Various interplays between relation and cylindric algebras

Tarek Sayed Ahmed

May 24, 2013

Abstract

Using model theoretic techniques that proved that the class of $n$ neat reducts of $m$ dimensional cylindric algebras, $\mathfrak{N}_n\text{CA}_m$, is not elementary, we prove the same result for $\text{RaCA}_k$, $k \geq 5$, and we show that $\text{RaCA}_k \subset S_r\text{RaCA}_k$ for all $k \geq 5$. Conversely, using the rainbow construction for cylindric algebra, we show that several classes of algebras, related to the class $\mathfrak{N}_n\text{CA}_m$, $n$ finite and $m$ arbitrary, are not elementary. Our results apply to many cylindric-like algebras, including Pinter’s substitution algebras and Halmos’ polyadic algebras with and without equality. The techniques used are essentially those used by Hirsch and Hodkinson, and later by Hirsch in [23] and [30]. In fact, the main result in [23] follows from our more general construction. Finally we blow up a little the blow up and blur construction of Andréka nd Németi, showing that various constructions of weakly representable atom structures that are not strongly representable, can be formalized in our blown up, blow up and blur construction, both for relation and cylindric algebras. Two open problems are discussed, in some detail, proposing ideas. One is whether class of subneat reducts are closed under completions, the other is whether there exists a weakly representable $\omega$ dimensional atom structure, that is not strongly representable. For the latter we propose a lifting argument, due to Monk, applied to what we call anti-Monk algebras (the algebras, constructed by Hirsch and Hodkinson, are atomic, and their atom structure is strongly representable.)

Introduction

Relation algebras and cylindric algebras introduced by Tarski are cousins. The concrete version of relation algebras are algebras of binary relations, with the binary operation of composition and unary one of taking converses, while the
concrete version of cylindric algebras of dimension $n$ are algebras of $n$ ary relations with $n$ unary operations of cylindrifiers or projections and constants, the diagonal elements reflecting equality.

There is an endless interplay between relation algebras and cylindric algebras. One can construct a relation algebra from a $\mathcal{CA}_n$, and conversely from an RA, that has what Maddux calls an $n$ dimensional cylindric basis, one can construct a $\mathcal{CA}_n$, and even without having a basis, as shown by Hodkinson, though in the latter construction one loose that original relation algebra is embeddable into the $\mathcal{Ra}$ reduct of the new $\mathcal{CA}_n$.

They have a lot of common features and manifestations. For example the representable algebras (in both cases, for $\mathcal{CA}_n$ $n \geq 3$) are unruly and wild, being resilient to any simple axiomatizations; it is undecidable to tell whether a finite algebra is representable or not. It is an entertaining practice among algebraic logicians to transfer results from RA to CA and vice versa, but for a statement that applies to both, it is usually easier to prove it for RAs (like the non finite-axiomatizability of the equational axiomatizations of the representables and the undecidability problem for finite algebras, as to wether they are representable or not). Indeed, this has been mostly the case historically in a temporal sense, but of course there are exceptions, for example the class of neat reduct was proved non-elementary before the class of $\mathcal{Ra}$ reducts [30]. However, even in this case, the CA analogue of some results proved in the latter reference will only be proved here.

There are also types of constructions that apply to both. In this paper we will be concerned with the rainbow construction, to prove new results concerning various subclasses that are related to neat reducts one way or another, and the so called Blow up and Blur construction, a construction invented by Andréka and Németi to show that there are weakly representable atom structures that are not strongly representable; not only that, but the term algebra can be a $k$- neat reduct for any pre assigned finite $k$. (It cannot be in $\mathcal{Ntr}_n\mathcal{CA}_\omega$ for in this case, having only countable many atoms it will be completely representable, forcing the complex algebra to be representable as well).

We start by give a unified model theoretic proof to several deep results that are scattered in the literature in a serious of publications dating back to the seventees of the last century. We follow the notation of [28] and [17]. In particular, for ordinals $\alpha < \beta$, $\mathcal{Ntr}_\alpha\mathcal{CA}_\beta$ denotes the class of $\alpha$ neat reducts of $\mathcal{CA}_\beta$s. Henkin et all [28] showed that $\mathcal{Ntr}_1\mathcal{CA}_\beta$ is a variety for any $\beta \geq 1$. Németi [12] showed that for any pair of ordinals $1 < \alpha, \beta$, the class $\mathcal{Ntr}_\alpha\mathcal{CA}_\beta$, though closed under homomorphic images and products, is not closed under forming subalgebras answering problem 2.11 in [28], hence is not a variety. Németi then posed the question as to whether it is closed under elementary subalgebras. Being closed under ultraproducts, as proved by Henkin et all [28], this amounts to asking whether it is elementary or not. Andréka and Németi proved that is
for the lowest dimension 2, namely $\mathfrak{Nt}_2 \mathcal{CA}_\beta$ is not elementary for any $\beta > 2$.

The result was extended to all dimension by the author, and a model theoretic proof was given in [7]; the same proof reported in some detail in the survey article [13]. This is the method we use here.

Later the problem was investigated for cylindric like algebras, like substitution algebras $\mathcal{SC}$ of Pinter, and polyadic algebras of Halmos $\mathcal{PA}$. The problem was solved for any class $K$ between $\mathcal{SC}_2$ and $\mathcal{PEA}_2$ in [6]; the proof hence generalizes that of Andréka and Németi. This result was extended to finite dimensions in [5], and infinite dimensions in [16].

In the context of cylindric algebras, closure under complete neat embeddings and complete representability was proved equivalent for countable atomic algebras by the author [9] The characterization also works for relation algebras, using the same method, which is a Baire category argument at heart [17]; later reproved by Robin Hirsch using games [30]. It was also proved that all three conditions cannot be omitted, atomicity, countability and complete embeddings. There are examples, that show that such conditions are necessary.

Hirsch and Hodkinson [24] prove that the class of completely representable $\mathcal{CA}_n$s is not elementary, for any $n \geq 3$.

For our results concerning neat reducts, we use techniques of Hirsch’s in [30] that deal with relation algebras, and those of Hirsch and Hodkinson in [23] on complete representations. The results in the latter had to do with investigating the existence of complete representations for cylindric algebras and for this purpose, an infinite (atomic) game that tests complete representability was devised, and such a game was used on a rainbow relation algebra. The rainbow construction has a very wide scope of applications, and it proved to be a nut cracker in solving many hard problems for relation algebras, particularly for constructing counterexamples distinguishing between classes that are very subtly related, or rather unrelated.

Unfortunately, relation rainbow algebras do not possess cylindric basis for $n \geq 4$ (so it seems that we cannot have our cake and eat it), so to prove the analogous result for cylindric algebra the construction had to be considerably modified to adapt the new situation, starting anew, though the essence of the two constructions is basically the same. Instead of using atomic networks, in the cylindric algebra case games are played on coloured graphs. On the one hand, such graphs have edges which code the relation algebra construction, but they also have hyperedges of length $n - 1$, reflecting the cylindric algebra structure.

It seems that there is no general theorem for rainbow constructions when it comes to cylindric like algebras, namely one relating winning strategies for pebble games on two structures or graphs $A, B$, to winning strategies for $\exists$ in the cylindric rainbow algebra based $A$ and $B$, [25].

Nevertheless, in the latebook on cylindric algebra [17], a general rainbow
construction is given in [24] in the context of building algebras from graphs giving rise to a class of models, from which the rainbow atom structure is defined, but just referring to one graph as a parameter, rather than two structures as done in their earlier book [25]. The second graph is fixed to be the greens; these are the set of colours that \( \exists \) never uses. (In our class the class of models will be coloured graphs, viewed as structures for a natural signature).

For cylindric algebras, we take the \( n \) neat reducts of algebras in higher dimension, ending up with a \( \text{CA}_n \), but we can also take relation algebra reducts, getting instead a relation algebra. The class of relation algebra reducts of cylindric algebras of dimension \( n \geq 3 \), denoted by \( \text{RaCA}_n \). The Ra reduct of a \( \text{CA}_n \), \( \mathfrak{A} \), is obtained by taking the 2 neat reduct of \( \mathfrak{A} \), then defining composition and converse using one space dimension. For \( n \geq 4 \), \( \text{RaCA}_n \subseteq \text{RA} \). Robin Hirsch dealt primarily with this class in [30]. This class has also been investigated by many authors, like Monk, Maddux, Németi and Simon (A chapter in Simon’s dissertation is devoted to such a class, when \( n = 3 \)). After a list of results and publications, Simon proved \( \text{RaCA}_3 \) is not closed under subalgebras for \( n = 3 \), with a persucor by Maddux proving the cases \( n \geq 5 \), and Monk proving the case \( n = 4 \).

In [30], Hirsch deals only the relation algebras proving that the Ra reducts of \( \text{CA}_k \), \( k \geq 5 \), is not elementary, and he ignored the CA case, probably because of analogous results proved by the author on neat reducts [26].

But the results in these two last papers are not identical (via a replacement of relation algebra via a cylindric algebra and vice versa). There are differences and similarities that are illuminating for both. For example in the RA case Hirsch proved that the elementary subalgebra that is not an Ra reduct is not a complete subalgebra of the one that is. In the cylindric algebra case, the elementary subalgebra that is not a neat reduct constructed is a complete subalgebra of the neat reduct.

Hirsch [30] also proved that any \( K \), such that \( \text{RaCA}_{\omega} \subseteq K \subseteq S_c\text{RaCA}_k \), \( k \geq 5 \) is not elementary; here, using a rainbow construction for cylindric algebras, we prove its CA analogue. In the same paper [30]. In op.cit Robin asks whether the inclusion \( \text{RaCA}_n \subseteq S_c\text{RaCA}_n \) is proper, the construction in [26], shows that for \( n \) neat reducts, it is.

Besides giving a unified proof of all cylindric like algebras for finite dimensions, we show that the inclusion is proper given that a certain \( \text{CA}_n \) term exists. (This is a usual first order formula using \( n \) variables). And indeed using the technique in [26] we prove an analogous result for relation algebras, answering the above question of Hirsch’s in [30]. We show that there is an \( \mathfrak{A} \in \text{RaCA}_{\omega} \) with a an elementary subalgebra \( \mathfrak{B} \in S_c\text{RaCA}_{\omega} \), that is not in \( \text{RaCA}_k \) when \( \leq 5 \). In particular, \( \text{RaCA}_k \subseteq S_c\text{RaCA}_5 \), for \( k \geq 5 \).

Now it is worthwhile to reverse the deed, and generalize Hirsch’s construction using rainbow cylindric algebras, to more results than that obtained for
cylindric algebras on neat reducts in [26]. For example, our construction here will give the following result not proved in op.cit: There is an algebra $\mathfrak{A} \in \mathfrak{N}_{r_n} \mathcal{CA}_\omega$ with an elementary subalgebra, that is not completely representable. But since the algebra $\mathfrak{A}$ has countably many atoms, then it is completely representable. This gives the result in [23].

The transfer from results on relation algebras to cylindric algebras is not mechanical at all. More often than not, this is not an easy task, indeed it is far from being trivial.

We use essentially the techniques in [30], together with those in [23], extending the rainbow construction to cylindric algebra. But we mention a very important difference.

In [23] one game is used to test complete representability. In [30] three games were devised testing different neat embedability properties. (An equivalence between complete representability and special neat embeddings is proved in [26])

Here we use only two games adapted to the CA case. This suffices for our purposes. The main result in [23], namely, that the class of completely representable algebras of dimension $n \geq 3$, is non elementary, follows from the fact that $\exists$ cannot win the infinite length game, but he can win the finite ones.

Indeed a very useful way of characterizing non elementary classes, say $K$, is a Koning lemma argument. The idea is to devise a game $G$ on atom structures such that for a given algebra atomic $\mathfrak{A}$ $\exists$ has a winning strategy on its atom structure for all games of finite length, but $\forall$ wins the $\omega$ round game. It will follow that there a countable cylindric algebra $\mathfrak{A}'$ such that $\mathfrak{A}' \equiv \mathfrak{A}$ and $\exists$ has a winning strategy in $G(\mathfrak{A}')$. So $\mathfrak{A}' \in K$. But $\mathfrak{A} \not\in K$ and $\mathfrak{A} \preceq \mathfrak{A}'$. Thus $K$ is not elementary.

To obtain our result we use two distinct games, both having $\omega$ rounds. Of course the games are very much related.

In this new context $\exists$ can also win a finite game with $k$ rounds for every $k$. Here the game used is more complicated than that used in Hirsch and Hodkinson, because in the former case we have three kinds of moves which makes it harder for $\exists$ to win.

Another difference is that the second game, call it $H$, is actually played on pairs, the first component is an atomic network (or coloured graph) defined in the new context of cylindric algebras, the second is a set of hyperlabels, the finite sequences of nodes are labelled, some special ones are called short, and neat hypernetworks or hypergraphs are those that label short hyperedges with the same label. And indeed a winning strategy for $\exists$ in the infinite games played on an atom structure forces that this is the atom structure of a neat reduct; in fact an algebra in $\mathfrak{N}_{r_n} \mathcal{CA}_\omega$. However, unlike complete representability, does not exclude the fact, in principal, there are other representable algebras having the same atom structure can be only subneat reducts.
On the other hand, $\forall$ can win another pebble game, also in $\omega$ rounds (like in \cite{23} on a red clique), but there is a finiteness condition involved in the latter, namely is the number of nodes 'pebbles' used, which is $k \geq n + 2$, and $\forall$'s winning strategy excludes the neat embeddability of the algebra in $k$ extra dimensions. This game will be denoted by $F^k$.

And in fact the Hirsch Hodkinson’s main result in \cite{30}, can be seen as a special case, of our construction. The game $F^k$, without the restriction on number of pebbles used and possibly reused, namely $k$ (they have to be reused when $k$ is finite), but relaxing the condition of finiteness, $\forall$ does not have to reuse node, and then this game is identical to the game $H$ when we delete the hyperlabels from the latter, and forget about the second and third kinds of move. So to test only complete representability, we use only these latter games, which become one, namely the one used by Hirsch and Hodkinson in \cite{23}. In particular, our algebra $\mathfrak{A}$ constructed below is not completely representable, but is elementary equivalent to one that is. This also implies that the class of completely representable atom structures are not elementary, the atom structure of the former two structures are elementary equivalent, one is completely representable, the other is not. Since an atom structure of an algebra is first order interpretable in the algebra, hence, the latter also gives an example of an atom structure that is weakly representable but not strongly representable, showing that the class $\text{CRA}_{n}$ is not elementary.

Concerning the blow up and blur construction, we give a simpler proof of a result of Hodkinson as an instance of such (blur and blow) constructions arguing that the idea at heart is similar to that adopted by Andréka et all \cite{27}. The idea is to blow up a finite structure, replacing each 'colour or atom' by infinitely many, using blurs to represent the resulting term algebra, but the blurs are not enough to blur the structure of the finite structure in the complex algebra. Then, the latter cannot be representable due to a finite-infinite contradiction. This structure can be a finite clique in a graph or a finite relation algebra or a finite cylindric algebra. This theme gives example of weakly representable atom structures that are not strongly representable. This is the essence too of construction of Monk like-algebras, one constructs graphs with finite colouring (finitely many blurs), converging to one with infinitely many, so that the original algebra is also blurred at the complex algebra level, and the term algebra is completely representable, yielding a representation of its completion the complex algebra.

