We consider sample covariance matrices of the form \( Q = (\Sigma^{1/2} X)(\Sigma^{1/2} X)^* \), where the sample \( X \) is an \( M \times N \) random matrix whose entries are real independent random variables with variance \( 1/N \) and where \( \Sigma \) is an \( M \times M \) positive-definite deterministic matrix. We analyze the asymptotic fluctuations of the largest rescaled eigenvalue of \( Q \) when both \( M \) and \( N \) tend to infinity with \( N/M \to d \in (0, \infty) \). For a large class of populations \( \Sigma \) in the sub-critical regime, we show that the distribution of the largest rescaled eigenvalue of \( Q \) is given by the type-1 Tracy-Widom distribution under the additional assumptions that (1) either the entries of \( X \) are i.i.d. Gaussians or (2) that \( \Sigma \) is diagonal and that the entries of \( X \) have a subexponential decay. We follow a new approach to the edge universality of deformed Wigner matrices introduced in [28].

1. Introduction

Covariance matrices are fundamental objects in multivariate statistics whose study is an integral part of various fields such as signal processing, genomics, financial mathematics, etc.. The sample covariance matrix is one of the simplest estimators of the population covariance matrix: The population covariance matrix of a mean-zero random variable \( y = (y_1, \ldots, y_M)' \in \mathbb{R}^M \) is given by \( \Sigma = \mathbb{E}yy' = (\mathbb{E}y_i y_j)_{i,j=1}^M \). Performing \( N \) independent measurements of \( y \) resulting in the samples \( (y_1, \ldots, y_N) \), \( \Sigma \) may be estimated through the sample covariance matrix \( \frac{1}{N} \sum_{i=1}^N y_i y_i' \). Changing the variables as \( y_i = (N\Sigma)^{1/2} x_i \), we are led to consider
\[
Q := (\Sigma^{1/2} X)(\Sigma^{1/2} X)^* ,
\]
where \( X = (x_1, \ldots, x_N) \). Note that \( \mathbb{E}Q = \Sigma \), thus for fixed \( M \), \( Q \) converges almost surely to \( \Sigma \) as \( N \) tends to infinity. Yet, in many modern applications the population size \( M \) may be as large or even larger than \( N \) and, hence, one may take \( M \) and \( N \) simultaneously to infinity in an asymptotic analysis.
In the present paper, we are interested in the limiting fluctuations of the largest eigenvalues of the matrix $Q$ as both $N$ and $M$ tend to infinity. Note that the matrix, $Q$, defined by
\[ Q := X^*\Sigma X , \] (1.2)
shares the same non zero eigenvalues as $Q$. For simplicity, we also refer to $Q$ as a sample covariance matrix in the following.

Motivated by the above, we consider random matrices of the form (1.2), where the “sample”, $X$, is an $M \times N$ matrix whose entries are a collection of independent real random variables of variance $1/N$ and where the general population covariance, $\Sigma$, is an $M \times M$ real positive-definite deterministic matrix. We are interested in the high-dimensional case, where $N/M \to d \in (0, \infty)$, as $N \to \infty$. For detailed discussions and applications of this model, we refer to, e.g., [3, 26, 12, 6]. We denote the eigenvalues of $Q$ and $\Sigma$ in decreasing order by $(\mu_i)_{i=1}^N$ and $(\sigma_m^m)_{m=1}^M$ respectively.

The main results of this paper show that the limiting distribution of the largest rescaled eigenvalue of $Q$, respectively $Q$, is given by the Tracy-Widom distribution, i.e.,
\[ \lim_{N \to \infty} \mathbb{P} \left( \gamma_0 N^{2/3} (\mu_1 - E_+) \leq s \right) = F_1(s), \quad (s \in \mathbb{R}), \] (1.3)
where $\gamma_0 \equiv \gamma_0(N)$ and $E_+ = L_+(N)$ depend only on the sequence $(\sigma_m^m)_{m=1}^M$ and the ratio $d$. Here $F_1$ denotes the cumulative distribution function (CDF) of the type-1 Tracy-Widom distribution [38, 39] which arises as the limiting CDF of the largest rescaled eigenvalue of the Gaussian orthogonal ensemble (GOE). More precisely, we show that (1.3) holds in the “sub-critical regime” where the largest eigenvalues of $\Sigma$ are close to the bulk of the spectrum of $\Sigma$ (for a precise statement see Assumption 2.3 below) and if either of the followings holds:

(1) the entries of $X$ are i.i.d. Gaussians (Corollary 2.8), or

(2) the general population $\Sigma$ is diagonal and the entries of $X$ have a subexponential decay (Theorem 2.5).

Our results may be easily extended to the complex case. In that setting, one replaces $F_1$ in (1.3) by $F_2$, the CDF of the type-2 Tracy-Widom distribution which arises as the limiting CDF of the largest eigenvalue of the Gaussian unitary ensemble (GUE).

To situate our result in the literature, we first recall that the limiting spectral distribution of the model (1.2) was derived for general $\Sigma$ by Marchenko and Pastur [30]. In the simplest case, where $\Sigma$ is the identity matrix and $X$ has Gaussian entries, $Q$ is called a Wishart matrix. For the Wishart ensemble it is well known that the limiting distribution of the largest rescaled eigenvalue coincides with the corresponding distribution of the GOE and GUE respectively: (1.3) was obtained in [26] for real Wishart matrices and in [25] for complex Wishart matrices.

In the “non-null” case, where $\Sigma$ is not a multiple of the identity matrix, the first results were obtained for “spiked population models” introduced in [26], where $\Sigma$ is a finite rank perturbation of the identity matrix. Complex spiked Wishart matrices were studied in detail in [4], where an interesting phase transition in the asymptotic behavior of the largest rescaled eigenvalue as a function of the spikes was observed. In particular, it was shown that the largest rescaled eigenvalue follows the Tracy-Widom distribution $F_2$ in the sub-critical regime, i.e., for small finite rank perturbations. These results rely on an explicit formula—the Baik-Ben Arous-Johansson-Péché (BBJP)-formula—for the joint eigenvalue distribution of deformed complex Wishart matrices. For full-rank deformations of complex Wishart matrices, sufficient conditions for the validity of (1.3) in the sub-critical regime were given in [12] for the non-singular case $d \in (0, \infty)$, $d \neq 1$, and [32] for the singular case $d = 1$. These results also rely on the BBJP-formula for complex deformed Wishart matrices.

For real deformed Wishart matrices, the counterpart of the BBJP-formula is not available, due to the lack of an analogue of the Harish-Chandra-Itzykson-Zuber integral for the orthogonal group. Relying on quite different methods, almost sure convergence of the largest eigenvalues was derived in [5] and Tracy-Widom fluctuations of the largest eigenvalue of spiked population models were obtained in [20]. The equivalent of the above mentioned phase transition were obtained in the real setting in [7, 8, 29, 23].
The aforementioned results are believed to be universal, i.e., independent of the details of the distributions of the entries of $X$ (provided they decay sufficiently fast). This phenomenon is often referred to as edge universality. It was established in the null case in [36, 33, 21] for symmetric distributions and subsequently in [41] for distributions with vanishing third moment. This third moment condition was removed in [34]. For spiked sample covariance matrices, universality results were obtained in [20] under the assumption that the entries’ distribution of $X$ are symmetric. This condition has been removed in [10]. For full rank deformed populations matrices $\Sigma$, universality results have recently been obtained in [6] under the assumption that $\Sigma$ is either diagonal, or that the first four moments of the entries’ distribution of $X$ match those of the standard Gaussian distribution in case $\Sigma$ is non-diagonal. Once the edge universality for full rank deformed covariance matrices has been established in the complex setting, the limiting CDF of the rescaled largest eigenvalues may then be identified with $F_2$ via the results of [12, 32]. Our main results allow to extend these conclusions to the real setting with $F_1$ replacing $F_2$.

Our proof of (1.3) is based on the Green function comparison method, which has been successful in proving the edge universality of Wigner matrices [19] and of null sample covariance matrices [34]. However, as for the deformed Wigner matrices considered in [28], a direct application of the conventional Green function comparison via Lindeberg’s replacement strategy [37] does not work for the sample covariance matrices in the non-null case. We thus adopt the new approach to the Green function comparison developed in [28]: we consider a continuous interpolation between the given sample covariance matrix and a null sample covariance matrix. We follow the associated flow of the Green function and estimate its change over time. This change is then offset by renormalizing the matrix, as discussed in Section 6. We remark that the same tactic can also be applied to the complex sample covariance matrices to prove analogous results.

Our analysis requires as an a priori ingredient a local law for the Green function of $Q$, i.e., an optimal estimate on the entries of the Green function of $Q$ on scales slightly below $N^{-2/3}$ at the upper edge. Optimal local laws in the bulk and at the edges of the spectrum were obtained for Wigner matrices in [18, 17, 15]. Using a similar approach, optimal local laws for sample covariance matrices with $\Sigma = 1$ were obtained in [34]; see also [9, 16]. Recently, these results were extended to sample covariance matrices with general population in [6] for energies close to the upper edge of the spectrum; see Lemma 3.3 below for a precise statement.

This paper is organized as follows: In Section 2, we precisely define the model and present the main results of the paper. In Sections 3, we collect the tools and known results we need in our proofs. In Section 4, we prove the main theorems using our essential new technical result, Proposition 4.1, the Green function comparison theorem at the edge. In Sections 5 and 6, we outline the ideas of the proof of the Green function comparison theorem. Its technical details can be found in the Sections 7-9. Some results required in these sections are adaptations from [28].

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2. Definition and Main Result

2.1. Sample covariance matrix with general population.

**Definition 2.1.** Let $X = (x_{ij})$ be an $M \times N$ matrix whose entries $\{x_{ij} : 1 \leq i \leq M, 1 \leq j \leq N\}$ are a collection of independent real random variables such that

$$\mathbb{E} x_{ij} = 0, \quad \mathbb{E} |x_{ij}|^2 = \frac{1}{N}. \quad (2.1)$$

Moreover, we assume that $(\sqrt{N}x_{ij})$ have a subexponential tail, i.e., there are $C$ and $\vartheta > 0$ such that

$$\mathbb{P}(|\sqrt{N}x_{ij}| > t) \leq Ce^{-t^\vartheta}, \quad (2.2)$$

for all $i, j$. 


Further, $M \equiv M(N)$ with
\[ \hat{d} = \frac{N}{M} \to d \in (0, \infty), \]
(2.3)
as $N \to \infty$. For simplicity, we assume that $N/M$ is constant, hence we use $d$ instead of $\hat{d}$.

Let $\Sigma$ be an $M \times M$ real positive-definite deterministic matrix. We denote by $\hat{\rho}$ the empirical eigenvalue distribution of $\Sigma$, i.e., if we let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_M \geq 0$ be the eigenvalues of $\Sigma$, then
\[ \hat{\rho} := \frac{1}{M} \sum_{j=1}^{M} \delta_{\sigma_j}. \]
(2.4)

**Definition 2.2.** A sample covariance matrix, $Q$, with sample $X$ and general population $\Sigma$ is an $N \times N$ matrix defined by
\[ Q := X^* \Sigma X, \]
(2.5)
with $X$ as in Definition 2.1.

In most applications, it is natural to assume that the columns of $X$ are identically distributed. For our results to hold this is not needed and we hence do not assume it in Definition 2.1.

2.2. **Deformed Marchenko Pastur law.** If the empirical spectral distribution $\hat{\rho}$ of $\Sigma$ converges weakly to some distribution $\rho$, it was shown in [30] that the empirical eigenvalue distribution of $Q$ converges weakly in probability to a deterministic distribution, $\rho_{fc}$, referred to as the “deformed Marchenko-Pastur law” below, which depends on $\rho$ and the ratio $d$. It can be described in terms of its Stieltjes transform: For a (probability) measure $\omega$ on the real line we define its Stieltjes transform, $m_\omega$, by
\[ m_\omega(z) := \int \frac{d\omega(v)}{v - z}, \quad (z = E + i\eta \in \mathbb{C}^+). \]
(2.6)

Here and below, we write $z = E + i\eta$, with $E \in \mathbb{R}, \eta \geq 0$. Note that $m_\omega$ is an analytic function in the upper half plane and that $\text{Im} m_\omega(z) \geq 0, \text{Im} z > 0$. Assuming that $\omega$ is absolutely continuous with respect to Lebesgue measure, we can recover the density of $\omega$ from $m_\omega$ by the inversion formula
\[ \omega(E) = \lim_{\eta \downarrow 0} \frac{1}{\pi} \text{Im} m_\omega(E + i\eta), \quad (E \in \mathbb{R}). \]
(2.7)

We use the same symbols to denote measures and their densities.

Choosing $\omega$ to be the standard Marchenko-Pastur law $\rho_{MP}$, the Stieltjes transform $m_{\rho_{MP}} \equiv m_{MP}$ can be computed explicitly and one checks that $m_{MP}$ satisfies the relation
\[ m_{MP}(z) = \frac{1}{-z + d^{-1} m_{MP}(z) + 1}, \quad \text{Im} m_{MP}(z) \geq 0, \quad (z \in \mathbb{C}^+). \]
(2.8)

The deformed Marchenko-Pastur law $\rho_{fc}$ is defined as follows. Assume that $\hat{\rho}$ converges weakly to $\rho$ as $N$ goes to infinity. Then the Stieltjes transform of the deformed Marchenko-Pastur law, $m_{fc}$, is obtained as the unique solution to the self-consistent equation
\[ m_{fc}(z) = \frac{1}{-z + d^{-1} \int_{m_{fc}(\xi) + 1}^{1} \text{d}\rho(t)}, \quad \text{Im} m_{fc}(z) \geq 0, \quad (z \in \mathbb{C}^+). \]
(2.9)

It is well known [30] that the functional equation (2.9) has a unique solution that satisfies
\[ \limsup_{\eta \downarrow 0} \text{Im} m(E + i\eta) < \infty, \]
for all $E \in \mathbb{R}$. Then the density of the deformed Marchenko-Pastur law $\rho_{fc}$ is obtained from $m_{fc}$ by the Stieltjes inversion formula (2.7). The measure $\rho_{fc}$ has been studied in detail in [35], e.g., it was shown that the density of $\rho_{fc}$ is an analytic function inside its support.
The measure $\rho_{\Sigma}$ is also called the multiplicative free convolution of the Marchenko-Pastur law and the measure $\rho$; we refer to, e.g., [40, 1] for details.

For finite $N$, we let $\hat{m}_{\Sigma}$ denote the unique solution to
$$
\hat{m}_{\Sigma}(z) = \frac{1}{-z + \sigma^{-1} \int \frac{t}{\hat{m}_{\Sigma}(z)+1} d\hat{\rho}(t)}, \quad \Im \hat{m}_{\Sigma}(z) \geq 0, \quad (z \in \mathbb{C}^+),
$$
and let $\hat{\rho}_{\Sigma}$ denote the measure obtained from $\hat{m}_{\Sigma}(z)$ through (2.7). It is easy to check that $\hat{\rho}_{\Sigma}$ is a well-defined probability measure with a continuous density.

The rightmost endpoint of the support of $\rho_{\Sigma}$ can be determined as follows. Define $\xi_+$ as the largest solution to
$$
\int \left( \frac{t \xi_+}{1 - t \xi_+} \right)^2 d\hat{\rho}(t) = d,
$$
with $d = \frac{N}{M}$. Note that $\xi_+$ is unique and that $\xi_+ \in [0, \sigma_1^{-1}]$. We also introduce $E_+$ by setting
$$
E_+ := \frac{1}{\xi_+} \left( 1 + \sigma^{-1} \int \frac{t \xi_+}{1 - t \xi_+} d\hat{\rho}(t) \right).
$$

Considering the imaginary part of (2.10) in the limit $\eta \searrow 0$, one sees that the rightmost edge of $\hat{\rho}_{\Sigma}$, i.e., the rightmost endpoint of the support of $\hat{\rho}_{\Sigma}$, is given by $E_+$ and that
$$
\xi_+ = - \lim_{\eta \to 0} \hat{m}_{\Sigma}(E_+ + i\eta) = -\hat{m}_{\Sigma}(E_+);
$$
see [35].

The following assumption on $\Sigma$ is required to establish our main results. It appeared previously in [12, 6].

**Assumption 2.3.** Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_M$ denote the eigenvalues of $\Sigma$. Then, we assume that \( \liminf_N \sigma_M > 0, \limsup_N \sigma_1 < \infty \) and
$$
\limsup_{N} \sigma_1 \xi_+ < 1. \tag{2.14}
$$

Further we assume that $d = N/M \in (0, \infty)$.

**Remark 2.4.** We remark that Assumption 2.3 was used in [6] to derive the local deformed Marchenko-Pastur law for $Q$. The inequality (2.14) guarantees that the distribution $\hat{\rho}_{\Sigma}(E)$ exhibits a square-root type behavior at the rightmost endpoint of its support; see Lemma 3.2 below.

2.3. **Main result.** The main result of this paper is as follows:

**Theorem 2.5.** Let $Q = X^{*} \Sigma X$ be an $N \times N$ sample covariance matrix with sample $X$ and population $\Sigma$, where $X$ is a real random matrix satisfying the assumptions in Definition 2.1 and $\Sigma$ is a real diagonal deterministic matrix satisfying Assumption 2.3. Recall that $F_1$ denotes the cumulative distribution function of the type-1 Tracy-Widom distribution.

Let $\mu_1$ be the largest eigenvalue of $Q$. Then, there exist $\gamma_0 \equiv \gamma_0(N)$ depending only on the empirical eigenvalue distribution $\hat{\rho}$ of $\Sigma$ and the ration $d$ such that the distribution of the largest rescaled eigenvalue of $Q$ converges to the Tracy-Widom distribution, i.e.,
$$
\lim_{N \to \infty} P \left( \gamma_0 N^{2/3} (\mu_1 - E_+) \leq s \right) = F_1(s),
$$
for all $s \in \mathbb{R}$, where $E_+ \equiv E_+(N)$ is given in (2.12).

