THE VANISHING OF THE CONTACT INVARIANT IN THE PRESENCE OF TORSION

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Abstract. We prove that the Ozsváth-Szabó contact invariant of a closed 3-manifold with positive $2\pi$–torsion vanishes.

In 2002, Ozsváth and Szabó [OSz1] defined an invariant of a closed, oriented, contact 3-manifold $(M, \xi)$ as an element of the Heegaard Floer homology group $\widehat{HF}(-M)$. The definition of the contact invariant was made possible by the work of Giroux [Gi3], which related contact structures and open book decompositions. The Ozsváth-Szabó contact invariant has undergone an extensive study, e.g., [LS1, LS2]. Recently, Honda, Kazez and Matić [HKM3] defined an invariant of a contact 3-manifold with convex boundary as an element of Juhász’ sutured Floer homology [Ju1, Ju2]. The goal of this paper is to use this relative contact invariant to prove a vanishing theorem in the presence of torsion.

Recall that a contact manifold $(M, \xi)$ has positive $n\pi$-torsion if it admits an embedding $(T^2 \times [0, 1], \eta_{n\pi}) \hookrightarrow (M, \xi)$, where $(x, y, t)$ are coordinates on $T^2 \times [0, 1] \simeq \mathbb{R}^2 / \mathbb{Z}^2 \times [0, 1]$ and $\eta_{n\pi} = \ker(\cos(n\pi t)dx - \sin(n\pi t)dy)$. The torsion was an essential ingredient for distinguishing tight contact structures on toroidal 3-manifolds (see for example [Gi1]), and is a source of non-finiteness of the number of isotopy classes of tight contact structures ([CGH, Co, HKM1]).

Theorem 1 (Vanishing Theorem). If a closed contact 3-manifold $(M, \xi)$ has positive $2\pi$-torsion, then its contact invariant $c(M, \xi)$ in $\widehat{HF}(-M)$ vanishes.

The coefficient ring of $\widehat{HF}(-M)$ is $\mathbb{Z}$ in Theorem 1. The behavior of the contact invariant with twisted coefficients in presence of torsion is the subject of a forthcoming paper by the first two authors [GH].

Theorem 1 was first conjectured in [Gh2, Conjecture 8.3], and partial results were obtained by [Gh1], [Gh2], and [LS3]. The corresponding vanishing result for the contact class in monopole Floer homology has recently been announced by Mrowka and Rollin (and is motivated by [Ga]). Theorem 1 together with a non-vanishing result of the contact invariant proved by Ozsváth and Szabó [OSz2, Theorem 4.2], implies that a contact manifold with positive...
2π-torsion is not strongly symplectically fillable. This non-fillability result was conjectured by Eliashberg, and first proved by Gay [Ga].

In this paper, a contact structure ξ on a compact, oriented 3-manifold N with convex boundary ∂N and dividing set Γ on ∂N will be denoted (N, Γ, ξ). We will write the invariant for a closed contact 3-manifold (M, ξ) as c(M, ξ) ∈ \(\overline{HF}(-M)\) and the invariant for a compact contact 3-manifold (N, Γ, ξ) as c(N, Γ, ξ) ∈ SFH(−N, −Γ), where SFH(−N, −Γ) is the sutured Floer homology of (−N, −Γ), and Γ ⊂ ∂N is now viewed as a balanced suture. Strictly speaking, the contact invariants have a ±1 ambiguity, but this will not complicate matters in this paper. The key property of the relative contact invariant which we use in this paper is the following theorem from [HKM3]:

**Theorem 2** ([HKM3 Theorem 4.5]). Let (M, ξ) be a closed contact 3-manifold and N ⊂ M be a compact submanifold (without any closed components) with convex boundary and dividing set Γ. If c(N, Γ, ξ|N) = 0, then c(M, ξ) = 0.

The behavior of the contact invariant under contact surgery will also play a fundamental role in the proof of Theorem 1.

**Lemma 3.** If (N′, Γ′, ξ′) is obtained by contact (+1)-surgery on a Legendrian curve in (N, Γ, ξ), then the contact (+1)-surgery gives rise to a natural map:

\[
\Phi : SFH(-N, -\Gamma) \to SFH(-N', -\Gamma'),
\]

which satisfies \(\Phi(c(N, \Gamma, \xi)) = c(N', \Gamma', \xi')\).

**Proof.** If (N′, ξ′) is obtained from (N, ξ) by contact (+1)-surgery, then (N, ξ) is obtained from (N′, ξ′) by contact (−1)-surgery (i.e., Legendrian surgery); see [DG1 Proposition 8]. The proof that the contact invariant is natural with respect to Legendrian surgery is the same as in the closed case, provided we use the reformulation of the contact invariant given by Honda, Kazez, and Matić [HKM2]. The proof in the closed case is given in [HKM2 Proposition 3.7]. See also [HKM3 Proposition 4.4].

