The Network Observability Problem:
Detecting nodes and connections and the role of graph symmetries

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Abstract

Reconstructing the connections between the nodes of a network is a problem of fundamental importance in the study of neuronal and genetic networks. An underlying related problem is that of observability, i.e., identifying the conditions under which such a reconstruction is possible. In this paper we consider observability of complex dynamical networks, for which we aim at identifying both node and edge states. We use a graphical approach, which we apply to both the Node Inference Diagram (NID) and the Node Edge Inference Diagram (NEID) of the network. We investigate the relationship between the observability of the NID and that of the NEID network representations and conclude that the latter can be derived from the former, under general assumptions. We further consider the effects of graph symmetries on observability and we show how a minimal set of outputs can be selected to obtain observability in the presence of graph symmetries.

Observability of nonlinear complex systems was studied in [18] through a graphical approach. This method was found to be computationally convenient when dealing with large complex networks. However, it fails in the presence of symmetries in the state variables. In this paper, we further expound on a graphical approach to compute observability of complex networks, for which both node states and edge states are to be estimated. We focus on the problem of estimating edge states, which is relevant to several applications, e.g., in the context of neuronal [35, 31] and genetic networks [8, 13, 12, 20, 11, 21], but also engineering networks [29, 30, 2, 5, 15]. We also consider an extension of the graphical approach method to the case in which graph symmetries are present. For this case, we introduce a simple algorithm for selecting the minimal set of output states for observability. This algorithm is tested on several examples of real world network topologies displaying symmetries.

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1 Introduction

Observability is the ability to deduce the state variables of a system for which only the outputs, typically coinciding with a subset of the states, can be measured. There are different methods that can be used to determine observability, ranging from an algebraic implementation to a graphical approach (GA). The latter allows to determine a set of output states from which one can infer information about the rest of the states by analyzing the properties of an inference graph. In [18] this method was applied to complex systems which can be represented as networks, for which accessibility to all of the nodes is typically unavailable. Examples of interest are genetic and neural networks, which could be reconstructed by observing a limited number of output states if these provide observability.

In biological networks, such as genetic or neuronal networks, an important problem is that of reconstructing the network architecture and the time evolution of the connections from existing data. For these applications, a typical limitation is the cost and the possibility of accessing the states of the individual nodes. The network inference problem and its application to the genome [8, 13, 9, 12, 20] has received significant attention over the last decade. References [11, 21] review and compare the more popular existing approaches. Connectivity between neuronal cells is a subject of investigations in neuroscience. A map of the nervous system of the nematode, Caenorhabditis elegans, with 302 neurons has been fully elucidated in [35]. A large ongoing collaborative effort aims at uncovering the brain connectome, with ultimate goal understanding in detail the architecture of the brain connectivity [31]. Moreover, an important question that is relevant to both genetic and neuronal networks, is what is the minimal amount of information needed to be able to reconstruct the state of these networks.

Some studies have considered the problem of estimating the structure of dynamical networks from limited measurements. Reference [23] considers a constrained optimization technique for inference of the network connections. A strategy for tracking links and connection strengths over time in neuronal networks from spike train data was presented in [4]. An adaptive strategy to estimate both node states and edge states from knowledge of the dynamical time evolution of only one node was presented in [2]. A Bayesian approach to reconstruct the structure of an undirected unweighted network was proposed in [15]. In this paper we will be concerned with the conditions for observability of a complex dynamical network, which is conditio sine qua non for implementing any type of estimation strategy. In particular, we will be concerned with the problem of characterizing the minimal set of sensor nodes for observability. Observability of small network motifs was studied in [34]. Reference [36] considered observability of large networks, for which a phase transition was observed and characterized as the number of measurement nodes is increased.

2 Observability

Since the vast majority of systems of practical interest are nonlinear, observability of nonlinear systems is of great relevance. Consider a nonlinear system
represented by,
\[ \dot{x} = f(x, u), \]  
(1)
where \( x \in \mathbb{R}^N \) is the system state, \( u \in \mathbb{R}^P \) is the system input, \( f : \mathbb{R}^N \times \mathbb{R}^P \to \mathbb{R}^N \) and
\[
y = \begin{bmatrix} h_1(x) \\ h_2(x) \\ \vdots \\ h_M(x) \end{bmatrix} \]  
(2)
is the system output, \( y \in \mathbb{R}^M, h_i : \mathbb{R}^N \to \mathbb{R} \). If the initial-state vector \( x(0) \) can be found from \( u(t) \) and \( y(t) \) measured over a finite interval of time, the system is said to be observable.

The Lie derivative of \( h \) with respect to \( f \) is defined to be
\[ L_fh = \frac{\partial h}{\partial x} f, \]  
(3)
where the function \( f \) is of the form:
\[ f = \begin{bmatrix} f_1(x) \\ \vdots \\ f_N(x) \end{bmatrix}. \]  
(4)
Then the Lie derivative is:
\[ L_fh = \left[ \frac{\partial h}{\partial x_1} f_1(x), \ldots, \frac{\partial h}{\partial x_N} f_N(x) \right] \]  
(5)
and the result of this derivative operation is a scalar.

Two states \( x_0 \) and \( x_1 \) are distinguishable if there exists an input function \( u^* \) such that:
\[ y(t, x_0) \neq y(t, x_1), \]  
(6)
where \( y(t, a) \equiv y(t) \) for \( x(0) = a \). This implies that the system is locally observable about \( x_0 \) if there exists a neighborhood of \( x_0 \) such that every \( x \) in that neighborhood other than \( x_0 \) is distinguishable from \( x_0 \).

