Detecting Simultaneous Integer Relations for Several Real Vectors

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Abstract

An algorithm which either finds a nonzero integer vector $m$ for given $t$ real $n$-dimensional vectors $x_1, \cdots, x_t$ such that $x^T m = 0$ or proves that no such integer vector with norm less than a given bound exists is presented in this paper. The cost of the algorithm is at most $O(n^4 + n^2 \log \lambda(X))$ exact arithmetic operations in dimension $n$ and the least Euclidean norm $\lambda(X)$ of such integer vectors. It matches the best complexity upper bound known for this problem. Experimental data show that the algorithm is better than an already existing algorithm in the literature. In application, the algorithm is used to get a complete method for finding the minimal polynomial of an unknown complex algebraic number from its approximation, which runs even faster than the corresponding Maple built-in function.

Keywords: integer relation, PSLQ, HJLS, algebraic number, minimal polynomial

1. Introduction

Given a real vector $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n$, say a nonzero vector $m = (m_1, \cdots, m_n)^T \in \mathbb{Z}^n$ is an integer relation for $x$ if $x^T m = 0$. How to detect an integer relation for a given real vector is an old problem. This is solved, for instance, by the PSLQ algorithm\textsuperscript{[8]} that together with related lattice reduction schemes such as LLL\textsuperscript{[14]}, was named one of ten "algorithms of the twentieth century" by the publication Computing in Science and Engineering (see \textsuperscript{[6]}). This paper considers a generalization of the problem. Let $x_1, \cdots, x_t$ be $t$ vectors in $\mathbb{R}^n$, and denote $(x_1, \cdots, x_t)$ by $X$. A \textit{simultaneous integer relation} (SIR) for $x_1, \cdots, x_t$ is a vector $m \in \mathbb{Z}^n \setminus \{0\}$ such that $X^T m = 0$, i.e. $x_i^T m = 0$ for $i = 1, \cdots, t$. For short, we also call $m$ an SIR for $X$. An algorithm which either finds an SIR for $t$ real $n$-dimensional vectors or proves that no SIR with norm less than a given bound exists is presented in this paper.

When $t = 1$, the problem of detecting integer relations for one rational or real vector is quite old. For two numbers $(a_1, a_2)$, the venerable Euclidean algorithm does the job by computing the ordinary continued fraction expansion of the real number $a_1/a_2$. For $n \geq 3$, many detecting algorithms under the names generalized Euclidean algorithm and multidimensional continued fraction algorithm were proposed. We refer the reader to \textsuperscript{[11, 8]} for comprehensive surveys. Among these integer relation algorithms, the LLL-based HJLS algorithm\textsuperscript{[11]} and the PSLQ algorithm\textsuperscript{[8]} have been used frequently.

To authors’ known, the first algorithm to detect SIRs for several real vectors ($t \geq 2$) was presented in \textsuperscript{[11, 8]}, in which J. Hastad, B. Just, J. C. Lagarias, and C. P. Schnorr not only presented the HJLS algorithm to find integer relations for one real vector, gave the first rigorous proof of a ‘polynomial time’ bound for a relation finding algorithm, but also proposed a simultaneous relations algorithm. Unfortunately HJLS has a serious drawback: it is extremely unstable numerically (see \textsuperscript{[1, 8]}). In their draft \textsuperscript{[16]}, C. Rössner and C. P. Schnorr proposed an algorithm which computes for real vectors $x_1, x_2$ simultaneous diophantine approximation to the plane spanned by the vectors $x_1, x_2$ by using a modified HJLS algorithm. It can be seen as a special case $t = 2$ of the aforementioned problem. But for the moment, it is still in a preliminary state with some open problems.

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The PSLQ algorithm \cite{2} is now extensively used in Experimental Mathematics, with applications such as identification of multiple zeta constants, a new formula for $\pi$, quantum field theory and so on (see \cite{1, 3, 2}). PSLQ employs a numerically stable matrix reduction procedure, so it is numerically stable in contrast to other integer relation algorithms. Moreover, it can be generalized to the complex number field and the Hamiltonian quaternion number field, but the corresponding outputs are in Gaussian integer ring and Hamilton integer ring respectively. For example, PSLQ will output $(1, i, -i)^T$ for the complex vector $(1 + i, 1 + 2i, 2 + i)^T$, where $i = \sqrt{-1}$. The reason is that Hermite reduction in PSLQ produces some Gaussian integers in the reducing matrix (see Section 2.2). Thus PSLQ can not be used to detect SIRs (in $\mathbb{Z}^n$) for several real vectors.

An algorithm to detect SIRs for $r$ real vectors is presented in this paper. It uses a technique, similar to that in HJLS, to construct the hyperplane matrix and a method, generalized from PSLQ, for matrix reduction. The algorithm either finds an SIR for $X$ if one exists or proves that there are no SIRs for $X$ of norm less than a given size. The cost of the algorithm is at most $O(n^4 + n^3 \log \lambda(X))$ exact arithmetic operations to detect an SIR for $X$, where $n$ is the dimension of the input real vectors and $\lambda(X)$ represents the least Euclidean norm of SIRs for $X$. Although the same theoretic complexity as obtained for the HJLS simultaneous relations algorithm is proved, experiments show that the algorithm in this paper often performs better in practice. Furthermore, in contrast to PSLQ, our algorithm can be applied to detect an integer relation in $\mathbb{Z}^n$ (rather than in the Gaussian or Hamiltonian integer rings) for complex or Hamiltonian vectors. Consequently, a complete method to find the minimal polynomial of an approximately complex algebraic number is obtained by applying our algorithm.

