Matrix algorithm for determination of the elementary paths and elementary circuits using exotic semirings

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Abstract. We propose a new method for determining the elementary paths and elementary circuits in a directed graph. Also, the Hamiltonian paths and Hamiltonian circuits are enumerated.

1 Introduction

Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, that is on replacing numerical fields by idempotent semirings. Exotic semirings such as the max-plus algebra \( \mathbb{R}_{\max} \) or concatenation semiring \( \mathcal{P}(\Sigma^*) \) have been introduced in connection with various fields: graph theory, Markov decision processes, language theory, discrete event systems theory, see [1], [2], [6], [1].

In this paper we will have to consider various semirings, and will universally use the notation \( \oplus, \otimes, \varepsilon, e \) with a context dependent meaning (e.g. \( \oplus := \max \) in \( \mathbb{R}_{\max} \) but \( \oplus := \cup \) in \( \mathcal{P}(\Sigma^*) \), \( \varepsilon := -\infty \) in \( \mathbb{R}_{\max} \) but \( \varepsilon := \emptyset \) in \( \mathcal{P}(\Sigma^*) \)).

In many fields of applications, the graphs are used widely for modelling of practical problems. This paper will focus on two algebraic path problems, namely: the elementary path problem (EPP) and the elementary circuit problem (ECP).

For a directed graph \( G = (V, E) \) (\(|V| = n\)) and \( u, v \in V \), (EPP) and (ECP) are formulated as follows:

- (EPP) enumerate the elementary paths from \( u \) to \( v \) of length \( k \) (\( 1 \leq k \leq n - 1 \));
- (ECP) enumerate the elementary circuits starting in \( u \) of length \( k \) (\( 1 \leq k \leq n \)).

For to solve the above problems we give a method based on a \( n \times n \) matrix with entries in the semiring of distinguished languages.

These algebraic path problems are applied into large domains: combinatorial optimization, traffic control, Internet routing etc.

The paper is organized as follows. In Section 2 we construct a special idempotent semiring denoted by \( \mathcal{P}^*(\Sigma_{du}^*) \) and named the semiring of distinguished languages. The semiring of matrices with entries in \( \mathcal{P}^*(\Sigma_{du}^*) \) is presented in Section 3. This algebraic tool is used to establishing of a one-to-one correspondence between the distinguished words and the elementary paths in a directed graph. In Section 4 we give an algorithm to determine all the elementary paths and elementary circuits in a directed graph. This new practical algorithm is based on the latin composition of distinguished languages.

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2 Semiring of distinguished formal languages

We start this section by recalling some necessary backgrounds on semirings for our purposes (see [6], [1], [5] and references therein for more details).

Semirings. Let \( S \) be a nonempty set endowed with two binary operations, addition (denoted with \( \oplus \)) and multiplication (denoted with \( \otimes \)). The algebraic structure \((S, \oplus, \otimes, \varepsilon, e)\) is a semiring, if it fulfills the following conditions:

1. \((S, \oplus, \varepsilon)\) is a commutative monoid with \( \varepsilon \) as the neutral element for \( \oplus \);
2. \((S, \otimes, e)\) is a monoid with \( e \) as the identity element for \( \otimes \);
3. \( \otimes \) distributes over \( \oplus \);
4. \( \varepsilon \) is an absorbing element for \( \otimes \), that is \( a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon, \forall a \in S \).

A semiring where addition is idempotent (that is, \( a \oplus a = a, \forall a \in S \)) is called an idempotent semiring. If \( \otimes \) is commutative, we say that \( S \) is a commutative semiring.

In the following we introduce the monoid of distinguished words over an alphabet. An alphabet is a finite set \( \Sigma \) of symbols. An word over \( \Sigma \) is a finite sequence of symbols of the alphabet \( \Sigma \). The number of symbols of a word \( x \) is called the length of \( x \) and its length is represented by \( |x| \). The empty word, denoted with \( \lambda \), is the word with length zero (that is, it has no symbols).

A simple word of length \( k \) with \( 1 \leq k \leq n \) over the alphabet \( \Sigma \) (\(|\Sigma| = n\)) is a word of the form \( a = a_1a_2...a_{k-1}a_k \) such that \( a_i \neq a_j \) for \( i \neq j \) and \( i, j = 1, k \).

A simple cyclic word of length \( k + 1 \) with \( 1 \leq k \leq n \) over \( \Sigma \) is a word of the form \( b = a_1a_2...a_{k-1}a_1a_k \) such that \( a = a_1a_2...a_{k-1}a_k \) is a simple word. The simple cyclic words of length 2 over \( \Sigma \) are the words of the form \( u = aa \) for \( a \in \Sigma \).

By a distinguished word over \( \Sigma \) we mean a word \( w \) over \( \Sigma \) which is a simple word of length \( k \) or a simple cyclic word of length \( k + 1 \) with \( 1 \leq k \leq n \). Denote the set of distinguished words over \( \Sigma \) completed with the word \( \lambda \) by \( \Sigma_{dw}^* \).

For example, if \( \Sigma = \{a, b\} \), then \( \Sigma_{dw}^* = \{\lambda, a, b, aa, bb, ab, ba, aba, bab\} \).

