Multi-Receiver Online Bayesian Persuasion

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Abstract
Bayesian persuasion studies how an informed sender should partially disclose information to influence the behavior of a self-interested receiver. Classical models make the stringent assumption that the sender knows the receiver’s utility. This can be relaxed by considering an online learning framework in which the sender repeatedly faces a receiver of an unknown, adversarially selected type. We study, for the first time, an online Bayesian persuasion setting with multiple receivers. We focus on the case with no externalities and binary actions, as customary in offline models. Our goal is to design no-regret algorithms for the sender with polynomial iteration running time. First, we prove a negative result: for any \(0 < \alpha < 1\), there is no polynomial-time no-\(\alpha\)-regret algorithm when the sender’s utility function is supermodular or anonymous. Then, we focus on the case of submodular sender’s utility functions and we show that, in this case, it is possible to design a polynomial-time no-\(\frac{1}{2}\)-regret algorithm. To do so, we introduce a general online gradient descent scheme to handle online learning problems with a finite number of possible loss functions. This requires the existence of an approximate projection oracle. We show that, in our setting, there exists one such projection oracle which can be implemented in polynomial time.

1. Introduction
Bayesian persuasion was originally introduced by Kamenica & Gentzkow (2011) to model multi-agent settings where an informed sender tries to influence the behavior of a self-interested receiver through the strategic provision of payoff-relevant information. Agents’ payoffs are determined by the receiver’s action and some exogenous parameters collectively termed the state of nature, whose value is drawn from a common prior distribution and observed by the sender only. Then, the sender decides how much of her/his private information has to be revealed to the receiver, according to a public randomized policy known as signaling scheme. From the sender’s perspective, this begets a decision-making problem that is essentially about controlling “who gets to know what”. This kind of problems are ubiquitous in application domains such as auctions and online advertising (Bro Miltersen & Sheffet, 2012; Emek et al., 2014; Badanidiyuru et al., 2018), voting (Alonso & Cámara, 2016; Cheng et al., 2015; Castiglioni et al., 2020a; Castiglioni & Gatti, 2021), traffic routing (Vasserman et al., 2015; Bhaskar et al., 2016; Castiglioni et al., 2021), recommendation systems (Mansour et al., 2016), security (Rabinovich et al., 2015; Xu et al., 2016), and product marketing (Babichenko & Barman, 2017; Candogan, 2019). The classical Bayesian persuasion model by Kamenica & Gentzkow (2011) makes the stringent assumption that the sender knows the receiver’s utility exactly. This is unreasonable in practice. Recently, Castiglioni et al. (2020b) propose to relax the assumption by framing Bayesian persuasion into an online learning framework, focusing on the basic single-receiver problem. In their model, the sender repeatedly faces a receiver whose type during each iteration—determining her/his utility function—is unknown and adversarially selected beforehand. In this work, we extend the model by Castiglioni et al. (2020b) to multi-receiver settings, where the (unknown) type of each receiver is adversarially selected before each iteration of the repeated interaction. We consider the case in which the sender has a private communication channel towards each receiver, which is commonly studied in multi-receiver models (see, e.g., (Babichenko & Barman, 2016)). Dealing with

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1Persuasion was famously attributed to a quarter of the GDP in the United States by McCloskey & Klamer (1995), with a more recent estimate placing this figure at 30% (Antioch et al., 2013).
2A recent work by Babichenko et al. (2021) relaxes the assumption in the offline setting. In that work, the goal is minimizing the sender’s regret over a single iteration, and the authors provide positive results for the case in which the sender knows the ordinal preferences of the receiver over states of nature. The authors study the case of a single receiver with a binary action space, and an arbitrary (unknown) utility function.

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multiple receivers introduces the additional challenge of correlating information disclosure across them and requires different techniques from those used in the single-receiver setting.

As customary when studying multi-receiver Bayesian persuasion problems (Dughmi & Xu, 2017; Xu, 2020), we address the case in which there are no inter-agent externalities, where each receiver’s utility does not depend on the actions of the other receivers, but only on her/his own action and the state of nature. Moreover, we focus on the commonly-studied setting with binary actions (Babichenko & Barman, 2016; Arieli & Babichenko, 2019), and we analyze different scenarios depending on whether the sender’s utility function is supermodular, submodular, or anonymous. Despite its simplicity, this basic model encompasses several real-world scenarios. For instance, think of a marketing problem in which a firm (sender) wants to persuade some potential buyers (receivers) to buy one of its products. Each buyer has to take a binary decision as to whether to buy a unit of the product or not, while the firm’s goal is to strategically disclose information about the product to the buyers, so as to maximize the number of units sold. In this example, the sender’s utility is anonymous, since it only depends on the number of buyers who decide to purchase (and not on their identities). Moreover, submodular sender’s utilities represent diminishing returns in the number of items sold, while supermodular ones encode decreasing production costs.

1.1. Original Contributions

Our goal is to design online algorithms for the sender that recommend a signaling scheme at each iteration of the repeated interaction, guaranteeing a sender’s expected utility close to that of the best-in-hindsight signaling scheme. In particular, we look for no-$\alpha$-regret algorithms, which collect an overall utility that is close to a fraction $\alpha$ of what can be obtained by the best-in-hindsight signaling scheme. In this work, we assume full-information feedback, which means that, after each iteration, the sender observes each receiver’s type during that iteration. Moreover, we are interested in no-$\alpha$-regret algorithms having a per-iteration running time polynomial in the size of the problem instance. To this end, we assume that the number of possible types of each receiver is fixed, otherwise polynomial-time no-$\alpha$-regret algorithms cannot be obtained even in the degenerate case of only one receiver (Castiglioni et al., 2020b).

In Section 4, we prove a negative result: for any $0 < \alpha \leq 1$, there is no polynomial-time no-$\alpha$-regret algorithm when the sender’s utility function is supermodular or anonymous. Thus, in the rest of the work, we focus on the case in which the sender’s utility function is submodular, where we provide a polynomial-time no-$\left(1 - \frac{1}{r}\right)$-regret algorithm.

As a first step in building our algorithm, in Section 5 we introduce a general online gradient descent (OGD) scheme to handle online learning problems with a finite number of possible loss functions. This can be applied to our setting, as we have a sender’s utility function (or, equivalently, negative loss function) for every combination of receivers’ types obtained as feedback. The OGD scheme works in a modified decision space whose dimensionality is the number of observed loss functions, and it is not affected by the dimensionality of the original space. This is crucial in our setting, as it avoids dealing with the set of sender’s signaling schemes, whose dimensionality grows exponentially in the number of receivers. Any OGD algorithm requires a projection oracle. Since in our setting an exact oracle cannot be implemented in polynomial time, we build our OGD scheme so that it works having access to a suitably-defined approximate projection oracle, which, as we show later, can be implemented in polynomial time in our model.

In Section 6, we build a polynomial-time approximate projection oracle. First, we formulate the projection problem as a convex linearly-constrained quadratic program, which has exponentially-many variables and polynomially-many constraints. Next, we show how to compute in polynomial time an approximate solution to this program by applying the ellipsoid algorithm to its dual. Since the dual has polynomially-many variables and exponentially-many constraints, the algorithm needs access to a particular (problem-dependent) polynomial-time separation oracle. Unfortunately, we do not have this in our setting, and, thus, our algorithm must rely on an approximate separation oracle. In general, running the ellipsoid method with an approximate separation oracle does not give any guarantee on the approximation quality of the returned solution. In order to make the ellipsoid algorithm return the desired approximate solution by only using an approximate separation oracle, we employ some ad-hoc technical tools suggested by a non-trivial primal-dual analysis. As a preparatory step towards our main result, at the beginning of Section 6, we use a derivation similar to that described so far to design a polynomial-time approximation algorithm for the offline version of our multi-receiver Bayesian persuasion problem, which may be of independent interest.

In Section 7, we conclude the construction of the no-$\left(1 - \frac{1}{r}\right)$-regret algorithm by showing how to implement in polynomial time an $(1 - \frac{1}{r})$-approximate separation oracle for settings in which the sender’s utility is submodular.

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3Our result is tight, as there is no poly-time no-$\alpha$-regret algorithm with $\alpha > 1 - \frac{1}{r}$. Indeed, it is NP-hard to approximate the sender’s optimal utility within a factor $1 - \frac{1}{r}$, even in the basic (offline) multi-receiver model of Babichenko & Barman (2016).
All the proofs omitted from the paper are in the Appendix.

1.2. Related Works

Most of the computational works on Bayesian persuasion study (offline) models in which the sender knows the receiver’s utility function exactly. Dughmi & Xu (2016) initiate these studies with the single receiver case, while Arieli & Babichenko (2019) extend their work to multiple receivers without inter-agent externalities, with a focus on private signaling. In particular, they focus on settings with binary actions for the receivers and a binary space of states of nature. They provide a characterization of the optimal signaling scheme in the case of supermodular, anonymous submodular, and super-majority sender’s utility functions. Babichenko & Barman (2016) extend this latter work by providing tight \((1−\frac{1}{e})\)-approximate signaling schemes for monotone submodular sender’s utilities and showing that an optimal private signaling scheme for anonymous utility functions can be found efficiently. Dughmi & Xu (2017) generalize the previous model to settings with an arbitrary number of states of nature. There are also some works focusing on public signaling with no inter-agent externalities, see, among others, (Dughmi & Xu, 2017) and (Xu, 2020).

The only computational work on Bayesian persuasion in an online learning framework is that of Castiglioni et al. (2020b), which, however, is restricted to the single-receiver case. The results and techniques in Castiglioni et al. (2020b) are different from those in our paper. In particular, they show that there are no polynomial-time no-\(\alpha\)-regret algorithms even in settings with a single receiver, when the number of receiver’s types is arbitrary. In contrast, we focus on settings in which the number of receivers’ types is fixed. Moreover, the main goal of Castiglioni et al. (2020b) is to design a (necessarily exponential-time) no-regret algorithm in the partial-information feedback setting in which the sender only observes the actions played by the receiver (and not her/his types). This is accomplished by providing slightly-biased estimators of the sender’s utilities for different signaling schemes. In our work, we assume full-information feedback, and, thus, our main focus is dealing with multiple receivers. This also introduces the additional challenge of correlating information disclosure across the receivers and working with an exponential number of possible feedbacks (tuples specifying a type for each receiver).

Our work is also related to the research line on online linear optimization with approximation oracles. In such setting, Kakade et al. (2009) show how to design a no-\(\alpha\)-regret algorithm relying on an \(\alpha\)-approximate linear optimization oracle, while Garber (2017) and Hazan et al. (2018) obtain analogous results with a better query complexity. The approach of these works to design approximate projection oracles is fundamentally different from ours, since they have access to a linear optimization oracle working in the learner’s decision space. On the other hand, our OGD scheme works on a modified decision space, and the approximate projection oracle can only rely on an approximate separation oracle dealing with the original sender’s decision space.

2. Preliminaries

There is a finite set \(\mathcal{R} := \{r_1\}_{i=1}^n\) of \(n\) receivers, and each receiver \(r \in \mathcal{R}\) has a type chosen from a finite set \(\mathcal{K}_r := \{k_{r,i}\}_{i=1}^{m_r}\) of \(m_r\) different types (let \(m := \max_{r \in \mathcal{R}} m_r\)). We introduce \(\mathcal{K} := \times_{r \in \mathcal{R}} \mathcal{K}_r\) as the set of type profiles, which are tuples \(k \in \mathcal{K}\) defining a type \(k_r \in \mathcal{K}_r\) for each receiver \(r \in \mathcal{R}\). Each receiver \(r \in \mathcal{R}\) has two actions available, defined by \(\mathcal{A}_r := \{a_0, a_1\}\). We let \(\mathcal{A} := \times_{r \in \mathcal{R}} \mathcal{A}_r\) be the set of action profiles specifying an action for each receiver. Sender and receivers’ payoffs depend on a random state of nature, which is selected from a finite set \(\Theta := \{\theta_i\}_{i=1}^d\) of \(d\) states. The payoff of a receiver also depends on the actions played by her/him, while it does not depend on the actions played by the other receivers, since there are no inter-agent externalities. Formally, a receiver \(r \in \mathcal{R}\) of type \(k \in \mathcal{K}_r\) has a utility \(u^{r,k} : \mathcal{A}_r \times \Theta \rightarrow [0,1]\). For the ease of notation, we let \(u^r := u^{r,k}(a_0, \theta) − u^{r,k}(a_1, \theta)\) be the payoff difference for a receiver \(r\) of type \(k\) when the state of nature is \(\theta \in \Theta\). The sender’s utility depends on the actions played by all the receivers, and it is defined by \(u^s : \mathcal{A} \times \Theta \rightarrow [0,1]\). For the ease of presentation, for every state \(\theta \in \Theta\), we introduce the function \(f_\theta : 2^\mathcal{R} \rightarrow [0,1]\) such that \(f_\theta(\mathcal{R})\) represents the sender’s utility when the state of nature is \(\theta\) and all the receivers in \(\mathcal{R} \subseteq \mathcal{R}\) play action \(a_1\), while the others play \(a_0\).

