Integrable Supersymmetry Breaking Perturbations of $N=1,2$ Superconformal Minimal Models

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Abstract: We display a new integrable perturbation for both $N=1$ and $N=2$ superconformal minimal models. These perturbations break supersymmetry explicitly. Their existence was expected on the basis of the classification of integrable perturbations of conformal field theories in terms of distinct classical KdV type hierarchies sharing a common second Hamiltonian structure.
0. Posing the problem

In two dimensional quantum field theory, integrability singles out the class of tractable models. These can be efficiently formulated as integrable perturbations of some conformal field theory \[^1\]. There exist few guiding principles which can be used to classify the full set of integrable perturbations of a given (extended) conformal field theory, but the most powerful and universal one appears to be the following: the number of integrable perturbations is given by the number of integrable hierarchies of the KdV type, whose second Hamiltonian structure is associated to the extended conformal algebra, and which have distinct first Hamiltonian structures\[^2\]\[^3\]\[^4\]. A one-to-one correspondence between perturbating fields and the KdV hierarchies can be obtained via the associated Toda systems\[^5\]\[^6\]. In the Feigin-Fuchs representation, the perturbating field is represented by the part of the Toda Hamiltonian which is not a screening operator.

For the usual Virasoro minimal models, there are three integrable perturbations (\(\phi_{1,3}\), \(\phi_{1,2}\) and \(\phi_{2,1}\)), corresponding to the existence of three integrable hierarchies sharing the second Poisson structure of the KdV equation but having distinct Poisson brackets for the first Hamiltonian structure. These are the KdV hierarchy itself and the two reductions of the Boussinesq hierarchy\[^2\]. Their Toda system is related to the affine \(su(2)\) and twisted \(su(3)\) algebras respectively (the asymmetry of the later giving rise to two KdV type hierarchies). For the N=1 superconformal minimal models, the only supersymmetric integrable perturbation is \(\hat{\phi}_{1,3}\) (the hat denotes a superfield). It corresponds to the unique (space) supersymmetric KdV type system whose second Hamiltonian structure is the classical form of the N=1 superconformal algebra\[^7\]. The underlying affine algebra is twisted \(osp(2, 2)\)\[^8\]. In \[^4\], this approach was used to predict the existence of three supersymmetric integrable perturbations for the N=2 minimal models (also found in \[^9\]), given that there are three integrable N=2 (space) supersymmetric KdV hierarchies\[^10\]\[^11\]. The corresponding perturbating fields are the chiral superfields \(\Phi^l\) with \(l = 1, 2, K\), where \(K\) is related to the central charge by \(K = 2c/(3 - c)\).\(^1\).

However, it is known that there exists one\(^2\) integrable fermionic (but not supersymmetric) extension of the KdV equation whose second Hamiltonian structure is the N=1 superconformal algebra. It is the “super KdV” equation of Kupershmidt\[^13\]. Since this

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\(^1\) These ideas have also been extended to parafermionic models via the non-linear Schrödinger equation in disguised form \[^12\].

\(^2\) See the appendix about the one.
equation is not actually supersymmetric, we call it the KuperKdV equation. Its underlying algebraic structure is $osp(1,2)$. Furthermore, its $o(2)$ integrable extension (connected to $osp(2,2)$) turns out to be related to the N=2 superconformal algebra $[10]$. Hence, according to the above organizing principle, one expects these hierarchies to be related to supersymmetry breaking integrable perturbations of the N=1,2 superconformal minimal models. Here we show that this is indeed the case.

Without using the Feigin-Fuchs representation, it is possible to guess which perturbing field is associated to each hierarchy: the most natural relevant supersymmetry breaking perturbation is simply the lowest component of the superfield whose top component yields the integrable supersymmetry preserving perturbation. To be more precise, we consider an N=1 superfield

$$\hat{\phi} = \phi + \theta \psi .$$

(0.1)

The supersymmetric transformations of the component fields are

$$\delta \phi = \eta \psi \quad \delta \psi = \eta \partial \phi$$

(0.2)

where $\eta$ is a constant anticommuting parameter. The superintegral $\int dz \, d\theta \hat{\phi}$ is just $\int dz \psi$, which is manifestly supersymmetric. However, $\int dz \phi$ is not supersymmetric invariant. $\phi$ is referred to as the lower component of the superfield. If $\psi$ is relevant (with scaling dimensions smaller than two), then so is $\phi$. Indeed, perturbing the N=1 superconformal minimal models with the lower component of $\hat{\phi}_{1,3}$ leads to a (presumably) infinite sequence of conservation laws whose classical limit agrees with the KuperKdV conserved integrals.