A test for local observability about \( x_0 \) is that the matrix:
\[
O(x_0, u^*) = \begin{bmatrix} \frac{\partial L_1^0 h_1}{\partial x_1} & \frac{\partial L_1^0 h_1}{\partial x_2} & \cdots & \frac{\partial L_1^0 h_1}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_M^0 h_1}{\partial x_1} & \frac{\partial L_M^0 h_1}{\partial x_2} & \cdots & \frac{\partial L_M^0 h_1}{\partial x_N} \\ \frac{\partial L_1^{N-1} h_1}{\partial x_1} & \frac{\partial L_1^{N-1} h_1}{\partial x_2} & \cdots & \frac{\partial L_1^{N-1} h_1}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_M^{N-1} h_1}{\partial x_1} & \frac{\partial L_M^{N-1} h_1}{\partial x_2} & \cdots & \frac{\partial L_M^{N-1} h_1}{\partial x_N} \\ \frac{\partial L_1^{N-1} h_M}{\partial x_1} & \frac{\partial L_1^{N-1} h_M}{\partial x_2} & \cdots & \frac{\partial L_1^{N-1} h_M}{\partial x_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_M^{N-1} h_M}{\partial x_1} & \frac{\partial L_M^{N-1} h_M}{\partial x_2} & \cdots & \frac{\partial L_M^{N-1} h_M}{\partial x_N} \end{bmatrix} \]  
(7)
is of rank \( N \). If the system is locally observable about any \( x_0 \in \mathbb{R}^N \), the system is said to be observable. Note that testing observability may require that the rank observability test is repeated for potentially infinitely many points \( x_0 \). Alternatively, one may recur to using a symbolic method for calculating the rank of (7).
3 Structural observability

The algebraic methods introduced in Sec. 2 to determine observability of a system can get quite complicated as \( N \), the number of states, increases. For example, the complexity of symbolic calculations of the rank of a matrix may become prohibitive for large \( N \). In this paper we will follow an alternative approach based on the concept of structural observability, which has been shown to provide computational advantages \[16, 22, 27, 14, 32, 17, 18\]. As we will see, structural observability reduces to the analysis of an inference diagram that describes the interactions between the state variables. Through this approach, the network observability problem is reduced to a purely topological one. Note that there are particular conditions under which structural observability does not imply observability \[16, 22, 27, 17, 18\]. Consider for example symmetries between two or more state variables that make these states undistinguishable from the system output. Such symmetries will make the system unobservable, though detecting them is typically beyond the scope of structural observability analysis.

In what follows, we will review the graphical approach proposed in \[17, 18\] to analyze controllability and observability of complex networks and will extend this approach to the case in which one wants to reconstruct both the states of the network nodes and connections. Moreover, we will exploit information contained in the inference diagram on the graph symmetries. We will employ available software to detect these symmetries \[25\] and we will show that under appropriate assumptions, by taking this information into account, it is possible to identify a minimal set of output states for observability. In particular, we will proceed under the assumption that each individual system is observable, i.e., that from measuring its output, we are able to observe its full state and by doing so, we will be able to focus on the effects of the network structure on observability.

Here we briefly review the graphical approach (GA) method for structural observability, described in \[13\]. This approach was used in \[15\] to predict and identify the minimum set of output states required for observability from an inference diagram. The inference diagram is a graphical representation of the interactions between state variables: consider two variables \( A \) and \( B \), if variable \( A \) appears in the dynamical equation for the evolution of variable \( B \) (\( A \) affects \( B \)), then there will be a directed edge from state \( A \) to state \( B \) in the inference diagram. Note here that the convention in this paper to construct the inference diagram is different from that in Ref. \[18\], namely if state \( A \) affects state \( B \) we draw a directed edge from \( A \) to \( B \) in the inference diagram (while in \[18\] we would have a directed edge from \( B \) to \( A \)).
The inference diagram is used to identify strongly connected components (SCCs) and root strongly connected components (root SCCs). In the figure (non-root) SCCs are delimited by dashed ovals and root SCCs by solid ovals. In order to achieve observability, at least one node needs to be selected from each root SCC.

The inference diagram contains information that can be used to identify the sensor nodes required to achieve observability. It corresponds to a directed unweighted graph. Consider a directed unweighted graph \( G = (V; E) \), where \( V \) is the set of nodes and \( E \subseteq V \times V \) is the set of edges. A path from node \( v_0 \) to node \( v_k \) in \( G \) is an alternating sequence of the type \((v_0, (v_0, v_1), v_1, (v_1, v_2), ..., (v_k, v_k), v_k)\) of nodes and edges that belong to \( V \) and \( E \), respectively. Two nodes \( v \) and \( w \) in \( G \) are path equivalent if there is a path from \( v \) to \( w \) and a path from \( w \) to \( v \). Path equivalence partitions \( V \) into maximal disjoint sets of path equivalent nodes. These sets are called the strongly connected components (SCCs) of the graph [24].

This means that, in a SCC, there exists a path from any node to all other nodes in that SCC. The sensor nodes predicted from the GA analysis are chosen from the root SCCs. A root SCC is a SCC which has no outgoing links, meaning that their information cannot be concluded from any other node in the inference diagram. However, root SCCs may have incoming links from other SCCs. This implies that information of nodes within other SCCs may be inferred from the root SCCs. In order to achieve observability it is required that at least one sensor node (any node) is selected from each root SCC [18]. In the case presented in Fig. 1, since there are three root SCCs, three sensor nodes will have to be selected to achieve observability. Note that if the diagram is undirected and connected, choosing any node in the graph as sensor node would provide observability.

When applied to the inference diagram of a network of coupled dynamical systems, GA may be used to identify a subset of sensor nodes [18] to be monitored to yield observability. In [18] it was argued that typically, for nonlinear systems, the sensor set predicted by GA is necessary and sufficient for observability, based on the assumption that for large randomly generated directed
networks, the probability of developing symmetries in the states variables approaches zero.

Hence Secs. 4 and 5, based on the results of [18] we will also assume that GA provides necessary and sufficient conditions for observability. Note that, for illustrative purposes, in Sec. 5 we will sometimes present examples of small networks which display symmetries. However, this should not distract the reader, as in Sec. 5 our focus is not on symmetries but rather on extending the approach of [18] to the problem of observing the states associated with the connections of a given dynamical network. The case in which symmetries are present, will be treated in detail in Sec. 6.

