Exponential convergence of distributed optimization for heterogeneous linear multi-agent systems

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Abstract

In this work we study a distributed optimal output consensus problem for heterogeneous linear multi-agent systems where the agents aim to reach consensus with the purpose of minimizing the sum of private convex costs. Based on output feedback, a fully distributed control law is proposed by using the proportional-integral (PI) control technique. For strongly convex cost functions with Lipschitz gradients, the designed controller can achieve convergence exponentially in an undirected and connected network. Furthermore, to remove the requirement of continuous communications, the proposed control law is then extended to periodic and event-triggered communication schemes, which also achieve convergence exponentially. Two simulation examples are given to verify the proposed control algorithms.

Keywords: Distributed convex optimization, multi-agent systems, event-triggered communication

1. Introduction

In recent few decades, distributed optimization has been attracting more and more research interests because of its wide applications in multi-agent systems, smart grids, machine learning and so on. Specifically, the purpose of each node in a network is to minimize the sum of private costs under constraints only by exchanging local information with neighbors. Various distributed algorithms have been proposed in this field (Nedic and Ozdaglar, 2009; Yang et al., 2019; Zhang et al., 2017; Tang et al., 2019; Zhao et al., 2017; Li et al., 2020a, 2019b).

In practical physical systems such as AGVs and UAVs, the implementation of distributed strategies must consider the dynamics of each agent. Therefore, in recent years, interest has been attracted increasingly in distributed optimization combined with physical systems. This problem requires each of a group of continuous-time physical systems to achieve the best performance. Generally, there are two ways to solve such problems. The first one is based on a “separative” way: treating it as a standalone distributed optimization problem for cost functions and simultaneously tracking the optimized variables for complex systems. In Zhang et al. (2017) where the system dynamics are Euler-Lagrange systems, two distributed algorithms are developed for the case without parametric uncertainties and

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it can only ensure that the algorithm converges to a neighbor-
hood of the optimal solution. Reference Yu and Chen (2020)
studies the Zeno behavior of the first-order multi-agent systems
and provides sufficient conditions for its existence in a finite
time consensus.

Aiming at the optimal output consensus problem of het-
erogeneous linear multi-agent systems, this paper designs a
proportional-integral (PI) controller to solve the problem, in
which the proportional term drives all the agents to the con-
sensus space and the integral term eliminates errors (Qiu et al.,
2019). The main contributions of this paper are as follows.

1) Through the feedback combination of own state and neigh-
bor output information, a PI based control law is designed,
which is shown to have an exponential convergence rate.
In comparison, the most related work (Li et al., 2020b)
only gives the result of asymptotic convergence, and is
based on a stronger assumption than that in this paper.

2) To reduce the communication overhead, this paper further
introduces periodic communication and event-triggered
communication mechanisms for the above-proposed al-
gorithm, which are both proven to guarantee exponen-
tial convergence. Besides the established exponential rate
here, compared with the gradually decreasing communi-
cation interval in Li et al. (2020b), the proposed algorithm
can clearly give a lower bound of the communication in-
terval, thus excluding the Zeno behavior.

The overall structure of this paper is as follows. Prelimi-
naries are given in Section 2. In Section 3, the heterogeneous
multi-agent systems under investigation are described math-
eematically, the optimal output consensus problem is defined and
some useful lemmas are given. Following that, three control
laws with continuous, periodic, and event-triggered communi-
cation are proposed respectively and their exponential con-
gerence is established in Section 4. Then two simulation ex-
amples are provided to verify the effectiveness of the algorithms
in Section 5. Finally, conclusions and future works are discussed
in Section 6.

2. Preliminaries

2.1. Notations

Let \( \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{m \times n} \) be the sets of real numbers, real vectors of
dimension \( n \) and real matrices of dimension \( m \times n \), re-
spectively. \( I_n \) denotes the \( n \)-dimensional identity matrix. \( I_n \)
and \( 0_n \) denote \( n \)-dimensional all-one and all-zero column vec-
tors, respectively. For a matrix \( A \in \mathbb{R}^{m \times n} \), \( A^\top \) is its trans-
pose, and \( \text{diag}(A_1, \ldots, A_n) = \text{blkdiag}(A_1, \ldots, A_n) \) denotes a
block diagonal matrix with diagonal blocks of \( A_1, \ldots, A_n \).
\( \text{col}(x_1, \ldots, x_n) = (x_1^\top, \ldots, x_n^\top)^\top \) is a column vector by stack-
ing vectors \( x_1, \ldots, x_n \). \( \|A\| \) and \( \|x\| \) are the induced 2-norm of
matrix \( A \) and the Euclidean norm of vector \( x \) respectively. \( A \otimes B \)
represents the Kronecker product of matrices \( A \) and \( B \).

