Reconstruction of the interaction term between dark matter and dark energy using SNe Ia

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Abstract. We apply a parametric reconstruction method to a homogeneous, isotropic and spatially flat Friedmann-Robertson-Walker (FRW) cosmological model filled of a fluid of dark energy (DE) with constant equation of state (EOS) parameter interacting with dark matter (DM). The reconstruction method is based on expansions of the general interaction term and the relevant cosmological variables in terms of Chebyshev polynomials which form a complete set orthonormal functions. This interaction term describes an exchange of energy flow between the DE and DM within dark sector. To show how the method works we do the reconstruction of the interaction function expanding it in terms of only the first six Chebyshev polynomials and obtain the best estimation for the coefficients of the expansion assuming three models: (a) a DE equation of the state parameter $w = -1$ (an interacting cosmological $\Lambda$), (b) a DE equation of the state parameter $w = \text{constant}$ with a dark matter density parameter fixed, (c) a DE equation of the state parameter $w = \text{constant}$ with a free constant dark matter density parameter to be estimated, and using the Union2 SNe Ia data set from "The Supernova Cosmology Project" (SCP) composed by 557 type Ia supernovae. In both cases, the preliminary reconstruction shows that in the best scenario there exist the possibility of a crossing of the noninteracting line $Q = 0$ in the recent past within the $1\sigma$ and $2\sigma$ errors from positive values at early times to negative values at late times. This means that, in this reconstruction, there is an energy transfer from DE to DM at early times and an energy transfer from DM to DE at late times. We conclude that this fact is an indication of the possible existence of a crossing behavior in a general interaction coupling between dark components.

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1. Introduction

In the last years the accelerated expansion of the universe has now been confirmed by several independent observations including those of high redshift ($z \leq 1$) type Ia Supernovae (SNeIa) data at cosmological distances [1]-[2]. This has been verified by precise measurements of the power spectrum of the cosmic microwave background (CMB) anisotropies [3]-[4], the galaxy power spectrum detection and the baryon acoustic peak in the large-scale correlation function of luminous red galaxies in the experiment Sloan Digital Sky Survey (SDSS) [5]-[6]. To explain these observations, it has been postulated the existence of a new and enigmatic component of the universe so-called dark energy (DE) [7]-[30] from which the cosmological constant is the simplest model [22], [31]-[37]. Recent observations [2], [4], [6], [38]-[39] show that if it is assumed a dark energy (DE) equation of state (EOS) with constant parameter $w = P_{DE}/\rho_{DE}$, then there remains little room for departure of DE from the cosmological constant. In addition these observations indicate that our universe is flat and it consists of approximately 70% of Dark Energy (DE) in the form of a cosmological constant, 25% of Dark Matter and 5% of baryonic matter.

However the cosmological constant model has two serious problems: the first of them is the cosmological constant problem [20], [22], [31]-[37] which consists in why the observed value of the Cosmological Constant $\rho_{\Lambda}^{\text{obs}} \sim (10^{-12} \text{ Gev})^4$ is so-small compared with the theoretical value $\rho_{\Lambda}^{\text{Pl}} \sim (10^{18} \text{ Gev})^4$ predicted from local quantum field theory if we are confident in its application to the Planck scale?. The second problem is the so named The Cosmic Coincidence problem [12]-[30] consisting in why, in the present, the energy density of DE is comparable with the density of dark matter (DM) while the first one is subdominant during almost all the past evolution of the universe?.

In the last decade, in order to solve the The Cosmic Coincidence problem, several researchers have considered a possible phenomenological interaction between the DE and DM components [40]-[87]. As far as we know, the first models of dark energy coupled with dark matter were proposed by Wetterich [9], [11] in the framework of a scalar field with an exponential potential (named The Cosmon) coupled with the matter. Some years later, with the discovery of the recent accelerated expansion of the universe [1]-[2] and in order to solve the coincidence problem, several authors put forward the idea of a coupled scalar field with dark matter named Coupled Quintessence [40]-[46], [56]-[58], [66], [71]-[72], [82]. On the other hand, the theory of dynamical systems have been applied to different models of coupled dark energy in order to clarify the cosmological evolution of the solutions of every model with emphasis in the study of the critical points [88]-[94].

Some recent studies have claimed that, for reasonable and suitably chosen interaction terms, the coincidence problem can be significantly ameliorated in the sense that the rate of densities $r \equiv \rho_{DM}/\rho_{DE}$ either tends to a constant or varies more slowly than the scale factor, $a(t)$, in late times [66], [76]. However, the existence or not of some class of interaction between dark components is to be discerned observationally. To this
recently, constraints on the strength of such interaction have been put using different observations [96]-[134].

Recently, it has been suggested that an interacting term $Q(z)$ dependent of the redshift crosses the noninteracting line $Q(z) = 0$ [133]-[134]. In [133], this conclusion have been obtained using observational data samples in the range $z \in [0, 1.8]$ in order to fit a scenario in which the whole redshift range is divided into a determined numbers of bins and the interaction function is set to be a constant in each bin. They found an oscillatory behavior of the interaction function $Q(z)$ changing its sign several times during the evolution of the universe. On the other hand, in [134] is reported a crossing of the noninteracting line $Q(z) = 0$ under the assumption that the interacting term $Q(z)$ is a linearly dependent interacting function of the scale factor with two free parameters to be estimated. They found a crossing from negative values at the past (energy transfers from dark matter to dark energy) to positive values at the present (energy transfers from dark energy to dark matter) at $z \simeq 0.2 - 0.3$.

While it is not totally clear if an interaction term can solved the The Cosmic Coincidence problem or if such crossing really exists, we can yet put constraints on the size of such assumed general interaction and on the probability of existence of such crossing using recent cosmological data. We will do this postulating the existence of an general nongravitational interaction between the two dark components. We introduce phenomenologically this general interaction term $Q$ into the equations of motion of DE and DM, which describes an energy exchange between these components [40]-[87]. In order to reconstruct the interaction term $Q$ as a function of the redshift we expand it in terms of Chebyshev Polynomials which constitute a complete orthonormal basis on the finite interval [-1,1] and have the nice property to be the minimax approximating polynomial (this technique has been applied to the reconstruction of the DE potential in [137]-[138]). At the end, we do the reconstruction using the observations of “The Supernova Cosmology Project” (SCP) composed by 557 type Ia supernovae [2].

Due to that in this paper our principal goals are: (i) the development of the formalism of reconstruction of the interaction and (ii) the recent reconstruction of the evolution of that interaction, we do not include another data sets like CMB anisotropies, the galaxy power spectrum or the baryon acoustic peak (BAO) measured in the experiment SDSS. Clearly the use of these data sets implies special considerations such as the application of the cosmological perturbation theory in the reconstruction method which is beyond the scope of this paper. We will do the total reconstruction in the evolution of the interaction in our future work.

In our reconstruction process we assume two interacting models: (a) a DE equation of the state parameter $w = -1$ (an interacting cosmological $\Lambda$) and (b) a DE equation of the state parameter $w = \text{constant}$ (as far as know the only reference proposing a reconstruction process of coupled dark energy using parameterizations of the coupling function is [139]).

The organization of this paper is a follow. In the second section we introduce the general equations of motion of the DE model interacting with DM. In the third
2. General equations of motion for dark energy interacting with dark matter.

We assume an universe formed by four components: the baryonic matter fluid \((b)\), the radiation fluid \((r)\), the dark matter fluid \((DM)\) and the dark energy fluid \((DE)\). Moreover all these constituents are interacting gravitationally and additionally only the dark components interact nongravitationally through an energy exchange between them mediated by the interaction term defined below.

