Abstract

In this review paper we discuss the different interpretations of the concept of connection in a fiber bundle and in a jet bundle, and its properties. We relate it with first and second-order systems of partial differential equations (PDE’s) and multivector fields. As particular cases we analyze the concepts of linear connections and connections in a manifold, and their properties and characteristics.

Key words: Jet bundles, Connections, Jet fields, Multivector fields, Partial differential equations.
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1 Introduction

This review paper recovers the contents of several talks given in an interdisciplinary seminar on themes of Theoretical Physics and Applied Mathematics. The main aim is to introduce the concept, characterizations and properties of connections in fiber bundles. In particular:

1. To show the different but equivalent interpretations of the idea of connection in a fiber bundle, including its relation with multivector fields.

2. To establish the relation between connections in fiber bundles and systems of partial differential equations and, in particular, between connections in jet bundles and second-order partial differential equations.

3. To analyze the characteristics of some particular kinds of connections: linear connections and, as a special case, connections on a manifold; as well as other concepts and properties related to them.

This is a brief review on all these subjects and all the material presented here is standard and can be found in many books and dissertations. For more information on these and other related topics we address, for instance, to the references [1, 2, 3, 4, 5].

All the manifolds are real, second countable and $C^\infty$. The maps and the structures are $C^\infty$. Sum over repeated indices is understood.

2 Connections and jet fields in fiber bundles

In this section, we present the basic elements concerning to first-order jet bundles and the theory of jet fields and connections in fiber bundles (see [5] for details).

2.1 First-order jet bundles

Let $M$ be a differentiable manifold and $\pi: E \rightarrow M$ a differentiable fiber bundle with typical fiber $F$. We denote by $\Gamma(M, E)$ or $\Gamma(\pi)$ the set of global sections of $\pi$. In the same way, if $U \subset M$ is an open set, let $\Gamma_U(\pi)$ be the set of local sections of $\pi$ defined on $U$. Let $\dim M = m$ and $\dim E = n$.

For every $y \in E$, the fibers of $J^1 \pi$ are denoted $J^1_y \pi$ and their elements by $\bar{y}$. If $\phi: U \rightarrow E$ is a representative of $\bar{y} \in J^1_y \pi$, we write $\phi \in \bar{y}$ or $\bar{y} = T_{\pi(y)}\phi$. In addition, the map $\bar{\pi}^1 = \pi \circ \pi^1: J^1 \pi \rightarrow M$ defines another structure of differentiable bundle. It can be proved [4] that $\pi^1: J^1 \pi \rightarrow E$ is an affine bundle modelled on the vector bundle $E = \pi^*T^*M \otimes_E V(\pi)$ (This notation denotes the tensor product of two vector bundles over $E$). Therefore, the rank of $\pi^1: J^1 \pi \rightarrow E$ is $mn$.

We denote by $V(\pi)$ and $V(\pi^1)$ the vertical bundles associated with $\pi$ and $\pi^1$ respectively; that is $V(\pi) = \text{Ker} T\pi$ and $V(\pi^1) = \text{Ker} T\pi^1$. We denote by $\mathcal{X}^V(\pi)(E)$ and $\mathcal{X}^{V(\pi^1)}(J^1 \pi)$ the corresponding set of sections; that is, the vertical vector fields. In the same way we denote by $\mathcal{X}(E)$ (resp. $\mathcal{X}(J^1 \pi)$) the set of vector fields in $E$ (resp. of $J^1 \pi$) and by $\Omega^k(E)$ (resp. $\Omega^k(J^1 \pi)$) the set of differential forms of degree $k$ in $E$ (resp. in $J^1 \pi$).

Sections of $\pi$ can be lifted to $J^1 \pi$ in the following way: let $\phi: U \subset M \rightarrow E$ be a local section of $\pi$, for every $x \in U$, the section $\phi$ defines an element of $J^1 \pi$; the equivalence class of $\phi$ in $x$, which is denoted $(j^1 \phi)(x)$. Therefore we can define a local section $j^1 \phi$ of $\bar{\pi}^1$ and a map $j^1: \Gamma_U(\pi) \rightarrow \Gamma_U(\bar{\pi}^1)$.
as follows
\[ j^1\phi : U \rightarrow J^1\pi_x \quad ; \quad j^1 : \Gamma_U(\pi) \rightarrow \Gamma_U(\pi^1) \quad \phi \mapsto j^1(\phi) \equiv j^1\phi. \]

The section \( j^1\phi \) is called the canonical lifting or the canonical prolongation of \( \phi \) to \( J^1\pi \). A section of \( \pi^1 \) which is the canonical extension of a section of \( \pi \) is called a holonomic section.

Let \( x^\mu, \mu = 1, \ldots, m, \) be a local system in \( M \) and \( y^i, i = 1, \ldots, n, \) a local system in the fibers; that is, \( \{ x^\mu, y^i \} \) is a coordinate system adapted to the bundle. In these coordinates, a local section \( \phi : U \rightarrow E \) is written as \( \phi(x) = (x^\mu, \phi^i(x)) \), that is, \( \phi(x) \) is given by functions \( y^i = \phi^i(x) \). These local systems \( x^\mu \), \( y^i \) allows us to construct a local system \( (x^\mu, y^i, y^i_\mu) \) in \( J^1\pi \), where \( y^i_\mu \) are defined as follows: if \( \bar{y} \in J^1\pi \), with \( \pi^1(\bar{y}) = y \) and \( \pi(y) = x \), let \( \phi : U \rightarrow E, \ y^i = \phi^i \), be a representative of \( \bar{y} \), then
\[ y^i_\mu(\bar{y}) = \left( \frac{\partial \phi^i}{\partial x^\mu} \right)_x. \]

These coordinate systems are called natural local systems in \( J^1\pi \). In them we have
\[ j^1\phi(x) = (x^\mu(x), \phi^i(x), \frac{\partial \phi^i}{\partial x^\mu}(x)). \]

### 2.2 Connections in fiber bundles and jet fields

In order to set the main definition, first we prove the following statement:

**Theorem 1.** Let \( \pi : E \rightarrow M \) be a fiber bundle and \( \pi^1 : J^1\pi \rightarrow E \) the corresponding first-order jet bundle. The following elements can be canonically constructed one from the other:

1. A \( \pi \)-semibasic 1-form \( \nabla \) on \( E \) with values in \( TE \); that is, an element of \( \Gamma(E, \pi^*T^*M) \otimes \Gamma(E, TE) \), such that \( \nabla^*\alpha = \alpha \), for every \( \pi \)-semibasic form \( \alpha \in \Omega^1(E) \).
2. A subbundle \( H(\nabla) \) of \( TE \) such that
\[ TE = V(\pi) \oplus H(\nabla). \quad (1) \]
3. A (global) section of \( \pi^1 : J^1\pi \rightarrow E \); that is, a mapping \( \Psi : E \rightarrow J^1\pi \) such that \( \pi^1 \circ \Psi = \text{Id}_E \).

(Proof) \( (1 \Rightarrow 2) \) First, observe that \( \nabla : \mathfrak{x}(E) \rightarrow \mathfrak{x}(E) \) is a \( C^\infty(E) \)-map which acts on the vertical vector fields. Its transposed map is \( \nabla^* : \Omega^1(E) \rightarrow \Omega^1(E) \), which is defined as usually by \( \nabla^*\beta := \beta \circ \nabla \), for every \( \beta \in \Omega^1(E) \). Moreover, as \( \nabla \) is \( \pi \)-semibasic, so is \( \nabla^* \beta \), then \( \nabla^*(\nabla^*\beta) = \nabla^*\beta \) and hence \( \nabla \circ \nabla = \nabla \). Therefore, \( \nabla \) and \( \nabla^* \) are projection operators in \( \mathfrak{x}(E) \) and \( \Omega^1(E) \) respectively. So we have the splittings
\[ \mathfrak{x}(E) = \text{Im}\nabla \oplus \text{Ker}\nabla ; \quad \Omega^1(E) = \text{Im}\nabla^* \oplus \text{Ker}\nabla^*. \]

Now, if \( S \) is a submodule of \( \mathfrak{x}(E) \), the incident or annihilator of \( S \) is defined as the set of 1-forms \( S^l := \{ \alpha \in \Omega^1(E) \mid \alpha(X) = 0, \forall X \in S \} \). Therefore we have the natural identifications
\[ (\text{Im}\nabla)' = \text{Ker}\nabla^* ; \quad (\text{Ker}\nabla)' = \text{Im}\nabla^*. \quad (2) \]

Taking this into account, for every \( y \in E \), the map \( \nabla_y : T_yE \rightarrow T_yE \) induces the splittings
\[ T_yE = \text{Im}\nabla_y \oplus \text{Ker}\nabla_y ; \quad T_yE = \text{Im}\nabla_y^* \oplus \text{Ker}\nabla_y^*. \quad (3) \]
Next we must prove that $V_y(\pi) = \text{Ker}\nabla_y$. But $V_y(\pi) \subseteq \text{Ker}\nabla_y$, and $\text{Im}\nabla^*_y$ is the set of $\pi$-semibasic forms at $y \in E$, then we have $V_y(\pi) = \text{Ker}\nabla_y$ and hence

$$\text{Ker}\nabla = \Gamma(E, V(\pi)) \equiv \mathfrak{X}^{V(\pi)}(E) \ ; \ \text{Im}\nabla^* = \Gamma(E, \pi^*T^*M) .$$

So we define

$$H(\nabla) := \bigcup_{y \in E} \{ \nabla_y(u) \mid u \in T_yE \} .$$

As a consequence of this, the first splitting in (3) leads to

$$TE = H(\nabla) \oplus V(\pi) ,$$

and it allows us to introduce the projections

$$\mathfrak{h} : TE \longrightarrow H(\nabla) \ ; \ \mathfrak{v} : TE \longrightarrow V(\pi) ,$$

whose transposed maps

$$\mathfrak{h}^* : H^*(\nabla) \longrightarrow T^*E \ ; \ \mathfrak{v}^* : V^*(\pi) \longrightarrow T^*E$$

are injections which lead to the splitting

$$T^*E = H^*(\nabla) \oplus V^*(\pi) ;$$

then, taking into account the second equality of (2) and (3), in a natural way we have the identifications $H(\nabla)^t$ with $V^*(\pi)$ and $V(\pi)^t$ with $H^*(\nabla)$.

