Canonical double covers of circulants

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Abstract

The canonical double cover $B(X)$ of a graph $X$ is the direct product of $X$ and $K_2$. If $\text{Aut}(B(X)) \cong \text{Aut}(X) \times \mathbb{Z}_2$ then $X$ is called stable; otherwise $X$ is called unstable. An unstable graph is nontrivially unstable if it is connected, non-bipartite and distinct vertices have different neighborhoods. Circulant is a Cayley graph on a cyclic group. Qin et al. conjectured in [J. Combin. Theory Ser. B 136 (2019), 154-169] that there are no nontrivially unstable circulants of odd order. In this paper we prove this conjecture.

Keywords: circulant, canonical double cover, unstable graph, automorphism group

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1 Introduction

All groups considered in this paper are finite and all graphs are finite, simple and undirected. For a graph $X$, we denote by $V(X)$, $E(X)$ and $\text{Aut}(X)$ the vertex set, the edge set and the automorphism group of $X$, respectively. The canonical double cover (also called the bipartite double cover or the Kronecker cover) of a graph $X$, denoted by $B(X)$, is the direct product $X \times K_2$ (where $K_2$ denotes the complete graph on two vertices). This means that $V(B(X)) = V(X) \times \mathbb{Z}_2$ and $E(B(X)) = \{\{(x,0),(y,1)\} | \{x,y\} \in E(X)\}$. Canonical double covers have been studied by several authors, see for example [3, 10, 16, 25, 27]. It is well-known that $B(X)$ is connected if and only if $X$ is connected and non-bipartite, see [5]. It can also be easily seen that $\text{Aut}(B(X))$ contains a subgroup isomorphic to $\text{Aut}(X) \times \mathbb{Z}_2$. However, determining the full automorphism group of $B(X)$ is not as trivial. Hammack and Imrich [4] investigated vertex-transitivity of direct product of graphs, and proved that for a non-bipartite graph $X$ and a bipartite graph $Y$, direct product $X \times Y$ is vertex-transitive if and only if both $B(X)$ and $Y$ are vertex-transitive. Hence, the problem of vertex-transitivity of direct product of graphs reduces to the problem of vertex-transitivity of canonical double covers. If $\text{Aut}(B(X))$ is isomorphic to $\text{Aut}(X) \times \mathbb{Z}_2$ then the graph $X$ is called stable, otherwise it is called unstable. This concept was first defined by Marušić et al. [14] and studied later most notably by Surowski [22, 23], Wilson [26], Lauri et al. [11] and Qin et al. [19]. A graph is said to be irreducible (or vertex-determining) if distinct vertices have different neighbours, and reducible otherwise. It is easy to see that the following graphs are all unstable: disconnected graphs, bipartite graphs with non-trivial automorphism group, and reducible graphs. An unstable graph is said to be nontrivially unstable if it is non-bipartite, connected, and irreducible.

For a group $G$ and an inverse closed subset $S \subseteq G \setminus \{1_G\}$, the Cayley graph $\text{Cay}(G,S)$ on $G$ with connection set $S$ is defined as the graph with vertex set $G$, with two vertices $x, y \in G$ being adjacent if and only if $x^{-1}y \in S$. A Cayley graph on a cyclic group is called circulant. Sabidussi [21] proved that a graph $X$ is a Cayley graph on a group $G$ if and only if $\text{Aut}(X)$ has a regular subgroup isomorphic to $G$. It is easy to see that if $X$ is a Cayley graph on a group $G$,
then its canonical double cover $B(X)$ is a Cayley graph on $G \times \mathbb{Z}_2$. The converse is not true in general, that is $B(X)$ can be a Cayley graph even if the graph $X$ is not Cayley. The problem of characterizing graphs with Cayley canonical double covers was studied by Marušić, Scapellato and Zagaglia Salvi [15], who introduced the class of generalized Cayley graphs (see also [7, 8]) and proved that if the canonical double cover of a generalized Cayley graph $X$ is a stable graph, then $X$ is a Cayley graph. The characterization of graphs whose canonical double covers are Cayley graphs was given by the second author in [6].