Our main contributions in this paper are the following:

- We present a new algorithm to detect SIRs for several real vectors and show that its complexity matches the best one known for this problem (HJLS simultaneous relation algorithm).

- We implement our algorithm in Maple by two schemes. The one uses software floating arithmetic (multiprecision floating point arithmetic) in all steps, and the other partially uses software floating arithmetic and mainly uses hardware floating arithmetic. Then we report many experimental results, which shows that our algorithm is relevant.

- We successfully apply our algorithm to find the minimal polynomial of an approximately complex algebraic number. This strategy is different from some known LLL-based methods, such as \cite{3, 2}, and is for all complex algebraic numbers rather than mere for real algebraic numbers in \cite{15}. We also present many experiments, which shows that this newly complete method is efficient and even better than the Maple built-in function PolynomialTools:-MinimalPolynomial.

The reminder of this paper is organized as follows. In Section 2 we both preliminaries and main results of this paper are presented. The cost of our algorithm is analyzed in Section 3. Some empirical studies, further discussions and an application of our algorithm are included in Section 4.

### 2. The Main Algorithm

#### 2.1. Notations and Assumptions

Throughout this paper, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ stand for the sets of integers, real numbers, and complex numbers respectively. For $c \in \mathbb{R}$, $[c]$ denotes an arbitrary integer closest to $c$, i.e. $[c] = \lfloor c + \frac{1}{2} \rfloor$. All vectors in this paper are column vectors, and will be denoted in bold. If $x \in \mathbb{R}^n$, then $\|x\|_2$ represents its Euclidean norm, i.e. $\|x\|_2 = \sqrt{x \cdot x}$, where $(\cdot, \cdot)$ is the inner product of two vectors. We denote the $n \times n$ identity matrix by $I_n$. Given a matrix $A = (a_{i,j})$, we denote its transpose by $A^T$, its trace by $\text{tr}(A)$, its determinant by $|A|$, and its Frobenius norm by $\|A\|_F = (\text{tr}(AA^T))^{1/2} = (\sum a_{i,j}^2)^{1/2}$. We say that a matrix $A$ is lower trapezoidal if $a_{i,j} = 0$ for $i < j$. The group of $n \times n$ unimodular matrices with entries in $\mathbb{Z}$ are denoted by $GL(n, \mathbb{Z})$. 

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In what follows we always suppose that $x_1, \cdots, x_t \in \mathbb{R}^n$ are linearly independent, where $x_i = (x_{i,1}, \cdots, x_{i,n})^T$ and $t < n$. Obviously, every $x_i$ is nonzero. Let $X \in \mathbb{R}^{n \times t}$ be the matrix $(x_1, \cdots, x_t)$ and suppose that $X$ satisfies

$$
\begin{vmatrix}
    x_{1,1-t+1} & x_{2,1-t+1} & \cdots & x_{n,1-t+1} \\
    x_{1,1-t+2} & x_{2,1-t+2} & \cdots & x_{n,1-t+2} \\
    \vdots & \vdots & \ddots & \vdots \\
    x_{1,n} & x_{2,n} & \cdots & x_{n,n} \\
\end{vmatrix} \neq 0. \tag{1}
$$

If $X \in \mathbb{R}^{n \times t}$ does not satisfy (1), we can always (since $X$ has rank $t$) exchange some rows of $X$ to produce $X' = CX$ such that $X'$ satisfying (1), where $C$ is an appropriate matrix in $GL(n, \mathbb{Z})$. In this case, we detect an SIR for $X'$. If $m$ is detected as an SIR for $X'$, then $C^Tm$ is an SIR for $X$.

2.2. A Method to construct a Hyperplane Matrix

**Definition 2.1** (Hyperplane Matrix). Let $X = (x_1, \cdots, x_t) \in \mathbb{R}^{n \times t}$. A hyperplane matrix with respect to $X$ is any matrix $H \in \mathbb{R}^{n \times (n-t)}$ such that $X^TH = 0$ and the columns of $H$ span $X^\perp = \{y \in \mathbb{R}^n : x_i^Ty = 0, i = 1, \cdots, t\}$.

Given $X = (x_1, \cdots, x_t) \in \mathbb{R}^{n \times t}$ satisfying (1), we now present a method to construct a hyperplane matrix for $X$. The basic idea is from HJLS [11]. The same strategy was also used in PSLQ, based on a partial-sum-of-squares vector and a lower-quadrature matrix factorization, instead of Gram-Schmidt orthogonalization.

Let $b_1, \cdots, b_n$ form the standard basis of $\mathbb{R}^n$, i.e. the $i$-th entry of $b_i$ is 1 and others are 0. Perform the process of standard Gram-Schmidt orthogonalization to $x_1, \cdots, x_t, b_1, \cdots, b_n$ in turn producing $x_1', \cdots, x_t', b_1', \cdots, b_n'$. Note that, since $X$ satisfies (1), we have $b_n', \cdots, b_1' = 0$.