2.2. Graph Theory

A communication network of \( N \) agents is modeled by an
undirected graph \( G = (\mathcal{V}, \mathcal{E}, \mathcal{A}) \), where \( \mathcal{V} = \{v_1, \ldots, v_N\} \)
is a node set, \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is an edge set and \( \mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N} \)
is the adjacency matrix. If information exchange can occur be-
tween \( v_i \) and \( v_j \), then \((v_i, v_j) \in \mathcal{E} \). If there exists a path from
any node to any other node in \( \mathcal{V} \), then \( G \) is called connected,
otherwise disconnected. The out-degree of node \( v_j \) is denoted
by \( d_j^\text{out} = \sum_{i=1}^N a_{ij} \). Denote \( L = D_{\text{out}} - A \) as the Laplacian matrix
of \( G \), where \( D_{\text{out}} = \text{diag}(d_1^{\text{out}}, \ldots, d_N^{\text{out}}) \) is the out-degree matrix
of \( G \).

Lemma 1 (Godsil and Royle, 2001). If \( G \) is undirected and
connected, all eigenvalues of \( L \) are real and except for a sin-
gle eigenvalue 0, the rest are positive numbers, denoted as
\( 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N \).

Lemma 2 (Li et al., 2020b). If \( G \) is undirected and
connected, there exists a positive definite matrix \( \Gamma \in \mathbb{R}^{N \times N} \) such
that \( \Gamma L = \Gamma L = \Pi \), where \( \Pi = I_N = \frac{1}{N} I_N I_N^\top \). Moreover, the
eigenvalues of \( \Gamma \) are \( \lambda_1 > \frac{1}{\lambda_2} = \cdots = \frac{1}{\lambda_N} \), where \( \lambda_1 > 0 \) can be any
positive constant and \( \lambda_2, \ldots, \lambda_N \) are defined in Lemma 1.

3. Problem Formulation

Consider a multi-agent system with \( N \) heterogeneous agents,
and the \( i \)-th agent has the linear dynamics:
\[
\begin{align*}
\dot{x}_i &= A_i x_i + B_i u_i, \\
y_i &= C_i x_i,
\end{align*}
\]
(1)
where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^p \) and \( y_i \in \mathbb{R}^q \) are the state, input and
output variables respectively. \( A_i \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^{n \times p} \), and
\( C_i \in \mathbb{R}^{q \times n} \) are the state, input and output matrices which are
constant.

The objective of this paper is to design a controller \( u_i(t) \) for
each agent by using only local interaction and information such
that all agents cooperatively reach the optimal outputs that solve
the following convex optimization problem:
\[
\min_{\{y_i\}_{i=1}^N} \sum_{i=1}^N f_i(y_i), \text{ s.t. } y_1 = \cdots = y_N,
\]
(2)
where \( f_i : \mathbb{R}^q \rightarrow \mathbb{R} \) is the local cost function which is only
known to the \( i \)-th agent.

Assumption 1. The communication network \( G \) is undirected
and connected.

Assumption 2. The local objective function \( f_i \) is differentiable
and its gradient is \( w_i \)-Lipschitz in \( \mathbb{R}^q \):
\[\|\nabla f_i(x) - \nabla f_i(y)\| \leq w_i \|x - y\|, \forall x, y \in \mathbb{R}^q, w_i > 0.\]
Denote \( \bar{w} := \max\{w_1, \ldots, w_N\} \).
**Assumption 3.** The local objective function $f_i$ is $m_i$-strongly convex:

$$(x - y)^T (\nabla f_i(x) - \nabla f_i(y)) \geq m_i \|x - y\|^2, \forall x, y \in \mathbb{R}^q, m_i > 0.$$  

Let $m := \min\{m_1, \ldots, m_N\}$.

**Remark 1.** Because of the strong convexity of $f_i, f$ is strongly convex, which guarantees the uniqueness of the optimal solution to (2).

**Assumption 4.** $(A_i, B_i)$ is controllable, and

$$\text{rank}(C_iB_i) = q, i = 1, \ldots, N.$$  

**Lemma 3.** Under Assumption 4, the matrix equations

$$(4a) C_iB_iK_{a_i} = C_iA_i,$$  

$$(4b) C_iB_iK_{b_i} = I_q,$$  

exist solutions $K_{a_i}, K_{b_i}$.

**Proof.** From (3), we can get

$$\text{rank}(C_iB_i, C_iA_i) = q = \text{rank}(C_iB_i),$$  

$$\text{rank}(C_iB_i, I_q) = q = \text{rank}(C_iB_i).$$  

Thus (4a) and (4b) have solutions $K_{a_i}, K_{b_i}$.

