Ergotropy from coherences in an open quantum system

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We show that it is possible to have non-zero ergotropy in the steady-states of an open quantum system consisting of qubits that are collectively coupled to a thermal bath at a finite temperature. The dynamics of our model leads the qubits into a steady-state that has coherences in the energy eigenbasis when the number of qubits in the system is more than one. We observe that even though the system do not have inverted populations, it is possible to extract work from the coherences and analytically show that in the high temperature limit, ergotropy per unit energy is equal to the $l_1$ norm of coherence for the two qubit case. Further, we analyze the scaling of coherence and ergotropy as a function of the number of qubits in the system for different initial states. Our results suggest that one can design a quantum battery that is charged by a dissipative thermal bath in the weak coupling regime.

I. INTRODUCTION

The field of quantum thermodynamics generalizes the definitions of quantities such as heat, work, and entropy that are made for macroscopic systems, to the realm of microscopic quantum systems and their dynamics [1–4]. In other words, it is a field that analyzes the thermodynamics of non-equilibrium quantum processes.

One of the most striking features of quantum systems is the fact that they can be in coherent superposition of possible available states. Despite its significance for the quantum systems, a robust scheme to quantify it has only been recently available states. It has been shown in many works that quantum coherence can be used as an advantage or a resource for different thermodynamic processes [7–20] (see also [6, 21]). Nevertheless, the inevitable interaction of a quantum system with its environment results in the loss of genuine quantum features, such as coherence [22], and the dynamics of these systems are treated in the well-established formalism of open quantum systems [23, 24].

Within the realm of quantum thermodynamics one topic that attracts a significant attention is quantum batteries. Quantum batteries are systems that contain certain amount of energy, from which a finite amount of can be extracted as work through unitary cyclic processes [25, 26]. There are mainly two sides to the literature in quantum batteries: the charging and the work extraction [11–13, 26–45]. In both of these processes the main goal is to desgin or find dynamics that the energy is imparted or extracted from the battery in a feasible way according to certain figures of merit such as the total energy involved, power, etc. While it is naturally the case for work extraction processes, vast majority of the charging protocols also rely on unitary evolution of the quantum battery with only a number of works take the interaction with an environment into account[46–53]. Among the literature on quantum batteries, [51, 52] and [11–13] particularly stand out for their relation to the present work, since they consider charging processes that only involve interaction with an environment and effect of coherences and/or correlations on ergotropy, respectively.

In this work, we consider a collection of two-level systems (qubits) that are collectively interacting with a thermal environment. It has been shown that the steady-state of such a system can maintain coherences in its energy eigenbasis, if the number of qubits are larger than one [15, 54], due to indistinguishably of the constituents of the system to the bath in the collective coupling regime. We show that the presence of these coherences results in a finite ergotropy of these steady-states starting from two qubit systems. We present analytical results on the coherence and ergotropy for arbitrary number of qubits when they are initiated in their ground state, and numerically analyze their behavior for random initial states up to seven qubits. Our findings present a robust way to store ergotropy in a quantum battery through coherences in an open system setting by totally thermal means together with only assuming a weak, but collective, coupling between the battery and its environment.

The remainder of the paper is organized as follows. In Sec. II we introduce the figures of merit that we will be mainly interested throughout this work. Sec. III describes the dynamical model which generates the coherences and the steady-states with finite ergotropy that we will use in our discussion. We look at the amount of ergotropy of the steady-states of our model for two qubits and how it compares with the coherence contained in them at different temperatures and initial states in Sec. IV A. In Sec. IV B we discuss how ergotropy and coherence scales with the number qubits in the system for different two classes of initial states. We conclude in Sec. V.

II. PRELIMINARIES

Assume that we are given a quantum state $\rho$ with its internal Hamiltonian $H$ such that they have the following spectral decomposition

$$\rho = \sum_i r_i |r_i\rangle\langle r_i| \quad H = \sum_i \varepsilon_i |\varepsilon_i\rangle\langle \varepsilon_i|,$$

where ordering of the eigenvalues for $\rho$ and $H$ is in decreasing, $r_1 \geq r_2 \geq \ldots$, and increasing $\varepsilon_1 \leq \varepsilon_2 \leq \ldots$ order, respectively. The maximum amount of work that can be extracted

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from this system through a unitary cyclic process is called ergotropy and can be calculated as [25]

$$W = \sum_{\mu} r_{\mu} \varepsilon_{\mu}(|(r_{\mu}|\varepsilon_{\mu})|^2 - \delta_{\mu}).$$

A state that has its ergotropy, $W$, equal to zero is called a passive state and any non-passive state is called an active state. Combination of passive states can allow work extraction from them through some collective process [29–32], with thermal states being an exception. No combination of thermal states end up in an active state, therefore they are called completely passive states [55].