(2 $\Rightarrow$ 1) Given the subbundle $H(\nabla)$ and the splitting $TE = V(\pi) \oplus H(\nabla)$, the projections $\mathfrak{h}$ and $\mathfrak{v}$ induce the corresponding projection operators $\mathfrak{h} \mathfrak{v} \mathfrak{h}$ and $\mathfrak{v} \mathfrak{v} \mathfrak{h}$ in $\mathfrak{X}(E)$ and the splitting $X = \mathfrak{h} \mathfrak{v} \mathfrak{h}(X) + \mathfrak{v} \mathfrak{v} \mathfrak{h}(X)$, for every $X \in \mathfrak{X}(E)$. Then we can define the map

$$\nabla : \mathfrak{X}(E) \longrightarrow \mathfrak{X}(E) \ ; \ X \mapsto \nabla \mathfrak{h}(X) ,$$

which is a $C^\infty(E)$-morphism and satisfies trivially the following properties:

1. $\nabla$ vanishes on the vertical vector fields and therefore $\nabla \in \Gamma(E, \pi^*T^*M) \otimes \mathfrak{X}(E)$.

2. $\nabla \circ \nabla = \nabla$, since $\nabla$ is a projection.

3. if $\alpha \in \Gamma(E, \pi^*T^*M)$ and $X \in \mathfrak{X}(E)$ we have

$$(\nabla^*\alpha)X = \alpha(\nabla(X)) = \alpha(h(X)) = \alpha(\mathfrak{h}\mathfrak{v}\mathfrak{h}(X)) + \mathfrak{v}\mathfrak{v}\mathfrak{h}(X)) = \alpha(X)$$

because $\alpha$ is $\pi^1$-semibasic. Therefore $\nabla^*\alpha = \alpha$.

(2 $\Rightarrow$ 3) Suppose that $TE$ splits as $TE = H(\nabla) \oplus V(\pi)$. Then, there is a natural way of constructing a section of $\pi^1 : J^1\pi \rightarrow E$. In fact, consider $y \in E$ with $\pi(y) = x$, we have $T_yE = H_y(E) \oplus V_y(\pi)$ and $T_y\pi|_{H_y(E)}$ is an isomorphism between $H_y(E)$ and $T_xM$. Let $\phi_y : U \rightarrow E$ be a local section defined in a neighbourhood of $x$, such that

$$\phi_y(x) = y , \ T_x\phi_y = (T_y\pi|_{H_y(E)})^{-1} ,$$

then we have a section

$$\Psi : E \longrightarrow J^1\pi$$

$$y \mapsto (j^1\phi_y)(\pi(y)) ,$$
which is differentiable because the splitting $T_yE = H_y(E) \oplus V_y(\pi)$ depends differentiably on $y$.

(\(\Rightarrow\) 2) Let $\Psi: E \to J^1\pi$ be a section and $\bar{y} \in J^1\pi$, with $\bar{y} \xrightarrow{\pi} y \xrightarrow{\phi} x$. Observe that $\Psi(y) \in J^1\pi$ is an equivalence class of sections $\phi: M \to E$, with $\phi(x) = y$, but the subspace $\text{Im} T_x\phi$ does not depend on the representative $\phi$, provided it is in this class. Then, for every $\bar{y} \in J^1\pi$ and being $\phi$ a representative of $\bar{y} = \Psi(y)$, we define

$$H_y(\nabla) := \text{Im} T_x\phi \quad \text{and} \quad H(\nabla) := \bigcup_{y \in E} H_y(\nabla).$$

\[\tag*{\square}\]

**Definition 1.** A connection in the bundle $\pi: E \to M$ is any of the equivalent elements of Theorem 6. Then, a (global) section $\Psi: E \to J^1\pi$ is said to be a jet field in the bundle $\pi^1: J^1\pi \to E$. The $\pi$-semibasic form $\nabla$ is called the connection form or Ehresmann connection. The subbundle $H(\nabla)$ is called the horizontal subbundle of $TE$ associated with the connection and the sections of $H(\nabla)$ are the horizontal vector fields. It is also denoted $D(\Psi)$ and is called the distribution associated with $\Psi$.

A jet field $\Psi: E \to J^1E$ (resp. an Ehresmann connection $\nabla$) is said to be orientable if $D(\Psi)$ is an orientable distribution on $E$. If $M$ is orientable, then every connection in $E$ is also orientable.

**Remarks:**

- If $\bar{y} \in J^1_yE$, with $x = \pi(y)$, and $\phi: U \to E$ is a representative of $\bar{y}$, we have the split

$$T_yE = \text{Im} T_x\phi \oplus V_y(\pi).$$

Hence the sections of $\pi^1$ are identified with connections in the bundle $\pi: E \to M$, since they induce a horizontal subbundle of $TE$. Observe that it is reasonable to write $\text{Im} \bar{y}$ for an element $\bar{y} \in J^1\pi$.

- Any global section of an affine bundle can be identified with its associated vector bundle. In particular:
  - Let $\pi: E \to M$ be a trivial bundle; that is $E = M \times F$. A section of $\pi^1$ can be chosen in the following way: denoting by $\pi_1: M \times F \to M$ and $\pi_2: M \times F \to F$ the canonical projections, for a given $y_0 \in M \times F$, $y_0 = (x_0, v_0) = (\pi_1(y_0), \pi_2(y_0))$, we define the section $\phi_{y_0}(x) = (x, \pi_2(y_0))$, for every $x \in M$. From a section of $\pi$ we construct another one of $\pi^1$ as follows:
    $$z(y) := (j^1\phi_y)(\pi_1(y)) \ ; \ y \in E,$$
    which is taken as the zero section of $\pi^1$. In this case, $J^1\pi$ is a vector bundle over $E$.
  - If $\pi: E \to M$ is a vector bundle with typical fiber $F$, let $\phi: M \to E$ be the zero section of $\pi$ and $j^1\phi: M \to J^1\pi$ its canonical lifting. We construct the zero section of $\pi^1$ in the following way:
    $$z(y) := (j^1\phi)(\pi(y)) \ ; \ y \in E;$$
    thereby, in this case $\pi: J^1\pi \to M$ is a vector bundle.

### 2.3 Local expressions and properties

Let $(x^\mu, y^i)$ be a local system of coordinates in an open set $U \subset E$. The most general local expression of a semibasic 1-form on $E$ with values in $TE$ is

$$\nabla = f_\mu dx^\mu \otimes \left( g^\nu \frac{\partial}{\partial x^\nu} + h^i \frac{\partial}{\partial y^i} \right).$$
As $\nabla^*$ is the identity on semibasic forms, it follows that $\nabla^* dx^\mu = dx^\mu$, so the local expression of the connection form $\nabla$ is

$$\nabla = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} \right),$$

where $\Gamma^i_\mu \in C^\infty(U)$. In this system the jet field $\Psi$ is expressed as

$$\Psi = (x^\mu, y^i, \Gamma^i_\mu(x^\mu, y^i)) .$$

Let $\phi$ be a representative of $\Psi(y)$ with $\phi = (x^\mu, f^i(x^\mu))$. Therefore $\phi(x) = y$, $T_x\phi = \Psi(y)$ and we have

$$y = \phi(x) = (x^\mu, f^i (x^\mu)) = (x^\mu, y^i) .$$

The matrix of $T_x\phi$ is

$$\begin{pmatrix} \frac{\partial f^i}{\partial x^\mu} \\
\frac{\partial}{\partial x^\mu} \end{pmatrix},$$

therefore

$$\frac{\partial f^i}{\partial x^\mu} = \Gamma^i_\mu(x^\mu, y^i) .$$

Now, taking $\left\{ \frac{\partial}{\partial x^\mu} \right\}$ as a basis of $T_xM$, we obtain

$$\text{Im} T_x\phi = \left\{ (T_x\phi) \left( \frac{\partial}{\partial x^\mu} \right) \right\} = \left\{ \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu(y) \frac{\partial}{\partial y^i} \right\} ,$$

hence, $H(\nabla)$ is locally spanned by

$$\left\{ \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu(y) \frac{\partial}{\partial y^i} \right\} .$$

As final remarks, notice that the splitting (4) induces a further one

$$X(E) = \text{Im} \nabla \oplus \Gamma(E, \nabla);$$

so every vector field $X \in X(E)$ splits into its horizontal and vertical components:

$$X = X^H + X^V = \nabla(X) + (X - \nabla(X)) ;$$

that is, $\mathfrak{H} \equiv \nabla$ and $\mathfrak{V} \equiv \text{Id} - \nabla$. Locally, this splitting is given by

$$X = f^\mu \frac{\partial}{\partial x^\mu} + g^i \frac{\partial}{\partial y^i} = f^\mu \left( \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} \right) + (g^i - f^\mu \Gamma^i_\mu) \frac{\partial}{\partial y^i},$$

since $\frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i}$ and $\frac{\partial}{\partial y^i}$ generate locally $\Gamma(E, H(\nabla))$ and $\Gamma(E, V(\pi))$, respectively. Observe that, if $X$ is an horizontal vector field, then $\nabla(X) = X$.

In an analogous way the splitting (6) induces the following one

$$\Omega^1(E) = \Gamma(E, \pi^* T^* M) \oplus \text{Ker} \nabla^* = \Gamma(E, \pi^* T^* M) \oplus (\text{Im} \nabla)',$$

then, for every $\alpha \in \Omega^1(E)$, we have

$$\alpha = \alpha^H + \alpha^B = \nabla^* \alpha + (\alpha - \nabla^* \alpha),$$

whose local expression is

$$\alpha = F^\mu dx^\mu + G_i dy^i = (F^\mu + G_i \Gamma^i_\mu)dx^\mu + G_i(\Delta y^i - \Gamma^i_\mu dx^\mu),$$

since $dx^\mu$ and $dy^i - \Gamma^i_\mu dx^\mu$ generate locally $\Gamma(E, H^*(\nabla))$ and $\Gamma(E, V^*(\pi))$, respectively.

