Ultradiscrete Bifurcations for One Dimensional Dynamical Systems

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Abstract

Bifurcations of one dimensional dynamical systems are discussed based on some ultradiscretized equations. The ultradiscrete equations are derived from the normal forms of one-dimensional nonlinear differential equations, each of which has saddle-node, transcritical, or pitchfork bifurcations. An additional bifurcation, which is similar to flip bifurcation, is also discussed. Dynamical properties of these ultradiscrete bifurcations can be characterized with graphical analysis. As an example of application of our treatment, we focus on an ultradiscrete equation of FitzHugh-Nagumo model, and discuss its dynamical properties.

Key Words: ultradiscritization, bifurcation, normal forms, discrete dynamical system

1 Introduction

Ultradiscretization is a limiting procedure converting a difference equation into other type of difference equation subject to max-plus algebra[1]. In this procedure, first, a positive variable \( u_n \) of a difference equation is transformed into \( U_n \) as \( u_n = \exp(U_n/\varepsilon) \), where \( \varepsilon \) is a positive parameter. After the transformation, the formulae

\[
\begin{align*}
\lim_{\varepsilon \to 0^+} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) &= \max(A, B, \ldots), \\
\lim_{\varepsilon \to 0^+} \varepsilon \log(e^{A/\varepsilon} \cdot e^{B/\varepsilon} \cdot \ldots) &= A + B + \ldots,
\end{align*}
\]

(1)
are adopted. Then, we can obtain a new difference equation called an ultradiscrete equation of $U_n$. This limiting procedure brings piecewise linearization of the original equation, and an obtained ultradiscrete equation is easy to be associated with a cellular automaton. Ultradiscretization has been successfully applied to integrable systems[2]. Actually, some cellular automata have been derived from soliton equations such as KdV equation through ultradiscretization[1] [3] [4] [5]. Lately, ultradiscretization has been applied to non-integrable non-equilibrium dissipative systems such as reaction diffusion systems, which are also expressed by non-linear differential equations[6] [7] [8] [9] [10] [11] [12].

For discretization of non-linear differential equations, Murata proposed the tropical discretization[7]. Now we consider the equation of $u$ :

$$\frac{du}{dt} = f(u) - g(u). \quad (2)$$

Here, $u = u(t) > 0$ and $f(u), g(u) \geq 0$. The tropical discretization leads Eq.(2) to the discrete equation

$$u_{n+1} = u_n \frac{u_n + \Delta t f(u_n)}{u_n + \Delta t g(u_n)}, \quad (3)$$

where $\Delta t$ is the discretized time interval. $n$ shows the number of iteration steps and is non-negative integer. Then, $u_n = u(n\Delta t)$. In order to obtain an ultradiscrete equation for Eq.(2), the variable transformations are adopted to Eq.(3): $\Delta t = e^{T/\varepsilon}$, $u_n = e^{U_n/\varepsilon}$, $f(u_n) = e^{F(U_n)/\varepsilon}$, and $g(u_n) = e^{G(U_n)/\varepsilon}$. After these transformation, we obtain the ultradiscretized equation by eq.(1) as

$$U_{n+1} = U_n + \max\{U_n, T + F(U_n)\} - \max\{U_n, T + G(U_n)\}. \quad (4)$$

So far, relationships between solutions of original non-linear differential equations and those of their ultradiscrete equations have been discussed for non-integrable dissipative systems [7] [8]...
Meanwhile, these differential equations have been widely studied from the viewpoint of dynamical systems\cite{13,14,15,16}. For instance, stability of fixed points and bifurcation phenomena have been treated. And frameworks for their treatments have been established. We consider that application of ultradiscretization to the nonlinear dynamical systems is important but seems to be insufficient now.

In this paper, we discuss dynamical properties of some ultradiscrete equations for the nonlinear dynamical systems. We focus on normal forms of saddle node, transcritical, and supercritical pitchfork bifurcations\cite{14,15}. In the next section, ultradiscrete equations of these normal forms are derived. It is noted that ultradiscrete bifurcations, namely bifurcations of ultradiscrete equations, are characterized by the piecewise linearity with max-plus algebra\cite{17}. To elucidate essence of the ultradiscrete bifurcations, we introduce some simple max-plus discrete equations in Section 3. These equations have the same piecewise linearity as the ultradiscrete equations. By introducing the simple max-plus discrete equations, it is easy to sketch the piecewise linear graphs and easy to grasp dynamical properties of the bifurcations; this approach is known as a graphical analysis\cite{18}. Discussion and conclusion are given in Sections 4 and 5, respectively.

