Knot invariants from rational conformal field theories

P. Ramadevi, T.R. Govindarajan and R.K. Kaul
The Institute of Mathematical Sciences,
C.I.T.Campus, Taramani,
Madras-600 113, INDIA.

Abstract

A framework for studying knot and link invariants from any rational conformal field theory is developed. In particular, minimal models, superconformal models and $W_N$ models are studied. The invariants are related to the invariants obtained from the Wess-Zumino models associated with the coset representations of these models. Possible Chern-Simons representation of these models is also indicated. This generalises the earlier work on knot and link invariants from Chern-Simons theories.

email: rama, trg, kaul@imsc.ernet.in
1. Introduction

In the last few years there has been enormous interest in obtaining the invariants for knots and links embedded in 3 manifolds through Chern Simons theories\cite{1} - \cite{8}. It was Schwarz who first conjectured that Jones polynomials\cite{9} may be related to Chern-Simons theories\cite{1}. Witten in his original papers set up a general framework to study knots and links through Chern-Simons theories. He has shown that Jones polynomials are given by the expectation values of the Wilson loop operators carrying spin $1/2$ representation in three-dimensional $SU(2)$ Chern-Simons theory. The two-variable generalization\cite{10} of Jones invariants are obtained in an $SU(N)$ Chern-Simons theory when N-dimensional representations are placed on all the component knots\cite{2}. These results were demonstrated by proving that these invariants satisfy the same Alexander-Conway skein relations as the respective polynomials. Unfortunately the generalized skein relations cannot be recursively solved for arbitrary knots/links when other than defining representations are living on the strands. For this purpose general methods of obtaining these link invariants directly have been developed. One such method for links made up of upto four strands in $SU(2)$ Chern-Simons theory has been presented in \cite{5}. Generalization to arbitrary compact semi-simple groups, in particular $SU(N)$ has also been obtained\cite{6}. A complete solution for links made up of arbitrary strands has also been developed in \cite{7}. Intimate relation between Chern-Simons theories and two-dimensional Wess-Zumino conformal field theory plays an important role in these studies. The Chern-Simons functional integral over 3-manifold with boundary are expressed in terms of vectors in the Hilbert space of the conformal blocks of the associated Wess-Zumino conformal field theories on the boundary. Since primary fields of the Wess-Zumino conformal fields are in one to one correspondence with the irreducible representation of quantum groups, there is also a close relationship of Chern-Simons theories with the quantum groups. The deformation parameter $q$ of these quantum groups is related to the coupling constant $k$ of our Chern-Simons theories through $q(k) = \exp\frac{2\pi i}{k+C_v}$, where $C_v$ is the quadratic casimir of the adjoint representation. Just like Chern-Simons (and equivalently Wess-Zumino conformal field theories) yield link
invariants, there are invariants also associated with other rational conformal field theories. This can be done by using the representations of braid generators and an explicit definition of trace with Markov invariances as presented in ref.\cite{11} for such models. Isidro, Labastida and Ramallo also have studied invariants for toral knots from the minimal models in ref.\cite{3}. The braiding matrices have been shown to be factorised in terms of $SU(2)$ Wess-Zumino model braid matrices. The method of obtaining link invariants presented in refs.\cite{5, 6, 7} can readily be generalised to any rational conformal field theory. This will allow us to obtain invariants not only for toral knots but for any arbitrary multicoloured link. In this paper, we shall present the results for minimal, superconformal and $W_N$ theories. The invariants for any arbitrary link in these models will be shown to be expressible in a simple manner in terms of the invariants of Wess-Zumino models associated with the coset representation of these conformal field theories. The underlying quantum group structure will also be emphasised.

In sec.2 we shall take up the minimal conformal field theories. The braiding matrices as well as duality matrices will be shown to be factorised in terms of those of Wess-Zumino models in the coset representation $(SU(2)_k \otimes SU(2)_1)/SU(2)_{k+1}$ of these models. The link invariants will also be shown to factorise in the same manner. In sec.3, we present a brief discussion of these ideas for the superconformal and $W_N$ models through their coset representation $(SU(2)_k \otimes SU(2)_2)/SU(2)_{k+2}$ and $(SU(N)_k \otimes SU(N)_1)/SU(N)_{k+1}$ respectively. In sec.4, we give the possible Chern-Simons description of these general $(G_k \otimes G_l)/G_{(k+l)}$ coset models.

2. Link invariants from minimal models

The minimal model fields admit a coset construction \cite{12, 13} based on $G/H = (SU(2)_k \otimes SU(2)_1)/SU(2)_{k+1}$. The central charge $c$ is given for these models by $c = 1 - \frac{6}{(k+2)(k+3)}$. The primary fields $\Phi_{(r,s)}$ are labelled by two integers $r$ and $s$ and their conformal weights are given by

$$h_{r,s} = \frac{[(k+3)r - (k+2)s]^2 - 1}{4(k+2)(k+3)}, \quad 1 \leq r \leq k+1, \quad 1 \leq s \leq k+2 \quad (1)$$

There is a doubling of fields through this labelling which can be distinguished through the
condition whether \( r + s \) is even or odd. It is convenient for our future discussion to rewrite these conformal weights as:

\[
h_{r,s} = h_r^{(k)} - h_s^{(k+1)} + \frac{(r - s)^2}{4}
\]

where \( h_r^{(k)} \) and \( h_s^{(k+1)} \) are the conformal weights of \( r \) and \( s \) dimensional representations of \( SU(2)_k \) and \( SU(2)_{k+1} \) WZW models respectively and \( h_r^{(k)} = \frac{r^2 - 1}{4(k+2)} \). An explicit labelling of the primary fields which also involves \( SU(2)_1 \) description, is done with the help of the label \( \epsilon \) (\( \epsilon = 1, 2 \)) corresponding to the primary fields of \( SU(2)_1 \). Incorporating the conformal weight of this level 1 field (\( \epsilon \)) in the conformal dimension of the minimal model, we write (2) as:

\[
h_{r,s} = h_r^{(k)} + h_\epsilon^{(1)} - h_s^{(k+1)} + \frac{(r - s)^2 - (\epsilon - 1)^2}{4}
\]

where \( \epsilon = 1 \) for \( r + s \) even and \( \epsilon = 2 \) for \( r + s \) odd.

