Black String in Massive Gravity

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ABSTRACT: We will analyze a black string in dRGT massive theory of gravity. Considering different approaches, we will study the critical behavior, phase transition and thermal stability for such a black string solution. We will also analyze the Van der Waals behavior for this system, and observe that the Van der Waals behavior depends on the graviton mass. It will be observed that the thermal fluctuations can modify the behavior of this system. We will explicitly analyze the effect of such fluctuations on the stability of this black string solution. This will be done using the Hessian matrix for this system.

KEYWORDS: Thermodynamics; Massive Gravity; Black String.
1 Introduction

Even though the general relativity is one of the most well tested theories [1], it is possible to be corrected in both the IR [2] and the UV [3–5]) limits. This has motivated the construction of a massive theory of gravity, in which the gravitons have a mass in IR limit [6, 7, 7–9]. The results from the LIGO collaboration have constraint the graviton mass to \( m_g < 1.2 \times 10^{-21} \text{eV}/c^2 \) [8, 10]. However, below this limit, it is possible for the gravitons to be massive. It may be noted if gravitons are massive, then it is possible to obtain a cosmological constant, which can explain the accelerated expansion of the universe [11–13]. So, there is a strong motivation to study such a massive theory of gravity.

Even though it is important to study a theory of massive gravitons, the original Fierz-Pauli theory of massive gravity [14] is not well-defined for the vanishing limit of graviton mass, due to the vDVZ discontinuity [15–17]. However, using the Vainshtein mechanism it is possible to obtain a non-linear theory of massive gravity (Stueckelberg trick) [18, 19]. Even though this non-linear theory of massive gravity does not have the vDVZ discontinuity, it has ghosts and so is not a physical theory [20]. It is possible to obtain a ghost free massive theory of gravity, without the vDVZ discontinuity, and this theory is called as the dRGT gravity [6]. Several black hole solutions have been constructed using the dRGT gravity, and it has been observed that the graviton mass can produce important modifications to such solutions [21–30]. It has also been observed that the graviton mass can have important consequences for the thermodynamics of various black hole solutions [31–34].

It is possible to study of thermodynamics of AdS black holes in an extended phase space, and this is done by identifying the cosmological constant with the thermodynamic pressure [35, 36]. It is also possible to study a Van der Waals like behavior for such
asymptotically AdS black holes [37, 38]. In fact, triplet point [39–41] and reentrant phase transitions [42–44] for such AdS black hole solution have been studied using such a Van der Waals behavior. It has been also been observed that the graviton mass in massive gravity can produce new interesting phase transitions in black hole solutions [31, 45–47]. So, it is important to study Van der Waals behavior for black hole solutions in massive gravity.

It is possible to extended the black hole solution to a black string solution [48]. It is also possible to consider black string solutions in asymptotically AdS space-time, and hence the CFT dual to such solutions can also be studied [49, 50]. It may be noted that the Hawking radiation and greybody factor of black strings in massive gravity has also been studied, and it was observed that this greybody factor depends on the graviton mass in this theory of massive gravity [51]. In fact, a solution for rotating black strings has also been constructed in massive gravity, and the dependence of the thermodynamics on the mass of graviton has also been studied, for such a solution [52]. It may be noted that black strings have an interesting behavior, and it has been observed that they can even produce effects, which are not observed in ordinary black holes [53–71]. As it is possible to study the Van der Waals behavior for AdS black hole in massive gravity [31, 45], we will analyze the Van der Waals like behavior for a black string in massive gravity.

It may be noted that the thermodynamics of black holes can get corrected due to thermal fluctuations [72, 73]. It has also been observed that these thermal fluctuations are in the thermodynamics of the black holes are produced by the quantum fluctuation in the geometry of a black hole [74–78]. It has been possible to study the effects of such thermal fluctuations on the Van der Waals like behavior of black holes [79, 80]. Thus, it is possible that these thermal fluctuations can change the behavior of various black hole solutions. As black strings are important solutions, we will analyze the effects of thermal fluctuations on black strings in massive gravity. The effects of thermal fluctuations on the thermodynamics of black holes has been studied in massive gravity [46, 47]. Furthermore, the stability of black hole solutions, and the effects of thermal fluctuations on the stability of black holes can be studied using the Hessian matrix [81–83]. So, we will analyze the effects of thermal fluctuations on the stability of a black string in massive gravity using this Hessian matrix.

