Generalized-lush spaces revisited

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Abstract

We study geometric properties of GL-spaces. We demonstrate that every finite-dimensional GL-space is polyhedral; that in dimension 2 there are only two, up to isometry, GL-spaces, namely the space whose unit sphere is a square (like $\ell_2^\infty$ or $\ell_2^1$) and the space whose unit sphere is an equilateral hexagon. Finally, we characterise the spaces $E = (\mathbb{R}^n, \| \cdot \|_E)$ with absolute norm such that for every collection $X_1, \ldots, X_n$ of GL-spaces their $E$-sum is a GL-space.

Keywords Tingley’s problem · Mazur–Ulam property · Polyhedral space · GL-space · Ultraproduct

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1 Introduction

In 1987, Tingley [13] posed the following question: let $f$ be a bijective isometry between the unit spheres $S_X$ and $S_E$ of real Banach spaces $X,$ $E$ respectively. Is it true that $f$ extends to a linear isometry $F : X \to E$ of the corresponding spaces?

Considering Tingley’s question, Cheng and Dong [4] introduced the following useful terminology:

Definition 1.1 A Banach space $X$ is said to have the Mazur–Ulam property provided that for every Banach space $E,$ every surjective isometry $f : S_X \to S_E$ is the restriction of a linear isometry between $X$ and $E.$
There is a number of publications devoted to Tingley’s problem (say, Zentralblatt Math. shows 57 related papers published from 2002 to 2019) and, in particular, it is known [8] that finite-dimensional polyhedral spaces (i.e. those spaces whose unit ball is a polyhedron) have the Mazur–Ulam property. Surprisingly, for general spaces the innocently-looking Tingley’s question remains unanswered even in dimension two.

In 2013 Tan et al. [12] have made a substantial advance. In order to explain it let us give some definitions.

The distance from a point $x$ of a normed space $X$ to a non-empty subset $A \subset X$ is defined to be the infimum of the distances from $x$ to the elements of $A$:

$$\rho(x, A) = \inf_{a \in A} \|x - a\|.$$

**Definition 1.2** A closed slice of the unit ball $B_X$ of a Banach space $X$ is a subset of $B_X$ of the form

$$S(x^*, \alpha) = \{x \in B_X : x^*(x) \geq 1 - \alpha\},$$

where $x^* \in S_{X^*}$ and $\alpha \in (0, 1)$.

**Definition 1.3** ([12]) A Banach space $X$ is said to be generalized-lush (GL-space for short) if for every $x \in S_X$ and every $\varepsilon > 0$ there exists a slice $S(x^*, \varepsilon)$ with $x^* \in S_{X^*}$ such that $x \in S(x^*, \varepsilon)$ and

$$\rho(y, S(x^*, \varepsilon)) + \rho(-y, S(x^*, \varepsilon)) < 2 + \varepsilon$$

for all $y \in S_X$. A Banach space $E$ is said to be a local-GL-space if for every separable subspace $Y \subset E$ there is a GL-subspace $X \subset E$ such that $Y \subset X \subset E$.

These definitions are motivated by the concept of lush spaces, introduced in [2] in relation to the numerical index of operators, see Sect. 1.5 in the recent monograph [9] for the definition, examples and the history of the subject.

In [12] it is demonstrated that all local-GL-spaces (and consequently all GL-spaces, all lush spaces and in particular all $C(K)$ and $L_1(\mu)$ spaces) enjoy the Mazur–Ulam property. Also it is shown that the class of GL-spaces is stable under $c_0$-, $\ell_1$- and $\ell_\infty$-sums and that the space $C(K, X)$ is a GL-space whenever $X$ is a GL-space, which gives a number of examples of spaces with the Mazur–Ulam property. In the same vein Hardtke [5] demonstrated that the class of GL-spaces is stable under ultraproducts and under passing to a large class of F-ideals, in particular to M-ideals.

In this article we, at first, demonstrate that every finite-dimensional GL-space is polyhedral (Theorem 2.8)—that is, in finite dimensions the results of [12] do not give new examples of Mazur–Ulam spaces comparing to [8]. At second, in Theorem 2.11 we show that in dimension 2 there are only two, up to isometry, GL-spaces, namely the space whose unit sphere is a square (like $\ell_\infty^2$ or $\ell_1^2$) and the space whose unit sphere is an equilateral hexagon. The above results are collected in Sect. 2. Finally, in Sect. 3 we address the question what are the spaces $E = (\ell^n, \|\cdot\|_E)$ with absolute norm such that for every collection $X_1, \ldots, X_n$ of GL-spaces their $E$-sum is a GL-space.

In the exposition we will use the standard Banach space theory notations. In particular, for Banach space $X$ we denote, as we already have done above, by $B_X$, $S_X$ and
$X^*$ the closed unit ball, unit sphere and the dual space respectively. In our paper we consider only real Banach spaces and real linear functionals (otherwise the definition of slice should be modified). All unexplained functional analysis terminology can be found in Ref. [7].

2 Polyhedrality of finite-dimensional GL-spaces

Recall that the Hausdorff distance between two not empty closed subsets $A$ and $B$ of a metric space $X$ is defined as

$$
\rho_H(A, B) = \max \left\{ \sup_{b \in B} \rho(b, A), \sup_{a \in A} \rho(a, B) \right\}.
$$

According to the Blaschke selection theorem (see [14, Theorem 2.5.14]) the collection of nonempty closed convex subsets of a given bounded subset of a finite-dimensional normed space forms a compact in the Hausdorff metric.

Definition 2.1 A face of the unit ball of a Banach space $X$ is a nonempty set of the form

$$
\mathcal{F}(x^*) = \{ x \in B_X : x^*(x) = 1 \},
$$

where $x^* \in S_{X^*}$.

Definition 2.2 We call a subset $A \subset B_X$ plump if for every $y \in S_X$ there are $a_1, a_2 \in A$ such that

$$
\|y - a_1\| + \|y + a_2\| \leq 2.
$$

Remark that in the case of compact $A$ the distances attain, so the inequality $\rho(y, A) + \rho(-y, A) \leq 2$ implies automatically the existence of corresponding $a_1, a_2 \in A$; that if $A \subset B \subset B_X$ and $A$ is plump then $B$ is plump as well; and if for every $\varepsilon > 0$ every point $x \in S_X$ is contained in a plump slice $S(x^*, \varepsilon)$ then $X$ is a GL-space.

