Toda hierarchies and their applications

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Abstract

The 2D Toda hierarchy occupies a central position in the family of integrable hierarchies of the Toda type. The 1D Toda hierarchy and the Ablowitz–Ladik (aka relativistic Toda) hierarchy can be derived from the 2D Toda hierarchy as reductions. These integrable hierarchies have been applied to various problems of mathematics and mathematical physics since 1990s. A recent example is a series of studies on models of statistical mechanics called the melting crystal model. This research has revealed that the aforementioned two reductions of the 2D Toda hierarchy underlie two different melting crystal models. Technical clues are a fermionic realization of the quantum torus algebra, special algebraic relations therein called shift symmetries, and a matrix factorization problem. The two melting crystal models thus exhibit remarkable similarity with the Hermitian and unitary matrix models for which the two reductions of the 2D Toda hierarchy play the role of fundamental integrable structures.

Keywords: Toda lattice, integrable hierarchy, free fermion, melting crystal, quantum torus, shift symmetry, factorization problem

1. Introduction

In 1967, Morikazu Toda introduced a one-dimensional lattice mechanical system with exponential interactions nowadays called the Toda lattice [1]. Though designed to have a periodic solution written in terms of elliptic functions [2], this nonlinear lattice was soon shown to have a solution with colliding solitons. This suggested remarkable similarity with the KdV equation, hence integrability.

Integrability of the Toda lattice was established by the middle of 1970s after the construction of exact N-soliton solutions [3], first integrals in involution [4, 5], Lax pairs for the inverse scattering method [6, 7] and finite-band integration of the periodic problem [8, 9]. These results were extended to a system of two-dimensional relativistic fields with exponential interactions among the components of the fields. By the end of 1970s, this 2D Toda
field equation was proved to be integrable by the Lie group theory [10], the inverse scattering method [11, 12] and the bilinearization method [13]. In the beginning of 1980s, a fully 3D discretization was proposed in a bilinear form along with $N$-soliton solutions [13]. A description of more general solutions of this discrete system was soon presented in the language of a 2D complex free fermion system [14].

The Toda hierarchies [15] were developed as a Toda version of the KP hierarchy [16, 17] and its various relatives [18]. One of its prototypes is an unpublished result of van Moerbeke that is quoted in the work of Adler [19]. This result explains how to construct an integrable hierarchy of Lax equations for a difference operator $L$. In particular, the integrable hierarchy for the Jacobi operator

$$L = e^{\partial_s} + b + ce^{-\partial_s},$$

referred to as the 1D Toda hierarchy, contains the equation of motion of the Toda lattice as the lowest member of the Lax equations with time variables $t = (t_1, t_2, \ldots)$. In view of the construction of the KP hierarchy with a pseudo-differential Lax operator, it is natural to extend this construction to a “pseudo-difference operator” of the form

$$L = e^{\partial_s} + u_0 + u_1 e^{-\partial_s} + \cdots.$$ 

This extension, however, is not enough to accommodate the 2D Toda field equation. To this end, another Lax operator of the form

$$\bar{L}^{-1} = u_0 e^{-\partial_s} + \bar{u}_1 + \bar{u}_2 e^{\partial_s} + \cdots$$

has to be introduced along with another set $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots)$ of time variables. The 2D Toda hierarchy consists of Lax equations for these two Lax operators $L, \bar{L}$ with respect to the two sets $t, \bar{t}$ of time variables. The whole system of these Lax equations turns out to be equivalent to a system of Zakharov–Shabat equations for difference operators. The 2D Toda field equation is contained therein as the lowest member.

The 2D Toda hierarchy can be reformulated as a system of bilinear equations of the Hirota form for a single tau function $\tau(s, t, \bar{t})$ (in which the lattice coordinate $s$ is treated on an equal footing with the other independent variables). These bilinear equations can be cast into (and derived from) a generating functional form. One can thereby deduce [15] that $\tau(s, t, \bar{t})$, up to a sign factor, coincides with the tau function of the two-component KP hierarchy with charge $(s, -s)$ [20]. This leads to a fermionic formula of $\tau(s, t, \bar{t})$. Actually, $\tau(s, t, \bar{t})$ has another fermionic formula [21–23] that is directly related to a matrix factorization problem for solving the 2D Toda hierarchy in the Lax formalism [24]. This fermionic formula is a very powerful tool for studying various special solutions of the 2D Toda hierarchy including those that we consider in this paper.

Many applications of the 2D Toda hierarchy and its relatives have been found in mathematics and mathematical physics. In the 1990s, the 1D and 2D Toda hierarchies were applied to 2D gravity [25–28] and $c = 1$ string theory [29–34] as well as mathematical aspects of random matrices and orthogonal polynomials [35–37]. This is also the place where the Ablowitz–Ladik hierarchy [38] (aka the relativistic Toda hierarchy [39]) plays a role. These studies also revealed new features of the 2D Toda hierarchy itself such as the Orlov–Schulman operators, additional symmetries and dispersionless analogues [40, 41]. Researches on the dispersionless 2D Toda hierarchy revived later on when a relation to interface dynamics and complex analysis was pointed out [42, 43].

Sources of new researches were discovered in the early 2000s in enumerative geometry of $\mathbb{C}P^1$ and $\mathbb{C}^2$ [44–52] and 4D $\mathcal{N} = 2$ supersymmetric gauge theories [53–56]. For example, a generating function of the double Hurwitz numbers of $\mathbb{C}P^1$ was shown to be a tau function of
the 2D Toda hierarchy [45]. This tau function falls into a class of special tau functions called ‘hypergeometric tau functions’ that was introduced around 2000 in a quite different context [57–59]. Intersection numbers of the Hilbert scheme of points on $\mathbb{C}^2$, too, give a hypergeometric tau function [50, 51]. On the other hand, Gromov–Witten invariants of $\mathbb{C}P^1$ yield a different kind of tau functions [47, 48].

This paper reviews our work in the last ten years on integrable structures of the melting crystal models [60–64]. We focus on two typical cases among these models of statistical mechanics. The first case is a statistical model of random 3D Young diagrams [65] (hence referred to as a ‘crystal model’). Its partition function may be also thought of as the simplest instanton partition function of 5D supersymmetric gauge theories [66]. The second case is a slight modification of the first case, and related to enumerative geometry [67] and topological string theory [68] of a Calabi–Yau threefold called the ‘resolved conifold’. Our work have proved that the 1D Toda hierarchy and the Ablowitz–Ladik hierarchy underlie these two melting crystal models. Let us mention that such a relation between the resolved conifold and the Ablowitz–Ladik hierarchy was pointed out first by Brini [69]. It is remarkable that these two integrable hierarchies, both of which are reductions of the 2D Toda hierarchy [15, 36, 70], are also known to be the integrable structures of two typical random matrix models, namely the Hermitian and unitary random matrix models [25–28, 35–37]. Technical clues of our work are the quantum torus algebra in the fermionic formalism, special algebraic relations in this algebra referred to as ‘shift symmetries’, and the matrix factorization problem in the Lax formalism.

This paper is organized as follows. Section 2 is a review of the 2D Toda hierarchy formulated in the Lax and bilinear forms. Fundamental building blocks of the 2D Toda hierarchy such as the Lax operators, the dressing operators, the wave functions and the tau function are introduced along with various equations. The 1D Toda and Ablowitz–Ladik hierarchies are shown to be reductions of the 2D Toda hierarchy. The matrix factorization problem is also commented. Section 3 is a review of the fermionic formalism of the 2D Toda hierarchy. The fermionic formula of the tau function and its relation to the matrix factorization problem are explained. Relevant combinatorial notions such as partitions, Young diagrams and the Schur functions are also introduced here. The fermionic formula is illustrated for hypergeometric tau functions, in particular, the generating function of the double Hurwitz numbers. Sections 4 and 5 are devoted to the melting crystal models. In section 4, the two melting crystal models are introduced. The partition functions are defined as sums of the Boltzmann weights over the set of all partitions. Fermionic expressions of these partition functions are also derived. In section 5, integrable structures of the two melting crystal models are identified. The quantum torus algebra and its shift symmetries are reviewed. With the aid of these algebraic tools, the partition functions are converted to tau functions of the 2D Toda hierarchy. The first model thus turns out to be related to the 1D Toda hierarchy. The second model is further examined in the Lax formalism, and shown to be related to the Ablowitz–Ladik hierarchy. Section 6 concludes these reviews.

2. 2D Toda hierarchy

2.1. Difference operators and infinite matrices

The Lax formalism of the 2D Toda hierarchy is formulated by difference operators in the lattice coordinate $s$ [15]. These operators are linear combinations of the shift operators $e^{\varepsilon_0 \partial_s}$ (symbolically expressed as the exponential of $\partial_s = \partial / \partial s$) that act on functions of $s$ as $e^{\varepsilon_0 \partial_s} f(s) = f(s + n)$. A genuine difference operators is a finite linear combination
of the shift operators. To formulate the 2D Toda hierarchy, we further use semi-infinite linear combinations of the form

\[ A = \sum_{n=0}^{N} a_n(s) e^{n\partial_s} \quad \text{(operator of } [M, N] \text{ - type)} \]

and

\[ A = \sum_{n=M}^{\infty} a_n(s) e^{n\partial_s} \quad \text{(operator of } [M, \infty) \text{ type)} \]

These ‘pseudo-difference operators’ are analogues of pseudo-differential operators in the Lax formalism of the KP hierarchy [16, 17]. Let \((A)_{\geq 0}\) and \((A)_{< 0}\) denote the projection to the \([0, \infty)\) and \((-\infty, -1] \) parts.

These difference operators are also represented by \(\mathbb{Z} \times \mathbb{Z}\) matrices. The shift operators \(e^{n\partial_s}\) correspond to the shift matrices

\[ A^n = (\delta_{i-j-n})_{i,j \in \mathbb{Z}}. \]

The multiplication operators \(a(s)\) are represented by the diagonal matrices

\[ \text{diag}(a(s)) = (a(i)\delta_{ij})_{i,j \in \mathbb{Z}}. \]

Thus a general difference operator of the form

\[ A = A(s, e^{\partial_s}) = \sum_{n \in \mathbb{Z}} a_n(s) e^{n\partial_s} \]

is represented by the infinite matrix

\[ A(\Delta, \Lambda) = \sum_{n \in \mathbb{Z}} \text{diag}(a_n(s))\Lambda^n = \sum_{n \in \mathbb{Z}} (a_n(i)\delta_{i-j-n})_{i,j \in \mathbb{Z}}. \]

The shift operator \(e^{\partial_s}\) and the multiplication operator \(s\) satisfy the twisted canonical commutation relation

\[ [e^{\partial_s}, s] = e^{\partial_s}. \quad (2.1) \]

This commutation relation can be translated to the language of matrices as

\[ [\Lambda, \Delta] = \Lambda, \quad (2.2) \]

where \(\Delta\) denotes the diagonal matrix

\[ \Delta = \text{diag}(s) = (i\delta_{ij})_{i,j \in \mathbb{Z}} \]

that represents the multiplication operator \(s\).
2.2. Lax and Zakharov–Shabat equations

The Lax formalism of the 2D Toda hierarchy uses two Lax operators \( L, \bar{L} \) of type \((-\infty, 1]\) and \([1, \infty)\). From the point of view of symmetry, it is better to consider \( L \) and \( \bar{L} \) rather than \( L \) and \( \bar{L}^{-1} \). These operators admit freedom of gauge transformations \( L \to e^{-f} \cdot L \cdot e^f \), \( \bar{L} \to e^{-f} \cdot \bar{L} \cdot e^f \). We mostly use the gauge in which the leading coefficient of \( L \) is equal to 1:

\[
    L = e^{\partial_s} + \sum_{n=1}^{\infty} u_n e^{(1-n)\partial_s},
\]

\[
    \bar{L}^{-1} = \bar{u}_0 e^{-\partial_s} + \sum_{n=1}^{\infty} \bar{u}_n e^{(n-1)\partial_s}.
\]

The coefficients \( u_n \) and \( \bar{u}_n \) are functions \( u_n(s, t, \bar{t}) \) and \( \bar{u}_n(s, t, \bar{t}) \) of \( s \) and the time variables \( t, \bar{t} \). To simplify notations, however, we shall frequently suppress \( t \) and \( \bar{t} \) as \( u_n = u_n(s) \) and \( \bar{u}_n = \bar{u}_n(s) \).

