OPTIMAL INVESTMENT AND RISK CONTROL PROBLEMS
WITH DELAY FOR AN INSURER IN DEFAULTABLE MARKET

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Abstract. This paper addresses a investment and risk control problem with a delay for an insurer in the defaultable market. Suppose that an insurer can invest in a risk-free bank account, a risky stock and a defaultable bond. Taking into account the history of the insurer’s wealth performance, the controlled wealth process is governed by a stochastic delay differential equation. The insurer’s goal is to maximize the expected exponential utility of the combination of terminal wealth and average performance wealth. We decompose the original optimization problem into two subproblems: a pre-default case and a post-default case. The explicit solutions in a finite dimensional space are derived for an illustrative situation, and numerical illustrations and sensitivity analysis for our results are provided.

1. Introduction. The study of investment and risk control strategies for insurers is becoming an area of mainstream research in actuarial science. Browne [5], a pioneer, examined the investment optimization problem for an insurance firm, where an insurer chooses the investment strategy that maximizes expected utility or minimizes the probability of ruin. Using stochastic control theory and the Hamilton-Jacobi-Bellman equation, he obtained explicit optimal investment policies under two criteria. Motivated by Browne [5], Yang and Zhang [25] studied the investment optimization problem with a jump-diffusion risk process for an insurer. Xu et al. [24] concluded that an insurer can invest in a financial market and also purchase a reinsurance contract to transfer the claim risk. The optimal investment and reinsurance strategy are derived explicitly. Since then, more studies have addressed the optimal investment–reinsurance problem under more general and more

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realistic model assumptions: see Schmidli [23], Luo et al. [16] and Liang and Bai [13] for ruin probability minimization; Liang et al. [15], Huang et al. [11] and Chen and Yang [7] for exponential utility maximization; and Zeng and Li [26], Li et al. [18] and Li et al. [19] for mean variance criteria. In addition, Zou and Cadenillas [29] proposed that the insurer can manage the risk by controlling the number of policies. The insurer’s risk is modeled by a controllable jump-diffusion process. They derived the optimal investment and risk control strategies under three types of expected utility goals, respectively.

In recent years, the problem of optimal investment-reinsurance with a defaultable bond has aroused much more attention. In this paper, we assume that the insurer can invest into a defaultable bond, which represents a significant portion of the market. Bielecki and Jang [3] considered the portfolio optimization problem with a defaultable security, and the default time $\tau$ was modeled as the first jump in a Poisson process. They found that by dividing the original problem into two sub-problems, pre-default and post-default, the explicit expression of the optimal strategies for pre- and post-default cases are obtained. Similarly, Bo et al. [6] solved the typical Merton problem with a defaultable bond. For the reinsurance-investment problem, Zhu et al. [28] and Deng et al. [9] extended the insurer’s investment opportunity with three assets, a risk-free bond, a stock and a defaultable bond, and derived the optimal reinsurance and investment rules for an insurer under an exponential utility-maximization objective. Zhao et al. [27] examined a time-inconsistent reinsurance-investment problem with a defaultable asset for an insurer.

However, the majority of these studies have modeled the wealth process with a Markovian process, in which decisions are made only on the basis of current information. In the real world, the insurer would tend to look at the historic performance(delay) of their wealth before they made an investment or risk control decision. Hence, the delay stochastic process has emerged as a popular approach to incorporate the impact of historical performance on present wealth. For example, Shen and Yang [21] assume that if wealth has substantially increased recently, the insurer will distribute the excess wealth as dividend to shareholders, and otherwise, the insurer will raise the capital by issuing equity. The insurer’s wealth process is modeled by a stochastic process with a delay. By using the maximum-principle approach, they derived the optimal reinsurance and investment strategy for the insurer under the mean-variance maximization criterion. A and Li [1] examined the investment and excess-of-loss reinsurance problem with a delay for an insurer under Heston’s stochastic volatility model; they obtained a closed-form expression of the optimal investment and reinsurance policy under certain conditions. For more detailed discussion of the investment reinsurance problem, see Chang et al. [8], Shen et al. [22] and Pang and Hussain [20].

In this paper, we claim that the wealth process of an insurer is governed by a stochastic delay differential equation. The insurer’s portfolio contains a risk-free bank account, a risky stock and a defaultable bond. The optimal investment and risk control strategy is explicitly derived in a finite-dimensional space, under certain conditions. We find that when delay (bounded memory) is considered, the insurer will choose a more cautious strategy than when no delay (bound memory) is involved. Moreover, the investment of the defaultable bond can improve the expected exponential utility. The sensitivities of the optimal investment and risk control strategy regarding delay parameters are studied.
Our paper is different from Zou and Cadenillas [29] in at least two respects. First, we consider the presence of insurer’s bounded memory behavior (delay) while Zou and Cadenillas [29] did not. Second, our portfolio set contains three assets: a risk free bond, a defaultable bond and a risky stock, while only a risk free bond and a stock in Zou and Cadenillas [29]. This paper is also different from the A and Li [1] and Zhu et al. [28], where the risk control method is purchasing the reinsurance contract from the reinsurer. Yet our paper regulate the insurer’s risk by controlling the number of policies, and the wealth process is governed by the stochastic delay process to model the bounded memory behavior of the insurer.

The rest of this paper is organized as follows. In Section 2, we formulate the optimal investment and risk control strategies with a delay and default risk for an insurer. In section 3, we discuss optimal strategies before and after default of the defaultable bond and derive the closed form for the optimal results. Section 4 provides sensitivity analysis for the model parameters to illustrate our results. Finally, we conclude the paper in Section 5.