The gravitational equations of motion are the Einstein field equations

\[
G_{\mu\nu} = 8\pi G \left[ T^b_{\mu\nu} + T^r_{\mu\nu} + T^{DM}_{\mu\nu} + T^{DE}_{\mu\nu} \right],
\]

whereas that the equations of motion for each fluid are

\[
\nabla^\nu T^b_{\mu\nu} = 0,
\]

\[
\nabla^\nu T^r_{\mu\nu} = 0,
\]

\[
\nabla^\nu T^{DM}_{\mu\nu} = -F_\mu,
\]

\[
\nabla^\nu T^{DE}_{\mu\nu} = F_\mu,
\]

where the respective energy-momentum tensor for the fluid \(i\) is defined as \((i = b, r, DM, DE)\),

\[
T^i_{\mu\nu} = \rho_i u_\mu u_\nu + (g_{\mu\nu} + u_\mu u_\nu) P_i
\]

here \(u_\mu\) is the velocity of the fluids (assumed to be the same for each one) where as \(\rho_i\) and \(P_i\) are respectively the density and pressure of the fluid \(i\) measured by an observer with velocity \(u^\mu\). \(F_\mu\) is the cuadrivector of interaction between dark components and its form is not known a priori because in general we do not have fundamental theory, in case of existing, to predict its structure.

We project the equations \((2)-(5)\) in a part parallel to the velocity \(u^\mu\),

\[
u^\mu \nabla^\nu T^b_{\mu\nu} = 0,
\]

\[
u^\mu \nabla^\nu T^r_{\mu\nu} = 0,
\]

\[
u^\mu \nabla^\nu T^{DM}_{\mu\nu} = -u^\mu F_\mu,
\]

section, we write the cosmological equations for both interacting dark components. In the forth section we develop the reconstruction scheme of the interaction term in terms of a expansion of Chebyshev polynomials. In the fifth section, we briefly describe the application of the type Ia Supernova data cosmological test and the priors used on the free parameters of the reconstruction together with a brief discussion of the results of our reconstruction and the best estimated values of the parameters fitting the observations. Finally, in the last section we discuss our main results and present our conclusions.
and in other part orthogonal to the velocity using the projector $h_{\beta\mu} = g_{\beta\mu} + u_{\beta}u_{\mu}$ acting on the hypersurface orthogonal to the velocity $u^{\mu}$,

\begin{align*}
    h_{\mu\beta} \nabla^{\nu} T^{b}_{\mu\nu} &= 0, \quad (11) \\
    h_{\mu\beta} \nabla^{\nu} T^{r}_{\mu\nu} &= 0, \quad (12) \\
    h_{\mu\beta} \nabla^{\nu} T^{DM}_{\mu\nu} &= -h^{\mu\beta} F_{\mu}, \quad (13) \\
    h_{\mu\beta} \nabla^{\nu} T^{DE}_{\mu\nu} &= h^{\mu\beta} F_{\mu}, \quad (14)
\end{align*}

using (6) in (11)-(14) we obtain the mass energy conservation equations for each fluid,

\begin{align*}
    u^{\mu} \nabla_{\mu} \rho_{b} + (\rho_{b} + P_{b}) \nabla_{\mu} u^{\mu} &= 0, \quad (15) \\
    u^{\mu} \nabla_{\mu} \rho_{r} + (\rho_{r} + P_{r}) \nabla_{\mu} u^{\mu} &= 0, \quad (16) \\
    u^{\mu} \nabla_{\mu} \rho_{DM} + (\rho_{DM} + P_{DM}) \nabla_{\mu} u^{\mu} &= u^{\mu} F_{\mu}, \quad (17) \\
    u^{\mu} \nabla_{\mu} \rho_{DE} + (\rho_{DE} + P_{DE}) \nabla_{\mu} u^{\mu} &= -u^{\mu} F_{\mu}, \quad (18)
\end{align*}

at the other hand it introducing (6) in (11)-(14) it permits to have the Euler equations for every fluid,

\begin{align*}
    h^{\mu\beta} \nabla_{\mu} P_{b} + (\rho_{b} + P_{b}) u^{\mu} \nabla_{\mu} u^{\beta} &= 0, \quad (19) \\
    h^{\mu\beta} \nabla_{\mu} P_{r} + (\rho_{r} + P_{r}) u^{\mu} \nabla_{\mu} u^{\beta} &= 0, \quad (20) \\
    h^{\mu\beta} \nabla_{\mu} P_{DM} + (\rho_{DM} + P_{DM}) u^{\mu} \nabla_{\mu} u^{\beta} &= -h^{\mu\beta} F_{\mu}, \quad (21) \\
    h^{\mu\beta} \nabla_{\mu} P_{DE} + (\rho_{DE} + P_{DE}) u^{\mu} \nabla_{\mu} u^{\beta} &= h^{\mu\beta} F_{\mu}, \quad (22)
\end{align*}

Finally we closed the system of equations assuming the following state equations for the respectively baryonic, dark matter, radiation components,

\begin{align*}
    P_{b} &= 0 \quad (23) \\
    P_{DM} &= 0 \quad (24) \\
    P_{r} &= \frac{1}{3} \rho_{r} \quad (25)
\end{align*}

while for the dark energy we assume a state equation with constant parameter $w$,

\begin{align*}
    P_{DE} &= w \rho_{DE} \quad (26)
\end{align*}
3. Cosmological Equations of motion for dark energy interacting with dark matter.

We assumed that the background metric is described by the flat Friedmann-Robertson-Walker (FRW) metric written in comoving coordinates as supported by the anisotropies of the cosmic microwave background (CMB) radiation measured by the WMAP experiment [3]

\[ ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2 \right), \]

where \( a(t) \) is the scale factor and \( t \) is the cosmic time.

In these coordinates we choose for the normalized velocity,

\[ u^\mu = (1, 0, 0, 0) \]  

and therefore we have,

\[ \nabla_\mu u^\mu = 3 \frac{\dot{a}}{a} \equiv 3H \]

\[ u^\nu \nabla_\nu u^\beta = 0 \]

where \( H \) is the Hubble parameter and the point means derivative respect to the cosmic time. In congruence with the symmetries of spatial isotropy and homogeneity of the FRW spacetime, the densities and pressures of the fluids are depending only of the cosmic time, \( \rho_i(t), P_i(t) \), and at the same time the parallel and orthogonal components of the quadrivector of interaction with respect to the velocity are respectively,

\[ u^\mu F_\mu = Q(a) \]

\[ h^\mu_\beta F_\mu = 0 \]

where \( Q(a) \) is known as the interaction function depending on the scale factor. The introduction of the state equations (23)-(26), the metric (27) and the expressions (28)-(32) in the equations of mass energy conservation for the fluids (15)-(18) produces,

\[ \dot{\rho}_b + 3H\dot{\rho}_b = 0, \]

\[ \dot{\rho}_r + 4H\rho_r = 0, \]

\[ \dot{\rho}_{DM} + 3H\rho_{DM} = Q, \]

\[ \dot{\rho}_{DE} + 3(1+w)H\rho_{DE} = -Q, \]

At the other hand, the Euler equations (19)-(22) are satisfied identically and do not produce any new equation. From the Einstein equation (1) we complete the equations of motion with the first Friedmann equation,

\[ H^2(a) = \frac{8\pi G}{3} (\rho_b + \rho_r + \rho_{DM} + \rho_{DE}). \]

Its convenient to define the following dimensionless density parameters \( \Omega_i^* \), for \( i = b, r, DM, DE \), as the energy densities normalized by the critical density at the actual epoch,

\[ \Omega_i^* \equiv \frac{\rho_i}{\rho_{crit}}, \]
and the corresponding dimensionless density parameters at the present,

\[ \Omega_i^0 \equiv \frac{\rho_i^0}{\rho_{\text{crit}}^0}, \] (39)

where \( \rho_{\text{crit}}^0 \equiv 3H_0^2/8\pi G \) is the critical density today and \( H_0 \) is the Hubble constant. Solving (33) and (34) in terms of the redshift \( z \), defined as \( a = 1/(1 + z) \), we obtain the known solutions for the baryonic matter and radiation density parameters respectively:

\[ \Omega_b^*(z) = \Omega_b^0 (1 + z)^3, \] (40)

\[ \Omega_r^*(z) = \Omega_r^0 (1 + z)^4, \] (41)