The quantification of the amount of coherence in a given quantum system is made by some well-defined measures of coherence [5, 6]. One of these measures introduced in [5] is called as the $l_1$ norm of coherence and it is given as

$$C_l = \sum_{i \neq j} |\rho_{ij}|,$$

which is just the absolute sum of the off-diagonal elements of a given density matrix.

In what follows, these two quantities will be our main interest.

### III. MODEL

The model that we are going to discuss throughout this work is the many-particle generalization of the well-known quantum optical master equation [23, 56–58], which has been widely used in the literature [15, 54, 59, 60]. We will assume that there are $N$ number of two-level systems (qubits from now on), which act like point-like dipoles and assumed to have identical dipole moments, embedded in a thermal electromagnetic field environment. Depending on the spatial configuration of the system particles, they can be individually or collectively coupled to the environment, which we will discuss in a more detailed manner in what follows. The master equation governing the dynamics of such a system can be derived in usual Born, Markov approximations and assuming a weak coupling to the bath, which is given as follows (in units of $\hbar = 1$) [15, 54, 59, 60]

$$\dot{\rho} = -i[(H_0 + H_d),\rho] + D_>(\rho) + D_=(\rho) = \mathcal{L}(\rho).$$

The first term on the right-hand side of the above equation accounts for the unitary evolution of the qubits where $H_0 = \omega \sum_i \sigma_i^+ \sigma_i^-$ is the free Hamiltonian of the qubits and $H_d = f_{ij} \sum_{i,j} \sigma_i^+ \sigma_j^-$ is the dipole-dipole interaction between them with $f_{ij}$ being its strength and $\sigma_i^+ = |e_i\rangle\langle g_i|$ and $\sigma_i^- = |g_i\rangle\langle e_i|$ are the raising and lowering operators for the $i$th atom, respectively. Second and third terms are describing the spontaneous and thermally induced emission (dissipation), and thermally induced absorption (incoherent driving) processes, respectively, whose explicit forms are as follows

$$D_>(\rho) = \sum_{i,j=1}^{N} \gamma_{ij} \hat{n} (\sigma^+_j \rho \sigma^-_i - \frac{1}{2}(\sigma^+_i \sigma^-_j, \rho)), \quad D_=(\rho) = \sum_{i,j=1}^{N} \gamma_{ij} \rho (\sigma^+_j \sigma^-_i - \frac{1}{2}(\sigma^+_i \sigma^-_j, \rho)),$$

where $\hat{n} = (\exp(\beta \hbar \omega) - 1)^{-1}$ is the mean number of thermal photons at the transition frequency of the qubits at an inverse temperature $\beta$.

The model above described by Eq. (4), has two extreme limits. First one is when the qubits are far apart from each other compared to the wavelength of the photons in the environment. In this case, the qubits in the system behave as if they are individually coupled to the environment and mathematically this corresponds to $f_{ij} \approx 0$ and $\gamma_{ij} = \gamma_0 \delta_{ij}$ with $\gamma_0 = \omega^3 d^2 / 3 \pi \hbar \epsilon_0 c^3$. Since the surrounding bath is a thermal one, naturally, all individual qubits thermalize with the temperature of the bath. The opposite extreme is the case when the qubits are closely packed such that the spatial separation between them is much smaller than the wavelength of the electromagnetic field in the environment. This is called the collective coupling limit with $f_{ij} \approx f$ and $\gamma_{ij} \approx \gamma_0$, and it has been shown that in this situation the total system evolves into a non-trivial steady-state that has coherences in the energy eigenbasis [15, 54], even when the system is initiated in an incoherent state. The mechanism underlying the generation of these steady-state coherences (SSC) is the indistinguishability of the qubits to the bath in the collective coupling regime. Since it is not possible to know which qubit absorbed or emitted a photon and changed its state, overall system enters into a superposition state of such possible configurations. It is also important to note that in the collective coupling regime, the presented model do not admit a unique steady-state.