**Remark 2.** The controllability in Assumptions 4 is quite standard in dealing with the problem for linear systems. And the requirement (3) is employed to guarantee the solvability of matrix equations (4), which is strictly weaker than the assumption (i.e., $\text{rank}\left[\begin{array}{cc} C_iB_i & 0 \\ -A_iB_i \\ B_i \end{array}\right] = n_i + q, i = 1, \ldots, N$) employed in (Li et al., 2020b; Zhang et al., 2020).

4. Main Results

4.1. Continuous Communication

A PI controller for the $i$th agent is proposed as

$$u_i = -K_{a_i}x_i + K_{b_i}(-\nabla f_i(y_i) - \sum_{j=1}^{N} a_{ij}(y_i - y_j) - \eta_i),$$  

(6a)

$$\eta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \quad \eta_i(0) = 0_n,$$  

(6b)

where $K_{a_i}, K_{b_i}$ are feedback matrices and $a_{ij}$ is the weight corresponding to the edge $(i, j)$.

As shown in Figure 1, the compact form of the closed-loop system is

$$\dot{x} = (A - BK_a)x + BK_b(-\nabla f(y)) - (L \otimes I_q)y - \eta,$$  

(7a)

$$\dot{\eta} = (L \otimes I_q)y, \quad \eta(0) = 0_{Nq},$$  

(7b)

$$y = Cx,$$  

(7c)

where $x = \text{col}(x_1, \ldots, x_N), \quad \eta = \text{col}(\eta_1, \ldots, \eta_N), \quad y = \text{col}(y_1, \ldots, y_N), \quad A = \text{diag}(A_1, \ldots, A_N), \quad B = \text{diag}(B_1, \ldots, B_N), \quad C = \text{diag}(C_1, \ldots, C_N), \quad K_a = \text{diag}(K_{a_1}, \ldots, K_{a_N}), \quad K_b = \text{diag}(K_{b_1}, \ldots, K_{b_N}),$ and $\nabla f(y) = \text{col}(\nabla f_1(y_1), \ldots, \nabla f_N(y_N)).$

**Theorem 1.** Supposing that Assumptions 1-4 hold, for linear multi-agent system (1) with control protocol (6), the problem (2) is solved and $y_i(t)$ converges to $y^*$ exponentially as $t \to \infty$ for $i = 1, \ldots, N$, where $y^* \in \mathbb{R}^q$ is the optimal solution to (2), and the feedback matrices $K_{a_i}$ and $K_{b_i}$ are solutions of equations (4a) and (4b), respectively.

**Proof.** First, we discuss the relationship between the equilibrium point of (7) and the optimal solution of (2). Pre-multiplying (7a) by $C$, we get

$$\dot{y} = -C \nabla f(y) - (L \otimes I_q)y - \eta,$$  

(8a)

$$\dot{\eta} = (L \otimes I_q)y, \quad \eta(0) = 0_{Nq},$$  

(8b)

Because $\eta(0) = 0_{Nq}$ and $\gamma_{1N}L = \gamma_{N1}I_q$, we can get $(\gamma_{1N} \otimes I_q)\gamma(t) = 0, \forall t > 0$. Pre-multiplying (8a) by $(\gamma_{1N} \otimes I_q)$, and let the derivatives of (8) be equal to 0. Then the equilibrium point $(\bar{y}, \bar{\eta})$ satisfies

$$\sum_{i=1}^{N} \nabla f_i(y_i) = 0, \quad (9a)$$

$$(L \otimes I_q)\bar{y} = 0_{Nq}, \quad (9b)$$

where $\bar{y} = \text{col}(\bar{y}_1, \ldots, \bar{y}_N)$.

From (9b) we know all $\bar{y}_i$ reach consensus, which together with (9a) ensures that the equilibrium point is the optimal solution to (2).

To proceed, taking $\rho = y - \bar{y}, \quad \sigma = \eta - \bar{\eta}$, one has

$$\dot{\rho} = -h - (L \otimes I_q)p - \sigma,$$  

(10a)

$$\dot{\sigma} = (L \otimes I_q)p,$$  

(10b)

where $h := \nabla f(y) - \nabla f(\bar{y})$.

Next we only need to discuss the convergence of (10). Select a Lyapunov candidate as

$$V = \frac{\xi}{2} p^T p + \sigma^T \rho + \frac{1}{2} \sigma^T \sigma + \frac{\xi}{2} (\Gamma \otimes I_q)\sigma,$$  

(11)

where $p := \text{col}(\rho, \sigma), \xi$ is a parameter to be determined, $\Gamma$ is given in Lemma 2, and $\tilde{\lambda}_e$ is the maximum eigenvalue of

$$E := \frac{1}{2} \left( \begin{array}{cc} I_{Nq} & \xi \Gamma_{1N} \end{array} \right).$$
It is easy to verify that $E$ is a positive definite matrix for $\xi \geq 1$ by Schur complement.

