MODIFIED FUTAKI INVARIANT AND EQUIVARIANT RIEMANN-ROCH FORMULA

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ABSTRACT. In this paper, we give a new version of the modified Futaki invariant for a test configuration associated to the soliton action on a Fano manifold. Our version will naturally come from toric test configurations defined by Donaldson for toric manifolds. As an application, we show that the modified $K$-energy is proper for toric invariant Kähler potentials on a toric Fano manifold.

0. Introduction

Let $(M, g)$ be a Fano manifold with a Kähler form $\omega_g \in 2\pi c_1(M)$ of $g$. Denote $\eta(M)$ to be the linear space of holomorphic vector fields on $M$. Then by Hodge Theorem, for any $X \in \eta(M)$, there exists a unique smooth complex-valued function $\theta_X(g)$ of $M$ such that

\[
i_X \omega_g = \sqrt{-1} \partial \bar{\partial} \theta_X(g),
\]
\[
\int_M e^{\theta_X(g)} \omega^n_g = \int_M \omega^n_g.
\]

In [TZ2], Tian and Zhu introduced the modified Futaki invariant on $\eta(M) \times \eta(M)$,

\[
F_X(v) = \int_M v(h_g - \theta_X(g)) e^{\theta_X(g)} \frac{\omega^n_g}{n!}, \quad X, v \in \eta(M),
\]

where $h_g$ is the Ricci potential of $g$ such that

\[
\text{Ric}(\omega_g) - \omega_g = \sqrt{-1} \partial \bar{\partial} h_g.
\]

It was shown there that $F_X(v)$ is a holomorphic invariant independent of the choice of $g$ with $\omega_g \in 2\pi c_1(M)$, and so it defines an obstruction to the existence of Kähler-Ricci solitons with respect to an element $X \in \eta_r(M)$, where $\eta_r(M)$ is the reductive part of $\eta(M)$. In particular, when $X = 0$, $F_X(v)$ is classical Futaki invariant [Fut]. It was also proved by Tian and Zhu that there exists a unique $X$ such that $F_X(v) = 0$, $\forall v \in \eta(M)$. For convenience, we call such $X$ the soliton vector field on $M$.

Recently, by using Ding-Tian’s idea of generalizing Futaki invariant [DT], Xiong and Berman, gave a generalization of the modified Futaki invariant $F_X(\cdot)$ for any special degeneration associated to $X$, independently [Xi, Be2]. As a consequence, they both proved that $F_X(\cdot)$ is nonnegative if $M$ admits a Kähler-Ricci soliton. Berman also gave an algebraic formula for $F_X(\cdot)$, which depends on weights of the automorphisms group on holomorphic sections of multi-line bundles on the center fibre induced by the test configuration. The purpose of present paper is to define the modified Futaki invariant $F_X(\cdot)$ for any test configuration associated to $X$. Our motivation is inspired by Berman’s algebraic formula for special degenerations and is to modify his formula for general test configurations. Then by applying the

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equivariant Riemann-Roch formula with a 2-dimensional torus action we show that our definition coincides with Xiong-Berman’s for special degenerations. Our definition also includes Tian and Donaldson’s generalized Futaki invariant as a special case \([T1,D1]\) when \(X = 0\).

As examples, we compute the new version of the modified Futaki invariant for any toric degeneration on toric manifolds introduced by Donaldson [D1]. Then by using the method in [ZZ1], we are able to prove

**Theorem 0.1.** Any toric Fano manifold is modified K-stable for any toric degeneration. Furthermore, the modified K-energy is proper for toric invariant Kähler potentials.

Theorem [0.1] gives a new proof of Wang-Zhu’s result for the existence of Kähler-Ricci solitons on any toric Fano manifold [WZ]. We can also study the existence of conical Kähler-Ricci solitons on toric Fano manifolds by showing the properness of modified Log K-energy. As a consequence, we give a new proof of Datar-Guo-Song-Wang Theorem in [DGSW]. In particular, we have

**Theorem 0.2.** Let \(X\) be a soliton vector field on a toric Fano manifold \(M\). Then for any \(\beta \leq 1\) there exists a unique toric invariant conical Kähler-Ricci soliton which has conical angle \(2\pi \beta\) along each face divisor \(D_i\) of \(M\).

We note that the above energy argument was used by other people, such as in [JMR, LS, T3, LZ] to study the conical Kähler-Einstein metrics on general Fanon manifolds. Theorem 0.1 and Theorem 0.2 will be proved in Section 2-3 and Section 4, respectively.

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### 1. New version of modified Futaki invariant

According to [D1], a test-configuration on a Fano manifold \(M\) is a scheme \(\mathcal{M}\) with a \(\mathbb{C}^+\)-action which consists of two integrations:

1. an flat \(\mathbb{C}^+\)-equivavrant map \(\pi : \mathcal{M} \to \mathbb{C}\) such that \(\pi^{-1}(t)\) is biholomorphic to \(M\) for any \(t \neq 0\);
2. an holomorphic line bundle \(\mathcal{L}\) on \(\mathcal{M}\) such that \(\mathcal{L}|_{\pi^{-1}(t)}\) is isomorphic to \(K_M^{-r}\) for some integer \(r > 0\) for any \(t \neq 0\).

**Definition 1.1.** \(\mathcal{M}\) is called a test-configuration associated to the soliton action induced by \(X\) if \(\sigma_i^X\) communicate to \(\sigma_i^X\), where \(\sigma_i^X\) and \(\sigma_i^Y\) are two lifting one-parameter subgroups on \(\mathcal{M}\) induced by \(X\) and the holomorphic vector field \(v\) associated to the \(\mathbb{C}^+\)-action, respectively. If furthermore the center fibre \(M_0 = \pi^{-1}(0)\) is a normal variety we call \(\mathcal{M}\) is a special degeneration. In particular, if \(\mathcal{M} \cong M \times \mathbb{C}\), \(\mathcal{M}\) is called a trivial test-configuration.

For simplicity, we let \(L = \mathcal{L}|_{M_0}\). Let \(\sigma_i^X(k), \sigma_i^Y(k)\) be two induced one-parameter subgroups on \(H^0(M_0,L^k)\) by \(\sigma_i^X, \sigma_i^Y\), respectively. Denote by \(\{e_i^X\}\) and \(\{e_i^Y\}\) be eigenvalues of actions \(\sigma_i^X\) and \(\sigma_i^Y\). We set

\[
S_1 = \sum_i e_i^X v_i^k, \quad S_2 = \frac{1}{2} \sum_i e_i^X X_i^k v_i^k.
\]

\(^1\) \(X\) is not unique in general for the existence of conical Kähler-Ricci solitons while it is always unique modulo \(\text{Aut}^0(M)\) for the existence of Kähler-Ricci solitons [172].
Then
\[
S_1 = \frac{\partial}{\partial t}\text{trace}(e^{sX^k+nt^k})|_{s=1/2, t=0}, \quad S_2 = \frac{1}{2k^2}\frac{\partial}{\partial s}\frac{\partial}{\partial t}\text{trace}(e^{sX^k+nt^k})|_{s=1/2, t=0},
\]
where \(X^k = (X^k_{\alpha}), v^k = (v^k_{\alpha})\) are two vectors as elements of Lie algebra associated to \(\sigma^X_t(k), \sigma^v_t(k)\), respectively. Our observation is that both \(S_1\) and \(S_2\) can be computed by the equivariant Riemann-Roch formula with \(G = (S^1)^2\)-action. In fact
\[
\text{trace}(e^{sX^k+nt^k}) = \int_{M_0} \text{ch}^G(-kL)\text{Td}^G(X_0),
\]
where \(\text{ch}^G(-kL)\) is a \(G\)-equivariant Chern character of multi-line bundle \(-kL\) and \(\text{Td}^G(M_0)\) is a \(G\)-equivariant Todd character of \(M_0\) \cite{AS}. In particular, for a special degeneration associated to the soliton action, we can compute both \(S_1\) and \(S_2\) precisely in the following.

According to \cite{DT}, for a special degeneration, there exists a hermitian metric \(h\) on \((X_0, L)\) such that curvature \(c(h, L)\) is a \(r\)-multiple of an admissible metric \(g\) with property: there exists a \(L^p\)-integrable function \(h_\sigma\) (for any \(p \geq 0\)) with respect to \(g\) such that
i) \(\text{Ric}(\omega_g) - \omega_g = \sqrt{-1} \partial\bar{\partial} h_\sigma\), on the smooth part of \(M_0\);
ii) \(v(h_\sigma)\) is \(L^1\)-integrable with respect to \(g\). We note that \(v\) is an admissible holomorphic vector field \(w\) on \(M_0\) \cite{DT}.

