A Proof for the Collatz Conjecture

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Abstract
It is well known that the following Collatz Conjecture is one of the unsolved problems in mathematics.

Collatz Conjecture: For any positive integer \( n > 1 \), the following recursive algorithm will converge to 1 by a finite number of steps.
A) If \( n \) is an even number then \( \frac{n}{2} \rightarrow n \).
B) If \( n \) is an odd number then \( 3n + 1 \rightarrow n \).

This paper proposes a proof for the Collatz Conjecture by the elementary mathematical induction.

Keywords: proof, Collatz Conjecture, mathematical induction.

1 Introduction
For any positive integer \( n > 1 \), we consider a recursive algorithm by repeating the following two steps A) and B).
A) If \( n \) is an even number then \( \frac{n}{2} \rightarrow n \),
B) If \( n \) is an odd number then \( 3n + 1 \rightarrow n \).

For example, let \( n = 7 \), this algorithm generates the following sequence and terminates to 1.

\[
7 \rightarrow 22 \rightarrow 11 \rightarrow 34 \rightarrow 17 \rightarrow 52 \rightarrow 26 \rightarrow 13 \rightarrow 40 \\
\rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1,
\]

In 1937, German mathematician Lothar Collatz shown a famous conjecture: For any positive integer \( n > 1 \), the above recursive algorithm always terminates to 1 by a finite number of steps [1][2]. It is also known as the 3n + 1 problem, the Ulam conjecture, Kakutani’s problem, the Thwaites conjecture, Hasse’s algorithm, or the Syracuse problem [2]-[4]. It seems that the Collatz conjecture is still an unsolved problem up to now. Based on the supercomputer simulation, it has been confirmed that the Collatz conjecture is correct for all positive integer \( 2 \leq n \leq 2^{68} \approx 2.95 \times 10^{20} \) [6]. In December of 2019, Terence Tao presented an important research paper, he proved that almost all orbits of the Collatz map attain almost bounded values [5]. It is to say that the Collatz conjecture is almost correct for almost all positive integers.

In the previous research works, we have presented some approximate formulas for the number of average steps in the Collatz recursive algorithm by using the statistical method [12][13]. And also shown two expansions \( 5n + 1 \) problem and \( 7n + 1 \) problem for the Collatz conjecture [14]. Recently, we derive a
high precision formula of the average number of multiplications and divisions $T(n) \approx 3 \log_4 n - 12.87$ for the Collatz problem based on analysis of average computational complexity of the recursive algorithm [15]. This paper proposes a proof for the Collatz conjecture by using the elementary mathematical induction.

## Lemma

From the Collatz recursive algorithm, if $n$ is an even number then $\frac{n}{2} \rightarrow n$, if $n$ is an odd number then $3n + 1 \rightarrow n$, because $3n + 1$ is even number, so we can consider the following recursive algorithm:

- $f_1$) If $n$ is an even number then $\frac{n}{2} \rightarrow n$,
- $f_2$) If $n$ is an odd number then $\frac{3n + 1}{2} \rightarrow n$.

First, in order to discuss some properties of the above transform $f_1$ and $f_2$, we show the following Lemma.

**Lemma.** Let function

$$f_1(x) = \frac{x}{2},$$

$$f_2(x) = \frac{3x + 1}{2},$$

then for any real number $x > 0$, we have the following properties for $f_1(x)$ and $f_2(x)$.

1. For any real number $0 < x < y$,

$$f_1f_2(x) < f_1f_2(y),$$

$$f_2f_1(x) < f_2f_1(y).$$

2. For any positive integer $\alpha \geq 1$,

$$(f_1f_2)^\alpha(x) < (f_2f_1)^\alpha(x).$$

3. For any real number $x > 1$ and any positive integer $\alpha \geq 1$,

$$(f_1f_2)^\alpha(x) = \left(\frac{3}{4}\right)^\alpha x + 1 - \left(\frac{3}{4}\right)^\alpha < x.$$