\( L \) and \( \bar{L} \) satisfy the Lax equations

\[
\begin{align*}
    \frac{\partial L}{\partial t_n} &= [B_n, L], & \frac{\partial L}{\partial \bar{t}_n} &= [\bar{B}_n, L], \\
    \frac{\partial \bar{L}}{\partial t_n} &= [B_n, \bar{L}], & \frac{\partial \bar{L}}{\partial \bar{t}_n} &= [\bar{B}_n, \bar{L}],
\end{align*}
\]

where \( B_n \) and \( \bar{B}_n \) are defined as

\[
    B_n = (L^n)_{>0}, \quad \bar{B}_n = (L^{-n})_{<0}.
\]

\( B_n \) and \( \bar{B}_n \), in turn, satisfy the Zakharov–Shabat equations

\[
\begin{align*}
    \frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_m, B_n] &= 0, \\
    \frac{\partial B_n}{\partial \bar{t}_m} - \frac{\partial B_m}{\partial \bar{t}_n} + [B_m, B_n] &= 0, \\
    \frac{\partial \bar{B}_n}{\partial t_m} - \frac{\partial \bar{B}_m}{\partial t_n} + [\bar{B}_m, \bar{B}_n] &= 0, \\
    \frac{\partial \bar{B}_n}{\partial \bar{t}_m} - \frac{\partial \bar{B}_m}{\partial \bar{t}_n} + [\bar{B}_m, \bar{B}_n] &= 0.
\end{align*}
\]

Actually, the Lax equations and the Zakharov–Shabat equations are equivalent [15].

Since

\[
    B_1 = e^{\partial_s} + u_1, \quad \bar{B}_1 = \bar{u}_0 e^{-\partial_s},
\]

the lowest \((m = n = 1)\) member of the third set of the Zakharov–Shabat equation reduces to the equations

\[
\begin{align*}
    \frac{\partial u_1(s)}{\partial t_1} + \bar{u}_0(s + 1) - \bar{u}_0(s) &= 0, \\
    -\frac{\partial \bar{u}(s)}{\partial \bar{t}_1} + \bar{u}_0(s)(u_1(s) - u_1(s - 1)) &= 0.
\end{align*}
\]

Upon parametrizing \( u_1 \) and \( \bar{u}_0 \) with new dependent variable \( \phi(s) = \phi(s, t, \bar{t}) \) as

\[1\]

In the earliest work [15], these Lax operators were denoted by \( L, M \). These notations have been changed to \( L, \bar{L} \) so as to use \( M \) for the Orlov–Schulman operators. Also note that the bar \( \bar{\cdot} \) of \( L, \bar{L}, \bar{u}, \) etc does not mean complex conjugation.
\( u_1(s) = \frac{\partial \phi(s)}{\partial t_1}, \quad u_0(s) = e^{\phi(s)-\phi(s-1)}, \)

these equations yields the 2D Toda field equation

\[
\frac{\partial^2 \phi(s)}{\partial t_1 \partial \bar{t}_1} + e^{\phi(s+1) - \phi(s)} - e^{\phi(s) - \phi(s-1)} = 0. \tag{2.5}
\]

2.3. Dressing operators and wave functions

The Lax operators \( L, \bar{L} \) can be converted to the undressed form \( e^{\partial \partial s} \) as

\[
L = We^\partial W^{-1}, \quad \bar{L} = \bar{W}e^\partial \bar{W}^{-1} \tag{2.6}
\]

by dressing operators of the form

\[
W = 1 + \sum_{n=1}^{\infty} w_n e^{-n\partial}, \quad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n e^{n\partial}, \quad \bar{w}_0 \neq 0.
\]

One can further choose \( W, \bar{W} \) to satisfy the Sato equations

\[
\frac{\partial W}{\partial t_k} = B_k W - We^{k\partial}, \quad \frac{\partial W}{\partial \bar{t}_k} = \bar{B}_k \bar{W}.
\]

Upon substituting the expression

\[
B_k = (We^{k\partial} W^{-1})_{\geq 0}, \quad \bar{B}_k = (We^{-k\partial} \bar{W}^{-1})_{\leq 0}
\]

for \( B_k \)'s and \( \bar{B}_k \)'s, the Sato equations (2.7) turn into the closed system of evolution equations

\[
\frac{\partial W}{\partial t_k} = - (We^{k\partial} W^{-1})_{<0} W, \quad \frac{\partial W}{\partial \bar{t}_k} = (We^{-k\partial} \bar{W}^{-1})_{<0} W,
\]

\[
\frac{\partial \bar{W}}{\partial t_k} = (We^{k\partial} W^{-1})_{\geq 0} W, \quad \frac{\partial \bar{W}}{\partial \bar{t}_k} = - (We^{-k\partial} \bar{W}^{-1})_{\geq 0} W \tag{2.8}
\]

for \( W \) and \( \bar{W} \). These equations and may be thought of as yet another formulation of the 2D Toda hierarchy, from which the Lax equations (2.3) can be recovered through the relation (2.6).

The dressing operators can be used to define the wave functions

\[
\Psi = \left( 1 + \sum_{k=1}^{\infty} w_k z^{-k} \right) z^\xi(t, z), \quad \bar{\Psi} = \left( \sum_{k=0}^{\infty} \bar{w}_k z^k \right) z^{\bar{\xi}(\bar{t}, z^{-1})}. \tag{2.9}
\]

where

\[
\xi(t, z) = \sum_{k=1}^{\infty} t_k z^k, \quad \bar{\xi}(\bar{t}, z^{-1}) = \sum_{k=1}^{\infty} \bar{t}_k z^{-k}.
\]

The wave functions satisfy the auxiliary linear equations

\[
L \Psi = z \Psi, \quad \bar{L} \bar{\Psi} = z \bar{\Psi} \tag{2.10}
\]
and

$$\frac{\partial \psi}{\partial t_k} = B_k \psi, \quad \frac{\partial \bar{\psi}}{\partial t_k} = \bar{B}_k \bar{\psi},$$

$$\frac{\partial \psi}{\partial t_k} = B_k \psi, \quad \frac{\partial \bar{\psi}}{\partial t_k} = \bar{B}_k \bar{\psi}. \quad (2.11)$$

### 2.4. Tau functions and bilinear equations

The tau function $\tau = \tau(s, \tau, \bar{\tau})$ of the 2D Toda hierarchy is related to the wave functions as:

$$\Psi(s, t, \bar{t}, z) = \frac{\tau(s - 1, t - [z^{-1}], \bar{t})}{\tau(s - 1, t, \bar{t})} e^{\xi(t, z)},$$

$$\bar{\Psi}(\bar{t}, t, \bar{t}, z) = \frac{\tau(s, t - [z])}{\tau(s - 1, t, \bar{t})} e^{\xi(\bar{t}, z^{-1})}, \quad (2.12)$$

where

$$[z] = (z, z^2/2, \ldots, z^k/k, \ldots).$$

Given the pair $\Psi, \bar{\Psi}$ of wave functions, one can define the tau function as a kind of potential that satisfies these relations.

The tau function satisfies an infinite number of Hirota equations. The first three members of these Hirota equations read

$$D_1 \bar{D}_1 \tau(s, t, \bar{t}) \cdot \tau(s, t, \bar{t}) + 2 \tau(s + 1, t, \bar{t}) \tau(s - 1, t, \bar{t}) = 0,$$

$$(\bar{D}_2 + \bar{D}_1^2) \tau(s + 1, t, \bar{t}) - \tau(s, t, \bar{t}) = 0,$$

$$(\bar{D}_2 + \bar{D}_1^2) \tau(s, t, \bar{t}) - \tau(s + 1, t, \bar{t}) = 0, \quad (2.13)$$

where we have used Hirota’s notation

$$P(D_1, D_2, \ldots, \bar{D}_1, \bar{D}_2, \ldots) f(t, \bar{t}) \cdot g(t, \bar{t}) = P(\partial' = \partial, \partial_2 = \partial, \ldots, \bar{\partial}' = \bar{\partial}, \bar{\partial}_2 = \bar{\partial}, \ldots) f(t', \bar{t}') g(t, \bar{t}) \bigg|_{\tau = 0},$$

where $\partial, \partial', \bar{\partial}, \bar{\partial}'$ denote the derivatives $\partial_k = \partial/\partial t_k, \partial'_k = \partial/\partial t'_k, \bar{\partial}_k = \partial/\partial \bar{t}_k, \bar{\partial}'_k = \partial/\partial \bar{t}'_k$.

The first equation of (2.13) amounts to the 2D Toda field equation (2.5). The infinite system of Hirota equations can be encoded to (and decoded from) the single bilinear equation

$$\int \frac{dz}{2\pi i} e^{\xi(t' - tz)} \tau(s', t' - [z^{-1}], \bar{t}') \tau(t + [z^{-1}], \bar{t})$$

$$\quad = \int \frac{dz}{2\pi i} e^{\xi(t' - tz^{-1})} \tau(s + 1, t', \bar{t} - [z]) \tau(s - 1, t, \bar{t} + [z]), \quad (2.14)$$

where the symbol $\int$ means extracting the ‘residue’ of a (formal) Laurent series:

$$\int \sum_{n=-\infty}^{\infty} \frac{dz}{2\pi i} a_n e^{nz} = a_{-1}.$$

Analytically, this symbol on the left side of the equation is understood to be the contour integral along a sufficiently large circle $|z| = R$, and that of the right side is the contour integral along a sufficiently small circle $|z| = R^{-1}$.

These relations differ from those commonly used in the literature. We have replaced $\tau(s, t, \bar{t})$ therein by $\tau(s - 1, t, \bar{t})$ so as to consistent with the convention of our fermionic formalism.
Various bilinear equations for the tau function can be derived from (2.14) by specialization of \( t', \bar{t}' \) and \( s' \). The Hirota equations (2.13) are obtained by Taylor expansion of (2.14) at \( t' = t \) and \( \bar{t}' = \bar{t} \) to low orders upon letting \( s' = s, s \pm 1 \). More systematic derivation of Hirota equations uses the polynomials \( S_n(t) \), \( n = 0, 1, \ldots \), defined by the generating function

\[
\sum_{n=0}^{\infty} S_n(t) z^n = \exp \left( \sum_{k=1}^{\infty} \frac{t_k z^k}{k} \right).
\]  

(2.15)

These polynomials are building blocks the Schur functions as well (we refer to Macdonald’s book [71] for the notions of the Schur functions, partitions and Young diagrams). Thus a complete set of Hirota equations can be obtained in the generating functional form

\[
\sum_{n=0}^{\infty} S_n(t) S_{n+x-s+1}(\tilde{D}_t)e^{(aD_t)\tau(s,t,\bar{t})} = \sum_{n=0}^{\infty} S_n(t) S_{n-x+s-1}(\tilde{D}_t)e^{(aD_t)\tau(s-1,t,\bar{t})} \cdot \tau(s'+1, t, \bar{t}),
\]

(2.16)

where \( a = (a_1, a_2, \ldots) \) and \( \bar{a} = (\bar{a}_1, \bar{a}_2, \ldots) \) are auxiliary variables, \( \langle a, D_t \rangle \) and \( \langle \bar{a}, \bar{D}_t \rangle \) are the linear combinations

\[
\langle a, D_t \rangle = \sum_{k=1}^{\infty} a_k D_k, \quad \langle \bar{a}, \bar{D}_t \rangle = \sum_{k=1}^{\infty} \bar{a}_k \bar{D}_k
\]

of \( D_t \)'s and \( \bar{D}_t \)'s, and \( S_n(\tilde{D}_t) \) and \( S_n(\bar{D}_t) \) are defined by substituting the variables \( t \) of \( S_n(t) \) for the Hirota bilinear operators

\[
\tilde{D}_t = (D_1, D_2/2, \ldots, D_n/k, \ldots), \quad \bar{D}_t = (\bar{D}_1, \bar{D}_2/2, \ldots, \bar{D}_k/k, \ldots).
\]

### 2.5. Orlov–Schulman operators

Following the idea of Orlov and Schulman [72], one can introduce a Toda version of the Orlov–Schulman operator of the KP hierarchy. Actually, we need two Orlov–Schulman operators of the form

\[
M = \sum_{k=1}^{\infty} k t_k L^k + s + \sum_{n=1}^{\infty} v_n L^{-n},
\]

\[
\tilde{M} = -\sum_{k=1}^{\infty} k \bar{t}_k \bar{L}^{-k} + s + \sum_{n=1}^{\infty} \bar{v}_n \bar{L}^n,
\]

where \( v_n \) and \( \bar{v}_n \) are new dependent variables. These operators are defined in terms of the dressing operators as

\[
M = W \left( s + \sum_{k=1}^{\infty} k t_k e^{k \partial_t} \right) W^{-1},
\]

\[
\tilde{M} = \bar{W} \left( s - \sum_{k=1}^{\infty} k \bar{t}_k e^{-k \partial_t} \right) \bar{W}^{-1},
\]

(2.17)

and satisfy the Lax equations
\[
\frac{\partial M}{\partial t_n} = [B_n, M], \quad \frac{\partial M}{\partial \bar{t}_n} = [\bar{B}_n, M], \\
\frac{\partial \bar{M}}{\partial t_n} = [B_n, \bar{M}], \quad \frac{\partial \bar{M}}{\partial \bar{t}_n} = [\bar{B}_n, \bar{M}]
\]  

(2.18)

of the same form as the Lax equations (2.3) for \(L, \bar{L}\). Moreover, the twisted canonical commutation relations

\[
[L, M] = L, \quad [L, \bar{M}] = \bar{L}
\]  

(2.19)

are satisfied as a result of the commutation relation (2.1) of \(e^\partial s\) and \(s\).

