ADIABATIC VACUUMSTATES OF THE DIRAC-FIELD ON A CURVED SPACETIME

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Abstract. In this article we review the quantization of the Dirac-field on a curved spacetime. For that purpose we describe the construction of the local observable algebras in the algebraic approach to quantum field theory. Among the possible states we single out the so called Hadamard-states, which are the ones relevant for physics. Finally, as an example, we give a definition for an adiabatic vacuum state of the Dirac-field on a Robertson-Walker spacetime. We believe that these states are physical in the sense that they have the singularity structure of Hadamard-form, although we cannot give a formal proof of this conjecture.

1. Introduction

In this paper we are concerned with quantum field theory on a curved spacetime (QFT on CST). Among the vast literature in this area on the Klein-Gordon-field only little work is done for the Dirac-field. The reason for this are not conceptual problems but the greater technical complexity involved in the description of multicomponent fields.

After Hawkings discovery that a black hole not only absorbs the energy incident on it, but also loses energy through the emission of pairs of elementary particles there were a great interest in the treatment of quantized fields in a classical background. This semiclassical description of quantum fields propagating in a non flat background can serve as an approximation to the yet non existing theory of quantum gravity. The range of validity goes down to the Planck-length, where quantum gravity effects are expected to become important.

One of the main questions in this branch was concerned with the characterization of physical states on the various spacetime models. It is now accepted that all physical quantum states must have a singularity structure of Hadamard-form. For these states there is a well defined prescription to regularize the energy-momentum tensor and the semiclassical Einstein-equations

\[ G_{ab} = 8\pi \langle T_{ab} \rangle \]

which govern the backreaction of the quantized fields on the spacetime geometry make sense. Recently there was made progress in the formulation of the spectrum condition on a curved spacetime. There the Hadamard-states play an important role too, see [2].

2. Dirac-field

To set our notation we give in this section a short description of spinors in arbitrary curved spacetimes. Furthermore we recall the existence- and uniqueness-results of fundamental solutions of the Dirac-equation. At the end we mention Dimock’s result that the Cauchy-Problem is well posed [4].

By curved spacetime we mean a 4-dimensional \( C^\infty \)-manifold which is endowed with a metric, a covariant tensor-field of type \((0,2)\), with Lorentzian signature \((+,-,-,-)\). The metric describes locally the causal structure of our spacetime.
We first introduce the complex Clifford-algebra of Dirac-matrices on Minkowski-spacetime and then we use the tetrad-formalism to lift them on our spacetime manifold. We denote the Minkowski-spacetime by \((\mathbb{R}^4, \eta_{ab} = \text{diag}(1, -1, -1, -1))\). Let \(\mathcal{C}(\mathbb{R}^4, \eta)\) be the real, associative algebra with unit \(1\) which is generated by the elements \(\{c(v), v \in \mathbb{R}^4\}\) and the relation

\[
\{c(v), c(w)\} = 2\eta(v, w)1 \tag{1}
\]

This is the Clifford-algebra associated to Minkowski-spacetime \((\mathbb{R}^4, \eta)\). In the description of the Dirac-field on Minkowski-spacetime the so called Dirac-matrices play a prominent role. We obtain these matrices when we look at a faithful representation \(\rho\) of the complexified Clifford-algebra \(\mathcal{C}(\mathbb{R}^4, \eta)\) on \(M_C(4)\). The resulting Dirac-matrices are denoted by \(\gamma^a := \rho(c(e^a))\), \(a = 0, \ldots, 3\). The vectors \(\{e_a\}\) built a standard basis of \(\mathbb{R}^4\). For two different representations \(\rho_1\) and \(\rho_2\) of \(\mathcal{C}(\mathbb{R}^4, \eta)\) we can always find a nonsingular matrix \(T\) which transforms the two corresponding sets of Dirac-matrices \(\gamma^a_{\rho_1}\) and \(\gamma^a_{\rho_2}\) into each other, i.e.

\[
T\gamma^a_{\rho_1}T^{-1} = \gamma^a_{\rho_2}, \quad a = 0, \ldots, 3 \tag{2}
\]

In the following we assume a form of the Dirac-matrices with the additional property

\[
\gamma^0 = \gamma_0, \quad \gamma^j = -\gamma_j \tag{3}
\]

The * means hermitian conjugation. Dirac-matrices with this property are said to be in standard representation. Next we consider the Lie-group \(\text{Spin}(1, 3)\) of matrices \(S\) with the properties

\[
\det S = 1 \quad \quad \quad \quad S\gamma_a S^{-1} = \gamma_b \Lambda^b_a \tag{4, 5}
\]

Together with the anticommutation relations of the Dirac-matrices one sees that the matrix \(\Lambda\) must be an element of the full Lorentz-group \(\mathcal{L}\). The mapping \(S \rightarrow \Lambda(S)\) defined by (5) is a 2-1 homomorphism of \(\text{Spin}(1, 3)\) onto \(\mathcal{L}\). In the following we restrict ourselves to \(\text{Spin}_0(1, 3)\) the connected component of the identity of \(\text{Spin}(1, 3)\). The image under the above mapping is the group of proper orthochronous Lorentz transformations \(\mathcal{L}^+\).

Now let \((M, g)\) the spacetime under consideration. We suppose \((M, g)\) to be time- and space-orientable, which means that we can make a continuous distinction between future and past as well as left and right throughout \(M\), and choose such orientations. Let \(TM\) denote the tangent bundle over \(M\). To a given coordinate basis \(\{e_\mu, \mu = 0, \ldots, 3\}\) at the point \(p \in M\) we associate an orthonormal basis \(\{e_a, a = 0, \ldots, 3\}\) which is a linear combination of the coordinate basis vectors \(e_\mu\) according to

\[
e_a = e^\mu_a e_\mu \quad \quad \quad g(e_a, e_b) = \eta_{ab} \quad \quad \quad g_{\mu\nu}(p)e^\mu_0 e^\nu_0 \geq 0, \quad e^0_0 \geq 0, \quad p \in M \tag{6, 7, 8}
\]

The matrix \((e^\mu_a)\) shall be an element of \(GL(4, \mathbb{R})\) and the last condition is necessary to preserve the orientations. The collection of all this orthonormal bases constitute the so called orthonormal frame bundle \(F(M, g)\) associated to \(TM\). Under a change of frame the tetrad \(e^\mu_a\) transforms as

\[
e^\mu_b = e^\mu_a \Lambda^a_b, \quad \Lambda \in \mathcal{L}^+_+ \tag{9}
\]
If we perform a local change of coordinates around \( p \in M \) we obtain the following transformation law for the tetrad

\[
e^{\mu}{}_a = \left( \frac{\partial x^\mu}{\partial y^a} \right)_p e^\nu {}_a
\]

(10)

In this way \( F(M, g) \) admits the structure of a \( \mathcal{L}^1_\ast \)-principal fiberbundle. If we consider \( \text{Spin}_0(1, 3) \) instead of \( \mathcal{L}^1_\ast \), we obtain a \( \text{Spin}_0(1, 3) \)-principal fiberbundle which is called a spin structure \( S(M, g) \) for \( (M, g) \), if we have in addition a bundlehomomorphism \( \varphi : S(M, g) \to F(M, g) \) which satisfies

\[
\varphi \circ R_S = R_{\Lambda(S)} \circ \varphi, \quad S \in \text{Spin}_0(1, 3)
\]

(11)

The symbol \( R \) denotes the right action with \( \Lambda \) resp. \( S(\Lambda) \) on orthonormal basis vectors in \( F(M, g) \) resp. \( S(M, g) \). If a given spacetime possesses spin structures is determined by the topological properties of the underlying manifold. Here we consider only globally hyperbolic spacetimes for which the manifold is so well behaved that the existence of spin structures is always guaranteed, but they are not uniquely determined by the topology.

Now we are ready to define Dirac-spinors. On our spacetime \( (M, g) \) with spin structure \( (S(M, g), \varphi) \) we take a standard representation \( \rho \) of the Clifford-algebra \( \mathcal{C}(\mathbb{R}^4, \eta)_C \) together with the group \( \text{Spin}_0(1, 3) \). The bundle of Dirac-spinors is now the associated vectorbundle \( D_\rho(M, g) \) to the \( \text{Spin}_0(1, 3) \)-principal fiberbundle \( S(M, g) \), the representation \( \rho \) and the vectorspace \( \mathbb{C}^4 \):

\[
D_\rho(M, g) := (S(M, g) \times \rho \mathbb{C}^4)/\text{Spin}_0(1, 3)
\]

(12)

In the following we abbreviate \( D_\rho(M, g) \) with \( DM \). A spinorfield is now a smooth section in \( DM \). The collection of all smooth spinorfields is denoted by \( C^\infty(DM) \). Those which have in addition compact support are denoted by \( C^\infty_0(DM) \). With \( D^*M \) we denote the dual spinorbundle. A \( v \in D^*M \) for \( p \in M \) an element of the dual space to the fiber over \( p \). The smooth sections in \( D^*M \) are called co-spinorfields. The set of all smooth co-spinorfields are denoted by \( C^\infty(D^*M) \). Let \( \Sigma \) be a submanifold of \( M \). By \( D'M \) we denote the vectorbundle \( \pi^{-1}_{DM}(\Sigma) \) over \( \Sigma \) with projection \( \pi_{DM|\Sigma} = \pi_{DM|\pi^{-1}_{DM}(\Sigma)} \), where \( \pi_{DM} \) is the projection in \( DM \). For a spinor \( u \in C^\infty(DM) \) and a co-spinor \( v \in C^\infty(D^*M) \) we have a natural dual pairing

\[
v(u)|_p = v_A u^A, \quad p \in M
\]

