THE MONOGENIC HUA-RADON TRANSFORM AND ITS INVERSE

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Abstract. The monogenic Hua-Radon transform is defined as an orthogonal projection on holomorphic functions in the Lie sphere. Extending the work of Sabadini and Sommen, J. Geom. Anal., 29 (2019), 2709-2737, we determine its reproducing kernel. Integrating this kernel over the Stiefel manifold yields a linear combination of the zonal spherical monogenics. Using the reproducing properties of those monogenics we obtain an inversion for the monogenic Hua-Radon transform.

Contents

1. Introduction 1
2. Preliminaries 2
2.1. Clifford algebras 2
2.2. Clifford analysis 3
2.3. The Lie ball 4
3. The monogenic Hua-Radon transform 5
4. Technical lemmas 9
5. The kernel of the monogenic Hua-Radon transform 12
6. Inversion of the monogenic Hua-Radon transform 16
7. Conclusions 22
Appendix A. Computation of $\lambda_k^{\alpha}$ from Theorem 5.2 22
Appendix B. Computation of the constants $\gamma_{\alpha,k}$ from Proposition 6.4 26
Acknowledgements 29
References 29

1. Introduction

The Radon transform is a well-known integral transform with many applications in both pure and applied mathematics. It was extended to the Clifford setting by Sommen in [20, 21, 23], see also [3].

The Szegö-Radon transform, a variant of the Clifford Radon transform, was introduced in [5] by Colombo, Sabadini and Sommen. It was originally defined as an orthogonal projection of a Hilbert module, but it can equally be written as an integral transform over the unit sphere $S^{m-1}$ with respect to a reproducing kernel. Like the Clifford Radon transform, it is a projection from a space of monogenic functions in $m$ variables over the real Clifford algebra $\mathbb{R}_m$ onto a space of monogenic functions in 2 variables over the complex Clifford algebra $\mathbb{C}_m$.

Applying the dual transform $\tilde{R}$ to the reproducing kernel of the Szegö-Radon transform, i.e. taking the integral over a Stiefel manifold of the reproducing kernel, will result in a linear combination of the zonal spherical monogenics. Consequently, when acting on a monogenic polynomial $M_k(x)$ of degree $k$ with the composition of the Szegö-Radon transform with its dual, one obtains a scalar multiple of $M_k(x)$, where the proportionality constant $\theta_k$ depends on $k$. As this scalar $\theta_k$ can be accounted for by

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2. PRELIMINARIES

In this section we introduce all notations and preliminary results that will be useful for the paper. We mostly follow the notations from [16].

2.1. Clifford algebras. Consider the vector space $\mathbb{R}^m$ with canonical orthonormal basis $(e_1, e_2, \ldots, e_m)$. The Clifford algebra $\mathbb{R}_m$, respectively $\mathbb{C}_m$, is the real, respectively complex, algebra generated by the basis elements $e_1, e_2, \ldots, e_m$ under the relations

\[
e_j^2 = -1, \quad j \in \{1, \ldots, m\},
\]
\[
e_j e_k + e_k e_j = 0, \quad j \neq k.
\]

Any element $\alpha$ in $\mathbb{R}_m$ (or $\mathbb{C}_m$) is of the form

\[
\alpha = \sum_{A \subset \{1, \ldots, m\}} \alpha_A e_A
\]

where $\alpha_A \in \mathbb{R}$ (or $\alpha_A \in \mathbb{C}$), $A = \{i_1, \ldots, i_l\}$ where $i_1 < \ldots < i_l$ is a multi-index such that $e_A = e_{i_1} \ldots e_{i_l}$ and $e_0 = 1$. The elements of $\mathbb{R}_m$ (or $\mathbb{C}_m$) which are linear combinations of only the basis vectors $e_j$ are called 1-vectors and will be underlined, e.g. $\underline{v} = \sum_{j=1}^m v_j e_j$. The scalar part $\alpha_0$ of an element $\alpha$ will be denoted by $[\alpha]_0$.

The Hermitian conjugation is an automorphism on $\mathbb{C}_m$ defined for $\alpha, \beta \in \mathbb{C}_m$ as

\[
(\alpha \beta)^\dagger = \beta^\dagger \alpha^\dagger,
\]
\[
(\alpha + \beta)^\dagger = \alpha^\dagger + \beta^\dagger,
\]
\[
(\alpha_A e_A)^\dagger = \overline{\alpha_A} e_A^\dagger,
\]
\[
e_j^\dagger = -e_j \quad j \in \{1, \ldots, m\}.
\]

where $\overline{\alpha_A}$ stands for the complex conjugate of the complex number $\alpha_A$. 

the Gamma operator $\Gamma_x$, one can invert the Szegö-Radon transform by applying the operator $\theta - \Gamma_x \tilde{R}$. These ideas were established in [5].

It was shown in e.g. [11, 13, 17] that monogenic functions on the unit ball admit a holomorphic extension in the Lie ball. Following this idea, Sabadini and Sommen defined several Radon-type transforms over the Lie sphere in [16]. They showed that each of these can be written as an integral transform over the Lie sphere with respect to a certain kernel. One of these is the monogenic Hua-Radon transform, which we will study in great detail.

In this paper we complete the work of Sabadini and Sommen, by determining explicitly the kernel of the monogenic Hua-Radon transform, which is defined as an orthogonal projection onto a space of holomorphic functions generated by a specific basis. Moreover, we will obtain an inversion for this transform, along the lines of [5]. At this point we require that the dimension $m \geq 3$, which can be justified by the fact that most of the basis is annihilated in case $m < 3$. It turns out that using the dual transform will not produce a full inversion for the monogenic Hua-Radon transform, but it will yield a projection operator of a holomorphic function onto the term in its Fischer decomposition corresponding to a certain power of $\bar{z}$. Hence we will introduce a total monogenic Hua-Radon transform, for which the outlined process leads to a complete inversion.

The structure of the paper is as follows. In Section 2 we give the necessary preliminaries on Clifford algebras, the Lie sphere and Clifford analysis that will be needed in this paper. In Section 3 we define the monogenic Hua-Radon transform as an orthogonal projection. Section 4 contains some technical lemmas required in order to compute the kernel of the monogenic Hua-Radon transform in Section 5, more precisely in Theorem 5.2. The inversion of the monogenic Hua-Radon transform will be discussed in Section 6. Finally, the more tedious calculations involving hypergeometric series and combinatorial identities will be performed in Appendix A and B.
2.2. Clifford analysis. The norm of a 1-vector $v$ in either $\mathbb{R}_m$ or $\mathbb{C}_m$ is defined as

$$|v|^2 = \sum_{j=1}^{m} v_j^2 = -v^2.$$  

In the real case, we can use $|v| = \sqrt{\sum_{j=1}^{m} v_j^2}$, whereas in the complex case we will be working with $|v|^2 = -v^2$, i.e. a complexified version of the square of the real norm.

The open unit ball with center at the origin in $\mathbb{R}_m$ will be denoted by $B(0, 1)$, while $S^{m-1}_m$ will denote the unit sphere, i.e. $S^{m-1}_m = \{ v \in \mathbb{R}_m \mid |v|^2 = 1 \}$. The area of the unit sphere is given by

$$A_m = \frac{2 \pi^{m/2}}{\Gamma \left( \frac{m}{2} \right)}.$$  

where $\Gamma$ is the gamma function.

We define the scalar product of two 1-vectors $u$ and $v$ as

$$\langle u, v \rangle = \sum_{j=1}^{m} u_j v_j.$$  

The wedge product of $u$ and $v$ is defined as

$$u \wedge v = \sum_{1 < j} (u_j v_i - u_i v_j) e_i e_j.$$  

Easy calculations show

$$u \cdot v = -\langle u, v \rangle + u \wedge v.$$  

We will need the following result which was proven in [5].

Lemma 2.1. Let $t, s \in S^{m-1}_m$ be such that $\langle t, s \rangle = 0$ and let $\tau = t + is \in \mathbb{C}_m$. Then $
abla \tau = -t + is$ and

(i) $\tau \tau^\dagger \tau = 4\tau,$

(ii) $\tau^\dagger \tau^\dagger = (\tau^\dagger)^2 = 0,$

(iii) $\tau \tau^\dagger + \tau^\dagger \tau = 4.$

We will be working in a complex setting, hence we will use the complexified Dirac operator

$$\partial_\tau = \sum_{j=1}^{m} e_j \partial_{t_j}.$$  

Its square satisfies $\partial^2_\tau = -\Delta_\tau$, where $\Delta_\tau = \sum_{j=1}^{m} \partial^2_{t_j}$ is the complexified Laplace operator. If we use the real Laplace operator, we will denote it by $\Delta_\tau = \sum_{j=1}^{m} \partial^2_{x_j}$. The symbol of the Dirac operator $\partial_\tau$ is denoted by the vector variable

$$\hat{\tau} = \sum_{j=1}^{m} c_j \tau_j$$  

which is a complex-valued variable. If we are working with real variables, we will denote them by $\hat{\tau}$. By complexifying the variables, the square of the norm and the Dirac operator, we can use the necessary results of real Clifford analysis.

Definition 2.2. A function $f : \Omega \subset \mathbb{C}_m \rightarrow \mathbb{C}_m$ which is continuously differentiable in the open set $\Omega$ is called (left) monogenic in $\Omega$ if $f$ is holomorphic and $f$ is in the kernel of the complexified Dirac operator $\partial_\tau$, i.e. $\partial_\tau f = \sum_{j=1}^{m} e_j \partial_{t_j} f = 0$. The right $\mathbb{C}_m$-module of (left) monogenic functions in $\Omega$ is denoted by $\mathcal{M}(\Omega)$.

A function $f : \Omega \subset \mathbb{C}_m \rightarrow \mathbb{C}_m$ which is continuously differentiable in the open set $\Omega$ is called harmonic in $\Omega$ if $f$ is holomorphic and $f$ is in the kernel of the complexified Laplace operator $\Delta_\tau = -\partial^2_\tau$. The right $\mathbb{C}_m$-module of harmonic functions in $\Omega$ is denoted by $\mathcal{H}(\Omega)$.

Looking at the set of $k$-homogeneous polynomials $\mathcal{P}_k(\Omega)$ with $\Omega \subset \mathbb{R}_m$ we define the set of $k$-homogeneous monogenic and harmonics as

$$\mathcal{M}_k(\Omega) = \mathcal{M}(\Omega) \cap \mathcal{P}_k(\Omega),$$

$$\mathcal{H}_k(\Omega) = \mathcal{H}(\Omega) \cap \mathcal{P}_k(\Omega).$$
It is a well known fact, (see e.g. [8]), that for each \( H_k(x) \in H_k(\Omega) \) there exist unique monogenic polynomials \( M_k(x), M_{k-1}(x) \in M_k(\Omega), M_{k-1}(x) \in M_{k-1}(\Omega) \) such that

\[
H_k(x) = M_k(x) + xM_{k-1}(x)
\]

Moreover these monogenics are determined by

\[
M_k(x) = \left( 1 + \frac{1}{2k + m - 2} x \partial_x \right) H_k(x),
\]

\[
M_{k-1}(x) = - \frac{1}{2k + m - 2} \partial_x.
\]

2.3. The Lie ball. As we are complexifying our variables, we will also complexify the space over which we are working. To this end we will consider the Lie sphere \( LS^{m-1} \) and Lie ball \( LB(0, 1) \), instead of the unit sphere \( S^{m-1} \) and unit ball \( B(0, 1) \).

**Definition 2.3.** The Lie ball can be defined as

\[
LB(0, 1) = \{ z = x + iy \in \mathbb{C}^m \mid S_{x,y} \subset B(0, 1) \}
\]

where \( S_{x,y} \) is the codimension 2 sphere:

\[
S_{x,y} = \{ u \in \mathbb{R}^m \mid |u - x| = |y|, (u - x, y) = 0 \}.
\]

**Remark 2.4.** Another way to introduce the Lie ball (see [15, 22]) is to consider the Lie norm

\[
L(z)^2 = L(x + iy)^2 = \sup_{u \in S_{x,y}} |u|^2 = |x|^2 + |y|^2 + 2|x \wedge y|
\]

where \( z = x + iy \in \mathbb{C}^m \). Using the Lie norm we define the Lie ball as

\[
LB(0, 1) = \{ z \in \mathbb{C}^m \mid L(z) < 1 \}.
\]

**Definition 2.5.** The Lie sphere \( LS^{m-1} \) is the set of points \( z = x + iy \in \mathbb{C}^m \) for which \( S_{x,y} \subset S^{m-1} \) or equivalently

\[
LS^{m-1} = \{ e^{i\theta} \omega \mid \omega \in S^{m-1}, \theta \in [0, \pi) \}.
\]

Since monogenic functions \( f(x) \) on \( B(0, 1) \) admit a holomorphic extension \( f(z) \) in the Lie ball \( LB(0, 1) \) (see e.g. [11, 13, 17]), we will be interested in the following space of holomorphic functions.