The fusion rules satisfied by these primary fields is

\[
\Phi_{(r_1,s_1)} \otimes \Phi_{(r_2,s_2)} = \sum_{r_3 = |r_1 - r_2| + 1}^{\min(r_1 + r_2 - 1, 2k + 3 - r_1 - r_2)} \sum_{s_3 = |s_1 - s_2| + 1}^{\min(s_1 + s_2 - 1, 2k + 5 - s_1 - s_2)} \Phi_{(r_3,s_3)}
\]

The modular transformation matrix for the minimal model is given by

\[
S_{rs'}^{s'} = \sqrt{\frac{8}{(k+2)(k+3)}} (-1)^{(r+s)(r'+s')} \frac{\pi rr'}{k+2} \frac{\pi ss'}{k+3}
\]

The modular transformation is related to the unknot polynomials for any conformal field theory\[^8, 14\]. This unknot invariant associated with representation \( \Phi_{(r,s)} \) is given by \( V_{(r,s)}[U] = S_{rs}^{11}/S_{11}^{11} \). The factorised form of this modular transformation above imply

\[
V_{(r,s)}[U] = V_r^{(k)}[U] V_s^{(k+1)}[U]
\]

where the two unknot invariants on the right hand side correspond to \( r \) and \( s \) dimensional representations of \( SU(2)_k \) and \( SU(2)_{k+1} \) Wess-Zumino models respectively. Notice that \( V_\epsilon^{(1)}[U] = 1 \) for both \( \epsilon = 1 \) and 2. This factorised form of the unknot invariant is consistent with the fact that link invariant for two cabled knots (such as unknots here) obey the
fusion rules[3] of the theory.

\[ V_{(r_1,s_1)}[U]V_{(r_2,s_2)}[U] = \sum_{r_3,s_3} V_{(r_3,s_3)}[U] \] (7)

To study the link invariants, we need the braiding properties of the correlator-conformal blocks. The eigenvalues of the braiding matrix \( B[(r_1,s_1); (r_2,s_2)] \) as depicted below are given by

\[
\lambda_{(r_3,s_3)}[(r_1,s_1); (r_2,s_2)] = \Omega \exp i\pi (h_{r_{1,s_1}} + h_{r_{2,s_2}} - h_{r_{3,s_3}}) (8)
\]

where \( \Omega = \pm 1 \) and we choose it to be \((-1)^{f_1+f_2-f_3}\) where \( f_i \) is an integer given in terms of the dimensions of the representations as \( f_i = \frac{1}{2} (r_i - s_i + \epsilon_i - 1)(\frac{r_i - s_i - \epsilon_i + 1}{2} + 1) \).

Using eqn (3), these eigenvalues can be rewritten in explicitly factorized form in terms of the WZW model braiding eigen values as shown:

\[
\lambda_{(r_3,s_3)}[(r_1,s_1); (r_2,s_2)] = \lambda^{(k)}_{r_3}(r_1,r_2) (\lambda^{k+1}_{s_3}(s_1,s_2))^{-1} \lambda^{(1)}_{\epsilon_3}(\epsilon_1,\epsilon_2) (9)
\]

where \( \lambda_{r_3}(r_1,r_2) = (-1)^{\frac{1}{2}((r_1-1)+(r_2-1)-(r_3-1))} \exp i\pi (h^{(k)}_{r_1} + h^{(k)}_{r_2} - h^{(k)}_{r_3}) \) and \( \epsilon_i \) are the \( SU(2)_1 \) labels of the primary fields \( \Phi_{(r_i,s_i)} \) respectively. The knots and links have to be specified with some framing convention. Usual convention corresponds to standard framing where linking number of the knot and its frame is zero. Twisting strands does not leave this framing unaltered. However, we shall here use the vertical framing [15] where braiding does not alter the framing. In this framing, link invariants reflect only regular isotopy invariance instead of invariance under all the Reidmeister moves. This is reflected in the following equation for the
The corresponding equation for minimal models involves the conformal weights \((h_{r,s})\) of the primary fields of these models:

\[
\begin{align*}
\begin{array}{c}
\text{❅❅} \\
\text{✒} \\
\text{❅❅} \\
\text{❅❅}
\end{array}
= (-1)^{r} e^{-i\pi h_{r}}
\end{align*}
\]

\(r\)

\(\Phi_{(r_{1},s_{1})}\)

where \(h_{r,s}\) has been written in terms of eqn(3).