These equations form an extended Lax formalism of the 2D Toda hierarchy. One can thereby formulate additional symmetries of \(W_{1+\infty}\) type \([40, 41]\). These additional symmetries play a central role in the so called ‘string equations’ for various special solutions \([29–34, 73]\). Moreover, general solutions of the 2D Toda hierarchy, too, can be captured by the generalization

\[
L = f(\bar{L}, M), \quad M = g(\bar{L}, \bar{M})
\]  

(2.20)

of those string equations \([41, 74]\).

### 2.6. Two reductions of 2D Toda hierarchy

#### 2.6.1. 1D Toda hierarchy

The 1D Toda hierarchy is a reduction of the 2D Toda hierarchy in which all dynamical variables depend on the time variables \(t, \bar{t}\) through the difference \(t - \bar{t}\). In the Lax formalism, the 1D reduction can be achieved by imposing the condition

\[
L = L^{-1}. 
\]  

(2.21)

Both sides of this equation become a difference operator of the form

\[
\mathcal{L} = e^{\partial t} + b + ce^{-\partial \bar{t}}, \quad b = u_1, \quad c = \bar{u}_0,
\]  

(2.22)

which satisfies the Lax equations

\[
\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad \frac{\partial \mathcal{L}}{\partial \bar{t}_k} = [\bar{B}_k, \mathcal{L}].
\]

Since (2.21) implies that \(B_k, \bar{B}_k\) and \(\mathcal{L}\) are linearly related as

\[
B_k + \bar{B}_k = \mathcal{L}^k,
\]  

(2.23)

the time evolutions with respect to \(t\) and \(\bar{t}\) are also linearly related as

\[
\frac{\partial \mathcal{L}}{\partial t_k} + \frac{\partial \mathcal{L}}{\partial \bar{t}_k} = [B_k, \mathcal{L}] + [\bar{B}_k, \mathcal{L}] = 0.
\]

Thus the reduced system has just one set of independent Lax equations

\[
\frac{\partial \mathcal{L}}{\partial t_k} = [B_k, \mathcal{L}], \quad B_k = (\mathcal{L}^k)_{\geq 0}.
\]

#### 2.6.2. Ablowitz–Ladik hierarchy

The reduction to the Ablowitz–Ladik hierarchy is a kind of ‘rational reduction’ \([70]\). This is achieved by assuming that \(L\) and \(L^{-1}\) are quotients

\[\text{In the earliest work [15], a condition of the form } L + L^{-1} = \bar{L} + \bar{L}^{-1} \text{ is proposed for the 1D reduction. This condition is related to the structure of soliton solutions of the Toda lattice [3, 6].}\]
\[
L = BC^{-1}, \quad L^{-1} = CB^{-1}
\]
of two difference operators of the form
\[
B = e^{\partial_t} - b, \quad C = 1 - ce^{-\partial_t}.
\]

\(B^{-1}\) and \(C^{-1}\) are understood to be difference operators of type \([0, \infty)\) and \((-\infty, 0]\). More explicitly,
\[
B^{-1} = -\sum_{k=0}^{\infty} (b(s)e^{\partial_t})^k b^{-1} = -b(s)^{-1} - \sum_{k=1}^{\infty} b(s)^{-1} \cdots b(s+k)^{-1} s e^{\partial_t},
\]
\[
C^{-1} = 1 + \sum_{k=1}^{\infty} (ce^{-\partial_t})^k = 1 + \sum_{k=1}^{\infty} c(s)c(s-1) \cdots c(s-k+1) s e^{-\partial_t},
\]
where \(b(s)\) and \(c(s)\) are abbreviations of \(c(s, t, t)\) and \(c(s, t, t)\). Under this interpretation, \(BC^{-1}\) is not the inverse of \(BC^{-1}\). Thus trivial situation where \(L = L = e^{\partial_t}\) can be avoided.

The Lax equations (2.3) of the 2D Toda hierarchy can be reduced to (and derived from) the equations
\[
\begin{align*}
\frac{\partial B}{\partial t_k} &= \left((BC^{-1})^k\right)_{\geq 0} B - B \left((C^{-1}B)^k\right)_{\geq 0}, \\
\frac{\partial C}{\partial t_k} &= \left((BC^{-1})^k\right)_{\geq 0} C - C \left((C^{-1}B)^k\right)_{\geq 0}, \\
\frac{\partial B}{\partial t_k} &= \left((CB^{-1})^k\right)_{< 0} B - B \left((B^{-1}C)^k\right)_{< 0}, \\
\frac{\partial C}{\partial t_k} &= \left((CB^{-1})^k\right)_{< 0} C - C \left((B^{-1}C)^k\right)_{< 0}.
\end{align*}
\]
(2.25)

Note that this is a closed system of evolution equations for \(B\) and \(C\). This implies that the reduced form (2.24) of \(L\) and \(L^{-1}\) is preserved by the time evolutions of the 2D Toda hierarchy.

The reduction condition to the Ablowitz–Ladik hierarchy can be reformulated in the alternative form
\[
L = \bar{C}^{-1}\bar{B}, \quad L^{-1} = \bar{B}^{-1}\bar{C},
\]
(2.26)
where \(\bar{B}\) and \(\bar{C}\) are difference operators of the form
\[
\bar{B} = e^{\partial_t} - \bar{b}, \quad \bar{C} = 1 - \bar{c}e^{-\partial_t}.
\]

Just like \(B^{-1}\) and \(C^{-1}\) in (2.24), \(\bar{B}^{-1}\) and \(\bar{C}^{-1}\) are understood to be difference operators of type \([0, \infty)\) and \((-\infty, 0]\). The Lax equations (2.3) of the 2D Toda hierarchy can be reduced to the equations
\[
\begin{align*}
\frac{\partial \bar{B}}{\partial t_k} &= \left((\bar{B}\bar{C}^{-1})^k\right)_{\geq 0} \bar{B} - \bar{B} \left((\bar{C}^{-1}\bar{B})^k\right)_{\geq 0}, \\
\frac{\partial \bar{C}}{\partial t_k} &= \left((\bar{B}\bar{C}^{-1})^k\right)_{\geq 0} \bar{C} - \bar{C} \left((\bar{C}^{-1}\bar{B})^k\right)_{\geq 0}, \\
\frac{\partial \bar{B}}{\partial t_k} &= \left((\bar{C}\bar{B}^{-1})^k\right)_{< 0} \bar{B} - \bar{B} \left((\bar{B}^{-1}\bar{C})^k\right)_{< 0}, \\
\frac{\partial \bar{C}}{\partial t_k} &= \left((\bar{C}\bar{B}^{-1})^k\right)_{< 0} \bar{C} - \bar{C} \left((\bar{B}^{-1}\bar{C})^k\right)_{< 0}.
\end{align*}
\]
(2.27)
for these operators as well.
The second reduction condition (2.26) is directly related to an auxiliary linear problem of the relativistic Toda hierarchy [39]. If the Lax operators are factorized in that form, the linear equations (2.10) for the wave functions can be converted to the ‘generalized eigenvalue problem’

\[ \tilde{B}\Psi = z\tilde{C}\Psi, \quad \tilde{B}\bar{\Psi} = z\tilde{C}\bar{\Psi}. \] (2.28)

A generalized eigenvalue problem of this form is used in Bruschi and Ragnisco’s scalar-valued Lax formalism [75] of the relativistic Toda lattice. Moreover, as pointed out by Kharchev et al [36], this generalized eigenvalue problem can be derived from the traditional $2 \times 2$ matrix-valued Lax formalism [38] of the Ablowitz–Ladik hierarchy.

### 2.7. Matrix factorization problem

General solutions of the 2D Toda hierarchy can be captured by a factorization problem [24] of the form

\[ \exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1} \bar{W}, \] (2.29)

where $U$ is a given (invertible) constant $Z \times Z$ matrix. The problem is to find two $Z \times Z$ matrices $W = W(t, \bar{t})$ and $\bar{W} = \bar{W}(t, \bar{t})$ that are triangular matrices of the form

\[ W = 1 + \sum_{n=1}^{\infty} \text{diag}(w_n(s))\Lambda^{-n}, \quad \bar{W} = \sum_{n=0}^{\infty} \text{diag}(\bar{w}_n(s))\Lambda^n, \quad \bar{w}_0 \neq 0. \]

Note that $W$ and $\bar{W}$ amount to the dressing operators of the last section by the correspondence

\[ A(s, e^{\partial s}) = \sum_{n \in Z} a_n(s)e^{n\partial s} \leftrightarrow A(\Delta, \Lambda) = \sum_{n \in Z} \text{diag}(a_n(s))\Lambda^n \]

of difference operators and $Z \times Z$ matrices.

Since $W$ and $\bar{W}$ are lower and upper triangular matrices, the factorization problem (2.29) is an infinite dimensional analogue of the gauss decomposition for finite matrices. If $W$ and $\bar{W}$ satisfy the factorization problem (2.29), one can readily derive the equations

\[ \frac{\partial W}{\partial t_k} W^{-1} + W\Lambda^k W^{-1} = \frac{\partial \bar{W}}{\partial t_k} \bar{W}^{-1}, \]

\[ \frac{\partial W}{\partial \bar{t}_k} W^{-1} = \frac{\partial \bar{W}}{\partial \bar{t}_k} \bar{W}^{-1} + \bar{W}\Lambda^{-k} \bar{W}^{-1}. \] (2.30)

Splitting these equations to the $(\cdot)_{\geq 0}$ and $(\cdot)_{< 0}$ parts, one can see that these equations are equivalent to the Sato equations (2.8). Thus the factorization problem yields a solution of the 2D Toda hierarchy.

In analogy with the procedure of the gauss decomposition for finite matrices, one can express the matrix elements of $W$ and $\bar{W}$ as quotients of semi-infinite minors of

\[ U(t, \bar{t}) = (U_{ij}(t, \bar{t}))_{i,j \in Z} = \exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right). \] (2.31)
The common denominator of these quotients is a principal minor of $U(t, \bar{t})$, and can be identified with the tau function$^4$:

$$\tau(s, t, \bar{t}) = \det(U_{ij}(t, \bar{t}))_{i, j \leq s}.$$ (2.32)

The determinant expression of the matrix elements of $W$ and $\bar{W}$ reproduces the generating functional expression

$$1 + \sum_{n=1}^{\infty} w_n z^{-n} = \frac{\tau(s-1, t - [z^{-1}], t)}{\tau(s-1, t, t)},$$

$$\sum_{n=0}^{\infty} \bar{w}_n z^n = \frac{\tau(s, t, \bar{t} - [z])}{\tau(s-1, t, \bar{t})}$$

of $w_n$’s and $\bar{w}_n$’s, which implies the relation (2.12).

These formal computations can be justified rigorously$^2$ in the case where $U$ is given by the quotient $U = W^{-1} W_0$ (2.33) of two triangular matrices of the same form as $W$ and $\bar{W}$. In this case, $W_0$ and $\bar{W}_0$ can be identified with the initial values of $W$ and $\bar{W}$:

$$W_0 = W\big|_{t=\bar{t}=0}, \quad \bar{W}_0 = \bar{W}\big|_{t=\bar{t}=0}.$$ (2.34)

In other words, the factorization problem (2.29) in this setup solves the initial value problem of the Sato equations (2.8).

The determinant formula (2.32) has many implications. First, this is an analogue of the determinant formula of the tau functions of the KP hierarchy. Since $U$ is an element of the ‘group’ $GL(\infty)$, it is $GL(\infty)$ itself that plays the role of the infinite-dimensional Grassmann manifold in the case of the KP hierarchy. More precisely, the true phase space lies in the product of two flag manifolds in which $W$ and $\bar{W}$ live. Second, the generating matrix $U$ is related to the generalized string equation (2.20). These equations are a consequence of the algebraic relations

$$\Lambda U = Uf(\Delta, \Lambda), \quad \Delta U = Ug(\Delta, \Lambda)$$ (2.35)

satisfied by $\Lambda$, $\Delta$ and $U$. Third, the determinant formula (2.32) can be translated to the language of a 2D complex free fermion system. Let us turn to this fermionic formalism of the 2D Toda hierarchy.

### 3. Fermionic formalism

#### 3.1. Complex free fermion system

Let

$$\psi(z) = \sum_{n \in \mathbb{Z}} \psi_n z^{-n-1}, \quad \psi^*(z) = \sum_{n \in \mathbb{Z}} \psi^*_n z^{-n}$$

$^4$This is a place where the aforementioned modification of the definition of $\tau(s, t, \bar{t})$ affects the outcome. In the earlier literature, the right hand side of this formula is the minor for $i, j < s$ rather than $i, j \leq s$.

