Poisson brackets, quasi-states and symplectic integrators

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Abstract

This paper is a fusion of a survey and a research article. We focus on certain rigidity phenomena in function spaces associated to a symplectic manifold. Our starting point is a lower bound obtained in an earlier paper with Zapolsky for the uniform norm of the Poisson bracket of a pair of functions in terms of symplectic quasi-states. After a short review of the theory of symplectic quasi-states, we extend this bound to the case of iterated Poisson brackets. A new technical ingredient is the use of symplectic integrators. In addition, we discuss some applications to symplectic approximation theory and present a number of open problems.

Contents

1 Introduction and main results 3
  1.1 Quasi-morphisms and quasi-states  4
  1.2 Lower bound on higher Poisson brackets  8
  1.3 Application to symplectic approximation theory  8
  1.4 Discussion and open problems  11
  1.5 Symplectic integrators  13

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2 Proof of the main theorem

2.1 Using symplectic integrators ........................................ 14
2.2 Estimating the remainder .............................................. 16
1 Introduction and main results

We discuss certain aspects of function theory on symplectic manifolds related to the so-called C⁰-rigidity of the Poisson bracket which can be described as follows.

Let \((M, \omega)\) be a closed symplectic manifold. Denote by \(C^\infty(M)\) the space of smooth functions on \(M\) and by \(\| \cdot \|\) the standard uniform norm (also called the \(C^0\)-norm) on it: \(\|F\| := \max_{x \in M} |F(x)|\). The Poisson bracket on \(C^\infty(M)\), induced by \(\omega\), will be denoted by \(\{\cdot, \cdot\}\). Applying the Poisson bracket repeatedly we get so-called iterated Poisson brackets of functions from \(C^\infty(M)\).

Note that the definition of an iterated Poisson bracket of two smooth functions \(F, G \in C^\infty(M)\) involves partial derivatives of the functions. Thus \textit{a priori} one does not expect any restrictions on possible changes of the Poisson bracket under \(C^0\)-small perturbations of \(F\) and \(G\). Surprisingly, such restrictions do exist (see \([3, 8, 19, 11, 5, 7, 2]\)).

In this paper we will discuss such restrictions coming from the theory of \textit{symplectic quasi-states} and our starting point is a lower bound on the \(\|\{F, G\}\|\) in terms of certain symplectic quasi-states obtained in \([8]\). We recall that symplectic quasi-states are functionals on \(C^\infty(M)\) introduced in \([4]\) which obey a convenient set of axioms of an algebraic flavor and, furthermore, are Lipschitz with respect to the uniform norm, so that the above-mentioned lower bound is robust with respect to \(C^0\)-perturbations of the functions and thus provides a restriction of the needed sort. Let us also note that when \(\dim M \geq 4\), only currently known symplectic quasi-states come from Floer theory, see Example 1.1 below for a brief discussion.

In the present paper we extend the bound from \([8]\) to the case of iterated Poisson brackets. A new technical ingredient of the paper is an unexpected use of \textit{symplectic integrators} whose origins lie in numerical Hamiltonian dynamics.

In addition, we discuss some applications to symplectic approximation theory, which are in the spirit of \([8]\) but presented under a somewhat different angle, and formulate a number of open problems.
1.1 Quasi-morphisms and quasi-states

A homogeneous quasi-morphism on a group \( G \) is a function \( \mu : G \to \mathbb{R} \) satisfying

(i) There exists a constant \( C \) such that for any \( g, h \in G \)

\[
|\mu(gh) - \mu(g) - \mu(h)| \leq C.
\]

(ii) For any \( g \in G \) and \( n \in \mathbb{Z} \) one has \( \mu(g^n) = n\mu(g) \).

Nowadays quasi-morphisms are intensively studied due to their importance in group theory as well as their applications in geometry and dynamics (see e.g. [12] for a brief introductory survey).

In this paper we are concerned with quasi-morphisms on a certain important group appearing in symplectic geometry. Let us recall its definition along with a few preliminaries (see [13, 14] for more details).

Given a time-dependent Hamiltonian \( H : M \times [0, 1] \to \mathbb{R} \), denote \( H(t) := H(\cdot, t) : M \to \mathbb{R} \). We say that \( H(t) \) is normalized if it has zero mean for all \( t \). The set of all normalized (time-dependent) Hamiltonian functions is denoted by \( \mathcal{F} \). Symplectomorphisms of \((M, \omega)\) which can be included in a Hamiltonian flow (i.e. the flow of a time-dependent Hamiltonian vector field) generated by a Hamiltonian from \( \mathcal{F} \) form a group \( \text{Ham}(M, \omega) \). Let \( \widetilde{\text{Ham}}(M, \omega) \) be the universal cover of \( \text{Ham}(M, \omega) \) with the base point at the identity. This is precisely the group of interest for us and we will consider quasi-morphisms on this group. Given a (time-dependent) Hamiltonian \( H \) on \( M \), we denote by \( \phi_H^t \) the Hamiltonian flow generated by \( H \). We write \( \phi_H^t \) for the element of \( \widetilde{\text{Ham}}(M, \omega) \) represented by the path \( \{\phi_H^t\}, t \in [0, 1] \).

Convention: Let \( \text{proj} : \widetilde{\text{Ham}}(M, \omega) \to \text{Ham}(M, \omega) \) be the natural projection. In what follows for the sake of brevity we write \( F \circ \phi \) instead of \( F \circ \text{proj}(\phi) \), where \( \phi \in \widetilde{\text{Ham}}(M, \omega) \) and \( F \in C^\infty(M) \).