In the N=2 case, three choices are possible. But again, a naturalness criterion would select the lower component of $\Phi^K$ as being the correct choice. Indeed, $\Phi^K$ is the N=2 analog of $\phi_{1,3}$ (or $\hat{\phi}_{1,3}$), and in any conformal field theory, the appropriate generalization of $\phi_{1,3}$ is an integrable perturbation. This makes the $\Phi^K$ perturbation ‘more fundamental’ than the other two. As a matter of fact, the massive theories obtained by perturbing the $N = 2$ minimal models with the lower component of $\Phi^K$ appear to be integrable and their conservation laws are exactly the quantum generalization of the $o(2)$ KuperKdV ones.

When treating a problem in which supersymmetry is not preserved, one needs to work out everything in terms of components. In such a case, the presence of the $u(1)$ symmetry induces a little catch. It appears at first sight that perturbing N=2 superconformal models with one component or the other of any chiral primary field (they are all relevant) produces non-trivial conservation laws (even without using the degeneracy equations of the
perturbating field). These conservation laws all happen to be expressible solely in terms of a twisted energy-momentum tensor \( \tilde{T} \). Actually, a closer look shows that switching on the perturbation does not induce any \( \bar{z} \) dependence on \( \tilde{T} \), which suggests that the critical point has not been left. In fact, this is exactly what happens. With respect to \( \tilde{T} \), all perturbating fields become marginal. This simple looking observation, when transposed in the context of perturbed \( \hat{su}(2)_k \) models, accounts for most of the conservation laws found in [14] (for a perturbation with an arbitrary primary field). We will report on this problem elsewhere.

1. \( N=1 \)

The procedure for computing integrals of motion in perturbed conformal field theory (to first order) is by now rather standard. In terms of the operators

\[
\Gamma_n = \frac{1}{2i\pi} \oint dzz^n \phi_{1,3}(z) ,
\]

\[
\Lambda_n = \frac{1}{2i\pi} \oint dzz^{n+1/2} \psi_{1,3}(z) ,
\]

(1.1)

where \( \phi_{1,3} \) and \( \psi_{1,3} \) are the components of the superfield \( \hat{\phi}_{1,3} \) (cf eq. (0.1)), \( F_s \), the conserved quantity of spin \( s \), (a differential polynomial in \( T(z) = L_{-2}I \) and \( G(z) = G_{-3}I \)) is characterized by the fact that \( \Gamma_0F_s \) is a total derivative. To show this, it is necessary to use the degeneracy equation of the perturbing field, whose component form reads [3]

\[
L_{-1} \psi(0) | 0 > = \frac{1}{2}(2h + 1)G_{-\frac{3}{2}} \phi(0) | 0 > ,
\]

\[
L_{-1}^2 \phi(0) | 0 > = (2h + 1)[L_{-2} - \frac{1}{2}G_{-\frac{3}{2}}] \phi(0) | 0 > ,
\]

(1.2)

with \( (\psi, \phi, h) = (\psi_{1,3}, \phi_{1,3}, h_{1,3}) \). With the central charge parametrized as

\[
c = \frac{3}{2}(1 - \frac{8}{p(p+2)}) ,
\]

(1.3)

\( h_{1,3} \) reads

\[
h_{1,3} = \frac{(p-2)}{2(p+2)} ,
\]

(1.4)
The vectors associated with the first few conserved densities are

\[ F_2 = L_{-2} \mid 0 \rangle, \]
\[ F_4 = (L_{-2}^2 - 2 \frac{h - 1}{2h + 1} G_{\frac{-3}{2}} G_{\frac{-5}{2}}) \mid 0 \rangle, \]
\[ F_6 = (L_{-2}^3 + 6 \frac{h - 1}{2h + 3} L_{-2} G_{\frac{-3}{2}} G_{\frac{-5}{2}} + 8 \frac{(h - 1)^3}{(2h + 1)(2h + 3)} G_{\frac{-9}{2}} G_{\frac{-11}{2}} + \frac{4h^3 + 4h^2 - 31h + 8}{4(2h + 1)(2h + 3)} L_{-3}^2) \mid 0 \rangle, \]
\[ F_8 = (L_{-2}^4 + a_1 L_{-2}^2 L_{-2} + a_2 L_{-4}^2 + a_3 G_{\frac{-9}{2}} G_{\frac{-7}{2}} + a_4 G_{\frac{-7}{2}} G_{\frac{-5}{2}} + a_5 L_{-2}^2 G_{\frac{-3}{2}} G_{\frac{-5}{2}} + a_5 L_{-4} G_{\frac{-3}{2}} G_{\frac{-5}{2}}) \mid 0 \rangle, \]

