Weighted estimates for the bilinear maximal operator on filtered measure spaces

Wei Chen\textsuperscript{a}, Yong Jiao\textsuperscript{b}

\textsuperscript{a}School of Mathematical Sciences, Yangzhou University, Yangzhou 225002, China
\textsuperscript{b}School of Mathematics and Statistics, Central South University, Changsha 410075, China

Abstract
Assuming the bilinear reverse Hölder’s condition, we characterize weighted inequalities for the bilinear maximal operator on filtered measure spaces. We also obtain Hytönen-Pérez type weighted estimates for the bilinear maximal operator, which is even new in the linear case. Our approaches are mainly based on the new construction of bilinear versions of principle sets and the new Carleson embedding theorem on filtered measure spaces.

Keywords: Filtered measure space, Bilinear maximal operator, Weighted inequality, Reverse Hölder’s condition, Hytönen-Pérez type estimate.

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1. Introduction

Let \( \mathbb{R}^n \) be the \( n \)-dimensional real Euclidean space and \( f \) a real valued measurable function, the classical Hardy-Littlewood maximal operator is defined by

\[
Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|dy,
\]

where \( Q \) is a non-degenerate cube with its sides parallel to the coordinate axes and \( |Q| \) is the Lebesgue measure of \( Q \).

Let \( u, v \) be two weights, i.e., positive measurable functions. As is well known, for \( p \geq 1 \), Muckenhoupt \cite{21} showed that the inequality

\[
\lambda^p \int_{\{Mf>\lambda\}} u(x)dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)dx, \quad \lambda > 0, \quad f \in L^p(v)
\]

holds if and only if \( (u, v) \in A_p \), i.e., for any cube \( Q \) in \( \mathbb{R}^n \) with sides parallel to the coordinates

\[
\left( \frac{1}{|Q|} \int_Q u(x)dx \right) \left( \frac{1}{|Q|} \int_Q v(x)^{\frac{1}{p-1}}dx \right)^{p-1} < C, \quad p > 1;
\]

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Email addresses: weichen@yzu.edu.cn (Wei Chen), jiaoyong@csu.edu.cn (Yong Jiao)
\[
\frac{1}{|Q|} \int_{Q} u(x) dx \leq C \text{ess inf}_{Q} v(x), \quad p = 1.
\]

Suppose that \(u = v\) and \(p > 1\), Muckenhoupt \[21\] also proved that
\[
\int_{\mathbb{R}^n} (Mf(x))^p v(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \quad \forall f \in L^p(v)
\]
holds if and only if \(v\) satisfies
\[
\left( \frac{1}{|Q|} \int_{Q} v(x) dx \right) \left( \frac{1}{|Q|} \int_{Q} v(x)^{p-1} dx \right)^{p-1} < C, \quad \forall Q \subset \mathbb{R}^n.
\]

The crucial step is to show that if \(v\) satisfies \(A_p\), then there is an \(\varepsilon > 0\) such that \(v\) also satisfies \(A_{p-\varepsilon}\). However, the problem of finding all \(u\) and \(v\) such that
\[
\int_{\mathbb{R}^n} (Mf(x))^p u(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x) dx, \quad \forall f \in L^p(v)
\]
is much hard and complicated. In order to solve the problem, Sawyer \[25\] established the testing condition \(S_{p,q}\), i.e. for any cube \(Q\) in \(\mathbb{R}^n\) with sides parallel to the coordinates
\[
\left( \frac{1}{|Q|} \int_{Q} (M(\chi_Q v^{1-p'})(x))^q u(x) dx \right)^{1/q} \leq C \left( \frac{1}{|Q|} \int_{Q} v(x)^{1-p'} dx \right)^{1/p'}, \quad \forall Q \subset \mathbb{R}^n,
\]
where \(1 < p \leq q < \infty\) and \(\chi_Q\) is the characteristic function of \(Q\). The condition \(S_{p,q}\) is a sufficient and necessary condition such that the weighted inequality
\[
\left( \int_{\mathbb{R}^n} (Mf(x))^p u(x) dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) dx \right)^{1/p}, \quad \forall f \in L^p(v)
\]
holds. In this case, the method of proof is very interesting. Motivated by \[21\] and \[22\], the theory of weights developed so rapidly that it is difficult to give its history a full account here (see \[6\] and \[3\] for more information).

Weighted estimates for the maximal operator \(\prod_{j=1}^{m} M_{f_j}\) (\(m\)-fold product of \(M\)) in the multilinear setting were studied in \[10\] and \[24\]. Recently, the new multilinear maximal function
\[
\mathcal{M}(f_1, \ldots, f_m)(x) := \sup_{x \in \mathbb{R}^n} \prod_{i=1}^{m} \frac{1}{|Q|} \int_{Q} |f_i(y_i)| dy_i, \quad x \in \mathbb{R}^n
\]
associated with cubes with sides parallel to the coordinate axes was first defined and the corresponding weight theory was studied in \[17\]. The importance of this operator is that it is strictly smaller than the \(m\)-fold product of \(M\). Moreover, it generalizes the Hardy–Littlewood maximal function (case \(m = 1\)) and in several ways it controls the class of multilinear Calderón–Zygmund operators as shown in \[17\]. The relevant class of multiple weights for \(\mathcal{M}\) is given by the condition.
A [17, Definition 3.5]: for \( p = (p_1, p_2, \cdots, p_m) \), \( \omega = (\omega_1, \omega_2, \cdots, \omega_m) \) and a weight \( v \), the weight vector \((v, \omega) \in A_p^m\) if

\[
\sup_Q \left( \frac{1}{|Q|} \int_Q v(x) dx \right) \prod_{i=1}^m \left( \frac{1}{|Q_i|} \int_{Q_i} \omega_i(x) - \frac{1}{p_i} dx \right)^{\frac{1}{p_i}} < \infty,
\]

where \( \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} \) and \( 1 \leq p_1, p_2, \cdots, p_m < \infty \). The more general case was extensively discussed in [9, 8]. Using a dyadic discretization technique, Damián, Lerner and Pérez [6] and Li, Moen and Sun [18] proved some sharp weighted norm inequalities for the multilinear maximal operator \( \mathcal{M} \). In order to establish the generalization of Sawyer’s theorem to the multilinear setting, a kind of monotone property and a reverse Hölder’s condition on the weights were introduced in [19] and [3], respectively. Note that if \( v = \prod_{i=1}^m \omega_i^{\frac{1}{p_i}} \), then the condition \((v, \omega) \in A_p^m\) implies the reverse Hölder’s condition \( \omega \in RH_p \) [1, Proposition 2.3]. In addition, Chen and Damián investigated a bound \( B_p^m \) [3, Theorem 2] and a mixed bound \( A_p^m - W_p^m \) [3, Theorem 3] for the multilinear maximal operator, which are the multilinear versions of one-weight norm estimates [14, Theorem 4.3]. Still more recently, the multilinear fractional maximal operator and the multilinear fractional strong maximal operator associated with rectangles were studied in [3, Theorem 2] and [3, Theorem 3], respectively.

On the other hand, Tanaka and Terasawa [29] very recently developed a theory of weights for positive (linear) operators and the generalized Doob’s maximal operators on a filtered measure space. In particular, two-weight norm inequalities [29, Theorem 4.1] and one-weight norm estimates of Hytönen-Pérez type [29, Theorem 5.1] for Doob’s maximal operator were established by the use of the Carleson embedding theorem and the construction of principal set, respectively. Note that if a filtered measure space naturally contains a filtered probability space with a filtration indexed by \( N \) and a Euclidean space with a dyadic filtration. It also contains a doubling metric measure space with dyadic lattice constructed by Hytönen and Kairema [15]. From this perspective, Dyadic Harmonic Analysis on the Euclidean space and Martingale Harmonic Analysis on a probability space can be unified on a filtered (infinite) measure space, as treated in [13, 26, 28]. We also mention that the Haar shift operators were studied by Lacey, Petermichl and Reguera in [16] and played an important role in the resolution of the so-called \( A_2 \) conjecture in [12]. On a filtered measure space, these operators could be seen from the point-of-view of martingale theory.

Motivated by the works above, the purpose of this paper is to develop a theory of weights for multilinear Doob’s maximal operator on a filtered measure space. For simplicity of notations, we only consider the bilinear case and all results can be extended to the multilinear case without essential difficulty.

The following theorem is our first main result, which gives the weights for which the bilinear maximal operator \( \mathcal{M} \) is bounded from \( L^{p_1}(\omega_1) \times L^{p_2}(\omega_2) \) to \( L^p(\omega_1^{\frac{1}{p_1}}, \omega_2^{\frac{1}{p_2}}) \). All unexplained notations can be found in Section 2.
Theorem 1.1. Let $\omega_1, \omega_2$ be weights and $1 < p_1, p_2 < \infty$. Suppose that

\[ v = \omega_1^{1/p_1} \omega_2^{1/p_2} \quad \text{and} \quad \sigma_i = \omega_i^{1/n_i}, \quad i = 1, 2, \quad 1/p = 1/p_1 + 1/p_2. \]

(1) If $(v, \omega_1, \omega_2) \in A^\sigma_\sigma$, then there exists a positive constant $C$ such that for all $f_1 \in L^{p_1}(\sigma_1)$, $f_2 \in L^{p_2}(\sigma_2)$ we have

\[ \|M(f_1 \sigma_1, f_2 \sigma_2)\|_{L^p(v)} \leq C \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)}. \]

(1.3)

We denote the smallest constant $C$ in (1.3) by $\|M\|$. Then it follows that \[\|M\| \leq 16 \cdot 4(q'-1)p_1p_2 \|v, \omega_1, \omega_2\|_{A^\sigma_\sigma}^{q'}, \quad \text{where} \quad q = \min\{p_1, p_2\}.

(2) Let $(\omega_1, \omega_2) \in RH^\sigma$. If there exists a positive constant $C$ such that for all $f_1 \in L^{p_1}(\sigma_1)$, $f_2 \in L^{p_2}(\sigma_2)$ we have

\[ \|M(f_1 \sigma_1, f_2 \sigma_2)\|_{L^p(v)} \leq C \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)}, \]

(1.4)

then $(v, \omega_1, \omega_2) \in A^\sigma_\sigma$. We denote the smallest constant $C$ in (1.4) by $\|M\|$. Then it follows that \[\|M\| \leq \|\omega_1, \omega_2\|_{RH^\sigma}^{1/p}. \]

First, Theorem 1.1 (more precisely, Corollary 5.1 below) is a bilinear analogue of [29, Corollary 4.5]. Remark that in the linear case, the reverse Hölder’s condition $RH^\sigma$ is automatically true. Second, in the multilinear setting, the theorem above is corresponding to [17, Theorem 3.7]. It is clear that Theorem 1.1 can be regarded as an extension of [17, Theorem 3.7] to a filtered measure space. We emphasise that the methods of proofs are quite different. Indeed, in [17], they showed that the multilinear $A^\sigma_\sigma$ condition has interesting characterization in terms of the linear $A_p$ classes [17, Theorem 3.6]. Then their proof was based on the Reverse Hölder’s inequality for linear $A_p$ classes. However, they are invalid on a filtered measure space (even on a filtered probability spaces without regularity condition [20, p.262]). We also note that in order to prove Theorem 1.1 one cannot apply some basic technique used in the linear case. For example, it is well known [6, p.137] that the distribution functions of $M$ and the dyadic maximal function $M^d$ are comparable, but it is easy to see that in the multilinear case $M$ and its dyadic version $M^d$ are not.

