Cosmic Duality in Quintom Universe

Yi-Fu Cai\textsuperscript{a}, Hong Li\textsuperscript{a,b}, Yun-Song Piao\textsuperscript{a} and Xinmin Zhang\textsuperscript{a}

\textsuperscript{a}Institute of High Energy Physics, Chinese Academy of Sciences, P.O. Box 918-4, Beijing 100049, P. R. China
\textsuperscript{b}Department of Astronomy, Peking University, Beijing 100871, P. R. China and
\textsuperscript{c}College of Physical Sciences, Graduate School of Chinese Academy of Sciences, Beijing 100049, China

In this paper we study the duality in two-field Quintom models of Dark Energy. We find that an expanding universe dominated by Quintom-A field is dual to a contracting universe with Quintom-B field.

The recent data from type Ia supernovae and cosmic microwave background (CMB) radiation and so on\textsuperscript{1,2,3,4} have provided strong evidences for a spatially flat and accelerated expanding universe at the present time. In the context of Friedmann-Robertson-Walker cosmology, this acceleration is attributed to the domination of a component, dubbed dark energy (DE). Theoretically, the simplest candidate for DE is a small positive cosmological constant, but it suffers from the difficulties associated with the fine tuning and the coincidence problems. So many physicists are attracted by the idea that dark energy is due to a dynamical component, such as the Quintessence, K-essence, Phantom or Quintom.

With the accumulated observational data (e.g. SN1a, Wilkinson Microwave Anisotropy Probe observations (WMAP), galaxy clustering (SDSS) and so on) it becomes possible in the recent years to probe the recent and early behavior of DE by using some parameterizations for its equation of state (EOS) and constrain the models of dark energy. Especially, the new released 3-year WMAP data (WMAP3\textsuperscript{5}) have given so far the most precise probe on the Cosmic Microwave Background (CMB) Radiations. Although the recent fits to the data in combination of the WMAP3 with the other cosmological observational data show remarkably the consistence of the cosmological constant, it is worth noting that a class of dynamical models with equation of state across $-1$ Quintom is mildly favored\textsuperscript{6,7}. In the literature there have been a lot of studies on this class of models\textsuperscript{8-19}. The similar work applied in scalar-tensor theory is also studied in Ref\textsuperscript{21}. In the context of string theory it has been shown that the crossing $w = -1$ can be realized as low energy limit of the dynamical behavior of slow-rolling tachyon on a non-BPS D3-brane\textsuperscript{22}.

The simplest Quintom model consists of two scalar fields, one is Quintessence-like and another is Phantom-like\textsuperscript{9,23}. The Quintom models differ from the Quintessence or Phantom in the determination of the evolution and fate of the universe, for example, as shown in Ref\textsuperscript{10}, Quintom model can give rise to a new scenario of the evolution of the universe. In this model with an oscillating equation of state the early inflation and current acceleration are unified. Moreover, with Quintom dark energy it might be possible to avoid the singularities of the universe like big rip or big crunch.

In terms of the two-field Quintom models, it has been shown recently that there exists two category\textsuperscript{19}: one is Quintom-A where at early time the Quintessence dominates with $w > -1$ and later the Phantom dominates with $w < -1$; and the other is Quintom-B for which the equation of state is arranged to change from below -1 to above -1. It is trivial to realize Quintom-A in model building; but to achieve Quintom-B, one may need not only to add more degrees of freedom, but also to fine tune the potentials of Quintom fields(see Ref\textsuperscript{19}), or to construct non-canonical field with high derivative terms(see Ref\textsuperscript{12}), or even to add interactive terms which provide a transition from Phantom-like to Quintessence-like(see Ref\textsuperscript{8}). Both Quintom-A and Quintom-B are consistent with the current observational data.