(13)

with respect to bases \( \{ E_A, A = 0, \ldots 3 \} \) resp. \( \{ E^A, A = 0, \ldots 3 \} \) in \( C^\infty(DM) \) resp. \( C^\infty(D^*M) \). In the above expression we use the Einstein summation convention. We can now generate arbitrary spinor-tensor fields, i.e. smooth sections in the fiberwise tensor products of \( TM, T^*M, DM, D^*M \). An element \( f \) in \( (\bigotimes_p TM) \otimes (\bigotimes_q T^*M) \otimes (\bigotimes_p DM) \otimes (\bigotimes_q D^*M) \) is given by specifying the family of \( \mathbb{C} \)-valued functions \( f^{a_1 \ldots a_p b_1 \ldots b_q}_{A_1 \ldots A_r B_1 \ldots B_s} \) on \( M \), such that

\[
f = (f^{a_1 \ldots a_p b_1 \ldots b_q}_{A_1 \ldots A_r B_1 \ldots B_s} e_{a_1} \otimes \ldots \otimes e_{a_p} \otimes \ldots \otimes E^{B_1} \otimes \ldots \otimes E^{B_s})
\]

(14)

Of special interest is the spinor-tensor \( \gamma \in T^*M \otimes DM \otimes D^*M \) which has the components

\[
\gamma^A_a B = (\gamma_a)^A_B
\]

(15)

with the Dirac-matrices \( \gamma_a, a = 0, \ldots 3 \) in standard representation. For a vector field \( k \in C^\infty(TM) \) we denote by \( \bar{k} \) the contraction of \( k \) with \( \gamma \), i.e. \( \bar{k}^A_B := k^a \gamma_a^A B \). Next we introduce the notion of an adjoint spinor \( u^\gamma \) for a spinor \( u \). This is the co-spinor with the following components: \( (u^\gamma)_B := u^\gamma \gamma_{0AB} \), where the bar means complex conjugation. We can now use the spinor-tensor \( \gamma \) to lift the Levi-Civita-derivative of the metric \( g \) to mixed spinor-tensor fields in the following way.
The usual covariant derivative $\nabla : C^\infty(TM) \to C^\infty(T^*M \otimes TM)$ is defined by $(\nabla k)_a^b := \partial_a k^b + T^b_{ac} k^c$, $k \in C^\infty(TM)$. We extend this definition to spinor fields as follows: $\nabla : C^\infty(DM) \to C^\infty(T^*M \otimes DM)$ is defined through the specification of the components of $\nabla f$, $f \in C^\infty(DM)$ with respect to bases $(E^A)$ and $(e_a)$ as $(\nabla f)_a^B := \partial_a f^B + \sigma_a^B A f^A$, where the so called spin connection coefficients are given by

$$\sigma_a^B_A := -\frac{1}{4} \Gamma_{ca}^b \gamma^c_{bD} \gamma^e_D$$

This defines indeed a covariant derivative on $M$. In demanding the Leibnitz-rule and commutativity with contractions, we extend this definition to all spinor-tensor fields. As an easy consequence of the above definitions we obtain the following lemma

**Lemma 1.** The spinor-tensor $\gamma$ is covariant constant, i.e. $\nabla \gamma = 0$.

**Proof.** One can see this immediately by using the anticommutation relations of the Dirac-matrices.

We can now write down the Dirac-equations for spinors and cospinors, which are

$$(-i\nabla + m)u = 0, u \in C^\infty(DM)$$

$$(i\nabla + m)v = 0, v \in C^\infty(D^*M)$$

In these equations $m$ is a fixed real number. The Dirac-operator obeys Lichnerowicz’ identity:

$$(-i\nabla + m)(i\nabla + m)u = (\Box - \frac{1}{4} R + m^2)u, u \in C^\infty(DM)$$

where $\Box = g^{ab} \nabla_a \nabla_b$ is the spinorial wave operator. The operator on the right side of (19) is known as the spinorial Klein-Gordon-operator. Next we consider the classical solutions to the Dirac-equation on a globally hyperbolic spacetime. We remind ourselves that a spacetime with given time- and space orientations is said to be globally hyperbolic, if it admits a Cauchy-surface. A Cauchy-surface is a smooth spacelike hypersurface $\Sigma \subset M$ which is intersected by every endless causal curve exactly once. A globally hyperbolic spacetime has the structure $\mathbb{R} \times \Sigma$, i.e. we can foliate $M$ into smooth hypersurfaces. We now introduce distributions on the spaces $C^\infty_0(DM)$ and $C^\infty(D^*M)$. The natural pairing $\langle v, u \rangle$ between $v \in C^\infty(D^*M)$ and $u \in C^\infty_0(DM)$ can be integrated over $M$ in the obvious way:

$$\langle v, u \rangle_M := \int_M v(u)(p)d\mu(p), u \in C^\infty_0(DM), v \in C^\infty(D^*M)$$

Here $d\mu(p)$ is the volume element associated with the metric $g$. In addition to that we demand continuity of $v$, seen as a distribution over $C^\infty_0(DM)$, with respect to certain Sobolev-norms, see [17] for details. In this way we have an embedding of $C^\infty(D^*M)$ in $C^\infty_0(DM)'$, the space of cospinor-valued distributions over $M$. Analogously $C^\infty_0(DM)$ is embedded in the space of spinor-valued distributions over $M$ with compact support, $C^\infty(D^*M)'$. The next Theorem gives us the existence of unique fundamental solutions to the Dirac operators.

**Theorem 1** (Dimock, 1982). The operator $(-i\nabla + m)$ on $C^\infty_0(DM)$ has unique retarded resp. advanced fundamental solutions $S^\pm : C^\infty_0(DM) \to C^\infty(DM)$ with

$$(-i\nabla + m)S^\pm = S^\pm (-i\nabla + m) = \text{id} \text{ on } C^\infty_0(DM)$$
and \( \text{supp}(S^\pm f) \subset J^\pm(\text{supp}(f)) \). The operator \((i\nabla + m)\) on \( C^\infty(D^*M) \) has unique retarded resp. advanced fundamental solutions \( S^\pm : C_0^\infty(D^*M) \to C^\infty(D^*M) \) with

\[
(i\nabla + m)S^\pm = S^\pm (i\nabla + m) = \text{id} \quad \text{on} \quad C_0^\infty(D^*M)
\]

and \( \text{supp}(S^\pm f) \subset J^\pm(\text{supp}(f)) \).

The difference \( S \) between retarded \( (S^+) \) and advanced \( (S^-) \) fundamental solution is called Dirac-Propagator for spinors: \( S = S^+ - S^- \). \( S^\pm = S^+ - S^- \) is called Dirac-Propagator for cospinors.

The Cauchy problem for the Dirac-equation is now well posed:

**Theorem 2** (Dimock, 1982). Let \( u_0 \in C_0^\infty(DM_\Sigma) \). Then there is a unique \( u \in C^\infty(DM) \) with

\[
(-i\nabla + m)u = 0 \quad \text{and} \quad u|_\Sigma = u_0
\]

In addition to that we have \( \text{supp}(u) \subset J^+(\text{supp}(u_0)) \cup J^-(\text{supp}(u_0)) \).

These two theorems lie at the heart of the quantization of the Dirac-field which is described in the next section.

### 3. Quantization

In this section the quantization of the Dirac-field on an arbitrary curved spacetime is described. Since on a general curved spacetime we do not have time translation symmetry, we have no natural criterion to single out a Hilbert space of solutions, oscillating with positive frequency, which play the role of a one-particle space in Minkowski-spacetime. Therefore we adopt the algebraic approach here, in which only the algebraic structure of the theory is specified, i.e. we construct a \( C^*-\)algebra and in a subsequent step certain subalgebras which are associated to open regions in spacetime [20]. These local algebras constitute the so called observable algebras in the sense of Haag and Kastler [4]. A state is then specified as a positive normalized linear functional on the observables. The connection to the usual operator approach is given by the GNS-construction.

Let \( S \) be the space of smooth complex-valued solutions to the Dirac-equation, which have compact support initial data:

\[
S = \{ u \in C^\infty(DM) \mid (-i\nabla + m)u = 0 \quad \text{and} \quad u|_\Sigma = u_0 \in C_0^\infty(DM_\Sigma) \}
\]

On \( S \) we introduce the following scalar product \((.,.) : S \times S \to \mathbb{C} \), given by

\[
(u_1, u_2) := \int_\Sigma \hat{u}_1^+(\hat{f} \hat{u}_2)(p) d\mu_\Sigma(p)
\]

In this expression \( \hat{u} \) is the restriction of the spinorfield \( u \) to \( \Sigma \). \( n \) is the unit normal vectorfield on \( \Sigma \) and \( \mu \) is the volume element induced by the three-dimensional Riemannian metric \( h_{ab} = g_{ab} + n_a n_b \) on \( \Sigma \). This scalar product is independent of the choice of Cauchy-surface \( \Sigma \). We complete the space \( S \) in the norm induced by this scalar product and obtain in this way a complex Hilbert space \( H \). Now we can associate to \( H \) an abstract \( C^*-\)algebra \( \mathcal{F}[H] \) by the antilinear mapping \( B : H \to \mathcal{F}[H] \) such that \( \{ B(h) : h \in H \} \) together with the unit \( 1 \) generate \( \mathcal{F}[H] \) algebraically and fulfill the canonical anticommutation relations:

\[
\{ B(h), B(g) \} = 0
\]

\[
\{ B(h), B(g)^* \} = (h, g) \cdot 1, \quad \text{for all} \quad h, g \in H
\]

[1] Here for \( N \subset M J^\pm(N) \) is the set of points in \( M \), which are connected to \( N \) through a future directed (+) resp. past directed (-) causal curve.
The algebra $\mathcal{F}[H]$ has a unique $C^*$-norm $\|\cdot\|_{C^*}$. Moreover it follows by the use of
the CAR-relations that the mapping $B$ is an isometry, i.e. we have
$$\|B(f)\|_{C^*} = \|f\|, \; f \in H$$
(28)

By assigning to every open relative compact subset $O \subset M$ the $C^*$-subalgebra
$\mathcal{F}(O)$ which is generated by elements $\{B(f) : f \in C_0^\infty(DM, O)\}$, where we denote
by $C_0^\infty(DM, O)$ spinors whose support lies in $O$, we obtain a net of field algebras
$O \to \mathcal{F}(O)$. The even part hereof, i.e. elements of the form $\{B(h)^* B(f) : h, f \in C_0^\infty(DM, O)\}$ built the net of observable algebras $O \to \mathcal{A}(O)$ in the sense of Haag
and Kastler [5].