As a final remark, we analyze the structure of the set of connections in $\pi: E \to M$. Then, let $\nabla_1, \nabla_2$ be two connection forms. The condition $\nabla_1^i \alpha = \nabla_2^i \alpha = 0$, for every semibasic 1-form $\alpha$, means that $(\nabla_1 - \nabla_2)^i \alpha = 0$; that is $\nabla_1 - \nabla_2 \in \Gamma(E, \pi^* T^* M) \otimes_E \Gamma(E, V(\pi))$. However, let $\nabla$ be a connection on $E \to M$ and $\gamma \in \Gamma(E, \pi^* T^* M) \otimes_E \Gamma(E, V(\pi))$, then $\nabla + \gamma$ is another connection form. So we have:
Proposition 1. The set of connection forms on $\pi: E \to M$ is an affine “space” over the module of semibasic differential 1-forms on $E$ with values in $\mathcal{V}(\pi)$.

In a local canonical system, if $\nabla = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} \right)$ and $\gamma = \gamma^i_\mu dx^\mu \otimes \frac{\partial}{\partial y^i}$, then

$$\nabla + \gamma = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + (\Gamma^i_\mu + \gamma^i_\mu) \frac{\partial}{\partial y^i} \right).$$

2.4 Integrability of jet fields and connections. First-order partial differential equations

Definition 2. The curvature of a connection $\nabla$ is a $(2,1)$-tensor field in $E$ which is defined as follows:

for every $Z_1, Z_2 \in \mathfrak{X}(E)$,

$$\mathcal{R}(Z_1, Z_2) := (\text{Id} - \nabla)([\nabla(Z_1), \nabla(Z_2)]) = i([\nabla(Z_1), \nabla(Z_2)])(\text{Id} - \nabla).$$

Using the coordinate expressions of the connection form $\nabla$ or the jet field $\Psi$, a simple calculation leads to

$$\mathcal{R} = \frac{1}{2} \left( \frac{\partial \Gamma^j_\mu}{\partial x^\mu} - \frac{\partial \Gamma^j_\nu}{\partial x^\nu} + \Gamma^i_\mu \frac{\partial \Gamma^j_\nu}{\partial y^i} - \Gamma^i_\nu \frac{\partial \Gamma^j_\mu}{\partial y^i} \right) (dx^\mu \wedge dx^\nu) \otimes \frac{\partial}{\partial y^j}.$$ 

Definition 3. Let $\Psi: E \to J^1\pi$ be a jet field associated with a connection $\nabla$.

1. A section $\phi: M \to E$ is an integral section of $\Psi$ (resp. of $\nabla$) if $\Psi \circ \phi = j^1\phi$.

2. $\Psi$ is an integrable jet field (resp. $\nabla$ is an integrable connection) if it admits integral sections.

One may readily check that, if $(x^\mu, y^i, y^i_\mu)$ is a natural local system in $J^1\pi$ and, in this system, $\Psi = (x^\mu, y^i, \Gamma^i_\rho(x^\mu, y^i))$ and $\phi = (x^\mu, f^i(x^\nu))$, then $\phi$ is an integral section of $\Psi$ if, and only if, $\phi$ is a solution of the following system of partial differential equations

$$\frac{\partial f^i}{\partial x^\mu} = \Gamma^i_\mu \circ \phi. \quad (9)$$

The integrable jet fields and connections can be characterized as follows:

Proposition 2. The following assertions on a jet field $\Psi$ are equivalent:

1. The jet field $\Psi$ is integrable.

2. The curvature of the connection form $\nabla$ associated with $\Psi$ is zero.

3. $\mathcal{D}(\Psi)$ is an involutive distribution.

(Proof) (1 $\iff$ 2) Notice that if $\phi$ is an integral section of $\Psi$, then the distribution $\mathcal{D}(\Psi)$ is tangent to the image of $\phi$, and conversely.

(2 $\iff$ 3) From the definition (2) we obtain that, if $\mathcal{R} = 0$, then

$$\nabla([\nabla(Z_1), \nabla(Z_2)]) = [\nabla(Z_1), \nabla(Z_2)],$$

hence, the horizontal distribution $\mathcal{D}(\Psi)$ is involutive. Conversely, if $\mathcal{D}(\Psi)$ is involutive, as $\nabla$ is the identity on $\mathcal{D}(\Psi)$, the last equation follows, and then $\mathcal{R} = 0$.

Remark: According to this proposition, from the local expression of $\mathcal{R}$ we obtain the local integrability conditions of the equations (9).
2.5 Connections and multivector fields

**Definition 4.** A $k$-multivector field in $E$ is a section of $\Lambda^m(TE) = TE \wedge \ldots \wedge TE$ or, what is equivalent, a skew-symmetric contravariant tensor of order $k$ in $E$. The set of $k$-multivector fields in $E$ is denoted $\mathfrak{X}^k(E)$.

A $k$-multivector field $X \in \mathfrak{X}^k(E)$ is said to be locally decomposable if, for every $y \in E$, there is an open neighbourhood $U_y \subset E$ and $X_1, \ldots, X_k \in \mathfrak{X}(U_y)$ such that $X|_{U_y} = X_1 \wedge \ldots \wedge X_k$.

**Definition 5.** If $\Omega \in \Omega^r(E)$ and $X \in \mathfrak{X}^k(E)$, the contraction between $X$ and $\Omega$ is defined as the natural contraction between tensor fields. In particular, for locally decomposable multivector fields,

$$i(X)\Omega|_U = i(X_1 \wedge \ldots \wedge X_k)\Omega = i(X_1) \ldots i(X_k)\Omega.$$

**Definition 6.** A $k$-multivector field $X \in \mathfrak{X}^k(E)$ is $\pi$-transverse if, for every $\beta \in \Omega^k(M)$ such that $\beta(\pi(y)) \neq 0$, at every point $y \in E$, we have that $(i(X)(\pi^*\beta))_y \neq 0$.

Let $D$ be a $k$-dimensional distribution in $E$; that is, a $k$-dimensional subbundle of $TE$. Obviously sections of $\Lambda^mD$ are $k$-multivector fields in $E$. The existence of a non-vanishing global section of $\Lambda^kD$ is equivalent to the orientability of $D$. Therefore:

**Definition 7.** A non-vanishing $k$-multivector field $X \in \mathfrak{X}^k(E)$ and a $k$-dimensional distribution $D \subset TE$ are locally associated if there exists a connected open set $U \subseteq E$ such that $X|_U$ is a section of $\Lambda^mD|_U$.

As a consequence of this definition, if $X, X' \in \mathfrak{X}^k(E)$ are non-vanishing multivector fields locally associated with the same distribution $D$, on the same connected open set $U$, then there exists a non-vanishing function $f \in C^\infty(U)$ such that $X' \simeq_U fX$. This fact defines an equivalence relation in the set of non-vanishing $k$-multivector fields in $E$, whose equivalence classes are denoted by $\{X\}_U$. Then:

**Theorem 2.** There is a bijective correspondence between the set of $k$-dimensional orientable distributions $D$ in $TE$ and the set of the equivalence classes $\{X\}_E$ of non-vanishing, locally decomposable $k$-multivector fields in $E$.

(Proof) Let $\omega \in \Omega^k(E)$ be an orientation form for $D$. If $y \in E$ there exists an open neighbourhood $U_y \subset E$ and $X_1, \ldots, X_k \in \mathfrak{X}(U_y)$, with $i(X_1 \wedge \ldots \wedge X_k)\omega > 0$, such that $D|_{U_y} = \langle X_1, \ldots, X_k \rangle$. Then $X_1 \wedge \ldots \wedge X_k$ is a representative of a class of $k$-multivector fields associated with $D$ in $U_y$. But the family $\{U_y ; y \in E\}$ is a covering of $E$; let $\{U_\alpha ; \alpha \in A\}$ be a locally finite refinement and $\{\rho_\alpha ; \alpha \in A\}$ a subordinate partition of unity. If $X_1^\alpha, \ldots, X_k^\alpha$ is a local basis of $D$ in $U_\alpha$, with $i(X_1^\alpha \wedge \ldots \wedge X_k^\alpha)\omega > 0$, then $X = \sum_{\alpha} \rho_\alpha X_1^\alpha \wedge \ldots \wedge X_k^\alpha$ is a global representative of the class of non-vanishing $k$-multivector fields associated with $D$ in $E$.

The converse is immediate since, if $X|_U = X_1^1 \wedge \ldots \wedge X_k^1 = X_1^2 \wedge \ldots \wedge X_k^2$, for different sets $\{X_1^1, \ldots, X_k^1\}$ and $\{X_1^2, \ldots, X_k^2\}$, then $\langle X_1^1, \ldots, X_k^1 \rangle = \langle X_1^2, \ldots, X_k^2 \rangle$.

**Definition 8.** A $k$-multivector field $X \in \mathfrak{X}^k(E)$ is integrable if its associated distribution $D(X)$ is integrable. Then the integral submanifolds of $X \in \mathfrak{X}^k(E)$ are the integral submanifolds of $D(X)$.

If $X \in \mathfrak{X}^k(E)$ is locally decomposable, then $X$ is $\pi$-transverse if, and only if, $T_y\pi(D(X)) = T_{\pi(y)}M$, for every $y \in E$. (Remember that $D(X)$ is the $k$-distribution associated to $X$).
Theorem 3.  

1. Let \( X \in \mathfrak{X}^n(E) \) be integrable. Then \( X \) is \( \pi \)-transverse if, and only if, its integral manifolds are local sections of \( \pi : E \to M \).