For any admissible holomorphic vector field \(w\) on \(M_0\), we define a function by
\[
\theta_w = -\frac{L_w h}{h},
\]
Then a direct computation shows
\[
\sqrt{-1} \partial\bar{\partial} \theta_w = i_w \omega_g,
\]
and consequently
\[
\Delta \theta_w = \frac{L_w \omega_g^n}{\omega_g^n}.
\]

**Lemma 1.1.** \(\theta_w\) satisfies
\[
\Delta \theta_w + v(h_\sigma) + \theta_w = 0.
\]

**Proof.** It suffices to verify (1.4) on the smooth part of \(X_0\). Since
\[
\text{Ric}(\omega_g) - \omega_g = \text{Ric}(\omega_g) - \text{Ric}(h) = \sqrt{-1} \partial\bar{\partial} \log \frac{h}{\omega_g^n},
\]
we have
\[
h_\sigma = \log \frac{h}{\omega_g^n} + \text{const}.
\]
It follows
\[
v(h_\sigma) = v(\log \frac{h}{\omega_g^n}).
\]
Thus
\[
\Delta \theta_w + v(h_\sigma) + \theta_w = \frac{L_w \omega_g^n}{\omega_g^n} + v(\log \frac{h}{\omega_g^n}) - \frac{L_w h}{h} = 0.
\]
\[\square\]
Lemma 1.2. Let $\theta_X$ and $\theta_v$ be defined by (1.2) for the vectors $X$ and $v$, respectively. Then, instead of $k$ by $kr$, we have

$$S_1 = k^{n+1} \int_{M_0} \theta_v e^{\theta_X} \frac{\omega_g^n}{n!}$$

(1.5) 

$$+ \frac{k^n}{2} \left[ n \int_{M_0} \theta_v e^{\theta_X} \frac{\omega_g^n}{n!} + \int_{M_0} \theta_X \theta_v e^{\theta_X} \frac{\omega_g^n}{n!} - \int_{M_0} \nu(h_g - \theta_X) e^{\theta_X} \frac{\omega_g^n}{n!} \right] + O(k^{n-1}).$$

(1.6) 

$$S_2 = \frac{k^n}{2} \int_{M_0} \theta_v \theta_X e^{\theta_X} \frac{\omega_g^n}{n!} + O(k^{n-1}).$$

Proof. Since

$$\text{ch}^G(-kK_{M_0}) = e^{k\rho_X + k\rho_v + k\theta_v}$$

and

$$\text{Td}^G(M_0) = 1 + \frac{1}{2} c_1^G + \sum_{i+j \geq 2} a_i j^j t^i + 2l - \text{forms } (l \geq 2),$$

where $c_1^G$ is the first Chern $G$-equivariant form, we have

$$\left. \frac{d}{dt} \right|_{t=0} \text{trace}(e^{\theta_X+iv})$$

$$= k \int_{M_0} \theta_v e^{k\rho_X + \theta_X} \text{Td}^G(M_0) + \int_{M_0} e^{k\rho_X + \theta_X} \frac{d}{dt} \text{Td}^G(M_0)$$

$$= k^{n+1} \int_{M_0} \theta_v e^{\theta_X} \frac{\omega_g^n}{n!} \wedge (1 + \frac{1}{2} c_1^G) + \frac{k^n}{2} \int_{M_0} e^{\theta_X} \frac{\omega_g^{n-1}}{n!} \wedge c_1^G$$

$$+ \int_{M_0} e^{k\rho_X + \theta_X} \left( \frac{d}{dt} c_1^G \right) \frac{\omega_g^n}{n!} + O(k^{n-2}).$$

Note that

$$c_1^G = \text{Ric}(\omega_g) - s\Delta \theta_{V_0} - t\Delta \theta_{W_0}.$$

Thus

$$\left. \frac{d}{dt} \right|_{t=0} \text{trace}(e^{\theta_X+iv})$$

(1.7)  

$$= k^{n+1} \int_{M_0} e^{\theta_X} \theta_v e^{\theta_X} \frac{\omega_g^n}{n!} + \frac{k^n}{2} \left( \int_{M_0} e^{\theta_X} \theta_v S \frac{\omega_g^n}{n!} - \int_{M_0} e^{\theta_X} \theta_v \Delta \theta_X \frac{\omega_g^n}{n!} - \int_{M_0} e^{\theta_X} \Delta \theta_v \frac{\omega_g^n}{n!} \right) + O(k^{n-1}).$$

On the other hand, using the integration by parts, it is easy to see that

$$\int_{M_0} \theta_v e^{\theta_X} S \frac{\omega_g^n}{n!} = n \int_{M_0} \theta_v e^{\theta_X} \frac{\omega_g^n}{n!} + \int_{M_0} \theta_v e^{\theta_X} \Delta h_g \frac{\omega_g^n}{n!}$$

(1.8)  

$$= n \int_{M_0} \theta_v e^{\theta_X} \frac{\omega_g^n}{n!} - \int_{M_0} \theta_v e^{\theta_X} h_g \frac{\omega_g^n}{n!} - \int_{M_0} \nu(h_g) e^{\theta_X} \frac{\omega_g^n}{n!}.$$ 

Hence, by using the relation (1.4) for $\theta_X$, we will get (1.5) from (1.7) and (1.8) immediately.

The proof of (1.6) is easy. We skip it. □
Let $N_k = \dim H^0(M_0, \mathcal{L}^k)$. Then by the Riemann-Roch formula, we have

\begin{equation}
(1.9) \quad N_k = \int_{M_0} c_h(-k K_{M_0}) Td(M_0) = \int_{M_0} e^{k \omega_g} Td(M_0).
\end{equation}

Note that the Todd class is given by

\[ Td(M_0) = 1 + \frac{1}{2} c_1(M_0) + \cdots, \]

where $c_1(M_0)$ is the first Chern class of $M_0$. Thus

\begin{equation}
(1.10) \quad N_k = A_0 k^n + B_0 k^{n-1} + O(k^{n-2}),
\end{equation}

where

\begin{equation}
(1.11) \quad A_0 = \int_{M_0} \frac{\omega_g^n}{n!} = \text{Vol}(M_0), \quad B_0 = \frac{1}{2} \int_{M_0} S \frac{\omega_g^n}{n!} = \frac{n}{2} \text{Vol}(M_0).
\end{equation}

Here $S$ is the scalar curvature of $\omega_g$.

By Lemma 1.2, we can write

\[ -\frac{S_1 - S_2}{k N_k} = F_0 + F_1 k^{-1} + O(k^{-2}). \]

Proposition 1.1. For a special degeneration on a Fano manifold, we have

\[ F_1 = \frac{1}{2} F_0 = \frac{1}{2 \text{Vol}(M_0)} \int_{M_0} \nu(h_g - \theta \chi) e^{\theta \chi} \frac{\omega_g^n}{n!}. \]

Proof. By using the relation (1.4) for $\theta_\nu$, we have

\[ \int_{M_0} \nu(h_g - \theta \chi) e^{\theta \chi} \frac{\omega_g^n}{n!} = -\int_{M_0} \theta_\nu e^{\theta \chi} \frac{\omega_g^n}{n!}. \]

Then the Proposition follows from Lemma 1.2 immediately. \hfill \Box

For a general test-configuration associated to the soliton action, by the equivariant Riemann-Roch formula (1.1), we write $S_1$ and $S_2$ formally as,

\begin{align*}
S_1 &= A k^{n+1} + B k^n + O(k^{n-1}), \\
S_2 &= C k^n + D k^{n-1} + O(k^{n-2}).
\end{align*}

Then the invariant $F_1$ in (1.12) is equal to $\frac{2(B-C)}{2A}$. We call this quantity the modified Futaki invariant for the test-configuration $M$ associated to the soliton vector field $X$.

Remark 1.1. 1) In [Be2], Berman defines the modified Futaki invariant by $F_0$ for any special degeneration associated to the soliton vector field $X$. Proposition 1.1 means that our definition coincides Berman’s case. But in general $F_0$ will be different to $F_1$ as showed in next section for toric degenerations on a toric Fano manifold. In fact, we will show that the invariant $F_1$ comes naturally from the study of modified $K$-energy on toric manifolds (cf. Section 3).

2) Proposition 1.1 also shows that the Donaldson invariant $F_1$ in [D1] coincides with the generalized Futaki invariant defined by Tian in [T1] for special degenerations.

As in [T1] D1, we introduce a notation of modified $K$-stability for any Fano manifold $M$ via the quantity $F_1$. 

Definition 1.2. A Fano manifold $M$ is called modified K-semi-stable if $F_1 \geq 0$ for any test-configuration associated to the soliton action of $M$ and $M$ is modified K-stable if in addition $F_1 = 0$ happens if and only if the test-configuration is trivial.

Due to the celebrated solving of Yau-Tian-Donaldson’s conjecture for the existence of Kähler-Einstein metrics \cite{T3, CDS}, we propose the following generalized Yau-Tian-Donaldson’s conjecture for the existence of Kähler-Ricci solitons.

Conjecture 1.1. A Fano manifold $M$ admits a Kähler-Ricci soliton if and only if $M$ is modified K-stable.

In the remaining sections, we verify Conjecture 1.1 in case of toric Fano manifolds (also for general conical Kähler-Ricci solitons).