2 Ultradiscretization of one dimensional normal forms

The one-dimensional normal forms of the saddle-node, transcritical, and supercritical pitchfork bifurcations are generally given as the following nonlinear equations, respectively \cite{14,15};

\[(\text{saddle-node}) \quad \frac{du}{dt} = r + u^2, \quad (5)\]
\[(\text{transcritical}) \quad \frac{du}{dt} = ru - u^2, \quad (6)\]
\[(\text{supercritical pitchfork}) \quad \frac{du}{dt} = ru - u^3, \quad (7)\]

where \(r\) is a bifurcation parameter. The bifurcation points of these three normal forms are \(r = 0\). In this section, we first derive ultradiscrete equations from eqs. (5)-(7). Then, we show
the bifurcation properties of the obtained ultradiscretized normal forms.

2.1 Saddle-node bifurcation

Now we focus on eq.(5), the normal form of saddle-node bifurcation. In eq.(5), there is no fixed point for \( r > 0 \). For \( r < 0 \), the two fixed points exist, and one of the points is positive and another one is negative. Note that, in general, existence of negative values makes it difficult to apply ultradiscretization. To avoid this difficulty for eq.(5), we consider the following equation instead of eq.(5).

\[
\frac{du}{dt} = c + (u - \alpha)(u - \beta), \tag{8}
\]

where \( \alpha, \beta (> \alpha) \), and \( c \) are positive. \( c \) is the bifurcation parameter, and the saddle-node bifurcation occurs at \( c = c_0 \equiv \frac{(\alpha - \beta)^2}{4} \). If \( c < c_0 \), eq.(8) has the two fixed points \( u_- \) and \( u_+ \), where \( u_\pm = \frac{\alpha + \beta \pm \sqrt{((\alpha - \beta)^2 - 4c)}}{2} \). Note that \( u = u_- \) and \( u_+ \) are stable and unstable, respectively. If \( c = c_0 \), \( u = (\alpha + \beta)/2 \) is half-stable. If \( c > c_0 \), there is no fixed point.

By using the tropical discretization, the discrete equation of eq.(8) is obtained as

\[
u_{n+1} = \frac{u_n + \Delta t[(u_n)^2 + \alpha \beta + c]}{1 + \Delta t(\alpha + \beta)}. \tag{9}\]

It is easy to check that eq.(9) shows the saddle-node bifurcation by changing the parameter \( c \).

The bifurcation point of eq.(9) is the same as that of eq.(8). After the variable transformations

\[
\begin{cases}
\Delta t = e^{T/\epsilon}, & u_n = e^{U_n/\epsilon}, \\
\alpha = e^{A/\epsilon}, & \beta = e^{B/\epsilon}, & c = e^{C/\epsilon},
\end{cases} \tag{10}
\]

and ultradiscretization by eq.(1), we obtain the ultradiscrete equation

\[
U_{n+1} = \max\{U_n, T + \max(2U_n, A + B, C)\} - \max\{0, T + \max(A, B)\}, \tag{11}
\]

where \( A < B \). Assuming \( T \geq \max\{0, -B, -(A + B)\} \), the following ultradiscrete equation can be derived:

\[
U_{n+1} = \max(2U_n - B, A, C - B). \tag{12}
\]
(For derivation of eq. (12), see Appendix A.1.) Dynamical properties of $U_n$ in Eq. (12) are summarized as follows, depending on the value of $C$ (see Appendix B.1 for details).

**SN(i)** When $C > 2B$, $U_{n+1} > U_n$ holds for any $n$; there is no fixed point.

**SN(ii)** When $C = 2B$, $U_n = B$ is a fixed point. If $U_n \leq B$, $U_{n+1} = B$. If $U_n > B$, $U_{n+1} > U_n$.

Hence, $U_n = B$ is half-stable.

**SN(iii)** We set $C < 2B$.

**SN(iii)-1** When $A + B < C < 2B$, $U_n = C - B$ and $U_n = B$ are fixed points.

(a) If $U_n < C - B$ or $C - B < U_n \leq C/2$, $U_{n+1} = C - B$.

(b) If $C/2 < U_n < B$, $C - B \leq U_{n+1} < U_n$.

(c) If $U_n > B$, $U_{n+1} > U_n$.

Hence, $U_n = C - B$ is stable and $U_n = B$ is unstable.

**SN(iii)-2** When $C \leq A + B$, $U_n = A$ and $U_n = B$ are fixed points.

(a) If $U_n < A$ or $A < U_n \leq (A + B)/2$, $U_{n+1} = A$.

(b) If $(A + B)/2 < U_n < B$, $A \leq U_{n+1} < U_n$.

(c) If $U_n > B$, $U_{n+1} > U_n$.

Hence, $U_n = A$ is stable and $U_n = B$ is unstable.