As it is customary in Bayesian persuasion, we assume that the state of nature is drawn from a common prior distribution \(\mu \in \text{int}(\Delta_\Theta)\), which is explicitly known to both the sender and the receivers. The sender can commit to a signaling scheme \(\phi\), which is a randomized mapping from states of nature to signals for the receivers. In this work, we focus on private signaling, where each receiver has her/his own signal that is privately communicated to her/him. Formally, there is a finite set \(\mathcal{S}_r\) of possible signals for each receiver \(r \in \mathcal{R}\). Then, \(\phi : \Theta \rightarrow \mathcal{S}_r\), where \(\mathcal{S} := \times_{r \in \mathcal{R}} \mathcal{S}_r\) is the set of signal profiles, which are tuples \(s \in \mathcal{S}\) defining a signal \(s_r \in \mathcal{S}_r\) for each receiver \(r \in \mathcal{R}\). We denote with \(\phi_\theta\) the probability distribution employed by \(\phi\) when the state of nature is \(\theta \in \Theta\), with \(\phi_\theta(s)\) being the probability of sending a signal profile \(s \in \mathcal{S}\). The one-shot

\(^4\)All vectors and tuples are denoted by bold symbols. For any vector (tuple) \(x\), the value of its \(i\)-th component is denoted by \(x_i\).

\(^5\)\(\text{int}(X)\) is the interior of a set \(X\), while \(\Delta_X\) is the set of all the probability distributions over a set \(X\).
interaction between the sender and the receivers goes on as follows: (i) the sender commits to a publicly known signaling scheme $\phi$; (ii) she/he observes the realized state of nature $\theta \sim \mu$; (iii) she/he draws a signal profile $s \sim \phi_0$ and communicates to each receiver $r \in \mathcal{R}$ signal $s_r$; and (iv) each receiver $r \in \mathcal{R}$rationally updates her/his prior belief over $\Theta$ according to the Bayes rule and selects an action maximizing her/his expected utility. We remark that, given a signaling scheme $\phi$, a receiver $r \in \mathcal{R}$ of type $k \in \mathcal{K}_r$ observing a private signal $s \in \mathcal{S}_r$ experiences an expected utility $\sum_{\theta \in \Theta} \mu(\theta) \sum_{s \in \mathcal{S}_r} s \phi_0(s) u^r(k, \theta)$ (up to a normalization constant) when playing action $a \in \mathcal{A}_r$. Assuming the receivers’ type profile is $k \in \mathcal{K}$, the goal of the sender is to commit to an optimal signaling scheme $\phi$, which is one maximizing her/his expected utility $f(\phi, k) := \sum_{\theta \in \Theta} \mu(\theta) \sum_{s \in \mathcal{S}} s \phi_0(s) f_\phi(R^k_s)$, where we let $R^k_s \subseteq \mathcal{R}$ be the set of receivers who play $a_1$ after observing their private signal $s$, in $s$, under signaling scheme $\phi$.

Assumptions  In the rest of this work, we assume that the sender’s utility is monotone non-decreasing in the set of receivers playing $a_1$. Formally, for each state $\theta \in \Theta$, we let $f_{\theta}(R) \leq f_{\theta}(R')$ for every $R \subseteq R' \subseteq \mathcal{R}$, while $f_{\theta}(\emptyset) = 0$ for the ease of presentation. Moreover, we assume that the number of types $m_r$ of each receiver $r \in \mathcal{R}$ is fixed; in other words, the value of $m$ cannot grow arbitrarily large.\(^6\)

Direct Signaling Schemes  By well-known revelation-principle-style arguments (Kamenica & Gentzkow, 2011; Arieli & Babichenko, 2019), we can restrict our attention to signaling schemes that are direct and persuasive. In words, a signaling scheme is direct if signals correspond to recommendations of playing actions, while it is persuasive if the receivers do not have any incentive to deviate from the recommendations prescribed by the signals they receive. In our setting, a direct signal sent to a receiver specifies an action recommendation for each receiver’s type; thus, we let $S_r := 2^{K_r}$ for every $r \in \mathcal{R}$. A signal $s \in S_r$ for a receiver $r \in \mathcal{R}$ is encoded by a subset of her/his types, namely $s \subseteq \mathcal{K}_r$. Intuitively, $s$ can be interpreted as the recommendation to play action $a_1$ when the receiver has type $k \in \mathcal{K}_r$ such that $k \in s$, while $a_0$ otherwise. Given a direct and persuasive signaling scheme $\phi$, for a signal profile $s \in \mathcal{S}$ and a type profile $k \in \mathcal{K}$, the set $R^k_s$ appearing in the definition of the sender’s expected utility $f(\phi, k)$ can be formally expressed as $R^k_s := \{ r \in \mathcal{R} \mid k_r \in s_r \}$.

\(^6\)The monotonicity assumption is w.l.o.g. for this work, since our main positive result (Theorem 7) relies on it. Instead, assuming a fixed number of types is necessary, since, even in single-receiver settings, designing no-regret algorithms with running time polynomial in $m$ is intractable (Castiglioni et al., 2020b).

Set Functions and Matroids  In Section 7, we show how to implement our approximate separation oracle by optimizing functions $f_\theta$ over suitably defined matroids (representing signals). Next, we introduce the necessary definitions on set functions and matroids. For the ease of presentation, we consider a generic function $f : 2^\Theta \to [0, 1]$ for a finite set $\mathcal{G}$. We say that $f$ is submodular, respectively supermodular, if for $I, I' \subseteq \mathcal{G}$: $f(I \cap I') + f(I' \cup I') \leq f(I) + f(I')$, respectively $f(I \cap I') + f(I' \cup I') \geq f(I) + f(I')$. The function $f$ is anonymous if $f(I) = f(I')$ for all $I, I' \subseteq \mathcal{G}$: $|I| = |I'|$. A matroid $\mathcal{M} := (\mathcal{G}, \mathcal{I})$ is defined by a finite ground set $\mathcal{G}$ and a collection $\mathcal{I}$ of independent sets, i.e., subsets of $\mathcal{G}$ satisfying some characterizing properties (see Schrijver, 2003) for a detailed formal definition. We denote by $\mathcal{B}(\mathcal{M})$ the set of the bases of $\mathcal{M}$, which are the maximal sets in $\mathcal{I}$.

3. Multi-Receiver Online Bayesian Persuasion

We consider a multi-receiver generalization of the online setting introduced by Castiglioni et al. (2020b). The sender plays a repeated game in which, at each iteration $t \in [T]$, she/he commits to a signaling scheme $\phi_t$, observes the realized state of nature $\theta_t \sim \mu$, and privately sends signals determined by $s_t \sim \phi_0$, to the receivers.\(^7\) Then, each receiver (whose type is unknown to the sender) selects an action maximizing her/his expected utility given the observed signal (in the one-shot interaction at iteration $t$).

We focus on the problem of computing a sequence $\{\phi^t\}_{t \in [T]}$ of signaling schemes maximizing the sender’s expected utility when the sequence of receivers’ types $\{k^t\}_{t \in [T]}$, with $k^t \in \mathcal{K}$, is adversarially selected beforehand. After each iteration $t \in [T]$, the sender gets payoff $f(\phi^t, k^t)$ and receives a full-information feedback on her/his choice at $t$, which is represented by the type profile $k^t$. Therefore, after each iteration, the sender can compute the expected utility $f(\phi, k^t)$ guaranteed by any signaling scheme $\phi$ she/he could have chosen during that iteration.

We are interested in an algorithm computing $\phi^t$ at each iteration $t \in [T]$. We measure the performance of one such algorithm using the $\alpha$-regret $R^\alpha_t$. Formally, for $0 < \alpha \leq 1$,

$$R^\alpha_t := \max_{\phi} \sum_{t \in [T]} f(\phi, k^t) - \mathbb{E} \left[ \sum_{t \in [T]} f(\phi^t, k^t) \right],$$

where the expectation is on the randomness of the algorithm. The classical notion of regret is obtained for $\alpha = 1$.

Ideally, we would like an algorithm that returns a sequence $\{\phi^t\}_{t \in [T]}$ with the following properties:

- the $\alpha$-regret is sublinear in $T$ for some $0 < \alpha \leq 1$;

\(^7\)Throughout the paper, the set $\{1, \ldots, x\}$ is denoted by $[x]$. 

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- the number of computational steps it takes to compute $\phi^t$ at each iteration $t \in [T]$ is $\text{poly}(T, n, d)$, that is, it is a polynomial function of the parameters $T$, $n$, and $d$.

An algorithm satisfying the first property is called a no-$\alpha$-regret algorithm (it is no-regret if it does so for $\alpha = 1$). In this work, we focus on the weaker notion of $\alpha$-regret since, as we discuss next, requiring no-regret is oftentimes too limiting in our setting (from a computational perspective).

4. Hardness of Being No-$\alpha$-Regret

We start with a negative result. We show that designing no-$\alpha$-regret algorithms with polynomial per-iteration running time is an intractable problem (formally, it is impossible unless $\text{NP} \subseteq \text{RP}$) when the sender’s utility is such that functions $f_\theta$ are supermodular or anonymous. This hardness result is deeply connected with the intractability of the offline version of our multi-receiver Bayesian persuasion problem that we formally define in the following Section 4.1. Then, Section 4.2 collects all the hardness results.

4.1. Offline Multi-Receiver Bayesian Persuasion

We consider an offline setting where the receivers’ type profile $k \in K$ is drawn from a known probability distribution (rather then being selected adversarially at each iteration). Given a subset of possible type profiles $K \subseteq K$ and a distribution $\lambda \in \text{int}(\Delta_K)$, we call $\text{BAYESIAN-OPT-SIGNAL}$ the problem of computing a signaling scheme that maximizes the sender’s expected utility. This can be achieved by solving the following LP of exponential size.\(^8\)

\[
\max_{\theta} \sum_{k \in K} \lambda_k \sum_{\theta \in \Theta} \sum_{s \in S} \phi_\theta(s) f_\theta(R^k_s) \\
\text{s.t.} \sum_{\theta \in \Theta} \sum_{s \in S : s_r = s} \phi_\theta(s) \mu_r k \geq 0 \\
\forall r \in R, \forall s \in S_r, \forall k \in K_r : k \in s \quad (1b) \\
\sum_{s \in S} \phi_\theta(s) = 1 \quad \forall \theta \in \Theta \quad (1c) \\
\phi_\theta(s) \geq 0 \quad \forall \theta \in \Theta, \forall s \in S. \quad (1d)
\]

4.2. Hardness Results

First, we study the computational complexity of finding an approximate solution to $\text{BAYESIAN-OPT-SIGNAL}$. In particular, given $0 < \alpha \leq 1$, we look for an $\alpha$-approximate solution in the multiplicative sense, i.e., a signaling scheme providing at least a fraction $\alpha$ of the sender’s optimal expected utility (the optimal value of LP (1)). Theorem 1 provides our main hardness result, which is based on a reduction from the promise-version of $\text{LABEL-COVER}$ (see Appendix A for its definition and the proof of the theorem).

**Theorem 1.** For every $0 < \alpha < 1$, it is $\text{NP}$-hard to compute an $\alpha$-approximate solution to $\text{BAYESIAN-OPT-SIGNAL}$, even when the sender’s utility is such that, for every $\theta \in \Theta$, $f_\theta(R) = 1$ iff $|R| \geq 2$, while $f_\theta(R) = 0$ otherwise.

Notice that Theorem 1 holds for problem instances in which functions $f_\theta$ are anonymous. Moreover, the reduction can be easily modified so that functions $f_\theta$ are supermodular and satisfy $f_\theta(R) = \max\{0, |R| - 1\}$ for $R \subseteq R$. Thus:

**Corollary 1.1.** For $0 < \alpha \leq 1$, it is $\text{NP}$-hard to compute an $\alpha$-approximate solution to $\text{BAYESIAN-OPT-SIGNAL}$, even when the sender’s utility is such that functions $f_\theta$ are supermodular or anonymous for every $\theta \in \Theta$.

By using arguments similar to those employed in the proof of Theorem 6.2 by Roughgarden & Wang (2019), the hardness of computing an $\alpha$-approximate solution to the offline problem can be extended to designing no-$\alpha$-regret algorithms in the online setting. Then:

**Theorem 2.** For every $0 < \alpha \leq 1$, there is no polynomial-time no-$\alpha$-regret algorithm for the multi-receiver online Bayesian persuasion problem, unless $\text{NP} \subseteq \text{RP}$, even when functions $f_\theta$ are supermodular or anonymous for all $\theta \in \Theta$.

In the rest of the work, we show how to design a polynomial-time no-(1 $- \frac{1}{\alpha}$)-regret algorithm for the case in which the sender’s utility is such that functions $f_\theta$ are submodular.

5. An Online Gradient Descent Scheme with Approximate Projection Oracles

As a first step in building our polynomial-time algorithm, we introduce our OGD scheme with an approximate projection oracle. Intuitively, it works by transforming the multi-receiver online Bayesian persuasion setting into an equivalent online learning problem whose decision space does not need to explicitly deal with signaling schemes (thus avoiding the burden of having an exponential number of possible signal profiles). The OGD algorithm is then applied on this new domain. In our setting, we do not have access to a polynomial-time (exact) projection oracle, and, thus, we design and analyze the algorithm assuming access to an approximate one only. As we show later in Sections 6 and 7, such approximate projection oracle can be implemented in polynomial time when the functions $f_\theta$ are submodular.

Let us recall that the OGD scheme that we describe in this
5.1. A General Approach

Consider an online learning problem in which the learner receives feedbacks at each iteration $t \in [T]$. Let $\mathcal{F}$ be the set of all possible feedbacks. The learner observes a feedback $f_t \in \mathcal{F}$ at iteration $t$. The reward (or negative loss) of a decision $y \in \mathcal{Y}$ given feedback $f_t$ is defined as $w(y, f_t)$.

