ON SET SYSTEMS WITHOUT A SIMPLEX-CLUSTER AND THE JUNTA METHOD

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Abstract. A family \(\{A_0, \ldots, A_d\}\) of \(k\)-element subsets of \([n] = \{1, 2, \ldots, n\}\) is called a simplex-cluster if \(A_0 \cap \cdots \cap A_d = \emptyset\), \(|A_0 \cup \cdots \cup A_d| \leq 2k\), and the intersection of any \(d\) of the sets in \(\{A_0, \ldots, A_d\}\) is nonempty. In 2006, Keevash and Mubayi conjectured that for any \(d + 1 \leq k \leq n\), the largest family of \(k\)-element subsets of \([n]\) that does not contain a simplex-cluster is the family of all \(k\)-subsets that contain a given element. We prove the conjecture for all \(k \geq \zeta n\) for an arbitrarily small \(\zeta > 0\), provided that \(n \geq n_0(\zeta, d)\).

We call a family \(\{A_0, \ldots, A_d\}\) of \(k\)-element subsets of \([n]\) a \((d, k, s)\)-cluster if \(A_0 \cap \cdots \cap A_d = \emptyset\) and \(|A_0 \cup \cdots \cup A_d| \leq s\). We also show that for any \(\zeta n \leq k \leq \frac{d+1}{d+2}n\) the largest family of \(k\)-element subsets of \([n]\) that does not contain a \((d, k, (\frac{d+1}{d+2} + \zeta k))\)-cluster is again the family of all \(k\)-subsets that contain a given element, provided that \(n \geq n_0(\zeta, d)\).

Our proof is based on the junta method for extremal combinatorics initiated by Dinur and Friedgut and further developed by Ellis, Keller, and the author.

1. Introduction

Throughout the paper, we denote \([n] = \{1, \ldots, n\}\), we write \(\binom{[n]}{k}\) for the family of all \(k\)-element subsets of \([n]\), and given a set \(S\), we write \(\mathcal{P}(S)\) for the power set of \(S\). A family \(\mathcal{F} \subseteq \mathcal{P}([n])\) is called intersecting if the intersection of any two sets in \(\mathcal{F}\) is nonempty. A star is the family of all sets that contain a given element.

Intersection problems for finite sets study the problem: ‘how large can a family of subsets of \([n]\) be given some restrictions on the unions and intersections of its elements?’ The earliest result of this class is the Erdős-Ko-Rado (EKR) Theorem [7] from 1961.

Theorem 1.1 ([7]). Let \(k \leq \frac{n}{2}\), and let \(\mathcal{F} \subseteq \binom{[n]}{k}\) be an intersecting family. Then \(|\mathcal{F}| \leq \binom{n-1}{k-1}\). If \(k < \frac{n}{2}\), then \(|\mathcal{F}| = \binom{n-1}{k-1}\) if and only if \(\mathcal{F}\) is a star.

Intersection problems for finite sets have become a prolific research field in extremal combinatorics, and numerous generalizations of the EKR theorem were obtained. (See the excellent survey of Frankl and Tokushige [11]). Let us mention one of these generalizations that we will use in the sequel:

A family \(\mathcal{F} \subseteq \binom{[n]}{k}\) is said to be \(s\)-wise intersecting if it does not contain \(s\) sets whose intersection is empty. The following theorem, proved by Frankl [3], generalizes the EKR theorem to \(s\)-wise intersecting families.

Theorem 1.2 (Frankl, 1976). Let \(k \leq \frac{(s-1)n}{s}\), and let \(\mathcal{F} \subseteq \binom{[n]}{k}\) be an \(s\)-wise intersecting family. Then \(|\mathcal{F}| \leq \binom{n-1}{k-1}\). If \(k < \frac{(s-1)n}{s}\), then \(|\mathcal{F}| = \binom{n-1}{k-1}\) if and only \(\mathcal{F}\) is a star.

1.1. Set families without a cluster. We shall be concerned with a generalization of the EKR Theorem, where the forbidden configuration is known as a \((d, k, s)\)-cluster.

Definition 1.3. A family \(\{A_0, \ldots, A_d\} \subseteq \binom{[n]}{k}\) is called a \((d, k, s)\)-cluster if \(|A_0 \cup \cdots \cup A_d| \leq s\) and \(A_0 \cap \cdots \cap A_d = \emptyset\).
We write \( f(d, k, s, n) \) for the largest size of a family \( \mathcal{F} \subseteq \binom{[n]}{k} \) that does not contain a \((d, k, s)\)-cluster. Note that \( f(d, k, s, n) \geq \binom{n-1}{k-1} \), since the star does not contain a \((d, k, s)\)-cluster.

In this paper we study the following problem.

**Problem 1.4.** For what values of \( d, k, s, n \) do we have \( f(d, k, s, n) = \binom{n-1}{k-1} \)?

This problem generalizes several questions that were studied extensively, and are still open. Before we discuss the history of the problem, we give a few basic observations.

1. Problem 1.4 makes sense only for \( s \geq \frac{d+1}{d} k \), since no \((d, k, s)\)-cluster exists in \( \binom{[n]}{k} \) if \( s < \frac{d+1}{d} k \).

2. A \((d + 1)\)-wise intersecting family in \( \binom{[n]}{k} \) is free of a \((d, k, s)\)-cluster for any value of \( s \). On the other hand, if \( s \geq \min((d + 1) k, n) \), then any \( d + 1 \) sets \( A_1, \ldots, A_d \in \binom{[n]}{k} \) whose intersection is empty form a \((d, k, s)\)-cluster. Hence, for such values of \( d, k, s, n \), a family that does not contain a \((d, k, s)\)-cluster is the same as a \((d + 1)\)-wise intersecting family. Thus, in this case the problem is settled by Theorem 1.2.

3. The function \( f \) is decreasing in \( s \). Combining this fact with the inequality \( f(d, k, s, n) \geq \binom{n-1}{k-1} \), we obtain that for any \( s_1 < s_2 \) such that \( f(d, k, s_1, n) = \binom{n-1}{k-1} \) we have \( f(d, k, s_2, n) = \binom{n-1}{k-1} \).

As mentioned above, different special cases of Problem 1.4 were studied in numerous works. In 1980, Katona considered the problem of determining \( f(d, k, s, n) \) in the case where \( d = 2 \):

**Problem 1.5** (Katona, 1980). How large can a family \( \mathcal{F} \subseteq \binom{[n]}{k} \) be if \( \mathcal{F} \) does not contain sets \( A_1, A_2, A_3 \), such that \( A_1 \cap A_2 \cap A_3 = \emptyset \) and \( |A_1 \cup A_2 \cup A_3| \leq s \)?

In 1983, Frankl and Füredi \([9]\) gave the following example that shows that the answer to our Problem 1.4 is negative if \( s < 2k \) and \( k \leq c \log n \) for a sufficiently small constant \( c \).

