On the gradient of Schwarz symmetrization of functions in Sobolev spaces

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Sunto. Sia $S$ uno spazio di Sobolev o Orlicz-Sobolev di funzioni non necessariamente nulle al bordo del dominio. Si danno condizioni sufficienti su una funzione non negativa in $S$ affinché la sua simmetrizzazione di Schwarz appartenga ancora ad $S$. Questi risultati sono ottenuti per mezzo di disuguaglianze isoperimetriche relative e generalizzano in un certo senso un noto teorema di Polya-Szego. Si dimostra anche che il riarrangiamento di una qualsiasi funzione in $S$ è localmente in $S$.

Abstract. Let $S$ be a Sobolev or Orlicz-Sobolev space of functions not necessarily vanishing at the boundary of the domain. We give sufficient conditions on a nonnegative function in $S$ in order that its spherical rearrangement ("Schwartz symmetrization") still belongs to $S$. These results are obtained via relative isoperimetric inequalities and somewhat generalize a well-known Polya-Szego's theorem. We also prove that the rearrangement of any function in $S$ is locally in $S$.

If $u$ is a nonnegative function in $H^1,2(\mathbb{R}^n)$, $u$ has compact support, and $\bar{u}$ denotes the Schwarz symmetrization of $u$, then a well known theorem by Polya-Szego states that $\bar{u}$ belongs to $H^1,2(\mathbb{R}^n)$ and:

$$
\int |D\bar{u}|^2 \, dx \leq \int |Du|^2 \, dx.
$$

(Henceforth, we will indicate with $D$ the gradient of a function of $n$ variables or the derivative of a function of one real variable).

In particular, this formula holds for $u \in H^1,2(\Omega)$, where $\Omega$ is a bounded domain of $\mathbb{R}^n$, the first integral is taken on the ball $\Omega$ having the same measure of $\Omega$ and the second is taken on $\Omega$.

If $u$ is a function in $H^1,2(\Omega)$, not necessarily vanishing at the boundary, or if $u$ belongs to $H^1,2(\Omega)$ but assumes also negative values (and so does $\bar{u}$), then inequality (*) can actually fail, and $\bar{u}$ does not necessarily belong to $H^1,2(\Omega)$ (see examples below). So, a natural question is under which additional assumptions a nonnegative function in $H^1,2(\Omega) \setminus H^1,2(\Omega)$ has Schwarz symmetrization in $H^1,2(\Omega)$. In section 1 we will prove some different sufficient conditions (in terms of the size of the set on which $u$ vanishes) in order to a Polya-Szego-type estimate holds, that is:

$$
\int_{\Omega} |D\bar{u}|^2 \, dx \leq (\text{const.}) \int_{\Omega} |Du|^2 \, dx.
$$

Moreover, we will prove that whenever $u$ is an $H^1,2(\Omega)$ function (even of changing sign), $\bar{u}$ belongs to $H^1,2(\Omega)$ and for any ball $\Omega$, concentric to $\Omega$ and with measure $|\Omega| - \epsilon$, one has:

$$
\int_{\Omega_\epsilon} |D\bar{u}|^2 \, dx \leq c(\epsilon) \int_{\Omega} |Du|^2 \, dx.
$$

where $c$ does not depend on $u$. (See section 2). All these results can naturally be generalized to Orlicz-Sobolev spaces. This will be done in section 3.

The interest in studying properties of the rearrangement of functions in $H^1,2(\Omega)$, or vanishing on part of the boundary, comes from the application of symmetrization techniques to elliptic or parabolic P.D.E. with boundary conditions of Neumann or mixed type: so thm. 2.1 and corollary 2.2 have been used in investigating parabolic Neumann problems, see [2]. We also mention [8], in which a similar result to thm. 1.3 is stated, in a different context: this result is related to the study of elliptic mixed problems, which is carried out in [13].

Some notations and examples

If $u$ is a real measurable function defined on $\Omega$, we define:

the distribution function of $u$:

$$
\mu(t) = \left| \left\{ x \in \Omega : u(x) > t \right\} \right| \quad \text{for } t \in \mathbb{R}
$$

( $\cdot$ denotes Lebesgue measure);
the decreasing rearrangement of $u$:

$$u^*(s) = \inf \left\{ \mu(t) : \mu(t) \leq s \right\} \text{ for } s \in [0, |\Omega|];$$

(0.2)

the Schwarz symmetrization of $u$:

$$\bar{u}(x) = u^* \left( c_n \left| x \right| \right) \text{ for } x \in \bar{\Omega},$$

(0.3)

where $\bar{\Omega}$ is the sphere centred at the origin with the same measure of $\Omega$; $c_n$ is the measure of the unit ball in $\mathbb{R}^n$.

For general properties of these functions, see [12]; note that, in our definition, $u^*$ and $\bar{u}$ assume also negative values, if $u$ is a function of changing sign, whereas rearrangements are sometimes defined for $|u|$.

From (0.3) it follows:

$$\left| \nabla \bar{u}(x) \right| = n c_n \left| \nabla u^* \left( c_n \left| x \right| \right) \right| \cdot \left| x \right|^{n-1}$$

$$\int_{\Omega} \left| \nabla \bar{u}(x) \right|^2 dx = (n c_n^{1/n})^2 \int_0^{\Omega} \left| \nabla u^*(s) \right|^2 s^{2-2/n} ds. \quad (0.4)$$

Hence, if $\bar{u} \in H^{1,2}(\bar{\Omega}), u^* \in H^{1,2}(\epsilon, |\Omega|)$ for any $\epsilon > 0$, so that $u^* \in AC(\epsilon, |\Omega|)$ for any $\epsilon > 0$.

