Transport coefficients of $O(N)$ scalar field theories close to the critical point

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Abstract

We investigate the critical dynamics of $O(N)$-symmetric scalar field theories to determine the critical exponents of transport coefficients as a second-order phase transition is approached from the symmetric phase. A set of stochastic equations of motion for the slow modes is formulated, and the long wavelength dynamics is examined for an arbitrary number of field components, $N$, in the framework of the dynamical renormalization group within the $\varepsilon$ expansion. We find that for a single component scalar field theory, $N = 1$, the system reduces to the model C of critical dynamics, whereas for $N > 1$ the model G is effectively restored owing to dominance of $O(N)$-symmetric charge fluctuations. In both cases, the shear viscosity remains finite in the critical region. On the other hand, we find that the bulk viscosity diverges as the correlation length squared, for $N = 1$, while it remains finite for $N > 1$. 
I. INTRODUCTION

In recent decades transport coefficients of the quantum chromodynamics (QCD) have attracted much interest in the context of Relativistic Heavy Ion Collider (RHIC) experiment, which aims at creating and studying a quark-gluon plasma. One of the interesting findings emerging from the experimental program at RHIC, the large elliptic flow $v_2$ observed in high energy non-central collisions, implies that the spatial anisotropy of the initial state created in the collision is efficiently converted during the expansion to a transverse momentum anisotropy of the observed hadrons [1–3]. These experimental results are well described by ideal hydrodynamics with vanishing viscosity [4–6]. Thus, the large elliptic flow observed in such collisions implies that the matter created in collisions behaves as an almost perfect fluid.

Although the transport coefficients in viscous hydrodynamics are phenomenological parameters, they can, in principle, be computed from a microscopic theory. Since the shear viscosity, one of the transport coefficients, has a direct influence on the elliptic flow, the experimental results have triggered numerous theoretical efforts to unravel its behavior as a function of thermodynamic variables. In general, these are performed in the framework of kinetic theory, e.g. using the Boltzmann equation, applied to effective theories of QCD [7–14] and to perturbative QCD [15–19]. Furthermore, some results on the temperature dependence of the transport coefficients have been obtained in lattice simulations [20–26].

The results of the RHIC experiments have motivated recent work on a field theoretical approach to evaluate transport coefficients. The $O(N)$ scalar field theory offers a testing ground for developing computational methods before facing the complications of a full QCD calculation. The scalar field theory has in fact long been studied as a prototype theory in many contexts of physics. Moreover, for $N = 4$, the $O(N)$ model serve as a low energy chiral effective theory for two flavor QCD [27]. A general Lagrangian density for the $O(N)$ scalar field theory is given by

\[\mathcal{L} = \frac{1}{2} (\partial \phi_i)^2 - \frac{1}{2} r \phi_i^2 - \frac{1}{4} u (\phi_i^2)^2,\]  

(1)

where $r$ is the mass parameter, $u$ is the coupling constant, and the implicit summation over $i$ runs from $i = 1$ to $N$. Recently, the dynamical properties of the $O(N)$ scalar theory, in particular behavior of the transport coefficients, formulated microscopically in the Green-Kubo-Nakano linear response theory, have been explored in several theoretical studies.
The shear viscosity, $\eta$, of the scalar field theory was first studied by Hosoya et. al. \cite{28} and Jeon and Yaffe \cite{29,30} in thermal field theory. Later on, the large $N$ behavior was examined by Aarts and Resco \cite{31,32}. These calculations demonstrated that $\eta$ is an increasing function of temperature, $T$. At high temperatures,

$$\eta \sim \frac{N^2 u^2}{T^3}. \quad (2)$$

The cubic power in temperature can be understood on dimensional grounds, and the factor $N^2$ is attributed to the scaling of the coupling constant $u$ with $1/N$. The inverse power of the coupling constant in Eq. (2) implies that the shear viscosity is a non-perturbative quantity. The precise numerical factor in Eq. (2) can be obtained by a resummation of ladder type diagrams. In Refs. \cite{29,30}, it was found that the ladder resummation is equivalent to the linearized Boltzmann equation with a thermal mass term. Some systematic approaches for computing higher order corrections are presented in Refs. \cite{33–35}, and relevant issues on the transport equation are discussed in Refs. \cite{36,37}.

In the present work, we discuss the critical behavior of the shear viscosity and other transport coefficients in the $O(N)$ scalar field theory. As demonstrated by Wilson using the renormalization group approach, there is a second-order phase transition in the $O(N)$ scalar field theory. Kinetic approaches employed for computing the transport coefficients (see e.g. \cite{38,39} and a discussion in Ref. \cite{40}) rely heavily on Boltzmann-like approximations, which take only the single particle distribution into account and neglect higher order correlations. Although these correlations, may be unimportant far from the critical point, they play an important role in the critical region.

In our study of the critical transport properties, we employ the dynamical renormalization group (DRG) combined with the epsilon expansion * developed by Hohenberg and Halperin (for a review, see Ref. \cite{45}). Within this approach we examine the scale evolution of a stochastic equation of motion, which describes the critical dynamics of slow modes. These include fluctuations of the order parameter and of conserved quantities, which are relevant variables when addressing the long-wave length behavior of the system near the critical point.

* An alternative non-perturbative approach to the epsilon expansion would be a direct application of the functional renormalization group (see e.g. Ref. \cite{41} for a review) to a quantum-field model constructed to be equivalent to the stochastic equations of motion \cite{42,43}. This method was tested for model A in Ref. \cite{44}.
Since the transport coefficients are obtained from the corresponding response functions by taking the limit of both frequency and momentum to zero, they characterize the dynamics of the system in the low energy limit.

In analogy to the static case, the flow equations for transport coefficients derived from the DRG admit non-trivial fixed points, from which the dynamical critical exponent, \( z \), and the dynamical scaling relations can be derived. The dynamical critical exponent, \( z \), defines the characteristic frequency of the most relevant slow mode \( \omega \sim k^z \), and the scaling relations link the singular contribution to the transport coefficients. From these properties one can deduce the singular behavior of the transport coefficients, in particular whether they diverge or remain finite at the critical point. Based on the universal behavior, i.e. on the dynamical critical exponents and scaling laws, one identifies each system with a dynamical universality class. In contrast to the static case, the dynamical universality class is governed not only by the dimensionality, locality, and the symmetries of the system under consideration, but, in addition, by the properties of the relevant slow modes. Thus, the conservation or non-conservation of an order parameter, and the existence of mode-mode couplings among the slow modes affect the dynamical universality class. Therefore, even if two systems belong to the same static universality class, their dynamic universality class † may be different ‡.

† In what follows, we will frequently refer to the universality classes that were defined in Ref. [45]. Here we provide some properties of the relevant universality classes:

| Model | Slow mode(s) | Dynamical critical exponent in d=3 |
|-------|--------------|-----------------------------------|
| A     | Non-conserved field | \( z_A = 2 + 0.7621 \eta' \) |
| B     | Conserved field | \( z_B = 4 - \eta' \) |
| C     | N-component non-conserved field coupled to one component conserved field | \( z_C = 2 + \alpha/\nu \) for \( N = 1 \) \( z_C = 2 + 0.7621 \eta' \) for \( N > 1 \) |
| H     | Conserved field coupled to conserved transverse vector field | \( z_H = 4 - 18/19 \) |
| G     | N-component non-conserved field coupled to \( N(N-1)/2 \)-component conserved field | \( z_G = 3/2 \) |

Here \( \alpha \) and \( \nu \) are the static critical exponents and \( \eta' \) is the anomalous dimension.

‡ An example of such a situation is given by the models A and B of critical dynamics (see Refs. [45, 46] for further details). In the static case, both models, exhibiting \( Z(2) \) symmetry and belong to the static universality class of the Ising model in three dimensions. The dynamical universality class is, however, different. This difference, arises from the non-conservation (conservation) of the order parameter in model A (B), and results in different long wavelength behavior characterized by the dynamical critical exponent: \( z_A = 2 + \text{const} \cdot \eta' \) and \( z_B = 4 - \eta' \) where \( \eta' \) is the anomalous dimension. Another nontrivial example is the \( O(N) \) model for \( N = 1 \) (non-conserved order parameter) and model H (conserved order
this article, we determine the dynamical universality class of the $O(N)$ scalar field theory and show how the dynamical universality class depends on the number of components, $N$, and on the dimensionality, $d$.

The paper is organized as follows: in the next section we identify the slow modes in the $O(N)$ scalar theory, and construct an effective Hamiltonian for them. In section III we review the static universality classification of the theory, and show that a non-trivial fixed point exists. In section IV we introduce the stochastic equation of motion, which describes the dynamics of the slow modes in the critical region. We then implement the DRG to find the fixed points of the stochastic equations of motion, and determine the dynamical universality class. We close this section with a brief discussion of the critical behavior of the bulk viscosity. Section V is devoted to summary, discussion, and outlook. Details on the derivation of the stochastic equation of motion and on the calculation of the response function are given in two appendices.

II. CONSTRUCTION OF THE EFFECTIVE HAMILTONIAN

Before considering the dynamics of the theory, we have to build an effective Hamiltonian for the slow modes $A_l$. The probability distribution for the modes $A_l$ is given by the exponential of the Hamiltonian $\langle A_l \rangle$, $e^{-\mathcal{H}(\{A_l\})}$. The effective Hamiltonian defines the static critical behavior of the theory, and will later on be incorporated in the equations of motion, from which we finally find the dynamical properties of the system close to the critical point. Although, the effective Hamiltonian, and the equations of motion for the slow modes have a microscopic origin, it is in general a very challenging problem to derive them starting from the microscopic Lagrangian. Therefore, in the present work, we formulate the effective Hamiltonian and the equations of motion on a phenomenological basis. The guiding principles in such a formulation are similar to those of Ginzburg-Landau theory. Note that, in our case, the slow variables in the Ginzburg-Landau Hamiltonian are all fluctuations, i.e.,

\begin{itemize}
  \item parameter) of critical dynamics. These models also share the same static universality class, while the dynamic universality classes differ. This implies a completely different behavior of quantities such as the shear viscosity, close to the critical point. In model H, the shear viscosity diverges, while, as will be shown in this article, it is always finite in the $O(N)$ scalar field theory.
\end{itemize}
deviations of variables from their equilibrium values.

Candidates for the slow mode of the theory are the fluctuations of the order parameter $\phi_i$, the energy-momentum density, $E$ and $\vec{J}$, and the $O(N)$ charge density, $Q_{ij}$. Owing to the symmetry $Q_{ij} = -Q_{ji}$, there are $N(N - 1)/2$ charges associated with generators of the $O(N)$ group. The order parameter of the theory is not conserved, while the remaining variables (energy, momentum and $O(N)$ charge) are conserved quantities.

In the present work, we consider a system approaching the critical point from the symmetric phase. In this case, it is straightforward to construct the effective Hamiltonian for the slow modes

$$\mathcal{H} = \int d^d x \left[ \mathcal{H}_\phi + \frac{\gamma_0}{2} \phi_i^2 E + \frac{1}{2} C_0^{-1} E^2 + \frac{1}{2} \vec{J}^2 + \frac{1}{2} \chi_0^{-1} Q_{ij}^2 + \mathcal{H}_s \right],$$

$$\mathcal{H}_\phi = \frac{1}{2} \left( \nabla \phi_i \right)^2 + \frac{r_0}{2} \phi_i^2 + \frac{u_0}{4} \left( \phi_i^2 \right)^2,$$

$$\mathcal{H}_s = -\phi_i h_i - \vec{J} \cdot \vec{H} + \beta E - \mu_{ij} Q_{ij},$$

where $\mathcal{H}_s$ is the source term, which is introduced for later convenience. We follow the convention that repeated indices imply summations, e.g., $Q_{ij}^2 \equiv \frac{1}{2} \sum_{ij=1}^N Q_{ij} Q_{ij}$. The effective Hamiltonian (3) includes all possible candidates for slow modes in an $O(N)$ scalar field theory.