It is easy to prove that, if \( \Sigma \) is an alphabet with \(|\Sigma| = n\) symbols, then \( \Sigma_{dw}^* \) is a finite set with \( \sigma_n \) elements, where

\[
\sigma_n = 1 + 2n! + 2 \sum_{k=1}^{n-1} \frac{n!}{(n-k)!}. \tag{2.1}
\]

On the set \( \Sigma_{dw}^* \) we introduce the binary operation \( \circ_{\ell} \) given as follows:

1. for all distinguished word \( x \in \Sigma_{dw}^* \) and simple cyclic word \( c \in \Sigma_{dw}^* \), we have \( \lambda \circ_{\ell} x = x \circ_{\ell} \lambda = \lambda \) and \( c \circ_{\ell} x = x \circ_{\ell} c = \lambda \); (2.2)

2. Let \( x = a_1a_2...a_{k-1}a_k \) and \( y = b_1b_2...b_{r-1}b_r \) be two simple words of the lengths \( k \) and \( r \) with \( 1 \leq k, r \leq n \). The word \( x \circ_{\ell} y \in \Sigma_{dw}^* \) is defined by:
Matrix algorithm for determination of the elementary paths and . . .

\[ x \circ_\ell y = \begin{cases} 
  a_1 \ldots a_k b_2 \ldots b_{r-1} b_r, & \text{if } a_k = b_1, \{a_1, \ldots, a_k\} \cap \{b_2, \ldots, b_r\} = \emptyset \\
  a_1 \ldots a_k b_2 \ldots b_{r-1} a_1, & \text{if } a_k = b_1, b_r = a_1, \{a_1, \ldots, a_k\} \cap \{b_2, \ldots, b_{r-1}\} = \emptyset \\
  \lambda, & \text{otherwise.}
\end{cases} \]

The operation \( \circ_\ell \) is called the \textit{latin composition of distinguished words.}

\textbf{Example 2.1.} The set \( \Sigma_{dw}^* \) of distinguished words over \( \Sigma = \{1, 2, 3, 4\} \) has \( \sigma_4 = 129 \) elements. If \( x = 123, y = 31, z = 1, c_1 = 22 \) and \( c_2 = 343 \), then
\[
\begin{align*}
  c_1 \circ_\ell x &= 22 \circ_\ell 123 = \lambda, \quad x \circ_\ell c_2 = 123 \circ_\ell 343 = \lambda, \quad x \circ_\ell y = 123 \circ_\ell 31 = 1231, \\
  y \circ_\ell x &= 31 \circ_\ell 123 = 3123, \quad z \circ_\ell x = 1 \circ_\ell 123 = 123. \quad \text{Note that } x \circ_\ell y \neq y \circ_\ell x. \quad \square
\end{align*}
\]

For all \( a, b, c \in \Sigma_{dw}^* \), we have \( (a \circ_\ell b) \circ_\ell c = a \circ_\ell (b \circ_\ell c) \). Then \( (\Sigma_{dw}^*, \circ_\ell, \lambda) \) is a monoid, called the \textit{monoid of distinguished words generated by alphabet} \( \Sigma \).

A \textit{distinguished language} \( L \) over alphabet \( \Sigma \) is a subset of distinguished words over \( \Sigma \) completed with the empty word \( \lambda \), that is \( L \subseteq \Sigma_{dw}^* \) and \( \lambda \in L \).

\textbf{Convention.} (i) If \( L = \{\lambda\} \) or \( L = \{a\} \) where \( a \) is a distinguished word, then we shall use the notations \( \lambda \) and \( a \), respectively.

(ii) If \( L \neq \{\lambda\} \), then we enumerate only its distinguished words of length \( k \geq 1. \square \)

The monoid \( \Sigma_{dw}^* \) contains two special distinguished languages, namely: the language \( \{\lambda\} \) (it contains only the empty word \( \lambda \)) and the language \( \Sigma \) (it contains all symbols of the alphabet and the empty word \( \lambda \)).

The \textit{set of distinguished languages over an alphabet} \( \Sigma \) is \( \mathcal{P}^*(\Sigma_{dw}^*) \).

Since distinguished languages are sets, all the set operations can be applied to distinguished languages. Then the union and intersection of two distinguished languages are distinguished languages.

The \textit{latin composition} of the distinguished languages \( L_1 \) and \( L_2 \) is the distinguished language \( L_1 \circ_\ell L_2 \) defined by
\[
L_1 \circ_\ell L_2 = \{ x \circ_\ell y \mid x \in L_1 \text{ and } y \in L_2 \},
\]
that is, \( L_1 \circ_\ell L_2 \) is the set of distinguished words obtaining by the latin composition of words of \( L_1 \) with those of \( L_2 \).

For example, if \( L_1, L_2 \subseteq \Sigma_{dw}^* \) are distinguished languages over \( \Sigma = \{1, 2, 3, 4\} \), where \( L_1 = \{2, 412\} \) and \( L_2 = \{11, 23\} \), then
\[
L_1 \circ_\ell L_2 = \{2, 412\} \circ_\ell \{11, 23\} = \{2 \circ_\ell 11, 412 \circ_\ell 11, 2 \circ_\ell 23, 412 \circ_\ell 23\} = \{23, 4123\}.
\]

\textbf{Proposition 2.1.} (i) Let \( L, L_1, L_2, L_3 \in \mathcal{P}(\Sigma_{dw}^*) \). Then:
\[
L \circ_\ell \lambda = \lambda \circ_\ell L = \lambda, \quad L \circ_\ell \Sigma = \Sigma \circ_\ell L = L, \quad (L_1 \circ_\ell L_2) \circ_\ell L_3 = L_1 \circ_\ell (L_2 \circ_\ell L_3). \quad (2.5)
\]

(ii) The set \( \mathcal{P}^*(\Sigma_{dw}^*) \) endowed with multiplication \( \circ_\ell \) has a structure of monoid.
Proof. (i) Using the definitions one easily verify that (2.5) holds.