2 Black String Solution

Now we give a brief review of the field equation and rotating solutions of dRGT nonlinear massive gravity, which is free of the ghost. The modification of the Einstein-Maxwell field equations from the graviton mass can be written as [7, 84]

\[ G_{\mu \nu} + m_g^2 \mathcal{X}_{\mu \nu} = \frac{1}{2} g_{\mu \nu} F^{\alpha \beta} F_{\alpha \beta} + 2 F_{\mu \lambda} F^{\lambda \nu} \quad \text{(2.1)} \]

\[ \nabla_\mu F^{\mu \nu} = 0, \quad \text{(2.2)} \]
where $m_g$ is the graviton mass parameter, $F_{\mu \nu}$ denotes the Faraday tensor which is constructed using the gauge field $A_\nu$ as $F_{\mu \nu} = \partial_\mu A_\nu$ and

$$X_{\mu \nu} = \mathcal{K}_{\mu \nu} - \alpha \left( \mathcal{K}_{\mu \nu}^2 - \mathcal{K} \mathcal{K}_{\mu \nu} \right) + 3 \beta \left( \mathcal{K}_{\mu \nu}^3 - \mathcal{K} \mathcal{K}_{\mu \nu}^2 + \frac{K^2 - [K^2]}{2} \mathcal{K}_{\mu \nu} \right)
- g_{\mu \nu} \left( \mathcal{K} + \frac{\alpha}{2} \left( \mathcal{K}^2 - K^2 \right) + \beta \left( \mathcal{K}^3 - 3 \mathcal{K} [K^2] + 2 [K^3] \right) \right). \quad (2.3)$$

Here $\mathcal{K}_{\mu \nu}^\mu = \delta_{\mu}^\nu - \left( \sqrt{g^{-1} \tilde{f}} \right)_\nu$ and $[\mathcal{K}^\alpha] = (\mathcal{K}^\alpha)_{ij}$. Now $\tilde{f}$ is the trace of the metric $\tilde{f}_{\mu \nu} = f_{ab} \partial_a \phi \partial_b \phi$, with $f_{ab}$ as the reference (fiducial) metric and $\phi$ as the St"uckelberg scalars (which are introduced to restore general covariance of the theory). The choices of the reference metric can be used to obtain different subclasses of dRGT massive gravity. In this paper, the unitary gauge $\phi^a = x^a \delta_{\mu}^a$ [84] is chosen so that the observable (physical) metric tensor $g_{\mu \nu}$ can describe the five degrees of freedom of the massive graviton.

Now, we begin with the following metric of rotating black string space-time in dRGT massive gravity

$$ds^2 = \left[ \frac{r^2 \omega_0^2}{l^2} - f(r) \right] dt^2 + \frac{dr^2}{f(r)} + 2l \omega_0 \left[ f(r) - \frac{r^2}{l^2} \right] dt d\varphi + r^2 \left[ 1 - \frac{l^2 \omega_0^2}{r^2} f(r) \right] d\varphi^2 + r^2 d\mathbf{z}^2, \quad (2.4)$$

where $z = z/l$ is the dimensionless coordinates along the black string and $\omega_0$ is rotation parameter (one can obtain the static line element with $\omega_0 = 0$). We will work in the cylindrical coordinates, with $-\infty < t < +\infty$, $0 \leq r < +\infty$, $-\infty < z < +\infty$ and $0 \leq \varphi < 2\pi$. The consistent gauge potential is defined as $A_\mu = [h(r), 0, -h(r) \omega_0, 0]$, where $h(r)$ is an arbitrary function of the radial coordinate $r$, and its explicit functional form can be obtained from the Maxwell equations (2.2). The exact solutions of the field equations with the mentioned metric (2.4) are given by [52]

$$f(r) = -\frac{4m}{r} + \frac{4q^2}{r^2} - \frac{\Lambda r^2}{3} - c_1 r + c_0, \quad (2.5)$$
$$h(r) = -\frac{2q}{r}, \quad (2.6)$$

where two integration constants $m$ and $q$ are related to the ADM mass and the electric charge per unit $z$, along $z$ direction. We can define the following constants in this theory of massive gravity

$$3m_b^2 (1 + \alpha + \beta) \equiv -\Lambda, \quad m_b^2 b_0 (1 + 2 \alpha + 3 \beta) \equiv c_1, \quad m_b^2 b_0^2 (\alpha + 3 \beta) \equiv c_0. \quad (2.7)$$

To obtain this exact solution of the field equations, the reference metric can be written as

$$f_{\mu \nu} = \begin{pmatrix} \frac{b_0^2 \omega_0}{l^2} & 0 & -\frac{b_0^2 \omega_0}{l^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{b_0^2 \omega_0}{l^2} & 0 & b_0^2 & 0 \\ 0 & 0 & 0 & \frac{\omega_0}{l^2} \end{pmatrix}. \quad (2.8)$$

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In order to investigate the geometric properties of the solutions, we can calculate the curvature scalars. Considering the metric (2.4), we can obtain

\[ R_{\alpha\beta\gamma\delta} = \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{df(r)}{dr} + \frac{4}{r^2} \left( \frac{df(r)}{r} + \frac{f(r)}{r^2} \right)^2, \]

\[ R_{\alpha\beta} = \frac{1}{2} \left( \frac{d^2 f(r)}{dr^2} + \frac{2}{r} \frac{df(r)}{dr} \right)^2 + 2 \left( \frac{1}{r} \frac{df(r)}{dr} + \frac{f(r)}{r^2} \right)^2, \]

\[ R = \frac{d^2 f(r)}{dr^2} + \frac{4}{r} \frac{df(r)}{dr} + \frac{2f(r)}{r^2}. \]