**Example 1.6.** Partition \([n]\) into sets \( X_1, \ldots, X_k \) of equal size. Let \( \mathcal{G} \subseteq \binom{[n]}{k} \) be the family of all sets that intersect each \( X_i \) in a single vertex. Then \( \mathcal{G} \) is easily seen to be free of any \((d, k, s)\)-cluster for any \( s < 2k \). Note that \( |\mathcal{G}| = \binom{n}{k} \), and that \( \binom{n}{k} \geq \binom{n-1}{k-1} \), provided that \( k \leq c \log n \) for a sufficiently small constant \( c \). Hence, for such \( k \) and any \( s < 2k \) we have \( f(d, k, s, n) > \binom{n-1}{k-1} \).

Frankl and Füredi also showed that \( f(2, k, 2k, n) = \binom{n-1}{k-1} \) for any \( n \geq k^2 + 3k \) and conjectured that \( f(2, k, 2k, n) = \binom{n-1}{k-1} \) for all \( k \leq \frac{2n}{3} \). In 2006 Mubayi \([21]\) proved this conjecture. He also made the following more general conjecture.

**Conjecture 1.7** (Mubayi 2006). Let \( d + 1 \leq k \leq \frac{3n}{d+1} \). Then \( f(d, k, 2k, n) = \binom{n-1}{k-1} \).

In 2007, Mubayi \([22]\) proved his conjecture in the case where \( d = 3 \), \( k \) is fixed and \( n \) is sufficiently large. He has also showed a stability result for general fixed \( d \). Specifically, he proved that if \( k \), \( d \) are fixed and \( n \) tends to infinity, then any family \( \mathcal{F} \subseteq \binom{[n]}{k} \) that is free of \((d, k, 2k)\)-cluster and whose size is \( \binom{n-1}{k-1} (1 - o(1)) \) must satisfy \( |\mathcal{F} \setminus S| = o(n^{(k-1)/d}) \) for some star \( S \). In 2009, Mubayi and Ramadurai \([24]\) applied Mubayi’s stability result and proved that Conjecture 1.7 holds for any fixed \( k \) and \( d \), provided that \( n \) is sufficiently large. In 2009, Füredi and Özkahya \([14]\) gave a different proof of the result of Mubayi and Ramadurai and showed that if \( k \) and \( d \) are fixed and \( n \) is sufficiently large, then any \( \mathcal{F} \subseteq \binom{[n]}{k} \) whose size is greater than \( \binom{n-1}{k-1} \) contains a special kind of a \((d, k, 2k)\)-cluster. Finally, Keevash and Mubayi \([17]\) showed that for a fixed \( d \) and an arbitrarily small \( \zeta \), there exists some \( T = T(d, \zeta) \), such that Conjecture 1.7 holds for any \( \zeta n \leq k \leq \frac{n}{2} - T \).
While Example 1.6 shows that we cannot hope to have $f(d, k, s, n) = \binom{n-1}{k-1}$ if $k \leq c \log n$, this seems to be a little bit too pessimistic if $k \geq C \log n$ for a sufficiently large constant $C$. Indeed, for such values of $k$ the family given in Example 1.6 is smaller than the star, and so the equality $f(d, k, s, n) = \binom{n-1}{k-1}$ becomes possible also for $\frac{d+1}{d}k \leq s \leq 2k$. (See Conjecture 6.1 and Example 6.2 below). We show that for $k$ linear in $n$, the equality $f(d, k, s, n) = \binom{n-1}{k-1}$ holds for any $s \geq (\frac{d+1}{d} + \zeta)k$ for an arbitrarily small constant $\zeta > 0$, provided that $n$ is sufficiently large.

**Theorem 1.8.** For any $d \in \mathbb{N}, \zeta > 0$ there exists $n_0 = n_0(d, \zeta)$, such that the following holds. Let $n > n_0$, let $\zeta n \leq k \leq \left(\frac{d}{d+1} - \zeta\right)n$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family that does not contain a $(d, k, (\frac{d+1}{d} + \zeta)k)$-cluster. Then $|\mathcal{F}| \leq \binom{n}{k-1}$, with equality if and only if $\mathcal{F}$ is a star. In particular, Conjecture 1.7 holds for any $\zeta n \leq k \leq \frac{d+1}{d}n$, provided that $n \geq n_0$.

This means that for $k$ linear in $n$, not only any family that is larger than the star must contain a $(d, k, 2k)$-cluster as conjectured by Mubayi, but actually it must contain a $(d, k, (\frac{d+1}{d} + \zeta)k)$-cluster, which is almost the ‘strongest’ cluster we can obtain, due to the first observation above.

Let $\mathcal{F}, \mathcal{G} \subseteq \binom{[n]}{k}$. We say that $\mathcal{F}$ is $\epsilon$-essentially contained in $\mathcal{G}$ if $|\mathcal{F}| \leq \epsilon|\mathcal{G}|$ (in words, if a random set in $\binom{[n]}{k}$ lies in $\mathcal{F}$ and not in $\mathcal{G}$ with probability at most $\epsilon$).

We also prove a stability result for Theorem 1.8 above.

**Theorem 1.9.** For any $d \in \mathbb{N}$, and an arbitrarily small $\zeta > 0$, there exists $C > 0$, such that the following holds. Let $\zeta n \leq k \leq \left(\frac{d}{d+1} - \zeta\right)n$, and let $\epsilon \geq 0$. Suppose that $\mathcal{F} \subseteq \binom{[n]}{k}$ does not contain a $(d, k, (\frac{d+1}{d} + \zeta)k)$-cluster, and that $|\mathcal{F}| \geq \binom{n}{k-1}(1 - \epsilon)$. Then $\mathcal{F}$ is $Ce^{1+\frac{1}{d}+\frac{1}{k}}$-essentially contained in a star.

### 1.2. Set families without a simplex-cluster and the Erdős-Chvátal conjecture.

We use Theorem 1.8 to partially resolve a conjecture of Keevash and Mubayi [17] on set families without a simplex-cluster, and to obtain a new proof for the Erdős-Chvátal simplex conjecture for $k$ linear in $n$.

A $d$-simplex is a family of $d+1$ sets, such that the intersection of all of them is empty and the intersection of any $d$ of them is nonempty. A $d$-simplex-cluster is a $(d, k, 2k)$-cluster which is also a $d$-simplex. The Erdős-Chvátal simplex conjecture [2] states the following.

**Conjecture 1.10** (Erdős and Chvátal, 1974). Let $d < k \leq \frac{d+1}{d}n$. Then any family $\mathcal{F} \subseteq \binom{[n]}{k}$ that does not contain a $d$-simplex satisfies $|\mathcal{F}| \leq \binom{n}{k-1}$. Moreover, equality holds if and only if $\mathcal{F}$ is a star.

In 1976, Frankl [8] showed that the conjecture holds if $k \geq \frac{d+1}{d}n$. In 1987, Frankl and Füredi [10] proved the conjecture in the case where $d$ and $k$ are fixed and $n \geq n_0(k,d)$. In 2005, Mubayi and Verstraëte [24] settled the case $d = 2$ for any values of $k$ and $n$. In 2010, Keevash and Mubayi [17] settled the case $\zeta n \leq k \leq \frac{n}{2} - T$, for any $\zeta > 0$, provided that $T = T(\zeta, d)$ is sufficiently large. Finally, Keller and the author [18] gave a 70 pages long proof that Conjecture 1.10 holds for any $k$ in the range (i.e., $d < k \leq \frac{d+1}{d}n$), provided that $n \geq n_0(d)$.