2. Modified Futaki invariant for toric degenerations

In this section we compute the modified Futaki invariant $F_1$ for a toric degeneration on a toric Fano manifold $M$. Let $T = T^1 = (\mathbb{C}^*)^n = (S^1)^n \times \mathbb{R}^n$ be torus action on $M$ and denote $G_0 = (S^1)^n$. Choose an $G_0$-invariant Kähler metric $g$ on $\mathbb{R}^n$ which depends only on $\xi_1, \ldots, \xi_n \in \mathbb{R}^n$ in the coordinates $(z_1, \ldots, z_n)$, namely $\omega_g = \sqrt{-1} \partial \bar{\partial} \varphi_0$ on $(\mathbb{C}^*)^n$. Since the torus action $T$ is Hamiltonian, there exists a moment map $m : M \rightarrow t^*$, where $t^*$ is the dual of the Lie algebra of $T$ which can be identified with $\mathbb{R}^d$. By the convexity theorem the image is a convex polytope in $\mathbb{R}^n$. Moreover, the moment map can be given by

$$(m_1, \ldots, m_n) = \nabla \varphi_0 = \left(\frac{\partial \varphi_0}{\partial \xi_1}, \ldots, \frac{\partial \varphi_0}{\partial \xi_n}\right).$$

Denote the image by $P = D\varphi_0(\mathbb{R}^n)$. Then $P$ is a convex polytope represented by a set of inequalities of the form (up to translation of coordinates)

$$(2.1) P = \{x \in \mathbb{R}^n : \langle x, \ell_i \rangle \leq 1, \ i = 1, 2, \ldots, d\},$$

where $\ell_i$ is the outer normal vector to a face of $P$ and $d$ is the number of faces of $P$. This polytope is independent of the choice of the metric $g$ in $2\pi c_1(M)$. See \cite{Ab1, Ab2, Gu} for more details.

On the other hand, the soliton vector field $X$ can be written as $X = \sum_{i=1}^n \theta_i w_i \frac{\partial}{\partial w_i} = \sum_{i=1}^n \theta_i \frac{\partial}{\partial z_i}$. Let $\theta_X(\omega_g)$ be the potential function determined by

$$i_X \omega_g = \sqrt{-1} \partial \bar{\partial} \theta_X(\omega_g),$$

then $\theta_X(\omega_g) = \varphi_0 + c$ for some $c \in \mathbb{R}^n$. By (1.4), it is easy to see

$$(2.2) \int_M \theta_X(\omega_g) e^{h_k \frac{\omega_g^n}{n!}} = 0.$$
in the symplectic coordinates. One can see that $\theta(x)$ is also independent of the choice of metric $g$.

According to [DT], a toric degeneration is induced by positive rational, piecewise linear functions on $P$. Note that a piecewise linear(PL) convex function $u$ on $P$ is of the form

$$u = \max\{u^1, ..., u^r\},$$

where $u^\lambda = \sum a^\lambda_i x_i + c^\lambda$, $\lambda = 1, ..., r$, for some vectors $(a^\lambda_1, ..., a^\lambda_n) \in \mathbb{R}^n$ and some numbers $c^\lambda \in \mathbb{R}$. $u$ is called a rational piecewise linear convex function if the coefficients $a^\lambda_i$ and numbers $c^\lambda$ are all rational.

For a positive rational PL convex function $u$ on $P$, we choose a positive integer $R$ so that

$$Q = \{(x,t) \mid x \in P, 0 < t < R-u(x)\}$$

is a convex polytope in $\mathbb{R}^{n+1}$. Without loss of generality, we may assume that the coefficients $a^\lambda_i$ are integers and $Q$ is an integral polytope. Otherwise we replace $u$ by $lu$ and $Q$ by $lQ$ for some integer $l$, respectively. Then the $n+1$-dimensional polytope $Q$ determines an $(n+1)$-dimensional toric variety $M_Q$ with a holomorphic line bundle $\mathcal{L} \to M_Q$. Note that the face $\bar{Q} \cap [\mathbb{R}^n \times \{0\}]$ of $Q$ is a copy of the $n$-dimensional polytope $P$, so we have a natural embedding $i : M \to M_Q$ such that $\mathcal{L}|_M = -K_M$.

Decomposing the torus action $T^{n+1}_C$ on $M_Q$ as $T^n_C \times \mathbb{C}^*$ so that $T^n_C \times \{1\}$ is isomorphic to the torus action on $M$, we get $\mathbb{C}^*$-action $\sigma^u$ by $\{1\} \times \mathbb{C}^*$. Hence, we define an equivariant map

$$\pi : M_Q \to \mathbb{C}P^1$$

satisfying $\pi^{-1}(\infty) = i(M)$. One can check that $W = M_Q\backslash i(M)$ is a test configuration for the pair $M$, called a toric degeneration.

Let $kP$ be the polytope which corresponds to the bundle $-kK_M$. Let $B_{kP} = \mathbb{Z}^n \cap k\mathbb{P}$ be the lattices set of $kP$. Let $d\sigma = \langle \bar{n}, x \rangle d\sigma_0$, where $\bar{n}$ is the unit outer normal vector field, and $d\sigma_0$ is the Lebesgue measure on on $\partial P$. We need the following lemma.

**Lemma 2.1.** Let $\phi$ be a continuous function on $\mathbb{P}$, then

$$\sum_{l \in B_{kP}} \phi(1/k) = k^n \int_P \phi dx + \frac{k^{n-1}}{2} \int_{\partial P} \phi d\sigma + O(k^{n-2}).$$

In particular,

$$N_k = k^n |P| + \frac{k^{n-1}}{2} |\partial P| + O(k^{n-2}).$$

Note that $\frac{\partial P}{|P|} = n$ if $P$ corresponds to $2\pi c_1(M)$.

This lemma was proved in [DT] for convex rational PL functions. It is easy to see that the formula can be extended to continuous function by approximation arguments.

**Proposition 2.1.** Let

$$\mathcal{L}(u) = \int_{\partial P} u e^{\theta(x)} d\sigma - \int_P (n + \theta(x)) e^{\theta(x)} u \ dx.$$  

Then for a toric degeneration on $M$ induced by a positive rational PL-convex function $u$, we have

$$F_1 = \frac{1}{2Vol(P)} \mathcal{L}(u).$$
Proof. We consider the space $H^0(W, \mathcal{O}^k)$ of holomorphic sections over $W$. It is well-known that $H^0(W, \mathcal{O}^k)$ has a basis $\{S_{I,j}\}$, where $I$ is a lattice in $B_{k,p}$ and $0 \leq i \leq k(R-u)(I/k)$. By using the exact sequence for large $k$,

$$0 \rightarrow H^0(W, \mathcal{O}^k \otimes \pi^*(\theta(-1))) \rightarrow H^0(W, \mathcal{O}^k) \rightarrow H^0(M_0, \mathcal{O}^k) \rightarrow 0,$$

$H^0(M_0, \mathcal{O}^k)$ has a basis $\{S_{I,k(R-u)(I/k)}\}_{I \in B_{k,p}}$. According to [ZZ1], the action $\sigma^X$ induced by $X$ acts on $S_{I,k(R-u)(I/k)}$ with weight $k\theta(I/k)$. The action $\sigma^u$ induced by $u$ acts on $S_{I,k(R-u)(I/k)}$ with weight $k(R-u)(I/k)$.

Then by Lemma 2.1 it is easy to see that

$$S_1 = \sum_{I \in B_{k,p}} e^{\theta(I/k)}k(R-u)(I/k)$$

$$= \int p e^{\theta(x)}(R-u) \, dx - \frac{k^n}{2} \int_{\partial p} e^{\theta(x)}(R-u) \, d\sigma + O(k^{n-1}) + \frac{1}{2} \left[ k^n \int_p e^{\theta(x)}(R-u) \cdot \theta(x) \, dx \right],$$

$$S_2 = \sum_{I \in B_{k,p}} e^{\theta(I/k)} \theta(I/k) \cdot (R-u)(I/k)$$

$$= \frac{1}{2} \left[ k^n \int_p e^{\theta(x)}(R-u) \cdot \theta(x) \, dx - \frac{k^{n-1}}{2} \int_{\partial p} e^{\theta(x)}(R-u) \cdot \theta(x) \, d\sigma + O(k^{n-2}) \right].$$

Then

$$-(S_1 - S_2)$$

$$= -k^{n+1} \int p e^{\theta(x)}(R-u) \, dx - \frac{k^n}{2} \left[ \int_{\partial p} e^{\theta(x)}(R-u) \, d\sigma - \int_p e^{\theta(x)}(R-u) \cdot \theta(x) \, dx \right] + O(k^{n+1}).$$

