]. For infinite-dimensional control systems, this duality has been proved by Szymon Dolecki and David Russell in \[10\]. For more details, see \[8, 3, 4\]. A detailed description of the RHUM approach is given by Lions in \[23\]. Both techniques used by Lions, RHUM and the penalization method were the subject of these papers \[17, 18\] in the case of fractional systems.

The remaining of the paper is organized as follows: In section 2, we introduce the notion of constrained controllability of parabolic systems in the global case, we give some definitions and properties related to this notion and we compute the control steering our system from an initial state to a desired one between two prescribed functions in the whole domain \( \Omega \). In section 3, we solve the problem of minimum energy control using RHUM approach in a subregion \( \omega \) of the evolution domain \( \Omega \). We end with Section 4 of conclusions and some interesting open questions that deserve further investigations.

2. Global case \( \omega = \Omega \). In this section, we compute the control steering the studied system from an initial state to a final one assumed unknown between two prescribed functions in the whole geometric domain \( \Omega \).

2.1. Problem setting. Let \( \Omega \) be an open bounded of \( \mathbb{R}^n \) \((n = 1, 2, 3)\), with a regular boundary \( \partial \Omega \). For \( T > 0 \) we denote \( Q = \Omega \times [0, T] \), where \( 0 < T < +\infty \) and \( \Sigma = \partial \Omega \times [0, T] \) its lateral. We consider the following parabolic linear system:

\[
\begin{aligned}
\frac{\partial y(x, t)}{\partial t} &= Ay(x, t) + Bu(t) & Q \\
y(\xi, t) &= 0 & \Sigma \\
y(x, 0) &= y_0(x) & \Omega,
\end{aligned}
\] (1)
where $\mathcal{A}$ is a second order differential linear operator which generates a strongly continuous semi-group $(S(t))_{t \geq 0}$ on Hilbert space $L^2(\Omega)$. The control operator $B$ is linear and defined from $\mathbb{R}^p$ to the state space $L^2(\Omega)$. The initial datum $y_0$ is in $L^2(\Omega)$ and the control $u \in U = L^2(0, T; \mathbb{R}^p)$.

Let $y_u$ be the mild solution of (1) excited by the control $u$. We have $y_u \in L^2(0, T; L^2(\Omega))$ (see [24]). And it is written as follows:

$$y_u(t) = S(t)y_0 + \int_0^t S(t - \tau)Bu(\tau)d\tau.$$  

Let $L_i$ be the operator from $U \rightarrow L^2(\Omega)$ defined by:

$$L_iu = \int_0^T S(T - s)Bu(s)ds \quad \forall u \in U.$$  

Let $\alpha(\cdot)$ and $\beta(\cdot)$ be two functions in $L^2(\Omega)$ such that $\alpha(\cdot) \leq \beta(\cdot)$ a.e. in $\Omega$. Throughout the paper we set:

$$[\alpha(\cdot), \beta(\cdot)] = \{y(\cdot) \in L^2(\Omega) \mid \alpha(\cdot) \leq y(\cdot) \leq \beta(\cdot) \text{ a.e. in } \Omega\}.$$  

**Definition 2.1.** We say that system (1) is $[\alpha(\cdot), \beta(\cdot)]$-controllable (or globally $[\alpha(\cdot), \beta(\cdot)]$-controllable) if

$$\alpha(\cdot) \leq y_u(T) \leq \beta(\cdot).$$

**Theorem 2.2.** System (1) is $[\alpha(\cdot), \beta(\cdot)]$-controllable in $\Omega$ if

$$([\alpha(\cdot), \beta(\cdot)] - \{S(T)y_0\}) \cap \text{Im}(L_i) \neq \emptyset$$

**Proof.** We suppose that system (1) is globally $[\alpha(\cdot), \beta(\cdot)]$-controllable, which is equivalent to say that there exist a control $u$ such that $y_u(T) \in [\alpha(\cdot), \beta(\cdot)]$. We have $y_u(T) = S(T)y_0 + L_iu$.

