Effective Gauss-Bonnet Interaction in
Randall-Sundrum Compactification

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**Abstract**

The effective gravitational interaction below the Planck scale in the Randall-Sundrum world is shown to be the Gauss-Bonnet term. In this theory we find that there exists another static solution with a positive bulk cosmological constant. Also, there exist solutions for positive visible sector cosmological constant, which is needed for a later Friedman-Robertson-Walker universe.

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In the last several years, the brane world has been one of the major research fronts toward probing the fundamental laws of nature \cite{1-3}. In many of these theories, the observable matter fields live in the brane(s) and are forbidden to propagate in the bulk. The bulk is populated mainly by the gravity related fields \cite{2,3}. Among these, the Randall-Sundrum(RS) compactification touches upon the gauge hierarchy problem (“Why is $M_W/M_P$ so small?”), and obtains an exponentially small Higgs boson mass parameter compared to the input mass parameter (assumed to be at the Planck scale) of the five dimensional theory. An exponential warp factor suppresses the soft mass in the visible brane, and it is possible to obtain this small ratio because the Higgs mass term is a dimension two operator. Thus, in the Randall-Sundrum(RS) world, one changes the traditional gauge hierarchy problem to a problem in geometry.

However, the RS model is not complete toward a solution of the gauge hierarchy problem. The most obvious problem is the appearance of unwanted high dimensional operators such as the one triggering proton decay. Generally, the TeV scale is the mass parameter scale suppressing the high dimensional ($d > 4$) operators in the RS compactification. Thus one has to make sure that the theory has a high degree of symmetry to suppress the unwanted operators up to the experimental limit. Another problem is the problem of inflation. Generally, the unwanted inflation is unavoidable unless one fine-tune the bulk cosmological constant and the brane tensions \cite{4-6}. Nevertheless, this scenario is so interesting that it is worthwhile to pursue the implications of the RS compactification further. The most interesting point of the RS world is the interplay of the bulk and the brane world. In particular, the bulk cosmological constant ($k$) and the brane tensions ($k_i$ ($i = 1, 2$)) must be related, $k_1 = -k = -k_2$ and $k_1 > 0$. Note, however, that the observed expansion rate of the observable universe is measured by the Hubble parameter which is vanishing, given the above condition, neglecting the matter field Lagrangian. These $k$’s are the appropriately defined from the original bulk cosmological constant $\Lambda_b$ and the brane tensions $\Lambda_1, \Lambda_2$ of the two branes 1 and 2 where the hidden fields live in Brane 1 (B1) and the observable fields live in Brane 2 (B2) \cite{7}. For simplicity, we will call these $\Lambda$’s the ‘cosmological constants’, be-
low. Therefore, we can hope for a possibility of understanding the old cosmological constant problem in this new setting.

In this paper, we consider the leading effective interactions below the Planck scale with the setting of two branes located at $y = 0$ and $y = 1/2$ and with an orbifold symmetry $S_1/Z_2$. We find that two static solutions exist only if the effective interaction is the Gauss-Bonnet term. One of these solutions reduces to the RS one in the limit of the vanishing effective interaction with a negative cosmological constant in the observable brane. Another solution is a new one with the possibility of positive cosmological constants in the observable brane.

We proceed to discuss the effective interaction and its solution with an $S_1/Z_2$ orbifold compactification. The space-time is assumed to be five dimensional ($M, N = 0, 1, \cdots, 4$) with the four dimensional brane worlds ($\mu, \nu = 0, 1, \cdots, 3; y = 0, 1/2$) and the bulk ($M, N = 0, \cdots, 4; 0 < y < 1/2$), where $y = x^4$. The $S_1/Z_2$ orbifold is used to locate the two branes at $y = 0$ and $y = 1/2$. Thus, the periodicity is $y \rightarrow y + 1$.

Below the Planck scale, the gravity effects can be added as additional effective interaction terms. Since we are neglecting the matter interactions, the possible terms in the Lagrangian is, up to $O(g_{MN}^2)$

$$S = \int d^5 x \sqrt{-g} \left( -\frac{M^3}{2} R - \Lambda_b + \frac{1}{2} \alpha R^2 + \frac{1}{2} \beta M R_{MN} R^{MN} + \frac{1}{2} \gamma M R_{MNPQ} R^{MNPQ} \right) + \sum_{i=1,2 \text{ branes}} \int d^4 x \sqrt{-g^{(i)}} (\mathcal{L}_i - \Lambda_i)$$

(1)

where $g, g^{(i)}$ are the determinants of the metrics in the bulk and the branes, $M$ is the five dimensional gravitational constant, $\Lambda_b$ and $\Lambda_i$ are the bulk and brane cosmological constants, and $\alpha, \beta, \gamma$ are the effective couplings. We assume that the three dimensional space is homogeneous and isotropic, and hence the metric is parametrized by $n, a,$ and $b$

$$ds^2 = -n^2(\tau, y)d\tau^2 + a^2(\tau, y)\delta_{ij}dx^i dx^j + b^2(\tau, y)dy^2.$$  

(2)

where the lower case Roman characters $i, j$ denotes the space indices 1, 2, and 3.