To understand the possible connections among the dark energy models, it is useful to study the cosmic duality. The dualities in field theory and string theory have been widely studied and in fact it predicts a lot of interesting phenomena\textsuperscript{24}. The authors of Ref\textsuperscript{25,26} have considered a possible transformation with the Hubble parameter and studied the relevant issues with the cosmic duality\textsuperscript{27,28}. Specifically Ref\textsuperscript{29} has shown a link between a standard cosmology with quintessence matter and a contracting cosmology with phantom. Later on this duality has been generalized into studies with more complicated dark energy models and it has been shown to exist for these various dark energy models\textsuperscript{30,31,32,33}. In Ref\textsuperscript{34,35} the authors have studied this type of duality and its connection to the fates of the universe. In Ref\textsuperscript{36}, the author has also discussed the possibility of realizing this cosmic duality in the braneworlds. In all of these studies, the dark energy models will not be able to give $w$ crossing -1. Further, with phantom dominant the universe will reach a big rip or big sudden in the future\textsuperscript{37,38}, or expands forever approaching to a de Sitter, so a contracting universe is unstable. In this paper, we study the implications of the cosmic duality in the Quintom models of dark energy. By studying the behavior of the equation of state we find a dual of the Quintom-A to the Quintom-B.

To begin with the discussion, we consider a model where the universe is filled with Quintom dark energy and we neglect the contribution of the components of matter and radiation. The quintom model we will study in this paper consists of two fields, one being quintessence-like,
another phantom-like with the lagrangian given by
\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi_1 \partial^{\mu} \phi_1 - \frac{1}{2} \partial_{\mu} \phi_2 \partial^{\mu} \phi_2 - V_1(\phi_1) - V_2(\phi_2), \] (1)
where \( V_1 \) and \( V_2 \) are the potential terms and we have neglected the interaction between the \( \phi_1 \) and \( \phi_2 \) for simplicity of the discussion.

In the framework of FRW cosmology, the Einstein equations are
\[ 3H^2 = \frac{1}{2} \dot{\phi}_1^2 - \frac{1}{2} \dot{\phi}_2^2 + V_1 + V_2, \] (2)
\[ \ddot{\phi}_1 + 3H \dot{\phi}_1 + \frac{dV_1}{d\phi_1} = 0, \] (3)
\[ \ddot{\phi}_2 + 3H \dot{\phi}_2 - \frac{dV_2}{d\phi_2} = 0. \] (4)

Obviously we can construct a form-invariant transformation by defining a group of new quantities \( \bar{H}, \bar{\rho}, \bar{\rho}, \bar{\phi} \) which keep the Einstein equations invariant. Following the similar work of Ref. [29] there is a form-invariant transformation as follow:
\[ \bar{\rho} = \bar{\rho}(\rho), \] (5)
\[ \bar{H} = -\frac{(\bar{\rho})^2}{\rho} \bar{H}. \] (6)

Under this transformation, we obtain the corresponding changes for the pressure \( p \) and the equation of state \( w \),
\[ \bar{p} = -\bar{\rho} - \left( \frac{\bar{\rho}}{\rho} \right)^2 (\rho + p) \frac{d\bar{\rho}}{d\rho}, \] (7)
\[ \bar{w} = -1 - \left( \frac{\bar{\rho}}{\rho} \right)^2 \frac{d\bar{\rho}}{d\rho} (1 + w). \] (8)

From eqs. (7) and (8) one can see that for a positive \( \frac{d\bar{\rho}}{d\rho} \), one would be able to establish a connection between the Quintom-A and Quintom-B of the Quintom model of the dark energy.

For the specific model of quintom dark energy we consider in this paper the energy density and the pressure of system are given by
\[ \rho = \frac{1}{2} \frac{\dot{\phi}_1^2}{\phi_1^2} - \frac{1}{2} \frac{\dot{\phi}_2^2}{\phi_2^2} + V_1 + V_2, \] (9)
\[ p = \frac{1}{2} \frac{\dot{\phi}_1^2}{\phi_1^2} - \frac{1}{2} \frac{\dot{\phi}_2^2}{\phi_2^2} - V_1 - V_2. \] (10)

Taking \( \bar{\rho} = \rho \) in (7) and (8) as an example of detailed discussion without loss of the generality of the physical conclusion and information, we can obtain the dual transformation
\[ \bar{H} = -H, \] (11)
\[ \bar{p} = -2\rho - p, \] (12)
\[ \bar{w} = -2 - w. \] (13)

