Hot band sound

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Chaotic lattice models at high temperature are generically expected to exhibit diffusive transport of all local conserved charges. Such diffusive transport is usually associated with overdamped relaxation of the associated currents. Here we show that by appropriately tuning the inter-particle interactions, lattice models of chaotic fermions at infinite temperature can be made to cross over from an overdamped regime of diffusion to an underdamped regime of “hot band sound”. We study a family of one-dimensional spinless fermion chains with long-range density-density interactions, in which the damping time of sound waves can be made arbitrarily long even as an effective interaction strength is held fixed. Our results demonstrate that underdamped sound waves of charge density can arise within a single band, far from integrability, and at very high temperature.

Introduction. Understanding the transport properties of systems of strongly interacting fermions continues to pose a substantial challenge for theory. One prominent open problem is understanding the resistivity growth of so-called “bad metals” at temperatures above the Mott-Ioffe-Regel limit, where the mean free path is of order the lattice spacing, so a semiclassical description is no longer a good first approximation\cite{Sachdev2011}. While there is still no firm consensus on the origin of bad metal physics, strong electron-electron interactions are expected to play a significant role\cite{Lee1985,Andraka2020}.

Identifying the specific consequences of strong electron-electron interactions in realistic condensed matter systems is often complicated by the presence of disorder and phonons. Recently, however, it has become possible to isolate purely “electronic” effects through cold-atom realizations of strongly interacting fermions in optical lattices, which have neither phonons nor disorder. One recent such experiment explored bad metal physics in the two-dimensional single-band Fermi-Hubbard model, extending to temperatures that are large compared to the hopping strength\cite{Jotzu2014}. An unexpected finding in this experiment was the observation of a transiently ballistic regime of underdamped “band sound” for low enough temperatures and short enough wavelengths.

What do we mean by “band sound”? We mean underdamped density oscillations of interacting quantum lattice particles moving in a single band. Motivated by the above-mentioned experimental results, this paper provides some theoretical justification for the phenomenon of such underdamped and thus transiently ballistic density modes in fermion lattice models. In particular, we provide detailed analytical and numerical arguments that such underdamped band sound can occur in systems that are quantum chaotic and at very high (even infinite) temperature.

Our results are surprising because the naïve expectation for the hydrodynamics of conserved charges in quantum chaotic lattice systems at very high temperature is normal diffusion. This should be contrasted with continuum systems, for which momentum conservation generically gives rise to ballistic sound modes. Similarly, quantum integrable systems on the lattice are well-known to exhibit ballistic and anomalous regimes of transport\cite{Huse2016,Tikhonov2016}, while chaotic lattice systems at low temperature can also exhibit anomalous transport by virtue of their proximity to integrable points\cite{Huse2016,Stefanucci2012}. However, for systems that are on a lattice, far from integrability and at high temperature, no such behaviour is expected.

Our main result is the construction of a family of chaotic, interacting spinless fermion chains in which the decay rate of the charge current can be made arbitrarily small as the interaction range is increased, all the while keeping the magnitude of the interaction term fixed. We explain this rather unusual phenomenology through a quantum Boltzmann equation treatment, which demonstrates that these models exhibit strong but localized scattering in pseudomomentum space. This both yields an efficient pathway to chaos and implies that Umklapp scattering is suppressed, so that the single-particle phase-space distribution function itself behaves like a locally conserved, hydrodynamic degree of freedom for long times. The kinetic theory of this “phase-space hydrodynamics” qualitatively resembles kinetic theories of quasiparticle dynamics in quantum integrable systems\cite{Huse2016} and the Landau kinetic equation that describes zero sound in cold Fermi liquid\cite{Landau1937}. A novel prediction of this work is that such phase-space hydrodynamics can emerge both far from integrability and at high temperature.

The paper is structured as follows. We first introduce the spinless fermion chains that will be considered in this paper and discuss their charge transport properties. We then formulate a variational problem for minimizing the decay of the charge current in this family of models, and present a class of exact solutions to this variational problem. Finally, we simulate the resulting “optimal models” numerically, and show that they exhibit signatures of both hot band sound and many-body quantum chaos.