**Definition 2.6.** Let \( \mathcal{O}L^2(LB(0, 1)) \) be the right \( \mathbb{C}_m \)-module consisting of holomorphic functions \( f : LB(0, 1) \to \mathbb{C}_m \) whose boundary value \( f(e^{i\theta} \omega) \) belongs to \( L^2(LS^{m-1}) \), i.e.

\[
\left[ \int_{S^{m-1}} \int_0^\pi f^1(e^{i\theta} \omega) f(e^{i\theta} \omega) d\theta dS(\omega) \right]^0 < \infty.
\]

We can equip \( \mathcal{O}L^2(LB(0, 1)) \) with the following inner product

\[
(f, g)_{\mathcal{O}L^2(LB(0, 1))} = \int_{S^{m-1}} \int_0^\pi f^1(e^{i\theta} \omega) g(e^{i\theta} \omega) d\theta dS(\omega)
\]

For ease of notation we will denote this by \( \langle \cdot, \cdot \rangle_{\mathcal{O}L^2} \).
3. The monogenic Hua-Radon transform

In this section, we will define the monogenic Hua-Radon transform and complete some proofs of Section 6 of [16]. Let us first remind the reader of the functions that we will be working with.

Let $\tau = \xi + i\eta \in \mathbb{C}_m$ with $\xi, \eta \in \mathbb{S}^{m-1}$ and $\xi \perp \eta$, i.e. $\langle \xi, \eta \rangle = 0$. We first define the following functions:

$$\psi_{\tau, 2s, k}(z) = \tau(z, \overline{z})^{s+k}(\overline{z}, \overline{\tau})^s$$

$$\psi_{\tau, 2s+1, k}(z) = \tau^1(z, \overline{z})^{s+k+1}(\overline{z}, \overline{\tau})^s$$

for $s, k \in \mathbb{N} = \{0, 1, 2, \ldots \}$. Note that for each $\alpha, k \in \mathbb{N}$ we have that $\psi_{\tau, \alpha, k}(z)$ is homogeneous of degree $\alpha + k$ in $z$.

The functions $\psi_{\tau, \alpha, k}(z)$ admit the following properties shown in [16].

**Proposition 3.1.** The functions $\psi_{\tau, 0, k}(z) = \tau(z, \overline{z})^k$ are monogenic for each $k \in \mathbb{N}$. Moreover for each $s, k \in \mathbb{N}$ we have

$$\partial_\tau \psi_{\tau, 2s+1, k}(z) = (s + 1)\psi_{\tau, 2s+1, k}(z)$$

$$\partial_\tau \psi_{\tau, 2s+2, k}(z) = (s + 1)\psi_{\tau, 2s+1, k}(z).$$

We can extend these functions to

$$\psi_{\tau, j, 2s, k}(z) = \tau^j M[\psi_{\tau, 2s, k}](z)$$

$$\psi_{\tau, j, 2s+1, k}(z) = \tau^j M[\psi_{\tau, 2s+1, k}](z)$$

where $j, s, k \in \mathbb{N}$ and $M[\cdot]$ stands for the monogenic projection which can be defined as follows. For each $k$-homogeneous polynomial $P_k(z)$ there exist harmonic polynomials $H_{k-2\ell}(z) \in \mathcal{H}(\Omega)$ of degree $k - 2\ell$ for $\ell = 0, \ldots, \lfloor \frac{k}{2} \rfloor$ such that

$$P_k(z) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} (-z^2)^\ell H_{k-2\ell}(z),$$

see e.g. [8]. Using (2.2) we can refine (3.1) to

$$P_k(z) = \sum_{j=0}^{k} z^j M_{k-j}(z).$$

The monogenic projection is now defined as $M[P_k](z) = M_k(z)$.

**Definition 3.2.** The right $\mathbb{C}_m$-module generated by $\{ \psi_{\tau, \alpha, k} \mid k \in \mathbb{N} \}$ will be denoted by $\mathfrak{M}^{\tau, \alpha}(\mathbb{C}_m)$.

**Definition 3.3.** The monogenic Hua-Radon transform $\mathcal{M}_{\tau, j}$ is now defined as the orthogonal projection of $\mathcal{OL}^2(LB(0, 1))$ onto the orthogonal direct sum $\bigoplus_{\alpha \in \mathbb{N}} \mathfrak{M}^{\tau, \alpha}(\mathbb{C}_m)$ (see [16]).

The aim of Sections 3 and 5 will be to write the monogenic Hua-Radon transform as an integral transform

$$\mathcal{M}_{\tau, j}[f](z) = \int_0^\pi \int_{\mathbb{S}^{m-1}} K^j(z, e^{-i\theta} \omega) f(e^{-i\theta} \omega) dS(\omega) d\theta$$

with respect to a certain reproducing kernel $K^j$ that is to be determined.

In order to determine the monogenic projection of $\psi_{\tau, \alpha, k}$, we will first compute its harmonic projection.
Remark 3.4. Note that if we are working in the 2-dimensional case $\psi_{\alpha,k}^j = 0$ for all $j, k$ and $\alpha \neq 0, 1$. Indeed if $m = 2$, then $z = e^{i\varphi}(e_1 \pm ie_2)$ for some $\varphi \in [0, 2\pi)$. This implies

$$\psi_{\alpha,k}^j = \frac{1}{2} e^{i\varphi} \left( z_1 \pm i z_2 \right)^{s+k}(z_1 \pm i z_2)^s = (1)^s \left( \frac{1}{2} \right)^s e^{i\varphi} \left( z_1 \pm i z_2 \right)^k,$$

$$\psi_{\alpha,k}^{j+1} = \frac{1}{2} e^{i\varphi(k+1)} \left( z_1 \pm i z_2 \right)^{s+k+1}(z_1 \pm i z_2)^s = (1)^s \left( \frac{1}{2} \right)^s e^{i\varphi(k+1)} \left( z_1 \pm i z_2 \right)^{k+1}.$$

We can now see that $M[\psi_{\alpha,k}^j] = 0 = M[\psi_{\alpha,k}^{j+1}]$ whenever $s \neq 0$. Hence from now on we will assume $m \geq 3$.

We can project $P_k$ in (3.1) onto each of its harmonic components $H_{k-2j}$ using the following projection operator (see e.g. [1] and [18]).

**Proposition 3.5.** The projection operator of a $k$-homogeneous polynomial onto its harmonic component of degree $k - 2\ell$ is given by the following operator

$$\sum_{j=0}^{\ell} \alpha_j (-z^2)^j \Delta_{\ell}^j,$$

where

$$\alpha_j = \frac{(-1)^j (\frac{m}{2} + k - 2\ell - 1) \Gamma(\frac{m}{2} + k - 2\ell - j)}{4^{j+\ell} j!}. \Gamma(\frac{m}{2} + k - 2\ell - 1)$$

The projection operator in Proposition 3.5 projects onto the harmonic components, but we need to project onto the monogenic components. Hence we will rewrite the previous operator using the Dirac operator and compose it with the projection onto its monogenic component (2.2):

**Proposition 3.6.** The projection operator of a $k$-homogeneous polynomial onto its monogenic component of degree $k - 2\ell$ is given by the following operator

$$\sum_{j=0}^{\ell} \beta_{j,2\ell} z^{j+2\ell}$$

where

$$\beta_{j,2\ell} = \frac{(-1)^j \Gamma(\frac{m}{2} + k - 2\ell - j)}{4^{j+\ell} j! \Gamma(\frac{m}{2} + k - 2\ell - 1)},$$

$$\beta_{j+1,2\ell} = \frac{(-1)^j \Gamma(\frac{m}{2} + k - 2\ell - j - 1)}{2^{j+\ell+1} j! \Gamma(\frac{m}{2} + k - 2\ell - 1)}.$$

**Proof.** Using the projection operator of a harmonic polynomial onto its monogenic component, we get the following operator projecting a $k$-homogeneous polynomial onto its monogenic component of degree $k - 2\ell$

$$\sum_{j=0}^{\ell} \alpha_j (-z^2)^j \Delta_{\ell}^j$$

Now observe that for any function $f$ we have

$$\partial_{\bar{z}} \left[ z^{2j} f \right] = -2jz^{2j-1}f + z^{2j} \partial_{\bar{z}} [f]$$

as $z^{2j}$ is a scalar and hence will commute with all $e_j$ for $j = 1, \ldots, m$. Consequently, we can rewrite the projection operator as follows:
where with the coefficients \( \beta \)

Using Proposition 3.6 we get

Proof. For suitable constants \( \alpha \)

Lemma 3.7. For suitable constants \( \mu_{l,a,k}, l = 1, \ldots, \alpha \) we have

\[
M[\psi_{z,a,k}](z) = \sum_{j=0}^{\alpha} \mu_{j,a,k} z^j \psi_{z,a-k}(z).
\]

where

\[
\begin{align*}
\mu_{2j,2s,k} &= (-1)^j \frac{\Gamma(s + 2s + k)}{j! \Gamma(s + k + 1)} \frac{1}{\Gamma(j + 1) \Gamma(s + k - j + 1)} \\
\mu_{2j+1,2s,k} &= (-1)^j \frac{\Gamma(s + 2s + k + 1)}{j! \Gamma(s + k + 2)} \frac{1}{\Gamma(j + 1) \Gamma(s + k - j + 2)} \\
\mu_{2j,2s+1,k} &= (-1)^j \frac{\Gamma(s + 2s + k + 2)}{j! \Gamma(s + 2s + k + 1)} \frac{1}{\Gamma(j + 1) \Gamma(s + k - j + 2)} \\
\mu_{2j+1,2s+1,k} &= 2(-1)^j \frac{\Gamma(s + 2s + k + 1)}{j! \Gamma(s + 2s + k + 1)} \frac{1}{\Gamma(j + 1) \Gamma(s + k - j + 2)}
\end{align*}
\]

Proof. Using Proposition 3.6 we get

\[
M[\psi_{z,a,k}](z) = \text{proj}_{l}^{\alpha+k}(\psi_{z,a,k})(z)
\]

where

\[
\beta_{2j,0} = \frac{(-1)^j}{4^j} \frac{\Gamma(s + \alpha + k - j)}{j! \Gamma(s + \alpha + k)}
\]

\[
\beta_{2j+1,0} = \frac{(-1)^j}{4^j} \frac{\Gamma(s + \alpha + k - j - 1)}{2j! \Gamma(s + \alpha + k)}
\]

If \( \alpha = 2s \) is even, then using Proposition 3.1 we get
and thus the sum reduces to
\[
M[\psi_{z,2s,k}(z)] = \psi_{z,2s,k}(z) + \mu_{1,2s,1} \psi_{z,2s-1,k}(z) + \ldots + \mu_{2s,2s,k} z^{2s} \psi_{z,0,k}(z)
\]
with
\[
\mu_{2j,2s,k} = \beta_{2j,0} \frac{\Gamma(s+1)}{\Gamma(s-j+1)} \frac{\Gamma(s+k+1)}{\Gamma(s+k-j+1)}
\]
\[
= (-1)^j \frac{j! (\frac{m}{2} + 2s + k - j)}{\Gamma(\frac{m}{2} + 2s + k)} \frac{\Gamma(s+1)}{\Gamma(s-j+1)} \frac{\Gamma(s+k+1)}{\Gamma(s+k-j+1)}
\]
\[
\mu_{2j+1,2s,k} = \beta_{2j+1,0} \frac{\Gamma(s+1)}{\Gamma(s-j)} \frac{\Gamma(s+k+1)}{\Gamma(s+k-j+1)}
\]
\[
= (-1)^{j+1} \frac{\Gamma(\frac{m}{2} + 2s + k - j - 1)}{2j! \Gamma(\frac{m}{2} + 2s + k)} \frac{\Gamma(s+1)}{\Gamma(s-j)} \frac{\Gamma(s+k+1)}{\Gamma(s+k-j+1)}.
\]
The case \(\alpha = 2s + 1\) is done in a similar way.

