The reciprocals of some characteristic 2
“theta series”

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Abstract

Suppose $l = 2m + 1$, $m > 0$. We introduce $m$ “theta-series”, $[1], \ldots, [m]$, in $\mathbb{Z}/2[[x]]$. It has been conjectured that the $n$ for which the coefficient of $x^n$ in $1/[i]$ is 1 form a set of density 0. This is probably always false, but in certain cases, for $n$ restricted to certain arithmetic progressions, it is true. We prove such zero-density results using the theory of modular forms, and speculate about what may be true in general.

1 Introduction

Throughout $L$ is a field of fractions of $\mathbb{Z}/2[[x]]$, viewed as the field of Laurent series with coefficients in $\mathbb{Z}/2$.

Definition 1.1. For $g \neq 0$ in $\mathbb{Z}/2[[x]]$, $B(g)$ is the set of $n$ in $\mathbb{Z}$ for which the co-efficient of $x^n$ in $1/g$ is 1. Note that only finitely many elements of $B(g)$ can be $< 0$.

Fix $l = 2m + 1$ with $m > 0$. We define certain “theta series” $[i]$ in $\mathbb{Z}/2[[x]]$.

Definition 1.2. $[i] = \sum x^{n^2}$, the sum extending over all $n$ in $\mathbb{Z}$ with $n \equiv i$ (l). (Note that $[0] = 1$, and that $[i] = [j]$ whenever $i \equiv \pm j$ (l). So the ring $S$ generated over $\mathbb{Z}/2$ by all the $[i]$ is just $\mathbb{Z}/2[[1], \ldots, [m]]$.)

In this note we study the sets $B([r])$ for fixed $l$ and $r$ with $r$ prime to $l$. Note that each $j$ in $B([r])$ is $\equiv -r^2$ (l) and that consequently $B([r])$ has (upper) density at most $1/l$ in the positive integers.

In [1], Cooper, Eichhorn and O’Bryant conjectured, in a slightly different language, that each $B([r])$ has density 0. I think this is never true, but we’ll show that for certain $l$ and $r$ and in certain congruence classes mod a power
of 2, \( B([r]) \) indeed has relative density 0. For example when \( l = 3 \) the relative density is 0 in the classes \( n \equiv 0 \pmod{2} \), \( n \equiv 1 \pmod{4} \) and \( n \equiv 3 \pmod{8} \). I’ll now describe more precisely, what perhaps is true in general, and the small part of it I’m able to prove.

**Definition 1.3.** Fix \( l, k < 0 \) is “\( l \)-exceptional” if \( k \) is in some \( B([r]) \) with \( r \) prime to \( l \). A “basic congruence class” is a congruence class of the form \( n \equiv k \pmod{8q} \), where \( k \) is \( l \)-exceptional and \( q \) is the largest power of 2 dividing \( k \).

**Definition 1.4.** An integer \( n \geq 0 \) is in \( U \) if it is in some basic congruence class, and in \( U^* \) otherwise.

**Example 1.** Suppose \( l = 3 \). Then \( 1/\{1\} = x^{-1} + \cdots \). So the only 3-exceptional \( k \) is \(-1\) and the only basic class is \( n \equiv -1 \pmod{8} \). \( U^* \) consists of the integers \( n \geq 0 \) with \( n \equiv 0 \pmod{2} \), \( n \equiv 1 \pmod{4} \), or \( n \equiv 3 \pmod{8} \).

**Example 2.** Suppose \( l = 9 \). The only \([r]\) we need consider are \([1\), \([2\) and \([4\). Now \( 1/\{1\} = x^{-1} + \cdots \), \( 1/\{2\} = x^{-4} + \cdots \) and \( 1/\{4\} = x^{-16} + x^{-7} + \cdots \). So the basic classes are \( n \equiv 1 \) or \(-1 \pmod{8} \), \( n \equiv -4 \pmod{32} \) and \( n \equiv -16 \pmod{128} \). Then \( U \) consists of the integers \( \geq 0 \) lying in \( 16 + 16 + 4 + 1 = 37 \) congruence classes to the modulus 128, and \( U^* \) of the integers \( \geq 0 \) in the remaining 91 classes.

It seems to me plausible that when \( r \) is prime to \( l \) then \( B([r]) \) has relative density 0 in \( U^* \). I’ll show that this holds for \( l \leq 11 \). When \( l = 13 \) or 15, then \( U^* \) is the union of 83 mod 128 congruence classes, and I’ll prove that \( B([r]) \) has relative density 0 in each of these classes, with the possible exception of the class \( n \equiv 48 \pmod{128} \). Unfortunately the proof is not unified—we have to write \( U^* \) as a union of congruence classes and examine each class in turn. To this end we now give the (easily proved) description of \( U^* \) as a union of congruence classes for each \( l \leq 15 \).

| \( l \) mod 2 | \( l \) mod 4 | \( l \) mod 8 | \( l \) mod 16 | \( l \) mod 32 | \( l \) mod 64 | \( l \) mod 128 |
|-------------|------------|-------------|-------------|-------------|-------------|-------------|
| 3           | 0          | 1           | 3           |              |              |              |
| 5           | 1, 2       | 0, 3        | 4           | 12          |              |              |
| 7           | 1          | 0, 2, 3     | 4, 6        | 12          |              |              |
| 9           | 2          | 3, 5        | 4, 8        | 0, 12       | 16           | 48           |
| 11          | 1, 3, 6    | 4, 8, 10    | 0, 12       | 16           | 48           |              |
| 13          | 2, 3, 5    | 4, 8, 14    | 0, 12       | 16           | 48           |              |
| 15          | 1, 2, 3    | 4, 6, 8     | 0, 12       | 16           | 48           |              |
Here’s a rough description of how our proofs proceed. Fix $l$ and $[r]$ and a congruence class $j \mod q$ where $q$ is a power of 2. We’ll construct a $g$ in $\mathbb{Z}/2[[x]]$, depending on $l$, $r$, $j$ and $q$, with the following properties:

1. There are integers $c_0, c_1, \ldots$ such that:
   
   \begin{enumerate}
   \item $\sum c_n e^{2\pi inz}$ converges in $\text{Im}(z) > 0$ to a modular form of integral weight for a congruence group.
   \item $g$ is the mod 2 reduction of $\sum c_n x^n$
   \end{enumerate}

2. Suppose that $g/[r]^q$ is itself the mod 2 reduction of some $\sum d_n x^n$ where $\sum d_n e^{2\pi inz}$ converges to a modular form as in 1(A) above. Then $B([r])$ has density 0 in the congruence class $j \mod q$.

$g$ is in fact the image of $[r]^{q-1}$ under a certain projection operator $p_{q,j}$ which we describe in the next section. The fact that $g$ is “the reduction of a modular form” comes from a corresponding result for $[r]$; $[r]$ is the reduction of a weight 1 modular form. (The proof of (2) is deeper, coming from a result of Deligne and Serre on the reduction of modular forms.) Once (1) and (2) are established we still need to show that for each of our choices of $l$, $[r]$, and the congruence class $j \mod q$ lying in $U^*$, the power series $g/[r]^q$ satisfies the condition (2) above. This is true, for example, whenever $g/[r]^q$ lies in the ring $S$ of Definition 1.2. In certain cases, extensive computer calculations tell us that $g/[r]^q$ lies in $S$.