Our proof is mainly based on the bilinear construction of principle sets on filtered measure spaces. The germ of principle set appeared as the sparse family on $\mathbb{R}^n$ (see [14, 6] for more information) and was successfully constructed on the filtered measure space in [29, p.942-943]. Remark that one new property appears in our construction (see Section 3, P.3) which plays a key role.

Theorem 1.1 also completes the bilinear version of one-weight theory in the martingale setting [4, Proposition 1.15]. In fact, in [4] only the second part of Theorem 1.1 on a filtered probability space was proved. In addition, it is clear that (1.3) implies the condition $S^\sigma_\sigma$. Then, it follows from Theorem 1.1 that the condition $A^\sigma_\sigma$ implies the condition $S^\sigma_\sigma$. Hence, Theorem 1.1 is a bilinear analogue of the equivalence between $A_p$ and $S_p$ in [11].
Our second main purpose is to characterize two-weight inequalities for the bilinear maximal operator on the filtered measure space. Assuming the reverse Hölder’s condition, Theorem 1.2 below is a bilinear version of Sawyer’s result [25, Theorem A] on filtered measure spaces.

**Theorem 1.2.** Let \( v, \omega_1, \omega_2 \) be weights and \( 1 < p_1, p_2 < \infty \). Suppose that \( 1/p = 1/p_1 + 1/p_2 \) and \( (\omega_1, \omega_2) \in RH_P \), then the following statements are equivalent:

1. There exists a positive constant \( C \) such that for all \( f_1 \in L^{p_1}(\sigma_1), f_2 \in L^{p_2}(\sigma_2) \) we have

   \[
   \| M(f_1 \sigma_1, f_2 \sigma_2) \|_{L^p(v)} \leq C \| f_1 \|_{L^{p_1}(\sigma_1)} \| f_2 \|_{L^{p_2}(\sigma_2)},
   \]

   where \( \sigma_i = \omega_i^{1/(n-1)}, i = 1, 2 \).

2. The triple of weights \( (v, \omega_1, \omega_2) \) satisfies the condition \( S_P \).

Moreover, we denote the smallest constant \( C \) in (1.3) by \( \| M \| \). Then it follows that

\[
[v, \vec{\omega}]_{S_P} \leq \| M \| \leq 32p_1'p_2' \| v, \vec{\omega} \|_{B_P} \| \omega_1, \omega_2 \|_{RH_P}^{\frac{1}{p}}.
\]

We also obtain Hytönen-Pérez type weighted estimates [14, theorem 4.3] for the bilinear maximal operator on filtered measure spaces. To be precise, we prove the following Theorems 1.3 and 1.4. They are even new in the linear case on filtered measure spaces.

**Theorem 1.3.** If \( (v, \vec{\omega}) \in B_P \), then the following statements are valid:

1. There exists a positive constant \( C \) such that for all \( f \in L^{p_1}(\omega_1), g \in L^{p_2}(\omega_2) \) we have

   \[
   \| M(f_1, f_2) \|_{L^p(v)} \leq C \| f_1 \|_{L^{p_1}(\omega_1)} \| f_2 \|_{L^{p_2}(\omega_2)}.
   \]

2. There exists a positive constant \( C \) such that for all \( f \in L^{p_1}(\sigma_1), g \in L^{p_2}(\sigma_2) \) we have

   \[
   \| M(f_1 \sigma_1, f_2 \sigma_2) \|_{L^p(v)} \leq C \| f_1 \|_{L^{p_1}(\sigma_1)} \| f_2 \|_{L^{p_2}(\sigma_2)}.
   \]

Moreover, we denote the smallest constants \( C \) in (1.6) and (1.7) by \( \| M \| \) and \( \| M \|' \), respectively. Then it follows that \( \| M \| = \| M \|' \leq 32(2e)^{\frac{1}{p}}p_1'p_2'[v, \vec{\omega}]_{B_P}^{\frac{1}{p}} \).

**Theorem 1.4.** If \( (v, \omega_1, \omega_2) \in A_P \) and \( (\omega_1, \omega_2) \in W^\infty_P \), then the following statements are valid:

1. There exists a positive constant \( C \) such that for all \( f \in L^{p_1}(\omega_1), g \in L^{p_2}(\omega_2) \) we have

   \[
   \| M(f_1, f_2) \|_{L^p(v)} \leq C \| f_1 \|_{L^{p_1}(\omega_1)} \| f_2 \|_{L^{p_2}(\omega_2)}.
   \]
There exists a positive constant $C$ such that for all $f \in L^p_1(\sigma_1)$, $g \in L^p_2(\sigma_2)$, we have
\[
\|M(f_1\sigma_1, f_2\sigma_2)\|_{L^p(v)} \leq C\|f_1\|_{L^p_1(\sigma_1)}\|f_2\|_{L^p_2(\sigma_2)}.
\]

Moreover, we denote the smallest constants $C$ in $(1.8)$ and $(1.9)$ by $\|M\|$ and $\|M\|'$, respectively. Then it follows that
\[
\|M\| = \|M\|' \leq 32 \cdot 2^{\frac{p_1}{p_1'}} [v, \omega_1, \omega_2]_{A_{p_1}}^{\frac{1}{p_1}} [\omega_1, \omega_2]_{W_{p_1'}}^{\frac{1}{p_1'}}.
\]

**Remark 1.5.** Using a dyadic discretization technique, Chen and Damián [3] investigated Theorems 1.2, 1.3 and 1.4 on $\mathbb{R}^n$. In addition, Cao and Xue [1] and Sehba [27] gave the similar theorems for bilinear fractional maximal function on $\mathbb{R}^n$, respectively.

To prove Theorems 1.2, 1.3 and 1.4, the key ingredient is the bilinear version of Carleson embedding theorem associated with the collection of principal sets developed in Section 4. Note that Hytönen and Pérez gave the dyadic Carleson embedding theorem [14, Theorem 4.5] (see [23] for more information), and Chen and Damián obtained its multilinear analogue [3, Lemma 3] on $\mathbb{R}^n$. In order to provide some two-weight norm estimates for multilinear fractional maximal function, Sehba [27] extensively discussed the more general Carleson embedding theorem. Tanaka and Terasawa [29, Section 3] introduced a refinement of the Carleson embedding theorem on a filtered measure space. In the present paper, our Carleson embedding theorem associated with the collection of principal sets is very different from [29, Theorem 3.1]; see Theorem 4.1 in Section 4.

**Remark 1.6.** As treated on filtered probability spaces [3] and on Euclidean spaces [1, 3, 27], we do not know if the reverse Hölder condition $RH_{\vec{p}}$ in the theorems above is essential.

The article is organized as follows. In Section 2 we state some preliminaries. We construct bilinear versions of principle sets and Carleson embedding theorem in Section 3 and Section 4, respectively. In Section 5 we provide the proofs of the above theorems.

The letter $C$ will be used for constants that may change from one occurrence to another.

2. Preliminaries

This section consists of the preliminaries for this paper.

2.1. Bilinear maximal operator on filtered measure spaces

In this subsection we introduce the bilinear maximal operator on filtered measure spaces, which are standard [29]. Let a triplet $(\Omega, \mathcal{F}, \mu)$ be a measure space. Denote by $\mathcal{F}^0$ the collection of sets in $\mathcal{F}$ with finite measure. The
measure space \((\Omega, \mathcal{F}, \mu)\) is called \(\sigma\)-finite if there exist sets \(E_i \in \mathcal{F}^0\) such that \(\Omega = \bigcup_{i=0}^{\infty} E_i\). In this paper all measure spaces are assumed to be \(\sigma\)-finite. Let \(\mathcal{A} \subset \mathcal{F}^0\) be an arbitrary subset of \(\mathcal{F}^0\). An \(\mathcal{F}\)-measurable function \(f : \Omega \to \mathbb{R}\) is called \(\mathcal{A}\)-integrable if it is integrable on all sets of \(\mathcal{A}\), i.e., \(\chi_E f \in L^1(\mathcal{F}, \mu)\) for all \(E \in \mathcal{A}\). Denote the collection of all such functions by \(L^1_\mathcal{A}(\mathcal{F}, \mu)\).

Fix a \(\mathcal{F}\)-measurable function \(f : \Omega \to \mathbb{R}\). By the convention, we will denote the set of all weights by \(\Omega = \{\omega\} \cup \{\omega = \infty\} \cup \{-\infty\}\) whenever \(i < j\).

Notice that \(\chi_E f \in L^1(\mathcal{F}, \mu)\), which exists uniquely in \(L^1_\mathcal{A}(\mathcal{F}, \mu)\) due to \(\sigma\)-finiteness of \((\Omega, \mathcal{F}, \mu)\).

A family of \(\sigma\)-algebras \((\mathcal{F}_i)_{i \in \mathbb{Z}}\) is called a filtration of \(\mathcal{F}\) if \(\mathcal{F}_i \subset \mathcal{F}_j \subset \mathcal{F}\) whenever \(i, j \in \mathbb{Z}\) and \(i < j\). We call a quadruplet \((\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i \in \mathbb{Z}})\) a \(\sigma\)-finite filtered measure space. As remarked in Section 1, it contains a filtered probability space with a filtration indexed by \(\mathbb{N}\), a Euclidean space with a dyadic filtration and doubling metric space with dyadic lattice.

We write

\[
\mathcal{L} := \bigcap_{i \in \mathbb{Z}} L^1_{\mathcal{F}_i}(\mathcal{F}, \mu).
\]

Notice that

\[
L^1_{\mathcal{F}_i}(\mathcal{F}, \mu) \supset L^1_{\mathcal{F}_j}(\mathcal{F}, \mu)
\]

whenever \(i < j\). For a function \(f \in \mathcal{L}\) we will denote \(\mathbb{E}(f|\mathcal{F}_i)\) by \(\mathbb{E}_i(f)\). By the tower rule of conditional expectations, a family of functions \(\mathbb{E}_i(f) \in L^1_{\mathcal{F}_i}(\mathcal{F}, \mu)\) becomes a martingale.

By a weight we mean a nonnegative function which belongs to \(\mathcal{L}\) and, by a convention, we will denote the set of all weights by \(\mathcal{L}^+\).

Let \((\Omega, \mathcal{F}, \mu; (\mathcal{F}_i)_{i \in \mathbb{Z}})\) be a \(\sigma\)-finite filtered measure space. Then a function \(\tau : \Omega \to \{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}\) is called a stopping time if for any \(i \in \mathbb{Z}\), we have \(\{\tau = i\} \in \mathcal{F}_i\). The family of all stopping times is denoted by \(\mathcal{T}\). Fixing \(i \in \mathbb{Z}\), we denote \(\mathcal{T}_i = \{\tau \in \mathcal{T} : \tau \geq i\}\).

Suppose that functions \(f \in \mathcal{L}\) and \(g \in \mathcal{L}\), the maximal operator and bilinear maximal operator are defined by

\[
Mf = \sup_{i \in \mathbb{Z}} |\mathbb{E}_i(f)| \quad \text{and} \quad M(f, g) = \sup_{i \in \mathbb{Z}} |\mathbb{E}_i(f)| |\mathbb{E}_i(g)|,
\]

respectively. Fix \(i \in \mathbb{Z}\), we define the tailed maximal operator and tailed bilinear maximal operator by

\[
^*M_i f = \sup_{j \geq i} |\mathbb{E}_j(f)| \quad \text{and} \quad ^*M_i(f, g) = \sup_{j \geq i} |\mathbb{E}_j(f)| |\mathbb{E}_j(g)|,
\]

respectively.

