Defects in Kitaev models and bicomodule algebras

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We construct a Kitaev lattice model, consisting of a Hamiltonian as the sum of commuting local projectors, for surfaces with boundaries and defects of dimension 0 and 1. More specifically, we show that one can consider cell decompositions of surfaces whose 2-cells are labeled by semisimple Hopf algebras, 1-cells are labeled by semisimple bicomodule algebras and 0-cells are labeled by representations of algebras which specialize to the Drinfeld double of a Hopf algebra in the absence of defects. This generalizes the standard case without defects or boundaries, where all 1-cells and 2-cells are labeled by a single Hopf algebra and where point defects are labeled by representations of its Drinfeld double. In the standard case, commuting local projectors are constructed using the Haar integral for semisimple Hopf algebras. In this paper we find that, in the presence of defects, the suitable generalization of the Haar integral is given by the unique symmetric separability idempotent for a semisimple bicomodule algebra.

1 Introduction

The Kitaev model has been constructed as a simple model for topological quantum computing, using a degenerate ground-state space as the code space and a set of commuting local projectors to correct local errors. It is also known as the quantum double model, surface code or toric code \cite{Kit, BMCA}. The algebraic input datum for such a construction is, in the simplest situation, a finite-dimensional semisimple complex Hopf algebra; for the toric code it is the group algebra of the group with two elements. The ground states of this model are described by a three-dimensional topological field theory of Turaev-Viro type \cite{BK}, which provides links to quantum topology.

On the other hand, it is interesting to consider such models not just on surfaces, but on surfaces with additional structure. In terms of physics, we want to allow for defects and boundaries; in mathematical terms, we consider the theories on a suitable class of stratified manifolds called \textit{defect surfaces} in the sense of \cite{FSS19}, but see also e.g. \cite{CMS}. (Here we study models on oriented surfaces, whereas in \cite{FSS19} surfaces with 2- framings are considered.) Defects in topological field theories are known to lead to higher-dimensional ground-state spaces and more interesting mapping class group representations of the underlying surfaces on these; see e.g. \cite{BJQ, FS, LLW}. This is, in particular, relevant for applications to topological quantum
computing, where quantum gates are implemented by mapping class group actions on the code space \([FLW]\). There have been already several approaches to include defects or boundaries in Kitaev models based on group algebras \([BK, BMD, BSW, CCW]\), but our approach deals with the more general case of semisimple Hopf algebras.

The main result of this paper is the construction of a Kitaev-type model, consisting of a commuting-projector Hamiltonian, for surfaces with general defects and boundaries, using general Hopf-algebraic and representation-theoretic input data.

For our construction it is necessary to realize the data labeling the defects, which are known for Turaev-Viro theory in a category-theoretic language, concretely in Hopf-algebraic and representation-theoretic terms. Specifically, topological field theories of Turaev-Viro type are parameterized by spherical fusion categories \([BW2]\). The data for defects separating two such theories are semisimple bimodule categories \([KK, FSV, FSS19]\). The idea for obtaining the data for a Kitaev construction is to invoke Tannaka-Krein duality \([D]\). It states that a semisimple Hopf algebra is equivalent to specifying a fusion category (the representation category of the Hopf algebra, admitting a canonical spherical structure) together with a monoidal fibre functor valued in finite-dimensional vector spaces (the forgetful functor assigning to a representation its underlying vector space). This recovers semisimple Hopf algebras as the input datum for the Kitaev models without defects, which we think of here as the labels for the two-dimensional strata of the defect surface.

We extend this idea and develop, for the bimodule categories labeling line defects on the surface in Turaev-Viro theory, the appropriate bimodule versions of fibre functors. By a bimodule version of Tannaka-Krein duality, which we explain in Subsection 2.1, this realizes these categories as the representation categories of bimodule algebras over Hopf algebras. We thus identify bimodule algebras as the labels for line defects and, as a special case, comodule algebras for boundaries.

Having established the algebraic data for line defects of the surface, we turn our attention to vertices where such line defects can join. They are labeled by objects in a category which serves as possible labels for generalized Wilson lines in a corresponding three-dimensional topological field theory, including boundary Wilson lines and Wilson lines at the intersection of surface defects. This category has been determined as a suitable generalization \([FSS14, FSS19]\) of the Drinfeld center for a spherical fusion category, which labels bulk Wilson lines. Here, in Subsection 2.3, this category is realized as a representation category as follows: For a vertex at which line defects meet, the bimodule algebras of the line defects and the algebras dual to the Hopf algebras attached to the adjacent two-dimensional strata naturally assemble into an algebra, defined in Definition \([5]\). This algebra, which in this paper we call vertex algebra, generalizes the Drinfeld double of the Hopf algebra, whose representations label point-like excitations in the Kitaev model without defects.

The category of possible labels for such a vertex is then the category of modules over this algebra.

Furthermore, a choice of cell decomposition on the underlying surface enters the construction of the Kitaev model. In the standard Kitaev model without defects, every 1-cell (or edge) of the cell decomposition is labeled by a single Hopf algebra. In our setting this should be seen as the regular bicomodule algebra and we consider this label as the transparent defect. In our case, edges of the cell decomposition are either transparently labeled or they constitute a non-trivial defect and are labeled by an arbitrary bicomodule algebra.

Our construction proceeds in the following steps – mirroring the construction of the standard Kitaev model without defects, as in e.g. \([BMCA, BK]\). We first define in Subsection 3 a vector space with local degrees of freedom for each edge and each 0-cell (or vertex) of the cell decomposition. Then we show in Subsection 3.1 that this vector space admits, locally with
respect to the cell decomposition, the structure of a bimodule over the algebras attached to the vertices. This is analogous to the representations of the Drinfeld double for each site, a pair of a vertex and an adjacent 2-cell (or plaquette), in the standard Kitaev model without defects. In this case one then proceeds to use the Haar integral for any semisimple Hopf algebra to define local projectors via these local representations. One of our main insights, established in Subsection 3.2, is that, in the presence of defects, the suitable generalization of the Haar integral to semisimple bicomodule algebras is given by the symmetric separability idempotent, see Definition 14. The symmetric separability idempotent of a semisimple algebra is unique, which we recall in Proposition 16. Furthermore, we show in Proposition 18 that for a semisimple (bi-)comodule algebra, the symmetric separability idempotent satisfies a compatibility with the (bi-)comodule structure which generalizes a basic property of the Haar integral of a semisimple Hopf algebra. In the absence of defects, the symmetric separability idempotent reduces to the Haar integral, as we show in Example 17.

Using such separability idempotents, in Subsection 3.3 we finally construct projectors for each vertex, as usual called vertex operators, and for each plaquette, as usual called plaquette operators. Our main result, Theorem 24, is that all vertex operators and plaquette operators commute – giving rise to an exactly solvable Hamiltonian defined as a sum of commuting projectors, which project to the ground states of the model.

Concerning the ground states, our construction can be seen as a concrete representation-theoretic realization of the category-theoretic construction in [FSS19]. While in [FSS19] more general categories than representation categories of Hopf algebras and bicomodule algebras are considered, for us the additional structure of fibre functors on the categories is necessary in order to define a larger vector space which contains the pre-block space and block space as subspaces. Moreover, while for the construction in [FSS19] no semisimplicity is required, in this paper semisimplicity is essential for the construction of commuting local projectors, since we define them in terms of the symmetric separability idempotents.

(See [KMS] for some progress on projectors for non-semisimple Hopf algebras.) Lastly, since semisimple Hopf algebras have an involutive antipode, they have a canonical trivial pivotal structure. Hence, we can define our model on any surface with orientation. The approach in [FSS19] is to assume no pivotal structure on the tensor categories, but instead to assume more structure, namely a 2-framing, on the surfaces.

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2 Hopf-algebraic and representation-theoretic labels for surfaces with cell decomposition

Following the discussion in the introduction, we will explain in this section the input data for our construction.
Let $\Sigma$ be a compact oriented surface together with a cell decomposition $(\Sigma^0, \Sigma^1, \Sigma^2)$ with non-empty sets of 0-cells (or vertices), 1-cells (or edges) and 2-cells (or plaquettes), respectively. This can be thought of as an embedding of a graph $(\Sigma^0, \Sigma^1)$ into $\Sigma$ such that its complement in $\Sigma$ is the disjoint union of a set $\Sigma^2$ of disks. Furthermore, let the edges be oriented, i.e. there are source and target maps $s, t : \Sigma^1 \rightarrow \Sigma^0$. If the surface $\Sigma$ has a boundary, then we require that the 1-skeleton of the cell decomposition be contained in the boundary.

For the construction of a Kitaev model one needs as a further input Hopf-algebraic and representation-theoretic data labeling the various strata of the cell decomposition. In the ordinary Kitaev model without defects as in [BMCA], all edges of the cell decomposition are labeled by a single semisimple Hopf algebra $H$, and wherever point-like excitations are considered [BK], a vertex is labeled by a representation of the Drinfeld double $D(H)$ of the Hopf algebra $H$. In this paper we consider more general labels for the edges, thereby implementing arbitrary line defects (also known as domain walls in condensed matter theory) and boundaries in the Kitaev model. Accordingly we also consider more general labels for vertices, implementing point defects (also known as point-like excitations) inside defect lines or boundaries. For the remainder of this section we will specify the three types of Hopf-algebraic and representation-theoretic data that label the plaquettes, edges and vertices of a cell decomposition.

### 2.1 Bicomodule algebras over Hopf algebras for line defects

We fix once and for all an algebraically closed field $\mathbb{k}$ of characteristic zero. For the necessary background on Hopf algebras and conventions regarding the notation, see [M] [Ka] [BMCA].

**Definition 1.**

- Let $H_1$ and $H_2$ be Hopf algebras over $\mathbb{k}$. An $H_1\text{-}H_2$-bicomodule algebra $K$ is a $\mathbb{k}$-algebra $K$ together with an $H_1\text{-}H_2$-bicomodule structure, i.e. with co-associative co-action written in Sweedler notation for comodules as

$$K \rightarrow H_1 \otimes K \otimes H_2,$$

$$k \mapsto k_{(-1)} \otimes k_{(0)} \otimes k_{(1)},$$

which is required to be a morphism of algebras. If $H_1 = \mathbb{k}$ or $H_2 = \mathbb{k}$, then $K$ is just a left or right comodule algebra, respectively.

A semisimple bicomodule algebra is a bicomodule algebra whose underlying algebra is semisimple.

- Let $\Sigma$ be an oriented surface with a cell decomposition with oriented edges. A label $H_p$ for a plaquette $p \in \Sigma^2$ is a semisimple Hopf algebra $H_p$ over $\mathbb{k}$.

For any edge $e \in \Sigma^1$ let $p_1 \in \Sigma^2$ and $p_2 \in \Sigma^2$ be the labeled plaquettes on the left and on the right of $e$, respectively, with respect to the orientation of $e$ relative to the orientation of $\Sigma$. Then a label $K_e$ for the edge $e$ is a finite-dimensional semisimple $H_{p_1}\text{-}H_{p_2}$-bicomodule algebra $K_e$ over $\mathbb{k}$.