At the end of the paper we’ll speculate on the relative density of $B([r])$ in the basic classes. Though we are unable to prove anything, computer calculations suggest that each $B([r])$ has relative density $1/(2l)$ in each basic class.

2 The operators $p_{q,j}$ and the case $l = 3$

If $q$ is a power of 2, let $L^{[a]} \subset L$ consist of all $q$th powers of elements of $L$. $L$ is the direct sum of the $L^{[a]}$ vector-spaces $x^j L^{[a]}$, $0 \leq j < q$.

**Definition 2.1.** $p_{q,j} L \to x^j L^{[a]}$ is the $L^{[a]}$-linear projection map attached to the above direct sum decomposition.

Note that $p_{q,j}(FG) = \sum_{a+b=j} p_{q,a}(F)p_{q,b}(G)$, the sum extending over all pairs $(a, b)$ with $a + b \equiv j \mod (q)$. Furthermore $p_{2q,2j}(F^2) = (p_{q,j}(F))^2$. We’ll use these facts often.

**Lemma 2.2.** Fix $l = 2m + 1$. Then:

1. $p_{2,0}( [2i] ) = [i]^4$
(2) The subring $S$ of $L$ generated over $\mathbb{Z}/2$ by all the $[i]$ is stabilized by the operators $p_{8,0}, \ldots, p_{8,7}$.

**Proof.** Since $[2i] = \sum_{n=2i} (t_n x^n)$, $p_{2,0}(2i) = \sum_{k=0} x^{4k^2} = [i]^4$.

In view of the formula for $p_{8,3}(FG)$, to prove (2) it suffices to show that $p_{8,0}([i]), \ldots, p_{8,7}([i])$ are all in the subring. Now if $j \neq 0, 1$ or 4, each $p_{8,j}([i])$ is 0. Since every odd square is $\equiv 1 \pmod{8}$, $p_{8,1}(2i) = p_{2,1}(2i) = [2i] + [i]^4$. Also $p_{8,0}(4i) = p_{8,0}p_{2,0}(4i) = p_{8,0}(2i)^4 = (p_{2,0}(2i))^4 = [i]^{16}$. Similarly, $p_{8,4}(4i) = (p_{2,1}(2i))^4 = [2i]^4 + [i]^{16}$. □

Suppose for the rest of this section that $l = 3$. In this case the proofs of zero-density in $U^*$ are much easier than the proofs for $l > 3$, requiring neither modular forms nor computer calculations. Observe that if 3 doesn’t divide $i$, then $[i] = 1$.

**Definition 2.3.** $a = [1] = [2]$. Note that $p_{2,0}(a) = a^4$.

**Theorem 2.4.** Suppose $n \equiv 0 \pmod{2}$ and $n$ is in $B(a)$. Then $n/2$ is a square.

**Proof.** $p_{2,0}(\frac{1}{n}) = \frac{1}{a}p_{2,0}(a) = a^2$. Since $n$ is in $B(a)$ and is even, the coefficient of $x^n$ in $a^2$ is 1, giving the result. □

**Theorem 2.5.** Suppose $n \equiv 1 \pmod{4}$ and $n$ is in $B(a)$. Then the number of pairs $(s_1, s_2)$ with $s_1$ and $s_2$ squares, and $s_1 + 4s_2 = n$ is odd. Furthermore $n$ is the product of a prime and a square.

**Proof.** $p_{4,1}(\frac{1}{a}) = \frac{1}{a}p_{4,1}(a^3) = \frac{1}{a}p_{4,1}(a)p_{4,1}(a^2) = \frac{1}{a^7} (a + a^4) a^8 = a^5 + a^8$. Since $n$ is in $B(a)$ and is $\equiv 1 \pmod{4}$, the coefficient of $x^n$ in $a^5 + a^8$ is 1, and so the coefficient in $a^5 = a \cdot a^4$ is 1. So the number of pairs $(r_1, r_2)$ with $r_1 \equiv r_2 \equiv 1 \pmod{3}$ and $r_1^2 + 4r_2^2 = n$ is odd. To each such pair attach the pair $(s_1, s_2)$ with $s_1$ and $s_2$ squares, $s_1 + 4s_2 = n$, by setting $s_i = r_i^2$. The function from pairs $(r_1, r_2)$ to pairs $(s_1, s_2)$ is 1-1. Since $n$ is in $B(a)$, $n \equiv -1 \pmod{3}$. So whenever we have a pair $(s_1, s_2)$ as above, $s_1$ and $s_2$ are $\equiv 1 \pmod{3}$ and have square roots $\equiv 1 \pmod{3}$. So the function $(r_1, r_2) \rightarrow (s_1, s_2)$ is onto, and we get the first assertion of the theorem. A little arithmetic in $\mathbb{Z}[i]$ gives the second assertion. □

**Lemma 2.6.** If $n \equiv 3 \pmod{8}$, $n$ is in $B(a)$ if and only if the number of triples $(r_1, r_2, r_3)$ with $r_1 \equiv r_2 \equiv r_3 \equiv 1 \pmod{3}$ and $r_1^2 + 2r_2^2 + 8r_3^2 = n$ is odd.

**Proof.** $p_{8,3}(\frac{1}{a}) = \frac{1}{a}p_{8,3}(a \cdot a^2 \cdot a^4) = \frac{1}{a^5}p_{8,1}(a)p_{8,2}(a^2)p_{8,0}(a^4) = \frac{1}{a^5} (a + a^4) (a + a^4)^2 a^{16} = a^{11} + a^{14} + a^{17} + a^{20}$. Since $n \equiv 3 \pmod{8}$ the coefficients of $x^n$ in
$a^{14}$, $a^{20}$, and $a^{17} = a \cdot a^{16}$ are evidently 0. So $n$ is in $B(a)$ if and only if the coefficient of $x^n$ in $a^{11} = a \cdot a^2 \cdot a^8$ is 1, giving the lemma. \hfill \Box

**Lemma 2.7.** If $n \equiv 11 \ (24)$ the number of triples $(s_1, s_2, s_3)$ where the $s_i$ are squares and $s_1 + s_2 + s_3 = n$ is 3 \cdot (\text{the number of triples } (r_1, r_2, r_3) \text{ as in Lemma 2.6}).

**Proof.** If the $s_i$ are as above, two of them are $\equiv 1 \ (3)$ while 3 divides the third. So our lemma states that the number of triples $(s_1, s_2, s_3)$ with the $s_i$ squares, $s_1 + s_2 + s_3 = n$ and $s_3 \equiv 0 \ (3)$ is the number of triples $(r_1, r_2, r_3)$ as in Lemma 2.6. If we have a triple $(r_1, r_2, r_3)$ let $s_1 = r_1^2$, $s_2 = (r_2 - 2r_3)^2$, $s_3 = (r_2 + 2r_3)^2$. Then the $s_i$ are squares, $s_3 \equiv 0 \ (3)$ and $s_1 + s_2 + s_3 = r_1^2 + 2r_2^2 + 8r_3^2 = n$. That $(r_1, r_2, r_3) \rightarrow (s_1, s_2, s_3)$ is 1–1 is easily seen. To prove ontoness suppose we’re given $(s_1, s_2, s_3)$. Then $s_1$ and $s_2$ are $\equiv 1 \ (3)$ and have square roots, $\sqrt{s_1}$ and $\sqrt{s_2}$, that are $\equiv 1 \ (3)$. Also, since $n \equiv 3 \ (8)$, the $s_i$ are odd. So we can find a square-root, $\sqrt{s_3}$ of $s_3$ with $\sqrt{s_3} \equiv \sqrt{s_2} \ (4)$. Then the triple $(\sqrt{s_1}, -\sqrt{s_2-\sqrt{s_3}}, \sqrt{s_2+\sqrt{s_3}})$ has its entries $\equiv 1 \ (3)$ and maps to $(s_1, s_2, s_3)$. \hfill \Box

**Theorem 2.8.** Suppose $n \equiv 3 \ (8)$ and $n$ is in $B(a)$. Then the number of pairs $(s_1, s_2)$ with $s_1$ and $s_2$ squares and $s_1 + 2s_2 = n$ is odd. Furthermore, $n$ is the product of a prime and a square.

