A GENERALIZATION OF THE TRISTRAM-LEVINE KNOT SIGNATURES AS A SINGULAR FURUTA-OHTA INVARIANT FOR TORI

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Abstract. Given a knot $K$ inside an integer homology sphere $Y$, the Casson-Lin-Herald invariant can be interpreted as a signed count of conjugacy classes of irreducible representations of the knot complement into $SU(2)$ which map the meridian of the knot to a fixed conjugacy class. It has the interesting feature that it determines the Tristram-Levine signature of the knot associated to the conjugacy class chosen.

Turning things around, given a 4-manifold $X$ with the integral homology of $S^1 \times S^3$, and an embedded torus $T$ inside $X$ such that $H_1(T; \mathbb{Z})$ surjects onto $H_1(X; \mathbb{Z})$, we define a signed count of conjugacy classes of irreducible representations of the torus complement into $SU(2)$ which satisfy an analogous fixed conjugacy class condition to the one mentioned above for the knot case. Our count recovers the Casson-Lin-Herald invariant of the knot in the product case, thus it can be regarded as implicitly defining a Tristram-Levine signature for tori.

This count can also be considered as a singular Furuta-Ohta invariant, and it is a special case of a larger family of Donaldson invariants which we also define. In particular, when $(X, T)$ is obtained from a self-concordance of a knot $(Y, K)$ satisfying an admissibility condition, these Donaldson invariants are related to the Lefschetz number of an Instanton Floer homology for knots which we construct. Moreover, from these Floer groups we obtain Frøyshov invariants for knots which allows us to assign a Frøyshov invariant to an embedded torus whenever it arises from such a self-concordance.

1. Introduction

An observation of some analogies: defining $\lambda_{FO}(X, T, \alpha)$.

The main impetus behind this work stems from the following observation: given an integer homology sphere $Y$, Casson defined [1] an invariant $\lambda_C(Y) \in \mathbb{Z}$ which morally can be regarded as $1/2$ of a signed count of conjugacy classes of irreducible representations $\pi_1(Y) \to SU(2)$. Moving one dimension up, if $X$ is a four manifold with the same integral homology as $S^1 \times S^3$, i.e., $H_*(X; \mathbb{Z}) \cong H_*(S^1 \times S^3; \mathbb{Z})$, Furuta and Ohta defined [27] a Casson-type invariant $\lambda_{FO}(X) \in \mathbb{Z}$ which again can morally be interpreted as a signed count of conjugacy classes of irreducible representations $\pi_1(X) \to SU(2)$. Strictly speaking, $X$ must satisfy additional conditions besides being a homology $S^1 \times S^3$, but it will always be the case that if one takes $X = S^2 \times Y$, then $\lambda_{FO}(X)$ is always well defined, and can be made to agree with $\lambda_C(Y)$.

Now, if we consider the case of a knot $K$ inside $Y$, one can follow the same strategy and try to define a signed count of conjugacy classes of irreducible representations $\pi_1(Y \setminus K) \to SU(2)$. As will become clear soon, in this case it is natural to fix the conjugacy class where the meridian of the knot $\mu_K$ is being sent, regarded as the generator of the reducible representations $H_1(Y \setminus K) \cong \mathbb{Z}[\mu_K] \to SU(2)$. This conjugacy class is specified by a (holonomy) parameter $\alpha \in (0, 1/2)$, and whenever $\Delta_K(e^{-4\pi i\alpha}) \neq 0$, where $\Delta_K$ is the Alexander polynomial of the knot, Herald showed [30, Theorem 0.1] that a Casson-type count of irreducible representations can be made, which we will denote as $\lambda_{CLH}(Y, K, \alpha) \in \mathbb{Z}$. Here the $L$ in $\lambda_{CLH}$ stands for Lin, who had studied before Herald...
the case where the meridian is sent to a trace zero matrix, which corresponds to the parameter \( \alpha = 1/4 \) (and using a symplectic rather than gauge theoretic approach).

The invariant \( \lambda_{CLH} \) recovers the Tristram-Levine knot signatures \( \sigma_K \), in the sense that (with our orientation conventions) \( \lambda_{CLH}(Y, K, \alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha}) \). This clearly begs the question:

**Question 1.** Given an embedded torus \( T \) inside a four manifold \( X \) with the integer homology of \( S^1 \times S^3 \) and a number \( \alpha \in (0, 1/2) \), which conditions must be imposed on \( T \) and \( \alpha \) so that one can define an invariant \( \lambda_{FO}(X, T, \alpha) \in \mathbb{Z} \) that satisfies the property that for any knot \( K \subset Y \) and \( \alpha \) such that \( \Delta_K(e^{-4\pi i\alpha}) \neq 0 \), \( \lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha) \) coincides with \( \lambda_{CLH}(Y, K, \alpha) \) (perhaps up to rescaling)?

Before stating such conditions, we recall quickly why it is possible to define \( \lambda_{CLH}(Y, K, \alpha) \), since the reader may be less aware of this construction. In fact, as far as we can tell there was no standard notation for this invariant, which is why we decided to name it \( \lambda_{CLH}(Y, K, \alpha) \). More context and motivation can be found in an “annotated” version of this paper posted on the author’s website [18].

Given a knot \( K \) inside \( Y \), as we mentioned before we want to define a (signed) count of irreducible representations of \( \pi_1(Y \setminus K) \) into \( SU(2) \), analogous to the definition of the Casson invariant \( \lambda_C(Y) \). For any Casson type count, we need to understand first the reducible representations, given that they correspond to singular points in the character variety \( \mathcal{R}(Y \setminus K, SU(2)) = \text{hom}(\pi_1(Y \setminus K), SU(2))/SU(2) \) (or the space of flat \( SU(2) \) connections mod gauge depending on one’s preference).

The reducible representations are determined by \( H_1(Y \setminus K; \mathbb{Z}) \cong \mathbb{Z}[\mu_K] \). In particular, a reducible representation \( \rho \) is completely specified by \( A = \rho(1) \in SU(2) \), and up to conjugacy is determined by its trace \( \text{tr}(A) \in [-2, 2] \). Thus the reducible representations inside \( \mathcal{R}(Y \setminus K, SU(2)) \) are in bijection with \( [-2, 2] \).

This means that inside \( \mathcal{R}(Y \setminus K, SU(2)) \) the reducible representations are not isolated. In order to isolate them, we can decompose this space as \( \mathcal{R}(Y \setminus K, SU(2)) = \bigcup_\alpha \mathcal{R}_\alpha(Y \setminus K, SU(2)) \), where \( \mathcal{R}_\alpha(Y \setminus K, SU(2)) \) consists of conjugacy classes of representations which map the meridian \( \mu_K \) to the conjugacy class of the matrix

\[
\mu_K \to \begin{pmatrix} e^{-2\pi i\alpha} & 0 \\ 0 & e^{2\pi i\alpha} \end{pmatrix}
\]

Since the trace of this matrix is \( 2 \cos(2\pi \alpha) \), it suffices to take \( \alpha \in [0, 1/2] \) to exhaust all the possible conjugacy classes where the meridian can be sent. We will assume that \( \alpha \in (0, 1/2) \) since the endpoints correspond to representations of \( \pi_1(Y) \) into \( SU(2) \) or \( SO(3) \) (so \( K \) is not involved in those cases). Inside each \( \mathcal{R}_\alpha(Y \setminus K, SU(2)) \) there will be only one reducible representation \( \theta_\alpha \), so trying to make a (signed) count \( \lambda_{CLH}(Y, K, \alpha) \) of the irreducible elements inside \( \mathcal{R}_\alpha(Y \setminus K, SU(2)) \) is very similar to the problem of defining \( \lambda_C(Y) \) as a (signed) count of irreducible elements inside \( \mathcal{R}(Y, SU(2)) \), which also contains only one reducible \( \theta \), the trivial representation of \( \pi_1(Y) \) into \( SU(2) \).

The only caveat with this analogy is that contrary to the case of \( Y \), where \( \theta \) is automatically isolated from the irreducible representations, it may be the case that \( \theta_\alpha \) is not isolated from the irreducible representations inside \( \mathcal{R}_\alpha(Y \setminus K, SU(2)) \), thus making it impossible to define \( \lambda_{CLH}(Y, K, \alpha) \). In fact, as we will review in section 4 whenever \( \Delta_K(e^{-4\pi i\alpha}) \neq 0 \), where \( \Delta_K \) is the Alexander polynomial of the knot \( K \), we can define \( \lambda_{CLH}(Y, K, \alpha) \), i.e., for such \( \alpha \) the reducible \( \theta_\alpha \) is isolated.

If we think of the representation \( \theta_\alpha \) as corresponding to a flat \( SU(2) \) connection \( A_\alpha \) on the trivial bundle \( E = (Y \setminus K) \times \mathbb{C}^2 \to Y \setminus K \), which is the point of view we will take, the condition
that \( \Delta_K(e^{-4\pi i \alpha}) \neq 0 \) is equivalent to the vanishing of the cohomology group with local coefficients \( H^1(Y\setminus K; L_\alpha^{\otimes 2}) \), where \( A_\alpha \) determines a complex line bundle \( L_\alpha \to Y\setminus K \) and a splitting \( E = L_\alpha \oplus L_\alpha^{-1} \).

Going back to the case of an embedded torus \( T \) inside \( X \), we will explain in Section 3 the analogue of the spaces \( \mathcal{R}_\alpha(Y\setminus K, SU(2)) \), which we denote \( \mathcal{R}_\alpha(X\setminus T, SU(2)) \). The point is that we need to understand again the reducible representations of \( \pi_1(X\setminus T) \) to \( SU(2) \), which are completely determined by \( H_1(X\setminus T; \mathbb{Z}) \). The natural condition for our problem is to assume that \( H_1(T; \mathbb{Z}) \) surjects onto \( H_1(X; \mathbb{Z}) \), so that homologically \( T \) behaves like \( S^1 \) times a knot. This condition also tells us that \( H_1(X\setminus T; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z} \), although for fixed \( \alpha \) only one of these \( \mathbb{Z} \) factors will be of importance. This means that inside \( \mathcal{R}_\alpha(X\setminus T, SU(2)) \) the reducible representations should form a one dimensional family, which is exactly what happens when one is trying to define \( \lambda_{FO}(X) \) using \( \mathcal{R}(X, SU(2)) \). Given these preliminaries we can finally state our answer to Question 1.

**Theorem 2.** Suppose that \( X \) is a closed oriented four manifold with the integral homology of \( S^1 \times S^3 \). Let \( T \) denote an embedded and oriented torus such that \( H_1(T; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \) is a surjection. Consider a rational value \( \alpha \in \mathbb{Q} \cap (0,1/2) \) such that for all reducible representations \( \rho : \pi_1(X\setminus T) \to SU(2) \) satisfying the holonomy condition 4 determined by \( \alpha \), i.e., elements of \( \mathcal{R}_\alpha(X\setminus T, SU(2)) \), we have that \( H^1(X\setminus T; L_\rho^{\otimes 2}) \) vanishes. Here \( E = L_\rho \oplus L_\rho^{-1} \) is the decomposition induced by the flat connection \( A_\rho \) associated to \( \rho \), and \( E \) is the trivial \( SU(2) \) bundle over \( X\setminus T \).

Then we can define a degree-zero Donaldson invariant, denoted the **singular Furuta-Ohta invariant** \( \lambda_{FO}(X,T,\alpha) \), which takes even values, i.e, \( \lambda_{FO}(X,T,\alpha) \in 2\mathbb{Z} \).

It has the property that for \( X = S^1 \times Y \), \( T = S^1 \times K \), for an oriented knot \( K \) inside an integer homology sphere \( Y \), and a value of \( \alpha \in \mathbb{Q} \cap (0,1/2) \) such that \( \Delta_K(e^{-4\pi i \alpha}) \neq 0 \), then \( \lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha) \) can be defined, and \( \lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha) = 2\lambda_{CLH}(Y,K,\alpha) \).

**Remark 3.** The condition that \( H^1(X\setminus T; L_\rho^{\otimes 2}) \) vanishes can be equivalently stated in terms of the Alexander polynomial \( \Delta_T = \Delta_{X\setminus T} \) of the knot complement. Namely, we would require that \( \Delta_T(\hat{\rho}) \neq 0 \) for all the characters associated to the reducible representations \( \rho \) which satisfy the holonomy condition determined by \( \alpha \). The notation \( \Delta_T(\hat{\rho}) \neq 0 \) is explained in the Appendix of the annotated version [13] although we will not use the interpretation in terms of the Alexander polynomial in this paper.

It is important to point out that in order to study the representations of \( \pi_1(X\setminus T) \) into \( SU(2) \), we will use the framework of Kronheimer and Mrowka’s papers [37] [39] on singular gauge theory. We will review the novel features of this approach in the next section, but the key points are as follows. For pairs \( (X, \Sigma) \) consisting of a closed, oriented four manifold \( X \) and an oriented surface \( \Sigma \), the \( SU(2) \) bundles \( E \) over \( (X, \Sigma) \) are now classified by a pair of two integers \( (k,l) \), called the instanton and monopole numbers respectively.

Write \( E(k,l) \) for the corresponding \( SU(2) \) bundle associated to the pair \( (k,l) \). Then for each such \( E(k,l) \), we can study the space of connections \( \mathcal{A}(E(k,l),\alpha) \) which have a prescribed singular behavior along \( \Sigma \). As before \( 0 < \alpha < 1/2 \), and the solutions of the anti-self-dual connections \( F_A^+ = 0 \) for \( A \in \mathcal{A}(E(0,0),\alpha) \) can again be interpreted (mod gauge) as corresponding to representations of \( \pi_1(X\setminus T) \) to \( SU(2) \) which map the “meridian” of the torus \( \mu_T \) to a specific conjugacy class, if we take \( \Sigma = T \). In general, the solutions of \( F_A^+ = 0 \) for \( A \in \mathcal{A}(E(k,l),\alpha) \) will be called \( \alpha \)-ASD connections. The moduli space of (perturbed) \( \alpha \)-ASD connections modulo gauge will be denoted \( \mathcal{M}(X, \Sigma, k,l,\alpha) \). Finally, a gauge equivalence class will normally be denoted as \( [A] \), instead of \( A \).

For technical reasons, we need to use an orbifold metric along \( \Sigma \). This means that there will be an integer parameter \( \nu \) such that the orbifold metric along \( \Sigma \) has a cone angle of \( 2\pi/\nu \). Each
\(\alpha\) determines a set of allowable cone parameters \(\{\nu\}\), but there is no cone parameter that works simultaneously for all values of \(\alpha\). Moreover, it is convenient to take the parameter \(\alpha\) as a rational number, i.e., \(\alpha \in \mathbb{Q} \cap (0, 1/2)\), so that we can talk about orbifold connections.

From this point of view, \(\lambda\) \((X, T, \alpha)\) corresponds to a signed count of the irreducible elements inside the moduli space \(M(X, \Sigma, 0, 0, \alpha)\). Strictly speaking, we should write \(\lambda\) \((X, T, \alpha, \nu)\) and \(M(X, \Sigma, 0, 0, \alpha, \nu)\) since we must always choose first a cone angle compatible with \(\alpha\) before defining our invariants. However, as we will mention near the end of the introduction we expect the invariants to be independent of the cone angle used which is why we will suppress the dependence on \(\nu\).

In section 8 we will give some examples of \(\lambda\) \((X, T, \alpha)\), which arise from the mapping tori associated to finite group actions on homology spheres, as well as certain circle bundles over a 3-manifold with the homology of \(S^1 \times S^2\). It will become clear that these are the orbifold versions of the calculations of \(\lambda\) \((X)\) done in 58 59.

**An observation of more analogies: defining \(HI(Y, K, \alpha)\) and the splitting formula.**

Now we describe the second motivation for this project. Thanks to Taubes 65, it is well known that for an integer homology sphere \(Y\), the instanton Floer homology \(HI(Y)\) categorifies the Casson invariant in the sense that \(\chi(HI(Y)) = 2\lambda_C(Y)\).

Moreover, Frøyshov defined 21 a refinement of \(HI(Y)\) which is known as the reduced instanton Floer homology \(HI_{red}(Y)\). From it one can define the Frøyshov \(h\)-invariant \(h(Y) = \frac{1}{2}(\chi(HI_{red}(Y)) - \chi(HI(Y)))\), which is the precursor to the \(d\)-invariant in Heegaard Floer homology and the Frøyshov invariant in monopole Floer homology. We should notice that our conventions are slightly different from those of Frøyshov, since we are working with the homology (not cohomology) version of instanton Floer homology.

By the TQFT-like features of instanton Floer homology, a homology cobordism \(W : Y_1 \rightarrow Y_2\) between two integer homology spheres will induce a map between the corresponding Floer homologies \(HI(W) : HI(Y_1) \rightarrow HI(Y_2)\). In particular, for a homology cobordism \(W : Y \rightarrow Y\) from \(Y\) to itself, there are maps \(HI(W) : HI(Y) \rightarrow HI(Y)\) and \(HI_{red}(W) : HI_{red}(Y) \rightarrow HI_{red}(Y)\) for the unreduced and reduced instanton Floer homologies. This is an interesting case since closing up \(W\) one obtains a four manifold \(X\) which is a homology \(S^1 \times S^3\) and for which \(\lambda\) \((X)\) can be defined. In this case, Anvari proved 2 Theorem A] a splitting formula for \(\lambda\) \((X)\) in terms of the Lefschetz number \(\text{Lef}(HI(W))\) of the cobordism map, which reads (in our conventions)

\[
\lambda\) \((X) = \frac{1}{2} \text{Lef}(HI(W)) = \frac{1}{2} \text{Lef}(HI_{red}(W)) - h(Y)
\]

and is the analogue of the splitting formula 48 Theorem A] Lin, Ruberman and Saveliev proved for a similarly-constructed invariant \(\lambda_{SW}(X)\) which is defined using the Seiberg-Witten equations instead. Moreover, an argument due to Frøyshov 24 Theorem 8] for the case of \(\lambda_{SW}(X)\) (and which is readily adapted to \(\lambda\) \((X)\) shows that if \(X\) is obtained as the closure of a different homology cobordism \(W' : Y' \rightarrow Y''\) then \(h(Y') = h(Y'')\) and thus it makes sense to talk about the \(h\)-invariant of \(X\), which we denote as \(h(X)\). In this way the splitting formula for \(\lambda\) \((X)\) reads

\[
\lambda\) \((X) + h(X) = \frac{1}{2} \text{Lef}(HI_{red}(W))
\]

This naturally leads to the following question:

**Question 4.** Is it possible to define instanton Floer homologies \(HI(Y, K, \alpha), HI_{red}(Y, K, \alpha)\) for a value of \(\alpha\) satisfying \(\Delta_K(e^{-4\pi i \alpha}) \neq 0\,\), and corresponding Frøyshov \(h\)-invariants \(h(Y, K, \alpha)\) for the knot \(K\) so that whenever \((W, \Sigma) : (Y, K) \rightarrow (Y, K)\) is a self-concordance of \(K\), there is a splitting
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A formula involving $\lambda_{FO}(X,T,\alpha)$ and $\text{Lef}(HI(Y,Y,\alpha))$ (respectively $\text{Lef}(HI_{red}(Y,K,\alpha))$, $h(Y,K,\alpha)$)? Moreover, can one define $HI(Y,K,\alpha)$ in such a way that $\chi(HI(Y,K,\alpha)) = \lambda_{CLH}(Y,K,\alpha)$?

As the reader may expect, the answers to all of these questions are mostly in the affirmative, however, we need to point out some details first. Our analytical framework will be based again on the singular gauge theory developed by Kronheimer and Mrowka. In particular this means that we will use a metric with a cone angle along the knot which will give rise to the structure of an orbifold.

The reader may be aware that Collin and Steer [11] had developed almost two decades ago a similar version of instanton Floer homology for knots. Many aspects of our construction are identical to theirs, however, as Kronheimer and Mrowka point out in [41, Section 1.2.5], unless similar version of instanton Floer homology for knots. Many aspects of our construction are identical to theirs. However, we need to point out some details first. Our analytic framework will be based again on the singular gauge theory developed by Kronheimer and Mrowka. In particular this means that we will use a metric with a cone angle along the knot which will give rise to the structure of an orbifold.

Furthermore, can one define $HI(Y,K,\alpha)$ in such a way that $\chi(HI(Y,K,\alpha)) = \lambda_{CLH}(Y,K,\alpha)$?

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The reader may be aware that Collin and Steer [11] had developed almost two decades ago a similar version of instanton Floer homology for knots. Many aspects of our construction are identical to theirs, however, as Kronheimer and Mrowka point out in [41, Section 1.2.5], unless $\alpha = 1/4$ (which corresponds to the case $k/n = 1/4$ in Collin and Steer’s paper), the differential needed to define the chain complex which gives rise to the Floer groups may be ill-defined, since certain energy bounds for the moduli spaces cannot be guaranteed, which are needed to appeal directly to some compactness theorems which will be recalled in Section 4. This is usually referred as saying that for $\alpha \neq 1/4$ we are in a non-monotone situation. Fortunately, as Kronheimer and Mrowka also point out in that same section of their paper, there is a way to get out of this conundrum provided one is willing to work with an appropriate local coefficient system. The finite dimensional analogue of this situation was first studied by Novikov in [55], where an analogue of Morse theory for the case of circle valued functions on a finite dimensional manifold was developed.

Although the Chern-Simons functional is typically circle valued (on the space of connections mod gauge that is), the reason why instanton Floer homology is typically referred as an infinite dimensional Morse theory, rather than an infinite dimensional Morse-Novikov theory, is that the monotonicity condition usually holds, so the behavior of the Chern Simons functional is more similar to the case of ordinary Morse theory and not the slightly more complicated Morse-Novikov theory. On the other hand, the use of Floer-Novikov theories is certainly not a new thing on the symplectic versions of Floer homologies, and was first investigated by Hofer and Salamon [31] (see also [46, 69, 25] for more recent, but in no way exhaustive references).

As will be explained in Section 3, there are at least three natural choices of local (Novikov) systems we could use in our situation, although most of the time we will stick with what we call the Universal Novikov/Local system, since it seems to require the fewest amount of extra choices (for functoriality purposes, that is). The end result will be that rather than defining the instanton chain complex over a vector space, like $\mathbb{Q}$ or $\mathbb{C}$, we will need to define it over a much larger vector space $\Lambda$ (the Novikov field), but in the end the groups $HI(Y,K,\alpha)$ we will produce continue to be finite dimensional over $\Lambda$, so formally many statements continue to hold. For example, the Euler characteristic $\chi_{\Lambda}(HI(Y,K,\alpha))$ of $HI(Y,K,\alpha)$ with respect to the field $\Lambda$ recovers $\lambda_{CLH}(Y,K,\alpha)$.

What seems to be new is the idea that one can also define a reduced version $HI_{red}(Y,K,\alpha)$ of these groups, although for the case of $\alpha = 1/4$, Christopher Scaduto and Aliakbar Daemi had also realized this independently (and earlier as well [14]). We expect that their techniques could be used to further understand the family of groups $HI(Y,K,\alpha)$, for arbitrary values of $\alpha$. In any case, here is the answer to Question 4.

**Theorem 5.** Suppose that $K$ is an oriented knot inside an integer homology sphere $Y$ and that a parameter $\alpha \in \mathbb{Q} \cap (0,1/2)$ is chosen so that $\Delta_K(e^{-4\pi \alpha}) \neq 0$.

There is a family of vector spaces $HI_i(Y,K,\alpha)$ for $i \in \mathbb{Z}/4\mathbb{Z}$, which are finite dimensional over a Novikov field $\Lambda$, and which we will call the instanton Floer-Novikov knot homology groups of the knot $K$. They recover the Casson-Lin-Herald invariant (and hence the Tristram-Levine
knot signatures), in the sense that
\[ \chi_A(HI(Y, K, \alpha)) = \lambda_{CLH}(Y, K, \alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i\alpha}) \]
where we are using an absolute \( \mathbb{Z}/2\mathbb{Z} \) grading of these Floer groups in order to compute the Euler characteristic with respect to \( \Lambda \). Moreover, each such \( HI(Y, K, \alpha) \) admits a refinement \( HI_{red,i}(Y, K, \alpha) \), which again will be a finite dimensional vector space over the Novikov field \( \Lambda \), and which we will call the \textit{reduced instanton Floer-Novikov knot homology groups of the knot} \( K \). Given these two Floer groups one can define the \textit{Frøyshov knot-invariants}
\[ h(Y, K, \alpha) = \chi_A(HI^{red}(Y, K, \alpha)) - \chi_A(HI(Y, K, \alpha)) \]
For the case of \( \alpha = 1/4 \), no Novikov field is needed and in fact the Floer groups can be defined over \( \mathbb{Q} \) (for example).

Remark 6. In order to be completely accurate, we should specify the cone parameter being used in the resulting groups, and write \( HI(Y, K, \alpha, \nu), HI_{red}(Y, K, \alpha, \nu) \) and \( h(Y, K, \alpha, \nu) \). However, we expect all of these to be independent of \( \nu \), which is why we will continue to suppress \( \nu \) from our notation. In fact, as we will discuss in the last section of the introduction, we also expect some independence regarding the values of \( \alpha \) being used.

Some examples/properties of these groups and the knot \( h \)-invariants will be discussed in Section 8, but now we have to answer the other part of Question 4, namely, what is the relation between \( \lambda_{FO}(X, T, \alpha) \) and \( \text{LeF}(HI(Y, K, \alpha)) \) in the case of a self-concordance of a knot?

First of all, we must observe that currently the functoriality properties of the Floer groups \( HI(Y, K, \alpha) \) are weaker than their non-singular counterparts \( HI(Y) \). By this we mean that a homology concordance \( (W, \Sigma) : (Y_1, K_1) \to (Y_2, K_2) \) for which both of \( HI(Y_1, K_1, \alpha) \) and \( HI(Y_2, K_2, \alpha) \) are defined, may not induce a cobordism map between the Floer groups. The reason for this has to do once again with the reducible connections on the cobordism. In the case where \( \Sigma \) is not present, there are maps between \( HI(Y_1) \) and \( HI(Y_2) \) essentially because the trivial connection \( \theta_W \) on the cobordism, which is the unique reducible up to gauge since \( H_1(W; \mathbb{Z}) = 0 \), is automatically isolated and non-degenerate given that \( H^1(W; T) = H^1(W; \mathbb{R}) \otimes \mathbb{R}^3 = 0 \) and \( H^{2+}(W; T) = H^{2+}(W; \mathbb{R}) \otimes \mathbb{R}^3 = 0 \).

In the case of a homology concordance there is still a unique reducible connection \( \theta_{W,\alpha} \) after we have fixed a choice of \( \alpha \) since \( H_1(W\backslash \Sigma; \mathbb{Z}) = \mathbb{Z} \), but now it is a priori not immediate that it will be isolated or non-degenerate. The cobordisms for which this will happen will be called \( \alpha \)-admissible, and after we explain more of the setup we are using in sections 2 and 3 it will become clear that the condition we are after is the following.

**Definition 7.** A homology concordance \( (W, \Sigma) : (Y_1, K_1) \to (Y_2, K_2) \) between two knots \( K_1 \subset Y_1 \) and \( K_2 \subset Y_2 \) is a homology cobordism \( W : Y_1 \to Y_2 \) together with an embedded annulus \( \Sigma : K_1 \to K_2 \).

The pair \( (W, \Sigma) \) will be called \( \alpha \)-admissible for \( \alpha \in (0, 1/2) \), if \( \Delta_{K_1}(e^{-4\pi i\alpha}) \neq 0, \Delta_{K_2}(e^{-4\pi i\alpha}) \neq 0 \) and moreover \( H^1(W\backslash \Sigma; L^{\otimes 2}_{\theta_{W,\alpha}}) = 0 \). Here \( \theta_{W,\alpha} \) denotes the unique reducible (up to gauge) compatible with the holonomy condition \( \alpha \), and \( E = L_{\theta_{W,\alpha}} \oplus L^{-1}_{\theta_{W,\alpha}} \) denotes the decomposition of the trivial \( SU(2) \) bundle over \( W\backslash \Sigma \) induced by \( \theta_{W,\alpha} \).

We will discuss to what extent this condition on the cobordism is really needed at the end of this section, but for now let’s assume that it holds. In that case it is straightforward to see that we have cobordism maps \( HI(W, \Sigma, \alpha) \) between \( HI(Y_1, K_1, \alpha) \) and \( HI(Y_2, K_2, \alpha) \) as explained in Section 4.
In order to obtain a splitting formula analogous to the one \( \lambda_{\mathrm{FO}}(X) \) satisfies we need an additional piece of data. In general, it is more accurate to regard \( \lambda_{\mathrm{FO}}(X, T, \alpha) \) as a degree zero Donaldson invariant, associated to the moduli space \( \mathcal{M}(X, T, 0, 0, \alpha) \), since in practice one needs to perturb the \( \alpha \)-ASD equation \( F^+_\alpha = 0 \), not the \( \alpha \)-flat equation \( F_\alpha = 0 \), in order to define \( \lambda_{\mathrm{FO}}(X, T, \alpha) \).

Interestingly enough, in the case that \( \alpha \neq 1/4 \), there are other moduli spaces \( \mathcal{M}(X, T, k, l, \alpha) \) whose expected dimension is zero (and are a priori non-empty), which means that there are additional candidates for degree zero Donaldson invariants. As we will explain in more detail in Section \ref{sec:additional-candidates} whenever \( k \) is an integer such that \( k(1 - 4\alpha) \geq 0 \), the moduli space \( \mathcal{M}(X, T, k, -2k, \alpha) \) is of expected dimension zero, and can in fact be used to define a Donaldson-type invariant \( D_0(X, T, \alpha, k) \) (see definition \ref{def:D_0} for a precise statement). In particular, we can create a formal power series

\[
\sum_{k \in \mathbb{Z}} D_0(X, T, \alpha, k) T^{-\varepsilon_{\mathrm{top}}(X, T, k, -2k, \alpha)}
\]

where \( \varepsilon_{\mathrm{top}}(X, T, k, -2k, \alpha) \) denotes the topological energy of the moduli space \( \mathcal{M}(X, T, k, -2k, \alpha) \). In fact, this energy is equal to the quantity \( k(1 - 4\alpha) \) which explains the restriction \( k(1 - 4\alpha) \geq 0 \), since negative energy moduli spaces of \( \alpha \)-ASD instantons are always empty, just as in the ordinary case where no holonomy condition is present. Now, the formal power series \( \mathbb{F} \) is in fact the sort of object a Novikov field is equipped to handle, in other words, we can think of the series \( \mathbb{F} \) as an element of \( \Lambda \).

This is a good thing, since the Lefschetz number of a degree-preserving linear transformation \( L : V \to V \) between two finite dimensional vector spaces over some field \( \mathbb{F} \) will be an element of \( \mathbb{F} \). In the case of a self-concordance \( (W, \Sigma) : (Y, K) \to (Y, L) \) which is \( \alpha \)-admissible we should think of \( V \) as being either \( HI(Y, K, \alpha) \) (or \( HI_{\mathrm{red}}(Y, K, \alpha) \)), \( \mathbb{F} \) as the Novikov field \( \Lambda \) and \( L \) as the map on corresponding Floer groups induced by the cobordism. Therefore, one would expect that the Lefschetz number of the map induced by the self-concordance equals \( \lambda \).