Consequently, the dual form of the Quintom dark energy to (11) is given by
\[ \bar{\mathcal{L}} = \bar{T} - \bar{V} \]
\[ = \frac{1}{2} \partial_{\mu} \bar{\phi}_2 \partial^{\mu} \bar{\phi}_2 - \frac{1}{2} \partial_{\mu} \bar{\phi}_1 \partial^{\mu} \bar{\phi}_1 - \delta \mathcal{L}_1(\phi_1) - \delta \mathcal{L}_2(\phi_2), \] (14)
where \( \bar{T} \) and \( \bar{V} \) denote for the kinetic and potential energy terms of dual form, and \( \delta \mathcal{L}_1 \) and \( \delta \mathcal{L}_2 \) are:
\[ \delta \mathcal{L}_1 = V_1 + \frac{\dot{\bar{\phi}}_1^2}{\phi_1^2}, \] (15)
\[ \delta \mathcal{L}_2 = V_2 - \frac{\dot{\bar{\phi}}_2^2}{\phi_2^2}. \] (16)

Comparing the quintom model to (11) and the dual form of the model in (14), one can see that with the dual transformation, if the original lagrangian is for a Quintom-A model the dual one is for Quintom-B model, or vice versa. With this duality, one expects a general connection among different fates of the universe, and it might be possible that the early universe be linked to other epochs of the universe.

For a detailed discussion on the duality connecting Quintom-A and Quintom-B in our note, we take a special form of the potentials in the unit of planck mass
\[ V_1 \propto -3\sqrt{2}\phi_1 + 2e^{-\sqrt{2}\phi_1}, \] (17)
\[ V_2 \propto \frac{3}{2} \phi_2^2 + 4\phi_2. \] (18)

Now we study its semi-analytic solution and discuss more details of this duality.

Solving explicitly the Einstein equation (2) and the equations of motion for two scalar fields (3,4) together, we will study specifically the two periods of the universe evolution with this quintom model. For the first period that \(|t| \ll 1\), we take the initial conditions by fixing \( \phi_1 \to -\infty \) and \( \phi_2 \to 0 \). With these initial conditions one can see that the dominant component in the energy density of the quintom model is the exponential term of \( V_1 \) in eq. (17), namely the contribution from the Phantom potential in eq. (15) and the linear part of Quintessence potential in eq. (16) can be neglected. Therefore the evolution of the universe behaves like the one dominant by Quintessence component \( \phi_1 \) and the universe is evolving with the approximate solution given by explicitly:
\[ \phi_1 \sim \sqrt{2} \ln |t|, \phi_2 \sim \frac{1}{2} t^2, H \sim \frac{1}{t}. \] (19)
and thus one can see that the scale factor in this period would variate with respect to time of form \( a \propto t \) in which the signal is determined by the positive definite form of the scale factor. Therefore the scale factor here is corresponding to the big bang or big crunch of Quintessence-dominant universe.

The dual form of the solution above is a description of one universe dominant by a Phantom component with a lagrangian given by eq. (14) and
\[ \delta \mathcal{L}_1 + \delta \mathcal{L}_2 = \frac{\dot{\bar{\phi}}_1^2}{\phi_1^2} - \frac{\dot{\bar{\phi}}_2^2}{\phi_2^2} + V_1 + V_2 \]
\[ \sim -3\sqrt{2}\phi_1 + 4e^{-\sqrt{2}\phi_1} + 2\phi_2 + \frac{3}{2} \phi_2^2. \] (20)
correspondingly the dual hubble parameter is of form $H \sim -\frac{1}{t}$ and the scale factor is of form $a \propto -\frac{1}{t}^\frac{3}{2}$. Accordingly, the scale factor of the dual form is tending forward to infinity in the beginning or the end of universe. From what we have investigated in the above, one can see that, for the positive branch there is a duality between an expanding universe with initial singularity at $t = 0^-$ and a contracting one that begins with an infinite scale factor at $t = 0^+$. However, for the negative branch there is a duality between a contracting universe ending in a big crunch at $t = 0^-$ and an expanding one that ends in a final big rip at $t = 0^-$. The latter is justly dominated by a phantom component. Besides, in general with phantom dominant the contracting solution is not stable, because the Phantom universe will run into big rip, big sudden or expands forever approaching to a de Sitter, but will not be able to stay in the contracting phase forever\cite{37, 38}. This problem, however, can be avoided in Quintom model since in the dual universe with Quintom-B dark energy the increase of kinetic energy of Phantom during the contraction can be set off by that of Quintessence at late time.