### IV. ERGOTROPY OF THERMAL COHERENCES

Throughout this work, we will be interested in the coherent steady-states that are generated in the under the dynamics dictated in the collective coupling limit. Intuitively, one would expect the steady-state of a system in contact with a thermal environment to be in a Gibbs state, which is a passive, even a completely passive state. However, due to the coherences generated and/or sustained as a result of the collective coupling of our system to the bath, we get a steady-state that do not have inverted populations, but still an active one such that it is possible to extract work from them through a unitary cyclic process [10–12, 61, 62]. We would like to emphasize again that the coherences are generated as a result of the indistinguishability of two-level systems to the bath. Therefore, in order to obtain active states in the present setting one needs to have at least a pair of such subsystems; a single system would simply end up in a Gibbs (passive) state.

#### A. Two qubits

Since our model do not admit a unique steady-state, the steady-state that a given system reaches depend on its initial
state. Recently, the analytical solution of Eq. (4) in the collective coupling regime for an arbitrary initial state of a two qubit system is given in [54] as

\[
\rho_{ss}(\beta, c) = (1 - c) |\psi_-\rangle \langle \psi_-| + c Z_{ss}^{-1}(\beta) \left( e^{-2\omega\beta} |\psi_1\rangle \langle \psi_1| + e^{-2\omega\beta} |\psi_+\rangle \langle \psi_+| + |\psi_0\rangle \langle \psi_0| \right),
\]

where \( |\psi_0\rangle = |00\rangle, |\psi_1\rangle = |11\rangle, |\psi_+\rangle = |01\rangle + |10\rangle / \sqrt{2}, \)
\( c = \langle \psi_0 | \rho_0 | \psi_0 \rangle + \langle \psi_1 | \rho_0 | \psi_1 \rangle + \langle \psi_+ | \rho_0 | \psi_+ \rangle, \) and \( Z_{ss}(\beta) = 1 + e^{-2\omega\beta} + e^{-2\omega\beta}. \) We would like to note that in the present case \( |\psi_-\rangle \) is stationary throughout the dynamics such that \( \mathcal{L}(|\psi_-\rangle \langle \psi_-|) = 0. \) The thermodynamics of these steady-states have been extensively discussed in [54, 63].

where \( Z(\beta) = 1 + 2e^{-\omega\beta} + e^{-2\omega\beta} \) is the partition function of the two qubit system. From the expression above, it is straightforward to determine the point at which \( C_i \) vanishes as \( c = Z_s / Z \) (cf. Fig 1 (a)). This implies that when the sum of the overlap of the initial state with \( |\psi_0\rangle, |\psi_+\rangle \) and \( |\psi_1\rangle \) is equal to its thermal value, i.e., the initial state is a thermal one, it is not possible for the dynamics to end up in a coherent steady-state. Note that as the temperature decreases the point of \( C_i = 0 \) shifts towards \( c = 1, \) since the thermal state at that temperature approaches ground state. Therefore, in order to reach a steady-state with coherences one needs to initiate the system in a state that has not equilibrated with the surrounding environment.

Next, we move on to the calculation of ergotropy defined in Eq. (2) for the steady-states we have at hand. It is straightforward to calculate the spectrum of the self-Hamiltonian of the qubits in our system, \( H_0, \) and in the ascending order it is given as \([0, \omega, \omega, 2\omega]\) with their corresponding eigenvectors \([|0, 0, 1\rangle^T, |0, 0, 1\rangle^T, |1, 0, 0\rangle^T, |1, 0, 0\rangle^T]\). On the other hand, the ordering of the spectrum of \( \rho_{ss}(\beta, c) \) heavily depends on the parameters characterizing the state. Below, we present the eigenvalues and corresponding eigenvectors and we will elaborate on the ordering problem later

\[
W(c) = \frac{c e^{2\beta \omega}}{1 + e^{\beta \omega} + e^{2\beta \omega}} + \frac{c e^{2\beta \omega}}{1 + e^{\beta \omega} + e^{2\beta \omega}} + \frac{c}{1 + e^{\beta \omega} + e^{2\beta \omega}} \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 0, -1, 1 \end{pmatrix}^T, \begin{pmatrix} 0, 0, 0 \end{pmatrix}^T, \frac{1}{\sqrt{2}} \begin{pmatrix} 1, 0, 0 \end{pmatrix}^T, \begin{pmatrix} 1, 0, 0 \end{pmatrix}^T \right).
\]

