A novel mathematical setup for fault tolerant control systems with state-dependent failure process

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Abstract. In this paper, we consider a fault tolerant control system (FTCS) with state-dependent failures and provide a tractable mathematical model to handle the state-dependent failures. By assuming abrupt changes in system parameters, we use a jump process modelling of failure process and the fault detection and isolation (FDI) process. In particular, we assume that the failure rates of the failure process vary according to which set the state of the system belongs to.

1. Introduction
In fault tolerant control system (FTCS), apart from the actuator, plant, sensors, input, outputs, there are two major modules in general: a fault detection and identification (FDI) module and a reconfiguration controller. In the event of faulty behaviour reported by the FDI module, the reconfiguration controller provides a necessary feedback action such that overall control system is fault tolerant, which is the reason for naming it as FTCS. One can find myriads of contributions of FTCS in the literature for instance in [1], [4] etc.

In the presence of abrupt changes or multiple modes that involve randomness, FTCS with Markov processes have been investigated in [11], [6], [7], [2] and the references therein. In these works, the authors primarily assumed different state spaces for a failure process and an FDI process to incorporate fault detection delays, false alarms that may result from various errors in the FDI module. FTCS modelling in this situation can be given by

\[
\begin{align*}
\dot{x}(t) &= A_{\theta(t)} x(t) + B_{\theta(t)} u(\eta(t), x(t), t), \\
x(0) &= x_0,
\end{align*}
\]

(1)

with the state vector \(x(t) \in \mathbb{R}^n\), the initial state \(x_0\), the control input \(u(\cdot, \cdot, \cdot) \in \mathbb{R}^m\), and the initial state \(x_0 \in \mathbb{R}^n\). Let \(A_{\theta(t)} \in \mathbb{R}^{n \times n}\), \(B_{\theta(t)} \in \mathbb{R}^{n \times m}\), be the system matrices that depend on the mode \(\theta(t)\), and are assumed to be known. Here \(\{\theta(t), t \geq 0\} \in S\) and \(\{\eta(t), t \geq 0\} \in T\) represent the failure process and the FDI process with finite state-spaces \(S := \{1, 2, \cdots, M\}\) and \(T := \{1, 2, \cdots, N\}\) respectively. An illustration of the overall FTCS is given in figure 1, where the controller reconfiguration depends on the FDI process at each time. In an ideal scenario of FDI, the FDI process \(\{\eta(t), t \geq 0\}\) should match exactly with the failure process \(\{\theta(t), t \geq 0\}\), which is not the case in practise, thus the reason for different state spaces \(S\) and \(T\) for \(\{\theta(t), t \geq 0\}\) and \(\{\eta(t), t \geq 0\}\).
We begin with the failure process.

In the presence of abrupt changes in system parameters that may cause failures in various components, the failure process \( \{\theta(t), t \geq 0\} \) has been modelled as time-homogeneous Markov Process, which implies the constant failure rates, which is a quite restrictive assumption. In practice, the failure rates depend on the usage and/or the current state of the component. For example, it was examined in [8] that the failure rate of a component is dependent on its state at age \( t \), where the state variable may be an amount of wear, stress etc. Also, for instance, in [9], a state-dependent Markov process was used to describe the random breakdown of cylinder lines in a heavy-duty marine diesel engine. Thus, we consider that the failure process \( \theta(t) \) depends on the state variable. Next we move to the FDI process.

In an ideal case, by using various measurements of the FTCS, the FDI process should give accurate results whether a faulty behaviour has occurred in the system, which is used as a feedback to reconfigure the overall the FTCS to correct faulty behaviour. Thus, the FDI process depend directly on the failure process. But, in the presence of noisy measurements, it is difficult for the FDI process to provide accurate behaviour of the system. If the measurement process contain white noise, and memoryless tests are used to characterize the FDI process, then the FDI process can be modelled as a Markov jump process with finite space as examined in [11], [6] etc. In our case, for simplicity, we consider that the FDI process is Markovian. It is a reasonable assumption, because even if the FTCS contain state-dependent failure process, one can still employ memoryless tests for FDI process, which just depend on the measurements from the system. Thus, we consider the FDI process, which is Markovian.

Thus, we consider state-dependent failure process, which is also Markovian in some suitably defined sets, and the FDI process as a pure Markov process which just depend on the failure process. The presence of state-dependent failure process makes the FTCS modelling (1) far more complex. To the best of author’s knowledge, a tractable the FTCS modelling (1) with state-dependent failure process has not been addressed, which forms a theme of this paper.

In the next section, we provide a tractable mathematical model of FTCS with state-dependent failure process.

2. Mathematical model of the FTCS with state-dependent failure process

Consider a mathematical model of FTCS:

\[
\begin{aligned}
\dot{x}(t) &= A_{\theta(t)} x(t) + B_{\theta(t)} u(\eta(t), x(t), t), \\
x(0) &= x_0.
\end{aligned}
\]  
(2)
Since we assume the Markovian modelling of the FDI process \( \{ \eta(t), t \geq 0 \} \in T \) that depends on the failure process \( \{ \theta(t), t \geq 0 \} \in S \), its evolutions for \( m, n \in T, m \neq n \) and \( z \in S \) can be considered as:

\[
\Pr\{\eta(t + \triangle) = n|\eta(t) = m, \theta(t) = z\} = \delta_{mn,z} + o(\triangle),
\]

where \( \triangle > 0 \) and \( \lim_{\triangle \to 0} \frac{o(\triangle)}{\triangle} = 0 \). Regarding the transition rates of \( \eta(t) \), we make the following assumption:

**Assumption 1.** For each \( z \in S, m, n \in T \), \( \delta_{mn,z} \) is the transition rate of \( \eta(t) \) from mode \( m \) to mode \( n \) with \( \delta_{mn,z} \geq 0 \) for \( m \neq n \), and \( \delta_{mn,z} = -\sum_{n=1,m \neq n}^{N} \delta_{mn,z} \). We further assume that \( \delta_{mn,z} \) is bounded for all \( z \in S, m, n \in T \).

