Abstract. We consider the XXX Bethe equation associated with integral dominant weights of a Kac-Moody algebra and introduce a generating procedure constructing new solutions starting from a given one. The family of all solutions constructed from a given one is called a population. We describe properties of populations.

1. Introduction

The XXX Bethe equation is a system of algebraic equations associated to a Kac-Moody algebra $\mathfrak{g}$, a non-zero step $h \in \mathbb{C}$, distinct complex numbers $z_1, \ldots, z_n$, integral dominant $\mathfrak{g}$-weights $\Lambda_1, \ldots, \Lambda_n$ and a $\mathfrak{g}$-weight $\Lambda_\infty$, see [OW] and Section 3 below.

In the theory of the Bethe ansatz one associates a Bethe equation to an integrable model. Then each solution of the Bethe equation gives an eigenvector of commuting Hamiltonians of the model. The general conjecture is that the constructed vectors form a basis in the space of states of the model. There is vast literature on this subject, see e.g. [BIK, F, FT].

The first step to the above mentioned conjecture is to count the number of solutions of the Bethe equation. One can expect that the number of solutions is equal to the dimension of the space of states of the model.

In an attempt to approach that counting problem, in this note we relate to each solution of the Bethe equation an object called the population. We hope that it might be easier to count populations than individual solutions. For instance, if the Kac-Moody algebra is $\mathfrak{sl}_{r+1}$, then each population corresponds to a point of intersection of suitable Schubert varieties in a suitable Grassmannian variety. Then the Schubert calculus allows us to count the populations, see [MV1], [MV2].

The populations in the case of $\mathfrak{sl}_{r+1}$ were introduced in [MV1]. The construction of a population goes as follows. First one defines $r$ polynomials $T_1(x), \ldots, T_r(x)$, $T_i(x) = \prod_{s=1}^n \prod_{p=1}^{(\Lambda_s, \alpha_i)} (x - z_s - (\Lambda_1, \alpha_i)h + 2ph)$, where $\alpha_1, \ldots, \alpha_r$ are simple roots and $(,)$ is the standard scalar product. A solution of the Bethe equation is an $r$-tuple of polynomials $y = (y_1(x), \ldots, y_r(x))$, each considered up to multiplication by a non-zero number. The fact that $y$ is a solution is formulated as follows. The tuple $y$ is a solution, if for every
i = 1, \ldots, r$, there exists a polynomial $\tilde{y}_i(x)$ satisfying the equation
\[
\tilde{y}_i(x + 2h)y_i(x) - \tilde{y}_i(x)y_i(x + 2h) = T_i(x)y_{i-1}(x + h)y_{i+1}(x + h)
\]
(we also need some requirements on $y$ to be generic). It turns out that for all complex numbers $c$ (with finitely many exceptions) the tuples $(y_1(x), \ldots, \tilde{y}_i(x) + cy_i(x), \ldots, y_r(x))$ are also solutions of the Bethe equation. The collection of all $r$-tuples which can be constructed starting from $y$ by iteration of the above procedure is called the population of solutions originated at $y$.

In this paper we suggest a similar generation procedure for an arbitrary solution of a Bethe equation of a non-homogeneous $XXX$ model associated with an arbitrary Kac-Moody algebra. The new feature of such a generation procedure is that the shift in the $i$-th difference equation depends on the length of the $i$-th simple root of the Kac-Moody algebra.

A population of solutions is an interesting object. It is an algebraic variety. It is a finite-dimensional irreducible algebraic variety, if the Weyl group of the Kac-Moody algebra is finite. In this short article we present only elementary properties of populations, see Section 5. The main open questions are:

- Describe a population as an algebraic variety,
- Compute the number of populations originated at solutions of a given Bethe equation.

The populations related to the Gaudin model of a Kac-Moody algebra $\mathfrak{g}$ were introduced in [MV2]. In [MV3], see also [Fr], it is proved that in the case of semi-simple Lie algebras every $\mathfrak{g}$-population is isomorphic to the flag variety of the Langlands dual Lie algebra $\mathfrak{g}^{\ell}$. It is conjectured in [MV1] that the same is true for all Kac-Moody algebras. It is plausible that every $\mathfrak{g}$-population introduced in this paper is isomorphic to the flag variety of the Langlands dual Kac-Moody algebra $\mathfrak{g}^{\ell}$. In [MV1], [MV4] this conjecture is proved for Lie algebras of types A,D,E.

