Unitarization of uniformly bounded subgroups in finite von Neumann algebras

Martín Miglioli

Abstract

This note presents a new proof of the fact that every uniformly bounded group of invertible elements in a finite von Neumann algebra is similar to a unitary group. In 1974, Vasilescu and Zsido proved this result using the Ryll–Nardzewsky fixed point theorem [Vasilescu and Zsido, ‘Uniformly bounded groups in finite W∗-algebras’, Acta Sci. Math. (Szeged) 36 (1974) 189–192]. This new proof involves metric geometric arguments in the non-positively curved space of positive invertible operators of the algebra, which yield a more explicit unitarizer.

1. Geometry of the cone of positive invertible operators in a finite algebra

The metric geometry of the cone of positive invertible operators in a finite von Neumann algebra was studied in [2,4]. In this section, we recall some facts from these papers.

Let $A$ be a von Neumann algebra with a finite (normal, faithful) trace $\tau$. Denote by $A_{sa}$ the set of self-adjoint operators, by $G$ the group of invertible operators, by $U$ the group of unitary operators and by $P$ the set of positive invertible operators in $A$.

Since $P$ is an open subset of $A_{sa}$ in the norm topology, we can regard $P$ as a submanifold of $A_{sa}$. Therefore, the tangent spaces of $P$ identify with $A_{sa}$ endowed with the uniform norm, which we denote by $\|\cdot\|$. The pre-Hilbert norm $\|x\|_2 = \tau(x^2)^{1/2}$ on $A_{sa}$ is used to give $P$ a Finslerian length structure (see [3, Section 2]). The admissible paths of the length structure are the piecewise smooth curves in $P$ and for each $a \in P$ the tangent space $T_aP \simeq A_{sa}$ is endowed with the norm given by $\|x\|_{a,2} = \|a^{-1/2}xa^{-1/2}\|_2$ for $x \in A_{sa}$. For $a, b \in P$, the unique geodesic $\gamma_{a,b} : [0,1] \to P$ between $a$ and $b$ is given by $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^{t}a^{1/2}$ and has length equal to $d(a,b) = \text{Length}(\gamma_{a,b}) = \|\ln(a^{-1/2}ba^{-1/2})\|_2$, see [2, Theorems 3.1 and 3.2]. The interior metric space associated with this length structure will be denoted by $(P,d)$.

If $A$ is finite-dimensional and therefore a sum of matrix spaces, then the metric $d$ is the well-known non-positively curved Riemannian metric on the set of positive-definite matrices [6]. If $A$ is of type $II_1$, then the trace inner product is not complete, so $P$ is not a Hilbert–Riemann manifold and $(P,d)$ is not a complete metric space; see [4, Remark 3.21].

The semi-parallelogram law, which states that if $a, b, c \in P$, then

$$d(a,b)^2 + 4d(c,\gamma_{a,b}(\frac{1}{2}))^2 \leq 2(d(c,a)^2 + d(c,b)^2)$$

holds in the metric space $(P,d)$; see [4, Theorem 4.4]. Hence the midpoint between $a$ and $b$ is $\gamma_{a,b}(\frac{1}{2})$. See the beginning of [5, Chapter XI, Section 3] for further discussion on the semi-parallelogram law and midpoint maps.

The action of $G$ on $P$ given by $I_g(a) = gag^*$ sends geodesic segments to geodesic segments and is isometric, that is, $I_g \circ \gamma_{a,b} = \gamma_{I_g(a),I_g(b)}$ and $d(I_g(a),I_g(b)) = d(a,b)$ for all $a, b \in P$ and $g \in G$; see the introduction of [2].
2. Uniformly bounded subgroups

A subset $C \subseteq P$ is geodesically convex if $\gamma_{a,b}(t) \in C$ for every $a,b \in C$ and $t \in [0,1]$, and is midpoint convex if $\gamma_{a,b}(\frac{1}{2}) \in C$ for every $a,b \in C$. Note that a geodesically convex set is midpoint convex.