After inserting \( f(r) \) into the above relations, we can find that near the origin \((r \to 0^+)\)

\[ R_{\alpha\beta\gamma\delta} \approx r^{-8}, \]
\[ R_{\alpha\beta} \approx r^{-8}, \]
\[ R \approx r^{-2}, \]

while far from the black object \((r \to \infty)\), we have

\[ R_{\alpha\beta\gamma\delta} \approx \frac{8\Lambda^2}{3}, \]
\[ R_{\alpha\beta} \approx \frac{4\Lambda^2}{3}, \]
\[ R \approx \frac{4\Lambda}{3}. \]

Thus, it seems that there is a curvature singularity along \(z\)-direction \((r = 0)\). Due to the Cosmic Censorship Conjecture, there should be an event horizon for such a singularity, and we can interpret such an event horizon as a black string. Using \( g^{rr} = 0 \), we can find the location of the event horizon as

\[ g^{rr} = -\frac{\Lambda}{3} - c_1 r^*_+ + c_0 - \frac{4m}{r^*_+} + \frac{4q^2}{r^2_{+}} = 0. \] (2.9)

Here the event horizon is the largest real positive root of Eq. (2.9), with positive slope. Since we cannot obtain an analytical form for \( r^*_+ = r^*_+(c_0, c_1, m, q, \Lambda) \), we plot Fig. 1 (left panel) to confirm the existence of such an event horizon. According to this figure, one finds the curvature singularity may be covered by an event horizon and the radius of the event horizon depends on the free parameters in the theory.

Another interesting properties of black holes is the infinite redshift surface. Although for static black holes the surface of event horizon coincides the static limit surface (infinite redshift), the situation is different for the rotating solutions. In order to find the static limit surface with radius \( r_i \), one has to consider \( g_{tt} = 0 \)

\[ g_{tt} = -\frac{\Lambda}{3} + \frac{\omega^2}{l^2} r^2_i + c_1 r_i - c_0 + \frac{4m}{r_i} - \frac{4q^2}{r^2_i} = 0. \] (2.10)

which is different from Eq. (2.9). Considering the right panel of Fig. 1, it is clear that the radius of static limit surface (the largest real positive root of Eq. (2.10)) is affected by the rotation parameter.
Figure 1. \(g^{rr}\) (left) and \(-g_{tt}\) (right) versus \(r\) for \(c = c_0 = c_1 = q = m = 1\). **Left panel:** \(\Lambda = -1\) (continuous line) and \(\Lambda = -0.5\) (dashed line). **Right panel:** \(\Lambda = -0.5\) and \(l = 1\) with \(\omega_0 = 0.1\) (continuous line) and \(\omega_0 = 0.2\) (dashed line).

Taking into account the dominant term of the metric function for asymptotically large \(r\) \((\Delta r^2/3\) term), one may deduce that the rotating black string approach the AdS space-time. However, we should note that although the behavior of the curvature invariant support such a statement, the asymptotic symmetry group is not necessarily that of pure AdS (see [85] for a counterexample).

### 3 Thermodynamics

Here, we are going to calculate thermodynamic and conserved quantity for the charged rotating black string. It is obvious that the cylindrically symmetric metric \((2.4)\) is extended infinitely along \(z\)–direction. So, both the mass and charge are distributed along \(z\)–direction, and would be infinite. So, to obtain the physical finite quantities, we can calculate the mass and charge linear densities, which are the mass and charge per unit length along \(z\)–direction.

Applying the Brown-York approach [86], and rewriting the metric in the canonical form, one can calculate the conserved quantities for this system. So, using the two known Killing vectors \(\partial/\partial t\) and \(\partial/\partial \varphi\) of the rotating metric \((2.4)\), we can calculate their corresponding conserved charges. These conserved charges are mass and angular momentum (per unit \(z\) along \(z\)–direction). They are associated with the time translation and rotation invariance of the system.

\[
M = \frac{|1 - \omega_0^2|}{l} m, \quad (3.1)
\]
\[
J = \frac{|1 - \omega_0^2|}{l} \omega_0, \quad (3.2)
\]

where these relations reduce to \(M = m\) and \(J = 0\) for the static case \((\omega_0 = 0)\). Using the axial symmetric of the rotating metric \((2.4)\), one can obtain \(\xi^\mu = (1, 0, \Omega_H, 0)\) as the
Killing vector, for which the angular velocity at the horizon is

$$\Omega_H = - \frac{g_{t \phi}}{g_{\phi \phi}} \bigg|_{r=r_+} = \frac{\omega_0}{T}, \quad (3.3)$$

Now, we can calculate the electric charge and the electric potential for this system. The electric charge density (per unit $z$ along $z-$direction) of black string can be calculated using the Gauss law as

$$Q = q. \quad (3.4)$$

The electric potential measured at infinity, with respect to the horizon, is given by

$$\Phi = A_\mu \xi^\mu \big|_{r \to \infty} - A_\mu \xi^\mu \big|_{r \to r_+} = \frac{2 |1 - \omega_0^2|}{r_+} q. \quad (3.5)$$

