Entanglement swapping between discrete and continuous variables

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We experimentally realize “hybrid” entanglement swapping between discrete-variable (DV) and continuous-variable (CV) optical systems. DV two-mode entanglement as obtainable from a single photon split at a beam splitter is robustly transferred by means of efficient CV entanglement and operations, using sources of squeezed light and homodyne detections. The DV entanglement after the swapping is verified without post-selection by the logarithmic negativity of up to 0.28±0.01. Furthermore, our analysis shows that the optimally transferred state can be post-selected into a highly entangled state that violates a Clauser-Horne-Shimony-Holt inequality by more than four standard deviations, and thus it may serve as resources for quantum teleportation and quantum cryptography.

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Quantum entanglement can be created between two distant quantum systems that have never directly interacted. This effect, called entanglement swapping [1–3], is a building block for quantum communication and computation [3–12]. It was originally proposed and demonstrated for discrete-variable (DV) optical systems [1–4]. The protocol starts with two entangled pairs, A-B and C-D, each represented either by twin photons or by a single photon split into two distinct modes [Fig. 1(a)]. A joint projective measurement of B and C onto one of the four two-qubit Bell states then leads to an entangled state for A and D, even though A and D never directly interact with each other. Entanglement swapping can also be interpreted as the transfer of one half of an entangled state, either from B to D or from C to A, by quantum teleportation [12]. It is a key element for quantum networking [5], quantum computing [6–10], and especially long-distance quantum communication by means of quantum repeaters [11, 12]. However, in this DV setting solely based upon single photons, due to the heralded conditional entangled-state generation and the probabilistic linear-optics Bell-state measurement (BSM), successful entanglement swapping events occur very rarely. As a result, in a quantum repeater for example, long-distance entangled-pair creation rates would be correspondingly low and requirements on the coherence times of the local quantum memories at each repeater station impractically high. In addition, the observation and verification of the final entanglement between A and D in the DV scheme typically requires post-selection.

Entanglement swapping was later extended to continuous-variable (CV) systems, where the pairs A-B and C-D each correspond to the two modes of a two-mode squeezed, quadrature-entangled state χ [3, 6] [Fig. 1(b)]. Since such entangled states are available on demand and a linear-optics BSM in the quadrature basis can be performed without failure, entanglement can be swapped deterministically and verified without post-selection [4, 5]. However, due to the finite squeezing of both initial entanglement sources, the final entanglement after swapping in the CV scheme is inevitably degraded by excess noise. In fact, the entanglement drops exponentially [11, 12] and in practice, CV entanglement swapping can always be replaced by a direct transmission through a lossy channel [10]. Moreover, purification techniques for this type of degraded entanglement are not so advanced at present [11, 12].

Our scheme combines the best features of the above two approaches, making use of both DV and CV entanglement at the same time [Fig. 1(c)]. By means of CV teleportation [13, 20], using squeezed-state entanglement, homodyne-based BSM, and feedforward by phase-space displacement, DV entanglement is transferred from A-B to A-D. Once the initial single-photon entanglement is conditionally prepared in modes A and B, all the remaining steps of our scheme are unconditional, achieving a highly efficient transfer of the DV entanglement. Be-

FIG. 1: Schematic of entanglement swapping. (a) DV entanglement swapping. (b) CV entanglement swapping. (c) Hybrid entanglement swapping.
cause optical CV quantum teleportation runs in a deterministic fashion (as opposed to optical DV quantum teleportation) and DV entanglement is robust against loss (as opposed to CV entanglement), entanglement is efficiently and reliably transferred only in this “hybrid” setting. Furthermore, as will be explained below, a maximally entangled state can be obtained after the swapping through post-selection, even though only finitely squeezed resources are used.