That will be the case, modulo a final caveat. There is an action of \( H^1(X; \mathbb{Z}/2) \) on the moduli spaces \( \mathcal{M}(X, T, k, -2k, \alpha) \), which has been studied ad nauseam in other situations involving Instanton Floer homology \cite{FO0, FO1, FO2, FO3, FO4, FO5}. Unless the action of \( H^1(X; \mathbb{Z}/2) \) is free on the irreducible part \( \mathcal{M}^*(X, T, k, -2k, \alpha) \) of the moduli space, one cannot expect a relationship between the formal power series and the Lefschetz number to hold, since there will be an ambiguity when solving the gluing problem, as explained in \cite[Section 5]{FO5}. In fact, it will turn out that the action of \( H^1(X; \mathbb{Z}/2) \) on \( \mathcal{M}^*(X, T, k, -2k, \alpha) \) is free, in which case the splitting (or Lefschetz) formula \( \lambda \) is essentially a consequence of \cite[Proposition 5.5]{FO5}, which in Kronheimer and Mrowka’s situation comes from the assumption that the subgroup \( \phi^* \subset H^1(W^*, S^*, P^*) \) (in their notation) satisfies a non-integral condition.

With these remarks in place, we can finish answering Question \ref{question:splitting}.

**Theorem 8.** Let \( K \subset Y \) an oriented knot inside an oriented integer homology sphere and \( (W, \Sigma) : (Y, K) \to (Y, L) \) a self-concordance of \( K \). Consider \( \alpha \in \mathbb{Q} \cap (0, 1/2) \) such that \( \Delta_K(e^{-4\pi i \alpha}) \neq 0 \).

If \((W, \Sigma)\) is \( \alpha \)-admissible then there are degree-preserving maps

\[
HI(W, \Sigma, \alpha) : HI(Y, K, \alpha) \to HI(Y, K, \alpha)
\]

\[
HI_{\mathrm{red}}(W, \Sigma, \alpha) : HI_{\mathrm{red}}(Y, K, \alpha) \to HI_{\mathrm{red}}(Y, K, \alpha)
\]

induced by the cobordism \((W, \Sigma)\).

If \((X, T)\) is the pair obtained by closing up \((Y, K)\) then for \( k \neq 0 \) the invariants \( D_0(X, T, \alpha, k) \) are always well defined. Moreover, \( \lambda_{\mathrm{FO}}(X, T, \alpha) = D_0(X, T, \alpha, 0) \) can be defined if and only if \((W, \Sigma)\)
is $\alpha$-admissible, and the following splitting formula holds
\[ \sum_{k \in \mathbb{Z}} D_0(X, T, \alpha, k) T^{-\varepsilon(X, T, k, -2k, \alpha)} = 2\text{Lef}(HI(W, \Sigma, \alpha)) = 2\text{Lef}(HI_{\text{red}}(W, \Sigma, \alpha)) - 2h(Y, K, \alpha) \]

Finally, if $(X, T)$ arises as the closure of another self-concordance $(W', \Sigma') : (Y', K') \to (Y', K')$ and $\Delta_{K'}(e^{-4\pi i \alpha}) \neq 0$ as well, then
\[ \text{Lef}(HI_{\text{red}}(W, \Sigma, \alpha)) = \text{Lef}(HI_{\text{red}}(W', \Sigma', \alpha)) \]
and hence
\[ h(Y, K, \alpha) = h(Y', K', \alpha) \]

In particular, we can define a Frohlov torus invariant $h(X, T, \alpha)$ for the embedded torus as $h(Y, K, \alpha)$ given an arbitrary “slice” $(Y, K)$.

**Remark** 9. a) Again, a dependence on the cone angle is implicit.

b) The statement regarding the equality of the Lefschetz numbers will follow the strategy employed by Frohlov in the case of $\lambda_{SW}(X)$, which as we mentioned before can be found in [24, Theorem 8]. The argument is essentially the same, the only thing one needs to verify is a suitable version of [24, Lemma 10] for the case of matrices with coefficients in a Novikov field.

c) Besides the case of $\lambda_{FO}(X)$ and $\lambda_{SW}(X)$, there are other instances where a Lefschetz formula has appeared in a similar context, in fact, in some ways more closely related to our situation. In [33], Juhász and Zemke compute the effect of concordance surgery on the Ozsváth-Szabó 4-manifold invariant. One is given a smooth closed oriented 4-manifold $X$ with $b_2^+(X) \geq 2$ and a homologically essential torus $T \subset X$ with trivial self-intersection. If $(I \times Y, \Sigma)$ is a self-concordance of a knot $K \subset Y$, then there is a natural torus $T_C$ inside $S^1 \times Y$ which they use to construct a 4-manifold $X_C$ generalizing the Fintushel and Stern knot surgery [19]. Namely, one forms the 4-manifold $X_C = (X \setminus N(T)) \cup_{\phi} W_C$ where $N(T)$ denotes a tubular neighborhood of $T$ and $W_C = (S^1 \times Y) \setminus N(T_C)$, while $\phi$ is a gluing diffeomorphism. In (knot) Heegaard-Floer homology there is a concordance map
\[ \hat{F}_C : \hat{HF}(Y, K) \to \hat{HF}(Y, K) \]
and an associated graded Lefschetz number
\[ \text{Lef}_t(C) = \sum_{i \in \mathbb{Z}} \text{Lef}\left( \hat{F}_C |_{\hat{HF}(Y, K, i)} : \hat{HF}(Y, K, i) \to \hat{HF}(Y, K, i) \right) t^i \]

Then Theorem 1.1 in [19] shows that $\Phi_{X_C, \omega} = \text{Lef}_t(C)\Phi_{X, \omega}$, where $\Phi_{X, \omega}$ denotes a version of the Ozsváth-Szabó 4-manifold invariant twisted by a certain collection of closed 2-forms.

d) Another situation very close to our splitting formula is the one Kronheimer and Mrowka found for the Seiberg-Witten invariants on a closed 4-manifold [43, Section 32.1]. More precisely, one is given a closed oriented 4-manifold with $b_2^+(X) \geq 2$ and within it a separating hypersurface $Y$, so that $X = X_1 \cup X_2$, $\partial X_1 = Y$ and $\partial X_2 = -Y$. If $\omega_X$ is a 2-form used to perturb the Seiberg-Witten equations on $X$ and $\omega$ is the restriction of $\omega_X$ to $X$, then for a torsion spin-c structure $\mathfrak{s}$ on $Y$, in general $\omega$ will induce a non-balanced perturbation in the sense of [43, Chapter 32], which essentially means that a Novikov system $\Lambda$ is required to define the corresponding monopole Floer homology groups $HM(Y, \mathfrak{s}, \omega)$. Proposition 32.1.1 in [43] shows that
\[ \sum_{\mathfrak{s}_X | Y = \mathfrak{s}} T^{-\mathcal{E}^{\text{top}}_{\omega_X}(\mathfrak{s}_X)} \text{SW}(X, \mathfrak{s}_X) = \langle \psi_+, \psi_- \rangle_{\omega_X} \]

Here the sum is taking place over all spin-c structures on $X$ which restrict to the given one on $Y$, $\text{SW}(X, \mathfrak{s}_X)$ denotes the Seiberg-Witten invariant associated to such a spin-c structure, $\mathcal{E}^{\text{top}}_{\omega_X}$ is a
perturbed topological energy, and the right hand side denotes a pairing of two relative invariants ($\psi_\pm$ being an element of $HM(\pm Y, S, \omega)$), the pairing taking place with respect to the Novikov ring. Notice that one can interpret our splitting formula as an analogue of the pairing formula when the 3-manifold is non-separating (rather than separating), and with the Donaldson invariants (rather than the Seiberg-Witten invariants). From this perspective, the different bundles $E(k, -2k)$ are playing a role analogous to the one the different isomorphism classes of spin-c structures play in the Seiberg-Witten context.

In section 8 we will give some applications of the splitting formula including a proof that $\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha) = 2\lambda_{CLH}(Y, K, \alpha)$. But as a way to entice the reader, we now prove the following.

**Theorem 10.** Suppose that $K \subset Y$ and $K' \subset Y'$ are knots and there exists a concordance $(C, A) : (Y, K) \to (Y', K')$ where $C$ is a homology $[0, 1] \times S^3$ and $A$ an embedded annulus. Then for any $\alpha \in \mathbb{Q} \cap (0, 1/2)$ such that $(C, A)$ is $\alpha$-admissible (if any), we have that $h(Y, K, \alpha) = h(Y', K', \alpha)$, i.e, the knot $h$-invariants are $\alpha$-concordance invariants.

In particular, $h(Y', K', \alpha)$ will vanish whenever $K'$ is $\alpha$-slice in the sense that there is an $\alpha$-concordance $(C, A) : (S^3, \circ) \to (Y', K')$, where $\circ$ is the unknot.

**Proof.** Observe that we can form a self-concordance of the knot $(Y, K)$ obtained by “stacking” $(C, A)$ with the opposite concordance $(\bar{C}, \bar{A}) : (Y', K') \to (Y, K)$ obtained by reversing orientations,

$$(W, \Sigma) = (\bar{C}, \bar{A}) \circ (C, A) : (Y, K) \to (Y, K)$$

From $(W, \Sigma)$ we can close it up to obtain $(X, T)$. The corresponding closed 4-manifold $(X, T)$ can also be obtained from doing the stacking in the opposite order, namely $(C, A) \circ (\bar{C}, \bar{A})$, so the last statement in the splitting formula (Theorem 8) implies that the knot $h$-invariants are the same, i.e,

$$h(Y, K, \alpha) = h(Y', K', \alpha)$$

The last claim follows from the fact that for the unknot, for all $\alpha \in \mathbb{Q} \cap (0, 1/2)$, we have $HI(S^3, \circ, \alpha) = HI_{red}(S^3, \circ, \alpha) = 0$, and hence $h(S^3, \circ, \alpha) = 0$. □

We conclude this section by mentioning the “flip symmetry” property our Floer groups enjoy (in the context of singular gauge theory on 4-manifolds it was introduced by Kronheimer and Mrowka, [377, Lemma 2.12]).

**Theorem 11.** Let $K \subset Y$ be an oriented knot inside an oriented integer homology sphere and choose $\alpha \in \mathbb{Q} \cap (0, 1/2)$ such that $\triangle_K(e^{-4\pi i \alpha}) \neq 0$. Then there is a flip isomorphism

$$\mathcal{F} : HI(Y, K, \alpha) \to HI(Y, K, \frac{1}{2} - \alpha)$$

which is grading preserving, with an analogous isomorphism for the reduced groups $HI_{red}(Y, K, \alpha)$. In particular,

$$h(Y, K, \alpha) = h\left(Y, K, \frac{1}{2} - \alpha\right)$$

Likewise, for a pair $(X, T)$ we have for all $k \neq 0$ that

$$D_0(X, T, k, \alpha) = D_0\left(X, T, -k, \frac{1}{2} - \alpha\right)$$

and a similar statement holds for $\lambda_{FO}(X, T, \alpha)$ whenever it can be defined.
Some Speculations and Further Directions of Work. We finish this introduction by mentioning remaining problems related to the invariants and Floer homologies we have described so far. The author hopes to address many of these questions in the near future. More details on these questions can be found in [18].

(1) Independence of cone angle: we would like to show that $HI(Y, K, \alpha, \nu)$, $HI_{red}(Y, K, \alpha, \nu)$ and $\lambda_{\text{FO}}(X, T, \alpha, \nu)$ do not depend on the cone parameter $\nu$ being used. To a large extent the analytical machinery from the original papers by Kronheimer and Mrowka on singular gauge theory was developed without using the orbifold language, and it is useful to point out that Gerard developed some of the technical aspects needed if one wants to construct the Floer groups with an arbitrary cone angle, using the theory of elliptic edge operators [52].

(2) Irrational values of holonomy parameter $\alpha$: we would also like to define versions of $HI(Y, K, \alpha)$, $HI_{red}(Y, K, \alpha)$ and $\lambda_{\text{FO}}(X, T, \alpha)$ for the case of irrational values of the holonomy $\alpha$.

(3) Dependence of the invariants on the holonomy parameter $\alpha$: It is also reasonable to expect that $HI(Y, K, \alpha)$, $HI_{red}(Y, K, \alpha)$ remain the same on the subintervals of $(0, 1/2)$ determined by the solutions of the equation $\triangle_K(e^{-4\pi i \alpha}) = 0$. At least $\lambda_{\text{CLH}}(Y, \alpha)$ does enjoy this property.

(4) Geometric meaning of $\lambda_{\text{FO}}(X, T, \alpha)$ and $D_0(X, T, k, \alpha)$ (when $k \neq 0$): in the case that no perturbations are needed, the meaning of $\lambda_{\text{FO}}(X, T, \alpha)$ is clear, but the meaning of the remaining invariants $D_0(X, T, k, \alpha)$ is less clear. This setup is not the only one where degree-zero Donaldson-type invariants can be defined, and where their geometric importance is not obvious [66]. In fact, if we could carry out the necessary deformation arguments on $\alpha$ and $\nu$, we would expect that for $k \neq 0$, all the additional Donaldson invariants $D_0(X, T, k, \alpha)$ vanish. The reason for this, as we explain in Section 8 is that the bundles $E(k, -2k)$ used to define the invariants $D_0(X, T, k, \alpha)$ carry no reducibles $\alpha$-ASD connections. Therefore, one would expect to be able to deform these moduli spaces to the case of holonomy $\alpha = 1/4$, where the $D_0(X, T, k, 1/4)$ vanish automatically (for $k \neq 0$), so the $D_0(X, T, k, \alpha)$ would have to vanish as well.

(5) Instanton thin knots: one can also ask if it is possible to characterize those knots with vanishing $HI(Y, K, \alpha)$ (or vanishing $HI_{red}(Y, K, \alpha)$). Following the analogy with Heegaard Floer homology [51], we will call these knots instanton thin knots. This question is mostly interesting from two extreme positions: which knots, besides the unknot $\sigma \subset S^3$, have the property that $HI(Y, K, \alpha)$ (or $HI_{red}(Y, K, \alpha)$) vanishes for all values of $\alpha$ where these Floer homologies can be defined? Likewise, which knots, besides the unknot $\sigma \subset S^3$, have the property that $HI(Y, K, 1/4)$ (or $HI_{red}(Y, K, 1/4)$) vanish? The specific value of $\alpha = 1/4$ is interesting not only because Novikov systems can be avoided but also because for knots inside $S^3$ we always have $\triangle_K(e^{-4\pi i(1/4)}) = \triangle_K(-1) = 0$ which means that $HI(S^3, K, 1/4)$ is well defined for any knot. As we will see in Section 8 for the case of the figure-eight knot $4_1$, $HI(S^3, 4_1, 1/4)$ vanishes identically (and so does $HI_{\text{red}}(S^3, 4_1, 1/4)$), despite the fact that there are two irreducible representations satisfying this holonomy condition. So, $HI(Y, K, 1/4)$ is not an unknot detector, as opposed to the case of the different knot Floer homology $F^2(K)$ Kronheimer and Mrowka defined in [40].

(6) Relation between $\lambda_{\text{FO}}(X, T, \alpha)$, $\lambda_{\text{FO}}(X)$ and the tori signature: Recall that $\lambda_{\text{CLH}}(Y, K, \alpha) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i \alpha})$. This can be rewritten as $\lambda_{\text{FO}}(S^1 \times Y, S^1 \times K, \alpha) = 8\lambda_{\text{FO}}(S^1 \times Y) + \sigma_K(e^{-4\pi i \alpha})$. This suggests that if one could find a non-gauge theoretic definition of a tori signature $\sigma_T$ for $T \subset X$, then the formula

\[
\lambda_{\text{FO}}(X, T, \alpha) = 8\lambda_{\text{FO}}(X) + \sigma_T(e^{-4\pi i \alpha})
\]

should hold [assuming that $\sigma_T$ is normalized so that $\sigma_{S^1 \times K} = \sigma_K$]. For every example where we can compute $\lambda_{\text{FO}}(X, T, \alpha)$, the underlying 4-manifold $X$ has a well defined Furuta-Ohta invariant
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\[ \lambda_{FO}(X) \]. We expect this will always be the case, but currently we do not have an argument saying that if there is an embedded torus \( T \) and a parameter \( \alpha \) for which \( \lambda_{FO}(X, T, \alpha) \) can be defined, then \( \lambda_{FO}(X) \) can be defined as well. If there are cases where this does not happen, then the equation would not make much sense to being with. However, one could still take equation as a definition of what we mean by \( \sigma_T \), at least when \( \lambda_{FO}(X, T, \alpha) \) and \( \lambda_{FO}(X) \) can be both computed. Then the more important problem would be to give a topological interpretation of \( \sigma_T \) (if possible) to facilitate computations of the invariant. It should be pointed out that the expression Herald found for \( \lambda_{CLH}(Y, K, \alpha) \) in terms of \( \lambda_C(Y) \) and \( \sigma_K(e^{-4\pi i \alpha}) \) relied on the fact that the Tristram-Levine signatures enjoy multiple interpretations, so the question becomes if the same could be done for \( \sigma_T \). It could also be the case that a reasonable topological definition of \( \sigma_T \) requires working on 4-manifolds where \( \lambda_{FO}(X) \) is well defined (since some condition on the infinite cyclic cover of \( X \) may be needed).

(7) \( \alpha \)-admissible cobordisms: we would also like to know to what extent one can relax condition that the cobordism is \( \alpha \)-admissible. As we mention in the statement of Theorem \[ 5 \] having an \( \alpha \)-admissible cobordism in the case of a self-concordance \( (W, \Sigma) : (Y, K) \to (Y, K) \) is equivalent to having a well defined singular Furuta-Ohta invariant \( \lambda_{FO}(X, T, \alpha) \) when we close the concordance up. Therefore, if it were possible to introduce holonomy perturbations (or other techniques) in order to deal with the degeneracy of the reducible connection on the cobordisms, it might also be possible to relax the conditions needed for defining \( \lambda_{FO}(X, T, \alpha) \), perhaps by adding some correction terms similar to the ones needed for the definition of \( \lambda_{SW}(X) \) in the case of the Seiberg-Witten analogue of \( \lambda_{FO}(X) \).

Outline of the paper: In section 2 we review the main features of singular gauge theory as developed by Kronheimer and Mrowka.

Sections 3, 4, 5 discuss how the monotonicity issue arises, the definition of \( HI(Y, K, \alpha) \) and of \( HI_{red}(Y, K, \alpha) \) respectively.

Sections 6 defines \( \lambda_{FO}(X, T, \alpha) \) as well as its relatives \( D_0(X, T, \alpha) \). Section 7 discusses the splitting formula \( \lambda_{FO}(X, T, \alpha) \) and \( D_0(X, T, \alpha) \) satisfy. Finally, section 8 includes some calculations of \( \lambda_{FO}(X, T, \alpha) \) for certain tori inside mapping tori as well as some properties of the groups \( HI(Y, K, \alpha) \) and \( HI_{red}(Y, K, \alpha) \), like the usual duality isomorphisms under orientation reversal as well as the flip symmetry property.

In order to reduce the length of the paper, we deleted an appendix which includes a summary of cohomology with local coefficients and the Alexander polynomial for \( CW \) complexes, which can still be found in the annotated version of the paper [18].

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2. Review of the Orbifold Setup

As we mentioned in the introduction, the Floer homology we will construct for knots is based on an orbifold approach, which was pioneered by Kronheimer and Mrowka in their papers on gauge theory and embedded surfaces [37, 39]. Slightly different versions of this setup have been employed by them over the years [42, 41, 40], so the main purpose of this section is to give a brief review of their orbifold construction, as well as discussing the advantages and disadvantages of different strategies for building Floer homologies for knots.

Let \( X \) be a smooth, oriented, closed four manifold and \( \Sigma \) any oriented, smoothly embedded surface inside \( X \). Similarly, suppose that \( Y \) is a smooth, oriented, closed three manifold and \( K \) is an oriented knot or link inside \( Y \). For either of the pairs \((X, \Sigma)\), \((Y, K)\), we want to do gauge theory using connections on an \( SU(2) \) bundle which are singular along \( \Sigma \) (or \( K \)). The nature of the singularity is precisely what distinguishes one approach from the other.

A natural strategy would be to work on either of the complements \( X \setminus \Sigma \), \( Y \setminus K \) and to choose a riemannian metric which is simply the restriction to one of these complements of a smooth metric \( g \) defined on all of \( X \) (or \( Y \)). Analytically, this means that one is working on an open manifold with an incomplete metric, which from the Sobolev package point of view is not a nice situation. However, provided one works with weighted Sobolev spaces, one can still define a reasonable family of function spaces [37, Section 3 (i)].

The next issue is whether one wants to work with an \( SU(2) \) bundle \( E \) which is defined only on the complement \( X \setminus \Sigma \) (respectively \( Y \setminus K \)). In this case the major problem is how to recover useful topological information. This was called the extension problem by Kronheimer and Mrowka, and discussed in some detail in [37, Section 2 (iv)].

The setup used in [37, 39, 41], which will follow as well, is one for which the bundle \( E \) is defined over the entire manifold \( X \) (or \( Y \)), subject to the condition that topologically the bundle \( E \) admits a reduction to a \( U(1) \) bundle along \( \Sigma \) (or \( K \)). For example, this means that for \((X, \Sigma)\) an \( SU(2) \) bundle \( E \) should decompose as

\[
E |_{\nu(\Sigma)} = L \oplus L^*
\]

where \( \nu(\Sigma) \) is a closed tubular neighborhood of \( \Sigma \), and \( L \to \nu(\Sigma) \) some complex line bundle compatible with the hermitian metric. In particular, this allows us to define two topological invariants, the instanton number \( k \)

\[
k = c_2(E)[X]
\]

as well as the monopole number \( l \)

\[
l = -c_1(L|_{\Sigma})[\Sigma]
\]

Notice that the latter invariant is not present when \( \Sigma = \emptyset \). The reader may be interested in knowing that for their “Khovanov is an unknot-detector” paper Kronheimer and Mrowka do work with bundles which in principle are not defined over the entire manifold [40 Section 2.1].

For the 3-manifold case, the bundle topology is not very interesting in our situation so we can consider \( E \) simply as the trivial \( SU(2) \) bundle over \( Y \), i.e. \( E = Y \times SU(2) \), together with the trivial \( U(1) \) sub-bundle corresponding to the diagonal embedding of \( U(1) \) in \( SU(2) \). In both cases what we will be more interesting is the space of connections we will use, which we now describe.

These connections have the following singular behavior: after choosing a riemannian metric we consider \( \nu(\Sigma) \) as being diffeomorphic to the unit disk bundle of the normal bundle, and we choose a connection 1-form \( \eta \) for the circle bundle \( \partial \nu(\Sigma) \), so that it coincides with the 1-form \( d\theta \) on each
circle fibre, where \((r, \theta)\) are polar coordinates in some local trivialization of the disk bundle (\(dr \wedge d\theta\) orients the normal planes). By radial projection \(\eta\) is extended to \(\nu(\Sigma)\setminus \Sigma\).

For each real number \(0 < \alpha < 1/2\), we consider the matrix 1-form with values in \(\text{su}(2)\) which behaves near \(\Sigma\) as

\[
i \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} d\theta
\]

Locally, the holonomy of a connection \(A\) on \(X \setminus \Sigma\) whose matrix connection coincides with the previous matrix 1-form near \(\Sigma\) on the positively-oriented small circles of constant \(r\) is approximately

\[
\exp \begin{pmatrix} -2\pi i\alpha & 0 \\ 0 & 2\pi i\alpha \end{pmatrix}
\]

We are excluding the values \(\alpha = 0, 1/2\) because those cases give no new information, i.e., they essentially correspond to the situation where the connections extend smoothly across the singularity (as \(SU(2)\) or \(SO(3)\) connections more generally).

For a more global description, using the reduction \((6)\), choose any smooth \(SU(2)\) connection \(A^0\) on \(E\) which reduces in the same way, i.e.,

\[
A^0 \mid_{\nu(\Sigma)} = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}
\]

where \(b\) is a smooth connection on \(L\). Define the **model connection** \(A^\alpha\) on \(E \mid_{X \setminus \Sigma}\) as

\[
A^\alpha = A^0 + i\beta(r) \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \eta
\]

where \(\beta\) is a smooth cut-off function equal to 1 in a neighborhood of 0 and equal to 0 for \(r \geq 1/2\). The model connection has holonomy around small linking circles asymptotically equal to \((9)\). The connections Kronheimer and Mrowka consider can be written as \(A = A^\alpha + a\), where \(a \in \Omega^1(X \setminus \Sigma, \text{su}(2))\), and as always necessary Sobolev spaces are needed [37, Eq. 2.2, 2.3].

**Remark 12.** In the 3-manifold case we can take \(A^0\) to be the (usual) trivial connection \(\theta\) on the trivial \(SU(2)\) bundle \(E\) from before. In the nomenclature of [40, Section 3.6], our situation corresponds to one where \(\Delta\) is trivial, where \(\Delta\) is a local system determining the possible extensions of the bundle across the singularity.

Most of the usual gauge theory technology can be extended to this setup: for example, one can define moduli spaces of \(ASD\) connections with the singular behavior described before, compute the expected dimension of the moduli space, as well as the energy of the connection in terms of topological quantities. These moduli spaces are indexed by the instanton and monopole number, so we will write them typically as \(\mathcal{M}(X, \Sigma, k, l, \alpha)\) or \(\mathcal{M}(k, l)\), depending on the context.

We will remind the reader about the precise formulas once we explain the monotonicity condition usually required before defining Floer homologies. For now, it suffices to say that there is one key aspect missing by working in the previous setup described. Namely, Kronheimer and Mrowka were not able to show that if one has a sequence of \(ASD\) connections \(A_i\) belonging to some moduli space \(\mathcal{M}(X, \Sigma, k, l, \alpha)\), then the limiting connection \(A_\infty\), which in principle belongs to a different moduli space \(\mathcal{M}(X, \Sigma, k', l', \alpha)\) because of Uhlenbeck bubbling, must belong to a moduli space of lower expected dimension. The fact that this should happen, that is, that after bubbling one should land in a moduli space of smaller dimension, would follow if Conjecture 8.2 in [37] were proven true. Fortunately, they were able to prove this fact by modifying slightly the previous setup and working instead with *orbifolds*. 
On a first level, this means that instead of using the restriction of a smooth metric on $X$ (or $Y$) to $X \setminus \Sigma$ (or $Y \setminus K$), we consider metrics which have a cone-like singularity along the surface (knot) $\Sigma$ (Section 2, iii)]. This means that near the surface $\Sigma$, the metric is modeled on

$$ds^2 = du^2 + dv^2 + dr^2 + \left(\frac{1}{\nu^2}\right)r^2d\theta^2$$

where $u, v$ are coordinates on $\Sigma$, and $\nu \geq 1$ is a real parameter. To obtain a global metric on $\nu(\Sigma)$ of this form replace $du^2 + dv^2$ by the pull-back of any smooth metric on $\Sigma$, and replace the form $d\theta$ by the 1-form $\eta$. The metric is then patched to a smooth one on the complement of $\nu(\Sigma)$ and extended to the rest of $X$. The resulting metric has a cone-angle of angle $\frac{2\pi}{\nu}$ in the normal planes to $\Sigma$. When $\nu$ is an integer greater than 1 the metric is an orbifold metric: locally there is a $\nu$-fold branched cover on which the metric is smooth.

An advantage of the orbifold perspective is that it allow us to compute certain quantities (like gradings for the generators of the Floer complex) in terms of equivariant indices on appropriate branched covers along the knot (and/or surface). Moreover, in the orbifold setup one can still use a Coulomb slice determined explicitly by $\ker \tilde{d}_A^*\eta$, where $\tilde{d}$ emphasizes that we are thinking of this operator as being defined on an orbifold. If we were to use instead the non-orbifold setup (with the smooth metric and the weighted Sobolev spaces mentioned before), then the construction of the slices is more subtle because of the lack of the usual $d_A^*$ operator [37, Lemma 5.5].

However, as we pointed out in the introduction, a drawback of the orbifold approach is that currently there is no way to show that the groups one obtains are independent of the choice of orbifold structure (cone angle) one uses. Moreover, each value of holonomy $\alpha$ determines an allowable set of possible orbifold cone angles $\nu$ compatible with $\alpha$ one can use for defining the Floer groups. These values are described in Proposition 4.8, Lemma 4.9 (and the remark after it), as well as Proposition 4.17 of [37].

In practice we can think of the allowable $\nu$ in the following way [41, Remark p.894]. If our holonomy parameter is a rational number $\alpha \in \mathbb{Q} \cap (0,1/2)$, then $\nu > 0$ can be taken to be any integer satisfying the property that

$$(11) \quad \exp \left( -\frac{2\pi i\alpha}{\nu} \begin{pmatrix} 0 & 0 \\ 2\pi i\alpha & 0 \end{pmatrix} \right) \in Z(SU(2)) = \pm Id_{2 \times 2}$$

Therefore, if we write $\alpha$ as $\frac{p}{q}$, where $p, q$ are relatively prime, it is not difficult to see from this description that taking $\nu = q$ suffices. However, if $q$ happens to be even, then in fact $\nu = \frac{q}{2}$ also satisfies the condition. This is why for $\alpha = \frac{1}{2}$ one can choose $\nu = 2$, so that the metric has a cone angle $\pi$ along the knot (a bifold in the terminology of Kronheimer and Mrowka [42]). If one is annoyed by this dependence on the cone angle, it is reasonable to choose the smallest cone parameter $\nu$ that works for $\alpha$ as the “canonical” cone angle. Incidentally, notice that the this canonical cone angle works simultaneously for $\alpha$ and $\frac{1}{2} - \alpha$ so that the flip symmetry $\alpha \rightarrow \frac{1}{2} - \alpha$ can be analyzed with this common choice.

**Remark 13.** Besides describing the singular metric in the orbifold setup, one can also adapt the notion of bundle and connections to this situation. The definitions given in [42] are particularly convenient for our purposes (it also has the advantage of being suitable for the case that the bundle does not extend across the singularity, which as we said is more general than what we will consider).

Every point in an orbifold has a neighborhood $U$ which is the codomain of an orbifold chart

$$\phi : \tilde{U} \rightarrow U$$
The map $\phi$ is a quotient map for an action of a finite group, which in our case will be either trivial or $\mathbb{Z}_m$. A $C^\infty$ SU(2) orbifold connection with respect to $(X, \Sigma)$ (or $(Y, K)$) means an oriented $\mathbb{C}^2$ vector bundle $E$ over $X \setminus \Sigma$ (respectively $Y \setminus K$) with an SU(2) connection $A$ having the property that the pull-back of $(E, A)$ via any orbifold chart $\phi : \hat{U} \to U$ extends to a smooth pair $(\hat{E}, \hat{A})$ on $\hat{U}$.

If we have two SU(2) bifold connections $(E, A)$ and $(E', A')$, then an isomorphism between them is a bundle map $\tau : E \to E'$ over $X \setminus \Sigma$ (respectively $Y \setminus K$) such that $\tau^*(A') = A$. The group $\Gamma_{E,A}$ of automorphisms of $(E, A)$ can be identified with the group of parallel sections of the associated bundle. It is isomorphic to either the trivial group, the circle $U(1)$ embedded as a maximal torus inside SU(2), or all of SU(2).