The model. For concreteness, we consider translation invariant spinless fermion chains with density-density in-
terations
\[ H = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = - \sum_{x,x' = 1}^L t_{|x-x'|} |c_x, \hat{c}_{x'}], \]
\[ \hat{V} = \sum_{x,x' = 1}^L U_{|x-x'|} (\hat{n}_{x'} - 1/2)(\hat{n}_x - 1/2) \tag{1} \]
where \( t_0 = U_0 = 0 \), we assume periodic boundary conditions \( x \equiv x + L \), and we set \( t_{|x-x'|} = U_{|x-x'|} = 0 \) for \( |x-x'| \geq [L/2] \) to avoid double counting. For site \( x \), the onsite charge density \( \hat{n}_x \) satisfies the operator-valued continuity equation
\[ \partial_t \hat{n}_x + \hat{j}_{x+1} - \hat{j}_x = 0, \tag{2} \]
where the charge current operator
\[ \hat{j}_x = i \sum_{r > 0}^{x+r-1} \sum_{x' = x} \tau_r (\hat{c}_{x-r}^{\dagger}, \hat{c}_x^{\dagger} - \hat{c}_{x-r}^{\dagger}, \hat{c}_{x-r}^{\dagger} - \hat{c}_{x-r}^{\dagger}, \hat{c}_{x-r}^{\dagger}) \tag{3} \]
is independent of the interaction strength. Let us define total charges and currents
\[ \hat{N} = \sum_x \hat{n}_x, \quad \hat{J} = \sum_x \hat{j}_x. \tag{4} \]
In general \( \hat{N} \) is conserved but \( \hat{J} \) is not, leading to diffusive transport of charge. Nevertheless, if \( \hat{J} \) decays very slowly, the possibility arises of an underdamped and thus transiently ballistic “sound mode” of lattice fermions at high temperature and short enough wavelengths.

In this paper, we realize this regime by tuning the form of the interactions. It will be useful to write the charge current explicitly in terms of Fourier modes
\[ \hat{c}_x = \frac{1}{\sqrt{N}} \sum_k e^{ikx} \hat{c}_k \]
as
\[ \hat{j}_x = i \sum_{r > 0} \sum_{x' = x} \tau_r (\hat{c}_{x-r}^{\dagger}, \hat{c}_x^{\dagger} - \hat{c}_{x-r}^{\dagger}) \sum_{k} v_k \hat{c}_k^{\dagger} \hat{c}_k \tag{5} \]
where the group velocity \( v_k = \sum_{r > 0} 2rt_r \sin kr \). This operator is manifestly conserved under the non-interacting dynamics due to \( \hat{H}_0 \), but decays due to the interaction term \( \hat{V} \). In real space, we find that
\[ \hat{j}_x = - \sum_{r > 0} 2rt_x \sum_{x} \sum_{y \neq x,x+r} (U_{|y-x|} - U_{|y-x+r|}) \left( \hat{n}_y - 1/2 \right) (\hat{c}_{x-r}^{\dagger}, \hat{c}_{x-r}^{\dagger}, \hat{c}_{x-r}^{\dagger}, \hat{c}_{x-r}^{\dagger}). \tag{6} \]