The problem of characterizing vertex-transitive unstable graphs was posed in [2, Problem 5.7]. However, the problem is difficult even when restricted to the class of circulant graphs. The problem of classification of nontrivially unstable circulants was posed in [19, Problem 1.2]. Qin et al proved that every circulant of odd prime order is stable (see [19, Theorem 1.4], and posed the following conjecture.

Conjecture 1.1. [19, Conjecture 1.3] There is no nontrivially unstable circulant of odd order.

In this paper we prove Conjecture 1.1, that is we prove the following theorem.

Theorem 1.2. Let $X$ be a connected irreducible circulant of odd order and let $B(X)$ be its canonical double cover. Then $\text{Aut}(B(X)) \cong \text{Aut}(X) \times \mathbb{Z}_2$.

Remark 1.3. Using Theorem 1.2, the automorphism group of a canonical double cover of any circulant $X$ of odd order can be determined in terms of the automorphism group of $X$. Namely, if $X$ is a connected reducible circulant, then there exist a positive integer $d > 1$ and a connected irreducible circulant $Y$, such that $X \cong Y \wr K_d$. It is now easy to see that $B(X) \cong (B(Y) \wr K_d)$. Using the result of Sabidussi [20] on the automorphism group of wreath product of graphs, we obtain $\text{Aut}(B(X)) \cong \text{Aut}(B(Y) \wr S_d) \cong (\text{Aut}(Y) \times \mathbb{Z}_2) \wr S_d$.

2 Preliminaries

Following [18], an automorphism $\gamma$ of a bipartite graph $Y$ is called a strongly switching involution if $\gamma$ is an involution that swaps the two colour classes and fixes no edge. For a graph $X$ and a partition $P$ of $V(X)$, the quotient graph of $X$ with respect to $P$ is the graph whose vertex set is $P$ with $B_1, B_2 \in P$ being adjacent if and only if there exist $x \in B_1$ and $y \in B_2$ such that $\{x, y\} \in E(X)$. The following result is proved in [18, Proposition 2.4].

Lemma 2.1. A bipartite graph $Y$ is isomorphic to the canonical double cover of a graph $X$ if and only if $\text{Aut}(Y)$ contains a strongly switching involution $\gamma$ such that $X$ is isomorphic to the quotient graph of $Y$ with respect to the orbits of $\langle \gamma \rangle$.

The following lemma gives a necessary and sufficient condition for the stability of a graph, and it has been proved in [6, Corollary 2.5]. However, we give the proof here for the sake of completeness.

Lemma 2.2. Let $X$ be a connected non-bipartite graph, and let $B(X)$ be its canonical double cover. Let $\tau$ be the automorphism of $B(X)$ defined by $\tau(x, i) = (x, i + 1)$. Then $X$ is stable if and only if $\tau$ is central in $\text{Aut}(B(X))$.

Proof. Suppose first that $X$ is stable. Then $\text{Aut}(B(X)) \cong \text{Aut}(X) \times \langle \tau \rangle$, hence $\tau$ is central in $\text{Aut}(B(X))$. Suppose now that $\tau$ is central in $\text{Aut}(B(X))$. Let $\alpha \in \text{Aut}(B(X))$ be arbitrary.
Suppose first that $\alpha$ fixes the color classes of $B(X)$. Let $g$ be the permutation of $V(X)$ such that $\alpha(x, 0) = (g(x), 0)$. Since $\tau$ commutes with $\alpha$ it follows that $\alpha(x, 1) = \alpha(\tau(x, 0)) = \tau(\alpha(x, 0)) = (g(x), 1)$. Let $\{x, y\} \in E(X)$. Then $\{x, 0\} \in E(B(X))$, and since $\alpha \in \text{Aut}(B(X))$ it follows that $\{g(x), 0\}, \{g(y), 1\} \in E(B(X))$. By definition of the canonical double cover, it follows that $\{g(x), g(y)\} \in E(X)$. It follows that $g \in \text{Aut}(X)$. Hence, $\alpha$ is induced by the automorphism of $X$, that is $\alpha \in \text{Aut}(X) \times \mathbb{Z}_2$. If $\alpha$ permutes the color classes of $B(X)$, then we apply the above arguments for $\alpha \tau$ which fixes the color classes, and conclude that $\alpha \tau \in \text{Aut}(X) \times \mathbb{Z}_2$, hence $\alpha \in \text{Aut}(X) \times \mathbb{Z}_2$. This finishes the proof.