(ii) From (i) it follows that \((\mathcal{P}^*(\Sigma_{dw}^*), \circ, \Sigma)\) is a monoid. \(\square\)

On the set \(\mathcal{P}^*(\Sigma_{dw}^*)\) of distinguished languages we define the binary operations:

\[
L_1 \oplus L_2 := L_1 \cup L_2 \quad \text{and} \quad L_1 \otimes L_2 := L_1 \circ L_2, \quad \forall L_1, L_2 \in \mathcal{P}^*(\Sigma_{dw}^*). \tag{2.6}
\]

**Proposition 2.2.** \((\mathcal{P}^*(\Sigma_{dw}^*), \cup, \circ, \lambda, \Sigma)\) is an idempotent semiring.

**Proof.** For to verify the conditions from definition of an idempotent semiring we apply the properties of the union of sets and Proposition 2.1. \(\square\)

We call \((\mathcal{P}^*(\Sigma_{dw}^*), \cup, \circ, \lambda, \Sigma)\) the semiring of distinguished languages over \(\Sigma\).

### 3 Matrices over semirings and directed graphs

Let \((S, \oplus, \otimes, \varepsilon, e)\) be an (idempotent) semiring. For each positive integer \(n\), let \(M_n(S)\) be the set of \(n \times n\) matrices with entries in \(S\). The operations \(\oplus\) and \(\otimes\) on \(S\) induce corresponding operations on \(M_n(S)\) in the obvious way. Indeed, if \(A = (A_{ij}), B = (B_{ij}) \in M_n(S)\) then we have:

\[
A \oplus B = ((A \oplus B)_{ij}) \quad \text{and} \quad A \otimes B = ((A \otimes B)_{ij}), \quad i, j = 1, n \quad \text{where}
\]

\[
(A \oplus B)_{ij} := A_{ij} \oplus B_{ij} \quad \text{and} \quad (A \otimes B)_{ij} := \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}. \tag{3.1}
\]

The set \(M_n(S)\) contains two special matrices with entries in \(S\), namely the zero matrix \(O_{\oplus n}\), which has all its entries equal to \(\varepsilon\), and the identity matrix \(I_{\otimes n}\), which has the diagonal entries equal to \(e\) and the other entries equal to \(\varepsilon\).

It is easy to check that the following proposition holds.

**Proposition 3.1.** \((M_n(S), \oplus, \otimes, O_{\oplus n}, I_{\otimes n})\) is an idempotent semiring, where the operations \(\oplus\) and \(\otimes\) are given in (3.1). \(\square\)

We call \((M_n(S), \oplus, \otimes, O_{\oplus n}, I_{\otimes n})\) the semiring of \(n \times n\) matrices with entries in \(S\). In particular, if \(S := (\mathcal{P}^*(\Sigma_{dw}^*), \cup, \circ, \lambda, \Sigma)\), then \((M_n(\mathcal{P}^*(\Sigma_{dw}^*))), \cup, \circ, O_{\oplus n}, I_{\otimes n})\) is called the semiring of \(n \times n\) matrices over \(\mathcal{P}^*(\Sigma_{dw}^*)\). The operation \(\otimes := \circ\) is called the multiplication of matrices based on latin composition of words.

**Example 3.1.** Let be semiring \((M_2(\mathcal{P}^*(\Sigma_{dw}^*))), \cup, \circ, O_{\oplus 2}, I_{\otimes 2}\) with \(\Sigma = \{a, b, c\}\). The product \(A \circ B\) of the \(A, B\) with entries in the semiring \(\mathcal{P}^*(\Sigma_{dw}^*)\) is

\[
A \circ B = \begin{pmatrix} ab & \varepsilon \\ bca & bc \end{pmatrix} \circ \begin{pmatrix} b & ab \\ c & \varepsilon \end{pmatrix} = \begin{pmatrix} (ab \circ \varepsilon) b \oplus (\varepsilon \circ \varepsilon) c \\ (bc \circ \varepsilon) b \oplus (bc \circ \circ) c \end{pmatrix} = \begin{pmatrix} (ab \circ \varepsilon) b \oplus (\varepsilon \circ \varepsilon) c \\ (bc \circ \varepsilon) b \oplus (bc \circ \varepsilon) c \end{pmatrix} = \begin{pmatrix} ab \varepsilon \oplus \varepsilon \\ bc \varepsilon \oplus \varepsilon \end{pmatrix} = \begin{pmatrix} ab \cup \varepsilon & \varepsilon \cup \varepsilon \\ bc \cup \varepsilon & bcab \cup \varepsilon \end{pmatrix} = \begin{pmatrix} ab \varepsilon & bc \varepsilon \varepsilon \varepsilon \end{pmatrix}. \quad \square
\]
A directed graph is a pair $G = (V, E)$ where $V$ is a finite set of vertices of the graph $G$ and $E \subseteq V \times V$ is a set of arcs of $G$. A typical arc $(u, v) \in E$ is thought of as an arrow directed from $u$ to $v$.

Let $G = (V, E)$ be a directed graph with $|V| = n$. A path from $u$ to $v$ of length $k$ $(k \geq 1)$ in $G$ is a sequence of vertices $p = (v_1, v_2, \ldots, v_k, v_{k+1})$ with $v_1 = u, v_{k+1} = v$ such that $(v_i, v_{i+1}) \in E$ for all $i = 1, k; v_1$ is called the starting vertex and $v_{k+1}$ the end-vertex of $p$, respectively. The length of path $p$ will be denoted by $\ell(p)$.

