Rethinking the Micro-Foundation of Opinion Dynamics: Rich Consequences of an Inconspicuous Change

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ABSTRACT

Nowadays public opinion formation faces unprecedented challenges such as opinion radicalization, echo chambers, and information manipulation. Realistic and predictive mathematical models play a fundamental role in obtaining reliable understanding of the mechanisms behind opinion formation processes. Although most opinion dynamics models are built on the common assumption that individuals update their opinions by averaging others’ opinions, researchers might need to rethink this micro-foundation. We point out that the weighted-averaging mechanism features a non-negligible unrealistic implication. By resolving this unrealistic feature in the framework of cognitive dissonance theory, we propose a novel opinion dynamics model based on a weighted-median mechanism instead. Experimental data validation indicates that, compared with the averaging mechanism, predictions of individual opinion shifts by the median mechanism enjoys significantly lower error rates. Moreover, theoretical analysis reveals that such an inconspicuous change in microscopic mechanism, from weighted-averaging to weighted-median, leads to dramatic macroscopic consequences. Compared to other widely-studied models, our new model, despite its simplicity in form, predicts various important realistic features of opinion dynamics while the other models fail to, e.g., the vulnerability of socially marginalized individuals to opinion radicalization, the formation of steady multi-polar opinion distributions, and the vanishing consensus probability in larger and more clustered social groups. In addition, our model exhibits richer consensus-disagreement phase transition behavior dependent on more delicate and robust network structures. The novel weighted-median model renovates our understanding of opinion formation processes and extends the applicability of opinion formation models to the setting of ordered multiple-choice issues, which are prevalent in modern-day public debates and elections.

1 Introduction

The key discourse in democratic society starts from exchanges of opinions in deliberative groups, over public debates, or via social media, to eventually reaching agreements or disagreements. Nowadays public opinion formation is deeply influenced by social networks and faces unprecedented challenges such as opinion radicalization, echo chambers, and misinformation. Mathematical modeling of opinion dynamics plays a fundamental role in gaining reliable understanding of how empirically observed macroscopic sociological phenomena emerge from certain microscopic social-influence mechanisms and social network structures. Realistic, predictive, and quantitative models also help to answer some practically important questions, e.g., what drives some online social media users to join terrorism organizations, and how robust is our society to political propaganda, fake news, or opinion manipulation? Interpersonal influences are highly complicated processes involving various cognitive and sociopsychological mechanisms. Therefore, the key challenge in building predictive and mathematically tractable models of opinion dynamics is to identify the “salient features” that govern the interpersonal influence processes, i.e., the microfoundation of opinion dynamics.

Most existing deterministic opinion dynamics models originate from the classic DeGroot model\textsuperscript{1,2}, in which individuals’ opinions on the issue being discussed are denoted by real numbers and are updated by taking some weighted average opinions of those they are influenced by (referred to as their social neighbors). In a social group, the interpersonal relations on “who influences whom” are described by an influence network. Despite its mathematical elegance and widespread use, the DeGroot model is limited to opinions that are continuous by nature and leads to overly-simplified and unrealistic macroscopic predictions. For example, according to the DeGroot model, a group of individuals reach consensus as long as the influence network is strongly connected and aperiodic. Arguably, this is a bold prediction under a very mild connectivity condition.

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To capture the phenomenon of persistent disagreement, various extensions have been proposed by introducing additional model assumptions and parameters. These extensions are still based on weighted averaging of real-valued opinions. Among them the most widely studied are the DeGroot model with absolutely stubborn individuals\(^3\), the bounded-confidence model with interpersonal influences truncated according to opinion distances\(^4\), the Friedkin-Johnsen model with prejudice, i.e., persistent attachments to initial conditions\(^5\). These models and some further extensions, e.g., see\(^6\)-\(^10\), provide different explanations for persistent disagreement and deepen in various aspects our understanding of possible socio-psychological mechanisms involved in opinion dynamics, as well as their implications.

However, despite being sufficiently sophisticated or even mathematically intractable, none of the aforementioned models captures other prominent features of opinion dynamics supported by sociological literature and everyday experience, such as the connection between social marginalization and opinion radicalization\(^11\), diverse public opinion distributions\(^12\), and lower likelihoods of consensus in larger groups\(^13\). In addition, the network topology in these models, as long as satisfying some mild connectivity conditions, barely plays a role in determining the consensus-disagreement phase transition.

The bottleneck in predictive power met by the DeGroot models and its extensions inspires us to retrospect the very foundation of opinion dynamics. In this paper, we point out that the weighted-averaging opinion update, adopted by all the aforementioned models as their micro-foundation, features a long-overlooked but non-negligible unrealistic implication. By resolving this unrealistic implication in the framework of network games and cognitive dissonance theory in psychology, we derive a new micro-foundation for opinion dynamics models, namely the weighted-median mechanism. In this paper, a complete set of studies are conducted on the proposed weighted-median mechanism. Empirical data collected from human-subject experiments indicate that, compared with the weighted-averaging mechanism, our weighted-median mechanism enjoys significantly lower errors in terms of predicting individual opinion shifts under social influences. Moreover, comparative numerical studies indicate that, our weighted-median model, despite being arguably the simplest in form, is able to replicates various prominent features of real-world opinion formation processes mentioned in last paragraph, which the most widely studied extensions of the DeGroot model fail to capture. Finally, dynamic behavior of the weighted-median model is rigorously analyzed. We fully characterize the set of equilibria and establish the almost-sure convergence of individual opinions. Furthermore, we provide network-topology conditions for the phase-transition behavior between consensus and persistent disagreement. Analytical results indicate that our weighted-median model exhibits sophisticated phase-transition behavior dependent on some delicate network structures such as cohesive sets and decisive links, which will be specified in a later section. To sum up, results obtained in this paper imply that the weighted-median mechanism can be adopted as a reasonable new micro-foundation for further modeling of opinion dynamics.

### 2 A Widely Overlooked Unrealistic Implication of Weighted-Averaging

The mathematical form of the DeGroot model is:

$$x_i(t + 1) = \text{Mean}_i(x(t); W) = \sum_{j=1}^{n} w_{ij} x_j(t),$$

(1)

for any individual \(i \in \{1, 2, \ldots, n\}\) in a group of \(n\) individuals. Here \(x_i(t)\) denotes individual \(i\)’s opinion at time \(t\), and \(w_{ij}\) is the weight individual \(i\) assigns to individual \(j\)’s opinion. The matrix \(W = (w_{ij})_{n \times n}\) is referred to as the influence matrix. By definition, \(W\) should satisfy: 1) \(w_{ij} \geq 0\) for any \(i, j\); 2) \(\sum_{j=1}^{n} w_{ij} = 1\) for any \(i\). We refer to matrices satisfying these two properties as row-stochastic matrices. The matrix \(W\) induces a directed and weighted graph, referred to as the influence network and denoted by \(G(W)\). In \(G(W)\), each node is an individual and each \(w_{ij} > 0\) corresponds to a directed link from \(i\) to their social neighbor \(j\) with weight \(w_{ij}\). See Fig. 1(a) as an example of the correspondence between the influence matrix and the influence network.

As the micro-foundation of the DeGroot model and its extensions, the weighted-averaging mechanism shown in equation (1) features a non-negligibly unrealistic implication. This unrealistic implication is manifested by the following example and is visually presented in Fig. 1(b): Suppose an individual \(i\)’s opinion is influenced by individuals \(j\) and \(k\) via the weighted-averaging mechanism, i.e.,

$$x_i(t + 1) = x_i(t) + w_{ik} (x_k(t) - x_i(t)) + w_{ij} (x_j(t) - x_i(t)).$$

The equation above implies that whether individual \(i\)’s opinion moves towards \(x_i(t)\) or \(x_j(t)\) is determined by whether \(w_{ik} |x_k(t) - x_i(t)|\) is larger than \(w_{ij} |x_j(t) - x_i(t)|\). That is, the “attractive force” of any opinion \(x_j(t)\) to individual \(i\) is proportional to the opinion distance \(|x_j(t) - x_i(t)|\), or equivalently, the more distant an opinion, the more attractive it is. (Here we appeal to an analogy between social interaction and physical forces, as in the seminal works on “social forces”\(^14, 15\). Note that, different from the physical forces, the “attractive forces” of opinions directly apply to the change of opinions rather than the second-order difference of opinions.)

Since the weighted-averaging mechanism implies overly large “attractive forces” between individuals holding different opinions, neither the individuals nor the influence network
structure, as long as well connected, is able to resist such huge attractions driving the system to consensus. An immediate unrealistic consequence of the weighted-averaging mechanism is that social groups have no resistance to opinion manipulation. For example, the DeGroot model predicts that, if one individual’s opinion is manipulated, this individual alone can drive all the other individuals’ opinions to arbitrarily extreme positions by moving their own opinion arbitrarily far. See Fig. 1(c) for an example. Moreover, this unrealistic feature of the weighted-averaging mechanism is inherited by all the extensions of the DeGroot model, though blended with other effects introduced by these extensions.

3 Derivation and Set-up of the Weighted-Median Opinion Dynamics

We resolve the unrealistic feature of the weighted-averaging mechanism and propose a new micro-foundation of opinion dynamics in the framework of network games and the cognitive dissonance theory. According to the seminal psychological logical theory on cognitive dissonance and its experimental validations, individuals in a group experience cognitive dissonance from disagreement and attempt to reduce such dissonance by changing their opinions. Given a row-stochastic influence matrix $W$, for any individual $i$, given their opinion $x_i$ and the other individuals’ opinions, denoted by $x_{-i}$, such cognitive dissonance could be modelled as their cost function in a network game. The individuals are the players and their strategies are the opinions they take. For each individual $i$, the most parsimonious form of their cognitive dissonance, i.e., their cost function, is written as

$$C_i(x_i, x_{-i}) = \sum_{j=1}^{n} w_{ij}|x_i - x_j|^\alpha,$$

Here $\alpha > 0$ is a parameter. Individuals’ opinion updates could be in turn modelled as the best responses to minimize their cost functions, i.e.,

$$x_i(t+1) = \arg\min_z C_i(z, x_{-i}(t)) = \arg\min_z \sum_{j=1}^{n} w_{ij}|z - x_j(t)|^\alpha,$$

for any $i \in \{1, \ldots, n\}$. For example, $\alpha = 2$ for the DeGroot model (1)\textsuperscript{18}. The parameter $\alpha$ has a clear sociological interpretation: An exponent $\alpha > 1$ ($\alpha < 1$ resp.) implies that individuals are more sensitive to distant (nearby resp.) opinions. In the absence of any widely-accepted psychological theory in favor of $\alpha > 1$ or $\alpha < 1$, we adopt the neutral hypothesis $\alpha = 1$. We point out that, for generic weights $W$, $\arg\min C_i(z, x_{-i}(t))$ with $\alpha = 1$ is unique and the best-response dynamics lead to the weighted-median opinion updates, i.e., $x_i(t+1)$ is the weighted-median of $(x_1(t), \ldots, x_n(t))$ associated with the non-negative weights $(w_{11}, \ldots, w_{nn})$. See Lemma 3.1 in\textsuperscript{19} for the proof. A detailed argument is also provided in the Supplementary Material\textsuperscript{4}. As will be manifested in the rest of this text, this inconspicuous and subtle change in microscopic mechanism from weighted-averaging to weighted-median leads to dramatic macroscopic consequences.