Suppose that $X = \text{Cay}(\mathbb{Z}_k, S)$ is a circulant of odd order $k$. Then it is straightforward to verify that $B(X)$ is a Cayley graph on $\mathbb{Z}_k \times \mathbb{Z}_2$, with connection set $S \times \{1\}$. Since the map $\alpha : \mathbb{Z}_k \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ defined by $\alpha(x, i) = 2x + ki$ is an isomorphism, it follows that $B(X) \cong \text{Cay}(\mathbb{Z}_k, k + 2S)$ (where elements of $S$ are now considered as elements of $\mathbb{Z}_2$), hence $B(X)$ is a circulant graph of order $2k$. Observe that the mapping $k_L : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$ defined by $k_L(x) = x + k$ is the automorphism of $B(X)$ that corresponds to the map $\tau$ from Lemma 2.2. Hence, for circulants of odd order we have the following result.

**Lemma 2.3.** Let $X = \text{Cay}(\mathbb{Z}_k, S)$ be a connected circulant of odd order $k$. Then $X$ is stable if and only if the permutation $k_L$ is central in the automorphism group of $\text{Cay}(\mathbb{Z}_2, k + 2) = B(X)$.

We will now give another characterization of stable vertex-transitive graphs.

**Lemma 2.4.** Let $X$ be a connected non-bipartite vertex-transitive graph. Let $B(X)$ be the canonical double cover of $X$, and let $A = \text{Aut}(B(X))$. Then $X$ is stable if and only if $A(v, 0) = A(v, 1)$ for some $v \in V(X)$.

**Proof.** Suppose first that $X$ is stable. Then by Lemma 2.2 it follows that the map $\tau$ is central in $A$. If $\varphi \in A(v, 0)$ then we have $\varphi(v, 1) = \varphi(\tau(v, 0)) = \tau(\varphi(v, 0)) = (v, 1)$. This shows that $A(v, 0) \leq A(v, 1)$. Since $B(X)$ is vertex-transitive, all stabilizers are of the same order, hence $A(v, 0) = A(v, 1)$.

Suppose now that $A(v, 0) = A(v, 1)$ for some $v \in V(X)$. Let $w \in V(X)$ be arbitrary, let $g$ be an automorphism of $X$ that maps $v$ into $w$, and let $\beta$ be the automorphism of $B(X)$ defined by $\beta(x, i) = (g(x), i)$. Then $A(w, 0) = \beta A(v, 0) \beta^{-1} = \beta A(v, 1) \beta^{-1} = A(w, 1)$.

Let $\alpha$ be an automorphism of $B(X)$. We will prove that $\alpha$ and $\tau$ commute. Without loss of generality, we may assume that $\alpha$ fixes the color classes of $B(X)$. Suppose that $\alpha(w, 0) = (w, 0)$, for some $w \in V(X)$. Let $h$ be an automorphism of $X$ that maps $w$ to $w$, and let $\gamma$ be the automorphism of $B(X)$ defined by $\gamma(x, i) = (h(x), i)$. Observe that $\gamma \alpha \in A(w, 0) = A(w, 1)$, hence $\gamma(\alpha(w, 1)) = (w, 1)$. It follows that $\alpha(w, 1) = \gamma^{-1}(w, 1) = (u, 1)$. It is now straightforward to conclude that $\alpha \tau(w, 0) = \tau \alpha(w, 0)$. Since $w \in V(X)$ was arbitrary, it follows that $\alpha$ and $\tau$ commute. We conclude that $\tau$ is central in $\text{Aut}(B(X))$. By Lemma 2.2 it follows that $X$ is stable.

For circulants, the above lemma implies the following.

**Lemma 2.5.** Let $X$ be a connected circulant of odd order $k$, let $B(X)$ be its canonical double cover and $A = \text{Aut}(B(X))$. Then $X$ is stable if and only if $A_0 = A_k$.

