EXAMPLES OF DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUPS

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Abstract. We give an example of a bounded, pseudoconvex, circular domain in $\mathbb{C}^n$ for any $n \geq 3$ with smooth real-analytic boundary and non-compact automorphism group, which is not biholomorphically equivalent to any Reinhardt domain. We also give an analogous example in $\mathbb{C}^2$, where the domain is bounded, non-pseudoconvex, and not equivalent to any Reinhardt domain, and the boundary is smooth real-analytic at all points except one.

Let $D$ be a bounded or, more generally, a hyperbolic domain in $\mathbb{C}^n$. Denote by $\text{Aut}(D)$ the group of biholomorphic self-mappings of $D$. The group $\text{Aut}(D)$, with the topology given by uniform convergence on compact subsets of $D$, is in fact a Lie group [Kob].

A domain $D$ is called Reinhardt if the standard action of the $n$-dimensional torus $\mathbb{T}^n$ on $\mathbb{C}^n$,

$$z_j \mapsto e^{i\phi_j}z_j, \quad \phi_j \in \mathbb{R}, \quad j = 1, \ldots, n,$$

leaves $D$ invariant. For certain classes of domains with non-compact automorphism groups, Reinhardt domains serve as standard models up to biholomorphic equivalence (see e.g. [R], [W], [BP], [GK1], [Kod]).

It is an intriguing question whether any domain in $\mathbb{C}^n$ with non-compact automorphism group and satisfying some natural geometric conditions is biholomorphically equivalent to a Reinhardt domain. The history of the study of domains with non-compact automorphism groups shows that there were expectations that the answer to this question would be positive (see [Kra]). In this note we give examples that show that the answer is in fact negative.

While the domain that we shall consider in Theorem 1 below has already been noted in the literature [BP], it has never been proved that this domain is not biholomorphically equivalent to a Reinhardt domain. Note that this domain is circular, i.e. it is invariant under the special rotations

$$z_j \mapsto e^{i\phi}z_j, \quad \phi \in \mathbb{R}, \quad j = 1, \ldots, n.$$

Our first result is the following
Theorem 1. There exists a bounded, pseudoconvex, circular domain $\Omega \subset \mathbb{C}^3$ with smooth real-analytic boundary and non-compact automorphism group, which is not biholomorphically equivalent to any Reinhardt domain.

Proof. Consider the domain

$$\Omega = \{ |z_1|^2 + |z_2|^4 + |z_3|^4 + (\overline{z_3}z_3 + \overline{z_2}z_2)^2 < 1 \}.$$ 

The domain $\Omega$ is invariant under the action of the two-dimensional torus $T^2$

$$z_1 \mapsto e^{i\phi_1}z_1, \quad \phi_1 \in \mathbb{R},$$

$$z_j \mapsto e^{i\phi_2}z_j, \quad \phi_2 \in \mathbb{R}, \quad j = 2, 3,$$

and therefore is circular. It is also a pseudoconvex, bounded domain with smooth real-analytic boundary. The automorphism group $\text{Aut}(\Omega)$ is non-compact since it contains the following subgroup

$$z_1 \mapsto z_1 - a \overline{z_1},$$

$$z_2 \mapsto \frac{(1 - |a|^2)\frac{4}{3}z_2}{(1 - |a|^2)^{\frac{1}{2}}},$$

$$z_3 \mapsto \frac{(1 - |a|^2)\frac{4}{3}z_3}{(1 - |a|^2)^{\frac{1}{2}}},$$

for a complex parameter $a$ with $|a| < 1$.

We are now going to explicitly determine $\text{Aut}(\Omega)$. Let $F = (f_1, f_2, f_3)$ be an automorphism of $\Omega$. Then, since $\Omega$ is bounded, pseudoconvex and has real-analytic boundary, $F$ extends smoothly to $\overline{\Omega}$ [BL]. Therefore, $F$ must preserve the rank of the Levi form $L_{\partial \Omega}(q)$ of $\partial \Omega$ at every $q \in \partial \Omega$. The only points where $L_{\partial \Omega} \equiv 0$ are those of the form $(e^{i\alpha}, 0, 0)$, $\alpha \in \mathbb{R}$. These points must be preserved by $F$. This observation implies that $f_j(e^{i\alpha}, 0, 0) = 0$ for all $\alpha \in \mathbb{R}$, $j = 2, 3$. Restricting $f_2$, $f_3$ to the unit disc $\Omega \cap \{ z_2 = z_3 = 0 \}$, we see that $f_j(z_1, 0, 0) = 0$ for all $|z_1| \leq 1$, $j = 2, 3$. Therefore, $F(0) = (b, 0, 0)$ for some $|b| < 1$. Taking the composition of $F$ and the automorphism $G$ of the form (1) with $a = b$, we find that the mapping $G \circ F$ preserves the origin. Since $\Omega$ is circular, it follows from a theorem of H. Cartan [C] that $G \circ F$ must be linear. Therefore, any automorphism of $\Omega$ is the composition of a linear automorphism and an automorphism of the form (1).