We formally define our novel weighted-median opinion dynamics as follows: Consider a group of $n$ individuals on an influence network $G(W)$, where the influence matrix $W$ is row-stochastic, and denote by $x(t) = (x_1(t), \ldots, x_n(t))$ the individuals’ opinions at time $t$. Starting with some initial condition $x(0) = (x_1(0), \ldots, x_n(0))$, at each time step $t+1$ ($t = 0, 1, 2, \ldots$), one individual $i$ is randomly selected and updates their opinion according to the following equation:

$$x_i(t+1) = \text{Med}_i(x(t); W).$$

Here $\text{Med}_i(x(t); W)$ denotes the weighted median of the $n$-tuple $x(t) = (x_1(t), \ldots, x_n(t))$ associated with the weights $(w_{11}, w_{22}, \ldots, w_{nn})$. Such a weighted median is in turn defined as

$$\text{Med}_i(x(t); W) = x^* \in \{x_1(t), \ldots, x_n(t)\}$$

satisfying

$$\sum_{j: x_j(t) < x^*} w_{ij} \leq \frac{1}{2} \quad \text{and} \quad \sum_{j: x_j(t) > x^*} w_{ij} \leq \frac{1}{2}.$$

For generic weights $W = (w_{ij})_{n \times n}$, the weighted median $\text{Med}_i(x(t); W)$ is unique for any individual $i \in \{1, \ldots, n\}$. If the weighted medians of $(x_1(t), \ldots, x_n(t))$ associated with the weights $(w_{11}, \ldots, w_{nn})$ are not unique, we assume that $\text{Med}_i(x(t); W)$ takes the value of the weighted median that is the closest to $x_i(t)$ and thereby the uniqueness of $\text{Med}_i(x(t); W)$ is guaranteed. See the Supplementary Material\textsuperscript{20} for a detailed discussion.

Fig. 1(d) provides an example of the aforementioned cognitive dissonance function, with $\alpha = 1$, of an individual in an influence network, and how the weighted-median opinion is computed given their social neighbors’ opinions. Intuitively, in our weighted-median model, since the cognitive dissonances generated by distant opinions are much less than those in the case of $\alpha > 1$, the “attractive forces” by distant opinions in our model are not overly strong and thereby social groups may not always be driven to consensus, even when the influence networks are connected. This intuitive speculation is confirmed later in the theoretical analysis section. Since the attractions by distant opinions in our new model are weaker than in the DeGroot model, the individual opinions in social groups are more resilient to opinion manipulation than in the

\textsuperscript{1}See Supplemental Material at [URL will be inserted by publisher] for extended technical details, detailed simulation set-ups, and additional supportive numerical and empirical results. All files related to a published paper are stored as a single deposit and assigned a Supplemental Material URL. This URL appears in the article’s reference list.
DeGroot model, see Fig. 1(e) for an example. As also indicated by Fig. 1(e), the weighted-median model is robust to outliers as well.

Besides the interpretations in the context of network games and cognitive dissonance theory, the weighted-median mechanism is also grounded in the psychological theory of extreme-ness aversion\(^\text{20}\), according to which, people’s preferences are not always stable but can be altered depending on what alternatives they are exposed to. Moreover, given multiple options with certain ordering, people tend to choose the median option, which directly supports our weighted-median mechanism.

## 4 Empirical Validation of the Weighted-Median Mechanism

Empirical validation on a longitudinal dataset\(^\text{21}\) shows that the weighted-median mechanism enjoys significantly lower errors than the weighted-averaging mechanism in predicting individual opinion shifts.

This dataset\(^\text{21}\) is collected in a set of online human-subject experiments. Every single experiment involves 6 anonymous individuals, who sequentially answer 30 questions within tightly limited time. The questions are either guessing the proportion of a certain color in a given image (gauging game), or guessing the number of dots in certain color in a given image (counting game), see Fig. 2(a) for two examples. A common feature these two types of questions share is that the answers are numerical by nature and based mainly on subjective guessing, given limited time. For each question, the 6 participants give their answers for 3 rounds. After each round, they will see the answers of all the 6 participants as feedback and possibly alter their opinions based on this feedback. The dataset records, for each experiment, the individuals’ opinions in each round of the 30 questions. See Fig. 2(b) as a sample of the dataset.

Our objective is to investigate whether the weighted-median mechanism is more accurate than the weighted-averaging mechanism in predicting individuals’ opinion shifts after being confronted with the others’ opinions. Since in these experiments the individuals are anonymous, it is reasonable to assume that the participants uniformly assign weights to each other when they update their opinions. Therefore, what we aim to compare are the following two hypothesis: (H1) Individuals update their opinions by taking the median of all the
participants’ current opinions; (H2) Individuals update their opinions by taking the average of all the participants’ current opinions. In addition, for Hypothesis (H1), if the medians are not unique, we assume that the individuals take the median closest to their own current opinions.

Here we report the data analysis results regarding the individuals’ opinion shifts from the 2nd round to the 3rd round of each question. Results regarding the opinion shifts from the 1st rounds to the 2nd rounds yield to quantitatively similar conclusions and are provided in Supplementary Material [20]. For counting games, we randomly sample 18 experiments from the dataset, in which 71 participants give answers to all the 30 questions at each round. For each of these 71 participants, we apply Hypothesis (H1) and (H2) respectively to predict their answers in the 3rd round of each question, based on the participants’ answers in the 2nd round, and then compare the error rates of the predictions, defined as

$$\text{error rate} = \frac{|\text{prediction} - \text{true value}|}{\text{true value}}.$$

For the gauging games, we randomly sampled 21 experiments, in which 55 participants answer all the 30 questions at each round. Since these answers are already in percentages, we directly compare the magnitudes of errors between the predictions by Hypothesis (H1) and (H2). Fig. 2(c) presents the scatter plots between the predictions and the observed values (71 × 30 = 2130 pairs of data points for the counting games and 55 × 30 = 1650 data pairs for the gauging games) for both hypothesis. As Fig. 2(d) shows, regarding the counting games, the median error rate of the predictions by median (H1) is 0.0714, which is a stunning 46.36% lower than that of the predictions by average (H2). Regarding the gauging games, the median error of the predictions by median is even 50% lower than the median error of the predictions by average. The predictions by median also enjoy significantly lower mean error rate (MER) or mean absolute-value error (MAE) than the
predictions by average, in both counting games and gauging games.

In addition, we also consider some meaningful extensions of the weighted-median and weighted-average mechanisms by introducing individual inertia or attachments to initial opinions3. The data analysis results are reported in the Supplementary Material [20]. For any of these set-ups, the model based on median is more accurate than the model based on averaging in predicting participants’ opinion shifts. Moreover, these extensions to the weighted-median mechanism achieve remarkably low prediction errors by introducing additional parameters. However, despite being useful for fitting the models, these parameters do not reflect intrinsic attributes of the individuals, nor are they stable over time. Hence, we refrain from such extensions and focus on the core issue, namely the mean v.s. the median mechanisms.

5 Comparative Numerical Studies and Sociological Relevance

Comparative numerical studies indicate that the weighted-median opinion dynamics (2), despite being arguably the simplest in form, replicate various non-trivial realistic features of opinion dynamics whereas the DeGroot model and its extensions fail to. The models in comparison include the DeGroot model with absolutely stubborn individuals3, the Friedkin-Johnsen model8, and the networked bounded-confidence model10, all with randomized model parameters. The detailed simulation set-ups are provided in the Supplementary Material [20]. Note that the widely-studied bounded-confidence model has been proposed and analyzed only for all-to-all networks23 and thus not comparable to the weighted-median model. The bounded-confidence model built on arbitrary networks, which is included here for comparison, has barely been rigorously analyzed in previous literature, due to its mathematical intractability and fragile convergence properties10.

5.1 Social marginalization and opinion radicalization

Extreme ideologies such as terrorism are among the most serious challenges our modern society faces. Previous sociological studies, via empirical, conceptual, and case studies11,24–26, identify social marginalization as an important cause of opinion radicalization. However, such a connection has barely been captured by quantitative models of opinion dynamics.

Among all the opinion dynamics models compared in this section, our weighted-median model (2) is the only one showing that extreme opinions tend to reside in peripheral areas of social networks. Fig. 3(a) provides a visualized illustration of this feature. As the quantitative comparisons presented in Fig. 3(d) indicate, among all the models in comparison, only our weighted-median model exhibits the feature that the in-degree centrality distributions of opinions with different levels of extremeness are clearly separated, and the empirical probability density of the most extreme opinions decays the fastest as the in-degree increases. Simulations regarding other notions of centralities, e.g., closeness centrality and betweenness centrality, lead to qualitatively the same result and are presented in Supplementary Material [20]. To avoid the risk of bias due to the higher probability of being absolutely stubborn (self-weight > 1/2) in the weighted-median model when the in-degree is small, we have performed a second experiment on graphs without self-weight, and obtained similar results, see the Supplementary Material [20].

Further simulation results on the weighted-median model indicate a mechanistic explanation for the cause of opinion radicalization. We simulate the weighted-median opinion dynamics on a scale-free network with 2000 nodes for 1000 times and record the individuals’ extremist focuses, i.e., the ratio of their social neighbors holding extreme opinions at final steady states. As shown in Fig. 3(b), compared to the entire population, the extreme opinion holders tend to have low in-degrees but relatively high extremist focus. This result implies that radicalized individuals form small-size clusters. Such clustered micro-structures are believed to develop powerful cohesion and are characteristic of terrorists cells11. According to the weighted-median opinion dynamics, individuals inside such radicalized small clusters stick to extreme opinions because the extreme opinions constitute their main information sources, i.e., the weighted-median opinions. This explanation is supported by previous sociological literature, e.g. see the case analysis29 and the empirical study30. These two studies lead to a common conclusion that socially marginalized individuals could adopt extreme opinions by yielding to social influence if extreme opinions are dominant among their social contacts. On the other hand, radicalization is less likely for individuals with more social relations, which implies potentially more diverse information.

Remarkably, the in-degree-extremist-focus distribution for the extremists presented in Fig. 3(b) resembles the empirical data on the in-degree-ISIS-focus distributions of randomly sampled Twitter users, see Figure 5 in28, cited as Fig. 3(c) in this paper. Other models in comparison do not capture this feature, see the Supplementary Material [20].