A Cayley graph $\Gamma = \text{Cay}(G, S)$ is said to be normal if $G_L$ is a normal subgroup of $\text{Aut}(\Gamma)$, or equivalently if $\text{Aut}(\Gamma)_0 = \text{Aut}(G, S)$, where $\text{Aut}(G, S) = \{ \varphi \in \text{Aut}(G) \mid \varphi(S) = S \}$. 

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Lemma 2.6. Let $G$ be an abelian group of odd order, and let $X = \text{Cay}(G, S)$ be a connected Cayley graph on $G$. If $B(X) = \text{Cay}(G \times \mathbb{Z}_2, S')$ is a normal Cayley graph then $X$ is stable.

Proof. Since $B(X)$ is normal Cayley graph, it follows that each element of $\text{Aut}(B(X))$ is a composition of some element of $(G \times \mathbb{Z}_2)_L$ with some element of $\text{Aut}(G \times \mathbb{Z}_2)$. Let $\tau$ be the automorphism of $B(X)$ defined by $\tau(x, i) = (x, i + 1)$. Let $t$ be the unique element of order 2 in $G \times \mathbb{Z}_2$. Observe that $\tau = t_L$. Since $G$ is abelian, it follows that each element of $(G \times \mathbb{Z}_2)_L$ commutes with $\tau$. Let $\varphi \in \text{Aut}(G \times \mathbb{Z}_2)$. Then $\varphi(t) = t$, since group automorphisms preserve the order of elements, and $t$ is the unique element of order 2 in $G \times \mathbb{Z}_2$. It follows that $(\varphi t_L)(x) = \varphi(tx) = t\varphi(x) = (t_L\varphi)(x)$. This shows that $\tau = t_L$ commutes with every element of $\text{Aut}(G)$. The result now follows by Lemma 2.3.

The following lemma tells that a normal circulant of even order not divisible by four has a unique regular cyclic subgroup.

Lemma 2.7. [24, Theorem 5.2.2] Let $k$ be an odd positive integer, and let $X = \text{Cay}(\mathbb{Z}_{2k}, S)$. Let $A = \text{Aut}(X)$ admitting a normal cyclic regular subgroup $H$. Then $H$ is the unique regular cyclic subgroup contained in $A$.

The wreath (lexicographic) product $\Sigma \wr \Gamma$ of a graph $\Gamma$ by a graph $\Sigma$ is the graph with vertex set $V(\Sigma) \times V(\Gamma)$ such that $\{(u_1, v_2), (v_1, v_2)\}$ is an edge if and only if either $\{u_1, v_1\} \in E(\Sigma)$, or $u_1 = v_1$ and $\{u_2, v_2\} \in E(\Gamma)$. Observe that $\Sigma \wr \Gamma$ is the graph obtained by substituting a copy of $\Gamma$ for each vertex of $\Gamma$.

The deleted wreath (deleted lexicographic) product of a graph $\Sigma$ and $\overline{K_d}$, denoted by $\Sigma \wr_d \overline{K_d}$, is the graph with vertex set $V(\Sigma) \times \mathbb{Z}_d$, such that $\{(u_1, i), (v_1, j)\}$ is an edge if and only if $\{u_1, v_1\} \in E(\Sigma)$ and $i \neq j$. Observe that $\Sigma \wr_d \overline{K_d}$ can be obtained from $\Sigma \wr \overline{K_d}$ by removing $d$ disjoint copies of $\Sigma$. Observe that the canonical double cover of a graph $X$ is isomorphic to $X \wr_d \overline{K_2}$ (see [19, Example 2.1]). The following result gives a characterization of all arc-transitive circulants.

Lemma 2.8. [9, 12] Let $\Gamma$ be a connected arc-transitive circulant of order $n$. Then one of the following holds:

(i) $\Gamma \cong K_n$;

(ii) $\Gamma = \Sigma \wr \overline{K_d}$, where $n = md$, $m, d > 1$ and $\Sigma$ is a connected arc-transitive circulant of order $m$;

(iii) $\Gamma = \Sigma \wr_d \overline{K_d}$ where $n = md$, $d > 3$, $\text{gcd}(d, m) = 1$ and $\Sigma$ is a connected arc-transitive circulant of order $m$;

(iv) $\Gamma$ is a normal circulant.

