On the Renormalizations of Circle Homeomorphisms with Several Break Points

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Abstract
Let \( f \) be an orientation preserving homeomorphism on the circle with several break points, that is, its derivative \( Df \) has jump discontinuities at these points. We study Rauzy-Veech renormalizations of piecewise smooth circle homeomorphisms by considering such maps as generalized interval exchange maps of genus one. Suppose that \( Df \) is absolutely continuous on each interval of continuity and \( D \ln Df \in L^p \) for some \( p > 1 \). We prove that under certain combinatorial assumptions on \( f \), renormalizations \( R^n(f) \) are approximated by piecewise Möbius functions in \( C^{1+L^1} \)-norm, that means, \( R^n(f) \) are approximated in \( C^1 \)-norm and \( D^2 R^n(f) \) are approximated in \( L^1 \)-norm. In particular, if the product of the sizes of breaks of \( f \) is trivial, then the renormalizations are approximated by piecewise affine interval exchange maps.

Keywords
Interval exchange map · Rauzy-Veech induction · Renormalization · Dynamical partition · Martingale · Homeomorphism on the circle · Approximation

Mathematics Subject Classification 37C05 · 37C15 · 37E05 · 37E10 · 37E20 · 37B10

1 Introduction

One of the most studied classes of dynamical systems are orientation-preserving homeomorphisms of the circle \( S^1 = \mathbb{R}/\mathbb{Z} \). Poincaré (1885) noticed that the orbit structure of an orientation-preserving diffeomorphism \( f \) is determined by some irrational mod 1, the \( \text{rota-} \)
The rotation number \( \rho = \rho(f) \) of \( f \), in the following sense: for any \( x \in \mathbb{S}^1 \), the mapping \( f^j(x) \to j\rho \mod 1 \), \( j \in \mathbb{Z} \), is orientation-preserving. Denjoy proved, that if \( f \) is an orientation-preserving \( C^1 \)-diffeomorphism of the circle with irrational rotation number \( \rho \) and \( \log f' \) has bounded variation then the orbit \( \{f^j(x)\}_{j \in \mathbb{Z}} \) is dense and the mapping \( f^j(x) \to j\rho \mod 1 \) can therefore be extended by continuity to a homeomorphism \( h \) of \( \mathbb{S}^1 \), which conjugates \( f \) to the linear rotation \( f_{\rho} : x \to x + \rho \mod 1 \). In this context it is a natural question to ask, under what conditions the conjugation is smooth. The first local results, that is the results requiring the closeness of diffeomorphism to the linear rotation, were obtained by Arnold [2] and Moser [25]. Next Herman [7] obtained a first global result (i.e. not requiring the closeness of diffeomorphism to the linear rotation) asserting regularity of conjugation of the circle diffeomorphism. His result was developed by Yoccoz [27], Stark [26], Khanin & Sinai [15–17], Katznelson & Ornstein [9], Khanin & Teplinsky [19]. They have shown, that if \( f \) is \( C^3 \) or \( C^{2+\nu} \) and \( \rho \) satisfies certain Diophantine condition, then the conjugation will be at least \( C^1 \). Notice that the renormalization approach used in [16,17] and [26] is more natural in the spirit of Herman’s theory. In this approach regularity of the conjugation can be obtained by using the convergence of renormalizations of sufficiently smooth circle diffeomorphisms. In fact, the renormalizations of a smooth circle diffeomorphism converge exponentially fast to a family of linear maps with slope 1. Such a convergence together with the condition on the rotation number (of Diophantine type) imply the regularity of conjugation.

The bottom of the scale of smoothness for a circle diffeomorphism \( f \) was first considered by Herman in [8]. He proved that if \( Df \) is absolutely continuous, \( D \log Df \in L_p \) for some \( p > 1 \), the rotation number \( \rho = \rho(f) \) is irrational of bounded type (meaning that the entries in the continued fraction expansion of \( \rho \) is bounded), and \( f \) is close to the linear rotation \( f_{\rho} \), then the conjugating map \( h \) (between \( f_{\rho} \) and \( f \)) is absolutely continuous. Later, using a martingale approach and not requiring the closeness of \( f \) to the linear rotation, Katznelson and Ornstein [10] gave a different proof of Herman’s theorem on absolute continuity of conjugacy. The latter condition on smoothness for \( f \) (that is, \( Df \) is absolutely continuous and \( D \log Df \in L_p, p > 1 \)) will be called the Katznelson and Ornstein’s (KO, for short) smoothness condition.

A natural generalization of diffeomorphisms of the circle are homeomorphisms with break points, i.e., those circle diffeomorphisms which are smooth everywhere with the exception of finitely many points at which their derivatives have jump discontinuities. Circle homeomorphisms with breaks were investigated by Herman [7] in the piecewise-linear (PL) case. The studies of more general (non PL) circle diffeomorphisms with a unique break point started with the work of Khanin & Vul [20]. It turns out that, the renormalizations of circle homeomorphisms with break points are rather different from those of smooth diffeomorphisms. Indeed, the renormalizations of such a circle diffeomorphism converge exponentially fast to a two-parameter family of Möbius transformations. Applications of their result are very wide in many branches of one dimensional dynamics, examples are the investigation of the invariant measures, nontrivial scalings and prevalence of periodic trajectories in one parameter families. In particular they investigated also the renormalization in the case of rational rotation number. Using convexity of the renormalization they analysed positions of periodic trajectories of one parameter family of circle maps and proved that the rotation number is rational for almost all parameter values. Moreover, the investigation of the Möbius transformations in [11,18] and [21] showed that the renormalization operator in that space possesses hyperbolic properties analogous to those predicted by Lanford [22] in the case of critical rotations. The result of Khanin and Vul is also at the core of the so-called rigidity problem, which concerns the smoothness of conjugacy between two dynamical systems, which a priori
are only topologically equivalent. The rigidity problem for circle homeomorphisms with a break point has recently been completely solved in [12–14,18].

The next problem concerning the rigidity problem is to study the regularity properties of the conjugacy for circle maps with several break points. Circle maps with several break points can be considered as generalized interval exchange transformations of genus one. Marmi, Moussa and Yoccoz introduced in [24] generalized interval exchange transformations, obtained by replacing the affine restrictions of generalized interval exchange transformations in each subinterval with smooth diffeomorphisms. They showed that sufficiently smooth generalized interval exchange transformations of a certain combinatorial type, which are deformations of standard interval exchange transformations and tangent to them at the points of discontinuities, are smoothly linearizable.

Recently Cunha and Smania studied in [5] and [6] the Rauzy-Veech renormalizations of piecewise $C^{2+\nu}$-smooth circle homeomorphisms with several break points by considering such maps as generalized interval exchange transformations of genus one. They proved that Rauzy-Veech renormalizations of $C^{2+\nu}$-smooth generalized interval exchange maps satisfying certain combinatorial condition are approximated by piecewise Möbius transformations in $C^2$-norm. Using convergence of renormalizations of two generalized interval exchange maps with the same bounded-type combinatorics and zero mean nonlinearities they proved in [6] that these maps $C^1$-smoothly conjugate to each other.

The purpose of the present work is to study the behavior of Rauzy-Veech renormalizations of generalized interval exchange maps of genus one and low smoothness. We prove that Rauzy-Veech renormalizations $R^n(f)$ of piecewise KO-smooth generalized interval exchange maps of genus one and satisfying certain combinatorial assumptions, are approximated by piecewise Möbius functions in $C^{1+L_1}$-norm, that means, the $R^n(f)$ are approximated in $C^1$-norm and the $D^2R^n(f)$ are approximated in $L_1$-norm. In particular, if $f$ has zero mean nonlinearity, then the renormalizations are approximated by piecewise affine interval exchange maps.

Our main tool in this paper is an argument from real analysis which is used for $C^{2+\nu}$-smooth circle maps in [16,17,20] and for the KO-smooth case in [3]. Note also that our proofs are based on considerations from the theory martingales, which for circle dynamics have been used by Katznelson and Ornstein in [10].

### 2 Rauzy-Veech Renormalization

To describe the combinatorial assumptions of our results, we will introduce the Rauzy-Veech renormalization scheme. Let $I$ be an open bounded interval and $A$ be an alphabet with $d \geq 2$ symbols. Consider the partition of $I$ into $d$ subintervals indexed by $A$, that is, $\mathcal{P} = \{I_\alpha, \alpha \in A\}$. Let $f : I \to I$ be a bijection. We say that the triple $(f,A,\mathcal{P})$ is a generalized interval exchange map with $d$ intervals (for short g.i.e.m.), if $f|_{I_\alpha}$ is an orientation-preserving homeomorphism for all $\alpha \in A$. Here and later, all intervals will be bounded, closed on the left and open on the right.

If $f|_{I_\alpha}$ is a translation, then $f$ is called a standard interval exchange map (for short s.i.e.m.). When $d = 2$, identifying the endpoints of $I$, standard i.e.m.’s correspond to linear rotations of the circle and generalized i.e.m.’s to homeomorphisms of the circle with two break points.
Now we formulate some conditions on the combinatorics for g.i.e.m and define the renormalization scheme. Note that the combinatorial conditions and the renormalization scheme are the same for generalized and standard i.e.m. cases.

The order of the subintervals $I_\alpha$ before and after the map, constitutes the combinatorial data for $f$, which will be explicitly defined as follows.

Given two intervals $J$ and $U$, we will write $J < U$, if their interiors are disjoint and $x < y$, for every $x \in J$ and $y \in U$. This defines a partial order in the set of all intervals.

Let $f : I \to I$ be a g.i.e.m. with alphabet $A$ and $\pi_0$, $\pi_1 : A \to \{1, \ldots, d\}$, be bijections such that

$$\pi_0(\alpha) < \pi_0(\beta), \quad \text{iff} \quad I_\alpha < I_\beta,$$

and

$$\pi_1(\alpha) < \pi_1(\beta), \quad \text{iff} \quad f(I_\alpha) < f(I_\beta).$$

We call pair $\pi = (\pi_0, \pi_1)$ the combinatorial data associated to the g.i.e.m. $f$, and $p = \pi_1^{-1} \circ \pi_0 : \{1, \ldots, d\} \to \{1, \ldots, d\}$ the monodromy invariant of the pair $\pi = (\pi_0, \pi_1)$. When appropriate we will also use the notation $\pi = (\pi(1), \pi(2), \ldots, \pi(d))$ for the combinatorial data of $f$. We always assume that the pair $\pi = (\pi_0, \pi_1)$ is irreducible, that is, for all $j \in \{1, \ldots, d-1\}$ we have: $\pi_0^{-1}(1, \ldots, j) \neq \pi_1^{-1}(1, \ldots, j)$.

Let $\pi = (\pi_0, \pi_1)$ be the combinatorial data associated to the g.i.e.m. $f$. For each $\varepsilon \in \{0, 1\}$, denote by $\alpha(\varepsilon)$ the last symbol in the expression of $\pi_\varepsilon$, that is $\alpha(\varepsilon) = \pi_\varepsilon^{-1}(d)$. Let us assume that the intervals $I_{\alpha(0)}$ and $f(I_{\alpha(1)})$ have different lengths. Then the g.i.e.m. $f$ is called Rauzy-Veech renormalizable (renormalizable, for short). If $|I_{\alpha(0)}| > |f(I_{\alpha(1)})|$ we say that $f$ is renormalizable of type 0. When $|I_{\alpha(0)}| < |f(I_{\alpha(1)})|$ we say that $f$ is renormalizable of type 1. In either case, the letter corresponding to the largest of these intervals is called winner and the one corresponding to the shortest is called the loser of $\pi$. Let $I^{(1)}$ be the subinterval of $I$ obtained by removing the loser, that is, the shortest of these two intervals:

$$I^{(1)} = \begin{cases} I \setminus f(I_{\alpha(1)}) & \text{if type 0}, \\ I \setminus I_{\alpha(0)} & \text{if type 1}. \end{cases}$$

Since the loser is the last subinterval on the right of $I$, the intervals $I$ and $I^{(1)}$ have the same left endpoint.

