Computing Maximum Entropy Distributions

Everywhere

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Abstract

We study the problem of computing the maximum entropy distribution with a specified expectation over a large discrete domain. Maximum entropy distributions arise and have found numerous applications in economics, machine learning and various sub-disciplines of mathematics and computer science. The key computational questions related to maximum entropy distributions are whether they have succinct descriptions (polynomial-size in the input) and whether they can be efficiently computed. Here we provide positive answers to both of these questions for very general domains and, importantly, with no restriction on the expectation vector. This completes the picture left open by the prior work on this problem by \cite{38} which requires that the given expectation vector is polynomially far in the interior of the convex hull of the domain. As a consequence of our result we obtain a general algorithmic tool and show how it can be applied to derive several old and new results in a unified manner. In particular, our results imply that certain recent continuous optimization formulations, for instance, for discrete counting and optimization problems, the matrix scaling problem, and the worst case Brascamp-Lieb constants in the rank-1 regime, are efficiently computable. Attaining these implications requires reformulating the underlying problem as a version of maximum entropy computation where optimization also involves the expectation vector and, hence, cannot be assumed to be sufficiently deep in the interior. The key new technical ingredient in our work is a polynomial bound on the bit complexity of near-optimal dual solutions to the maximum entropy convex program. This result is obtained by a geometrical reasoning that involves convex analysis and polyhedral geometry, avoiding combinatorial arguments based on the specific structure of the domain. We also provide a lower bound on the bit complexity of near-optimal solutions showing the tightness of our results.
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1 Introduction

We consider the problem of computing maximum entropy (max-entropy) distributions over discrete sets. For a finite set of vectors $F \subseteq \mathbb{Z}^m$ and a point $\theta \in \mathbb{R}^m$, the max-entropy distribution for $\theta$ is defined to be the one that maximizes the entropy $\sum_{\alpha \in F} q_\alpha \log \frac{1}{q_\alpha}$ among all distributions $\{q_\alpha\}_{\alpha \in F}$ whose expectation is $\theta$, i.e., for which $\mathbb{E}[X] = \theta$ when $X$ is distributed according to $q$. This notion is natural when one has an unknown distribution $q$ on the set $F$ and the only information available is its expectation vector $\theta := \sum_{\alpha \in F} q_\alpha \cdot \alpha$. Then, according to the max-entropy principle [21, 22], the best guess for $q$ is the max-entropy distribution with expectation $\theta$. The rationale behind this choice is that maximizing entropy yields a distribution with the least amount of prior information built-in.

Max-entropy distributions arise and have found numerous applications in information theory [21, 23], machine learning [33, 31], economics [6, 41], physics [29] and statistics [39, 37]. Max-entropy distributions have recently also been studied in Theoretical Computer Science, specifically in approximation algorithms and randomized rounding. For instance, sampling algorithms for max-entropy distributions over spanning trees in a graph have been used to derive approximation algorithms for the TSP [1, 32] and max-min fair allocation [5].

In these numerous applications of max-entropy distributions, one is interested in finding their computationally efficient representations. Such representations, ideally, allow efficient sampling and, thus, give a method to compute marginals or infer other useful statistics. Note that the task of finding such a representation is easy when $F$ is specified explicitly as a list of vectors, since the entropy function is concave and thus can be maximized efficiently using convex optimization tools. However, in the more common and interesting case – when $F$ is exponential in $m$ and is specified implicitly, for example as the set of all spanning trees of a graph or the support of a multivariate polynomial – the max-entropy distribution is an exponential-sized object.

An important fact about max-entropy distributions is that there exists a vector $\gamma \in \mathbb{R}^m_{>0}$ such that the max-entropy distribution $q^*$ satisfies $q^*_\alpha \propto \prod_{i=1}^m \gamma_i^{a_i}$ for all $\alpha \in F$. This is a consequence of convex duality; indeed the max-entropy program is convex and $\gamma$ is the optimal solution to its dual. However, such a vector $\gamma$ may not be succinct and its representation may even require infinitely many bits. A first step towards addressing this problem with max-entropy distributions was taken by [38] who showed that, under mild assumptions on $F$, whenever the point $\theta$ is polynomially far from the boundary of the marginal polytope $P := \text{conv}(F)$, i.e., the convex hull of $F$, then $\gamma$ can be represented with polynomially many bits. This allows the max-entropy distribution to be efficiently computable for a wide class of polytopes when $\theta$ is sufficiently deep inside $P$. The question of what happens when $\theta$ is either very close to or on the boundary of $P$ was left open.

In this paper we complete the picture and prove that, under the same conditions as in [38], max entropy distributions can be computed everywhere – for all $\theta \in P$. As a consequence, we provide a general tool which is applicable wherever max-entropy computations arise, and can also be applied for solving more complicated continuous optimization problems that also optimize over $\theta$. In particular, we derive polynomial time algorithms for solving recently introduced [40, 3] convex relaxations for a large class of counting and optimization problems involving polynomials. A polynomial time algorithm for computing worst case Brascamp-Lieb constants [28, 12] in the rank-1 setting also follows from our main result. Another corollary that can be derived from our result is a polynomial bit complexity bound on the solution of a matrix scaling problem recently studied by [1, 14]. The starting point of these applications is to first formulate them as optimization problems involving max-entropy distributions. We believe that the general tool we provide in this paper will find other applications and inspire future work on entropy-based methods.
2 Our Results

In this section we introduce the problems we study and state our main results. For the purpose of clarity, we provide simplified and slightly informal variants of our theorems here and refer the reader to Sections 5 and 6 for a detailed and formal exposition.

The Max-Entropy Program. Consider a finite subset \( F \subseteq \mathbb{Z}^m \) of the integer lattice. Given a vector \( \theta \), the following max-entropy convex program solves for a probability distribution over \( F \) that has maximum entropy with expectation \( \theta \):

\[
\begin{align*}
\max & \quad \sum_{\alpha \in F} q_{\alpha} \log \frac{1}{q_{\alpha}}, \\
\text{s.t.} & \quad \sum_{\alpha \in F} q_{\alpha} \cdot \alpha = \theta, \\
& \quad \sum_{\alpha \in F} q_{\alpha} = 1, \\
& \quad q \geq 0.
\end{align*}
\]

While an explicit solution \( q^* \) to this problem would require at least \( \Omega(|F|) \) (possibly exponential) space, by analyzing the dual program to (1) one can prove that there exists a vector \( \gamma \in \mathbb{R}^m > 0 \) that satisfies \( q^*_{\alpha} \propto \prod_{i=1}^m \gamma_i^{\alpha_i} \) (see [38, 43]). However, there is no guarantee that the number of bits required to store \( \gamma \) is small (poly(\( m \))) and in fact can be infinite.

From the work of [38], it follows that the efficient computability of such representations essentially relies on the two following conditions:

1. a polynomial bound on the bit complexity of \( \gamma \) and
2. the existence of an efficient counting oracle for \( F \).

To explain the latter we consider the function \( g_F(x) := \sum_{\alpha \in F} x^{\alpha} \) where \( x^{\alpha} := \prod_{i=1}^m x_i^{\alpha_i} \). A counting oracle for \( F \) is then an algorithm which, given a positive \( x > 0 \), outputs \( g_F(x) \). Note in particular, that when we plug in the all-ones vector we obtain \( g_F(1) = |F| \). In fact, [38] show that existence of efficient counting oracles is also necessary for max-entropy computations, as being able to find the optimal value of the max-entropy program efficiently for a given family \( F \) implies (roughly) an efficient method for evaluating \( g_F(x) \). Thus, when computing max-entropy distributions one can assume without loss of generality that an oracle for answering queries \( x \mapsto g_F(x) \) is available.

Consequently, the problem of computing efficient representations of max-entropy distributions boils down to proving upper bounds on the bit complexity of optimal dual solutions, and is the main focus of this paper. The main result of [38] states that whenever the point \( \theta \) lies in \( P := \text{conv}(F) \) and also a ball of radius \( \eta > 0 \) centered at \( \theta \) is contained in \( P \), then the bit complexity of \( \gamma \) is bounded by \( O(\log |F|/\eta) \). In particular, this bound deteriorates as \( \eta \to 0 \) and it can be shown that the bit complexity of \( \gamma \) goes to infinity as \( \eta \to 0 \).

However, in applications an arbitrarily good approximate solution to the max-entropy program suffices and, in the light of the discussion above, one may ask if we can compute a distribution that is \( \epsilon \)-close to the max-entropy distribution in time that is proportional to \( \log 1/\epsilon \).

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1 For brevity, in this section we assume that \( F \) has a diameter polynomial in \( m \), see Section 5 for details.

2 In fact [38] show that even approximate counting oracles (with suitable precision) suffice to compute max-entropy distributions. In this paper, we choose to work with exact counting oracles to improve readability, however the results extend straightforwardly to the approximate setting.
2.1 Computability of Maximum Entropy Distributions

Our main result answers the above question affirmatively under mild conditions on the polytope corresponding to \( \mathcal{F} \). In particular, we assume that this polytope has a low unary facet complexity, as formalized below\(^3\).

**Definition 2.1 (Informal; see Definition 5.1)** Let \( P \subseteq \mathbb{R}^m \) be a convex polytope with integer vertices. The **unary facet complexity** of \( P \), denoted by \( \text{fc}(P) \), is the smallest number \( M \in \mathbb{N} \), such that \( P \) can be described by linear inequalities with coefficients in the set \( \{-M, \ldots, 0, \ldots, M\} \).

This class is very general and includes most polytopes of interest. Consequently, as our first result, we close the problem of computability of max-entropy distributions for polytopes with polynomial unary facet complexity for all points \( \theta \in \text{conv}(\mathcal{F}) \), which was left open in the work of [38].

**Theorem 2.1 (Simplified; see Theorem 6.1)** Let \( \mathcal{F} \) be any finite subset of \( \mathbb{Z}^m \) and assume that the unary facet complexity of \( P := \text{conv}(\mathcal{F}) \) is polynomial in \( m \). Then, there exists an algorithm such that given an evaluation oracle for \( g_{\mathcal{F}} \), a \( \theta \in P \) and an \( \varepsilon > 0 \), computes a vector \( y \in \mathbb{R}^m \) with \( \|y\| \leq \text{poly}(m, \log \frac{1}{\varepsilon}) \) such that
\[
\|q^\gamma - q^\alpha\|_1 < \varepsilon,
\]
where \( q^\gamma \) is the optimal solution to (1) and \( q^\alpha \) is a distribution over \( \mathcal{F} \) defined as
\[
q^\alpha := \frac{e^{(\alpha,y)}}{\sum_{\beta \in \mathcal{F}} e^{(\beta,y)}}
\]
for some \( \alpha \in \mathcal{F} \). The running time of this algorithm is polynomial in \( m \) and \( \log \frac{1}{\varepsilon} \).

Note that above, a vector \( \gamma \) given by \( \gamma_i = e^{y_i} \) for all \( i \in [m] \), approximately satisfies \( q^\gamma \propto \gamma^\alpha \). Thus, such a \( \gamma \) has a polynomial bit complexity as the norm of \( y \) is polynomially bounded. This result allows us to sample from a distribution that is \( \varepsilon \)-close in total variation distance to the optimal distribution, hence, adding at most an \( \varepsilon \)-error to any further use of such samples. We remark that, as in [38], a suitable approximate evaluation oracle for \( g_{\mathcal{F}} \) suffices; we omit the details.

There are many natural examples of families \( \mathcal{F} \) for which Theorem 2.1 yields polynomial running time bounds, i.e., when the polytope \( P \) has a small unary facet complexity and an efficient counting oracle is available for \( \mathcal{F} \). It turns out that many practically relevant cases fall into this category – below we discuss two classes of examples: combinatorial polytopes and polytopes arising from supports of polynomials. For both, one condition for efficient computability is easily satisfied, while the second requires additional assumptions.

**Combinatorial Polytopes.** We consider families of subsets of \([m]\) that represent certain combinatorial structures. In our setting these can be seen as subsets of the hypercube \( \mathcal{F} \subseteq \{0,1\}^m \). For instance let \( \mathcal{F} \subseteq \{0,1\}^m \) be a family of independent sets of a matroid; then the convex hull of \( \mathcal{F} \) has the form
\[
P = \text{conv}(\mathcal{F}) = \{x \in [0,1]^m : \forall S \subseteq [m] \sum_{i \in S} x_i \leq r(S)\},
\]
where \( r : \{0,1\}^m \rightarrow \mathbb{N} \) is the rank function of the corresponding matroid [36]. Note that the coefficients in the linear inequalities describing \( P \) are all in \( \{0,1\} \) hence, trivially, \( \text{fc}(P) = 1 \). Similarly, consider the matroid intersection problem – if the family \( \mathcal{F} \) arises as the set of common bases (or common independent sets) of two matroids then their convex hull is the intersection of two polytopes (as in [2]) [15] and hence a constant bound on the unary facet complexity follows. For many

\(^3\)Note that this condition is different from the **binary facet complexity** defined in [17]. Incidentally, the unary facet complexity being polynomially bounded is also sufficient for a poly-sized ball to be contained in \( P \) and was used in a different context in [38].
combinatorial polytopes (such as the matching polytope) and totally unimodular polytopes, the facets are described by inequalities with small integer coefficients (see [36] for numerous examples).