**Proof.** Consider the set of triples $(s_1, s_2, s_3)$ where the $s_i$ are squares and $s_1 + s_2 + s_3 = n$. Since $n$ is in $B(a)$, and $n \equiv 3 \ (8)$, $n \equiv 11 \ (24)$. Lemmas 2.6 and 2.7 then show that the number of such triples is odd. Now $(s_1, s_2, s_3) \rightarrow (s_1, s_3, s_2)$ is an involution on the set of such triples whose fixed points identify with the pairs $(s_1, s_2)$ as in the statement of the theorem. This gives the first assertion of the theorem, and a little arithmetic in $\mathbb{Z}[\sqrt{-2}]$ gives the second. \hfill \Box

**Theorem 2.9.**

1. Every element $n$ of $B(a)$ that lies in $U^*$ is the product of a prime and a square.
2. The number of elements of $B(a)$ that are $\leq x$ and lie in $U^*$ is $O(x/\log x)$.

**Proof.** The elements of $U^*$ are $\equiv 0 \ (2)$, 1 (4), or 3 (8), and we use Theorems 2.4, 2.5 and 2.8 to get (1). (2) is an immediate consequence. \hfill \Box

**Remark 1.** The proof of Theorem 2.9 is easier than that of a similar result in Monsky [2], which makes use of results of Gauss on representations by sums of 3 squares.

**Remark 2.** The set $B(a + a^4)$ has been more extensively studied. One sees immediately that $a + a^4 = \sum x^{1+24s}$, where $s$ runs over the generalized pen-
tagonal numbers 0, 1, 2, 5, 7, 12, 15, . . . So the elements of $B(a + a^4)$ are all \(\equiv -1\) (24). The mod 2 reduction of a famous identity of Euler tells us that 24\(k - 1\) is in $B(a + a^4)$ if and only if the number of partitions, $p(k)$, of $k$ is odd. Large-scale computer calculations suggest very strongly that the $k$ for which $p(k)$ is odd have density 1/2, so that $B(a + a^4)$ has relative density 1/2 in the congruence class $n \equiv -1$ (24). It’s tempting to believe that $B(a)$ also has relative density 1/2 in this congruence class. This would be in line with the (modest) computer calculations that have been made; see our final section.

3 Enter modular forms. The quintic theta relations

In the proofs of section 2 we expressed $p_{2,0}(\frac{1}{a})$, $p_{4,1}(\frac{1}{a})$, and $p_{8,3}(\frac{1}{a})$ as elements of $\mathbb{Z}/2[a]$, and were able to deduce that $B(a)$ has density 0 in the congruence classes $n \equiv 0$ (2), $n \equiv 1$ (4) and $n \equiv 3$ (8). (Note that $p_{8,7}(\frac{1}{a})$ is not in $\mathbb{Z}/2[a]$. Indeed $p_{8,7}(\frac{1}{a}) = x^{-1} + \cdots$ and is not even in $\mathbb{Z}/2[[x]]$). In our treatment of larger $l$ we’ll use a similar idea, but in most cases we’ll have to rely on a deep result on modular forms due to Deligne and Serre. My thanks go to David Rohrlich for telling me about this result.

The following is well-known; for a more general theorem on definite quadratic forms in an even number of variables see Schöneberg [4].

**Theorem 3.1.** \(\sum \sum e^{2\pi i(m^2 + n^2)z}\), the sum extending over all pairs $(m, n)$ with $m$ and $n$ in $\mathbb{Z}$ and $n \equiv$ some $j$ mod $l$, converges in $\text{Im}(z) > 0$ to a weight 1 modular form for a congruence group.

**Corollary 3.2.** Fix $l$. Let $u = \sum a_s x^s$ be a product of powers of various $[j]$. Then there are integers $c_0, c_1, \ldots$ such that:

(A) \(\sum c_s e^{2\pi i z} \) converges in $\text{Im}(z) > 0$ to a modular form of integral weight for a congruence group.

(B) The mod 2 reduction of $c_s$ is $a_s$.

**Proof.** It’s enough to show this when $u = [j]$. We take our modular form to be that of Theorem 3.1. If we write this form as $\sum c_s e^{2\pi i s z}$, then (A) is satisfied. Furthermore $c_s$ is the number of pairs $(m, n)$ with $n \equiv j \ (l)$ and $m^2 + n^2 = s$. $(m, n) \to (-m, n)$ is an involution on this set of pairs. There is one fixed point if $s$ is the square of some $n \equiv j \ (l)$, and no fixed point otherwise. It follows that the mod 2 reduction of $c_s$ is $a_s$. \(\square\)

Now fix $l$. Recall that $S$ is the subring of $\mathbb{Z}/2[[x]]$ generated over $\mathbb{Z}/2$ by all the $[j]$. 6
Theorem 3.3. If \( u = \sum a_n x^n \) is in \( S \), then the set of \( n \) for which \( a_n \) is 1 has density 0.

Proof. We may assume that \( u \) is a product of powers of various \([j]\). As we’ve seen, there are \( c_n \) in \( \mathbb{Z} \), with \( c_n \) reducing to \( a_n \) mod 2, such that \( \sum c_n e^{2\pi i n z} \) converges in \( \text{Im}(z) > 0 \) to a modular form of integral weight for a congruence group. A theorem of Serre [5], based on results of Deligne attaching Galois representations to Hecke eigenforms, shows that the \( n \) divide \( c \) form a set of density 0. \( \square \)

Corollary 3.4. Suppose that \( p_{q,j}(1/\lfloor r \rfloor) \) is in \( S \), or more generally in \( p_{q,j}(S) \). Then \( B([r]) \) has relative density 0 in the congruence class \( j \ mod \ q \).

Now \( p_{q,j}(1/\lfloor r \rfloor) = (1/\lfloor r \rfloor^q) p_{q,j}(\lfloor r \rfloor^{q-1}) \). But to show that this quotient lies in \( p_{q,j}(S) \) for various choices of \( j \) and \( q \) seems very difficult. There is however a technique for showing that a quotient of two elements of \( S \) lies in \( S \) that makes use of certain “quintic theta relations”.

Lemma 3.5. \( p_{2,0}([2i][2j]) = [i+j]^2[i-j]^2 \).