If the edge $e$ lies in the boundary of $\Sigma$ and hence only has a plaquette $p$ on one side (left or right), then $K_e$ is just a left or right $H_p$-comodule algebra, respectively.
Examples 2.

1. Let $H$ be a Hopf algebra. The regular $H$-bimodule algebra is the algebra underlying the Hopf algebra $H$ together with left and right co-action given by the co-multiplication of $H$. Note that the regular $H$-bimodule algebra is semisimple if and only if the Hopf algebra $H$ is semisimple, since both are defined by the semisimplicity of the underlying algebra.

2. Let $G$ be a finite group and $kG$ its group algebra, which has a basis $(b_g)_{g \in G}$ parametrized by $G$ and multiplication induced by the group multiplication. $kG$ is a semisimple Hopf algebra with comultiplication given by the diagonal map $b_g \mapsto b_g \otimes b_g$ for all $g \in G$. Further, let $U \subseteq G$ be a subgroup and $\zeta \in Z^2(U, k^*)$ a group 2-cocycle. Then the cocycle-twisted group algebra $kU_{\zeta}$ with multiplication $b_u \cdot b_v := \zeta(u, v)b_{uv}$ for all $u, v \in U$ is a $kG$-comodule algebra with co-action given by the diagonal map $b_u \mapsto b_u \otimes b_u$.

Let us explain the emergence of bimodule algebras from the point of view of Tannaka-Krein duality, as outlined in the Introduction. It is well known [EGNO] that the data of a finite-dimensional Hopf algebra $H$ over $k$ is equivalent to the data of the category of finite-dimensional vector spaces. More precisely, the Hopf algebra $H$ can be reconstructed as the algebra of natural endo-transformations of the fiber functor $\omega$ and the tensor structure on the fiber functor $\omega$ induces the additional coalgebra structure on the algebra $H$, such that $A \cong H\mod$. Conversely, the category $H\mod$ of finite-dimensional left $H$-modules for any finite-dimensional Hopf algebra over $k$ is a finite $k$-linear tensor category and the forgetful functor $H\mod \to \text{vect}(k)$ is a monoidal fiber functor.

We extend this idea to bimodule categories as follows. Let $(A_1, \omega_1 : A_1 \to \text{vect}(k))$ and $(A_2, \omega_2 : A_2 \to \text{vect}(k))$ be finite $k$-linear tensor categories together with monoidal fiber functors. Consider $\text{vect}(k)$ as an $A_1$-$A_2$-bimodule category via the monoidal functors $\omega_1$ and $\omega_2$. Let $M$ be a finite $k$-linear $A_1$-$A_2$-bimodule category. Then we define a bimodule fiber functor $\omega : M \to \text{vect}(k)$ for $M$ to be an exact and faithful $A_1$-$A_2$-bimodule functor from $M$ to the category of finite-dimensional vector spaces. Let $H_1$ and $H_2$ be the corresponding finite-dimensional Hopf algebras over $k$ corresponding to $(A_1, \omega_1)$ and $(A_2, \omega_2)$. Then, by the same argument as for tensor categories mutatis mutandis, the bimodule structure on the fiber functor $\omega$ induces the structure of an $H_1$-$H_2$-bimodule algebra $K$ on the algebra of natural endo-transformations of $\omega$, such that $\omega$ induces an equivalence of bimodule categories $M \cong K\mod$.

Hence, bimodule algebras emerge naturally as algebraic input data for Kitaev models, if one equips the category-theoretic data underlying the corresponding topological field theories.
or modular functors with suitable fiber functors in order to obtain concrete Hopf-algebraic or representation-theoretic data.

2.2 Half-edges and sites

It remains to determine the possible labels for the vertices of the cell decomposition. This is the content of Subsection 2.3. Before that, in this Subsection 2.2, we first introduce suitable notation and terminology in order to extract and conveniently speak about the combinatorial information contained in the cell decomposition.

Fix a vertex \( v \in \Sigma^0 \). Then let \( \Sigma^0_v \) be the set of half-edges incident to \( v \). This is the set of incidences of an edge with the given vertex \( v \in \Sigma^0 \). (A loop at \( v \) yields two half-edges incident to \( v \).) Note that we have a map \( \Sigma^0_v \rightarrow \Sigma^1 \), assigning to any half-edge its underlying edge, which is in general not injective due to the possible existence of loops. We will denote by \( \Sigma^1_v \) its image in \( \Sigma^1 \), that is the set of edges starting or ending at the given vertex \( v \).

We will say that \( e \in \Sigma^0_v \) is directed away from \( v \) if \( v = s(e) \) and, that \( e \in \Sigma^0_v \) is directed towards \( v \) if \( v = t(e) \). Then for any half-edge \( e \in \Sigma^0_v \) incident to the vertex \( v \), let the sign \( \varepsilon(e) \in \{+1, -1\} \) be positive if the half-edge \( e \in \Sigma^0_v \) is directed away from the vertex \( v \): and negative if \( e \in \Sigma^0_v \) is directed towards \( v \):

![Figure 2: A half-edge \( e \in \Sigma^0_v \) incident to \( v \) with sign \( \varepsilon(e) := +1 \)](image)

![Figure 3: A half-edge \( e \in \Sigma^0_v \) incident to \( v \) with sign \( \varepsilon(e) := -1 \)](image)

Let \( p \in \Sigma^2 \) be the plaquette on the left of the half-edge \( e \in \Sigma^0_v \), as seen from the vertex \( v \in \Sigma^0 \), and let \( p' \in \Sigma^2 \) be the plaquette on the right, as in Figure 4.

![Figure 4: A half-edge \( e \) at \( v \) with neighboring plaquettes \( p \) and \( p' \)](image)

What we have not represented in the figure is that the half-edge \( e \) comes with an orientation, expressed by the sign \( \varepsilon := \varepsilon(e) \). By our assignment of labels, if the half-edge \( e \) is directed away from the vertex \( v \), i.e. \( \varepsilon = +1 \), then it is labeled with an \( H_p \cdot H_{p'} \)-bicomodule algebra \( K_e \), with co-action written in Sweedler notation for comodules:

\[
K_e \rightarrow H_p \otimes K_e \otimes H_{p'} \quad k \mapsto k_{(-1)} \otimes k_{(0)} \otimes k_{(1)} \quad \text{if } \varepsilon(e) = +1.
\]
If, on the other hand, the half-edge $e$ points towards $v$, that is $\varepsilon = -1$, then $K_e$ is an $H_{\rho'}^-H_p^-$-bicomodule algebra:

$$K_e \rightarrow H_{\rho'}^- \otimes K_e \otimes H_p^-, \quad k \mapsto k_{(-1)} \otimes k_{(0)} \otimes k_{(1)}$$

if $\varepsilon(e) = -1$.

We shall introduce a notation which allows us to write uniformly about the cases $\varepsilon = +1$ and $\varepsilon = -1$. Let

$$K_e^{\pm 1} := K_e, \quad K_e^{-1} := K_e^{\text{op}},$$

where $K_e^{\text{op}}$ is the algebra with opposite multiplication. Moreover, let

$$H_p^{\pm 1} := H_p, \quad H_p^{-1} := H_p^{\text{cop}}$$

where $H_p^{\text{cop}}$ is the Hopf algebra with opposite multiplication and opposite comultiplication. If $K_e$ is a left (or right, respectively) $H_p$-comodule algebra, then $K_e^{-1}$ is canonically a left (or right, respectively) $H_p^{-1}$-comodule algebra.

Hence, in both above cases we can write that $K_e^\varepsilon$ is an $H_{\rho'}^-H_{\rho'}^\varepsilon$-bicomodule algebra, with co-action in Sweedler notation:

$$K_e^\varepsilon \rightarrow H_p^\varepsilon \otimes K_e^\varepsilon \otimes H_p^\varepsilon, \quad k \mapsto k_{(-\varepsilon)} \otimes k_{(0)} \otimes k_{(\varepsilon)}.$$

Denote by $\Sigma^\text{sit}_v$ the set of sites incident to $v$. These are incidences of a plaquette $p \in \Sigma^2$ with the given vertex $v \in \Sigma^0$. Dually, for a plaquette $p \in \Sigma^2$ denote by $\Sigma^\text{sit}_p$ the set of sites incident to $p$. These are incidences of a vertex $v \in \Sigma^0$ with the given plaquette $p$. It is justified to use the name site for both notions: To any site $p \in \Sigma^\text{sit}_v$ at a vertex $v \in \Sigma^0$ corresponds a unique site $\bar{v} \in \Sigma^\text{sit}_p$ with underlying vertex $v$ at the plaquette that underlies the site $p \in \Sigma^\text{sit}_v$.

Now let $p \in \Sigma^\text{sit}_v$ be such a site at the vertex $v \in \Sigma^0$. There is a half-edge $e'_p \in \Sigma^{0,5}_v$ bounding $p$ on the left as seen from the vertex $v$ and there is a half-edge $e_p \in \Sigma^{0,5}_v$ bounding $p$ on the right. For an example consider Figure 5.

![Figure 5: A site $p \in \Sigma^\text{sit}_v$ with neighboring half-edges $e'_p$ and $e_p$.](image)

Then, in consideration of the respective signs $\varepsilon := \varepsilon(e_p)$ and $\varepsilon' := \varepsilon(e'_p)$ of the half-edges $e_p$ and $e'_p$, we have by our assignment of labels that $K_{e_p}^{\varepsilon'}$ is a right $H_p^{\varepsilon'}$-comodule algebra and that $K_{e'p}^{\varepsilon}$ is a left $H_{\rho'}^{\varepsilon'}$-comodule algebra. In other words, we have a left $((H_{\rho'}^{\varepsilon'})^{\text{cop}} \otimes H_{\rho'}^{\varepsilon})$-comodule structure on the algebra

$$K_{\{e_p,e'_p\}} := \bigotimes_{e \in \{e_p,e'_p\} \subseteq \Sigma^{0,5}_v} K_e^{\varepsilon(e)} = \begin{cases} K_{e_p}^{\varepsilon'} \otimes K_{e_p}^{\varepsilon}, & e_p \neq e'_p \in \Sigma^{0,5}_v, \\ K_{e'_p}^{\varepsilon} & e_p = e'_p \in \Sigma^{0,5}_v. \end{cases}$$

(1)
Next we introduce, for a fixed site \( p \in \Sigma^\text{sit}_v \), a canonical left \(((H_p^{e^{'}})^{\text{cop}} \otimes H_p^e)\)-module algebra, which we think of as associated to the site \( p \):

**Definition 3.** Let \( v \in \Sigma^0 \) be a vertex and \( p \in \Sigma^\text{sit}_v \) a site at \( v \) with neighboring half-edges \( e_p, e'_p \in \Sigma^0_v \) with signs \( \varepsilon, \varepsilon' \in \{+1, -1\} \) as before.