For better understanding the problem of assuring integrability of $\left| \nabla \bar{u} \right|^2$, let us consider the case of a radially symmetric and increasing function $u$ defined on a ball $\Omega$, i.e.:

$$u(x) = u^* \left( \left| \Omega \right| - c_n \left| x \right| \right). \quad (0.5)$$

In this case one has:

$$\int_{\Omega} \left| \nabla u(x) \right|^2 dx = \frac{(n c_n^{1/n})^2}{4} \int_0^{\Omega} \left| \nabla u^*(s) \right|^2 \left( \left| \Omega \right| - s \right)^{2-2/n} ds. \quad (0.6)$$

Comparing (0.4) and (0.6) one sees how it may happen that $u \in H^{1,2}(\Omega)$ but $\bar{u} \notin H^{1,2}(\bar{\Omega})$. Take, for instance, $u^*(s) = \sqrt{\left| \Omega \right|} - s$ and $u$ as in (0.5). Then:

$$\int_{\Omega} \left| \nabla u(x) \right|^2 dx = \frac{(n c_n^{1/n})^2}{4} \int_0^{\Omega} s^{2-2/n} ds < \infty \text{ for every } n \geq 2,$$

while:

$$\int_{\Omega} \left| \nabla \bar{u}(x) \right|^2 dx = \frac{(n c_n^{1/n})^2}{4} \int_0^{\Omega} \frac{s^{2-2/n}}{|\Omega|^2} ds = \infty \text{ for every } n.$$

Similarly, if one defines: $u^*(s) = \sqrt{\left| \Omega \right|} - s - \sqrt{\left| \Omega \right|} \text{ and } u$ as in (0.5), one has an example of a (negative) function $u \in H^{1,2}_0(\Omega)$ such that $\bar{u} \notin H^{1,2}(\bar{\Omega})$.

**Remark 0.1.** The above example works for $n \geq 2$. If $n = 1$ inequality (*) can actually be proved for any nonnegative function in $H^{1,2}(\Omega)$. (See [6], p.35). So in this paper we will always consider $n \geq 2$.

**1. Isoperimetric inequalities and $L^2$ norm of the gradient of $\bar{u}$**

Here we want to obtain a proof of integrability of $\left| \nabla \bar{u} \right|^2$ without assuming that $u$ vanishes at the boundary of $\Omega$. In what follows $u$ will be a nonnegative function defined on $\Omega$. A first basic tool we need is Federer's "coarea formula", as appears in [11]:

if $f \in L^1(\mathbb{R}^n)$ and $v$ is a nonnegative Lipschitz function with compact support, then:

$$\int_{\mathbb{R}^n} f(x) \left| \nabla v(x) \right| dx = \int_0^{\infty} dt \int_{\left\{ x : v(x) = t \right\}} f(x) dH_{n-1}(x). \quad (1.1)$$

(Here and below, $H_{n-1}$ stands for (n − 1)-dimensional Hausdorff measure).

Let us consider a nonnegative Lipschitz function $u$ defined on $\Omega$. If $\Omega$ is Lipschitz, we can extend $u$ to a compact supported Lipschitz function on $\mathbb{R}^n$. Then, if $f \in L^1(\Omega)$ and we put $f \equiv 0$ outside $\Omega$, (1.1) becomes:

$$\int_{\Omega} f(x) \left| \nabla u(x) \right| dx = \int_0^{\infty} dt \int_{\left\{ x : u(x) = t \right\}} f(x) dH_{n-1}(x). \quad (1.2)$$

From (1.2) it follows in particular:

$$\int_{\left\{ x \in \Omega : u(x) > t \right\}} \left| \nabla u(x) \right| dx = \int_t^{\infty} \left| H_{n-1} \left\{ x \in \Omega : u(x) = \xi \right\} \right| d\xi. \quad (1.3)$$
Note that:
\[ \{ x \in \Omega: \{ x \in \Omega: \{ u(x) = \xi \} \} \} \ni \Omega, \text{ and:} \]
\[ H_{n-1}\{ x \in \Omega: u(x) = \xi \} \geq P_{\Omega}\{ x \in \Omega: u(x) > \xi \}. \] (1.4)

Here \( P_{\Omega} \) stands for the perimeter, in the sense of De Giorgi, relative to \( \Omega \). For a definition of this concept in the general case, see [9]. However, we will only use the fact that \( P_{\Omega}(E) \leq H_{n-1}(\partial E) \cap \Omega \) for every measurable subset \( E \) of \( \Omega \), and, if \( \partial E \) is sufficiently smooth, this is an equality. (See [4]). The perimeter of \( E \), \( P(E) \), is equal to \( P_{\Omega}(E) \) when \( \Omega = \mathbb{R}^n \). We recall De Giorgi's isoperimetric inequality in \( \mathbb{R}^n \):
\[ P(E) \geq n c_{n}^{1/n} | E |^{1-1/n}. \]

The next theorem points out the role of isoperimetric inequalities in Polya-Szego-type estimates.