The following definition is motivated by [5, Proposition 2.2] and by analogous concepts of ultra-lush spaces from [3] and rigid narrow operator with respect to a subset from [1].

Definition 2.3 A normed space $X$ is said to be ultra-GL with respect to a subspace $W \subset X^*$ (ultra-GL($W$)-space) if for every $x \in S_X$ there exists an $x^* \in S_W$ such that $x \in \mathcal{F}(x^*)$ and $\mathcal{F}(x^*)$ is plump. In the case of $W = X^*$ the space $X$ is said to be ultra-GL.

Lemma 2.4 Let $X$ be a finite-dimensional normed space. Then the following conditions are equivalent:

1. $X$ is a GL-space;
2. $X$ is ultra-GL.
3. for every \( x \in S_X \) there exists a convex plump subset \( B \subset S_X \) such that \( x \in B \).

**Proof** (3) \( \Rightarrow \) (2) Let the third condition hold. Then for given \( x \in S_X \) there is a convex plump subset \( B \subset S_X \) containing \( x \). This \( B \) can be separated from the open unit ball by a norm-one functional (Hahn–Banach), so there exists a functional \( x^* \in S_{X^*} \) such that \( B \subset F(x^*) \). Then \( x \in F(x^*) \) and \( F(x^*) \) is plump.

(2) \( \Rightarrow \) (1) Take for given \( x \in S_X \) the corresponding \( x^* \in S_{X^*} \) that generates a plump face containing \( x \). For every \( \varepsilon > 0 \) consider the slice \( S(x^*, \varepsilon) \). It is obvious that \( F(x^*) \subset S(x^*, \varepsilon) \subset B_X \), so \( S(x^*, \varepsilon) \) is plump.

(1) \( \Rightarrow \) (3) Observe that (1) means for every \( n \in \mathbb{N} \) there exists \( x_n^* \in S_{X^*} \) such that

\[
\rho(y, S_n) + \rho(y, -S_n) < 2 + \frac{1}{n}
\]

for every \( y \in S_X \). The Blaschke selection theorem implies the existence of a subsequence \( S_{n_k} \) that converges in the Hausdorff metrics to a closed convex set

\[
B = \lim_{k \to \infty} S_{n_k} \subset B_X.
\]

Obviously, the inclusion \( S_n \subset \{ x \in B_X : \|x\| \geq 1 - \frac{1}{n} \} \) implies that the limiting set \( B \) lies on the unit sphere. Since for a fixed \( y \in S_X \) the distance \( \rho(y, S) \) depends continuously in the Hausdorff metrics on the variable \( S \), (2.1) gives us the desired inequality \( \rho(y, B) + \rho(y, -B) \leq 2 \).

Note the following obvious corollary.

**Corollary 2.5** Let \( X \) be a finite-dimensional GL-space. Then \( S_X \) is the union of its plump faces.

**Lemma 2.6** Let \( X \) be a normed space, \( x^* \in S_{X^*} \). Then the following conditions are equivalent:

1. \( F(x^*) \) is plump;
2. \( \rho(y, F(x^*)) + \rho(-y, F(x^*)) \leq 2 \) for every \( y \in B_X \) and the distances are attained, i.e., there exist \( u, v \in F(x^*) \) such that \( \|u - y\| + \|v + y\| \leq 2 \);
3. \( \rho(y, F(x^*)) = 1 - x^*(y) \) for every \( y \in B_X \) and the distance is attained.
4. \( \rho(y, F(x^*)) = 1 - x^*(y) \) for every \( y \in S_X \) and the distance is attained.

**Proof** The implications (2) \( \Rightarrow \) (1) and (3) \( \Rightarrow \) (4) are obvious, so it remains to prove the implications (1) \( \Rightarrow \) (2), (2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (1).

(1) \( \Rightarrow \) (2). For a given \( y \in B_X \) denote \( \hat{y} = \frac{y}{\|y\|} \). Using plumpness of \( F(x^*) \) let us choose \( u, v \in F(x^*) \) such that \( \|u - \hat{y}\| + \|v + \hat{y}\| \leq 2 \). Denote \( \lambda = \|y\| \). Then we have

\[
\|u - y\| + \|v + y\| = \|\lambda(u - \hat{y}) + (1 - \lambda)u\| + \|\lambda(v + \hat{y}) + (1 - \lambda)v\|
\]
The unit ball of every finite-dimensional GL-space is a polyhedron whose faces are plump.

**Theorem 2.8** Let $X$ be a GL-space, and let $\ker x^*$ be the minimal $(n-1)$-dimensional linear subspace of $X$. Fix a plump face $F$ in an $k$-dimensional space $E$. Then $\rho(y, F(x^*)) = 1 - x^*(y)$ and $\rho(-y, F(x^*)) = 1 + x^*(y)$. Consequently, $\rho(y, F(x^*)) + \rho(-y, F(x^*)) = 2$. This may happen only if $\|u - y\| = 1 - x^*(y)$ and $\|v + y\| = 1 + x^*(y)$.

**Lemma 2.7** Let $x^* \in S_{X^*}$ and $F(x^*)$ be plump. Then $F(x^*) - F(x^*) \supset B_X \cap \ker x^*$.

**Proof** Fix $y \in B_X \cap \ker x^*$. Our goal is to show that $y \in F(x^*) - F(x^*)$. Indeed, select $v \in F(x^*)$ with $\|v + y\| = \rho(-y, F(x^*))$. According to item (3) of Lemma 2.6 $\|v + y\| = x^*(v + y) = 1$. Consequently, $v + y \in F(x^*)$, and $y = (v + y) - v \in F(x^*) - F(x^*)$.