Define $H_X$ to be the $n \times (n-t)$ matrix $(b_1', \cdots, b_{n-t}')$. From the following lemma, $H_X = (b_1', \cdots, b_{n-t}')$ is a hyperplane matrix with respect to $X \in \mathbb{R}^{n \times t}$.

**Lemma 2.2.** Let $X \in \mathbb{R}^{n \times t}$ and $H_X$ be as above. Then

1. $H_X^TH_X = I_{n-t}$.
2. $\|H_X\|_F = \sqrt{n-t}$.
3. $(x_1', \cdots, x_t', H_X)$ is an orthogonal matrix.
4. $X^TH_X = 0$, i.e. $H_X$ is a hyperplane matrix of $X$.
5. $H_X$ is a lower trapezoidal matrix and every diagonal element of $H_X$ is nonzero.

**Proof.** Since every two columns of $H_X$ are orthogonal, part 1 follows. And part 2 follows from part 1. Let $X' = (x_1', \cdots, x_t')^T$. Obviously, $(x_1', \cdots, x_t', H_X)$ is an orthogonal matrix. From part 3 and standard Gram-Schmidt orthogonalization we have $X'^TH_X = 0$ and $X = X'Q$ respectively, where $Q$ is an appropriate $t \times t$ invertible matrix. Thus $X^TH_X = Q^TX'^TH_X = 0$ and hence that part 4 follows. We now prove part 5. Denote the $k$-th element of $b_1'$ by $b_{1,k}$. The diagonal elements of $H_X$ are $b_{i,i}$ for $i = 1, \cdots, n-t$. Before normalizing $b_1'$ we have $b_{i,i}' = 1 - \sum_{k=1}^{t} x_{k,i}'^2 - \sum_{j=1}^{t-1} b_{j,i}'^2$, and at the same time,

$$0 \neq \|b_1'\|^2_2 = (b_1', b_1') = 1 - \sum_{k=1}^{t} x_{k,1}'^2 - \sum_{j=1}^{t-1} b_{j,1}'^2.$$

Thus all the diagonal elements of $H_X$ are nonzero. Now we only need to show that $H_X$ is lower trapezoidal. From standard Gram-Schmidt orthogonalization, we can check that $b_{i,k}' = (b_j', b_k') = 0$ holds for $i > k$. This completes the proof. \[\Box\]

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\[\footnote{Our assumption $t < n$ is based on the following fact: Any SIR $m$ is in the orthogonal complement space of span($x_1, \cdots, x_t$). Since $x_1, \cdots, x_t$ are linearly independent vectors in $\mathbb{R}^n$ we have $t \leq n$. So if $t = n$, then the dimension of the linear space span($x_1, \cdots, x_t$) is $n$, hence that there exists no simultaneous integer relations for $x_1, \cdots, x_t$.}

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2.3. Generalized Hermite Reduction

We now study how to reduce the hyperplane matrix \( H_X \). First we recall (modified) Hermite reduction as presented in [8].

**Definition 2.3** (Modified Hermite reduction). Let \( H = (h_{i,j}) \) be a lower trapezoidal matrix with \( h_{ij} \neq 0 \) and set \( D := I_n \). For \( i \) from 2 to \( n \), and for \( j \) from \( i - 1 \) to 1 by step \(-1\), set \( q := |h_{ij}/h_{ij}| \); then for \( k \) from 1 to \( n \), replace \( d_{ik} \) by \( d_{ik} - qd_{jk} \). We say \( DH \) is the modified Hermite reduction of \( H \) and \( D \) is the reducing matrix of \( H \).

If the entries of \( H \) are complex numbers, then \( q = |h_{ij}/h_{ij}| \) may be a Gaussian integer. Thus for a complex vector, PSLQ can only gives a Gaussian integer relation.

Hermite reduction is also presented in [8], and is equivalent to modified Hermite reduction for a lower triangular matrix \( H \) with \( h_{ij} \neq 0 \). Both of the two equivalent reductions have the following properties:

1. The reducing matrix \( D \in GL(n, \mathbb{Z}) \).
2. For all \( k > i \), the (modified) Hermite reduced matrix \( H' = (h'_{ij}) = DH \) satisfies \( |h'_{s,t}| \leq |h'_{s,t}|/2 = |h_{s,t}|/2 \).

Unfortunately, (modified) Hermite reduction is not suitable to detect SIRs any more because it does not deal with the last \( t - 2 \) rows of \( H_X \) when \( 2 < t < n \). In order that the reduced and reducing matrices of \( H_X \in \mathbb{R}^{n \times (n - t)} \) satisfy the two properties above, we generalize the Hermite reduction as follows.

**Definition 2.4** (Generalized Hermite Reduction). Let \( H \) be a lower trapezoidal matrix with \( h_{ij} \neq 0 \) and set \( D := I_n \), \( H' = (h'_{ij}) := DH \). For \( i \) from 2 to \( n \), for \( j \) from \( \min(i - 1, n - t) \) by \(-1\) to \( 1 \), \( q := |h'_{ij}/h'_{ij}| \); for \( k \) from 1 to \( j \), \( h'_{ik} = h'_{ik} - qh'_{jk} \); for \( k \) from 1 to \( n \), \( d_{ik} := d_{ik} - qd_{jk} \). For every two integers \( s_1, s_2 \in \{n - t + 1, \ldots, n\} \) satisfying \( s_1 < s_2 \), \( h'_{s_1,n-1} = 0 \) and \( h'_{s_2,n-1} \neq 0 \), exchange the \( s_1 \)-th row and the \( s_2 \)-th row of \( D \). We call \( DH \) the generalized Hermite reduction of \( H \) and \( D \) the reducing matrix.