where

\[ a_1 = \frac{4h^3 - 4h^2 - 67h + 4}{(2h + 1)(2h + 5)}, \]
\[ a_2 = \frac{24h^5 + 28h^4 - 1386h^3 + 1713h^2 + 215h + 225}{15(2h + 1)^2(2h + 5)}, \]
\[ a_3 = \frac{24(h - 4)(h - 1)^2(16h^2 - 6h + 13)}{5(2h + 1)^2(2h + 5)}, \]
\[ a_4 = \frac{16(h - 1)^2(2h - 5)}{(2h + 1)(2h + 5)}, \]
\[ a_5 = \frac{12(h - 1)}{2h + 5}, \]
\[ a_6 = \frac{4(h - 1)(-6h^2 + 19h + 5)}{(2h + 1)(2h + 5)}. \]

We have checked their commutativity (using the Mathematica package of [13]). With the rescalings

\[ T = -\frac{c}{6} u, \quad G = \frac{c}{3} \xi, \]

these conserved quantities can be checked to reduce to those of the KuperKdV equations [13]

\[ u_t = -u_{xxx} + 6uu_x + 3\xi \xi_{xx}, \]
\[ \xi_t = -4\xi_{xxx} + 6\xi_x u + 3\xi u_x. \]
in the limit of \( c \to \infty \) (which can be realized by \( p \to -2 \) or \( h \to -\infty \)). The first few conserved densities for the system (1.7) are

\[
\begin{align*}
    h_2 &= u, \\
    h_4 &= u^2 - 4\xi \xi_{xx}, \\
    h_6 &= u^3 + \frac{1}{2} uu_x + 12u\xi_x\xi + 8\xi_{xx}\xi_x, \\
    h_8 &= u^4 + 2u^2 u + \frac{1}{5} u_{xx}^2 + 24u^2 \xi_x\xi - 24u_{xx}\xi_x\xi + 32u\xi_{xx}\xi_x + \frac{64}{5}\xi_{xxx}\xi_{xx}.
\end{align*}
\]

(1.8)

On the other hand, no conserved densities with odd spins have been found and these do not exist even in the classical case.

2. \( N=2 \)

An (anti)-chiral superfield \( \tilde{\Phi} \) satisfies the constraints \( D^- \tilde{\Phi} = 0 \), which implies a component expansion of the form

\[
\tilde{\Phi} = \varphi + \frac{1}{2} \theta^- \psi^+ - \theta^+ \theta^- \partial \varphi. \tag{2.1}
\]

Let \( \tilde{\Phi} \) stand for \( \Phi^K \) where \( K \) refers to the following parametrization of the central charge

\[
c = 3(1 - \frac{2}{K + 2}). \tag{2.2}
\]

The conformal dimension and the \( u(1) \) charge of \( \tilde{\Phi} \) are given by

\[
h = -q = \frac{K}{2(K + 2)} \tag{2.3}
\]

(which fixes our relative normalization for the \( u(1) \) current) while its degeneracy equation in component form reads