The above argument also shows that any linear automorphism of $\Omega$ can be written as

$$z_1 \mapsto e^{i\phi_1}z_1,$$

$$z_2 \mapsto az_2 + bz_3,$$

$$z_3 \mapsto cz_3 + dz_3,$$

where $\phi_1 \in \mathbb{R}$, $a, b, c, d \in \mathbb{C}$, and the transformation in the variables $(z_2, z_3)$ is an automorphism of the section $\Omega \cap \{ z_1 = 0 \}$. Further, since the only points of $\partial \Omega$ where rank $L_{\partial \Omega} = 1$ are those of the form $(z_1, w, \pm w)$ with $w \neq 0$ and since automorphisms of $\Omega$ preserve such points, it follows that any linear automorphism of $\Omega$ is in fact given by

$$z_1 \mapsto e^{i\phi_1}z_1,$$

$$z_2 \mapsto e^{i\phi_2}z_{\sigma(2)},$$

$$z_3 \mapsto e^{i\phi_3}z_{\sigma(3)},$$
where $\phi_1, \phi_2 \in \mathbb{R}$, and $\sigma$ is a permutation of the set $\{2, 3\}$.

The preceding description of $\text{Aut}(\Omega)$ implies that $\dim \text{Aut}(\Omega) = 4$. That is to say, each of the four connected components of $\text{Aut}(\Omega)$ is parametrized by the point $a$ from the unit disc and by the rotation parameters $\phi_1, \phi_2$.

Suppose now that $\Omega$ is biholomorphically equivalent to a Reinhardt domain $D \subset \mathbb{C}^3$. Since $\Omega$ is bounded, it follows that $D$ is hyperbolic. It follows from [Kru] that any hyperbolic Reinhardt domain $G \subset \mathbb{C}^n$ can be biholomorphically mapped onto its normalized form $\tilde{G}$ for which the identity component $\text{Aut}_0(\tilde{G})$ of $\text{Aut}(G)$ is described as follows. There exist integers $0 \leq s \leq t \leq p \leq n$ and $n_i \geq 1$, $i = 1, \ldots, p$, with $\sum_{i=1}^p n_i = n$, and real numbers $\alpha_i, \beta_i, \gamma_i, \delta_i$ such that if we set $z^i = (z_{n_1+\ldots+n_{i-1}+1}, \ldots, z_{n_1+\ldots+n_i})$, $i = 1, \ldots, p$, then $\text{Aut}_0(\tilde{G})$ is given by the mappings

$$
\begin{align*}
    z^i &\mapsto \frac{A^i z^i + b^i}{c^i z^i + d^i}, & i = 1, \ldots, s, \\
    z^j &\mapsto B^j z^j + e^j, & j = s + 1, \ldots, t, \\
    z^k &\mapsto C^k \frac{\prod_{j=s+1}^t \exp \left( -\beta_j \left( 2\exp(z^j) + |e^j|^2 \right) \right) z^k}{\prod_{i=1}^s (c^i z^i + d^i)^2 \alpha_i}, & k = t + 1, \ldots, p,
\end{align*}
$$

where

$$
\begin{align*}
    &\begin{pmatrix} A^i & b^i \\ c^i & d^i \end{pmatrix} \in SU(n_i, 1), & i = 1, \ldots, s, \\
    &B^j \in U(n_j), & e^j \in \mathbb{C}^{n_j}, & j = s + 1, \ldots, t, \\
    &C^k \in U(n_k), & k = t + 1, \ldots, p.
\end{align*}
$$