Let’s denote by $z = y_u(T) - S(T)y_0 = L_iu$, which leads to $z \in \text{Im}(L_i)$ and $z \in [\alpha(\cdot), \beta(\cdot)] - \{S(T)y_0\}$. And we prove that $([\alpha(\cdot), \beta(\cdot)] - \{S(T)y_0\}) \cap \text{Im}(L_i) \neq \emptyset$.

Conversely, we suppose that

$$([\alpha(\cdot), \beta(\cdot)] - \{S(T)y_0\}) \cap \text{Im}(L_i) \neq \emptyset$$

then, there exists $z \in [\alpha(\cdot), \beta(\cdot)] - \{S(T)y_0\}$ such that $z \in \text{Im}(L_i)$. Hence $\exists u \in U$ such that $z = L_iu$, which proves that $z = S(T)y_0 + L_iu \in [\alpha(\cdot), \beta(\cdot)]$. Thus we prove the global $[\alpha(\cdot), \beta(\cdot)]$-controllability.  

\[\square\]

2.2. RHUM approach. In this section we adjust the Hilbert Uniqueness Method introduced in the linear case by Lions [25] and El Jai et al. [12] which allows the characterization of a control $u^*$ that steers system (1) from an initial state to a final one between two prescribed functions.

Let $G_{[\alpha(\cdot), \beta(\cdot)]}$ be the smallest closed sub-space of $X = L^2(\Omega)$ which contains all the functions in $[\alpha(\cdot), \beta(\cdot)]$. The problem of constrained controllability can be presented as follows:

**Problem:**

Is there a control $u \in U$ solution of the following problem:

$$\begin{align*}
\inf_{u \in U} & \|u\|_2 \\
y_u(T) & \in G_{[\alpha(\cdot), \beta(\cdot)]}.
\end{align*}$$  

(3)
Let $U_{ad} = \{ u \in U, \text{ such that } y_u(T) \in G_{[\alpha(\cdot), \beta(\cdot)]} \}$. So problem (3) becomes:

$$\inf_{u \in U_{ad}} \|u\|^2,$$

(4)

$U_{ad}$ is a closed and convex set, so if there is a solution, it is unique.

For $\varphi_0 \in H$ ($H$ will be defined later), we consider the backward system of (1):

$$\begin{cases}
\frac{\partial \varphi(x,t)}{\partial t} = -A^* \varphi(x,t) & Q \\
\varphi(\xi, T) = 0 & \Sigma \\
\varphi(x, T) = \varphi_0(x) & \Omega,
\end{cases}$$

(5)

admits a unique solution $\varphi(.) \in C^1(0, T; L^2(\Omega))$ given by:

$$\varphi(T) = S^*(T-t)\varphi_0.$$

Let the mapping:

$$\varphi_0 \mapsto \|\varphi_0\|_H = \left[ \int_0^T \|B^* \varphi(t)\|^2 dt \right]^{1/2}$$

(6)

defines a semi-norm on $H$.

Let $F_{[\alpha(\cdot), \beta(\cdot)]}$ be the complete of $G_{[\alpha(\cdot), \beta(\cdot)]}$ by (6).

$$(\varphi_1, \varphi_2) = \int_0^T \langle B^* S^*(T-t)\varphi_1, B^* S^*(T-t)\varphi_2 \rangle dt \quad \forall \varphi_1, \varphi_2 \in H.$$  

(7)

For $\varphi_0 \in H$, We denote $H = F'_{[\alpha(\cdot), \beta(\cdot)]}$ ($F'_{[\alpha(\cdot), \beta(\cdot)]}$ is the dual of $F_{[\alpha(\cdot), \beta(\cdot)]}$).