The derivative of (11) is

$$
V = -\xi \rho^\top \dot{h} - (\xi - 1) \rho^\top (L \otimes I_q) \rho - \sigma^\top \sigma - \sigma^\top h - \xi \rho^\top (I_N - \Pi) \otimes I_q) \sigma,
$$

where $\Pi$ is defined in Lemma 2.

By the strong convexity of $f_i$, we have

$$
\rho^\top h \geq \frac{m}{2} \rho^\top \rho.
$$

(13)

Because the gradients of local objective functions are Lipschitz, we have $[h] \leq \overline{w} \|\rho\|$. Then one has

$$
-\sigma^\top h \leq \frac{1}{2} \sigma^\top \sigma + \frac{1}{2} \overline{w}^2 \rho^\top \rho.
$$

(14)

From the fact that $(I_N \otimes I_q) \eta = 0$, it can be obtained that $(I_N - \Pi) \otimes I_q) \sigma = 0$. Since the matrix $L \otimes I_q \eta = 0$ and $\xi \geq 1$, we can get $\xi - 1) \rho^\top (L \otimes I_q) \rho \geq 0$. Substituting (13), (14) into (12), one has

$$
\dot{V} \leq -\xi m \rho^\top \rho + \frac{1}{2} \overline{w}^2 \rho^\top \rho - \frac{1}{2} \sigma^\top \sigma
$$

$$
= -p^\top \dot{F}_1 p,
$$

with

$$
F_1 := \left( \left( \xi m - \frac{1}{2} \overline{w}^2 \right) \otimes I_{Nq} \quad 0_{Nq} \right) \frac{1}{2} I_{Nq}.
$$

For $\xi > \frac{\overline{w}^2}{\overline{w}^2}$, the matrix $F_1$ is positive definite. Note that $A_{F_1}^* := \min \{\xi m - \frac{1}{2} \overline{w}^2, \frac{1}{2}\}$ is the smallest eigenvalues of $F_1$,

$$
\dot{V} \leq -\frac{A_{F_1}^*}{2} p^\top \rho.
$$

(15)

Finally, setting $\xi > \max\{1, \frac{\overline{w}^2}{\overline{w}^2}\}$, by Theorem 4.10 in Khalil and Grizzle (2002), we can conclude the global exponential stability of system (8), and the variable $y$ satisfies

$$
\|y(t) - y^\star\| \leq c_1 e^{-\frac{\lambda_2}{2} t},
$$

(16)

where $c_1 > 0$ is some constant and $c_2 := \frac{A_{F_1}^*}{T}$.\n
\textbf{Remark 3.} For each agent, the parameters in algorithm (6) only depend on its own information, so the proposed algorithm is fully distributed. In comparison, an exponential convergence rate is established, while the most related work (Li et al., 2020b) only provides an asymptotic convergence without analysis of the convergence speed building upon a stronger assumption than Assumption 4 here. Meanwhile, compared with Zhang et al. (2020), where an exponential convergence rate is obtained for homogeneous linear multi-agent systems under fixed directed graphs, the exponential convergence is established here for heterogeneous linear multi-agent systems based on a strictly weaker Assumption 4, including homogeneous linear multi-agent systems as a special case. It should be also noted that the algorithm in Theorem 1 can be extended to fixed directed graphs just like Zhang et al. (2020).

\textbf{Remark 4.} For the convergence rate, $c_2$ reaches the maximum value $\overline{c}_2 := \frac{\xi + \frac{\xi}{4} + 1 + \frac{1}{4} \xi - 4 \lambda_2 + 5}{\sqrt{\frac{2}{2} + \frac{\xi}{4} + 1 + \frac{1}{4} \xi - 4 \lambda_2 + 5}}$ when choosing $\xi = \frac{\overline{w}^2 + 1}{2m}$ in (16), where $\lambda_2$ is the second smallest eigenvalue of $L$, called the algebraic connectivity of $G$. To compare with the algorithm in Kia et al. (2015) for first-order integrator systems, by setting global parameters $\alpha = \beta = 1$, there, its variable $x$ has a similar convergence to (16): $|x(t) - x^\star| \leq c_3 e^{-\frac{\lambda_1}{2} t}$, where $c_3 > 0$ is some constant and $c_4 := \frac{\min\{\alpha + 1, \beta + 1\}}{4m}$

$$
\phi > \max\{1, \frac{\overline{w}^2}{2m} - 1\}. \quad \text{when } \overline{w}^2 > 4m - 1, \text{ by choosing } \phi = \frac{\overline{w}^2 - 2m + 1}{2m}, \text{ $c_4$ achieves its maximum value which is the same as $\overline{c}_2$, that is, the same convergence rate is obtained for both algorithms. When } \overline{w}^2 < 4m - 1, \text{ $c_4$ reaches the maximum value } \overline{c}_4 := \frac{\overline{c}_2}{\frac{\overline{w}^2 - 2m + 1}{2m}}, \text{ then it is easy to verify that } \overline{c}_2 > \overline{c}_4, \text{ which shows that the established convergence rate for algorithm (6) in this paper is faster than that of the algorithm in Kia et al. (2015).}