From these dynamical properties of $U_n$, it is found that the ultradiscrete bifurcation point is $C = 2B$ and that this point corresponds to the bifurcation point $c = c_0$ for eq. (8). Thus, the ultradiscrete equation (12) reproduces a similar saddle-node bifurcation to the original normal form.
2.2 Transcritical bifurcation

Next we consider eq. (6), the normal form for the transcritical bifurcation. To apply ultradiscretization, the translation $u \to u - \alpha$ is adopted to eq. (6), and we obtain

$$\frac{du}{dt} = (u - \alpha)(c - u), \quad (13)$$

where $\alpha$ and $c (= r + \alpha)$ are assumed to be positive. In eq. (13), $c$ is the bifurcation parameter and the transcritical bifurcation occurs at $c = \alpha$. By the tropical discretization for eq. (13), we obtain

$$u_{n+1} = u_n \frac{u_n + \Delta t(\alpha + c)u_n}{u_n + \Delta t[(u_n)^2 + \alpha c]}. \quad (14)$$

Then ultradiscretization of eq. (14) produces

$$U_{n+1} = U_n + \max\{U_n, T + U_n + \max(A, C)\} - \max\{U_n, T + \max(2U_n, A + C)\}, \quad (15)$$

where we set the variable transformations

$$\Delta t = e^{T/\epsilon}, \quad x_n = e^{U_n/\epsilon}, \quad \alpha = e^{A/\epsilon}, \quad c = e^{C/\epsilon}. \quad (16)$$

Now we assume $T \geq \max\{0, -(A+C), -\max(A, C)\}$. Then, eq. (15) can be simplified as follows.

$$U_{n+1} = 2U_n + \max(A, C) - \max(2U_n, A + C). \quad (17)$$

(For derivation of eq. (17), see Appendix A.2) Dynamical properties of $U_n$ by eq. (17) are summarized as follows (see Appendix B.2 for further explanation).

**TC(i)** When $C > A$, $U_n = A$ and $U_n = C$ are fixed points.

(a) If $U_n < A$, $U_{n+1} < U_n$.

(b) If $A < U_n < \frac{A+C}{2}$, $U_{n+1} > U_n$.

(c) If $U_n > C$ or $\frac{A+C}{2} \leq U_n < C$, $U_{n+1} = C$.

Hence, $U_n = A$ is unstable and $U_n = C$ is stable.
TC(ii) When $C = A$, $U_n = A$ is a fixed point.

(a) If $U_n < A$, $U_{n+1} = A$.

(b) If $U_n < A$, $U_{n+1} < U_n$.

Then, $U_n = A$ is the half-stable point.

TC(iii) When $C < A$, $U_n = A$ and $U_n = C$ are fixed points. As in the case of TC(i),

(a) If $U_n < C$, $U_{n+1} < U_n$.

(b) If $C < U_n < \frac{C+A}{2}$, $U_{n+1} > U_n$.

(c) If $U_n > A$ or $\frac{C+A}{2} \leq U_n < A$, $U_{n+1} = A$.

Hence, $U_n = A$ is stable and $U_n = C$ is unstable.

These dynamical properties are similar to those of the transcritical bifurcation by eq.(13). Actually, TC(i), (ii), and (iii) correspond to the cases of $c > \alpha$, $c = \alpha$, and $c < \alpha$ in eq.(13), respectively.

2.3 Supercritical pitchfork bifurcation

Applying the variable translation $u \to u - \beta$ and setting $r = c - 1$, eq.(7) becomes

$$\frac{du}{dt} = (c-1)(u-\beta) - (u-\beta)^3$$

$$= \begin{cases} 
-(u-\alpha)(u-\beta)(u-\gamma) & \text{for } c \geq 1, \\
-(u-\beta)^3 - \eta(u-\beta) & \text{for } c < 1,
\end{cases}$$

(18)

where $\alpha = \beta - \sqrt{c-1}$, $\gamma = \beta + \sqrt{c-1}$, and $\eta = 1 - c$. Here we assume $\alpha > 0$ and $\beta > 1$. Then, the discrete equation of eq.(18) by the tropical discretization is

$$u_{n+1} = \frac{u_n + \Delta t\{(\alpha + \beta + \gamma)u_n^2 + \alpha \beta \gamma\}}{1 + \Delta t\{(u_n)^2 + \alpha \beta + \beta \gamma + \alpha \gamma\}} \quad \text{for } c \geq 1,$$

(19)

$$u_{n+1} = \frac{u_n + \Delta t\{3\beta u_n^2 + \beta^3 + \eta \beta\}}{1 + \Delta t\{(u_n)^2 + 3\beta^2 + \eta\}} \quad \text{for } c < 1.$$  

(20)

For the ultradiscretization, we set

$$\begin{cases} 
\Delta t = e^{T/\varepsilon}, & u_n = e^{U_n/\varepsilon}, & c = e^{C/\varepsilon}, \\
\alpha = e^{A/\varepsilon}, & \beta = e^{B/\varepsilon}, & \gamma = e^{T/\varepsilon}, & \eta = e^{E/\varepsilon}.
\end{cases}$$

(21)
Note that $A \leq B \leq \Gamma$ and $B > 0$. Assuming $T \geq \max\{0, -(A + B + \Gamma)\}$, the following ultradiscrete equations are obtained from eq.(19) and eq.(20).

