New properties of Cauchy and event horizons

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Abstract

We present several recent results concerning Cauchy and event horizons. In the first part of the paper we review the differentiability properties of the Cauchy and the event horizons. In the second part we discuss compact Cauchy horizons and summarize their main properties.

1 Introduction

Cauchy horizons and event horizons play an important role in relativity. The consequence of causality which is one of the basic postulates of relativity is that data on an initial surface \( S \) determine the evolution of the relativistic equations in the domain of dependence which is the set of points \( p \) such that all past-directed causal curves from \( p \) intersect \( S \). The future boundary of domain of dependence, if non-empty, is called the Cauchy horizon and it is a null surface. To the future of a Cauchy horizon space-time cannot be predicted from the initial surface \( S \). A fundamental unresolved problem in classical relativity posed by Roger Penrose is whether there is a "cosmic censor" that ensures existence of an initial surface from which the whole of space-time is predictable and a Cauchy horizon does not occur. Hence it is of interest to study the properties of Cauchy horizons and conditions on space-times under which they can or cannot arise. The event horizon is the boundary of the region of space-time to the past of the boundary at infinity. The event horizon like a Cauchy horizon is also a null surface. The region of space-time to the future of the event horizon is the black hole region from which observers cannot escape and which may contain a space-time singularity at which curvature grows unboundedly. This indicates breakdown of classical general relativity theory.
No satisfactory theory exists till now to describe these singular regions. Black hole regions should arise as a result of the collapse of stars and galaxies and the astronomical observations show that almost certainly this is the case. Thus study of the properties of event horizon has attracted great interest.

2 Preliminaries

In this Section we shall recall several basic definitions and properties, all of which can be found, e.g., in [1,2].

Definition 1 A space-time \((M, g)\) is a separable smooth \(n\)-dimensional, Hausdorff manifold \(M\) with a pseudo-Riemannian metric \(g\) of signature \((-,-,+,...,+\) and a time orientation.

Let \(I^+(S,U), I^-(S,U)\) be respectively chronological future and chronological past of set \(S\) relative to set \(U\) and let \(J^+(S,U), J^-(S,U)\) be the relative causal future and past. When the set \(U\) is omitted the chronological and causal pasts and futures are relative to space-time manifold.

A set \(S\) is said to be achronal respectively acausal if there are no two points of \(S\) with timelike respectively causal separation. The edge, \(\text{edge}(A)\), of the achronal set \(A\) consists of all points \(p\) in \(\overline{A}\) such that every neighborhood \(U\) of \(p\) contains a timelike curve from \(I^-(p,U)\) to \(I^+(p,U)\) which does not meet \(A\). The set \(A\) is said to be edgeless if \(\text{edge}(A) = \emptyset\).

Definition 2 A partial Cauchy surface \(S\) is a connected, acausal, edgeless \((n-1)\) dimensional submanifold of \((M,g)\).

We give definitions and state our results in terms of the future horizon \(H^+(S)\), but similar results hold for any past Cauchy horizon \(H^-(S)\).

Definition 3 The future Cauchy development \(D^+(S)\) consists of all points \(p \in M\) such that each past endless and past directed causal curve from \(p\) intersects the set \(S\). The future Cauchy horizon is \(H^+(S) = \overline{(D^+(S))} - I^-(D^+(S))\).

The future Cauchy horizon is generated by null geodesic segments. Let \(p\) be a point of the Cauchy horizon; then there is at least one null generator of \(H^+(S)\) containing \(p\). When a null generator of \(H^+(S)\) is extended into the past it either has no past endpoint or has a past endpoint on \(\text{edge}(S)\) [see [1], p. 203]. However, if a null generator is extended into the future it may have a last point on the horizon which then said to be an endpoint of the horizon. We define the multiplicity [see [3]] of a point \(p\) in \(H^+(S)\) to be the number of null generators containing \(p\). Points of the horizon which are not endpoints must have multiplicity one. The multiplicity of an endpoint may be any positive.
integer or infinite. We call the set of endpoints of multiplicity two or higher the crease set, compare [5].

Let the spacetime $(M, g)$ be strongly future asymptotically predictable and let $\mathcal{J}^+$ be the future null infinity (see [1] Chapter 9). The set $B = M - J^-(\mathcal{J}^+)$ is called the black hole region and $E = \mathcal{J}^-(\mathcal{J}^+)$ is the event horizon. The event horizon like a Cauchy horizon is generated by null geodesic segments however the generators of the event horizon have no future endpoints but may have past endpoints.