**Theorem 2.8** The unit ball of every finite-dimensional GL-space is a polyhedron whose faces are plump.

**Proof** Let $X = (\mathbb{R}^n, \| \cdot \|)$ be a GL-space, and let $r > 0$ be the minimal $(n-1)$-dimensional volume of intersections of $B_X$ with $(n-1)$-dimensional linear subspaces of $X$. Fix a plump face $F(x^*)$ and $x \in F(x^*)$. Then $F(x^*) - x \subset \ker x^*$. Remark that due to the previous Lemma

$$(F(x^*) - x) - (F(x^*) - x) = F(x^*) - F(x^*) \supset B_X \cap \ker x^*.$$ 

According to the Rogers–Shephard theorem [11, Theorem 1], for every convex body $K$ in an $m$-dimensional space $E$

$$\vol_m(K - K) \leq \binom{2m}{m} \vol_m(K).$$

Applying that theorem to $K = F(x^*) - x \subset E := \ker x^*$ we obtain that

$$\vol_{n-1}(F(x^*)) = \vol_{n-1}(F(x^*) - x)$$
\[\frac{2m}{m}^{-1} \max_{\lambda n^{-1}} \left( (|x^*| - x) - (|x^*| - x) \right) \]
\[\geq \left( \frac{2m}{m} \right)^{-1} \max_{\lambda n^{-1}} \left( B_x \cap \ker x^* \right) \geq \left( \frac{2m}{m} \right)^{-1} r.\]

So the \((n - 1)\)-dimensional volumes of plump faces are bounded below by a positive number, so the set of plump faces of \(B_x\) is finite. Enumerate \(\{x^*_k\}_{k=1}^N, x^*_k \in S_{x^*}\), all plump faces. Our Corollary 2.5 ensures that \(S_x = \bigcup_{k=1}^N \mathcal{F}(x^*_k)\), which means that \(B_x = \{ x \in \mathbb{R}^n : |x^*_k(x)| \leq 1 \text{ for all } k = 1, 2, \ldots, N \}. \]

**Corollary 2.9** The length of each edge of a real two-dimensional GL-space is not smaller than 1.

**Proof** The proof of this fact follows from Theorem 2.8 and Lemma 2.7. \(\square\)

Before moving to our second main theorem we will set out the following result, needed for the proof.

**Proposition 2.10** ([10, Proposition 20]) On the unit sphere of a Minkowski plane there are at most three pairs of segments of length at least 1. If there are three pairs of segments of length at least 1, then the unit disc must be a hexagon with vertices \(\pm x_1, \pm x_2, \pm \lambda(x_1 + x_2)\) for some \(\lambda \in \left(\frac{1}{2}, 1\right]\), and at least two pairs are of length exactly 1.

It is known [12] that two-dimensional spaces \(\mathbb{R}^2\) with the norm

\[\|(a_1, a_2)\|_* = \max\{|a_1|, |a_2|\} \text{ or } \|(a_1, a_2)\|_1 = |a_1| + |a_2|\]

(theses space are called \(\ell^2_*\) and \(\ell^1_1\) respectively), and \(\mathbb{R}^2\) equipped with the norm \(\|(a_1, a_2)\| = \max\{|a_2|, |a_1| + \frac{1}{2}|a_2|\}\) (let us denote it \(E\)) are GL-spaces. The first two spaces are mutually isometric (their unit spheres are parallelograms), so as Banach spaces they are indistinguishable, and the unit sphere of \(E\) is an equilateral, in the metric of \(E\), hexagon. Our next theorem says that there are no other (up to isometry) two-dimensional GL-spaces.

**Theorem 2.11** Let \(X\) be a real two-dimensional GL-space. Then \(X\) is isometric either to the space \(E\) or to \((\mathbb{R}^2, \| \cdot \|_1)\).

**Proof** According to Theorem 2.8, the unit ball of \(X\) is a polygon whose edges are plump. Due to Corollary 2.9 the length of each edge is not smaller than 1. Application of Proposition 2.10 implies, that the unit ball of the space \(X\) either is a parallelogram (and then the space is isometric to two-dimensional space \(\ell^2_1\)), or a hexagon with vertices \(\pm x_1, \pm x_2, \pm \lambda(x_1 + x_2)\) for some \(\lambda \in \left(\frac{1}{2}, 1\right]\). It remains to consider the case of hexagon. Let us choose the basis vectors \(e_1 = x_1, e_2 = x_2\). The coordinates of the vertices in this basis are: \((\pm 1, 0), (0, \pm 1)\) and \((\pm \lambda, \lambda)\). So \(X\) may be considered as \(\mathbb{R}^2\) equipped with the norm \(\|(u, v)\| = \|ue_1 + ve_2\|\). Now we may explicitly write down an expression for the norm in this space. For every \(x = (u, v) \in (\mathbb{R}^2, \| \cdot \|)\)

\[\|x\| = \max_{1 \leq i \leq 3} \{|f_i(x)|\},\]
where \( f_i(x), 1 \leq i \leq 3 \) are linear functionals of the form: 
\[ f_1(x) = u + \frac{1-\lambda}{\lambda}v, \quad f_2(x) = \frac{1-\lambda}{\lambda}u + v \text{ and } f_3(x) = -u + v. \]
Now, let us consider the edge \([\lambda(x_1 + x_2), x_2] = \mathcal{F}(f_2)\) and compute \( \rho(x_1, \mathcal{F}(f_2)) \).

\[
\rho(x_1, \mathcal{F}(f_2)) = \inf_{\alpha \in [0, 1]} \rho(x_1, \alpha \lambda x_1 + \alpha \lambda x_2 + (1 - \alpha)x_2)
\]

\[= \inf_{\alpha \in [0, 1]} \| (1 - \alpha \lambda)x_1 + (\alpha - 1 - \alpha \lambda)x_2 \|
\]

\[= \inf_{\alpha \in [0, 1]} \max_{1 \leq i \leq 3} \{ |f_i(1 - \alpha \lambda, \alpha - 1 - \alpha \lambda)| \}
\]

\[= \inf_{\alpha \in [0, 1]} f_3(1 - \alpha \lambda, \alpha - 1 - \alpha \lambda)
\]

\[= \inf_{\alpha \in [0, 1]} (2 - \alpha) = 1.
\]

On the other hand, since \( \mathcal{F}(f_2) \) is plump, \( \rho(x_1, \mathcal{F}(f_2)) = 1 - f_2(x_1) = 1 - \frac{1-\lambda}{\lambda}, \)
so \( \lambda = 1 \). Summarizing, in the hexagonal case our space is isometric to \( \mathbb{R}^2 \) endowed with the norm

\[
\|x\| = \|(u, v)\| = \max\{|u|, |v|, |u - v|\}.
\]

This space is isometric to \( \tilde{E} \). \( \square \)

The following property of \( \tilde{E} \) will be helpful in the next section.