$$\|\varphi_0\|_{F'_{[\alpha(\cdot), \beta(\cdot)]}}^2 = \int_0^T \|B^* S^*(T-t)\varphi_0\|^2 dt$$

$$\leq \left( \|B^*\|^2 \int_0^T \|S^*(T-t)\|^2 dt \right) \|\varphi_0\|^2_X.$$  

So

$$\exists k > 0 \text{ such that } \|\varphi_0\|_{F'_{[\alpha(\cdot), \beta(\cdot)]}} \leq k \|\varphi_0\|_X \quad \forall \varphi_0 \in H$$

(8)

Then we deduce that

$$\begin{cases}
X \subset F'_{[\alpha(\cdot), \beta(\cdot)]} \text{ with continuous injection,} \\
X \text{ dense in } F'_{[\alpha(\cdot), \beta(\cdot)]}.
\end{cases}$$

For $\varphi_0 \in F'_{[\alpha(\cdot), \beta(\cdot)]}$ and $\varphi$ solution of (5) we define $\Psi(\cdot) \in C^1(0, T; L^2(\Omega))$ by:

$$\begin{cases}
\frac{\partial \Psi(x,t)}{\partial t} = A\Psi(x,t) + BB^* \varphi(t) & Q \\
\Psi(\xi, t) = 0 & \Sigma \\
\Psi(x, 0) = 0 & \Omega,
\end{cases}$$

(9)

We denote $\Pi \varphi_0 = \Psi(T)$. The control $u(t) = B^* \varphi(t) \in L^2(0, T)$, the solution $\Psi(T)$ verifying $\Psi(T) \in L^2(\Omega)$. Which defines the operator $\Pi : F_{[\alpha(\cdot), \beta(\cdot)]} \rightarrow F'_{[\alpha(\cdot), \beta(\cdot)]}$.
we have:

$$
\Psi(T) = \int_0^T S(T-t)BB^*\varphi(t)dt
= \int_0^T S(T-t)BB^*S^*(T-t)\varphi_0 dt, \quad \varphi_0 \in F'_{[\alpha(\cdot),\beta(\cdot)]}.
$$

For $\varphi_1, \varphi_2 \in F'_{[\alpha(\cdot),\beta(\cdot)]}$, we have:

$$
\langle \Pi \varphi_1, \varphi_2 \rangle = \int_0^T \langle S(T-t)BB^*S^*(T-t)\varphi_1, \varphi_2 \rangle dt
= \int_0^T \langle B^*S^*(T-t)\varphi_1, B^*S^*(T-t)\varphi_2 \rangle dt
= \langle \varphi_1, \varphi_2 \rangle_1.
$$

Then we have:

$$
\langle \Pi \varphi_1, \varphi_2 \rangle = (\varphi_1, \varphi_2)_1 \quad \forall \varphi_1, \varphi_2 \in F'_{[\alpha(\cdot),\beta(\cdot)]}. \quad (10)
$$

For $\varphi_1 = \varphi_2 = \varphi_0$

$$
\langle \Pi \varphi_0, \varphi_0 \rangle = \|\varphi_0\|^2_{F'_{[\alpha(\cdot),\beta(\cdot)]}} \quad \forall \varphi_0 \in F'_{[\alpha(\cdot),\beta(\cdot)]}. \quad (11)
$$

Thus, we have the following main result:

**Theorem 2.3.** We suppose that (6) is a norm on $F'_{[\alpha(\cdot),\beta(\cdot)]}$. Then $\Pi$ is extended by continuity on $F'_{[\alpha(\cdot),\beta(\cdot)]}$, to an isomorphism from $F_{[\alpha(\cdot),\beta(\cdot)]}$ to $F'_{[\alpha(\cdot),\beta(\cdot)]}$ denoted also by $\Pi$ and verifying:

$$
\langle \Pi \varphi_1, \varphi_2 \rangle = (\varphi_1, \varphi_2)_1 \quad \forall \varphi_1, \varphi_2 \in F'_{[\alpha(\cdot),\beta(\cdot)]}. \quad (12)
$$

Furthermore, the control

$$
u^*(t) = B^*\varphi(t)
$$

ensures the controllability into $G_{[\alpha(\cdot),\beta(\cdot)]}$.