Proof. It suffices to show that the coefficients of \( x^{2n} \) on the two sides are equal. On the left one has the mod 2 reduction of the number of pairs \((r,s)\) with \( r \equiv 2i \ (l) \), \( s \equiv 2j \ (l) \) and \( r^2 + s^2 = 2n \). On the right one has the mod 2 reduction of the number of pairs \((t,u)\) with \( t \equiv i+j \ (l) \), \( u \equiv i-j \ (l) \) and \( t^2 + u^2 = n \). Clearly \((r,s) \rightarrow (\frac{r+s}{2}, \frac{r-s}{2}) \) gives the desired bijection. \( \square \)

Theorem 3.6. \( [i]^4[2j] + [j]^4[2i] + [2i][2j] + [i+j]^2[i-j]^2 = 0 \).

Proof. \( p_{2,0}([2i][2j]) = p_{2,0}([2i])p_{2,0}([2j]) + p_{2,1}([2i])p_{2,1}([2j]) = [i]^4[j]^4 + ([i]^4 + [2i]) ([j]^4 + [2j]) \). Now use Lemma 3.5. \( \square \)

Let \( x_1, \ldots, x_m \) (where \( l = 2m + 1 \)) be indeterminates over \( \mathbb{Z}/2 \).

Definition 3.7. If \( r \) is prime to \( l \), \( \phi_r \) is the homomorphism \( \mathbb{Z}/2[x_1, \ldots, x_m] \rightarrow S \) taking \( x_k \) to \( [rk] \).

Note that each \( \phi_r \) is onto. We’ll use Theorem 3.6 to construct \( \frac{m(m-1)}{2} \) elements of \( \mathbb{Z}/2[x_1, \ldots, x_m] \) lying in the kernel of each \( \phi_r \).

Theorem 3.8. Suppose that \( m \geq i > j \geq 1 \). For \( 1 \leq k \leq m \) define \( x_{1-k} \) to be \( x_k \), so that we have elements \( x_1, \ldots, x_{2m} \) of \( \mathbb{Z}/2[x_1, \ldots, x_m] \). Then if we define \( R_{i,j} \) to be \( x_i^4 x_{2j} + x_j^4 x_{2i} + x_2 x_{2j} + x_i^2 x_{i+j} x_{i-j}^2 \), each \( R_{i,j} \) is in the kernel of each \( \phi_r \).
Proof. The definition of \( x_{m+1}, \ldots, x_{2m} \) shows that \( \phi_r(x_k) = [rk] \) for \( k = 1, \ldots, 2m \). The result now follows from Theorem 3.6 on replacing \( i \) and \( j \) by \( ri \) and \( rj \) throughout.

**Theorem 3.9.** Let \( u \) and \( v \) be elements of \( \mathbb{Z}/2[x_1, \ldots, x_m] \), and \( N \) the ideal in this ring generated by the \( R_{i,j} \). Suppose that the ideals \( (N, v) \) and \( (N, u, v) \) are the same. Then the element \( \phi_r(u)/\phi_r(v) \) of the field of fractions of \( S \) in fact lies in \( S \).

Proof. \( u \) is in \( (N, v) \). Applying \( \phi_r \) and using Theorem 3.9 we find that in \( S \), \( \phi_r(u) \) lies in the principal ideal \( \phi_r(v) \).

**Remark.** Commutative algebra computer programs such as Macaulay 2 use Gröbner bases to decide whether 2 ideals in a polynomial ring are equal. We shall use such a program to show that in many cases of interest the quotient \( \phi_r(u)/\phi_r(v) \) lies in \( S \).

There is one further simple result that we’ll use frequently in the calculations to follow.

**Lemma 3.10.** Suppose that for some \( a \) and \( b \), \( p_{2,0}(a) = b^4 \). Then:

1. \( p_{2,0} \left( \frac{1}{a} \right) = \frac{b^4}{a^4} \)
2. \( p_{4,0} \left( \frac{1}{a} \right) = \frac{b^{12}}{a^{12}} \)
3. \( p_{8,0} \left( \frac{1}{a} \right) = \frac{b^8}{a^8} (p_{2,0}(ab))^4 \)

Proof. \( p_{2,0} \left( \frac{1}{a} \right) = \frac{1}{a^4} p_{2,0}(a) = \frac{b^4}{a^4} \). Then \( p_{4,0} \left( \frac{1}{a} \right) = p_{4,0} p_{2,0} \left( \frac{1}{a} \right) = p_{4,0} \left( \frac{b^4}{a^4} \right) = b^4 \left( p_{2,0} \left( \frac{1}{a} \right) \right)^2 = \frac{b^{12}}{a^{12}} \). Furthermore, \( p_{8,0} \left( \frac{1}{a} \right) = p_{8,0} p_{4,0} \left( \frac{1}{a} \right) = p_{8,0} \left( \frac{b^{12}}{a^{12}} \right) = \frac{b^8}{a^8} p_{8,0}(ab^4) \), giving the last result.

**4 \( l = 5 \)**

In this section \( l = 5 \), so that \( m = 2 \). Then the ideal \( N \) of Theorem 3.9 is generated by the single element \( R_{2,1} = x_2 x_4 + x_4 x_2 + x_3 x_3 = x_1^5 + x_2^5 + x_1 x_2 + x_2^2 x_3^2 \). Now let \( r = 1 \) or \( 2 \) and set \( a = [r], \ b = [2r] \). Then \( p_{2,0}(a) = b^4, \ p_{2,0}(b) = a^4 \) and we have the quintic relation \( a^5 + b^5 + ab + a^2 b^2 = 0 \).

We’ll use the techniques sketched in the last section to show that \( p_{4,1} \left( \frac{1}{a} \right), \ p_{4,2} \left( \frac{1}{a} \right), \ p_{8,0} \left( \frac{1}{a} \right), \ p_{8,3} \left( \frac{1}{a} \right), \ p_{16,4} \left( \frac{1}{a} \right) \) and \( p_{32,12} \left( \frac{1}{a} \right) \) are all in \( S \). Corollary 3.4 in conjunction with the description of \( U^* \) given in the introduction when \( l = 5 \) then tells us that \( B(a) \) has relative density 0 in \( U^* \).
Theorem 4.1. \( p_{8,0} \left( \frac{1}{a} \right) = b^{16} \).

Proof. By Lemma 3.10, \( p_{8,0} \left( \frac{1}{a} \right) = \frac{b^8}{a^4} (p_{2,0}(ab))^4 \). Now \( p_{2,0}(ab) = p_{2,0}([4r][2r]) = [3r]^2 \cdot [r]^2 = a^2b^2 \).

Theorem 4.2. \( p_{4,2} \left( \frac{1}{a} \right), \ p_{4,1} \left( \frac{1}{a} \right) \) and \( p_{8,3} \left( \frac{1}{a} \right) \) are in \( S \).

Proof. We first write these power series as quotients of elements of \( S \).

1. \( p_{4,2} \left( \frac{1}{a} \right) = p_{2,0} \left( \frac{1}{a} \right) + p_{4,0} \left( \frac{1}{x} \right) = \frac{b^4}{a^4} + \frac{b^{12}}{a^{12}} = \left( \frac{b^4}{a^4} \right) (a^2 + b^8) \).
2. \( p_{4,1} \left( \frac{1}{a} \right) = \left( \frac{1}{a^4} \right) p_{4,1}(a)p_{4,0}(a^2) = \left( \frac{1}{a^4} \right) p_{2,1}(a) (p_{2,0}(a))^2 = \left( \frac{b^4}{a^4} \right) (a + b^4) \).
3. \( p_{8,3} \left( \frac{1}{a} \right) = \left( \frac{1}{a^4} \right) p_{8,1}(a)p_{8,2}(a^2)p_{8,0}(a^4) = \left( \frac{1}{a^4} \right) p_{2,1}(a) (p_{2,1}(a))^2 (p_{2,0}(a))^4 = \left( \frac{b^{16}}{a^4} \right) (a + b^4)^3 \).