The normalized form $\tilde{G}$ is written as

$$
G = \left\{ |z^1| < 1, \ldots, |z^n| < 1, \begin{array}{c}
    \frac{z^{t+1}}{\prod_{i=1}^s (1 - |z^i|^2)^{\alpha_i}} \\
    \prod_{j=s+1}^t \exp \left( -\beta_j \left( 2\exp(z^j) + |e^j|^2 \right) \right)
\end{array}, \ldots, \begin{array}{c}
    \frac{z^p}{\prod_{i=1}^s (1 - |z^i|^2)^{\alpha_p}} \\
    \prod_{j=s+1}^t \exp \left( -\beta_j \left( 2\exp(z^j) + |e^j|^2 \right) \right)
\end{array} \right\} \subset \tilde{G}_1,
$$

where $\tilde{G}_1 := \tilde{G} \cap \{ z^i = 0, i = 1, \ldots, t \}$ is a hyperbolic Reinhardt domain in $\mathbb{C}^{n+1} \times \ldots \times \mathbb{C}^{n_p}$.

It is now easy to see that, for any hyperbolic Reinhardt domain $D \subset \mathbb{C}^3$ written in a normalized form $\tilde{D}$, $\text{Aut}_0(\tilde{D})$ given by formulas (2) cannot have dimension equal to 4.

This completes the proof.

\[ \blacksquare \]

**Remark.** The theorem can be easily extended to $\mathbb{C}^n$ for any $n \geq 3$ (just replace $|z^i|^2$ in the defining function of $\Omega$ by $\sum_{i=1}^{n-2} |z^i|^2$, $\alpha_i$ by $\alpha_{i+2}$, $\beta_i$ by $\beta_{i+2}$).
There is considerable evidence that, in complex dimension two, an example such as that constructed in Theorem 1 does not exist. Certainly the example provided above depends on the decoupling, in the domain $\Omega$, of the variables $z_2, z_3$ from the variable $z_1$. Such decoupling is not possible when the dimension is only two.

The work of Bedford and Pinchuk (see [BP] and references therein) suggests that the only smoothly bounded domains in $\mathbb{C}^2$ with non-compact automorphism groups are (up to biholomorphic equivalence) the complex ellipsoids $\Omega_\alpha = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2\alpha} < 1 \}$, where $\alpha$ is a positive integer. It is also a plausible conjecture that any bounded domain in $\mathbb{C}^2$ with non-compact automorphism group and a boundary of finite smoothness $C^k$ for $k \geq 1$, is biholomorphically equivalent to some $\Omega_\alpha$, where $\alpha \geq 1$ and is not necessarily an integer. Of course all the domains $\Omega_\alpha$ are pseudoconvex and Reinhardt.

However, as the following theorem shows, if we allow the boundary to be non-smooth at just one point, then the domain may be non-pseudoconvex and be non-equivalent to any Reinhardt domain.

**Theorem 2.** There exists a bounded, non-pseudoconvex domain $\Omega \subset \mathbb{C}^2$ with non-compact automorphism group such that $\partial \Omega$ is smooth real-analytic everywhere except one point (this exceptional point is an orbit accumulation point for the automorphism group action), and such that $\Omega$ is not biholomorphically equivalent to any Reinhardt domain.

For the proof of Theorem 2, we first need the following lemma.

**Lemma A.** If $\Omega \subset \mathbb{C}^2$ is a bounded, non-pseudoconvex, simply-connected domain such that the identity component $\text{Aut}_0(\Omega)$ of the automorphism group $\text{Aut}(\Omega)$ is non-compact, then $\Omega$ is not biholomorphically equivalent to any Reinhardt domain.

**Proof of Lemma A.** Suppose that $\Omega$ is biholomorphically equivalent to a Reinhardt domain $D$. Since $\Omega$ is bounded, it follows that $D$ is hyperbolic. Also, since $\text{Aut}_0(\Omega)$ is non-compact, then so is $\text{Aut}_0(D)$. We are now going to show that any such domain $D$ is either pseudoconvex, or not simply-connected, or cannot be biholomorphically equivalent to a bounded domain. This result clearly implies the lemma.

We can now assume that the domain $D$ is written in its normalized form $\tilde{D}$ as in (3), and $\text{Aut}_0(\tilde{D})$ is given by formulas (2). Then, since $\text{Aut}_0(\tilde{D})$ is non-compact, it must be that $t > 0$. Next, if $p = t$, then $\tilde{D}$ is either non-hyperbolic (for $s < t$), or (for $s = t$) is the unit ball or the unit polydisc and therefore is pseudoconvex. Thus we can assume that $t = 1$, $p = 2$, $n_1 = n_2 = 1$.