3 Non-differentiable Cauchy and event horizons

By a basic Proposition due to Penrose [[1], Prop. 6.3.1] $H^+(S)$ is an $n - 1$ dimensional Lipschitz topological submanifold of $M$ and is achronal. Since a Cauchy horizon is Lipschitz it follows from a theorem of Rademacher that it is differentiable almost everywhere (i.e. differentiable except for a set of $n - 1$ dimensional measure zero). This does not exclude the possibility that the set of non-differentiable points is a dense subset of the horizon. In this section we shall give examples of such a behaviour.

$H^+(S)$ is differentiable if it is a differentiable submanifold of $M$. Thus if $H^+(S)$ is differentiable at the point $p$, then there is a well defined 3-dimensional linear subspace $N_0$ in the tangent space $T_p(M)$ such that $N_0$ is tangent to the 3-dimensional surface $H^+(S)$ at $p$.

**Theorem 4** (Chruściel and Galloway [5])

There exists a connected set $U \subset \mathbb{R}^2 = \{t = 0\} \subset \mathbb{R}^{2,1}$, where $\mathbb{R}^{2,1}$ is a 2 + 1 dimensional Minkowski space-time, with the following properties:

1. The boundary $\partial U = \bar{U} - \text{int} U$ of $U$ is a connected, compact, Lipschitz topological submanifold of $\mathbb{R}^2$. $U$ is the complement of a compact set in $\mathbb{R}^2$.
2. There exists no open set $\Omega \subset \mathbb{R}^{2,1}$ such that $\Omega \cap H^+(U)$ is a differentiable submanifold of $\mathbb{R}^{2,1}$.

A further study has shown that the densely nondifferentiable Cauchy horizons are quite common [6]. Let $\mathbb{R}^{2,1}$ be the 3-dimensional Minkowski space-time. Let $\Sigma$ be the surface $t = 0$, and let $K$ be a compact, convex subset of $\Sigma$. Let $\partial K$ denote the boundary of $K$.

Let $\mathcal{H}$ be the set of Cauchy horizons arising from compact convex sets $K \subset \Sigma$. The topology on $\mathcal{H}$ is induced by the Hausdorff distance on the set of compact and convex regions $K$. 

3
Theorem 5 Let $H$ be the set of future Cauchy horizons $H^+(K)$ where $K$ are compact and convex regions of $\Sigma$. The subset of densely nondifferentiable horizons is dense in $H$.

The above theorem generalizes to the 3-dimensional case.

It also possible to construct examples of densely nondifferentiable Cauchy horizons of partial Cauchy surfaces and also the existence of densely non-differentiable black hole event horizons.

Example 1: A rough wormhole.

Let $\mathbb{R}^{3,1}$ be the 4-dimensional Minkowski space-time and let $K$ be a compact subset of the surface $\{t = 0\}$ such that its Cauchy horizon is nowhere differentiable in the sense of Theorem 5. We consider a space-time obtained by removing the complement of the interior of the set $K$ in the surface $t = 0$ from the Minkowski space-time. Let us consider the partial Cauchy surface $S = \{t = -1\}$. The future Cauchy horizon of $S$ is the future Cauchy horizon of set $K - \text{edge}(K)$, since $\text{edge}(K)$ has been removed from the space-time. Thus the future Cauchy horizon is nowhere differentiable and it is generated by past-endless null geodesics. The interior of the set $K$ can be thought of as a “wormhole” that separates two “worlds”, one in the past of surface $\{t = 0\}$ and one in its future.

Example 2: A transient black hole.

Let $\mathbb{R}^{3,1}$ be the 4-dimensional Minkowski space-time and let $K$ be a compact subset of the surface $\{t = 0\}$ such that its past Cauchy horizon is nowhere differentiable in the sense of Theorem 5. We consider a space-time obtained by removing from Minkowski space-time the closure of the set $K$ in the surface $t = 0$. Let us consider the event horizon $E = J^{-}(J^+)$. The event horizon $E$ coincides with $H^-(K) - \text{edge}(K)$ and thus it is not empty and nowhere differentiable. The event horizon disappears in the future of surface $\{t = 0\}$ and thus we can think of the black hole (i.e. the set $B := \mathbb{R}^{3,1} - J^{-}(J^+)$) in the space-time as “transient”.

The following results summarize differentiability properties of Cauchy horizons.

Theorem 6 (Beem and Królak [3], Chruciel [4]) A Cauchy horizon is differentiable at all points of multiplicity one. In particular, a Cauchy horizon is differentiable at an endpoint where only one null generator leaves the horizon.
**Proposition 7** (Beem and Królak [3]) Let $W$ be an open subset of the Cauchy horizon $H^+(S)$. Then the following are equivalent:

1. $H^+(S)$ is of class $C^r$ on $W$ for some $r \geq 1$.
2. $H^+(S)$ has no endpoints on $W$.
3. All points of $W$ have multiplicity one.

