The attributes of the Spin-Charge-Family theory giving hope that the theory offers the next step beyond the Standard Model

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Abstract. The assumptions of the standard model are still waiting for an explanation. The spin-charge-family theory is promising in offering not only the explanation for the standard model postulates for quarks and leptons and vector and scalar gauge fields, but also for the cosmological observations, like there are the appearance of the dark matter, the matter-antimatter asymmetry, making several predictions. This theory assumes that the internal space of fermions (spins, handedness and all the charges) are described by the Clifford algebra in \(d \geq (13+1)\)-dimensional space, representing in \(d = (3 + 1)\) all by the standard model required properties of quarks and leptons and antiquarks and antileptons, with families included. Fermions interact with gravity only (the vielbeins and the two kinds of the spin connection fields), manifesting in \(d = (3+1)\) as all the vector gauge fields and the scalar gauge fields (higgs scalars and Yukawa couplings). In this talk I overview shortly the achievements of the spin-charge-family theory so far, explaining in particular the new way of the second quantization of fermions, offered by the description of the internal space of fermions with the anticommuting Clifford algebra objects of the odd character.

1. Introduction
Let me start with the motivation for the spin-charge-family theory.

The standard model offered 50 years ago an elegant new step towards understanding elementary fermion and boson fields by postulating:

a. The existence of massless fermion family members with the spins and charges in the fundamental representation of the groups, a.i. the quarks as colour triplets and colourless leptons, a.ii the left handed members as the weak doublets, the right handed weak chargeless members, a.iii. the left handed quarks differing from the right handed leptons in the hyper charge, a.iv. all the right handed members differing among themselves in hyper charges, a.v. antifermions carry the corresponding anticharges of fermions and opposite handedness, a.vi. the number of massless families, determined by experiments (there is no right handed neutrino postulated, since it would carry none of the so far observed charges, and correspondingly there is also no left handed antineutrino allowed).

b. The existence of massless vector gauge fields to the observed charges of quarks and leptons, carrying charges in the adjoint representations of the corresponding charged groups.

c. The existence of the massive weak doublet scalar higgs, c.i. carrying the weak charge \(\pm \frac{1}{2}\) and the hypercharge \(\mp \frac{1}{2}\) (as it would be in the fundamental representation of the two groups), c.ii.
gaining at some step of the expanding universe the nonzero vacuum expectation value, c.iii.
breaking the weak and the hyper charge and correspondingly breaking the mass protection,
c.iv. taking care of the properties of fermions and of the weak bosons masses, c.v. as well as
of the Yukawa couplings.
d. The presentation of fermions and bosons as second quantized fields.
e. The gravitational field in $d = (3 + 1)$ as independent gauge field.

The standard model assumptions have been confirmed without raising any doubts so far, but
also by offering no explanations for the assumptions. The last among the fields postulated by
the standard model, the scalar higgs, was detected in June 2012, the gravitational waves were
detected in February 2016.

The standard model has in the literature several explanations, mostly with many new not
explained assumptions. The most popular seem to be the grand unifying theories [18, 19, 20,
21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]. At least $SO(10)$ offers the explanation for
the postulates from a.i. to a.iv. partly to b. — but does not explain the assumptions a.v. up
to a.vi., c. and d., and does not connect gravity with gauge vector and scalar fields.

What questions should one ask to be able to find a trustworthy next step beyond the standard
models of elementary particle physics and cosmology, which would offer understanding of not
yet understood phenomena?
i. Where do fermions, quarks and leptons, originate and why do they differ from the boson
fields in spins, charges and statistics?

ii. Why are charges of quarks and leptons so different, why have the left handed family members
so different charges from the right handed ones and why does the handedness relate charges to
anticharges?

iii. How can one describe the internal degrees of fermions to explain the Dirac's postulates of
the second quantization?

iv. Where do families of quarks and leptons originate and how many families do exist?
v. Why do family members – quarks and leptons — manifest so different masses if they all start
as massless?

vi. How is the origin of the scalar field (the Higgs's scalar) and the Yukawa couplings connected
with the origin of families and how many scalar fields determine properties of the so far (and
others possibly be) observed fermions and masses of weak bosons? (The Yukawa couplings
certainly speak for the existence of several scalar fields with the properties of Higgs's scalar.)

Why is the Higgs's scalar, or are all scalar fields of similar properties as the higgs, if there are
several, doublets with respect to the weak and the hyper charge?

vii. Do possibly exist also scalar fields with the colour charges in the fundamental representation
and where, if they are, do they manifest?

viii. Where do the so far observed (and others possibly non observed) vector gauge fields
originate? Do they have anything in common with the scalar fields and the gravitational fields?

ix. Where does the dark matter originate?

x. Where does the "ordinary" matter-antimatter asymmetry originate?

xi. Where does the dark energy originate and why is it so small?

xii. What is the dimension of space? $(3 + 1)$?, $((d - 1) + 1)$?, $\infty$?

And many others.

My working hypotheses is that a trustworthy next step must offer answers to several open
questions, the more answers to the above open questions the step covers the greater the
possibilities of the theory being the right next step.

I am proposing the spin-charge-family theory [1, 2, 6, 7, 9, 10, 11, 13, 12, 14], offering so far
the answers from i. to x. of the above questions; The more work is invested in this theory the
more answers to the above open questions the theory offers.
Let me make in what follows a short introduction into the spin-charge-family theory to show briefly up the way the theory is offering the answers to the above mentioned open questions. A more detailed presentation of the theory and its achievements are presented in Sect. 2.

The spin-charge-family theory is a kind of the Kaluza-Klein like theories [35, 36, 37, 38, 39, 40, 41, 42, 12] due to the assumption that in $d \geq 5$ — in the spin-charge-family theory $d \geq (13 + 1)$ — fermions interact with the gravity only $^1$, treating consequently all the vector gauge fields, the scalar gauge fields, and the gravity in an equivalent way, offering answers to the above questions vi. and viii.

In the spin-charge-family theory the fermion internal space is described by the "basis vectors", which are the superposition of the odd products of the Clifford algebra objects [57, 58]. There are two kinds of the Clifford algebra objects [1, 2, 16, 45, 43, 44]. In $d = (13 + 1)$-dimensional space the odd Clifford algebra objects of one kind offer in $d = (3 + 1)$ the description of the spins and all the charges of fermions and antifermions, since both — fermions and antifermions — appear in the same irreducible representation of one of the two Lorentz groups in the internal space of fermions, what consequently explains the connection among the spins, handedness and charges of fermions, answering the questions i. and iii.

The other kind takes care of the family quantum numbers of fermions, distinguishing among different irreducible representations [6, 7, 11, 13], and offering the answer to iv.

The creation operators, creating the single particle states, are tensor products of the superposition of the finite number of the Clifford odd "basis vectors" describing the internal space of fermions and of the infinite basis in the momentum space. The "basis vectors" of the internal space transfer their oddness to the creation operators and correspondingly guarantees the oddness of the single fermion states, since the vacuum state has an even Clifford character.

The Hilbert space of fermions is formed from all possible tensor products of any number of single fermion creation operators, operating on the vacuum state [16].

The spin-charge-family theory offers correspondingly answers to the questions from i. to iv., explaining the common origin of spins and charges of fermions and antifermions, of all the quantum numbers of quarks and leptons and antiquarks and antileptons postulated by the standard model, as well as of the origin of families. The theory explains as well the Dirac postulates of the second quantization of the fermion fields.

Fermions interact with the vielbeins and the two kinds of the spin connection fields, the gauge fields of the momenta and of the two kinds of the generators of the Lorentz transformations, determined by the two kinds of the Clifford algebra objects [6, 7, 9, 10, 11, 13, 12, 14, 16].

The spin connection fields of one kind manifest in $d = (3+1)$ as the vector gauge fields of the charges of fermions, as the gravitational fields and also as the scalar gauge fields [9], to which also the scalar fields which are the gauge field of the second kind of the spin connection fields contribute. These offer answers to the questions vi. and viii., while explaining the common origin of the gravity, the vector gauge fields of the charges and the scalar gauge fields. The scalar gauge fields of both origins — of both generators of the Lorentz transformations in internal space of fermions — determine the scalar higgs and the Yukawa couplings, which are in the standard model postulated.

$^1$ Correspondingly the spin-charge-family theory shares with the Kaluza-Klein like theories their weak points, at least: a. Not yet solved the quantization problem of the gravitational field. b. The spontaneous break of the starting symmetry, which would at low energies manifest the observed almost massless fermions [36]. Concerning this second point we proved on the toy model of $d = (5+1)$ that the break of symmetry can lead to (almost) massless fermions [62, 63, 64]. The breaks of symmetry are so far assumed (although they follow dynamically from the degrees of freedom of the simple starting action): i. The break at $E \geq 10^{16}$ is caused with the appearance of the right handed neutrinos with the family quantum numbers which distinguish from the family quantum numbers of the so far observed families and also of the by the spin-charge-family theory predicted fourth family, which is coupled to the observed three. ii. The electroweak break is caused with the nonzero vacuum expectation values of the scalar (with respect to $d = (3+1)$) gauge fields with the weak and hyper charges of the Higgs's scalar.
The two kinds of the Clifford algebra objects require the existence of the two groups of four families of quarks and leptons and antiquarks and antileptons. The two groups distinguish from each other with respect to the family quantum numbers and correspondingly with respect to the interaction with the different two groups of the scalar gauge fields, which determine masses of these two groups of families after the break of the weak and hyper charge symmetries. Consequently: a. To the observed three families of quarks and leptons and antiquarks and antileptons there must exist the fourth family [46, 48, 50, 51, 6, 13]. b. The second group of the four families offers the explanation for the existence of the dark matter [49, 56].

The quantum numbers of the weak charge and the hyper charge of the scalar fields, taking care of the masses of the two groups of four families, depend on the space index of the scalar fields. The scalar fields with the space indexes 7 and 8 do carry the weak and the hyper charge as assumed by the standard model, explaining the origin of scalar higgs and Yukawa couplings [46, 48, 50, 51, 6, 13], what adds the explanation to the question vi.

There appear in the spin-charge-family the scalar fields with the space indexes 9 – 14, which are the colour triplets [7, 56]. They cause the transitions of antiquarks and antileptons into quarks and back. In the expanding universe under the non equilibrium conditions they offer the explanation of today’s dominance of ordinary matter in the observed part of the universe, answering the question vii.

It remains to tell how does in the spin-charge-family appear the spontaneous breaking of the starting symmetry in \( d = (13 + 1) \), first with the appearance of the condensate of two right handed neutrinos [7, 6, 13], and then when scalar fields with space index (7, 8) obtain nonzero vacuum expectation values [6, 9, 10].

More detailed, although still short, overview of the spin-charge-family theory is presented in Sects. 2 and 2.1.2.

2. Short presentation of the spin-charge-family theory

The spin-charge-family theory assumes a simple starting action for fermions, coupled in \( d \geq (13 + 1) \)-dimensional space to only gravitational field through the vielbeins \( f_{\alpha a} \), the gauge fields of momenta, and the two kinds of the spin connection fields, \( \omega_{\alpha a b} \) and \( \bar{\omega}_{\alpha a b} \), the gauge fields of the two kinds of the generators of the Lorentz transformations of the two Clifford algebras \(^2\), and with the internal space of fermions described by the anticommuting ”basis vectors” of one of the two Clifford algebras, while the second kind of the Clifford algebra objects equips each irreducible representation of the ”basis vectors” with the family quantum numbers

\[
\mathcal{A} = \int d^d x \ E \frac{1}{2} (\bar{\psi} \gamma^a p_{0a} \psi) + h.c. + \int d^d x \ E \ (\alpha R + \tilde{\alpha} \tilde{R}) , \\
p_{0a} = f^a_{\alpha p_{0a}} + \frac{1}{2E} \{ p_{0a}, E f^a \} - , \\
p_{0a} = p_{\alpha} - \frac{1}{2} S_{ab} \omega_{\alpha a b} - \frac{1}{2} \bar{S}_{ab} \bar{\omega}_{\alpha a b} , \\
R = \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} (\omega_{\alpha a b} \omega_{\beta b} - \omega_{\beta b} \omega_{\alpha a b}) \} + h.c. , \\
\tilde{R} = \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} (\tilde{\omega}_{\alpha a b} \tilde{\omega}_{\beta b} - \tilde{\omega}_{\beta b} \tilde{\omega}_{\alpha a b}) \} + h.c. .
\]

\(^2\)Starting with the Grassmann algebra of the anticommuting objects \( \theta^a \)'s and their derivatives \( \frac{\partial}{\partial \theta^a} \), one can define two kinds of the Clifford algebra objects, anticommuting with each other and defining two independent algebras, Eqs. (A.3, 2).
Here \( f_a^b[a f^b] = f^{a[b} f^{c]} - f^{a[c} f^{c]} \).

As written in the introduction, the tensor products of the superposition of the finite number of anticommuting "basis vectors" and of the infinite basis in the momentum space offer the description of the fermion creation and annihilation anticommuting operators. The creation and annihilation operators explain the Dirac postulates for the second quantized fermions, Sect. 2.1.2.1.2, 2.1.3.

The single fermion states manifest in \( d = (3 + 1) \) space the spins and all the charges of the observed quarks and leptons and antiquarks and antileptons, as well as families. Each irreducible representation of the Lorentz group in the internal space of fermions, described by one kind of the Clifford algebra objects (represented by \( \gamma^a \)‘s), presented in Table D1, contains namely all the quarks, leptons, antiquarks and antileptons, explaining the spins, charges and handedness of fermions and antifermions. Another kind of the Clifford algebra objects (represented by \( \tilde{\gamma}^a \)‘s) equips each irreducible representation of the first kind of the Clifford algebra objects with the family quantum numbers. Table D2 presents two groups of four families, predicting the fourth family quantum numbers. Table D2 presents two groups of four families, predicting the fourth family quantum numbers. Table D2 presents two groups of four families, predicting the fourth family quantum numbers. Table D2 presents two groups of four families, predicting the fourth family quantum numbers. Table D2 presents two groups of four families, predicting the fourth family quantum numbers.

The spin connection gauge fields manifest in \( d = (3 + 1) \) as the ordinary gravity, the known vector gauge fields, the scalar gauge fields [9] with the properties of higgs explaining the higgses and the Yukawa couplings, predicting new vector and scalar fields, which offer explanation for the dark matter [49, 56].

To be in agreement with the observations in \( d = (3 + 1) \) the manifold \( M^{13+1} \) must break first into \( M^{(7+1)} \times M^{(6)} \) (which manifests as \( SU(7, 1) \times SU(3) \times U(1) \)), affecting both internal degrees of freedom - the one represented by \( \gamma^a \) and the one represented by \( \tilde{\gamma}^a \) [6].

There is a scalar condensate of two right handed neutrinos with the family quantum numbers of the group of four families which does not include the observed three families (in Table D2 appear both groups. the first four families include the observed three, the condensate of neutrinos belong to the second group of four families), bringing masses of the scale \( 10^{16} \) GeV or higher to all the vector and scalar gauge fields, which interact with the condensate [7].

Since the left handed spinors couple differently (with respect to \( M^{(7+1)} ) \) to scalar fields than the right handed ones, the break can leave massless and mass protected \( 2(7)^2 = 3\times16 \) families [62].

The rest of families get heavy masses [4].

There is additional breaking of symmetry: The manifold \( M^{(7+1)} \) breaks further to \( M^{(3+1)} \times SU(2) \times SU(2) \) included in \( M^{(4)} \). These electroweak break is caused by the scalar fields with the space index \( (7, 8) \). They carry due to the space index the weak charge and hyper charge [7, 6].

I shall shortly present the influence of the breaks with the condensate and with the scalar fields (the electroweak break) when presenting properties of fermions and vector and scalar gauge fields in \( d = (3 + 1) \).

Let me add that my talk at the 23rd Bled workshop "What comes beyond the standard models" 4-12 July 2020, published DMFA Založnštvo, Ljubljana, December 2020 is in great deal similar to this talk. The largest difference is in added proofs concerning the second quantization of fermions and antifermions. Another kind of the Clifford algebra objects (represented by \( \tilde{\gamma}^a \)‘s), presented in Table D2, contains namely all the quarks, leptons, antiquarks and antileptons, explaining the spins, charges and handedness of fermions and antifermions. Another kind of the Clifford algebra objects (represented by \( \tilde{\gamma}^a \)‘s) equips each irreducible representation of the first kind of the Clifford algebra objects with the family quantum numbers. Table D2 presents two groups of four families, predicting the fourth family quantum numbers.

\[ \gamma^a \]
of fermion fields while using the Clifford algebras.

2.1. Properties of fermion fields in the spin-charge-family theory

Let me start with the properties of the fermion fields in the spin-charge-family theory.

Fermion fields, presented as the superposition of tensor products of the anticommuting "basis vectors" describing fermion internal space and of commuting basis in the momentum (coordinate) space, manifest the anticommuting properties already on the single fermion level [17], demonstrating that the first quantized fermions are the approximation to the second quantized fields.

There are two kinds of the anticommuting algebras [1, 2, 13, 6, 16]: The Grassmann algebra with objects $\theta^a$'s and the derivatives with respect to $\theta^a$'s $- \frac{\partial}{\partial \theta^a}$ — and the two kinds of the Clifford objects, $\gamma^a$'s and $\tilde{\gamma}^a$'s.

$$\{\theta^a, \theta^b\}_+ = 0, \quad \{\frac{\partial}{\partial \theta^a}, \frac{\partial}{\partial \theta^b}\}_+ = 0, \quad \{\theta_a, \frac{\partial}{\partial \theta_a}\}_+ = \delta_{ab},$$

$$\langle \theta^a \rangle^\dagger = \eta^{aa} \frac{\partial}{\partial \theta_a}, \quad \langle \frac{\partial}{\partial \theta_a}\rangle^\dagger = \eta^{aa} \theta^a,$$

$$\{\gamma^a, \gamma^b\}_+ = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+, \quad \{\gamma^a, \tilde{\gamma}^b\}_+ = 0,$$

$$\langle \gamma^a \rangle^\dagger = \eta^{aa} \gamma^a, \quad \langle \tilde{\gamma}^a \rangle^\dagger = \eta^{aa} \tilde{\gamma}^a,$$

$$\langle a, b \rangle = \langle 0, 1, 2, 3, 5, \cdots, d \rangle.$$

They are expressible with one another

$$\gamma^a = \langle \theta^a + \frac{\partial}{\partial \theta_a} \rangle, \quad \tilde{\gamma}^a = i \langle \theta^a - \frac{\partial}{\partial \theta_a} \rangle,$$

$$\theta^a = \frac{1}{2} (\gamma^a - i \tilde{\gamma}^a), \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2} (\gamma^a + i \tilde{\gamma}^a).$$

(3)

Either the Grassmann or the two Clifford algebras offer in $d$-dimensional space $2 \cdot 2^d$ operators (the Grassmann algebra has $2^d$ products of $\theta^a$'s with identity included and $2^d$ products of $\frac{\partial}{\partial \theta_a}$'s with the identity included, each of the two Clifford algebras has $2^d$ products of $\gamma^a$'s with the identity included and $2^d$ products of $\tilde{\gamma}^a$'s with the identity included) [16, 17]. The identity is the self adjoint member. The signature $\eta^{ab} = diag\{1, -1, -1, \cdots, -1\}$ is assumed.