In view of (1), (2) and (3) it will suffice to show that \( \frac{b^2}{a^2}(a + b^4) \) and \( \frac{b^4}{a^4}(a + b^4) \) are each in \( S \). This can be done by hand, but in the mechanized spirit of the paper I'll give a computer argument. First let \( u = x_2^3(x_1 + x_2^2) \) and \( v = x_1^2 \). Macaulay 2 tells us that \( (N, v) = (N, u, v) \). So by Theorem 3.9, \( \phi_r(u)/\phi_r(v) \) is in \( S \). But \( \phi_r(u)/\phi_r(v) = \frac{b^2}{a^4}(a + b^4) \). For the second result we argue similarly taking \( u = x_2^3(x_1 + x_2^2) \) and \( v = x_1^4 \).

Lemma 4.3. \( p_{8,4} \left( \frac{1}{a} \right) + \left( p_{2,1} \left( \frac{1}{a} \right) \right)^4 = a^4 + b^{16} \).

Proof. \( p_{8,4} \left( \frac{1}{a} \right) = p_{4,0} \left( \frac{1}{a} \right) + p_{8,0} \left( \frac{1}{a} \right) = \frac{b^{12}}{a^{12}} + b^{16} \), by Lemma 3.10 and Theorem 4.1. Furthermore \( p_{2,1} \left( \frac{1}{b} \right) = \frac{1}{b} + p_{2,0} \left( \frac{1}{b} \right) = \frac{1}{b} + \frac{a^4}{b^4} \). So the left hand side in the statement of Lemma 4.3 is \( b^{16} + \left( \frac{b^2}{a} + \frac{1}{b} + \frac{a^4}{b^4} \right)^4 \). But the quintic relation \( a^5 + b^5 + ab + a^2b^2 = 0 \) tells us that \( \frac{b^5}{a} + \frac{1}{b} + \frac{a^4}{b^4} = \frac{1}{ab^4} (b^5 + ab + a^5) = a \).

Theorem 4.4. \( p_{16,4} \left( \frac{1}{a} \right) \) and \( p_{32,12} \left( \frac{1}{a} \right) \) are in \( S \).

Proof. Applying \( p_{16,4} \) to the identity of Lemma 4.3 we find that \( p_{16,4} \left( \frac{1}{a} \right) + \left( p_{4,1} \left( \frac{1}{a} \right) \right)^4 = (p_{4,1}(a))^4 = a^4 + b^{16} \). But Theorem 4.2 (with \( r \) replaced by \( 2r \)) tells us that \( p_{4,1} \left( \frac{1}{a} \right) \) is in \( S \). Applying \( p_{32,12} \) to the identity of Lemma 4.3 we find that \( p_{32,12} \left( \frac{1}{a} \right) + \left( p_{8,3} \left( \frac{1}{a} \right) \right)^4 = (p_{8,3}(a))^4 = 0 \). And Theorem 4.2 (with \( r \) replaced by \( 2r \)) shows that \( p_{8,3} \left( \frac{1}{a} \right) \) is in \( S \).
In this section $l = 7$. Then $m = 3$ and the ideal $N$ is generated by $x_1^3 + x_2^3x_2 + x_1x_2 + x_1^2x_3^2, x_2 + x_1^2x_3 + x_2x_3 + x_2^2x_4^2$ and $x_1^3 + x_2x_3 + x_3x_1 + x_1^2x_2^3$. Let $r$ be 1, 2 or 3, $a = [r], b = [4r], c = [2r]$. Then $p_{2,0}$ takes $a, b$ and $c$ to $b^4, c^4$ and $a^4$. Lemma 3.5 shows that $p_{2,0}$ takes $ab, bc$ and $ac$ to $a^2b^2, a^2b^2$ and $b^2c^2$. We'll prove that $B(a)$ has relative density 0 in $U^*$ by showing that each of $p_{4,1}\left(\frac{1}{a}\right), p_{8,0}\left(\frac{1}{a}\right), p_{8,3}\left(\frac{1}{a}\right), p_{16,4}\left(\frac{1}{a}\right), p_{16,6}\left(\frac{1}{a}\right)$ and $p_{32,12}\left(\frac{1}{a}\right)$ is in $S$.

Remark. In this case, as in the case $l = 5$, $N$ is the kernel of each $\phi_r$. This is not true when $l = 9$. Whether it holds for all prime $l$ is an interesting question.

Theorem 5.1. $p_{8,0}\left(\frac{1}{a}\right) = b^8c^8$, and $p_{8,2}\left(\frac{1}{a}\right) = (a^2 + b^8)c^8$.

Proof. By Lemma 3.10, $p_{8,0}\left(\frac{1}{a}\right) = \left(\frac{b^8}{a^8}\right)(p_{2,0}(ab))^4 = b^8c^8$. Also, $p_{8,2}\left(\frac{1}{a}\right) = \frac{1}{a^2}p_{2,2}(a)(p_{2,0}(ab))^2 = \frac{1}{a^2}p_{2,1}(a)(p_{2,0}(ab))^2 = (a + b^4)b^4$.

Theorem 5.2. $p_{4,1}\left(\frac{1}{a}\right), p_{8,3}\left(\frac{1}{a}\right)$ and $p_{16,6}\left(\frac{1}{a}\right)$ are in $S$.

Proof. Again we first write these power series as quotients of elements in $S$.

1. The proof of Theorem 4.2 shows that $p_{4,1}\left(\frac{1}{a}\right) = \frac{b^8}{a^8}(a + b^4)$, and that $p_{8,3}\left(\frac{1}{a}\right) = \frac{b^{16}}{a^8}(a + b^4)^3$.

2. $p_{16,6}\left(\frac{1}{a}\right) = p_{16,6}p_{2,0}\left(\frac{1}{a}\right) = p_{16,6}\left(\frac{b^4}{a^4}\right) = \left(p_{8,3}\left(\frac{b^2}{a}\right)\right)^2$. Now $p_{8,3}\left(\frac{b^2}{a}\right) = \frac{1}{a^2}p_{8,1}(a)p_{8,2}(ab) = \frac{1}{a^2}p_{2,1}(a)(p_{2,0}(ab))^2 = \frac{1}{a^2}p_{2,1}(a)(p_{2,0}(ab))^2 = \frac{a^4 + b^4 + a^2c^2}{a^4}$. So $p_{8,3}\left(\frac{b^2}{a}\right) = \frac{a^4 + b^4 + a^2c^2}{a^4}$.

We can now use the technique of the last section to prove the theorem. It suffices to show that $\left(\frac{b^8}{a^8}\right)(a + b^4)$ and $\left(\frac{b^{16}}{a^8}\right)(a + b^4)(a^2b^2 + a^4c^4)$ are in $S$. To prove the second result we take $u$ to be $x_3^3(x_1 + x_3^4)(x_1^2x_3 + x_1^4x_3^2)$, and $v$ to be $x_3^8$. Macaulay 2 verifies that $(N, v) = (N, u, v)$. So $\phi_r(u)/\phi_r(v)$ is in $S$, as desired. The first result is proved similarly.