**Proposition 2.12** Let \( \tilde{E} = \mathbb{R}^2 \) equipped with the norm \( \|(a_1, a_2)\| = \max\{|a_1|, |a_2| + \frac{1}{2}|a_2|\} \), \( \check{e}_1 = (1, 0) \), \( \check{e}_2 = (\frac{1}{2}, 1) \), \( \check{e}_3 = (-\frac{1}{2}, 1) \), \( \check{e}_2^* \in S_{\check{E}^*} \) be the second coordinate functional on \( \mathbb{R}^2 \): \( \check{e}_2^*(a, b) = b \). Then \( \mathcal{F}(\check{e}_2^*) \) is equal to the edge of the hexagon \( B_{\check{E}} \) that connects \( \check{e}_2 \) and \( \check{e}_3 \), and the vertex \( \check{e}_1 \) belongs to the kernel of \( \check{e}_2^* \). Let \( t \in [0, 1] \) and let \( \tilde{y} = t\check{e}_2 + (1 - t)\check{e}_1 \) be such an element of the edge that connects \( \check{e}_2 \) and \( \check{e}_1 \) that \( \check{e}_2^*(\tilde{y}) = t \). Then if for a given \( \alpha > 0 \) there is \( \tilde{x} \in \mathcal{F}(\check{e}_2^*) \) such that

\[
\|\tilde{x} - \alpha \tilde{y}\| = 1 - \alpha t
\]

then \( \alpha \leq 1 \).

**Proof** Indeed, assume that \( \alpha > 1 \). Let us write \( \alpha \tilde{y} \) in coordinate form \( \alpha \tilde{y} = (a_1, a_2) \), \( a_1, a_2 \geq 0 \). Remark that (2.2) implies that \( a_2 = \check{e}_2^*(\alpha \tilde{y}) = \alpha t \leq 1 \), so \( 1 < \alpha = \|\alpha \tilde{y}\| = a_1 + \frac{1}{2}a_2 = a_1 + \frac{1}{2}\alpha t \), in particular

\[
a_1 > 1 - \frac{1}{2}\alpha t \geq \frac{1}{2}.
\]

The vector \( \tilde{x} \) is of the form \( \tilde{x} = (b_1, 1) \) with \( |b_1| \leq \frac{1}{2} \). So \( \tilde{x} - \alpha \tilde{y} = (b_1 - a_1, 1 - \alpha t) \), and \( \|\tilde{x} - \alpha \tilde{y}\| \geq |b_1 - a_1| + \frac{1}{2}(1 - \alpha t) = a_1 - b_1 - \frac{1}{2} + \frac{1}{2}(1 - \alpha t) \geq a_1 - \frac{1}{2} + \frac{1}{2}(1 - \alpha t) = a_1 - \frac{1}{2}\alpha t > 1 - \alpha t. \) \( \square \)
3 Direct sums of GL-spaces

Let \( E = (\mathbb{R}^n, \| \cdot \|_E) \) be a normed space, and denote \( e_k, k = 1, \ldots, n \), the elements of the canonical basis: the \( k \)-th coordinate of \( e_k \) is equal to 1, and the remaining ones are zeros. The norm \( \| \cdot \|_E \) is called absolute if it satisfies the following conditions:

(i) \( \|e_k\|_E = 1, k = 1, \ldots, n \);
(ii) for every \( a = (a_1, \ldots, a_n) \) the vector \( |a| := (|a_1|, \ldots, |a_n|) \) has the same norm as \( a \):

\[
\|(a_1, \ldots, a_n)\|_E = \|(|a_1|, \ldots, |a_n|)\|_E.
\]

The above properties imply that the norm is monotone in the following sense: if \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) satisfy \( 0 \leq a_k \leq b_k, k = 1, \ldots, n \), then \( \|a\|_E \leq \|b\|_E \).

The dual \( E^* \) to \( E \) we identify in the standard way with \( (\mathbb{R}^n, \| \cdot \|_{E^*}) \), where a functional \( b = (b_1, \ldots, b_n) \in E^* \) acts on \( a = (a_1, \ldots, a_n) \in E \) by the formula \( b(a) = b_1a_1 + \ldots + b_na_n \). Remark that the norm \( \| \cdot \|_{E^*} \) is also absolute.

Let \( X_1, \ldots, X_n \) be normed spaces, and \( E = (\mathbb{R}^n, \| \cdot \|_E) \) be a space with absolute norm. The \( E \)-sum of the spaces \( X_k \) (we will denote it \( E(X_k)_1^n \)) is the vector space of all \( n \)-tuples \( x = (x_1, \ldots, x_n) \), \( x_k \in X_k, k = 1, \ldots, n \), equipped with the norm

\[
\|x\| = (\|x_1\|, \ldots, \|x_n\|)_E.
\]

Recall, that condition (ii) from the definition of absolute norm guaranties that the expression (3.1) satisfies the triangle inequality. It also gives the following natural property: if \( X_k = \mathbb{R}, k = 1, \ldots, n \), then \( E(X_k)_1^n = E \). The condition (i) is not so essential because it can be easily achieved by rescaling, but it is usually assumed for sake of convenience. In order to shorten the notation, for \( x = (x_1, \ldots, x_n) \in E(X_k)_1^n \) we denote \( N(x) = (\|x_1\|, \ldots, \|x_n\|) \). In this notation, \( \|x\| = \|N(x)\|_E \). The dual space to \( E(X_k)_1^n \) is \( E^*(X_k)_1^n \), where \( f = (f_1, \ldots, f_n) \in E^*(X_k)_1^n \) acts on \( x = (x_1, \ldots, x_n) \in E(X_k)_1^n \) by the rule \( f(x) = f_1(x_1) + \ldots + f_n(x_n) \).

**Definition 3.1** A space \( E = (\mathbb{R}^n, \| \cdot \|_E) \) with absolute norm is said to be GL-respecting (GLR-space for short) if for every collection \( X_1, \ldots, X_n \) of GL-spaces the corresponding \( E \)-sum \( E(X_k)_1^n \) is generalized lush.