Let \(B \in \mathcal{F}\), \(w \in \mathcal{L}^+\), we always denote \(\int_\Omega \chi_B d\mu\) and \(\int_\Omega \chi_B \omega d\mu\) by \(|B|\) and \(|B|_\omega\), respectively.
2.2. Bilinear Weights

In this subsection we define several kinds of bilinear weights.

**Definition 2.1.** Let $\omega_1$, $\omega_2$ be weights and $1 < p_1$, $p_2 < \infty$. Suppose that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Denote that $\vec{p} = (p_1, p_2)$ and $\sigma_s = \omega_s^{-\frac{1}{p_s-1}} \in \mathcal{L}^+$, $s = 1, 2$. We say that the couple of weights $(\omega_1, \omega_2)$ satisfies the reverse Hölder’s condition $RH_{\vec{p}}$, if there exists a positive constant $C$ such that for all $i \in \mathbb{Z}$ and $\tau \in T_i$, we have

$$
\left( \int_{\{\tau < +\infty\}} \sigma_1 d\mu \right)^{\frac{p_1}{p}} \left( \int_{\{\tau < +\infty\}} \sigma_2 d\mu \right)^{\frac{p_2}{p}} \leq C \int_{\{\tau < +\infty\}} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu. \quad (2.1)
$$

We denote by $[\omega_1, \omega_2]_{RH_{\vec{p}}}$ the smallest constant $C$ in (2.1).

**Remark 2.2.** In literatures there exist many inverse Hölder’s inequalities of the type

$$
\|f\|_p \|g\|_q \leq C \|fg\|_1,
$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $C$ is a constant and the functions $f$ and $g$ are subjected to suitable restrictions. The suitable restrictions can be found in [24, 30]. In our paper, we find that the reverse Hölder’s condition is useful for bilinear weighted theory.

**Definition 2.3.** Let $v$, $\omega_1$ and $\omega_2$ be weights and $1 < p_1$, $p_2 < \infty$. Suppose that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Denote that $\vec{p} = (p_1, p_2)$ and $\sigma_s = \omega_s^{-\frac{1}{p_s-1}} \in \mathcal{L}^+$, $s = 1, 2$. We say that the triple of weights $(v, \omega_1, \omega_2)$ satisfies the condition $A_{p}$, if there exists a positive constant $C$ such that

$$
\sup_{j \in \mathbb{Z}} \mathbb{E}_j(v) \mathbb{E}_j(\omega_1^{1-p'_1})^{\frac{p_1}{p}} \mathbb{E}_j(\omega_2^{1-p'_2})^{\frac{p_2}{p}} \leq C, \quad (2.2)
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $s = 1, 2$. We denote by $[v, \omega_1, \omega_2]_{A_{p}}$ the smallest constant $C$ in (2.2).

**Definition 2.4.** Let $v$, $\omega_1$ and $\omega_2$ be weights and $1 < p_1$, $p_2 < \infty$. Suppose that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Denote that $\vec{p} = (p_1, p_2)$ and $\sigma_s = \omega_s^{-\frac{1}{p_s-1}} \in \mathcal{L}^+$, $s = 1, 2$. We say that the triple of weights $(v, \omega_1, \omega_2)$ satisfies the condition $S_{p}$, if

$$
[v, \omega]_{S_{p}} := \sup_{i \in \mathbb{Z}, \tau \in T_i} \left( \int_{\{\tau < +\infty\}} M(\sigma_1 \chi_{\{\tau < +\infty\}}, \sigma_2 \chi_{\{\tau < +\infty\}})^{\frac{p}{p_1}} d\mu \right)^{\frac{1}{p}} < \infty,
$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, $s = 1, 2$.

**Definition 2.5.** Let $v$, $\omega_1$ and $\omega_2$ be weights and $1 < p_1$, $p_2 < \infty$. Suppose that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Denote that $\vec{p} = (p_1, p_2)$ and $\sigma_s = \omega_s^{-\frac{1}{p_s-1}} \in \mathcal{L}^+$, $s = 1, 2$. 

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We say that the couple of weights \((\omega_1, \omega_2)\) satisfies the condition \(B_{\overline{p}}\), if there exists a positive constant \(C\) such that for all \(i \in \mathbb{Z}\) we have

\[
\mathbb{E}_i(v)\mathbb{E}_i(\sigma_1)^{p}\mathbb{E}_i(\sigma_2)^{p} \leq C \exp \left( \mathbb{E}_i(\log(\frac{\mu_1}{\sigma_1} \frac{\mu_2}{\sigma_2})) \right).
\]  

(2.4)

We denote by \([v, \omega_1, \omega_2]_{B_{\overline{p}}}\) the smallest constant \(C\) in (2.4).

**Definition 2.6.** Let \(\omega_1\) and \(\omega_2\) be weights and \(1 < p_1, p_2 < \infty\). Suppose that \(\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\). Denote that \(\overline{p} = (p_1, p_2)\) and \(\sigma_s = \omega_s \frac{\mu_{1-s}}{\mu_1} \in \mathcal{L}^+, \ s = 1, 2\).

We say that the couple of weights \((\omega_1, \omega_2)\) satisfies the condition \(W_{\overline{p}}^\infty\), if there exists a positive constant \(C\) such that for all \(i \in \mathbb{Z}\) and \(\tau \in \mathcal{T}_i\) we have

\[
\int_{(\tau < +\infty)} M(\sigma_1 \chi(\tau < +\infty))^{\overline{p}_1} M(\sigma_2 \chi(\tau < +\infty))^{\overline{p}_2} d\mu \leq C \int_{(\tau < +\infty)} \sigma_1^{\overline{p}_1} \sigma_2^{\overline{p}_2} d\mu.
\]

(2.5)

We denote by \([\omega_1, \omega_2]_{W_{\overline{p}}^\infty}\) the smallest constant \(C\) in (2.4).

**Remark 2.7.** If \(p_1 = p_2\) and \(\omega_1 = \omega_2\) in the above definitions, we obtain the linear ones.

3. The construction of principal sets

Let \(i \in \mathbb{Z}, \ h_1 \in \mathcal{L}^+\) and \(h_2 \in \mathcal{L}^+\). Fixing \(k \in \mathbb{Z}\), we define a stopping time

\[
\tau := \inf\{j \geq i : \mathbb{E}(h_1|\mathcal{F}_j)\mathbb{E}(h_2|\mathcal{F}_j) > 4^{k+1}\}.
\]

For \(\Omega_0 \in \mathcal{F}_i^0\), we denote that

\[
P_0 := \{4^{k-1} < \mathbb{E}(h_1|\mathcal{F}_j)\mathbb{E}(h_2|\mathcal{F}_j) \leq 4^k\} \cap \Omega_0,
\]

(3.1)

and assume \(\mu(P_0) > 0\). It follows that \(P_0 \in \mathcal{F}_i^0\). We write \(\mathcal{K}_1(P_0) := i\) and \(\mathcal{K}_2(P_0) := k\). We let \(\mathcal{P}_1 := \{P_0\}\) which we call the first generation of principal sets. To get the second generation of principal sets we define a stopping time

\[
\tau_{P_0} := \tau\chi_{P_0} + \infty\chi_{P_0^c},
\]

where \(P_0^c = \Omega \setminus P_0\). We say that a set \(P \subset P_0\) is a principle set with respect to \(P_0\) if it satisfies \(\mu(P) > 0\) and there exists \(j > i\) and \(l > k + 1\) such that

\[
P = \{4^{l-1} < \mathbb{E}(h_1|\mathcal{F}_j)\mathbb{E}(h_2|\mathcal{F}_j) \leq 4^l\} \cap \{\tau_{P_0} = j\} \cap P_0
\]

\[
= \{4^{l-1} < \mathbb{E}(h_1|\mathcal{F}_j)\mathbb{E}(h_2|\mathcal{F}_j) \leq 4^l\} \cap \{\tau = j\} \cap P_0.
\]

Noticing that such \(j\) and \(l\) are unique, we write \(\mathcal{K}_1(P) := j\) and \(\mathcal{K}_2(P) := l\). We let \(\mathcal{P}(P_0)\) be the set of all principal sets with respect to \(P_0\) and let \(\mathcal{P}_2 := \mathcal{P}(P_0)\) which we call the second generalization of principal sets.

We now need to verify that

\[
\mu(P_0) \leq 2\mu(E(P_0)),
\]
where
\[ E(P_0) := P_0 \cap \{ \tau_{P_0} = \infty \} = P_0 \cap \{ \tau = \infty \} = P_0 \setminus \bigcup_{P \in \mathcal{P}(P_0)} P. \]

Indeed, we have
\[
\begin{align*}
\mu(P_0 \cap \{ \tau_{P_0} < \infty \}) &\leq (4^{-k-1})^{\frac{1}{2}} \int_{P_0 \cap \{ \tau_{P_0} < \infty \}} \mathbb{E}(h_1 | \mathcal{F}_{\tau_{P_0}})^{\frac{1}{2}} \mathbb{E}(h_2 | \mathcal{F}_{\tau_{P_0}})^{\frac{1}{2}} d\mu \\
&= (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \mathbb{E}(h_1 | \mathcal{F}_{\tau_{P_0}})^{\frac{1}{2}} \mathbb{E}(h_2 | \mathcal{F}_{\tau_{P_0}})^{\frac{1}{2}} \chi_{\tau_{P_0} < \infty} d\mu \\
&= (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \sum_{j \geq i} \mathbb{E}(h_1 | \mathcal{F}_{\tau_{P_0}})^{\frac{1}{2}} \mathbb{E}(h_2 | \mathcal{F}_{\tau_{P_0}})^{\frac{1}{2}} \chi_{\tau_{P_0} = j} d\mu \\
&= (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \sum_{j \geq i} \mathbb{E}(h_1 | \mathcal{F}_j)^{\frac{1}{2}} \mathbb{E}(h_2 | \mathcal{F}_j)^{\frac{1}{2}} \chi_{\tau_{P_0} = j} d\mu.
\end{align*}
\]

It follows from the H"older's inequality for sum that
\[
\begin{align*}
\mu(P_0 \cap \{ \tau_{P_0} < \infty \}) &\leq (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \left( \sum_{j \geq i} \mathbb{E}(h_1 \chi_{\tau_{P_0} = j} | \mathcal{F}_j) \right)^{\frac{1}{2}} \left( \sum_{j \geq i} \mathbb{E}(h_2 \chi_{\tau_{P_0} = j} | \mathcal{F}_j) \right)^{\frac{1}{2}} d\mu \\
&= (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \mathbb{E}_i \left( \sum_{j \geq i} \mathbb{E}(h_1 \chi_{\tau_{P_0} = j} | \mathcal{F}_j) \right)^{\frac{1}{2}} \left( \sum_{j \geq i} \mathbb{E}(h_2 \chi_{\tau_{P_0} = j} | \mathcal{F}_j) \right)^{\frac{1}{2}} d\mu.
\end{align*}
\]