In this hybrid setting, the DV entanglement can be transferred for any nonzero squeezing, as is theoretically shown in [21, 22]. Our setup (Fig. 2) uses the DV entanglement in the form of a photon split at a beam splitter with reflectivity \( R \), described by \( |\psi\rangle_{AB} = \sqrt{1-R} |1\rangle_A |0\rangle_B + \sqrt{R} |0\rangle_A |1\rangle_B \) in the two-mode photon number basis. This state is maximally entangled when \( R = 0.5 \). In contrast, the CV entanglement is a two-mode squeezed state, \( \sqrt{1-g_{opt}} \sum_{n=0}^{\infty} g_{opt}^n |n\rangle_C |n\rangle_D \) with \( g_{opt} \equiv \tanh r \), where \( r \) is the squeezing parameter. Though this state is non-maximally entangled for finite \( r \), the DV entanglement can be transferred for any \( r > 0 \) by tuning the feedforward gain to \( g_{opt} \), when the final state of \( D \) is an imperfect version of the initial state of \( B \) attenuated by a factor \( 1 - g_{opt}^2 \) [21, 22]. At this gain, the initial entangled state at \( R = 0.5 \) is swapped and transformed according to

\[
\hat{\rho}_{AB} \equiv |\psi\rangle_{AB} \langle \psi| \\
\rightarrow \hat{\rho}_{AD} = \frac{1+g_{opt}^2}{2} |\psi\rangle_{AD} \langle \psi| + \frac{1-g_{opt}^2}{2} |0\rangle_A |0\rangle_B |0\rangle_D , \tag{1}
\]

where \( |\psi\rangle_{AD} = (|1\rangle_A |0\rangle_D + g_{opt} |0\rangle_A |1\rangle_D ) / (1 + g_{opt}^2)^{1/2} \). Here, the initial maximally entangled state \( |\psi\rangle_{AB} \) is converted into a non-maximally entangled state \( |\psi\rangle_{AD} \) mixed with an extra two-mode vacuum term. When \( g_{opt} > 0 \), \( \hat{\rho}_{AD} \) violates the positivity after partial transposition; this shows that DV entanglement remains present after teleportation for any \( r > 0 \) by optimal gain tuning [23]. Since no additional photons are created in \( \hat{\rho}_{AD} \), it can be used for teleportation [24], swapping [3], and purification protocols [25].

The present experimental setup (Fig. 3) is an extended version of the setup in [20]. We use a continuous-wave titanium-sapphire laser at 860 nm. A heralded single photon with a half-width at maximum (HWHM) of 6.2 MHz is created from a non-degenerate optical parametric oscillator (OPO) at a rate of 7000 s\(^{-1}\) [26]. The photon when incident on a beam splitter of reflectivity \( R = 0.50 \) or \( R = 0.67 \) yields a DV entangled state \( |\psi\rangle_{AB} \). The CV entangled state is deterministically generated by combining at a beam splitter two squeezed vacua each produced from a degenerate OPO with a HWHM of 12 MHz. A CV BSM is performed jointly on the two corresponding halves of these two entangled states by combining them at a 50:50 beam splitter and then measuring the orthogonal quadratures of the output modes by homodyne detection. The measurement results are multiplied by a factor \( g \) and used for displacing the other half of the CV entangled state. Tomographic reconstruction of the initial and final states, \( \hat{\rho}_{AB} \) and \( \hat{\rho}_{AD} \), are performed by two homodyne measurements with local oscillators’ phases \( \theta_1 \) and \( \theta_2 \). For these particular states, the sum \( \theta_1 + \theta_2 \) does not affect the homodyne statistics in theory [22]. Thus, we first confirm the sum independence of the homodyne statistics and then scan only the relative phase \( \theta_1 - \theta_2 \) for tomography [27]. For every state, 100,000 sets of quadrature and phase values are acquired and used for a maximum likelihood algorithm without compensation of the measurement inefficiency [28].

