AN INTEGRATED INVENTORY MODEL WITH VARIABLE HOLDING COST UNDER TWO LEVELS OF TRADE-CREDIT POLICY

MAGFURA PERVIN AND SANKAR KUMAR ROY

Department of Applied Mathematics with Oceanology and Computer Programming
Vidyasagar University, Midnapore-721102, West Bengal, India

GERHARD WILHELM WEBER
2Faculty of Engineering Management
Chair of Marketing and Economic Engineering
Poznan University of Technology
Ul. Strzelecka 11, 60-965 Poznan, Poland

ABSTRACT. This paper presents an integrated vendor-buyer model for deteriorating items. We assume that the deterioration follows a constant rate with respect to time. The vendor allows a certain credit period to buyer in order to promote the market competition. Keeping in mind the competition of modern age, the stock-dependent demand rate is included in the formulated model which is a new policy to attract more customers. Shortages are allowed for the model to give the model more realistic sense. Partial backordering is offered for the interested customers, and there is a lost-sale cost during the shortage interval. The traditional parameter of holding cost is considered here as time-dependent. Henceforth, an easy solution procedure to find the optimal order quantity is presented so that the total relevant cost per unit time will be minimized. The mathematical formation is explored by numerical examples to validate the proposed model. A sensitivity analysis of the optimal solution for important parameters is also carried out to modify the result of the model.

2010 Mathematics Subject Classification. Primary: 90B05, 91B70; Secondary: 91B24.

Key words and phrases. Integrated inventory model, Trade credit, Stock-dependent demand, Variable holding cost, Deterioration, Partial backorder.

The reviewing process of this paper was handled by Associate Editors A. (Nima) Mirzazadeh, Kharazmi University, Tehran, Iran, and Gerhard-Wilhelm Weber, Middle East Technical University, Ankara, Turkey. “This paper was for the occasion of The 12th International Conference on Industrial Engineering (ICIE 2016), which was held in Tehran, Iran during 25-26 January, 2016”. The author, Magfura Pervin is very much thankful to University Grants Commission (UGC) of India for providing financial support to continue this research work under [MANF(UGC)] scheme: Sanctioned letter number [F1-17.1/2012-13/MANF-2012-13-MUS-WES-19170/(SA-III/Website)] dated 28/02/2013.

The research of Gerhard-Wilhelm Weber (Institute of Applied Mathematics, Middle East Technical University, 06800, Ankara, Turkey) is partially supported by the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), through the CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/ 04106/2013.

* Corresponding Author: sankroy2006@gmail.com.
1. **Introduction.** Nowadays, inventory is attracting many researchers because an inventory control policy is mainly dependent on various companies which are the backbone for developing a country. *Inventory* is nothing but a stock of idle resources having an economic value. Most of the companies found that they can obtain more advantages by establishing a long-term relationship between supplier and retailer. Recently, Reliance-Jio sim card is making a huge market from customers in India, where the sim card is delivered by the retailer to the customer, not by the supplier to the customer. But, if there is a misunderstanding between the supplier and retailer, it will be impossible to enjoy such a profit from the customers. So, *coordination* between partners is a powerful promotional tool to increase the profit and also an efficient key to achieve a global optimality of the system. Hence, an integrated inventory model is an excellent idea to obtain the minimum total relevant cost with a purpose of greater success, rather than by acting separately.

In a traditional *Economic Order Quantity* (EOQ) model, it is tacitly assumed that the customer must pay for the items as soon as it is received from the retailer. But in reality, the retailer offers a certain grace period to the customer for settling the account. The customer does not have to pay any amount within the fixed period but after that time schedule; if the customer is unable to pay, the retailer charges an extra amount as interest. This process makes a great economic sense to the customer, because the customer can earn revenue and interest within the last moment of that period. On the other hand, the retailer can attract more customers by allowing that period which seems to be more profitable to him. Hence, permissible delay in payments can play a major role in inventory control for both retailer and customer.

In classical inventory models, it is assumed that the demand rate of an item be either time-dependent or constant. But in reality, one can observe that demand rate is hugely influenced by the stock level. As an example, nowadays, people prefer shopping from a supermarket rather than a small shop. In a supermarket, a customer gets an available amount of various items at a time from one shop. It consumes their time and attracts them with the idea of freshness, popularity, variety and quality. On the other hand, a low stock level in shelves can raise a question of quality and freshness of the original items. Therefore, a large amount of stock level has a dashing importance in determining an inventory model. Variable *holding cost* is attracting researchers day by day. In general, it is assumed that holding cost is known and constant. But when inventory is stored for future usage, then it is essential to maintain the physical status of the inventory at its present situation. Especially, when companies are handling deteriorating items, it is quite difficult to maintain the utility forever. In general, some products have negligible deterioration rates and, henceforth, holding cost for such a product can be neglected. However, highly deteriorative products may be subject to a significant amount of holding cost. Hence, holding cost must be taken into account for such a type of products and have an explicit importance in determining an inventory model.

In a conventional model, it is assumed that items can be stored for an infinite time interval. But in practice, every items enjoy some sort of deterioration over a particular time. The term *deterioration* is a process that prevents items from being used in their original condition. It is defined as pilferage, evaporation, pinch, vaporization, decay, spoilage, degradation, collapse or loss of its utility or original value.
It is highly found for fruits, vegetables, medicine, alcohol, blood bank and radioactive materials, but it is rarely found for toys, hardware, glassware, etc. Therefore, deterioration of items plays a major role in the determination of inventory model and must be taken into consideration.

Shortage is a natural phenomenon in an inventory model. Backlogging occurs during a shortage period due to unsatisfied demand. Partially backlogging is offered because when a shortage occurs, then customers react differently. Regarding a highly fashionable product, branded items, particular medicine, etc., customers wait for a backorder; however, there are some impatient and needy customer who think that it will be better to purchase the item from somewhere else. Moreover, the rate of partial backlogging is highly dependent on the length of the delivery interval, whereas the complete backlogging case is mainly applicable for a monopolistic market. Therefore, partial backlogging is an important consideration in an inventory management.

Hence, each term described in our model has a vital role to construct our paper and the main motivations of our paper are as follows:

- Here, we use the integrated vendor-buyer model to calculate the minimum total cost of the system.
- The selling items are perishable such as gasoline, fruits, fresh fishes, photographic films, vegetables, etc., over time and the deterioration follows a constant rate with respect to time which has a physical importance in reality.
- The parameters of holding cost are assumed as a time-dependent (particularly linear function of time) because as the changes in time, value of money and holding cost cannot remain constant over time.
- The rate of replenishment is finite.
- Stock-dependent demand approach is considered here because a large number of items in shelf attract more customers, and that interesting fact reflects more real-life situations.
- Shortages are allowed for our model to give the model in more realistic senses because we are dealing with a high competitive era.
- Partial backlogging is permitted during shortage period for those customers who are interested to wait for the backordered items.