The \( \varepsilon'\cdot \varepsilon \)-balancing algebra \( H_p^{e^{'}} \), or more explicitly \((H_p^{e^{'}})^\ast\), is the left \(((H_p^{e^{'}})^{\text{cop}} \otimes H_p^e)\)-module algebra, whose underlying \( \mathbb{k} \)-algebra is the dual algebra of the Hopf algebra \( H_p \), with the following action.

\[
((H_p^{e^{'}})^{\text{cop}} \otimes H_p^e) \otimes H_p^* \longrightarrow H_p^*,
\]

where

\[
a^{(\varepsilon)} := \begin{cases} a, & \varepsilon = +1 \\ S(a), & \varepsilon = -1 \end{cases}
\]

and where \( S : H_p \longrightarrow H_p \) denotes the antipode.

Together, the \(((H_p^{e^{'}})^{\text{cop}} \otimes H_p^e)\)-comodule algebra \( K_{\{e_p, e'_p\}} \), associated to the half-edges \( e_p \in \Sigma^0_v \) and \( e'_p \in \Sigma^0_v \), and the \(((H_p^{e^{'}})^{\text{cop}} \otimes H_p^e)\)-module algebra \( H_p^* \), associated to the site \( p \in \Sigma^\text{sit}_v \) situated between the edges \( e_p \) and \( e'_p \), can be **coupled** into a single \( \mathbb{k} \)-algebra, denoted by

\[
H_p^* \otimes K_{\{e_p, e'_p\}}
\]

which has underlying vector space \( H_p^* \otimes K_{\{e_p, e'_p\}} \) and which is an instance of the following general construction. For related constructions see [M].

**Definition 4.** Let \( H \) be a Hopf algebra over \( \mathbb{k} \), let \( A \) be a left \( H \)-module algebra and let \( K \) be a left \( H \)-comodule algebra. Then the **crossed product algebra** \( A \otimes K \) is the \( \mathbb{k} \)-algebra with underlying vector space \( A \otimes K \) and multiplication

\[
(a \otimes k) \cdot (a' \otimes k') := a(k_{(-1)} a', k_{(0)} k') \quad \text{for} \quad (a \otimes k), (a' \otimes k') \in A \otimes K.
\]

In particular, the algebra \( H_p^* \otimes K_{\{e_p, e'_p\}} \) contains \( H_p^* \) and \( K_{\{e_p, e'_p\}} \) as subalgebras and the commutation relation between these is

\[
k \cdot f = f(k_{(e'_p)} \cdot k_{(-e^{'})}) \cdot k_{(0)} \quad \forall f \in H_p^*, k \in K_{\{e_p, e'_p\}};
\]

the so-called **straightening formula**. This generalizes the straightening formula of the Drinfeld double of a Hopf algebra, see Example 6.

### 2.3 Vertex algebras and their representations as labels for vertices

In this subsection we introduce, for each vertex \( v \in \Sigma^0 \), an algebra over \( \mathbb{k} \), which is constructed from the algebraic labeling in the neighbourhood of the vertex \( v \). The representations of this algebra will serve as possible labels for the vertex \( v \). In a corresponding three-dimensional topological field theory these are the possible labels for generalized Wilson lines.

Let us collect the algebras \( K_e^{e'(v)} \) of all half-edges \( e \in \Sigma^0_v \) incident to the vertex \( v \in \Sigma^0 \) into a tensor product

\[
K_{\Sigma^0} := \bigotimes_{e \in \Sigma^0_v} K_e^{e'(v)}.
\]

With the notation of the previous subsection, for each site \( p \in \Sigma^\text{sit}_v \) with neighboring half-edges \( e_p \) and \( e'_p \) as in Figure 5 the algebra \( K_{\{e'_p, e_p\}} \) is a left comodule over

\[
(H_p^{e'_p})^{\text{cop}} \otimes H_p^{e_p}.
\]

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This trivially extends to an \( ((H_p^{e_p})^{\text{cop}} \otimes H_p^{e_p}) \)-comodule structure on the tensor product \( K_{\Sigma_v} \) of \( K_{v} \) with the algebras attached to the remaining half-edges in \( \Sigma_v \). The co-actions on \( K_{\Sigma_v} \) for different sites commute with each other, because they come from the bicomodule structures of the tensor factors \( (K_e)_{e \in \Sigma_v} \), making \( K_{\Sigma_v} \) a left comodule algebra over the tensor product of Hopf algebras

\[
\bigotimes_{p \in \Sigma_v^{\text{sit}}} (H_p^{e_p})^{\text{cop}} \otimes H_p^{e_p}.
\]

For each site \( p \in \Sigma_v^{\text{sit}} \) we want to couple the balancing algebra \( H_p^* \) to \( K_{\Sigma_v} \), similarly as in \( (2) \). For this we collect the balancing algebras of the sites around the vertex \( v \) into a tensor product

\[
H_{\Sigma_v}^* := \bigotimes_{p \in \Sigma_v^{\text{sit}}} H_p^*.
\]

This is a left module algebra over the tensor product of Hopf algebras as in \( (4) \). Now we have all the ingredients to introduce:

**Definition 5.** Let \( v \in \Sigma \). The \( \mathbf{k} \)-algebra \( C_v \) associated to the vertex \( v \), or vertex algebra, is defined as follows. For any site \( p \in \Sigma_v^{\text{sit}} \) denote by \( e_p \) and \( e_p \in \Sigma_v^{0.5} \) the half-edges bounding \( p \) on the left and on the right, respectively, from the perspective of the vertex \( v \), as illustrated in Figure 5. Then let

\[
C_v := H_{\Sigma_v}^* \otimes K_{\Sigma_v} = \left( \bigotimes_{p \in \Sigma_v^{\text{sit}}} H_p^* \right) \otimes \left( \bigotimes_{e \in \Sigma_v^{0.5}} K_e^{e(e)} \right)
\]

be the crossed product algebra, as introduced in Definition \( (4) \) for the left module algebra \( H_{\Sigma_v}^* \) and the left comodule algebra \( K_{\Sigma_v} \) over the tensor product \( (4) \) of Hopf algebras.

In particular, the algebra contains \( H_{\Sigma_v}^* = \bigotimes_{p \in \Sigma_v^{\text{sit}}} H_p^* \) and \( K_{\Sigma_v} = \bigotimes_{e \in \Sigma_v^{0.5}} K_e^{e(e)} \) as subalgebras and, for each site \( p' \in \Sigma_v^{\text{sit}} \), we have the commutation relation \( (3) \); so in other words,

\[
H_{p'}^* \otimes K_{\{e_p, e_{p'}\}} \subseteq \left( \bigotimes_{p \in \Sigma_v^{\text{sit}}} H_p^* \right) \otimes \left( \bigotimes_{e \in \Sigma_v^{0.5}} K_e^{e(e)} \right) = C_v
\]

is a subalgebra of \( C_v \).

**Example 6.** Let us consider the situation where the vertex \( v \in \Sigma \) has precisely one half-edge \( e \), which is directed away from the vertex and which is labeled by the regular \( H \)-bicomodule algebra \( H \), the transparent label.

\[\text{Figure 6: A vertex } v \text{ with a single half-edge transparently labeled by } H; \]
the associated algebra \( C_v \) is the Drinfeld double \( D(H) \)

Then for the algebra \( C_v \) at the vertex \( v \) we have \( H_{\Sigma_v}^* \otimes K_{\Sigma_v} = H^* \otimes H \) and the commutation relation \( (3) \) gives

\[
h \cdot f = f(S(h_{(3)}) \cdot h_{(1)}) \cdot h_{(2)}.
\]

This is precisely the so-called straightening formula of the Drinfeld double \( D(H) \) of a semisimple Hopf algebra \( H \) \( \text{[Ka]} \). In the Kitaev model without defects as in \( \text{[BMCA] [BK]} \), representations of the Drinfeld double \( D(H) \) label point-like excitations.
Up to this point we have explained how, for a given vertex \( v \in \Sigma^0 \), the algebraic labeling of the edges and plaquettes and the combinatorial structure of the cell decomposition around that vertex gives rise to the \( k \)-algebra \( C_v = H_{\Sigma^1}^* \otimes K_{\Sigma^0}^* \).

**Definition 7.** We declare the category of possible labels for a vertex \( v \in \Sigma^0 \) for the Kitaev construction to be the \( k \)-linear category \( C_v \text{-mod} \) of finite-dimensional left modules over the \( k \)-algebra \( C_v \).

Indeed, in [FSS19], the category-theoretic data assigned to a vertex \( v \in \Sigma^0 \) is as follows. In the language of [FSS19], a vertex \( v \) corresponds to a boundary circle \( L_v \) with marked points on which defect lines end. A plaquette \( p \in \Sigma^2 \) is labeled by a finite tensor category; in our context this is the representation category \( H_p \text{-mod} \) of a finite-dimensional Hopf algebra \( H_p \). An edge \( e \in \Sigma^1 \) is labeled by a finite bimodule category; in our context this is the representation category \( K_e \text{-mod} \) of a bicomodule algebra \( K_e \). Then according to [FSS19] Definitions 3.4 and 3.9 the category of possible labels of a vertex \( v \in \Sigma^0 \) is given by the category \( T(L_v) \) of so-called balancings on the Deligne tensor product \( \boxtimes_{e \in \Sigma^1} (K_e(e) \text{-mod}) \) of the bimodule categories labeling the half-edges around the vertex \( v \). There is a canonical equivalence of \( k \)-linear categories

\[
T(L_v) \cong C_v \text{-mod}
\]

between the category assigned by the modular functor \( T \), constructed in [FSS19], to the circle \( L_v \) with marked points corresponding to the half-edges incident to \( v \) and the representation category of the algebra \( C_v \). We relegate the detailed proof of this statement to an upcoming updated version of the present paper.

Furthermore, in the case that the edges incident to the vertex \( v \) are labeled transparently by a single Hopf algebra \( H \) seen as the regular \( \Box \)-bicomodule algebra, then the category \( C_v \text{-mod} \) is equivalent to the Drinfeld center \( Z(H \text{-mod}) \) [FSS19] Remarks 3.5 (iii) and 5.23, which is equivalent to the category of representations of the Drinfeld double \( D(H) \). These are also the possible labels for point-like excitations in the Kitaev model without defects, cf. [BK].

### 3 Construction of a Kitaev model with defects

Having specified in the preceding subsections the algebraic input data for the Kitaev model and, in particular, having identified the possible labels for vertices, we are now in a position to construct, for any oriented surface \( \Sigma \) with labeled cell decomposition, the vector space and local projectors of the model.

We recall that we have for each plaquette \( p \in \Sigma^2 \) a semisimple Hopf algebra \( H_p \), for each edge \( e \in \Sigma^1 \) a semisimple algebra \( K_e \) with a compatible bicomodule structure over the Hopf algebras of the incident plaquettes, and for each vertex \( v \in \Sigma^0 \) a left module \( Z_v \) over the algebra \( C_v = H_{\Sigma^1}^* \otimes K_{\Sigma^0}^* \), introduced in Definition 5. We abbreviate

\[
K_{\Sigma^1} := \bigotimes_{e \in \Sigma^1} K_e, \quad Z_{\Sigma^0} := \bigotimes_{v \in \Sigma^0} Z_v,
\]

for the tensor products as vector spaces over \( k \). More precisely, \( K_{\Sigma^1} \) enters our construction of the local projectors and the Hamiltonian of the model not only as a vector space, but together with its structure as the regular \( \bigotimes_{e \in \Sigma^1} K_e \)-bicomodule and its various co-actions with respect to the Hopf algebras labeling the plaquettes. Similarly, we will regard \( Z_{\Sigma^0} \) together with its \( C_v \)-module structure for every vertex \( v \in \Sigma^0 \).

