A result on Ulam stability impulsive fractional integro-differential equation

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Abstract. This article deals with stability results for a class of impulsive fractional integro-differential equation. The order is taken between one and two and hypothesis of the sectorial operator is utilized to represent the solution. First we prove that our problem is Asymptotically stable using well-known Gronwall’s inequality. Next we establish the generalized Ulam-Hyers-Rassias stability. Finally, an example is exhibited to help the applicability of the main result.

Keywords: Fractional integro-differential equation, Impulsive condition, asymptotic stability, Ulam-Hyers stability.

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1. Introduction

Fractional differential equation have various application in mathematics and physics to create a mathematical model of a physical event, control systems and Dynamic Engineering[3, 6].

Impulsive equations or impulsive integro-differential equations are known to make an essential subject for various mathematical analysis and comprise a huge branch of differential equations. It has several applications in real modeling for example, neural networks, scourge, optimization control, population flow, economics, designing and so on. Many authors have made essential contributions in this area using topology, functional analysis and fixed point theorems[1, 10].

In 1940, Ulam[11] originally raised the stability for the functional equations. In 1941, Hyers[5] gave answer to Ulam question in Banach Space. In 1978, Rassias[9] extended Hyers’s result. In 1993, Obloza[8] discussed Ulam type stability for differential equations. Later Wang[12] devoted to the stability for fractional and impulsive differential equations. In [4], the authors studied Ulam stability results for random impulsive differential equations. Recently in 2017, Zada[13] et al. studied Bielecki-Ulam’s stabilities of differential equations with impulses. Niazi[7] et al. studied Ulam’s stability for nonlinear fractional differential by applying Picard operator.

In spite of all these contributions to this area of study, Ulam-Hyer’s type stabilities have not been studied for second order fractional integro differential equation with impulsive condition. In this manuscript, we analyze stablility for a class of impulsive fractional integro-differential...
is sectorial type linear densely defined operator on a complex Banach space $Y$
\[ D_0^\alpha z(l) = A z(l) + D_0^{\alpha-1} G_1(l, z(l), K z(l)), \quad l \in \mathbb{S} = [0, a], \quad l \neq l_j, \]
\[ K z(l) = \int_{-\infty}^l k(l-s)h_1(s, z(s))ds, \]
\[ \Delta z(l_j) = I_j(z(l_j)), \quad j = 1, 2, \ldots, m, \tag{1.1} \]
where $1 < \alpha < 2$, $k$ satisfy $|k(l)| \leq C_k e^{-\mu l}$ for some positive constant $C_k, \mu$, $A : D(A) \subset Y \to Y$
is sectorial type linear densely defined operator on a complex Banach space $Y$, $I_j : Y \to Y$, $G_1 : \mathbb{S} \times Y \times Y \to Y$ and $h_1 : \mathbb{S} \times Y \to Y$. $\Delta z(l)|_{l=l_j} = z(l_j^+) - z(l_j^−), j = 1, 2, \ldots, m, 0 = l_0 < l_1 < \ldots < l_m$.

2. Asymptotic stability

In this section, we investigate the asymptotic stability for the model(1.1). The symbol $C(\mathbb{S}, Y)$ denotes a Banach space of all continuous functions $l : \mathbb{S} \to Y$ and $PC(\mathbb{S}, Y) = \{ z : \mathbb{S} \to Y : z$ is continuous for every $l \notin \{ l_j \}, \lim_{h \to 0} z(l_j + h) = z(l_j^+), \lim_{h \to 0} z(l_j - h) = z(l_j^-)$ exist and $z(l_j^+) = z(l_j^-) \}$. Now we give some necessary results.

**Definition 1.** The Caputo sense fractional derivative of $f \in \mathbb{S}$ of order $\alpha > 0$ is defined as
\[ D_0^\alpha f(l) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{l} (l-s)^{n-\alpha-1} f^{(n)}(s)ds, \quad n-1 < \alpha < n, \quad \forall n \in \mathbb{N}. \]

**Remark 1.** From [2, Theorem 1], an important estimate of sectorial operator is as follows:
\[ \|E_{\alpha}(l)\| \leq \frac{C(\theta, \alpha)M}{1 + |\theta|^l^n}, \quad l \geq 0. \tag{2.1} \]

**Definition 2.** A function $z \in PC(\mathbb{S}, Y)$ is said to be a mild solution of Eqn.(1.1) if
\[ z(l) = E_{\alpha}(l)z(0) + \int_{0}^{l} E_{\alpha}(l-s) G_1(s, z(s), Kz(s))ds + \sum_{0 < l_j < l} E_{\alpha}(l-l_j) I_j(z(l_j)). \tag{2.2} \]

To prove our main result we need the subsequent assumptions:

(H1) $h_1 : \mathbb{S} \times Y \to Y$. Let $L_{h_1} > 0$ be such that $\|h_1(l, z_1) - h_1(l, z_2)\| \leq L_{h_1} \|z_1 - z_2\|$ for all $(l, z_1), (l, z_2) \in \mathbb{S} \times Y$.