2. \( X \in \mathfrak{X}^n(E) \) is integrable and \( \pi \)-transverse if, and only if, for every point \( y \in E \), there exists a local section \( \phi : U \subset M \to E \) such that \( \phi(\pi(y)) = y \), and a non-vanishing function \( f \in C^\infty(E) \) such that \( \Lambda^mT\phi = fX \circ \phi \circ \Lambda^kT_U \).

(Proof)

1. Consider \( y \in E \), with \( \pi(y) = x \). In a neighbourhood of \( y \) there exist \( X_1, \ldots, X_k \in \mathfrak{X}(E) \) such that \( X_1, \ldots, X_m \) span \( \mathcal{D}(X) \) and \( X = X_1 \wedge \cdots \wedge X_m \). But, as \( X \) is \( \pi \)-transverse, \( (i(X)(\pi^*(\omega)))_y \neq 0 \), for every \( \omega \in \Omega^m(M) \) with \( \omega(x) \neq 0 \). Thus, taking into account the second comment above, \( \mathcal{D}(X) \) is a \( \pi \)-transverse distribution and \( X_\mu \not\in \mathfrak{X}^\pi(E) \) at any point, for every \( X_\mu \). Now, let \( S \hookrightarrow E \) be the integral manifold of \( \mathcal{D}(X) \) passing through \( y \), then \( T_yS = \langle (X_1)_y, \ldots, (X_m)_y \rangle \). As a consequence of all of this, and again taking into account the second comment above, for every point \( y \in S \), \( T_y\pi(\mathcal{D}(X)) = T_y\pi(y)M \), then \( \pi|S \) is a local diffeomorphism and \( S \) is a local section of \( \pi \). The converse is obvious.

2. If \( X \) is integrable and \( \pi \)-transverse, then by theorem 3 for every \( y \in E \), with \( \pi(y) = x \), there is an integral local section \( \phi : U \subset M \to E \) of \( X \) at \( y \) such that \( \phi(x) = y \). Then, as a consequence of the definition of integrability, \( X_\mu \) spans \( \Lambda^mT_y\phi \), and hence the relation in the statement holds.

Conversely, if the relation holds, then \( \text{Im } \phi \) is an integral manifold of \( X \) at \( y \), then \( X \) is integrable and, as \( \phi \) is a section of \( \pi \), \( X \) is necessarily \( \pi \)-transverse.

In this case, if \( \phi : U \subset M \to E \) is a local section with \( \phi(x) = y \) and \( \phi(U) \) is the integral manifold of \( X \) through \( y \), then \( T_y(\text{Im } \phi) = \mathcal{D}_y(X) \).

Theorem 4. There is a bijective correspondence between the set of orientable jet fields \( \Psi : E \to J^1\pi \) (that is, the set of orientable Ehresmann connection \( \nabla \) in \( \pi : E \to M \)) and the set of the equivalence classes of locally decomposable and \( \pi \)-transverse multivector fields \( \{X\} \subset \mathfrak{X}^n(E) \). They are characterized by the fact that \( \mathcal{D}(\Psi) = \mathcal{D}(X) \).

In addition, the orientable jet field \( \Psi \) is integrable if, and only if, so is \( X \), for every \( X \in \{X\} \).

(Proof) If \( \Psi \) is an orientable jet field in \( J^1\pi \), let \( \mathcal{D}(\Psi) \) its horizontal distribution. Then, taking \( \mathcal{D} \equiv \mathcal{D}(\Psi) \), we construct \( \{X\} \) by applying theorem 2 and, since the distribution \( \mathcal{D}(\Psi) \) is \( \pi \)-transverse, the result follows immediately. The proof of the converse statement is similar.

Moreover, \( \Psi \) is integrable if, and only if, \( \mathcal{D}(\Psi) = \mathcal{D}(X) \) is also. Therefore it follows that \( X \) is also integrable, for \( X \in \{X\} \), and conversely.

Reminding the local expressions of Section 2.3, we have that the local expression for a particular representative multivector field \( X \) of the class \( \{X\} \subset \mathfrak{X}^k(E) \) associated with a jet field \( \Psi \) (or a connection form \( \nabla \)) is

\[
X \equiv \bigwedge_{\mu=1}^k X_\mu = \bigwedge_{\mu=1}^k \left( \frac{\partial}{\partial x^\mu} + \Gamma^A_\mu \frac{\partial}{\partial y^\nu} \right)
\]

Then, \( \phi = (x^\mu, f^i(x^\nu)) \) is an integral section of \( X \) if, and only if, \( \phi \) is a solution of the system of partial differential equations (9).
3 Connections and jet fields in jet bundles

The geometrical framework for treating with systems of second order partial differential equations are the jet bundles $J^1\pi$ and $J^1J^1\pi$. Next we analyze this topic.

3.1 Connections in $J^1\pi$ and jet fields in $J^1J^1\pi$

Consider the bundle $\pi^1: J^1\pi \to M$. The jet bundle $J^1J^1\pi$ is obtained by defining an equivalence relation on the local sections of $\pi^1$. Hence, the elements of $J^1J^1\pi$ are equivalence classes of these local sections and $J^1J^1\pi$ is an affine bundle over $J^1\pi$, modelled on the vector bundle $\pi^1 T^* M \otimes J^1\pi V(\pi^1)$. So, we have the commutative diagram

\[
\begin{array}{ccc}
J^1J^1\pi & \xrightarrow{\pi^1} & J^1\pi \\
\downarrow & & \downarrow \pi^1 \\
M & \xrightarrow{\pi^1} & E
\end{array}
\]

(10)

Let $\mathcal{Y}: J^1\pi \to J^1J^1\pi$ be a jet field in $J^1J^1\pi$. As we know $\mathcal{Y}$ induces a connection form $\nabla$ and a horizontal $n+1$-subbundle $H(\nabla)$ such that

\[
T J^1\pi = V(\pi^1) \oplus \text{Im} \nabla = V(\pi^1) \oplus H(\nabla),
\]

where $H_y = \text{Im} T_{\pi^1(y)} \psi$, for $y \in J^1\pi$ and $\psi: M \to J^1\pi$ a representative of $\mathcal{Y}(y)$. We denote by $D(\mathcal{Y})$ the $C^\infty(J^1\pi)$-module of sections of $H(\nabla)$.

If $(x^\mu, y^i, \nu^\mu, z_i, \nu^\mu_\nu)$ is a natural system of coordinates in $J^1J^1\pi$, we have the following local expressions for these elements

\[
\mathcal{Y} = (x^\mu, y^i, \nu^\mu, F^i_\nu(x^\rho, y^j, \nu^\rho), G^i_\nu(x^\rho, y^j, \nu^\rho)),
\]

\[
H(J^1\pi) = \left\{ \frac{\partial}{\partial x^\mu} + F^i_\nu(y^j), \frac{\partial}{\partial y^j} + G^i_\nu(y^j) \right\},
\]

\[
\nabla = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + F^i_\nu \frac{\partial}{\partial y^j} + G^i_\nu \frac{\partial}{\partial y^j} \right).
\]

From these local expressions we obtain that a representative $m$-multivector field $X$ of the class $\{X\} \subset \mathcal{X}^m(J^1\pi)$ associated with the jet field $\mathcal{Y}$, has the local expression

\[
X \equiv \bigwedge_{\mu=1}^m X^\mu = \bigwedge_{\mu=1}^m \left( \frac{\partial}{\partial x^\mu} + F^i_\nu \frac{\partial}{\partial y^j} + G^i_\nu \frac{\partial}{\partial y^j} \right).
\]

If $\mathcal{Y}: J^1\pi \to J^1J^1\pi$ is a jet field then a section $\psi: M \to J^1\pi$ is an integral section of $\mathcal{Y}$ if $\mathcal{Y} \circ \psi = j^1\psi$. $\mathcal{Y}$ is an integrable jet field if it admits integral sections. One may readily check that, in a natural system of coordinates in $J^1J^1\pi$, if $\psi = (x^\mu, f^i(x^\nu), g^\mu_\nu(x^\nu))$, then it is an integral section of $\mathcal{Y}$ if, and only if, $\psi$ is a solution of the following system of differential equations

\[
\frac{\partial f^i}{\partial x^\mu} = F^i_\nu \circ \psi \quad \frac{\partial g^\mu_\nu}{\partial x^i} = G^i_\nu \circ \psi.
\]

Remember that if $\psi$ is an integral section of $\mathcal{Y}$, then the distribution $D(\mathcal{Y})$ is tangent to the image of $\psi$ and conversely. Hence, $\mathcal{Y}$ is integrable if, and only if, $D(\mathcal{Y})$ is an involutive distribution or, what is
consider the diagram (10), we see that there is another natural projection from Proposition 3.

sections are canonical prolongations of sections of the projection \( \phi \) then \( \pi_j \).

Remark A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy: The idea of this Section is to characterize the integrable jet fields in \( J^1 J^1 \pi \) such that their integral sections are canonical prolongations of sections of the projection \( \pi \).