A reverse of this process exists in the literature; that can be called anti-Monk algebras, it builds algebras with infinite blurs converging to one with finite blurs. This idea due to Hirsch and Hodkinson, uses probabilistic methods of Erdos to construct a sequence of graphs with infinite chromatic number one that is 2 colourable. This construction, which works for both relation and cylindric algebras, further shows that the class of strongly representable atom
structures is not elementary. Using such algebras we will give an idea of how to construct weakly representable infinite dimensional atom structure that is not strongly representable.

**Layout** In section 1, we prove the Ra analogue of the results on neat reducts using model theoretic techniques in [26]. In section 2, we prove the CA analogue of the results in [30] to cylindric algebras, using the rainbow construction. In the last section, we give a general form of the so called Blow up and Blur construction, a term and construction invented by Andréka and Németi, and we present many constructions in the literature proving the existence of weakly representable algebras that are not strongly representable, as an instance of our general framework.

1 **The class RaCA\(n\)**

Here we manifest yet another interplay between relation algebras and cylindric algebras. Using a construction for cylindric algebra given in [7], and given in some detail in [13], we prove a result on relation algebras, and reprove several results for cousins of cylindric algebras. Our model-theoretic proof, unifies results and proofs in [6], [15], [26], using the methods in [7]. The methods used in the former three references are more basic. The advantage in the method used in [7], is that it uses rather sophisticated methods in Model theory, namely, Fraisse’s methods of constructing homogeneous models that admit elimination of quantifiers, by amalgamating its smaller parts.

We will not give the complete proof in detail; the proof is complete modulo the existence of two terms, a cylindric algebra term, and a relation algebra term, whose definition we omit, but their properties will be clearly stated; and we hope that the general idea will be clear modulo this omission.

1.1 **Neat and Ra reducts of cylindric algebras**

We shall prove (the second item (modulo the existence of a \(k\) witness) answers a question of Hirsch [30].)

**Theorem 1.1.** Let \(K\) be any of cylindric algebra, polyadic algebra, with and without equality, or Pinter’s substitution algebra. We give a unified model theoretic construction, to show the following:

1. For \(n \geq 3\) and \(m \geq 3\), \(\mathfrak{Nt}_nK_m\) is not elementary, and \(S_c\mathfrak{Nt}_nK_\omega \nsubseteq \mathfrak{Nt}_nK_m\).

2. Assume that there exists a \(k\)-witness. For any \(k \geq 5\), \(RaCA_k\) is not elementary and \(S_cRaCA_\omega \nsubseteq RaCA_k\).
A \( k \) witness which is a CA\(_k\) term with special properties will be defined below. For CA and its relatives the idea is very much like that in [7], the details implemented, in each separate case, though are significantly distinct, because we look for terms not in the clone of operations of the algebras considered; and as much as possible, we want these to use very little spare dimensions, hopefully just one.

The relation algebra part is more delicate. We shall construct a relation algebra \( \mathfrak{A} \in \text{RaCA}_\omega \) with a complete subalgebra \( \mathfrak{B} \), such that \( \mathfrak{B} \notin \text{RaCA}_k \), and \( \mathfrak{B} \) is elementary equivalent to \( \mathfrak{A} \). (In fact, \( \mathfrak{B} \) will be an elementary subalgebra of \( \mathfrak{A} \).)

We work with \( n = 3 \). One reason, is that for higher dimensions the proof is the same. Another one is that in the relation algebra case, we do not need more dimensions.

Roughly, the idea is to use an uncountable cylindric algebra \( \mathfrak{A} \in \text{Nr}_3 \text{CA}_\omega \), hence \( \mathfrak{A} \) is representable, together with a finite atom structure of another simple cylindric algebra, that is also representable.

The former algebra will be a set algebra based on a homogeneous model, that admits elimination of quantifiers (hence will be a full neat reduct).

Such a model is constructed using Fraisse’s methods of building models by amalgamating smaller parts. The Boolean reduct of \( \mathfrak{A} \) can be viewed as a finite direct product of the of disjoint Boolean relativizations of \( \mathfrak{A} \). Each component will be still uncountable; the product will be indexed by the elements of the atom structure. The language of Boolean algebras can now be expanded by constants also indexed by the atom structure, so that \( \mathfrak{A} \) is first order interpretable in this expanded structure \( \mathfrak{P} \) based on the finite Boolean product. The interpretation here is one dimensional and quantifier free.

The Ra reduct of \( \mathfrak{A} \) be as desired; it will be a full Ra reduct of a full neat reduct of an \( \omega \) dimensional algebra, hence an Ra reduct of an \( \omega \) dimensional algebra, and it has a complete elementary equivalent subalgebra not in \( \text{RaCA}_k \).

(This is the same idea for CA, but in this case, and the other cases of its relatives, one spare dimension suffices.)

This elementary subalgebra is obtained from \( \mathfrak{P} \), by replacing one of the components of the product with an elementary countable Boolean subalgebra, and then giving it the same interpretation. First order logic will not see this cardinality twist, but a suitably chosen term \( \tau_k \) not term definable in the language of relation algebras will, witnessing that the twisted algebra is not in \( \text{RaCA}_k \).

For CA’s and its relatives, as mentioned in the previous paragraph, we are lucky enough to have \( k \) just \( n + 1 \), proving the most powerful result.

**Definition 1.2.** Let \( k \geq 4 \). A \( k \) witness \( \tau_k \) is \( m \)-ary term of CA\(_k\) with rank \( m \geq 2 \) such that \( \tau_k \) is not definable in the language of relation algebras (so that \( k \) has to be \( \geq 4 \)) and for which there exists a term \( \tau \) expressible in the
language of relation algebras, such that $\mathcal{A}_k \models \tau_k(x_1, \ldots, x_m) \leq \tau(x_1, \ldots, x_m)$. (This is an implication between two first order formulas using $k$-variables).

Furthermore, whenever $\mathcal{A} \in \mathcal{C}_k$ (a set algebra of dimension $k$) is uncountable, and $R_1, \ldots, R_m \in A$ are such that at least one of them is uncountable, then $\tau^\mathcal{A}_k(R_1 \cdots R_m)$ is uncountable as well.

The following lemma, is available in [9] with a sketch of proof; it is fully proved in [7]. If we require that a (representable) algebra be a neat reduct, then quantifier elimination of the base model guarantees this, as indeed illustrated in our fully proved next lemma.

**Lemma 1.3.** Let $V = (\mathcal{A}, \equiv, d^i_{ij})_{i,j<3}$ be a finite cylindric atom structure, such that $|\mathcal{A}| \geq |3^3|$. Let $L$ be a signature consisting of the unary relation symbols $P_0, P_1, P_2$ and uncountably many tenary predicate symbols. For $u \in V$, let $\chi_u$ be the formula $\bigwedge_{v \in V} P_{u}(x_i)$. Then there exists an $L$-structure $\mathcal{M}$ with the following properties:

1. $\mathcal{M}$ has quantifier elimination, i.e. every $L$-formula is equivalent in $\mathcal{M}$ to a boolean combination of atomic formulas.

2. The sets $P^\mathcal{M}_i$ for $i < n$ partition $M$, for any permutation $\tau$ on 3, $\forall x_0 x_1 x_2 [R(x_0, x_1, x_2) \leftrightarrow R(x_{\tau(0)}, x_{\tau(1)}, x_{\tau(2)})]$.

3. $\mathcal{M} \models \forall x_0 x_1 (R(x_0, x_1, x_2) \rightarrow \bigvee_{u \in V} \chi_u)$, for all $R \in L$.

4. $\mathcal{M} \models \exists x_0 x_1 x_2 (\chi_u \land R(x_0, x_1, x_2) \land \neg S(x_0, x_1, x_2))$ for all distinct tenary $R, S \in L$, and $u \in V$.

5. For $u \in V$, $i < 3$, $\mathcal{M} \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \leftrightarrow \bigvee_{v \in V, v \equiv u} \chi_v)$.

6. For $u \in V$ and any $L$-formula $\phi(x_0, x_1, x_2)$, if $\mathcal{M} \models \exists x_0 x_1 x_2 (\chi_u \land \phi)$ then $\mathcal{M} \models \forall x_0 x_1 x_2 (\exists x_i \chi_u \leftrightarrow \exists x_i (\chi_u \land \phi))$ for all $i < 3$.

**Proof.** [7] \hfill \Box

**Lemma 1.4.**

1. For $\mathcal{A} \in \mathcal{C}_3$ or $\mathcal{A} \in \mathcal{S}_3$, there exist a unary term $\tau_4(x)$ in the language of $\mathcal{S}_3$ and a unary term $\tau(x)$ in the language of $\mathcal{C}_3$ such that $\mathcal{A}_4 \models \tau_4(x) \leq \tau(x)$, and for $\mathcal{A}$ as above, and $u \in \mathcal{A} = 3^3$, $\tau^\mathcal{A}_4(\chi_u) = \chi_{\tau^\mathcal{A}_4(u)}$.

2. For $\mathcal{A} \in \mathcal{P}_3$ or $\mathcal{A} \in \mathcal{P}_3$, there exist a binary term $\tau_4(x, y)$ in the language of $\mathcal{S}_3$ and another binary term $\tau(x, y)$ in the language of $\mathcal{C}_3$ such that $\mathcal{P}_3 \models \tau_4(x, y) \leq \tau(x, y)$, and for $\mathcal{A}$ as above, and $u, v \in \mathcal{A} = 3^3$, $\tau^\mathcal{A}_4(\chi_u, \chi_v) = \chi_{\tau^\mathcal{A}_4(u, v)}$. 

\hfill 9
Proof. (1) For all reducts of polyadic algebras, these terms are given in [6], and [7]. For cylindric algebras \( \tau_4(x) = 3s(0,1)x \) and \( \tau(x) = s_0^1c_1x.s_0^1c_0x. \) For polyadic algebras, it is a little bit more complicated because the former term above is definable. In this case we have \( \tau(x,y) = c_1(c_0x.s_0^1c_1y).c_1x.c_0y, \) and \( \tau_4(x,y) = c_3(s_3^1c_3x.s_3^0c_3y). \)

(2) We omit the construction of such terms. But from now on, we assume that they exist.

\[\square\]

**Theorem 1.5.** (1) There exists \( \mathfrak{A} \in \mathcal{N}_3\text{QEA}_\omega \) with an elementary equivalent cylindric algebra, whose SC reduct is not in \( \mathcal{N}_3\text{SC}_4. \) Furthermore, the latter is a complete subalgebra of the former.

(2) Assume that there is \( k \) witness. Then there exists a relation algebra \( \mathfrak{A} \in \text{RaCA}_k, \) with an elementary equivalent relation algebra not in \( \text{RaCA}_k. \) Furthermore, the latter is a complete subalgebra of the former.

**Proof.** Let \( \mathfrak{L} \) and \( \mathfrak{M} \) as above. Let \( \mathfrak{A}_\omega = \{ \phi^M : \phi \in \mathfrak{L} \}. \) Clearly \( \mathfrak{A}_\omega \) is a locally finite \( \omega \)-dimensional cylindric set algebra.

The proof for CAs; and its relatives are very similar. Let us prove it for PEA. Here we have to add a condition to our constructed model. Assume that the relation symbols are indexed by an uncountable set \( I. \) We assume that there is a group structure on \( I, \) and that \( R_i \circ R_j = R_{i+j}. \) We take \( \text{At} = (3^3, \equiv_i, \equiv_j, d_{ij}), \) where for \( u, v \in \text{At}, u \equiv_i v \text{ iff } u \text{ and } v \text{ agree off } i \) and \( v \equiv_j u \text{ iff } u \circ [i,j] = v. \) We denote \( 3^3 \) by \( V. \)

By the symmetry condition we have \( \mathfrak{A} \) is a PEA3, and \( \mathfrak{A} \cong \mathcal{N}_3\mathfrak{A}_\omega, \) the isomorphism is given by \( \phi^\mathfrak{M} \mapsto \phi^\mathfrak{M}. \) Quantifier elimination in \( \mathfrak{M} \) guarantees that this map is onto, so that \( \mathfrak{A} \) is the full neat reduct. For \( u \in V, \) let \( \mathfrak{A}_u \) denote the relativisation of \( \mathfrak{A} \) to \( \chi^\mathfrak{M}_u \) i.e \( \mathfrak{A}_u = \{ x \in A : x \leq \chi^\mathfrak{M}_u \}. \) Then \( \mathfrak{A}_u \) is a Boolean algebra. Also \( \mathfrak{A}_u \) is uncountable for every \( u \in V \) because by property (iv) of the above lemma, the sets \( (\chi_u \land R(x_0, x_1, x_2)^\mathfrak{M}) \), for \( R \in L \) are distinct elements of \( \mathfrak{A}_u. \) Define a map \( f : \mathfrak{B} \text{t} \mathfrak{A} \to \prod_{u \in V} \mathfrak{A}_u, \) by \( f(a) = \langle a \cdot \chi_u \rangle_{u \in V}. \)

We expand the language of the Boolean algebra \( \prod_{u \in V} \mathfrak{A}_u \) by constants in such a way that \( \mathfrak{A} \) becomes interpretable in the expanded structure (see the next proof for a detailed description of these constants)

\( \mathfrak{P} \) denotes the structure \( \prod_{u \in V} \mathfrak{A}_u \) for the signature of Boolean algebras expanded by constant symbols \( 1_u \) for \( u \in V \) and \( d_{ij} \) for \( i, j \in 3 \) as in [13]. The closed terms corresponding to substitutions are given by \( h_S = \{ v : \exists u \in S : v \equiv_{ij} u \}. \)

In more detail let \( \mathfrak{P} \) denote the following structure for the signature of Boolean algebras expanded by constant symbols \( 1_u \) for \( u \in V \) and \( d_{ij} \) for \( i, j \in \alpha: \)
(1) The Boolean part of $\mathfrak{P}$ is the boolean algebra $\prod_{u \in V} \mathfrak{A}_u$.

(2) $1^\mathfrak{P}_u = f(\chi_u^\mathfrak{P}) = \langle 0, \cdots, 0, 1, 0, \cdots \rangle$ (with the 1 in the $u^{th}$ place) for each $u \in V$.

(3) $d_{ij}^\mathfrak{P} = f(d_{ij}^\mathfrak{P})$ for $i, j < \alpha$.

Define a map $f : \mathfrak{B} \mathfrak{A} \rightarrow \prod_{u \in V} \mathfrak{A}_u$, by

$$f(a) = \langle a \cdot \chi_u \rangle_{u \in V}.$$  

Then there are quantifier free formulas $\eta_i(x, y)$ and $\eta_{ij}(x, y)$ such that $\mathfrak{P} \models \eta_i(f(a), b)$ iff $b = f(c^\mathfrak{P}_i a)$ and $\mathfrak{P} \models \eta_{ij}(f(a), b)$ iff $b = f(s_{[i, j]} a)$.

Now, like the CA case, $\mathfrak{A}$ is interpretable in $\mathfrak{P}$, and indeed the interpretation is one dimensional and quantifier free. For $v \in V$, let $\mathfrak{B}_v$ be a complete countable elementary subalgebra of $\mathfrak{A}_v$. Then proceed like the CA case, except that we take a different product, since we have a different atom structure, with relations for substitutions: Let $u_1, u_2 \in V$ and let $v = \tau(u_1, u_2)$, as given in the above lemma. Let $J = \{ u_1, u_2, s_{[i, j]} v : i, j < 3 \}$. Let $\mathfrak{B} = \mathfrak{A}_{u_1} \times \mathfrak{A}_{u_2} \times \mathfrak{B}_v \times \prod_{i, j \leq 3, i \neq j} \mathfrak{B}_{s_{[i, j]} v} \times \prod_{u \in V \sim J} \mathfrak{A}_u$ inheriting the same interpretation. Notice that here we made all the permuted versions of $\mathfrak{B}_v$ countable, so that $B_v$ remains countable, because substitutions corresponding to transpositions are present in our signature, so if one of the permuted components is uncountable, then $\mathfrak{B}_v$ would be uncountable, and we do not want that.