The search for efficient counting oracles for such polytopes is a large and important area of research [24, 34]. Counting oracles are known for several classes of matroids, such as uniform and partition matroids, spanning tree matroids and, more generally, regular matroids. An approximate counting oracle for perfect matchings in the case when the underlying graph is bipartite follows from the work by [25], however for non-bipartite graphs it is an open problem to construct such oracles. Thus, while not all combinatorial polytopes are known to have polynomial time counting oracles, Theorem 6.1 asserts that progress on the underlying counting problems implies progress on the computability of max-entropy distributions.

Polytopes arising from polynomials. An important set of examples where Theorem 2.1 applies comes from the study of polynomials. Given a polynomial $p \in \mathbb{R}[x_1, \ldots, x_m]$ with non-negative coefficients, we can write it in the form $p(x) = \sum_{\alpha \in \mathbb{N}^m} p_{\alpha} x^{\alpha}$ where $x^{\alpha} := \prod_{i=1}^m x_i^{\alpha_i}$ and $p_{\alpha} \geq 0$ for all $\alpha \in \mathbb{N}^m$. Then the support of $p$ is the finite set $\mathcal{F} \subseteq \mathbb{N}^m$ consisting of all $\alpha$ such that $p_{\alpha} \neq 0$. Note that an evaluation oracle for $\mathcal{F}$ is then essentially an algorithm to compute $p(x)$ given $x > 0$.

Thus, given appropriate access to a polynomial implies a counting oracle for its support. On the other hand, bounds on the facet complexity of its Newton polytope\(^5\) do not follow generally, and depend on the example at hand.

For instance, consider the family of real stable polynomials (see Section 8.1 for a definition). It is known [11, 9] that if $\mathcal{F}$ is the support of a real stable polynomial then the polytope $\text{conv}(\mathcal{F})$ can be described by inequalities whose coefficients lie in $\{-1, 0, 1\}$; thus, for every such $\mathcal{F}$ the unary facet complexity is $\text{fc}(\text{conv}(\mathcal{F})) = 1$. This, in particular, applies to supports of determinantal polynomials of the form $p(x) = \det(\sum_{i=1}^m x_i A_i)$ where $A_i \in \mathbb{R}^{d \times d}$ are symmetric positive semidefinite matrices and thus implies an $O(d^3)$ time counting oracle for them.

2.2 Bit Complexity of Entropy Maximization

The proof of Theorem 2.1 is based on a structural result on the bit complexity of close-to-optimal dual solutions and comprises the technical core of the paper. Before we state it, let us introduce the dual problem to (1). For $\theta \in \text{conv}(\mathcal{F})$ and any $y \in \mathbb{R}^m$ define $h(\theta, y) := \log \left( \sum_{\alpha \in \mathcal{F}} e^{(\alpha - \theta, y)} \right)$. The dual problem is parametrized by $\theta$ and is to minimize $h(\theta, y)$ over all $y \in \mathbb{R}^m$, i.e.,

$$g(\theta) := \inf_{y \in \mathbb{R}^m} h(\theta, y) := \inf_{y \in \mathbb{R}^m} \log \left( \sum_{\alpha \in \mathcal{F}} e^{(\alpha - \theta, y)} \right). \quad (3)$$

Theorem 2.2 (Main Structural Result - Simplified; see Theorem 5.1) There exists a bound $R = R(m, \log \frac{1}{\epsilon}) \in \mathbb{R}_{>0}$, polynomial in $m$ and $\log \frac{1}{\epsilon}$ such that for every set $\mathcal{F} \subseteq \mathbb{Z}^m$ with unary facet complexity of $\text{conv}(\mathcal{F})$ polynomially bounded\(^4\), for every $\epsilon > 0$ we have

$$\forall \theta \in \text{conv}(\mathcal{F}) \exists y \in B(0, R) \quad h(\theta, y) \leq g(\theta) + \epsilon,$$

where $h$ and $g$ are defined as in (3).

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\(^4\)More precisely, for this all the coefficients would need to be 1. However, generalized max-entropy programs and such counting oracles are studied in Section 6.\(^5\)The Newton polytope of $p(x) = \sum_{\alpha} p_{\alpha} x^{\alpha}$ is the convex hull of its support, i.e., $\text{conv}\{\alpha \in \mathbb{N}^m : p_{\alpha} \neq 0\}$.\(^6\)Specifically, we require the bound on the unary complexity to be $m^c$ for some constant $c > 0$ fixed in advance.
Here $B(0, R)$ denotes the unit $\ell_2$-ball of radius $R$ around the origin. Note that the bound we provide is *uniform* over the whole polytope $P$, meaning that the radius $R$ does not depend on $\theta$ – this is crucial for the subsequent applications.

Below we present a result complementing Theorem 2.2 that asserts that the conclusion of Theorem 2.2 does not always hold and that some assumptions on $F$ are necessary. More precisely, for certain sets $F$, the length of $y$ which yields a solution $\varepsilon$-close to the optimal one cannot depend logarithmically on $\log \frac{1}{\varepsilon}$ as in Theorem 2.2.

**Theorem 2.3 (Lower Bound)** For every bound $R = R\left(m, \log \frac{1}{\varepsilon}\right) \in \mathbb{R}_{>0}$, polynomial in $m$ and $\log \frac{1}{\varepsilon}$ there exists an $m \in \mathbb{N}$, an $\varepsilon > 0$ and a set $F \subseteq \mathbb{Z}^m$ with $\|\alpha\| \leq O\left(m^{3/2}\right)$ for every $\alpha \in F$ such that

$$\exists_{\theta \in \text{conv}(F)} \forall_{y \in B(0,R)} h(\theta, y) > g(\theta) + \varepsilon.$$ 

The above result relies on existence of “flat” $0$–$1$ polytopes by Alon and Vu [2], i.e., full-dimensional polytopes whose vertices are in $\{0,1\}^m$ and which fit between two $e^{-\Theta(m \log m)}$-close hyperplanes. It is an intriguing question whether a similar lower bound as in Theorem 2.3 holds also for families $F \subseteq \{0,1\}^m$.

### 2.3 Applications and Connections

Max-entropy distributions appear in various guises in many distinct fields of mathematics, physics and computer science. As a consequence, we are able to present a host of disparate looking results from a unified point of view and sometimes obtain new corollaries.

In machine learning, Roughgarden and Kearns [35] consider various computational problems involving graphical models and prove (using the result of [38]) that if the underlying counting problem is polynomial time solvable for a given family of graphical models, then the max-entropy problem on the same family of graphical models is solvable in polynomial time for all polynomially feasible instances [8]. A simple consequence of our results is that the polynomial feasibility assumption can be omitted for a large class of graphical models, broadening the scope of the results of [35].

In optimization, the dual (3) of the max-entropy program can be viewed as an instance of geometric programming. Geometric programs have been widely studied and appear in a number of applications; see the survey by Boyd *et al.* [10]. The crucial difference is that in the standard description of a Geometric Program the posynomial (the objective) is given explicitly as a list of monomials and thus the problem can be solved directly, e.g., using a variant of the Newton method, because both the primal and dual program are expressed using a polynomial number of variables. However it is not clear how to extend such algorithms to the case when the objective is not given explicitly, but only as an evaluation oracle, and could consist of exponentially many terms. Our results demonstrate that certain classes of such exponentially sized geometric programs can also be solved in polynomial time.

Now we consider some of the less obvious applications of our results to problems of current interest in TCS and mathematics. These problems require a fair bit of notation to describe and also require more work to reinterpret them in the language of max-entropy distributions. For these reasons, below we only provide a list and defer their discussion to Section 4.

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7A variant of Theorem 2.2 which applies to families $F \subseteq \mathbb{N}^m$ that are spanning trees of undirected graphs was obtained in [4] and, for matroids (and more generally – jump systems) in [3]. However, these arguments heavily rely on the combinatorial fact that such families admit greedy optimization algorithms and do not seem to generalize beyond this setting.

8The polynomial feasibility assumption essentially says that the corresponding marginal vector $\theta$ should be at least inverse-polynomially far from the boundary of the marginal polytope.
• **Bounds for the matrix scaling problem.** For the doubly-stochastic (and more general) matrix scaling problem we prove polynomial bit complexity bounds on the scaling factors using Theorem 5.1. Such bounds are then useful in the study of computational aspects of this problem (see the recent work of [11, 12]).

• **Computability of recent continuous relaxations for counting and optimization problems.** The convex relaxations for counting and optimization problems involving polynomials studied in [40, 3] can be solved in polynomial time whenever the supports of the underlying polynomials have small unary facet complexity.

• **Computability of worst-case Brascamp-Lieb constants.** We consider the question of computability of Brascamp-Lieb constants, and show that in the rank-1 regime one can compute worst-case constants (over the Brascamp-Lieb polytope) in polynomial time.

**Organization of the rest of the paper.** The remainder of the paper is organized as follows. Section 3 is a technical overview of our results. In Section 4 we give an overview of our applications, their proofs can be found in Section 8. The proof of a generalized Theorem 2.2 is presented in Section 5, the proofs of Theorems 2.1 and 2.3 appear in Sections 6 and 7 respectively. The proofs of the applications discussed in Section 4 can be found in Section 8.

### 3 Technical Overview

Since Theorem 2.2 is the technical core of the paper we start by discussing its proof and then show how Theorem 2.1 can be derived from it.

**Theorem 2.2 – previous work and possible approaches.** In the context of Theorem 2.2, the result of [38] can be restated as follows: whenever a ball of radius \( \eta > 0 \) centered at \( \theta \) is contained in \( P := \text{conv}(\mathcal{F}) \) the optimal solution \( y^* \) of (3) exists and its length \( \|y^*\| \) is bounded by \( O\left(\frac{m \log m}{\eta}\right) \).

The proof is simple: the term \( O(m \log m) \) above is really a bound on \( \log |\mathcal{F}| \), and the \( \frac{1}{\eta} \) term comes from the fact that after scaling by \( \log |\mathcal{F}| \), the optimal solution \( y^* \) of the dual program belongs to the polar of the radius-\( \eta \) ball around \( \theta \). (The polar of radius-\( \eta \) ball is itself a ball of radius \( 1/\eta \).)

While it is not clear whether the dependency on \( \eta \) is optimal, one can see that \( \|y^*\| \to \infty \) when \( \theta \) tends to the boundary of \( P \) and \( y^* \) might not even exist when \( \theta \) is on the boundary. For this reason, Theorem 2.2 cannot be derived from the results of [38]. Moreover, any approach to bound the norm of the optimal solution would be unsuccessful; this is why we focus on approximately optimal solutions.

One could consider the following two approaches to prove Theorem 2.2 that try to take advantage of the above mentioned bound: centering and projection. Both are based on slightly moving \( \theta \) to a new point \( \theta' \) so that the result by [38] is applicable to \( \theta' \) and then reasoning that the small shift does not affect the optimal dual solution. Centering is based on moving the point more towards the interior, for example, by taking the centroid of the polytope \( P \) (which is far from the boundary) and taking a small step from \( \theta \) towards \( \theta' \). One can then prove that \( \theta' \) is well in the interior of the polytope, and hence a suitable bound for it follows. The second idea is based on projections: start with any point \( \theta \), if \( \theta \) is already (inverse-polynomially) far from the boundary of \( P \), then the result of [38] implies a suitable bound. Otherwise, project the point \( \theta \) onto the closest facet of \( P \) and continue recursively. By doing this, either the resulting new point \( \theta' \) will end up in a vertex of \( P \), or on a lower-dimensional face of \( P \), where it is far from the boundary.

In both approaches, proving the result for the new point \( \theta' \) is easy, given [38]; what remains is to bound the error introduced when moving from \( \theta \) to \( \theta' \). Proving such a bound on the error turns out to be a non-trivial task. In the case of centering one has to pick an appropriate direction.
along which the point \( \theta \) is moved. Similarly, the first challenge in the projection approach is to even define a suitable “projection operator” on a facet which would behave as expected and do not cause the point \( \theta \) to land outside of the polytope \( P \) (as Euclidean projections might do). An issue which concerns both approaches is to deal with the behavior of the function \( g \) close to the boundary, where it can be shown to be non-Lipschitz and hence very susceptible to local perturbations. More precisely: moving from \( \theta \) to \( \theta' \) guarantees that the gradient \( \nabla_y h(\theta, y) \) at \( y = y^* \) – the dual optimal solution at \( \theta' \) – is small, however it is a challenge to derive a guarantee on the gap \( g(\theta) - h(\theta, y^*) \) as the function \( y \mapsto h(\theta, y) \) is not strongly convex\(^9\). In fact, even a weaker claim that \( g(\theta) \approx g(\theta') \) does not follow from \([78]\), since the convexity argument (as in Lemma 7.5 in the full version of \([78]\)) only allows one to prove that \( g(\theta') \geq g(\theta) - O(\varepsilon) \), where \( \varepsilon \) denotes the distance between \( \theta \) and its “centering” \( \theta' \), but not in the opposite direction, as required.