Now that we have explained the geometric setup, we will describe of the space of connections and gauge transformations that we will be working with, in the spirit of Appendix I. Suppose there is an oriented link $K$ inside an integer homology sphere $Y$. For $\alpha \in \mathbb{Q} \cap (0, 1/2)$ consider the model connection $B^\alpha$ described before. We will typically use $B$ when referring to connections defined on three-dimensional manifolds from now on. At the same time we choose an orbifold metric $g^\nu$ compatible with $\alpha$ in the sense we mentioned before.

We will usually follow the convention of Kronheimer and Mrowka, and use the notation when we want to emphasize that the orbifold metric is being used in a specific construction. Hence, $L^2_{m,B^\alpha}$ will denote the Sobolev spaces using the Levi-Civita derivative of $g^\nu$ and the covariant derivative of $B^\alpha$ on $\mathfrak{g}_p$. Taking $p = 2$ and an integer $m > 2$, the space of connections to be considered is

$$\mathcal{C}(Y, K, \alpha) = \{ B = B^\alpha + b \mid b \in L^2_{m,B^\alpha}(T^*Y \mid (\gamma \otimes \mathfrak{g}_p)) \}$$

The gauge group $\mathcal{G}(Y, K, \alpha)$ consists of those automorphisms of $E$ which preserve the reducibility of the model connection $B^\alpha$. In fact, $\mathcal{G}(Y, K, \alpha)$ turns out to be independent of $\alpha$ (and cone angle used) so we will just denote it as $\mathcal{G}(Y, K)$. Here is a useful way to think about $\mathcal{G}(Y, K)$ Appendix I:

Let $\mathcal{G}^K(Y, K)$ denote the subgroup of $\mathcal{G}(Y, K)$ consisting of those gauge transformations which are the identity over $K$. Let $\mathcal{G}_K(Y, K)$ denote the space of gauge transformations of the line bundle $L \to K$, that is, the space of maps from $K$ to $U(1)$. Then there is an exact sequence

$$1 \to \mathcal{G}^K(Y, K) \to \mathcal{G}(Y, K) \to \mathcal{G}_K(Y, K) \to 1$$

(12)

In particular, we can think of $\mathcal{G}(Y, K)$ as the space of maps $g : Y \to SU(2)$ satisfying $g(K) \subset U(1) \subset SU(2)$. Moreover, $\mathcal{G}^K(Y, K)$ is weakly homotopy equivalent to the space of smooth gauge transformations which are the identity over a sufficiently small neighborhood of the knot $K$. From this we can see that for knots the topological information of the map $g$ is carried by two integers, corresponding to the degrees of $g : Y \to SU(2)$ and $g |_K : K \to U(1)$.

More generally, if the link $K$ has $r$ components, the space of components $\pi_0(\mathcal{G}(Y, K))$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}^r$.

Also, there is a preferred homomorphism

$$d : \mathcal{G}(Y, K) \to \mathbb{Z} \oplus \mathbb{Z}$$

(13)

We denote this integer by $m$ instead of $k$ to avoid confusion with the instanton number $k$. 

[1] We denote this integer by $m$ instead of $k$ to avoid confusion with the instanton number $k$. 

A Generalization of the Tristram-Levine Knot Signatures as a Singular Furuta-Ohta Invariant for Tori
where the map to the second factor is obtained by taking the sum of all the entries in the factor \( \mathbb{Z}^r \). After introducing Sobolev completions, we will take our gauge group to be

\[
G(Y, K) = \{ g \mid g \in \tilde{L}^2_{m+1,B^\alpha}(\text{Aut}(P)) \}
\]

The space of connections \( \mathcal{C}(Y, K, \alpha) \) is an affine space, and on the tangent space \( T_B \mathcal{C}(Y, K, \alpha) \) we define an inner product (independent of \( B \)) by

\[
\langle b, b' \rangle_{L^2} = \int_Y -\text{tr}(\text{ad}(b) \wedge \text{ad}(b'))
\]

where we are using the Killing form to contract the Lie algebra indices, and the Hodge star on \( Y \) and the wedge product to contract the form indices. It is important to notice that the Hodge star is the one defined by the orbifold metric \( g^\nu \).

The \textbf{Chern-Simons functional} on \( \mathcal{C}(Y, K, \alpha) \) is defined to be the unique function

\[
CS : \mathcal{C}(Y, K, \alpha) \to \mathbb{R}
\]

satisfying \( CS(B^\alpha) = 0 \) and having gradient (with respect to the above inner product)

\[
(\text{grad} CS)_B = *F_B
\]

From this one obtains

\[
CS(B^\alpha + b) = \langle *F_{B^\alpha}, b \rangle_{L^2} + \frac{1}{2} \langle *d_{B^\alpha}b, b \rangle_{L^2} + \frac{1}{6} \langle *[b, b], b \rangle_{L^2}
\]

The critical points \( B \) of \( CS \) then satisfy \( F_B = 0 \). When restricted to \( Y \setminus K \), this gives rise to a representation (here \( y_0 \) is a base-point in the complement of the knot or link)

\[
\rho : \pi_1(Y \setminus K, y_0) \to SU(2)
\]

satisfying the constraint that for each oriented meridian \( \mu_K \), \( \rho(\mu_K) \) is conjugate to

\[
(14) \quad \rho(\mu_K) \sim \begin{pmatrix} \exp(-2\pi i \alpha) & 0 \\ 0 & \exp(2\pi i \alpha) \end{pmatrix}
\]

There is one such conjugacy class for each component of \( K \), once the components are oriented.

As usual in the gauge theory context, we will need to perturb the flatness equation (in fact, the Chern-Simons functional) to obtain the necessary transversality properties for the moduli spaces. The perturbations that we will use are described in section 3.2 of [41]. As we will see soon, the support of these perturbations must stay away from the knot if we want to appeal to the relationship Herald found between the count these flat connections (modulo conjugacy) and the knot signatures [30]. Kronheimer and Mrowka had to address a similar problem in their paper on Tait colorings [42], although for different purposes.

Before discussing this support condition, we will explain next how the monotonicity issue arises when one tries to define a version of Instanton Floer homology for an arbitrary value of \( \alpha \).

\[2\text{Notice that there is a factor of } \frac{1}{2} \text{ missing in the last term of the formula in [41, Eq. 67].} \]
3. Monotonicity and Novikov Systems

As usual, we would like to do “Morse theory” on

\[ B(Y, K, \alpha) = C(Y, K, \alpha) / G(Y, K) \]

Notice that since \( G(Y, K) \) is independent of \( \alpha, \nu \) and \( C(Y, K, \alpha) \) is contractible, the homotopy type of the space \( B(Y, K, \alpha) \) is independent of \( \alpha, \nu \). As in the case when \( K \) is not present, the Chern-Simons functional is no longer single-valued on \( B(Y, K, \alpha) \). In fact \( CS \) is invariant only under the identity component of the gauge group. If \( B \in C(Y, K, \alpha) \) is a connection and \( g \) a gauge transformation, we can use the latter to form the bundle \( S^1 \times_g E \) over \( S^1 \times Y \) together with its reduction over \( S^1 \times K \), defined by \( B^g \). Then the map \( d \) from (13) becomes

\[ d(g) = (k, l) \in \mathbb{Z} \oplus \mathbb{Z} \]

where \( k \) and \( l \) are the instanton and monopole numbers respectively. In this case we have \([11\text{ p. 874}]

\[ CS(B) - CS(g(B)) = 32\pi^2(k + 2\alpha l) \]

More generally, we define for a path \( \gamma : [0, 1] \rightarrow C(Y, K, \alpha) \) the topological energy as twice the drop in the Chern Simons functional, so that the last equation implies that a path from \( B \) to \( g(B) \) has topological energy

\[ E(k, l) = 64\pi^2(k + 2\alpha l) \]

For a path that formally solves the downward gradient-flow equation for \( CS \) on \( C(Y, K, \alpha) \), the topological energy coincides with the modified path energy

\[ \int_0^1 \left( \|\dot{\gamma}(t)\|^2 + \|\text{grad} CS(\gamma(t))\|^2 \right) dt \]

Before comparing \( E(k, l) \) with the formula for the grading of two flat connections, we discuss the perturbations needed to achieve transversality, which involve using holonomy perturbations. These will be the same as in \([11\text{ Section 3.2}]\), and we will give more details about them in the next section, where we will need to keep track on certain estimates involving their norms. The basic idea is that each holonomy perturbation gives rise to a cylinder function

\[ f : C(Y, K, \alpha) \rightarrow \mathbb{R} \]

which depends on an \( l \)-tuple

\[ q = (q_1, \cdots, q_l) \]

of immersions \( q_i : S^1 \times D^2 \rightarrow Y \setminus K, i = 1, \cdots, l \) and an \( SU(2) \)-invariant function \( h \) on \( SU(2)^\times l \) via the formula

\[ f(B) = \int_{D^2} h(\text{Hol}_q(B)) \mu \]

Here \( \mu \) is a non-negative 2-form supported in the interior of \( D^2 \) and having integral 1. We assume that \( h \) is invariant under the action of \( SU(2) \) on each of the \( l \) factors separately (see the remark after definition 3.3 in \([11]\)). It follows that each cylinder function is invariant under the full gauge group, not just the identity component.

The space of perturbations one uses to achieve transversality involves taking a countable collection of cylinder functions with an \( l^1 \) notion of convergence \([11\text{ Definition 3.6}]\). If \( P \) denotes the
Banach space of perturbations, then \(p \in \mathcal{P}\) induces a function \(f_p\), and the perturbed Chern-Simons functional we consider is

\[
CS_p = CS + f_p
\]

The critical points are now connections satisfying the equation

\[
*F_B + V_p(B) = 0
\]

where \(V_p\) is the formal gradient of \(f_p\) with respect to the \(L^2\) inner product. Proposition 3.10 in [41] states that there is a residual subset of \(\mathcal{P}\) such that, for all \(p\) all the irreducible critical points of \(CS + f_p\) are non-degenerate (we will return to the reducibles momentarily). The **perturbed topological energy** is then defined for a path \(\gamma : [0, 1] \to \mathcal{C}(Y, K, \alpha)\) as

\[
\mathcal{E}_p(\gamma) = 2((CS + f_p)(B(t_0)) - (CS + f_p)(B(t_1)))
\]

Notice that because the perturbations are invariant under the full gauge group, for a path from \(B\) to \(g(B)\), the term \(f_p(B) - f_p(g(B))\) will cancel, so the perturbed topological energy \(\mathcal{E}_p\) on loops has the same expression as the unperturbed case, namely, equation (17).

Returning to the grading question, define the (perturbed) **Hessian** of a connection \(B \in \mathcal{C}(Y, K, \alpha)\) as

\[
\text{Hess}_{B,p}(b) = *d_B b + DV |_{B}(b)
\]

where \(b \in T_B \mathcal{C}(Y, K, \alpha)\) and the (perturbed) extended **Hessian** [41] p. 881

\[
\text{Hess}_{B,p} = \begin{pmatrix}
0 & -d_B^* \\
-d_B & \text{Hess}_{B,p}
\end{pmatrix} : \hat{L}_2^2(Y; g_p) \oplus \mathcal{T}_j \to \hat{L}_{j-1}^2(Y; g_p) \oplus \mathcal{T}_{j-1}
\]

To clarify the notation, here \(j \leq m\) is an integer, \(\hat{L}_2^2(Y; g_p)\) is a shorthand for \(\hat{L}_{j,B^0}^2(Y \setminus K; g_p |_{Y \setminus K})\), and \(\mathcal{T}_j\) is shorthand for sections of \(T_B \mathcal{C}(Y, K, \alpha)\) with regularity Sobolev \(\hat{L}_{j-1,B^0}^2\).

For two irreducible, non-degenerate (perturbed) flat connections \(B_0, B_1 \in \mathcal{C}(Y, K, \alpha)\), the **relative grading**

\[
gr(B_0, B_1) \in \mathbb{Z}
\]

will be defined by taking a path \(B(t)\) in \(\mathcal{C}(Y, K, \alpha)\) from \(B_0\) to \(B_1\) and letting \(\text{gr}(B_0, B_1)\) be equal to the spectral flow of the 1-parameter family \(\text{Hess}_{B(t)}\). This number only depends on the endpoints and not the path since \(\mathcal{C}(Y, K, \alpha)\) is contractible. If

\[
[\beta_0] = [B_0], \quad [\beta_1] = [B_1]
\]

are the corresponding gauge equivalence classes in \(B = B(Y, K, \alpha)\), then \([B(t)]\) determines a path \(\zeta\) from \([\beta_0]\) to \([\beta_1]\), defining a relative homotopy class \(z \in \pi_1([\beta_0], B, [\beta_1])\). This relative homotopy class only depends on \(B_0, B_1\). Conversely, for a relative homotopy class of paths from \([\beta_0]\) to \([\beta_1]\), we define

\[
gr_z([\beta_0], [\beta_1]) \in \mathbb{Z}
\]

via the above procedure.

For the case of a closed loop \(z\) based at a point \([\beta] \in \mathcal{B}(Y, K, \Phi)\), we obtain an element inside \(\pi_1(B(Y, K, \alpha)) \simeq \pi_0(\mathcal{G}(Y, K, \alpha))\). Lemma 3.14 in [41] shows that

\[
gr_z([\beta], [\beta]) = 8k + 4l
\]

where we think of \(z\) as being specified by a gauge transformation \(g\) as before, and \(d(g) = (k, l)\).

From this grading formula we can see that \(\text{gr}_z([\beta], [\beta]) \equiv 0 \mod 4\), which means that the Floer homology we will define is \(\mathbb{Z}/4\mathbb{Z}\) graded. We will eventually show how to promote this relative
grading to an absolute grading. More importantly, comparing the energy and grading formulas on loops we can see that

\[ \mathcal{E}(k, l) = 8\pi^2 \text{gr}_z([\beta], [\beta]) + 32\pi^2 l(4\alpha - 1) \]

which means that the only way for \( \mathcal{E}(k, l) \) and \( \text{gr}_z([\beta], [\beta]) \) to be proportional regardless of the value of \( (k, l) \) is if \( \alpha = \frac{1}{4} \).

To explain what happens when \( \alpha \neq 1/4 \), we recall Proposition 3.23 in [11]. Let \( \mathcal{M}_z([\beta_0], [\beta_1]) \) denote a connected component of the moduli space of ASD connections on \( \mathbb{R} \times Y \) limiting to \( [\beta_0] \) as \( t \to -\infty \) and \( [\beta_1] \) as \( t \to \infty \) (with suitable perturbations thrown into the picture to guarantee transversality).

**Proposition 14.** ([11] Proposition 3.23) Given any \( C > 0 \), there are only finitely many \([\beta_0], [\beta_1]\) and \( z \) for which the moduli space \( \mathcal{M}_z([\beta_0], [\beta_1]) \) is non-empty and has topological energy \( \mathcal{E}_p(\gamma) \) at most \( C \).

Now suppose one wants to define the differential via the usual formula

\[ \partial [\beta_0] = \sum_{[\beta_1] \in \pi \mathcal{M}_z([\beta_0], [\beta_1])} \sum_{z} n_z([\beta_0], [\beta_1]) [\beta_1] \]

where we are taking the sum over all moduli spaces \( \mathcal{M}_z([\beta_0], [\beta_1]) \) which are one-dimensional and \( n_z([\beta_0], [\beta_1]) \in \mathbb{Z} \) denotes the number (with orientations included) of trajectories inside the zero-dimensional (compact) space \( \mathcal{M}_z([\beta_0], [\beta_1]) = \mathcal{M}_z([\beta_0], [\beta_1]) / \mathbb{R} \). The issue is that for fixed \([\beta_1]\), the sum

\[ \sum_{z} n_z([\beta_0], [\beta_1]) [\beta_1] \]

could be infinite! To see why, suppose that \( z, z' \) represent two trajectories from \([\beta_0]\) to \([\beta_1]\) with \( \mathcal{M}_z([\beta_0], [\beta_1]) \) and \( \mathcal{M}_z([\beta_0], [\beta_1]) \) one dimensional. Then \( z^{-1} \circ z' \) represents a loop based at \([\beta_0]\) with \( \text{gr}_{z^{-1} \circ z'}([\beta_0], [\beta_0]) = 0 \). From (21) we see that the topological energy of this loop must be \( 32\pi^2 l(4\alpha - 1) \), which is a priori unbounded since \( \alpha \neq 1/4 \) and the monopole number \( l \) can in principle be any integer. Since \( \mathcal{E}_p(z) - \mathcal{E}_p(z') \) differ (up to sign) by \( 32\pi^2 l(4\alpha - 1) \), we see there is no way to guarantee the assumptions in Proposition 14.

The way to get out of this conundrum is via a Novikov system, but first we recall a few facts about the construction of homology groups using local coefficients. A good reference for the general construction is [13] Chapter 5. We will use \( \Gamma \) to denote a local systems of coefficients. Also, if \([\beta] \in \mathcal{B}(Y, K, \alpha)\), then \( o([\beta]) \) will denote the 2-element of orientations for \([\beta]\) ([11] Section 3.6).

Suppose that to each \([\beta]\) we assign an abelian group \( \Gamma_{[\beta]} \) and for each homotopy class of paths \( z \) from \([\beta_0]\) to \([\beta_1]\) there is an isomorphism \( \Gamma_z \) from \( \Gamma_{[\beta_0]} \) to \( \Gamma_{[\beta_1]} \) satisfying the usual composition law for paths.

Define the chain group \( C_* (Y, K, \alpha; \Gamma) \) generated by irreducible critical points \([\beta]\) of \( CS_p \) as

\[ C_* (Y, K, \alpha; \Gamma) = \bigoplus_{[\beta] \in \mathcal{E}} \mathbb{Z} o([\beta]) \otimes \Gamma_{[\beta]} \]

\footnote{The notation Kronheimer and Mrowka use for the system of orientations is \( \Lambda([\beta]) \) but we will use \( \Lambda \) for the Novikov field so a sacrifice had to be made somewhere.}
and the boundary map as
\begin{equation}
\partial = \sum_{([\alpha],[\beta],z)} \epsilon[\tilde{A}] \otimes \Gamma_z
\end{equation}
where \([A] \in \mathcal{M}_z([\beta_0],[\beta_1])\) belongs to a 1-dimensional space, \([\tilde{A}]\) denotes the connection modulo the \(\mathbb{R}\) action, and
\[
\epsilon[\tilde{A}] : o[\beta_0] \rightarrow o[\beta_1]
\]
is an isomorphism obtained by comparing orientations. Written differently, we can think of the differential as
\[
\partial(e) = \sum_{[\beta_1] \in \mathcal{E}} \sum_z n_z([\beta_0],[\beta_1]) \Gamma_z(e), \quad e \in \Gamma_{[\beta_0]}
\]
which we will occasionally write as
\[
\partial([\beta_0]) = \sum_{[\beta_1] \in \mathcal{E}} \sum_z n_z([\beta_0],[\beta_1]) \Gamma_z[\beta_1]
\]
After choosing trivializations for \(o([\beta_0])\) and \(o([\beta_1])\), the contribution for a given pair of critical points in the differential takes the form
\[
\sum_z n_z \Gamma_z
\]
where \(z\) runs through all relative homotopy classes satisfying the conditions
\[
gr_z([\beta_0],[\beta_1]) = 1
\]
Now, define the support
\[
\text{sup}([\beta_0],[\beta_1]) = \{z \mid n_z \neq 0\}
\]
By Proposition (14), for all \(C\), the intersection
\[
\text{sup}([\beta_0],[\beta_1]) \cap \{z \mid E_p(z) \leq C\}
\]
is finite. In the notation of definition 30.2.2 in [43], we will define a \(c\)-complete local system of coefficients which will allow us to make sense of infinite sums like \(\sum_z n_z \Gamma_z\).

A definition of the \(c\)-complete condition can be found in our annotated version [18, Definition 18], as well as some additional details. For our purposes we will be content with describing the \(c\)-complete local system we will be using throughout the paper, based on [43 Section 30.2], [31 Section 4], [46 Section 2.2], [68] and [69 Section 2.1].

In fact, there are a couple of natural \(c\)-complete systems one could use, but in order to minimize the number of auxiliary choices we have to make, we will stick with a particular local system which we will call the universal Novikov field/local system [69 Section 4]. Moreover, the universal Novikov field/local system has the advantage that the fibers \(\Gamma_{[\beta]}\) assigned to each \([\beta]\) is independent of \(\alpha\) (which does not happen for the other candidates), so it might be a more suited candidate if one wanted to compare the Floer homologies corresponding to different values of \(\alpha\) (though \(\Gamma_z\) will depend on \(\alpha\) which is to be expected as we will see in a moment).

First of all, if \(\mathbb{F}\) is a ground field and \(\Gamma \leq \mathbb{R}\) an additive subgroup, then the Novikov field \(A^{\mathbb{F},\Gamma}\) associated to \(\mathbb{F}, \Gamma\) is
\[
A^{\mathbb{F},\Gamma} = \left\{ \sum_{r \in \Gamma} a_r T^r \mid a_r \in \mathbb{F} \text{ and } \# \{r \mid a_r \neq 0, r > C\} < \infty \text{ for all } C \in \mathbb{R} \right\}
\]
In other words, we allow infinitely many terms in the negative direction. The example we will have in mind most of the time is \( \mathbb{F} = \mathbb{Q} \) or \( \mathbb{F} = \mathbb{C} \).

Now write \( \pi_1(\mathcal{B}) = \pi_1(\mathcal{B}(Y,K,\alpha)) = \mathbb{Z} \oplus \mathbb{Z} \), and consider the universal covering \( \tilde{\mathcal{B}}(Y,K,\alpha) \) of \( \mathcal{B}(Y,K,\alpha) \). This is the space we obtain after taking the quotient of \( \mathcal{C}(Y,K,\alpha) \) using only gauge transformations \( g \in \mathcal{G}(Y,K) \) which satisfy \( d(g) = (0,0) \) (so they are in the connected component of the identity gauge transformation). The Chern-Simons functional \( \text{CS} \) becomes real valued on \( \tilde{\mathcal{B}}(Y,K,\alpha) \) so we can regard it as a map

\[
\text{CS} : \tilde{\mathcal{B}}(Y,K,\alpha) \to \mathbb{R}
\]

In this context \( \tilde{\beta} \in \tilde{\mathcal{B}}(Y,K,\alpha) \) will denote a lift of \([\beta] \in \mathcal{B}(Y,K,\alpha)\). On \( \pi_1(\mathcal{B}(Y,K,\alpha)) \) we can also define the period homomorphism

\[
\Delta_{\text{CS}} : \pi_1(\mathcal{B}(Y,K,\alpha)) \to \mathbb{R}
\]

\[
g \to 64\pi^2(k + 2\alpha l)
\]

as well as the spectral flow map

\[
\text{sf} : \pi_1(\mathcal{B}(Y,K,\alpha)) \to \mathbb{Z}
\]

\[
z \to \text{sf}(z)
\]

whose kernel is the annihilator

\[
\text{Ann} = \ker \text{sf} \subset \pi_1(\mathcal{B}(Y,K,\alpha))
\]

We also define the additive subgroup of \( \mathbb{R} \)

\[
I = \text{im} \Delta_{\text{CS}} \subset \mathbb{R}
\]

The different local systems we will now explain depend on how they are related to \( I \), and each has their own set of advantages and disadvantages.

- **Minimal Novikov field/Local System**: here we work with Novikov field \( \Lambda^{F,I'} \), where \( I' \) is defined as

\[
I' = \Delta_{\text{CS}}(\text{Ann})
\]

Recalling the formula for the grading, this means that we must evaluate \( \Delta_{\text{CS}} \) on those loops for which

\[
8k + 4l = 0
\]

so in fact

\[
I' = \{64\pi^2k(1 - 4\alpha) \mid k \in \mathbb{Z}\}
\]

The appealing feature of \( \Lambda^{F,I'} \) is that when \( \alpha = 1/4 \) (the monotone case), \( I' \) collapses to \{0\}, in which case working over \( \Lambda^{F,I'} \) is the same as working over \( \mathbb{F} \), as should be the case in the monotone situation. The local system associated to \( \Lambda^{F,I'} \) will assign to each configuration \([\beta]\) a copy of \( \Lambda^{F,I'} \), i.e, \( I_{[\beta]} = \Lambda^{F,I'} \). To specify what happens on paths, i.e, what the maps \( \Gamma_z : I_{[\beta_0]} \to I_{[\beta_1]} \) should be, observe that since \( \Delta_{\text{CS}} : \pi_1(\mathcal{B}(Y,K,\alpha)) \to \mathbb{R} \) is a homomorphism whose domain is a finitely generated abelian group and whose image is torsion free, there is an exact sequence

\[
\ker \Delta_{\text{CS}} \to \pi_1(\mathcal{B}(Y,K,\alpha)) \to I
\]

which splits, so \( \pi_1(\mathcal{B}(Y,K,\alpha)) \) can be identified with \( \ker \Delta_{\text{CS}} \oplus I \), i.e,

\[
(23) \quad \pi_1(\mathcal{B}(Y,K,\alpha)) \simeq \ker \Delta_{\text{CS}} \oplus I
\]
By the same token, there is an exact sequence
\[ \ker \mathcal{f} \to \pi_1(\mathcal{B}(Y,K,\alpha)) \to 4\mathbb{Z} \]
which splits, so \( \pi_1(\mathcal{B}(Y,K,\alpha)) \) can be identified with \( \text{Ann} \oplus 4\mathbb{Z} \), i.e
\[ (24) \quad \pi_1(\mathcal{B}(Y,K,\alpha)) \simeq \text{Ann} \oplus 4\mathbb{Z} \]
This allows us to construction a projection
\[ p : I \to I' \]
as follows. Regard \( I \) as a subset of \( \pi_1(\mathcal{B}(Y,K,\alpha)) \) under the first identification. Then using the second identification project \( I \) onto \( \text{Ann} \) and let \( p \) be \( \Delta_{CS} \) of this element. In other words
\[ p(i) = \Delta_{CS}(\pi_{Ann}(i)) \]
Now, we can define \( \Gamma_z \) on loops as multiplication by \( T^{p(-\varepsilon_{top}(z))} \). Fixing the map \( \Gamma_z \) on loops is enough to pin down the local system up to isomorphism. The “issue” with this option is that it is not obvious how the resulting Floer groups depend on the choice of projection \( p \). In fact, we were unable to write explicitly a projection \( p \), so if one ever wanted to compute explicitly the resulting homology groups, it is not clear what to do in this circumstance.

- **Strongly c-complete Novikov field/Local System:** we call this intermediate case strongly c-complete in analogy with the examples given by Kronheimer and Mrowka in [13] Section 30.2. In this case we use \( A^F,I \) (so \( I \) instead of \( I' \)) as the Novikov system and a local system which assigns to \( [\beta] \) a copy of \( A^F,I \), i.e, \( \Gamma_{[\beta]} = A^F,I \). On loops \( \Gamma_z \) will act as multiplication by \( T^{-\varepsilon_{top}(z)} \). Notice that this system has the advantage that we no longer need to choose a projection \( p \). However, in order to specify actual groups (and not groups up to isomorphisms), further choices are needed, since knowing how \( \Gamma_z \) is defined on loops is not enough to specify a unique formula for how it should act on paths. More precisely, for a path \( z : [\beta_0] \to [\beta_1] \) we would like \( \Gamma_z : \Gamma_{[\beta_0]} \to \Gamma_{[\beta_1]} \) to be multiplication by \( T^{-\varepsilon_{top}(z)} \). However, notice that even for a simple element inside \( \Gamma_{[\beta_0]} \) like \( T^2_{\text{CS}(g)} \), for \( g \in \pi_1([\beta_0],\mathcal{B}(Y,K,\alpha),[\beta_0]) \), there is no reason why we must have that \( \Gamma_z T^2_{\text{CS}(g)} = T^{-\varepsilon_{top}(z)} T^2_{\text{CS}(g')} \) for some \( g' \in \pi_1([\beta_1],\mathcal{B}(Y,K,\alpha),[\beta_1]) \). However, with additional choices we make sense of this formula. Namely, suppose that we choose a preferred lift \( \tilde{\beta} \) for each \( [\beta] \in \mathcal{B}(Y,K,\alpha) \). Then from formula [15] we have that \( \Delta_{CS}(g) = 2 \left( CS(\beta) - CS(g'\tilde{\beta}) \right) \), which means that an element inside \( \Gamma_{[\beta_0]} \) can be rewritten as a formal power series
\[
\sum_{r=\Delta_{CS}(g)} c_r T_{\Delta_{CS}(g)} = T^{2CS(\tilde{\beta}_0)} \sum_{r=\Delta_{CS}(g)} c_r T^{-2CS(g'\tilde{\beta}_0)}
\]
Then \( \Gamma_z = \Gamma_{z,\tilde{\beta}_0,\tilde{\beta}_1} \) can be taken to be
\[
\Gamma_{z,\tilde{\beta}_0,\tilde{\beta}_1} \left( \sum_{r=\Delta_{CS}(g)} c_r T_{\Delta_{CS}(g)} \right) \equiv T^{2CS(\tilde{\beta}_1)} \sum_{r=\Delta_{CS}(g)} c_r T^{-2CS(zg'\tilde{\beta}_0)}
\]
Notice that we added the “basepoints” in our notation to make it clear that an additional choice is needed. Here \( z \cdot g \cdot \tilde{\beta}_0 \) means the following: regard \( g \cdot \tilde{\beta}_0 = \text{an element in } \mathcal{B}(Y,K,\alpha) \), projecting to \( [\beta_0] \in \mathcal{B}(Y,K,\alpha) \), and use the path \( z \) to determine a (unique) element \( z \cdot g \cdot \tilde{\beta}_0 \), projecting to \( [\beta_1] \in \mathcal{B}(Y,K,\alpha) \). Moreover, this formula uses implicitly the fact that if \( r = \Delta_{CS}(g) = \Delta_{CS}(g') \), i.e, \( CS(g'\tilde{\beta}) = CS(g'\tilde{\beta}) \) or equivalently \( g' \circ g^{-1} \in \ker \Delta_{CS} \), then \( CS(z \cdot g \cdot \tilde{\beta}_0) = CS(z \cdot g' \cdot \tilde{\beta}_0) \). In any case, these extra choices of basepoints will not make our lives any easier and do not gives us
much of an advantage, so we will prefer to use the universal Novikov/ local system, which we will describe next.

- **Universal Novikov field/Local System:** as the name suggests, here we are working with $A^{F,R}$. The local system assigns $[\beta]$ a copy of $A^{F,R}$, i.e., $\Gamma_{[\beta]} = A^{F,R}$. In order to define $\Gamma_z$ for a path $z : [\beta_0] \to [\beta_1]$, we lift $z$ to a path $\tilde{z} : \tilde{\beta}_0 \to \tilde{\beta}_1$ on the universal cover $\tilde{B}(Y,K,\alpha)$ (where again $CS$ is real valued), and define

$$\Gamma_z \equiv \text{multiplication by } T^{-2(CS(\tilde{\beta}_0) - CS(\tilde{\beta}_1))}$$

The factor of 2 in the exponent is chosen so that on loops it acts as multiplication by $T^{-E_{exp}(z)}$. Because of the gauge invariance of the perturbation $p$ we could also write this as

$$T^{-2(CS\rho(\tilde{\beta}_0) - CS\rho(\tilde{\beta}_1))}T^{-2(f_p([\beta_1]) - f_p([\beta_0]))}$$

Notice that the second term is independent of the lift of $[\beta_0], [\beta_1]$. In a way, it would have been more tempting to let $\Gamma_z$ depend on the perturbation being used and define $\Gamma_{z,p}$ as multiplication by $T^{-2(CS\rho(\tilde{\beta}_0) - CS\rho(\tilde{\beta}_1))}$. This would give the advantage that for gradient flow-lines the exponent $-2(CS\rho(\tilde{\beta}_0) - CS\rho(\tilde{\beta}_1))$ would always be negative, hence making automatic the condition that the differential is well defined on the chain complex. However, once we analyze the case of cobordisms, it is more tricky to figure out what the right notion of perturbed topological energy is, given that we have to use many different kinds of holonomy perturbations, so this monotonicity property would not be automatic anyways. In other words, there is no way to avoid the fact that some control on the perturbations has to be imposed, and this is precisely what we will do in the next section (based on ideas from [13]).