$^5$This notation is used here in a loose sense and not intended to denote a true group.
denote the conjugate pair of 2D complex free fermion fields. For convenience, we use integers rather than half-integers for the labels of Fourier modes $\psi_n, \psi_n^\dagger$. The Fourier modes satisfy the anti-commutation relations

$$\psi_m^\dagger \psi_n + \psi_n \psi_m = \delta_{m+n,0}, \quad \psi_m^\dagger \psi_n^\dagger = 0, \quad \psi_m^\dagger \psi_n^\dagger + \psi_n \psi_m = 0.$$ 

$\psi^\dagger$'s and $\psi^\dagger$'s are understood to be linear operators on the fermionic Fock spaces. They act on the Fock space $\mathcal{F}$ from the left side and on its dual space $\mathcal{F}^*$ from the right side. These Fock spaces are decomposed to charge-$s$ sectors $\mathcal{H}_s, \mathcal{H}_s^*$, $s \in \mathbb{Z}$. Let $\langle s \rangle$ and $|s\rangle$ denote the ground states in $\mathcal{H}_s$ and $\mathcal{H}_s^*$:

$$\langle s \rangle \equiv \langle -\infty | \cdots \psi_{s-1}^\dagger \psi_s^\dagger, \quad |s\rangle = \psi_{-s}^\dagger \psi_{-s+1}^\dagger \cdots | -\infty \rangle.$$ 

Excited states are labelled by partitions $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n, 0, 0, \cdots)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$, of arbitrary length as

$$\langle \lambda, s \rangle = \langle \lambda, s \rangle \psi_{-s-1}^\dagger \cdots \psi_{s-1}^\dagger \psi_{s-1}^\dagger \cdots \psi_{-s}^\dagger \psi_{-s+1}^\dagger \cdots \psi_{-\lambda_1}^\dagger \cdots \psi_{-\lambda_n}^\dagger \cdots | s \rangle.$$ 

$\langle s \rangle$ and $|s\rangle$ are identified with $\langle 0, s \rangle$ and $|0, s\rangle$. $\langle \lambda, s \rangle$ and $|s, \lambda \rangle$ represent a state in which the semi-infinite subset $\{\lambda_1 - i + 1 + s\}_{i=1}^\infty$ of the set $\mathbb{Z}$ of all energy levels are occupied by particles. These vectors form dual bases of $\mathcal{H}_s$ and $\mathcal{H}_s^*$:

$$\langle \lambda, r|\mu, s \rangle = \delta_{\lambda \mu} \delta_{s, r}.$$  

The normal ordered fermion bilinears

$$:\psi_{-i} \psi_{-j}^\dagger : = \psi_{-i} \psi_{-j}^\dagger - |0\rangle \langle 0| \psi_{-i} \psi_{-j}^\dagger, \quad i, j \in \mathbb{Z},$$

where

$$|0\rangle \langle 0| \psi_{-i} \psi_{-j}^\dagger = \begin{cases} 1 & \text{if } i = j \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

span the one-dimensional central extension $\hat{\mathfrak{gl}}(\infty)$ of the Lie algebra $\mathfrak{gl}(\infty)$ of $\mathbb{Z} \times \mathbb{Z}$ matrices $[76, 77]$. $\mathfrak{gl}(\infty)$ consists of infinet matrices $A = (a_{ij})_{i,j \in \mathbb{Z}}$ that correspond to difference operators of finite type (i.e. of $[M,N]$-type for a pair of integers $M, N$ that can depend on $A$). For such a matrix $A \in \mathfrak{gl}(\infty)$, the fermion bilinear

$$\hat{A} = \sum_{i,j \in \mathbb{Z}} a_{ij} \psi_{-i} \psi_{-j}^\dagger,$$

becomes a well-defined linear operator on the Fock space, and preserves the charge in the sense that

$$\langle \lambda, r|\hat{A}|\mu, s \rangle = 0 \quad \text{if } r \neq s.$$  

The elements of $\hat{\mathfrak{gl}}(\infty)$ satisfy the commutation relation

$$[\hat{A}, \hat{B}] = [\hat{A}, \hat{B}] + \gamma(A, B)$$

with the $c$-number cocycle

$$\gamma(A, B) = \sum_{i>j \geq 0} (a_{ij} b_{ji} - b_{ij} a_{ij}).$$

The shift of $s$ in (2.12) and (2.32) is related to this definition of the ground states.
3.2. Vertex operators and Schur functions

We here introduce the special fermion bilinears
\[ J_m = \hat{\Lambda}_m = \sum_{n \in \mathbb{Z}} \psi_{m-n} \psi^*_{n}, \quad m \in \mathbb{Z}, \]
which satisfy the commutation relations
\[ [J_m, J_n] = m \delta_{m+n} \]
(3.5)
of the Heisenberg algebra. These operators are used to construct vertex operators. The matrix elements of such a vertex operator with respect to the vectors \( \langle \lambda, s \rangle \) and \( |\mu, s\rangle \) are related to the Schur and skew Schur functions. Actually, there are two different types of vertex operators that correspond to different formulations of these functions.

Vertex operators of the first type are given by the product
\[ \Gamma^\pm (x) = \prod_{i \geq 1} \Gamma^\pm (x_i), \quad x = (x_1, x_2, \ldots), \]
of the elementary vertex operators
\[ \Gamma^\pm (x) = \exp \left( \sum_{k=1}^{\infty} \frac{x^k}{k} J^{\pm k} \right). \]
The matrix elements of these operators are the skew Schur functions \( s_{\lambda/\mu}(x) \) in the sense of symmetric functions of \( x \) [65]:
\[ \langle \lambda, s | \Gamma^- (x) | \mu, s \rangle = \langle \mu, s | \Gamma^+ (x) | \lambda, s \rangle = s_{\lambda/\mu}(x). \]
(3.6)
In particular, if \( \mu = \emptyset \), the matrix elements become the Schur functions \( s_{\lambda}(x) \):
\[ \langle \lambda, s | \Gamma^- (x) | s \rangle = \langle s | \Gamma^+ (x) | \lambda, s \rangle = s_{\lambda}(x). \]
(3.7)
Vertex operators of the second type are defined as
\[ \gamma^\pm (t) = \exp \left( \sum_{k=1}^{\infty} t_k J^{\pm k} \right). \]
It is these operators \( \gamma^\pm (t) \) that are commonly used in the fermionic formula of tau functions of the KP and 2D Toda hierarchies [23, 77]. The matrix elements of \( \gamma^\pm (t) \) are the skew Schur functions \( S_{\lambda/\mu}(t) \) of the \( t \)-variables:
\[ \langle \lambda, s | \gamma^- (t) | \mu, s \rangle = \langle \mu, s | \gamma^+ (t) | \lambda, s \rangle = S_{\lambda/\mu}(t). \]
(3.8)
These functions are defined by the determinant formula
\[ S_{\lambda/\mu}(t) = \det (S_{\lambda-\mu_i-i+j}(t))_{i,j=1}^{n}. \]
(3.9)
for partitions of the form \( \lambda = (\lambda_1, \ldots, \lambda_n, 0, 0, \ldots), \mu = (\mu_1, \ldots, \mu_n, 0, 0, \ldots) \). \( S_n(t) \)'s are the polynomials defined by the generating function (2.15).
\( \gamma^\pm (t) \) can be converted to \( \Gamma^\pm (x) \) by substituting
\[ t_k = \frac{1}{k} \sum_{i \neq 1} x_i^k. \]
(3.10)
By the same transformation of variables, the polynomials $S_n(t)$ in $t$ turn into the homogeneous symmetric function $h_n(x)$ of $x$. The determinant formula \((3.9)\) of $S_{\lambda/\mu}(t)$ thereby reproduces the Jacobi–Trudi formula of $s_{\lambda/\mu}(x)$.

3.3. Fermionic formula of tau functions

In terms of the foregoing vertex operators, the fermionic formula of Toda tau functions \([21–23]\) reads\(^7\)

$$
\tau(s, t, \bar{t}) = \langle s|\gamma_+(t)g^{-1}\gamma_-(\bar{t})|s \rangle,
$$

\[(3.11)\]

where $g$ is an element of the ‘group’ $\hat{\text{GL}}(\infty)$ of Clifford operators (typically, the exponential $e^A$ of a fermion bilinear $A$) \([76, 77]\). Such a Clifford operator induces a linear transformation on the linear span of $\psi_i$s and $\psi_i^*$s by the adjoint action:

$$
g\psi_j g^{-1} = \sum_{i \in \mathbb{Z}} \psi_i U_{ij}, \quad g\psi_j^* g^{-1} = \sum_{i \in \mathbb{Z}} \psi_i^* \tilde{U}_{ij}. \quad (3.12)
$$

The coefficients $U_{ij}$ and $\tilde{U}_{ij}$ satisfy the orthogonality condition

$$
\sum_{k \in \mathbb{Z}} U_{ik} \tilde{U}_{kj} = \sum_{k \in \mathbb{Z}} U_{ki} \tilde{U}_{kj} = \delta_{ij}. \quad (3.13)
$$

The fermionic formula \((3.11)\) corresponds to the determinant formula \((2.32)\) of the factorization problem \((2.29)\) for the matrix $U = (U_{ij})_{i,j \in \mathbb{Z}}$.

An immediate consequence of \((3.11)\) is the Schur function expansion

$$
\tau(s, t, \bar{t}) = \sum_{\lambda, \mu \in \mathcal{P}} \langle \lambda, s|g|\mu, s \rangle S_\lambda(t) S_\mu(-\bar{t}),
$$

\[(3.14)\]

where $\mathcal{P}$ denotes the set of all partitions. This expansion is obtained by inserting the partition of unity

$$
1 = \sum_{\lambda \in \mathcal{P}, s \in \mathbb{Z}} |\lambda, s\rangle \langle \lambda, s| \quad (3.15)
$$

to the two places among $\gamma_+(t)$, $g$ and $\gamma_-(\bar{t})$. This amounts to applying the Cauchy–Binet formula to the determinant formula \((2.32)\). The three factors $\langle \lambda, s|g|\mu, s \rangle$, $S_\lambda(t)$, $S_\mu(-\bar{t})$ may be thought of as minors of the three matrices on the right side of \((2.31)\). As regards $S_\lambda(t)$ and $S_\mu(-\bar{t})$, this is indeed a consequence of the special case

$$
S_\lambda(t) = \det(S_{\lambda-i+j}(t))_{i,j=1}^n \quad (3.16)
$$

of the determinant formula \((3.9)\).

3.4. Hypergeometric tau functions

Let us illustrate the fermionic formula \((3.11)\) in the case of hypergeometric tau functions \([57–59]\). This is the case where the generating operator $g$ of the tau function \((3.11)\) corresponds to a diagonal matrix in $\text{GL}(\infty)$.

Let $U = (e^{T_i} \delta_{ij})_{i,j \in \mathbb{Z}}$ be such a diagonal matrix. The associated generating operator can be expressed as

\(^7\)A prototype of this formula can be found in the work of Jimbo and Miwa \([18]\).

\(^8\)This notation, too, is used here in a loose sense just like $\text{GL}(\infty)$.
This operator, too, is diagonal with respect to the basis \( \{|\lambda,s\rangle\}_{\lambda \in P, s \in \mathbb{Z}} \) of the Fock space. Thus the tau function becomes a single sum over all partitions:

\[
\tau(s,t,\bar{t}) = \sum_{\lambda \in P} \langle \lambda,s | g | \lambda,s \rangle S_{\lambda}(t) S_{\lambda}(\bar{t}).
\]  

(3.18)

The diagonal elements of \( g \) takes the so called 'contents product' form:

\[
\langle \lambda,s | g | \lambda,s \rangle = \prod_{(i,j) \in \lambda} e^{r_{i+1} - r_{i} + s},
\]  

(3.19)

where \((i,j) \in \lambda\) means that \((i,j)\) runs over the cells of the Young diagram of shape \( \lambda \), and \( r_n \)'s are defined as

\[
r_n = e^{T_n - T_{n-1}}.
\]

The \( \lambda \)-independent factor \( \langle s|g|s \rangle \) can be expressed as

\[
\langle s|g|s \rangle = \prod_{i=1}^{\infty} e^{T_{i+1} - i + s} = \begin{cases} 
1 & \text{if } s = 0, \\
\prod_{i=1}^{s} e^{-T_{i+1}} & \text{if } s < 0.
\end{cases}
\]  

These tau functions are called 'hypergeometric' after the work of Orlov and Scherbin [57–59], because their work aimed at applications to multivariate hypergeometric functions. Actually, specialization of the parameters \( \{ T_n \}_{n \in \mathbb{Z}} \) yields a variety of examples other than hypergeometric functions. Earliest examples of these tau functions can be found in the studies of random matrix models [78–81] and \( c = 1 \) string theory [29, 33, 34]. Another source of examples is enumerative geometry of \( \mathbb{C}P^1 \) and \( \mathbb{C}^2 \) [45, 50, 51]. Recent researches of this class of tau functions are focussed on the double Hurwitz numbers [73, 82–84] and their variants [85–88].