It is known [12, Theorem 2.11] that \( \ell_1^n \) and \( \ell_\infty^n \) are GL-respecting (formally speaking, that theorem deals with infinite sums, but it remains valid for finite sums as well). Our goal in this section is to find out, if there are other examples of GLR-spaces. We start with an evident remark.

**Proposition 3.2** Every GLR-space is a GL-space.

**Proof** Just substitute into the definition of GLR-space \( X_k = \mathbb{R}, k = 1, \ldots, n \). \( \square \)

To tell the truth, when we started to think about a possible description of GLR-spaces, we expected that the converse to Proposition 3.2 should be true, that is every
GL-space $E = (\mathbb{R}^n, \| \cdot \|_E)$ with absolute norm should be GLR. Surprisingly, this is not the case. Before we formulate another necessary condition for being a GLR-space, let us make some preliminary observations. From Proposition 3.2 and Theorem 2.8 it follows that the unit ball of every GLR-space $E$ is a polyhedron. Then the unit ball of $E^*$ is a polyhedron as well. Denote $\text{ext}(B_{E^*})$ the set of extreme points of $B_{E^*}$. The set $\text{ext}(B_{E^*})$ is finite, the number of elements in $\text{ext}(B_{E^*})$ is the same as the number of faces of $B_E$ and each $d \in \text{ext}(B_{E^*})$ corresponds to a face $F(d)$ of $B_E$. Since the norm $\| \cdot \|_E$ is absolute, $\text{ext}(B_{E^*})$ is mirror-symmetric with respect to each coordinate hyperplane, that is $d \in \text{ext}(B_{E^*})$ if and only if $|d| \in \text{ext}(B_{E^*})$.

**Definition 3.3** Let $E = (\mathbb{R}^n, \| \cdot \|_E)$ be a space with absolute norm, $d^* = (d_1, \ldots, d_n) \in \text{ext}(B_{E^*})$ with $d_k \geq 0$. The face $F(d^*) \subset S_E$ is said to be monotone plump if, denoting $D = \{k : d_k \neq 0\}$, for every $a = (a_1, \ldots, a_n) \in S_E$ with $a_k \geq 0$ and every $z = (z_1, \ldots, z_n) \in B_E$ with

$$0 \leq z_k \leq a_k \text{ for } k \in D$$

(3.2)

there is $b = (b_1, \ldots, b_n) \in F(d^*)$ such that $\|b - z\| = 1 - d^*(z)$ and $b_k \geq a_k$ for $k \in D$. The space $E$ is said to be GL-monotone (GLM-space for short) if for every $d^* \in \text{ext}(B_{E^*})$ with $d_k \geq 0$ the corresponding face $F(d^*) \subset S_E$ is monotone plump.

**Proposition 3.4** In the above notation let $d^* = (d_1, \ldots, d_n) \in \text{ext}(B_{E^*})$, $d_k \geq 0$ generate a monotone plump face. Then

1. all the coordinates $d_k$ belong to $\{0, 1\}$;
2. the property formulated in Definition 3.3 remains valid for every $z = (z_1, \ldots, z_n) \in B_E$ that satisfies the following weaker version of (3.2): $|z_k| \leq a_k$ for $k \in D$.

**Proof** Since $F(d^*)$ is an $(n-1)$-dimensional affine subset of the sphere, it cannot be contained in the kernel of a coordinate functional. So the set

$$\{x = (x_1, \ldots, x_n) \in F(d^*) : \exists k \in [1, \ldots, n] \ x_k = 0\}$$

is nowhere dense in $F(d^*)$. Take an $x = (x_1, \ldots, x_n) \in F(d^*)$ with all non-zero coordinates and consider $a = (a_1, \ldots, a_n) \in S_E$ with $a_k = |x_k| > 0$, $k = 1, \ldots, n$. Then

$$1 = d^*(x) \leq d^*(a) \leq \|a\| = \|x\| = 1,$$

which means that $a \in F(d^*)$. Fix a $j \in D$. Our goal is to demonstrate that $d_j = 1$. Apply Definition 3.3 to such a $z = (z_1, \ldots, z_n) \in B_E$ that $z_j = 0$, and $z_k = a_k$ for $k \neq j$. We get a $b = (b_1, \ldots, b_n) \in S_E$ with $d^*(b) = 1$, $\|b - z\| = 1 - d^*(z)$ and $b_k \geq a_k$ for $k \in D$. Remark that

$$1 = d^*(a) = \sum_{k \in D} d_k a_k \leq \sum_{k \in D} d_k b_k = d^*(b) = 1,$$
which implies that \( b_k = a_k \) for \( k \in D \), and in particular the \( j \)-th coordinate of \( b - z \) is equal to \( a_j \). Finally,

\[
a_j \geq d_j a_j = \sum_{k \in D} d_k (a_k - z_k) = \sum_{k \in D} d_k (b_k - z_k) = 1 - d^*(z) = \|b - z\| \geq a_j.
\]

Consequently, \( d_j = 1 \). This completes the proof of the statement (1).

For the statement (2), remark first that for every \( u = (u_1, \ldots, u_n) \) such that \( u_k \geq 0 \) for \( k \in D \) and \( u_k = 0 \) for \( k \notin D \) the following equality holds:

\[
d^*(u) = \|u\|.
\]

Indeed, thanks to (1) we have

\[
\|u\| \geq d^*(u) = \sum_{k \in D} u_k = \sum_{k \in D} u_k \|e_k\| \geq \|u\|.
\]

Now consider \( a = (a_1, \ldots, a_n) \in S_E \) with \( a_k \geq 0 \), a \( z = (z_1, \ldots, z_n) \in B_E \) with \( |z_k| \leq a_k \), \( k \in D \). Define \( w = (w_1, \ldots, w_n) \in S_E \) with \( w_k = |z_k| \) for \( k \in D \) and \( w_k = z_k \) for remaining values of \( k \). Find a \( b = (b_1, \ldots, b_n) \in S_E \) that serves in Definition 3.3 for \( w \), that is such that \( d^*(b) = 1 \), \( \|b - w\| = 1 - d^*(w) \) and \( b_k \geq a_k \) for \( k \in D \). Let us demonstrate that the same \( b \) serves for \( z \): \( \|b - z\| \leq \|b - w\| + \|w - z\| = 1 - d^*(w) + d^*(w - z) = 1 - d^*(z) = d^*(b - z) \leq \|b - z\| \).