The rest of the paper is organized as follows: In Section 2, literature review of the papers is presented. The notations and assumptions for the development of the model are provided in Subsections 3.1 and 3.2, respectively. The formulation of the model separately for vendor’s and buyer’s are described in Subsections 4.1 and 4.2, respectively. In Section 5, solution procedure of the formulated model is presented. Section 6 consists of numerical examples to illustrate our developed model. In Section 7, sensitivity analysis with respect to various parameters is carried out. Finally, the summary and the future direction of research are given in Section 8.

2. Literature review. The integrated vendor-buyer model has received much attention in the last few decades. Among them, Goyal [11] suggested a lot-for-lot policy and that the vendors economic production quantity should be an integer multiple of buyer purchase quantity. Later, Liao et al. [20] considered a probabilistic inventory model and solved the model by allowing lead time as a decision variable. After that, Ouyang et al. [23] addressed a production inventory model with lead time for a single-vendor single-buyer supply chain. Huang et al. [13] proposed a single-vendor single-buyer by allowing a permissible delay in payment.
An integrated production inventory model for ameliorating and deteriorating items is described comprehensively by Law and Wee [18].

In traditional inventory model, it is usually assumed that the retailer must pay for the items to the supplier as soon as the items are received but, in practice, the supplier is willing to offer the retailer a certain credit period without interest to promote the market competition. In recent two decades, the effect of a permissible delay in payments on the optimal inventory system has received more attention from numerous researchers. Goyal [10] firstly explored a single item EOQ model under permissible delay in payments. Aggarwal and Jaggi [1] extended Goyal’s [10] model to the case with deteriorating items. Jaggi et al. [14] established an EOQ inventory model with defective items under allowable shortages and trade credit. Mahata [22] designed an EPQ model for deteriorating items under trade-credit policy. Many researchers such as Goswami and Chaudhuri [9], Chung and Liao [4] have studied and published many articles in this area which extended a new horizon to implement the research. Pervin et al. [25] described a deteriorating EOQ model with variable demand and holding cost under the effect of trade-credit policy.

In practical situations, deterioration of items is a common phenomenon. During the replenishment period, the inventory decreases continuously due to the combined effect of demand and deterioration. Thus, while determining the EOQ model, loss may occur due to deterioration which cannot be ignored. Ghare and Schrader [6] considered no shortage inventory model with constant deterioration rate while Harris [12] was first to generate an EOQ model. Liao [19] developed an inventory control model with instantaneous receipt and exponential deteriorating item under two levels of trade-credit policy. Tripathy and Pandey [30] introduced an inventory model for deteriorating items with Weibull distribution deterioration and time-dependent demand under trade-credit policy. Lo et al. [21] presented an integrated production inventory model for imperfect items with Weibull distribution deterioration in very close detail.

In classical inventory model, we assume that demand rate is either constant or time-dependent. But in real-life situations, we see that demand rate may go up or down with respect to the stock level. Several researchers are engaged to publish articles on stock-dependent demand rate. Balkhi and Benkherouf [2] developed an inventory model for deteriorating items with stock-dependent and time-varying demand rates over a finite planning horizon. Datta and Paul [5] analyzed a multi-period EOQ model with stock-dependent, and price-sensitive demand rate. An EPQ model for deteriorating items with price and stock dependent demand is derived by Teng and Chang [29]. Recently, Pervin et al. [24] presented a deteriorating and decaying inventory model under trade-credit policy.

In traditional inventory model, it is assumed that holding cost is pre-determined and constant with time. But holding cost may not always be constant. Goh [8] first considered a stock-dependent demand model with variable holding cost and assumed the unit holding cost as a nonlinear continuous function of time. In few inventory models like Giri et al. [7], the holding cost as well as the demand function were considered as time-dependent. Pervin et al. [26] analyzed a deteriorating inventory model with stock-dependent demand and variable holding cost.

In an inventory model, whenever shortages occur, some customers may think about backorder, but other customers who experience the stock-out may leave the system and meet their demand from another system. The length of the waiting-time
interval is managed smartly behind the backlogging; the longer the length of waiting time, the less the amount of backorder. An inventory models with allowable shortages under partial backlogging condition were proposed by Chang and Dye [3]. Later, a deteriorating EOQ model with planned backorder was presented by Widyadana et al. [31]. Thereafter, Jolai et al. [15] derived a lot-size deteriorating model with inflation and partial backlogging. Then, Khalilpourazari et al. [17] discussed a multi-product EPQ model with partial backordering and some constraints. San José et al. [27] studied a model with constant and known demand and with partially backorder. Recently, Khalilpourazari and Pasandideh [16] described an EOQ model for multi-items with nonlinear holding cost and partial backordering.

Table 1 represents the research works of various authors related to this area.

Table 1: Previous works of different authors in this field including our work.

| Author(s)           | I | II | III | IV | V | VI |
|---------------------|---|----|-----|----|---|----|
| Aggarwal and Jaggi  | ✓ |    |     |    |   |    |
| Ouyang et al. [23]  |   | ✓ |     |    |   |    |
| Huang et al. [13]   | ✓ | ✓ |     |    |   |    |
| Tripathi and Pandey  |   | ✓ | ✓   |    |   |    |
| Mahata [22]         | ✓ |   |     |    |   |    |
| Goswami and Chaudhuri [9] | ✓ |     |     |    |   |    |
| Teng and Chang [29] |   | ✓ | ✓   |    |   |    |
| Law and Wee [18]    | ✓ | ✓ | ✓   |    |   |    |
| Jolai et al. [15]   | ✓ | ✓ | ✓   |    |   |    |
| Lo et al. [21]      | ✓ |   |     |    |   |    |
| Pervin et al. [24]  |   | ✓ |     |    |   |    |
| Pervin et al. [25]  | ✓ | ✓ | ✓   |    |   |    |
| Our paper           | ✓ | ✓ | ✓   | ✓  | ✓ | ✓  |

where “I” represents Integrated inventory model, “II” represents Trade-credit policy, “III” represents Stock dependent demand, “IV” represents Deteriorations, “V” represents Time varying costs, “VI” represents Partial backorder.