**Theorem 1.1.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), \( n \geq 2 \), \( u \in \text{Lip}(\Omega) \), \( u \geq 0 \) in \( \Omega \), and assume that \( u \) satisfies:
\[ P_{\Omega}\{ x \in \Omega: u(x) > \xi \} \geq \gamma(1/t)^{1-1/n} \] (1.5)
for some positive constant \( \gamma \), any \( t \geq 0 \). (Here and below, \( \mu \) is the distribution function of \( u \), defined in (0.1)). Then \( \bar{u} \in \text{Lip}(\overline{\Omega}) \), and:
\[ \int_{\Omega} | D\bar{u} |^2 \, dx \leq \left( \frac{u_{\infty}}{\gamma} \right)^2 \int_{\Omega} | Du |^2 \, dx. \] (1.6)

**Proof.** (Here we revise an argument of [11]). Let us prove that \( \bar{u} \) is Lipschitz. If \( L \) is a constant such that \( | Du(x) | \leq L \) in \( \Omega \), and \( t, h \) such that \( 0 < h < t \), then:
\[ L(\mu(t-h) - \mu(t)) \geq \int_{\{ x \in \Omega: \bar{u}(x) < t \}} | Du(x) | \, dx = (\text{by (1.3), (1.4)}) \]
\[ = \int_{t-h}^{t} P_{\Omega}\{ x \in \Omega: u(x) > \xi \} \, d\xi \geq (\text{by (1.5)}) \gamma \int_{t-h}^{t} \mu(\xi)(1-1/n) \, d\xi \geq \gamma h \mu(t)^{1-1/n}. \]

Hence \( \mu \) is strictly decreasing in \( (0, \| u \|_{\infty}) \), so that \( u^* \) is continuous and satisfies:
\[ u^*(s) - u^*(s+k) \leq \frac{L}{k} s^{-1+1/n} \cdot k \]
for any \( k > 0 \), \( s + k \in (0, \| \Omega \|) \). Therefore \( u^* \in AC(\epsilon, \| \Omega \|) \) for any \( \epsilon > 0 \) and:
\[ 0 \leq - \frac{du}{ds}(s) \leq \frac{L}{k} s^{-1+1/n}. \] (1.7)

By the definition of \( \bar{u} \) and (1.7) one can compute:
\[ | \bar{u}(x) - \bar{u}(y) | = | \int_{|x-y|}^{\epsilon |x-y|} \frac{du}{ds}(s) \, ds | \leq L \frac{\epsilon^{1/n}}{\gamma} \cdot | y - x | \]
that is \( \bar{u} \) is Lipschitz in \( \Omega \).

Let us prove now that (1.6) holds. From (1.3)-(1.4) it follows:
\[ - \frac{d}{dt} \int_{\{ x \in \Omega: u(x) > t \}} | Du(x) | \, dx = P_{\Omega}\{ x \in \Omega: u(x) > t \} \geq (\text{by (1.5)}) \gamma \mu(t)^{1-1/n}. \] (1.8)

From (1.2) it follows that:
\[ \varphi(t) \equiv \int_{\{ x \in \Omega: u(x) > t \}} | Du(x) |^2 \, dx = \int_{t}^{+\infty} d\xi \int_{\{ x \in \Omega: u(x) = \xi \}} | Du | \, dH_{n-1}(x) \]
from which one reads that \( \varphi \) is absolutely continuous, so that:
\[ \int_{\Omega} | Du |^2 \, dx = \varphi(0) = \int_{0}^{+\infty} \varphi(t) \, dt. \] (1.9)

Writing differential quotients and applying Holder's inequality one has:
\[ -\varphi'(t) \geq \frac{1}{\mu(t)} \left[ -\frac{d}{dt} \int_{\{x \in \Omega: u(x) > t\}} |Du(x)| \, dx \right]^2. \] \hspace{1cm} (1.10)

From (1.8), (1.9), (1.10) it follows:

\[ \int_{\Omega} |Du|^2 \, dx \geq \gamma^2 \int_0^{\infty} \frac{\mu(t)^{2-2n}}{\mu(t)} \, dt. \] \hspace{1cm} (1.11)

Now consider \( \bar{u} \). Since its level sets are balls, in (1.10) the equal sign holds, and (1.8) becomes:

\[ -\frac{d}{dt} \int_{\{x \in \Omega: \bar{u}(x) > t\}} |D\bar{u}| \, dx = n \, c_n^{1/n} \mu(t)^{1-1/n} \]

Hence:

\[ \int_{\Omega} |D\bar{u}|^2 \, dx = \left( n \, c_n^{1/n} \right)^2 \int_0^{\infty} \frac{\mu(t)^{2-2n}}{\mu(t)} \, dt. \] \hspace{1cm} (1.12)

From (1.11)-(1.12) it follows estimate (1.6).

Now we are interested in discussing sufficient conditions in order that (1.5) holds. In the following the function \( u \) is still supposed nonnegative and Lipschitz in \( \overline{\Omega} \).

(i) If \( u = 0 \) on \( \partial\Omega \), we obtain Polya-Szego's theorem, since:

\[ P_\Omega \left\{ x \in \Omega: u(x) > t \right\} = P \left\{ x: u(x) > t \right\} \geq n \, c_n^{1/n} \mu(t)^{1-1/n}, \]

by the isoperimetric inequality in \( \mathbb{R}^n \). So \( \gamma = n \, c_n^{1/n} \), and (1.6) holds with constant equal to 1.

(ii) Suppose that: \( \text{support of } u \leq \frac{\|\Omega\|}{2} \). The relative isoperimetric inequality of \( \Omega \) says that:

\[ Q \cdot P_\Omega(E) \geq \min \left( |E|, \frac{\|\Omega\|}{2} - |E| \right)^{1/n} \] \hspace{1cm} (1.13)

for some constant \( Q > 0 \), any measurable set \( E \subseteq \Omega \). (Such an inequality certainly holds if \( \Omega \) is Lipschitz). Then:

\[ P_\Omega \left\{ x \in \Omega: u(x) > t \right\} \geq Q^{-1} \mu(t)^{1-1/n}, \] \hspace{1cm} (1.14)

and (1.5) holds with \( \gamma = Q^{-1} \).