So, the entropy of a black object equals one-quarter of its horizons area. As this is a universal relation between area and entropy, it holds for all black objects, including black strings [87–92]. The entropy of the black string per unit $z$ along $z-$direction can be written as

$$S = \frac{A}{4} = \frac{\pi r_+^2}{2}. \quad (3.6)$$

The Hawking temperature of black holes can be calculated from its surface gravity. Using the vanishing of the metric function at the event horizon, we obtain

$$T = \frac{1}{2 \pi} \sqrt{-\frac{1}{2} (\nabla_\mu \xi_\nu) (\nabla^\mu \xi^\nu)} = \frac{1 - \omega_0^2}{4 \pi} \frac{df(r)}{dr} \bigg|_{r=r_+}$$

$$= \frac{1}{4 \pi r_+} \left( -\Lambda r_+^2 - 2c_1 r_+ + c_0 - \frac{4q^2}{r_+^2} \right). \quad (3.7)$$

Using these conserved quantities, it is straightforward to write the first law of thermodynamics as [52],

$$dM = TdS + \Omega_H dJ + \Phi dQ. \quad (3.8)$$

### 3.1 Phase Transition

In the usual form for the first law of black hole thermodynamics, the work term $(P - V)$ is absent. However, it is possible to consider such a work term in extended phase space [93]. This is done by identifying the negative cosmological constant with the thermodynamic pressure

$$P = \frac{-\Lambda}{8 \pi} |1 - \omega_0^2|. \quad (3.9)$$

Using the Eq. (3.7), with such a pressure, we can obtain the following equation of state

$$P = \frac{1}{8 \pi r_+^3} \left[ \left( 2c_1 r_+ - c_0 + \frac{4q^2}{r_+^2} \right) |1 - \omega_0^2| + 4\pi r_+ T \right]. \quad (3.10)$$

Here the conjugate variable to pressure is interpreted as the thermodynamic volume

$$V = \frac{2\pi r_+^3}{3}. \quad (3.11)$$
This is equivalent to \( \int 4Sdr_+ = \frac{2\pi r^3}{3} \).

Such an identification leads to an extended version of the first law of black hole thermodynamics with a volume-pressure term

\[
dM = TdS + \Phi dQ + PdV + C dc_1, \tag{3.12}
\]

where \( C = \left( \frac{\partial M}{\partial c_1} \right)_{S,Q,V} = \frac{-|1-\omega_0^2|}{4} r^2. \)

\[\begin{align*}
\text{Left panel: } & T > T_c \text{ (continuous black line), } T = T_c \text{ (bold blue line) and } T < T_c \text{ (dashed red line).} \\
\text{Middle and right panels: } & P > P_c \text{ (continuous black line), } P = P_c \text{ (bold blue line) and } P < P_c \text{ (dashed red line).}
\end{align*}\]

In order to investigate the Van der Waals like behavior of the black string solution, we can focus on the isothermal \( P-V \) diagram (see left panel of Fig. 2). As the inflection point of isothermal curves in \( P-V \) diagram determine the critical point, we can use the following relations

\[
\left( \frac{\partial P}{\partial v} \right)_T = 0, \quad \left( \frac{\partial^2 P}{\partial v^2} \right)_T = 0. \tag{3.13}
\]

Here the specific volume \( v \) is proportional to \( r_+ \) (here \( v = 2r_+ \) in relativistic unit with \( l_P = 1 \) \cite{94}). Using the both relations of Eq. (3.13), simultaneously, we can analytically find the following critical quantities,

\[
T_c = \frac{\sqrt{6} |1-\omega_0^2| \left( \frac{c_1^2}{c_0^2} - 3\sqrt{6}qc_1 \right)}{36\pi q},
\]

\[
r_c = \frac{2\sqrt{6}q}{\sqrt{c_0}},
\]

\[
P_c = \frac{c_0^2 |1-\omega_0^2|}{384\pi q^2}.
\]

In order to obtain the physical (positive) critical quantities, one has to satisfy the following inequality

\[
c_1 \leq \frac{\sqrt{6c_0^3} \left( \frac{c_1^2}{c_0^2} - 3\sqrt{6}qc_1 \right)}{18q}.
\]
It is also notable that unlike $T_c$, both critical pressure and horizon radius are independent of the rotation parameter and $c_1$. Furthermore, using Eq. (3.10), one can write the pressure as

$$P = \frac{1}{8\pi r_+^2} \left[ \left( \frac{4q^2}{r_+^2} - c_0 \right)|1 - \omega_0^2| + 4\pi r_+ T_{\text{eff}} \right],$$

(3.14)

where the effective temperature is defined as

$$T_{\text{eff}} = T + \frac{|1 - \omega_0^2| c_1}{2\pi}.$$

Here the new effective critical temperature is given by

$$T_{c_{\text{eff}}} = \frac{\sqrt{6} |1 - \omega_0^2| c_1^{\frac{3}{2}}}{36\pi q}.$$

Now we can use the Gibbs free energy to study of phase transitions. Its behavior as a function of pressure and temperature can be used to analyze the phase transition in this system, along with its equilibrium state. Although the Gibbs free energy has a different functional dependence on its variables in different phases, the equilibrium state is the one with the lowest Gibbs free energy for given values of $T$ and $P$. Furthermore, the differences in the properties of the phases appear as discontinuities in some derivatives of the Gibbs free energy. Strictly speaking if the $n^{th}$ order derivatives are discontinuous, the phase transition is called $n^{th}$ order. The Gibbs free energy in the extended phase space is obtained as

$$G = M - TS = \left( c_0 r_+^2 + 12q^2 \right)|1 - \omega_0^2| - \frac{\pi}{3} P r_+^3.$$

(3.15)

The qualitative behavior of Gibbs free energy as an implicit function of temperature is shown in the middle panel of Fig. 2. Evidently, the swallow-tail characteristic demonstrates the first order phase transition (as indicated before in $P - V$ diagram of Van der Waals like behavior).