We assume to have access to an approximate projection oracle for $\mathcal{F}$, which we denote in the following. By letting $\phi : \mathcal{F} \rightarrow \hat{\mathcal{F}}$, we define an algorithm that operates by letting $\phi(f) = \text{arg min}_{\hat{f} \in \hat{\mathcal{F}}} ||f - \hat{f}||_2$. We assume that $\phi$ is $(1, \delta)$-approximate projection whenever $\delta < 1/2$. The algorithm is a two-phase algorithm: an offline phase and an online phase.

### Offline Phase

#### Step 1: Projections

We consider a family of projections $\phi : \mathcal{F} \rightarrow \hat{\mathcal{F}}$. For any $f_t \in \mathcal{F}$, let $\hat{f}_t = \phi(f_t)$. The set $\hat{\mathcal{F}}$ is the set of all possible feedbacks. The reward (or negative loss) of a decision $y \in \mathcal{Y}$ given feedback $f_t$ is defined as $\phi^{-1}(w(y, f_t))$.

#### Step 2: Algorithm

Let $\mathcal{X}_t = \{x \in \mathcal{X} \mid \phi^{-1}(w(y, f_t)) \leq \gamma \}$ be the set of all possible signaling schemes. For every $y \in \mathcal{Y}$, let $\mathcal{X}_t(y) = \{x \in \mathcal{X} \mid \phi^{-1}(w(y, f_t)) \leq \gamma \}$. The learner observes a feedback $f_t \in \mathcal{F}$ at iteration $t$. The reward (or negative loss) of a decision $y \in \mathcal{Y}$ given feedback $f_t$ is defined as $w(y, f_t)$.

We transform this general online learning problem to a new one in which the learner's decision set is $\mathcal{X} \subseteq \{0, 1\}^p$.

5.2. A Particular Setting: Multi-Receiver Online Bayesian Persuasion

Multi-Receiver Online Bayesian Persuasion

Consider a multi-receiver online Bayesian persuasion game, where there are $n$ receivers and $m$ senders. Each receiver $i$ has a type $\theta_i \in \Theta_i$ and a value $v_i \in V_i$. The sender's goal is to persuade each receiver to choose a decision $y_i \in \mathcal{Y}_i$. The reward for the sender is given by $w(y_i, \theta_i)$, where $w$ is a function mapping any vector $x \in \mathcal{X}$ to a scalar.
are zero in all the components corresponding to feedbacks \( e \notin E \). Since it is the case that \( |E| \leq t \), the procedure in Algorithm 1 attains a per-iteration running time that is independent of the number of possible feedbacks \( p \).

**Algorithm 1 OGD-APO**

| Input: |
|--------|
| - approximate projection oracle \( \varphi_\alpha \) |
| - learning rate \( \eta \in (0, 1] \) |
| - approximation error \( \epsilon \in [0, 1] \) |

Initialize \( y^0 \in Y, E^0 \in \varnothing \), and \( x^1 \leftarrow 0 \in X_E \)

for \( t = 1, \ldots, T \) do

Take decision \( y^t \)

Observe feedback \( e^t \in E \) and reward \( u(y^t, e^t) = x^t_{e^t} \)

\( E^t \leftarrow E^t \cup \{ e^t \} \)

\( y^{t+1} \leftarrow x^t + \eta E^t \)

\( (x^{t+1}, y^{t+1}) \leftarrow \varphi_\alpha (E^t, y^{t+1}, \epsilon) \)

end for

Next, we bound the \( \alpha \)-regret incurred by Algorithm 1.

**Theorem 3.** Given an oracle \( \varphi_\alpha \) (as in Definition 1) for some \( 0 < \alpha \leq 1 \), a learning rate \( \eta \in (0, 1] \), and an approximation error \( \epsilon \in [0, 1] \), Algorithm 1 has \( \alpha \)-regret

\[
R_\alpha^T \leq \frac{|E_T|}{2\eta} + \frac{\eta T}{2} + \epsilon T
\]

with a per-iteration running time \( \text{poly}(t) \).

By setting \( \eta = \frac{1}{\sqrt{T}}, \epsilon = \frac{1}{T} \), we get \( R_\alpha^T \leq \sqrt{T} \left( 1 + \frac{|E_T|}{2} \right) \).

Notice that the bound only depends on the number of observed feedbacks \( |E_T| \), while it is independent of the overall number of possible feedbacks \( p \). This is crucial for the multi-receiver online Bayesian persuasion case, where \( p \) is exponential in the number of receivers \( n \). On the other hand, as \( T \) goes to infinity, we have \( |E_T| \leq p \), so that the regret bound is sublinear in \( T \).

### 6. Constructing a Poly-Time Approximate Projection Oracle

The crux of the OGD-APO algorithm (Algorithm 1) is being able to perform the approximate projection step. In this section, we show that, in the multi-receiver Bayesian persuasion setting, the approximate projection oracle \( \varphi_\alpha \) required by OGD-APO can be implemented in polynomial time by an appropriately-engineered ellipsoid algorithm. This calls for an approximate separation oracle \( \mathcal{O}_\alpha \) (see Definition 2).

We proceed as follows. In Section 6.1, we define an appropriate notion of approximate separation oracle, and show how to find, in polynomial time, an \( \alpha \)-approximate solution to the offline problem BAYESIAN-OPT-SIGNAL. This is a preparatory step towards the understanding of our main result in this section, and it may be of independent interest. Then, in Section 6.2, we exploit some of the techniques introduced for the offline setting in order to build \( \varphi_\alpha \) starting from an approximate separation oracle \( \mathcal{O}_\alpha \).

#### 6.1. Warming Up: The Offline Setting

An approximate separation oracle \( \mathcal{O}_\alpha \) finds a signal profile \( s \in S \) that approximately maximizes a weighted sum of the \( f_\theta \) functions, plus a weight for each receiver which depends on the signal \( s_r \) sent to that receiver. Formally:

**Definition 2** (Approximate separation oracle). Consider a state \( \theta \in \Theta \), a subset \( K \subseteq \mathcal{K} \), a vector \( \lambda \in \mathbb{R}^{|K|} \), weights \( w = (w_{r,s})_{r \in \mathcal{R}, s \in \mathcal{S}_r} \), with \( w_{r,s} \in \mathbb{R} \) and \( w_{r,s} = 0 \) for all \( r \in \mathcal{R} \), and an approximation error \( \epsilon \in \mathbb{R}_+ \). Then, for any \( 0 < \alpha \leq 1 \), an approximate oracle \( \mathcal{O}_\alpha(\theta, K, \lambda, w, \epsilon) \) is an algorithm returning an \( s \in S \) such that:

\[
\sum_{k \in K} \lambda_k f_\theta(R_k^s) + \sum_{r \in \mathcal{R}} w_{r,s_r} \geq \max_{s^* \in S} \left\{ \alpha \sum_{k \in K} \lambda_k f_\theta(R_k^{s^*}) + \sum_{r \in \mathcal{R}} w_{r,s_r^*} \right\} - \epsilon.
\]

As a preliminary result, we show how to use an oracle \( \mathcal{O}_\alpha \) to find in polynomial time an \( \alpha \)-approximate solution to BAYESIAN-OPT-SIGNAL (see Section 4). This problem is interesting in its own right, and allows us to develop a line of reasoning that will be essential to prove Theorem 5.

**Theorem 4.** Given \( \epsilon \in \mathbb{R}_+ \) and an approximate separation oracle \( \mathcal{O}_\alpha \), with \( 0 < \alpha \leq 1 \), there exists a polynomial-time approximation algorithm for BAYESIAN-OPT-SIGNAL returning a signaling scheme with sender’s utility at least \( \alpha \text{OPT} - \epsilon \), where \( \text{OPT} \) is the value of an optimal signaling scheme. Moreover, the algorithm works in time \( \text{poly}(\frac{1}{\epsilon}) \).

**Proof Overview.** The dual of LP (1) has a polynomial number of variables and an exponential number of constraints, and a natural way to prove polynomial-time solvability would be via the ellipsoid method (see, e.g., [Khachiyan, 1980; Grötschel et al., 1981]). However, in our setting, we can only rely on an approximate separation oracle, which renders the traditional ellipsoid method unsuitable for our problem. We show that it is possible to exploit a binary search scheme on the dual problem to find a value \( \gamma^* \in [0, 1] \) such that the dual problem with objective \( \gamma^* \) is feasible, while the dual with objective \( \gamma^* - \beta, \beta \geq 0 \), is infeasible. That algorithm runs in \( \log(1/\beta) \) steps. At each iteration of the algorithm, we solve a feasibility problem through the ellipsoid method equipped with an appropriate approximate separation oracle which we design. In order to build a
poly-time separation oracle we have to carefully manage all the settings in which $O_\alpha$ would not run in polynomial time, according to Definition 2. Specifically, we need to properly manage large values of the weights $w$, since $O_\alpha$ is polynomial in $\max_{r,s} |w_{r,s}|$. Once we do that, the approximate separation oracle is guaranteed to find a violated constraint, or to certify that all constraints are approximately satisfied. Finally, we show that the approximately feasible solution computed via bisection allows one to recover an approximate solution to the original problem.

6.2. From an Approximate Separation Oracle to an Approximate Projection Oracle

Now, we show how to design a polynomial-time approximate projection oracle $\varphi_\alpha$ using an approximate separation oracle $O_\alpha$. The proof employs a convex linearly-constrained quadratic program that computes the optimal projection on $X$, the ellipsoid method, and a careful primal-dual analysis.

Theorem 5. Given a subset $K \subseteq \cal K$, a vector $y \in [0,2]^{|K|}$ such that $y_k = 0$ for all $k \notin K$, and an approximation error $\epsilon \in \mathbb{R}_+$, for any $0 < \alpha \leq 1$, the approximate projection oracle $\varphi_\alpha(K, y, \epsilon)$ can be computed in polynomial time by querying the approximate separation oracle $O_\alpha$.

Proof Overview. We start by defining a convex minimization problem, which we denote by (\ref{eq:general}), for computing the projection of $y$ on $X_K$. Then, we work on the dual of (\ref{eq:general}), which we suitably simplify by reasoning over the KKT conditions of the problem. As in the proof of Theorem 4, we proceed by repeatedly applying the ellipsoid method on a feasibility problem obtained from the dual, decreasing the required objective $\gamma^*$ by a small additive factor $\beta$. The ellipsoid method is equipped with the approximate separation oracle that employs the oracle in Definition 2 and carefully manages the cases in which $O_\alpha$ would not run in polynomial time. In this case, the problem is complicated by the fact that we have to determine an approximate projection over $\alpha X_K$, rather than an approximate solution to (\ref{eq:general}). We found two dual problems such that one dual problem with objective $\gamma^*$ is feasible, while the second one with objective $\gamma^* + \beta$ is infeasible. From these problems, we define a new convex optimization problem that is a modified version of (\ref{eq:general}) and has value at least $\gamma^*$. Then, we show that a solution to this problem is close to a projection on a set which includes $\alpha X_K$. Finally, we restrict (\ref{eq:general}) to the primal variables corresponding to the set of (polynomially-many) violated dual constraints determined during the last application of the ellipsoid method that returns unfeasible, i.e., where the ellipsoid method for feasibility problem is run with objective $\gamma^* + \beta$. We conclude the proof by showing that a solution to this restricted problem is precisely an approximate projection on a superset of $\alpha X_K$.

7. A Poly-Time No-$\alpha$-Regret Algorithm for Submodular Sender’s Utilities

In this section, we conclude the construction of our polynomial-time no-$(1 - \frac{1}{e})$-regret algorithm for settings in which sender’s utilities are submodular. The last component that we need to design is an approximate separation oracle $O_\alpha$ (see Definition 2) running in polynomial time. Next, we show how to obtain this by exploiting the fact that functions $f_\theta$ are submodular in the set of receivers playing action $a_1$.

First, we establish a relation between direct signals $S$ and matroids. We define a matroid $M_S := (G_S, I_S)$ such that:

- the ground set is $G_S := \{(r, s) \mid r \in \mathbb{R}, s \in S_r\}$;
- a subset $I \subseteq G_S$ belongs to $I_S$ if and only if $I$ contains at most one pair for each receiver $r \in \mathbb{R}$.

The elements of the ground set $G_S$ represent receiver, signal pairs. However, sets $I \in I_S$ do not characterize signal profiles, as they may not define a signal for each receiver. Indeed, direct signal profiles are captured by the basis set $B(M_S)$ of the matroid $M_S$. Let us recall that $B(M_S)$ contains all the maximal sets in $I_S$, and, thus, a subset $I \subseteq I_S$ belongs to $B(M_S)$ if and only if $I$ contains exactly one pair for each receiver $r \in \mathbb{R}$. Intuitively, a basis $I \in B(M_S)$ defines a direct signal profile $s \in S$ in which, for each receiver $r \in \mathbb{R}$, all the receiver’s types in $s \in S_r$ such that $(r, s) \in I$ are recommended to play action $a_1$, while the others are told to play $a_0$.