Since the Hamiltonian includes up to quadratic terms in $E$, $\vec{J}$ and $Q_{ij}$, the original Hamiltonian density for the order parameter fluctuation, $\mathcal{H}_\phi$, is recovered after integrating out these variables and performing a suitable redefinition of the couplings. This implies that the critical statics of the Hamiltonian $\mathcal{H}$ is the same as that of $\mathcal{H}_\phi$.

The coefficients of the Hamiltonian (3), are given by the static susceptibilities of the slow modes. Since the susceptibility of the momentum current $\vec{J}$ always remains finite, we have absorbed the coefficient of $\vec{J}^2$ by a redefinition of the field $\vec{J}$. The $O(N)$ charge susceptibility, $\chi_0$, also remains finite for zero net charge (i.e., zero chemical potential). In the case of Bose-Einstein condensation with a finite $O(N)$ charge, however, $\chi_0$ diverges at the critical point. We do not consider this situation, but keep $\chi_0$ explicitly in the Hamiltonian for later convenience.

There are two contributions to the fluctuations of the energy density, $\delta E = T \delta S + h_i \delta \phi_i$, where $S$ is the entropy density. Consequently, the static correlation with the order

\footnote{In the remainder of this section, we explicitly denote the fluctuation of a variable $X$ by $\delta X$, in order to}
parameter fluctuation is given by

$$\langle \delta E \delta \phi_i \rangle = T \langle \delta S \delta \phi_j \rangle + h_j \langle \delta \phi_i \delta \phi_j \rangle.$$  \hspace{1cm} (6)

In an $O(N)$ symmetric system (no explicit symmetry breaking), $h_j = 0$ at the physical point. Therefore, the last term in Eq. (6) does not contribute to $\langle \delta E \delta \phi_i \rangle$. The correlation $\langle \delta S \delta \phi_i \rangle$ is nonzero at temperatures below $T_c$, where the symmetry is spontaneously broken in a specific direction of the field. Thus, for this component of the field $\phi_i$: $\langle \delta S \delta \phi_i \rangle \sim -\partial_T \langle \phi_i \rangle \neq 0$. Moreover, this quantity diverges close to the critical point in the broken phase, since $\langle \delta S \delta \phi_i \rangle \sim t^{\beta-1}$, where $t = (T - T_c)/T_c$ is the reduced temperature, and the critical exponent $\beta \leq 1/2$. However, at temperatures above $T_c$, the correlation function $\langle \delta S \delta \phi_i \rangle$ vanishes due to symmetry in the absence of the external field $h_i$. Indeed, since for $t > 0$ and $h_i \to 0$ the order parameter scales like $\langle \phi_i \rangle \sim t^{-\gamma} h_i$, which implies that $\langle \delta S \delta \phi_i \rangle \sim t^{-\gamma-1} h_i$. Thus, in the effective Hamiltonian, there is no bilinear contribution of the form $\sim \phi_i E$ for $t > 0$.

Now consider the autocorrelation function of the energy fluctuations

$$\langle \delta E \delta E \rangle = T^2 \partial_T E,$$ \hspace{1cm} (7)

which is proportional to the specific heat, $C$. Near the critical point, the singular part of $C$ scales as $\sim t^{-\alpha} \sim \xi^{\alpha/\nu}$ where $\xi$ is the correlation length. The specific heat is related to the static susceptibility of the energy, $C \propto \chi_E(|\vec{k}| = 0; T)$, up to some dimensionful factor. The sign and numerical value of the critical exponent $\alpha$ depends on the number of field components, $N$, and the dimensionality, $d$ (see e.g. [47]).

In the effective Hamiltonian, we have dropped the spatial derivative terms, i.e. terms of the form $(\nabla_m A_l)^2$, for all fields $A_l$ which turn out to be irrelevant for long wavelength physics, except for the order parameter. Consider for instance the term involving derivatives of the energy density, i.e. $(\nabla E)^2$. For negative $\alpha$, the specific heat remains finite at the critical point. Hence, the coefficient of the $E^2$ term in the Hamiltonian scales as $C^{-1} \sim \xi^0$. Using standard renormalization group arguments, one then finds that the derivative term $(\nabla E)^2$ is irrelevant. Also for positive $\alpha$, when the specific heat diverges as $\xi^{\alpha/\nu}$, the term is irrelevant as long as $\alpha/\nu < 2$. This inequality is in general satisfied, since $\alpha/\nu$ is small, $O(\varepsilon)$, where $\varepsilon = 4 - d$. In the case of interest, where the critical point is approached from the
symmetric phase, i.e. $T \rightarrow (T_c)^+$ for $h = 0$, all derivative terms of the conserved quantities are, by the same reasoning, negligible. Consequently, for static properties, contributions at the scale $\sim \Lambda$ are due only to loop corrections involving fluctuations of the order parameter. The corresponding derivative term is relevant, yielding nontrivial contributions to the critical exponents through the nonzero anomalous dimension.

III. CRITICAL STATICS

A. Critical exponents and scaling hypothesis

In this section we review the critical statics at continuous/second-order phase transitions [47, 48]. A general effective theory for the order parameter of a continuous phase transition was developed by Landau. This theory provides a mean-field description of the phase transition. The Ginzburg criterion defines the region of applicability of the mean-field approximation. Close to the critical point, in the critical region, the Ginzburg criterion is violated and mean-field theory breaks down. As the critical temperature is approached, low-energy fluctuations of the order parameter diverge owing to the flatness of the potential. Consequently, naive perturbation theory for loop corrections fails. One finds by dimensional analysis in terms of the correlation length $\xi$, that higher order interaction terms, e.g., the 4-point coupling, diverge as the critical point is approached for $d < 4$, in particular in three dimensions. Therefore, a systematic analysis of the loop contributions in the critical region is in general difficult.

In spite of these complications, various scaling relations have been found among the critical exponents. These relations imply that there are only a few independent critical exponents. In the $O(N)$ theory, there are two independent exponents associated with the reduced temperature and the external field. Except for the hyper scaling relations, the scaling relations hold for empirically determined exponents in critical region, but also for the Landau mean-field theory. Scaling relations are easily derived, once the general assumption of homogeneity is made for the singular part of the thermodynamic potential density:

$$F_s(t, h) = L^{-d}F_s(L^{\Delta_t}t, L^{\Delta_h}h),$$

where $L$ is an arbitrary number not much greater than unity, $d$ is the number of spatial dimensions, and $\Delta_{t,h}$ is the scaling dimension of the reduced temperature $t$ and the external
field, $h$. This hypothesis was established more rigorously by Kadanoff using block spin transformations for the Ising model. Later on Wilson developed a systematic method, applicable to any system, for evaluating scaling dimensions $\Delta_{\ell,h}$ explicitly. The latter is known as the renormalization group (RG) method with the epsilon expansion about the critical dimension [47, 48].

The RG method consists of two steps: i) integrating out a high momentum shell $\Lambda/b < k < \Lambda$ with the parameter $b > 1$, and ii) rescaling the unit length $k \to bk$ and other variables accordingly. Consecutive implementation of these procedures yields a flow of the renormalized Hamiltonian (thermodynamic potential), i.e., a flow under the RG transformation in the full parameter space $V$.

A critical point of a continuous phase transition corresponds to a fixed point of the flow, where the length scale $\xi$ goes to infinity. Let $V^*$ be a fixed point, and $v = V - V^*$ the deviation from it. Then the thermodynamic potential density can be written as

$$F(V) = F(V^*; \{v\}).$$

(9)

Consider a system at a point in the parameter space, which is not a fixed point. After a single renormalization step, we obtain

$$F(V^*; \{v_r\}, \{v_{ir}\}) \to b^{-d}F(V^*; \{b^{\Delta_{vr}}v_r\}, \{b^{\Delta_{vir}}v_{ir}\}),$$

(10)

where the deviations $\{v\}$ can be classified into relevant $\{v_r\}$ and irrelevant parameters $\{v_{ir}\}$ according to their scaling dimension. By definition, the relevant (irrelevant) parameters have positive (negative) scaling dimension $\Delta_{vr} > 0$ ($\Delta_{vir} < 0$), and ones with vanishing scaling dimension are called marginal parameters. The factor $b^{-d}$ in front of $F$ stems from the rescaling and reflects the dimensionality of $F$. Repeating this procedure $n$ times results in the substitution $b \to b^n$.

Note that in general, a constant term appears in $F$ after the renormalization procedures. This term, which breaks homogeneity, originates from integrating out the higher momentum shells. However, since this term is non-singular, it can be dropped. The remainder obeys the homogeneous relation: $F_s(V^*; \{v_r\}, \{v_{ir}\}) \simeq b^{-d}F_s(V^*; \{b^{\Delta_{vr}}v_r\}, \{b^{\Delta_{vir}}v_{ir}\})$. The homogeneous scaling relation [8] holds only close to the critical point, in the so called scaling region. Here the irrelevant variables are very small, and can be put to zero, since $b^{n\Delta_{vir}} \ll 1$. The singular behavior near the critical point is controlled only by the relevant parameters, and
various scaling relations are obtained naturally, provided the system is sufficiently close to the critical point. The relevant variables \( \{v_r\} \) again, can be identified with the temperature and the magnetic field, \( v_{r1} \propto t \) and \( v_{r2} \propto h \).

**B. Static critical phenomena**

We first discuss renormalization of the effective potential \( \Omega = -(\ln Z)/V \) per volume to define the static properties of the \( O(N) \) scalar field theory in the low energy limit. The partition function is defined by \( Z = \sum e^{-\mathcal{H}} \) with the dimensionless reduced Hamiltonian \( \mathcal{H} \). The static renormalization group aims at tracing the evolution of the coefficients in the Hamiltonian \( [4] \), under the RG transformation.

The theory is defined with a finite ultraviolet cutoff \( \Lambda \). This means that the \( O(N) \) scalar field theory is an effective one, which can be applied only at scales below \( \Lambda \). Since the soft modes are treated explicitly, the theory possesses the correct infrared behavior. We follow the renormalization group procedure developed by Wilson and Kogut [48]. This involves the two steps mentioned above: integration over the momentum shell \( \Lambda/b \leq k \leq \Lambda \) in loops corrections with a parameter \( b > 1 \), and rescaling the variables and fields

\[
\begin{align*}
x & \rightarrow x/b, \\
\Lambda & \rightarrow b\Lambda, \\
\phi_i & \rightarrow b^{a\phi} \phi_i.
\end{align*}
\]

The scaling dimension of the order parameter field, \( a_\phi \), is determined by the requirement that the rescaling leaves the auto-correlation function of \( \phi_i \) unchanged, i.e., keeping the derivative term of \( \phi_i \) to be marginal: \( a_\phi = \frac{1}{2}(d - 2 + \eta') \). Here \( \eta' \) is the anomalous dimension, not be confused with the shear viscosity, \( \eta \).