The energy conservation equations (35) and (36) for both dark components are rewritten in terms of the redshift as:

\[ \frac{d\rho_{DM}}{dz} - \frac{3}{1 + z} \rho_{DM} = -\frac{Q(z)}{(1 + z) \cdot H(z)}, \] (42)

\[ \frac{d\rho_{DE}}{dz} - \frac{3(1 + w)}{1 + z} \rho_{DE} = \frac{Q(z)}{(1 + z) \cdot H(z)}. \] (43)

Phenomenologically, we choose to describe the interaction between the two dark fluids as an exchange of energy at a rate proportional to the Hubble parameter:

\[ Q(z) \equiv \rho_{\text{crit}}^0 \cdot (1 + z)^3 \cdot H(z) \cdot I_Q(z), \] (44)

The term \( \rho_{\text{crit}}^0 \cdot (1 + z)^3 \) has been introduced by convenience in order to mimic a rate proportional to the behavior of a matter density without interaction. Let be note that the dimensionless interaction function \( I_Q(z) \) depends of the redshift and it will be the function to be reconstructed. With the help of (44) we rewrite the equations for the dark fluids (42)-(43) as,

\[ \frac{d\Omega_{DM}^*}{dz} - \frac{3}{1 + z} \Omega_{DM}^* = -(1 + z)^2 \cdot I_Q(z), \] (45)

\[ \frac{d\Omega_{DE}^*}{dz} - \frac{3(1 + w)}{1 + z} \Omega_{DE}^* = (1 + z)^2 \cdot I_Q(z), \] (46)

4. General Reconstruction of the interaction using Chebyshev polynomials.

We do the parametrization of the dimensionless coupling \( I_Q(z) \) in terms of the Chebyshev polynomials, which form a complete set of orthonormal functions on the interval \([-1, 1]\). They also have the property to be the minimax approximating polynomial, which means that has the smallest maximum deviation from the true function at any given order [137-138]. Without loss of generality, we can then expand the coupling \( I_Q(z) \) in the redshift representation as:

\[ I_Q(z) \equiv \sum_{n=0}^{N} \lambda_n \cdot T_n(z), \] (47)
where \( T_n(z) \) denotes the Chebyshev polynomials of order \( n \) with \( n \in [0, N] \) and \( N \) a positive integer. The coefficients of the polynomial expansion \( \lambda_n \) are real free dimensionless parameters. Then the interaction function can be rewritten as

\[
Q(z) = \rho_{crit}^0 \cdot (1 + z)^3 \cdot H(z) \cdot \sum_{n=0}^{N} \lambda_n \cdot T_n(z),
\]

We introduce (47) in (45)-(46) and integrate both equations obtaining the solutions,

\[
\Omega_{DM}^*(z) = (1 + z)^3 \left[ \Omega_{0, DM}^0 - \frac{z_{\text{max}}}{2} \sum_{n=0}^{N} \lambda_n \cdot K_n(x, 0) \right],
\]

\[
\Omega_{DE}^*(z) = (1 + z)^{3(1+w)} \left[ \Omega_{0, DE}^0 + \frac{z_{\text{max}}}{2} \sum_{n=0}^{N} \lambda_n \cdot K_n(x, w) \right],
\]

where we have defined the integrals

\[
K_n(x, w) = \int_{1}^{x} \frac{T_n(\tilde{x})}{(a + b\tilde{x})(1+3w)} d\tilde{x},
\]

and the quantities,

\[
x \equiv \frac{2z}{z_{\text{max}}} - 1,
\]

\[
a \equiv 1 + \frac{z_{\text{max}}}{2},
\]

\[
b \equiv \frac{z_{\text{max}}}{2},
\]

here \( z_{\text{max}} \) is the maximum redshift at which observations are available so that \( x \in [-1, 1] \) and \( |T_n(x)| \leq 1 \), for all \( n \in [0, N] \).

Finally, using the solutions (49)-(50) and (49)-(50) we rewrite the Friedmann equation (37) as

\[
H^2(z) = H_0^2 \left[ \Omega_{b}^0(1 + z)^3 + \Omega_{\gamma}^0(1 + z)^4 + \Omega_{DM}^0(z) + \Omega_{DE}^0(z) \right],
\]

The Hubble parameter depends of the parameters \((H_0, \Omega_{b}^0, \Omega_{\gamma}^0, \Omega_{DM}^0, \Omega_{DE}^0, w)\) and the dimensionless coefficients \( \lambda_n \). However one of the parameters depends of the others due to the Friedmann equation evaluated at the present,

\[
\Omega_{DE}^0 = 1 - \Omega_{b}^0 - \Omega_{\gamma}^0 - \Omega_{DM}^0
\]

At the end, for the reconstruction, we have the five parameters \((H_0, \Omega_{b}^0, \Omega_{\gamma}^0, \Omega_{DM}^0, w)\) and the dimensionless coefficients \( \lambda_n \).

To do a general reconstruction in (49)-(50) we must take \( N \rightarrow \infty \) and to obtain the solutions in a closed form. The details of the calculation of the integrals \( K_n(x, w) \) in the right hand side of (49)-(50) are shown in detail in the Appendix A which shows the closed forms (A.9)-(A.10) for the integrals with odd and even integer \( n \) subindex, and valid for \( w \neq n/3 \) where \( n \geq 0 \).

Finally, we point out the formula we use for the reconstruction of other important cosmological property of the universe:
5. Reconstruction of the interaction up to order $N = 5$ using the type Ia Supernovae test.

To simplify our analysis and to show how the method works, in this section we reconstruct the coupling function $I_Q(z)$ to different orders ($N = 1, 2, 3, 4, 5$), up to order $N = 5$, using the type Ia Supernovae test. The details of this reconstruction are described in the Appendix B. We test and constrain the coupling function $I_Q(z)$ using the "Union2" SNe Ia data set from “The Supernova Cosmology Project” (SCP) composed by 557 type Ia supernovae [2]. As it is usual, we use the definition of luminosity distance $d_L$ (see [1]) in a flat cosmology,$$d_L(z) = c(1 + z) \int_0^z \frac{dz'}{H(z')} (58)$$where $H(z, X)$ is the Hubble parameter, i.e., the expression (55), ",c" is the speed of light given in units of km/sec and $X$ represents the parameters of the model,$$X \equiv (H_0, \Omega_b^0, \Omega_r^0, \Omega_{DM}^0, w, \lambda_1, ... , \lambda_N) (59)$$The theoretical distance moduli for the $k$-th supernova with redshift $z_k$ is defined as$$\mu_{th}^{\text{th}}(z_k, X) \equiv m(z) - M = 5 \log_{10} \left[ \frac{d_L(z_k, X)}{\text{Mpc}} \right] + 25 (60)$$where $m$ and $M$ are the apparent and absolute magnitudes of the SNe Ia respectively, and the superscript “th” stands for “theoretical”. We construct the statistical $\chi^2$ function as$$\chi^2(X) = \sum_{k=1}^{n} \frac{[\mu_{th}^{\text{th}}(z_k, X) - \mu_k]^2}{\sigma_k^2} (61)$$where $\mu_k$ is the observational distance moduli for the $k$-th supernova, $\sigma_k^2$ is the variance of the measurement and $n$ is the amount of supernova in the data set. In this case $n = 557$, using the “Union2” SNe Ia data set [2].

With this $\chi^2$ function we construct the probability density function (pdf) as$$\text{pdf}(X) = A \cdot e^{-\chi^2 / 2} (62)$$where $A$ is an integration constant.