The experimental density matrix of the initial DV entangled state, \( \hat{\rho}_{AB} \), at \( R = 0.5 \) is shown in Fig. 3a). This state includes 80.6 ± 0.3% of the ideal \( |\psi\rangle_{AB} \), 18.3 ± 0.3% of vacuum, and 1.1 ± 0.2% of multi-photon terms. The density matrices of the swapped states, \( \hat{\rho}_{AD} \), at \( r = 0.71 \), \( g = 0.63 \) \( (g_{opt} = 0.61) \) and \( r = 1.01 \), \( g = 0.79 \) \( (g_{opt} = 0.77) \) are also shown in Figs. 3b) and (c), respectively [24]. It can be seen that only one mode of the entangled state is attenuated by a factor close to \( 1 - g_{opt}^2 \), but the off-diagonal elements \( (0,1)(1,0) \) and \( (1,0)(0,1) \) are still preserved, indicating that the DV entanglement remains present after the swapping. The amount of entanglement can be assessed by the logarithmic negativity, \( E(\hat{\rho}) = \log_2 ||\hat{\rho}^\Gamma|| \), where \( ||\hat{\rho}|| \equiv \text{Tr}(\hat{\rho}^\Gamma) \) is the trace norm and \( \Gamma \) denotes partial transposition with regards to one of the subsystems [21]. The gain dependence of \( E(\hat{\rho}_{AD}) \) at \( r = 0 \) is also plotted in Fig. 3d). The positive values of \( E(\hat{\rho}_{AD}) \) at \( r > 0 \) clearly demonstrate the successful entanglement swapping. As expected, no entanglement is observed for \( r = 0 \). These results also confirm the quantum nature of CV quantum teleportation [19] in transfer-
ring DV systems [20]. The transferred entanglement from $A-B$ to $A-D$ $[E(\hat{\rho}_{AD}) = 0.28 \pm 0.01$ at the maximum] is much greater than in the previous swapping experiments for discrete variables [2-4], which post-selectively transferred the initial entanglement with a probability less than 1%, corresponding to $E(\hat{\rho}_{AD}) < 0.01$ without post-selection. We also performed the experiment for $R = 0.67$ and observed $E(\hat{\rho}_{AD}) > 0$ for $r = 0.71$ and $r = 1.01$ [31].

The final state, $\hat{\rho}_{AD}$, is contaminated with an extra vacuum due to the finite squeezing $r$; however, it can be, in principle, purified to a maximally entangled state post-selectively [12]. Remarkably, this also works for any $r > 0$, even though the pure-state component in $\hat{\rho}_{AD}$ is itself only a non-maximally entangled state for finite squeezing (as opposed to the maximally entangled Bell-state fractions in the scheme of [12]). The purification is achieved by first preparing two copies of $\hat{\rho}_{AD}$, written as $\hat{\rho}_{A_1D_1} \otimes \hat{\rho}_{A_2D_2}$, and then projecting this state onto the subspace with one photon in each location, corresponding to $\{|1\}_{A_1}|0\rangle_{A_2}, |0\rangle_{A_1}|1\rangle_{A_2}\rangle \otimes \{|1\rangle_{D_1}|0\rangle_{D_2}, |0\rangle_{D_1}|1\rangle_{D_2}\rangle \equiv \{|A_1\rangle, |A_2\rangle\rangle \otimes \{|D_1\rangle, |D_2\rangle\rangle$. When $\hat{\rho}_{AD}$ has the form of Eq. (1), this projection leads to a maximally entangled state, $(|A_1\rangle |D_2\rangle + |A_2\rangle |D_1\rangle)/\sqrt{2}$, regardless of the squeezing level $r > 0$. In other words, in principle, the hybrid setting allows for transferring maximally entangled states by means of finitely squeezed resources with a finite success probability, which in our experiment is already an order of magnitude larger compared to Refs. [2-4] (see below) and can be further increased for higher squeezing.