Obviously, generalized Hermite reduction is equivalent to modified Hermite reduction when \( t = 1 \). In addition, we can easily check that generalized Hermite reduction retains the two properties mentioned above when \( 1 \leq t < n \).

There are two main differences between the (modified) Hermite reduction and the generalized Hermite reduction. Firstly, the last \( t - 1 \) rows of \( H \) will also be reduced by the first \( n - t \) rows of \( H \) in the generalized Hermite reduction, while the (modified) Hermite reduction not. Secondly, generalized Hermite reduction exchanges the \( s_1 \)-th row and the \( s_2 \)-th row of \( D \) if \( s_1 < s_2 \), \( h'_{s_1,n-1} = 0 \) and \( h'_{s_2,n-1} \neq 0 \) hold. This implies that if \( h_{n-t+1,n-1} = 0 \) after generalized Hermite reduction then \( h_{n-t+2,n-1} = \cdots = h_{n,n-1} = 0 \).

2.4. The Algorithm Description

Based on the method to construct the hyperplane matrix and the generalized Hermite reduction, an algorithm to detect SIR for real vectors is proposed as follows.

**Algorithm 1** (Simultaneous Integer Relation Detection).

**Input:** \( (x_1, \ldots, x_d) = X \in \mathbb{R}^{d \times d} \) satisfying \( 1 \) and a parameter \( \gamma > 2/\sqrt{5} \).

**Initialization.**

- Compute the hyperplane matrix \( H_X \) and set \( H := H_X, B := I_n \).
- Reduce the hyperplane matrix \( H \) by the generalized Hermite reduction producing the reducing matrix \( D \). Set \( X^T := X^T D^{-1}, H := DH, B := BD^{-1} \).

**Iteration.**

1. Exchange. Let \( H = (h_{ij}) \). Choose an integer \( r \) such that \( \gamma' |h_{ij}| \geq \gamma |h_{ij}| \) for \( 1 \leq i \leq n - t \). Let

\[
\alpha := h_{ir}, \quad \beta := h_{r+1,i}, \quad \lambda := h_{r+1,r+1}, \quad \delta := \sqrt{\beta^2 + \lambda^2}.
\]

Define the permutation matrix \( R \) to be the identity matrix with the \( r \) and \( r + 1 \) rows exchanged. Update \( X^T := X^T R, H := RH, B := BR \).
2. Corner. Let $Q := I_{n-t}$. If $r < n - t$, then let the submatrix of $Q$ consisting of the $r$-th and $(r + 1)$-th rows of columns $r$ and $r + 1$ be

$$
\begin{bmatrix}
\beta/\delta & -\lambda/\delta \\
\lambda/\delta & \beta/\delta
\end{bmatrix},
$$

Update $H := HQ$.

3. Reduction. Reduce $H$ by the generalized Hermite reduction producing $D$. Set $X^T := X^TD^{-1}$, $H := DH$, $B := BD^{-1}$.

4. Termination. Compute $G := 1/\|H\|_F$. Then there exists no SIR whose Euclidean norm is less than $G$. Denote $B = (B_1, \cdots, B_n)$, where $B_j \in \mathbb{R}^n$. If $X^TB_j = 0$ for some $1 \leq j \leq n$, or $h_{r-t,n-t} = 0$ then

**Output:** the corresponding SIR for $X$.

**Remark 2.5.** The description of Algorithm 1 is similar to PSLQ. But the biggest difference is that Algorithm 1 used the generalized Hermite reduction. The (modified) Hermite reduction used in PSLQ is not suitable to detect SIRs for several real vectors, as mentioned early. PSLQ may be viewed as a particular case of Algorithm 1 when $t = 1$.

**Remark 2.6.** Given a complex vector $z = x + yI$ in $\mathbb{C}^n$ where $x, y \in \mathbb{R}^n$ and $I = \sqrt{-1}$, finding an integer relation (in $\mathbb{Z}^n$) for $z$ is equivalent to finding an SIR for $(x, y)$. Thus Algorithm 1 can be used. For instance, let $z = (2 + 3I, 4 + 9I, 8 + 27I, 16 + 81I, 32 + 243I)^T$. For finding an integer relation for $z$, first let $x = (2, 4, 8, 16, 32)^T$ and $y = (3, 9, 27, 81, 243)^T$. Running Algorithm 1 with $\gamma = 1.16$, and $x, y$ as its input vectors gives a SIR $(6, 7, -9, 2, 0)^T$ for $(x, y)$, which, of course, also is an integer relation for $z$. This is one of the biggest differences between Algorithm 1 and PSLQ since PSLQ can only give a Gaussian integer relation in $\mathbb{Z}[I]^n$ for $z$ rather than an integer relation in $\mathbb{Z}^n$.