**Remark 3.8.** Lemma 3.7 also shows that \(M[\psi_{z,2s,k}] = 0 = M[\psi_{z,2s+1,k}]\) if \(s \geq 1\) and \(m = 2\) just as we have shown in Remark 3.4. This might not be obvious at first, but using Lemma 3.7 we have
\[
M[\psi_{z,2s,k}] = \sum_{j=0}^{2s} \mu_{j,2s,k} z^{2j} \psi_{z,2s-j,k}(z)
\]
\[
= \sum_{j=0}^{s} \mu_{2j,2s,k} z^{2j} \psi_{z,2s-2j,k}(z) + \sum_{j=0}^{s-1} \mu_{2j+1,2s,k} z^{2j+1} \psi_{z,2s-2j-1,k}(z).
\]
Using the calculation done in Remark 3.4, we get
\[
z^{2j} \psi_{z,2s-2j,k}(z) = z^{2j} z^{2s-2j} e^{i\phi k} (z_1 \pm iz_2)^k,
\]
\[
z^{2j+1} \psi_{z,2s-2j-1,k}(z) = z^{2j+1} z^{2s-2j-2} e^{i\phi (k+1)} (z_1 \pm iz_2)^{k+1}
\]
\[
= z^{2s-2} e^{i\phi (k+1)} (2z_1 e_1 e_2 + 2i e_2 (z_1^2 + z_2^2))(z_1 \pm iz_2)^k
\]
\[
= z^{2s-2} e^{i\phi k} (2z_1 e_1 e_2 + 2i e_2 (z_1^2 + z_2^2))(z_1 \pm iz_2)^k
\]
\[
= -2z^{2s-2} e^{i\phi k} (z_1 \pm iz_2)^k.
\]
Thus now we can rewrite (3.3) as a hypergeometric series:
\[
(3.3) = \left( \sum_{j=0}^{s} \mu_{2j,2s,k} - 2 \sum_{j=0}^{s-1} \mu_{2j+1,2s,k} \right) z^{2s} e^{i\phi k} (z_1 \pm iz_2)^k
\]
\[
= \binom{-s}{2 F_1 \left( \begin{array}{c} [-s, -s - k], \left(-2s - k\right); 1 \\ 2 F_1 \left( \begin{array}{c} [-s + 1, -s - k], \left(-2s - k - 1\right); 1 \\ \end{array} \right) \right) z^{2s} e^{i\phi k} (z_1 \pm iz_2)^k.
\]
where \( {}_2F_1 \) denotes the hypergeometric series given by

\[
{}_2F_1 ([a, b], [c]; w) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{w^n}{n!}
\]

with \((a)_n = a(a+1) \ldots (a+n-1)\) the Pochhammer symbol. Now using Chu-Vandermonde identity (see [2]), we have

\[
\begin{align*}
{}_2F_1 ([s, s-k], [-2s-k]; 1) &= \frac{(-s)_s}{(-2s-k)_s} \frac{s!(s+k)!}{(2s+k)!}, \\
{}_2F_1 ([s+1, s-k], [-2s-k+1]; 1) &= \frac{(-s+1)_{s-1}}{(-2s-k+1)_{s-1}} \frac{(s-1)!(s+k)!}{(2s+k-1)!},
\end{align*}
\]

which shows that \( M[\psi_{\mathbb{Z}, 2s,k}] = 0 \). The case \( \alpha = 2s + 1 \) is done analogously.

4. Technical lemmas

In order to determine the kernel of the monogenic Hua-Radon transform in Section 5, we will need the following technical lemmas. The first result was already proven in [5].

**Lemma 4.1.** Let \( t, \bar{s} \in \mathbb{R}^m \) be such that \(|t| = |\bar{s}| = 1\) and \((t, \bar{s}) = 0\), let \( \tau = t + i\bar{s} \in \mathbb{C}^m \) and \( \omega \in \mathbb{S}^{m-1} \). Then for \( k \neq l \) we have

\[
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \bar{\tau} \rangle^l dS(\omega) = 0
\]

and

\[
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \bar{\tau} \rangle^k dS(\omega) = (-1)^k 2\pi \frac{\Gamma(k+1)}{\Gamma(k+\frac{m}{2})}.
\]

We can write the integral in Lemma 4.1 as the inner product \( A_m ((-1)^k \langle \varphi, \tau \rangle^k, \langle \varphi, \bar{\tau} \rangle^l)_{\mathbb{S}^{m-1}} \) with

\[
\langle P, Q \rangle_{\mathbb{S}^{m-1}} = \frac{1}{A_m} \int_{\mathbb{S}^{m-1}} \overline{P(\omega)} Q(\omega) dS(\omega)
\]

where \( \overline{P(\omega)} \) is the complex conjugate of \( P(\omega) \). It has been proven (see e.g. [6]) that

\[
2^k \frac{\Gamma \left( k + \frac{m}{2} \right)}{\Gamma \left( \frac{m}{2} \right)} \langle H_k, P_l \rangle_{\mathbb{S}^{m-1}} = \left[ H_k(\partial_{\tau}) P_l(x) \right]_{x=0}
\]

where \( H_k \in \mathcal{H}_k \), \( P_l \) is an \( \ell \)-homogeneous polynomial and \( H_k(\partial_{\tau}) \) is the operator obtained by substituting \( \partial_{x_i} \) for \( x_i \) in \( H(x) \). Moreover if \( k \neq \ell \) then

\[
\langle H_k, H_\ell \rangle_{\mathbb{S}^{m-1}} = 0
\]

where \( H_k \in \mathcal{H}_k, H_\ell \in \mathcal{H}_\ell \).

**Remark 4.2.** We can use (4.2) to get a similar result for monogenic polynomials, namely whenever \( k \neq \ell \)

\[
\begin{align*}
\langle M_k, M_\ell \rangle_{\mathbb{S}^{m-1}} &= 0 \\
\langle M_k, \bar{x} M_\ell \rangle_{\mathbb{S}^{m-1}} &= 0 \\
\langle x M_k, M_\ell \rangle_{\mathbb{S}^{m-1}} &= 0,
\end{align*}
\]

where \( M_k \in \mathcal{M}_k \) and \( M_\ell \in \mathcal{M}_\ell \).

Now we can easily prove the following:
Lemma 4.3. Let $t, s \in \mathbb{R}^m$ be such that $|t| = |s| = 1$ and $(t, s) = 0$, let $\tau = t + is \in \mathbb{C}^m$ and $\omega \in \mathbb{S}^{m-1}$. Then for $k \neq \ell + 1$ one has

$$
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \tau \rangle^\ell \tau^t \omega dS(\omega) = 0
$$

and

$$
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^{\ell+1} \langle \omega, \tau \rangle^t \tau^t \omega dS(\omega) = (-1)^{\ell} \frac{\Gamma(\ell+2)}{\Gamma(\ell+2 + \frac{m}{2})} \tau^t.
$$

Proof. We have

$$
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \tau \rangle^\ell \tau^t \omega dS(\omega) = A_m((-1)^{\ell+1} \langle x, \tau^t \rangle^\ell x, \langle x, \tau^t \rangle^k)_{\mathbb{S}^{m-1}}.
$$

Using (4.1), it is easy to see that

$$
\langle (-1)^{\ell+1} \langle x, \tau^t \rangle^\ell x, \langle x, \tau^t \rangle^k \rangle_{\mathbb{S}^{m-1}} = \frac{1}{2 (k + \frac{m}{2})} \langle (-1)^{\ell+1} \langle x, \tau^t \rangle^\ell x, \partial_\ell \langle x, \tau^t \rangle^k \rangle_{\mathbb{S}^{m-1}}
$$

$$
= \frac{1}{2 (k + \frac{m}{2})} \langle (-1)^{\ell+1} \langle x, \tau^t \rangle^\ell x, k \langle x, \tau^t \rangle^{k-1} \rangle_{\mathbb{S}^{m-1}}
$$

$$
= \frac{1}{2 (k + \frac{m}{2})} \langle (-1)^{\ell+1} \langle x, \tau^t \rangle^\ell x, k \langle x, \tau^t \rangle^{k-1} \rangle_{\mathbb{S}^{m-1}}.
$$

The result now follows from Lemma 4.1.

Remark 4.4. Since $\tau^t$ and $\tau$ have the same properties as $-t \in \mathbb{S}^{m-1}$ and $-t \perp s$, we can easily replace $\tau$ by $\tau^t$. Moreover, we have

$$
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \tau \rangle^\ell \tau^t \omega dS(\omega) = -\left( \int_{\mathbb{S}^{m-1}} (-1)^{k+\ell} \langle \omega, \tau \rangle^k \langle \omega, \tau \rangle^\ell \tau^t dS(\omega) \right)^t
$$

$$
= \begin{cases} 0 & \text{for } k \neq \ell + 1, \\ (-1)^{\ell+1} \frac{\Gamma(\ell+2)}{\Gamma(\ell+1 + \frac{m}{2})} \tau^t & \text{for } k = \ell + 1. \end{cases}
$$

In order to prove the following lemma, we will need to use Pizzetti’s formula (see e.g. [7, 14]), which provides an easy way to calculate integrals over the unit sphere using the Laplacian. If $f$ is a polynomial, then

$$
(4.3) \quad \int_{\mathbb{S}^{m-1}} f(\omega) \ dS(\omega) = \sum_{k=0}^{\infty} \frac{2\pi^{m/2}}{4^k k! \Gamma(k + m/2)} (\Delta^k f)(0).
$$

This will allow us to show the following.

Lemma 4.5. Let $t, s \in \mathbb{R}^m$ be such that $|t| = |s| = 1$ and $(t, s) = 0$, let $\tau = t + is \in \mathbb{C}^m$ and $\omega \in \mathbb{S}^{m-1}$. Then for $k \neq \ell$ we have

$$
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \tau \rangle^\ell \omega \tau^t dS(\omega) = 0
$$

and

$$
\int_{\mathbb{S}^{m-1}} \langle \omega, \tau \rangle^k \langle \omega, \tau \rangle^k \omega \tau^t \omega dS(\omega) = (-1)^{k+1} \frac{\Gamma(\ell+2)}{\Gamma(\ell+2 + \frac{m}{2})} (4\tau \wedge \tau^t - m \tau \tau^t - 2k \tau^t \tau).
$$
Proof. To prove this, we will use Pizzetti's formula. It can be shown, using Lemma 2.1 and some straightforward calculations, that for $j \leq \min(k, \ell)$ one has

$$
\Delta_k^j ((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^k) = \Delta_k^j ((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^k)_{\mathbf{x} \tau^\dagger \mathbf{z} \tau^\dagger} \\
+ 8kj \Delta_k^{j-1}((\mathbf{x}, \tau^\dagger)^{k-1}, (\mathbf{z}, \tau^\dagger)^k)_{\mathbf{x} \tau^\dagger} \\
+ 8\ell j \Delta_k^{j-1}((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^{k-1})_{\mathbf{x} \tau^\dagger} \\
- 4j(j-1) \Delta_k^{j-1}((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^k)_{\mathbf{x} \tau^\dagger} \\
+ 2j \Delta_k^{j-1}((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^k)T
$$

where $T = \sum_{i=1}^m e_i \tau^\dagger e_i$. So now it is easy to see that if $k \neq \ell$, our integral will vanish, whereas if $k = \ell$, we have

$$
\Delta_k^{\ell+1} ((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^k) = 2(-4)^k \ell!(k+1)! (T - 2k \tau^\dagger).$$

Hence after applying Pizzetti's formula we get

$$
\int_{\mathbb{R}^{m-1}} \langle \omega, \tau^\dagger \rangle^{l+1} \langle \omega, \tau^\dagger \rangle \omega \tau \tau^\dagger \omega dS(\omega) = \sum_{j=0}^{2m/2} \frac{2\pi^{m/2}}{\sqrt{j! \Gamma(j + m/2)}} \Delta_k^j ((\mathbf{x}, \tau^\dagger)^k, (\mathbf{z}, \tau^\dagger)^k)_{\mathbf{x} \tau^\dagger} |_{\mathbf{x} = 0} \\
= (-1)^k \pi^{m/2} \frac{k!}{\Gamma(k+1+m/2)} (T - 2k \tau^\dagger).$$

We can now rewrite $T$ to get a neater result. Using (2.1) we get

$$
T = \sum_{i=1}^m e_i \tau^\dagger e_i = \sum_{i=1}^m e_i (-\langle \mathbf{x}, \tau^\dagger \rangle + \mathbf{z} \wedge \tau^\dagger) e_i \\
= -\langle \mathbf{x}, \tau^\dagger \rangle (-m) + \sum_{i=1}^m e_i \mathbf{x} \wedge \tau^\dagger e_i = m \langle \mathbf{x}, \tau^\dagger \rangle + (4-m) \mathbf{z} \wedge \tau^\dagger \\
= 4 \mathbf{z} \wedge \tau^\dagger - m (-\langle \mathbf{x}, \tau^\dagger \rangle + \mathbf{z} \wedge \tau^\dagger) = 4 \mathbf{z} \wedge \tau^\dagger - m \mathbf{z} \wedge \tau^\dagger
$$

where we used Lemma 4.3 of [4].}

Remark 4.6. As $\mathbf{z} \wedge \tau^\dagger$ is a bivector and $\lbrack \mathbf{z} \tau^\dagger \rbrack_0 = \lbrack \tau^\dagger \tau \rbrack_0 = 2$, we have that the scalar part of the right-hand side of (4.4) yields
\[-\frac{1}{\pi^\frac{m}{2}} \frac{k!}{\Gamma(k + \frac{m}{2})}.\]

5. THE KERNEL OF THE MONOGENIC HUA-RADON TRANSFORM

In order to determine the kernel of the monogenic Hua-Radon transform, we will need the following proposition.