**Proposition 3.5** Let \( E = (\mathbb{R}^n, \| \cdot \|_E) \) be GL-respecting. Then \( E \) is GL-monotone.

**Proof** Let us consider \( d^* = (d_1, \ldots, d_n) \in \text{ext}(B_{E^*}) \) with \( d_k \geq 0 \) and demonstrate that \( \mathcal{F}(d^*) \subset S_E \) is monotone plump. As before, we denote \( D = \{ k : d_k \neq 0 \} \). Substitute into the definition of GLR-space \( n \) isometric copies of the hexagonal space \( \tilde{E} \). Denote these copies \( X_k, k = 1, \ldots, n \). Fix \( t_k a_k = z_k \) and for each \( k = 1, \ldots, n \) take a functional \( x_k^* \in \text{ext}(B_{X_k^*}) \) being a copy of the second coordinate functional on \( \tilde{E} \), and take an element \( y_k \in S_{X_k^*} \) belonging to the copy of the edge connecting \( \hat{e}_2 \) and \( \hat{e}_1 \) in \( \tilde{E} \) such that \( x_k^*(y_k) = t_k \) (like \( \hat{y} \) from Proposition 2.12). The following Property A that will be used below is a direct consequence of Proposition 2.12: if for a given \( \alpha > 0 \) there is \( x_k \in \mathcal{F}(x_k^*) \) such that \( \|x_k - \alpha y_k\| = 1 - \alpha t_k \), then \( \alpha \leq 1 \).

By definition of GLR-space, the space \( X = E(X_k)^n \) is generalized lush. Consider \( x^* = (d_1 x_1^*, \ldots, d_n x_n^*) \in \text{ext}(B_{X^*}) \). Since \( X \) is finite-dimensional, the corresponding face \( \mathcal{F}(x^*) \) of \( B_X \) must be plump. Applying the reformulation of plumpness given in (4) of Lemma 2.6 to \( y = (a_1 y_1, \ldots, a_n y_n) \in S_X \) with arbitrary \( a = (a_1, \ldots, a_n) \in S_E \), \( a_k \geq 0 \), we find an \( x = (b_1 x_1, \ldots, b_n x_n) \in \mathcal{F}(x^*) \) with \( x_k \in S_{X_k^*}, b_k \geq 0, b = (b_1, \ldots, b_n) \in S_E \) such that \( \|x - y\| = 1 - x^*(y) \). The condition \( x \in \mathcal{F}(x^*) \) implies that

\[
1 \geq \sum_{k=1}^n d_k b_k \geq \sum_{k=1}^n d_k b_k x_k^*(x_k) = x^*(x) = 1.
\]
This means that $\sum_{k=1}^{n} d_k b_k = 1$ and for every $k \in D$ we have $b_k x_k^*(x_k) = b_k$. Using the properties listed above we obtain the following chain of inequalities
\[
1 - x^*(y) = \sum_{k=1}^{n} d_k (x_k^*(b_k x_k) - x_k^*(a_k y_k)) \\
\leq \| (|x_1^*(b_1 x_1) - x_1^*(a_1 y_1)|, \ldots, |x_n^*(b_n x_n) - x_n^*(a_n y_n)|) \|_E = \| x - y \| = 1 - x^*(y).
\]

From this and the fact that the first equality holds in every coordinate we deduce that for every $k \in D$ we have $\|b_k x_k - a_k y_k\| = x_k^*(b_k x_k - a_k y_k)$. The Property A of hexagonal norm gives us $a_k \leq b_k$ for every $k \in D$. On the other hand,
\[
1 - x^*(y) = \sum_{k=1}^{n} d_k (x_k^*(b_k x_k) - x_k^*(a_k y_k)) \\
\leq \| (|x_1^*(b_1 x_1) - x_1^*(a_1 y_1)|, \ldots, |x_n^*(b_n x_n) - x_n^*(a_n y_n)|) \|_E \\
\leq \| (|b_1 x_1 - a_1 y_1|, \ldots, |b_n x_n - a_n y_n|) \|_E = \| x - y \| = 1 - x^*(y),
\]

So, $\sum_{k=1}^{n} d_k (x_k^*(b_k x_k) - x_k^*(a_k y_k))$ is equal to
\[
\| (|x_1^*(b_1 x_1) - x_1^*(a_1 y_1)|, \ldots, |x_n^*(b_n x_n) - x_n^*(a_n y_n)|) \|_E.
\]

Substituting $x_k^*(x_k) = 1$, $x_k^*(y_k) = t_k$ we obtain
\[
\sum_{k=1}^{n} d_k (b_k - t_k a_k) = \| (|b_1 - t_k a_1|, \ldots, |b_n - t_k a_n|) \|_E,
\]
that is $d^*(b - z) = \| b - z \|$. \hfill \qed

**Corollary 3.6** The only two-dimensional GLR-spaces are $\ell_1^2$ and $\ell_\infty^2$.

**Proof** Proposition 3.2 and Theorem 2.11 imply that in dimension 2 there are no other candidates for being GLR-space except for those spaces whose unit ball is either parallelogram or hexagon. Let $X = (\mathbb{R}^2, \| \cdot \|)$ be a GLR-space. Since by definition the norm on $X$ is absolute, for every extreme point $(x, y)$ of $B_X^*$ the point $(|x|, |y|)$ is also extreme. Propositions 3.4 and 3.5 imply that the only possible $(x, y) \in \text{ext}(B_X^*)$ with non-negative coordinates are $(0, 1), (1, 1)$ and $(1, 0)$. Let us consider two cases.

**Case 1** $(1, 1) \in \text{ext}(B_X^*)$. Then by symmetry all four points $(\pm 1, \pm 1) \in \text{ext}(B_X^*)$, i.e. $X^* = \ell_\infty^2$ and consequently $X = \ell_1^2$.

**Case 2** $(1, 1) \notin \text{ext}(B_X^*)$. Then $(1, 0), (-1, 0), (0, 1), (0, -1) \in \text{ext}(B_X^*)$ are the only extreme points, i.e. $X^* = \ell_1^2$ and consequently $X = \ell_\infty^2$. \hfill \qed

Now we are starting preparations for the inverse to Proposition 3.5 statement.