Note that the three parts of Proposition 7 are logically equivalent for an open set $W$, but that, in general, they are not equivalent for sets which fail to be open. Using the equivalence of parts (1) and (3) of Proposition 7, it now follows that near each endpoint of multiplicity one there must be points where the horizon fails to be differentiable. Hence, each neighborhood of an endpoint of multiplicity one must contain endpoints of higher multiplicity. This yields the following corollary.

**Corollary 8** (Beem and Królak [3])

If $p$ is an endpoint of multiplicity one on a Cauchy horizon $H^+(S)$, then each neighborhood $W(p)$ of $p$ on $H^+(S)$ contains points where the horizon fails to be differentiable. Hence, the set of endpoints of multiplicity one is in the closure of the crease set.

However the following conjecture remains so far unresolved.

**Conjecture.** The set of all endpoints of a Cauchy horizon must have measure zero.

### 4 Compact and compactly generated Cauchy horizons

In this section we shall assume that space-time manifold is 4-dimensional. We shall say that a Cauchy horizon $H$ is **almost smooth** if it contains an open set $G$ where it is $C^2$ and such that complement of $G$ in $H$ has measure zero. Throughout the rest of this section we shall assume that all Cauchy horizons are almost smooth. At the end of this section we shall mention recent results by which it may be possible to relax the above differentiability assumption.

Let $t$ be an affine parameter on a null geodesic $\lambda$, $k^a$ be components of the tangent vector to $\lambda$, $\theta$ be expansion, $\sigma$ be shear and $R_{ab}$ be components of the Ricci tensor. Then we have the following ordinary non-linear equation of Riccati type for $\theta$.

$$\frac{d\theta}{dt} = -\frac{1}{2} \theta^2 - 2\sigma^2 - R_{ab}k^ak^b,$$  \hspace{1cm} (1)
The quantity $\theta$ describes the expansion of congruences of null geodesics infinitesimally neighboring $\lambda$ and it is defined as $\theta = \frac{1}{A} \frac{dA}{dt}$ where $A$ is cross-section area of the congruence. The above equation is used extensively in geometrical techniques to study the large-scale structure of space-time developed by Geroch, Hawking, and Penrose and we shall call it Raychaudhuri-Newman-Penrose (RNP) equation.

We shall first introduce the following two conditions.

**Condition 9 (Null convergence condition)** We say that the null convergence condition holds if $R_{ab}k^ak^b \geq 0$ for all null vectors $k$.

Let $R_{abcd}$ be components of Riemann tensor. We say that an endless null geodesic $\gamma$ is **generic** if for some point $p$ on $\gamma$, $k^cR_{[a|b]d[|e]k} [f] \neq 0$ where $k$ is a vector tangent to $\gamma$ at $p$.

**Condition 10 (Generic condition)** All endless null geodesics in space-time are generic.

By the Einstein equations the null convergence condition means that the local energy density is non-negative and it is satisfied by all reasonable classical matter models. The generic condition means that every null geodesic encounters some matter or radiation that is not pure radiation moving in the direction of the geodesic tangent. The above two purely geometrical conditions have a very clear physical interpretation and they are reasonable to impose for any classical matter fields like gravitational and electromagnetic fields.

**Definition 11** Let $\mathcal{S}$ be a partial Cauchy surface. A future Cauchy horizon $H^+(\mathcal{S})$ is compactly generated if all its generators, when followed into their past, enter and remain in a compact subset $C$.

The above class of Cauchy horizons has been introduced by Hawking [9] to describe a situation in which a Cauchy horizon arises as a result of causality violation rather than singularities or timelike boundary at infinity.

**Remark:** It is clear that every compact Cauchy horizon is compactly generated where the set $C$ in the above definition is the horizon itself.

**Theorem 12** If the null convergence condition holds then a compactly generated Cauchy horizon that is non-compact cannot arise.

The proof of Theorem 12 will be outlined later in the paper.

**Corollary 13 (Hawking 1992[9])** If null convergence condition holds then a compactly generated Cauchy horizon $H^+(\mathcal{S})$ cannot arise from a non-compact partial Cauchy surface $\mathcal{S}$.
Outline of the proof: The result follows from Theorem 12 because if a partial Cauchy surface $\mathcal{S}$ is noncompact then the future Cauchy horizon $H^+(\mathcal{S})$ cannot be compact. □

Thus under a very mild - from physical point of view - restriction on space-time a nontrivial class of Cauchy horizons is ruled out.