Keevash and Mubayi [17] gave the following conjecture that strengthens both Chvátal’s conjecture and Conjecture 1.7.

**Conjecture 1.11.** Let $d < k \leq \frac{d+1}{d}n$. Then any family $\mathcal{F} \subseteq \binom{[n]}{k}$ that does not contain a $d$-simplex-cluster satisfies $|\mathcal{F}| \leq \binom{n}{k-1}$. Moreover, equality holds only if $\mathcal{F}$ is a star.
Hence, if \( s < \frac{d}{d-1}k \) then the intersection of each \( d \) sets in a \((d, k, s)\)-cluster is of size larger than \( k - (d - 1) \left( \frac{d}{d-1}k - k \right) = 0 \). Thus, for such \( s \), any \((d, k, s)\)-cluster is a \( d \)-simplex. Therefore, as an immediate corollary of Theorem 1.8 we obtain that Conjecture 1.11 holds for all \( k \geq \zeta n \), provided that \( n \geq n_0 (\zeta, d) \).

**Theorem 1.12.** For each \( d \in \mathbb{N}, \zeta > 0 \), there exists \( n_0 = n_0 (d, \zeta) \), such that the following holds. Let \( n \geq n_0 (\zeta, d) \), and let \( \zeta n < k \leq \frac{d}{d-1}n \). Then any family \( F \subseteq \binom{[n]}{d} \) that does not contain a \( d \)-simplex-cluster, satisfies \( |F| \leq \binom{n-1}{k-1} \). Moreover, equality holds if and only if \( F \) is a star.

Since Conjecture 1.11 strengthens the Erdős-Chvátal conjecture 1.10, this paper gives a relatively short proof of the fact that the Erdős-Chvátal conjecture holds for all \( k \geq \zeta n \), provided that \( n \geq n_0 (\zeta, d) \).

### 1.3. The proof method.

The main tool we use in our proof is the ‘junta method’, initiated by Dinur and Friedgut [3] and further developed by Keller and the author [18], and by Ellis, Keller, and the author [6]. One of our goals in writing this paper is to make this recent technique more accessible, by providing a shorter paper that follows the framework of the junta method.

#### 2. Juntas and proof sketch

Let \( j < k < n \). A family \( J \subseteq \binom{[n]}{k} \) is said to be a \( j \)-junta if there exists a set \( J \) of size \( j \), and a family \( G \subseteq \mathcal{P}(J) \), such that a set \( A \) is in \( J \) if and only if \( A \cap J \) is in \( G \). Informally, a family is a junta if it is a \( j \)-junta for a constant \( j \) independent of \( k \) and \( n \).

The notion ‘junta’ originates in the field known as ‘analysis of Boolean functions’, where it plays a central role (see e.g., Bourgain [1], Dinur et al. [4], Friedgut [12], and Kindler-Safra [19]).

They were introduced to extremal combinatorics by Dinur and Friedgut [3]. They showed that any intersecting family is essentially contained in an intersecting junta.

**Theorem 2.1** (Dinur and Friedgut 2009). For any \( r \) there exists a \( C = C (r) \), \( j = j (r) \), such that any intersecting family \( J \subseteq \binom{[n]}{k} \) is \( C (k/n)^r \) -essentially contained in an intersecting \( j \)-junta.

Note that Theorem 2.1 is trivial if \( k/n = \Theta (1) \). In that regime, they managed to show a slightly weaker version of the following recent result of Friedgut and Regev [13].

**Theorem 2.2** ([13]). For each \( \zeta, \varepsilon > 0 \) there exists \( j = j (\zeta, \varepsilon) \in \mathbb{N} \), such that the following holds. Let \( \zeta n < k < (\frac{1}{2} - \zeta) n \) and let \( F \subseteq \binom{[n]}{k} \) be an intersecting family. Then \( F \) is \( \varepsilon \)-essentially contained in an intersecting \( j \)-junta.

These results inspired the works of Ellis, Keller, and the author [6] [18] who developed a method to show that the extremal family that is free of a certain forbidden configuration is some junta \( J_{\text{ex}} \). The method is combined of the following ingredients.

- **Ingredient 1:** Show that any family is essentially contained in a junta that does not contain the forbidden configuration.
• **Ingredient 2**: Show that $J_{\text{ex}}$ is the largest junta that does not contain the forbidden configuration, and prove a stability result of this statement. I.e. if $J$ is a junta that does not contain the forbidden configuration and whose size is close to $|J_{\text{ex}}|$, then $J$ is essentially contained in $J_{\text{ex}}$.

• **Ingredient 3**: Show that if $F$ is a small alteration of $J_{\text{ex}}$ that does not contain the forbidden configuration, then $|F| \leq |J_{\text{ex}}|$.

These ingredients fit together to show that $J_{\text{ex}}$ is the extremal family that does not contain the forbidden configuration. Indeed, suppose that $F$ is the extremal family. Then the first ingredient yields that $F$ is essentially contained in a junta $J$. In particular, the size of $J$ is not much smaller than the size of $F$, which is greater or equal to the size of $J_{\text{ex}}$. The second ingredient implies that $J$ is essentially contained in $J_{\text{ex}}$, and hence $F$ is essentially contained in $J_{\text{ex}}$. The third ingredient implies that $|F| \leq |J_{\text{ex}}|$, and therefore $J_{\text{ex}}$ is the extremal family.

In our case, the forbidden configuration is a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster, and the junta $J_{\text{ex}}$ is a star.

**Showing that the largest junta free of a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster is a star, and proving stability.**

We observe that any $j$-junta that does not contain a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster is actually $(d+1)$-wise intersecting, provided that $k \geq k_0(j)$. Then, Ingredient 2 amounts to proving a stability result for Frankl’s Theorem (Theorem 12), i.e. to showing that a $(d+1)$-wise intersecting family whose size is close to $\binom{n-1}{k-1}$ is close to a star. This was proved by Ellis, Keller, and the author [13].

**Showing that any family free of a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster is essentially contained in a $(d+1)$-wise intersecting junta.**

The proof is based on the regularity method and it goes as follows.

1. Note that each set $J$ of constant size decomposes the sets in $F$ into $2^{|J|}$ parts according to their intersection with $J$. The first step is to show that we may find a set $J$ of constant size, such that $F$ is a union of parts that satisfy a certain quasirandomness notion and a sufficiently small remainder that can be ignored.

2. We then take our approximating junta to consist of the union of the parts that satisfy the quasirandomness notion. The small size of the remainder translates into the fact that $F$ is essentially contained in $J$, and our goal becomes to show that $J$ is $(d+1)$-wise intersecting. The second step is to turn this task into a statement about the quasirandom parts. Namely, we obtain that it is enough to show that if $F_0, \ldots, F_d$ are quasirandom families, then they mutually contain a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster, provided that $n \geq n_0(\zeta)$, i.e. it is enough to show that there exists sets $A_0 \subseteq F_0, \ldots, A_d \subseteq F_d$ whose intersection is empty, such that $|A_0 \cup \cdots \cup A_d| \leq (\frac{d+1}{d} + \zeta) k$. The next steps concern this new task.