It is well known that there are two natural projections from \( TTQ \) to \( TQ \). In the same way, if we consider the diagram (10), we see that there is another natural projection from \( J^1 J^1 \pi \) to \( J^1 \pi \). Let \( y \in J^1 J^1 \pi \) with \( y \mapsto y \mapsto y \mapsto x \) and \( \psi: M \to J^1 \pi \) a representative of \( y \), that is, \( y = T_x \psi \). Consider now the section \( \phi = \pi^1 \circ \psi: M \to E \), then \( j^1 \phi(x) \in J^1 \pi \) and we have:

**Proposition 3.** The following projection is a differentiable map:

\[
j^1 \pi^1: J^1 J^1 \pi \longrightarrow J^1 \pi \quad y \mapsto j^1(\pi^1 \circ \psi)(\pi^1_1(y))
\]

(Proof) Let \((x^\mu, y^i, y^i_{\mu}, z^i_{\mu}, z^i_{\nu \mu})\) be a natural coordinate system in a neighbourhood of \( y_0 \in J^1 J^1 \pi \) and \( y_0 = (x_0^\mu, y_0^i, y_0^i_{\mu}, z_0^i_{\mu}, z_0^i_{\nu \mu}) \). We have \( \pi^1(y_0) = (x_0^\mu, y_0^i, y_0^i_{\mu}) \). Let \( \psi: M \to J^1 \pi \) be a representative of \( y_0 \); locally \( \psi = (x^\nu, f^i(x^\nu), g^i_{\mu}(x^\nu)) \) with

\[
f^i(x_0) = y_0^i, \quad g^i_{\mu}(x_0) = y_0^i_{\mu} ; \quad \frac{\partial f^i}{\partial x^\nu}(x_0) = z_0^i_{\nu \mu} , \quad \frac{\partial g^i_{\mu}}{\partial x^\nu}(x_0) = z_0^i_{\nu \mu}
\]

then \( \phi = \pi^1 \circ \psi = (x^\mu, f^i(x^\nu)) \), and

\[
j^1 \phi = \left( x^\mu, f^i(x^\nu), \frac{\partial f^i}{\partial x^\nu}(x^\nu) \right)
\]

Hence \( j^1 \pi^1(y_0) = (x_0^\mu, y_0^i, z_0^i_{\nu \mu}) \) and the result follows.

**Remark:** Observe that \( j^1 \pi^1 \) and \( \pi^1 \) exchange the coordinates \( y^i_{\mu} \) and \( z^i_{\nu \mu} \).

**Corollary 1.** If \( \psi: M \to J^1 \pi \) is a section of \( \pi^1 \), then \( j^1 \pi^1 \circ j^1 \psi = j^1(\pi^1 \circ \psi) \).

(Proof) In a coordinate system \((x^\mu, y^i, y^i_{\mu}, z^i_{\nu \mu}, z^i_{\nu \mu})\), we have \( \psi = (x^\mu, f^i(x^\nu), g^i_{\mu}(x^\nu)) \) and

\[
j^1 \psi = \left( x^\mu, f^i(x^\nu), g^i_{\mu}(x^\nu), \frac{\partial f^i}{\partial x^\nu}(x^\nu), \frac{\partial g^i_{\mu}}{\partial x^\nu}(x^\nu) \right)
\]

but \( j^1 \pi^1 \circ j^1 \psi = \left( x^\mu, f^i(x^\nu), \frac{\partial f^i}{\partial x^\nu}(x^\nu) \right) = j^1(\pi^1 \circ \psi) \).

**Definition 9.** A jet field \( \mathcal{Y}: J^1 J^1 \pi \to J^1 J^1 \pi \) is a Second Order Partial Differential Equation (SOPDE) (or also that verifies the SOPDE condition) if it is a section of the projection \( j^1 \pi^1 \) or, what is equivalent,

\[
j^1 \pi^1 \circ \mathcal{Y} = \text{Id}_{j^1 \pi}.
\]
Now, we are going to characterize SOPDE integrable jet fields. First we define:

**Definition 10.** A section \( \psi: M \to J^1E \) of \( \bar{\pi}^1 \) is said to be holonomic if it is the canonical lifting of a section \( \phi: M \to E \) of \( \pi \); that is, \( \psi = j^1\phi \).

**Proposition 4.** Let \( \mathcal{Y}: J^1\pi \to J^1J^1\pi \) be an integrable jet field. The necessary and sufficient condition for \( \mathcal{Y} \) to be a SOPDE is that its integral sections are holonomic.

**Proof** \( \implies \) If \( \mathcal{Y} \) is a SOPDE then \( j^1\pi^1 \circ \mathcal{Y} = \text{Id}_{J^1\pi} \). Let \( \psi: M \to J^1\pi \) be an integral section of \( \mathcal{Y} \); that is, \( \mathcal{Y} \circ \psi = j^1\psi \), then

\[
\psi = j^1\pi^1 \circ \mathcal{Y} \circ \psi = j^1\pi^1 \circ j^1\psi = j^1(\pi^1 \circ \psi) \equiv j^1\phi,
\]

and \( \psi \) is a canonical prolongation.

**Proof** \( \impliedby \) Now, let \( \mathcal{Y} \) be an integrable jet field whose integral sections are canonical prolongations. Take \( \bar{y} \in J^1\pi \) and \( \phi: M \to E \) a section such that \( j^1\phi: M \to J^1\pi \) is an integral section of \( \mathcal{Y} \) with \( j^1\phi(\bar{\pi}^1(\bar{y})) = \bar{y} \). We have

\[
(j^1\pi^1 \circ \mathcal{Y})(\bar{y}) = (j^1\pi^1 \circ \mathcal{Y})(j^1\phi(\bar{\pi}^1(\bar{y}))) = (j^1\pi^1 \circ j^1\phi)(\bar{\pi}^1(\bar{y})) = (j^1\pi^1 \circ \mathcal{Y} \circ \psi)(\bar{\pi}^1(\bar{y}))
\]

\[
= (j^1\pi^1 \circ j^1\psi)(\bar{\pi}^1(\bar{y})) = j^1(\pi^1 \circ \psi)(\bar{\pi}^1(\bar{y})) = j^1\phi(\bar{\pi}^1(\bar{y})) = \bar{y},
\]

and \( \mathcal{Y} \) is a SOPDE. \( \blacksquare \)

**Remarks:**

- In coordinates, the condition \( j^1\pi^1 \circ \mathcal{Y} = \text{Id}_{J^1\pi} \) is expressed as follows: the jet field \( \mathcal{Y} = (x^\mu, y^i, y^i_\mu, F^i_\nu, G^i_{\nu\mu}) \) is a SOPDE if, and only if, \( F^i_\nu = y^i_\nu \).

- If \( \mathcal{Y} = (x^\mu, y^i, y^i_\mu, y^i_\nu, F^i_\nu, G^i_{\nu\mu}) \) is a SOPDE, then \( j^1\phi = \left( x^\mu, f^i, \frac{\partial f^i}{\partial x^\nu} \right) \) is an integral section of \( \mathcal{Y} \) if, and only if, \( \phi \) is the solution of the system of (second order) PDE’s

\[
G^i_{\nu\mu} \left( x^\rho, f^j, \frac{\partial f^i}{\partial x^\gamma} \right) = \frac{\partial^2 f^i}{\partial x^\nu \partial x^\mu}, \tag{11}
\]

which justifies the terminology. Sometimes, SOPDE jet fields which are not integrable are also called semi-holonomic jet fields.

If \( \mathcal{Y} \) is a SOPDE, then a representative \( m \)-multivector field \( \mathbf{X} \) of the class \( \{ \mathbf{X} \} \subset \mathfrak{X}^m(J^1\pi) \) associated with \( \mathcal{Y} \), has the local expression

\[
\mathbf{X} \equiv \bigwedge_{\mu=1}^m \left( \frac{\partial}{\partial x^\mu} + y^i_\mu \frac{\partial}{\partial y^i} + G^i_{\nu\mu} \frac{\partial}{\partial y^\nu} \right),
\]

and, if \( \mathbf{X} \) is integrable, its (holonomic) integral sections are the solutions of (11). It is important to remark that, since the integrability of a class of multivector fields is equivalent to demanding that the curvature \( \mathcal{R} \) of the connection associated with this class vanishes everywhere; the system (11) has solution if, and only if, the following additional system of equations holds

\[
0 = G^i_{\nu\mu} - G^i_{\mu\nu},
\]

\[
0 = \frac{\partial G^i_{\nu\rho}}{\partial x^\mu} + v^i_\mu \frac{\partial G^i_{\nu\rho}}{\partial y^j} + G^i_{\nu\mu} \frac{\partial G^i_{\nu\rho}}{\partial y^\gamma} - \frac{\partial G^i_{\nu\rho}}{\partial x^\mu} - v^i_\mu \frac{\partial G^i_{\nu\rho}}{\partial y^j} - G^i_{\nu\gamma} \frac{\partial G^i_{\nu\rho}}{\partial y^\gamma}.
\]
4 Connections in a vector bundle

As special cases of connections in fiber bundles, we study connections in a vector bundle, in particular linear connections and the related notion of connections on a manifold.

4.1 Structures in a vector bundle

Linear connections are a particular type of connections, which can be defined only on vector bundles. In order to give their different characterizations, first we need to introduce some previous concepts.

First, remember that, if $E$ is a vector space and $u \in E$, there is a natural identification between $E$ and $T_u E$, which is given by

$$E \rightarrow T_u E,$$

where $D_w(u)$ is the directional derivative with respect the vector $w$ at the point $u$; that is, for every differentiable function $f : E \rightarrow \mathbb{R}$,

$$D_w(u)f := \lim_{t \rightarrow 0} \frac{1}{t}(f(u + tw) - f(u)).$$

So, if $z^1, \ldots, z^n$ are coordinates in $E$ and $w \equiv (\lambda^1, \ldots, \lambda^n)$, then $D_w(u) = \lambda^i \frac{\partial}{\partial z^i}|_u$, and the identification is immediate.

In this way, if $\pi : E \rightarrow M$ is a vector bundle, for every section $\phi : M \rightarrow E$ and $p \in M$, we denote by $\sharp \phi(p) : V_{\phi(p)}(\pi) \rightarrow T_p E$, the natural identification between the vector space $E_p$ and $V_{\phi(p)}(\pi) = T_{\phi(p)}E_p$.

**Definition 11.** Let $\phi(p) \equiv (p, u) \in E$, with $u \in V_{\phi(p)}(\pi) = T_u E_p$. The map

$$E \rightarrow T E = V(\pi),
(p, u) \mapsto ((p, u), D_w(u))$$

defines a $\pi$-vertical vector field $\Delta \in \mathfrak{X}(E)$ which is called the Liouville vector field of the vector bundle $\pi : E \rightarrow M$.