Now we come to the consideration of states on our CAR-algebra $\mathcal{F}[H]$. A state
is a positive, normalized linear functional $\omega$ on $\mathcal{F}[H]$, by which we mean a linear
map $\omega : \mathcal{F}[H] \to \mathbb{C}$ with the properties
$$\omega(B(f)^* B(f)) \geq 0 \quad \text{(positivity)}$$
$$\omega(1) = 1 \quad \text{(normalization)}$$
(29)
(30)

As usual we focus attention on quasifree states, which have the form
$$\omega(B(f_1) \ldots B(f_{2n})) = (-1)^{n(n-1)} \sum \text{sgn}(\sigma) \prod_{j=1}^n \omega(B(f_{\sigma(j)}) B(f_{\sigma(j+n)})), \; n \in \mathbb{N}$$
(31)

where the sum runs through all permutations $\sigma$ of $\{1, \ldots, 2n\}$ with $\sigma(1) < \sigma(2) < \ldots < \sigma(n)$ and $\sigma(j) < \sigma(j+n)$, $j = 1, \ldots, n$. sgn$(\sigma)$ is $+1(-1)$, if $\sigma$ is an even (odd)
permutation of $\{1, \ldots, 2n\}$. In addition to that $\omega$ vanishes on all odd monomials
$B(f_1) \ldots B(f_{2n+1}), n \in \mathbb{N}_0$.

Let $P$ be an orthogonal projection in $H$. A quasifree state $\omega_P$ on $\mathcal{F}[H]$ can then
be specified in the following way. We choose an orthonormal basis $\{f_i\}_{i \in \mathbb{N}}$ for $H$
consisting of eigenspinors for $P$. Then every element $h \in H$ has an expansion,
$$h = \sum_{i \in \mathbb{N}} (f_i, h) f_i.$$ We define an operator $\Gamma$ on $H$ by $\Gamma h = \sum_{i \in \mathbb{N}} (f_i, h) f_i$. $\Gamma$ is a
well defined involution and it is antiunitary, i.e.
$$(\Gamma g, \Gamma h) = (h, g), \quad \forall g, h \in H$$
(32)

Moreover $\Gamma$ commutes with $P$. Then the spaces $PH$ and $(I - P)H$ together span
the entire Hilbertspace $H$ and are orthogonal w.r.t. $(\cdot, \cdot)$. Therefore we can write
every element $h \in H$ in a unique way as
$$h = h^+ + h^-, \quad h^+ \in PH, h^- \in (I - P)H$$
(33)

Now we take the antisymmetric Fockspace $\mathcal{F}_\wedge(H)$ over $H$ with annihilation operator
$a(f) : \wedge^{n+1}H \to \wedge^nH$ defined by
$$a(f) \Omega = 0$$
(34)

and creation operator $a(f)^* : \wedge^nH \to \wedge^{n+1}H$ given by
$$a(f)^* \Omega = f$$
$$a(f)^* (f_1 \wedge \ldots \wedge f_n) = f \wedge f_1 \wedge \ldots \wedge f_n, \; \forall f, f_1, \ldots, f_n \in H$$
(35)
(36)

where $\Omega$ is the vacuum vector in $\mathcal{F}_\wedge(H)$. With $\wedge^nH$ we denote the antisymmetric
tensor product $P_{\wedge}(\otimes^nH)$ defined by
$$P_{\wedge}(f_1 \otimes \ldots \otimes f_n) = (n!)^{-1} \sum_{\sigma \in S_n} \chi_\wedge(\sigma) \cdot f_{\sigma(1)} \otimes \ldots \otimes f_{\sigma(n)}$$
where the sum runs through all permutations $\sigma$ in the permutation group $S_n$ and
$\chi_\wedge$ is the sign of the permutation $\sigma$. Furthermore we have used the notation
$$f_1 \wedge \ldots \wedge f_n = (n!)^{\frac{1}{2}} P_{\wedge}(f_1 \otimes \ldots \otimes f_n)$$
The operators \( a(f) \) and \( a(f)^* \) are bounded operators on \( \mathcal{F}_a(\mathcal{H}) \) and they have anticommutation relations given by

\[
\{ a(f), a(g)^* \} = (f, g) \cdot 1
\]  

(37)

This is a concrete \( \mathcal{C}^* \)-algebra, the so called Fock-representation of the CAR. We take the Dirac-field operator to be

\[
\Psi_P(f) = a(f^+) + a(\Gamma f^-)^*, \quad f = f^+ + f^- \quad \text{where} \quad f^+ \in \mathcal{P} \mathcal{H}, \ f^- \in (I - \mathcal{P})\mathcal{H}
\]  

(38)

Then we have the anticommutation relation

\[
\{ \Psi(f), \Psi(g)^* \} = (f, g) \cdot 1
\]  

(39)

This is known as the quasifree representation of the CAR-algebra \( \mathcal{F}[\mathcal{H}] \). The state \( \omega_P \) is the quasifree state determined by the two-point function

\[
\Lambda_2^{\omega_P}(f, g) = \langle \Omega, \Psi(f)\Psi(g)^*\Omega \rangle
\]  

(40)

where \( \langle ., . \rangle \) is the inner product in \( \mathcal{F}_a(\mathcal{H}) \).

4. Hadamard states

In this section we are looking at those states for which the two-point function has the singularity structure of Hadamard form [3]. For these states it has been proven in the case of the Klein-Gordon field on a curved spacetime that the resulting operator description, which was sketched above, gives a physical theory in the sense that the principle of local definiteness is satisfied [6, 16]. We define now Hadamard states for the Dirac-field which are believed to be equally important here as they are for the scalar case. For the introduction of these states we need the concept of bi-spinors and some technicalities concerning the causal structure of spacetime which we relegate in an appendix.

Let \((M, g)\) be a globally hyperbolic spacetime which Cauchy-surface \(\Sigma\). The square root of the bi-spinor valued VanVleck-Morette determinant is defined by

\[
U^{A}_{B'} = \Delta^{1/2}(q, q') \mathcal{J}^{A}_{B'}
\]  

(41)

where \(\Delta^{1/2}(q, q') = g(q)^{1/2} \det(\sigma(q, q')_{\mu\nu}) g(q')^{1/2}\) is the scalar VanVleck-Morette determinant and \(\sigma(q, q')\) is one half of the square of the geodesic distance \(s(q, q')\). The bi-spinor of the parallel transport is defined by

\[
\sigma^\mu \mathcal{J}^{A}_{B';\mu} = 0
\]  

(42)

\[
\mathcal{J}^{A}_{B'}(q, q) = 1^{A}_{B'}
\]  

(43)

Let \(V_m^{A}_{B'}, m \in \mathbb{N}_0\) be a sequence of bi-spinors which are determined by the Hadamard recurrence relations [14]. Both \(U^{A}_{B'}\) and \(V_m^{A}_{B'}\) are well defined on \(X\), the set of points which can be joined by a unique geodesic (see the appendix for a precise definition of \(X\)). Furthermore we define for all \(n \in \mathbb{N}\)

\[
V^{(n)}^{A}_{B'}(q, q') := \sum_{m=0}^{n} V_m^{A}_{B'}(q, q')(s(q, q'))^m, \quad (q, q') \in X
\]  

(44)

Finally we take a global time function \(T : M \to \mathbb{R}\) and set

\[
r^T_{\epsilon}(q, q') := -2i\epsilon(T(q) - T(q')) - \epsilon^2, \quad q, q' \in M, \ \epsilon > 0
\]  

(45)

When we look at the Dirac-operator we observe that it is not a hyperbolic operator in the sense of Leray [12]. We use the fact that, due to Lichnerowicz’ identity, the product of the Dirac-operator times it’s complex conjugate is an hyperbolic operator. In the next definition we will introduce the singularity structure of Hadamard form for the spinor Klein-Gordon operator which underlies the definition of Hadamard state for the Dirac-field.
Definition 1. Let $\Lambda$ be a sesquilinear form on $C_0^\infty(DM)$. $\Lambda$ has Hadamard form (for the spinor Klein-Gordon operator), if the following data exist:

1. a causal normal neighborhood $N$ of a Cauchy-surface $\Sigma$
2. a $N$-regularising function $\chi$
3. a smooth future directed global time function $T$ on $M$
4. a sequence $H^{(n)} \in C^n(DN \otimes D^*N)$, $n \in \mathbb{N}$, so that for all $n \in \mathbb{N}$ and all $f, h \in C_0^\infty(DM, N)$

\[ \Lambda(f, h) = \lim_{\epsilon \to 0} \int (A^{T,n}_\epsilon)^A_B(q, q') f A(q) h B'(q') d\mu(q) d\mu(q'), \]

where

\[ (A^{T,n}_\epsilon)^A_B(q, q') := \chi(q, q')(G^{T,n}_\epsilon)^A_B(q, q') + H^{(n)}A_B'(q, q') \]

\[ (G^{T,n}_\epsilon)^A_B(q, q') := \frac{1}{4\pi^2} \left( \frac{U^{A_B}(q, q')}{(s(q, q') + r^T_\epsilon(q, q'))^2} + V^{(n)A_B}(q, q') \ln(s(q, q')) + r^T_\epsilon(q, q')) \right) \]

for $(q, q') \in X \cap (N \times N)$; $\ln$ is the main branch of the logarithm.

We now view the two-point function $\omega(B(f_1)^*B(f_2))$ as a hermitian form $f_1, f_2 \mapsto (f_1, Q f_2)$ over $H$. This form is a solution of the Dirac equation in both arguments, i.e. we have

\[ ((-i\nabla + m)f_1, Q f_2) = (f_1, Q(-i\nabla + m)f_2) = 0 \]  \hspace{1cm} (46)

With the next definition we arrive at the Hadamard form of a state of the Dirac-field.