This procedure provides an evolution of the system under successive changes of the length scale and decimation of shorter wavelength modes. This process generates all couplings including higher order ones allowed by the symmetry of the system. The theory approaches a low-energy effective theory for the long wavelength modes.

After repeating the renormalization procedure \( l \) times, one obtains the well-known recur-
sion relations for the coefficients, to leading order in the coupling \( u \),
\[
  r_{l+1} = b^{d-2a_{\phi}} \left[ r_l + 2(N + 2)\Omega_4 u_l \left\{ \Lambda^2 \left( 1 - b^{-2} \right) - 2 r_l \ln b \right\} \right] \tag{14}
\]
\[
  u_{l+1} = b^{d-4a_{\phi}} u_l \left[ 1 - 4(N + 8)\Omega_4 u_l \ln b \right]. \tag{15}
\]

The above relations are obtained for \( d = 4 - \varepsilon \) dimensions. The factor \( \Omega_d = 2^{1-d} \pi^{-d/2} \Gamma(d/2) \) originates from the solid angle integration in \( d \) dimensions, divided by \( (2\pi)^d \), with \( \Gamma(x) \) being the Gamma function. In the right hand side of the recursion relations, the factors of \( b \) with exponents stem from the rescaling, while the terms proportional to \( \Omega_4 \) arise in the decimation of shorter wavelength modes. These two contributions play a competitive role in the RG evolution. This makes an appearance of non-trivial fixed points possible. A simple dimensional analysis shows that interaction terms higher than quartic are irrelevant under the renormalization. The recursion relations in fact admit a non-trivial critical fixed point,
\[
  r^* = \frac{1}{2} \varepsilon \frac{N + 2}{N + 8} \Lambda^2 + O(\varepsilon^2), \tag{16}
\]
\[
  u^* = \frac{\varepsilon}{4\Omega_4(N + 8)} + O(\varepsilon^2), \tag{17}
\]

implying that the system undergoes a second-order phase transition with infinite correlation length \( \xi \).

One can extract the scaling dimensions by observing how the coupling parameters behave near the fixed point. To do this, it is sufficient to linearize the recursion relations in terms of \( \delta r_l \equiv (r^* - r_l) / \Omega_4 \Lambda^2 \) and \( \delta u_l \equiv u^* - u_l \):
\[
\begin{pmatrix}
  \delta r_{l+1} - \delta r_l \\
  \delta u_{l+1} - \delta u_l
\end{pmatrix} \simeq \ln b \begin{pmatrix}
  2 - \frac{N + 2}{N + 8} \varepsilon & 4(N + 2) \left[ 1 + \frac{N + 2}{2(N + 8)} \varepsilon \right] \\
  0 & -\varepsilon
\end{pmatrix} \begin{pmatrix}
  \delta r_l \\
  \delta u_l
\end{pmatrix}. \tag{18}
\]

Then eigenvalue problem of the above matrix tells that only \( r \) is the relevant parameter and \( \delta r \propto b^{\Delta_r} \) with \( \Delta_r = 2 - \frac{N + 2}{N + 8} \varepsilon + O(\varepsilon^2) \) being the scaling dimension, while \( u \) is irrelevant, with a negative scaling dimension \( \Delta_u = -\varepsilon + O(\varepsilon^2) \).

We also note that in the long-wave length limit, the self-interaction of the field \( \phi_i \) vanishes in \( d = 4 \) because \( u^* \sim O(\varepsilon) \). Thus, the perturbative expansion in the coupling constant \( u \) is equivalent to an expansion in \( \varepsilon \). This expansion is valid near the fixed point in a dimension slightly below four. In dimensions higher than four, the fluctuation contribution to the renormalization of 4-point coupling is negligible, i.e. the mean-field description remains valid. The physical correspondence of dimensionful quantity \( r - r^* \) with thermodynamic variables is introduced by hand, e.g., \( r - r^* \propto T - T_c \) near the critical point.
Let us now examine the interaction term $\gamma_0 \phi_i^2 E$. Since only this term provides the static coupling between order parameter and energy density fluctuations, its critical behavior is crucial in the subsequent analyses of critical dynamics. A system with a non-conserved order parameter coupled to the conserved energy was classified by Hohenberg and Halperin (see Ref. \[45\] and references therein), as model C. The recursion relations for $\gamma_0$ and $C_0$ are given by

$$C_{i+1}^{-1} = b^{d-2a_E} C_i^{-1} [1 - 2Nv_1 \ln b],$$

$$v_{l+1} = b^{d-4a_0} v_l [1 - 8(N + 2)\Omega_4 u_l \ln b - 2Nv_l \ln b],$$

where $v_0 \equiv \Omega_4 \gamma_0^2 C_0$ is the dimensionless three-point coupling, and $d-2a_E = \tilde{\alpha}/\nu = \alpha \theta(\alpha)/\nu$ with $\alpha$ being the exponent of specific heat $C \sim (T - T_c)^{-\alpha} \sim t^{-\alpha}$. Here $\theta(\alpha)$ is the unit step function and $\alpha$ is the critical exponent of the specific heat. The fixed point of the coupling $v$ is given by $v^* = \tilde{\alpha} + O(\epsilon^2)$, which vanishes for negative $\alpha$. The sign and value of the critical exponent $\alpha$ depends on $N$ and $d$, as noted above. We return to this point in the subsequent section.

For $\alpha > 0$, fluctuations of the energy can become critical, i.e. the corresponding mass (the inverse of the specific heat) vanishes at the critical point. Thus, also the critical dynamics may be affected by energy fluctuations. On the other hand, for $\alpha < 0$ the mass term remains finite and fluctuations of the energy do not affect the static critical properties of other variables. Nevertheless, since the order parameter always exhibits critical fluctuations at a second-order transition, it is possible that these fluctuations affect other variables through dynamical effects, like mode-mode couplings.

\section{Critical Dynamics}

\subsection{The stochastic equation of motion}

To address the critical dynamics of a system, one needs the equations of motion \[49, 51\]. The low energy and long-wave length dynamics in the critical region is dominated by slow modes, i.e. fluctuations of the order parameter and the conserved quantities. We describe such modes by fields $A_l(t, \vec{x})$ varying in space and time, and introduce a stochastic equation
of motion to describe the dynamics of the fields. In the mixed Fourier representation

\[ \partial_t A_l(t, \vec{k}) = L_{lm}(\vec{k}) \frac{\delta \mathcal{H}}{\delta A_m(t, \vec{k})} - [A_l, A_m]_{PB} \frac{\delta \mathcal{H}}{\delta A_m(t, \vec{k})} + \Theta_l(t, \vec{k}). \]  

(21)

Here \( \mathcal{H} = \mathcal{H} (\{ A_l \}) \) is a reduced effective Hamiltonian, which is a functional of the slow modes, and \( e^{-\mathcal{H}} \) is proportional to the probability for a particular configuration of the fields \( A_l \). The first term on the right side involves transport coefficients \( L_{lm}(\vec{k}) \), which are responsible for the damping of fluctuations. Hence, this term describes irreversible processes. Owing to this term and the noise term \( \Theta_l \), the system eventually reaches an equilibrium state where \( \delta \mathcal{H}/\delta A_l = 0 \). The noise term satisfies the fluctuation-dissipation relation,

\[ \langle \Theta_l(t, \vec{k}) \Theta_m(t', \vec{k}') \rangle = 2L_{lm}(\vec{k})\delta(t - t')\delta(\vec{k} - \vec{k}'), \]  

(22)

which is valid for Gaussian noise. The cross terms with \( l \neq m \) originate from a possible bilinear mixing among the variables, \( \sim A_l A_m \), in the effective Hamiltonian \( \mathcal{H} \).

The second term, with the Poisson bracket, \([\cdots]_{PB}\), yields non-linear interactions, the mode-mode couplings \([52, 53]\). These describe the non-dissipative (reversible) processes, which are responsible for the large amplitude collective fluctuations induced by the critical behavior of the order parameter. Consequently, this term contributes to the singularities, which define the critical dynamics. The mode-mode couplings are formulated in terms of the generators of the relevant symmetries, and thus preserve the invariances of the original equations of motions.

The equation of motion can be derived from the Liouville equation by the projection method in the Markovian approximation under some reasonable assumptions. The derivation is reviewed in appendix A. Further details can be found in Ref. \([54]\). The presence of the Poisson bracket implies that the equations were derived from Hamilton’s equations of the classical theory, which is valid for slow modes (see also discussion in Ref. \([55]\)).
B. Response functions and transport coefficients

A set of stochastic equations of motion for the slow modes \( A_l = \{ \phi_i, E, \vec{J}, Q_{ij} \} \) is obtained from Eq. (21), given the effective Hamiltonian constructed above (4),

\[
\begin{align*}
\frac{\partial \phi_i}{\partial t} &= -\lambda_0 \frac{\delta H}{\delta \phi_i} - g_0 \vec{\nabla} \phi_i \cdot \frac{\delta H}{\delta \vec{J}} + \tilde{g}_0 [\phi_i, Q_{jk}]_{PB} \frac{\delta H}{\delta Q_{jk}} + \theta_i, \\
\frac{\partial E}{\partial t} &= \Gamma_0 \vec{\nabla}^2 \frac{\delta H}{\delta E} - g_0 \vec{\nabla} E \cdot \frac{\delta H}{\delta \vec{J}} + \theta_E, \\
\frac{\partial \vec{J}}{\partial t} &= \mathcal{T} \cdot \left[ \eta_0 \vec{\nabla}^2 \frac{\delta H}{\delta \vec{J}} + g_0 \vec{\nabla} \phi_i \frac{\delta H}{\delta \phi_i} + g_0 \vec{\nabla} E \frac{\delta H}{\delta E} + g_0 \vec{\nabla} Q_{ij} \frac{\delta H}{\delta Q_{ij}} + \tilde{\theta}_j \right], \\
\frac{\partial Q_{ij}}{\partial t} &= \Pi_0 \vec{\nabla}^2 \frac{\delta H}{\delta Q_{ij}} + \tilde{g}_0 [Q_{ij}, Q_{kl}]_{PB} \frac{\delta H}{\delta Q_{lk}} + \tilde{g}_0 [Q_{ij}, \phi_k]_{PB} \frac{\delta H}{\delta \phi_k} \\
&- g_0 \vec{\nabla} Q_{ij} \cdot \frac{\delta H}{\delta \vec{J}} + \theta_{ij},
\end{align*}
\]

where \( g_0 \) and \( \tilde{g}_0 \) are the mode-mode couplings associated with the translation and \( O(N) \) symmetries and \( \mathcal{T} = 1 - \frac{\vec{\nabla} \vec{\nabla}}{\vec{\nabla}^2} \) is the projection operator on the transverse direction. Fluctuations of the transverse momentum describe diffusive modes, while fluctuations of the longitudinal momentum coupled with energy fluctuation describe sound waves, which have a linear dispersion relation. The latter are not taken into account because the sound mode corresponds to fast dynamics, which does not affect late time evolution. The longitudinal momentum, however, has to be considered in order to address the critical behavior of the bulk viscosity. This will be discussed in section IV.