5.1. Priors on the the probability density function (pdf).

In the models I, II and III shown in the Table 1 we marginalize the parameters $Y = (H_0, \Omega_{DM}^0, \Omega_b^0, \Omega_r^0)$ in the pdf (62) choosing priors on them. In order to it, we must compute the following integration,
\[ \text{pdf}(V) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \text{pdf}(X) \text{pdf}(Y) \, dH_0 \, d\Omega_{DM}^0 \, d\Omega_b^0 \, d\Omega_r^0 \]  

where \( V = (w, \lambda_1, ..., \lambda_N) \) represents the nonmarginalized parameters, \( \text{pdf}(X) \) is given by (62) and \( \text{pdf}(Y) \) is the prior probability distribution function for the parameters \( (H_0, \Omega_{DM}^0, \Omega_b^0, \Omega_r^0) \) which are chosen as Dirac delta priors around the specific values \( \tilde{Y} = (\tilde{H}_0, \tilde{\Omega}_{DM}^0, \tilde{\Omega}_b^0, \tilde{\Omega}_r^0) \) measured by some other independent observations,

\[ \text{pdf}(Y) = \delta(H_0 - \tilde{H}_0) \cdot \delta(\Omega_{DM}^0 - \tilde{\Omega}_{DM}^0) \cdot \delta(\Omega_b^0 - \tilde{\Omega}_b^0) \cdot \delta(\Omega_r^0 - \tilde{\Omega}_r^0) \]  

Introducing (64) in (63) it produces,

\[ \text{pdf}(V) = A \cdot e^{-\tilde{\chi}^2/2} \]  

where we have defined a new function \( \tilde{\chi}^2 \) depending only on the parameters \( V = (w, \lambda_1, ..., \lambda_N) \) as,

\[ \tilde{\chi}^2(V) \equiv \sum_{k=1}^{n} \left[ \frac{\mu_{th}(z_k, V, \tilde{Y}) - \mu_k}{\sigma_k^2} \right]^2 \]  

The specific values chosen for the Dirac delta priors are,

- \( \tilde{H}_0 = 72 \) (km/s)Mpc\(^{-1} \) as suggested by the observations of the Hubble Space Telescope (HST) [140].
- \( \tilde{\Omega}_{DM}^0 = 0.233 \)
- \( \tilde{\Omega}_b^0 = 0.0462 \)
- \( \tilde{\Omega}_r^0 = 4.62 \times 10^{-5} \)

Once constructed the function \( \tilde{\chi}^2 \), we numerically minimize it to compute the "best estimates" for the free parameters of the model: \( V = (w, \lambda_1, ..., \lambda_N) \). The minimum value of the \( \tilde{\chi}^2 \) function gives the best estimated values of \( V \) and measures the goodness-of-fit of the model to data.

For the Model IV, we leave too the parameter \( \Omega_{DM}^0 \) free to vary and estimated it from the minimization of the \( \tilde{\chi}^2 \) function. In this case, the parameters to be marginalized are \( Y = (H_0, \Omega_b^0, \Omega_r^0) \). Then, the marginalization will be as,

\[ \text{pdf}(V) = \int_0^\infty \int_0^\infty \int_0^\infty \text{pdf}(X) \text{pdf}(Y) \, dH_0 \, d\Omega_b^0 \, d\Omega_r^0 \]  

where now \( V = (w, \Omega_{DM}^0, \lambda_1, ..., \lambda_N) \) represents the nonmarginalized parameters to be estimated, \( \text{pdf}(X) \) is given by (62) and \( \text{pdf}(Y) \) is the prior probability distribution function for the parameters \( (H_0, \Omega_b^0, \Omega_r^0) \) which are chosen as Dirac delta priors around the specific values \( \tilde{Y} = (\tilde{H}_0, \tilde{\Omega}_b^0, \tilde{\Omega}_r^0) \) given above.

In the models II, III and IV the interaction function \( I_Q(z) \) will be reconstructed up to order \( N = 5 \) in the expansion in terms of Chebyshev polynomials.
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5.2. Results of the reconstruction of the interaction function.

Now, we present the results of the fit of the models listed in the Table 1 with the the “Union2” SNe Ia data set [2] and the priors described in the Section 5.1. For the noninteracting model I, the only free parameter to be estimated is \( \theta = \{ w \} \), whilst for the interacting models II, III and IV the free parameters are \( \theta = \{ \lambda_0, ..., \lambda_N \} \), \( \theta = \{ w, \Omega^0_{DM}, \lambda_0, ..., \lambda_N \} \) respectively, where \( N \) is taking the values \( N = 1, 2, 3, 4, 5 \). In every case, we obtain the best fit parameters and the corresponding \( \tilde{\chi}^2_{\text{min}} \).

The Figure 1 show the reconstruction of the dimensionless interaction function \( I_Q(z) \) as a function of the redshift for the models II (corresponding to a dark energy EOS parameter \( w = -1 \)), III (corresponding to a dark energy EOS parameter \( w = \text{constant} \) and \( \Omega^0_{DM} \text{fixed} \)) and IV (corresponding to a dark energy EOS parameter \( w = \text{constant} \) and \( \Omega^0_{DM} \) as a free parameter to be estimated) respectively.

The Figure 2 show the reconstruction of the the dark matter and dark energy density parameters \( \Omega^*_{DM}(z), \Omega^*_{DE}(z) \) as a function of the redshift for the models II, III and IV described above. The same applies to the Figure 3 which shows the reconstruction of the deceleration parameter \( q(z) \).

All the Figures show the superposition of the best estimates for every cosmological variable using the expansion of the interaction function \( I_Q(z) \) in terms of the parameters \( \lambda_n \) corresponding to the Chebyshev polynomial expansion (47) ranging from \( N = 1 \) to 5. The Tables 3, 5 and 7 show the best fit parameters and the minimum of the function \( \chi^2_{\text{min}} \) for the models II, III and IV respectively. From these tables, we can note the fast convergence of the best estimates when the numbers of parameters \( N \) is increased in the expansion (47).

From the Figure 1 we first notice that, for all interacting models, the best estimates for the interaction function \( I_Q(z) \) cross marginally the noninteracting line \( I_Q(z) = 0 \) during the present cosmological evolution (at around \( z \approx 0.08 \)) changing sign from positive.
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values at the past (energy transfers from dark energy to dark matter) to negative values at the present (energy transfers from dark matter to dark energy). However, taking into account the errors corresponding to the fit using three parameters ($N = 2$), we see that within the $1\sigma$ and $2\sigma$ errors, it exists the possibility of crossing of the noninteraction line in the recent past at around the range $z \in (0.04, 0.16)$.

Crossings of the noninteracting line $Q(z) = 0$ have been recently reported at the references [133] (with an interacting term $Q(z)$ proportional to the Hubble parameter) and [134] (where the assumption that the interacting term $Q(z)$ is proportional to the Hubble parameter is abandoned), although the direction of the change found in our work is contrary to the results published by Li and Zhang in the reference [134] where a crossing (from negative values at the past to positive values at the present) was found at $z \simeq 0.2 - 0.3$ with a linearly dependent interacting function of the scale factor with two free parameters. On the other hand, we did not find the oscillatory behavior of the interaction function found by Cai and Su [133] who, using observational data samples in the range $z \in [0, 1.8]$, fitted a scheme in which the whole redshift range is divided into a determined numbers of bins and the interaction function set to be a constant in each bin.

Let’s mention that, recently, it has been mentioned that a crossing of the noninteracting line $Q(z) = 0$ implying a transfer of energy from DM to DE may well conflict with the second law of thermodynamics [135]. Too, in the context of bouncing coupled dark energy scenarios, a possible inversion of the flux of energy from DM to DE has been found in terms of a change of sign of the gradient of a scalar field [136] which, of course, it is not the case considered here.