Definition 2. A quasifree state $\omega$ on $\mathcal{F}[H]$ is called a Hadamard state of the Dirac-field, if there is a solution $\Lambda$ of the spinorial Klein-Gordon equation of Hadamard form, so that the two-point function of $\omega$ has the following form

\[ \omega(B(f_1)^*B(f_2)) = \Lambda((i\nabla + m)f_1, f_2), f_1, f_2 \in H \]  \hspace{1cm} (47)

$\Lambda$ is a solution of the Dirac equation in the first argument, since it fulfills Lichnerowicz’ identity and in the second argument because of hermiticity. The singular part of $\Lambda$ is determined by geometric quantities like metric and spinor connection only. Therefore a state is characterized, if we fix the smooth part of $\Lambda$.

Since the work of Radzikowski [13] there is another possibility to characterize the singularity structure of a Hadamard state. This method uses Hörmander’s wavefront sets to describe the singular behavior of the numerical distributions

\[ f_1, \ldots, f_n \mapsto \Lambda_n^\omega(f_1, \ldots, f_n) := \langle \Omega, \Psi(f_1) \ldots \Psi(f_n)\Omega \rangle \]  \hspace{1cm} (48)

We will here only give a brief introduction to this powerful theory; for more information see [15] [14] [13]. The idea in using wavefront sets is to localize the distributions around the singular support and then to analyze the directions in Fourier space which causes these singularities. The advantage of this approach is that only local concepts are used. In the following we will give the definition of the wavefront set of a distribution and recall the main result of Radzikowski’s thesis.

Definition 3. Let $(M, g)$ an $n$-dimensional manifold with associated cotangent bundle $T^*M$. Let $u \in D'(M)$. A point $(x_0, k_0) \in T^*M \setminus \{0\}$ is called a regular directed point of $u \in D'(M)$ if and only if for all $\lambda_0 \in \mathbb{R}^n$ and for every function $\phi \in C^\infty(M \times \mathbb{R}^n, \mathbb{R})$ with $d_\phi(x_0, \lambda_0) = k_0$ there exists a neighborhood $U$ of $x_0$ in $M$ and a neighborhood $\Lambda$ of $\lambda_0$ in $\mathbb{R}^n$, so that for all $\rho \in C^\infty(U)$ and all $N \geq 0$ uniformly in $\lambda \in \Lambda$:

\[ |\langle u, \rho \exp^{-itr\phi(\cdot, \lambda)} \rangle| = o(\tau^{-N}) \quad \text{for} \quad \tau \to \infty \]  \hspace{1cm} (49)
Theorem 3

The wavefront set $WF(u)$ is then the complement in $T^*M \setminus \{0\}$ of the set of regular directed points of $u$.

The wavefront set of a distribution $u$ has some remarkable properties:

1. The projection of $WF(u)$ on the first factor gives $\text{singsupp}(u)$.
2. $WF(u)$ is a closed subset of $T^*M \setminus \{0\}$, since the complement of $WF(u)$ are open sets.
3. For all testfunctions $\phi$ with compact support it holds
   \[ WF(\phi u) \subset WF(u) \]
4. For a differential operator $D$ with $C^\infty$ coefficients we have
   \[ WF(Du) \subset WF(u) \]

The last relation means that the application of a linear differential operator $D$ on $u$ does not enlarge the wavefront set of $u$.

For the wavefront set of a tensor product $u \otimes v$ of two distributions $u$ and $v$ on $M$ the following formula holds (see Hörmander [8]):

\[
WF(u \otimes v) \subseteq WF(u) \times WF(v) \\
\quad \cup ((\text{supp}(u) \times \{0\}) \times WF(v)) \\
\quad \cup (WF(u) \times (\text{supp}(v) \times \{0\})) \quad (50)
\]

With the help of wavefront sets we are now able to characterize the Hadamard states of the Dirac field. For this purpose we first look at the wavefront set of the two-point function of the Klein-Gordon field. As was explained previously we obtain the two-point function of the Dirac field through the application of the adjoint Dirac-operator on the scalar two-point function. Due to the fact that this procedure does not enlarge the wavefront set (since the Dirac-operator is a linear differential operator) it is sufficient to determine the wavefront set of the two-point function of the Klein-Gordon field. A completely local description of Hadamard states of the Klein-Gordon field was given in [15]:

**Theorem 3** (Radzikowski, 1992). A quasifree state $\omega$ of the Klein-Gordon field on a globally hyperbolic spacetime $(M, g)$ is a Hadamard state, if and only if its two-point function $\Lambda^\omega_{2,KG}$ has the following wavefront set:

\[
WF(\Lambda^\omega_{2,KG}) = \{(x_1, \xi_1; x_2, -\xi_2) \in T^*M^2 \setminus \{0\}; (x_1, \xi_1) \sim (x_2, \xi_2), \xi_{1}^0 \geq 0 \} \quad (51)
\]

Here $(x_1, \xi_1) \sim (x_2, \xi_2)$ means that $x_1$ and $x_2$ are connected by a lightlike geodesic $\gamma$ in the way that $\xi_1^\mu$ is tangential to $\gamma$ in $x_1$ and $\xi_2^\mu$ is $\xi_1^\mu$ parallely transported to $x_2$. On the diagonal $(x_1, \xi_1) \sim (x_2, \xi_2)$ means that $\xi_1 = \xi_2$ and $(\xi_1)^2 = 0$. We obtain the following result as a corollary to the above theorem:

**Corollary 1.** A state $\omega$ on $\mathcal{F}[H]$ is a Hadamard state for the Dirac-field, if and only if its two-point function $\Lambda^\omega_2$ has the following wavefront set

\[
WF(\Lambda^\omega_2) = \{(x_1, \xi_1; x_2, -\xi_2) \in T^*M^2 \setminus \{0\}; (x_1, \xi_1) \sim (x_2, \xi_2), \xi_{1}^0 \geq 0 \} \quad (52)
\]

**Proof.** This is an immediate consequence of property 4. above. □

One finds the wavefront set of an arbitrary $m$-point function with the help of (50). When we denote the kernel of $\Lambda^\omega_m$ by $\Lambda^\omega_m(x_1, \ldots, x_m)$ we obtain

\[
WF(\Lambda^\omega_m(x_1, \ldots, x_m)) \subseteq \bigcup_{\sigma} \bigoplus_{j=1}^{n} WF(\Lambda^\sigma(j)), \ n = m/2 \quad (53)
\]
where $\Lambda_2^{\sigma(j)}$ is the two-point function in the variables $x_{\sigma(j)}, x_{\sigma(j+n)}$ viewed as a distribution over $M^m$, with wavefront set

$$\text{WF}(\Lambda_2^{\sigma(j)}) = \{(x_1, 0; \ldots; x_{\sigma(j)}, k_{\sigma(j)}; \ldots; x_{\sigma(j+n)}, k_{\sigma(j+n)}; \ldots; x_m, 0)| (x_{\sigma(j)}, k_{\sigma(j)}; x_{\sigma(j+n)}, k_{\sigma(j+n)}) \in \text{WF}(\Lambda_2^{\sigma(j)})\} \quad (54)$$

That this expression is the wavefront set of $\Lambda_m^\omega$ is clear, since $\Lambda_m^\omega$ is a sum over tensor products of two-point functions.

5. Dirac-equation on Robertson-Walker spaces

In this section we consider homogeneous and isotropic spacetimes. It is known that the requirement of homogeneity and isotropy leads to the Robertson-Walker spacetimes of the form $M = \mathbb{R} \times \Sigma^c$, where $\Sigma^c$ is a homogeneous Riemannian manifold. The parameter $\kappa$ can take the values $+1, 0$ or $-1$. Correspondingly $\Sigma^c$ is a space of constant positive, vanishing or constant negative curvature. We focus attention to the Dirac-equation for which we construct explicit solutions by the method of separation of variables due to Villalba and Percoco [18]. We will use these solutions in the next section to define adiabatic vacuum states of the Dirac-field in this spacetime.

The metric in chronometrical coordinates $\tau, \chi, \theta, \varphi$ takes the form

$$ds^2 = e^{\alpha(\tau)}[d\tau^2 - d\chi^2 - \xi^2(\chi)d\Omega^2] \quad (55)$$

where $d\Omega^2$ is given by

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 \quad (56)$$

($\theta \in [0, \pi], \varphi \in [0, 2\pi], \chi \in [0, \infty)$ for $\kappa = 0, -1$ and $\chi \in [0, \pi)$ for $\kappa = +1$) and $\alpha(\tau)$ is a real-valued smooth function. The function $\xi(\chi)$ is given by

$$\xi(\chi) = \begin{cases} 
\sin \chi & \text{falls } \kappa = +1 \\
\chi & \text{falls } \kappa = 0 \\
\sinh \chi & \text{falls } \kappa = -1 
\end{cases} \quad (57)$$

To calculate the spinor connection we use a vierbein basis $e^a_\mu$ defined by

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu} \quad (58)$$

The formula for the Christoffel symbols in this basis is given by

$$\Gamma^{b}_{\; ac} = \frac{\delta^b_c}{e_a^\mu} \left( \partial_\mu e^\nu_c + e^\lambda_a \Gamma^{\nu}_{\; \mu \lambda} \right) \quad (59)$$