**Proof.** Let $\varphi \in G_{[\alpha(\cdot),\beta(\cdot)]}$, $\Pi \varphi$ is identified to the linear form:

$$
\Pi \varphi : g \in G_{[\alpha(\cdot),\beta(\cdot)]} \to \langle \Pi \varphi, g \rangle.
$$

According to (7), we have:

$$
|\langle \Pi \varphi, g \rangle| = |\langle \varphi, g \rangle| \leq \|\varphi\|_{F_{[\alpha(\cdot),\beta(\cdot)]}} \|g\|_{F'_{[\alpha(\cdot),\beta(\cdot)]}} \quad \forall g \in G_{[\alpha(\cdot),\beta(\cdot)]}.
$$

So $\Pi \varphi$ is continuous on $G_{[\alpha(\cdot),\beta(\cdot)]}$ and $\|\Pi \varphi\| = \|\varphi\|_{F_{[\alpha(\cdot),\beta(\cdot)]}}$. Then we can extend $\Pi \varphi$ by continuity on $F'_{[\alpha(\cdot),\beta(\cdot)]}$ which gives $\Pi \varphi \in F'_{[\alpha(\cdot),\beta(\cdot)]}$.

In the other hand, the operator $\Pi$ could be identified to the application:

$$
\varphi \in G_{[\alpha(\cdot),\beta(\cdot)]} \to \Pi \varphi \in F'_{[\alpha(\cdot),\beta(\cdot)]}
$$

This mapping is linear and continuous on $(G_{[\alpha(\cdot),\beta(\cdot)]}, \|\cdot\|_{F_{[\alpha(\cdot),\beta(\cdot)]}})$ and

$$
\|\Pi \varphi\|_{F'_{[\alpha(\cdot),\beta(\cdot)]}} = \|\varphi\|_{F_{[\alpha(\cdot),\beta(\cdot)]}}, \quad \forall \varphi \in G_{[\alpha(\cdot),\beta(\cdot)]}
$$

So we can extend $\Pi$ by continuity on $F_{[\alpha(\cdot),\beta(\cdot)]}$, and the relation (12) derived from (11) and it shows that $\Pi$ is an isomorphism from $F_{[\alpha(\cdot),\beta(\cdot)]}$ to $F'_{[\alpha(\cdot),\beta(\cdot)]}$.

In order to prove that the semi-norm defined in (6) is a norm on $F'_{[\alpha(\cdot),\beta(\cdot)]}$ we have the following theorem:
We prove that (13) is a norm.

If \( G_{(\alpha),\beta} \) is dense in \( L^2(\Omega) \) then (6) is a norm on \( F_{(\alpha),\beta}^\prime \).

Proof. In this case, we consider that system (1) is excited by an internal zone actuator \((D,f)\) with \( D \in \Omega \) and \( f \in L^2(\Omega) \). So we have \( Bu(t) = \chi_\alpha f(x)u(t) \). Then the semi-norm (6) will be rewritten as follows:

\[
\varphi_0 \in G_{(\alpha),\beta} \rightarrow \|\varphi_0\|^2_{G_{(\alpha),\beta}} = \int_0^T \langle f, \varphi(t) \rangle^2_{L^2(D)} dt \tag{13}
\]

We prove that (13) is a norm.

For that we suppose that \( G_{(\alpha),\beta} \) is dense in \( L^2(\Omega) \). Furthermore, if \( \varphi_0 \in G_{(\alpha),\beta} \) verify \( \|\varphi_0\|_{G_{(\alpha),\beta}} = 0 \), then \( \langle f, \varphi(t) \rangle_{L^2(D)} = 0 \) a.e in \([0,T]\).