Let me add the generators of the Lorentz transformations in the internal space described by the Grassmann algebra and in the internal spaces described by both Clifford algebras

$$S^{ab} = i \langle \theta^a \frac{\partial}{\partial \theta^b} - \theta^b \frac{\partial}{\partial \theta^a} \rangle, \quad \langle S^{ab} \rangle^\dagger = \eta^{aa} \eta^{bb} S^{ab},$$

$$\tilde{S}^{ab} = i \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a), \quad \tilde{S}^{ab} = i \frac{1}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a),$$

$$\tilde{S}^{ab} = S^{ab} + \tilde{S}^{ab}, \quad \{S^{ab}, \tilde{S}^{cd}\} = 0,$$

$$\{S^{ab}, \gamma^c\} = i \langle \eta^{bc} \gamma^a - \eta^{ac} \gamma^b \rangle, \quad \{S^{ab}, \tilde{\gamma}^c\} = 0,$$

$$\{\tilde{S}^{ab}, \gamma^c\} = i \langle \eta^{bc} \tilde{\gamma}^a - \eta^{ac} \tilde{\gamma}^b \rangle, \quad \{\tilde{S}^{ab}, \tilde{\gamma}^c\} = 0.$$

(4)

The Grassmann algebra offers the description of the integer spin fermions, with the charges in the adjoint representations [16, 17]). Both Clifford algebras offer the description of the half integer spin fermions with charges in the fundamental representations. The Grassmann algebra and the two Clifford algebras, can be separated into odd and even parts with odd and even number of products of algebra elements. Only odd number of products of algebra elements have...
the anticommuting properties needed to describe the second quantized fermion fields in either the Grassmann or in the two Clifford algebras cases.

While in the Grassmann algebra the Hermitian conjugated partners of products of $\theta^a$'s are the corresponding products of $\bar{\theta}^a$'s, Eq. (2), and opposite, in the Clifford algebras the Hermitian conjugated partners are less transparent, due to the relations $\gamma^a \vec{\gamma} = \eta^{aa} \gamma^a$ and $\vec{\gamma} \gamma = \eta^{aa} \gamma^a$, Eq. (2).

In order to resolve the problem of the Hermitian conjugated partners in the Clifford case and also to be able to make predictions of the theory to be compared with the experimental results, let us arrange products of either $\gamma^a$'s or $\vec{\gamma}^a$'s into irreducible representations with respect to the Lorentz group with the generators [2] presented in Eq. (4) and to arrange the members of each irreducible representation to be eigenstates of the Cartan subalgebra

$$ S^{03}, S^{12}, S^{56}, \ldots, S^{d-1 \ d}, $$
$$ S^{03}, S^{12}, S^{56}, \ldots, S^{d-1 \ d}, $$

The easiest way to achieve this is to find the eigenstates of each member of the Cartan subalgebra

$$ S^{ab} (k) = \frac{k}{2} (ab) = \frac{1}{2} ( \gamma^a + \eta^{aa} \gamma^b ), \quad (ab) = 0 \quad (k) = \eta^{aa} (-k), $$

$$ S^{ab} [k] = \frac{k}{2} [ab] = \frac{1}{2} (1 + \frac{i}{k} \gamma^a \gamma^b ), \quad [ab] = 0 \quad [k] = \eta^{aa} [/k]. $$

The eigenstates of the corresponding $\vec{S}^{ab}$ follow by replacing $\gamma^a$'s with $\vec{\gamma}^a$'s.

The notation $(ab) [k]$ and correspondingly $(k) (ab)$ is introduced to simplify the discussions [43, 44]. The Clifford "vectors" — nilpotents $(k) (ab)$ and $(k) (k) = 0$, $(k)(k) = 0$ and projectors $[k] (ab)$ and $[k] (k) = [k], [k] [k] = [k]$ — of both algebras are normalized up to a phase [2, 16, 17].

Both have half integer spins. The "eigenvalues" of the operator $S^{03}$ for the "eigenvectors" $\frac{1}{2} (\gamma^0 \mp \gamma^3)$, for example, are equal to $\pm \frac{1}{2}$, respectively, for the "vectors" $\frac{1}{2} (1 \pm \gamma^3)$ are $\pm \frac{i}{2}$, respectively, while all the rest "eigenvectors" have "eigenvalues" $\pm \frac{1}{2}$. One finds equivalently for the "eigenvectors" of the operator $S^{03}$: for $\frac{1}{2} (\gamma^0 \mp \gamma^3)$ the "eigenvalues" $\pm \frac{1}{2}$, respectively, and
for the "eigenvectors" $\pm (1 \pm \gamma^0 \gamma^3)$ the "eigenvalues" \( k = \pm \frac{i}{2} \), respectively, while all the rest "eigenvectors" have \( k = \pm \frac{1}{2} \).

The "basis vectors", which are eigenvectors of all the members of a particular kind of the two Clifford subalgebras, are products of $\frac{d}{2}$ eigenvalues of the Cartan subalgebra members, either nilpotents or projectors or of both. For the description of the internal space of fermions only those "basis vectors" which are products of an odd number of nilpotents (if \( d = 2(2n + 1) \) only nilpotents, or $\frac{d}{2} - 2$ nilpotents and two projectors, or $\frac{d}{2} - 4$ nilpotents and four projectors, up to one nilpotent and $2n$ projectors, if $d = 4n$ we must start with one projector and the rest nilpotents) and the rest projectors are acceptable, since they anticommutate algebraically, what we expect for the single fermion states of the second quantized fields.

To clarify what does the anticommutation of the "basis vectors" mean, let us start with the first "basis vector", denoting it as $\hat{b}^m_{f=1 \dagger}$, with $f$ defining different irreducible representations and $m$ a member in the representation $f$. Let us make a choice of the starting "basis vector" for the Clifford algebra of the kind $\gamma^a$'s with an odd products of the nilpotents

$$
\hat{b}^m_{f=1 \dagger} = \begin{pmatrix}
0.3 & 1.2 & 5.6 & 7.8 & 9.10 & 11.12 & 13.14 & d-3d-2d-1d
\end{pmatrix}
$$

The rest products in $\cdots$ are assumed to be all projectors with $k = -1, [-]$. Its Hermitian conjugated partner $\hat{b}^m_{f=1}$ is presented in the second line. In Table D1 the same "basis vector", but with $d = (13 + 1)$ (with seven products of nilpotents and projectors), represents u-quark of right handness, with spin $\frac{1}{2}$ and with the colour $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{3}})$.

All the rest members of this irreducible representation are reachable by the application of $S^{ab}$. There are $2^d - 1$ members in each irreducible representation $f$.

Let us see how do $S^{ab}$'s transform the "basis vectors".

$$
\begin{align*}
S^{ac}_{\begin{pmatrix} k \end{pmatrix} [k]} & = \frac{i}{2} \eta^{ab} \eta^{cc} [-k][-k], & S^{ac}_{\begin{pmatrix} k \end{pmatrix} [k]} & = \frac{i}{2} [-k][-k], \\
S^{ac}_{\begin{pmatrix} k \end{pmatrix} [k]} & = \frac{i}{2} \eta^{ab} [-k][-k], & S^{ac}_{\begin{pmatrix} k \end{pmatrix} [k]} & = \frac{i}{2} \eta^{cc} [k][k].
\end{align*}
$$

We learn from Eq. (8) that $S^{10}$ transforms $\hat{b}^m_{f=1 \dagger}$ into, let us call it $\hat{b}^m_{f=1}$.

$$
\begin{pmatrix}
0.3 & 1.2 & 5.6 & 7.8 & 9.10 & 11.12 & 13.14 & d-3d-2d-1d
\end{pmatrix}
$$

In Table D1 the same "basis vector", with $d = (13 + 1)$, represents u-quark of right handness, with spin $\frac{1}{2}$ and with the colour charge equal to $(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{3}})$.

The application of all possible $S^{ab}$ generates $2^d - 1$ members of a particular Clifford irreducible representation. To each irreducible representation the Hermitian conjugated irreducible representation belongs. Starting with a Clifford odd "basis vector", the whole irreducible representation has an odd Clifford character, since $S^{ab}$ are even operators.

The Hermitian conjugated partner of the starting "basis vector" of an odd product of nilpotents obviously belong to another Clifford odd irreducible representation, since it is not reachable by $S^{ab}$. Each $S^{cd}$ namely transforms a pair of projectors into a pair of nilpotents, a pair of nilpotents transforms under $S^{cd}$ into a pair of projectors, and a pair of a nilpotent and a projector transforms into a pair of a projector and a nilpotent, changing in each member of a pair its $k$ into $-k$. The Hermitian conjugation transforms an Clifford odd $\hat{b}^m_{f \dagger}$, a product of
an odd number of nilpotents, each carrying its own \( k \), into the same number of nilpotents, each carrying \( -k \) \(^5\). while projectors are self conjugate.

The starting member \( \hat{b}^{m=1}_{f=1} \) of the next irreducible representation can be obtained from \( \hat{b}^{m=1}_{f=1} \) by replacing, for example, \((+i)[+]\) in \( \hat{b}^{m=1}_{f=1} \) with \([+i][+]\). This new "basis vector" does not belong to either the starting irreducible representation, or to the Hermitian conjugated partners of the starting irreducible representation, due to the way it is created: \( S^{\gamma \delta} \) transforms \((+i)[+\]) into \([-i][-\]) and the Hermitian conjugation transforms \((i)[-\]) into \([-i][+] \) \(^6\).

Exchanging all possible pairs in the starting "basis vector" by keeping the same \( k \)'s while transforming a pair of nilpotents into a pair of projectors, a pair of projectors into a pair of nilpotents and a pair of a nilpotent and a projector into a pair of the projector and the nilpotent, we generate \( 2^d-1 \) irreducible representations with \( 2^d-1 \) members each.

The Hermitian conjugation then generates \( 2^d-1 \cdot 2^d-1 \) partners to the \( 2^d-1 \) members of each of the \( 2^d-1 \) irreducible representations.

One can find that the algebraic product of \( \hat{b}^{m*}_{f*} \hat{b}^{m}_{f} \) is the same for all \( m \) of a particular irreducible representation \( f \). The proof can be found in Statement 4 of App. Appendix A.

Each irreducible representation contributes different algebraic product \( \hat{b}^{m*}_{f*} \hat{b}^{m}_{f} \) as it is proved in Statement 5 of App. Appendix A.

Let us define the vacuum state \( |\psi_{oc} > \) for the "basis vectors" space determined by \( \gamma^{a} \)'s as a sum of all different products of \( \sum_{f=1}^{2^d-1} \hat{b}^{m*}_{f*} \hat{b}^{m}_{f}, \forall m, \) and for \( d = 2n + 1 \), one obtains

\[
|\psi_{oc} > = \sum_{f=1}^{2^d-1} \hat{b}^{m*}_{f*} \hat{b}^{m}_{f}, \forall m, \quad \text{for } d = 2(2n + 1).
\]

(9)

Let me add that the application of any member of the Cartan subalgebras on the vacuum state gives zero: \( S^{\gamma \delta} |\psi_{oc} > = 0, S^{ab} |\psi_{oc} > = 0, \forall S^{ab} \) and \( S^{ab} \) belonging to Cartan subalgeras of Eq. (5).

One finds that all the members of all the irreducible representations fulfill together with their Hermitian conjugated partners the relations

\[
\hat{b}^{m*}_{f*} |\psi_{oc} > = 0 \cdot |\psi_{oc} > ,
\]

\[
\hat{b}^{m}_{f*} |\psi_{oc} > = |\psi_{oc} > ,
\]

\[
\{ \hat{b}^{m*}_{f*}, \hat{b}^{m'}_{f} \}_{*a+} |\psi_{oc} > = 0 |\psi_{oc} > ,
\]

\[
\{ \hat{b}^{m}_{f*}, \hat{b}^{m'}_{f} \}_{*a+} |\psi_{oc} > = \delta^{mm'} |\psi_{oc} > ,
\]

\[
\{ \hat{b}^{m*}_{f*}, \hat{b}^{m'}_{f} \}_{*a+} |\psi_{oc} > = 0 \cdot |\psi_{oc} > ,
\]

(10)

for each \( f \). \( *A \) represents the algebraic multiplication of \( \hat{b}^{m*}_{f*} \)'s and \( \hat{b}^{m'}_{f} \)'s among themselves and with the vacuum state \( |\psi_{oc} > \) of Eq.(9). For the proof see Statement 7 of App. Appendix A.

The relations of Eq. (10) almost manifest the anticommutation relations for the second quantized fermion fields postulated by Dirac [61]. It is pointed out almost, since the relation

\[
\{ \hat{b}^{m}_{f}, \hat{b}^{m'}_{f} \}_{*a+} |\psi_{oc} > = \delta^{mm'} \delta^{ff'} |\psi_{oc} >
\]

(11)

\(^5\) The "basis vectors" with an even number of nilpotents have in even dimensional spaces the property that there is one member of each irreducible representation, which is self adjoint, the one which is the product of only projectors.

\(^6\) We shall learn in Subsect. 2.1.1 that the requirement of Eq. (2.1.1) enables that the application of \( \tilde{S}^{\gamma \delta} \) on \( \hat{b}^{m=1}_{f=1} \) transforms \( \hat{b}^{m=1}_{f=1} \) into \( \hat{b}^{m=1}_{f=2} \).
is not fulfilled. There are, namely, besides $\hat{b}^m_f$, additional $2^d - 1$ members of the Hermitian
conjugated partners, belonging each to a different irreducible representation, which give a
nonzero contribution (they do not give an identity or zero as $\hat{b}^m_f (\hat{b}^m_f)^\dagger = \delta^mm'|\psi_{oc} >$)
does, they give instead an even "basis vector") when multiplying $\hat{b}^m_f$ from the left hand side:
$\hat{b}^m_{f*,A} \hat{b}^m_f \neq 0$ for $2^d - 1$ different $f' \neq f$, while $\hat{b}^m_{f*,A} \hat{b}^m_f = |\psi_{oc} >$.

Let me illustrate this on the example of $\hat{b}^{m=1\dagger}_{f=1} = (+(1)[+][+][+](+)(−)(−)(−)(−) \cdots (+)(−))$ of
Eq. (7). Besides

$$
\begin{align*}
(\hat{b}^{m=1\dagger}_{f=1})^\dagger &= \hat{b}^{m=1}_{f=1} = \\
&= \begin{bmatrix}
  d-1 & d-3 & d-2 & 13141112910 & 78 & 56 & 12 & 03 \\
  -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix},
\end{align*}
$$

also

$$
\begin{align*}
d-1 & d-3 & d-2 & 13141112910 & 78 & 56 & 12 & 03 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix},
$$

$$
\begin{align*}
d-1 & d-3 & d-2 & 13141112910 & 78 & 56 & 12 & 03 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix},
$$

etc (12)

applied on $\hat{b}^{m=1\dagger}_{f=1}$, give a nonzero contributions, what can be checked by taking into account
Eq. (C.1), and App. Appendix A, Statement 5.

But index $f$ determines different irreducible representations and we can not expect that
the algebraic anticommutation relations will be fulfilled also among different irreducible
representations. Different irreducible representations should be treated in tensor products and
should carry the additional quantum number, the quantum number of the particular irreducible
representation.

All the discussions about the Clifford algebra with $\gamma^*\alpha$'s, started by Eq. (7) and continuing
up to this point, can be as well repeated also for the Clifford algebra with $\hat{\gamma}^*\alpha$'s.

To describe fermions in the second quantized picture, we must take into account the basis
in the ordinary space (momentum or coordinate) as well. The Dirac’s postulates for the second
quantized fermion fields include the infinite basis in momentum space, while we treated so far
the finite dimensional internal space of fermions. Before extending the vector basis space by
making the tensor product of internal space and the momentum space let us recognize that the
observed quarks and leptons and antiquarks and antileptons do not at all suggest that there
might be two different internal spaces, which could be described by the two kinds of the Clifford
algebra objects. Let us therefore first reduce the Clifford space by the postulate, which leave
only $\gamma^*\alpha$'s as the algebra describing the internal degrees of freedom of fermions, while $\hat{\gamma}^*\alpha$'s are
used to give quantum numbers to different irreducible representations.

2.1.1. Reduction of the Clifford space  We want to give to each irreducible representation of
the Lorentz transformations in the internal space of fermions the quantum number, which will
distinguish among the $2^d - 1$ different irreducible representations. If we keep the Clifford algebra
of the $\gamma^*\alpha$'s kind to describe the internal space of fermions, then $\gamma^*\alpha$'s, or rather $\hat{S}^{ab}$'s, can be
used to determine "family" quantum number of each irreducible representation of the Lorentz
algebra in the Clifford space of $\gamma^*\alpha$'s.

7 Let us demonstrate this on the case of $d = (5 + 1)$, presented on Table B1. Besides \(56, 12, 03 \),
and all the rest of the fourth column of odd $H$, give a nonzero contribution, when applying on $\hat{b}^{m=1\dagger}_{f=1}$:

---

\[ (-)(-)(-)+(+) \] in addition to $\hat{b}^{m=1\dagger}_{f=1}$ and $\hat{b}^{m=2\dagger}_{f=1}$

and two more cases.
We want that at the same time all the relations among $\tilde{\gamma}^a$’s and $\hat{\gamma}^a$’s, presented in Eq. (2), remain unchanged, while we expect that the eigenvalues of the Cartan subalgebra of $\tilde{S}^{ab}$ change.

The postulate [2, 44, 11, 13, 14, 16]

$$\tilde{\gamma}^a B = (-)^B i B \gamma^a,$$

with $(-)^B = -1$, if $B$ is a function of an odd product of $\gamma^a$’s, otherwise $(-)^B = 1$ [44], does just that. It is not difficult to check that the relations in Eq. (2), concerning $\tilde{\gamma}^a$’s are still valid, (See Statement 3a, Sect. Appendix A): $\{\gamma^a, \gamma^b\} = 2\eta^{ab} = \{\tilde{\gamma}^a, \tilde{\gamma}^b\} + \{\tilde{\gamma}^a, \gamma^b\} = 0, (\gamma^a)^\dagger = \eta^{aa} \gamma^a, (\tilde{\gamma}^a)^\dagger = \eta^{aa} \tilde{\gamma}^a$.

After this postulate the vector space of $\tilde{\gamma}^a$’s is ”frozen out” and also the Grassmann algebra space is now reduced to $\theta^a = \gamma^a$ and $\frac{\partial}{\partial \theta^a} = 0$. Statement 3. of App. Appendix A demonstrates, how does the Grassmann space loose the Hermitian conjugated partners to $\theta^a$’s, while the whole space reduces to $\gamma^a$’s.

After this postulate no vector space of $\tilde{\gamma}^a$’s exists any longer, what is in agreement with the observed properties of fermions. While the anticommutation relations among $\gamma^a$’s and $\tilde{\gamma}^a$’s remain the same as in Eq. (2), it follows for the eigenvalues of $\tilde{S}^{ab}$, the proof is presented in App. Appendix A, Statement 3b,

$$S^{ab} (k) = \frac{k}{2} (ab) \, , \quad \tilde{S}^{ab} (k) = \frac{k}{2} (ab) \, ,$$

$$S^{ab} (k) = \frac{k}{2} (ab) \, , \quad \tilde{S}^{ab} (k) = \frac{k}{2} (ab) \, ,$$

showing that the eigenvalues of $S^{ab}$ on the nilpotents and projectors of $\gamma^a$’s differ from the eigenvalues of $\tilde{S}^{ab}$ on the nilpotents and projectors of $\gamma^a$’s. The members of the Cartan subalgebra of $\tilde{S}^{ab}$, Eq. (5), can now be used to give to the irreducible representations of $S^{ab}$ the ”family” quantum numbers.

Let me mention that if one arranges the space of odd products of $\gamma^a$’s with respect to $S^{ab}(= S^{ab} + \tilde{S}^{ab})$, these new ”basis vector” will form multiplets with integer spins and charges in adjoint representations. Like the ”basis vectors” expressed by Grassmann algebra do in Ref. [17], Table 1, but after the reduction of space with $\theta^a$’s replaced by $\gamma^a$’s and $S^{ab}$ replaced by $\tilde{S}^{ab} + S^{ab}$, with $\tilde{S}^{ab} = \frac{1}{2} \gamma^a \gamma^b$, with the application of $\tilde{\gamma}^a$ defined by Eq. (13).

It is useful to notice that $\gamma^a$ transform $(k)$ into $[-k]$, never to $[k]$, while $\tilde{\gamma}^a$ transform $(k)$ into $ab[k]$, never to $ab[-k]$

$$\gamma^a (k) = \eta^{aa} [-k], \quad \gamma^b (k) = -ik [-k],$$

$$\gamma^a (k) = \eta^{aa} [-k], \quad \gamma^b (k) = -ik \eta^{ab} (-k),$$

$$\tilde{\gamma}^a (k) = -i \eta^{aa} [k], \quad \tilde{\gamma}^b (k) = -k [ab],$$

$$\tilde{\gamma}^a (k) = -i \eta^{aa} [k], \quad \tilde{\gamma}^b (k) = -k [ab].$$

Some additional applications of $\tilde{\gamma}^a$’s and $\tilde{S}^{ab}$’s on nilpotents and projectors expressed by the $\gamma^a$’s can be found in App. Appendix C.