In this paper we assume that the reader is familiar with the terminology introduced in [H1], such as basic slice, standard neighborhood of a Legendrian curve, Legendrian ruling curve, and minimally twisting.

Let Γ be the following suture/dividing set on the boundary of \(T^2 \times [0, 1]\): \#Γ₀ = \#Γ₁ = 2, slope(Γ₀) = −1, and slope(Γ₁) = −2. Here \# denotes the number of connected components, \(T_i = T^2 \times \{i\}\), the slope is calculated with respect to a fixed oriented identification \(T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2\), and the orientation of \(T_i\) is inherited from that of \(T^2\). (Hence \(\partial(T^2 \times [0, 1]) = T_1 \cup -T_0\).)

Let ζ₀ be a tight contact structure so that \((T^2 \times [0, 1], \Gamma, \zeta_0)\) is a basic slice. There are two possible isotopy classes rel boundary, and \(\zeta_0\) can be in either one.

**Lemma 4.** Let L be a Legendrian ruling curve with infinite slope on a parallel copy \(T_0\) of \(T_0\) with the same dividing set, inside the basic slice \((T^2 \times [0, 1], \Gamma, \zeta_0)\). Then there is an embedding \(i\) of \((T^2 \times [0, 1], \Gamma, \zeta_0)\) into the standard tight \((S^3, \xi_{std})\), so that \(i(L)\) is an unknot with Thurston-Bennequin invariant −1.
defined by the same contact form as
also contains a slightly larger submanifold (N, ζ)
boundary of
1tori with rational slopes
a contact structure in the neighborhood of a pre-Lagrangian torus. By direct computation,
we can choose ε
Bennequin number
M, ξ implies that (M, ξ) has positive 2π-torsion if and only if there exists an embedding of (T^2 × [0, 1], Γ, ζ_1) into (M, ξ), where (T^2 × [0, 1], Γ, ζ_1) is not minimally twisting and is homotopic relative to the boundary to a basic slice (T^2 × [0, 1], Γ, ζ_0).

Proof. From the classification of tight contact structures on T^2 × [0, 1] (see Theorem 2.2 as well as the discussion in Section 5.2 in [H1], an equivalent result is given in [Gi2]) it follows that, if ζ_1 is not minimally twisting and is homotopic to a basic slice, then (T^2 × [0, 1], ζ_1) has positive 2π-torsion. Therefore the existence of an embedding of (T^2 × [0, 1], ζ_1) into (M, ξ) implies that (M, ξ) has positive 2π-torsion.

Assume (M, ξ) contains a contact submanifold isomorphic to (T^2 × [0, 1], η_2π). Then it also contains a slightly larger submanifold (N, ζ'), where N = T^2 × [−ε_0, 1 + ε_1], and ζ' is defined by the same contact form as η_2π. This can be easily seen from the normal form of a contact structure in the neighborhood of a pre-Lagrangian torus. By direct computation, we can choose ε_0, ε_1 ≥ 0 so that the tori T^2 × {−ε_0} and T^2 × {1 + ε_1} are pre-Lagrangian tori with rational slopes s_1, s_2 forming an integer basis of H_1(T^2). Then we can perturb the boundary of N to make it convex, so that the boundary tori have #Γ = 2 and slopes s_1, s_2;

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Positive and negative stabilizations of the Legendrian unknot in S^3 with Thurston–Bennequin number −1.}
\end{figure}
see for example [Gh3, Lemma 3.4]. Let $\zeta_1$ be the resulting contact structure: the contact manifold $(N, \zeta_1)$ constructed in this way is clearly non-minimally-twisting. After a change of coordinates in $N$, we can make its boundary slopes $-1$ and $-2$. The contact structure is homotopic to a basic slice by a standard explicit computation (see [Gh2, Proposition 6.1]).

Proof of Theorem 6.1. By Theorem 2 and Lemma 5, it suffices to prove that $c(N, \Gamma, \zeta_1) = 0$, where $N = T^2 \times [0, 1]$ and $\Gamma, \zeta_1$ are as defined above. This proof is modeled on the argument in [Gh2].

Take a parallel copy $T_0$ of $T_0$ in the interior of $N$ with the same dividing set, and let $L$ be a Legendrian ruling curve on $T_2$ with slope $\infty$. The Legendrian curve $L$ has twisting number $-1$ with respect to the framing coming from $T_0$. Now apply a contact (+1)-surgery to $N$ along $L$; see for example [DG2]. As the surgery coefficient is 0 with respect to the framing induced by the torus $T_0$, the resulting 3-manifold is $N' = (S^1 \times D^2) \# (S^1 \times D^2)$. Next write $\Gamma'$ as $\Gamma'_1 \cup \Gamma'_2$, where $\Gamma'_i$ is the dividing set on the $i$th connect summand $S^1 \times D^2$. Since each component of $\Gamma'_i$ intersects the meridian once geometrically, we may take $\Gamma'_i$ to have slope $\infty$, after diffeomorphism. (Here the slope of the boundary of a solid torus is defined by setting the meridian to have slope 0 and choosing some longitude.)

