Hyponormality on General Bergman Spaces

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Abstract. A bounded operator $T$ on a Hilbert space is hyponormal if $T^*T - TT^*$ is positive. We give a necessary condition for the hyponormality of Toeplitz operators on weighted Bergman spaces, for a certain class of radial weights, when the symbol is of the form $f + \overline{g}$, where both functions are analytic and bounded on the unit disk. We give a sufficient condition when $f$ is a monomial.

1. Introduction

Let $w(r)$ be a nonnegative measurable function defined on $(0, 1)$, and assume $0 < \int_0^1 rw(r)dr < \infty$. Define the Hilbert space $L^2_{a,w}$ to be the space of analytic functions on the unit disk $U$ such that

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2 rw(r) \frac{d\theta}{\pi} < \infty.$$ 

We set $\alpha_n = 2 \int_0^1 r^{2n+1}w(r)dr$. Then

$$L^2_{a,w} = \{ f = \sum a_n z^n \text{ analytic on the unit disk such that } \|f\|^2 = \sum \alpha_n |a_n|^2 < \infty \}$$

and its orthonormal basis is given by $e_n = \frac{z^n}{\sqrt{\alpha_n}}$. Toeplitz operators on $L^2_{a,w}$ are defined by $Tf(k) = Pfk$, with $f$ bounded measurable on $U$, $k$ in $L^2_{a,w}$, and $P$ the orthogonal projection on $L^2_{a,w}$. Hankel operators are defined by $Hf(k) = (I - P)fk$ where $f$ and $k$ are as before. A bounded operator on a Hilbert space is said to be hyponormal if $T^*T - TT^*$ is positive. Unweighted Bergman spaces are considered in [2, 3, 11]. Hyponormality on the Hardy space was first considered in [4, 5]. The first results on hyponormality of Toeplitz operators on Bergman spaces are in [10] and the necessary condition is improved in [1]. All the known results on hyponormality on weighted Bergman spaces consider particular types of polynomials as a symbol. We cite for example [8] and [9]. In this work we consider hyponormality of Toeplitz operators on $L^2_{a,w}$. Under a condition on the weight we give a general necessary condition for the hyponormality of Toeplitz operators on $L^2_{a,w}$ with a symbol of the form $f + \overline{g}$, where $f$ and $g$ are bounded analytic on the unit disk. We give sufficient conditions for hyponormality when $f$ is a monomial and $g$ is a polynomial. A necessary and sufficient condition for normality of $T_{f + \overline{g}}$, when $f$ and $g$ are analytic in an open set containing $U$, is also obtained as a consequence.

2. Basic properties of Toeplitz operators and equivalent forms of hyponormality

These properties are known on the Bergman space and they hold also for weighted Bergman spaces. We assume $f, g$ are in $L^\infty(U)$. Then we have:

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1. \( T_{f+g} = T_f + T_g \).
2. \( T_f^* = T_f \).
3. \( T_f^*T_g = T_gT_f \) if \( f \) or \( g \) analytic on \( U \).

The use of these properties leads to describing hyponormality in more than one form. These are easy to prove, and one of the forms uses Douglas lemma [6].

**Proposition 2.1.** Let \( f, g \) be bounded and analytic on \( U \). Then the following are equivalent:
1. \( T_{f+g} \) is hyponormal.
2. \( H_f^*H_g \leq H_f^*T_f \).
3. \( \|(I-P)(gk)\| \leq \|(I-P)(f\bar{k})\| \) for any \( k \) in \( L^2_{m,w} \).
4. \( \|f\|^2 \leq \|f\|^2 + \|P(f\bar{k})\|^2 \) for any \( k \) in \( L^2_{m,w} \).
5. \( H_f = LH_f \) where \( L \) is of norm less than or equal to one.

We also need the following two lemmas. The symbols \( m, n, p \) etc denote nonnegative integers.