**Proposition 5.1.** The following equalities hold:

\[(5.1) \quad \langle \psi_{\alpha, k}^j, \psi_{\alpha', k'}^j \rangle_{\mathcal{O}L^2} = \langle M[\psi_{\alpha, k}], M[\psi_{\alpha', k'}] \rangle_{\mathcal{O}L^2} = \langle M[\psi_{\alpha, k}], \psi_{\alpha', k'} \rangle_{\mathcal{O}L^2}.\]

Moreover if \(\alpha + k \neq \alpha' + k'\) then the three quantities in (5.1) will vanish.

**Proof.** Using the homogeneity of the functions we get

\[
\langle \psi_{\alpha, k}^j, \psi_{\alpha', k'}^j \rangle_{\mathcal{O}L^2} = \int_0^\pi e^{i\theta(\alpha'+k'-\alpha-k)} d\theta \\
\times \int_{S_{m-1}} [M[\psi_{\alpha, k}](\omega)]^\dagger (-\omega^2)^j M[\psi_{\alpha', k'}](\omega) dS(\omega) \\
= \langle M[\psi_{\alpha, k}], M[\psi_{\alpha', k'}] \rangle_{\mathcal{O}L^2},
\]

where we used that \(\omega^\dagger = -\omega\) and \(-\omega^2 = 1\). We can now use (3.2) to decompose \(\psi_{\alpha, k'}(z)\):

\[
\psi_{\alpha, k'}(z) = \sum_{\ell=0}^{\alpha' + k'} z^\ell M_{\alpha' + k' - \ell}(z)
\]

with \(M_{\alpha' + k' - \ell}(z)\) a monogenic function for each \(\ell\) and \(M_{\alpha' + k'}(z) = M[\psi_{\alpha, k'}](z)\). So now we have

\[
\langle M[\psi_{\alpha, k}], \psi_{\alpha', k'} \rangle_{\mathcal{O}L^2} = \int_0^\pi e^{i\theta(\alpha'+k'-\alpha-k)} d\theta \\
\times \int_{S_{m-1}} [M[\psi_{\alpha, k}](\omega)]^\dagger \sum_{\ell=0}^{\alpha' + k'} \omega^\ell M_{\alpha' + k' - \ell}(\omega) dS(\omega) \\
= \sum_{\ell=0}^{\alpha' + k'} \int_0^\pi e^{i\theta(\alpha'+k'-\alpha-k)} d\theta \\
\times \left\langle \left[ M[\psi_{\alpha, k}](\omega) \right]^\dagger z^\ell, M_{\alpha' + k' - \ell}(z) \right\rangle_{S_{m-1}}.
\]

If we now use (4.1) we see that

\[
\left\langle \left[ M[\psi_{\alpha, k}](\omega) \right]^\dagger z^\ell, M_{\alpha' + k' - \ell}(z) \right\rangle_{S_{m-1}} = \left\langle \left[ M[\psi_{\alpha, k}](\omega) \right]^\dagger z^{\ell-1}, \partial_z M_{\alpha' + k' - \ell}(z) \right\rangle_{S_{m-1}} = 0
\]

for each \(\ell \neq 0\), hence the last equality of (5.1) is proven. Moreover, if \(\alpha + k \neq \alpha' + k'\) we have

\[
\left\langle \left[ M[\psi_{\alpha, k}](\omega) \right]^\dagger, M_{\alpha' + k' - \ell}(z) \right\rangle_{S_{m-1}} = 0.
\]

Furthermore, \(\partial_z M[\psi_{\alpha, k}](z) = 0\) and hence \(\Delta_z M[\psi_{\alpha, k}](z) = 0\). But as \(\Delta_z\) is a real-valued operator, we also have \(\Delta_z \left[ M[\psi_{\alpha, k}](\omega) \right]^\dagger = 0\). So now using (4.2) we get the anticipated result. \(\Box\)
Theorem 5.2. The kernel of the monogenic Hua-Radon transform is given by

\[ K^j(z, e^{-i\theta} \omega) = \sum_{\alpha \in \mathbb{N}} K^{j, \alpha}(z, e^{-i\theta} \omega), \]

where

\[ K^{j, \alpha}(z, e^{-i\theta} \omega) = z^j L^\alpha(z, e^{-i\theta} \omega)(e^{i\theta} \omega)^{-j} \]

with

\[ L^\alpha(z, e^{-i\theta} \omega) = \sum_{k=0}^{\infty} \lambda_k^\alpha M[\psi_{z, \alpha, k}](z) \left[ M[\psi_{z, \alpha, k}](e^{i\theta} \omega) \right]^\dagger \]

and

\[ \lambda_k^\alpha = \begin{cases} \pi A_m \left( \frac{m}{2} - 1 \right) & \frac{\Gamma(\frac{m}{2} + 2s + k)^2}{4s!(s + k)!\Gamma(s + m + \frac{m}{2})\Gamma(m + s + k)} \quad \alpha = 2s, \\
\pi A_m \left( \frac{m}{2} - 1 \right) & \frac{16s!(s + k + 1)\Gamma(s + \frac{m}{2} + 2s + k + 1)^2}{\Gamma(m + s + k)!\Gamma(s + m + \frac{m}{2})}\quad \alpha = 2s + 1. \end{cases} \]

Proof. In order to prove this result, we will prove that \( K^{j, \alpha} \) reproduces the basis elements \( \psi_{z, \alpha, k}^j \) and hence it will reproduce each element of \( \oplus_{\alpha \in \mathbb{N}} \mathcal{M}^{j, \alpha} \). Moreover, due to the way \( K^{j, \alpha} \) is constructed, we get that the monogenic Hua-Radon transform is indeed represented by an integral transform with kernel \( K^{j, \alpha} \). Thus we need to prove that

\[ \psi_{z, \alpha, k}^j(z) = \int_{S_m-1}^\pi \int_0^\pi K^{j, \alpha}(z, e^{-i\theta} \omega) \psi_{z, \alpha, k}^j(e^{i\theta} \omega) dS(\omega) d\theta. \]  

The theorem will then follow using the orthogonality relations of \( \psi_{z, \alpha, k}^j \), i.e. if \( \alpha \neq \alpha' \) or \( k \neq k' \) then \( \langle \psi_{z, \alpha, k}^j, \psi_{z, \alpha', k'}^j \rangle_{\mathcal{C}^2} = 0 \), which was shown in [16]. We can rewrite equation (5.2) using the orthogonality relations of \( M[\psi_{z, \alpha, k}] \) to the following identity

\[ \lambda_k^\alpha M[\psi_{z, \alpha, k}](z) \int_{S_m-1}^\pi \int_0^\pi M[\psi_{z, \alpha, k}](e^{i\theta} \omega) M[\psi_{z, \alpha, k}](e^{i\theta} \omega) dS(\omega) d\theta = M[\psi_{z, \alpha, k}](z). \]

We will now show that the \( \lambda_k^\alpha \) stated in the Theorem will be the required coefficient in order to reproduce \( M[\psi_{z, \alpha, k}] \). We have, using Lemma 3.7

\[ \int_{S_m-1}^\pi \int_0^\pi M[\psi_{z, \alpha, k}](e^{i\theta} \omega) M[\psi_{z, \alpha, k}](e^{i\theta} \omega) dS(\omega) d\theta = \int_{S_m-1}^\pi \int_0^\pi M[\psi_{z, \alpha, k}](e^{i\theta} \omega) \psi_{z, \alpha, k}^j(e^{i\theta} \omega) dS(\omega) d\theta = \sum_{j=0}^{\infty} J_{j, \alpha, k} \int_{S_m-1}^\pi \int_0^\pi \psi_{z, \alpha-j, k}(e^{i\theta} \omega)^\dagger (e^{i\theta} \omega)^{-j} \psi_{z, \alpha, k}(e^{i\theta} \omega) dS(\omega) d\theta. \]

Let us now set

\[ \Phi_{j, \alpha, k} = \int_{S_m-1}^\pi \int_0^\pi \psi_{z, \alpha-j, k}(e^{i\theta} \omega)^\dagger (e^{i\theta} \omega)^{-j} \psi_{z, \alpha, k}(e^{i\theta} \omega) dS(\omega) d\theta. \]

Using the fact that \( \psi_{z, \alpha, k} \) has degree of homogeneity \( \alpha + k \), we get
\[
\Phi_{j,\alpha,k} = \int_{S^{m-1}} \int_0^{\pi} e^{-i\theta(\alpha-j+k)} (\psi_{\alpha-j,k}(\omega))^\dagger (e^{i\theta} \omega)^{\alpha-j} e^{i\theta(\alpha+k)} \psi_{\alpha,k}(\omega) dS(\omega) d\theta
\]
\[
= \pi \int_{S^{m-1}} (\psi_{\alpha-j,k}(\omega))^\dagger (\omega)^{\alpha-j} \psi_{\alpha,k}(\omega) dS(\omega).
\]
First we consider the case \( \alpha = 2s \) even. For \( j = 2l \) we have
\[
\Phi_{2l,2s,k} = (-1)^{-l} \pi \int_{S^{m-1}} (\psi_{\alpha-j,k}(\omega))^\dagger \psi_{\alpha,k}(\omega) dS(\omega)
\]
\[
= (-1)^{k-l} \pi^{\dagger} \int_{S^{m-1}} (\omega, \tau)^{k-l+2s}(\omega, \tau)^{k-l+2s} dS(\omega).
\]
Hence by Lemma 4.1, we get
\[
\Phi_{2l,2s,k} = (-1)^{k-l} \pi^{\dagger} \Gamma(2s-l+1) \Gamma(\frac{m}{2} + k - l + 2s) \]
\[
= 2\pi^{\dagger} \frac{\Gamma(2s-l+1)}{\Gamma(\frac{m}{2} + k - l + 2s)}.
\]
Now defining
\[
\phi_{2l,2s,k} = 2\pi^{\dagger} \frac{\Gamma(2s-l+1)}{\Gamma(\frac{m}{2} + k - l + 2s)},
\]
we have \( \Phi_{2l,2s,k} = \phi_{2l,2s,k} \tau^{\dagger} \tau \).
For \( j = 2l+1 \), we have by Lemma 4.3
\[
\Phi_{2l+1,2s,k} = (-1)^{-l+1} \pi \int_{S^{m-1}} (\psi_{\alpha-j,k}(\omega))^\dagger \omega \psi_{\alpha,k}(\omega) dS(\omega)
\]
\[
= (-1)^{k-l+1} \pi \int_{S^{m-1}} (\omega, \tau)^{2s-l+k}(\omega, \tau)^{2s-l+k} \tau^{\dagger} \tau \omega \tau dS(\omega)
\]
\[
= -4\pi^{\dagger} \frac{\Gamma(2s-l+k+1)}{\Gamma(2s-l+k+\frac{m}{2})} \tau^{\dagger} \tau.
\]
Write
\[
\phi_{2l+1,2s,k} = -4\pi^{\dagger} \frac{\Gamma(2s-l+k+1)}{\Gamma(2s-l+k+\frac{m}{2})} \tau^{\dagger} \tau,
\]
then we have that \( \Phi_{2l+1,2s,k} = \phi_{2l+1,2s,k} \tau^{\dagger} \tau \). Note that \( \Phi_{j,2s,k} \) is a multiple of \( \tau^{\dagger} \tau \) for each \( j \). Hence we have
\[
(5.3) \quad \lambda_k^s M[\psi_{\alpha,k}(\bar{z})] \left( \sum_{l=0}^{s} \mu_{2l,2s,k} \phi_{2l,2s,k} + \sum_{l=0}^{s-1} \mu_{2l+1,2s,k} \phi_{2l+1,2s,k} \right) \tau^{\dagger} \tau = M[\psi_{\alpha,k}(\bar{z})].
\]
Now note that each of the terms in the decomposition of \( M[\psi_{2s,k}] \) shown in Lemma 3.7 is a multiple of \( \tau^{\dagger} \tau \) since
\[
\psi_{\alpha,2s-2j,k}(\bar{z}) = (\bar{z}, \tau)^{s-j+k} (\bar{z}, \tau)^{s-j} \tau,
\]
\[
\psi_{\alpha,2s-2j-1,k}(\bar{z}) = (\bar{z}, \tau)^{s-j+k-1} (\bar{z}, \tau)^{s-j} \tau.
\]
Thus if we use Lemma 2.1 (i), i.e. \( \tau^{\dagger} \tau = 4\tau \), we have
\[
4\lambda_k^{2s} \left( \sum_{l=0}^{s} \mu_{2l,2s,k} \phi_{2l,2s,k} + \sum_{l=0}^{s-1} \mu_{2l+1,2s,k} \phi_{2l+1,2s,k} \right) = 1
\]
Summarizing the case \( \alpha = 2s \) even, we have
$$\frac{1}{4}(\lambda_k^{2s})^{-1} = \sum_{l=0}^{s} \mu_{2l,2s,k} \phi_{2l,2s,k} + \sum_{l=0}^{s-1} \mu_{2l+1,2s,k} \phi_{2l+1,2s,k}$$