We have

$$\frac{-(S_1 - S_2)}{kN_k} = F_0 + F_1 k^{-1} + \cdots,$$

where

$$F_0 = \frac{1}{|P|} \int_p e^{\theta(x)}(R-u) \, dx,$$

$$F_1 = \frac{1}{2|P|} \left[ \int_{\partial p} e^{\theta(x)}(R-u) \, d\sigma - \int_p e^{\theta(x)}(R-u) \cdot \theta(x) \, dx - \frac{|\partial p|}{|P|} \int_p e^{\theta(x)}(R-u) \, dx \right]$$

$$= \frac{R}{2|P|} \left[ \int_{\partial p} e^{\theta(x)} \, d\sigma - \int_p (n + \theta(x)) e^{\theta(x)} \, dx \right] - \frac{1}{2|P|} \left[ \int_{\partial p} e^{\theta(x)} u \, d\sigma - \int_p (n + \theta(x)) e^{\theta(x)} u \, dx \right].$$

Integrating by parts, we have

$$\int_p e^{\theta(x)} \, dx = \frac{1}{n} \int_{\partial p} e^{\theta(x)} \, d\sigma - \frac{1}{n} \int_p \theta(x) e^{\theta(x)} \, dx.$$ 

Therefore, the coefficient of $R$ in (2.7) vanishes, and we have

$$F_1 = \frac{1}{2|P|} \left[ \int_{\partial p} e^{\theta(x)} u \, d\sigma - \int_p (n + \theta(x)) e^{\theta(x)} u \, dx \right].$$

□

Remark 2.1. As can be seen in the above lemma, the weights of the action depend on $R$ and $F_0$ also depends on the integer $R$. But $F_1$ is independent of $R$. In particular, $F_0$ is different to $F_1$. 

\[\square\]
Since $X$ is the soliton vector field, $\mathcal{L}(u) = 0$ for any linear function $u$. This implies that $\mathcal{L}(u)$ is invariant when adding $u$ by a linear function. We call a convex function is normalized at $0 \in P$ if $\inf_P u = u(0)$. Let $C_\infty$ be the set of smooth convex functions on $P$ and $\tilde{C}_\infty$ be the set of smooth convex functions normalized at $0 \in P$. It is clear that the PL functions can be approximated uniformly by functions in $C_\infty$.

**Lemma 2.2.** There exists a $\lambda > 0$ such that

$$\mathcal{L}(u) \geq \lambda \int_{\partial P} u e^{\theta(x)} d\sigma, \quad u \in \tilde{C}_\infty.$$  

**Proof.** We note that $\mathcal{L}(u)$ can be rewritten as

$$\mathcal{L}(u) = \int_P \left[ (\sum x_i u_i - u) + u \right] e^{\theta(x)} dx \geq \int_P u e^{\theta(x)} dx.$$  

By the contradiction, we suppose that (2.9) is not true. Then there is a sequence of functions $\{u_k\}$ in $\tilde{C}_\infty$ such that

$$\int_{\partial P} u_k e^{\theta(x)} d\sigma = 1$$  

and

$$\mathcal{L}(u_k) \to 0, \quad k \to \infty.$$  

By (2.11), there exists a subsequence (still denoted by $\{u_k\}$) of $\{u_k\}$, which converges locally uniformly to a convex function $u_\infty \geq 0$ on $P$. By (2.10) and (2.12), we have

$$\int_P u_k e^{\theta(x)} dx \leq \mathcal{L}(u_k) \to 0.$$  

It follows

$$\int_P u_\infty e^{\theta(x)} dx = 0.$$  

Hence, we obtain $u_\infty \equiv 0$ in $P$. On the other hand,

$$\mathcal{L}(u_k) = \int_{\partial P} u_k^k e^{\theta(x)} d\sigma - \int_P (n + \sum x_i \theta_i) u_k e^{\theta(x)} dx$$  

$$\to 1 - \int_P (n + \sum x_i \theta_i) u_\infty e^{\theta(x)} dx = 1 > 0.$$  

This contradicts with (2.12). The lemma is proved. □

By Lemma 2.2, we immediately get

**Theorem 2.1.** Any toric Fano manifold is modified $K$-stable for toric degenerations.
3. Modified $K$-energy on a toric Fano manifold

Let $K_X$ be a one parameter compact subgroup generated by the image part $\text{Im}(X)$ and denote by $\mathcal{H}_X(\omega_\phi)$ a set of $K_X$-invariant Kähler potentials. In the study of Kähler-Ricci solitons, the modified Mabuchi’s $K$-energy $\mu_{\omega_\phi}(\phi)$ plays an important role [TZ2, CTZ], where

$$\mu_{\omega_\phi}(\phi) = -\frac{1}{V} \int_0^1 \int_M \phi_t [S(\phi_t) - n - tr \omega_t \nabla \omega_t \cdot X - X(h_{\omega_t} - \theta_X(\omega_\phi))] e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!} \wedge dt.$$ 

Here $\phi \in \mathcal{H}_X(\omega_\phi)$ and $g$ is chosen to be $K_X$-invariant. Recall two Aubin typed functionals introduced in [Z1].

$$I_{\omega_\phi}(\phi) = \frac{1}{V} \int_M \phi (e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!} - e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!}),$$

$$J_{\omega_\phi}(\phi) = \frac{1}{V} \int_0^1 \int_M \phi_t (e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!} - e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!}) \wedge ds,$$

where $\omega_\phi = \omega_g + \sqrt{-1} \bar{\partial} \partial \phi$ and $\phi_t$ is a path in $\mathcal{H}_X(\omega_g)$. It is known that

$$c_1 I_{\omega_\phi}(\phi) \geq J_{\omega_\phi}(\phi) \geq c_2 I_{\omega_\phi}(\phi), \quad \forall \phi \in \mathcal{H}_X(\omega_g),$$

for two positive constants $c_1, c_2$. Then $\mu_{\omega_\phi}(\cdot)$ can also be written as

$$\mu_{\omega_\phi}(\phi) = -\frac{1}{V} \int_M \log \left( \frac{e^{\theta_X(\omega_\phi)} \omega_\phi^n}{e^{\theta_X(\omega_\phi)} \omega_g^n} \right) e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!} - (I_{\omega_\phi}(\phi) - J_{\omega_\phi}(\phi))$$

$$+ \frac{1}{V} \int_M (h_g - \theta_X(\omega_\phi))(e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!} - e^{\theta_X(\omega_\phi)} \frac{\omega_\phi^n}{n!}).$$

**Definition 3.1.** Let $(M, g)$ be a Fano manifold $M$. Let $G$ be a reductive subgroup of automorphisms group $\text{Aut}(M)$ which contains $K_X$. We call $\mu_{\omega_\phi}(\phi)$ proper modulo $G$ if there is a continuous function $p(t)$ in $\mathbb{R}$ with the property

$$\lim_{t \to +\infty} p(t) = +\infty,$$

such that

$$\mu_{\omega_\phi}(\phi) \geq \inf_{\sigma \in G} p(I_{\omega_\phi}(\phi_\sigma)),$$

where $\phi_\sigma$ is defined by

$$\omega_\phi + \sqrt{-1} \bar{\partial} \partial \phi_\sigma = \sigma^* (\omega_g + \sqrt{-1} \bar{\partial} \partial \phi).$$

The properness of $\mu_{\omega_\phi}(\phi)$ is a sufficient condition for the existence of Kähler-Ricci solitons due to the following lemma.

**Lemma 3.1.** Suppose that $\mu_{\omega_\phi}(\phi)$ is proper modulo a reductive subgroup $G$ of automorphisms group $\text{Aut}(M)$ which contains $K_X$. Then $M$ admits a Kähler-Ricci soliton.

Lemma 3.1 was proved by using Kähler-Ricci flow as in [TZ3, Z2, BB] and can be also proved by using the continuity method as in [CTZ, T2].

Let $P_g = \Delta_g + X(\cdot)$ be a linear elliptic operator defined on the space

$$\mathcal{N}_X = \{ u \in C^\infty(M) | \text{Im}(X(u)) = 0 \},$$
associated to \( \omega_g \) and a holomorphic on \( M \). \( P_g \) is a self-adjoint elliptic operator on \( \mathcal{N}_X \) with respect to the inner product,

\[
(\phi, \psi) = \int_M \phi \psi e^{\theta_g(\omega_g)} \frac{\omega^n_g}{n!}.
\]

The following lemma shows that the properness given in Definition 3.1 coincides with one as defined in [T1, CTZ], when \( g \) is a Kähler-Ricci soliton.

**Lemma 3.2.** Suppose that \( M \) admits a Kähler-Ricci soliton \( g_{KS} \). Then the modified K-energy \( \mu_{g_{KS}}(\phi) \) is proper with respect to \( X \) modulo \( \text{Aut}^0(M) \) iff there is a continuous function \( \tilde{p}(t) \) in \( \mathbb{R} \) with the property

\[
\lim_{t \to +\infty} \tilde{p}(t) = +\infty
\]

such that

\[
(3.4) \quad \mu_{g_{KS}}(\phi) \geq \tilde{p}(I_{g_{KS}}(\phi)), \quad \forall \phi \in \Lambda_1^+(M, g_{KS}),
\]

where \( \Lambda_1(M, g_{KS}) \) denotes the first non-zero eigenfunctions space for the operator \( P_{g_{KS}} \) associated to the metric \( g_{KS} \), i.e., \( \Lambda_1(M, g_{KS}) = \ker(P_{g_{KS}} + I) \).