**Lemma 2.2.** For \( m \) and \( n \) integers we have \( P\left(z^n\bar{z}^m\right) = \begin{cases} 0, & \text{if } n < m \\ \frac{\alpha_m}{\sqrt{\alpha_m}}z^{n-m}, & \text{if } n \geq m. \end{cases} \)

**Proof.** If \( n < m \) we have \( \left(P\left(z^n\bar{z}^m\right), z^n\right) = \langle z^n\bar{z}^m, z^n \rangle = \langle z^n, z^{n+m} \rangle = 0 \) for any integer \( p \). Thus \( P\left(z^n\bar{z}^m\right) = 0 \). For \( n \geq m \), \( \left(P\left(z^n\bar{z}^m\right), z^n\right) = \langle z^n, z^{n+m} \rangle = 0 \) if \( p \neq n - m \). So \( P\left(z^n\bar{z}^m\right) = \lambda z^{n-m} \) and \( \left(P\left(z^n\bar{z}^m\right), z^n \right) = \lambda \|z^{n-m}\|^2 = \lambda \alpha_{n-m}. \)

Since \( \left(P\left(z^n\bar{z}^m\right), z^n \right) = \langle z^n, z^n \rangle = \alpha_n \), we deduce that \( \lambda = \frac{\alpha_n}{\alpha_{n-m}} \) and the result follows.

**Lemma 2.3.** For \( f = \sum a_nz^n \) bounded and analytic on the unit disk. The matrix of \( T_f^*T_f - T_fT_f^* \) in the orthonormal basis \( \{e_n\} \) is given by

\[
\Lambda_i = \sum_{m \geq j, m \geq 0} a_{m-i, j} \alpha_{i-m} - \sum_{j \geq m \geq 0} a_{m, j} \alpha_m \sqrt{\alpha_j} \sqrt{\alpha_i}.
\]

**Proof.** We have \( T_f(e_j) = \frac{1}{\sqrt{\alpha_j}} \sum_n a_n z^{n+j} \) and \( T_f^*T_f(e_j) = \frac{1}{\sqrt{\alpha_j}} P\left(\sum_{m,n} a_{m,n}z^{n+j}z^\ast\right) = \frac{1}{\sqrt{\alpha_j}} \sum_{m-n \leq j} a_{m,n} \alpha_m \alpha_{i-m} \) which can be written

\[
T_f^*T_f(e_j) = \frac{1}{\sqrt{\alpha_j}} \sum_{p \geq 0, m \geq j} \frac{\alpha_{m+p}}{\alpha_m} a_{m+p} \bar{a}_m z^{n+j}.
\]

We deduce that

\[
\langle T_f^*T_f(e_j), e_i \rangle = \frac{1}{\sqrt{\alpha_i}} \sum_{m \geq j, m \geq 0} a_{m-i,j} \alpha_{i-m} \sqrt{\alpha_i} \sqrt{\alpha_j}.
\]

Similarly, we show that

\[
\langle T_fT_f^*(e_j), e_i \rangle = \sum_{i \leq m \geq 0} a_{m, j} \alpha_m \sqrt{\alpha_i} \alpha_{i-m}
\]

and the proof is complete.

**Corollary 2.4.** The following holds

\[
\Lambda_{i+n, j+n+p} = \sum_{l \geq i+n} \frac{(\alpha_{l+i+n+p}(\alpha_{l+i+n} - 2\alpha_{i+n} \alpha_{i+n+p}) + \alpha_{i+n+p} \alpha_{l+i+n+p}) \sqrt{\alpha_l} \sqrt{\alpha_i}}{\sqrt{\alpha_{i+n} \sqrt{\alpha_{i+n+p}}}} + \sum_{l \geq i+n} \frac{\alpha_{l+i+n+p} \sqrt{\alpha_{l+i+n+p}}}{\sqrt{\alpha_{i+n} \sqrt{\alpha_{i+n+p}}}}.
\]

**Proof.** This follows from the previous lemma by putting \( m = p + l \) in the first sum and \( m = l \) in the second.
3. The results

Denote by \((\theta_{i,j})\) the matrix of the, possibly unbounded, Toeplitz operator on \(H^2\) with symbol \(|f|^2\). Our main result uses the following lemma, where \(C\) denotes a constant.