Let us briefly recall the tau function of the double Hurwitz numbers [45]. The generating operator takes such a form as

\[
g = Q e^{3K/2},
\]  

(3.21)

where \( Q \) and \( \beta \) are constants, and \( L_0 \) and \( K \) are the special fermion bilinears

\[
L_0 = \hat{\Delta} = \sum_{n \in \mathbb{Z}} n : \psi_{-n} \psi_{n}^* :,
\]

\[
K = (\Delta - 1/2)^2 = \sum_{n \in \mathbb{Z}} (n - 1/2)^2 : \psi_{-n} \psi_{n}^* :.
\]

The diagonal matrix elements of these fermion bilinears can be computed as follows:

\[
\langle \lambda,s | L_0 | \lambda,s \rangle = |\lambda| + \frac{s(s+1)}{2},
\]

\[
\langle \lambda,s | K | \lambda,s \rangle = \kappa(\lambda) + 2s|\lambda| + \frac{4s^3 - s}{12},
\]  

(3.22)
where

$$|\lambda| = \sum_{i=1}^{\infty} \lambda_i, \quad \kappa(\lambda) = \sum_{i=1}^{\infty} \lambda_i(\lambda_i - 2i + 1).$$

This implies that

$$\langle \lambda, s|Q^{|\lambda|+s(s+1)/2}\rangle = e^{\beta\kappa(\lambda)/2 + s|\lambda| + (4s^3-s)/24},$$

Consequently, the tau function has a Schur function expansion of the form

$$\tau(s, t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} e^{\beta\kappa(\lambda)/2 + s|\lambda| + (4s^3-s)/24} S_{\lambda}(t) S_{\lambda}(-\bar{t}).$$

(3.23)

Its specialization

$$\tau(0, t, \bar{t}) = \sum_{\lambda \in \mathcal{P}} Q^{|\lambda|} e^{\beta\kappa(\lambda)/2} S_{\lambda}(t) S_{\lambda}(-\bar{t})$$

(3.24)

to $s = 0$ is a genuine generating function of the double Hurwitz numbers.

Further specialization to $\bar{t} = (-1, 0, 0, \ldots)$ becomes a generating function of the single Hurwitz numbers. The special value of the second Schur function at this point can be computed by the combinatorial formula [71]

$$S_{\lambda}(1, 0, 0, \ldots) = \frac{\dim \lambda}{|\lambda|!} = \prod_{(i,j) \in \lambda} h(i,j)^{-1},$$

(3.25)

where $h(i,j)$ is the hook length of the cell $(i,j)$ in the Young diagram of shape $\lambda$, and $\dim \lambda$ is the number of standard tableau therein (i.e. the dimension of the associated irreducible representation of the symmetric group $S_N$, $N = |\lambda|$). The doubly specialized tau function

$$\tau(0, t, -1, 0, 0, \ldots) = \sum_{\lambda \in \mathcal{P}} \frac{\dim \lambda}{|\lambda|!} Q^{|\lambda|} e^{\beta\kappa(\lambda)/2} S_{\lambda}(t)$$

(3.26)

reproduces a generating function of the single Hurwitz numbers. Note that this is a tau function of the KP hierarchy with the fermionic expression

$$\tau(0, t, -1, 0, 0, \ldots) = \langle 0|\gamma_+ (t) Q^{|\lambda|} e^{\beta K/2} e^{J_{-1}} |0\rangle.$$  

(3.27)

4. Melting crystal models

4.1. Statistical model of random 3D Young diagrams

The simplest melting crystal model [65] has a single parameter $q$ in the range $0 < q < 1$ (or just a formal variable). The partition function is the sum

$$Z = \sum_{\pi \in \mathcal{PP}} q^{\pi \cdot \pi}$$

(4.1)

of the Boltzmann weight $q^{\pi \cdot \pi}$ over the set $\mathcal{PP}$ of all plane partitions. The plane partition
\[ \pi = (\pi_{ij})_{i,j=1}^{\infty} = \begin{pmatrix} \pi_{11} & \pi_{12} & \cdots \\ \pi_{21} & \pi_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad \pi_{i+1,j} \leq \pi_{ij} \geq \pi_{i,j+1}, \]

represent a 3D Young diagram in the first octant of the xyz-space. \( \pi_{ij} \) is the height of the stacks of unit cubes on the unit square \([i-1,i] \times [j-1,j] \) of the xy-plane. \(|\pi|\) denotes the volume of the 3D Young diagram, i.e.

\[ |\pi| = \sum_{i,j=1}^{\infty} \pi_{ij}. \]

By the method of diagonal slicing \([65]\), the sum (4.1) over the set of plane partitions can be converted to the sum

\[ Z = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{-\rho})^2 \]

over the set of ordinary partitions. The building block \( s_{\lambda}(q^{-\rho}) \) of the Boltzmann weight is the special value of the infinite-variate Schur function \( s_{\lambda}(x) \) at

\[ x = q^{-\rho} = (q^{1/2}, q^{3/2}, \ldots, q^{(-1)/2}, \ldots). \]

This is a kind of ‘principal specialization’ of \( s_{\lambda}(x) \) \([71]\), and can be computed by the hook-length formula

\[ s_{\lambda}(q^{-\rho}) = \frac{q^{-\kappa(\lambda)/4}}{\prod_{(i,j) \in \lambda} (q^{-h(i,j)/2} - q^{h(i,j)/2})}, \]

(4.3)

Note that this formula is a \( q \)-analogue of (3.25) for \( S_{\lambda}(1,0,0,\ldots) \).

Let us introduce another parameter \( Q \), a discrete variable \( s \in \mathbb{Z} \) and an infinite number of continuous variables \( t = (t_1, t_2, \ldots) \), and deform (4.2) as

\[ Z(s,t) = \sum_{\lambda \in \mathcal{P}} s_{\lambda}(q^{-\rho})^2 Q^{\lambda[s+(s+1)/2]} e^{\phi(\lambda,s,t)}. \]

(4.4)

\( Q^{\lambda[s+(s+1)/2]} \) is the same factor as inserted in the tau function (3.23) of the double Hurwitz numbers. \( \phi(\lambda,s,t) \) is a linear combination

\[ \phi(\lambda,s,t) = \sum_{k=1}^{\infty} t_k \phi_k(\lambda,s) \]

of the external potentials

\[ \phi_k(\lambda,s) = \sum_{i=1}^{\infty} \left( q^{k(\lambda_i-s+i+s)} - q^{k(-i+s+i+s)} \right) + \frac{1 - q^{ks}}{1 - q^s q^t}, \]

(4.5)

and \( t_k's \) play the role of coupling constants of these potentials. Note that the sum on the right hand side of (4.5) is a finite sum, because only a finite number of \( \lambda_i's \) are non-zero. (4.4) is related to 5D \( \mathcal{N} = 1 \) supersymmetric U(1) Yang–Mills theory \([66]\). The external potentials represent the contribution of Wilson loops along the fifth dimension therein \([60]\).

Let us mention that these external potentials are obtained from the apparently divergent (as far as \( |q| < 1 \) expression
\[ \phi_k(\lambda, s) = \sum_{i=1}^{\infty} q^{k(\lambda_i-i+1+s)} - \sum_{i=1}^{\infty} q^{k(-i+1)} \]  

by recombination of terms as

\[ \phi_k(\lambda, s) = \sum_{i=1}^{\infty} \left( q^{k(\lambda_i-i+1+s)} - q^{k(-i+1+s)} \right) + \sum_{i=1}^{\infty} q^{k(-i+1+s)} - \sum_{i=1}^{\infty} q^{k(-i+1)}. \]

The difference of the last two sums, too, thereby becomes a finite sum:

\[ \sum_{i=1}^{\infty} q^{k(-i+1+s)} = \begin{cases} 
\sum_{i=1}^{\infty} q^{k(-i+1)} & \text{if } s > 0 \\
0 & \text{if } s = 0 \\
-\sum_{i=1}^{\infty} q^{k(-i+1)} & \text{if } s < 0 
\end{cases} = \frac{1 - q^{ks}}{1 - q^s}. \]

A similar prescription is used in the computation (3.20) of the factor \( \langle s|g|s \rangle \) in hypergeometric tau functions. These computations are related to normal ordering of fermion bilinears.

It is this deformed partition function \( Z(s, t) \) that is shown to be related to the 1D Toda hierarchy. To this end, we use a fermionic expression of \( Z(s, t) \). Before showing this expression, let us present another melting crystal model.

### 4.2. Modified melting crystal model

The second model is obtained by replacing the main part of the Boltzmann weight as

\[ s^\lambda(q^{-\rho})^2 \rightarrow s^\lambda(q^{-\rho})s^{t\lambda}(q^{-\rho}), \]

where \( t^\lambda \) denotes the conjugate (or transposed) partition of \( \lambda \). Namely, in place of (4.2) or its \( Q \)-deformed version

\[ Z = \sum_{\lambda \in \mathcal{P}} s^\lambda(q^{-\rho})^2 Q^{\lambda}, \]

we here consider the modified partition function

\[ Z' = \sum_{\lambda \in \mathcal{P}} s^\lambda(q^{-\rho})s^{t\lambda}(q^{-\rho})Q^{\lambda} \]

and its deformations by external potentials. In view of the relation

\[ s^{\lambda}(q^{-\rho}) = q^{\kappa(\lambda)/2}s^{\lambda}(q^{-\rho}) \]

that can be derived from (4.3), one can rewrite \( Z' \) as

\[ Z' = \sum_{\lambda \in \mathcal{P}} s^\lambda(q^{-\rho})^2 q^{\kappa(\lambda)/2} Q^{\lambda}. \]

These partition functions originate in Gromov–Witten/topological string theory of special local Calabi–Yau threefolds called ‘local \( \mathbb{C}P^2 \) geometry’ [67, 68]. In particular, \( Z' \) is related to the ‘resolved conifold’, for which Brini pointed out a relation to the Ablowitz–Ladik hierarchy [69].
Let us mention that one can use the homogeneity
\[ s_\lambda(Q_{x_1}, Q_{x_2}, \ldots) = Q^{\lambda} s_\lambda(x_1, x_2, \ldots) \]
and the Cauchy identities
\[
\sum_{\lambda \in P} s_\lambda(x_1, x_2, \ldots) s_\lambda(y_1, y_2, \ldots) = \prod_{i,j \geq 1} \left( 1 - x_i y_j \right)^{-1},
\]
\[
\sum_{\lambda \in P} s_\lambda(x_1, x_2, \ldots) s^\lambda(y_1, y_2, \ldots) = \prod_{i,j \geq 1} \left( 1 + x_i y_j \right)
\]
of the Schur functions to convert these partition functions to an infinite product form:
\[
Z = \prod_{i,j=1}^{\infty} (1 - Q q^{i+j-1})^{-1} = \prod_{n=1}^{\infty} (1 - Q q^n)^{-n},
\]
\[
Z' = \prod_{i,j=1}^{\infty} (1 + Q q^{i+j-1}) = \prod_{n=1}^{\infty} (1 + Q q^n)^{-n}.
\]
These functions are referred to as the ‘MacMahon function’ in the literature of combinatorics and mathematical physics.

We deform \( Z' \) by two sets of external potentials \( \phi_{\pm k}(\lambda, s), k = 1, 2, \ldots \), with coupling constants \( t = (t_1, t_2, \ldots) \) and \( \bar{t} = (\bar{t}_1, \bar{t}_2, \ldots) \) as
\[
Z'(s, t, \bar{t}) = \sum_{\lambda \in P} s_\lambda(q^{-\rho}) s\lambda(q^{\rho}) Q^{\lambda + \bar{t} s + \bar{t} \lambda} e^{\phi(\lambda, s t, \bar{t})},
\]
\[
\phi(\lambda, s t, \bar{t}) = \sum_{k=1}^{\infty} t_k \phi_k(\lambda, s) + \sum_{k=1}^{\infty} t_k \phi_{-k}(\lambda, s).
\]
\( \phi_{-k}(\lambda, s) \)'s are defined by the same formula as (4.5) with \( k \) replaced by \( -k \). As it turns out, \( t \) and \( \bar{t} \) correspond to the two sets of time variables of the 2D Toda hierarchy.

### 4.3. Fermionic expression of partition functions

To translate \( Z(s, t) \) and \( Z'(s, t) \) to the language of the complex free fermion system, we need some more operators on the Fock space.