2. Model formulation. We consider a financial market that consists of a risk-free bond, a risky asset (i.e., a stock) and a defaultable zero-coupon corporate bond. The price processes are denoted by \( \{R(t)\}_{t \geq 0} \), \( \{S(t)\}_{t \geq 0} \) and \( \{p(t, T_1)\}_{t \geq 0} \), where \( T_1 \) is a fixed time horizon. The processes are defined in a completely filtered probability space \( (\Omega, \mathcal{G}, \mathbb{P}) \), where \( \mathbb{P} \) is the real-world probability measure and \( \mathcal{G} := \{\mathcal{G}_t\}_{t \geq 0} \) is an enlarged filtration given by \( \mathcal{G}_t = \mathcal{F}_t \vee \sigma\{H_s: 0 \leq s \leq t\} \) (the filtrations \( \mathcal{F}_t \) and \( \mathcal{H}_t \) will be introduced later). Let \( W(t) \), \( W_1(t) \) and \( \hat{W}(t) \) be three standard Brownian motions and \( N(t) \) be a Poisson process with intensity \( \lambda > 0 \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \). The natural filtration \( \mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0} \) satisfies the usual hypotheses of completeness and right continuity.

2.1. Financial market. The insurer can invest its idle capital into a financial market that consists of risk-free bond, a risky asset (a stock) and a defaultable zero-coupon corporate bond. The price process of the risk-free asset is given by

\[
dR(t) = rR(t)dt,
\]

where \( r > 0 \) is a constant denoting the risk-free rate of return. The stock price process \( S(t) \) follows

\[
dS(t) = S(t)(\mu dt + \sigma dW(t)),
\]

where \( W(t) \) is a standard Brownian motion and \( \mu > r > 0 \) and \( \sigma > 0 \) are two constants that represent the expected instantaneous rate of return of the risky asset and the volatility of the risky asset price, respectively.

Next, we derive the price process for a zero-coupon corporate bond under a risk-neutral measure \( \mathbb{Q} \), which is equivalent to the real-world probability measure \( \mathbb{P} \). We assume that the maturity date of corporate bond is \( T_1 \) and adopt the market-value recovery scheme at the time of default, following Duffie and Singleton [10].

Let \( \tau \) denote the default time of the corporate bond. The default indicator process is denoted by \( H(t) = I_{\{\tau \leq t\}} \), which is assumed to be a Poisson point process with a fixed intensity \( h^Q \). Let \( \mathcal{G}_t \) be the smallest filtration containing the reference filtration \( \mathcal{F}_t \), under which \( \tau \) is a stopping time; this means that \( \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \mathcal{F}_t \vee \sigma\{H_s: 0 \leq s \leq t\} \). Such an information structure is standard in
the reduced-form approach. We formulate this model under the martingale invariance property, which is generally called the \((\text{H})\) hypothesis (see Section 6.1.1 in Bielecki and Rutkowski [2], and Proposition 1 in Blanchet-Scalliet and Jeanblanc [4]): under the real-world probability measure \(P\), every square-integrable \(\mathcal{F}_t\)-martingale is also a square-integrable martingale under the enlarged filtration \(\mathcal{G}_t\).

In the case of default, the investor recovers a fraction of the market value of the defaultable bond just prior to default. Let \(\zeta \in (0, 1)\) denote the constant loss rate of the corporate bond. Consequently, the price process of the corporate zero-coupon bond with maturity \(T_1\) is given by:

\[
p(t, T_1) = \mathbf{I}_{\{\tau > t\}} E^Q \left[ e^{-(r+h^Q)(t-T_1)} \mathbf{1}\{\tau > T_1\} \right]
\]

where \(\mathbf{1}\{\tau > t\} = (1 - \zeta)p(t-, T_1)\) if \(t < \tau \leq T_1\), and \(p(t-, T_1) = (1 - \zeta)p(t-, T_1)\) if \(t < \tau \leq T_1\).

As shown in Bielecki and Jang [3], although (4) can reduce the pricing of the defaultable bond to that of default-free bonds, it is only applicable prior to the default event, i.e., \(t < 0, \tau \leq T_1\). In deriving the drift term of the return on the defaultable bond for \(t \in [0, \tau \wedge T_1]\), one cannot avoid the questions of what happens to bond value at default and how to deal with the recovered amount in terms of optimal-asset allocation. To solve this problem, Bielecki and Jang utilize an expression for the price process of the auxiliary security and obtain the dynamic price process of the defaultable bond under the real-world measure \(P\) (see Lemma 3 in Bielecki and Jang [3]). We list this result as follows without proof.

**Proposition 2.1.** The dynamics of the corporate bond price process \(p(t, T_1)\) under the real-world measure \(P\) is given by

\[
dp(t, T_1) = p(t-, T_1) \left[ rd \tau + (1 - H(t-))\eta(1 - \Delta) dt - (1 - H(t-))\Delta \zeta dM^P(t) \right] \quad \text{for} \quad t \in [0, \tau \wedge T_1],
\]

where we use

\[
\begin{align*}
p(t, T_1) &= e^{-(r+\eta)(T_1-t)} \quad \text{if} \quad t \in [0, \tau \wedge T_1], \\
p(t, T_1) &= (1 - \zeta)e^{-(r+\eta)(T_1-t)}e^r(t-\tau) \quad \text{if} \quad t \in [\tau \wedge T_1, T_1].
\end{align*}
\]

\(M^P(t) = H(t) - h^Q \int_0^t \Delta(1 - H(u-)) du\) is a \(\mathcal{G}\)-martingale under \(P\), \(\eta = h^Q\zeta\) is the credit spread under the real-world probability measure, and \(\frac{1}{\Delta} \geq 1\) denotes the constant default-risk premium. The arrival intensity of the default under the \(Q\) measure is then given by \(h^Q = h^P/\Delta\).