The Rauzy-Veech induction of $f$ is the first return map $R(f)$ to the subinterval $I^{(1)}$. We want to see $R(f)$ is again g.i.e.m. with the same alphabet $A$. For this we need to associate to this map an $A$-indexed partition of its domain. Denote by $I^{(1)}_\alpha$ the subintervals of $I^{(1)}$. Let $f$ be renormalizable of type 0. Then the domain of $R(f)$ is the interval $I^{(1)} = I \setminus f(I_{\alpha(1)})$ and we have

$$I^{(1)}_\alpha = \begin{cases} I_\alpha, \quad \text{for } \alpha \neq \alpha(0), \\ I_{\alpha(0)} \setminus f(I_{\alpha(1)}), \quad \text{for } \alpha = \alpha(0). \end{cases}$$

These intervals form a partition of the interval $I^{(1)}$ and denoted by $\mathcal{P}^{(1)} = \{I^{(1)}_\alpha, \alpha \in A\}$. Since $f(I_{\alpha(1)})$ is the last interval on the right of $f(\mathcal{P})$, we have $f(I^{(1)}_\alpha) \subset I^{(1)}$ for every $\alpha \neq \alpha(1)$. This means that, $R(f) := f$ restricted to these $I^{(1)}_\alpha$. On the other hand, due to $I^{(1)}_\alpha = I_{\alpha(1)}$, we have

$$f\left(I^{(1)}_\alpha\right) = f\left(I_{\alpha(1)}\right) \subset I_{\alpha(0)}, \quad \text{and so } f^2\left(I^{(1)}_\alpha\right) \subset f\left(I_{\alpha(0)}\right) \subset I^{(1)}.$$
Then \( R(f) := f^2 \) restricted to \( I_{α(1)}^{(1)} \). Thus,

\[
R(f)(x) = \begin{cases} 
  f(x), & \text{if } x \in I_{α(1)}^{(1)} \text{ and } α \neq α(1), \\
  f^2(x), & \text{if } x \in I_{α(1)}^{(1)}. 
\end{cases}
\]

If \( f \) is renormalizable of type 1, the domain of \( R(f) \) is the interval \( I^{(1)} = I \setminus I_{α(0)} \) and we have

\[
I_{α}^{(1)} = \begin{cases} 
  I_α, & \text{for } α \neq α(0), α(1), \\
  f^{-1}(I_{α(0)}), & \text{for } α = α(0), \\
  I_{α(1)} \setminus f^{-1}(I_{α(0)}), & \text{for } α = α(1). 
\end{cases}
\]

Then \( f(I_{α}^{(1)}) \subset I^{(1)} \) for every \( α \neq α(0) \), and so \( R(f) = f \) restricted to these \( I_{α}^{(1)} \). On the other hand,

\[
f^2(I_{α(0)}^{(1)}) = f(I_{α(0)}) \subset I^{(1)},
\]

and, so \( R(f) = f^2 \) restricted to \( I_{α(0)}^{(1)} \). Thus,

\[
R(f)(x) = \begin{cases} 
  f(x), & \text{if } x \in I_{α(1)}^{(1)} \text{ and } α \neq α(0), \\
  f^2(x), & \text{if } x \in I_{α(0)}^{(1)}. 
\end{cases}
\]

It is easy to see, that \( R(f) \) is a bijection on \( I^{(1)} \) and an orientation-preserving homeomorphisms on each \( I_{α}^{(1)} \). Moreover, the alphabet \( A \) for \( f \) and \( R(f) \) remains the same.

The triple \((R(f), A, P^1)\) is called the **Rauzy-Veech renormalization** of \( f \). If \( f \) is renormalizable of type \( ε \in \{0, 1\} \), then the combinatorial data \( π^1 = (π₀^1, π₁^1) \) of \( R(f) \) are given by

\[
π_{ε} := π_{ε}, \quad \text{and} \quad π_{1-ε}^{(1)}(α) = \begin{cases} 
  π_{1-ε}(α), & \text{if } π_{1-ε}(α) ≤ π_{1-ε}(α(ε)), \\
  π_{1-ε}(α) + 1, & \text{if } π_{1-ε}(α(ε)) < π_{1-ε}(α) < d, \\
  π_{1-ε}(α(ε)) + 1, & \text{if } π_{1-ε}(α) = d.
\end{cases}
\]

We say that a g.i.e.m. \( f \) is **infinitely renormalizable**, if \( R^n(f) \) is well defined for every \( n \in \mathbb{N} \). Let \( I^{(n)} \) be the domain of \( R^n(f) \). It is clear that, \( R^n(f) \) is the first return map for \( f \) to the interval \( I^{(n)} \). Similarly, \( R^n(f)^{-1} = R^n(f^{-1}) \) is the first return map for \( f \) to the interval \( I^{(n)} \).

For every interval of the form \( J = [a, b] \) we put \( \partial J := \{a\} \).

**Definition 2.1** We say that g.i.e.m. \( f \) has **no connection**, if

\[
f^m(\partial I_{α}) \neq \partial I_{β}, \quad \text{for all} \quad m \geq 1 \quad \text{and} \quad α, β \in A \quad \text{with} \quad π₀(β) \neq 1. \tag{5}
\]

It is clear that in case \( π₀(β) = 1 \) then \( f(\partial I_{α}) = \partial I_{β} \) for \( α = π^{-1}_1(1) \). Notice that the no connection condition is a necessary and sufficient condition for \( f \) to be infinitely renormalizable. Condition (5) means that the orbits of the left end point of the subintervals \( I_{α} \), \( α \in A \) are disjoint when ever they can be.

Let \( ε_n \) be the type of the \( n \)-th renormalization and let \( α_n(ε_n) \) the winner and \( α_n(1 - ε_n) \) be the loser of the \( n \)-th renormalization.

**Definition 2.2** We say that g.i.e.m. \( f \) has **k- bounded combinatorics**, if for each \( n \in \mathbb{N} \) and \( β, γ \in A \) there exist \( n_1, p \geq 0 \) with \( |n - n_1| < k \) and \( |n - n_1 - p| < k \) such that

\[
α_{n_1}(ε_{n_1}) = β, \quad α_{n_1+p}(1 - ε_{n_1+p}) = γ, \quad \text{and} \quad α_{n_1+i}(1 - ε_{n_1+p}) = α_{n_1+i+1}(ε_{n_1+i}), \quad \text{for every} \quad 0 ≤ i < p.
\]
We say that g.i.e.m. $f : I \to I$ has **genus one** (or belongs to the rotation class), if $f$ has at most two discontinuities. Note that every g.i.e.m. with either two or three intervals has genus one. The genus of g.i.e.m. is invariant under renormalization.

**Remark 2.3** Every orientation-preserving homeomorphism of the circle when viewed as a g.i.e.m. with $d \geq 2$ intervals, has genus one.

### 3 Main Results

Denote by $\mathbb{B}^{KO}$ the set of g.i.e.m. satisfying the following conditions:

(i) for each $\alpha \in \mathcal{A}$ we can extend $f$ to $\tilde{T}_\alpha$ as an orientation-preserving diffeomorphism satisfying the Katznelson and Ornstein’s (KO, for short) smoothness condition: $f'$ is absolutely continuous and $f'' \in L_p$, for some $p > 1$;

(ii) the map $f$ has no connection;

(iii) the map $f$ has $k$-bounded combinatorics and has genus one.

The main idea of the renormalization group method is to study the behaviour of the renormalization map $R^n(f)$ as $n \to \infty$. For this usually rescaling of the coordinates is used.

Let $H$ be a non-degenerate interval and $g : H \to \mathbb{R}$ be a diffeomorphism. We define the **Zoom** (renormalized coordinate) $Z_H(g)$ of $g$ in $H$ as follows:

$$Z_H(g) = \tau^{-1} \circ g \circ \tau,$$

where $\tau : [0, 1] \to H$ is an orientation-preserving affine map.

Denote by $q^n_\alpha \in \mathbb{N}$ the first return time of the interval $I^{(n)}_\alpha$ to the interval $f^{(n)}$, that is, $R^n(f) | I^{(n)}_\alpha = f^{q^n_\alpha} | I^{(n)}_\alpha$, for some $q^n_\alpha \in \mathbb{N}$. Define the fractional linear map $F_n : [0, 1] \to [0, 1]$ as follows:

$$F_n(x) = \frac{x m_n}{1 + x(m_n - 1)}, \quad \text{where} \quad m_n = \exp \left\{ - \sum_{i=0}^{q^n_\alpha-1} \frac{f''(i)}{2f'(i)} dt \right\}. \quad (6)$$

Whenever necessary, we will use $D^m f$ instead of the $m^{th}$ derivative of $f$. The first result of our present paper is the following

**Theorem 3.1** Let $f \in \mathbb{B}^{KO}$. Then for all $\alpha \in \mathcal{A}$ the following bounds hold:

$$\| Z_{I^{(n)}_\alpha}(R^n(f)) - F_n \|_{C^1([0,1])} \leq \delta_n, \quad \| Z_{I^{(n)}_\alpha}(D^2 R^n(f)) - D^2 F_n \|_{L_1([0,1],d\ell)} \leq \delta_n,$$

where $\delta_n = O(\lambda^n + \eta_n)$, $\lambda \in (0, 1)$ and $\eta_n \in l_2$.

Denote by $\mathbb{B}^{KO}_*$ the subset of functions $f \in \mathbb{B}^{KO}$ satisfying zero mean nonlinearity condition:

$$\int_{[0,1]} \frac{f''(t)}{f'(t)} dt = 0.$$

Our second result is a consequence of Theorem 3.1.

**Theorem 3.2** Let $f \in \mathbb{B}^{KO}_*$. Then for all $\alpha \in \mathcal{A}$ the following bounds hold:

$$\| Z_{I^{(n)}_\alpha}(R^n(f)) - 1d \|_{C^1([0,1])} \leq \delta_n, \quad \| Z_{I^{(n)}_\alpha}(D^2 R^n(f)) \|_{L_1([0,1],d\ell)} \leq \delta_n,$$

where $\delta_n = O(\lambda^{\sqrt{n}} + \eta_n)$, $\lambda \in (0, 1)$ and $\eta_n \in l_2$. 

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Remark 3.3 The sequence $\eta_n$ in Theorems 3.1 and 3.2 has an explicit form and is given in Proposition 4.6.

Remark 3.4 The class $B^{K,O}$ is wider than $B^{2+\nu}$ considered in [5]. However, the rate of approximations in Theorems 3.1 and 3.2 is not exponential, contrary to the class $B^{2+\nu}$.

The structure of the paper is as follows. In Section 4 we formulate some facts on dynamical partitions generated by interval exchange maps. Following Katznelson and Ornstein [10] we define a sequence of piecewise constant functions which generate a finite martingale. In Section 5 and Section 6, using the martingale expansion for the nonlinearity of $f$, we obtain some estimates for the sum of integrals of the nonlinearities of $f$. Finally, in Section 7 we prove our main theorems.

4 The Dynamical Partition and a Martingale

Let $(f, A, \mathcal{P})$ be a g.i.e.m. with $d$ intervals and $\mathcal{P} = \{I_\alpha : \alpha \in A\}$ be the initial $A$-indexed partition of $I$. For specificity we take $I = [0, 1)$. Suppose that $f$ is infinitely renormalizable. Let $I^{(n)}$ be the domain of $R^n(f)$. Note that $I^{(n)}$ is the nested sequence of subintervals, with the same left endpoint of $I$. We want to construct the dynamical partition of $I$ associated to the domain of $R^n(f)$.

As mentioned above, $R(f)$ is g.i.e.m. with $d$ intervals and the intervals $I^{(1)}_\alpha$ generate an $A$-indexed partition of $I^{(1)}$, denoted by $\mathcal{P}^1$. By induction one can check, that $R^n(f)$ is g.i.e.m. with $d$ intervals. Let $\mathcal{P}^n = \{I^{(n)}_\alpha : \alpha \in A\}$ be the $A$-indexed partition of $I^{(n)}$, generated by $R^n(f)$. We call $\mathcal{P}^n$ the fundamental partition and $I^{(n)}_\alpha$ the fundamental segments of rank $n$.

Since $R^n(f)$ is the first return map for $f$ to the interval $I^{(n)}$, each fundamental segment $I^{(n)}_\alpha \in \mathcal{P}^n$ returns to $I^{(n)}$ under certain iterates of the map $f$. Until returning, these intervals will be in the interval $I \setminus I^{(n)}$ for some time. Consequently the system of intervals (their interiors are mutually disjoint)

$$\xi_n = \left\{ f^i(I^{(n)}_\alpha), \ 0 \leq i \leq q^n_\alpha - 1, \ \alpha \in A \right\}$$

cover the whole interval and form a partition of $I$.