Our proof of Theorem 2.2 is based on a purely geometric reasoning and bypasses the above obstacles by working entirely in the dual space (working in which we believe is necessary). This allows us to appropriately capture the geometry of sub-level sets of \( h(\theta, y) \) and as a consequence, understand effects of seemingly large perturbations in \( y \) which lead to only small changes in the function value. When tracked in the primal domain, the proof resembles the “centering” idea, however the implicit direction to move \( \theta \) along is not easy to come up with (or analyze) just from the primal perspective.

**Theorem 2.2 – proof overview.** At a high level, in the proof we consider the optimal dual solution \( y^* \) and first identify a vertex \( \alpha^* \) such that \( y^* \) belongs to \( C_{\alpha^*} \) – the normal cone at \( \alpha^* \). Subsequently, a projection operation of \( y^* \) with respect to the cone \( C_{\alpha^*} \) is used to find a vector \( y^0 \) – a witness for a short and close to optimal solution of the dual problem. The proof can be decomposed naturally into the following three steps, which we subsequently explain in more detail.

1. **Identify a “good” basis for the dual space with respect to \( \theta \).**
2. **Truncate the optimal dual solution with respect to the basis in (a).**
3. **Establish a bound on the length of the truncated solution.**

In what follows, we assume for simplicity that the polytope \( \text{conv}(\mathcal{F}) \) is full-dimensional. Given any \( \theta \in P \) and a point \( y^* \in \mathbb{R}^m \) which satisfies \( h(\theta, y^*) \leq g(\theta) + \varepsilon / 2 \) (i.e., is close to optimal\(^10\)) we aim to find a point \( y^0 \) whose length is polynomial in \( m \) and \( \log 1 / \varepsilon \) such that \( h(\theta, y^0) \leq h(\theta, y^*) + \varepsilon / 2 \).

**Step (a).** We first identify a subset \( I_0 \subseteq I \) such that \( \{a_i : i \in I_0\} \) is a basis of \( \mathbb{R}^m \) and \( y^* \) is expressed as a nonnegative linear combination in this basis, i.e., \( y^* = \sum_{i \in I_0} \beta_i a_i \) with \( \beta \geq 0 \). The basis \( I_0 \) is chosen as a basis of tight constraints at a point \( \alpha^* \in \mathcal{F} \), i.e., \( \alpha^* \) satisfies \( (a_i, \alpha^*) = b_i \) for \( i \in I_0 \). This follows by selecting an \( \alpha \) that maximizes \( \langle \alpha, y^* \rangle \) over all \( \alpha \in \mathcal{F} \) and invoking Farkas’ lemma and Caratheodory’s theorem. This part of the reasoning does not make any assumptions on the polytope and only relies on the convexity of \( P \).

**Step (b).** Subsequently we prove that the point \( y^0 = \sum_{i \in I_0} \min(\Delta_i, \beta_i) a_i \) satisfies the claim stated above, for a suitable choice of \( \Delta \), polynomial in the considered parameters. The bound \( h(\theta, y^0) \leq h(\theta, y^*) + \varepsilon / 2 \) is proved by replacing \( \beta_i \) by \( \min(\Delta, \beta_i) \) one by one and showing that the value \( h(\theta, \cdot) \) does not increase by more than \( \varepsilon / 2m \). This relies on a careful analysis of the effect such a perturbation has on the function and crucially uses the fact that the coefficients \( \beta_i \) for \( i \in I_0 \) are nonnegative. Most importantly, we rely on the fact that the coefficients of the inequalities defining

\(^9\)In the simple case when \( \mathcal{F} := \{0, 1\} \), the function \( g : [0, 1] \to \mathbb{R} \) is of the form \( g(\theta) = -\theta \log \theta - (1 - \theta) \log (1 - \theta) \). One can see that on every interval \( (\varepsilon, 1 - \varepsilon) \) for \( \varepsilon > 0 \) the function \( g \) is Lipschitz, but it is not Lipschitz on \([0, 1]\).

\(^10\)One can again consider the case of \( \mathcal{F} := \{0, 1\} \) – the function \( h(0, y) \) is then of the form \( h(0, y) = \log (1 + e^y) \).

Note that \( h(0, y) \) is a convex function of \( y \), but \( \frac{d}{dy} h(0, y) \to 0 \) whenever \( y \to \pm \infty \), hence the function is essentially “flat” at infinity, and not strongly convex.

\(^11\)Note that there might not exist a point \( y^* \) such that \( g(\theta) = h(\theta, y^*) \) when \( \theta \) is on the boundary of \( P \); that is why we allow a slack of \( \varepsilon / 2 \).
$P$ are integral; and hence for any point $\alpha \in F$ which does not lie on a facet $\langle a_i, x \rangle = b_i$ for some $i \in I_0$ we have $\langle \alpha^* - \alpha, a_i \rangle \geq 1$.

**Step (c).** To bound the length of $y^\circ$ note first that all the vectors $\{a_i\}_{i \in I_0}$ are short, i.e., $\|a_i\|_\infty \cdot m \leq fc(P) \cdot m = \text{poly}(m)$, because we assume that the unary facet complexity of $P$ is polynomially bounded. Further, from the triangle inequality, we obtain $\|y^\circ\| \leq m \cdot \Delta \cdot \text{poly}(m) = \text{poly}(m, \log 1)$. Thus, we conclude the proof of Theorem 2.2.

**Theorem 2.1 – proof overview.** Given $\theta \in P$ and $y \in \mathbb{R}^m$ we first relate the closeness of $q^y$ to the max-entropy distribution $q$ with the suboptimality gap $h(\theta, y) - g(\theta)$. For this, even though we would like to bound the $\ell_1$ distance between $q^y$ and $q^\star$, the right distance function to consider turns out to be the KL-divergence $KL(p, q) := -\sum_{\alpha \in F} p_\alpha \log \frac{p_\alpha}{q_\alpha}$. This is the case as one can show that $h(\theta, y) - g(\theta)$ is equal to $KL(q^\star, q^y)$. Further, using Pinsker’s inequality ($\|q - p\|_1 \leq \sqrt{2 \cdot KL(p, q)}$) we can recover a bound on the $\ell_1$ distance.

Consequently, to find a $y \in \mathbb{R}^m$ whose induced distribution $q^y$ is close to the max-entropy distribution $q^\star$ it is enough to ensure that the dual suboptimality gap $h(\theta, y) - g(\theta)$ is small. Hence, the problem boils down to finding an approximate solution $y$ to the dual problem at $\theta$. This can be accomplished using the ellipsoid algorithm; the crucial part of the implementation is to provide a bounding box, i.e., a bound on the norm of the solution we are looking for — such a bound follows from Theorem 2.2.

**Theorem 2.3 – proof overview.** The proof of Theorem 2.3 is based on existence of so called flat 0-1 polytopes. These are polytopes of the form $\text{conv}\{\alpha_0, \ldots, \alpha_m\}$ with $\alpha_0, \ldots, \alpha_m \in \{0, 1\}^m$ such that the distance from $\alpha_0 = 0$ to the affine subspace $H$ generated by $\alpha_1, \ldots, \alpha_m$ is exponentially small. Existence of such configurations was proved in [2]. Given such a polytope we consider the lattice generated on $H$ by the points $\alpha_1, \ldots, \alpha_m$ and construct a new polytope by taking a certain finite subset of such lattice points and the point 0. The vertices of the new polytope are still integral and have relatively small entries (polynomial in $m$). Moreover, the projection of 0 onto $H$ lies within the opposite facet. The family $F$ is defined to be all the vertices of the newly constructed polytope, $\theta$ is chosen to be 0 and we pick $\varepsilon \approx e^{-m}$.

To prove that for every short $y \in \mathbb{R}^m$ the gap between $h(\theta, y) - g(\theta)$ is significant we consider the gradient $\nabla_y h(\theta, y)$. Intuitively, if the gradient is large (in magnitude) at a point $y$, then $y$ cannot be an approximately optimal solution, hence it is enough to show that every short vector $y$ admits a suitably long gradient. For this one can show that the gradient at $y$ is given by $\theta - y^\circ$ where $y^\circ$ is the expectation of a distribution defined by $y$.

In order to make $\theta - y^\circ$ $\varepsilon$-close to $\theta = 0$ one has to ensure that $\langle 0, y \rangle - \langle \alpha, y \rangle \gtrsim \Omega(1)$ for all $\alpha \in F \setminus \{0\}$. However, by introducing an auxiliary optimization problem (see Fact 7.1) we show that this can happen only when $\|y\|$ is roughly, inverse-proportional to the distance from 0 to $\text{conv}(F \setminus \{0\})$. Since the distance is exponentially small, we arrive at a lower-bound on $\|y\|$.

## 4 Discussion of Applications

**Bounds for the matrix scaling problem.** Consider the $(r, c)$—matrix scaling problem, where one is given a nonnegative square matrix $A \in \mathbb{R}^{n \times n}$ and two vectors $r, c \in \mathbb{N}^n$ (with $\|r\|_1 = \|c\|_1$) and the goal is to find a scaling: two positive vectors $x, y \in \mathbb{R}_{>0}^n$ such that for $B$ defined as $B := XAY$ (with $X = \text{Diag}(x)$ and $Y = \text{Diag}(y)$) it holds that $B \mathbf{1} = r$ and $B^\top \mathbf{1} = c$ where $\mathbf{1} \in \mathbb{R}^n$ is the all-one vector. In other words, we want the row-sums of the matrix $B$ to be equal to $r$ and column sums to be equal to $c$.

For applications of matrix scaling we refer to [11, 13], where recently fast algorithms for matrix scaling were recently derived. Here let us only mention the problem of approximating the permanent.
of a nonnegative matrix. One can show that \( \text{per}(XAY) = \text{per}(A) \cdot \prod_{i=1}^{n} x_i \cdot \prod_{i=1}^{n} y_i \) and thus, by scaling (with \( r = c = 1 \)), one can reduce the problem of computing the permanent of a nonnegative matrix to the problem of computing the permanent of a doubly-stochastic matrix which is better understood. In particular several useful bounds, such as the Van der Waerden’s bound (see [19]), and others [20], are known for the permanent of doubly stochastic matrices.

One can prove that if such a scaling exists even asymptotically (i.e., a sequence of scalings exists such that they satisfy the scaling condition in the limit), then it can be recovered from the optimal solution to the following convex program

\[
\inf_{z \in \mathbb{R}^n} \frac{1}{2} \sum_{i=1}^{n} r_i \log \left( \sum_{j=1}^{n} A_{i,j} e^{z_j} \right) - \langle c, z \rangle; \tag{4}
\]

see [19] [26] [14]. Indeed, the scaling is recovered as \( x_i := r_i \cdot \left( \sum_{j=1}^{n} A_{i,j} e^{z_j} \right)^{-1} \) for \( i = 1, 2, \ldots, n \), and \( y_j := e^{z_j} \) for \( i = 1, 2, \ldots, n \), where \( z^* \) stands for the optimal (or approximately optimal) solution to (4). The question which arises naturally is: does the optimal (or approximately optimal) scaling \((x, y)\) have polynomial bit complexity? Or in other words: can we prove that the vector \( z^* \) has polynomially bounded entries: \( \|z\|_\infty = \text{poly}(n, L_A) \)? Here \( L_A \) denotes the bit complexity of the matrix \( A \), i.e., \( L_A := \max_{i,j\in[n]} |\log A_{i,j}| \).

We interpret the optimization problem (4) as an instance of entropy maximization which will allow us to apply Theorem 2.2 and deduce appropriate bounds. To this end we rewrite the problem (4) as

\[
\inf_{z \in \mathbb{R}^n} \frac{1}{2} \log \prod_{i=1}^{n} \left( \sum_{j=1}^{n} A_{i,j} e^{z_j} \right)^{r_i} - \langle c, z \rangle.
\]

Hence, there is a polynomial \( p \in \mathbb{R}_{\geq 0}[x_1, x_2, \ldots, x_n] \) such that the above optimization problem is equivalent to

\[
\inf_{z \in \mathbb{R}^n} \log \frac{p(e^{z_1}, e^{z_2}, \ldots, e^{z_n})}{e^{\langle z, c \rangle}} = \inf_{z \in \mathbb{R}^n} \log \left( \sum_{\alpha} p_{\alpha} e^{\langle \alpha - c, z \rangle} \right),
\]

where the summation is over the support of \( p \), which in this case (if, say, all the entries \( A_{i,j} \) are positive) is equal to \( F = \left\{ \alpha \in \mathbb{N}^n : \sum_{j=1}^{n} \alpha_j = \|r\|_1 \right\} \). Hence (4) is essentially a generalized max-entropy program over the set \( F \) with expectation \( e \in \text{conv}(F) \). This allows us to analyze the scaling problem by applying Theorem 5.1 (a generalization of Theorem 2.2) and prove polynomial bounds on the bit complexity. We refer to Section 8.1 for details.

**Computability of recent continuous relaxations for counting and optimization problems.** The recent works [10] and [9] study counting and optimization problems involving polynomials and provide approximation algorithms for them. In the discussion below, for concreteness, we follow [10] – the setting of [3] is similar.