In a natural set of coordinates $(x^\mu, y^i)$ of $E$, the local expression of $\Delta$ is $\Delta = y^i \frac{\partial}{\partial y^i}$. Then, its integral curves are the solution of the system of differential equations

$$\frac{dx^\mu}{dt} = 0, \quad \frac{dy^i}{dt} = y^i;$$

that is, $x^\mu(t) = A^\mu$ and $y^i(t) = B^i e^t (A^\mu, B^i \in \mathbb{R})$. Thus, $\Delta$ generates dilatations along the fibres of the vector bundle.

**Remark:** Observe that the flow of $\Delta$ is the map

$$\Phi : \mathbb{R} \times E \rightarrow E,
(t, (x, y)) \mapsto (x, e^t y);$$

which, by a suitable reparametrization, can be equivalently defined as

$$\Phi : \mathbb{R}^+ \times E \rightarrow E,
(t, (x, y)) \mapsto (x, ty).$$

This map allows to define the following one: for every $t \in \mathbb{R}^+$, the map $\Phi_t : E \rightarrow E$ is given by $\Phi_t(y) = ty$, for every $y \in E$.

If a connection is given in a vector bundle, then we can establish the following:
Definition 12. Let $\nabla$ be a connection on a vector bundle $\pi: E \to M$. The covariant derivative induced by $\nabla$ is the map $\tilde{\nabla}: \Gamma(\pi) \to \Gamma(\pi) \otimes \Omega^1(M)$ defined as follows: if $\phi: M \to E$ is a section, $Z \in \mathfrak{X}(M)$ and $p \in M$, 

$$(\tilde{\nabla} \phi)(p; Z) := \tilde{\phi}(p)(v(T_p\phi(Z_p))) = \tilde{\phi}(p)[T_p\phi(Z_p) - \nabla(\phi(p); T_p\phi(Z_p))] .$$

(It is usual to write $(\tilde{\nabla}_Z \phi)(p)$ instead of $(\tilde{\nabla} \phi)(p; Z)$.)

4.2 Linear connections

Now we can prove the following equivalence:

Theorem 5. Let $\pi: E \to M$ be a vector bundle. Consider a connection given by the equivalent elements $\nabla$, $\mathbb{H}(\nabla)$ or $\Psi$. Then the following conditions are equivalent:

1. The connection form is invariant under the Liouville vector field:

$$L(\Delta)\nabla = 0 ,$$

or, what is the same thing, the vertical projection operator $\mathfrak{V} \equiv \text{Id} - \nabla$ is invariant by $\Delta$:

$$L(\Delta)\mathfrak{V} = 0 .$$

2. The Liouville vector field preserves the horizontal subbundle; that is, for every $t \in \mathbb{R}^+$ and every $y \in E$, we have

$$T_y\Phi_t(H_y(\nabla)) = H_{\Phi_t(y)}(\nabla) .$$

3. If $(x^\mu, y^i)$ is a bundle system of coordinates in the vector bundle $\pi: E \to M$, then the functions $\Gamma^\mu_\nu$ which characterize the connection are linear on the fibers and their expressions are $\Gamma^\mu_\nu = \pi^*(\Gamma^1_{\mu \nu})y^j$, where $\Gamma^1_{\mu \nu} \in \mathcal{C}^\infty(M)$ are the Christoffel symbols of the linear connection.

4. The jet field $\Psi$ is a vector bundle morphism. (Notice that if $E \to M$ is a vector bundle, so is $J^1\pi \to M$).

5. For every $f \in \mathcal{C}^\infty(M)$ and every section $\phi: M \to E$,

$$\tilde{\nabla}(f\phi) = df \otimes \phi + f\tilde{\nabla}\phi .$$

(Proof) $(1 \Rightarrow 2)$ If $L(\Delta)\nabla = 0$ then $\Phi_t_\ast \nabla = \nabla$, for every $t \in \mathbb{R}^+$. If $u \in H_y(E)$, then there exists $v \in T_yE$ such that $\nabla(v) = u$ and we have

$$T_y\Phi_t(u) = T_y\Phi_t(\nabla(v)) = (\Phi_t_\ast \nabla)(\Phi_t_\ast v) = \nabla(\Phi_t_\ast v) ,$$

which implies that $T_y\Phi_t(u) \in H_{\Phi_t(y)}(\nabla)$ and, since $\Phi_t$ is a diffeomorphism the result follows.

$(2 \Rightarrow 1)$ If $T_y\Phi_t(H_y(\nabla)) = H_{\Phi_t(y)}(\nabla)$, for every $t \in \mathbb{R}^+$, then the splitting $T_yE = H_y(E) + V_y(\pi)$ implies that

$$T_{\Phi_t(y)}E = H_{\Phi_t(y)}(\nabla) + V_{\Phi_t(y)}(\pi) = T_y\Phi_t(H_y(\nabla)) + V_{\Phi_t(y)}(\pi) = T_y\Phi_tH_y(\nabla) + T_y\Phi_t(V_y(\pi)) ;$$

that is, if $u \in T_yE$, writing $u = h(u) + v(u) := \nabla(u) + v(u)$, we have

$$T_y\Phi_t(u) = \nabla(T_y\Phi_t(u)) + v(T_y\Phi_t(u)) = T_y\Phi_t(\nabla(u)) + T_y\Phi_t(v(u)) ,$$

that is, the splitting $T_yE = H_y(E) + V_y(\pi)$ implies that

$$T_y\Phi_t(u) = \nabla(T_y\Phi_t(u)) + v(T_y\Phi_t(u)) = T_y\Phi_t(\nabla(u)) + T_y\Phi_t(v(u)) ,$$
and hence $\nabla \circ T_y \Phi_t = T_y \Phi_t \circ \nabla$, so $\nabla$ is invariant by $\Phi_t$ and, therefore, $L(\Delta)\nabla = 0$.

\[ (1 \iff 3) \quad \text{Locally we have } \nabla = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + \Gamma^i_\mu \frac{\partial}{\partial y^i} \right), \text{then } L(\Delta)\nabla = 0 \text{ implies that the functions } \Gamma^i_\mu \text{ are homogeneous of degree 1 (Euler’s theorem) and, as they are differentiable at the origin, then they are also linear in the variables } y^i. \text{ So } \Gamma^i_\mu = \pi^*( -\Gamma^i_{j\mu} ) y^j, \text{ where } \Gamma^i_{j\mu} \in C^\infty( M ) \text{ (they are functions of the coordinates } x^\mu). \]

The converse is immediate.

\[ (3 \iff 4) \quad \text{Taking into account that the local expression of } \Psi \text{ is } \Psi(x^\mu, y^i) = (x^\mu, y^i, \Gamma^i_\mu(x, y)), \text{ the assertion is immediate since } \Psi \text{ is a vector bundle morphism if, and only if, the functions } \Gamma^i_\mu \text{ are linear of the coordinates } y^i. \]

\[ (3 \iff 5) \quad \text{On the one hand we have that} \]

\[
(df \otimes \phi + f \nabla \phi)(p; Z) = Z_p(f)\phi(p) + f(p)(\nabla \phi)(p; Z) = Z_p(f)\phi(p) + f(p)\nabla_i \phi (p)(T_p\phi)Z_p = Z_p(f)\phi(p) + f(p)\nabla_i \phi (p)(T_p\phi)Z_p - (\nabla(\phi(p)); T_p\phi Z_p),
\]

and, on the other hand,

\[
\nabla (f \phi)(p; Z) = Z_f (f \phi)(p; Z) - \nabla ((f \phi)(p); T_p(f \phi)Z_p).
\]

In a local natural system of coordinates in the vector bundle, if $Z = g^\mu \frac{\partial}{\partial x^\mu}$ and $\phi = (x^\mu, \phi^i)$, then

\[
T_p(f \phi)Z_p = g^\mu(p) \frac{\partial}{\partial x^\mu} \bigg|_{(f \phi)(p)} + f(p) \frac{\partial \phi^i}{\partial x^\mu} \bigg|_p g^\mu(p) \frac{\partial}{\partial y^i} \bigg|_p (f \phi)(p) + g^\mu(p) \frac{\partial f}{\partial x^\mu} \bigg|_p \phi^i(p),
\]

\[
\nabla (f \phi)(p; Z) = \left( p; g^\mu(p) \left( f(p) \frac{\partial \phi^i}{\partial x^\mu} \bigg|_p + \frac{\partial f}{\partial x^\mu} \bigg|_p \phi^i(p) - \Gamma^i_\mu(f \phi(p)) \right) \right),
\]

\[
(df \otimes \phi + f \nabla \phi)(p; Z) = \left( p; g^\mu(p) \left( f(p) \frac{\partial \phi^i}{\partial x^\mu} \bigg|_p + \frac{\partial f}{\partial x^\mu} \bigg|_p \phi^i(p) - f(p) \Gamma^i_\mu(\phi(p)) \right) \right);
\]

therefore, the local condition in order that item 5 holds is

\[
\Gamma^i_\mu(f \phi(p)) = f(p) \Gamma^i_\mu(\phi(p)),
\]

for every $f \in C^\infty(M)$. This proves the assertion.

Then we define

**Definition 13.** A connection in the vector bundle $\pi: E \to M$ is a linear connection if the equivalent conditions in Theorem 5 hold.

Finally, we can state the following concept:

**Definition 14.** Let $\nabla$ be a linear connection on a vector bundle $\pi: E \to M$, and $\phi: M \to E$ a section. Then $\nabla \phi$ is called the covariant differential of $\phi$. It is a map $\nabla \phi: \mathfrak{X}(M) \to \Gamma(\pi)$, which is an element of $\Omega^1( M ) \otimes_M \Gamma(\pi)$.