Lemma 5.3. $p_{8,4}\left(\frac{1}{a}\right) + \left(p_{4,2}\left(\frac{1}{c}\right)\right)^2 = u^4$ for some $u$ in $S$.

Proof. $p_{8,4}\left(\frac{1}{a}\right) = p_{4,0}\left(\frac{1}{a}\right) + p_{8,0}\left(\frac{1}{a}\right)$. By Lemma 3.10 and Theorem 5.1, this is $\frac{b^{12}}{a^8} + b^8c^8$. And $p_{4,2}\left(\frac{1}{c}\right) = p_{2,0}\left(\frac{1}{c}\right) + p_{4,0}\left(\frac{1}{c}\right) = \frac{a^4}{c^4} + \frac{a^2}{c^2}$. So the left-hand side in the statement of the lemma is the fourth power of $\frac{b^8}{a^8} + \frac{b^2c^2}{a^4} + \frac{a^2}{c^2}$. To
show that $\frac{b^2}{c} + \frac{a^2}{c} + x^6$ is in $S$, we write it as a quotient, $\frac{b^2c^2 + a^3c + a^7}{ac^2}$, and use our Macaulay 2 technique.

**Theorem 5.4.** $p_{16,4} \left( \frac{1}{a} \right)$ and $p_{32,12} \left( \frac{1}{a} \right)$ are in $S$.

**Proof.** Applying $p_{16,4}$ to the identity of Lemma 5.3 we find that $p_{16,4} \left( \frac{1}{a} \right) + \left( p_{8,2} \left( \frac{1}{a} \right) \right)^2 = (p_{4,1}(u))^4$. Now Theorem 5.1 (with $r$ replaced by $2r$) shows that $p_{8,2} \left( \frac{1}{a} \right)$ is in $S$. Since $S$ is stable under $p_{4,1}$, $p_{16,4} \left( \frac{1}{a} \right)$ is in $S$. Similarly, applying $p_{32,12}$ to the identity, we find that $p_{32,12} \left( \frac{1}{a} \right) + \left( p_{16,6} \left( \frac{1}{a} \right) \right)^2 = (p_{8,3}(u))^4$. Theorem 5.2 shows that $p_{16,6} \left( \frac{1}{c} \right)$ is in $S$, and we use the fact that $p_{8,3}$ stabilizes $S$.  

6 $l = 9$

Now $l = 9$. Then $m = 4$ and $N$ is generated by $x_1^5 + x_4^2x_2 + x_1x_2 + x_2^2x_3^2$, $x_2^5 + x_4^2x_2 + x_1x_2 + x_2^2x_3^2$, $x_3^5 + x_4^2x_2 + x_1x_2 + x_2^2x_3^2$, $x_4^5 + x_4^2x_2 + x_1x_2 + x_2^2x_3^2$, $x_1^5 + x_3^2x_1 + x_1x_2 + x_2^2x_3^2$, $x_2^5 + x_3^2x_1 + x_1x_2 + x_2^2x_3^2$, $x_3^5 + x_3^2x_1 + x_1x_2 + x_2^2x_3^2$, and $x_4^5 + x_3^2x_1 + x_1x_2 + x_2^2x_3^2$. Let $r$ be 1, 2 or 4, $a = [r]$, $b = [4r]$, $c = [2r]$ and $d = [3r] = [6r]$. Then $p_{2,0}(d) = d^4$, and $p_{2,0}$ takes $a, b$ and $c$ to $b^4, c^4$ and $a^4$. Lemma 3.5 shows that $p_{2,0}$ takes $ab, bc$ and $ac$ to $c^2d^2, a^2d^2$ and $b^2d^2$, and that it takes $ad, bd$ and $cd$ to $a^2c^2, a^2b^2$ and $b^2c^2$. We'll prove that $B(a)$ has relative density 0 in $U^*$ by showing that each of $p_{4,2} \left( \frac{1}{a} \right)$, $p_{8,3} \left( \frac{1}{a} \right)$, $p_{8,5} \left( \frac{1}{a} \right)$, $p_{16,4} \left( \frac{1}{a} \right)$, $p_{16,8} \left( \frac{1}{a} \right)$, $p_{32,0} \left( \frac{1}{a} \right)$, $p_{64,16} \left( \frac{1}{a} \right)$ and $p_{128,48} \left( \frac{1}{a} \right)$ is in $S$.

**Theorem 6.1.** $p_{4,2} \left( \frac{1}{a} \right)$, $p_{8,3} \left( \frac{1}{a} \right)$, $p_{8,5} \left( \frac{1}{a} \right)$, $p_{16,4} \left( \frac{1}{a} \right)$ and $p_{16,8} \left( \frac{1}{a} \right)$ are in $S$.

**Proof.** Again we first write these power series as quotients of elements in $S$.

1. The proof of Theorem 4.2 shows that $p_{4,2} \left( \frac{1}{a} \right) = \left( \frac{b^4}{a^3} \right) (a^2 + b^2)$ while $p_{8,3} \left( \frac{1}{a} \right) = \left( \frac{b^4}{a^3} \right) (a + b)^3$.
2. $p_{8,5} \left( \frac{1}{a} \right) = \left( \frac{1}{a^3} \right) p_{8,0}(a)p_{8,4}(a^2)p_{8,4}(a^4) = \left( \frac{1}{a^3} \right) p_{2,1}(a) \left( p_{4,0}(a) \right)^2 \left( p_{2,1}(a) \right)^4 = \left( \frac{b^4}{a^3} \right) (a + b)^5$.
3. $p_{16,4} \left( \frac{1}{a} \right) = p_{16,4}p_{4,0} \left( \frac{1}{a} \right) = \left( p_{4,1} \left( \frac{b^4}{a^3} \right) \right)^4$. And $p_{4,1} \left( \frac{b^4}{a^3} \right) = \left( \frac{1}{a^3} \right) p_{4,1}(ab)p_{4,0}(a^2b^2)$
   \[
   = \left( \frac{1}{a^3} \right) p_{2,1}(ab) \left( p_{2,0}(ab) \right)^2 = \frac{c^4d^2}{a^3}(ab + c^2d^2).
   \]
4. $p_{8,0} \left( \frac{1}{a} \right) = \left( \frac{b^4}{a^3} \right) \left( p_{2,0}(ab) \right)^4 = \left( \frac{b^4c^4d^2}{a^3} \right)^8$. If follows that $p_{16,8} \left( \frac{1}{a} \right) = p_{16,8}p_{8,0} \left( \frac{1}{a} \right) = \left( \frac{b^4c^4d^2}{a^3} \right)^8$. Now $p_{2,1} \left( \frac{b^4c^4d^2}{a^3} \right) = \left( \frac{1}{a^3} \right) p_{2,1} \left( (ab)(cd) \right) = \left( \frac{1}{a^3} \right) (ab^3c^2 + c^3d^3)$. 

We conclude with our by now standard computer procedure. For example to show that \( \left( \frac{1}{a} \right) (ab^3c^2 + c^3d^3) \) is in \( S \) we set \( u = x_1x_4^2x_2^2 + x_2^3x_3^3, v = x_1^2 \) and use Macaulay 2 to verify that \( (N, v) = (N, u, v) \). \(\square\)

**Lemma 6.2.** \( p_{16,0} \left( \frac{1}{a} \right) \) is the sixteenth power of \( \frac{d(ab^2 + bc^2 + ca^2)}{a} \).