\textbf{4.2 Periodic Communication}

In order to avoid continuous communication and reduce communication overhead, we next discuss the case of discrete communication.

Suppose that $t_k^i$ is the $k$th communication instant of the $i$th agent, and denote $\tilde{y}_i(t) := \tilde{y}_i(t_k^i), \forall t \in [t_k^i, t_{k+1}^i)$ as the latest known output of agent $i \in V$ transmitted to its neighbors. The communication instant sequence of the $i$th agent $[t_1^i, \ldots, t_k^i, \ldots]$ will be determined later. We define a measurement error $e_i := \tilde{y}_i - y_i(t)$, and it is clear that $e_i = 0$ at any instant $t_k^i$.

Consider the next implementation of the algorithm (6) with discrete-time communication,

$$
u_i = -K_a x_i + K_b (\nabla f_i(y_i) - \sum_{j=1}^N a_{ij} (\tilde{y}_j - \tilde{y}_i) - \eta_i),
$$

(17a)

$$\eta_i = \sum_{j=1}^N a_{ij} (\tilde{y}_j - \tilde{y}_i), \eta_i(0) = 0_q,
$$

(17b)

where $K_a, K_b$ are feedback matrices and $a_{ij}$ is the weight corresponding to the edge $(j,i)$.

The following is the conclusion of the periodic communication control law.

\textbf{Theorem 2.} Supposing that Assumptions 1–4 hold, for linear multi-agent system (1) with discrete control protocol (17), the problem (2) is solved and $y_i(t)$ converges to $y^\star$ exponentially as $t \to \infty$ for $i = 1, \ldots, N$, if the communication instant is set as $t_{k+1}^i = t_k^i + \Delta, \forall \Delta \in (0, \tau_0]$, where $y^\star \in \mathbb{R}^d$ is the optimal solution to (2).

$$
\tau_0 := \frac{1}{\overline{w}} \ln \left(1 + \frac{(\overline{w} + 1)\epsilon}{\overline{w} + 1 + \sqrt{2} \lambda_2 + \sqrt{2} \lambda_2 \epsilon}\right).
$$

(18)
with \( \epsilon := \frac{1}{2 \sqrt{(\xi - 1)^2 + 1}} \) and \( \xi > \frac{4\pi^2 + 2 \lambda_j^2 + 1}{8\mu} \), and the feedback matrices \( K_{\alpha} \) and \( K_{\beta} \) are solutions of equations (4a) and (4b), respectively.

Proof. The compact form of the closed-loop system is

\[
\dot{x} = (A - BK_x)x + BK_y(-\nabla f(y) - (L \otimes I_q)y - \eta), \tag{19a}
\]

\[
\dot{y} = (L \otimes I_q)y, \quad \eta(0) = 0_{Nq}, \tag{19b}
\]

\[
y = Cx, \tag{19c}
\]

where \( y = col(\hat{y}_1, \ldots, \hat{y}_N) \).

Using the same state transformation as in the previous section and letting \( \hat{\rho} := \hat{y} - \hat{y} \), the dynamics (19) can be written as

\[
\begin{align*}
\dot{\hat{\rho}} &= -h - (L \otimes I_q)\hat{\rho} - \sigma, \tag{20a} \\
\dot{\sigma} &= (L \otimes I_q)\hat{\rho}. \tag{20b}
\end{align*}
\]

And because \( e_i = \hat{y}_i(t) - y_i(t) \), we have \( \hat{\rho} = \rho + e \) with \( e = col(e_1, \ldots, e_N) \):

\[
\begin{align*}
\dot{\hat{\rho}} &= -h - (L \otimes I_q)\rho - \sigma + e \tag{21a} \\
\dot{\sigma} &= (L \otimes I_q)\rho + e. \tag{21b}
\end{align*}
\]

Selecting the Lyapunov candidate as in (11), its derivative is

\[
\begin{align*}
\dot{V} &= -\xi \sigma^T h - (\xi - 1) \rho^T (L \otimes I_q) \rho - \sigma^T \sigma - \sigma^T h \\
&\quad - (\xi - 1) \rho^T (L \otimes I_q) e + \xi \sigma^T (\Pi \otimes I_q) e,
\end{align*}
\]

where \( \Pi \) is defined in Lemma 2.