\[
\begin{align*}
    (L_{-1} + 2J_1)\varphi(0) & | 0 > = 0, \\
    (L_{-1} + 2J_1)\psi^+(0) & | 0 > = -2G^+_{-\frac{3}{2}}\varphi(0) | 0 >, \\
    G^-_{-\frac{3}{2}}\varphi(0) & | 0 > = 0, \\
    (L_{-1}^2 + 2J_{-1} - 2L_{-2})\varphi(0) & | 0 > = \frac{1}{2} G^-_{-\frac{5}{2}} \psi^+(0) | 0 >. \tag{2.4}
\end{align*}
\]
We consider the perturbation $\int dz \varphi$ (ignoring the antiholomorphic part). For this we introduce the quantities
\[
\Gamma_n = \frac{1}{2i\pi} \oint dz z^n \varphi(z) ,
\]
\[
\Lambda_n^+ = \frac{1}{2i\pi} \oint dz z^{n+1/2} \psi^+(z) ,
\]
whose commutators with the generators of the $N = 2$ superconformal algebra are
\[
[\Gamma_m, L_n] = [-h(n + 1) + (m + n + 1)]\Gamma_{m+n} ,
\]
\[
[\Gamma_m, G^+_n] = \frac{1}{2} \Lambda^+_{m+n} ,
\]
\[
[\Gamma_m, G^-_n] = 0 ,
\]
\[
[\Gamma_m, J_n] = h\Gamma_{m+n} ,
\]
\[
[\Lambda^+_m, L_n] = [-(n + 1)(h + \frac{1}{2}) + (m + n + \frac{3}{2})] \Lambda^+_{m+n} ,
\]
\[
\{\Lambda^+_m, G^+_n\} = 0 ,
\]
\[
\{\Lambda^+_m, G^-_n\} = [-4h(2n + 1) + 4(m + n + 1)]\Gamma_{m+n} ,
\]
\[
[\Lambda^+_m, J_n] = (h - \frac{1}{2})\Lambda^+_{m+n} .
\]
\[
(2.5)
\]
Notice that $\Lambda^+_{m+n}$ is odd and that
\[
\Gamma_{-m-1}I = \frac{1}{m!} L_{-1}^m \varphi , \quad \Lambda^+_{-m-\frac{1}{2}}I = \frac{1}{m!} L_{-1}^m \psi^+ .
\]
\[
(2.6)
\]
$I$ being the identity field. Proceeding as for $N = 1$, the first few conserved densities with integer spins are found to be
\[
F_2 = L_{-2} \mid 0 > ,
\]
\[
F_3 = (L_{-2}J_{-1} - \frac{2}{3h} J^{-3} - \frac{h}{2} G^-_{-\frac{3}{2}} G^+_{-\frac{1}{2}}) \mid 0 > ,
\]
\[
F_4 = (L_{-2}^2 + a_1 L_{-3}J_{-1} + a_2 J_{-2}^2 + a_3 L_{-2}J_{-1}^2 + a_4 J_{-1}^4
\]
\[
+ a_5 G^-_{-\frac{5}{2}} G^+_{-\frac{1}{2}} + a_6 G^-_{-\frac{3}{2}} G^+_{-\frac{1}{2}} J_{-1}) \mid 0 > ,
\]
where
\[
a_5 - a_6(h - 1) = 0 ,
\]
\[
-2(h - 1) + a_1 h - a_3 h(h - 1) + a_6(2h - 1) = 0 ,
\]
\[
a_1(h - 1) - 2a_2 h + a_3(h - 1)^2 + 2a_4 h(h^2 - 3h + 1) = 0 .
\]
\[
(2.8)
\]
At this point, we see that there are three arbitrary parameters in $\tilde{F}_4$. This unusual feature will be commented on shortly. Nevertheless, $\tilde{F}_3$ is sufficient to make an unambiguous contact, in the classical limit, with the $o(2)$ KuperKdV equation. This equation can be written exactly like the usual KuperKdV equation but with the replacement $[10]$

$$u \to q = u - w^2 ,$$

$$\xi \to \Psi = e^{\sigma(\partial^{-1} w)} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} , \quad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

(2.9)

together with $w_t = 0$. To investigate the classical limit, we set

$$T = -\frac{1}{6 cu} , \quad G^\pm = \frac{i}{3} \xi^{\pm} , \quad J = -\frac{i}{6} cw ,$$

(2.10)

where $\xi^{\pm} = (\xi_1 \pm i\xi_2)/\sqrt{2}$, and let $c = 6h \to \infty$. $\tilde{F}_3$ reduces then to the product $\xi_1 \xi_2$ (up to a multiplicative factor). In terms of $\Psi$ it reads $\Psi^T \sigma \Psi$ ($T$ stands for transpose) and this is easily checked to be conserved for the $o(2)$ KuperKdV equation.  

Now this equation is somewhat ‘degenerate’ in that the field associated to the $u(1)$ current is time independent. This reflects itself in the fact that with respect to the first Hamiltonian structure, the Poisson bracket of $w$ with itself is zero. As a consequence, the recursive generation of the classical conservation laws does not fix all the parameters $[10]$. In particular, for $\int \tilde{h}_3$, three parameters are left undetermined. Similarly, the commutativity of $\int \tilde{h}_3$ and $\int \tilde{h}_4$ leaves two parameters undetermined.

So apparently, this degeneracy extends to the quantum case, given that the direct determination of $\tilde{F}_4$ contains three free parameters. Moreover, if we start from a generic form for $\tilde{F}_4$ and impose (using the package of $[15]$)

$$[\int dz \tilde{F}_3 , \int dz \tilde{F}_4] = 0 ,$$

(2.11)

we find that four coefficients get determined exactly

$$a_1 = -a_6 = \frac{-4}{2h - 1} , \quad a_3 = \frac{-2(2h - 3)}{h(2h - 1)} , \quad a_5 = \frac{4(h - 1)}{2h - 1} ,$$

(2.12)

3 A similar situation is observed for the perturbation $\int dz \psi^+$ treated in terms of components. But in this case, we can rely on supersymmetry to fix all undetermined coefficients. In this way, the ambiguity discussed below is bypassed.