$$= \frac{2\pi \varpi^{s+1} \Gamma(s+1)\Gamma(s+k+1)}{\Gamma(\frac{m}{2} + 2s + k)} \left( \sum_{l=0}^{s} (-1)^l \frac{\Gamma(2s + k - l + 1)}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s+k-l+1)} \right) - \sum_{l=0}^{s-1} (-1)^l \frac{\Gamma(2s + k - l + 1)}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s+k-l+1)(\frac{m}{2} + 2s + k - l - 1)} \right).$$

Now consider the case where $\alpha = 2s + 1$ is odd. For $j = 2l$ we have

$$\Phi_{2l,2s+1,k} = (-1)^{-l} \pi \int_{S^m-1} (\psi_{\omega,\alpha-k} (\omega))^j \psi_{\omega,\alpha,k} (\omega) d\omega$$

$$= (-1)^{-l-1} \pi \varpi \Gamma(2s + k + l + 1) \Gamma(\frac{m}{2} + 2s + k + l + 1)$$

Hence by Lemma 4.1, we get

$$\Phi_{2l,2s+1,k} = (-1)^{-l-1} \pi \varpi \Gamma(k + l + 2) \Gamma(\frac{m}{2} + k - l + 2s + 1).$$

If we now define

$$\phi_{2l,2s+1,k} = \frac{\Gamma(k + l + 2) \Gamma(\frac{m}{2} + k - l + 2s + 1)}{\Gamma(\frac{m}{2} + 2s + k + l + 1)},$$

we have $\Phi_{2l,2s+1,k} = \phi_{2l,2s+1,k} \varpi \varpi$.

For $j = 2l + 1$, we have by Lemma 4.3

$$\Phi_{2l+1,2s+1,k} = (-1)^{-l-1} \pi \int_{S^m-1} (\psi_{\omega,\alpha-k} (\omega))^j \psi_{\omega,\alpha,k} (\omega) d\omega$$

$$= (-1)^{-l-1} \pi \varpi \Gamma(2s + k + l + 1) \Gamma(\frac{m}{2} + 2s + k + l + 1)$$

$$= -4\pi \varpi^{s+1} \Gamma(2s - l + k + 2) \Gamma(2s + k + 1 + \frac{m}{2}) \varpi \varpi$$

We can now define

$$\phi_{2l+1,2s+1,k} = -4\pi \varpi^{s+1} \Gamma(2s - l + k + 2) \Gamma(2s + k + 1 + \frac{m}{2}),$$

so that $\Phi_{2l+1,2s+1,k} = \phi_{2l+1,2s+1,k} \varpi \varpi$.

Summarizing the case $\alpha = 2s + 1$ odd, we have

$$\frac{1}{4}(\lambda_k^{2s+1})^{-1} = \sum_{l=0}^{s} \mu_{2l,2s+1,k} \phi_{2l,2s+1,k} + \sum_{l=0}^{s} \mu_{2l+1,2s+1,k} \phi_{2l+1,2s+1,k}$$

$$= 8\pi \varpi \Gamma(s + 1) \Gamma(s + k + 1) \Gamma(\frac{m}{2} + 2s + k + 1) \left( \sum_{l=0}^{s} (-1)^l \frac{\Gamma(2s + k - l + 2)}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s+k-l+2)} \right) - \sum_{l=0}^{s} (-1)^l \frac{\Gamma(2s + k - l + 2) \Gamma(\frac{m}{2} + 2s + k - l)}{\Gamma(l+1)\Gamma(s-l+1)\Gamma(s+k-l+1)\Gamma(\frac{m}{2} + 2s + k - l + 1)}.$$
6. Inversion of the monogenic Hua-Radon transform

Recall that the zonal spherical monogenics $C_k(x, y)$ can be written in terms of Gegenbauer polynomials $C_k^{m-1}$ and $C_k^{-1}$ (see [19])

\begin{equation}
C_k(x, y) = \frac{|x||y|^k}{m-2} \left( (k + m - 2)C_k^{m-1}(t) + (m - 2)\frac{x \wedge y}{|x||y|}C_k^{-1}(t) \right)
\end{equation}

with $t = \langle x, y \rangle /(|x||y|)$. Moreover they exhibit the following properties:

(i) $C_k(x, y)$ is a homogeneous polynomial of degree $k$ in $x$ and $y$,
(ii) $\partial_x C_k(x, y) = 0$ and $C_k(x, y)\partial_y = 0$, where $C_k(x, y)\partial_y = \sum_{j=1}^{m}(\partial_y C_k(x, y))e_j$,
(iii) For every $\sigma \in \text{Spin}(m) = \{\prod_{i=1}^{2r} \sigma_i | \sigma_i \in S^{m-1}\}$ the functions $C_k(x, y)$ have the invariance property

$$\sigma C_k(\sigma x, \sigma y)\sigma = C_k(x, y).$$

where we note that $\text{Spin}(m)/\{-1, 1\} \simeq SO(m)$, see [9, 10].

Recall the following lemma proven in [5] which proves that the zonal spherical monogenics $C_k(x, y)$ are, up to a constant, the unique Clifford algebra valued functions which admit these properties.

**Lemma 6.1.** Let $F(x, y)$ be a Clifford valued function satisfying the properties (i) – (ii) – (iii) above. Then there exist complex constants $\lambda$ and $\mu$ such that

$$F(x, y) = (\lambda + \mu e_{1\ldots m})C_k(x, y).$$

The functions $M[\psi_{\sigma, \alpha, k}]$ that we defined earlier in Section 3, meet (i) and (ii). But they do not satisfy property (iii). However they do admit the following Lemma.

**Lemma 6.2.** Let $\sigma \in \text{Spin}(m)$, then it holds that

$$\sigma M[\psi_{\sigma, \alpha, k}](\sigma x, \sigma y)\sigma = M[\psi_{\sigma, \sigma x, \alpha, k}](x).$$

**Proof.** Using Lemma 3.7 we have

$$M[\psi_{\sigma, \alpha, k}](x) = \psi_{\sigma, \alpha, k}(x) + \mu_1 x\psi_{\sigma, \alpha, 1, k}(x) + \ldots + \mu_n x^n \psi_{\sigma, 0, k}(x).$$

Hence it suffices to look at $\sigma \psi_{\sigma, \alpha, k}(\sigma x, \sigma y)\sigma$. Now observing that $\sigma \sigma = \sigma \sigma = 1$ and that $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$, we get

$$\sigma \psi_{\sigma, \alpha, k}(\sigma x, \sigma y)\sigma = \psi_{\sigma x, \sigma y, \alpha, k}(x)$$

and thus the statement is proven.

We now have the following.

**Proposition 6.3.** For $\alpha, k \in \mathbb{N}$ we have

\begin{equation}
\int_{S^{m-1}} \int_{S^{m-2}} M[\psi_{\sigma, \alpha, k}](x) M[\psi_{\sigma, \alpha, k}](y) dS(x) dS(y) = \gamma_{n, k} C_{\alpha + k}(x, y)
\end{equation}

for a suitable real or complex constant $\gamma_{n, k}$ provided in Proposition 6.4.
Proof. If we define the integral on the left of (6.2) as $L_{\alpha,k}(x,y)$, then we see that this integral is a homogeneous polynomial of degree $\alpha + k$ in both $x$ and $y$. Moreover it is left monogenic in $x$ and right monogenic in $y$. Furthermore, let $\sigma \in \text{Spin}(m)$, then by using Lemma 6.2 we get

$$\sigma L_k(\overline{\sigma} \sigma, x, y) d\sigma = \int_{S_{m-1}} \int_{S_{m-2}} \sigma \mathcal{M}(\psi_{\overline{\sigma} \sigma}) \mathcal{M}(\psi_{\overline{\sigma} \sigma})^\dagger \sigma dS(x) dS(y).$$

Hence $L_{\alpha,k}(x,y)$ is Spin-invariant, so all the conditions of Lemma 6.1 are fulfilled. Now noting that $L_{\alpha,k}(x,x)$ is the sum of a scalar $a_{\alpha,k}(x)$ and a bivector $b_{\alpha,k}(x)$, $\mathcal{C}_{\alpha+k}(x,x)$ is a scalar, we get

$$a_{\alpha,k}(x) + b_{\alpha,k}(x) = (\lambda + \mu e_1 \ldots e_m) \mathcal{C}_{\alpha+k}(x,x).$$

If we use the fact that $m \geq 3$ (see Remark 3.4), we see that $b_{\alpha,k}(x) = 0$ and $\mu = 0$. Hence we have

$$L_{\alpha,k}(x,y) = \gamma_{\alpha,k} \mathcal{C}_{\alpha+k}(x,y).$$

□

It is possible to compute the constants $\gamma_{\alpha,k}$ in Proposition 6.3 explicitly. They are as follows:

**Proposition 6.4.** For $s, k \in \mathbb{N}$ we have

$$\gamma_{2s,k} = \frac{s(s + k) \Gamma \left( \frac{m}{2} + s - 1 \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( s + k \right) \left( m - 2 \right) \frac{\Gamma(m - 1)\Gamma(2s + k + 1)}{\Gamma(2s + k + m - 1)}$$

and

$$\gamma_{2s+1,k} = \frac{s(s + k + 1) \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( s + k \right) \left( m - 2 \right) \frac{\Gamma(m - 1)\Gamma(2s + k + 2)}{\Gamma(2s + k + m)}.$$

Proof. As the equality in Proposition 6.3 must hold for each $x$ and $y$, it suffices to calculate $\gamma_{\alpha,k}$ for $x = y = \omega \in S^{m-1}$ and take the scalar part of the equation. The right-hand side of (6.2) will now be equal to

$$\gamma_{\alpha,k} \frac{\alpha + k + m - 2}{m - 2} \mathcal{C}_{\alpha+k}(1).$$

First we make the following observation for the left hand side of (6.2). There exists $\sigma \in \text{Spin}(m)$ such that $x = \sigma (e_1 + ie_2) \sigma$. For ease of notation let us write $\mathcal{Y} = e_1 + ie_2$. Using Lemma 6.2 we get

$$\mathcal{M}(\psi_{\overline{\sigma} \sigma}) \mathcal{M}(\psi_{\overline{\sigma} \sigma})^\dagger \sigma = \mathcal{M}(\psi_{\overline{\sigma} \sigma}) \mathcal{M}(\psi_{\overline{\sigma} \sigma})^\dagger \sigma.$$