The reproduction procedure for the Bethe ansatz type equations was studied in some previous works, see [MV1], [MV4]. Note that the equations of the B, C type in [MV1] differ from the B, C type Bethe equations of the present paper, and the equations associated to a Kac-Moody algebra in [MV4] differ from the Bethe equations of the present paper if and only if there is an non-zero non-diagonal entry of the Cartan matrix different from $-1$.

We remark that all the constructions and statements of the present paper can be repeated for the $XXY$ Bethe equation.

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2. Kac-Moody algebras

Let $A = (a_{ij})_{i,j=1}^r$ be a generalized Cartan matrix, $a_{ii} = 2$, $a_{ij} = 0$ if and only $a_{ji} = 0$, $a_{ij} \in \mathbb{Z}_{\leq 0}$ if $i \neq j$. We assume that $A$ is symmetrizable, there is a diagonal matrix
Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra (see $[K]$, §1.2), $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The associated scalar product is non-degenerate on $\mathfrak{h}^*$ and $\dim \mathfrak{h} = r + 2d$, where $d$ is the dimension of the kernel of the Cartan matrix $A$.

Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i = 1, \ldots, r$, be the sets of simple roots, coroots, respectively. We have

\[ d_i = (\alpha_i, \alpha_i)/2, \quad a_{ij} = 2(\alpha_i, \alpha_j)/\langle \alpha_i, \alpha_i \rangle = \langle \alpha_j, \alpha_i^\vee \rangle, \quad \langle \lambda, \alpha_i^\vee \rangle = 2(\lambda, \alpha_i)/\langle \alpha_i, \alpha_i \rangle, \]

where $\lambda \in \mathfrak{h}^*$. Let $\mathcal{P} = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \}$. Let $\mathcal{P}^+ = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \}$. Elements of $\mathcal{P}^+$ are called dominant integral weights.

The Weyl group $W \in \text{End}(\mathfrak{h}^*)$ is generated by reflections $s_i$, $i = 1, \ldots, r$,

\[ s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i, \quad \lambda \in \mathfrak{h}^*. \]

Fix $\rho \in \mathfrak{h}^*$ such that $\langle \rho, \alpha_i^\vee \rangle = 1$, $i = 1, \ldots, r$. We have $(\rho, \alpha_i) = (\alpha_i, \alpha_i)/2$. We use the notation $w \cdot \lambda$ for the shifted action of the Weyl group given by

\[ w \cdot \lambda = w(\lambda + \rho) - \rho, \quad w \in W, \lambda \in \mathfrak{h}^*. \]

3. The Bethe equations

Let $\Lambda = (\Lambda_i)_{i=1}^n \in (\mathcal{P}^+)^n$, $z = (z_i)_{i=1}^n \in \mathbb{C}^n$, $l = (l_i)_{i=1}^r \in \mathbb{Z}_{\geq 0}^r$, $t = (t_{ij})_{i=1,\ldots,r}$. Set $\tilde{\Lambda} = (\Lambda_1, \ldots, \Lambda_n, \Lambda_\infty)$, where

\[ \Lambda_\infty = \sum_{i=1}^n \Lambda_i - \sum_{i=1}^r l_i \alpha_i \in \mathcal{P}. \]

Fix a non-zero complex number $h$. The XXX Bethe equation associated with $z, \tilde{\Lambda}$ is the following system of algebraic equations for variables $t$, see $[OW]$:

\[ \prod_{s=1}^n t_{ij}^{(s)} - z_s + (\Lambda_s, \alpha_i)h \prod_{m=1}^r \prod_{k=1}^l t_{ij}^{(m)} - t_{ij}^{(m)} - (\alpha_m, \alpha_i)h = -1, \]

where $i = 1, \ldots, r$, $j = 1, \ldots, l_i$.