**Remark 2.6.** The scaling factor $\gamma_0 \equiv \gamma_0(N)$ is given by [12]
$$
\frac{1}{\gamma_0} = \frac{1}{d} \int \left( \frac{t}{1 - t \xi_+} \right)^3 d\hat{\rho}(t) + \frac{1}{\xi_+}.
$$
It follows from Assumption 2.3 that $\gamma_0 = O(N^0)$. 

Remark 2.7. Theorem 2.5 can be extended to correlation functions of the extremal eigenvalues as follows: Let $W^{\text{GOE}}$ be an $N \times N$ random matrix belonging to the Gaussian Orthogonal Ensemble (GOE); see [1, 31]. The joint distributions of $\mu_1^{\text{GOE}} \geq \mu_2^{\text{GOE}} \geq \cdots \geq \mu_N^{\text{GOE}}$, the eigenvalues of $W^{\text{GOE}}$, are explicit and the joint distribution of the $k$ largest eigenvalues can be written in terms of the Airy kernel [22] for any fixed $k$.

The generalization of (2.15) to the $k$ largest eigenvalues of $Q$ then reads

$$
\lim_{N \to \infty} \mathbb{P} \left( \gamma_0 N^{2/3} (\mu_1 - E_+) \leq s_1, \gamma_0 N^{2/3} (\mu_2 - E_+) \leq s_2, \ldots, \gamma_0 N^{2/3} (\mu_k - E_+) \leq s_k \right) = \lim_{N \to \infty} \mathbb{P} \left( N^{2/3} (\mu_1^{\text{GOE}} - 2) \leq s_1, N^{2/3} (\mu_2^{\text{GOE}} - 2) \leq s_2, \ldots, N^{2/3} (\mu_k^{\text{GOE}} - 2) \leq s_k \right),
$$

(2.17)

for all $s_1, s_2, \ldots, s_k \in \mathbb{R}$.

If the sample matrix $X$ has Gaussian entries, the result in Theorem 2.5 holds for general, non-diagonal $\Sigma$.

Corollary 2.8. Let $Q = X^* \Sigma X$ be an $N \times N$ sample covariance matrix with sample $X$ and general population $\Sigma$, where $X$ is a real random matrix with independent Gaussian entries satisfying the assumptions in Definition 2.1 and $\Sigma$ is a real positive-definite deterministic matrix satisfying Assumption 2.3. Let $\mu_1$ be the largest eigenvalue of $Q$.

Then the distribution of the largest rescaled eigenvalue of $Q$ converges to the type-1 Tracy-Widom distribution, i.e.,

$$
\lim_{N \to \infty} \mathbb{P} \left( \gamma_0 N^{2/3} (\mu_1 - E_+) \leq s \right) = F_1(s),
$$

(2.18)

for all $s \in \mathbb{R}$, where $E_+ \equiv E_+(N)$ is given in (2.12) and $\gamma_0 = \gamma_0(N)$ is given in (2.16).

Remark 2.9. For non-Gaussian $X$ and general off-diagonal $\Sigma$, we can combine our results with Theorem 1.3 of [6] to identify the Tracy-Widom distribution for the largest eigenvalues under a four moment matching condition. In other words, assuming that the first four moments of the entries $(x_{ij})$ of $X$ agree with the moment of the standard normal distribution, (2.18) and its generalization to the $k$ largest eigenvalues hold true.

Remark 2.10. For deformed Wigner matrices, the edge universality was obtained in [28] via Green function comparison. Combining an entropy estimate in Proposition 5.3 of [27] with [11] this result may also be obtained via the local ergodicity of Dyson’s Brownian motion (DBM) at the edge [11]. The strength of that approach is that it allows varying variances, i.e., generalized Wigner matrices.

For sample covariance matrices with general population, however, this approach seems to be difficult to implement since the evolution of the eigenvalues under the DBM is not decoupled from the evolution of the eigenvectors unless $\Sigma$ is a multiple of the identity. Our approach, on the other hand, does not allow for general varying variances in $X$. Yet, for the statistics problems of interest to us, all columns of $X$ have the same distribution so varying variances only occur along the rows and they can therefore be absorbed into the population matrix $\Sigma$.

3. Preliminaries

3.1. Notations. We first introduce a notation for high-probability estimates which is suited for our purposes. A slightly different form was first used in [13].

Definition 3.1. Let

$$
X = (X^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)}), \quad Y = (Y^{(N)}(u) : N \in \mathbb{N}, u \in U^{(N)})
$$

(3.1)

be two families of nonnegative random variables where $U^{(N)}$ is a possibly $N$-dependent parameter set. We say that $Y$ stochastically dominates $X$, uniformly in $u$, if for all (small) $\epsilon > 0$ and (large) $D > 0$,

$$
\sup_{u \in U^{(N)}} \mathbb{P} \left[ X^{(N)}(u) > N^\epsilon Y^{(N)}(u) \right] \leq N^{-D},
$$

(3.2)
for sufficiently large $N \geq N_0(\epsilon, D)$. If $Y$ stochastically dominates $X$, uniformly in $u$, we write $X \prec Y$. If for some complex family $X$ we have $|X| \prec Y$ we also write $X = \mathcal{O}(Y)$.

The relation $\prec$ is a partial ordering: it is transitive and it satisfies the arithmetic rules of an order relation, e.g., if $X_1 \prec Y_1$ and $X_2 \prec Y_2$ then $X_1 + X_2 \prec Y_1 + Y_2$ and $X_1 X_2 \prec Y_1 Y_2$. Further assume that $\Phi(u) \geq N^{-C}$ is deterministic and that $Y(u)$ is a nonnegative random variable satisfying $\mathbb{E}[Y(u)]^{2} \leq N^{C'}$ for all $u$. Then $Y(u) \prec \Phi(u)$, uniformly in $u$, implies $\mathbb{E}[Y(u)] \prec \Phi(u)$, uniformly in $u$.

We use the symbols $O(\cdot)$ and $o(\cdot)$ for the standard big-O and little-o notation. The notations $O$,$o$, $\ll$, $\gg$, refer to the limit $N \to \infty$ unless otherwise stated. Here $a \ll b$ means $a = o(b)$. We use $c$ and $C$ to denote positive constants that do not depend on $N$, usually with the convention $c \leq C$. Their value may change from line to line. We write $a \sim b$, if there is $C \geq 1$ such that $C^{-1}|b| \leq |a| \leq C|b|$.

Finally, we use double brackets to denote index sets, i.e.,
\[
[n_1, n_2] := [n_1, n_2] \cap \mathbb{R},
\]
for $n_1, n_2 \in \mathbb{R}$.

3.2. Local deformed Marchenko-Pastur law. For small positive $\epsilon, \epsilon$ and sufficiently large $C_+$, $(C_+ > E_+)$, we define the domain, $\mathcal{D}(\epsilon, \epsilon)$, of the spectral parameter $z$ by
\[
\mathcal{D}(\epsilon, \epsilon) := \{z = E + \im \eta \in \mathbb{C}^+ : E_+ - \epsilon \leq E \leq C_+, N^{-1/2} \eta \leq \eta \leq 1\}.
\] (3.3)

Let $\kappa \equiv \kappa_\epsilon := |E - E_+|$. Then we have the following results:

**Lemma 3.2** (Theorem 3.1 in [6]). Under Assumption 2.3, there is $c > 0$ such that
\[
\hat{\rho}_\kappa(E) \sim \sqrt{E_+ - E}, \quad (E \in [E_+ - 2c, E_+]).
\] (3.4)

The Stieltjes transform $\hat{m}_\kappa(z)$ of $\hat{\rho}_\kappa$ satisfies the following.

i. For $z \in \mathcal{D}(\epsilon, 0)$,
\[
|\hat{m}_\kappa(z)| \sim 1.
\] (3.5)

ii. For $z \in \mathcal{D}(\epsilon, 0)$,
\[
\text{Im} \hat{m}_\kappa(z) \sim \begin{cases} 
\frac{\eta}{\sqrt{\kappa + \eta}}, & \text{if } E \geq E_+ + \eta, \\
\frac{\eta}{\sqrt{\kappa + \eta}}, & \text{if } E \in [E_+ - \epsilon, E_+ + \eta].
\end{cases}
\] (3.6)

We introduce the $z$-dependent control parameter, $\Psi(z)$, by setting
\[
\Psi \equiv \Psi(z) := \left(\frac{\text{Im} \hat{m}_\kappa(z)}{N\eta}\right)^{1/2} + \frac{1}{N\eta}.
\] (3.7)

We remark that, for $z = E + \im \eta$ with $\kappa_E \leq N^{-2/3+\epsilon}$ and $\eta = N^{-2/3-\epsilon}$, we have
\[
\Psi \leq C N^{-2/3+\epsilon}.
\]

Define the Green function $G_Q = ((G_Q)_{ij})$ by
\[
G_Q(z) := (Q - z)^{-1}, \quad (z \in \mathbb{C}^+),
\] (3.8)
and denote its average by
\[
m_Q(z) := \frac{1}{N} \text{Tr} G_Q(z), \quad (z \in \mathbb{C}^+).
\] (3.9)

Recall that $\mu_1$ denotes the largest eigenvalue of the sample covariance matrix $Q$. We have the following local law from [6].
Lemma 3.3 (Theorem 3.2 and Theorem 3.3 in [6]). Under the Assumption 2.3, we have, for any sufficiently small \( \epsilon > 0 \),

\[
|m_Q(z) - \hat{m}_k(z)| < \frac{1}{N\eta}, \quad \max_{i,j} |(G_Q)_{ij}(z) - \delta_{ij}\hat{m}_k(z)| \prec \Psi(z),
\]

uniformly in \( z \) on \( D(\epsilon, c) \), where \( c \) is the constant in Lemma 3.2. Moreover, we have

\[
|\mu_1 - E_+| \prec N^{-2/3},
\]

where \( E_+ \) is given in (2.12).

3.3. Density of states. In this subsection, we explain how the distribution of the largest eigenvalues of \( Q \) can be related to \( m_Q(z) \) for appropriately chosen \( z \). The arguments given here are small modifications of the methods presented in [19, 34, 28].

Recall the definition of the scaling factor \( \gamma_0 \) in (2.16). We set \( T := \gamma_0 \Sigma \),

\[
\tilde{Q} := X^*TX.
\]

We denote by \( \tilde{m}_Q \) the averaged Green function of \( \tilde{Q} \), i.e.,

\[
\tilde{m}_Q(z) := \frac{1}{N}
\text{Tr}((\tilde{Q} - z)^{-1}), \quad (z \in \mathbb{C}^+).
\]

Let \( \tilde{\mu}_1 \geq \tilde{\mu}_2 \geq \cdots \geq \tilde{\mu}_N \) be the eigenvalues of \( \tilde{Q} \). Let \( L_+ := \gamma_0 E_+ \) and observe that from Lemma 3.3 we have

\[
|\tilde{\mu}_1 - L_+| \prec N^{-2/3}.
\]

Thus, we may assume in (2.15) that \( |s| < 1 \).

Fix \( E_+ \) such that

\[
E_+ - L_+ < N^{-2/3}, \quad \mathbb{1}(\mu_1 > E_+) \prec 0.
\]

We note that the choice of \( E_+ \) guarantees that the probability of the event \( \{\mu_1 > E_+\} \) is negligible. For \( E \) satisfying

\[
|E - L_+| \prec N^{-2/3},
\]

we let

\[
\chi_E := \mathbb{1}[E, E_+].
\]

We also define the Poisson kernel, \( \theta_\eta \), for \( \eta > 0 \),

\[
\theta_\eta(x) := \frac{\eta}{\pi(x^2 + \eta^2)} = \frac{1}{\pi} \text{Im} \frac{1}{x - i\eta}.
\]

Introduce a smooth cutoff function \( K : \mathbb{R} \rightarrow \mathbb{R} \) satisfying

\[
K(x) = \begin{cases} 
1 & \text{if } x \leq 1/9, \\
0 & \text{if } x \geq 2/9.
\end{cases}
\]

Let \( \mathcal{N}(E_1, E_2) \) be the number of the eigenvalues in \( (E_1, E_2) \), i.e.,

\[
\mathcal{N}(E_1, E_2) := |\{\alpha : E_1 < \tilde{\alpha} \leq E_2\}|,
\]

and define the density of states in the interval \( (E_1, E_2) \) by

\[
n(E_1, E_2) := \frac{1}{N} \mathcal{N}(E_1, E_2).
\]

In order to estimate \( \mathbb{P}(\tilde{\mu}_1 \leq E) \), we consider the following approximations:

\[
\mathbb{P}(\tilde{\mu}_1 \leq E) = \mathbb{E}K(\mathcal{N}(E, \infty)) \simeq \mathbb{E}K(\mathcal{N}(E, E_+)) \simeq \mathbb{E}K \left( N \int_{E}^{E_+} \text{Im} m_Q(y + i\eta) \, dy \right),
\]

(3.17)
with $\eta \sim N^{-2/3-\epsilon'}$, for some small $\epsilon' > 0$. The first approximation in (3.17) follows from Lemma 3.3, the rigidity of the eigenvalues, and the second from

$$N(E, E_*) = \text{Tr} \chi_E(H) \simeq \text{Tr} \chi_E * \theta_{\eta}(H) = \frac{1}{\pi} N \int_{E} \text{Im} m_Q(y + i\eta) \, dy.$$ 

The following lemma shows that the approximations in (3.17) indeed hold.

**Lemma 3.4.** Suppose that $E$ satisfies (3.15). For $\epsilon > 0$, let $\ell := \frac{1}{2} N^{-2/3-\epsilon}$ and $\eta := N^{-2/3-9\epsilon}$. Recall that $K$ is a smooth function satisfying (3.16). Then, for any sufficiently small $\epsilon > 0$ and any (large) $D > 0$, we have

$$\text{Tr} (\chi_{E+\ell} * \theta_{\eta}(H)) - N^{-\epsilon} \leq n(E, \infty) \leq \text{Tr} (\chi_{E-\ell} * \theta_{\eta}(H)) + N^{-\epsilon} \quad (3.18)$$

and

$$\mathbb{E} K \left( \text{Tr} (\chi_{E-\ell} * \theta_{\eta}(H)) \right) \leq \mathbb{P} (\tilde{\eta}_1 \leq E) \leq \mathbb{E} K \left( \text{Tr} (\chi_{E+\ell} * \theta_{\eta}(H)) \right) + N^{-D}, \quad (3.19)$$

for any sufficiently large $N \geq N_0(\epsilon, D)$.

**Proof.** We may follow the proof of Corollary 6.2 of [34]. Note that the estimates on $|m_Q(E + i\ell) - \tilde{m}_k(E + i\ell)|$ and $\text{Im} \tilde{m}_k(E - \kappa + i\ell)$, which replace similar estimates with respect to $m_c$ in the proof of Corollary 6.2 in [34], are already proved in Lemma 3.2 and Lemma 3.3. \qed

### 4. Green Function Comparison and Proof of the Main Result

Having established Lemma 3.4, the proof of Theorem 2.5 directly follows from our main technical result: the Green function comparison theorem at the edge, Proposition 4.1 below. It compares the expectations of functions of the averaged Green functions of $Q$ and $X^*X$. More precisely, we let

$$W := \sqrt{d} (1 + \sqrt{d})^{-4/3} X^* X, \quad (4.1)$$

and introduce

$$m_W(z) := \frac{1}{N} \text{Tr}(W - z)^{-1}, \quad (z \in \mathbb{C}^+).$$

It is well known that the distribution of the rescaled largest eigenvalue of $W$ converges to the Tracy-Widom distribution; see [34].

Our main technical result is as follows. Recall that we write $L_+ = \gamma_0 E_+$, with $E_+$ given in (2.12) and with $\gamma_0$ given in (2.16).

**Proposition 4.1 (Green function comparison).** Let $\epsilon > 0$ and set $\eta = N^{-2/3-\epsilon}$. Denote by $M_+$ the upper edge of the Marchenko-Pastur law $\rho_{MP}$ for $W = \sqrt{d}(1 + \sqrt{d})^{-4/3} X^* X$. Let $E_1, E_2 \in \mathbb{R}$ satisfy $E_1 < E_2$ and

$$|E_1|, |E_2| \leq N^{-2/3+\epsilon}. \quad (4.2)$$

Let $F : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying

$$\max_x |F^{(\ell)}(x)(|x| + 1)^{-C} \leq C, \quad \ell = 1, 2, 3, 4. \quad (4.3)$$

Then, there exists a constant $\phi > 0$ such that, for any sufficiently large $N$ and for any sufficiently small $\epsilon > 0$, we have

$$\left| \mathbb{E} F \left( N \int_{E_1}^{E_2} \text{Im} m_Q(x + L_+ + i\eta) \, dx \right) - \mathbb{E} F \left( N \int_{E_1}^{E_2} \text{Im} m_W(x + M_+ + i\eta) \, dx \right) \right| \leq N^{-\phi}. \quad (4.4)$$

We outline the proof of Proposition 4.1 in the Sections 7, 8 and 9.
Remark 4.2. Proposition 4.1 can be extended as follows: Let \( \epsilon > 0 \) and set \( \eta = N^{-2/3-\epsilon} \). Let \( E_0, E_1, \ldots, E_k \in \mathbb{R} \) satisfy \( E_1 < E_2 < \cdots < E_k \) and
\[
|E_0| \leq N^{-2/3+\epsilon}, \quad |E_1 - E_0| \leq N^{-2/3+\epsilon}, \quad \ldots, \quad |E_k - E_0| \leq N^{-2/3+\epsilon}.
\]

Let \( F : \mathbb{R}^k \to \mathbb{R} \) be a smooth function satisfying
\[
\max_x |F^{(\ell)}(x)|(|x| + 1)^{-C} \leq C, \quad \ell = 1, 2, 3, 4.
\]
Then, there exists a constant \( \phi > 0 \) such that, for any sufficiently large \( N \) and for any sufficiently small \( \epsilon > 0 \), we have
\[
\left| \mathbb{E} F \left( N \int_{E_0}^{E_1} \text{Im} m_Q(x + L_+ + i\eta) \, dx, \ldots, N \int_{E_0}^{E_k} \text{Im} m_Q(x + L_+ + i\eta) \, dx \right) \right| \leq N^{-\phi}.
\]
The proof of (4.5) is similar to that of Proposition 4.1 and will be omitted.