Let $\tilde{D}_1 \subset \mathbb{C}$ be the hyperbolic Reinhardt domain analogous to $\tilde{G}_1$ that was defined above (see (3)). Clearly, there are the following possibilities for $\tilde{D}_1$:

(i) $\tilde{D}_1 = \{ 0 < |z_2| < R \}, 0 < R < \infty$;
(ii) $\tilde{D}_1 = \{ r < |z_2| < R \}, 0 < r < R \leq \infty$;
(iii) $\tilde{D}_1 = \{ |z_2| < R \}, 0 < R < \infty$.

For the cases (i), (ii), $\tilde{D}$ is always not simply-connected, and therefore we will concentrate on the case (iii). If $s = 0$, then $\tilde{D}$ is not hyperbolic since it contains the complex line $\{ z_2 = 0 \}$. Thus we can assume that $s = 1$. Next observe that, for
\[ \alpha_1^2 \geq 0, \] the domain \( \tilde{D} \) is always pseudoconvex. Thus we may take \( \alpha_1^2 < 0 \). Then the domain \( \tilde{D} \) has the form

\[
\tilde{D} = \left\{ |z_1| < 1, |z_2| < \frac{R}{(1 - |z_1|^2)\gamma} \right\}, \quad \gamma > 0.
\]

We will now show that the above domain \( \tilde{D} \) cannot be biholomorphically equivalent to a bounded domain. More precisely, we will show that any bounded holomorphic function on \( \tilde{D} \) is independent of \( z_2 \).

Let \( f(z_1, z_2) \) be holomorphic on \( \tilde{D} \) and \( |f| < M \) for some \( M > 0 \). For every \( \rho \) such that \( |\rho| \leq \frac{R}{2} \), the disc \( \Delta_\rho = \{ |z_1| < 1, z_2 = \rho \} \) is contained in \( \tilde{D} \). We will show that \( \partial f/\partial z_2 \equiv 0 \) on every such \( \Delta_\rho \), which implies that \( \partial f/\partial z_2 \equiv 0 \) everywhere in \( \tilde{D} \).

Fix a point \((\mu, \rho) \in \Delta_\rho \) and restrict \( f \) to the disc \( \Delta'_\mu = \{ z_1 = \mu, |z_2| < R_\mu \} \), where \( R_\mu = R/2(1 - |\mu|^2)^{\gamma} \). Clearly, \((\mu, \rho) \in \Delta'_\mu \) and \( \Delta'_\mu \subset \tilde{D} \). By the Cauchy Integral Formula

\[
f(\mu, z_2) = \frac{1}{2\pi i} \int_{\partial \Delta'_\mu} \frac{f(\mu, \zeta)}{\zeta - z_2} d\zeta,
\]

for \( |z_2| < R_\mu \), and therefore

\[
\frac{\partial f}{\partial z_2}(\mu, \rho) = \frac{1}{2\pi i} \int_{\partial \Delta'_\mu} \frac{f(\mu, \zeta)}{(\zeta - \rho)^2} d\zeta.
\]

Hence

\[
\left| \frac{\partial f}{\partial z_2}(\mu, \rho) \right| \leq \frac{MR_\mu}{(R_\mu - |\rho|)^2}.
\]

Letting \( |\mu| \to 1 \) and taking into account that \( R_\mu \to \infty \), we see that \( |\partial f/\partial z_2(\mu, \rho)| \to 0 \) as \( |\mu| \to 1 \). Therefore, \( \partial f/\partial z_2 \equiv 0 \) on \( \Delta_\rho \).

The lemma is proved.

**Proof of Theorem 2.** We will now present a domain that satisfies the conditions of the lemma. Set

\[
\Omega = \left\{ |z_1|^2 + |z_2|^4 + 8|z_1| - 1|^2 \left( \frac{z_2^2}{z_1 - 1} - \frac{3}{2} \frac{|z_2|^2}{|z_1| - 1} + \frac{2z_2^2}{z_1 - 1} \right)^2 < 1 \right\}.
\]