Introduce

\[ h_i = (\alpha_i, \alpha_i)h. \]

Then the Bethe equation $[3.2]$ can be written in the form

\[ \prod_{s=1}^n t_{ij}^{(s)} - z_s + (\Lambda_s, \alpha_i^\vee)h_i/2 \prod_{m=1}^r \prod_{k=1}^l t_{ij}^{(m)} - t_{ij}^{(m)} - a_{im}h_i/2 \prod_{k=1}^l t_{ij}^{(m)} - t_{ij}^{(m)} + h_i = 1. \]

The product of symmetric groups $S_l = S_{l_1} \times \cdots \times S_{l_N}$ acts on the set of solutions of $[3.2]$ permuting the coordinates with the same upper index. An $S_l$ orbit of solutions of $[3.2]$ such that

\[ D = \text{diag}\{d_1, \ldots, d_r\} \text{ with relatively prime positive integers } d_i \text{ such that } B = DA \text{ is symmetric.} \]
Bethe solution associated to $t$ where $p$ and $\bar{t}$ are equivalent:

**Theorem 4.1.** Let $t$ be a generic $r$-tuple of polynomials of a variable $x$, $y = (y_1, \ldots, y_r)$, $y_i(x) = \prod_{j=1}^{t(i)} (x - t_j(i))$. We say that $y$ represents $t$.

We are interested only in the zeros of $y_i$, and we consider those polynomials up to multiplication by a non-zero complex number, so the $r$-tuple $y$ is an element of $(\mathbb{P} \mathbb{C}[x])^r$, where $\mathbb{P} \mathbb{C}[x]$ is the projectivization of the space of polynomials in $x$.

For $i = 1, \ldots, r$, introduce polynomials $T_i(x)$:

$$T_i(x) = \prod_{s=1}^{n} \prod_{p=1}^{\frac{r}{2}} (x - z_s - \langle \Lambda_s, \alpha_i^\vee \rangle h_i/2 + ph_i).$$

An $r$-tuple of polynomials $y$ is called generic if $y_i(x)$ has no multiple roots and no common roots with polynomials $y_i(x + h_i), T_i(x), y_m(x + a_{im} h_i/2 + ph_i)$ for $m = 1, \ldots, r$ and $p = 1, \ldots, -a_{im}$. If $y$ represents a Bethe solution, then $y$ is generic.

**Theorem 4.1.** Let $y$ be a generic $r$-tuple of polynomials. The following three properties are equivalent:

(i) $y$ represents a Bethe solution associated to $z, \Lambda$,

(ii) for $i = 1, \ldots, r$, the polynomial

$$
\left( \prod_{s=1}^{n} (x - z_s - \langle \Lambda_s, \alpha_i^\vee \rangle h_i/2) \prod_{m, a_{im} < 0} y_m(x + a_{im} h_i/2) \right) y_i(x + h_i) + \\
\left( \prod_{s=1}^{n} (x - z_s + \langle \Lambda_s, \alpha_i^\vee \rangle h_i/2) \prod_{m, a_{im} < 0} y_m(x - a_{im} h_i/2) \right) y_i(x - h_i)
$$

is divisible by $y_i(x)$,

(iii) there exist polynomials $\tilde{y}_1, \ldots, \tilde{y}_r$ such that for $i = 1, \ldots, r$,

$$y_i(x + h_i)\tilde{y}_i(x) - y_i(x)\tilde{y}_i(x + h_i) = T_i(x) \prod_{m=1}^{r} \prod_{p=1}^{\frac{r}{2}} y_m(x + a_{im} h_i/2 + ph_i/2).$$
Note that the product over \( p \) in the part (iii) equals to 1 in the case of \( a_{im} = 0 \) and in the case of \( i = m \).

**Proof.** Since \( y \) has no multiple roots, condition (ii) is equivalent to vanishing of the polynomial (1.1) at the roots of \( y \). This condition is exactly equation (3.2). Thus (i) and (ii) are equivalent, cf. Lemma 2.2 in [MV1].