A homogeneous quasi-morphism \( \mu \) on \( \widetilde{\text{Ham}}(M, \omega) \) is called stable, if for any (time-dependent) Hamiltonian functions \( F, G \in \mathcal{F} \)

\[
\int_0^1 \min_M(F(t) - G(t))dt \leq \mu(\phi_G) - \mu(\phi_F) \leq \int_0^1 \max_M(F(t) - G(t))dt .
\]
In particular, for a stable $\mu$ one has

$$\|\mu(\phi_F) - \mu(\phi_G)\| \leq \int_0^1 \|F(t) - G(t)\| dt.$$  \hfill (2)

Though the definition of a stable quasi-morphism is fairly simple, it is not easy to prove its existence. Nowadays such quasi-morphisms are constructed for certain symplectic manifolds (for instance, for complex projective spaces and their products) with the help of the Floer theory. However, for instance, it is unknown whether such a quasi-morphism exists on the standard symplectic torus – see [6] for a detailed discussion of symplectic manifolds admitting quasi-morphisms constructed by means of the Floer theory.

**Example 1.1.** To give the reader a feeling of what we are talking about, let us very briefly outline construction of a stable quasi-morphism on $\mathcal{Ham}(M, \omega)$ where $(M, \omega)$ is the complex projective space $\mathbb{C}P^n$ equipped with the standard Fubini-Study form, normalized so that the total symplectic volume of $\mathbb{C}P^n$ is 1. Let $\Lambda$ be a covering of the free loop space of $M$ whose elements are equivalence classes of pairs $(\gamma, u)$, where $\gamma : S^1 \to M$ is a loop and $u : D^2 \to M$ is a disc with $u|_{\partial D^2} = \gamma$. The equivalence relation is defined as follows: $(\gamma_1, u_1) \sim (\gamma_2, u_2)$ whenever $\gamma_1 = \gamma_2$, while the discs $u_1$ and $u_2$ are homotopic with fixed boundary. Every Hamiltonian $H \in \mathcal{F}$ defines the classical action functional

$$\mathcal{A}_H : \Lambda \to \mathbb{R}, \quad [(\gamma, u)] \mapsto \int_0^1 H(\gamma(t), t) dt - \int_{D^2} u^* \omega.$$  

Its critical points correspond to 1-periodic orbits of the Hamiltonian flow $\phi_H^t$, or in other words to fixed points of its time-one map. Floer theory is an infinite-dimensional version of the Morse-Novikov homology theory for the action functional $\mathcal{A}_H$ on $\Lambda$. In particular, given $\alpha \in \mathbb{R}$, it enables one to define Floer homology groups of the sublevel sets $\{\mathcal{A}_H < \alpha\}$ which we denote $HF(\alpha)$. These groups come with natural inclusions $j_\alpha : HF(\alpha) \to HF(\infty)$. The group $HF(\infty)$ can be identified via the Piunikhin-Salamon-Schwarz isomorphism with another remarkable object, the quantum homology algebra of $M$. This is an algebra with the unity element $e$. The next construction (due to Oh, Schwarz and Viterbo) is a Floer homological analogue of the standard min-max: put

$$c(H) := \inf\{\alpha : e \in \text{Image}(j_\alpha)\}.$$
It turns out that the value $c(H)$ depends only on the time-one map $\phi_H \in \tilde{\text{Ham}}(M,\omega)$ and not on the Hamiltonian $H$ itself. Writing $c(\phi_H) := c(H)$, we get a map $c : \tilde{\text{Ham}}(M,\omega) \to \mathbb{R}$. The specific algebraic structure of the quantum homology algebra of $\mathbb{C}P^n$ guarantees that

$$|c(\phi \psi) - c(\phi) - c(\psi)| \leq \text{const} \quad \forall \phi, \psi \in \tilde{\text{Ham}}(M,\omega)$$

for some constant independent of $\phi, \psi$. Therefore the homogenization (with the opposite sign)

$$\mu(\phi) := -\lim_{k \to \infty} \frac{c(\phi^k)}{k}$$

of the function $\phi \mapsto c(\phi)$ is a homogeneous quasi-morphism. A deep fact of the Floer theory is that $\mu$ is stable. We refer the reader to [13] and to references therein for the relevant preliminaries on Floer theory and quantum homology.

Any stable homogeneous quasi-morphism $\mu$ on $\tilde{\text{Ham}}(M,\omega)$ induces a functional $\zeta : C^\infty(M) \to \mathbb{R}$ as follows. From now on assume for simplicity that the symplectic volume $\int_M \omega^n$ of $M$ equals 1. For $F \in C^\infty(M)$, set

$$\zeta(F) = \int_M F\omega^n - \mu(\phi_F).$$

The functional $\zeta$ – which is, in general, non-linear – satisfies the following system of axioms [8]:

(i) $\zeta(1) = 1$.

(ii) $F \geq G \Rightarrow \zeta(F) \geq \zeta(G)$. 

(iii) $\zeta(aF + bG) = a\zeta(F) + b\zeta(G)$ for any $F, G \in C^\infty(M)$ such that $\{F, G\} \equiv 0$ (that is $F,G$ commute with respect to the Poisson bracket) and any $a, b \in \mathbb{R}$.

The functionals satisfying (i)-(iii) were introduced in [4] and are called symplectic quasi-states. An important feature of symplectic quasi-states is the they are Lipschitz in the uniform norm $||F|| = \max |F|$ on $C^\infty(M)$:

$$\zeta(F) = \zeta(G + F - G) \leq \zeta(G + ||F - G||) = \zeta(G) + ||F - G||. \quad (3)$$
The theory of symplectic quasi-states lies on the borderline between symplectic geometry and functional analysis and its origins lie in mathematical foundations of quantum mechanics. In particular, any symplectic quasi-state extends to a topological quasi-state in the sense of Aarnes [1] on the space of continuous functions $C(M)$: This means that it is linear on any singly-generated subalgebra of $C(M)$. For further discussion on quasi-morphisms and quasi-states on symplectic manifolds, see [4, 6, 8, 9, 19, 20].