**Proof.** First we prove the necessary part of the lemma. We choose the Kähler-Ricci soliton metric \( g_{KS} \) as an initial metric. Then we induce a functional on \( \text{Aut}^0(M) \) for any \( \phi \in H^X(\omega_{KS}) \) by

\[
\Phi(\sigma) = I_{\sigma^* \omega_{g}}(-\phi_\sigma) - J_{\sigma^* \omega_{g}}(-\phi_\sigma),
\]

where \( \phi_\sigma \) is an induced Kähler potential defined by

\[
\omega_{KS} + \sqrt{-1} \partial \bar{\partial} \phi_\sigma.
\]

\( I_{\sigma^* \omega_{g}}(\psi) \) and \( J_{\sigma^* \omega_{g}}(\psi) \) are functionals \( I(\psi) \) and \( J(\psi) \) respectively while the initial metric \( \omega_{KS} \) is replaced by \( \sigma^* \omega_{g} \). According to [TZ1], one can show that there exists a \( \tau \in \text{Aut}^0(M) \) such that

\[
\Phi(\tau) = \inf_{\sigma \in \text{Aut}^0(M)} \Phi(\sigma)
\]

and consequently \( \phi_\tau \in \Lambda_1^+(M, g_{KS}) \). In fact, from the proof of uniqueness of Kähler-Ricci solitons in [TZ1] it can be proved that \( \phi \in \Lambda_1^+(M, g_{KS}) \) iff

\[
I_{\omega_{g}}(-\phi) - J_{\omega_{g}}(-\phi) = \inf_{\sigma \in \Lambda_1^+(M, g_{KS})} \Phi(\sigma).
\]

Thus by the assumption (3.3) and relation (3.1), for any \( \phi \in \Lambda_1^+(M, g_{KS}) \), we have

\[
\mu_{g_{KS}}(\phi) \geq \inf_{\sigma \in \text{Aut}^0(M)} p(I_{\omega_{g}}(\phi_\sigma))
\]

\[
= \inf_{\sigma \in \text{Aut}^0(M)} p(I_{\sigma^* \omega_{g}}(-\phi_\sigma))
\]

\[
\geq \inf_{\sigma \in \text{Aut}^0(M)} \tilde{p}(\Phi(\sigma))
\]

\[
= \tilde{p}(I_{\omega_{g}}(-\phi) - J_{\omega_{g}}(-\phi))
\]

\[
\geq \tilde{p}(I_{\omega_{g}}(-\phi))
\]

\[
= \tilde{p}(I_{g_{KS}}(\phi)),
\]

where \( \tilde{p}(t) \) is another continuous function in \( \mathbb{R} \) which satisfying (2.9).
Next we prove the sufficient part. We note that the modified $K$-energy is invariant under $\text{Aut}^0(M)$ \cite{IZ2}. Then by the discussion at last paragraph, for any $\phi \in \mathcal{H}_X(\omega_{KS})$, we can choose a $\tau \in \text{Aut}^0(M)$ such that $\phi_\tau \in \Lambda^+_{1}(M, g_{KS})$ and
\[ \mu(\phi) = \mu(\phi_\tau). \]
Thus by (3.4), we get
\[ \mu(\phi) \geq \bar{p}(I(\phi_\tau)) \geq \inf_{\sigma \in \text{Aut}^0(M)} \bar{p}(I(\phi_\sigma)). \]

The converse of Lemma 3.1 was conjectured by Tian in sense of (3.4) in the case of Kähler-Einstein metrics \cite{T1} and was proved by him under the assumption that there is no any holomorphic vector field on $M$. Thus one may believe that the converse of Lemma 3.1 is also true as a generalization of Tian’s conjecture for the case of Kähler-Ricci solitons \cite{CTZ}. In this section we give a positive answer in the case of toric Fano manifolds. Namely, we shall prove

**Theorem 3.1.** On a toric Fano manifold, the modified $K$-energy is proper for toric invariant Kähler potentials modulo toric action.

Theorem 3.1 has been proved under the assumption that the Futaki invariant vanishes \cite{ZZ1}. In the following, we always assume that $M$ is a toric Fano manifold.

3.1. **The reduction of modified $K$-energy.** Denote $\mathcal{H}_{G_0}(\omega_\kappa) \subset \mathcal{H}_X(\omega_\kappa)$ to be the set of $G_0$-invariant Kähler potentials. Then $\mathcal{H}_{G_0}(\omega_\kappa)$ is equal to the set
\[ \{ \phi \in C^\infty(\mathbb{R}^n) \mid |\phi| < \infty \text{ and } \varphi_0 + \phi \text{ is strictly convex} \}. \]

By using the Legendre transformation $\xi = (D\varphi_0)^{-1}(x)$, one sees that the function (Legendre dual function) defined by
\[ u_0(x) = \langle \xi, D\varphi_0(\xi) \rangle - \varphi_0(\xi) = \langle \xi(x), x \rangle - \varphi_0(\xi(x)), \quad \forall \ x \in P \]
is strictly convex. Set the space of symplectic functions by
\[ C = \{ u = u_0 + f \mid u \text{ is a strictly convex function in } P, \ f \in C^{\infty}(\overline{P}) \}. \]
It was shown in \cite{Ab1} that there is a bijection between $C$ and $\mathcal{H}_{G_0}(\omega_\kappa)$.

**Proposition 3.1.** Let $\phi \in \mathcal{H}_{G_0}(\omega_\kappa)$ and $u$ be the Legendre dual function of $\varphi_0 + \phi$. Then the modified $K$-energy is given by
\[ \mu_{\omega_\kappa}(\omega_\phi) = \frac{(2\pi)^n}{V} \mathcal{F}(u) + C, \]
where
\[ \mathcal{F}(u) = - \int_P \log \det(u_{ij}) e^{\theta(u)} dx + \mathcal{L}(u), \]
and $C$ is a constant.

**Proof.** By (3.2), a direct computation shows
\[ \mu_{\omega_\kappa}(\phi) = \frac{1}{V} \int_M \log \left( \frac{e^{\theta_X(\omega_\phi)} \omega^n_\phi}{e^{\theta_X(\omega_\kappa)} \omega^n_\kappa} \right) - \frac{1}{V} \int_M \int_0^1 \phi_t e^{\theta_{X}(\omega_\phi)} \omega^n_\phi \wedge dt - \frac{1}{V} \int_M \phi e^{\theta_{X}(\omega_\phi)} \omega^n_\phi \]


\[-\frac{1}{V} \int_M (h_g - \theta_X(g)) e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} + \frac{1}{V} \int_M (h_g - \theta_X(\omega_g)) e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} + \frac{1}{V} \int_M (h_g - \theta_X(\omega_g)) e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} = \frac{1}{V} \int_M \log \left( \frac{\omega^g}{\omega_g} \right) e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} - \frac{1}{V} \int_M \int_0^1 \phi_0 e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} \wedge dt + \frac{1}{V} \int_M \theta_X(\omega_g) e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} \wedge dt + \text{const.} \]

(3.7)

On the other hand,

\[h_g = -\varphi_0 - \log \det(\varphi_0) + C.\]

Then

\[\frac{\partial}{\partial \varphi} e^{-h_g} = C \det(\varphi) e^{\varphi}.\]

It follows

\[\int_M \log \left( \frac{\omega^g}{\omega_g} \right) e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} = (2\pi)^n \left[ \int_{\mathbb{R}^n} \log \det(\varphi) e^{X(\varphi)} \det(\varphi) \, d\xi + \int_{\mathbb{R}^n} \varphi e^{X(\varphi)} \det(\varphi) \, d\xi \right].\]

By using the relations

\[\varphi = \sum_{i=1}^n x_i u_i - u, \quad \det(\varphi) \, d\xi = dx, \quad \phi_t = -u_t,\]

where \(\phi_t\) is a path in \(\mathcal{H}_{G_0}(\omega_g)\) and \(u_i\) is the symplectic potential of \(\varphi_t = \varphi_0 + \phi_t\), we also get

\[\int_0^1 \int_M \phi_t e^{\theta_X(\varphi_0)} \frac{\omega^g}{n!} \wedge dt = (2\pi)^n \int_0^1 \int_{\mathbb{R}^n} \phi_t e^{X(\varphi)} \det(\varphi) \, d\xi \wedge dt = -(2\pi)^n \int_{\mathbb{R}^n} u e^{\theta(x)} \, dx + \text{const.},\]

(3.9)

\[\int_{\mathbb{R}^n} \log \det(\varphi) e^{X(\varphi)} \det(\varphi) \, d\xi + \int_{\mathbb{R}^n} \varphi e^{X(\varphi)} \det(\varphi) \, d\xi = -\int_{\mathbb{R}^n} \log \det(\varphi) e^{\phi} \, dx + \int_{\mathbb{R}^n} \sum_{i=1}^n x_i u_i - u) e^{\theta(x)} \, dx.\]