Each irreducible representation has now the ”family” quantum number, determined by $\tilde{S}^{ab}$ of the Cartan subalgebra of Eq. (5). Now we can replace the fourth equation in Eq. (10) $\{b^m, b^m_\dagger\}_{+A+}\psi_{oc} = \delta^{mm_\dagger}\psi_{oc}$ with the relation in Eq. (11) $\{b^m_\dagger, b^m_\dagger\}_{+A+}\psi_{oc} = \delta^{mm_\dagger}\delta_{ff_\dagger}\psi_{oc}$. 

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Each family contributes in even dimensional spaces one summand of $\frac{d}{2}$ projectors to the vacuum state $|\psi_{\text{vac}}\rangle$ of fermions, App. Appendix A.

Correspondingly the ”basis vectors” and their Hermitian conjugated partners fulfill algebraically the anticommutation relations of Dirac’s second quantized fermions: Different irreducible representations carry different ”family” quantum numbers and to each ”family” quantum number only one Hermitian conjugated partner with the same ”family” quantum number belongs. Each summand of the vacuum state, Eq. (9), belongs to a particular ”family”.

One can easily check that each ”basis vector” $\hat{b}_{m}^{f \dagger}$, applied algebraically on $|\psi_{\text{vac}}\rangle$, gives nonzero contribution on the summand obtained by $\hat{b}_{m}^{f} \hat{b}_{m}^{f \dagger}$, which is the same for all $m$ of particular $f$, representing therefore the corresponding state $|\psi_{m}^{f}\rangle$, while on all other summands $\hat{b}_{m}^{f \dagger}$ gives zero, $\hat{b}_{m}^{f}$ applying on $|\psi_{\text{vac}}\rangle$ gives zero for all $f$ and all $m$, Statement 5., App. Appendix A.

To define creation and annihilation operators, which determine on the vacuum state the single fermion states, we ought to make the tensor products of the $2 \frac{d}{2} - 1 \times 2 \frac{d}{2} - 1$ ”basis vectors”, describing the internal space of fermions and of infinite basis of momenta.

The oddness of the products of the odd number of $\gamma$’s guarantees the anticommuting properties of all the objects which include an odd number of $\gamma$’s.

The creation and annihilation operators, derived as tensor products of the ”basis vectors” and the basis in momentum space, will fulfill the Dirac’s postulates of the second quantized fermions without postulating them, as Dirac did. They follow by themselves from the fact that the creation and annihilation operators are superposition of odd products of $\gamma$’s.

### 2.1.2. Second quantized fermion fields

Let us try to recognize what properties should the single particle states have to form the Hilbert space of the second quantized fields.

In the references [14, 16, 17] the properties of the single fermion states, the tensor products among which form the Hilbert space, are discussed in details. In this talk I am presenting this topic from the point of view of the spin-charge-family theory. This theory offers, as written in the introduction, the explanation for the appearance of the spin (and handedness in the case of massless fermions), of all the charges, as well as of the families of fermions. The number of families depends on the way how does the symmetry of the space breaks from $d = (13 + 1)$ to $d = (3 + 1)$.

In Table D1 one irreducible representation of $SO(13 + 1)$ of one family (belonging to one of the two groups of four families, to the one which includes the so far observed three families) is presented. The first ”basis vector” describes the internal degrees of freedom of the right handed quark $\hat{u}_{R}^{1\dagger}$, of the first family with $(S^{03}, S^{12}, S^{56}, S^{78})$ equal to $(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, presented in Table D2 as $\hat{u}_{R}^{1\dagger}$. The ”basis vector” $\hat{b}_{m=1}^{f \dagger}$, Eq. 7, represents for $d = (13 + 1)$ just this $\hat{u}_{R}^{1\dagger}$ quark, and $\hat{b}_{m=1}^{f \dagger}$ is its Hermitian conjugated partner.

The ”basis vector” $\hat{b}_{m=2}^{f \dagger}$ represents for $d = (13 + 1)$ the right handed $u$-quark with all the properties of $\hat{u}_{R}^{1\dagger}$ except for the family quantum numbers, which are now equal to $(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$. One can read in Table D1 that the spin of this right handed quark $\hat{u}_{R}^{1\dagger}$ is $+\frac{1}{2}$, the weak $SU(2)$ charge is zero, the colour charge is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. It carries the additional $SU(2)$ charge equal to $\frac{1}{2}$ and the ”fermion” quantum number — $\tau^4$ charge — equal to $\frac{1}{2}$.

When solving the equations of motion for free massless fermions, which follow from the action, presented in Eq. (1), under the assumption, that at low energies the momentum of this right handed quark is $p^\alpha = (p^0, p^1, p^2, p^3, 0, \cdots, 0)$, the solution $s = 1$ is the superposition

$$
\hat{u}_{R}^{c\dagger}(p) = \beta(\hat{u}_{R}^{c\dagger} + \frac{p^1 + ip^2}{p^0 + |p|^2} \hat{u}_{R}^{c\dagger}),
$$

(16)
with \(|p^0| = |\vec{p}|\), with \(\uparrow, \downarrow\) denoting spin \(\pm \frac{1}{2}\), respectively, and with \(\beta^* \beta = \frac{|p^0|^2 + |p^3|^2}{2|p^0|}\) normalizing the state.

There are steps from the \(d = (13 + 1)\) dimensional space to the step where momentum in higher dimensions do not contribute to dynamics in \(d = (3 + 1)\), while the break of symmetry makes the internal degrees of freedom (spins and families) to manifest as the spin and charges as presented in Table D1 and families as presented at Table D2. One finds the detailed presentations in Ref. [6, 9, 7, 13, 46, 49, 64] and the references therein).

Let us here represent the general solutions of equations of motion for free massless fermions with the internal space of fermions described by the "basis vectors" \(\hat{b}_{\alpha}^{m}\) fulfilling the relations of Eq. (10), for each family \(f\) separately, and also with respect to different families, \(\hat{b}_{f,\alpha}^{m} \hat{b}_{f',\beta}^{m'} = \delta^{m m'} \delta_{f f'}\).

\[
\hat{b}_{sf}^{\dagger}(\vec{p})|_{p^0 = |\vec{p}|}\overset{\text{def}}{=} \sum_{m} \epsilon_{sf}^{m} (\vec{p}, |p^0| = |\vec{p}|) \hat{b}_{f,\alpha}^{m};
\]

\[
\hat{b}_{sf}^{\dagger}(\vec{p}, \vec{x}) \overset{\text{def}}{=} \sum_{m} \epsilon_{sf}^{m} (\vec{p}, \vec{x})|_{p^0 = |\vec{p}|} = \delta_{ss'} \delta_{ff'} ,
\]

\((s)\) represents different orthonormalized solutions of the equations of motion, \(\epsilon_{sf}^{m}(\vec{p}, |p^0| = |\vec{p}|)\) are coefficients, depending on momentum \(|\vec{p}|\) with \(|p^0| = |\vec{p}|\). For the case of the right handed \(u\)-quarks of Eq. (16) the two nonzero coefficients are \(\beta\) and \(\beta \frac{p^0 + ip^3}{|p^0| + |\vec{p}|}\).

Creation operators of an odd Clifford character \(\hat{b}_{sf}^{\dagger}(\vec{p})\) create the single particle states, \(<x|\psi_{sf}(\vec{p}, \textbf{p}^0) > |p_0 = |\vec{p}|\rangle\), manifesting the oddness of the creation operators

\[
<x|\psi_{sf}(\vec{p}, \textbf{p}^0) > |p_0 = |\vec{p}|\rangle = \int dp^0 \delta(p^0 - |\vec{p}|) \hat{b}_{sf}^{\dagger}(\vec{p}) e^{-i p^0 x^a} A |\psi_{oc} > = (\hat{b}_{sf}^{\dagger}(\vec{p}) \cdot e^{-i p^0 x^a - \vec{p} \vec{x}})|_{p^0 = |\vec{p}|} A |\psi_{oc} > ,
\]

with the property

\[
\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} \psi_{sf'}(\vec{p}', \textbf{p}^0 = |\vec{p}'|)|x > < x|\psi_{sf}(\vec{p}, \textbf{p}^0 = |\vec{p}|) >= 0
\]

\[
\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} e^{i p^0 x^a} |p^0 = |\vec{p}|\rangle e^{-i p^0 x^a} |p^0 = |\vec{p}|\rangle = \delta_{ss'} \delta_{ff'} \delta(\vec{p} - \vec{p}') <\psi_{oc}|\psi_{oc} > ,
\]

where it is taken into account that \(\int \frac{d^{d-1}x}{(\sqrt{2\pi})^{d-1}} e^{i p^0 x^a} e^{-i p^0 x^a} = \delta(\vec{p} - \vec{p}')\). We normalize \(<\psi_{oc}|\psi_{oc} > to one.

One further finds the single particle fermion states in the coordinate representation

\[
|\psi_{sf}(\vec{x}, \textbf{x}^0) > = \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} (\hat{b}_{sf}^{\dagger}(\vec{p}) e^{-i (p^0 x^0 - \vec{p} \vec{x})})|_{p^0 = |\vec{p}|} A |\psi_{oc} > = \sum_{m} \hat{b}_{f}^{m}|\psi_{oc} > \int_{-\infty}^{+\infty} \frac{d^{d-1}p}{(\sqrt{2\pi})^{d-1}} (\epsilon_{sf}^{m}(\vec{p}) e^{-i (p^0 x^0 - \vec{p} \vec{x})})|_{p^0 = |\vec{p}|} = \sum_{m} \hat{b}_{f}^{m}|\psi_{oc} > \epsilon_{sf}^{m}(\vec{x}) = (-i \frac{\partial}{\partial x^a}) \delta(\vec{x}) ,
\]
where it is taken into account that \( \hat{b}^s_{sf}(\vec{p}) | p^0 = | \vec{p} \rangle | \psi_{oc} > = \sum m c^{sf m} (\vec{p}, | p^0 \rangle = | \vec{p} \rangle \hat{b}^{sf}_{\upsilon} | \psi_{oc} > >

Eq. (17), \( \varepsilon = \pm 1 \), depending on handedness and spin of solutions.

Taking into account the above derivations, leading to \( \int dp^0 \delta(p^0 - | \vec{p} \rangle) e^{i(\vec{p} \cdot \vec{x}^0)} = 1 \) and \( < \psi_{oc} \hat{b}^{sf}(\vec{p}, p^0) \hat{b}^{sf}_{\upsilon} (\vec{p}, p^0) | \psi_{oc} > = \delta^{ss'} \delta_{\upsilon \upsilon'} | \psi_{oc} > >

one finds

\[
< \psi^{sf}(\vec{x}, x^0) | \psi^{sf}_{\upsilon}(\vec{x}', x^0) > =
\]

\[
= \int^{+\infty}_{-\infty} \frac{d^dp}{(2\pi)^d-1} \int^{+\infty}_{-\infty} \delta(p^0 - | \vec{p} \rangle) < \psi^{sf}(\vec{x}, x^0) | \vec{p} > < \vec{p} | \psi^{sf}_{\upsilon}(\vec{x}', x^0) >
\]

\[
= \int^{+\infty}_{-\infty} \frac{d^dp}{(2\pi)^d-1} e^{-ip \cdot \vec{x}} e^{ip \cdot \vec{x}'} \int dp^0 \delta(p^0 - | \vec{p} \rangle)
\]

\[
< \psi_{oc} \hat{b}^{sf}(\vec{p}, p^0) \hat{b}^{sf}_{\upsilon}(\vec{p}, p^0) \hat{b}^{sf}_{\upsilon'}(\vec{p}, p^0) \hat{b}^{sf}_{\upsilon'}(\vec{p}, p^0) | \psi_{oc} > =
\]

\[
= \delta^{ss'} \delta_{\upsilon \upsilon'} \delta(\vec{x} - \vec{x}').
\]

We took into account that \(< \psi_{oc} | \psi_{oc} > \) is normalized to 1.

The scalar product \( < \psi^{sf}(\vec{x}, x^0) | \psi^{sf}_{\upsilon}(\vec{x}', x^0) > \) has obviously the desired properties of the second quantized states.

The new creation operators \( \hat{b}^s_{sf}(\vec{p}, \vec{x}) \), which are generated on the tensor products of both spaces, internal and momentum, fulfill obviously the below anticommutation relations when applied on \( \psi_{oc} >

\[
\{ \hat{b}^s_{sf}(\vec{p}, \vec{x}), \hat{b}^s_{sf}(\vec{p}', \vec{x}) \} + \leftrightarrow | \psi_{oc} > = \delta^{ss'} \delta_{\upsilon \upsilon'} \delta(\vec{p} - \vec{p}') | \psi_{oc} > ,
\]

\[
\{ \hat{b}^s_{sf}(\vec{p}, \vec{x}), \hat{b}^s_{sf}(\vec{p}', \vec{x}) \} + \leftrightarrow | \psi_{oc} > = 0 \cdot | \psi_{oc} > ,
\]

\[
\{ \hat{b}^s_{sf}(\vec{p}, \vec{x}), \hat{b}^s_{sf}(\vec{p}', \vec{x}) \} + \leftrightarrow | \psi_{oc} > = 0 \cdot | \psi_{oc} > ,
\]

\[
\hat{b}^s_{sf}(\vec{p}, \vec{x}) \hat{b}^s_{sf}(\vec{p}, \vec{x}) | \psi_{oc} > = | \psi^{sf}_{\upsilon}(\vec{p}, \vec{x}) > ,
\]

\[
\hat{b}^s_{sf}(\vec{p}, \vec{x}) \hat{b}^s_{sf}(\vec{p}, \vec{x}) | \psi_{oc} > = 0 \cdot | \psi_{oc} > ,
\]

\[
| p^0 | = | \vec{p} | .
\]

It is not difficult to show that \( \hat{b}^s_{sf}(\vec{p}, \vec{x}) \) and \( \hat{b}^s_{sf}(\vec{p}, \vec{x}) \) manifest the same anticommutation relations also on tensor products of an arbitrary chosen products of sets of single fermion states [17].

2.1.3. Hilbert space of fermion fields. The tensor products of any number of any sets of the single fermion creation operators \( \hat{b}^s_{sf}(\vec{p}, \vec{x}) \) (fulfilling together with their Hermitian conjugated partners annihilation operators \( \hat{b}^s_{sf}(\vec{p}, \vec{x}) \) the anticommutation relations of Eq. (22)) form the Hilbert space of the second quantized fermion fields. The number of the sets is infinite. The internal space, defined by \( \hat{b}^s_{sf} \), contributes in \( d \)-dimensional space for each chosen momentum \( \vec{p} \) (and for a parameter \( \vec{x} \)) the finite number, \( 2^{2^{d-1}} \cdot 2^{d-1} \), of such sets, the total Hilbert space has, due to the infinite basis in the momentum (or coordinate) space, the infinite number of sets

\[
N_H = \prod_{\vec{p}} 2^{2^{d-2}}.
\]

The number operator is defined as

\[
N^s_{\vec{p}} = \hat{b}^s_{sf}(\vec{p}, \vec{x}) \hat{b}^s_{sf}(\vec{p}, \vec{x}) \psi_{oc} > ,
\]

\[
N^s_{\vec{p}} | \psi_{oc} > = 0 \cdot | \psi_{oc} > .
\]
The vacuum state contains no fermions.

The Clifford odd objects $\hat{b}_{\text{tot}}^{sf}(\vec{p}, \vec{x})$ demonstrate their oddness also with respect to the whole Hilbert space $\mathcal{H}$, that is with respect to any tensor product of members of any sets of creation operators $\hat{b}_{\text{tot}}^{sf}(\vec{p}', \vec{x})$. Correspondingly the anticommutation relations follow also for the application of $\hat{b}_{\text{tot}}^{sf}(\vec{p}, \vec{x})$ and $\hat{b}_{\text{tot}}^{sf}(\vec{p}', \vec{x})$ on $\mathcal{H}$

\[
\{\hat{b}_{\text{tot}}^{sf}(\vec{p}, \vec{x}), \hat{b}_{\text{tot}}^{sf}(\vec{p}', \vec{x})\}_{*T} + \mathcal{H} = \delta^{ss'} \delta_{jj'} \delta(\vec{p} - \vec{p}') \mathcal{H},
\]

\[
\{\hat{b}_{\text{tot}}^{sf}(\vec{p}, \vec{x}), \hat{b}_{\text{tot}}^{sf}(\vec{p}', \vec{x})\}_{*T} + \mathcal{H} = 0 \cdot \mathcal{H},
\]

\[
\{\hat{b}_{\text{tot}}^{sf}(\vec{p}, \vec{x}), \hat{b}_{\text{tot}}^{sf}(\vec{p}', \vec{x})\}_{*T} + \mathcal{H} = 0 \cdot \mathcal{H}.
\]

I presented in this talk the derivation of the creation and annihilation operator of the second quantized fermion fields, which obey the Dirac’s postulates for the second quantized fermion fields without postulating them, just by analyzing properties of creation and annihilation operators obtained as tensor products of the "basis vectors" of an odd Clifford algebra and of the basis in either momentum or coordinate space. In Ref. [15, 14, 16, 17] the relation between the creation and annihilation operators, postulated by Dirac and the ones presented in this talk are discussed.

2.1.4. Properties of fermions in $d = (3 + 1)$ This section follows quite a lot Refs. [6, 7, 8]. With respect to the last years I have not succeeded to improve much the part presented in this subsection. I have been working on the symmetries of the spin-charge-family theory and in particular on how can the theory, using the Clifford algebra to describe all the internal properties of fermions — spins, charges and families — help to explain the assumptions of the second quantized fermion fields. I shall therefore review the other achievements of the theory very briefly.

In Eq. (1) the starting action is presented for fermion and boson fields in $d = (13 + 1)$. Let me point out again that this simple starting action has all the degrees of freedom, needed to describe all the fermion, boson and scalar fields appearing in the standard model, as well as those, needed to explain the observed phenomena. Knowing the starting boundary conditions, the theory should lead to observable world in $(3 + 1)$-dimensional space. Instead, in order that predictions of the spin-charge-family theory are in agreement with the observed properties of quarks and leptons and antiquarks and antileptons, of the vector gauge fields and of the scalar gauge fields (manifesting as the higgs and Yukawa couplings), the manifold $\mathcal{M}^{(13+1)}$ ought to break first into $\mathcal{M}^{(7+1)} \times \mathcal{M}^{(6)}$ (which manifests as $SO(7, 1) \times SU(3) \times U(1)$), affecting fermions, vector gauge fields and scalar gauge fields.

This first break is assumed to be caused by the scalar condensate of two right handed neutrinos, presented in Table D3, Sect. Appendix D, which interact with all the scalar gauge fields (with the gauge fields with the space index $(5, 6, 7, \cdots, 14)$, as well as with those vector gauge fields (the gauge fields with the space index $(0, 1, 2, 3)$, which couple to the condensate. The only vector gauge fields, which remain massless, since they do not interact with the condensate, are the weak charge, colour charge and hyper charge vector gauge fields.

Since the left handed fermions couple differently to scalar fields than the right handed ones (as can be seen in Table D1), the right handed fermions have the weak charge equal to zero, while they are doublets with respect to the second $SU(2)$ charge, the left handed fermions are doublets with respect to the weak charge and have the second $SU(2)$ charge equal to zero.

It is worthwhile to notice the properties of antifermions appearing in the same irreducible representation as fermions, as seen in Table D1, they have all the properties as postulated by the standard model, with the handedness included. No assumption is needed with respect to handedness what must be done when using $SO(10)$ [19] to describe charges of fermions.
the break can leave massless and mass protected $2^i[(7+1)/2−1]$ families [62]. The rest of families get heavy masses.