The integral in (6.2) can be interpreted as the mean of (6.3) over the Stiefel manifold. But using the notation $\tau = \sigma \mathcal{Y} \sigma$, we can also interpret it as a mean over the Spin-group Spin$(m)$ (see e.g. [5]). Hence using Lemma 6.2 we get

$$\frac{1}{A_m A_{m-1}} \int_{S_{m-1}} \int_{S_{m-2}} \mathcal{M}(\psi_{\overline{\sigma} \sigma}) \mathcal{M}(\psi_{\overline{\sigma} \sigma})^\dagger \sigma dS(x) dS(y)$$

where

$$\mathcal{M}(\psi_{\overline{\sigma} \sigma}) = \frac{1}{\text{Vol}(\text{Spin}(m))} \int_{\text{Spin}(m)} \mathcal{M}(\psi_{\overline{\sigma} \sigma}) \sigma dS(\sigma).$$
Since the action of the Spin element on a vector is a rotation, it is easy to see that it is an automorphism on \(k\)-vectors, see [9, 10]. Hence we get that the scalar part of (6.4) is equal to

\[
\frac{1}{\text{Vol}({\text{Spin}(m)})} \int_{{\text{Spin}(m)}} \left[ M[{\psi}_0, \alpha, k] (\sigma \omega \vec{\sigma}) M[{\psi}_0, \alpha, k] (\sigma \omega \vec{\sigma}) \right]_0 dS(\sigma) = \frac{1}{A_m} \int_{S^{m-1}} \left[ M[{\psi}_0, \alpha, k] (\omega) M[{\psi}_0, \alpha, k] (\omega) \right]_0 dS(\omega).
\]

Expanding our integrand yields

\[
M[{\psi}_0, \alpha, k] (\omega) M[{\psi}_0, \alpha, k] (\omega) = \sum_{j,l=0}^{\alpha} \mu_{j, \alpha, k} \mu_{l, \alpha, k} \omega^j {\psi}_0, \alpha - j, k(\omega) {\psi}_0, \alpha - l, k(\omega) (\omega) + \omega^j {\psi}_0, \alpha - j, k(\omega) {\psi}_0, \alpha - l, k(\omega) (\omega) +\]

In order to continue, we need to know the parity of \(\alpha - j\) and \(\alpha - l\), so we will distinguish the cases where \(\alpha\) is even and where \(\alpha\) is odd. Suppose \(\alpha = 2s\) even, then we have

\[
\sum_{j,l=0}^{\alpha} \mu_{j, \alpha, k} \mu_{l, \alpha, k} \omega^j {\psi}_0, \alpha - j, k(\omega) {\psi}_0, \alpha - l, k(\omega) (\omega) + \omega^j {\psi}_0, \alpha - j, k(\omega) {\psi}_0, \alpha - l, k(\omega) (\omega) +\]

where we used that \(\omega^2 = -1\) and \(\omega^1 = -\omega\) for \(\omega \in S^{n-1}\). So now we have 4 different double sums we need to integrate. We can calculate the integrals using the previous lemmas and then we can take the scalar part in order to simplify the next calculations.

- Rewriting the terms of the first sum, gives us

\[
\mu_{j, \alpha, k} \mu_{l, \alpha, k} (-1)^{j+l} (\omega \eta)^{s-j+k} (\omega \eta)^{s-l+k} (\omega \eta)^{s-l} = \mu_{j, \alpha, k} \mu_{l, \alpha, k} (-1)^{j+l} (\omega \eta)^{2s-j-l+k} (\omega \eta)^{2s-j-l+k} (\omega \eta)^{2s-j-l+k}.
\]

Hence using Lemma 4.1 we get

\[
\frac{1}{A_m} \int_{S^{m-1}} \mu_{j, \alpha, k} \mu_{l, \alpha, k} (-1)^{j+l} {\psi}_0, \alpha - j, k(\omega) {\psi}_0, \alpha - l, k(\omega) (\omega) dS(\omega) = \frac{\Gamma(2s - j - l + k + 1)}{\Gamma(2s - j - l + k + \frac{m}{2})} \frac{\Gamma(2s - j + k + 1)}{\Gamma(2s - j + k + \frac{m}{2})}.
\]

Hence the scalar part will be equal to

\[
\frac{1}{A_m^2} \mu_{j, \alpha, k} \mu_{l, \alpha, k} 2\pi \frac{\Gamma(2s - j - l + k + 1)}{\Gamma(2s - j - l + k + \frac{m}{2})}.
\]
• For the terms of our second sum we get

\[ \mu_{2j+1,\alpha,k}\mu_{2l,\alpha,k}(-1)^{j+l}\omega(\omega,\eta)^{s+j+k}\langle \omega,\eta^\dagger \rangle^{s-j-1}\eta^\dagger \langle \omega,\eta \rangle^{s-l+k}\langle \omega,\eta^\dagger \rangle^{s-l}. \]

Now using Remark 4.4 we obtain

\[ \frac{1}{A_m} \int_{S_{m-1}} 4\mu_{2j+1,\alpha,k}\mu_{2l,\alpha,k}(-1)^{j+l+k}\omega(\omega,\eta)^{2s-j-l+k}\langle \omega,\eta^\dagger \rangle^{2s-j-l+k-1}\eta^\dagger \omega^\dagger dS(\omega) \]

\[ = -\frac{1}{A_m} 4\mu_{2j+1,\alpha,k}\mu_{2l,\alpha,k} \pi \frac{\Gamma(2s-j-l+k+1)}{\Gamma(2s-j-l+k+m_2)} \eta^\dagger \eta. \]

So now the scalar part yields

\[ -\frac{1}{A_m} 8\mu_{2j+1,\alpha,k}\mu_{2l,\alpha,k} \pi \frac{\Gamma(2s-j-l+k+1)}{\Gamma(2s-j-l+k+m_2)}. \]

• For the terms of our third sum we get

\[ \mu_{2j,\alpha,k}\mu_{2l+1,\alpha,k}(-1)^{j+l+1}\omega(\omega,\eta)^{s+j+k}\langle \omega,\eta^\dagger \rangle^{s-j-1}\eta^\dagger \langle \omega,\eta \rangle^{s-l+k}\langle \omega,\eta^\dagger \rangle^{s-l-1}\omega. \]

Now using Lemma 4.3 we get

\[ \frac{1}{A_m} \int_{S_{m-1}} 4\mu_{2j,\alpha,k}\mu_{2l+1,\alpha,k}(-1)^{j+l+k}\omega(\omega,\eta)^{2s-j-l+k-1}\langle \omega,\eta^\dagger \rangle^{2s-j-l+k}\eta^\dagger \omega dS(\omega) \]

\[ = -\frac{1}{A_m} 4\mu_{2j,\alpha,k}\mu_{2l+1,\alpha,k} \pi \frac{\Gamma(2s-j-l+k+1)}{\Gamma(2s-j-l+k+m_2)} \eta^\dagger \eta. \]

Hence the scalar part becomes

\[ -\frac{1}{A_m} 8\mu_{2j,\alpha,k}\mu_{2l+1,\alpha,k} \pi \frac{\Gamma(2s-j-l+k+1)}{\Gamma(2s-j-l+k+m_2)}. \]

Note that this is equal to the terms of the second sum where we interchange \( j \) and \( l \).

• Lastly, for the terms of the fourth sum, we have

\[ \mu_{2j+1,\alpha,k}\mu_{2l+1,\alpha,k}(-1)^{j+l+1}\omega(\omega,\eta)^{s-j+k}\langle \omega,\eta^\dagger \rangle^{s-j-1}\eta^\dagger \langle \omega,\eta \rangle^{s-l+k}\langle \omega,\eta^\dagger \rangle^{s-l-1}\omega. \]

Using Lemma 4.5 and taking the scalar part, the terms of this sum become

\[ \frac{1}{A_m} \mu_{2j+1,\alpha,k}\mu_{2l+1,\alpha,k} 16\pi \frac{(2s-j-l+k-1)!}{\Gamma(2s-j-l+k-1+m_2)}. \]

So now we have
Theorem 6.6. Let $M_\ell(z)$ be a monogenic polynomial of degree $\ell$. We have

$$\tilde{R}[[z^n M_\ell(z)]] = 0 \quad \text{if} \quad n \neq j$$

and for $f(z) = z^j M_\ell(z)$ we have

$$\tilde{R}[[z^j M_\ell(z)]] = \partial_{j,\ell} z^j M_\ell(z)$$

with

$$\partial_{j,\ell} = \frac{(\ell + 1)!(m - 2)!}{2(\ell + m - 2)!}.$$
Proof. We have

\[
\tilde{R}[\mathcal{M}_{z,j}(z^n M_\ell)] = \tilde{R} \left[ \int_{S_{m-1}}^{m} \int_{0}^{\pi} K^j(z, e^{-i\theta} \omega) (e^{i\theta} \omega)^n M_\ell(e^{i\theta} \omega) dS(\omega) d\theta \right]
\]

\[
= \tilde{R} \left[ \int_{S_{m-1}}^{m} \int_{0}^{\pi} \sum_{n \in \mathbb{N}} z^j L^n(z, e^{-i\theta} \omega) (e^{i\theta} \omega)^{-j} (e^{i\theta} \omega)^n M_\ell(e^{i\theta} \omega) dS(\omega) d\theta \right]
\]

\[
= \int_{S_{m-1}}^{m} \sum_{n \in \mathbb{N}} z^j \tilde{R} \left[ L^n(z, e^{-i\theta} \omega) \right] (e^{i\theta} \omega)^{-j} M_\ell(e^{i\theta} \omega) dS(\omega) d\theta.
\]

Using Proposition 6.3, we get

\[
\tilde{R} \left[ L^n(z, e^{-i\theta} \omega) \right] = \tilde{R} \left[ \sum_{k=0}^{\infty} \lambda_k^\alpha M[\psi_\alpha,\alpha, k](z) M[\psi_\alpha,\alpha, k](e^{i\theta} \omega)^\uparrow \right]
\]

\[
= \sum_{k=0}^{\infty} \lambda_k^\alpha \tilde{R} \left[ M[\psi_\alpha,\alpha, k](z) M[\psi_\alpha,\alpha, k](e^{i\theta} \omega)^\uparrow \right]
\]

\[
= \sum_{k=0}^{\infty} \lambda_k^\alpha \gamma_{\alpha, k} C_{\alpha+k}(z, e^{-i\theta} \omega).
\]

Hence

\[
(6.5) \quad \tilde{R}[\mathcal{M}_{z,j}(z^n M_\ell)] = z^j \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \int_{0}^{\pi} e^{i\theta(n+\ell-j-k)} d\theta
\]

\[
\times \int_{S_{m-1}}^{m} \lambda_k^\alpha \gamma_{\alpha, k} C_{\alpha+k}(z, \omega)^{-j} M_\ell(\omega) dS(\omega).
\]

Using Remark 4.2, we get that (6.5) is non-zero if and only if \( n = j \). If \( n = j \), we can use the reproducing properties of the zonal spherical monogenics so that

\[
\tilde{R}[\mathcal{M}_{z,j}(z^j M_\ell)] = z^j \pi A_m \left( \sum_{2s+k=\ell} \lambda_k^{2s} \gamma_{2s,k} + \sum_{2s+1+k=\ell} \lambda_k^{2s+1} \gamma_{2s+1,k} \right) M_\ell(z)
\]

\[
= z^j \pi A_m \left( \sum_{2s+k=\ell} \frac{1}{\pi A_m} \frac{(2s+k)!((m-2)!}{2(2s+k+m-2)!} \right) M_\ell(z)
\]

\[
+ \sum_{2s+1+k=\ell} \frac{1}{\pi A_m} \frac{(2s+k+1)!((m-2)!}{2(2s+k+m-1)!} \right) M_\ell(z)
\]

\[
= z^j (\ell + 1) \frac{\ell!(m-2)!}{2(\ell+m-2)!} M_\ell(z)
\]

\[
= z^j \frac{(\ell + 1)!((m-2)!}{2(\ell+m-2)!} M_\ell(z).
\]

\[
\square
\]

Now using the gamma operator \( \Gamma_z = -z \wedge \partial_z \) we get

\[
\Gamma_z (z^j M_\ell(z)) = -\ell z z^j M_\ell(z),
\]

\[
\Gamma_z (z^{j+1} M_\ell(z)) = (\ell + m - 1) z z^{j+1} M_\ell(z).
\]

Using this result we can write \( \psi_{j, \ell} \) calculated in Theorem 6.6 as an operator:
\[ \vartheta_{2j, \ell} = \vartheta_{2j, -\Gamma_z}, \]
\[ \vartheta_{2j+1, \ell} = \vartheta_{2j+1, \Gamma_z - m+1}. \]