4. For any real number $x > 2$ and any positive integer $\alpha \geq 1$.

$$(f_2f_1)^\alpha(x) = \left(\frac{3}{4}\right)^\alpha x + 2\left(1 - \left(\frac{3}{4}\right)^\alpha\right) < x.$$ 

5. For any real number $x > 1$ and any positive integer $\alpha, \beta \geq 1$.

$$f_2^\beta(f_1f_2)^\alpha(x) = \left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta x - 1 + 2\left(\frac{3}{2}\right)^\beta - 1.$$ 

**Proof of Lemma.**

1. For any real number $x > 0$,

$$f_1f_2(x) = f_1\left(\frac{3x + 1}{2}\right) = \frac{3x + 1}{4},$$
\[ f_2 f_1(x) = f_1 \left( \frac{f_2(x)}{2} \right) = \frac{3\left( \frac{\sqrt{2}}{4} \right) + 1}{2} = \frac{3x + 2}{4}. \]

It is obvious that \( f_1 f_2(x) \) and \( f_2 f_1(x) \) are the monotonically increasing functions. So, for any real number \( 0 < x < y \),

\[ f_1 f_2(x) < f_1 f_2(y), \]

\[ f_2 f_1(x) < f_2 f_1(y). \]

(2) For any real number \( x > 0 \), from

\[ \frac{3x + 1}{4} < \frac{3x + 2}{4}, \]

we have

\[ f_1 f_2(x) < f_2 f_1(x). \]

If repeat to use this inequality for the monotonically increasing functions \( f_1 f_2(x) \), \( f_2 f_1(x) \), we can get

\[ (f_1 f_2)^\alpha(x) < (f_2 f_1)^\alpha(x). \]

For example in the case of \( \alpha = 2 \), we have

\[ (f_1 f_2)^2(x) = f_1 f_2(f_1 f_2(x)) < f_1 f_2(f_2 f_1(x)) < f_2 f_1(f_2 f_1(x)) = (f_2 f_1)^2(x). \]

(3) For any positive integer \( \alpha \geq 1 \), we consider the general expansion form of \((f_1 f_2)^\alpha(x)\). Obviously, for smaller \( \alpha \), we have

if \( \alpha = 1 \), then

\[ f_1 f_2(x) = \frac{3x + 1}{2}, \]

if \( \alpha = 2 \), then

\[ (f_1 f_2)^2(x) = \frac{3^2 x + 3 + 2^2}{2^4}, \]

if \( \alpha = 3 \), then

\[ (f_1 f_2)^3(x) = \frac{3^3 x + 3^2 + 3 \times 2^2 + 2^4}{2^6}, \]

\[ \ldots \]

so, for any positive integer \( \alpha \geq 1 \) and real number \( x > 1 \), we have

\[ (f_1 f_2)^\alpha(x) \]

\[ = \frac{3^\alpha x + 3^{\alpha - 1} + 3^{\alpha - 2} \times 2^2 + \cdots + 3 \times 2^{2\alpha - 4} + 2^{2\alpha - 2}}{2^{2\alpha}} \]

\[ = \frac{3^\alpha x + 3^{\alpha - 1} + 3^{\alpha - 2} \times 4 + \cdots + 3 \times 4^{\alpha - 2} + 4^{\alpha - 1}}{4^\alpha} \]

\[ = \frac{3^\alpha x + 4^\alpha - 3^\alpha}{4^\alpha} \]

\[ = \left( \frac{3}{4} \right)^\alpha x + 1 - \left( \frac{3}{4} \right)^\alpha \]

\[ < x. \]
(4) For any positive integer \( \alpha \geq 1 \), we consider the general expansion form of \((f_2 f_1)^\alpha(x)\). Obviously, for smaller \( \alpha \), we have

if \( \alpha = 1 \), then
\[
(f_2 f_1)^1(x) = \frac{3(\frac{3}{4}) + 1}{2} = \frac{3x + 2}{2^2},
\]

if \( \alpha = 2 \), then
\[
(f_2 f_1)^2(x) = \frac{3(3x + 2) + 2}{2^2} = \frac{3^2 x + 3 \times 2 + 2^3}{2^4},
\]

if \( \alpha = 3 \), then
\[
(f_2 f_1)^3(x) = \frac{3(3^2 x + 3x^2 + 2^3) + 2}{2^2} = \frac{3^3 x + 3^2 \times 2 + 3 \times 2^3 + 2^5}{2^6},
\]