4 Graphical approach to observe nodes and connections of a complex network

The graphical approach is a powerful method to analyze and determine whether a system is observable or not. With this method one can select some of the states of a network to be monitored to collect information about the entire system. In many applications, it is often the case that one wants to reconstruct the structure of a network of coupled dynamical systems. Frequently, the goal is not only that of estimating the states corresponding to the network nodes, but also those corresponding to the connections, which is important e.g., in neural and genetic networks. Faults occurring in large technological networks can be dynamically detected by monitoring the dynamics of the connections between the nodes, e.g., a fault may affect the state of one or more connections [2]. In sensor networks, knowledge about the time evolution of the connections between sensors can be used to survey an area of interest, e.g. to detect the movement of an object through the area [29, 30]. In networks of coupled mobile platforms, the time evolution of the connections can provide information on the environment, e.g. on the presence of obstacles [5]. Moreover, in the absence of information about the edge states, it is impossible to reconstruct the network from partial measurements of the nodes. Hence, in a complex network, the two problems of identifying node states and edge states are typically intertwined.

In this paper, we will explicitly take into account a situation for which the network states include both node states and edge states. We will generally assume, without loss of generality, that node states are vectors composed of several variables, \( x_i \in \mathbb{R}^q, i = 1, ..., n \), while edge states are scalars. We will assume to have access to a subset of these states (i.e., both a subset of the nodes and of the edge states) and will analyze the conditions under which observability of the whole network is achieved.

In the following sections, we will extend the GA method presented in [18] and [26] to the case that the states corresponding to the network connections are explicitly taken into account. The following two cases will be considered: static connections (they do not evolve in time or they evolve in time independently of the network states but on a time scale that is much longer than the time scale on which the individual nodes evolve) and adaptively evolving connections.
(they respond to the dynamics of the network nodes). A typical example of such adaptive dynamics is synaptic plasticity in neural networks, which is a process by which neural activity can actively modify the strength or efficacy of synaptic transmission [3].

In what follows we will explicitly consider the two cases of directed and undirected networks. While for the case of undirected networks, we will consider that the network states are \( \{x_i\} , i = 1, \ldots, n \) and \( \{A_{ij} = A_{ji}\} , i = 1, \ldots, n , j = (i+1), \ldots, n \), for the case of directed networks, the states will be \( \{x_i\} , i = 1, \ldots, n \), and \( \{A_{ij}\} , i, j = 1, \ldots, n , j \neq i \). Therefore, the total number of network states is \( N = N_n + N_e \), where \( N_n \) is the number of node states, \( N_n = nq \), and \( N_e \) is the total number of edge states, \( N_e = \frac{n(n-1)}{2} \) if the network is undirected and \( N_e = n(n-1) \) if the network is directed.

![Network Diagram](image)

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Figure 2: a) Three coupled tank network. b) Node Inference Diagram (NID) of the network in a). c) Node and Edge Inference Diagram (NEID) of the network in a). Node states are represented as yellow (light gray) circles and edge states as red (dark gray) circles.

As an example, Fig. 2(a) depicts the physical representation of a network consisting of three coupled tanks interconnected one another. The states variables \( x_1, x_2, \) and \( x_3 \) represent the volume of liquid in tank 1,2, and 3, respectively; \( A_{12} \) and \( A_{23} \) determine the liquid flow rates between tanks 1 and 2 and tanks 2 and 3, respectively, and are functions of the sections and of the lengths of the pipes that connect the tanks. Fig. 2(b) shows the Node Inference Diagram (NID) of the network. In a NID representation, there is a one-to-one correspondence between nodes of the NID and those of the corresponding network. Note that GA can be applied to the NID in Fig. 2(b) to determine observability. However, for a situation in which the connections (in this case the sections and the
lengths of the pipes) are unknown, we introduce an extended inference graph, which represents both node and edge states. An example of such a graph is shown in Fig. 2(c). We call the graph in Fig. 2(c) a Node and Edge Inference Diagram (NEID). In this alternative description, the diagram nodes can either correspond to the network nodes (shown in yellow or light gray in the figure), or to the network connections (shown in red or dark gray in the figure). Note that this is the case of interest for neural and genetic networks. In the next section we will apply GA to the NEID representation in Fig. 2(c) for several examples of dynamics on complex networks. In doing so, we will be able to characterize the observability of complex dynamical networks.

5 Observability of complex dynamical networks

In this section, we will present GA and its application to both the NID and NEID representations of a network presented in Sec. 4. We will employ this method to determine the strongly connected components (SCCs), root SCCs, and the sensor nodes for several examples of dynamics on complex networks. In general, knowledge about whether the network is directed or undirected is of fundamental importance. As we will see, our approach to determining observability will depend on this information.

In what follows we will classify the networks in four different classes: undirected and directed networks with constant connections, and undirected and directed networks with adaptive (time-evolving) connections.

5.1 Undirected networks with constant connections

The first case we consider is that of undirected networks with constant connections. A mathematical description for such networks is the following,

\[
\dot{x}_i = f(x_i) + g(y_1, ..., y_k, ..., y_n, A_{i1}, ..., A_{ik}, ..., A_{in}), \quad i = 1, ..., n, \quad k \neq i, \quad (8)
\]

\[
y_i = h(x_i), \quad (9)
\]

\[
A_{ij} = A_{ji}, \quad \dot{A}_{ij} = \dot{A}_{ji} = 0, \quad i = 1, ..., n, \quad j = (i + 1). \quad (10)
\]

In Eq. (8) the function \(f(x_i)\) determines the uncoupled dynamics of node \(i\), the function \(h(x_i)\) is the output of node \(i\), and the function \(g\) represents the coupling with the other nodes in the network. A large class of complex dynamical networks can be described by equations (8-10).