\[ U_{n+1} = \max\{2U_n + \Gamma, A + B + \Gamma\} - \max\{2U_n, B + \Gamma\}, \quad \text{for } C \geq 0, \quad (22) \]

\[ U_{n+1} = \max\{2U_n, 3B, B + E\} - \max\{2U_n, 2B, E\} 
\quad = B, \quad \text{for } C < 0. \quad (23) \]

(For derivation of eq.(22) and eq.(23), see Appendix A.3.) Note that $A$, $\Gamma$, and $E$ depend on $C$ due to $c$ dependence of $\alpha$, $\beta$, and $\eta$. Furthermore when $C = 0$, it is necessary that eq.(22) is equal to eq.(23), namely $A = \Gamma = B$ and $E = 0$. Then if we set

\[ A = \min(B, B - C), \quad \Gamma = \max(B, B + C), \quad E = \max(0, -C), \quad (24) \]

eq.(22) and eq.(23) can be rewritten as the following single equation.

\[ U_{n+1} = \max\{2U_n + \Gamma, A + B + \Gamma, B + E\} - \max\{2U_n, B + \Gamma, E\}. \quad (25) \]

$C$-dependence of dynamical properties of eq.(25) with eq.(24) are summarized as follows (see Appendix B.3 for details);

**SP(i)** When $C \geq 0$, the following three values of $U_n$ are fixed points: $U_n = A(= B - C)$, $B$, and $\Gamma(= B + C)$.

(a) If $U_n < B - C$ or $B - C < U_n \leq B - \frac{C}{T}$, $U_{n+1} = B - C$.

(b) If $B - \frac{C}{T} < U_n < B$, $U_{n+1} < U_n$.

(c) If $B < U_n < B + \frac{C}{T}$, $U_{n+1} > U_n$.

(d) If $U_n > B + C$ or $B + \frac{C}{T} \leq U_n < B + C$, $U_{n+1} = B + C$.

From (a) $\sim$ (d), $U_n = A$ and $\Gamma$ are stable, and $B$ is unstable.

**SP(ii)** When $C < 0$, $U_n = B$ is the only one fixed point. If $U_n \neq B$, $U_{n+1} = B$. Hence, $B$ is stable.
From the above properties \( SP(i) \) and \( SP(ii) \), it is found that eq.(25) exhibits a bifurcation similar to the supercritical pitchfork bifurcation. Note that for \( C \geq 0 \), eq.(22) is the same form as the ultradiscrete Allen-Cahn equation [7] without diffusion effect.

3 Max-plus normal forms and graphical analysis

In the previous section, we have shown the bifurcation properties of the ultradiscrete equations (12), (17), (25) with the bifurcation parameter \( C \). In this section, we propose some simpler max-plus discrete equations, which exhibit the same properties of the ultradiscrete equations. Especially, by using a graphical analysis, their dynamical properties can be visualized. The graphical analysis is well known as a method to intuitively understand one dimensional discrete iterated dynamics [18, 19, 20]. In general, time evolution of \( U_n \) can be described as \( U_{n+1} = f(U_n, U_{n-1}, \cdots; C) \). Here we consider the case where \( U_{n+1} \) is determined only by \( U_n \): \( U_{n+1} = f(U_n; C) \).

3.1 Saddle-node bifurcation

First let us consider the following max-plus equation with the bifurcation parameter \( C \):

\[
U_{n+1} = \max(PU_n, C).
\]  

(26)

Here, we set \( P > 1 \). Figure 1 shows the graphs of eq.(26) for (a) \( C > 0 \), (b) \( C = 0 \), and (c) \( C < 0 \). For (a) \( C > 0 \), the graph of eq.(26) does not touch the diagonal \( U_{n+1} = U_n \). Then, eq.(26) has no fixed point. If \( U_n < C/P \), \( U_{n+1} = C \) and \( U_{n+2} \) increases along \( U_{n+2} = PU_{n+1} > U_{n+1} \). For (b) \( C = 0 \), eq.(26) touches the diagonal at the origin of the graph. Then, \( U_n = 0 \) is the only fixed point and it is half-stable. For (c) \( C < 0 \), the graph intersects the diagonal at the two points \( U_n = 0 \) and \( U_n = C \), which are the unstable and stable fixed points, respectively. In fact, when \( U_n \leq C/P, U_{n+1} = C \). When \( C/P < U_n < 0 \), \( U_n \) tends to \( C/P \) first along \( U_{n+1} = PU_n \), and after that \( U_n \) finally arrives at \( C \). If \( U_n > 0 \), \( U_n \) goes to positive infinity along \( U_{n+1} = PU_n \).
Then this bifurcation is saddle-node. The bifurcation diagram is shown in Fig. 2. In the diagram, the solid arrows show the transition of $U_n$ to the stable point just at the next step. The dotted arrows represent the transition satisfying $U_{n+1} = PU_n$.