(iii) More generally, suppose that:

\[ \left| \left\{ x \in \Omega: u(x) = 0 \right\} \right| = \epsilon \]

with \( 0 < \epsilon < \frac{\|\Omega\|}{2} \). Fix \( t > 0 \). If \( \mu(t) \leq \frac{\|\Omega\|}{2} \), (1.14) still holds. Otherwise, from (1.13) we get:

\[ Q \cdot P_\Omega \left\{ x \in \Omega: u(x) > t \right\} \geq \left( \frac{\|\Omega\|}{2} - \mu(t) \right)^{1-1/n} \geq \left[ \alpha \mu(t) \right]^{1-1/n} \]

with \( \alpha = \frac{\epsilon}{\|\Omega\| - \epsilon} \). Hence (1.5) holds with:

\[ \gamma = Q^{-1} \alpha^{1-1/n} = Q^{-1} \left( \frac{\epsilon}{\|\Omega\| - \epsilon} \right)^{1-1/n} \]

and (1.6) holds with constant:

\[ \left( \frac{Q \, n \, c_n^{1/n}}{\alpha^{1-1/n}} \right)^2. \]

(iv) Now, suppose that:

\[ H_{n-1} \left\{ x \in \partial\Omega: u(x) = 0 \right\} = \epsilon > 0. \]

We also suppose that \( \Omega \) satisfies the following geometric property (this already appears in [10]):

\[ H_{n-1} \left( \partial E \cap \partial \Omega \right) \leq C \cdot P_\Omega(E) \] \hspace{1cm} (1.15)
for some positive constant \( c \), for any measurable \( E \subseteq \Omega \) such that \( |E| \leq \frac{|\Omega|}{2} \). (If \( \Omega \) is Lipschitz, (1.15) actually holds).

Fix \( t > 0 \). Again, we consider the case \( \mu(t) > \frac{|\Omega|}{2} \); then, by (1.15):

\[
H_{n-1}\left( \partial \left\{ x \in \Omega : u(x) \leq t \right\} \right) \cap \partial \Omega \leq C \cdot \mathcal{P}_\Omega \left\{ x \in \Omega : u(x) \leq t \right\}.
\]

Hence:

\[
\mathcal{P}_\Omega \left\{ x \in \Omega : u(x) > t \right\} = \mathcal{P}_\Omega \left\{ x \in \Omega : u(x) \leq t \right\}
\geq \frac{1}{\varepsilon} H_{n-1}\left( \partial \left\{ x \in \Omega : u(x) \leq t \right\} \right) \cap \partial \Omega \geq \frac{1}{\varepsilon} H_{n-1}\left\{ x \in \partial \Omega : u(x) = 0 \right\} = \varepsilon \geq \varepsilon \left( \frac{\mu(t)}{|\Omega|} \right)^{\frac{1}{1-1/n}}.
\]

So (1.5) holds with:

\[
\gamma = \min \left( Q^{-1}, \frac{\varepsilon}{c^{1/n}} \right)
\]

and (1.6) holds with constant:

\[
\max \left\{ \left( Q \cdot c^{1/n} \right), \left( \frac{Q \cdot c^{1/n} |\Omega|^{1-1/n}}{\varepsilon} \right) \right\}^2.
\]

Note that:

\[
\frac{nc^{1/n} |\Omega|^{1-1/n}}{\varepsilon} \leq \frac{H_{n-1}(\partial \Omega)}{\varepsilon},
\]

which is a more expressive ratio.

(v) Suppose that \( E = \left\{ x \in \Omega : u(x) = 0 \right\} \) is such that its projection on at least one hyperplane has positive \((n - 1)\)-dimensional Hausdorff measure, in symbols:

\[
H_{n-1}\left( \Pi(E) \right) = \epsilon > 0 \text{ for some projection } \Pi.
\]

For any \( t > 0 \), the set \( A = \{ u \leq t \} \) contains \( E \), so:

\[
\mathcal{P}(A) = H_{n-1}(\partial A) \geq H_{n-1}\left( \Pi(A) \right) \geq H_{n-1}\left( \Pi(E) \right) = \epsilon.
\]

Now, if \( \mu(t) > \frac{|\Omega|}{2} \), one has:

\[
\mathcal{C} \cdot \mathcal{P}_\Omega \{ u > t \} \geq H_{n-1}\left( \partial \left\{ u \leq t \right\} \right) \cap \partial \Omega).
\]

Hence:

\[
\mathcal{P}_\Omega \{ u > t \} \geq \frac{1}{\epsilon^{1/n}} \mathcal{P}\{ u \leq t \} \geq \frac{\epsilon^{1/n}}{\epsilon^{1/n}} \geq \frac{\epsilon^{1/n}}{\epsilon^{1/n} \mu(t)^{1/n}} \quad \frac{\mu(t)^{1/n}}{\mu(t)^{1/n}}
\]

So (1.5) holds with:

\[
\gamma = \min \left( Q^{-1}, \frac{\epsilon}{(c^{1/n})^{1-1/n}} \right).
\]

Now we state separately the results obtained from (iii)-(iv)-(v).