4 This system admits conservation laws at all integer degrees. In the reduction $\xi_2 = w = 0$, even degree densities reduce to those of the KuperKdV equation, while those at odd degrees vanish.
while $a_2$ and $a_4$ remain undetermined, precisely as in the classical case. The above values are compatible with the conditions (2.8) and they reduce, in the classical limit, to the coefficients of $\tilde{h}_4$. The undetermined coefficients can be fixed only from the commutation with the higher order conservation laws.

In the perturbed theory, there are also conservation laws with half-integer degrees, whose first few related vectors are

\[
\tilde{F}^2_3 = G^{-\frac{1}{2}} | 0 > , \\
\tilde{F}^2_5 = J_{-1} G^{-\frac{3}{2}} | 0 > , \\
\tilde{F}^2_7 = [J_{-1}^2 - hJ_{-2} + a(L_{-2} - \frac{h-1}{h}J_{-2})]G^{-\frac{5}{2}} | 0 > , \tag{2.13}
\]

where $a$ is undetermined. At first sight, this is somewhat surprising because such conservation laws are not present at the classical level, and this would be regarded as a strong indication that they should not be there either in the quantum case. However, a closer look shows that they do not provide integrals of motion for the quantum $0(2)$ KuperKdV equation, simply because they do not commute with $\int dz \tilde{F}_4$, the defining Hamiltonian of the system when formulated canonically. They do not commute either among themselves. Their interest, if any, is thus rather limited.

For other perturbations (in particular with the lowest component of the chiral superfield $\Phi^l$, $l = 1, 2$) no conservation laws were found except for a trivial sequence which we now discuss. From the commutators (2.6), it follows that

\[
[\Gamma_0, L_{-n} - (n - 1)\frac{h-1}{h} J_{-n}] = 0 . \tag{2.14}
\]

This means that the perturbation commutes with the field

\[
\tilde{T} = T - \frac{h-1}{h} J' , \tag{2.15}
\]

and any of its derivatives. It thus implies that any differential polynomial in $\tilde{T}$ commutes with $\Gamma_0$. This result only uses the fact that $\varphi$ is the lower component of a chiral field but it is independent of the degeneracy equation of the field under consideration. This implies that (2.14) holds for the perturbation by the lower component of any chiral field$^5$. However,

$^5$ For supersymmetry preserving perturbation, where $\Gamma_0$ is replaced by $\Lambda^+_\frac{1}{2}$, the analog of (2.14) is

\[
[\Lambda^+_\frac{1}{2}, L_{-n} - (n - 1)J_{-n}] = 0 .
\]
with respect to $\tilde{T}$, the conformal dimension of the lower component of any chiral primary fields is one, which means that the perturbation is marginal. It thus acts as a simple twist, a supersymmetry breaking term. But it does not drive the system off-criticality.

Appendix A.

The most general local fermionic extension of the KdV equation is

$$u_t = -u_{xxx} + 6uu_x - 3\xi\xi_{xx},$$
$$\xi_t = -a\xi_{xxx} + bu\xi_x + cu_x\xi,$$  \hspace{1cm} (A.1)

where $\xi$ is a fermionic field. The coefficients of the two nonlinear terms in the first equation can of course be modified by a rescaling the two fields. However, they have been fixed in order to exclude their possible vanishing, since in that case the system becomes trivial. The canonical formulation of the system in terms of the Poisson structure which is the classical limit of the superconformal algebra, forces the relations $c = 3$ and $b = 2 + a$. The supersymmetric KdV equation corresponds to $a = 1$ while $a = 4$ for the Kupershmidt system. In [16], it was shown that the one parameter family of systems related to the superconformal algebra, is integrable only for these two values of $a$. The same conclusion was obtained from the Painlevé analysis of the above more general three parameters systems [17]. However, it has been argued in [18] that the system $a = 1, b = c = 6$ is also Painlevé admissible (see also [19] for further support on integrability). But this latter system is trivial in the sense that the change of variables $u = v + \xi(\partial^{-1}\xi)$ transforms the first equation into the usual KdV equation (i.e. the fermionic field decouples) without affecting the second one.

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