**Example 1.2.** In the case when $M = \mathbb{C}P^1 = S^2$ and the symplectic form is the standard area form normalized so that the total area of $S^2$ is 1, the quasi-state $\zeta$ associated to the stable quasi-morphism from Example 1.1 can be described in elementary combinatorial terms. Let $F$ be a generic Morse function on $M$. Its *Reeb graph* $\Gamma$ is obtained by collapsing connected components of the level sets of $F$ to points. It is easy to see that $\Gamma$ is a tree. Let $\sigma$ be the push-forward of the symplectic area from $M$ to $\Gamma$. One can show that there exists unique point $m \in \Gamma$, called *the median*, so that $\sigma(Y) \leq 1/2$ for every connected component $Y$ of $\Gamma \setminus m$. It turns out that $\zeta(F)$ equals to the value of $F$ on the connected component of its level set which corresponds to $m$.

Given a quasi-state $\zeta$, we can define a functional

$$\Pi(F, G) = |\zeta(F + G) - \zeta(F) - \zeta(G)|$$

on the space $\mathcal{H} := C^\infty(M) \times C^\infty(M)$. Let $d$ be the uniform distance on $\mathcal{H}$:

$$d((F, G), (F', G')) := ||F - F'|| + ||G - G'|| .$$

It follows from (3) that $\Pi$ is Lipschitz with respect to $d$:

$$|\Pi(F, G) - \Pi(F', G')| \leq 2d((F, G), (F', G')) .$$

(4)

*From now on we assume that $\zeta$ is a symplectic quasi-state associated to a stable homogeneous quasi-morphism on $\text{Ham}(M, \omega)$.*
1.2 Lower bound on higher Poisson brackets

There is another class of functionals on $\mathcal{H} := \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M)$ of a more classical nature. Denote by $\mathcal{P}_N$ the set of Lie monomials in two variables involving $N$-times-iterated Poisson brackets (i.e. $\mathcal{P}_1$ consists of $\{F,G\}$, $\mathcal{P}_2$ of $\{\{F,G\}, F\}$ and $\{\{F,G\}, G\}$ and so on). For $F, G \in \mathcal{C}^\infty(M)$ set

$$Q_N(F, G) = \sum_{p \in \mathcal{P}_{N-1}} ||p(F, G)||.$$ 

Our main result is:

**Theorem 1.3.** Let $\zeta: \mathcal{C}(M) \rightarrow \mathbb{R}$ be a symplectic quasi-state induced by a stable homogeneous quasi-morphism, and let

$$\Pi(F, G) = |\zeta(F + G) - \zeta(F) - \zeta(G)|.$$ 

Then there exist constants $C_N$ for any $N \in \mathbb{N}$ so that

$$\Pi(F, G) \leq C_N \cdot Q_N(F, G)^{1/N} \quad (5)$$

for any $F, G \in \mathcal{C}^\infty(M)$.

Theorem 1.3 is another display of $C^0$-rigidity of (iterated) Poisson brackets mentioned in the introduction: since the definition of $p(F, G), p \in \mathcal{P}_{N-1}$, involves partial derivatives of functions $F$ and $G$, there are no a priori restrictions on the changes of $Q_N(F, G)$ under perturbations of $F$ and $G$ in the uniform norm. At the same time the functional $\Pi$ is Lipschitz in the uniform norm, and hence inequality (5) yields such a restriction.

For the case $N = 2$ inequality (5) had been proved in an earlier paper [8].

1.3 Application to symplectic approximation theory

The discussion below was initiated in [8], though here we slightly change the viewpoint. The problem we are going to deal with can be roughly stated as follows.

**Problem 1.4.** Given a pair of functions on a symplectic manifold, what is its optimal uniform approximation by a pair of (almost) Poisson-commuting functions?
The value $\Pi(F,G)$ can be considered as a fancy measure of non-commutativity of functions $F, G \in C^\infty(M)$. We illustrate this as follows: Note that inequality (4) implies that

$$\Pi(F,G) \leq 2 \max(||F||, ||G||).$$

This motivates the following definition: We say that a pair of functions $F, G$ is $\zeta$-extremal if $||F|| = ||G|| = 1$ and the previous inequality is an equality:

$$\Pi(F,G) = 2.$$

We denote by $E \subset H$ the subset of all $\zeta$-extremal pairs and by $K \subset H$ the set of all Poisson-commuting pairs: $K := \{(F,G) \in H | \{F,G\} \equiv 0\}$.

**Example 1.5.** Let $M = S^2 \times \ldots \times S^2$ be the product of $n$ copies of the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$ in $\mathbb{R}^3$. Each sphere is equipped with the standard area form $\theta$, normalized so that the total area of $S^2$ is 1, and the symplectic structure on $M$ equals $\theta \oplus \ldots \oplus \theta$. Denote by $x_i, y_i, z_i$ the standard Euclidean coordinates on the $i$-th factor of $M$. It is known $[4]$ that $M$ admits a symplectic quasi-state $\zeta$ associated to a stable quasi-morphism so that the functions $F = 1 - 2x_i^2$ and $G = 1 - 2y_i^2$ form a $\zeta$-extremal pair. In the case when $n = 1$, the quasi-state $\zeta$ is described in Example 1.2 above.

Our starting observation is that

$$d((F,G), K) = 1 \ \forall (F,G) \in E. \tag{6}$$

Indeed, if $F'$ and $G'$ Poisson-commute we have that $\Pi(F',G') = 0$. Given any $\zeta$-extremal pair $(F,G)$, we apply inequality (4) and get

$$2 \leq 2d((F,G), (F',G')),$$

and thus $d((F,G), K) \geq 1$. Since $(F,0) \in K$, we get the opposite inequality, and thus (6) follows.