3. Notations and Assumptions. This model is developed on the basis of the following assumptions and notations:

3.1. Notations.

\( A \): ordering cost per order,

\( c \): unit purchasing cost per item,

\( T \): length of cycle time,

\( s \): unit selling price per item,

\( c_1 \): unit shortage cost per item and per unit time,

\( c_2 \): unit backorder cost per item and per time interval,

\( \delta \): fraction of the demand during the stock-out period that will be back ordered, where \( 0 \leq \delta \leq 1 \),

\( I(t) \): inventory level at time \( t \),

\( I_{1}(t) \): inventory level that changes with time \( t \) during production period,

\( I_{2}(t) \): inventory level that changes with time \( t \) during non-production period,

\( I_{3}(t) \): inventory level that changes with time \( t \) during shortage period,

\( I_e \): interest, which can be earned per unit of time (i.e., in $ per year) by the buyer,

\( I_c \): interest charges in stocks per unit of time (i.e., per $ in stocks per year) by the vendor,

\( \theta \): constant deterioration rate, where \( 0 \leq \theta < 1 \),

\( M \): credit period in years offered by the vendor,

\( TC_V(T) \): total cost function per unit of time for vendor,

\( TC_B(T) \): total cost function per unit of time for buyer,
The total cost function per unit of time for the integrated inventory model, \( D(t) \): demand rate is considered as:

\[
D(t) = \begin{cases} 
\alpha + \beta I(t), & \text{if } I(t) > 0, \\
\alpha, & \text{if } I(t) \leq 0,
\end{cases}
\]

where \( \alpha \) and \( \beta \) are positive constants, \( \alpha > \beta \), \( 0 \leq t \leq T \).

The holding cost per item per time-unit is time dependent and is assumed as \( h(t) = a + bt \), where \( a > 0, 0 < b < 1 \).

### 3.2. Assumptions.
1. Annual demand \( D(t) \) is stock dependent.
2. Replenishment rate is finite.
3. \( I_c \geq I_e \).
4. The lead time is negligible.
5. Shortages are allowed for the buyer’s model and the shortages are backordered in a partial amount with a backlogging rate \( \delta \).
6. The fixed credit period \( M \) offered by the vendor to the buyer.
7. If \( T \geq M \), then buyer settles the account at time \( M \) and pays for the interest charges on items in stock with rate \( I_c \) over the interval \([M, T]\). If \( T \leq M \), then the buyer settles the account at time \( M \) and there is no interest charge in stock during the whole cycle.
8. There is no repair or replacement of deteriorated units during planning horizon. The item will be withdrawn from stock immediately as they become deteriorated.

### 4. Mathematical Formulation.
Here, we consider our model from vendor’s and buyer’s perspectives. Since our work is on a two level supply chain system, therefore, the formulated models are depicted in two separate subsections.

#### 4.1. Vendor’s Model.
We assume that, at time \( t = 0 \), the cycle starts with zero stock level at supply rate \( k \). The replenishment or supply continues up to time \( t_1 \). During the time period \([0, t_1]\), inventory piles up by adjusting the demand in market. This accumulated inventory level at time \( t_1 \) gradually diminishes during the period \([t_1, t_2]\) due to those reasons of market demand and deterioration of items and ultimately falls to zero at time \( t = t_2 \). After the scheduling period, the cycle repeats itself.

Now, the differential equations involved the instantaneous states of the inventory level in the interval \([0, t_2]\) are given by

\[
\frac{dI_1(t)}{dt} + \theta(t)I_1(t) = k - D(t) = k - [\alpha + \beta I_1(t)] \quad (t \in [0, t_1])
\]

with \( I_1(0) = 0 \).

\[
\frac{dI_2(t)}{dt} + \theta(t)I_2(t) = -\alpha \quad (t \in [t_1, t_2])
\]

with \( I_2(t_2) = 0 \). Now the solution of Equation 1 using the boundary condition becomes

\[
I_1(t) = \frac{k - \alpha}{\beta + \theta} \left[ e^{(\beta + \theta)(t_1 - t)} - 1 \right] \quad (t \in [0, t_1]).
\]
Also, the solution of Equation 2 using boundary condition becomes

\[ I_2(t) = \frac{\alpha}{\beta + \theta} \left[ e^{(\beta + \theta)(t_2 - t)} - 1 \right] \quad (t \in [t_1, t_2]). \]

**Figure 1.** Graphical representation of Inventory model for a Vendor.

The elements comprising the vendor’s total cost function per cycle are listed below:

1. Annual ordering cost \((OC) = A\).
2. Annual stock holding cost \((HC)\) is defined as follows:

\[
\int_0^{t_1} h(t)I_1(t)dt + \int_{t_1}^{t_2} h(t)I_2(t)dt
= \int_0^{t_1} (a + bt) \left[ \frac{k - \alpha}{\beta + \theta} \left( e^{(\beta + \theta)(t_1 - t)} - 1 \right) \right] dt
+ \int_{t_1}^{t_2} (a + bt) \left[ \frac{\alpha}{\beta + \theta} \left( e^{(\beta + \theta)(t_2 - t)} - 1 \right) \right] dt
= \frac{a(k - \alpha)}{(\beta + \theta)^2} \left( e^{(\beta + \theta)t_1} - (\beta + \theta)t_1 - 1 \right) + \frac{b(k - \alpha)}{(\beta + \theta)^3} \left( e^{(\beta + \theta)t_1} - 1 \right) - \frac{b\alpha t_1}{(\beta + \theta)^2}
+ \frac{a\alpha}{(\beta + \theta)^2} \left( e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) - 1 \right)
+ \frac{b\alpha}{(\beta + \theta)^2} \left( e^{(\beta + \theta)(t_2 - t_1)} - 1 \right) - \frac{b\alpha(t_2 - t_1)}{(\beta + \theta)^2}
= \frac{a}{(\beta + \theta)^2}[(k - \alpha)\left( e^{(\beta + \theta)t_1} - (\beta + \theta)t_1 \right) + \alpha(\beta + \theta)(t_2 - t_1) - (\beta + \theta)(t_2 - t_1)]
- k] + \frac{b}{(\beta + \theta)^3} \left[ (k - \alpha)e^{(\beta + \theta)t_1} + \alpha e^{(\beta + \theta)(t_2 - t_1)} - k \right] - \frac{b\alpha t_2}{(\beta + \theta)^2}.\]
3. Deteriorating cost (DC) is calculated as follows:

\[
c_\theta \left[ \int_{0}^{t_1} I_1(t) dt + \int_{t_1}^{t_2} I_2(t) dt \right]
\]

\[
= c_\theta \left[ \int_{0}^{t_1} \left( \frac{k - \alpha}{\beta + \theta} e^{(\beta + \theta)(t_1 - t)} - 1 \right) dt + \int_{t_1}^{t_2} \left( \frac{\alpha}{\beta + \theta} e^{(\beta + \theta)(t_2 - t)} - 1 \right) dt \right]
\]

\[
= \frac{c(k - \alpha)\theta}{(\theta + \beta)^2} \left[ e^{(\beta + \theta)t_1} - (\beta + \theta)t_1 - 1 \right]
\]

\[
+ \frac{c\alpha\theta}{(\theta + \beta)^2} \left[ e^{(\beta + \theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) - 1 \right]
\]

\[
= \frac{c\theta}{(\theta + \beta)^2} \left[ (k - \alpha)e^{(\beta + \theta)t_1} + \alpha e^{(\beta + \theta)(t_2 - t_1)} - (k - \alpha)(\beta + \theta)t_1 - \alpha(\beta + \theta)(t_2 - t_1) - k \right].
\]