Besides framing, we shall also put orientation on the lines in a link. Any link can be thought of as closure of oriented braids. For this purpose we need to distinguish braiding in parallel and antiparallel strands. The eigenvalues for these braiding matrices are related. For example for \(SU(2)_{k}\) theory, the eigenvalues of a right handed half twist in parallel (+) and antiparallel (-) strands are given for standard framing as in ref([5, 6, 7]). In vertical framing these are simply related; they are each others inverse:

\[
\lambda_{r_{3}}^{(+)}(r_{1}, r_{2}) = \lambda_{r_{3}}(r_{1}, r_{2}) = [\lambda_{r_{3}}^{(-)}(r_{1}, r_{2})]^{-1}.
\] (10)

Same is true for the eigenvalues of braiding matrices in minimal models.

\[
\lambda_{(r_{3},s_{3})}^{(+)}[(r_{1}, s_{1}); (r_{2}, s_{2})] = \lambda_{(r_{3},s_{3})}[(r_{1}, s_{1}); (r_{2}, s_{2})] = [\lambda_{r_{3},s_{3}}^{(-)}(r_{1}, s_{1}; r_{2}, s_{2})]^{-1}
\] (11)

Factorisability of braid matrices [16, 17, 18] can also be directly justified from general properties of link invariants. This we shall discuss later. The method of obtaining link invariants for \(SU(2)\) and \(SU(N)\) Chern- Simons theories presented in ref([3, 6]) can be immediately generalised to any rational conformal field theory. We shall now present a brief discussion of this formalism in the context of minimal models.
Consider a 3-ball having 4-punctures with two strands going into the boundary ($S^2$) carrying primary fields $\Phi_{(r_1,s_1)}$ and $\Phi_{(r_2,s_2)}$ of the minimal models and the two strands coming out of the boundary carrying the appropriate primary fields with no crossing as shown in the fig. 1. With this diagram, we associate a state $|\psi_0\rangle$ in the Hilbert space of conformal blocks of four-point correlators on $S^2$. This state can be written in a convenient basis. We express it in terms of the eigen-basis of braid operators twisting central two strands or side two strands as:

$$|\psi_0\rangle = \sum_{r_3,s_3} \mu_{(r_3,s_3)} |\phi_{(r_3,s_3)}^{\text{cent}}\rangle = \sqrt{[r_1][r_2][s_1][s_2]} |\phi_{(1,1)}^{\text{side}}\rangle$$

The summation over $r_3$ and $s_3$ runs over values allowed by fusion rules eqn.(4). Here square brackets denote q-number defined as $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$. The corresponding state in the dual Hilbert-space representing the same fig.1 with oppositely oriented boundary is given by

$$\langle \psi_0 | = \sum_{r_3,s_3} \mu_{(r_3,s_3)} \langle \phi_{(r_3,s_3)}^{\text{cent}} | = \sqrt{[r_1][r_2][s_1][s_2]} \langle \phi_{(1,1)}^{\text{side}} |$$

Glueing the diagrams of fig.1 onto its dual along the boundary gives us two disjoint unknots. This is represented by the inner product of the associated states as:

$$V_{(r_1,s_1)}[U] V_{(r_2,s_2)}[U] = \langle \psi_0 | \psi_0 \rangle = \sum_{r_3,s_3} \sum_{s_3} \mu_{(r_3,s_3)}^2$$

In view of eqn (4), this implies $\mu_{(r_3,s_3)} = \mu_{r_3} \mu_{s_3}$ where $\mu_{r_3}$ and $\mu_{s_3}$ are the corresponding coefficients for the $SU(2)_k$ and $SU(2)_{k+1}$ WZW models: $\mu_r = \sqrt{[2r+1]}$. In order to appreciate factorisation of braid matrices, consider the state $|\psi_m\rangle$ drawn in fig.2, representing $m$ half-twists in central two strands in parallel orientation. This state expanded in terms of the above basis can be written as:

$$|\psi_m\rangle = (B)^m |\psi_0\rangle = \sum_{r_3,s_4} \mu_{(r_3,s_3)} \lambda_{(r_3,s_3)}^{(+) \left[(r_1,s_1);(r_2,s_2)\right] \rangle}^{-m} |\phi_{(r_3,s_3)}^{\text{cent}}\rangle$$

where $\lambda_{(r_3,s_3)}^{(+) \left[(r_1,s_1);(r_2,s_2)\right]}$ are the eigenvalues of the braid matrix for parallely oriented strands. Similarly, we can associate a state $|\psi'_0\rangle$ with the fig.3 and writing in terms of eigen basis of braid operator twisting side two strands or central two strands we get:

$$|\psi'_0\rangle = \sum_{r_3,s_3} \mu_{(r_3,s_3)} |\phi_{(r_3,s_3)}^{\text{side}}\rangle = \sqrt{[r_1][r_2][s_1][s_2]} \langle \phi_{(1,1)}^{\text{cent}} |$$
We can write $|\psi'_m\rangle$ representing $m$-half twists in the side two antiparallel strands as shown in fig.4 as

$$|\psi'_m\rangle = (B)^m|\psi_0\rangle = \sum_{r_3,s_3} \mu_{(r_3,s_3)}(\lambda_{(r_3,s_3)}^{-1}((r_1,s_1),(r_2,s_2)))^m|\phi_{(r_3,s_3)}^{side}\rangle$$  \hspace{1cm} (17)

Now look at the closure of two-strand braid carrying same primary field $\Phi_{(r_1,s_1)}$ with one half twist in the central two strands. This represents unknot and is given by

$$V_{(r_1,s_1)}[U](-1)^{2h_1}\exp(\pm i\pi 2h_{r_1,s_1}) = \langle \psi_0|\psi_1\rangle = \sum_{r_3,s_3} [2r_3+1][2s_3+1](\lambda_{(r_3,s_3)}^{+1}((r_1,s_1),(r_1,s_1)))^{\pm 1}$$  \hspace{1cm} (18)

This equation is consistent with factorisation of the braid eigenvalues in terms of $SU(2)_k$, $SU(2)_{k+1}$, $SU(2)_1$ eigenvalues.