Here and in what follows, we will proceed under the assumption that each individual system is observable, i.e., that from measuring \(y_i\), we are able to observe the full state \(x_i\) of node \(i\). Thus we assume observability of the individual systems and focus on the conditions under which the network is observable from measuring a subset of its states. Note that this assumption is important as it allows us to identify each node of the inference diagram with either a node or a connection of the network, without having to worry about the several state variables that form each node of the network. The question we are interested in is whether by measuring a subset of the states (either nodes or connections), we will obtain enough information to reconstruct the whole network.
As an example, Fig. 3 shows the NID and NEID representations of a small ($n = 3$) undirected network with constant connections. The states for this network are the following: the node states, $x_1$, $x_2$, and $x_3$, and the edge states, $A_{12} = A_{21}$, $A_{13} = A_{31}$, and $A_{23} = A_{32}$.

Figure 3: a) NID and corresponding b) NEID for an undirected network with constant connections. Non-root SCCs are delimited by dashed ovals and root SCCs by solid ovals.

For this case, equations (8) and (10) become:

$$
\dot{x}_1 = f(x_1) + g(y_2, y_3, A_{12}, A_{13}),
$$
$$
\dot{x}_2 = f(x_2) + g(y_1, y_3, A_{21}, A_{23}),
$$
$$
\dot{x}_3 = f(x_3) + g(y_1, y_2, A_{31}, A_{32}),
$$
$$
\dot{A}_{12} = \dot{A}_{21} = 0,
$$
$$
\dot{A}_{13} = \dot{A}_{31} = 0,
$$
$$
\dot{A}_{23} = \dot{A}_{32} = 0.
$$

Note that the time derivative of the edges states is equal to zero because of the assumption of constant connections. We first apply GA to the NID representation (Fig. 3 a)) from which we obtain that the whole inference diagram forms a root SCC as there is a path from any of its nodes to any other node. We wish to emphasize that for any undirected connected graph, this is always the case. Hence, selecting any node in the graph to be a sensor guarantees NID observability.

Fig. 3b) illustrates the corresponding NEID representation, where, the connections are also shown as network states. We see from the GA analysis that, the nodes in the root SCC (those associated with the nodes) are the same in the NID and NEID representations in Fig. 3 a) and b), while the edge states form additional SCCs in the NEID representation.
5.2 Directed networks with constant connections

The next case we consider is that of directed networks with constant connections. A mathematical description for such networks is the following,

\[ \dot{x}_i = f(x_i) + g(y_1, ..., y_k, ..., y_n, A_{i1}, ..., A_{ik}, ..., A_{in}), \quad i = 1, ..., n, \quad k \neq i, \]  

(11)

\[ y_i = h(x_i), \]  

(12)

\[ \dot{A}_{ij} = 0, \quad i, j = 1, ..., n, \quad j \neq i. \]  

(13)

A large class of complex dynamical networks can be described by equations (11-13). As an example, consider synchronization between coupled oscillators, arranged to form a network, see e.g. Ref [1].

Also in this case, we employ GA to determine SCCs, root SCCs, and the output states to be selected to achieve observability.

Consider a directed network with constant connections, \( n = 3 \), node states \( x_1, x_2, \) and \( x_3 \), and edge states \( A_{12}, A_{13}, A_{21} = 0, A_{23}, A_{31} = 0, \) and \( A_{32} \), described by the following set of equations,

\[ \dot{x}_1 = f(x_1) + g(y_2, y_3, A_{21}, A_{31}), \]  

\[ \dot{x}_2 = f(x_2) + g(y_1, y_3, A_{12}, A_{32}), \]  

\[ \dot{x}_3 = f(x_3) + g(y_1, y_2, A_{13}, A_{23}), \]  

\[ \dot{A}_{12} = 0, \dot{A}_{21} = 0, \dot{A}_{13} = 0, \dot{A}_{23} = 0, \dot{A}_{32} = 0. \]

The edge states are constant in time. Hence, the time evolution of the node states depends on the edge states but not vice versa.
Note that the NID is shown in Fig. 4 a) and the corresponding NEID in Fig. 4 b). From Fig. 4 a) we see that GA reveals that node \( x_1 \) forms a root SCC and the other nodes form SCCs. Hence, by monitoring \( x_1 \), one can potentially infer information on the remaining nodes in the network. Fig. 4 b), shows the corresponding NEID. The GA analysis shows that the SCCs and root SCCs are the same in the NID and NEID representations in Fig. 4 a) and b), with the difference that the edge states constitute extra SCCs in Fig. 4 b). Thus for both the NID and NEID representations, GA provides the same answer, that in order to achieve observability, \( x_1 \) needs to be monitored.

To conclude, for (either undirected or directed) networks for which the connections are constant, GA tells us that the root SCCs and the SCCs of the NEID coincide with those of the NID, except for extra SCCs corresponding to the edge states. The edge states do not form root SCCs, as observing the constant connections does not provide any benefit in terms of observability. In what follows, we will analyze networks with time evolving adaptive connections, i.e., for which the couplings evolve under the influence of the nodes they connect.

### 5.3 Networks with undirected adaptive couplings (evolving connections)

In this section we consider networks with adaptive connections, i.e., such that they evolve in time depending on the states of the nodes. We first consider the case of undirected adaptive edges. In particular, we assume that the time evolution of each connection \( A_{ij} \) depends on the nodes \( i \) and \( j \) at its end points.