This yields an inversion formula for the monogenic Hua-Radon transform for functions of the type \( \tilde{z}^j M_\ell(z) \) given in the following Theorem:

**Theorem 6.7.** For any \( j, \ell \in \mathbb{N} \), we have
\[ \tilde{z}^{2j} M_\ell(z) = \vartheta^{-1}_{2j, -\Gamma_z} \tilde{R}[M_{\tilde{z}, 2j}(\tilde{z}^{2j} M_\ell(z))], \]
\[ \tilde{z}^{2j+1} M_\ell(z) = \vartheta^{-1}_{2j+1, \Gamma_z - m+1} \tilde{R}[M_{\tilde{z}, 2j+1}(\tilde{z}^{2j+1} M_\ell(z))]. \]

If we were to consider a total monogenic Hua-Radon transform defined as the direct sum
\[ \oplus_{j=0}^{\infty} M_{\tau, j} : \mathcal{O} \mathcal{L}^2(LB(0, 1)) \to \bigoplus_{j=0}^{\infty} \bigoplus_{\alpha \in \mathbb{N}} M^{j, \alpha}(\tau) : f \mapsto \sum_{j=0}^{\infty} M_{\tau, j}(f) \]
we can define an inversion operator for all \( \mathcal{O} \mathcal{L}^2(LB(0, 1)) \) as
\[ \bigoplus_{j=0}^{\infty} \left( \vartheta^{-1}_{2j, -\Gamma_z} \tilde{R}[M_{\tilde{z}, 2j}(\cdot)] + \vartheta^{-1}_{2j+1, \Gamma_z - m+1} \tilde{R}[M_{\tilde{z}, 2j+1}(\cdot)] \right). \]

7. Conclusions

In Theorem 5.2 we completed the computations of the kernel of the monogenic Hua-Radon transform. Following the techniques used to invert the Szegő-Radon transform in [5], we obtained a projection operator mapping a holomorphic function onto the term of the form \( \tilde{z}^j M_\ell(z) \) in its Fischer decomposition. This process leads to a full inversion formula for the total monogenic Hua-Radon transform defined in Theorem 6.7.
THE MONOGENIC HUA-RADON TRANSFORM AND ITS INVERSE

APPENDIX A. COMPUTATION OF $\lambda_k^s$ FROM THEOREM 5.2

In order to determine the coefficients $\lambda_k^s$ from Proposition 5.2 as a closed formula, we first look at the following sums:

$$\sum_{l=0}^s (-1)^l \frac{\Gamma(2s + k - l + 1)}{\Gamma(l + 1)\Gamma(s - l + 1)\Gamma(s + k - l + 1)} = \sum_{l=0}^s (-1)^l \frac{(2s + k - l)!}{l!(s - l)!(s + k - l)!} s!$$

$$= \sum_{l=0}^s (-1)^l \binom{s}{l} \frac{(2s + k - l)}{s + k - l},$$

$$\sum_{l=0}^s (-1)^l \frac{\Gamma(2s + k - l + 2)}{\Gamma(l + 1)\Gamma(s - l + 1)\Gamma(s + k - l + 2)} = \sum_{l=0}^s (-1)^l \frac{(2s + k - l + 1)!}{l!(s - l)!(s + k - l + 1)!} s!$$

$$= \sum_{l=0}^s (-1)^l \binom{s}{l} \frac{(2s + k - l + 1)}{s + k - l + 1}.$$

Note that the last sum is just the first sum by shifting $k$ by 1. This leads to the following:

**Proposition A.1.** For each $k$ we have

(A.1) \hspace{1cm} \sum_{l=0}^s (-1)^l \binom{s}{l} \frac{(2s + k - l)}{s + k - l} = 1.

**Proof.** First of all we recall Pascal’s formula:

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1}$$

Hence for the sum in (A.1) we get

$$\sum_{l=0}^s (-1)^l \binom{s}{l} \frac{(2s + k - l)}{s + k - l} = \binom{s}{0} \frac{(2s + k)}{s + k} + (-1)^s \binom{s}{s} \frac{(s + k)}{k}$$

$$+ \sum_{l=1}^{s-1} (-1)^l \left( \frac{(s-1)}{l} \binom{s-1}{l-1} \right) \frac{(2s + k - l)}{s + k - l}$$

$$= \binom{s-1}{0} \frac{(2s + k)}{s + k} - (-1)^{s-1} \binom{s-1}{s-1} \frac{(s + k)}{k}$$

$$+ \sum_{l=1}^{s-1} (-1)^l \left( \frac{(s-1)}{l} \binom{s-1}{l-1} \right) \frac{(2s + k - l)}{s + k - l}$$

$$= \sum_{l=0}^{s-1} (-1)^l \binom{s-1}{l} \left( \frac{(2s + k - l)}{s + k - l} - \frac{(2s + k - l - 1)}{s + k - l - 1} \right)$$

(A.2) \hspace{1cm} = \sum_{l=0}^{s-1} (-1)^l \binom{s-1}{l} \frac{(2s + k - l - 1)}{s + k - l}.
Hence we see that (A.2) is equal to sum (A.1) under the transform \( s \mapsto s - 1 \) and \( k \mapsto k + 1 \). Hence repeating this process allows us to lower the upper index of the sum, so that we end up with

\[
\sum_{l=0}^{s} (-1)^{l} \binom{s}{l} \frac{2s + k - l}{s + k - l} = \sum_{l=0}^{s-1} (-1)^{l} \binom{s-1}{l} \frac{s+1}{s+k-l+1} \frac{2s + k - l}{s + k - l},
\]

\[
= \binom{0}{0} \binom{s + k - 0}{s + k},
\]

1.

Hence by Proposition A.1 we have

\[
\begin{align*}
(\lambda_{2}^{s+1})^{-1} &= \frac{8\pi^{3-s} \Gamma(s+k+1)}{2s+k+1} \left(1 - \sum_{l=0}^{s-1} (-1)^{l} \frac{s}{\frac{m}{2} + 2s + k - l - 1} \binom{s-1}{l} \frac{2s + k - l}{s + k - l} \right), \\
(\lambda_{k}^{2+s+1})^{-1} &= \frac{32\pi^{3-s} \Gamma(s+k+1)}{2s+k+1} \left(1 - \sum_{l=0}^{s} (-1)^{l} \frac{s + 1}{\frac{m}{2} + 2s + k - l} \binom{s}{l} \frac{2s + k - l + 1}{s + k - l} \right).
\end{align*}
\]

We see that for the remaining sums, if we shift \( s \) by 1 and \( k \) by \(-1\) in the first sum, we get the second sum. Hence it suffices to calculate the first sum.

**Proposition A.2.** We have

\[
\sum_{l=0}^{s-1} (-1)^{l} \frac{s}{\frac{m}{2} + 2s + k - l - 1} \binom{s-1}{l} \frac{2s + k - l}{s + k - l} = 1 - \frac{\Gamma(s-1 + \frac{m}{2}) \Gamma(s + k + 1)}{\Gamma(\frac{m}{2} - 1) \Gamma(\frac{m}{2} + 2s + k)},
\]

\[
= 1 - \frac{(\frac{m}{2} - 1)_s}{(\frac{m}{2} + s + k)_s},
\]

where \((a)_n = a(a+1)\ldots(a+n-1)\) denotes the Pochhammer symbol.

**Proof.** First of all, we see that we can rewrite this series as a multiple of a hypergeometric function \( _{3}F_{2} \):

\[
\sum_{l=0}^{s-1} (-1)^{l} \frac{s}{\frac{m}{2} + 2s + k - l - 1} \binom{s-1}{l} \frac{2s + k - l}{s + k - l} = \frac{s \Gamma(2s + k + 1)}{\Gamma(\frac{m}{2} + 2s + k - 1) \Gamma(s + k + 1) \Gamma(s + 1)} \times _{3}F_{2}[1 - s, -k - s, -\frac{m}{2} - 2s - k + 1; \frac{m}{2} - 2s - k + 2; 1].
\]

In order to determine the value of this hypergeometric function, we will use the following result from [12, p873, appendix, equation (II)]:

\[
_{3}F_{2}[(a, b, -N), (d, e); 1] = (-1)^{N} \frac{(1 + a - d - N - e + b)_{N}}{(d)_{N}} \times _{3}F_{2}[(e - a, e - b, -N), (c - b - a + d, e); 1].
\]

If we now apply this to our hypergeometric series with
\[ N = s - 1 \quad b = -k - s \quad a = -\frac{m}{2} - 2s - k + 1 \]

\[ d = -\frac{m}{2} - 2s - k + 2 = a + 1 \quad e = -k - 2s = b - N - 1, \]
we get that

\[
\phi = (-1)^N \frac{N!}{(a + 1)_N} \binom{a + 1}{N} b \{b - N - a - 1, -N - 1, -N\}, \quad [-N, b - N - 1]; 1 \]

\[
= (-1)^N \frac{N!}{(a + 1)_N} \sum_{j=0}^{N} \frac{N}{(b - N - a - 1)_j (-N)_j} (-N) j \]

\[
= (-1)^N \frac{N!}{(a + 1)_N} \sum_{j=0}^{N+1} (b - N - a - 1)_j (-N - 1)_j \]

\[
= (-1)^N \frac{N!}{(a + 1)_N} \binom{a + 1}{N} \binom{a + 1}{N + 1} b \{b - N - a - 1, -N - 1, -N\}, \quad [-N, b - N - 1]; 1 \]

We can now use Gauss hypergeometric theorem [2, Chapter 15] so that we end up with

\[
\phi = (-1)^N \frac{N!}{(a + 1)_N} \binom{a + 1}{N} \binom{a + 1}{N + 1} b \{b - N - a - 1, -N - 1, -N\}, \quad [-N, b - N - 1]; 1 \]

\[
= (-1)^N \frac{N!}{(a + 1)_N} \binom{a + 1}{N} \binom{a + 1}{N + 1} b \{b - N - a - 1, -N - 1, -N\}, \quad [-N, b - N - 1]; 1 \]

\[
= (-1)^N \frac{N!}{(a + 1)_N} \binom{a + 1}{N} \binom{a + 1}{N + 1} b \{b - N - a - 1, -N - 1, -N\}, \quad [-N, b - N - 1]; 1 \]

\[
= (-1)^N \frac{N!}{(a + 1)_N} \binom{a + 1}{N} \binom{a + 1}{N + 1} b \{b - N - a - 1, -N - 1, -N\}, \quad [-N, b - N - 1]; 1 \]

Thus now our sum becomes

\[
\sum_{l=0}^{s-1} (-1)^l \frac{m/2 + 2s + k - l - 1}{2s + k - l} \left( \frac{s - 1}{l} \right) \frac{2s + k - l}{s + k - l} \]

\[
= \frac{1}{m/2 + 2s + k - 1} \left( \frac{m/2 + 2s + k - 1}{m/2 + s + k} \right) \]

\[
= 1 - \frac{(m/2 - 1)_s}{(m/2 + s + k)_s} \]

\[
= 1 - \frac{\Gamma(s - 1 + m/2) \Gamma(m/2 + s + k)}{\Gamma(m/2 - 1) \Gamma(m/2 + 2s + k)} .
\]

\[ \square \]
Appendix B. Computation of the constants $\gamma_{a,k}$ from Proposition 6.4

Plugging in the explicit expressions for $\mu_{j,a,l}$ (see Lemma 3.7), the summations at the end of the proof of Proposition 6.4 become

(B.1) \[
\sum_{j=0}^{s} \sum_{l=0}^{s} \mu_{2j} \mu_{2l} \frac{\Gamma(2s - j - l + k + 1)}{\Gamma(2s - j - l + k + \frac{m}{2})}
\]

(B.2) \[
= \frac{(s!)^2((s + k)!)^2}{\Gamma(\frac{m}{2} + 2s + k)^2} \sum_{j=0}^{s} (-1)^j \frac{\Gamma(\frac{m}{2} + 2s + k - j)}{j!(s - j)!}(s + k - j)!
\]

(B.3) \[
\times \sum_{l=0}^{s} \frac{\Gamma(\frac{m}{2} + 2s + k - l) \Gamma(2s - j - l + k + 1)}{(s - l)!(s + k - l)! \Gamma(2s - j - l + k + \frac{m}{2})},
\]

(B.4) \[
\sum_{j=0}^{s-1} \sum_{l=0}^{s} \mu_{2j+1} \mu_{2l+1} \frac{\Gamma(2s - j - l + k + 1)}{\Gamma(2s - j - l + k + \frac{m}{2})}
\]