Consider the following equation for a function \( c_i(x) \):

\[
c_i(x + h_i) - c_i(x) = \frac{T_i(x) \prod_{m=1}^{r} \prod_{p=1}^{a_{im}} y_m(x + a_{im} h_i / 2 + p h_i / 2)}{y_i(x) y_i(x + h)}.
\]

Condition (iii) is equivalent to the existence of the solution of the form \( c(x) = \tilde{y}_i(x) / y(x) \), where \( \tilde{y}_i(x) \) is a polynomial. This is equivalent to the condition on residues at zeroes of \( y \):

\[
\left( \text{Res}_{x=t_i^{(i)}} + \text{Res}_{x=t_i^{(i)} - h_i} \right) \left( \frac{T_i(x) \prod_{m=1}^{r} \prod_{p=1}^{a_{im}} y_m(x + a_{im} h_i / 2 + p h_i / 2)}{y_i(x) y_i(x + h)} \right) = 0,
\]

which coincides with equation (3.2). Thus (i) and (iii) are equivalent, cf. Lemmas 2.3 and 2.4 in [MV1].

An \( r \)-tuple \( y \) satisfying condition (iii) of the theorem is called *fertile with respect to \( z, \Lambda \)*. An \( r \)-tuple \( y \) represents a Bethe solution if and only if it is fertile and generic. If \( y \) is fertile, then the \( r \)-tuples of the form \( y^{(i)} = (y_1, \ldots, \tilde{y}_i, \ldots, y_r) \) are called the *immediate descendants of \( y \) in the \( i \)-th direction*.

Let \( \mathbb{C}[x] \) be the space of all polynomials of degree at most \( d \). The set of fertile tuples is closed in \( (\mathbb{P} \mathbb{C}[x])^r \):

**Lemma 4.2.** Assume that a sequence of fertile tuples of polynomials \( y_k \), \( k = 1, 2, \ldots \), has a limit \( y_\infty \) in \( (\mathbb{P} \mathbb{C}[x])^r \) as \( k \) tends to infinity. Then the limiting tuple \( y_\infty \) is fertile.

Assume in addition, that for some \( i \), all immediate descendents of all \( y_k \) in the \( i \)-th direction are in \( (\mathbb{P} \mathbb{C}[x])^r \). If \( y^{(i)}_\infty \) is an immediate descendant of \( y_\infty \) in the \( i \)-th direction, then \( y^{(i)}_\infty \in (\mathbb{P} \mathbb{C}[x])^r \) and there exist immediate descendents \( y_k^{(i)} \) of \( y_k \) in the \( i \)-th direction such that \( y^{(i)}_\infty \) is the limit of \( y_k^{(i)} \).

**Proof.** (Cf. Lemma 3.4 in [MV1].) For each \( k \) we have a plane of polynomials \( \mathcal{P}^{(i)}_k \) spanned by \( (y_k)_i \) and \( (y_k^{(i)})_i \). Since the Grassmannian variety of planes in \( \mathbb{C}[x] \) is compact, there is a limiting plane \( \mathcal{P}^{(i)}_\infty \). This plane \( \mathcal{P}^{(i)}_\infty \) obviously contains \( (y_\infty)_i \) and the lemma follows.

Now we are ready to prove the main result of this section.

**Theorem 4.3.** Let \( y \) represent a Bethe solution associated to \( z, \Lambda \), let \( i \in \{1, \ldots, r\} \) and let an \( r \)-tuple \( y^{(i)} = (y_1, \ldots, \tilde{y}_i, \ldots, y_r) \) be an immediate descendant in the direction \( i \). Assume that \( y^{(i)} \) is generic. Then \( y^{(i)} \) represents a Bethe solution associated either to \( z, \Lambda \) if \( \deg \tilde{y}_i = \deg y_i \) or to \( z, (\Lambda_1, \ldots, \Lambda_n, s_i \cdot \Lambda_\infty) \) if \( \deg \tilde{y}_i \neq \deg y_i \).
Proof. Denote \( \tilde{t}_j^{(i)} \) the zeroes of \( \tilde{y}_i \). We have to check equation (3.2) for \( y^{(i)} \).

Equation (3.2) with respect to \( t_j^{(k)} \) where \( a_{ik} = 0 \) is the same for \( y \) and \( y^{(i)} \) and therefore it is satisfied.

Equation (3.2) with respect to \( \tilde{t}_j^{(i)} \) is equivalent to the existence of a polynomial \( \tilde{y}_i \) satisfying the \( i \)-th Wronskian identity as in (1.2), see Theorem 4.1. We can simply take \( \tilde{y}_i = -y_i \).