Applying the H"older's inequality for conditional expectations, we have
\[
\begin{align*}
\mu(P_0 \cap \{ \tau_{P_0} < \infty \}) &\leq (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \left( \sum_{j \geq i} \mathbb{E}_i (\mathbb{E}(h_1 \chi_{\tau_{P_0} = j} | \mathcal{F}_j)) \right)^{\frac{1}{2}} \left( \sum_{j \geq i} \mathbb{E}_i (\mathbb{E}(h_2 \chi_{\tau_{P_0} = j} | \mathcal{F}_j)) \right)^{\frac{1}{2}} d\mu \\
&= (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \left( \sum_{j \geq i} \mathbb{E}_i (h_1 \chi_{\tau_{P_0} = j}) \right)^{\frac{1}{2}} \left( \sum_{j \geq i} \mathbb{E}_i (h_2 \chi_{\tau_{P_0} = j}) \right)^{\frac{1}{2}} d\mu \\
&= (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \mathbb{E}_i (h_1)^{\frac{1}{2}} \mathbb{E}_i (h_2)^{\frac{1}{2}} d\mu \\
&\leq (4^{-k-1})^{\frac{1}{2}} \int_{P_0} \mathbb{E}_i (h_1)^{\frac{1}{2}} \mathbb{E}_i (h_2)^{\frac{1}{2}} d\mu \leq 4^{-\frac{1}{2}} \mu(P_0) = \frac{1}{2} \mu(P_0).
\end{align*}
\]

This clearly implies
\[
\mu(P_0) \leq 2 \mu(E(P_0)).
\]

For any \( P'_0 \in (P_0 \cap \mathcal{F}_0^0) \), there exists a set \( \Omega'_0 \in \mathcal{F}_0^0 \) such that
\[
P'_0 = P_0 \cap \Omega'_0 = \{ 4^{k-1} < \mathbb{E}(h_1 | \mathcal{F}_j) \mathbb{E}(h_2 | \mathcal{F}_j) \leq 4^k \} \cap \Omega_0 \cap \Omega'_0.
\]
Taking \( \Omega' = \Omega_0 \cap \Omega''_0 \), we have \( P' = \{ 4^{k-1} < E(h_1|F_i)E(h_2|F_i) \leq 4^k \} \cap \Omega' \). Using \( \Omega'_0 \) instead of \( \Omega_0 \) in (3.1), we deduce that
\[
\mu(P'_0) \leq 2\mu(E(P'_0)).
\]
Moreover, we obtain that
\[
\int_{P'_0} \chi_{P_0} d\mu = \mu(P'_0 \cap P_0) = \mu(P'_0) \leq 2\mu(E(P'_0)) = 2\mu(P'_0 \cap \{ \tau = \infty \})
\]
\[
= 2\mu(P'_0 \cap \{ \tau = \infty \}) = 2 \int_{P'_0} \chi_{E(P_0)} d\mu
\]
\[
= 2 \int_{P'_0} E_i(\chi_{E(P_0)}) d\mu.
\]
Since \( P'_0 \) is arbitrary, we have \( \chi_{P_0} \leq 2E_i(\chi_{E(P_0)})\chi_{P_0} \).

The next generalizations are defined inductively,
\[
P_{n+1} := \bigcup_{P \in P_n} P(P),
\]
and we define the collection of principal sets \( P \) by
\[
P := \bigcup_{n=1}^{\infty} P_n.
\]
It is easy to see that the collection of principal sets \( P \) satisfied the following properties:

(P. 1) The set \( E(P) \) where \( P \in P \), are disjoint and \( P_0 = \bigcup_{P \in P} E(P) \);

(P. 2) \( P \in F_{K_1(P)} \);

(P. 3) \( \chi_{P} \leq 2E(\chi_{E(P)}|F_{K_1(P)})\chi_{P} \);

(P. 4) \( 4^{K_2(P)-1} < E(h_1|F_{K_1(P)})E(h_2|F_{K_1(P)}) \leq 4^{K_2(P)} \) on \( P \);

(P. 5) \( \sup_{j \geq i} E_j(h_1\chi_P)E_j(h_2\chi_P) \leq 4^{K_2(P)+1} \) on \( E(P) \).

Then we use the principal sets to represent the tailed bilinear maximal operator and obtain the following Lemma 3.1.

**Lemma 3.1.** Let \( i \in \mathbb{Z}, h_1 \in \mathcal{L}^+ \) and \( h_2 \in \mathcal{L}^+ \). Fixing \( k \in \mathbb{Z} \) and \( \Omega_0 \in \mathcal{F}_i^0 \), we denote
\[
P_i : = \{ 4^{k-1} < E(h_1|F_i)E(h_2|F_i) \leq 4^k \} \cap \Omega_0.
\]
If \( \mu(P_0) > 0 \), then
\[
\ast M_i(h_1,h_2) \chi_{P_0} = \ast M_i(h_1\chi_{P_0},h_2\chi_{P_0}) \chi_{P_0}
\]
\[
= \sum_{P \in P} \ast M_i(h_1\chi_{P_0},h_2\chi_{P_0}) \chi_{E(P)}
\]
\[
\leq 16 \sum_{P \in P} 4^{(K_2(P)-1)} \chi_{E(P)}.
\]
4. Carleson embedding theorem associated with the collection of principal sets

For \( \omega_1 \in \mathcal{L}^+ \) and \( \omega_2 \in \mathcal{L}^+ \), we set \( \sigma_1 := \omega_1^{-\frac{1}{p_1 \cdot r}} \in \mathcal{L}^+ \) and \( \sigma_2 := \omega_2^{-\frac{1}{p_2 \cdot r}} \in \mathcal{L}^+ \). Suppose that \( f_1^p \omega_1 \in L_1^+ \) and \( f_2^p \omega_2 \in L_1^+ \). It follows from Hölder’s inequality that \( f_1 \in \mathcal{L}^+ \) and \( f_2 \in \mathcal{L}^+ \). Let \( h_1 = f_1 \) and \( h_2 = f_2 \). Fixing \( k \in \mathbb{Z} \), \( i \in \mathbb{Z} \) and \( \Omega_0 \in \mathcal{F}_i \) such that \( \mu(\{4^{k+i} < \mathbb{E}(h_1|\mathcal{F}_i)\mathbb{E}(h_2|\mathcal{F}_i) \leq 4^k\} \cap \Omega_0) > 0 \), we apply the construction of principal sets to give the following Carleson embedding theorem.

**Theorem 4.1.** For \( P \in \mathcal{P} \) and \( l \in \mathbb{Z} \), let

\[
A_P^l := P \cap \{2^l < \mathbb{E}(\sigma_1 | \mathcal{F}_{k_1}(P))| \mathbb{E}(\sigma_2 | \mathcal{F}_{k_1}(P)) \leq 2^{l+1}\}. \tag{4.1}
\]

We denote \( \mathcal{Q} := \bigcup_{P \in \mathcal{P}} \bigcup_{l \in \mathbb{Z}} A_P^l \). If the nonnegative numbers \( a_Q \) and non-negative function \( \sigma_1^{p_1} \sigma_2^{p_2} \) satisfy

\[
\sum_{Q \subseteq \{\tau < +\infty\}} a_Q \leq A \int_{\{\tau < +\infty\}} \sigma_1^{p_1} \sigma_2^{p_2} \, d\mu, \forall Q \in \mathcal{Q}, \tau \in T_i, \tag{4.2}
\]

where \( A \) is an absolute constant, then

\[
\sum_{A_P^l \in \mathcal{Q}} \text{essinf} \left( \mathbb{E}^{\sigma_1}(h_1 \sigma_1^{-1}|\mathcal{F}_{k_1}(P))| \mathbb{E}^{\sigma_2}(h_2 \sigma_2^{-1}|\mathcal{F}_{k_1}(P)) \right) a_{A_P^l} \leq A(p_1 p_2)' \left( \int_{P_0} h_1^{p_1} \omega_1 \, d\mu \right)^{\frac{p_1}{p_1 p_2}} \left( \int_{P_0} h_2^{p_2} \omega_2 \, d\mu \right)^{\frac{p_2}{p_1 p_2}},
\]

where \( \mathbb{E}^{\sigma_s}(\cdot|\mathcal{F}_{k_1}(P)) \) is the conditional expectation with respect to \( \mathcal{F}_{k_1}(P), \sigma_s \, d\mu \) in place of \( d\mu, s = 1, 2 \).

**Proof.** We view the sum

\[
\sum_{A_P^l \in \mathcal{Q}} \text{essinf} \left( \mathbb{E}^{\sigma_1}(h_1 \sigma_1^{-1}|\mathcal{F}_{k_1}(P))| \mathbb{E}^{\sigma_2}(h_2 \sigma_2^{-1}|\mathcal{F}_{k_1}(P)) \right) a_{A_P^l}
\]

as an integral on a measure space \((\mathcal{Q}, 2^\mathcal{Q}, \nu)\) built over \( \mathcal{Q} \), assigning to each \( Q \in \mathcal{Q} \) the measure \( a_Q \). Thus

\[
\sum_{A_P^l \in \mathcal{Q}} \text{essinf} \left( \mathbb{E}^{\sigma_1}(h_1 \sigma_1^{-1}|\mathcal{F}_{k_1}(P))| \mathbb{E}^{\sigma_2}(h_2 \sigma_2^{-1}|\mathcal{F}_{k_1}(P)) \right) a_{A_P^l} = \int_0^\infty p \lambda^{p-1} \nu(D_\lambda) \, d\lambda,
\]

where \( D_\lambda = \{A_P^l \in \mathcal{Q} : \text{essinf} \left( \mathbb{E}^{\sigma_1}(h_1 \sigma_1^{-1}|\mathcal{F}_{k_1}(P))| \mathbb{E}^{\sigma_2}(h_2 \sigma_2^{-1}|\mathcal{F}_{k_1}(P)) \right) > \lambda \} \).

Let

\[
\tau = \inf \left\{ n \geq i : \mathbb{E}^{\sigma_1}(h_1 \sigma_1^{-1}|\mathcal{F}_n)| \mathbb{E}^{\sigma_2}(h_2 \sigma_2^{-1}|\mathcal{F}_n) \chi_{P_0} > \lambda \right\}.
\]
Then $\tau \in T_i$ and $\bigcup_{Q \in \mathcal{D}_\lambda} Q \subset \{ * \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) > \lambda \} = \{ \tau + \infty \}$, where

$$* \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) := \sup_{j \geq i} \mathbb{E}^{\sigma_1} (h_1 \sigma_1^{-1} \chi_{P_0} | \mathcal{F}_j) \mathbb{E}^{\sigma_2} (h_2 \sigma_2^{-1} \chi_{P_0} | \mathcal{F}_j).$$

Thus

$$\nu (\mathcal{D}_\lambda) = \sum_{Q \in \mathcal{D}_\lambda} a_Q \leq A \int_{* \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) > \lambda} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu \leq A \int_{* \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) > \lambda} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu,$$

which implies that

$$\sum_{A_p \in Q} \left( \mathbb{E}^{\sigma_1} (h_1 \sigma_1^{-1} | \mathcal{F}_{K_i} (P)) \mathbb{E}^{\sigma_2} (h_2 \sigma_2^{-1} | \mathcal{F}_{K_i} (P)) \right)^p a_{A_p} \leq A \int_0^\infty p \lambda^{p-1} \int_{* \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) > \lambda} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu,$$

where

$$* \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) := \sup_{j \in \mathbb{Z}} \mathbb{E}^{\sigma_1} (h_1 \sigma_1^{-1} \chi_{P_0} | \mathcal{F}_j) \mathbb{E}^{\sigma_2} (h_2 \sigma_2^{-1} \chi_{P_0} | \mathcal{F}_j).$$