4.2. **Buyer’s Model.** In this subsection, the cycle starts with stock level \(k\) at time 0. During the time period \([0, t_2]\), inventory level decreases by joint effect of demand and deterioration in the market and at time \(t_2\), the inventory level becomes 0. During the period \([t_2, T]\), shortages occur because of market demand of items, and ultimately, the inventory falls to its lowest level at time \(t = T\). After the scheduling period \(T\), inventory level increases and after some time period, the cycle repeats itself.

Now, the differential equations involved the instantaneous states of the inventory level in the interval \([0, T]\) are given by

\[
\frac{dI_2(t)}{dt} + \theta(t)I_2(t) = k - D(t) = k - [\alpha + \beta I_2(t)] \quad (t \in [0, t_2]) \tag{3}
\]

with \(I_2(t_2) = 0\), and

\[
\frac{dI_3(t)}{dt} + \theta(t)I_3(t) = -\alpha \delta \quad (t \in [t_2, T]) \tag{4}
\]

with \(I_3(t_2) = 0\). Now the solution of Equation 3 using the boundary condition becomes

\[
I_2(t) = \frac{k - \alpha}{\beta + \theta} \left[ e^{(\beta + \theta)(t_2 - t)} - 1 \right], \quad t \in [0, t_2].
\]

Again, the solution of Equation 4 using boundary condition becomes

\[
I_3(t) = -\alpha \delta (T - t_2), \quad t \in [t_2, T].
\]

Maximum inventory level is

\[
I_0 = \frac{k - \alpha}{\beta + \theta} e^{(\beta + \theta)t_1} - 1.
\]

The elements taking part in the buyer’s cost function per cycle are listed below:

1. Annual ordering cost (OC) = \(A\).
2. Annual stock holding cost \((HC)\) is defined as follows:

\[
\int_0^{t_2} h(t)I_2(t)dt = \frac{a}{(\beta + \theta)^2} \left[ (k - \alpha)(e^{(\beta + \theta)t_2} - (\beta + \theta)t_2) \right] \\
+ \frac{b}{(\beta + \theta)^3} \left[ (k - \alpha)e^{(\beta + \theta)t_2} + \alpha e^{(\beta + \theta)t_2} - k \right].
\]

3. Deteriorating cost \((DC)\) is calculated as follows:

\[
c\theta \left[ \int_0^{t_2} I_2(t)dt \right] = c\theta \left[ \frac{a}{(\beta + \theta)^2} \left[ (k - \alpha)(e^{(\beta + \theta)t_2} - (\beta + \theta)t_2) \right] \\
+ \frac{b}{(\beta + \theta)^3} \left[ (k - \alpha)e^{(\beta + \theta)t_2} + \alpha e^{(\beta + \theta)t_2} - k \right].
\]

4. Shortage occurred during time interval \([t_2, T]\). Therefore, the shortages cost \((SC)\) is expressed as:

\[
c_1 \int_{t_2}^T I_3(t)dt = \frac{c_1\alpha \delta}{2} (T - t_2)^2.
\]

5. Partial backordering is allowed during the shortage period. Then, the backordering cost \((BC)\) is calculated as:

\[
c_2\delta \int_{t_2}^T I_3(t)dt = \frac{c_2\alpha \delta^2}{2} (T - t_2)^2.
\]
6. Due to shortage during time interval \([t_2, T]\), loss may occur in profit. Hence, the lost sale cost \((LSC)\) is:

\[
s \int_{t_2}^{T} (1 - \delta)D(t)dt = s \int_{t_2}^{T} (1 - \delta)\alpha dt = s\alpha(1 - \delta)(T - t_2).
\]

7. Purchase cost \((PC)\) becomes

\[
c(I_0 + \int_{t_2}^{T} \delta D(t)dt) = c\left[k - \alpha\frac{\theta}{\beta + \theta} + \theta(e^{(\beta + \theta)T} - 1) + \alpha\delta(T - t_2)\right].
\]

From our considered assumptions, there are two cases for both interest earned and interest charged, which are listed below:

**Case 1:** \(T \leq M\).

8. Interest earned \((IE)\): Here the trade-credit period \(M\) is greater than the total cycle time \(T\). So, the total interest gained by the buyer is

\[
sI_e \left[\int_0^{T} D(t)dt + (M - T)\int_0^{T} D(t)dt\right] = sI_e \left[\frac{\alpha T^2}{2} - \frac{\beta (k - \alpha)}{(\beta + \theta)^2} (Te^{(\beta + \theta)T} + \frac{1}{\beta + \theta}e^{(\beta + \theta)T} - T) + (M - T)\alpha T\right].
\]

9. Interest charged \((IC)\): Here also the trade-credit period \(M\) is greater than the total cycle time \(T\). So, the buyer has not charged any interest by the vendor; therefore, the interest charged is 0.

**Case 2:** \(M \leq T\).

10. Interest earned \((IE)\): In this case, the trade-credit period \(M\) is less than the total cycle time \(T\). Hence, the customer will pay the interest only on demand basis and the interest earned by the buyer is

\[
sI_e \left[\int_0^{T} D(t)dt + (M - T)\int_0^{T} D(t)dt\right] = sI_e \left[\frac{\alpha M^2}{2} - \frac{\beta (k - \alpha)}{(\beta + \theta)^2} (Me^{(\beta + \theta)M} + \frac{1}{\beta + \theta}e^{(\beta + \theta)M} - M)\right].
\]

11. Interest charged \((IC)\): In this case, the trade-credit period \(M\) is less than the total cycle time \(T\). So, the interest charged by the vendor from the buyer is given by

\[
cI_c \left[\int_{T}^{M} I_2(t)dt\right] = cI_c \left[\frac{\alpha}{\beta + \theta} (T - M - \frac{1}{\beta + \theta}e^{(\beta + \theta)(M - T)})\right].
\]