Let now \( k \) be such that \( a_{ik} < 0 \). To obtain the Bethe equation (3.2) with respect to \( t_j^{(k)} \) for \( y^{(i)} \) from the Bethe equation for \( y \) with respect to the same variable \( t_j^{(k)} \) it is enough to show that

\[
\prod_{s=1}^t \frac{t_j^{(k)} - t_s^{(i)} - a_{ki}h_k/2}{t_j^{(k)} - t_s^{(i)} + a_{ki}h_k/2} = \prod_{s=1}^t \frac{\tilde{t}_j^{(k)} - \tilde{t}_s^{(i)} - a_{ki}h_k/2}{\tilde{t}_j^{(k)} - \tilde{t}_s^{(i)} + a_{ki}h_k/2}.
\]

This condition can be rewritten as:

\[
\frac{y_i(t_j^{(k)} - a_{ki}h_k/2)}{y_i(t_j^{(k)} + a_{ki}h_k/2)} = \frac{\tilde{y}_i(t_j^{(k)} - a_{ki}h_k/2)}{\tilde{y}_i(t_j^{(k)} + a_{ki}h_k/2)}. \tag{4.3}
\]

By substituting \( x = t_j^{(k)} - a_{ki}h_k/2 - ph_i/2 \) where \( p = 1, \ldots, -a_{ik} \) in the \( i \)-th Wronskian identity (1.2), we observe that the right hand side vanishes and therefore we obtain

\[
\frac{y_i(t_j^{(k)} - a_{ki}h_k/2 - (p-1)h_i/2)}{y_i(t_j^{(k)} - a_{ki}h_k/2 - ph_i/2)} = \frac{\tilde{y}_i(t_j^{(k)} - a_{ki}h_k/2 - (p-1)h_i/2)}{\tilde{y}_i(t_j^{(k)} - a_{ki}h_k/2 - ph_i/2)}. \tag{4.4}
\]

The identity (4.3) is obtained by the product of identities (4.4) for \( p = 1, \ldots, -a_{ik} \), since \( a_{ik}h_i = a_{ki}h_k = 2(\alpha_i, \alpha_k)h \).

Finally, if \( \deg \tilde{y}_i \neq \deg y_i \) then using the \( i \)-th Wronskian identity (1.2) once again we obtain a relation

\[
\deg y_i + \deg \tilde{y}_i = 1 + \deg T_i - \sum_{m,a_{im} < 0} a_{im} \deg y_m,
\]

which implies the relation between the weights at infinity stated in the theorem. \( \square \)

Note that if the polynomial \( \tilde{y}_i \) satisfies (1.2) then for any complex number \( c \), the polynomial \( \tilde{y}_i + cy_i \) also satisfies (1.2). For all but finitely many values of \( c \) the \( r \)-tuple \( (y_1, \ldots, \tilde{y}_i + cy_i, \ldots, y_r) \) is generic. Thus starting from a Bethe solution we constructed a family of \( r \)-tuples of polynomials, all but finitely many of which represent Bethe solutions. Then by Lemma 1.2 all these \( r \)-tuples are fertile.

We call this construction the simple reproduction in the \( i \)-th direction.

5. Populations of Bethe solutions

Let \( y_0 \) represent a Bethe solution associated to \( z, \Lambda \).

We apply the simple reproduction procedure in each direction \( i \) and obtain a family of fertile \( r \)-tuples \( y_0^{(i)} \). Then we apply the simple reproduction procedure in each direction
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...j to these tuples again and obtain a larger family of r-tuples \((y_0^{(i)})^{(j)}\). We continue this procedure which we call the reproduction procedure. The set of all r-tuples obtained by the reproduction procedure from \(y_0\) is called the populations of Bethe solutions originated at \(y_0\) and is denoted \(P(y_0)\).

More formally, the population of \(y_0\) is the set of all r-tuples \(y \in (\mathbb{P} \mathbb{C}[x])^r\) such that there exist r-tuples \(y_1, y_2, \ldots, y_k = y\), where \(y_i\) is an immediate descendant of \(y_{i-1}\) for all \(i = 1, \ldots, k\).