It follows from Fubini's theorem that

$$\int_0^\infty p \lambda^{p-1} \int_{* \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0}) > \lambda} \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu = \int_\Omega * \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0})^p \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu \quad (4.3)$$

Applying H"older's inequality and Doob's maximal inequality, we obtain that

$$\int_\Omega * \mathcal{M}_i^{\sigma_1, \sigma_2} (h_1 \sigma_1^{-1} \chi_{P_0}, h_2 \sigma_2^{-1} \chi_{P_0})^p \sigma_1^{\frac{p}{p_1}} \sigma_2^{\frac{p}{p_2}} d\mu \leq \int_\Omega \left( \left( * \mathcal{M}_i^{\sigma_1} (h_1 \sigma_1^{-1} \chi_{P_0}) \right)^{p_1} \sigma_1^{\frac{p}{p_1}} \int_\Omega \left( * \mathcal{M}_i^{\sigma_2} (h_2 \sigma_2^{-1} \chi_{P_0}) \right)^{p_2} \sigma_2^{\frac{p}{p_2}} d\mu \right)^{\frac{p}{p_1}} \left( \left( * \mathcal{M}_i^{\sigma_2} (h_2 \sigma_2^{-1} \chi_{P_0}) \right)^{p_2} \sigma_2^{\frac{p}{p_2}} d\mu \right)^{\frac{p}{p_2}}$$

$$\leq \left( \int_\Omega \left( * \mathcal{M}_i^{\sigma_1} (h_1 \sigma_1^{-1} \chi_{P_0}) \right)^{p_1} \sigma_1 d\mu \right)^{\frac{p}{p_1}} \left( \int_\Omega \left( * \mathcal{M}_i^{\sigma_2} (h_2 \sigma_2^{-1} \chi_{P_0}) \right)^{p_2} \sigma_2 d\mu \right)^{\frac{p}{p_2}} \leq \left( \int_\Omega \left( * \mathcal{M}_i^{\sigma_1} (h_1 \sigma_1^{-1} \chi_{P_0}) \right)^{p_1} \sigma_1 d\mu \right)^{\frac{p}{p_1}} \left( \int_\Omega \left( * \mathcal{M}_i^{\sigma_2} (h_2 \sigma_2^{-1} \chi_{P_0}) \right)^{p_2} \sigma_2 d\mu \right)^{\frac{p}{p_2}}$$

$$\leq (p_1 p_2')^p \left( \int_{P_0} h_1^{p_1} \sigma_1^{-1-p_1} d\mu \right)^{\frac{p}{p_1}} \left( \int_{P_0} h_2^{p_2} \sigma_2^{-1-p_2} d\mu \right)^{\frac{p}{p_2}} = (p_1 p_2')^p \left( \int_{P_0} h_1^{p_1} \omega_1 d\mu \right)^{\frac{p}{p_1}} \left( \int_{P_0} h_2^{p_2} \omega_2 d\mu \right)^{\frac{p}{p_2}},$$

where

$$* \mathcal{M}_i^{\sigma} (h_s \sigma_s^{-1} \chi_{P_0}) := \sup_{j \in \mathbb{Z}} \mathbb{E}^{\sigma} (h_s \sigma_s^{-1} \chi_{P_0} | \mathcal{F}_j), \quad s = 1, 2.$$

Combining (4.3), (4.4) and the inequalities above, we conclude this proof. □
Remark 4.2. Using $A'_p := E(P) \cap \{2^l < E(\sigma_1 | F_{K_1(P)})E(\sigma_2 | F_{K_1(P)}) \leq 2^{l+1}\}$ instead of (4.1), we still have Lemma 4.2.

5. Main results and their proofs

5.1. Bilinear Version of one-weight Inequalities

Proof of Theorem 4.1. Let $i \in \mathbb{Z}$ be arbitrarily chosen and fixed. For $k \in \mathbb{Z}$ and $\Omega_0 \in \mathcal{F}_i$, we denote

$$P_i = \{4^{k-1} < E(f_1 \sigma_1 | F_i)E(f_2 \sigma_2 | F_i) \leq 4^k \} \cap \Omega_0.$$  

We claim that

$$\left( \int_{P_i} *M_i(f_1 \sigma_1 \chi_{P_0}, f_2 \sigma_2 \chi_{P_0})^p v d\mu \right)^\frac{1}{p} \leq 16 \cdot 4^{(q-1)p_1'p_2'[v, \omega_1, \omega_2]} A_P \left( \int_{P_0} f_1^{p_1} \chi_P d\mu \right)^{\frac{1}{p_1'}} \left( \int_{P_0} f_2^{p_2} \chi_P d\mu \right)^{\frac{1}{p_2'}}$$

where $q = \min\{p_1, p_2\}$. To see this, denote $h_1 = f_1 \sigma_1 \chi_{P_0}$ and $h_2 = f_2 \sigma_2 \chi_{P_0}$.

Without loss of generality assume that $1 < p_1 \leq p_2 < \infty$. We now estimate $\int_{E(P)} v d\mu$ as follows:

$$\int_{E(P)} v d\mu \leq \int_P v d\mu = \int_P \mathbb{E}(v | F_{K_1(P)}) d\mu$$

$$= \int_P \mathbb{E}(v | F_{K_1(P)})^{p_1'} \mathbb{E}(v | F_{K_1(P)})^{1-p_1'} \mathbb{E}(\sigma_1 | F_{K_1(P)})^{\frac{p_1}{p_1'}} \mathbb{E}(\sigma_2 | F_{K_1(P)})^{\frac{p_1}{p_1'}} \mathbb{E}(\sigma_1 | F_{K_1(P)})^{1-p_1'} \mathbb{E}(\sigma_2 | F_{K_1(P)})^{\frac{p_1}{p_1'}} d\mu$$

$$= \int_P \mathbb{E}(v | F_{K_1(P)})^{p_1'} \mathbb{E}(v | F_{K_1(P)})^{1-p_1'} \mathbb{E}(\sigma_1 | F_{K_1(P)})^{\frac{p_1}{p_1'}} \mathbb{E}(\sigma_2 | F_{K_1(P)})^{\frac{p_1}{p_1'}} \mathbb{E}(\sigma_1 | F_{K_1(P)})^{1-p_1'} \mathbb{E}(\sigma_2 | F_{K_1(P)})^{\frac{p_1}{p_1'}} d\mu.$$
It follows from the definition of $A_0$ and the property of principle sets that

\[
\int_{E(P)} vd\mu \\
\leq \quad [v, \omega_1, \omega_2] A_P \int_P [v] \mathbb{E}(v|F_{K_1}(P))^{1-p'_1} \mathbb{E}({\sigma}_1|F_{K_1}(P))^{-\frac{pp'_1}{p_1}} \mathbb{E}({\sigma}_2|F_{K_1}(P))^{-\frac{pp'_1}{p_1}} d\mu \\
\leq \quad 2^{2p(p'_1-1)} [v, \omega_1, \omega_2] A_P \int_P [v] \mathbb{E}(v|F_{K_1}(P))^{1-p'_1} \mathbb{E}({\sigma}_1|F_{K_1}(P))^{-\frac{pp'_1}{p_1}} \\
\times \mathbb{E}({\sigma}_2|F_{K_1}(P))^{-\frac{pp'_1}{p_2}} \mathbb{E}(\chi_{E(P)}|F_{K_1}(P))^{2p(p'_1-1)} d\mu \\
= \quad 2^{2p(p'_1-1)} [v, \omega_1, \omega_2] A_P \int_P [v] \mathbb{E}(v|F_{K_1}(P))^{1-p'_1} \mathbb{E}({\sigma}_1|F_{K_1}(P))^{-\frac{pp'_1}{p_1}} \\
\times \mathbb{E}({\sigma}_2|F_{K_1}(P))^{-\frac{pp'_1}{p_2}} \mathbb{E}(\chi_{E(P)}|F_{K_1}(P))^{2p(p'_1-1)} d\mu.
\]

Applying Hölder’s inequality for the conditional expectation, we have

\[
\int_{E(P)} vd\mu \leq 2^{2p(p'_1-1)} [v, \omega_1, \omega_2] A_P \int_P [v] \mathbb{E}(v|F_{K_1}(P))^{1-p'_1} \mathbb{E}({\sigma}_1|F_{K_1}(P))^{-\frac{pp'_1}{p_1}} \\
\times \mathbb{E}({\sigma}_2|F_{K_1}(P))^{-\frac{pp'_1}{p_2}} \mathbb{E}(v\chi_{E(P)}|F_{K_1}(P))^{\frac{1}{p}} \mathbb{E}(\sigma_1\chi_{E(P)}|F_{K_1}(P))^{\frac{pp'_1}{p_2}} \\
\times \mathbb{E}(\sigma_2\chi_{E(P)}|F_{K_1}(P))^{2p(p'_1-1)} d\mu.
\]

It follows from $p'_1 \geq p_2$ that $\frac{1}{2p_2} 2^{2p(p'_1-1)} - \frac{p}{p_2} = \frac{pp'_1}{p_2} - p \geq 0$, $s = 1, 2$. Then

\[
\mathbb{E}(\sigma_s\chi_{E(P)}|F_{K_1}(P))^{\frac{pp'_1}{p_2} - p} = \mathbb{E}(\sigma_s\chi_{E(P)}|F_{K_1}(P))^{\frac{pp'_1}{p_2} - p} \\
\leq \quad \mathbb{E}(\sigma_s|F_{K_1}(P))^{\frac{pp'_1}{p_2} - p}, \quad s = 1, 2.
\]

Thus

\[
\int_{E(P)} vd\mu \leq 2^{2p(p'_1-1)} [v, \omega_1, \omega_2] A_P \int_P [v] \mathbb{E}(v\chi_{E(P)}|F_{K_1}(P))^{1-p'_1} \mathbb{E}({\sigma}_1|F_{K_1}(P))^{-\frac{pp'_1}{p_1}} \\
\times \mathbb{E}({\sigma}_2|F_{K_1}(P))^{-\frac{pp'_1}{p_2}} \mathbb{E}(v\chi_{E(P)}|F_{K_1}(P))^{\frac{1}{p}} \mathbb{E}(\sigma_1\chi_{E(P)}|F_{K_1}(P))^{\frac{pp'_1}{p_2}} \\
\times \mathbb{E}(\sigma_2\chi_{E(P)}|F_{K_1}(P))^{2p(p'_1-1)} d\mu \\
\leq \quad 2^{2p(p'_1-1)} [v, \omega_1, \omega_2] A_P \int_P [v] \mathbb{E}(\sigma_1|F_{K_1}(P))^{-p} \mathbb{E}(\sigma_2|F_{K_1}(P))^{-p} \\
\times \mathbb{E}(\sigma_1\chi_{E(P)}|F_{K_1}(P))^{\frac{pp'_1}{p_1}} \mathbb{E}(\sigma_2\chi_{E(P)}|F_{K_1}(P))^{\frac{pp'_1}{p_2}} d\mu.
\]
Noting that $E(P) \subset P$ and $4^{K_2(P)-1} < E(h_1|\mathcal{F}_{K_1(P)})E(h_2|\mathcal{F}_{K_1(P)})$ on $P$, we obtain that