The set $K$ possesses a natural system of “tubular neighborhoods”

$$K_N(\epsilon) := \{(F,G) \in H : Q_N(F,G) < \epsilon\}.$$
This viewpoint is justified by a symplectic version of the Landau-Hadamard-Kolmogorov inequality proved in [7] which implies (for $N \geq 2$) that

$$\|\{F,G\}\| \leq \text{const}(N) \cdot \min(\|F\|,\|G\|) \cdot \frac{N^2}{N-1} \cdot \|p(F,G)\|^{\frac{1}{N-1}},$$

for some $p \in \mathcal{P}_{N-1}$. Since $\|p(F,G)\| \leq Q_N(F,G)$, we get that for any integer $N \geq 2$ there exists a constant $a_N > 0$ so that

$$\|\{F,G\}\| \leq a_N \cdot \min(\|F\|,\|G\|) \cdot \frac{N^2}{N-1} \cdot Q_N(F,G) \quad \forall F,G \in C^\infty(M). \quad (7)$$

In particular, $(F,G) \in \mathcal{K}$ provided $Q_N(F,G) = 0$.

Finally, we arrive to the following problem: given an extremal pair $(F,G)$, explore the behavior of the function

$$D_N(\epsilon) := d((F,G), \mathcal{K}_N(\epsilon))$$

as $\epsilon \to 0$. Noticing that for some real $b_N = b_N(F,G)$, and all positive $\epsilon < 1$ the pair $(F, \epsilon b_N G)$ lies in $\mathcal{K}_N(\epsilon)$, we get an obvious upper bound for $D_N(\epsilon)$:

$$D_N(\epsilon) \leq 1 - b_N \epsilon \quad \forall 0 < \epsilon < 1. \quad (8)$$

As far as the lower bound is concerned, we claim that

$$D_N(\epsilon) \geq 1 - \frac{1}{2} C_N \epsilon^{\frac{1}{N}} \quad \forall \epsilon > 0,$$

where $C_N$ is the constant from Theorem 1.3.

This inequality immediately follows from the following corollary of Theorem 1.3 (cf. [8] for the case $N = 2$).

**Corollary 1.6.** For any $F, G, F', G' \in C^\infty(M)$ we have

$$\frac{\Pi(F,G)}{2} - d((F,G), (F',G')) \leq \frac{1}{2} C_N Q_N(F',G')^{\frac{1}{\epsilon}}.$$

**Proof.** By inequality (4),

$$\Pi(F',G') \geq \Pi(F,G) - 2d((F,G), (F',G')).$$

Applying Theorem 1.3 to $(F',G')$ we get the desired inequality. \qed
1.4 Discussion and open problems

The gap between upper and lower bounds (8) and (9) suggests the following question.

**Problem 1.7.** What is the actual asymptotical behavior of function $D_N(\epsilon)$ as $\epsilon \to 0$? The answer is unknown even for the specific extremal pair of functions described in Example 1.5 above.

Another question related to inequality (9) is as follows:

**Problem 1.8.** Given a specific extremal pair of functions (say, as in Example 1.5 above), can one prove lower bound (9) for $D_N(\epsilon)$ without methods of “hard” symplectic topology (like Floer theory)? (See [10] for a discussion about the dichotomy between “hard” and “soft” symplectic topology).

Let us emphasize once again that our proof of (9) uses the fact that the symplectic manifold $M = S^2 \times \ldots \times S^2$ admits a symplectic quasi-state associated to a stable quasi-morphism on $\widehat{\text{Ham}}(M)$ which follows from Floer-homological considerations.

Let us also note that in dimension $\text{dim } M = 2$ there exist alternative constructions of symplectic quasi-states (see e.g. [1], [4], [19], [20], [17]) which do not involve Floer homology. None of those quasi-states is known to be induced by a stable quasi-morphism. For instance, it is shown in [17] that Py’s quasi-morphism [15] gives rise to a quasi-state, but it is unknown whether this quasi-morphism is stable. On the other hand, Zapolsky [19, 20] proved that for a wide class of quasi-states $\zeta$ on surfaces one has inequality

$$\Pi(F, G) := |\zeta(F + G) - \zeta(F) - \zeta(G)| \leq \sqrt{||F, G||_{L_1}}.$$  

Note that this is a sharper version of inequality (5) in Theorem 1.3 for $N = 2$, where the uniform norm is replaced by the $L_1$-norm. Interestingly enough, Zapolsky’s argument does not involve quasi-morphisms: it is based on methods of two-dimensional topology. This discussion leads to the following problem.
Problem 1.9. Can one extend Zapolsky’s inequality to the case of iterated Poisson brackets? More precisely, given a closed 2-dimensional symplectic manifold \((M, \omega)\) and a quasi-state \(\zeta\), is it true that for \(N \geq 3\)
\[
\Pi(F, G) \leq \text{const}(N) \cdot \left( \sum_{p \in \mathcal{P}_{N-1}} \|p(F, G)\|_{L_1} \right)^{\frac{1}{N}} ?
\]
In case the answer is affirmative, it would be interesting to develop an \(L_1\)-version of symplectic approximation theory on surfaces along the lines of Section 1.3 above.