The contradiction follows from the fact that had $\mathfrak{B}$ been a neat reduct, say $\mathfrak{B} = \mathfrak{N} \mathfrak{A} \mathfrak{D}$ then the term $\tau_3$ as in the above lemma, using 4 variables, evaluated in $\mathfrak{D}$ will force the component $\mathfrak{B}_v$ to be uncountable, which is not the case by construction.

For the second part; for relation algebras. The Ra reduct of $\mathfrak{A}$ is a generalized reduct of $\mathfrak{A}$, hence $\mathfrak{P}$ is first order interpretable in $\text{Ra}\mathfrak{A}$, as well. It follows that there are closed terms and a formula $\eta$ built out of these closed terms such that

$$\mathfrak{P} \models \eta(f(a), b, c) \iff b = f(a \circ_{\text{Ra}\mathfrak{A}} c),$$

where the composition is taken in $\text{Ra}\mathfrak{A}$. Here $\text{At}$ defined depends on $\tau_k$ and $\tau$, so we will not specify it any further, we just assume that it is finite.

As before, for each $u \in \text{At}$, choose any countable Boolean elementary complete subalgebra of $\mathfrak{A}_u$, $\mathfrak{B}_u$ say. Le $u_i : i < m$ be elements in $\text{At}$, and let $v = \tau(u_1, \ldots, u_m)$. Let

$$Q = \left( \prod_{u : i < m} \mathfrak{A}_{u_i} \times \mathfrak{B}_v \times \times \mathfrak{B}_v \times \prod_{u \in V \setminus \{ u_1, \ldots, u_m, v \}} \mathfrak{A}_u \right)_{u, v \in V, i, j \leq 3} \equiv (\prod_{u \in V} \mathfrak{A}_u, 1_{u, v}, d_{ij})_{u \in V, i, j \leq 3} = \mathfrak{P}.$$
Let $\mathcal{B}$ be the result of applying the interpretation given above to $Q$. Then $\mathcal{B} \equiv \text{Ra}\mathcal{A}$ as relation algebras, furthermore $\mathcal{B}|\mathcal{A}$ is a complete subalgebra of $\mathcal{B}|\mathcal{A}$. Now we use essentially the same argument. We force the $\tau(u_1, \ldots, u_m)$ component together with its permuted versions (because we have converse) countable; the resulting algebra will be a complete elementary subalgebra of the original one, but $\tau_k$ will force our twisted countable component to be uncountable, arriving at a contradiction.

In more detail, assume for contradiction that $\mathcal{B} = \text{Ra}\mathcal{D}$ with $\mathcal{D} \in \text{CA}_k$. Then $\tau_\mathcal{D}(f(x_{u_1}), \ldots, f(x_{u_m}))$, is uncountable in $\mathcal{D}$. Because $\mathcal{B}$ is a full RA reduct, this set is contained in $\mathcal{B}$. For simplicity assume that $\tau_\mathcal{C} \mathcal{M} = \text{Id}$. On the other hand, for $x_i \in \mathcal{B}$, with $x_i \leq \chi_{u_i}$, let $\bar{x}_i = (0 \ldots x_i, \ldots)$ with $x_i$ in the $u$th place. Then we have

$$\tau_\mathcal{D}(\bar{x}_1, \ldots, \bar{x}_m) \leq \tau(\bar{x}_1 \ldots, \bar{x}_m) \in \tau(f(x_{u_1}), \ldots, f(x_{u_m})) = f(\tau_{u_1 \ldots, u_m}) = f(\text{Id}).$$

But this is a contradiction, since $\mathcal{B}_{\text{Id}} = \{x \in \mathcal{B} : x \leq \chi_{\text{Id}}\}$ is countable and $f$ is a Boolean isomorphism.

**1.2 Neat atom structures**

We note that the construction here is actually stronger than the one given for finite dimensions, since it provides atomic algebras $\mathcal{A}$ and $\mathcal{B}$, so that we can talk about their atom structures, and it also encompasses the finite dimensional case.

$R$ be an uncountable set and let $CofR$ be set of all non-empty finite or cofinite subsets $R$. Let $\alpha$ be an ordinal. For $k$ finite, $k \geq 1$, let

$$S(\alpha, k) = \{i \in ^{\alpha}(\alpha + k)^{\text{Id}} : \alpha + k - 1 \in Rgi\},$$

$$\eta(X) = \bigvee \{C_r : r \in X\},$$

$$\eta(R \sim X) = \bigwedge \{-C_r : r \in X\}.$$

We give a construction for cylindric algebras for all dimensions $> 1$. Let $\alpha > 1$ be any ordinal. $(W_i : i \in \alpha)$ be a disjoint family of sets each of cardinality $|R|$. Let $M$ be their disjoint union, that is $M = \bigcup W_i$. Let $\sim$ be an equivalence relation on $M$ such that $a \sim b$ iff $a, b$ are in the same block. Let $T = \prod W_i$. Let $s \in T$, and let $V = \alpha M(s)$. For $s \in V$, we write $D(s)$ if $s_i \in W_i$, and we let $\mathcal{C} = \varnothing(V)$.

**Lemma 1.6.** There are $\alpha$-ary relations $C_r \subseteq \alpha M(s)$ on the base $M$ for all $r \in R$, such that conditions (i)-(v) below hold:

(i) $\forall s(s \in C_r \implies D(s))$
(ii) For all \( f \in {}^\alpha W^{(s)} \) for all \( r \in R \), for all permutations \( \pi \in {}^\alpha (\text{Id}) \), if \( f \in C_r \) then \( f \circ \pi \in C_r \).

(iii) For all \( 1 \leq k < \omega \), for all \( v \in {}^{\alpha+k-1}W^{(s)} \) one to one, for all \( x \in W \), \( x \in W^m \) say, then for any function \( g : S(\alpha, k) \to \text{Cof}^+ R \) for which \( \{ i \in S(\alpha, k) : |\{ g(i) \neq R \}| < \omega \} \), there is a \( v_{\alpha+k-1} \in W^m \setminus R^v \) such that and

\[
\bigwedge \{ D(v_{ij})_{j<\alpha} \Rightarrow \eta(g(i))[\{v_{ij}\}] : i \in S(\alpha, k) \}.
\]

(iv) The \( C_r \)'s are pairwise disjoint.

If an atom structure has one completely representable algebra, then all algebras based on this atom structure are completely representable. Here we show that in contrast, there is an atom structure \( \mathcal{A} \) and \( \mathfrak{A} \in \mathfrak{M}_{\alpha} K_{\alpha+\omega}, \mathfrak{B} \notin \mathfrak{M}_{\alpha} K_{\alpha+1} \), such that \( \mathcal{A} \mathfrak{A} = \mathcal{A} \mathfrak{B} = \mathcal{A} \). Furthermore \( \mathfrak{A} \) and \( \mathfrak{B} \) are not elementary equivalent.

**Theorem 1.7.** For every ordinal \( \alpha > 1 \), there exists an atom structure that is not neat.

**Proof.** Let \( \alpha > 1 \) and \( F \) is field of characteristic 0. Let

\[ V = \{ s \in {}^\alpha F : |\{ i \in \alpha : s_i \neq 0 \}| < \omega \}, \]

Note that \( V \) is a vector space over the field \( F \). We will show that \( V \) is a weakly neat atom structure that is not strongly neat. Indeed \( V \) is a concrete atom structure \( \{ s \} \equiv_i \{ t \} \) if \( s(j) = t(j) \) for all \( j \neq i \), and \( \{ s \} \equiv_{ij} \{ t \} \) if \( s \circ [i, j] = t \).

Let \( \mathfrak{C} \) be the full complex algebra of this atom structure, that is

\[ \mathfrak{C} = (\varphi(V), \cup, \cap, \sim, \emptyset, V, c_i, d_{ij}, s_{ij})_{i,j \in \alpha}. \]

Then clearly \( \varphi(V) \in \mathfrak{M}_{\alpha} \mathfrak{C}_{\alpha+\omega} \). Indeed let \( W = {}^{\alpha+\omega} F^{(0)} \). Then \( \psi : \varphi(V) \to \mathfrak{M}_{\alpha} \varphi(W) \) defined via

\[ X \mapsto \{ s \in W : s \upharpoonright \alpha \in X \} \]

is an isomomorphism from \( \varphi(V) \) to \( \mathfrak{M}_{\alpha} \varphi(W) \). We shall construct an algebra \( \mathfrak{A} \) such that \( \mathcal{A} \mathfrak{A} \cong V \) but \( \mathfrak{A} \notin \mathfrak{M}_{\alpha} \mathfrak{C}_{\alpha+1} \).

Let \( y \) denote the following \( \alpha \)-ary relation:

\[ y = \{ s \in V : s_0 + 1 = \sum_{i>0} s_i \}. \]

Note that the sum on the right hand side is a finite one, since only finitely many of the \( s_i \)'s involved are non-zero.

**Theorem 1.8.** If \( \tau_k \) exists then \( \mathfrak{A} \) and \( \mathfrak{B} \) can be chosen to be atomic.
2 Neat reducts and games

We start by characterizing the class $\mathfrak{Nr}_n \mathbf{CA}_\omega$ using games, or rather the atomic algebras in $\mathfrak{Nr}_n \mathbf{CA}_\omega$ using games. Therefore, the devised games will be played on atom structures. Admittedly, games played on atom structures of neat reduct miss something, for not all neat reducts are atomic, which is not the case for example with complete representations. But such games can go very deeply into the analysis distinguishing between various classes that are intimately related, and hard to distinguish. So we basically use games that are oriented to constructing counterexamples.

Our treatment in this part follows very closely [30]. The essential difference is that in the games devised we deal with $n$ dimensional networks (as opposed to 2 dimensional networks or basic matrices) and triangle moves are replaced by what we call cylindrifier moves in the games. Therefore, we will be rather sketch referring to the RA analogues of our results proved by Hirsch [30]. We need some preliminaries.

**Definition 2.1.** Let $n$ be an ordinal. An $s$ word is a finite string of substitutions $(s_i^j)$, a $c$ word is a finite string of cylindrifications $(c_k)$. An $sc$ word is a finite string of substitutions and cylindrifications. Any $sc$ word $w$ induces a partial map $\hat{w} : n \to n$ by

- $\hat{e} = \text{Id}$
- $\hat{w}^j_i = \hat{w} \circ [i|j]$  
- $\hat{w}c_i = \hat{w} \mid (n \sim \{i\})$

If $\bar{a} \in ^{<n-1}n$, we write $s_{\bar{a}}$, or more frequently $s_{a_0...a_{k-1}}$, where $k = |\bar{a}|$, for an an arbitrary chosen $sc$ word $w$ such that $\hat{w} = \bar{a}$. $w$ exists and does not depend on $w$ by [24 definition 5.23 lemma 13.29]. We can, and will assume [24 Lemma 13.29] that $w = sc_{n-1}c_n$. [In the notation of [24 definition 5.23, lemma 13.29], $s_i^j$ for example is the function $n \to n$ taking 0 to $i$, 1 to $j$ and 2 to $k$, and fixing all $l \in n \setminus \{i,j,k\}$.] Let $\delta$ be a map. Then $\delta[i \to d]$ is defined as follows. $\delta[i \to d](x) = \delta(x)$ if $x \neq i$ and $\delta[i \to d](i) = d$. We write $\delta^i_j$ for $\delta[i \to \delta_j]$.  

**Definition 2.2.** From now on let $2 \leq n < \omega$. Let $\mathfrak{C}$ be an atomic $\mathbf{CA}_n$. An atomic network over $\mathfrak{C}$ is a map

$$N : ^n\Delta \to \text{At} \mathfrak{C}$$

such that the following hold for each $i,j < n$, $\delta \in ^n\Delta$ and $d \in \Delta$:

- $N(\delta^j_i) \leq d_{ij}$
- $N(\delta[i \to d]) \leq c_i N(\delta)$

14
Note than $N$ can be viewed as a hypergraph with set of nodes $\Delta$ and each hyperedge in $\Delta$ is labelled with an atom from $\mathcal{C}$. We call such hyperedges atomic hyperedges.

For relation algebras an atomic network, is just a basic matrix in the sense of Maddux, which is a map from a set of ordered pairs to the atoms of a relation algebra. What we have defined can be viewed as a hyppernet or, if you like, a basic tensor. We write $\text{nodes}(N)$ for $\Delta$. But it can happen let $N$ stand for the set of nodes as well as for the function and the network itself. Context will help.

Define $x \sim y$ if there exists $\tilde{z}$ such that $N(x, y, \tilde{z}) \leq d_{01}$. Define an equivalence relation $\sim$ over the set of all finite sequences over $\text{nodes}(N)$ by $\tilde{x} \sim \tilde{y}$ iff $|\tilde{x}| = |\tilde{y}|$ and $x_i \sim y_i$ for all $i < |\tilde{x}|$. (It can be checked that this indeed an equivalence relation.)

(3) A hypernetwork $N = (N^a, N^h)$ over $\mathcal{C}$ consists of a network $N^a$ together with a labelling function for hyperlabels $N^h : <^\omega \text{nodes}(N) \rightarrow \Lambda$ (some arbitrary set of hyperlabels $\Lambda$) such that for $\tilde{x}, \tilde{y} \in <^\omega \text{nodes}(N)$

IV. $\tilde{x} \sim \tilde{y} \Rightarrow N^h(\tilde{x}) = N^h(\tilde{y})$.

If $|\tilde{x}| = k \in N$ and $N^h(\tilde{x}) = \lambda$ then we say that $\lambda$ is a $k$-ary hyperlabel. $(\tilde{x})$ is referred to a a $k$-ary hyperedge, or simply a hyperedge. (Note that we have atomic hyperedges and hyperedges, context will help which one we intend.) When there is no risk of ambiguity we may drop the superscripts $a, h$. Th labelling function for hyperlabels, labels sequences of nodes of arbitrary lengths by a set of hyperlabels. The idea, here is that a neat reduct can be viewed as a two sorted structure, the hypernetwork has to do with the first sort, and the hyperlabels adjusts the algebra in higher dimensions in which the first sort, namely, the neat reduct embeds.

The following notation is defined for hypernetworks, but applies equally to networks.

(4) If $N$ is a hypernetwork and $S$ is any set then $N \upharpoonright_S$ is the $n$-dimensional hypernetwork defined by restricting $N$ to the set of nodes $S \cap \text{nodes}(N)$. For hypernetworks $M, N$ if there is a set $S$ such that $M = N \upharpoonright_S$ then we write $M \subseteq N$. If $N_0 \subseteq N_1 \subseteq \ldots$ is a nested sequence of hypernetworks then we let the limit $N = \bigcup_{i<\omega} N_i$ be the hypernetwork defined by $\text{nodes}(N) = \bigcup_{i<\omega} \text{nodes}(N_i)$, $N^a(x_0, \ldots x_{n-1}) = N^a_i(x_0, \ldots x_{n-1})$ if $x_0 \ldots x_{n-1} \in \text{nodes}(N_i)$, and $N^h(\tilde{x}) = N^h_i(\tilde{x})$ if $\text{rng}(\tilde{x}) \subseteq \text{nodes}(N_i)$. This is well-defined since the hypernetworks are nested and since hyperedges $\tilde{x} \in <^\omega \text{nodes}(N)$ are only finitely long.