**Remark 2.1.** In order to derive the defaultable bond price \(p(t, T_1)\) under real-world probability measure \(P\), we should state the following Girsanov’s theorem (See Bielecki and Jang [3]): A probability \(P\) is equivalent to \(Q\) on the filtration \(\mathcal{G}\) if and only if there exists progressively measurable, \(\mathbb{R}\)-valued process \(\phi(t)\) and predictable
\[ \mathbb{R}^+ \triangleq [0, \infty] \text{-valued process } \Delta(t) > 0 \text{ for } t \in [0, T] \text{ such that} \\
1. \ E_P(\Xi(T)) = 1, \text{ where} \\
\Xi(t) = \Xi_1(t)\Xi_2(t), \\
\Xi_1(t) = \exp\{\int_0^t \phi(u)dW^Q(u) - \frac{1}{2} \int_0^t \phi^2(u)du\}, \\
\Xi_2(t) = \exp\{\int_0^t \ln(\Delta(u))dH(u) - h^Q \int_0^{t\wedge \tau} [\Delta(u) - 1]du\}, \forall t \in [0, T], \\
2. \ \frac{dP}{dQ} = \Xi(t). \text{ Moreover, the process } W^P(t) = W^Q(t) - \int_0^t \phi(u)du \text{ is a } \mathcal{G}\text{-Brownian motion under measure } P \text{ and the process } M^P(t) = H(t) - \int_0^t h^Q \Delta(1 - H(u-))du \text{ is a } \mathcal{G}\text{-martingale under measure } P. \\

2.2. Dynamics of wealth processes. Following Zou and Cadenillas [29], the insurer’s risk (per policy) is given as follows:
\[ dU(t) = cdt + ad\check{W}(t) + \theta dN(t), \ U(0) = 0, \] (7)
where \( \check{W}(t) \) is a standard Brownian motion and \( N(t) \) is a Poisson process with intensity \( \lambda \) defined by \( (\Omega, \mathcal{F}, P) \), and \( c, a, \theta \) are all positive constants. We assume that \( E[dW(t)d\check{W}(t)] = pdt \), i.e., \( W(t) = \rho W(t) + \sqrt{1 - \rho^2} \check{W}(t) \). The average premium per policy for the insurer is \( p \), and \( p \geq c > 0 \). At time \( t \), an insurer invested in the risk-free bond, risky stock asset and defaultable bond. The investment horizon is \([0, T]\) and \( T < T_1 \). As the same as Bielecki and Jiang [3], we also assume that the defaultable bond is not traded after default. The insurer chooses to invest an amount of her/his wealth \( \gamma(t) \) and \( k(t) \) in the stock \( S(t) \) and the defaultable bond \( p(t, T_1) \), respectively. The total written insurance policies at time \( t \) denotes as \( l(t) \). Let \( \pi(t) = (\gamma(t), k(t), l(t)) \). Then, the insurer’s wealth process is governed by
\[ dX^\pi(t) = \left[rX^\pi(t-\mu)(1 - H(t-))\eta(1 - \Delta) + (p - c)l(t)\right]dt - \theta l(t)dN(t) + (\gamma(t)\sigma - \rho a l(t))dW(t) - a \sqrt{1 - \rho^2} l(t)d\check{W}(t) - k(t)\zeta(1 - H(t-))dM^P(t). \] (8)
In the real world, if the wealth history of an insurer is good, the insurer could choose to distribute some idle wealth as a dividend for its share holders, which leads to capital outflow. On the other hand, if the wealth history of an insurer is bad, the insurer would need an additional equity injection to boost its operations, which leads to capital inflow. Then, we define two new processes as follows:
\[ Y(t) = \int_t^0 e^{\alpha s}X^\pi(t + s-)ds, \] (9)
\[ Z(t) = X^\pi(t - u), \forall t \in [0, T]. \] (10)
\( Y(t) \) represents the average performance of the wealth process during the period \([t - u, u]\), where \( \alpha > 0 \) is a weighting parameter and \( u > 0 \) is a delay (memory) parameter. \( Z(t) \) represents the wealth of the insurer at time \( t - u \). Let \( g(t, X^\pi(t) - Y(t), X^\pi(t) - Z(t)) \) be the exogenous capital inflow or outflow of the insurer and \( X^\pi(t) - Y(t) \) the average performance of the wealth process during the period \([t - u, t]\).
Following Shen and Zeng [21] and A and Li [1], the wealth process with a delay is given by:

\[
dX^\pi(t) = [rX^\pi(t-1) + (\mu - r)\gamma(t) + k(t)(1 - H(t-1))\eta(1 - \Delta) + (p - c)l(t)]dt \\
- \theta l(t)dN(t) + (\gamma(t)\sigma - \rho a(t))dW(t) - a\sqrt{1 - \rho^2}l(t)dW_1(t) \\
-k(t)\zeta(1 - H(t-1))dM^P(t) - g(t, X^\pi(t-1) - Y(t-1), X^\pi(t-1) - Z(t))dt
\]  

(11)

For convenience, we also assume that the capital inflow or outflow is proportional to the historical performance of the wealth process, i.e.:

\[
g(t, X^\pi(t) - Y(t), X^\pi(t) - Z(t)) = B(X^\pi(t) - Y(t)) + C(X^\pi(t) - Z(t)),
\]

(12)

where \( B, C > 0 \) are constants. This implies that if the insurer’s wealth is greater than the average wealth in the given history period, the insurer distributes a proportion \( B \) of the excess wealth to its share holders, and if the insurer’s current wealth is lower than the average wealth, the insurer needs to obtain additional equity financing. Then the wealth process with bounded memory can be rewritten as:

\[
dX^\pi(t) = [\eta(1 - \Delta) + (p - c)l(t) + AX^\pi(t-1)] \\
+ BY(t-1) + CZ(t-1)]dt - \theta l(t)dN(t) + (\gamma(t)\sigma - \rho a(t))dW(t) - a\sqrt{1 - \rho^2}l(t)dW_1(t) \\
\times l(t)dW_1(t) - k(t)\zeta(1 - H(t-1))dM^P(t),
\]

(13)

where \( A = r - B - C \). Define \( X(t) = x, \forall t \in [-u, 0] \); i.e., we assume that the insurer takes an exogenous initial wealth \( x > 0 \) at time \(-u\) and does not have investment or liability operations until time \( 0 \). At time \( 0 \), the average performance of the historical wealth process is then \( Y(0) = x(1 - e^{-\alpha u}/\alpha) \).