A path $p = (v_1, v_2, \ldots, v_k, v_{k+1})$ is called a circuit if $v_{k+1} = v_1$ and $k \geq 1$. In particular, for $k = 1$ we obtain the circuit $(v_1, v_1)$ of length $1$.

We denote with $P(v_i, v_j, k) (k \geq 1)$, the set of all paths of length $k$ from the starting vertex $v_i \in V$ to end-vertex $v_j \in V$. In particular, when $v_i = v_j$, $C(v_i, k) = P(v_i, v_i, k)$ $(k \geq 1)$ is the set of all circuits of length $k$ starting at vertex $v_i$.

A path $p = (v_1, v_2, \ldots, v_k, v_{k+1})$ is called an elementary path from $v_1$ to $v_{k+1}$, if $k \geq 1$ and $v_i \neq v_j$ for $i \neq j$ and $i, j = 1, k + 1$. A circuit $c = (u_1, u_2, \ldots, u_k, u_1)$ with $\ell(c) = k$ is called an elementary circuit, if $(u_1, u_2, \ldots, u_k)$ is an elementary path.

We denote with $P_{\text{elem}}(v_i, v_j, k) (k \geq 1)$, the set of all elementary paths length $k$ from $v_i \in V$ to $v_j \in V$. In particular, when $v_i = v_j$, then $C_{\text{elem}}(v_i, k) = P_{\text{elem}}(v_i, v_i, k)$ $(k \geq 1)$ is the set of all elementary circuits of length $k$ starting at $v_i$.

A Hamiltonian path (resp., circuit) is a path (resp., circuit) that contains each vertex exactly once. Hence, a Hamiltonian path (resp., circuit) is an elementary path $p_H$ with $\ell(p_H) = n - 1$ (resp., an elementary circuit $c_H$ with $\ell(c_H) = n$).

A weighted directed graph is a graph $G = (V, E)$ with a mapping $w : E \rightarrow S$ that assigns each arc $(u, v) \in E$ a weight $w(u, v)$ from the semiring $(S, \oplus, \otimes, \varepsilon, e)$. A weighted directed graph with the cost function $w$ is denoted by $G = (V, E, w)$.

The weight or cost of path $p = (v_1, \ldots, v_k, v_{k+1})$ is the element $w(p) \in S$ where

$$w(p) = \bigotimes_{i=1}^{k} w(v_i, v_{i+1}).$$

(3.2)

To each given weighted directed graph $G = (V, E, w)$ with $V = \{v_1, v_2, \ldots, v_n\}$ we can associate an $n \times n$ matrix $M_w(G)$ with entries in a semiring $(S, \oplus, \otimes, \varepsilon, e)$ as follows. For this, we define the matrix $M_w(G) = (M_{ij}) \in M(n, S)$ where

$$M_{ij} = \begin{cases} w(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ \varepsilon & \text{if } (v_i, v_j) \notin E \end{cases}$$

(3.3)

To each directed graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$ we can associate two weight functions in the following way.

- Let be the numbering semiring $(\mathbb{N}, +, \cdot, 0, 1)$ of natural numbers endowed with the usual addition and multiplication. Consider the weight function $w_a : E \rightarrow \mathbb{N}$ defined by $w_a(v_i, v_j) = 1$ for all $(v_i, v_j) \in E$. The matrix $M_{w_a}(G) \in M(n, \mathbb{N})$, denoted with $A$, is called the adjacency matrix of graph $G$. 

Let be the idempotent semiring \((\mathcal{P}^*(\Sigma_{dw}^*), \cup, \circ, \emptyset, \Sigma)\) of distinguished languages over alphabet \(\Sigma = V\). Define the weight function \(w_\ell : E \to \mathcal{P}^*(\Sigma_{dw}^*)\) given by \(w_\ell(v_i, v_j) = v_i v_j\) for all \((v_i, v_j) \in E\) (that is, \(w_\ell(v_i, v_j)\) is the distinguished language which contains only the distinguished word \(v_i v_j\) of length 1). The matrix \(M_{w_\ell}(G) \in M(n, \mathcal{P}^*(\Sigma_{dw}^*))\) is denoted with \(L\) and is called the latin matrix of \(G\).

More precisely, the adjacency matrix \(A = (A_{ij}) \in M(n, \mathbb{N})\) and latin matrix \(L = (L_{ij}) \in M(n, \mathcal{P}^*(\Sigma_{dw}^*))\) associated to graph \(G\) are given by

\[
A_{ij} = \begin{cases} 
1 & \text{if } (v_i, v_j) \in E \\
0 & \text{if } (v_i, v_j) \notin E
\end{cases}
\quad \text{and} \quad
L_{ij} = \begin{cases} 
v_i v_j & \text{if } (v_i, v_j) \in E \\
\varepsilon & \text{if } (v_i, v_j) \notin E
\end{cases}
\tag{3.4}
\]

4 Matrix algorithm for enumerating of elementary paths and elementary circuits in a directed graph

Consider the latin matrix \(L = (L_{ij}) \in M(n, \mathcal{P}^*(\Sigma_{dw}^*))\) associated to graph \(G = (V, E)\) (\(|V| = n\)), defined by (3.4). Using the multiplication of matrices based on latin composition, we define by recurrence the power \(L[k]\) of matrix \(L\) in the following way:

\[
L[1] = L, \quad L[2] = L \circ_\ell L, \quad \ldots, \quad L[k] = L \circ_\ell L[k-1] \quad \text{for } k \geq 2.
\tag{4.1}
\]