Then static solutions exist only when the Gauss-Bonnet relations $\beta = -4\alpha$ and $\gamma = \alpha$ are satisfied. Namely, the higher $y$ derivative terms, $a'''$, $a''a'$, and $a''a''$ (where prime denotes the
derivative with respect to \( y \) are absent in the l.h.s. of the Einstein equation. This condition is necessary since the r.h.s. of the Einstein equation contains only one power of the Dirac delta function, and the higher derivative terms diverge more rapidly than the delta function at the branes. It is miraculous that there exists such a solution, since the number\((=3)\) of conditions we must satisfy are more than the number\((=2)\) of independent ratios of the couplings \( \alpha, \beta \) and \( \gamma \). Thus, we use the Gauss-Bonnet effective interaction below. This condition leads to ghost-free nontrivial gravitational self interactions for dimensions higher than four \[8\].

Variations of the above action with the Gauss-Bonnet term gives, apart from those for the brane Lagrangian,

\[
\sqrt{-g} \left[ R_{MN} - \frac{1}{2} g_{MN} R \right. \\
\left. - \frac{1}{2} \alpha M^{-2} g_{MN} \left( R^2 - 4 R_{PQ} R^{PQ} + R_{STPQ} R^{STPQ} \right) \right. \\
+ 2 \alpha M^{-2} \left( R R_{MN} - 4 R_{MP} R_{NP}^\mu + R_{MQSP} R_N^{QSP} \right) + 2 \alpha M^{-2} \left( g_{MN} R_{;P}^{;P} - R_{;M;N} \right) \\
- 4 \alpha M^{-2} \left( g_{MN} R_{;P}^{;P} + R_{MN;P}^{;P} - R_M^P N^{P;N} - R_N^P M^{P;M} \right) \\
\left. + 2 \alpha M^{-2} \left( R_M^P N^{Q;P} - R_M^P N^{Q;Q} \right) \right] \\
= -M^{-3} \left[ \Lambda_b \sqrt{-g} g_{MN} + \Lambda_1 \sqrt{-g((1)} g_{\mu}^{(1)} \delta_{M}^\mu \delta_{N}^\nu \delta(y) + \Lambda_2 \sqrt{-g((2)} g_{\mu(2)} \delta_{M}^\mu \delta_{N}^\nu \delta(y - \frac{1}{2}) \right]
\]

where 1 refers to the brane of the hidden world B1 and 2 refers to the observable brane B2. The l.h.s. of the above equation contains the extra term due to the Gauss-Bonnet term, \( \sqrt{-g} X_{MN} \), in addition to the familiar Einstein tensor \( \sqrt{-g} G_{MN} \). The \( G_{MN} \) and \( X_{MN} \) are

\[
G_{00} = -3 \left( \frac{n^2 a''}{ab^2} + \frac{n^2 a'^2}{a^2 b^2} - \frac{n^2 a' b'}{ab^3} \right) + \cdots \tag{4}
\]

\[
G_{ij} = \left( \frac{n'' a^2}{nb^2} + 2 \frac{a a''}{b^2} + 2 \frac{n' a a'}{nb^2} - \frac{n' a'^2}{nb^2} + \frac{a'^2}{b^2} - 2 \frac{a a' b'}{b^3} \right) \delta_{ij} + \cdots \tag{5}
\]

\[
G_{55} = 3 \left( \frac{n' a'}{n a} + \frac{a^2}{a^2} \right) + \cdots \tag{6}
\]

\[
G_{05} = \cdots \tag{7}
\]
\[ X_{00} = 12\alpha \left( \frac{a'^2 a'' n'^2}{a^3 b^4} - \frac{a'^3 b' n'^2}{a^3 b^5} \right) + \cdots \] (8)

\[ X_{ii} = -4\alpha \left( \frac{2 a' a'' n' + n'' a'^2}{n b^4} - 3 \frac{a'^2 n' b'}{n b^5} \right) + \cdots \] (9)

\[ X_{55} = -12\alpha \frac{a'^3 n'}{a^3 n b^2} + \cdots \] (10)

\[ X_{05} = \cdots \] (11)

where \( t \) denotes the derivative with respect to \( y \) and \( \cdots \) denote terms containing derivatives with respect to \( \tau \).