**Proposition 3.7** Let $E = (\mathbb{R}^n, \| \cdot \|_E)$ be a GL-monotone space. Then for every collection $X_1, \ldots, X_n$ such that each $X_k$ is ultra-GL with respect to a subspace $W_k \subset X_k^*$...
the corresponding E-sum $X = E(X_k)^n$ is ultra-GL with respect to the subspace $W = E^*(W_k)^n \subset X^*$. In particular, if $X_1, \ldots, X_n$ are ultra-GL then $X$ is ultra-GL as well.

**Proof** According to Definition 2.3, for a given $x = (x_1, \ldots, x_n) \in S_X$ we must find an $x^* = (x_1^*, \ldots, x_n^*) \in S_W$ such that $x \in F(x^*)$ and $F(x^*)$ is plump. For each $k = 1, \ldots, n$, using that $X_k$ is ultra-GL with respect to $W_k$, we may take $w_k^* \in S_{W_k}$ such that $F(w_k^*) \subset S_{X_k}$ is plump in $X_k$ and $x_k \in \|x_k\| F(w_k^*)$, that is

$$w_k^*(x_k) = \|x_k\|.$$  \hfill (3.3)

Also, monotone GL-ness of $E$ applied to $N(x) = (\|x_1\|, \|x_2\|, \ldots, \|x_n\|)$ gives us a $d^* = (d_1, \ldots, d_n) \in \text{ext}(B_{E^*})$, $d_k \geq 0$, such that $N(x) \in F(d^*) \subset S_E$,

$$\sum_{k=1}^n d_k \|x_k\| = 1 \quad \hfill (3.4)$$

and $F(d^*)$ is monotone plump in $E$. Let us demonstrate that $x_k^* := d_k w_k^*, k = 1, \ldots, n$, generate the functional $x^* = (x_1^*, \ldots, x_n^*) \in S_W$ we need. Indeed, the conditions (3.3) and (3.4) imply that $x \in F(x^*)$, so it remains to show that $F(x^*)$ is plump. As above, denote $D = \{k : d_k \neq 0\}$. Consider an arbitrary $y = (a_1 y_1, \ldots, a_n y_n) \in S_X, y_k \in S_{X_k}, a_k \geq 0$. For $z = (z_1, \ldots, z_n) = (a_1 w_1^*(y_1), \ldots, a_n w_n^*(y_n)) \in B_E$ there is an element $b = (b_1, \ldots, b_n) \in F(d^*)$ such that $\|b - z\|_E = 1 - d^*(z)$ and $b_k \geq a_k$ for all $k \in D$.

According to (3) of Lemma 2.6, for every $k \in D$ such that $b_k \neq 0$ there is a $w_k \in F(w_k^*)$ such that $\|w_k - \frac{a_k y_k}{b_k}\| = |1 - a_k w_k^*(\frac{y_k}{b_k})|$. If $k \in D$ and $b_k = a_k = 0$ take $w_k \in F(w_k^*)$ arbitrarily. With such an election, for every $k \in D$ we have

$$\|b_k w_k - a_k y_k\| = |b_k - a_k w_k^*(y_k)|.$$  \hfill (3.5)

For $k \notin D$ the election of $w_k^*$ does not affect the value of $x_k^* = d_k w_k^* = 0$. So, for $k \notin D$ we can take as $w_k^*$ a supporting functional at $y_k$ and take $w_k = y_k$. Then $\|b_k w_k - a_k y_k\| = |b_k - a_k| = |b_k - a_k w_k^*(y_k)|$, so (3.5) remains valid for all $k$. Put $\tilde{x} = (b_1 w_1, \ldots, b_n w_n)$. Then $\tilde{x} \in F(x^*)$ and

$$1 - x^*(y) = \sum_{k=1}^n d_k \|b_k - a_k w_k^*(y_k)\| = 1 - d^*(z) = \|b - z\|_E$$

$$= \|(b_1 - a_1 w_1^*(y_1), \ldots, b_n - a_n w_n^*(y_n))\|_E$$

$$= \|(b_1 w_1 - a_1 y_1, \ldots, b_n w_n - a_n y_n)\|_E = \|\tilde{x} - y\|.$$

In order to complete our paper it remains to apply the standard ultraproduct technique. We refer for instance to [7, Ch. 16] for properties of filters and ultrafilters and to classical paper [6] for introduction to ultrapowers of Banach spaces. Let us recall the basic definitions.
Definition 3.8 A family $F$ of subsets of the set $\mathbb{N}$ is called a filter on $\mathbb{N}$ if it satisfies the following axioms:

1. $F$ is not empty;
2. $\emptyset \not\in F$;
3. if $A, B \in F$, then $A \cap B \in F$;
4. if $A \in F$ and $A \subset B \subset \mathbb{N}$, then $B \in F$.

Definition 3.9 Let $Y$ be a topological space, and $F$ be a filter on $\mathbb{N}$. The point $y$ in $Y$ is called the limit of the sequence of $y_n \in Y, n = 1, 2, \ldots$ with respect to the filter $F$ (denoted $y = \lim_{F} y_n$), if for any neighborhood $U$ of $y$ there exists an element $A \in F$ such that $y_n \in U$ for all $n \in A$.

Definition 3.10 An ultrafilter on $\mathbb{N}$ is a filter on $\mathbb{N}$ that is maximal with respect to inclusion. An ultrafilter is nontrivial, if all its elements are infinite.

Observe that the existence of nontrivial ultrafilters is guaranteed by Zorn’s lemma and the limit with respect to the ultrafilter exists for any bounded sequence of reals.