5.2 Empirically observed steady public opinion distributions

Empirical evidences suggest that public opinions usually form into certain steady distributions. One particular interesting opinion distribution is the multi-modal distribution, which is frequently observed in real data, e.g., see the Supplementary Material [20] for the longitudinal survey on Europeans’ at-
Nodes’ frequencies of finally adopting the extreme opinions

Figure 3. Simulation results on the relations between opinion extremeness and in-degree centrality (defined as the sum of incoming link weights). In each simulation, the initial opinions are independently randomly generated from the uniform distribution on $[-1, 1]$ and opinions are classified into 4 categories: extreme ($[-1, -0.75) \cup (0.75, 1]$), radical ($[-0.75, -0.5) \cup (0.5, 0.75]$), biased ($[-0.5, -0.25) \cup (0.25, 0.5]$), and moderate ($[-0.25, 0.25]$). Panel (a) visualizes the spatial distribution of nodes adopting extreme opinions on a scale-free network with 1500 nodes. The layout of the nodes is arranged as follows: For each node $i$ with in-degree $d_i$, its radius from the center of the figure is $r_i = (\max_k d_k - d_i)^5$ and its angle is randomly generated. Panel (b) shows the 2-dimension distributions over the in-degree and the extremist-focus, for the the entire population and the extreme opinion holders respectively, in 1000 independent simulations of the weighted-median model on a randomly generated scale-free network with 2000 nodes. Among these simulations, 37254 individuals in total eventually adopt extreme opinions. Panel (c) is Figure 5 in\textsuperscript{28}, licensed under Creative Commons CC0 public domain dedication (CC0 1.0). This figure plots the empirical distribution of randomly sampled Twitter users over in-degree and the ISIS focus (the ratio of their social neighbors whose Twitter accounts get suspended for posting pro-ISIS terrorism contents). Panel (d) shows different models’ predictions of the in-degree centrality distributions for individuals with various levels of extremeness at the steady states. The empirical probability density curves are plotted by simulating different opinion dynamics models for 1000 times on the scale-free network shown in Panel (a).
Multi-modal opinion distributions constitute the premise of multi-party political systems and sociologists have long been interested in what mathematical assumptions are needed to model the formation of steady multi-modal opinion distributions along opinion dynamics. Our weighted-median opinion dynamics (2) offer perhaps the simplest answer to this open problem. As shown in Fig. 4, the weighted-median model (2) naturally generate various types of non-trivial steady opinion distributions that are frequently observed empirically, while the other models, without deliberately tuning their parameters, only predict some of them.

### 5.3 Vanishing likelihood of reaching consensus in large and clustered networks

One could easily conclude from everyday experience that it is usually more difficult for groups with larger sizes to reach consensus, see also the empirical evidence. However, most of the previous opinion dynamics models do not capture this obvious feature. As Fig. 5(a) and 5(b) indicates, among all the models in comparison, only the weighted-median model and the networked bounded-confidence model reflect the realistic feature that larger groups have lower likelihoods of reaching consensus. Moreover, as shown by Fig. 5(c), with fixed network sizes and link densities, our weighted-median model predicts that the likelihoods of reaching consensus increase as the networks become less clustered. Such a feature is not clearly reflected by the network bounded-confidence model, see Fig. 5(d). For the other opinion dynamics based on weighted-averaging, network features such as size and clustering coefficient play no role in determining the probability of reaching consensus. Instead, these models predict either almost-sure consensus or almost-sure disagreement, as shown in Fig. 5(b).

### 5.4 Physical intuitions behind the advantages exhibited by weighted-median mechanism

As indicated by the numerical comparisons presented in this section, our weighted-median opinion dynamics model reflects various realistic features of public opinion formation processes, which the other models in comparison fail to capture. The physical intuition behind these advantages is that the weighted-median model is built on a more realistic micro-foundation. For opinion dynamics based on weighted-averaging opinion updates, in order to resist the overly large attractions by distant opinions, which force the individuals to reach consensus, additional assumptions have to be introduced. These additional assumptions are either individual-level dynamics, such as individual stubbornness and persistent attachment to prior belief, which are irrelevant to any network structure, or discontinuous sudden truncations of the attractions by distant opinions, e.g., the bounded-confidence model, which are somehow artificial and barely mathematically tractable. Despite these additional assumptions being added, the roles of the influence network structure in these models are still not well captured. Different from these widely-studied models, our weighted-median opinion dynamics resolve the problem of overly large attractions by distant opinions. As the consequences, some non-trivial opinion distributions, e.g., the bimodal and multi-modal distributions, can present as steady distributions, and the effects of some delicate network structures on the model’s dynamical behavior naturally emerge.

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**Figure 4.** Distributions of the initial opinions and the final opinions predicted by different models. All simulations are run on the same scale-free network with 5000 nodes and starting with the same randomly generated initial conditions. Comparisons conducted on a small-world network indicate similar conclusions and are provided in the Supplementary Material [20].
In the next section, we investigate some important delicate network structures and how they determine the behavior of the weighted-median opinion dynamics.

6 Analytical results: equilibria, convergence, and phase transition

Theoretical analysis of the weighted-median opinion dynamics indicates that, despite its simplicity in form, the weighted-median model exhibits rich dynamical behavior that depends on some delicate and robust influence network structures. In this section, we mathematically establish the set of equilibria, and the phase-transition behavior between eventual consensus and persistent disagreement. The salient features responsible for the numerical observations in last section, as well as our key analysis tools, are the notions of cohesive sets and decisive links described below.

6.1 Important concepts: cohesive sets and decisive links

The definition of cohesive sets is given in previous literature on contagion processes and applied in the linear-threshold network diffusion model. To put it simply, a cohesive set is a subset of individuals on the influence network, of which each individual assigns more weights to the insiders than the outsiders. Intuitively, according to the weighted-median mechanism, if all the individuals in a cohesive set hold the same opinion, they will never change their opinions. A maximal cohesive set is a cohesive set of individuals such that adding any single outsider to this set makes it non-cohesive. The formal definitions of cohesive sets and maximally cohesive sets are given as follows: Given an influence network $G(W)$ with nodes set $V = \{1, \ldots, n\}$, a cohesive set $M \subseteq V$ is a subset of nodes that satisfies

$$\sum_{j \in M} w_{ij} \geq \frac{1}{2}, \quad \text{for any } i \in M.$$

A cohesive set $M$ is a maximal cohesive set if there does not exist $i \in V \setminus M$ such that $\sum_{j \in M} w_{ij} > 1/2$. A visualized example of (maximal) cohesive set is provided in Fig. 6(a). Cohesive sets are intricately related to the weighted-median dynamics, and their salient properties are presented and proved in Appendix A.

Cohesive set as defined above can be interpreted as a characterization of the so-called echo-chambers. In news media, echo chamber is a metaphorical description of a situation in which beliefs are amplified or reinforced by communication and repetition inside a closed system. According to the weighted-median mechanism, whenever all the individuals in a cohesive set adopt the same opinion, this cohesive set becomes an echo chamber in the sense that the individuals in this cohesive set will never change their opinion. If an influence network have multiple cohesive sets, these cohesive sets might prevent the system from converging to consensus.

The concepts of decisive/indecisive links are novel. A link from $i$ to $j$ in the influence network $G(W)$ is indecisive if there is no circumstances under which the opinion of $j$ makes any difference to the update opinion of $i$, and is decisive otherwise. Their formal definitions are given as follows: Given an influence network $G(W)$ with the node set $V$, define the out-neighbor set of each node $i$ as $N_i = \{j \in V | w_{ij} \neq 0\}$. A link $(i,j)$ is a decisive out-link of node $i$, if there exists a subset $\theta \subset N_i$ such that the following three conditions hold:

1) $j \in \theta$; 2) $\sum_{k \in \theta} w_{ik} \geq \frac{1}{2}$; 3) $\sum_{k \in \theta \setminus \{j\}} w_{ik} < \frac{1}{2}$.

Otherwise, the link $(i,j)$ is an indecisive out-link of node $i$. Visualized examples of decisive and indecisive links are provided in Fig. 6(b).

6.2 Set of equilibria

Recall from Section III that our weighted-median opinion dynamics are derived from a network-game set-up, where the individuals are the players, with the opinions they take as their
strategies, and, for each individual \( i \), the cost function is their cognitive dissonance, given by

\[
C_i(x_i, x_{-i}) = \sum_{j=1}^{n} w_{ij} |x_i - x_j|.
\]

Here \( x_{-i} \) denotes all the other individuals’ opinions. In this subsection, we establish that the equilibria of the weighted-median opinion dynamics are equivalent to the Nash equilibria of this network game. Moreover, we show that the possible configurations of these equilibria are determined by the cohesive sets in the influence network. Consider the weighted-median opinion dynamics on an influence network \( G(W) \) with \( n \) individuals and let \( \mathbb{R}^n \) be the set of all the \( n \)-dimensional vectors of real numbers. An opinion vector \( x^* \in \mathbb{R}^n \) is an equilibrium of the weighted-median opinion dynamics if \( x^*_i = \text{Med}_i(x^*; W) \) for any \( i \in \{1, \ldots, n\} \), i.e., no individual can change their opinion via the weighted-median mechanism. For the corresponding network game, \( x^* \) is a Nash equilibrium if, for any \( i \in \{1, \ldots, n\} \), the inequality

\[
C_i(x^*_i, x^*_{-i}) \leq C_i(x_i, x^*_{-i})
\]

holds for any \( x_i \in \mathbb{R} \). Given any generic influence matrix \( W \), the following statements are equivalent:

(i) \( x^* \in \mathbb{R}^n \) is an equilibrium of the weighted-median opinion dynamics;

(ii) \( x^* \) is an Nash equilibrium of the corresponding network game;

(iii) \( x^* \) is either a consensus state, i.e., \( x^*_i = x^*_j \) for any \( i \) and \( j \); or satisfy the following condition: for any \( y \) with \( \min_i x^*_i \leq y < \max_i x^*_i \), both the node set

\[
\{ i \in \{1, \ldots, n\} \mid x^*_i \geq y \}
\]

and the node set

\[
\{ i \in \{1, \ldots, n\} \mid x^*_i < y \}
\]

are maximal cohesive sets on \( G(W) \).

The proof is provided in Appendix B. Actually, one could infer from this proof that the equivalence between statement (i) and statement (iii) holds for any row-stochastic matrix \( W \). Statement (iii) above explicitly characterizes how the influence network structure confines the possible configurations of the equilibria of the weighted-median opinion dynamics. Apparently, any consensus state \( x^* \) is an equilibrium, referred to as a consensus equilibrium; For any \( x^* \) that is not a consensus states, in order for \( x^* \) to be an equilibrium of the weighted-median opinion dynamics, it must satisfy the following constraint: As long as the node set of \( G(W) \) is partitioned into two disjoint “factions” \( V_1 \) and \( V_2 \) such that the opinion \( x^*_i \) of any individual \( i \in V_1 \) is smaller than the opinion \( x^*_j \) of any individual \( j \in V_2 \), the sets \( V_1 \) and \( V_2 \) must both be maximal cohesive sets. In this case, \( x^* \) is referred to as a disagreement equilibrium.

6.3 Convergence and consensus-disagreement phase transition

Given the influence network \( G(W) \), denote by \( G_{\text{decoive}}(W) \) the influence network with all the indecisive out-links in \( G(W) \) removed. In addition, we say a node on a given network is globally reachable if any other node on this network has at least one directed path connecting to this node. The main results on the dynamical behavior of the weighted-median model are summarized as follows: Consider the weighted-
median opinion dynamics on an influence network $G(W)$ with the node set $V = \{1, \ldots, n\}$. The following statements hold:

(i) For any initial condition $x_0 \in \mathbb{R}^n$, the solution $x(t)$ almost surely reaches an equilibrium $x^*$ in finite time;

(ii) If the only maximal cohesive set of $G(W)$ is $V$ itself, then, for any initial condition $x_0 \in \mathbb{R}^n$, the solution $x(t)$ almost surely converges to a consensus equilibrium;

(iii) If $G(W)$ has a maximal cohesive set $M \neq V$, then there exists a subset of initial conditions $X_0 \subset \mathbb{R}^n$, with non-zero measure in $\mathbb{R}^n$, such that for any $x_0 \in X_0$ there is no update sequence along which the solution converges to a consensus equilibrium;

(iv) If $G_{\text{decisive}}(W)$ does not have a globally reachable node, then, for any generic initial condition $x_0 \in \mathbb{R}^n$, of which the values of its entries are all different, the solution $x(t)$ almost surely reaches a disagreement equilibrium in finite time.