**Theorem 1.2.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), \( n \geq 2 \); let \( u \in H^{1,2}(\Omega) \), \( u \geq 0 \) in \( \Omega \), and suppose that \( \text{support of } u = |\Omega| - \epsilon \) for some \( \epsilon > 0 \). Then \( \tilde{u} \in H^{1,2}_0(\Omega) \) and:

\[
\int_\Omega |D \tilde{u}|^2 \, dx \leq L^2 \int_\Omega |Du|^2 \, dx \quad (1.16)
\]

with \( L = \left( \frac{Q \cdot c^{1/n}}{\varepsilon} \right) \), where \( Q \) is as in (1.13) and: \( \alpha = \frac{\epsilon}{|\Omega|^{1/n}} \) if \( \epsilon \leq \frac{|\Omega|}{2} \); \( \alpha = 1 \) otherwise.
Theorem 1.3. Let \( \Omega \) be as above, let \( u \geq 0, u \in \mathcal{H} \), where \( \mathcal{H} \) is the closure in \( H^{1,2} \)-norm of the space:

\[
\mathcal{H} = \{ \varphi \in \text{Lip}(\Omega) : \text{supp} \varphi \cap F = \emptyset \}
\]

and \( F \) is a fixed closed subset of \( \partial \Omega \) with \( H_{n-1}(F) = \epsilon > 0 \). Then \( \bar{u} \in H^{1,2}_0(\Omega) \) and (1.16) holds with:

\[
L = \max \left\{ \left( Q n c_n^{1/n} \right), \left( \frac{(C + 1) n c_n^{1/n} \| \Omega \|^{-1/n}}{\epsilon} \right) \right\}
\]

(1.17)

and \( C \) as in (1.15).

Theorem 1.4. Let \( \Omega \) be as above, let \( u \geq 0, u \in \mathcal{H} \), where \( \mathcal{H} \) is the closure in \( H^{1,2} \)-norm of the space:

\[
\mathcal{H} = \{ \varphi \in \text{Lip}(\Omega) : \text{supp} \varphi \subseteq \Omega \setminus F \}
\]

where \( F \) is a closed subset of \( \Omega \) with the property stated in (\( \nu \)). Then \( \bar{u} \in H^{1,2}_0(\Omega) \) and (1.16) holds with:

\[
L = \max \left\{ \left( Q n c_n^{1/n} \right), \left( \frac{(C + 1) n c_n^{1/n} \| \Omega \|^{-1/n}}{\epsilon} \right) \right\}
\]

Remark 1.5. We note that the spaces \( \mathcal{H} \) defined in thms. 1.3-1.4 are properly contained in \( H^{1,2}(\Omega) \) whenever \( F \) has positive capacity. This is the case, in particular, if \( F \) has positive \((n-1)\)-measure. Moreover, if \( F \) has (positive and) finite \((n-1)\)-measure and is a regular set in the sense of geometric measure theory (that is a.e. \((H_{n-1}) \) point of \( F \) is a density point in sense \( H \)) then property (\( \nu \)) is certainly satisfied. (See \([5]\), p.87).

Proof of theorem 1.2. If \( u \in H^{1,2}(\Omega), u \geq 0 \) and \( \Omega \) is Lipschitz, \( u \) may be approximated in \( H^{1,2} \)-norm with smooth functions \( u_n \) in \( \Omega \). (See \([1]\), thm. 3.18). Moreover, if the support of \( u \) has measure \( \| \Omega \| - \epsilon \), then for any \( \epsilon_1 \in (0,\epsilon) \{u_n\} \) can be chosen such that:

\[
|\{x \in \Omega : u_n(x) = 0\}| \geq \epsilon_1.
\]

Hence, for every \( m, u_n \) satisfies (1.16) (with \( \epsilon \) replaced by \( \epsilon_1 \)), so that \( \{u_n\} \) is a bounded sequence in \( H^{1,2}_0(\Omega) \). Let \( \bar{u}_n \) be a subsequence converging to some \( v \in H^{1,2}_0(\Omega) \) weakly in \( H^{1,2} \) and strongly in \( L^2 \). By [3], \( u_n \rightharpoonup u \) in \( L^2(\Omega) \) implies \( \bar{u}_n \rightharpoonup \bar{u} \) in \( L^2(\Omega) \), so \( v \equiv \bar{u} \) and \( \bar{u} \in H^{1,2}_0(\Omega) \). Then from weak convergence it follows that \( u \) satisfies (1.16) for any \( \epsilon_1 < \epsilon \), and hence for \( \epsilon, \) too.

Proof of theorem 1.3. If \( u_n \in \mathcal{H}, u_n \rightharpoonup u \) in \( H^{1,2}(\Omega) \), then \( u_n \) satisfies (1.16)-(1.17). Hence arguing as above, it follows that these hold for \( u \). Note that the condition \( \text{supp} u_n \cap F = \emptyset \) implies that \( |\text{supp} u_n| < \| \Omega \| \) and so does \( u \).

In a similar way it follows theorem 1.4. Incidentally, we note that a Sobolev embedding theorem for functions vanishing on part of the boundary can be derived from thm. 1.3:

Corollary 1.6. Let \( u \in \mathcal{H} \), where \( \mathcal{H} \) is as in theorem 1.3 or 1.4. Then the following estimate holds:

\[
\| u \|_{L^p(\Omega)} \leq \text{const.} \| D u \|_{L^{r}(\Omega)}.
\]

(1.18)

Proof. It is sufficient to prove (1.18) for \( u \geq 0 \). Then \( \bar{u} \in H^{1,2}_0(\Omega) \), so by Sobolev's embedding theorem and theorem 1.3 (or 1.4) one has:

\[
\| u \|_{L^p(\Omega)} = \| u \|_{L^p(\Omega)} \leq C \| D u \|_{L^{r}(\Omega)} \leq C \| D u \|_{L^{r}(\Omega)}
\]

2. Local integrability of \( | D u |^2 \) for \( u \in H^{1,2}(\Omega) \)

Theorem 1.2 allows us to prove the following result, which holds for \emph{any} function \( u \in H^{1,2}(\Omega) \) (even assuming negative values):