The following result is a direct consequence of [2, Theorem 5.3].

Lemma 2.9. Let $d > 3$ be an integer, let $\Sigma$ be an irreducible vertex-transitive graph whose order is not divisible by $d$, and let $\Gamma = \Sigma \wr_d \overline{K_d}$. Then $\text{Aut}(\Gamma) \cong \text{Aut}(\Sigma) \times S_d$.

Proof. Since $\Sigma$ is vertex-transitive, and $\text{Aut}(\Gamma)$ contains a subgroup isomorphic to $\text{Aut}(\Sigma) \times S_d$, it follows that $\Gamma$ is vertex-transitive. Since $\Sigma$ is reducible, by [2, Theorem 5.3] it follows that $\text{Aut}(\Gamma) \not\cong \text{Aut}(\Sigma) \times S_d$ if and only if the conditions (i) – (iii) of [2, Theorem 5.3] hold. However, since $|V(\Sigma)|$ is not divisible by $d$, it follows that condition (iii) of [2, Theorem 5.3] doesn’t hold. We conclude that $\text{Aut}(\Gamma) \cong \text{Aut}(\Sigma) \times S_d$. 


Lemma 3.1. Let $d > 3$ it follows that $\Sigma$ is also bipartite, hence order of $\Sigma$ is even.

Proof. By Lemma 2.10 it follows that $\varphi(\langle K \rangle) = \langle K \rangle$ for every $\varphi \in A_1$. Moreover, if $K$ is inverse closed, then $\varphi$ induces an automorphism of $\text{Cay}(\langle K \rangle, K)$.

The following lemma gives a partial generalization of [19, Theorem 1.6], where it is proved that there is no arc-transitive nontrivially unstable circulant. It is easy to see that $\varphi(M) = M$ for every $\varphi \in A_1$. Observe that $(xL)^{-1} \omega xL \in A_1$. By the assumption on $A_1$ and $M$ it follows that $(xL)^{-1} \omega xL(M) = M$. Therefore, $\omega(xM) = xM$ for every $x \in G$ and every $\omega \in A_x$.

We claim that $\varphi(K^t) = K^t$ for every positive integer $t$. The proof is by induction on $t$. If $t = 1$, the claim follows from the hypothesis. Suppose that $\varphi(K^t) = K^t$. Let $\varphi \in A_1$ and let $k_1 \in K$ be arbitrary. Observe that $\langle (k_1k_2^{-1})L, \varphi \rangle \in A_{k_1}$. Applying the first part of the proof with $M = K^t$ and $x = k_1$, it follows that $\langle (k_1k_2^{-1})L, \varphi(k_1K^t) \rangle = k_1K^t$. This implies that $\varphi(k_1K^t) = k_2K^t = \varphi(k_1)K^t \subseteq K^{t+1}$. Since this is true for every $k_1 \in K$, it follows that $\varphi(K^{t+1}) = K^{t+1}$.

Since $K = K \cup K^2 \cup \ldots \cup K^t$ for some positive integer $t$, it follows that $\varphi(\langle K \rangle) = \langle K \rangle$. Moreover, let $e$ be an arbitrary edge of $\text{Cay}(\langle K \rangle, K)$. Then $e = \{k_1, k_1k_2\}$ for some $k_1, k_2 \in K$. As we proved that $\varphi(k_1K^t) = \varphi(k_1)K^t$ for every positive integer $t$, it follows that $\varphi(k_1K) = \varphi(k_1)K$. Since $\varphi(k_1k_2) \in \varphi(k_1K) = \varphi(k_1)K$ we conclude that $\varphi(k_1k_2) = \varphi(k_1)k$ for some $k \in K$. This shows that $\varphi(e)$ is an edge of $\text{Cay}(\langle K \rangle, K)$, hence $\varphi$ induces an automorphism of $\text{Cay}(\langle K \rangle, K)$. □

3 Main result

The following lemma gives a partial generalization of [19, Theorem 1.6], where it is proved that there is no arc-transitive nontrivially unstable circulant. It is easy to see that $B(X)$ is arc-transitive if $X$ is arc-transitive.