4. Floer-Novikov Homology and its Relationship To the Knot Signature

Recall that for the construction of the usual instanton Floer homology groups on an integer homology sphere $Y$ [20], the chain complex $CI(Y)$ is generated by (perturbed) irreducible flat connections and the proof that the differential $\partial$ on $CI(Y)$ squares to zero requires showing that no broken flow lines can factorize through the trivial flat connection $\theta$, which cannot be eliminated through the use of perturbations.

Moreover, the proof that the homology $HI(Y)$ one obtains does not depend on the choice of metric and perturbations, requires using the fact that as we deform this data, we do not meet reducibles, since once again they are singular points in the space of connections mod gauge.

To show these properties one uses the fact that the stabilizer of the trivial connection $\theta$ is positive dimensional and that the trivial connection is isolated from the irreducible (perturbed) flat connections. Even if there were no need for using local coefficients suitable analogues of these results are needed in our situation, which is what we will proceed to discuss next.

Recall that the critical points $B$ of the unperturbed Chern-Simons functional $CS$ on $(Y,K,\alpha)$ can be interpreted as flat connections on $Y \setminus K$, which modulo gauge will correspond to conjugacy classes of representations

$$\rho : \pi_1(Y \setminus K, y_0) \to SU(2)$$

Since $H_1(Y \setminus K ; \mathbb{Z}) \simeq \mathbb{Z}[\mu_K]$ regardless of the pair $(Y,K)$, the abelian representations of $\pi_1(Y \setminus K, y_0)$ are completely determined by their action on an oriented meridian $\mu_K$ of the knot $K$, which we
already required to be conjugate to the matrix

\[ \rho(\mu_K) \sim \begin{pmatrix} \exp(-2\pi i\alpha) & 0 \\ 0 & \exp(2\pi i\alpha) \end{pmatrix} \]

In other words, for any fixed choice of \( \alpha \), there is only one reducible critical point of the unperturbed Chern-Simons functional \( CS \), which we denoted previously as \( [\theta_\alpha] \).

In general, a representation \( \rho : \pi_1(Y\setminus K, y_0) \to SU(2) \) will determine a local coefficient system \( g_\rho \) on \( Y\setminus K \), with fiber \( g = \mathfrak{su}(2) \). In turn this gives rise to cohomology groups \( H^1(Y\setminus K; g_\rho) \). If we identify \( \rho \) with a flat connection \( B \), then its gauge orbit \([B]\) being isolated among the set of critical points \( \mathcal{C} \) of \( CS \) is equivalent to \( B \) being non-degenerate, that is, the kernel of the map

\[ \ker : H^1(Y\setminus K; g_\rho) \to H^1(\mu_K; g_\rho) \]

is zero, where here \( \mu_K \) is a collection of loops representing the meridians of all the components of \( K \) (which will be one since we are focusing on the case of a knot).

Here is an explanation of this condition as well of the notation. The map \( (24) \) is simply the one induced in cohomology by the inclusion of \( \mu_K \hookrightarrow Y\setminus K \). When \( B \) is an irreducible flat connection, Lemma 3.13 in [41] shows that this condition is equivalent to the extended Hessian \( \tilde{\text{Hess}}_B \) defined in [19] being invertible.

At the reducible flat connection \( \theta_\alpha \), one can still use \( (25) \) as the criterion for being isolated, the only thing that changes is that this condition does not imply that the extended Hessian \( \tilde{\text{Hess}}_{\theta_\alpha} \) is invertible, since at reducible connections it will always have a non-trivial kernel [17], Section 2.5.4]. To better understand the requirement \( (25) \) at the reducible \( \theta_\alpha \), notice that the Lie algebra \( \mathfrak{g}_{\theta_\alpha} \simeq \mathfrak{su}(2) \simeq \mathfrak{so}(3) \) decomposes as

\[ \mathfrak{g}_{\theta_\alpha} \simeq \mathbb{R} \oplus L^{\otimes 2}_\alpha \]

where \( E = L_\alpha \oplus L^{-1}_\alpha \) is the decomposition of the \( SU(2) \) bundle induced by \( \theta_\alpha \). Hence

\[ H^1(Y\setminus K; \mathfrak{g}_{\theta_\alpha}) \simeq H^1(Y\setminus K; \mathbb{R}) \oplus H^1(Y\setminus K; L^{\otimes 2}_\alpha) = \mathbb{R} \oplus H^1(Y\setminus K; L^{\otimes 2}_\alpha) \]

One should think of the \( \mathbb{R} \) summand as being the directions in the Zariski tangent space obtained by deforming the value of the holonomy \( \alpha \). Given that \( \alpha \) is fixed for our problem and \( H^1(Y\setminus K; \mathbb{R}) \to H^1(\mu_K; g_\rho) \) is an injection since \( H^1(\mu_K; g_\rho) \simeq H^1(\mu_K; \mathbb{R}) \) [18], Lemma 63], the condition \( (25) \) captures that we only need to worry about about the factor \( H^1(Y\setminus K; L^{\otimes 2}_\alpha) \). As we mentioned in the introduction, the condition \( (25) \) is only satisfied for certain values of \( \alpha \) determined by the Alexander polynomial \( \triangle_K(t) \) of \( K \).

**Lemma 15.** Suppose that \( K \) is a knot and that \( \triangle_K(e^{-4\pi i\alpha}) \neq 0 \). Then the reducible connection \( [\theta_\alpha] \) is isolated from the irreducible flat connections.

**Proof.** Again, it suffices to guarantee that \( H^1(Y\setminus K; L^{\otimes 2}_\alpha) \) vanishes. In the notation of [18], Appendix], the reducible connection \( [\theta_\alpha] \) defines a local coefficient system with fiber \( \mathbb{C} \) where the monodromy map \( \tilde{\rho}_\alpha : H_1(Y\setminus K; \mathbb{Z}) \to \mathbb{C}^* \) maps the meridian \( \mu_K \) to \( e^{-4\pi i\alpha} \). The vanishing of the cohomology \( H^1(Y\setminus K; L^{\otimes 2}_\alpha) \) now follows from [18], Corollary 65].

For completeness sake, we will now give more details to Lemma 3.13 in [11], since our method of proof will generalize to the case of an embedded torus \( T \) inside a four manifold \( X \) with the homology of \( S^1 \times S^3 \).
Lemma 16. Suppose that \( \rho \) is a flat connection corresponding to a critical point of \( CS \) and \( \mathcal{G}_\rho \) is the corresponding local system. Then the first (orbifold) cohomology \( \hat{H}^1(\hat{Y}; \mathcal{G}_\rho) \) can be identified with \( \ker : H^1(Y\setminus K; \mathcal{G}_\rho) \to H^1(\mu_K; \mathcal{G}_\rho) \).

Proof. For the proof we will follow the suggestion of Lemma 3.13 in [41] and use the Mayer-Vietoris sequence for cohomology with local coefficients. More precisely, for \( \epsilon > 0 \) write \( \hat{Y} = (\hat{Y}\setminus \hat{\nu}_\epsilon(K)) \cup \hat{\nu}(K) \) where \( \hat{\nu}_\epsilon(K), \hat{\nu}(K) \) are tubular neighborhoods of \( K \), and \( \hat{\nu}_\epsilon(K) \subset \hat{\nu}(K) \). Then the first terms in the Mayer-Vietoris sequence for this decomposition read

\[
\cdots \to \hat{H}^0((\hat{Y}\setminus \hat{\nu}_\epsilon(K)) \cap \hat{\nu}(K); \mathcal{G}_\rho) \\
\to \hat{H}^1(\hat{Y}; \mathcal{G}_\rho) \\
\to (i_{\hat{Y}\setminus \hat{\nu}_\epsilon(K)}^*; i_{\hat{\nu}(K)}^*) \hat{H}^1(\hat{Y}\setminus \hat{\nu}_\epsilon(K); \mathcal{G}_\rho) \oplus \hat{H}^1(\hat{\nu}(K); \mathcal{G}_\rho) \\
\to \hat{H}^1((\hat{Y}\setminus \hat{\nu}_\epsilon(K)) \cap \hat{\nu}(K); \mathcal{G}_\rho) \\
\to \hat{H}^2(\hat{Y}; \mathcal{G}_\rho) \\
\to \cdots
\]

Here \( i_{\hat{Y}\setminus \hat{\nu}_\epsilon(K)}^* \) and \( i_{\hat{\nu}(K)}^* \) denote the pullback in cohomology of the inclusions \( i : \hat{Y}\setminus \hat{\nu}_\epsilon(K) \hookrightarrow \hat{Y} \) and \( i : \hat{\nu}(K) \hookrightarrow \hat{Y} \). Now using the usual properties of invariance of the cohomology groups under deformation retractions we can rewrite the previous sequence as

\[
\cdots \to \hat{H}^0(T_\epsilon; \mathcal{G}_\rho) \\
\to \hat{H}^1(\hat{Y}; \mathcal{G}_\rho) \\
\to (i_{\hat{Y}\setminus \hat{\nu}_\epsilon(K)}^*; i_{\hat{\nu}(K)}^*) \hat{H}^1(Y\setminus K; \mathcal{G}_\rho) \oplus \hat{H}^1(\hat{\nu}(K); \mathcal{G}_\rho) \\
\to i_{\hat{Y}\setminus \hat{\nu}_\epsilon(K)}^*H^1(T_\epsilon; \mathcal{G}_\rho) \\
\to \cdots
\]

where \( T_\epsilon = \partial \hat{\nu}_\epsilon(K) \). Notice that we have dropped the \( ^* \) notation on the regions where the orbifold singularity is not present.

Observe first of all that since \( T_\epsilon \) has abelian fundamental group then the restriction of \( \mathcal{G}_\rho \) to \( T_\epsilon \) automatically becomes reducible and we can write

\[
\mathcal{G}_\rho \mid_{T_\epsilon} = \mathbb{R} \oplus L_\rho^{\otimes 2}
\]

Now, because of the holonomy condition, for \( \epsilon \) sufficiently small the local system defined by \( L_\rho^{\otimes 2} \) is non-trivial, which means in particular that

\[
H^\bullet(T_\epsilon; L_\rho^{\otimes 2}) \equiv 0
\]
by \cite{ES} Lemma 63. If we write $\mu_K$ and $\lambda_K$ for the meridian and longitudes of $K$, then the Mayer-Vietoris sequence simplifies to

$$\cdots \rightarrow \tilde{H}^1(\tilde{Y}; \mathfrak{g}_p) \rightarrow \tilde{H}^1(Y \setminus K; \mathfrak{g}_p) \oplus \tilde{H}^1(\tilde{\nu}(K); \mathfrak{g}_p) \rightarrow \tilde{H}^2(\tilde{Y}; \mathfrak{g}_p) \rightarrow \cdots$$

(26)

where we regard $\tilde{D}^2$ as an orbifold. In fact, we can think of $\tilde{D}^2$ as $D^2/\mathbb{Z}_\nu$, where $\mathbb{Z}_\nu$ denotes a cyclic action determining the cone angle $2\pi/\nu$. If $p: D^2 \setminus \{0\} \rightarrow \tilde{D}^2 \setminus \{0\}$ denotes the quotient map, then as long as the cone angle is sufficiently sharp and $\alpha$ is a rational value, the pullback of the flat connection $\rho$ to $D^2 \setminus \{0\}$ extends smoothly to a flat connection $p^*\rho$ on all of $D^2$, and in fact we can identify $\tilde{H}^1(\tilde{D}^2; \mathfrak{g}_p)$ with the equivariant cohomology $H^{1,\nu}(D^2; p^*\mathfrak{g}_p)$. Hence

$$\tilde{H}^1(\tilde{\nu}(K); \mathfrak{g}_p) = H^0(S^1; \mathfrak{g}_p) \otimes H^{1,\nu}(D^2; p^*\mathfrak{g}_p) \oplus H^1(S^1; \mathfrak{g}_p) \otimes H^{0,\nu}(D^2; p^*\mathfrak{g}_p)$$

(27)

Now, the factor $H^{1,\nu}(D^2; p^*\mathfrak{g}_p)$ will vanish since $D^2$ equivariantly retracts to the origin so that

$$H^{1,\nu}(D^2; p^*\mathfrak{g}_p) \simeq H^{1,\nu}(\{0\}, p^*\mathfrak{g}_p) = 0$$

Likewise, we have that

$$H^{0,\nu}(D^2; p^*\mathfrak{g}_p) \simeq H^{0,\nu}(\{0\}; p^*\mathfrak{g}_p) \simeq \mathbb{R}$$

since only one factor of $\mathbb{R}^3 \simeq p^*\mathfrak{g}_p |_{\{0\} \subset D^2}$ is preserved under the group action. To compute $H^1(S^1; \mathfrak{g}_p)$ we use again the fact that $S^1$ has abelian fundamental group which in particular means that $\mathfrak{g}_p$ reduces to the system

$$\mathfrak{g}_p |_{S^1} \simeq \mathbb{R} \oplus L^\otimes_\rho$$

In this case, $L^\otimes_\rho$ may or may not be the trivial system which means that

$$H^1(S^1; \mathfrak{g}_p) = \begin{cases} H^1(S^1; \mathbb{R}) & \text{if } L^\otimes_\rho |_{S^1} \text{ non-trivial} \\ H^1(S^1; \mathbb{R}) \otimes \mathbb{R}^3 & \text{if } L^\otimes_\rho |_{S^1} \text{ trivial} \end{cases}$$

(28)

In any case that means that

$$\hat{H}^1(\hat{\nu}(K); \mathfrak{g}_p) \simeq H^1(S^1_{\lambda_K}; \mathfrak{g}_p)$$

(29)

so the maps in the Mayer-Vietoris sequence become

(30) $\hat{H}^1(\hat{Y}; \mathfrak{g}_p) \rightarrow \hat{H}^1(\hat{Y} \setminus K; \mathfrak{g}_p) \oplus \hat{H}^1(\hat{\nu}(K); \mathfrak{g}_p)
\hat{\omega} \mapsto (\hat{\omega} |_{Y \setminus K}, \hat{\omega} |_{\tilde{\nu}(K)})$

and

(31) $H^1(Y \setminus K; \mathfrak{g}_p) \oplus \hat{H}^1(\hat{\nu}(K); \mathfrak{g}_p) \rightarrow \tilde{i}_{\nu,K}^* \mathfrak{g}_p \oplus \tilde{i}_{\nu,K}^* \mathfrak{g}_p
(\omega, \tilde{\omega}) \mapsto \left(\omega |_{S^1_{\mu_K}, [\mu_K]}, (\omega - \tilde{\omega}) |_{S^1_{\lambda_K}, [\lambda_K]}\right)$. 

\[\begin{array}{c}
\right)\end{array}\]
where the notation means that we are restricting the forms to the \( S^1 \) factors generated by the meridian and longitude respectively, and then using the pairing coming from the isomorphisms we established previously. In particular, since Mayer-Vietoris is an exact sequence this means that for any \( \tilde{\omega} \in \check{H}^1(\check{Y}; g_p) \) we have
\[
\left( i^*_{T_\omega(K)} - i^*_{T_{\omega,Y\setminus K}} \right) \circ (i^*_{Y\setminus K, i^*_{\check{Y},K}}) \tilde{\omega} = 0
\]
which according to the formulas 30 and 31 imply that
\[
\left\langle \tilde{\omega} \mid S^1_{\mu_K}, [\mu_K] \right\rangle = 0
\]
In other words, there is a well defined map
\[
i^*: \check{H}^1(\check{Y}; g_p) \to \left\{ \omega \in H^1(Y \setminus K; g_p) \mid \left\langle \omega \mid S^1_{\mu_K}, [\mu_K] \right\rangle = 0 \right\}
\]
or more succinctly \( i^*: \check{H}^1(\check{Y}; g_p) \to \ker( H^1(Y \setminus K; g_p) \to H^1(m; g_p) ) \) as we wrote in the statement of the lemma. We just need to show that this map is an isomorphism.

First we address the surjectivity of the map. Suppose that \( \omega \in H^1(Y \setminus K; g_p) \) and that \( \left\langle \omega \mid S^1_{\mu_K}, [\mu_K] \right\rangle = 0 \). Using the isomorphism [29] we can consider the element
\[
\left( \omega, \omega \mid S^1_{\mu_K} \right) \in H^1(Y \setminus K; g_p) \oplus \check{H}^1(\check{Y}; g_p)
\]
Because \( \left\langle \omega \mid S^1_{\mu_K}, [\mu_K] \right\rangle = 0 \) we have that \( \left( \omega, \omega \mid S^1_{\mu_K} \right) \in \ker \left( i^*_{T_{\omega,Y\setminus K}} - i^*_{T_{\omega,Y\setminus K}} \right) \) which means by exactness of the Mayer-Vietoris sequence that \( \left( \omega, \omega \mid S^1_{\mu_K} \right) \in \text{im}(i^*_{T_{\omega,Y\setminus K}}) \). In other words, there is \( \tilde{\omega} \in \check{H}^1(\check{Y}; g_p) \) such that \( i^*(\tilde{\omega}) = \omega \), which is what we wanted to show.

For injectivity we simply observe that if \( i^*(\tilde{\omega}) = [0] \) then \( \omega = \tilde{\omega} \mid Y \setminus K \) is exact, that is, \( \omega = d_B \xi \) for some \( \xi \in \Omega^0(Y \setminus K; g_p) \) and \( B \) is the connection representative of the flat connection \( \rho \). Now we can compute the norm of \( \tilde{\omega} \) in the following way, which is essentially the same computation as in [28] Proposition 2.10
\[
\|\tilde{\omega}\|_{L^2(Y)}^2 = - \lim_{\epsilon \to 0} \int_{Y \setminus \phi_\epsilon(K)} \text{tr}(*\omega \wedge \omega)
\]
\[
= - \lim_{\epsilon \to 0} \int_{Y \setminus \phi_\epsilon(K)} \text{tr}(*\omega \wedge d_B \xi)
\]
\[
= \lim_{\epsilon \to 0} \left[ \int_{Y \setminus \phi_\epsilon(K)} \text{tr}(d_B(*\omega) \wedge \xi) - \int_{T_\epsilon} \text{tr}(*\omega \wedge \xi) \right]
\]
\[
= - \lim_{\epsilon \to 0} \int_{T_\epsilon} \text{tr}(*\omega \wedge \xi)
\]
\[
= 0
\]
where we have used that \( d_B(*\omega) = *d_B^* \omega = 0 \) since \( \omega \in \ker \tilde{\Delta}_B = \ker(d_B^*) \cap \ker(d_B) \). That \( \lim_{\epsilon \to 0} \int_{T_\epsilon} \text{tr}(*\omega \wedge \xi) = 0 \) vanishes is essentially a restatement that on orbifolds one can use integration by parts [15] Section 2.1. Alternatively, one can follow the approximation scheme in the proof of [37] Proposition 8.3. Hence \( \|\tilde{\omega}\|_{L^2(Y)}^2 = 0 \) which also means that \( \tilde{\omega} = 0 \), which is what we needed to show for the proof of injectivity. \( \square \)
Using values of \( \alpha \in \mathbb{Q} \cap (0, 1/2) \) which satisfy \( \triangle_K(e^{-4\pi i \alpha}) \neq 0 \), and with a local system which satisfies the properties discussed in the previous section, we can define the chain complex

\[
(C_*(Y, K, \alpha; \Gamma, p), \partial)
\]

where we are also indicating the dependence of the chain complex on the perturbation used to achieve transversality. Again, the local coefficient system we will have in mind is the universal Novikov field/local system described in the previous section, in order to minimize additional choices to get an actual Floer group, rather than a group up to isomorphism.

We will choose our perturbation \( p \) as

\[
p = p_{\text{crit}} + p_{\partial}
\]

where \( p_{\text{crit}} \) and \( p_{\partial} \) mean the following. If \( p_{\text{crit}} \) is a perturbation which makes the critical set \( \mathcal{C}_{p_{\text{crit}}} \) non-degenerate, then we can approximate \( p_{\text{crit}} \) by a finite sum of holonomy perturbations. Since the non-degeneracy of the compact set \( \mathcal{C}_{p_{\text{crit}}} \) is an open condition, we can arrange that \( f_{p_{\text{crit}}} \) is a finite sum by truncating \( p_{\text{crit}} \) after finitely many terms. This guarantees that the support of the holonomy perturbations do not meet a neighborhood of the knot \( K \). This argument appears in Proposition 3.1 of [42], although the setup in Kronheimer and Mrowka’s paper is more complicated since they also need to guarantee certain finiteness condition for the moduli spaces of flow lines of dimension less or equal to 2. Then \( p_{\partial} \) is the remaining perturbation needed to cut out the moduli spaces of trajectories transversely.

Because of this support condition, we can appeal to Herald’s result [30, Theorem 0.1], which relates the signed count of elements in the (finite) set \( \mathcal{C}_{p_{\text{crit}}} \) to the Casson invariant and the knot signature of \( K \).

**Theorem 17.** Suppose that \( \alpha \in \mathbb{Q} \cap (0, 1/2) \) satisfies \( \triangle_K(e^{-4\pi i \alpha}) \neq 0 \). Then the signed count of \( \mathcal{C}_{p_{\text{crit}}} \), which we denote \( \#_s[\mathcal{C}_{p_{\text{crit}}}] \), equals

\[
\#_s[\mathcal{C}_{p_{\text{crit}}}] = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i \alpha})
\]

where \( \lambda_C(Y) \) is the Casson invariant of \( Y \) and \( \sigma_K(e^{-4\pi i \alpha}) \) the Tristram-Levine knot signature of \( K \) evaluated at \( e^{-4\pi i \alpha} \).

Moreover, as vector spaces over the Novikov field \( \Lambda \), we have that

\[
\chi_A(HI(Y, K, \alpha)) = 4\lambda_C(Y) + \frac{1}{2}\sigma_K(e^{-4\pi i \alpha})
\]

Notice that the second part of previous statement implicitly assumed:

a) A choice for the orientation of the moduli spaces involved in the definition of \( \mathcal{C}_{p_{\text{crit}}} \) and \( HI(Y, K, \alpha) \).

b) The claim that there is a well defined differential \( \partial \) which squares to zero so that we can take \( HI(Y, K, \alpha) = \ker \partial/\im \partial \).

c) A proof that \( HI(Y, K, \alpha) \) is independent of the perturbations used.

We will start by discussing the orientation of the moduli spaces, which we have kept under the rug until this point. For our conventions we will follow closely the exposition in [63 Section 4.3], which in turn are based on those used in [17 [40 [41]. They also agree with the conventions used by Collin and Steer in their paper [11], which we used to pin down the signs in the formula [42].

The orientation set \( o([\beta]) \) of a critical point \( \mathcal{C}_p \) refers to the 2-element set of orientations of the real line \( \det D_A \), where \( A \) is a connection on \( \mathbb{R} \times Y \) such that \( A \big|_{\{t\} \times Y} \) is (gauge) equivalent to \( \theta_\alpha \) for \( t \) sufficiently negative and (gauge) equivalent to \( \beta \) for \( t \) sufficiently large. Here \( D_A \) is the
Fredholm operator \(-d_A^* \oplus d_A^+\), after suitable Sobolev completions have been introduced. In general any reference connection would work, but the advantage of using the reducible connection \(\theta_a\) as one of the limiting connections is that it will automatically gives us an absolute \(\mathbb{Z}/4\mathbb{Z}\) grading as we will explain momentarily.

In general, given two connections \(\beta_1 \in C(Y_1, K_1, \alpha)\) and \(\beta_2 \in C(Y_2, K_2, \alpha_2)\) and a cobordism

\[(W, \Sigma, \alpha) : \emptyset \to (-Y_1, -K_1, \alpha) \sqcup (Y_2, K_2, \alpha)\]

one can consider the operator (for \(\epsilon > 0\) sufficiently small)

\[D'_A = -d_A^* \oplus d_A^+ : \tilde{L}_m^p(A^1 \otimes g_P) \to \tilde{L}_m^{p-1,\phi'_t}((A^0 \oplus A^+) \otimes g_P)\]

where \(A \in C(W^*, \Sigma^*, \alpha)\) is a connection on the completion

\[W^* = (\mathbb{R}^- \times Y_1) \cup W \cup (\mathbb{R}^+ \times Y_2)\]
\[\Sigma^* = (\mathbb{R}^- \times K_1) \cup W \cup (\mathbb{R}^+ \times K_2)\]

which is asymptotic to \(\beta_1\) for \(t\) sufficiently negative and to \(\beta_2\) for \(t\) sufficiently positive. Here \(\tilde{L}_m^p, \phi'_t = e^{\phi'_t} \tilde{L}_m^p\) denotes a weighted Sobolev space, weighted by a real function \(e^{\phi'_t}\), where \(\phi'_t\) is a non-positive smooth function equal to \(et\) for \(t\) sufficiently negative on \(\mathbb{R}^- \times Y_1\), to \(-et\) for \(t\) sufficiently large on \(\mathbb{R}^+ \times Y_2\), and equal to 0 on \(W\).

When \(\beta_1\) is a perturbed instanton on \((Y_1, K_1, \alpha, p_1)\), with respect to the perturbation \(p_1\), and similarly for \(\beta_2\) with respect to the perturbation \(p_2\) on \((Y_2, K_2, \alpha, p_2)\), then the moduli space \(\mathcal{M}(\beta_1, W, [\beta_2])\) of perturbed \(\alpha\)-ASD instantons, which we will denote as \(\mathcal{M}(\beta_1, W, [\beta_2])\) will decompose into connected components

\[\mathcal{M}(\beta_1, W, [\beta_2]) = \bigcup_z \mathcal{M}_z(\beta_1, W, [\beta_2])\]

and the dimension of \(\mathcal{M}_z(\beta_1, W, [\beta_2])\) can be computed as \(\text{ind}D'_A\), where \(A\) is an appropriate representative of an element \([A] \in \mathcal{M}_z(\beta_1, W, [\beta_2])\). For a composite cobordism, one has the additivity relation (assuming the critical points are non-degenerate) [33 eq. 4.3]

\[
\dim \mathcal{M}_{z_0\circ z}(\beta_0, W', [\beta_2]) = \dim \mathcal{M}_z(\beta_0, W, [\beta_1]) + \dim \mathcal{M}_{z'}(\beta_1, W', [\beta_2])
\]

(33)

When \([\beta_0], [\beta_1]\) are irreducible (perturbed) flat connections on \((Y, K, \alpha)\), we had defined in Section 3 a relative grading \(\text{gr}([\beta_0], [\beta_1]) \in \mathbb{Z}/4\mathbb{Z}\) in terms of the spectral flow of the extended Hessian. Since the spectral flow of this operator can be interpreted as the index of the operator \(D'_A\) on the cylinder \(\mathbb{R} \times Y\), this means that the relative grading can be interpreted as the dimension (mod-4) of the moduli space of flow-lines between \([\beta_0]\) and \([\beta_1]\):

\[\text{gr}([\beta_0], [\beta_1]) = \dim \mathcal{M}([\beta_0], [\beta_1]) \mod 4\]

where again we are looking at moduli spaces asymptotic to \([\beta_0]\) as \(t \to -\infty\) and to \([\beta_1]\) as \(t \to \infty\).

To make this an absolute grading, set

\[\text{gr}([\beta]) = -1 - \dim \mathcal{M}([\theta_\alpha], [\beta]) \mod 4 = \dim \mathcal{M}([\beta], [\theta_\alpha]) \mod 4\]

(34)
In other words, the candidate for the differential \( (36) \) does make sense as an element \( (35) \) of
\[
\int_{\theta} \omega = \dim \mathcal{M}(\theta, [\beta_1]) - \dim \mathcal{M}(\theta, [\beta_0]) \quad \text{mod 4}
\]
\[
= \dim \mathcal{M}(\theta, [\beta_1]) + \dim \mathcal{M}(\theta, [\beta_0]) + 1 \quad \text{mod 4}
\]
\[
= \dim \mathcal{M}(\theta, [\beta_1]) \quad \text{mod 4}
\]
\[
= \text{gr}(\beta_0, [\beta_1])
\]

With respect to this absolute grading, if \( \mathcal{E}_{p,i} \) denotes the set of critical points associated to the perturbation \( p \) whose absolute grading is \( i \in \mathbb{Z}/4\mathbb{Z} \), then an expression like \( (32) \) means
\[
\# \mathcal{E}_{p, \text{crit}} = \# \mathcal{E}_{p, \text{crit}, 0} + \# \mathcal{E}_{p, \text{crit}, 2} - \# \mathcal{E}_{p, \text{crit}, 1} - \# \mathcal{E}_{p, \text{crit}, 3}
\]

Now we must turn to a discussion of the properties of the perturbations used in order to guarantee that we have a well defined differential and that the homology groups we obtain are independent of the perturbations used. First we need to understand what is at stake. Recall that we are assigning to each element \( [\beta] \in \mathcal{B}(Y, K, \alpha) \) a vector space \( \mathcal{I}_{[\beta]} \) which is a copy of the universal Novikov field \( \Lambda^\mathbb{Q, \mathbb{R}} \). In other words, an element of \( \mathcal{I}_{[\beta]} \) is a formal power series
\[
(35) \sum_{r \in \mathbb{R}} a_r T^r
\]
where \( a_r \in \mathbb{Q} \), and \( \# \{ r \mid a_r \neq 0, r > C \} < \infty \) for all \( C \in \mathbb{R} \). Therefore, the power series can extend indefinitely in the negative direction, but not the positive one. Now, the differential \( \partial \) on the chain complex \( C_*(Y, K, \alpha, F; p) \) should be defined by the formula (for \( [\beta_1] \in \mathcal{E}(Y, K, \alpha, p) \))
\[
(36) \partial [\beta_1] = \sum_{[\beta_2] \in \mathcal{E}(Y, K, \alpha, p) \mid \text{gr}(\beta_1, [\beta_2]) = 1 \mod 4} \sum_{x} n_x([\beta_1], [\beta_2]) T^{-\mathcal{E}_{\text{top}}(x)} [\beta_2]
\]
Here
\[
\mathcal{E}_{\text{top}}(z) = \frac{1}{8\pi^2} \int_{\mathbb{R} \times Y} \text{tr}(F_A \wedge F_A)
\]
where \( [A] \in \mathcal{M}_z([\beta_1], [\beta_2]) \). In the absence of perturbations, i.e, \( p = 0 \) then \( \mathcal{E}_{\text{top}}(z) \) would equal
\[
\frac{1}{8\pi^2} \| F_A^+ \|_{L^2} = 2(CS(\beta_1) - CS(\beta_2)),
\]
since \( A \) would solve the \( \alpha \)-ASD equation \( F_A^+ = 0 \). Therefore, the exponents in \( T^{-\mathcal{E}_{\text{top}}(z)} \) would always be non-positive and do not accumulate on any finite subinterval, since they must always differ from each other by some integer combination of \( 64\pi^2 \) and \( 128\pi^2 \alpha \) (this follows from the formula \( (10) \) for the topological energy in the closed 4-manifold case). In other words, the candidate for the differential \( (36) \) does make sense as an element \( (35) \) of \( \Lambda^\mathbb{Q, \mathbb{R}} \).