(H2) $I_j : Y \to Y$. Let $L_{I_j} > 0$ be such that $\|I_j(z_1) - I_j(z_2)\| \leq L_{I_j} \|z_1 - z_2\|$ for all $z_1, z_2 \in Y$.

(H3) $G_1 : \mathbb{S} \times Y \times Y \to Y$. Let $L_{G_1} > 0$ be such that $\|G_1(l, z_1, z_2) - G_1(l, z_3, z_4)\| \leq L_{G_1} \|z_1 - z_3\| + \|z_2 - z_4\|$ for all $(l, z_1, z_2), (l, z_3, z_4) \in \mathbb{S} \times Y \times Y$.

(H4) $I_j : Y \to Y$. Let $L_I > 0$ be such that $\|I_j(z_1) - I_j(z_2)\| \leq L_I \|z_1 - z_2\|$ for all $z_1, z_2 \in Y$.

(H5) $G_1 : \mathbb{S} \times Y \times Y \to Y$. Let $L_{G} > 0$ be such that $\|G_1(l, z_1, z_2) - G_1(l, z_3, z_4)\| \leq L_{G} \|z_1 - z_3\| + \|z_2 - z_4\|$ for all $(l, z_1, z_2), (l, z_3, z_4) \in \mathbb{S} \times Y \times Y$.

(H6) $h_1 : \mathbb{S} \times Y \to Y$. Let $L_h > 0$ be such that $\|h_1(l, z_1) - h_1(l, z_2)\| \leq L_h \|z_1 - z_2\|$ for all $(l, z_1), (l, z_2) \in \mathbb{S} \times Y$.

**Theorem 1.**
Assumptions (H1) - (H3) hold. Then the mild solution of Eqn.(1.1) is asymptotically stable.
Proof. By (2.2), the mild solutions $z_1(l)$ and $z_2(l)$ are

$$z_1(l) = E_\alpha(l)z_1(0) + \int_0^l E_\alpha(l-s)G_1(s, z_1(s), Kz_1(s))ds + \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(z_1(l_j))$$

$$z_2(l) = E_\alpha(l)z_2(0) + \int_0^l E_\alpha(l-s)G_1(s, z_2(s), Kz_2(s))ds + \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(z_2(l_j)).$$

Now

$$\|z_1(l) - z_2(l)\| \leq \|E_\alpha(l)\|\|z_1(0) - z_2(0)\|$$

$$+ \int_0^l \|E_\alpha(l-s)\|\|G_1(s, z_1(s), Kz_1(s)) - G_1(s, z_2(s), Kz_2(s))\|ds$$

$$+ \sum_{0<l_j<l} \|E_\alpha(l-l_j)\|\|I_j(z_1(l_j)) - I_j(z_2(l_j))\|$$

$$\leq \frac{CM}{1 + |\omega|(l)^\alpha}\|z_1(0) - z_2(0)\| + \int_0^l \frac{CMLG_1}{1 + |\omega|(l-s)^\alpha}\|z_1(s) - z_2(s)\|ds$$

$$+ \|Kz_1(s) - Kz_2(s)\|ds + \sum_{0<l_j<l} \frac{CML^*_j}{1 + |\omega|(l-l_j)^\alpha}\|(z_1(l_j)) - (z_2(l_j))\|$$

$$\leq \frac{CM}{1 + |\omega|(l)^\alpha}\|z_1(0) - z_2(0)\| + \int_0^l \frac{CMLG_1}{1 + |\omega|(l-s)^\alpha}\|z_1(s) - z_2(s)\|$$

$$+ C_k \int_{-\infty}^l e^{-\mu(l-u)}L_{h_1}^s \|z_1(u) - z_2(u)\|du|ds + \sum_{0<l_j<l} \frac{CML^*_j}{1 + |\omega|(l-l_j)^\alpha}\|(z_1(l_j)) - (z_2(l_j))\|.$$
3. Generalized Ulam-Hyers-Rassias stability