The key to the proof is a so-called “monkey-typewriter” argument. That is, to put it in a vivid way, a monkey hitting keys at random on a typewriter keyboard for an infinite amount of time will almost surely type any given text, such as the complete works of William Shakespeare. This idea has proved to be a quite useful mathematical method in analyzing dynamical processes with uncertainty by transforming them to control design problems, e.g., see\textsuperscript{37}. According to the definition of the weighted-median opinion dynamics, at each time step, one individual is randomly picked and updates their opinion. Therefore, the system almost surely converges to an equilibrium in finite time as long as one can manually construct an update sequence for each initial state such that, along the constructed update sequence, the system reaches an equilibrium in finite time. Based on this argument, we first discuss the construction of update sequences when there exist only two different opinions in the network, and then extend the analysis to the general case with generic initial opinions. The detailed proof is provided in Appendix C.

The weighted-model exhibits more sophisticated phase-transition behavior between asymptotic consensus and persistent disagreement, while many averaging-based models, e.g., the DeGroot model, the DeGroot model with absolutely stubborn individuals, and the Friedkin-Johnsen model, predict either almost-sure consensus or almost-sure disagreement. Moreover, different from the DeGroot model, in which the consensus-disagreement phase transition is determined only by the network connectivity, in the weighted-median model, such a phase transition depends on the initial condition as well as a more delicate network structure, i.e., the non-trivial maximal cohesive sets. Compared to network connectivity, the non-existence of non-trivial maximal cohesive set implies a more strict and thereby more realistic condition for almost-sure consensus.

Compared with the DeGroot model, our weighted-median model enjoys higher robustness to structural changes, i.e., perturbations of influence networks coming from random noises or model imprecision. For the DeGroot model, one infinitesimal perturbation, e.g., adding one social link with very small weight, could completely change the connectivity property of the influence network and thus the prediction about consensus or disagreement. In the weighted-median model, in generic cases, adding one link with very small weight has no effect on the system’s dynamical behavior, since very likely the added link will be an indecisive link. See Fig. 6(c) and 6(d) for an example showing the resilience of the weighted-median model and DeGroot model to network perturbation.

7 Discussion and Conclusion

Occam’s razor in opinion dynamics: The weighted-median opinion dynamics model (2) is a splendid application of the principle of the Occam’s razor in social science (One way to state the principle of Occam’s razor is that “Entities should not be multiplied unnecessarily.”) In terms of microscopic mechanism, the weighted-median model is as simple as the classic DeGroot model. Despite its simplicity in form, the weighted-median model replicates various realistic features of opinion dynamics, which DeGroot model and its widely studied more complex extensions fail to fully capture, such as vulnerability of socially marginalized individuals to opinion radicalization, the formation of various steady public opinion distributions, and the effects of group size and clustering on the likelihood of reaching consensus.

Broader applicability and fundamental advantage in the representation of opinions: Our new model broadens the applicability of opinion dynamics to the scenarios of ordered multiple-choice issues. The weighted median operation is well-defined as long as opinions are ranked and the weighted-median opinions are always chosen among the opinions of the individuals’ social neighbors. Therefore, the opinion evolution is discrete and the “ordered multiple choices” are preserved. Debates and decisions about ordered multiple-choice issues are prevalent in reality. For example, in modern societies, many political issues are evaluated along one-dimension ideology spectra and political solutions often do not lend themselves to a continuum of viable choices. At a fundamental level, our weighted-median model has an advantage that it is independent of numerical representations of opinions. Such representations may be non-unique and artificial for any issue where the opinions are not intrinsically quantitative. Obviously, a nonlinear opinion rescaling leads to major changes in the evolution of the averaging-based opinion dynamics. It is notable that the human mind often perceives and manipulates quantities in a nonlinear fashion, e.g., the perception of probability according to prospect theory\textsuperscript{38}.

Influence networks with state-dependent weights: In the
classic DeGroot model and its widely-studied extensions, link weights in influence networks are usually assumed to be fixed and independent of the opinion evolution. With fixed weights, the weighted-averaging mechanism leads to the implication that attractiveness of opinions are proportional to opinion distances. One natural way to resolve this unrealistic feature is considering weighted-averaging models with state-dependent weights, e.g., weights that somehow decrease with the opinion distance. In terms of sociological interpretation, fixed weights $w_{ij}$ may describe a stable social structure among individuals and be therefore exogenous to the opinion formation process, while state-dependent weights may be formed upon listening to the arguments of the individuals and be therefore endogenous. The cognitive mechanisms leading to the establishment of endogenous weights are wide-ranging, complex, and in general hard to model, e.g., see the paper [39]. As shown by theoretical analysis in last section, our weighted-median model exhibits a robustness to the network weights. Thus, it is less sensitive to state-dependent or uncertain graphs. In addition, the weighted-median model itself can be interpreted as a special weighted-averaging mechanism, in which the weights are highly non-linear functions of individuals’ current states. That is, at any time, each individual assign all their weights to the social neighbor that currently sits right in the weighted-median position and assign zero weight to any other social neighbor’s opinion.

A new line of research inspired by the weighted-median model: The weighted-median opinion dynamics proposed in this paper could inspire the readers to rethink the microfoundation of opinion dynamics and open up a new line of research on the mathematical modeling of opinion formation processes. All the previous meaningful extensions of the classic DeGroot, e.g., persistent attachments to initial opinions, time-varying graphs, and antagonistic relations, can be introduced to the weighted-median model to further improve its predictive power and enrich its dynamical behavior. In addition, since the weighted-median mechanism with inertia exhibits remarkably high accuracy in quantitatively predicting individual opinion shifts, it would be of great research value to study the properties and efficient estimations of individual inertia, as well as the dynamical behavior of the weighted-median opinion dynamics with inertia.

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Appendix

A Properties of cohesive sets

Before presenting the properties of cohesive sets, we first define another related concept, namely the cohesive expansion. Given an influence network $G(W)$ with node set $V$ and a subset of nodes $M \subseteq V$, the cohesive expansion of $M$, denoted by $\text{Expansion}(M)$, is the subset of $V$ constructed via the following iteration algorithm: Let $M_0 = M$:

(i) For $k = 0, 1, 2, \ldots$, if there exists $i \in V \setminus M_k$ such that $\sum_{j \in M_k} w_{ij} > 1/2$, then let $M_{k+1} = M_k \cup \{i\}$;

(ii) Terminate the iteration at step $k$ as long as there does not exist any $i \in V \setminus M_k$ such that $\sum_{j \in M_k} w_{ij} > 1/2$, and let $\text{Expansion}(V) = M_k$.

The following lemma presents some important properties of cohesive sets and cohesive expansions.

Lemma A.1 (Properties of cohesive sets/expansions). Given an influence network $G(W)$ with node set $V$, the following statements hold:

(i) For any $M \subseteq V$, the cohesive expansion of $M$ is unique, i.e., independent of the order of node additions;

(ii) For any $M, \tilde{M} \subseteq V$, if $M \subseteq \tilde{M}$, then $\text{Expansion}(M) \subseteq \text{Expansion}(\tilde{M})$;

(iii) For any $M, \tilde{M} \subseteq V$, $\text{Expansion}(M) \cup \text{Expansion}(\tilde{M}) \subseteq \text{Expansion}(M \cup \tilde{M})$;

(iv) If $M \subseteq V$ is a cohesive set, then $\text{Expansion}(M)$ is also cohesive and is the smallest maximal cohesive set that contains $M$, that is, for any maximal cohesive set $\tilde{M}$ such that $M \subseteq \tilde{M}$, we have $\text{Expansion}(M) \subseteq \tilde{M}$;

(v) If $M \subseteq V$ is a cohesive set, then either $\text{Expansion}(M) = V$ or both $\text{Expansion}(M)$ and $V \setminus \text{Expansion}(M)$ are non-empty and maximally cohesive.

Proof. For any cohesive set $M \subseteq V$, suppose that $E_1 = M \cup \{i_1, \ldots, i_k\}$ and $E_2 = M \cup \{j_1, \ldots, j_l\}$ are both cohesive expansions of $M$ and $E_1 \neq E_2$. Here $(i_1, \ldots, i_k)$ means the ordered set containing $i_1, \ldots, i_k$. If $E_1 \subseteq E_2$, let $s = \min \{r \mid j_r \notin (i_1, \ldots, i_k)\}$ and then we have $M \cup \{j_1, \ldots, j_{s-1}\} \subseteq E_1$ (For convenience we let $(j_1, \ldots, j_{s-1}) = \emptyset$ if $s = 1$). According to the expansion of $M$ to $E_2$, we have

$$\sum_{r \in E_1} w_{jr} \geq \sum_{r \in M \cup \{j_1, \ldots, j_{s-1}\}} w_{jr} > 1/2.$$ 

Therefore, $E_1$ can be further expanded to $E_1 \cup \{j_s\}$, which contradicts the assumption that $E_1$ is already a cohesive expansion of $M$. We conclude that $E_1 \subseteq E_2$ can not be true. Following the same argument, we have that $E_2 \subseteq E_1$ can not be true. Since neither $E_1 \subseteq E_2$ nor $E_2 \subseteq E_1$ is true, there exists $j_{s_0}$, where $s_0 \in \{1, \ldots, l\}$, such that $j_{s_0} \notin (i_1, \ldots, i_k)$. First of all,
all, \( s_0 \) cannot be 1, otherwise
\[
\sum_{r \in E_1} w_{j_1,r} \geq \sum_{r \in M} w_{j_1,r} > 1/2
\]
implies that \( E_1 \) can be further expanded to \( E_1 \cup \{ j_1 \} \). Secondly, there must exist \( s_1 \in \{ 1, \ldots, s_0 - 1 \} \) such that \( j_i \notin (i_1, \ldots, i_k) \), otherwise \( M \cup (j_1, \ldots, j_{s_0-1}) \subseteq E_1 \) and
\[
\sum_{r \in E_1} w_{j_{s_0},r} \geq \sum_{r \in M \cup (j_1, \ldots, j_{s_0-1})} w_{j_{s_0},r} > 1/2,
\]
which implies that \( E_1 \) can be further expanded to \( E_1 \cup \{ j_{s_0} \} \). As the same argument goes on, we will obtain that \( j_{s_1} \notin (i_1, \ldots, i_k) \). But we have already shown that \( j_1 \notin (i_1, \ldots, i_k) \) cannot be true. Therefore, it must not hold that \( E_1 \neq E_2 \). This concludes the proof of Statement (i).

For any set of nodes \((i_1, \ldots, i_k)\) and node \( i_{k+1} \), let \( V_k = M \cup (i_1, \ldots, i_k) \) and \( \tilde{V}_k = \tilde{M} \cup (i_1, \ldots, i_k) \). Suppose \( M \subseteq \tilde{M} \). If \( \sum_{j \in \tilde{V}_k} w_{j,i_{k+1}} > 1/2 \), then, since \( M \subseteq \tilde{M} \), we have
\[
\sum_{j \in \tilde{V}_k} w_{i_{k+1},j} = \sum_{j \in V_k} w_{i_{k+1},j} + \sum_{j \in \tilde{M} \setminus M} w_{i_{k+1},j} > 1/2.
\]
Therefore, \( \text{Expansion}(M) \subseteq \text{Expansion}(\tilde{M}) \). This concludes the proof of Statement (ii).