Assuming the validity of Proposition 4.1, we now prove our main results.

Proof of Theorem 2.5. We follow the proof of Theorem 1.10 of [34]. Let \( \mu_1^W \) be the largest eigenvalue of \( W \) (see (4.1)) and denote by \( M_+ \) the upper edge of the rescaled Marchenko-Pastur law \( \rho_{\text{MP}} \). We notice that the distribution of \( N^{2/3} (\mu_1^W - M_+) \) converges to the Tracy-Widom law \( F_1 \). (See [36, 33, 21, 34].) Thus, in order to prove (2.15), it suffices to show that
\[
\mathbb{P} \left[ N^{2/3} (\mu_1^W - M_+) \leq s \right] - N^{-\phi} \leq \mathbb{P} \left[ N^{2/3} (\tilde{\mu}_1 - L_+) \leq s \right] \leq \mathbb{P} \left[ N^{2/3} (\mu_1^W - M_+) \leq s \right] + N^{-\phi},
\]
for some \( \phi > 0 \).

Fix \( |s| < 1 \) and let \( E := L_+ + sN^{-2/3} \). Let \( \ell := \frac{1}{2} N^{-2/3-\epsilon} \) and \( \eta := N^{-2/3-9\epsilon} \). For any sufficiently small \( \epsilon > 0 \), we have from Lemma 3.4 that
\[
\mathbb{P}(\tilde{\mu}_1 \leq E) \geq \mathbb{E} \left[ K \left( \text{Tr} \left( \chi_{E-\ell} \ast \theta_\eta(H) \right) \right) \right].
\]
From Proposition 4.1, we find that
\[
\mathbb{E} \left[ K \left( \text{Tr} \left( \chi_{E-\ell} \ast \theta_\eta(H) \right) \right) \right] \geq \mathbb{E} \left[ K \left( \text{Tr} \left( \chi_{E-(L_+-M_+)-\ell} \ast \theta_\eta(W) \right) \right) \right] - N^{-\phi},
\]
for some \( \phi > 0 \). Finally, we have from Corollary 6.2 of [34] that
\[
\mathbb{E} \left[ K \left( \text{Tr} \left( \chi_{E-(L_+-M_+)-\ell} \ast \theta_\eta(W) \right) \right) \right] \geq \mathbb{P} \left( \mu_1^W \leq E - (L_+ - M_+) \right) - N^{-\phi}.
\]
Altogether, we have shown that
\[
\mathbb{P}(\tilde{\mu}_1 \leq E) \geq \mathbb{P} \left( \mu_1^W \leq E - (L_+ - M_+) \right) - 2N^{-\phi},
\]
which proves the first inequality of (4.6). The second inequality can be proved similarly. 

Proof of Corollary 2.8. Let \( U \) be an \( M \times M \) orthogonal matrix that diagonalizes \( \Sigma \), i.e., there exists an \( M \times M \) real diagonal matrix \( D \) such that \( \Sigma = U^* D U \). Then, \( UX \) is a real random matrix with Gaussian entries, satisfying the assumptions in Definition 2.1. Furthermore, \( D \) is a real diagonal deterministic matrix satisfying Assumption 2.2. Thus, applying Theorem 2.5 with \( X^* \Sigma X = (UX)^* D(UX) \), we get the desired result.
In this section, we recall a well-known formalism that simplifies the computations in the proof of Proposition 4.1 considerably. Instead of working with the product matrices \( Q = X^* TX \) or \( T^{1/2} X X^* T^{1/2} \), we may “linearize” the problem by introducing an \((N + M) \times (N + M)\) matrix \( H \), whose respective entries are either \((x_{\alpha \beta}), (t_{\alpha}^{-1}), \) or simply zero. The inverse of \( H \) is then related to the Green function of \( X^* TX \), respectively of \( T^{1/2} X X^* T^{1/2} \), through Schur’s complement formula or the Feshbach map. For similar applications in random matrix theory see, e.g., [2, 24].

The linearization of \( \tilde{Q} \) is established in Subsection 5.1. In the Subsections 5.2, 5.3 and 5.4, we collect useful technical results on the inverse of \( H \).

### 5.1. Schur complement

Suppose that \( X \) and \( \Sigma \) satisfy the assumptions in Theorem 2.5. Let \( z \) be as in the previous section. We define an \((N + M) \times (N + M)\) matrix \( H \) as follows. Let \( P \) be the projection on the first \( N \) coordinates in \( \mathbb{C}^{N+M} \) and set \( \overline{P} := \mathbb{1} - P \). Then, we write

\[
H = PHP + PHP\overline{P} + \overline{P}PH + \overline{P}H\overline{P},
\]

where

\[
PHP := -z\mathbb{1}, \quad PHP\overline{P} := X^*, \quad \overline{P}PH := X, \quad \overline{P}H\overline{P} := -T^{-1},
\]

with \( T = \gamma_0 \Sigma \). Note that \( H(z) \) is invertible for \( z \in \mathbb{C}^+ \): Assuming that \( v, v \neq 0 \), is in the kernel of \( H(z), \) \( \text{Im} z > 0 \), and writing \( v_P := Pv \) and \( v_{\overline{P}} := \overline{P}v \), we must have

\[
-zv_P + X^* v_{\overline{P}} = 0, \quad XV_P - T^{-1}v_{\overline{P}} = 0.
\]

Thus,

\[
XX^* v_{\overline{P}}/z = T^{-1}v_{\overline{P}}.
\]

Hence, taking the inner product with \( v_{\overline{P}} \), we find that the left side is not real while the right side is. We thus get a contradiction allowing us to conclude that \( v = 0 \).

We define the “Green function”, \( G \), of \( H \equiv H(z) \) by

\[
G(z) := H(z)^{-1}, \quad (z \in \mathbb{C}^+),
\]

and the averages, \( m \) and \( \bar{m} \), of \( G \) by

\[
m(z) := \frac{1}{N} \sum_{a=1}^{N} G_{aa}(z), \quad \bar{m}(z) := \frac{1}{M} \sum_{\alpha=N+1}^{M+N} G_{\alpha\alpha}(z), \quad (z \in \mathbb{C}^+).
\]

Note that by Schur’s complement formula we have

\[
PG(z)P = \frac{1}{PH(z)P - PH(z)\overline{P} \frac{1}{PH(z)\overline{P}} PH(z)P} = \frac{1}{-zP + X^* TX},
\]

so that

\[
G_{ab}(z) = \left((\bar{Q} - z)^{-1}\right)_{ab},
\]

for any \( a, b \in [1, N] \). In particular,

\[
m(z) = m_{\bar{Q}}(z).
\]

Also note that

\[
z^{-1}\overline{P}G(z)\overline{P} = \frac{z^{-1}}{\overline{P}H\overline{P} - \overline{P}HP \frac{1}{PH\overline{P}} PH\overline{P}} = \frac{1}{-zT^{-1} - XX^*}.
\]

In the following we use lowercase Roman letters for indices in \([1, N]\), Greek letters for indices in \([N + 1, M + N]\) and uppercase Roman letters for indices in \([1, N + M]\).
5.2. Green function, minors and partial expectations. Recall the definitions of the $(N + M) \times (N + M)$ matrix $H \equiv H(z)$ in (5.1) and of the Green function $G$ in (5.3).

Let $T \subset [1, N + M]$. We then define $H^{(T)}$ as the $(N + M - |T|) \times (N + M - |T|)$ minor of $H$ obtained by removing all columns and rows of $H$ indexed by $T$. We do not change the names of the indices of $H$ when defining $H^{(T)}$. More specifically, we define an operation $\pi_A$, $A \in [1, N + M]$, on the probability space by

$$
(\pi_A(H))_{BC} := 1(B \neq A)1(C \neq A)b_{BC}.
$$

(5.7)

Then, for $T \subset [1, N + M]$, we set $\pi_T := \prod_{A \in T} \pi_A$ and define

$$
H^{(T)} := ((\pi_T(H))_{BC})_{B,C \notin T}.
$$

(5.8)

The Green functions $G^{(T)}$, are defined in an obvious way using $H^{(T)}$. Moreover, we use the shorthand notations

$$
\sum_a^{(T)} := \sum_{a=1}^{N}, \quad \sum_{a \neq b}^{(T)} := \sum_{a=1, b=1}^{N} \{a \notin T, a \neq b \notin T\}, \quad \sum_{a=1}^{N+M} \{a \notin T\}, \quad \sum_{a \neq b}^{N+M} \{a \notin T, b \notin T\}
$$

(5.9)

and abbreviate (A) = ({A}), (TA) = (T \cup {A}). In Green function entries $(G^{(T)}_{AB})$ we refer to {A, B} as lower indices and to $T$ as upper indices.

We further set

$$
m^{(T)} := \frac{1}{N} \sum_a^{(T)} G^{(T)}_{aa}, \quad \tilde{m}^{(T)} := \frac{1}{M} \sum_a^{(T)} G^{(T)}_{aa}.
$$

(5.10)

Note that we use the normalizations $N^{-1}$ and $M^{-1}$ here since they are more convenient in computations.

Finally, we denote by $E_a$, $E_\alpha$ the partial expectation with respect to the variables $(x_{aa})_{\alpha = M+1}^{N+M}$, respectively $(x_{aa})_{a=1}^{N}$.

5.3. Green function identities. The next lemma collects the main identities between the matrix elements of $G$ and $G^{(T)}$.

**Lemma 5.1.** Let $G \equiv G(z)$, $z \in \mathbb{C}^+$, be defined in (5.3). Assume that the matrix $T$ is diagonal. Then, for $a, b \in [1, N]$, $\alpha, \beta \in [N + 1, N + M]$, $A, B, C \in [1, N + M]$, the following identities hold:

- Schur complement/Feshbach formula: For any $a$ and $\alpha$,

$$
G_{aa} = \frac{1}{z - \sum_{a,\beta} x_{aa} G^{(a)}_{a\beta} x_{\beta a}}, \quad G_{aa} = \frac{1}{-(T^{-1})_{aa} - \sum_{a,b} x_{aa} G^{(a)}_{ab} x_{ab}}.
$$

(5.11)

- For $a \neq b$,

$$
G_{ab} = -G_{aa} \sum_{\alpha} x_{aa} G^{(a)}_{ab} = -G_{bb} \sum_{\beta} G^{(b)}_{ab} x_{\beta b}.
$$

(5.12)

- For $\alpha \neq \beta$,

$$
G_{\alpha \beta} = -G_{aa} \sum_{\alpha} x_{aa} G^{(a)}_{a\beta} = -G_{bb} \sum_{\beta} G^{(\beta)}_{\alpha b} x_{\beta b}.
$$

(5.13)

- For any $a$ and $\alpha$,

$$
G_{aa} = -G_{aa} \sum_{\beta} x_{\beta a} G^{(a)}_{\beta a} = -G_{aa} \sum_{\alpha} G^{(a)}_{ab} x_{ab}.
$$

(5.14)
Lemma 5.3. Then, under Assumption 2.3, the Green function from [15], provides useful large deviation estimates.

\[ G_{ij} = G_{aa}^{(a)} \sum_\alpha \sum_\beta x_{aa} G_{\alpha \beta}^{(ab)} x_{\beta b} , \quad G_{\alpha \beta} = G_{aa} G_{\alpha \beta}^{(a)} \sum_a \sum_b x_{aa} G_{ab}^{(ab)} x_{\beta b} , \quad (5.15) \]

- For \( A, B \neq C, \)

\[ G_{AB} = G_{AC}^{(C)} + \frac{G_{AC} G_{CB}}{G_{CC}} . \quad (5.16) \]

- Ward identity: For any \( a, \)

\[ \sum_b |G_{ab}|^2 = \frac{\Im G_{aa}}{\eta} . \quad (5.17) \]

For a proof we refer to, e.g., [14].

5.4. Local law for \( H \) at the edge. Consider two families of random variables \((X_i)\) and \((Y_i)\), with \( i \in [1, N] \), satisfying

\[ \mathbb{E} Z_i = 0 , \quad \mathbb{E} |Z_i|^2 = 1 , \quad \mathbb{E} |Z_i|^p \leq c_p , \quad (p \geq 3) , \quad (5.18) \]

\( Z_i = X_i Y_i, \) for all \( p \in \mathbb{N} \) and some constants \( c_p, \) uniformly in \( i \in [1, N] \). The following lemma, taken from [15], provides useful large deviation estimates.

Lemma 5.2. Let \((X_i)\) and \((Y_i)\) be independent families of random variables and let \((a_{ij})\) and \((b_i)\), \( i, j \in [1, N], \) be families of complex numbers. Suppose that all entries \((X_i)\) and \((Y_i)\) are independent and satisfy (5.18). Then we have the bounds:

\[ \left| \sum_i b_i X_i \right| < \left( \sum_i |b_i|^2 \right)^{1/2} , \quad (5.19) \]

\[ \left| \sum_i \sum_j a_{ij} X_i Y_j \right| < \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2} , \quad (5.20) \]

\[ \left| \sum_i \sum_j a_{ij} X_i Y_j - \sum_i a_{ii} \right| < \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2} . \quad (5.21) \]

If the coefficients \( a_{ij} \) and \( b_i \) depend on an additional parameter \( u, \) then all of these estimates are uniform in \( u, i.e., the threshold \( N_0 = N_0(\epsilon, D) \) in the definition of \( \sim \) depends only on the family \( (c_p) \) from (5.18); in particular, \( N_0 \) does not depend on \( u. \)

From the large deviation estimates in Lemma 5.2 and the local law in Lemma 3.3, we obtain the following estimates.

Lemma 5.3. Let \( G = G(z) , \) \( z \in \mathbb{C}^+, \) be defined in (5.3). Suppose that \( T \) is diagonal, i.e., \( T = \text{diag}(t_\alpha) \).

Then, under Assumption 2.3, the Green function \( G \) satisfies the following bounds uniformly in \( z \) on \( \mathcal{D}(\epsilon, c) \) (with \( \epsilon, c > 0 \) as in Lemma 3.3):

i. For any \( \alpha \in [N + 1, N + M], \)

\[ |G_{\alpha\alpha}(z)| \sim 1 , \quad \Im G_{\alpha\alpha}(z) \sim \Psi . \quad (5.22) \]

ii. For any \( a \in [1, N] \) and \( \alpha \in [N + 1, N + M], \)

\[ |G_{\alpha a}(z)| \sim \Psi . \quad (5.23) \]

iii. For any \( \alpha, \beta \in [N + 1, N + M] \) with \( \alpha \neq \beta, \)

\[ |G_{\alpha\beta}(z)| \sim \Psi . \quad (5.24) \]
Proof. From Schur’s complement formula \((5.11)\), we obtain that
\[
\frac{1}{G_{\alpha\alpha}} = -t_{\alpha}^{-1} - \sum_{k,l} x_{\alpha k} G_{kl}^{(\alpha)} x_{\alpha l}.
\]
Further, from the large deviation estimate \((5.21)\) and the Ward identity \((5.17)\), we find
\[
\left| m^{(\alpha)} - \sum_{k,l} x_{\alpha k} G_{kl}^{(\alpha)} x_{\alpha l} \right| \prec \left( \frac{1}{N^2} \sum_{k,l} |G_{kl}^{(\alpha)}|^2 \right)^{1/2} = \frac{\text{Im} m^{(\alpha)}}{N\eta} \prec \Psi,
\] (5.25)
uniformly in \(z \in D(\epsilon, c)\).

We next claim that \(|m - m^{(\alpha)}| \prec \Psi\). To prove this claim, we first let
\[
\tilde{Q} := T^{1/2} X X^* T^{1/2}.
\]
We notice that the averaged Green function \(m\) can be written in terms of \(\tilde{Q}\) by
\[
m(z) = m \tilde{Q}(z) = \frac{1}{N} \left( \text{Tr}(\tilde{Q} - z)^{-1} + \frac{N - M + 1}{z} \right).
\]
Next, we consider the minor \(\tilde{Q}^{(\alpha)}\), which is obtained by removing all columns and rows of \(\tilde{Q}\) indexed by \(\alpha\). Then,
\[
m^{(\alpha)}(z) = \frac{1}{N} \left( \text{Tr}(\tilde{Q}^{(\alpha)} - z)^{-1} + \frac{N - M + 1}{z} \right).
\]
By Cauchy’s eigenvalue interlacing property, we get
\[
\left| \text{Tr}(\tilde{Q} - z)^{-1} - \text{Tr}(\tilde{Q}^{(\alpha)} - z)^{-1} \right| \leq C\eta^{-1}.
\]
(See the proof of Lemma 5.1 in [6].) This proves the desired claim.