In this expression $\Gamma^{\nu}_{\; \mu \lambda}$ are the Christoffel-symbols in a coordinate basis. Greek letters will always denote indices with respect to a coordinate basis, whereas Latin indices correspond to a vierbein basis. We choose the vierbein basis in the following way:

$$e^a_\mu = e^{\alpha/2}(\delta^a_0 \delta_{\mu 0} + \delta^a_1 \delta_{\mu 1} + \xi(\chi) \delta^a_0 \delta_{\mu 2} + \xi(\chi) \sin \theta \delta^a_0 \delta_{\mu 3}) \quad (60)$$

$$e^b_\nu = e^{-\alpha/2}(\delta^0_\nu \delta_{0 0} + \delta^1_\nu \delta_{1 1} + \xi^{-1}(\chi) \delta^2_\nu \delta_{2 2} + \xi^{-1}(\chi) \sin^{-1} \theta \delta^3_\nu \delta_{3 3}) \quad (61)$$
The spinor connection is given by

\[ \Gamma^b_{ac} = \frac{1}{2} g^{\alpha/2} \left\{ \alpha \delta^{00} \left[ \delta_{a1} \delta_{c1} + \delta_{a2} \delta_{c2} + \delta_{a3} \delta_{c3} \right] 
+ \delta^{01} \left[ \delta_{a1} \delta_{c0} \alpha - 2 \delta_{a2} \delta_{c2} \xi^{-1}(\chi) \xi'(\chi) - 2 \delta_{a3} \delta_{c3} \xi^{-1}(\chi) \xi'(\chi) \right] 
+ \delta^{02} \left[ \delta_{a2} \delta_{c0} \alpha + 2 \delta_{a2} \delta_{c1} \xi^{-1}(\chi) \xi'(\chi) - 2 \delta_{a3} \delta_{c3} \xi^{-1}(\chi) \xi'(\chi) \cot \theta \right] 
+ \delta^{03} \left[ \delta_{a3} \delta_{c0} \alpha + 2 \delta_{a3} \delta_{c1} \xi^{-1}(\chi) \xi'(\chi) + 2 \delta_{a3} \delta_{c2} \xi^{-1}(\chi) \xi'(\chi) \cot \theta \right] \right\} \] (62)

The spinor connection is given by

\[ \sigma^B_A := -\frac{1}{4} \Gamma^b_{ca} g^b_{BC} D_C D_A \] (63)

The \( \gamma_a \) are the components of the spinor-tensor \( \gamma \) w.r.t. the used moving frame. The explicit expression for the spinor connection coefficients can now be given:

\[ \sigma^a = -\frac{1}{4} g^{-\alpha/2} \left\{ \alpha \left[ \delta_{a1} \gamma_0 \gamma^1 + \delta_{a2} \gamma_0 \gamma^2 + \delta_{a3} \gamma_0 \gamma^3 \right] 
- 2 \delta_{a2} \gamma_1 \gamma_2 \xi^{-1}(\chi) \xi'(\chi) - 2 \delta_{a3} \gamma_1 \gamma_3 \xi^{-1}(\chi) \xi'(\chi) \right\} \] (64)

Here the anticommutation relations of the Dirac-matrices were used.

The Dirac-equation reads

\[ (-i \gamma^a \nabla_a + m) \Psi(x) = (-i \gamma^a e^\mu_a \nabla_\mu + m) \Psi(x) = 0 \] (65)

Here the covariant derivative is lifted to the spinor bundle \( DM \), as was explained previously:

\[ \nabla_a = \partial_a + \sigma_a \] (66)

Inserting the expressions for the spinor connection coefficients and the vierbeins we obtain the following explicit form of the Dirac-equation:

\[ \left\{ -ie^{-\alpha/2} \left[ \gamma^0 \left( \partial_0 - \frac{3}{4} \alpha \right) + \gamma^1 \left( \partial_1 - \xi^{-1}(\chi) \xi'(\chi) \right) \right] 
+ \gamma^2 \xi^{-1}(\chi) \left( \partial_2 - \frac{1}{2} \cot \theta \right) \right. 
+ \left. \gamma^3 \xi^{-1}(\chi) \sin^{-1} \theta \partial_3 \right) + m \right\} \Psi(x) = 0 \] (67)

In order to arrive at a more simpler equation we make the following transition to the new spinor \( \Phi \) defined by:

\[ \Phi(x) = \xi^{-1}(\chi) \sin^{-1/2} \theta e^{-3\alpha/4} \Psi(x), \quad \chi \neq 0, \theta \neq 0, \pi, 2\pi \] (68)

The Dirac-equation for \( \Phi \) now reads:

\[ \left\{ -ie^{-\alpha/2} \left[ \gamma^0 \partial_0 + \gamma^1 \partial_1 + \gamma^2 \xi^{-1}(\chi) \partial_2 + \gamma^3 \xi^{-1}(\chi) \sin^{-1} \theta \partial_3 \right] + m \right\} \Phi(x) = 0 \] (69)

We now want to solve this equation by the method of separation of variables. This was already done in the paper by Villalba and Percoco [18], so we shall outline here the main steps of this calculation only. All of the missing details are completely straightforward.
First of all we decompose the Dirac-operator into two commuting differential operators which can be dealt with separately. We use the Minkowski-space Dirac-matrices in the following representation:

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3
\]

The Pauli-spinmatrices \(\sigma^j\) are given by

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

The Dirac-equation can be written in the form:

\[
[D^1(\theta, \varphi) + D^2_\kappa(\tau, \chi)] \Phi(x) = 0 \tag{70}
\]

Here we have defined \(\Phi = \gamma^1 \gamma^0 \Phi\) and the operators \(D^1(\theta, \varphi)\) and \(D^2_\kappa(\tau, \chi)\) are given by:

\[
D^1(\theta, \varphi) = \gamma^2 \partial_2 + \gamma^3 \sin^{-1} \theta \partial_3 \gamma^1 \gamma^0 \tag{71}
\]

\[
D^2_\kappa(\tau, \chi) = \xi(\chi) \left[ \gamma^0 \partial_0 + \gamma^1 \partial_1 + \text{ime}^{\alpha/2} \right] \gamma^1 \gamma^0 \tag{72}
\]

We observe that the operator \(D^1(\theta, \varphi)\) is selfadjoint with respect to the scalar product:

\[
\langle \Psi, \Phi \rangle := \int_0^\pi d\theta \int_0^{2\pi} d\varphi \Psi_A(\theta, \varphi) \Phi_A(\theta, \varphi) \tag{73}
\]

Now we have the following lemma:

**Lemma 2.** \(D^1\) and \(D^2_\kappa\) are commuting operators on \(C^\infty(\mathbb{C})\), i.e. we have

\[
[D^1(\theta, \varphi), D^2_\kappa(\tau, \chi)] = 0 \tag{74}
\]

**Proof.** We evaluate the product \(D^1 D^2_\kappa\):

\[
D^1 D^2_\kappa = \gamma^2 \gamma^1 \gamma^0 \partial_2 + \gamma^3 \gamma^1 \gamma^0 \sin^{-1} \theta \partial_3 \xi(\chi) \left[ \gamma^0 \partial_0 + \gamma^1 \partial_1 + \text{ime}^{\alpha/2} \right] \gamma^1 \gamma^0
\]

\[
= \gamma^1 \gamma^0 \left[ \gamma^2 \partial_2 + \gamma^3 \sin^{-1} \theta \partial_3 \right] \xi(\chi) \left[ \gamma^0 \partial_0 + \gamma^1 \partial_1 + \text{ime}^{\alpha/2} \right] \gamma^1 \gamma^0
\]

\[
= - \gamma^1 \gamma^0 \xi(\chi) \left[ \gamma^0 \partial_0 + \gamma^1 \partial_1 - \text{ime}^{\alpha/2} \right] \left[ \gamma^2 \partial_2 + \gamma^3 \sin^{-1} \theta \partial_3 \right] \gamma^1 \gamma^0
\]

\[
= \xi(\chi) \left[ \gamma^0 \partial_0 + \gamma^1 \partial_1 + \text{ime}^{\alpha/2} \right] \gamma^1 \gamma^0 \left[ \gamma^2 \partial_2 + \gamma^3 \sin^{-1} \theta \partial_3 \right] \gamma^1 \gamma^0
\]

\[
= D^2_\kappa D^1
\]

Because of this lemma it should be possible to find a common set of eigenspinors for these two operators. Therefore we make the following ansatz to obtain a solution to equation (70):

\[
D^1(\theta, \varphi) \Phi(x) = \lambda \Phi(x) = -D^2_\kappa(\tau, \chi) \Phi(x), \quad \lambda \in \mathbb{C} \tag{75}
\]

First we look at the angle dependent part \(D^1(\theta, \varphi)\):

\[
D^1(\theta, \varphi) \Phi = \lambda \Phi \iff \left\{ \gamma^2 \partial_2 + \gamma^3 \sin^{-1} \theta \partial_3 \right\} \gamma^1 \gamma^0 \Phi - \lambda \Phi = 0 \tag{76}
\]

A solution to this equation can be obtained in terms of orthogonal polynomials, i.e. we have the result

**Proposition 1.** 1. The eigenfunctions of the operator \(D^1\) have the following form:

\[
\Phi(\theta, \varphi) = U^{-1} \Xi(\theta, \varphi) \tag{77}
\]
where \( \Xi = \left( \begin{array}{c} \Xi_1 \\ \Xi_2 \end{array} \right) \) is a spinor with components
\[
\Xi_1(\theta, \varphi) = \begin{pmatrix} C_{nl}^1(\theta, \varphi) \\ C_{nl}^2(\theta, \varphi) \end{pmatrix}, \quad \Xi_2(\theta, \varphi) = \begin{pmatrix} -C_{nl}^1(\theta, \varphi) \\ C_{nl}^2(\theta, \varphi) \end{pmatrix}
\]
(78)
with \( l \in \mathbb{N}, \ n = \frac{1}{2}(2m + 1), \ m \in \mathbb{N} \) and scalar valued functions
\[
C_{nl}^1(\theta, \varphi) = \sin^n \theta \cos(\theta/2)G_{nl}(\cos \theta)W_n(\varphi)
\]
(79)
\[
C_{nl}^2(\theta, \varphi) = \sin^n \theta \sin(\theta/2)F_{nl}(\cos \theta)W_n(\varphi)
\]
(80)
\( G_{nl} \) and \( F_{nl} \) are the Jacobi-polynomials \( P_{l}^{(\alpha, \beta)} \) in the variable \( \cos \theta \) of order \( l \), i.e.
\[
G_{nl}(\cos \theta) = \frac{1}{\sqrt{N_{nl}}} P_{l}^{(\beta_1, \beta_2)}(\cos \theta)
\]
(81)
\[
F_{nl}(\cos \theta) = (-1)^l \frac{1}{\sqrt{N_{nl}}} P_{l}^{(\beta_2, \beta_1)}(\cos \theta)
\]
(82)
The parameters \( \beta_1 \) and \( \beta_2 \) are given by
\[
\beta_1 = n - \frac{1}{2}, \quad \beta_2 = n + \frac{1}{2}
\]
and \( N_{nl} \) is a normalization constant. The scalar valued function \( W_n \) is given by
\[
W_n(\varphi) = \exp(in\varphi)
\]
(83)
And the unitary matrix \( U \) is given by
\[
U = \frac{1}{2} \left( \begin{array}{cc} \gamma^2 \gamma^1 + \gamma^3 \gamma^2 + \gamma^1 \gamma^3 + 1_4 \end{array} \right)
\]
(84)
2. The eigenvalues \( \lambda \) are given by
\[
\lambda = l + n + \frac{1}{2} \in \mathbb{N} \setminus \{0\}
\]
(85)
The Jacobi-polynomials are defined in the proof.