It was proved in [HKM3] that $SFH(-N, -\Gamma) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where each $\mathbb{Z}$-summand corresponds to a distinct relative Spin$^c$-structure. As for $SFH(-N', -\Gamma')$, Juhász [Ju1, Proposition 9.15] proved that the sutured Floer homology of a connected sum of two balanced sutured manifolds is the sutured Floer homology of their tensor product, tensored with an extra $\mathbb{Z}^2$ factor. Since each $(S^1 \times D^2, \Gamma'_i)$ is product disk decomposable, $SFH(S^1 \times D^2, \Gamma'_i) \cong \mathbb{Z}$, and hence $SFH(-N', -\Gamma') \cong (\mathbb{Z} \otimes \mathbb{Z}) \otimes \mathbb{Z}^2 \cong \mathbb{Z}^2$.

Let $\mathfrak{s}$ be the relative Spin$^c$-structure induced by $\zeta_1$. We claim that the map $\Phi$ induced by the surgery is injective on the direct summand $SFH(-N, -\Gamma, \mathfrak{s}) \cong \mathbb{Z}$; that is the content of Lemma 6 below. In Lemma 7 we will prove that applying contact (+1)-surgery to $(N, \Gamma, \zeta_1)$ along $L$ yields an overtwisted contact structure $\zeta_1'$ on $N'$. Therefore, $\Phi(c(N, \Gamma, \zeta_1)) = c(N', \Gamma', \zeta_1') = 0$, and by the injectivity of $\Phi$ on the appropriate $\mathbb{Z}$-summand it follows that $c(N, \Gamma, \zeta_1) = 0$.

Lemma 6. Let $\mathfrak{s}$ be the relative Spin$^c$-structure induced by $(\Gamma, \zeta_1)$ and $\mathfrak{s}'$ be that induced by $(\Gamma', \zeta_1')$. Then the map

$$\Phi: SFH(-N, -\Gamma, \mathfrak{s}) \rightarrow SFH(-N', -\Gamma', \mathfrak{s}')$$

given by Equation 1 is injective.

Proof. Recall that $\zeta_0$ and $\zeta_1$ have the same relative Spin$^c$-structure $\mathfrak{s}$. By Lemma 4, $(N, \Gamma, \zeta_0)$ can be embedded in $(S^3, \xi_{std})$, which has nonzero contact invariant. Hence, by Theorem 2 the contact invariant $c(N, \Gamma, \zeta_0) \in SFH(-N, -\Gamma, \mathfrak{s})$ is nonzero. Since $SFH(-N, -\Gamma, \mathfrak{s}) \cong \mathbb{Z}$ (since it is nonzero) and $SFH(-N', -\Gamma') \cong \mathbb{Z}^2$, it suffices to prove that $\Phi(c(N, \Gamma, \zeta_0)) \neq 0$.

By Lemma 3, the cobordism map $\Phi$ takes the contact class $c(N, \Gamma, \zeta_0)$ to $c(N', \Gamma', \zeta_0')$, where $\zeta_0'$ is the contact structure obtained from $\zeta_0$ by contact (+1)-surgery along $L$. Now consider the embedding $i: (N, \Gamma, \zeta_0) \hookrightarrow (S^3, \xi_{std})$ from Lemma 4. Legendrian (+1)-surgery along the unknot $i(L)$ with Thurston-Bennequin invariant $-1$ inside $(S^3, \xi_{std})$ yields the
unique tight contact structure on $S^1 \times S^2$, which has nonzero contact invariant: for example, see [LS2] Lemma 3.7. Hence $c(N', \Gamma', \zeta'_0) \neq 0$, and it follows that $SFH(-N, -\Gamma, s)$ maps injectively into $SFH(-N', -\Gamma')$.

**Lemma 7.** Applying contact $(+1)$-surgery to $(N, \Gamma, \zeta_1)$ along $L$ yields an overtwisted contact structure $\zeta'_1$ on $N'$.

**Proof.** For any $s \in \mathbb{Q} \cup \{\infty\}$, there is a convex torus (in standard form) with slope $s$ in $(N, \Gamma, \zeta_1)$ parallel to the boundary, according to [H1] Proposition 4.16. In particular, there is a standard torus whose Legendrian divides have the same slope as the Legendrian ruling curve $L$ we are doing surgery on. After the surgery, this Legendrian divide bounds an overtwisted disk in $N'$.

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