A mathematical description for these networks is the following,

\[
\dot{x}_i = f(x_i) + g(y_1, ..., y_k, ..., y_n, A_{i1}, ..., A_{ik}, ..., A_{in}), \quad i = 1, ..., n, \quad k \neq i, \quad (14)
\]

\[
y_i = h(x_i), \quad (15)
\]

\[
A_{ij} = A_{ji}, \quad (16)
\]

\[
\dot{A}_{ij} = \ell(x_i, x_j), \quad \text{if } (i, j) \in E, \quad \text{i.e., if the connection exists} \quad (17)
\]

and \( A_{ij} \) is constantly equal to zero otherwise, \( i = 1, ..., n, \ j = (i + 1), ..., n \). In this case \( \ell : R^q \times R^q \to R \) is a symmetric function of the two variables \( x_i \) and \( x_j \) that determines the time evolution of the edge states. A large class of networks can be described by equations (14 - 17). Examples are synchronizing adaptive networks [10] for which the coupling between nodes can be adaptively varied to obtain synchronization, and formation control networks [7] for which the strengths of the connections may evolve in time to guarantee that the distances between the units remain equal to a certain desired value.
Figure 5: a) Undirected network for which the edge states $\dot{A}_{ij} = \ell(x_i)$ (the case $\dot{A}_{ij} = \ell(x_j)$ is analogous). b) Undirected network for which $\dot{A}_{ij} = \ell(x_i, x_j)$. In both cases a) and b), the nodes along with the edge that connect them fall into the same root SCC. c) Directed network in which the synapsis only depends on the postsynaptic neuron, i.e., $\dot{A}_{ij} = \ell(x_i)$. In this case the synapsis $A_{ij}$ falls into the same root SCC as $x_i$ while $x_j$ forms an isolated SCC. d) Directed network in which the synapsis only depends on the presynaptic neuron, i.e., $\dot{A}_{ij} = \ell(x_j)$. In this case, $A_{ij}$ and $x_j$ form isolated SCCs while $x_i$ forms a root SCC. e) Directed network in which the synapsis depends on both the presynaptic and postsynaptic neurons, i.e., $\dot{A}_{ij} = \ell(x_i, x_j)$. Here, the synapsis $A_{ij}$ falls into the same root SCC as $x_i$ while $x_j$ forms an isolated SCC.

We are now going to answer the question, by comparing the NID and NEID representations, in Fig. 5 a) and b), whether the NEID edge states form isolated SCCs or they fall into the SCCs determined by the NID analysis. As it can be seen, in the case of undirected adaptive networks, edge states fall into the same NID SCCs as the nodes that they connect. Hence, if the original network is connected, all the node states as well as the edge states (for the existing connections) become part of a unique root SCC. Another exemplificative case is shown in Fig. 6.
Figure 6: NID (a) and corresponding NEID (b) for an undirected network with adaptive connections.

The NID in Fig. 6 a) is undirected and connected and thus forms a unique SCC. In Fig. 6 b), also the edge states (for the existing edges) fall into a unique root SCC. An important observation is that for undirected networks with adaptive connections not only the nodes can be sensors, but also the edges. This is due to the fact that the edges evolve in time based on the time evolution of the nodes. Therefore, for this case, either any node state or any edge state can be monitored to achieve observability.

5.4 Networks with directed adaptive couplings (evolving connections)

In this section we still consider adaptive couplings but in the context of directed networks. In particular we focus on the relevant example of neuronal networks, for which each directed connection corresponds to a synapsis. We consider the following three alternative assumptions:

1. The evolution of the connections (synapses) depends on the presynaptic neuron only [6].

2. The evolution of the connections (synapses) depends on the postsynaptic neuron only.

3. The evolution of the connections (synapses) depends on both the presynaptic and postsynaptic neurons [28].

We will investigate these three cases separately.

The mathematical description for these networks is the following,

\[ \dot{x}_i = f(x_i) + g(y_1, \ldots, y_n, A_{i1}, \ldots, A_{ik}, \ldots, A_{in}), \quad i = 1, \ldots, n, \quad k \neq i, \quad (18) \]

\[ y_i = h(x_i), \quad (19) \]

\[ \dot{A}_{ij} = \ell(x_i, x_j), \quad \text{if } (i, j) \in E, \quad \text{i.e., if the connection exists,} \quad (20) \]
and $A_{ij}$ is constantly equal to zero otherwise; $i, j = 1, \ldots, n, j \neq i$. In particular note that for case 1, $\ell(x_i, x_j) \equiv \ell(x_j)$ and for case 2, $\ell(x_i, x_j) \equiv \ell(x_i)$. The three cases are illustrated in Fig. 5 c), d), and e).

As it can be seen from Fig. 5 c) and e), in case the synapsis depends either on the postsynaptic neuron or on both the presynaptic and postsynaptic neurons, it falls into the same (root) SCC as its postsynaptic neuron. Whether this is a SCC or a root SCC will be determined by the NID GA analysis, i.e. whether the postsynaptic neuron belongs to a SCC or a root SCC in the NID representation. However, in the case that the synapsis depends on the presynaptic neuron only, it will form an isolated SCCs.

6 Network symmetries and observability

In this section we will analyze the effects of graph symmetries on observability. Note that Ref. [18] discussed symmetries but did not consider the specific effects of graph symmetries on observability. To start with, we will introduce some assumptions on the network dynamics. We will then consider several examples of inference diagrams with symmetries; for each individual diagram, we will use GA to determine SCCs as well as root SCCs and will methodically examine the conditions for observability. We will then propose a simple algorithm for selecting a minimal set of sensor nodes for observability in networks with graph symmetries. This algorithm will be tested on several examples of real world network topologies.

While we do not attempt to present here an exhaustive treatment of the effects of symmetries on the observability of networks, we will present several examples and we will show that: (i) the GA method presented in the previous section fails in the presence of graph symmetries and (ii) full observability can be restored by selecting appropriate additional sensor nodes. Our main goal in this section is to show that GA can be conveniently exploited not only to find (root) SCCs, but also to detect symmetries. Several softwares are available that given a graph, return a list of all the graph symmetries. Hence, here we propose that SCCs analysis in conjunction with the analysis of symmetries of the inference diagram may enable us to identify the minimal set of sensor nodes for observability.
Figure 7: NID representation of four networks displaying symmetries. SCCs and root SCCs are delimited by dashed ovals and solid ovals, respectively. For each network, a minimal set of sensor nodes to achieve observability is determined (the sensor nodes are those pointed by small purple arrows).