In the presence of perturbations \( p \), the inequality that is satisfied is
\[
\mathcal{E}_{\text{top}}(z) + f_p([\beta_1]) - f_p([\beta_2]) \geq 0
\]
Since \( f_p \) is fully gauge equivariant as we mentioned when we introduced the holonomy perturbations, the lower bound for \( \mathcal{E}_{\text{top}}(z) \)
\[
(37) \mathcal{E}_{\text{top}}(z) \geq f_p([\beta_2]) - f_p([\beta_1])
\]
is independent of the specific moduli space \( \mathcal{M}_z([\beta_1], [\beta_2]) \) we are looking at, so the expression \( (36) \) continues to be well defined as an element of \( \Lambda^\mathbb{Q, \mathbb{R}} \).
After we know that $\partial$ makes sense, proving $\partial^2 = 0$ is in this setting is no different than the situation for monopole Floer homology with local coefficients [13, Section 30], except for two main differences:

1. In the instanton setup, bubbling in general can play a role.
2. We have excluded the reducible flat connection from our chain group, so we need to guarantee that the broken trajectories considered to show that $\partial^2 = 0$ do not include factorizations through this reducible connection.

Regarding the first point, this is not a problem for showing that $\partial^2 = 0$, since bubbles drop the dimension of the moduli spaces involve by at least 4 [41, Proposition 3.22], and our transversality assumptions for the moduli space of flow lines guarantee that negative dimensional moduli spaces are empty.

The second point involves analyzing the non-degeneracy of the reducible connection and some index formulas.

- Even after perturbations, there is still one reducible connection $[\theta_\alpha]$ up to gauge, the same one as the unperturbed case, since $[\theta_\alpha]$ is isolated from the irreducible connections so one can choose the holonomy perturbations in such a way that they vanish near $[\theta_\alpha]$. Notice that $[\theta_\alpha]$ is unobstructed since by Poincare duality $H^2(Y; g_\alpha) \simeq H^1(Y; g_\rho) = 0$.
- If $M_2([\beta_0], [\beta_2])$ represents a d-dimensional moduli space, and there is a broken trajectory belonging to $M_2([\beta_0], [\theta_\alpha]) \times M_2([\theta_\alpha], [\beta_2])$ with dimensions $d_1, d_2$ respectively, the fact that $[\theta_\alpha]$ has one dimensional stabilizer implies that

$$\dim M_2([\beta_0], [\beta_2]) = \dim M_2([\beta_0], [\theta_\alpha]) + \dim M_2([\theta_\alpha], [\beta_2]) + 1$$

so in particular the right hand side is bounded from below by 3, since each $d_1, d_2$ must be at least one dimensional. Hence moduli spaces admitting $M_2([\beta_0], [\beta_2])$ such factorizations through the reducible connection $[\theta_\alpha]$ must be at least three dimensional, so they can be ignored for the definition of the differential. Notice that this is the same as what happens in the ordinary case of Instanton Floer homology for integer homology spheres, but in that case the bound for the dimension is 5, since the stabilizer of the trivial connection is three-dimensional.

This means that we can define the **Instanton Floer-Novikov homology for knots $H1(Y, K, \alpha)$** as $\ker \partial / \text{im} \partial$. Notice that our notation makes implicit that the homology we obtain is independent of the choice of perturbation $\pi \in \mathcal{P}$. To see why this is true, we adapt the proof of independence in [17, Section 5.3] to our situation, and discuss more generally the functoriality properties of these instanton Floer-Novikov groups.

Suppose that $(W, \Sigma) : (Y_1, K_1) \to (Y_2, K_2)$ is a concordance of the knots $K_1, K_2$. By this we mean that $Y_1, Y_2$ will both be integer homology spheres and $W$ homology cobordism, i.e $H_*(W; \mathbb{Z}) = H_*(Y_i; \mathbb{Z})$ for $i = 1, 2$. Moreover, $\Sigma$ will be an embedded annulus with $\partial \Sigma = -K_1 \sqcup K_2$.

Finally, suppose that the cobordism $(W, \Sigma)$ is $\alpha$-admissible in the sense of Definition 17. Recall that this means that that for the unique reducible up to gauge $\theta_{W, \alpha}$ we have $H^1(W; \mathcal{P}_{\theta_{W, \alpha}}) = 0$. Since $\theta_{Y_1, \alpha}$ and $\theta_{Y_2, \alpha}$ also satisfy $H^1(Y_i \setminus K_i; L_{\theta_{Y_i, \alpha}}^{\otimes 2}) = 0$ for $i = 1, 2$, it is immediate that on the completion $H^1(W \setminus \Sigma^*; L_{\theta_{W, \alpha}}^{\otimes 2}) = 0$, and an straightforward adaptation of Lemma 16 will imply that for $\alpha$-admissible cobordisms the reducible $\theta_{W, \alpha}$ is isolated (and non-degenerate as well).

We want to define a cobordism map

$$m_{(W, \Sigma)} : C_*(Y_1, K_1, \alpha, \Gamma_p, p_1) \to C_*(Y_2, K_2, \alpha, \Gamma_p, p_2)$$
4.3] shows that should interact with the gradings $H^I(Y, K, \alpha)$ of the Floer groups. Before showing that $\tilde{\beta}$ is well defined, let’s discuss first how the cobordism maps should interact with the gradings $H^I(Y, K, \alpha)$ of the Floer groups.

If $\text{gr}([\beta])$ denotes the (absolute) $\mod 4$ grading of $[\beta]$ then [40] Proposition 4.4, [11] Section 4.3 shows that

\begin{equation}
\dim M_\pm([\beta], W, [\beta]) = \text{gr}([\beta]) - \text{gr}([\beta_2]) - \frac{3}{2}(\chi(W) + \sigma(W)) - \chi(\Sigma) \mod 4
\end{equation}

In particular, given our assumptions on $(W, \Sigma)$, we can see that the cobordism map $m_{(W, \Sigma)}$ is a sum over all elements $[\beta_2]$ whose relative grading is the same as $[\beta_1]$. What is left to see is why the formula for $m_{(W, \Sigma)}[\beta_1]$ defines an element in $\Lambda^Q, R$.

This requires further discussion of how the $ASD$ equation is perturbed on a cobordism. Here we follow [41] Section 3.8, [13] Section 2.2, [38] Section 3.1 and [53] Section 4. On the cobordism $W$ we choose a collar neighborhood of each boundary component, and if $t$ denotes the coordinate (say in the collar $[0,1] \times Y_1$), we choose a $t$-dependent holonomy perturbation $p_t$ equal to $p_1$ on the near $[0,1/4] \times Y_1$. On $(3/4, 1] \times Y_1$ the perturbation $p_t$ vanishes and then we interpolate on $[1/4, 3/4] \times Y_1$ choosing an auxiliary perturbation $p_1 \in \mathcal{P}_{Y_1}$. The net effect is that on $[0,1] \times Y_1$ the perturbed equations take the form

$$F^+_A + \beta_1(t)U_1(A) + \tilde{\beta}_1(t)\tilde{U}_1(A) = 0$$

where $\beta_1, \tilde{\beta}_1$ denote suitable cut-off functions and $U_1, \tilde{U}_1$ denote the perturbation terms associated to $\beta_1, \tilde{\beta}_1$. Similar remarks apply to $Y_2$. These were the perturbations used in [41] Section 3.8. To deal with transversality issues involving flat connections on the cobordism, we also need interior holonomy perturbations, supported on a compact subset of $W \setminus \text{nbd}(\Sigma \cup \partial W)$, which were introduced for closed 4-manifolds in [53], although analogous constructions appear for example in [16] [22].

These are constructed in a similar way to how cylinder functions were constructed on a 3-manifold. We sketch the construction given in [41] Section 3.8, which also appears in [53] Definition 4.2. Namely, one chooses a closed ball $B \subset W \setminus \text{nbd}(\partial W)$ and a finite collection of smooth submersions $q_i : S^1 \times B \to W \setminus \text{nbd}(\Sigma \cup \partial W)$ so that $q_i(1, b) = b$ and $q_i(-, b)$ is an immersion for all $1 \leq i \leq n$ and $b \in B$. Choose also a self dual two form $\omega$ whose support is contained in $B$. Then for $\mathbf{q} = (q_1, \cdots, q_n)$ we have a section

$V_{\mathbf{q}, \omega} : C(W, \Sigma, \alpha) \to \Omega^{2+, +}(W, g_E)$

given by

$$V_{\mathbf{q}, \omega}(A)(x) = \text{Hol}_q(A) \otimes \omega(x)$$

Again, after introducing suitable completions one constructs a Banach space of secondary holonomy perturbations with a notion of $L^1$ convergence [53] p.51]. Therefore, on the cobordism with cylindrical ends $W^*$ the perturbed $\alpha$-ASD equations we must consider are of the form

\begin{equation}
F^+_A + U_p(A) + V_\omega(A) = 0
\end{equation}
where the term $U_p(A)$ is generic notation for the cylindrical holonomy perturbations and $V_\omega(A)$ denotes the interior holonomy perturbations. The important property we need to know about $U_p$ and $V_\omega$ is that their $L^\infty$ norms are uniformly bounded: that is, there exists $K > 0$ such that

- $\|U_p(A)\|_{L^\infty} \leq K\|p\|_p$ (this is statement iii) in [41 Proposition 3.7].
- $\|V_\omega(A)\|_{L^\infty} \leq K\|\omega\|_\rho$ (this is statement 2) in [33 Proposition 4.4], which first appeared in [36 Section 3.2]).

In fact, for transversality purposes, which is the reason why these perturbations were introduced in the first place, for any given $\epsilon > 0$, we can assume that we chose perturbations $p$ and $\omega$ satisfying $\|p\|_p, \|\omega\|_\rho < \epsilon$, so in particular that will mean that $|f_p([\beta])| < \epsilon$

for all $[\beta] \in B(Y, K, \alpha)$. To bound $E(z)$ from below, we follow the proof of [13 Proposition 2.15] and split $W^*$ into three regions

$$\frac{1}{8\pi^2} \int_{W^*} \text{tr}(F(A) \wedge F(A))$$

$$= \frac{1}{8\pi^2} \int_{\mathbb{R} \times \tilde{Y}_1} \text{tr}(F(A) \wedge F(A)) + \frac{1}{8\pi^2} \int_{\mathbb{R} \times Y_2} \text{tr}(F(A) \wedge F(A)) + \frac{1}{8\pi^2} \int_{\mathbb{R} \times \tilde{Y}_2} \text{tr}(F(A) \wedge F(A))$$

$$\geq \left[ f_p([A \mid_{(0) \times \tilde{Y}_1}]) - f_p([\beta_1]) \right] + \frac{1}{8\pi^2} \int_{W} \left( |F^-(A)|^2 - |F^+(A)|^2 \right) + \left[ f_p([\beta_2]) - f_p([A \mid_{(0) \times \tilde{Y}_2}]) \right]$$

$$\geq -2\epsilon + f_p([\beta_2]) - f_p([\beta_1]) - \frac{1}{8\pi^2} \int_{W} |U_p(A) + V_\omega(A)|^2$$

$$\geq -2\epsilon + f_p([\beta_2]) - f_p([\beta_1]) - \frac{3\epsilon^2}{8\pi^2}$$

In these steps we used the inequality [37] to deal the first and third integrals while we used the equation [40] on the second integral. The specific bound is not that important, only knowing that it does not depend on the component of the moduli space that is being analyzed. Therefore $m_{(W, \Sigma)}[\beta_1]$ will define an element of $A^{Q,R}$ once we know that the numbers $m_{\tau}([\beta_1], W, [\beta_2])$ are well defined.

For that we follow [17 Proposition 5.9]. That is, we want to show that the 0-dimensional moduli spaces $\mathcal{M}_z([\beta_1], W, [\beta_2])$ with topological energy $E(z)$ are compact. This is because if we start with a sequence $[A_i]_{i=1}^\infty$ in $\mathcal{M}_z([\beta_1], W, [\beta_2])$ (which is defined using a bundle $P([\beta_1], [\beta_2])$, then it would converge weakly to

- An ideal instanton $([A_\infty], x_\infty)$ on a bundle $Q$ over $W^*$, asymptotic on each cylindrical end to $[\beta'_1], [\beta'_2]$ respectively.
- A broken trajectory $([A_1], x_1)$ over $\mathbb{R} \times Y_1$ connecting $[\beta_1]$ and $[\beta'_1]$, and a broken trajectory $([A_2], x_2)$ over $\mathbb{R} \times Y_2$ connecting $[\beta'_2]$ and $[\beta_2]$.

Additivity of the index says that (again because the critical points are non-degenerate)

$$\text{ind} P([\beta_1], [\beta_2]) = \text{ind}[A_\infty] + \text{ind}[A_1] + \text{ind}[A_2] + \dim \tilde{H}^0(Y_1; g_{\beta'_1}) + \dim \tilde{H}^0(Y_2; g_{\beta'_2}) + 4(|x_\infty| + |x_1| + |x_2|)$$

- If $[A_\infty]$ is irreducible then by transversality $\text{ind}[A_\infty] \geq 0$ and since $[A_1], [A_2]$ are both irreducible given that at least one of their limits is irreducible, then $\text{ind}[A_1] \geq 0, \text{ind}[A_2] \geq 0$ by transversality. Then the only way for the above equality to hold is if $[\beta_1] = [\beta'_1], [\beta_2] = [\beta'_2]$ and there were no bubbles, i.e., $|x_\infty| = |x_1| = |x_2| = 0$. 
• If $[A_\infty]$ is reducible then our assumptions on homology imply that $H_1(W \setminus \Sigma; \mathbb{Z}) \cong H_1(W^* \setminus \Sigma^*; \mathbb{Z}) \cong \mathbb{Z}$ and because of the holonomy condition $[A_\infty]$ is determined to be the unique (up to gauge) reducible with $S^1$ stabilizer which is asymptotic to the reducibles $[\theta_{a,Y_1}]$ and $[\theta_{a,Y_2}]$ respectively. In this case $\text{ind}[A_\infty]$ can be computed from the dimension formula [33] with the caveat that the formula as written only works assuming that the limits are irreducible connections. When the limits are reducible, one needs to take into account the stabilizer in the formula and one concludes that $\text{ind}[A_\infty] = -1$. Since $\dim \tilde{H}^0(Y_1; g_{\theta_{a,Y_1}}) = \tilde{H}^0(Y_2; g_{\theta_{a,Y_2}}) = 1$ one finds in [31] that 

$$-1 = \text{ind}[A_1] + \text{ind}[A_2] + 4(|x_\infty| + |x_1| + |x_2|) \geq 0$$

which is impossible. Therefore, $m_z([\beta_1], W, [\beta_2])$ is well defined.

Finally, to verify the chain property one needs to compute

$$\nabla_{\nu}^2 m_W + m_W \partial_1 [\beta_1]$$

$$= \sum_{[\beta_2'] \in \{\beta_2\}} \sum_{\beta_2} \sum_{2} m_{z}([\beta_1], \beta_2) n_{z z}([\beta_2], [\beta_2'], \beta_2') \tilde{T}^{\top} T^{-\tilde{\varepsilon}_{top}(z)} T^{-\tilde{\varepsilon}_{top}(z)} [\beta_2']$$

$$+ \sum_{[\beta_2]} \sum_{\beta_1} \sum_{\beta_2} \sum_{z_1} m_{z_1}([\beta_1], [\beta_1']) m_{w}([\beta_1'], W, [\beta_2]) \tilde{T}^{\top} T^{-\tilde{\varepsilon}_{top}(z)} T^{-\tilde{\varepsilon}_{top}(w)} [\beta_2]$$

and show that it vanishes. Again, the usual argument will still work as long as one remember that the topological energy is additive under concatenation of paths.

Similar arguments can be applied for showing that the composition law for $\alpha$-admissible cobordisms holds, and in this way we have verified that $HI(Y,K,\alpha)$ is a topological invariant of the data $(Y,K,\alpha)$, together with a choice of cone parameter $\nu$, which has been omitted from our notation.

5. Reduced Version

Now we will define a reduced version $H_{I,\text{red}}(Y,K,\alpha)$ of the singular instanton Floer-Novikov homology for knots $HI(Y,K,\alpha)$ we just constructed, following [21]. As we also mentioned in the introduction, the case of $\alpha = 1/4$ , where local coefficients are not needed, was defined by Daemi and Scaduto earlier [14].

Our conventions will differ slightly from those of Frøyshov, since we are defining the homology version of the Floer groups, which means that our grading of the groups differ, and in fact are closer to the ones used in [17] Section 3.3.2, [63] Section 9. The key difference between the reduced and unreduced versions of instanton Floer homologies is that the reduced version takes into account the flow-lines between the critical points and the reducible flat connection. Moreover, in the reduced version we can define a $U$-map, which in the situation of knots will arise from the $\mu$-map evaluated at a point $x \in K$.

Now we want to define maps which take into account the interaction with the reducible connection $[\theta_\alpha]$. Consider a critical point $[\beta]$ with $\text{gr}([\beta]) = 1$. From the grading formula [31], we can see that in principle there are non-empty moduli spaces $\mathcal{M}_1([\beta], [\theta_\alpha])$ asymptotic to $[\beta]$ as $t \to -\infty$ and to $[\theta_\alpha]$ as $t \to \infty$. After taking the quotient by the $\mathbb{R}$ action, we get a 0-dimensional moduli space $\mathcal{M}_1([\beta], [\theta_\alpha])$. We would like to define a map $\delta_1[\beta]$ obtained by counting the points in this 0-dimensional moduli space. However, because of non-monotonicity there can be a priori infinitely many components of this moduli space

$$\mathcal{M}_1([\beta], [\theta_\alpha]) = \bigcup_z \mathcal{M}_{1,z}([\beta], [\theta_\alpha])$$
Therefore, we define

$$\delta_1 : CI_1(Y, K, \alpha, p) \to \Lambda$$

$$[\beta] \to \sum_z \# \mathcal{M}_{1,z}([\beta], [\theta_\alpha]) T^{-\varepsilon_{top}(z)}$$

Recall that $\Lambda$ is the Novikov field, while $CI_1(Y, K, \alpha, p)$ denotes the chain complex of $p$-perturbed flat connections whose absolute grading is 1. The formula for $\delta_1[\beta]$ does give a well defined element in $\Lambda$ for exactly the same reasons as the differential $\partial$ from the previous section being well defined. Just as in the case where $K$ is absent, it is straightforward to see that $\delta_1$ descends to a map in homology, that is:

**Lemma 18.** The map $\delta_1$ satisfies $\delta_1 \partial = 0$, so it induces a map in homology $\delta_1 : HI_1(Y, K, \alpha) \to \Lambda$.

Likewise, suppose that $\text{gr}(\beta) = 2$, so that a priori there are non-empty moduli spaces $\mathcal{M}_1([\theta_\alpha], [\beta])$. Analogous to $\delta_1$, define an element $\delta_2 \in CI_2(Y, K, \alpha, p)$ by the formula

$$\delta_2 = \sum_{[\beta] \in CI_2} \sum_z \# \mathcal{M}_{1,z}([\theta_\alpha], [\beta]) T^{-\varepsilon_{top}(z)} [\beta]$$

As before, it is straightforward to check that:

**Lemma 19.** The element $\delta_2$, descends to an element in homology, i.e, $\partial \delta_2 = 0$ so that $\delta_2 \in HI_2(Y, K, \alpha)$.

The next map to define is the $\mu$-map, which we will denote $\mu_K$, to emphasize the fact that it is not the ordinary $\mu$-map. First we need to understand the homotopy type of the space of connections mod gauge, i.e, $B(Y, K, \alpha)$, since for $x \in K$, $\mu_K(x)$ will be a degree 2 element in $H^*(B(Y, K, \alpha); \mathbb{Q})$.

The main idea is to take advantage of the fact that singular connections have a stronger notion of framing than ordinary connections. We will follow the discussion in [56, Section 5] and [35, Section 4]. Recall that if $G$ is a compact Lie group that acts on a topological space $Z$, then the homotopy quotient $Z//G$ is defined as $Z \times_G EG$, where $EG$ is a contractible space with a free $G$ action. The natural map $Z//G \to Z/G$ induces a map $H^*(Z//G, \mathbb{Z}) \to H^*(Z//G, \mathbb{Z})$, which is an isomorphism when $G$ acts freely. Lemma 5.1 in [56] shows the following:

**Lemma 20.** If $U(1)$ acts on $Z$ and the stabilizer of every point in $Z$ is $\{ \pm 1 \}$, then the pull-back map $H^*(Z//U(1), \mathbb{Q}) \to H^*(Z//U(1), \mathbb{Q})$ is an isomorphism.

When $V$ is a complex vector bundle over $Z$ with a lift of the $G$ action to $V$, we can also define $V//G = V \times_G EG$ and the $G$-equivariant Chern classes of $V$ as $c_i(G)(V) = c_i(V//G) \in H^{2i}(Z//G, \mathbb{Z})$. These are the pull-backs of the Chern classes on $Z//G$.

For our setup, when we are working with the pair $(X, \Sigma)$, the bundle $E$ decomposes near $\Sigma$ as $E = L \oplus L^{-1}$. Moreover, we wrote an exact sequence [12] for the gauge group, which in particular implies that over $\Sigma$ the $SU(2)$ gauge is broken to an $U(1)$ gauge. If $x \in \Sigma$ is a base-point and $G_x$ the gauge transformations which act trivially on $E_x = L_x \oplus L_x^{-1}$, then we have the framed configuration space

$$B^o(X, \Sigma, \alpha) = C(X, \Sigma, \alpha)/G_x(X, \Sigma)$$

Since $G/G_x \cong U(1)$, a residual gauge group isomorphic to $U(1)$ acts on $B^o(X, \Sigma, \alpha)$, and $B^o(X, \Sigma, \alpha)/U(1) = B(X, \Sigma, \alpha)$. Now we are ready to define the universal $SO(3)$ bundle and the corresponding $\mu$ map.
**Definition 21.** Define the universal $SO(3)$ bundle

\[ E^{ad} = \mathcal{C}(X, \Sigma, \alpha) \times_{G_s} g_E \to (\mathcal{B}^*(X, \Sigma, \alpha) \times X) \]

Moreover, the $U(1)$ bundle

\[ L^0 = \mathcal{C}(X, \Sigma, \alpha) \times_{G_s} L \to \mathcal{B}^0(X, \Sigma, \alpha) \times \Sigma \]

descends to the **universal $U(1)$ bundle**

\[ L \otimes^2 \to (\mathcal{B}^*(X, \Sigma, \alpha) \times \Sigma) \]

Define for $\eta \in H_4(X; \mathbb{Q})$ and $\eta_{\Sigma} \in H_2(\Sigma; \mathbb{Q})$ the $\mu$-maps

\[
\mu(\eta) = -\frac{1}{4}p_1(E^{ad})/\eta \in H^{4-d}(\mathcal{B}^*(X, \Sigma, \alpha); \mathbb{Q}) \\
\mu_{\Sigma}(\eta_{\Sigma}) = -\frac{1}{2}(L \otimes^2)/\eta_{\Sigma} \in H^{2-d}(\mathcal{B}^*(X, \Sigma, \alpha); \mathbb{Q})
\]

**Remark 22.**

1) We follow the sign conventions of Kronheimer for $\mu_{\Sigma}$ [35 Section 2.1].

2) Notice that since the homotopy type of $\mathcal{B}^*(X, \Sigma, \alpha)$ is independent of $\alpha$, the $\mu$-maps corresponding to different values of $\alpha$ can be identified with each other, which is why we do not indicate the value of $\alpha$ in our notation.

3) We can also view these as elements in $\mathcal{B}(X, \Sigma, \alpha)$, since the reducible connections form a stratum of infinite codimension in $\mathcal{B}(X, \Sigma, \alpha)$.

4) Notice that our construction singles out one of the line bundles in the decomposition $E = L \oplus L^{-1}$. In fact, had we used $L^{-1}$ instead of $L$, then $\mu_{\Sigma}$ would differ only in sign.

Clearly a similar procedure can be used to define $\mu_K$ in the case of $(Y, K)$: a cheap way to do this is to consider $X = [0, 1] \times Y$ and $\Sigma = [0, 1] \times K$. Therefore, for $x \in K$ we let

\[ u_K(x) : Cl_s(Y, K, \alpha, p) \to Cl_{s-2}(Y, K, \alpha, p) \]

\[ [\beta_0] \to \sum_{[\beta_1]|\text{gr}([\beta_0],[\beta_1])=2} \sum_z <u_K(x), M_{2, z}([\beta_0],[\beta_1])> T^{-\mathcal{E}_{top}(z)}[\beta_1] \]

Contrary to the case of $\delta_1, \delta_2$, $u_K(x)$ will not descend to a map between the Floer homology groups. To see why this is the case, observe that the maps

\[ \partial u_K - u_K \partial \]

involve considering three dimensional moduli spaces, for which we had said factorizations through the reducibles can occur. More precisely, notice that for any $[\beta_0]$, we have [here we denote for convenience $<u_K(x), M_{2, z}([\beta_0],[\beta_1])>$ as $U_z([\beta_0],[\beta_1])]$}

\[ (\partial u_K - u_K \partial)[\beta_0] \]

\[ = \sum_{[\beta_2]|\text{gr}([\beta_2],[\beta_1])=1} \sum_{z_{12}} \sum_{[\beta_1]|\text{gr}([\beta_0],[\beta_1])=2} \sum_{z_{01}} U_{z_{01}}([\beta_0],[\beta_1])n_{z_{12}}([\beta_2]) T^{-\mathcal{E}_{top}(z_{01})} T^{-\mathcal{E}_{top}(z_{12})}[\beta_2] \]

\[ - \sum_{[\beta_2]|\text{gr}([\beta_2],[\beta_1])=2} \sum_{w_{12}} \sum_{[\beta_1]|\text{gr}([\beta_1],[\beta_0])=1} \sum_{w_{01}} n_{w_{01}}([\beta_0],[\beta_1])U_{w_{12}}([\beta_1],[\beta_2]) T^{-\mathcal{E}_{top}(w_{01})} T^{-\mathcal{E}_{top}(w_{12})}[\beta_2] \]
The typical argument would look at a three dimensional moduli space $\mathcal{M}_3([\beta_0], [\beta_2])$ and consider the possible ends of this moduli space. Some of the ends correspond to the terms in $\mathcal{M}_3$ but when $\text{gr}([\beta_0]) = 1$, a priori it is also possible to have factorizations of the form

$$\mathcal{M}_1([\beta_0], [\theta_x]) \times \mathcal{M}_1([\theta_x], [\beta_2])$$

which needs to be accounted for. In fact, we have the analogue of [21, Theorem 4], [24, Proposition 8] and [17, Lemma 7.6].

**Lemma 23.** The map $u_K$ satisfies the relation

$$\partial u_K - u_K \partial - \frac{1}{2} \delta_1 \otimes \delta_2 = 0 \quad (44)$$

**Proof.** Of the references cited above, the closest to our argument is in fact [24, Proposition 8], since the monopole case also uses a universal $U(1)$ bundle to define the corresponding $u$-map. Therefore we obtain the same formula as the one Frøyshov writes for the monopole case, except for the difference in conventions for the constants in front of the $\mu$-maps.

In fact, the only place where one needs to be careful with the previous argument is that any of the proofs quoted above use the holonomy of a connection $A$ along the path $\mathbb{R} \times \{x\} \subset \mathbb{R} \times Y$.

More precisely, consider a 3-dimensional moduli space $\mathcal{M}_{3,z}([\beta_0], [\beta_2])$, and choose a representative $A \in \mathcal{M}_{3,z}([\beta_0], [\beta_2])$ whose “centre of mass” is 0, i.e.,

$$\int_{\mathbb{R} \times Y} t|F_A|^2 = 0$$

If $\text{ad}\beta_0, \text{ad}\beta_1$ are the corresponding (perturbed) flat $SO(3)$ bundles corresponding to $[\beta_0], [\beta_1]$, then we can choose a base point $y \in Y$, which is close to $x \in K$, without being equal to it. Using a normal neighborhood $\nu(K)$ of $K$, we may assume that $y$ belongs to a normal disk to $K$, centered at $x$, for which $y$ has polar coordinates $(r, \theta)$. Since we are away from the knot, there is no controversy as to what we mean by

$$h_A(r, \theta) = \text{hol}_A (\mathbb{R} \times \{y\})$$

That is, the holonomy of $A$ along the path $\mathbb{R} \times \{y\}$, where $y$ has coordinates $(r, \theta)$. Comparing the frames for the fibres of the bundles $\text{ad}\beta_0, \text{ad}\beta_1$, we get an element in $SO(3)$, i.e, $h_A(r, \theta) \in SO(3)$. Now, for fixed $\theta$, as $r$ decreases the decomposition $E = L \oplus L^{-1}$, $\text{ad}E = \mathbb{R} \oplus L^{\otimes 2}$, becomes asymptotically parallel with respect to $A$. Therefore, we obtain an element

$$h_A(\theta) = \lim_{r \to 0} \text{hol}_A (\mathbb{R} \times \{y\}) \in U(1)$$

which is obtained by comparing the frames for the fibres of the $U(1)$ bundles $\text{ad}\beta_0, \text{ad}\beta_1$, over the $U(1)$ line bundle $L^{\otimes 2} \to \Sigma$ [compare with the description of the universal bundle 42]. The existence of this limit follows for example from [64], or one can also use the fact that we are working with orbifold connections, as Kronheimer and Mrowka do in [44, Section 3.1].

However, we still need to analyze what happens as we vary the angle at which we approach the point $x$. It is not difficult to see that as we vary the angle by a full revolution, i.e, $\theta \to \theta + 2\pi$, then the holonomy picks out $h_A(\theta)$ picks out the asymptotic holonomy factor $e^{-4\pi i \alpha}$, in other words

$$h_A(\theta + 2\pi) = e^{-4\pi i \alpha} h_A(\theta)$$

So in the case of rational holonomy, we can take

$$\begin{align*}
   h_A(x) &\equiv [h_A(\theta)]^q & \alpha = \frac{2p+1}{2q} \\
   h_A(x) &\equiv [h_A(\theta)]^{2q+1} & \alpha = \frac{2p}{2q+1}
\end{align*}$$
as our desired holonomy map. For example, the case of $\alpha = \frac{1}{2} = \frac{1}{20}$ implies that we should square the limiting holonomy maps, which is exactly what Kronheimer and Mrowka do in [44, Section 3.1]. The difference with their construction is mainly stylistically, since they pass first to a local cover of a neighborhood of $x$, where the pull-backs of the connections extend smoothly.

Once we know how to take the holonomy along a point on the knot $x \in K$, the proof follows in exactly the same way as in [24, Proposition 8], where the coefficient of $\delta_1 \otimes \delta_2$ is the Euler number of the rank 1 Hermitian vector bundle over $S^2 = D^2 \cup_{S^1} D^2$ whose “clutching map” $S^1 \to U(1)$ has degree 1.