Since \(|\xi_\gamma + m| \prec \Psi\) and \(t_{\alpha}^{-1} \geq \tilde{\gamma}_0 (1 + c)\xi_\gamma\) for some \(c > 0\) (see \((2.14)\)), we get \(1 \prec |G_{\alpha\alpha}|^{-1}\), hence \(|G_{\alpha\alpha}| \prec 1\). Moreover, using once more Schur’s complement formula \((5.11)\), we have
\[
\left| \text{Im} G_{\alpha\alpha} \right| \prec \left| \text{Im} \sum_{k,l} x_{\alpha k} G_{kl}^{(\alpha)} x_{\alpha l} \right| \prec \text{Im} m^{(\alpha)} + \left| m^{(\alpha)} - \sum_{k,l} x_{\alpha k} G_{kl}^{(\alpha)} x_{\alpha l} \right| \prec \Psi,
\]
uniformly in \(D(\epsilon, c)\), where we used \((5.25)\) and that \(\text{Im} m^{(\alpha)} \prec \text{Im} m_{\alpha} \sim \sqrt{\kappa + \eta}\). This proves statement \(i\).

From the Green function identity \((5.14)\) and statement \(i\), we have
\[
|G_{\alpha\alpha}| = \left| G_{\alpha\alpha} \sum_{k} x_{\alpha k} G_{ka}^{(\alpha)} \right| \prec \left| \sum_{k} x_{\alpha k} G_{ka}^{(\alpha)} \right|,
\]
where we used the local law of Lemma 3.3. Thus, applying the large deviation estimate \((5.19)\) and the local law \(|G_{ka}^{(\alpha)}| \prec \Psi\), we get
\[
|G_{\alpha\alpha}| \prec \left( \frac{1}{N} \sum_{k} |G_{ka}^{(\alpha)}|^2 \right)^{1/2} \prec \Psi,
\]
uniformly in \(z \in D(\epsilon, c)\), which proves statement \(ii\) of the lemma.

Similarly, we have from the Green function identity \((5.13)\) that
\[
|G_{\alpha\beta}| = \left| G_{\alpha\beta} \sum_{k} x_{\alpha k} G_{k\beta}^{(\alpha)} \right| \prec \left| \sum_{k} x_{\alpha k} G_{k\beta}^{(\alpha)} \right| \prec \left( \frac{1}{N} \sum_{k} |G_{k\beta}^{(\alpha)}|^2 \right)^{1/2} \prec \Psi,
\]
where we used \(|G_{k\beta}| \prec \Psi\) to get the last inequality. This proves statement \(iii\) of the lemma. \(\square\)
We conclude this section by giving estimates on expectations of monomials of Green functions entries.

**Lemma 5.4.** Let \( P \equiv P(z) \) be a monomial in the Green function entries \((G_{AB}(z))\), with \( z \in \mathcal{D}(\varepsilon, c) \), for some \( \varepsilon, c > 0 \). Then, there exists a universal constant \( C \), such that

\[
\mathbb{E}|P(z)|^2 \lesssim N^{Cn},
\]

where \( n \) is the degree of \( P \). In particular, if \( |P(z)| \lesssim \Psi(z)^k \), uniformly in \( \mathcal{D}(\varepsilon, c) \), then \( \mathbb{E}|P(z)| \lesssim \Psi(z)^k \), uniformly in \( \mathcal{D}(\varepsilon, c) \). (See the paragraph after Definition 3.1.)

Moreover, the same conclusions hold with \( G^{(T)} \) replacing \( G \) for any \( T \).

**Proof.** First, we note that \( |G_{ab}| \leq \frac{1}{n}, a, b \in [1, N] \), as follows from the self-adjointness of \( X^*TX \) and the spectral calculus.

Second, to bound \( |G_{\alpha\beta}|, \alpha, \beta \in [N + 1, M + N] \), we recall that \( T^{-1} \) is a strictly positive operator by Assumption 2.3. Thus

\[
\text{Im}(v, (zT^{-1} - XX^*)v) = \eta(v, T^{-1}v) \geq c\eta \|v\|^2, \quad \forall v \in \mathbb{C}^M,
\]

for some \( c > 0 \) independent of \( v \), where \( \langle \cdot, \cdot \rangle \) denotes the canonical inner product in \( \mathbb{C}^M \). Since \( z^{-1}T\overline{G}(z)\overline{T} = (-zT^{-1} + XX^*)^{-1} \), \( |z| > 0 \), we get \( |G_{\alpha\beta}| \leq \frac{C|z|}{\eta} \).

Third, to bound \( \mathbb{E}|G_{aa}|^p, a \in [1, N], \alpha \in [N + 1, N + M], \) \( p \geq 0 \), we note that by (5.14) we have

\[
|G_{aa}| = |G_{aa}| \sum_{\beta} |x_{\beta\alpha}G_{\beta\alpha}^{(a)}| \leq \frac{C|z|}{\eta^2}N|x_{\alpha\alpha}|,
\]

by the estimates above. From the moment bounds in Assumption 2.2, we then conclude that \( \mathbb{E}|G_{aa}|^p \leq C^pN^{cp} \), where we also used that \( \eta \gg N^{-1}, 0 < |z| < C \) by assumption.

The lemma now easily follows from Hölder’s inequality.

In the rest of the paper, we prove Proposition 4.1 with the formalism outlined in this section. The actual calculation will be done for the simple case \( F' \equiv 1 \); the proof for general \( F' \) is basically the same, though the computations are much longer for this case. The details for \( F' \neq 1 \) can be found in the Appendices A, B and C of [28].

### 6. Green Function Flow

The key idea of our proof of Proposition 4.1 is similar to the one of the proof of Proposition 5.2 in [28] for deformed Wigner matrices: We consider a continuous interpolation between the sample covariance matrices \( \overline{Q} \) and \( W \) by introducing a time evolution that deforms \( T \) continuously to the identity. We then track the associated flow of the Green function for sufficiently long time. The outcome is an estimate on the time derivative of the Green function which is sufficiently accurate to prove Proposition 4.1.

**6.1. Preliminaries.** Suppose that \( T = \gamma_0\Sigma \) is diagonal, i.e., \( T = \text{diag}(t_\alpha) \). We interpolate between \( \Sigma = \text{diag}(\sigma_\alpha) \) and the identity matrix \( \mathbb{1} \) by introducing the time evolution \( t \mapsto (\sigma_\alpha(t)) \) defined by

\[
\frac{1}{\sigma_\alpha(t)} = e^{-t} \frac{1}{\sigma_\alpha(0)} + (1 - e^{-t}), \quad \Sigma(t) = \text{diag}(\sigma_\alpha(t)), \quad (t \geq 0).
\]

We let \( z = z(t) \) be time-dependent and define the \((N + M) \times (N + M)\) matrix \( H(t) = H(z, t) \) by

\[
PH(t)P := -z(t), \quad PH(t)\overline{P} := X^*, \quad \overline{P}H(t)P := X, \quad \overline{P}H(t)\overline{P} := -T^{-1}(t),
\]

with \( T(t) = \gamma(t)\Sigma(t), T(0) = \gamma_0\Sigma \), for some time-dependent scaling factor \( \gamma(t) \in \mathbb{R} \) (see (6.5) below for the definition of \( \gamma(t) \)). We also let

\[
G(z, t) := H(z, t)^{-1}, \quad m(z, t) = \frac{1}{N} \text{Tr} G(z, t), \quad (z \in \mathbb{C}^+).
\]
From (2.11), it is natural to let \( \xi_+(t) \) be the largest solution to
\[
\frac{1}{M} \sum_\alpha \left( \frac{\sigma_\alpha(t)\xi_+(t)}{1 - \sigma_\alpha(t)\xi_+(t)} \right)^2 = d,
\]
with \( \xi_+(0) = \xi_+ \). We then choose the scaling factor \( \gamma \equiv \gamma(t) \) to be given by
\[
\gamma(t) = \left( \frac{1}{N} \sum_\alpha \left( \frac{\sigma_\alpha(t)}{1 - \sigma_\alpha(t)\xi_+(t)} \right)^3 + (\xi_+(t))^{-3} \right)^{-1/3}, \quad \gamma_0 = \gamma(0),
\]
and we also introduce
\[
\tau \equiv \tau(t) := \frac{\xi_+(t)}{\gamma(t)}.
\]
For simplicity, we often omit the \( t \)-dependence in the notation for \( T(t), \gamma(t) \) and \( \tau(t) \) in the following.

Note that we have from (6.4), (6.5) and (6.6) that
\[
\frac{1}{N} \sum_\alpha \left( \frac{1}{t_\alpha - \tau} \right)^2 = \frac{1}{\tau^2}, \quad \frac{1}{N} \sum_\alpha \left( \frac{1}{t_\alpha - \tau} \right)^3 + \frac{1}{\tau^3} = 1.
\]
In the following, we refer to the identities in (6.7) as “sum rules”.

We now consider the evolution of the Green function \( G \equiv G(t) \) under the evolution governed by (6.1). For the diagonal Green function entries \( G_{ii}, i \in [1, N] \), we get
\[
\mathbb{E} \frac{\partial G_{ii}}{\partial t} = \frac{\partial}{\partial t} \sum_\alpha \mathbb{E}[G_{ia}G_{ai}] - \sum_\alpha \frac{\partial t_\alpha}{\partial t} \mathbb{E}[G_{ia}G_{ai}].
\]

**Remark 6.1.** Let \( \tilde{m}_{\ell\ell}(z, t) \) be the solution to
\[
\tilde{m}_{\ell\ell}(z, t) = -z + \frac{1}{dM} \sum_\alpha \frac{t_\alpha}{t_\alpha - m_\ell(z, t) \gamma_0}, \quad \text{Im} \tilde{m}_{\ell\ell}(z, t) \geq 0, \quad (z \in \mathbb{C}^+, t \geq 0).
\]
Setting \( \tilde{\rho}_\ell(E, t) := \lim_{\eta \searrow 0} \pi^{-1} \mathbb{E} \tilde{m}_{\ell\ell}(E + i\eta, t) \), we note that the rightmost point of the support of the measure \( \tilde{\rho}_\ell(t) \), denoted by \( L_+ \equiv L_+(t) \), is given by \( L_+ = \gamma E_+ \), or equivalently,
\[
L_+ = \frac{1}{\tau} + \frac{1}{dM} \sum_\alpha \frac{t_\alpha}{1 - t_\alpha \tau} = \frac{1}{\tau} + \frac{1}{N} \sum_\alpha \frac{1}{t_\alpha - \tau}.
\]
In fact, the rescaling by \( \gamma(t) \) assures that
\[
\tilde{\rho}(E, t) = \frac{1}{\pi} \sqrt{L_+ - E (1 + O(L_+ - E))}, \quad (t \geq 0),
\]
as \( E \nearrow L_+ \), as may be checked by an explicit computation.

### 6.2. Proof of Proposition 4.1

In this subsection, we give the proof of Proposition 4.1, which is based on two technical lemmas, Lemma 6.2 and Lemma 6.3 below. For simplicity, we choose \( F^a = 1 \). Recall the definition of the deterministic control parameter \( \Psi \) in (3.7).

The main ingredient of the proof of the Green function comparison theorem, Proposition 4.1, is the estimate \( \text{Im} \mathbb{E} [\partial_t G_{ii}(z)] = \mathcal{O}(\Psi^2) \), for appropriately chosen \( z \). (The naive size of \( \mathbb{E} [\partial_t G_{ii}] \) is \( \mathcal{O}(\Psi^2) \) as one can easily see from (6.8).) Once we have established the estimate \( \text{Im} \mathbb{E} [\partial_t G_{ii}(z)] = \mathcal{O}(\Psi^5) \), we can integrate it from \( t = 0 \) to \( t = 2 \log N \) to compare \( \text{Im} m_{\ell\ell} \) with \( \text{Im} m_{\ell\ell}|_{t=2\log N} \). The comparison between \( \text{Im} m_{\ell\ell}|_{t=2\log N} \) and \( m_{\ell\ell} \) can easily be done, since \( \Sigma(t) \) is close enough to the identity at \( t = 2 \log N \).

To show that the imaginary part of (6.8) is much smaller than its naive size, we use, in a first step, the following “decoupling” lemma.
Lemma 6.2. Under the assumptions of Proposition 4.1 the following holds true.

Let $z(t) \equiv z = L_+(t) + y + i\eta$, with $\eta = N^{-2/3-\epsilon}$, $y \in [-N^{-2/3+\epsilon},N^{-2/3+\epsilon}]$ and with $L_+(t)$ as in (6.10).

Then, there are $z$-dependent random variables $X_{22}, X_{32}, X_{33}, X_{42}, X_{43}, X_{44}$ and $X_{44}'$, satisfying

\[ X_{22} = O(\Psi^2), \quad X_{32}, X_{33} = O(\Psi^3), \quad X_{42}, X_{43}, X_{44}, X_{44}' = O(\Psi^4), \tag{6.11} \]

such that

\[ E\alpha[G_{ia}G_{\alpha i}] = \frac{1}{(\alpha^{-1} - \tau)^2} X_{22} - \frac{2}{(\alpha^{-1} - \tau)^3} X_{32} - \frac{2}{(\alpha^{-1} - \tau)^3} X_{33} \]
\[ + \frac{3}{(\alpha^{-1} - \tau)^4} X_{42} + \frac{6}{(\alpha^{-1} - \tau)^4} X_{43} + \frac{12}{(\alpha^{-1} - \tau)^4} X_{44} + \frac{3}{(\alpha^{-1} - \tau)^4} X_{44}' + O(\Psi^5), \tag{6.12} \]

uniformly in $t \geq 0$. The random variables above are explicitly given by

\[ X_{22} = \frac{1}{N} \sum_s G_{is} G_{si}, \quad X_{32} = (m + \tau) \frac{1}{N} \sum_s G_{is} G_{si}, \]

\[ X_{33} = \frac{1}{N^2} \sum_{r,s} G_{ir} G_{rs} G_{si}, \quad X_{42} = (m + \tau)^2 \frac{1}{N} \sum_s G_{is} G_{si}, \]

\[ X_{43} = (m + \tau) \frac{1}{N} \sum_{r,s} G_{ir} G_{rs} G_{si}, \quad X_{44} = \frac{1}{N^3} \sum_{r,s,t} G_{ir} G_{rs} G_{si} G_{ti}, \]

\[ X_{44}' = \frac{1}{N^3} \sum_{r,s,t} G_{is} G_{si} G_{ir} G_{tr}, \tag{6.13} \]

where $G \equiv G(z(t), t)$, $m \equiv m(z(t), t) = \frac{1}{N} \sum_s G_{is}(z(t), t)$ and $\tau(t)$ is defined in (6.6).

We refer to Lemma 6.2 as a “decoupling” lemma, since on the right side of (6.12) the Greek index $\alpha$ is, up to the error $O(\Psi^5)$, decoupled from the Green functions which only have Roman indices as lower indices. Lemma 6.2 is proven in Section 7 below.

Taking the time derivative of (6.10) we get

\[ \dot{z} = \dot{L}_+ = -\frac{\dot{\tau}}{\tau} + \frac{1}{dm} \sum_\alpha \frac{t_{\alpha}^2 \dot{\tau}}{(1 - t_\alpha \tau)^2} + \frac{1}{dm} \sum_\alpha \frac{\partial t_\alpha}{1 - t_\alpha \tau} + \frac{1}{dm} \sum_\alpha \tau t_\alpha (\partial t_\alpha) \tau. \tag{6.14} \]

From (6.6) we observe that the first two terms on the right side of (6.14) cancel. Thus, simplifying the last two terms in (6.14), we obtain

\[ \dot{z} = \frac{1}{dm} \sum_\alpha \frac{\partial t_\alpha}{1 - t_\alpha \tau^2} = \frac{1}{N} \sum_\alpha \frac{\partial t_\alpha}{t_\alpha (1 - t_\alpha \tau)^2}. \tag{6.15} \]

Hence, plugging (6.12) into (6.8) we find

\[ E \frac{\partial G_{ii}}{\partial t} = \dot{z} \sum_\alpha E[G_{ia}G_{\alpha i}] - \sum_\alpha \frac{\partial t_\alpha}{t_\alpha (1 - t_\alpha \tau)^2} \frac{1}{N} \sum_\alpha E[G_{ia}G_{\alpha i}] \]
\[ + \sum_\alpha \frac{\partial t_\alpha}{t_\alpha^2} E \left[ \frac{2}{(\alpha^{-1} - \tau)^3} X_{32} - \frac{2}{(\alpha^{-1} - \tau)^3} X_{33} \right] \]
\[ - \sum_\alpha \frac{\partial t_\alpha}{t_\alpha^2} E \left[ \frac{3}{(\alpha^{-1} - \tau)^4} X_{42} + \frac{6}{(\alpha^{-1} - \tau)^4} X_{43} + \frac{12}{(\alpha^{-1} - \tau)^4} X_{44} + \frac{3}{(\alpha^{-1} - \tau)^4} X_{44}' \right] \]
\[ + O(M\Psi^5). \tag{6.16} \]
Note that the first two terms in (6.16) cancel by (6.15) and that we have
\[
\sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha^2} \mathbb{E} \left[ \frac{2}{(t_\alpha^{-1} - \tau)^3} X_{32} + \frac{2}{(t_\alpha^{-1} - \tau)^3} X_{33} \right]
- \sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha^2} \mathbb{E} \left[ \frac{3}{(t_\alpha^{-1} - \tau)^4} X_{42} + \frac{6}{(t_\alpha^{-1} - \tau)^4} X_{43} + \frac{12}{(t_\alpha^{-1} - \tau)^4} X_{44} + \frac{3}{(t_\alpha^{-1} - \tau)^4} X_{44}' \right]
+ O(M \Psi^5) \quad (6.17)
\]

To complete the proof of Proposition 4.1, we are going to show that the imaginary part of the right side of (6.17) is of \(O(M \Psi^5)\) as is noted in the next lemma.