We define a graph symmetry as a permutation between tuples of nodes for which the dynamics of the network is equivariant. In what follows, we will specify the network equations to be,

\[
\dot{x}_i = f(x_i) + g(A_{i1}y_1, A_{i2}y_2, \ldots, A_{in}y_n), \quad i = 1, \ldots, n,
\]

with \( y_i = h(x_i), \)

where \( g : R^m \times R^m \times \ldots \times R^m \rightarrow R^q \) is a symmetric function of its variables. While in general, symmetries may appear both in the equations of the individual nodes and in the couplings between them (graph symmetries), we will proceed under the same assumption introduced in the previous section, that each individual system is observable, i.e., that from measuring \( y_i \), we are able to observe the full state \( x_i \) of each node \( i \). Therefore, in what follows we will focus on graph symmetries.

In Fig. 7 a) nodes 2 and 3 can be swapped without modifying the dynamics of node 1. In Fig. 7 b) any permutation of the ordered pair of nodes (2,3), (4,5), (6,7) and (8,9) leaves the dynamics of node 1 unaltered. In Fig. 7 c) a permutation of the ordered triplets of the nodes (2,3,4) and (5,6,7) leaves the dynamics of node 1 unaltered. Fig. 7 d) shows four independent symmetries occurring in the same diagram, namely, swapping nodes 4 and 5 would leave the dynamics of the rest of the network unaltered, as well as swapping nodes 2 and 3, and 6 and 7. Moreover, another symmetry is generated by the pairs (2,3) and (6,7).
Hereafter, we will discuss how the symmetries of the diagrams in Fig. 7 limit their observability and how additional sensor nodes can be selected to restore it. In Fig. 7 a) one can see a network composed of three nodes 1, 2, and 3, where node 1 forms a root SCC and nodes 2 and 3 form individual SCCs. Moreover, nodes 2 and 3 are symmetric. This means that monitoring node 1 alone is not sufficient to achieve observability, while monitoring node 1 and either node 2 or 3 will provide observability. In Fig. 7 b), the symmetric ordered pairs (2,3), (4,5), (6,7), (8,9) form individual SCCs. To achieve observability we need to monitor node 1 along with at least three other nodes and each of these extra nodes need to be selected from three different symmetric pairs. The third case shown in Fig. 7 c) is that of a network in which the ordered triples (2,3,4) and (5,6,7) are symmetric. Similarly to the case presented in Fig. 7 b), for observability, we need to monitor node 1 and at least one other node from either ordered triplet. In the forth case, Fig. 7 d), there are symmetries both inside the root SCC, as well as in the other two SCCs. A minimal set of sensor nodes for this case will be: either node 4 or 5, either node 2 or 3 and either node 6 or 7. An algorithm to identify the minimal sensor set in the presence of graph symmetries will be presented in what follows.

Figure 8: a) NID representation of a network with symmetries between two SCCs. b) NID representation of a network with symmetries between two root SCCs. c) NID representation of a network with symmetries between the SSCs. d) NID representation of a network with symmetries between two SCCs. e) NEID representation of the network in d). f) NEID representation of the network in d) for the case of adaptive connections.
We continue to analyze cases in which graph symmetries occur. We see from Fig. 8 a) that nodes 2 and 3 are symmetric. In this case, for observability, we need to monitor either one of the nodes in the root SCC along with either node 2 or node 3. Fig. 8 a) shows the case of a symmetry between two nodes in two different SCCs. By definition of SCC and root SCC, a node in a SCC and a node in a root SCC cannot be symmetric. However, there can be symmetries between nodes that belong to the same root SCC and also between nodes in different root SCCs (as shown in Fig. 8 b)). In Fig. 8 b) the ordered triplets (1,2,3) and (4,5,6) are symmetric and there exist symmetries within each root SCC, nodes 2 and 3 are symmetric as well as nodes 5 and 6. In this case, by monitoring a node from each root SCC, the symmetries inside each root SCC are resolved and observability is achieved. The same symmetries are present in Fig. 8 c), but in this case the triplets (1,2,3) and (4,5,6) contain nodes from both SCCs and root SCCs. However, nodes belonging to non-root (root) SCCs can only be swapped with other nodes belonging to non-root (root) SCCs.

We will now analyze graph symmetries for a case with both NID and NEID representations shown in Fig. 8 d), Fig. 8 e) and Fig. 8 f), respectively. In Fig. 8 d) nodes 2 and 4 are symmetric. Hence, as in the previous cases, monitoring only the root SCC is not sufficient for observability. Because of the symmetry generated by nodes 2 and 4, we need to monitor the root SCC node 1 and either node 2 or 4. By looking at the NEID representation of the network in Fig. 8 d), it can be seen that both the edge states $A_{32}$ and $A_{43}$ and the nodes 2 and 4 are symmetric. In general, symmetries in the node states generate symmetries in the edge states. However, being able to resolve the node symmetry, allows to resolve the edge symmetry too. In fact, in the case of Fig. 8 e), observability can be obtained by monitoring node 1 and either one of nodes 2 or 4 or either one of the edges $A_{32}$ or $A_{43}$. Fig. 8. f) is the NEID equivalent of Fig. 8 d), but for the case that the connections evolve adaptively. The analysis of symmetries in Fig. 8 f) leads to the same conclusions as those obtained for Fig. 8 e).

In the cases presented in Figs. 7 and 8, in which graph symmetries are present, monitoring one node from each root SCC is not sufficient to obtain observability. In general, additional nodes need to be monitored. Here we are going to present an algorithm that, given a NID with symmetries, returns a minimal set of sensor nodes $S$ to obtain observability. First we present the algorithm for the case that there are no symmetries. Consider a diagram with $z$ root SCCs. Each root SCC can be described by a set as follows,

$$
\Pi^1 = [\Pi^1_1, \Pi^1_2, \ldots, \Pi^1_{n_1}], \Pi^2 = [\Pi^2_1, \Pi^2_2, \ldots, \Pi^2_{n_2}], \ldots, \Pi^z = [\Pi^z_1, \Pi^z_2, \ldots, \Pi^z_{n_z}], \tag{22}
$$