**Definition 2.1.** A triple process \( \{\pi(t)\}_{t \in [0,T]} \) is called an “admissible strategy” if

(1) \( \pi(t) \) is a \( \mathcal{G}_t \)-progressively measurable process;

(2) \( \int_0^T [\gamma(t)^2\sigma^2 + I(t)^2(a^2 + \theta^2\lambda) + k^2(t)\Delta^2\zeta^2h^2]dt < \infty \); and

(3) Under \( \pi \), the SDE (13) has a unique strong solution.

Let \( \Pi \) denotes the space of all admissible strategies.

### 3. Solution to the Optimization Problem with a Delay

Since the insurer is concerned about his terminal wealth and the performance measured by the average historical wealth, the objective of the insurer is modeled as follows:

\[
U(X^\pi(T), Y(T)) = -\frac{1}{\nu}e^{-\nu(X^\pi(T) + \beta Y(T))},
\]

(14)

where \( \nu > 0 \) is the risk-aversion parameter and \( 1 > \beta > 0 \) is the weight parameter. This utility indicates that \( X^\pi(T) + \beta Y(T) \) is the actual “terminal wealth” that the insurer is concerned with. This “combined terminal wealth” is consistent with the approaches of A and Li [1] and Pang and Hussain [20].

The objective of the insurer is to choose a investment and risk control strategy such that the expected utility from the combined terminal wealth \( X^\pi(T) + \beta Y(T) \) is maximized. For any admissible strategy \( \pi(t) = (\gamma(t), k(t), l(t)) \), define the reward function of the insurer under strategy \( \pi(t) \) as

\[
V^\pi(t, x, y, z, h) = E[U(X^\pi(T), Y(T)) | (X^\pi(t), Y(t), Z(t), H(t)) = (x, y, z, h)].
\]

(15)

Then, the value function of the optimization problem is

\[
V(t, x, y, z, h) = \sup_{\pi \in \Pi} V^\pi(t, x, y, z, h), \quad 0 \leq t < T,
\]

(16)
with the boundary condition $V(T, x, y, z, h) = \mathcal{U}(x, y)$. The objective of the insurer is to search for the optimal investment and liability strategy $\pi^*(t) = (\gamma^*(t), k^*(t), l^*(t))$, such that $V^{\pi^*}(t, x, y, z, h) = V(t, x, y, z, h)$.

3.1. Hamilton-Jacobi-Bellman Equation with a delay. We now proceed to solve the investment and risk control problem in (16) by applying the stochastic control method. The following two conditions follow A and Li [1]:

(I) $C = \beta e^{-\alpha u}$, \hspace{1cm} (17)

(II) $B = (A + \beta + \alpha)C$. \hspace{1cm} (18)

We assume these conditions in order to solve the optimization problem in finite dimensions, as the optimal control problem with a delay is, in general, infinitely dimensional. The two conditions also have a corresponding economic interpretation, which will be discussed in later sections. At the same time, analytic solutions for the investment and risk control problem with a delay and CARA preference can be derived with these two conditions. Similar to that in Chang et al. [8], we can also show that the value function $V$ just only depends on $(t, x, y, h)$ in this case, i.e.,

$$V(t, x, y, z, h) = V(t, x, y, h).$$

By the principle of optimal stochastic control, we can derive the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\pi(t) \in \Pi(t)} \mathcal{L}V(t, x, y, h) = 0,$$ \hspace{1cm} (19)

where

$$\mathcal{L}V(t, x, y, h) = V_t(t, x, y, h) + \left[ (\mu - r)\gamma(t) + k(t)(1 - h)\eta + (p - c)\ell(t) \right] + Ax$$

$$+ By + Cz]V_x(t, x, y, h) + \frac{1}{2}(\gamma(t)^2\sigma^2 + \ell(t)^2\sigma^2 - 2\rho a\sigma l(t)\gamma(t))V_{xx}(t, x, y, h)$$

$$+(x - \alpha y - e^{-\alpha u}z)V_y(t, x, y, h) + \lambda(V(t, x - \theta l(t), y, h) - V(t, x, y, h))$$

$$+(V(t, x - k(t)z, y, h + 1) - V(t, x, y, h))hP(1 - h),$$ \hspace{1cm} (20)

with the boundary condition $V(T, x, y, h) = \mathcal{U}(x, y)$.

To solve the HJB equation, we can find the explicit solution to this nonlinear HJB equation with two steps. First, we can decompose the original value function into two parts.

$$V(t, x, y, h) = \begin{cases} 
V(t, x, y, 0), & \text{if } h = 0 \ (\text{the pre-default case}), \\
V(t, x, y, 1), & \text{if } h = 1 \ (\text{the post-default case}). 
\end{cases}$$

Second, we reduce the HJB equation (19) into two simple HJB equations that are satisfied by the pre-default value function and the post-default value function, in which the pre-default value function is dependent on the post-default value function. We can then straightforwardly solve the post-default case by the usual approach, and by applying the expression of the post-default value function, the pre-default case can also be solved.

3.2. Optimal results after default. In this subsection, we derive the optimal investment and risk control strategies to characterize the insurer’s bounded memory after default of the defaultable bond. In order to solve the optimization problem,
we will first give the following lemma.