The Figure 2 shows that, for all the interacting models studied in this work, the best estimates for the dark energy density parameter $\Omega_{DE}^*(z)$ become to be definite positive at all the range of redshifts considered in the data sample. However, this statement is conclusive which shows that within the $1\sigma$ and $2\sigma$ errors for the fit with three parameters ($N = 2$), the $\Omega_{DE}^*(z)$ becomes positive in all the range of redshifts considered (remember that we are using only the SNe Ia data set which has been attributed to have systematic errors [141]-[144]. However, let us mention that $1\sigma$ error becomes greater if the number of free parameter to be fitted increases, a fact already described in [145] as the cost of the compression.

The Figure 3 shows the reconstruction for the deceleration parameter $q(z)$ for the fit with $N = 2$ parameters and their respective $1\sigma$ and $2\sigma$ constraints. It also shows that, for all models, a transition from a deceleration era at early times dominated by the dark and baryonic matter density to an acceleration era at late times corresponding to the present domain of the dark energy density. At the present, the dark energy density parameter becomes $\Omega_{DE}^0 \approx 0.7$ as we can see from the Figure 2 which is sufficiently large to generate a non negligible dimensionless interacting term of the or-
**Model I:** \( w = \text{constant}. \)

Best estimate for the EOS parameter \( w \).

| Errors | \( \pm 1\sigma \) | \( \pm 2\sigma \) |
|--------|-----------------|-----------------|
| \( \omega \) | \(-1.24^{+0.03}_{-0.05}\) | \(-1.24^{+0.06}_{-0.07}\) |

**Table 2.** The best estimate of the dark energy EOS parameter \( w \) for the Model I. The best estimate was computed through a Bayesian statistical analysis using the “Union2” SNe Ia data set giving \( \chi^2_{\text{min}} = 562.51 \).

**Model II:** \( w = -1 \).

Best estimates for the parameters \( \lambda_n \).

| \( N \) | \( \lambda_0 \) | \( \lambda_1 \) | \( \lambda_2 \) | \( \lambda_3 \) | \( \lambda_4 \) | \( \lambda_5 \) | \( \chi^2_{\text{min}} \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( N = 1 \) | \(-1.46 \times 10^{-2}\) | \(2.47 \times 10^{-1}\) | \(0.0\) | \(0.0\) | \(0.0\) | \(544.80\) |
| \( N = 2 \) | \(-1.43 \times 10^{-2}\) | \(2.48 \times 10^{-1}\) | \(1.810 \times 10^{-3}\) | \(0.0\) | \(0.0\) | \(544.72\) |
| \( N = 3 \) | \(-1.40 \times 10^{-2}\) | \(2.50 \times 10^{-1}\) | \(1.800 \times 10^{-3}\) | \(1.07 \times 10^{-5}\) | \(0.0\) | \(544.58\) |
| \( N = 4 \) | \(-1.37 \times 10^{-2}\) | \(2.51 \times 10^{-1}\) | \(2.0 \times 10^{-3}\) | \(1.07 \times 10^{-6}\) | \(0.0\) | \(544.48\) |
| \( N = 5 \) | \(-1.32 \times 10^{-2}\) | \(2.52 \times 10^{-1}\) | \(1.801 \times 10^{-3}\) | \(3.29 \times 10^{-6}\) | \(1.0 \times 10^{-7}\) | \(544.36\) |

**Table 3.** Summary of the best estimates of the dimensionless coefficients \( \lambda_n \) of the expansion of the interaction function for the Model II corresponding to an interacting dark energy EOS parameter \( w = -1 \). The best estimates were computed through a Bayesian statistical analysis using the “Union2” SNe Ia data set. The number \( N \) in the top of every column indicates the maximum number of Chebyshev polynomials used in the expansion of the coupling function \( I_Q(z) \) starting from \( N = 1 \). From the Figure 1 to Figure 3 show the reconstruction of several cosmological variables using the best estimates for \( N = 5 \). Moreover, they also show the reconstruction of several cosmological variables using the best estimates for \( N = 2 \) and their confidence intervals at 1\(\sigma \) and 2\(\sigma \).

In order to study the coincidence problem, in the Figure 5 we plot the best estimates for the rate between dark density parameters \( \Omega_{DE}^*(z)/\Omega_{DM}^*(z) \) for the model II (left above...
panel), III (right above panel) and IV (left below panel).

Finally, in the Figure 4 we plot the 1σ and 2σ constraints on all the pair of parameters taken from the set \{λ₀, λ₁, λ₂\} for the Model II marginalizing on one of them. The same is plotted in the Figure 5 for the model III but with the parameters taken from the set \{w, λ₀, λ₁, λ₂\} and marginalizing on two of them. Finally, in the Figure 6 a similar procedure is applied to the model IV with the parameters taken from the set \{w, Ω_{DM}^{0}, λ₀, λ₁, λ₂\} and marginalizing on three of them.

We noted that for the model III, from the Figure 1 to Figure 3, show the reconstruction of several cosmological variables using the best estimates for \(N = 2\) and their confidence intervals at 1σ and 2σ.

### Model III: \(w = \text{constant}\).

| \(N\) | \(N = 1\) | \(N = 2\) | \(N = 3\) | \(N = 4\) | \(N = 5\) |
|-------|------------|------------|------------|------------|------------|
| \(\lambda₀\) | \(-2.32 \times 10^{-2}\) | \(-1.85 \times 10^{-2}\) | \(-1.80 \times 10^{-2}\) | \(-1.70 \times 10^{-2}\) | \(-1.58 \times 10^{-2}\) |
| \(\lambda₁\) | \(2.46 \times 10^{-1}\) | \(2.48 \times 10^{-1}\) | \(2.49 \times 10^{-1}\) | \(2.50 \times 10^{-1}\) | \(2.51 \times 10^{-1}\) |
| \(\lambda₂\) | 0.0 | \(1.80 \times 10^{-3}\) | \(1.78 \times 10^{-3}\) | \(1.70 \times 10^{-3}\) | \(1.62 \times 10^{-3}\) |
| \(\lambda₃\) | 0.0 | 0.0 | \(1.06 \times 10^{-5}\) | \(1.07 \times 10^{-5}\) | \(1.05 \times 10^{-5}\) |
| \(\lambda₄\) | 0.0 | 0.0 | 0.0 | \(1.0 \times 10^{-6}\) | \(1.19 \times 10^{-6}\) |
| \(\lambda₅\) | 0.0 | 0.0 | 0.0 | 0.0 | \(1.0 \times 10^{-7}\) |
| \(w\) | \(-1.080\) | \(-1.068\) | \(-1.065\) | \(-1.064\) | \(-1.063\) |
| \(\chi^2_{\text{min}}\) | 543.52 | 543.42 | 543.40 | 543.38 | 543.36 |

### Table 5. The same as the Table 4 for the Model III corresponding to a interacting dark energy EOS parameter \(w = \text{constant}\) and \(\Omega_{DM}^{0} = 0.233\). From the Figure 1 to Figure 3 show the reconstruction of several cosmological variables using the best estimates for \(N = 5\). Moreover, they also show the reconstruction of several cosmological variables using the best estimates for \(N = 2\) and their confidence intervals at 1σ and 2σ.