The following Theorem 6 provides a polynomial-time approximation oracle $O_{1 - \frac{1}{e}}$ for instances in which $f_\theta$ is submodular for each state of nature $\theta \in \Theta$. The core idea of its proof is that $\sum_{k \in K} \lambda_k f_\theta(R_k^a)$ (see Equation (3)) can be seen as a submodular function defined for the ground set $G_S$ and optimizing over direct signal profiles $s \in S$ is equivalent to doing that over the bases $B(M_S)$ of the matroid $M_S$. Then, the result is readily proved by exploiting some results concerning the optimization over matroids.9

Theorem 6. If the sender’s utility is such that function $f_\theta$ is submodular for each $\theta \in \Theta$, then there exists a polynomial-time separation oracle $O_{1 - \frac{1}{e}}$.

In conclusion, by letting $\mathcal{K}^T \subseteq \mathcal{K}$ be the set of receivers’ type profiles observed by the sender up to iteration $T$, the following Theorem 7 provides our polynomial-time no-$(1 - \frac{1}{e})$-regret algorithm working with submodular sender’s utilities.

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9The separation oracle provided in Theorem 6 guarantees the desired approximation factor with arbitrary high probability. It is easy to see that, since the algorithm fails with arbitrary small probability, this does not modify our regret bound except for an (arbitrary small) negligible term.
Theorem 7. If the sender’s utility is such that function $f_\theta$ is submodular for each $\theta \in \Theta$, then there exists a no-$\frac{1}{\epsilon^2}$-regret algorithm having $(1 - \frac{1}{10})$-regret

$$R^T_{1-\frac{1}{10}} \leq O \left( \sqrt{T} |K^T| \right),$$

with a per-iteration running time polynomial in $T, n, d$.

Proof. We can run Algorithm 1 on an instance of our multi-receiver online Bayesian persuasion problem. By Theorem 3, if we set $\eta = \frac{1}{\sqrt{T}}, \epsilon = \frac{1}{10}$, and $\alpha = 1 - \frac{1}{10}$, we get the desired regret bound (notice that the set of observed feedbacks is $E^T = K^T$ in our setting). Algorithm 1 employs an approximate projection oracle $\varphi_{1-\frac{1}{10}}$ that we can implement in polynomial time by using the algorithm provided in Theorem 5. This requires access to a polynomial-time approximate separation oracle $O_{1-\frac{1}{10}}$, which can be implemented by using Theorem 6, under the assumption that the sender’s utility is such that functions $f_\theta$ are submodular.

Notice that the regret bound only depends on the number $|K^T|$ of receivers’ type profiles observed up to iteration $T$, while it is independent of the overall number of possible type profiles $|K| = m^n$, which is exponential in the number of receivers. Thus, the $(1 - \frac{1}{10})$-regret is polynomial in the size of the problem instance provided that the type profiles received as feedbacks by the sender are polynomially many (though the sender does not have to know which are these type profiles in advance). This is reasonable in many practical applications, where not all the type profiles can occur, since, e.g., receivers’ types are highly correlated. On the other hand, let us remark that, as $T$ goes to infinity, we have $|K^T| \leq m^n$, so that the regret is sublinear in $T$.

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A. Proofs Omitted from Section 4

In this section, we provide the complete proof of the hardness result in Theorem 1. This is based on a reduction from the promise-problem version of LABEL-COVER, which we define next.

The following is the formal definition of an instance of the LABEL-COVER problem.

Definition 3 (LABEL-COVER instance). An instance of LABEL-COVER consists of a tuple \((G, \Sigma, \Pi)\), where:

- \(G := (U, V, E)\) is a bipartite graph defined by two disjoint sets of nodes \(U\) and \(V\), connected by the edges in \(E \subseteq U \times V\), which are such that all the nodes in \(U\) have the same degree;
- \(\Sigma\) is a finite set of labels; and
- \(\Pi := \{\Pi_e : \Sigma \rightarrow \Sigma \mid e \in E\}\) is a finite set of edge constraints.

Definition 4 (Labeling). Given an instance \((G, \Sigma, \Pi)\) of LABEL-COVER, a labeling of the graph \(G\) is a mapping \(\pi : U \cup V \rightarrow \Sigma\) that assigns a label to each vertex of \(G\) such that all the edge constraints are satisfied. Formally, a labeling \(\pi\) satisfies the constraint for an edge \(e = (u, v) \in E\) if \(\pi(v) = \Pi_e(\pi(u))\).

The classical LABEL-COVER problem is the search problem of finding a valid labeling for a LABEL-COVER instance given as input. In the following, we consider a different version of the problem, which is the promise problem associated with LABEL-COVER instances, defined as follows.

Definition 5 (GAP-LABEL-COVER\(_{c,b}\)). For any pair of numbers \(0 < b < c < 1\), we define GAP-LABEL-COVER\(_{c,b}\) as the following promise problem.

- **Input**: An instance \((G, \Sigma, \Pi)\) of LABEL-COVER such that either one of the following is true:
  - there exists a labeling \(\pi : U \cup V \rightarrow \Sigma\) that satisfies at least a fraction \(c\) of the edge constraints in \(\Pi\);
  - any labeling \(\pi : U \cup V \rightarrow \Sigma\) satisfies less than a fraction \(b\) of the edge constraints in \(\Pi\).
- **Output**: Determine which of the above two cases hold.

In order to prove Theorem 1, we make use of the following result due to Raz (1998) and Arora et al. (1998).

Theorem 8 (Raz (1998); Arora et al. (1998)). For any \(\epsilon > 0\), there exists a constant \(k_\epsilon \in \mathbb{N}\) that depends on \(\epsilon\) such that the promise problem GAP-LABEL-COVER\(_{1,\epsilon}\) restricted to inputs \((G, \Sigma, \Pi)\) with \(|\Sigma| = k_\epsilon\) is \(\text{NP}\)-hard.

Next, we provide the complete proof of Theorem 1.

Theorem 1. For every \(0 < \alpha \leq 1\), it is \(\text{NP}\)-hard to compute an \(\alpha\)-approximate solution to BAYESIAN-OPT-SIGNAL, even when the sender’s utility is such that, for every \(\theta \in \Theta\), \(f_\theta(R) = 1\) iff \(|R| \geq 2\), while \(f_\theta(R) = 0\) otherwise.

Proof. We provide a reduction from GAP-LABEL-COVER\(_{1,\epsilon}\). Our reduction maps an instance \((G, \Sigma, \Pi)\) of LABEL-COVER to an instance of BAYESIAN-OPT-SIGNAL with the following properties:

- **(completeness)** if the LABEL-COVER instance admits a labeling satisfying all the edge constraints (recall \(c = 1\)), then the BAYESIAN-OPT-SIGNAL instance has a signaling scheme with sender’s expected utility \(\geq \left(1 - \frac{\epsilon}{|\Sigma|}\right) \frac{1}{|\Sigma|} \geq \frac{1}{2|\Sigma|}\);
- **(soundness)** if the LABEL-COVER instance is such that any labeling satisfies at most a fraction \(\epsilon\) of the edge constraints, then an optimal signaling scheme in the BAYESIAN-OPT-SIGNAL instance has sender’s expected utility at most \(\frac{2\epsilon}{|\Sigma|}\).

By Theorem 8, for any \(\epsilon > 0\) there exists a constant \(k_\epsilon \in \mathbb{N}\) that depends on \(\epsilon\) such that GAP-LABEL-COVER\(_{1,\epsilon}\) restricted to inputs \((G, \Sigma, \Pi)\) with \(|\Sigma| = k_\epsilon\) is \(\text{NP}\)-hard. Given \(0 < \alpha \leq 1\), by setting \(\epsilon = \frac{\alpha}{2|\Sigma|}\) and noticing that \(\frac{2\epsilon}{|\Sigma|} = 4\epsilon = \alpha\), we can conclude that it is \(\text{NP}\)-hard to compute an \(\alpha\)-approximate solution to BAYESIAN-OPT-SIGNAL.
Construction Given an instance \((G, \Sigma, \Pi)\) of LABEL-COVER defined over a bipartite graph \(G := (U, V, E)\), we build an instance of BAYESIAN-OPT-SIGNAL as follows.

- For each label \(\sigma \in \Sigma\), there is a corresponding state of nature \(\theta_\sigma \in \Theta\). Moreover, there is an additional state \(\theta_0 \in \Theta\). Thus, the total number of possible states is \(d = |\Sigma| + 1\).
- The prior distribution is \(\mu \in \text{int}(\Delta_\Theta)\) such that \(\mu_{\sigma} = \frac{\epsilon}{|\Sigma|}\) for every \(\theta_\sigma \in \Theta\) and \(\mu_{\theta_0} = 1 - \frac{\epsilon}{|\Sigma|}\).
- For every vertex \(v \in U \cup V\) of the graph \(G\), there is a receiver \(r_v \in \mathcal{R}\). Thus, \(n = |U \cup V|\).
- Each receiver \(r_v \in \mathcal{R}\) has \(m_{r_v} = |\Sigma| + 1\) possible types. The set of types of receiver \(r_v\) is \(\mathcal{K}_{r_v} = \{k_\sigma \mid \sigma \in \Sigma\} \cup \{k_0\}\).
- A receiver \(r_v \in \mathcal{R}\) of type \(k_\sigma \in \mathcal{K}_{r_v}\) has utility such that \(u_{\theta_\sigma, k_\sigma} = \frac{1}{\mu_{\theta_\sigma}}\) and \(u_{\theta_0, k_0} = -1\) for all \(\theta_\sigma \in \Theta : \theta_\sigma \neq \theta_0\), while \(u_{\theta_0, k_0} = -\frac{1}{\mu_{\theta_0}}\). Moreover, a receiver \(r_v \in \mathcal{R}\) of type \(k_0\) has utility such that \(u_{\theta_0, k_0} = -1\) for all \(\theta_0 \in \Theta\).
- The sender’s utility is such that, for every \(\theta \in \Theta\), the function \(f_\theta : 2^\mathcal{R} \rightarrow [0, 1]\) satisfies \(f_\theta(R) = 1\) if and only if \(R \subseteq \mathcal{R} : |R| \geq 2\), while \(f_\theta(R) = 0\) otherwise.
- The subset \(K \subseteq \mathcal{K}\) of type profiles that can occur with positive probability is \(K := \{k_{uv, \sigma} \mid e = (u, v) \in E, \sigma \in \Sigma\}\), where, for every edge \(e = (u, v) \in E\) and label \(\sigma \in \Sigma\), the type profile \(k_{uv, \sigma} \in \mathcal{K}\) is such that \(k_{uv, \sigma} = k_{\sigma}\), \(k_{uv, \sigma} = k_{\sigma'}\) with \(\sigma' = \Pi_e(u)\), and \(k_{uv, \sigma} = k_{0}\) for every \(\sigma' \in \mathcal{R} : r_{\sigma'} \notin \{r_u, r_v\}\).
- The probability distribution \(\lambda \in \text{int}(\Delta_K)\) is such that \(\lambda_k = \frac{1}{|E| \cdot |\Sigma|}\) for every \(k \in K\).

Notice that, in the BAYESIAN-OPT-SIGNAL instances used for the reduction, the sender’s payoff is 1 if and only if at least two receivers play action \(a_1\), while it is 0 otherwise. Let us also recall that direct signals for a receiver \(r_v \in \mathcal{R}\) are defined by the set \(\mathcal{S}_{r_v} := 2^{K_{r_v}}\), with a signal being represented as the set of receiver’s types that are recommended to play action \(a_1\).

Completeness Let \(\pi : U \cup V \rightarrow \Sigma\) be a labeling of the graph \(G\) that satisfies all the edge constraints. We define a corresponding direct signaling scheme \(\phi : \Theta \rightarrow \Delta_\Sigma\) as follows. For any label \(\sigma \in \Sigma\), let \(s^\sigma \in \mathcal{S}\) be a signal profile such that the signal sent to receiver \(r_v \in \mathcal{R}\) is \(s^\sigma = \{k_\sigma\}\) if \(\sigma \in \Sigma\) is only one signal profile (with probability one). As a first step, we prove that the signaling scheme \(\phi\) is persuasive. Let us fix a receiver \(r_v \in \mathcal{R}\). After receiving a signal \(s = \{k_\sigma\} \in \mathcal{S}_{r_v}\) with \(\sigma \in \Sigma : \sigma \neq \pi(v)\), by definition of \(\phi\), the receiver’s posterior belief is such that state of nature \(\theta_\sigma\) is assigned probability one. Thus, if the receiver has type \(k_{\sigma}\), then she/he is incentivized to play action \(a_1\), since \(u_{\theta_\sigma, k_\sigma} = \frac{1}{\mu_{\theta_\sigma}} > 0\) (recall that \(u_{\theta_\sigma, k_\sigma}\) is the utility different “action \(a_1\) minus action \(a_0\)“ when the state is \(\theta_\sigma\)). Instead, if the receiver has type \(k \in \mathcal{K}_{r_v} : k \neq k_{\sigma}\), then she/he is incentivized to play action \(a_0\), since either \(k = k_0\) and \(u_{\theta_0, k_0} = -1 < 0\) or \(k = k_{\sigma'}\) with \(\sigma' \in \Sigma : \sigma' \neq \sigma\) and \(u_{\theta_0, k_0} = -1 < 0\). After receiving a signal \(s = \{k_\sigma\} \in \mathcal{S}_{r_v}\) with \(\sigma = \pi(v)\), the receiver’s posterior belief is such that the states of nature \(\theta_\sigma\) and \(\theta_0\) are assigned probabilities proportional to their corresponding prior probabilities, respectively \(\mu_{\theta_\sigma}\) and \(\mu_{\theta_0}\) (she/he cannot tell whether \(\sigma_\sigma\) or \(\sigma_\sigma\) has been selected by the sender). Thus, if the receiver has type \(k_{\sigma}\), then she/he is incentivized to play action \(a_1\), since her/his expected utility difference “action \(a_1\) minus action \(a_0\)“ is the following:

\[
\frac{\mu_{\theta_\sigma}}{\mu_{\theta_\sigma} + \mu_{\theta_0}} u_{\theta_\sigma, k_\sigma} + \frac{\mu_{\theta_0}}{\mu_{\theta_\sigma} + \mu_{\theta_0}} u_{\theta_0, k_0} = \frac{1}{\mu_{\theta_\sigma} + \mu_{\theta_0}} \left[ \frac{\epsilon}{|\Sigma|^2} - \left(1 - \frac{\epsilon}{|\Sigma|^2}\right) \right] > \frac{1}{\mu_{\theta_\sigma} + \mu_{\theta_0}} \left[ \frac{\epsilon}{2|\Sigma|^2} - \frac{\epsilon}{2|\Sigma|^2}\right] = 0.
\]