**Lemma 3.1.** If the condition

\[ p - c + \frac{(\mu - r)\rho a}{\sigma} > \lambda \theta, \tag{21} \]

holds, then the equation for variable \( s \)

\[ \lambda \theta e^{K_1(t)s} + K_2(t)s + K_3(t) = 0, \text{ for } t \in [0, T] \text{ and } s \in \mathbb{R}. \tag{22} \]

there exists a unique positive solution, where \( K_1(t) = \nu \theta e^{(A+\beta)(T-t)}, K_2(t) = \nu \rho a^2(1 - \rho^2)e^{(A+\beta)(T-t)} \), \( K_3 = -(p - c) - \frac{(\mu - r)\rho a}{\sigma} \).

**Proof.** Let \( \hat{K}(s) = \lambda \theta e^{K_1(t)s} + K_2(t)s + K_3 \). Then we have the following:

\[ \hat{K}'(s) = \lambda \theta K_1(t)e^{K_1(t)s} + K_2(t). \]

Since \( K_1(t) = \nu \theta e^{(A+\beta)(T-t)} > 0 \) and \( K_2(t) = \nu \rho a^2(1 - \rho^2)e^{(A+\beta)(T-t)} > 0 \) for all \( t \in [0, T] \), we have \( \hat{K}'(s) > 0 \). According to condition (21), this implies that \( \hat{K}(0) = \lambda \theta + K_3 < 0 \). Moreover, \( K_3 \) is a bounded constant with \( K_3 < \infty \), then \( \hat{K}(s) > 0 \) holds when \( s \) is large enough for all \( t \in [0, T] \). Hence, the function \( \hat{K}(s) \) has a unique and positive zero point.

**Theorem 3.1 (Post-default)** For any \( t \in [\tau \land T, T] \), if the conditions (17), (18) and (21) hold, the optimal investment strategy is given by

\[ \gamma^*(t) = \frac{\mu - r}{\nu \sigma^2} G^{-1}(t) + \frac{\rho a}{\sigma} t^*(t), \tag{23} \]

\[ k^*(t) = 0, \tag{24} \]

and the optimal risk control policy \( t^*(t) \) is defined by Eq.(32). The post-default value function is then given by

\[ V(t, x, y, 1) = -\frac{1}{\nu} e^{-\nu G(t)(x+\beta y)+f(t)}, \tag{25} \]

where

\[ G(t) = e^{(A+\beta)(T-t)}, \tag{26} \]

\[ f(t) = \int_t^T \left[ -\frac{1}{2} \left( \frac{\mu - r}{\sigma^2} \right)^2 + (1 - \rho^2)A^2(t)(s) \right] \nu^2 G(s) - [(p - c) \frac{(\mu - r)\rho a}{\sigma} t^*(s) + \lambda(\nu G(s)\theta l^*(s) - 1)] ds. \tag{27} \]

**Proof.** When \( H(t) = 1 \), the HJB equation reduces to

\[ 0 = \sup_{\pi(t)\in \Pi(t)} \left\{ V_\pi(t, x, y, 1) + [(\mu - r)\gamma(t) + (p - c)l(t)] + Ax + By + Cz \right\} \]

\[ \times V_\pi(t, x, y, 1) + \frac{1}{2} (\gamma(t))^2 \sigma^2 + l(t)^2 \alpha^2 - 2\rho \sigma l(t)\gamma(t) )V_\pi(t, x, y, 1) + (x - \alpha y) \]

\[ -e^{-\alpha u z} )V_\pi(t, x, y, 1) + \lambda( V(t, x - l(t), y, 1) - V(t, x, y, 1) \right\}. \tag{28} \]

We posit the form of the post-default value, followed by

\[ V(t, x, y, 1) = -\frac{1}{\nu} e^{-\nu G(t)(x+\beta y)+f(t)}, \tag{29} \]
with bounded condition $V(T, x, y, 1) = -\frac{1}{\nu}e^{-\nu(x+\beta y)}$ and the functions $G(t)$ and $f(t)$ to be determined later. Thus, we have

$$0 = -\nu G'(t)(x + \beta y) + f'(t) - (Ax + By + Cz)\nu G(t) - (x - \alpha y - e^{-\alpha u})\nu \beta G(t) + \inf_{\pi(t)} \left\{ -(\mu - r)\gamma(t)\nu G(t) + \left[ \frac{1}{2} \gamma^2(t)\sigma^2 - \rho \sigma \gamma(t)(l(t))\nu^2 G^2(t) - (p - c)l(t)\nu G(t) + \lambda(e^{\nu G(t)}\theta l(t) - 1) + \frac{1}{2}(t)\hat{a}^2\nu^2 G^2(t) ) \right\} \right. \right) \}.$$  

We take partial derivatives of the first order of (30) with respect to $\gamma(t)$ and $l(t)$ and obtain the optimal strategies $\pi^*(t)$ as follows:

$$\gamma^*(t) = \frac{\mu - r}{\nu \sigma^2} G^{-1}(t) + \frac{\rho \alpha}{\sigma} l^*(t).$$  

We can show that the optimal investment strategy $l^*(t), t \in [\tau \wedge T, T]$ exists using Lemma 3.1 and is the solution of the equation

$$\lambda \theta e^{K_1(t)l^*(t)} + K_2(t)l^*(t) + K_3(t) = 0,$$

with $K_1(t) = \nu \theta G(t), K_2(t) = \nu \sigma^2(1 - \rho^2)G(t), K_3(t) = -(p - c) - \frac{(\mu - r)\rho a}{\sigma}.$ Substituting $\gamma^*(t)$ and $l^*(t)$ into Eq. (30), we obtain

$$0 = -\nu \gamma'(t)(x + \beta y) - [(A + \beta)x + (B - \alpha \beta)y]{\nu G(t) - (C - \beta e^{-\alpha u})\nu G(t)}$$

$$- \frac{(\mu - r)^2}{2\sigma^2} + \frac{1}{2}(1 - \rho^2)\alpha^2 l^*(t)^2 G^2(t) - [(p - c) + \frac{(\mu - r)\rho a}{\sigma}] l^*(t)\nu G(t)$$