**Remark 2.7.** Generally, $t$ real $n$-dimensional vectors may have 0, 1, or up to $n - t$ linearly independent SIRs. One can follow the strategy in [8, Section 6] to find them.

**Theorem 2.8.** Let $X = (x_1, \cdots, x_t)$ satisfy (1) and $\lambda(X)$ be the least Euclidean norm of SIRs for $X$. Suppose there exists an SIR for $X$. Then

1. An SIR for $X$ will appear as a column of $B$ after no more than

$$\left(\begin{array}{c} n \\ 2 \end{array}\right) - \left(\begin{array}{c} t \\ 2 \end{array}\right) \frac{\log(\gamma^{n-t})\lambda(X)}{4\log \left(\frac{\lambda(X)}{\gamma^{n-t}}\right)}.$$

iterations in Algorithm 1.

2. If after a number of iterations no SIR has yet appeared in a column of $B$, then there are no SIRs of norm less than the bound $1/\|H\|_F$.

From this theorem, Algorithm 1 either finds an SIR $m$ for given real vectors $x_1, \cdots, x_t$ such that $x_1^T m = 0$ or proves that no small simultaneous integer relation with Euclidean norm less than $1/\|H\|_F$ exists.

Moreover, it can be proved that the norm of the SIR for $X$ output by Algorithm 1 is no greater than $\gamma^{n-t} \lambda(X)$. This is an important property of Algorithm 1, and the proof is similar to that of Theorem 3 in [8].

**Corollary 2.9.** If $X \in \mathbb{R}^{n \times t}$ has SIRs, then there exists a $\gamma$ such that Algorithm 1 can find an SIR for $X$ in polynomial time $O(n^4 + n^3 \log \lambda(X))$.

**Proof.** Let $\gamma = 2$. Then Algorithm 1 constructs an SIR for $X$ in no more than

$$(n - t)^3(n + t - 1) + (n - t)(n + t - 1) \log \lambda(X)$$

iterations. Algorithm 1 takes $O(n - t)$ exact arithmetic operations per iteration, and hence that $O(n - t)^3 + (n - t)^3 \log \lambda(X))$ exact arithmetic operations are enough to produce an SIR for $X$. Since $t < n$, the proof is complete.

**Remark 2.10.** All conclusions above also hold for complex numbers with $\gamma > \sqrt{2}$, but the outputs of the corresponding variation of Algorithm 1 are in Gaussian integer ring.
3. Proof of Theorem 2.8

Given \( X = (x_1, \cdots, x_t) \in \mathbb{R}^{n \times t} \), let \( H_X \) be the hyperplane matrix obtained by the method introduced in section 2.2 and let \( P_X = H_X H_X^T \). By expanding this expression, it follows that \( P_X = I_t - \sum_{i=1}^{t} x_i x_i^T \). Let \( m \in \mathbb{Z}^n \) be an SIR for \( X \). Then it can be seen that \( P_X m = m \) and \( \|P_X\|_F = \sqrt{n-t} \). For any matrix \( D \in GL(n, \mathbb{Z}) \) and \((n-t) \times (n-t)\) orthogonal matrix \( Q \),

\[
1 \leq \|Dm\|_2 = \|DP_X m\|_2 \leq \|DP_X\|_F \cdot \|m\|_2 = \|DH_X Q\|_F \cdot \|m\|_2,
\]

where \( \|DP_X\|_F = \|DH_X\|_F \) follows from \( P_X^T = P_X \) and \( P_X^2 = P_X \). From (3), the part 2 of Theorem 2.8 follows.

Let \( H(k) \) be the result after \( k \) iterations of Algorithm 1.

**Lemma 3.1.** If \( h_{j,k}(0) = 0 \) for some \( 1 \leq j \leq n-t \) and no smaller \( k \), then \( j = n-t \) and an SIR for \( X \) must appear as a column of the matrix \( B \).

**Proof.** By the hypothesis on \( k \), all diagonal elements of \( H(k-1) \) are not zero. Now, suppose the \( r \) chosen in the Exchange step is not \( n-t \). Since generalized Hermite reduction does not introduce any new zeros on the diagonal, and from the Exchange step and the Corner step, we have that no diagonal element of \( H(k) \) is zero. This contradicts the hypothesis on \( k \), and hence that our assumption that \( r < n-t \) was false. Thus \( r = n-t \) after the \((k-1)\)-th iteration.

Next we show that there must be an SIR for \( X \) appeared as a column of the matrix \( B \). We have \( X^T H_X = 0 \) from Lemma 2.2 and hence that \( 0 = X^T BH_1^T H_X = X^T BH_1^T H_X Q = X^T BH(k-1), \) where \( Q \) is an appropriate orthogonal \((n-t) \times (n-t)\) matrix. Let \( (z_1, \cdots, z_t)^T = X^T B \), where \( z_i = (z_{i1}, \cdots, z_{it})^T \). Then

\[
\begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\end{bmatrix}
= X^T BH(k-1) = \begin{bmatrix}
\vdots & \vdots \\
0 & \cdots & 0 \\
\end{bmatrix} H(k-1)
\]

We know \( h_{n-t+1,j,0}(1) = 0 \) and \( h_{n-t,j,0}(k-1) \neq 0 \) from \( h_{n-t,j,0}(k) = 0 \). From Definition 2.4 and \( h_{n-t+1,j,0}(k-1) = 0 \) we have \( h_{n-t+1,j,0}(k-1) = \cdots = h_{n-t,j,0}(k-1) = 0 \) which implies the last equality in (4). Since \( h_{n-t,j,0}(k-1) \neq 0 \), it follows that \( z_{n-t+1} = \cdots = z_{n-t} = 0 \). Thus the \((n-t)\)-th column of \( B \) is an SIR for \( X \).