3. We choose an $l = k(1 + \zeta')$ for a small constant $\zeta' > 0$, and we write $F_i^{|l|}$ for the family of all sets in $\binom{[n]}{l}$ that contain a set in $F_i$. The third step is to show that the probability that a random set in $\binom{[n]}{l}$ is in $F_i^{|l|}$ is close to 1.

4. The fourth step is to give a simple union bound that shows that the families $F_0^{|l|}, \ldots, F_d^{|l|}$ mutually contain a random $(d, l, (\frac{d+1}{d} l))$-cluster with positive probability.

5. The last step is to deduce from the $(d, l, (\frac{d+1}{d} l))$-cluster appearing in the families $F_0^{|l|}, \ldots, F_{d+1}^{|l|}$, that a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster appears in the quasirandom families $F_0, \ldots, F_l$.

Showing that the star is the largest family free of a $(d, k, (\frac{d+1}{d} + \zeta) k)$-cluster in its neighborhood.
Finally, we shall give an argument based on the Kruskal-Katona Theorem \[15, 20\] to accomplish the third ingredient, i.e. we show that if a family that does not contain a \((d, k, \left(\frac{d+1}{d} + \zeta\right) k)\)-cluster is close to a star, then its size must be smaller than it. Let \(\mathcal{F} \subseteq \binom{[n]}{k}\) be a family close to a star \(\mathcal{S}\). We start by decomposing \(\mathcal{F}\) into the large family \(\mathcal{F}_1 := \mathcal{F} \cap \mathcal{S}\) inside the star and the small family \(\mathcal{F}_0 := \mathcal{F} \setminus \mathcal{S}\) outside of it. One can think of \(\mathcal{F}\) as a family constructed from \(\mathcal{S}\) by first adding the element of \(\mathcal{F}_0\) into \(\mathcal{F}\), thereby unavoidably putting \((d, k, \left(\frac{d+1}{d} + \zeta\right) k)\)-clusters inside it, and then removing the elements of \(\mathcal{S} \setminus \mathcal{F}_1\) out of \(\mathcal{F}\) to destroy all of these copies. Our goal then becomes to show that \(\mathcal{F}_0\) is negligible compared to \(|\mathcal{S} \setminus \mathcal{F}_1|\). The proof follows the following steps:

1. We choose \(l\) slightly larger than \(k\), and we use the Kruskal-Katona Theorem (Theorem 3.2) bellow to give a lower bound on \(\left|\mathcal{F}_1\right|\) in terms of \(\left|\mathcal{F}_0\right|\).
2. We observe that the families \(\mathcal{F}_1^l, \ldots, \mathcal{F}_0^l\) do not mutually contain a \((d, l, \frac{d+1}{d} l)\)-cluster, and we use this fact to deduce an upper bound on \(\left|\mathcal{F}_1^l\right|\) in terms of \(\left|\mathcal{F}_0^l\right|\).
3. We apply the Kruskal-Katona Theorem again in order to upper bound the size of \(\mathcal{F}_0\) in terms of \(\left|\mathcal{F}_1\right|\).

Combining all these upper bound we obtain an upper bound of \(\left|\mathcal{F}_0\right|\) in terms of \(\left|\mathcal{F}_1\right|\). It turns out that this upper bound is sufficient to complete the proof.

2.1. Notations. We use bold letters to denote random variables. Let \(X\) be some set. We write \(A \sim \binom{X}{k}\) to denote that \(A\) is a uniformly random \(k\)-set in \(X\). Let \(\mathcal{F} \subseteq \binom{X}{k}\) be some family. We write

\[
\mu(\mathcal{F}) = \frac{|\mathcal{F}|}{\binom{|X|}{k}} = \Pr_{A \sim \binom{X}{k}}[A \in \mathcal{F}].
\]

Given a set \(J \subseteq X\), and \(B \subseteq J\), we write \(\mathcal{F}_J^B\) for the family \(\{A \in \binom{X \setminus J}{k-|B|} | A \cup B \in \mathcal{F}\}\).

We therefore have

\[
\mu(\mathcal{F}_J^B) = \Pr_{A \sim \binom{X}{k}}[A \in \mathcal{F} | A \cap J = B].
\]

Let \(J \subseteq [n]\), and let \(G \subseteq \mathcal{P}(J)\) be some family. We write \(\langle G \rangle\) for the \(|J|\)-junta of all the sets \(A \in \binom{[n]}{k}\) such that \(A \cap J\) is in \(G\). We call \(\langle G \rangle\) the junta generated by \(G\).

A family \(A\) is said to be monotone if for any \(A \in A\) and any \(B \supseteq A\) we have \(B \in A\). The monotone closure of \(\mathcal{F}\), denoted by \(\mathcal{F}^*\), is the monotone family of all sets in \(\mathcal{P}(n)\) that contain a set in \(\mathcal{F}\). Hence, \(\mathcal{F}^* = \mathcal{F}^1 = \mathcal{F}^\top \cap \binom{[n]}{k}\).

The \(p\)-biased measure is a probability distribution on sets \(A \sim \mathcal{P}(\binom{[n]}{k})\), where we put each element \(i\) in \(A\) independently with probability \(p\). For a family \(A \subseteq \mathcal{P}(\binom{[n]}{k})\) we write

\[
\mu_p(A) = \Pr_{A \sim \mu_p}[A \in \mathcal{A}].
\]

3. Consequences of the Kruskal-Katona Theorem

The Kruskal-Katona Theorem gives us a lower bound on \(\left|\mathcal{F}^\top\right|\) in terms of \(\left|\mathcal{F}\right|\). Before stating it we shall give a trivial lower bound that would also be useful to us.

**Lemma 3.1.** Let \(k < l < n\) be some natural numbers and let \(\mathcal{F} \subseteq \binom{[n]}{k}\) be some family. Then \(\mu(\mathcal{F}) \leq \mu\left(\mathcal{F}^\top\right)\).

**Proof.** Choose a set \(A \sim \binom{[n]}{k}\) and choose a set \(B \sim \binom{[n] \setminus A}{l-k}\). We have

\[
\mu\left(\mathcal{F}^\top\right) = \Pr[\mathcal{A} \cup B \in \mathcal{F}^\top] \geq \Pr[\mathcal{A} \in \mathcal{F}] = \mu(\mathcal{F}).
\]

\[\square\]
The lexicographically ordering on \((\binom{n}{k})\) is the ordering on sets defined by \(A <_L B\) if \(\min \{A \Delta B\} \in A\). We let \(L_i(k, n)\) be the family of the \(i\) sets in \((\binom{n}{k})\) that are first in the lexicographic ordering. Thus, \(L_{\binom{n-1}{k-1}, k, n}\) is the star of all sets that contain the element 1. The Kruskal-Katona \cite{16,20} Theorem is known to be equivalent to the following:

**Theorem 3.2 (Kruskal-Katona).** Let \(k < l < n\), and let \(i \leq \binom{n}{k}\). Suppose that \(|F| \geq |L_{i, k, n}|\). Then \(|F^{\uparrow l}| \geq \left(\binom{L_{i, k, n}}{\uparrow l}\right)|\).

We shall make use of the following corollaries of the Kruskal-Katona theorem.

**Corollary 3.3.** Let \(k < l < n\), let \(\epsilon > 0\), let \(F \subseteq \binom{n}{k}\) be some family, and let \(G = F^{\uparrow l}\).