Proof. Eq.(76) is a system of partial differential equations in two variables and the aim of the following calculation is to separate these two variables. The matrix \( U \) transforms the Dirac-matrices in the following way:
\[
U\gamma^0 U^{-1} = \gamma^0, \quad U\gamma^1 U^{-1} = \gamma^3, \quad U\gamma^2 U^{-1} = \gamma^1, \quad U\gamma^3 U^{-1} = \gamma^2
\]
The application of \( U \) to equation (76) then gives
\[
\left\{ \gamma^1 \partial_2 + \gamma^2 \sin^{-1} \theta \partial_3 \right\} \gamma^3 \gamma^0 \Xi - \lambda \Xi = 0
\]
(86)
Inserting the above form of the Dirac-matrices we obtain
\[
\left\{ \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} \partial_2 + \begin{pmatrix} -\sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix} \sin^{-1} \theta \partial_3 \right\} \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} - \lambda \begin{pmatrix} \Xi_1 \\ \Xi_2 \end{pmatrix} = 0
\]
(87)
The variable \( \varphi \) enters this system of equations in the derivative operators only. Therefore we make the following ansatz for the spinor \( \Xi \):
\[
\Xi_1 = a_1(\tau)b_1(\chi) \begin{pmatrix} \chi_1(\theta) \\ \chi_2(\theta) \end{pmatrix} W(\varphi), \quad \Xi_2 = a_3(\tau)b_3(\chi) \begin{pmatrix} \chi_3(\theta) \\ \chi_4(\theta) \end{pmatrix} W(\varphi)
\]
This is the most general ansatz for the spinor \( \Xi \), for which one can separate the variables \( \theta \) and \( \varphi \) in eq.(87). This form is also compatible with the structure of the
operator $D^2_\kappa(\tau, \chi)$ as we will see later. When we insert this into equation (87), we see that we have

$$\chi_1 = \chi_3 \quad \chi_2 = -\chi_4$$

so we obtain only two coupled linear independent partial differential equations in the variables $\theta$ and $\varphi$. Now we can separate this system in the following way:

$$\sin \theta \chi_2^{-1}(i\lambda \chi_1 - \partial_\theta \chi_2) = n = -iW^{-1}\partial_\varphi W$$  \hspace{1cm} (88)

$$\sin \theta \chi_1^{-1}(i\lambda \chi_2 + \partial_\theta \chi_1) = n = -iW^{-1}\partial_\varphi W$$  \hspace{1cm} (89)

The solution for $W$ reads:

$$W_n(\varphi) = \exp(in\varphi)$$  \hspace{1cm} (90)

We demand the spinor $\Psi$ to change the sign when rotated about an angle $2\pi$ in the direction $\varphi$, so $n$ varies in the range $n = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$.

Next we have to solve the remaining system of equations in the variable $\theta$:

$$\left(\partial_\theta + \frac{n}{\sin \theta}\right)\chi_2 - \tilde{\lambda} \chi_1 = 0$$  \hspace{1cm} (91)

$$\left(\partial_\theta - \frac{n}{\sin \theta}\right)\chi_1 + \tilde{\lambda} \chi_2 = 0$$  \hspace{1cm} (92)

with $\tilde{\lambda} := i\lambda$. We make for $\chi_1$ and $\chi_2$ the ansatz

$$\chi_{1n}(\theta) = \sin^n \theta \cos(\theta/2)g(\theta)$$  \hspace{1cm} (93)

$$\chi_{2n}(\theta) = \sin^n \theta \sin(\theta/2)f(\theta)$$  \hspace{1cm} (94)

where $f$ and $g$ are unknown scalar functions, still to be determined. Inserting this into equation (92) we obtain after a little manipulation

$$\cos(\theta/2)\left[ng(\theta)\left(\cos \theta - 1\right) - \sin^2 \theta \dot{g}(\theta)\right] +$$

$$\sin(\theta/2)\left[\sin \theta \left(\tilde{\lambda} f(\theta) - \frac{1}{2} g(\theta)\right)\right] = 0$$  \hspace{1cm} (95)

where

$$\dot{g}(\theta) = \frac{dg}{d\cos \theta} \frac{d\cos \theta}{d\theta} \equiv -\dot{g}(\theta) \sin \theta$$  \hspace{1cm} (96)

With the identity

$$\cot(\theta/2) = \frac{1 + \cos \theta}{\sin \theta}$$  \hspace{1cm} (97)

we finally obtain

$$-ng(\theta) - (1 + \cos \theta)\dot{g}(\theta) + \tilde{\lambda} f(\theta) - \frac{1}{2} g(\theta) = 0$$  \hspace{1cm} (98)

With equation (91) we proceed in the same way and find

$$nf(\theta) - (1 - \cos \theta)\dot{f}(\theta) + \frac{1}{2} f(\theta) - \tilde{\lambda} g(\theta) = 0$$  \hspace{1cm} (99)

With $x := \cos \theta$ these two equations become

$$\left(1 + x\right)\frac{dG}{dx} + \left(\frac{1}{2} + n\right) G = \tilde{\lambda} F$$  \hspace{1cm} (100)

$$\left(x - 1\right)\frac{dF}{dx} + \left(\frac{1}{2} + n\right) F = \tilde{\lambda} G$$  \hspace{1cm} (101)

The functions $F$ and $G$ are defined by

$$F(x) := f(\theta)$$  \hspace{1cm} (102)

$$G(x) := g(\theta)$$  \hspace{1cm} (103)
When we now insert \text{eq.}(100)\ into \text{eq.}(101)\ we\ obtain\ a\ differential\ equation\ which
is\ solved\ by\ the\ Jacobi-polynomials\ \cite{1}:

\begin{equation}
(1 - x^2)\dddot{G} + (\beta_2 - \beta_1 - (\beta_1 + \beta_2 + 2)x)\dot{G} + l(l + \beta_1 + \beta_2 + 1)G = 0 \tag{104}
\end{equation}

Here we have made the definitions:

\begin{align*}
\beta_1 & := n - \frac{1}{2} \\
\beta_2 & := n + \frac{1}{2} \\
l & := |\tilde{\lambda}| - n - \frac{1}{2}
\end{align*}

The Jacobi-polynomials of order \(l\) are defined by

\begin{equation}
P_l^{(\beta_1,\beta_2)}(x) = 2^{-l} \sum_{m=0}^{l} c_m(\beta_1,\beta_2)g_m(x), \quad l \in \mathbb{N} \quad \text{where}
\end{equation}

\begin{align*}
c_m(\beta_1,\beta_2) & = \binom{l + \beta_1}{m} \binom{l + \beta_2}{l - m} \\
g_m(x) & = (x - 1)^{l-m}(x+1)^m
\end{align*}

The parameters \(\beta_1,\beta_2\) are restricted to the values

\begin{align*}
\beta_1 > -1, \quad \beta_2 > -1
\end{align*}

This restrict the values of the parameter \(n\) further to \(n = \frac{1}{2}(2m + 1), \quad m \in \mathbb{N}\)

The solution for \(G\) now reads

\begin{equation}
G_{nl}(x) = \frac{1}{\sqrt{N_{nl}}} P_l^{(\beta_1,\beta_2)}(x) \tag{105}
\end{equation}

where \(N_{nl}\) is a normalization constant. The eigenvalues \(|\tilde{\lambda}|\), which are related to
the order of the Jacobi-polynomials \(l\) and to \(n\) via \(l = |\tilde{\lambda}| - n - \frac{1}{2}\) must be integer,
since \(l\) is an integer. With \(G\) we easily obtain the solution for \(F\):

\begin{equation}
F_{nl}(x) = \frac{1}{\sqrt{N_{nl}}} (-1)^l P_l^{(\beta_2,\beta_1)}(x) \tag{106}
\end{equation}

If we choose \(N_{nl}\) to be

\begin{equation}
N_{nl} = \frac{2^{2n+3}\pi \Gamma(l + n + 1/2)\Gamma(l + n + 3/2)}{(2l + 2n + 1)!} \tag{107}
\end{equation}

then the eigenfunctions \(\tilde{\Phi}(\theta,\varphi)\) of \(D^1\) are orthonormal, i.e. we have the relation:

\begin{equation}
\int_0^\pi d\theta \int_0^{2\pi} d\varphi \tilde{\Phi}^+_n(\theta,\varphi)_A \tilde{\Phi}_n^{(\nu)}(\theta,\varphi)^A = \delta_{nn'}\delta_{\nu\nu'} \tag{108}
\end{equation}

We now consider the operator \(D^2_n(\tau,\chi)\). Equation (75) reads

\begin{align*}
D^2_n(\tau,\chi) \tilde{\Phi} + \lambda \tilde{\Phi} = \\
\left[\gamma^0 \partial_0 + \gamma^1 \partial_1 + im e^{\alpha/2}\right] \gamma^1 \gamma^0 \tilde{\Phi} + \lambda \xi^{-1} \tilde{\Phi} = 0
\end{align*}

After the application of \(U\), this equation becomes

\begin{equation}
\left[-\gamma^3 \partial_0 - \gamma^0 \partial_1 + \gamma^3 \gamma^0 im e^{\alpha/2} + \lambda \xi^{-1}\right] \Xi = 0 \tag{109}
\end{equation}