The system of intervals $\xi_n$ is called the $n$-th dynamical partition of $I$. The dynamical partitions $\xi_n$ are refined with increasing $n$, where $\xi_{n+1} \supset \xi_n$ means that any element of the preceding partition is a union of a number of elements of the next partition, or belongs to the next partition. Denote by $\xi_{n+1}^{pr}$ the system of preserved intervals of $\xi_n$. More precisely, if $R^n f$ has type 0

$$\xi_{n+1}^{pr} = \left\{ f^i(I^{(n)}_\alpha), \ 0 \leq i \leq q^n_\alpha - 1, \ \text{for } \alpha \neq \alpha(0) \right\},$$

and if $R^n f$ has type 1

$$\xi_{n+1}^{pr} = \left\{ f^i(I^{(n)}_\alpha), \ 0 \leq i \leq q^n_\alpha - 1, \ \text{for } \alpha \neq \alpha(1) \right\}.$$
Let \( \xi_{n+1}^{ln} \) be the set of elements of \( \xi_{n+1} \) which are properly contained in some element of \( \xi_n \). Therefore if \( R^n \) has type 0

\[
\xi_{n+1}^{ln} = \left\{ f^i(I_0^{(n+1)}), \ 0 \leq i < q_{a(0)}^n \right\} \cup \left\{ f^i(I_1^{(n+1)}), \ 0 \leq i < q_{a(1)}^n \right\}
\]

and if \( R^n \) has type 1

\[
\xi_{n+1}^{ln} = \sqcup_{i=0}^{q_{a(1)}^n-1} f^i \left( I_0^{(n+1)} \setminus f^{-q_{a(1)}^n}(I_0^{(n+1)}) \right) \cup \sqcup_{i=0}^{q_{a(1)}^n-1} f^i (I_1^{(n+1)} \setminus f^{-q_{a(1)}^n}(I_1^{(n+1)})).
\]

So, the partition \( \xi_{n+1} \) consists of the preserving elements of \( \xi_n \) and the images of two (new) intervals for defining \( R^{n+1} \), that is, \( \xi_{n+1} = \xi_{n+1}^{pr} \cup \xi_{n+1}^{ln} \). Note also that for the first return time \( q_{a}^n \), we have:

1. If \( \alpha = a^n(\varepsilon) \), then \( q_{a^n(\varepsilon)}^{n+1} = q_{a^n(\varepsilon)}^n \);
2. If \( \alpha = a^n(1-\varepsilon) \), then \( q_{a^n(1-\varepsilon)}^{n+1} = q_{a^n(1-\varepsilon)}^n + q_{a^n(\varepsilon)}^n \).

**Martingale** Now we define a martingale generated by the dynamical partitions associated to \( \text{g.i.e.m.} \), and give some properties which will be used in the proof of our results. A similar martingale generated by dynamical partitions associated to circle maps was considered in [3] and [9].

Let \( g : I \to I \) be a function of class \( L_p(I, d\ell), \ p > 1 \). Using the dynamical partitions \( \xi_n \), we define a sequence of piecewise constant functions \( \Phi_n : I \to R^1, \ n \geq 1 \), on \( I \) as follows

\[
\Phi_n(x) := \frac{1}{|\Delta^{(n)}|} \int_{\Delta^{(n)}} g(y) dy, \ x \in \Delta^{(n)},
\]

where \( \Delta^{(n)} \) is an interval of the partition \( \xi_n \).

**Theorem 4.1** Let \( g \in \mathbb{L}_p(I, d\ell) \), \( p > 1 \). Then the sequence of piecewise functions \( \{\Phi_n(x), \ n \geq 1\} \) generate a finite martingale with respect to the dynamical partition \( \xi_n \).

**Proof** Note that each \( \Phi_n(x) \) is a step function, which takes constant values on each element \( I_\alpha^{(n)} \) of the partition \( \xi_n \). It follows that \( \Phi_n(x) \) is \( \xi_n \)-measurable. Therefore, it is enough to show that

\[
E(\Phi_{n+1}/\xi_n) = \Phi_n, \text{ for all } n \geq 1,
\]

where \( E(\Phi_{n+1}/\xi_n) \) is a conditional expectation of the random variable \( \Phi_{n+1} \) with respect to the partition \( \xi_n \). Define the characteristic functions on the elements of \( \xi_n \):

\[
X_\alpha^{(n)}(x) = \begin{cases} 1, & \text{if } x \in f^i \left( I_\alpha^{(n)} \right), \\ 0, & \text{if } x \notin f^i \left( I_\alpha^{(n)} \right). \end{cases}
\]
where \( \alpha \in A \) and \( 0 \leq i \leq q^n_{\alpha} - 1 \). By definition of conditional expectation with respect to the partition, we have

\[
E(\Phi_{n+1}/\xi_n) = \sum_{\alpha \in A} q^n_{\alpha} \sum_{i=0}^{q^n_{\alpha}-1} E\left(\Phi_{n+1}/f^i(I^{(n)}_{\alpha})\right) X^{(n)}_{\alpha,i}(x). \tag{8}
\]

Recall, that the partition \( \xi_{n+1} \) consists of the preserving elements of \( \xi_n \) and the images of two (new) intervals for defining \( R^{n+1}(f) \), that is, \( \xi_{n+1} = \xi^{pr}_{n+1} \cup \xi^{tn}_{n+1} \). Split the sum (8) into two sums corresponding to \( \xi_{n+1}^{pr} \) and \( \xi_{n+1}^{tn} \):

\[
E(\Phi_{n+1}/\xi_n) = \sum_{J_i \in \xi_{n+1}^{pr}} E(\Phi_{n+1}/J_i) X^{(n)}_{\alpha,i}(x) + \sum_{J_i \in \xi_{n+1}^{tn}} E(\Phi_{n+1}/J_i) X^{(n)}_{\alpha,i}(x), \tag{9}
\]

where \( J_i = f^i(I^{(n)}_{\alpha}) \). Consider first the sum corresponding to \( \xi_{n+1}^{pr} \) in (9). Then

\[
E(\Phi_{n+1}/J_i) = \int_{[0,1]} \Phi_{n+1}(x) \ell(dx/f^i(I^{(n)}_{\alpha})) = \frac{1}{|f^i(I^{(n)}_{\alpha})|} \int_{f^i(I^{(n)}_{\alpha})} \Phi_{n+1}(x) dx
\]

\[
= \frac{1}{|f^i(I^{(n)}_{\alpha})|} \int_{f^i(I^{(n)}_{\alpha})} g(y) dy. \tag{10}
\]

Next we consider the sum corresponding to \( \xi_{n+1}^{tn} \) in (9). Let \( J_i := \bigcup I^{(n+1)}_{\alpha} \), where \( I^{(n+1)}_{\alpha} \in \xi_{n+1}^{tn} \). Then we obtain

\[
E(\Phi_{n+1}/J_i) = \frac{1}{|f^i(I^{(n)}_{\alpha})|} \int_{f^i(I^{(n)}_{\alpha})} \Phi_{n+1}(x) dx = \frac{1}{|f^i(I^{(n)}_{\alpha})|} \sum_{I^{(n+1)}_{\alpha} \in J_i} \int_{I^{(n+1)}_{\alpha}} \Phi_{n+1}(x) dx
\]

\[
= \frac{1}{|f^i(I^{(n)}_{\alpha})|} \sum_{I^{(n+1)}_{\alpha} \in J_i} \left( \int_{I^{(n+1)}_{\alpha}} \frac{1}{|I^{(n+1)}_{\alpha}|} \int_{I^{(n+1)}_{\alpha}} g(y) dy \right) dx
\]

\[
= \frac{1}{|f^i(I^{(n)}_{\alpha})|} \int_{f^i(I^{(n)}_{\alpha})} g(y) dy.
\]

This, and equations in (9), (10) imply the result. \( \square \)

Denote by \( \|f\|_p \) the norm of \( f \) in \( L_p(I, d\ell) \), \( p > 1 \).

**Theorem 4.2** Let \( g \in L_p(I, d\ell) \), \( p > 1 \). Then

\[
\lim_{n \to \infty} \|g - \Phi_n\|_2 = 0.
\]

**Proof** Note that the functions of the class \( L_p \) are well approximated by continuous functions, that is, if \( g \in L_p(I, d\ell) \), \( p > 1 \) then for any \( \varepsilon > 0 \), there exist an uniformly continuous function \( \omega_{\varepsilon} \) and a summable function \( \psi_{\varepsilon} \) such that

\[
g(x) = \omega_{\varepsilon}(x) + \psi_{\varepsilon}(x), \quad x \in I, \text{ and moreover, } \|\psi_{\varepsilon}\|_2 < \varepsilon.
\]
Consider the partition $\xi_n$. Then $[0, 1] = \bigcup_{\alpha \in \mathcal{A}} \bigcup_{i=0}^{\ell_n-1} f^i(I_{\alpha}^{(n)})$. Using the above expansion for $g$ we get

$$
\|g - \Phi_n\|_{L^2}^2 = \int_{[0,1]} |g(x) - \Phi_n(x)|^2 dx = \sum_{\alpha \in \mathcal{A}} \sum_{i=0}^{\ell_n-1} \int_{f^i(I_{\alpha}^{(n)})} |\Phi_n(x) - \omega_\varepsilon(x) + \psi_\varepsilon(x)|^2 dx
$$

$$
\leq 2 \sum_{\alpha \in \mathcal{A}} \sum_{i=0}^{\ell_n-1} \int_{f^i(I_{\alpha}^{(n)})} \frac{1}{|f^i(I_{\alpha}^{(n)})|} \int_{f^i(I_{\alpha}^{(n)})} |\omega_\varepsilon(y) - \omega_\varepsilon(x)|^2 dy dx
$$

$$
+ 2 \sum_{\alpha \in \mathcal{A}} \sum_{i=0}^{\ell_n-1} \int_{f^i(I_{\alpha}^{(n)})} \frac{1}{|f^i(I_{\alpha}^{(n)})|} \int_{f^i(I_{\alpha}^{(n)})} |\psi_\varepsilon(y) - \psi_\varepsilon(x)|^2 dy dx := M_n^{(1)} + M_n^{(2)}.
$$

It is clear that

$$
M_n^{(2)} \leq 2 \sum_{\alpha \in \mathcal{A}} \sum_{i=0}^{\ell_n-1} \int_{f^i(I_{\alpha}^{(n)})} \frac{1}{|f^i(I_{\alpha}^{(n)})|} \int_{f^i(I_{\alpha}^{(n)})} |\psi_\varepsilon(y)|^2 dx + \int_{f^i(I_{\alpha}^{(n)})} |\psi_\varepsilon(x)|^2 dx \leq 4 \|\psi_\varepsilon\|_{L^2}^2.
$$

By assumption, $\omega_\varepsilon$ is uniformly continuous. This means, that for all $x, y : |x - y| < \delta$ inequality $|\omega_\varepsilon(x) - \omega_\varepsilon(y)| < \varepsilon$ is fulfilled. On the other hand, for each $f^i(I_{\alpha}^{(n)}) \in \xi_n$, we have $\max_{\alpha, i} |f^i(I_{\alpha}^{(n)})| \leq \lambda^n$, $\lambda \in (0, 1)$ (see for instance (4.9)). It follows that for all $x, y \in f^i(I_{\alpha}^{(n)})$, the inequality $|\omega_\varepsilon(x) - \omega_\varepsilon(y)| < \varepsilon$ is fulfilled. Then

$$
M_n^{(1)} \leq 2 \sum_{\alpha \in \mathcal{A}} \sum_{i=0}^{\ell_n-1} \int_{f^i(I_{\alpha}^{(n)})} \frac{1}{|f^i(I_{\alpha}^{(n)})|} \int_{f^i(I_{\alpha}^{(n)})} (\omega_\varepsilon(y) - \omega_\varepsilon(x)) dy \leq \varepsilon^2.
$$

The estimates for $M_n^{(1)}$ and $M_n^{(2)}$ imply the assertion of Theorem 4.2. \(\square\)

Set $\Phi_0(x) = \int_{[0,x]} g(y) dy$, forall $x \in I$. Define $h_n := \Phi_n - \Phi_{n-1}$, $n \geq 1$.

**Theorem 4.3** Let $g \in L_p(I, d\ell)$, $p > 1$. Then

1. $g - \Phi_0 = \sum_{n=1}^{\infty} h_n$ (in $L_2$ -- norm);

2. for any interval $\Delta^{(n-1)}$ of the partition $\xi_{n-1}$ and for all $n \geq 1$, we have

$$
\int_{\Delta^{(n-1)}} h_n(x) d\ell = 0.
$$

**Proof** Assertion (1) immediately follows from Theorem 4.2. We’ll prove the second assertion. Consider the partition $\xi_{n-1}$. Recall, that $\xi_n = \xi_n^{pr} \cup \xi_n^{tn}$. Let $\Delta^{(n-1)} \in \xi_{n-1}$. If $\Delta^{(n-1)} \in \xi_{n}^{pr}$,
then we have
\[ \int_{\Delta^{(n-1)}} h_n(x) d\ell = \int_{\Delta^{(n-1)}} \Phi_n(x) dx - \int_{\Delta^{(n-1)}} \Phi_{n-1}(x) dx = 0. \]

Suppose that \( \Delta^{(n-1)} \notin \xi^{pr}_n \). Let \( I^{(n)}_a \in \Delta^{(n-1)} \) and \( I^{(n)}_a \in \xi^{ln}_n \). Then we obtain
\[
\int_{\Delta^{(n-1)}} h_n(x) d\ell = \int_{\Delta^{(n-1)}} \Phi_n(x) dx - \int_{\Delta^{(n-1)}} \Phi_{n-1}(x) dx \\
= \sum_{I^{(n)}_a \in \Delta^{(n-1)}} \int_{I^{(n)}_a} \left( \frac{1}{|I^{(n)}_a|} \int_{I^{(n)}_a} g(y) dy \right) dx - \int_{\Delta^{(n-1)}} \left( \frac{1}{|\Delta^{(n-1)}|} \int_{\Delta^{(n-1)}} g(y) dy \right) dx \\
= \sum_{I^{(n)}_a \in \Delta^{(n-1)}} \int_{I^{(n)}_a} g(y) dy - \int_{\Delta^{(n-1)}} g(y) dy = 0.
\]

We are done. \( \square \)

The following theorem plays an important role for our result.