$$+ \lambda(e^{\nu G(t)}\theta l^*(t) - 1) + f'(t).$$  

Applying the conditions $C = \beta e^{-\alpha u}$ and $(A + \beta)\beta = B - \alpha \beta$, we can derive that

$$0 = [G'(t) + (A + \beta)G(t)](x + \beta y),$$

$$0 = -\frac{(\mu - r)^2}{2\sigma^2} + \frac{1}{2}(1 - \rho^2)\alpha^2 l^*(t)^2 G^2(t) - [(p - c) + \frac{(\mu - r)\rho a}{\sigma}] l^*(t)\nu G(t)$$

$$+ \lambda(e^{\nu G(t)}\theta l^*(t) - 1) + f'(t).$$  

Then, we can obtain the closed form for the functions $G(t)$ and $f(t)$ in Eqs. (26) and (27).

3.3. Optimal results before default. In this subsection, we derive the pre-default optimal strategies and value function by using the post-default value function.

**Theorem 3.2 (Pre-default)** For any $t \in [0, \tau \wedge T]$, if the conditions (17), (18) and (21) hold, the optimal investment strategy $k^*(t)$ is given by

$$k^*(t) = \frac{(\ln(\frac{1}{\Delta}) + \Delta - 1)\nu \zeta (T-t) - \Delta + 1}{\nu \zeta} G^{-1}(t).$$  

The optimal investment strategy $\gamma^*(t)$ and risk control policy $l^*(t)$ are defined by Eq. (23) and Eq. (32), respectively. The pre-default value function is then given by

$$V(t, x, y, 0) = -\frac{1}{\nu} e^{-\nu G(t)(x+\beta y)+f(t)+m(t)} = e^{m(t)} V(t, x, y, 1),$$  

where

$$m(t) = \ln(\frac{1}{\Delta}) + \Delta - 1)\nu \zeta (T-t) - \ln(\frac{1}{\Delta}) - \Delta + 1.$$  


we utilize the ansatz with Proof. When $H(t) = 0$, the HJB equation (19) reads as

$$
0 = V_t(t, x, y, 0) + \sup_{\pi \in \Pi} \left\{ \left[ (\mu - r)\gamma(t) + (p - c)l(t) + k(t)(1 - h)\eta \right] + Ax + By + Cz \right\} V_x(t, x, y, 0) + \frac{1}{2}(\gamma(t)^2\sigma^2 + l(t)^2a^2 - 2\rho a\sigma l(t)\gamma(t))V_{xx}(t, x, y, 0)
$$

$$(x - \alpha y - e^{-\alpha u}z)V_y(t, x, y, 0) + \lambda(V(t, x - \theta(t), y, 0) - V(t, x, y, 0)) + \lambda f(t, (t, x, y, 0))h^P \right\},
$$

with boundary condition $V(T, x, y, 0) = -\frac{1}{2}e^{-\nu(x + \beta y)}$. Following Browne (1995), we utilize the ansatz

$$
V(t, x, y, 0) = -\frac{1}{\nu}e^{-\nu G_0(t)(x + \beta y)} + f_0(t), \text{ for } t \in [0, \tau \wedge T],
$$

with $G(T) = 1$ and $f_0(0) = 0$. Putting (40) into the equation (39) and rearranging the terms yields:

$$
0 = -\nu G_0'(t)(x + \beta y) + f_0'(t) - (Ax + By + Cz)\nu G_0(t) - (x - \alpha y - e^{-\alpha u})
$$

$$
\times \nu G_0(t) + \inf_{\pi(t)} \left\{ \left[ (\mu - r)\gamma(t)\nu G_0(t) + \frac{1}{2}\gamma^2(t)\sigma^2 - \rho a\sigma l(t)\gamma(t)\nu^2 G_0^2(t) \right]
$$

$$
- (p - c)l(t)\nu G_0(t) + \lambda(e^{\nu G_0(t)\theta(t)} - 1) + \frac{1}{2}l^2(t)\alpha^2 G_0^2(t) - \nu G_0(t)k(t)\eta
$$

$$
+ (e^{-\nu[1 - G_0(t)]}(x + \beta y) + G_0(t)k(t)\xi + f_1(t) - f_0(t) - 1)h^P \right\}. \tag{41}
$$

Using the first-order conditions for a regular interior minimizer of (41), we have

$$
\gamma^*(t) = \frac{\mu - r}{\nu \sigma^2} G_0^{-1}(t) + \frac{\rho a \sigma}{\sigma} l^*(t), \tag{42}
$$

$$
k^*(t) = \frac{\ln \left( \frac{G_0'(t)}{G(t)} \right) + (G(t) - G_0(t))(x + \beta y) + f_1(t) - f_0(t)}{\nu \zeta} G_0^{-1}(t), \tag{43}
$$

and risk control policy $l^*(t)$ is defined by

$$
\lambda \theta e^{K_{01}(t)l^*(t)} + K_{02}(t)l^*(t) + K_{03} = 0, \tag{44}
$$

with $K_{01}(t) = \nu \theta G_0(t), K_{02}(t) = \nu a^2(1 - \rho^2) G_0(t)$, and $K_{03} = -(p - c) - \frac{(\mu - r)\rho a}{\sigma}$. Substituting these optimal rules into (41), we obtain

$$
0 = -\nu G_0'(t)(x + \beta y) - [(A + \beta)x + (B - \alpha \beta)y]\nu G_0(t) - (C - \beta e^{-\alpha u})\nu G_0(t)z
$$

$$
- \frac{(\mu - r)^2}{2\sigma^2} + \frac{1}{2}(1 - \rho^2)\alpha^2 G_0^2(t) + \frac{1}{2}l^2(t)\alpha^2 G_0^2(t) - [(p - c) + \frac{(\mu - r)\rho a}{\sigma}]l^*(t)\nu G_0(t)
$$