If the receiver has a type different from \(k_{\sigma}\), simple arguments show that the expected utility difference is negative, incentivizing action \(a_0\). This proves that the signaling scheme \(\phi\) is persuasive. Next, we bound the sender’s expected utility in \(\phi\). Notice that, when the state of nature is \(\theta_0\), if the receivers’ type profile is \(k_{uv, \sigma} \in K\) with \(\sigma = \pi(u)\) for some edge \(e = (u, v) \in E\), then both receivers \(r_u\) and \(r_v\) play action \(a_1\). This is readily proved since \(k_{uv, \sigma} = k_{\sigma}\) and \(k_{uv, \sigma} = k_{\sigma'}\) with \(\sigma = \pi(u)\) and \(\sigma' = \pi(v)\) (recall that \(\pi(v) = \Pi_e(u)\) as \(\phi\) satisfies all the edge constraints), and, thus, both \(r_u\) and \(r_v\) are recommended to play \(a_1\) when the state is \(\theta_0\). As a result, under signaling scheme \(\phi\), when the receivers’ type profile
is $k_{uv, \sigma} \in K$, then the sender’s resulting payoff is one (recall the definition of functions $f_\theta$). By recalling that each type profile $k_{uv, \sigma} \in K$ with $\sigma = \pi(u)$ (for each edge $e = (u, v) \in E$) occurs with probability $\lambda_{k_{uv, \sigma}} = \frac{1}{|E||\Sigma|}$, we can lower bound the sender’s expected utility (see the objective of Problem (1)) as follows:

$$\sum_{k \in K} \lambda_k \sum_{\theta \in \Theta} \mu_{\theta} \sum_{s \in S} \phi_{\theta}(s) f_\theta (R_k^s) \geq \mu_{\theta_0} \sum_{k_{uv, \sigma} \in K : \sigma = \pi(u)} \lambda_{k_{uv, \sigma}} = \mu_{\theta_0} \frac{1}{|\Sigma|} \left(1 - \frac{\epsilon}{|\Sigma|}\right) \frac{1}{|\Sigma|}.$$  

**Soundness** By contradiction, suppose that there exists a direct and persuasive signaling scheme $\phi : \Theta \to \Delta_S$ that provides the sender with an expected utility greater than $\frac{1}{|\Sigma|}$. Since the sender can extract an expected utility at most $\frac{1}{|\Sigma|}$ from states of nature $\theta \in \Theta$ with $\theta \neq \theta_0$ (as $\sum_{\theta \in \Theta : \theta \neq \theta_0} \mu_{\theta} = \frac{1}{|\Theta|}$ and the maximum value of functions $f_\theta$ is one), then it must be the case that the expected utility contribution due to state $\theta_0$ is greater than $\frac{1}{|\Sigma|}$. Let us consider the distribution over signal profiles $\phi_{\theta_0} \in \Delta_S$ induced by state of nature $\theta_0$. We prove that, for each signal profile $s \in S$ such that $\phi_{\theta_0}(s) > 0$ and each receiver $r_v \in R$, it must hold that $|s_v| \leq 1$, i.e., at most one type of receiver $r_v$ is recommended to play $a_1$. First, notice that a receiver of type $k_0$ cannot be incentivized to play $a_1$, since $u_\theta^{r_v, k_0} = -1$ for all $\theta \in \Theta$. By contradiction, suppose that there are two receiver’s types $k_\sigma, k_\sigma' \in K_{r_v}$ with $k_\sigma \neq k_\sigma'$ such that $k_\sigma, k_\sigma' \in s_v$ (i.e., they are both recommended to play $a_1$). By letting $\xi \in \Delta \Theta$ be the posterior belief of receiver $r_v$ induced by $s_v$, for type $k_\sigma$ it must be the case that:

$$\xi_{\theta_0} u^{r_v, k_\sigma}_\theta + \sum_{\theta \in \Theta : \theta \neq \theta_0} \xi_{\theta \sigma'} u^{r_v, k_\sigma'}_\theta + \xi_{\theta_0} u^{r_v, k_\sigma}_0 = \frac{1}{2} \xi \theta_0 - \sum_{\theta \in \Theta : \theta \neq \theta_0} \xi_{\theta \sigma'} - \frac{\epsilon}{|\Sigma|} \xi_{\theta_0} > 0,$$

since the signaling scheme is persuasive, and, thus, a receiver of type $k_\sigma$ must be incentivized to play action $a_1$. This implies that $\xi_{\theta_0} > \frac{2}{\sum_{\theta \in \Theta : \theta \neq \theta_0} \xi_{\theta \sigma'}} \geq 2 \xi_{\theta_0}$. Analogous arguments for type $k_\sigma'$ imply that $\xi_{\theta_0} > 2 \xi_{\theta_0}$, reaching a contradiction. This shows that, for each $s \in S$ such that $\phi_{\theta_0}(s) > 0$ and each $r_v \in R$, it must be the case that $|s_v| \leq 1$. Next, we provide the last contradiction proving the result. Let us recall that, by assumption, the sender’s expected utility contribution due to $\theta_0$ is $\sum_{k \in K} \lambda_k \sum_{s \in S} \phi_{\theta_0}(s) f_\theta (R_k^s) \geq \frac{\epsilon}{|\Sigma|}$. By an averaging argument, this implies that there must exist a signal profile $s \in S$ such that $\phi_{\theta_0}(s) > 0$ and $\sum_{k \in K} \lambda_k f_\theta (R_k^s) \geq \frac{\epsilon}{|\Sigma|}$. Let $s \in S$ be such signal profile. Let us define a corresponding labeling $\sigma : U \cup V \to \Sigma$ of the graph $G$ such that, for every vertex $v \in U \cup V$, it holds $\sigma(v) = \sigma$, where $\sigma \in \Sigma$ is the label corresponding to the unique type $k_\sigma$ of receiver $r_v$ that is recommended to play action $a_1$ under $s$ (if any, otherwise any label is fine). Since $\sum_{k \in K} \lambda_k f_\theta (R_k^s) \geq \frac{\epsilon}{|\Sigma|}$ and it holds $\lambda_k = \frac{1}{|E||\Sigma|}$ and $f_\theta (R_k^s) \in [0, 1)$ for every $k \in K$, it must be the case that there are at least $\epsilon |E|$ type profiles $k \in K$ such that $f_\theta (R_k^s) = 1$. Since a receiver of type $k_0$ cannot be incentivized to play action $a_1$, the value of $f_\theta (R_k^s)$ can be one only if there are at least two receivers with types different from $k_0$ that play action $a_1$. Thus, it must hold that $f_\theta (R_k^s) = 0$ for all the type profiles $k_{uv, \sigma} \in K$ such that $\sigma \neq \pi(u)$ (as $k_{uv, \sigma}$ would be equal to $k_0$ with $\sigma \neq \pi(u)$ and $k_0 \notin s_v$). For the type profiles $k_{uv, \sigma} \in K$ such that $\sigma = \pi(u)$ (one per edge $e = (u, v) \in E$ of the graph $G$), the value of $f_\theta (R_k^s)$ is one if and only if $\pi(v) = \Pi_1(u)$, so that both receivers $r_u$ and $r_v$ are told to play action $a_1$. As a result, this implies that there must be at least $\epsilon |E|$ edges $e \in E$ for which the labeling $\pi$ satisfies the corresponding edge constraint $\Pi_1$, which is a contradiction. 

\[ \square \]

**B. Proofs Omitted from Section 5**

**Theorem 3.** Given an oracle $\varphi_\alpha$ (as in Definition 1) for some $0 < \alpha \leq 1$, a learning rate $\eta \in (0, 1]$, and an approximation error $\epsilon \in [0, 1]$, Algorithm 1 has $\alpha$-regret

$$R^* \leq \frac{|E|^2}{2\eta} + \frac{\eta T}{2} + \frac{\epsilon T}{2\eta},$$

with a per-iteration running time $poly(t)$.

**Proof.** First, we bound the per-iteration running time of Algorithm 1. For any $t \in [T]$, we have $E^t = \bigcup_{t' \in [t]} E^{t'}$, which represents the set of feedbacks observed up to iteration $t$. Thus, it holds $|E^t| \leq t$. At iteration $t \in [T]$, the algorithm works with vectors $x^t$ and $y^{t+1}$. The first one belongs to $X^{E^t-1}$ (as it is returned by $\varphi_\alpha$ at iteration $t-1$), and, thus, it has at most $t-1$ non-zero components. Similarly, since $y^{t+1} = x^t + \eta \lambda x_t$, it holds that $y^{t+1} \in [0, 2]^p$ and $y_{x^t} = 0$ for all $e \notin E^t$, which implies that $y^{t+1}$ has at most $t$ non-zero components. As a result, we can sparsely represent vectors $x^t$ and $y^{t+1}$ so that Algorithm 1 has a per-iteration running time bounded by $t$ for any iteration $t \in [T]$, independently of the actual size $p$ of the vectors. Moreover, notice that $y^{t+1}$ satisfies the conditions required by the inputs of the oracle $\varphi_\alpha$. 

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Next, we bound the $\alpha$-regret of Algorithm 1. For the ease of notation, in the following, for any vector $x \in \mathcal{X}$ and subset $E \subseteq \mathcal{E}$, we let $x_E := r_E(x)$. Moreover, for any $t \in [T]$, we let $\mathbb{1}_t := \mathbb{1}\{ e^t \notin E^{t-1} \}$, which is the indicator function that is equal to 1 if and only if $e^t \notin E^{t-1}$, i.e., when the feedback $e^t$ at iteration $t$ has never been observed before. Fix $x \in \alpha \mathcal{X}$. Then, the following relations hold:

$$
||x_{E^t} - x^{t+1}||^2 \leq ||x_{E^t} - y^{t+1}||^2 + \epsilon \tag{4a}
$$

$$
= ||x_{E^t} - x^t - \eta 1_{E^t}||^2 + \epsilon \tag{4b}
$$

$$
= ||x_{E^{t-1}} + \mathbb{1}_t x_{e^t} 1_{e^t} - x^t||^2 + \eta^2 - 2\eta 1_{e^t}^T (x_{E^{t-1}} + \mathbb{1}_t x_{e^t} 1_{e^t} - x^t) + \epsilon \tag{4c}
$$

$$
= ||x_{E^{t-1}} - x^t||^2 + \mathbb{1}_t ||x_{e^t} - x^t||^2 + \eta^2 - 2\eta 1_{e^t}^T (x_{E^{t-1}} + \mathbb{1}_t x_{e^t} 1_{e^t} - x^t) + \epsilon \tag{4d}
$$

Notice that Equation (4b) holds by definition of $\varphi_\alpha$ since $x_{E^t} \in \alpha \mathcal{X}$, Equation (4d) follows from $x_{E^t} = x_{E^{t-1}} + \mathbb{1}_t x_{e^t} 1_{e^t}$, while Equation (4e) can be derived by decomposing the first squared norm in the preceding expression. By using the last relation above, we can write the following:

$$
\sum_{t \in [T]} 1_{e^t}^T (x - x^t) = \sum_{t \in [T]} 1_{e^t}^T (x_{E^{t-1}} + \mathbb{1}_t x_{e^t} 1_{e^t} - x^t) \tag{5a}
$$

$$
\leq \frac{1}{2\eta} \sum_{t \in [T]} \left( ||x_{E^{t-1}} - x^t||^2 - ||x_{E^t} - x^{t+1}||^2 + \mathbb{1}_t + \eta^2 + \epsilon \right) \tag{5b}
$$

$$
= \frac{1}{2\eta} \sum_{t \in [T]} \left( \mathbb{1}_t + \eta^2 + \epsilon \right) \tag{5c}
$$

$$
= \frac{1}{2\eta} \left( |E^T| + T \eta^2 + T \epsilon \right), \tag{5d}
$$

where Equation (5c) is obtained by telescoping the sum. Then, the following concludes the proof:

$$
R^T_\alpha := \alpha \max_{y \in \mathcal{Y}} \sum_{t \in [T]} u(y, e^t) - \sum_{t \in [T]} u(y^t, e^t) \leq \alpha \max_{x \in \mathcal{X}} \sum_{t \in [T]} x_{e^t} - \sum_{t \in [T]} x^t = \alpha \max_{x \in \mathcal{X}} \sum_{t \in [T]} 1_{e^t}^T (x - x^t) = \max_{x \in \alpha \mathcal{X}} \sum_{t \in [T]} 1_{e^t}^T (x - x^t) \leq \frac{1}{2\eta} \left( |E^T| + T \eta^2 + T \epsilon \right).
$$

**C. Proofs Omitted from Section 6.1**

**Theorem 4.** Given $\epsilon \in \mathbb{R}_+$ and an approximate separation oracle $O_\alpha$, with $0 < \alpha \leq 1$, there exists a polynomial-time approximation algorithm for \textsc{Bayesian-opt-signal} returning a signaling scheme with sender’s utility at least $\alpha \text{OPT} - \epsilon$, where $\text{OPT}$ is the value of an optimal signaling scheme. Moreover, the algorithm works in time $\text{poly}(\frac{1}{\epsilon})$.