We perform this purification protocol by analytically extracting the corresponding subspace from two copies of the experimental $\hat{\rho}_{AD}$ [31]. The renormalized density matrices after post-selection, $\hat{\rho}_{AD}^{ps}$, calculated from $\hat{\rho}_{AD}$ in Figs. 3(b) and (c), are shown in Figs. 4(a) and (b), respectively. The probability for the state being projected onto $\hat{\rho}_{AD}^{ps}$ is $12.5 \pm 0.2\%$ and $16.0 \pm 0.3\%$, respectively, which is calculated as the trace of the post-selected subspace. It can be seen that both states are almost purified to the maximally entangled state $(|A_1\rangle |D_2\rangle + |A_2\rangle |D_1\rangle)/\sqrt{2}$. For the ideal $\hat{\rho}_{AD}$ in Eq. (1), the $|A_1D_1\rangle, |A_1D_1\rangle$, and $|A_2D_2\rangle, |A_2D_2\rangle$ elements should be zero after post-selection. The small contributions of these terms in Fig. 4(a) and (b) originate from the multi-photon term $|1\rangle_1|1\rangle_1$ in $\hat{\rho}_{AD}$, and this term is mainly attributed to the impurity of squeezing. The values of the logarithmic negativity, $E(\hat{\rho}_{AD}^{ps}) = 0.67 \pm 0.02$ for Fig. 4(a) and $E(\hat{\rho}_{AD}^{ps}) = 0.75 \pm 0.02$ for Fig. 4(b), are greater than those without post-selection, demonstrating the purification of the entanglement. In addition, the post-selected state can be used for measuring violations of Bell’s inequality by the setup shown in Fig. 4(c) [12]. Our calculation shows that the post-selected state in Figs. 4(a) and (b) can, in principle, violate the Clauser-Horne-Shimony-Holt inequality [32] by the estimated $S$ parameters of $S = 2.08 \pm 0.05 > 2$ and $S = 2.21 \pm 0.05 > 2$, respectively (the latter indicates the violation by more than four standard deviations). This shows that the entanglement shared between $A$ and $D$ is of a quality high enough for further use in quantum communication protocols with post-selection, such as quantum teleportation.

![FIG. 3: Experimental results. (a)-(c) Density matrices of $\hat{\rho}_{AB}$ [(a)], $\hat{\rho}_{AD}$ at $r = 0.71$ and $g = 0.63$ [(b)], and $\hat{\rho}_{AD}$ at $r = 1.01$ and $g = 0.79$ [(c)]. The absolute value of each matrix element is plotted. (d) Gain-dependence of the logarithmic negativity for $r = 0$ (blue diamond), $r = 0.71$ (green triangle), and $r = 1.01$ (red circle). Theoretical curves are also plotted in the same colors.](image-url)
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FIG. 4: Analysis with post-selection. (a) $\rho_{AD}^{\text{ps}}$ calculated from $\rho_{AD}$ at $r = 0.71$ and $g = 0.63$ [Fig. 3(b)]. (b) $\rho_{AD}^{\text{ps}}$ calculated from $\rho_{AD}$ at $r = 1.01$ and $g = 0.79$ [Fig. 3(c)]. (c) Schematic setup for the realization of Bell’s inequality detection with post-selection. The values of the $S$ parameter estimated in the main text correspond to those values that would be measured in this setup. The setup consists of two entangled pairs $(A_1-D_1, A_2-D_2)$, phase shifters $(\theta, \phi)$, 50:50 beam splitters (BS), and photon detections (PD$_1$-PD$_4$), and quantum cryptography.[12]

In conclusion, we demonstrated a “hybrid” entanglement swapping scheme, transferring robust DV entanglement in the form of a split photon by means of efficient CV entanglement and operations. By tuning the feedforward gain of the teleporter, entanglement is reliably and efficiently transferred, and then verified unconditionally. Moreover, despite the finite squeezing of the CV entanglement resource, the DV states after the swapping can always be post-selected into highly entangled states that violate Bell’s inequality and may serve as resources for advanced quantum information protocols. These results imply many possibilities for near-future applications of “hybrid” quantum networks, where more general forms of DV entanglement may be efficiently transferred or manipulated with the help of CV techniques.