**Proof.** Arguing as in the above calculation of \( p_{16,8} \left( \frac{1}{a} \right) \) we find that \( p_{16,0} \left( \frac{1}{a} \right) \) is the eighth power of \( p_{2,0} \left( \frac{bcd}{a} \right) = \frac{abcd}{a^2} + \frac{ab^3c^2 + c^3d^3}{a^2} \). So it suffices to show that \( (abcd + ab^3c^2 + c^3d^3) + d^2(a^2b^4 + b^2c^4 + c^2a^4) = 0 \). To do this, set \( u = (x_1x_4x_2x_3 + x_1x_4^3x_2^2 + x_2^3x_3^3) + x_3^2(x_1^2x_4^4 + x_2^2x_4^2 + x_2^2x_4^1) \). Macaulay 2 shows that \( (N, u) = N \). So \( u \) is in \( N \) and applying \( \phi_r \) gives the result. \(\square\)

**Lemma 6.3.** \( p_{16,0} \left( \frac{1}{a} \right) + (p_{8,4} \left( \frac{1}{b} \right))^4 = u^{16} \) for some \( u \) in \( S \).

**Proof.** \( p_{8,4} \left( \frac{1}{b} \right) = p_{8,0} \left( \frac{1}{b} \right) + p_{4,0} \left( \frac{1}{b} \right) \). Using Lemma 3.10 we find that this is \( \left( \frac{acd}{b} \right)^8 + \left( \frac{b^2}{c} \right)^4 \). So the left-hand side in the statement of the lemma is the sixteenth power of \( u = \frac{d(ab^2 + bc^2 + ca^2)}{a} + \frac{a^2b^2c^2}{b^2} + \frac{c^3}{b} \). It remains to show that this \( u \) is in \( S \). This is established using Macaulay 2 in the usual way. \(\square\)

**Theorem 6.4.** \( p_{32,0} \left( \frac{1}{a} \right) \) and \( p_{64,16} \left( \frac{1}{a} \right) \) are in \( S \).

**Proof.** Applying \( p_{32,0} \) to the identity of Lemma 6.3 we find that \( p_{32,0} \left( \frac{1}{a} \right) = (p_{2,0}(u))^{16} \) with \( u \) in \( S \). Applying \( p_{64,16} \) to the identity we find that \( p_{64,16} \left( \frac{1}{a} \right) + \left( p_{16,4} \left( \frac{1}{b} \right) \right)^4 = (p_{4,1}(u))^{16} \). But Theorem 6.1 shows that \( p_{16,4} \left( \frac{1}{b} \right) \) is in \( S \). \(\square\)

**Theorem 6.5.** \( p_{32,12} \left( \frac{1}{a} \right) \) and \( p_{128,48} \left( \frac{1}{a} \right) \) are in \( S \).

**Proof.** We show how the second result follows from the first. Applying \( p_{128,48} \) to the identity of Lemma 6.3 we find that \( p_{128,48} \left( \frac{1}{a} \right) + \left( p_{32,12} \left( \frac{1}{b} \right) \right)^4 = (p_{8,3}(u))^{16} \).

Since \( p_{32,12} \left( \frac{1}{b} \right) \) is in \( S \) and \( p_{8,3} \) stabilizes \( S \) we get the second result. To prove the first result we once again express our element as a quotient of two elements of \( S \). \( p_{32,12} \left( \frac{1}{a} \right) = p_{32,12}p_{4,0} \left( \frac{1}{a} \right) = p_{32,12} \left( \frac{b^2}{a^2} \right) = \left( p_{8,3} \left( \frac{b^2}{a} \right) \right)^4 \). So it’s enough to show that \( p_{8,3} \left( \frac{b^2}{a} \right) \) is in \( S \). Now \( p_{8,3} \left( \frac{b^2}{a} \right) = \left( \frac{1}{a^2} \right) p_{8,3} ((a^2b^2)(ab)(a^4)) = \left( \frac{1}{a^2} \right) p_{8,3}(ab) p_{8,5}(ab) p_{8,4}(a^4) \). Now modulo \( a^8 \), \( p_{8,1}(ab) = p_{8,0}(a)p_{8,1}(b) = c^6(b + c^4) \). Also \( p_{4,1}(ab) = p_{2,1}(ab) = ab + c^2d^2 \). So modulo \( a^8 \), \( p_{8,5}(ab) = p_{4,1}(ab) + c^6(b + c^4) = ab + c^2d^2 + c^6(b + c^4) \). We conclude that \( p_{8,3} \left( \frac{b^2}{a^2} \right) \) is the sum of an element of \( S \) and \( \frac{1}{a^2}(a^2b^2 + c^4d^4) \). A Macaulay 2 calculation shows that this last element is in \( S \). \(\square\)
7 \ l = 11, 13 and 15

We state the results for these \( l \) with very brief indications of proofs.

**Lemma 7.1.** Let \( a = [r] \) with \( r \) prime to \( l \). Then \( p_{8,k} \left( \frac{1}{a} \right), p_{16,2k} \left( \frac{1}{a} \right), p_{32,4k} \left( \frac{1}{a} \right) \) and \( p_{64,8k} \left( \frac{1}{a} \right) \) are all quotients of elements of \( S \) by powers of \( a \).

**Proof.** \( p_{8,k} \left( \frac{1}{a} \right) = \frac{1}{a^k} p_{8,k}(a^7) \), and we use Lemma 2.2. For the remaining results we may assume that \( r = 4s \). Let \( b = [2s], c = [s], e = [3s] \) so that \( p_{2,0}(a) = b^4, p_{2,0}(ab) = c^2e^2 \). Then \( p_{64,8k} \left( \frac{1}{a} \right) = p_{64,8k}p_{8,0} \left( \frac{1}{a} \right) \). By Lemma 3.10 this is the eighth power of \( p_{8,k} \left( \frac{bce}{a} \right) \), and we use the fact that \( p_{8,k}(a^7bce) \) is in \( S \). \( p_{16,2k} \left( \frac{1}{a} \right) \) and \( p_{32,4k} \left( \frac{1}{a} \right) \) are treated similarly.

**Theorem 7.2.** Let \( a = [r] \) with \( r \) prime to \( l \).

(1) When \( l = 11 \), \( p_{8,1}, p_{8,3}, p_{8,6}, p_{16,4}, p_{16,8}, p_{16,10}, p_{32,0}, p_{32,12} \) and \( p_{64,16} \) all take \( \frac{1}{a} \) to an element of \( S \).

(2) When \( l = 13 \), \( p_{8,2}, p_{8,3}, p_{8,5}, p_{16,4}, p_{16,8}, p_{16,14}, p_{32,0}, p_{32,12} \) and \( p_{64,16} \) all take \( \frac{1}{a} \) to an element of \( S \).

(3) When \( l = 15 \), \( p_{8,1}, p_{8,2}, p_{8,3}, p_{16,4}, p_{16,6}, p_{16,8}, p_{32,0}, p_{32,12} \) and \( p_{64,16} \) all take \( \frac{1}{a} \) to an element of \( S \).

**Idea of proof.** By Lemma 7.1 each \( p_{q,j} \left( \frac{1}{a} \right) \) is the quotient of an element of \( S \) by a power of \( a \). It’s clear that one can write down such a representation explicitly. In each case the Macaulay 2 argument using the ideal \( N \) of quintic relations shows that \( p_{q,j} \left( \frac{1}{a} \right) \) is in fact in \( S \).