In natural coordinates the local expression of $\nabla \phi$ is

\[
\nabla \phi = dx^\mu \otimes \left( x^\mu \frac{\partial \phi^i}{\partial x^\mu} + \Gamma^i_{j\mu} \phi^i \right).
\]
In relation to the structure of the set of linear connections, if $\nabla_1, \nabla_2$ are linear connections, then $\nabla_1 - \nabla_2 : \Gamma(\pi) \to \mathfrak{X}(M) \otimes \Gamma(\pi)$ is a $C^\infty(M)$-linear map, so that

$$\nabla_1 - \nabla_2 \in \Gamma(\pi)^* \otimes \Omega^1(M) \otimes \Gamma(\pi) = \Omega^1(M) \otimes \text{End}(\pi) = \Omega^1(M) \otimes_M \Gamma(M, L_E),$$

where $L_E$ is the bundle of endomorphisms of $E$. On the other hand, if $\Upsilon \in \Omega^1(M, L_E) = \Omega^1(M) \otimes \Gamma(M, L_E)$ and $\nabla$ is a linear connection, then $\nabla + \Upsilon$ is another linear connection because

$$(\nabla + \Upsilon)(f\phi) = df \otimes (\nabla + \Upsilon) + f(\nabla + \Upsilon)\phi,$$

since the action of $\Upsilon$ on sections is the following: writing $\Upsilon = \alpha_i \otimes v^i$, with $\alpha_i \in \Omega^1(M)$ and $v^i \in \Gamma(M, L_E)$, and taking $Z \in \mathfrak{X}(M), \phi : M \to E$ and $p \in M$, then

$$\Upsilon(p; Z, \phi) = \alpha_i(p; Z) v^i(\phi(p))$$

verifying that

$$\Upsilon(p; Z, f\phi) = f(p) \Upsilon(p; Z, \phi),$$

because $\Upsilon$ is $C^\infty(M)$-linear. Therefore we can state:

**Proposition 5.** The set of linear connections on $\pi : E \to M$ is an affine “space” modelled on the $C^\infty(M)$-module of 1-forms on $M$ with values on the bundle of endomorphisms of $E, L_E$.

As you can observe, there exists a canonical injection of the module $\Omega^1(M) \otimes_M \Gamma(M, L_E)$ into $\Gamma(E, \pi^*\Gamma^*M) \otimes_E \Gamma(E, V(\pi))$ defined as follows: $\alpha \otimes \Upsilon \mapsto \pi^*\alpha \otimes (\sharp^{-1} \circ \Upsilon)$, and that this injection is a morphism of $C^\infty(M)$-modules.

## 5 Connections in a manifold

### 5.1 Basic definitions and properties and covariant derivative

The concept of connection on a manifold is closely related to that of linear connection. Let $M$ be an $m$-dimensional differentiable manifold.

**Definition 15.** A connection on $M$ is a linear connection in the tangent bundle $T^1M$.

In this case, the sections of the bundle are vector fields, hence we can define:

**Definition 16.** Let $\nabla$ be a connection on the manifold $M$.

1. If $X, Y \in \mathfrak{X}(M)$, the map

$$\mathcal{T}(X, Y) := \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

is called the torsion associated to this connection. $\mathcal{T}$ is an antisymmetric tensor field on $M$ with values on $TM$; so that $\mathcal{T} \in \Omega^2(M) \otimes \mathfrak{X}(M)$.

2. $\nabla$ is a torsion-free or symmetric connection if $\mathcal{T}$ is zero.

Let $(x^\mu, v^\mu)$ be a natural system of coordinates of $TM$. If the local expression of $\nabla$ is

$$\nabla = dx^\mu \otimes \left( \frac{\partial}{\partial x^\mu} + \Gamma^\rho_{\mu\nu} v^\nu \frac{\partial}{\partial x^\rho} \right),$$
then, for every vector field in $M$ (a section of the vector bundle $TM \to M$) we have
\[
\hat{\nabla}_{\frac{\partial}{\partial x^\mu}} \left( g^{\eta\nu} \frac{\partial}{\partial x^\eta} \right) = \left( \frac{\partial g^{\eta\nu}}{\partial x^\rho} + \Gamma^\eta_{\rho\nu} g^{\nu\rho} \right) \frac{\partial}{\partial x^\mu},
\]
and the local expression of $\mathcal{T}$ is
\[
\mathcal{T} = (-\Gamma^\mu_{\rho\eta} + \Gamma^\mu_{\eta\rho}) \, dx^\rho \otimes dx^\eta \otimes \frac{\partial}{\partial x^\mu}.
\]
According to this we can state the following Proposition (which justifies the name given to these kinds of connections):

**Proposition 6.** The necessary and sufficient condition for the connection $\nabla$ on $M$ to be torsion-free is that the following relation holds for every system of coordinates,
\[
\Gamma^\mu_{\rho\eta} = \Gamma^\mu_{\eta\rho}.
\]

If $\nabla_1, \nabla_2$ are symmetric connections on $M$, this implies that, for every $X, Y \in \mathfrak{X}(M)$,
\[
\nabla_1 X Y - \nabla_1 Y X = \nabla_2 X Y - \nabla_2 Y X,
\]

therefore $(\nabla_1 - \nabla_2)_X Y = (\nabla_1 - \nabla_2)_Y X$ and hence $\nabla_1 - \nabla_2$ is symmetric too; that is, it is an element of $S_2(M) \otimes \mathfrak{X}(M)$, where $S_2(M)$ denotes the module of symmetric 2-degree tensor fields on $M$. (Remember that an element of $S_2(M) \otimes \mathfrak{X}(M)$ is a section of the bundle $\sqrt{2}T^* M \otimes TM \to M$, where $\sqrt{2}T^* M$ is the symmetric product of $T^* M$ with itself. If $(x^\mu)$ is a local system of coordinates in $M$, then a local section of this bundle is locally expressed as $\Gamma^\gamma_{\mu\nu} dx^\mu \otimes dx^\nu \otimes \frac{\partial}{\partial x^\gamma}$, with $\Gamma^\gamma_{\mu\nu} = \Gamma^\gamma_{\nu\mu}$.

Conversely, if $\nabla$ is a symmetric connection on $M$ and $\Sigma \in S_2(M) \otimes \mathfrak{X}(M)$, then $\nabla + \Sigma$ is another symmetric connection on $M$. Therefore:

**Proposition 7.** The set of symmetric connections on a manifold $M$ is an affine space modelled on the $C^\infty(M)$-module $S_2(M) \otimes_M \mathfrak{X}(M)$.

Observe that this module is a submodule of $\Omega^1(M) \otimes_M \Gamma(M, L_E) = \Omega^1(M) \otimes_M \Omega^1(M) \otimes_M \mathfrak{X}(M)$.

### 5.2 Covariant derivative along a path and parallel transport

Here we recall some elementary constructions on a manifold $M$ with a connection $\nabla$. Let $\tau: TM \to M$ be the natural projection.

Let $\sigma: I = (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M$ be a smooth curve. A **vector field along** $\sigma$ is a smooth mapping $V : I \to \mathfrak{X}(M)$ such that $\tau \circ V = \sigma$. The set of all vector fields along $\sigma$ is denoted by $\mathfrak{X}(M; \sigma)$.

With the natural operations, $\mathfrak{X}(M; \sigma)$ is module over the algebra of smooth real functions defined on the interval $I \subset \mathbb{R}$. The elements in $\mathfrak{X}(M; \sigma)$ can be understood as curves in $TM$. If $V \in \mathfrak{X}(M; \sigma)$ we can say that $V$ is a lifting of $\sigma$ from $M$ to $TM$.

As it is known, the covariant derivative $\nabla_X Y \in \mathfrak{X}(M)$, for $X, Y \in \mathfrak{X}(M)$, can be extended to a covariant derivative of vector fields along a curve with respect to the tangent vector to the curve, that is $\nabla_{\dot{\sigma}} V$, for $V \in \mathfrak{X}(M; \sigma)$, obtaining another element of $\mathfrak{X}(M; \sigma)$ with the natural properties with respect to the real functions defined on the interval $I \subset \mathbb{R}$.

An element $V \in \mathfrak{X}(M; \sigma)$ is said to be **parallel along the curve** if $\nabla_{\dot{\sigma}} V = 0$. Given $u_p \in T_p$, with $p = \sigma(0)$, there exists only one element $V \in \mathfrak{X}(M; \sigma)$ satisfying the conditions
1. $\nabla_\sigma V = 0$.
2. $V(0) = u_p$.

With this in mind it is easy to prove that the set of vector fields along a curve $\sigma$, that is, $\mathcal{X}(M; \sigma)$ is a module over the real functions defined on the same interval than the curve with finite rang equal to the dimension of the manifold $M$.

### 5.3 Horizontal liftings and covariant derivatives

Let $\nabla$ be a connection in $M$.

**Definition 17.** Let $p \in M$, $u \in T_pM$, and $\sigma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M$ a smooth curve with $\sigma(0) = p$. The horizontal lifting of $\sigma$ to the point $u_p = (p, u) \in TM$ is the curve $\mathcal{X} : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to TM$, which is solution to the initial value problem given by

\[
(i) \quad \nabla_{\sigma} \mathcal{X} = 0 \quad , \quad (ii) \quad \mathcal{X}(0) = u_p
\]

that is, $\mathcal{X}$ is parallel along $\sigma$ and coincides with $u_p$ at $p$. (Observe that $\mathcal{X}$ is actually a vector field along $\sigma$; i.e., $\mathcal{X} \in \mathcal{X}(M; \sigma)$).