Here, we recall eq. (12) : $U_{n+1} = \max(2U_n - B, A, C - B)$, where $A < B$. In the vicinity of the bifurcation point $C = 2B$, we only consider the case $C > A + B$. Then eq. (12) becomes $U_{n+1} = \max(2U_n - B, C - B)$. It is readily confirmed that this equation exhibits the same piecewise linear graph as shown in Fig. 1. Therefore, eq. (26) has the same bifurcation properties as eq. (12). Actually, Fig. 1 (a), (b), and (c) correspond to graphical descriptions of SN(i), (ii), and (iii) in Sec. 2.1.

### 3.2 Transcritical bifurcation

Next we focus on the following max-plus equation with the parameter $C$:

$$U_{n+1} = PU_n + \max(0, C) - \max(PU_n, C),$$

(27)

where $P > 1$. The graphs of eq. (27) with three different cases of $C$ are shown in Fig. 3. For (a) $C > 0$, the graph intersects the diagonal at the two points $U_n = 0$ and $U_n = C$, which are unstable and stable fixed points, respectively. Actually if $U_n < 0$, $U_{n+1} = PU_n < U_n$ goes to negative infinity. If $0 < U_n \leq C/P$, there is the iteration step $m$, at which $U_m \geq C/P$ and $U_{m+1} = C$. If $U_n > C/P$, $U_{n+1} = C$. For (b) $C = 0$, there exists only one half-stable fixed point.
In the case of ultradiscrete transcritical bifurcation shown by eq.(17), a similar piecewise linear graph to the case of eq.(27) can be drawn. In fact, Fig.3 (a), (b), and (c) become graphical representations of TC(i), (ii), and (iii), respectively. Note that eq.(17) and eq.(27)
are exactly the same when we put $P = 2, A = 0$.

### 3.3 Supercritical pitchfork bifurcation

Here we consider the following max-plus equation:

$$U_{n+1} = \max\{PU_n + \max(0, C), 0\} - \max\{PU_n, \max(0, C)\},$$

(28)

where $P > 1$. Figure 5 (a) and (b) show the graph of eq. (28) for $C > 0$ and $C \leq 0$, respectively.

Note that $U_{n+1}$ of eq. (28) is an odd function of $U_n$ as shown in Fig. 5. For (a) $C > 0$, the graph of eq. (28) intersects the diagonal at the three fixed points $U_n = 0, \pm C$. If $U_n \geq C/P (U_n \leq -C/P)$, $U_{n+1} = +C (U_{n+1} = -C)$. If $-C/P < U_n < 0$, there exists a certain $m$ at which $U_m \leq -C/P$ and $U_{m+1} = -C$. In the same way, $0 < U_n < C/P$ finally goes to $C$. Therefore, $U_n = \pm C$ are stable and $U_n = 0$ is unstable. As $C$ decreases, the two stable fixed points $U_n = \pm C$ coalesce when $C = 0$. Then, $U_n = 0$ is only the stable fixed point for $C \leq 0$, and $U_1 = 0$ for any initial $U_0$. Figure 6 represents the bifurcation diagram of eq. (28). Note that the dynamical properties of the cases (a) and (b) in Fig. 5 correspond to SP(i) and SP(ii), respectively.

### 4 Discussion
As a max-plus equation which produces another bifurcation, let us consider the following equation,

$$U_{n+1} = \max\{PU_n, \max(0, C)\} - \max\{PU_n + \max(0, C), 0\}, \quad (29)$$

where $P > 1$. The difference between eqs. (28) and (29) is the sign of $U_{n+1}$; we can obtain eq. (29) by changing $U_{n+1} \to -U_{n+1}$ in eq. (28). Fig. 7 shows the graphs of eq. (29), $U_{n+1}$ is an odd function of $U_n$ as in the case of eq. (28). Note that eq. (29) has the following bifurcation properties:

**FL(i)** When $C > 0$, $U_n = 0$ becomes the unstable fixed point and there is a cycle $\mathcal{C} = \{C, -C\}$ with period 2, which is the attractor of $U_n$. The properties of $\mathcal{C}$ are shown as follows.

**FL(i)-1** $\mathcal{C}$ surrounds the unstable fixed point: $U_n = 0$. 
\textbf{FL(i)-2} Whenever $|U_0| > C$, $U_n \in C$ for any $n \geq 1$.

\textbf{FL(i)-3} If $U_0$ satisfies $0 < |U_0| < C$, $U_n$ leaves from 0 oscillating around 0 and reaches the point $U_m$ such that $C/P \leq |U_m|$. After that, $U_n \in C$ for $n > m$.

\textbf{FL(ii)} When $C \leq 0$, $U_n = 0$ is stable fixed point such that $U_1 = 0$ for any $U_0 \neq 0$. Then, the bifurcation occurs at $C = 0$. From FL(i)-2 and FL(i)-3, any $U_n$ starting from $U_0 \neq 0$ is finally absorbed by the cycle $C$. These bifurcation is similar to flip bifurcation \[20\].