The following lemma is straightforward.

Lemma 5.1. All r-tuples in the population \(P(y_0)\) are fertile with respect to \(z, \Lambda\). □

For an r-tuple \(y\), define the corresponding weight at infinity by formula (3.1) where \(l_i = \deg y_i\). From Theorem 4.3, we obtain

Lemma 5.2. The set of weights at infinity of the r-tuples in the population \(P(y_0)\) coincides with the orbit of \(\Lambda_\infty\) under the shifted action of the Weyl group of \(g\). □

In particular, if \(y \in P(y_0)\) is generic then it represents a Bethe solution associated to \(z, (\Lambda_1, \ldots, \Lambda_n, w \cdot \Lambda_\infty)\) for some element \(w\) of the Weyl group \(W\).

If \(y\) represents a Bethe solution associated to \(z, \tilde{\Lambda}\), then we say that the population \(P(y)\) is associated to \(z, \Lambda\). Clearly,

Lemma 5.3. If two populations associated to \(z, \Lambda\) intersect then they coincide. □

Let \(y\) and \(\tilde{y}\) represent Bethe solutions. We say that \(y \equiv \tilde{y}\) if and only if \(y\) and \(\tilde{y}\) belong to the same population. Then by Lemma 5.3 \(\equiv\) is an equivalence relation.

The following lemma follows from the general standard arguments, cf. Corollaries 3.13 and 3.14 in [MV2].

Lemma 5.4. For any \(d \in \mathbb{Z}_{\geq 0}\), the intersection of any population of Bethe solutions with \((\mathbb{P} \mathbb{C}[x])^r\) is an algebraic variety. In particular, if the Weyl group of \(g\) is finite then the population of Bethe solutions is an irreducible algebraic variety. □

In conclusion we describe situations in which the number of populations can be computed by general arguments.

The r-tuple \((1, \ldots, 1) \in (\mathbb{P} \mathbb{C}[x])^r\) is the unique r-tuple of non-zero polynomials of degree 0. The weight at infinity of \((1, \ldots, 1)\) is \(\Lambda_\infty(1, \ldots, 1) = \sum_{s=1}^n \Lambda_s\). Let \(P_{(1, \ldots, 1)}\) be the population associated to the initial data and originated at \((1, \ldots, 1)\).

The next lemma says that if \(\Lambda_\infty(1, \ldots, 1)\) is in the orbit of \(\Lambda_\infty\) then there is exactly one population of Bethe solutions, cf. Corollary 4.4 in [MV1] and Corollary 3.17 in [MV2].

Lemma 5.5. Let \(y\) represent a Bethe solution such that \(\Lambda_\infty\) has the form \(w \cdot \Lambda_\infty(1, \ldots, 1)\) for some \(w \in W\). Then \(y\) belongs to the population \(P_{(1, \ldots, 1)}\).

Proof. This lemma follows directly from Lemmas 5.2 and 5.3. □

The last two lemmas give some sufficient conditions for the absence of Bethe solutions, cf. Corollary 4.3 in [MV1] and Corollaries 3.15, 3.16 in [MV2].
Lemma 5.6. Suppose there is an element $w$ of the Weyl group such that $\sum_{s=1}^n \Lambda_s - w \cdot \Lambda_\infty$ does not belong to the cone $\mathbb{Z}_{\geq 0} \alpha_1 \oplus \ldots \oplus \mathbb{Z}_{\geq 0} \alpha_r$. Then there are no Bethe solutions associated to $z, \hat{\Lambda}$.

Proof. The Lemma follows from Lemma 5.2 and the absence of polynomials of negative degrees. □

Lemma 5.7. Suppose that $s_i \cdot \Lambda_\infty = \Lambda_\infty$ for some simple reflection $s_i$. Then there are no Bethe solutions associated to $z, \hat{\Lambda}$.

Proof. If $y$ represents a Bethe solution, then it is fertile in all directions. Then there is a polynomial $\tilde{y}_i$ satisfying (4.2) which has a degree different from the degree of $y_i$. Therefore the corresponding weights at infinity for $y$ and $y^{(i)}$ are different. This contradicts the assumption of the lemma in view of Theorem 4.3. □

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