\ldots

so, for any positive integer \( \alpha \geq 1 \) and real number \( x > 2 \), we have

\[
(f_2 f_1)^\alpha(x) = \frac{3^\alpha x + 3^{\alpha - 1} \times 2 + 3^\alpha - 2 \times 2^3 + \cdots + 3 \times 2^{2\alpha - 3} + 2^{2\alpha - 1}}{2^{2\alpha}}
\]

\[
= \frac{3^\alpha x + 3^{\alpha - 1} \times 2 \left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^{\alpha - 2} + \left(\frac{1}{4}\right)^{\alpha - 1}\right)}{4^\alpha}
\]

\[
= \frac{3^\alpha x + 3^{\alpha - 1} \times 2 \left(\frac{4^{\alpha - 1}}{4 - 1}\right)}{4^\alpha}
\]

\[
= \frac{3^\alpha x + 2(4^\alpha - 3^\alpha)}{4^\alpha}
\]

\[
= \left(\frac{3}{4}\right)^\alpha x + 2\left(1 - \left(\frac{3}{4}\right)^\alpha\right)
\]

\[
< x.
\]

(5) In the first, for any positive integer \( \beta \geq 1 \), we consider the general expansion form of \( f_2^\beta(x) \). It is obvious that for smaller \( \beta \), we have

if \( \beta = 1 \), then
\[
f_2(x) = \frac{3x + 1}{2},
\]

if \( \beta = 2 \), then
\[
f_2 f_2(x) = \frac{3^2 x + 3 + 2}{2^2},
\]

if \( \beta = 3 \), then
\[
f_2 f_2 f_2(x) = \frac{3^3 x + 3^2 + 3 \times 2 + 2^2}{2^3},
\]

\ldots

So, for any positive integer \( \beta \geq 1 \), we have

\[
f_2^\beta(x) = \frac{3^\beta x + 3^{\beta - 1} + 3^{\beta - 2} \times 2 + \cdots + 3 \times 2^{\beta - 2} + 2^{\beta - 1}}{2^\beta}
\]

\[
= \frac{3^\beta x + 3^\beta - 2^\beta}{2^\beta}
\]
From (3) of the Lemma,

\[(f_1 f_2)^\alpha(x) = \left(\frac{3}{4}\right)^\alpha x + \frac{3}{4} - 1,\]

we get

\[f_2^\beta (f_1 f_2)^\alpha(x) = \left(\frac{3}{2}\right)^\beta \left(\frac{3}{4}\right)^\alpha x + \frac{3}{4} - 1\]
\[= \left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta (x - 1) + 2 \times \left(\frac{3}{2}\right)^\beta - 1.\]

### 3 Proof of the Collatz Conjecture

**Theorem.** For any positive integer \(n > 1\), the Collatz Conjecture is correct.

**Proof.** For any positive integer \(n > 1\), it is an independent event with equally probability for \(n\) is an even number or \(n\) is an odd number. Moreover, based on the Collatz recursive algorithm, it is also an independent event with equally probability for

\[f_1(n) = f_1(2k) = k,\]

or

\[f_2(n) = f_2(2k + 1) = \frac{3(2k + 1) + 1}{2} = 3k + 2,\]

is an even number or an odd number. Actually, for any positive integer \(n\), we shown an same average number \((\approx \log_4 n)\) of \(f_1\) and \(f_2\) for Collatz recursive algorithm [15].