Lagrangian for minimizing current decay. We would like to minimize the decay rate of \( \hat{J} \) at infinite temperature. This is slightly complicated by the fact that \( \langle \hat{J}(t) \rangle_{\beta = 0} = 0 \) and
\[ \langle \hat{J}\hat{J}(t) \rangle_{\beta = 0} = \langle \hat{J}(-t)\hat{J} \rangle_{\beta = 0} = \langle \hat{J}\hat{J}(-t) \rangle_{\beta = 0}. \tag{7} \]
by cyclicity of the trace. The latter equation is a simple case of the Kubo-Martin-Schwinger relation, and here implies time-reversal symmetry and vanishing of the correlation function \( \langle \hat{J}\hat{J}(t) \rangle_{\beta = 0} = 0 \). We therefore instead consider the quantity
\[ ||\hat{J}(t)||^2 = \langle \hat{J}(t)\hat{J}(t) \rangle_{\beta = 0} = \langle \hat{J}\hat{J} \rangle_{\beta = 0} \tag{8} \]
which is (up to rescaling) the square of the Hilbert-Schmidt norm of \( \hat{J}(t) \) and invariant under unitary time evolution. We would like to minimize this decay rate over all possible density-density interactions \( U \). To avoid obtaining the trivial solutions with no interactions or no hopping, we also fix the variance of the interaction operator \( \hat{V} \) to equal a characteristic fluctuation scale \( \sigma_F^2 = L^2/4 \), and the variance of the current operator \( \hat{J} \) to equal a characteristic scale \( \sigma_J^2 = L^2/2 \). (Note that both scales are set by unit nearest-neighbour hoppings and interactions \( t_1 = U_1 = 1 \), and that both \( \hat{V} \) and \( \hat{J} \) are traceless.) This yields the Lagrangian
\[ \mathcal{L}(t_r, U_r, \lambda_1, \lambda_2) = \langle \hat{J}\hat{J} \rangle_{\beta = 0} + \lambda_1 \left( \langle \hat{V}^2 \rangle_{\beta = 0} - \sigma_F^2 \right) + \lambda_2 \left( \langle \hat{J}^2 \rangle_{\beta = 0} - \sigma_J^2 \right), \tag{9} \]
which is a function of the hopping and interaction strengths at each range \( r \).

Explicit expressions for the terms in the Lagrangian are given by
\[ \langle \hat{J}\hat{J} \rangle_{\beta = 0} = \frac{1}{2} \sum_{r > 0} \sum_{x} \sum_{y \neq x,x-r} (U_{|y-x|} - U_{|y-x+r|})^2 \tag{10} \]
for the objective function and
\[ \langle \hat{V}^2 \rangle_{\beta = 0} = \frac{L}{4} \sum_{r > 0} U_r^2, \quad \langle \hat{J}^2 \rangle_{\beta = 0} = \frac{L}{2} \sum_{r > 0} t_r^2 \tag{11} \]
for the constraints.

Solving for optimal models. Solving first for stationary points with \( U_r = t_r = 0 \) for \( r > 2 \), we find that the only non-trivial solutions have \( t_2 = 0 \) and \( |t_1| = 1 \). This suggests that the most effective method for attaining minimal current decay subject to the chosen constraints is through hopping that is only nearest-neighbour in combination with long-range interactions. We therefore set \( t_1 = t = 1 \), \( t_r = 0 \) for all \( r \neq 1 \), and consider Hamiltonians with non-zero density-density interactions up to some specified interaction range \( 1 \leq R < [L/2] \).

In this case the Lagrangian for current decay can be written as a function of the vector of non-zero couplings
\[ \vec{U} = (U_1, U_2, \ldots, U_R), \text{ as} \]
\[ \mathcal{L}(t, \vec{U}, \lambda_1, \lambda_2) = \Lambda t^2 \left( \sum_{n=1}^{R-1} (U_{n+1} - U_n)^2 + U_R^2 \right) \]
\[ + \frac{\Lambda \lambda_1}{4} \left( \sum_{n=1}^{R} U_n^2 - 1 \right) + \frac{\Lambda \lambda_2}{2} (t^2 - 1). \]

Remarkably, this specific instance of the optimization problem posed above is exactly solvable. The key observation is that the stationary condition in \( \vec{U} \) can be written as a matrix eigenvalue equation \( A \vec{U} = \alpha \vec{U} \), where \( \alpha = -\frac{\lambda}{4t^2} \) and the \( R \)-by-\( R \) matrix \( A \)
\[ A = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{pmatrix}. \]

The eigenvalue equation fixes the value of \( \lambda \) and the direction of the vector \( \vec{U} \). The stationary condition for \( \lambda_1 \) imposes unit normalization of \( \vec{U} \), while the stationary conditions for \( \lambda_2 \) and \( t \) fix \( t^2 = 1 \) and the numerical value of \( \lambda_2 \) respectively.