Applying (3.1) and replacing \(\oplus\) and \(\otimes\) with the correspondent operations of the semiring \(M(n, \mathcal{P}^*(\Sigma_{dw}^*))\), we have \(L[k] = (L_{ij}^k)\) for \(i, j = 1, n\), where

\[
L_{ij}^k = \bigcup_{m=1}^n (L_{im} \circ_\ell L_{mj}^{k-1}), \quad k \geq 2.
\tag{4.2}
\]

It is easy to prove that:

\[
L[n] \quad \text{is a diagonal matrix and} \quad L[n+q] = O_\otimes n \quad \text{for all } q \geq 1.
\tag{4.3}
\]

**Theorem 4.1.** Let \(G = (V, E)\) be a directed graph with \(V = \{v_1, v_2, \ldots, v_n\}\) and the latin matrix \(L = (L_{ij}) \in M(n, \mathcal{P}^*(\Sigma_{dw}^*))\). If \(L[k] = (L_{ij}^k)\), then:

\[
L_{ij}^k = \text{P}_{\text{elem}}(v_i, v_j, k), \quad i, j = 1, n, \quad i \neq j, \quad 1 \leq k \leq n - 1;
\tag{4.4}
\]

\[
L_{ii}^k = \text{C}_{\text{elem}}(v_i, k), \quad i = 1, n, \quad 1 \leq k \leq n.
\tag{4.5}
\]

**Proof.** We proceed by induction on \(k\). Choose \(v_i\) and \(v_j\) arbitrarily.

(1) **Case** \(i \neq j\). For \(k = 1\), the relation (4.4) holds. Indeed, we have that the only elementary path in \(\text{P}_{\text{elem}}(v_i, v_j, 1)\) is the arc \((v_i, v_j)\), hence \(L_{ij}^{[1]} = v_i v_j\). Therefore, \(\text{P}_{\text{elem}}(v_i, v_j, 1) = \{(v_i, v_j)\}\). Note that if \(L_{ij} = \varepsilon\), then \(\text{P}_{\text{elem}}(v_i, v_j, 1) = \emptyset\).

Now assume the relation (4.4) holds true for \(k\). Then \(L_{ij}^{[k]} = \text{P}_{\text{elem}}(v_i, v_j, k)\), that is \(L_{ij}^{[k]}\) represent the set of all elementary paths of length \(k\) from \(v_i\) to \(v_j\).
Using (4.2) the \((i, j)\) entry of the matrix \(L^{[k+1]} = L \circ L^{[k]}\) is given explicitly by
\[
L^{[k+1]}_{ij} = (L_{i1} \circ L_{1j}) \cup \ldots \cup (L_{im} \circ L_{mj}) \cup \ldots \cup (L_{in} \circ L_{nj}), \quad k \geq 1. \tag{4.6}
\]

We first evaluate the term \(L_{im} \circ L^{[k]}_{mj}\) from the equality (4.6), for an fixed integer \(m\) with \(1 \leq m \leq n\). We have the following situations.

- If \(L_{im} = \varepsilon\) or \(L^{[k]}_{mj} = \varepsilon\), then \(L_{im} \circ L^{[k]}_{mj} = \varepsilon\), that is the set of elementary paths of length \(k + 1\) from \(v_i\) to \(v_j\) stopping at vertex \(v_m\) is empty set.

- If \(L_{im} \neq \varepsilon\) and \(L^{[k]}_{mj} \neq \varepsilon\). By induction hypothesis we have
\[
L^{[k]}_{mj} = P_{\text{elem}}(v_m, v_j, k) = \{p^1_{mj}, p^2_{mj}, \ldots, p^{r-1}_{mj}, p^r_{mj}\},
\]
where \(p^s_{mj}\) is an elementary path of length \(k\) from \(v_m\) to \(v_j\) for \(1 \leq s \leq r\). Since \(L_{im} = v_i v_m\) it follows that
\[
L_{im} \circ L^{[k]}_{mj} = v_i v_m \circ L^{[k]}_{mj} = v_i v_m \circ L^{[k]}_{mj} = v_i v_m \circ L^{[k]}_{mj} = \{p^1_{mj}, p^2_{mj}, \ldots, p^{r-1}_{mj}, p^r_{mj}\}
\]
where \(p^s_{mj}\) is regarded as a distinguished word, having the set of vertices
\[
X^s_{mj} = \{v_m = v_{m_0,j}, v_{m_0+k-1,j}, v_{m_0+k,j} = v_j\}.\]
The element \(v_i v_m \circ L^{[k]}_{mj}\)
can be taken the following values:
- \(v_i v_m \circ L^{[k]}_{mj}\) \(= \{v_i v_m v_{m_0+1,j} \ldots v_{m_0+k-1,j} v_j\}\), if \(\{v_i\} \cap X^s_{mj} = \emptyset\),
that is \(v_i v_m \circ L^{[k]}_{mj}\) is an elementary path of length \(k + 1\) from \(v_i\) to \(v_j\);  
- \(v_i v_m \circ L^{[k]}_{mj}\) \(= \{\lambda\}\), if \(\{v_i\} \cap X^s_{mj} \neq \emptyset\).

Then \(L_{im} \circ L^{[k]}_{mj} = \{\lambda\}\) or \(L_{im} \circ L^{[k]}_{mj}\) is a set which contains \(t\) \((1 \leq t \leq s)\) elementary paths of length \(k + 1\) from \(v_i\) to \(v_j\) stopping at the vertex \(v_m\).