where $n_i$ is the number of elements in the set $i = 1, \ldots, z$. Then as we need to choose a node from each root SCC, and as SCCs are disjoint sets, the minimal set of sensor nodes $S = [\Pi^1_s, \Pi^2_s, \ldots, \Pi^z_s]$, where $\Pi^i_s$ is any element from the set $\Pi^i$. In the case of a NID with $c > 0$ symmetries, let a symmetry $S_i$ be represented by the following list of $m_i$ $l_i$-tuples,

$$
S_i = ((t^i_1), (t^i_2), \ldots, (t^i_{m_i})), \tag{23}
$$

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\[ i = 1, \ldots, c. \] Here, \( m_i \) is the total number of tuples in \( S_i \) (note that any permutation of tuples belonging to the same symmetry is equivariant with respect to the network dynamics) and \( l_i \) is the number of elements of the tuples in \( S_i \). We call \( m_i \) the number of symmetry \( S_i \) and \( l_i \) the length of symmetry \( S_i \). Without loss of generality, we label the symmetries so that \( l_1 \leq l_2 \leq \ldots \leq l_c \), i.e., in order of nondecreasing length.

Now we introduce the definition of internal and external symmetry. A symmetry \( S_i \) is internal to another symmetry \( S_j \), \( j \neq i \) (or, equivalently, \( S_j \) is external to \( S_i \)) if there exists a tuple of \( S_j \), say \( t^i_j \), such that all the entries of the tuples \( t^i_k \) are also entries of \( t^i_j \), \( k = 1, \ldots, m_i \).

In order to achieve observability, we need to include in \( S \): (A) at least one element from each set \( \Pi_i, i = 1, \ldots, z \) and (B) at least one element from \((m_i - 1)\) different tuples of \( S_i \), \( t^i_1, t^i_2, \ldots, t^i_{m_i} \), \( i = 1, \ldots, c \). Given a set of sensor nodes \( S \), if condition (A) is verified for a given \( i \), we say that the root SCC \( \Pi_i \) is resolved, and if condition (B) is verified for a given \( i \), we say that symmetry \( S_i \) is resolved.

We now present an algorithm to obtain a minimal set \( S \) which provides observability, in the presence of graph symmetries. Our algorithm is based on the observation that if a symmetry does not present internal symmetries, it will be resolved by choosing any \((m_i - 1)\) elements from \((m_i - 1)\) different tuples. Otherwise, if a symmetry presents internal symmetries, then resolving the internal symmetries ensures that the external symmetry is resolved as well. As the algorithm proceeds by considering symmetries of nondecreasing length, it first resolves internal symmetries, which automatically also resolves the external ones.

The algorithm consists of the following steps:

1. Consider symmetries in the order of nondecreasing length. For each symmetry that has not been already resolved, select any \((m_i - 1)\) elements from \((m_i - 1)\) different tuples and add them to \( S \). Terminate when all symmetries have been resolved.

2. Consider one by one all the root strongly connected components. For each root strongly connected component that has not been already resolved, select any element and add it to \( S \).

As an example, we apply our algorithm to the NID representation in Fig. 9. This network is directed and is composed of 6 non-root SCCs and 1 root SCC. We computed the graph symmetries by using the SAGE routine AUTOMORPHISM_GROUP. The symmetries for this network are given by \( S_1 = ((3),(4)), S_2 = ((6),(7)), S_3 = ((9),(10)), S_4 = ((12),(13)), S_5 = ((14,15),(16,17)), \) and \( S_6 = ((2,3,4),(5,6,7),(8,9,10),(11,12,13)) \). In this case, \( m_1 = m_2 = m_3 = m_4 = m_5 = 2 \) and \( m_6 = 4 \); moreover, \( l_1 = l_2 = l_3 = l_4 = 1, l_5 = 2, \) and \( l_6 = 3 \). Note that the symmetries \( S_1, S_2, S_3, \) and \( S_4 \) are all internal to \( S_6 \). According to our algorithm, we resolve the internal symmetries first. To do that, we choose one node from each one of the symmetries \( S_1, S_2, S_3, \) and \( S_4 \) and add them to \( S \). For example, a possible choice is to set \( S = [3,6,9,12] \). We see that with this \( S \), we have automatically resolved the external symmetry \( S_6 \). In fact, \( S \) contains \((m_6 - 1) = 3 \) different nodes from \((m_6 - 1) = 3 \) different tuples of \( S_6 \). We then need to resolve \( S_1 \) by choosing one node from either the tuple \((14,15)\) or \((16,17)\). Say we choose 14, then the updated set \( S \) is \([3,6,9,12,14] \). The
network has one root SCC. According to step 2 of the algorithm, we need to
choose one node from this root SCC, unless one is already included in $S$. Here
node 1 is the only option and it has not been already selected. Then, the final
set is $S = [1, 3, 6, 9, 12, 14]$. Note that this set is minimal, as eliminating any
node from $S$ would correspond to loss of observability.

We now apply our algorithm to three different real networks. First we consider
the Zachary’s Karate club which is a social network of friendships between 34
members [37]. Since the network is undirected and connected, the whole network
coincides with a root SCC (see Fig. 10). Graph symmetries for this example have
been evidenced in Fig. 10, where same shapes (triangle, rectangle and oval) have
been used to indicate nodes that are symmetric. By applying the algorithm, we
find a minimal set of nodes for observability $S = [5, 15, 16, 18, 19, 21]$.

Next we consider a social network of frequent associations between 62 dolphins
in a community living off Doubtful Sound, New Zealand [19] (Fig. 11). As this
network is also undirected and connected, the whole network corresponds to a
root SCC. This network also displays graph symmetries and by applying the
algorithm, we see that a minimal set of two dolphins (named Croos and MN23)
provides observability.