The spectral problem for \( A \) can be solved in closed form, and amounts to a boundary value problem for a second-order difference equation, explicitly
\[ (1 - \alpha) U_1 - U_2 = 0, \]
\[ (2 - \alpha) U_n - U_{n-1} - U_{n+1} = 0, \quad 1 < n < R, \]
\[ (2 - \alpha) U_R - U_{R-1} = 0. \]

We find \( R \) linearly independent normalized solutions, which can be written as
\[ U_n^{(m)} = \frac{2}{\sqrt{2R + 1}} \cos k_m (n - 1/2), \quad 1 \leq n \leq R, \]
with
\[ k_m = \frac{(2m + 1)\pi}{2R + 1}, \quad m = 0, 1, \ldots, R - 1, \]
and associated eigenvalues \( \alpha_m = 2 - 2 \cos k_m \). The eigenvalue \( \alpha_m \) is related to the decay rate of the current via \( \langle \dot{J}^2 \rangle = \alpha_m L \), and we deduce that the solution with the slowest current decay, subject to the constraints, is given by setting \( m = 0 \) in the above expressions. Explicitly, for each interaction range \( R \geq 1 \), a solution to the optimization problem posed above with the slowest current decay is given by
\[ U_n^{\ast}(R) = \frac{2}{\sqrt{2R + 1}} \cos \frac{\pi(n - 1/2)}{(2R + 1)}, \quad 1 \leq n \leq R, \]
\[ \alpha^\ast(R) = 2 - 2 \cos \frac{\pi}{2(R + 1)}. \]

Some of the resulting optimal solutions are tabulated for small values of \( R \) in the Supplemental Material.

For \( R = 1 \), this is an integrable spinless fermion model equivalent to the spin-1/2 Heisenberg chain\[21\] For \( R > 1 \) we obtain a family of models (henceforth “optimal models”) for which the rate of current decay decreases with \( R \). In the regime of interactions at a large but finite range, \( R \gg 1 \), an estimate for the relaxation rate of the charge current is given by
\[ \tau_{\text{eff}}^{-1} = \sqrt{\langle \dot{J}^2 \rangle / \langle J^2 \rangle} \sim \frac{\pi}{2R + 1} \to 0, \quad R \to \infty. \]

Thus the decay of the charge current in this family of models can be made arbitrarily slow as the interaction range \( R \) is increased. In fact, this expression yields an upper bound on the true relaxation rate; a more detailed quantum Boltzmann equation treatment\[22\] predicts even slower relaxation due to local diffusion in pseudomomentum space, which occurs at a rate \( \tau^{-1} \propto 1/R^2 \).

**Numerical results.** To verify that the optimal models defined above indeed exhibit underdamped sound modes, we first examine the relaxation of weak density modulations in real space. Therefore consider the initial condition
\[ \dot{\rho}(0) = \frac{1}{Z} \left( 1 + \epsilon \sum_{x=1}^{L} \sin(qx)(\hat{n}_x - \langle \hat{n}_x \rangle_{\beta=0}) \right), \]
with \( \epsilon = 0.01 \) and \( q = 2\pi/L \), let it evolve numerically under Schrödinger evolution \( \dot{\rho}(t) = i[H, \rho(t)]e^{i\rho L t} \) and look at the time evolution of the lowest Fourier mode of

![FIG. 1. Decay of an initial density modulation in a system of \( L = 14 \) sites at half-filling, near infinite temperature and for various optimal models with \( 1 \leq R < [L/2] \). We include the integrable model with \( R = 1 \) for comparison. It is clear that the level of damping decreases as \( R \) increases.](image-url)
FIG. 2. Decay and growth of pseudomomentum correlations \( \text{Re}[\langle \hat{c}_{k=\pm \frac{\pi}{L}}^{\dagger} \hat{c}_{0}(t) \rangle]/\epsilon \) (top) and \( \text{Re}[\langle \hat{c}_{k=\pm \frac{\pi}{L}}^{\dagger} \hat{c}_{k=\mp \frac{\pi}{L}}(t) \rangle]/\epsilon \) (bottom) in the time evolution of the initial state Eq. (24), for an \( L = 12 \) site system. The presence of interactions in the optimal models leads to a coherent transfer of correlations from small to large pseudomomenta whose magnitude increases with \( R \). This behaviour differs markedly from the non-interacting case.