In this section we establish the Ulam-Hyers’ stability for the model (1.1). Let $\varphi : \mathbb{R} \to R^+$ and $\epsilon > 0$ be a continuous function. Assume the following inequalities hold

\begin{align}
\left| D_t^\alpha \varphi_1(z(t)) - \varphi_2(l, z(l)) \right| &\leq \epsilon, \quad l \in \mathbb{R}, \\
\left| \triangle z(l_j) - I_j(z(l_j)) \right| &\leq \epsilon, \quad j = 1, 2, ..., m. \tag{3.1}
\end{align}

\begin{align}
\left| D_t^\alpha \varphi_1(z(l)) - \varphi_2(l, z(l)) \right| &\leq \varphi(l), \quad l \in \mathbb{R}, \\
\left| \triangle z(l_j) - I_j(z(l_j)) \right| &\leq \varphi(l), \quad j = 1, 2, ..., m. \tag{3.2}
\end{align}

\begin{align}
\left| D_t^\alpha \varphi_1(z(l)) - \varphi_2(l, z(l)) \right| &\leq \epsilon \varphi(l), \quad l \in \mathbb{R}, \\
\left| \triangle z(l_j) - I_j(z(l_j)) \right| &\leq \epsilon \varphi(l), \quad j = 1, 2, ..., m. \tag{3.3}
\end{align}

where

$$
\varphi_1(l) = z(l), \quad \varphi_2(l, z(l)) = A(z(l)) + D_t^{\alpha - 1}G(l, z(l), Kz(l))
$$

Definition 3. [12] If there occurs $\epsilon > 0, \ C_{G_1, m} > 0$ such that for every mild solution $r \in \mathcal{P}(\mathbb{R}, Y)$ to (3.1) there occurs a mild solution $z \in \mathcal{P}(\mathbb{R}, Y)$ to Eqn.(1.1) with $|r(l) - z(l)| \leq C_{G_1, m} \epsilon, \ l \in \mathbb{R}$, then Eqn.(1.1) is Ulam-Hyers stable.

Definition 4. [12] If there occurs $D_{G_1, m} \in \mathcal{P}(\mathbb{R}, Y), D_{G_1, m}(0) = 0$ such that for every mild solution $r \in \mathcal{P}(\mathbb{R}, Y)$ to (3.1) there occurs a mild solution $z \in \mathcal{P}(\mathbb{R}, Y)$ to Eqn.(1.1) with $|r(l) - z(l)| \leq D_{G_1, m}(\epsilon), \ l \in \mathbb{R}$, then Eqn.(1.1) is generalized Ulam-Hyers stable.

Definition 5. [12] If there occurs $C_{G_1, m, \varphi} > 0, \ \epsilon > 0$ such that for every mild solution $r \in \mathcal{P}(\mathbb{R}, Y)$ to (3.3) there occurs a mild solution $z \in \mathcal{P}(\mathbb{R}, Y)$ to Eqn.(1.1) with $|r(l) - z(l)| \leq C_{G_1, m, \varphi} \epsilon \varphi(l), \ l \in \mathbb{R}$, then Eqn.(1.1) Ulam-Hyers-Rassias stable with respect to $\varphi$.

Definition 6. [12] If there occurs $C_{G_1, m, \varphi} > 0$ such that for every mild solution $r \in \mathcal{P}(\mathbb{R}, Y)$ to (3.2) there occurs a mild solution $z \in \mathcal{P}(\mathbb{R}, Y)$ to Eqn.(1.1) with $|r(l) - z(l)| \leq C_{G_1, m, \varphi} \varphi(l), \ l \in \mathbb{R}$, then Eqn.(1.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi$.

Remark 2. The necessary and sufficient condition for a function $r \in \mathcal{P}(\mathbb{R}, Y)$ to be mild solution to (3.1) is that there occurs a sequence $g_{s_j}, j = 1, 2, ..., m$ and a function $g_s \in \mathcal{P}(\mathbb{R}, Y)$ such that

(i) $|g_s(l)| \leq \epsilon, \ l \in \mathbb{R}$ and $|g_{s_j}| \leq \epsilon, \ j = 1, 2, ..., m$.

(ii) $D_t^\alpha \varphi_1(l, z(l)) = \varphi_2(l, z(l)) + g_s(l), \ l \in \mathbb{R}$.

(iii) $\triangle z(l_j) = I_j(z(l_j)) + g_{s_j}, \ j = 1, 2, ..., m$.

Similarly we get the remarks to (3.2) and (3.3).