The first thing we construct is the vector space, on which subsequently the local commuting projectors and the Hamiltonian will be defined.
each site is therefore fully determined by a $H$ is the tensor product of the algebras

$$\Delta \in C$$ is the co-multiplication pulled back to a left $C$ of $C$ of $C$.

In general not Hopf algebras and we only obtain a semisimple Hopf algebra $D$.

Next, we exhibit on the vector space $k$.

3. Note that we are only defining a vector space over $k$, and not a Hilbert space, i.e. we do not consider a scalar product here. Accordingly, when we speak of projectors on this vector space we always mean idempotent endomorphisms. By a Hamiltonian we mean a diagonalizable endomorphism.

3.1 Local representations of the vertex algebras on the state space

Next, we exhibit on the vector space $\mathcal{H}$ a natural $C_v$-bimodule structure for each vertex $v \in \Sigma^0$, that is local in the sense that it acts non-trivially only on the local degrees of freedom in a neighborhood of the vertex $v \in \Sigma^0$. This is analogous to the existence of local actions of the Drinfeld double $D(H)$ on the state space in the ordinary Kitaev model without defects for a semisimple Hopf algebra $H$. In our construction, however, the algebras $C_v$ are in general not Hopf algebras and we only obtain bimodule structures on $\mathcal{H}$. (A $C_v$-bimodule structure is equivalent to a left $(C_v \otimes C_v^{\text{op}})$-action, where $C_v^{\text{op}}$ has the opposite multiplication of $C_v$. Whenever $C_v$ is a Hopf algebra, such as $D(H)$, any $C_v$-bimodule structure can be pulled back to a left $C_v$-action via the algebra map $(\text{id} \otimes S) \circ \Delta : C_v \to C_v \otimes C_v^{\text{op}}$, using the co-multiplication $\Delta$ and the antipode $S$ of the Hopf algebra.)

Let $v \in \Sigma^0$ be any vertex. Recall from Subsection 2.3 that the algebra

$$C_v = H_{\Sigma^0}^{v,\text{sit}} \otimes K_{\Sigma^0,5}$$

is a crossed product of $H_{\Sigma^0}^{v,\text{sit}}$ and $K_{\Sigma^0,5}$ and contains these as subalgebras, and that

$$H_{\Sigma^0}^{v,\text{sit}} = \bigotimes_{p \in \Sigma^0} H_p^*$$

is the tensor product of the algebras $H_p^*$ for each site $p \in \Sigma^0$. A $C_v$-bimodule structure on $\mathcal{H}$ is therefore fully determined by a $K_{\Sigma^0,5}$-bimodule structure and $H_p^*$-bimodule structures for each site $p \in \Sigma^0$, provided that for each $p \in \Sigma^0$ the left and right actions of $K_{\Sigma^0,5}$ and $H_p^*$
each satisfy the straightening formula \(3\) of the crossed product algebra \(H_p^* \otimes K_{\Sigma_0^p}\), which we prove in Theorem 12.

We start by exhibiting a \(K_{\Sigma_0^p}\)-bimodule structure on the vector space \(H\). This is the analogon of the action of the Hopf algebra \(H\) for every vertex in the ordinary Kitaev model for a semisimple Hopf algebra \(H\).

**Definition 10.** Let \(v \in \Sigma^0\). The \(K_{\Sigma_0^v}\)-bimodule structure on \(H\)

\[
\tilde{A}_v : K_{\Sigma_0^v} \otimes K_{\Sigma_0^v}^\text{op} \otimes \mathcal{H} \longrightarrow \mathcal{H},
\]

is defined on the vector space of linear maps \(\mathcal{H} = \text{Hom}(K_{\Sigma_1}, Z_{\Sigma_0})\) in the standard way by pre-composing with the left action on \(K_{\Sigma_1}\) and post-composing with the left action on \(Z_{\Sigma_0}\), which are defined as follows:

- Firstly, the vector space \(K_{\Sigma_1}\) becomes a left \(K_{\Sigma_0^v}\)-module as follows. Restrict the regular \(K_{\Sigma_1}\)-bimodule structure of \(K_{\Sigma_1}\), seen as a left \((K_{\Sigma_1} \otimes K_{\Sigma_1}^\text{op})\)-action, to the subalgebra \(K_{\Sigma_0^v} \subseteq K_{\Sigma_1} \otimes K_{\Sigma_1}^\text{op}\).

- Secondly, the vector space \(Z_{\Sigma_0}\) becomes a left \(K_{\Sigma_0^v}\)-module as follows. Restrict the given \(C_v\)-module structure on \(Z_v\) to the subalgebra \(K_{\Sigma_0^v} \subseteq \bigotimes_{v \in \Sigma^0}(H_{\Sigma_0^v}^* \otimes K_{\Sigma_0^v}) = C_v\) and extend the action trivially to the vector space \(Z_{\Sigma_0} = Z_v \otimes \bigotimes_{w \in \Sigma^0 \setminus \{v\}} Z_w\).

Next we will exhibit, for any site \(p \in \Sigma_\text{sit}^v\) incident to a vertex \(v \in \Sigma^0\), an \(H_p^*\)-bimodule structure on \(\mathcal{H}\).

Recall that \(\Sigma_\text{sit}^p\) denotes the set of incidences of a vertex with a given plaquette \(p\) (which we also call *sites*) and denote by \(\Sigma_\text{pl}^1\) the set of incidences of an edge with the given plaquette \(p\) (which we call *plaquette edges*). We consider their union \(\Sigma_\text{sit}^p \cup \Sigma_\text{pl}^1\) together with a cyclic order on it, given by the clockwise direction along the boundary of \(p\) with respect to the orientation of \(\Sigma\), as illustrated in Figure 7.

![Figure 7: Cyclic order on the set \(\Sigma_\text{sit}^p \cup \Sigma_\text{pl}^1\) of sites and plaquette edges of a plaquette \(p\)](image)

Furthermore, for any plaquette edge \(e \in \Sigma_\text{pl}^1\) at the plaquette \(p\), let the sign \(\varepsilon_p(e) \in \{+1, -1\}\) be positive if the plaquette edge \(e \in \Sigma_\text{pl}^1\) is clockwise directed around the plaquette \(p\):

![Figure 8: A plaquette edge \(e\) with sign \(\varepsilon_p(e) := +1\)](image)

and negative if \(e \in \Sigma_\text{pl}^1\) is directed counter-clockwise around \(p\):

![Figure 9: A plaquette edge \(e\) with sign \(\varepsilon_p(e) := -1\)](image)
Recall that, attached to each plaquette $p \in \Sigma^2$, there is a Hopf algebra $H_p$. Now, depending on choice of a site $v \in \Sigma^\text{sit}_p$ at $p$, we define an $H^*_p$-bimodule structure on the vector space $\mathcal{H}$. This is the analogon of the action of the dual Hopf algebra $H^*$ for every site in the ordinary Kitaev model for a semisimple Hopf algebra $H$.

**Definition 11.** Let $p \in \Sigma^2$. We define, for each site $v \in \Sigma_p^\text{sit}$, the $H^*_p$-bimodule structure on $\mathcal{H}$, or left action of the enveloping algebra $H^*_p \otimes (H^*_p)^{\text{op}}$, 

$$\tilde{B}_{(p,v)} : H^*_p \otimes (H^*_p)^{\text{op}} \otimes \mathcal{H} \longrightarrow \mathcal{H},$$

by the following left and right $H^*_p$-actions on $\mathcal{H}$.

- We start by declaring that $H^*_p$ acts from the left on $\mathcal{H} = (\bigotimes_{e \in \Sigma^1} K^*_e) \otimes (\bigotimes_{w \in \Sigma^0} Z_w)$ by the action of $H^*_p \subseteq H^*_{\Sigma^0} \otimes K_{\Sigma^0.5}$ on the $(H^*_{\Sigma^0} \otimes K_{\Sigma^0.5})$-module $Z_v$ and by acting as the identity on the remaining tensor factors of $\mathcal{H}$.

- For the right action of $H^*_p$ on $\mathcal{H}$, we use the total order on the set $(\Sigma^\text{sit}_p \cup \Sigma^1_p) \setminus \{v\}$ starting right after $v \in \Sigma^\text{sit}_p$ in $\Sigma^\text{sit}_p \cup \Sigma^1_p$ with respect to the cyclic order declared above, given by the clockwise direction around the plaquette $p$. We first exhibit individual right $H^*_p$-actions on the tensor factors of $(\bigotimes_{e \in \Sigma^1_p} K^*_e) \otimes (\bigotimes_{w \in \Sigma^0 \setminus \{v\}} Z_w)$:

  - For any $e \in \Sigma^1_p$, the vector space $K^*_e$ becomes a right $H^*_p$-module as follows. $K^*_e$ is a right $H^*_p\varepsilon_{\varepsilon}\varepsilon(e)$-comodule and, hence, a left $(H^*_p)^{\varepsilon_{\varepsilon}}(e)$-module. Thus the vector space dual $K^*_e$ becomes a right $(H^*_p)^{\varepsilon_{\varepsilon}(e)}$-module, and finally, by pulling back along the algebra isomorphism $\phi^{\varepsilon_{\varepsilon}(e)} : H^*_p \rightarrow H^*_p\varepsilon_{\varepsilon}(e)$, a right $H^*_p$-module.

Recall that $\phi^{(\varepsilon_{\varepsilon}(e))} \overset{\text{def}}{=} \text{id}_{H^*_p}$ and $\phi^{(\varepsilon_{\varepsilon}(e))} \overset{\text{def}}{=} S$, the antipode of $H^*_p$. Explicitly, this right $H^*_p$-action is given by

$$K^*_e \otimes H^*_p \rightarrow K^*_e,$$

$$\varphi \otimes f \longmapsto \left( k \mapsto \varphi \left( k(0) f \left( k^{(\varepsilon_{\varepsilon}(e))} \right) \right) \right),$$

- For any $w \in \Sigma^\text{sit}_p \setminus \{v\}$, the vector space $Z_w$ becomes a right $H^*_p$-module as follows. The $(H^*_{\Sigma^0} \otimes K_{\Sigma^1})$-module $Z_w$ comes with a left $H^*_p$-action since $H^*_p \subseteq H^*_{\Sigma^0} \otimes K_{\Sigma^1}$ is a subalgebra. We let $H^*_p$ act on $Z_w$ from the right by pulling back this left action along the antipode $\phi = S : H^*_p \rightarrow H^*_p$.