Theorem 2.1. Let \( \Omega \) be as in theorem 1.2, \( u \in H^{1,2}(\Omega) \). Then \( \bar{u} \in H^{1,2}_0(\Omega) \) and, for any \( \epsilon > 0 \), one has:
\[ \int_{\tilde{\Omega}_r} |D\mu|^2 \, dx \leq c(\epsilon) \left( Q_n c_n^{1/n} \right)^2 \int_{\Omega} |Du|^2 \, dx \]

where \( \tilde{\Omega}_r \) is the sphere centred at the origin with measure \( | \Omega | - \epsilon \) and:

\[ c(\epsilon) = \left( \frac{|\Omega| - \epsilon}{\epsilon} \right)^{2-2/n} \quad \text{if} \quad \frac{|\Omega|}{2} \leq |\Omega| - \epsilon, \quad c(\epsilon) = 1 \quad \text{otherwise}. \]

Moreover, \( u^* \in AC(\epsilon, | \Omega | - \epsilon) \).

**Proof.** Put \( h = u \left( \frac{|\Omega|}{2} \right) \), and let \( u_1, u_2 \) be the positive and negative parts of \((u - h)\). Then \( u_i \in H^{1,2}(\Omega) \), \( |\text{supp } u_i| \leq \frac{|\Omega|}{2} \) \((i = 1, 2)\). So by theorem 1.2 \( \tilde{u}_i \in H^{1,2}_0(\tilde{\Omega}) \) and:

\[ \int_{\tilde{\Omega}} |D\tilde{u}_i|^2 \, dx \leq \left( Q_n c_n^{1/n} \right)^2 \int_{\Omega} |Du_i|^2 \, dx. \]

In particular, \( u^*_i \in AC(\epsilon, | \Omega | - \epsilon) \) for any \( \epsilon > 0 \). Now, noting that:

\[
(v^+)^+(s) = (v^*)^+(s) \tag{2.1}
\]
\[
(v^-)^+(s) = (v^*)^-(s) \tag{2.2}
\]

one has:

\[
(u^* - h)^+ \in AC(\epsilon, | \Omega |), \quad (u^* - h)^- \in AC(0, | \Omega | - \epsilon), \quad \text{so that:}
\]

\[ u^* \in AC(\epsilon, | \Omega | - \epsilon) \quad \text{for any } \epsilon > 0. \tag{2.3} \]

Note also that:

\[
(u - h)^+ = (u - h)^+	ag{2.4}
\]

whereas the same is *not* true for the negative part. To handle the gradient of \((u - h)^-\), let us observe that, for any \( \epsilon > 0 \), one has, by (0.4):

\[
\int_{\tilde{\Omega}} |D(u - h)^-|^2 \, dx = \left( n c_n^{1/n} \right)^2 \int_{\tilde{\Omega}} \left| \frac{|\Omega| - \epsilon}{|\Omega|} \right| s^{2-2/n} \, |Du^*(s)|^2 \, ds \tag{2.5}
\]

while, by (0.4) and (2.2):

\[
\int_{\tilde{\Omega}} |D(u - h)^-|^2 \, dx = \left( n c_n^{1/n} \right)^2 \int_{\tilde{\Omega}} \left| \frac{|\Omega| - \epsilon}{|\Omega|} \right| s^{2-2/n} \, |Du^*(s)|^2 \, ds = \tag{2.6}
\]

Comparing (2.5) and (2.6) we can write:

\[
\int_{\tilde{\Omega}} |D(u - h)^-|^2 \, dx \leq \left( \frac{|\Omega| - \epsilon}{\epsilon} \right)^{2-2/n} \int_{\tilde{\Omega}} |D(u - h)^-| \, dx. \tag{2.7}
\]

Finally, we can estimate:

\[
\int_{\tilde{\Omega}} |D\mu(x)|^2 \, dx \leq \int_{\tilde{\Omega}} |D\mu(x)|^2 \, dx \leq \int_{\tilde{\Omega}} |D\mu(x)|^2 \, dx + \int_{\tilde{\Omega}} |D\mu(x)|^2 \, dx \leq b(y) \tag{2.8}, (2.7)
\]

\[
\leq \left( Q n c_n^{1/n} \right)^2 \cdot \max \left\{ 1, \left( \frac{|\Omega| - \epsilon}{\epsilon} \right)^{2-2/n} \right\} \left\{ \int_{\tilde{\Omega}} |Du_1|^2 \, dx + \int_{\tilde{\Omega}} |Du_2|^2 \, dx \right\} = \tag{2.9}
\]

\[
\left( Q n c_n^{1/n} \right)^2 \cdot \max \left\{ 1, \left( \frac{|\Omega| - \epsilon}{\epsilon} \right)^{2-2/n} \right\} \int_{\tilde{\Omega}} |Du|^2 \, dx. \]

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So the theorem is completely proved.

From the previous theorem it follows the next estimate, giving an approximation result for rearrangements:

**Corollary 2.2.** Let \( \Omega \) be as above, \( u, v \in H^{1,2}(\Omega) \). Then for any \( \epsilon > 0 \) one has:

\[
\sup_{s \in (\epsilon, |\Omega| - \epsilon)} | (u^* - v^*)(s) | \leq c_1(n, Q, |\Omega|) \| u - v \|_{2} + \\
+ c_2(\epsilon, n, Q, |\Omega|) \| u - v \|_{2}^{1/2} \left\{ \| Du \|_{2} + \| Dv \|_{2} \right\}^{1/2}.
\]

In particular, if \( u_m \) is a sequence of \( H^{1,2} \) functions converging to \( u \) in \( H^{1,2}(\Omega) \), then \( u_m^* \) converges to \( u^* \) uniformly in \((\epsilon, |\Omega| - \epsilon)\) for any \( \epsilon > 0 \).