**Theorem 4.4** (see. [10]) Suppose \( g \in L_p(I, \, d\ell), \, 1 < p \leq 2 \). Let \( \{\Phi_n(x), \, n \geq 1\} \) be a \( \mathbb{L}_p \)-bounded martingale w.r.t. the partition \( \xi_n \). Then the sequence \( \{||h_n||_p, \, n \geq 1\} \) belongs to \( l_2 \).

We need the following lemma which can be checked easily.

**Lemma 4.5** Let \( \{r_n, \, n \geq 1\} \in l_2 \) be a sequence of positive numbers and let \( \lambda \in (0, 1) \) be a constant. Set \( \epsilon_n := \sum_{j=1}^{\infty} \lambda^{j-n} r_j, \, n \geq 1 \). Then \( \sum_{n=1}^{\infty} \epsilon_n^2 < \infty \).

As we know, in case of KO smoothness, the function \( f'' \) is defined almost everywhere. Whenever necessary, let’s conditionally call the derivative \( f'' \) the **nonlinearity** of \( f \).

Next we define the a sequence of piecewise constant functions for \( g = \frac{f''}{f} \) in a similar way as in (7):
\[ \Phi_n(x) := \frac{1}{|\Delta^{(n)}|} \int_{\Delta^{(n)}} \frac{f''(t)}{f'(t)} dt, \quad x \in \Delta^{(n)}, \quad \alpha \in A \]
where \( \Delta^{(n)} \) is an interval of the partition \( \xi_n \). Set \( h_n = \Phi_n - \Phi_{n-1} \). Similar to results in Theorem 4.1, Theorem 4.4 and Lemma 4.5 we obtain the following

**Proposition 4.6** Let \( f \in B^{KO} \) and \( \eta_n = \sum_{m=n}^{\infty} \lambda^{m-n} ||h_m||_p \). Then \( \{\eta_n\} \in l_2 \).

**Bounded Geometry or Denjoy Type Inequalities** Denote by \( B^{1+bv} \) the set of g.i.e.m \( f : I \rightarrow I \) satisfying the conditions \((ii) - (iii)\), which are piecewise \( C^1 \)- smooth and have bounded variation of the first derivative.

From now on we will denote by \( C \) constants, which depend only on the original map \( f \). Put \( x_i = f^i(x), \, i \geq 0 \) and \( x_0 := x \). The following lemma plays a key role in studying metrical properties of the dynamical partition \( \xi_n \).
Lemma 4.7 (see [5]) Let \( f \in \mathbb{B}^{1+bv} \). Put \( \theta := \text{Var}_I \log f' \). Then there is a constant \( C > 0 \) such that
\[
e^{-C\theta} \leq \prod_{i=0}^{q_n^n-1} Df(x_i) \leq e^{C\theta}, \quad \text{for all } x \in I_{\alpha}^{(n)}.
\]
Define the norm of the dynamical partition \( \xi_n \) by
\[
\|\xi_n\| = \max \{|f^i(I^{(n)}_{\alpha})|\}, \quad \text{where the maximum is taken for all } \alpha \in A \text{ and } 0 \leq i \leq q_n^n - 1.
\]
Using lemma 4.7, it has been shown in [5] that the intervals of the dynamical partition \( \xi_n \) have exponentially small length.

Lemma 4.8 (see [5]) Let \( f \in \mathbb{B}^{1+bv} \). Then for sufficiently large \( n \) there is \( \lambda \in (0, 1) \) such that \( \|\xi_{n+k}\| \leq \lambda \|\xi_n\| \).

The following corollary follows from Lemma 4.8.

Corollary 4.9 Let \( f \in \mathbb{B}^{1+bv} \). Then for sufficiently large \( n \) and \( m \) with \( m - n > k \), there is \( \lambda \in (0, 1) \) such that
\[
\|\xi_n\| \leq \lambda^{\frac{n}{k}} - 1 \quad \text{and} \quad \|\xi_m\| \leq \lambda^{\frac{m-n}{k}} - 1 \|\xi_n\|.
\]

Consider the sequence of dynamical partitions \( \xi_n \). We recall the following definition introduced in [9].

Definition 4.10 An interval \( J = (a, b) \subset [0, 1] \) is called \( q_n \)-small and its end points \( a, b \) are \( q_n \)-close, if the system of intervals \( f^i(J), \quad 0 \leq i \leq q_n \) are disjoint.

The following lemmas are modification of similar ones used in [9] and [10] for circle maps.

Lemma 4.11 Suppose that \( f \in \mathbb{B}^{1+bv} \). Let \( I_{\alpha}^{(n)} \) be \( q_n \)-small and \( m < n - k \), then
\[
\ell \left( \bigcup_{i=0}^{q_{m+1}-1} f^i(I_{\alpha}^{(n)}) \right) \leq C\lambda_1^{n-m}, \quad \text{where } \lambda_1 = \lambda^{1/k}.
\]

Proof Let \( I_{\beta}^{(m+1)} \) be \( q_{m+1} \)-small and assume that it contains the interval \( I_{\alpha}^{(n)} \). The second inequality in (11) implies:
\[
|f^i(I_{\alpha}^{(n)})| \leq C\lambda_1^{n-m} |f^i(I_{\beta}^{(m)})|, \quad 0 \leq i \leq q_{m+1}.
\]

Then, we get
\[
\ell \left( \bigcup_{i=0}^{q_{m+1}-1} f^i(I_{\alpha}^{(n)}) \right) = \sum_{i=0}^{q_{m+1}-1} |f^i(I_{\alpha}^{(n)})| \leq C\lambda_1^{n-m} \sum_{i=0}^{q_{m+1}-1} |f^i(I_{\beta}^{(m)})| \leq C\lambda_1^{n-m}.
\]

Put \( Df^i(t) := (f^i)'(t) \).
Lemma 4.12 Suppose that \( f \in \mathbb{B}^{1+b} \). Let \( x \) and \( y \) be \( q_n \)-close. Then for any \( 0 \leq l \leq q_n - 1 \) the following inequality holds:
\[
e^{-\theta} \leq \frac{Df^l(x)}{Df^l(y)} \leq e^\theta.
\]

Proof Take any two \( q_n \)-close points \( x, y \in [0, 1] \) and \( 0 \leq m \leq q_n - 1 \). Denote by \( I_a^{(n)} \) the open interval with endpoints \( x \) and \( y \). Since the intervals \( f^i(I_a^{(n)}) \), \( 0 \leq i < q_n \) are disjoint, we obtain
\[
| \log Df^{m}(x) - \log Df^{m}(y) | \leq \sum_{s=0}^{q_n-1} | \log Df(f^s(x)) - \log Df(f^s(y)) | \leq \theta.
\]
From this, we obtain the result.

Consider an arbitrary fundamental segment \( I_a^{(n)} \) of the \( n \)-th basic partition \( \mathcal{P}^{(n)} \). Put \( I_a^{(n)} = [a, b] \). For each \( 0 \leq i \leq q_n^n \), we introduce the relative coordinates \( z_i : [f^i(a), f^i(b)] \to [0, 1] \) as:
\[
z_i := \frac{f^i(x) - f^i(a)}{f^i(b) - f^i(a)}, \quad x \in [a, b]. \tag{12}
\]
We consider the relative coordinates \( z_i \) as functions of the variable \( z_0 \).

Lemma 4.13 Suppose that \( f \in \mathbb{B}^{K,O} \). Then for all \( i = 0, 1, ..., (q_n^n - 1) \) the following inequalities hold:
\[
e^{-2\theta} \leq \frac{z_0(1 - z_0)}{z_i(1 - z_i)} \leq e^{2\theta}, \quad e^{-\theta} \leq \frac{dz_i}{dz_0} \leq e^{\theta}, \quad \int_0^1 \left| \frac{d^2z_i}{dz_0^2} \right| dz_0 \leq C \left\| \frac{f''}{f'} \right\|_1. \tag{13}
\]

Proof Using (12) we get
\[
z_0(1 - z_0) = \frac{x - a}{f^i(x) - f^i(a)} \cdot \frac{b - x}{f^i(b) - f^i(a)} \cdot \left( \frac{f^i(b) - f^i(a)}{b - a} \right)^2 = \frac{Df^i(t_0)}{Df^i(t_1)} \cdot \frac{Df^i(t_1)}{Df^i(t_2)},
\]
where \( t_0 \in [a, b], \ t_1 \in [a, x], \ t_2 \in [x, b] \). Note that both of the pairs \( \{t_0, t_1\} \) and \( \{t_0, t_2\} \) are \( q_n \)-close. Applying Lemma 4.12, we obtain the first inequality in (13).

Using (12) we find for \( \frac{dz_i}{dz_0} \):
\[
\frac{dz_i}{dz_0} = \frac{dz_i}{dx_i} \cdot \frac{dx_i}{dz_0} = \frac{|I_a^{(n)}|}{|I_{a,i}^{(n)}|} \cdot Df^i(x) = \frac{Df^i(x)}{Df^i(t_0)}, \quad \text{where} \ t_0 \in I_a^{(n)}.
\]

Then due to Lemma 4.12, we get the second inequality in (13).

Note, that the functions \( \frac{d^2z_i}{dz_0^2} \) are defined almost everywhere. We can estimate the functions \( \frac{d^2z_i}{dz_0^2} \) in the integral norm. According to the relations \( x = a + z_0(b - a) \) and \( x_i = f^i(x) \), we...
find for $\frac{d^2z_i}{dz_0^2}$:

$$\frac{d^2z_i}{dz_0^2} = \frac{d}{dx_i} \left( \frac{|f_a^{(n)}|}{|f_{a,i}^{(n)}|} \prod_{j=0}^{i-1} f'(x_j) \right) \cdot \frac{dx_i}{dx} \cdot \frac{dx}{dz_0} = \frac{|f_a^{(n)}|^2}{|f_{a,i}^{(n)}|} \cdot Df^i(x) \cdot \sum_{j=0}^{i-1} f''(x_j) \cdot Df^j(x)$$

$$= \frac{dz_i}{dz_0} \cdot \left( \sum_{j=0}^{i-1} \frac{f''(x_j)}{f'(x_j)} \cdot Df^j(x) \right) \cdot |f_a^{(n)}| = \frac{dz_i}{dz_0} \cdot \left( \sum_{j=0}^{i-1} \frac{f''(x_j)}{f'(x_j)} \cdot |f_a^{(n)}| \cdot \frac{dz_j}{dz_0} \right).$$

This together with the first and second inequalities in (13) imply

$$\int_0^1 \left| \frac{d^2z_i}{dz_0^2} \right| dz_0 \leq c^3 \int_0^1 \left( \sum_{j=0}^{i-1} \left| \frac{f''(x_j)}{f'(x_j)} \right| \cdot |f_a^{(n)}| \right) dz_i.$$

Substituting $z_i = \frac{x_i - a_i}{b_i - a_i}$ in the last integral, we obtain

$$\int_0^1 \left| \frac{d^2z_i}{dz_0^2} \right| dz_0 \leq C \sum_{j=0}^{b_j - 1} \int_{a_j}^{b_j} \left| f''(x_j) \right| dx_j \leq C \left\| f'' \right\|_1,$$

as we claimed. \qed

## 5 Approximations of the Nonlinearity for $\mathbb{B}^{KO}$ Maps with a Martingale

In the low smoothness case considered here, we still have not known, how to obtain the necessary bounds for the integral of $\frac{f}{f'}$ on any interval of the dynamical partition. For this reason we had to consider the sum of these integrals over all the intervals of dynamical partition.