Remark 3. Let $1 < \alpha < 2$, if $r \in \mathcal{P}(\mathbb{R}, Y)$ is a mild solution to (3.1) then $r$ is the mild solution of subsequent inequality

$$
|r(l) - E_{\alpha}(l)r(0)| - \int_0^l E_{\alpha}(l - s)G_1(s, r(s), Kr(s))ds - \sum_{0 < l_j < l} E_{\alpha}(l - l_j)I_j(r(l_j)) \leq \left( \int_0^l |E_{\alpha}(l - s)|ds + \sum_{0 < l_j < l} |E_{\alpha}(l - l_j)| \right) \epsilon
$$
By remark(2), we have $r(l) = E_\alpha(l)r(0) + \int_0^l E_\alpha(l-s)G_1(s,r(s),Kr(s))ds + \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(r(l_j)) + \int_0^l E_\alpha(l-s)g_*(s)ds + \sum_{0<l_j<l} E_\alpha(l-l_j)g_{*j}$.

It follows that
\[ |r(l) - E_\alpha(l)r(0)| \leq \int_0^l E_\alpha(l-s)G_1(s,r(s),Kr(s))ds + \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(r(l_j)) |\]
\[ \leq \int_0^l E_\alpha(l-s)g_*(s)ds + \sum_{0<l_j<l} E_\alpha(l-l_j)g_{*j} |\]
\[ \leq (\int_0^l |E_\alpha(l-s)|ds + \sum_{0<l_j<l} |E_\alpha(l-l_j)|)\epsilon. \]

**Theorem 2.** Assumptions [H4]-[H6] hold. Suppose there occurs $\lambda_\varphi > 0$ and $\mu_\varphi > 0$ such that $\int_0^l |E_\alpha(l-s)|\varphi(s)ds \leq \lambda_\varphi\varphi(l)$, $\sum_{0<l_j<l} |E_\alpha(l-l_j)|\varphi(l_j) \leq \mu_\varphi\varphi(l)$ for every $l \in \mathbb{R}$, where $\varphi : \mathbb{R} \to \mathbb{R}^+$ is non-decreasing. Then mild solution to Eqn.(1.1) is generalised Ulam-Hyers-Rassias Stable.

**Proof.** Assume $r \in PC(\mathbb{R},Y)$ is mild solution to the inequality (3.2) and $z \in PC(\mathbb{R},Y)$ is the mild solution to (1.1). Then we have
\[ |r(l) - E_\alpha(l)r(0)| \leq (\lambda_\varphi + \mu_\varphi)\varphi(l). \]

It follows
\[ |r(l) - z(l)| = |r(l) - E_\alpha(l)z(0) - \int_0^l E_\alpha(l-s)G_1(s,z(s),Kz(s))ds - \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(z(l_j))| \]
\[ \leq |r(l) - E_\alpha(l)r(0) - \int_0^l E_\alpha(l-s)G_1(s,r(s),Kr(s))ds - \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(r(l_j))| \]
\[ - \sum_{0<l_j<l} E_\alpha(l-l_j)I_j(r(l_j)) + |E_\alpha(l)||r(0) - z(0)| \]
\[ + \int_0^l |E_\alpha(l-s)||G_1(s,r(s),Kr(s)) - G_1(s,z(s),Kz(s))|ds \]
\[ + \sum_{0<l_j<l} |E_\alpha(l-l_j)||I_j(r(l_j)) - I_j(z(l_j))| \]
\[ \leq (\lambda_\varphi + \mu_\varphi)\varphi(l) + |E_\alpha(l)||r(0) - z(0)| + LG \int_0^l |E_\alpha(l-s)||r(s) - z(s)| \]
\[ + |Kr(s) - Kz(s)|ds + Ll \sum_{0<l_j<l} |E_\alpha(l-l_j)||r(l_j) - z(l_j)| \]

\[ \leq (\lambda_\varphi + \mu_\varphi) \int_0^l |E_\alpha(l-s)||r(s) - z(s)|ds + Ll \sum_{0<l_j<l} \int_0^l |E_\alpha(l-s)||r(l_j) - z(l_j)|ds, \]
which implies that
\[
1 - |E_\alpha(l)| - L_G \left( \int_0^l |E_\alpha(l - s)| \left[ 1 + \frac{L_k C_k}{\mu} \right] ds \right) - L_l \sum_{0<l_j<l} |E_\alpha(l - l_j)| \|r - z\| \leq (\lambda_\varphi + \mu_\varphi) \varphi(l).
\]
That is
\[
|r(l) - z(l)| \leq \|r - z\| \leq C_{G_1, m, \varphi} \varphi(l),
\]
where
\[
C_{G_1, m, \varphi} = \frac{(\lambda_\varphi + \mu_\varphi)}{1 - |E_\alpha(l)| - L_G \left( \int_0^l |E_\alpha(l - s)| \left[ 1 + \frac{L_k C_k}{\mu} \right] ds \right) - L_l \sum_{0<l_j<l} |E_\alpha(l - l_j)|}.
\]
Thus we conclude Eqn.(1.1) is a generalized Ulam-Hyers-Rassians stable with respect to \( \varphi \) on \( \mathbb{R} \).