**Proof of theorem 4.** The dual problem of LP (1) reads as follows:

$$
\min_{z,d} \sum_{\theta \in \Theta} d_\theta \tag{6a}
$$

$$
\text{s.t. } \mu_\theta \sum_{r \in \mathcal{R}} \sum_{k \in \mathcal{K}_r} u_{\theta}^k z_{r,s,r,k} + d_\theta \geq \sum_{k \in \mathcal{K}} \lambda_k f_\theta (R^k_s) \quad \forall \theta \in \Theta, \forall s \in \mathcal{S} \tag{6b}
$$

$$
z_{r,s,k} \leq 0 \quad \forall r \in \mathcal{R}, \forall s \in \mathcal{S}_r, \forall k \in \mathcal{K}_r : k \in s, \tag{6c}
$$
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where \( \mathbf{d} \in \mathbb{R}^{|\Theta|} \) is the vector of dual variable corresponding to the primal Constraints (1c), and \( \mathbf{z} \in \mathbb{R}_-^{\lceil R \times S_r \times K_r \rceil} \) is the vector of dual variable corresponding to Constraints (1b) in the primal. We rewrite the dual LP (6) so as to highlight the relation between an approximate separation oracle for Constraints (6b) and the oracle \( O_\alpha \). Specifically, we have

\[
\min_{\mathbf{d} \geq 0} \sum_{\theta \in \Theta} d_\theta \quad \text{s.t. } d_\theta \geq \mu_\theta \left( \sum_{k \in K} \lambda_k f_\theta(R^k_a) + \sum_{r \in R} \sum_{k \in S_r} u^r_{\theta,k} z_{r,s_r,k} \right) \quad \forall \theta \in \Theta, \forall s \in S. \tag{7a}
\]

Now, we show that it is possible to build a binary search scheme to find a value \( \gamma^* \in [0, 1] \) such that the dual problem with objective \( \gamma^* \) is feasible, while the dual with objective \( \gamma^* - \beta \) is infeasible. The constant \( \beta \geq 0 \) will be specified later in the proof. The algorithm requires \( \log(\beta) \) steps and works by determining, for a given value \( \bar{\gamma} \in [0, 1] \), whether there exists a feasible pair \((\mathbf{d}, z)\) for the following feasibility problem \( \mathcal{P} \):

\[
\begin{align*}
\mathcal{P} : & \quad \sum_{\theta \in \Theta} d_\theta \leq \bar{\gamma} \\
& \quad d_\theta \geq \mu_\theta \left( \sum_{k \in K} \lambda_k f_\theta(R^k_a) + \sum_{r \in R} \sum_{k \in S_r} u^r_{\theta,k} z_{r,s_r,k} \right) \quad \forall \theta \in \Theta, \forall s \in S \\
& \quad z \geq 0.
\end{align*}
\]

At each iteration of the bisection algorithm, the feasibility problem \( \mathcal{P} \) is solved via the ellipsoid method. The algorithm is initialized with \( l = 0, h = 1 \), and \( \bar{\gamma} = \frac{1}{2} \). If \( \mathcal{P} \) is infeasible for \( \bar{\gamma} \), the algorithm sets \( l \leftarrow (l + h)/2 \) and \( \bar{\gamma} \leftarrow (h + \bar{\gamma})/2 \). Otherwise, if \( \mathcal{P} \) is (approximately) feasible, it sets \( h \leftarrow (l + h)/2 \) and \( \bar{\gamma} \leftarrow (l + \bar{\gamma})/2 \). Then, the procedure is repeated with the updated value of \( \bar{\gamma} \). The bisection procedure terminates when it determines a value \( \gamma^* \) such that \( \mathcal{P} \) is feasible for \( \gamma = \gamma^* \), while it is infeasible for \( \gamma = \gamma^* - \beta \). In the following, we present the approximate separation oracle which is employed at each iteration of the ellipsoid method.

**Separation Oracle** Given a point \((\mathbf{d}, z)\) in the dual space, and \( \bar{\gamma} \in [0, 1] \), we design an approximate separation oracle to determine if the point \((\mathbf{d}, z)\) is approximately feasible, or to determine a constraint of \( \mathcal{P} \) that is violated by such point. For each \( \theta \in \Theta, r \in R \), and \( s \in S_r \), let

\[
u^r_{\theta,s} := \mu_\theta \sum_{k \in K} u^r_{\theta,k} z_{r,s_r,k}.
\]

When the magnitude of the weights \( |\nu^r_{\theta,s}| \) is small, we show that it is enough to employ the optimization oracle \( O_\alpha \) in order to find a violated constraint, or to certify that all the constraints are approximately satisfied. On the other hand, when the weights \( |\nu^r_{\theta,s}| \) are large (in particular, when the largest weight has exponential size in the size of the problem instance), the optimization oracle \( O_\alpha \) loses its polynomial time guarantees (see Definition 2). We show how to handle those specific settings in the following case analysis:

- Equation (7b) implies that \( d_\theta \geq 0 \) for each \( \theta \in \Theta \). Then, if there exists a \( \theta \in \Theta \) such that \( \bar{d}_\theta < 0 \), we return the violated constraint \((\theta, \varnothing)\) (that is, \( d_\theta \geq 0 \)).

- If there exists \( \theta \in \Theta \) such that \( \bar{d}_\theta > 1 \), then the first constraint of \( \mathcal{P} \) must be violated as \( \bar{\gamma} \in [0, 1] \).

- If there exists a receiver \( r \in R \) and a signal \( s \in S_r \) such that \( \nu^r_{\theta,s} > 1 \), then the constraint of \( \mathcal{P} \) corresponding to the pair \((\theta, s)\) is violated, because \( d_\theta \leq 1 \).

- If no violated constraint was found in the previous steps, we proceed by checking if there exists a state \( \theta' \in \Theta \), a receiver \( r' \in R \), and a signal \( s' \in S_r \), such that \( \nu^r_{\theta',s'} \leq -|R| \). If this is the case, we observe that for any pair \((\theta', s')\), with \( s \in S : s_r = s' \), the corresponding constraint in \( \mathcal{P} \) reads

\[
\mu_\theta \sum_{k \in K} \lambda_k f_{\theta'}(R^k_a) + \sum_{r \in R \setminus \{r'\}} \nu^r_{\theta,s} + \nu^r_{\theta',s'} \leq 0,
\]

since \( \bar{d} \geq 0 \) if the current step is reached. For \( \nu^r_{\theta',s'} \leq -|R| \), the above constraints are trivially satisfied, and therefore we can safely manage (for the current iteration of the ellipsoid method) any such constraint by setting \( \nu^r_{\theta',s'} = -|R| \).
If none of the previous steps returned a violated constraint, we can safely assume that \( 0 \leq d_0 \leq 1 \) and \(-|R| \leq w^\theta_{r,s} \leq 1\), for each \( \Theta \in \Theta, r \in R, \) and \( s \in S_r \). Moreover, we observe that, by definition, for each \( r \in R \) and \( \Theta \in \Theta \), it holds \( w^\theta_{r,\emptyset} = 0 \).

Since the magnitude of the weights is guaranteed to be small (that is, weights are guaranteed to be in the range \([-|R|, 1]\)), for each \( \Theta \in \Theta \) we can invoke \( O, (\Theta, K, \lambda, w^0, \delta) \) to determine an \( s^0 \in S \) such that

\[
\mu_\Theta \sum_{k \in K} \lambda_k f_\Theta(R^k_s) + \sum_{r \in R} w^\theta_{r,s} \geq \max_{s \in S} \left\{ \alpha \mu_\Theta \sum_{k \in K} \lambda_k f_\Theta(R^k_s) + \sum_{r \in R} w^\theta_{r,s} \right\} - \delta,
\]

where \( \delta \) is an approximation error that will be defined in the following. If at least one \( s^0 \) is such that \( (\Theta, s^0) \) is violated, we output that constraint, otherwise the algorithm returns that the LP is feasible.

### Putting It All Together

The bisection algorithm computes a \( \gamma^* \in [0, 1] \) and a pair \((d, z)\) such that the approximate separation oracle does not find a violated constraint. The following lemma defines a modified LP and shows that \((d, z)\) is a feasible solution for this problem and has value at most \( \gamma^* \).

**Lemma 9.** The pair \((d, z)\) is a feasible solution to the following LP and has value at most \( \gamma^* \):

\[
\begin{align*}
\min_{\Theta} & \quad \sum_{\Theta} d_\Theta \\
\text{s.t.} & \quad d_\Theta \geq \alpha \mu_\Theta \sum_{k \in K} \lambda_k f_\Theta(R^k_s) + \mu_\Theta \sum_{r \in R} \sum_{k \in K} u^\Theta_{r,k} - \delta & & \forall \Theta \in \Theta, \forall s \in S.
\end{align*}
\]

**Proof.** The value is at most \( \gamma^* \) by assumption (that is, the separation oracle does not find a violated constraint for \((d, z)\) in \( \mathcal{P} \) with objective \( \gamma^* \)). Analogously, it holds that \( d_\Theta \in [0, 1] \) for each \( \Theta \in \Theta \), and \( w^\theta_{r,s} \leq 1 \) for each \( r \in R \), \( s \in S_r \), and \( \Theta \in \Theta \). Suppose, by contradiction, that \( (\Theta, s') \) is a violated constraint of the modified LP above. Then, given \( d \), oracle \( O_\Theta \) would have found an \( s \in S \) such that

\[
\mu_\Theta \sum_{k \in K} \lambda_k f(R^k_s) + \mu_\Theta \sum_{r \in R} \sum_{k \in K} u^\theta_{r,k} - \delta > d_\Theta,
\]

where the first inequality follows by Definition 2, and the second from the assumption that the modified dual is infeasible. Hence, \( O_\Theta \) would return a violated constraint, reaching a contradiction.

The dual problem of the LP of Lemma 9 reads as follows:

\[
\begin{align*}
\max_{\phi} & \quad \sum_{s \in S} \sum_{\Theta} \phi_\Theta(s) \left( \alpha \mu_\Theta \sum_{k \in K} \lambda_k f_\Theta(R^k_s) - \delta \right) \\
\text{s.t.} & \quad \sum_{\Theta} \mu_\Theta \sum_{s: s' = s} \phi_\Theta(s) u^\Theta_{r,k} \geq 0 & & \forall r \in R, \forall s' \in S_r, \forall k \in K_r : k \in s' \\
& \quad \sum_{s \in S} \phi_\Theta(s) = 1 & & \forall \Theta \in \Theta \\
& \quad \phi_\Theta(s) \geq 0 & & \forall \Theta \in \Theta, s \in S.
\end{align*}
\]

By strong duality, Lemma 9 implies that the value of the above problem is at most \( \gamma^* \). Then, let \( \text{OPT} \) be value of the optimal solution to LP (1). The same solution is feasible for the LP we just described, where it has value

\[
\alpha \text{OPT} - |\Theta| \delta \leq \gamma^*.
\]

Now, we show how to find a solution to the original problem (LP (1)) with value at least \( \gamma^* - \beta \). Let \( H \) be the set of constraints returned by the ellipsoid method run on the feasibility problem \( \mathcal{P} \) with objective \( \gamma^* - \beta \).

**Lemma 10.** LP (1) with variables restricted to those corresponding to dual constraints \( H \) returns a signaling scheme with value at least \( \gamma^* - \beta \). Moreover, the solution can be determined in polynomial time.
Proof. By construction of the bisection algorithm, \( \text{APX} \) is infeasible for value \( \gamma^* - \beta \). Hence, the following LP has value at least \( \gamma^* - \beta \):

\[
\min_{z \geq 0, d} \sum_{\theta \in \Theta} d_\theta \sum_{k \in K} \lambda_k f_\theta(R^k_a) + \sum_{r \in R} \sum_{k \in s_r} u_{r,k} z_{r,s_r,k} \geq \forall (\theta, s) \in \mathcal{H}.
\]

Notice that the primal of the above LP is exactly LP (1) with variables restricted to those corresponding to dual constraints in \( \mathcal{H} \), and that the former (restricted) LP has value at least \( \gamma^* - \beta \) by strong duality. To conclude the proof, the ellipsoid method guarantees that \( \mathcal{H} \) is of polynomial size. Hence, the LP can be solved in polynomial time.

D. Proofs Omitted from Section 6.2

Theorem 5. Given a subset \( K \subseteq K \), a vector \( y \in \{0, 2\}^{|K|} \) such that \( y_k = 0 \) for all \( k \notin K \), and an approximation error \( \epsilon \in \mathbb{R}_+ \), for any \( 0 < \alpha \leq 1 \), the approximate projection oracle \( \varphi_\alpha(K, y, \epsilon) \) can be computed in polynomial time by querying the approximate separation oracle \( O_\alpha \).