$$
\int_{E(P)} 4^p(K_2(P)-1)vd\mu \leq 2^{2p(p_1'-1)}[v,\omega_1,\omega_2]^P \int_P \left( E(f_1\sigma_1|\mathcal{F}_{K_1(P)}) E(f_2\sigma_2|\mathcal{F}_{K_1(P)}) \right)^p \\
\times E(\sigma_1|\mathcal{F}_{K_1(P)})^{-p} E(\sigma_2|\mathcal{F}_{K_1(P)})^{-p} \\
\times E(\chi_{E(P)}\sigma_1|\mathcal{F}_{K_1(P)})^\frac{p}{p_1} E(\chi_{E(P)}\sigma_2|\mathcal{F}_{K_1(P)})^\frac{p}{p_2} d\mu \\
= 2^{2p(p_1'-1)}[v,\omega_1,\omega_2]^P \int_P \left( E^{\sigma_1}(f_1|\mathcal{F}_{K_1(P)}) E^{\sigma_2}(f_2|\mathcal{F}_{K_1(P)}) \right)^p \\
\times E(\chi_{E(P)}\sigma_1|\mathcal{F}_{K_1(P)})^\frac{p}{p_1} E(\chi_{E(P)}\sigma_2|\mathcal{F}_{K_1(P)})^\frac{p}{p_2} d\mu,
$$

where the last equality uses a standard fact that $E(f\sigma|\mathcal{F}) = E(f|\mathcal{F})E(\sigma|\mathcal{F})$. Using Hölder’s inequality, we get

$$
\int_{E(P)} 4^p(K_2(P)-1)vd\mu \\
\leq 2^{2p(p_1'-1)}[v,\omega_1,\omega_2]^P \left( \int_P E^{\sigma_1}(f_1|\mathcal{F}_{K_1(P)})^{p_1} E(\chi_{E(P)}\sigma_1|\mathcal{F}_{K_1(P)}) d\mu \right)^\frac{p}{p_1} \\
\times \left( \int_P E^{\sigma_2}(f_2|\mathcal{F}_{K_1(P)})^{p_2} E(\chi_{E(P)}\sigma_2|\mathcal{F}_{K_1(P)}) d\mu \right)^\frac{p}{p_2} \\
= 2^{2p(p_1'-1)}[v,\omega_1,\omega_2]^P \left( \int_P E^{\sigma_1}(f_1|\mathcal{F}_{K_1(P)})^{p_1} \chi_{E(P)}\sigma_1 d\mu \right)^\frac{p}{p_1} \\
\times \left( \int_P E^{\sigma_2}(f_2|\mathcal{F}_{K_1(P)})^{p_2} \chi_{E(P)}\sigma_2 d\mu \right)^\frac{p}{p_2} \\
\leq 2^{2p(p_1'-1)}[v,\omega_1,\omega_2]^P \left( \int_P M^{\sigma_1}(f_1\chi_{P_0})^{p_1} \chi_{E(P)}\sigma_1 d\mu \right)^\frac{p}{p_1} \\
\times \left( \int_P M^{\sigma_2}(f_2\chi_{P_0})^{p_2} \chi_{E(P)}\sigma_2 d\mu \right)^\frac{p}{p_2} \\
= 2^{2p(p_1'-1)}[v,\omega_1,\omega_2]^P \left( \int_{E(P)} M^{\sigma_1}(f_1\chi_{P_0})^{p_1} \sigma_1 d\mu \right)^\frac{p}{p_1} \left( \int_{E(P)} M^{\sigma_2}(f_2\chi_{P_0})^{p_2} \sigma_2 d\mu \right)^\frac{p}{p_2}.
$$
Hence, the estimation (5.1) is proved. Consequently, it follows from (5.2) that
\[
\int_{\Omega_0} \mathcal{M}_i(f_1\sigma_1, f_2\sigma_2)^p \nu d\mu \\
\leq 16^{p'p(p_1'-1)}[v, \omega_1, \omega_2]_{A_{p'}} \sum_{p \in P} \left( \int_{E(P)} M^{\sigma_1}(f_1\chi_{\Omega_0})^{p_1} \sigma_1 d\mu \right)^{\frac{p'}{p_1}} \\
\times \left( \int_{E(P)} M^{\sigma_2}(f_2\chi_{\Omega_0})^{p_2} \sigma_2 d\mu \right)^{\frac{p'}{p_2}} \\
\leq 16^{p'p(p_1'-1)}[v, \omega_1, \omega_2]_{A_{p'}} \left( \sum_{p \in P} \int_{E(P)} M^{\sigma_1}(f_1\chi_{\Omega_0})^{p_1} \sigma_1 d\mu \right)^{\frac{p'}{p_1}} \\
\times \left( \sum_{p \in P} \int_{E(P)} M^{\sigma_2}(f_2\chi_{\Omega_0})^{p_2} \sigma_2 d\mu \right)^{\frac{p'}{p_2}} \\
\leq 16^{p'p(p_1'-1)} \left( p_1' \right)^p [v, \omega_1, \omega_2]_{A_{p'}} \left( \int_{\Omega_0} f_1^{p_1'} \sigma_1 d\mu \right)^{\frac{p'}{p_1}} \left( \int_{\Omega_0} f_2^{p_2'} \sigma_2 d\mu \right)^{\frac{p'}{p_2}}.
\]
Hence, the estimation (5.1) is proved. Consequently,
\[
\left( \int_{\Omega_0} \mathcal{M}_i(f_1\sigma_1, f_2\sigma_2)^p \nu d\mu \right)^{\frac{1}{p}} \\
= \sum_{k \in Z} \left( \int_{\{4^{k-1} < E(f_1\sigma_1|F_i) E(f_2\sigma_2|F_i) \leq 4^k \} \cap \Omega_0} \mathcal{M}_i(f_1\sigma_1, f_2\sigma_2)^p \nu d\mu \right)^{\frac{1}{p}} \\
\leq 16 \cdot 4^{(p_1'-1) p_1' p_2'} [v, \omega_1, \omega_2]_{A_{p'}} \\
\times \left( \sum_{k \in Z} \left( \int_{\{4^{k-1} < E(f_1\sigma_1|F_i) E(f_2\sigma_2|F_i) \leq 4^k \} \cap \Omega_0} f_1^{p_1'} \sigma_1 d\mu \right)^{\frac{1}{p_1'}} \right) \\
\times \left( \sum_{k \in Z} \left( \int_{\{4^{k-1} < E(f_1\sigma_1|F_i) E(f_2\sigma_2|F_i) \leq 4^k \} \cap \Omega_0} f_2^{p_2'} \sigma_2 d\mu \right)^{\frac{1}{p_2'}} \right) \\
\leq 16 \cdot 4^{(p_1'-1) p_1' p_2'} [v, \omega_1, \omega_2]_{A_{p'}} \left( \int_{\Omega_0} f_1^{p_1'} \sigma_1 d\mu \right)^{\frac{1}{p_1'}} \left( \int_{\Omega_0} f_2^{p_2'} \sigma_2 d\mu \right)^{\frac{1}{p_2'}}.
\]
Since the measure space \((\Omega, \mathcal{F}, \mu)\) is \(\sigma\)-finite, we have
\[
\left( \int \mathcal{M}_i(f_1 \sigma_1, f_2 \sigma_2)^p v \, d\mu \right)^{\frac{1}{p}}
\leq 16 \cdot 4^{(p'-1)} p_1' p_2'[v, \omega_1, \omega_2]_{1/p} \left( \int f_1^{p_1'} \sigma_1 d\mu \right)^{\frac{1}{p_1'}} \left( \int f_2^{p_2'} \sigma_2 d\mu \right)^{\frac{1}{p_2'}}.
\]

Using the monotone convergence theorem, we obtain that
\[
\left( \int \mathcal{M}(f_1 \sigma_1, f_2 \sigma_2)^p v \, d\mu \right)^{\frac{1}{p}}
\leq 16 \cdot 4^{(p'-1)} p_1' p_2'[v, \omega_1, \omega_2]_{1/p} \left( \int f_1^{p_1'} \sigma_1 d\mu \right)^{\frac{1}{p_1'}} \left( \int f_2^{p_2'} \sigma_2 d\mu \right)^{\frac{1}{p_2'}}.
\]

\[2\] Fix \(i \in \mathbb{Z}\). For \(B \in \mathcal{F}_0\), set \(f_1 = \chi_B\) and \(f_2 = \chi_B\). Then
\[
\mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}})\mathbb{E}_i(\omega_2^{-\frac{1}{p_2-1}}) \chi_B \leq \mathcal{M}(f_1 \sigma_1, f_2 \sigma_2) \chi_B.
\]

It follows from the assumption that
\[
\left( \int_B \mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}})^p \mathbb{E}_i(\omega_2^{-\frac{1}{p_2-1}})^p v \, d\mu \right)^{\frac{1}{p}}
\leq C \left( \int \omega_1^{-\frac{1}{p_1-1}} \chi_B d\mu \right)^{\frac{1}{p_1'}} \left( \int \omega_2^{-\frac{1}{p_2-1}} \chi_B d\mu \right)^{\frac{1}{p_2'}}.
\]

Since \((\omega_1, \omega_2) \in RH_{\frac{p}{p'}}\), we have
\[
\int_B \mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}})^p \mathbb{E}_i(\omega_2^{-\frac{1}{p_2-1}})^p v \, d\mu
\leq \|\mathcal{M}\|^{p}[\omega_1, \omega_2]_{RH_{\frac{p}{p'}}} \left( \int_B \omega_1^{-\frac{1}{p_1-1}} \omega_2^{-\frac{1}{p_2-1}} d\mu \right)^{\frac{p}{p'}}.
\]

Thus
\[
\mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}})^p \mathbb{E}_i(\omega_2^{-\frac{1}{p_2-1}})^p \mathbb{E}_i(v)
\leq \|\mathcal{M}\|^{p}[\omega_1, \omega_2]_{RH_{\frac{p}{p'}}} \mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}} \omega_2^{-\frac{1}{p_2-1}})
\leq \|\mathcal{M}\|^{p}[\omega_1, \omega_2]_{RH_{\frac{p}{p'}}} \mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}} \omega_2^{-\frac{1}{p_2-1}})^{\frac{1}{p_1'}} \mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}} \omega_2^{-\frac{1}{p_2-1}})^{\frac{1}{p_2'}}
\]
where we have used Hölder’s inequality for conditional expectations. Then, we obtain
\[
\mathbb{E}_i(v)^{\frac{1}{p}} \mathbb{E}_i(\omega_1^{-\frac{1}{p_1-1}})^{\frac{1}{p_1'}} \mathbb{E}_i(\omega_2^{-\frac{1}{p_2-1}})^{\frac{1}{p_2'}} \leq \|\mathcal{M}\| \mathbb{E}_i(\omega_1, \omega_2)^{\frac{1}{p}}.
\]

**Corollary 5.1.** Let \(\omega_1, \omega_2\) be weights and \(1 < p_1, p_2 < \infty\). Suppose that \(
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}\) and \(v = \omega_1^{\frac{1}{p_1}} \omega_2^{\frac{1}{p_2}}\).
Corollary 5.2. Let \( f_1 \in L^{p_1}(\omega_1), \ f_2 \in L^{p_2}(\omega_2) \) we have
\[
\|M(f_1, f_2)\|_{L^p(v)} \leq C \|f_1\|_{L^{p_1}(\omega_1)} \|f_2\|_{L^{p_2}(\omega_2)},
\] (5.3)
We denote the smallest constant \( C \) in (5.3) by \( \|M\| \). Then it follows that
\[
\|M\| \leq 16 \cdot 4^{(q'-1)} p_1' p_2' [v, \omega_1, \omega_2]_{A_p'},
\]
where \( q = \min\{p_1, p_2\} \).