According to Statement (ii), since \( M \subseteq M \cup \tilde{M} \) and \( \tilde{M} \subseteq M \cup \tilde{M} \), we have \( \text{Expansion}(M) \subseteq \text{Expansion}(M \cup \tilde{M}) \) and \( \text{Expansion}(\tilde{M}) \subseteq \text{Expansion}(M \cup \tilde{M}) \). Therefore, \( \text{Expansion}(M) \cap \text{Expansion}(\tilde{M}) \subseteq \text{Expansion}(M \cup \tilde{M}) \). This concludes the proof of Statement (iii).

If \( M \) is cohesive, for any \( i \in M \), obviously we have
\[
\sum_{k \in \text{Expansion}(M)} w_{ik} \geq \sum_{k \in M} w_{ik} \geq \frac{1}{2}.
\]
For any \( i \in \text{Expansion}(M) \setminus M \), if any, suppose the node \( i \) is added at some step \( t \) in the cohesive expansion process. We have
\[
\sum_{k \in \text{Expansion}(M)} w_{ik} \geq \sum_{k \in M_{t-1}} w_{ik} \geq \frac{1}{2},
\]
where \( M_{t-1} \) is as defined in the definition of cohesive expansions. This proves the statement that \( \text{Expansion}(M) \) is cohesive. From the definitions of maximal cohesive sets and cohesive expansions, a cohesive set \( \tilde{M} \) is maximal if and only if \( \text{Expansion}(\tilde{M}) = \tilde{M} \). Consider a cohesive set \( M \) and a maximal cohesive set \( \tilde{M} \) such that \( M \subseteq \tilde{M} \). By statement (ii) and the previous observation, we have \( \text{Expansion}(M) \subseteq \text{Expansion}(\tilde{M}) = \tilde{M} \), which concludes the proof of statement (iv).

The proof of statement (v) is straightforward by definitions of cohesive expansion and maximal cohesive set. □

**B Set of equilibria of the weighted-median opinion dynamics**

Consider any generic influence matrix \( W = (w_{ij})_{n \times n} \). Here by “generic” we mean that, for any \( i \in \{ 1, \ldots, n \} \), there does not exist any node subset \( \theta \subset \{ 1, \ldots, n \} \) such that \( \sum_{j \in \theta} w_{ij} \) is exactly 1/2. This condition almost surely holds if the weights \( w_{ij} \) are randomly generated from some continuous distributions, or are perturbed by some continuous random noises. By carefully examining the definition of weighted median, one could conclude that, for any such generic \( W \) and any opinion vector \( x \in \mathbb{R}^n \), the weighted-median opinion \( \text{Med}_t(x;W) \) is unique. Moreover, according to Lemma 3.1 in\(^{19}\),

\[
\text{Med}_t(x;W) = \arg\min_{y} \sum_{j=1}^{n} w_{ij} |z - x_j|, \quad \text{for any} \ i.
\]

Therefore, given any \( x^* \in \mathbb{R}^n \), we have that \( x^*_i = \text{Med}_t(x^*;W) \) for any \( i \) if and only if \( x^*_i = \arg\min_{y} \sum_{j=1}^{n} w_{ij} |z - x_j| \) for any \( i \), i.e., \( C(x^*_i, x^*_{-i}) \leq C(x_i, x^*_{-i}) \) for any \( i \). This leads to the equivalence between statements (i) and (ii) in Section VI.B.

Now we prove the equivalence between statements (i) and (iii) in Section VI.B. We first show that statement (iii) leads to statement (i). If \( x^* \) is a consensus vector, apparently \( x^*_i = \text{Med}_t(x^*;W) \) for any \( i \). Now consider the case when \( x^* \) is not a consensus vector but satisfies that, for any \( y \in (\min_i x^*_i, \max_i x^*_i) \), \( \{ j \mid x^*_j < y \} \) and \( \{ j \mid x^*_j \geq y \} \) are both maximal cohesive sets. For any given \( i \), from the definition of \( \text{Med}_t(x^*;W) \), one could infer that, for any \( y \in \mathbb{R} \),
\[
\sum_{j \mid x^*_j \leq y} w_{ij} \geq 1/2 \quad \Leftrightarrow \quad \text{Med}_t(x^*;W) \geq y,
\]
and
\[
\sum_{j \mid x^*_j < y} w_{ij} \geq 1/2 \quad \Leftrightarrow \quad \text{Med}_t(x^*;W) < y.
\]
Now let \( y = x^*_i \). Since \( i \in \{ j \mid x^*_j \geq x^*_i \} \) and \( \{ j \mid x^*_j \geq x^*_i \} \) is a maximal cohesive set, we have
\[
\sum_{j \mid x^*_j \geq x^*_i} w_{ij} \geq 1/2 \quad \Rightarrow \quad \text{Med}_t(x^*;W) \geq x^*_i.
\]
Let \( \bar{y} = \min \{ x^*_i \mid \text{any} \ k \text{ such that } x^*_k > x^*_i \} \). Since \( i \in \{ j \mid x^*_j < \bar{y} \} \), which is a maximal cohesive set, we have
\[
\sum_{j \mid x^*_j < \bar{y}} w_{ij} \geq 1/2 \quad \Rightarrow \quad \text{Med}_t(x^*;W) < \bar{y}.
\]
Since \( x^*_i \leq \text{Med}_t(x^*;W) < \bar{y} \) leads to \( x^*_i = \text{Med}_t(x^*;W) \), we have that statement (iii) implies statement (i).

Now we prove by contradiction that statement (i) implies statement (iii). Suppose \( x^* \) is not a consensus vector and there
exists \( y \in (\min x_i^*, \max x_i^*) \) such that either \( \{ j|x_j^* < y \} \) or \( \{ j|x_j^* \geq y \} \) is not a maximal cohesive set. Since these two sets form a disjoint partition of the node set \( \{1, \ldots, n\} \), one of them must not be cohesive. Without loss of generality, suppose \( \{ j|x_j^* \geq y \} \) is not cohesive. As a direct consequence, there exists \( i \) with \( x_i^* \geq y \) but

\[
\sum_{j:x_j^*<y} w_{ij} > \frac{1}{2},
\]

which in turn implies that \( \text{Med}_i(x^*; W) < y \leq x_i^* \). Therefore, such \( x^* \) cannot be an equilibrium of the weighted-median opinion dynamics. This concludes the proof that statements (i) and (iii) are equivalent.

### C Analysis of convergence and consensus-disagreement phase transition

As mentioned in Section VI.C, the main analytical results on the convergence and the consensus-disagreement phase transition of the weighted-median opinion dynamics are obtained by leveraging the so-called “monkey-typewriter” argument, formalized as the following lemma.

**Lemma C.1** (Transforming randomness to control design). Consider the weighted-median opinion dynamics defined in Section III. If, for any \( x \), there exists some \( T_x \in \{1,2,\ldots\} \) and some update order \( i_1, \ldots, i_T \) such that the solution to the weighted-median opinion dynamics starting from \( x \) reaches an equilibrium at time step \( T_x \) by adopting this update order, then the solution to the weighted-median opinion dynamics almost surely converges to an equilibrium in finite time, for any initial condition \( x(0) \).

**Proof.** For any given \( x(0) \in \mathbb{R}^n \), due to the definition of weighted-median, we have \( x(t) \in \Omega = \{ x_1(0), \ldots, x_n(0) \}^n \) along any update sequence. Here \( \Omega \) is a finite set of \( n \)-dimensional vectors in \( \mathbb{R}^n \). Since, for any \( x \in \Omega \),

\[
P[x(t+1) = x^{(i)} | x(t) = x] = 1/n
\]

for any \( x^{(i)} \in \Omega \) satisfying \( x^{(i)}_i = \text{Med}_i(x; W) \) and \( x^{(i)}_j = x_j \) for any \( j \neq i \), the weighted-median opinion dynamics is a Markov chain over the finite state space \( \Omega \). This Markov chain has absorbing states, e.g., all the consensus states. Moreover, for any \( x \in \Omega \), there exists at least one update sequence along which the trajectory \( x(t) \) starting from \( x \) reaches a fixed point. Therefore, the weighted-median opinion dynamics is an absorbing Markov chain. According to Theorem 11.3 in the textbook\(^4\), \( x(t) \) starting from \( x(0) \) almost surely converges to a fixed point. Since the stochastic process \( x(t) \) is a finite-state Markov chain, \( x(t) \) reaches a fixed point almost surely in finite time.

In what follows, we prove statements (i)-(iv) in Section VI.C. Firstly, according to Lemma C.1, the following two claims are equivalent:

1. For any initial state \( x(0) \), the solution \( x(t) \) to the weighted-median opinion dynamics almost surely converges to an equilibrium state \( x^* \) in finite time;
2. For any initial state \( x(0) \), there exists an update sequence \( \{i_1, \ldots, i_T\} \) such that the solution \( x(t) \) reaches an equilibrium after \( T \) steps of update if node \( i_t \) is updated at time step \( t \) for any \( t \in \{1, \ldots, T\} \).

Now we prove that claim (2) is true. We first consider the case in which there are only two different opinions initially in the network. Without loss of generality, let the two opinions be \( y_1 \) and \( y_2 \). Due to the weighted-median update rule, for any initial state \( x(0) \in \{y_1, y_2\}^n \), the solution \( x(t) \) satisfies \( x(t) \in \{y_1, y_2\}^n \) for any \( t \geq 0 \).

For any non-negative integer \( t \). We neglect the trivial cases when \( V(0) = V \) or \( V(0) = V \), otherwise the system is already at fixed points. We construct an update sequence as follows:

(i) For any time step \( t+1, t = 0, 1, 2, \ldots \), if there exists some \( i_{t+1} \in V(t) \) such that

\[
\sum_{j \in V(t)} w_{i_{t+1}, j} > \frac{1}{2},
\]

then update node \( i_{t+1} \) at time step \( t+1 \) and thereby

\[
V(t+1) = V(t) \setminus \{i_{t+1}\} \quad \text{and} \quad V(t+1) = V(t) \cup \{i_{t+1}\};
\]

(ii) The update stops at time step \( T \) if there does not exists any \( i \in V(T) \) such that \( \sum_{j \in V(T)} w_{ij} > 1/2 \).

By updating the system along the sequence \( \{i_1, \ldots, i_T\} \) we obtain two sets \( V_1(T) \) and \( V_2(T) \), with \( V_1(T) = V \setminus V_2(T) \), and all the individuals in \( V_1(T) \) \( V_2(T) \) resp.) hold the opinion \( y_1 \) \( y_2 \) resp.). Note that \( V_2(T) \) is the cohesive expansion of \( V(0) \). However, since \( V_2(T) \) is not necessarily cohesive, \( V_2(T) \) is not necessarily cohesive.