**Lemma 6.3.** Under the assumptions of Proposition 4.1 with the notation of Lemma 6.2, we have
\[
\sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha^2} \text{Im} \mathbb{E} \left[ \frac{2}{(t_\alpha^{-1} - \tau)^3} X_{32} + \frac{2}{(t_\alpha^{-1} - \tau)^3} X_{33} \right]
- \sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha^2} \text{Im} \mathbb{E} \left[ \frac{3}{(t_\alpha^{-1} - \tau)^4} X_{42} + \frac{6}{(t_\alpha^{-1} - \tau)^4} X_{43} + \frac{12}{(t_\alpha^{-1} - \tau)^4} X_{44} + \frac{3}{(t_\alpha^{-1} - \tau)^4} X_{44}' \right]
= O(M \Psi^5) \quad (6.18)
\]
uniformly in \(t \geq 0\).

We remark that the naive size of the right side of (6.18) is \(O(M \Psi^3)\), but for our choice of \(\gamma\) the terms cancel up to errors of \(O(M \Psi^5)\). Similar to the discussion in [28], the sum rules in (6.7) have crucial roles in this cancellation mechanism. Lemma 6.3 is proven in Section 9.

**Proof of Proposition 4.1.** For simplicity we choose \(F' \equiv 1\). From (6.17) and Lemma 6.3, we find that
\[
\mathbb{E} \left[ \text{Im} \frac{\partial G_{ii}}{\partial t} \right] = O(\Psi^2) \quad (6.19)
\]
Integrating both sides of (6.19) from \(t = 0\) to \(t = 2 \log N\) we obtain that
\[
\left| \mathbb{E} \left[ N \int_{E_2} \text{Im} m(x + L_+ + i\eta) \big|_{t=0} \, dx \right] - \mathbb{E} \left[ N \int_{E_1} \text{Im} m(x + L_+ + i\eta) \big|_{t=2 \log N} \, dx \right] \right|
\leq N^{1/3 + C'}
\]
for some constant \(C' > 0\).

At \(t = \infty\), we have \(\sigma_\alpha(\infty) = 1\), for all \(\alpha \in [N + 1, N + M]\), hence by definition
\[
\xi_+(\infty) = \frac{\sqrt{d}}{1 + \sqrt{d}}, \quad \gamma(\infty) = \sqrt{d} (1 + \sqrt{d})^{-4/3}.
\]
In particular,
\[
m(x + L_+ + i\eta) \big|_{t=\infty} = m_W(x + M_+ + i\eta)
\]
Let \(T_1 := 2 \log N\). At \(t = T_1\), we have \(\sigma_\alpha(T_1) = 1 + O(N^{-2})\). Using the result at \(t = \infty\), it can be easily seen that
\[
\gamma(T_1) = \gamma(\infty) + O(N^{-2})
\]
Similarly, we also have that \(z(T_1) = z(\infty) + O(N^{-2})\). Thus, the matrix \(H(T_1) - H(\infty)\) is a diagonal matrix whose entries are \(O(N^{-2})\).

Using the resolvent identity
\[
G(T_1) - G(\infty) = -G(T_1) (H(T_1) - H(\infty)) G(\infty) \quad (6.21)
\]
we can now bound
\[ |G_{ii}(T_i) - G_{ii}(\infty)| = \left| \sum_A -G_{iA}(T_i) \left( H_{AA}(T_i) - H_{AA}(\infty) \right) G_{Ai}(\infty) \right| \lesssim N^{-5/3}, \tag{6.22} \]
and we thus have
\[ \left| \mathbb{E} \left[ N \int_{E_1}^{E_2} \text{Im} \, m(x + L_+ + i\eta) \big|_{t=2 \log N} \, dx \right] - \mathbb{E} \left[ N \int_{E_1}^{E_2} \text{Im} \, m(x + L_+ + i\eta) \big|_{t=\infty} \, dx \right] \right| \lesssim N^{-4/3+C''}. \tag{6.23} \]
Since \( m(x + L_+ + i\eta) \big|_{t=0} = m_q(x + L_+ + i\eta) \), we get the desired result from (6.20) and (6.23). \( \square \)

7. Proof of Lemma 6.2

In this section we prove Lemma 6.2. We start expanding \( \mathbb{E} [G_{i\alpha} G_{\alpha i}] \) in the random variables indexed by the Greek index \( \alpha \). The following expansion follows closely the expansions used in the Appendix A of [28].

**Proof of Lemma 6.2.** Using the formula for \( G_{i\alpha} \) in (5.12), i.e.,
\[ G_{i\alpha} = -G_{\alpha} \sum_k x_{ak} G_{ik}^{(\alpha)} , \tag{7.1} \]
we expand \( G_{i\alpha} G_{\alpha i} \) in the lower index \( \alpha \) as
\[ G_{i\alpha} G_{\alpha i} = G_{\alpha}^2 \sum_{k,l} G_{ik}^{(\alpha)} x_{ak} x_{\alpha l} G_{\alpha l}^{(\alpha)} . \tag{7.2} \]
Note that, by Schur’s complement formula (5.11),
\[ G_{\alpha} = \frac{1}{h_{\alpha} - \sum_{p,q} G_{pq}^{(\alpha)} h_{qa}} = -t_{\alpha}^{-1} \frac{1}{\sum_{p,q} x_{ap} G_{pq}^{(\alpha)} x_{aq}} . \tag{7.3} \]
(The use of Roman letters \( p, q \) can be justified since \( h_{ap} = 0 \) for \( p \in [N + 1, N + M] \) and \( p \neq \alpha \).)

We next expand \( G_{\alpha} \) around \((-t_{\alpha}^{-1} + \tau)^{-1}\). (Note that \( \limsup \tau < 1 \), thus \(-t_{\alpha}^{-1} - \tau > c > 0 \) for some constant \( c \) independent of \( N \).) From the large deviation estimates in Lemma 5.2 and the Ward identity (5.17), we have
\[ \left| \sum_{p,q} x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau \right| \lesssim \Psi . \tag{7.4} \]
Returning to (7.3), we thus have
\[ G_{\alpha} = \frac{1}{-t_{\alpha}^{-1} + \tau} + \frac{1}{(-t_{\alpha}^{-1} + \tau)^2} \left( \sum_{p,q} x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau \right) + \frac{1}{(-t_{\alpha}^{-1} + \tau)^3} \left( \sum_{p,q} x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau \right)^2 \]
\[ + O(\Psi^3) , \]
respectively,
\[ G_{\alpha}^2 = \frac{1}{(t_{\alpha}^{-1} - \tau)^2} - \frac{2}{(t_{\alpha}^{-1} - \tau)^3} \left( \sum_{p,q} x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau \right) + \frac{3}{(t_{\alpha}^{-1} - \tau)^4} \left( \sum_{p,q} x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau \right)^2 \]
\[ + O(\Psi^3) . \]
Hence, from the resolvent identity (7.2), obtain the following expansion of $G_{is} G_{ai}$ in the lower index $\alpha$,

$$G_{ia} G_{ai} = \frac{1}{(t_\alpha^{-1} - \tau)^2} \sum_{s,t} G_{is}^{(\alpha)} x_{as} x_{at} G_{ti}^{(\alpha)} - \frac{2}{(t_\alpha^{-1} - \tau)^3} \left( \sum_{p,q} (x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau) \right) \sum_{s,t} G_{is}^{(\alpha)} x_{as} x_{at} G_{ti}^{(\alpha)}$$

$$+ \frac{3}{(t_\alpha^{-1} - \tau)^4} \left( \sum_{p,q} (x_{ap} G_{pq}^{(\alpha)} x_{aq} + \tau) \right)^2 \sum_{s,t} G_{is}^{(\alpha)} x_{as} x_{at} G_{ti}^{(\alpha)} + \mathcal{O}(\Psi^5).$$

Taking the partial expectation $\mathbb{E}_\alpha$ we get

$$\mathbb{E}_\alpha \left[ G_{ia} G_{ai} \right] = \frac{1}{(t_\alpha^{-1} - \tau)^2} \frac{1}{N} \sum_s G_{is}^{(\alpha)} G_{si}^{(\alpha)}$$

$$- \frac{2}{(t_\alpha^{-1} - \tau)^3} \left( m^{(\alpha)} + \tau \right) \frac{1}{N} \sum_s G_{is}^{(\alpha)} G_{si}^{(\alpha)} - \frac{4}{(t_\alpha^{-1} - \tau)^4} \frac{1}{N^2} \sum_{s,t} G_{is}^{(\alpha)} G_{st}^{(\alpha)} G_{ti}^{(\alpha)}$$

$$+ \frac{3}{(t_\alpha^{-1} - \tau)^4} \left( m^{(\alpha)} + \tau \right) \frac{2}{N} \sum_s G_{is}^{(\alpha)} G_{si}^{(\alpha)} + \frac{12}{(t_\alpha^{-1} - \tau)^4} \frac{1}{N^2} \sum_{s,t} G_{is}^{(\alpha)} G_{st}^{(\alpha)} G_{ti}^{(\alpha)}$$

$$+ \frac{6}{(t_\alpha^{-1} - \tau)^4} \frac{1}{N^3} \sum_{s,t} G_{is}^{(\alpha)} G_{si}^{(\alpha)} G_{pq}^{(\alpha)} + \frac{24}{(t_\alpha^{-1} - \tau)^4} \frac{1}{N^3} \sum_{s,t} G_{is}^{(\alpha)} G_{si}^{(\alpha)} G_{pq}^{(\alpha)} + \mathcal{O}(\Psi^5).$$

(7.5)

In a next step, we expand (7.5) in the upper index $\alpha$ by using the resolvent formula (5.16), i.e.,

$$G_{is}^{(\alpha)} = G_{is} - \frac{G_{is} G_{as} G_{as}}{G_{aa}}.$$  

(7.6)

In other words, using (7.6), we can remove the upper index $\alpha$ from the Green functions entries in (7.5) at the expense of higher order terms containing $\alpha$ as a lower index in the Green function entries. We obtain for the first term in (7.5) that

$$G_{is}^{(\alpha)} x_{si}^{(\alpha)} = G_{is} G_{si} - \frac{G_{is} G_{as} G_{as}}{G_{aa}} G_{si}^{(\alpha)} - \frac{G_{is} G_{as} G_{as}}{G_{aa}} \frac{G_{as} G_{ai}}{G_{aa}} G_{si}^{(\alpha)}$$

$$= G_{is} G_{si} - \frac{G_{is} G_{as} G_{as}}{G_{aa}} G_{si}^{(\alpha)} - G_{is}^{(\alpha)} G_{as} G_{ai}^{(\alpha)} - G_{is}^{(\alpha)} G_{as} G_{ai}^{(\alpha)} - G_{is}^{(\alpha)} G_{as} G_{ai}^{(\alpha)}.$$  

(7.7)

We stop expanding the first term on the right side of (7.7), since it does not contain the index $\alpha$, and we set

$$X_{22} := \frac{1}{N} \sum_s G_{is} G_{si}.$$  

(7.8)

Using (5.14), the partial expectation of the second term on the right side of (7.7) can be expanded in the lower index $\alpha$ to get

$$\mathbb{E}_\alpha \left[ \frac{G_{is} G_{as} G_{si}^{(\alpha)}}{G_{aa}} \right] = \mathbb{E}_\alpha \left[ G_{aa} \sum_{k,l} G_{ik}^{(\alpha)} x_{ak} x_{al} G_{i_k}^{(\alpha)} G_{si}^{(\alpha)} \right]$$

$$= - \frac{1}{t_\alpha^{-1} - \tau} \frac{1}{N} \sum_k G_{ik}^{(\alpha)} G_{ks} G_{si}^{(\alpha)} + \frac{1}{(t_\alpha^{-1} - \tau)^2} \left( m^{(\alpha)} + \tau \right) \frac{1}{N} \sum_k G_{ik}^{(\alpha)} G_{ks} G_{si}^{(\alpha)}$$

$$+ \frac{2}{(t_\alpha^{-1} - \tau)^2} \frac{1}{N^2} \sum_{k,l} G_{ik}^{(\alpha)} G_{kl}^{(\alpha)} G_{il}^{(\alpha)} G_{si}^{(\alpha)} + \mathcal{O}(\Psi^5).$$  

(7.9)
Expanding the first term in the right side of (7.9) further using (5.16), we get
\[ G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} = G_{ik} G_{ks} G_{si} - \frac{G_{io} G_{ak}}{G_{aa}} G_{ks} G_{si} - G_{ik} \frac{G_{ka} G_{aa}}{G_{aa}} G_{si} - G_{ik} G_{ks} \frac{G_{sa} G_{ai}}{G_{aa}}, \] (7.10)
We stop expanding the first term on the right side of (7.10), since it does no more contain the index \( a \), and we let
\[ X_{33} := \frac{1}{N^2} \sum_{k,s} G_{ik} G_{ks} G_{si} . \] (7.11)
Expanding the remaining terms on the right side of (7.10) in the lower index \( \alpha \) using (5.14), we obtain
\[ \mathbb{E}_x \left[ \frac{G_{io} G_{ak}}{G_{aa}} G_{ks} G_{si} \right] = - \frac{1}{t_{aa} - \tau} \mathbb{E}_x \left[ \sum_{l,m} G_{il}^{(a)} x_{al} x_{cm} G_{mk}^{(a)} G_{ks} G_{si} \right] + \mathcal{O}(\Psi^5) \]
\[ = - \frac{1}{t_{aa} - \tau} \frac{1}{N} \sum_{l} G_{il}^{(a)} G_{ik}^{(a)} G_{ks} G_{si} + \mathcal{O}(\Psi^5) \]
\[ = - \frac{1}{t_{aa} - \tau} \frac{1}{N} \sum_{l} G_{il} G_{ik} G_{ks} G_{si} + \mathcal{O}(\Psi^5) \]
and, similarly,
\[ \mathbb{E}_x \left[ G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} \right] = - \frac{1}{t_{aa} - \tau} \frac{1}{N} \sum_{l} G_{ik} G_{kl} G_{ls} G_{si} + \mathcal{O}(\Psi^5) \]
respectively,
\[ \mathbb{E}_x \left[ G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} \right] = - \frac{1}{t_{aa} - \tau} \frac{1}{N} \sum_{l} G_{ik} G_{ks} G_{sl} G_{li} + \mathcal{O}(\Psi^5) . \]
Thus, setting
\[ X_{44} := \frac{1}{N^3} \sum_{k,l,s} G_{ik} G_{kl} G_{ls} G_{si} , \] (7.12)
we have
\[ \mathbb{E}_x \left[ \sum_{k,s} G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} \right] = X_{33} + \frac{3}{t_{aa} - \tau} X_{44} + \mathcal{O}(\Psi^5) . \] (7.13)
Next, we consider the \( \mathcal{O}(\Psi^4) \) terms on the right side of (7.9). Let
\[ X_{43} := (m + \tau) \frac{1}{N^2} \sum_{k,s} G_{ik} G_{ks} G_{si} . \] (7.14)
Then, we have for the second term on the right side of (7.9) that
\[ (m^{(a)} + \tau) \frac{1}{N^2} \sum_{k,s} G_{ik}^{(a)} G_{ks}^{(a)} G_{si}^{(a)} = X_{43} + \mathcal{O}(\Psi^5) . \] (7.15)
The last term on the right side of (7.9) is simply estimated by
\[ \frac{1}{N^3} \sum_{k,l,s} G_{ik}^{(a)} G_{kl}^{(a)} G_{ls}^{(a)} G_{si}^{(a)} = X_{44} + \mathcal{O}(\Psi^5) . \] (7.16)
In sum, we find
\[ \mathbb{E}_x \left[ \frac{1}{N} \sum_{s} G_{io} G_{ak} G_{si} \right] = - \frac{1}{t_{aa} - \tau} X_{33} + \frac{1}{(t_{aa} - \tau)^2} X_{43} - \frac{1}{(t_{aa} - \tau)^2} X_{44} + \mathcal{O}(\Psi^5) . \] (7.17)
Similarly, we also have
\[
\mathbb{E}_\alpha \left[ \frac{1}{N} \sum_s G_{is}^{(a)} G_{sa} G_{a\alpha} \right] = -\frac{1}{t_\alpha - \tau} X_{33} + \frac{1}{(t_\alpha - \tau)^2} X_{43} - \frac{1}{(t_\alpha - \tau)^2} X_{44} + O(\Psi^5). \tag{7.18}
\]

For the last term in (7.7), we obtain
\[
\frac{G_{ia} G_{as} G_{sa} G_{a\alpha}}{G_{a\alpha}} = G_{a\alpha}^2 \sum_{k,l,p,q} G_{ik}^{(a)} x_{ak} x_{ai} G_{is}^{(a)} G_{sp}^{(a)} x_{ap} x_{aq} G_{qi}^{(a)}.
\]
Hence, denoting
\[
X'_{44} := \frac{1}{N^2} \sum_{k,l,s} G_{is} G_{si} G_{kl} G_{lk}, \tag{7.19}
\]
we find
\[
\mathbb{E}_\alpha \left[ \frac{1}{N} \sum_s G_{ia} G_{as} G_{sa} G_{a\alpha} \right] = \frac{2}{(t_\alpha - \tau)^2} X_{44} + \frac{1}{(t_\alpha - \tau)^2} X'_{44} + O(\Psi^5). \tag{7.20}
\]
Thus, from (7.7), (7.17), (7.18) and (7.20) we obtain
\[
\mathbb{E}_\alpha \left[ \frac{1}{N} \frac{1}{(t_\alpha - \tau)^2} \frac{1}{N} \sum_s G_{is}^{(a)} G_{si}^{(a)} \right] = \frac{1}{(t_\alpha - \tau)^2} X_{22} + \frac{2}{(t_\alpha - \tau)^2} X_{32} - \frac{2}{(t_\alpha - \tau)^2} X_{43} - \frac{1}{(t_\alpha - \tau)^2} X_{44} + \frac{2}{(t_\alpha - \tau)^2} X_{44} + O(\Psi^5), \tag{7.21}
\]
which completes the expansion of the first term in (7.5). The calculation and the result coincide with those in the deformed Wigner case in the Appendix A of [28], except the sign of the \(X_{33}\) term. The discrepancy is due to the sign difference in the coefficient \((t_\alpha - \tau)^{-1}\).