In a local chart of coordinates $(x^\mu)$ in $M$, if $\sigma = (\sigma_1, \ldots, \sigma^m)$ is the local expression of $\sigma$ in this chart, and

\[
\mathcal{X}(t) = \mathcal{X}^\mu(t) \frac{\partial}{\partial x^\mu} |_{\sigma(t)} \quad , \quad \dot{\sigma}(t) = \dot{\mathcal{X}}^\mu(t) \frac{\partial}{\partial x^\mu} |_{\sigma(t)}
\]

taking into account that $\nabla_{\dot{\sigma}} \left( \frac{\partial}{\partial x^\nu} \right) = \Gamma^\nu_{\mu\rho} \frac{\partial}{\partial x^\rho}$, then the local expression of $\nabla_{\dot{\sigma}} \mathcal{X} = 0$ is

\[
\nabla_{\dot{\sigma}} \mathcal{X} = \dot{\mathcal{X}}^\rho(t) + \Gamma^\rho_{\mu\nu}(\sigma(t)) \mathcal{X}^\mu(t) \dot{\mathcal{X}}^\nu(t) = 0
\]

and the unique solution $\mathcal{X}^\rho(t)$ gives the horizontal lifting of $\sigma$. Thus, in a natural chart of coordinates $(x^\mu, v^\rho)$ in $TM$, the curve $\mathcal{X}$ has the expression $\mathcal{X}(t) = (\mathcal{X}^\mu(t), \mathcal{X}^\rho(t))$, and its tangent vector at every point is

\[
\dot{\mathcal{X}}(t) = (\dot{\mathcal{X}}^\mu(t), \dot{\mathcal{X}}^\rho(t)) = (\dot{\mathcal{X}}^\mu(t), -\Gamma^\rho_{\mu\nu}(\sigma(t)) \mathcal{X}^\mu(t) \dot{\mathcal{X}}^\nu(t))
\]

In particular,

\[
\dot{\mathcal{X}}(0) = (\dot{\mathcal{X}}^\mu(0), -\Gamma^\rho_{\mu\nu}(p) u^\mu(0) \dot{\mathcal{X}}^\nu(0)) = \dot{\mathcal{X}}^\mu(0) \frac{\partial}{\partial x^\rho} |_{u_p} - \Gamma^\rho_{\mu\nu}(p) u^\mu(0) \frac{\partial}{\partial v^\rho} |_{u_p}
\]

which depends only on the tangent vector to the curve at $p$.

As a consequence, given $v \in T_pM$, if the curve $\sigma$ is a representative of $v$ (that is, $\sigma(0) = p, \dot{\sigma}(0) = v$), then $\dot{\mathcal{X}}(0) \in T_{u_p}(TM)$ depends only on $v$ and not on the selected representative curve.

**Definition 18.** The horizontal lifting of the vector $v \in T_pM$ to $u \in T_pM$ is the vector tangent at $u_p \in T_pM$ to the horizontal lifting of any curve representative of $v$.

The map that implements this operation is denoted $h^{u_p}_v : T_pM \to T_{u_p}(TM)$.

Locally, if $v = v^\rho \frac{\partial}{\partial x^\rho} |_{u_p}$ and $u = u^\rho \frac{\partial}{\partial x^\rho} |_{u_p}$, we have that

\[
h^{u_p}_v(v) = v^\rho \frac{\partial}{\partial x^\rho} |_{u_p} - \Gamma^\rho_{\nu\mu}(p) u^\mu(0) v^\nu \frac{\partial}{\partial v^\rho} |_{u_p}
\]
**Proposition 8.** The map $h^u_p : T_p M \to T_u(TM)$ has the following properties:

1. It is a linear map.
2. It is an injective map.
3. $\text{Im } h^u_p$ is an $m$-dimensional vector subspace of $T_u(TM)$.
4. $T_u(TM) = V_u(\tau) \oplus \text{Im } h^u_p$.

(Proof) These properties are an immediate consequence of the local expression (12). In particular:

1. It is a consequence of the linearity of all the operations.
2. It holds because $T_u\tau \circ h^u_p = \text{Id}_{T_pM}$.
3. It is a consequence of the above items (1) and (2).
4. It is a consequence of the above items (1), (2) and (3). \hfill \Box

**Definition 19.** $\text{Im } h^u_p$ is the horizontal subspace in $u \in T_pM$ associated with the connection $\nabla$, and it is denoted $H^{u_p}\tau$.

A basis for $H^{u_p}\tau$ is given by taking a basis $\left\{ \frac{\partial}{\partial x^\nu} \bigg|_p \right\}$ in $T_pM$ and obtaining the corresponding horizontal liftings (from (12)):

$$h^u_p \left( \frac{\partial}{\partial x^\nu} \bigg|_p \right) = \frac{\partial}{\partial x^\nu} \bigg|_u - \Gamma^\rho_{\nu\mu}(p) u^\mu \frac{\partial}{\partial v^\rho} \bigg|_{u_p} = \frac{\partial}{\partial x^\nu} \bigg|_{u_p} - \Gamma^\rho_{\nu\mu}(p) v^\mu (u_p) \frac{\partial}{\partial v^\rho} \bigg|_{u_p}.$$

Observe also that $T_u\tau : H^{u_p} \to T_pM$ is an isomorphism.

The expression of $h^u_p$ depends differentially on $p, u_p$ and $v_p$; then the subspace $H^{u_p}\tau$ depends differentially on $u_p$ and hence it defines a subbundle of $T(TM)$ of rank $m$, which is denoted $H(\nabla) := \bigcup_{u_p \in T_pM} H^{u_p}\tau$, and is called the horizontal subbundle associated with the connection $\nabla$. Obviously we have that $T(TM) = V(\tau) \oplus H(\nabla)$.

A local basis of $H(\nabla)$ is given by the vector fields

$$\left\{ \frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\nu\mu} u^\mu \frac{\partial}{\partial v^\rho} \right\};$$

hence, comparing this local expression with (7), and taking $\Gamma^\rho_{\nu\lambda} = -\Gamma^\rho_{\lambda\nu} v^\mu$ (as stated in the item 3 of Theorem 5), we see that this horizontal subspace is just the one introduced in (4).

In this way, we have the two projections (5):

$$\mathfrak{h} : T(TM) \longrightarrow H(\nabla) ; \quad \mathfrak{v} : T(TM) \longrightarrow V(\tau),$$

whose extension to vector fields $\mathfrak{h}$ and $\mathfrak{v}$ have the local expressions are obtained from (8) and, in this case, if $\left\{ \frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial v^\nu} \right\}$ is a local basis for $\mathfrak{X}(TM)$, are explicitly given by

$$\mathfrak{h} \left( \frac{\partial}{\partial x^\nu} \right) = \frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\nu\mu} v^\mu \frac{\partial}{\partial v^\rho} \quad , \quad \mathfrak{v} \left( \frac{\partial}{\partial x^\nu} \right) = \Gamma^\rho_{\nu\mu} v^\mu \frac{\partial}{\partial v^\rho};$$

$$\mathfrak{h} \left( \frac{\partial}{\partial v^\nu} \right) = 0 \quad , \quad \mathfrak{v} \left( \frac{\partial}{\partial x^\nu} \right) = \frac{\partial}{\partial x^\nu}.$$
Taking into account the linearity of these operators, these expressions allows us to compute the splitting of any vector field in \( T^*M \).

Furthermore, for every \( X \in \mathfrak{X}(M) \) we can obtain its horizontal lifting by doing \((\mathfrak{H}(X))(u_p) = h(X(p))\), for every \( u_p \in TM \). Locally, if \( X = X^\nu \frac{\partial}{\partial x^\nu} \), we have that
\[
\mathfrak{H}(X) = X^\nu \frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\nu\mu} X^\nu v^\mu \frac{\partial}{\partial v^\rho} = X^\nu \left( \frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\nu\mu} v^\mu \frac{\partial}{\partial v^\rho} \right) .
\]

Finally, for every \( X,Y \in \mathfrak{X}(M) \), the local expression of the covariant derivative \( \nabla X Y \) is
\[
\nabla X Y = X^\nu \frac{\partial Y^\rho}{\partial x^\nu} \frac{\partial}{\partial x^\rho} + \Gamma^\rho_{\nu\mu} X^\nu Y^\mu \frac{\partial}{\partial v^\rho},
\]
and it can be expressed by means of the horizontal-vertical splitting as follows:

**Proposition 9.** For every \( p \in M \), \((\nabla X Y)(p) = [T_p \tau (\mathfrak{h} \circ Y^C \circ X)](p)\); where \( Y^C \) denotes the complete lift of \( Y \) to \( TM \).

**Proof** Taking into account that
\[
Y^C = Y^\nu \frac{\partial}{\partial x^\nu} + \frac{\partial Y^\rho}{\partial x^\mu} v^\mu \frac{\partial}{\partial v^\rho} = Y^\nu \left( \frac{\partial}{\partial x^\nu} - \Gamma^\rho_{\nu\mu} v^\mu \frac{\partial}{\partial v^\rho} \right) + \left( \frac{\partial Y^\rho}{\partial x^\mu} v^\mu + \Gamma^\rho_{\nu\mu} Y^\nu v^\mu \right) \frac{\partial}{\partial v^\rho},
\]
as \( h^X_p : T_p M \to T_{X_p}(TM) \) is an isomorphism, we have that
\[
(h^X_p)^{-1}(\mathfrak{H}(Y^C)|_{X_p}) = \left( \frac{\partial Y^\rho}{\partial x^\mu} X^\mu + \Gamma^\rho_{\nu\mu} Y^\nu X^\mu \right) \frac{\partial}{\partial v^\rho} |_{X_p}.
\]
and the results follows.

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**References**

[1] A. Echeverría-Enríquez, M.C. Muñoz-Lecanda, N. Román-Roy, “Multivector fields and connections: Setting Lagrangian equations in field theories”, *J. Math. Phys.* 39(9) (1998) 4578-4603. (doi: 10.1063/1.532525).

[2] W. Greub, S. Halpering, S. Vanstone, *Connections, curvature and cohomology*, Pure Appl. Math. 47, Acad. Press, New York, 1972.

[3] D. Husemoller, *Fibre Bundles*, McGraw-Hill, New York, 1966. (ISBN 0-387-94087-1).

[4] R. Ouzilou, “Expression symplectique des problèmes variationnels”, *Symp. Math.* 14 (1973) 85-98.

[5] D.J. Saunders, *The Geometry of Jet Bundles*, London Math. Soc. Lect. Notes Ser. 142, Cambridge, Univ. Press, 1989. (ISBN 13: 978-0521369480).