\[ \square \]

4. Example
Consider the following problem:
\[
\begin{cases}
\partial_t^\alpha v(l) = \partial_y^2 (v(l) - \mu v(l)) + \partial_t^{\alpha-1}(\beta v(l)(\cos l + \cos \sqrt{2}l) + \beta e^{-l}\sin(v(l)) + \sin(\int_0^l e^{(l-s)} h(s, v(s)) ds), & l \in \mathbb{R}, l \neq l_j, \\
\Delta v(l_j) = I_j(v(l_j)), & j = 1, 2, \ldots, m.
\end{cases}
\]
Here \( G_1 = \beta v(l)(\cos l + \cos \sqrt{2}l) + \beta e^{-l}\sin(v(l)) + \sin(\int_0^l e^{(l-s)} h(s, v(s)) ds) \), it is clear that the function \( G_1 \) with respect to \( l \) is a continuous function. Define the operators \( Av = \frac{\partial^2}{\partial y^2} - \mu v, \quad D(A) = \{ v \in L^2(0,1) : v, v' \text{ are absolutely continuous and } v, v', v'' \in L^2(0,1) \} \) with \( C_k = \mu = 1 \) and \( \omega = -1 \). The hypothesis \( (H_4), (H_5) \) and \( (H_6) \) are satisfied with \( L_1 = L_G = L_h = \frac{1}{5} \) and set \( \lambda_\varphi = \mu_\varphi = M = C(\theta, \alpha) = 1 \). Thus, the model (4.1) is generalized Ulam-Hyers-Rassias stable with \( |r(l) - z(l)| \leq C_{G_1, m, \varphi} \varphi(l), l \in [0,1], \) where \( C_{G_1, m, \varphi} \geq 7.143 > 0 \).

5. Conclusion
We present some new results about stability of a class of impulsive fractional integro-differential equation with Caputo fractional derivative. We discuss generalized Ulam-Hyers-Rassias stability, which is a new way for mathematical analysis. It is applied to the SIS epidemic model, logistic equation (both difference and differential), Cournot model in economics and a reaction diffusion equation.

References
[1] Bainov D D and Simeonov P S 1995 Impulsive Differential Equations: Asymptotic properties of the solutions World Scientific Singapore
[2] Cuesta E 2007 Asymptotic behaviour of the solutions of fractional integrodifferential equations and some time discretizations Discrete Continuum Dynamics Systems pp 277-285
[3] Glockle W G and Nonnenmacher T F 1995 A fractional calculus approach of self-similar protein dynamics Biophysical Journal 68 pp 46-53
[4] Gowrisankar M, Mohankumar P and Vinodkumar A 2014 Stability results of random impulsive semilinear differential equations Acta Mathematica Scientia 34 4 pp 1055-1071
[5] Hyers D H 1941 On the stability of the linear functional equation Proceedings of the National Academy of Sciences of the United States of America 27 pp 222-224

[6] Metzler F, Schick W, Kilian H and Nonnenmacher T F 1995 Relaxation in filled polymers: a fractional calculus approach The Journal of Chemical Physics 103 pp 7180-7186

[7] Azmat Ullah Khan Niazi, Jiang Wei, Mujeeb Ur Rehman and Du Jun, Ulam-Hyers-Stability for Nonlinear Fractional Neutral Differential Equations Hacettepe University Bulletin of Natural Sciences and Engineering Series B Doi: 10.15672/HJMS.2017.518

[8] Obloza M 1993 Hyers stability of the linear differential equation Rocznik Nauk. Dydakt. Prace Mat 13 pp 259-270

[9] Rassias T M 1978 On the stability of the linear mapping in Banach spaces Proceedings of the American Mathematical Society 72 pp 297-300

[10] Samoilenko A M and Perestyuk N A 1995 Impulsive differential equations World Scientific Singapore

[11] Ulam S M 1960 A Collection of the Mathematical Problems Interscience New York

[12] Wang J R, Yong Zhou and Michal Feckan 2012 Nonlinear impulsive problems for fractional differential equations and Ulam stability Computers and Mathematics with Applications 64 pp 3389-3405

[13] Akbar Zada, Wajid Ali and Syed Farina 2017 Hyers-Ulam stability of nonlinear differential equations with fractional integrable impulses Mathematical methods in the Applied Sciences 40 pp 5502-5504