For hypernetworks $M, N$ and any set $S$, we write $M \equiv_S N$ if $N \upharpoonright_S = M \upharpoonright_S$. For hypernetworks $M, N$, and any set $S$, we write $M \equiv N$ if the symmetric difference $\Delta(\text{nodes}(M), \text{nodes}(N)) \subseteq S$ and $M \equiv \text{nodes}(M) \cup \text{nodes}(N) \setminus_S N$. We write $M \equiv_k N$ for $M \equiv_{\{k\}} N$.  

Let $N$ be a network and let $\theta$ be any function. The network $N\theta$ is a complete labelled graph with nodes $\theta^{-1}(\text{nodes}(N)) = \{ x \in \text{dom}(\theta) : \theta(x) \in \text{nodes}(N) \}$, and labelling defined by $(N\theta)(i_0, \ldots, i_{\mu-1}) = N(\theta(i_0), \theta(i_1), \theta(i_{\mu-1}))$, for $i_0, \ldots, i_{\mu-1} \in \theta^{-1}(\text{nodes}(N))$. Similarly, for a hypernetwork $N = \langle N^a, N^h \rangle$, we define $N\theta$ to be the hypernetwork $(N^a \theta, N^h \theta)$ with hyperlabelling defined by $N^h \theta(x_0, x_1, \ldots) = N^h(\theta(x_0), \theta(x_1), \ldots)$ for $(x_0, x_1, \ldots) \in ^{<\omega}\theta^{-1}(\text{nodes}(N))$.

Let $M, N$ be hypernetworks. A partial isomorphism $\theta : M \rightarrow N$ is a partial map $\theta : \text{nodes}(M) \rightarrow \text{nodes}(N)$ such that for any $i_1, \ldots, i_{\mu-1} \in \text{dom}(\theta) \subseteq \text{nodes}(M)$ we have $M^a(i_1, \ldots, i_{\mu-1}) = N^a(\theta(i_1), \ldots, \theta(i_{\mu-1}))$ and for any finite sequence $\bar{x} \in ^{<\omega}\text{dom}(\theta)$ we have $M^h(\bar{x}) = N^h(\theta(\bar{x}))$. If $M = N$ we may call $\theta$ a partial isomorphism of $N$.

Hirsch played games only on relation algebra atom structures. We will play games that apply to cylindric algebra for every finite dimension, so that in fact we are dealing with infinitely many cases. We are in front of two choices, either explicitly refer to the dimension in our notation (so that in some cases we will need two 'indices' one for the dimension of the algebra, the other for the number of rounds played on the atom structure of the algebra), or else fix it throughout. We choose the latter alternative. To simplify notation, fix $n \geq 3$. $n$ will only appear as the dimension. It will not appear in the notation of games played; since it will be clear from context. This simplifies notation considerably, and definitely permits better readability.

The next definition is crucial.

**Definition 2.3.** A hyperedge $\bar{x} \in ^{<\omega}\text{nodes}(N)$ of length $m$ is short, if there are $y_0, \ldots, y_{n-1} \in \text{nodes}(N)$, such that $N(x_j, y_i, z) \leq d_{01}$, for some $j < m$, $i < n$, for some (equivalently for all) $z$. Otherwise, it is called long. A hypernetwork is called $\lambda$ neat if $N(\bar{x}) = \lambda$ for all short hyperedges.

Short hyperedges have to do with the atoms of the small algebra the neat $n$ reduct, which will actually be the hypernetworks. If $\mathfrak{A} = \mathfrak{R}_n \mathfrak{B}$, and $\mathfrak{A}$ is atomic, then we want the atoms of the $n$ neat reduct to be no smaller than the atoms of the big algebra, of which they are a neat reduct. This is the role of the $\lambda$ neat hypernetworks, labelling short hyperedges. This will enable us to prove that a given atomic $n$ dimensional atomic cylindric algebra is the full neat reduct of an $\omega$ dimensional one.

**Definition 2.4.** Let $2 \leq n < \omega$. For any $\text{CA}_n$ atom structure $\alpha$, and $n \leq m \leq \omega$, we define two-player games $F^m(\alpha)$, and $H(\alpha)$, each with $\omega$ rounds, and for $m < \omega$ we define $H_m(\alpha)$ with $m$ rounds.

- Let $m \leq \omega$. This is a typical $m$ pebble game. In a play of $F^m(\alpha)$ the two players construct a sequence of networks $N_0, N_1, \ldots$ where $\text{nodes}(N_i)$ is a finite subset of $m = \{ j : j < m \}$, for each $i$. In the initial
round of this game $\forall$ picks any atom $a \in \alpha$ and $\exists$ must play a finite network $N_0$ with nodes$(N_0) \subseteq m$, such that $N_0(\vec{d}) = a$ for some $\vec{d} \in ^{\mu} \text{nodes}(N_0)$. In a subsequent round of a play of $F_m(\alpha)$ $\forall$ can pick a previously played network $N$ an index $l < n$, a “face” $F = \langle f_0, \ldots, f_{n-2} \rangle \in \text{nodes}(N)$, $k \in m \setminus \{f_0, \ldots, f_{n-2}\}$, and an atom $b \in \alpha$ such that $b \leq c_i N(f_0, \ldots, f_i, x, \ldots f_{n-2})$. (the choice of $x$ here is arbitrary, as the second part of the definition of an atomic network together with the fact that $c_i(\vec{c}_i x) = \vec{c}_i x$ ensures that the right hand side does not depend on $x$). This move is called a cylindrical move and is denoted $(N, \langle f_0, \ldots, f_{n-2} \rangle, k, b, l)$ or simply $(N, F, k, b, l)$. In order to make a legal response, $\exists$ must play a network $M \supseteq N$ such that $M(f_0, \ldots, f_{i-1}, k, f_i, \ldots f_{n-2}) = b$ and nodes$(M) = \text{nodes}(N) \cup \{k\}$.

$\exists$ wins $F_m(\alpha)$ if she responds with a legal move in each of the $\omega$ rounds. If she fails to make a legal response in any round then $\forall$ wins. The more pebbles we have, the easier it is for $\forall$ to win.

- Fix some hyperlabel $\lambda_0$. $H(\alpha)$ is a game the play of which consists of a sequence of $\lambda_0$-neat hypernetworks $N_0, N_1, \ldots$ where nodes$(N_i)$ is a finite subset of $\omega$, for each $i < \omega$, so that short hyperedges are all labelled by $\lambda_0$. In the initial round $\forall$ picks $a \in \alpha$ and $\exists$ must play a $\lambda_0$-neat hypernetwork $N_0$ with nodes contained in $\mu$ and $N_0(\vec{d}) = a$ for some nodes $\vec{d} \in ^{\mu} N_0$. At a later stage $\forall$ can make any cylindrical move $(N, F, k, b, l)$ by picking a previously played hypernetwork $N$ and $F \in \text{nodes}(N)$, $l < n$, $k \in \omega \setminus \text{nodes}(N)$ and $b \leq c_i N(f_0, \ldots f_{i-1}, x, \ldots f_{n-2})$.

[In $H(\alpha)$ we require that $\forall$ chooses $k$ as a ‘new node’, i.e. not in nodes$(N)$, whereas in $F_m$ for finite $m$ it was necessary to allow $\forall$ to ‘reuse old nodes’. This makes the game easier as far as he is concerned.) For a legal response, $\exists$ must play a $\lambda_0$-neat hypernetwork $M \equiv_k N$ where nodes$(M) = \text{nodes}(N) \cup \{k\}$ and $M(f_0, f_{i-1}, k, f_{n-2}) = b$. Alternatively, $\forall$ can play a transformation move by picking a previously played hypernetwork $N$ and a partial, finite surjection $\theta : \omega \rightarrow \text{nodes}(N)$, this move is denoted $(N, \theta)$. $\exists$ must respond with $N\theta$. Finally, $\forall$ can play an amalgamation move by picking previously played hypernetworks $M, N$ such that $M \equiv_{\text{nodes}(M) \cap \text{nodes}(N)} N$ and $\text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$. This move is denoted $(M, N)$. To make a legal response, $\exists$ must play a $\lambda_0$-neat hypernetwork $L$ extending $M$ and $N$, where nodes$(L) = \text{nodes}(M) \cup \text{nodes}(N)$.

Again, $\exists$ wins $H(\alpha)$ if she responds legally in each of the $\omega$ rounds, otherwise $\forall$ wins.

- For $m < \omega$ the game $H_m(\alpha)$ is similar to $H(\alpha)$ but play ends after $m$ rounds, so a play of $H_m(\alpha)$ could be

$$N_0, N_1, \ldots, N_m$$
If \( \exists \) responds legally in each of these \( m \) rounds she wins, otherwise \( \forall \) wins.

**Definition 2.5.** For \( m \geq 5 \) and \( \mathcal{C} \in \text{CA}_m \), if \( \mathfrak{A} \subseteq \mathfrak{Mr}_n(\mathcal{C}) \) is an atomic cylindric algebra and \( N \) is an \( \mathfrak{A} \)-network then we define \( \hat{N} \in \mathcal{C} \) by

\[
\hat{N} = \prod_{i_0, \ldots, i_{n-1} \in \text{nodes}(N)} s_{i_0, \ldots, i_{n-1}} N(i_0 \ldots i_{n-1})
\]

\( \hat{N} \in \mathcal{C} \) depends implicitly on \( \mathcal{C} \).

We write \( \mathfrak{A} \subseteq_c \mathfrak{B} \) if \( \mathfrak{A} \in S_c(\mathfrak{B}) \).

**Lemma 2.6.** Let \( n < m \) and let \( \mathfrak{A} \) be an atomic \( \text{CA}_n \), \( \mathfrak{A} \subseteq_c \mathfrak{Mr}_n \mathcal{C} \) for some \( \mathcal{C} \in \text{CA}_m \). For all \( x \in \mathcal{C} \setminus \{0\} \) and all \( i_0, \ldots, i_{n-1} < m \) there is a \( a \in \text{At}(\mathfrak{A}) \) such that \( s_{i_0, \ldots, i_{n-1}} a = x \).

**Proof.** We can assume, see definition \( \ref{2.1} \) that \( s_{i_0, \ldots, i_{n-1}} \) consists only of substitutions, since \( c_m \ldots c_{m-1} \ldots c_n x = x \) for every \( x \in \mathfrak{A} \). We have \( s_j \) is a completely additive operator (any \( i, j \)), hence \( s_{i_0, \ldots, i_{n-1}} \) is too (see definition \( \ref{2.1} \)). So \( \sum \{ s_{i_0, \ldots, i_{n-1}} a : a \in \text{At}(\mathfrak{A}) \} = s_{i_0, \ldots, i_{n-1}} \sum \text{At}(\mathfrak{A}) = s_{i_0, \ldots, i_{n-1}} - 1 = 1 \), for any \( i_0, \ldots, i_{n-1} < n \). Let \( x \in \mathcal{C} \setminus \{0\} \). It is impossible that \( s_{i_0, \ldots, i_{n-1}} a = x \) for all \( a \in \text{At}(\mathfrak{A}) \) because this would imply that \( 1 - x \) was an upper bound for \( \{ s_{i_0, \ldots, i_{n-1}} a : a \in \text{At}(\mathfrak{A}) \} \), contradicting \( \sum \{ s_{i_0, \ldots, i_{n-1}} a : a \in \text{At}(\mathfrak{A}) \} = 1 \). \( \square \)

We now prove two theorems relating neat embeddings to the games we defined:

**Theorem 2.7.** Let \( n < m \), and let \( \mathfrak{A} \) be an atomic \( \text{CA}_m \). If \( \mathfrak{A} \in S_c \mathfrak{Mr}_n \text{CA}_m \), then \( \exists \) has a winning strategy in \( F^m(\text{At}\mathfrak{A}) \). In particular, if \( \mathfrak{A} \) is countable and completely representable, then \( \exists \) has a winning strategy in \( F^\omega(\text{At}\mathfrak{A}) \).

**Proof.** For the first part, if \( \mathfrak{A} \subseteq \mathfrak{Mr}_n \mathcal{C} \) for some \( \mathcal{C} \in \text{CA}_m \) then \( \exists \) always plays hypernetworks \( N \) with \( \text{nodes}(N) \subseteq n \) such that \( \hat{N} \neq 0 \). In more detail, in the initial round, let \( \forall \) play \( a \in \text{At}\mathfrak{A} \). \( \exists \) play a network \( N \) with \( N(0, \ldots, n-1) = a \). Then \( \hat{N} = a \neq 0 \). At a later stage suppose \( \forall \) plays the cylindrifier move \( \langle N, \langle f_0, \ldots, f_{n-2} \rangle, k, b, l \rangle \) by picking a previously played hypernetwork \( N \) and \( f_i \in \text{nodes}(N) \), \( l < \mu, k \notin \{ f_i : i < n - 2 \} \), and \( b \leq c_t N(f_0, \ldots, f_{n-2}) \). Let \( \bar{a} = \langle f_0 \ldots f_{l-1}, k, f_{n-2} \rangle \). Then \( c_k \hat{N} \cdot s_b \neq 0 \). Then there is a network \( M \) such that \( M c_k \hat{N} \cdot s_b \neq 0 \). Hence \( M(f_0, \ldots, k, f_{n-2}) = b \).

For the second part, we have from the first part, that \( \mathfrak{A} \in S_c \mathfrak{Mr}_n \text{CA}_\omega \), the result now follows. \( \square \)

**Theorem 2.8.** Let \( \alpha \) be a countable \( \text{CA}_n \) atom structure. If \( \exists \) has a winning strategy in the infinite game \( H(\alpha) \), then there is a representable cylindric algebra \( \mathcal{C} \) of dimension \( \omega \) such that \( \mathfrak{Mr}_n \mathcal{C} \) is atomic and \( \text{At}\mathfrak{Mr}_n \mathcal{C} \cong \alpha \); in other words \( \alpha \) is a neat atom structure.
Proof. We shall construct a generalized atomic weak set algebra of dimension \( \omega \) such that the atom structure of its full \( n \) neat reduct is isomorphic to the given atom structure.

Suppose \( \exists \) has a winning strategy in \( H(\alpha) \). Fix some \( a \in \alpha \). We can define a nested sequence \( N_0 \subseteq N_1 \ldots \) of neat hypernetworks where \( N_0 \) is \( \exists \)'s response to the initial \( \forall \)-move \( a \), requiring that

1. If \( N_r \) is in the sequence and and \( b \leq \operatorname{c}_i N_r((f_0, f_{n-2}) \ldots, x, f_{n-2}) \). then there is \( s \geq r \) and \( d \in \text{nodes}(N_s) \) such that \( N_s(f_0, f_{i-1}, d, f_{n-2}) = b \).

2. If \( N_r \) is in the sequence and \( \theta \) is any partial isomorphism of \( N_r \) then there is \( s \geq r \) and a partial isomorphism \( \theta^+ \) of \( N_s \) extending \( \theta \) such that \( \text{rng}(\theta^+) \supseteq \text{nodes}(N_r) \).