If \( V_1(T) \) is empty, then the system is already at a fixed point where all the nodes hold opinion \( y_2 \). If \( V_1(T) \) is not empty, then, for any \( i \in V_1(T) = V \setminus V_2(T) \), since \( V_2(T) \) is already the cohesive expansion of \( V(0) \), we have \( \sum_{j \in V_2(T)} w_{ij} \leq 1/2 \), which implies that

\[
\sum_{j \in V_1(T)} w_{ij} = \sum_{j \in V_2(T)} w_{ij} = 1 - \sum_{j \in V_2(T)} w_{ij} \geq 1/2.
\]

Therefore, \( V_1(T) \) is cohesive. Denote by \( E_1 = V_1(T) \cup \{j_1, \ldots, j_k\} \)
the cohesive expansion of $V_1(T)$, and the nodes are added to $V_1(T)$ along the sequence $j_1, \ldots, j_k$. Now we obtain the update sequence $i_1, \ldots, i_{r}, j_1, \ldots, j_k$. If $E_1 = V$, then along the update sequence $i_1, \ldots, i_{r}, j_1, \ldots, j_k$ the system reaches the fixed point where all the nodes adopt opinion $y_1$. If $E_1 \neq V$, then along such update sequence the system reaches the state in which all the nodes in $E_1$ adopt opinion $y_1$ while all the nodes in $V \setminus E_1$ adopt opinion $y_2$. According to Lemma A.1, $E_1$ and $V \setminus E_1$ are both maximally cohesive sets. Therefore, the system reaches a fixed point along the update sequence $i_1, \ldots, i_{r}, j_1, \ldots, j_k$.

Now we consider the case of any arbitrary initial condition $x_0 \in \mathbb{R}^n$. Without loss of generality, suppose there are $r \leq n$ different values among the entries of $x_0$, denoted by $\{y_1, \ldots, y_r\}$ with $y_1 < y_2 < \cdots < y_r$. Define two subsets of opinions $A_1 = \{y_1\}$ and $B_1 = \{y_2, \ldots, y_r\}$. Due to the weighted-median update rule, whether a node switch from state $A_1$ to $B_1$ only depends on which neighbors of this node are in state $B_1$. It is irrelevant what specific opinions in $B_1$ those neighbors hold. Therefore, repeating the argument in the two-opinion case, along some update sequence $i_{11}, \ldots, i_{1k_1}$, the system reach a state in which the nodes are divided into two nodes sets $E_1$ and $V \setminus E_1$. Due to Lemma A.1, all the nodes in $E_1$ hold the opinion $y_1$ and $E_1$ is a maximal cohesive set. Therefore, after the update sequence $i_{11}, \ldots, i_{1k_1}$, nodes in $E_1$ never switch their opinion from $y_1$ to the other opinions, while nodes in $V \setminus E_1$ never switch their opinions to $y_1$.

Let $A_2 = \{y_1, y_2\}$ and $B_2 = \{y_3, \ldots, y_r\}$. Since the set of nodes that hold opinion $y_1$ no longer changes after the update sequence $i_{11}, \ldots, i_{1k_1}$, for all the nodes in $V \setminus E_1$, it makes no difference to their opinion updates whether the nodes in $E_1$ hold opinion $y_1$ or $y_2$. Therefore, in the sense of determining the behavior of the nodes in $V \setminus E_1$, the opinions $y_1$ and $y_2$ can be considered as the same opinion. As the consequence and following the same line of argument in the previous paragraph, there exists another update sequence $i_{21}, \ldots, i_{2k_2}$, right after the sequence $i_{11}, \ldots, i_{1k_1}$, such that, after these two sequences of updates, the nodes are partitioned into two sets $E_2$ and $V \setminus E_2$, where $E_2$ is the set of all the nodes that hold either opinion $y_1$ or opinion $y_2$, and $E_2$ is a maximal cohesive set.

Repeating the argument in the previous paragraph, we obtain the sets $E_1, \ldots, E_{r-1}$, which are all maximal cohesive sets, and the entire update sequence $i_{11}, \ldots, i_{1k_1}, \ldots, i_{r-1,1}, \ldots, i_{r-1,k_{r-1}}$.

Define $V_1 = E_1$ and

$$V_1 = E_1,$$
$$V_l = E_l \setminus \bigcup_{s=1}^{l-1} E_s, \text{ for any } l = 2, \ldots, r - 1;$$
$$V_r = V \setminus \bigcup_{s=1}^{r-1} E_s$$

The way we construct $E_1, \ldots, E_{r-1}$ implies that, after the update sequence $i_{11}, \ldots, i_{1k_1}, \ldots, i_{r-1,1}, \ldots, i_{r-1,k_{r-1}}$, the system reaches a state in which, for any $s \in \{1, \ldots, r\}$, all the nodes in $V_s$ hold the opinion $y_s$ and will not switch to any other opinion. Therefore, the system is at a fixed point. This concludes the proof of statement (i) in Section VI.C.

Now we proceed to prove statement (ii) in Section VI.C. If the only maximal cohesive set in $G(W)$ is $V$ itself, then according to Lemma A.1, the cohesive expansion of any cohesive set is $V$ itself. Therefore, for any initial condition, following the same construction of update sequences in the proof of statement (i), the system will end up being at a state in which all the nodes hold the same opinion, i.e., the consensus equilibrium. This concludes the proof of statement (ii).

Statement (iii) in Section VI.C is proved by constructing the set $X_0$ of initial conditions as

$$X_0 = \left\{ x_0 \in \mathbb{R}^n \left| \begin{array}{c}
\max_{j \in M} x_{0,j} < \min_{k \in V \setminus M} x_{0,k}, \\
\text{or } \min_{j \in M} x_{0,j} > \max_{k \in V \setminus M} x_{0,k} \end{array} \right. \right\}.$$
Supplementary Material

S1 Definition and uniqueness of weighted median

The formal definition of weighted median is given as follows:

Definition S1.1 (Weighted median). Given any n-tuple of real numbers \( x = (x_1, \ldots, x_n) \) and the associated n-tuple of nonnegative weights \( w = (w_1, \ldots, w_n) \), where \( \sum_{i=1}^{n} w_i = 1 \), the weighted median of \( x \), associated with the weights \( w \), is denoted by \( \text{Med}(x; w) \) and defined as the real number \( x^* \in \{x_1, \ldots, x_n\} \) such that

\[
\sum_{i : x_i < x^*} w_i \leq \frac{1}{2}, \quad \text{and} \quad \sum_{i : x_i > x^*} w_i \leq \frac{1}{2}.
\]

By carefully examining this definition, one could observe that, associated with certain specific weights \( w \), there might exist multiple weighted medians of \( x \) satisfying the definitions above. Here we point out the following facts:

Fact 1: The weighted median of \( x \) associated with \( w \) is unique if and only if there exists \( x^* \in \{x_1, \ldots, x_n\} \) such that

\[
\sum_{i : x_i < x^*} w_i < \frac{1}{2}, \quad \sum_{i : x_i = x^*} w_i > 0, \quad \text{and} \quad \sum_{i : x_i > x^*} w_i < \frac{1}{2}.
\]

In this case, \( x^* \) is the unique weighted median;

Fact 2: The weighted medians of \( x \) associated with \( w \) are NOT unique if and only if there exists \( z \in \{x_1, \ldots, x_n\} \) such that \( \sum_{i : x_i < z} w_i = \sum_{i : x_i > z} w_i = 1/2 \). Among all these weighted medians of \( x \), the smallest one, denoted by \( x^* \), satisfies

\[
\sum_{i : x_i < x^*} w_i < \frac{1}{2}, \quad \sum_{i : x_i = x^*} w_i > 0, \quad \text{and} \quad \sum_{i : x_i > x^*} w_i = \frac{1}{2},
\]

while the largest weighted median, denoted by \( x^\circ \), satisfies

\[
\sum_{i : x_i < x^\circ} w_i = \frac{1}{2}, \quad \sum_{i : x_i = x^\circ} w_i > 0, \quad \text{and} \quad \sum_{i : x_i > x^\circ} < \frac{1}{2}.
\]

Moreover, if there exists any \( \hat{x} \in \{x_1, \ldots, x_n\} \) such that \( x^* < \hat{x} < x^\circ \), then \( \hat{x} \) is also a weighted median and it must hold that \( \sum_{i : x_i = \hat{x}} w_i = 0 \).

For generic weights, e.g., if \( w_1, \ldots, w_n \) are independently randomly generated from some continuous probability distributions, the case in Fact 2 never occurs since almost surely there does not exist any \( \theta \in \{1, \ldots, n\} \) such that \( \sum_{i \in \theta} w_i = 1/2 \). Therefore, given generic weights \( w \), the weighted median of \( x \) is unique.

In order to avoid unnecessary mathematical complexity, we would like to make each individual’s opinion update well-defined and deterministic. Therefore, in the weighted-median opinion dynamics, we slightly change the definition of weighted median when it is not unique according to Definition S1.1. Consider a group of \( n \) individuals discussing certain issue. Denote by \( x_i(t) \) the opinion of individual \( i \) at time \( t \) and let \( x(t) \) be the \( n \)-tuple \( (x_1(t), \ldots, x_n(t)) \). The interpersonal influences are characterized by the influence matrix \( W = (w_{ij})_{n \times n} \), which is entry-wise non-negative and satisfies \( \sum_{j=1}^{n} w_{ij} = 1 \) for any \( i \in \{1, \ldots, n\} \). The formal definition of weighted-median opinion dynamics is given as follows:

Definition S1.2 (Weighted-median opinion dynamics). Consider a group of \( n \) individuals discussing on some certain issue, with the influence matrix given by \( W = (w_{ij})_{n \times n} \). The weighted-median opinion dynamics is defined as the following process: At each time \( t + 1 \), one individual \( i \) is randomly picked and update their opinion according to the following equation:

\[
x_i(t + 1) = \text{Med}_i(x(t); W),
\]

where \( \text{Med}_i(x(t); W) \) is the weighted median of \( x(t) \) associated with the weights given by the \( i \)-th row of \( W \), i.e., \((w_{i1}, w_{i2}, \ldots, w_{in})\). \( \text{Med}_i(x(t); W) \) is well-defined if such a weighted-median is unique. If the weighted-median is not unique, then let \( \text{Med}_i(x(t); W) \) be the weighted median that is the closest to \( x_i(t) \).
This set-up guarantees the uniqueness of $\text{Med}_i(x;W)$ since only one of the following 3 cases can occur when the weighted medians are not unique:

i) $x_i \leq x^*$, where $x^*$ is the smallest weighted median of $x$ associated with the weights $(w_1, \ldots, w_n)$. In this case, $\text{Med}_i(x;W) = x^*$ is unique;

ii) $x_i \geq x^*$, where $x^*$ is the largest weighted median of $x$ associated with the weights $(w_1, \ldots, w_n)$. In this case, $\text{Med}_i(x;W) = x^*$ is unique;

iii) $x^* < x_i < x^*$. According to Fact 2 for the weighted median in last paragraph, this must imply that $\sum_{j:x_j = x_i} w_{ij} = 0$ and $x_i$ is also a weighted median of $x$ associated with the weights $(w_1, \ldots, w_n)$. Therefore, in this case, $\text{Med}_i(x;W) = x_i$ is also unique.