The resulting data for optimal models with interaction ranges \( R \in \{1, 2, 3, 6\} \) is shown in Fig. 1 for half-filled chains on \( L = 14 \) sites. There is a clear decrease in the level of damping as one moves to larger interaction ranges \( R \), as expected.

We next present numerical evidence that underdamped charge relaxation in these models is associated with interactions, rather than proximity to a non-interacting system as Eq. (21) might suggest. To probe this physics, we consider optimal models on \( L = 12 \) sites, initialized in a state that has weak coherence between pseudomomenta modes with \( k = \pm 2\pi/L \) and \( k = 0 \), but no other correlations between distinct pseudomomenta, namely

\[
\hat{\rho}(0) = \frac{1}{2L} \left( 1 + \epsilon (\hat{c}_{k=0}^{\dagger} \hat{c}_{k=\pm \frac{\pi}{L}} + \hat{c}_{k=\mp \frac{\pi}{L}} + \text{h.c.}) \right) \tag{24}
\]

where we again set the small parameter \( \epsilon = 0.01 \). In the absence of interactions, the time-evolution of this density matrix is given simply by

\[
\hat{\rho}(0) = \frac{1}{2L} \left( 1 + \epsilon e^{i\omega t} (\hat{c}_{k=0}^{\dagger} \hat{c}_{k=\pm \frac{\pi}{L}} + \hat{c}_{k=\mp \frac{\pi}{L}} + \text{h.c.}) \right) \tag{25}
\]

where \( \omega = 2(1 - \cos 2\pi/L) \sim 1/L^2 \) for large \( L \). Thus the expectation value \( \langle \hat{c}_{k=\pm \frac{\pi}{L}}^{\dagger} \hat{c}_{k=0}(t) \rangle = \frac{\epsilon}{4} e^{i\omega t} \) while for higher-pseudomomentum correlations, e.g. for \( L k/2\pi = 2, 3 \) we have \( \langle \hat{c}_{k=\pm \frac{\pi}{L}}^{\dagger} \hat{c}_{k=\mp \frac{\pi}{L}}(t) \rangle \rho(t) = 0 \).

In the presence of interactions, scattering between pseudomomenta should deplete correlations between the lowest lying pseudomomentum modes and enhance correlations between higher pseudomomentum modes. A coherent transfer of such correlations, indicative of normal scattering rather than Umklapp scattering, is demonstrated numerically in Fig. 2. Note that the resulting underdamped sound mode in the interacting system has a substantially higher frequency than the oscillations of the non-interacting system produced by this initial state.

We next turn to the question of whether these optimal models are quantum chaotic. To this end, we compute the \( \langle r \rangle \) statistic (23) for the optimal models at a relatively large numerically accessible system size, \( L = 16 \). To avoid additional symmetries arising at half-filling, we work near half-filling in the sector with \( M = 7 \) fermions. With periodic boundary conditions, the optimal models have both lattice translation symmetry and reflection symmetry. To eliminate spurious near-level-crossings arising from these symmetries, we therefore project into the sector with zero pseudomomentum and positive parity under reflections. The results are plotted in Fig. 3 and strongly suggest that the optimal models are chaotic for \( R > 1 \), since the \( \langle r \rangle \) statistic jumps from its expected Poisson value \( \langle r \rangle \approx 0.38 \) at \( R = 1 \), indicating quantum integrability, to its Gaussian Orthogonal Ensemble (GOE) value \( \langle r \rangle \approx 0.53 \) for \( R > 1 \), indicating quantum chaos.

Conclusion. We have constructed a family of fermionic lattice models that exhibit both infinite-temperature band sound and signatures of many-body quantum chaos at numerically accessible system sizes. We have further shown that the decay rate of the charge current in these models can be made arbitrarily small by increasing the range of the density-density interaction.
We expect similar conclusions to hold for the corresponding models one could make in higher spatial dimensions \( d > 1 \).