We now consider the Caenorhabditis elegans neuronal networks formed of 279
neurons [33]. These can be linked through connections of two different types,
either chemical synapses or gap junctions. Both these networks are weighted
but for simplicity, here we consider them as unweighted (namely, if at least
a connection exists between neuron $i$ and $j$, then we set the corresponding
entry of the adjacency matrix equal to 1). The chemical synapses network is
directed. This network is not fully connected, it has 42 SCCs, 11 of which
are root SCCs. This network also presents graph symmetries. We used our
algorithm to find a minimal set of neurons $S$ to achieve observability of the
chemical synapses network and presented the results of our computations in
Table 1. On the other hand, the gap junction network is undirected. This
network is not fully connected and is formed of 29 SCCs, all of which are root
SCCs. This network also presents some graph symmetries. A set of sensor nodes
to achieve observability of the gap junction network is presented in Table 2.
Finally, we consider the overall neuronal network of the Caenorhabditis elegans, formed of both chemical synapses and electric gap junctions. For this case we considered that if there exists at least a gap junction or a chemical synapsis between neuron $i$ and $j$, then a connection exists and the corresponding entry of the adjacency matrix is set to 1 (otherwise it is a 0). This network is directed and does not present any graph symmetry. For this case, we have 6 SCCs and 4 root SCCs. A set of sensor nodes to achieve observability of this network is presented in Table 3.
Figure 11: Social network of 62 dolphins, Doubtful Sound \[19\]. The triangles and ovals are presenting symmetries.

| No. of nodes | Name of The Neurons | Neuron Label No. |
|--------------|---------------------|-----------------|
| 1            | IL2DL               | 1               |
| 2            | IL2DR               | 6               |
| 3            | ASIL                | 76              |
| 4            | ASIR                | 90              |
| 5            | AINL                | 120             |
| 6            | SDQR                | 179             |
| 7            | DB05                | 204             |
| 8            | AS08                | 220             |
| 9            | PVDR                | 227             |
| 10           | DVB                 | 253             |
| 11           | PLNR                | 260             |
| 12           | PHCR                | 274             |
| 13           | PLML                | 279             |

Table 1: The list of sensor nodes to achieve observability of chemical synapse network of C. elegans (directed network).
| No. of nodes | Name of The Neurons | Neuron Label No. |
|--------------|---------------------|------------------|
| 1            | IL2DL               | 1                |
| 2            | IL2VL               | 2                |
| 3            | IL2L                | 3                |
| 4            | URADL               | 4                |
| 5            | IL2DR               | 6                |
| 6            | RIPL                | 15               |
| 7            | IL2VR               | 18               |
| 8            | URADR               | 20               |
| 9            | URAVL               | 22               |
| 10           | URAVR               | 34               |
| 11           | RIAL                | 49               |
| 12           | RIAR                | 57               |
| 13           | AWCL                | 69               |
| 14           | AWCR                | 82               |
| 15           | ASEL                | 99               |
| 16           | SIADL               | 102              |
| 17           | ASER                | 105              |
| 18           | RMFR                | 108              |
| 19           | ASJL                | 114              |
| 20           | SIADR               | 124              |
| 21           | AIMR                | 135              |
| 22           | BDUR                | 176              |
| 23           | BDUL                | 177              |
| 24           | VA06                | 202              |
| 25           | HSNL                | 211              |
| 26           | AS08                | 220              |
| 27           | DA06                | 225              |
| 28           | PVDL                | 226              |
| 29           | PVDR                | 227              |
| 30           | VA10                | 234              |
| 31           | AS10                | 237              |
| 32           | VDI1                | 239              |
| 33           | VDI2                | 242              |
| 34           | DD06                | 246              |
| 35           | VDI3                | 251              |
| 36           | PLNR                | 260              |
| 37           | ALNR                | 267              |
| 38           | PVWL                | 271              |
| 39           | PLNL                | 273              |

Table 2: The list of sensor nodes to achieve observability of gap junction network of C. elegans (undirected network).

We wish to emphasize that all the examples of real networks we have considered present symmetries, which is in accordance with the observation that
| No. of nodes | Name of The Neurons | Neuron Label No. |
|--------------|---------------------|-----------------|
| 1            | IL2DL               | 1               |
| 2            | IL2DR               | 6               |
| 3            | PVDR                | 227             |
| 4            | PLNR                | 260             |

Table 3: The list of sensor nodes to achieve observability of the coexistence network (directed network).

symmetries are a common feature of several complex networks [25]. We conclude that the method proposed in this paper to identify the minimal set of sensor nodes for observability may be useful under general circumstances.

7 Conclusions

In this paper we have considered the problem of reconstructing both nodes and edges of complex dynamical networks and shown that a graphical approach (GA) can be successfully used to determine the output states for observability. We have shown that the GA method presented in [18] can be extended to the case that both nodes and edges of a complex network need to be estimated. We have presented two inference diagrams, NID and NEID, which should be used in the cases that the nodes of the network and the nodes and edges of the network need to be estimated, respectively.

We showed that by assuming minimal knowledge about the underlying network (i.e., directed vs. undirected, constant vs. adaptive), it is possible to obtain SCCs and root SCCs for the NEID representation directly from knowledge of the NID representation. The NEID analysis provides conditions to observe the structural connectivity patterns of unknown complex dynamical networks by monitoring only some of their states (either nodes or connections). We have found that monitoring edge states may be convenient in the case of adaptive connections, while it does not provide benefits in terms of observability in the case of static connections.

Our method based on a graphical approach could be used to uncover the conditions required for the observation of neural and genetic networks, for which access to all the network states is limited and knowledge about the connectivity pattern is an important step. We conclude that the GA analysis can be conveniently applied to infer information on the connections of a complex dynamical network. This is relevant to estimation of neuronal [35, 31] and genetic networks [8, 13, 12, 20, 11, 21] and several engineering applications [2, 6, 15, 29, 30].

Moreover, while the GA method has been shown to be useful in the case that the network inference diagram does not display symmetries [18], here we have proposed an extension of this method to NIDs and NEIDs with graph symmetries. We have shown that in the presence of graph symmetries, observability can be achieved by a proper selection of the sensor nodes, which takes into account the symmetries. We have also proposed a simple algorithm for selecting the minimal set of output nodes for observability. This algorithm has been
tested on several examples of real world network topologies displaying symmetries. Finally, we would like to emphasize that our results can be generalized to the dual problem of controllability of networks.

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