The fermion families are arranged into twice two groups of massless four families, with respect to family quantum numbers as presented in Table D2 in Sect. Appendix D, each group manifesting $SU(2) \subset SO(3,1) \times SU(2) \subset SO(4)$ symmetry. One group manifests the $SU(2)_L \times SU(2)_L$ symmetry, the other $SU(2)_R \times SU(2)_R$ symmetry.

The nonzero vacuum expectation values of the scalar fields with the space index $(7, 8)$, which carry the weak and hyper charges, break the mass protection and make families massive [11, 13].

The breaks of the standing symmetry make the spins in higher dimensions to manifest as charges in $d = (3 + 1)$.

The superposition of the Lorentz members of the Clifford algebra, manifesting in $d = (3 + 1)$ the spins, Eq. (D.1), charges, Eqs. (D.2, D.3) and families, Eqs (D.4, D.5), are presented in Sect. Appendix D.

Let me rewrite the fermion part of the action, Eq. (1), by taking into account the degrees of freedom the action manifests in $d = (3 + 1)$ in the way that we can clearly see that the action does manifest in the low energy regime by the standard model required properties of fermions, of vector gauge fields and of scalar gauge fields [6, 8, 11, 13, 65, 66, 1, 2, 48, 49, 50].

$$\mathcal{L}_f = \bar{\psi} \gamma^m m(p_m) - \sum_{A_i} g(A_i \bar{A}_i) A_i m) \bar{\psi} + \{ \sum_{s=7,8} \bar{\psi} \gamma^s P_{0s} \psi \} + \{ \sum_{t=5,6,9,...,14} \bar{\psi} \gamma^t P_{0t} \psi \},$$

where $p_{0s} = p_s - \frac{1}{2} S^{s's'} s' + \frac{1}{2} S^{ab} \omega_{abs}$, $p_{0t} = p_t - \frac{1}{2} S^{t't'} t' + \frac{1}{2} S^{ab} \omega_{abs}$, with $m \in (0, 1, 2, 3)$, $s \in (7, 8)$, $(s', s'') \in (5, 6, 7, 8)$, $(a, b)$ (appearing in $S^{ab}$) run within either $(0, 1, 2, 3)$ or $(5, 6, 7, 8)$, $t$ runs in $(5, ..., 14)$, $(t', t'')$ run either in $(5, 6, 7, 8)$ or in $(9, 10, ..., 14)$. The spinor function $\psi$ represents all family members of all the $2^{7+1}-1 = 8$ families.

a. The first line of Eq. (26) determines in $d = (3 + 1)$ the kinematics and dynamics of fermion fields, coupled to the vector gauge fields [9, 13, 6]. The vector gauge fields are the superposition of the spin connection fields $\omega_{stm}$, $m \in (0, 1, 2, 3)$, $(s, t) = (5, 6, ..., 13, 14)$, the gauge fields of $S^m$. They are shortly presented in Sect. 2.2.1.

The operators $\tau^{A_i}$ ($\tau^{A_i} = \sum_{a,b} c_{A_i}^{a,b} S^{a,b}$, $S^{a,b}$ are the generators of the Lorentz transformations in the Clifford space of $\gamma^a$'s) are presented in Eqs. (D.2, D.3) of Sect. Appendix D. They represent the colour charge, $\tilde{\tau}^3$, the weak charge, $\tilde{\tau}^1$, and the hyper charge, $Y = \tau^1 + \tau^{23}$, $\tau^4$ is the fermion charge, originating in $SO(6) \subset SO(13,1)$, $\tau^{24}$ belongs together with $\tilde{\tau}^1$ of $SU(2)_{weak}$ to $SO(4)$ ($\subset SO(13,1)$).

One fermion irreducible representation of the Lorentz group contains, as seen in Table D1, quarks and leptons and antiquarks and antileptons, belonging to the first family in Table D2. One can notice that the $SO(7,1)$ subgroup content of the $SO(13,1)$ group is the same for the quarks and leptons and the same for the antiquarks and antileptons. Quarks distinguish from leptons, and antiquarks from antileptons, only in the $SO(6) \subset SO(13,1)$ part, that is in the colour ($\tau^{33}, \tau^{38}$) part and in the fermion quantum number $\tau^4$. The quarks distinguish from antiquarks, and leptons from antileptons, in the handedness, in the colour part and in the $\tau^4$ part, explaining the relation between handedness and charges of fermions and antifermions.

9 Ref. [12] points out that the connection between handedness and charges for fermions and antifermions, both appearing in the same irreducible representation, explains the triangle anomalies in the standard model with no need to connect “by hand” the handedness and charges of fermions and antifermions.
The vector gauge fields, which interact with the condensate, presented in Table D3, become massive. The vector gauge fields not interacting with the condensate — the weak, colour and hyper charged vector gauge fields — remain massless, in agreement with by the standard model assumed gauge fields before the electroweak break of the mass protection.

After the electroweak break, caused by the scalar fields, the only conserved charges are the colour and the electromagnetic charge $Q = \tau^{13} + Y$, $Y = \tau^4 + \tau^{23}$.

b. The second line of Eq. (26) is the mass term, responsible in $d = (3 + 1)$ for the masses of fermions. The interaction of fermions with the superposition of the spin connection fields with the space index $s = (7, 8)$, which gain nonzero vacuum expectation values, cause the electroweak break, bringing masses to fermions and antifermions and to the weak vector gauge fields. They are superposition of either $\omega_{s't's}$ or $\bar{\omega}_{abs}$. These scalar fields explain the appearance of the higgs and Yukawa couplings of the standard model. Their properties are shortly presented in Subsect. 2.2.2.

These scalar gauge fields split into two groups of four families, one group manifesting the symmetry $-SU(2)_{SO(3,1),L} \times SU(2)_{SO(4),L} \times U(1)$ — and the other the symmetry $-SU(2)_{SO(3,1),R} \times SU(2)_{SO(4),R} \times U(1)$, Eq. (36). The scalar gauge fields, manifesting $SU(2)_{L,R} \times SU(2)_{L,R}$, are the superposition of the gauge fields $\bar{\omega}_{abs}$, $s = (7, 8), (a, b) = \text{(either (0, 1, 2, 3) or (5, 6, 7, 8)}, manifesting as twice two triplets interacting each with one of the two groups of four families, presented in Table D2. The three $U(1)$ singlet scalar gauge fields are superposition of $\omega_{s't's}$, $s = (7, 8), (s', t') = (5, 6, 7, 8, \ldots, 14)$, with the sum of $S^{s't'}$ arranged into superposition of $\tau^{13}, \tau^{23}$ and $\tau^4$. The three triplets interact with both groups of quarks and leptons and antiquarks and antileptons.

Each of the two groups have well defined symmetry of mass matrices, what limits the number of free parameters [54].

To one of the groups of four families the observed quarks and leptons belong [48, 51, 54, 55]. We predict the mixing matrices for quarks, taking as the input the masses of the fourth family, since the elements for the $3 \times 3$ submatrix of the $4 \times 4$ mixing matrix are (far) not accurately enough measured $^{10}$, that we could predict masses of the fourth family quarks [48, 51, 12]. The newer are the experimental data the better is the agreement of the measured mixing matrix elements with our predictions [51, 55] at least so far.

The stable of the upper four families offers the explanation for the dark matter appearance and it is so far in agreement with experimental evidences of the dark matter [49, 56].

I discuss predictions of the spin-charge-family theory for the properties of the lower four families and of the dark matter in Sect. 3.

c. The third line of Eq. (26) represents the scalar fields, which cause transitions from antileptons and antiquarks into quarks and leptons and back, offering the explanation for the matter/antimatter asymmetry in the expanding universe at non equilibrium conditions [7]. They are colour triplets with respect to the space index equal to $(9, 10, 11, 12, 13, 14)$, while they carry the quantum numbers with respect to the superposition of $S_{ab}^k$ in adjoint representations, as can be seen in Table 2 and in Fig. 1 of Subsect. 2.2.2. I discuss properties of these scalar fields, offered by the spin-charge-family theory, in Sect. 3.

$^{10}$ The $3 \times 3$ submatrix of the unitary $4 \times 4$ matrix determines $4 \times 4$ matrix uniquely.
2.2. Properties of vector and scalar gauge fields in spin-charge-family theory

In the starting action, Eq. (1), the second line represents the action for gauge fields in $d = (13+1)$-dimensional space, with the index $gf$ denoting gauge fields, vector and scalar ones,

$$\mathcal{A}_{gf} = \int d^d x \, E (\alpha R + \tilde{\alpha} \tilde{R}),$$

$$R = \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} (\omega_{ab\alpha \beta} - \omega_{cab} \omega^{c \beta b}) \} + h.c.,$$

$$\tilde{R} = \frac{1}{2} \{ f^{\alpha [a} f^{\beta b]} (\tilde{\omega}_{ab\alpha \beta} - \tilde{\omega}_{cab} \tilde{\omega}^{c \beta b}) \} + h.c.,$$

(27)

which in the spin-charge-family theory manifests after the break of the starting symmetry in $d = (3 + 1)$ as the action for all observed vector and scalar gauge fields. Here $f^{\beta a}$ and $e^{a \alpha}$ are vielbeins and inverted vielbeins respectively

$$e^{a \alpha} f^{\beta a} = \delta_\alpha^\beta, \quad e^{a \alpha} f^{\alpha b} = \delta_b^\alpha,$$

(28)

$$E = \det(e^{a \alpha}).$$

Varying the action of Eq. (27) with respect to the spin connection fields, the expression for the spin connection fields $\omega^{\alpha}_{ab}$ follows

$$\omega^{\alpha}_{ab} = \frac{1}{2E} \{ e^{\alpha \beta} \partial_\beta(E f^{\alpha [a} f^{\beta b]}) - e^{\alpha \alpha} \partial_\beta(E f^{\alpha [b} f^{\beta e]})$$

$$- e^{\beta \alpha} \partial_\beta(E f^{\alpha [e} f^{\beta a]}) \}$$

$$+ \frac{1}{4} \{ \Psi (\gamma^e S^{ab} - \gamma^{[a} S_{b]}^e) \Psi \}$$

$$- \frac{1}{d-2} \{ \delta^{\alpha}_{a} \{ \frac{1}{E} e^{d \alpha} \partial_\beta(E f^{\alpha [d} f^{\beta b]}) + \bar{\Psi} \gamma_{d} S^{d \beta}_{b} \Psi \}$$

$$- \delta^{\alpha}_{b} \{ \frac{1}{E} e^{d \alpha} \partial_\beta(E f^{\alpha [d} f^{\beta a]}) + \bar{\Psi} \gamma_{d} S^{d \alpha}_{a} \Psi \} \}.$$

(29)

If replacing $S^{ab}$ in Eq. (29) with $\tilde{S}^{ab}$, the expression for the spin connection fields $\tilde{\omega}^{\alpha}_{ab}$ follows.

In Ref. [9] it is proven that in spaces with the desired symmetry the vielbein can be expressed with the gauge fields, if only one of the two spin connection fields are present

$$f^{\alpha}_{m} = \sum A \tilde{\tau}^{A \sigma} \tilde{A}^{A}_{m},$$

(30)

with

$$A^{Ai}_{m} = \sum_{st} c^{Ai}_{st} \tilde{\omega}^{st}_{m},$$

$$\tilde{\tau}^{Ai \sigma} = \sum_{st} c^{Ai}_{st} (e_{s \tau} f^{\sigma}_{1} - e_{t \tau} f^{\sigma}_{s}) x^{\tau},$$

$$\tilde{\tau}^{Ai} = \sum_{st} c^{Ai}_{st} S^{st}.$$

(31)

If fermions are not present them spin connections of both kinds are uniquely determined by vielbeins, as can be noticed from Eq. (29). If fermions are present, carrying both — family members and family quantum numbers — then vielbeins and both kinds of spin connections are influenced by the presence of fermions, which carry different family and family members quantum numbers.
The scalar (gauge) fields, carrying the space index \(s = (5, 6, \ldots, d)\), offer in the spin-charge-family for \(s = (7, 8)\) the explanation for the origin of the Higgs's scalar and the Yukawa couplings of the standard model, while scalars with the space index \(s = (9, 10, \ldots, 14)\) offer the explanation for the proton decay, as well as for the matter/antimatter asymmetry in the universe.

The notation

\[
\tau^{Ai} = \sum_{a,b} c^{Ai}_{ab} S^{ab},
\]

\[
\{\tau^{Ai}, \tau^{Bj}\} = i\delta^{AB} f_{ijk} \tau^{Ak},
\]

\[
A^{Ai}_a = \sum_{s,t} c^{Ai}_{st} \omega^{st}_a,
\]

is used, where \(a = m = (0, 1, 2, 3)\) denotes the vector gauge fields and \(a = s = (5, 6, \ldots, 14)\) denotes scalar gauge fields.

The explicit expressions for \(c^{Ai}_{ab}\), and correspondingly for \(\tau^{Ai}\), and \(A^{Ai}_a\), are written in Sect. Appendix D.

2.2.1. Vector gauge fields in \(d = (3+1)\) In the spin-charge-family theory there are besides the gravity, the colour and the weak \(SU(2)\) vector gauge fields, also the second \(SU(2)_{II}\) and the \(U(1)_{\tau^4}\) vector gauge fields, all originating in the simple starting action, Eq. (1). The \(U(1)_{\tau^4}\) vector gauge field is the vector gauge field of the standard model is the superposition of the third component of the second \(SU(2)_{II}\) vector gauge fields and the \(U(1)_{\tau^4}\) vector gauge field \((A^Y_m = \cos \theta_2 A^\tau_{m} + \sin \theta_2 A^{23} m, \theta_2\) is the angle of the break of the \(SU(2)_{II} \times U(1)_{\tau^4}\) symmetry to \(U(1)_{Y}\) at the scale \(\geq 10^{16}\) or higher, [13] and references therein). After the appearance of the condensate, presented in Table D3, there are namely only the gravity, the colour, the weak \(SU(2)\) and the \(U(1)_{Y}\) hyper charge vector gauge fields, which remain massless, since they do not couple to the condensate. The two components of the second \(SU(2)_{II}\) vector gauge fields and the superposition \(A^\tau_m = - \sin \theta_2 A^\tau_m + \cos \theta_2 A^{23}_m\), which is the gauge field of \(Y^l = - \tan^2 \theta_2 \tau^4 + \tau^{23}\) gain high masses due to the interaction with the condensate. All the vector gauge fields are expressible with the spin connection fields \(\omega^{st}_m\),

\[
A^{Ai}_m = \sum_{s,t} c^{Ai}_{st} \omega^{st}_m. \tag{33}
\]

Let me present expressions for the two \(SU(2)\) vector gauge fields, \(SU(2)_I\) and \(SU(2)_{II}\)

\[
\vec{A}^I_m = \vec{A}^I_m = (\omega_{58m} - \omega_{67m}, \omega_{57m} + \omega_{68m}, \omega_{56m} - \omega_{78m}),
\]

\[
\vec{A}^2_m = \vec{A}^2_m = (\omega_{58m} + \omega_{67m}, \omega_{57m} - \omega_{68m}, \omega_{56m} + \omega_{78m}). \tag{34}
\]

The reader can similarly construct all the other vector gauge fields from the coefficients for the corresponding charges, or find the expressions in Refs. [11, 7, 13] and references therein.

The electroweak break, caused by the non zero expectation values of the scalar gauge fields, carrying the space index \(s = (7, 8)\), makes the weak and the hyper charge massive. The only vector gauge fields which remains massless are the electromagnetic and the colour vector gauge fields — the observed two.

2.2.2. Scalar gauge fields in \(d = (3+1)\) There are in the spin-charge-family theory scalar fields taking care of the masses of quarks and leptons: They have the space index \(s = (7, 8)\) and carry
with respect to the space index the weak charge $\tau^{13} = \pm \frac{1}{2}$ and the hyper charge $Y = \mp \frac{1}{2}$. With respect to $\tau^A_i = \sum_{ab} c_{ab}^i \hat{S}_{ab}$ and $\bar{\tau}^A_i = \sum_{ab} c_{ab}^i \tilde{S}_{ab}$ they carry charges and family charges in adjoint representations, Table 1, Eq. (38).

There are scalar fields transforming antileptons and antiquarks into quarks and leptons and back. They carry space index $s = (9, 10, \ldots, 14)$. They are with respect to the space index colour triplets, while they carry charges $\tau^A_i$ and $\bar{\tau}^A_i$ in adjoint representations.

The infinitesimal generators $S_{ab}^\tau$ apply on the spin connections $\omega_{bde} = (f^a_{\tau e} \omega_{bdea})$ and $\tilde{\omega}_{bde} (= f^a_{\tau e} \tilde{\omega}_{bdea})$, on either the space index $e$ or any of the indices $(b, d, \bar{b}, \bar{d})$, as follows

$$S_{ab}^\tau A^{d..e.g} = i (\eta^{ae} A^{d..b..g} - \eta^{be} A^{d..a..g}),$$

(35)

(see Section IV. and Appendix B in Ref. [13]).

**Scalar gauge fields determining scalar higgs and Yukawa couplings**

Let me introduce a common notation $A_{a}^{A_i}$ for all the scalar gauge fields with $s = (7, 8)$, independently of whether they originate in $\omega_{abs}$ — in this case $A_i = (Q, Q', Y')$ — or in $\tilde{\omega}_{abs}$ — in this case all the family quantum numbers of all eight families contribute. All these gauge fields contribute to the masses of the quarks and leptons and the antiquarks and antileptons after gaining nonzero vacuum expectation values.

$$A_{a}^{A_i} \quad \text{represents} \quad (A_{s}^{Q}, A_{s}^{Q'}, A_{s}^{Y'}, \vec{A}_{s}^{1}, \vec{A}_{s}^{N_L}, \vec{A}_{s}^{2}, \vec{A}_{s}^{N_R}) ,$$

$$\tau^A_i \quad \text{represents} \quad (Q, Q', Y', \vec{F}_L, \vec{F}_R, \vec{N}_L, \vec{N}_R).$$

(36)

Here $\tau^A_i$ represent all the operators, which apply on the fermions. These scalars, the gauge scalar fields of the generators $\tau^A_i$ and $\bar{\tau}^A_i$, are expressible in terms of the spin connection fields (Ref. [13], Eqs. (10, 22, A8, A9)).

Let me demonstrate [13] that all the scalar fields with the space index $(7, 8)$ carry with respect to this space index the weak and the hyper charge $(\mp \frac{1}{2}, \mp \frac{1}{2})$, respectively. This means that all these scalars have properties as required for the Higgs in the standard model.

We need to know the application of the operators $\tau^{13} = \frac{1}{2} (S^{56} - S^{78})$, $Y = (\tau^4 + \tau^{23})$ and $Q = (\tau^{13} + Y)$, Eqs. (D.2, D.3, D.7), with $S_{ab}^{\tau}$ defined in Eq. (35), on the scalar fields with the space index $s = (7, 8)$.

To compare the properties of the scalar fields with those of the Higgs’s scalar of the standard model I let the scalar fields be eigenstates of $\tau^{13} = \frac{1}{2} (S^{56} - S^{78})$.

I rewrite for this purpose the second line of Eq. (26) as follows, ignoring the momentum $p_s$, $s = (5, 6, \ldots, d)$, since it is expected that solutions with nonzero momenta in higher dimensions do not contribute to the masses of fermion fields at low energies in $d = (3 + 1)$. We pay correspondingly no attention to the momentum $p_s$, $s \in (5, 6, \ldots, 8)$, when having in mind the lowest energy solutions, manifesting at low energies.

$$\sum_{s=(7,8),s} \bar{\psi} \gamma^\nu (-\tau^A_i A_{s}^{A_i}) \psi =$$

$$-\bar{\psi} \frac{78}{78} (\tau^A_i (A_{s}^{A_i} - i A_{s}^{A_i}) + (-) (\tau^A_i (A_{s}^{A_i} + i A_{s}^{A_i}) ) \psi ,$$

$$= \frac{1}{2} (\gamma^7 \pm i \gamma^8), \quad A_{s}^{A_i} := (A_{s}^{A_i} \pm i A_{s}^{A_i}),$$

(37)

with the summation over $A$ and $i$ performed, since $A_{s}^{A_i}$ represent the scalar fields $(A_{s}^{Q}, A_{s}^{Q'}, A_{s}^{Y'})$ determined by $\omega_{a,s''}, a''$, and those determined by $(\tilde{\omega}_{a,b,s}, \vec{A}_{s}^{1}, \vec{A}_{s}^{1}, \vec{A}_{s}^{N_L} and \vec{A}_{s}^{N_R})$. 