Then we declare $H^*_p$ to act from the right on the tensor product $(\bigotimes_{e \in \Sigma^1_p} K^*_e) \otimes (\bigotimes_{w \in \Sigma^0 \setminus \{v\}} Z_w)$ by applying the co-multiplication on $H^*_p$ suitably many times and then acting individually on the tensor factors in the sequence given by the image of the clockwise linear order that we have prescribed on the set $(\Sigma^\text{sit}_p \cup \Sigma^1_p) \setminus \{v\}$ under the map $(\Sigma^\text{sit}_p \cup \Sigma^1_p) \setminus \{v\} \rightarrow (\Sigma^\text{sit}_p \cup \Sigma^1_p) \setminus \{v\}$ that assigns to a site its underlying vertex and to a plaquette edge its underlying edge. Finally, this gives a right $H^*_p$-action on $\mathcal{H} = (\bigotimes_{e \in \Sigma^1_p} K^*_e) \otimes (\bigotimes_{w \in \Sigma^0} Z_w)$ by acting with the identity on all remaining tensor factors.

So far we have defined, in Definitions 10 and 11, on the vector space $\mathcal{H}$ an $K_{\Sigma^0.5}$-bimodule structure $\tilde{A}_v$ for each vertex $v \in \Sigma^0$ and an $H^*_p$-bimodule structure $\tilde{B}_{(p,v)}$ for each site $p \in \Sigma^\text{sit}_v$. These are analogous to the actions of the Hopf algebra $H$ and the dual Hopf algebra $H^*$ defined for each site in the ordinary Kitaev model without defects. Just as the latter are shown to interact with each other non-trivially, giving a representation of the Drinfeld double $D(H)$ at each site $\text{BMCA}$, we will now proceed to study how the bimodule structures $\tilde{A}_v$ and $\tilde{B}_{(p,v')}$ of $K_{\Sigma^0.5}$ and $H^*_p$ for various $v$ and $(p, v')$ interact with each other.
In order to simplify the proof we will make a certain regularity assumption on the cell decomposition of the surface \( \Sigma \): We call a cell decomposition regular if it has no looping edges, i.e. there is no edge which has the same source vertex as target vertex and if the Poincaré-dual cell decomposition also has no looping edges, i.e. in the original cell decomposition there is no plaquette that has two incidences with one and the same edge (on its two sides).

**Theorem 12.** Let \( \mathcal{H} \) be the vector space defined in Definition 3 for an oriented surface \( \Sigma \) with a labeled cell decomposition. Recall from Definitions 10 and 11 the \( K_{\Sigma^{0,5}} \)-bimodule structure \( \tilde{A}_v \) on \( \mathcal{H} \) for every vertex \( v \in \Sigma^0 \), and the \( H^*_p \)-bimodule structure \( \tilde{B}_{(p,v)} \) on \( \mathcal{H} \) for every plaquette \( p \in \Sigma^2 \) together with incident site \( v' \in \Sigma^{0,\text{eff}}_v \). Then

- For any pair of vertices \( v_1 \neq v_2 \in \Sigma^0 \), the actions \( \tilde{A}_{v_1} \) and \( \tilde{A}_{v_2} \) commute with each other.
- For any pair of sites \((p_1 \in \Sigma^2, v_1 \in \Sigma^{0,\text{eff}}_{p_1})\) and \((p_2 \in \Sigma^2, v_2 \in \Sigma^{0,\text{eff}}_{p_2})\) such that \( p_1 \neq p_2 \), the actions \( \tilde{B}_{(p_1,v_1)} \) and \( \tilde{B}_{(p_2,v_2)} \) commute with each other.
- Assume that the cell decomposition of \( \Sigma \) is regular. For any site \((p \in \Sigma^2, v \in \Sigma^{0,\text{eff}}_{p})\), the actions \( \tilde{A}_v \) and \( \tilde{B}_{(p,v)} \) compose to give on \( \mathcal{H} \) a bimodule structure over the crossed product algebra \( H^*_p \otimes K_{\Sigma^{0,5}} \),

\[
\tilde{B}_{(p,v)} \tilde{A}_v : H^*_p \otimes K_{\Sigma^{0,5}} \otimes (H^*_p \otimes K_{\Sigma^{0,5}})^{\text{op}} \otimes \mathcal{H} \rightarrow \mathcal{H},
\]

\[
f \otimes k \otimes f' \otimes k' \otimes x \mapsto \tilde{B}_{(p,v)}^{f \otimes f'} \tilde{A}_v^{k \otimes k'}(x).
\]

**Proof.**

- The left \( K_{\Sigma^{0,5}} \) and \( K_{\Sigma^{0,5}} \)-actions act as the identity on all tensor factors of \( \mathcal{H} \) except on \( Z_{v_1} \) and \( Z_{v_2} \), respectively. It is thus clear that they commute for \( v_1 \neq v_2 \).

The right \( K_{\Sigma^{0,5}} \) and \( K_{\Sigma^{0,5}} \)-actions only have a common tensor factor on which they do not act by the identity for every edge \( e \in \Sigma^1 \) that joins the vertices \( v_1 \) and \( v_2 \). Such an edge is directed away from one of the vertices and directed towards the other. Hence, the action for one of the vertices comes from left multiplication of \( K_e \) and the other one from right multiplication, so they commute.

- The left \( H^*_p \) and \( H^*_p \)-actions act as the identity on all tensor factors of \( \mathcal{H} \) except on \( Z_{v_1} \) and \( Z_{v_2} \), respectively. It is thus clear that they commute for \( v_1 \neq v_2 \). In the remaining case \( v_1 = v_2 = v \), \( H^*_p \) and \( H^*_p \) are commuting subalgebras in \( C_v \). Since their actions on \( Z_v \) are by Definition 11 the restrictions of the \( C_v \)-action that \( Z_v \) comes with, they must therefore commute.

The right \( H^*_p \) and \( H^*_p \)-actions only have a common tensor factor on which they do not act by the identity for every vertex \( v \in \Sigma^0 \) and for every edge \( e \in \Sigma^1 \) that lies in the boundaries of both plaquettes \( p_1 \) and \( p_2 \). For any such vertex \( v \), the two actions come from the \( (H^*_p \otimes K_{\Sigma^{0,5}}) \)-action on \( Z_v \) restricted to the two subalgebras \( H^*_p \) and \( H^*_p \), respectively. These subalgebras commute inside \( H^*_p \otimes K_{\Sigma^{0,5}} \), therefore showing the claim.

- The left \( K_{\Sigma^{0,5}} \) and \( K_{\Sigma^{0,5}} \)-actions on \( \mathcal{H} \) are simply the restrictions of the left \( C_e \)-action on \( Z_v \) to \( K_{\Sigma^{0,5}} \) and \( H^*_p \), respectively, and the identity on all other tensor factors of \( \mathcal{H} \). Hence, by construction they satisfy the commutation relations of the crossed product algebra \( H^*_p \otimes K_{\Sigma^{0,5}} \subseteq C_v \), see also (5).

The right \( K_{\Sigma^{0,5}} \) and \( H^*_p \)-actions on \( \mathcal{H} \) are non-trivial only on the tensor factors \( \bigotimes_{e \in \Sigma^1} K_e \) and \( (\bigotimes_{e \in \Sigma^1} K_e^*) \otimes (\bigotimes_{w \in \Sigma^0 \setminus \{v\}} Z_w) \), respectively. We can therefore restrict our attention
to the vector space \((\bigotimes_{e \in \Sigma_0 \cup \Sigma_1} K_e^*) \otimes (\bigotimes_{w \in \Sigma_0^p \setminus \{v\}} Z_w)\), on which \(K_{\Sigma_0^p}\) and \(H_p^*\) act from the right.

For convenience, for the remainder of the proof we now switch to the dual vector space \((\bigotimes_{e \in \Sigma_0 \cup \Sigma_1} K_e) \otimes (\bigotimes_{w \in \Sigma_0^p \setminus \{v\}} Z_w^*)\), with the corresponding left actions of \(K_{\Sigma_0^p}\) and \(H_p^*\). With the notation of Subsection 2.2 let \(e_p, e_p' \in \Sigma_0^p\) be the half-edges at \(v\) on the two sides of the site \(p \in \Sigma_{\text{sit}}\), with signs \(\varepsilon := \varepsilon(e_p)\) and \(\varepsilon' := \varepsilon(e_p')\). The \(K_{\Sigma_0^p}\)- and \(H_p^*\)-actions only overlap on the tensor factors \((K_e)_{e \in \Sigma_0 \cup \Sigma_1}\) corresponding to the edges underlying the half-edges \(e_p, e_p' \in \Sigma_0^p\). Due to our regularity assumption on the cell decomposition, the half-edges \(e_p\) and \(e_p'\) have distinct underlying edges. Then the action of \(\tilde{K}_{\Sigma_0^p} = (K_{e_p}^\varepsilon \otimes K_{e_p'}^{\varepsilon'}) \otimes \bigotimes_{e \in \Sigma_0^p \setminus \{e_p,e_p'\}} K_e(\varepsilon)\) on \(\bigotimes_{e \in \Sigma_1} K_e\), which is a tensor product of algebras, decomposes into a tensor product of the action of \(K_{e_p}^\varepsilon \otimes K_{e_p'}^{\varepsilon'}\) on \(K_e(\varepsilon)\) on the latter vector space, \(H_p^*\) does not act non-trivially by our regularity assumption on the cell decomposition. Hence, it remains to consider the interactions of the left actions of \(K_{e_p}^\varepsilon \otimes K_{e_p'}^{\varepsilon'}\) and \(H_p^*\) on the vector space \(K_{e_p} \otimes K_{e_p'} \otimes \bigotimes_{e \in \Sigma_0^p \setminus \{e_p,e_p'\}} K_e(\varepsilon)\). We abbreviate by \(V := (\bigotimes_{e \in \Sigma_0^p \setminus \{e_p,e_p'\}} K_e) \otimes (\bigotimes_{w \in \Sigma_0^p \setminus \{v\}} Z_w^*)\) the tensor factor on which only \(H_p^*\) acts non-trivially. Furthermore, without loss of generality, we write the left \(H_p\)-action on \(V\) in terms of the Sweedler notation for the corresponding right \(H_p\)-coaction, \(V \rightarrow V \otimes H_p, v \mapsto v(0) \otimes v(1)\):

\[
H_p^* \otimes V \rightarrow V, v \mapsto f.v := f(v(1))v(0).
\]