For \( u \in U \), we apply green formula for \( y(u) \) and \( \varphi \)

\[
\int_0^T \int_\Omega \chi_\alpha f(x)u(t)\varphi(x,t)dxdt = \langle y(T), \varphi_0 \rangle
\]

Hence,

\[
\langle y(T), \varphi_0 \rangle = \int_0^T u(t)\langle f, \varphi(t) \rangle_{L^2(D)} dt = 0, \quad \forall u \in U
\]

Thus \( \varphi \in G_{(\alpha),\beta}^\perp \) and \( \varphi = 0 \). \( \square \)

3. **Regional case** \( \omega \subset \Omega \). We consider a given sub-region \( \omega \), a subset of \( \Omega \), for which we study the controllability of system (1) instead of the whole domain \( \Omega \) [12, 11].

3.1. **Problem setting.** We define the restriction operator and its adjoint by the following expressions:

\[
\chi_\omega : L^2(\Omega) \rightarrow L^2(\omega) \quad y \rightarrow \chi_\omega y = y|_\omega. \tag{14}
\]

Its adjoint \( \chi_\omega^\ast \) is given by:

\[
(\chi_\omega^\ast y)(x) = \begin{cases} 
  y(x) & x \in \omega \\
  0 & x \in \Omega \setminus \omega.
\end{cases} \tag{15}
\]

Let \( \overline{\alpha}(\cdot) \) and \( \overline{\beta}(\cdot) \) be two functions in \( L^2(\omega) \) such that \( \overline{\alpha}(\cdot) \leq \overline{\beta}(\cdot) \) a.e. in \( \omega \).

Throughout the paper we set:

\[
[\overline{\alpha}(\cdot),\overline{\beta}(\cdot)] = \left\{ y(\cdot) \in L^2(\Omega) \mid \overline{\alpha}(\cdot) \leq \chi_\omega y(\cdot) \leq \overline{\beta}(\cdot) \text{a.e. in } \omega \right\}.
\]

**Definition 3.1.** We say that system (1) is \([\overline{\alpha}(\cdot),\overline{\beta}(\cdot)]\)-controllable in \( \omega \) if

\[
Im(\chi_\omega L) \cap \left[ [\overline{\alpha}(\cdot),\overline{\beta}(\cdot)] - \{S(T)y_0\} \right] \neq 0
\]
Remark 1. 1. The above definition is equivalent to say that, system (1) is $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega$ at the time $T$ if there exists a control $u \in U$ such that

$$\tilde{\alpha}(\cdot) \leq \chi_\omega y_u(T) \leq \tilde{\beta}(\cdot).$$

2. If system (1) is $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega_1$ then it’s $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega_2 \subset \omega_1$.

3. A system which is exactly controllable in $\omega$ is $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega$. Indeed, if system (1) is exactly controllable in $\omega$ then $Im(\chi_\omega L_i) = L^2(\omega)$ which implies that

$$Im(\chi_\omega L_i) \cap [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)] \neq \emptyset.$$

Which allow us to say that system (1) is $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega$.

4. There exist systems which are not weakly controllable but are $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable [30].

**Proposition 1.** System (1) is $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega$ if and only if

$$(Ker \chi_\omega + Im L_i) \cap [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)] \neq \emptyset.$$

**Proof.** We suppose that $(Ker \chi_\omega + Im L_i) \cap [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)] \neq \emptyset$ then there exists $z \in [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$ such that $z \in (Ker \chi_\omega + Im L_i)$. So $z = z_i + z_2$ where $z_i|\omega = 0$ and $\exists u \in U$ such that $z_2 = L_i u$. Hence, $\exists u | y_u(T) \in [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$.

Conversely, suppose that (1) is $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$-controllable in $\omega$ which means $\exists u \in U$ such that $\chi_\omega y_u(T) = \chi_\omega z \in [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$ in $\omega$. Let’s consider $z_2 = z - y_u(T)$ and $z_2 = y_u(T)$, then $z = z_i + z_2$ where $z_i \in Ker \chi_\omega$ and $z_2 \in Im L_i$ which proves that $z \in (Ker \chi_\omega + Im L_i)$.