Here, we focus on the ultradiscrete equation of FitzHugh-Nagumo model, which is given as the following reaction-diffusion system\[21\]:

$$
\begin{align*}
\varepsilon \frac{du}{dt} &= u(1-u)(u-a) - v + i, \\
\frac{dv}{dt} &= \kappa u - \lambda v
\end{align*}
$$

where $\varepsilon, a, i, \kappa,$ and $\lambda$ are positive and we set $0 < a < 1$. The ultradiscrete equation of FitzHugh-Nagumo model has been proposed by Sasaki et.al\[22\] as the following one-dimensional ultradiscrete equation

$$
U_{n+1} = \max\{2U_n, I\} - \max\{2U_n, B\}. \tag{31}
$$

By numerical calculation, it was found that eq.\,(31) has some characteristic solutions by changing the parameters $I$ and $B$ ($B > 0$). For instance, a cyclic solution was found for $I > 3B/2$. In the graphical analysis, the dynamics of eq.\,(31) for $B > 0$ can be visualized as shown in Fig. 8 (a)-(d). Figure 8 (a) shows the case of $I > 3B/2$. We can find the cyclic solution $C'$ surrounding the unstable fixed point $U_n = I/3$ which satisfies the properties of FL(i) stated above. Then, $C'$ is composed of the two points; $C' = \{0, I - B\}$ for $I \geq 2B$, $C' = \{2B - I, I - B\}$ for $I < 2B$.

Note that it is easily confirmed that eq.\,(31) with $I = 3B (> 3B/2)$ is the same as eq.\,(29) with $C = B$ by the variable translation $U_n \rightarrow U_n + B$. Note that if $U_0 \geq I/2$, it is verified that $U_n = 0$ when $n$ is odd and $U_n = I - B$ when $n$ is even.
For $B < I \leq 3B/2$, Figure 8 (b) shows that $U_n = I - B$ is the only fixed point and it is stable. According to the graphical analysis, when $U_0 \leq B/2$, $U_1 = I - B$. On the other hand, when $U_0 > B/2$, $U_1 < I - B$, and $U_2 = I - B$. Note that this time evolution is similar to that of excitability. In Figs. 8 (c) and (d), all of the initial values finally go to the unique stable fixed point. For $I = B$, Fig. 8 (c) is the same as Fig. 8 (b). For $I < B$, it is confirmed that any $U_0$ finally arrives at the stable point $U_n = I - B$ according to eq.(31); Fig. 8 (d) shows the case of $I < 0$. The graphical analysis can also be applied to eq.(31) for $B < 0$ as shown in Fig. 9 (a)-(d). In Fig. 9 (a), $U_n = I - B$ and $U_n = 0$ are stable and half-stable, respectively. In Fig. 9 (b), $U_n = I - B, U_n = 0$ are stable and $U_n = B$ is unstable. In Fig. 9 (c), $U_n = I/3$ is the unstable fixed point, and a cyclic solution is obtained. In Fig. 9 (d), $U_n = 0$ is stable in which the system produces the excitability like behavior when $I/2 \lesssim 0$.

Here, we comment the above cyclic solution of the ultradiscrete equation (31). It is noted that
Figure 9: The graphs of eq. (31) with $B \leq 0$. (a) $I < B = 0$, (b) $I < B < 0$, (c) $B < 0 < I$, and (d) $B < I \leq 0$.

eq. (31) is composed of one-variable discrete dynamics for $U_n$, although the original FitzHugh-Nagumo model (30) shows the two-variable dynamics for $u$ and $v$. This reduction of the variables occurs in the derivation of eq. (31) from eq. (30) through tropical discretization in which the variable $v$ in the first differential equation of eq. (30) is discritized by $v_{n+1}$ instead of $v_n$.

Since a one-dimensional differential equation can not have periodic solutions and the limit cycle solution of (30) occur due to the two-dimensional dynamics, the above periodic solution $C'$ of eq. (31) for $I > 3B/2$ is not associated with the limit cyclic solution of the original FitzHugh-Nagumo model, but just caused by the discretization of eq. (30). Actually eq. (31) can be also derived from the one dimensional differential equation of $u$, $\frac{du}{dt} = u(1 - u)(u - a) - \frac{\kappa}{\lambda}u + i$, which is obtained from eq. (30) by setting $\frac{dv}{dt} = 0$. Note that this periodicity is similar to the well-known relation between logistic map and logistic differential equation; the former has the periodic solution whereas the latter has no periodic solution.
Several developments of the current study are expected as the next steps. (i) The cases in higher dimensions, such as stability and bifurcations of simultaneous ultradiscrete equations\cite{11,12}. (ii) Other topics for the one-dimensional discrete dynamical systems such as chaos\cite{23}. (iii) Application to ultradiscrete equations with spatial variables.