Our current impression is that Theorem 1.3 lies on a rather narrow borderline between “soft” and “hard” symplectic topology, but on the “hard” side. To illustrate this, let us compare the inequalities
\[
\Pi(F, G) \leq C_N \cdot Q_N(F, G)^{1/N} \quad (\bigstar_N)
\]
for the previously known case \(N = 2\) and the new case \(N \geq 3\). Assume that \(\|F\| = \|G\| = 1\). By the symplectic Landau-Hadamard-Kolmogorov inequality (7) (which is proved by elementary calculus)
\[
Q_2(F, G)^\frac{1}{2} \leq a'_N Q_N(F, G)^{\frac{1}{2(N-2)}}.
\]
Thus inequality \((\bigstar_2)\) yields
\[
\Pi(F, G) \leq C'_N Q_N(F, G)^{\frac{1}{2(N-2)}}.
\]
Since \(2N - 2 > N\) for \(N \geq 3\), the latter inequality is weaker than \((\bigstar_N)\) provided \(N \geq 3\) and \(Q_N(F, G)\) is small enough. In fact, it can be shown [16] that on every symplectic manifold of dimension \(\geq 4\) there exist sequences of functions \(A_\epsilon\) and \(B_\epsilon\) so that \(\|A_\epsilon\| = \|B_\epsilon\| = 1\) and
\[
Q_N(A_\epsilon, B_\epsilon) = \beta_N \cdot \epsilon^{2N-2},
\]
where \(\beta_N, N \geq 2\), is a sequence of positive numbers, and \(\epsilon \in (0, \epsilon_0)\) for some \(\epsilon_0 > 0\) independent of \(N\). Thus for any fixed \(N\) we have that \(Q_N(A_\epsilon, B_\epsilon)^\frac{1}{N} \sim \epsilon^{2-\frac{2}{N}}\) as \(\epsilon \to 0\). Therefore, for a pair of integers \(N > L \geq 2\) the upper bound for \(\Pi(A_\epsilon, B_\epsilon)\) given by \((\bigstar_N)\) is sharper than the one given by \((\bigstar_L)\) provided \(\epsilon\) is small enough.

12
Let us mention finally that the functionals $Q_N$ have been studied within function theory on symplectic manifolds from a different viewpoint. It is known that $Q_2$ and $Q_3$ are lower semi-continuous with respect to the uniform metric $d$ on $\mathcal{H}$, and in fact have rather tame local behavior [5, 2, 7]. It would be interesting to investigate the lower semi-continuity of $Q_N$ for $N > 3$.

1.5 Symplectic integrators

Our proof of Theorem 1.3 for $N \geq 3$ follows closely the lines of [8] with one technical innovation: we use symplectic integrators for the proof of our main theorem. Their appearance was somewhat unexpected to us: symplectic integrators have been designed for the purposes of numerical Hamiltonian dynamics [18], the subject which is seemingly remote from the theme of the present work.

Let $\Phi_t$, $\Psi_t$, $t \in (-\varepsilon, \varepsilon)$, be two smooth families of diffeomorphisms of a closed manifold $M$. We say that they are equivalent modulo $t^N$, denoted

$$\Phi_t = \Psi_t \mod t^N,$$

if for any $f \in C^\infty(M)$ and any $x \in M$

$$|f(\Phi_t x) - f(\Psi_t x)| = O(t^N) \text{ as } t \to 0.$$

Equivalently, for any $x \in M$, one should have $\text{dist} (\Phi_t(x), \Psi_t(x)) = O(t^N) \text{ as } t \to 0$ for any Riemannian metric defined around $x$.

**Definition 1.10.** A symplectic integrator of order $N$ is a set of real numbers $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m$ such that for any $F, G \in C^\infty(M)$,

$$\phi^t_{F+G} = \phi^t_{\alpha_1 F} \phi^t_{\beta_1 G} \cdots \phi^t_{\alpha_m F} \phi^t_{\beta_m G} \mod t^{N+1}. \quad (10)$$

Note that taking $G = 0$ in (10) yields

$$\phi^t_F = \phi^t_{\sum_{i=1}^m \alpha_i F} \mod t^{N+1}.$$

and hence $\sum_{i=1}^m \alpha_i = 1$. Similarly, $\sum_{i=1}^m \beta_i = 1$.

A crucial fact used in the next section is the existence of symplectic integrators of all orders. This was proved by Yoshida [18] (see also [16] for a detailed proof).
2 Proof of the main theorem

2.1 Using symplectic integrators

We start with the following estimate. Let \( \{\alpha_i, \beta_i\}_{i=1}^m \) be a symplectic integrator of order \( N - 1 \), where \( N \geq 2 \). Let \( F, G \in C^\infty(M) \) be Hamiltonian functions with zero mean. Set

\[
\Psi^t_N = \phi_{\alpha_1 F}^t \phi_{\beta_1 G}^t \cdots \phi_{\alpha_m F}^t \phi_{\beta_m G}^t
\]

and note that, by the cocycle formula (see e.g. [14]), the Hamiltonian flow \( \Psi^t_N \) is generated by the Hamiltonian

\[
K_N = \alpha_1 F + \beta_1 G \circ \phi_{\alpha_1 F}^{-t} + \ldots + \\
+ \alpha_m F \circ \phi_{\beta_{m-1} G}^{-t} \circ \phi_{\alpha_m F}^{-t} \circ \ldots \circ \phi_{\beta_1 G}^{-t} \circ \phi_{\alpha_1 F}^{-t} + \\
+ \beta_m G \circ \phi_{\alpha_m F}^{-t} \circ \phi_{\beta_{m-1} G}^{-t} \circ \phi_{\alpha_{m-1} F}^{-t} \circ \ldots \circ \phi_{\beta_1 G}^{-t} \circ \phi_{\alpha_1 F}^{-t}.
\] (11)

Proposition 2.1. There exists a constant \( \kappa > 0 \) independent of \( F \) and \( G \) such that

\[
||K_N(t) - (F + G)|| \leq \kappa \cdot Q_N(F, G)t^{N-1}
\] (12)

for all \( t \in [0, 1] \).

Together with (2) Proposition 2.1 immediately yields the following corollary.

Corollary 2.2. Let \( \mu : \widetilde{\text{Ham}}(M, \omega) \to \mathbb{R} \) be a stable homogeneous quasi-morphism. Then, under the hypothesis of Proposition 2.1,

\[
|\mu(\phi_{F+G}) - \mu(\Psi^1_N)| \leq kQ_N(F, G),
\] (13)

where \( k := \frac{\kappa}{N} \).

The proof of Proposition 2.1 is given in Section 2.2.