Let $I_{a}^{(n)}$ be an arbitrary fundamental segment of the $n$-th basic partition $P^{(n)}$. Let $I_{a}^{(n)} = [a, b]$. For the iteration of the interval $I_{a}^{(n)}$ and its endpoints we use the following notations:

$$I_{a,i}^{(n)} = f^i(I_{a}^{(n)}) = [a_i, b_i], \quad 0 \leq i \leq q_a^n - 1,$$

where $a_0 = a$, $b_0 = b$ and

$$a_i = f^i(a), \quad b_i = f^i(b), \quad x_i = f^i(x) \in [a_i, b_i].$$

For simplicity of the notation put $q_n := q_a^n$. Next define

$$S_n^{(1)} := \sum_{i=0}^{q_a^n-1} \int_{a_i}^{x_i} \frac{f''(t)}{f'(t)} \left( \frac{1}{x_i - a_i} - \frac{1}{2} \right) dt.$$

**Proposition 5.1** Let $f \in \mathbb{B}^{KO}$. Then we have $S_n^{(1)} = O(\lambda^n + \eta_n)$, where $\lambda \in (0, 1)$ and $\eta_n \in l_2$ is from Proposition 4.6.
The first assertion of Theorem 4.3 implies that
\[
\sum_{i=0}^{n-1} \int_{a_i}^{x_i} \left( f''(t) - \Phi_0 - \sum_{m=1}^{N} h_m(t) \right) \left( \frac{t - a_i}{x_i - a_i} - \frac{1}{2} \right) dt + \sum_{i=0}^{n-1} \int_{a_i}^{x_i} \sum_{m=1}^{N} h_m(t) \left( \frac{t - a_i}{x_i - a_i} - \frac{1}{2} \right) dt.
\]
(14)

It is easy to see that the absolute value of the first sum in (14) is not greater than
\[
C \| f'' - \Phi_0 - \sum_{m=1}^{N} h_m \|_1.
\]
The first assertion of Theorem 4.3 implies that
\[
\lim_{N \to \infty} \| f'' - \Phi_0 - \sum_{m=1}^{N} h_m \|_1 = 0.
\]
Then we can choose a sufficiently large number \( N \) such that
\[
\left\| f'' - \Phi_0 - \sum_{m=1}^{N} h_m \right\|_1 \leq \lambda^N.
\]
Hence, the absolute value of the first sum in (14) is bounded above by \( C \lambda^N \).

Recall, that the point \( x_i = f^i(x) \) belongs to the interval \([a_i, b_i]\). Next choose \( r_0 > 0 \) minimal, such that for \([a_i, b_i]\) one has
\[
I_{\beta}^{(n+r_0)} \subset [a_i, x_i] \subset I_{\alpha}^{(n+r_0)} \subset [a_i, b_i],
\]
(15)

where \( I^{(n+r_0+s)} \in \xi_{n+s} \), for \( s = 0, 1 \).

To estimate the last sum in (14), we split the sum in the integrand into three terms corresponding to the summations over \( 1 \leq m \leq n + r_0 \), \( n + r_0 + 1 \leq m \leq n + r_0 + k \) and \( n + r_0 + k + 1 \leq m \leq N \). Consider the first sum. By definition, the function \( h_m(t) \) takes constant values on the atoms of the dynamical partition \( \xi_m \). On the other hand, when passing from partition \( \xi_m \) to \( \xi_{m+1} \), the elements of the partition \( \xi_m \) are preserved, or divided in two subintervals. This together with \([a_i, x_i] \subset I^{(n+r_0)} \subset I_{\alpha, i}^{(n)} \in \xi_n \) imply that the function \( h_m(t) \) takes constant values on the intervals \([a_i, b_i] \), i.e. \( h_m([a_i, b_i]) = h_m(a_i) \), \( i = 0, 1, ..., q_n \).

Using these remarks, we get
\[
\sum_{i=0}^{q_n-1} \int_{a_i}^{x_i} h_m(t) \left( \frac{t - a_i}{x_i - a_i} - \frac{1}{2} \right) dt = \sum_{m=1}^{n+r_0} \sum_{i=0}^{q_n-1} \int_{a_i}^{x_i} h_m(a_i) \left( \frac{t - a_i}{x_i - a_i} - \frac{1}{2} \right) dt = 0.
\]
Consider the sum over \( n + r_0 + 1 \leq m \leq n + r_0 + k \). Then we have
\[
\sum_{m=n+r_0+1}^{n+r_0+k} \sum_{i=0}^{q_n-1} \int_{a_i}^{x_i} h_m(t) \left( \frac{t - a_i}{x_i - a_i} - \frac{1}{2} \right) dt \leq \sum_{m=n+r_0+1}^{n+r_0+k} \sum_{i=0}^{b_i} \int_{a_i}^{x_i} h_m(t) dt \leq \sum_{m=n+r_0+1}^{n+r_0+k} \| h_m \|_p.
\]
It is easy to see, that the last sum also belongs to the class \( l_2 \).
Next we consider the sum over \( n + r_0 + k + 1 \leq m \leq N \) and denote this sum by \( P_n \).
Since \( m \geq n + r_0 + k + 1 \), each atom \([a_i, b_i] \in \xi_n\) is the union of intervals of the partition \( \xi_{n-1} \). Define a piecewise constant function \( L^{m,i}(y) \) on \([a_i, b_i]\) which takes constant values on the atoms of the partition \( \xi_{m-1} \), such that

\[
L^{m,i}|_{[c^{(k-1)}, d^{(k-1)}]} = \frac{c^{(k-1)} - a_i}{x_i - a_i} - \frac{1}{2},
\]

if \([c^{(k-1)}, d^{(k-1)}] \in \xi_{m-1}\) and \([c^{(k-1)}, d^{(k-1)}] \subset [a_i, x_i]\). Then we rewrite the sum \( P_n \) as follows

\[
P_n = \sum_{m=n+r_0+k+1}^N \sum_{i=0}^{q_n-1} \int_{a_i}^{x_i} h_m(t) \left[ \frac{t - a_i}{x_i - a_i} - \frac{1}{2} - L^{m,i}(t) \right] dt
\]

\[
+ \sum_{m=n+r_0+k+1}^N \sum_{i=0}^{q_n-1} \int_{a_i}^{x_i} h_m(t) L^{m,i}(t) dt.
\]

Denote by \( P_n^{(1)} \) and \( P_n^{(2)} \) the last two sums over \( m \), respectively. First we estimate the sum \( P_n^{(2)} \). Since \([a_i, x_i] \subset [a_i, b_i] \in \xi_n\), the interval \([a_i, x_i]\) is covered by intervals of the partition \( \xi_{m-1} \). Denote by \( I_i^{(m-1)} \) the interval of the partition \( \xi_{m-1} \) containing the point \( x_i \). If there are two such intervals then we consider the left one. Applying the second assertion of Theorem 4.3, we obtain:

\[
|P_n^{(2)}| \leq \sum_{m=n+r_0+k+1}^N \sum_{i=0}^{q_n-1} L^{m,i}(I_i^{(m-1)}) \int_{I_i^{(m-1)}} |h_m(t)| dt
\]

\[
+ \sum_{m=n+r_0+k+1}^N \sum_{i=0}^{q_n-1} \int_{I_i^{(m-1)}} |h_m(t)| dt \leq C \sum_{m=n+r_0+k+1}^\infty \int_{I_i^{(m-1)}} |h_m(t)| dt,
\]

where \( U_m = \bigcup_{i=0}^{q_n-1} I_i^{(m-1)} \). Lemma 4.11 implies that \( \ell(U_m) \leq \lambda_1^{m-n-1} \). We have

\[
\sum_{m=n+r_0+k+1}^\infty \int_{U_m} |h_m(t)| dt \leq \sum_{m=n+k+1}^\infty \|h_m\|_{L^p}(\ell(U_m))^{\frac{1}{q}} \leq C \sum_{m=n+1}^\infty \lambda_2^{m-n-1} \|h_m\|_{L^p} = \eta_n,
\]

where \( \lambda_2 = \lambda_1^{1/kq} \). Finally, \( |P_n^{(2)}| \leq \eta_n \) and \( \{\eta_n\} \in l_2 \), due to Proposition 4.6.

Since \( m \geq n + r_0 + k + 1 \), Corollary 4.9 implies that

\[
|t - a_i| < \frac{1}{2} - L^{m,i}(t) = \left| \frac{t - c^{(m-1)}}{x_i - a_i} \right| \leq \left| \frac{d^{(m-1)} - c^{(m-1)}}{x_i - a_i} \right| \leq \lambda_1^{m-n-r_0-2}, \tag{16}
\]

\( \square \) Springer
for all \( t \in [c^{(m-1)}, d^{(m-1)}] \subset [a_i, x_i] \), with \( \lambda_1 = \frac{1}{k} \). Using this estimate, we obtain:

\[
|P_n^{(1)}| \leq \sum_{m=n+r_0+k+1}^{N} \lambda_1^{m-n-r_0-2} \sum_{i=0}^{q_n-1} \left( \int_{t^*(\gamma_0)}^t |h_m(t)| dt \right)^{1/p} \leq C \sum_{m=n}^{\infty} \lambda_2^{m-n} \|h_m\|_p = \eta_n,
\]

where we have used

\[
U_{r_0} = \bigcup_{i=0}^{q_n-1} I(i), \quad \text{and} \quad \ell(U_{r_0}) < \lambda_1^{n+r_0-n} = \lambda_1^{r_0}.
\]

By Proposition 4.6, \( \{\eta_n\} \in l_2 \). This completes the proof. \( \square \)

Set

\[
E_n := \sum_{i=0}^{q_n-1} \left( 1 - z_i \right) \int_{x_i}^{x_i+1} \frac{f''(t)}{f'(t)} \frac{t - a_i}{x_i - a_i} dt - z_i \int_{x_i}^{x_i+1} \frac{f''(t)}{f'(t)} \frac{b_i - t}{b_i - x_i} dt.
\]

**Remark 5.2** Using the same arguments for estimating \( |S_n^{(1)}| \), one can show that \( |E_n| = O(\lambda^n + \eta_n) \).

Now we define

\[
Q_n := \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left[ \frac{f''(t)}{f'(t)} \frac{t - a_i}{(x_i - a_i)^2} dt - \left( \Phi_0 - \sum_{m=1}^{N} h_m(t) \right) \frac{b_i - t}{(b_i - x_i)^2} \right] dx_i.
\]

**Proposition 5.3** Let \( f \in \mathbb{R}_K^O \). Then we have \( |Q_n| = O(\lambda^n + \eta_n) \), where \( \lambda \in (0, 1) \) and \( \eta_n \in l_2 \) is from Proposition 4.6.

**Proof** It is clear that

\[
Q_n \leq \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left[ \frac{f''(t)}{f'(t)} \frac{t - a_i}{(x_i - a_i)^2} dt - \left( \Phi_0 - \sum_{m=1}^{N} h_m(t) \right) \frac{b_i - t}{(b_i - x_i)^2} \right] dx_i + \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \frac{1}{x_i - a_i} \sum_{m=1}^{N} h_m(t) \left( \frac{a_i - d_i}{x_i - a_i} \right) dt - \frac{1}{b_i - x_i} \sum_{m=1}^{N} h_m(t) \left( \frac{b_i - t}{b_i - x_i} \right) dt + \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left( \Phi_0 - \sum_{m=1}^{N} h_m(t) \right) \frac{b_i - t}{(b_i - x_i)^2} \right] dx_i.
\]

Denote by \( Q_n^{(1)} \) and \( Q_n^{(2)} \) the last two sums over \( i \) in (5), respectively. Let us first estimate \( Q_n^{(1)} \). Using Hölder’s inequality for the integrals over \([a_i, x_i] \) and \([x_i, b_i] \) in \( Q_n^{(1)} \) we get

\[
Q_n^{(1)} \leq \frac{4}{\sqrt{3}} \sum_{i=0}^{q_n-1} \left( \int_{a_i}^{b_i} \left( \frac{f''(t)}{f'(t)} \frac{t - a_i}{(x_i - a_i)^2} \right)^2 dt \right)^{1/2} \cdot \sqrt{b_i - a_i}.
\]
Again using Hölder’s inequality for the last sum we obtain:

\[
Q_n^{(1)} \leq \frac{4}{\sqrt{3}} \left\| \frac{f''}{f'} - \Phi_0 - \sum_{m=1}^{N} h_m \right\|^2_2.
\]

The first assertion of Theorem 4.3 implies that

\[
\lim_{N \to +\infty} \left\| \frac{f''}{f'} - \Phi_0 - \sum_{m=1}^{N} h_m \right\|_2 = 0.
\]

Then we choose sufficiently a large number \( N \) such that

\[
\left\| \frac{f''}{f'} - \Phi_0 - \sum_{m=1}^{N} h_m \right\|_2 \leq \lambda_2^n.
\]

Hence \( Q_n^{(1)} \) is bounded above by \( C\lambda_2^n \).

To estimate \( Q_n^{(2)} \), we split the integrand into three terms with summations over \( 1 \leq m \leq n + r_0 \), \( n + r_0 + 1 \leq m \leq n + r_0 + k \), and \( n + r_0 + k + 1 \leq m \leq N \), where \( r_0 \) was defined in (15). Denote the corresponding sums by \( T_1, T_2, T_3 \). Then \( Q_n^{(2)} \leq T_1 + T_2 + T_3 \). Consider first the sums over \( m \) from 1 to \( n + r_0 \). The piecewise constant function \( h_m(t) \) takes constant values on the atoms of the partition \( \xi_m \). Since \([a_i, x_i] \subset I^{(n+r_0)}_{\alpha, i} \subset I_{\alpha, i} \in \xi_n\), the function \( h_m(t) \) takes constant values on the intervals \([a_i, b_i]\), \( 1 \leq m \leq n + r_0 \), i.e. \( h_m([a_i, b_i]) = h_m(a_i) \). Then we have

\[
T_1 = \sum_{i=0}^{n-1} \sum_{m=1}^{b_i} h_m(a_i) \int_{a_i}^{x_i} \left[ \int_{a_i}^{x_i} h_m(y) \frac{y-a_i}{(x_i-a_i)^2} dy \right] dx_i
\]

\[
= \sum_{i=0}^{n-1} \sum_{m=1}^{n+r_0} h_m(a_i) \int_{a_i}^{x_i} \left[ \int_{a_i}^{x_i} h_m(y) \frac{y-a_i}{(x_i-a_i)^2} dy \right] dx_i = 0,
\]

where we used, that the difference of two integrals in the last sum vanishes.