Let us introduce the new fermion bilinears
\[
H_k = q^{\Delta_k} = \sum_{n \in \mathbb{Z}} q_j^{m_n} \psi_{a_n} \psi_{a_n}^*, \quad k \in \mathbb{Z}.
\]
These operators are diagonal with respect to the basis \( \{ |\lambda, s\rangle \}_{\lambda \in P, s \in \mathbb{Z}} \), and the matrix elements are nothing but the external potentials \( \phi_k(\lambda, s) \):
\[ \langle \lambda, s | H_k | \lambda, s \rangle = \phi_k(\lambda, s). \]
This explains the origin of the formal expression (4.6) and its interpretation (4.5). The exponential factors in (4.4) and (4.11) can be thereby expressed as
\[ e^{\phi(\lambda, s t)} = \langle \lambda, s | e^{H(t)} | \lambda, s \rangle, \quad e^{\phi(\lambda, s \bar{t})} = \langle \lambda, s | e^{H(\bar{t})} | \lambda, s \rangle, \]
where

\[ H(t) = \sum_{k=1}^{\infty} t_k H_k, \quad H(t, \bar{t}) = \sum_{k=1}^{\infty} t_k H_k + \sum_{k=1}^{\infty} \bar{t}_k H_{-k}. \]

The other building blocks of \( Z(s, t) \) are similar to those of the tau function (3.23) of the double Hurwitz numbers:

\[ s_\lambda(q^{-\rho}) = \langle s|\Gamma_+(q^{-\rho})|\lambda, s \rangle = \langle \lambda, s|\Gamma_-(q^{-\rho})|s \rangle, \]

\[ Q^{\lambda + s(s+1)/2} = \langle \lambda, s|Q^s|\lambda, s \rangle. \]

These building blocks are glued together by the partition of unity (3.15) to construct the following fermionic formula of \( Z(s, t) \):

\[ Z(s, t) = \langle s|\Gamma_+(q^{-\rho})Q^s e^{H(t)}\Gamma_-(q^{-\rho})|s \rangle. \]  

(4.14)

To derive a similar fermionic formula of \( Z'(s, t, \bar{t}) \), we use the following variants of \( \Gamma_\pm(x) \) [89]:

\[ \Gamma'_\pm(x) = \prod_{i \geq 1} \Gamma'_\pm(x_i), \quad x = (x_1, x_2, \ldots), \]

\[ \Gamma'_\pm(z) = \exp \left( -\sum_{k=1}^{\infty} \frac{(-z)^k}{k} J_{kk} \right). \]

The matrix elements of these modified vertex operators, too, are related to the skew Schur functions except that they are labelled by conjugate partitions:

\[ \langle \lambda, s|\Gamma'_\pm(x)|\mu, s \rangle = \langle \mu, s|\Gamma'_\pm(x)|\lambda, s \rangle = s_{\lambda/\mu}(x). \]  

(4.15)

Thus the following fermionic formula of \( Z'(s, t, \bar{t}) \) can be obtained in the same way as the case of \( Z(s, t) \):

\[ Z'(s, t, \bar{t}) = \langle s|\Gamma_+(q^{-\rho})Q^s e^{H(t, \bar{t})}\Gamma'_-(q^{-\rho})|s \rangle. \]  

(4.16)

These fermionic formulae resemble the fermionic expression of the stationary Gromov–Witten invariants of \( \mathbb{CP}^1 \) [47, 48] and the instanton partition functions of 4D \( \mathcal{N} = 2 \) supersymmetric gauge theories [53–56]. We use these formulae to show that \( Z(s, t) \) and \( Z'(s, t, \bar{t}) \) are related to tau functions of the 2D Toda hierarchy.

5. Integrable structures of melting crystal models

5.1. Quantum torus algebra and shift symmetries

Although the fermionic formulae (4.14) and (4.16) of the partition functions of the melting crystal mode resemble the fermionic formula (3.11) of Toda tau functions, they have manifestly different structures. In particular, it is \( H_k \)'s rather than \( J_k \)'s that generate deformations of the partition functions. We use special algebraic relations connecting \( H_k \)'s and \( J_k \)'s to convert the partition functions to Toda tau functions. These algebraic relations, referred to as ‘shift symmetries’, are formulated in the language of a subalgebra in \( \hat{\mathfrak{gl}}(\infty) \).

This subalgebra is spanned by the fermion bilinears
\[ V_m^{(k)} = q^{-k m/2} N_m q^k \Delta = q^{-k m/2} \sum_{n \in \mathbb{Z}} q^{\Delta} \psi_{m-n} \psi_n^*; \quad k, m \in \mathbb{Z}. \]

This is substantially the same fermionic realization of the quantum torus algebra that are used in the work of Okounkov and Pandharipande on \( \mathbb{CP}^1 \) Gromov–Witten theory \cite{47, 48}. \( V_m^{(k)} \)'s satisfy the commutation relations

\[ [V_m^{(k)}, V_n^{(l)}] = (q^{(k-m)n/2} - q^{(k-l)m/2}) \left( V_{m+n}^{(k+l)} - \frac{q^{k+l}}{1 - q^{k+l}} \delta_{m+n,0} \right) \]  

(5.1)

for \( k \) and \( l \) with \( k + l \neq 0 \) and

\[ [V_m^{(k)}, V_n^{(-k)}] = (q^{-k(m+n)} - q^{k(m-n)}) V_{m+n}^{(0)} + m \delta_{m+n,0}. \]  

(5.2)

\( H_k \)'s and \( J_k \)'s are particular elements among \( V_m^{(k)} \)'s:

\[ H_k = V_0^{(k)}, \quad J_k = V_k^{(0)}. \]

We have the following three types of shift symmetries \cite{60, 61, 63}:

(i) For \( k > 0 \) and \( m \in \mathbb{Z} \),

\[ \Gamma_- (q^{-\rho}) \Gamma_+ (q^{-\rho}) \left( V_m^{(k)} - \frac{q^k}{1 - q^k} \delta_{m,0} \right) = (-1)^k \left( V_{m+k}^{(k)} - \frac{q^k}{1 - q^k} \delta_{m+k,0} \right) \Gamma_- (q^{-\rho}) \Gamma_+ (q^{-\rho}). \]  

(5.3)

(ii) For \( k > 0 \) and \( m \in \mathbb{Z} \),

\[ \Gamma'_- (q^{-\rho}) \Gamma'_+ (q^{-\rho}) \left( V_m^{(-k)} + \frac{1}{1 - q^k} \delta_{m,0} \right) = \left( V_{m+k}^{(-k)} + \frac{1}{1 - q^k} \delta_{m+k,0} \right) \Gamma'_- (q^{-\rho}) \Gamma'_+ (q^{-\rho}). \]  

(5.4)

(iii) For \( k, m \in \mathbb{Z} \),

\[ V_m^{(k)} q^{K/2} = q^{-m/2} q^{K/2} V_m^{(k+m)}. \]  

(5.5)

Note that the indices of \( V_m^{(k)} \)'s are literally shifted after exchanging the order of operator product with \( \Gamma_- (q^{-\rho}) \Gamma_+ (q^{-\rho}), \Gamma'_- (q^{-\rho}) \Gamma'_+ (q^{-\rho}) \) and \( q^{K/2} \).

In the earlier work \cite{60, 61, 63}, we used the slightly different fermion bilinear

\[ W_0 = \sum_{n \in \mathbb{Z}} n \psi_{-n} \psi_n^*; \]

and the algebraic relation

\[ V_m^{(k)} q^{W_0/2} = q^{W_0/2} V_m^{(k+m)} \]

in place of \( K \) and (5.5). This difference does not affect the essential part of the whole story.

5.2. \( Z(s, t) \) as tau function

Let us explain how to convert the partition function \( Z(s, t) \) of the first melting crystal model to a tau function of the 1D Toda hierarchy with the aid of the foregoing shift symmetries \cite{60, 61}.
The first step is to insert apparently redundant operators among $\langle s \rangle$, $|s\rangle$ and the operator product in between:
\[
Z(s,t) = q^{-(4s^2 - s)/12} \langle s | q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) e^{H(t)} \times Q^2 \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{K/2} | s \rangle, \tag{5.6}
\]
This is based on the identities
\[
\langle s | q^{K/2} = q^{(4s^2 - s)/24} \langle s |, \quad \langle s | \Gamma_-(q^{-\rho}) = \langle s |, \quad q^{K/2} | s \rangle = \langle s |
\]
that can be derived from (3.22) and the fact that $\langle s | J_{-k} = 0$ and $J_k | s \rangle = 0$ for $k > 0$. Also note that the order of $Q^m$ and $e^{H(t)}$, which are commutative, is reversed.

The second step is to apply the shift symmetries. The first set (5.3) of shift symmetries, specialized to $m = 0$ and $k > 0$, yields the identity
\[
\Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \left( H_k - \frac{q^k}{1 - q^k} \right) = (-1)^k V_k^{(k)} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho})
\]
that connects $V_0^{(k)} = H_k$ and $V_k^{(k)}$. The third set (5.5) of shift symmetries imply the relation
\[
V_k^{(k)} = q^{1/2} q^{-K/2} J_k q^{K/2}
\]
between $V_k^{(k)}$ and $V_k^{(0)} = J_k$. Thus $H_k - q^k/(1 - q^k)$ and $J_k$ turn out to satisfy the intertwining relation
\[
q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \left( H_k - \frac{q^k}{1 - q^k} \right) = (-q^{1/2})^k J_k q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}).
\]
This relation can be exponentiated as
\[
q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) \exp \left( \sum_{k=1}^{\infty} t_k (H_k - \frac{q^k}{1 - q^k}) \right) = \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^k t_k J_k \right) q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}).
\]
We can thus rewrite the first half of the operator product in (5.6) as
\[
q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) e^{H(t)} = \exp \left( \sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) \times \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^k t_k J_k \right) q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}). \tag{5.7}
\]
Plugging (5.7) into (5.6), we obtain the following expression of $Z(s,t)$:
\[
Z(s,t) = q^{-(4s^2 - s)/12} \exp \left( \sum_{k=1}^{\infty} \frac{q^k t_k}{1 - q^k} \right) \times \langle s | \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^k t_k J_k \right) g | s \rangle, \tag{5.8}
\]
where

\[ g = q^{K/2} \Gamma - (q^{-\rho}) \Gamma_+ (q^{-\rho}) Q^\rho \Gamma_-(q^{-\rho}) \Gamma_+ (q^{-\rho}) q^{K/2}. \]  
(5.9)

Let us note that this expression is slightly different from the one presented in the previous papers [60, 61], because we use \( K \) in place of \( W_0 \) in (5.6).

In much the same way, moving \( e^{\eta(t)} \) to the right of \( \Gamma_- (q^{-\rho}) \Gamma_+ (q^{-\rho}) q^{K/2} \) in (5.6), we can derive another expression of \( Z(s,t) \):

\[ Z(s,t) = q^{-(4\rho - s)/12} \exp \left( \sum_{k=1}^{\infty} \frac{q^{2t_k}}{1 - q^2} \right) \times \langle g \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^{t_k} J_{-k} \right) |s\rangle. \]  
(5.10)

Actually, as one can show with the aid of the shift symmetries, the operator (5.9) connects \( J_k \)'s and \( J_{-k} \)'s as

\[ J_k g = g J_{-k}, \quad k = 1, 2, \ldots. \]  
(5.11)

This explains why \( Z(s,t) \) has the two apparently different expressions (5.8) and (5.10).

Apart from the prefactors and the rescaling \( t_k \rightarrow (-q^{1/2})^{t_k} \) of the time variables, the essential part of the right side of (5.8) and (5.10) is the function

\[ \tau(s,t) = \langle g \gamma_+(t) g |s\rangle = \langle s g \gamma_-(t) |s\rangle. \]  
(5.12)

By the symmetry (5.11) of \( g \), the associated 2D Toda tau function reduces to this function:

\[ \tau(s,t) = \langle s \gamma_+(t) g \gamma_-(t) |s\rangle = \tau(s,t - \tau). \]  
(5.13)

This means that \( \tau(s,t) \) is a tau function of the 1D Toda hierarchy.