Lemma 3.1. Let \(f = \sum a_n z^n\) be bounded on \(U\). Assume \(f' \in H^2\) and \((a_n)\) satisfies the following conditions:

\[ n^2 \left| \frac{\alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \right| \leq C(l + p), \quad l \leq i + n \quad (1) \]

\[ n^2 \left( \frac{\alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \right) \to l(l + p) \quad (2) \]

Then

\[ n^2 \Lambda_{i+n,i+n+p} \to n \to \theta_{i,i+p} \]

Proof. We have \(\sum |l| < \infty\) since \(f' \in H^2\). Using the previous lemma we have

\[ n^2 \Lambda_{i+n,i+n+p} = \sum_{l \leq i+n} n^2 \left( \frac{\alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \right) \frac{\alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}}} + \sum_{l > i+n} n^2 \left( \frac{\alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \right) \frac{\alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}}} \]

Set \(h_n(l) = \frac{n^2 \alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \). From (1) we have \(|h_n(l)| \leq (C/2)(l^2|a|^2 + (l + p)^2|a_{l+p}|^2) = l(l + p)\) and \(\int_0^\infty l(l)(l)\text{d}v(l) < \infty\), where \(v\) is the counting measure. Using (2) and the dominated convergence theorem we obtain

\[ \lim_{n \to \infty} \sum_{l \leq i+n} n^2 \left( \frac{\alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \right) \frac{\alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}}} = \sum_{l > i+n} l(l + p)|a_{l+p}|^2. \]

We also have, for \(l > i + n\)

\[ \frac{n^2 \alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \leq 1/2(l^2|a|^2 + (l + p)^2|a_{l+p}|^2). \]

By the dominated convergence theorem we see that

\[ \sum_{l > i+n} n^2 \left( \frac{\alpha_{i+n+p} \alpha_{i+n-1} - \alpha_{i+n} \alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}} \sqrt{\alpha_{i+n-1}}} \right) \frac{\alpha_{i+n+p}}{\sqrt{\alpha_{i+n+p}}} \to 0. \]

The result follows since \(\theta_{i,i+p} = \sum l(l + p)|a_{l+p}|^2. \)

Remark 3.2. Examples of weights satisfying conditions (1) and (2) of the previous lemma are: \(w(r) = r^s,\) \(s > -\frac{1}{2}\), \(w(r) = |\log r|\), and \(w(r) = 1 - r^2\).

From now on we assume \((a_n)\) satisfies the hypotheses of the previous lemma. We state our main result.

Theorem 3.3. Let \(f\) and \(g\) be bounded analytic functions on \(U\), and assume \(f' \in H^2\). If \(T_{f,g}\) is hyponormal on \(L^2_{a,w}\) then \(g' \in H^2\) and \(|g'| \leq |f'|\) a.e on the unit circle.
Proof. Denote by \((T_n)\) the matrix of \(T^n T - T^n T^n\) and put \(g = \sum b_n z^n\). Hyponormality of \(T_{f*g}\) leads to the inequality \(n^2 \lambda_{i+n,i+n} \leq n^2 \lambda_{i+n,i+n} + \sum_{l \leq i+n} n^2 (a_{i+1+n, a_{i+1-n}} - (a_{i+1,n})^2)\). We deduce that
\[
\sum_{l \leq i+n} n^2 (a_{i+1+n, a_{i+1-n}} - (a_{i+1,n})^2) \leq \sum_{l \leq i+n} n^2 (a_{i+1+n, a_{i+1-n}} - (a_{i+1,n})^2) + \sum_{l \leq i+n} n^2 (a_{i+1+n, a_{i+1-n}})\]
Write the left hand side as an integral \(\int \frac{u_n(l)}{dv(l)}\). By Fatou’s lemma, condition (2) of the previous lemma and taking the limit on both sides we get
\[\sum \int n^2 [b_n]^2 \leq \sum \int n^2 [b_n]^2\]
Thus \(g' \in H^2\). From the previous lemma we deduce that \(n^2 (\lambda_{i+n,i+n} - \lambda_{i+n,i+n}) \to \theta_{i+n} - \phi_{i+n}\) where \((\phi_{i+1})\) is the matrix of the Hardy space Toeplitz operator \(T_{g'f}\). Hyponormality leads to the positivity of \(T_{g'f}\) and a property of Toeplitz forms [7] implies that \(|g'| \leq |f'|\) a.e on the unit circle. The proof is complete.