For the case of compact Cauchy horizons we have the following result.

**Theorem 14** If null convergence condition holds and at least one of the null geodesic generators of a Cauchy horizon $H$ is generic then $H$ cannot be compact.

The proof of the above results relies on several lemmas which summarize basic properties of compact and compactly generated Cauchy horizons.

**Lemma 15 (Hawking and Ellis 1973[1])** Let $H^+(\mathcal{S})$ be a compact future Cauchy horizon for a partial Cauchy surface $\mathcal{S}$, then the null geodesic generating segments of $H^+(\mathcal{S})$ are geodesically complete in the past direction.

Outline of the proof: One shows that when a null generator $\gamma$ of the future Cauchy horizon is past incomplete, $\gamma$ can be varied into the past to give a past-intextendible timelike curve imprisoned in a compact subset of the interior of the future domain of dependence $D^+(\mathcal{S})$. This is a contradiction as all timelike past-intextendible curves in $\text{int}D^+(\mathcal{S})$ must intersect $\mathcal{S}$. □

**Lemma 16 (Hawking and Ellis 1973[1])** Let the null convergence condition hold. Then the expansion $\theta$ and the shear $\sigma$ of null geodesic generators of a compact Cauchy horizon $H^+(\mathcal{S})$ are zero.

Outline of the proof: On space-time manifold $M$ one can introduce a Riemannian positive definite metric $\hat{g}$. Let $G$ be the set defined at the beginning of the section. Through every point of $G$ there passes a unique generator of $H^+(\mathcal{S})$. Following Hawking we introduce a map $u_t : G \rightarrow G$, see [[9], p. 606] which moves each point of $G$ a proper distance $t$ in the metric $\hat{g}$ into the past along a generator of $H^+(\mathcal{S})$. We have the equation

$$\frac{d}{dt} \int_{u_t(G)} dA = 2 \int_{u_t(G)} \theta dA \tag{2}$$

Notice that since $H^+(\mathcal{S})$ is compact we find that $\int_{u_t(G)} dA$ is finite. The derivative of $\int_{u_t(G)} dA$ cannot be positive since the set $G$ is mapped into itself. Thus, the right hand side of Equation 2 is nonpositive. On the other hand $\theta \geq 0$ since otherwise by null convergence condition and past-completeness from the RNP equation there would be a past endpoint on a generator of $H^+(\mathcal{S})$. Thus
the right hand side of Equation 2 is nonnegative. Since by null convergence condition and RNP equation \( \theta \) is a monotonic function the only possibility is that \( \theta = 0 \). Consequently from RNP equation and the null convergence condition it follows that \( \sigma = 0 \) as well. \( \Box \)

Let \( \lambda : I \to M \) be a continuous curve defined on an open interval, \( I \in \mathbb{R}^1 \), which may be infinite or semi-infinite. We say that point \( x \in M \) is a terminal accumulation point of \( \lambda \) if for every open neighborhood \( O \) of \( x \) and every \( t_0 \in I \) there exists \( t \in I \) with \( t > t_0 \) such that \( \lambda(t) \in O \). When \( \lambda \) is a causal curve, we call \( x \) a past terminal accumulation point if it is a terminal accumulation point when \( \lambda \) is parametrized so as to make it past-directed.

**Definition 17** Let \( H^+(S) \) be a compactly generated future Cauchy horizon. The base set \( B \) is defined by \( B = \{ x \in H^+(S) : \text{there exists a null generator, } \lambda \text{ of } H^+(S) \text{ such that } x \text{ is a past terminal accumulation point of } \lambda \} \)

We have the following proposition.

**Proposition 18 (Kay et al.\cite{10})** The base set \( B \) of any compactly generated Cauchy horizon, \( H^+(S) \), always is nonempty. In addition, all the generators of \( H^+(S) \) asymptotically approach \( B \) in the sense that for each past-directed generator, \( \lambda \), of \( H^+(S) \) and each open neighbourhood \( O \) of \( B \), there exists a \( t_0 \in I \) (where \( I \) is the interval of definition of \( \lambda \)) such that \( \lambda(t) \in O \) for all \( t > t_0 \). Finally, \( B \) is comprised by null geodesic generators, \( \gamma \), of \( H^+(S) \) which are contained entirely within \( B \) and are both past and future inextendible.