From the analysis above and Lemma 3.1, the correctness of Algorithm 1 is proved. From the iteration of Algorithm 1, \( \|H(k)\|_F \) is decreasing with respect to \( k \). Thus if there exist SIRs for \( X \), Algorithm 1 can always find one.

**Definition 3.2** (\( \Pi \) function). Let \( \lambda(X) \) be the least norm of SIRs for \( X \). For the \( k \)-th iteration in Algorithm 1, define

\[
\Pi(k) = \prod_{1 \leq j \leq n-t} \min \left\{ \gamma^{n-t} \lambda(X), \frac{1}{|h_{j,k}(k)|} \right\}^{n-j}.
\]

**Lemma 3.3.** For \( k > 1 \) we have

1. \( (\gamma^{n-t} \lambda(X)) \left( \begin{array}{c} n \\ 2 \\ \frac{t}{2} \end{array} \right) \geq \Pi(k) \geq 1 \).

2. \( \Pi(k) \geq \sqrt{\frac{4 \gamma^2}{\gamma^2 + 4}} \Pi(k-1) \).

The routine of analyzing the number of iterations in [8] can be carried over here with redefining the \( \Pi \) function as above, so we state Lemma 3.3 directly without proof. From this lemma, it follows that the \( \Pi \) function is increasing with respect to \( k \) and has an upper bound for a fixed \( \gamma \in (2/\sqrt{3}, +\infty) \). From Definition 3.2 we can infer \( \Pi(0) \geq 1 \). And from Lemma 3.3 we know that

\[
\left( \gamma^{n-t} \lambda(X) \right) \left( \begin{array}{c} n \\ 2 \\ \frac{t}{2} \end{array} \right) \geq \Pi(k) \geq \sqrt{\frac{4 \gamma^2}{\gamma^2 + 4}} k.
\]

Solving \( k \) from this inequality gives the part 1 of Theorem 2.8 as was to be shown.
4. Empirical Study and Further Discussion

4.1. Implementation

All discussions above are based on exact arithmetic operation, i.e. uses the Blum-Shub-Smale model [5, 4] of computation. The reason is that Algorithm 1 involves real numbers. Thus Algorithm 1 can only be implemented using floating point arithmetic on computer. Both Algorithm 1 and the HJLS simultaneous relations algorithm when \( t = 2 \), i.e. detecting an SIR for two real vectors, were implemented in Maple 13 under multiprecision floating point arithmetic (one level scheme). Like PSLQ (see [1, Section 5]), it is possible to perform most iterations using hardware floating point arithmetic, with only occasional depending on multiprecision arithmetic. So the two level implementation of Algorithm 1 is also developed by the first author. It partially uses software floating arithmetic and mainly uses hardware floating arithmetic (Cf. [1, Section 5] for details).

It is well known that the Gram-Schmidt orthogonalization algorithm is numerically unstable [10], so in our implementations we construct the hyperplane matrix by using QR decomposition, instead of Gram-Schmidt orthogonalization. In contrast to HJLS, the iteration step in Algorithm 1 is not based on the Gram-Schmidt orthogonalization, but on LQ decomposition (this is equivalent to QR decomposition). Householder transformations are used in our implementations to compute these decompositions. Thus our implementations of Algorithm 1 is numerically stable.

4.2. Experimental Result

In theory, the cost of Algorithm 1 (in Corollary 2.9) matches the best complexity upper bound known for this problem (Cf. [11, section 5]), whereas in practice Algorithm 1 usually needs fewer iterations. For \( x_1 = (11, 27, 31)^T \) and \( x_2 = (1, 2, 3)^T \), HJLS outputs \( (19, -2, -5)^T \) after 5 iterations while Algorithm 1 outputs the same SIR after only 3 iterations.

| No. | \( n \) | \( \text{itr}_{\text{HJLS}} \) | \( \text{itr}_{\text{SIRD}} \) | \( t_{\text{HJLS}} \) | \( t_{\text{SIRD}} \) |
|-----|-----|-----|-----|-----|-----|
| 1   | 4   | 15  | 8   | 0.063 | 0.   |
| 2   | 4   | 13  | 6   | 0.062 | 0.   |
| 3   | 4   | 21  | 11  | 0.094 | 0.015|
| 4   | 5   | 25  | 12  | 0.109 | 0.016|
| 5   | 5   | 27  | 7   | 0.141 | 0.   |
| 6   | 5   | 21  | 10  | 0.094 | 0.   |
| 7   | 54  | 34  | 9   | 2.203 | 0.453|
| 8   | 79  | 34  | 5   | 4.860 | 0.625|
| 9   | 97  | 37  | 5   | 7.438 | 1.047|
| 10  | 118 | 45  | 5   | 13.765| 1.687 |
| 11  | 149 | 29  | 2   | 19.016| 1.610 |
| 12  | 173 | 26  | 3   | 26.812| 2.421 |
| 13  | 192 | 29  | 5   | 34.218| 3.563 |
| 14  | 278 | 28  | 5   | 85.797| 8.860 |
| 15  | 290 | 35  | 4   | 95.636| 8.328 |
| 16  | 293 | 23  | 4   | 98.062| 8.750 |
| 17  | 305 | 22  | 3   | 109.187| 8.063 |
| 18  | 316 | 19  | 3   | 120.187| 8.766 |
| 19  | 325 | 18  | 2   | 129.031| 6.953 |