Inserting the Dirac-matrices we obtain the following system of equations:

\begin{align*}
-\sigma^3 \partial_0 \Xi_2 - \partial_1 \Xi_1 - \sigma^3 im e^{\alpha/2} \Xi_2 + \lambda \xi^{-1} \Xi_1 = 0 \\
\sigma^3 \partial_0 \Xi_1 + \partial_1 \Xi_2 - \sigma^3 im e^{\alpha/2} \Xi_1 + \lambda \xi^{-1} \Xi_2 = 0
\end{align*}

(110)
As in the angle dependent case we are looking for a separation of the variables $\tau$ and $\varphi$. Therefore we enlarge our ansatz for the spinor $\Xi$ in the following way:

$$\Xi_1 = \left( \begin{array}{c} a_1(\tau) b_1(\chi) C_{nl}^1(\theta, \varphi) \\ a_2(\tau) b_2(\chi) C_{nl}^2(\theta, \varphi) \end{array} \right) \quad (111)$$

$$\Xi_2 = \left( \begin{array}{c} a_3(\tau) b_3(\chi) C_{nl}^1(\theta, \varphi) \\ -a_4(\tau) b_4(\chi) C_{nl}^2(\theta, \varphi) \end{array} \right) \quad (112)$$

where $a_i, b_j$, $i, j = 1, \ldots, 4$ are unknown scalar valued functions, to be determined in the following. As was mentioned earlier in the proof of proposition 1, to be compatible with both operators $D_2$ and $D_3$ we must have

$$a_1 = a_2, \quad b_1 = b_2$$

$$a_3 = a_4, \quad b_3 = b_4$$

This is indeed the case, as can be seen when eqs. (111), (112) are inserted in (110). Therefore we have to solve only the reduced system

$$a_3^{-1} \partial_\theta a_1 + b_1^{-1} \partial_\theta b_3 - ime^{\alpha/2} a_1 a_3^{-1} + \tilde{\lambda}^{-1} b_3 b_1^{-1} = 0 \quad (113)$$

$$-a_1^{-1} \partial_\theta a_3 - b_3^{-1} \partial_\theta b_1 - ime^{\alpha/2} a_3 a_1^{-1} + \tilde{\lambda}^{-1} b_1 b_3^{-1} = 0 \quad (114)$$

Introducing the two constants $z_1, z_2 \in \mathbb{C}$ we obtain for the variable $\tau$:

$$\left( \frac{d}{d\tau} - ime^{\alpha/2} \right) a_1(\tau) = z_1 a_3(\tau) \quad (115)$$

$$\left( \frac{d}{d\tau} + ime^{\alpha/2} \right) a_3(\tau) = z_2 a_1(\tau) \quad (116)$$

And for the variable $\chi$:

$$\left( \frac{d}{d\chi} + \tilde{\lambda}^{-1}(\chi) \right) b_3(\chi) = -z_1 b_1(\chi) \quad (117)$$

$$\left( \frac{d}{d\chi} - \tilde{\lambda}^{-1}(\chi) \right) b_1(\chi) = -z_2 b_3(\chi) \quad (118)$$

The equations in the radial coordinate $\chi$ can be solved by hypergeometric functions.

**Proposition 2.** For $\kappa = 0$ the solutions to the eqn. (117) and (118) can be expressed through the functions

$$b_1(\chi) = B_{1w}(y) = B_{k}(d_1 y j_k(y) + d_2 y y_k(y)) \quad (119)$$

$$b_3(\chi) = B_{3w}(y) = B_{k}(d_1 y j_{k-1}(y) + d_2 y y_{k-1}(y)) \quad (120)$$

where $d_1, d_2 \in \mathbb{R}$, $k \in \mathbb{Z}$, $y = \pm \sqrt{w} \chi$, with $w \in \mathbb{R}_+$

**Proof.** First of all we take $\kappa = 0$ in (117) and (118) and combine them to the single equation of second order

$$b_1'' - b_1 \left( \tilde{\lambda}^2 \chi^{-2} - \tilde{\lambda} \chi^{-2} + z \right) = 0 \quad (121)$$

where $z = z_1 z_2$ and a prime denotes a derivative with respect to $\chi$. For $z \in \mathbb{R}_+: = \{ z \in \mathbb{R} | z < 0 \}$ we set $-z := w > 0$ and define a new variable $y$ by $y := \pm \sqrt{w} \chi$. The derivative w.r.t $y$ is denoted by a dot. Then equation (121) becomes

$$y^2 \ddot{B}_1(y) + B_1(y) \left( \tilde{\lambda}(1 - \tilde{\lambda}) + y^2 \right) = 0 \quad (122)$$

\footnote{On physical grounds there are only negative values for $z$ of interest, since it follows from the treatment of the equations (115) and (116) that $-z$ is the absolute value squared of the wavevector $k$.}
where the function $B_1$ is defined by $B_1(y) := b_1(\chi)$. If we finally set $\lambda = -k = -k(l,n)$ we obtain the so called Riccati-Bessel differential equation [6]:

$$y^2 \ddot{B}_1(y) + B_1(y)[y^2 - k(k + 1)] = 0$$  \hspace{1cm} (123)

The solutions to this equation are the spherical Bessel-functions of the first and second kind $j_k(y)$ and $y_k(y)$ respectively. The general solution is a linear combination of these two:

$$B_{1w}(y) = B_k(d_1 j_k(y) + d_2 y_k(y)), \quad k \in \mathbb{Z}, \ d_1, d_2 \in \mathbb{R}$$  \hspace{1cm} (124)

where $B_k$ is a normalization constant. The solution for $b_{3w}$ is, expressed through the function $B_{3w}(y) := b_{3w}(\chi)$,

$$B_{3w}(y) = B_k \sqrt{\frac{\pi}{2}} y^{-1/2} (d_1 j_{k-1/2}(y) + d_2 y_{k-1/2}(y))$$  \hspace{1cm} (125)

where the $j_{k+1/2}(y)$ and $y_{k+1/2}(y)$ are the usual Bessel-functions of rational order of first resp. second kind. They are related to the spherical ones by

$$j_k(y) = \sqrt{\frac{\pi}{2}} y^{-1/2} J_{k+1/2}(y), \quad y_k(y) = \sqrt{\frac{\pi}{2}} y^{-1/2} Y_{k+1/2}(y)$$  \hspace{1cm} (126)

We summarize the above calculation in the following two statements.

**Proposition 3.** The solution to the Dirac-equation eq.(67) has the following spinor structure:

$$\Psi_{wnl}(x) = -\bar{w}^{3/4} \chi \sin^{1/2} \theta M \Xi_{wnl}(x)$$  \hspace{1cm} (127)

where $M$ is the matrix $M = \gamma^0 \gamma^1 U^{-1}$ and $\Xi_{wnl}(x)$ is given by the equations (111) and (112).

**Proposition 4.** The spatial part of the solution of the Dirac-equation (67) in the coordinates $\chi, \theta, \varphi$ on a flat Robertson-Walker spacetime has the form:

$$\Psi(\vec{x}) = \int dw \sum_{l,n} \left( a_{nl}(w) \Psi_{wnl}(\vec{x}) + a^*_{nl}(w) \overline{\Psi_{wnl}(\vec{x})} \right)$$  \hspace{1cm} (128)

with basis $\Psi_{wnl}$ given by

$$\Psi_{wnl}(\vec{x}) = -\chi \sin^{1/2} \theta M \Xi_{wnl}(\vec{x})$$

where

$$\Xi_{wnl}(\vec{x}) = \begin{pmatrix} \Xi_{wnl}(\vec{x}) \\ \Xi_{wnl}(\vec{x}) \end{pmatrix}$$

with

$$\Xi_{wnl}(\vec{x}) = b_{1w}(\chi) \begin{pmatrix} C_{nl}^1(\theta, \varphi) \\ C_{nl}^2(\theta, \varphi) \end{pmatrix}$$

$$\Xi_{2w}(\vec{x}) = b_{3w}(\chi) \begin{pmatrix} C_{nl}^1(\theta, \varphi) \\ -C_{nl}^2(\theta, \varphi) \end{pmatrix}$$

The coefficient function $a_{nl}(w)$ in eq.(128) must be chosen with compact support to make the integral convergent. The solution to the equations (117) and (118) in the cases $\kappa = \pm 1$ can be given equally in terms of more advanced hypergeometric functions but there is only little to learn of these formulas, so we skip them here.

The treatment of the equations (115) and (116) requires the specification of the function $\alpha$, which corresponds to choosing a certain universe model.
6. Adiabatic Vacuum States

We come now to the discussion of adiabatic vacuum states on $\mathcal{F}[\mathcal{H}]$. Associated with each homogeneous space $\Sigma^\kappa$ there is a group of isometries $G^\kappa$. This group acts through $g(t, \bar{x}) = (t, g\bar{x})$, $g \in G^\kappa$ on $M$. In the case of constant positive curvature ($\kappa = +1$) this is the rotation group $SO(4)$. If we take flat Cauchy-surfaces we have $G^\kappa = E(3)$. In the case of constant negative curvature ($\kappa = -1$) the symmetries are the proper orthochronous Lorentz-transformations $L^\kappa$. See ref. [13] for more details.