**Proof.** We start by noting that if \( \varphi \) is an absolutely continuous function on \([a,b]\), then:

\[
\varphi(s) = \left( u_m^*(s) - u^*(s) \right)^2
\]

for every \( s \in [a,b] \). Applying this formula to the function:

\[
\varphi(s) = \left( u_m^*(s) - u^*(s) \right)^2
\]

we have, by Hölder's inequality:

\[
\left( u_m^*(s) - u^*(s) \right)^2 \leq \frac{1}{|\Omega| - 2\epsilon} \int_\Omega |u_m^*(\sigma) - u^*(\sigma)|^2 \, d\sigma + \\
+ 2 \left( \int_\Omega |u_m^*(\sigma) - u^*(\sigma)| \, d\sigma \right)^{1/2} \left( \int_\Omega |\varphi'(\sigma)| \, d\sigma \right)^{1/2} \\
\equiv A_m + 2B_m C_m.
\]

Now:

\[
A_m \leq \frac{1}{|\Omega| - 2\epsilon} \| u_m - u \|_{2}^{2}
\]

and:

\[
B_m \leq \epsilon^{-1+1/n} \| u_m - u \|_{2},
\]

while:

\[
C_m \leq \left( \int_\Omega |u_m^*(\sigma) - u^*(\sigma)| \, d\sigma \right)^{1/2} \left( \int_\Omega |\varphi'(\sigma)| \, d\sigma \right)^{1/2} \\
= \left( \int_\Omega |D u_m| \, d\sigma \right)^{1/2} \left( \int_\Omega |D u| \, d\sigma \right)^{1/2} \leq (by thm. 2.1) \\
\leq c(n, Q, |\Omega|) \left\{ \| D u_m \|_{2} + \| D u \|_{2} \right\}.
\]

Collecting (2.8), (2.9), (2.10), (2.11) one gets the result.

3. Extension to Orlicz-Sobolev spaces

Let \( A: [0, +\infty) \rightarrow [0, +\infty) \) be an "N-function" (see [7]), that is \( A \) is an increasing continuous convex function, such that:

\[
\lim_{t \to 0} \frac{A(t)}{t} = 0; \quad \lim_{t \to +\infty} \frac{A(t)}{t} = +\infty.
\]

By Jensen's inequality, we can repeat the proof of theorem 1.1 and obtain, under the same assumptions:

\[
\int_\Omega \ A( |Du(x)| ) \, dx \geq \int_0^{+\infty} A \left( \frac{\gamma(t)}{\mu(t)} \right) ( - \mu(t) ) \, dt
\]

and:

\[
\int_\Omega \ A( |D\bar{u}(x)| ) \, dx = \int_0^{+\infty} A \left( \frac{\alpha(t)}{\mu(t)} \right) ( - \mu(t) ) \, dt
\]

We can rewrite (3.1)-(3.2) replacing \( A(t) \) with \( A( \lambda t ) \) for any fixed \( \lambda > 0 \).
Then, choosing $\lambda_0 = \frac{n \epsilon^{1/n}}{\gamma}$ we get:

$$\int_\Omega A(\frac{|Du|}{\lambda_0}) \, dx \leq \int_\Omega A(\frac{|Du|}{\gamma}) \, dx. \quad (3.3)$$

Now, recall that the natural norm in the Orlicz space:

$$L^*_A(\Omega) \equiv \left\{ u: \Omega \to \mathbb{R}, u \text{ measurable such that } \int_\Omega A(\frac{|u|}{\lambda}) \, dx < +\infty \text{ for some } \lambda > 0 \right\}$$

is: $\| u \|_A \equiv \inf \left\{ \lambda > 0 : \int_\Omega A(\frac{|u|}{\lambda}) \, dx \leq 1 \right\}$.

Rewriting again (3.3) with $A(r)$ replaced by $A(\frac{r}{\gamma})$, and choosing $\lambda = \| Du \|_A$ we get:

$$\int_\Omega A(\frac{|Du|}{\lambda}) \, dx \leq 1. \text{ Hence:}$$

$$\| Du \|_{L^*_A(\Omega)} \leq \left( \frac{n \epsilon^{1/n}}{\gamma} \right) \| Du \|_{L^*_A(\Omega)}. \quad (3.4)$$

So we have proved the following:

**Theorem 3.1.** Let $\Omega$ and $A$ be as above, let $u$ be a nonnegative Lipschitz function in $\Omega$, such that one of the following holds:

(i) $u = 0$ in $E \subseteq \Omega$ with $\| E \| = \epsilon > 0$

(ii) $u = 0$ in $F \subseteq \partial \Omega$ with $H_{n-1}(F) = \epsilon > 0$

(iii) $u = 0$ in $G \subseteq \Omega$ with $H_{n-1}(\Pi(G)) = \epsilon > 0$ for some projection $\Pi$ (see section 1).

Then $u \in \text{Lip}(\Omega)$ and (3.4) holds, with $\gamma$ possibly depending on $n, C, Q, |\Omega|, \epsilon$.

**Remark 3.2.** We did not state the previous theorem for $u \in H^1L^*_A(\Omega)$ because to apply a limit process as in the proof of theorems (1.2)-(1.3)-(1.4) we have to know that a bounded sequence in $H^1L^*_A(\Omega)$ has a weakly converging subsequence. This cannot be assured without further assumptions on $A$. To discuss this fact, we recall some results from the theory of Orlicz-Sobolev spaces. (See [1]).