Now we interpret the equation (44) to find out how the reduced Floer groups should be defined.

- Case when $[\beta] \in H_{I_1}(Y, K, \alpha)$ and $\delta_1[\beta] = 0$: then $\partial u_K[\beta] - u_K \partial[\beta] = 0$, which means that $u_K$ descends to a map

  $$u_K : \ker \delta_1 \subset H_{I_1}(Y, K, \alpha) \to H_{I_{-1}}(Y, K, \alpha) = H_{I_3}(Y, K, \alpha)$$

- Case when $[\beta_0] \in H_{0}(Y, K, \alpha)$: abusing notation write a representative of $[\beta_0]$ as $[\beta] + \partial[\beta_1]$, where $\partial[\beta] = 0$ and $[\beta_1] \in H_{I_1}(Y, K, \alpha)$. Then

  $$u_K[\beta_0] = u_K[\beta] + u_K \partial[\beta_1] = u_K[\beta] + \partial u_K[\beta_1] - \frac{1}{2} \delta_1([\beta_1]) \delta_2$$

Therefore, we must identify elements which differ by an element on the “ray” $\Lambda \delta_2$, in other words, we get a map

$$u_K : H_{I_0}(Y, K, \alpha) \to \text{coker}(\delta_2) = H_{I_2}(Y, K, \alpha)/\langle \Lambda \delta_2 \rangle$$

- On the other summands $H_{I_2}(Y, K, \alpha)$ and $H_{I_1}(Y, K, \alpha)$, the $u_K$ map is actually well defined without additional considerations so we get maps

$$u_K : H_{I_2}(Y, K, \alpha) \to H_{I_0}(Y, K, \alpha)$$
$$u_K : H_{I_3}(Y, K, \alpha) \to H_{I_1}(Y, K, \alpha)$$

The definition the reduced Floer homology groups is now identical to [21, Definition 1], since as long as we work with $\alpha$-admissible cobordism it is straightforward to adapt section 3 of [21] (which analyzes the behavior of $\delta_1, \delta_2, u_K$ under cobordisms) to obtain:

**Theorem 24.** Let $K \subset Y$ be a knot and $\alpha \in \mathbb{Q} \cap (0, 1/2)$ be such that $\Delta_K(e^{-4\pi i \alpha}) \neq 0$. The reduced Instanton Floer homology groups $H_{I_0}^{red}(Y, K, \alpha)$

\[
\begin{align*}
H_{I_0}^{red}(Y, K, \alpha) &= H_{I_0}(Y, K, \alpha)/\big( \sum \text{im}(u_K^{2l+1}) \delta_2 \big) \\
H_{I_1}^{red}(Y, K, \alpha) &= \cap_{l \geq 0} \ker(\delta_1 u_K^{2l}) \subset H_{I_1}(Y, K, \alpha) \\
H_{I_2}^{red}(Y, K, \alpha) &= H_{I_2}(Y, K, \alpha)/\big( \sum \text{im}(u_K^{2l}) \delta_2 \big) \\
H_{I_3}^{red}(Y, K, \alpha) &= \cap_{l \geq 0} \ker(\delta_1 u_K^{2l+1}) \subset H_{I_3}(Y, K, \alpha)
\end{align*}
\]

are topological invariants of the data $(Y, K, \alpha)$, together with the choice of cone parameter $\nu$ used to define the groups.

Moreover, the Frøyshov knot invariants

\[
b(Y, K, \alpha) = \chi_A(H^{red}(Y, K, \alpha)) - \chi_A(HI(Y, K, \alpha)) \in \mathbb{Z}
\]

where $\chi_A$ denotes the Euler characteristic with respect to the Novikov field $\Lambda$, are also invariants of $(Y, K, \alpha)$ (and the cone parameter).
Remark 25. The usual $h$-invariants have a factor $\frac{1}{2}$ in front of the difference in Euler characteristics. We choose not to include this factor because in our case the Euler characteristics of the groups may be odd, since these groups need not be 2-periodic as opposed to the 4-periodic instanton Floer homology groups on $Y$, and we prefer to obtain an integer rather than a half-integer.

6. Singular Orbifold Furuta-Ohta and Tori Signature

In this section we define an analogue of the Furuta-Ohta invariant $\lambda_{FO}(X)$ [27] to the case of an embedded torus $T$ inside $X$ satisfying certain topological conditions.

In other words, we want to define an invariant $\lambda_{FO}(X,T,\alpha)$, which in the best case scenario can be interpreted as a signed count of irreducible representations $\pi_{1}(X\setminus T) \to SU(2)$ (modulo conjugacy) satisfying a certain holonomy condition determined by the parameter $\alpha$. As usual, perturbations of the flatness equation will be needed, so the interpretation of $\lambda_{FO}(X,T,\alpha)$ as a count of flat connections is slightly more complicated. In any case, our construction will be cooked up in such a way that when we take $X = S^{1} \times Y$ and $T = S^{1} \times K$ then $\lambda_{FO}(X,T,\alpha)$ agrees with $2\lambda_{CH}(Y,K,\alpha)$.

First we want to explain why we consider only torus complements $X \setminus T$, and not more general surface complements $X \setminus \Sigma$. Moreover, we will discuss what conditions are needed to make the definition of $\lambda_{FO}(X,T,\alpha)$ work.

Recall from section 2 that $SU(2)$ bundles $E \to X$ are characterized by two topological invariants, the instanton number $k$ and the monopole number $l$. The moduli space of $\alpha$-ASD connections $\mathcal{M}(X, \Sigma, k, l, \alpha)$ satisfying the asymptotic condition [9] has the expected dimension [37, Eq. 1.6]

$$\dim \mathcal{M}(X, \Sigma, k, l, \alpha) = 8k + 4l - 3(b_{2}^{+} - b^{1} + 1) - (2g - 2)$$

while the formula for the topological energy is [37, Eq. 1.7]

$$\mathcal{E}(X, \Sigma, k, l, \alpha) = \frac{1}{8\pi^{2}} \int_{X} \text{tr}(F_{A} \wedge F_{A}) = k + 2\alpha l - \alpha^{2} \Sigma \cdot \Sigma$$

Since $H_{*}(X; \mathbb{Z}) \simeq H_{*}(S^{1} \times S^{3}; \mathbb{Z})$ these formulas simplify to

$$\dim \mathcal{M}(X, \Sigma, k, l, \alpha) = 8k + 4l - (2g - 2)$$

$$\mathcal{E}(X, \Sigma, k, l, \alpha) = k + 2\alpha l$$

Notice that $\alpha$-flat connections (i.e., flat connections satisfying [9]) are equivalent to energy zero $\alpha$-ASD instantons (i.e., ASD instantons satisfying [9]). In particular, this means that $\alpha$-flat connections only exist a priori on bundles whose monopole and instanton numbers are related as

$$k + 2\alpha l = 0$$

Clearly $k = l = 0$ is always a solution of this equation. Since $k, l$ must always be integers, for irrational values of $\alpha$, $k = l = 0$ is the only solution. In fact, the next lemma shows that this continue to hold regardless of the value of $\alpha$.

Lemma 26. Suppose that $E$ is an $SU(2)$ bundle over $(X, \Sigma)$ with instanton and monopole numbers $(k,l)$. If $E$ supports an $\alpha$-flat connection then $(k,l) = (0,0)$ and thus $E$ is the trivial bundle over $X$.

Proof. As discussed before, any $\alpha$-flat connection can only exist on a bundle $E$ for which

$$k + 2\alpha l = 0$$

If $\alpha$ is irrational we already explained that $k = l = 0$ is automatic.
For $\alpha$ rational, it suffices to show that $l$ vanishes, since the previous equation will force $k$ to vanish and we will be done. To understand why $l$ vanishes, we must use the fact that $l$ can be computed as \[50\]

Here we use the fact the orbifold connection has a locally well defined restriction as an abelian connection on $\Sigma$, so the curvature decomposes as

$$F_A = \begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix}$$

when regarded as a 2-form on $\Sigma$. The quantity $\lambda$ can then be computed as

$$\lambda = \frac{i}{2\pi} \int_{\Sigma} (-\omega)$$

In the case of a flat connection it is clear that $\omega = 0$ thus $\lambda = 0$. Since $\Sigma \cdot \Sigma = 0$ equation \[50\] now implies that $l = 0$, as desired. \hfill \Box

This means that for counting $\alpha$-flat connections we can simply concentrate on the case $k = l = 0$, for which the expected dimension of the moduli space is

$$\dim \mathcal{M}(X, \Sigma, 0, 0, \alpha) = -2(g - 1)$$

So it is now clear that the expected dimension of the moduli space of $\alpha$-flat connections is zero dimensional if and only if $g = 1$, i.e, $\Sigma$ must be an embedded torus $T$. In this case the energy and dimension formulas become

$$\dim \mathcal{M}(X, T, k, l, \alpha) = 8k + 4l$$
$$\mathcal{E}(X, T, k, l, \alpha) = k + 2\alpha l$$

Our next objective will be to analyze which further hypothesis on $T$ must be made in order to have a well defined invariant $\lambda_{FO}(X, T, \alpha)$.

But first, we also analyze what are the possible $\alpha$-ASD instantons which are reducible on $(X, \Sigma)$.

**Lemma 27.** Suppose that $E$ is an $SU(2)$ bundle over $(X, \Sigma)$ with instanton and monopole numbers $(k, l)$. If $E$ supports an $\alpha$-ASD connection which is reducible then $(k, l) = (0, 0)$ and thus $E$ is the trivial bundle over $X$.

**Proof.** We follow the remarks after Proposition 5.9 in [37], more specifically the remark “iii) Transversality”. For an $\alpha$-reducible ASD connection the bundle $E$ must first of all split globally as $E = L \oplus L^{-1}$, where $L$ is a complex line bundle which admits an $\alpha$-ASD connection. This in turn can be represented by a smooth, harmonic, anti-self-dual 2-form $\omega$ whose cohomology class represents $c_1(L) + \alpha[\Sigma]$. Again, because $b_2(X) = 0$ this means in fact that $\omega$ represents the class 0, i.e, it must be an $\alpha$-flat twisted connection. But we already know from Lemma 26 that this forces $k$ and $l$ to be both zero so we are done. \hfill \Box

Back to the case where $\Sigma = T$, we need to discuss under which conditions we can expect to define a count of $\alpha$-representations $\pi_1(X \setminus T) \to SU(2)$. As in the case of a knot $K$ inside a homology sphere $Y$, we have to guarantee that the $\alpha$-reducible representations are isolated from the $\alpha$-irreducible representations. To analyze the $\alpha$-reducible representations we need to consider the first homology of the torus complement, i.e, $H_1(X \setminus T; \mathbb{Z})$. In fact, we are interested in the case where $T$ is homologically indistinguishable from the product situation $S^1 \times K \subset S^1 \times Y$, so we will make the following assumption.
**Assumption 28.** $T$ will be an embedded torus inside $X$ such that the natural map $H_1(T; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is a surjection.

More precisely, if we take our torus $T$ to be the image of an embedding $i_T : S^1 \times S^1 \to X$, then we will assume that $\partial = (i_T)_*(S^1 \times \{pt\})$ is a generator of $H_1(X; \mathbb{Z})$. A tubular neighborhood of $T$ looks like $T \times D^2$, and if we use polar coordinates $(r, \theta)$ for the second factor, then

$$H_1(X \setminus T; \mathbb{Z}) \simeq \mathbb{Z}[\partial] \oplus \mathbb{Z}[\mu_T]$$

where $\mu_T$ is a “meridian” for the torus $T$, and can be represented for example as $(i_T \times D^2)_*(\{pt\} \times \partial D^2)$.

If $\rho$ is an $\alpha$ flat connection then we want to understand the Zariski tangent space $\hat{H}^1(\hat{X}; g_\rho)$. Just as in Lemma 16 we have:

**Lemma 29.** Suppose $\alpha \in \mathbb{Q} \cap (0, 1/2)$ and that $\rho$ is an $\alpha$ flat connection on the orbifold $\hat{X}$. Then the first (orbifold) cohomology $\hat{H}^1(\hat{X}; g_\rho)$ can be identified with $\ker : H^1(X \setminus T; g_\rho) \to H^1(\mu_T; g_\rho)$.

**Proof.** The argument is completely analogous. Namely, after applying a Mayer-Vietoris decomposition to $\hat{X} = (\hat{X} \setminus \hat{\nu}_\alpha(T)) \cup \hat{\nu}_\alpha(T)$

we end up with the analogue of equations 30 and 31

$$\hat{H}^1(\hat{X}; g_\rho) \to (i_{\hat{X} \setminus T} \cdot i_{\nu_\alpha(T)})^* H^1(X \setminus T; g_\rho) \oplus (\mathbb{R}[\lambda_T] \oplus \mathbb{R}[\theta]) \to i_{\nu_\alpha(T)}^* \mathbb{R}[\nu_\alpha(T)] \oplus \mathbb{R}[\lambda_T] \oplus \mathbb{R}[\theta]$$

The composition being 0 now says that $\omega \to \langle \omega, Q_{\nu_\alpha(T)} \rangle$ vanishes, so there is a map $\hat{H}^1(\hat{X}; g_\rho) \to \ker(H^1(X \setminus T; g_\rho) \to H^1(\mu_T; g_\rho))$. The surjectivity and injectivity of this map are proven in exactly the same way as before. \hfill \Box

In particular, for an $\alpha$-reducible representation $\rho$ we have $\hat{g}_\rho = \mathbb{R} \oplus L^\otimes 2$ and thus

$$\hat{H}^1(\hat{X}; \hat{g}_\rho) \simeq \ker(H^1(X \setminus T; \mathbb{R}) \oplus H^1(X \setminus T; L^\otimes 2) \to H^1(\mu_T; \mathbb{R}) \oplus H^1(\mu_T; L^\otimes 2))$$

$$\simeq \ker(\mathbb{R}[\theta] \oplus \mathbb{R}[\mu_T] \oplus H^1(X \setminus T; L^\otimes 2) \to \mathbb{R}[\mu_T])$$

$$\simeq \mathbb{R}[\theta] \oplus H^1(X \setminus T; L^\otimes 2)$$

So at an $\alpha$-reducible representation $\rho$, $\hat{H}^1(\hat{X}; \hat{g}_\rho)$ is at least one dimensional, and for $\rho$ to be isolated from the irreducible representations it suffices to assume that $H^1(X \setminus T; L^\otimes 2)$ vanishes. Using [13] Corollary 65, we can characterize this in terms of the Alexander polynomial of the torus complement $X \setminus T$.

**Corollary 30.** Let $\alpha \in \mathbb{Q} \cap (0, 1/2)$ and $T$ be an embedded oriented torus such that $H_1(T; \mathbb{Z}) \to H_1(X; \mathbb{Z})$. Then $H^1(X; g_\rho)$ is one dimensional for every $\alpha$-reducible representation $\rho$ if and only if $H^1(X \setminus T; L^\otimes 2)_\rho = 0$, where $\hat{g}_\rho = \mathbb{R} \oplus L^\otimes 2$. Equivalently, for every $\alpha$-reducible representation we have $\Delta_{X \setminus T}(\hat{\rho}) \neq 0$, where $\hat{\rho} \in \pi_1(X \setminus T)$ is the character determined by the local system $H^1(X \setminus T; L^\otimes 2)$.

**Remark 31.** For the examples of embedded tori we will analyze, we find it easier to verify directly that $H^1(X \setminus T; L^\otimes 2) = 0$ vanishes, rather than using the condition on the Alexander polynomial, since our constructions will arise from some operation on a knot inside an integer homology sphere. It would be interesting to study an example of an embedded torus where it is easier to verify the condition on the Alexander polynomial directly. It would probably need to be more four dimensional in nature.
Notice that whenever the previous condition is satisfied then the obstruction space (i.e., the second cohomology group $\hat{H}^2(X; A_\rho)$ of the deformation complex defined by $\rho$ as a solution of the $\alpha$-ASD equations) must vanish. This is because the Euler characteristic of the deformation complex is (minus) the virtual dimension of the moduli space, which is zero in our situation, so
\[
\dim \hat{H}^0(X; A_\rho) - \dim \hat{H}^1(X; A_\rho) + \dim \hat{H}^2(X; A_\rho) = 0
\]
The first factor is 1 because that is the dimension of the stabilizer of $A_\rho$ and the one in the middle is also 1 by assumption so that forces $\hat{H}^2(X; A_\rho)$ to vanish.

Now we are finally ready to give a definition $\lambda_{FO}(X, T, \alpha)$.

**Definition 32.** Suppose that $\alpha \in \mathbb{Q} \cap (0, 1/2)$ and $T$ be an oriented embedded torus such that $H_1(T; \mathbb{Z}) \to H_1(X; \mathbb{Z})$. Suppose moreover that for every $\alpha$-reducible representation $\rho$ we have $H^1(X \setminus T; L^{\otimes 2}) = 0$, where $g_\rho = \mathbb{R} \oplus L^{\otimes 2}$. Choose a homology orientation for $(X, T)$, that is, an orientation of $H^1(X; \mathbb{Z})$, which in turn is determined by the orientation of $T$. Given this homology orientation, there is an orientation of the moduli spaces $M(X, T, k, \alpha)$ [39 Section 2.i]. We define the singular Furuta-Ohta invariant $\lambda_{FO}(X, T, \alpha)$ as follows.

Choose the trivial $SU(2)$ bundle $E \to X$ corresponding to the instanton and monopole numbers $k = l = 0$. Then, after perturbations if necessary, the irreducible $\alpha$-ASD connections $\mathcal{M}^*(X, T, 0, 0, \alpha)$ will form a 0-dimensional compact moduli space. We define $\lambda_{FO}(X, T, \alpha) \in \mathbb{Z}$ as the signed count of elements inside $\mathcal{M}^*(X, T, 0, 0, \alpha)$.

**Remark 33.** The perturbations we have in mind for the statement of the previous theorem are exactly the same as the interior holonomy perturbations that were needed to define the cobordism maps for the Floer groups $HI(Y, K, \alpha)$.

Despite the fact that $\lambda_{FO}(X, T, \alpha)$ is morally defined as a count of flat connections, it is important to notice that it is the equation $F_A^+ = 0$ which is perturbed, not the flatness equation $F_A = 0$. In other words, $\lambda_{FO}(X, T, \alpha)$ is better interpreted as a degree 0 Donaldson invariant.

This raises the question of whether this is the only degree 0 Donaldson invariant which can be defined for the embedded torus $T$. In fact, it is possible to define additional invariants $D_0(X, T, \alpha, k)$ where $k \in \mathbb{Z}$ is an integer, and provided $\alpha \neq 1/4$. Interestingly enough, for $k \neq 0$, the invariants $D_0(X, T, \alpha, k)$ can be defined for any embedded torus, independent of whether it is null-homologous or not. To see why this is the case, we need to go back to the formulas [51] for the energy and dimension of the moduli spaces
\[
\dim \mathcal{M}(X, T, k, l, \alpha) = 8k + 4l \\
\mathcal{E}(X, T, k, l, \alpha) = k + 2\alpha l
\]
Notice that $\mathcal{M}(X, T, k, l, \alpha)$ is zero-dimensional whenever
\[
l = -2k
\]
The corresponding energy of this moduli space is
\[
\mathcal{E}(X, T, k, -2k, \alpha) = k(1 - 4\alpha)
\]
In particular, when $\alpha = 1/4$, the energy of $\mathcal{M}(X, T, k, -2k, 1/4)$ is zero, which means it can only consist of $\alpha$-flat connections. But we already know from Lemma [26] that this can only happen when $k = 0$, which means that
\[
k \neq 0 \implies \mathcal{M}(X, T, k, -2k, 1/4) = \emptyset
However, when $\alpha \neq 1/4$, the moduli spaces $\mathcal{M}(X, T, k, -2k, \alpha)$ are a priori non-empty, at least provided that $\mathcal{E}(X, T, k, -2k, \alpha) \geq 0$. A more important question is whether they are compact.

Since they are already 0 dimensional, the only way for $\mathcal{M}(X, T, k, -2k, \alpha)$ to be non-compact is if a sequence of $\alpha$-ASD connections $[A_i]$ inside $\mathcal{M}(X, T, k, -2k, \alpha)$ bubbles off and converges weakly to an $\alpha$-ASD connection $[A_\infty]$ on some moduli space $\mathcal{M}(X, T, k', l', \alpha)$ of negative dimension. In fact, since bubbles drop dimensions by 4, $\dim \mathcal{M}(X, T, k', l', \alpha) \leq -4$. Fortunately, by Lemma 27, $\mathcal{M}(X, T, k', l', \alpha)$ can admit no reducible $\alpha$-ASD connections, so for generic perturbations there is no risk in assuming that $\mathcal{M}(X, T, k', l', \alpha)$ is empty (this is explained in great detail in Sections 3 and 5).

Therefore, for generic perturbations $\mathcal{M}(X, T, k, -2k, \alpha)$ is in fact compact, and a count of signed points in $\mathcal{M}(X, T, k, -2k, \alpha)$ will be independent of the perturbation chosen, because a path of perturbations will generically miss non-empty negative dimensional moduli spaces. Notice that for $k \neq 0$, $\alpha \neq 1/4$, $\mathcal{M}(X, T, k, -2k, \alpha)$ has no reducibles to begin with, which in particular means that the count of signed points inside $\mathcal{M}(X, T, k, -2k, \alpha)$ can be made regardless of whether $T$ is null-homologous or not.

These observations allows us to define the additional invariants $D_0(X, T, k, \alpha)$ we promised earlier.

**Definition 34.** Suppose that $\alpha \in \mathbb{Q} \cap (0, 1/2)$ and $k \in \mathbb{Z} \setminus \{0\}$ is a non-zero integer such that $k(1 - 4\alpha) \geq 0$. Let $T$ be a oriented embedded torus inside $X$ (null-homologous or not). After choosing an orientation of $H^1(X; \mathbb{R})$, define $D_0(X, T, k, \alpha)$ as the signed count of points inside the moduli space $\mathcal{M}(X, T, k, -2k, \alpha) = \mathcal{M}^\ast(X, T, k, -2k, \alpha)$. When $\alpha = 1/4$, set $D_0(X, T, k, 1/4) = 0$.

**Remark 35.** Notice that the choice of homology orientation is no longer determined in a canonical way by an orientation of $T$, if we allow $T$ to be null-homologous.

Also, it is not all clear what is the geometric meaning of the invariants $D_0(X, T, \alpha, k)$. It is possible they may not contain any interesting topological information about the torus $T$. For example, if we could solve the issue of the implicit dependence of the invariants on the cone angle being used, and moreover if we succeeded in defining them for irrational values of $\alpha$ as well, then one could try to use a deformation argument to show that for $k \neq 0$, $D_0(X, T, \alpha, k)$ must vanish, since in this case $D_0(X, T, \alpha, k)$ could be compared to $D_0(X, T, 1/4, k)$, which we already know vanishes.

Likewise, we can construct an analogue of the invariants $D_0(X, T, \alpha, k)$ for the case of an embedded sphere $S^2 \hookrightarrow X$. Namely, the expected dimension of the moduli spaces $\mathcal{M}(X, S^2, k, -2k, \alpha)$ is now 2, as can be seen from the formula 49. As long as $\alpha \neq 1/4$ and $k \neq 0$, some of these moduli spaces could be non-empty, and they will be compact and free of reducibles by similar argument. Hence, by pairing them with $\mu_{S^2}(x)$ for $x \in S^2$, we can define a degree-two Donaldson invariant $D_2(X, S^2, k, \alpha)$. However, the fact that these invariants cannot be defined when $\alpha = 1/4$ (which is the best value $\alpha$ could take from many points of view), suggests to us these invariants $D_2(X, S^2, k, \alpha)$ will probably end up giving no interesting topological information.

We finish this section by analyzing the action of $H^1(X; \mathbb{Z}/2)$ on the moduli spaces $\mathcal{M}(X, T, k, -2k, \alpha)$, as was promised in the introduction. We follow [60] Section 4.6] and [62] Section 3] in order to describe this action.

First of all, $H^1(X; \mathbb{Z}/2) = \text{hom}(\pi_1(X); \mathbb{Z}/2)$ parametrizes isomorphism classes of complex line bundles (with connection) $\chi$ with holonomy $\{\pm 1\}$ (along the loop $\vartheta$ in our case). Since $\chi$ lifts to an integral homology class, the bundle $L_\chi$ is trivial and thus for any $(k, l)$, the bundles $E(k, l)$ and
The first factor of this case \( H \) mines a Klein-4 representation. Call these (real) line bundles \( \rho \) corresponds to \( Q \) \( X \) distinct representations is a homological phenomenon. Namely, they exist whenever one can find three \( A \) induces a connection \( V \) \( \pi \) opposed to the \( T \) \( M \) \( \Gamma \) \( \chi \) are isomorphic. Thus the action of \( H^1(X; \mathbb{Z}_2) \) on \( M(X, T, k, -2k, \alpha) \) can be regarded as the one which sends a connection \([A]\) to \([A \otimes \chi]\).

In general this action may or may not be free. We only care about the freeness of the action on the irreducible part of the moduli space \( M^*(X, T, k, -2k, \alpha) \) (again, when \( k \neq 0 \), this coincides with the entire moduli space \( M(X, T, k, -2k, \alpha) \)), which we will show in the next lemma.

Here we only need to analyze the action on the unperturbed moduli spaces, since the idea is that once we know the action is free in the unperturbed case, one can find perturbations that are \( H^1(X; \mathbb{Z}_2) \) equivariant and still guarantee transversality for the moduli spaces [60] section 4.6.

**Lemma 36.** Suppose that the (unperturbed) moduli space \( M^*(X, T, k, -2k, \alpha) \) is non-empty. Then \( H^1(X; \mathbb{Z}/2) \) acts freely on \( M^*(X, T, k, -2k, \alpha) \).

**Proof.** Let \([A] \in M^*(X, T, k, -2k, \alpha)\) be an irreducible \( \alpha \)-ASD connection. The connection \( A \) induces a connection \( A^{ad} \) on the adjoint bundle \( E^{ad}(k, -2k) \) of \( E(k, -2k) \). On the adjoint bundle it makes sense to consider the gauge group \( G_{SO(3)}(X, T) \) of all \( SO(3) \) gauge transformations (as opposed to the \( SU(2) \) gauge transformations which are the ones we have been working with). As in the non-singular case (i.e., when \( T \) is not present), it is still the case that [40] Section 5.1

\[
G_{SO(3)}(X, T)/\{\pm 1\}) \simeq H^1(X; \mathbb{Z}/2)
\]

Therefore, the action of \( H^1(X; \mathbb{Z}/2) \) is free on \([A]\) if and only if the stabilizer of \( A^{ad} \) with respect to the full gauge group \( G_{SO}(X, T) \) is trivial. In general, since \( A \) is irreducible with respect to \( G(X, T) \), \( stab_{SO(3)} A^{ad} \) can only be one of three possibilities: 1, \( \mathbb{Z}_2 \) or the Klein-4 group \( V_4 \). Therefore we must rule out that \( \mathbb{Z}_2 \) and \( V_4 \) can arises as potential stabilizers.

The case of \( V_4 \) is easy: a connection \( A \) with stabilizer \( V_4 \) must be flat [62] Section 4], and thus cannot belong to \( M(X, T, k, -2k, \alpha) \) for \( k \neq 0 \), since these a priori do not support any flat connections.

In the case that \( k = 0 \) but \( \alpha \neq 1/4 \), we just need to use the fact that every \( \alpha \)-representation \( \rho \) such that \( \rho^{ad} \) has stabilizer \( V_4 \) also has holonomy \( V_4 \), hence \( \rho^{ad} \) will correspond in general to representations of \( \pi_1(X \setminus T) \) with image into \( V_4 \subset SO(3) \) [62] Section 4], [42] Examples 2.9].

Given that every element of \( V_4 \) has order 2, the only value of \( \alpha \) compatible with \( V_4 \) representations corresponds to \( \alpha = 1/4 \). Notice that \( \rho \) will then have image contained in the quaternionic subgroup \( Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \) when we identify \( SU(2) \) with the unit quaternions.

For the case of \( k = 0 \) and \( \alpha = 1/4 \), we need to use the fact that the existence of Klein-4 representations is a homological phenomenon. Namely, they exist whenever one can find three nontrivial real line bundles which are distinct [62] Section 3] on the manifold. In our case, the manifold to consider is \( X \setminus \text{nbd}(T) \), where \( \text{nbd}(T) \) is an open neighborhood of the torus \( T \). Since \( H^1(X \setminus \text{nbd}(T); \mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), one can find three distinct nontrivial real line bundles which determines a Klein-4 representation. Call these (real) line bundles \( \epsilon_1, \epsilon_2, \epsilon_3 = \epsilon_1 + \epsilon_2 \), where \( \epsilon_1 \) generates the first factor of \( H^1(X \setminus \text{nbd}(T); \mathbb{Z}_2) \), while \( \epsilon_2 \) generates the second factor. It is easy to see that in this case

\[
w_2(E^{ad} |_{X \setminus \text{nbd}(T)})
= w_2(\epsilon_1 \oplus \epsilon_2 \oplus \epsilon_3)
= w_1(\epsilon_1)w_1(\epsilon_2) + w_1(\epsilon_1)w_1(\epsilon_3) + w_1(\epsilon_2)w_1(\epsilon_3)
= w_1(\epsilon_1)w_1(\epsilon_2) + [w_1(\epsilon_1)]^2 + w_1(\epsilon_1)w_1(\epsilon_2) + w_1(\epsilon_2)w_1(\epsilon_1) + [w_1(\epsilon_2)]^2
= w_1(\epsilon_2)w_1(\epsilon_1) \neq 0
\]
In other words, the Klein 4 representation we found exists on a bundle with non-trivial $w_2$. However, the bundle $E^d(0,0)$ over $X$ has vanishing $w_2$ (since $X$ is an integral homology $S^1 \times S^3$, or alternatively, because $E(0,0)$ was the trivial bundle to begin with), which means that its restriction to the torus complement should have vanishing $w_2$ as well by naturality of the Stiefel-Whitney classes. Therefore, $\mathcal{M}^*(X,T,0,0,1/4)$ is also free of connections with stabilizer $V_4$ (with respect to $G_{SO(3)}(X,T)$).

The case of $Z_2$ stabilizer corresponds to the so-called twisted reducibles [38, Section 2 i]): these are those connections which preserve a splitting $g_E = \lambda \oplus P$, where now $\lambda$ is a non-orientable real line bundle and $P$ is a non-orientable real two plane bundle with orientation bundle isomorphic to $\lambda$. Now we can use the fact that our connections have a prescribed model near the surface $T$: as explained before Lemma 2.22 in [38], if $A$ were a twisted reducible, then $P$ would have to coincide with $\pm L^\otimes 2$ in a tubular neighborhood of $T$, and therefore $\lambda$ must be trivial on $T$.

Now, because $H_1(T;\mathbb{Z}_2)$ maps onto $H_1(X;\mathbb{Z}_2)$, then $\lambda$ will be trivial on all of $X$, which means that it is orientable, thus we obtain a contradiction. Notice that the paragraph we refer to from [38] starts by stating that $A$ must be a non-flat connection. However, this condition is not used in this argument, rather it was assumed by Kronheimer and Mrowka because they were interested in obtaining a generic metrics theorem in the presence of twisted reducibles, and in general this cannot be achieved whenever there are flat connections.