### Model III: \(w = \text{constant}\).

| \(\lambda₀\) | \(-1.85 \times 10^{-2}\) | \(-1.85 \times 10^{-2}\) |
| \(\lambda₁\) | \(+2.48 \times 10^{-1}\) | \(+2.48 \times 10^{-1}\) |
| \(\lambda₂\) | \(+1.80 \times 10^{-3}\) | \(+1.80 \times 10^{-3}\) |
| \(\omega\) | \(-1.068^{+0.070}_{-0.001}\) | \(-1.068^{+0.114}_{-0.006}\) |

### Table 6. Summary of the 1σ and 2σ errors of the best estimate for \(N = 2\).
Model IV: \( w = \text{constant and } \Omega_{DM}^0 = \text{constant} \)

Best estimates for the parameters \( \lambda_n, w \) and \( \Omega_{DM}^0 \).

|    | \( N = 1 \)                          | \( N = 2 \)                          | \( N = 3 \)                          | \( N = 4 \)                          | \( N = 5 \)                          |
|----|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| \( \lambda_0 \) | \(-1.0041 \times 10^{-2}\) | \(-1.0045 \times 10^{-2}\) | \(-1.0050 \times 10^{-2}\) | \(-1.0046 \times 10^{-2}\) | \(-1.0042 \times 10^{-2}\) |
| \( \lambda_1 \) | \(2.41 \times 10^{-1}\)      | \(2.38 \times 10^{-1}\)      | \(2.351 \times 10^{-1}\)      | \(2.3509 \times 10^{-1}\)      | \(2.3500 \times 10^{-1}\)      |
| \( \lambda_2 \) | \(0.0\)                         | \(2.17 \times 10^{-3}\)       | \(2.25 \times 10^{-3}\)       | \(2.292 \times 10^{-3}\)       | \(2.294 \times 10^{-3}\)       |
| \( \lambda_3 \) | \(0.0\)                         | \(0.0\)                        | \(1.06 \times 10^{-5}\)       | \(1.06 \times 10^{-5}\)       | \(1.90 \times 10^{-5}\)       |
| \( \lambda_4 \) | \(0.0\)                         | \(0.0\)                        | \(0.0\)                        | \(1.0 \times 10^{-6}\)        | \(1.80 \times 10^{-6}\)       |
| \( \lambda_5 \) | \(0.0\)                         | \(0.0\)                        | \(0.0\)                        | \(0.0\)                        | \(1.0 \times 10^{-7}\)       |
| \( w \)          | \(-0.979\)                      | \(-0.973\)                     | \(-0.968\)                     | \(-0.964\)                     | \(-0.96\)                       |
| \( \Omega_{DM}^0 \) | \(0.21\)                       | \(0.207\)                      | \(0.205\)                      | \(0.204\)                      | \(0.202\)                       |
| \( \chi^2_{min} \) | 542.77                         | 542.73                         | 542.71                         | 542.70                         | 542.69                         |

Table 7. The same as the Table 8 for the Model IV corresponding to a interacting dark energy EOS parameter \( w = \text{constant} \) and \( \Omega_{DM}^0 \) as a free parameter to be estimated. From the Figure 1 to Figure 8 show the reconstruction of several cosmological variables using the best estimates for \( N = 5 \). Moreover, They also show the reconstruction of several cosmological variables using the best estimates for \( N = 2 \) and their confidence intervals at 1\( \sigma \) and 2\( \sigma \).

Model IV: \( w = \text{constant and } \Omega_{DM}^0 = \text{constant} \).

| Errors       | \( \pm 1\sigma \) | \( \pm 2\sigma \) |
|--------------|------------------|------------------|
| \( \lambda_0 \) | \(-1.00 \times 10^{-2} + 0.07 \times 10^{-2}\) | \(-1.00 \times 10^{-2} + 0.04 \times 10^{-2}\) |
| \( \lambda_1 \) | \(+2.38 \times 10^{-1} + 0.25 \times 10^{-1}\) | \(+2.38 \times 10^{-1} + 0.36 \times 10^{-1}\) |
| \( \lambda_2 \) | \(+2.17 \times 10^{-3} + 0.13 \times 10^{-3}\) | \(+2.17 \times 10^{-3} + 0.22 \times 10^{-3}\) |
| \( \omega \)    | \(-0.973 \pm 0.045\) | \(-0.973 \pm 0.079\) |
| \( \Omega_{DM}^0 \) | \(+0.207 \pm 0.013\) | \(+0.207 \pm 0.031\) |

Table 8. Summary of the 1\( \sigma \) and 2\( \sigma \) errors of the best estimate for \( N = 2 \).

panel of Figure 7 show that the 1\( \sigma \) and 2\( \sigma \) constraints imply that the preferred region for the EOS parameter \( w \) is totally contained in the phantom region. This is due to the fact that we have considered only the fit to the SNIa observations which are affected by systematic errors [141]-[142], [144]. In fact, some studies show that there exists the possibility of a crossing of the phantom divide line [143]. By the contrary, for the model IV, the above panels of Figure 8 show that the 1\( \sigma \) and 2\( \sigma \) constraints on the EOS parameter \( w \) contain high probability of being in the quintessence region. Finally we note the broad dispersion on the dark matter density parameter measured at the present \( \Omega_{DM}^0 \).

The Figure 4 shows the superposition of the best estimates for the dimensionless interaction function \( I_Q(z) \) (left above panel), the density parameters \( \Omega_{DM}^0(z) \), \( \Omega_{DE}^0(z) \) (right above panel) and the deceleration parameter \( q(z) \) (left below panel) as a function of the redshift for the models I, II, III and IV respectively.
Figure 1. Reconstruction for the dimensionless interaction function $I_{Q}(z)$ as a function of the redshift for the models II (left above panel), III (right above panel) and IV (left below panel) corresponding to a dark energy equation of state parameter (II) $w = -1$, (III) $w = \text{constant}$ (both with $\Omega_{0}^{DM} = 0.233$), and (IV) $w = \text{constant}$, $\Omega_{0}^{DM} = \text{constant}$, respectively. The curves with different colours show the best estimates using the expansion of $I_{Q}(z)$ in terms of the parameters $\lambda_n$ corresponding to the Chebyshev polynomial expansion ranging from $N = 1$ to $5$. Note the fast convergence of the curves when the number of polynomials $N$ involved in the expansion increases. We show the best estimates (red lines), the $1\sigma$ (green lines) and $2\sigma$ errors (blue lines) for the dimensionless interaction function $I_{Q}(z)$ as a function of the redshift. The reconstruction is derived from the best estimation obtained using the type Ia Supernova SCP Union2 data set sample. Note that the best estimates of the strength of the interaction cross marginally the noninteracting line $I_{Q}(z) = 0$ only at the present changing sign from positive values at the past (energy transfers from dark energy to dark matter) to negative values almost at the present (energy transfers from dark matter to dark energy). Moreover, note that within the $1\sigma$ and $2\sigma$ errors it could be the possibility that the crossing the noninteracting line $I_{Q}(z) = 0$ happens before the present and the dark energy density is positive in all the redshift range.

6. Conclusions.

In this paper, we developed theoretically a novel method for the reconstruction of the interaction function between dark matter and dark energy assuming an expansion of the
general interaction term proportional to the Hubble parameter in terms of Chebyshev polynomials which form a complete set of orthonormal functions. To show how the method works, we applied it to the reconstruction of the interaction function expanding it in terms of only the first $N$ Chebyshev polynomials (with $N = 1, 2, 3, 4, 5$) and fitted for the coefficients of the expansion assuming two models: (a) a DE equation of the state parameter $w = -1$ (an interacting cosmological $\Lambda$) and (b) a DE equation of the state parameter $w = \text{constant}$. The fit of the free parameters of every model is done using the Union2 SNe Ia data set from “The Supernova Cosmology Project” (SCP) composed by 557 type Ia supernovae [2].

Our principal results can be summarized as follows:

(i) Our fitting results show the fast convergence of the best estimates for the several cosmological variables considered in this paper when the numbers of parameters $N$
is increased in the expansion (47).

(ii) The best estimates for the interaction function $I_Q(z)$ prefer to cross the noninteracting line $I_Q(z) = 0$ during the present cosmological evolution. This conclusion is independent of the numbers of coefficients (up to $N = 5$ in this work) used in the expansion of $I_Q(z)$. The crossing implies a change of sign of $I_Q(z)$ from positive values at the past (energy transfers from dark energy to dark matter) to negative values at the present (energy transfers from dark matter to dark energy). The decay direction is contrary to the results found in the recent literature [134] and in disagreement with the oscillatory behavior reported in [133].