Table 1. The two scalar weak doublets, one with $\tau^3 = -\frac{1}{2}$ and the other with $\tau^3 = +\frac{1}{2}$, both with the "fermion" quantum number $\tau^4 = 0$, are presented. In this table all the scalar fields carry besides the quantum numbers determined by the space index also the quantum numbers $A$ and $i$ from Eq. (36). The table is taken from Ref. [13].

| name | superposition | $\tau^{14}$ | $\tau^{23}$ | spin | $\tau^4$ | $Q$ |
|------|---------------|-------------|-------------|------|--------|-----|
| $A_{78}^{Ai}$ | $A_{78}^{Ai} + iA_{8i}^{Ai}$ | $\pm \frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | 0 |
| $A_{56}^{Ai}$ | $A_{56}^{Ai} + iA_{6i}^{Ai}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | -1 | 1 |
| $A_{78}^{Ai}$ | $A_{78}^{Ai} - iA_{8i}^{Ai}$ | $-\frac{1}{2}$ | $+\frac{1}{2}$ | 0 | 0 | 0 |
| $A_{56}^{Ai}$ | $A_{56}^{Ai} - iA_{6i}^{Ai}$ | $+\frac{1}{2}$ | $+\frac{1}{2}$ | 0 | +1 | 1 |

The application of the operators $\tau^{13}$, $Y = \frac{1}{\sqrt{3}}(S^{56} + S^{78}) - \frac{1}{\sqrt{3}}(S^{90} + S^{1112} + S^{1314})$ and $Q$ on the scalar fields ($A_{78}^{Ai} + iA_{8i}^{Ai}$) with respect to the space index $s = (7,8)$, by taking into account Eq. (35) to make the application of the generators $S^{ab}$ on the space indexes, gives

$$
\tau^{13} (A_{78}^{Ai} + iA_{8i}^{Ai}) = \mp \frac{1}{2} (A_{78}^{Ai} + iA_{8i}^{Ai}),
$$

$$
Y (A_{78}^{Ai} + iA_{8i}^{Ai}) = \mp \frac{1}{2} (A_{78}^{Ai} + iA_{8i}^{Ai}),
$$

$$
Q (A_{78}^{Ai} + iA_{8i}^{Ai}) = 0.
$$

(38)

Since $\tau^4$, $Y$, $\tau^{13}$ and $\tau^{1+}, \tau^{1-}$ give zero if applied on ($A_{78}^Q$, $A_{8i}^Q$ and $A_{8i}^{Y'}$) with respect to the quantum numbers ($Q$, $Q'$, $Y'$), and since $Y$ and $\tau^{13}$ commute with the family quantum numbers, one sees that the scalar fields $A_{78}^{Ai}$ ( =($A_{78}^{Q}$, $A_{8i}^{Y}$, $A_{8i}^{Y'}$, $A_{78}^{1}$, $A_{56}^{1}$, $A_{8i}^{7}$, $A_{6i}^{7}$, $A_{8i}^{56}$, $A_{8i}^{78}$)) rewritten as $A_{78}^{Ai}$ ($A_{78}^{Ai} = iA_{8i}^{Ai}$), are eigenstates of $\tau^{13}$ and $Y$, having the quantum numbers of the standard model Higgs’s scalar.

These superposition of $A_{78}^{Ai}$ are presented in Table 1 as two doublets with respect to the weak charge $\tau^{13}$, with the eigenvalue of $\tau^{23}$ (the second $SU(2)_{11}$ charge) equal to either $-\frac{1}{2}$ or $+\frac{1}{2}$, respectively. The operators $\tau^{12}$ transform one member of a doublet from Table 1 into another member of the same doublet, keeping $\tau^{23}$ ( = $\frac{1}{2} (S^{56} + S^{78})$) unchanged, clarifying the above statement.

It is not difficult to show that the scalar fields $A_{78}^{Ai}$ are triplets as the gauge fields of the family quantum numbers ($\vec{N}_R$, $\vec{N}_L$, $\vec{\tau}_{22}$, $\vec{\tau}_{11}$; Eqs. (D.4, D.5, 35)) or singlets as the gauge fields of $Q = \tau^{13} + Y$, $Q' = -\tan^2 \theta_Y Y + \tau^{13}$ and $Y' = -\tan^2 \theta_Y Y + \tau^{23}$.

Let us do this for $\vec{N}_{78}^{Li}$ and for $A_{78}^{Q}$, taking into account Eq. (D.1) (where we replace $S^{ab}$ by $S^{ab}$ and Eq. (35), and recognizing that $\vec{N}_{78}^{Li} = \vec{A}_{78}^{Li} + i \vec{A}_{78}^{Li} = \vec{A}_{78}^{Li} + i \vec{A}_{78}^{Li}$, and further

$$
\vec{A}_{78}^{L1} = \{ (\vec{\omega} \gamma_{23(z)} + i \vec{\omega} \gamma_{01(z)} ) \} \vec{A}_{78}^{L1} + \{ (\vec{\omega} \gamma_{31(z)} + i \vec{\omega} \gamma_{02(z)} ) \};
$$

$$
\vec{A}_{78}^{L2} = \{ (\vec{\omega} \gamma_{12(z)} + i \vec{\omega} \gamma_{03(z)} ) \},
$$

$$
A_{78}^{Q} = \omega \gamma_{56(z)} - (\omega \gamma_{910(z)} + \omega \gamma_{1112(z)} + \omega \gamma_{1314(z)}).
$$
One finds

\[ \tilde{N}_L \tilde{A}^{N_L} \tilde{A}_{\tilde{q}} = \tilde{A}^{N_L} \tilde{A}_{\tilde{q}}, \quad \tilde{N}_L \tilde{A}^{N_L} = 0, \]

\[ Q \tilde{A}^Q = 0. \]  \( (40) \)

with \( Q = S^{56} + \tau^4 = S^{56} - \frac{1}{3}(S^{10} + S^{11} + S^{13}), \) and with \( \tau^4 \) defined in Eq. (D.3), if replacing \( S^{ab} \) by \( S^{ab} \) from Eq. (35). Similarly, one finds properties with respect to the \( \bar{a}_i \) quantum numbers for all the scalar fields \( \tilde{A}_{\bar{a}_i} \).

After the appearance of the condensate (Table D3), which breaks the \( SU(2)_{II} \) symmetry and brings masses to all the scalar fields, the weak \( \bar{\tau} \) and the hyper charge \( Y \) remain the conserved charges.

At the electroweak scale, the scalar gauge fields with the space index \( (7, 8) \), with the Lagrange density

\[ L_{sg} = E \sum_{A_i} \left\{ (p_m A^{Ai}_{s})^+ (p_m A^{Ai}_{s}) - (\lambda^{Ai} + (m'_{Ai})^2) A^{Ai}_{s} A^{Ai}_{s} \right\} + \sum_{B,J} A^{AiBj} A^{Ai}_{s} A^{Ai}_{s} A^{Bj}_{s} A^{Bj}_{s}, \]  \( (41) \)

gain nonzero vacuum expectation values and cause the electroweak break. The above Lagrange density needs to be studied. At this stage, it is just postulated.

The two groups of four families became massive. The mass matrices manifest either \( \tilde{SU}(2)_{SO(3,1)}L \times \tilde{SU}(2)_{SU(4)}L \times U(1) \) symmetry, this is the case for the lower four families, presented in Table D2, or \( \tilde{SU}(2)_{SO(3,1)R} \times \tilde{SU}(2)_{SU(4)R} \times U(1) \) symmetry, this is the case for the higher four families, presented in Table D2. The same three \( U(1) \) singlet fields contribute to the masses of both groups, the two \( SU(2) \) triplet fields are for each of the two groups different, although manifesting the same symmetries.

The mass matrix of family members — quarks and leptons — are \( 4 \times 4 \) matrices. The observed three families of quarks and leptons form the \( 3 \times 3 \) submatrices of the \( 4 \times 4 \) matrices. The symmetry of the mass matrices, manifesting in all orders [54], limits the number of free parameters.

All the scalars, the two triplets and the three singlets, are doublets with respect to the weak charge, contributing to the weak and the hyper charge of the fermions so that they transform the right-handed members into the left-handed ones.

\[ M^\alpha = \begin{pmatrix} -a_1 - a & e & d & b \\ e^* & -a_2 - a & b & d \\ d^* & b^* & a_2 - a & e \\ b^* & d^* & e^* & a_1 - a \end{pmatrix}^\alpha, \]

\( (42) \)

with \( \alpha \) representing family members — quarks and leptons of left and right handedness [46, 47, 48, 50, 51, 55].

The mass matrices of the upper four families have the same symmetry as the mass matrices of the lower four families, but the scalar fields determining the masses of the upper four families have different properties (nonzero vacuum expectation values, masses and coupling constants)

\[ ^{11} \] The expression for the Lagrange density of Eq. (41) is only estimated, more or less guessed, I have no estimate yet for the constants.
than those of the lower four, giving to quarks and leptons of the upper four families much higher masses in comparison with the lower four families of quarks and leptons, what offers the explanation for the appearance of the dark matter, studied at Refs. [49, 56].

Scalar fields transforming antiquarks and antileptons into quarks and leptons

I follow in this part to a great deal similar part in Ref. [6].

To the matter-antimatter asymmetry the terms contribute, which cause transitions from antileptons into quarks and from antiquarks into quarks and back. These are terms included into the third line of Eq. (26). Let me rewrite this part of the fermion action $L_{fi} = \psi^{\dagger} \gamma^{0} \gamma^{t} \sum_{t=1}^{9,10\ldots.14} \left[ \bar{\psi}_{t} - \left( \frac{1}{2} S^{s_{s}s_{t}t} \psi_{s_{s}s_{t}t} + \frac{1}{2} S^{s_{s}t} \psi_{s_{s}t} + \frac{1}{2} \bar{S}_{ab} \tilde{\omega}_{ab} \right) \right] \psi$, as follows

\[
L_{fi} = \psi^{\dagger} \gamma^{0} (-) \left( \sum_{t, t'} \frac{1}{2} S^{s_{s}s_{t}t} \omega_{s_{s}s_{t}t} + \omega_{s_{s}t} \right) \left( \sum_{t, t'} \frac{1}{2} S^{s_{s}s_{t}t} \omega_{s_{s}s_{t}t} \right),
\]

where $(t, t')$ run in pairs over $[9,10,11,12,13,14]$ and the summation goes also over $+$ and $-$ of $t'$.

In Eq. (43) the relations below are used

\[
\sum_{t, t'} \omega_{s_{s}t} = \omega_{s_{s}t}, \quad \omega_{s_{s}t} = \left( \omega_{s_{s}t} \right),
\]

where $(t, t')$ run in pairs over $[9,10,11,12,13,14]$ and the summation goes also over $+$ and $-$ of $t'$.

The rest of expressions in Eq. (44) are obtained in a similar way. They are presented in Eq. (E.3).

The scalar fields with the scalar index $s = (9,10,\ldots,14)$, presented in Table 2, carry one of the triplet colour charges and the "fermion" charge equal to twice the quark "fermion" charge, or
Table 2. Quantum numbers of the scalar gauge fields carrying the space index $t = (9, 10, \cdots, 14)$, appearing in Eq. (26), are presented. The space degrees of freedom contribute one of the triplets values to the colour charge of all these scalar fields. These scalars are with respect to the two $SU(2)$ charges, $(\tau^1\bar{3}$ and $\tau^2\bar{3}$), and the two $SU(2)$ charges, $(\bar{\tau}^1\bar{3}$ and $\bar{\tau}^2\bar{3}$), triplets (that is in the adjoint representations of the corresponding groups), and they all carry twice the "fermion" number ($\tau^4$) of the quarks. The quantum numbers of the two vector gauge fields, the colour and the $SU(1\bar{1})$ ones, are added.

| Field | prop. | $\tau^1\bar{3}$ | $\tau^2\bar{3}$ | $(\tau^1\bar{3}, \tau^2\bar{3})$ | $Y$ | $Q$ | $\tau^1\bar{3}$ | $\tau^2\bar{3}$ | $\tau^1\bar{3}$ | $\tau^1\bar{3}$ |
|-------|-------|-----------------|-----------------|-----------------|-----|-----|----------------|----------------|----------------|----------------|
| $A_{10}^{T1}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T2}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T3}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T4}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T5}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T6}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T7}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T8}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T9}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |
| $A_{10}^{T10}\bar{3}$ | scalar | $\frac{1}{3}$ | 0 | $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $0$ | $0$ | $0$ | $0$ |

the antitriplet colour charges and the "antifermion" charge. They carry in addition the quantum numbers of the adjoint representations originating in $S_{10}^{ab}$ or in $S_{10}^{ab}$.\footnote{Although carrying the colour charge in one of the triplet or antitriplet states, these fields can not be interpreted as superpartners of the quarks since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.} If the antiquark $\bar{u}_{L}^{2}$, from the line 43 presented in Table D1, with the "fermion" charge $\tau^4 = -\frac{1}{2}$, the weak charge $\tau^{13} = 0$, the second $SU(2)_{II}$ charge $\tau^{23} = -\frac{1}{2}$, the colour charge $(\tau^{33}, \tau^{38}) = (\frac{1}{2}, -\frac{1}{2\sqrt{3}})$, the hyper charge $Y = \tau^4 + \tau^{23} = -\frac{2}{3}$ and the electromagnetic charge \footnote{Although carrying the colour charge in one of the triplet or antitriplet states, these fields can not be interpreted as superpartners of the quarks since they do not have quantum numbers as required by, let say, the $N = 1$ supersymmetry. The hyper charges and the electromagnetic charges are namely not those required by the supersymmetric partners to the family members.}
Figure 1. The birth of a "right handed proton" out of an positron $\bar{e}_L^\uparrow$, antiquark $\bar{u}_L^{a2}$ and quark (spectator) $u_R^{c2}$. The family quantum number can be any.

$Q(=Y+\tau^{13})=-\frac{2}{3}$ submits the $A_{(6)}^{2\Xi}$ scalar field, it transforms into $u_R^{c3}$ from the line 17 of Table D1, carrying the quantum numbers $\tau^4=\frac{1}{6}$, $\tau^{13}=0$, $\tau^{23}=\frac{1}{2}$, $(\tau^{33},\tau^{38})=(0,-\frac{1}{\sqrt{3}})$, $Y=\frac{2}{3}$ and $Q=\frac{2}{3}$. These two quarks, $d_R^1$ and $u_R^{c3}$ can bind together with $u_R^{c2}$ from the $9^{th}$ line of the same table (at low enough energy, after the electroweak transition, and if they belong to a superposition with the left handed partners to the first family) -into the colour chargeless baryon - a proton. This transition is presented in Figure 1.

The opposite transition at low energies would make the proton decay.

(Let me add that these colour triplets scalar fields (which do not gain the constant values at the electroweak phase transition) have properties of lepto-quarks.)

3. Achievements and conclusions

Let me in conclusions shortly review the general properties of the spin-charge-family theory.

a. The spin-charge-family theory assumes a simple starting action in $d \geq (13+1)$, Eq. (1), for fermions and vielbeins [7, 11, 13, 6, 1, 2, 4, 3, 5].

a.i. The internal space of fermions is described by the "basis vectors", which are superposition of an odd number of the Clifford algebra objects $\gamma^a$'s, manifesting therefore the anticommuting properties of the "basis vectors", and transferring correspondingly the anticommuting properties to tensor products of finite number of "basis vectors" of internal space and continuously infinite number of basis in ordinary (coordinate and momentum) space. The second kind of the Clifford algebra objects, $\tilde{\gamma}^a$'s, which anticommute with $\gamma^a$'s, determines the family quantum numbers of the irreducible representations of the Lorentz algebra in internal space. Fermions and antifermions, appearing together in one irreducible representation of the Lorentz algebra of
One irreducible representation of the Lorentz algebra of \( \mathbb{R} \)\(^{1,3} \) lies \([65, 66, 15, 16, 17]\), explaining how and why the charges are connected with handedness \([12]\) to the vector gauge fields \([9]\) of the standard model charges of the standard model in fields, explaining correspondingly the Dirac’s postulates.

The description of the internal space of fermions offers the new way of second quantization of fermion operators fulfill the Dirac’s anticommutation relations on the whole Hilbert space. The creation and annihilation of fermions with the Clifford odd algebra objects of one kind, while enabling the second kind of the Clifford algebra objects to describe family quantum numbers of the irreducible representations of the Lorentz transformations of the first kind, makes the creation operators (defined as tensor Clifford algebra objects) to describe family quantum numbers of the irreducible representations of the Lorentz algebra of \( \mathbb{R} \)\(^{1,3} \) the known vector gauge fields, the scalar gauge fields (offering explanation for the origin of the Higgs’s scalar and the Yukawa couplings).

To see in \( d = (3 + 1) \) at low energies the observed properties of fermion, vector and scalar gauge fields the starting symmetry \( \mathit{SO}(13, 1) \) must be spontaneously broken two times: At \( E \geq 10^{16} \text{ GeV} \) \( \mathit{SO}(13, 1) \) breaks into \( \mathit{SO}(7, 1) \times \mathit{SU}(3) \times \mathit{U}(1) \) by the condensate of two right handed neutrinos, to which couple all the vector gauge fields except the hyper, colour and weak gauge fields, and all the scalar gauge fields. And the electroweak scale, when \( \mathit{SO}(7, 1) \times \mathit{SU}(3) \times \mathit{U}(1) \) and the symmetry must further break at the electroweak scale (in which the scalar fields with the space index \((7, 8)\) gain nonzero vacuum expectation values).

Correspondingly the simple starting action offers the explanations for all the assumptions of the standard model, as well as for several observed phenomena, predicting new gauge fields and new families.

Let me present first some of the achievements of the spin-charge-family theory, many of them only briefly presented in this talk, which all follow from the simple starting action.

The decision \([1, 2, 7, 43, 44, 65, 66, 15, 16, 17]\) to describe the internal space of fermions with the Clifford odd algebra objects of one kind, while enabling the second kind of the Clifford algebra objects to describe family quantum numbers of the irreducible representations of the Lorentz transformations of the first kind, makes the creation operators (defined as tensor products of the finite number of "basis vectors" in internal space and continuously infinite basis in ordinary space) and their Hermitian conjugated partners to fulfill, when applying on the vacuum state, the anticommutation relations postulated by Dirac for the second quantized fields.

The single fermion states have therefore by themselves the anticommuting character. Tensor products of any number and any kind of the single fermion creation operators define the second quantized fermion fields forming the whole Hilbert space. The creation and annihilation operators fulfill the Dirac’s anticommutation relations on the whole Hilbert space. The description of the internal space of fermions offers the new way of second quantization of fermion fields, explaining correspondingly the Dirac’s postulates.

The choice of the space \( d \geq (13 + 1) \) enables unification of spins and charges and families \([65, 66, 15, 16, 17]\), explaining how and why the charges are connected with handedness \([12]\).

One irreducible representation of the Lorentz algebra of \( S^{ab} \) in internal space manifests in \( d = (3 + 1) \) all the quarks and leptons and antiquarks and antileptons with the spins and charges of the standard model. Eight irreducible representations define in \( d = (3 + 1) \) (after the reduction of the Clifford algebra from the two kinds of algebras to only one kind) two times at low energies decoupled four families, which remain massless. The spin connection fields of one kind with the space index \( m = (0, 1, 2, 3) \) manifest in \( d = (3 + 1) \) all the vector gauge fields \([9]\) of the standard model. The scalar gauge fields with respect to \( d = (3 + 1) \), those with the space index \( s = (7, 8) \), of two kinds, carry the weak and the hyper charge \( \pm \frac{1}{2} \) and \( \mp \frac{1}{2} \), respectively, form two groups of scalar fields, each manifesting the \( \mathit{SU}(2) \times \mathit{SU}(2) \times \mathit{U}(1) \) symmetry (the two \( \mathit{SU}(2) \) triplet scalar fields are the gauge fields of \( \tilde{S}^{ab} \)’s).