From the above results, the annual total relevant cost per unit of time for the vendor can be expressed as \(TC_V = [OC + HC + DC + IC - IE]\); with some simplification,
the following results are achieved:

\[
TC_V = \frac{A}{T} + \frac{a(k - \alpha)}{(\beta + \theta)^2} \left( e^{(\beta+\theta)t_1} - (\beta + \theta)t_1 - 1 \right) + \frac{b(k - \alpha)}{(\beta + \theta)^3} \left( e^{(\beta+\theta)t_1} - 1 \right) - \frac{b\alpha t_1}{(\beta + \theta)^2} + \frac{a\alpha}{(\beta + \theta)^2} \left( e^{(\beta+\theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) - 1 \right) + \frac{b\alpha}{(\beta + \theta)^3} \left( e^{(\beta+\theta)(t_2 - t_1)} - 1 \right) - \frac{b\alpha(t_2 - t_1)}{(\beta + \theta)^2} + \frac{a}{(\beta + \theta)^2} \left[ (k - \alpha)\left( e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2 - t_1)} - k \right) - \frac{b\alpha t_2}{(\beta + \theta)^2} \right] + \frac{c\theta}{T(\theta + \beta)^2} \left[ (k - \alpha)\left( e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2 - t_1)} \right) - (k - \alpha)(\beta + \theta)(t_2 - t_1) - k \right] - \frac{sI_c}{T} \left[ \alpha M^2 - \beta(k - \alpha)M + \frac{1}{\beta + \theta} e^{(\beta+\theta)M} - M \right].
\]

(5)

Also, the annual total relevant cost per unit of time for the buyer can be expressed as

\[
TC_B = [OC + HC + DC + SC + BC + LSC + IC - IE].
\]

With some simplification, the subsequent results are attained:

\[
TC_B = \frac{A}{T} + \frac{a(k - \alpha)}{(\beta + \theta)^2} \left( e^{(\beta+\theta)t_2} - (\beta + \theta)t_2 - 1 \right) + \frac{b(k - \alpha)}{(\beta + \theta)^3} \left( e^{(\beta+\theta)t_1} - 1 \right) + \frac{a\alpha}{(\beta + \theta)^2} \left( e^{(\beta+\theta)(t_2 - t_1)} - (\beta + \theta)(t_2 - t_1) - 1 \right) + \frac{b\alpha}{(\beta + \theta)^3} \left( e^{(\beta+\theta)(t_2 - t_1)} - 1 \right) - \frac{b\alpha(t_2 - t_1)}{(\beta + \theta)^2} + \frac{a}{(\beta + \theta)^2} \left[ (k - \alpha)\left( e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2 - t_1)} - k \right) - \frac{b\alpha t_2}{(\beta + \theta)^2} \right] + \frac{c\theta}{T(\theta + \beta)^2} \left[ (k - \alpha)\left( e^{(\beta+\theta)t_1} + \alpha e^{(\beta+\theta)(t_2 - t_1)} \right) - (k - \alpha)(\beta + \theta)t_1 - \alpha(\beta + \theta)(t_2 - t_1) - k \right] + \frac{s}{T} \left[ \alpha(1 - \delta)(T - t_2) + \frac{c_1\alpha\delta}{2} (T - t_2)^2 \right] + \frac{c_2\alpha\delta^2}{2} (T - t_2)^2 + \frac{cI_c}{T} \left[ \frac{\alpha}{\beta + \theta} \left( (T - M) - \frac{1}{\beta + \theta} e^{(\beta+\theta)(M - T)} \right) \right] - \frac{sI_c}{T} \left[ \frac{\alpha T^2}{2} - \beta(k - \alpha)M + \frac{1}{\beta + \theta} e^{(\beta+\theta)M} - (M - T)\alpha T \right].
\]

(6)

Then the total cost \(TC\) for the integrated vendor-buyer model can be defined as \(TC = TC_V + TC_B\) and can be expressed as
and which vanish at that point are given as follows:

\[ \alpha(T - t_2) \]

\[ + \frac{c_2 \alpha T^2}{2} (T - t_2)^2 + \frac{c I e}{T} \left( \frac{\alpha}{(\beta + \theta)^2} \left( (T - M) - \frac{1}{\beta + \theta} e^{\beta T (M - T)} \right) \right) \]

\[ - s I \left( \frac{\alpha M^2}{2} - \frac{\alpha M^2}{(\beta + \theta)^2} (Me^{(\beta + \theta)M} + \frac{1}{\beta + \theta} e^{(\beta + \theta)M} - M) \right). \]  

\( (7) \)

5. Solution procedure. Taking first-order partial derivatives for total cost \( TC \) with respect to \( T \) and \( M \) and which vanish at that point are given as follows:

\[ \frac{\partial TC}{\partial T} = 0 \]  

\( (8) \)

and

\[ \frac{\partial TC}{\partial M} = 0. \]  

\( (9) \)

They are evaluated and yield a minimizer \( (T^*, M^*) \) provided that they satisfies the following second-order sufficient conditions at that point:

\[ \left( \frac{\partial^2 TC}{\partial T^2} \right) \left( \frac{\partial^2 TC}{\partial M^2} \right) - \left( \frac{\partial^2 TC}{\partial T \partial M} \right)^2 > 0 \]  

\( (10) \)

and

\[ \left( \frac{\partial^2 TC}{\partial T^2} \right) > 0, \left( \frac{\partial^2 TC}{\partial M^2} \right) > 0. \]  

\( (11) \)

Now, by solving Equations \( 8 \) and \( 9 \), we obtain the values of \( T \) and \( M \) which satisfies Equations \( 10 \) and \( 11 \) and utilizing the values of \( T \) and \( M \) into Equation \( 7 \), we evaluate a solution that provides the minimum total inventory cost per unit time for the integrated inventory system. The minimizer is an isolated local one and, up to restriction, it is a global one also because the Hessian matrix is a positive one which ensure the convexity of \( TC \).
6. **Numerical Example.** The following numerical examples are given to illustrate the validity of the proposed problem.

**Example 1** Let us assume $A = $250/order, $s = $0.9/unit, $c = $80/unit, $a = $0.6/unit/year, $b = $0.8/unit/year, $c_1 = $40/unit, $I_c = $0.15$/year, $I_e = $0.19$/year, $k = 2000/unit/year, $c_2 = $60/unit, $\alpha = 300$, $\beta = 0.7$, $\theta = 0.9$, $\delta = 0.6$. By using Mathematica, we obtain the maximum solution of $TC(M, T)$ as $M = 0.110$ years, $T = 0.0872$ years, $TC_V = $4822.62, $TC_B = $4371.29 and $TC = $3,575.17.

**Example 2** Using the same data as those in **Example 1** except $k = 1000/unit/year, we calculate the following results: $M = 0.134$ years, $T = 0.0843$ years, $TC_V = $4725.50, $TC_B = $4166.04 and $TC = $3,845.76.