Consider next the sum \( T_2 \). Using Holder’s inequality for the integral and for the sum we obtain:

\[
T_2 \leq \sum_{m=n+r_0+1}^{n+r_0+k} \sum_{i=0}^{b_i} \int_{a_i}^{x_i} \left( \int_{a_i}^{x_i} |h_m(t)|^p dt \right)^{\frac{1}{p}} (x_i - a_i)^{\frac{1}{q}-1} dx_i
\]

\[
+ \sum_{m=n+r_0+1}^{n+r_0+k} \sum_{i=0}^{b_i} \int_{a_i}^{x_i} \left( \int_{a_i}^{x_i} |h_m(t)|^p dt \right)^{\frac{1}{p}} (b_i - x_i)^{\frac{1}{q}-1} dx_i
\]

\[
\leq 2q \sum_{m=n+r_0+1}^{n+r_0+k} \sum_{i=0}^{b_i} \left( \int_{a_i}^{x_i} |h_m(t)|^p dt \right)^{\frac{1}{p}} (b_i - a_i)^{\frac{1}{q}} \leq 2q \sum_{m=n+r_0+1}^{n+r_0+k} \|h_m\|_p.
\]

Since \( \|h_m\|_p \in l_2 \) and \( k \) is a fixed number, the last sum also belongs to \( l_2 \).

Next we consider the sum \( T_3 \), i.e. the sum over \( n + r_0 + k + 1 \leq m \leq N \). Notice, that each interval \([a_i, b_i]\) in \( \xi_n \) is the union of a finite number of intervals of the partition \( \xi_{m-1} \). Define
piecewise constant functions $\tilde{L}^{m,i}(t)$ and $\tilde{M}^{m,i}(t)$ on $[a_i, x_i]$ and $[x_i, b_i]$, respectively, which are approximations of the integrands $Q_n^{(2)}$ in the corresponding intervals and take constant values on the atoms of $\xi_{m-1}$, as follows

$$\tilde{L}^{m,i}|_{[c^{(m-1)}, d^{(m-1)}]} = \frac{c^{(m-1)} - a_i}{x_i - a_i},$$

if $[c^{(m-1)}, d^{(m-1)}] \in \xi_{m-1}$ and $[c^{(m-1)}, d^{(m-1)}] \subset [a_i, x_i]$, respectively

$$\tilde{M}^{m,i}|_{[c^{(m-1)}, d^{(m-1)}]} = \frac{b_i - d^{(m-1)}}{b_i - x_i},$$

if $[c^{(m-1)}, d^{(m-1)}] \in \xi_{m-1}$ and $[c^{(m-1)}, d^{(m-1)}] \subset [x_i, b_i]$.

Then we have

$$T_3 \leq \sum_{m=n+r_0+k+1}^{N} \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{x_i - a_i} \int_{a_i}^{x_i} h_m(t) \left( \frac{t - a_i}{x_i - a_i} - \tilde{L}^{m,i}(t) \right) dt \right| \sum_{m=n+r_0+k+1}^{N} \int_{n}^{r_1} h_m(t) \left| \tilde{M}^{m,i}(t) \right| dx_i$$

$$+ \sum_{m=n+r_0+k+1}^{N} \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{b_i - x_i} \int_{x_i}^{b_i} h_m(t) \left( \frac{b_i - t}{b_i - x_i} - \tilde{M}^{m,i}(t) \right) dt \right| \sum_{m=n+r_0+k+1}^{N} \int_{x_i}^{b_i} h_m(t) \left| \tilde{M}^{m,i}(t) \right| dx_i$$

$$+ \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{x_i - a_i} \int_{a_i}^{x_i} h_m(t) \left( \tilde{L}^{m,i}(t) \right) dt \right| \sum_{m=n+r_0+k+1}^{N} \int_{x_i}^{b_i} h_m(t) \left| \tilde{M}^{m,i}(t) \right| dx_i$$

$$+ \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{b_i - x_i} \int_{x_i}^{b_i} h_m(t) \left( \tilde{M}^{m,i}(t) \right) dt \right| \sum_{m=n+r_0+k+1}^{N} \int_{a_i}^{b_i} h_m(t) \left| \tilde{M}^{m,i}(t) \right| dx_i$$

(18)

Denote by $J_n^{(1)}$, $J_n^{(2)}$, $J_n^{(3)}$ the three double sums in (18). Consider first the sum $J_n^{(3)}$. Recall, that $[a_i, x_i] \subset [a_i, b_i]$ and the interval $[a_i, x_i]$ is covered by intervals of the partition $\xi_{m-1}$. If $x_i$ lies on the boundary of one of the intervals of $\xi_{m-1}$, then by the second assertion of Theorem 4.3 $J_n^{(3)}$ vanishes. If the point $x_i$ lies inside of some interval of $\xi_{m-1}$, we denote this interval by $I^{(m-1)}(x_i)$ and get

$$J_n^{(3)} \leq \sum_{m=n+r_0+k+1}^{N} \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{x_i - a_i} \sum_{m=n+1}^{r_1} h_m(t) \right| \sum_{m=n+1}^{r_1} \tilde{L}^{m,i}(t) \int_{x_i}^{b_i} \left| \tilde{M}^{m,i}(t) \right| dx_i$$

$$+ \sum_{m=n+r_0+k+1}^{N} \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{b_i - x_i} \sum_{m=n+1}^{r_1} h_m(t) \right| \sum_{m=n+1}^{r_1} \tilde{M}^{m,i}(t) \int_{a_i}^{b_i} \left| \tilde{L}^{m,i}(t) \right| dx_i$$

$$+ \sum_{m=n+r_0+k+1}^{N} \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{x_i - a_i} \sum_{m=n+1}^{r_1} h_m(t) \right| \sum_{m=n+1}^{r_1} \tilde{M}^{m,i}(t) \int_{a_i}^{b_i} \left| \tilde{L}^{m,i}(t) \right| dx_i$$

$$+ \sum_{m=n+r_0+k+1}^{N} \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left| \frac{1}{b_i - x_i} \sum_{m=n+1}^{r_1} h_m(t) \right| \sum_{m=n+1}^{r_1} \tilde{M}^{m,i}(t) \int_{a_i}^{b_i} \left| \tilde{L}^{m,i}(t) \right| dx_i$$

$$= (I) + (II) + (III) + (IV).$$

By the second assertion of Theorem 4.3 the sums (I) and (III) are equal to zero. We estimate only sum (II), the sum (IV) is estimated analogously. Note that the step function $\tilde{L}^{m,i}$ is
bounded above by 1. Using Hölder’s inequality for the second (interior) integral in (II), we obtain:

\[
\int_{\tilde{I}^{(m-1)}(x_i)} |h_m(t)| dt \leq \left( \int_{\tilde{I}^{(m-1)}(x_i)} |h_m(t)|^p dt \right)^{1/p} \cdot \left( \tilde{I}^{(m-1)}(x_i) \right)^{1/q}.
\]

We take the maximum of the integral

\[
\int_{\tilde{I}^{(m-1)}(x_i)} |h_m(t)|^p dt
\]

over \(x_i\). Then, after multiplying with \(|\tilde{I}^{(m-1)}(x_i)|^{1/q}\), we simplify as follows

\[
\frac{|\tilde{I}^{(m-1)}(x_i)|^{1/q}}{x_i - a_i} \leq \left| \frac{|\tilde{I}^{(m-1)}(x_i)|^{1/q}}{|I^{n \cap r_0}|^{1/q}} \cdot (x_i - a_i) \right|^{1/q - 1} \leq \lambda_2^{\frac{m-n-r_0}{2}} \cdot (x_i - a_i)^{\frac{1}{q} - 1},
\]

where we have used Corollary 4.9 and the definition of \(r_0\). After these preparations we have

\[
(II) \leq q \sum_{m=n+1}^{N} \lambda_2^{\frac{m-n-r_0}{2}} \sum_{i=0}^{q_n-1} \max_{a_i \leq t \leq b_i} \left( \int_{\tilde{I}^{(m-1)}(x_i)} |h_m(t)|^p dt \right)^{1/p} (b_i - a_i)^{\frac{1}{q} - 1} \leq C \sum_{m=n}^{\infty} \lambda_2^{\frac{m-n}{2}} \|h_m\|_p \leq C \sum_{m=n}^{\infty} \lambda_2^{\frac{m-n}{2}} \|h_m\|_p = \eta_n.
\]

Finally, due to the Proposition 4.6, \(|J_n^{(3)}| \leq \eta_n\) and \(\{\eta_n\} \in l_2\).

We next estimate \(J_n^{(1)}\) in (18), \(J_n^{(2)}\) is estimated analogous. Using inequality (16) and Hölder’s inequality for the interior integral over \([a_i, x_i]\) in \(J_n^{(1)}\), we obtain

\[
\frac{b_i}{x_i - a_i} \int_{a_i}^{x_i} h_m(t) \left( \frac{t - a_i}{x_i - a_i} - \tilde{L}^{m,i}(t) \right) dt \, dx_i \leq \lambda_1^{m-n-r_0-3} \left( \int_{a_i}^{b_i} |h_m(t)|^p dt \right)^{1/p} \left( \int_{a_i}^{b_i} (x_i - a_i)^{1/q - 1} dx \right) \]

\[
= q \lambda_1^{m-n-r_0-3} \left( \int_{a_i}^{b_i} |h_m(t)|^p dt \right)^{1/p} (b_i - a_i)^{\frac{1}{q} - 1}.
\]

Then, using Hölder’s inequality for the sum over \(i\) in \(J_n^{(1)}\), we get

\[
|J_n^{(1)}| \leq C \sum_{m=n+1}^{\infty} \lambda_1^{\frac{m-n-r_0}{1}} \left( \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} |h_m(t)|^p dt \right)^{1/p} \left( \sum_{i=0}^{q_n-1} (b_i - a_i) \right)^{1/q} \leq \eta_n.
\]

We are done. \(\square\)

Set

\[
U_n := \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} \left[ \int_{x_i}^{x_i} \left( f''(x_i) - f''(t) \right) \frac{t - a_i}{(x_i - a_i)^2} \, dt + \int_{x_i}^{b_i} \left( f''(x_i) - f''(t) \right) \frac{b_i - t}{(b_i - x_i)^2} \, dt \right] dx_i.
\]
\textbf{Remark 5.4} Using the same arguments for estimating $Q_n$, one can show, that $|U_n| = O(\lambda^n + \eta_n)$. Note, that here the differences of $f''$ in $U_n$ allow us to use the martingale expansion.

\section{6 Estimates for $\tau_n(z_0)$}

In this section we will obtain some estimates for the sum $\tau_n(z_0)$ defined in (20). More specifically, the estimates for $\tau_n(z_0)$ are reduced to the estimates in Propositions 5.1, 5.3 and Remarks 5.2, 5.4.

Define

$$A_i := -\frac{1}{f''(a_i)(x_i-a_i)} \int_{a_i}^{x_i} f''(t)(t-a_i)dt + \frac{1}{f''(b_i)(b_i-x_i)} \int_{x_i}^{b_i} f''(t)(b_i-t)dt$$

$$1 + \frac{1}{f''(a_i)(b_i-a_i)} \int_{a_i}^{b_i} f''(t)(b_i-t)dt$$

$$N_i := \int_{a_i}^{b_i} f''(t) \frac{1}{2 f''(t)} dt, \quad \psi_i(z_0) = N_i - \log \left( \frac{1 + A_i z_i}{1 + A_i (z_i - 1)} \right), \quad \tau_n(z_0) := \sum_{i=0}^{n-1} \psi_i(z_0).$$

(20)

\textbf{Proposition 6.1} Suppose that $f \in B^{K,0}$. Then the following estimates hold for $\tau_n(z_0)$ and its derivatives

$$\max_{0 \leq z_0 \leq 1} |\tau_n(z_0)| \leq \delta_n, \quad \max_{0 \leq z_0 \leq 1} |(z_0 - z_0^2)\tau_n'(z_0)| \leq \delta_n,$$

(21)

$$\int_{[0,1]} |\tau_n'(z_0)| dz_0 \leq \delta_n, \quad \int_{[0,1]} |(z_0 - z_0^2)|^{\tau_n''(z_0)}|d z_0 \leq \delta_n,$$

(22)

where $\delta_n = O(\lambda^n + \eta_n)$, $\lambda \in (0, 1)$ and $\eta_n \in l_2$ is from proposition 4.6.