3.2. RHUM approach. The aim of this subsection is to compute the control $u^*$ that steers system (1) from an initial state to a state between two prescribed functions in $L^2(\omega)$. Let $y_\omega \in [\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$ the target state. We set the problem of steering system (1) from $y_\omega$ to a desired state $y_d$ between $\tilde{\alpha}(\cdot)$ and $\tilde{\beta}(\cdot)$ with a minimum cost.

Let $G$ be the smallest sub-space of $L^2(\omega)$ which contains $[\tilde{\alpha}(\cdot), \tilde{\beta}(\cdot)]$. The problem of regional controllability becomes

**Problem:**

Is there a control $u \in U$ such that $\chi_\omega y_u(T) \in G$?

$u$ is solution of the following minimization problem:

$$\begin{align*}
\inf_{u \in U} ||u||^2 \\
\chi_\omega y_u(T) \in \mathcal{G}.
\end{align*}$$

Let $U_{ad} = \{u \in U, \text{ such that } \chi_\omega y_u(T) \in \mathcal{G}\}$. So problem (16) becomes:

$$\begin{align*}
\inf_{u \in U_{ad}} ||u||^2 \\
\chi_\omega y_u(T) \in \mathcal{G}.
\end{align*}$$

We suppose that system (1) is excited by an internal pointwise actuator, so the control is written as follows $Bu(t) = \delta(x-b)u(t)$, where $b \in \Omega$. System (1) shall be
rewritten in this form:
\[
\begin{cases}
\frac{\partial y(x,t)}{\partial t} = Ay(x,t) + \delta(x-b)u(t) & Q \\
y(\xi,t) = 0 & \Sigma \\
y(x,0) = y_0(x) & \Omega \ .
\end{cases}
\] (18)

For \( \varphi_0 \in H \) (\( H \) will be defined later), we consider the backward system:
\[
\begin{cases}
\frac{\partial \varphi(x,t)}{\partial t} = -A^* \varphi(x,t) & Q \\
\varphi(\xi,t) = 0 & \Sigma \\
\varphi(x,T) = \varphi_0(x) & \Omega,
\end{cases}
\] (19)

which admits a unique solution \( \varphi \in L^2(0,T;L^2(\Omega)) \) (see [24]).

Let’s define the semi-norm by the following mapping:
\[
\varphi_0 \mapsto ||\varphi_0||_H = \left[ \int_0^T \varphi^2(b,t)dt \right]^{1/2},
\] (20)
it induces a norm.

Let \( F_{[\tilde{\alpha}(\cdot),\tilde{\beta}(\cdot)]} \) be the complete of \( \mathcal{G} \) by (20). Then \( H = F'_{[\tilde{\alpha}(\cdot),\tilde{\beta}(\cdot)]} \) where \( F'_{[\tilde{\alpha}(\cdot),\tilde{\beta}(\cdot)]} \) is the dual of \( F_{[\tilde{\alpha}(\cdot),\tilde{\beta}(\cdot)]} \).

Let’s consider then the following system:
\[
\begin{cases}
\frac{\partial \Psi(x,t)}{\partial t} = A\Psi(x,t) + \varphi(b,t)\delta(x-b) & Q \\
\Psi(\xi,t) = 0 & \Sigma \\
\Psi(x,0) = y_0(x) & \Omega \ .
\end{cases}
\] (21)

We define then the operator \( M \) from \( \mathcal{G} \) onto \( (\mathcal{G})' \) (where \( (\mathcal{G})' \) is the dual of \( \mathcal{G} \)) by:
\[
M \varphi_0 = \chi_\omega \Psi(T).
\]