Adapting the expansion procedure of the Appendix A of [28] we conclude, with the definitions
\[
X_{32} := (m + \tau) \frac{1}{N} \sum_s G_{is} G_{si}, \quad X_{42} := (m + \tau)^2 \frac{1}{N} \sum_s G_{is} G_{si}, \tag{7.22}
\]
that
\[
\mathbb{E}_\alpha [G_{ia} G_{ai}] = \frac{1}{(t_\alpha - \tau)^2} X_{22} - \frac{2}{(t_\alpha - \tau)^3} X_{32} - \frac{2}{(t_\alpha - \tau)^3} X_{33} \tag{7.23}
\]
\[
+ \frac{3}{(t_\alpha - \tau)^4} X_{42} + \frac{6}{(t_\alpha - \tau)^4} X_{43} + \frac{12}{(t_\alpha - \tau)^4} X_{44} + \frac{3}{(t_\alpha - \tau)^4} X'_{44} + O(\Psi^5).
\]
This shows (6.12) and hence completes the proof of Lemma 6.12. \(\square\)

Before we move on to the next section, we introduce some more notation. For \(k \in \mathbb{N}\), let
\[
A_k := \frac{1}{N} \sum_{\rho} \frac{1}{(t_\rho - \tau)^k}. \tag{7.24}
\]
We remark that from (6.7), we have
\[
A_2 = \tau^{-2}, \quad A_3 + \tau^{-3} = 1. \tag{7.25}
\]
Finally, averaging (7.23) over \(\alpha\), we have in this notation
\[
\frac{1}{N} \sum_\alpha \mathbb{E}_\alpha [G_{ia} G_{ai}] = A_2 X_{22} - 2 A_3 (X_{32} + X_{33}) + 3 A_4 (X_{42} + 2 X_{43} + 4 X_{44} + X'_{44}) + O(\Psi^5). \tag{7.26}
\]
This concludes the current section.
8. Optical theorems

In this section, we establish the following "optical theorem".

**Lemma 8.1.** Under the assumptions of Proposition 4.1 with the notation of Lemma 6.2, we have

\[
2E[X_{32} + X_{33}] - \frac{1}{N} = 3(A_4 - \tau^{-4})E[X_{12} + 2X_{43} + 4X_{44} + X'_{44}] + O(\Psi^3),
\]

uniformly in \( t \geq 0 \).

Lemma 8.1 is an example of what we call optical theorems: optical theorems assure that the expectations of certain linear combinations of the random variables introduced in Lemma 6.2 are smaller than their naive sizes obtained from power counting using the local laws in Lemma 3.2 and Lemma 5.3. Such estimates were key technical inputs in the proof of edge universality for deformed Wigner matrices in [28]. As in [28], the optical theorems used in this paper are obtained by combining expansions of random variables, e.g., \( X_{22} \) or \( X_{33} \), with the sum rules in (6.7). In the rest of the section, we derive the required optical theorems.

The proof of Lemma 8.1 is given in the Subsection 8.4 based on estimates obtained in the Subsections 8.1, 8.2 and 8.3.

### 8.1. Optical theorem from \( X_{22} \)

To derive the first optical theorem, we consider

\[
\sum_s G_{is}G_{si} = G_{ii}^2 + \sum_s G_{is}G_{si}.
\]

(8.2)

Similar to the expansion of \( G_{\alpha\alpha} \), we now expand \( G_{ss} \) around \(-\tau\). We notice that

\[
\left| \tau^{-1} - z - \sum_{\gamma,\delta} x_{\gamma\delta} G_{\gamma\delta}^{(s)} x_{\delta s} \right| < \Psi,
\]

which can be checked from (6.10) and the estimate

\[
\left| G_{\alpha\alpha} - \frac{1}{\tau_{\alpha} + \tau} \right| < \Psi.
\]

Thus, using Schur’s complement formula (5.11), we obtain the following expansion of \( G_{ss} \) in the lower index \( s \),

\[
G_{ss} = \frac{1}{h_{ss} - \sum_{\gamma,\delta} h_{\gamma\delta} G_{\gamma\delta}^{(s)} h_{\delta s}} = \frac{1}{-\tau^{-1} + \tau^{-1} - z - \sum_{\gamma,\delta} x_{\gamma\delta} G_{\gamma\delta}^{(s)} x_{\delta s}}
\]

\[
= -\tau - \tau^2 \left( \tau^{-1} - z - \sum_{\gamma,\delta} x_{\gamma\delta} G_{\gamma\delta}^{(s)} x_{\delta s} \right) \tau^3 \left( \tau^{-1} - z - \sum_{\gamma,\delta} x_{\gamma\delta} G_{\gamma\delta}^{(s)} x_{\delta s} \right)^2 + O(\Psi^3). \quad (8.3)
\]

Using the resolvent formula (5.12) we therefore get the following expansion of \( G_{is}G_{si} \) in the lower index \( s \), for \( s \neq i \),

\[
G_{is}G_{si} = G_{ss}^2 \sum_{\rho,\sigma} G_{ip}^{(s)} x_{\rho s} x_{\sigma i} G_{si}^{(s)}
\]

\[
= \tau^2 \sum_{\rho,\sigma} G_{ip}^{(s)} x_{\rho s} x_{\sigma i} G_{si}^{(s)} + 2\tau^3 \left( \tau^{-1} - z - \sum_{\gamma,\delta} x_{\gamma\delta} G_{\gamma\delta}^{(s)} x_{\delta s} \right) \sum_{\rho,\sigma} G_{ip}^{(s)} x_{\rho s} x_{\sigma i} G_{si}^{(s)}
\]

\[
+ 3\tau^4 \left( \tau^{-1} - z - \sum_{\gamma,\delta} x_{\gamma\delta} G_{\gamma\delta}^{(s)} x_{\delta s} \right)^2 \sum_{\rho,\sigma} G_{ip}^{(s)} x_{\rho s} x_{\sigma i} G_{si}^{(s)} + O(\Psi^3). \quad (8.4)
\]
Taking the partial expectation $\mathbb{E}_s$, we obtain, for $s \neq i$, 

$$
\mathbb{E}_s[G_{is}G_{si}] = \frac{\tau^2}{N} \sum_\rho G_{ip}^{(s)} G_{pi}^{(s)} + \frac{2\tau^3}{N} \left( \tau^{-1} - z - \frac{\bar{m}(s)}{d} \right) \sum_\rho G_{ip}^{(s)} G_{pi}^{(s)} - \frac{4\tau^3}{N^2} \sum_{\rho,\sigma} G_{ip}^{(s)} G_{\rho\sigma}^{(s)} G_{\sigma i}^{(s)}
$$

$$
+ \frac{3\tau^4}{N} \left( \tau^{-1} - z - \frac{\bar{m}(s)}{d} \right)^2 \sum_\rho G_{ip}^{(s)} G_{pi}^{(s)} - \frac{12\tau^4}{N^2} \left( \tau^{-1} - z - \frac{\bar{m}(s)}{d} \right) \sum_{\rho,\sigma} G_{ip}^{(s)} G_{\rho\sigma}^{(s)} G_{\sigma i}^{(s)}
$$

$$
+ \frac{6\tau^4}{N^3} \sum_{\rho,\sigma,\gamma} G_{ip}^{(s)} G_{\rho\sigma}^{(s)} G_{\gamma i}^{(s)} G_{\gamma i}^{(s)} + \frac{24\tau^4}{N^3} \sum_{\rho,\sigma,\gamma} G_{ip}^{(s)} G_{\rho\gamma}^{(s)} G_{\gamma \sigma}^{(s)} G_{\sigma i}^{(s)} + O(\Psi^5).
$$

Using the resolvent formula (5.6) to remove the upper indices $s$ in (8.5), we get, for $s \neq i$, 

$$
\mathbb{E}_s[G_{is}G_{si}] = \frac{\tau^2}{N} \sum_\rho G_{ip} G_{pi} + \frac{2\tau^3}{N} \left( \tau^{-1} - z - \frac{\bar{m}}{d} \right) \sum_\rho G_{ip} G_{pi} - \frac{2\tau^3}{N^2} \sum_{\rho,\sigma} G_{ip} G_{\rho\sigma} G_{\sigma i}
$$

$$
+ \frac{3\tau^4}{N} \left( \tau^{-1} - z - \frac{\bar{m}}{d} \right)^2 \sum_\rho G_{ip} G_{pi} - \frac{6\tau^4}{N^2} \left( \tau^{-1} - z - \frac{\bar{m}}{d} \right) \sum_{\rho,\sigma} G_{ip} G_{\rho\sigma} G_{\sigma i}
$$

$$
+ \frac{12\tau^4}{N^3} \sum_{\rho,\sigma,\gamma} G_{ip} G_{\rho\sigma} G_{\gamma i} G_{\gamma i} + \frac{3\tau^4}{N^3} \sum_{\rho,\sigma,\gamma} G_{ip} G_{\rho\gamma} G_{\sigma \gamma} G_{\sigma i} + O(\Psi^5).
$$

We next expand all terms on the right side of (8.6) except the first one to change Greek indices into Roman indices. Recall from (7.24) that 

$$
A_k = \frac{1}{N} \sum_\rho \frac{1}{(t_{\rho}^{-1} - \tau)^k}.
$$

The last two terms on the right side of (8.6) are easy to convert. For example, 

$$
G_{ip} G_{\rho\sigma} G_{\gamma \sigma} G_{\gamma i} = G_{ip} G_{\rho\sigma} G_{\rho(i)}^{(p)} G_{\gamma \gamma}^{(i)} + O(\Psi^5)
$$

$$
= \frac{1}{(t_{\rho}^{-1} - \tau)^2} \sum_{j,k} G_{ij}^{(p)} x_{pj} x_{pk} G_{\rho\sigma}^{(p)} G_{\gamma \gamma}^{(i)} + O(\Psi^5),
$$

which shows that 

$$
\mathbb{E}_s[G_{ip} G_{\rho\sigma} G_{\gamma \sigma} G_{\gamma i}] = \frac{1}{(t_{\rho}^{-1} - \tau)^2} \frac{1}{N} \sum_j G_{ij} G_{j\sigma} G_{\gamma \sigma} G_{\gamma i} + O(\Psi^5).
$$

Repeating the argument once more we also find, using (6.7), that 

$$
\mathbb{E} \left[ \frac{12\tau^4}{N^3} \sum_{\rho,\sigma,\gamma} G_{ip} G_{\rho\sigma} G_{\gamma \sigma} G_{\gamma i} \right] = \frac{12\tau^4}{N^3} A_2 \mathbb{E} \left[ \sum_{j,k,l} G_{ij} G_{jk} G_{kl} G_{li} \right] + O(\Psi^5)
$$

$$
= 12\tau^{-2} \mathbb{E} [X_{44}] + O(\Psi^5).
$$

Similarly, 

$$
\mathbb{E} \left[ \frac{3\tau^4}{N^3} \sum_{\rho,\sigma,\gamma} G_{ip} G_{\rho\sigma} G_{\gamma \sigma} G_{\gamma i} \right] = 3\tau^{-2} \mathbb{E} [X'_{44}] + O(\Psi^5).
$$
The other fourth order terms in (8.6) require more treatment. We first consider

\[ \tau^{-1} - z - \frac{\tilde{m}}{d} = \tau^{-1} - z - \frac{1}{N} \sum_{\beta} G_{\beta \beta} \]

\[ = \tau^{-1} - z + \frac{1}{N} \sum_{\beta} \frac{1}{t_{\beta}^{-1} - \tau} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) + \mathcal{O}(\Psi^2) \]

\[ = (L+ - z) - \frac{1}{N} \sum_{\beta} \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) + \mathcal{O}(\Psi^2) \]

\[ = \frac{1}{N} \sum_{\beta} \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) + \mathcal{O}(\Psi^2). \quad (8.11) \]

We then obtain for the fifth term on the right side of (8.6) that

\[ \mathbb{E}\left[ -\frac{6\tau^4}{N^2} \left( \tau^{-1} - z - \frac{\tilde{m}}{d} \right) \sum_{\rho,\sigma} G_{i\rho} G_{\rho\sigma} G_{\sigma i} \right] \]

\[ = \mathbb{E}\left[ \frac{6\tau^4}{N^2} \left( \frac{1}{N} \sum_{\beta} \frac{1}{t_{\beta}^{-1} - \tau} \right) \left( m^{(\beta)} + \tau \right) \sum_{\rho,\sigma} G_{i\rho}^{(\beta)} G_{\rho\sigma}^{(\beta)} G_{\sigma i}^{(\beta)} \right] + \mathcal{O}(\Psi^5) \]

\[ = 6\tau^{-2} \mathbb{E}\left[ (m + \tau) \frac{1}{N^2} \sum_{k,l} G_{ik} G_{kl} G_{ki} \right] + \mathcal{O}(\Psi^5) \]

\[ = 6\tau^{-2} \mathbb{E}[X_{43}] + \mathcal{O}(\Psi^5). \quad (8.12) \]

Similarly, we have for the forth therm on the right side of (8.6) that

\[ \mathbb{E}\left[ \frac{3\tau^4}{N} \left( \tau^{-1} - z - \frac{\tilde{m}}{d} \right)^2 \sum_{\rho} G_{i\rho} G_{\rho i} \right] = 3\tau^{-2} \mathbb{E}[X_{42}] + \mathcal{O}(\Psi^5). \quad (8.13) \]

This completes the discussion of the fourth order terms in (8.6).

We move on to the third order terms on the right side of (8.6). Adapting the expansion method above, we note that

\[ \frac{1}{N} \left( \tau^{-1} - z - \frac{\tilde{m}}{d} \right) \sum_{\rho} G_{i\rho} G_{\rho i} \]

\[ = -\frac{1}{N^2} \sum_{\beta} \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) \sum_{\rho} G_{i\rho} G_{\rho i} \]

\[ + \frac{1}{N^2} \sum_{\beta} \frac{1}{(t_{\beta}^{-1} - \tau)^3} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right)^2 \sum_{\rho} G_{i\rho} G_{\rho i} + \frac{L+ - z}{N} \sum_{\rho} G_{i\rho} G_{\rho i} + \mathcal{O}(\Psi^5). \quad (8.14) \]
Taking the partial expectation $E_\beta$, we get for the summand in the first term on the right side of (8.14) that

$$E_\beta \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) G_{ip} G_{\rho i} \right]$$

$$= \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( m^{(\beta)} + \tau \right) G_{ip} G_{\rho i} + E_\beta \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) \frac{G_{i\beta} G_{\beta p} G_{\rho i}}{G_{i\beta}} \right]$$

$$+ E_\beta \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) G_{i\rho} \frac{G_{i\beta} G_{\beta k} G_{\rho i}}{G_{i\beta}} \right] + O(\Psi^5). \quad (8.15)$$

Expanding the first term on the right side of (8.15) with respect to the upper index $\beta$, we find

$$E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( m^{(\beta)} + \tau \right) G_{ip} G_{\rho i}^{(\beta)} \right]$$

$$= E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} (m + \tau) G_{ip} G_{\rho i} \right] + E \left[ \frac{2}{(t_{\beta}^{-1} - \tau)^2} (m + \tau) \sum_k G_{ik} G_{kp} G_{\rho i} \right]$$

$$+ E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \sum_{k,l} G_{kl} G_{ik} G_{ip} G_{\rho i} \right] + O(\Psi^5). \quad (8.16)$$

Similarly, we get for the expectation of the second term on the right side of (8.15) that

$$E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) \frac{G_{\rho i} G_{\beta \rho p}}{G_{i\beta}} \right]$$

$$= -E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} (m + \tau) \sum_k G_{ik} G_{kp} G_{\rho i} \right] - E \left[ \frac{2}{(t_{\beta}^{-1} - \tau)^2} \frac{1}{N^2} \sum_{k,l} G_{ik} G_{kl} G_{ip} G_{\rho i} \right] + O(\Psi^5) \quad (8.17)$$

and for the expectation of the third term on the right side of (8.15) that

$$E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) G_{i\rho} \frac{G_{i\beta} G_{\beta k}}{G_{i\beta}} \right]$$

$$= -E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} (m + \tau) \sum_k G_{ik} G_{kp} G_{\rho i} \right] - E \left[ \frac{2}{(t_{\beta}^{-1} - \tau)^2} \frac{1}{N^2} \sum_{k,l} G_{ik} G_{kl} G_{ip} G_{\rho i} \right] + O(\Psi^5). \quad (8.18)$$

Thus, from (8.15), (8.16), (8.17) and (8.18) we find

$$E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) G_{ip} G_{\rho i} \right] = E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} (m + \tau) G_{ip} G_{\rho i} \right]$$

$$= E \left[ \frac{1}{(t_{\beta}^{-1} - \tau)^2} (m^{(\rho)} + \tau) \sum_{k,l} G_{ik} x_{pk} G_{\rho j} G_{ti} + \frac{1}{N} \sum_j G_{ip} G_{\rho j} G_{pp} \sum_{k,l} G_{ik} x_{pk} G_{\rho l} G_{ti} + O(\Psi^5). \quad (8.19)$$