\textbf{Proof} Denote by $V_i$ the second term in the denominator of $A_i$. Using Hölder’s inequality we get

$$|V_i| \leq \frac{1}{f''(a_i)(b_i-a_i)} \int_{a_i}^{b_i} |f''(t)|(b_i-t)dt \leq C(b_i-a_i)^{1/q}, \quad \text{where} \quad q = \frac{p}{p-1}.$$

In analogy one can show, that the absolute values of both terms of the numerator of $A_i$ are bounded by $C(b_i-a_i)^{1/q}$. Since $[a_i, b_i]$ is an interval of the partition $\xi_n$, by corollary 4.9 its length is not larger than $C \lambda^{\frac{1}{q}}$. Hence

$$|V_i| = O(\lambda_1^n), \quad |A_i| = O(\lambda_1^n), \quad \text{where} \quad \lambda_1 = \lambda^{\frac{1}{q}}.$$

We rewrite $\tau_n(z_0)$ as follows

$$\tau_n(z_0) = \sum_{i=0}^{n-1} N_i - \sum_{i=0}^{n-1} \log \left( \frac{1 + A_i z_i}{1 + A_i (z_i - 1)} \right) = - \log m_n - \sum_{i=0}^{n-1} A_i - \sum_{i=0}^{n-1} O(A_i^2).$$

(23)
To estimate the last sum in (23) as follows
\[ \left| \sum_{i=0}^{q_n-1} A_i^n \right| \leq 2 \sum_{i=0}^{q_n-1} \frac{1}{(1+V_i)^2} \left\{ \left( \int_{x_i}^{x_i} f''(t)(t-a_i) f'(a_i)(x_i-a_i) \, dt \right)^2 + \left( \int_{x_i}^{x_i} f''(t)(b_i-t) f'(a_i)(b_i-x_i) \, dt \right)^2 \right\} \]
\[
\leq C \| f'' \|_\rho \sum_{i=0}^{q_n-1} \left\{ (x_i-a_i) \int_{a_i}^{x_i} |f''(t)| \, dt + (b_i-x_i) \int_{x_i}^{b_i} |f''(t)| \, dt \right\} 
\]
\[
\leq C \cdot \max_{0 \leq i \leq q_n} |b_i-a_i| \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} |f''(t)| \, dt \leq C \lambda_1^n. \tag{24}
\]

To estimate the sum \( \sum_{i=0}^{q_n-1} A_i \) we rewrite it in the following form:
\[
\sum_{i=0}^{q_n-1} A_i = - \sum_{i=0}^{q_n-1} \frac{1}{1+V_i} \left\{ \int_{x_i}^{x_i} f''(t)(t-a_i) f'(a_i)(x_i-a_i) \, dt + \int_{x_i}^{b_i} f''(t)(b_i-t) f'(a_i)(b_i-x_i) \, dt \right\} 
\]
\[
= - \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} f''(t) \, dt - \sum_{i=0}^{q_n-1} \int_{a_i}^{b_i} f''(t) \, dt 
\]
\[
+ \sum_{i=0}^{q_n-1} \frac{V_i}{1+V_i} \left[ \int_{a_i}^{x_i} f''(t)(t-a_i) f'(a_i)(x_i-a_i) \, dt + \int_{x_i}^{b_i} f''(t)(b_i-t) f'(a_i)(b_i-x_i) \, dt \right] 
\]
\[
+ \frac{1}{f'(a_i)(b_i-x_i)} \int_{x_i}^{b_i} f''(t)(b_i-t) \, dt. \tag{25}
\]

The first sum after the second equality sign gives \((- \log m_n)\). Since \(|V_i| = O(\lambda_1^n)\), the absolute value of the last sum in (25) is bounded above by \(C \| f'' \|_\lambda_1^n\). Denote by \(S_n\) respectively \(\bar{S}_n\), the second and third sums after the last equality sign in (25). Then we obtain
\[
\sum_{i=0}^{q_n-1} A_i = - \log m_n - S_n - \bar{S}_n + O(\lambda_1^n). \]

We rewrite the sum \(S_n\) in the following form:
\[
S_n = \sum_{i=0}^{q_n-1} \left[ \int_{a_i}^{x_i} f''(t) \frac{t-a_i}{f'(t) x_i-a_i} \, dt - \int_{a_i}^{x_i} f''(t) \frac{t}{2 f''(t)} \, dt \right] 
\]
\[
+ \sum_{i=0}^{q_n-1} \int_{a_i}^{x_i} f''(t)(t-a_i) f'(a_i)(x_i-a_i) \, dt \int_{a_i}^{t} f''(s) \, ds \equiv S_n^{(1)} + S_n^{(2)}. \]
Using Holder’s inequality for the integral we obtain:

\[ |S_n^{(2)}| \leq C \cdot \sum_{i=0}^{q_n-1} \left( \int_{a_i}^{b_i} |f''(t)| dt \right)^2 \leq C \|f''\|_p \cdot \max_{0 \leq i \leq q_n} |b_i - a_i|^{1/q} \sum_{i=0}^{q_n-1} \left( \int_{a_i}^{b_i} |f''(t)| dt \right) \leq C\lambda_2^n. \]

where \( \lambda_2 = \lambda^{\frac{1}{2}}. \) This together with Proposition 5.1 imply that \( |S_n| \leq \delta_n. \) Analogously one can show, that \( |\bar{S}_n| \leq \delta_n. \) So we get the first estimate in (21).

As seen from their definitions in (19) and (20), the functions \( A_i \) and \( \psi_i \) depend on the variable \( x_i \), which is linear in the variable \( z_i \). Therefore \( A_i, \psi_i \) themselves depend on \( z_i \). Calculating the derivatives of \( \psi_i \) and \( A_i \) we get

\[ \frac{d\psi_i}{dz_i} = \frac{A_i^2 - A_i'}{(1 + A_i z_i)(1 + A_i (z_i - 1))}, \]

\[ A_i' = \frac{dA_i}{dz_i} = (b_i - a_i) \frac{dA_i}{dx_i}, \] (26)

where

\[ \frac{dA_i}{dx_i} = \frac{1}{f'(a_i)(x_i - a_i)^2} \int_{a_i}^{x_i} f''(t)(t - a_i) dt - \frac{1}{f'(a_i)(b_i - x_i)^2} \int_{x_i}^{b_i} f''(t)(b_i - t) dt \]

\[ 1 + \frac{1}{f'(a_i)(b_i - a_i)} \int_{a_i}^{b_i} f''(t)(b_i - t) dt \]

(28)

Consider now

\[ |(z_0 - z_0^2) \frac{d\tau_n(z_0)}{dz_0}| = |(z_0 - z_0^2) \sum_{i=0}^{q_n-1} \frac{d\psi_i}{dz_i} \cdot \frac{dz_i}{dz_0}| \]

Since \( |A_i| = O(\lambda_1^n) \), the denominator of the right hand side in (26) is bounded. Relation (24) implies, that the sum corresponding \( A_i^2 \) is not greater than \( C\lambda_1^n. \) As in rewriting \( S_n \), we change the \( f'' \) (in the integrals in the numerator of \( A_i' \) in (28)) to \( \frac{f''}{f'} \) in the sum \( E_n \) (see Remark 5.2). Then relations (27)-(28), and Lemma 4.13 imply that

\[ |(z_0 - z_0^2) \frac{d\tau_n(z_0)}{dz_0}| \leq C\lambda_1^n + C \sum_{i=0}^{q_n-1} (z_i - z_i^2) \left| \frac{dA_i}{dz_i} \right| \leq C\lambda_2^n + E_n. \]

This together with Remark 5.2 imply the second relation in (21).

It is clear that

\[ \int_0^1 |\tau_n'(z_0)|dz_0 = \int_0^1 \left| \sum_{i=0}^{q_n-1} \frac{d\psi_i}{dz_i} \cdot \frac{dz_i}{dz_0} \right| dz_0 \leq C\lambda_1^n + \int_0^1 \left| \sum_{i=0}^{q_n-1} \frac{dA_i}{dz_i} \right| dz_0. \]

As when rewriting \( S_n \), we change the \( f'' \) under the integrals in the numerator of \( A_i' \) to \( \frac{f''}{f'} \) in the last sum. Then using second relation in (13), together with (27)-(28) and substituting \( z_i = \frac{x_i - a_i}{b_i - a_i} \), we get \( \int_0^1 |\tau_n'(z_0)|dz_0 = O(Q_n + \lambda_1^n) \). The latter equality and Proposition 5.3 imply the first inequality in (22).
Differentiating (26), (27), (28) we obtain:

\[
\frac{d^2 \psi_i}{dz_i^2} = \frac{2A_i A'_i - A''_i}{(1 + A_i z_i)(1 + A_i (z_i - 1))} - \frac{2(A'_i z_i + A_i)}{1 + A_i z_i} \cdot \frac{d \psi_i}{d z_i} - \left( \frac{d \psi_i}{d z_i} \right)^2.
\]

(29)

\[
A''_i = \frac{d^2 A_i}{dx_i^2} = (b_i - a_i)^2 \frac{d^2 A_i}{dx_i^2}.
\]

(30)

where

\[
\frac{d^2 A_i}{dx_i^2} = \int_{a_i}^{b_i} \frac{2(f''(x_i) - f''(t)(t - a_i))}{(x_i - a_i)^2} dt + \int_{x_i}^{b_i} \frac{2(f''(x_i) - f''(t)(b_i - t))}{(b_i - x_i)^2} dt.
\]

(31)

We have

\[
\tau''_n(z_0) = \sum_{i=0}^{q_n-1} \left( \frac{d^2 \psi_i}{dz_i^2} \cdot \left( \frac{d \psi_i}{d z_i} \right)^2 + \frac{d \psi_i}{d z_i} \cdot \frac{dz_i^2}{d z_0^2} \right).
\]

The first relation in (22), relation (29) and Lemma 4.13 imply that

\[
\int_0^1 \left| (z_0 - z_0^2) \tau''_n(z_1) \right| dz_0 \leq C \int_0^1 \left| (z_i - z_i^2) \sum_{i=0}^{q_n-1} (b_i - a_i)^2 \frac{d^2 A_i}{dx_i^2} \right| dz_0 + \delta_n
\]

\[
\leq C \int_0^1 \left| (z_i - z_i^2) \sum_{i=0}^{q_n-1} \left( \frac{b_i - a_i}{x_i - a_i} \right)^2 \int_{a_i}^{x_i} \left[ f''(x_i) - f''(t) \right] \frac{t - a_i}{x_i - a_i} dt \right| dz_0
\]

\[
+ C \int_0^1 \left| (z_i - z_i^2) \sum_{i=0}^{q_n-1} \left( \frac{b_i - a_i}{x_i - a_i} \right)^2 \int_{x_i}^{b_i} \left[ f''(x_i) - f''(t) \right] \frac{b_i - t}{b_i - x_i} dt \right| dz_0 + \delta_n
\]

Hence, by Lemma 4.13 and substituting \( z_i = \frac{x_i - a_i}{b_i - a_i} \) in the last integral, we get

\[
\int_0^1 \left| (z_0 - z_0^2) \tau''_n(z_1) \right| dz_0 = O(U_n + \delta_n).
\]

This and Remark 5.4 imply the second relation in (22).

\[\square\]

7 Proofs of Main Theorems

Before giving the proof of the main results, we approximate relative coordinates \( z_{q_n} \) by Möbius functions. Consider an arbitrary fundamental segment \( I_{\alpha}^{(n)} = [a; b] \) of the \( n \)-th basic partition \( P^{(n)} \). Recall, that we have introduced the relative coordinates \( z_i : [f^i(a), f^i(b)] \rightarrow \)
Lemma 7.1 Suppose that \( f \in \mathbb{R}^{K,O} \). Then the following approximations hold

\[
\|z_{q_n} - F_n\|_{C^1([0,1])} \leq \delta_n, \quad \|z_{q_n}' - F_n''\|_{L_1([0,1], dt)} \leq \delta_n,
\]

where \( F_n \) is defined in (6) and \( \{\delta_n\} \in l_2 \).