Another interesting consistency condition is obtained by looking at a Hopf (anti Hopf) link in two equivalent representations as shown in fig.5. In the first one, we have braiding in parallelly oriented two strands whereas in the second antiparallel strands are braided. The two ways of writing this invariant have to be equal:

$$\sum_{r_3,s_3} [2r_3+1][2s_3+1](\lambda_{(r_3,s_3)}^{+1}((r_1,s_1),(r_2,s_2)))^{\pm 2} = \sum_{r_3,s_3} [2r_3+1][2s_3+1](\lambda_{(r_3,s_3)}^{-1}((r_1,s_1),(r_2,s_2)))^{\pm 2}$$  \hspace{1cm} (19)

There are two copies of each primary field in the minimal model which needs to be identified up to a phase: $\Phi_{(r_1,s_1)} = \exp\left\{\frac{\pi i}{4}(r_1-s_1-\epsilon_1-1)(\epsilon_1-\epsilon_2)\right\}\Phi_{(k+2-r_1,k+3-s_1)}$ where $\epsilon_2 = 3 - \epsilon_1$ and accordingly the braid operator $B$ operator is transformed to $\tilde{B}$ such that

$$\langle \Phi_{(r_1,s_1)}\Phi_{(r_2,s_2)}\Phi_{(r_3,s_3)}|B|\Phi_{(r_2,s_2)}\Phi_{(r_1,s_1)}\phi_{(r_3,s_3)}\rangle = \langle \Phi_{(r'_1,s'_1)}\Phi_{(r'_2,s'_2)}\Phi_{(r'_3,s'_3)}|\tilde{B}|\Phi_{(r'_2,s'_2)}\Phi_{(r'_1,s'_1)}\phi_{(r'_3,s'_3)}\rangle$$  \hspace{1cm} (20)

where the primed states are related to the unprimed ones by the above transformation along with the requirement that the respective three fields satisfy the fusion rules. While the braid matrix $B$ for the unprimed fields factorise in terms of the corresponding $SU(2)$ braid matrices, $\tilde{B}$ also factorise in terms of braid matrices associated with primed $SU(2)$ fields.

Not only do braid matrices factorise, so do the duality matrices $[[\Pi]]$. The duality matrices
relate the four-point correlators in different basis as shown below:

\[ \Phi_{(r_1,s_1)} \Phi_{(r_2,s_2)} \Phi_{(r_3,s_3)} \Phi_{(r_4,s_4)} a_{(r,s);(r',s')} \begin{pmatrix} (r_1, s_1) & (r_2, s_2) \\ (r_3, s_3) & (r_4, s_4) \end{pmatrix} \Phi_{(r_1,s_1)} \Phi_{(r_2,s_2)} \Phi_{(r_3,s_3)} \Phi_{(r_4,s_4)} \Phi_{(r',s')} \]

These conformal blocks refer to the bases \(|\phi^{\text{side}}_{(r,s)}\rangle\) and \(|\phi^{\text{cent}}_{(r',s')}\rangle\) of eqns.\((12),(13)\). Thus,

\[ \langle \phi^{\text{side}}_{(r,s)} | \phi^{\text{cent}}_{(r',s')} \rangle = a_{(r,s);(r',s')} \begin{pmatrix} (r_1, s_1) & (r_2, s_2) \\ (r_3, s_3) & (r_4, s_4) \end{pmatrix} \]

(21)

The factorisation of these duality matrices in terms of \(SU(2)\) duality matrices can be justified in the following manner. Consider the fig.2 and the fig.4 with \(m\) being 1. These have been redrawn in fig.6. Clearly, the states representing them are equal. Use eqn(14), (17) and (21) to write:

\[ (\lambda^{(+)\ast}_{r',s'}[(r_1, s_1); (r_2, s_2)])^{1\pm} a_{(r,s);(1,1)} \begin{pmatrix} (r_1, s_1) & (r_2, s_2) \\ (r_3, s_3) & (r_1, s_1) \end{pmatrix} = \]

\[ \sum_{r,s} a_{(1,1);(r,s)} \begin{pmatrix} (r_2, s_2) & (r_2, s_2) \\ (r_1, s_1) & (r_1, s_1) \end{pmatrix} (\lambda^{(-\ast)}_{r,s}[(r_1, s_1); (r_2, s_2)])^{1\pm} a_{(r,s);(r,s)} \begin{pmatrix} (r_2, s_2) & (r_1, s_1) \\ (r_2, s_2) & (r_1, s_1) \end{pmatrix} \]

(22)

This equation is explicitly satisfied by the factorised duality matrix:

\[ a_{(r,s);(r',s')} \begin{pmatrix} (r_1, s_1) & (r_2, s_2) \\ (r_3, s_3) & (r_4, s_4) \end{pmatrix} = a_{r,r'}^{(k)} \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} a_{s,s'}^{(k+1)} \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} a_{\epsilon,\epsilon'}^{(1)} \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix} \]

(23)

Here \(a_{r,r'}^{(k)} \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} \), \(a_{s,s'}^{(k+1)} \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \) and \(a_{\epsilon,\epsilon'}^{(1)} \begin{pmatrix} \epsilon_1 & \epsilon_2 \\ \epsilon_3 & \epsilon_4 \end{pmatrix} \) are the duality matrices for \(SU(2)_k, SU(2)_{k+1}\) and \(SU(2)_1\) theories respectively and are given in terms of \(6-j\) symbols.