**Remark 1.** The exponential functions in (5.8) and (5.10) can be absorbed by redefinition of the tau function replacing

\[ g \rightarrow \tilde{g} = \exp \left( \sum_{k=1}^{\infty} \frac{(-q^{1/2})^{k}}{k(1 - q^2)} J_{-k} \right) g \exp \left( \sum_{k=1}^{\infty} \frac{(-q^{1/2})^{k}}{k(1 - q^2)} J_k \right). \]

This is a consequence of the identities

\[ \exp \left( \sum_{k=1}^{\infty} \frac{q^{t_k}}{1 - q^2} \right) \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^{t_k} J_k \right) = \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^{t_k} J_k \right) \exp \left( \sum_{k=1}^{\infty} \frac{(-q^{1/2})^{k}}{k(1 - q^2)} J_{-k} \right), \]

\[ \exp \left( \sum_{k=1}^{\infty} \frac{q^{t_k}}{1 - q^2} \right) \exp \left( \sum_{k=1}^{\infty} (-q^{1/2})^{t_k} J_{-k} \right) = \exp \left( \sum_{k=1}^{\infty} \frac{(-q^{1/2})^{k}}{k(1 - q^2)} J_{-k} \right) \exp \left( \sum_{k=1}^{\infty} \frac{q^{t_k}}{1 - q^2} J_k \right). \]
that can be deduced from the commutation relations (3.5) of $J_{\pm k}$'s. Note that the new generating operator $\tilde{g}$, too, satisfies the 1D reduction condition

$$J_{\pm k} = \tilde{g}J_{\pm k}, \quad k = 1, 2, \ldots$$

It is also remarkable that the two operators in the transformation $g \to \tilde{g}$ are related to the vertex operators:

$$\exp\left(\sum_{k=1}^{\infty} \frac{(-q^{1/2})^k}{k(1-q^k)} J_{\pm k}\right) = \Gamma'_{\pm}(q^{-\rho})^{-1}.$$

**Remark 2.** There is another way to avoid the exponential factors in (5.8) and (5.10). These factors disappear if the external potentials $\phi_k(\lambda)$ are modified as

$$\phi_k(\lambda, s) = \sum_{i=1}^{\infty} \left( q^{k(\lambda - i + 1 + i)} - q^{k(-i + 1 + i)} \right) - \frac{q^{k}}{1-q^k} q^s,$$

namely, if the constant term $q^k/(1-q^k)$ is subtracted from $\phi_k(\lambda)$. This amounts to modifying the definition (4.12) of $H_k$ as

$$H_k = q^\Delta - \frac{q^k}{1-q^k}.$$

The foregoing computations with the aid of the shift symmetries, too, can be slightly simplified by this redefinition of $H_k$'s. Note that the prefactor $q^{-(4s^3 - s)/12}$ cannot be removed by this modification.

5.3. $Z'(s, t, \bar{t})$ as tau function

The partition function $Z'(s, t, \bar{t})$ of the second melting crystal model can be treated in a parallel manner. Let us show an outline of the computations [63].

The first step is to rewrite the fermionic expression (4.16) as follows:

$$Z'(s, t, \bar{t}) = \langle s | q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\sigma}) e^{H(t)} \times \mathcal{U}_e e^{\bar{H}(\bar{t})} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\sigma}) q^{-K/2} | s \rangle,$$

where

$$\bar{H}(\bar{t}) = \sum_{k=1}^{\infty} \bar{t}_k J_{-k}.$$ 

Note that we have split $e^{H(t)}$ into $e^{\bar{H}(\bar{t})}$ and $e^{\bar{H}(\bar{t})}$, and inserted $\Gamma'_-(q^{-\rho}) q^{-K/2}$ to the right end of the operator product.

The second step is to transfer $e^{\bar{H}(\bar{t})}$ and $e^{\bar{H}(\bar{t})}$ to the left and right ends, respectively, with the aid of the shift symmetries. Computations for $e^{\bar{H}(\bar{t})}$ are exactly the same as the case of $Z(s, t)$.

To transfer $e^{\bar{H}(\bar{t})}$, we combine the shift symmetries of the second type (5.4) and the third type (5.5). This yields the relation

$$\left( H_{-k} + \frac{1}{1-q^k} \right) \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\sigma}) q^{-K/2} = \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\sigma}) q^{-K/2} H_{-k}.$$
connecting $H_{-k} + 1/(1 - q^k)$ and $J_{-k}$. Exponentiating this relation, we obtain the following counterpart of (5.17):

$$e^{H(t_\bar{t})} \Gamma_-(q^{-\rho}) \Gamma_+ (q^{-\rho}) q^{-K/2} = \exp \left( - \sum_{k=1}^{\infty} \frac{t_k}{1 - q^k} \right) \times \Gamma_-(q^{-\rho}) \Gamma_+ (q^{-\rho}) q^{-K/2} \exp \left( \sum_{k=1}^{\infty} q^{-k/2} t_k J_{-k} \right), \quad (5.17)$$

Plugging (5.7) and (5.17) into (5.16), we can rewrite $Z'(s, t, \bar{t})$ as

$$Z'(s, t, \bar{t}) = e^{\sum_{k=1}^{\infty} q^{k/2} t_k - \frac{t_k}{1 - q^k}} \times \langle s | \exp \left( \sum_{k=1}^{\infty} \left(-q^{1/2}\right)^k t_k J_k \right) g \exp \left( \sum_{k=1}^{\infty} q^{-k/2} t_k J_{-k} \right) | s \rangle \quad (5.18)$$

where

$$g = q^{K/2} \Gamma_-(q^{-\rho}) \Gamma_+ (q^{-\rho}) Q^\rho \Gamma_-(q^{-\rho}) \Gamma_+ (q^{-\rho}) q^{-K/2} \quad (5.19)$$

Thus, apart from the exponential prefactor and the rescaling $t_k \to (-q^{1/2})^k t_k$, $\bar{t}_k \to q^{-k/2} \bar{t}_k$ of the time variables, $Z'(s, t, \bar{t})$ is a tau function of the 2D Toda hierarchy generated by the operator (5.19).

One can find no symmetry like (5.11) for the generating operator (5.19). The associated tau function is a genuine 2D Toda tau function. Actually, this special solution of the 2D Toda hierarchy falls into the Ablowitz–Ladik hierarchy [63].

**Remark 3.** The exponential prefactor in (5.18) can be absorbed by replacing

$$g \to \tilde{g} = \exp \left( \sum_{k=1}^{\infty} \left(-q^{1/2}\right)^k \right) g \exp \left( - \sum_{k=1}^{\infty} \frac{q^{k/2}}{k(1 - q^k)} J_k \right).$$

Alternatively, one can remove this prefactor by subtracting the constant terms $q^{\pm k}/(1 - q^{\pm k})$ from the external potentials $\phi_{\pm k}(\lambda)$ as shown in (5.14). The operators $H_{\pm k}$ are accordingly modified as shown in (5.15).

### 5.4. Shift symmetries in matrix formalism

We here turn to a digression on the quantum torus algebra and the shift symmetries. This is not just a digression, but closely related to the subsequent consideration in the perspective of the Lax formalism.

The foregoing quantum torus Lie algebra and shift symmetries can be translated to the language of infinite matrices by the correspondence $A \leftrightarrow \hat{A}$ between $\mathbb{Z} \times \mathbb{Z}$ matrices and fermion bilinears. This matrix formalism enables us to use the associative product of matrices as well. In particular, the matrix representation $V_{m}^{(k)}$ of $V_{m}^{(k)}$ are expressed in term of $\Lambda$ and $\Delta$ as

$$V_{m}^{(k)} = q^{-km/2} \Lambda^m q^{k} \Delta. \quad (5.20)$$
Moreover, the commutation relations
\[ [V_m^{(k)}, V_n^{(l)}] = (q^{(lm-ka)/2} - q^{(kn-la)/2}) V_{m+n}^{(k+l)} \]  
(5.21)
of the centerless quantum torus Lie algebra can be derived from the so called quantum torus relation
\[ \Lambda q^{-\Lambda} = q^{\Lambda} \Lambda \]  
(5.22)satisfied by \( \Lambda \) and \( q^{\Lambda} \), which generate an associative quantum torus algebra.

Moreover, the vertex operators \( \Gamma_\pm (q^{-\rho}) \) and \( \Gamma'_\pm (q^{-\rho}) \) reveals a hidden link with the notion of quantum dilogarithmic functions [90, 91] through the matrix representation. Such a Clifford operator, too, have the associated matrix representation through the exponentiation \( e^A \leftrightarrow e^A \) of the Lie algebraic correspondence \( A \leftrightarrow A \). The fundamental vertex operators \( \Gamma_\pm (x) \) and \( \Gamma'_\pm (x) \) thereby correspond to the following matrices:
\[ \Gamma_\pm (x) = \exp \left( \sum_{k=1}^\infty \frac{x^k}{k} \Lambda^\pm k \right) = (1 - x\Lambda^\pm 1)^{-1}, \]
\[ \Gamma'_\pm (x) = \exp \left( - \sum_{k=1}^\infty \frac{(-x)^k}{k} \Lambda^\pm k \right) = (1 + x\Lambda^\pm 1). \]  
(5.23)

Consequently, the matrix representation of \( \Gamma_\pm (q^{-\rho}) \) and \( \Gamma'_\pm (q^{-\rho}) \) become an infinite product of these matrices specialized to \( x = q^{-1/2} \):
\[ \Gamma_\pm (q^{-\rho}) = \prod_{i=1}^\infty (1 - q^{-1/2}\Lambda^\pm 1)^{-1}, \quad \Gamma'_\pm (q^{-\rho}) = \prod_{i=1}^\infty (1 + q^{-1/2}\Lambda^\pm 1). \]  
(5.24)

These infinite products may be thought of as matrix-valued quantum dilogarithmic functions in the sense of Faddeev et al.

We thus find the following matrix analogues of the shift symmetries:

(i) For \( k > 0 \) and \( m \in \mathbb{Z} \),
\[ \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) V_m^{(k)} = (-1)^k V_m^{(k)} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}). \]  
(5.25)

(ii) For \( k > 0 \) and \( m \in \mathbb{Z} \),
\[ \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}) V_m^{(-k)} = V_m^{(-k)} \Gamma'_-(q^{-\rho}) \Gamma'_+(q^{-\rho}). \]  
(5.26)

(iii) For \( k, m \in \mathbb{Z} \),
\[ V_m^{(k)} q^\pm (A^{-1/2})^2 = q^{-m/2} q^\pm (A^{-1/2})^2 V_m^{(k+m)}. \]  
(5.27)

These matrix analogues of the shift symmetries can be derived from the matrix representation (5.20) and (5.24) of \( V_m^{(k)} \)'s and the vertex operators by straightforward computations using the quantum torus relation (5.22) [63].

### 5.5. Perspectives in Lax formalism

Let us return to the melting crystal models, and consider the associated special solutions of the 2D Toda hierarchy in the Lax formalism. The goal is to show that the Lax operators \( L, \bar{L} \) satisfy the reduction conditions (2.21) and (2.26) to the 1D Toda and Ablowitz–Ladik hierarchies [63]. The reasoning can be outlined as follows.
1. It is enough to show that the initial values of the Lax operators at \( t = \bar{t} = 0 \) satisfy the reduction condition (2.21) and (2.26), because these factorized forms are preserved by the time evolutions of the 2D Toda hierarchy.

2. One can explicitly solve the factorization problem (2.29) for these cases at the initial time. The initial values of the dressing operators are written in terms of the matrix representation (5.24) of the vertex operators and some other simple matrices.

3. The initial values of the Lax operators can be computed with the aid of these matrices, and turn out to take the forms as shown in (2.21) and (2.26).

5.5.1. First melting crystal model. The generating operator (5.9) in this case corresponds to a matrix of the form

\[
U = q^{(\Delta - 1/2)^2/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q^\Delta \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{(\Delta - 1/2)^2/2}. \tag{5.28}
\]

One can use the identities

\[
Q^\Delta \Lambda^\rho Q^{-\Delta} = Q^{-\rho} \Lambda^\rho, \quad Q^{-\Delta} \Lambda^\rho Q^\Delta = Q^\rho \Lambda^\rho \tag{5.29}
\]

to rewrite the triple product in the middle as

\[
U = q^{(\Delta - 1/2)^2/2} \Gamma_-(q^{-\rho}) \Gamma_-(Qq^{-\rho}) Q^\Delta \Gamma_+(Qq^{-\rho}) \Gamma_+(q^{-\rho}) q^{(\Delta - 1/2)^2/2}.
\]

This matrix is already factorized to a product of lower and upper triangular matrices as

\[ U = W_0^{-1} \bar{W}_0, \]

where

\[
W_0 = q^{(\Delta - 1/2)^2/2} \Gamma_-(Qq^{-\rho})^{-1} \Gamma_-(q^{-\rho})^{-1} q^{-(\Delta - 1/2)^2/2}, \quad \bar{W}_0 = q^{(\Delta - 1/2)^2/2} Q^\Delta \Gamma_+(Qq^{-\rho}) \Gamma_+(q^{-\rho}) q^{(\Delta - 1/2)^2/2}. \tag{5.30}
\]

This means that \( W_0 \) and \( \bar{W}_0 \) are the initial values \( W|_{t=\bar{t}=0} \), \( \bar{W}|_{t=\bar{t}=0} \) of the dressing operators determined by the generating matrix (5.28).