Therefore, \(L^{[k+1]}_{ij} = \bigcup_{m=1}^{n} (L_{im} \circ L^{[k]}_{mj}) = P_{\text{elem}}(v_i, v_j, k + 1)\). Hence, (4.4) holds for \(k + 1\). This completes the inductive step and proves the assertion in the case \(i \neq j\).

(2) Case \(i = j\). If \(L^{[1]}_{ii} = v_i v_i\), then \(C_{\text{elem}}(v_i, 1) = \{v_i, v_i\}\). Also, if \(L_{ii} = \varepsilon\), then \(C_{\text{elem}}(v_i, 1) = \emptyset\). Hence, (4.5) holds for \(k = 1\). Assume that (4.5) holds true for \(k\). Then \(L^{[k]}_{ii} = C_{\text{elem}}(v_i, k)\), that is \(L^{[k]}_{ii}\) represent the set of all elementary circuits of length \(k\) starting at \(v_i\).

The \((i, i)\) entry of the matrix \(L^{[k+1]}\) is \(L^{[k+1]}_{ii} = \bigcup_{m=1}^{n} (L_{im} \circ L^{[k]}_{mi})\), \(k \geq 1\).

We evaluate the term \(L_{im} \circ L^{[k]}_{mi}\) for an fixed integer \(m\) with \(1 \leq m \leq n\). We have the following situations.

- If \(L_{im} = \varepsilon\) or \(L^{[k]}_{mi} = \varepsilon\), then \(L_{im} \circ L^{[k]}_{mi} = \varepsilon\.

- If \(L_{im} \neq \varepsilon\) and \(L^{[k]}_{mi} \neq \varepsilon\). Applying (4.4) we have \(L^{[k]}_{mi} = P_{\text{elem}}(v_m, v_i, k) = \{q^1_{mi}, q^2_{mi}, \ldots, q^{r-1}_{mi}, q^r_{mi}\}\), where \(q^s_{mi}\) is an elementary path of length \(k\) from \(v_m\) to \(v_i\) for \(1 \leq s \leq r\). Since \(L_{im} = v_i v_m\) it follows that
\[
L_{im} \circ L^{[k]}_{mi} = v_i v_m \circ L^{[k]}_{mi} = v_i v_m \circ L^{[k]}_{mi} = \{q^1_{mi}, q^2_{mi}, \ldots, q^{r-1}_{mi}, q^r_{mi}\}
\]
where \(q^s_{mi}\) is regarded as a distinguished word, having the set of vertices
$Y_{mi}^* = \{v_m = v_{m_1,i}, v_{m_1+1,i}, \ldots, v_{m_1+k-1,i}, v_{m_1+k,i} = v_i\}$. The element $v_i v_m \circ \ell q_{mi}^*$ can be taken as following values:

- $v_i v_m \circ \ell q_{mi}^* = \{v_i v_m v_{m_1+1,i} \ldots v_{m_1+k-1,i} v_i\}$, if $\{v_i\} \cap (Y_{mi}^* \setminus \{v_i\}) = \emptyset$.

That is $v_i v_m \circ \ell q_{mi}^*$ is an elementary circuit from $v_i$ to $v_i$ of length $k + 1$;

- $v_i v_m \circ \ell q_{mi}^* = \{\lambda\}$, if $\{v_i\} \cap (Y_{mi}^* \setminus \{v_i\}) \neq \emptyset$.

Then $L_{im} \circ \ell L_{mi}^k = \{\lambda\}$ or $L_{im} \circ \ell L_{mi}^{k+1}$ is a set which contains $t_1 (1 \leq t_1 \leq s)$ elementary circuits of length $k + 1$ from $v_i$ to $v_i$ stopping at $v_m$. Therefore, $L_{ii}^{k+1} = C_{	ext{elem}}(v_i, k + 1)$. Hence, (4.5) holds for $k + 1$. This completes the inductive step and proves the assertion in the case $i = j$. □

Applying Theorem 4.1 we give an answer of (EPP) and (ECP) for a directed graph. In this purpose we give a new method based on latin composition of distinguished languages. We will called the algorithm of latin composition of distinguished languages (shortly, LCDL-algorithm).

For a directed graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, the LCDL-algorithm consists from the following steps:

**Step 1.** Associate the latin matrix $L \in M(n, P^*(\Sigma_{dv}^*))$ to graph $G$;

**Step 2.** For each $k (1 \leq k \leq n)$ compute the matrix $L^k = (L_{ij}^k)$;

**Step 3.** (i) For each pair $(i, j)$ and each $k (1 \leq k \leq n - 1)$ enumerate the elementary paths of length $k$ from $v_i$ to $v_j$ in $G$ (we apply the relation (4.4));

(ii) For each $i$ and each $k (1 \leq k \leq n)$ enumerate the elementary circuits of length $k$ starting at $v_i$ in $G$ (we apply the relation (4.5)).

**Remark 4.1.** Let $G = (V, E)$ be a directed graph with $V = \{v_1, v_2, \ldots, v_n\}$.

(i) (1) For $1 \leq k \leq n - 2$ and $i, j = 1, n$ with $i \neq j$, the elements $L_{ij}^k$ indicates the elementary paths which are formed by $k$ arcs, so that $L_{ij}^{n-1}$ determines all Hamiltonian paths in $G$ between $v_i$ and $v_j$.