(B.5) \[
= \frac{(s!)^2((s + k)!)^2}{\Gamma(\frac{m}{2} + 2s + k)^2} \sum_{j=0}^{s-1} (-1)^j \frac{\Gamma(\frac{m}{2} + 2s + k - j - 1)}{2j!(s - j - 1)!}(s + k - j)!
\]

(B.6) \[
\times \sum_{l=0}^{s-1} \frac{(-1)^l \Gamma(\frac{m}{2} + 2s + k - l - 1) \Gamma(2s - j - l + k)}{2!(s - l - 1)!(s + k - l) \Gamma(2s - j - l + k + \frac{m}{2})}.
\]

Now note that the right-hand sides of (B.2) and (B.4) are identical and if we replace $s$ by $s - 1$ and $k$ by $k + 1$, the right-hand side of (B.2) transforms into (B.6). Moreover, if we write out the sums in the odd case, the summations with respect to $l$ are equal to (B.2) (if necessary change $k$ to $k + 1$). Moreover (B.3) and (B.5) can be achieved by changing $s$ to $s - 1$ and $k$ to $k + 1$ in (B.1) and the same can be done for the remaining sums in the odd case. Hence it suffices to only calculate sum (B.2) and (B.1). We have the following:

Proposition B.1.

\[
\sum_{l=0}^{s} \frac{(-1)^l \Gamma(\frac{m}{2} + 2s + k - l) \Gamma(2s - j - l + k + 1)}{l!(s - l)!(s + k - l)! \Gamma(2s - j - l + k + \frac{m}{2})} = 1.
\]

Proof. We have

\[
\sum_{l=0}^{s} \frac{(-1)^l \Gamma(\frac{m}{2} + 2s + k - l) \Gamma(2s - j - l + k + 1)}{l!(s - l)!(s + k - l)! \Gamma(2s - j - l + k + \frac{m}{2})} = \frac{\Gamma(\frac{m}{2} + 2s + k)}{s!(s + k)!} \frac{\Gamma(2s - j + k + 1)}{\Gamma(\frac{m}{2} + 2s + k - j)} \phi,
\]

where

\[
\phi = {}_3F_2([a, b, -N], [d, e]; 1)
\]
Using (A.3), we have

\[ a = -s - k \quad b = j - k - 2s - \frac{m}{2} + 1 \quad N = s \]
\[ d = -2s + j - k = a - N + j \quad e = -\frac{m}{2} - 2s - k + 1 = b - j. \]

Using (A.3), we have

\[
\phi = (-1)^N \frac{(1 + a + b - d - e - N)_N}{(d)_N} \binom{3F_2}{[e - a, e - b, -N], [e - a - b + d, e]; 1} \\
= (-1)^N \frac{(1)_N}{(d)_N} \binom{3F_2}{[e - a, -j, -N], [-N, e]; 1} \\
= (-1)^N N! \sum_{l=0}^{j} \frac{(e - a)_l((-j)_l((-N)_l)}{(N)_l(e)_l} \\
= \frac{\Gamma(-d + N + 1)N!}{\Gamma(-d + 1)} \sum_{l=0}^{j} \frac{(e - a)_l((-j)_l}{(e)_l} \\
= \frac{\Gamma(-d + N + 1)N!}{\Gamma(-d + 1)} \binom{2F_1}{[e - a, -j], [e]; 1}
\]

where we used that \( j \leq s = N \) and hence the sum will terminate at \( j \) instead of \( N \). Now using Chu-Vandermonde identity (see [2]) we have

\[ \binom{2F_1}{[e - a, -j], [e]; 1} = \frac{(a)_j}{(e)_j} \]
\[ = \frac{(-s - k)_j}{(-\frac{m}{2} - 2s - k + 1)_j} \\
= \frac{a(a + 1)\ldots(a + j - 1)}{e(e + 1)\ldots(e + j - 1)} \\
= \frac{(-a)(-a - 1)\ldots(-a - j + 1)}{((-e)(-e - 1)\ldots(-e - j + 1)} \\
= \frac{\Gamma(-a + 1)\Gamma(-e - j + 1)}{\Gamma(-a - j + 1)\Gamma(-e + 1)} \\
\]

as \(-a - j + 1 \geq 0, -a \geq 0, -e \geq 0 \) and \(-e - j + 1 \geq 0 \). Thus for our total sum, we have

\[
\sum_{l=0}^{s} (-1)^l \frac{\Gamma\left(\frac{m}{2} + 2s + k - l\right)}{l!(s - l)!(s + k - l)!} \frac{\Gamma(2s - j - l + k + 1)}{\Gamma(2s - j - l + k + \frac{m}{2})} \\
= \frac{\Gamma\left(\frac{m}{2} + 2s + k\right)}{s!(s + k)!} \frac{\Gamma(2s - j + k + 1)}{\Gamma\left(\frac{m}{2} + 2s + k - j\right)} \frac{\Gamma(-d + N + 1)N!\Gamma(-a + 1)\Gamma(-e - j + 1)}{\Gamma(-d + 1)\Gamma(-a - j + 1)\Gamma(-e + 1)} \\
= \frac{\Gamma(-e + 1)}{\Gamma(-d + 1)} \frac{\Gamma(-d + N + 1)N!\Gamma(-a + 1)\Gamma(-e - j + 1)}{\Gamma(-e - j + 1)} \\
= \frac{\Gamma(-d - N + 1)}{\Gamma(-a - j + 1)} \\
= \frac{\Gamma(-d - N + 1)}{\Gamma(-d - N + 1)} \\
= 1.
\]

\[ \Box \]
Proposition B.2. The right-hand side of (B.1) is equal to
\[ \sum_{j=0}^{s} (-1)^j \frac{\Gamma \left( \frac{m}{2} + 2s + k - j \right)}{j! \Gamma(s - j + 1) \Gamma(s + k - j + 1)} = \frac{\Gamma \left( \frac{m}{2} + s \right) \Gamma \left( \frac{m}{2} + s + k \right)}{s! (s + k) \Gamma \left( \frac{m}{2} \right)}. \]

Proof. Using Chu-Vandermonde identity (see [2]), we have
\[ \sum_{j=0}^{s} (-1)^j \frac{\Gamma \left( \frac{m}{2} + 2s + k - j \right)}{j! \Gamma(s - j + 1) \Gamma(s + k - j + 1)} = \frac{\Gamma \left( \frac{m}{2} + 2s + k \right)}{\Gamma(s + 1) \Gamma(s + k + 1)} \]
\[ \times 2F1 \left( \left[ -s, -s - k \right], \left[ -\frac{m}{2} - 2s - k + 1 \right]; 1 \right) \]
\[ = \frac{\Gamma \left( \frac{m}{2} + 2s + k \right)}{\Gamma(s + 1) \Gamma(s + k + 1)} \left( -\frac{m}{2} - 2s - k + 1 \right) \]
\[ = \frac{\Gamma \left( \frac{m}{2} + s \right) \Gamma \left( \frac{m}{2} + s + k \right)}{\Gamma(s + 1) \Gamma(s + k + 1) \Gamma \left( \frac{m}{2} \right)}. \]

Now combining the results of Proposition B.1 and Proposition B.2 we get the following:

Proposition B.3. The constants \( \gamma_{\alpha,k} \) are given by
\[ \gamma_{2s,k} = \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s - 1 \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \frac{\Gamma(m - 1) \Gamma(2s + k + 1)}{\Gamma(2s + k + m - 1)}, \]
\[ \gamma_{2s+1,k} = \frac{4s!(s + k + 1)! \Gamma \left( \frac{m}{2} + s - 1 \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \frac{\Gamma(m - 1) \Gamma(2s + k + 1)}{\Gamma(2s + k + m - 1)}. \]

Proof. Combining Proposition B.1 and Proposition B.2 we get
\[ \gamma_{2s,k} \frac{2s + k + m - 2}{m - 2} C_{2s,k}^{-1}(1) = 2\Gamma \left( \frac{m}{2} \right) \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \]
\[ \times \frac{\Gamma(m - 1) \Gamma(2s + k + 1)}{\Gamma(2s + k + m - 1)} \]
\[ = 2\Gamma \left( \frac{m}{2} \right) \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \]
\[ \times \left[ \frac{\Gamma(m - 1) \Gamma(2s + k + 1)}{\Gamma(2s + k + m - 1)} - \frac{\Gamma \left( \frac{m}{2} + s - 1 \right) \Gamma \left( \frac{m}{2} + s + k \right)}{\Gamma(s) \Gamma(s + k + 1) \Gamma \left( \frac{m}{2} \right)} \right] \]
\[ = 2\Gamma \left( \frac{m}{2} \right) \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \]
\[ \times \left[ \frac{\Gamma(m - 1) \Gamma(2s + k + 1)}{\Gamma(2s + k + m - 1)} - \frac{\Gamma \left( \frac{m}{2} + s - 1 \right) \Gamma \left( \frac{m}{2} + s + k \right)}{\Gamma(s) \Gamma(s + k + 1) \Gamma \left( \frac{m}{2} \right)} \right] \]
\[ = 2\Gamma \left( \frac{m}{2} \right) \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \]
\[ \times \left[ \frac{\Gamma(2s + k) \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)} - \frac{\Gamma \left( \frac{m}{2} + s - 1 \right) \Gamma \left( \frac{m}{2} + s + k \right)}{\Gamma(s) \Gamma(s + k + 1) \Gamma \left( \frac{m}{2} \right)} \right] \]
\[ = 2\Gamma \left( \frac{m}{2} \right) \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \]
\[ \times \left[ \frac{\Gamma(2s + k) \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)} - \frac{\Gamma \left( \frac{m}{2} + s - 1 \right) \Gamma \left( \frac{m}{2} + s + k \right)}{\Gamma(s) \Gamma(s + k + 1) \Gamma \left( \frac{m}{2} \right)} \right] \]
\[ = 2\Gamma \left( \frac{m}{2} \right) \frac{s!(s + k)! \Gamma \left( \frac{m}{2} + s \right)}{\Gamma \left( \frac{m}{2} + 2s + k \right)^2} \Gamma \left( \frac{m}{2} + s + k \right) (m - 2) \]
and thus
\(\gamma_{2s,k} = \frac{s!(s+k)!}{\Gamma\left(\frac{m}{2} + s + k\right)} \frac{\Gamma\left(\frac{m}{2} + s + k + 1\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{\Gamma\left(\frac{m}{2} + s - 1\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{\Gamma\left(\frac{m}{2} + s + k\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{m - 2}{(2s + k + m - 2)C_{2s+k}^{m-1}}(1)
\)
\[
= \frac{s!(s+k)!}{\Gamma\left(\frac{m}{2} + s + k\right)} \frac{\Gamma\left(\frac{m}{2} + s + k + 1\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \Gamma\left(\frac{m}{2} + s - 1\right) \frac{\Gamma\left(\frac{m}{2} + s + k\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{m - 2}{(2s + k + m - 2)\Gamma(2s + k + 1)}
\]
\[
= \frac{s!(s+k)!}{\Gamma\left(\frac{m}{2} + s + k\right)} \frac{\Gamma\left(\frac{m}{2} + s + k + 1\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \Gamma\left(\frac{m}{2} + s - 1\right) \frac{\Gamma\left(\frac{m}{2} + s + k\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{m - 2}{(2s + k + m - 2)\Gamma(2s + k + 1)}
\]

Analogously we get for \(\alpha = 2s + 1\):

\(\gamma_{2s+1,k} = \frac{s!(s+k+1)!}{\Gamma\left(\frac{m}{2} + s + k + 1\right)} \frac{\Gamma\left(\frac{m}{2} + s + k + 2\right)}{\Gamma\left(\frac{m}{2} + 2s + k+1\right)} \Gamma\left(\frac{m}{2} + s - 1\right) \frac{\Gamma\left(\frac{m}{2} + s + k\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{m - 2}{(2s + k + m - 1)\Gamma(2s + k + 2)}
\)
\[
= \frac{s!(s+k+1)!}{\Gamma\left(\frac{m}{2} + s + k + 1\right)} \frac{\Gamma\left(\frac{m}{2} + s + k + 2\right)}{\Gamma\left(\frac{m}{2} + 2s + k+1\right)} \Gamma\left(\frac{m}{2} + s - 1\right) \frac{\Gamma\left(\frac{m}{2} + s + k\right)}{\Gamma\left(\frac{m}{2} + 2s + k\right)} \frac{m - 2}{(2s + k + m - 1)\Gamma(2s + k + 2)}
\]

\(\square\)

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