We can schedule these requirements to extend. To find \( k \) and \( N_{r+1} \supseteq N_r \) such that \( N_{r+1}(f_0, k, f_{n-2}) = b \) then let \( k \in \omega \setminus \text{nodes}(N_r) \) where \( k \) is the least possible, and let \( N_{r+1} \) be \( \exists \)'s response using her winning strategy, to the \( \forall \)-move \( N_r, (f_0, \ldots f_{n-1}), k, b, l \).

For an extension of the other type, let \( \tau \) be a partial isomorphism of \( N_r \) and let \( \theta \) be any finite surjection onto a partial isomorphism of \( N_r \) such that \( \text{dom}(\theta) \cap \text{nodes}(N_r) = \text{dom}(\tau) \). \( \exists \)'s response to \( \forall \)'s move \( (N_r, \theta) \) is necessarily \( N\theta \). Let \( N_{r+1} \) be her response, using her winning strategy, to the subsequent \( \forall \)-move \( (N_r, N_r, \theta) \).

Now let \( N_\omega \) be the limit of this sequence. This limit is well-defined since the hypernetworks are nested. We shall show that \( N_\omega \) is the base of a weak set algebra having unit \( \omega N_a^{(p)} \), for some fixed sequence \( p \in \omega N \), for which there exists a homomorphism \( h \) from \( \mathfrak{A} \rightarrow \varphi(N_a) \) such that \( h(a) \neq 0 \).

Let \( \theta \) be any finite partial isomorphism of \( N_\omega \) and let \( X \) be any finite subset of \( \text{nodes}(N_\omega) \). Since \( \theta, X \) are finite, there is \( i < \omega \) such that \( \text{nodes}(N_i) \supseteq X \cup \text{dom}(\theta) \). There is a bijection \( \theta^+ \supseteq \theta \) onto \( \text{nodes}(N_i) \) and \( j \geq i \) such that \( N_j \supseteq N_i, N_i\theta^+ \). Then \( \theta^+ \) is a partial isomorphism of \( N_j \) and \( \text{rng}(\theta^+) = \text{nodes}(N_i) \supseteq X \). Hence, if \( \theta \) is any finite partial isomorphism of \( N_\omega \) and \( X \) is any finite subset of \( \text{nodes}(N_\omega) \) then

\[ \exists \text{ a partial isomorphism } \theta^+ \supseteq \theta \text{ of } N_\omega \text{ where } \text{rng}(\theta^+) \supseteq X \] (1)

and by considering its inverse we can extend a partial isomorphism so as to include an arbitrary finite subset of \( \text{nodes}(N_\omega) \) within its domain. Let \( L \) be the signature with one \( n \)-ary predicate symbol \( (b) \) for each \( b \in \alpha \), and one \( k \)-ary predicate symbol \( (\lambda) \) for each \( k \)-ary hyperlabel \( \lambda \). We are working in usual first order logic. Here we have a sequence of variables of order type \( \omega \), the \( n \) predicate symbols uses only \( n \) variables, and roughly the \( n \) variable formulas built up out of the first \( n \) variables will determine the neat reduct, the \( k \) ary predicate symbols will determine algebras of higher dimensions as \( k \) gets larger.
Then, $D$; that is, $\forall x \in D$.

Then this is the universe of the following weak set algebra $\phi, \psi$

For fixed $f_a \in \text{"nodes}(N_a)$, let $U_a = \{ f \in \text{"nodes}(N_a) : \{ i < \omega : g(i) \neq f_a(i) \} \text{ is finite} \}$. Notice that $U_a$ is weak unit (a set of sequences agreeing cofinitely with a fixed one.)

We can make $U_a$ into the universe an $L$ relativized structure $N_a$; here relativized means that we are only taking those assignments agreeing cofinitely with $f_a$, we are not taking the standard square model. However, satisfiability for $L$ formulas at assignments $f \in U_a$ is defined the usual Tarskian way, except that we use the modal notation, with assignments on the left:

For fixed $f, \pi$, let $N_a, f \models \{ \pi \}$.

This process will be interpreted in an infinite weak set algebra with base $N_a$, whose elements are those assignments satisfying such formulas.

Then set out to prove our claim. We shall show that $\alpha$

Note that unit of $C$ coinfinitely with a fixed one.

For fixed $f, \pi$ the projection defined by $\pi_i(x_0, ..., x_{i-1}, x_i)$ is $\Sigma$-formula using only variables belonging to $\{ x_0, ..., x_{i-1} \}$. Let $f, g \in U_a$ (some $a \in \alpha$) and suppose that $\{(f(i_j), g(i_j) : j \leq k) \}$ is a partial isomorphism of $N_a$, then one can easily prove by induction over the quantifier depth of $\phi$ and using (1), that

For any $L$-formula $\phi$, write $\phi^{N_a}$ for the set of all $n$-ary assignments satisfying it; that is $\{ f \in \text{"nodes}(N_a) : N_a, f \models \phi \}$. Let $D_a = \{ \phi^{N_a} : \phi \text{ is an } L\text{-formula} \}$.

Then this is the universe of the following weak set algebra

$$D_a = (D_a, \cup, \sim, D_{ij}, C_i)_{i,j<\omega}$$

then $D_a \in \text{RCA}_\omega$. (Weak set algebras are representable).

Let $\phi(x_{i_0}, x_{i_1}, ..., x_{i_k})$ be an arbitrary $L$-formula using only variables belonging to $\{ x_{i_0}, ..., x_{i_k} \}$. Let $f, g \in U_a$ (some $a \in \alpha$) and suppose that $\{ (f(i_j), g(i_j) : j \leq k) \}$ is a partial isomorphism of $N_a$, then one can easily prove by induction over the quantifier depth of $\phi$ and using (1), that

$$N_a, f \models \phi \iff N_a, g \models \phi$$ (2)

Let $\mathcal{C} = \prod_{a \in \alpha} D_a$. Then $\mathcal{C} \in \text{RCA}_\omega$, and $\mathcal{C}$ is the desired generalized weak set algebra. Note that unit of $\mathcal{C}$ is the disjoint union of the weak spaces. We set out to prove our claim. We shall show that $\alpha \cong \text{At} \mathcal{F}_\alpha \mathcal{C}$.

This is exactly like the corresponding proof for relation algebras; we include it for the sake of completeness and for the readers convenience. An element $x$ of $\mathcal{C}$ has the form $\langle x_a : a \in \alpha \rangle$, where $x_a \in D_a$. For $b \in \alpha$ let $\pi_b : \mathcal{C} \to D_b$ be the projection defined by $\pi_b(x_a : a \in \alpha) = x_b$. Conversely, let $\iota_a : D_a \to \mathcal{C}$ be the embedding defined by $\iota_a(y) = (x_b : b \in \alpha)$, where $x_a = y$ and $x_b = 0$ for $b \neq a$. Evidently $\pi_b(\iota_a(y)) = y$ for $y \in D_b$ and $\pi_b(\iota_a(y)) = 0$ if $a \neq b$.

Suppose $x \in \mathcal{F}_a \mathcal{C} \setminus \{0\}$. Since $x \neq 0$, it must have a non-zero component $\pi_a(x) \in D_a$, for some $a \in \alpha$. Say $\emptyset \neq \phi(x_{i_0}, ..., x_{i_k})^{D_a} = \pi_a(x)$ for some
Theorem 3.1. Let $L$-formula $\phi(x_{i_0}, \ldots, x_{i_k})$. We have $\phi(x_{i_0}, \ldots, x_{i_k})^{D_a} \in \mathcal{R}_\mu D_a$. Pick $f \in \phi(x_{i_0}, \ldots, x_{i_k})^{D_a}$ and let $b = N_a(f(0), f(1), \ldots f(n-1)) \in \alpha$. We will show that $b(x_0, x_1, \ldots x_{n-1})^{D_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{D_a}$. Take any $g \in b(x_0, x_1 \ldots x_{n-1})^{D_a}$, so $N_a(g(0), g(1), \ldots g(n-1)) = b$. The map $\{(f(0), g(0)), (f(1), g(1)) \ldots (f(n-1), g(n-1))\}$ is a partial isomorphism of $N_a$. By (1) this extends to a finite partial isomorphism $\theta$ of $N_a$ whose domain includes $f(i_0), \ldots, f(i_k)$. Let $g' \in U_a$ be defined by

$$g'(i) = \begin{cases} \theta(i) & \text{if } i \in \text{dom}(\theta) \\ g(i) & \text{otherwise} \end{cases}$$

By (2), $N_a, g' \models \phi(x_{i_0}, \ldots, x_{i_k})$. Observe that $g'(0) = \theta(0) = g(0)$ and similarly $g'(n-1) = g(n-1)$, so $g$ is identical to $g'$ over $\mu$ and it differs from $g'$ on only a finite set of coordinates. Since $\phi(x_{i_0}, \ldots, x_{i_k})^{D_a} \in \mathcal{R}_n(\mathcal{C})$ we deduce $N_a, g \models \phi(x_{i_0}, \ldots, x_{i_k})$, so $g \in \phi(x_{i_0}, \ldots, x_{i_k})^{D_a}$. This proves that $b(x_0, x_1 \ldots x_{n-1})^{D_a} \subseteq \phi(x_{i_0}, \ldots, x_{i_k})^{D_a} = \pi_a(x)$, and so

$$\iota_a(b(x_0, x_1, \ldots x_{n-1})^{D_a}) \leq \iota_a(\phi(x_{i_0}, \ldots, x_{i_k})^{D_a}) \leq x \in \mathcal{C} \setminus \{0\}.$$ 

Hence every non-zero element $x$ of $\mathcal{R}_n \mathcal{C}$ is above an atom $\iota_a(b(x_0, x_1, \ldots, x_n)^{D_a})$ (some $a, b \in \alpha$) of $\mathcal{R}_n \mathcal{C}$. So $\mathcal{R}_n \mathcal{C}$ is atomic and $\alpha \simeq \mathsf{At}\mathcal{R}_n \mathcal{C}$ — the isomorphism is $b \mapsto (b(x_0, x_1, \ldots x_n)^{D_a} : a \in A)$.

3 The Rainbow construction, non elementary classes

We can use such games to show that for $n \geq 3$, there is a representable $\mathfrak{A} \simeq \mathcal{CA}_n$ with atom structure $\alpha$ such that $\forall$ can win the game $F^{n+2}(\alpha)$. However, $\exists$ has a winning strategy in $H_k(\alpha)$, for any $k < \omega$. It will follow that there a cylindric algebra $\mathfrak{A}'$ such that $\mathfrak{A}' \equiv \mathfrak{A}$ and $\exists$ has a winning strategy in $H(\mathfrak{A}')$. So let $K$ be any class such that $\mathcal{R}_n \mathcal{CA}_\omega \subseteq K \subseteq S_c \mathcal{R}_n \mathcal{CA}_{n+2}$. $\mathfrak{A}'$ must belong to $\mathcal{R}_n(\mathcal{RCA}_\omega) = \mathcal{R}_n \mathcal{CA}_\omega$, hence $\mathfrak{A}' \in K$. But $\mathfrak{A} \not\in K$ and $\mathfrak{A} \not\succeq \mathfrak{A}'$. Thus $K$ is not elementary.

From this it easily follows that the class of completely representable cylindric algebras is not elementary, and that the class $\mathcal{R}_n \mathcal{CA}_{n+k}$ for any $k \geq 0$ is not elementary either. Furthermore, the constructions works for many variants of cylindric algebras like Halmos’ polyadic equality algebras and Pinter’s substitution algebras. In fact, we shall prove:

**Theorem 3.1.** Let $3 \leq n < \omega$. Then the following hold:

(i) Any $K$ such that $\mathcal{R}_n \mathcal{CA}_\omega \subseteq K \subseteq S_c \mathcal{R}_n \mathcal{CA}_{n+2}$ is not elementary.

(ii) The inclusions $\mathcal{R}_n \mathcal{CA}_\omega \subseteq S_c \mathcal{R}_n \mathcal{CA}_\omega \subseteq S \mathcal{R}_n \mathcal{CA}_\omega$ are all proper
(ii) Follows from the first part of the paper. The $A$ constructed there, is in $\mathcal{M}_n^*\text{CA}_\omega$, and $B \in S_c\mathcal{M}_n^*\text{CA}_\omega$ but $B \notin \mathcal{M}_n^*\text{CA}_{n+1}$. For the last inclusion take a countable atomic algebra in $\text{RCA}_n$ that is not completely representable. Then $A \in S\mathcal{M}_n^*\text{CA}_\omega$, but $A \notin S_c\mathcal{M}_n^*\text{CA}_\omega$, because it is atomic and not completely representable.

In what follows we prove the first item. Fix finite $n > 2$. We use a rainbow construction for cylindric algebras. The main difficulty here, is that atoms of a cylindric algebra cannot be coded simply as binary relations. This makes them hard to visualize. The way to get round this is to code the atoms as coloured graphs, where almost all the information is coded in colours of binary relations. This makes the part of the proof dealing with labeling edges very similar to the relation algebra case, almost identical. However, one range of colours, namely the shades of yellow, is reserved to to label $n - 1$ tuples in the graph (Notice that if $n = 3$ then the construction is the same as relation algebras.) This also confines the $n - 1$ ary coding to only one part of the construction.

The rainbow construction for cylindric algebras is an instance of the very general method of what Hirsch and Hodkinson called constructing atom structures, hence algebras from graphs. One fixes a graph $\Gamma$. Then a class $K$ of structures in the signature $\Gamma \times n$, is defined by viewing every node of such graph as a relation symbol of arity $< n$. This condition is very fortunate, because it allows all results proved for cylindric algebras easily transferred to polyadic equality algebras and diagonal free reducts of cylindric algebras.

The atom structure will actually consist of all functions $f : n \rightarrow M$, where $M \in K$, this class will be factored out suitably, to give a set, and the equivalence class of $f$ will be denoted by $[f]$. One can define the polyadic operations in an absolutely straightforward manner.

Properties of the graph are reflected in the complex algebra of this atom structure, for example $C_{\text{mol}}(\Gamma)$ is representable, iff $\Gamma$ has infinite chromatic number. This is a very general construction, and achieving such equivalences at this very abstract level is definitely an achievement. Our construction will be more tangible. Our class of modes will be coloured graphs.

We shall construct a cylindric atom structure based on finite coloured graphs, in the sense that these will constitute the atoms of the algebra. The rainbow algebra for relation algebras was invented by by Hirsch [30]. Here, following Hodkinson, we modify the construction, by allowing shades of yellow colours for $n - 1$ tuples. This will complicate matters a little, because such colours create cones, which are particular coloured graphs, and the only part of the construction dealing the labelling of $n - 1$ tuples, will depend on whether certain nodes are apexes of the same cone or not.

The relation algebra constructed by Hirsch does not have an $n$ dimensional cylindric algebras. So basis matrices are replaced by $n$-coloured graphs,
meaning that they have at most \( n \) nodes.

So taking \( n \) coloured graphs as our the atom structure, this codes the relation algebra constructed by Robin Hirsch \[30\]. We next show that the results proved for this relation algebra atom structure lifts to cylindric algebras.

So let \( \mathbb{Z} \) denotes the set of integers. Let \( P \) be the set of partial order preserving functions \( f : \mathbb{Z} \to \mathcal{N} \) with \( |\text{dom}(f)| \leq 2 \).

In the following the colours, the edge colours, namely the greens, whites, yellows, black and reds, are like the relation algebra case, so that we ensure that the rainbow part of the relation algebra construction is faithfully coded, or is actually the part of the construction when we deal only of labeling edges.

The shades of yellow are for labeling \( n - 1 \) hyperedges.