(2) For $1 \leq k \leq n - 1$ and $i = 1, n$, the elements $L_{ii}^k$ indicates the elementary circuits of length $k$, so that $L_{ii}^n$ determines all Hamiltonian circuits in $G$ starting at $v_i$.

(ii) The necessity to determine the elementary paths and elementary circuits of maximum length in a directed graph arises. The LCDL-algorithm determines the Hamiltonian paths (resp., Hamiltonian circuits) when there exist or determines the elementary paths (resp., circuits) of maximum length when we have no Hamiltonian paths (resp., circuits). □

**Remark 4.2.** ([P]) The powers of the adjacency matrix $A \in M(n; \mathbb{N})$ associated to graph $G = (V, E)$ are used to find the number of distinct paths and distinct circuits between two vertices in $V$ (two paths or circuits of length $k$ are distinct if they visit a different sequence of vertices). More precisely:

Let $A^k = (A_{ij}^k)$ be the $k$-th power of the adjacency matrix $A$. Then:

$$A_{ij}^k = |P(v_i, v_j, k)| \quad \text{and} \quad A_{ii}^k = |C(v_i, k)| \quad \text{for all} \ k \geq 1. \quad \Box$$
Example 4.1. Let $G=(V,E)$ be a directed graph with vertex set $V = \{v_1,v_2,v_3,v_4\}$ and the adjacency matrix $A \in M_4(\mathbb{N})$ of $G$ where

$$A = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$ 

The geometric representation of graph $G$ is given in Figure 4.1.

Fig. 4.1.

Computing the power $A^3$ of the adjacency matrix $A$ in terms of numbering semiring $(\mathbb{N}, +, \cdot, 0, 1)$ we have $A^3_{14} = 5$ and $A^3_{22} = 1$. Then:

- Between $v_1$ and $v_4$ there exist 5 paths of length 3, namely: $p_1^3 = (v_1,v_2,v_3,v_4)$, $p_2^3 = (v_1,v_1,v_1,v_4)$, $p_3^3 = (v_1,v_2,v_2,v_4)$, $p_4^3 = (v_1,v_2,v_3,v_4)$, $p_5^3 = (v_1,v_1,v_3,v_4)$;
- $G$ has one circuit of length 3 starting at $v_2$, namely $c_1^3 = (v_2,v_2,v_2)$.

(ii) We consider the alphabet $\Sigma = \{v_1,v_2,v_3,v_4\}$. For to find the elementary paths and elementary circuits in graph $G$ we apply LCDL-algorithm.

The latin matrix $L \in M_4(\mathcal{P}(\Sigma_{dv}^*)$ associated to graph $G$ is

$$L = \begin{pmatrix}
v_1v_1 & v_1v_2 & v_1v_3 & v_1v_4 \\
v_2v_1 & v_2v_2 & v_2v_3 & v_2v_4 \\
v_3v_1 & v_3v_2 & v_3v_3 & v_3v_4 \\
v_4v_1 & v_4v_2 & v_4v_3 & v_4v_4
\end{pmatrix}. $$

We compute the powers of the latin matrix $L$ in terms of the semiring of distinguished languages. We first compute the matrix $L^{[2]} = (L^{[2]}_{ij})$. We have

$$L^{[2]} = L \circ_\ell L = \begin{pmatrix}
\varepsilon & \varepsilon & v_1v_2v_3 & \{v_1v_2v_4, v_1v_3v_4\} \\
\varepsilon & \varepsilon & \varepsilon & v_2v_3v_4 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}. $$
For example, \( L_1^2 \) is computed as follows:

\[
L_1^2 = (L_{11} \circ L_{14}) \cup (L_{12} \circ L_{24}) \cup (L_{13} \circ L_{44}) = (v_1 v_1 \circ v_1 v_4) \cup (v_1 v_2 \circ v_2 v_4) \cup (v_1 v_3 \circ v_3 v_4) = \varepsilon \cup \{v_1 v_2 v_4\} \cup \{v_1 v_3 v_4\} = \{v_1 v_2 v_4, v_1 v_3 v_4\}.
\]

Since \( L_1^2 = \{v_1 v_2 v_4, v_1 v_3 v_4\} \) it follows that there exist two elementary paths of length 2 from \( v_1 \) to \( v_4 \) and we have \( P_{\text{elem}}(v_1, v_4) = \{(v_1, v_2), (v_1, v_3, v_4)\} \).

The matrices \( L^3 = L \circ L^2 \) and \( L^4 = L \circ L^3 \) are given by

\[
L^3 = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & v_1 v_2 v_3 v_4 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}
\]

and

\[
L^4 = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}.
\]

Using the matrices \( L^3 \) and \( L^4 \) one obtains the following results:

- \( L_1^3 = \{v_1 v_2 v_3 v_4\} \). Then \( P_{\text{elem}}(v_1, v_4, 3) = \{(v_1, v_2, v_3, v_4)\} \) and \( G \) has only one Hamiltonian path. The set of elementary paths of maximum length from \( v_2 \) to \( v_4 \) is \( P_{\text{elem}}(v_2, v_4, 2) = \{(v_2, v_3, v_4)\} \), since \( L_{24}^2 = \{v_2 v_3 v_4\} \) and \( L_{24}^3 = \varepsilon \).

- \( L_{ii}^k = \varepsilon \) for \( 2 \leq k \leq 4 \) and \( i = 1, 4 \). Then \( C_{\text{elem}}(v_i, v_i, k) = \emptyset \). The elementary circuits of maximum length are those of length 1. \( \square \)

**Application.** The finding of Hamiltonian paths and Hamiltonian circuits of minimal cost in a weighted directed graph.

The **LCDL-algorithm** can be used to solve the following two problems in a weighted directed graph \( G = (V, E) \):

(i) find a Hamiltonian path of minimal cost between two vertices in \( G \);

(ii) for each \( v \in G \), find a Hamiltonian circuit of minimal cost starting at \( v \).