Our study was motivated by the experimental observation of a transient ballistic mode of charge density in a cold atom, single-band Fermi-Hubbard model\[2\]. One outstanding question is how the physics of hot band sound connects to these experimental observations for the 2D Fermi-Hubbard model. We believe that kinematic constraints on two-particle scattering are ultimately responsible for both phenomena and that these constraints are enhanced for the models studied above, which is why they exhibit band sound at longer wavelengths and much higher temperatures than the Fermi-Hubbard model. We defer a detailed discussion of the microscopic mechanism giving rise to band sound in the 2D Fermi-Hubbard model to future work\[23\].

The spinless fermion chains constructed in this paper are equivalent by a Jordan-Wigner transformation to spin-1/2 chains with nearest-neighbour exchange and long-range ZZ interactions, in which band sound appears as long-lived ballistic modes of the local magnetization \( \langle S_z^f \rangle \) at high temperature. We note that similar “quasiballistic” behaviour was recently observed numerically in spin-1/2 chains with both long-range exchange and long-range interactions\[24\], although the latter models have no fermionic counterpart and therefore do not exhibit band sound in the sense of this paper. Systems of trapped ions naturally realize such long-range interacting spin-1/2 chains\[25\]. While constructing the specific interaction graph corresponding to an optimal model requires a high degree of tunability, such fine control is possible in principle\[27\], and the qualitative effects we find should not require such precise optimality. This might provide one near-term experimental realization of the physics in this paper.

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[1] V. J. Emery and S. A. Kivelson, Phys. Rev. Lett. 74, 3253 (1995).
[2] O. Gunnarsson, M. Calandra, and J. E. Han, Rev. Mod. Phys. 75, 1085 (2003).
[3] N. E. Hussey, K. Takenaka, and H. Takagi, Philosophical Magazine 84, 2847 (2004) https://doi.org/10.1080/14786430410001716944.
[4] S. A. Hartnoll and A. P. Mackenzie, Planckian dissipation in metals (2021), arXiv:2107.07802.
[5] P. W. Phillips, N. E. Hussey, and P. Abbamonte, Science 377, eabb4273 (2022).
[6] P. T. Brown, D. Mitra, E. Guardado-Sanchez, R. Nourafkan, A. Reyimbaut, C.-D. Hébert, S. Bergeron, A.-M. S. Tremblay, J. Kokalj, D. A. Huse, P. Schauß, and W. S. Bakr, Science 363, 379 (2019) https://www.science.org/doi/pdf/10.1126/science.aat4134.
[7] Z. F. Zotos, F. Nae, and P. Prelovsek, Physical Review B 55, 11029 (1997).
Quantum Boltzmann equation for hot band sound

In this Appendix, we argue that the dynamics of optimal models for $R \gg 1$ and infinite temperature is approximated by a linearized Boltzmann equation of the form

$$\partial_t \delta \rho_k + v_k \partial_x \delta \rho_k = D \partial^2 \delta \rho_k,$$

where the effective pseudomomentum diffusion constant $D \propto \frac{1}{R^2}$. Note that since we are in a band, $k$-space is finite and periodic. This equation locally conserves the distribution function $\delta \rho_k(x,t)$ in phase space, and also conserves fluctuations in the local particle density $n(x,t) = \int dk \delta \rho_k(x,t)$ and in the local momentum density $p(x,t) = \int dk k \delta \rho_k(x,t)$, which implies that it supports ballistic sound modes. Eq. (26) further predicts a $k$-space diffusion time $\tau \propto R^2$ at which Umklapp processes become important. At this level of approximation, energy is not fully conserved; work on more complete descriptions of this novel hydrodynamics is in progress.\[23\]