\[ a_{r,r'}^{(k)} \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} = (-1)^{\frac{1}{2}(r_1+r_2+r_3+r_4)} \sqrt{[r][r']} \begin{pmatrix} \frac{r_1-1}{2} & \frac{r_2-1}{2} & \frac{r_3-1}{2} \\ \frac{r_1-1}{2} & \frac{r_2-1}{2} & \frac{r_3-1}{2} \end{pmatrix} \]

(24)
The $6−j$ symbol is given by:

$$
\begin{vmatrix}
{j_1} & {j_2} & {j_{12}} \\
{j_3} & {j_4} & {j_{23}}
\end{vmatrix} = \Delta(j_1, j_2, j_{12})\Delta(j_3, j_4, j_{12})\Delta(j_1, j_4, j_{23})\Delta(j_3, j_2, j_{23})
$$

\[
\sum_{m \geq 0} (-1)^m \frac{[m + 1]!}{[m - j_1 - j_2 - j_{12}]![m - j_3 - j_4 - j_{12}]![m - j_1 - j_4 - j_{23}]![m - j_3 - j_2 - j_{23}]!} \right
\]

and

$$
\Delta(a, b, c) = \left(\frac{[-a + b + c]![a - b + c]![a + b - c]!}{[a + b + c + 1]!}\right)^{-1}
$$

Here $[a]! = [a][a - 1] \ldots [2][1]$. The spin triplets $(j_1, j_2, j_{12}), (j_3, j_4, j_{12}), (j_2, j_3, j_{23})$ and $(j_1, j_4, j_{23})$ satisfy the fusion rules of $SU(2)_k$ Wess-Zumino model.

Since $\Phi_{(r,s)}$ and $\Phi_{(k+2-r,k+3-s)}$ fields of the minimal models are identified, the corresponding duality matrices $\tilde{a}$ for the shifted fields should also factorise in the same manner as does the braiding matrix $\tilde{B}$ of eqn(21). This can be easily verified by using the following properties of the $SU(2)_k$ duality matrices under the shifts:

$$
\begin{align*}
\tilde{a}_{rr'} \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix} &= (-1)^{\frac{1}{2}(r_1 + r_2 + r_3 + r_4 - 2k)} \tilde{a}_{rr'} \begin{pmatrix} k + 2 - r_1 & k + 2 - r_2 \\ k + 2 - r_3 & k + 2 - r_4 \end{pmatrix} = \\
(-1)^{\frac{r_1 + r_3 - r - r'}{2}} \tilde{a}_{(k+2-r)(k+2-r')} \begin{pmatrix} k + 2 - r_1 & r_2 \\ k + 2 - r_3 & r_4 \end{pmatrix} &= \\
(-1)^{\frac{r' - 1 + r_2 + r_3}{2}} \tilde{a}_{(k+2-r)r'} \begin{pmatrix} k + 2 - r_1 & r_2 \\ r_3 & k + 2 - r_4 \end{pmatrix}
\end{align*}
$$

Factorisation of braid matrices and duality matrices has an important and immediate implication. This implies that the link invariants also factorise. We formulate this main result in the form of a theorem as follows:
Theorem 1: The invariant for any knot/Link \((L)\) carrying primary fields of the minimal model \(\Phi_{r_1,s_1}, \Phi_{r_2,s_2}, \ldots, \Phi_{r_n,s_n}\) on the \(n\) component knots is given by:

\[
V_{(r_1,s_1),\ldots,(r_n,s_n)}[L] = V_{r_1,\ldots,r_n}[L] \ V_{s_1,\ldots,s_n}^{(k+1)}[\bar{L}] \ V_{\epsilon_1,\ldots,\epsilon_n}^{(1)}[L]
\]  

where \(V_{r_1,r_2,\ldots,r_n}[L]\) is the invariant from the \(SU(2)_k\) WZW models for the link \(L\) carrying \(r_1, r_2, \ldots, r_n\) dimensional representations on the component knots. \(V_{s_1,\ldots,s_n}^{(k+1)}[\bar{L}]\) is the corresponding invariant for \(SU(2)_{k+1}\) theory for the mirror link \(\bar{L}\) and \(V_{\epsilon_1,\ldots,\epsilon_n}^{(1)}[L]\) is the invariant for the \(SU(2)_1\) model. Here \(\epsilon_i = 1\) or \(2\) for \((r_i - s_i)\) even or odd respectively.

Our discussion above also reflects the quantum group structure for the minimal model to be \(SU_q(k) (2) \otimes SU_q(1) (2) \otimes SU_q(k+1) (2)\). This is in agreement with the conjecture for a general \(G/H\) models\([16, 17]\). Exploiting the properties of vertex operators, this quantum group structure has already been pointed out by Lashkevich\([18]\).