1. If \(|F| \geq \binom{n-1}{k-1}(1 - \epsilon)\), then \(|G| \geq \binom{n-1}{k-1}(1 - \epsilon)\).
2. If \(F^{\uparrow l}\) is contained in \(L_{\binom{n-1}{k-1}, k, n}\), then \(|G| \geq \binom{n-1}{k-1}(1 - \epsilon)\).
3. Consequently, for each \(\zeta > 0\) there exists a constant \(C > 0\), such that the following holds. Suppose that \(k, l \in (\zeta n, (1 - \zeta) n)\), and that \(l - k > \zeta n\). If \(|F| \geq \binom{n-1}{k-1}(1 - \epsilon)\), then \(|G| \geq \binom{n-1}{k-1} - C\epsilon^{1+\frac{1}{p}}\).

**Proof.** The Kruskal-Katona Theorem implies that it suffices to prove the corollary in the case where \(F = L(i, k, n)\).

**Proving (1).** Note that we may suppose that \(|F| \leq \binom{n-1}{k-1}\). Let \(S\) be the family of all elements containing 1. Since any set in \(S\) is (lexicographically) smaller than any set not in \(S\), the family \(F\) is contained in \(S\).

Furthermore, \(\mu\left(F^{\uparrow l}_{(i)}\right) \geq 1 - \epsilon\), and so Lemma 3.1 implies that \(\mu\left(\left(F^{\uparrow l}_{(i)}\right)^{(l-1)}\right) \geq 1 - \epsilon\).

Note that we have \(A \in \left(F^{\uparrow l}_{(i)}\right)^{(l-1)}\) if and only if \(A \cup \{1\} \in G\). Hence,

\[
|G| = \left|\left(F^{\uparrow l}_{(i)}\right)^{(l-1)}\right| = \left(\binom{n-1}{l-1}\right)\mu\left(\left(F^{\uparrow l}_{(i)}\right)^{(l-1)}\right) \geq \left(\binom{n-1}{l-1}\right)(1 - \epsilon),
\]

as desired.

**Proving (2).** Again we may assume that \(|F| \leq \binom{n-1}{k-1}\). Write \(i = \binom{n-1}{k-1} - \binom{n-m}{k-1}\), and note that \(L(i, k, n)\) is the family

\[
\left\{A \in \binom{n}{k} | 1 \in A, A \cap [2, \ldots, m] \neq \emptyset\right\}.
\]

Since \(|F| > i\) we obtain that \(F \supseteq L(i, k, n)\). Additionally, the intersection of any sets in \(F \backslash L(i, k, n)\) with the set \([m]\) is the set \(\{1\}\). Therefore,

\[
\mu\left(F^{\uparrow l}_{[m]}\right) = \frac{|F| - i}{\binom{n-m}{k-1}} \geq \epsilon.
\]

By Lemma 3.1 \(\mu\left(\left(F^{\uparrow l}_{[m]}\right)^{(l-1)}\right) \geq \epsilon\).

Write \(j = \binom{n-1}{l-1} - \binom{n-m}{l-1}\). Note that, similarly to the family \(F\), the family \(G\) contains the family

\[
L(j, l, n) = \left\{A \in \binom{n}{l} | 1 \in A, A \cap [2, \ldots, m] \neq \emptyset\right\}.
\]

Moreover, all the elements of \(G \backslash L(j, l, n)\) are the elements of the form \(A \cup \{1\}\), where \(A \in \left(F^{\uparrow l}_{[m]}\right)^{(l-1)}\). Therefore,

\[
\epsilon \leq \mu\left(\left(F^{\uparrow l}_{[m]}\right)^{(l-1)}\right) = \mu\left(G^{\uparrow l}_{[m]}\right) = \frac{|G| - j}{\binom{n-m}{l-1}}.
\]
Rearranging and substituting the value of \( j \), we have

\[
|G| \geq \frac{(n-1)}{l-1} - (1 - \epsilon) \left( \frac{n-m}{l-1} \right).
\]

This completes the proof of (2).

**Deducing (3) from (2).** Let \( m \) be maximal with \( |F| \geq \frac{(n-1)}{k-1} - \frac{(n-m)}{k-1} \), and write

\[
|F| = \frac{(n-1)}{k-1} - (1 - \epsilon') \left( \frac{n-m}{k-1} \right) = \left( \frac{n-1}{k-1} \right) \left( 1 - \frac{(n-m)}{k-1} \right).
\]

By (2),

\[
|G| \geq \left( \frac{n-1}{l-1} \right) - (1 - \epsilon') \left( \frac{n-m}{l-1} \right) = \left( \frac{n-1}{l-1} \right) \left( 1 - \frac{(n-m)}{l-1} \right).
\]

Hence, to complete the proof we must show that

\[
\frac{(n-m)}{(l-1)} \frac{(1 - \epsilon')}{(n-1)} \leq C \left( \frac{1 - \epsilon'}{(n-1)} \right)^{1+\frac{1}{2}} \tag{3.1}
\]

provided that \( C = C(\zeta) \) is sufficiently large.

**Getting rid of \( \epsilon' \).** We shall now show that the \( (1 - \epsilon') \)-terms of (3.1) get swallowed by the constant \( C \), i.e \( (1 - \epsilon') = O(1) \). We may assume that \( n-m \geq l-1 \), for otherwise the left hand side of (3.1) is 0. By the definition of \( m \),

\[
(1 - \epsilon') \left( \frac{n-m}{k-1} \right) \geq \left( \frac{n-m-1}{k-1} \right) = \left( \frac{l-k}{n-m} \right) \left( \frac{n-m}{k-1} \right).
\]

Hence,

\[
1 - \epsilon' \geq 1 - \frac{k-1}{n-m} \geq 1 - \frac{k-1}{l-1} = \frac{l-k}{l-1} \geq \frac{l-k}{n} \geq \zeta, \tag{3.2}
\]

This completes the proof that \( (1 - \epsilon') = O(1) \), and so it is enough to show that

\[
\frac{(n-m)}{(l-1)} \leq C \left( \frac{(n-m)}{(k-1)} \right)^{1+\frac{1}{2}} \tag{3.3}
\]

provided that \( C \) is sufficiently large.

**Showing (3.3).** Rearranging (3.3), our goal becomes to show that

\[
\frac{(n-m)}{(k-1)} \leq C \left( \frac{(n-m)}{(k-1)} \right)^{\frac{1}{2}} \tag{3.4}
\]

This would follow once we show that that:

\[
\frac{(n-m)}{(k-1)} = \left( 1 - \frac{k-1}{n-1} \right) \left( 1 - \frac{k-1}{n-2} \right) \cdots \left( 1 - \frac{k-1}{n-m+1} \right) \geq C'' \tag{3.4}
\]

and

\[
\frac{(n-m)}{(l-1)} \leq C \left( \frac{(n-m)}{(k-1)} \right)^{\frac{1}{2}} \tag{3.5}
\]

where \( 0 < C', C'' < 1 \) are constants depending only on \( \zeta \).
Now note there are \( m - 1 \) terms in the middle of \((5.5)\) and each is greater than 
\[ 1 - \frac{k - 1}{n - m}, \]
which is greater than \( \zeta \) by \((3.2)\). Similarly, there are \( m - 1 \) terms in the middle of \((3.5)\), and each term satisfies
\[ 1 - \frac{l - k}{n - 1} = 1 - \frac{l - k}{n - 1} \leq 1 - \frac{l - k}{n} \leq 1 - \zeta. \]
This completes the proof of the lemma. \( \Box \)

4. Proof of the approximation by junta result and of the stability result

In this section we shall prove a stability result that says that any family that does not contain a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster whose size close to that of a star must in itself be close to a star.