For the case that ($|t| \gg 1$), Phantom component in the quintom model (1) will dominant and the universe will expand. For the specific potentials we consider in eqs. 17 and 18 the mass term in $V_2$ will gradually play an important role in the the evolution of dark energy. With the analysis similar to above we have

$$\phi_1 \sim \sqrt{2} \ln |t|, \phi_2 \sim \sqrt{2t}, H \sim t,$$  \hspace{1cm} (21)

where we note that the scale factor is $a \propto \exp(\sqrt{2}t)$. Consequently, the scale factor would increase forward to the infinity rapidly for the positive branch while start from an infinity for the negative branch.

Then the transformed lagrangian now is \cite{14} with

$$\delta \mathcal{L}_1 + \delta \mathcal{L}_2 \sim -3\sqrt{2}\phi_1 + 4e^{-\sqrt{2}\phi_1} + 4\phi_2 + \frac{3}{2}\phi_2^2 - 2.$$  \hspace{1cm} (22)

The universe is evolving with the hubble parameter like $H \sim -t$, and the scale factor like $a \propto \exp(-\sqrt{2}t)$ which is close to singularity related to the origin and the fate of universe. Finally, the component which dominates the evolution of the universe is $\phi_2$, the scalar field resembling the quintessence in \cite{14}. Consequently we obtain the conclusion that, for the positive branch, there is a dual relation between an expanding universe with a fate of expanding for ever with $t \rightarrow +\infty$ and a contracting universe with a destiny of shrinking for ever with $t \rightarrow +\infty$; meanwhile, for the negative one, there is a duality connecting a contracting universe starting from near infinity with $t \rightarrow -\infty$ and a expanding universe origin from infinity with $t \rightarrow -\infty$.

Having present the analytical arguments for the duality between the quintom A and quintom B model of the dark energy we study in detail the numerical solutions. In Fig.1 we plot the evolution of the equation of state of the quintom model and its dual. One can see from this figure that under the framework of the duality studied in this paper, the equation of state of the quintom model and its dual are symmetric around $w = -1$. Accordingly, in this case the Quintom A model of dark energy is dual to a Quintom B model rigorously, which supports for our analytical arguments above.

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig1.png}
\caption{Plot of the equation of state $w$ of the Quintom model and its dual as a function of the scale factor $\ln a$ for $V = -3\sqrt{2}M^3\phi_1 + 2M^4e^{-\sqrt{2}\phi_1} + \frac{3}{2}M^2\phi_2^2 + 4M^3\phi_2$ and $M$ is the planck mass.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.45\textwidth]{fig2.png}
\caption{Plot of the $w$ of the Quintom model and its dual as a function of the scale factor $\ln a$ for $V = V_0(e^{-\sqrt{2}\phi_1} + e^{-\sqrt{2}\phi_2})$ and $M$ is the planck mass.}
\end{figure}

In Fig.2 we take the potentials $V_1$ and $V_2$ to be exponential and one can see that the equation of state for quintom A approaches to a fixed value which corresponds to the attractor solution of this type of model\cite{8}.
Through the duality one can see that there exists a corresponding attractor of the Quintom B model dual to the former one. In Fig. 3 we provide another examples for the duality.

In summary, we have studied the cosmic duality in the quintom models. In general we have shown the Quintom model has its dual partner, specifically the Quintom A model is dual to the Quintom B. These two models describe two different behaviour of the universe evolution with one in the expanding phase and another in the contracting depending on the initial conditions we choose.

The cosmic duality which connects the two totally different scenarios of universe evolution keeps the energy density of the universe unchanged, but transforms the hubble parameter. With this duality and keeping Einstein equations form-invariant, we have found that Quintom A model is dual to Quintom B. As is know that with different type of dark energy, the fate of the universe will be different. Our study in this paper helps understand the properties of various dark energy models and their connections to the evolution and the fate of the universe.

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\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{Plot of the $w$ of the Quintom model and its dual as a function of the scale factor $\ln a$ for $V = m_1^2 \phi_1^2 + m_2^2 \phi_2^2$ where $m_1$ corresponds to the Quintessence component mass and $m_2$ the Phantom component mass.}
\end{figure}
[39] H. M. Sadjadi, and M. Alimohammadi, Phys. Rev. D74, 043506 (2006), gr-qc/0605143.