The domain \( \Omega \) is plainly bounded since the third term on the left is non-negative. Next, the identity component \( \text{Auto}_0(\Omega) \) of its automorphism group is non-compact since it contains the subgroup

\[
\begin{align*}
    z_1 &\mapsto \frac{z_1 - a}{1 - az_1}, \\
    z_2 &\mapsto \frac{(1 - a^2)^{\frac{1}{2}} z_2}{(1 - az_1)^{\frac{1}{2}}},
\end{align*}
\]

where \( a \in (-1, 1) \).
Further, $\Omega$ is simply-connected, since the family of mappings $F_\tau(z_1, z_2) = (z_1, \tau z_2)$, $0 \leq \tau \leq 1$, retracts $\Omega$ inside itself, as $\tau \to 0$, to the unit disc $\{|z_1| < 1, z_2 = 0\}$ (which is simply-connected).

To show that $\Omega$ is not pseudoconvex, consider its unbounded realization. Namely, under the mapping

$$
\begin{align*}
z_1 &\mapsto \frac{z_1 + 1}{z_1 - 1}, \\
z_2 &\mapsto \frac{\sqrt{2}z_2}{\sqrt{z_1 - 1}},
\end{align*}
$$

the domain $\Omega$ is transformed into the domain

$$
\Omega' = \left\{ \Re z_1 + \frac{1}{4}|z_2|^4 + 2 \left( z_2^2 - \frac{3}{2}|z_2|^2 + \frac{3}{2}z_2^2 \right)^2 < 0 \right\}.
$$

It is easy to see that at the boundary point $(-\frac{3}{4}, 1) \in \partial\Omega'$ the Levi form of $\partial\Omega'$ is equal to $-|z_2|^2$, and thus is negative-definite. Therefore, $\Omega$ is non-pseudoconvex.

Hence, by Lemma A, $\Omega$ is not biholomorphically equivalent to any Reinhardt domain.

Next, if $\phi$ denotes the defining function of $\Omega$, the following holds at every boundary point of $\Omega$ except $(1, 0)$:

$$
\frac{\partial \phi}{\partial z_1} = \frac{1}{z_1 - 1} \left( -\frac{z_2}{2} \frac{\partial \phi}{\partial z_2} + 1 - \frac{1}{z_1} \right),
$$

and therefore $\text{grad} \phi$ does not vanish at every such point. Hence, $\partial \Omega$ is smooth real-analytic everywhere except at $(1, 0)$.

The theorem is proved.

Remarks.

1. The hypothesis of simple connectivity in Lemma A is automatically satisfied if, for example, the boundary of the domain is locally variety-free and smooth near some orbit accumulation point for the automorphism group of the domain (see e.g. [GK2]). For a smoothly bounded domain it would follow from a conjecture of Greene/Krantz [GK3].

2. Tedious calculations show that the boundary of the domain $\Omega$ in Theorem 2 is quite pathological near the exceptional point $(1, 0)$. It is not Lipschitz-smooth of any positive degree. It would be interesting to know whether there is an example with Lipschitz-1 boundary at the bad point.

In fact, many more examples similar to that in Theorem 2 can be constructed in the following way. Let

$$\Omega' = \{ (z_1, z_2) \in \mathbb{C}^2 : \Re z_1 + P(z_2) < 0 \},$$

where $P = |z_2|^{2m} + Q(z_2)$ is a homogeneous non-plurisubharmonic polynomial, $m$ is a positive integer, and $Q(z_2)$ is positive away from the origin. Then, by a mapping analogous to (4), $\Omega'$ can be transformed into a bounded domain $\Omega$. The domain $\Omega$ is simply-connected, non-pseudoconvex, $\text{Aut}(\Omega)$ is non-compact, and $\partial\Omega$ is smooth.
real-analytic everywhere except at the point \((1, 0)\). For all such examples, \(\partial \Omega\) is not Lipschitz-smooth of any positive degree at \((1, 0)\).

It is also worth noting that, in the example contained in Theorem 2, the point \((-1, 0)\) is also an orbit accumulation point, but \(\partial \Omega\) is smooth real-analytic at this point.

3. It is conceivable that the domain \(\Omega\) as in Theorem 2 has an alternative, smoothly bounded realization, but it looks plausible that if in formula (5) we allow \(P(z_2)\) to be an arbitrary homogeneous polynomial positive away from the origin with no harmonic term, then domain (5) does not have a bounded realization with \(C^1\)-smooth boundary, unless \(P(z_2) = c|z_2|^{2m}\), where \(c > 0\) and \(m\) is a positive integer.

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