5 Conclusion

Ultradiscrete equations are derived from the normal forms of saddle node, transcritical, and supercritical pitchfork bifurcations. These derived equations exhibit ultradiscrete bifurcations, namely the similar bifurcation properties to the original normal forms. We also consider another ultradiscrete bifurcation, similar to the flip bifurcation, where there is a stable cycle around an unstable fixed point. With the aid of graphical analysis, these ultradiscrete bifurcations can be characterized with the much simpler max-plus equations. As an example of application, we can grasp essential dynamical features of the ultradiscrete equation for FitzHugh-Nagumo model, which was previously proposed in \cite{22}, based on our treatment for the ultradiscrete bifurcations,

A Derivation of the ultradiscrete equations

A.1 Saddle-node bifurcation: eq.(11) $\rightarrow$ eq.(12)

We set $T \geq \max\{0, -B, -(A + B)\}$. From $A < B$ and $T + B \geq 0$, eq.(11)

$$U_{n+1} = \max\{U_n, T + \max(2U_n, A + B, C)\} - \max\{0, T + \max(A, B)\}$$

becomes

$$U_{n+1} = \max\{U_n, T + \max(2U_n, A + B, C)\} - (T + B).$$

(32)

If $U_n \geq 0$, $U_n \leq 2U_n \leq 2U_n + T$ since $T \geq 0$. If $U_n < 0$, $U_n < 0 \leq T + A + B$ since $T \geq -(A + B)$.

Hence, $\max\{U_n, T + \max(2U_n, A + B, C)\} = T + \max(2U_n, A + B, C)$, and eq.(12) is obtained
from eq.\((32)\) as follows:

\[
U_{n+1} = T + \max\{2U_n, A + B, C\} - (T + B)
\]

\[
= \max\{2U_n - B, A, C - B\}.
\]

**A.2 Transcritical bifurcation : eq.\((15)\) \(\rightarrow\) eq.\((17)\)**

We set \(T \geq \max\{0, -(A+C), -\max(A,C)\}\). From \(T \geq -\max(A,C)\), \(U_n \leq U_n + T + \max(A,C)\).

If \(U_n \leq 0\), \(U_n \leq 0 \leq T + A + C \leq T + \max(2U_n, A + C)\) since \(T \geq -(A + C)\). If \(U_n > 0\), \(0 < U_n < 2U_n + T \leq T + \max(2U_n, A + C)\) since \(T \geq 0\). Thus, eq.\((17)\) is derived from eq.\((15)\) as follows:

\[
U_{n+1} = U_n + \max\{U_n, T + U_n + \max(A,C)\} - \max\{U_n, T + \max(2U_n, A + C)\}
\]

\[
= U_n + T + U_n + \max(A,C) - T - \max(2U_n, A + C)
\]

\[
= 2U_n + \max(A,C) - \max(2U_n, A + C).
\]  \(\text{(33)}\)

**A.3 Supercritical pitchfork bifurcation: eqs.\((19)\) and \((20)\) \(\rightarrow\) eqs.\((22)\) and \((23)\)**

From eqs. \((19)\) and \((20)\) with eq. \((21)\), the following ultradiscretized equations are obtained through the ultradiscretization shown by eq.\((1)\):

\[
U_{n+1} = \max\{U_n, T + \max(2U_n + \Gamma, A + B + \Gamma)\} - \max\{0, T + \max(2U_n, B + \Gamma)\}, \quad \text{(34)}
\]

\[
U_{n+1} = \max\{U_n, T + \max(2U_n + B, 3B, B + E)\} - \max\{0, T + \max(2U_n, 2B, E)\}. \quad \text{(35)}
\]

Here we set \(T \geq \max\{0, -(A + B + \Gamma)\}\). For eq.\((34)\), we obtain \(0 \leq T + \max(2U_n, B + \Gamma)\) from \(0 < B < \Gamma\) and \(T \geq 0\). If \(U_n \leq 0\), \(U_n \leq 0 \leq T + A + B + \Gamma \leq T + \max(2U_n + \Gamma, A + B + \Gamma)\) since \(T \geq -(A + B + \Gamma)\). If \(U_n > 0\), \(U_n < 2U_n + T \leq 2U_n + \Gamma\) since \(T \geq 0\) and \(0 < \Gamma\). Therefore, eq. \((22)\)

\[
U_{n+1} = \max\{2U_n + \Gamma, A + B + \Gamma\} - \max\{2U_n, B + \Gamma\}
\]
is derived from eq. (34). Similarly for eq. (35), we obtain \( 0 \leq T + \max(2U_n, 2B, E) \) from \( 0 < B \) and \( T \geq 0 \). If \( U_n \leq 0, U_n \leq 0 \leq T + 3B \leq T + \max(2U_n + B, 3B, B + E) \). If \( U_n > 0 \), \( U_n < 2U_n < T + 2U_n + B \) since \( T \geq 0 \) and \( 0 < B \). Therefore, eq. (23)

\[
U_{n+1} = \max\{2U_n + B, 3B, B + E\} - \max\{2U_n, 2B, E\} = B
\]

is derived from eq. (35).