(2) Let \((\omega_1, \omega_2) \in RH_{\infty}. If there exists a positive constant \( C \) such that for all \( f_1 \in L^{p_1}(\omega_1), \ f_2 \in L^{p_2}(\omega_2) \) we have
\[
\|M(f_1, f_2)\|_{L^p(v)} \leq C \|f_1\|_{L^{p_1}(\omega_1)} \|f_2\|_{L^{p_2}(\omega_2)},
\] (5.4)
then \((v, \omega_1, \omega_2) \in A_p\). We denote the smallest constant \( C \) in (5.4) by \( \|M\| \). Then it follows that \( [v, \omega_1, \omega_2]_{A_p} \leq \|M\|^{p, \omega_1, \omega_2}_{RH_{\infty}} \).

Corollary 5.2. Let \( \omega_1, \omega_2 \) be weights and \( 1 < p_1, p_2 < \infty \). Suppose that \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \) and \( v = \omega_1^{\frac{1}{p_1}} \omega_2^{\frac{1}{p_2}} \). If \((v, \omega_1, \omega_2) \in A_p\), then \((v, \omega_1, \omega_2) \in S_p\) and \([v, \omega]_{S_p} \leq 16 \cdot 4^{(q'-1)} p_1' p_2' [v, \omega_1, \omega_2]_{A_p'} \).

5.2. Bilinear Version of Two-weight Inequalities

Proof of Theorem 1.2. To prove (1) \( \Rightarrow \) (2). Fix \( i \in \mathbb{Z} \). For \( \tau \in \mathcal{T}_i \), set \( f_1 = \chi_{(\tau, \infty]} \) and \( f_2 = \chi_{(\tau, \infty]} \). It follows from (5.3) that
\[
\|M(\sigma_1 \chi_{(\tau, \infty]}, \sigma_2 \chi_{(\tau, \infty]})\|_{L^p(v)} \leq C \|\chi_{(\tau, \infty]}\|_{L^{p_1}(\sigma_1)} \|\chi_{(\tau, \infty]}\|_{L^{p_2}(\sigma_2)}.
\]
Thus \([v, \omega]_{S_p} \leq \|M\| \).

To prove (2) \( \Rightarrow \) (1), as we do in Theorem 1.1, we have
\[
\int_{P_0} \ast M_i(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \leq 16^p \sum_{P \in \mathcal{P}} \int_{E(P)} 4^{p(K_{2}(P)-1)} v d\mu. \] (5.5)
This also implies that
\[
\int_{P_0} \ast M_i(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \leq 16^p \sum_{P \in \mathcal{P}} \sum_{I \in \mathbb{Z}} \int_{A_{I}'_p} 4^{p(K_{2}(P)-1)} v d\mu,
\]
where \( A_{I}'_p := E(P) \cap \{2^I < E(\sigma_1 | \mathcal{F}_{K_1}(P)) E(\sigma_2 | \mathcal{F}_{K_1}(P)) \leq 2^{I+1}\} \). By the properties of principal sets, we have
\[
4^{p(K_{2}(P)-1)} \chi_{A_{I}'_p} \leq E(\sigma_1 | \mathcal{F}_{K_1}(P)) E(\sigma_2 | \mathcal{F}_{K_1}(P)) \chi_{A_{I}'_p} = E^\sigma_1 (h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P)) E_{\sigma_2}(h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P)) E(\sigma_1 | \mathcal{F}_{K_1}(P)) E(\sigma_2 | \mathcal{F}_{K_1}(P)) \chi_{A_{I}'_p} \leq E^\sigma_1 (h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P)) E_{\sigma_2}(h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P)) 2^{I+1} \chi_{A_{I}'_p}.
\]
It follows that
\[ 4^{(K_2(P) - 1)} \chi_{A'_p} \leq \text{essinf}_{A'_p} \left( \mathbb{E}^{\sigma_1 (h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P))} \mathbb{E}^{\sigma_2 (h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P))} \right) 2^{l+1} \chi_{A'_p} \]
\[ \leq 2 \text{essinf}_{A'_p} \left( \mathbb{E}^{\sigma_1 (h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P))} \mathbb{E}^{\sigma_2 (h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P))} \right) \times \mathbb{E}(h_1 \sigma_1|\mathcal{F}_{K_1}(P)) \mathbb{E}(h_2 \sigma_2|\mathcal{F}_{K_1}(P)) \chi_{A'_p}. \]

For simplicity, we denote
\[ a_{A'_p} := \int_{A'_p} \left( \mathbb{E}(h_1 \sigma_1|\mathcal{F}_{K_1}(P)) \mathbb{E}(h_2 \sigma_2|\mathcal{F}_{K_1}(P)) \right)^p \mu dp. \]

Then
\[ \int_{P_0} ^{\ast} M_i (f_1 \sigma_1, f_2 \sigma_2)^p \mu dp \leq 16^p \sum_{P \in \mathcal{P}} \sum_{l \in \mathbb{Z}} \int_{A'_p} 4^{p(K_2(P) - 1)} \mu dp \]
\[ \leq 32^p \sum_{P \in \mathcal{P}} \sum_{l \in \mathbb{Z}} \text{essinf}_{A'_p} \left( \mathbb{E}^{\sigma_1 (h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P))} \mathbb{E}^{\sigma_2 (h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P))} \right)^p \times \int_{A'_p} \left( \mathbb{E}(h_1 \sigma_1|\mathcal{F}_{K_1}(P)) \mathbb{E}(h_2 \sigma_2|\mathcal{F}_{K_1}(P)) \right)^p \mu dp \]
\[ = 32^p \sum_{P \in \mathcal{P}} \sum_{l \in \mathbb{Z}} \text{essinf}_{A'_p} \left( \mathbb{E}^{\sigma_1 (h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P))} \mathbb{E}^{\sigma_2 (h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P))} \right)^p a_{A'_p}. \]

Now we claim that
\[ \left( \int_{P_0} ^{\ast} M_i (f_1 \sigma_1 \chi_{P_0}, f_2 \sigma_2 \chi_{P_0})^p \mu dp \right)^{\frac{1}{p}} \]
\[ \leq 32^p [p_1, p_2]_{S^{\tau}} [\omega_1, \omega_2]_{R^{\tau}} \left( \int_{P_0} ^{p_1} f_1 \mu dp \right)^{\frac{1}{p_1}} \left( \int_{P_0} ^{p_2} f_2 \mu dp \right)^{\frac{1}{p_2}}. \quad (5.6) \]

To see this, we apply the Carleson embedding theorem to these $a_{A'_p}$. By Theorem 4.1 it suffices to prove
\[ \sum_{A'_p \subseteq \{ \tau < +\infty \}} a_{A'_p} \leq A \int_{\{ \tau < +\infty \}} ^{\ast} \sigma_1^{\frac{p_1}{p}} \sigma_2^{\frac{p_2}{p}} d\mu, \quad \tau \in T_i. \quad (5.7) \]
Moreover, we denote the smallest constant $C$ by $\|M\|$.

For $\tau \in T_1$, we have

$$
\sum_{A'_\rho \subseteq \tau} a_{A'_\rho} \leq \sum_{A'_\rho \subseteq \tau} \int_{A'_\rho} \left( E(\sigma_1 | F_{K_\rho}(P)) E(\sigma_2 | F_{K_\rho}(P)) \right)^p v d\mu
$$

$$= \sum_{A'_\rho \subseteq \tau} \int_{A'_\rho} \left( E(\sigma_1 \chi_{\{\tau < +\infty\}} | F_{K_\rho}(P)) E(\sigma_2 \chi_{\{\tau < +\infty\}} | F_{K_\rho}(P)) \right)^p v d\mu
$$

$$\leq \sum_{A'_\rho \subseteq \tau} \int_{A'_\rho} M(\sigma_1 \chi_{\{\tau < +\infty\}}, \sigma_2 \chi_{\{\tau < +\infty\}})^p v d\mu
$$

$$\leq \int_{\{\tau < +\infty\}} M(\sigma_1 \chi_{\{\tau < +\infty\}}, \sigma_2 \chi_{\{\tau < +\infty\}})^p v d\mu
$$

$$\leq [v, \omega]^p \int_{\{\tau < +\infty\}} \sigma_1 d\mu \left( \int_{\{\tau < +\infty\}} \sigma_2 d\mu \right)^{p \over 2}
$$

$$\leq [v, \omega]^p [\omega_1, \omega_2]_{RH_\rho} \int_{\{\tau < +\infty\}} \sigma_1 \sigma_2 d\mu.
$$

Therefore, (5.7) is proved and (5.6) immediately follows. Similarly to the argument in the proof of Theorem 1.1, it follows that

$$
\left( \int_{\Omega} M(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \right)^{1 \over p}
$$

$$\leq 32 p' p_2' [v, \omega]_{S_\rho} \int_{\Omega} f_1 \sigma_1 d\mu \left( \int_{\Omega} f_2 \sigma_2 d\mu \right)^{p' \over 2}.
$$

\[\square\]

**Corollary 5.3.** Let $v, \omega_1, \omega_2$ be weights and $1 < p_1, p_2 < \infty$. Suppose that $1/p = 1/p_1 + 1/p_2$ and $(\omega_1, \omega_2) \in RH_\rho$, then the following statements are equivalent:

1. There exists a positive constant $C$ such that

$$
||M(f_1, f_2)||_{L^p(v)} \leq C ||f_1||_{L^{p_1}(\omega_1)} ||f_2||_{L^{p_2}(\omega_2)}, \forall f \in L^{p_1}(\omega_1), g \in L^{p_2}(\omega_2);
$$

2. The triple of weights $(v, \omega_1, \omega_2)$ satisfies the condition $S_\rho$.

Moreover, we denote the smallest constant $C$ in (1.3) by $\|M\|$. Then it follows that

$$[v, \omega]_{S_\rho} \leq ||M|| \leq 32 p' p_2' [v, \omega]_{S_\rho} [\omega]^p_{RH_\rho}.
$$

**Proof of Theorem 1.3.** It is clear that (1) $\Leftrightarrow$ (2), so $\|M\| = ||M||'$. To prove (2), noting that (5.2) and $E(P) \subset P$, we have

$$
\int_{P_\rho} \mathcal{M}_i(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \leq 16p \sum_{P \in P} \int_P 4^{p(K_2(P)-1)} v d\mu.
$$

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It follows that
\[ \int_{P} \mathcal{M}_1(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \leq 16^p \sum_{P \in \mathcal{P}} \sum_{l \in \mathbb{Z}} \int_{A_P} 4^{p(K_2(P)-1)} v d\mu, \]
where \( A_P' = P \cap \{ 2^l < E(\sigma_1 | \mathcal{F}_{K_1}(P))E(\sigma_2 | \mathcal{F}_{K_1}(P)) \leq 2^{l+1} \} \). Similarly to the argument used in the proof of Theorem 1.2 we denote
\[ a_{A'_P} = \int_{A'_P} \left( E(\sigma_1 | \mathcal{F}_{K_1}(P))E(\sigma_2 | \mathcal{F}_{K_1}(P)) \right)^p v d\mu. \]

Then
\[ \int_{P} \mathcal{M}_1(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \leq 32^p \sum_{P \in \mathcal{P}} \sum_{l \in \mathbb{Z}} \text{essinf} \left( E^{\sigma_1}(h_1 \sigma_1^{-1} | \mathcal{F}_{K_1}(P))E^{\sigma_2}(h_2 \sigma_2^{-1} | \mathcal{F}_{K_1}(P)) \right)^p a_{A'_P}. \] (5.8)