**Outline of the proof:** The first two properties follow easily from the definitions. To prove the last property one chooses a point \( x \in B \) and considers a past inextendible generator \( \lambda \) such that \( x \) is a past terminal accumulation point of \( \lambda \). One takes a sequence of points \( p_i \) on \( \lambda \) and a sequence of tangent vectors \( k_i \) to \( \lambda \) at each \( p_i \). By compactness of the Cauchy horizon there exists a tangent vector \( k \) at \( x \) such that \( \{(p_i,k_i)\} \) converges to \( \{(p,k)\} \). This determines an inextendible null geodesic \( \gamma \) through \( x \). One then shows that \( \gamma \) is contained in \( B \) and that it is inextendible using the fact that for an arbitrary point \( y \) on \( \gamma \) there is a sequence of points on \( \lambda \) converging to \( y \). \( \Box \)

**Remark.** The proof that a compact Cauchy horizon must contain an endless null generator has independently been given by Borde \cite{11} and Hawking \cite{9}.

**Lemma 19 (Beem and Królik 1998\cite{3})** Let the null convergence condition hold. Then all null geodesic generators of a compact future Cauchy horizon \( H^+(S) \) are endless.

**Outline of the proof:** Assume that \( H^+(S) \) has an endpoint \( p \) of a null generator \( \gamma \). Even if \( \gamma \) does not lie in \( G \), the horizon will be differentiable on the part of
γ in the past of p. Choose some q on γ in the past of p and some u on γ in the future of p. Then for some small neighborhood W(q) of q on \( H^+(S) \), all null generators of the horizon through points r of W(q) will have directions close to the direction of γ at q. Recall that given a compact domain set in the t-axis, geodesics with close initial conditions remain close on the compact domain set. Thus, by choosing W(q) sufficiently small all null generators through points of W(q) will come arbitrarily close to u. Therefore for sufficiently small W(q) all null generators intersecting W(q) must leave \( H^+(S) \) in the future. Since W(q) is open in \( H^+(S) \) it must have a nontrivial intersection with the set G. Thus, \( u_t(G) \) cannot be all of G for some positive values of t and this yields a negative for some values of t on the left hand side of Equation 2, in contradiction. Thus, \( H^+(S) \) has no endpoints and by Proposition 7 must be (at least) of class \( C^1 \) at all points.

**Outline of the proof of Theorem 12** One can show that Lemma 15 can be adapted to the case of a compactly generated Cauchy horizon. The past-complete and future-endless generators of \( H^+(S) \) are then contained in the compact set C in Definition 11 [9]. One can also introduce the map \( u_t \) described in the proof of Lemma 16 restricting it to the set C where the integral formula 2 holds as well. Suppose that the null convergence condition holds then by past-completeness of the null generators of the horizon in the set C, \( \theta \geq 0 \) and the right hand side of Eq. 2 cannot be negative. However if the compactly generated horizon is noncompact it will not lie completely in the compact set C and consequently the left hand side of the Eq. 2 has to be strictly negative. This gives a contradiction.

**Proof of Theorem 14** Suppose that \( H^+(S) \) is a compact Cauchy horizon. By generic condition and the last part of Proposition 18 it follows that there will be an endless null generator γ of H such that for some point p on γ \( k^a R_{b[cd]} k^d \neq 0 \) where k is a vector tangent to γ at p. By RNP equation and the null convergence condition this is would mean that the expansion \( \theta \) is non-zero somewhere on λ and this contradicts Lemma 16.

By Lemma 19, a compact Cauchy horizon cannot contain even one generic generator. Thus we see that under Conditions 9 and 10 modulo certain differentiability assumptions compact Cauchy horizons are ruled out.

The above results are proved under the assumption of \( C^2 \) differentiability (modulo a set of measure zero) of the Cauchy horizon. Chruściel et al. [12] using the methods of geometric measure theory have shown that it was possible to define expansion \( \theta \) on non-differentiable event and Cauchy horizons. Consequently they were able to give a proof of Hawking’s black hole area theorem applicable to any event horizon in an asymptotically flat smooth space-time. They were also able to prove a remarkable theorem (Theorem 6.18 [12]) that when \( \theta \) vanishes on a Cauchy horizon and generators are endless then the hori-
zon is as smooth as the metric allows. If one were able to actually prove that \( \theta \) is always zero on a compact Cauchy horizon and that generators are end-less (without almost smoothness assumption) one would have a fundamental result that any compact Cauchy horizon in a smooth space-time is smooth.

The results given above were stated for the case of compact and compactly generated future Cauchy horizons. The time reverse versions hold for the case of compact and compactly generated past Cauchy horizons.

5 Acknowledgments

This work was supported by the Polish Committee for Scientific Research through grants 2 P03B 130 16 and 2 P03B 073 15.

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