So far the masslessness of twice four families after the break of symmetry is for the spin-charge-family theory assumed, but we show for the toy model in \( d = (5 + 1) \) \([62]\) how this can happen.
differing in the family quantum numbers if they belong to two different groups of four families, the three \( U(1) \) singlet scalar fields are the gauge fields of \( S^{ab} \)'s and are the same for both groups of four families) and offer the explanation for the Higgs's scalar and Yukawa couplings of the standard model. Gaining at the electroweak break nonzero vacuum expectation values, they give masses to two groups of four families. c.v. The existence of the lower four families predict the fourth family of quarks and leptons coupled to the observed three \([47, 48, 50, 51, 55, 54]\). The stable of the upper four families offers the explanation for the dark matter \([49, 56]\). c.vi. Both groups of four families together spread masses from almost zero to \( \geq 10^{16} \) GeV. c.vii. The scalar gauge fields manifesting as colour triplets and antitriplets offer the explanation for the matter/antimatter asymmetry of the ordinary matter \([7]\).

Let me point out some of the predictions in more details.

d. Prediction of the fourth family to the observed three families, Subsect. 2.1.4. Taking into account the experimental data for masses of the observed families of quarks and the corresponding mixing matrix we fit 6 parameters of the two quark mass matrices, presented in Eq. (42), to twice 3 measured massess of quarks and to 6 measured parameters of the mixing matrix.

Although any accurate \( 3 \times 3 \) submatrix of the \( 4 \times 4 \) unitary matrix determines the \( 4 \times 4 \) matrix completely, neither the quark nor the lepton mixing matrix is measured accurately enough that it would be possible to determine three complex phases of the \( 4 \times 4 \) quark mixing matrix and the mixing matrix elements of the fourth family quarks to the other three family members. We therefore assume that mass matrices are symmetric and real, while making a choice for the mass matrices of the fourth family.

Results are presented for two choices of \( m_{u_4} = m_{d_4} \), Ref. \([51]\), [arxiv:1412.5866]:

1. \( m_{u_4} = 700 \) GeV, \( m_{d_4} = 700 \) GeV.....new1
2. \( m_{u_4} = 1200 \) GeV, \( m_{d_4} = 1200 \) GeV.....new2

\[
|V_{u_4d_4}| =
\begin{pmatrix}
0.97425 \pm 0.00022 & 0.2253 \pm 0.0008 & 0.00413 \pm 0.00049 \\
0.97423(4) & 0.22539(4) & 0.00299 \\
0.97422(4) & 0.22538(4) & 0.00299 & 0.00791(4)6
\end{pmatrix}
\]

One can see that the above results for the mixing matrices of the lower three families are in agreement with what Ref. \([52]\) requires, namely that

\( V_{u_1d_1} > V_{u_1d_3} \), \( V_{u_2d_4} < V_{u_1d_4} \), and \( V_{u_3d_1} < V_{u_1d_4} \).

Since we have not yet fit the mass matrix of Eq. (42) to the newest experimental data \([53]\), which appear after IARD 2020, the evaluation for our \( 4 \times 4 \) quark mixing matrix with the new data and correspondingly a new prediction is not yet offered.

Let me repeat the discussion of Ref. \([55]\) that the existence of the fourth family to the observed three is still not in disagreement with the latest experimental data although some phenomenologists say different.

e. The spin-charge-family theory predicts in the low energy regime (up to \( 10^{16} \) GeV or higher) the existence of two decoupled groups of four families, which at the electroweak break become massive \([49]\). The stable family of the upper group of four families (with almost zero Yukawa couplings to the lower group of four families) is the candidate for the dark matter, Subsect. 2.1.4.

I review here briefly the estimations done in Ref. \([49]\). We used the simple hydrogen-like model to evaluate properties of the fifth family heavy baryons, taking into account that for masses of the order of a few TeV or larger the force among the constituents of the fifth family...
baryons is determined mostly by one gluon exchange. The fifth family neutron is estimated as the most stable nucleon. The “nuclear interaction” among these baryons is found to have very interesting properties. We studied scattering amplitudes among fifth family neutrons and with the ordinary matter.

We followed the behaviour of the fifth family quarks and antiquarks in the plasma of the expanding universe, through the freezing out procedure, solving the Boltzmann equations, through the colour phase transition, while forming neutrons, up to the present dark matter, taking into account the cosmological evidences, the direct experimental evidences and all others known properties of the dark matter.

The cosmological evolution suggested the limits for the masses of the fifth family quarks

\[ 10 \text{ TeV} < m_{q^5} c^2 < \text{a few} \cdot 10^2 \text{ TeV} \]  \hspace{1cm} (46)

and for the scattering cross sections

\[ 10^{-8} \text{ fm}^2 < \sigma_{c5} < 10^{-6} \text{ fm}^2, \]  \hspace{1cm} (47)

while the measured density of the dark matter does not put much limitation on the properties of heavy enough clusters.

The direct measurements limit the fifth family quark mass to

\[ \text{several} 10 \text{ TeV} < m_{q^5} c^2 < 10^5 \text{ TeV}. \]  \hspace{1cm} (48)

We also find that our fifth family baryons of the mass of several 10 TeV/c^2 have for a factor more than 100 times too small scattering amplitude with the ordinary matter to cause a measurable heat flux on the Earth’s surface.

f. The spin-charge-family theory predicts several scalar fields with the weak and the hyper charge of the Higgs’s scalar \((\pm \frac{1}{2}, \mp \frac{1}{2})\) — two triplets and three singlets — offering the explanation for the existence of the Higgs’s scalar and Yukawa couplings, Subsect. 2.2.2 [11, 13].

The additional two triplets and the same three singlets determine properties of the upper four families of quarks and leptons, Subsect. 2.2.2.

g. The spin-charge-family theory predicts several scalar fields which are colour triplets or antitriplets, offering the explanation for the matter/antimatter asymmetry in the (nonequilibrium) expanding universe as well as the proton decay [7], Subsect. 2.2.2.

h. The mass matrices of the two fourth family groups are close to democratic one, causing spreading of the fermion masses from \(10^{-8} \text{ MeV}\) to \(10^{16} \text{ GeV}\) or even higher [6].

I conclude by saying that there are still a lot of open problems to be solved. Some of them, like the quantization of gravitational fields, are common to other theories, like the Kaluza-Klein-like theories, the others require to extract as much as possible from the offer of the theory. We need collaborators, since the more work is put into the spin-charge-family theory the more explanations for the observed phenomena follow.

Acknowledgments
The author thanks Department of Physics, FMF, University of Ljubljana, Society of Mathematicians, Physicists and Astronomers of Slovenia, for supporting the research on the spin-charge-family theory, and Matjaž Breskvar of Beyond Semiconductor for donations, in particular for sponsoring the annual workshops entitled ”What comes beyond the standard models” at Bled.
Appendix A. Statements and proofs of statements

There are two kinds of the Clifford algebra objects, $\gamma^a$’s and $\tilde{\gamma}^a$’s, both expressible with the Grassmann algebra objects, $\theta^a$’s and their derivatives $\frac{\partial}{\partial \theta_a}$’s. In Grassmann $d$-dimensional space there are $d$ anticommuting operators $\theta^a$, $\{\theta^a, \theta^b\}_+ = 0$, $a = (0, 1, 2, 3, 5, \ldots, d)$, and $d$ anticommuting derivatives with respect to $\theta^a$, $\frac{\partial}{\partial \theta_a}$, $\{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_+ = 0$, offering together $2 \cdot 2^d$ operators, the half of which are superposition of products of $\theta^a$ and another half corresponding superposition of $\frac{\partial}{\partial \theta_a}$.

Defining [14]

$$\{\theta^a, \theta^b\}_+ = 0, \quad \{\frac{\partial}{\partial \theta_a}, \frac{\partial}{\partial \theta_b}\}_+ = 0,$$

$$\{\theta_a, \frac{\partial}{\partial \theta_b}\}_+ = \delta_{ab}, (a, b) = (0, 1, 2, 3, 5, \ldots, d).$$

(A.1)

One can define new operators, expressed with $\theta^a$’s and $\frac{\partial}{\partial \theta_a}$’s.

$$\gamma^a = (\theta^a + \frac{\partial}{\partial \theta_a}) , \quad \tilde{\gamma}^a = i (\theta^a - \frac{\partial}{\partial \theta_a}) ,$$

$$\theta^a = \frac{1}{2} (\gamma^a - i \tilde{\gamma}^a) , \quad \frac{\partial}{\partial \theta_a} = \frac{1}{2} (\gamma^a + i \tilde{\gamma}^a) ,$$

(A.3)

One can make $2^d$ products of superpositions of $\gamma^a$’s and $2^d$ products of superposition of $\tilde{\gamma}^a$’s, all together $2 \cdot 2^d$ objects.

Statement 1. $\gamma^a$’s and $\tilde{\gamma}^a$’s define two independent Clifford algebras.

To prove this statement one only needs to take into account Eqs. (A.2, A.3).

$$\{\gamma^a, \gamma^b\}_+ = 2 \eta^{ab} = \{\gamma^a, \tilde{\gamma}^b\}_+ ,$$

$$\{\gamma^a, \tilde{\gamma}^b\}_+ = 0 , \quad (a, b) = (0, 1, 2, 3, 5, \ldots, d) ,$$

$$\gamma^a \dagger = \eta^{aa} \gamma^a , \quad (\tilde{\gamma}^a) \dagger = \eta^{aa} \tilde{\gamma}^a .$$

(A.4)

Eq. (A.4) demonstrates that these two Clifford algebra objects obviously define two independent “basis vectors”. Either $\gamma^a$’s or $\tilde{\gamma}^a$’s are, up to a sign, self adjoint operators.

The generators of the Lorentz transformations in the Grassmann algebra space are defined as follows

$$S^{ab} = i (\theta^a \frac{\partial}{\partial \theta_b} - \theta^b \frac{\partial}{\partial \theta_a}) , \quad (S^{ab}) \dagger = \eta^{aa} \eta^{bb} S^{ab} .$$

(A.5)

Statement 2. The sum of the generators of the Lorentz transformations in each of the two Clifford algebra spaces, $S^{ab} = i \frac{1}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a)$ and $\tilde{S}^{ab} = i \frac{1}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a)$, respectively, are equal to the generators of the Lorentz transformation in the Grassmann algebra space $S^{\dagger \dagger} = \tilde{S}^{\dagger \dagger} + \tilde{S}^{\dagger \dagger}$.

To prove this statement one only has to express in the sum $S^{ab} + \tilde{S}^{ab}$ first $S^{ab}$ with $\gamma^a$ and $\gamma^b$ and $\tilde{S}^{ab}$ with $\tilde{\gamma}^a$ and $\tilde{\gamma}^b$, and then $\gamma^a$ and $\gamma^b$ with $\theta^a$ and $\theta^b$ and $\frac{\partial}{\partial \theta_a}$, using Eq. (A.3).

One obtains that $S^{ab} = i \frac{1}{2} (\frac{\partial}{\partial \theta_a} \theta^b + \theta^a \frac{\partial}{\partial \theta_b})$ and $\tilde{S}^{ab} = i \frac{1}{2} (\theta^a \frac{\partial}{\partial \theta_a} + \theta^b \frac{\partial}{\partial \theta_b})$, leading to $S^{ab} + \tilde{S}^{ab} = S^{ab}$. 

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We conclude
\[ S_{ab} = \frac{i}{4} (\gamma^a \gamma^b - \gamma^b \gamma^a), \quad \tilde{S}_{ab} = \frac{i}{4} (\tilde{\gamma}^a \tilde{\gamma}^b - \tilde{\gamma}^b \tilde{\gamma}^a), \]
\[ S_{ab} = S_{ab} + \tilde{S}_{ab}, \quad \{S_{ab}, \tilde{S}_{ab}\} = 0, \]
\[ \{S_{ab}, \gamma^c\} = i (\eta^{bc} \gamma^a - \eta^{ac} \gamma^b), \]
\[ \{\tilde{S}_{ab}, \tilde{\gamma}^c\} = i (\eta^{bc} \tilde{\gamma}^a - \eta^{ac} \tilde{\gamma}^b), \]
\[ \{S_{ab}, \tilde{\gamma}^c\} = 0, \quad \{\tilde{S}_{ab}, \gamma^c\} = 0. \quad (A.6) \]

**Statement 2a.** The eigenvectors of the operators \(S_{ab} = i (\theta^a \frac{\partial}{\partial \theta^b} - \theta^b \frac{\partial}{\partial \theta^a})\) can be written as follows
\[ S_{ab} \frac{1}{\sqrt{2}} (\theta^a + \frac{\eta^{aa}}{ik} \theta^b) = k \frac{1}{\sqrt{2}} (\theta^a + \frac{\eta^{aa}}{ik} \theta^b), \]
\[ S_{ab} \frac{1}{\sqrt{2}} (1 + i \frac{k}{\sqrt{2}} \theta^a \theta^b) = 0, \quad (A.7) \]
while the corresponding eigenvectors of \(\tilde{S}_{ab}\) and \(\tilde{S}_{ab}\) in each of the two spaces are
\[ S_{ab} \frac{1}{2} (\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b) = k \frac{1}{2} (\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b), \]
\[ \tilde{S}_{ab} \frac{1}{2} (\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b) = k \frac{1}{2} (\gamma^a + \frac{\eta^{aa}}{ik} \gamma^b), \]
\[ \tilde{S}_{ab} \frac{1}{2} (1 + i \frac{k}{\sqrt{2}} \gamma^a \gamma^b) = k \frac{1}{2} (1 + i \frac{k}{\sqrt{2}} \gamma^a \gamma^b), \quad (A.8) \]
with \(k^2 = \eta^{aa} \eta^{bb}\).

The Eq. (A.7) can be proven by applying \(S_{ab}\) on \(\frac{1}{\sqrt{2}} (\theta^a + \frac{\eta^{aa}}{ik} \theta^b)\), leading to
\[ i \frac{1}{\sqrt{2}} (-\theta^b \eta^{ab} + \frac{\eta^{ab} \eta^{bb}}{ik} \theta^a) = \frac{i k^2}{ik \sqrt{2}} (\theta^a - \frac{\eta^{aa} \theta^b}{k^2}). \]

The application of \(S_{ab}\) on either a constant or on a constant-\(\theta^a \theta^b\) gives 0. In all these cases it is assumed that \(a \neq b\).

The proof of Eq. (A.8) goes similarly, again \(a \neq b\) is assumed.
\[ i \frac{1}{\sqrt{2}} (-\theta^b \eta^{ab} + \frac{\eta^{ab} \eta^{bb}}{ik} \theta^a) = \frac{i k^2}{ik \sqrt{2}} (\theta^a - \frac{\eta^{aa} \theta^b}{k^2}), \]
\[ i \frac{1}{\sqrt{2}} (\gamma^a \gamma^b - \frac{\eta^{ab} \eta^{bb}}{ik} \gamma^a) = \frac{i k^2}{ik \sqrt{2}} \gamma^a \gamma^b, \]
Repeating \(S_{ab}\) with \(S_{ab}\), and \(\gamma^a\)’s with \(\tilde{\gamma}^a\)’s goes through the same steps.

**Statement 2b.** The members of any irreducible representation of \(S_{ab}\) follow from the starting one by the application of \(S_{cd}\), which do not belong to the Cartan subalgebra of the Lorentz algebra.

The proof follows if we apply \(\gamma^a\) on nilpotents and projectors, since \(S_{ab} = i \gamma^a \gamma^b\).
\[ \gamma^a (k) = \eta^{aa} [-k], \quad \gamma^b (k) = -ik [\gamma^n - k], \quad \gamma^a [k] = (\gamma^a [k]), \quad \gamma^b [k] = (-ik \gamma^{ab} [-k]). \quad (A.9) \]
Correspondingly the any transformations on ”basis vectors” of the kind, which do not change sign as is required in Eq. (A.9), lead to another irreducible representations.

**Statement 3.** Postulating that \(\tilde{\gamma}^a\)’s operate on \(\gamma^a\)’s as follows \([44, 2, 11, 13, 14]\)
\[ \tilde{\gamma}^a B = (-)^B i B \gamma^a, \]
with \((-)^B = -1\), if \(B\) is (a function of) an odd product of \(\gamma^a\)'s, otherwise \((-)^B = 1\) [44], the reduction of the Clifford space and correspondingly also the reduction of the Grassmann space follows.

Eq. (13) requires
\[
[\bar{\gamma}^a(a_0 + a_{bc} \gamma^b \gamma^c + a_{bce} \gamma^b \gamma^c \gamma^d \gamma^e + \cdots) = i(a_0 + a_{bc} \gamma^b \gamma^c + a_{bce} \gamma^b \gamma^c \gamma^d \gamma^e + \cdots) \gamma^a)]|_{\psi_{oc}} >, \\
[\bar{\gamma}^a(a_{\gamma} \gamma^b + a_{bcd} \gamma^b \gamma^c \gamma^d + \cdots) = -i(a_{\gamma} \gamma^b + a_{bcd} \gamma^b \gamma^c \gamma^d + \cdots) \gamma^a)]|_{\psi_{oc}} >, 
\]

To prove Statement 3, let us evaluate what does Eq. (13) require when we use Eq. (A.3) on \(|\psi_{oc}|: \gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a})\) and \(\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a})\), with \(|\psi_{oc}| >\) expressed as well with \(\theta^a\)'s and \(\frac{\partial}{\partial \theta^a}\).

Let us point out that \(|\psi_{oc}| >\), expressed in terms of \(\theta^a\)'s and \(\frac{\partial}{\partial \theta^a}\), is an even function of \(\theta^a\)'s, \(|\psi_{oc}| > = (1 + \theta^a \theta^b + \cdots)\), while \(\partial \frac{\partial}{\partial \theta^a}\), applying on identity, gives zero.

The proof is needed for any even and any odd summand of \(B\), appearing in Eq. (13) and for an arbitrary \(|\psi_{oc}| >\).

Let us start with \([\tilde{\gamma}^a a_0 = ia_0 \gamma^a]|_{\psi_{oc}} >\), with \(|\psi_{oc}| > = (1 + \theta^a \theta^b + \cdots)\) and \(a_0\) an arbitrary constant. This relation requires that \([i(\theta^a - \frac{\partial}{\partial \theta^a}) = i(\theta^a + \frac{\partial}{\partial \theta^a})]|_{\psi_{oc}} >\), leading to \(-2i \frac{\partial}{\partial \theta^a}|_{\psi_{oc}} > = 0, \forall \frac{\partial}{\partial \theta^a}\) this last relation can only be true if \(\frac{\partial}{\partial \theta^a}|_{\psi_{oc}} > = 0, \forall \frac{\partial}{\partial \theta^a}\).

Evaluating \([\bar{\gamma}^a a_{bc} \gamma^b \gamma^c = i a_{bc} \gamma^b \gamma^c \gamma^a]|_{\psi_{oc}} >\) we end up again with the requirement \(\frac{\partial}{\partial \theta^a}|_{\psi_{oc}} > = 0, \forall \frac{\partial}{\partial \theta^a}\). Applying \(\tilde{\gamma}^a\) on any even products of \(\gamma^a\)'s we end up with the same requirement \(\frac{\partial}{\partial \theta^a}|_{\psi_{oc}} > = 0, \forall \frac{\partial}{\partial \theta^a}\).