**Corollary 3.4.** Let \(f\) and \(g\) be analytic and univalent in an open set containing \(U\). Then \(T_{f*g}\) is normal if and only if \(g = cf + d\) for some constants \(c\) and \(d\) with \(|c| = 1\).

**Proof.** If \(g = cf + d\) with \(|c| = 1\), it is easy to see that \(T_{f*g}\) is normal. Conversely if \(T_{f*g}\) is normal then \(|g'| = |f'|\) on the circle and a maximum modulus argument shows that \(g' = cf + d\) with \(|c| = 1\). Thus \(g = cf + d\).

We now find a sufficient condition for hyponormality when \(f = z^p\). We begin with the case \(g = \lambda z^q\). We set
\[\mu_1 = \min \left\{ \frac{\alpha_{i+1}}{\alpha_i} : 0 \leq i < q \right\}, \quad \mu_2 = \min \left\{ \frac{\alpha_{i+1}}{\alpha_i} : q \leq i < p \right\}, \quad \mu_3 = \inf \left\{ \frac{\alpha_{i+1}}{\alpha_i} : p \leq i \right\}.
\]

**Proposition 3.5.** Assume \(p > q\). The operator \(T_{z^p,z^q}\) is hyponormal if and only if \(|\lambda| \leq \lambda_{p,q} = \min(\mu_1, \mu_2, \mu_3)\).

**Proof.** In this case hyponormality is equivalent to \(|\lambda|^2 H^2_{z^p} \leq H^2_{z^q}\). A computation shows that the matrix of \(H^2_{z^p} H^2_{z^q}\) is diagonal and its diagonal term is given by:
\[D_i = \frac{\alpha_{i+1}}{\alpha_i} \text{ if } m > i, \quad D_i = \frac{\alpha_{i+1}}{\alpha_i} \text{ if } m \leq i.
\]

Hyponormality is thus equivalent to the following inequalities:
\[i) |\lambda|^2 \frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\alpha_{i+1}}{\alpha_i} 0 \leq i < q\]
\[ii) |\lambda|^2 \frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\alpha_{i+1}}{\alpha_i} \leq \frac{\alpha_{i+1}}{\alpha_i} q \leq i < p\]
\[iii) |\lambda|^2 \left( \frac{\alpha_{i+1}}{\alpha_i} - \frac{\alpha_{i+1}}{\alpha_i} \right) \leq \frac{\alpha_{i+1}}{\alpha_i} \frac{\alpha_{i+1}}{\alpha_i} p \leq i \]

Obviously inequality i) is equivalent to \(|\lambda| \leq \mu_1 = \min \left\{ \frac{\alpha_{i+1}}{\alpha_i} : 0 \leq i < q \right\}, \text{ and ii) is equivalent to } |\lambda| \leq \mu_2 = \min(\sqrt{\alpha_{i+1},0 \leq i < q} q \leq i < p\). The last inequality is equivalent to \(|\lambda| \leq \mu_3 = \inf(\sqrt{\alpha_{i+1},p \leq i} q \leq i < p\). Thus hyponormality of \(T_{z^p,z^q}\) is equivalent to \(|\lambda| \leq \lambda_{p,q} = \min(\mu_1, \mu_2, \mu_3)\).

**Remark 3.6.** If \(p = q\) then clearly hyponormality of \(T_{z^p,z^p}\) is equivalent to \(|\lambda| \leq 1\). Thus if \(p \geq q\) from the previous theorem \(|\lambda_{p,q} | \leq \frac{q}{p}\).