The Poisson bracket between \( Q_{ij} \) and \( \phi_i \) are deduced from the quantum commutation relations

\[
\begin{align*}
[\phi_i, Q_{jk}]_{PB} &= -\phi_j \delta_{ki} + \phi_k \delta_{ij}, \\
[Q_{ij}, Q_{kl}]_{PB} &= -Q_{jl} \delta_{ik} + Q_{jk} \delta_{il} + Q_{il} \delta_{jk} - Q_{ik} \delta_{jl}.
\end{align*}
\]

In line with (22), the noise term correlation functions satisfy

\[
\begin{align*}
\langle \theta_i(t, x)\theta_j(t', x') \rangle &= 2\lambda_0 \delta_{ij} \delta^d(x - x') \delta(t - t'), \\
\langle \theta_E(t, x)\theta_E(t', x') \rangle &= -2\Gamma_0 \vec{\nabla}^2 \delta^d(x - x') \delta(t - t'), \\
\langle \tilde{\theta}_j(t, x)\tilde{\theta}_j(t', x') \rangle &= -2\eta_0 \vec{\nabla}^2 \delta^d(x - x') \delta(t - t'), \\
\langle \theta_{ij}(t, x)\theta_{kl}(t', x') \rangle &= -2\Pi_0 \left( \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \right) \vec{\nabla}^2 \delta^d(x - x') \delta(t - t').
\end{align*}
\]
The transport coefficients are obtained from low energy limit of the dynamical response functions. In frequency and momentum space, they are given by

\[
\frac{1}{\lambda} = i \lim_{k \to 0} \frac{\partial \chi^{-1}_\phi(k)}{\partial \omega},
\]

\[
\frac{1}{\Gamma} = i \lim_{k \to 0} \vec{k}^2 \frac{\partial \chi^{-1}_E(k)}{\partial \omega},
\]

\[
\frac{1}{\eta} = i \lim_{k \to 0} \vec{k}^2 \frac{\partial \chi^{-1}_J(k)}{\partial \omega},
\]

\[
\frac{1}{\Pi} = i \lim_{k \to 0} \vec{k}^2 \frac{\partial \chi^{-1}_Q(k)}{\partial \omega},
\]

where we use the short hand notation \( k \equiv \{ \omega, \vec{k} \} \). The limit is taken first with respect to frequency and then to momentum. The response functions are obtained from the solution of the stochastic equations of motion, after averaging over the noise \( \langle \cdots \rangle_\theta \),

\[
\chi_\phi(k)_{ij} = \left\langle \frac{\delta \phi_i(k)}{\delta h_i(k)} \right\rangle_\theta \delta_{ij},
\]

\[
\chi_E(k) = - \left\langle \frac{\delta E(k)}{\delta \beta(k)} \right\rangle_\theta,
\]

\[
\chi_J(k)_{T ij} = \left\langle \frac{\delta \vec{J}_i(k)}{\delta \vec{H}_j(k)} \right\rangle_\theta,
\]

\[
\chi_Q_{ij}(k) = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \left\langle \frac{\delta Q_{ij}(k)}{\delta \mu_{kl}(k)} \right\rangle_\theta.
\]

The source terms are put to zero, after the variations in the above equations.

**C. Dynamical renormalization group and dynamic scaling**

In this section we investigate the fixed points of the stochastic equations of motion for the \( O(N) \) scalar field theory, using the dynamical renormalization group (DRG) **. We thus determine the universal properties of the critical dynamics. The procedure of the DRG is very similar to the static one: i) in loop corrections, the momentum shell \( \Lambda/b \leq |\vec{k}| \leq \Lambda \) is integrated out, while the frequency is integrated over the whole domain of definition \( -\infty \leq k_0 \leq \infty \), and ii) rescaling of all variables. A difference from the static renormalization is that there appears a frequency scale, and its scaling in length units is assumed to be

** In analogy to the static case, a fixed point of the equations of motion is necessary to be able to sort out the critical dynamics.
\( \omega \rightarrow b^z\omega \) where \( z \) is the dynamical critical exponent for the slowest mode. The rescaling factors in the \( O(N) \) scalar field theory follow from dimensional analysis of the stochastic equations of motion and of the effective Hamiltonian, \( \mathcal{H} \), which is a dimensionless quantity,

\begin{align*}
  x & \rightarrow b^{-1}x, \\
  \Lambda & \rightarrow b\Lambda, \\
  t & \rightarrow b^{-z}t, \\
  \phi & \rightarrow b^{a_\phi}\phi, \\
  E & \rightarrow b^{a_E}E, \\
  \vec{J} & \rightarrow b^{a_Q}\vec{J}, \\
  Q & \rightarrow b^{a_Q}Q.
\end{align*}

The exponents for the fields are \( a_E = (d - \bar{\alpha}/\nu)/2 \), \( a_J = d/2 \), \( a_Q = d/2 \), and as in the static case \( a_\phi = (d - 2 + \eta')/2 \). We have set the scaling dimension of \( \chi_0 \) to zero.

In evaluating the dynamical response function, we employ a loop expansion in terms of the deviation from the upper critical dimension \( \varepsilon = 4 - d \) in the same way as in the static case. For instance, the response function of the order parameter is expressed as

\[ \chi_\phi(k) = \left< \frac{\delta \phi(k)}{\delta h(k)} \right|_{h \rightarrow 0} = G_\phi(k) \left[ \lambda + \Sigma_\phi(k) \right], \]

where \( \Sigma \) represents the loop corrections integrated over the momentum shell and the bare propagator is given by

\[ G_\phi(k) = \frac{1}{-i\omega + \lambda_0 \left( r_0 + k^2 \right)}. \]

A renormalized relaxation rate for the order parameter fluctuation, \( \lambda \), is derived from the response function \( \chi_\phi \). This procedure corresponds to a single renormalization operation. Thus, the recursion relation for \( \lambda \) reads

\[ \lambda_{t+1}^{-1} = b^{2-z-\eta'}\lambda_t^{-1} \left[ 1 + \bar{\Sigma}_\phi(\lambda_t, \Gamma_t, \cdots ; b) \right], \]

where \( \bar{\Sigma} \) is a dimensionless loop function, and the overall rescaling factor in \( b \) can be determined from the rescaling factors of the other variables using the equation of motion. The recursion relations for the remaining transport coefficients follow the same procedure. See the following sections and Appendix B for details.
V. RESULTS FOR TRANSPORT COEFFICIENTS

A. Flow equation

In the dynamical renormalization procedure presented in the previous sections, we derived a set of recursion relations for transport coefficients to one loop order

\[
\lambda_{l+1} = b^{z-2+\nu} \lambda_l \left[ 1 - \frac{4\gamma^2 C_l \lambda_l}{\lambda_l + \Gamma_l/C_l} \Omega_4 \ln b + \frac{h_l^2 (N - 1)}{\lambda_l \chi_l (\lambda_l + \Pi_l/\chi_l)} \Omega_4 \ln b \right],
\]

\[
\Gamma_{l+1} = b^{z-2-\alpha/\nu} \Gamma_l \left[ 1 + \frac{3}{4} \frac{g_l^2}{\Gamma_l C_l^{-1} (\Gamma_l C_l^{-1} + \eta_l) \Omega_4^2 \lambda^2} (1 - b^{-2}) \right],
\]

\[
\eta_{l+1} = b^{z-2} \eta_l \left[ 1 + \frac{g_l^2}{24 \lambda_l \eta_l \Omega_4^2 (1 - b^{-2})} \right],
\]

\[
\Pi_{l+1} = b^{z-2} \Pi_l \left[ 1 + \frac{3g_l^2}{4 (\Pi_l + \eta_l) \Omega_4 \Lambda^2 (1 - b^{-2})} + \frac{g_l^2}{2 \lambda_l \Pi_l \Omega_4 \ln b} \right].
\]

The corresponding relations for the mode-mode couplings and static coefficients read

\[
g_{l+1} = b^{z-3+\epsilon/2} g_l,
\]

\[
\tilde{g}_{l+1} = b^{z-2+\epsilon/2} \tilde{g}_l,
\]

\[
C_{l+1}^{-1} = b^{d-2a_E} C_l^{-1} \left[ 1 - 2NC_l \gamma_l^2 \Omega_4 \ln b \right],
\]

\[
\gamma_{l+1} = b^{d-2a_E} \gamma_l \left[ 1 - 4 (N + 2) u_l^2 \Omega_4 \ln b - 2N \gamma_l^2 C_l \right],
\]

\[
\chi_{l+1} = \chi_l,
\]

where \(a_\phi\) and \(a_E\) were defined above. Note, that the mode-mode couplings \(g\) and \(\tilde{g}\) exhibit only trivial scaling without loop corrections. This follows from Ward identities for the higher order response functions. This can be also understood from Galilean invariance and invariance of the equations of motion under \(O(N)\) rotations. Using these recursion relations, we find the fixed points of the equations of motion, and extract dynamical critical exponents and scaling relations.
In the continuum limit, \( b \to 1 \), the recursion relations yield the flow equations,

\[
\begin{align*}
\partial \lambda &= \lambda \left[ z - 2 + \eta' - \frac{4v^*}{1 + \omega_1} - (N - 1) \frac{f_2}{1 + \omega_2} \right], \\
\partial \Gamma &= \Gamma \left[ z - 2 - \frac{\bar{\alpha}}{\nu} + \frac{3}{4} \frac{f_3}{1 + \omega_3} \right], \\
\partial \eta &= \eta \left[ z - 2 + \frac{1}{24} \frac{f_1}{1 + \omega_4} \right], \\
\partial \Pi &= \Pi \left[ z - 2 + \frac{3}{21 + \omega_4} + \frac{1}{2} \frac{f_2}{1 + \omega_2} \right],
\end{align*}
\]

where \( \partial = \frac{\partial}{\partial \nu} \) with \( b = 1 + \delta \). We introduce effective vertices for the mode-mode couplings:

\[
f_1 = \frac{g^2}{\eta_A} \Omega_4 \Lambda^2, \quad f_2 = \frac{g^2}{\eta_B} \Omega_4, \quad f_3 = \frac{g^2}{\eta_C} \Omega_4 \Lambda^2, \quad \text{and} \quad f_4 = \frac{g^2}{\eta_D} \Omega_4 \Lambda^2,
\]

\[
\begin{align*}
\partial f_1 &= f_1 \left[ -2 + \varepsilon - \eta' + \frac{4v^*}{1 + \omega_1} - (N - 1) \frac{f_2}{1 + \omega_2} - \frac{1}{24} \frac{f_1}{1 + \omega_4} \right], \\
\partial f_2 &= f_2 \left[ \varepsilon - \eta' + \frac{4v^*}{1 + \omega_1} - (N - 1) \frac{f_2}{1 + \omega_2} - \frac{1}{2} \frac{f_2}{1 + \omega_4} - \frac{3}{2} \frac{f_4}{1 + \omega_4} \right], \\
\partial f_3 &= f_3 \left[ -2 + \varepsilon + 2Nv^* - \frac{3}{4} \frac{f_3}{1 + \omega_3} - \frac{1}{24} \frac{f_1}{1 + \omega_4} \right], \\
\partial f_4 &= f_4 \left[ -2 + \varepsilon - \frac{3}{2} \frac{f_4}{1 + \omega_4} - \frac{1}{2} \frac{f_2}{1 + \omega_4} - \frac{1}{24} \frac{f_1}{1 + \omega_4} \right].
\end{align*}
\]

and ratios of the transport coefficients, \( \omega_1 = \frac{\Gamma c^{-1}}{\chi} \), \( \omega_2 = \frac{\lambda}{\pi \chi} \), \( \omega_3 = \frac{\Gamma c^{-1}}{\eta} \), and \( \omega_4 = \frac{\mu_{c^{-1}}}{\eta} \),

\[
\begin{align*}
\partial \omega_1 &= \omega_1 \left[ -\eta' - 2Nv^* + \frac{3}{4} \frac{f_3}{1 + \omega_3} + \frac{4v^*}{1 + \omega_1} - (N - 1) \frac{f_2}{1 + \omega_2} \right], \\
\partial \omega_2 &= \omega_2 \left[ \eta' - \frac{3}{2} \frac{f_4}{1 + \omega_4} - \frac{1}{2} \frac{f_2}{1 + \omega_2} - \frac{4v^*}{1 + \omega_1} + (N - 1) \frac{f_2}{1 + \omega_2} \right], \\
\partial \omega_3 &= \omega_3 \left[ -2Nv^* + \frac{3}{4} \frac{f_3}{1 + \omega_3} - \frac{1}{24} \frac{f_1}{1 + \omega_2} \right], \\
\partial \omega_4 &= \omega_4 \left[ \frac{3}{2} \frac{f_4}{1 + \omega_4} + \frac{1}{2} \frac{f_2}{1 + \omega_2} - \frac{1}{24} \frac{f_1}{1 + \omega_2} \right].
\end{align*}
\]

Here the static fixed point of the three point function \( v^* = \alpha/(2N\nu) \) has been inserted. The three point vertex is of order \( \varepsilon \), with \( \alpha/\nu = (4 - N)\varepsilon/(N + 8) + O(\varepsilon^2) \) near four dimensions.