**Example 3** Utilizing the same data as considered in **Example 1** except $b = 0.6/unit/year and $s = $0.8/unit, we derive the following results: $M = 0.1821$ years, $T = 0.0981$ years, $TC_V = $4603.58, $TC_B = $4123.14 and $TC = $3,981.60.

7. **Sensitivity analysis.** Now, the effects of changes in the system parameters $A$, $s$, $c$, $a$, $b$, $I_c$, $c_1$, $c_2$, $\delta$, $\alpha$ and $\beta$, on the optimal values of $T$, $M$ and the optimal cost $TC(T)$ are analyzed. The sensitivity analysis is performed by changing each of the parameter by +50%, +25%, +10%, taking one parameter at a time and keeping the remaining parameters unchanged. The results based on Example 1 are shown in Table 2 and, on the basis of the results, the observations are taken into account.

The convexity of the integrated cost function with respect to **Example 1, Example 2** and **Example 3** are shown in Figure 5, Figure 6 and Figure 7, respectively. From Figure 5, we get the optimal values as $T = 0.0872$ and $M = 0.110$, which is compatible with the values which we receive by solving Equations 8 and 9. Similarly, from Figure 6 and Figure 7, we get $T = 0.0843$, $M = 0.134$, and $T = 0.0981$, $M = 0.1821$, respectively, which is also agreeable with the obtained value from Equations 8 and 9.

![Figure 3. TC vs. T at M = 0.1091.](image-url)
The convexity of the integrated cost function are shown in Figure 3 and Figure 4.

The following observations are made on the basis of Table 2:

(i) As ordering cost $A$, increase, the replenishment cycle time, $T^*$, increase as well as the value of $M^*$ and total optimal cost, $TC(T^*)$, decrease.

### Table 2: Sensitivity analysis for different parameters involved in Example 1.

| Parameter | parametric value after % change | $T$ | $M$ | $TC$ | $TC_0$ | $TC^*$ |
|-----------|---------------------------------|-----|-----|------|--------|--------|
| $A$       | 375                             | 0.0910 | 0.1129 | 4527.31 | 4632.05 | 3461.12 |
|           | 312.5                           | 0.0873 | 0.1091 | 4578.44 | 4113.10 | 3478.69 |
|           | 275                             | 0.0821 | 0.1051 | 4629.73 | 4150.03 | 3510.85 |
| $s$       | 1.35                            | 0.0767 | 0.1104 | 4665.28 | 4197.46 | 3582.35 |
|           | 1.125                           | 0.0763 | 0.1096 | 4022.38 | 3810.14 | 3412.19 |
|           | 0.99                            | 0.0791 | 0.1052 | 4065.19 | 3843.52 | 3487.71 |
| $c$       | 120                             | 0.0863 | 0.1027 | 4122.16 | 3874.00 | 3560.31 |
|           | 100                             | 0.0724 | 0.1035 | 4203.23 | 3890.64 | 3619.41 |
|           | 88                              | 0.0812 | 0.1037 | 4247.11 | 3915.26 | 3670.10 |
| $a$       | 0.9                             | 0.0690 | 0.1366 | 3877.35 | 3682.34 | 3510.94 |
|           | 0.75                            | 0.0728 | 0.1380 | 3852.55 | 3650.34 | 3422.30 |
|           | 0.66                            | 0.0749 | 0.1418 | 3796.21 | 3614.57 | 3392.62 |
| $b$       | 1.2                             | 0.0467 | 0.1510 | 3785.04 | 3654.87 | 3473.22 |
|           | 1.0                             | 0.0511 | 0.1572 | 3751.39 | 3627.95 | 3376.99 |
|           | 0.88                            | 0.0585 | 0.1618 | 3728.45 | 3567.53 | 3108.04 |
| $c_1$     | 60                              | 0.0814 | 0.1136 | 4244.33 | 3935.10 | 3630.44 |
|           | 50                              | 0.0889 | 0.1219 | 4407.61 | 3846.72 | 3780.00 |
|           | 44                              | 0.0926 | 0.1305 | 4521.17 | 3918.34 | 3822.35 |
| $c_2$     | 90                              | 0.0672 | 0.1158 | 4176.88 | 3839.15 | 3610.23 |
|           | 75                              | 0.0779 | 0.1227 | 4366.08 | 3882.46 | 3679.14 |
|           | 66                              | 0.0837 | 0.1293 | 4541.00 | 3918.27 | 3746.22 |
| $\alpha$  | 450                             | 0.0832 | 0.1745 | 3983.52 | 3728.61 | 3555.22 |
|           | 375                             | 0.0811 | 0.1659 | 3875.00 | 3894.17 | 3487.20 |
|           | 330                             | 0.0768 | 0.1633 | 3854.33 | 3729.26 | 3390.38 |
| $\beta$   | 1.05                            | 0.0577 | 0.1594 | 4539.74 | 4218.40 | 3641.99 |
|           | 0.875                           | 0.0597 | 0.1677 | 4487.23 | 4179.30 | 3647.19 |
|           | 0.77                            | 0.0649 | 0.1626 | 4435.41 | 4097.64 | 3557.73 |
| $\delta$  | 0.285                           | 0.0855 | 0.1547 | 4923.07 | 4580.68 | 3759.08 |
|           | 0.2375                          | 0.0732 | 0.1588 | 4866.31 | 4483.47 | 3921.39 |
|           | 0.209                           | 0.0611 | 0.1470 | 4736.58 | 4375.07 | 3982.63 |
| $\theta$  | 1.35                            | 0.0769 | 0.1720 | 4739.13 | 4460.53 | 3699.28 |
|           | 1.125                           | 0.0732 | 0.1693 | 4688.57 | 4379.24 | 3642.57 |
|           | 0.99                            | 0.0684 | 0.1647 | 4635.14 | 4316.20 | 3571.26 |
| $\delta$  | 0.9                             | 0.0592 | 0.1281 | 4037.45 | 3658.74 | 3429.50 |
|           | 0.75                            | 0.0633 | 0.1357 | 4168.70 | 3791.26 | 3557.35 |
|           | 0.66                            | 0.0748 | 0.1414 | 4231.05 | 3834.00 | 3658.10 |
Figure 5. The convexity of the integrated cost described in Example 1. Included are $T$, $M$ and the integrated cost $TC$, along the $x$-axis, the $y$-axis and the $z$-axis, respectively.

Figure 6. The convexity of the integrated cost showed in Example 2. Represented are $T$, $M$ and the integrated cost $TC$, along the $x$-axis, the $y$-axis and the $z$-axis, respectively.

(ii) $TRC$, $T$ and $M$ are more sensitive with respect to changes of the values of $\alpha$ and $\beta$.