In the following proposition we assume \(q \geq 2\) (the case \(q = 1\) being trivial). We set
\[\tau_1 = \min(\sqrt{\alpha_{i+1},0 \leq i < p}, \tau_2 = \min(\sqrt{\alpha_{i+1},0 \leq i < q}, p \leq i < q) \text{ and } \tau_3 = \inf(\sqrt{\alpha_{i+1},p \leq i} q \leq i < p)\).

Proposition 3.7. Assume $p < q$ then $T_{f^+, \tau}$ is hyponormal if and only if $|\lambda| \leq \sigma_{p,q} = \min\{\tau_1, \tau_2, \tau_3\}$.

The proof, being similar to the proof given above, is omitted. We set $\sigma_{q,q} = 1$. Note that hyponormality of $T_{f^+, \tau}$ implies that $\|g\| \leq \|f\|$ in particular $\sigma_{p,q} \leq \sqrt{\frac{p}{q}}$. In what follows we give a sufficient condition for the hyponormality of $T_{f^+, \tau}$. We denote by $B_1$ the unit ball of $L_{\alpha,\omega}^2$.

Definition 3.8. For $f \in L_{\alpha,\omega}^2$, set

$$G_f = \left\{ g \in L_{\alpha,\omega}^2 : \sup \{ \| \langle g, k \rangle, u \| : u \in B_1 \} \leq \sup \{ \| \langle f, k \rangle, u \| : u \in B_1 \} \text{ for any } k \in H^\infty \right\}.$$ 

By the density of $H^\infty$ in $L_{\alpha,\omega}^2$, we see that $g \in G_f$ is equivalent to $T_{f^+, \tau}$ is hyponormal. We list the properties of $G_f$ in the following proposition:

Proposition 3.9. Let $f \in L_{\alpha,\omega}^2$, the following holds:

i) $G_f$ is convex and balanced.

ii) If $g \in G_f$, and $c$ is a constant the $g + c \in G_f$.

iii) $f \in G_f$.

iv) $G_f$ is weakly closed.

Proof. i), ii) and iii) follow from the definition of $G_f$. For the proof of iv) assume $(g_n)$ is a net in $G_f$ such that $g_n \rightarrow g$. We have for $v \in B_1$ and $k \in H^\infty,$ $\| \langle g_n, k \rangle, v \| \leq \sup \{ \| \langle f, k \rangle, u \| : u \in B_1 \}$. Taking the limit we get $\| \langle g, k \rangle, v \| \leq \sup \{ \| \langle f, k \rangle, u \| : u \in B_1 \}$. Taking the supremum on the left hand side we get: $\sup \{ \| \langle g, k \rangle, u \| : u \in B_1 \} \leq \sup \{ \| \langle f, k \rangle, u \| : u \in B_1 \}$ for any $v \in B_1$. Thus (iv) completes the proof.

Corollary 3.10. Assume $(\lambda_n)$ is a sequence of complex numbers satisfying $\sum |\lambda_n| \leq 1$. Then $T_{z^1 \sum \lambda_n z^n, \sum \lambda_n |z|^m}$ is hyponormal.

Proof. Set $g_N = \sum_{n=1}^{N} \lambda_n |z|^m$ for $N \geq q + 1$ and let $h = \sum_{n=1}^{\infty} h_n z^n$ be in $L_{\alpha,\omega}^2$. We have the following inequalities for $M > N \geq q + 1$

$$|\langle g_M - g_N, h \rangle| \leq \sum_{N} |\lambda_n| |h_n| |\alpha_m| \leq \left( \sum_{N} |\lambda_n|^2 |\alpha_m|^2 \right)^{1/2} \left( \sum_{N} |h_n|^2 |\alpha_m|^2 \right)^{1/2}.$$

Thus $(g_n)$ converges weakly and a similar argument shows that the limit is $\sum_{n=1}^{q} \lambda_n |z|^m + \sum_{q+1}^{\infty} \lambda_n |h_n| |z|^m$.

The result follows from the previous proposition.

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