- greens: \( g_i \) (\( 1 \leq i < n - 2 \)), \( g^i_0 \), \( i \in \mathbb{Z} \).
- whites: \( w, w_f : f \in P \)
- yellow: \( y \)
- black: \( b \)
- reds: \( r_{ij} \) (\( i, j \in \mathcal{N} \)),
- shades of yellow: \( y_S : S \subseteq \omega \mathcal{N} \) or \( S = \mathcal{N} \)

The above choice of atoms is very similar to the one based on \( \Gamma \), as defined in Hirsch Hodkinson, with \( \Gamma \) being an infinite red clique, with a notational difference concerning the indices of the greens with superscript 0, and the whites are coded by partial functions in \( P \). These functions will help \( \exists \) choose the suitable whites in her game during labelling edges. Note that any cylindric algebra based on this atom structure will be representable, because the chromatic number of the red clique on which it is based is infinite, so that its complex algebra is representable.

We should also point out that the greens are different than the relation algebra case; we have \( n - 2 \) new greens. These will be used to define cones, which are particular coloured graphs, they are labels for edges in a cone. Such cones will play an essential role in the labeling of \( n - 1 \) tuples. But first we define general coloured graphs:

Using the colours above,

**Definition 3.2.** A *coloured graph* is an undirected irreflexive graph \( \Gamma \) such that every edge of \( \Gamma \) is coloured by a unique edge colour and some \( n - 1 \) tuples have a unique colour too, so it is really a hypergraph.
Definition 3.3. Let \( i \in \mathbb{Z} \), and let \( \Gamma \) be a coloured graph consisting of \( n \) nodes \( x_0, \ldots, x_{n-2}, z \). We call \( \Gamma \) an \( i \)-cone if \( \Gamma(x_0, z) = g^0_i \) and for every \( 1 \leq j \leq n-2 \) \( \Gamma(x_j, z) = g_j \), and no other edge of \( \Gamma \) is coloured green. \((x_0, \ldots, x_{n-2})\) is called the centre of the cone, \( z \) the apex of the cone and \( i \) the tint of the cone.

We define a class \( J \) consisting of coloured graphs with the following properties.

(1) \( \Gamma \) is a complete graph.

(2) \( \Gamma \) contains no triangles (called forbidden triples) of the following types:

\[
\begin{align*}
(g, g, g^*), (g_i, g_i, w), & \quad \text{any } i \in n-1 \\
(g^i_j, y, w_f), & \quad \text{unless } f \in P, i \in \text{dom}(f) \\
(g^i_j, g^k_0, w_0), & \quad \text{any } j, k \in \mathbb{Z} \\
(g^i_0, g^i_0, r_{kl}), & \quad \text{unless } \{(i, k), (j, l)\} \text{ is an order-preserving partial function } \mathbb{Z} \to \mathbb{N} \\
(y, y, y), (y, y, b), & \quad \text{(7)} \\
(r_{ij}, r_{j'k'}, r_{i^*k^*}), & \quad \text{unless } i = i^*, j = j' \text{ and } k' = k^* \quad \text{(8)}
\end{align*}
\]

and no other triple of atoms is forbidden. This part is the coding of binary relations in the graph. The next two conditions have to do with labeling \( n-1 \) tuples, and imposing conditions on shades of yellow used to label the base of an \( i \)-cone; \( i \) has to belong to the indexing set.

Edges are labelled like the relation algebra case, \( n-1 \) tuples are labelled by shades of yellow, the interaction of the two is pinned down to colouring the cones.

(3) If \( a_0, \ldots, a_{n-2} \in \Gamma \) are distinct, and no edge \((a_i, a_j)\) \( i < j < n \) is coloured green, then the sequence \((a_0, \ldots, a_{n-2})\) is coloured a unique shade of yellow. No other \((n-1)\) tuples are coloured shades of yellow.

(4) If \( D = \{d_0, \ldots, d_{n-2}, \delta\} \subseteq \Gamma \) and \( \Gamma \upharpoonright D \) is an \( i \)-cone with apex \( \delta \), inducing the order \( d_0, \ldots, d_{n-2} \) on its base, and the tuple \((d_0, \ldots, d_{n-2})\) is coloured by a unique shade \( y_S \) then \( i \in S \).

This is the class of structures \( K \) we are dealing with, every element \( M \) in is a coloured graph. The defining relations above can be coded in first order logic, more precisely, every green, white, black, red, atom corresponds to a binary relation, and every \( n-1 \) colour is coded as an \( n-1 \) relations, and the coloured graphs are defined as the first order structures, of a set of \( L_{\omega_1, \omega} \), as presented in [24].
We define a cylindric algebra of dimension $n$. We first specify our atom structure which will consists of surjections to finite coloured graphs, or rather a factroing out of this set, identifying two coloured graphs the obvious way.

Let

\[ K = \{ a : a \text{ is a surjective map from } n \text{ onto some } \Gamma \in J \text{ with nodes } \Gamma \subseteq \omega \}. \]

We write $\Gamma_a$ for the element of $K$ for which $a : n \to \Gamma$ is a surjection. Let $a, b \in K$ define the following equivalence relation: $a \sim b$ if and only if

- $a(i) = a(j)$ and $b(i) = b(j)$
- $\Gamma_a(a(i), a(j)) = \Gamma_b(b(i), b(j))$ whenever defined
- $\Gamma_a(a(k_0) \ldots a(k_{n-2})) = \Gamma_b(b(k_0) \ldots b(k_{n-1}))$ whenever defined

Let $\mathcal{C}$ be the set of equivalences classes. Then define

\[ [a] \in E_{ij} \text{ iff } a(i) = a(j) \]

\[ [a]T_i[b] \text{ iff } a \upharpoonright n \sim \{i\} = b \upharpoonright n \sim \{i\}. \]

This defines a $\mathcal{CA}_n$ atom structure. Let $3 \leq n < \omega$. The idea is to show that $\mathcal{C}_n$ be the complex algebra over $\mathcal{C}$. Using the games devised above, we will show that $\mathcal{C}_n$ is not in $S_c \mathfrak{N}_n \mathcal{CA}_{n+2}$ but an elementary extension of $\mathfrak{A}$ belongs to $\mathfrak{N}_n \mathfrak{CA}_\omega$.

The games above were formulated for networks on atom structures of cylindric algebras. A network has a set of nodes, and every $n$ tuple is labelled by an atom, that is a surjection from $n$ to a coloured graph. It is very hard to deal with such networks, so what we do next, is to translate our games defined above on networks to games on coloured graphs. First a general remark; the coloured graph and the corresponding unique network will have the same set of nodes.

Let $N$ be an atomic $\mathfrak{C}_n$ network, that is $N$ maps $n$ tuples, to surjections form $n$ to coloured graphs. Assume that $N : \Delta \to K$. We want to associate a coloured complete graph. The nodes are the same as $N$. Informally, we start by labelling edges. Let $x, y$ be two distinct nodes in $\Delta$, and $\bar{z}$ be any tuple in which they occur. We know that $N(\bar{z})$ is an atom of $\mathfrak{C}_n$, namely, a surjective map from $n$ to a finite coloured graph; or rather the class of this map. This defines an edge colour of $x, y$. Using the fact that the dimension is at least $3$, the edge colour depends only on $x$ and $y$ not on the other elements of $\bar{z}$ or the positions of $x$ and $y$ in $\bar{z}$. So actually in the resulting coloured graph every edge is labelled by a finite surjection from $n$ to a coloured graph.

Similarly, $N$ defines shades of yellow for certain $(n - 1)$ tuples. In this way $N$ translates into a coloured graph.

More precisely:
Definition 3.4. Let $\Gamma \in \mathbf{J}$ be arbitrary. Define the corresponding network $N_\Gamma$ on $\mathfrak{C}_n$, whose nodes are those of $\Gamma$ as follows. For each $a_0, \ldots, a_{n-1} \in \Gamma$, define $N_\Gamma(a_0, \ldots, a_{n-1}) = [\alpha]$ where $\alpha : n \to \Gamma \upharpoonright \{a_0, \ldots, a_{n-1}\}$ is given by $\alpha(i) = a_i$ for all $i < n$. Then, as easily checked, $N_\Gamma$ is an atomic $\mathfrak{C}_n$ network. Conversely, let $N$ be any non-empty atomic $\mathfrak{C}_n$ network. Define a complete colored graph $\Gamma_N$ whose nodes are the nodes of $N$ as follows:

- For all distinct $x, y \in \Gamma_N$ and edge colours $\eta$, $\Gamma_N(x, y) = \eta$ if and only if for some $\bar{z} \in \mathfrak{C}_N$, $i, j < n$, and atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_i = x$ $z_j = y$ and the edge $(\alpha(i), \alpha(j))$ is coloured $\eta$ in the graph $\alpha$.

- For all $x_0, \ldots, x_{n-2} \in \mathfrak{C}_N$ and all yellows $y_S$, $\Gamma_N(x_0, \ldots, x_{n-2}) = y_S$ if and only if for some $\bar{z} \in \mathfrak{C}_N$, $i_0, \ldots, i_{n-2} < n$ and some atom $[\alpha]$, we have $N(\bar{z}) = [\alpha]$, $z_{i_j} = x_j$ for each $j < n - 1$ and the $n - 1$ tuple $(\alpha(i_0), \ldots, \alpha(i_{n-2}))$ is coloured $y_S$. Then $\Gamma_N$ is well defined and is in $\mathbf{J}$.

The following is then, though tedious and long, easy to check:

Theorem 3.5. For any $\Gamma \in \mathbf{J}$, we have $\Gamma_{N_\Gamma} = \Gamma$, and for any $\mathfrak{C}_n$ network $N$, $N_{\Gamma_N} = N$.

This translation makes the following equivalent formulation of the games $F^m(\text{At}\mathfrak{C}_n)$, originally defined on networks.

Definition 3.6. The new game builds a nested sequence $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ of colored graphs. $\forall$ picks a graph $\Gamma_0 \in \mathbf{J}$ with $|\Gamma_0| = m$. $\exists$ makes no response to this move. In a subsequent round, let the last graph built be $\Gamma_i$. $\forall$ picks

- a graph $\Phi \in \mathbf{J}$ with $|\Phi| = m$
- a single node $k \in \Phi$
- a colored graph embedding $\theta : \Phi \sim \{k\} \to \Gamma_i$ Let $F = \phi \setminus \{k\}$. Then $F$ is called a face. $\exists$ must respond by amalgamating $\Gamma_i$ and $\Phi$ with the embedding $\theta$. In other words she has to define a graph $\Gamma_{i+1} \in \mathfrak{C}$ and embeddings $\lambda : \Gamma_i \to \Gamma_{i+1}$ $\mu : \phi \to \Gamma_{i+1}$, such that $\lambda \circ \theta = \mu \upharpoonright F$.

Let us halt for a minute to take our breath, because we have so many notions involved in our construction, then we discuss possibilities. We started by a set of colours (atoms), then defined colored graphs, which are complete graphs excluding certain triangles corresponding to forbidden triples in defining atom structures of relation algebras. But in addition to the relation algebra part, which is not enough to code $n - 1$ ary tuples, colored graphs have also hyperedges which are colored by shades of yellow, but with a restriction, namely, we have an $i$ cone, that happens to be a subgraph of a colored graph.
\( \Gamma \) having a base a set cardinality \( n - 1 \), then \( i \in S \) where \( y_S \) is the label of the base. (Note that the networks have hyperedges of length \( n \) that are labelled by atoms, and coloured graphs also have hyperedges, which we, from now on, refer to as \( n - 1 \) tuples, to avoid confusion. These last are coloured by shades of yellow.

The cones themselves are special coloured graphs whose sides are labelled by the greens, and their base consisting of an \( n - 1 \) tuple is coloured by shades of yellow, as indicated above. Then we defined a cylindric algebra atom structure consisting of certain maps, each such map, is a surjection from \( n \) to a coloured graph.

But there is a crucial difference here that has to be pointed out between networks and coloured graphs. Coloured graphs have edges that has to be labelled by colours. Networks have have only \( n - 1 \) tuples that has to be labelled by surjections from \( n \) to coloured graphs. But the equivalence established above basically follows from the fact that the dimension is > 2, so that in labelling edges for a coloured graph arising from a network, we just take any tuple in the network containing this edge; this will be well defined. If \( N \) is a network, and \((x, y)\) is an edge then \( \Gamma_N(x, y) \) will be \( \eta \), if \( \eta \) is the colour of the edge \((\alpha(i), \alpha(j))\) where \([\alpha]\) is the image of \( N \) at any tuple \( \bar{z} \) such that \( z_i = x \) and \( z_j = y \). This does not depend neither on the representative \( \alpha \) nor \( z \). So for nodes \( x, y \) in a coloured graph edges are labelled by colours, and not surjections from \( n \) to a coloured graph. So this makes the colouring of edges in coloured graphs identical to the relation algebra case, where every edge in a network has a unique label. Had we played with networks, then it would have been really hard, to extend a given network to a larger one. Because, in such a case, we would have to label every new \( n \) tuple, by a coloured graph having at most \( n \) nodes, the old tuples are labelled as they were, but if we have a new node, and hence new \( n \) tuples, then we would have had to label every such new \( n \) tuple by a surjection from \( n \) to a finite coloured graph, responding to every eventuality imposed by \( \forall \)'s moves. Fortunately, Hirsch and Hoskinson simplified the game considerably by dealing with coloured graphs, with no restriction on their size, except that they are finite, rather than dealing with labellings that involve surjective maps into coloured graphs of a fixed size.

Now let us consider the possibilities. There may be already a point \( z \in \Gamma_i \) such that the map \((k \to z)\) is an isomorphism over \( F \). In this case does not need to extend the graph \( \Gamma_i \), she can simply let \( \Gamma_{i+1} = \Gamma_i \) \( \lambda = Id_{\Gamma_i} \), and \( \mu \upharpoonright F = Id_F \), \( \mu(\alpha) = z \). Otherwise, without loss of generality, let \( F \subseteq \Gamma_i \), \( k \notin \Gamma_i \). Let \( \Gamma^*_i \) be the colored graph with nodes \( \text{nodes}(\Gamma_i) \cup \{k\} \), whose edges are the combined edges of \( \Gamma_i \) and \( \Phi \), such that for any \( n - 1 \) tuple \( \bar{x} \) of nodes of \( \Gamma^*_i \), the color \( \Gamma^*_i(x) \) is

- \( \Gamma_i(x) \) if the nodes of \( x \) all lie in \( \Gamma \) and \( \Gamma_i(\bar{x}) \) is defined
\begin{itemize}
  \item $\phi(\bar{x})$ if the nodes of $\bar{x}$ all lie in $\phi$ and $\phi(\bar{x})$ is defined
  \item undefined, otherwise.
\end{itemize}

$\exists$ has to complete the labeling of $\Gamma_i^*$ by adding all missing edges, colouring each edge $(\beta, k)$ for $\beta \in \Gamma_i \sim \Phi$ and then choosing a shade of white for every $n - 1$ tuple $\bar{a}$ of distinct elements of $\Gamma_i^*$ not wholly contained in $\Gamma_i$ nor $\Phi$, if non of the edges in $\bar{a}$ is coloured green. She must do this on such a way that the resulting graph belongs to $J$. If she survives each round, $\exists$ has won the play. Notice that $\exists$ has a winning strategy in the in $F^m(At(\mathcal{C}_n))$ if and only if she has a winning strategy in the graph games defined above. This is tedious and rather long to verify but basically routine.