**Proposition 4.1.** For each \( \zeta, \epsilon > 0 \) there exists \( \delta > 0, n_0 \in \mathbb{N} \), such that the following holds. Let \( n > n_0 \), let \( \zeta < \frac{k}{n} < \frac{d}{d+1} - \zeta \), and let \( F \subset \binom{[n]}{k} \) be some family that does not contain a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster. If \( |F| \geq \binom{n-1}{k-1} \) \( (1 - \delta) \), then \( F \) is \( \epsilon \)-essentially contained in a star.

Note that Proposition 4.1 is a weaker version of Theorem 1.9. However, we shall show that the ‘weak’ Proposition 4.1 can be bootstrapped into the stronger Theorem 1.9 in Section 3.

This section is divided into three parts.

1. We first show that a junta is free of a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster if and only if it is \((d+1)\)-wise intersecting. This part is needed only for motivational purposes and we shall not use this fact.

2. We then show that any family that is free of a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster is essentially contained in a \((d+1)\)-wise intersecting junta.

3. Finally, we shall apply a stability result of Theorem 1.2 by \([5]\) to deduce Proposition 4.1.

4.1. **Any junta that is free of a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster is \((d+1)\)-wise intersecting.** We now show that a junta does not contain a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster if and only if it is \((d+1)\)-wise intersecting.

**Proposition 4.2.** Let \( j > 0 \), let \( s \geq \frac{d+1}{d} k + j \), and let \( n \geq s \). Then a \( j \)-junta \( J \subset \binom{[n]}{k} \) is free of a \((d, k, s)\)-cluster if and only if it is \((d+1)\)-wise intersecting.

**Proof.** Note that any \((d+1)\)-wise intersecting family is free of a \((d, k, s)\)-cluster. So suppose on the contrary that \( J \) is a \( j \)-junta that does not contain a \((d, k, s)\)-cluster and is not \((d+1)\)-wise intersecting. Let \( J \) be some \( j \)-set and let \( G \subset \mathcal{P}(J) \) be a family, such that a set \( A \) is in \( G \) if and only if \( A \cap J \) is in \( G \). Let \( A_0, \ldots, A_d \in J \) be some sets whose intersection is empty, and let \( S \subset [n] \setminus J \) be some set of size \( s - j \). Since \( |S| \geq \frac{d+1}{d} k \), it is easy to see that there exists sets

\[
B_0 \in \binom{S}{k - |A_0 \cap J|}, B_1 \in \binom{S}{k - |A_1 \cap J|}, \ldots, B_d \in \binom{S}{k - |A_d \cap J|},
\]
such that \( B_0 \cap \cdots \cap B_d = \varnothing \). Now the sets \( B_0 \cup (A_0 \cap J), \ldots, B_d \cup (A_d \cap J) \) form a \((d, k, s)\)-cluster in \( J \), contradicting the hypothesis that \( J \) does not contain a \((d, k, s)\)-cluster.

Our goal will now be to prove that any family that does not contain a \((d, k, (\frac{d+1}{d} + \zeta) k)\)-cluster is essentially contained in a \((d+1)\)-wise intersecting junta. Our first ingredient is the following regularity lemma of \([6]\).
Suppose on the contrary that we will need to show that if \( \mu(F^B) > 4/5 \).

The regularity lemma of \([6]\) allows us to find a set \( J \) that decomposes our family into some \((r, \epsilon)\)-regular parts and some ‘negligible’ parts that together contribute very little to the measure of \( F \).

**Theorem 4.3** (\([6]\) Theorem 1.7). For each \( \delta, \epsilon, \zeta > 0 \) there exists \( j \in \mathbb{N} \), such that the following holds. Let \( \zeta n < k \leq (1 - \zeta)n \) and let \( F \subseteq \binom{[n]}{k} \) be a family. Then there exists a set \( J \) of size \( j \) and a family \( G \subseteq \mathcal{P}(J) \) such that the following holds.

1. For each \( B \in J \), the family \( F^B \) is \((\left\lceil \frac{1}{3} \right\rceil, \delta)\)-regular and \( \mu(F^B) > 4/5 \).

2. The family \( F \) is \( \epsilon \)-essentially contained in the \( j \)-junta \( \langle G \rangle \).

**Lemma 4.4.** For each \( \zeta > 0 \), there exists \( \delta > 0 \) such that the following holds. Let \( \zeta < \frac{1}{2} < 1 - \zeta \), let \( \frac{1}{n} < \frac{1}{2} - \frac{\delta}{2} \), and suppose that \( F \subseteq \binom{[n]}{k} \) is a \((\left\lceil \frac{1}{3} \right\rceil, \delta)\)-regular family. Then either \( \mu(F) \) is close to 0 or \( \mu(F^*) > 1 - \epsilon \).

One of the main tools is the following well known corollary of Friedgut’s Junta Theorem \([12]\) and Russo’s Lemma \([25]\).

**Theorem 4.5** (Friedgut’s junta Theorem for monotone families). For each \( \epsilon, \zeta, C > 0 \) there exists \( j \in \mathbb{N} \), such that the following holds. Let \( F \subseteq \mathcal{P}([n]) \) be a monotone family, let \( p \in (\zeta, 1 - \zeta) \), and suppose that \( \frac{d\mu(F)}{dp} \leq C \). The there exists a \( j \)-junta \( J \), such that \( \mu_p(F \Delta J) < \epsilon \).

**Proof of Lemma 4.4.** Note that we may assume that \( n \) is sufficiently large by decreasing \( \delta \) if necessary. By Lemma 3.11 we have \( \mu(F^\tau) \leq \mu(F^\tau) \) for any \( r \leq l \). Hence, for any \( p \leq \frac{k + l}{2n} \) we have

\[
\mu_p(F^\tau) = \sum_{r=0}^{n} p^r (1 - p)^{n-r} \binom{n}{r} \mu(F^\tau) \\
\leq \sum_{r=0}^{l} p^r (1 - p)^{n-r} \binom{n}{r} \mu(F^\tau) + \Pr_{r \sim \text{Bin}(n, p)} [r \geq l] \\
\leq \mu(F^\tau) + \Pr_{r \sim \text{Bin}(n, p)} [r \geq l].
\]

Suppose on the contrary that \( \mu(F^\tau) \geq 1 - \epsilon \). A simple Chernoff bound implies that \( \Pr_{r \sim \text{Bin}(n, p)} [r \geq l] < \frac{\epsilon}{2} \), provided that \( n \geq n_0(\zeta) \). Thus,

\[
\mu_p(F^\tau) \leq 1 - \epsilon + \frac{\epsilon}{2} = 1 - \frac{\epsilon}{2}
\]

for any \( p \leq \frac{k + l}{2n} \). The Mean Value Inequality imply that there exists \( q \in (\frac{k}{n} + \frac{\zeta}{8}, \frac{k}{n} + \frac{2\epsilon}{5}) \), such that

\[
\frac{d\mu_q(F^\tau)}{dq} \leq \frac{\mu_{\frac{k + l}{2n}}(F^\tau) - \mu_{\frac{k + l}{2n}}(F^\tau)}{(\zeta/5)} \leq \frac{5}{\zeta}.
\]

By Friedgut’s junta Theorem, there exists a set \( J \subseteq [n] \) with \( |J| = O(1) \) and a family \( G \subseteq \mathcal{P}(J) \), such that \( \mu_q(F^\tau \Delta \langle G \rangle) < \frac{\epsilon^2}{16} \).
Claim 4.6. $\mu_q \left( (F^\circ_j)^\circ \right) < \frac{\varepsilon}{4^4}$.