Note that, if the entries of $W$ are randomly generated from some continuous distributions, then, for any subset of the links on the influence network $G(W)$, the sum of their weights is almost surely not equal to 1/2. As a consequence, the weighted median for each individual at any time is almost surely unique. Therefore, for generic influence networks, the weighted-median opinion dynamics defined by Definition S1.2 follows a simple rule and is consistent with the formal definition of weighted median given in Definition S1.1. In the rest of this article, by weighted-median opinion dynamics, or weighted-median model, we mean the dynamical system described by Definition S1.2. According to Definition S1.2, for any given initial condition $x(0) = (x_{0,1}, \ldots, x_{0,n})^T$, the solution $x(t)$ to the weighted-median opinion dynamics satisfies $x_i(t) \in \{x_{0,1}, \ldots, x_{0,n}\}$ for any $i \in \{1, \ldots, n\}$ and any $t \geq 0$. Moreover, according to Definition S1.2, for each node $i$,

$$x_i(t + 1) > x_i(t) \text{ if and only if } \sum_{j:x_j > x_i(t)} w_{ij} > 1/2, \text{ and } x_i(t + 1) < x_i(t) \text{ if and only if } \sum_{j:x_j < x_i(t)} w_{ij} > 1/2.$$

### S2 Absolute-Value Cognitive Dissonance Function and Weighted-Median Opinion Update

Consider an influence network $G(W)$ with $n$ individuals. Given the opinion vector $x$, each individual $i$’s cognitive dissonance generated by disagreeing with others can be modelled as

$$C_i(x_i, x_{-i}) = \sum_{j=1}^{n} w_{ij} |x_i - x_j|^\alpha,$$

and individual $i$’s opinion update can be modelled as the best response to minimize the cognitive dissonance $C_i(x_i, x_{-i})$. That is, the updated opinion of individual $i$, denoted by $x_i^+$, satisfies

$$x_i^+ = \arg\min_{z \in \mathbb{R}} \sum_{j=1}^{n} w_{ij} |z - x_j|^\alpha. \quad (S1)$$

We use equality here in the sense that the right-hand side of the equation above is unique for generic weights $w_{ij}$’s. The following proposition states the relation between the system given by equation (S1) and the weighted-median opinion update, when we set the value of the parameter $\alpha = 1$.

**Proposition S2.1** (Weighted-median update as best-response dynamics). *Given the row-stochastic influence matrix $W = (w_{ij})_{n \times n}$ and the vector $x = (x_1, \ldots, x_n)^T$, the following statements holds: for any $i \in \{1, \ldots, n\}$,*

i) *If there exists $x^* \in \{x_1, \ldots, x_n\}$ such that*

$$\sum_{j:x_j < x^*} w_{ij} < \frac{1}{2}, \text{ and } \sum_{j:x_j > x^*} w_{ij} < \frac{1}{2},$$

*then*

$$\text{Med}_i(x;W) = x^* = \arg\min_{z \in \mathbb{R}} \sum_{j=1}^{n} w_{ij} |z - x_j|;$$

ii) *If there does not exist such $x^*$, then the set*

$$M_i(x;W) = \left\{ y \in \{x_1, \ldots, x_n\} : \sum_{j:x_j \leq y} w_{ij} \leq \frac{1}{2}, \sum_{j:x_j > y} w_{ij} \leq \frac{1}{2} \right\}$$
answers and 30 observed 2nd-round (3rd-round resp.) answers regarding Hypothesis 1 and 2. For Hypothesis 1 and 2, since they are parameter-free, we directly apply them to predict the participants’ answers at the 2nd round given the participants’ answers at the 1st round. For Hypothesis 3 and 4, with the parameters $\tilde{\gamma}(t)$ and $\tilde{\beta}(t)$, characterize the effects of individual prejudice, i.e., the persistent attachment to initial opinions. We apply these hypotheses above to predict individuals’ answers at the $(t+1)$-th round given the participants’ answers at the $t$-th round, for $t = 1$ and 2 respectively. For Hypothesis 1 and 2, since they are parameter-free, we directly apply them to predict the participants’ answers at the $(t+1)$-th round based on their answers at the $t$-th round. For Hypothesis 3-6, in practice, for each participant $i$ in a given experiment, the parameters $\gamma(t)$, $\beta(t)$, $\tilde{\gamma}(t)$ and $\tilde{\beta}(t)$ are estimated by least-square linear regression based on her/his answers in the first 20 questions as the training set. Then these estimated parameters are used to predict the her/his answers in the remaining 10 questions. Therefore, for each participant in a given experiment, we obtain 30 predictions of the 2nd-round (3rd-round resp.) answers and 30 observed 2nd-round (3rd-round) answers regarding Hypothesis 1 and 2. For Hypothesis 3-6, we obtain 10 predictions of the 2nd-round (3rd-round resp.) answers and 10 observed 2nd-round (3rd-round) answers respectively.

Regarding the opinion shifts from the first round to the second round, Hypotheses 5 and 6 are equivalent to Hypotheses 3 and 4 respectively. For counting games, we randomly sample 18 experiments from the dataset, in which 71 participants give answers to all the 30 questions at each round. For each of these 71 participants, we apply Hypothesis 1-4 respectively to predict their answers to each question in the first round, and then compare the error rates of the predictions. The error rate is defined as:

$$\text{error rate} = \frac{\text{prediction} - \text{observed value}}{\text{observed value}}.$$ 

The results are presented in Panel (a) of Fig. S1. For gauging games, we randomly sampled 21 experiments, in which 55 participants answers all the 30 questions at each round. Since the answers to gauging games are already in percentages, we...
Figure S1. Empirical analysis results for the dataset collected in an online human-subject experiment. Here Hypothesis 1-6 correspond to median, average, median with inertia, average with inertia, median with prejudice, and average with prejudice, respectively, as defined in Section S4. The acronym “MAE” in these tables is short for “mean absolute-value error” and “MER” is short for “mean error rate”.

As the data analysis results indicate, in any of the three set-ups (parameter-free, inertia, prejudice), the model with median predicts the opinion shifts with smaller errors than the predictions by the model with average. Remarkably, as for the parameter-free models, the predictions by median enjoy significantly smaller median error (rates), mean error rate, and mean absolute-value error, compared with the predictions by average. For counting games, the predictions of the 2nd-round (3rd-round resp.) answers by median (i.e., Hypothesis 1) enjoy a 37.35% (46.36% resp.) lower median error rate than the corresponding predictions by average (i.e., Hypothesis 2). For gauging games, the predictions of the 2nd-round (3rd-round resp.) answers by median enjoy a 40.00% (50.00% resp.) lower median absolute-value error than the corresponding predictions by average.

In addition, the parameters $\gamma(t)$, $\tilde{\gamma}(t)$, $\beta(t)$, $\tilde{\beta}(t)$ in Hypothesis 3-6 and estimated by mean-square linear regression are not stable and thereby might not reflect any intrinsic personal attribute of the participants. We note that some individuals participated in multiple experiments and their parameters vary significantly among different experiments. For example, the parameter $\gamma(2)$ of an individual with anonymous ID 22 in three different experiments are 0.3052, 0.5158, and 0.976 respectively.

S4 Empirical data on steady multi-modal opinion distributions

Empirical observations indicate that, contrasting to the prediction of consensus by DeGroot model, persistent disagreement is quite common in social groups. Moreover, in large-scale social networks, we often observe steady-state opinion distributions and the distribution can be either uni-modal or multi-modal. Fig. S2 provide a longitudinal empirical data on European people’s attitude towards the effect of immigration of local culture. The data is obtained from the European Social Survey website: http://nesstar.ess.nsd.uib.no/webview/.
Figure S2. Longitudinal data of the distribution of European people’s attitudes, in the years of 2004, 2008, 2012 and 2016, towards the following statement: “Country’s cultural life is undermined by immigrants”. In the opinion spectrum, 0 stands for strongly agree, while 10 represents strongly disagree.

S5 Set-ups and Additional Results on the Numerical Comparisons

In this section we compare by simulations the differences in predictions between the weighted-median opinion dynamics and some of the extensions of the DeGroot model based on the weighted-average opinion updates. We focus on the following aspects of model predictions: (1) the relation between initial opinion distribution and the final steady opinion distribution; (2) the centrality distributions for opinions with distinct levels of extremeness; (3) the effects of group size and clustering on the probability of reaching consensus. The simulation results indicate that the weighted-median model predicts realistic features of opinion dynamics in all of those aspects, which can not be achieved by the other models without deliberately tuning their parameters.

S5.1 Set-up of the models in comparison

Before presenting the simulation results, we first specify what models we compare with the weighted-median opinion dynamics.

**DeGroot model with absolutely stubborn agents:** Since the assumption of absolute stubbornness is often too strong and there is no widely-accepted statistical result on the proportion of “absolutely stubborn individuals” in real society, we assume that the social system we consider has 5% absolutely stubborn agents. Given an influence network \( G(W) \) with no absolutely stubborn individuals, we randomly pick 5% of the individuals and let them be absolutely stubborn, i.e., for each of the picked individuals, let \( w_{ii} = 1 \) and \( w_{ij} = 0 \) for any \( j \neq i \).

**Friedkin-Johnsen model:** The equation for Friedkin-Johnsen model is given by

\[
x(t + 1) = AWx(t) + (I - A)x(0),
\]

where \( A = \text{diag}(a_1, \ldots, a_n) \). The Friedkin-Johnsen model itself does not specify what the values of \( a_1, \ldots, a_n \) are. We assume that each \( a_i \) is independently randomly generated from the uniform distribution \( \text{Unif}[0,1] \).

**The networked bounded-confidence model:** Given the influence network \( G(W) \) and the individual confidence radii \( r_1, \ldots, r_n \), the networked bounded-confidence model\(^{10}\) is given below:

\[
x_i(t + 1) = \frac{\sum_{j \in N_i; \mid x_j(t) - x_i(t)\mid < r_i} w_{ij}x_j(t)}{\sum_{j \in N_i; \mid x_j(t) - x_i(t)\mid < r_i} w_{ij}},
\]

for any \( i \). In addition, we assume that, if the initial opinions are randomly generated from the uniform distribution \( \text{Unif}[0,1] \), then the individual confidence radii are independently randomly generated from the uniform distribution \( \text{Unif}[0,0.5] \); if the initial opinions are randomly generated from the uniform distribution \( \text{Unif}[-1,1] \), then the individual confidence radii are independently randomly generated from the uniform distribution \( \text{Unif}[0,1] \). As a result, the most closed-minded individuals are absolutely stubborn and the most open-minded individuals are open to any opinion.

Since the Altafini model with negative weights is not based on the same concept of influence network as the other models mentioned in this article, it is not included in the comparison.
S5.2 Simulation study 1: centrality distribution for opinions with different levels of extremeness

We investigate the centrality distributions of opinions with different levels of extremeness predicted by all the models in comparison. Let the individual initial opinions be randomly generated from the uniform distribution \( \text{Unif}[-1, 1] \) and classify the opinions into four categories: the moderate opinions correspond to those in the interval \([-0.25, 0.25]\); the biased opinions correspond to those in \([-0.5, -0.25) \cup (0.25, 0.5]\); the radical opinions correspond to those in \([-0.75, -0.5) \cup (0.5, 0.75]\); the extreme opinions correspond to those in \([-1, -0.75) \cup (0.75, 1]\).

Figure S3. Comparisons among the weighted-median model, the Friedkin-Johnsen model, and the DeGroot model with absolutely stubborn agents, on their predictions of the two-dimension distributions of the final opinions, over the extremists focus and the indegree centrality. Panel (a) is Fig. 5 in a previous paper\(^2\), licensed under Creative Commons CC0 public domain dedication (CC0 1.0). This figure plots the empirical distribution of randomly sampled Twitter users over in-degree and the ISIS focus (the ratio of social neighbors who support the ISIS terrorists). Panel (b)-(d) are the three aforementioned models’ predictions respectively. Among these three models, only the two-dimension distribution predicted by the weighted-median model resembles the real data in Panel (a).