Lemma 3.1. Let $X$ be a nontrivially unstable circulant of odd order $m$. Then $B(X)$ is not arc-transitive.

Proof. Suppose that $B(X)$ is an arc-transitive circulant. Since $B(X)$ is also a connected circulant, by Lemma 2.8 it follows that $B(X)$ is a complete graph, normal circulant, wreath product or a deleted wreath product. If $B(X)$ is a complete graph, then it must be isomorphic to $K_1$ or $K_2$, since they are the only bipartite complete graphs. However, it is easy to see that none of them is a canonical double cover of some graph. Lemma 2.9 implies that $B(X)$ is not a normal circulant. If $B(X)$ is a wreath product, then $B(X)$ is reducible, hence by Lemma 2.10 it follows that $X$ is also reducible, contrary to the assumption that $X$ is nontrivially unstable. Therefore, we may assume that $B(X)$ is the deleted wreath product. Suppose that $B(X) \cong \Sigma \wr \overline{K}_d$, where $2m = td, d > 3$, $\gcd(d, t) = 1$ and $\Sigma$ is a connected arc-transitive circulant of order $t$. Since $B(X)$ is bipartite and $d > 3$ it follows that $\Sigma$ is also bipartite, hence order of $\Sigma$ is even.

If $\Sigma$ is reducible, then $\Sigma \cong \Sigma_1 \wr \overline{K}_d$, where $\Sigma_1$ is an irreducible circulant. Then by [2, Proposition 4.5], it follows that $B(X) \cong \Sigma_1 \wr \overline{K}_d \cong (\Sigma_1 \wr \overline{K}_d) \wr \overline{K}_d$, implying that $B(X)$ is reducible. By Lemma 2.10 it follows that $X$ is also reducible, a contradiction.

Suppose now that $\Sigma$ is irreducible. Then by Lemma 2.9 it follows that $\text{Aut}(B(X)) \cong \text{Aut}(\Sigma) \times S_d$. Recall that $\Sigma$ is a connected arc-transitive bipartite circulant of even order. If $\Sigma$ is a wreath
product, then again $B(X)$ is a wreath product, hence $B(X)$ is reducible, and consequently also $X$ is reducible, contrary to the assumption that $X$ is nontrivially unstable. We conclude that $\Sigma$ is a normal circulant, or $\Sigma \cong \Sigma_1 \wr \overline{K_{d_1}}$, where $\Sigma_1$ is an arc-transitive circulant. Applying the same arguments for $\Sigma_1$, we conclude that $\Sigma_1$ is a normal circulant, or a deleted wreath product. As $\Gamma$ is finite, this process has to terminate, so eventually we will get $B(X) = (\Sigma_1 \wr \overline{K_{d_1}}) \wr \ldots \wr \overline{K_{d}}$, where $\Sigma_i$ is a normal circulant of even order and $\text{Aut}(B(X)) \cong \text{Aut}(\Sigma_i) \times S_{d_1} \times \ldots \times S_d$.

Observe that $S_{d_1} \times \ldots \times S_d$ is a normal subgroup of $\text{Aut}(B(X))$, hence its orbits form a system of imprimitivity for $\text{Aut}(B(X))$. Let $P$ denote the set of orbits of $S_{d_1} \times \ldots \times S_d$ on $V(B(X))$. Observe that the quotient graph of $B(X)$ with respect to $P$ is isomorphic to $\Sigma_i$. The group $(\mathbb{Z}_{2m})_L$ projects into cyclic regular subgroup of $\text{Aut}(\Sigma_i)$. For $g \in \text{Aut}(B(X))$, let $g/P$ denote the permutation induced by the action of $g$ on $P$. Since $\Sigma_i$ is a normal circulant, by Lemma 2.7 it follows that $(\mathbb{Z}_{2m})_L/P$ is a normal cyclic regular subgroup of $\text{Aut}(\Sigma_i)$, hence $m_L/P$ is central in $\text{Aut}(\Sigma_i)$. If follows that $m_L$ is central in $\text{Aut}(\Sigma_i) \times S_{d_1} \times \ldots \times S_d$. The result now follows by Lemma 2.3.