Recall that in order to calculate the ergotropy we need to order the eigenvalues of the state of the quantum system in the descending order. The ordering of the eigenvalues given above is in the descending order only when \( c \leq 1/2. \) Moreover, when \( c = 1, \) it is straightforward to arrange the ordering, since the first eigenvalue becomes zero and is the last one in the descending order. Since the correct ordering is guaranteed for \( c \leq 1/2 \) and \( c = 1, \) we can obtain an analytical expression for the ergotropy in these regimes and it is given as

\[
W = \begin{cases} \phi(1 - c \frac{3(c + 3\cosh(\beta \omega)\sinh(\beta \omega))}{1 + 2\cosh(\beta \omega)}) & 0 \leq c < 1/2 \\ \phi(1 - c) & c = 1, \end{cases}
\]

In the region \( 1/2 < c < 1 \) the first eigenvalue gradually becomes smaller than the second, third and the fourth eigenvalue as \( c \) increases up to 1. We give the conditions that change the ordering of the eigenvalues in Appendix A. Although it is involved to get an analytical expression for ergotropy for all \( c \) at an arbitrary temperature due to the reasons stated above, we can obtain analytical expressions for low and high temperature limits.

**Low temperature limit** - In this regime we have \( \beta \to \infty, \) thus the eigenvalues given in Eq. (9) reduces to \([1 - c, c, 0, 0]. \) Clearly, the ordering of the eigenvalues change at the point \( c = 1/2, \) yielding a stepwise behavior in ergotropy of the form

\[
W = \begin{cases} \phi(1 - 2c) & 0 \leq c < 1/2 \\ 0 & 1/2 \leq c \leq 1, \end{cases}
\]

which is consistent with the behavior we observe in Fig. 1 (b) for \( \beta = 10. \) Note that \( 1 - c \) measures the overlap that the initial
state has with the anti-symmetric Bell state $|\psi_-\rangle$. Therefore, the result above imples that one needs to initiate the system in a state that has at least $1/2$ overlap with $|\psi_-\rangle$ to get a finite ergotropy at the steady-state in low temperatures.

**High temperature limit** - This regime is characterized by $\beta \to 0$ which results in the eigenvalues given in Eq. (9) to have the form $[1 - c, c/3, c/3, c/3, c/3]$. Clearly, we have a change in the ordering of the eigenvalues at the point $c = 3/4$ that again gives rise to a stepwise behavior in the ergotropy given as

$$W = \begin{cases} 
\omega \left(1 - \frac{c}{3}\right) & 0 \leq c < 3/4 \\
0 & c = 3/4 \\
\omega \left(\frac{4c}{3} - 1\right) & 3/4 < c \leq 1,
\end{cases} \quad (12)$$

which is consistent with Fig. 1 (b) for $\beta = 0.01$. At this point we would like to point an interesting connection. The $l_1$ norm of coherence for our two qubit system given in Eq. (8), reduces to $C_l = |\omega|\bar{1}_l$ in the present limit. This is exactly the same as the ergotropy per unit energy given in Eq. (12), therefore in the high temperature limit we have the following equality $W/\omega = C_l$.

Related with the presented analysis we would like to discuss two relevant and natural questions. First, do quantum correlations give us any further insight about the origin of the finite ergotropy in these steady-states? Focusing on entanglement and quantum discord, we can conclude that the answer is negative. The entanglement content of the steady-states quickly go to zero with increasing temperature and remain finite only for small $c$, i.e., when the initial state has a sufficiently large overlap with $|\psi_-\rangle$. On the other hand, behavior of the quantum discord qualitatively follows the same trend with $C_l$ for different temperatures while quantitatively it is always smaller than that, with equality attained when $C_l$ becomes zero and at $c = 1$.