To find static solutions, let us redefine \( y \) such that the metric takes the following form,

\[ ds^2 = e^{-2\sigma(\phi)} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2 \] (12)

where the length parameter \( r_c \) is a constant, and the angle variable is defined as \( \phi = 2\pi y \), with the periodicity \( \phi \to \phi + 2\pi \). Note that the Einstein equation for the \((00)\) component is identical to the \((ii)\) component. Thus the \((00)\), \((ii)\), and \((55)\) components of Eq. (3) lead to

\[ \frac{3\sigma''}{r_c^2} \left( 1 - \frac{4\alpha}{M^2 r_c^2} (\sigma')^2 \right) = \frac{\Lambda_1}{M^3 r_c} \delta(\phi) + \frac{\Lambda_2}{M^3 r_c} \delta(\phi - \pi) \] (13)

\[ \frac{6(\sigma')^2}{r_c^2} \left( 1 - \frac{2\alpha}{M^2 r_c^2} (\sigma')^2 \right) = -\frac{\Lambda_b}{M^3} \] (14)

There exist two solutions of Eq. (14), consistent with the orbifold symmetry \( \phi \to -\phi \),

\[ \sigma^\pm = r_c |\phi| \left[ \frac{M^2}{4\alpha} \left( 1 \pm \left( 1 + \frac{4\alpha\Lambda_b}{3M^5} \right)^{\frac{1}{2}} \right) \right]^{\frac{1}{2}} \equiv k_\pm r_c |\phi| . \] (15)

These solutions exist for:

(i) \( \alpha < 0 \) and \( \Lambda_b < 0 \) allows only \( \sigma^- \), and

(ii) \( \alpha > 0 \) allows both \( \sigma^\pm \) solutions. The \( \sigma^+ \) solution is possible for both \( \Lambda_b > 0 \) and \( \Lambda_b < 0 \). The \( \sigma^- \) solution is possible only for \( \Lambda_b < 0 \). In any case, there exists the lower limit of \( \alpha\Lambda_b \), \( \alpha\Lambda_b \geq -3M^5/4 \).
Considering the discontinuities at the branes given in Eq. (13), we obtain two solutions if the following relations among the brane cosmological constants are satisfied

\[ \Lambda_1^\pm = -\Lambda_2^\pm = \mp 6k_\pm M^3 \sqrt{1 + \frac{4\alpha \Lambda_b}{3M^5}} \]

\[ = \mp 6M^3 \left[ \frac{M^2}{4\alpha} \left( 1 \pm \left( 1 + \frac{4\alpha \Lambda_b}{3M^5} \right)^{\frac{1}{2}} \right) \left( 1 + \frac{4\alpha \Lambda_b}{3M^5} \right) \right]^{\frac{1}{2}} \tag{16} \]

where \( k_\pm > 0 \). The RS solution is obtained by taking \( \alpha \to 0 \) in the – solution.

Possible solutions are depicted in Fig. 1 as a function of the Gauss-Bonnet coupling \( \alpha \). The vertical axis (\( \equiv \lambda_2 \)) is the solution for \( \Lambda_2 \) in the visible brane in units of \( \sqrt{6M^3|\Lambda_b|} \), and the horizontal axis (\( \equiv \alpha \Lambda \)) is defined as \( 4\alpha \Lambda_b/(3M^5) \).

In the literature, it has been argued that \( \Lambda_{\text{vis}} = \Lambda_2 \) is better to be positive \( \text{[10]} \). The argument is the following. The hubble parameter is expressed in terms of the cosmological constant and the energy density, \( H_{\pm, \text{vis}} = \sqrt{(k_{\pm, \text{vis}})^2 - k^2} \text{[9]} \), where \( k^2 = k^2_\pm \) for the static solutions and the two parameters corresponding to + and – solutions at the visible brane, \( k_{\pm, \text{vis}} \), are given by

\[ k_{\pm, \text{vis}} = \frac{\pm \left( \Lambda_2^\pm + \rho_{\text{vis}} \right)}{6M^3 \sqrt{1 + (4\alpha \Lambda_b/3M^5)}} \tag{17} \]

Thus the Hubble parameter at B2 is given by

\[ H_{\pm, \text{vis}} = \frac{\sqrt{\rho_{\text{vis}}(\rho_{\text{vis}} + 2\Lambda_2^\pm)}}{6M^3 \sqrt{1 + (4\alpha \Lambda_b/3M^5)}} \tag{18} \]