Consider a multiaffine polynomial \( p \in \mathbb{R}[x_1, x_2, \ldots, x_m] \) with nonnegative coefficients and a family of sets \( B \subseteq 2^{[m]} \). Let us denote by \( p_{\alpha} \) the coefficient of the monomial \( x^\alpha := \prod_{i=1}^{m} x_i^{\alpha_i} \) in \( p \). Then the problems considered are to compute

\[
p_B := \sum_{\alpha \in B} p_{\alpha} \quad \text{and} \quad p_B^{\max} := \max_{\alpha \in B} p_{\alpha}. \tag{5}
\]

\(^{12}\)The polynomial in this problem is provided as an evaluation oracle. Similarly the family \( B \) is given implicitly in the input, as a separation oracle for the convex hull of \( B \).
One particular application where one is required to solve such problems is when dealing with constrained Determinantal Point Processes \[ \text{27, 13}. \] There, the polynomial \( p(x) \) is of the form 
\[
\sum_{S \subseteq [m]} \det(L_{S,S})x^S,
\]
where \( L \in \mathbb{R}^{m \times m} \) is a PSD matrix and \( L_{S,S} \) denotes the submatrix of \( L \) corresponding to rows and columns in \( S \subseteq [m] \). Solving such counting and optimization problems allows one to draw samples from constrained DPPs, i.e., distributions where the probability of a set \( S \) is proportional to \( \det(L_{S,S}) \) when \( S \in \mathcal{B} \) and is 0 when \( S \notin \mathcal{B} \). Such distributions have various interesting uses in data summarization and fair and diverse sampling (see \[ \text{13} \]).

To tackle the counting problem of computing \( p_B \) (as in \( \text{(5)} \)) the following relaxation is considered:

\[
\log \operatorname{Cap}_B(p) := \sup_{\theta \in P(\mathcal{B})} \inf_{x > 0} \frac{p(x)}{\prod_{i=1}^m x_i^{\theta_i}}. \tag{6}
\]

The relaxation for maximization is derived as a small modification of the above, hence we focus on \( \operatorname{Cap}_B(p) \) in this discussion. \( \operatorname{Cap}_B(p) \) approximates \( p_B \) provably well under the assumption that (roughly) the polynomial \( p \) is real stable (see \[ \text{12} \]) and that the family \( \mathcal{B} \) has a matroid structure. However, the question of efficient computability of these relaxations was not established in \[ \text{40} \]. The results of this paper allow us to deduce polynomial time algorithms for this relaxation in a fairly general setting.\[ \text{13} \] We interpret this relaxation as a variant of a max-entropy program which then allows us to apply Theorem \[ 2.1 \] to reason about its computability.

After taking the logarithm and replacing the variables \( x_i > 0 \) by \( e^{y_i} \) with \( y_i \in \mathbb{R} \) for \( i = 1, 2, \ldots, m \) in \( \operatorname{Cap}_B(p) \) we get

\[
\log \operatorname{Cap}_B(p) = \sup_{\theta \in P(\mathcal{B})} \inf_{y \in \mathbb{R}^m} \log \left( \sum_{\alpha} p_\alpha e^{\langle y, \alpha - \theta \rangle} \right),
\]

where the summation in the inner optimization problem runs over \( \alpha \in \operatorname{supp}(p) \), i.e., the set of all monomials \( \alpha \in \mathbb{N}^m \) whose coefficient \( p_\alpha \) is non-zero. Note that the inner optimization problem is the dual of a generalized max-entropy program, as in \[ \text{35} \]. Hence, the inner optimization problem searches for a probability distribution over monomials of the polynomial \( p \), whose expectation is \( \theta \) and has the smallest possible KL-distance to the distribution given by the coefficients of \( p \).

By taking into account the outer optimization over \( \theta \), one can see that \( \log \operatorname{Cap}_B(p) \) minimizes the KL-distance of a distribution over monomials of \( p \) restricting its expectation to be in the convex hull of \( \mathcal{B} \). This makes it suitable to apply Theorem \[ 6.1 \] (a generalization of Theorem \[ 2.1 \]) and deduce polynomial time computability. Finally, we note that having a max-entropy solver for all \( \theta \) is crucial here. If \( \mathcal{F} \subseteq \mathbb{N}^m \) denotes the support of \( p \), then for any \( \theta \notin \operatorname{conv}(\mathcal{F}) \) the value of the entropy maximization problem is \( -\infty \). Hence, in the outer optimization problem, the variable \( \theta \) is constrained to be in \( P(\mathcal{B}) \cap \operatorname{conv}(\mathcal{F}) \). Thus, the optimal solution \( \theta^* \) of such an optimization problem might lie at the boundary of \( \operatorname{conv}(\mathcal{F}) \) or very close to it, hence the result of \[ \text{35} \] does not yield polynomial bit complexity bounds in this setting. For details we refer the reader to Section \[ 8.2 \].

Computability of worst-case Brascamp-Lieb constants. Brascamp-Lieb inequalities are an ultimate generalization of many inequalities used in analysis and all of mathematics, such as the Hölder inequality and Loomis-Whitney \[ \text{12, 28} \]. Recently, these inequalities have been studied from the computational point of view \[ \text{10} \], where the main problem is to compute quantities of the following form — called Brascamp-Lieb constants. Given a collection of matrices \( B = (B_1, B_2, \ldots, B_m) \)

\[ \text{14} \]While the computability result of \[ \text{3} \] for a similar relaxation requires the support of the polynomial to be a jump system (a generalization of a matroid), in this paper, we just need it to have low facet complexity.
with $B_j \in \mathbb{R}^{n_j \times n}$ and a point $p \in \mathbb{R}_0^m$, compute

$$BL(B, p) = \inf \left\{ \frac{\det \left( \sum_{j=1}^m p_j B_j^\top X_j B_j \right)}{\prod_{j=1}^m \det(X_j)^{p_j}} : X_j \in \mathbb{R}^{n_j \times n_j}, X_j \succeq 0, j = 1, 2, \ldots, m \right\}.$$  \hspace{1cm} (7)

The constant $BL(B, p)$ is non-zero whenever $p$ belongs to the so called Brascamp-Lieb polytope $P_B \subseteq \mathbb{R}^m$ (see Section 8.3 for details).

Recently [16] gave a method for calculating the Brascamp-Lieb constant in polynomial time when the vector $p$ is rational and given in unary. Note that this does not imply a polynomial time algorithm in the classical sense (when the vector $p$ is given in binary). Here we consider the special, but already non-trivial, case when the matrices are of rank 1; i.e., $B_j \in \mathbb{R}^{1 \times n}$ for every $j = 1, 2, \ldots, m$. By interpreting Brascamp-Lieb constants in the rank-1 regime as solutions to certain entropy-maximization problems we can deduce from Theorem 2.1 that they can be calculated, up to a multiplicative precision $\varepsilon > 0$, in time polynomial in $m$ and $\log \frac{1}{\varepsilon}$.

Moreover, our entropy interpretation along with Theorem 2.2 leads to an algorithm for computing worst-case Brascamp-Lieb constants over the whole Brascamp-Lieb polytope, i.e., $C(B) := \sup_{p \in P_B} BL(B, p)$. This quantity can be used as a universal constant (for any $p \in P_B$) for the reverse Brascamp-Lieb inequality [7]. To see how the optimization problem in (7) can be seen as entropy maximization let us denote the (only) row of $B_j$ by $v_j \in \mathbb{R}^n$ and replace the matrix variables $X_j \succeq 0$ by scalar variables $x_j \geq 0$

$$C(B) = \sup_{p \in P_B} \inf_{x \succ 0} \frac{\det \left( \sum_{j=1}^m p_j x_j v_j v_j^\top \right)}{\prod_{j=1}^m x_j^{p_j}}.$$  

Since $x \mapsto \det \left( \sum_{j=1}^m p_j x_j v_j v_j^\top \right)$ is a polynomial function\(^{14}\), we can conclude that the inner optimization problem for log $C(B)$ is the dual of a max-entropy program over the set of monomials of the determinantal polynomial, which in this case is $F \subseteq \{\alpha \in \{0, 1\}^m : \sum_{j=1}^m \alpha_j = n\}$. This observation along with concavity of the objective of the outer optimization problem (with respect to $p \in P_B$) allows us to deduce efficient computability of log $C(B)$ from Theorem 2.2.

5 Proof of the Bit Complexity Upper Bound

We start by reintroducing several notions and restating the structural result, as in Section 2 only simplified or informal definitions and theorem statements were given. Consider a finite subset $F \subseteq \mathbb{Z}^m$ of the integer lattice and a positive function $p : F \to \mathbb{R}_{>0}$. We will use $p_\alpha$ to denote the value of the function at a point $\alpha \in F$, thus treating $p$ as an $|F|$-dimensional vector with coordinates indexed by $F$. For any $\theta \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ we define the following generalized max-entropy program

\(^{14}\)We have from the Cauchy-Binet formula that $\det \left( \sum_{j=1}^m p_j x_j v_j v_j^\top \right) = \sum_{S \subseteq [m], |S| = n} \det(V_S V_S^\top) \prod_{i \in S} x_i$, where $V \in \mathbb{R}^{m \times n}$ is a matrix collecting $v_1, v_2, \ldots, v_m$ as its rows and $V_S$ is a submatrix of $V$ consisting of rows in $S$. 

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max \sum_{\alpha \in F} q_\alpha \log \frac{p_\alpha}{q_\alpha},
\text{s.t. } \sum_{\alpha \in F} q_\alpha \cdot \alpha = \theta,
\sum_{\alpha \in F} q_\alpha = 1,
q \geq 0. \tag{8}

In the case when \( p \) is normalized so that \( \sum_{\alpha \in F} p_\alpha = 1 \), \( \triangleq \) describes the distribution closest to \( p \) (in \( KL \)-distance), whose expectation is equal to \( \theta \). In the case when \( p \equiv 1 \) (corresponds to the uniform distribution) the above program asks simply for a max-entropy distribution with expectation \( \theta \), as in Section 2. Let us also extend the definition of the dual program, as introduced in Section 2 to capture general functions \( p \) not only \( p \equiv 1 \).

\[ g(\theta) := \inf_{y \in \mathbb{R}^m} h(\theta, y) := \inf_{y \in \mathbb{R}^m} \log \left( \sum_{\alpha \in F} p_\alpha e^{\langle \alpha - \theta, y \rangle} \right). \tag{9} \]

The following notion of facet complexity of a polytope plays an important role in our main result.

**Definition 5.1** Let \( P \subseteq \mathbb{R}^m \) be a convex polytope with integer vertices. Let \( M \in \mathbb{N} \) be the smallest integer such that \( P \) has a description of the form

\[ P = \{ x \in \mathbb{R}^m : \langle a_i, x \rangle \leq b_i, \ \text{for } i \in I \} \cap H \tag{10} \]

where \( I \) is a finite index set, \( a_i \in \mathbb{Z}^m \), \( \| a_i \|_\infty \leq M \) and \( b_i \in \mathbb{R} \) for \( i \in I \), and \( H \) is a linear subspace of \( \mathbb{R}^m \). Then we call \( M \) the unary facet complexity of \( P \) and denote \( \text{fc}(P) = M \).

Observe that in a description of a polytope \( P \) as in the definition above we could have also included \( H \) in the first term of (10), by adding \( \langle c, x \rangle \leq d \) and \( \langle c, x \rangle \geq d \), for every equation \( \langle c, x \rangle = d \) defining \( H \). However, the unary facet complexity defined as above might be significantly lower than one measured with respect to all (including equality) constraints and, as it turns out, this is the right measure to study the bit complexity of close to optimal solutions of (8).

To state our main result also a notion of bit complexity of a function \( p : F \to \mathbb{R}_{>0} \) is required. We denote it by \( L_p \) and define as

\[ L_p := \max_{\alpha \in F} | \log p_\alpha |. \]

Note that \( L_p \) is finite, because we assume that \( p_\alpha > 0 \) for every \( \alpha \in F \). It represents, roughly, the maximum number of bits required to store the binary representation of any \( p_\alpha \) (for \( \alpha \in F \)).

The last definition we need to state our structural result is the diameter of the set \( F \) (or equivalently \( P \)), it is simply \( \text{diam}(F) := \max\{ \| \alpha_1 - \alpha_2 \| : \alpha_1, \alpha_2 \in F \} \). \[ ^{15} \]

**Theorem 5.1 (Main Structural Result)** Let \( F \) be any finite subset of \( \mathbb{Z}^m \) and let \( d \in \mathbb{R}_{>0} \) be its diameter, let \( M \) be the unary facet complexity of \( \text{conv}(F) \). Then, for every function \( p : F \to \mathbb{R}_{>0} \) and for every \( \varepsilon > 0 \) there exists a number \( R > 0 \) which is polynomial in \( m, \log d, M, L_p \) and \( \log \frac{1}{\varepsilon} \)

such that

\[ \forall \theta \in P \exists y \in B(0, R) \quad h(\theta, y) \leq g(\theta) + \varepsilon, \]

\[ ^{15} \text{Importantly, the complexity measure } L_p \text{ is the maximum – not the total – number of bits required to store any of the coefficients. The latter is always at least } |F|, \text{ hence typically exponential in } m. \]

\[ ^{16} \text{In this paper we use } \|x\| \text{ to denote the Euclidean } \ell_2 \text{ norm of } x. \text{ This choice of a norm is by any means essential, as we tend to ignore factors polynomial in the dimension } m. \]
where \( h \) and \( g \) are defined as in \([5] \).