Let $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$, $X$ be a Banach space. The ultrapower $X^{\mathcal{U}}$ is the quotient space of $\ell_\infty(X)$ over the subspace of those $x = (x_n) \in \ell_\infty(X)$ for which $\lim_{\mathcal{U}} \|x_n\| = 0$. The ultrapower $X^{\mathcal{U}}$ we will view as the space of bounded sequences $x = (x_n)$ with $x_n \in X$, equipped with the norm $\|x\| = \lim_{\mathcal{U}} \|x_n\|$ under the convention that $x = (x_n)$ and $y = (y_n)$ are considered to be the same element of the ultrapower if $\lim_{\mathcal{U}} \|x_n - y_n\| = 0$. The ultrapower $(X^{\mathcal{U}})^*$ can be identified with a subspace of $(X^{\mathcal{U}})^*$ as follows: every $x^* = (x_n^*)$, $x_n^* \in X^*$, $\sup_n \|x_n^*\| < \infty$ is a linear functional on $X^{\mathcal{U}}$ that acts on every $x = (x_n)$ by the rule $x^*(x) = \lim_{\mathcal{U}} x_n^*(x_n)$. For a functional $x^* = (x_n^*)$ its norm is the same as its norm as an element of the ultrapower $(X^*)^{\mathcal{U}}$: $\|x^*\| = \lim_{\mathcal{U}} \|x_n^*\|$. The theorem below is modeled after a similar result about narrow operators [1, Lemma 2.6]. An analogous result for lush spaces was demonstrated in [3, Corollary 4.4]. The implication $(1) \Rightarrow (2)$ is almost contained in demonstration of [5, Proposition 2.2].

Theorem 3.11 Let $X$ be a Banach space, and $\mathcal{U}$ be a nontrivial ultrafilter on $\mathbb{N}$. Then the following assertions are equivalent:

1. $X$ is a GL-space,
2. $X^{\mathcal{U}}$ is ultra-GL with respect to the subspace $W = (X^*)^{\mathcal{U}}$.

Proof $(1) \Rightarrow (2)$. Our goal is to demonstrate that for every $x = (x_n) \in S_X^{\mathcal{U}}$, $x_n \in X$, there exists an $x^* = (x_n^*) \in S_W$ such that $x \in F(x^*)$ and $F(x^*)$ is plump. Remark that $\|x\| = \lim_{\mathcal{U}} \|x_n\| = 1$, so $\lim_{\mathcal{U}} \left\| \frac{x_n}{\|x_n\|} - x_n \right\| = 0$, so $(x_n) = \left( \frac{x_n}{\|x_n\|} \right)$ as elements of the ultrapower. Substituting if necessary $x_n$ by $\frac{x_n}{\|x_n\|}$ we may assume that $x_n \in S_X$, $n \in \mathbb{N}$. Since by $(1)$ $X$ is a GL-space, for each $n \in \mathbb{N}$ there is $x_n^* \in S_{x_n}$ such that $x_n^* \in S(x_n^*, \frac{1}{n}) = S_n$ and $(2.1)$ holds true for every $y \in S_X$. Denote $x^* = (x_n^*)$. By our construction,

\[ \|x^*\| = \lim_{\mathcal{U}} \|x_n^*\| = 1 = \lim_{\mathcal{U}} x_n^*(x_n) = x^*(x) . \]
This means that $x^* \in S_W$ and $x \in F(x^*)$. It remains to show that the corresponding face $F(x^*)$ is plump. Let $y = (y_n) \in S_{X^*}$, $y \in S_X$. For every $n \in \mathbb{N}$, using (2.1), pick $u^1_n, u^2_n \in S_n$ in such a way that

$$||y_n - u^1_n|| + ||y_n + u^2_n|| \leq 2 + \frac{1}{n}.$$  

Denote $u_i = (u^j_n) \in S_{X^*}$, $i = 1, 2$. We have $u_i \in F(x^*)$ and $||y - u_1|| + ||y + u_2|| \leq 2$ which by definition 2.2 means that indeed $F(x^*)$ is plump.

(2) $\Rightarrow$ (1). This time let $x \in S_X$ be an arbitrary element. Assume to the contrary that there is an $\varepsilon > 0$ such that for every $x^* \in S_{X^*}$ with $x^*(x) > 1 - \varepsilon$ there is a $y \in S_X$ with

$$\rho(y, S(x^*, \varepsilon)) + \rho(-y, S(x^*, \varepsilon)) \geq 2 + \varepsilon.$$  

Take $\tilde{x} = (x, x, x, \ldots) \in S_{X^*}$ and let us show that for every $x^* = (x^*_n) \in S_W$ such that $\tilde{x} \in F(x^*)$ the $F(x^*)$ is not plump. Indeed, denote $A = \{n \in \mathbb{N} : x^*_n(x) > 1 - \varepsilon\}$. The condition $1 = x^*(\tilde{x}) = \lim_U x^*_n(x)$ implies that $A \in \mathcal{U}$. Using our assumption, for every $n \in A$ take a $y_n \in S_X$ with

$$\rho(y_n, S(x^*, \varepsilon)) + \rho(-y_n, S(x^*, \varepsilon)) \geq 2 + \varepsilon. \quad (3.6)$$  

Denote $\tilde{y} = (y_n) \in S_{X^*}$. If the face $F(x^*)$ were plump, there would be $u_i = (u^j_n) \in F(x^*)$, $i = 1, 2$, such that $||\tilde{y} - u_1|| + ||\tilde{y} + u_2|| \leq 2$. This contradicts (3.6).

Now, finally, the main result of the section.

**Theorem 3.12** A space $E = (\mathbb{R}^n, \| \cdot \|_E)$ with absolute norm is GL-respecting if and only if it is GL-monotone.

**Proof** The “only if” part is demonstrated in Proposition 3.5, so it remains to demonstrate the “if” part. Let $E$ be GL-monotone, and let $X_1, \ldots, X_n$ be a collection of GL-spaces. Due to the previous theorem, for a fixed nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$ all $X_k^{\mathcal{U}}$, $k = 1, \ldots, n$, are ultra-GL with respect to the corresponding subspaces $(X^*_k)^{\mathcal{U}}$. By Proposition 3.7 the $E$-sum $E((X^*_k)^{\mathcal{U}})_1^n$ is ultra-GL with respect to the subspace $E^* \left( (X^*_k)^{\mathcal{U}}_1^n \right)$. Using the natural isometry between $E((X^*_k)^{\mathcal{U}}_1^n)$ and $E((X^*_k)^{\mathcal{U}}_1^n)$ we deduce that $(E((X^*_k)^{\mathcal{U}}_1^n))^{\mathcal{U}}$ is ultra-GL with respect to the subspace $E^* ((X^*_k)^{\mathcal{U}}_1^n)$. Another application of the previous theorem gives us the desired generalized lushness of $(E(X_k)_1^n)_{\mathcal{U}}$.

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