With \( \rho_{\text{vis}} = 0 \), we obtain the previous static solution. But with \( \Lambda_2^\pm = \Lambda_b < 0 \), there exists a possibility that \( \rho_{\text{vis}}(2\Lambda_2^\pm + \rho_{\text{vis}}) < 0 \) at a sufficiently low temperature, and hence it is difficult to obtain a real Hubble parameter \( \text{[10]} \). But with a positive \( \Lambda_2^+ \), there does not exist such a problem. This is possible for our + solution for \( \alpha > 0 \).

Inflationary solutions were obtained for a flat bulk geometry \( \text{[12]} \) and for an AdS bulk geometry \( \text{[4,5]} \). In our case, inflationary solutions exist also for a positive bulk cosmological constant \( \text{[9]} \). In general, inflation occurs if parameters \( \alpha, \Lambda_b, \Lambda_1, \) and \( \Lambda_2 \) do not satisfy the two relations implied by Eq. (16) \( \text{[5,6,9]} \). We find that in general there exist terms with higher
(more than two) time derivatives if the parameters of Eq. (1) satisfy $16\alpha + 5\beta + 4\gamma = 0$. But the inflationary solutions in Ref. [5] (with a separable metric) and Ref. [6] (with a nonseparable metric) are still valid with these higher time derivatives if $16\alpha + 5\beta + 4\gamma = 0$ is satisfied [9].

The Planck constant at B2 (observable sector) is given by [3]

$$M_{Pl}^2 = M^3 r_c \int_{-\pi}^{\pi} d\phi e^{-2k\pm r_c |\phi|} = \frac{M^3}{k_{\pm}} [1 - e^{-2k\pm r_c \pi}]$$

$$= M^2 \left[ \frac{1}{4\alpha} \left( 1 \pm 1 + \frac{4\alpha \Lambda_b}{3M^5} \right)^{\frac{1}{2}} \right]^{-\frac{1}{2}} [1 - e^{-2k\pm r_c \pi}]$$

(19)

The Higgs boson mass parameter at the observable sector is obtained by redefining the Higgs field such that the kinetic energy term of the Higgs boson takes a standard form [3]. Thus the Higgs mass parameter is given by

$$m \equiv e^{-k\pm r_c \pi} m_0$$

$$= m_0 \exp \left( - r_c \pi \left[ \frac{M^2}{4\alpha} \left( 1 \pm 1 + \frac{4\alpha \Lambda_b}{3M^5} \right)^{\frac{1}{2}} \right] \right)$$

(20)

where $m_0$ is the mass given in the fundamental Lagrangian, before redefining the Higgs field. For $k\pm r_c \simeq 12$, the + solution gives a needed large mass hierarchy through the warp factor $e^{-k\pm r_c \pi}$ from the input mass parameter($M$) of order $10^{19}$ GeV, leading to a TeV scale observable mass. For $k- r_c \simeq 12$, the – solution has the same behavior. This small warp factor [3] makes it possible to generate a TeV scale mass from the fundamental parameter of $O(M_{Pl})$. But these TeV scale masses also appear in the other mass parameters of the effective operators, in particular those leading to proton decay are also parametrized by a TeV scale mass. Therefore, one has to suppress sufficiently the low dimensional proton decay operators such that it is sufficiently long-lived ($\tau_p > 10^{32}$ years), implying $D > 14$.

In this paper, we studied the Randall-Sundrum compactification with an additional effective Gauss-Bonnet term with the coupling $\alpha$. It is miraculous that the Gauss-Bonnet term is allowable for a static solution, since it is not at all obvious to expect such an effective Lagrangian. This is because the conditions we must satisfy are more than the independent
ratios of the effective couplings. With this effective Gauss-Bonnet term, we find two solutions. There exist solutions for negative and positive cosmological constants in the visible sector. In particular, we obtained a solution for a positive cosmological constant in the visible sector in case $\alpha > 0$, which can introduce a positive matter energy density in the visible Friedman-Robertson-Walker universe.

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Fig. 1. Possible solutions for $\lambda_2 \equiv \Lambda_2 / \sqrt{6M^3|\Lambda_b|}$ as a function of $\alpha_\Lambda \equiv 4\alpha\Lambda_b/(3M^5)$. The star point is the RS solution. The four quadrants have different sets of signs of $\alpha$ and $\Lambda_b$, denoted as (sign of $\alpha$, sign of $\Lambda_b$).