3. Superconformal and \(W_N\) models

Our discussion above is also valid for other conformal field theories. We shall discuss the \(N = 1\) superconformal minimal models now. These models are obtained through the coset construction\([12, 19]\) based on \(G/H = (SU_k(2) \otimes SU_2(2))/SU_{k+2}(2)\). The central charges for these models are given by \(c = \frac{3}{2} \left\{ 1 - \frac{8}{(k+2)(k+4)} \right\} \). The primary fields are again labelled through two integers \(r\) and \(s\) and the conformal weight of the \(\Phi_{(r,s)}\) primary field is given by

\[
h_{r,s} = \frac{[(k+4)r - (k+2)s]^2 - 4}{8(k+2)(k+4)} + \frac{\alpha}{8}
\]  

NS sector is defined through \(r - s = 2Z\) and \(\alpha = 0\). Ramond sector is obtained through \(r - s = 2Z + 1\) and \(\alpha = \frac{1}{2}\). The conformal weights can also be rewritten as

\[
h_{r,s} = h_r^{(k)} - h_s^{(k+2)} + \frac{(r-s)^2}{8} + \frac{\alpha}{8}
\]  

Like in the minimal models in Sec.2, we incorporate the level 2 field labelled by \(\epsilon\) in the conformal dimensions to get:

\[
h_{r,s} = h_r^{(k)} + h_\epsilon^{(2)} - h_s^{(k+2)} + \frac{(r-s)^2 - (\epsilon - 1)^2}{8}
\]
where for $(r - s)$ even, $\epsilon$ is given by $\epsilon = 1 + [(r - s) \mod 4]$ and for $(r - s)$ odd, $\epsilon = 2$. The fusion algebra is given by

$$\Phi_{(r_1,s_1)} \otimes \Phi_{(r_2,s_2)} = \sum_{r_3 = |r_1 - r_2| + 1}^{\min(r_1 + r_2 - 1, 2k + 3 - r_1 - r_2)} \sum_{s_3 = |s_1 - s_2| + 1}^{\min(s_1 + s_2 - 1, 2k + 7 - s_1 - s_2)} \Phi_{(r_3,s_3)}$$

(29)

The modular transformation matrix for the NS superconformal model is given by [19, 20]

$$S_{rs'}^{r's'} = \frac{4}{(\sqrt{(k + 2)(k + 4)})} \sin \frac{\pi rr'}{(k + 2)} \sin \frac{\pi ss'}{(k + 4)}$$

(30)

Like in the minimal models of sec.2, this implies that the unknot polynomial $V_{(r,s)}[U]$ for the strands carrying the superconformal NS field $\Phi_{(r,s)}$ to be

$$V_{(r,s)}[U] = V_r^{(k)}[U] V_s^{(k+2)}[U]$$

(31)

Note that the level two field in the NS sector are $\epsilon = 1 \text{ or } 3$ and hence its unknot polynomial $(V_{(2)}^\epsilon[U])$ is 1. For the Ramond field, the unknot polynomial will contain a factor $V_{(2)}^\epsilon[U]$ with $\epsilon = 2$. Thus in general:

$$V_{(r,s)}[U] = V_r^{(k)}[U] V_{(2)}^\epsilon[U] V_s^{(k+2)}[U]$$

(32)

The braiding eigenvalues are given by:

$$\lambda_{(r_3,s_3)}[(r_1,s_1);(r_2,s_2)] = \Omega \exp \pi i (h_{r_1,s_1} + h_{r_2,s_2} - h_{r_3,s_3})$$

(33)

where $\Omega = \pm 1$ explicitly given by $\Omega = (-1)^{f_1 + f_2 - f_3}$ and $f_i$ are integers and are given in terms of the dimensions of the fields is given by

$$f_i = (\frac{r_i - s_i + \epsilon_i - 1}{2})(\frac{r_i - s_i - \epsilon_i + 1}{4} + 1)$$

This eigenvalue can be rewritten in terms of WZW model braiding eigen values as shown:

$$\lambda_{(r_3,s_3)}[(r_1,s_1);(r_2,s_2)] = \lambda_{r_3}^{(k)}(r_1,r_2) \lambda_{s_3}^{(k+2)}(s_1,s_2)^{-1} \lambda_{\epsilon_3}^{(2)}(\epsilon_1,\epsilon_2)$$

(34)

Like in the minimal model in sec.2, we shall use the vertical framing for the knots. The eigenvalues for braiding matrices associated with parallel and antiparallel strands are again
inverse of each other. Further duality matrices again factorise in terms of $SU(2)_k$, $SU(2)_{k+2}$ and $SU(2)_2$ duality matrices. This leads us to the result:

**Theorem 2:** The invariant for a link $L$, made up of $n$ knots, carrying representations $\Phi_{(r_1,s_1)}, \ldots, \Phi_{(r_n,s_n)}$ respectively is given in terms of $SU(2)_k$, $SU(2)_{k+2}$ and $SU(2)_2$ invariants as follows:

$$V_{(r_1,s_1),\ldots,(r_n,s_n)}[L] = V^{(k)}_{r_1,\ldots,r_n}[L] V^{(k+2)}_{s_1,\ldots,s_n}[-L] V^{(2)}_{e_1,\ldots,e_n}[L]$$

(35)

where $[-L]$ stands for the mirror link and $\epsilon_i = 1, 3$ if $r_i - s_i = 4Z, 4Z + 2$ respectively and $\epsilon_i = 2$ if $r_i - s_i = 2Z + 1$.

Similar discussion can be extended to $W_N$ models [21], which have a coset representation in terms of $SU(N)_k \otimes SU(N)_1/SU(N)_{k+1}$. The central charge for these models is given by

$$c = (N-1)(1 - \frac{N(N+1)}{(k+N)(k+N+1)})$$

(36)

The primary fields are labelled by two weight vectors $\mu, \nu$ of $SU(N)$ and the conformal weights for the $\Phi_{(\mu,\nu)}$ field is given by

$$h_{\mu,\nu} = \frac{[(k+N+1)n_i - (k+N)m_i + 1]g_{ij}[(k+N+1)n_j - (k+N)m_j + 1] - \rho^2}{2(k+N)(k+N+1)}$$

(37)

where $\rho$ is the Weyl vector, $g_{ij}$ is the cartan matrix and the weights in terms of fundamental weights $\Lambda^i$ of $SU(N)$ are $\mu = \sum_{i=1}^{N-1} n_i \Lambda^i$, $\nu = \sum_{i=1}^{N-1} m_i \Lambda^i$. Further $\mu$ and $\nu$ are restricted to $\mu.\psi \leq k$, $\nu.\psi \leq k + 1$, where $\psi$ is the longest root.