The flow equations for the mode-mode couplings show that, except for \( f_2 \), there are contributions of order \( O(\varepsilon^0) \) on the right hand side. Since these equations admit only trivial stable fixed points, i.e. \( f_1^* = f_3^* = f_4^* = 0 \), these mode-mode couplings vanish in the long wavelength limit.

In the classification of the dynamical universality class we must, as implied by the discussion above, consider two cases, depending on the sign of \( \alpha \). The critical number \( N_c \), where
α changes sign, is given by Fischer \cite{47}: α is positive for $N < N_c$, with $N_c \simeq 4(1 - \varepsilon)$ near four dimensions, and $N_c \simeq 1.8$ for $d = 3$.

B. Fixed point for $N = 1$

We first consider the $N = 1$ case, where the symmetry is reduced to the discrete $Z_2$ symmetry, and the energy fluctuation must be taken into account, owing to the small but positive exponent $\alpha > 0$. The fixed points can be found by setting the right hand side of the flow equations to zero. To leading order in $\varepsilon$ we find:

\begin{align}
\lambda \left[ z - 2 - \frac{2\alpha}{\nu} \frac{1}{1 + \omega_1} \right] &= 0, \\
\Gamma \left[ z - 2 - \frac{\alpha}{\nu} \right] &= 0, \\
\frac{\omega_1 \alpha}{\nu} \left[ \frac{2}{1 + \omega_1} - 1 \right] &= 0,
\end{align}

which admit a stable fixed point,

\begin{align}
\omega_1^* &= 1, \\
z &= 2 + \frac{\alpha}{\nu},
\end{align}

The last equation defines the dynamical critical exponent, which was deduced from the condition that $\Gamma$ and $\lambda$ each have a finite non-trivial fixed point. Thus, the long-wavelength dynamics of the system is, up to order $\varepsilon$, governed by fluctuations of the energy and the order parameter on equal footing. In the critical limit $\xi \to \infty$ (keeping $\xi k$ finite) we obtain the following relaxation rates

\begin{align}
\delta \phi(t) &\sim \exp(-\lambda \chi_\phi(k)^{-1} t) \sim \exp(-k^{2+\alpha/\nu} t), \\
\delta E(t) &\sim \exp(-\Gamma C^{-1} k^2 t) \sim \exp(-k^{2+\alpha/\nu} t),
\end{align}

where we have used the fact that $\lambda \sim \xi^{2-z-\eta'} \sim \xi^{-\alpha/\nu + O(\varepsilon^2)}$, the order parameter susceptibility $\chi_\phi(k) \sim k^{-2+\eta'}$, and $\Gamma \sim \xi^{2-z+\alpha/\nu} \sim \xi^{0+O(\varepsilon^2)}$. This result\footnote{The dependence of transport coefficients (and of other physical quantities) on the coherence length (temperature) are derived as follows: let $\{a\}$ be the full set of parameters (static coefficients) including relevant and irrelevant ones. Now we pick up only one relevant parameter $a_1$, e.g., the reduced temperature $a_1 \propto t$, and set the other relevant parameters on the critical surface, i.e., to zero. Since $\xi = \xi(\{a\})$, a transport} can also be obtained from the
fixed point of $\omega^*_1$, which is the ratio of these two fluctuating modes, i.e., $\omega_1 \sim \Gamma C^{-1}/\lambda \sim \xi^{\eta'}$. Thus, to leading order in $\varepsilon$, the critical exponent of $\lambda$ is smaller than that of $\Gamma$ by $\alpha/\nu$.

It follows from the discussion above that, owing to the dominance of the fluctuations of the non-conserved order parameter and the conserved energy, the single component scalar theory belongs to the dynamic universality class of model C. In Ref. [55], the same conclusion was drawn based on the solution of a classical relativistic $\phi^4$ theory on the lattice in $d = 2$ spatial dimensions.

An important point, which was not discussed so far is the renormalization flow of the shear viscosity. In the regime where $z = 2 + \alpha/\nu$, $\eta$ does not reach a finite stable fixed point. This means that the critical dynamics of the order-parameter does not affect the shear fluctuations. Only short wavelength processes (rapid processes) contribute. Consequently, the shear viscosity remains finite, in contrast to model H, where a finite fixed point of a mode-mode coupling provides the scaling relation between the exponents of heat conductivity and shear viscosity.

In order to obtain a finite fixed point of the flow equation for $\eta^*$, $\partial \eta = \eta [z - 2]$, we have to set $z = 2$, which is smaller than that of the fluctuating modes of the order parameter and the energy. Thus, long-wavelength fluctuation of the transverse momentum diffuses faster than the other modes, since $\delta J(t) \sim \exp(-\eta_0 \chi^{-1}_\eta k^2 t) \sim \exp(-k^2 t)$, where we have used a bare shear viscosity $\eta_0 \sim \xi^0$ and susceptibility $\chi_\eta \sim \xi^0$. Therefore, fluctuations of the transverse momentum correspond to a faster mode and decouple in the long-wavelength dynamics inside the critical region.

### C. Fixed point for $N > 1$

For $N > 1$ the static coupling between the energy density and the order parameter vanishes at the fixed point, as shown by Hohenberg and Halperin [45] (more precisely the coefficient $\Gamma = \Gamma(\{a\}) = \Gamma(\xi, \{\bar{a}\})$, where $\{\bar{a}\}$ represents the irrelevant parameters. We drop the irrelevant parameters assuming that the system is very close to the critical point, and that the RG flow is sufficiently developed so that the parameters are in the immediate vicinity of the corresponding fixed point. Then, an RG transformation changes $\xi \rightarrow \xi/b$ and $\Gamma \rightarrow b^X \Gamma$. One thus finds the scaling relation $b^X \Gamma(\{a\}) = \Gamma(\xi/b, \{b^\Delta \bar{a}\})$ which leads to $\Gamma \sim \xi^{-X}$. 
effective three-body coupling $\gamma_0^2 C_0$ vanishes)\footnote{As explained earlier, the critical dynamics is governed by the sign of the critical exponent, $\alpha$. The absolute value of $\alpha$ is small for not too large $N$. Consequently, the sign of $\alpha$ is very sensitive to the approximations used. It is well known, that, to leading order, the epsilon expansion results in spurious sign of $\alpha$ in the range $2 < N < 4$ \cite{56}. To obtain a physically correct result, we use the input from non-perturbative methods according to which $\alpha$ in $d = 3$ is positive only for $N=1$, and negative otherwise.}. Therefore, the critical fluctuations of the order parameter do not directly affect the energy-momentum dynamics in the static case. In the dynamic case, such a coupling could be induced by the mode-mode coupling $f_1$, which, however, vanishes in the long-wavelength limit. Moreover, critical fluctuations of the order parameter couple to the $O(N)$ charge density only via the mode-mode coupling $f_2$. Thus, for $N > 1$ one expects the energy modes to be irrelevant, while the $O(N)$ charge fluctuations affect the critical dynamics owing to the mode-mode coupling $f_2$. Taking these arguments into account, we find the fixed points of the flow equations in the same way as for $N = 1$ case:

$$\lambda \left[ z - 2 + \frac{N - 1}{1 + \omega_2} f_2 \right] = 0, \quad (79)$$

$$\Pi \left[ z - 2 + \frac{1}{2} f_2 \right] = 0, \quad (80)$$

$$f_2 \left[ \varepsilon - \frac{N - 1}{1 + \omega_2} f_2 - \frac{1}{2} f_2 \right] = 0, \quad (81)$$

$$\omega_2 f_2 \left[ \frac{N - 1}{1 + \omega_2} - \frac{1}{2} \right] = 0. \quad (82)$$

These equations yield the following stable fixed point and dynamical exponent,

$$\omega_2^* = 2(N - 1) - 1, \quad (83)$$

$$f_2^* = \varepsilon, \quad (84)$$

$$z = 2 - \varepsilon = \frac{d}{2}. \quad (85)$$

The dynamical exponent is obtained by requiring that $\lambda$ and $\Pi$ have a non-trivial fixed point. At the critical point, the transport coefficients scale as $\lambda \sim \xi^{\varepsilon/2}$ and $\Pi \sim \xi^{\varepsilon/2}$ to leading order in $\varepsilon$. These results are consistent with the fixed point of the mode-mode coupling: $f_2 \sim \xi^{-\varepsilon + \eta'}$. Long-wavelength fluctuations of the order parameter and of the $O(N)$ charge fall off with a characteristic frequency $\omega_k \sim k^{d/2}$.

Fluctuations of energy and transverse momentum are governed by the flow equations $\partial \eta = \eta (z - 2)$ and $\partial \Gamma = \Gamma (z - 2)$. The fixed point at $z = 2$ implies that these fluctuations
are slower than those of the order-parameter and the $O(N)$ charge with $z = d/2 < 2$ in $d < 4$ dimensions. However, from the fixed point analysis we see that the critical fluctuations of the order parameter do not affect the energy and transverse momentum fluctuations in the long wavelength limit. Thus, although they participate in the critical dynamics at finite wavelengths, they decouple at late times. Consequently, owing to the dominance of the $O(N)$ charge fluctuations the critical dynamics of the multicomponent $O(N)$ theory is described by the dynamical universality class of model G.

D. Bulk viscosity

Before summarizing the main result of this work we briefly discuss the behavior of the bulk viscosity at the phase transition. The properties of the bulk viscosity in a slowly relaxing fluid and its possible singular behavior were first addressed in Ref. [57] (see also Ref. [58]). The behavior of the bulk viscosity in system with a single component non-conserved order parameter was considered in Ref. [59]. Here, however, the critical exponent was not evaluated, but rather it was guessed based on input from experiment.

In contrast to the shear viscosity, the bulk viscosity can diverge at the critical point in the $O(N)$ model depending on the value of $N$. For the case of the single component scalar theory in $d = 4 - \varepsilon$ spatial dimensions, the bulk viscosity tends to infinity as $\zeta \sim \xi^{z - \alpha/\nu} = \xi^2$ (to leading order in $\varepsilon$), while for the multicomponent $N > 1$ theory the bulk viscosity remains finite $\zeta \sim \xi^0$. Here $z$ is the dynamical critical exponent, which was determined from the slowest mode as a function of $N$.