**B Supplementaries for dynamical properties of the ultradiscrete bifurcations**

For simplicity, we set \( n = 0 \) without loss of generality.

**B.1 Saddle node bifurcation**

**SN(i)** For \( C > 2B \), \( U_1 = \max(2U_0 - B, C - B) \) from eq. (12). If \( U_0 \leq C/2, U_1 = \max(2U_0 - B, C - B) = C - B > C/2 \geq U_0 \). If \( U_0 > C/2, U_1 = \max(2U_0 - B, C - B) = 2U_0 - B > U_0 \).

**SN(ii)** For \( C = 2B \), eq. (12) becomes \( U_1 = \max(2U_0 - B, B) \) since \( A < B \). Then, \( U_0 = B \) is a fixed point. If \( U_0 < B, U_1 = \max(2U_0 - B, B) = B > U_0 \). If \( U_0 > B, U_1 = \max(2U_0 - B, B) = 2U_0 - B > U_0 \). Hence, \( B \) is half-stable.

**SN(iii)** We set \( C < 2B \). For \( A + B < C < 2B \), eq. (12) becomes \( U_1 = \max(2U_0 - B, C - B) \).

Then, \( U_0 = C - B \) and \( U_0 = B \) are fixed points. If \( U_0 \leq C/2, U_1 = C - B \). If \( C/2 < U_0 < B, U_1 = 2U_0 - B < U_0 \). If \( B < U_0, U_1 = 2U_0 - B > U_0 \). Hence, \( U_0 = C - B \) and \( U_0 = B \) are stable and unstable, respectively. For \( C \leq A + B \), \( U_1 = \max(2U_0 - B, A) \). Then, \( U_0 = A \) and \( U_0 = B \) are fixed points. If \( U_0 \leq (A + B)/2, U_1 = A \). If \( (A + B)/2 < U_0 < B, U_1 = 2U_0 - B < U_0 \). If \( B < U_0, U_1 = 2U_0 - B > U_0 \). Hence, \( U_0 = A \) and \( U_0 = B \) are stable and unstable, respectively.
B.2 Transcritical bifurcation

**TC(i)** For $C > A$, eq. (17) becomes $U_1 = 2U_0 + C - \max(2U_0, A + C)$. Then, $U_0 = A$ and $U_0 = C$ are fixed points. If $U_0 < A$, $U_1 = 2U_0 + C - (A + C) = 2U_0 - A < U_0$. If $A < U_0 < (A + C)/2$, $U_1 = 2U_0 - A < U_0$. If $(A + B)/2 \leq U_0 < C$, $U_1 = 2U_0 + C - 2U_0 = C$. If $C < U_0$, $U_1 = C$. Hence, $C$ and $A$ are stable and unstable, respectively.

**TC(ii)** For $C = A$, it follows from eq. (17) that $U_1 = 2U_0 + A - 2 \max(U_0, A)$. Then, $U_0 = A$ is the only one fixed point. If $U_0 < A$, $U_1 = 2U_0 - A < U_0$. If $U_0 > A$, $U_1 = A$. Hence, $A$ is half-stable.

**TC(iii)** For $C < A$, eq. (17) becomes $U_1 = 2U_0 + A - \max(2U_0, A + C)$. Then, $U_0 = A$ and $U_0 = C$ are fixed points. If $U_0 < C$, $U_1 = 2U_0 - C < U_0$. If $C < U_0 < (A + C)/2$, $U_1 = 2U_0 - C > U_0$. If $U_0 \geq (A + C)/2$, $U_1 = A$. Hence, $U_0 = A$ and $U_0 = C$ are stable and unstable, respectively.

B.3 Supercritical pitchfork bifurcation

**SP(i)** For $C \geq 0$, the stability of eq. (22) have already been studied. If $U_0 \leq (A + B)/2$, $U_1 = A$. If $(B + \Gamma)/2 \leq U_0$, $U_1 = \Gamma$. If $(A + B)/2 < U_0 < B$, $U_1 = 2U_0 - B < U_0$ and there is a finite time $m(> 0)$ at which $U_m \leq (A + B)/2$ and $U_{m+1} = A$. Similarly if $B < U_0 < (B + \Gamma)/2$, there is a finite time $m(> 0)$ at which $(B + \Gamma)/2 \leq U_m$ and $U_{m+1} = \Gamma$.

In conclusion, $A, \Gamma$ are stable fixed points, $B$ is an unstable fixed point.

**SP(ii)** For $C < 0$, it follows from eq. (23) with (24) that $U_1 = B$ for any $U_0$. $B$ is a stable fixed point.

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