Proof of Theorem 1.3. Evidently it is enough to consider the case when \( F \) and \( G \) have zero mean. Let \( \Psi^1_N \) be as above and let \( \mu \) be a stable homogeneous quasi-morphism inducing the symplectic quasi-state \( \zeta \). Then

\[
\Psi^1_N = \phi_{\alpha_1 F} \phi_{\beta_1 G} \cdots \phi_{\alpha_m F} \phi_{\beta_m G}
\]
and hence, since $\mu$ is a quasi-morphism,

$$|\mu(\Psi_N^1) - \sum_{i=1}^m \mu(\phi_{\alpha_i F}) - \sum_{i=1}^m \mu(\phi_{\beta_i G})| \leq 2m \cdot C.$$ 

Combining this with (13) we obtain

$$|\zeta(F + G) - \sum_{i=1}^m \zeta(\alpha_i G) - \sum_{i=1}^m \zeta(\beta_i G)|$$

$$= |\mu(\phi_{F+G}) - \sum_{i=1}^m \mu(\phi_{\alpha_i F}) - \sum_{i=1}^m \mu(\phi_{\beta_i G})| \leq 2m \cdot C + k \cdot Q_N(F, G).$$

Note that

$$\sum_{i=1}^m \zeta(\alpha_i F) = \left(\sum_{i=1}^m \alpha_i\right) \zeta(F) = \zeta(F).$$

Similarly,

$$\sum_{i=1}^m \zeta(\beta_i G) = \left(\sum_{i=1}^m \beta_i\right) \zeta(g) = \zeta(G).$$

Therefore

$$\Pi(F, G) = |\zeta(F + G) - \zeta(F) - \zeta(G)| \leq 2m \cdot C + k \cdot Q_N(F, G). \quad (14)$$

Finally, note that, by the homogeneity of $\zeta$, for any $E > 0$

$$\Pi(EF, EG) = E \cdot \Pi(F, G).$$

On the other hand,

$$Q_N(EF, EG) = E^N Q_N(F, G).$$

Substituting both into (14) and dividing by $E$ we obtain

$$\Pi(F, G) \leq \frac{2m \cdot C}{E} + k Q_N(F, G) E^{N-1}.$$ 

The right-hand side is minimized by

$$E_{min} = \left(\frac{2m \cdot C}{k(N - 1)Q_N(F, G)}\right)^{1/N}.$$
which yields the inequality
\[ \Pi(F,G) \leq C_N \cdot Q_N(F,G)^{1/N} \]
for
\[ C_N = N(N - 1)^{\frac{1-N}{N}}(2mC)^{\frac{N-1}{N}} k^\frac{1}{N}. \]

### 2.2 Estimating the remainder

In order to prove Proposition 2.1, we need to write a finite order expansion for a composition of several Hamiltonian flows. We use the following notation:

Given smooth functions \( H_1, \ldots, H_n, A \in C^\infty(M) \) and non-negative integers \( i_1, \ldots, i_{n-1} \), such that \( \sum_{j=1}^{n-1} i_j \leq N - 1 \) denote \( h_k^{(k)} := \phi_{H_k} \) for any \( k \) and \( l := N - \sum_{j=1}^{n-1} i_j \). We denote \( \text{ad}_H F := \{F, H\} \).

Set
\[
I_N^{(n)}(H_1, \ldots, H_n, i_1, \ldots, i_{n-1}, A, t) := \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_l (\text{ad}_{H_n})^{i_n} \cdots (\text{ad}_{H_1})^{i_1} A \circ h_k^{(n)}
\]
and
\[
I_N^{(1)}(H_1, A, t) = \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{N-1}} ds_N (\text{ad}_{H_1})^{N} A \circ h_k^{(1)}.
\]

With this notation the expansion takes the following form:

**Proposition 2.3.** Let \( A, H_1, \ldots, H_n \in C^\infty(M) \). Then
\[
A \circ h_k^{(1)} \circ \cdots \circ h_k^{(n)} = \sum_{i_1 + \cdots + i_n \leq N-1} \frac{1}{i_1!} \cdots \frac{1}{i_n!} (\text{ad}_{H_n})^{i_n} \cdots (\text{ad}_{H_1})^{i_1} A \cdot t^{i_1 + \cdots + i_n} + R_N^{(n)},
\]
where
\[
R_N^{(n)} = \sum_{i_1 + \cdots + i_{n-1} \leq N-1} \frac{1}{i_1!} \cdots \frac{1}{i_{n-1}!} I_N^{(n)}(H_1, \ldots, H_n, i_1, \ldots, i_{n-1}, A, t) \cdot t^{i_1 + \cdots + i_n} + R_N^{(n-1)} \circ h_k^{(n)}
\]
\[ R_N^{(1)} = I_N^{(1)}(H_1, A, t). \]

**Remark 2.4.** While this formula may seem complicated, its importance lies in the fact that for any \( n \) the term \( I_N^{(n)} \), and hence the remainder \( R_N^{(n)} \), contains Lie monomials in variables \( H_1, \ldots, H_n, A \) involving \( N \)-times-iterated Poisson brackets. Furthermore, recalling that \( l = \sum_{j=1}^{n-1} i_j \), note that

\[ I_N^{(n)}(H_1, \ldots, H_n, i_1, \ldots, i_{n-1}, A, t) = O(t^l) \quad \text{as} \quad t \to 0 \]

because of the multiple integral of multiplicity \( l \) appearing in the definition of \( I_N^{(n)} \). Hence, for any \( n \) we have (by induction) that \( R_N^{(n)} = O(t^N) \) as \( t \to 0 \).