**Proof** In the following we use the following notations:

\[
a_i = f^i(a), \quad b_i = f^i(b), \quad x_i = f^i(x) \in [a_i, b_i].
\]

The points \( f^i(x) \in I^{(n)}_{\alpha,i} \) are mapped by \( f \) to the points \( f^{i+1}(x) \in I^{(n)}_{\alpha,i+1} \), with relative coordinates \( z_{i+1} \). Then one has for the relative coordinates \( z_i \) and \( z_{i+1} \) of the points \( f^i(x) \) respectively \( f^{i+1}(x) \) in the interval \([a_i, b_i]\) respectively \([a_{i+1}, b_{i+1}]\):

\[
z_i = \frac{x_i - a_i}{b_i - a_i}, \quad z_{i+1} = \frac{x_{i+1} - a_{i+1}}{b_{i+1} - a_{i+1}}.
\]

It is clear, that

\[
a_{i+1} = f(a_i), \quad x_{i+1} = f(x_i) = f(a_i) + f'(a_i)(x_i - a_i) + \int_{a_i}^{x_i} f''(t)(x_i - t) dt,
\]

\[
b_{i+1} = f(b_i) = f(a_i) + f'(a_i)(b_i - a_i) + \int_{a_i}^{b_i} f''(t)(b_i - t) dt.
\]

Using this, we rewrite \( z_{i+1} \) as follows

\[
z_{i+1} = \frac{f'(a_i)(x_i - a_i) + \int_{a_i}^{x_i} f''(t)(x_i - t) dt}{f'(a_i)(b_i - a_i) + \int_{a_i}^{b_i} f''(t)(b_i - t) dt}.
\]

\[
= \frac{x_i - a_i}{b_i - a_i} \left( 1 + \frac{(b_i - a_i) \int_{a_i}^{x_i} f''(t)(x_i - t) dt - (x_i - a_i) \int_{a_i}^{b_i} f''(t)(b_i - t) dt}{f'(a_i)(b_i - a_i)(x_i - a_i) + (x_i - a_i) \int_{a_i}^{b_i} f''(t)(b_i - t) dt} \right)
\]

\[
= z_i(1 + A_i(z_i - 1)),
\]

where \( A_i \) was defined in (19). It follows that

\[
\frac{1 - z_{i+1}}{z_{i+1}} = \frac{1 - z_i}{z_i} \cdot \frac{1 + A_i z_i}{1 + A_i(z_i - 1)} = \frac{1 - z_i}{z_i} \exp[N_i] \exp[-\psi_i].
\]
Iterating this equation we obtain
\[
\frac{1 - z_{qn}}{z_{qn}} = \frac{1 - z_0}{z_0} \exp \left\{ \frac{q_n - 1}{\sum_{i=0}^{q_n-1} N_i} \right\} \exp \left\{ - \frac{q_n - 1}{\sum_{i=0}^{q_n-1} \psi_i} \right\} = \frac{1 - z_0}{z_0} \cdot \frac{1}{m_n \exp(\tau(q_n)(z_0))}.
\]

Solving for \(z_{qn}\) we get
\[
z_{qn} = \frac{z_0 m_n e^{\tau_n(z_0)}}{1 + z_0 (m_n e^{\tau_n(z_0)} - 1)}.
\]

A not too hard calculation show, that
\[
z_{qn}'(z_0) = \frac{m_n \exp(\tau_n(z_0))(1 + z_0 (1 - z_0) \tau_n'(z_0))}{(1 + z_0 (m_n \exp(\tau_n(z_0)) - 1))^2}, \quad F_n'(z_0) = \frac{m_n}{(1 + z_0 (m_n - 1))^2}.
\]

Then, using the estimates for \(\tau_n(z_0)\) in Proposition 6.1, we get the first relation in (32). Similarly,
\[
z_{qn}''(z_0) = \frac{m_n \exp(\tau_n(z_0))(z_0 - z_0^2 \tau_n''(z_0))}{(1 + z_0 (m_n \exp(\tau_n(z_0)) - 1))^2} + \frac{2m_n \exp(\tau_n(z_0))(1 - z_0^2 - (2z_0 - z_0^2)m_n \exp(\tau_n(z_0))) \tau_n'(z_0)}{(1 + z_0 (m_n \exp(\tau_n(z_0)) - 1))^3}
\]
\[+ \frac{(1 - 2z_0 m_n \exp(\tau_n(z_0)))(z_0 - z_0^2 \tau_n'(z_0))^2}{(1 + z_0 (m_n \exp(\tau_n(z_0)) - 1))^3} - \frac{2m_n \exp(\tau_n(z_0))(m_n \exp(\tau_n(z_0)) - 1)}{(1 + z_0 (m_n \exp(\tau_n(z_0)) - 1))^3},
\]
\[
F_n''(z_0) = -\frac{2m_n (m_n - 1)}{(1 + z_0 (m_n - 1))^3}.
\]

It is clear that the expression \(1 + z_0 (m_n \exp(\tau_n(z_0)) - 1)\) is bounded and
\[
\int_0^1 |(z_0 - z_0^2)(\tau_n'(z_0))^2| dz_0 \leq C \int_0^1 |\tau_n'(z_0)| dz_0.
\]

Then, using Proposition 6.1 and the expression for \(z_{qn}''\), we get the result. \(\Box\)

**Proof of Theorem 3.1.** By definitions of Zoom and the relative coordinates we have
\[
z_{qn}(z_0) = Z_{f^{(n)}}(R^n(f))(z_0),
\]
where \(x = a + z_0(b - a), \ z_0 \in [0, 1].\) Then due to Lemma 7.1, we get Theorem 3.1. \(\Box\)

**Proof of Theorem 3.2.** In [5] an ergodic theorem for the random process, corresponding to a symbolic representation for the elements of partition \(\xi_n\), has been proven. Note, that this theorem is also true in our (KO smoothness) case. It follows, that for any \(\alpha, \beta \in A\)
\[
\left| \sum_{i=1}^{q_n} f^i(I_A^\alpha) |f^i(I_B^\beta)| - \sum_{i=1}^{q_n} f^i(I_B^\beta) |f^i(I_A^\alpha)| \right| \leq C\lambda \sqrt{n}.
\]
For simplicity of notion we use $f_n$ to denote $R^n(f)$. It is clear that

$$- \log \sqrt{m_n} = \int_{I_n^a} \frac{D^2 f_n(t)}{Df_n(t)} \, dt. \quad (34)$$

Set $r := \lfloor \frac{n}{2} \rfloor$. We rewrite the last integral as follows:

$$\int_{I_n^a} \frac{D^2 f_n(t)}{Df_n(t)} \, dt = \sum_{\beta \in A} \sum_{j=1}^{q^\beta_n} \sum_{I^i_n \subset f^j(I^r_{\beta})} \int_{I^i_n} \frac{D^2 f(t)}{Df(t)} \, dt$$

Put

$$\Lambda_n = \left| \int_{I_n^a} \frac{D^2 f_n(t)}{Df_n(t)} \, dt - \sum_{i=1}^{q^\beta_n} \frac{|f^i(I^a_n)|}{|I|} \int_{[0,1]} \frac{D^2 f(t)}{Df(t)} \, dt \right|.$$

Then we have

$$\Lambda_n = \left| \sum_{\beta \in A} \sum_{j=1}^{q^\beta_n} \sum_{I^i_n \subset f^j(I^r_{\beta})} \int \frac{D^2 f(y)}{Df(y)} \, dy - \sum_{i=1}^{q^\beta_n} \frac{|f^i(I^a_n)|}{|I|} \int_{f^j(I^r_{\beta})} \frac{D^2 f(t)}{Df(t)} \, dt \right|$$

$$+ \left| \sum_{\beta \in A} \sum_{j=1}^{q^\beta_n} \left( \sum_{I^i_n \subset f^j(I^r_{\beta})} \frac{|f^i(I^a_n)|}{|f^j(I^r_{\beta})|} - \sum_{i=1}^{q^\beta_n} \frac{|f^i(I^a_n)|}{|I|} \right) \int_{f^j(I^r_{\beta})} \frac{D^2 f(t)}{Df(t)} \, dt \right|$$

$$= \Lambda^{(1)}_n + \Lambda^{(2)}_n.$$

Due to the relation (33) we obtain: $\Lambda^{(2)}_n = \mathcal{O}(\lambda \sqrt{n})$. We estimate next the sum $\Lambda^{(1)}_n$. Denote the endpoints of intervals $f^i(I^a_n)$, $f^j(I^r_{\beta})$ and the ratio of its lengths by

$$f^i(I^a_n) = [a_i, b_i], \quad f^j(I^r_{\beta}) = [c_j, d_j], \quad \rho_{i,j} = \frac{b_i - a_i}{d_j - c_j}, \quad 0 \leq \rho_{i,j} \leq 1.$$

We change the variable $y \in [a_i, b_i]$ over the first integral in $\Lambda^{(1)}_n$ to $t \in [c_j, d_j]$ by the formula: $y = a_i + \rho_{i,j}(t - c_j)$. Then we have

$$\Lambda^{(1)}_n = \left| \sum_{\beta \in A} \sum_{j=1}^{q^\beta_n} \frac{\sum_{I^i_n \subset f^j(I^r_{\beta})} |f^i(I^a_n)|}{|f^j(I^r_{\beta})|} \int_{f^j(I^r_{\beta})} \left( \frac{D^2 f(a_i + \rho_{i,j}(t - c_j))}{Df(a_i + \rho_{i,j}(t - c_j))} - \frac{D^2 f(t)}{Df(t)} \right) dt \right|.$$

We use the first assertion of Theorem 4.3 to get

$$\frac{D^2 f}{Df} - \Phi_0 = \sum_{m=1}^{\infty} h_m \quad \text{(in } L_1 \text{- norm)}.$$
By definition, the function $h_m(t)$ takes constant values on the atoms of the dynamical partition $\xi_m$. On the other hand, $[a_i, b_i] \subset [c_i, d_i]$. This together with $[a_i, b_i] \in \xi_n$, $[c_j, d_j] \in \xi_r$, $r = \lfloor \frac{n}{2} \rfloor < n$ imply, that $h_m(a_i + \rho_i, j(t - c_j)) = h_m(t)$, $t \in [c_j, d_j]$. Next subtracting and adding the sum $\Phi_0 + \sum_{m=1}^{N} h_m(t)$ in the integrand in $\Lambda_n^{(1)}$, we obtain:

$$\Lambda_n^{(1)} \leq 2 \sum_{\beta \in \mathcal{A}} \sum_{j=1}^{q^0_{\beta}} \int_{f^j(f^j_\beta)} \left| \frac{D^2 f(t)}{Df(t)} - \Phi_0 - \sum_{m=1}^{N} h_m(t) \right| dt \leq C \left\| \frac{D^2 f}{Df} - \Phi_0 - \sum_{m=1}^{N} h_m \right\|_1$$

Since

$$\lim_{N \to \infty} \left\| \frac{D^2 f}{Df} - \Phi_0 - \sum_{m=1}^{N} h_m \right\|_1 = 0,$$

we can choose a sufficiently large number $N$ such that

$$\left\| \frac{D^2 f}{Df} - \Phi_0 - \sum_{m=1}^{N} h_m \right\|_1 \leq \lambda_n^2.$$

Consequently, due to relation (34), we obtain: $|\log mn| = O(\lambda \sqrt{n})$.

Next define the map $\mathcal{M}_a : [0, 1] \mapsto [0, 1]$ as

$$\mathcal{M}_a(x) = \frac{xe^{-a}}{1 + x(e^{-a} - 1)},$$

One can show that the inequality $\|\mathcal{M}_a - \mathcal{M}_b\|_{C^2} \leq C|a - b|$ is fulfilled for every $a, b \in \mathbb{R}$ with $|a|, |b| \leq C$. Using this inequality for $F_n$ defined in (6), we obtain

$$\left\| F_n - Id \right\|_{C^2} \leq C|\log mn - 0| \leq C\lambda \sqrt{n}.$$ 

The last inequality and Theorem 3.1 imply the assertions of Theorem 3.2. □

8 Afterthought

At the end of the paper, we would like to comment on the further extensions and applications of our results.

(1) It is clear that the set of the irrational rotation numbers of GIEM’s which has $k$-bounded combinatorics has zero Lebesgue measure. We believe that our main Theorems 3.1 and 3.2 can be extended for Roth-type of GIEM’s (for the definition see [23]) which has a full Lebesgue measure of irrational rotation numbers.

(2) Akhadkulov et. al [1] proved that renormalizations of circle diffeomorphisms with a break satisfying certain Zygmund smoothness condition are approximated by Möbius transformations. Theorems 3.1 and 3.2 can be extended for GIEM’s satisfying the Zygmund smoothness condition defined in [1]. We expend that in this case, the estimates of Theorems 3.1 and 3.2 will be an algebraic type.

(3) Note that Katznelson and Ornstein [10] proved the absolute continuity of the conjugation between a circle diffeomorphism satisfying KO conditions with a irrational rotation number of bounded type and the rigid rotation. As we have mentioned in the Introduction,
in the proving of absolute continuity of conjugation of two circle diffeomorphisms, the basic factors are the convergence of the Renormalizations and Linearization. Recently, in [4], we have shown the convergence of the renormalizations of two GIEM’s with zero mean nonlinearities. We believe that an analogue of Katznelson and Ornstein’s result, that is, the absolute continuity of the conjugations of two GIEM’s can be shown by using Theorem 3.2 and the convergence of their renormalizations that was shown in [4].

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