Since \( M \) is an affine operator, we can decompose it in this way:
\[
M \varphi_0 = \chi_\omega (\Psi_0 + \Psi_1)(T),
\]
where \( \Psi_0 \) and \( \Psi_1 \) are solutions of:
\[
\begin{cases}
\frac{\partial \Psi_0(x,t)}{\partial t} = A\Psi_0(x,t) & Q \\
\Psi_0(\xi,t) = 0 & \Sigma \\
\Psi_0(x,0) = y_0(x) & \Omega,
\end{cases}
\] (22)
and
\[
\begin{cases}
\frac{\partial \Psi_1(x,t)}{\partial t} = A\Psi_1(x,t) + \varphi(b,t)\delta(x-b) & Q \\
\Psi_1(\xi,t) = 0 & \Sigma \\
\Psi_1(x,0) = 0 & \Omega.
\end{cases}
\] (23)

We define the operator \( \Pi \) from \( \mathcal{G} \) to \( (\mathcal{G})' \) by \( \Pi(\varphi_0) = \chi_\omega (\Psi_1(T)) \).

With this notations, the problem of constrained controllability accrue to the resolution of the equation
\[
\Pi(\varphi_0) = \chi_\omega (\Psi(T)) - \chi_\omega (\Psi_0(T)).
\] (24)
So $\Pi$ is an isomorphism from $\overline{G}$ to $(\overline{G})'$. And the equation (24) admits a unique solution. Then, we have the following result:

**Theorem 3.2.** With $[\tilde{\alpha}(),\tilde{\beta}()]\subset \overline{G} \subset L^2(\omega)$. We suppose that (20) defines a norm on $\overline{G}$ which is equivalent to the norm induced by $L^2(\Omega)$. Then, there exist constrained controllability (depending on $U$ and $\Pi$). So we just prove that $\|\varphi\| = 0 \Rightarrow \varphi = 0$.

We have $\|\varphi_0\| = 0 \Rightarrow \left( \int_0^T \varphi^2(b,t)dt \right)^{1/2} = 0 \Rightarrow \varphi(b,t) = 0$.

We consider $(\psi_i)_i$ a complete set of eigenfunctions in $L^2(\Omega)$ of the operator $A$ and $(\lambda_i)$ the eigenvalues associated to $(\psi_i)_i$, so $\|\varphi_0\| = 0$ gives:

$$\sum_{i=0}^{\infty} e^{\lambda_i(T-t)} \langle \varphi_0, \psi_i \rangle \psi_i(b) = 0 \text{ on } [0,T]$$

which leads to $\langle \varphi_0, \psi_i \rangle \psi_i(b) = 0 \quad \forall i$. Furthermore, $\psi_i(b) \neq 0 \quad \forall i$, which implies that $\varphi_0 \equiv 0$ and we prove that (20) is a norm.

- Then, we prove that (24) admits a unique solution, we just prove that $\Pi$ is an isomorphism. For that, we multiply the equation (23) by $\varphi$ and we integrate on $Q$. We have:

$$\int_Q \varphi(t) \frac{\partial \Psi_i(t)}{\partial t} dQ = \int_Q \varphi(t) A \Psi_i(t) dQ + \int_Q \varphi^2(b,t) dQ,$$

which gives:

$$\int_\Omega \varphi(t) \ installations(t) dx - \int_\Omega \varphi'(t) \Psi_i(t) dQ = \int_\Omega \varphi(t) A \Psi_i(t) dQ + \int_0^T \varphi^2(b,t) dt,$$

and

$$\int_\Omega \varphi(T) \Psi_i(T) dx - \int_\Omega \varphi(0) \Psi_i(0) dx$$

$$= \int_Q (\varphi(t) A \Psi_i(t) - A^* \varphi(t) \Psi_i(t)) dQ + \int_0^T \varphi^2(b,t) dt.$$

Using the boundary conditions $\Psi_i(0) = 0$ and green formula, we obtain:

$$\int_\Omega \varphi_i \Psi_i(T) dx = \int_{\Sigma} \left( \varphi(t) \frac{\partial \Psi_i(t)}{\partial n_A} - \Psi_i(t) \frac{\partial \varphi}{\partial n_{A^*}} \right) d\xi dt + \int_0^T \varphi^2(b,t) dt.$$