We start by a preliminary lemma which is then used in the proof of Theorem 5.1.

**Lemma 5.2** Let \( P \subseteq \mathbb{R}^m \) be a polytope \( P := \{ x \in \mathbb{R}^m : \langle a_i, x \rangle \leq b_i \text{ for } i \in I \} \cap H \), where \( I \) is a finite index set, \( H \subseteq \mathbb{R}^m \) is a linear subspace of \( \mathbb{R}^m \) and \( a_i \in H, b_i \in \mathbb{R} \) for \( i \in I \). Let \( y \in H \) be any vector, then there exists a vertex \( v \in P \) and a subset of the constraints \( I_0 \subseteq I \) of size \( |I_0| \leq \dim(H) \), such that \( \langle a_i, v \rangle = b_i \) for \( i \in I_0 \) and there exist non-negative numbers \( \{ \beta_i \}_{i \in I_0} \) satisfying

\[
\sum_{i \in I_0} \beta_i a_i = v.
\]

**Proof:** We start by picking \( v \in P \) which maximizes \( \langle x, y \rangle \) over \( x \in P \). Note that since \( P \) is a polytope (and is compact) such a maximum exists and, moreover without loss of generality we might assume that \( v \) is a vertex of \( P \). Given such a \( v \in P \) determine all inequalities which are tight at \( v \), i.e.,

\[
I^* = \{ i \in I : \langle a_i, v \rangle = b_i \}.
\]

Note that \( |I^*| \) might be arbitrarily large, not even polynomially bounded. We claim that there exist \( \{ \beta_i \}_{i \in I^*} \) with \( \beta_i \geq 0 \) for all \( i \in I^* \), such that

\[
y = \sum_{i \in I^*} \beta_i a_i.
\]

Suppose it is not the case. Then from the Farkas lemma, there exists a vector \( z \in \mathbb{R}^m \) such that:

\[
\forall_{i \in I^*} \quad \langle z, a_i \rangle \leq 0 \quad \text{and} \quad \langle z, y \rangle > 0.
\]

Note also that we may assume that \( z \in H \), by projecting \( z \) orthogonally onto \( H \) if necessary. Further, the above is true also for \( \delta \cdot z \) (in place of \( z \)) for an arbitrarily small \( \delta > 0 \). In other words we can take \( z \) of arbitrarily small norm. Hence we obtain that the cone

\[
C = \{ u \in H : \forall_{i \in I^*} \quad \langle u, a_i \rangle \leq 0 \}
\]

contains arbitrarily short vectors \( z \) with \( \langle z, y \rangle > 0 \). Note that since \( I^* \) is the collection of all inequalities tight at \( v \), it follows that every point in \( H \), which is sufficiently close to \( v \) and satisfies the inequalities in \( I^* \) belongs itself to \( P \). In other words there exists a \( \delta > 0 \) (might be exponentially small) such that

\[
(v + C) \cap B(v, \delta) \subseteq P,
\]

where \( v + C = \{ v + u : u \in C \} \) and \( B(v, \delta) = \{ x \in \mathbb{R}^m : \| x - v \| \leq \delta \} \). Combining this with our previous observation regarding the cone \( C \) it follows that there exists \( z \in H \) such that \( \mu := v + z \in P \) and \( \langle z, y \rangle > 0 \) and hence

\[
\langle \mu, y \rangle > \langle v, y \rangle.
\]

This contradicts our choice of \( v \in P \).

\[
\langle \alpha, y \rangle > \langle v, y \rangle.
\]

Knowing that \( y \) belongs to the cone generated by \( \{ a_i \}_{i \in I^*} \) we can apply Caratheodory’s theorem to reduce the number of nonzero coefficients in the resulting conic combination. Indeed there exist a set \( I_0 \subseteq I^* \), such that \( |I_0| \leq \dim(H) \) and non-negative \( \{ \beta_i \}_{i \in I_0} \) (possibly different \( \beta_i \)'s than obtained above) such that

\[
y = \sum_{i \in I_0} \beta_i a_i'.
\]
Proof of Theorem 5.1. Before we proceed with the argument let us first observe that one can assume without loss of generality that \(0 \in F\). This follows from the “shift invariance” of our problem. Indeed if we consider \(F\) and \(F' = F + \gamma = \{\alpha + \gamma : \alpha \in F\}\), then the corresponding functions \(h\) ad \(h'\) satisfy

\[
h(\theta, y) = h'(\theta + \gamma, y)
\]

for every \(y \in \mathbb{R}^m\). Hence, by shifting \(F\) by \(\gamma = -\alpha\) for some \(\alpha \in F\) we obtain an equivalent instance of our problem with \(0 \in F\). It follows in particular, that the affine subspace \(H\) on which \(P\) is full-dimensional is now a linear subspace of \(\mathbb{R}^m\).

Fix \(\theta \in P\) and let \(y^*\) be such that

\[
h(\theta, y^*) \leq g(\theta) + \frac{\varepsilon}{2}.
\]

Note that we may assume that \(y^* \in H\), by projecting it orthogonally onto \(H\) (which does not alter the value). Further, note that by denoting by \(a'_i \in H\) (for \(i \in I\)) the orthogonal projection of \(a_i\) onto \(H\), the polytope \(P\) can be equivalently written as

\[
P = \{x \in H : \langle a'_i, x \rangle \leq b_i\}.
\]

By applying Lemma 5.2 for \(y^*\) and \(P\) we obtain a vertex \(\alpha^* \in F\) and a subset of constraints \(I_0\) of size at most \(\dim(H) \leq m\), tight at \(\alpha^*\) such that

\[
y^* = \sum_{i \in I_0} \beta_i a'_i
\]

for some non-negative scalars \(\{\beta_i\}_{i \in I_0}\). We prove that by modifying the coefficients in the above conic combination we can obtain a point

\[
y' = \sum_{i \in I_0} \beta'_i a'_i
\]

(with \(\beta'_i \geq 0\) for \(i \in I_0\)) such that the norm of \(y'\) is small (polynomial in \(L_p, m, \log d, M\) and \(\log \frac{1}{\varepsilon}\)) and

\[
h(\theta, y') \leq h(\theta, y^*) + \frac{\varepsilon}{2} \leq g(\theta) + \varepsilon.
\]

Showing that will complete the argument. Let \(\Delta > 0\) be a certain number (to be specified later), polynomial in \(L_p\) and \(\log \frac{1}{\varepsilon}\). We define \(\beta'_i := \min(\Delta, \beta_i)\) and prove that the point \(y' = \sum_{i \in I_0} \beta'_i a'_i\) satisfies the above claim.

To this end, we prove that by changing one coordinate \(i_0\), from \(\beta_{i_0} > \Delta\) to \(\Delta\) we cause only a slight increase in the value of \(h(\theta, y)\). In other words, by taking \(y^*\) as before and \(y' = y^* - (\beta_{i_0} - \Delta) a_i\) we want to show that

\[
\log \left( \sum_{\alpha \in F} p_{\alpha} e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} \right) \leq \log \left( \sum_{\alpha \in F} p_{\alpha} e^{\langle \alpha, y^* \rangle - \langle \theta, y^* \rangle} \right) + \frac{\varepsilon}{2m}
\]

Towards this, define

\[
F_0 = \{\alpha \in F : \langle \alpha, a_{i_0} \rangle = b_{i_0}\}.
\]

Below we analyze separately the effect of changing \(y\) to \(y'\) on the terms \(p_\alpha e^{\alpha \cdot y}\) for \(\alpha \in F_0\) and for \(\alpha \in F \setminus F_0\).
Case 1: $F \setminus F_0$. Consider any $\alpha \in F \setminus F_0$. We have
\[
\langle \alpha, y' \rangle - \langle \alpha^*, y' \rangle = \sum_{i \in I_0} \beta_i \langle \alpha - \alpha^*, a_i' \rangle \\
\leq \Delta \langle \alpha - \alpha^*, a_{i_0} \rangle \\
\leq -\Delta.
\]
In the above we used the fact that $a_i'$ is the projection of $a_i$ onto $H$, $\langle \alpha - \alpha^*, a_i \rangle \leq 0$ for every $i \in I_0$ and that $\langle \alpha - \alpha^*, a_{i_0} \rangle$ is a negative integer. This implies in particular that
\[
\frac{p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle}}{p_{\alpha^*} e^{\langle \alpha^*, y' \rangle - \langle \theta, y' \rangle}} \leq \frac{p_\alpha}{p_{\alpha^*}} e^{-\Delta}
\]
and hence:
\[
\sum_{\alpha \in F} p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} = \sum_{\alpha \in F_0} p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} + \sum_{\alpha \in F \setminus F_0} p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} \\
\leq \left( \sum_{\alpha \in F_0} p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} \right) \left( 1 + |F \setminus F_0| \max_{\alpha \in F \setminus F_0} \frac{p_\alpha}{p_{\alpha^*}} e^{-\Delta} \right)
\]
Note that for any $\alpha \in F$ we have $e^{-L_p} \leq p_\alpha \leq e^{L_p}$, further $|F \setminus F_0| \leq |F| \leq \exp(\log d, m)$, hence we can pick $\Delta := \log(L_p, m, \log d, \log \frac{1}{\varepsilon})$ to guarantee
\[
|F \setminus F_0| \max_{\alpha \in F \setminus F_0} \frac{p_\alpha}{p_{\alpha^*}} e^{-\Delta} \leq \frac{\varepsilon}{2m}.
\]
For such a choice of $\Delta$ we have:
\[
h(\theta, y') \leq \log \left( \sum_{\alpha \in F_0} p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} \right) + \log \left( 1 + \frac{\varepsilon}{2m} \right) \leq \log \left( \sum_{\alpha \in F_0} p_\alpha e^{\langle \alpha, y^* \rangle - \langle \theta, y^* \rangle} \right) + \frac{\varepsilon}{2m}.
\]

Case 2: $F_0$. Consider now $\alpha \in F_0$, we have
\[
\langle \alpha, y' \rangle - \langle \theta, y' \rangle = \langle \alpha, y^* \rangle - \langle \theta, y^* \rangle - (\beta_i - \Delta) \langle \alpha - \theta, a_i \rangle \leq \langle \alpha, y^* \rangle - \langle \theta, y^* \rangle
\]
as $\langle \theta, a_i \rangle \leq \langle \alpha, a_i \rangle = b_i$ (because $\theta \in P$). Consequently, we obtain
\[
h(\theta, y') \leq \log \left( \sum_{\alpha \in F_0} p_\alpha e^{\langle \alpha, y' \rangle - \langle \theta, y' \rangle} \right) + \frac{\varepsilon}{2m} \\
\leq \left( \sum_{\alpha \in F_0} p_\alpha e^{\langle \alpha, y^* \rangle - \langle \theta, y^* \rangle} \right) + \frac{\varepsilon}{2m} \\
\leq h(\theta, y^*) + \frac{\varepsilon}{2m}.
\]
It remains to argue that after performing the above procedure, the norm of $y'$ is small. For this observe
\[
\|y'\| = \left\| \sum_{i \in I_0} \beta'_i a'_i \right\| \leq \sum_{i \in I_0} \beta'_i \|a'_i\| \leq m \cdot \Delta \cdot (\sqrt{m} \cdot M) = \text{poly} \left( m, \log \frac{1}{\varepsilon} \right).
\]
In the above we used the fact that since $a'_i$ is a projection of $a_i$ onto $H$ (for any $i \in I_0$) we have
\[
\|a'_i\| \leq \|a_i\| \leq \sqrt{m} \cdot M.
\]
\[\square\]
Below we present a useful corollary of Theorem 5.1 which is often convenient in applications.