To derive Eq. (26), let us consider an optimal model on $L$ sites with interactions of range $1 \ll R < |L/2|$, as defined in the main text. Discarding central terms, the interaction term can be written in terms of Fourier modes as

$$\hat{V} = \sum_{q,k,k'} U(q) \hat{c}_{k+q}^{\dagger} \hat{c}_{k'}^{\dagger} \hat{c}_{k'} \hat{c}_k$$

where a direct calculation reveals that

$$U(q) = \frac{2}{L \sqrt{2R+1}} \left[ \sin \left( \frac{\pi q}{R+1} \right) \left( \frac{R}{1} \right) \cos \left( \frac{\pi R}{2(2R+1)} + q \left( \frac{R+1}{2} \right) \right) - \sin \left( \frac{\pi q}{R+1} - \frac{R}{1} \right) \cos \left( \frac{\pi R}{2(2R+1)} - q \left( \frac{R+1}{2} \right) \right) \right].$$

(28)

In order to obtain a continuous long-wavelength limit of $U_n^*(R)$, we regulate its discontinuity at $n = 0$ by introducing the function

$$\tilde{U}_n(R) = U_n^*(R) + \frac{2}{\sqrt{2R+1}} \delta_{n0}.$$  

(29)

This implies that the long-wavelength ($L, R \gg 1$) behaviour due to interactions is controlled by the function

$$\tilde{U}(q) = U(q) + \frac{2}{L \sqrt{2R+1}}.$$  

(30)

Note that the function $\tilde{U}(q)$ becomes increasingly localized about $q = 0$ as $R$ is increased (intuitively speaking, $L\tilde{U}(q)$ tends to the “square root” of a Dirac delta function as $R \to \infty$).

At infinite temperature and for $L \gg 1$, it follows by Fermi’s Golden Rule that the quantum Boltzmann equation\[22\] describing collisions between quasiparticles of the non-interacting system due to $U$ is given by ($\hbar = 1$)

$$\partial_t \rho_k + v_k \partial_x \rho_k = 2\pi \nu \int \frac{dq}{2\pi} \frac{dk'}{2\pi} |U(q)|^2 \left[ |\rho_{k+q} \rho_{k'-q} - (1-\rho_k)(1-\rho_{k'}) - \rho_{k+q} \rho_{k'}(1-\rho_{k+q})(1-\rho_{k'-q})| \right],$$

(31)

where the integrals can be replaced by sums for finite $L$, and $\nu$ denotes the two-particle density of states of $\tilde{H}_0$, which we approximate as $\nu \approx L^2/2$ at infinite temperature. Considering small perturbations $\rho_k = \frac{1}{2} + \delta \rho_k$ about infinite temperature equilibrium yields the linearized kinetic equation

$$\partial_t \delta \rho_k + v_k \partial_x \delta \rho_k = \frac{\pi \nu}{2} \int \frac{dq}{2\pi} |U(q)|^2 \left( \delta \rho_{k+q} - \delta \rho_k \right).$$

(32)
To proceed further, it will be useful to approximate
\[ \nu |\bar{U}(q)|^2 \approx F(q) \]  
where the function \( F(q) \) is supported in a small region of width \( \epsilon \sim 1/R \) about the origin (we will justify this approximation below). By evenness of \( \bar{U}(q) \) in \( q \), this yields
\[ \partial_t \rho_k + v_k \partial_x \rho_k = \frac{\pi}{2} \int_{-\epsilon}^{\epsilon} dq \left( \frac{1}{2} q^2 \partial_{x}^2 \rho_k + O(q^3) \right). \]  
(34)

Keeping only the leading term in the derivative expansion in \( k \) yields an advection-diffusion equation in phase space as in Eq. (26), where the effective pseudomomentum-space diffusion constant
\[ D = \frac{1}{8} \int_{-\epsilon}^{\epsilon} dq q^2 F(q). \]  
(35)

Finally, we estimate the dependence of the diffusion constant \( D \) on the interaction range \( R \gg 1 \). We first note that for \( q \gg 1/R \), expanding \( \bar{U}(q) \) to leading order in \( 1/R \) yields
\[ L \bar{U}(q) = O(1/R^{3/2}), \]  
(36)
which is very small when \( R \gg 1 \). Thus \( \bar{U}(q) \) is approximately zero outside a region of width \( \sim 1/R \). However, \( \bar{U}(q) \) is sharply peaked at \( q = 0 \), with a value
\[ L \bar{U}(q = 0) \sim \frac{4\sqrt{2}}{\pi} \sqrt{R}, \quad R \to \infty. \]  
(37)