(iii) The statement above is conclusive because the existence of crossing of the noninteraction line $I_Q(z) = 0$ in some moment of the recent past is totally contained inside the $1\sigma$ and $2\sigma$ constraints given by SNe Ia observations.

(iv) For all the interacting models studied in this work, the best estimates for the dark
energy density parameter $\Omega_{DE}(z)$ becomes positive definite in the range of redshifts considered in this work. This statement is conclusive because, within the $1\sigma$ and $2\sigma$ errors for the fit with three parameters ($N = 2$), $\Omega_{DE}(z)$ becomes positive in all the range of redshifts considered in the sample of SNe Ia.

(v) The $1\sigma$ and $2\sigma$ confidence intervals, for the EOS parameter $w$ considered in the marginalized model III, are totally contained in the phantom region ($w < -1$). This is not totally true for the model IV which presents high regions of probability for the EOS parameter $w$ of being in the phantom or in the quintessence region. This is because we have performed the fit with only the SNe Ia data set which, for some data samples, is known to fit preferably in the phantom region [141]–[142], [144].
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Figure 5. Superposition of the best estimates for the rate between dark density parameters $\Omega_{DE}(z)/\Omega_{DM}(z)$ for the model II (left above panel), III (right above panel), IV (left below panel). In the above figures, the different colored curves show the best estimates using the expansion in terms of the first $N = 1, 2, 3, 4, 5$ Chebyshev polynomials respectively. For comparison purposes, the figure of the right below panel shows the best estimates for the same variable for models I, II, II, and IV using the expansion in terms of the first $N = 2$ Chebyshev polynomials and the corresponding curve for the LCDM model. Here we also show the best estimates (red lines), the $1\sigma$ (green lines) and $2\sigma$ errors (blue lines) for the rate between dark density parameters $\Omega_{DE}(z)/\Omega_{DM}(z)$ as a function of the redshift.

This last problem can be corrected if we use more cosmological observations that can provide information of the late and early universe as such observations of the Cosmic Microwave Background (CMB) anisotropies from Microwave Anisotropy Probe (WMAP) experiment and the large scale structure (LSS) from Sloan Digital Sky Survey (SDSS) experiment.

In order to clarify the last points mentioned above (the crossing or not of the noninteracting line and the appearance or not of phantom regions for the EOS parameter $w$) in our reconstruction, we must repeat our analysis fitting with more data samples covering a broader range of the history of the universe: the Cosmic Microwave Background (CMB) shift parameters from the test 7-year Wilkinson Microwave
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**Figure 6.** Best estimate and the 1σ, 2σ confidence intervals for the marginalized probability densities for the model II using an expansion in terms of the first $N = 2$ Chebyshev polynomials. Before marginalization, we have three free parameters $(\lambda_0, \lambda_1, \lambda_2)$. In every figure, we marginalized on one of the parameters.

Anisotropy Probe (WMAP) \[4\], the Baryon Acoustic Oscillation (BAO) from experiment SDSS DR7 \[6\], a spectroscopic catalog of red galaxies in galaxy clusters \[146\], the Hubble expansion rate (15 data) \[147\]-\[149\] and the X-ray gas mass fraction (42 data) \[150\]-\[155\]. We will go ahead with this analysis in our next future work \[156\].

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Appendix A. Calculation of the integrals $K_n(x, w)$ and $J_n(x, w)$.

**Appendix A.1. Calculation of the integrals $J_n(x, w)$.**

In order to calculate the integrals $K_n(x, w)$ defined in (51) we need to obtain, for $m \geq 0$, closed expressions for the integrals,

$$J_m(x, w) = \int_{-1}^{x} \frac{x^m}{(a + bx)(1+3w)} d\tilde{x}$$  \hspace{1cm} (A.1)

To this goal, we use the recurrence relation valid for integers $m \geq 1, m \neq 3w$,

$$J_m(x, w) = \frac{1}{(m-3w)b} \cdot \left\{ \left[ \frac{x^m}{(a+bx)^{3w}} - (-1)^m \right] - amJ_{m-1}(x, w) \right\}$$  \hspace{1cm} (A.2)

where we have for the initial integrals $J_0(x, w)$,

$$J_0(x, w) = \begin{cases} \frac{1}{b} \ln(a + bx) & \text{if } w = 0, \\ \frac{1}{3wb} \left[ 1 - \frac{1}{(a+bx)^{3w}} \right] & \text{if } w \neq 0. \end{cases}$$  \hspace{1cm} (A.3)
Figure 7. Best estimate and the 1σ, 2σ confidence intervals for the marginalized probability densities for the model III using an expansion in terms of the first $N = 2$ Chebyshev polynomials. Before marginalization, we have four free parameters ($\lambda_0, \lambda_1, \lambda_2, w$). In every figure, we marginalized on the last two remaining parameters. Note that the preferred region for the EOS parameter $w$ is the phantom region.
From (A.2) we can guess the series for $J_m(x, w)$ in terms of $J_0(x, w)$ and $m \geq 1,$

$$J_m(x, w) = \frac{1}{b} \cdot \sum_{l=1}^{m} (-1)^{m-l} \cdot \frac{m!}{l!} \cdot \left(\frac{a}{b}\right)^{m-l} \cdot \prod_{k=l}^{m} \frac{1}{(k-3w)} \cdot \left[\frac{x^l}{(a + bx)^3w} - (-1)^l\right]$$

$$+ (-1)^m \cdot \left(\frac{a}{b}\right)^m \cdot \prod_{k=1}^{m} \frac{k}{(k-3w)} \cdot J_0(x, w)$$

(A.4)

For the case $w = 0$, the above formula reduces to

$$J_m(x, 0) = \sum_{l=1}^{m} (-1)^{m-l} \cdot \frac{(a/b)}{lb} \cdot \left[\frac{x^l}{(a + bx)^3} - (-1)^l\right] + (-1)^m \left(\frac{a}{b}\right)^m \cdot J_0(x, 0)$$

(A.5)
Appendix A.2. Calculation of the integrals $K_n(x, w)$.

Now we calculate the integrals $K_n(x, w)$ defined in (51) by

$$K_n(x, w) \equiv \int_{-1}^{x} \frac{T_n(\tilde{x})}{(a + b\tilde{x})^{1+3w}} d\tilde{x}, \quad (A.6)$$

To this end we use the following representation for Chebyshev polynomials with odd and even integer subindex,

$$T_{2m+1}(\tilde{x}) = \frac{2m+1}{2} \cdot \sum_{l=0}^{m} \frac{(-1)\cdot (m+l)! \cdot (m+l)!}{(m-l)! \cdot (2l)!} \cdot (2\tilde{x})^{2l+1} \quad \text{for } m \geq 0, \quad (A.7)$$

$$T_{2m}(\tilde{x}) = m \cdot \sum_{l=0}^{m} \frac{(-1)^{m+l}}{(m-l)! \cdot (2l)!} \cdot (2\tilde{x})^{2l} \quad \text{for } m \geq 1 \quad (A.8)$$