In order to address the critical behavior of the bulk viscosity, the longitudinal component of the momentum current has to be considered in Eq. (25). In this case the projection operator on the transverse direction is dropped and an additional contribution owing to the bulk viscosity is added on the right hand side of Eq. (25). The critical behavior of the bulk viscosity can be deduced along the lines discussed in Ref. [60], where the dynamical critical exponent for the bulk viscosity in model H was computed. Also, the QCD critical end point, which is theoretically expected to exist at a finite density and temperature in the QCD phase diagram [61], belongs to the universality class of model H [62]. Recently the critical dynamics of the QCD critical end point was examined in a comprehensive manner based on DRG [63].
The results of Ref. [60] for the bulk viscosity can be immediately generalized to the single component scalar field theory since the (non)conservation of the order parameter does not affect the result as soon as the dynamical critical exponent is defined. Consequently, for $N = 1$ the bulk viscosity diverges at the critical point as $\zeta \sim \xi^{z-\alpha/\nu}$. Indeed, the bulk viscosity is given by (see e.g. Refs [60, 64])

$$\zeta = \frac{1}{d^2T} \lim_{\omega \to 0} \int_0^\infty dt \int d^d x e^{-i\omega t} \langle \Pi_{ii}(\vec{x}, t)\Pi_{jj}(0, 0) \rangle,$$

(86)

where $\Pi_{ij}$ is the stress tensor, which can be defined by comparing Eq. (25) with the Euler equation $\partial J_i/\partial t = \nabla_j \Pi_{ij}$. We are interested in only the dominant singular contribution to Eq. (86). As noted in Ref. [60], it arises from the part of the stress tensor that is proportional to $\gamma \phi^2$. The integral in Eq. (86) is taken over the domain with characteristic spatial extension of order $\xi$ and in the time direction of order $\xi^z$. Therefore, the dominant singular contribution to the bulk viscosity reads $\zeta \sim \xi^{z-d}\gamma^2\chi_\phi^2$, which reduces to $\zeta \sim \xi^{z-\alpha/\nu}$ after substitution of the renormalized quantities for $\gamma$ and $\chi_\phi$. In contrast to model H, one should, however, keep in mind that in this expression $z = 2 + \alpha/\nu$. Thus, extrapolating to $\varepsilon \to 1$, we find that the singularity of the bulk viscosity is given by $\zeta \sim \xi^2$, while in model H it is stronger, $\zeta \sim \xi^{2.8}$. Note, that in both cases the ratio of the singular part of the bulk viscosity to the relaxation time of the $O(N)$ charge fluctuations, $\tau$, vanishes at the critical point as $\zeta/\tau \sim \xi^{-\alpha/\nu} \sim C^{-1}$, in agreement with [14]. This is a consequence of the fact that the single component scalar field theory belongs to the same static universality class as the liquid-gas phase transition.

For $N > 1$ the above discussion does not apply because the energy fluctuation decouples from the order parameter in statics, i.e. $\gamma^* \to 0$, as we found in Section III. Owing to the vanishing mode-mode coupling $g^* \to 0$ in the long wave limit, the critical fluctuations do not couple dynamically to the current $J_i$ either.

Therefore, the bulk viscosity is finite $\zeta \sim \xi^0$ at the critical point. In this case, the ratio of the bulk viscosity to the relaxation time vanishes as $\zeta/\tau \sim O(\xi^{-z})$.

VI. SUMMARY

In this paper we have evaluated the critical exponents for the dynamics of the $O(N)$ scalar field theory with all possible slow modes. We showed that for the case of the single
component theory its dynamical universality class reduces to model C. The dynamical critical
exponent is given by \( z = 2 + \alpha / \nu \). On the other hand, for the multicomponent theory,
the critical dynamics is dominated by \( O(N) \) charge fluctuations. This drives the critical
exponent down to the value \( z = d / 2 \) and the theory belongs to the dynamic universality
class of model G. In both cases, \( N = 1 \) and \( N > 1 \), the shear viscosity remains finite at the
critical point, while the bulk viscosity diverges for \( N = 1 \), and remains finite for \( N > 1 \).

In QCD, the \( O(4) \) chiral symmetry in the light quark sector is broken by the finite \( u \) and \( d \)
quark masses. For high temperatures and small values of the chemical potential, the second-
order phase transition is replaced by a crossover. Our results imply that the singular part of
the shear and bulk viscosity remain finite also at the QCD phase transition. However, from
the present analysis within the DRG, we cannot draw any conclusions on the behavior of the
regular parts of the viscosities near a second-order or a crossover transition. This problem
can only be addressed in more microscopic approaches based on QCD or QCD-like models
\[38, 39, 65, 67\], or within the novel microscopic approach to critical dynamics, employing
the conjectured gravity dual description of conformal field theories \[68–70\].

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Appendix A: Derivation of stochastic equation of motion

In critical dynamics we are interested only in tracing the evolution of slow modes. From
microscopic point of view even if we start with a set of exact equation of motions for the slow
modes, they would be inevitably affected by all the other degrees of freedom (including fast
modes). Slow modes also would mix after finite elapse time. Therefore, we need a method to
extract time evolution of slow modes only from full microscopic equation of motion, where
the other degrees of freedom are fairly incorporated.

1. Master equation with projection

We first derive a master equation, i.e., an equation of motion for the distribution function
\( g_a(t) = \delta (A(t) - a) = \Pi_t \delta (A_t(t) - a_t) \), which defines a probability distribution for the
macroscopic variable $A_l(t)$ to take the value $a_l$ at the time moment $t$. We start with Liouville equation
\[
\frac{d}{dt}A_l(t) = iL A_l(t), \tag{A1}
\]
where since we are dealing with slow modes, the operator $L$ is supposed to be Poisson bracket with the classical Hamiltonian,
\[
iLA_l(t) = [H, A_l(t)]_{PB}. \tag{A2}
\]

We would like to split $g_a(t)$ into systematic and fluctuating parts. At initial time we start from a state defined by the slow variables. In general there is no a priori rule for the choice of slow variables. The integral of motion are, however, required to be included among slow modes. The slow variables at initial time $A_l(0)$ will be rotated in Hilbert space by the Liouville operator $\exp(itL)$. This would take $A_l(t)$ out of the subset of slow modes. By the systematic part of $g_a(t)$ we mean the amount of an overlap between initial and elapsed distributions at time $t$. Therefore, it is reasonable to define a projection onto initial state with equilibrium average $\langle \cdots \rangle$,
\[
P g_a(t) \equiv \sum_b \langle g_a(t) g_b(0) \rangle g_b(0) \tag{A3}
\]
with general properties of the projection operator such as $PP = P$, $P + \bar{P} = 1$ and $\langle P g_a(t) \bar{P} g_b(t) \rangle = 0$. The time evolution of $g_a(t) = P g_a(t) + \bar{P} g_a(t)$ is given by
\[
\frac{d}{dt}g_a(x, t) = -\sum_l \frac{\partial}{\partial a_l} [v_l(a) g_a(x, t)] + \int db \int ds \langle iLF_a(s); b \rangle g_b(x, t-s) + F_a(t), \tag{A4}
\]
where
\[
v_l(a) = \frac{\langle iL A_l(0) g_a(0) \rangle}{\langle g_a(0) \rangle} \equiv \langle iL A_l(0); a \rangle
\]
\[
F_a(t) = ie^{itPL}PLg_a(0). \tag{A5}
\]
In deriving the above equation, we have used the following decomposition of Liouville operator: $e^{itL} = e^{itPL} + i \int_0^t ds e^{isL} PLe^{i(t-s)PL}$, which can be verified by taking derivative from both sides.

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2. From Master equation to Langevin equation

One can derive non-linear Langevin equation for \( A_i(t) \) from Eq. (A4) by taking the first moment of the distribution function, \( A_i(t) = \int da \, a \, g_i(a) \),

\[
\frac{d}{dt} A_i(t) = v_i[A(t)] + \int_0^t ds \langle iLR_i(s); A(t-s) \rangle + R_i(t),
\]

\[
\simeq - \frac{\partial \langle [A(0), A_m(0)]_{PB}; A(t) \rangle}{\partial A_m(t)} + \langle \{A(0), A_m(0)\}_{PB}; A(t) \rangle \frac{\partial H[A(t)]}{\partial A_m(t)}
\]

\[
- L_{lm}[A(t)] \frac{\partial H[A(t)]}{\partial A_m(t)} + R_l(t),
\]

(A7)

where we introduced the effective Hamiltonian for macroscopic variables \( H(a) \equiv - \ln \langle g_a \rangle \) in units of \( k_B T = 1 \), and

\[
v_i[A(t)] = v_i(a)|_{a=A(t)} \equiv \langle iLA(0); a \rangle |_{a=A(t)},
\]

(A8)

\[
R_l(t) \equiv e^{itPL} \hat{P} \hat{A}_l(0) = ice^{itPL} \hat{P} \hat{L} \hat{A}_l(0),
\]

(A9)

\[
L_{lm}[A(t)] \equiv \int_0^\infty ds \frac{\langle R_l(s)R_m(0)g_a(0) \rangle}{\langle g_a(0) \rangle} |_{a=A(t)}.
\]

(A10)

Note that in derivation of Eq. (A7) from Eq. (A6) the Markovian approximation for the memory term \( \int_0^t ds \cdots A(t-s) \to \int_0^\infty ds \cdots A(t) \) was applied. We also assumed that the background transport coefficient \( L_{lm}(A) \) is approximately independent on \( A_i(t) \) at late times. Owing to properties of Poisson brackets the first term in Eq. (A7) vanishes in most cases.

Important point here is that since \( R_l(t) \propto \hat{P} \), the force \( R_l(t) \) is uncorrelated with any macroscopic variables by construction \( \langle G[A(0)]R_l(t) \rangle = 0 \) for any arbitrary function \( G[A(0)] \).

In this sense the force \( R_l(t) \) is a pure random force.