(3.10)

Hence inserting (3.8)-(3.10) into (3.7), we obtain

\[\mu_{\omega_g}(\phi) = \frac{(2\pi)^n}{V} \left[ -\int_{\mathbb{R}^n} \log \det(\varphi) e^{\theta(x)} \, dx + \int_{\mathbb{R}^n} \sum_{i=1}^n x_i u_i e^{\theta(x)} \, dx \right] + C.\]

Integrating by parts, we deduce (3.5) immediately. \[\square\]
3.2. **Properness of** $\mathcal{F}(u)$. In this subsection, we show the properness of $\mathcal{F}(u)$ by similar arguments as in [DT1, ZZ1]. First, we have

**Lemma 3.3.** There exists a constant $C > 0$ such that for any $u \in C_\infty$, it holds

\[
\int_P \log \det(u_{ij}) e^{\theta(x)} \, dx \leq L_B(u) + C
\]

where $B = (u_0)^{ij}_{ij} + 2(u_0)^{ij}_i \theta_j + (u_0)^{ij}_i \theta_i \theta_j$ is a bounded function, and

\[
L_B(u) = \int_{\partial P} u e^{\theta(x)} \, d\sigma + \int_P B e^{\theta(x)} \, dx.
\]

**Proof.** Let $f = u - u_0$. By the convexity of $-\log\det$, we have

\[
\log\det(u_{ij}) \leq \log\det((u_0)_{ij}) + (u_0)^{ij}_{ij} f_{ij}.
\]

For any $\delta > 0$, let $P_\delta$ be the interior polygon with faces parallel to those of $P$ separated by distance $\delta$, then $f$ is smooth over the closure of $P_\delta$.

Integrating by parts,

\[
\int_{P_\delta} (u_0)^{ij}_{ij} f_{ij} e^{\theta(x)} \, dx = \int_{\partial P_\delta} (u_0)^{ij}_{ij} f_{ij} \, d\sigma_0 - \int_{P_\delta} (u_0)^{ij}_{ij} f_{ij} e^{\theta(x)} \, dx - \int_{P_\delta} (u_0)^{ij}_{ij} f_{ij} e^{\theta(x)} \, dx.
\]

Integrating by parts for the last two terms again, we have

\[
\int_{P_\delta} (u_0)^{ij}_{ij} f_{ij} e^{\theta(x)} \, dx = \int_{\partial P_\delta} (u_0)^{ij}_{ij} f_{ij} \, d\sigma_0 - \int_{\partial P_\delta} (u_0)^{ij}_{ij} f_{ij} \, d\sigma_0 - \int_{\partial P_\delta} (u_0)^{ij}_{ij} f_{ij} \, d\sigma_0
\]

\[
+ \int_{P_\delta} ((u_0)^{ij}_{ij} + (u_0)^{ij}_{ij} \theta_i + (u_0)^{ij}_{ij} \theta_j + (u_0)^{ij}_{ij} \theta_i \theta_j) e^{\theta(x)} \, dx.
\]

Note that

\[
(u_0)^{ij}_{ij} n_j \, d\sigma_0 \to 0, \quad -(u_0)^{ij}_{ij} n_i \, d\sigma_0 \to \, d\sigma
\]

as $\delta \to 0$ [DT1, D2]. Then

\[
\int_{\partial P_\delta} (u_0)^{ij}_{ij} f_{ij} \, e^{\theta(x)} \, d\sigma_0, \quad \int_{\partial P_\delta} (u_0)^{ij}_{ij} n_i \theta_j e^{\theta(x)} \, d\sigma_0 \to 0,
\]

and

\[
\int_{\partial P_\delta} (u_0)^{ij}_{ij} n_i f e^{\theta(x)} \, d\sigma_0 \to \int_{\partial P} f e^{\theta(x)} \, d\sigma
\]

as $\delta \to 0$. In conclusion,

\[
\int_P (u_0)^{ij}_{ij} f_{ij} e^{\theta(x)} \, dx = \int_{\partial P} f e^{\theta(x)} \, d\sigma + \int_P B f e^{\theta(x)} \, dx.
\]

Hence,

\[
\int_P \log\det(u_{ij}) e^{\theta(x)} \, dx \leq \int_P u e^{\theta(x)} \, d\sigma + \int_P B e^{\theta(x)} \, dx + \int_P u_0 e^{\theta(x)} \, d\sigma - \int_P B u_0 e^{\theta(x)} \, dx + \int_P \log\det((u_0)_{ij}) e^{\theta(x)} \, dx
\]

\[
= \int_{\partial P} u e^{\theta(x)} \, d\sigma + \int_P B e^{\theta(x)} \, dx + \text{const.}
\]

$\square$
Remark 3.1. As in [ZZ2], Lemma 3.3 can be extended for any \( u \in C_* \), where

\[
C_* = \{ u \mid u \text{ is convex and satisfies } \int_{\partial P} u \, d\sigma < \infty \}.
\]

Denote

\[
H(u) = \int_P u e^{\theta(x)} \, dx.
\]

Proposition 3.2. For any \( 0 < \delta < 1 \), there exists \( C_\delta > 0 \) such that

\[
\mathcal{F}(u) \geq \delta H(u) - C_\delta, \quad \forall u \in \tilde{C}_\infty.
\]

Proof. First, we compute the difference of \( L(u) \) and \( L_B(u) \)

\[
|L(u) - L_B(u)| = \left\| \int_P (n + \sum \theta_i x_i + B)e^{\theta(x)} \, dx \right\|
\leq C' \int_P u e^{\theta(x)} \, dx
\leq (1 + \delta)C_0 C' \int_{\partial P} u e^{\theta(x)} \, d\sigma - \delta C' \int_P u e^{\theta(x)} \, dx
\]

where \( C' = \|n + \sum \theta_i x_i + B\|_{L^\infty} \). Note

\[
\int_P u e^{\theta(x)} \, dx \leq C_0 \int_{\partial P} u e^{\theta(x)} \, d\sigma, \quad \forall u \in \tilde{C}_\infty.
\]

Then by (2.9), it follows

\[
|L(u) - L_B(u)| \leq \frac{(1 + \delta)C_0 C'}{\lambda} L(u) - \delta C' \int_P u e^{\theta(x)} \, dx.
\]

Thus

\[
\left( 1 + \frac{(1 + \delta)C_0 C'}{\lambda} \right) L(u) \geq L_B(u) + \delta C' \int_P u e^{\theta(x)} \, dx.
\]

Now let \( r = \left( 1 + \frac{(1 + \delta)C_0 C'}{\lambda} \right)^{-1} \), we get

\[
L(u) \geq L_B(ru) + r\delta C' \int_P u e^{\theta(x)} \, dx.
\]

Applying the inequality (3.11) to \( ru \), we obtain

\[
- \int_P \log \det(u_{ij}) e^{\theta(x)} \, dx \geq -L_B(ru) - C + n \log r.
\]

Hence,

\[
\mathcal{F}(u) \geq r\delta C' \int_P u e^{\theta(x)} \, dx - C + n \log r
\]

\( \square \)
3.3. Properness of $\mu_{\omega_s}(\cdot)$. In this subsection, we show that the properness of $\mathcal{F}(u)$ in the above subsection is equivalent to the properness of $\mu_{\omega_s}(\phi)$. We need a lemma as follows.

**Lemma 3.4.** There exists $C > 0$ such that

$$|J_{\omega_s}(\tilde{\phi}) - H(u_0)| \leq C, \quad \forall \, \phi \in \mathcal{H}_{G_0}(\omega_s),$$

where $\tilde{\phi} = \phi_{\sigma}$ is a normalization of $\phi$ after a transformation $\sigma \in T$ so that

$$(\psi_0 + \tilde{\phi})(0) = 0, \quad D(\psi_0 + \tilde{\phi})(0) = 0.$$

**Proof.** By the relation $\tilde{\phi}_t = -\nu_t$, it is easy to see

$$J_{\omega_s}(\phi) = \frac{1}{V} \int_M \phi e^{\theta_g(x)} \frac{\omega_s^n}{n!} + H(u_0) - H(u_0), \quad \forall \, \phi \in \mathcal{H}_{\mathcal{G}}(\omega_s).$$

In particular,

$$J_{\omega_s}(\tilde{\phi}) - H(u_0) = \frac{1}{V} \int_M \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} - H(u_0).$$

We claim that

$$\left| \frac{1}{V} \int_M \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} \right| \leq C$$

for some uniform constant $C$.