Finally, it is left to analyze the interaction between the \(H_p^*\)-action

\[
H_p^* \otimes K_{e_p} \otimes K_{e_p'} \otimes V \rightarrow K_{e_p} \otimes K_{e_p'} \otimes V,
\]

\[
f \otimes x \otimes x' \otimes v \mapsto f(3).x \otimes f(1).x' \otimes f(2).v
\]

\[
= f \left( x'(e') v(1) x(-e) \right) x(0) \otimes x'(0) \otimes v(0),
\]

and the \((K_{e_p}^\varepsilon \otimes K_{e_p'}^{\varepsilon'})\)-action

\[
(K_{e_p}^\varepsilon \otimes K_{e_p'}^{\varepsilon'}) \otimes K_{e_p} \otimes K_{e_p'} \otimes V \rightarrow K_{e_p} \otimes K_{e_p'} \otimes V,
\]

\[
a \otimes a' \otimes x \otimes x' \otimes v \mapsto a.x \otimes a'.x' \otimes v
\]

\[
(a \cdot e x) \otimes (a' \cdot e' x') \otimes v,
\]

where \(\cdot e\) and \(\cdot e'\) denote the multiplication in \(K_{e_p}^\varepsilon\) and \(K_{e_p'}^{\varepsilon'}\), respectively, that is

\[
a \cdot e x := \begin{cases} ax, & \varepsilon = +1, \\
xa, & \varepsilon = -1.\end{cases}
\]

It remains to show that that these actions satisfy the straightening formula

\[
f(a'(\varepsilon') \cdot a(\varepsilon)).(a(0) \otimes a'(0)).(x \otimes x' \otimes v) = (a \otimes a').f.(x \otimes x' \otimes v),
\]

for all \(f \in H_p^*, a \otimes a' \in K_{e_p}^\varepsilon \otimes K_{e_p'}^{\varepsilon'}\) and \(x \otimes x' \otimes v \in K_{e_p} \otimes K_{e_p'} \otimes V\). Indeed, the following calculation, which is analogous to the calculation in the proof of [BMCA] Theorem 1 but more general and at the same time shorter, verifies this.

\[
f\left( a'(\varepsilon') \cdot a(\varepsilon) \right). (a(0) \otimes a'(0)) .(x \otimes x' \otimes v)
\]
Let $\ell.M$ be the central idempotent which projects to the integrals of $3.1$ to define commuting local projectors on the vector space $\mathcal{H}$. Before we proceed to use the bimodule structures on the state space together, we thus get, due to Theorem 12 on $\mathcal{H}$ a bimodule structure over the vertex algebra $C_v$. It is remarkable that this makes the crossed product algebra structure on $C_v$ show up naturally – analogous to the appearance of the algebra structure of the Drinfeld double in the commutation relation of the vertex and plaquette actions in the standard Kitaev model without defects.

\[
\begin{align*}
&= f\left(a'(\varepsilon') \cdot (a(x)) \cdot (a'(\varepsilon') \cdot (a(x)) \otimes v) \\
&= f\left(a'(\varepsilon') \cdot (a(0) \cdot (a(x)) \cdot (a'(\varepsilon') \cdot (a(x))) \otimes v) \\
&= f\left(a'(\varepsilon') \cdot (a'(\varepsilon') \cdot (a(0) \cdot (a(x)) \cdot (a'(\varepsilon') \cdot (a(x))) \otimes v) \\
&= f\left(x'(\varepsilon') \cdot (a(0) \cdot (a(x)) \cdot (a'(\varepsilon') \cdot (a(x))) \otimes v) \\
&= (a \otimes a'). \left(f\left(x'(\varepsilon') \cdot (a(0) \cdot (a'(\varepsilon') \cdot (a(x))) \otimes v)\right) \\
&= (a \otimes a'). \left(f.(x \otimes x' \otimes v)\right).
\end{align*}
\]

This proves that $H_p^*$ and $K_{r_p}^\varepsilon \otimes K_{r_p}^\varepsilon$ together give a representation of the crossed product algebra $H_p^* \otimes (K_{r_p}^\varepsilon \otimes K_{r_p}^\varepsilon)$, as claimed.

\[\square\]

**Remark 13.** Taking all sites $p \in \Sigma_v^{\text{st}}$ around a given vertex $v \in \Sigma^0$ together, we thus get, due to Theorem 12 on $\mathcal{H}$ a bimodule structure over the vertex algebra $C_v$. It is remarkable that this makes the crossed product algebra structure on $C_v$ show up naturally – analogous to the appearance of the algebra structure of the Drinfeld double in the commutation relation of the vertex and plaquette actions in the standard Kitaev model without defects.

### 3.2 Towards local projectors: Symmetric separability idempotents for bimodule algebras

Before we proceed to use the bimodule structures on the state space $\mathcal{H}$ defined in Subsection 3.3 to define commuting local projectors on the vector space $\mathcal{H}$, we need to invoke another algebraic ingredient.

The standard Kitaev construction for a semisimple Hopf algebra $H$ makes use of the Haar integrals of $H$ and of $H^*$, in order to define commuting local projectors on the state space via the actions of $H$ and $H^*$. The Haar integral of a semisimple Hopf algebra $H$ over $\mathbb{k}$ is the unique element $\ell \in H$ satisfying $x\ell = \varepsilon(x)\ell = \ell x$ for all $x \in H$ and $\varepsilon(\ell) = 1$. This means that $\ell$ is the central idempotent which projects to the $H$-invariants: for any $H$-module $M$, we have $\ell.\mathcal{M} = \mathcal{M}^H := \{m \in \mathcal{M} \mid h.m = \varepsilon(h)m \quad \forall h \in H\}$. Furthermore, $\ell \in H$ is cocommutative, i.e. $\ell(1) \otimes \ell(2) = \ell(2) \otimes \ell(1)$ in Sweedler notation. The idempotence, centrality and cocommutativity of the Haar integral are crucial in showing that the Haar integral gives rise to commuting local projectors in the standard Kitaev construction [BMCA].

In our setting, instead of a semisimple Hopf algebra acting on the state space, we have, for each vertex $v \in \Sigma^0$, a bimodule structure on the state space over a semisimple (bi-)comodule algebra $K_{\Sigma^v}$. Hence, we need a notion replacing the Haar integral, that works in this setting. Our main insight is that the suitable generalization of the Haar integral to our setting is the unique symmetric separability idempotent, which exists for any semisimple algebra over an algebraically closed field $\mathbb{k}$ with characteristic zero.

**Definition 14.** Let $A$ be an algebra over a field $\mathbb{k}$. A **symmetric separability idempotent** for $A$ is an element $p \in A \otimes A$, which we write as $p = p^1 \otimes p^2 \in A \otimes A$ omitting the summation symbol, satisfying

\[
(x \cdot p^1) \otimes p^2 = p^1 \otimes (p^2 \cdot x) \quad \forall x \in A,
\]  

(7)
\[
p^1 \cdot p^2 = 1, \quad (8)
\]
\[
p^1 \otimes p^2 = p^2 \otimes p^1, \quad \text{(symmetry)} \quad (9)
\]
where on both sides of equation (7) and in equation (8) we are using the multiplication in \(A\).

The properties (7) and (8) immediately imply that \(p^1 \otimes p^2\) is an idempotent when seen as an element of the enveloping algebra \(A \otimes A^{\text{op}}\).

Remarks 15.

1. The structure of a separability idempotent, i.e. an element \(p^1 \otimes p^2 \in A \otimes A\) satisfying (7) and (8), is equivalent to an \(A\)-bimodule map \(s : A \rightarrow A \otimes A\) that is a section of the multiplication \(m : A \otimes A \rightarrow A\), by defining \(s(x) := p^1 \otimes p^2 x\) for all \(x \in A\). An algebra endowed with such a structure is called separable and, in general, such a separability structure might not exist or be unique. A symmetric separability structure, however, is always unique – see the end of the proof of Proposition 16.

2. Representation-theoretically, a separability idempotent \(p^1 \otimes p^2 \in A \otimes A^{\text{op}}\) plays the role of projecting to the subspace of invariants for any \(A\)-bimodule \(M\). Indeed, due to property (7), one has

\[
p^1.M.p^2 = M^A := \{ m \in M \mid a.m = m.a \ \forall a \in A \} \subseteq M.
\]

This is in analogy to the Haar integral \(\ell \in H\) of a semisimple Hopf algebra \(H\) which projects to the invariants \(\ell.M = M^H := \{ m \in M \mid h.m = \varepsilon(h)m \ \forall h \in H \} \) of any left \(H\)-module \(M\).

Just as every finite-dimensional semisimple Hopf algebra over a field \(k\) has a unique Haar integral, for every finite-dimensional semisimple \(k\)-algebra there exists a unique symmetric separability idempotent:

Proposition 16 ([1]). Let \(A\) be a finite-dimensional semisimple algebra over a field \(k\) which is algebraically closed and of characteristic zero. Then there exists a unique symmetric separability idempotent \(p^1 \otimes p^2 \in A \otimes A^{\text{op}}\) for \(A\).

Proof. For a more detailed proof, see [1] Thm. 3.1, Cor. 3.1.1. Here we recall the main idea that the unique symmetric separability idempotent can be described in terms of the trace form on \(A\), because we will use this description in Proposition 18.

Due to semisimplicity, the following symmetric bilinear pairing on \(A\) is non-degenerate:

\[
T : A \otimes A \rightarrow k,
\]
\[
a \otimes b \mapsto t(a \cdot b) := \text{tr}_A(L_{a \cdot b}),
\]
defined in terms of the trace form \(t : A 
\rightarrow k, a \mapsto \text{tr}_A(L_a)\), where \(L_a\) denotes the left multiplication of \(A\). In fact, this non-degenerate bilinear pairing turns \(A\) into a symmetric special Frobenius algebra. Consider the isomorphism \(#_T : A \rightarrow A^{\text{op}}, a \mapsto t(a \cdot -)\), induced by this non-degenerate bilinear pairing. This is an isomorphism of \(A\)-bimodules. It induces an isomorphism \(A \otimes A \rightarrow A^{\text{op}} \otimes A \cong \text{End}_k(A)\). Consider the pre-image \(p \in A \otimes A\) of the identity \(\text{id}_A\) under this isomorphism. As usual, we write an element \(p \in A \otimes A\) as \(p = p^1 \otimes p^2\), omitting the summation symbol. In fact, if we choose a basis \((p_i^1)\) for \(A\) and let \((p_i^2)\), be its dual basis of \(A\) with respect to the non-degenerate pairing \(T\), then \(p^1 \otimes p^2\) is the sum \(\sum_i p_i^1 \otimes p_i^2\). With this definition of \(p^1 \otimes p^2 \in A \otimes A\) it is straightforward to verify the defining properties (7), (8) and (9) of a symmetric separability idempotent.
To prove that the symmetric separability idempotent is unique, let $p^1 \otimes p^2, q^1 \otimes q^2 \in A \otimes A^{op}$ be any two symmetric separability idempotents for $A$. Then they are equal by the following computation:

$$p^1 \otimes p^2 \otimes q^1 q^2 p^1 \otimes p^2 \otimes q^1 q^2 q^1 \otimes p^2 q^1 \otimes q^2 \otimes q^1,$$

using the defining properties (7), (8) and (9).

Example 17. Let $H$ be a finite-dimensional semisimple Hopf algebra over $k$ with Haar integral $\ell \in H$. Then the symmetric separability idempotent for $H$ is $\ell(1) \otimes S(\ell(2)) \in H \otimes H^{op}$.