Second, how does the ergotropy of the initial and final states compare? Depending on the initial state of the system, the considered dynamics do not always increase the ergotropy. For example, assume that we initiate our system in which both particles are in their excited state, $|\psi_+\rangle$, which corresponds to the steady-state with $c = 1$. The initial ergotropy of such an initial state is $2\omega$ while its final ergotropy is definitely going to be less than that value. However, another initial state that also correspond to the same steady-state with $c = 1$ is when both qubits are initiated in their ground states, $|\psi_0\rangle$. Clearly, this initial state has zero ergotropy but it is brought to a state with non-zero ergotropy for a large range of temperatures. All in all, the presented method of obtaining states with finite ergotropy may not be the one that has the highest yield, but it requires very little control (collective coupling) and resources (a thermal bath). Moreover, due to the fact that the ergotropy is stored in the steady-state, it is robust and stable, which is an important issue in quantum batteries [40, 43, 47].

Lastly, we would like to mention that local states of the qubits are diagonal with non-inverted populations. As a result, when processing these states to extract work, one must design a global process without discarding neither one of the qubits.

**B. Scaling with the number of particles**

In this section, we would like to analyze how the ergotropy in the steady-state of our model scales with the number of particles and, in particular, how it compares with the scaling of $C_l$ [15]. However, as stated before, the model under consideration do not have a unique steady-state and we only have the analytical solution in the two qubit case for arbitrary initial states. As a result, in what follows we will analyze the cases with fixing the initial state to (i) ground initial states and (ii) random initial states.

However, before going into these cases, we would like to briefly comment on a different class of initial states. As we have already seen in the two qubit case, the dynamics under consideration do not generate any coherence at the steady-state for a thermal initial state, and thus ergotropy of is also equal to zero. This behavior also continues for larger number of qubits.

1. **Ground initial states**

We begin our discussion on the scaling with the case in which all qubits are initiated in their ground states. Such an initial state contains no coherence and clearly do not have inverted populations. Therefore, any amount of coherence and ergotropy is generated by the dynamics described by Eq. (4).

Remarkably, in this case it is possible to find an analytical expression for the steady-state of the system, as presented in [20], and it has the following block diagonal structure

$$\rho = \begin{pmatrix} 
D_N & 0 & \ldots & 0 & 0 \\
0 & D_{N-1} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & D_1 & 0 \\
0 & 0 & \ldots & 0 & D_0
\end{pmatrix}, \quad (13)$$

where each block has the form of $(p_k \times p_k)$ with $p_k = C(N, k)$. The explicit form of these blocks are given as $D_k = d_k U_k$ with $U_k$ is matrix of ones and

$$d_k = \frac{(1-r)^k}{(1-r^{N+1})p_k}, \quad (14)$$

with $r = \bar{n}/\bar{n} + 1$. Full derivation of this result can be found in [20] (in particular see Appendix A of the mentioned reference).

The number of off-diagonal elements in a given block is $p_k^2 - p_k$ with all of them being equal to $d_k$. Then, we can analytically calculate the $l_1$ norm by adding these up over all blocks which results in

$$C_l = \sum_{k=0}^{N} d_k p_k (p_k - 1) = \frac{(r-1)(r+1)^N}{r^{N+1} - 1} - 1. \quad (15)$$

We can check a couple of relevant limits of the environment temperature for the above expression. In the low temperature limit we have $r \to 0$ and in turn $C_l \to 0$, which is expected...
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We present the details of this calculation in Appendix B. As
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block which corresponds
to the population of the double excited state. While this may
seem in contrast to our claims that the generated coherences
are the actual source of the ergotropy, the reason behind \( D_2 \)
may be extracted as work is indeed due to the presence of co-
herences in the density matrix of the system which modifies
the spectrum of the state.

The reduced density matrices of each individual qubit of
Eq. (13) are also diagonal, similar to the previous section.
However, when the number of qubits in the system is larger
than two, it is possible to have coherences in the bipartite
or larger sized reduced states. Therefore, although it is again
not possible to get a finite ergotropy from individual qubits,
it may be possible to extract work from combinations of the
local states. Naturally, the amount one can get is smaller than
that of the total, global state.