**Remark 37.** Since for $k \neq 0$, $\alpha \neq 1/4$, the moduli spaces $\mathcal{M}(X,T,k,-2k,\alpha)$ are free of reducible $\alpha$-ASD connections, free of $\alpha$-flat connections and free of twisted reducible connections, one can also obtain transversality for these moduli spaces using the generic metrics theorem, thanks to [38, Lemma 2.17]. Hence, one could avoid using holonomy perturbations for defining the invariants $D_0(X,T,k,\alpha)$ for $k \neq 0$.

Before discussing some examples and properties of $\lambda_{FO}(X,T,\alpha,k)$, we will prove the splitting formula for $\lambda_{FO}(X,T,\alpha,k)$, which was one of our main motivations for defining the invariant.

### 7. The Splitting Formula

Our first step for finding the splitting formula requires understanding how $\alpha$-admissibility for self-concordances $(W,\Sigma): (Y,K) \to (Y,K)$ is related to the reducible representations being isolated from the irreducible representations in the case of $(X,\Sigma)$. The next lemma says that in fact both notions are equivalent.

**Lemma 38.** Suppose that $(W,\Sigma): (Y,K) \to (Y,K)$ is a self-concordance of a knot and we choose a parameter $\alpha$ for which $\Delta_K(e^{-4\pi \alpha}) \neq 0$. Let $(X,T)$ be the closed 4-manifold obtained by closing up $(W,\Sigma)$. Then $\lambda_{FO}(X,T,\alpha)$ is well defined if and only if the cobordism $(W,\Sigma)$ is $\alpha$-admissible.

**Proof.** We can think of $(X,T)$ as being obtained from $(W,\Sigma)$ after attaching a tube $(I \times Y, I \times K)$ to the boundary of $(W,\Sigma)$, where $I$ is some interval. That is,

$$(X,T) = (W,\Sigma) \cup (I \times Y, I \times K)$$

Then we want to apply Mayer-Vietoris to this decomposition of $(X,T)$, where we enlarged $I$ a little bit so that the overlap of $(W,\Sigma)$ with $(I \times Y, I \times K)$ is the disjoint union

$$(W,\Sigma) \cap (I \times Y, I \times K) = (I_1 \times Y, I_1 \times K) \cup (I_2 \times Y, I_2 \times K)$$

where $I_1, I_2$ are two small subintervals. Recall that on the orbifold $\hat{W}$ there is only one $\alpha$-flat reducible $\theta_{W,\alpha}$, while on $Y$ we have the reducible $\theta_\alpha$. For any $\alpha$-flat reducible $A_\rho$ on $\hat{X}$, it must
restrict to \( \theta_{W,\alpha} \) and \( \theta_{\alpha} \) on \( \bar{W} \) and \( \bar{Y} \) respectively, which means that exact sequence for the (orbifold) cohomology groups reads

\[
0 \to \tilde{H}^0(\bar{X}; g_{\rho}) \\
\to \tilde{H}^0(\bar{W}; g_{\theta_{W,\alpha}}) \oplus \tilde{H}^0(I \times \bar{Y}; g_{\theta_{\alpha}}) \\
\to \tilde{H}^0((I_1 \sqcup I_2 \times \bar{Y}); g_{\theta_{\alpha}}) \\
\to \tilde{H}^1(\bar{X}; g_{\rho}) \\
\to \tilde{H}^1(\bar{W}; g_{\theta_{W,\alpha}}) \oplus \tilde{H}^1(I \times \bar{Y}; g_{\theta_{\alpha}}) \\
\to \tilde{H}^1((I_1 \sqcup I_2) \times \bar{Y}); g_{\theta_{\alpha}}) \\
\to \cdots
\]

Since \( \triangle_K(e^{-4\pi i\alpha}) \neq 0 \), we have that \( \tilde{H}^1(\bar{Y}; g_{\theta_{\alpha}}) = 0 \) so we can simplify the previous exact sequence into

\[
0 \to \mathbb{R} \\
\to \mathbb{R} \oplus \mathbb{R} \\
\to \tilde{H}^1(\bar{X}; g_{\rho}) \\
\to \tilde{H}^1(\bar{W}; g_{\theta_{W,\alpha}}) \\
\to 0 \\
\to \cdots
\]

From this we can conclude that the alternating sum of the dimensions of these vector spaces is zero, which means that

\[
\dim \tilde{H}^1(\bar{X}; g_{\rho}) = 1 + \dim \tilde{H}^1(\bar{W}; g_{\theta_{W,\alpha}})
\]

Thus, if \( (W, \Sigma) \) is \( \alpha \)-admissible (i.e., \( \dim \tilde{H}^1(\bar{W}; g_{\theta_{W,\alpha}}) = 0 \)) then \( \tilde{H}^1(\bar{X}; g_{\rho}) \) vanishes for all \( \alpha \)-reducible representations \( \rho \), and conversely, if \( \lambda_{FO}(X, T, \alpha) \) can be defined (i.e, \( \dim \tilde{H}^1(\bar{X}; g_{\rho}) = 1 \)) then \( \dim \tilde{H}^1(\bar{W}; g_{\theta_{W,\alpha}}) \) must vanish which is the condition for the cobordism to be \( \alpha \)-admissible. \( \Box \)

We are finally ready to state the splitting formula.

**Theorem 39.** (Splitting Theorem) Suppose that \( (W, \Sigma) : (Y, K) \to (Y, K) \) is a self-concordance of a knot and we choose a parameter \( \alpha \in \mathbb{Q} \cap (0, 1/2) \) for which \( \triangle_K(e^{-4\pi i\alpha}) \neq 0 \). Let \( (X, T) \) be the closed 4-manifold obtained by closing up \( (W, \Sigma) \). Suppose that \( \lambda_{FO}(X, T, \alpha) \) is well defined, or equivalently, that \( (W, \Sigma) \) is \( \alpha \)-admissible. Then we have the splitting formula

\[
\sum_k D_0(X, T, \alpha, k) T^{-E_{cusp}(X, T, k, -2k, \alpha)} = 2\text{Left}(W | HI(Y, K, \alpha)) = 2\text{Left}(W | HF^{red}(Y, K, \alpha)) - 2h(Y, K, \alpha)
\]

**Proof.** (first equality of Theorem 39) The argument is standard and analogous to the one given in [24, Section 11], [24, Section 11.1] and [43, Section 9]. In fact, since we already analyzed the action of \( H^1(X; \mathbb{Z}/2) \) on the moduli spaces, as we mentioned in the introduction, it can be regarded as a consequence of Proposition 5.5 in [40] and the remarks after it.

Namely, the idea is to compare the moduli spaces \( \mathcal{M}^+(X, T, k, -2k, \alpha) \) (\( k \in \mathbb{Z} \)) with the 0-dimensional moduli spaces \( \mathcal{M}_0([\beta]; W; [\beta]) \), i.e, those which are asymptotic to \([\beta]\) as \( t \to \pm \infty \).
Notice that because of the failure of monotonicity, specifying the dimension is not enough (when $\alpha \neq 1/4$), i.e., $\mathcal{M}_0([\beta], W, [\beta])$ needs to be furthermore indexed by the energy of the moduli space

$$\mathcal{M}_0([\beta], W, [\beta]) = \bigcup_{k \in \mathbb{Z}} \mathcal{M}_{0,E(k)}([\beta], W, [\beta])$$

Now, we can focus on one of the individual moduli spaces $\mathcal{M}^*(X, T, k, -2k, \alpha)$ and introduce a parameter $R$ which keeps track of the length of the cylinder in the usual stretching the neck argument, so that we are consider the manifold $X(R)$ where a cylinder of length $2R \ [-R, R] \times \tilde{Y}$ has been introduced along $\tilde{Y}$.

The gluing argument says that for $R$ sufficiently large, $\mathcal{M}^*(X, T, k, -2k, \alpha)$ can be identified with $\bigcup_{[\beta] \in \mathcal{C}^*(Y,K,\alpha)} \mathcal{M}_{0,E(k)}([\beta], W, [\beta])$. Notice that we do not need to worry about $\mathcal{M}_{0,E(k)}([\theta_\alpha], W, [\theta_\alpha])$ because of the $\alpha$-admissibility condition.

However, this correspondence is two to one, since when we are closing the bundle over $W^*$ to produce the bundle over $X$, there are two ways to do this since the stabilizer of $[\beta]$ is $\mathbb{Z}_2 = Z(SU(2))$, given that we are dealing with an irreducible connection [41 Remark p.893]. This explains the factor of 2 in the statement of the theorem. □

Now we will proof the second part of the Splitting Theorem. The proof is an adaptation word by word of the one Anvari gives in [2], so we will just illustrate one of the cases needed to proof this formula, the other one can be found in the annotated version [18].

**Proof.** (second equality of Theorem 39) We start by recalling the behavior of the maps

$$\delta_{1,n} = \delta_1u_k^\alpha : HI_{1+2n}(Y, K, \alpha) \rightarrow \Lambda$$

$$\delta_{2,n} = u_k^\alpha \delta_2 : \Lambda \rightarrow HI_{2-2n}(Y, K, \alpha)$$

which appear implicitly in our definition of the reduced Floer groups [15] In the case of a self-concordance $(W, \Sigma)$, the induced cobordism map $m_{W}^\alpha : HI_*(Y, K, \alpha) \rightarrow HI_*(Y, K, \alpha)$ acts on the $\delta_{1,n}$ and $\delta_{2,n}$ as follows [21 Theorem 7]: there are integers $a_{ij}, b_{ij}$ such that

$$\delta_{1,n}m_{W}^\alpha = \delta_{1,n} + \sum_{i=0}^{n-1} a_{in}\delta_{1,n}$$

$$m_{W}^\alpha \delta_{2,n} = \delta_{2,n} + \sum_{i=0}^{n-1} b_{in}\delta_{2,n}$$

Since $m_{W}^\alpha$ preserves gradings we can see that $a_{in} = 0$ and $b_{in} = 0$ whenever $i, n$ have opposite parity. We will also use the fact (which follows from Lemma [14] that either $\delta_1$ or $\delta_2$ must vanish. Finally, we also need the fact that for a commutative diagram of exact sequences of finite dimensional vector spaces over an arbitrary field $\Lambda$

$$0 \rightarrow V_0 \rightarrow V_1 \rightarrow V_2$$

$$\downarrow \alpha \quad \downarrow \beta \quad \downarrow \gamma$$

$$0 \rightarrow W_0 \rightarrow W_1 \rightarrow W_2$$

we have

$$\text{tr}(\beta) = \text{tr}(\alpha) + \text{tr}(\gamma)$$

To show that

$$h(Y, K, \alpha) = \text{Lef}(W \mid HI^{red}(Y, K, \alpha)) - \text{Lef}(W \mid HI(Y, K, \alpha))$$
we make cases based on the vanishing of $\delta_1, \delta_2$.

**Case $\delta_2 = 0$:** In this situation the reduced Floer groups simplify to
\[
\begin{aligned}
HI_1^{red}(Y, K, \alpha) &= \cap_l \ker(\delta_{1, 2l}) \\
HI_3^{red}(Y, K, \alpha) &= \cap_l \ker(\delta_{1, 2l+1}) \\
HI_5^{red}(Y, K, \alpha) &= HI_5(Y, K, \alpha) \\
HI_2^{red}(Y, K, \alpha) &= HI_2(Y, K, \alpha)
\end{aligned}
\]

Therefore the difference in Lefschetz numbers simplifies to
\[
\operatorname{Lef}(W | HI_1^{red}(Y, K, \alpha)) - \operatorname{Lef}(W | HI(Y, K, \alpha))
= \operatorname{Tr}(W | HI_1(Y, K, \alpha) \oplus HI_3(Y, K, \alpha)) - \operatorname{Tr}(W | HI_1^{red}(Y, K, \alpha) \oplus HI_3^{red}(Y, K, \alpha))
= \operatorname{Tr}(W | HI_1(Y, K, \alpha)) - \operatorname{Tr}(W | HI_1^{red}(Y, K, \alpha)) + \operatorname{Tr}(W | HI_5(Y, K, \alpha)) - \operatorname{Tr}(W | HI_5^{red}(Y, K, \alpha))
\]

In this case $h(Y, K, \alpha)$ simplifies to
\[
\begin{aligned}
\chi_A(HI_1^{red}(Y, K, \alpha)) - \chi_A(HI(Y, K, \alpha))
&= - \dim_A HI_1^{red}(Y, K, \alpha) - \dim_A HI_3^{red}(Y, K, \alpha) + \dim_A HI_1(Y, K, \alpha) + \dim_A HI_3(Y, K, \alpha) \\
&= \dim_A HI_1(Y, K, \alpha) - \dim_A HI_1^{red}(Y, K, \alpha) + \dim_A HI_3(Y, K, \alpha) - \dim_A HI_3^{red}(Y, K, \alpha) \\
&= \dim_A(HI_1(Y, K, \alpha)/ \cap_l \ker(\delta_{1, 2l})) + \dim_A(HI_3(Y, K, \alpha)/ \cap_l \ker(\delta_{1, 2l+1}))
\end{aligned}
\]

So it clearly suffices to show that
\[
\dim_A(HI_1(Y, K, \alpha)/ \cap_l \ker(\delta_{1, 2l})) = \operatorname{Tr}(W | HI_1(Y, K, \alpha)) - \operatorname{Tr}(W | HI_1^{red}(Y, K, \alpha))
\]
\[
\dim_A(HI_3(Y, K, \alpha)/ \cap_l \ker(\delta_{1, 2l+1})) = \operatorname{Tr}(W | HI_3(Y, K, \alpha)) - \operatorname{Tr}(W | HI_3^{red}(Y, K, \alpha))
\]

to obtain our result. The argument is completely analogous in both cases, so let us do verify the second identity. The result will be obtained from induction on the sequence of subspaces
\[
Z_{3,k} = \cap_{l=0}^k \delta_{1, 2l+1}
\]

From we can see that for $k = 0$ we have $\delta_1 m_{W} = \delta_1$, which in other words means that there is an exact sequence
\[
0 \to Z_{3,0} \to HI_3 \to \delta_1 \Lambda \\
\downarrow m_{3,0} \quad \downarrow \text{id} \quad \downarrow \text{id}
\]
\[
0 \to Z_{3,0} \to HI_3 \to \delta_1 \Lambda
\]

Here $m_{3,0}$ denotes the restriction of the cobordism map to $Z_{3,0}$. The additivity of the trace formula now says that
\[
\operatorname{tr}(m_W) = \operatorname{tr}(m_{3,0}) + \operatorname{tr}(\text{id}) = \operatorname{tr}(m_{3,0}) + 1 = \operatorname{tr}(m_{3,0}) + \dim_A(HI_3(Y, K, \alpha)/Z_{3,0})
\]

provided that $\delta_1 \neq 0$ (which in any case is the only interesting situation since we already assumed that $\delta_2 = 0$). An induction argument on $k$ [2 p. 7] based on the identities now says that
\[
\operatorname{tr}(m_W) = \operatorname{tr}(m_{3,k}) + \dim(HI_3(Y, K, \alpha)/Z_{3,k})
\]

Since the sequence of the $Z_{3,k}$ stabilizer once $k$ is large enough then we find that
\[
\operatorname{tr}(m_W | HI_3(Y, K, \alpha)) - \operatorname{tr}(m_W^{red} | HI_3^{red}(Y, K, \alpha)) = \dim(HI_3(Y, K, \alpha)/Z_{3,k})
\]

Again, the case for $HI_1(Y, K, \alpha)$ is completely analogous so has been verified in this situation.
Finally, the case $\delta_1 = 0$ is dealt with in a similar way, but the interested reader can find a proof in [18].

Now we will show that under the previous circumstances $(X, T)$ can be assigned an $h$-invariant.

**Theorem 40.** Suppose that $(X, T)$ can be written as the closed-up version of two different self-concordances $(W, \Sigma) : (Y, K) \rightarrow (Y, K)$ and $(W', \Sigma') : (Y', K') \rightarrow (Y', K')$ which satisfy $\Delta_K(e^{-4\pi i\alpha}) \neq 0$ and $\Delta_{K'}(e^{-4\pi i\alpha}) \neq 0$ for some $\alpha \in \mathbb{Q} \cap (0, 1/2)$. Suppose $\lambda_{FO}(X, T, \alpha)$ can be defined, or equivalently, either of the concordances (and hence the other) is $\alpha$-admissible. Then

$$\text{Lef}(W | HI^{red}(Y, K, \alpha)) = \text{Lef}(W | HI^{red}(Y', K', \alpha))$$

and thus we can use the splitting formula [52] to define $h(X, T, \alpha)$ as $h(Y, K, \alpha) = h(Y', K', \alpha)$.

**Proof.** The proof is indistinguishable from the one Frøyshov gives for monopole $h$-invariant in [24] Section 13]. First Frøyshov proves Lemma 10 in [24], which in our case translates to the following statement: let $A$, $B$ be $r \times r$ matrices with coefficients in the universal Novikov field $A^{C, R}$, and let $m$ be a natural number such that $\text{tr}(A^n) = \text{tr}(B^n)$ for all natural numbers $n$ satisfying $m \leq n < 2r + m$. Then $A$ and $B$ have the characteristic polynomial, and in particular $\text{tr}(A) = \text{tr}(B)$.

Frøyshov states this Lemma for matrices with coefficients over the complex field $\mathbb{C}$, but a quick inspection of the proof reveals that the only property used about $\mathbb{C}$ is that it is algebraically closed, so that the characteristic polynomial of $A$ (or $B$) has roots, which are the eigenvalues of $A$ (or $B$). That $A^{C, R}$ is algebraically closed is proven in [26 Lemma A.1]. Once we know this lemma holds, the argument Frøyshov gives is the same, just replace every occurrence of $A_{ FO}$ by $A_{ FO}^{red}$.

To give more details, as in [24] Section 13 one assumes that $Y, Y'$ are the inverse images of maps $f : X \to S^1, f' : X \to S^1$ in such a way that $f^{-1}(1) = Y, f'^{-1}(1) = Y'$. The important features of these maps is that they were homotopic through some map $F : [0, 1] \times X \to S^1$, since $Y, Y'$ represented the same class in $H_3(X; \mathbb{Z}) \simeq H^1(X; \mathbb{Z}) \simeq [X, S^1]$.

Now, if we let $X_T = X \setminus T$ denote the torus complement there is no problem in assuming that as maps $f_T = f : X_T \to S^1, f'_T = f' : X_T \to S^1$, we also have that $f_T^{-1}(1) = Y \setminus K, f'_T^{-1}(1) = Y' \setminus K'$. Since $T \cap Y = K$ and $T \cap Y' = K'$, by restricting $F$ to $[0, 1] \times X_T$, we find out that that $f_T, f'_T$ are homotopic as well (as elements of $[X_T, S^1]$), which means that the infinite cyclic covers they determine are in fact diffeomorphic. After that Frøyshov's argument goes through.

8. Some examples and Properties

**Product Case.**

As a corollary of the splitting theorem [52] we can verify our basic desiderata for $\lambda_{FO}(X, T, \alpha)$.

**Corollary 41.** Suppose that $(Y, K)$ is such that $\Delta_K(e^{-4\pi i\alpha}) \neq 0$ for $\alpha \in \mathbb{Q} \cap (0, 1/2)$. Then $\lambda_{FO}(X, T, \alpha)$ can be defined on $(X, T) = (S^1 \times Y, S^1 \times K)$.

Moreover, $\lambda_{FO}(X, T, \alpha) = 2\lambda_{CLH}(Y, K, \alpha)$ and the additional degree zero Donaldson invariants $D_0(X, T, \alpha, k)$ vanish for $k \neq 0$.

**Proof.** In this case the self-concordance is $(W, \Sigma) = ([0, 1] \times Y, [0, 1] \times K)$ which will clearly be $\alpha$-admissible.

According to the splitting formula [52] we have for $(X, T) = (S^1 \times Y, S^1 \times K)$

$$\sum_k D_0(X, T, \alpha, k) T^{-k} = 2\text{Lef}(Id | HI(Y, K, \alpha)) = 2\chi_A(HI(Y, K, \alpha)) = 2\lambda_{CLH}(Y, K, \alpha).$$
from which we will conclude that
\[
\begin{align*}
\lambda_{FO}(S^1 \times Y, S^1 \times K, \alpha) &= 2\lambda_{CH}(Y, K, \alpha) \\
D_0(X, T, \alpha, k) &= 0 \quad k \neq 0
\end{align*}
\]

\[\square\]

Remark 42. In particular, notice that one cannot use the invariants \(D_0(X, T, \alpha, k)\) (for \(k \neq 0\)) to obtain new invariants for knots, as one might have suspected all along.

**Flip symmetry.**

Now we briefly discuss the proof of Theorem 11 from the introduction, i.e., the flip symmetry \(\mathcal{F}\) obtained by changing the holonomy parameter from \(\alpha\) to \(\frac{1}{2} - \alpha\). This is defined similar to the action of \(H^1(X; \mathbb{Z}/2)\), in the sense that there is a natural map \(\mathcal{F} : \mathcal{M}(X, T, k, l, \alpha) \to \mathcal{M}(X, T, k + l, -l, \frac{1}{2} - \alpha)\) obtained by tensoring the bundle \(E(k, l)\) with a line bundle \(\chi\) whose holonomy on the small circles linking \(T\) is \(-1\). The new values for \(k\) and \(l\) after the flip symmetry is performed were computed in [37, Section 2, iv]).

Moreover, in [39, Appendix 1, ii)] they discuss the effect of \(\mathcal{F}\) on the orientation of the moduli spaces. In particular, \(\mathcal{F}\) preserves or reverses orientation according to the parity of \(\frac{1}{2}T \cdot T - (g - 1)\), which in our case vanishes, which means that \(\mathcal{F}\) is orientation preserving.

Recalling that \(D_0(X, T, k, \alpha)\) was computed using \(\mathcal{M}(X, T, k, -2k, \alpha)\), this means that \(D_0(X, T, k, \alpha) = D_0(X, T, -k, \frac{1}{2} - \alpha)\) as was claimed in Theorem 11.

The effect on the Floer homologies \(\mathcal{F} : HI(Y, K, \alpha) \to HI(Y, K, \frac{1}{2} - \alpha)\) can be analyzed in a similar way. The only thing to worry about is about the effect of \(\mathcal{F}\) on the gradings of a critical point before and after the flip has been performed. Clearly \(\mathcal{F}(\theta_\alpha) = \mathcal{F}(\theta_{\frac{1}{2} - \alpha})\) and more generally \(\mathcal{F}(\mathcal{M}([\beta], [\theta_\alpha])) = \mathcal{M}([\mathcal{F} [\beta]], [\theta_{\frac{1}{2} - \alpha}])\) which from the formula for the absolute grading [11] will imply that \(gr([\mathcal{F} [\beta]]) = gr([\beta])\), in other words, \(\mathcal{F}\) will be grading preserving. In particular, the Euler characteristic is preserved under the flip operation, which could also be checked directly from the formula for \(\lambda_{CLH}(Y, K, \alpha)\).

The effect of \(\mathcal{F}\) on the reduced Floer homology \(HI^{red}(Y, K, \alpha)\) can be analyzed in a similar way. The only thing to be aware of is that under \(\mathcal{F}\) the \(u\)-map \(\mu_K(x)\) changes sign, i.e., \(\mathcal{F}(\mu_K(x)) = -\mu_K(x)\), as is discussed in [55, Section 4.2]. However, changing the sign of the \(u\) map still is compatible with the definition of the reduced Floer groups [55] so the Euler characteristic of \(HI^{red}(Y, K, \alpha)\) is preserved under flip symmetries. From the formula for the Frøyshov knot invariants [40] it is immediate that \(h(Y, K, \alpha) = h(Y, K, \frac{1}{2} - \alpha)\), which was the first statement of Theorem 11.

**Duality and the figure eight knot.** It is well known that in general it is very difficult to compute the instanton Floer homology groups of some three manifold whenever there are non-trivial differentials between the critical points. Moreover, one can expect that computing the singular Floer groups \(HI(Y, K, \alpha)\) will be even harder, considering that the metric has a conical singularity and using local coefficients requires keeping track of additional data.

Therefore, we will illustrate indirect approaches for trying to compute the groups \(HI(Y, K, \alpha)\), using the pairing and duality properties Floer homologies are equipped with. This example first appeared in [7] for the case of \(\alpha = 1/4\), but as will become clear the argument is entirely formal so it can be readily adapted to other values of \(\alpha\) as long as the same properties about the character varieties hold true.

In order to do this we need to discuss how the Floer groups \(HI(Y, K, \alpha)\) are related to those of \(HI(-Y, -K, \alpha)\), where \((-Y, -K)\) denotes the pair \((Y, K)\) with the opposite orientation on both factors. When there is no knot present, the way to understand the Floer homology \(HI(-Y)\) in
terms of the Floer homology $HI(Y)$ is standard, what we need to do discuss is that happens in the presence of the knot $K$ as well as the effect on the local systems $\Gamma[\beta]$. We will follow the discussion in [43] Sections 22.5 and 32.1, [63] Section 7.4.

The Chern-Simons functionals $CS_{\hat{Y}}$ and $CS_{-\hat{Y}}$ on the orbifolds $\hat{Y}$ and $-\hat{Y}$ are related as

$$CS_{\hat{Y}} = -CS_{-\hat{Y}}$$

Therefore the critical points of the functionals can be identified in a natural way with each other.

The perturbation on $-\hat{Y}$ can be taken to be $-p$, while the vector field $\text{grad}CS_{-\hat{Y}}$ is the negative of $\text{grad}CS_{\hat{Y}}$. In other words, if $\gamma(t)$ is a trajectory on $\hat{Y}$ of $CS_{\hat{Y}}$ then $\gamma(-t)$ is a trajectory on $-\hat{Y}$ of $CS_{-\hat{Y}}$. The coefficient system on $-\hat{Y}$ can be described as a coefficient system on $B(Y, \alpha) = B(Y, K, \alpha)$, by saying that the fiber at each point is still $A^0, \mathbb{R}$, but now along paths $z$ from $[\beta_0]$ to $[\beta_1]$ one multiplies by $T^{+\epsilon_{top}(z)}$ instead of $T^{-\epsilon_{top}(z)}$.

To obtain the grading formula, consider the cylinders $\mathbb{R} \times \hat{Y}$ and $\mathbb{R} \times -\hat{Y}$. Let $[\beta] \in \mathfrak{C}(Y, K, \alpha)$ be a critical point and $[\beta]$ the corresponding class in $\mathfrak{C}(-Y, -K, \alpha)$. Recall that

$$\text{gr}([\beta]) = -1 - \dim \mathcal{M}([\theta_\alpha], [\beta]) \mod 4 = \dim \mathcal{M}([\beta], [\theta_\alpha]) \mod 4$$

A flow line on the moduli space $\mathcal{M}([\beta], [\theta_\alpha])$ can be identified with a flow line of the moduli space $\mathcal{M}([\theta_\alpha], [\beta])$, since the time direction has been reversed, which means

$$\text{gr}([\beta]) = \dim \mathcal{M}([\beta], [\theta_\alpha]) \mod 4 = \dim \mathcal{M}([\theta_\alpha], [\beta]) \mod 4 = -1 - \text{gr}([\beta])$$

Therefore we have found ([11] Proposition 4.3):

**Theorem 43.** For each $i \in \mathbb{Z}/4\mathbb{Z}$, there is an isomorphism

$$HI_i(-Y, -K, \alpha) \simeq HI_{-i-1}(Y, K, \alpha)$$

In order to extract some useful consequences from the above isomorphism we need to say some words about the mirror of a knot. We will use the terminology of [50] Definitions 2.3.2, 2.3.4.

**Definition 44.** Given a knot $(Y, K)$, the mirror image or the obverse is the knot $mK$ which corresponds to the data $(-Y, K)$, the reverse knot is $rK$ which corresponds to the data $(M, -K)$, and the inverse knot $-K$ is the knot $rmK$ which corresponds to the data $(-Y, -K)$.

For the case of a 3 manifold $Y$ admitting an orientation preserving diffeomorphism $i: Y \to Y$, then $i(mK)$ and $i(-K)$ can be regarded as knots in $Y$. In the case of $S^3$ every orientation preserving diffeomorphism is isotopic to the identity, so the choice of $i$ does not matter.

A knot $K$ in $S^3$ is positive amphicheiral if $K$ and $mK$ are isotopic, $K$ is negative amphicheiral if $K$ and $-K$ are isotopic, and reversible if $K$ and $rK$ are isotopic. A knot $K$ is called amphicheiral if it is either positive or negative amphicheiral.

The example we want to consider is the Figure 8 knot $K = 4_1$, which is both positive and negative amphicheiral [50] Exercise 2.3.5. In this case the duality isomorphism [50] implies that for all values of $\alpha$ such that $\Delta_{4_1}(e^{-4\pi i\alpha}) \neq 0$ we have the isomorphism

$$HI_i(S^3, 4_1, \alpha) \simeq HI_{-i-1}(S^3, 4_1, \alpha)$$

or in other words

$$\begin{cases} HI_0(S^3, 4_1, \alpha) \simeq HI_8(S^3, 4_1, \alpha) \\ HI_1(S^3, 4_1, \alpha) \simeq HI_7(S^3, 4_1, \alpha) \end{cases}$$

(57)

Now, the Alexander polynomial and character varieties of the figure 8 knot is well understood. We follow the summary given in [24] Section 5.2 for details on the character variety of $S^3 \setminus 4_1$. 

Some examples of tori inside mapping tori.

These examples can be considered as the orbifold version of [59]. Before writing a general statement, let's consider a toy model. Suppose that we have a knot $K'$ inside an integer homology sphere $Y'$ and we choose holonomy $\alpha' = \frac{1}{2}$ along the $K'$.

Moreover, assume that after taking the 3-fold branched cover along $K'$ we obtain a 3 manifold $Y$ which is still an integer homology sphere. Notice that $Y$ comes with a natural $\mathbb{Z}_3$ action $\tau$, whose fixed point set is a knot $K$. Now choose holonomy $\alpha = \frac{1}{5}$ along the knot $K$. Then the mapping torus

$$X_\tau = ([0,1] \times Y)/(\{0\} \times Y \sim^\tau \{1\} \times Y)$$

of $(Y, \tau)$ will be a homology $S^1 \times S^3$ with a natural torus $T_\tau$ obtained as the mapping torus of $K \subset Y$. In this situation the analogue of [59] Proposition 3.1 will tell us that there is a two to one correspondence between $\alpha$-representations on $(X_\tau, T_\tau)$ and $\alpha$-representations on $(Y, K)$ which are $\tau$-equivariant.

Clearly, every $\alpha'$-representation of $(Y', K')$ will pullback to an $\alpha$-representation on $(Y, K)$ which is $\tau$-equivariant. So the only question is whether this exhausts all the possibilities for being a $\tau$-equivariant representation. In fact, it does not! Suppose we had chosen holonomy $\alpha' = \frac{1}{6}$ along $K'$. Then the pull-back of an $\alpha'$ representation of $(Y', K')$ will have holonomy $6/5$ along $K$ upstairs.