**Corollary 5.1** Under the assumptions of Theorem 5.1, for every function $p : \mathcal{F} \to \mathbb{R}_{>0}$, for every $\varepsilon > 0$ there exists a number $R > 0$ which is polynomial in $m$, $\log d$, $M$, $L_p$ and $\log \frac{1}{\varepsilon}$ such that
\[\forall \theta \in \mathcal{P} \exists y \in B(0, R) \quad \left\| \sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)} \cdot \alpha \right\| < \varepsilon.\]

**Proof of Corollary 5.1** It is enough to establish it for $\theta \in \text{int}(P)$ (relative interior is meant here, i.e. interior of $P$ when restricted to $H$). To this end consider $y^*$ to be
\[y^* = \arg\min_{y \in \mathbb{R}^m} h(\theta, y) = g(\theta).\]
It is not hard to prove that such a $y^*$ exists, i.e., the minimum is attained, for $\theta$ in the relative interior of $P$. Consider now the gradient of $h$ with $\theta$ fixed:
\[\nabla_y h(\theta, y) = \frac{\sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)} \cdot \alpha}{\sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)}} - \theta.\]
(11)
To conclude the Corollary from Theorem 5.1 one has to prove that if the value at a point is close to optimal, then the gradient is short. Towards this we show that $y \mapsto h(\theta, y)$ is $L$–smooth (for some polynomially bounded $L$), i.e.
\[\|\nabla_y h(\theta, y_1) - \nabla_y h(\theta, y_2)\| \leq L \|y_1 - y_2\|.\]
This can be deduced from the fact that the Hessian matrix of $h$, $\nabla^2_y h(\theta, y)$ has polynomially bounded entries (and the bound does not depend neither on $y$ nor on $\theta$). Indeed, under the notation that $\alpha' = \alpha - \theta$ for $\alpha \in \mathcal{F}$ we obtain
\[
(\nabla^2_y h(\theta, y))_{i,j} = \frac{\sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)} \alpha'_i \alpha'_j}{\sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)}} - \frac{\left( \sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)} \alpha'_i \right) \left( \sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)} \alpha'_j \right)}{\left( \sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha, y)} \right)^2},
\]
hence it follows
\[
\left| (\nabla^2_y h(\theta, y))_{i,j} \right| \leq 2 \max_{\alpha \in \mathcal{F}} \|\alpha - \theta\|^2 \leq 2d^2.
\]
Hence the function $y \mapsto h(\theta, y)$ is $L := 2d^2$–smooth. Now, it is well known that for a convex, $L$–smooth function, we have (see e.g. [30]):
\[h \left( \theta, y + \frac{1}{2L} v \right) \leq h(\theta, y) - \frac{1}{4L} \|v\|^2,
\]
where $v = \nabla_y h(\theta, y)$. Hence, if $y$ is as in Theorem 5.1 then we obtain $\|\nabla_y h(\theta, y)\|^2 \leq 4L\varepsilon$ and consequently
\[\|\nabla_y h(\theta, y)\| \leq 4\varepsilon^{1/2}.
\]
The corollary follows by combining the above obtained bound with (11). \[\square\]
6 Proof of the Computability of Max-Entropy Distributions

Below we restate Theorem 2.1 in its fully general form. For definitions of the unary facet complexity, the complexity measure $L_p$ of $p$ and the diameter of $\mathcal{F}$ we refer to Section 5.

**Theorem 6.1** Let $\mathcal{F}$ be any finite subset of $\mathbb{Z}^m$ and let $d \in \mathbb{R}_{\geq 0}$ be its diameter, let $M$ be the unary facet complexity of $\text{conv}(\mathcal{F})$. Then, there exists an algorithm such that given a probability distribution $p$ on $\mathcal{F}$ (via an evaluation oracle for $g_p$), $\theta \in \mathcal{P}$ and an $\varepsilon > 0$, computes a vector $y \in \mathbb{R}^m$ with $\|y\| \leq \text{poly}(m, M, \log d, L_p, \log \frac{1}{\varepsilon})$ such that

$$\|q^y - q^*\|_1 < \varepsilon,$$

where $q^*$ is the optimal solution to (8) and $q^y$ is a distribution over $\mathcal{F}$ defined as

$$q^y = \frac{p^\alpha e^{\langle \alpha, y \rangle}}{\sum_{B \in \mathcal{F}} p_B e^{\langle y, B \rangle}},$$

for $\alpha \in \mathcal{F}$. The running time of the algorithm is polynomial in $m, M, d, L_p, \log \frac{1}{\varepsilon}$.

In the above $g_p$ is a generalized counting function – the function $g_{\mathcal{F}}$ introduced in Section 2 is a special case when $p$ is the uniform distribution over $\mathcal{F}$. An oracle for $g_p$ is then simply defined as a procedure that given an $x > 0$ outputs

$$g_p(x) = \sum_{\alpha \in \mathcal{F}} p^\alpha x^\alpha,$$

where $x^\alpha = \prod_{i=1}^m x_i^\alpha_i$.

**Proof:** For convenience without loss of generality we might assume that $\mathcal{F} \subseteq \mathbb{N}^m$ and in fact even $\mathcal{F} \subseteq [0, 2d]^m \cap \mathbb{N}^m$ where $d$ is the diameter of $\mathcal{F}$. This is because one can always shift the set $\mathcal{F}$ (together with $\theta$) and this operation does not affect the problem nor its parameters. To obtain a vector $y$, as required, we solve the dual program up to a desired precision (below we explain why this is enough). Recall that the dual program to (8) is given by

$$\inf \ h(\theta, y) = \log \left( \sum_{\alpha \in \mathcal{F}} p^\alpha e^{\langle \alpha - \theta, y \rangle} \right),$$

s.t. $y \in \mathbb{R}^m$.

By a direct calculation one can show that for every feasible solution $q$ of the primal problem (8)

$$KL(q, q^y) = h(\theta, y) - \sum_{\alpha \in \mathcal{F}} q^\alpha \log \frac{p^\alpha}{q^\alpha},$$

where $KL(\cdot, \cdot)$ denotes the KL-divergence. In particular for $q := q^*$

$$h(\theta, y) - g(\theta) = KL(q^*, q^y).$$

This means that in order to obtain a distribution $q^y$ being $\varepsilon-$close in the KL-distance to the max-entropy distribution $q^*$ it is enough to find an $\varepsilon-$optimal solution to the dual program. Moreover, from Pinsker’s inequality we have

$$\|q - q^*\|_1 \leq \sqrt{2 \cdot KL(q^*, q)},$$

hence it suffices to find a solution $y$ to the dual program which is $\delta := \Theta(\varepsilon^2)$-optimal, to guarantee that $\|q^* - q^y\|_1 < \varepsilon.$
To find a $\delta$-optimal solution to the dual problem we apply the ellipsoid method. First of all we note that $h(\theta, y)$ is a convex function of $y$ (this follows from Hölder’s inequality), which is the first requirement for the ellipsoid method to be applicable. It follows from Theorem 6.1 that in order to find a $\delta$-optimal solution to $\inf_{y \in \mathbb{R}^m} h(\theta, y)$ it is sufficient to solve
$$\min_{y \in B(0, R)} h(\theta, y),$$
where $R$ is a certain bound, polynomial in $m, M, \log d, L_p$ and $\log \frac{1}{\delta}$. Now, following the treatment of the ellipsoid method in [8] (Theorem 8.2.1) it remains to address the following issues.

1. Construct a first order oracle for $y \mapsto h(\theta, y)$, i.e. an efficient way to evaluate values $h(\theta, y)$ and gradients $\nabla_y h(\theta, y)$ of this function.

2. A bound on the gap between the maximum and minimum value of $h(\theta, \cdot)$ in $B(0, R)$. More precisely, defining $D := \max_{y \in B(0, R)} h(\theta, y) - \min_{y \in B(0, R)} h(\theta, y)$ we would like $\log D$ to be polynomially bounded.

3. Provide an outer ball – containing the domain of the considered optimization problem and an inner ball – contained in the domain. The radii of them should be of polynomial bit complexity.

4. Provide a separation oracle for the domain $B(0, R)$.

Point (1) We first note that $h$ can be equivalently written as
$$h(\theta, y) = \log \left( \sum_{\alpha \in F} p_{\alpha} e^{\langle \alpha, y \rangle} \right) - \langle \theta, y \rangle.$$
Thus $h(\theta, y) = \log g_p(e^y) - \langle \theta, y \rangle$, where $e^y = (e^{y_1}, e^{y_2}, \ldots, e^{y_m})$ and consequently $h(\theta, y)$ can be evaluated using just the evaluation oracle for $g_p$. For the case of gradients observe first that
$$\nabla_y h(\theta, y) = \frac{1}{g_p(e^y)} \sum_{\alpha \in F} \alpha p_{\alpha} e^{\langle \alpha, y \rangle} - \theta.$$
Since computing $g_p(e^y)$ is easy, it remains to deal with $\sum_{\alpha \in F} \alpha p_{\alpha} e^{\langle \alpha, y \rangle}$. For this note that the $i$th coordinate of the above is
$$\sum_{\alpha \in F} \alpha_i p_{\alpha} e^{\langle \alpha, y \rangle} = \frac{d}{dt} g_p(e^{y_1}, \ldots, e^{y_{i-1}}, e^{y_i + t}, e^{y_{i+1}}, \ldots, e^{y_m}).$$
The right hand side above is a univariate polynomial $h(t)$ of degree at most $2d$ (since $F \subseteq [0, 2d]$). To compute its derivative it is enough to learn all its coefficients (in fact it is enough to learn the coefficient of $t$ in $h(t)$). Towards this, note that the evaluation oracle for $g_p$ implies an evaluation oracle for $t \mapsto h(t)$, and hence we can simply evaluate $h$ at $2d + 1$ different points and recover its coefficients using polynomial interpolation. The running time of such a procedure is polynomial in $d$ – as required.

Note also that, importantly, to implement the first order oracle for $h(\theta, \cdot)$ the oracle $g_p$ is queried only on inputs of polynomial bit complexity, as for every $y \in B(0, R)$ the vector $e^y$ has polynomial bit complexity (since $R = \text{poly} \left( m, \log \frac{1}{\delta} \right)$). Thus the running time of these procedures is polynomial in $m, \log \frac{1}{\delta}$ and $d$. 

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Point (2) Note that from the Cauchy-Schwarz inequality, for every \( y \in B(0, R) \)
\[
(y, \alpha - \theta) \leq \|y\| \|\alpha - \theta\| \leq R \cdot d,
\]
hence, we have
\[
\max_{y \in B(0, R)} h(\theta, y) \leq \log \left( e^{L_p \cdot |\mathcal{F}| \cdot e^{R \cdot d}} \right) \leq L_p \cdot R \cdot d \cdot \log |\mathcal{F}|.
\]

Similarly, for the minimum
\[
\min_{y \in B(0, R)} h(\theta, y) \geq -L_p \cdot R \cdot d \cdot \log |\mathcal{F}|.
\]

Clearly the logarithm of the gap:
\[
\log \left( \max_{y \in B(0, R)} h(\theta, y) - \min_{y \in B(0, R)} h(\theta, y) \right) \leq \log(2L_p \cdot R \cdot d \cdot \log |\mathcal{F}|)
\]
is polynomially bounded (even without taking the logarithm it is still true), as \(|\mathcal{F}| \leq (2d)^m\).

Point (3) The outer ball is the domain itself: \( B(0, R) \); \( R \) is polynomially bounded (note that in fact we only need that \( \log R \) is polynomially bounded). For the inner ellipsoid we can just take \( B(0, 1) \).

Point (4) This is clear - given a vector \( y_0 \in \mathbb{R}^m \), if \( \|y\| \leq R \) we just report \( y_0 \) to be in the domain and if \( \|y_0\| = R' > R \) then \( \{y \in \mathbb{R}^m : \langle y, y_0 \rangle = \frac{R + R'}{2} \} \) is the required separating hyperplane.

\[\square\]

7 Proof of the Bit Complexity Lower Bound

We start by proving a technical fact which will be useful in establishing the large bit complexity example.

Fact 7.1 Let \( v_1, v_2, \ldots, v_N \subseteq \mathbb{R}^m \) be a set of vectors and denote \( \delta := \text{dist}(0, \text{conv}(v_1, v_2, \ldots, v_N)) \). Assume that \( \delta > 0 \) and consider the optimal value of the following optimization problem:
\[
\tau = \min \{\|y\| : y \in \mathbb{R}^m, \langle y, v_i \rangle \leq -1 \text{ for all } i = 1, 2, \ldots, N\},
\]
then \( \tau \geq \frac{1}{\delta} \).

Proof: We formulate the problem as a convex, quadratic program with linear constraints:
\[
\min \|y\|^2,
\text{ s.t. } \langle y, \alpha \rangle \leq -1 \text{ for all } \alpha \in \mathcal{F}'. \tag{13}
\]

To derive a lower bound on the optimal value of (13) we consider the dual program:
\[
\max \sum_{i=1}^N \lambda_i - \frac{1}{4} \left\| \sum_{i=1}^N \lambda_i v_i \right\|^2,
\text{ s.t. } \lambda_i \geq 0 \text{ for all } i = 1, 2, \ldots, N. \tag{14}
\]
From weak duality we know that the optimal value of (14) is a lower bound to (13) thus we just need to provide a feasible solution to the dual program. To this end let $v$ be the shortest vector in the convex hull of $v_1, v_2, \ldots, v_N$, i.e., $v \in \text{conv}(v_1, v_2, \ldots, v_N)$ and $\|v\| = \delta$. $v$ can be written as

$$v = \sum_{i=1}^{N} \mu_i v_i$$

for some $\mu \geq 0$ with $\sum_{i=1}^{N} \mu_i = 1$. Consider now $\lambda := \frac{2}{\delta^2} \mu \in \mathbb{R}_{\geq 0}^N$. The dual objective value for $\lambda$ is

$$\sum_{i=1}^{N} \lambda_i - \frac{1}{4} \left\| \sum_{i=1}^{N} \lambda_i v_i \right\|^2 = \frac{2}{\delta^2} - \frac{1}{4} \cdot \frac{4}{\delta^4} \left\| \sum_{i=1}^{N} \mu_i v_i \right\|^2 = \frac{1}{\delta^2}.$$ 

This provides us with a lower bound of $\frac{1}{\delta^2}$ on the optimal value of (13) and thus a lower bound of $\frac{1}{\delta}$ on the optimization problem as in the statement of the Fact.  

\[\square\]

**Remark 7.1** It is not hard to prove that in Fact 7.1 the value $\frac{1}{\delta}$ is not only a lower bound but is actually equal to the optimal value of the considered optimization problem. This can be established by plugging in an appropriate scaling of the shortest vector in $\text{conv}(v_1, v_2, \ldots, v_N)$ for $y$.