One way to solve the above problems consists of searching all possible Hamiltonian paths and Hamiltonian circuits (we apply **LCDL-algorithm**) and computing their cost (we apply the relation (3.1)).

**Example 4.2.** Let \( G = (V, E, w) \) be a weighted directed graph where \( V = \{1, 2, 3, 4, 5\} \), \( E = \{(1, 2), (1, 3), (1, 5), (2, 1), (2, 5), (3, 2), (4, 3), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4)\} \) and the cost function \( w_{\text{cost}} : E \rightarrow \mathbb{R} \), \( (i, j) \mapsto w_{\text{cost}}(i, j) = w_{ij} \) given by

| \( (i, j) \) | (1, 2) | (1, 3) | (1, 5) | (2, 1) | (2, 5) | (3, 2) | (4, 3) | (4, 5) | (5, 1) | (5, 2) |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| \( w_{ij} \) | 4     | 2     | 6     | 3     | 3     | 1     | 5     | 4     | 6     | 1     |

| \( (i, j) \) | (5, 3) | (5, 4) |
|----------|-------|-------|
| \( w_{ij} \) | 2     | 1     |
Matrix algorithm for determination of the elementary paths and . . .

The latin matrix $L \in M_5(\mathcal{P}(\Sigma_{d_{uw}}))$ ($\Sigma = \{1, 2, 3, 4, 5\}$) associated to graph $G$ is

$$L = \begin{pmatrix}
\varepsilon & 12 & 13 & \varepsilon & 15 \\
21 & \varepsilon & \varepsilon & \varepsilon & 25 \\
\varepsilon & 32 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 43 & \varepsilon & 45 \\
51 & 52 & 53 & 54 & \varepsilon
\end{pmatrix}.$$

Let us we compute the powers of the latin matrix $L$. We have

$$L^2 = L \circ_L L = \begin{pmatrix}
\{121,151\} & \{132,152\} & 153 & 154 & 125 \\
251 & \{212,252\} & \{213,253\} & 254 & 215 \\
321 & \varepsilon & \varepsilon & \varepsilon & 325 \\
451 & \{432,452\} & 453 & 454 & \varepsilon \\
521 & \{512,532\} & \{513,543\} & \varepsilon & \{515,525,545\}
\end{pmatrix}.$$

The matrix $L^3 = L \circ_L L^2$ has the following lines:

$L_1^3 : \{(1321,1521,1251), 1532, \{1543,1253\}, 1254, 1325\};$

$L_2^3 : \{\varepsilon, \{2512,2132,2532,2152\}, \{2513,2543,2153\}, 2154, \varepsilon\};$

$L_3^3 : \{3251, \varepsilon, \{3213,3253\}, 3254, 3215\};$

$L_4^3 : \{4321,4521\}, \{4512,4532\}, 4513, \varepsilon, 4325\};$

$L_5^3 : \{5321, \{5132,5432\}, 5213, \varepsilon, \{5215,5125,5325\}\}.$

The matrix $L^4 = L \circ_L L^3$ has the following lines:

$L_1^4 : \{(15321,13251), 15432, 12543, 13254, \varepsilon\};$

$L_2^4 : \{\varepsilon, \{25132,25432,21532\}, 21543, \varepsilon, \varepsilon\};$

$L_3^4 : \{\varepsilon, \varepsilon, \{32513,32543,32153\}, \varepsilon, \varepsilon\};$

$L_4^4 : \{45321,43251\}, 45132, 45213, 43254, 43215\};$

$L_5^4 : \{54321, \varepsilon, \varepsilon, \varepsilon, \{53215,51325,54325\}\}.$

Finally, we have

$$L^5 = L^4 \circ_L L = \begin{pmatrix}
154321 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & 215432 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 432154 & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 543215
\end{pmatrix}.$$

- From $L^4$ it follows that $G$ has 11 Hamiltonian paths. For example, $P_{elem}(4,1,4) = \{p_{1,H}^4 = (4,5,3,2,1), p_{2,H}^4 = (4,3,2,5,1)\}$ since $L_4^{11} = \{45321,43251\}$. We have
It follows that $p^4_{1,H}$ is a Hamiltonian path between the vertices 4 and 1 having the maximal cost equal to 15.

- From $L^{[5]}$ it follows that $G$ has 4 Hamiltonian circuits. For example, $C_{\text{elem}}(1,1,5) = L^{[5]}_{11} = \{c^5_{1,H} = (1,5,4,3,2,1)\}$. We have $w(c^5_{1,H}) = 16$. Hence $c^5_{1,H}$ is a Hamiltonian circuit starting at vertex 1 having the maximal cost equal to 16. \qed

Conclusions. The LCDL- algorithm can be easily programmed and gives an efficient solution of EPP and ECP for a finite directed graph. This can be seen as an improved version of Kaufmann’s algorithm.

Let us list some classes of algebraic path problems, which can be reduced to applying the LCDL-algorithm:

(i) enumeration of the elementary paths (resp., circuits);
(ii) determination of the elementary paths (resp., circuits) of maximum length;
(iii) testing a graph $G$ for having Hamiltonian paths or Hamiltonian circuits;
(iv) optimization (Hamiltonian path or Hamiltonian circuit of minimal cost). \qed

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