**Lemma 2.1.** If $C \subseteq P$ is geodesically convex, then its closure $\overline{C}$ in $(P,d)$ is geodesically convex.

**Proof.** By [2, Corollary 3.4], the distance between two geodesics is convex, that is,

$$t \mapsto d(\gamma_{a_1,b_1}(t), \gamma_{a_2,b_2}(t)), \quad [0,1] \rightarrow [0, +\infty)$$

is convex for all $a_1,b_1,a_2,b_2 \in P$. Hence, for a fixed $t \in [0,1]$, the function $(a,b) \mapsto \gamma_{a,b}(t)$ is $d$-continuous.

If $a,b \in \overline{C}$ and $t \in [0,1]$, let $(a_n)_n, (b_n)_n$ be sequences in $C$ such that $a_n \rightarrow a$ and $b_n \rightarrow b$. Since $C$ is geodesically convex $\gamma_{a_n,b_n}(t) \in C$ for all $n \in \mathbb{N}$. The $d$-continuity of $(a,b) \mapsto \gamma_{a,b}(t)$ implies that $\gamma_{a_n,b_n}(t) \rightarrow \gamma_{a,b}(t)$, so that $\gamma_{a,b}(t) \in \overline{C}$. 

**Lemma 2.2.** For $0 < c_1 < c_2$, the interval $P_{c_1,c_2} = \{a \in P : c_1 \leq a \leq c_2\}$ endowed with the metric $d$ is a complete and bounded metric space.

**Proof.** In $P_{c_1,c_2}$, the linear metric and the rectifiable distance are equivalent [4, Proposition 3.2], that is, there are $C,C' > 0$ such that $\|a-b\|_2 \leq C d(a,b)$ and $d(a,b) \leq C'|a-b\|_2$ for all $a,b \in P_{c_1,c_2}$.

Since $\| \cdot \|_2$ induces a complete metric on subsets of $A$ that are closed and bounded in the uniform norm, and $P_{c_1,c_2}$ is closed and bounded in the uniform norm, we conclude that $(P_{c_1,c_2}, d)$ is a complete metric space.

Also, $(P_{c_1,c_2}, d)$ is a bounded metric space because $d(a,b) \leq C'\|a-b\|_2 \leq C'|a-b| \leq 2C'c_2$ for all $a,b \in P_{c_1,c_2}$. 

**Theorem 2.3.** If $H \subseteq G$ is a subgroup such that $\sup_{h \in H} \|h\| = M < \infty$, then there exists $s \in P_{M^{-1},M}$ such that $s^{-1}Hs \subseteq U$.

**Proof.** Consider the isometric action $I : H \rightarrow \text{Isom}(P)$ given by $I_h(a) = hah^*$ for $h \in H$ and $a \in P$. We denote the action by $h \cdot a = I_h(a)$. Take $X_1 = H \cdot 1$ and define inductively $X_{n+1} = \{\gamma_{a,b}(t) : a,b \in X_n, t \in [0,1]\}$ for $n \geq 1$. Let

$$\text{conv}(H \cdot 1) = \bigcup_{n \in \mathbb{N}} X_n.$$ 

Since $P_{M^{-2},M^2}$ is geodesically convex [1] and the action sends geodesic segments to geodesic segments, if $X_n \subseteq P_{M^{-2},M^2}$, then $X_{n+1} \subseteq P_{M^{-2},M^2}$ for all $n \in \mathbb{N}$. Therefore, $\text{conv}(H \cdot 1) \subseteq P_{M^{-2},M^2}$ follows from $X_1 = H \cdot 1 = \{hh^*\}_{h \in H} \subseteq P_{M^{-2},M^2}$. Using the fact that $P_{M^{-2},M^2}$ is closed in $(P, d)$ we conclude that $\text{conv}(H \cdot 1) \subseteq P_{M^{-2},M^2}$.