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Supplemental material:  
Entanglement swapping between discrete and continuous variables

PURIFICATION BY POST-SELECTION

Here we will show how the entangled state after the entanglement swapping,
\[ \hat{\rho}_{AD} = \sum_{k,l,m,n} \rho_{klmn} |k\rangle_A |l\rangle_D |A\rangle |D\rangle |\langle n\rangle |\langle m\rangle \], \tag{2}
written in the photon number basis, is purified by the post-selection method given in Ref. [1]. First we consider the \{0\}_A \otimes |D\rangle \otimes |1\rangle_A \otimes |D\rangle \otimes |1\rangle_A \otimes |D\rangle \} subspace of Eq. (2), which has the matrix form [2]
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & p_{01} & d & 0 & 0 & 0 \\
0 & 0 & d^* & p_{10} & 0 & 0 \\
0 & 0 & 0 & p_{11} & 0 & 0
\end{pmatrix}, \tag{3}
\]
where \(p_{ij} \geq 0\) \((i,j = 0, 1)\) and \(|d|^2 \leq p_{01}p_{10}\). The quantity \(p_{ij}\) is the probability of finding \(i\) photons in mode \(A\) and \(j\) photons in mode \(D\), and the off-diagonal element \(d\) represents the coherence between \(|0\>_A |1\>_D \) and \(|1\>_A |0\>_D \). Next we assume that two copies of the entangled state of Eq. (2) are distributed between two locations through the same swapping channels. This state can be written as \(\hat{\rho}_{A_1 D_1} \otimes \hat{\rho}_{A_2 D_2}\), from which we post-select the cases when there is only one photon in each location. This operation extracts the subspace spanned by \{|1\>_A |0\>_A \otimes |0\>_A |1\>_A \otimes \rangle \}
\{(1\>_D \otimes |0\>_D \otimes |0\>_D \otimes |1\>_D \otimes |1\>_D \} \equiv \{|A\>_1 \otimes |D\>_2 \} \otimes \{|D\>_1 \otimes |A\>_2 \}
from the full density matrix \(\hat{\rho}_{A_1 D_1} \otimes \hat{\rho}_{A_2 D_2}\). The corresponding subspace can be written as
\[
\begin{pmatrix}
p_{00} & 0 & 0 & 0 & 0 & 0 \\
0 & p_{01} & d & 0 & 0 & 0 \\
0 & 0 & d^* & p_{10} & 0 & 0 \\
0 & 0 & 0 & p_{11} & 0 & 0
\end{pmatrix}, \tag{4}
\]
in the basis \{|A_1 D_1 \rangle, |A_1 D_2 \rangle, |A_2 D_1 \rangle, |A_2 D_2 \rangle\}. The density matrix of the post-selected state, \(\hat{\rho}_{AD}^{ps}\), is obtained by renormalizing the matrix of Eq. (4). The probability that the initial state \(\hat{\rho}_{A_1 D_1} \otimes \hat{\rho}_{A_2 D_2}\) is projected onto \(\hat{\rho}_{AD}^{ps}\) is given by the trace of Eq. (4) as \(P = 2(p_{00}p_{11} + p_{01}p_{10})\). When there is no \(|1, 1\rangle\) contribution in \(\hat{\rho}_{A_1 D_1}\) and \(\hat{\rho}_{A_2 D_2}\) \((p_{11} = 0)\), the \(p_{00}p_{11}\) terms in Eq. (4) vanish. In this case, only the original elements of the \{|0\>_A \otimes |D\rangle \otimes |1\>_A \otimes |D\rangle \} subspace in Eq. (3) are post-selected and constitute the elements in the \{|A_1 D_2 \rangle, |A_2 D_1 \rangle\} subspace in Eq. (4). Especially, if \(p_{11} = 0\) and \(|d|^2 = p_{01}p_{10}\), Eq. (4) becomes a pure maximally entangled state, \(|A_1 D_2 \rangle + |A_2 D_1 \rangle\)/\(\sqrt{2}\), even when \(p_{00} > 0\) and \(p_{01} \neq p_{10}\). This means that the initial mixture of a non-maximally entangled state with the vacuum is automatically purified through this post-selection to a pure, maximally entangled state without the vacuum. In other words, the entanglement is concentrated and purified at the same time. For example, if \(\hat{\rho}_{AD}\) is written as Eq. (1) of the main text, the state can be, in principle, purified to a maximally entangled state.