Introducing (A.7) and (A.8) in (A.6) and it doing the explicit integration we have the closed forms for the integrals with odd integer subindex, valid for $n \geq 0$, $w \neq (n+1)/3$,

$$K_{2n+1}(x, w) = \frac{(2n+1)}{2b} \cdot \sum_{m=0}^{n} \sum_{l=1}^{2m+1} \frac{(-1)^{n+3m-l+1} \cdot (n+m)! \cdot (2)^{2m+1}}{(n-m)! \cdot l!} \cdot \left(\frac{a}{b}\right)^{(2m+1-l)} \cdot \left[ \prod_{k=l}^{2m+1} \frac{1}{k - 3w} \right] \cdot \left[ \frac{x^l}{(a + bx)^3w} - (-1)^l \right] + \frac{(2n+1)}{2} \cdot \sum_{m=0}^{n} \frac{(-1)^{n+3m+1}}{(n-m)! \cdot (2m+1)!} \cdot \left(\frac{a}{b}\right)^{(2m)} \cdot \left[ \prod_{k=1}^{2m+1} \frac{k}{k - 3w} \right] \cdot K_0(x, w) \quad (A.9)$$

meanwhile the integrals with even integer subindex, valid for $n \geq 1$, $w \neq n/3$, are

$$K_{2n}(x, w) = \frac{n}{b} \cdot \sum_{m=1}^{n} \sum_{l=1}^{2m} \frac{(-1)^{n+3m-l} \cdot (n+m-1)! \cdot (2)^{2m}}{(n-m)! \cdot l!} \cdot \left(\frac{a}{b}\right)^{(2m-l)} \cdot \left[ \prod_{k=l}^{2m} \frac{1}{k - 3w} \right] \cdot \left[ \frac{x^l}{(a + bx)^3w} - (-1)^l \right] + \left[ 1 + \sum_{m=1}^{n} \frac{(-1)^{3m}(n+m-1)! \cdot (2)^{2m}}{(n-m)! \cdot (2m)!} \cdot \left(\frac{a}{b}\right)^{2m} \cdot \prod_{k=1}^{2m} \frac{k}{k - 3w} \right] \cdot (-1)^n \cdot K_0(x, w) \quad (A.10)$$
where the initial function $K_0(x, w)$ is given by

$$K_0(x, w) = \begin{cases} 
\frac{1}{b} \ln(a + bx) & \text{if } w = 0, \\
\frac{1}{3w} \left[ 1 - \frac{1}{(a + bx)^w} \right] & \text{if } w \neq 0.
\end{cases} \quad (A.11)$$

### Appendix B. Reconstruction of the interaction using Chebyshev polynomials up to order 5.

To simplify our analysis and to show how the method works we do the reconstruction taking a expansion in terms of Chebyshev polynomials up to order $N = 5$. The first step is to calculate the first fifth integrals $J_n(x, w)$:

$$J_1(x, w) = \frac{1}{b(1 - 3w)} \left\{ \left[ \frac{x}{(a + bx)^{3w}} + 1 \right] - aJ_0(x, w) \right\} \quad (B.1)$$

$$J_2(x, w) = \frac{1}{b(2 - 3w)} \left[ \frac{x^2}{(a + bx)^{3w}} - 1 \right] + \frac{2a}{b^2(2 - 3w)(1 - 3w)} \left\{ aJ_0(x, w) - \left[ \frac{x}{(a + bx)^{3w}} + 1 \right] \right\} \quad (B.2)$$

$$J_3(x, w) = \frac{1}{b(3 - 3w)} \left[ \frac{x^3}{(a + bx)^{3w}} + 1 \right] - \frac{3a}{b^2(3 - 3w)(2 - 3w)} \left[ \frac{x^2}{(a + bx)^{3w}} - 1 \right] + \frac{6a^2}{b^3(3 - 3w)(2 - 3w)(1 - 3w)} \left\{ \left[ \frac{x}{(a + bx)^{3w}} + 1 \right] - aJ_0(x, w) \right\} \quad (B.3)$$

$$J_4(x, w) = \frac{1}{b(4 - 3w)} \left[ \frac{x^4}{(a + bx)^{3w}} - 1 \right] - \frac{4a}{b^2(4 - 3w)(3 - 3w)} \left[ \frac{x^3}{(a + bx)^{3w}} + 1 \right] + \frac{12a^2}{b^3(4 - 3w)(3 - 3w)(2 - 3w)} \left[ \frac{x^2}{(a + bx)^{3w}} - 1 \right] + \frac{24a^3}{b^4(4 - 3w)(3 - 3w)(2 - 3w)(1 - 3w)} \left\{ aJ_0(x, w) - \left[ \frac{x}{(a + bx)^{3w}} + 1 \right] \right\} \quad (B.4)$$

$$J_5(x, w) = \frac{1}{b(5 - 3w)} \left[ \frac{x^5}{(a + bx)^{3w}} + 1 \right] - \frac{5a}{b^2(5 - 3w)(4 - 3w)} \left[ \frac{x^4}{(a + bx)^{3w}} - 1 \right] + \frac{20a^2}{b^3(5 - 3w)(4 - 3w)(3 - 3w)} \left[ \frac{x^3}{(a + bx)^{3w}} + 1 \right] - \frac{40a^3}{b^4(5 - 3w)(4 - 3w)(3 - 3w)(2 - 3w)} \left\{ aJ_0(x, w) - \left[ \frac{x}{(a + bx)^{3w}} + 1 \right] \right\} \quad (B.5)$$
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\[ \frac{60a^3}{b^4(5-3w)(4-3w)(3-3w)(2-3w)} \left( \frac{x^2}{(a+bx)^{3w}} - 1 \right) + \frac{120a^4}{b^5(5-3w)(4-3w)(3-3w)(2-3w)(1-3w)} \]

\[ \left\{ \frac{x}{(a+bx)^{3w}} + 1 \right\} - aJ_0(x, w) \]

(B.5)

where we have the definitions,

\[ b = \frac{z_{\text{max}}}{2} \]  

(B.6)

\[ a = 1 + \frac{z_{\text{max}}}{2} \]  

(B.7)

\[ x = \frac{2z}{z_{\text{max}}} - 1 \]  

(B.8)

\[ a + bx = 1 + z \]  

(B.9)

At the other hand, the first fifth Chebyshev polynomials are:

\[ T_0(x) = 1 \]  

(B.10)

\[ T_1(x) = x \]  

(B.11)

\[ T_2(x) = 2x^2 - 1 \]  

(B.12)

\[ T_3(x) = 4x^3 - 3x \]  

(B.13)

\[ T_4(x) = 8x^4 - 8x^2 + 1 \]  

(B.14)

\[ T_5(x) = 16x^5 - 20x^3 + 5x \]  

(B.15)

using these polynomials we find the relation between the integrals (A.1) and (A.6),

\[ K_0(x, w) = J_0(x, w) \]  

(B.16)

\[ K_1(x, w) = J_1(x, w) \]  

(B.17)

\[ K_2(x, w) = 2J_2(x, w) - J_0(x, w) \]  

(B.18)

\[ K_3(x, w) = 4J_3(x, w) - 3J_1(x, w) \]  

(B.19)

\[ K_4(x, w) = 8J_4(x, w) - 8J_2(x, w) + J_0(x, w) \]  

(B.20)

\[ K_5(x, w) = 16J_5(x, w) - 20J_3(x, w) + 5J_1(x, w) \]  

(B.21)

The general solutions (49)-(50) up to order \( N \) can be written as

\[ \Omega^{*}_{\text{DM}}(z) = (1 + z)^3 \left[ \Omega^0_{\text{DM}} - \frac{z_{\text{max}}}{2} \sum_{n=0}^{N} \lambda_n \cdot K_n(x, 0) \right] , \]  

(B.22)

\[ \Omega^{*}_{\text{DE}}(z) = (1 + z)^{3(1+w)} \left[ \Omega^0_{\text{DE}} + \frac{z_{\text{max}}}{2} \sum_{n=0}^{N} \lambda_n \cdot K_n(x, w) \right] , \]  

(B.23)

Finally, the Hubble parameter is written as,

\[ H^2(z) = H_0^2 \left[ \Omega^0_{\text{b}}(1 + z)^3 + \Omega^0_{\gamma}(1 + z)^4 + \Omega^*_\text{DM}(z) + \Omega^*_\text{DE}(z) \right] , \]  

(B.24)

With this formulation we do the reconstruction of the coupling function \( I_Q(z) \) for \( N = 1, 2, 3, 4, 5 \) respectively using the using the type Ia Supernova SCP Union2 data set sample.
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