We now want to characterize certain states on the CAR-Algebra $\mathcal{F}[\mathcal{H}]$ which are invariant under the action of the transformation group $G^\kappa$. The unitary representation $U$ of $G^\kappa$ on $\mathcal{C}_0^\infty(\mathcal{D}M)$, defined by

$$(U(g)f)(x) := f(g^{-1}x), \quad g \in G^\kappa$$

commutes with the Dirac-propagator for spinors $S : \mathcal{C}_0^\infty(\mathcal{D}M) \rightarrow \mathcal{C}_0^\infty(\mathcal{D}M)$. In addition to that it holds that the operators $U(g)$ commute with the antiunitary involution $\Gamma$ on $\mathcal{H}$. In this way they form a group of Bogoliubov-transformations on $\mathcal{H}$. Induced by these transformations there are automorphisms $\alpha_g$ of $\mathcal{F}[\mathcal{H}]$, i.e.

$$\alpha_g(B(f)) = B(U(g)f), \quad g \in G^\kappa$$

These so called Bogoliubov-automorphisms are now used to single out homogeneous and isotropic states of the Dirac-field. A state $\omega$ on $\mathcal{F}[\mathcal{H}]$ is said to be homogeneous and isotropic, if and only if it is invariant under all Bogoliubov-automorphisms $\alpha_g$, i.e.

$$\omega \circ \alpha_g = \omega, \quad \forall g \in G^\kappa$$

The restriction to quasifree states makes it possible to formulate an equivalent condition on the two-point function

$$\Lambda_2^\omega(f_1, f_2) := \omega(B(f_1)^* B(f_2))$$

**Definition 4.** A quasifree state $\omega$ on $\mathcal{F}[\mathcal{H}]$ is called homogeneous and isotropic, if and only if it’s two-point function satisfies

$$\Lambda_2^\omega(U(g)f_1, U(g)f_2) = \Lambda_2^\omega(f_1, f_2), \quad f_1, f_2 \in \mathcal{H}, \quad g \in G^\kappa$$

We now introduce adiabatic vacuum states for the Dirac-field on a Robertson-Walker spacetime. For the Klein-Gordon-field these states were invented by Lüders and Roberts [13], motivated by Parkers requirement that the particle production by the expanding universe should be minimal [4]. It has been shown that adiabatic vacuum states for the Klein-Gordon-field are the physically correct states, in the sense that they have the singularity structure of Hadamard-form [4]. This fact motivates the same procedure in the case of the Dirac-field. First of all we write the field operator according to proposition 4 as:

$$\hat{\Psi}(x) = \int d\nu \sum_{l,n} \left[ \hat{a}_{nl}(w) \Psi_{wnl}(x) + \hat{a}^+_{nl}(w) \Psi_{wnl}(x) \right]$$

where $\hat{a}_{nl}(w), \hat{a}^+_{nl}(w)$ are annihilation and creation operators on the antisymmetric Fock-space over $\mathcal{H}$. They have anticommutation relations

$$\{ \hat{a}_{nl}(w), \hat{a}_{n'l'}(w')^* \} = \delta(w - w') \delta_{nn'} \delta_{ll'}$$

We now consider the equations (115) and (116) for the time dependent part. Combining them we get

$$\left( \frac{d^2}{d\tau^2} - \frac{i}{2} \text{me}^{\alpha/2} + m^2 \text{e}^{\alpha} \right)a_{1w}(\tau) = -wa_{1w}(\tau)$$

(136)
In analogy to the Klein-Gordon case we make a WKB-type ansatz for a solution to equation (136)

$$a_{1w}(\tau) = e^{-3\alpha/4} (2\Omega_{w}(\tau))^{-1/2} \exp \left[ i \int_{0}^{\tau} d\tau' \Omega_{w}(\tau') \right] \quad (137)$$

where $\Omega_{w}$ is a complex valued smooth function with positive imaginary part. If we insert this in (136) we get an equation for $\Omega_{w}$:

$$\Omega_{w}(\tau)^{2} = \frac{9}{16} \alpha^{2} + \frac{3}{4} \frac{\dot{\Omega}_{w}(\tau)}{\Omega_{w}(\tau)} - \frac{3i}{2} \alpha \Omega_{w}(\tau) + \frac{1}{4} \left( \frac{\ddot{\Omega}_{w}(\tau)}{\Omega_{w}(\tau)} \right)^{2} - \frac{3}{4} \ddot{\alpha}$$

$$- \frac{1}{2} \frac{\dot{\Omega}_{w}(\tau)}{\Omega_{w}(\tau)} + \frac{1}{2} \frac{\dot{\Omega}_{w}(\tau)}{\Omega_{w}(\tau)^{2}} - \frac{i}{2} \dot{\alpha} \alpha e^{\alpha/2} + m^{2} e^{\alpha} + w \quad (138)$$

In a slowly varying universe we expect the derivative terms to be small, so we try an iterative solution to this equation:

$$(\Omega_{w}^{(0)}(\tau))^{2} = w + m^{2} e^{\alpha}$$

$$(\Omega_{w}^{(r+1)}(\tau))^{2} = w + m^{2} e^{\alpha} - \frac{i}{2} \dot{\alpha} \alpha e^{\alpha/2} + \frac{9}{16} \alpha^{2} - \frac{3}{4} \ddot{\alpha} + \frac{3}{4} \frac{\dot{\Omega}_{w}^{(r)}(\tau)}{\Omega_{w}^{(r)}(\tau)}$$

$$- \frac{3i}{2} \dot{\alpha} \dot{\Omega}_{w}^{(r)}(\tau) + \frac{1}{4} \left( \frac{\dot{\Omega}_{w}^{(r)}(\tau)}{\Omega_{w}^{(r)}(\tau)} \right)^{2} - \frac{1}{2} \frac{\ddot{\Omega}_{w}^{(r)}(\tau)}{\Omega_{w}^{(r)}(\tau)} + \frac{1}{2} \frac{\dddot{\Omega}_{w}^{(r)}(\tau)}{\Omega_{w}^{(r)}(\tau)}$$

The only freedom we have in the choice of a homogeneous and isotropic quasifree state is to specify initial data for the functions $a_{1w}$ and $a_{3w}$. Therefore we make the following definition:

**Definition 5.** An adiabatic vacuum state of the Dirac-field of order $r$ on a flat Robertson-Walker spacetime is specified by the following initial data for the function $a_{1w}$:

$$a_{1w}(\tau) = W_{w}^{(r)}(\tau) \quad (139)$$

$$\dot{a}_{1w}(\tau) = \dot{W}_{w}^{(r)}(\tau), \text{ where} \quad (140)$$

$$W_{w}^{(r)}(\tau) := e^{-3\alpha/4} (2\Omega_{w}^{(r)}(\tau))^{-1/2} \exp \left[ i \int_{0}^{\tau} d\tau' \Omega_{w}^{(r)}(\tau') \right] \quad (141)$$

The function $a_{3w}$ is then fixed by eq.(115). Now one calculates the two-point distribution to be

$$\Lambda_{2}(x_{1}, x_{2}) = \langle \Omega, \hat{\Psi}(x_{1}) \hat{\Psi}(x_{2}) \Omega \rangle$$

$$\int d\omega \sum_{nl} \Psi_{wn}(x_{1}) \overline{\Psi_{wn}(x_{2})} \quad (142)$$

where the anticommutation relations eq.(135) were used. To decide whether the quasifree state associated with this two-point distribution is an Hadamard-state we must look at the short distance behaviour of the expression (142). If we use the Taylor expansions of the Bessel-functions we arrive at a form which agrees with the one in definition 1. So we conjecture that we obtain in this way a Hadamard-state.

### 7. Appendix

In this appendix we define bi-spinors in curved spacetime and introduce the relevant notions of the causal structure of this spacetime for the definition of Hadamard states for the Dirac-field. In the following our spacetime is a smooth $C^{\infty}$-manifold on which there is defined a Lorentz metric $g$. In addition to that we assume that $(M, g)$ is globally hyperbolic.
Let $N$ be a open subset of $M$. Further let $DN \boxtimes D^*N$ be the outer tensor product of the bundles $DN$ and $D^*N$. This is the bundle over the product manifold $N \times N$. the fibres at $(p, p') \in N \times N$ are $(DN)_p \otimes (D^*N)_{p'}$. $\pi$ is the product-projection, i.e. $\pi((p, f), (p', f')) = (p, p')$. Smooth sections in $DN \boxtimes D^*N$ are called bi-spinors. With the choice of vierbeins $(E_A)_p$ at $p$ and $(E'^{B'})_p$ at $p'$, we construct a new vierbein 
\begin{equation}
(q, q') \mapsto (E \boxtimes E')_A^{B'}(q, q') := E_A(q) \otimes E'^{B'}(q'), A, B' = 0, \ldots, 3 \text{ in } DN \boxtimes D^*N \text{ at } (p, p').
\end{equation}
Smooth sections $v \in C^\infty(DN \boxtimes D^*N)$ are given by
\begin{equation}
v(q, q') = v^A_{\ B'}(q, q')(E \boxtimes E')_A^{B'}(q, q')
\end{equation}
with $C^\infty$-functions $v^A_{\ B'}$.

For the singularity structure of the two-point function to be well defined we need the notion of a causal normal neighborhood of a Cauchy-surface $\Sigma$, which we explain now. An open neighborhood $N$ of a Cauchy-surface $\Sigma$ in $(M, g)$ is called a causal normal neighborhood for $\Sigma$, if for every choice of $p, q \in N$ with $p \in J^+(q)$ there is a convex normal neighborhood \[ R \subset M, \text{ so that } J^-(p) \cap J^+(q) \subset R. \] It was shown by Kay and Wald [10] that to every Cauchy-surface there exists a causal normal neighborhood. Now let $X$ be the set of points $(p, q) \in M \times M$, such that $p$ and $q$ can be joined by a causal curve with $J^+(p) \cap J^-(q)$, if $q \in J^+(p)$ or $J^-(p) \cap J^+(q)$, if $p \in J^+(q)$ contained in a convex normal neighborhood. The square of the geodesic distance $s(p, q)$ of $p$ and $q$ is then well defined and smooth in $X \subset M \times M$.

Let $N$ be a causal normal neighborhood of a Cauchy-surface $\Sigma$. A smooth function $\chi$ on $N \times N$ is called a $N$-regularizing function, if it has the following properties: There is an open neighborhood $Y \subset N \times N$ of the set of causally connected points in $N$, such that $\overline{Y} \subset X$ and $\chi \equiv 1$ on $Y$ and $\chi \equiv 0$ outside of $X$. Then $\chi$ is well defined, since $Y$ and the complement of $X$ in $N \times N$ are disjoined closed subsets of $N \times N$. It can be shown that such a $N$-regularizing function always exists [15].

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