**Proof.** The proof will proceed by induction on \( n \). The case \( n = 1 \) is simply the Taylor expansion with the Lagrange remainder written as a multiple integral:

\[
A \circ h_1^{(1)} = A + \sum_{i=1}^{N-1} \frac{d^i}{dt^i}(A \circ h_1^{(1)})(0) \frac{t^i}{i!} + \\
+ \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{N-1}} ds_N \frac{d^N}{dt^N}(A \circ h_1^{(1)})
= A + \sum_{i=1}^{N-1} (\text{ad}_{H_1})^i A \cdot \frac{t^i}{i!} + \\
+ \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{N-1}} ds_N (\text{ad}_{H_1})^N A \circ h_1^{(1)}
= \sum_{i=0}^{N-1} \frac{1}{i!} (\text{ad}_{H_1})^i A \cdot t^i + I_N^{(1)}(H_1, A, t).
\]
Now, assume the result holds for \( n \). Then
\[
A \circ h_t^{(1)} \circ \ldots \circ h_t^{(n+1)} =
\left( \sum_{i_1 + \ldots + i_n \leq N-1} \frac{1}{i_1!} \ldots \frac{1}{i_n!} (\text{ad}_{H_n})^{i_n} \ldots (\text{ad}_{H_1})^{i_1} A \cdot t^{i_1 + \ldots + i_n} + R_N^{(n)} \right) \circ h_t^{(n+1)} =
\sum_{i_1 + \ldots + i_n \leq N-1} \frac{1}{i_1!} \ldots \frac{1}{i_n!} (\text{ad}_{H_n})^{i_n} \ldots (\text{ad}_{H_1})^{i_1} A \circ h_t^{(n+1)} \cdot t^{i_1 + \ldots + i_n} + R_N^{(n)} \circ h_t^{(n+1)}.
\]

Using the case \( n = 1 \) we get
\[
(\text{ad}_{H_n})^{i_n} \ldots (\text{ad}_{H_1})^{i_1} A \circ h_t^{(n+1)} =
\sum_{i_{n+1} = 0}^{N-1-i_1-\ldots-i_n} (\text{ad}_{H_{n+1}})^{i_{n+1} + i_n} \ldots (\text{ad}_{H_1})^{i_1} A \cdot \frac{t^{i_{n+1} + i_n}}{i_{n+1}!} + I^{(1)}_{N-i_1-\ldots-i_n} (H_{n+1}, (\text{ad}_{H_n})^{i_n} \ldots (\text{ad}_{H_1})^{i_1} A, t).
\]

Set \( l = \sum_{j=1}^n i_j \) and note that
\[
I^{(1)}_{N-l} (H_{n+1}, (\text{ad}_{H_n})^{i_n} \ldots (\text{ad}_{H_1})^{i_1} A, t) =
\int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{N-l-1}} ds_{N-l} (\text{ad}_{H_{n+1}})^{N-l} (\text{ad}_{H_n})^{i_n} \ldots .
\]
\[
(\text{ad}_{H_1})^{i_1} A \circ h_{s_{N-l}}^{(n+1)} =
I^{(n+1)}_N (H_1, \ldots, H_{n+1}, i_1, \ldots, i_n, A, t).
\]
Therefore

\[ A \circ h_t^{(1)} \circ \ldots \circ h_t^{(n+1)} = \]

\[ = \sum_{i_1+\ldots+i_n \leq N-1} \frac{1}{i_1!} \cdot \ldots \cdot \frac{1}{i_n!} \cdot \]

\[ \sum_{i_{n+1}=0}^{N-l-1} (\text{ad}_{H_{n+1}})^{i_{n+1}}(\text{ad}_{H_n})^{i_n} \cdot \ldots \cdot (\text{ad}_{H_1})^{i_1} A \cdot \frac{t^{i_{n+1}}}{i_{n+1}!} \]

\[ \cdot \left( \sum_{i_1+\ldots+i_n \leq N-1} \frac{1}{i_1!} \cdot \ldots \cdot \frac{1}{i_n!} \cdot I^{(n+1)}_{N}(H_1,\ldots,H_{n+1},i_1,\ldots,i_n,A,t) \cdot t^{i_1+\ldots+i_n} + \right. \]

\[ + R^{(n)}_{N} \circ h_{t}^{(n+1)} = \]

\[ = \sum_{i_1+\ldots+i_{n+1} \leq N-1} \frac{1}{i_1!} \cdot \ldots \cdot \frac{1}{i_{n+1}!} \cdot \]

\[ \cdot (\text{ad}_{H_{n+1}})^{i_{n+1}}(\text{ad}_{H_n})^{i_n} \cdot \ldots \cdot (\text{ad}_{H_1})^{i_1} A \cdot t^{i_1+\ldots+i_{n+1}} + R^{(n+1)}_{N}, \]

which is the desired result. \( \square \)

Finally, we need to relate the notions of equivalence modulo \( t^N \) for Hamiltonian flows and Hamiltonian functions.

**Proposition 2.5.** Let \( U(t), V(t) \) be smooth time-dependent Hamiltonian functions. Then

\[ \phi^t_U = \phi^t_V \mod t^N \iff U(t) - V(t) = O(t^{N-1}) \text{ as } t \to 0. \]

**Proof.** For any function \( H = H(t) \) depending on \( t \) we will denote by \( H^{(i)} \) the \( i \)-th derivative of \( H \) with respect to \( t \). Then

\[ \frac{d}{dt} \circ \phi^t_U = \{ \varphi, U(t) \} \circ \phi^t_U, \]

\[ \frac{d^2}{dt^2} \circ \phi^t_U = \{ \varphi, U^{(1)}(t) \} \circ \phi^t_U + \{ \{ \varphi, U(t) \}, U(t) \} \circ \phi^t_U \]

and, generally,

\[ \frac{d^n}{dt^n} \circ \phi^t_U = \{ \{ \varphi, U^{(n-1)}(t) \} + S_n(U, \varphi, t) \} \circ \phi^t_U, \]

19
where $S_n(U, \varphi, t)$ is a Lie polynomial involving Poisson brackets of $U^{(i)}(t)$ and $\varphi$ for $i < n$ only. Substituting $t = 0$ we get that

$$\frac{d^n}{dt^n} \bigg|_{t=0} \varphi \circ \phi^t_U = \{\varphi, U^{(n-1)}(0)\} + S_n(U, \varphi, 0). \quad (15)$$