Recall that we are using the normalization for the holonomy to be between 0 and 1/2, and since

$$\frac{6}{5} = \frac{1}{5} + 2 \cdot \frac{1}{2}$$
Theorem 46. Let \((Y', K')\) be a pair of an oriented knot inside an integer homology sphere and take \(\alpha' = \frac{s}{pq}\), where \(p, q, r\) are all odd integers, relatively prime and such that \(0 < \alpha' < \frac{1}{2}\). Suppose moreover that \(0 < \frac{r}{q} < \frac{1}{2}\) and assume that the \(p\)-fold branched cover along \(K'\) is an integer homology sphere \(Y\). Let \(K\) denote the fixed point set of the \(\mathbb{Z}_p\) action \(\tau\) on \(Y\), which will be another oriented knot. For \(\alpha = \frac{s}{q}\) consider the \(l = \left\lfloor \frac{r}{q} \right\rfloor + 1\) integers
\[n_0 = 0, n_1 = 1, \ldots, n_{l-1} = \left\lfloor \frac{p}{2} \right\rfloor\]
and the \(l\) holonomy values
\[\alpha_0 = \alpha', \alpha_1 = \alpha' + \frac{1}{p}, \ldots, \alpha_{l-1} = \alpha' + \frac{l-1}{p}\]
Assume furthermore that for each \(j = 0, \ldots, l-1, \Delta_{K'}(e^{-4\pi i \alpha_j'}) \neq 0\) and that the reducible representation \(\theta_{\alpha}\) is \(\tau\) non-degenerate, i.e., \(\bar{H}^{1, \tau}(\tilde{Y}; g_{\theta_{\alpha}}) = 0\). Denote \((X_\tau, T_\tau)\) the \(\tau\)-mapping torus of \((Y, K)\). Then \(\lambda_{FO}(X_\tau, T_\tau, \alpha)\) is well defined and moreover
\[\lambda_{FO}(X_\tau, T_\tau, \alpha) = \sum_{j=0}^{\left\lfloor \frac{r}{q} \right\rfloor} 2\lambda_{CLH}(Y', K', \alpha_j') + 8 \left( \left\lfloor \frac{p}{2} \right\rfloor + 1 \right) \lambda_C(Y') + \sum_{j=0}^{\left\lfloor \frac{r}{q} \right\rfloor} \sigma_{K'}(e^{-4\pi i \alpha_j'})\]
Remark 47. i) There are certainly infinitely many examples that satisfy our assumptions. For example, the Brieskorn spheres \(\Sigma(p, a, b)\) may be realized as the \(p\)-fold branched covering of \(S^3\) along a torus knot \(T(a, b)\) \([13]\). A way to guarantee \(\theta_{\alpha}\) is \(\tau\) isolated as an \(\alpha\) representation of
$\pi_1(\Sigma(p, a, b)\setminus K)$ is for it to be isolated in the ordinary sense, i.e, $H^1(\Sigma(p, a, b); g_{0a}) = 0$. Since the Alexander polynomial of both $K'$ and $K$ have at most a finite number of roots on the unit circle, there are infinitely many numbers of the form $\frac{p}{pq}$ which will serve our purposes.

ii) A statement involving one of the integers $p, q$ being even would be slightly more complicated, since the center of $SU(2)$ is $\mathbb{Z}_2$ and there can be a lifting issue when one tries to lift an action of $\mathbb{Z}_p$ for $p$ even on a 3-manifold to an $SU(2)$ bundle [3].

To keep the proof manageable we will break it into several pieces: first we will discuss the identification at the level of the critical sets, then we will address how the Zariski tangent spaces are related, third we will discuss how to relate the critical sets after perturbations have been introduced into the picture, and finally we will discuss how to relate the orientations of the moduli spaces. Our steps should be regarded as the orbifold version of the corresponding statements in [59].

First we need to review some brief facts about the orbifold fundamental group, as well as some aspects about equivariant gauge theory. Our main sources are [3, 9, 10, 59].

At this point it is a matter of preference whether one wants to think we are working over $Y\setminus K$ or the orbifold $\tilde{Y}$ for analyzing the action, so we will change perspectives whenever it is more convenient.

Over the manifold $Y\setminus K$ we have an action $\mathbb{Z}_p$ and a principal $SU(2)$ bundle $P \to Y\setminus K$. Let $\tau$ denote the generator of $\mathbb{Z}_p$. When one is trying to understand the action of a group on some principal bundle, one needs to make the assumption that

$$\tau^* (P) \simeq P$$

Usually on a closed-4 manifold, this means verifying that the characteristic numbers of the bundle are preserved ($c_2(\mathcal{P})$ in the $SU(2)$ case for example). On any 3-manifold the $SU(2)$ bundles are necessarily trivial, so one may think that [59] is automatically guaranteed. However, we now need to take into account that we are working with connections with a prescribed singularity along the knot, so we need to check that the action of $\tau$ preserves the model connection. Conversely, from the orbifold perspective, this is the same as checking that the isotropy data of the bundle is preserved [11].

Recall that using a tubular neighborhood for the knot $K$, the model connection [10] in a trivialization could be understood given by the matrix valued 1-form

$$i \left( \begin{array}{cc} \alpha & 0 \\ 0 & -\alpha \end{array} \right) d\theta$$

where $(r, \theta)$ are polar coordinates. We are choosing the action $\mathbb{Z}_p$ in such a way that it becomes an isometry and orientation preserving, and from the local model of a branched cover it is not difficult to see that $\mathbb{Z}_p$ acts on the coordinates as

$$\tau \cdot (r, \theta) = (r, \theta + 2\pi/p)$$

Clearly this will preserve $d\theta$ and thus the local model of the connection. In other words, we have verified the singular (orbifold) analogue of the condition [59]. If $\mathcal{G}(Y, K, \alpha)$ is the usual gauge group and $\mathcal{G}(Y, K, \alpha)$ the group of bundle automorphisms of $P$ covering an element of $\mathbb{Z}_p$, then we have an exact sequence

$$1 \to \mathcal{G}(Y, K, \alpha) \to \mathcal{G}(Y, K, \alpha) \to \mathbb{Z}_p \to 1$$

There is an action of $\mathcal{G}(Y, K, \alpha)$ by pullbacks on the space of connections $\mathcal{C}(Y, K, \alpha)$. Let $\tilde{\tau} : P \to P$ denote a lift of $\tau$. By [59] for any two lifts $\tilde{\tau}_1, \tilde{\tau}_2$ of $\tau$, there is a gauge transformation $g \in \mathcal{G}(Y, K, \alpha)$
such that
\[ \hat{\tau}_2 = \hat{\tau}_1 \cdot g \]

Thus there is a well defined action \( \tau^* \) on \( B^\tau(Y, K, \alpha) \). We will denote the fixed point set of \( \tau^* \) by \( B^\tau(Y, K, \alpha) \). Let \( B \in B^\tau(Y, K, \alpha) \). Then we can find a representative \( B \) and a lift \( \tilde{\tau} \in \mathcal{G}(Y, K, \alpha) \) such that
\[ \hat{\tau}^* B = B \]

If there were another such \( \tilde{\tau}' \) then \( \tilde{\tau}' \circ \tilde{\tau}^{-1} \) would be an element of \( \mathcal{G}(Y, K, \alpha) \) fixing \( B \), hence it would belong to the stabilizer of \( B \), which since we assume was irreducible must be \( \mathbb{Z}/2 \). In other words
\[ \tilde{\tau}' = \pm \hat{\tau} \]

which means \( \hat{\tau} \) is well defined up to a sign. Moreover, \( (\hat{\tau})^p \) is an element of \( \mathcal{G}(Y, K, \alpha) \) fixing \( B \), so by the same token
\[ (\hat{\tau})^p = \pm 1 \]

Therefore we can write
\[ B^\tau(Y, K, \alpha) = \bigsqcup_{\tilde{\tau} \in \mathcal{G}(Y, K, \alpha)} B^{\tilde{\tau}}(Y, K, \alpha) \]

where the disjoint union is over the lifts such that \( \hat{\tau}^p = \pm 1 \) and the equivalence relation is \( \hat{\tau}_1 \sim \hat{\tau}_2 \)

if and only if \( \hat{\tau}_2 = \pm g \cdot \hat{\tau}_1 \cdot g^{-1} \) for some gauge transformation \( g \in \mathcal{G}(Y, K, \alpha) \).

Each \( B^{\tilde{\tau}}(Y, K, \alpha) \) can be described as follows: for a fixed lift \( \tilde{\tau} \), let \( C^{\tilde{\tau}}(Y, K, \alpha) \) denote the irreducible connections \( B \)

such that \( \hat{\tau}^* B = B \). Define \( \mathcal{G}^\tau(Y, K, \alpha) = \{ g \in \mathcal{G}(Y, K, \alpha) \mid g \hat{\tau} = \pm \hat{\tau}g \} \). Then \( B^\tau(Y, K, \alpha) = C^\tau(Y, K, \alpha)/\mathcal{G}^\tau(Y, K, \alpha) \).

We are after the analogue of [59, Proposition 2.1], which will be useful for the discussion of the spectral flow calculations. Namely, any lift \( \tilde{\tau} : P \to P \) can be written in the base-fiber coordinates as
\[ \tilde{\tau}(y, f) = (\tau(y), \sigma(y)f) \]

where \( \sigma : Y \setminus K \to SU(2) \). Notice that when we regard it as an orbifold, then the automorphism \( \sigma \) must be \( S^1 \) valued along \( K \), because the automorphisms of \( P \), i.e \( \mathcal{G}(Y, K, \alpha) \), consists of maps \( Y \to SU(2) \) which restrict to \( S^1 \subset SU(2) \) along \( K \).

Therefore, we will say that the lift \( \tilde{\tau} \) is constant if there exists \( u \in S^1 \subset SU(2) \) such that \( \sigma(y) = u \) for all \( y \in \hat{Y} \). Notice for \( y \in \text{Fix}(\tau) \) we must have that
\[ \tilde{\tau}^p(y, f) = (y, \sigma^p(y)f) = (y, \pm f) \]

Since \( \sigma \mid_K \) is circle valued, for fixed \( y \in K \), \( \sigma(y) \) must be one of the \( p \)-th roots of \( \pm 1 \). Clearly this is a discrete set, and given that we can assume that the gauge transformations are continuous, this automatically says that \( \sigma \mid_K \) is constant.

Let \( u_j = \pm \left( e^{-2\pi ij/p}, \begin{bmatrix} 0 \\ e^{2\pi ij/p} \end{bmatrix} \right) \), for \( j = 0, \cdots, \left\lfloor \frac{p}{2} \right\rfloor \) and consider the constant lift
\[ \tilde{\tau}_u(y, f) = (\tau(y), uf) \]

The orbifold bundles \( P^{\text{ad}}/\tilde{\tau} \) and \( P^{\text{ad}}/\tilde{\tau}_u \) have the same holonomy along \( K' \), which is \( 2\alpha' \mp \frac{2i}{p} \) (modulo some twists to normalize it so that it belongs to the interval \( (0, 1/2) \) again), therefore they are isomorphic. Take any isomorphism and pull it back to an (equivariant) gauge transformation \( g^{\text{ad}} \in \mathcal{G}_{SO(3)}(Y, K, \alpha) \). Then as \( SO(3) \) orbifold bundles we have that
\[ B^{\text{ad}}_{ad}(Y, K, \alpha) = B^{nu}_{ad}(Y, K, \alpha) \]
Since $H^1(Y; \mathbb{Z}/2) = 0$ there is no obstruction to lifting $g^{ad}$ to a gauge transformation $g \in G(Y, K, \alpha)$, in fact there are two choices for such a lift. Since $G^\tau(Y, K, \alpha)$ incorporates the ambiguity of the lift already, we have as well that

$$B^\tau(Y, K, \alpha) = B^u(Y, K, \alpha)$$

We will start now discussing the relation between equivariant $\alpha$- representations on $(Y, K, \alpha)$ and the different $\alpha'_l$-representations on $(Y', K', \alpha')$.

**Lemma 48.** Assume the hypothesis of Theorem 46. Then there is a bijective correspondence between $\bigcup_{j=0}^{j-1} R(Y', K', \alpha')$ and $R^\tau(Y, K, \alpha)$. Likewise, there is a two to one correspondence between $R^\tau(X_\tau, T_\tau, \alpha)$ and $R^\tau(Y, K, \alpha)$.

**Proof.** From covering space theory, the covering $Y \setminus K \rightarrow Y' \setminus K'$ induces a homotopy exact sequence

$$1 \rightarrow \pi_1(Y \setminus K) \rightarrow \pi_1(Y' \setminus K') \rightarrow \mathbb{Z}_p \rightarrow 1$$

Recall that the orbifold fundamental group can be defined in terms of the knot complement as

$$\tau = \pi_1(Y \setminus K)/\langle \mu_K \rangle$$

Moreover, $\tau$ induces an action $\tau_\ast$ on $\pi_1(Y \setminus K)$ which we denote as $\tau \cdot h$ for $h \in \pi_1(Y \setminus K)$. The induced action on $R(Y \setminus K)$ is given by [3] Section 2]

$$\tau_\ast(h) = \rho(\tau \cdot h)$$

This action descends to an action on the orbifold group and we denote the fixed point as $\tilde{\pi}_1^\tau(Y, K, r/q)$. From the exact sequence we obtain a split exact sequence

$$1 \rightarrow \tilde{\pi}_1(Y, K, q) \rightarrow \tilde{\pi}_1(Y', K', pq) \rightarrow \mathbb{Z}_p \rightarrow 1$$

Notice that every $\alpha'_l$-representation of $(Y', K')$ (for any $l = 0, \cdots, [\frac{d}{2}]$) can be regarded as an element of $\tilde{\pi}_1(Y', K', pq)$.

What we need to check is that the pull-back of any $\alpha'_l$ irreducible representation of $(Y', K')$ continues to be an $\alpha$- irreducible representation of $(Y, K)$, and conversely, an $\tau$-equivariant $\alpha$-irreducible representation of $(Y, K)$ pushes down to an $\alpha'_l$-irreducible representation of $(Y', K')$, for some $\alpha'_l$. The second statement will follow from the first one, so let’s focus on the former. Since the sequence splits, we can write $\tilde{\pi}_1(Y', K', pq)$ as a semi-direct product of $\tilde{\pi}_1(Y, K, q)$ and $\mathbb{Z}_p$.

As discussed in [3] Section 7.4, the representations of a semi-direct product, like $\tilde{\pi}_1(Y', K', pq) \cong \tilde{\pi}_1(Y, K, q) \rtimes \mathbb{Z}_p$ are determined in terms of the representations of $\tilde{\pi}_1(Y, K, q)$ and $\mathbb{Z}_p$.

Let $\rho' : \tilde{\pi}_1(Y, K, q) \rightarrow SU(2)$ be an $\alpha'_l$ representation and denote by $\rho$ the induced $\alpha$-representation on $\tilde{\pi}_1(Y, K, q)$. If $t$ is a generator of $\mathbb{Z}_p$ (with unit 1) and $e$ is the identity of $\pi_1(Y \setminus K)$, then $\rho(h) = \rho'(h, 1)$ for $h \in \pi_1(Y \setminus K)$ and $u = \rho'(t) = \rho'(e, t) \in SU(2)$ determine the induced representations from the semidirect product $\rho'((h, t^m))$

$$\rho'((h, t^m)) = \rho'((e, t^m) \cdot (h, 1)) = \rho'((e, t^m))\rho'(h, 1) = u^m \rho(h)$$
The fact that $\rho$ must be $\tau$ equivariant implies that [32 Proposition 7.7]
\[ \tau^* \rho = u \rho u^{-1} \]
Now, if we assume that $\rho$ is abelian, then $\tau^* \rho$ is completely specified by how $\tau$ acts on $H_1(Y\setminus K; \mathbb{Z})$.
Since $\tau$ is a diffeomorphism of odd order, it is not difficult to check that $\tau_* = \text{id}$ on $H_1(Y\setminus K; \mathbb{Z})$.
Hence $u$ and $\rho$ commute. Therefore we have
\[ \rho'((h_1, t^{m_1}) \cdot (h_2, t^{m_2})) = u^{m_1} \rho(h_1) u^{m_2} \rho(h_2) = u^{m_2} \rho(h_2) u^{m_1} \rho(h_1) = \rho'((h_2, t^{m_2}) \cdot (h_1, t^{m_1})) \]
This means that $\rho'$ vanishes on commutators hence it must be an abelian representation, giving rise to a contradiction.

Therefore, irreducibility is preserved under pull back of connections and push-down of connections. Once we know this, that $\bigcup_{j=0}^{l-1} R^*(Y', K', \alpha')$ and $R^{*,*}(Y, K, \alpha)$ are in bijection is immediate.

The two to one correspondence between $R^*(X, T, \alpha)$ and $R^{*,*}(Y, K, \alpha)$ is similar and follows the proof of [29 Proposition 2.1]. First of all, we have a splitting exact sequence
\[ 1 \to \pi_1(Y, K, q) \to \pi_1(X, T, q) \to \mathbb{Z} \to 0 \]
The same argument we just gave applies to show that the pullback of an irreducible $\alpha$-representation $\rho_X$ of $(X, T)$ gives rise to an irreducible $\rho_Y$ $\alpha$-representation of $(Y, K)$ (or one can also think about this in terms of unique continuation coming from restricting the corresponding $\alpha$-flat connection on $(X, T)$ to a slice $(Y, K)$). Notice, that the pull back representation is equivariant, so we can write as before
\[ \tau^* \rho_Y = u \rho_Y u^{-1} \]
where $u = \rho_X(e, t)$ and $\rho_Y = \rho_X(h, 1)$. The correspondence is two to one because replacing $u$ by $-u$ gives rise to a new representation of $(X, T)$ which induces the same representation $\rho_Y$. This is in fact coming from the action of $H^1(X; \mathbb{Z}/2)$, and the fact that the new representation is a new one is granted by the freeness of this action, which is a true because of Lemma 30. That every equivariant representation on $(Y, K)$ induces one on $(X, T)$ is also clear ([60 Theorem 6.1] and [69 Proposition 3.1]).

Now we will discuss the non-degeneracy condition. First we will show that under the assumptions of Theorem 46 that $\lambda_{FO}(X, T, \alpha)$ is well defined. Likewise, we will show that the Zariski tangent spaces of the $\alpha$-representations on $(X, T)$ can be identified with the equivariant Zariski tangent spaces of the $\alpha$-representations of $(Y, K, \alpha)$.

**Lemma 49.** Suppose that the conditions of Theorem 46 hold. Then $\lambda_{FO}(X, T, \alpha)$ is well defined. Moreover, the unperturbed moduli space $M^*(X, T, 0, 0, \alpha)$ use to compute $\lambda_{FO}(X, T, \alpha)$ is non-degenerate if and only if $R^{*,*}(Y, K, \alpha)$ is non-degenerate.

**Proof.** Our proof is essentially the same as the one in [29 Proposition 3.3]. Let $\rho_X$ denote an $\alpha$-representation on $(X, T)$. By restriction to a slice it induces an $\alpha$ representation $\rho_Y$ on $(Y, K)$. To study the non-degeneracy of the representation we use Lemma 29. Therefore, we want to compute $H^1(X\setminus T; \mathcal{G}_{\rho_X})$. 


From the fibration $X \setminus T \to S^1$ with fiber $Y \setminus K$ we have by Leray-Serre a spectral sequence whose $E^2$ page is
\[ H^p(S^1, H^q(Y \setminus K; \mathfrak{g}_{\rho_Y})) \]
This spectral sequence collapses for all $p \geq 2$ so
\[ H^3(X \setminus T; \mathfrak{g}_{\rho_X}) = H^0(S^1, H^1(Y \setminus K; \mathfrak{g}_{\rho_Y})) \oplus H^1(S^1, H^0(Y \setminus K; \mathfrak{g}_{\rho_Y})) \]
(64)

Now we analyze this decomposition in two cases:

• Case when $\rho_X$ is a reducible $\alpha$-flat connection: in this situation $\mathfrak{g}_{\rho_X} \simeq \mathbb{R} \oplus L_{\rho_X}^{(2)}$ and we just need to analyze the $L_{\rho_X}^{(2)}$ part of the decomposition [64]. Since $\rho_X$ induces the reducible connection $\rho_{\theta_\alpha}$ on $Y \setminus K$ [44] becomes
\[ H^1(X \setminus T; L_{\rho_X}^{(2)}) = H^0(S^1, H^1(Y \setminus K; L_{\theta_\alpha}^{(2)})) \oplus H^1(S^1, H^0(Y \setminus K; L_{\theta_\alpha}^{(2)})) \]

Observe that $H^0(Y \setminus K; L_{\theta_\alpha}^{(2)})$ vanishes since $\dim H^0(Y \setminus K; \mathbb{R} \oplus L_{\theta_\alpha}^{(2)}) = \dim \text{stab} \theta_\alpha = 1$ and we already know that $\dim H^0(Y \setminus K; \mathbb{R}) = 1$. Therefore $H^1(S^1, H^0(Y \setminus K; L_{\theta_\alpha}^{(2)}))$ vanishes.

To compute $H^0(S^1, H^1(Y \setminus K; L_{\theta_\alpha}^{(2)}))$ notice that the generator of $\pi_1(S^1)$ acts on $H^1(Y \setminus K; L_{\theta_\alpha}^{(2)})$ as $\tau : H^1(Y \setminus K; L_{\theta_\alpha}^{(2)}) \to H^1(Y \setminus K; L_{\theta_\alpha}^{(2)})$, thus $H^0(S^1, H^1(Y \setminus K; L_{\theta_\alpha}^{(2)}))$ is the fixed point set of $\tau^*$, which is the equivariant cohomology $H^1, \tau(Y \setminus K; L_{\theta_\alpha}^{(2)})$. This term must vanish by the assumption on Theorem [40] that $H^1, \tau(Y; \mathfrak{g}_{\theta_\alpha}) = 0$.

In conclusion, $H^1(X \setminus T; L_{\rho_X}^{(2)})$ vanishes for all $\alpha$-reducible representations $\rho_X$ which is the condition needed for $\lambda_{FO}(X, T, \alpha)$ to be well defined.

• Case when $\rho_X$ is an irreducible $\alpha$-flat connection: since the restriction $\rho_Y$ is an irreducible $\alpha$-flat connection then $H^0(Y \setminus K; \mathfrak{g}_{\rho_Y})$ will vanish which means that the second term in [64] will vanish as well.

As in the previous case, $H^0(S^1, H^1(Y \setminus K; \mathfrak{g}_{\rho_Y}))$ can be identified with $H^1, \tau(Y \setminus K; \mathfrak{g}_{\rho_Y})$. Also, that the meridian $\mu_T$ restricts in a natural way to the meridian of $\mu_K$, and any local system restricted to either $\mu_T$ or $\mu_K$ always reduces (since the loops have abelian fundamental group so any flat connection becomes reducible). Because of the holonomy condition, it cannot be the trivial local system $\simeq \mathbb{R}^3$ in either case which means that
\[ \mathbb{R} \simeq H^1(\mu_T; \mathfrak{g}_\rho) \simeq H^1(\mu_K; \mathfrak{g}_{\rho_Y}) \simeq H^1, \tau(\mu_K; \mathfrak{g}_{\rho_Y}) \]
where the last isomorphism follows from the fact that $\tau$ restricts to the identity on the knot. Therefore, $\ker : H^1(X \setminus T; \mathfrak{g}_\rho) \to H^1(\mu_T; \mathfrak{g}_\rho)$ will vanish if and only if $\ker : H^1, \tau(Y \setminus K; \mathfrak{g}_{\rho_Y}) \to H^1(\mu_K; \mathfrak{g}_{\rho_Y})$ vanishes, which means that $M^\ast(X, T; 0, 0, \alpha)$ is non-degenerate if and only if $\mathcal{R}^\ast, \tau(Y, K, \alpha)$ is non-degenerate.

The case where perturbations are needed follow essentially the arguments in [59]. Namely, for computing $\lambda_{FO}(X, T, \alpha)$ as well as the different $\lambda_{CLH}(Y', K', \alpha')$ we can always use (finitely many) holonomy perturbations whose stays away from the singularity (i.e, the torus or knot). This was already discussed before the proof of Theorem [17] where we used this condition to identify the Euler characteristic of our Floer groups with the Casson Lin Herald invariant. Since the support of the holonomies do not meet the singularity, we can still find equivariant perturbations $p^\ast$ to achieve transversality for $\mathcal{R}^\ast, \tau(Y, K, \alpha, p^\ast)$ as in [59 Section 5.1]. Since the action is free away from the knot $K$, these equivariant holonomy perturbations can be pushed down to the quotient $(Y', K')$ in such a way that they guarantee transversality for the different moduli spaces $\bigcup_{j=1}^{\infty} \mathcal{R}(Y', K'; \alpha_j', p^\ast)$, as was done in [10 Section 3.8].
That the perturbations needed to achieve transversality for $R^* (Y, K, \alpha)$ and $R^* (X, T, \alpha)$ can be chosen in a consistent matter corresponds precisely to [59] Section 5.3]. Moreover, that the correspondence between the perturbed versions of $R^* (Y, K, \alpha)$ and $R^* (X, T, \alpha)$ continues to be 2 to 1 is proven in exactly the same way as [59] Proposition 5.3.

Finally, that the orientations of the moduli spaces for $\bigcup_{j=0}^{d-1} R (Y', K', \alpha_j', \tilde{p}^\tau)$, $R^* (Y, K, \alpha)$ and $R^* (X, T, \alpha)$ can be chosen in a consistent way is a consequence of the analysis in [59] Section 3.5], where we just need to use the existence of a constant lift $\tilde{\tau}$, which we already know exist from our previous discussion when we defined the spaces $B^2 (Y, K, \alpha)$.

Some tori inside circle bundles over homology $S^1 \times S^2$.

Now we will work out an orbifold version of the examples discussed in [60] Section 8]. There Ruberman and Saveliev analyzed a family of homologies $S^1 \times S^3$ for which $\lambda_{FO} (X)$ can be computed, and in fact vanishes identically. The four manifolds $X$ they considered arise as circle bundles $\pi : X \to Y_0$ over a 3 manifold with the integral homology of $S^1 \times S^2$.

The manifolds $Y_0$ can be obtained from doing 0 surgery on a knot $K$ in an integral homology sphere $Y$, and in order to guarantee that $\lambda_{FO} (X)$ is well defined, it is assumed that $\Delta_K (t) \equiv 1$ and moreover that the Euler class $e \in H^2 (Y; \mathbb{Z}) = \mathbb{Z}$ of the bundle satisfies $e = 1$. In order to be able to compute $\lambda_{FO} (X)$, they furthermore assume that $\pi_2 (Y) = 0$ so that the homotopy exact sequence of the $S^1$ bundle $\pi : X \to Y_0$ allows them to consider $\pi_1 X$ as a central extension of $\pi_1 Y_0$ by the integers

$$1 \to \mathbb{Z} \to \pi_1 X \to \pi_1 Y_0 \to 1$$

From here one can identify $M^* (X, SO(3))$ with $R^* (Y, SO(3))$ as well as the corresponding Zariski tangent spaces.

The natural tori inside $X$ that can be used for trying to compute $\lambda_{FO} (X, T, \alpha)$ are related to the Longitudinal Floer Homology Kronheimer and Mrowka define in [41] Section 4.4. First of all, observe that $Y_0$ contains a natural knot $K_0$, which is the core of the solid torus used in the surgery. Furthermore, $K_0$ represents a primitive element in the first homology of $Y_0$, so in particular $H_1 (K_0; \mathbb{Z})$ generates $H_1 (Y_0; \mathbb{Z})$.

Therefore, when we look at the inverse image of $K_0$ under the map $\pi : X \to Y_0$ it is clear that we obtain an embedded torus $T$ satisfying the condition that $H_1 (T; \mathbb{Z})$ surjects onto $H_1 (X; \mathbb{Z})$.

Example 50. Consider the unknot $K = \circ \subset S_3$. After doing 0 -surgery on $K$ we obtain $Y_0 = S^1 \times S^2$. A natural $S^1$ bundle over $Y_0$ satisfying the our requirements is $\pi : S^1 \times S^3 \to S^1 \times S^2$ where we are using the Hopf fibration $S^3 \to S^2$ on the second factor and the trivial projection on the first factor. Notice that in this case $K_0$ can be identified with $S^1 \times \{pt\}$ and therefore the natural torus $T$ is $T = S^1 \times S^1$, where the second factor is the standard circle as well (an unknot).

It is clear in this case that for any $\alpha$ we have $\lambda_{FO} (X, T, \alpha) = 0$, since the fundamental group of $X \setminus T$ is abelian.

In fact, our expectation is that the previous example is the norm in the following sense:

Conjecture 51. Let $\pi : X \to Y_0$ as before and consider the torus $T = \pi^{-1} (K_0)$. Then whenever $\lambda_{FO} (X, T, \alpha)$ can be defined it will vanish.

To give insight into why we are making this conjecture, we need to explain a bit more how $\lambda_{FO} (X)$ was shown to vanish by Ruberman and Saveliev and how this computation would be modified in the case of $\lambda_{FO} (X, T, \alpha)$.

As we mentioned before, Ruberman and Saveliev identified $M^* (X, SO(3))$ with $R^* (Y, SO(3))$ in an orientation preserving way. The action of $H^1 (X; \mathbb{Z}_2)$ continues to be free in this situation, so
in particular $\lambda_{FO}(X)$ can be computed as one half the signed count of elements in $R^*(Y, SO(3))$ (after perturbations if needed). This count ends up being the same as $\triangle''_K(1)$, which is zero in this case.

The interesting feature of this calculation is that $\triangle''_K(1)$ is the Euler characteristic of the Instanton Floer homology Floer defined for a homology $S^1 \times S^2$. Therefore, we expect that $\lambda_{FO}(X, T, \alpha)$ is related to the Euler characteristic of an orbifold version of Instanton Floer homology on a homology $S^1 \times S^2$. But this is precisely the Longitudinal Floer homology $HIL(Y_0, K_0)$ Kronheimer and Mrowka defined in [11]. Here the connections are allowed to have a singularity along the knot $K_0$, where they used holonomy $\alpha = 1/4$ in order to avoid the compactness issues due to non-monotonicity.

If one is willing to use local coefficients, there is no difficulty in obtaining a version $HIL(Y_0, K_0, \alpha)$ which is always well defined, since there are no reducible $\alpha$-flat connections to worry about in this situation. They conjectured that $\chi(HIL(Y_0, K_0))$ should be $2\triangle''_K(1)$ in general, which in our situation would imply that $\chi(HIL(Y_0, K_0))$ vanishes.

For the other values of $\alpha$ there is a brief discussion in [8, Section 4.3], where it is promised that a definition of what we are defining as $HIL(Y_0, K_0, \alpha)$ would be constructed eventually. In any case, based on the Property 3 Collin states in the survey we conjecture the following:

**Conjecture 52.** For $\alpha \in \mathbb{Q} \cap (0, 1/2)$ the Euler characteristic of $HIL(Y_0, K_0, \alpha)$ over the Novikov ring $\Lambda$ equals $2\triangle''_K(1)$. In particular, for the tori we are considering arising as $\pi^{-1}(K_0)$ we would have that $\lambda_{FO}(X, T, \alpha)$ vanishes in all of these cases.

**Remark 53.** Besides the fact that conjecture 52 might be used for computing $\lambda_{FO}(X, T, \alpha)$ for this family of examples, it could be of independent interest to compute the Euler characteristic of $HIL(Y_0, K_0, \alpha)$. In fact, we plan to verify this in the future, adapting the ideas of Floer as described in [5].

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