For the simulation presented in Fig. 3(a) in the main text, we construct 1000 realizations of the weighted-median opinion dynamics on the same scale-free network with 1500 nodes. The scale-free network is randomly generated according to the Barabási-Albert model\(^2\), with the degree distribution \( f_D[d] \sim ad^{-b} \), where \( a = 3866 \) with the 95% confidence bound \((3633, 4098)\) and \( b = -2.356 \) with the 95% confidence bound \((-2.429, -2.283)\). Each realization starts with a different randomly generated initial condition. For each individual, we compute the frequency of finally adopting an extreme opinion over the 1000 independent realizations.

For the simulation results presented in Fig. 3(b) in the main text, we construct a scale-free network with 2000 nodes and run 1000 independent simulations of the weighted-median opinion dynamics. The initial opinions are randomly generated from the uniform distribution on the interval \([-1, 1]\). For the final steady state in each simulation, we compute the extremists focus, defined as the ratio of neighbors adopting extreme opinions, and the indegree centrality for each individual. Then we plot the 2-dimension distributions over the extremists focus and the indegree for the extremists and the entire population respectively.

Here we further simulated the Friedkin-Johnsen model and the DeGroot model with absolutely stubborn agents on the
same scale-free network as in last paragraph. The reason why the networked bounded-confidence model is not included in this comparative numerical study is that the convergence time of the networked bounded-confidence model is too long for simulations on networks with 2000 nodes. We simulated the Friedkin-Johnsen model and the DeGroot model with absolutely stubborn agents on the same scale-free network as in last paragraph. For the Friedkin-Johnsen model, before each simulation, the model parameters, i.e., the individuals’ attachments to initial opinions, are randomly generated from the uniform distribution on \([0, 1]\). For the DeGroot model with absolutely stubborn agents, before each simulation, each individual has a 0.05 probability of being set to be absolutely stubborn. The two-dimension distributions for the final opinions over the extremists focus and the indegree centrality, of the weighted-median model, the Friedkin-Johnsen model, and the DeGroot model with absolutely stubborn agents are presented in Fig. S3.

The results presented in Fig. 3(d) in the main text is contained in Fig. S4, where we consider four types of centrality measure for the individuals in the influence network: the indegree centrality, the closeness centrality, the betweenness centrality, and the eigenvector centrality. Here the indegree centrality is defined as the sum of the weights of all the incoming links, including the self loop. We construct the simulations on scale-free networks with 1000 nodes and with the average degree equal to 4. The reason why we do not use small-world networks is that, the centrality distribution for small-world networks is not as heavy-tailed as scale-free networks, i.e., in small-world networks there are not enough individuals with very high centrality. We construct 500 realizations of different opinion dynamics models in comparison. For each realization we randomly generate a scale-free network with \(n = 1000\) nodes and randomly generate the initial opinions from the uniform distribution Unif\([-1, 1]\). Then we run different models and obtain their corresponding predicted final opinions. The probability density functions of individual centrality for the final opinion holders with different levels of extremeness are estimated based on the obtained data.

Simulation results shown in Fig. S4 indicate that, in the weighted-median model, the centrality distributions of different types of opinions are clearly separated, and, compared to the centrality distribution of the total population, the extreme opinions tend to concentrate more on the low-centrality nodes. Such features hold in the weighted-median model for indegree, closeness, and betweenness centralities, and are not observed in any of the other models.

Note that, according to the weighted-median mechanism, an individual is absolutely stubborn as long as their self weight is no less than 1/2, that is, this individual thinks that he or she is more important than all the other individuals together. Based on this observation, one might argue that, in the weighted-median model, individuals with fewer social neighbors are more vulnerable to extreme opinions just because they have higher likelihoods of being assigned no less than 1/2 self weights, when the link weights of the influence network are randomly generated, and as the consequence, they can never get rid of their initial opinions if they are extreme. In order to rule out such an effect of link-weight randomization, simulations with the same set-up as described in this subsection are done on a scale-free network with no self loop. The simulation results indicate that the same features presented in the previous paragraph are still preserved. See Fig. S5. Therefore, the tendency that relatively peripheral nodes in the influence network are more vulnerable to extreme opinions is not merely an effect of link-weight randomization, but due to some more profound effects related to both network structure and microscopic mechanism.

### S5.3 Simulation study 2: initial and final opinion distribution

In this numerical study, we compare the final steady opinion distributions predicted by different models under the same initial condition. We compare the model predictions on both the scale-free networks and small-world networks. The former are randomly generated according to the Barabási-Albert model\(^{27}\), while the latter are randomly generated according to the Watts-Strogatz small-world mode\(^{31}\). Given a randomly generated network, we add self loops to all the individuals. Weights are randomly assigned to all the links in the network and normalized such that, for each individual, the weights of their out-links sum up to 1. We consider five examples of initial opinion distributions: a uniform distribution, a uni-modal and symmetric distribution, an uni-modal and skewed distribution, a bi-modal distribution and a 3-modal distribution, defined as follows respectively:

1. **Regarding the uniform distribution**, we let the initial opinion of each individual be independently randomly sampled from the uniform distribution on \([0, 1]\), i.e., \(x_i(0) \sim \text{Unif}[0, 1]\) for any \(i \in \{1, \ldots, n\}\);
2. **Regarding the uni-modal distribution**, we let the initial opinion of each individual be independently randomly sampled from the Beta distribution Beta\((2, 2)\);
3. **Regarding the skewed distribution**, we let the initial opinion of each individual be independently randomly sampled from the Beta distribution Beta\((2, 7)\);
4. **Regarding the bimodal distribution**, each individual \(i\)'s initial opinion is independently generated in the following way: Firstly we generate a random sample \(Y\) from the Beta distribution Beta\((2, 10)\), and then let \(x_i(0) = Y\) or \(1 - Y\) with probability 0.5 respectively;
Regarding the 3-modal distribution, each individual i’s initial opinion is independently generated in the following way: Firstly we generate two random samples Y and Z from Beta(2, 17) and Beta(12, 12) respectively, and then let $x_i(0)$ be $Y$, $1 - Y$, or $Z$ with probabilities 0.33, 0.33, and 0.34 respectively.

For each initial opinion distribution, we randomly generate the initial opinion of each individual independently and let the models in comparison start with the same initial condition. When each of these models reaches a steady state, or is sufficiently close to a steady state, e.g., when $\sum_{i=1}^{n} (x_i(t + 1) - x_i(t))^2 < 0.001$, their final opinion distributions are computed respectively.

The randomly generated scale-free network is undirected and contains $n = 5000$ nodes (individuals). The distribution of individual degrees $d$ is $\text{Pr}[d] \sim ad^{-b}$, where $a = 12620$ with the 95% confidence bound $(12270, 12970)$ and $b = -2.333$ with the 95% confidence bound $(-2.367, -2.300)$. Simulation results shown in Fig. S6 indicate that our weighted-median opinion model is the only one that naturally generates various types of steady opinion distributions empirically observed in real society.

Numerical comparisons conducted on a small-world network, with average degree equal to 7 and the rewiring probability $\beta = 0.2$, indicates the same conclusion as on the scale-free network. See Fig. S7.

### S5.4 Simulation study 3: effects of group size and clustering on the probability of reaching consensus

In this subsection, we investigate the effects of group size and network clustering on the probability of reaching consensus. This numerical study is motivated by the everyday experience that it is usually more difficult for a large group, or a group containing many clusters, to reach consensus in discussions. Such phenomena is prominent but not predicted by any of the extensions of the DeGroot model: The DeGroot model itself always predicts consensus if the influence network satisfies some mild connectivity conditions. On the contrary, the DeGroot model with absolutely stubborn individuals predicts persistent disagreement whenever there are more than one absolutely stubborn individual holding different initial opinions. Similarly, the Friedkin-Johnsen model predicts persistent disagreement whenever there are more than one individuals with non-zero attachment to distinct initial opinions. Therefore, those models mentioned above are not eligible for comparison regarding the probability of reaching consensus. The only model we compare with the weighted-median model is the networked bounded-confidence model.

For the numerical study presented in Fig. 5 in the main text, we simulate different models on Watts-Strogatz small-world networks. This generative model has three parameters: the network size $n$, the individual degree $d$, and the rewiring probability $\beta$ of individuals’ out-links. When we investigate the effect of group size, we can fix the parameters $d$ and $\beta$ so that the network size changes without significantly changing the local structure of the network; When we investigate the effect of clustering, we can fix $n$, $d$ and change the parameter $\beta \in [0, 1]$. According to the Watts-Strogatz model, the smaller $\beta$, the more clustered the network is. For the simulations presented in Fig. 5(a) and 5(b) in the main text, we fix the rewiring probability as $\beta = 1$ and randomly generate small-world networks with different sizes and average degrees. For each pair of network size and average degree, we construct 5000 realizations. For each realization, different models start with the same initial condition that is independently randomly generated from the uniform distribution on $[0, 1]$. For each model we compute the frequency of finally achieving consensus over the 5000 realizations. For the simulations presented in Fig. 5(c) and 5(d) in the main text, we fix the network size as $n = 30$ and $n = 60$ respectively, and construct small-world networks with different rewiring probabilities $\beta$ and average degrees, as shown in the figures. For each pair of $\beta$ and average degree, we construct 5000 realizations of the weighted-median opinion dynamics (Fig. 5(c) in the main text) or the networked bounded-confidence model (Fig. 5(d) in the main text). Each realization starts with a different initial condition randomly sampled from the uniform distribution on $[0, 1]$. For each setting of the model, the rewiring probability, and the average degree, we compute the frequency of finally achieving consensus over the 5000 realizations.
The distributions are presented in the form of log probability density. Here the initial opinions be randomly generated from the interval $\left[-0.25, 0.25\right]$; the biased opinions correspond to those in $\left[-0.5, -0.25\right) \cup \left(0.25, 0.5\right]$; the radical opinions correspond to those in $\left[-0.75, -0.5\right) \cup \left(0.5, 0.75\right]$; the extreme opinions correspond to those in $\left[-1, -0.75\right) \cup \left(0.75, 1\right]$. 

**Figure S4.** Centrality distributions for moderate, biased, radical and extreme final opinions predicted by different models. The distributions are presented in the form of log probability density. Here the initial opinions be randomly generated from the uniform distribution $\text{Unif} \left[-1, 1\right]$ and classify the opinions into four categories: the moderate opinions correspond to those in the interval $\left[-0.25, 0.25\right]$; the biased opinions correspond to those in $\left[-0.5, -0.25\right) \cup \left(0.25, 0.5\right]$; the radical opinions correspond to those in $\left[-0.75, -0.5\right) \cup \left(0.5, 0.75\right]$; the extreme opinions correspond to those in $\left[-1, -0.75\right) \cup \left(0.75, 1\right]$.

Acronyms: **WM** = the weighted-median model; **DS** = the DeGroot model with absolutely stubborn agents; **F-J** = the Friedkin-Johnsen model; **NBC** = the networked bounded-confidence model.
Figure S5. Centrality distributions for moderate, biased, radical and extreme final opinions predicted by the weighted-median model, on a scale-free network with no self loop. The distributions are presented in the form of log probability density. The opinion spectrum is given by Panel (a). Panels (b)-(d) show the log probability distributions in terms of different measures of centrality.

Acronyms: WM = the weighted-median model; DS = the DeGroot model with absolutely stubborn agents; F-J = the Friedkin-Johnsen model; NBC = the networked bounded-confidence model.

Figure S6. Distributions of the initial opinions and the final opinions predicted by different models. The simulations are run on the same scale-free network with 5000 nodes.
Figure S7. Distributions of the initial opinions and the final opinions predicted by different models. The simulations are run on the same small-world network with 5000 nodes.
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