The second term in Eq. (A7), known also as mode-mode coupling, describes reversible process, and involve non-linear interactions among \( A_i(t) \), responsible for critical dynamics.
3. Application to $O(N)$ model

Substituting the slow mode candidates in $O(N)$ model and their effective Hamiltonian to the above Langevin equation, we obtain

\[
\frac{\partial \phi_i}{\partial t} = -\lambda_0 \left( r_0 \phi_i - \nabla^2 \phi_i + u_0 \phi_i^2 \phi_i + 2\gamma_0 \phi_i E - h_i \right) - g_0 \left( \nabla \phi_i \right) \cdot \left( \vec{J} - \vec{H} \right)
+ 2g_0 \phi_j \left( \chi_Q^{-1} Q_{ij} - \mu_{ij} \right) + \theta_i, \quad (A11)
\]

\[
\frac{\partial E}{\partial t} = \Gamma_0 \nabla^2 \left( C_0^{-1} E - \nabla^2 E + \gamma_0 \phi_i^2 + \beta \right) - g_0 \left( \nabla E \right) \cdot \left( \vec{J} - \vec{H} \right) + \theta_E, \quad (A12)
\]

\[
\frac{\partial \vec{J}}{\partial t} = \mathbf{T} \cdot \left[ \eta_0 \nabla^2 \left( \vec{J} - \vec{H} \right) + g_0 \left( \nabla \phi_i \right) \left( r_0 \phi_i - \nabla^2 \phi_i + u_0 \phi_i^2 \phi_i + 2\gamma_0 \phi_i E - h_i \right)
+ g_0 \nabla E \left( C_0^{-1} E - \nabla^2 E + \gamma_0 \phi_i^2 + \beta \right) + g_0 \nabla Q_{AB} \left( \chi_Q^{-1} Q_{AB} - \mu_{AB} \right) + \vec{\theta}_J \right], \quad (A13)
\]

\[
\frac{\partial Q_{AB}}{\partial t} = \Pi_0 \nabla^2 \left( \chi_Q^{-1} Q_{AB} - \mu_{AB} \right) - g_0 \left( \nabla Q_{AB} \right) \cdot \left( \vec{J} - \vec{H} \right)
+ g_0 \left( \phi_B h_A - \phi_A h_B - \phi_A \nabla^2 \phi_B + \phi_B \nabla^2 \phi_A \right) + \theta_{AB}. \quad (A14)
\]
In Fourier space $k \equiv \{ \omega, \vec{k} \}$, the formal solution is given by

$$\phi_i(k) = G^{0}_\phi(k) \times \left[ \lambda_0 h_i(k) + \theta_i(k) - ig_0 \int \phi_i(k_1) \vec{k}_1 \cdot \left\{ \vec{J}(k-k_1) - \vec{H}(k-k_1) \right\} \right] - \lambda_0 u_0 \int \phi_i(k_1) \phi_i(k-k_1-k_2) - 2\lambda_0 \phi_i(k)E(k-k_1) \right] + 2g_0 \phi_i(k_1) \left[ \chi^{-1}_Q Q_{ij}(k-k_1) - \mu_{ij}(k-k_1) \right], \quad (A15)$$

$$E(k) = G^{0}_E(k) \times \left[ -\Gamma_0 \vec{k}^2 \beta(k) + \xi(k) - \Gamma_0 \gamma_0 \vec{k}^2 \int \phi_i(k_1) \phi_i(k-k_1) - ig_0 \int \phi_i(k_1) \vec{k}_1 \cdot \left\{ \vec{J}(k-k_1) - \vec{H}(k-k_1) \right\} \right], \quad (A16)$$

$$\vec{J}(k) = G^{0}_{ij}(k) \cdot \mathcal{T}_k \cdot \left[ \eta_0 \vec{k}^2 \vec{H}(k) + \vec{\zeta}(k) \right] + ig_0 \int \vec{k}_1 \phi_i(k_1) \left\{ \chi^{-1}_\phi(\vec{k}-\vec{k}_1) \phi_i(k-k_1) - h_i(k-k_1) + 2\gamma_0 \int \phi_i(k_2)E(k-k_1-k_2) \right\} + ig_0 \int \vec{k}_1 \phi_i(k_1) \phi_j(k_2) \phi_j(k_3) \phi_i(k-k_1-k_2-k_3) + ig_0 \int \vec{k}_1 E(k_1) \left\{ \chi^{-1}_E(\vec{k}-\vec{k}_1)E(k-k_1) + \beta(k-k_1) + \gamma_0 \int \phi_i(k_2) \phi_i(k-k_1-k_2) \right\} + ig_0 \int Q_{AB}(k_1) \vec{k}_1 \left[ \chi^{-1}_Q Q_{AB}(k-k_1) - \mu_{AB}(k-k_1) \right] \right], \quad (A17)$$

$$Q_{AB}(k) = G^{0}_Q(k) \left[ \Pi_0 \vec{k}^2 \mu_{AB}(k) + \theta_Q(k) - ig_0 \int Q_{AB}(k_1) \vec{k}_1 \cdot \left\{ \vec{J}(k-k_1) - \vec{H}(k-k_1) \right\} \right] + g_0 \int \{ \phi_B(k_1) h_A(k-k_1) - \phi_A(k_1) h_B(k-k_1) \} + \phi_A(k_1) \phi_B(k-k_1) \left( \vec{k}^2 - 2\vec{k} \cdot \vec{k}_1 \right), \quad (A18)$$

where $\chi^{-1}_\phi(\vec{k}) = r_0 + \vec{k}^2$ the inverse of static susceptibility, $\int_{12\ldots m} = \int \Pi_{n=1}^m d\omega_n d^d k_n / (2\pi)^{d+1}$ and $(\mathcal{T}_k)_{ij} = \delta_{ij} - k_i k_j / \vec{k}^2$. Bare propagators read

$$G^{0}_j(k) \cdot \mathcal{T}_k = \frac{1}{-i\omega + \eta \vec{k}^2} \mathcal{T}_k, \quad (A19)$$

$$G^{0}_E(k) = \frac{1}{-i\omega + \Gamma_0 \vec{k}^2 \left( C^{-1}_0 + \vec{k}^2 \right)}, \quad (A20)$$

$$G^{0}_\phi(k) = \frac{1}{-i\omega + \lambda_0 \left( r_0 + \vec{k}^2 \right)}, \quad (A21)$$

$$G^{0}_Q(k) = \frac{1}{-i\omega + \Pi_0 \chi^{-1}_Q \vec{k}^2}, \quad (A22)$$

Note that $G^{0}_\phi(k)$ and $G^{0}_Q(k)$ are of diagonal form in $O(N)$ space.
The noise-noise correlation functions satisfy the fluctuation-dispersion relations,

\[
\langle \theta_i(k) \theta_j(k') \rangle = 2 \lambda_0 \delta_{ij} \delta \left( \vec{k} + \vec{k}' \right), \quad (A23)
\]
\[
\langle \xi(k) \xi(k') \rangle = 2 \Gamma_0 \vec{k}^2 \delta \left( \vec{k} + \vec{k}' \right), \quad (A24)
\]
\[
\langle \zeta_i(k) \zeta_j(k') \rangle = 2 \eta_0 \vec{k}^2 \delta_{ij} \delta \left( \vec{k} + \vec{k}' \right). \quad (A25)
\]

The renormalized transport coefficients are determined from the response functions,

\[
\phi_i(\omega, \vec{k}) = \chi_{\phi}(\omega, \vec{k}) h_j(\omega, \vec{k}), \quad (A26)
\]
\[
E(\omega, \vec{k}) = -\chi_E(\omega, \vec{k}) \beta(\omega, \vec{k}), \quad (A27)
\]
\[
J_i(\omega, \vec{k}) = \chi_J(\omega, \vec{k}) \left( T_{\vec{k}} \right)_{ij} H_j(\omega, \vec{k}). \quad (A28)
\]

**Appendix B: Response function**

The response functions of slow modes are obtained from a set of stochastic equations of motions. The old-fashioned perturbation method, i.e., iteration of formal solution for different orders of interaction terms and taking average over noises in energy-momentum space, systematically generates loop corrections to response functions. In this article we perform only leading order calculations. In order to proceed with the calculations beyond the leading order, it is preferable to use the alternative field-theoretical approach (already mentioned in the text).

1. **Order parameter relaxation constant**

The response function for order parameter fluctuations is given as follows:

\[
\chi_{\phi}(k) = \left. \frac{\delta \phi_i(k)}{\delta h_i(k)} \right|_{h_i \to 0} = G_{\phi}(k) \left[ \lambda + \Sigma_{\phi}(k) \right], \quad (B1)
\]

where loop corrections are accounted for in $\Sigma_{\phi}(k)$.
FIG. 1: Leading order contributions to $\chi_\phi$. Solid line represents propagator of order parameter fluctuations, dashed – energy, wavy – transverse momentum, and double – $O(N)$ charge fluctuations, respectively. The solid circle indicates external field.

The diagrams for the leading order contributions are shown in Fig. 1 and the correspond-
where equations are given by

\[
\Sigma_a^\phi(k) = 2(N+2)(\lambda u)^2 \int \frac{3r + \vec{k}^2 + \vec{k}_1^2 + \left(\vec{k} - \vec{k}_1 - \vec{k}_2\right)^2}{r + \left(\vec{k} - \vec{k}_1 - \vec{k}_2\right)^2} \frac{1}{[r + \vec{k}_1^2] [r + \vec{k}_2^2]}
\]

\[
\times \left[-i\omega + \lambda \left(3r + \left(\vec{k} - \vec{k}_1 - \vec{k}_2\right)^2 + \vec{k}_1^2 + \vec{k}_2^2\right)\right] (B2)
\]

\[
\Sigma_b^\phi(k) = \left[-2\lambda \gamma\right]^2 \int \frac{2\Gamma (\vec{k} - \vec{k}_1)^2 G_E(k_1 - k)G_E(k - k_1)G_\phi(k_1)}{1 + \left(\vec{k} - \vec{k}_1\right)^2} \left[2\lambda G_{\phi}(k_1)G_{\phi}(-k_1)G_E(k - k_1)\right] G_\phi(k) \lambda
\]

\[
= 4G_{\phi}(k)\lambda^2\gamma^2 C \int \frac{1}{i r + \vec{k}_1^2 - i\omega + \lambda \left(r + \vec{k}_1^2\right)} + \Gamma/C\vec{k}_1^2, \quad (B3)
\]

\[
\Sigma_c^\phi(k) = 2\lambda_0 g_0 \int G_\phi^0(k_1)G_\phi^0(-k_1)G_J(k - k_1)\vec{k}_1 \cdot \vec{T}_{k-\vec{k}_1} \cdot \vec{k}_1 + \cdots
\]

\[
= \left[1 - \lambda \left(r + \vec{k}_1^2\right) G_\phi^0(k_1)\right] g^2 \int \frac{1}{i r + \vec{k}_1^2 - i\omega + \lambda \left(r + \vec{k}_1^2\right)} + \eta\vec{k}_1^2, \quad (B4)
\]

\[
\Sigma_d^\phi(k) = 2\chi_0^{-1} g^2 \lambda \sum_j \int G_{Q,j}(k - k_1)G_{\phi_j}(k_1)G_{\phi_j}(-k_1) + \cdots
\]

\[
= \frac{g^2(N - 1)}{\chi Q} \int \frac{1}{i r + \vec{k}_1^2 - i\omega + \lambda \left(r + \vec{k}_1^2\right)} + \Pi\chi_0^{-1} \vec{k}_1^2, \quad (B5)
\]

The renormalized order parameter relaxation constant to leading order is given by

\[
\lambda_{\text{ren}}^{-1} = \left. \frac{\partial \chi_0^{-1}}{\partial (-i\omega)} \right|_{k \to 0} = \lambda^{-1} \left[1 + 4\gamma^2 C \lambda / \chi + \Gamma/C\Omega_4 \ln b - \frac{g^2(N - 1)}{\lambda \chi Q (\lambda + \Pi/\chi Q)} \Omega_4 \ln b \right]. \quad (B6)
\]

where \( \vec{k}_\pm = \vec{k}_1 \pm \vec{k}/2 \). We evaluate the above equation near \( d = 4 \) and at the critical point, where a renormalized mass goes like \( r \sim 0 \) (note that non-trivial fixed point \( r^* \) can be adjusted to 0). Since \( \Sigma_\phi \sim \vec{k}^2 \), \( \lambda \) acquires no mode-mode coupling contribution of order of \( g^2 \) at zero momentum.