By the inequality \(-((\xi - 1) \rho^T (L \otimes I_q) \rho \leq \frac{1}{2} \rho^T \rho + (\xi - 1) \rho^T \rho \leq \frac{1}{4} \rho^T \rho + \xi e^T e\), one has

\[
\begin{align*}
\dot{V} &\leq -\xi \mu \rho^T \rho - \frac{1}{2} \sigma^T \sigma + \frac{1}{2} \omega^T \rho^T \rho + \frac{1}{4} \rho^T \rho \\
&\quad + (\xi - 1) \rho^T \rho + \frac{1}{4} \rho^T \rho + \xi e^T e \\
&= -p^T F_2 \rho - \frac{1}{8} (p^T p - e \cdot e^T e), \tag{23}
\end{align*}
\]

where \( p = col(\rho, \sigma) \), \( e = \frac{1}{2 \sqrt{(\xi - 1) \rho^T \rho}} \) and

\[
F_2 := \begin{pmatrix} \frac{\xi \mu}{2} - \frac{1}{\xi} - 1/2 & 0_{Nq} \\ 0_{Nq} & \frac{1}{4} \end{pmatrix}
\]

For \( \xi > \frac{4\pi^2 + 2 \lambda_j^2 + 1}{8\mu} \), the matrix \( F_2 \) is positive definite. Inspired by Tran et al. (2019), let \( q = \frac{\|h\|}{\|p\|} \), and then its derivative is

\[
\dot{q} = \frac{\|\rho\|}{\|p\|} \leq \frac{(1 + q)\|p\|}{\|\rho\|}.
\]

Because

\[
\begin{align*}
\|\rho\| &\leq \|\frac{\|\|e\|\|}{\|\|p\|\|} + \|\sigma\| + \sqrt{2}\lambda_N\|\|p\|\| + \sqrt{2}\lambda_N\|\|e\|\|
\leq (\|\| + \sqrt{2}\lambda_N\|\|p\|\| + \sqrt{2}\lambda_N\|\|e\|\|,
\end{align*}
\]

we can conclude

\[
\dot{q} \leq (\|\| + 1 + \sqrt{2}\lambda_N\|\|1 + q) + \sqrt{2}\lambda_N\|\|+ q
= (\|\| + 1 + \sqrt{2}\lambda_N\|\|1 + q^2.
\]

By \( \epsilon(t_k) = 0 \), one has \( q(t_k) = 0 \). Choosing the differential equation

\[
\mu = (\|\| + 1 + \mu) + \sqrt{2}\lambda_N\|\|1 + \mu^2, \tag{24}
\]

the solution \( \mu(t) \) of (24) with \( \mu(0) = 0 \) is

\[
\mu(t) = \frac{(e^{\|\|t\|\|} - 1)(\|\| + 1 + \sqrt{2}\lambda_N\|\|)}{\|\| + 1 + \sqrt{2}\lambda_N\|\|e^{\|\|t\|\|}. \tag{25}
\]

Based on the Lemma 3.4 in Khalil and Grizzle (2002), the solution \( q(t) \) satisfies \( q(t) \leq \mu(t), \forall t > 0 \). Thus, for \( t_k < t_k \leq t_{k+1} \), we have \( q(t) \leq \mu(t_k) \equiv \epsilon, \) in which \( t_0 \) is defined in (18). Therefore, we get \( p^T p - e \cdot e^T e \geq 0 \).

Note that \( \Delta_{F_2} \) is the smallest eigenvalues of \( F_2 \):

\[
\hat{V} \leq -\Delta_{F_2} p^T p. \tag{26}
\]

Finally, setting \( \xi > \max\{1, \frac{4\pi^2 + 2 \lambda_j^2 + 1}{8\mu} \} \), by Theorem 4.10 in Khalil and Grizzle (2002), we can conclude the global exponential stability of system (19).

4.3. Event-triggered Communication

When the outputs of agents do not change too much, periodic communication schemes will transmit a lot of unnecessary data. So we introduce an event-triggered mechanism to further reduce the communication overhead.

Theorem 3. Supposing that Assumptions 1-4 hold, for linear multi-agent system (1) with discrete control protocol (17), the problem (2) is solved and \( y(t) \) converges to \( y^* \) exponentially as \( t \to \infty \) for \( i = 1, \ldots, N \), if the communication instant is chosen as \( t_{k+1} = t_k + \max(t_k, \Delta) \), where

\[
t_k := \max\{t^* - t_k, \|e(t_k)\|\} \geq \frac{1}{4\|h\| + \kappa} \sum_{j=1}^N a_{ij}||y_i - y_j||^2, \tag{27}
\]

with the parameter \( \kappa > \max\{\frac{\pi^2}{8\mu} + \frac{1}{2} \}, \) and \( y^*, K_{\alpha}, K_{\beta}, \Delta \) are the same as in Theorem 2.