In the following lemma, we consider graphs that can be realised as a canonical double cover of some connected arc-transitive circulant of odd order, and derive certain important properties of their automorphism groups.

**Lemma 3.2.** Let $m$ be an odd positive integer, and let $\Gamma$ be a connected bipartite arc-transitive circulant of order $2m$ and even valency, and let $A = \text{Aut}(\Gamma)$. Then one of the following holds:

(i) $A_0 = A_m$, or

(ii) $\Gamma \cong \Gamma_1 \wr \overline{K_d}$ where $\Gamma_1$ is an irreducible arc-transitive circulant of even order $2m_1$ and $\text{Aut}(\Gamma_1)_0 = \text{Aut}(\Gamma_1)_{m_1}$.

**Proof.** Since $\Gamma$ is a circulant of order $2m$ and has even valency, it follows that $\Gamma \cong \text{Cay}(\mathbb{Z}_{2m}, S)$ where $S$ does not contain element $m$. This implies that the automorphism $m_L : x \mapsto m + x$ is a strongly switching involution of $\Gamma$. By Lemma 2.4 it follows that $\Gamma$ is the canonical double cover of the graph $X$ obtained as the quotient graph of $\Gamma$ with respect to the orbits of $m_L$. Observe that $X$ is a connected and non-bipartite circulant of order $m$, since $\Gamma$ is a connected circulant of order $2m$.

If $X$ is a stable graph, then by Lemma 2.5 it follows that $\Gamma$ satisfies condition (i), and we are done. As $\Gamma$ is arc-transitive, by Lemma 3.1 it follows that $X$ is not nontrivially unstable. Therefore, $X$ is trivially unstable, and since it is connected and non-bipartite, it follows that $X$ is reducible. We conclude that $X \cong \Sigma \wr \overline{K_d}$, where $\Sigma$ is a connected irreducible circulant of order $m_1$, with $m_1$ being odd.

Observe that $\Gamma \cong X \times K_2 \cong (\Sigma \wr \overline{K_d}) \times K_2 \cong (\Sigma \times K_2) \wr \overline{K_d}$. Let $\Gamma_1 = \Sigma \times K_2$. Since $\Gamma$ is an arc-transitive circulant, it follows that $\Gamma_1$ is an arc-transitive circulant (see 13, Remark 1.2]). Since $\Gamma_1$ is a canonical double cover of $\Sigma$, by Lemma 3.1 it follows that $\Sigma$ is not non-trivially unstable. Recall that $\Sigma$ is irreducible, connected and non-bipartite. We conclude that $\Sigma$ is stable, hence by Lemma 2.3 it follows that $\text{Aut}(\Gamma_1)_0 = \text{Aut}(\Gamma_1)_{m_1}$.

We are now ready to prove the main result of this paper. We will show that there is no non-trivially unstable circulant of odd order.

**Theorem 1.2 (restated).** Let $X$ be a connected irreducible circulant of odd order and let $B(X)$ be its canonical double cover. Then $\text{Aut}(B(X)) \cong \text{Aut}(X) \times \mathbb{Z}_2$. 