In the following, we take the same procedure to obtain the other transport coefficients renormalized to the leading order.
2. Energy diffusion constant

\[ \chi_E(k) = -\left( \frac{\delta E(k)}{\delta \beta(k)} \right)_{\beta \to 0} = G_E(k) \left[ \Gamma \vec{k}^2 + \Sigma_E(k) \right]. \quad (B7) \]

The response function for energy fluctuation is given by

\[ \chi_E(k) = -\left( \frac{\delta E(k)}{\delta \beta(k)} \right)_{\beta \to 0} = G_E(k) \left[ \Gamma \vec{k}^2 + \Sigma_E(k) \right]. \quad (B7) \]

Leading order contributions \( \Sigma_E \) are depicted in Fig. 2, and are given by

\[ \Sigma_a^E(k) = G_E(k) \sum_i \left( 2\lambda \Gamma \vec{k}^2 \right)^2 \int_{-1}^1 G_\phi(k_1) \left[ G_\phi(-k_1)G_\phi(k - k_1) + G_\phi(k - k_1)G_\phi(k_1 - k) \right] \]

\[ = G_E(k) N2\lambda \left( \Gamma \vec{k}^2 \right)^2 \int_{-1}^1 \frac{2 \left( r + \vec{k}_1^2 + \vec{k}^2 / 4 \right)}{\left( r + \vec{k}_1^2 \right) \left( r + \vec{k}_+^2 \right) - i\omega + \lambda \left( r + \vec{k}_+^2 \right) + \lambda \left( r + \vec{k}_-^2 \right)} \] \quad (B8)

\[ \Sigma_b^E(k) = -g^2 \int_{-1}^1 2\Gamma \vec{k}_1^2 G_E(k_1)G_E(-k_1)G_J(k - k_1) \vec{k}_1 \cdot \vec{T}_{-\vec{k}_1 \cdot \vec{k}} \]

\[ + g^2 \int_{-1}^1 \vec{k}_1 \cdot \vec{T}_{-\vec{k} - \vec{k}_1 \cdot \vec{k}} (-2\eta) \left( \vec{k} - \vec{k}_1 \right)^2 G_E(k_1)G_J(k_1 - k)G_J(k - k_1) \]

\[ = g^2 \left[ C - \Gamma \vec{k}^2 G_E(k) \right] \int_{-1}^1 \frac{\vec{k} \cdot \vec{T}_{\vec{k} \cdot \vec{k}}}{-i\omega + \Gamma C^{-1} \vec{k}_+^2 + \eta \vec{k}_-^2}, \quad (B9) \]

where \( \vec{k}_\pm = \vec{k}_1 \pm \vec{k} / 2 \).

Evaluating equations above near \( d = 4 \) and at the critical point:

\[ \Gamma_{\text{ren}}^{-1} = \vec{k}^2 \frac{\partial \chi_E^{-1}}{\partial (-i\omega)} \bigg|_{k \to 0} \approx \frac{1}{\Gamma} \left[ 1 - G_E^{-1} \frac{\Sigma_{b, \beta}'}{\Gamma \vec{k}^2} \right] \]

\[ = \frac{1}{\Gamma} \left[ 1 - \frac{3}{4} \frac{g^2}{\Gamma C^{-1} \left( \Gamma C^{-1} + \eta \right)} \frac{\Lambda^2}{2} \left( 1 - b^{-2} \right) \right]. \quad (B10) \]
3. Shear viscosity

**FIG. 3: Loop corrections to \( \chi_J \).**

The response function for the transverse momentum is of a tensor form because of the projection operator \( \mathcal{T} \), with leading order corrections \( \Sigma_J \) shown in Fig. 3,

\[
\chi_J(k) (\mathcal{T}_k)_{ij} = \left. \frac{\delta J_i(k)}{\delta H_j(k)} \right|_{H \to 0} = G_J(k) (\mathcal{T}_k)_{ij} \cdot \left[ \eta \vec{k}^2 + \Sigma_J(k) \right]_{jj}. \tag{B11}
\]

\[
\Sigma^a_J(k) = -2\lambda g^2 \int \mathcal{T}_k \cdot \vec{k}_1 G_\phi(k_1) \left[ r + \left( \vec{k} - \vec{k}_1 \right)^2 \right] G_\phi(k - k_1) \vec{k}_1 \cdot \mathcal{T}_k
\times \left[ G_\phi(k_1 - k) - G_\phi(-k_1) \right] + \cdots
= 2g^2 \left[ 1 - \eta \vec{k}^2 G_J(k) \right] \int \frac{\vec{k} \cdot \vec{k}_1}{i r + \vec{k}^2 - i\omega + \lambda_0 \left( r + \vec{k}_1^2 \right) + \lambda_0 \left( r + \vec{k}_2^2 \right)}, \tag{B12}
\]

\[
\Sigma^b_J(k) = \Sigma^c_J(k) = 0, \tag{B13}
\]

where \( \vec{k}_1 \to - \left( \vec{k}_1 - \vec{k}/2 \right) \) to reach the last equality in \( \Sigma^a_J \). The transverse part \( \Sigma_J \) is extracted by projection \( (\text{Tr} \mathcal{T}_k = d - 1) \),

\[
\Sigma_J(k) = \frac{2g^2_0}{d - 1} \left[ 1 - \eta \vec{k}^2 G_J(k) \right] \int \frac{\vec{k} \cdot \vec{k}_1}{i r + \vec{k}^2 - i\omega + \lambda_0 \left( r + \vec{k}_1^2 \right) + \lambda_0 \left( r + \vec{k}_2^2 \right)}. \tag{B14}
\]
For $r \sim 0$

\[
G_{J}^{-1} \sum_{j} (k) \left|_{k \to 0} \right. \simeq G_{J} \frac{2g_{0}^{2}}{d-1} \int \frac{d^{2}k}{k^{2}} \frac{\bar{k} \cdot \bar{k}}{k^{2}} \frac{\bar{k} \cdot T_{k} \cdot \bar{k}}{\lambda_{0} (\bar{k}^{2} + \bar{k}^{2})} \left|_{k \to 0} \right.
\]
\[\simeq \frac{2g_{0}^{2}}{3\eta} \Omega_{4} \int_{\Lambda/b}^{\Lambda} d^{3}k \bar{k}^{3} \frac{\langle (\hat{k} \cdot \hat{k}_{1})^{2} [1 - (\hat{k} \cdot \hat{k}_{1})^{2}] \rangle_{4}}{2\lambda_{0}k_{1}^{2}} \]
\[= \Omega_{4} \frac{g^{2}}{24\lambda \eta} \left( 1 - \frac{1}{b^{2}} \right), \tag{B15}\]

where $\langle \cdots \rangle_{4}$ implies taking an average on the solid angle at $d = 4$: a $d$-dimensional angle average of an even power of one momentum component is defined by

\[
\langle k^{2n} \rangle_{d} \equiv \frac{\int d^{d}k k^{2n} \delta (k^{2} - 1)}{\int d^{d}k \delta (k^{2} - 1)} = \frac{1 \cdot 3 \cdots (2n - 1)}{2^{n} (n + d/2 - 1)!}, \tag{B16}\]

For the present use $\langle k^{2} \rangle_{4} = \frac{1}{4}$ and $\langle k^{4} \rangle_{4} = \frac{1}{8}$. The renormalized $\eta$ eventually reads

\[
\eta^{-1}_{\text{ren}} = \bar{k}^{2} \left. \frac{\partial \chi^{-1}}{\partial (-i\omega)} \right|_{k \to 0} = \eta^{-1} \left[ 1 - \frac{g^{2}}{24\lambda \eta} \Omega_{4} \frac{\Lambda^{2}}{2} (1 - b^{-2}) \right]. \tag{B17}\]

4. $O(N)$ charges diffusion constant

The response function for the $O(N)$ charges diffusion constant is defined by

\[
\chi_{Q}(k) = \left. \frac{\delta Q_{ij}(k)}{\delta \mu_{ij}(k)} \right|_{\mu \to 0} = G_{Q}(k) \left[ \Pi \bar{k}^{2} + \Sigma_{Q}(k) \right]. \tag{B18}\]
where different contributions depicted in Fig. 4 read

\[ \Sigma_Q^a(k) = -2 \lambda g^2 \int \left( \hat{k}^2 - 2 \hat{k} \cdot \hat{k} \right) G_\phi(k_1) \left[ G_\phi(-k_1)G_\phi(k - k_1) - G_\phi(k_1 - k)G_\phi(k - k_1) \right] + \cdots \]

\[ = \bar{g}^2 \left[ 1 - \Pi \chi^{-1} \hat{k}^2 G_Q(k) \right] \int \frac{4 \left( \hat{k} \cdot \hat{k}_1 \right)^2}{(r + \hat{k}^2) (r + \hat{k}_1^2)} \frac{1}{-i \omega + \lambda \left( r + \hat{k}^2 \right) + \lambda \left( r + \hat{k}_1^2 \right)} \]

\[ \Sigma_Q^b(k) = 2 \Pi g^2 \int \frac{\hat{k}_1^2 \hat{k} \cdot \hat{k}_1 \hat{k}_1 G_Q(k_1) G_Q(-k_1) G_J(k - k_1) [\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} (-1)] + \cdots}{r + \hat{k}^2 + \eta \hat{k}_1^2} \]

\[ = 2 g^2 \int_{i - i \omega + \Pi \hat{k}_1^2 + \eta \hat{k}_1^2} \hat{k} \cdot \hat{k}_1 \hat{k} \cdot \hat{k}_1 \phi \]

The renormalized \( O(N) \) charge diffusion constant is given by

\[ \Pi^{-1}_{\text{ren}} = \bar{k}^2 \frac{\partial \chi^{-1}}{\partial (-i \omega)} \bigg|_{k = 0} = \Pi^{-1}_0 \left[ G_Q^{-1} \left( 1 - \frac{\Sigma_Q}{\Pi_0 \hat{k}^2} \right) + G_Q^{-1} \left( 1 - \frac{\Sigma_Q}{\Pi_0 \hat{k}^2} \right) \right] \]

\[ \simeq \Pi^{-1} \left[ 1 - \frac{3 g^2}{4 (\Pi + \eta) \Pi} \Omega^2 \lambda^2 (1 - b^{-2}) - \frac{\bar{g}^2}{2 \lambda \Pi} \Omega^4 \ln b \right] \]

where we have taken the solid angle average:

\[ \frac{\Sigma_Q}{\Pi_0 \hat{k}^2} = \frac{2 g^2}{(\Pi + \eta) \Pi_0} \int \frac{\hat{k} \cdot \hat{k}_1}{\hat{k}^2} \frac{4 \left( \hat{k} \cdot \hat{k}_1 \right)^2}{(r + \hat{k}^2) (r + \hat{k}_1^2)} \frac{1}{-i \omega + \lambda \left( r + \hat{k}^2 \right) + \lambda \left( r + \hat{k}_1^2 \right)} \approx \frac{2 g^2}{(\Pi + \eta) \Pi} \Omega^2 \lambda^2 \]

\[ \frac{G_Q^{-1} \Sigma_Q}{\Pi_0 \hat{k}^2} = \frac{\chi^{-1} g^2 G_Q}{(\Pi + \eta) \Pi_0} \int \frac{\hat{k} \cdot \hat{k}_1}{\hat{k}^4} \left( r + \hat{k}^2 \right) \frac{1}{-i \omega + \lambda \left( r + \hat{k}^2 \right) + \lambda \left( r + \hat{k}_1^2 \right)} \]

\[ \simeq \frac{2 g^2}{\lambda_0 \Pi_0} \frac{(\hat{k} \cdot \hat{k}_1)^2}{\hat{k}_1^4} \approx \frac{\bar{g}^2}{2 \lambda \Pi} \Omega^4 \ln b. \]

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