**Theorem 3.7.** $\forall$ has a winning strategy in $F^{n+2}(At\mathcal{C}_n)$.

**Proof.** For that we show $\forall$ can win the game $F^{n+2}(At\mathcal{C}_n)$. In his zeroth move, $\forall$ plays a graph $\Gamma \in J$ with nodes $0, 1, \ldots, n-1$ and such that $\Gamma(i, j) = w(i < j < n-1), \Gamma(i, n-1) = g_i(i = 1, \ldots, n), \Gamma(0, n-1) = g_0^0$, and $\Gamma(0, 1, \ldots, n-2) = y_\omega$. This is a 0-cone with base $\{0, \ldots, n-2\}$. In the following moves, $\forall$ repeatedly chooses the face $(0, 1, \ldots, n-2)$ and demands a node (possibly used before) $\alpha$ with $\Phi(i, \alpha) = g_i(i = 1, \ldots, n-2)$ and $\Phi(0, \alpha) = g_0^\alpha$, in the graph notation – i.e., an $\alpha$-cone on the same base. $\exists$, among other things, has to colour all the edges connecting nodes. The idea is that by the rules of the game the only permissible colours would be red. Using this, $\forall$ can force a win eventually for else we are led to a a decreasing sequence in $\mathcal{N}$.

In more detail, In the initial round $\forall$ plays a graph $\Gamma$ with nodes $0, 1, \ldots, n-1$ such that $\Gamma(i, j) = w$ for $i < j < n-1$ and $\Gamma(i, n-1) = g_i(i = 1, \ldots, n-2), \Gamma(0, n-1) = g_0^0$ and $\Gamma(0, 1, \ldots, n-2) = y_N$. $\exists$ must play a graph with $\Gamma_1(0, \ldots, n-1) = g_0$. In the following move $\forall$ chooses the face $(0, \ldots, n-2)$ and demands a node $\gamma$ with $\Gamma_2(i, \gamma) = g_i$ and $\Gamma_2(0, \gamma) = g_0^{-1}$. $\exists$ must choose a label for the edge $(n, n-1)$ of $\Gamma_2$. It must be a red atom $r_{mn}$. Since $-1 < 0$ we have $m < n$. In the next move $\forall$ plays the face $(0, \ldots, n-2)$ and demands a node $n+1$ such that $\Gamma_3(i, n+1) = g_i^{-2}$. Then $\Gamma_3(n+1, \gamma) \Gamma_3(n+1, n-1)$ both being red, the indices must match. $\Gamma_3(n+1, n) = r_m$ and $\Gamma_3(n+1, n-1) = r_{lm}$ with $l < m$. In the next round $\forall$ plays $(0, 1, \ldots, n-2)$ and reuses the node $n-2$ such that $\Gamma_4(0, n-2) = g_0^{-3}$. This time we have $\Gamma_4(n, n-1) = r_{jl}$ for some $j < l \in N$. Continuing in this manner leads to a decreasing sequence in $\mathcal{N}$. \hfill \Box

(Notice that here $\forall$ needed at least $n+2$ pebbles. The number of pebbles, $k > n$ say, necessary for $\forall$ to win the game, excludes complete neat embeddability of $\mathfrak{A}$ in an algebra with $k$ dimensions.)

**Corollary 3.8.** The algebra $\mathfrak{A}$ (definition above) is not in $S_c\mathfrak{N}_n CA_{n+2}$.  

28
Corollary 3.9. The algebra $\mathfrak{A}$ is not completely representable

Proof. The term algebra is countable, is not completely representable. Hence the complex algebra is not completely representable, and so is any algebra in between based on this atom structure.

But, even still, things get a little bit more complicated when we have hyperlabels. The network part is translated as above to coloured graphs. Since the graph and the network have the same nodes, then hyperlabels are simply labels for finite sequences of nodes of the graph. We refer to the graph and the hyperlabels together as a hypergraph.

Recall from definition 2.4 that $H_k(\alpha)$ is the hypernetwork game with $k$ rounds. So here we have hypernetwork. A hypernetwork consists of a network together with hyperlabels, functions from finite sequences of nodes to a set of labels, that is every hyperedge has a label. The translation of the games $H$ and $H_k$ to hypergraphs is as follows.

• Fix some hyperlabel $\lambda_0$. $H_k(\alpha)$ is a game the play of which consists of a sequence of $\lambda_0$-neat hypernetworks $N_0, N_1, \ldots$ where $\text{nodes}(N_i)$ is a finite subset of $\omega$, for each $i < \omega$.

Now let us translate the game $H(\alpha)$ to coloured graphs, which we now call hypergraphs. The first kind of moves is very similar to the game $F^m$ without the restriction of finitely many pebbles or nodes. The second and third moves, are easily translatable to coloured graphs.

$N^h$ are the hyperlabels; these we did not have in the game $F^m$, however, we will usually deal with those separately, and it will turn out that it is easier to work with them, in response to $\forall$'s moves.

Here as before a neat hypergraph means that it is constant on short hyperedges. A short hyperedge $\bar{x}$ consisting of nodes of the graph, is one such that there exists nodes $y_0, \ldots y_{n-1}$ such $\Gamma(x_i, y_j) \leq d_{01}$, for some $i, j$. Due to the correspondence established before between coloured graphs and networks, this is equivalent to the definition above given for networks.

We will play a $k$ rounded game on neat hypernetworks. Fix a hyperlabel $\lambda_0$ and a finite $k \geq 3$. A neat hypernetwork, now, is a pair $(\Gamma, N^h)$ with $\Gamma$ a coloured graph and $N^h$ is a set of functions from a finite sequences of nodes to a fixed set of labels.

Now the first move (baring in mind that we do not have only finitely many pebbles, and accordingly $\exists$ cannot reuse nodes), we get, and indeed this is reflected in the next game on coloured graphs.
Definition 3.10. ∀ picks a graph $\Gamma_0 \in J$ with $\Gamma_0 \subseteq \omega$ and here we do not require that $|\Gamma_0| = n$. ∃ make no response to this move. In a subsequent round, let the last graph built be $\Gamma_i$. ∀ picks

- a graph $\Phi \in J$ with $|\phi| = n$,
- a single node $k \in \Phi$,
- a coloured graph embedding $\theta : \Phi \sim \{k\} \rightarrow \Gamma_i$. Let $F = \phi \sim \{k\}$.

Then, as before, $F$ is called a face. ∃ must respond by amalgamating $\Gamma_i$ and $\phi$ with the embedding $\theta$ as before. In other words she has to define a graph $\Gamma_{i+1} \in C$ and embeddings $\lambda : \Gamma_i \rightarrow \Gamma_{i+1}$ $\mu : \phi \rightarrow \Gamma_{i+1}$, such that $\lambda \circ \theta = \mu \upharpoonright F$.

Now we may write $N_{\Gamma}$ or simply $N$ instead of $\Gamma$, but in all cases we are dealing with coloured graphs that is the translation of networks. That is when we write $N$ then, $N$ will be viewed as a coloured graph.

The other moves are exactly like the relation algebra case, since the nodes of a network on $C_n$ is the same as that of that of the corresponding coloured graphs.

Alternatively, ∀ can play a transformation move by picking a previously played coloured hypergraphs $N$ and a partial, finite surjection $\theta : \omega \rightarrow \text{nodes}(N)$, this move is denoted $(N, \theta)$. ∃ must respond with $N\theta$.

Finally, ∀ can play an amalgamation move by picking previously played hypergraphs $M, N$ such that $M \equiv \text{nodes}(M) \cap \text{nodes}(N) \neq \emptyset$ This move is denoted $(M, N)$. To make a legal response, ∃ must play a $\lambda_0$-neat hypergraph $L$ extending $M$ and $N$, where $\text{nodes}(L) = \text{nodes}(M) \cup \text{nodes}(N)$. Again, ∃ wins $H(\alpha)$ if she responds legally in each of the $\omega$ rounds, otherwise ∀ wins.

We can alter the rules of the game $H_k$ slightly, to make life easier. We impose certain restrictions on ∀(that are only apparent).

- ∀ is only allowed to play transformation moves $(N, \theta)$ if $\theta$ is injective.
- ∀ is only allowed to play an amalgamation move $(M, N)$ if for all $m \in \text{nodes}(M) \setminus \text{nodes}(N)$ and all $n \in \text{nodes}(N) \setminus \text{nodes}(M)$ the map $\{(m, n)\} \cup \{(x, x) : x \in \text{nodes}(M) \cap \text{nodes}(N)\}$ is not a partial isomorphism. I.e. he can only play $(M, N)$ if the amalgamated part is ‘as large as possible’.

If, as a result of these restrictions, ∀ cannot move at some stage then he loses and the game halts.

It is easy to check that ∀ has a winning strategy in $H(\alpha)$ iff he has a winning strategy with these restrictions to his moves. Also, if ∀ plays with
these restrictions to his moves, if $\exists$ has a winning strategy then she has a winning strategy which only directs her to play strict hypernetworks. The same holds when we consider $H_n(\alpha)$. We will assume that $\forall$ plays according to these restrictions.

We make two comments, before working out the details, to give a general idea of the essence of the construction, and why it actually works?

First $\exists$ cannot win the game $H$ with $\omega$ many rounds, because in this case $\forall$ has an ‘infinite space’ to use essentially the winning strategy he used before, namely, forcing a strictly decreasing sequences of indices of reds.

In fact, by some reflection, one can see that as far as $\exists$ is concerned, winning $H$ is harder than winning $F^\omega$, in fact strictly so, which, in turn, strictly harder than winning $F^{n+2}$. (As we saw before, $\forall$ can win the latter two games, in fact $\forall$ can win any game $F^m$ with $n+2 \leq m \leq \omega$.)

But if the game is truncated to any finite $k$, we will show that she can win the game $H_k$. (Notice that if $F^m$ was also restricted to finitely many rounds then $\exists$ would have won, by playing as he played before, remembering that she lost in this last game because there were infinitely many rounds.)

But why can $\exists$ win the finite games. Basically, because the only winning strategy for $\forall$ is to win on a red clique (like he played before), an a necessary condition for this is the existence of $\omega$ round. In the finite case, that is when we have only finitely many rounds, as we proceed to show, $\exists$ has enough colours, to respond to $\forall$ moves. She uses white, then black, then red. We shall see that these three colours suffice, in case of labelling edges, that are not apexes of the same cone. To label edges that are apexes of the same cone $\exists$ can only use reds, and this will not lead to a red clique (like in the game $F^m$, $m \geq 2$) because the game is finite.

However, to win the game $\exists$ has to respond to every possible move of $\forall$. A winning strategy here is complicated because $\forall$ has three kinds of moves, which makes it harder for $\exists$ to win; she has to respond to every such move. Besides corresponding to such moves $\exists$ has to label also the hyperedges, in the hypergraph produced by $\forall$. Furthermore, in the second move $\exists$ really has no choice.

Neat hyperedges are easy. In a play $\exists$ is required to play $\lambda_0$ neat hypernetworks, so she has no choice about the hyperlabels for short hyperedges, these are labelled by $\lambda_0$. In response to a cylindrifier move all long hyperedges not incident with $k$ necessarily keep the hyperlabel they had in $N$.

All long hyperedges incident with $k$ in $N$ are given unique hyperlabels not occurring as the hyperlabel of any other hyperedge in $N$.

We can assume, without loss of generality, that we have infinite supply of hyperlabels of all finite arities so this is possible. In response to an amalgamation move $(M,N)$ all long hyperedges whose range is contained in $\text{nodes}(M)$ have hyperlabel determined by $M$, and those whose range is contained in $\text{nodes}$
$N$ have hyperlabel determined by $N$. If $\bar{x}$ is a long hyperedge of $\exists$'s response $L$ where $\text{rng}(\bar{x}) \not\subseteq \text{nodes}(M)$, $\text{nodes}(N)$ then $\bar{x}$ is given a new hyperabel, not used in any previously played hypernetwork and not used within $L$ as the label of any hyperedge other than $\bar{x}$. This completes her strategy for labelling all hyperedges.

Now we turn to the real core of the construction. We shall deal now only with the graph part of the hypergraph. We need to label edges, and also label the $n-1$ tuples suitably by yellow shades. Now we give $\exists$'s strategy for edge labelling. This is very similar to the relation algebra case except that we deal with cylinder moves, instead of triangle ones. We need some notation and terminology taken from [30].

Every irreflexive edge of any hypergraph has an owner $\forall$ or $\exists$ namely the one who played this edge. We call such edges $\forall$ edges or $\exists$ edges. Each long hyperedge $\bar{x}$ in a hypergraph $N$ occurring in the play has an envelope $v_N(\bar{x})$ to be defined shortly. In the initial round of $\forall$ plays $a \in \alpha$ and $\exists$ plays $N_0$ then all irreflexive edges of $N_0$ belongs to $\forall$There are no long hyperedges in $N_0$. If in a later move, $\forall$ plays the transformation move $(N, \theta)$ and $\exists$ responds with $N\theta$ then owners and envelopes are inherited in the obvious way. If $\forall$ plays a cylinder move and $\exists$ responds with $M$ then the owner in $M$ of an edge not incident with $k$ is the same as it was in $N$ and the envelope in $M$ of a long hyperedge not incident with $k$ is the same as that it was in $N$. The edges $(f, k), (k, f)$ belong to $\forall$ in $M$ all edges $(l, k)(k, l)$ for $l \in \text{nodes}(N) \sim F$ (where $F$ is the face played in the cylinder move) belong to $\exists$ in $M$. if $\bar{x}$ is any long hyperedge of $M$ with $k \in \text{rng}(\bar{x})$, then $v_M(\bar{x}) = \text{nodes}(M)$. If $\forall$ plays the amalgamation move $(M, N)$ and $\exists$ responds with $L$ then for $m \neq n \in \text{nodes}(L)$ the owner in $L$ of a edge $(m, n)$ is $\forall$ if it belongs to $\forall$ in either $M$ or $N$, in all other cases it belongs to $\exists$ in $L$. If $\bar{x}$ is a long hyperedge of $L$ then $v_L(\bar{x}) = v_M(\bar{x})$ if $\text{rng}(\bar{x}) \subseteq \text{nodes}(M)$, $v_L(\bar{x}) = v_N(\bar{x})$ and $v_L(\bar{x}) = \text{nodes}(M)$, otherwise. This completes the definition of owners and envelopes. By induction on the number of rounds one can show

**Claim**: Let $M, N$ occur in a play of $H_k(\alpha)$ in which $\exists$ uses default labelling for hyperedges. Let $\bar{x}$ be a long hyperedge of $M$ and let $\bar{y}$ be a long hyperedge of $N$.

(1) For any hyperedge $\bar{x}'$ with $\text{rng}(\bar{x}') \subseteq_M (\bar{x})$, if $M(\bar{x}') = M(\bar{x})$ then $\bar{x}' = \bar{x}$.

(2) if $\bar{x}$ is a long hyperedge of $M$ and $\bar{y}$ is a long hyperedge of $N$, and $M(\bar{x}) = N(\bar{y})$ then there is a local isomorphism $\theta : v_M(\bar{x}) \rightarrow v_N(\bar{y})$ such that $\theta(x_i) = y_i$ for all $i < |x|$. 