Since the action sends geodesic segments to geodesic segments, if $X_n$ is invariant under the action $I$, then $X_{n+1}$ is invariant for all $n \in \mathbb{N}$. Since $X_1 = H \cdot 1$ is invariant, we conclude that $\text{conv}(H \cdot 1)$ is invariant. The action is also isometric, hence $\text{conv}(H \cdot 1)$ is invariant and we can restrict the action $I$ to this subset.
The space $(\overline{\text{conv}}(H \cdot 1), d)$ is midpoint convex and the semi-parallelogram law holds in $P$, hence this law also holds in $(\overline{\text{conv}}(H \cdot 1), d)$. Since $\overline{\text{conv}}(H \cdot 1)$ is a closed subset of the complete metric space $(P_{M-2,M^2}, d)$, the space $(\overline{\text{conv}}(H \cdot 1), d)$ is complete. We conclude that $(\overline{\text{conv}}(H \cdot 1), d)$ is a complete metric space in which the semi-parallelogram holds.

Since $(P_{M-2,M^2}, d)$ is a bounded metric space $\overline{\text{conv}}(H \cdot 1)$ is a bounded set. Therefore, the restricted action has bounded orbits, and the Bruhat–Tits fixed point theorem [5, Chapter XI, Theorem 3.2] states that there exists $a \in \overline{\text{conv}}(H \cdot 1)$ such that $I_h(a) = hah^* = a$ for all $h \in H$.

Then $1 = a^{-1/2}aa^{-1/2} = a^{-1/2}hah^*a^{-1/2} = (a^{-1/2}ha^{1/2})(a^{1/2}h^*a^{-1/2}) = (a^{-1/2}ha^{1/2})(a^{-1/2}ha^{1/2})^*$ for all $h \in H$.

Therefore, $a^{-1/2}Ha^{1/2} \subseteq U$, that is, $s = a^{1/2}$ is a unitarizer of $H$.

Since the square root is an operator monotone function and $a \in P_{M-2,M^2}$, it follows that $s = a^{1/2} \in P_{M-1,M}$.

**Remark 2.4.** A fixed point of the action $I$ in the above theorem is the circumcenter of the set $H \cdot 1 = \{hh^*\}_{h \in H}$ in $(\overline{\text{conv}}(H \cdot 1), d)$; see [5, Chapter XI, Theorems 3.1 and 3.2]. Therefore, a unitarizer $s$ is the square root of the circumcenter of $\{hh^*\}_{h \in H}$. The unitarizability of a uniformly bounded subgroup $H$ of the group of bounded linear invertible operators acting on a Hilbert space was obtained independently in the 1950s by Day, Dixmier, Nakamura and Takeda assuming that $H$ is amenable (see [7] and the references therein). In that context the unitarizer $s$ was obtained as the square root of the center of mass of $\{hh^*\}_{h \in H}$.

**References**

1. **E. Andruchow, G. Corach and D. Stojanoff**, ‘Geometrical significance of Löwner–Heinz inequality’, *Proc. Amer. Math. Soc.* 128 (2000) 1031–1037.
2. **E. Andruchow and G. Larotonda**, ‘Nonpositively curved metric in the positive cone of a finite von Neumann algebra’, *J. London Math. Soc.* (2) 74 (2006) 205–218.
3. **D. Burago, Y. Burago and S. Ivanov**, *A course in metric geometry* (American Mathematical Society, Providence, RI, 2001).
4. **C. Conde and G. Larotonda**, ‘Spaces of nonpositive curvature arising from a finite algebra’, *J. Math. Anal. Appl.* 368 (2010) 636–649.
5. **S. Lang**, *Fundamentals of differential geometry*, Graduate Texts in Mathematics 191 (Springer, New York, 1999).
6. **G. D. Mostow**, ‘Some new decomposition theorems for semi-simple groups’, *Mem. Amer. Math. Soc.* 14 (1955) 31–54.
7. **M. Nakamura and Z. Takeda**, ‘Group representation and Banach limit’, *Tohoku Math. J.* 3 (1951) 132–135.

Martín Miglioli  
Instituto Argentino de Matemática Alberto P. Calderón  
CONICET  
Saavedra 15, Piso 3  
1083 Buenos Aires  
Argentina  
martin.miglioli@gmail.com