We next remove the Greek index $\rho$ on the right side of (8.19). We note that

$$(m + \tau) G_{ip} G_{\rho i} = G_{pp}^{(\rho)} \left( m^{(\rho)} + \tau \right) \sum_{k,l} G_{ik} x_{pk} x_{\rho l} G_{ti} + \frac{1}{N} \sum_j G_{ip} G_{\rho j} G_{pp} \sum_{k,l} G_{ik} x_{pk} x_{\rho l} G_{ti}. \quad (8.20)$$
Thus, taking the partial expectation $E_\rho$, we find
\[
E_\rho [(m + \tau) G_{ip} G_{pi}] = \frac{1}{(t_\rho - \tau)^2} \left( m(\rho) + \tau \right) \frac{1}{N} \sum_k G_{ik}^{(\rho)} G_{ki}^{(\rho)} - \frac{2}{(t_\rho - \tau)^3} \left( m(\rho) + \tau \right) \frac{1}{N} \sum_k G_{ik}^{(\rho)} G_{ki}^{(\rho)}
\]
\[
- \frac{4}{(t_\rho - \tau)^3} \left( m(\rho) + \tau \right) \frac{1}{N^2} \sum_{k,l} G_{ik}^{(\rho)} G_{jl}^{(\rho)} G_{li}^{(\rho)} - \frac{2}{(t_\rho - \tau)^3} \frac{1}{N^3} \sum_{j,k,l} G_{ik}^{(\rho)} G_{jl}^{(\rho)} G_{li}^{(\rho)} + O(\Psi^5) .
\]

Expanding the right side of (8.20) with respect to the upper index $\rho$, we obtain
\[
E [(m + \tau) G_{ip} G_{pi}]
\]
\[
= \frac{1}{(t_\rho - \tau)^2} E [X_{32}] - \frac{1}{(t_\rho - \tau)^3} \left( 2E [X_{42}] + 2E [X_{43}] + 2E [X_{44}] \right) + O(\Psi^5) .
\]

We thus have for the first term on the right side of (8.14) that
\[
E \left[ \frac{1}{N^2} \sum_\beta \frac{1}{(t_\beta - \tau)^2} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right) \sum_\rho G_{ip} G_{pi} \right]
\]
\[
= -\tau^{-4} E [X_{32}] + \tau^{-2} A_3 \left( 2E [X_{42}] + 2E [X_{43}] + 6E [X_{44}] - E [X_{44}] \right) + O(\Psi^5) .
\]

The fourth order terms in (8.14) can easily be handled: we have
\[
E \left[ \frac{1}{N^2} \sum_\beta \frac{1}{(t_\beta - \tau)^3} \left( \sum_{p,q} x_{\beta p} G_{pq}^{(\beta)} x_{\beta q} + \tau \right)^2 \sum_\rho G_{ip} G_{pi} \right]
\]
\[
= \tau^{-2} A_3 \left( 2E [X_{42}] + 2E [X_{44}] + \right)
\]
\[
+ O(\Psi^5) ,
\]
respectively,
\[
E \left[ \frac{L_+ - z}{N} \sum_\rho G_{ip} G_{pi} \right] = \tau^{-2} (L_+ - z) E [X_{22}] + O(\Psi^5) .
\]

We thus obtain from (8.14), (8.22), (8.23) and (8.24) that
\[
E \left[ \frac{2 \tau^3}{N} \left( \tau^{-1} - z - \frac{m}{d} \right) \sum_\rho G_{ip} G_{pi} \right]
\]
\[
= -2 \tau^{-1} E [X_{32}] + 2A_3 \left( 3E [X_{42}] + 2E [X_{43}] + 6E [X_{44}] + E [X_{44}'] \right)
\]
\[
+ 2 \tau (L_+ - z) E [X_{22}] + O(\Psi^5) .
\]

The third term on the right side of (8.6), which is also $O(\Psi^5)$, can be expanded in a similar manner: We begin with
\[
G_{ip} G_{\rho \sigma} G_{\sigma i} = G_{ip} G_{\rho \sigma} G_{\sigma i}^{(\rho)} + G_{ip} G_{\rho \sigma} \frac{G_{ip} G_{\sigma \rho}}{G_{pp}} .
\]

The second term on the right side of (8.26) can easily be controlled: we have
\[
\frac{1}{N^2} \sum_{\rho, \sigma} E \left[ G_{ip} G_{\rho \sigma} \frac{G_{ip} G_{\sigma \rho}}{G_{pp}} \right] = -\tau^{-2} A_3 \left( 2E [X_{44}] + E [X_{44}'] \right) + O(\Psi^5) .
\]
Taking the partial expectation $E_\rho$, we have

$$E_\rho \left[ G_{ip} G_{\rho \sigma} G_{\sigma i}^{(\rho)} \right] = \frac{1}{(t_\rho^{-1} - \tau)^2} \frac{1}{N} \sum_k G_{ik}^{(\rho)} G_{k \rho}^{(\rho)} G_{\sigma i}^{(\rho)} - \frac{2}{(t_\rho^{-1} - \tau)^3} \frac{1}{N} \sum_k m^{(\rho)} G_{ik}^{(\rho)} G_{k \rho}^{(\rho)} G_{\sigma i}^{(\rho)}$$

Thus, expanding with respect to the upper index $\rho$, we obtain

$$E_\rho \left[ G_{ip} G_{\rho \sigma} G_{\sigma i}^{(\rho)} \right] = \frac{1}{(t_\rho^{-1} - \tau)^2} \frac{1}{N} \sum_k G_{ik} G_{k \rho} G_{\sigma i} - \frac{2}{(t_\rho^{-1} - \tau)^3} \frac{1}{N} \sum_k (m + \tau) G_{ik} G_{k \rho} G_{\sigma i}$$

Repeating the same procedure with $\sigma$ instead of $\rho$, we eventually find

$$E \left[ \frac{2\gamma^3}{N^2} \sum_{\rho, \sigma} G_{s \rho} G_{\rho \sigma} G_{\sigma s} \right] = 2\tau^{-1} E [X_{33}] - 2\tau A_3 (4E [X_{43}] + 6E [X_{44}] + 2E [X_{44}']) + O(\Psi^5).$$

We conclude from (7.26), (8.6), (8.9), (8.10), (8.12), (8.13), (8.25) and (8.30) that

$$\frac{1}{N} \sum_{\alpha} E [G_{ia} G_{\alpha i}]$$

$$= \frac{1}{N^2} \sum_s \sum_{\rho} E [G_{ip} G_{\rho i}] + \frac{\tau^{-2}}{N} E [G_{ii}^2] + 2\tau^{-1} (L_+ - z) E [X_{22}] - 2(A_3 + \tau^{-3}) E [X_{32} + X_{33}]$$

$$+ 3(A_4 + \tau^{-4} + 2\tau^{-1} A_3) E [X_{42} + 2X_{43} + 4X_{44} + X_{44}'] + O(\Psi^5).$$

Since

$$\frac{1}{N^2} \sum_s \sum_{\rho} E [G_{ip} G_{\rho i}] = \frac{1}{N} \sum_{\rho} E [G_{ip} G_{\rho i}] + O(\Psi^5),$$

we obtain the relation

$$2(A_3 + \tau^{-3}) E [X_{32} + X_{33}] = 3(A_4 + \tau^{-4} + 2\tau^{-1} A_3) E [X_{42} + 2X_{43} + 4X_{44} + X_{44}']$$

$$+ \frac{\tau^{-2}}{N} E [G_{ii}^2] + 2\tau^{-1} (L_+ - z) E [X_{22}] + O(\Psi^5).$$

Recalling that

$$G_{ii}^2 = \tau^2 + 2\tau^3 \left( \tau^{-1} - z - \sum_{\gamma, \delta} x_{\gamma i} G_{\gamma \delta}^{(i)} x_{\delta i} \right) + O(\Psi^2),$$

we find

$$E [G_{ii}^2] = \tau^2 + 2\tau^3 E \left[ \tau^{-1} - z - \frac{m}{d} \right] + O(\Psi^2) = \tau^2 - 2\tau E [m + \tau] + O(\Psi^2).$$
Thus, plugging (8.33) into (8.32) and recalling from (6.7) that $A_3 + \tau^{-3} = 1$, we find
\[
2\mathbb{E} [X_{32} + X_{33}] - \frac{1}{N} = 3(A_4 + \tau^{-4} + 2\tau^{-1}A_3)\mathbb{E} [X_{42} + 2X_{43} + 4X_{44} + X_{44}']
- \frac{2\tau^{-1}}{N} \mathbb{E} [m + \tau] + 2\tau^{-1}(L_+ - z)\mathbb{E} [X_{22}] + O(\Psi^5).
\] (8.34)

The identity (8.34) is the optical theorem derived from $X_{22}$. We remark that the second and third term on the right side of (8.34) are both $O(\Psi^4)$. In Subsection 8.3, we show that they can be written as linear combinations of $X_{42}, X_{43}, X_{44}$ and $X_{44}'$.

8.2. Optical theorems from $X_{32}$ and $X_{33}$. In a next step, we derive further optical theorems using the ideas presented in Subsection 8.1. We start by considering
\[
X_{32} = (m + \tau) \frac{1}{N} \sum_s G_{is} G_{si} = (m + \tau) \frac{1}{N} \sum_s G_{is} G_{si} + (m + \tau) \frac{1}{N} G_{ii}^2.
\] (8.35)

To estimate the first term on the very right side of (8.35), we consider, for $s \neq i$,
\[
(m + \tau)G_{is} G_{si} = (m^{(s)} + \tau)G_{is} G_{si} + \frac{1}{N} \sum_s G_{js} G_{sj} G_{is} G_{si} + O(\Psi^5).
\] (8.36)

We expand the first term on the right side of (8.36) with respect to the lower index $s$ to get
\[
(m^{(s)} + \tau)G_{is} G_{si} = (m^{(s)} + \tau)G_{is}^2 \sum_{\rho, \sigma} \sum_{\rho, \sigma} G_{ip}^{(s)} x_{\rho, \sigma} G_{\sigma i}^{(s)}
= \tau^2 (m^{(s)} + \tau) \sum_{\rho, \sigma} G_{ip}^{(s)} x_{\rho, \sigma} G_{\sigma i}^{(s)}
+ 2\tau^3 (m^{(s)} + \tau) \left( \tau^{-1} - z - \sum_{\gamma, \delta} x_{\gamma, \delta} G_{\gamma i}^{(s)} x_{\delta s} G_{\sigma i}^{(s)} \right) \sum_{\rho, \sigma} G_{ip}^{(s)} x_{\rho, \sigma} x_{\sigma i} G_{\gamma i}^{(s)} + O(\Psi^5).
\]

Taking the partial expectation $\mathbb{E}_s$, we obtain
\[
\mathbb{E}_s \left[ (m^{(s)} + \tau)G_{is} G_{si} \right]
= \frac{\tau^2}{N} (m^{(s)} + \tau) \sum_{\rho} G_{ip}^{(s)} G_{\rho i}^{(s)} + \frac{2\tau^3}{N} \left( m^{(s)} + \tau \right) \left( \tau^{-1} - z - \frac{m^{(s)}}{d} \right) \sum_{\rho} G_{ip}^{(s)} G_{\rho i}^{(s)}
- \frac{4\tau^3}{N^2} (m^{(s)} + \tau) \sum_{\rho, \sigma} G_{ip}^{(s)} G_{\rho \sigma} G_{\sigma i}^{(s)} + O(\Psi^5).
\]

Since
\[
\frac{\tau^2}{N} (m^{(s)} + \tau) \sum_{\rho} G_{ip}^{(s)} G_{\rho i}^{(s)}
= \frac{\tau}{N} (m + \tau) \sum_{\rho} G_{ip} G_{\rho i} + \frac{2\tau}{N} (m + \tau) \sum_{\rho} G_{is} G_{sp} G_{pi} + \frac{\tau}{N^2} \sum_{\rho} G_{js} G_{sj} G_{ip} G_{pi} + O(\Psi^5)
\]

and since
\[
\mathbb{E} \left[ \frac{\tau^3}{N^2} (m + \tau) \sum_{\rho, \sigma} G_{ip} G_{\rho \sigma} G_{\sigma i} \right]
= \mathbb{E} \left[ \frac{\tau}{N^2} (m + \tau) \sum_{k} \sum_{\sigma} G_{ik} G_{k \sigma} G_{\sigma i} \right] + O(\Psi^5),
\]
we obtain
\[
E \left[ (m^{(s)} + \tau) \frac{1}{N} \sum_{i} G_{is} G_{si} \right] = E \left[ \frac{\tau^2}{N} (m + \tau) \sum_{\rho} G_{ip} G_{\rho i} \right] - \tau^{-1} E \left[ 2X_{42} + 2X_{43} - X_{44}' \right] + O(\Psi^5).
\]
Moreover, we have that
\[
E \left[ \frac{1}{N} \sum_{j} \frac{G_{js} G_{sj}}{G_{ss}} G_{is} G_{si} \right] = -\tau^{-1} E \left[ 2X_{44} + X_{44}' \right] + O(\Psi^5).
\]
We thus find the relation
\[
E [X_{32}] - N^{-1} E \left[ (m + \tau) G_{ii}^2 \right] = \tau^2 E \left[ (m + \tau) \frac{1}{N} \sum_{\rho} G_{ip} G_{\rho i} \right] - 2\tau^{-1} E \left[ X_{42} + X_{43} + X_{44} \right] + O(\Psi^5).
\]
Applying (8.21), we obtain
\[
E [X_{32}] - N^{-1} E \left[ (m + \tau) G_{ii}^2 \right] = E [X_{32}] - 2(\tau^2 A_3 + \tau^{-1}) E \left[ X_{42} + X_{43} + X_{44} \right] + O(\Psi^5).
\]
Further, since
\[
N^{-1} E \left[ (m + \tau) G_{ii}^2 \right] = \tau^2 N^{-1} E [m + \tau] + O(\Psi^5),
\]
we obtain from (8.37) the identity
\[
N^{-1} E [m + \tau] = 2(\tau^3 + \tau^{-3}) E \left[ X_{42} + X_{43} + X_{44} \right] + O(\Psi^5),
\]
which is the optical theorem derived from \(X_{32}\).
We next derive the optical theorem obtained from
\[
X_{33} = \frac{1}{N^2} \sum_{k,s} G_{ik} G_{ks} G_{si}.
\]
Since the contributions to the sums in (8.39) from the cases \(i = k\) or \(s = k\) are negligible (of \(O(\Psi^2)\)), we assume that \(i, s \neq k\). Expanding the summand in (8.39) with respect to the lower index \(k\), we get
\[
G_{ik} G_{ks} G_{si} = G_{kk}^2 \sum_{\rho, \sigma} G_{ip}^{(k)} x_{\rho k x_{\sigma k}} G_{\alpha s}^{(k)} G_{\alpha s}^{(k)} + G_{ik} G_{ks} \frac{G_{sk} G_{ki}}{G_{kk}}.
\]
Taking the partial expectation \(E_k\), we find for the first term on the right side of (8.40) that
\[
E \left[ G_{kk}^2 \sum_{\rho, \sigma} G_{ip}^{(k)} x_{\rho k x_{\sigma k}} G_{\alpha s}^{(k)} G_{\alpha s}^{(k)} \right] = E \left[ \frac{\tau^2}{N} \sum_{\rho} G_{ip}^{(k)} G_{\rho i}^{(k)} G_{si}^{(k)} \right] + \frac{2\tau^3}{N} \left( \tau^{-1} - z - \frac{\bar{m}^{(k)}}{d} \right) \sum_{\rho} G_{ip}^{(k)} G_{\rho i}^{(k)} G_{si}^{(k)}
\]
Expanding further with respect to the upper index \(k\), we thus find from (8.40) that
\[
E [X_{33}] = E \left[ \frac{\tau^2}{N^2} \sum_{s} \sum_{\rho} G_{ip} G_{\rho s} G_{si} \right] - \tau^{-1} E \left[ 2X_{43} + 3X_{44} + X_{44}' \right] + O(\Psi^5).
\]
Expanding the summand in the first term on the right side of (8.41) with respect the lower Greek index \(\rho\), we obtain
\[
E [X_{33}] = E [X_{33}] - (\tau^2 A_3 + \tau^{-1}) E \left[ 2X_{43} + 3X_{44} + X_{44}' \right] + O(\Psi^5).
\]
that is, recalling $A_3 + \tau^{-3} = 1$ (see (7.25)),
\[
\tau^{-2}E[2X_{43} + 3X_{44} + X'_{44}] = O(\Psi^5),
\]
which is the optical theorem derived from $X_{33}$.

8.3. Optical theorem from $mX_{22}$. We return to the concluding remarks of Subsection 8.1. In the present subsection, we show that the terms $(L_+ - z)E[X_{22}]$ and $N^{-1}E[m + \tau]$, both appearing in (8.34), can be decomposed into linear combinations of $X_{12}$, $X_{43}$, $X_{44}$ and $X'_{44}$. The latter term, $N^{-1}E[m + \tau]$, can be handled by (8.38), while the former needs to be dealt with the optical theorem obtained from $mX_{22}$.