Application of \(\gamma^a\) on any odd products of \(\gamma^a\)'s, while \(|\psi_{oc}| > = (1 + \theta^a \theta^b + \cdots)\),
\[
[\bar{\gamma}^a(a_{\gamma} \gamma^b + a_{bcd} \gamma^b \gamma^c \gamma^d + \cdots) = -i(a_{\gamma} \gamma^b + a_{bcd} \gamma^b \gamma^c \gamma^d + \cdots) \gamma^a)]|_{\psi_{oc}} >, 
\]

it follows again, after expressing \(\tilde{\gamma}^a = i(\theta^a - \frac{\partial}{\partial \theta^a})\) and \(\gamma^a = (\theta^a + \frac{\partial}{\partial \theta^a})\) into Eq. A.10, that \(\frac{\partial}{\partial \theta^a}|_{\psi_{oc}} > = 0\) is the only solution, leading to.
\[
\theta^a \Rightarrow \gamma^a, 
\]

which does not mean that \(\theta^a\) is equal to \(\gamma^a\) but rather that the whole Grassmann algebra reduces to only one of the two Clifford algebras, the one, in which the "basis vectors" are superposition of products of (odd when describing fermions) number of \(\gamma^a\)'s. It also does not mean that \(\theta^a\)'s are equal to \(i \gamma^a\)'s, since the application of \(\tilde{\gamma}^a\)'s depend on properties of \(B(\gamma^a)\), on which \(\tilde{\gamma}^a\)'s apply.

**Statement 3a.** The relations of Eq. (A.4) remain valid also after the reduction of the Clifford space.

Let us check Eq. (A.4): \(\{\tilde{\gamma}^a, \tilde{\gamma}^b\}_+ = 2 \eta^{ab} \tilde{\gamma}^a \tilde{\gamma}^b + \tilde{\gamma}^a \tilde{\gamma}^b = \tilde{\gamma}^a i \gamma^b + \tilde{\gamma}^b i \gamma^a = i \gamma^b(-i) \gamma^a + i \gamma^a(-i) \gamma^b = 2 \eta^{ab}\). \(\{\tilde{\gamma}^a, \gamma^b\}_+ = 0 = \tilde{\gamma}^a \gamma^b + \gamma^b \tilde{\gamma}^a = \gamma^b(-i) \gamma^a + \gamma^a(i) \gamma^b = 0\). For a particular case one has \(\{\tilde{\gamma}^a, \gamma^a\}_+ = 0 = \tilde{\gamma}^a \gamma^a + \gamma^a \tilde{\gamma}^a = \gamma^a(-i) \gamma^a + \gamma^a(i) \gamma^a = 0\).

The application of \(\tilde{\gamma}^a\) obviously depends on the space on which it applies, Eq. (13) namely requires: \(\tilde{\gamma}^a(a_0 + a_{\gamma} \gamma^b + a_{bcd} \gamma^b \gamma^c \gamma^d + \cdots) (i a_0 \gamma^a + (-i) a_0 \gamma^a + i a_{bc} \gamma^b \gamma^c \gamma^a + \cdots)|_{\psi_{oc}}\).

**Statement 3b.** Taking into account in Eq. (13) required application of \(\tilde{\gamma}^a\)'s on the Clifford space of \(\gamma^a\)'s (causing the reduction of the Clifford space and at the same time as well the reduction of the Grassmann space), it follows that the eigenvalues of \(\hat{S}^{ab}\) on the eigenvectors of \(S^{ab}\) agree with the eigenvalues of \(S^{ab}\) on nilpotents, while the eigenvalues of \(\hat{S}^{ab}\) on projectors, which are eigenvectors of \(S^{ab}\), have opposite sign.

Let us check Eq. (14).
\[
\hat{S}^{ab} \frac{i}{2}(\gamma^a + \eta^{ac}_b \gamma^c) = \frac{i}{2} \bar{\gamma}^a \gamma^b \frac{i}{2}(\gamma^a + \eta^{ac}_b \gamma^c) = \frac{i}{2} \frac{1}{2}(\gamma^a + \eta^{ac}_b \gamma^c) \gamma^b \gamma^a = \frac{i}{2} \frac{1}{2}(-\eta^{ac}_b \gamma^b + \eta^{ac}_b \gamma^a) = \frac{k}{2} \frac{1}{2}(\gamma^a + \eta^{ac}_b \gamma^c), 
\]
\[
\hat{S}_{ab}^{\frac{1}{2}}(1 + \frac{i}{\gamma}a_{\gamma}b) = \frac{1}{2}\left(1 + \frac{i}{\gamma}a_{\gamma}b\right)\gamma^b\gamma^a = \frac{1}{2}\left((-\gamma^a_{\gamma}b + \frac{i}{\gamma}\eta_{\gamma}^{\alpha\beta}ab) = \frac{1}{2}\left(1 + \frac{i}{\gamma}a_{\gamma}b\right),
\]
where it is taken into account that \(k^2 = \eta^{ab}ab\). This proves Statement 3b.

**Statement 4.** The algebraic product of \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\) is the same for all \(m\) of a particular irreducible representation \(f\).

To prove this we take into account that \(\hat{b}_{f}^{m*}\) follows from \(\hat{b}_{f}^{m-1*}\) by the application of a particular \(2iS^{eq}\). Then \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*} = \hat{b}_{f}^{m-1*}(2S^{eq})\hat{b}_{f}^{m-1*} = \hat{m}_{f}^{m-1*}\hat{b}_{f}^{m-1*}\), due to the relation \((2iS^{ab})(-2iS^{ab}) = 1\). Repeating this procedure for each \(m\) proves the statement.

**Statement 5.** Each irreducible representation has its own algebraic product \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\).

We pay attention to the Clifford odd representations, but the proof is as well valid for the Clifford even representations.

To prove this statement let us start with \(d = 2(2n + 1)\) with the irreducible representation \(f\) and the member \(m\) equal to \(\hat{b}_{f}^{m*} = (+)(+)...(+).\) To obtain the \(2^d - 1\) rest irreducible representations we must transform each of possible pairs \((k)(k)\) into \([k][k]\). Let us start with the first two. One obtains \(\hat{b}_{f}^{m*} = [+][+]...(+).\) The two algebraic products, \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*} = [+]...[+]\) and \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*} = [+]...[-]\), distinguish in the first two projectors. When replacing a pair by a pair in \(\hat{b}_{f}^{m*}\), we end up with \(2^d - 1\) different \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\), differing in all possible pairs \([+][+]\) replacing \([-][-].\)

For \(d = 4n\) and the Clifford odd representations we must start with \(\hat{b}_{f}^{m*} = (+)(+)...[+]\), and then repeat all steps. We shall again obtain \(2^d - 1\) different \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\).

**Statement 5a.** There are \(2^d - 1\) different algebraic products \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\).

Since due to Statement 4. all the members of a particular irreducible representation have the same algebraic product \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\), we can conclude that there are \(2^d - 1\) different algebraic products \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\).

**Statement 5b.** Each creation operator \(\hat{b}_{f}^{m*}\) gives nonzero contribution when applied on \(|\psi_{oc}\rangle\).

There is one summand in \(|\psi_{oc}\rangle\), namely, \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\), \(\forall m\), on which \(\hat{b}_{f}^{m*}\) gives a nonzero contribution: \(\sum_{f'}^{2^d - 1}\hat{b}_{f'}^{m}\hat{b}_{f'}^{m*} = \sum_{f'}^{2^d - 1}\delta_{f'f}\hat{b}_{f'}^{m*}\).

**Statement 5c.** In odd representations the algebraic product of any two annihilation operators \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\) gives zero, as also the algebraic product of any two creation operators \(\hat{b}_{f}^{m*}\hat{b}_{f}^{m*}\) gives zero. Correspondingly the application of the annihilation operators on \(|\psi_{oc}\rangle\) gives nonzero contribution.

All annihilation operators are "orthogonal", as also all the creation operators are: \(\hat{b}_{f}^{m*}\hat{b}_{f'}^{m'} = 0, \hat{b}_{f}^{m*}\hat{b}_{f}^{m*} = 0\). Within the same irreducible representation at least one nilpotent of the two creation operators or of the two annihilation operators are the same. Among different irreducible representations of each kind separately, either one nilpotent is the same in both operators appearing in the product, or \((k)\) multiplies \([k]\) or \([-k]\) multiplies \((k)\) or \([k]\) multiplies \([-k]\), since one irreducible representation differs from the other in a pair \((k)(k')\) going to \([k][k']\) or \([k][k']\) going to \((k)(k')\).

**Statement 6.** The operator \(S_{cd}\), which does not belong to the Cartan subalgebra of Eq. (5), generates after the reduction of the Clifford space a new irreducible representation, carrying
different family quantum number.

The proof of Statement 5. contains also the proof for Statement 6.. All the members of one irreducible representation are reachable by the application of $S^{ab}$'s. Let us start in $d = 2(2n + 1)$ with the Clifford odd representation containing the member $\hat{b}^m_\alpha|\psi_{oc}\rangle = (+i)(+ \cdots (+ \cdots (+)$ $\cdots (+)$ (with the family quantum numbers $(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}, \cdots, \frac{1}{2})$), determined by $S^{ab}$'s from Eq. (5)). Operator $\hat{S}^{bd}$ transforms $\hat{b}^m_\alpha$ into $(+i)(+ \cdots [+ \cdots [+ \cdots (+)$. This new creation operator belongs to new irreducible representation (with the family quantum numbers $(\frac{1}{2}, \frac{1}{2}, \cdots, -\frac{1}{2}, \cdots, -\frac{1}{2}, \cdots, \frac{1}{2})$, since it is not reachable by $S^{ab}$ (which generate all the rest members of the same irreducible representation). Transforming all pairs into the new ones, one obtains $2^d - 1$ families.

For $d = 4n$ we start for odd representations by $\hat{b}^m_\alpha = (+i)(+ \cdots (+ \cdots (+)$ and repeat the above procedure (by taking into account that $\hat{S}^{bd}$ transforms $\cdots [+ \cdots (+)$ into $|\psi_{oc}\rangle$.

Similarly we can find all the families of the Clifford even representations, if taking into account Eq. (13)

Statement 7. Creation operators $\hat{b}^m_\alpha$ and their Hermitian conjugated partners annihilation operators, appearing in Eq. (10), have the properties

$$
\hat{b}^m_\alpha |\psi_{oc}\rangle = 0 \cdot |\psi_{oc}\rangle ,
\hat{b}^m_\alpha |\psi_{oc}\rangle = |\psi^m_\alpha\rangle ,
\{\hat{b}^m_\alpha, \hat{b}^m_\alpha\} \cdot |\psi_{oc}\rangle = 0 \cdot |\psi_{oc}\rangle ,
\{\hat{b}^m_\alpha, \hat{b}^m_\alpha\} \cdot |\psi_{oc}\rangle = \delta^{mm'} |\psi_{oc}\rangle ,
\{\hat{b}^m_\alpha, \hat{b}^m_\alpha\} \cdot |\psi_{oc}\rangle = 0 \cdot |\psi_{oc}\rangle .
$$

(A.11)

Let us prove this statement step by step:
a. The last line of Eq. (A.11) requires that $\{\hat{b}^m_\alpha, \hat{b}^m_\alpha\} = 0$ $|\psi_{oc}\rangle$.

a.i. The product of two equal creation operators is zero, since the product of two nilpotents of the same kind, $(|k\rangle)^2 = 0$, gives zero.

a.ii. All the creation operators can be obtained from the starting one by the application $S^{cd}$. Since at this application at least one nilpotent remains the same, it follows that all creation operators within the same irreducible representation are orthogonal $(\hat{b}^m_\alpha \cdot \hat{b}^m_\alpha = 0)$, by themselves and then also on $|\psi_{oc}\rangle$.

a.iii. The creation operators of two different irreducible representations $f \neq f'$ can be obtained from the starting one by replacing two nilpotents $(k) \cdot (k')$ by $[k] \cdot [k']$, a nilpotent $(k)$ and a projector $[k]$ by the projector $[k]$ and the nilpotent $(k)$, or two projectors $[k]$ $[k']$ by two nilpotents. Since at least one of the nilpotents remains the same, it follows that all the same members of different irreducible representations are orthogonal, $(\hat{b}^m_\alpha \cdot \hat{b}^m_\alpha = 0)$, by themselves and then also on $|\psi_{oc}\rangle$.

a.iv. The creation operators, belonging to two different irreducible representations $(f, f')$ and to two different members $(m, m')$ have the property $(\hat{b}^m_\alpha \cdot \hat{b}^m_\alpha = 0)$ (are orthogonal), due to the way how they are created $(\hat{b}^m_\alpha \cdot \hat{b}^m_\alpha = S^{mm'} \hat{b}^m_\alpha \cdot \hat{b}^m_\alpha = 0$, as it is proven under iii.).
the two have the same nilpotent or there appear a product of two projectors of the same type with opposite $k$ ($[k], [-k]$), or it appears ($[k]_A$ $[-k]$), or $(k)_A$ $[k]$, Eq. (C.1).

b. The first line requires $\hat{b}_f^m|\psi_{oc}> = 0$, which is equivalent to requiring $\hat{b}_f^m|\hat{b}_f^m| = 0$, $\forall m'$ and $f'$, since the vacuum state is equal to $|\psi_{oc}> = \sum_{f=1}^{2}\hat{b}_f^m|\psi_{oc}> = 0$, since $\hat{b}_f^m = (\hat{b}_f^m)^\dagger$.

c. The second line requires that $\hat{b}_f^{m',\dagger}|\psi_{oc}> = |\psi_{oc}>$. Namely, $\hat{b}_f^{m',\dagger}|\hat{b}_f^{m',\dagger}| = 0$, $\forall f \neq f'$, since the application of $\hat{b}_f^{m',\dagger}$ on $\hat{b}_f^{m',\dagger}$ gives zero due to the orthogonality of the members of different irreducible representations.

d. The third line requires of Eq. (A.11) requires that $|\psi_{oc}> = 0$, since $\hat{b}_f^{m'} = (\hat{b}_f^{m'})^\dagger$, the proof is the same as in the case a., e. The fourth line requires that $\{\hat{b}_f^m, \hat{b}_f^{m'}\}|_{A} = |\psi_{oc}>$. Since $\hat{b}_f^m = (\hat{b}_f^m)^\dagger$, this proofs follows from a. and b..

Allowing, however, that also $f \neq f'$, one finds that the term $\hat{b}_f^m|\hat{b}_f^{m'}| > 0$, gives an nonzero contribution — an Clifford even object instead of zero. But after reducing the Clifford algebra space, Eq. (13), the two irreducible representations ($f, f'$) carry two different family quantum numbers. The algebraic product has no meaning any longer: Two different families, reachable from each other by $S^{ab}$, are orthogonal in the sense of the tensor product.

(As an example let us demonstrate this on the case $d = (5 + 1)$, presented on Table B1:

$\hat{b}_f^{m,1} = I_{f} = (-) \{56, 12, 03\}, \hat{b}_f^{m,\dagger} = \{03, 12, 56\}$. One obtains $\hat{b}_f^{m,1}\hat{b}_f^{m,\dagger} = |\psi_{oc}> = (+) = 0.3$.

Let us show as an example, that $\hat{b}_f^m|\hat{b}_f^m| = 0$. Each annihilation operator of the same irreducible representation $f$ and different member $m'$ follows from $\hat{b}_f^m$, $\hat{b}_f^{m'} = (S^{m'm}\hat{b}_f^{m'})^\dagger = \hat{b}_f^m|\hat{b}_f^{m'}| = \hat{b}_f^m|\hat{b}_f^{m'}| = 0$.)

Appendix B. Example for creation and annihilation operators of single fermion states in $d = (5 + 1)$

In Table B1, taken from Ref. [15], the basic creation operators $\hat{b}_f^{m}\dagger$, $m = (ch, s)$, and their annihilation partners $\hat{b}_f^m$ in $d = (5 + 1)$ are presented for all four $(2^5 - 1 = 4)$ families, $f = (I, II, III, IV)$. Index $m$ is divided into $s$, determining the spin, and $ch$, determining the charge, to point out that $S^{56}$ represents the charge from the point of view of $d = (3 + 1)$, having two values, $+\frac{1}{2}$ and $-\frac{1}{2}$. The vacuum state, Eq. (9), is the sum of the selfadjoint operators $\hat{b}_f^m$, $\hat{b}_f^{m'} = (S^{m'm}\hat{b}_f^{m'})^\dagger = \hat{b}_f^m|\hat{b}_f^{m'}| = \hat{b}_f^m|\hat{b}_f^{m'}| = 0$.)

All the Clifford even "families" can be obtained as algebraic products, $A$, of the Clifford odd "basis vectors" of Table B1.
Table B1. The basic creation operators — $\hat{b}_{f}^{m=(ch,s)}$ (chl, charge), the eigenvalue of $S^{56}$, and $s$ (spin), the eigenvalues of $\hat{S}_{f}^{03}$ and $\hat{S}_{f}^{12}$, explain the index $m$ — and their annihilation partners — $\hat{b}_{f}^{m=(ch,s)}$ — are presented for $d = (5 + 1)$-dimensional case. The basic creation operators are the odd products of nilpotents and projectors, which are the "eigenstates" of the Cartan subalgebra generators, $(\hat{S}_{f}^{03}, \hat{S}_{f}^{12}, S^{56})$, $(\hat{S}_{f}^{03}, \hat{S}_{f}^{12}, S^{56})$, presented in Eq. (5). Operators $\hat{b}_{f}^{m=(ch,s)}$ and $\hat{b}_{f}^{m=(ch,s)}$ fulfill the anticommutation relations of Eq. (10).

| family $f$ | $m = (ch,s)$ | $\hat{b}_{f}^{m=(ch,s)}$ | $\hat{b}_{f}^{m=(ch,s)}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ | $S^{03}$ | $S^{12}$ | $S^{56}$ |
|-----------|-------------|-----------------|-----------------|---------|---------|---------|---------|---------|---------|
| I         | (\(\frac{1}{2}, \frac{1}{2}\) | +1) | +1) | +1) | +1) | +1) | +1) | +1) | +1) |
| I         | (\(\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| I         | (\(-\frac{1}{2}, \frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| I         | (\(-\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| II        | (\(\frac{1}{2}, \frac{1}{2}\) | +1) | +1) | +1) | +1) | +1) | +1) | +1) | +1) |
| II        | (\(\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| II        | (\(-\frac{1}{2}, \frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| II        | (\(-\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| III       | (\(\frac{1}{2}, \frac{1}{2}\) | +1) | +1) | +1) | +1) | +1) | +1) | +1) | +1) |
| III       | (\(-\frac{1}{2}, \frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| III       | (\(-\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| IV        | (\(\frac{1}{2}, \frac{1}{2}\) | +1) | +1) | +1) | +1) | +1) | +1) | +1) | +1) |
| IV        | (\(\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| IV        | (\(-\frac{1}{2}, \frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |
| IV        | (\(-\frac{1}{2}, -\frac{1}{2}\) | -1) | -1) | -1) | -1) | -1) | -1) | -1) | -1) |

Appendix C. Some useful relations

Some relations among nilpotents and projectors, taken from Ref. [6], are presented, following from the last three lines of Eq. (2)

$$ab_{ab}^{(k)[k]} = 0 , \quad ab_{ab}^{[k](-k)} = 0 , \quad \eta_{ab}^{ab} = 0 . \quad \text{(C.1)}$$

The same relations are valid also if one replaces $(k)$ with $(\bar{k})$ and $[k]$ with $[\bar{k}]$.