$$
+ \lambda(e^{\nu G_0(t)\theta(l^*)} - 1) + f_0'(t) - \frac{\eta}{\zeta} \ln \left( \frac{G_0'(t)}{G(t)} \right) + h^P \frac{G_0(t)}{G(t)} - 1)
$$

$$
+ \frac{(G(t) - G_0(t))(x + \beta y)}{\zeta} \eta + \frac{f(t) - f_0(t)}{\zeta} \eta, \tag{45}
$$
From assumptions (17) and (18), we can divide (45) into two differential equations:

\[ 0 = \left[ G_0'(t) + (A + \beta)G_0(t) \right] (x + \beta y) + \frac{(G(t) - G_0(t))(x + \beta y)}{\zeta}, \]  
\[ 0 = -\frac{(\mu - r)^2}{2\sigma^2} + \frac{1}{2}(1 - \rho^2)a^2 \nu^2 l^r(t)^2 G_0^2(t) + \lambda(e^{\psi G_0(t)} \theta^r(t) - 1) + f_0'(t) \]  
\[ -\frac{\eta}{\zeta} \ln(\frac{G_0(t)}{G(t)} \Delta) - [(p - c) + \frac{(\mu - r)\rho a}{\sigma}] l^r(t) \nu G_0(t) + h_p(\frac{G_0(t)}{G(t)} \Delta - 1) \]  
\[ + \frac{(G(t) - G_0(t))(x + \beta y)}{\zeta} \eta + \frac{f(t) - f_0(t)}{\zeta} \eta. \]  

(46)

(47)

Obviously, we find that \( G_0(t) = G(t) \). Thus, the pre-default optimal investment strategy \((\gamma^*(t))\) and risk control strategy \((l^*(t))\) are consistent with the post-default results in Theorem 3.1.

Next, we can derive the expression for the function \( f_0(t) \). Similar to Zhu et al. (2015), we define \( m(t) = f_0(t) - f(t) \). Then:

\[ m'(t) = f_0'(t) - f'(t) = \frac{f_0(t) - f(t)}{\zeta} \eta - \frac{\eta}{\zeta} \ln(\frac{1}{\Delta}) + h_p(\frac{1}{\Delta} - 1). \]  

(48)

Applying the boundary condition \( m(T) = f_0(T) - f(T) = 0 \), we have

\[ m(t) = (\ln(\frac{1}{\Delta}) + \Delta - 1)e^{-\frac{\Delta}{2}(T-t)} - \ln(\frac{1}{\Delta}) - \Delta + 1. \]  

(49)

The proof is completed.

Remark 3.1 Under the CARA utility function assumption, the risk process \( U(t) \) and price process \( S(t) \) are uncorrelated to the price process \( p(t, T_1) \) of the defaultable bond. Hence, the optimal investment strategy \((\gamma^*(t))\) and risk control policy \((l^*(t))\) are independent of the default time \( \tau \) of the defaultable bond.

Based on the results in Theorem 3.1 and Theorem 3.2, we can obtain the following theorem, 3.3, straightforwardly.

Theorem 3.3 For the investment and risk control optimization problem with a delay shown in Eq. (16), the optimal strategies for an CARA insurer are given by

\[ k^*(t) = \begin{cases} \ln(\frac{1}{\Delta}) + \Delta - 1, & t \in [0, \tau \wedge T], \\ \gamma, & t \in [\tau \wedge T, T]. \end{cases} \]  

(50)

The optimal investment strategy \((\gamma^*(t))\) and risk control policy \((l^*(t))\) are defined by Eq. (23) and Eq. (32), respectively. The value function is

\[ V(t, x, y, h) = -\frac{1}{\gamma} \exp(-\gamma G(t)(x + \beta y) + f(t) + m(t)(1 - h)), \text{ for } h = 0, 1, \]  

(51)

where the expressions \( G(t), f(t) \) and \( m(t) \) are given in Eq. (26), Eq. (27) and Eq. (38), respectively.

Remark 3.2 From the proof of Theorems 3.1 and 3.2, we find that the conditions (17) and (18) play key roles in deriving the closed solution for the optimal investment and risk control problem with a delay. This indicates that the insurer first selects the parameter \( \alpha \) and the delay time \( u \) to calculate the average performance of wealth \( Y(t) \) and the pointwise performance of wealth \( Z(t) \) at each \( t \in [0, T] \). Next, the insurer chooses the weight \( \beta \) between \( X \) and \( Y \) in the final performance measure and then uses Eqs. (17) and (18) to determine \( B \) and \( C \).

Corollary 3.1 (i) The pre-default value function \( V(t, x, y, 0) \) is always greater than the post-default value function \( V(t, x, y, 1) \).
(ii) When $\frac{1}{\Delta} = \frac{hQ}{hP} = 1$, then $V(t, x, y, 0) = V(t, x, y, 1)$ and $k^*(t) = 0$; whereas $k^*(t) > 0$ if $\frac{1}{\Delta} > 1$.

**Proof.** Using the expressions of the pre-default value function $V(t, x, y, 0)$ and post-default value function $V(t, x, y, 1)$, we can obtain these results straightforwardly. A detailed proof is omitted.

**Remark 3.3** The optimal risk control strategy $l^*(t)$ and the optimal investment strategy $\gamma^*(t)$ are independent of the default time $\tau$. This is because either the claim process of the insurer or the stock-price process is uncorrelated with the corporate bond’s price process. The insurer will move the recovered value from the default bond into the risk-free asset after the default of the defaultable bond.

**Remark 3.4** If $\beta = u = \alpha = 0$ and $\frac{1}{\Delta} = 1$, the optimal strategy $\pi^*(t) = (\gamma^*(t), k^*(t), l^*(t))$ is consistent with Theorem 5.1 in Zou and Cadenillas [29], without delay.