One can compute the initial values

\[ L_0 = L|_{t=\bar{t}=0} = W_0 \Lambda W_0^{-1}, \quad L_{-1} = L|_{t=\bar{t}=0} = \bar{W}_0 \Lambda^{-1} \bar{W}_0^{-1} \]

of the Lax operators from these explicit forms of \( W_0 \) and \( \bar{W}_0 \) as follows.

The first step for computing \( L_0 \) is to use the identity

\[
q^{-(\Delta - 1/2)^2/2} \Lambda q^{(\Delta - 1/2)^2/2} = q^\Delta \Lambda
\]

that is a consequence of (5.27). By this identity and the expression (5.30) of \( W_0 \), one can rewrite \( L_0 \) as

\[
L_0 = q^{(\Delta - 1/2)^2/2} \Gamma_-(Qq^{-\rho})^{-1} \Gamma_-(q^{-\rho})^{-1} q^\Delta \Lambda \Gamma_-(q^{-\rho}) \Gamma_-(Qq^{-\rho}) q^{-(\Delta - 1/2)^2/2}.
\]

Since \( \Gamma_-(q^{-\rho}) \) and \( \Gamma_-(Qq^{-\rho}) \) are matrices of the form

\[
\Gamma_-(q^{-\rho}) = \prod_{i=1}^{\infty} (1 - q^{-1/2} \Lambda^{-1})^{-1}, \quad \Gamma_-(Qq^{-\rho}) = \prod_{i=1}^{\infty} (1 - Qq^{-1/2} \Lambda^{-1})^{-1},
\]

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the matrix $\Lambda$ in front of these two matrices can be moved to the right side as
\[ \Lambda \Gamma_-(q^{-\rho}) \Gamma_-(Q q^{-\rho}) = \Gamma_-(q^{-\rho}) \Gamma_-(Q q^{-\rho}) \Lambda. \]

One can further use the identity
\[ q^\Delta \Lambda^{-1} q^{-\Delta} = q \Lambda^{-1} \]
to transfer the remaining $q^\Delta$ to the right as
\[ q^\Delta \Gamma_-(q^{-\rho}) \Gamma_-(Q q^{-\rho}) = q^\Delta \prod_{i=1}^{\infty} (1 - q^{-i/2} \Lambda^{-1})^{-1} \prod_{i=1}^{\infty} (1 - Q q^{-i/2} \Lambda^{-1})^{-1} \cdot q^\Delta \]
\[ = \prod_{i=1}^{\infty} (1 - q^{i+1/2} \Lambda^{-1})^{-1} \prod_{i=1}^{\infty} (1 - Q q^{i+1/2} \Lambda^{-1})^{-1} \cdot q^\Delta \]
\[ = \Gamma_-(q^{-\rho}) \Gamma_-(Q q^{-\rho}) (1 - Q q^{1/2} \Lambda^{-1}) (1 - q^{1/2} \Lambda^{-1}) q^\Delta. \]

The outcome reads
\[ L_0 = q^{(\Delta - 1/2)^2/2} (1 - Q q^{1/2} \Lambda^{-1}) (1 - q^{1/2} \Lambda^{-1}) q^\Delta \Lambda q^{-(\Delta - 1/2)^2/2}. \]

Lastly, by the identities
\[ q^{(\Delta - 1/2)^2/2} \Lambda q^{-(\Delta - 1/2)^2/2} = q^{-\Delta} \Lambda, \]
\[ q^{(\Delta - 1/2)^2/2} \Lambda^{-1} q^{-(\Delta - 1/2)^2/2} = \Lambda^{-1} q^{-\Delta} = q^{-1} q^{-\Delta} \Lambda^{-1} \]
one can rewrite the last expression of $L_0$ as
\[ L_0 = (1 - Q q^{-1/2} \Lambda^{-1}) (1 - q^{-1/2} \Lambda^{-1}) \Lambda \]
\[ = \Lambda - (Q + 1) q^{-1/2} \Delta + Q q^{-2} \Delta \Lambda^{-1}. \]

One can compute $L_0^{-1}$ in much the same way, and confirm that it coincides with the expression (5.31) of $L_0$. This implies that the reduction condition (2.21) to the 1D Toda hierarchy is indeed satisfied.

5.5.2. Second melting crystal model. The generating operator (5.19) in this case corresponds to the matrix
\[ U = q^{(\Delta - 1/2)^2/2} \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) Q \Delta \Gamma_-(q^{-\rho}) \Gamma_+(q^{-\rho}) q^{-(\Delta - 1/2)^2/2}. \]

This matrix can be factorized as
\[ U = W_0^{-1} \tilde{W}_0 \]
with
\[ W_0 = q^{(\Delta - 1/2)^2/2} \Gamma_+ (Q q^{-\rho})^{-1} \Gamma_-(q^{-\rho})^{-1} q^{-(\Delta - 1/2)^2/2}, \]
\[ \tilde{W}_0 = q^{(\Delta - 1/2)^2/2} Q \Delta \Gamma_+ (Q q^{-\rho}) \Gamma_+(q^{-\rho}) q^{-(\Delta - 1/2)^2/2}. \]

One can compute $L_0$ in much the same way as the previous case, starting from the expression
\[ L_0 = q^{(\Delta - 1/2)^2/2} \Gamma_+ (Q q^{-\rho})^{-1} \Gamma_-(q^{-\rho})^{-1} q^\Delta \Lambda \Gamma_-(q^{-\rho}) \Gamma_+(Q q^{-\rho}) q^{-(\Delta - 1/2)^2/2}. \]
This expression contains
\[ \Gamma'_{-}(Qq^{-\rho}) = \prod_{i=1}^{\infty} (1 + Qq^{i+1/2} \Lambda^{-1}) \]
in place of \( \Gamma_{-}(Qq^{-\rho}) \). Consequently, the foregoing transfer procedure of \( q^{\Delta} \) is modified as
\[ q^{\Delta} \Gamma_{-}(Qq^{-\rho}) \Gamma'_{-}(Qq^{-\rho}) = q^{\Delta} \prod_{i=1}^{\infty} (1 - q^{i+1/2} \Lambda^{-1})^{-1} \prod_{i=1}^{\infty} (1 + Qq^{i+1/2} \Lambda^{-1}) \]
\[ = \prod_{i=1}^{\infty} (1 - q^{i+1/2} \Lambda^{-1})^{-1} \prod_{i=1}^{\infty} (1 + Qq^{i+1/2} \Lambda^{-1}) \cdot q^{\Delta} \]
\[ = \Gamma_{-}(Qq^{-\rho}) \Gamma'_{-}(Qq^{-\rho})(1 + Qq^{1/2} \Lambda^{-1})^{-1} (1 - q^{1/2} \Lambda^{-1}) q^{\Delta}. \]
The final expression of \( L_{0} \) takes the quotient form
\[ L_{0} = (1 + Qq^{1/2} q^{\Delta} \Lambda^{-1})^{-1} (1 - q^{1/2} q^{\Delta} \Lambda^{-1}) \Lambda. \] (5.34)

One can compute \( \tilde{L}_{0}^{-1} \) in much the same (but slightly more complicated) way starting, (5.33) and the identity
\[ q^{-(\Delta-1)/2} q^{\Delta-1} q^{(\Delta-1)/2} = \Lambda^{-1} q^{\Delta} \]
imply that \( \tilde{L}_{0}^{-1} \) can be expressed as
\[ \tilde{L}_{0}^{-1} = q^{(\Delta-1)/2} q^{\Delta} \Gamma_{+} (Qq^{-\rho}) \Gamma'_{+} (q^{-\rho}) \]
\[ \times \Lambda^{-1} q^{-\Delta} \Gamma'_{+} (q^{-\rho})^{-1} \Gamma_{+} (Qq^{-\rho})^{-1} Q^{-\Delta} q^{-(\Delta-1)/2}. \]
The outcome of somewhat lengthy computations reads
\[ \tilde{L}_{0}^{-1} = (1 - q^{1/2} q^{\Delta} \Lambda)^{-1} (1 + Q^{-1} q^{1/2} q^{-\Delta} \Lambda) Q \Lambda^{-1}. \] (5.35)

It is easy to see that (5.34) and (5.35) can be rewritten as
\[ L_{0} = \tilde{C}_{0}^{-1} \tilde{B}_{0}, \quad \tilde{L}_{0}^{-1} = -\tilde{B}_{0}^{-1} \tilde{C}_{0}, \]
where
\[ \tilde{B}_{0} = \Lambda - q^{-1/2} q^{\Delta}, \quad \tilde{C}_{0} = 1 + Q^{-1/2} q^{\Delta} \Lambda^{-1}. \] (5.36)
This coincides with the reduced form of (2.26) except for the negative sign in the expression of \( L_{0}^{-1} \). The negative sign is harmless, because it can be absorbed by the time reversal \( t \to -t \).

Actually, one can express \( L_{0} \) and \( \tilde{L}_{0} \) in the form of (2.24) as well (again with an extra negative sign) [63]. Anyway, the reduction condition to the Ablowitz-Ladik hierarchy is satisfied in this case.

6. Conclusion

It is remarkable that the two melting crystal models repeat the same pattern of integrable structures as the Hermitian and unitary matrix models. A major difference is the fact that the partition functions of the matrix models are \( s \times s \) determinants (hence the lattice coordinate \( s \) therein take values in positive integers), whereas there is no such expression of the partition functions of the melting crystal models as determinants of finite size. The discrete variable \( s \)
of the melting crystal models enters the Boltzmann weights as a parameter. This is a main reason why we need an entirely different method to identify the underlying integrable structures.

On the other hand, the undeformed partition functions (4.7) and (4.9) of the two melting crystal models differs in just the single factor $q^{a(\lambda)/2}$. It is somewhat surprising that this tiniest modification leads to a drastic change in the underlying integrable structure. Of course this is rather natural from a geometric point of view, because the associated Calabi–Yau threefolds are different.

The shift symmetries of the quantum torus algebra lie in the heart of our method. These algebraic relations are used to transform the ‘diagonal’ Hamiltonians $H_k = V_{0}^{(\lambda)}$ to the ‘non-diagonal’ generators $J_m$ of time evolutions of the 2D Toda hierarchy. Let us mention two other approaches to this kind of unconventional time evolutions (see also section 3.5 of the review of Alexandrov and Zabrodin [23]).

The first one is Orlov’s approach [92] to a class of KP tau functions obtained from the hypergeometric functions (3.18) by specializing the second set $t$ of time variables to a particular point. The special value of $S_\lambda(-i)$ at that point $t = -a$ becomes a determinant of the Cauchy type. The Schur function expansion of $\tau(s,t,-a)$ can be thereby reorganized to an ‘$\infty$-soliton solution’ of the KP hierarchy in which the parameters $T = (T_1, T_2, \ldots)$ of the generating operator (3.17) play the role of time variables.

The second approach is developed by Bettelheim et al [93] in their research of a complex fermion system on the real line. Time evolutions of this system are generated by diagonal Hamiltonians similar to our $H_k$’s except that the coefficients $q^a$ of $\psi_e^* \psi_r$ are replaced by $n^a$. Bettelheim et al considered an analogue of KP and Toda tau functions in which $J_k$’s and the ground states $\langle s \rangle$ and $\langle \bar{s} \rangle$ are replaced by $H_k$’s and what they call ‘boundary states’, $\langle B_s \rangle$ and $\langle B_{\bar{s}} \rangle$. These boundary states are generated from the vacuum states $|0\rangle$ and $|0\rangle$ by ‘boundary operators’ $B_s$. The modified ‘tau functions’ are shown to satisfy the bilinear equations of the KP and Toda hierarchies. Unfortunately, it is difficult to compare the results of Bettelheim et al with ours literally, because the setup of the fermion system is different. Our complex free fermions live on a circle $|z| = R$ of the $z$-plane rather than the real axis. Nevertheless, it is obvious that the boundary operators $B$ play the same role as $\Gamma_-(q^{-\rho})|\Gamma_+(q^{-\rho})$ in our approach.

We believe that the shift symmetries will be useful beyond the scope of the melting crystal models. The results reviewed in this paper should be just a small piece of possible applications. In fact, we recently applied the shift symmetries to computations of topological string theory in a special case [94]. We are currently trying to find how the algebraic relations (5.3) and (5.4) are altered outside the range $k > 0$. Hopefully, the shift symmetries thus extended will become a new tool of computations for various purposes.

It is also true that the shift symmetries are a very special property of the vertex operators $\Gamma_\pm(q^{\rho})$ and $\Gamma_\pm(q^{-\rho})$. Until now, we have been unable to find a similar tool for the 4D version [53–56] of $Z(s,t)$ and $Z'(s,t)$. The aforementioned boundary operators of Bettelheim et al might be a clue to this problem. It seems more likely that another clue is hidden in the fermionic formalism of CP$^1$ Gromov–Witten theory developed by Okounkov and Pandharipande [47, 48].

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