Proof. Selecting the Lyapunov candidate (11), its derivative can be calculated as in (22).

Carrying out a transformation as in the proof of Theorem 2, one can get

\[
\rho \sigma^T (\Pi \otimes I_q) e \leq \frac{\xi^2}{4(\xi - 1)\kappa} \sigma^T \sigma + (\xi - 1)\kappa e^T e, \tag{28}
\]

in which \( \kappa \in \mathbb{R} \) is a parameter to be determined later.
Substituting (13), (14), (28) into (22) and noting $p = \text{col}(\rho, \sigma)$, one has

\[
\dot{V} \leq -\xi \rho^T \rho + \frac{\bar{m}^2}{2} \bar{\rho}^T \bar{\rho} - \frac{1}{2} \sigma^T \sigma - \xi - \frac{1}{2} \rho^T (L \otimes I_q)p
\]

\[
- (\xi - 1) \rho^T (L \otimes I_q)e + \frac{\bar{m}^2}{4(\xi - 1)} \sigma^T \sigma + (\xi - 1) \kappa \dot{e}^T e
\]

\[
\leq -p^T F_3 p - (\xi - 1) \bar{s}
\]

where

\[
F_3 := \left( \frac{(\xi m - \frac{1}{2} \bar{m}^2)}{2} \otimes I_{N_q} \right) \left( \frac{1}{2} - \frac{e^T}{\frac{\bar{m}^2}{4(\xi - 1)}} \right) \otimes I_{N_q}
\]

\[
\bar{s} := s - \kappa e^T e, \text{ and } s = \frac{1}{2} \rho^T (L \otimes I_q)p + \rho^T (L \otimes I_q)e.
\]

By using $I_y \dot{L} = 0$, and $p = y - \dot{y}$, we get

\[
s = \frac{1}{2} (y - \dot{y})^T (L \otimes I_q)(y - \dot{y}) + (y - \dot{y})^T (L \otimes I_q)e
\]

\[
= \frac{1}{2} \dot{y}^T (L \otimes I_q)y + \dot{y}^T (L \otimes I_q)e
\]

\[
= \frac{1}{2} \dot{y}^T (L \otimes I_q)y - \frac{1}{2} e^T (L \otimes I_q)e.
\]

Because $L = D_{out} - A$ and $D_{out} + A \succeq 0$, we can get $\frac{1}{2} e^T (L \otimes I_q)e \leq e^T (D_{out} \otimes I_q)e = \sum_{i=1}^N d_i^out e_i^T e_i$.

Because $\sum_{i=1}^N \sum_{j=1}^N a_{ij} (\dot{y}_i^T \dot{y}_j - \dot{y}_j^T \dot{y}_i) = 0$, one has

\[
\dot{y}^T (L \otimes I_q)\dot{y}
\]

\[
= \sum_{i=1}^N \sum_{j=1}^N a_{ij} \dot{y}_i \dot{y}_j - \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} (\dot{y}_i^T \dot{y}_j - \dot{y}_j^T \dot{y}_i)
\]

\[
= \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \|\dot{y}_i - \dot{y}_j\|^2.
\]

Therefore, it can be obtained that

\[
s \geq -\frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N a_{ij} \|\dot{y}_i - \dot{y}_j\|^2 - \sum_{i=1}^N d_i^out e_i^T e_i.
\]

By the triggering conditions (27), we get

\[
\bar{s} \geq \sum_{i=1}^N \left( \frac{1}{4} \sum_{j=1}^N a_{ij} \|\dot{y}_i - \dot{y}_j\|^2 - (d_i^out + \kappa) \|e_i\|^2 \right)
\]

\[
\geq 0,
\]

thereby implying

\[
\dot{V} \leq -p^T F_3 p.
\]

Choosing $\kappa > \max(\frac{\bar{m}^2}{4}, \frac{1}{2})$, there must be a parameter $\xi > 1$ that makes the matrix $F_3$ positive definite. Note that $\lambda_{min}$ is the smallest eigenvalues of $F_3$:

\[
\dot{V} \leq -\lambda_{min} F_3 p.
\]

By the Theorem 4.10 in Khalil and Grizzle (2002), we can conclude the global exponential stability of this system.

Remark 5. In Theorem 2 and Theorem 3, we have designed the discrete-time communication algorithms for the optimal output consensus of continuous heterogeneous linear multi-agent systems. Compared with the most related work (Li et al., 2020b) which only proves asymptotic convergence and has gradually decreasing communication interval, our algorithms guarantee exponential convergence and clearly give a lower bound of the communication interval, thus excluding the Zeno behavior.