By definition, $\phi^t_U = \phi^t_V \mod t^N$ if and only if for any $\varphi \in C^\infty(M)$ and any $n \leq N - 1$,

$$\frac{d^n}{dt^n} \bigg|_{t=0} \varphi \circ \phi^t_U = \frac{d^n}{dt^n} \bigg|_{t=0} \varphi \circ \phi^t_V.$$

Assume first that $\phi^t_U = \phi^t_V \mod t^N$. Thus the right-hand sides of the equations (15) for $U$ and for $V$ coincide for all $n = 1, \ldots, N - 1$. Observe that if an autonomous function $H \in F$ satisfies $\{\varphi, H\} = 0$ for all $\varphi \in C^\infty(M)$, then $H \equiv 0$. Thus, increasing $n$ from 1 to $N - 1$, we consecutively get that

$$U(0) = V(0), U^{(1)}(0) = V^{(1)}(0), \ldots, U^{(N-2)}(0) = V^{(N-2)}(0), \quad (16)$$

which is equivalent to the fact that $U(t) - V(t) = O(t^{N-1})$ as $t \to 0$.

Vice versa, (16) yields that the right-hand sides of the equations (15) for $U$ and for $V$ coincide for all $n = 1, \ldots, N - 1$ and hence $\phi^t_U = \phi^t_V \mod t^N$.

**Proof of Proposition 2.1.** By definition, $\Psi^t_N = \phi^t_{F+G} \mod t^N$, and hence, by Proposition 2.5, $K_N = F + G \mod t^{N-1}$ as $t \to 0$. Denote $H_N := K_N - (F + G)$. Thus $H_N(t) = O(t^{N-1})$. Applying Proposition 2.3 and Remark 2.4, we get that $H_N$ is the sum of the remainders $R_{N-1}$ of each term in the sum (11). In turn, by Remark 2.4, each such remainder is the sum of terms of the form $ct^{N-1}p(F, G) \circ h_t$, where $\{h_t\}$ is a path of Hamiltonian diffeomorphisms, $p$ is a monomial from $P_{N-1}$ and $c$ is a constant independent of $F$ and $G$. Thus, recalling that

$$Q_N(F, G) = \sum_{p \in P_{N-1}} ||p(F, G)||,$$

we get that

$$\|K_N(t) - (F + G)\| = \|H_N(t)\| \leq \kappa Q_N(F, G)t^{N-1},$$

where $\kappa$ is a constant depending only on $N$ and on the choice of the symplectic integrator. This finishes the proof of the proposition. \qed
References

[1] Aarnes, J., Quasi-states and quasi-measures, Adv. Math. 86:1 (1991), 41-67.

[2] Buhovsky, L., The 2/3-convergence rate for the Poisson bracket, preprint, 2008, to appear in Geom. and Funct. Analysis. ArXiv version: L.Buhovsky, math/0802.3792.

[3] Cardin, F., Viterbo, C., Commuting Hamiltonians and Hamilton-Jacobi multi-time equations, Duke Math. J. 144 (2008), 235-284.

[4] Entov, M., Polterovich, L., Quasi-states and symplectic intersections, Comm. Math. Helv. 81:1 (2006), 75-99.

[5] Entov, M., Polterovich, L., C^0-rigidity of Poisson brackets, preprint, arXiv:0712.2913, 2007. To appear in Proceedings of the Joint Summer Research Conference on Symplectic Topology and Measure-Preserving Dynamical Systems, Contemporary Mathematics, AMS.

[6] Entov, M., Polterovich, L., Symplectic quasi-states and semi-simplicity of quantum homology, in Toric Topology, pp. 47-70, Contemporary Mathematics 460, AMS, Providence, 2008.

[7] Entov, M., Polterovich, L., C^0-rigidity of the double Poisson bracket, Int. Math. Res. Notices. 2009; 2009: 1134-1158.

[8] Entov, M., Polterovich, L., Zapolsky, F., Quasi-morphisms and the Poisson bracket, Pure and Applied Math. Quarterly 3:4 (2007), 1037-1055.

[9] Entov, M., Polterovich, L., Zapolsky, F., An “anti-Gleason” phenomenon and simultaneous measurements in classical mechanics, Foundations of Physics 37:8 (2007), 1306-1316.

[10] Gromov, M., Soft and hard symplectic geometry, in Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), 81-98, Amer. Math. Soc., Providence, 1987.

[11] Humilière, V., Hamiltonian pseudo-representations, Comment. Math. Helv. 84:3 (2009), 571-585.
[12] Kotschick, D., *What is... a quasi-morphism?*, Notices Amer. Math. Soc. **51**:2 (2004), 208-209.

[13] McDuff, D., Salamon, D., *J-holomorphic curves and symplectic topology*, AMS, Providence, 2004.

[14] Polterovich, L., *The geometry of the group of symplectic diffeomorphisms*, Lectures in Mathematics – ETH Zürich, Birkhäuser, 2001.

[15] Py, P., *Quasi-morphismes et invariant de Calabi*, Ann. Sci. Ecole Norm. Sup. **39** (2006), 177-195.

[16] Rosen, D., *Master thesis*, Tel Aviv University, 2009.

[17] Rosenberg, M., *Py-Calabi quasi-morphisms and quasi-states on orientable surfaces of higher genus*, preprint, arXiv:0706.0028, 2007.

[18] Yoshida, H., *Construction of higher order symplectic integrators*, Phys. Lett. A **150** (1990), 262-268.

[19] Zapolsky, F., *Quasi-states and the Poisson bracket on surfaces*, J. of Modern Dynamics **1**:3 (2007), 465-475.

[20] Zapolsky, F., *Isotopy-invariant topological measures on closed orientable surfaces of higher genus*, preprint, arXiv:0903.2659, 2009.

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