3.2 Thermal stability

Now we will investigate the local stability of charged rotating black string solutions using the canonical ensemble. This can be done as the heat capacity can be used to determine not only the number of the phases for such a system, but also their thermal stability. It may be noted that a black hole is thermally stable (unstable) if its heat capacity is positive (negative). In addition, the discontinuities (divergences) in the heat capacity occur at phase transition. Using the canonical ensemble in the extended phase space, both the pressure and total charge are kept constant. So, the stability can be analyzed using positivity of heat capacity at constant pressure and electric charge. The heat capacity for black strings under these considerations is given by

$$C_{Q,P} = \left( T \frac{\partial S}{\partial T} \right)_{Q,P} = \frac{\pi r_+^2}{8\pi P r_+^4 + (12q^2 - c_0 r_+^2)} \frac{8\pi P r_+^4 - (2c_1 r_+^3 - c_0 r_+^2)}{1 - \omega_0^2}.\quad (3.16)$$
Plotting the behavior of $C_{Q,P}$ in the right panel of Fig. 2, one observes that for $P > P_c$, the heat capacity is positive, with $T > 0$, and so the system is stable. Now for $P = P_c$, $C_{Q,P}$ has only one divergence point. However, for $P < P_c$, there are two divergence points for $C_{Q,P}$, and the system undergoes a phase transition.

3.3 Geometrical Thermodynamics

Black hole’s phase transition can also be studied using Geometrical Thermodynamics (GT) method. In this approach, one can use different metrics in terms of thermodynamic quantities to build a geometrical space. Then, by calculating the Ricci scalar and its divergence points of those thermodynamic metrics, we can investigate the phase transition of interested black hole. It is expected that the curvature singularities of those spacetimes relate to the heat capacity divergences and root.

During past few years, there have been different attempts to introduce such a thermodynamic metric and the well known ones are Weinhold, Ruppeiner, Quevedo and HPEM metrics. Weinhold used the second derivative of the internal energy with respect to the entropy and other extensive parameters to build a thermodynamic metric on the equilibrium space \cite{95, 96}. Following his work, Ruppeiner introduced a metric using the minus second derivative of entropy with respect to the internal energy and other extensive parameters, which was conformally related to Weinhold’s metric \cite{97, 98}. The main problem of these metrics was that neither of them were Legendre invariant, making them not suitable for describing the phase transition of black holes, so Quevedo tried to introduce a metric which was invariant under Legendre transformation \cite{99, 100}. But, even though his proposed metric met the Legendre invariance, it couldn’t provide a flawless mechanism to study phase transition and thermodynamic properties of some certain black holes through geometrical thermodynamics \cite{101, 102}. In recent years another metric has been proposed in Ref. \cite{101} which seemed to solve the problems that other metrics faced. This new metric is Legendre invariant and also doesn’t have the problem of extra singularities that are not consistent with any of the phase transition points and roots of heat capacity. In the following, we aim to employ these thermodynamic metrics to study phase transition of our interested black hole and compare their results to see which metric works the best for case of study.

The Weinhold metric is given by

$$ds^2_w = M g^w_{ab} dx^a dx^b,$$

(3.17)

where $M$ is the mass, $g^w_{ab} = \frac{\partial^2 M(X^c)}{\partial X^a \partial X^b}$ in which $X^a \equiv X^a(S,N^i)$ and $N^i$’s are other extensive parameters. The quantity that we care about is the Ricci scalar and its divergences. Since the Ricci scalar’s numerator is a finite smooth function, we just care about its denominator to get the divergence points and compare them with the divergence points and root of the heat capacity. The Weinhold Ricci scalar denominator is

$$\text{denom}(R_W) = M^3 (M_{SS} M_{QQ} - M_{SQ}^2),$$

(3.18)

where $M_X = \frac{\partial M}{\partial X}$, $M_{XY} = \frac{\partial^2 M}{\partial X \partial Y}$ and $M_{XX} = \frac{\partial^2 M}{\partial X^2}$. Considering Eq. (3.18), we find that there is at least one divergence point where $M_{SS} M_{QQ} = M_{SQ}^2$, and it is obvious from Fig. 3, that this point doesn’t coincide with any of the divergences or root of the heat capacity.
The Ruppeiner metric is defined as

\[ ds_R^2 = -MT^{-1}g_{ab}dx^a dx^b, \]  

(3.19)

which can be seen that, it is conformally related to Weinhold metric. Again, we just care about the divergences of the Ruppeiner Ricci scalar, so its denominator is our center of attention, since its numerator is a finite smooth function

\[ \text{denom}(R_R) = MT^3 (M_{SS}M_{QQ} - M_{SQ}^2)^2. \]  

(3.20)

As it is obvious from Fig. 4, one can find that the divergence points of the Ruppeiner Ricci scalar don’t match with any of the divergences or root of the heat capacity. It can be concluded from Figs. 3 and 4 that the phase transition points reported by the Weinhold and Ruppeiner metrics are not consistent with phase transition points indicated by the heat capacity.