Before proceeding with the state-dependent failure process \( \{ \theta(t), t \geq 0 \} \), we make the following assumption regarding the different sets in \( \mathbb{R}^n \).

**Assumption 2.** Let \( K = \{1, 2, \cdots, K\} \). We assume that the sets \( \mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_K \subseteq \mathbb{R}^n \), \( \bigcup_{i=1}^{K} \mathcal{C}_i = \mathbb{R}^n \) and \( \mathcal{C}_p \cap \mathcal{C}_q = \phi \) for any \( p, q \in K \), \( p \neq q \). At any time \( t \geq 0 \), we assume that \( x(t) \) belongs to exactly one of the sets \( \mathcal{C}_i \), \( i \in K \). By its definition, \( \mathcal{C}_1, \mathcal{C}_2, \cdots \mathcal{C}_K \) are Borel sets in \( \mathbb{R}^n \).

For example, as an illustration, we can consider \( \mathbb{R}^2 \) space with three sets \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \), which can be defined as: \( \mathcal{C}_1 := \{x(t) \in \mathbb{R}^2, x_1^2(t) + x_2^2(t) \leq \xi_1\} \), \( \mathcal{C}_2 := \{x(t) \in \mathbb{R}^2, \xi_1 < x_1^2(t) + x_2^2(t) \leq \xi_2\} \) and \( \mathcal{C}_3 := \{x(t) \in \mathbb{R}^2, x_1^2(t) + x_2^2(t) \geq \xi_2\} \) for some \( \xi_1 > 0 \) and \( \xi_2 > 0 \). The illustration is given in figure 2, where the state variable \( x(t) \) can belong to exactly one of the sets at each time \( t \geq 0 \).

![Figure 2](image-url)

**Figure 2.** A schematic representation of the sets \( \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3 \) in \( \mathbb{R}^2 \)

Now, the evolutions of the state-dependent failure process \( \{ \theta(t), t \geq 0 \} \) for \( i, j \in S, i \neq j \) can be governed by:

\[
\Pr\{\theta(t + \triangle) = j|\theta(t) = i, x(t)\} = \begin{cases} 
\lambda_{ij}^{1} \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_1, \\
\lambda_{ij}^{2} \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_2, \\
& \vdots \\
\lambda_{ij}^{K} \Delta + o(\Delta), & \text{if } x(t) \in \mathcal{C}_K.
\end{cases}
\]
Regarding the transition rates of $\theta(t)$, we make the following assumption:

**Assumption 3.** Let $\mathcal{K} \triangleq \{1, 2, \cdots, K\}$. For each $m \in \mathcal{K}$, $\lambda_{ij}^m$ is the transition rate of $\theta(t)$ from mode $i$ to mode $j$ with $\lambda_{ij}^m \geq 0$ for $i \neq j$, and $\lambda_{ii}^m = -\sum_{j=1, j \neq i}^{N} \lambda_{ij}^m$. We further assume that $\lambda_{ij}^m$ is bounded for all $m \in \mathcal{K}$.

**Remark 1:** Notice that the evolution of $\theta(t)$ in (4) depends on the state variable $x(t)$, hence $\theta(t)$ is called as a state-dependent failure process. Also observe that the failure rates are different depending on which set the state of the system belongs to. It is a valid assumption, because, in this setup, one can predefine suitable sets $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_K$ and assign different failure rates in a suitable fashion, thus it is tractable. A more general jump system with state-dependent jump process without any other jump process (for example the FDI process in our case) is considered in [3] similar to (4), where sufficient conditions for stability and stabilizing controller in terms of linear matrix inequalities (LMIs) are obtained by utilizing techniques of probability theory and stochastic version of Lyapunov’s second method.

3. Challenges and future work

We would like to highlight a couple of challenges in the FTCS with state-dependent failure setup given above.

1. Since the overall FTCS (2) is nonlinear, it is necessary to provide its flow as a function of time.

2. For a simple state-feedback reconfiguration control of the FTCS, usually one design state-feedback gains obtained from Lyapunov’s second method, which requires the assumption of Markovian process of the joint variables of the Lyapunov candidate [5].

For point 1 above, one can describe the flows of the FTCS (2) when $x(t) \in \mathcal{C}_m$ for each $m \in \mathcal{K}$, which can be characterized in terms of suitably defined first exit times from different sets $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_K$. For point 2 above, one can prove that the joint process formed by $(x(t), \theta(t), \eta(t))$ is Markov process by using arguments of probability theory.

Both of these issues has already been investigated in [3] for a more general system without FTCS configuration, which we would like to extend to the FTCS framework, which can provide answers to the reconfiguration control in FTCS with state-dependent failure process. This forms a future extension of this paper.

4. Conclusions

We provided a tractable mathematical model for FTCS with state-dependent failure process. For the underlying state-dependent failure process, we explicitly considered different failure rates depending on the set to which the state of the system belongs. Finally, we presented a future course of direction.

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