BELL’S INEQUALITY DETECTION WITH POST-SELECTION

Next we derive the condition when Bell’s inequality is violated post-selectively with the setup of Fig. 4(c) in the main text. Suppose that the coincidences of photon detectors are registered only when one photon is detected at both sides. In this case, we only have to consider the post-selected elements of Eq. (4), which is renormalized to
\[
\hat{\rho}_{AD}^{ps} = \begin{pmatrix}
(1 - x)/2 & 0 & 0 & 0 & 0 & 0 \\
0 & x/2 & y/2 & 0 & 0 & 0 \\
0 & y/2 & x/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{5}
\]
where \(x = 2p_{01}p_{10}/P\) and \(y = 2|d|^2/P\). An appropriate setting of the beam splitters and phase shifters in Fig. 4(c) enables one to measure \(\rho_{AD}^{ps}\) in the rotated basis \(\{\cos \theta |A_1 \rangle + \sin \theta |A_2 \rangle, -\sin \theta |A_1 \rangle + \cos \theta |A_2 \rangle\} \otimes \{\cos \theta |D_1 \rangle + \sin \theta |D_2 \rangle, -\sin \theta |D_1 \rangle + \cos \theta |D_2 \rangle\}\). This measurement is equivalent to the measurement of \(R(\theta, \phi) = \langle A_1 D_1 | \hat{R}_\theta \hat{R}_\phi | A_2 D_2 \rangle\) in the basis \(|A_1 \rangle \otimes |D_1 \rangle \otimes |D_2 \rangle \).

This describes a \(\phi\) rotation in the \(|A_1 \rangle \otimes |A_2 \rangle\) basis and a \(\phi\) rotation in the \(|D_1 \rangle \otimes |D_2 \rangle\) basis. The probability of obtaining a coincidence between the detectors for \(|A_i\rangle\) and \(|D_j\rangle\) \((i, j = 1, 2)\) is given by
\[
P_{ij}(\theta, \phi, \theta_D) = \langle A_1 D_1 | \hat{R}_\theta \hat{R}_\phi | \hat{\rho}_{AD}^{ps} \hat{R}(\theta, \theta_D) | A_1 D_1 \rangle. \tag{7}
\]
By substituting Eqs. (5) and (6) into Eq. (7), we obtain
\[
P_{11}(\theta, \theta_D) = P_{22}(\theta, \theta_D) = \frac{1}{8} \{2 + (1 - 2x + y) \cos[2(\theta_A - \theta_D)] + (1 - 2x - y) \cos[2(\theta_A + \theta_D)]\}. \tag{8}
\]
\[
P_{12}(\theta, \theta_D) = P_{21}(\theta, \theta_D) = \frac{1}{8} \{2 - (1 - 2x + y) \cos[2(\theta_A - \theta_D)] - (1 - 2x - y) \cos[2(\theta_A + \theta_D)]\}. \tag{9}
\]
Eq. (11) to obtain negativity for the same colors.

For a detection of Bell’s inequality, we estimate a correlation function defined as

\[ E(\theta_A, \theta_D) = P_{11}(\theta_A, \theta_D) - P_{12}(\theta_A, \theta_D) - P_{21}(\theta_A, \theta_D) + P_{22}(\theta_A, \theta_D) \]

\[ = \frac{1}{2} \left( (1 - 2x + y) \cos[2(\theta_A - \theta_D)] + (1 - 2x - y) \cos[2(\theta_A + \theta_D)] \right), \quad (10) \]

and then assess the parameter

\[ S = |E(\theta_A, \theta_D) + E(\theta'_A, \theta_D) - E(\theta_A, \theta'_D) + E(\theta'_A, \theta'_D)|. \quad (11) \]

The Clauser–Horne–Shimony–Holt (CHSH) inequality implies that \( S \) should be below 2 for any local hidden variable theories. Here we set \((\theta_A, \theta'_A, \theta_D, \theta'_D) = (0, \pi/4, 3\pi/8, \pi/8)\), and then substitute Eq. (10) into Eq. (11) to obtain

\[ S = \sqrt{2} |2x + y - 1|. \quad (12) \]

This formula gives an estimated \( S \) parameter for a given experimental \( \hat{\rho}_{AD} \). \( S > 2 \) indicates that the state potentially violates the CHSH inequality post-selectively. In the ideal case of \( x = y = 1 \), Eq. (12) gives \( S = 2\sqrt{2} > 2 \), and thus the inequality can be violated.