Next, we examine the Quevedo metric which is defined as

\[ ds_Q^2 = (SM_S + QM_Q) (-M_{SS}dS^2 + M_{QQ}dQ^2). \]  

(3.21)

Like before for getting the divergences of the Ricci scalar, we isolate its denominator which is the form

\[ \text{denom}(R_Q) = M_{SS}^3M_{QQ}^3 (SM_S + QM_Q)^3. \]  

(3.22)

Existence of \( M_{SS} \) in the denominator ensures the consistence of the phase transition points of Ricci scalar with the corresponding points of heat capacity (see right panel of Fig. 5), but as it is shown in Fig. 7, one can see that there is another divergence point (related to the root of \( SM_S + QM_Q \)) that is not consistent with the results of heat capacity. In addition, left panel of Fig. 5 confirms that Quevedo’s Ricci scalar could not characterize the root of heat capacity.

Finally, we employ the HPEM metric which has the form of

\[ ds_{HPEM}^2 = (SM_S/M_{QQ}^3)(-M_{SS}dS^2 + M_{QQ}dQ^2), \]  

(3.23)

and the denominator of its Ricci scalar can be simplified as

\[ \text{denom}(R_{HPEM}) = 2S^3M_{SS}^3M_s^3. \]  

(3.24)

As it is evident from Eq. (3.24), we can see that there exist two sets of divergence points, one corresponding to \( M_S = 0 \) and the other one corresponds to \( M_{SS} = 0 \), resulting in consistence of it divergence points with both the root and phase transition points of the heat capacity. One interesting feature which can be seen from graphs in Fig. 6 is the behavior of HPEM metric Ricci scalar in the vicinity of its divergence points, meaning that the divergence points of Ricci scalar related to the roots of heat capacity is distinguishable from its divergence points related to divergences of heat capacity based on the Ricci scalar behavior.
Figure 3. Weinhold’s Ricci scalar (continuous line) and heat capacity (dashed line) versus $r_+$ for $q = 0.1$, $\omega = 0.2$, $c_0 = 1$, $c_1 = 0.1$ and $P < P_C$ (different scales).

Figure 4. Ruppeiner’s Ricci scalar (continuous line) and heat capacity (dashed line) versus $r_+$ for $q = 0.1$, $\omega = 0.2$, $c_0 = 1$, $c_1 = 0.1$ and $P < P_C$ (different scales).

4 Thermal Fluctuations

We will now discuss the effects of thermal fluctuations on the thermodynamics and stability of this black string solution. As this solution can be an AdS solution, it has to have a CFT dual, and this CFT can be used to obtain the microstates of this system [72]. So, let us assume that these microstates, which give rise to the entropy of this black string, are denoted by $\rho$. Now it is possible to write the partition function using such microstates $\rho$ as

$$Z = \int dE \rho e^{-E/T},$$  \hspace{1cm} (4.1)
Figure 5. Quevedo’s Ricci scalar (continuous line) and heat capacity (dashed line) versus $r_+$ for $q = 0.1, \omega = 0.2, c_0 = 1, c_1 = 0.1$ and $P < P_C$ (different scales).

Figure 6. HPEM’s Ricci scalar (continuous line) and heat capacity (dashed line) versus $r_+$ for $q = 0.1, \omega = 0.2, c_0 = 1, c_1 = 0.1$ and $P < P_C$ (different scales).

where $E$ is internal energy, which is related to Helmholtz free energy $F$ as $E = ST + F$. It may be noted that the Helmholtz free energy $F$ can be written as

$$F = - \int TdS. \quad (4.2)$$

Now using the relations (3.6) and (3.7), one can obtain,

$$F = |1 - \omega_0^2| \left( \frac{\Lambda}{12} r_+^3 + \frac{c_1}{4} r_+^2 - \frac{c_0}{4} r_+ - \frac{q^2}{r_+} \right). \quad (4.3)$$

Hence, the partition function of canonical ensemble for this black string can be constructed using these microstates. Having partition function for black strings, we can use the first
order Taylor expansion around the equilibrium to express these density of states as [72, 73],

$$ \rho = \frac{e^S}{\sqrt{2\pi \frac{d^2S}{d\beta_k^2}}} $$  
\hspace{1cm} (4.4)

where $\beta_k = \frac{1}{T}$ is the equilibrium temperature. Now we can use the statistical relation $\bar{S} = \ln \rho$ to obtain corrected entropy for black strings [73–76]

$$ \bar{S} = S - \frac{1}{2} \gamma \ln \left( 2\pi \frac{T^4 \frac{\partial^2 S}{\partial T^2} + 2T^3 \frac{\partial S}{\partial T}}{T S_3 + 6T S_2 + 6S_1} \right), $n\hspace{1cm} (4.5)$$

where $S$ is given by the equation (3.6). As the coefficient of the leading order correction term to the black hole entropy depends on the model used, we have added a free parameter $\gamma$ to measure the strength of this correction term [77]. It may be noted that we can set $\gamma = 1$ for the corrected thermodynamics. Furthermore, the corrected thermodynamics reduces to the ordinary thermodynamics at $\gamma = 0$. It has been demonstrated that these thermal fluctuation in the thermodynamics occurs due to quantum fluctuation of the metric, and becomes important when the black hole is small [78]. Now, we would like to study effect of such corrections on the thermodynamics of rotating black string in massive gravity.