5. Simulation

Example 1. Consider a network of six agents, where $A_{1,2} = [1, 0; 0, 1], A_{3,4} = [0, 1; -2, 1], A_{5,6} = [1, 1; 0, 1, 0, 1], B_{1,2} = [0, 1; 1, -2], B_{1,4} = [1, 1; 1, 0], B_{5,6} = [1, 0; 0, 1, 2, 0], C_{1,2} = [3, 0; 0, 1], C_{1,4} = [2, 2; -1, 1], C_{5,6} = [1, -1, 2, 1, 2, 2]$. The local objective functions are as follows with decision variable $y = (y_a, y_b)^T \in \mathbb{R}^2$:

\[
f_1(y) = (y_a - 5)^2 + 2(y_b - 3)^2;
\]

\[
f_2(y) = (2y_a + 5y_b - 9)^2 + \frac{0.2y_a}{\sqrt{y_a^2 + 2}};
\]

\[
f_3(y) = \ln(e^{y_a} + e^{y_b});
\]

\[
f_4(y) = (2y_a + 1)^2 + 2(y_b - 1)^2;
\]

\[
f_5(y) = (y_a + y_b)^2 + \ln(y_b + 3);
\]

\[
f_6(y) = \|y\|^2 + y_a + y_b.
\]

The communication network among these agents is depicted as Figure 4 with all the edge weights as 1.

It can be verified that Assumptions 1–4 hold. And we can calculate $\gamma^2 = (0.2622, 1.59614)^T$ by minimizing the global cost function $f(y) = \sum_{i=1}^6 f_i(y)$.

The parameters of each agent can be selected by the proposed algorithms, where $K_{a,12} = [2, 1; 0, 0], K_{a,34} = [2, -1; -2, 2], K_{a,56} = [0, 0; 0, 0, 0, 0], K_{b,12} = [0.667, 0.333, 0], K_{b,56} = [0.25, 0.5; 0, -1], K_{c,14} = [0.133, 0.0667; -0.333, 0.333], \text{ and } \Delta = 0.2$. The initial values $x_i(0)$ are randomly selected in $[-10, 10]$.

Figure 2 depicts the optimization errors $\sum_{i=1}^6 \|y_i(t) - y^*\|^2$ with three control laws of continuous, periodic, and event-triggered communication respectively. It can be seen that the outputs of all agents converge to the optimal value $y^*$ exponentially. Figure 3 shows the triggering instants of six agents with periodic and event-triggered communication control laws, from which we can observe that the communication among six agents is discrete and neither of them exhibits Zeno behavior. Compared with the periodic communication control law, the event-triggered mechanism can further reduce communication overhead.

Example 2. In order to verify the convergence speed of our algorithm, we compare it with the most related work (Li et al., 2020b) and the case of continuous communication. For convenience, we adopt the linear system parameters, objective functions and communication network that are consistent with the simulation in Li et al. (2020b). The parameters of our algorithm are selected as $K_{a,12} = [0, 1; 0, 0], K_{a,34} = [0.5, -0.5; 0, 0], K_{a,56} = [0.25, 1, -1.5; 0, 0, 0], K_{b,12} = [1; 0], K_{b,56} = [0.5; 0], K_{c,14} = [2, 2; -1, 1], C_{5,6} = [1, -1, 2, 1, 2, 2]$. The local objective functions are as follows with decision variable $y = (y_a, y_b)^T \in \mathbb{R}^2$:

\[
f_1(y) = (y_a - 5)^2 + 2(y_b - 3)^2;
\]

\[
f_2(y) = (2y_a + 5y_b - 9)^2 + \frac{0.2y_a}{\sqrt{y_a^2 + 2}};
\]

\[
f_3(y) = \ln(e^{y_a} + e^{y_b});
\]

\[
f_4(y) = (2y_a + 1)^2 + 2(y_b - 1)^2;
\]

\[
f_5(y) = (y_a + y_b)^2 + \ln(y_b + 3);
\]

\[
f_6(y) = \|y\|^2 + y_a + y_b.
\]
6. Results

This paper has investigated the optimal output consensus problem for heterogeneous linear multi-agent systems. A proportional-integral (PI) control law has been proposed, which can converge to the optimal solution exponentially. The proposed continuous algorithm does not require any global in-
formation, so it is fully distributed. Then, in order to avoid continuous communication among agents, the algorithm has been extended to periodic and event-triggered communication schemes. It was shown that the global exponential convergence is preserved and no Zeno behavior is exhibited.

Future works include extending the algorithms to the case of unbalanced directed and time-varying networks.

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