Recall that
\[
mx_{22} = \frac{1}{N^2} \sum_{\alpha,\beta} G_{\alpha\beta} G_{\alpha\beta}. \tag{8.44}
\]
Expanding the summand on the right side of (8.44) in the index $a$, we get
\[
mx_{22} = \frac{1}{N^2} \sum_{\alpha \neq \beta} G_{\alpha\beta} G_{\alpha} G_{\beta} + G_{\alpha} G_{\beta} G_{\alpha} + G_{\alpha} G_{\alpha} G_{\beta} + O(\Psi^5)
\]
\[
= \frac{1}{N^2} \sum_{\alpha \neq \beta} \left(G_{\alpha\beta} G_{\alpha} G_{\beta} - G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} + 2X_{33} + O(\Psi^5). \tag{8.45}\right.
\]
Expanding the second summand of the first term on the right side of (8.45) with respect the lower index $a$, we get
\[
\mathbb{E} [mx_{22}] = \mathbb{E} \left[ \frac{1}{N^2} \sum_{\alpha \neq \beta} G_{\alpha\beta} G_{\alpha} G_{\beta} \right] + \tau^{-1}E[2X_{44} + X'_{44}] + 2E[X_{33}] + O(\Psi^5). \tag{8.46}\]
We expand the summand of the first term on the right side of (8.46) further in the lower index $a$ to find
\[
\mathbb{E}_a [G_{\alpha\beta} G_{\alpha} G_{\beta}] = -\tau G_{\alpha} G_{\beta} + \tau^2 \left( \tau^{-1} - z - \frac{m^{(a)}}{d} \right) G_{\alpha} G_{\beta} - \tau^3 \left( \tau^{-1} - z - \frac{m^{(a)}}{d} \right)^2 G_{\alpha} G_{\beta} + O(\Psi^5). \tag{8.47}\]
Expanding the first term on the right side of (8.47) with respect the upper index $a$, we get
\[
G_{\alpha} G_{\beta} = G_{\alpha} G_{\beta} - G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} - G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} + O(\Psi^5). \tag{8.48}\]
We stop expanding the first term on the right side of (8.48) which will eventually, after averaging over $s$, become $X_{22}$. For the second term on the right side of (8.48), we have
\[
\mathbb{E}_a \left[ G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} G_{\alpha} G_{\beta} \right] = -\frac{\tau}{N} \sum_{\gamma} G_{\gamma} G_{\gamma} G_{\gamma} G_{\gamma} - \frac{\tau^2}{N} \left( \tau^{-1} - z - \frac{m^{(a)}}{d} \right) \sum_{\gamma} G_{\gamma} G_{\gamma} G_{\gamma} G_{\gamma} + O(\Psi^5). \tag{8.49}\]
Thus,
\[
\mathbb{E} \left[ \frac{\tau}{N^2} \sum_{a \neq s} G^{(a)}_{is} G_{sa} G_{ai} \right] = -\mathbb{E} \left[ \frac{\tau^2}{N^3} \sum_{a \neq s} \sum_{\gamma} G^{(a)}_{is} G^{(a)}_{s\gamma} G_{\gamma i} \right] + \tau^{-1} \mathbb{E} [X_{43} + 2X_{44}] + \mathcal{O}(\Psi^5)
\]
\[
= -\mathbb{E} \left[ \frac{\tau^2}{N^2} \sum_{s} G_{is} G_{s\gamma} G_{\gamma i} \right] + \tau^{-1} \mathbb{E} [X_{43} - X_{44}] + \mathcal{O}(\Psi^5).
\]
(8.50)

Following the calculation in (8.41)–(8.42), we obtain from (8.50) that
\[
\mathbb{E} \left[ \frac{\tau}{N^2} \sum_{a \neq s} G^{(a)}_{is} G_{sa} G_{ai} \right] = -\mathbb{E} [X_{33}] + \tau^2 A_3 \mathbb{E} [2X_{43} + 3X_{44} + X_{44}']
\]
\[+ \tau^{-1} \mathbb{E} [X_{43} - X_{44}] + \mathcal{O}(\Psi^5).
\]
(8.51)

The third term on the right side of (8.48) can be expanded in a similar manner. In sum, we get
\[
-\mathbb{E} \left[ \frac{\tau}{N^2} \sum_{a \neq s} G^{(a)}_{is} G^{(a)}_{si} \right] = -\tau \mathbb{E} [X_{22}] - 2\mathbb{E} [X_{33}] + 2\tau^2 A_3 \mathbb{E} [2X_{43} + 3X_{44} + X_{44}']
\]
\[+ \tau^{-1} \mathbb{E} [2X_{43} + X_{44}'] + \mathcal{O}(\Psi^5).
\]
(8.52)

We next consider the second term on the right side of (8.47). We note that
\[
\left( \tau^{-1} - z - \frac{\bar{m}^{(a)}}{d} \right) G^{(a)}_{is} G^{(a)}_{si} = \left( \tau^{-1} - z - \frac{\bar{m}}{d} \right) G_{is} G_{si} + \tau^{-1} \left( \tau^{-1} - z - \frac{\bar{m}}{d} \right) G_{ia} G_{as} G_{si}
\]
\[+ \tau^{-1} \left( \tau^{-1} - z - \frac{\bar{m}}{d} \right) G_{is} G_{sa} G_{ai} - \frac{\tau^{-1}}{N} \sum_{\gamma} G_{\gamma a} G_{a\gamma} G_{is} G_{si} + \mathcal{O}(\Psi^5).
\]
(8.53)

We expand the first term on the right side of (8.53) similar to (8.14) to get
\[
\left( \tau^{-1} - z - \frac{\bar{m}}{d} \right) G_{is} G_{si}
\]
\[= -\frac{1}{N} \sum_{\beta} \sum_{p,q} \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G^{(p)}_{pq} x_{\beta q} + \tau \right) G_{is} G_{si}
\]
\[+ \frac{1}{N} \sum_{\beta} \sum_{p,q} \frac{1}{(t_{\beta}^{-1} - \tau)^3} \left( \sum_{p,q} x_{\beta p} G^{(p)}_{pq} x_{\beta q} + \tau \right)^2 G_{is} G_{si} + (L_+ - z) G_{is} G_{si} + \mathcal{O}(\Psi^5).
\]
(8.54)

Taking the partial expectation \( \mathbb{E}_\beta \) and proceeding as in (8.15)–(8.18) we find for the first term on the right side of (8.54) that
\[
\mathbb{E} \left[ \frac{\tau^2}{N^3} \sum_{i,s} \sum_{\beta} \frac{1}{(t_{\beta}^{-1} - \tau)^2} \left( \sum_{p,q} x_{\beta p} G^{(p)}_{pq} x_{\beta q} + \tau \right) G_{is} G_{si} \right] = \mathbb{E} [X_{32}] + \tau^2 A_3 \mathbb{E} [X_{44} + 4X_{44}] + \mathcal{O}(\Psi^5).
\]
We thus have
\[
\mathbb{E} \left[ -\frac{\tau^2}{N^2} \sum_{a \neq s} \left( \tau^{-1} - z - \frac{\bar{m}^{(a)}}{d} \right) G^{(a)}_{is} G^{(a)}_{si} \right] = \mathbb{E} [X_{32}] - \tau^2 A_3 \mathbb{E} [2X_{43} + 4X_{44} + X_{44}']
\]
\[+ \tau^2 (L_+ - z) \mathbb{E} [X_{22}] + \tau^{-1} \mathbb{E} [2X_{43} + X_{44}] + \mathcal{O}(\Psi^5).
\]
(8.55)
From (8.47), (8.52) and (8.55) we find for the first term on the right side of (8.46) that

\[ E \left[ \frac{1}{N^2} \sum_{a \neq s} G_{aa} G_{is}^{(a)} G_{si}^{(a)} \right] \]

\[ = -\tau E [X_{22}] - 2E [X_{33}] + 2\tau^2 A_3 E [2X_{43} + 3X_{44} + X_{44}'] + \tau^{-1} E [2X_{43} + X_{44}'] \]

\[ + E [X_{32}] - \tau^2 A_3 E [X_{42} + 4X_{44} + X_{44}'] - \tau^2 (L_+ - z) E [X_{22}] + \tau^{-1} E [2X_{43} + X_{44}'] \]

\[ - \tau^{-1} E [X_{42} + 2X_{44}'] - \tau^{-1} E [2X_{44} + X_{44}'] + 2E [X_{33}] + O(\Psi^5). \] (8.56)

Plugging (8.56) into (8.46), we finally find

\[ E [mX_{22}] + \tau^2 (L_+ - z) E [X_{22}] = -\tau E [X_{22}] + E [X_{32}] + \tau^2 A_3 + \tau^{-1} E [-X_{42} + 4X_{43} + 2X_{44} + X_{44}'] + O(\Psi^5). \] (8.57)

Since \(X_{32} = (m + \tau)X_{22}\) by definition, we obtain

\[ (L_+ - z) E [X_{22}] = (A_3 + \tau^{-3}) E [-X_{42} + 4X_{43} + 2X_{44} + X_{44}'] + O(\Psi^5), \] (8.58)

which is the optical theorem obtained from \(mX_{22}\).

8.4. Proof of Lemma 8.1. In this subsection, we prove Lemma 8.1 based on the optical theorems derived in the subsections 8.1, 8.2 and 8.3.

Proof of Lemma 8.1. For simplicity set

\[ X_3 := 2(X_{32} + X_{33}), \quad X_4 := 3(X_{42} + 2X_{43} + 4X_{44} + X_{44}'). \] (8.59)

From (8.34), (8.38) and (8.58) we have

\[ E [X_3] - N^{-1} = (A_4 + \tau^{-4} + 2\tau^{-1} A_3) E [X_4] - 2\tau^{-1} N^{-1} E [m + \tau] + 2\tau^{-1} (L_+ - z) E [X_{22}] + O(\Psi^5) \]

\[ = (A_4 + \tau^{-4} + 2\tau^{-1} A_3) E [X_4] - \tau^{-1} E [6X_{42} - 4X_{43} - 2X_{44}] + O(\Psi^5). \] (8.60)

Subtracting 8-times (8.43) from (8.60), we obtain

\[ E [X_3] - N^{-1} = (A_4 + \tau^{-4} + 2\tau^{-1} A_3) E [X_4] - \tau^{-1} 6E [X_{32} + 2X_{43} + 4X_{44} + X_{44}'] + O(\Psi^5) \]

\[ = (A_4 + \tau^{-4} + 2\tau^{-1} A_3 - 2\tau^{-1}) E [X_4] + O(\Psi^5). \] (8.61)

Using \(A_4 + \tau^{-3} = 1\) (see (7.25)), we conclude that

\[ E [X_3] - N^{-1} = (A_4 - \tau^{-4}) E [X_4] + O(\Psi^5). \] (8.62)

This proves (8.1) and concludes the proof of Lemma 8.1.

\[ \square \]

9. Proof of Lemma 6.3

In this last section, we prove Lemma 6.3.

Proof of Lemma 6.3. In a first step of the proof of (6.18), we express \((\partial_t t_\alpha)/t_\alpha^2\) in terms of \(\gamma\) and \(\dot{\gamma}\).

From the time evolution of \(\Sigma = \text{diag}(\sigma_\alpha)\) in (6.1), we have

\[ \partial_t \frac{1}{\sigma_\alpha(t)} = -e^{-\gamma t} \frac{1}{\sigma_\alpha(0)} + e^{-t} = 1 - \frac{1}{\sigma_\alpha(t)}, \quad (t \geq 0). \]

Since \(t_\alpha = \gamma\sigma_\alpha\) by the definition of \(T\), we get

\[ \frac{\partial_t t_\alpha}{t_\alpha^2} = -\partial_t \frac{1}{t_\alpha(t)} = \left( \frac{\dot{\gamma}}{\gamma} + 1 \right) \frac{1}{t_\alpha(t)} - \frac{1}{\gamma}, \quad (t \geq 0). \]
Recalling the definitions of \((A_k)\) in (7.24) and that \(A_2 = \tau^{-2}\), we then obtain, dropping for simplicity the \(t\)-dependence from the notation,

\[
\frac{1}{N} \sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha^2} \left( \frac{1}{(t_\alpha^{-1} - \tau)^3} \right) = \left( \frac{\dot{\gamma}}{\gamma} + 1 \right) \tau^{-2} + \left( \frac{\dot{\gamma}}{\gamma} + 1 \right) \tau A_3 - \frac{1}{\gamma} A_4,
\]

(9.1)

respectively,

\[
\frac{1}{N} \sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha^3} \left( \frac{1}{(t_\alpha^{-1} - \tau)^4} \right) = \left( \frac{\dot{\gamma}}{\gamma} + 1 \right) A_3 + \left( \frac{\dot{\gamma}}{\gamma} + 1 \right) \tau A_4 - \frac{1}{\gamma} A_4.
\]

(9.2)

Using the short-hand notation

\[
X_3 = 2(X_{32} + X_{33}), \quad X_4 = 3(X_{42} + 2X_{43} + 4X_{44} + X_{44}'),
\]

(9.3)

(see (8.59)), we observe that (6.18) is proven, once we have established that

\[
[(\dot{\gamma} + \gamma) \tau^{-2} + (\dot{\gamma} \tau + \gamma \tau - 1) A_3] \text{Im} E[X_3] = [(\dot{\gamma} + \gamma) A_3 + (\dot{\gamma} \tau + \gamma \tau - 1) A_4] \text{Im} E[X_4] + \mathcal{O}(\Psi^5).
\]

(9.4)

Combining the following lemma with Lemma 8.1, it is straightforward to assure the validity (9.4).

**Lemma 9.1.** Let \(\gamma\) and \(\tau\) be defined in (6.5) and (6.6). Then we have

\[
(\dot{\gamma} + \gamma) \tau^{-2} + (\dot{\gamma} \tau + \gamma \tau - 1) A_3 = \gamma(\tau^{-2} A_4 - A_2^3),
\]

(9.5)

\[
(\dot{\gamma} + \gamma) A_3 + (\dot{\gamma} \tau + \gamma \tau - 1) A_4 = \gamma(\tau^{-2} A_4 - A_2^3)(A_4 - \tau^{-4}).
\]

(9.6)

Assuming the correctness of Lemma 9.1, we can recast (9.4) as

\[
\gamma(\tau^{-2} A_4 - A_2^3) \text{Im} E[X_3] = \gamma(\tau^{-2} A_4 - A_2^3)(A_4 - \tau^{-4}) \text{Im} E[X_4] + \mathcal{O}(\Psi^5).
\]

(9.7)

Since \(E[X_3] - 1/N = (A_4 - \tau^{-4})E[X_4] + \mathcal{O}(\Psi^5)\), by the optical theorem (8.1), we see that (9.7), respectively (9.4), indeed hold true. This in turn proves, by the discussion above, the claim in (6.18), i.e., Lemma 6.3.

It remains to prove Lemma 9.1:

**Proof of Lemma 9.1.** First, we differentiate the sum rule

\[
\frac{1}{N} \sum_{\alpha} \left( \frac{1}{t_\alpha^{-1} - \tau} \right)^2 = \frac{1}{\tau^2},
\]

(see (6.7)) with respect to \(t\) to find

\[
\dot{t} \tau^3 = \frac{1}{N} \sum_{\alpha} \frac{\partial t_\alpha}{t_\alpha} \left( \frac{1}{(t_\alpha^{-1} - \tau)^3} \right) = -\dot{\tau} A_3 - \gamma^{-1} A_3 + \left( \frac{\dot{\gamma}}{\gamma} + 1 \right) \frac{1}{N} \sum_{\alpha} \frac{t_\alpha^{-1}}{(t_\alpha^{-1} - \tau)^3},
\]

which yields

\[
(A_3 + \tau^{-3})\dot{\tau} = \gamma^{-1}[\dot{\gamma} + \gamma(\tau^{-2} + \tau A_4) - A_3].
\]

(9.8)

Using \(A_3 + \tau^{-3} = 1\), we hence get

\[
\dot{\tau} = \gamma^{-1}[\dot{\gamma} + \gamma \tau - A_3].
\]

(9.9)

Similarly, differentiating the sum rule

\[
\frac{1}{N} \sum_{\alpha} \left( \frac{1}{t_\alpha^{-1} - \tau} \right)^3 = \frac{1}{\tau^3},
\]

(see (6.7)) with respect to \(t\) we find

\[
(A_4 - \tau^{-4})\dot{\tau} = \gamma^{-1}[\dot{\gamma} + \gamma(A_3 + \tau A_4) - A_4].
\]
Combination with (9.9) yields

\[(A_4 - \tau^{-4})[(\dot{\gamma} + \gamma)\tau - A_3] = (\dot{\gamma} + \gamma)(A_3 + \tau A_4) - A_4,\]

hence

\[\dot{\gamma} + \gamma = \tau^{-4}A_3 - A_3A_4 + A_4 = \tau^{-4}A_3 + \tau^{-3}A_4.\]  \hspace{1cm} (9.10)

Thus, we can write the left side of (9.5) as

\[(\tau^{-4}A_3 + \tau^{-3}A_4)\tau^{-2} + (\tau^{-3}A_3 + \tau^{-2}A_4 - 1)A_3 = (\tau^{-3}A_3 + \tau^{-2}A_4) - A_3\]

\[= (\tau^{-3}A_3 + \tau^{-2}A_4) - A_3(A_3 + \tau^{-3})\]

\[= (\tau^{-2}A_4 - A_3^2).\]  \hspace{1cm} (9.11)

This proves (9.5).

Similarly, we have for the left side of (9.6)

\[(\tau^{-4}A_3 + \tau^{-3}A_4)A_3 + (\tau^{-3}A_3 + \tau^{-2}A_4 - 1)A_4\]

\[= (\tau^{-4}A_3 + \tau^{-3}A_4)A_3 + (\tau^{-3}A_3 + \tau^{-2}A_4 - (A_3 + \tau^{-3})^2)A_4\]

\[= \tau^{-4}A_3^2 + \tau^{-2}A_4^2 - A_3^2A_4 - \tau^{-6}A_4\]

\[= (A_4 - \tau^{-4})(\tau^{-2}A_4 - A_3^2).\]  \hspace{1cm} (9.12)

This proves (9.6) and hence finishes the proof of Lemma 9.1.

Having proven Lemma 9.1, we can conclude the proof of Lemma 6.3.

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