2. Random initial states

In this section, we present numerical scaling results for
10^5 initial states for each system size up to \( N = 7 \), to make
a comparison with Fig. 2. We again investigate the high-
temperature limit, since in this regime the initial states that
have no coherences can end up in steady-states with higher
amount of coherences, and therefore ergotropy, making this
it more interesting and relevant. In Fig. 3 (a) blue circles
and orange squares mark the mean value of coherence and er-

average the standard deviations

We observe that even though the mean coherence still grows more rapidly with the num-
ber of qubits as compared to the ergotropy, its growth rate is
slower than it was for ground initial states. On the other hand,

If \( C_{1} = (2^N - N - 1)/(N + 1) \), which shows a 2^N scaling with the number of particles. Note
that this limit gives us the highest amount of coherence that
can be generated with the considered initial states under the
considered dynamics. The physical mechanism behind this is
as follows: Since all the qubits are in their ground state the only way to create coherence in this system is through
the thermally induced absorption processes which happens at
a rate \( \gamma \tilde{n} \). Therefore, increasing \( \tilde{n} \), which corresponds to increasing \( \beta \), generates the highest amount coherence for this
initial state [15].

Even though it is a bit more involved as compared to the
calculation of \( C_{1} \), it is also possible to obtain an analytical expression for the ergotropy for arbitrary number of particles
which is given as

\[
W = \sum_{k=1}^{N-1} kp_{k+1}d_{k+1} \omega \omega \left( N + \frac{N + r}{p^{N+1} - 1} + \frac{r}{1 - r} \right).
\]

We present the details of this calculation in Appendix B. As
expected, in the zero temperature limit, \( r \to 0 \), ergotropy goes
to zero, since \( C_{1} \) is also zero in this limit, which is actually
our source of ergotropy in the present model. In the opposite
end, as \( r \to 1 \), we obtain \( W = \omega [(N(N - 1))/(2(N + 1))] \).
We can immediately observe the linear scaling in the ergotropy,
in contrast to the 2^N scaling in the coherence at the high tem-
perature limit. We present the scaling behavior in Fig. 2 at
the high temperature limit where we get the highest amount
of coherence and ergotropy up to \( N = 7 \).

Eq. (16) also allows us to comment on which blocks in
the density matrix of our system contribute to the ergotropy
together with the magnitude of their contribution. Note that
the all ground state, \( D_0 \), and single excitation subspace, \( D_1 \),
do not contribute to the ergotropy. While it is natural to ex-

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Eq. (16) also allows us to comment on which blocks in
the density matrix of our system contribute to the ergotropy
together with the magnitude of their contribution. Note that
the all ground state, \( D_0 \), and single excitation subspace, \( D_1 \),
do not contribute to the ergotropy. While it is natural to ex-
pect \( D_0 \) not having any effect on the ergotropy, it is notable
to see that coherences in the single excitation subspace also
do not contribute to the ergotropy. All remaining blocks have
a finite contribution which is proportional to their only non-
zero eigenvalue \( p_k d_k \). The value of these eigenvalues de-
crease as go up in the block number, i.e. \( p_k d_k > p_{k+1} d_{k+1} \)
(see Appendix B). This implies that coherences in the lower
blocks have higher impact on ergotropy as compared to higher
blocks, with the lowest two blocks being exceptions.

At this point, a remark on the \( N = 2 \) case is in order. Since
the lowest two blocks do not affect the ergotropy, in the case
of two-qubits only source of ergotropy, when the system is ini-
tiated in its ground state, is the \( D_2 \) block which corresponds
to the population of the double excited state. While this may
seem in contrast to our claims that the generated coherences
are the actual source of the ergotropy, the reason behind \( D_2 \)
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result in fact shows us that one does not gain much by trying to achieve it.

V. CONCLUSIONS

We analyzed the steady-states of a system of qubits (two-level) systems in contact with a thermal environment in a collective manner. As a result of the collective coupling, when the system is composed of more than one qubit, these steady-states admit coherences in the energy eigenbasis due to their indistinguishability to the bath. We showed that solely due to the presence of such coherences, the steady-states generated by this open system dynamics yield a finite amount of ergotropy. In the case of two qubits we obtained an analytical expression for the ergotropy for a large number of initial states and showed that the amount of ergotropy per unit energy in the high bath temperature limit is equal to the $l_1$ norm of coherence. Further, we looked at the scaling behavior of both coherence and ergotropy with the number of qubits for two different classes of initial states which are all ground and random initial states. In the former case, we presented analytical expressions for both coherence and ergotropy for arbitrary number of qubits and observe a $2^N/N$ and $N$ scaling, respectively. For the latter case we initiated our system in $10^5$ random initial states for all system sizes up to seven qubits and analyzed the mean values of coherence and ergotropy. We observed that coherence grow with a smaller rate while ergotropy shows a similar scaling with the number of particles, as compared to the ground initial state case.