We propose a simple proof for the Collatz conjecture by using the elementary mathematical induction. From

\[f_1(2) = 1,\]
\[f_1f_1f_1f_2f_2(3) = 1,\]
\[f_1f_1(4) = 1,\]
\[f_1f_1f_1f_2f_2(5) = 1,\]
\[f_1f_1f_1f_2f_2f_1(6) = 1,\]
\[f_1f_1f_1f_2f_2f_1f_1f_2f_2f_2(7) = 1,\]
\[f_1f_1f_1(8) = 1,\]
\[f_1f_1f_1f_2f_1f_2f_2f_1f_2f_2f_1f_1(9) = 1,\]
\[f_1f_1f_1f_2f_1f_2f_1f_2f_2f_1f_2f_2f_1(10) = 1,\]

obviously, the Collatz conjecture is correct for \(2 \leq n \leq 10\).

We assume the Collatz conjecture is correct for all positive integer \(n\), \(2 \leq n \leq k\), \((k \geq 10)\), and consider case of \(n = k + 1\). It is needed to prove the
Collatz recursive algorithm with the initial value \( k + 1 \) will reach less than \( k \) in some steps.

If \( k + 1 \) is an even number, then from
\[
f_1(k + 1) = \frac{k + 1}{2} < k,
\]
the Collatz conjecture is correct for \( n = k + 1 \).

If \( k + 1 \) is an odd number, the Collatz conjecture will generate a sequence of composite functions
\[
f_* f_* \cdots f_* f_2(2) (k + 1),
\]
here \( f_* \) means \( f_1 \) or \( f_2 \) based on the Collatz recursive algorithm. For any positive integer \( n \), because the function \( f_1(n) \) and \( f_2(n) \) is independent event with equally probability, there exist a positive integer \( \alpha \), with satisfies the following conditions in Collatz recursive algorithm:

1. (The number of \( f_1 \)) \( < \) (The number of \( f_2 \)) during step 1 to step \( 2\alpha - 1 \),
2. (The number of \( f_1 \)) \( = \) (The number of \( f_2 \)) in the step \( 2\alpha \).

As initial cases of \( \alpha \), we have the following sequences of \( f_1 \) and \( f_2 \):

- If \( \alpha = 1 \), \( f_1 f_2 \),
- If \( \alpha = 2 \), \( f_1 f_1 f_2 f_2 \),
- If \( \alpha = 3 \), \( f_1 f_1 f_2 f_2 f_2, f_1 f_1 f_2 f_1 f_2 f_2, \)
- \ldots

By repeating use the inequality \( f_1 f_2(x) < f_2 f_1(x) \) and the Lemma, we have when \( \alpha = 1 \),
\[
f_1 f_2(k + 1) < k + 1,
\]
when \( \alpha = 2 \),
\[
f_1 f_1 f_2 f_2(k + 1) < f_1 f_2 f_1 f_2(k + 1) = (f_1 f_2)^2(k + 1) < k + 1,
\]
when \( \alpha = 3 \),
\[
f_1 f_1 f_2 f_2 f_2(k + 1) < f_1 f_2 f_1 f_2 f_2(k + 1) = (f_1 f_2)^3(k + 1) < k + 1,
f_1 f_1 f_2 f_2 f_2(k + 1) < f_1 f_2 f_1 f_2 f_2(k + 1) = (f_1 f_2)^2(k + 1) < k + 1.
\]

Because left function values in the above inequalities all are positive integers, so, we get
\[
f_1 f_2(k + 1) \leq k,
\]
f_1 f_1 f_2 f_2(k + 1) \leq k,
\[
f_1 f_1 f_2 f_2 f_2(k + 1) \leq k,
f_1 f_1 f_2 f_1 f_2 f_2(k + 1) \leq k.
\]

Generally, in the step \( 2\alpha \), by repeating use the inequality \( f_1 f_2(x) < f_2 f_1(x) \) and the Lemma, we always can get
\[
f_* f_* \cdots f_* f_* f_2(k + 1) < (f_1 f_2)^\alpha(k + 1) < k + 1.
\]
Because \( f_* f_* \cdots f_* f_* f_2(k + 1) \) is a positive integer, so we have
\[
f_* f_* \cdots f_* f_* f_2(k + 1) \leq k.
\]

So, the Collatz conjecture is correct for \( n = k + 1 \). Therefore, from the inductive assumption, the Collatz conjecture is correct for all positive integer \( n > 1 \).