In this case, we can obtain corrected Helmholtz free energy,

$$ \bar{F} = - \int T d\bar{S}. $n\hspace{1cm} (4.6)$$

Using the corrected entropy (4.5), we can find,

$$ \bar{F} = F + \frac{\gamma}{2} \int \frac{T^2 S_3 + 6T S_2 + 6S_1}{T S_2 + 2S_1} dT, $$n\hspace{1cm} (4.7)$$

**Figure 7.** Quevedo’s Ricci scalar (continuous line) and heat capacity (dashed line) versus $r_+$ for $q = 0.1$, $\omega = 0.2$, $c_0 = 1$, $c_1 = 2$ and $P > P_C$ (mismatches are clear).
where, we have

\[ S_1 \equiv \frac{\partial S}{\partial T} = -\frac{4\pi^2 r_+^5}{|1 - \omega_0^2| (\Lambda r_+^4 - 12q^2 + c_0 r_+^2)}, \]
\[ S_2 \equiv \frac{\partial^2 S}{\partial T^2} = \frac{16\pi^3 r_+^8}{|1 - \omega_0^2|} (\Lambda r_+^4 - 60l^2 q^2 + 3c_0 r_+^2)^3 \]
\[ S_3 \equiv \frac{\partial^3 S}{\partial T^3} = \frac{-768\pi^4 r_+^{11}}{|1 - \omega_0^2|^3} ((8\Lambda q^2 + c_0^2) r_+^4 - 40c_0 q^2 r_+^2 + 480q^4). \]  

(4.8)

Figure 8. Corrected Helmholtz free energy \( \bar{F} \) in terms of horizon radius for \( c_0 = c_1 = l = q = 1 \), and \( \omega^2 = 0.5 \).

Now as the thermal fluctuations are important, when the black hole is small, \( (r_+ \ll 1 \), hence \( S_3 \ll S_2 \ll S_1 \)), otherwise \( \gamma \ll 1 \) and \( \bar{F} \approx F \). Therefore, neglecting \( S_3 \) and \( S_2 \), and keeping only \( S_1 \), we can obtain,

\[ \bar{F} \approx F + \frac{3\gamma}{8} \frac{-\Lambda r_+^4 - 4q^2 + c_0 r_+^2}{\pi r_+^2}. \]  

(4.9)

In the plots of the Fig. 8, we can see typical behavior of the Helmholtz free energy for different values of \( \Lambda \). Dashed blue lines show original \( F \), given by the equation (4.3), while solid red line represent the corrected quantity. In the cases of \( \Lambda \geq 0 \), the effect of thermal fluctuations are important for smaller \( r_+ \), hence the corrections can be neglected for the large black hole. In the case of \( \Lambda < 0 \), there is a critical radius, where the effect of thermal fluctuations vanishes. Before the critical radius the thermal fluctuations decrease the energy, and after the critical radius, we observe that the Helmholtz free energy increases.

We can also calculate the corrected specific heat using

\[ \bar{C} = T \frac{d\bar{S}}{dT} = C + C(\gamma), \]  

(4.10)

where \( C(\gamma) \) is the correction term, which depend on \( \gamma \), and

\[ C = \frac{\pi r_+^2 (\Lambda r_+^4 - 2c_1 r_+^3 + 4q^2 - c_0 r_+^2)}{(\Lambda r_+^4 - 12q^2 + c_0 r_+^2)}. \]  

(4.11)
in agreement with the equation (3.16). In the plots of the Fig. 9, we can observe the typical behavior of the specific heat for different values of $\Lambda$. Dashed blue lines show ordinary $C$ given in the Eq. (4.11), while solid red line represent the corrected quantity obtained in Eq. (4.10). As before, $\Lambda = -1$ is different from other values of $\Lambda \geq 0$. In the case of $\Lambda = -1$, the thermal fluctuations decrease the specific heat (see the left plot of the Fig. 9). In all cases, we can observe some instabilities at small $r_+$. We can interpret it as the black string becoming small, due to the hawking radiation, and then becoming unstable. So, there seems to be a minimum size for the stable black strings in massive gravity.

Now for $\Lambda \geq 0$, with thermal fluctuations, we observe that there is a second order phase transition (see the right plot of the Fig. 9). As expected, for the large $r_+$, there is no important differences between ordinary and corrected quantities. The thermal fluctuations modify the stability and phase transition of a charged rotating black string in massive gravity.