**Corollary 7.3.** Suppose \( l = 11, 13, \) or \( 15 \). Then in each of the mod 128 congruence classes constituting \( U^* \), with the possible exception of the congruence class \( n \equiv 48 \pmod{128} \), \( B(a) \) has relative density 0.

**Proof.** This follows from Theorem 7.2, Corollary 3.4 and the explicit description of \( U^* \) as a union of congruence classes.

I’ll now show that when \( l = 11 \) each \( B(a) \) in fact has relative density 0 in the congruence class 48 mod 128.

**Lemma 7.4.** When \( l = 11 \), \( p_{8,0} \left( \frac{1}{a} \right) + \left( p_{8,4} \left( \frac{1}{b} \right) \right)^4 = u^8 \) for some \( u \) in \( S \).

**Idea of proof.** As we noted in the proof of Lemma 7.1, \( p_{8,0} \left( \frac{1}{a} \right) = \left( \frac{bce}{a} \right)^8 \). Furthermore \( p_{8,4} \left( \frac{1}{b} \right) = p_{8,4}p_{2,0} \left( \frac{1}{b} \right) = \left( \frac{b}{a} \right)^{p_{8,4}(b^5c^4)}. \) This is the quotient of a
square in $S$ by $b^8$. It follows that the left-hand side in the statement of Lemma 7.4 is the eighth power of $\frac{w}{ab}$, for some $v$ in $S$. Our usual Macaulay 2 technique shows that $\frac{w}{ab}$ is in fact in $S$.

**Theorem 7.5.** When $l = 11$, $p_{128,48} \left( \frac{1}{a} \right)$ is in $p_{128,48}(S)$. In fact it’s the eighth power of an element of $p_{16,6}(S)$. Corollary 3.4 then shows that $B(a)$ has relative density 0 in the congruence class 48 mod 128, and consequently in $U^*$. 

**Proof.** Applying $p_{128,48}$ to the identity of Lemma 7.4 we find that $p_{128,48} \left( \frac{1}{a} \right) + \left( p_{32,12} \left( \frac{1}{6} \right) \right)^4 = (p_{16,6}(u))^8$. Now $p_{32,12} \left( \frac{1}{6} \right) = p_{32,12}p_{4,0} \left( \frac{1}{6} \right) = p_{32,12} \left( \frac{\frac{12}{b}}{b^7} \right)$, which is the square of $p_{16,6} \left( \frac{c}{12} \right)$. So $p_{128,48} \left( \frac{1}{a} \right)$ is the eighth power of $p_{16,6} \left( \frac{c}{12} \right) + p_{16,6}(u)$, and it will suffice to show that $p_{16,6} \left( \frac{c}{12} \right)$ is in $S$. In fact, $p_{8,3} \left( \frac{c^3}{12} \right)$ is in $S$; the Macaulay 2 calculations going into the proof of Theorem 7.2 show this.

**Remarks.** We’ve established various zero-density results when $l \leq 15$. If we take $l > 15$, computer trouble arises. Suppose for example we restrict ourselves to congruence classes to the modulus 8 that lie in $U^*$. Then necessarily $l \leq 21$ or $l = 25$. When $l = 17$, the classes $n \equiv 5 \pmod{8}$ and $n \equiv 6 \pmod{8}$ are in $U^*$. But the ideal $N$ in $\mathbb{Z}/2[x_1, \ldots, x_8]$ has 28 generators, and attempts, using Macaulay 2, to show that $p_{8,5} \left( \frac{1}{a} \right)$ (or $p_{8,6} \left( \frac{1}{a} \right)$) is in $S$ cause a computer crash. Indeed the computer seemed at its limit in handling the congruence class $n \equiv 16 \pmod{64}$ when $l = 15$; it was an all-day calculation.

For $l = 11$ I don’t know whether Theorem 7.5 can be strengthened to show that $p_{128,48} \left( \frac{1}{a} \right)$ is in $S$. When $l = 13$ or 15 it’s possible that, as in the case $l = 11$, $p_{128,48} \left( \frac{1}{a} \right)$ is the eighth power of an element of $p_{16,6}(S)$. But there’s no analogue of Lemma 7.4 that could be used to prove this.

### 8 The basic classes — a little computer evidence

Fix $l$ together with $r$ prime to $l$ and a basic congruence class $C$. All the elements of $B([r])$ are $\geq -r^2$ and are congruent to $-r^2 \mod l$. There is some evidence that $B([r])$ has density $\frac{1}{2l}$ in $C$, so that “half the elements of $C$ that are $\geq -r^2$ and are congruent to $-r^2 \mod l$ lie in $B([r])$.”

Suppose for example that $l \leq 9$ and we are looking at the basic classes to the modulus 8. These are:
Consider the first \( 2^{17} = 131,072 \) elements of \( C \) that are \( \geq -r^2 \) and congruent to \(-r^2 \mod l \). The number of these lying in \( B([r]) \) has been calculated by O’Bryant [3]. Here are his results.

(1) \( l = 3 \quad n \equiv 7 \pmod 8 \), \( r = 1 \quad 65,411 \)

(2) \( l = 5 \quad n \equiv 7 \pmod 8 \), \( r = 1 \quad 65,397 \quad r = 2 \quad 65,713 \)

(3) \( l = 7 \quad n \equiv 7 \pmod 8 \), \( r = 1 \quad 65,185 \quad r = 2 \quad 65,474 \quad r = 3 \quad 65,622 \)

(4) \( l = 9 \quad n \equiv 1 \pmod 8 \), \( n \equiv 7 \pmod 8 \), \( r = 1 \quad 65,877 \quad r = 2 \quad 65,579 \quad r = 4 \quad 65,813 \)

We may also consider the basic congruence class \( n \equiv 14 \pmod {16} \) when \( l = 7 \). Now if we consider the first 65,536 elements of the class that are \( \equiv -r^2 \mod 7 \) and \( \geq -r^2 \), the number in \( B([r]) \) is 32,673 when \( r = 1 \). It is 32,716 when \( r = 2 \) and 32,981 when \( r = 3 \). All this suggests the following:

**Speculation.** Suppose that \( \rho > \frac{1}{2} \). Consider a basic class \( C \) and the first \( X \) elements in the class that are \( \geq -r^2 \) and congruent to \(-r^2 \mod l \). Of these elements, the number in \( B([r]) \) is \( \frac{X}{2} + O(X^\rho) \).

We might go even further, speculating that this is true not only for the basic classes, but for any congruence class contained in a basic class.

It would be interesting to test these speculations further experimentally. But some caution is in order. Suppose for example that \( l = 9 \). Then the congruence class \( n \equiv 2 \pmod 4 \) is contained in \( U^* \), and as we’ve seen, \( B([1]), B([2]) \) and \( B([4]) \) all have relative density 0 in this class. Consider now the first \( 2^{18} = 262,144 \) elements of this class that are \( \geq -r^2 \) and congruent to \(-r^2 \mod 9 \). The number of these elements that lie in \( B([r]) \) is 102,284 when \( r = 1 \), and 110,034 when \( r = 2 \). This is in good accord with our zero-density result. But when \( r = 4 \) more than half of the elements are in \( B([r]) \)! (The number is 137,657.) So we are advised not to place too much predictive power in such computer counts unless the range over which we’re counting is considerably extended.
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