The primary fields of this model come in $N$ copies, which are identified up to a phase. These copies go into each other under $Z_N$ transformation. We can easily generalise the $Z_3$ transformation (see ref.[22])in $W_3$ to the case of $W_N$. This transformation $Z(Z^N = 1)$ acts on the weight vectors $(\mu, \nu) \equiv (\sum_i n_i \Lambda^i, \sum_i m_i \Lambda^i)$ of $SU(N)_k$ and $SU(N)_{k+1}$ respectively as

$$(n_1, n_2, \ldots, n_{N-1}) \xrightarrow{Z} \left( k - \sum_i n_i, n_1, \ldots, n_{N-2} \right)$$

(37)

$$(m_1, m_2, \ldots, m_{N-1}) \xrightarrow{Z} \left( k + 1 - \sum_i m_i, m_1, \ldots, m_{N-2} \right)$$

(38)

This transformation leaves the conformal weight $(h_{\mu,\nu})$ invariant.

The conformal weights can also be rewritten as

$$h_{\mu,\nu} = h_{\mu}^{(k)} - h_{\nu}^{(k+1)} + \frac{(\mu - \nu)^2}{2}$$
where $h^{(k)}_{\mu} = \frac{\mu(\mu+2\rho)}{2(k+N)}$ is the conformal weight of $SU(N)$ representation $\mu$ in the $SU(N)_k$ WZW model.

The $N$ copies of each primary field in $W_N$ models will be obvious if we rewrite the conformal weight incorporating the level 1 field labelled by $\epsilon$ as follows:

$$h_{\mu,\nu} = h^{(k)}_{\mu} + h^{(1)}_{\epsilon} - h^{(k+1)}_{\mu} + \frac{(\mu - \nu)^2 - (\epsilon)^2}{2}$$

(39)

where $\epsilon$ is such that $\epsilon.\psi \leq 1$ (i.e. $\epsilon$ is a singlet or any of the $(N-1)$ fundamental representations of $SU(N)$) and is given by $\mu - \nu = \epsilon \mod \text{root}$.

Two copies $\Phi_{(\mu_0,\nu_0)}$, $\Phi_{(\mu_1,\nu_1)}$ are related by the $Z$ transformation through a phase:

$$\Phi_{(\mu_0,\nu_0)} = \exp\left( (N - 1) \frac{\pi i}{2} (\mu_1 - \nu_1 + \epsilon_1) \right) \Phi_{(\mu_1,\nu_1)}$$

where $\mu_i - \nu_i = \epsilon_i \mod \text{root}$; and hence $(\mu_0 - \nu_0)$ = a root; and $\mu_1 = Z\mu_0$, $\nu_1 = Z\nu_0$, $\epsilon_1 = Z\epsilon_0$. The transformed weight vectors: $Z\mu_0$, $Z\nu_0$ are given by eqns.(37) and (38) respectively and $Z\epsilon_0$ is the corresponding transformed weight vector of $SU(N)_1$. Other copies are generated by transformation $Z^2$, $Z^3$, $\cdots$, $Z^{N-1}$ on $\Phi_{(\mu_0,\nu_0)}$.

The fusion rules satisfied by these primary fields are

$$\Phi_{(\mu_1,\nu_1)} \otimes \Phi_{(\mu_2,\nu_2)} = \sum_{\mu_3} \sum_{\nu_3} \Phi_{(\mu_3,\nu_3)}$$

(40)

where $\mu_3$ and $\nu_3$ are the representations allowed by the fusion rules of two $SU(N)_k$ and $SU(N)_{k+1}$ Wess-Zumino models respectively.

The braiding eigen values are given by

$$\lambda_{(\mu_3,\nu_3)}[(\mu_1,\nu_1);(\mu_2,\nu_2)] = \Omega \exp(\pi i h_{\mu_1,\nu_1} + h_{\mu_2,\nu_2} - h_{\mu_3,\nu_3})$$

(41)

where $\Omega = (-1)^{f_1+f_2-f_3}$ where $f_i$ is an integer given in terms of the weight vectors by

$$f_i = \frac{1}{2} (\mu_i - \nu_i + \epsilon_i). (\mu_i - \nu_i - \epsilon_i + 2\rho)$$

This braiding eigen value can be rewritten in terms of the $SU(N)$ WZW eigenvalues:

$$\lambda_{(\mu_3,\nu_3)}[(\mu_1,\nu_1);(\mu_2,\nu_2)] = \lambda^{(k)}_{\mu_3}(\mu_1,\mu_2) \left( \lambda^{(k+1)}_{\nu_3}(\nu_1,\nu_2) \right)^{-1} \lambda^{(1)}_{\epsilon_3}(\epsilon_1,\epsilon_2)$$

(42)
We expect that the duality matrix also factorises. Thus theorem 1 generalises to the $W_N$ model:

**Theorem 3:** The invariant for a link $L$, made up of $n$ knots, carrying representations $\Phi(\mu_1, \nu_1), \ldots, \Phi(\mu_n, \nu_n)$ respectively is given in terms of $SU(N)_k$, $SU(N)_{k+1}$ and $SU(N)_1$ invariants as follows:

$$V_{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)}[L] = V^{(k)}_{\mu_1, \ldots, \mu_n} [L] \ V^{(k+1)}_{\nu_1, \ldots, \nu_n} [\bar{L}] \ V^{(1)}_{\epsilon_1, \ldots, \epsilon_n} [L]$$

(43)

where $\mu_i - \nu_i = \epsilon_i \mod \text{root}$. 