![Figure 9](image_url)  

**Figure 9**. Corrected specific heat $\tilde{C}$ in terms of horizon radius for $c_0 = c_1 = l = q = 1$, and $\omega^2 = 0.5$.

It is argued that in the presence of electric charge, the heat capacity analysis is not sufficient to analyze the stability of system [81–83]. In such a system, we need the entire Hessian matrix of the Helmholtz free energy. This is constructed by second derivatives of Helmholtz free energy with respect to temperature and electric potential. So, the Hessian metric for the black string can be written using

\[
H_{11} \equiv \frac{\partial^2 \tilde{F}}{\partial T^2} = -\frac{8\pi^2 r_+^5}{|1 - \omega_0^2| (\Lambda r_+^4 - 12q^2 + c_0 r_+^2)} X, \\
H_{12} \equiv \frac{\partial^2 \tilde{F}}{\partial T \partial \Phi} = \frac{\alpha_q \pi r_+^3}{Q(\Lambda r_+^4 - 12q^2 + c_0 r_+^2)^2} X, \\
H_{21} \equiv \frac{\partial^2 \tilde{F}}{\partial \Phi \partial T} = -\frac{\alpha_q r_+}{2Q(\Lambda r_+^4 - 12q^2 + c_0 r_+^2)} Y, \\
H_{22} \equiv \frac{\partial^2 \tilde{F}}{\partial \Phi^2} = -\frac{|1 - \omega_0^2| \alpha_q^2}{16Q^2 r_+} Y, 
\]  

(4.12)
where

\[
X = \Lambda^2 r_+^8 + c_1 \Lambda r_+^5 + 2c_0 \Lambda r_+^3 + 3c_0 c_1 r_+^5 \\
- (40\Lambda q^2 + c_0^2) r_+^4 - 60c_1 q^2 r_+^3 + 24c_0 q^2 r_+^2 - 48q^4,
\]

\[
Y = -4\pi \Lambda r_+^6 - 6\pi c_1 r_+^5 + (3\gamma \Lambda + 2\pi c_0) r_+^4 + 36\gamma q^2.
\]

(4.13)

Now Hessian matrix $\mathcal{H}$ is given by,

\[
\mathcal{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.
\]

(4.14)

It is easy to check that determinant of above matrix is zero. Thus, one of its eigenvalues is zero. The other is given by the trace of matrix,

\[
\lambda = H_{11} + H_{22}.
\]

(4.15)

In the plot of the Fig. 10, we draw $\lambda$, and observe that the asymptotic points coincide with the Fig. 9. Hence our discussion about the stability are valid, and in agreement with analysis done using the specific heat of this system.

It may be noted that the extended first law is valid even with these corrected entropy terms (4.5). In fact, the extended first law be satisfied for such correction terms, as the corrected entropy can be write as [72, 73]

\[
\bar{S} = S + \gamma_1 S_1 + \gamma_2 S_2 + \cdots,
\]

(4.16)

where $\gamma_1 \equiv -\frac{1}{2}\gamma$ and $S_1 \equiv \ln \left(2\pi \left[T^4 \frac{\partial^2 S}{\partial T^2} + 2T^3 \frac{\partial S}{\partial T}\right]\right)$. In order to satisfy the extended first law, we should have,

\[
\gamma_1 dS_1 + \gamma_2 dS_2 + \cdots = 0.
\]

(4.17)

Above condition could be satisfied by suitable choice of coefficients $\gamma_1, \gamma_2, \ldots$, as the Lagrange multiplier [74–77]. Thus, the first law of thermodynamics is valid for black strings, even after thermal fluctuations are considered.

5 Conclusions

In this paper, we will analyze a rotating black string in the massive theory of gravity. In this massive theory of gravity, the gravitons have a mass, and these massive gravitons modify the black string solution. We will study thermal stability, critical behavior and phase transition for such a black string by applying different methods. We will also analyze the Van der Waals like behavior for this solution. Finally, we will analyze the effects of thermal fluctuations on the thermodynamics of such a solution. It is known that in the presence of electric charge, heat capacity analysis is not sufficient to analyze the stability of such a system. In fact, for such a system, the stability can be studied using the entire Hessian matrix of the Helmholtz free energy [81–83]. Such a Hessian matrix is obtained using second derivatives of Helmholtz free energy with respect to temperature and electric...
Figure 10. Trace of Hessian matrix in terms of horizon radius for $c_0 = c_1 = l = q = Q = \alpha_g = \gamma = 1$, and $\omega^2 = 0.5$.

potential. Here we have used such a Hessian matrix for analyzing the stability of black strings.

It is interesting to note that there are other black brane solutions, motivated by string theory [103–106]. It would be interesting to analyze such solutions in massive gravity. It would also be interesting to analyze the thermodynamics, and Van der Waals behavior for such black branes solutions using extended phase space. The stability of such black brane solutions can again be analyzed using the Hessian matrix. It is expected that here again the stability will depend on the mass of the graviton. The quantum fluctuations for such solutions can be analyzed using thermal fluctuations to the thermodynamics of such black holes [72, 73]. It would also be interesting to analyze the effect of such thermal fluctuations on the stability of such black brane solutions.

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