We think that it is interesting to see that it is possible to extract work due to the coherences in the steady-state of a quantum system that is coupled to a dissipative heat bath, whereas vast majority of proposals in the charging process of quantum batteries is composed of unitary dynamics. Naturally, the presented method provides neither the highest amount of ergotropy nor the highest power as compared to the cases of closed charging processes. However it shows an example of an open quantum battery that is cheap in terms of control and resource requirements and in which the ergotropy is robust, since it is stored in the steady-state of the dynamics.

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Appendix A: Ordering of eigenvalues between $1/2 < c < 1$

In this region the first eigenvalue appearing on Eq. (9) becomes gradually smaller than the second, third and fourth eigenvalue depending on $c$ and $\beta$. We present these conditions below

\[
1 - c < \frac{e^{\beta \omega}}{1 + e^{\beta \omega} + e^{2\beta \omega}} \quad \text{when} \quad \begin{cases} \beta > \log \left( \frac{\sqrt{c^2 + 10c - 3} - 2c + 1}{2c - 1} \right) & \text{for } 1/2 < c \leq 3/4 \\ \beta \geq 0 & \text{for } 3/4 < c < 1, \end{cases}
\]

\[
1 - c < \frac{ce^{\beta \omega}}{1 + e^{\beta \omega} + e^{2\beta \omega}} \quad \text{when} \quad 0 \leq \beta < \log \left( \frac{\sqrt{c^2 + 10c - 3} - 2c + 1}{2c - 1} \right) / \omega \quad \text{for } 3/4 < c < 1
\]

\[
1 - c < \frac{c}{1 + e^{\beta \omega} + e^{2\beta \omega}} \quad \text{when} \quad 0 \leq \beta < \log \left( \frac{1}{\omega} \left( \sqrt{\frac{3-7c}{c-1} - 1} \right) \right) \quad \text{for } 3/4 < c < 1
\]

Note that the ordering among the second, third and fourth eigenvalues remains the same for all parameters.

Appendix B: Calculation of ergotropy for ground initial states

The ingredients we need for the calculation of ergotropy for arbitrary number of particles initiated in their ground-states are naturally the spectrum of the self-Hamiltonian of the particles and the spectrum of their steady-state. For $N$ particles,
\[ H_0 = \omega \sum_{i=1}^{N} \sigma_i^+ \sigma_i^- \], and its spectrum in the increasing order given as

\[ \begin{align*}
\varepsilon_1 &= 0 \quad |\varepsilon_1\rangle = \left[0, \ldots, 1\right]^T \\
\varepsilon_2 &= \omega \quad |\varepsilon_2\rangle = \left[0, \ldots, 0, 1, 0\right]^T \\
\vdots & \quad \vdots \\
\varepsilon_{p_i} &= \omega \quad |\varepsilon_{p_i}\rangle = \left[0, \ldots, 0, 1, \ldots, 0\right]^T \\
\varepsilon_{p_i+p_1} &= 2\omega \quad |\varepsilon_{p_i+p_1}\rangle = \left[0, \ldots, 0, 1, \ldots, 0\right]^T \\
\vdots & \quad \vdots \\
\varepsilon_{p_i+p_1+p_2} &= 3\omega \quad |\varepsilon_{p_i+p_1+p_2}\rangle = \left[0, \ldots, 0, 1, \ldots, 0\right]^T \\
\vdots & \quad \vdots \\
\varepsilon_{2^N} &= N\omega \quad |\varepsilon_{2^N}\rangle = \left[1, \ldots, 0\right]^T
\end{align*} \]