From Eq. (15) it follows

$$S_{ac}^{ab} (k)[k] = -i \eta_{aa}^{ab} \eta_{cc}^{cd} [k] (k) [k] , \quad \tilde{S}_{ac}^{ab} (k)[k] = i \eta_{aa}^{ab} \eta_{cc}^{cd} [k] [k] ,$$

$$S_{ac}^{ab} (k)[k] = \frac{i}{2} \delta_{ab}^{cd} (k) (-k) , \quad \tilde{S}_{ac}^{ab} (k)[k] = -\frac{i}{2} \delta_{ab}^{cd} (k) [k] ,$$

$$S_{ac}^{ab} (k)[k] = -i \eta_{aa}^{ab} [k] (k) , \quad \tilde{S}_{ac}^{ab} (k)[k] = i \eta_{aa}^{ab} [k] [k] ,$$

$$S_{ac}^{ab} (k)[k] = \frac{i}{2} \eta_{cc}^{cd} (k) (-k) , \quad \tilde{S}_{ac}^{ab} (k)[k] = -\frac{i}{2} \eta_{cc}^{cd} (k) [k] . \quad \text{(C.2)}$$
By using Eq. (13) one finds the relations
\[ ab \tilde{\rho}(k) ab(k) = 0, \quad ab \tilde{\rho}(-k) ab(k) = -i \eta^{ab} [k], \]
\[ ab \tilde{\rho}(k) ab(k) = i (k), \quad \tilde{\rho}(k) [ab] = 0, \]
\[ ab \tilde{\rho}(k) ab(k) = (k), \quad [ab](-k) = 0, \]
\[ ab \tilde{\rho}(k) ab(k) = 0, \quad [ab](-k) = [k]. \quad (C.3) \]

Appendix D. One irreducible representation of internal space in \( d = (13 + 1) \),
described by Clifford algebra elements \( \gamma^a \), and families of irreducible representations
Below the subgroups of the starting groups \( SO(13,1) \) and \( \tilde{SO}(13,1) \) are presented, manifesting in \( d = (3 + 1) \) the spins, charges and families of fermions in the spin-charge-family theory. Table D1, representing one \( SO(13,1) \) irreducible representation of fermions — quarks and leptons and antiquarks and antileptons. The Clifford odd products of \( \gamma^a \)'s are expressed by products of an odd number of nilpotents and projectors.

a.i. The generators of the two \( SU(2) (\subset SO(3,1) \subset SO(7,1) \subset SO(13,1)) \) groups, describing spins of fermions
\[ \vec{N}_L, R := \frac{1}{2} (S_{23}^{\pm}, S_{31}^{\pm}, S_{12}^{\pm}, \tilde{S}^{01}, \tilde{S}^{02}, \tilde{S}^{03}), \quad (D.1) \]
are presented.

a.ii. The generators of the two \( SU(2) (SU(2) \subset SO(4) \subset SO(7,1) \subset SO(13,1)) \) groups, describing the two kinds of weak charges of fermions
\[ \vec{\tau}^1 := \frac{1}{2} (S_{58}^{\pm}, S_{57}^{\pm}, S_{56}^{\pm}, S_{57}^{\pm}, S_{58}^{\pm}, S_{56}^{\pm}, S_{57}^{\pm}, S_{58}^{\pm}), \quad (D.2) \]
are presented.

a.iii. The \( SU(3) \) and \( U(1) \) subgroups of \( SO(6) \subset SO(13,1) \), describing the colour charge and the "fermion" charge of fermions
\[ \vec{\tau}^3 := \frac{1}{2} \{ S_{9}^{12} - S_{10}^{11}, S_{10}^{12} + S_{9}^{11}, S_{9}^{11} + S_{10}^{12} - S_{11}^{12}, \]
\[ S_{10}^{12} - S_{11}^{12}, S_{11}^{12} + S_{10}^{12} - S_{12}^{13}, \]
\[ S_{10}^{14} + S_{11}^{12} + S_{13}^{14}, \sqrt{3} (S_{9}^{10} + S_{11}^{12} - 2S_{12}^{13}), \}
\[ \vec{\tau}^4 := -\frac{1}{3} (S_{9}^{10} + S_{11}^{12} + S_{13}^{14}), \quad (D.3) \]
are presented.

b.i. The two \( \tilde{SU}(2) \) subgroups of \( \tilde{SO}(3,1) (\subset \tilde{SO}(7,1) \subset \tilde{SO}(13,1)) \), describing families of fermions
\[ \tilde{N}_{L,R} := \frac{1}{2} (\tilde{S}_{23}^{\pm}, \tilde{S}_{31}^{\pm}, \tilde{S}_{31}^{\pm}, \tilde{S}_{12}^{\pm}, \tilde{S}_{12}^{\pm}, \tilde{S}^{03}), \quad (D.4) \]
are presented.

**b.ii.** The two $\widetilde{SU}(2)$ subgroups of $\widetilde{SO}(4) \subset SO(7,1) \subset SO(13,1)$), describing families of fermions

$$\tau^1 := \frac{1}{2}(\tilde{S}^{68} - \tilde{S}^{67}, \tilde{S}^{57} + \tilde{S}^{68}, \tilde{S}^{56} - \tilde{S}^{78}),$$

$$\tau^2 := \frac{1}{2}(\tilde{S}^{68} + \tilde{S}^{67}, \tilde{S}^{57} - \tilde{S}^{68}, \tilde{S}^{56} + \tilde{S}^{78}),$$

are presented.

**b.iii.** The group $\tilde{U}(1)$, the subgroup of $\widetilde{SO}(6) \subset \widetilde{SO}(13,1)$, describing family quantum numbers of fermions

$$\tau^4 := -\frac{1}{3}(\tilde{S}^{10} + \tilde{S}^{11} - \tilde{S}^{13}),$$

are presented.

**c.** Relations among the hyper, weak and the second $SU(2)$ charges

$$Y := \tau^4 + \tau^{23}, \quad Y' := -\tau^4 \tan^2 \vartheta_2 + \tau^{23}, \quad Q := \tau^1 + Y, \quad Q' := -Y \tan^2 \vartheta_1 + \tau^1,$$

$$\tilde{Y} := \tau^1 + \tau^{23}, \quad \tilde{Y}' := -\tau^4 \tan^2 \vartheta_2 + \tau^{23}, \quad \tilde{Q} := \tilde{Y} + \tau^{13}, \quad \tilde{Q}' := -Y \tan^2 \vartheta_1 + \tau^{13}$$

are presented.

Below are some of the above expressions written in terms of nilpotents and projectors

$$N^\pm = N_+^1 \pm i N_+^2 = -\left[\frac{3}{4}(\mp i)(\pm)\right], \quad N^\mp = N_-^1 \pm i N_-^2 = \left[\frac{3}{4}(\pm i)(\mp)\right],$$

$$\tilde{N}^\pm = -\left[\frac{3}{4}(\pm i)(\pm)\right], \quad \tilde{N}^\mp = \left[\frac{3}{4}(\mp i)(\mp)\right],$$

$$\tau^{1\pm} = (\mp) (\mp), \quad \tau^{2\mp} = (\mp) (\mp),$$

$$\tau^{1\mp} = (\mp) (\mp), \quad \tau^{2\pm} = (\mp) (\mp).$$

**Table D1.** The left handed ($\tau^{(13+1)} = -1$), multiplet of creation operators of fermions — the members of one particular fundamental representation of the $SO(13,1)$ group, manifesting the subgroup $SO(7,1)$ of the charge colour quarks and antiquarks and the colourless leptons and antileptons — is presented in the massless basis as odd products of nilpotents and the rest of projectors (together are $\frac{3}{4}$ = 7 nilpotents and projectors). The multiplet contains the left handed ($\tau^{(13+1)} = -1$ weak SU(2)J charged ($\tau^+ = \pm \frac{1}{3}$), ($\tau^2 = \frac{1}{2}(S^{68} - S^{67}, S^{57} + S^{68}, S^{56} - S^{78})$ and SU(2)J chargeless ($\tau^3 = 0$), $\tau^2 = \frac{1}{2}(S^{68} + S^{67}, S^{57} - S^{68}, S^{56} + S^{78})$) quarks and leptons and the right handed ($\tau^{(3+1)} = 1$), weak (SU2J) chargeless and SU(2)J chargeless ($\tau^3 = \pm \frac{1}{2}$) quarks and leptons, both with the spin $S^{12}$ up and down ($\frac{3}{4}$, $\frac{1}{2}$, respectively). The creation operators of quarks distinguish from those of leptons only in the SU(3) x U(1) part: Quarks are triplets of three colours ($\tau^{(13)}, \tau^{(8)}$) = [(1, -1, 1), (-1, 1, 1), (0, 0, 0)], ($\tau^{(3)} = \frac{1}{2}(S^{12} - S^{10}, S^{10} + S^{12}, S^{11} - S^{14}, S^{14} + S^{11}, S^{11} - S^{13}, S^{13} + S^{11}, S^{14} - S^{13}, S^{13} + S^{14}, S^{11} - S^{13}, S^{13} + S^{14}, S^{11} - S^{13}, S^{13} + S^{14})$, carrying the "fermion charge" ($\tau^4 = \pm \frac{1}{3}$). The colourless leptons carry the "fermion charge" ($\tau^4 = \mp \frac{1}{3}$). The same multiplet of creation operators the left handed weak (SU2J) chargeless and SU(2)J chargeless antiquarks and antileptons belong. Antiquarks distinguish from antileptons again only in the SU(3) x U(1) part: Anti-quarks are antitriplets, carry the "fermion charge" ($\tau^4 = \mp \frac{1}{3}$). The anticolourless antiquarks carry the "fermion" charge ($\tau^4 = \pm \frac{1}{3}$).

| $^N$ | $^{13+1} \quad ^{3+1}$ | $^{(13+1)}$ | $^{(13)}$ | $^{(8)}$ | $^{(3)}$ | $^{(1)}$ | $^Y$ | $^Q$ |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
| 1 | $^N_{1^+}$ | $^N_{1+}$ | $^N_{1+}$ | $^N_{1+}$ | $^N_{1+}$ | $^N_{1+}$ | $^N_{1+}$ | $^N_{1+}$ |
| 2 | $^N_{1^-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ |
| 3 | $^N_{1^0}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ |
| 4 | $^N_{1^-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ | $^N_{1-}$ |
| 5 | $^N_{1^0}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ | $^N_{10}$ |

(Continued on next page)
| i | $d_{\Lambda}^{21}$ | $d_{\Lambda}^{23}$ | $d_{\Lambda}^{25}$ | $d_{\Lambda}^{27}$ | $d_{\Lambda}^{29}$ | $d_{\Lambda}^{31}$ | $d_{\Lambda}^{33}$ | $d_{\Lambda}^{35}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

(Anti)jetlet, $T^{(1+)} = (−1) L, T^{(0)} = (1) − 1$ of (anti)quarks and (anti)leptons
Table D1 represents in the spin-charge-family theory the creation operators for the observed quarks and leptons and antiquarks and antileptons for a particular family, Table D2. Hermitian conjugation of the creation operators of Table D1 generates the corresponding annihilation operators, fulfilling together with the creation operators anticommutation relations for fermions of Eq. (22). Creation and annihilation operators are the tensor products of the finite "basis vectors" in internal space and the continuously infinite basis in ordinary space. The "basis vectors" in internal space transfer their oddness to creation and correspondingly to annihilation operators.

The condensate of two right handed neutrinos with the family quantum numbers of the upper four families, causing the break of the starting symmetry SO(13, 1) into SO(7, 1) × SU(3) × U(1), is presented in Table D3.

Appendix E. Expressions for scalar fields in term of $\omega_{s's''s}$ and $\tilde{\omega}_{abs}$

The scalar fields, responsible for masses of the family members and of the heavy bosons [10, 11] after gaining nonzero vacuum expectation values and triggering the electroweak break, are presented in the second line of Eq. (26). These scalar fields are included in the covariant derivatives as $-\frac{1}{2} \xi^{s} \omega_{s's''s} - \frac{1}{2} \xi^{ab} \tilde{\omega}_{abs}$. $s \in (7, 8), (a, b) \in (0, \ldots, 3), (5, \ldots, 8)$.

One can express the scalar fields carrying the quantum numbers of the subgroups of the family groups, expressed in terms of $\tilde{\omega}_{abs}$ (they contribute to mass matrices of quarks and leptons and
Table D2. Eight families of creation operators of $u_R^{\tau l}$ — the right handed $u$-quark with spin $\frac{1}{2}$ and the colour charge ($\tau^{33} = 1/2$, $\tau^{38} = 1/(2\sqrt{3})$), appearing in the first line of Table D1 — and of the colourless right handed neutrino $\bar{\nu}_R^\dagger$ — of spin $\frac{1}{2}$, appearing in the 25th line of Table D1 — are presented in the left and in the right column, respectively. Table is taken from [13]. Families belong to two groups of four families, one ($I$) is a doublet with respect to ($\bar{N}_L$ and $\bar{T}^{(1)}$) and a singlet with respect to ($\bar{N}_R$ and $\bar{T}^{(2)}$), the other ($II$) is a singlet with respect to ($\bar{N}_L$ and $\bar{T}^{(1)}$) and a doublet with respect to ($\bar{N}_R$ and $\bar{T}^{(2)}$), Eq. (D.1). All the families follow from the starting one by the application of the operators ($\bar{N}_{R,L}$, $\bar{T}^{(2,1)\pm}$), Eq. (D.8). The generators ($N^\pm_{R,L}$, $T^{(2,1)\pm}$), Eq. (D.8), transform $u_R^{\tau l}$ to all the members of one family of the same colour charge. The same generators transform equivalently the right handed neutrino $\bar{\nu}_R^\dagger$ to all the colourless members of the same family.

| $r$ | $\bar{N}_{R,L}$ | $\bar{T}^{(2,1)\pm}$ | $\bar{N}_{R,L}$ | $\bar{T}^{(2,1)\pm}$ |
|-----|-----------------|---------------------|-----------------|---------------------|
| $1$ | $\bar{N}_{R,L}$ | $\bar{T}^{(2,1)\pm}$ | $\bar{N}_{R,L}$ | $\bar{T}^{(2,1)\pm}$ |
| $2$ | $\bar{N}_{R,L}$ | $\bar{T}^{(2,1)\pm}$ | $\bar{N}_{R,L}$ | $\bar{T}^{(2,1)\pm}$ |

Table D3. The condensate of the two right handed neutrinos $\nu_R$, with the quantum numbers of the $VII^{th}$ family, coupled to spin zero and belonging to a triplet with respect to the generators $\tau^{23}$, is presented, together with its two partners. The condensate carries $\tau^l = 0$, $\tau^{23} = 1$, $\tau^4 = -1$ and $Q = 0 = Y$. The triplet carries $\bar{\tau}^4 = -1$, $\bar{\tau}^{23} = 1$ and $\bar{N}_R^3 = 1$, $\bar{Y} = 0$, $\bar{Q} = 0$. The family quantum numbers of quarks and leptons are presented in Table D2.

| state | $S^{12}$ | $\tau^{13}$ | $\tau^{23}$ | $\tau^4$ | $Y$ | $Q$ | $\bar{\nu}_R$ | $\bar{\nu}_R^\dagger$ |
|-------|----------|-----------|-----------|---------|-----|-----|----------------|----------------|
| $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ | $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ | $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ | $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ |
| $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ | $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ | $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ | $|\nu_{VIII}^{1R}| > 1$ | $|\nu_{VIII}^{2R}| > 1$ |

to masses of the heavy bosons), if taking into account Eqs. (D.4, D.5, D.7),

$$\sum_{a,b} -\frac{1}{2} \tilde{\omega}_{ab} \omega_{ab} = -(\bar{\tau}^1 \bar{A}_s^1 + \bar{\tau}^2 \bar{A}_s^2 + \bar{\tau}^3 \bar{A}_s^3 + \bar{N}_R \bar{N}_R),$$

$$\bar{A}_s^1 = (\bar{\omega}_{58s} - \bar{\omega}_{67s}, \bar{\omega}_{57s} + \bar{\omega}_{68s}, \bar{\omega}_{56s} - \bar{\omega}_{78s}),$$

$$\bar{\tau}^{23} \bar{A}_s^2 = (\bar{\omega}_{23s} + i \bar{\omega}_{01s}, \bar{\omega}_{31s} + i \bar{\omega}_{02s}, \bar{\omega}_{12s} + i \bar{\omega}_{03s}),$$

$$\bar{\tau}^{23} \bar{A}_s^3 = (\bar{\omega}_{58s} + \bar{\omega}_{67s}, \bar{\omega}_{57s} - \bar{\omega}_{68s}, \bar{\omega}_{56s} + \bar{\omega}_{78s}),$$

$$\bar{\tau}^{23} \bar{A}_s^3 = (\bar{\omega}_{23s} - i \bar{\omega}_{01s}, \bar{\omega}_{31s} - i \bar{\omega}_{02s}, \bar{\omega}_{12s} + i \bar{\omega}_{03s}),$$

$$s \in (7,8).$$

Scalars, expressed in terms of $\omega_{abc}$ (contributing as well to the mass matrices of quarks and
leptons and to masses of the heavy bosons) follow, if using Eqs. (D.2, D.3, D.7)

\[
\sum_{s',s''} -\frac{1}{2} S_{s's''}^{s''} \omega_{s's''} = - (g^{23} \tau_3 A_{s''}^{23} + g^{13} \tau_1 A_{s''}^{13} + g^4 \tau_4 A_{s''}^4),
\]

\[
g^{13} \tau_1 A_{s''}^{13} + g^{23} \tau_3 A_{s''}^{23} + g^4 \tau_4 A_{s''}^4 = g^Q \mathcal{A}_Q^Q + g^{Q'} \mathcal{A}_Q^{Q'} + g^Y \mathcal{A}_Y^Y',
\]

\[
A_{s''}^{13} = (\omega_{10,s} + \omega_{1112,s} + \omega_{1314,s}),
\]

\[
A_{s''}^{23} = (\omega_{56,s} + \omega_{78,s}),
\]

\[
A_{s''}^Q = \sin \vartheta_1 A_{s''}^{13} + \cos \vartheta_1 A_{s''}^Y, \quad A_{s''}^{Q'} = \cos \vartheta_1 A_{s''}^{13} - \sin \vartheta_1 A_{s''}^Y,
\]

\[
A_{s''}^Y = \cos \vartheta_2 A_{s''}^{23} - \sin \vartheta_2 A_{s''}^4,
\]

\[(s \in (7,8))\). (E.2)

Scalar fields from Eq. (E.1) interact with quarks and leptons and antiquarks and antileptons through the family quantum numbers, while those from Eq. (E.2) interact through the family members quantum numbers. In Eq. (E.2) the coupling constants are explicitly written in order to see the analogy with the gauge fields of the standard model.

Expressions for the vector gauge fields in terms of the spin connection fields and the vielbeins, which correspond to the colour charge ($\mathcal{A}_m^3$), the $SU(2)_{ll}$ charge ($\mathcal{A}_m^2$), the weak $SU(2)_l$ charge ($\mathcal{A}_m^1$) and the $U(1)$ charge originating in $SO(6)$ ($\mathcal{A}_m^0$), can be found by taking into account Eqs. (D.2, D.3). Equivalently one finds the vector gauge fields in the "tilde" sector, or one just uses the expressions from Eqs. (E.2, E.1), if replacing the scalar index $s$ with the vector index $m$.

The expression for $\sum_{t} \gamma^t \frac{1}{2} S_{ab} \bar{\omega}_{ab}$, used in Eq. (44) ($\bar{S}_{ab}$ are the infinitesimal generators of either $SO(3,1)$ or $SO(4)$, while $\bar{\omega}_{ab}$ belong to the corresponding gauge fields with $t = (9, \ldots, 14)$), and obtained by using Eqs. (D.4 - D.8), are

\[
\sum_{ab} \gamma^t \bar{S}_{ab} \bar{\omega}_{ab} = \sum_{+ - t \bar{\omega}_{ab}} \left( t' \right) \frac{1}{2} \bar{S}_{ab} \bar{\omega}_{ab} = \sum_{+ - t \bar{\omega}_{ab}} \left( t' \right) \frac{1}{2} \bar{S}_{ab} \bar{\omega}_{ab}.
\]

The term $\sum_{t' t''} \gamma^t \frac{1}{2} S_{t't''} \omega_{t't''}$ in Eq. (26) can be rewritten with respect to the generators $S_{t't''}$ and the corresponding gauge fields $\omega_{t't''}$ as one colour octet scalar field and one $U(1)_{II}$ singlet scalar field (Eq. D.3)

\[
\sum_{t't''} \gamma^t \frac{1}{2} S_{t't''} \omega_{t't''} = \sum_{+ - t'} \sum_{t''} \left( t' \right) \frac{1}{2} \left( t'' \right) \{ \tilde{\tau}^3 \cdot \tilde{\mathcal{A}}_{t'}^3 + \tau^4 \cdot \tilde{\mathcal{A}}_{t''}^4 \},
\]

\[(t, t') \in ((9,10), (11,12), (13,14)).\]
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