Proof. The problem of computing the projection of point \( y \) on \( X_K \) (see Equation (2)) can be formulated via the following convex programming problem, which we denote by \( \mathcal{P} \):

\[
\begin{aligned}
\min_{\phi, x} & \sum_{k \in K} (x_k - y_k)^2 \\
\text{s.t.} & \sum_{\theta \in \Theta} \mu_\theta \left( \sum_{s \in S'} \phi_\theta(s) u_0^{r,k} \right) \geq 0 \quad \forall r \in R, \forall s' \in S_r, \forall k \in K : k \in s' \\
& \sum_{s \in S} \phi_\theta(s) = 1 \quad \forall \theta \in \Theta \\
& \phi_\theta(s) \geq 0 \quad \forall \theta \in \Theta, \forall s \in S \\
& x_k \leq \sum_{\theta \in \Theta} \sum_{s \in S} \mu_\theta \phi_\theta(s) f_\theta(R^k_a) \quad \forall k \in K.
\end{aligned}
\]

Then, we compute the Lagrangian of \( \mathcal{P} \) by introducing dual variables \( z_{r,s,k} \leq 0 \) for each \( r \in R, s \in S_r \), and \( k \in s \), \( d_\theta \in \mathbb{R} \) for each \( \theta \in \Theta \), \( v_{0,s} \leq 0 \) for each \( \theta \in \Theta, s \in S \), and \( v_k \geq 0 \) for each \( k \in K \). Specifically, the Lagrangian of \( \mathcal{P} \) reads as follows

\[
L(\phi, x, z, v, \mu, d) := \sum_{k \in K} (x_k - y_k)^2 + \sum_{r \in R} \sum_{s' \in S_r} \sum_{k \in s'} z_{r,s,k} \left( \sum_{\theta \in \Theta} \mu_\theta \sum_{s : s_r = s'} \phi_\theta(s) u_0^{r,k} \right)
+
\sum_{\theta \in \Theta} v_{0,s} \phi_\theta(s) + \sum_{\theta \in \Theta} d_\theta \left( \sum_{s \in S} \phi_\theta(s) - 1 \right)
+
\sum_{k \in K} v_k \left( x_k - \sum_{\theta \in \Theta} \sum_{s \in S} \mu_\theta \phi_\theta(s) f_\theta(R^k_a) \right).
\]
We observe that Slater’s condition holds for \( \Phi \) (all constraints are linear, and by setting \( x = 0 \) any signaling scheme \( \phi \) constitutes a feasible solution). Therefore, by strong duality, an optimal solution must satisfy the KKT conditions. In particular, in order for stationarity to hold, it must be \( 0 \in \partial \phi_{\theta}(s) \) for each \( s \) and \( \theta \). Then, for each \( \theta \in \Theta \) and \( s \in S \), we have
\[
\partial \phi_{\theta}(s)(L) = \sum_{r \in R} \sum_{k \in s_r} \mu_{\theta} z_{r,s,k} u_{\theta}^{r,k} + v_{\theta,s} + d_{\theta} - \sum_{k \in K} \nu_{k} \mu_{\theta} f_{\theta}(R_{s}^{k}) = 0.
\]
Then, for each \( \theta \in \Theta \) and \( s \in S \), we obtain
\[
\sum_{r \in R} \sum_{k \in s_r} \mu_{\theta} z_{r,s,k} u_{\theta}^{r,k} + d_{\theta} - \sum_{k \in K} \nu_{k} \mu_{\theta} f_{\theta}(R_{s}^{k}) \geq 0. \tag{12}
\]
Moreover, stationarity has to hold with respect to variables \( x \). Formally, for each \( k \in K \),
\[
\partial x_k(L) = 2(x_k - y_k)\nu_k = 0.
\]
Therefore, for each \( k \in K \),
\[
x_k = y_k - \frac{\nu_k}{2}. \tag{13}
\]
By Equations (12) and (13), we obtain the following dual quadratic program
\[
\begin{align*}
\max_{z,v,u,v,d} & \quad \sum_{k \in K} \left( \nu_k y_k - \frac{v_k^2}{4} \right) - \sum_{\theta \in \Theta} d_{\theta} \\
\text{s.t.} & \quad d_{\theta} \geq \sum_{k \in K} \nu_k \mu_{\theta} f_{\theta}(R_{s}^{k}) + \sum_{r \in R} \sum_{k \in s_r} \mu_{\theta} z_{r,s,k} u_{\theta}^{r,k} \quad \forall \theta \in \Theta, \forall s \in S \\
& \quad z_{r,s,k} \geq 0 \quad \forall r \in R, \forall s \in S_r, \forall k \in K_r : k \in s \\
& \quad \nu_k \geq 0 \quad \forall k \in K,
\end{align*}
\]
in which the objective function is obtained by observing that each term \( \phi_{\theta}(s) \) in the definition of \( L \) is multiplied by \( \partial \phi_{\theta}(s)(L) \), which has to be equal to zero by stationarity. Similarly to what we did in the proof of Theorem 4, we repeatedly apply the ellipsoid method equipped with an approximate separation oracle to problem \( \Phi \). In this case, the analysis is more involved than what happens in Theorem 4, because we are interested in computing an approximate projection on \( \alpha \chi_K \) rather than an approximate solution of \( \Phi \). We proceed by casting \( \Phi \) as a feasibility problem with a certain objective (analogously to \( \Phi \) in Theorem 4). In particular, given objective \( \gamma \in [0,1] \), the objective function of \( \Phi \) becomes the following constraint in the feasibility problem
\[
\sum_{k \in K} \left( \nu_k y_k - \frac{v_k^2}{4} \right) - \sum_{\theta \in \Theta} d_{\theta} \geq \gamma. \tag{14}
\]
Then, given an approximation oracle \( O_\alpha \) which will be specified later, we apply to the feasibility problem the search algorithm described in Algorithm 2.

**Algorithm 2 Search Algorithm**

**Input:** Error \( \epsilon, \nu \in \mathbb{R}^{|K|} \), subspace \( K \subseteq K \).

1. **Initialization:** \( \beta \leftarrow \frac{\epsilon}{2} \), \( \delta \leftarrow \frac{\nu}{2|\Theta|} \), \( \gamma \leftarrow |K| + \beta \), and \( \mathcal{H} \leftarrow \emptyset \).
2. **repeat**
3. \( \gamma \leftarrow \gamma - \beta \)
4. \( \mathcal{H}_{\text{UNF}} \leftarrow \mathcal{H} \)
5. \( \mathcal{H} \leftarrow \{ \text{violated constraints returned by the ellipsoid method on } \Phi \text{ with objective } \gamma \text{ and constraints } \mathcal{H}_{\text{UNF}} \} \)
6. **until** \( \Phi \) is feasible with objective \( \gamma \) (see Equation (14))
7. **return** \( \mathcal{H}_{\text{UNF}} \)

At each iteration of the main loop, given an objective value \( \gamma \), Algorithm 2 checks whether the problem \( \Phi \) is approximately feasible or unfeasible, by applying the ellipsoid algorithm with separation oracle \( O_\alpha \). Let \( \mathcal{H} \) be the set of constraints returned by the separation oracle (the separating hyperplanes due to the linear inequalities). At each iteration, the ellipsoid
method is applied on the problem with explicit constraints in the current set \( \mathcal{H}_{\text{Unf}} \) (that is, each constraint in \( \mathcal{H}_{\text{Unf}} \) is explicitly checked for feasibility), while the other constraints are checked through the approximate separation oracle. Algorithm 2 returns the set of violated constraints \( \mathcal{H}_{\text{Unf}} \) corresponding to the last value of \( \gamma \) for which the problem was unfeasible. Now, we describe how to implement the approximate separation oracle employed in Algorithm 2. Then, we conclude the proof by showing how to build an approximate projection starting from the set \( \mathcal{H}_{\text{Unf}} \) computed as we just described.

**Approximate Separation Oracle** Let \((\bar{z}, \bar{v}, \bar{\nu}, \bar{d})\) be a point in the space of dual variables. Then, let, for each \( \theta \in \Theta \), \( r \in \mathcal{R} \), and \( s \in \mathcal{S}_r \),

\[
w^\theta_{r, s} := \sum_{k \in \mathcal{S}} \bar{z}_{r, s, k} \mu_{\theta} u^k_{\theta}.
\]

First, we can check in polynomial time if one of the constraint in \( \mathcal{H} \) is violated. If at least one of those constraints is violated, we output that constraint. Moreover, if the constraint corresponding to the objective is violated, we can output a separation hyperplane in polynomial time since the constraint has a polynomial number of variables. Then, by following the same rationale of the proof of Theorem 4 (offline setting), we proceed with a case analysis in which we ensure it is possible to output a violated constraint when \(|\nu_k| \) or \(|w^\theta_{r, s}| \) are too large to guarantee polynomial-time solvability by Definition 2.

- First, it has to hold \( d_\theta \in [0, 4|K|] \) for each \( \theta \in \Theta \). Indeed, if \( d_\theta < 0 \), then the constraint relative to \((\theta, \varnothing)\) would be violated. Otherwise, suppose that there exists a \( \theta \) with \( d_\theta > 4|K| \). Two cases are possible: (i) the constraint corresponding to the objective is violated, which allows us to output a separation hyperplane; (ii) it holds

  \[
  \sum_{k \in K} \left( \bar{v}_k y_k - \frac{2}{\bar{\nu}_k^2} \right) > 4|K|,
  \]

  which implies that there exists a \( k \in K \) such that \( \bar{v}_k y_k - \frac{2}{\bar{\nu}_k^2} > 4 \). However, we reach a contradiction since, by assumption, \( y_k \leq 2 \) for each \( k \in K \), and therefore it must hold \( \bar{v}_k y_k - \frac{2}{\bar{\nu}_k^2} \leq 2 \bar{\nu}_k - \frac{2}{\bar{\nu}_k^2} \leq 4 \).

- Second, we show how to determine a violated constraint when \( \bar{\nu}_k \notin [0, |K| + 10] \). Specifically, if there exists a \( k \in K \) for which \( \bar{\nu}_k < 0 \), then the objective is negative, and we can return a separation hyperplane (corresponding to Equation (14)). If there exists a \( \nu_k > |K| + 10 \), then

  \[
  \sum_{k' \in K} \left( \bar{v}_{k'} y_{k'} - \frac{2}{\bar{\nu}_{k'}^2} \right) \leq 2\nu_k - \frac{2\nu_k^2}{4} + \sum_{k' \in K \setminus \{k\}} \left( 2\bar{v}_{k'} - \frac{2\bar{\nu}_{k'}^2}{4} \right)
  \]

  \[
  \leq 2|K| + 20 - \frac{|K|^2}{4} - 5|K| - 25 + 4|K|
  \]

  \[
  = -\frac{|K|^2}{4} + |K| - 5
  \]

  \[
  < 0,
  \]

  where the first inequality follows by the assumption that \( y_k \leq 2 \) for each \( k \in K \), and the second inequality follows from the fact that \( 2\nu_k - \frac{2\nu_k^2}{4} \) has its maximum in \( \bar{\nu}_k = 4 \) and, when \( \bar{\nu}_k \geq |K| + 10 \), the maximum is at \( \bar{\nu}_k = |K| + 10 \) since the function is concave. Hence, we obtain that Constraint (14) is violated.

- Finally, suppose that there exists a \( \theta \in \Theta \), \( r \in \mathcal{R} \), \( s \in \mathcal{S}_r \) such that \( w^\theta_{r, s} > 4|K| \). Then, the constraint corresponding to \((\theta, s)\) is violated (because \( d_\theta \leq 4|K| \), otherwise we would have already determined a violated constraint in the first case of our analysis). If, instead, there exists a \( \theta \in \Theta \), \( r \in \mathcal{R} \), \( s \in \mathcal{S}_r \) such that \( w^\theta_{r, s} < -4|K||\mathcal{R}| - 10 \), then, for all the inequalities \((\theta, s')\) with \( s' = s \), it holds \( \bar{d}_\theta \geq 0 \) and

  \[
  \mu_{\theta} \sum_{k \in K} \bar{v}_k f_{\theta}(P^k_{s}) + \sum_{r' \in \mathcal{R} \setminus \{r\}} w^\theta_{r', s'} + w^\theta_{r, s} \leq 0.
  \]

In this last case, all the inequalities corresponding to \((\theta, s')\) with \( s' = s \) are guaranteed to be satisfied. Then, we can safely manage all the inequalities comprising of \( w^\theta_{r, s} \leq -4|K||\mathcal{R}| - 10 \) by setting \( w^\theta_{r, s} = -4|K||\mathcal{R}| - 10 \).
After the previous steps, it is guaranteed that \(|w_{r,s}^\theta| \leq 4|K||R| + 10\) for each \(\theta, r, s\), and \(\nu_k \in [0, |K| + 10]\) for each \(k\). Hence, we can employ an oracle \(O_\alpha\) with \(|w_{r,s}^\theta|\) and \(\lambda_k^\theta = \nu_k \mu_{\theta}\), which is guaranteed to be polynomial in the size of the instance by Definition 2. Let \(\delta\) be an error parameter which will be defined in the remainder of the proof. For each \(\theta \in \Theta\), we call the oracle \(O_\alpha(\theta, K, \{\nu_k\})\). Each query to the oracle returns an \(s^\theta\). If at least one of the constraints corresponding to a pair \((\theta, s^\theta)\) is violated, we output that constraint. Otherwise, if for each \(\theta \in \Theta\) the constraint \((\theta, s^\theta)\) is satisfied, we conclude that the point is in the feasible region.