Proof. Note that the fact that $F^\circ_j$ is monotone implies that $\mu_q \left( (F^\circ_j)^B_j \right) \geq \mu_q \left( (F^\circ_j)^\circ_j \right)$. Suppose for a contradiction that $\mu_q \left( (F^\circ_j)^\circ_j \right) \geq \frac{\varepsilon}{4}$. Then

$$\mu_q \left( F^\circ_j \setminus (G) \right) = \sum_{B \not\in G} q^{\left| B \right|} (1 - q)^{\left| J \right| - \left| B \right|} \mu_q \left( (F^\circ_j)^B_j \right) \geq \sum_{B \not\in G} q^{\left| B \right|} (1 - q)^{\left| J \right| - \left| B \right|} \frac{\varepsilon}{4} \geq \frac{\varepsilon}{4} (1 - \mu_q ((G))) \geq \frac{\varepsilon}{4} \left( \frac{\varepsilon}{2} - \frac{c^2}{10^2} \right) > \frac{c^2}{10^2},$$

a contradiction.

Note that $(F^\circ_j)^\circ = (F^\circ_j)^\circ_j$. Since $\mu \left( (F^\circ_j)^\circ_j \right) \geq \mu \left( (F^\circ_j)^\circ_j \right)$ for any $r \geq k$, we have

$$\mu_q \left( (F^\circ_j)^\circ_j \right) = \mathbb{E}_{r \sim \text{Bin}(n, q)} \left[ \Pr_{A \sim \mu_q} \left[ A \in (F^\circ_j)^\circ_j \mid |A| = r \right] \right] \geq \mathbb{E}_{r \sim \text{Bin}(n, q)} \left[ \mu \left( (F^\circ_j)^\circ_j \right) \right] \geq \frac{\mu \left( F^\circ_j \right)}{2} \frac{1}{2},$$

provided that $n$ is sufficiently large. Rearranging and using the claim, we obtain

$$\mu \left( F^\circ_j \right) < 2\mu_q \left( (F^\circ_j)^\circ_j \right) \leq \frac{\varepsilon}{2},$$

However, if $\delta$ is sufficiently small to satisfy $|J| < \left\lceil \frac{k}{\delta} \right\rceil$ and $\delta < \frac{\varepsilon}{2}$, then we obtain

$$\mu \left( F^\circ_j \right) \geq \mu (F) - \delta > \frac{\varepsilon}{2},$$

a contradiction. \qed

4.4. Showing that every family that does not contain a $(d, k, (\frac{4}{d} + \zeta) k)$-cluster is essentially contained in a $(d + 1)$-wise intersecting $j$-junta.

Proposition 4.7. For each $\varepsilon, \zeta > 0$ there exists $a \varepsilon > 0$, $n_0 \in \mathbb{N}$, such that the following holds. Let $n > n_0$, let $\zeta n < k < \left( \frac{d}{d+1} - \varepsilon \right) n$, and let $F \subseteq \mathcal{P}(J)$ be some family that does not contain a $(d, k, (\frac{4}{d+1} + \zeta) k)$-cluster. Then $F$ is $\varepsilon$-essentially contained in a $(d + 1)$-wise intersecting $j$-junta.

Proof. Let $\delta = \delta (\varepsilon, \zeta, d)$ be sufficiently small, and choose $j = j (\delta, \varepsilon, \zeta, d), n_0 = n_0 (j, \zeta)$ sufficiently large. By Theorem 4.3 there exists a set $J$ of size $j$ and a family $G \subseteq \mathcal{P}(J)$, such that $F$ is $e$-essentially contained in $(G), \mu_q (F_j^B)$ is $(\frac{1}{2^2}, \delta)$-regular and has measure $\geq \frac{\varepsilon}{2}$. If the junta $\langle G \rangle$ is $(d + 1)$-wise intersecting, then we are done. Otherwise, there exist sets $B_0, \ldots, B_d \in G$ whose intersection is empty.

Let $l = \left( 1 + \frac{\zeta}{2} \right) k$. Note that we may apply Lemma 4.3 with $\frac{\varepsilon}{2}$ instead of $\zeta, \min \left\{ \frac{\varepsilon}{2}, \frac{1}{d+1} \right\}$. Instead of $\varepsilon, \delta$, provided that $\delta$ is sufficiently small. Doing so, we obtain that

$$\mu \left( (F_j^B)^\circ_j \right) > 1 - \frac{1}{d+1}$$

for each $i$.\qed
Choose a set $S \subseteq [n] \setminus J$ of size $\left\lceil \frac{d+1}{d} l \right\rceil$ uniformly at random. Choose a uniformly random $(d, l, \left\lceil \frac{d+1}{d} l \right\rceil)$-cluster $(C_0, \ldots, C_d)$ in $(S \setminus J)$. Note that each $C_i$ is a uniformly random set in $\binom{\{n\} \setminus J}{i}$. Therefore,

$$\Pr\left[ C_i \notin \left( F_{B_i}^{B_i} \right)^\dagger \right] = 1 - \mu \left( \left( F_{B_i}^{B_i} \right)^\dagger \right) < \frac{1}{d+1}.$$ 

A union bound implies that the probability that $C_i \notin \left( F_{B_i}^{B_i} \right)^\dagger$ for some $i$ is at most

$$\sum_{i=0}^{d} \left( 1 - \mu \left( \left( F_{B_i}^{B_i} \right)^\dagger \right) \right) < 1.$$

Therefore, there exists sets $C_0 \in \left( F_{B_0}^{B_0} \right)^\dagger, \ldots, C_d \in \left( F_{B_d}^{B_d} \right)^\dagger$ that form a $(d, l, \left\lceil \frac{d+1}{d} l \right\rceil)$-cluster. By definition, this implies that there exists sets $A_0, \ldots, A_d \in \mathcal{F}$ such that $A_i \subseteq B_i \cup C_i$. Now the sets $A_0, \ldots, A_d$ form a $(d, k, \left\lceil \frac{d+1}{d} l \right\rceil + j)$-cluster. This is a contradiction since

$$\left\lceil \frac{d+1}{d} l \right\rceil + j = \left\lceil \frac{d+1}{d} \left( 1 + \frac{\zeta}{3} \right) k \right\rceil + j \leq \left( \frac{d+1}{d} + \zeta \right) k,$$

provided that $n_0$ is sufficiently large. □

4.5. Proof of Proposition 4.1. We shall need the following stability result for Frankl’s Theorem that was essentially proved by Ellis, Keller, and the author [15]. We shall use the following corollary of their work stated by Keller and the author in [18].