From these observations alone, we can estimate the \( R \)-dependence of the diffusion constant to be
\[ D \approx \frac{1}{4} \int_0^{1/R} dq q^2 F(q) \approx F(0) \int_0^{1/R} dq q^2 \propto \frac{1}{R^2}, \quad R \gg 1. \]  
(38)

**Optimal models for small \( R \)**

For reference, the density-density interaction strengths at range \( r \) for the optimal models, \( U^*_r \), are tabulated below in Table I for the values \( 1 \leq R \leq 7 \) of \( R \) simulated in this paper.

| Interaction range | \( U^*_1 \) | \( U^*_2 \) | \( U^*_3 \) | \( U^*_4 \) | \( U^*_5 \) | \( U^*_6 \) | \( \langle \hat{J}_r \rangle \) |
|-------------------|------------|------------|------------|------------|------------|------------|----------------|
| 1                 | 1          | 0          | 0          | 0          | 0          | 0          | 0.382L        |
| 2                 | 0.851      | 0.526      | 0          | 0          | 0          | 0          | 0.198L        |
| 3                 | 0.737      | 0.591      | 0.328      | 0          | 0          | 0          | 0.121L        |
| 4                 | 0.657      | 0.577      | 0.429      | 0.228      | 0          | 0          | 0.081L        |
| 5                 | 0.597      | 0.549      | 0.456      | 0.326      | 0.170      | 0          | 0.058L        |
| 6                 | 0.551      | 0.519      | 0.457      | 0.368      | 0.258      | 0.133      | 0.044L        |
| 7                 | 0.514      | 0.491      | 0.447      | 0.384      | 0.304      | 0.210      | 0.107         |

**TABLE I. Structure of optimal density-density interactions for small interaction ranges.**

**Behaviour of optimal models as \( R \to \infty \)**

In this Appendix, we discuss the behaviour of the optimal models as \( R \to \infty \). It will be useful to let \( L \to \infty \) and parameterize these models as finitely supported sequences
\[ s_R = (U^*_1(R), U^*_2(R), \ldots, U^*_R(R), 0, 0, \ldots) \]  
(39)
in the sequence space $S = \mathbb{R}^N$. Given that the effective lifetime $\tau_{\text{eff}}$ of the charge current diverges in this limit, it is natural to ask whether there is any meaningful sense in which the sequences $s_R$ “converge” to an integrable model as $R \to \infty$.

A mathematically precise answer to this question is that while the sequences $s_R$ converge pointwise to the zero sequence $(0, 0, \ldots)$ (i.e. the non-interacting limit) as $R \to \infty$, they do not converge to the zero sequence with respect to the $\ell^2$ norm $\| \cdot \|_2$ on $S$, because in this norm

$$\| s_R \|_2^2 = \sum_{n=1}^{\infty} U_n^*(R)^2 = 1$$ (40)

by the constraint on the interaction strength.

In more physical terms, one can argue that the optimal models are always in a strongly interacting regime because the Fourier transform $|\tilde{U}_q^*(R)|^2$, where $\tilde{U}_q^*(R) = \sum_n e^{-i q n} U_n^*(R)$, tends to a delta function of width $\sim 1/R$ about $q = 0$ as $R \to \infty$, as discussed above. Since $|\tilde{U}_q^*(R)|^2$ sets the rate of scattering by $\hat{V}$, that appears for example in the collision integral of the quantum Boltzmann equation\[28\], the optimal models always exhibit strong particle-particle scattering. Note that this is consistent with what is observed in Fig. 2 of the main text, and remains true even when the pseudomomentum transferred by these individual scattering events is very small.

Thus in some physically important aspects, these optimal models do not converge to their non-interacting limit as $R \to \infty$. It seems to us that the most natural expectation is that the optimal models remain chaotic for $R \gg 1$. 