Putting It All Together Algorithm 2 terminates at objective \(\gamma^*\). It is easy to see that the algorithm terminates in polynomial time because it must return feasible when \(\gamma = 0\). Our proof proceeds in two steps. First, we prove that a particular problem obtained from \(\mathcal{P}\) has value at least \(\gamma^*\). Then, we prove that the solution of \(\mathcal{P}\) with only variables in \(\mathcal{H}_{\text{UNF}}\) has value close to \(\gamma^*\). Finally, we show that the two solutions are, respectively, the projection and an approximate projection on a set that includes \(\alpha X_K\). This will complete the proof.

If the algorithm terminates at objective \(\gamma^*\), the following convex optimization problem is feasible (see Theorem 4).

\[
\begin{align*}
\min_{\phi, z} & \sum_{k \in K} (x_k - y_k)^2 + \delta \sum_{(\theta, s) \notin \mathcal{H}_{\text{UNF}}} \phi_\theta(s) \\
\text{s.t.} & \quad \sum_{\theta \in \Theta} \mu_\theta \left( \sum_{s': s' = s} \phi_\theta(s') u_{\theta}^{r,k} \right) \geq 0 \quad \forall r \in R, \forall s \in S_r, \forall k \in K_r : k \in s \\
& \quad \sum_{s \in S} \phi_\theta(s) = 1 \quad \forall \theta \in \Theta \\
& \quad \phi_\theta(s) \geq 0 \quad \forall \theta \in \Theta, \forall s \in S \\
& \quad x_k \leq \sum_{\theta \in \Theta} \left( \sum_{s' : (\theta, s') \in \mathcal{H}_{\text{UNF}}} \mu_\theta \phi_\theta(s') f_\theta(R_{s'}^{k}) + \alpha \sum_{s' : (\theta, s') \in \mathcal{H}_{\text{UNF}}} \mu_\theta \phi_\theta(s') f_\theta(R_{s'}^{k}) \right) \quad \forall k \in K.
\end{align*}
\]

Moreover, since the algorithm did not terminate at value \(\gamma^* + \beta\), problem \(\mathcal{P}\) with value \(\gamma^* + \beta\) is unfeasible when restricting the set of constraints to \(\mathcal{H}_{\text{UNF}}\). The primal problem \(\mathcal{P}\) restricted to primal variables corresponding to dual constraints in \(\mathcal{H}_{\text{UNF}}\) reads as follows

\[
\begin{align*}
\min_{\phi, z} & \sum_{k \in K} (x_k - y_k)^2 \\
\text{s.t.} & \quad \sum_{\theta \in \Theta} \mu_\theta \left( \sum_{s' : (\theta, s') \in \mathcal{H}_{\text{UNF}}} \phi_\theta(s') u_{\theta}^{r,k} \right) \geq 0 \quad \forall r \in R, s' \in S_r, \forall k \in K_r : k \in s' \\
& \quad \sum_{s' : (\theta, s') \in \mathcal{H}_{\text{UNF}}} \phi_\theta(s') = 1 \quad \forall \theta \in \Theta \\
& \quad \phi_\theta(s) \geq 0 \quad \forall \theta \in \Theta, \forall s \in \mathcal{H}_{\text{UNF}} \\
& \quad x_k \leq \sum_{\theta \in \Theta} \sum_{s' : (\theta, s') \in \mathcal{H}_{\text{UNF}}} \mu_\theta \phi_\theta(s') f_\theta(R_{s'}^{k}) \quad \forall k \in K.
\end{align*}
\]

\(\text{In the following, we will refer to the proof of Theorem 4 when the steps of the two proofs are analogous.}\)
By strong duality, the above problem has value at most $\gamma^* + \beta$. Moreover, it has a polynomial number of variables and constraints because the ellipsoid method returns a set of constraints $H_{\text{Unif}}$ of polynomial size. Therefore, the above problem can be solved in polynomial time.

A solution to the above problem is a feasible signaling scheme. Let $(x^*, \phi)$ be its solution. We have that $x^* \in \bar{X}_K$, with

$$\bar{X}_K = \left\{ x : x_k \leq \sum_{\theta \in \Theta} \left( \sum_{s=(\theta, a) \in H_{\text{Unif}}} \mu_{\theta} \phi_{\theta}(s) f_{\theta}^k(R^k_s) + \alpha \sum_{s=(\theta, a) \notin H_{\text{Unif}}} \mu_{\theta} \phi_{\theta}(s) f_{\theta}^k(R^k_s) \right) \right\} \forall k \in K, \phi \in \Phi.$$  

It holds $\alpha \bar{X}_K \subseteq \bar{X}_K$. Now, we show that $x^*$ is close to $x^*$, where $x^*$ is the projection of $y$ on $\bar{X}_K$ (that is the solution of (2) with $\delta = 0$). Since $x^*$ is a feasible solution of (2) and the minimum of (2) is at least $\gamma^*$, it holds $||x^* - y||^2 + \delta|\Theta| \geq \gamma^*$. Then,

$$||x^* - y||^2 + \delta|\Theta| + \beta \geq \gamma^* + \beta$$

$$\geq ||x^* - y||^2$$

$$= ||x^* - x^* + x^* - y||^2$$

$$= ||x^* - x^*||^2 + ||x^* - y||^2 + 2\langle x^* - x^* , x^* - y \rangle$$

$$\geq ||x^* - x^*||^2 + ||x^* - y||^2,$$

where the last inequality follows from $\langle x^* - x^* , x^* - y \rangle \geq 0$, because $x^*$ is the projection of $y$ on $\bar{X}_K$ and $x^* \in \bar{X}_K$. Hence, $||x^* - x^*||^2 \leq \delta|\Theta| + \beta$. Finally, let $x$ be a point in $\alpha \bar{X}_K$. Then,

$$||x^* - x||^2 \leq ||x^* - x^*||^2 + ||x^* - y||^2$$

$$\leq ||x^* - x^*||^2 + ||y - x||^2$$

$$\leq ||y - x||^2 + \delta|\Theta| + \beta,$$

where the second inequality follows from the fact that $x^*$ is the projection of $y$ on a superset of $\alpha \bar{X}_K$. Setting $\bar{\delta} = \frac{\delta}{2|\Theta|}$ and $\beta = \bar{\beta}$ concludes the proof.

### E. Proofs Omitted from Section 7

In this section, we provide the complete proof of Theorem 11.

First, we introduce some preliminary, known results concerning the optimization over matroids. Given a non-decreasing submodular set function $f : 2^G \rightarrow \mathbb{R}_+$ and a linear set function $\ell : 2^G \rightarrow I \mapsto \sum_{i \in I} w_i$ defined for finite ground set $G$ and weights $w = (w_i)_{i \in G}$ with $w_i \in \mathbb{R}$ for each $i \in G$, let us consider the problem of maximizing the sum $f(I) + \ell(I)$ over the bases $I \in \mathcal{B}(M)$ of a given matroid $M := (G, \mathcal{I})$. We make use of a theorem due to Sviridenko et al. (2017), which, by letting $v_f := \max_{I \in \mathcal{G}} f(I)$, $v_\ell := \max_{I \in \mathcal{G}} \ell(I)$, and $v := \max\{v_f, v_\ell\}$, reads as follows:

**Theorem 11** (Essentially Theorem 3.1 by Sviridenko et al. (2017)). For every $\epsilon > 0$, there exists an algorithm running in time $\text{poly}\left(|G|, \frac{1}{\epsilon}\right)$ that produces a basis $I \in \mathcal{B}(M)$ satisfying $f(I) + \ell(I) \geq (1 - \frac{1}{2}) f(I') + \ell(I') - O(\epsilon)v$ for every $I' \in \mathcal{B}(M)$ with high probability.

Next, we provide the complete proof of Theorem 11.

**Theorem 6.** If the sender’s utility is such that function $f_\theta$ is submodular for each $\theta \in \Theta$, then there exists a polynomial-time separation oracle $\mathcal{O}_{1 - \frac{1}{e}}$.

**Proof.** We show how to implement an approximation oracle $\mathcal{O}_\alpha(\theta, K, \lambda, \epsilon, \epsilon)$ (see Definition 2) running in time poly $(n, |K|, \max_{r,s} |w_{r,s}|, \max_k \lambda_k, \frac{1}{\epsilon})$ for $\alpha = 1 - \frac{1}{e}$. Let $M_S := (\mathcal{G}_S, \mathcal{I}_S)$ be a matroid defined as in Section 7 for direct signal profiles $S$. Let us recall that, given the relation between the bases of $M_S$ and direct signals, each direct signal profiles $s \in S$ corresponds to a basis $I \in \mathcal{B}(M_S)$, which is defined as $I := \{(r, s) \mid r \in \mathcal{R}\}$. In the following, given a subset $I \subseteq \mathcal{G}_S$ and a type profile $k \in K$, we let $R^k_I \subseteq \mathcal{R}$ be the set of receivers $r \in \mathcal{R}$ such that there exists a pair $(r, s) \in I$ (for some signal $s \in S_r$) with the receiver’s type $k_r$ being recommended to play $a_1$ under signal $s$; formally,

$$R^k_I := \{ r \in \mathcal{R} \mid \exists (r, s) \in I : k_r \in s \}.$$
First, we show that, when using matroid notation, the left-hand side of Equation (3) can be expressed as the sum of a non-decreasing submodular set function and a linear set function. To this end, let $f^\lambda_\theta : 2^\mathcal{G}_S \to \mathbb{R}_+$ be defined as $f^\lambda_\theta (I) = \sum_{k \in K} \lambda_k f_\theta (R^k_I)$ for every subset $I \subseteq \mathcal{G}_S$. We prove that $f^\lambda_\theta$ is submodular. Since $f^\lambda_\theta$ is a suitably defined weighted sum of the functions $f_\theta$, it is sufficient to prove that, for each type profile $k \in K$, the function $f_\theta : 2^\mathcal{R} \to [0, 1]$ is submodular in the sets $R^k_I$. For every pair of subsets $I \subseteq I' \subseteq \mathcal{G}_S$, and for every receiver $r \in \mathcal{R}$ and signal $s \in \mathcal{S}_r$, the marginal contribution to the value of function $f_\theta$ due to the addition of element $(r, s)$ to the set $I$ is:

\[
\begin{align*}
    f_\theta (R^k_{I \cup (r, s)}) - f_\theta (R^k_I) &= \mathbb{I} \{ k_r \in s \land \hat{\mu}(r, s') \in I : k_r \in s' \} \left( f_\theta (R^k_{I \cup \{ r \}}) - f_\theta (R^k_I) \right) \\
    &\geq \mathbb{I} \{ k_r \in s \land \hat{\mu}(r, s') \in I' : k_r \in s' \} \left( f_\theta (R^k_{I' \cup \{ r \}}) - f_\theta (R^k_I) \right) \\
    &\geq \mathbb{I} \{ k_r \in s \land \hat{\mu}(r, s') \in I' : k_r \in s' \} \left( f_\theta (R^k_{I' \cup \{ r \}}) - f_\theta (R^k_I) \right) = \\
    &= f_\theta (R^k_{I' \cup (r, s)}) - f_\theta (R^k_I),
\end{align*}
\]

where the last inequality holds since the functions $f_\theta$ are submodular by assumption. Since the last expression is the marginal contribution to the value of function $f_\theta$ due to the addition of element $(r, s)$ to the set $I'$, the relations above prove that the function $f^\lambda_\theta$ is submodular. Let $\ell^w : 2^{\mathcal{G}_S} \to \mathbb{R}_+$ be a linear function such that $\ell^w (I) = \sum_{r \in \mathcal{R}} w_{r, s_r}$ for every basis $I \subseteq \mathcal{B}(\mathcal{M}_S)$, with each $s_r \in \mathcal{S}_r$ being the signal of receiver $r \in \mathcal{R}$ specified by the signal profile corresponding to the basis, namely $(r, s_r) \in I$. Then, we have that finding a signal profile $s \in \mathcal{S}$ satisfying Equation (3) is equivalent to finding a basis $I \in \mathcal{B}(\mathcal{M}_S)$ of the matroid $\mathcal{M}_S$ (representing a direct signal profile) such that:

\[
f^\lambda_\theta (I) + \ell^w (I) \geq \max_{I^* \in \mathcal{B}(\mathcal{M}_S)} \left\{ \alpha \sum_{k \in K} f^\lambda_\theta (I^*) + \ell^w (I^*) \right\} - \epsilon.
\]

Notice that, for $\epsilon' > 0$, the algorithm of Theorem 11 by Sviridenko et al. (2017) can be employed to find a basis $I \in \mathcal{B}(\mathcal{M}_S)$ such that $f^\lambda_\theta (I) + \ell^w (I) \geq (1 - \frac{1}{2}) f^\lambda_\theta (I') + \ell^w (I') - O(\epsilon')v$ for every $I' \in \mathcal{B}(\mathcal{M})$ with high probability, employing time polynomial in $|\mathcal{G}_S|$ and $\frac{1}{\epsilon}$. Since $|\mathcal{G}_S|$ is polynomial in $n$ and $v$ is polynomial in $|\mathcal{K}|$, $\max_{r, s} |w_{r, s}|$ and $\max_k \lambda_k$, by setting $\epsilon' = O(\frac{\epsilon}{\sqrt{n}})$ and $\alpha = 1 - \frac{1}{2^{|\mathcal{K}|}}$, we get the result. $\square$