Indeed, the invariance property of the Haar integral, $x\ell = \varepsilon(x)\ell$ for all $x \in H$, implies the corresponding invariance property (7) of $\ell(1) \otimes S(\ell(2))$. The normalization $\varepsilon(\ell) = 1$ of the Haar integral implies the corresponding normalization property (8) for the separability idempotent. Finally, using that the Haar integral is two-sided, which implies $S(\ell) = \ell$, it can be shown that $\ell(1) \otimes S(\ell(2))$ is symmetric.

Hence we see that, in the sense of this example, the symmetric separability idempotent of a semisimple algebra generalizes the Haar integral of a semisimple Hopf algebra.

In our construction of a Kitaev model, however, we are not only dealing with semisimple algebras, but semisimple algebras together with a compatible bicomodule structure. On the other hand, the Haar integral $\ell \in H$ has the property of being cocommutative, $\ell(1) \otimes \ell(2) = \ell(2) \otimes \ell(1)$, which is crucial in showing that it gives rise to commuting projectors in [BMCA] and we have not exhibited an analogous property of the symmetric separability idempotent. In the following proposition we prove such a property, which holds for the symmetric separability idempotent of a semisimple (bi-)comodule algebra and which generalizes the cocommutativity of the Haar integral, see Example 19.

Proposition 18. Let $H$ be a semisimple Hopf algebra over $k$ and let $K$ be a semisimple right $H$-comodule algebra with symmetric separability idempotent $p^1 \otimes p^2 \in K \otimes K^{op}$. Consider the right $H$-coaction on the tensor product $K \otimes K^{op}$:

$$K \otimes K^{op} \longrightarrow K \otimes K^{op} \otimes H;$$

$$k \otimes k' \longmapsto k_{(0)} \otimes k'_{(0)} \otimes k_{(1)}k'_{(1)}.$$

Then $p^1 \otimes p^2 \in K \otimes K^{op}$ is an $H$-coinvariant element of $K \otimes K^{op}$, i.e. $p^1_{(0)} \otimes p^2_{(0)} \otimes p^1_{(1)}p^2_{(1)} = p^1 \otimes p^2 \otimes 1_H \in K \otimes K^{op} \otimes H$, and this is equivalent to

$$p^1_{(0)} \otimes p^1_{(1)} \otimes p^2 = p^1 \otimes S(p^2_{(1)}) \otimes p^2_{(0)} \in K \otimes H \otimes K^{op}. \quad (10)$$

Analogously, if $K$ is a left $H$-comodule algebra, then

$$p^1_{(0)} \otimes p^1_{(-1)} \otimes p^2 = p^1 \otimes S(p^1_{(-1)}) \otimes p^2_{(0)} \in K \otimes H \otimes K^{op}. \quad (11)$$

Proof. Without loss of generality we only show the case where $K$ is a right $H$-comodule algebra. Recall from the proof of Proposition 16 that the symmetric separability idempotent $p^1 \otimes p^2 \in K \otimes K^{op}$ for $K$ can be characterized in terms of the multiplication and the trace form $t : K \longrightarrow k$ on $K$, namely by $t(p^1 \cdot x)p^2 = x \forall x \in K$, as explained in the proof of Proposition 16. Another way of phrasing this is that the map $K^* \longrightarrow K$ defined by $f \longmapsto f(p^1)p^2$ is the inverse of the isomorphism $K \longrightarrow K^*, k \longmapsto t(\cdot k)$ induced by the non-degenerate pairing $t \circ \mu$, where $\mu : K \otimes K \longrightarrow K$ is the multiplication on $K$.

The crucial step for the present proof is the observation that the multiplication and the trace form on $K$ are morphisms of $H$-comodules if $K$ is an $H$-comodule algebra. For the multiplication
this means that \( x(0)y(0) \otimes x(1)y(1) = (xy)(0) \otimes (xy)(1) \), \( \forall x, y \in K \), which holds by definition of a comodule algebra, see Definition 1. As for the \( H \)-co-linearity of the trace form, note that \( t = ev_K \circ (\mu \otimes id_K) \circ (id_K \otimes coev_K) \), where \( \mu : K \otimes K \rightarrow K \) denotes the multiplication, and \( coev_K: k \longrightarrow K \otimes K^* \) and \( ev_K : K \otimes K^* \longrightarrow k \) are the standard coevaluation and evaluation morphisms for vector spaces. Due the involutivity of the antipode \( S \) of \( H \), both \( ev_K \) and \( coev_K \) are morphisms of right \( H \)-comodules for the \( H \)-comodule structure on the dual \( K^* \) given by \( K^* \longrightarrow K^* \otimes H, \varphi \longmapsto \varphi(0) \otimes \varphi(1) \), where \( \varphi(0)(x)\varphi(1) := \varphi(x(0))S(x(1)) \) for all \( x \in K \). (We are here implicitly using the canonical trivial pivotal structure on the tensor category of right \( H \)-comodules, which exists due to the involutivity of the antipode of \( H \).) Since therefore the trace form \( t \) is composed only of morphisms of right \( H \)-comodules, it is itself a morphism of right \( H \)-comodules, i.e.

\[
t(k(0))k(1) = t(k)1_H \quad \forall k \in K. \tag{12}
\]

As a consequence, the isomorphism \( K \longrightarrow K^*, k \longmapsto t(\cdot \cdot k) \) induced by the pairing \( t \circ \mu \) is an isomorphism of \( H \)-comodules. Indeed, for all \( x \in K \) one has \( t(xk(0))k(1) = t(x(0))S(x(2))x(1)k(1) \) \( \text{coev} \) \( t(x(0))kS(x(1)) = (t(\cdot \cdot k))(0)(x)(t(\cdot \cdot k))(1) \).

This immediately implies that the inverse map, \( K^* \longrightarrow K, \varphi \longmapsto \varphi(p(1))p(2) \), must also be a morphism of \( H \)-comodules, which spelled out means that \( \varphi(p(1))p(2) \otimes S(p(1)) = \varphi(0)(p(1))p(2) \otimes \varphi(1) = \varphi(p(1))p(2) \otimes p(1) \) for all \( \varphi \in K^* \). This implies the equation \( (10) \) of the claim. To show that this is equivalent to \( p^1 \otimes p^2 \in K \otimes K^{op} \) being \( H \)-coinvariant, we compute

\[
p^1(0) \otimes p^2(0) \otimes p^1(2) = p^1 \otimes p^2(0) \otimes S(p^1(1))p^2(2) = p^1 \otimes p^2(0) \otimes S(p^1(2))p^2(2) = p^1 \otimes p^2 \otimes 1_H.
\]

\( \square \)

**Example 19.** Let \( H \) be a semisimple Hopf algebra and consider it as the regular \( H \)-bicomodule algebra, as in Example 2.1. Recall that for \( H \) the symmetric separability idempotent is \( p^1 \otimes p^2 = \ell(1) \otimes S(\ell(2)) \in H \otimes H \). Let us spell out Proposition 18 for the left and right \( H \)-comodule structures on the regular bicomodule algebra \( H \). Equation \( (10) \) boils down to the equation \( (\ell(1))(1) \otimes (\ell(1))(2) \otimes S(\ell(3)) = \ell(1) \otimes S(S(\ell(2))(2)) \otimes S(\ell(2))(1) \). But due to \( S^2 = id_H \) both sides of the equation are equal to \( \ell(1) \otimes \ell(2) \otimes S(\ell(3)) \). On the other hand, equation \( (11) \) boils down to the equation \( (\ell(1))(2) \otimes (\ell(1))(1) \otimes S(\ell(3)) = \ell(1) \otimes S(S(\ell(2))(1)) \otimes S(\ell(2))(2) \), which in turn due to \( S^2 = id_H \) simplifies to \( \ell(2) \otimes \ell(1) \otimes S(\ell(3)) = \ell(1) \otimes \ell(2) \otimes S(\ell(2)) \). This is equivalent to the cocommutativity property \( \ell(1) \otimes \ell(2) = \ell(2) \otimes \ell(1) \).

Hence we have shown that the coinvariance property of the symmetric separability idempotent for a bicomodule algebra, proven in Proposition 18 is the appropriate analogue of the cocommutativity of the Haar integral. In the proof of Lemma 20 we will use it in a crucial way, on the way towards proving in Theorem 24 that symmetric separability idempotents allow for defining commuting projectors.

**Lemma 20.** Let \( H \) be a semisimple Hopf algebra over \( k \) and let \( K \) be a semisimple left \( H \)-comodule algebra and \( A \) a semisimple left \( H \)-module algebra. Let \( p^1 \otimes p^2 \in K \otimes K^{op} \) and \( p^1 \otimes p^2 \in A \otimes A^{op} \) be the symmetric separability idempotents for \( K \) and \( A \), respectively.

Then \( (1_A \otimes p^1) \otimes (1_A \otimes p^2) \) and \( (p^1 \otimes 1_K) \otimes (p^1 \otimes 1_K) \) commute in the algebra \( (A \otimes K) \otimes (A \otimes K)^{op} \), where \( A \otimes K \) is the crossed product algebra defined in Definition 4.

**Proof.** Due to the co-invariance of the symmetric separability idempotent of a semisimple comodule algebra over \( k \), proven in Proposition 18, we have

\[
p^{(0)}(1) \otimes p^{(0)}(0) \otimes p^{(1)} \otimes S(p^{(2)}(-1)) \otimes p^1 \otimes p^{(2)}(0)
\]

and

\[
(h.p^1) \otimes p^2 = p^1 \otimes (S(h).p^2)
\]
for all $h \in H$, where the latter can be derived from equation \([10]\) by regarding $A$ as a right $H^*$-comodule algebra, which is equivalent to a left $H$-module algebra \([M]\). By definition of the multiplication in $(A \otimes K) \otimes (A \otimes K)_{op}$ we have:

$$(1_A \otimes p^1) \otimes (1_A \otimes p^2) \cdot (\pi^1 \otimes 1_K) \otimes (\pi^2 \otimes 1_K) = (p^1_{(1)} \cdot \pi^1 \otimes p^1_{(0)}) \otimes (\pi^2 \otimes p^2)$$

and

$$(\pi^1 \otimes 1_K) \otimes (\pi^2 \otimes 1_K) \cdot (1_A \otimes p^1) \otimes (1_A \otimes p^2) = (\pi^1 \otimes p^1) \otimes (\pi^2 \otimes p^2_{(1)}) \cdot (\pi^1 \otimes p^2_{(0)})$$

But the right-hand sides of these equations are equal by the following computation:

$$(p^1_{(1)} \cdot \pi^1 \otimes p^1_{(0)}) \otimes (\pi^2 \otimes p^2) = (S(p^2_{(1)} \cdot \pi^1 \otimes p^1) \otimes (\pi^2 \otimes p^2_{(0)})) = (\pi^1 \otimes p^1) \otimes (S(p^2_{(1)} \cdot \pi^2 \otimes p^2_{(0)})) = (\pi^1 \otimes p^1) \otimes (p^2_{(1)} \cdot \pi^2 \otimes p^2_{(0)}).$$

3.3 Local commuting projector Hamiltonian from vertex and plaquette operators

In this subsection we define on the vector space $\mathcal{H}$ assigned to a surface $\Sigma$ with a labeled cell decomposition a set of commuting local projectors and finally, in the spirit of Kitaev lattice models, a Hamiltonian on $\mathcal{H}$ as the sum of commuting projectors.