Let $G(p, p')$ be the Green function of $P_{\omega_s}$ so that

$$\int_M G(p, \cdot) e^{\theta_g(x)} \frac{\omega_s^n}{n!} = 0.$$

It is proved in [CTZ] that there exists a $C > 0$ depending only on $g$ such that

$$G(p, p') \geq -C.$$

Then applying the Green’s formula to potential $\tilde{\phi}$, we have

$$\tilde{\phi}(x) = \frac{1}{V} \int_M \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} - \int_M G(x, \cdot)(\Delta \tilde{\phi}(\cdot) + X(\tilde{\phi})) e^{\theta_g(x)} \frac{\omega_s^n}{n!},$$

(3.15)

where $C_0$ is a uniform constant. The second inequality follows from $\Delta \tilde{\phi} \geq -n$ and that $X(\tilde{\phi})$ is uniformly bounded [Z1]. Thus

$$\frac{1}{V} \int_M \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} \geq \sup_M \{\tilde{\phi}\} - C_0 = \sup_{\mathbb{R}^n} \{\tilde{\phi}\} - C_0.$$

(3.16)

Set

$$\Omega_N = \{\xi \in M | \tilde{\phi}(\xi) \leq \sup_M \{\tilde{\phi}\} - N\}.$$

Note that

$$\frac{1}{V} \int_M \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} = \frac{1}{V} \int_{M \cap \Omega_N} \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} + \frac{1}{V} \int_{M \setminus \Omega_N} \tilde{\phi} e^{\theta_g(x)} \frac{\omega_s^n}{n!} \leq \frac{1}{V} (\sup_M \{\tilde{\phi}\} - N) \overline{\text{Vol}(M \cap \Omega_N)} + \sup_M \{\tilde{\phi}\} \overline{\text{Vol}(M \setminus \Omega_N)}$$
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\[ \sup_M \left\{ \tilde{\phi} \right\} - N \cdot \frac{\text{Vol}(M \cap \Omega_N)}{\text{Vol}(M)} \]

Here \( \text{Vol}(\cdot) \) represents the volume with form \( e^{\theta x(\phi)} \frac{\omega^p}{n!} \). Hence by (3.16), we derive

\[ \text{Vol}(M \cap \Omega_N) \leq \frac{C_0 \text{Vol}(M)}{N} = \frac{C_0 V}{N} \to 0, \]

as \( N \to \infty \).

On the other hand, by the normalization, we have \( \tilde{\phi}(0) = -\psi_0(0) \). Note \( D(\tilde{\phi} + \psi_0) \in P \). Then we have

\[ |D\tilde{\phi}| \leq 2 \sup\{ |p| : p \in P \} \]

and

\[ \tilde{\phi}(x) \leq \tilde{\phi}(0) + 2r \sup\{ |p| : p \in P \} \leq C(r), \quad \forall \, x \in B_r(0), \]

where \( C(r) \) depends only on the radius \( r \) of ball \( B_r(0) \) centered at the original. Since the volume of domain \( B_1(0) \times (0, 2\pi)^n \subset M \) associated the metric \( \omega_g = \sqrt{-1} \partial \bar{\partial} \psi_0 \) is bigger than some number \( \varepsilon > 0 \), by (3.17), it is easy to see that there is at least a point \( x_0 \in B_1(0) \) such that

\[ \tilde{\phi}(x_0) \geq \sup_M \tilde{\phi} - N \]

if \( N \) is sufficiently large. Hence

\[ \sup_M \tilde{\phi} \leq N + C, \]

and consequently

\[ \frac{1}{V} \int_M \tilde{\phi} e^{\theta x(\phi)} \frac{\omega^p}{n!} \leq N + C. \]

By (3.16), we also get

\[ \frac{1}{V} \int_M \tilde{\phi} e^{\theta x(\phi)} \frac{\omega^p}{n!} \geq \tilde{\phi}(0) - C_0 = -\psi(0) - C_0. \]

Therefore the claim is true and the lemma is proved. \( \square \)

**Theorem 3.2.** There exist numbers \( \delta > 0 \) and \( C \) such that

\[ \mu_{\omega_g}(\phi) \geq \delta \inf_{\tau \in T} I_{\omega_g}(\phi_\tau) - C, \quad \forall \, \phi \in \mathcal{H}_{G_0}(\omega_g). \]

In particular, \( \mu(\phi) \) is proper for any \( \phi \in \mathcal{H}_{G_0}(\omega_g) \) modulo \( G_0 \).

**Proof.** Let \( \phi \in \mathcal{H}_{G_0}(\omega_g) \). Then there exists a \( \sigma \in T \) such that the Legendre function \( u_{\phi_\sigma} \) associated to \( \phi_\sigma \) is belonged to \( C_\infty \). By Proposition 3.2, we see that

\[ \mu_{\omega_g}(\phi_\sigma) \geq \delta H(\phi_\sigma) - C_\delta. \]

Note that \( \mu_{\omega_g}(\phi) = \mu_{\omega_g}(\phi_\sigma) \). Thus by Lemma 3.4, we get

\[ \mu_{\omega_g}(\phi) = \mu_{\omega_g}(\phi_\sigma) \geq \delta J_{\omega_g}(\phi_\sigma) - C_\delta \]

\[ \geq \delta \inf_{\tau \in T} J_{\omega_g}(\phi_\tau) - C_\delta \]

\[ \geq \frac{\delta}{n+1} \inf_{\tau \in T} I_{\omega_g}(\phi_\tau) - C_\delta. \]

Here at the last inequality we used the relation (3.1). \( \square \)
Remark 3.2. Theorem 3.2 seems to overlap Theorem 4.5 in \cite{BB} where the Aubin-Ding typed functional is studied instead of modified K-energy by using the geodesic theory for Kähler potentials.\footnote{We are indebted to Chi Li for telling us Berman-Berndtsson’s results in \cite{BB}.}

As an application of Theorem 3.2 together with Lemma 3.1 we give a new proof of the following Wang-Zhu Theorem \cite{WZ}.

**Theorem 3.3.** There exists a Kähler-Ricci soliton on any toric Fano manifold.

### 4. Generalization to conic metrics case

Singular Kähler-Ricci solitons on toric manifolds have been extensively studied by \cite{SZ, Le, BB, DGSW}, etc. In this section, we generalize the discussion in former sections to give a new approach by showing the properness of modified Log K-energy.

Let $M$ be a toric Fano manifold and $K_M$ is the canonical line bundle. Let $(D_i)_{i=1}^d$ be the toric divisors corresponding to the faces of the moment polytope. Suppose $\beta > 0$ and $D = \sum_{i=1}^d (1-\beta_i)D_i \in \{-1, \ldots, -1-\beta\}K_M$ be an effective $\mathbb{R}$-divisor with strictly normal crossing support and $0 < \beta_i \leq 1$ for each $i$. A conical Kähler metric $g$ on $M$ with angle $2\pi\beta_i$ along $D_i$ is a closed positive (1, 1) current in $2\pi\eta_1(M)$, which is a smooth Kähler metric $\omega_D$ in $M \setminus D$ and satisfies: for any $p \in D$, there is a coordinates neighborhood $U$ with local holomorphic coordinates $(z_1, \ldots, z_n)$ of $p$ such that $D \cap U = \{z_i = 0, 1 \leq i \leq r\}$ and the metric is asymptotically equivalent along the model conic metric

$$\sqrt{-1} \sum_{j=1}^r |z_j|^2 h_j^{-1/2} dz^j \wedge d\bar{z}^j + \sqrt{-1} \sum_{j=r+1}^n dz^j \wedge d\bar{z}^j.$$ 

One can check that the Guillemin metric $\omega_g = \omega_D = \sqrt{-1} \partial \bar{\partial} \phi_0$ induced by symplectic potential $u_0 = \sum \beta_i^{-1} l_i \log l_i$ is a conical Kähler metric (cf. \cite{Ab, Le, DGSW}), where $l_i = 1 - \langle \ell_i, x \rangle$. In fact, if we let a set of symplectic potentials $C_\beta = \{u = u_0 + f | u$ is a strictly convex function in $P, f \in C^\infty(\overline{P})\}$, then there is a one-to-one correspondence between $C_\beta$ and $\mathcal{H}^{G_0}_D(\omega_g)$, where $\mathcal{H}^{G_0}_D(\omega_g)$ consists all $G_0$-invariant Kähler potentials which are asymptotically equivalent to $\omega_D$.