Thus, we have:

$$\langle \varphi_0, \Pi \varphi_0 \rangle = \int_0^T \varphi^2(b,t) dt = \|\varphi_0\|^2.$$

- Finally, we show that the control $u^*$ minimize the cost function given by (25). To prove that, it is sufficient to verify that:

$$J'(u^*)(u - u^*) = 0 \quad \forall u \in U_{ad}.$$
We have
\[ J'(u^*)(u - u^*) = \int_0^T u^*(t)(u(t) - u^*(t))dt = \int_0^T \varphi(b, t)(u(t) - u^*(t))dt. \]

Let multiply \( \varphi' \) by \( y_u(t) - y_{u^*}(t) \) and integrate by part
\[ \langle \varphi'(t), y_u(t) - y_{u^*}(t) \rangle_Q = (-A^*\varphi(t), y_u(t) - y_{u^*}(t))_Q, \]
which leads to:
\[
\begin{aligned}
[(\varphi(t), y_u(t) - y_{u^*}(t))]_0^T &= \langle \varphi(t), y'_u(t) - y'_{u^*}(t) \rangle_Q \\
&= (-A^*\varphi(t), y_u(t))_Q + (A^*\varphi(t), y_{u^*}(t))_Q.
\end{aligned}
\]

And
\[
\begin{aligned}
\langle \varphi(T), y_u(T) - y_{u^*}(T) \rangle_\alpha &= \langle \varphi(0), y_u(0) - y_{u^*}(0) \rangle_\alpha \\
&= \int_0^T \varphi(b, t) \left[ u(t) - u^*(t) \right] dt + \langle \varphi(t), A(y_u(t) - y_{u^*}(t)) \rangle_Q - \langle A^*\varphi(t), y_u(t) \rangle_Q \\
&\quad + \langle A^*\varphi(t), y_{u^*}(t) \rangle_Q \\
&= \int_0^T \varphi(b, t) \left[ u(t) - u^*(t) \right] dt + \langle \varphi(t), Ay_u(t) \rangle_Q - \langle \varphi(t), Ay_{u^*}(t) \rangle_Q \\
&\quad - \langle A^*\varphi(t), y_u(t) \rangle_Q + (A^*\varphi(t), y_{u^*}(t))_Q \\
&= \int_0^T \varphi(b, t) \left[ u(t) - u^*(t) \right] dt + \int_\Sigma \frac{\partial \varphi(t)}{\partial \nu_{\lambda_u}} y_u(t)d\Sigma - \int_\Sigma \frac{\partial \varphi(t)}{\partial \nu_{\lambda_u}} y_{u^*}(t)d\Sigma \\
&\quad - \int_\Sigma \varphi(t) \frac{\partial y_u(t)}{\partial \nu_{\lambda_u}} d\Sigma + \int_\Sigma \varphi(t) \frac{\partial y_{u^*}(t)}{\partial \nu_{\lambda_u}} d\Sigma \\
&= \int_0^T \varphi(b, t) \left[ u(t) - u^*(t) \right] dt.
\end{aligned}
\]

Using the boundary conditions we have
\[ \langle \varphi_u(T), y_u(T) - y_{u^*}(T) \rangle_\alpha = \int_0^T \varphi(b, t) \left[ u(t) - u^*(t) \right] dt. \]

Which means that \( J'(u^*)(u - u^*) = \langle \varphi_u(T), y_u(T) - y_{u^*}(T) \rangle_\alpha. \)

Moreover, \( \langle \varphi_u, y_u(T) - y_{u^*}(T) \rangle = 0, \) Hence:
\[ J'(u^*)(u - u^*) = 0 \quad \forall u \in U_{ad}. \]

4. Conclusion. We have solved the problem of constrained controllability for parabolic linear systems with RHUM approach and we characterized the optimal control in the general case and the regional one. It is evident that by modifying
- the boundary conditions (Dirichlet, Neumann, mixed),
- the nature of the control (distributed, boundary),
- the type of the state equation.

we can have many other open problems that need further investigations and which can be developed in future works.
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