Recall that in Subsection 3.1 we have defined on $\mathcal{H}$ a $K_{\Sigma^0,5}$-bimodule structure $\tilde{A}_v$ for each vertex $v \in \Sigma^0$ and a $H^*$-bimodule structure $\tilde{B}_{(p,v)}$ for each site $(p,v), p \in \Sigma^2, v \in \Sigma_{sit}^5$.

A $K_{\Sigma^0,5}$-bimodule structure is equivalent to a left $(K_{\Sigma^0,5} \otimes K_{\Sigma^0,5}^{op})$-action on $\mathcal{H}$, so that specifying an element of the so-called enveloping algebra $(K_{\Sigma^0,5} \otimes K_{\Sigma^0,5}^{op})$ determines an endomorphism of $\mathcal{H}$. By assumption, all bimodule algebras $K_v$ labeling the cell decomposition of $\Sigma$ are semisimple and, hence, the tensor product $K_{\Sigma^0,5}$ is semisimple and possesses a unique symmetric separability idempotent $p_v^1 \otimes p_v^2 \in (K_{\Sigma^0,5} \otimes K_{\Sigma^0,5}^{op})$ according to Proposition \[16\].

**Definition 21.** Let $v \in \Sigma^0$. The vertex operator for the vertex $v$ is the idempotent endomorphism of the state space $\mathcal{H}$

$$A_v := \tilde{A}_v(p_v^1 \otimes p_v^2) : \mathcal{H} \longrightarrow \mathcal{H}$$

given by acting with the unique symmetric separability idempotent

$$p_v^1 \otimes p_v^2 \in K_{\Sigma^0,5} \otimes K_{\Sigma^0,5}^{op}$$

via the $K_{\Sigma^0,5}$-bimodule structure $\tilde{A}_v$, defined in Definition \[10\].

This operator is local in the sense that it acts as the identity on all tensor factors in $\mathcal{H} = (\otimes_{e \in \Sigma^1} K_e) \otimes (\otimes_{w \in \Sigma^0} Z_w)$ except for those associated to the vertex $v \in \Sigma^0$ and to the edges $e \in \Sigma^1_v$ incident to $v$. Since the symmetric separability idempotent of a semisimple bicomodule algebra generalizes the Haar integral of a semisimple Hopf algebra, as explained in Subsection 3.2, we see that the vertex operator defined here provides a suitable analogon to the vertex operators in the ordinary Kitaev model for a semisimple Hopf algebra.

Next we want to define a projector on $\mathcal{H}$ for each plaquette $p \in \Sigma^2$ in analogy to the plaquette operators of the ordinary Kitaev model for a semisimple Hopf algebra $H$, which are defined by acting with the Haar integral of the dual Hopf algebra $H^*$. In our construction, we have defined in Definition \[11\] an $H^*$-bimodule structure $\tilde{B}_{(p,v)}$ on $\mathcal{H}$ for every plaquette $p \in \Sigma^2$ with incident site $v \in \Sigma_{sit}^5$ and we can again use this to define a projector $\tilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)}))$ on
\( \mathcal{H} \) by acting with the symmetric separability idempotent of the semisimple algebra \( H^*_p \), which is \( \lambda_{p(1)} \otimes S(\lambda_{p(2)}) \in H^*_p \otimes (H^*_p)^{\text{op}} \), see Example \[ \text{17} \]. However note that, as opposed to the vertex operator here it actually is not necessary to invoke the concept of the symmetric separability idempotent, since \( H^*_p \) is a Hopf algebra just as in the ordinary Kitaev model, and its symmetric separability idempotent is given by the Haar integral.

When considering the projector \( \tilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)})) \) on \( \mathcal{H} \), it seems that a priori it depends not only on the plaquette \( p \) but also on the site \( v \in \Sigma^\text{sit}_p \) that we had to choose in Definition \[ \text{11} \] in order to define the bimodule structure \( \tilde{B}_{(p,v)} \). Just like the plaquette operators in the ordinary Kitaev model, we will show that due to the properties of the Haar integral the projector only depends on the plaquette \( p \):

\[ \text{Lemma 22.} \text{ Let } p \in \Sigma^2. \text{ If } \lambda_p \in H^*_p \text{ is the Haar integral of } H^*_p, \text{ then the endomorphism} \]
\[ \tilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)})) : \mathcal{H} \rightarrow \mathcal{H} \]
\[ \text{does not depend on the choice of the site } v \in \Sigma^\text{sit}_p. \]

\[ \text{Proof.} \text{ The endomorphism } \tilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)})) \text{ is equal to the endomorphism of } \mathcal{H} \text{ obtained by acting with the Haar integral } \lambda \text{ via the left } H^*_p\text{-action } B'_{(p,v)} \text{ on } \mathcal{H} \text{ that is the pullback of the left } \left( H^*_p \otimes (H^*_p)^{\text{op}} \right)\text{-action } \tilde{B}_{(p,v)} \text{ along the algebra map } (\text{id}_{H^*_p} \otimes S) \circ \Delta : H^*_p \rightarrow H^*_p \otimes (H^*_p)^{\text{op}}. \text{ Next we observe that the action } B'_{(p,v)} \text{ is independent of } v \text{ for any cocommutative element } \lambda \text{ of the Hopf algebra } H^*_p. \text{ Indeed, looking carefully at Definition } \text{11} \text{ we extract from it that } B'_{(p,v)}(\lambda) \text{ acts with the multiple coproduct of } \lambda \text{ on the degrees of freedom of } \mathcal{H} \text{ in the boundary of the plaquette } p \text{ in a cyclic order starting at the vertex } v. \text{ Therefore, for a different vertex } v' \in \Sigma^\text{sit}_p, \text{ the endomorphism } B'_{(p,v')}(\lambda) \text{ will only differ by a cyclic shift in the multiple coproduct of } \lambda. \text{ But since } \lambda \text{ is cocommutative, any multiple coproduct of it is invariant under such cyclic shifts of its tensor factors.} \]

Thus we have shown that the following is well-defined.

\[ \text{Definition 23.} \text{ Let } p \in \Sigma^2. \text{ The plaquette operator for the plaquette } p \text{ is the idempotent endomorphism of the state space } \mathcal{H} \]
\[ B_p := \tilde{B}_{(p,v)}(\lambda_{p(1)} \otimes S(\lambda_{p(2)})) : \mathcal{H} \rightarrow \mathcal{H} \]
given by acting via the \( H^*_p \otimes (H^*_p)^{\text{op}} \)-action \( \tilde{B}_{(p,v)} \) introduced in Definition \[ \text{11} \] with the unique symmetric separability idempotent \( \lambda_{p(1)} \otimes S(\lambda_{p(2)}) \in H^*_p \otimes (H^*_p)^{\text{op}} \) for \( H^*_p \). Here \( \lambda_p \in H^*_p \) is the Haar integral for \( H^*_p \).

This operator is local in the sense that it acts as the identity on all tensor factors in \( \mathcal{H} = (\otimes_{e \in \Sigma^1} K_e^*) \otimes (\otimes_{v \in \Sigma^0} Z_v) \) except for those associated to the edges \( e \in \Sigma^1_p \) and the vertices \( v \in \Sigma^0_p \) incident to the plaquette \( p \).

We have thus defined a family of projectors \( (A_v)_{v \in \Sigma^0} \) and \( (B_p)_{p \in \Sigma^2} \) on the vector space \( \mathcal{H} \). We now finally reach our main result that they all commute with each other.

\[ \text{Theorem 24.} \text{ Let } \Sigma \text{ be an oriented compact surface with a regular cell decomposition labeled by semisimple Hopf algebras, semisimple bicomodule algebras and representations of the vertex algebras, and let } \mathcal{H} \text{ be the associated vector space defined in Definition } \text{8} \text{ with vertex and plaquette operators } \{(A_v)_{v \in \Sigma^0}, (B_p)_{p \in \Sigma^2}\} \text{ defined in Definitions } \text{21} \text{ and } \text{23}. \]

Then any pair of vertex or plaquette operators commutes.
Proof. Due to Theorem 12, the only non-trivial commutation relations between a $K_{\Sigma^0}$-action and an $H^*_p$-action on $\mathcal{H}$ may occur when $v$ and $p$ are incident to each other. In that case, the $K_{\Sigma^0}$-bimodule structure $\bar{A}_v$ and the $H^*_p$-bimodule structure $\bar{B}_{(p,v)}$ together form a bimodule structure over the crossed product algebra $H^*_p \otimes K_{\Sigma^0}$. However, due to Lemma 20 the symmetric separability idempotents for $K_{\Sigma^0}$ and $H^*_p$ commute in $(H^*_p \otimes K_{\Sigma^0}) \otimes (H^*_p \otimes K_{\Sigma^0})^{op}$ and, hence, the vertex operator $A_v$ and the plaquette operator $B_p$ commute with each other. 

This is completely analogous to the standard Kitaev model without defects: We have a family of commuting projectors on the state space. Since any family of commuting projectors is simultaneously diagonalizable, this allows for the definition of an exactly solvable Hamiltonian as the sum of commuting projectors. We thus conclude our construction of the Kitaev lattice model with defects as follows:

**Definition 25.** The Hamiltonian on the state space $\mathcal{H}$ assigned to an oriented surface $\Sigma$ with labeled cell decomposition as above is the diagonalizable endomorphism

$$h := \sum_{v \in \Sigma^0} (\text{id}_\mathcal{H} - A_v) + \sum_{p \in \Sigma^2} (\text{id}_\mathcal{H} - B_p) : \mathcal{H} \rightarrow \mathcal{H}.$$

The associated ground-state space is its kernel,

$$\mathcal{H}_0 := \ker h,$$

i.e. the simultaneous 1-eigenspace for all the projectors $\{(A_v)_{v \in \Sigma^0}, (B_p)_{p \in \Sigma^2}\}$.

Such a Hamiltonian is also called frustration-free, as its lowest eigenvalue is not lower than any eigenvalue of its summands.

**Remark 26.** The ground-state space $\mathcal{H}_0$ is isomorphic to the vector space that is category-theoretically realized by the modular functor constructed in [FSS19] for the defect surface $\Sigma$ labeled by the corresponding representation categories of the Hopf algebras and bicomodule algebras. We leave the detailed proof of this statement for a future update of this paper.

As a consequence, the ground-state space $\mathcal{H}_0$ is invariant under fusion of defects and independent of the transparently labeled part of the cell decomposition. Moreover, due to the results of [FSS19], there will be a mapping class group action on $\mathcal{H}_0$ that can be explicitly computed. This allows to define quantum gates on the ground-state space in terms of the mapping class group action, as has been proposed before, and to address questions of universality of such gates. We have thus constructed an explicit Hamiltonian model which offers the possibility for quantum computation, realizing a general framework for theories of the type discussed e.g. in [BJQ].

A detailed investigation of the above and related questions remain for future work.
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