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Proof. Let \( X = Cay(\mathbb{Z}_m, S) \) be a connected irreducible circulant of odd order \( m \). If \( X \) is stable, the result follows by the definition. It is clear that \( X \) is non-bipartite, as it is of odd order. Hence, we may assume that \( X \) is a nontrivially unstable. We have that \( B(X) = Cay(\mathbb{Z}_{2m}, S') \) is a circulant of order \( 2m \), where \( S' = m + 2S \). Let \( A = \text{Aut}(B(X)) \). If \( A_0 \) is transitive on \( S' \), then \( B(X) \) is arc-transitive, and the result follows by Lemma \ref{lem:trivial}. Let \( S_1, \ldots, S_k \) be the orbits of \( A_0 \) on \( S' \). Observe that \( S_i = -S_i \) (since mapping \( i : x \mapsto -x \) is contained in \( A_0 \)), and \( m \notin S_i \), for every \( i \in \{1, \ldots, k\} \). Let \( \Gamma_i = Cay(\langle S_i \rangle, S_i) \). Observe that \( \langle S_i \rangle \) is a subgroup of \( \mathbb{Z}_{2m} \) of even order, hence it contains the element of order 2, that is \( m \in \langle S_i \rangle \) for every \( i \in \{1, \ldots, k\} \). By Lemma \ref{lem:trivial} every element of \( A_0 \) fixes \( \Gamma_i \), hence it follows that every element of \( A_0 \) induces an automorphism of \( \Gamma_i \). As \( A_0 \) acts transitively on \( S_i \) it follows that \( \Gamma_i \) is arc-transitive. Therefore, \( \Gamma_i \) is an arc-transitive circulant of even order (not divisible by 4) and even valency. Hence we can apply Lemma \ref{lem:trivial} to each of the graphs \( \Gamma_i \). If for some \( i \) we have that \( \Gamma_i \) satisfies condition 1 of Lemma \ref{lem:trivial} it follows that every automorphism of \( B(X) \) that fixes 0, must also fix \( m \), hence \( A_0 = A_m \) and by Lemma \ref{lem:trivial} it follows that \( X \) is stable, a contradiction.

We can now assume that \( \Gamma_i \cong \Sigma_i \wr K_d, \) where \( \Sigma_i \) is an irreducible arc-transitive circulant of even order satisfying condition (i) from Lemma \ref{lem:trivial}. Let \( R \) be the equivalence relation of “having the same neighbourhood” defined on \( V(\Gamma_i) \). The equivalence classes of this relation are all of size \( d_i \), and form a system of imprimitivity for \( \text{Aut}(\Gamma_i) \). Let \( H_i \) be the kernel of the action of \( \langle S_i \rangle \) (which is a regular cyclic subgroup of \( \text{Aut}(\Gamma_i) \)) on the partition induced by \( R \). The permutation group \( \langle S_i \rangle_{\Gamma_i}/H_i \) induced by the action of \( \langle S_i \rangle_{\Gamma_i} \) on the classes of \( R \) is a cyclic regular group. Since \( H_i \) is semiregular on \( V(\Gamma_i) \), it follows that the equivalence classes of \( R \) coincide with the orbits of \( H_i \). It follows that \( S_i \) is a union of cosets of the subgroup \( H_i \) of \( \langle S_i \rangle \). Observe that the element of order 2 in the quotient group \( \langle S_i \rangle/H_i \) is \( m + H_i \).

Recall that \( \Sigma_i \) has the property that the stabilizer of the identity and the element of order 2 in the cyclic regular subgroup of \( \text{Aut}(\Sigma_i) \) are equal, which implies that every automorphism of \( \Gamma_i \) that fixes 0, fixes setwise coset \( m + H_i \). As every automorphism of \( \Gamma \) that fixes 0 induces automorphism of \( \Gamma_i \), it follows that every automorphism of \( \Gamma \) that fixes 0 fixes setwise \( m + H_i \). If \( \text{GCD}(d_1, \ldots, d_k) = d > 1 \), then \( H = H_1 \cap \ldots \cap H_k \) has order \( d \). It follows that \( S' \) is a union of cosets of \( H \), hence \( B(X) \) is a wreath product with \( K_d \). This shows that \( B(X) \) is reducible, and by Lemma \ref{lem:trivial} it follows that \( X \) is also reducible, contrary to the assumption that \( X \) is nontrivially unstable circulant.

If \( \text{GCD}(d_1, \ldots, d_k) = 1 \), then it follows that \( H_1 \cap \ldots \cap H_k = \{0\} \). As observed above, every automorphism of \( \Gamma \) fixes setwise each of the sets \( m + H_i \), for \( i \in \{1, \ldots, k\} \), hence it also fixes their intersection. Since \( (m + H_1) \cap \ldots \cap (m + H_k) = \{m\} \), by Lemma \ref{lem:trivial} it follows that \( X \) is stable, a contradiction. This finishes the proof. \( \square \)

For further research, we propose the following problem.

**Problem 3.3.** Does there exist a nontrivially unstable Cayley graph on an Abelian group of odd order?

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