Table 1: Comparison of performance results for HJLS and Algorithm 1

The purpose of the trials in Table 1 is to compare the performances of HJLS and Algorithm 1 when \( t = 2 \). All of the tests were run on AMD Athlon™ 7750 processor (2.70 GHz) with 2GB main memory.

In Table 1, \( n \) gives the dimension of the relation vector, \( \text{itr}_{\text{HJLS}} \) and \( \text{itr}_{\text{SIRD}} \) are the numbers of iterations of HJLS and Algorithm 1 respectively, and the columns headed \( t_{\text{HJLS}} \) and \( t_{\text{SIRD}} \) give the CPU run time respectively of the two algorithms in seconds. The 20 trials in Table 1 were constructed by Maple’s pseudo random number generator. The first 6 trials are for low dimension, and others for higher dimension.

The results show that Algorithm 1 appears to be more effective than HJLS. In all 20 trials, the number of iterations of Algorithm 1 is less than that of HJLS. It is still true that Algorithm 1 usually needs fewer iterations than HJLS for more tests. This leads that the running time of Algorithm 1 is much less than HJLS. With the dimension \( n \) increasing, the difference between the efficiencies of Algorithm 1 and HJLS is increasingly notable. On average, the running time of Algorithm 1 is less than 1/10 (based on the data in Table 1) of the running time of HJLS.

The package is available from \( \text{http://cid-5dbb16a211c63a9b.skydrive.live.com/self.aspx/}\).
To some extent, the number of iterations is related to the parameter $\gamma$. In practice we have found that for many examples, larger values of $\gamma$ are more effective in finding SIRs for $X$. For $x_1 = (86, 6, 8, 673)^T$ and $x_2 = (83, 5, 87, 91)^T$, if we choose $\gamma = 2$ then Algorithm 1 outputs $(-32, -747, 63, 10)^T$ after 10 iterations, however, if we choose $\gamma = 93$, Algorithm 1 outputs $(-35, -2624, 157, 26)^T$ after only 7 iterations. It is worth mentioning that, in the example above, both of the two different output vectors are SIRs for $(x_1, x_2)$ and they are linearly independent and hence that all SIRs for $(x_1, x_2)$ can be obtained from them. Going on choosing a $\gamma$ larger than 93 in this example, after many tests, the authors find that the number of iterations is always 7, and it will not decrease any more. Based on this observation, all results in Table 1 are obtained under the condition that $\gamma = 1000$.

In general, a larger $\gamma$ requires a higher precision in authors’ tests. Usually, a high precision leads a large height of the output SIR and a large cost of memory. If one sets $\gamma = 1.15470053838 (> 2/\sqrt{3})$, for about 60% of our whole tests, the height of the vector returned by Algorithm 1 is less than that of HJLS simultaneous relations algorithm, however the number of iterations turns large.

This means that we should try to find the balance between the number of iterations and the precision because both of them are relevant to the running time of Algorithm 1. So in practice, what are the best choices for $\gamma$ needs both further exploration.

4.3. An Application

We end this paper with an application of Algorithm 1 to find the minimal polynomial of a complex algebraic number from its approximation.

Example 4.1. Let $\alpha = 2 + \sqrt{3}i$. We know that the minimal polynomial of $\alpha$ in $\mathbb{Z}[x]$ is $7 - 4x + x^2$. Let $\bar{\alpha} = 2.000 + 1.732i$ be an approximation to $\alpha$ with four significant digits. Let $v_1 = (1, 2, 1)^T$ and $v_2 = (0, 1.732, 6.928)^T$ be the real part and the imaginary part of $(1, \bar{\alpha}, \bar{\alpha}^2)^T$ respectively. Feeding Algorithm 1 with $v_1, v_2$ as its input vectors gives an SIR for $v_1, v_2$ after 2 iterations. The corresponding matrices $B$ are

$$
\begin{bmatrix}
2 & 1 & 0 \\
-1 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
7 & 0 & 2 \\
-4 & 0 & -1 \\
1 & 1 & 0
\end{bmatrix}.
$$

It is obvious that the first column of the latter one is an SIR for $v_1$ and $v_2$ and corresponds to the coefficients of the minimal polynomial of $\alpha$. However, if one takes only 3 significant digits for the same data, after 3 iterations Algorithm 1 outputs $(1213, -693, 173)^T$, which is an exact SIR for $(1, 2, 1)^T$ and $(0, 1.73, 6.93)^T$, but does not correspond to the coefficients of the minimal polynomial of $\alpha$. For this reason, how to appropriately control the error also is an interesting problem.