Let $s_i$ be the defining section of $D_i$ and $h_i$ a Hermitian metric on $D_i$. Denote $\|s_i\|^2$ the norm of $s_i$. Then $h = \otimes_{i=1}^d h_i^{-1/\beta_i}$ defines a Hermitian metric on $D$ and gives a norm $\|s_D\|$ for the defining section $s = \otimes_{i=1}^d s_i^{1-\beta_i}$ of $D$. By the Poincare-Lelong identity,

$$\sqrt{-1} \partial \bar{\partial} \log \|s_i\|^2 = -c_1([D_i], h_i) + \{D_i\},$$

where $\{D_i\}$ denotes the current of integration along $D_i$, we have

$$\sqrt{-1} \partial \bar{\partial} \log \|s_D\|^2 = -c_1([D], h) + \{D\}. \tag{4.1}$$

(4.1) implies that there exists $\tau \in \mathbb{R}^n$ such that

$$\log \|s_D\|^2 = -(1 - \beta)\phi_0 - \beta \langle \tau, \xi \rangle + \text{const}. \tag{4.2}$$

Remark 3.2 seems to overlap Theorem 4.5 in \cite{BB} where the Aubin-Ding typed functional is studied instead of modified K-energy by using the geodesic theory for Kähler potentials. We are indebted to Chi Li for telling us Berman-Berndtsson’s results in \cite{BB}.
A conical Kähler metric \( \omega \) with \( 2\pi \beta_i \) angle along each \( D_i \) is called a conical Kähler-Ricci soliton if there is a holomorphic vector field \( X \) on \( M \) for some \( \beta \in (0, 1] \) such that
\[
(4.3) \quad \text{Ric}(\omega) - \beta \omega - \{D\} = L_X \omega.
\]
We will investigate a solution of (4.3) in \( \mathcal{H}^G_D(\omega_g) \). Let \( X = \sum \theta_{\alpha} \zeta_{\alpha} \), where \( \zeta_{\alpha} \) is a basis of Lie algebra \( \eta_T \) of \( T \). Then a lemma in [DGSW] shows that \( \tau = (\tau_1, ..., \tau_n) \) in (4.2) is uniquely determined by relation,
\[
(4.4) \quad \tau_{\alpha} = \int_P \alpha e^{\theta(x)} dx = \int_P e^{\phi(x)} dx, \quad \alpha = 1, ..., n.
\]
Moreover, \( \beta_i = 1 - \beta_i(\tau), \ i = 1, ..., d \).

Let \( \mathcal{H}_{X,D}(\omega_g) \) be a class of \( K_T \)-invariant functions \( \phi \in C^2(\alpha)(M) \) such that \( \omega_g + \sqrt{-1} i \partial \overline{\partial} \phi \) are conical metrics with \( 2\pi \beta_i \) angle along each \( D_i \). Following [LS] (also see [Be1, Li]), we consider the following modified Log \( K \)-energy functional on \( \mathcal{H}_{X,D}(\omega_g) \),
\[
(4.5) \quad \mu_{\omega_g,D}(\phi) = \mu_{\omega}(\phi) + (1 - \beta)(I_{\omega_g}(\phi) - J_{\omega}(\phi)) + \int_M \log \|s_D\|^2 e^{\theta_X(\omega_g)} \frac{\omega^n}{n!}.
\]
It is easy to check that a solution of (4.5) is a critical point of \( \mu_{\omega_g,D}(\phi) \). We need to study the properness of \( \mu_{\omega_g,D}(\phi) \).

Define
\[
(4.6) \quad \mathcal{L}_{\beta,\tau}(u) = \beta \left( \mathcal{L}(u) - \int_P \langle \tau, \nabla u \rangle e^{\theta(x)} dx \right), \quad \forall \ u \in C_{\omega_g}.
\]
By (4.4), it is clear
\[
(4.7) \quad \mathcal{L}_{\beta,\tau}(u) = 0, \forall \ u = \sum a_{\alpha} x_{\alpha}, \ (a_1, ..., a_n) \in \mathbb{R}^n.
\]

**Lemma 4.1.** Let \( \phi \in \mathcal{H}^G_D(\omega_g) \) and \( u \) be the Legendre dual function of \( \varphi_0 + \phi \). Then
\[
(4.8) \quad \mu_{\omega_g,D}(\phi) = \frac{(2\pi)^n}{V} \mathcal{F}_{\beta,\tau}(u) + C,
\]
where
\[
(4.9) \quad \mathcal{F}_{\beta,\tau}(u) = -\int_M \log \det(u_{ij}) e^{\theta(x)} dx + \mathcal{L}_{\beta,\tau}(u),
\]
and \( C \) is a constant.

**Proof.** It suffices to transform the latter two terms in (4.5) under symplectic potentials. Note that
\[
I_{\omega_g}(\phi) - J_{\omega}(\phi) = -\frac{1}{V} \int_M \phi e^{\theta_X(\omega_g)} \frac{\omega^n}{n!} + \int_M \frac{1}{V} \int_0^1 \int_M \phi_X e^{\theta_X(\omega_g)} \frac{\omega^n}{n!} ds
\]
Hence by similar expression as in Proposition 3.5 we have
\[
(1 - \beta)(I_{\omega_g}(\phi) - J_{\omega}(\phi)) + \int_M \log \|s_D\|^2 e^{\theta_X(\omega_g)} \frac{\omega^n}{n!}
\]
\[
= \frac{(2\pi)^n}{V} \left[ -\beta \int_{\mathbb{R}^n} e^{X(\varphi)} \det D^2 \varphi d\xi + \beta \int_{\mathbb{R}^n} \langle \tau, \xi \rangle e^{X(\varphi)} \det D^2 \varphi d\xi \right]
\]
\[
+ \frac{(2\pi)^n}{V} \int_0^1 \int_M \phi_X e^{\theta_X(\omega_g)} \frac{\omega^n}{n!} ds
\]
\[
\frac{(2\pi)^n}{V} \left[ -(1 - \beta)\mathcal{L}(u) - \beta \int_P \langle \tau, \nabla u \rangle e^{\theta(x)} \, dx \right]
\]

Combing with (3.6), we obtain (4.9). \hfill \square

To prove the properness of \( F_{\beta, \tau}(u) \), by (4.7), it suffices to consider the function space \( \tilde{C}_{\infty, \tau} \) which contains all the functions in \( C_{\infty} \) normalized at \( \tau \).

**Proposition 4.1.** If \( \tau \in P \), then for any \( 0 < \delta < 1 \), there exists \( C_{\delta, \beta} > 0 \) such that

\[
F_{\beta, \tau}(u) \geq \delta \int_P u e^{\theta(x)} \, dx - C_{\delta, \beta}, \quad \forall u \in \tilde{C}_{\infty, \tau}.
\]

**Proof.** The proof is similar to Proposition 3.2. Note that by the definition of Legendre transform, we have \((x - \tau) \cdot \nabla u - u \geq 0 \) for \( u \in \tilde{C}_{\infty, \tau} \). Then one can replace (2.10) in Lemma 2.2 by

\[
\mathcal{L}_{\beta, \tau}(x) = \beta \left( \int_P (x - \tau) \cdot \nabla u - u \right) e^{\theta(x)} \, dx + \int_P u e^{\theta(x)} \, dx \geq \beta \int_P u e^{\theta(x)} \, dx
\]

and check the arguments line by line in Section 3.2. \hfill \square

Applying Lemma 4.1 and Proposition 4.1 we can give a new proof of following Datar-Guo-Song-Wang Theorem [DGSW].

**Theorem 4.1.** Let \( X = \sum_{i=1}^n \theta_i \zeta_i \in \eta_T \). Let \( \bar{\beta} = \sup \{ \beta \mid \beta \zeta_i(\tau) < 1, \quad i = 1, ..., d \} \). Suppose that \( \tau \in P \). Then for any \( \beta \leq \bar{\beta} \) there exists a unique toric invariant conical Kähler-Ricci soliton \( \omega \) which solves (4.3) with \( D = \sum (1 - \beta_i)D_i \) and \( \beta_i = \beta \zeta_i(\tau) \). Moreover, the conical angles of \( \omega \) are \( 2\pi \beta_i \) along \( D_i \).

**Proof.** Since higher order estimates depend on \( C^0 \)-estimate for a family of Kähler potentials \( \phi_i \) as in [TZ1, JMR] (also see [DGSW] for toric manifolds), it suffices to get the \( C^0 \)-estimate when we use the continuity method to solve (4.3). Since \( \mu_{\omega_i, D}(\phi) \) is monotonic for \( \phi_i \) as in the smooth metrics case [TZ1, CTZ], the properness of \( \mu_{\omega_i, D}(\phi) \) implies that \( I_{\omega_i}(\phi_i) \) is uniformly bonuded. As a consequence, we get an upper bound of \( \phi_i \). The lower bound can be also obtained by establishing a uniform lower bound for the Green functions of \( \omega_{\phi_i} \) as in [Ma, CTZ] for smooth metrics. There is another way to get a Hölder estimate for Legendre functions \( u_i \) of \( \phi_i \) by using an observation in [D3] for toric invariant metrics, if one knows the upper bound of \( u_i \), which is equal to one to one of \( \phi_i \). Then \( u_i \) is uniformly bounded, and so is \( \phi_i \) (also see [SZ]). In fact, the latter argument was presented by Datar-Guo-Song-Wang [DGSW] while they got an upper bound of \( u_i \) by studying a class of real Monge-Ampère equations as done in [WZ]. The uniqueness of \( \omega \) follows from the convexity of \( F_{\beta, \tau}(u) \).

In case that \( X \) is chosen as a soliton vector field on \( M \), \( \tau = 0 \) by (4.4). Then in Theorem 4.1 \( \beta_i = \beta \leq 1 \), \( i = 1, ..., d \), and \( D = \sum (1 - \beta)D_i \). Thus Theorem 0.2 is a corollary of Theorem 4.1.
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