**Proof Theorem 2.3**: Our construction of $F$ is based on existence of “flat” 0-1 polytopes, as established by [2]. There exist $m+1$ affinely independent points $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_m \in \{0, 1\}^m$ such that if we let $H = \alpha_1 + \text{span}\{\alpha_2 - \alpha_1, \ldots, \alpha_m - \alpha_1\}$ (the $(m-1)$-dimensional affine subspace containing all points $\alpha_1, \ldots, \alpha_m$) then

$$\text{dist}(\alpha_0, H) = e^{-\Omega(m \log m)}.$$

Without loss of generality we assume that $\alpha_0 = 0$. Let $y \in H$ be the projection of $\alpha_0 = 0$ onto $H$, i.e., a point such that

$$\langle y, \alpha_i - y \rangle = 0 \quad \text{for every } i = 1, 2, \ldots, m.$$

Consider the lattice

$$L = \alpha_1 + \sum_{i=2}^{m} (\alpha_i - \alpha_1) \cdot \mathbb{Z},$$

with the origin at $\alpha_1$, with basis $\{(\alpha_i - \alpha_1)\}_{2 \leq i \leq m}$. Since the hyperplane $H$ is covered by disjoint copies of the fundamental parallelepiped

$$F := \left\{ \sum_{i=2}^{m} \beta_i (\alpha_i - \alpha_1) : \beta_2, \ldots, \beta_m \in [0, 1) \right\},$$

there is an integer translation of $F$ which contains the point $y$. More formally, there exists an integer vector $\gamma \in \mathbb{Z}^m$ such that

$$y \in \gamma + F \subseteq H.$$

Note now that by denoting $F' = \gamma + F$ we obtain

$$\text{diam}(F') = \text{diam}(F) \leq m^{3/2},$$

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since every vertex of \( F \) has integer coordinates in the range \([0, m]\). Let \( \alpha \in \mathbb{Z}^m \) be now any of the \( 2^m \) vertices of \( F' \). Since \( y \) belongs to \( F' \) we have

\[
\|\alpha\| \leq \|y\| + \text{diam}(F') \leq O(1) + m^{3/2}.
\]

Let \( F' \) be a subset of \( \mathbb{Z}^m \) consisting of all the vertices of \( F' \) and let \( F := F' \cup \{0\} \). Further, define \( \theta := 0 \). We prove that the conclusion of the Lemma holds under such a choice of \( \theta \).

Towards this we first note that affine hull of \( F' \) is equal to \( H \), as \( F' \subseteq H \) and its vertices clearly span \( H \). Moreover the point \( y \), which is the projection of 0 onto \( H \) belongs to the convex hull of \( F' \). Let \( \delta > 0 \) be the distance between 0 and \( H \) and let \( a \in \mathbb{R}^m \) (with \( \|a\| = 1 \)) be the normal vector of \( H \), in other words

\[
H = \{ x \in \mathbb{R}^m : a^\top x = \delta \}.
\]

Note that the gradient of \( h(\theta, y) \) with respect to \( y \) is given by:

\[
\nabla_y h(\theta, y) = \sum_{\alpha \in F} \alpha \cdot e^{(y, \alpha)}
\]

Since \( h \) is \( L \)-smooth for some \( L = \text{poly}(m) \) (see the proof of Corollary 5.1), we know that points with a large-magnitude gradient cannot be close to optimal. Quantitatively, we have

\[
|g(\theta) - h(\theta, y)| \geq \frac{\|\nabla_y h(\theta, y)\|^2}{L}.
\]

Thus to prove that \( |g(\theta) - h(\theta, y)| \geq \varepsilon \) (for some \( \varepsilon > 0 \)) it is enough to prove that \( \|\nabla_y h(\theta, y)\| \geq \varepsilon L^{-1} \). We pick \( \varepsilon \) to be \( \frac{\delta^2}{e^4 L} \), then \( \varepsilon = e^{-O(m \log m)} \). Moreover, the condition \( |g(\theta) - h(\theta, y)| < \varepsilon \) implies that \( \|\nabla_y h(\theta, y)\| < \frac{\delta}{e^2} \). We prove that the latter is possible only when \( \|y\| \geq e^{\Omega(m \log m)} \).

Indeed, assume that \( \|\nabla_y h(\theta, y)\| < \frac{\delta}{e^2} \). By the Cauchy-Schwarz inequality we have

\[
\langle a, \nabla_y h(\theta, y) \rangle \leq \|\nabla_y h(\theta, y)\| \cdot \|a\| = \|\nabla_y h(\theta, y)\|
\]

and moreover

\[
\langle a, \nabla_y h(\theta, y) \rangle = \frac{0 + \sum_{\alpha \in F'} \langle a, \alpha \rangle e^{(y, \alpha)}}{\sum_{\alpha \in F} e^{(y, \alpha)}} = \delta \cdot \frac{\sum_{\alpha \in F'} e^{(y, \alpha)}}{1 + \sum_{\alpha \in F'} e^{(y, \alpha)}}.
\]

It follows that

\[
\frac{\sum_{\alpha \in F'} e^{(y, \alpha)}}{1 + \sum_{\alpha \in F'} e^{(y, \alpha)}} < \frac{1}{e^2},
\]

and consequently

\[
\sum_{\alpha \in F'} e^{(y, \alpha)} < \frac{1}{e^2},
\]

which implies in particular that

\[
\forall_{\alpha \in F'} \langle y, \alpha \rangle \leq -1.
\] (15)

The question we would like to answer is: what is the shortest \( y \in \mathbb{R}^m \) which satisfies condition (15)? This will give us a lower bound on \( \|y\| \) satisfying \( |g(\theta) - h(\theta, y)| < \varepsilon \). To answer this question, we apply Fact 7.1 and conclude that every such \( y \) has length at least \( \frac{1}{e^2} \). As \( \delta = e^{-\Omega(m \log m)} \) we conclude that the optimal solution \( y^\star \) to (13) satisfies \( \|y^\star\| = e^{\Omega(m \log m)} \) and the Lemma follows by contraposition. \( \square \)
8 Applications

Preliminaries on Real Stability

In this section we define the concept of real stable polynomials that appear in some applications of our results. For a survey on real stable polynomials we refer the reader to [42].

Definition 8.1 A polynomial \( p \in \mathbb{C}[x_1, \ldots, x_m] \) is called real stable if all its coefficients are real numbers and the following condition holds

\[
\forall z \in \mathbb{C}^m \quad (\Re(z_i) > 0 \text{ for all } i = 1, 2, \ldots, m) \Rightarrow p(z) \neq 0.
\]

Fact 8.1 ([9] [11]) Let \( p \in \mathbb{R}[x_1, \ldots, x_m] \) be a real stable polynomial with non-negative coefficients. Then, there exists a “rank function” \( r : \{-1, 0, 1\}^m \to \mathbb{Z} \) is, such that the convex hull of \( \mathcal{F} \subseteq \mathbb{N}^m \) - the support of \( p \) can be described as:

\[
\text{conv}(\mathcal{F}) = \{x \in \mathbb{R}^m : \forall c \in \{-1, 0, 1\}^m \quad (x, c) \leq r(c)\}.
\]

Proof: The proof is a simple consequence of two results. It was proved in [11] that the support of a real stable polynomial is a jump system. Such sets were studied previously in [9], where a polyhedral characterization, as in the conclusion, was shown. \( \square \)

Fact 8.2 ([18]) Let \( p \in \mathbb{R}[x_1, \ldots, x_m] \) be a real stable polynomial with non-negative coefficients. Then the function \( x \mapsto \log p(x) \) is concave over \( x \in \mathbb{R}^m_0 \).

8.1 Bounds for the Matrix Scaling Problem

Consider the \((r, c)\)-matrix scaling problem as introduced in Section 4. Recall that in this problem we are interested in finding a scaling \( B := XAY \) (with \( X := \text{Diag}(x) \) and \( Y := \text{Diag}(y) \), with \( x, y \in \mathbb{R}_0^n \)) of a nonnegative matrix \( A \) so that the row-sums of \( B \) are \( r \in \mathbb{N}^m \) and the column sums are \( c \in \mathbb{N}^m \). A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be asymptotically \((r, c)\)-scalable, if for every \( \varepsilon > 0 \) there exist scalings \( x, y \in \mathbb{R}_{>0}^n \) such that for \( B := XAY \) it holds \( \|B - r\| < \varepsilon \) and \( \|B^\top - c\| < \varepsilon \).

As already discussed in Section 4, the problem of finding bounds on the bit complexity of the scaling \((x, y)\) boils down to proving that close to optimal solutions to the convex program below are polynomially bounded.

\[
\inf_{z \in \mathbb{R}^n} \sum_{i=1}^{n} r_i \log \left( \sum_{j=1}^{n} A_{i,j} e^{z_j} \right) - \langle c, z \rangle. \quad (16)
\]

We derive the following Corollary of Theorem 5.1. Below we use \( L_A \) for the bit complexity of the matrix \( A \), i.e., \( L_A := \max \{\log |A_{i,j}| : i, j \in \{1, 2, \ldots, n\}, A_{i,j} \neq 0\} \).

Corollary 8.1 Let \( A \in \mathbb{R}^{n \times n} \) be a nonnegative matrix which is asymptotically \((r, c)\)-scalable with \( \|r\|_1 = \|c\|_1 = h \). Then for every \( \varepsilon > 0 \) there exists a scaling \((x, y)\) such that:

\[
\sum_{k=1}^{n} x_i y_k A_{i,k} - r_i < \varepsilon \quad \text{for all } i \in [n],
\]

\[
\sum_{i=1}^{n} x_i y_j A_{i,j} - c_j < \varepsilon \quad \text{for all } j \in [n],
\]

and the bit complexities of all entries of \( x \) and \( y \), i.e., \( \max_{i \in [n]} |\log x_i| \) and \( \max_{i \in [n]} |\log y_i| \) are bounded by \( \text{poly}(n, L_A, h, \log \frac{1}{\varepsilon}) \).
Proof: Our strategy is to derive the result from Corollary 5.1. We first rewrite the objective (4) so that it matches the form as in Theorem 5.1.

\[
\sum_{i=1}^{n} r_i \log \left( \sum_{j=1}^{n} A_{i,j} e^{z_j} \right) - \langle c, z \rangle = \log \left( \prod_{i=1}^{n} \left( \sum_{j=1}^{n} A_{i,j} e^{z_j} \right)^{r_i} \right) - \langle c, z \rangle
\]

\[
= \log \left( \sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{\langle \alpha - c, z \rangle} \right)
\]

\[
= h(c, z)
\]

For some set \( \mathcal{F} \subseteq \mathbb{N}^m \) and some family of positive numbers \( \{p_{\alpha}\}_{\alpha \in \mathcal{F}} \).

The next step is to obtain bounds on the various quantities \( m, d, M, L \), which appear in Corollary (5.1). Clearly we choose \( m := n \). Next observe that \( \|\alpha\|_{\infty} \leq h \) for \( \alpha \in \mathcal{F} \), hence \( d \leq h \) and that \( |\log p_{\alpha}| \leq \text{poly}(h, L_A) \) for all \( \alpha \in \mathcal{F} \), hence \( L_p = \text{poly}(h, L_A) \).

Let us now prove that the polytope \( \text{conv}(\mathcal{F}) \) can be described by inequalities with small integer coefficients. To this end note that \( p \) can be naturally treated as a polynomial \( (p_{\alpha} \) is the coefficient of \( x^{\alpha} := \prod_{i \in [n]} x_i^{\alpha_i} \)).

\[
p(x) = \sum_{\alpha \in \mathcal{F}} p_{\alpha} x^\alpha = \prod_{i=1}^{n} \left( \sum_{j=1}^{n} A_{i,j} x_j \right)^{r_i}
\]

and the support of \( p \) is equal to \( \mathcal{F} \). Since \( p \) is a product of linear polynomials with nonnegative coefficients, it is a real stable polynomial (see preliminaries at the beginning of this section for some background). As shown in the preliminaries (Fact 8.1), the Newton polytope of a real stable polynomial can be described by inequalities of the form

\[
\langle a, x \rangle \leq b
\]

where \( a \in \{-1, 0, 1\}^n \). Hence we can take \( M = O(1) \) in the statement of Corollary 5.1.

We now apply Corollary 5.1 to obtain a point \( z^* \) and use it to define a scaling by the following formulas

\[
x_i = r_i \cdot \left( \sum_{j=1}^{n} A_{i,j} e^{z^*_j} \right)^{-1} \quad \text{and} \quad y_j = e^{z^*_j}, \quad \text{for } i, j = 1, 2, \ldots, n.
\]

(17)

Note that such a pair \((x, y)\) has bit complexity which is polynomial in \( n, L_A, h, \log \frac{1}{\varepsilon} \), hence it remains to reason about the precision of the resulting scaling.