(iii) For larger value of unit selling price, $s$, and unit cost price, $c$, we get a smaller value of optimal cycle time, $T^*$, $M^*$ and optimal annual total cost, $TC(T^*)$. 

Figure 7. The convexity of the integrated cost presented in Example 3. Followed are $T$, $M$ and the integrated cost $TC$, along the $x$-axis, the $y$-axis and the $z$-axis, respectively.

(iv) We can also see that under a higher value of the rate of interest earned, $I_e$, the annual total relevant cost, $TC(T^*)$, will be very much less.

(v) As holding cost, $h$, increase, the cycle time, $T^*$, decrease whereas the value of $M^*$ and the optimal annual total cost, $TC(T^*)$, increases.

(vi) When the rate of deterioration $\theta$ increases with respect to time, then the holding cost also increases with respect to deterioration, and the total cost of the integrated system will expand inevitably. Hence, the total gain for the retailer will reduce with respect to time.

(vii) An increase of the shortage cost $c_1$ and backorder cost $c_2$ will have an effect of a loss for the buyer as well as for the integrated system.

(vi) When the fraction of the backlogging rate $\delta$ increases, the total backorder cost also increases. This will cause a loss for the integrated system; however, a sufficient amount of backorder will increase the profit for the integrated system during the shortage period.

The sensitivity analysis with respect to respective parameters is expressed clearly in Figure 8, Figure 9 and Figure 10 by graphical representation. Figure 11 provides a schematic view of all parameters with respect to the objective function in one figure to show the significance of our sensitivity analysis.

8. Conclusions and future studies. In this paper, an integrated inventory model for perishable items with stock-dependent demand and time varying holding cost is developed. We have assumed integrated model because in that type of model a single user produces an item and supply it to several buyers. In addition of that, vendors production cycle time is calculated as an integer multiple of the function of buyers ordering time which helps to obtain the minimum cost easily. Here, a stock-dependent demand approach is considered, because a large pile of stocks in shelves attracts the customers to buy more. As an example, there are many
shopping malls (e.g., Shopping Hut, Big Bazar, City Life, City Mart, Style Bazar, Fashion Point, etc.), where the customer gets many items at a time and so, to save time, the customers are shopping everything from there, which increases the
profit for the decision maker. Time-dependent holding cost is chosen because, as
time increases, the rate of deterioration as well as the holding cost also increases.
The term deterioration add the model an extra benefit to attract the industrialist.
Shortages are allowed and only a portion of that product will be back-ordered to
match the work in a wide sense. Here, the vendor will offer to the buyer a trade-
credit period which will influence the buyer to gain more during that offered period.

The main difference between our proposed model with the existing ones is that,
in our model, we have determined the total cost for both vendor and buyer and,
then, the integrated system. In our section on numerical examples, three practical
numerical situations have been proposed to validate our model. From the sensitivity
analysis of our model, we have concluded that for a higher value of ordering cost,
the retailer has to order more at a time to reduce the cost. For a minimum value of the total cost of the integrated system, the backorder rate should be as high as possible. No infeasibility has been obtained from the existing data during the sensitivity analysis. Furthermore, we have observed that for a higher rate of the selling price, if the retailer wants a benefit from the trade-credit policy, he or she has to order less. We can see that when the holding cost increases, the retailer shortens the cycle time and reduces the order quantity to maximize the profit. To reduce the corresponding cost for the deteriorating items, the retailer will increase the maintaining facility for that type of items. By establishing a collaboration between vendor and buyer, companies’ focuses shift from a local perspective to a more global perspective. Therefore, companies will always try to optimize their global performances instead of their own performances.

The main purpose of this study is to simulate the market competition scenario, and to relax the previously assumed research direction. Partial backlogging is proposed in this paper to make the model more reasonable during a shortage period. In addition to the aforementioned, the proposed approach will be very useful for integrated inventory systems where both vendor and buyer address variable holding cost. Stock-dependent demand approach influences the industrialist and has a highest impact on an integrated model. Our model will further provide a valuable explanation for many organizations that could employ our methodology to improve their total operation costs. This model may have a strong use to determine an optimal inventory policy in situations such as stationary stores, fancy designable items and super-market bakeries, which could unveil the characteristics modeled. Finally, this paper considers a trade credit which plays an important practical role and is a major part of inventory control and a powerful tool to improve sales and financial return in an industry.

The proposed model can be further extended in several ways. For example, one can concentrate on the following: Stochastic demand instead of stock-dependent demand, Warehouses, Quantity discount, Stochastic inflation, and Imperfectness in the production. One can also consider multi-echelon supply chain model with set up cost reduction.

Acknowledgments. The authors are very much thankful to the anonymous referees for their valuable comments to modify our paper.

Appendix. The necessary optimality conditions for finding the optimal values of $T$ and $M$ are calculated by the following derivations:

$$\frac{\partial TC}{\partial T} = -A - \frac{a}{(\beta + \theta)^2}[(k - \alpha)(T(\beta + \theta)\frac{dt_1}{dT}(e^{(\beta+\theta)t_1} - 1) - (e^{(\beta+\theta)t_1} - (\beta + \theta)t_1))$$

$$+ \alpha(T(\beta + \theta)(\frac{dt_2}{dT} - \frac{dt_1}{dT})e^{(\beta+\theta)(t_2-t_1)} - 1) - e^{(\beta+\theta)(t_2-t_1)} - (\beta + \theta)(t_2 - t_1))]$$

$$+ \frac{b}{(\beta + \theta)^3}[(k - \alpha)e^{(\beta+\theta)t_1}(T(\beta + \theta)\frac{dt_1}{dT} - 1) + \alpha e^{(\beta+\theta)(t_2-t_1)}(T(\frac{dt_2}{dT} - \frac{dt_1}{dT}) - 1)]$$

$$- \frac{b\alpha}{(\beta + \theta)^2}(T\frac{dt_2}{dT} - t_2)] - \frac{c\theta}{(\theta + \beta)^2}[(k - \alpha)e^{(\beta+\theta)t_1}(1 - (\beta + \theta)T\frac{dt_1}{dT})$$