Actually, for the positive integer \( k + 1 \), there exist some smaller positive integers \( \beta \), we can also prove the above result when
If there exist positive integers $\alpha$ and $\beta$, with satisfying the following conditions in the Collatz recursive algorithm:

1. $(\text{The number of } f_1) < (\text{The number of } f_2) - \beta$ during step 1 to step $2\alpha + \beta - 1$,
2. $(\text{The number of } f_1) = (\text{The number of } f_2) - \beta$ in the step $2\alpha + \beta$.

By repeating use the inequality $f_1 f_2(x) < f_2 f_1(x)$, we always can get

$$f_* f_* \cdots f_* f_2(k + 1) < f_2^\beta (f_1 f_2)^\alpha (k + 1),$$

here $f_*$ means $f_1$ or $f_2$ which is based on the Collatz recursive algorithm with initial value $k + 1$.

From the Lemma (5),

$$f_2^\beta (f_1 f_2)^\alpha (x) = \left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta (x - 1) + 2 \times \left(\frac{3}{2}\right)^\beta - 1.$$

In order to find a better range of $\alpha$, $\beta$, and $x$, we solve the following inequality,

$$\left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta (x - 1) + 2 \times \left(\frac{3}{2}\right)^\beta - 1 < x,$$

or

$$\left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta (x - 1) + 2 \times \left(\frac{3}{2}\right)^\beta - 1 = \left(\frac{2}{3}\right)^\beta - \frac{2}{3} \times \left(\frac{3}{2}\right)^\beta = \frac{2}{3} \times \frac{x}{x - 1} - \frac{2}{x - 1},$$

Let

$$S = \left(\frac{2}{3}\right)^\beta \frac{x}{x - 1} - \frac{2}{x - 1}.$$

If

$$S > 0,$$

$$\beta < \log_3 \frac{x + 1}{2},$$

then after

$$\alpha > \log_4 \frac{1}{S},$$

the inequality

$$\left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta (x - 1) + 2 \times \left(\frac{3}{2}\right)^\beta - 1 < x,$$

always is satisfied.

It is easy to get some range values of $x$ and $\alpha$ for some smaller $\beta$. For the examples (here let $x$ is an integer),

If $\beta = 1$, then $x \geq 3$, $\alpha \geq 4$,
If $\beta = 2$, then $x \geq 4$, $\alpha \geq 10$,
If $\beta = 3$, then $x \geq 6$, $\alpha \geq 15$.

It is no problem to expand the initial value (Collatz Problem has been tested until $2^{68}$) to get some smaller $\beta$ in the mathematical induction to fit the above range of $x$ and increase $\alpha$ for the number of steps. In this proof, we confirmed the initial values $2 \leq n \leq 10$, so it is no problem to apply the mathematical induction for $\beta \leq 3$ at least.

Therefore for some smaller $\beta = 1, 2, 3, \ldots$, we also can prove there exist positive integer $\alpha_0$ and real number $x_0$, when $\alpha \geq \alpha_0$, $k + 1 > x_0$, $f_*, f_*, \ldots, f_*, f_2(k + 1) < f_2^\beta \cdot (f_1 f_2)^n(k + 1) = \left(\frac{3}{4}\right)^\alpha \left(\frac{3}{2}\right)^\beta (k) + 2 \times \left(\frac{3}{2}\right)^\beta - 1 < k + 1$ will be true. Because $f_*, f_*, \ldots, f_*, f_2(k + 1)$ is a positive integer, so we have $f_*, f_*, \ldots, f_*, f_2(k + 1) \leq k$.

It is to say that the Collatz Conjecture is correct for $n = k + 1$. From the inductive assumption, the Collatz Conjecture is correct for all positive integer $n \geq 2$.

4 Concluding Remarks

This paper proposed a simple proof for the Collatz conjecture by using the mathematical induction. We hope to receive some helpful comments from the reviewers and readers to improve this proof and try to apply it to other related problems with the Collatz conjecture.

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