Generally, for computing the minimal polynomial of an algebraic number $\alpha$ with degree $n$, we detect an integer relation for $(1, \alpha, \cdots, \alpha^n)^T$. If $\alpha \in \mathbb{C}$, we detect an SIR for $(1, \Re(\alpha), \cdots, \Re(\alpha^n))^T$ and $(0, \Im(\alpha), \cdots, \Im(\alpha^n))^T$ by Algorithm 1 under a proper decimal precision. The output vector corresponds to a polynomial of degree $n$, whose primitive part must be the minimal polynomial of $\alpha$.

As mentioned early, Algorithm 1 has been implemented in two schemes (one-level, two-level). Using our two level implementation of Algorithm 1, the authors obtain the following polynomial of degree 84 from an approximation to $\alpha = 3^{1/6} - 2^{1/7}i$ with $\text{Digits}:=1300$ in Maple 13. It is easy to check that this polynomial is the exact minimal polynomial of $\alpha$.

$$
\begin{align*}
5067001 + & 783962907 x^{16} + 21027764272536 x^{30} - 7504504 x^{42} \\
+ & 83639618394696 x^{34} - 36683081862336 x^{38} + 770305668258672 x^{42} \\
+ & 142394998636968 x^{28} + 1254656434122 x^{30} + 1370000831472 x^{10} \\
+ & 34207465357611 x^{12} - 284692059376032 x^{14} + 2475814167842 x^{16} \\
+ & 190959510258972 x^{18} + 2306173886216928 x^{20} + 99120704967648 x^{22} \\
+ & 6411149001809 x^{24} - 302925467658448 x^{26} + 250312437648 x^{44} \\
+ & 2189187 x^{28} - 112615776 x^{30} - 486486 x^{32} + 81081 x^{60} - 378550368 x^{2} \\
+ & 11935794528 x^{3} + 19043111046 x^{6} + 3293025660288 x^{8} + x^{34} \\
+ & 88074554904 x^{52} + 240 x^{56} + 1041237288 x^{58} + 1952496 x^{64} - 9828 x^{66} \\
+ & 24 x^{70} + 819 x^{72} - 42 x^{78} + 2212809521832 x^{46}
\end{align*}
$$
Table 2: Running times for the two implementations of Algorithm 1

| r | s | Dim. n | Digits | $itr_{SIRD}$ | $t_{SIRD}$ | $itr_{TLSIRD}$ | $t_{TLSIRD}$ |
|---|---|--------|--------|-------------|------------|---------------|--------------|
| 4 | 3 | 25     | 100    | 5685        | 75.937     | 5611          | 8.125        |
| 3 | 5 | 31     | 150    | 11792       | 356.890    | 11792         | 28.157       |
| 6 | 3 | 37     | 300    | 18927       | 556.031    | 18993         | 73.109       |
| 4 | 5 | 41     | 350    | 2432.727    | 2342.516   | 26192         | 134.360      |
| 5 | 5 | 51     | 50084  | 2642.079    | 6422.079   | 81758         | 1267.985     |
| 6 | 5 | 61     | 84677  | 50084       | 84677      | 81758         | 1267.985     |
| 4 | 9 | 73     | 1000   | 159326      | 4889.922   |               |              |
| 6 | 7 | 85     | 1300   | 234422      | 10658.735  |               |              |

Some performance results are reported in Table 2. In this table, $r$ and $s$ ($s$ is an odd integer number) define the constant $\alpha = 3^{1/r} - 2^{1/s}$, which is an algebraic number of degree $2rs$, and $n = 2rs + 1$. The column headed “Digits” gives a sufficient precision in decimal digits, while $itr$ and $t$ are the number of iterations and running times required for the correct output respectively, where the suffix SIRD is for one-level and TLSIRD for two-level. Every output vector in Table 2 corresponds to the coefficients of the exact minimal polynomial of $\alpha$.

From Table 2, it can be seen that the two-level program is much faster than the one-level program since the two-level program only involves multiprecision operation partially. The reason is that not all operations of Algorithm 1 have to use high precision. As a matter of fact, the two-level program performs most of the iterations using IEEE hardware arithmetic. Thus the running times can be dramatically reduced.

Using the case of $t = 1$ (it is PSLQ in fact) and $t = 2$ of Algorithm 1 for real and complex algebraic numbers, respectively, we get a new complete method (The method in [13] is only for real algebraic numbers.) to recover the minimal polynomial of an arbitrary algebraic number from its approximation and degree. It should be noted that since this method depends on integer relation detection that is based on a generalization of Euclidean algorithm [9], it is different from LLL-based algorithms, such as [13, 12].

In practice, the presented method is efficient. For $\alpha := \sqrt{21} + \sqrt{43}$, our procedure MiniPoly takes 1.062 seconds for outputting the exact minimal polynomial of $\alpha$, whereas the Maple built-in function MinimalPolynomial in PolynomialTools package that is LLL-based takes 6.032 seconds under the same decimal precision $Digits:=500$.

5. Conclusion

Using a method to construct a hyperplane matrix and the generalized Hermite reduction, a new SIRs detecting algorithm, Algorithm 1, is presented in this paper. It runs faster than the HJLS simultaneous relation algorithm through the authors' Maple package. Applying the algorithm, we obtain a complete method to find the minimal polynomial of an approximately algebraic number, which is even faster than the corresponding Maple built-in function.

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