Applying the Carleson embedding theorem to these \( a_{A'_P} \), we claim that
\[ \sum_{A'_P \subseteq \{ \tau < +\infty \}} a_{A'_P} \leq 2\varepsilon[v, \omega_1, \omega_2]_{B_p} \int_{\{ \tau < +\infty \}} \sigma_1^{\frac{p}{p-1}} \sigma_2^{\frac{p}{p-1}} d\mu, \; \tau \in T_\varepsilon. \] (5.9)

In fact, for \( \tau \in T_\varepsilon \), noting that \( A'_P \subseteq P \), we have
\[ \sum_{A'_P \subseteq \{ \tau < +\infty \}} a_{A'_P} = \sum_{A'_P \subseteq \{ \tau < +\infty \}} \int_{A'_P} E(\sigma_1 | \mathcal{F}_{K_1}(P))^p E(\sigma_2 | \mathcal{F}_{K_1}(P))^p v d\mu \]
\[ = \sum_{A'_P \subseteq \{ \tau < +\infty \}} \int_{A'_P} E(\sigma_1 | \mathcal{F}_{K_1}(P))^p E(\sigma_2 | \mathcal{F}_{K_1}(P))^p E(v | \mathcal{F}_{K_1}(P)) d\mu \]
\[ \leq [v, \omega_1, \omega_2]_{B_p} \sum_{A'_P \subseteq \{ \tau < +\infty \}} \int_{A'_P} \exp \left( E(\log(\sigma_1^{\frac{p}{p-1}} \sigma_2^{\frac{p}{p-1}}) | \mathcal{F}_{K_1}(P)) \right) d\mu \]
\[ = [v, \omega_1, \omega_2]_{B_p} \int_{A'_P} \exp \left( E(\log(\sigma_1^{\frac{p}{p-1}} \sigma_2^{\frac{p}{p-1}}) | \mathcal{F}_{K_1}(P)) \right) \chi_P d\mu \]
\[ =: [v, \omega_1, \omega_2]_{B_p} \cdot I. \]

It follows from the property of principle sets that \( I \) is controlled by
\[ 2 \sum_{A'_P \subseteq \{ \tau < +\infty \}} \int_{A'_P} \exp \left( E(\log(\sigma_1^{\frac{p}{p-1}} \sigma_2^{\frac{p}{p-1}}) | \mathcal{F}_{K_1}(P)) \right) E(\chi_{E(P)} | \mathcal{F}_{K_1}(P)) d\mu, \]
which is equal to
\[ 2 \sum_{A'_P \subseteq \{ \tau < +\infty \}} \int_{A'_P} \left( \exp \left( E(\log(\sigma_1^{\frac{p}{p-1}} \sigma_2^{\frac{p}{p-1}}) | \mathcal{F}_{K_1}(P)) \right) \right)^r E(\chi_{E(P)} | \mathcal{F}_{K_1}(P)) d\mu, \]
where \( r \) is an arbitrary real number and bigger than 1. Using Jensen’s inequality, we have

\[
\exp \left( E(\log(\sigma_1^{-p_1} \sigma_2^{-p_2} \chi_{A_p^0}) | \mathcal{F}_{K_0}(P)) \right) \leq E(\sigma_1^{-p_1} \sigma_2^{-p_2} \chi_{A_p} | \mathcal{F}_{K_0}(P)).
\]

Since for all \( P \in \mathcal{P} \) and \( l \in \mathbb{Z} \), \( E(P) \cap A_p^0 \) are disjoint sets, it follows that

\[
I \leq 2 \sum_{A_p \subseteq \{ \tau < +\infty \}} \int_{A_p} \left( E(\sigma_1^{-p_1} \sigma_2^{-p_2} \chi_{A_p^0} | \mathcal{F}_{K_0}(P)) \right)^r E(\chi_{E(P)} | \mathcal{F}_{K_0}(P)) d\mu
\]

\[
= 2 \sum_{A_p \subseteq \{ \tau < +\infty \}} \int_{A_p} \left( E(\sigma_1^{-p_1} \sigma_2^{-p_2} \chi_{A_p^0} | \mathcal{F}_{K_0}(P)) \right)^r \chi_{E(P)} d\mu
\]

\[
\leq 2 \sum_{A_p \subseteq \{ \tau < +\infty \}} \int_{E(P) \cap A_p} M(\sigma_1^{-p_1} \sigma_2^{-p_2} \chi_{\{ \tau < +\infty \}})^r d\mu
\]

\[
\leq 2 \int_{\Omega} M(\sigma_1^{-p_1} \sigma_2^{-p_2} \chi_{\{ \tau < +\infty \}})^r d\mu \leq 2 \left( \frac{r}{r - 1} \right) \int_{\{ \tau < +\infty \}} \sigma_1^{-p_1} \sigma_2^{-p_2} d\mu.
\]

Letting \( r \to +\infty \), we deduce that

\[
I \leq 2e \int_{\{ \tau < +\infty \}} \sigma_1^{-p_1} \sigma_2^{-p_2} d\mu.
\]

Therefore, the estimation (5.9) is proved. It follows from Theorem 1.3 and (5.8) that

\[
\left( \int_{P_0} \mathcal{M}_0(f_1 \sigma_1 \chi_{B_0}, f_2 \sigma_2 \chi_{B_0})^p d\mu \right)^{\frac{1}{p}}
\]

\[
\leq 32(2e)^{\frac{1}{p}} p_1' p_2' [v, \omega_1, \omega_2]_{B_{\mathbb{R}^d}} \left( \int_{P_0} f_1^{p_1} \sigma_1 d\mu \right)^{\frac{1}{p_1}} \left( \int_{P_0} f_2^{p_2} \sigma_2 d\mu \right)^{\frac{1}{p_2}}.
\]

Similarly to the proof of Theorem 1.1, it follows that

\[
\left( \int_{\Omega} \mathcal{M}(f_1 \sigma_1, f_2 \sigma_2)^p d\mu \right)^{\frac{1}{p}} \leq 32(2e)^{\frac{1}{p}} p_1' p_2' [v, \omega_1, \omega_2]_{B_{\mathbb{R}^d}} \| f_1 \|_{L^{p_1}(\sigma_1)} \| f_2 \|_{L^{p_2}(\sigma_2)}.
\]

The proof is complete. \( \square \)

**Proof of Theorem 1.4** This proof is similar to one of Theorem 1.3. For \( A_p^0 \) and \( a_{A_p^0} \) defined in the proof of Theorem 1.3 it suffices to check the Carleson embedding condition,

\[
\sum_{A_p^0 \subseteq \{ \tau < +\infty \}} a_{A_p^0} \leq 2 [v, \omega_1, \omega_2]_{A_{\mathbb{R}^d}} [\omega_1, \omega_2]_{\mathcal{W}_{\mathbb{R}^d}} \int_{\{ \tau < +\infty \}} \sigma_1^{-p_1} \sigma_2^{-p_2} d\mu, \quad \tau \in T_i.
\]
Indeed, for $\tau \in T_1$, it follows from the definitions of $A_\alpha$ that

$$
\sum_{A_{\alpha}^{c} \subseteq \{ \tau < \infty \}} a_{A_{\alpha}^{c}} = \sum_{A_{\alpha}^{c} \subseteq \{ \tau < \infty \}} \int_{A_{\alpha}^{c}} E(\sigma_1 | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(\sigma_2 | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(v | \mathcal{F}_{K_1(p)}) d\mu
$$

\begin{align*}
&\leq \left[ v, \omega_1, \omega_2 \right]_{A_\alpha} \sum_{A_{\alpha}^{c} \subseteq \{ \tau < \infty \}} \int_{A_{\alpha}^{c}} E(\sigma_1 | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(\sigma_2 | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(v, \omega) d\mu \\
&= \left[ v, \omega_1, \omega_2 \right]_{A_\alpha} \sum_{A_{\alpha}^{c} \subseteq \{ \tau < \infty \}} \int_{A_{\alpha}^{c}} E(\sigma_1 \chi_{A_{\alpha}^{c}} | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(\sigma_2 \chi_{A_{\alpha}^{c}} | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(v, \omega) d\mu \\
&= \left[ v, \omega_1, \omega_2 \right]_{A_\alpha} \cdot 1.
\end{align*}

It follows from the property of principle sets that I is controlled by

$$
2 \sum_{A_{\alpha}^{c} \subseteq \{ \tau < \infty \}} \int_{A_{\alpha}^{c}} E(\sigma_1 \chi_{A_{\alpha}^{c}} | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(\sigma_2 \chi_{A_{\alpha}^{c}} | \mathcal{F}_{K_1(p)})^{\frac{p}{2}} E(v, \omega) d\mu,
$$

which is smaller than

$$
2 \sum_{A_{\alpha}^{c} \subseteq \{ \tau < \infty \}} \int_{E(P) \cap A_{\alpha}^{c}} M(\sigma_1 \chi_{A_{\alpha}^{c}})^{\frac{p}{2}} M(\sigma_2 \chi_{A_{\alpha}^{c}})^{\frac{p}{2}} d\mu.
$$

It follows from the definition of $W_{\alpha, \rho}$ that

$$
1 \leq 2 \int_{\{ \tau < \infty \}} M(\sigma_1 \chi_{(\tau < \infty)})^{\frac{p}{2}} M(\sigma_2 \chi_{(\tau < \infty)})^{\frac{p}{2}} d\mu
$$

\begin{align*}
&\leq 2 \left[ v, \omega_1, \omega_2 \right]_{A_\alpha} \left[ v, \omega_1, \omega_2 \right]_{W_{\alpha, \rho}} \int_{\{ \tau < \infty \}} s_1^{\frac{p}{2}} s_2^{\frac{p}{2}} d\mu.
\end{align*}

Therefore, by (5.55) and Theorem 4.1, we obtain that

$$
\left( \int_{P_1^0}^{*} \mathcal{M}(f_1 \sigma_1 \chi_E, f_2 \sigma_2 \chi_E)^p v d\mu \right)^{\frac{1}{p}}
\leq 32 \cdot 2^p p_1^{\frac{p}{2}} p_2^{\frac{p}{2}} \left[ v, \omega_1, \omega_2 \right]_{A_\alpha} \left[ v, \omega_1, \omega_2 \right]_{W_{\alpha, \rho}} \left( \int_{P_0^0}^{\frac{p}{2}} f_1^{p_1} \sigma_1 d\mu \right)^{\frac{1}{p_1}} \left( \int_{P_0^0}^{\frac{p}{2}} f_2^{p_2} \sigma_2 d\mu \right)^{\frac{1}{p_2}}.
$$

Similarly to the proof of Theorem 4.1, it follows that

$$
\left( \int_{\Omega} \mathcal{M}(f_1 \sigma_1, f_2 \sigma_2)^p v d\mu \right)^{\frac{1}{p}}
\leq 32 \cdot 2^p p_1^{\frac{p}{2}} p_2^{\frac{p}{2}} \left[ v, \omega_1, \omega_2 \right]_{A_\alpha} \left[ v, \omega_1, \omega_2 \right]_{W_{\alpha, \rho}} \| f_1 \|_{L^{p_1}(\sigma_1)} \| f_2 \|_{L^{p_2}(\sigma_2)}.
$$

We conclude this proof. \hfill \Box

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