We say that $A$ satisfies a "global $\Delta_2$-condition" if:

$$A(2t) \leq \delta A(t) \text{ for some } \delta > 0, \text{ any } t > 0. \quad (3.5)$$

We say that $A$ satisfies a "$\Delta_2$-condition near infinity" if (3.5) holds only for any $t \geq t_0$, for some $t_0 > 0$. We say that $(A,\Omega)$ is $\Delta$-regular if: $A$ satisfies a global $\Delta_2$-condition, or: $A$ satisfies a $\Delta_2$-condition near infinity and $|\Omega| < +\infty$. If $(A,\Omega)$ is $\Delta$-regular, then $L^*_A(\Omega)$ and $H^1L^*_A(\Omega)$ are reflexive spaces; if $\Omega$ is Lipschitz then $C^\infty(\Omega)$ is dense in $H^1L^*_A(\Omega)$; if $|\Omega| < +\infty$ then $L^*_A(\Omega)$ is continuously embedded in $L^1(\Omega)$. Using these facts one can repeat the proofs of theorems (1.2)-(1.3)-(1.4) to get the following:

**Theorem 3.3.** Let $\Omega$, $A$ be as above. Suppose that $A$ satisfies a $\Delta_2$-condition near infinity, and let $u$ satisfy the assumptions of one of theorems 1.2, 1.3, 1.4, with $H^{1,2}(\Omega)$ replaced by $H^1L^*_A(\Omega)$. Then $u \in H^{1,2}_0L^*_A(\Omega)$ and (3.4) holds, with $\gamma$ possibly depending on $n, C, Q, |\Omega|, \epsilon$.

**Example.** An example of Orlicz-Sobolev space which does not reduce to a standard Sobolev space and satisfies the previous theorem is the one defined by $A(r) = r^p \log(1 + r)$ with $p \geq 1$.

Now we are interested in stating an analogue of theorem 2.1 for Orlicz-Sobolev spaces. We first consider the case of a Lipschitz function $u$. The analogue of formula (0.4) is:

$$\int_\Omega A(|Du(x)|) \, dx = \int_0^{[\Pi]} A \left( n c_n^{1/n} |Du^*(s)| s^{1-1/n} \right) ds.$$

Arguing as in section 2 one gets:

$$\int_\Omega A \left( |D(u - h)| \right) \, dx = \int_{[\Pi]-\epsilon}^{[\Pi]} A \left( n c_n^{1/n} |Du^*(s)| s^{1-1/n} \right) ds \quad (3.6)$$
\[
\int_{\Omega} \mathcal{A} \left( \left| D(u - h) - \right| \right) \, dx = \int_{\frac{1}{\epsilon} \Omega} \mathcal{A} \left( n c_n^{1/n} \left| Du^+ (s) \right| \left( \left| \Omega \right| - s \right)^{-1/n} \right) \, ds. \tag{3.7}
\]

Comparing (3.6)-(3.7) one can write:

\[
\int_{\Omega} \mathcal{A} \left( \frac{|D(u - h)|}{\lambda} \right) \, dx \leq \int_{\Omega} \mathcal{A} \left( \left| D(u - h) - \right| \right) \, dx \tag{3.8}
\]

with \( \lambda = \left( \frac{|\Omega|}{c_n} \right)^{1/n} \) (we take \( \epsilon < \frac{|\Omega|}{c_n} \), so \( \lambda > 1 \)).

Applying (3.3) to the positive and negative parts of \( (u - h) \) we get, by (3.8):

\[
\int_{\Omega} \mathcal{A} \left( \frac{|D(u - h)|}{\lambda} \right) \, dx = \int_{\Omega} \mathcal{A} \left( \frac{|D(u - h)|}{\lambda} \right) \, dx \leq \int_{\Omega} \left\{ \mathcal{A} \left( \left| D(u - h) \right| \right) + \mathcal{A} \left( \left| D(u - h)^- \right| \right) \right\} \, dx \leq \int_{\Omega} \left\{ 2 \lambda_0 \left| D(u - h)^- \right| \right\} \, dx = \int_{\Omega} \mathcal{A} (\lambda_0 \left| Du \right|) \, dx \quad \text{with} \quad \lambda_0 = Q n c_n^{1/n}.
\]

Again, rewriting the previous inequality for \( \mathcal{A}(\frac{x}{p}) \) instead of \( \mathcal{A}(x) \) and choosing \( \rho = \lambda_0 \| Du \|_{\mathcal{A}} \) we find:

\[
\| Du \|_{\mathcal{L}_{\mathcal{A}}(\tilde{\Omega})} \leq \left( Q n c_n^{1/n} \right) \left( \frac{|\Omega|}{\epsilon} \right)^{1/n-1} \| Du \|_{\mathcal{L}_{\mathcal{A}}(\Omega)} \tag{3.9}
\]

for every \( \epsilon \in (0, \frac{|\Omega|}{c_n}) \).

This holds for every Lipschitz function \( u \) defined in \( \Omega \). From this fact we get, by approximation with smooth functions:

**Theorem 3.4.** Let \( \Omega, A \) be as in theorem 3.3. If \( u \in H^1 \mathcal{L}_{\mathcal{A}}(\Omega) \) then \( \tilde{u} \in H^1_{\text{loc}} \mathcal{L}_{\mathcal{A}}(\Omega) \) and (3.9) holds. Moreover, \( u^+ \in AC_{\text{loc}}(\epsilon, |\Omega| - \epsilon) \) for every \( \epsilon > 0 \).

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