$$+ \alpha e^{(\beta+\theta)(t_2-t_1)}(1 - (\beta + \theta)T(\frac{dt_2}{dT} - \frac{dt_1}{dT}))(k - \alpha)(\beta + \theta)(T\frac{dt_1}{dT} - t_1)$$
\[ \begin{align*}
+ \alpha &+ (\beta + \theta)\left( T\left( \frac{dt_2}{dT} - \frac{dt_1}{dT} \right) - (t_2 - t_1) \right) - s[\alpha(1 - \delta)(T\frac{dt_2}{dT} - t_2)] \\
+ cI_c\left[ \frac{\alpha}{\beta + \theta} \left( M + \frac{1}{\beta + \theta} e^{(\beta + \theta)(M-T)} + Te^{(\beta + \theta)(M-T)} \right) \right] \\
+ sI_c \frac{1}{2} \alpha T^2 + \frac{\beta(k - \alpha)}{(\beta + \theta)^2} \left( Te^{(\beta + \theta)T} + T^2(\beta + \theta)e^{(\beta + \theta)T} - \frac{1}{\beta + \theta} e^{(\beta + \theta)T} \right) \\
+ 2MaT + \frac{1}{2} \alpha M^2 - \frac{\beta(k - \alpha)}{(\beta + \theta)^2} \left( Me^{(\beta + \theta)M} + \frac{1}{\beta + \theta} e^{(\beta + \theta)M} - M \right),
\end{align*} \]

\[ \frac{\partial TC}{\partial M} = \frac{a(k - \alpha)}{(\beta + \theta)} \left( (e^{(\beta + \theta)t_1} - 1) \frac{dt_1}{dM} \right) + \frac{b(k - \alpha)}{(\beta + \theta)^2} e^{(\beta + \theta)t_1} \frac{dt_1}{dM} + \frac{a\alpha}{(\beta + \theta)} e^{(\beta + \theta)(t_2 - t_1)} \]

\[ \frac{dt_2}{dM} - \frac{dt_1}{dM} \left( \frac{a}{(\beta + \theta)^2} e^{(\beta + \theta)(t_2 - t_1)} \left( \frac{dt_2}{dM} - \frac{dt_1}{dM} \right) - \frac{2b\alpha}{(\beta + \theta)^2} \frac{dt_2}{dM} \right) + \frac{a}{(\beta + \theta)^2} \left[ (k - \alpha)(\beta + \theta)(e^{(\beta + \theta)t_1} - 1) \frac{dt_1}{dM} + \alpha(\beta + \theta)(e^{(\beta + \theta)(t_2 - t_1)} - 1) \right] \]

\[ \frac{dt_1}{dM} - \frac{b}{(\beta + \theta)^2} \left[ (k - \alpha)(\beta + \theta)e^{(\beta + \theta)t_1} \frac{dt_1}{dM} + \alpha e^{(\beta + \theta)(t_2 - t_1)} \left( \frac{dt_2}{dM} - \frac{dt_1}{dM} \right) \right] \]

\[ \frac{\partial^2 TC}{\partial T^2} = \frac{a}{(\beta + \theta)^2} \left[ (k - \alpha)(\beta + \theta)^2 \frac{d^2t_1}{dT^2} \right] \left( 1 - e^{(\beta + \theta)t_1} \right) - T(\beta + \theta)^2 \left( \frac{dt_2}{dT} \right)^2 e^{(\beta + \theta)t_1} \]

\[ + \alpha T(\beta + \theta)^2 e^{(\beta + \theta)(t_2 - t_1)} \left( \frac{dt_2}{dT} - \frac{dt_1}{dT} \right)^2 + T(\beta + \theta)^2 \left( \frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2} \right) \]

\[ e^{(\beta + \theta)t_1} \frac{d^2t_1}{dT^2} \left( \frac{dt_1}{dT} \right)^2 + \alpha T(\beta + \theta)^2 e^{(\beta + \theta)(t_2 - t_1)} \frac{dt_2}{dT} - \frac{dt_1}{dT} \right)^2 \]

\[ + \frac{\alpha T(\beta + \theta)^2 e^{(\beta + \theta)(t_2 - t_1)} \frac{d^2t_2}{dT^2} - \frac{d^2t_1}{dT^2} \right)^2 + \frac{c\theta}{(\beta + \theta)^2} \left( k - \alpha \right) (\beta + \theta)^2 \]
The Hessian matrix of our model is represented as follows:

\[
\frac{\partial^2 TC}{\partial M^2} = a(k - \alpha)e^{(\theta + \beta)t_1}((\frac{dt_1}{dM})^2 + \frac{d^2 t_1}{dM^2}) + b(k - \alpha)\frac{e^{(\theta + \beta)t_1}}{(\beta + \theta)}((\frac{dt_1}{dM})^2 + \frac{d^2 t_1}{dM^2}) + ace^{(\beta + \theta)(t_2 - t_1)}((\frac{dt_2}{dM} - \frac{dt_1}{dM})^2 + (\frac{d^2 t_2}{dM^2} - \frac{d^2 t_1}{dM^2})) + \frac{b}{(\beta + \theta)(t_2 - t_1)}\]

\[
\frac{\partial^2 TC}{\partial T \partial M} = -\frac{a}{(\beta + \theta)^2}((k - \alpha)(T(\beta + \theta))\frac{d^2 t_1}{dTdM}e^{(\theta + \beta)t_1} - 1) + T(\beta + \theta)^2e^{(\theta + \beta)t_1}\frac{dt_1}{dT}\frac{dt_1}{dM} - (\beta + \theta)e^{(\theta + \beta)t_1}\frac{dt_1}{dM} - (\beta + \theta)\frac{dt_1}{dM} + \frac{b}{(\beta + \theta)}[(k - \alpha)(\beta + \theta)e^{(\theta + \beta)t_1}\frac{dt_1}{dM}]
\]

\[
= -\frac{a}{(\beta + \theta)^2}((k - \alpha)(T(\beta + \theta))\frac{d^2 t_1}{dTdM}(e^{(\theta + \beta)t_1} - 1) + T(\beta + \theta)^2e^{(\theta + \beta)t_1}\frac{dt_1}{dT}\frac{dt_1}{dM} - (\beta + \theta)e^{(\theta + \beta)t_1}\frac{dt_1}{dM} - (\beta + \theta)\frac{dt_1}{dM} + \frac{b}{(\beta + \theta)}[(k - \alpha)(\beta + \theta)e^{(\theta + \beta)t_1}\frac{dt_1}{dM}]
\]

The Hessian matrix of our model is represented as follows:

\[
H = \begin{pmatrix}
\frac{\partial^2 TC}{\partial M^2} & \frac{\partial^2 TC}{\partial T \partial M} \\
\frac{\partial^2 TC}{\partial T \partial M} & \frac{\partial^2 TC}{\partial T^2}
\end{pmatrix}
\]

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Received February 2017; 1st revision August 2017; final revision October 2017.

E-mail address: mamoni2014@rediffmail.com [For Magfura Pervin]
E-mail address: sankroy2006@gmail.com [For Sankar Kumar Roy]
E-mail address: gweber@metu.edu.tr [For Gerhard Wilhelm Weber]