These results reflect the quantum group structure of superconformal and $W_N$ models to be $SU(2)_{q(k)} \otimes SU(2)_{q(2)} \otimes SU(2)_{q^{*}(k+2)}$ and $SU(N)_{q(k)} \otimes SU(N)_{q(1)} \otimes SU(N)_{q^{*}(k+1)}$ respectively.

3. Concluding Remarks

In this paper we have argued following earlier works \cite{16, 17} and generalising our methods of computation from Chern-Simons theories \cite{2, 5, 6, 7} that factorisation of the braid and duality matrices completely simplifies the corresponding knot/link invariants for any rational conformal field theory admitting a coset representation. The invariants from such a theory are given in terms of Wess-Zumino theories based on the factors in the coset. In particular, minimal, superconformal and $W_N$ model invariants are given by the product of invariants of Wess-Zumino models as given in Theorems 1, 2 and 3. While one can completely write down the knot/link invariants for this type of coset models we also find that we do not get any more new invariants other than that given by Chern-Simons theories for compact semi-simple Lie groups.

At this stage it is natural to ask the question: Is there a Chern-Simons description of the invariants above obtained from conformal field theories? To answer this, let us focus our attention for definiteness on the above invariants obtained from $(G_k \otimes G_l)/G_{k+l}$ theories. These knot invariants can be given in terms of expectation values of certain definite Wilson operators in three Chern-Simons theories based on gauge groups $G$ and couplings $k, l$ and $k+l$. All possible Wilson link operators are given by the product of Wilson link operators of these
three theories. For a link \( L \) made up of \( n \) component knots \( K^{(i)}, K^{(2)}, \ldots, K^{(n)} \), we place the representations \((R_1^{(i)}, R_2^{(i)}, S^{(i)})\) of the three gauge groups \( G \) respectively on the component knot \( K^{(i)} \). Then the Wilson link operator are:

\[
W_{\{R_1\}\{R_2\}\{S\}}[L] = \prod_{i=1}^{n} W_{R_1^{(i)}}^{(k)}[K^{(i)}] \prod_{j=1}^{n} W_{R_2^{(j)}}^{(l)}[K^{(j)}] \prod_{m=1}^{n} W_{S^{(m)}}^{(k+l)}[K^{(m)}] \\
= W_{\{R_1\}}^{(k)}[L]W_{\{R_2\}}^{(l)}[L]W_{\{S\}}^{(k+l)}[L]
\]

where the three factors are the link operators for the Chern-Simons theories based on gauge group \( G \) with couplings respectively \( k, l, k + l \) and set of representation \( \{R_1\}, \{R_2\} \) and \( \{S\} \) living on the component knots. Of all possible representation \((R_1^{(i)}, R_2^{(i)}, S^{(i)})\), only a subset can be associated with the primary fields \( \Phi_{(R_1^{(i)}, R_2^{(i)}, S^{(i)})} \) of the conformal field theory based on \((G_k \otimes G_l)/G_{k+l}\). For this restricted but invariant set of representations, construct the following product of Wilson link operators:

\[
W_{\{R_1\}\{1\}\{1\}}[L]W_{\{1\}\{R_2\}\{1\}}[L]W_{\{1\}\{1\}\{S\}}[\bar{L}] = W_{\{R_1\}\{R_2\}\{S\}}^{(k)}[L]W_{\{\bar{L}\}}^{(l)}[L]W_{\{S\}}^{(k+l)}[\bar{L}] \\
(44)
\]

where \( \bar{L} \) is the mirror image of link \( L \) and 1 reflects that all component knots carry the singlet representation in the corresponding Chern-Simons theory. Then the invariant associated with link \( L \) obtained from the \((G_k \otimes G_l)/G_{k+l}\) conformal field theory with fields \( \Phi_{(R_1^{(i)}, R_2^{(i)}, S^{(i)})} \) placed on the component knots \( K^{(i)} \) is given by the functional average of the operator (44) in the product Chern-Simons theory:

\[
V_{(R_1^{(1)}, R_2^{(2)}, S^{(1)}), \ldots, (R_1^{(n)}, R_2^{(n)}, S^{(n)})}[L] = \langle W_{\{R_1\}\{R_2\}\{S\}}^{(k)}[L]W_{\{R_2\}}^{(l)}[L]W_{\{S\}}^{(k+l)}[\bar{L}] \rangle \\
(45)
\]

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Figure Captions:

Fig.1 Three-ball $B_1$ with $S^2$ boundary represented by $|\psi_0\rangle$.

Fig.2 Three-ball with $m$ half-twists in the central two parallel strands represented by $|\psi_m\rangle$.

Fig.3 Three-ball with $S^2$ boundary represented by $|\psi'_0\rangle$.

Fig.4 Three-ball with $m$ half-twists in the side strands represented by $|\psi'_m\rangle$.

Fig.5 Hopf links drawn in two different ways—closure of a parallel 2-braid and an antiparallel 2-braid with two half-twists.

Fig.6 Consistency condition for duality matrix factorisation.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9312215v1