**Theorem 4.8** ([18] Proposition 10.11). For each $\zeta, s > 0$, there exists $C > 0$, such that the following holds. Let $\zeta < \frac{k}{2} < \frac{k-1}{3} - \zeta$, let $\epsilon > 0$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an $s$-wise intersecting family. If $|\mathcal{F}| \geq \left( \frac{n}{k-1} \right)^s (1 - \epsilon)$, then $\mathcal{F}$ is $C\epsilon^{1+\frac{1}{k}}$-essentially contained in a star.

**Proof of Proposition 4.1.** Since the theorem becomes stronger as $\epsilon$ decreases, we may assume that $\epsilon$ is sufficiently small as a function of $\zeta, d$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family that does not contain a $(d, k, \left\lceil \frac{d+1}{d} + \zeta \right\rceil k)$-cluster and suppose that $|\mathcal{F}| \geq \left( \frac{n}{k-1} \right)^s (1 - \delta)$. By Proposition 4.7, there exists some $(d+1)$-wise intersecting family $\mathcal{J}$, such that $\mathcal{F}$ is $\frac{\epsilon}{2}$-essentially contained in $\mathcal{J}$. This implies that

$$|\mathcal{J}| \geq |\mathcal{F}| - \frac{\epsilon}{2} \frac{n-1}{k-1} \geq \left( 1 - \delta - \frac{\epsilon}{2k} \right) \frac{n-1}{k-1}.$$

By Theorem 1.8, the family $\mathcal{J}$ is $C\left( \delta + \frac{\epsilon}{2k} \right)^{1-C}$-essentially contained in a star, where $C = C(\zeta, d) > 1$. Therefore, $\mathcal{F}$ is $\frac{\epsilon}{2} + C\left( \delta + \frac{\epsilon}{2k} \right)^C$-essentially contained in a star.

This completes the proof since $\frac{\epsilon}{2} + C\left( \delta + \frac{\epsilon}{2k} \right)^C < \epsilon$, provided that $\delta, \epsilon$ are sufficiently small. □

5. The bootstrapping step: Proof of theorems 1.8 and 1.9

**Proof of Theorem 1.9.** Note that by increasing $C$ if necessary we may assume that $\epsilon$ is sufficiently small. Let $\epsilon' = \epsilon(\zeta, d)$ be sufficiently small. By Proposition 4.1, the family $\mathcal{F}$ is $\epsilon'$-essentially contained in a star, provided that $\epsilon$ is sufficiently small. Without loss of generality, it is the star $S$ of all sets that contain 1. Let $l = \left\lceil k \left( 1 + \frac{\epsilon}{2} \right) \right\rceil$. Similarly to the proof of Proposition 4.7, the families

$$\left( F_{(1)}^{(1)} \right)^\dagger, \ldots, \left( F_{(1)}^{(1)} \right)^\dagger, \left( F_{(1)}^{S} \right)^\dagger$$
do not mutually contain a \((d, l, \lceil \frac{d+1}{d}l \rceil)\)-cluster. Let \(\{A_0, \ldots, A_d\} \subseteq \binom{[n]}{l+1}\) be a uniformly random \((d, l, \lceil \frac{d+1}{d}l \rceil)\)-cluster. We have

\[
1 = \Pr[\exists i : A_i \notin F] \leq \sum_{i=0}^{d} \Pr[A_i \notin F] \tag{5.1}
\]

Write \(|F \cap S| = \binom{n-1}{k-1} (1 - \epsilon'')\). By Corollary 3.3,

\[
|\{F \cap S\}^{\uparrow l}| \geq \left( n - 1 \right) (1 - \epsilon'') \tag{5.2}
\]

Hence,

\[
\frac{k}{n} (1 - \epsilon) \leq \mu(F) = \mu(F \cap S) + \mu(F \setminus S) \leq \frac{k}{n} (1 - \epsilon'') + \min \left\{ 1 - \frac{k}{n}, dC' \epsilon' \right\}. \tag{5.3}
\]

In particular, \(\epsilon'' \leq \epsilon + \frac{1}{k} \epsilon'\). Since both \(\epsilon, \epsilon'\) may be assumed to be arbitrarily small as a function of \(d, \zeta, C'\), we may assume that \(\epsilon''\) is sufficiently small to have

\[
\frac{1 - \frac{k}{n}}{\frac{k}{n}} dC' \epsilon' \leq \frac{\epsilon''}{2}.
\]

Combining with (5.3), we have \(\epsilon'' \leq 2 \epsilon\). Hence, by (5.2)

\[
\mu(F \setminus S) \leq dC' \epsilon' \leq \Theta_{d, \zeta} (\epsilon) \leq C \epsilon' \leq \Theta_{d, \zeta} (\epsilon) \leq C \epsilon' \leq \Theta_{d, \zeta} (\epsilon) \leq C \epsilon',
\]

provided that \(C\) is sufficiently large. This completes the proof of the theorem.

\[\square\]

Theorem 1.8 now follows easily.

Proof of Theorem 1.8. The Theorem immediately follows from Theorem 1.8 if \(\frac{k}{n} \leq \frac{d}{d+1} - \zeta'\) for any fixed \(\zeta' = \zeta'(d, \epsilon) > 0\), by substituting \(\epsilon = 0\). On the other hand, it follows from Theorem 1.2 if \(n \leq \frac{d+1}{d} + \zeta \). Hence, the theorem follows by substituting \(\zeta' = \frac{d}{d+1} - \frac{1}{(d+1)\zeta'}\).
6. Open problem

We conjecture that Theorem 1.8 in fact holds for all $k \geq C \log n$ for a sufficiently large constant $C$.

**Conjecture 6.1.** For each $d, \epsilon > 0$ there exists a constant $C = C(d, \epsilon)$, such that the following holds. Let $C \log n \leq k \leq \frac{d}{d-1} n$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a family that does not contain a $(d, k, (\frac{d+1}{d} + \epsilon) k)$-cluster. Then $|\mathcal{F}| \leq \left(\frac{n}{k-1}\right)$, with equality only if and only if $\mathcal{F}$ is a star.

We would also like to mention that one cannot strengthen Conjecture 6.1 by replacing the requirement that $\mathcal{F}$ does not contain a $(d, k, (\frac{d+1}{d} + \epsilon) k)$-cluster by the requirement that $\mathcal{F}$ does not contain a $(d, k, \frac{d+1}{d} k)$-cluster. For instance, in the case where $d = 2$ one can take the following example known as the complete odd bipartite hypergraph.

**Example 6.2.** The family $\mathcal{F} := \big\{ A \in \binom{[n]}{k} \mid |A \cap [\frac{n}{2}]| \text{ is odd} \big\}$ does not contain a $(2, k, \frac{3}{2} k)$-cluster and its measure is asymptotically $\frac{1}{2}$. Therefore, for any $\zeta > 0$ and $\frac{k}{n} < \frac{1}{2} - \zeta$, we have $|\mathcal{F}| \geq \left(\frac{n}{k-1}\right)$, provided that $n \geq n_0(\zeta)$.

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