By a direct calculation one obtains that

\[
\left| \sum_{k=1}^{n} x_i y_k A_{i,k} - r_i \right| = 0 \quad \text{for all } i \in [n],
\]

hence it remains to prove a bound on the precision of the scaling with respect to columns. For this, note that

\[
(\nabla_z h(c, z))_j = \sum_{l=1}^{n} x_l y_j A_{l,j} - c_j.
\]

Since Corollary 5.1 implies that

\[
\|\nabla_z h(c, z^*)\|_2 < \varepsilon
\]

the bound on the scaling precision follows. \( \square \)
8.2 Computability of Recent Continuous Relaxations for Counting and Optimization Problems

Recall that in the setting introduced in Section 4 we are interested in solving the following optimization problem

\[
\text{Cap}_B(p) := \sup_{\theta \in P(B)} \inf_{x > 0} \frac{p(x)}{\prod_{i=1}^{m} x^\theta_i},
\]

where \( p \in \mathbb{R}[x_1, x_2, \ldots, x_m] \) is a polynomial with non-negative coefficients and \( B \subseteq \{0, 1\}^m \) is a certain family of sets (points in the binary cube). We prove the following

**Theorem 8.1** Let \( p \in \mathbb{R}[x_1, x_2, \ldots, x_m] \) be a polynomial with nonnegative coefficients with support \( F \subseteq \mathbb{N}^m \) and let \( B \subseteq \mathbb{N}^m \). Assume that the unary facet complexity of \( \text{conv}(F) \) is \( M \) and denote \( d = \max(\text{diam}(F), \text{diam}(B)) \). There is an algorithm which given an evaluation oracle for \( p \), a separation oracle for \( \text{conv}(F) \), a separation oracle for \( \text{conv}(B) \) and an \( \epsilon > 0 \) computes a number \( X \) such that

\[
1 - \epsilon < X < 1 + \epsilon
\]

in time \( \text{poly}(m, L_p, \log d, M, \log \frac{1}{\epsilon}) \).

Note that in the above theorem we require separation oracle access to the corresponding Newton polytope: this is actually not necessary in various cases, in particular when \( p \) is real stable, in which case such a separation oracle can be constructed given access to an evaluation oracle to \( p \) only (see [3]).

**Proof:** Denote by \( P \subseteq \mathbb{R}^m \) the convex hull of \( F \) – the support of \( p \). As in the discussion in Section 4 we observe that \( \text{Cap}_B(p) \) can be rewritten equivalently as

\[
\log \text{Cap}_B(p) = \sup_{\theta \in P(B)} \inf_{y \in \mathbb{R}^m} \log \left( \sum_{\alpha \in F} p_\alpha e^{(\alpha - \theta, y)} \right) = \sup_{\theta \in P(B)} \inf_{y \in \mathbb{R}^m} h(\theta, y),
\]

where \( F \subseteq \mathbb{N}^m \) denotes the support of \( p \). Thus we obtain a form of \( \log \text{Cap}_B(p) \) in terms of a function \( h \) as in (9). Let us denote

\[
g(\theta) = \inf_{y \in \mathbb{R}^m} \log \left( \sum_{\alpha \in F} p_\alpha e^{(\alpha - \theta, y)} \right).
\]

Hence the goal is to solve \( \sup_{\theta \in P(B)} g(\theta) \). In fact, since \( g(\theta) = -\infty \) whenever \( \theta \notin P \), we can rewrite it equivalently as

\[
\sup_{\theta \in P(B) \cap P} g(\theta).
\]

Importantly, since we are only interested in an additive \( \epsilon \)-approximation of the above quantity we can apply Theorem 5.1 to replace \( g \) by the following Lipschitz proxy:

\[
\tilde{g}(\theta) := \inf_{y \in B(0, R)} h(\theta, y)
\]

for an appropriate number \( R \) – polynomial in the input size and \( \log \frac{1}{\epsilon} \) (as in Theorem 5.1).

We now apply the ellipsoid method to find an additive \( \epsilon \)-approximation to

\[
\sup_{\theta \in P(B) \cap P} \tilde{g}(\theta).
\]

(19)

Firstly, observe that the function \( \tilde{g} \) is concave – as a pointwise infimum of affine functions. Now, following the treatment of the ellipsoid method in [3] (Theorem 8.2.1) we have to address the following requirements to obtain polynomial running time.
1. Construct a first order oracle for $\theta \mapsto \tilde{g}(\theta)$, i.e. an efficient way to evaluate values $\tilde{g}(\theta)$ and (sub)gradients of this function for $\theta \in P(B) \cap P$.

2. A bound on the gap between the maximum and minimum value of $\tilde{g}(\theta)$ in $P(B) \cap P$. More precisely, defining $D := \max_{\theta \in P(B) \cap P} \tilde{g}(\theta) - \min_{\theta \in P(B) \cap P} \tilde{g}(\theta)$ we would like $\log D$ to be polynomially bounded.

3. Provide an outer ball – containing the domain of the considered optimization problem and an inner ball – contained in the domain. The radii of them should be of polynomial bit complexity.

4. Provide a separation oracle for the domain $P(B) \cap P$.

**Point (1).** To obtain an evaluation oracle for $\tilde{g}$ note that Theorem 6.4 gives an algorithm to compute $\tilde{g}$ up to $\delta$ additive error in time polynomial in $\log \frac{1}{\delta}$. This provides an approximate (weak) evaluation oracle, which is still enough to run the ellipsoid method (using shallow cuts, see [17]).

Let us now discuss the oracle for the gradient of $\tilde{g}$. It is a standard fact in convex programming that for any $\theta \in P$ it holds that the point

$$y^* := \arg\min_{y \in B(0,R)} h(\theta, y)$$

is a subgradient of $\tilde{g}$ at $\theta$. Using Theorem 6.4 we can efficiently compute a vector $y$ which is a $\delta$-approximation to $y^*$ in the sense that $h(\theta, y) - h(\theta, y^*) \leq \delta$. We use this algorithm to implement an approximate subgradient oracle for $\tilde{g}$, i.e., for a given $\theta$ we output the vector $y$ as above, using the algorithm in Theorem 6.4.

It remains to justify why is such an approximate gradient enough to run the ellipsoid method. Note that if $\theta \in P$ and $y \in B(0,R)$ is such that $h(\theta, y) \leq \tilde{g}(\theta) + \delta$ then for any $\theta' \in P$ we have

$$g(\theta') \leq h(\theta', y) = h(\theta, y) + \langle \theta' - \theta, y \rangle \leq g(\theta) + \delta + \langle \theta' - \theta, y \rangle,$$

where the equality above holds because $h$ is an affine function of $\theta$. Hence, by applying such a $y$ as a gradient oracle, we obtain a separation hyperplane which never cuts out a point $\theta' \in P$ of value $g(\theta') > g(\theta) + \delta$. This, along with the fact that $y$ can be found in time polynomial in $\log \frac{1}{\delta}$, allows us to apply the shallow cut ellipsoid method. We also refer the reader to the full version of [38] and the proof of Theorem 2.11 therein where a detailed discussion of an ellipsoid algorithm for a related problem is provided.

**Point (2).** As in the proof of Theorem 6.4 we observe that for every $y \in B(0,R)$

$$-L_p \cdot R \cdot m \cdot d \log d \leq \log \left( \sum_{\alpha \in \mathcal{F}} p_{\alpha} e^{(\alpha \cdot \theta, y)} \right) \leq L_p \cdot R \cdot m \cdot d \log d,$$

and hence the gap (and hence clearly its logarithm as well) is polynomially bounded.

**Point (3).** An outer ball is easy to obtain since $P(B) \subseteq [0,1]^m$. For the inner ball, suppose first that $P(B) \cap P$ is full-dimensional. Then, since the vertices of both these polytopes have small integer entries, one can show using standard techniques that a ball of radius $\Omega(e^{-\text{poly}(m)})$ can be fit inside this polytope. In the non-full-dimensional case one has to work in a linear subspace $H \subseteq \mathbb{R}^m$ on which $P(B) \cap P$ is full-dimensional; $H$ can be found given separation oracles of $P(B)$ and $P$ using standard subspace identification techniques.

**Point (4).** This is clear as we are given separation oracles for both $P(B)$ and $P$. □
8.3 Computability of Worst-Case Brascamp-Lieb constants

Given a sequence of matrices $B_j \in \mathbb{R}^{n_j \times n}$ for $j \in [m]$ and a vector $p \in \mathbb{R}^m_{\geq 0}$ consider the following Brascamp-Lieb inequality:

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(B_j x)^{p_j} dx \leq C \prod_{j=1}^m \left( \int_{\mathbb{R}^{n_j}} f_j(x_j) dx_j \right)^{p_j},$$

(20)

for any sequence of integrable functions $f_j : \mathbb{R}^{n_j} \to \mathbb{R}_{\geq 0}$. These capture in particular the Hölder inequality and Loomis-Whitney inequalities as special cases [12, 28]. The best constant $C$ for which the above holds universally is called the Brascamp-Lieb constant and can be computed [28] as $\text{BL}(B, p)^{-2}$ ($B$ stands here for the collection of all matrices $B_1, B_2, \ldots, B_m$) where

$$\text{BL}(B, p) = \inf \left\{ \frac{\det \left( \sum_{j=1}^m p_j B_j^\top X_j B_j \right)}{\prod_{j=1}^m \det(X_j)^{p_j}} : X_j \in \mathbb{R}^{n_j \times n_j}, X_j \succeq 0, j = 1, 2, \ldots, m \right\}.$$

The constant $\text{BL}(B, p)$ is non-zero whenever $p$ belongs to the so called Brascamp-Lieb polytope, which can be described as follows

$$P_B = \left\{ p \in \mathbb{R}^m_{\geq 0} : \sum_{j=1}^m p_j \dim(B_j U) \geq \dim(U), \text{ for every lin. subspace } U \subseteq \mathbb{R}^n \right\}.$$

Recently [16] gave a method for calculating the Brascamp-Lieb constant in polynomial time when the vector $p$ is given in unary. Here we ask a question of computability of the worst possible Brascamp-Lieb constant over the whole Brascamp-Lieb polytope, i.e. what $\sup_{p \in P_B} \text{BL}(B, p)$ is. This quantity can be used in particular as a universal constant (for any $p \in P_B$) for the so called reverse Brascamp-Lieb inequality [7]. We prove that it can be computed efficiently when all the matrices $B_1, B_2, \ldots, B_m$ are of rank 1.

**Theorem 8.2** Consider a sequence of matrices $B_1, B_2, \ldots, B_m \in \mathbb{R}^{1 \times n}$. The worst-case Brascamp-Lieb constant $\sup_{p \in P_B} \text{BL}(B, p)$ can be computed up to precision $\varepsilon > 0$ in time polynomial in the description size of $B_1, B_2, \ldots, B_m$, $m$ and $\log \frac{1}{\varepsilon}$.

**Proof:** Let us denote by $v_j \in \mathbb{R}^n$ the only row of the matrix $B_j$ and by $V \in \mathbb{R}^{m \times n}$ the matrix collecting all the $v_j$’s as rows. Then the the Brascamp-Lieb constant (for the rank-1 case) can be computed as

$$\text{BL}(B, p) = \inf_{x > 0} \frac{\det \left( \sum_{j=1}^m p_j x_j v_j v_j^\top \right)}{\prod_{j=1}^m x_j^{p_j}}.$$  

(21)

Note that the numerator is simply a polynomial $r(x) = \sum_{S \subseteq [m],|S|=n} p^S x^S \det(V_S V_S^\top)$, where $V_S$ is the submatrix of $V$ corresponding to the index set $S \subseteq [m]$ and $x^S := \prod_{i \in S} x_i$.

Then, the problem of computing the (logarithm of the) worst-case constant can be reformulated as

$$\sup_{p \in P_B} \inf_{y \in \mathbb{R}^m} \log \left( \sum_{\alpha \in \mathcal{F}} r_\alpha(p) e^{\langle \alpha - p, y \rangle} \right),$$

28
where $F \subseteq \{0, 1\}^m$ is the support of $r$ and $r_\alpha(p)$ is the coefficient of $\alpha$ in $r$, as a function of $p \in P_B$. Thus, the inner optimization problem is the dual of a certain max-entropy program. Moreover, the function
\[
p \mapsto \log \left( \sum_{\alpha \in F} r_\alpha(p) e^{(\alpha - p,y)} \right) = \log \left( \sum_{S \subseteq [m], |S| = n} p^S x^S \det(V_S V_S^T) \right)
\]
is concave over $\mathbb{R}^m_{>0}$, as the polynomial $p \mapsto \sum_{S \subseteq [m], |S| = n} p^S x^S \det(V_S V_S^T)$ (treating $x > 0$ as a vector of positive constants) is real stable, see Fact 8.2. From this point on, the argument follows along the same lines as the proof of Theorem 8.1.

\[\square\]

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