We now turn our attention to the spectrum of the steady-state for ground initial states given in Eq. (13). The eigenvalues of a block diagonal matrix are the combination of the eigenvalues of each individual block. For \( D_k \) we only have a single non-zero eigenvalue given as \( p_k d_k \) with its corresponding eigenvector being \( \left[1, 1, \ldots, 1\right] / \sqrt{p_k} \) and rest of them equal to zero. As a result, in the spectrum of the steady-state we only have \( N+1 \) number of non-zero eigenvalues which can have a non-zero contribution to the ergotropy. The non-zero eigenvalues in the decreasing order with their corresponding eigenvectors is given as follows

\[ \begin{align*}
r_1 &= p_0 d_0 \quad |r_1\rangle = \left[0, \ldots, 1\right]^T \\
r_2 &= p_1 d_1 \quad |r_2\rangle = \left[0, \ldots, 0, 1, \ldots, 0\right]^T / \sqrt{p_1} \\
r_3 &= p_2 d_2 \quad |r_3\rangle = \left[0, \ldots, 0, 1, \ldots, 0, 0\right]^T / \sqrt{p_2} \\
\vdots & \quad \vdots \\
r_{N+1} &= p_N d_N \quad |r_{N+1}\rangle = \left[1, \ldots, 0\right]^T
\end{align*} \]

We continue by first focusing on the diagonal terms in the ergotropy, i.e. \( j = i \), and they are given as

\[ \begin{align*}
\sum_{j=1}^{N+1} r_j |\varepsilon_j\rangle (|\varepsilon_j\rangle |\langle \varepsilon_j|\rangle|^2 - 1) = 0 + r_2 \varepsilon_2 \left(\frac{1}{p_1} - 1\right) - r_3 \varepsilon_3 - r_4 \varepsilon_4 - \cdots - r_{N+1} \varepsilon_{N+1} \\
= \frac{\omega p_1 d_1}{p_1} - \omega p_1 d_1 - \omega p_2 d_2 - \cdots - \omega p_N d_N.
\end{align*} \]
On the other hand, the off-diagonal terms, i.e. \( j \neq i \), are

\[
\sum_{j \neq i} r_j \varepsilon_j |(r_j | \varepsilon_j)|^2 = \frac{r_2 \varepsilon_3}{p_1} + \frac{r_2 \varepsilon_4}{p_1} + \cdots + \frac{r_2 \varepsilon_{p_1+1}}{p_1}
\]

\[\left\{\begin{array}{ll}
\text{} & \text{\( p_1 - 1 \) terms (B4)} \\
\end{array}\right.\]

\[
+ \frac{r_3 \varepsilon_{p_1+2}}{p_2} + \cdots + \frac{r_3 \varepsilon_{p_1+p_2+1}}{p_2}
\]

\[\left\{\begin{array}{ll}
\text{} & \text{\( p_2 \) terms}
\end{array}\right.\]

\[
+ \frac{r_4 \varepsilon_{p_1+p_2+2}}{p_3} + \cdots + \frac{r_4 \varepsilon_{p_1+p_2+p_3+1}}{p_3}
\]

\[\left\{\begin{array}{ll}
\text{} & \text{\( p_3 \) terms}
\end{array}\right.\]

\[\vdots\]

\[
\right.
+ \frac{r_N \varepsilon_{p_1+p_2+\cdots+p_{N-2}+2}}{p_{N-1}} + \cdots + \frac{r_N \varepsilon_{p_1+p_2+\cdots+p_{N-1}+1}}{p_{N-1}}
\]

\[\left\{\begin{array}{ll}
\text{} & \text{\( p_{N-1} \) terms}
\end{array}\right.\]

\[
+ r_{N+1} \varepsilon_{2N} = [p_1 - 1] \left[ \frac{\omega p_1 d_1}{p_1} \right] + p_2 \left[ \frac{2 \omega p_2 d_2}{p_2} \right] + \cdots + p_{N-1} \left[ \frac{(N-1) \omega p_{N-1} d_{N-1}}{p_{N-1}} \right] + N \omega p_N d_N.
\]

Combining Eq. B3 and Eq. B4 together we obtain the expression in Eq. 16 as follows

\[
W' = \omega p_2 d_2 + 2 \omega p_3 d_3 + \cdots + (N-1) \omega p_N d_N
\]

\[\text{B5}\]

\[
= \sum_{k=1}^{N-1} k \omega p_{k+1} d_{k+1}.
\]