### 3.4. Verification theorem.

Using dynamic programming methods, we provide a verification theorem to guarantee that a solution of the HJB equation is indeed the value function. In this subsection, we give the verification theorem as follows.

**Theorem 3.3** Let $F(t, x, y, h) \in C^{1,2,1}$, $h = 0, 1$, be a classical solution to the HJB equation (19). The value function $V(t, x, y, h)$ is then given by (51), which coincides with $F(t, x, y, h)$. That is,

$$F(t, x, y, h) = V(t, x, y, h).$$

Furthermore, let $\pi^*(t) = (\gamma^*(t), k^*(t), l^*(t))$, such that

$$\mathcal{L}^\pi(t)V(t, x, y, h) = 0$$

holds for all $(t, x, y) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, $h = 0, 1$, $\gamma^*(t), k^*(t)$ and $l^*(t)$ are then the optimal strategies.

**Proof.** This proof is similar to the theorem in Zhu et al. [28]; we omit it here.

### 4. Sensitivity analysis.

To illustrate the sensitivity of the optimal strategies with respect to the model parameters, we provide numerical experiments in this section. We use the following model parameter values.

| Symbol | Value | Symbol | Value |
|--------|-------|--------|-------|
| $\alpha$ | 0.1 | $\nu$ | 1 |
| $u$ | 5 | $\lambda$ | 0.3 |
| $\beta$ | 0.3 | $\theta$ | 0.1 |
| $r$ | 0.05 | $\eta$ | 0.4 |
| $\zeta$ | 0.5 | $p$ | 1 |
| $\Delta$ | 0.25 | $c$ | 0.5 |
| $\mu$ | 0.15 | $\sigma$ | 0.2 |

We now examine the sensitivity of the optimal investment strategy $k^*(t)$ with respect to the delay parameters $u$, $\alpha$ and $\beta$, by varying the parameters within reasonable intervals. Fig. 1 shows a negative correlation between the optimal investment strategy $k^*(t)$ and the delay parameters $u$, $\alpha$ and $\beta$, respectively. A greater value of $u$ yields a smaller value of the optimal investment strategy $k^*(t)$; moreover, $k^*(t)$ decreases as the value of $\alpha(\beta)$ increases. These results indicate that the bounded memory behavior may make insurer’s investment decision becoming more cautious. Because the larger memory parameter($u, \alpha, \beta$) means that insurer care more about the performance of the historic wealth, and the larger delay parameter $u$ and $\alpha$. 
In this case, the insurer in order to maximize the probability of generating greater historical average terminal wealth $Y(T)$, the insurer will hold less defaultable bond shares to smooth the wealth process.

![Image of Figure 1](image1.png)

**Fig. 1.** Effect of delay parameters $u$, $\alpha$ and $\beta$ on the optimal investment strategy $k^*(t)$.

![Image of Figure 2](image2.png)

**Fig. 2.** Effect of delay parameters $u$, $\alpha$ and $\beta$ on the optimal investment strategy $\gamma^*(t)$.

Figs. 2 and 3 provide graphical illustrations of the effect of the delay parameters on the optimal investment strategy $\gamma^*(t)$ and risk control strategy $l^*(t)$; we also observe insurer's optimal investment strategy $\gamma^*(t)$ and liability strategy $l^*(t)$.
increase as the delay parameter $\alpha$ decreases. Note that a larger $\alpha$ implies that less weight is taken up by wealth earlier in time, and earlier wealth is more important in constructing average delay wealth $Y(t)$. In addition, the larger the parameter $u$ is, the more stable the average delayed wealth $Y(t)$. Hence, the insurer will reduce investment strategies and the liability ratio if $u$ is larger.

Fig. 3. Effect of delay parameters $u$, $\alpha$ and $\beta$ on the optimal risk control $l^*(t)$.

Fig. 4. Value functions with respect to $x$.

Fig. 5. Effect of delay parameters $\beta$ on the pre-default value function.
A more interesting finding from Figs. 1-3 is that the effects of delay parameters on optimal strategies are more prominent with greater distance from the terminal time. Thus, under a more uncertain environment, historical experience would have a higher impact on the insurer’s investment and risk control strategy.

Based on the results presented in Figs. 4-6, we observe the effects of model parameters on value function \( V(t, x, y, h) \). From Fig. 4, we find that the pre-default value function \( V(0, x, y, 0) \) is always larger than the post-default value function \( V(0, x, y, 0) \), which is in accordance with the result of Corollary 3.1. Moreover, this result demonstrates that the additional income result from the investment in the defaultable bond is positive. Fig. 5 shows that the value function is increasing in delay parameter \( \beta \) and historical average wealth \( y \). It indicates that the better the historical performance of insurers’ wealth is, the higher the utility of the insurer. Finally, Fig. 6 shows that the larger the default risk premium \( 1/\Delta \) or smaller the default loss rate \( \zeta \), the larger the pre-default value function \( V(0, x, y, 0) \). In this case, the default assets are becoming more attractive to insure, which will improve insurer’s utility. This result is also consistent with Zhu et al. [28] and Deng et al. [9].

5. Conclusion. In this study, we considered the determination of the optimal investment and risk control policies of an insurance firm when the stochastic dynamics of the underlying cash flow are characterizable as a jump-diffusion process with a delay. The insurer may invest in a financial market, which consists of a risk-free bond, a stock asset and a defaultable bond, and the insurer’s goal is to choose the optimal investment and risk control (liabilities/liability ratio) strategies to maximize his expected CARA utility of terminal wealth. Similar to Zhu et al. [28] and A and Li [1], we explicitly solved the optimization problem for a special case. The results show that the pre-default value function is larger than the post-default value function, which implies that the investment of the defaultable bond can improve the insurer’s expected CARA utility. Moreover, the insurer’s memory behavior (delay) has a negative effect on his investment and risk control decisions.

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