**EXPERIMENTAL RESULTS FOR \( R = 0.67 \)**

In our experiment, the beam splitter reflectivity \( R \) for generating the discrete-variable entangled state \( |\psi\rangle_{AB} = \sqrt{1 - R} |1\>_A |0\>_B + \sqrt{R} |0\>_A |1\>_B \) is set to two values, \( R = 0.50 \) and \( R = 0.67 \). Below we show the results for \( R = 0.67 \), which are not shown in the main text.

The experimental density matrices of the initial \( \hat{\rho}_{AB} \) and final \( \hat{\rho}_{AD} \) state as well as the logarithmic negativity of the final state are summarized in Fig. 5. It can be seen from Fig. 5(a)-(c) that one mode of the initial state is attenuated by a factor \( 1 - g_{\text{opt}}^2 \) through the entanglement swapping channel, as expected from theory. Positive values of the logarithmic negativity \( E(\hat{\rho}_{AD}) \) are obtained for \( r > 0 \), as can be seen in Fig. 5(d), which verifies the successful entanglement swapping without post-selection. In the case of \( R = 0.67 \), the initial discrete-variable entanglement is an asymmetric, non-maximally entangled state, and thus the logarithmic negativity of the initial state \( E(\hat{\rho}_{AB}) = 0.64 \pm 0.01 \) is slightly below the value for \( R = 0.50 \). As a result, the logarithmic negativity of the final state is also below the value for \( R = 0.50 \) at the same \( r \) and \( g \) values.

**FIG. 5:** Experimental results for \( R = 0.67 \). (a)-(c) Density matrices of \( \hat{\rho}_{AB} \) [(a)], \( \hat{\rho}_{AD} \) at \( r = 0.71 \) and \( g = 0.63 \) [(b)], and \( \hat{\rho}_{AD} \) at \( r = 1.01 \) and \( g = 0.79 \) [(c)]. The absolute value of each matrix element is plotted. (d) Gain-dependence of the logarithmic negativity for \( r = 0 \) (blue diamond), \( r = 0.71 \) (green triangle), and \( r = 1.01 \) (red circle). Theoretical curves are also plotted in the same colors.

**FIG. 6:** Analysis with post-selection for \( R = 0.67 \). (a) \( \hat{\rho}_{AD}^{\text{ps}} \) calculated from \( \hat{\rho}_{AD} \) at \( r = 0.71 \) and \( g = 0.63 \) [Fig. 5(b)]. (b) \( \hat{\rho}_{AD}^{\text{ps}} \) calculated from \( \hat{\rho}_{AD} \) at \( r = 1.01 \) and \( g = 0.79 \) [Fig. 5(c)].
We analytically perform the post-selection from \( \hat{\rho}_{AD} \) in Figs. 5(b) and (c). Renormalized density matrices after the post-selection, \( \hat{\rho}_{AD}^{\text{ps}} \), are shown in Figs. 6(a) and (b). The probability of the state being projected onto \( \hat{\rho}_{AD}^{\text{ps}} \) is 10.3 \( \pm \) 0.2\% and 13.4 \( \pm \) 0.3\%, respectively. The values of the logarithmic negativity, \( E(\hat{\rho}_{AD}^{\text{ps}}) = 0.70 \pm 0.04 \) for Fig. 6(a) and \( E(\hat{\rho}_{AD}^{\text{ps}}) = 0.77 \pm 0.02 \) for Fig. 6(b), are greater than those without post-selection, demonstrating the purification of the entanglement. In addition, the post-selected state can violate the CHSH inequality by the estimated \( S \) parameters of \( S = 2.11 \pm 0.08 > 2 \) and \( S = 2.26 \pm 0.04 > 2 \), respectively.

These results and our analysis further support the demonstration of unconditional entanglement swapping and the possibility of purification by post-selection.

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