(3) For any $x \in \text{nodes}(M) \sim v_M(\bar{x})$ and $S \subseteq v_M(\bar{x})$, if $(x, s)$ belong to $\forall$ in $M$ for all $s \in S$, then $|S| \leq 2$.
Now we define $\exists$’s strategy for choosing the labels for edges and yellow colours for $n - 1$ hyperedges. We proceed inductively. This part is taken from [30]. Let $N_0, N_1, \ldots, N_r$ be the start of a play of $H_k(\alpha)$ just before round $r + 1$. $\exists$ computes partial functions $\rho_s : Z \rightarrow N$, for $s \leq r$. These partial functions will help $\exists$ specify the suffix of the red atoms she has to choose in case whites and blacks do not work, in response to $\forall$’s move. It has to do only with labelling edges. Inductively for $s \leq r$ suppose

I. If $N_s(x, y)$ is green or yellow then $(x, y)$ belongs to $\forall$ in $N_s$.

II. $\rho_0 \subseteq \ldots \rho_r$.

III. $\text{dom}(\rho_s) = \{i \in Z : \exists t \leq s, x, y \in \text{nodes}(N_t), N_t(x, y) = g_i^1\}$

IV. $\rho_s$ is order preserving: if $i < j$ then $\rho_s(i) < \rho_s(j)$. The range of $\rho_s$ is widely spaced: if $i < j \in \text{dom}\rho_s$ then $\rho_s(i) - \rho_s(j) \geq 3^{n-r}$, where $n - r$ is the number of rounds remaining in the game.

V. For $u, v, x, y \in \text{nodes}(N_s)$, if $N_s(u, v) = r_{\mu, \delta}$, $N_s(x, u) = g_i^j$, $N_s(x, v) = g_j^i$ $N_s(y, u) = N_s(y, v) = y$ then
  (a) if $N_s(x, y) \neq w_f$ then $\rho_s(i) = \mu$ and $\rho_s(j) = \delta$
  (b) If $N_s(x, y) = w_f$ for some $f \in P$, the $\mu = f(i)$, $\delta = f(j)$.

VI. $N_s$ is a strict $\lambda_0$ neat hypernetwork.

To start with if $\forall$ plays a in the initial round then $\text{nodes}(N_0) = \{0, 1, \ldots, n - 1\}$, the hyperedge labelling is defined by $N_0(0, 1, \ldots, n) = a$.

In response to a cylindrifier move by $\forall$ for some $s \leq r$ and some $p \in Z$, $\exists$ must extend $\rho_r$ to $\rho_{r+1}$ so that $p \in \text{dom}(\rho_{r+1})$ and the gap between elements of its range is at least $3^{n-r-1}$. Inductively, $\rho_r$ is order preserving and the gap between its elements is at least $3^{n-r}$, so this can be maintained in a further round.

If $\forall$ chooses non green atoms, green atoms with the same suffix, or green atom whose suffixes already belong to $\rho_r$, there would be fewer elements to add to the domain of $\rho_{r+1}$, which makes it easy for $\exists$ to define $\rho_{r+1}$. Tis establishes properties II–IV for round $r + 1$.

Let us assume that $\forall$ played the cylindrifier move. $\exists$ has to choose labels for $\{(x, k), (k, x)\}$ $x \in \text{nodes}(N_s) \sim F$, where $F$ is the face, and also for $n - 1$ tuples so that the outcome is an $n$ coloured graph, the latter case will be dealt with separately. Let us start with edges. $\exists$ chooses labels for the edges $(x, k)$ one at a time and then determines the reverse edges $(k, x)$ uniquely. Property I is clear since in all cases the only atoms $\exists$ chooses white, black or red. She never chooses green.

We distinguish between two case.

(1) if $x$ and $k$ are both apexes of cones on $F$, then $\exists$ has no choice but to pick a red atom, that is not used before, and because the game is finite, she has enough reds; this cannot lead to an infinite clique. The colour she chooses is uniquely defined (as in the game $F^{n+2}$).
(2) Otherwise, this is not the case, so for some \( i < n - 1 \) there is no \( f \in F \) such that \( N_s(k, f), N_s(f, x) \) are both coloured \( g_i \) or if \( i = 0 \), they are coloured \( g_0 \) and \( g_0' \) for some \( l \) and \( l' \).

In the second case \( \exists \) uses the normal strategy in rainbow constructions. She chooses white if possible, else black and if both are not possible she chooses red. In the last choice, which is the most tricky, she uses \( \rho_s \) and the suffix \( f \) in \( w_f \) to help her choose the suffices of red atoms.

Now we distinguish between several subcases of the second case. We assumed that \( \forall \) played the cylindrifier move \((N_s, F)\), here \( F \) is the face, in round \( r + 1 \), that \( \exists \) survived till the \( r \)th round, and \( x \) and \( k \) are not appexes of the same cone.

This is similar to the Hirsch’s labelling edges, for networks \([30]\). Let \( i, j \in F \).

(1) Suppose that it is not the case that \( N_s(x, i) \) and \( N_s(x, j) \) are both green. Let \( S = \{ p \in \mathbb{Z} : (N_s(x, i) = g_0^p) \lor N_s(x, j) = g_0^p \lor N_s(x, j) = y \} \). Then \( |S| \leq 2 \). \( \exists \) lets \( N_{s+1}(x, k) = w_f \) for some \( f \) with \( \text{dom}(f) = S \).

Suppose that \( N_s(i, j) = r_{\beta, \mu}, N_s(x, i) = g_0^p, N_s(x, j) = g_0^q \) for some \( p, q \in \mathbb{Z} \). By property \((IV) f = \{(p, \beta), (q, \mu)\} \) is order preserving. \( \exists \) lets \( N_{s+1}(x, k) = w_f \) in this case.

In all other cases: \( N_s(i, j) \) is not red, or if it is then it is not the case that \( N_s(x, i) \) \( N_s(x, j) \) are both green, and it is not the case that \( N_s(x, i) = N_s(x, j) = y \), she lets \( f : S \rightarrow \mathfrak{R} \) an arbitrary order preserving function. The only forbidden triangles involving \( w_f \) are avoided. Since \( \exists \) does not change green or yellow atoms to label new edges and \( N_{r+1}(x, k) = w_f \), all triangles involving the new edge \( (x, k) \) are consistent in \( N_{r+1} \). Clearly property \( VI \) holds after round \( r + 1 \).

(2) Else it is not the case that \( N_s(x, i) = N_s(x, j) = y \), \( \exists \) lets \( N_r(x, k) = b \). Property \( V \) is not applicable in this case. The only forbidden triple involving the atom \( b \) is avoided, so all triangles \( (x, y, k) \) are consistent in \( N_{r+1} \) and property \( VI \) holds after round \( r + 1 \).

(3) If neither case above applies, either \( N_s(x, i) = g_0^p \) ad \( N_s(x, j) = y \) or \( N_s(x, i) = y \) and \( N_s(x, j) = g_0^p \). Assume the first case. There are two subcases.

\[(i) \ N_s(i, j) \neq w_f \] for all \( f \in P \). \( \exists \) lets \( \mu = \rho_{r+1}(p), \delta = \rho_{r+1}(q) \), maintaining property \( Va \). The only forbidden triples of atoms involving \( r_{\mu, \delta} \) are avoided. The triple of atoms form a triangle \( (x, y, k) \) will not be forbidden since the only green edge incident with \( k \) is \( (i, k) \) and since \( \rho_{r+1} \) is order preserving.
Concerning forbidden triples involving reds. Suppose that we have $N_s(x, y), N_{r+1}(y, k)$ are both red for some $y \in \text{nodes}(N_s)$. We have $y \notin \{i, j\}$ so $\exists$ chose the red label $N_{r+1}(y, k)$. By her strategy we have $N_s(i, y) = g_i$ and $N_s(j, y) = y$. By property $Va$ for $N_{r+1}$ we have $N_{r+1}(x, y) = r_{\rho_{r+1}(p)\rho_{r+1}(t)}$ and $N_{r+1}(y, k) = r$. The property $VI$ holds for $N_{r+1}$.

(ii) $N_s(i, j) = w_f$. By consistency of $N_s$, we have $p \in \text{dom}(f)$ and since $\forall$'s move we have $q \in \text{dom}(f)$. $\exists$ lets $\mu = f(p) \delta = f(q)$ maintaining property $V$ for round $r + 1$.

Concerning forbidden triples of atoms involving reds $r_{\mu, \delta}$. Since $f$ is order preserving and since the only green edge incident with $k$ is $(i, k)$ in $N_{r+1}$ triangles involving the new edge $(x, k)$ cannot give a forbidden triple.

For the other case (involving reds) let $y \in \text{nodes}(N_s)$ and suppose $N_{r+1}(x, y)$ and $N_{r+1}(y, k)$ are both red. As above, by her strategy we must have $N_s(y, i) = g_i$ for some $t$ and $N_s(y, j) = y$. By consistency of $N_s$ we have $t \in \text{dom}(f)$ and the current part of her strategy she lets $N_{r+1}(y, k) = r_{f(t), f(q)}$. By property $Vb$ for $N_s$ we have $N_{r+1}(x, y) = r_{f(p), f(t)}$. So the triple of atoms from the triangle $(x, y, k)$ is not forbidden. This establishes property $(VI)$ for $N_{r+1}$.

We have finished with cylindrifier moves. Now we move to amalgamation moves. Although our hypernetworks are all strict, it is not necessarily the case that hyperlabels label unique hyperedges - amalgamation moves can force that the same hyperlabel can label more than one hyperedge. However, within the envelope of a hyperedge $\bar{x}$, the hyperlabel $L(\bar{x})$ is unique.

We consider an amalgamation move $(N_s, N_t)$ chosen by $\forall$ in round $r + 1$. $\exists$ has to choose a label for each edge $(i, j)$ where $i \in \text{nodes}(N_s) \sim \text{nodes}(N_t)$ and $j \in \text{nodes}(N_t) \sim (N_s)$. This determines the label for the reverse edge. Also $\exists$ has to choose a $y$ for any $n - 1$ tuple $\bar{a}$, that is not contained completely in only one of $N_t$ or $N_s$.

Let $\bar{x}$ enumerate $\text{nodes}(N_s) \cap \text{nodes}(N_t)$. If $\bar{x}$ is short, then there are at most $n$ nodes in the intersection and this case is similar to the cylindrifier moves. If not, that is if $\bar{x}$ is long in $N_s$, then by the claim there is a partial isomorphism $\theta : v_{N_s}(\bar{x}) \to v_{N_t}(\bar{x})$ fixing $\bar{x}$. We can assume that $v_{N_s}(\bar{x}) = \text{nodes}(N_s) \cap \text{nodes}(N_t) = \text{rng}(\bar{x}) = v_{N_t}(\bar{x})$. It remains to label the edges $(i, j) \in N_{r+1}$ where $i \in \text{nodes}(N_s) \sim \text{nodes}(N_t)$ and $j \in \text{nodes}(N_t) \sim \text{nodes}(N_s)$. Her strategy is similar to the cylindrifier move. If $i$ and $j$ are apexes of the same cone she choose a red. If not she chooses white atom if possible, else the black atom if possible, otherwise a red atom. She never chooses a green atom, she lets $\rho_{r+1} = \rho_r$ and properties $II$, $III$, $IV$ remain true in round $r + 1$.

(1) There is no $x \in \text{nodes}(N_s) \cap \text{nodes}(N_t)$ such that $N_s(i, x)$ and $N_t(x, j)$
are both green. If there are nodes \( u, v \in \text{nodes}(N_s) \cap \text{nodes}(N_t) \) such that \( N_s(u, v) = r_{\beta, \mu}, N_s(i, u) = g^p_0, N_s(i, v) = g^q_0, N_t(u, j) = N_t(v, j) = y \) for some \( \beta, \mu \in N, p, q \in Z \) or the roles of \( i, j \) are swapped, she lets \( f = \{(p, \beta), (q, \mu)\} \) and sets \( N_{r+1}(i, j) = w_f \). Since all edges labelled by green or yellow atoms belong to \( \forall \) we can apply the above claim to show that the points \( u, v \) are unique so \( f \) is well defined. This is also true if \( x \) is short, since in this case there are only two nodes in \( \text{nodes}(N_s) \cap \text{nodes}(N_t) \).

If there are no such \( u, v \) as described then let \( S = \{p \in Z : \exists y \in \text{nodes}(N_s) \cap \text{nodes}(N_t), (N_s(i, y) = g^p_0 \land N_t(y, j) = y) \lor (N_s(i, y) = y \land N_t(y, j) = g^q_0)\} \). Then \( |S| \leq 2 \). Let \( f \) be any order preserving function and \( \exists \) let \( N_{r+1} = w_f \). Property \( (VI) \) holds for \( N_{r+1} \) as for triangle moves.

(2) Otherwise, if there is no such \( x \), then she lets \( N_r(i, j) = b \). As with cylindrfier moves all properties are maintained.

(3) Otherwise, there are \( x, y \in \text{nodes}(N_s) \cap \text{nodes}(N_t) \) such that \( N_s(i, x) = g_k, N_s(x, j) = g_l \) for some \( k, l \in N \) and \( N_s(i, y) = N_t(y, j) = y \). By the above proven claim \( x, y \) are unique. She labels \( (i, j) \) in \( N_r \) with a red atom \( r_{\beta, \mu} \) where

(i) If \( N_s(x, y) \neq w_f \) for all \( f \in P \), then \( \beta = \rho_{r+1}(k), \mu = \rho_{r+1}(l) \). This maintains property \( \forall a \).

(ii) Otherwise \( N_s(x, y) = w_f \) for some \( f \in F \) and \( \beta = f(k), \mu = f(l) \).

Now we turn to coloring of \( n - 1 \) tuples. For each tuple \( \bar{a} = a_0, \ldots, a_{n-2} \in N^{n-1} \) with no edge \( (a_i, a_j) \) coloured green, then \( \exists \) colours \( \bar{a} \) by \( y_s \), where

\[ S = \{i \in \mathfrak{N} : \text{there is an i cone in N with base } \bar{a}\} \]

We need to check that such labeling works.

Let us check that \( (n - 1) \) tuples are labeled correctly, by yellow colours. Let \( D \) be set of \( n \) nodes, and suppose that \( N \upharpoonright D \) is an \( i \) cone with apex \( \delta \) and base \( \{d_0, \ldots, d_{n-2}\} \), and that the tuple \( (d_0, \ldots, d_{n-2}) \) is labelled \( y_s \) in \( N \). We need to show that \( i \in S \). If \( D \subseteq N \), then inductively the graph \( N \) constructed so far is in \( \mathfrak{J} \), and therefore \( i \in S \). If \( D \subseteq \Phi \) then as \( \forall \) chose \( \Phi \) in \( \mathfrak{J} \) we get also \( i \in S \). If neither holds, then \( D \) contains \( \alpha \) and also some \( \beta \in N \sim \Phi \). \( \exists \) chose the colour \( N^+(\alpha, \beta) \) and her strategy ensures her that it is green. Hence neither \( \alpha \) or \( \beta \) can be the apex of the cone \( N^+ \upharpoonright D \), so they must both lie in the base \( \bar{d} \). This implies that \( \bar{d} \) is not yet labelled in \( N^* \), so \( \exists \) has applied her strategy to choose the colour \( y_s \) to label \( \bar{d} \) in \( N^+ \). But this strategy will have chosen \( S \) containing \( i \) since \( N^* \upharpoonright D \) is already a cone in \( N^* \). Also \( \exists \) never chooses a green edge, so all green edges of \( N^+ \) lie in \( N^* \).

That leaves one (hard) case, where there are two nodes \( \beta, \beta', \in N \), \( \exists \) colours both \( (\beta, \alpha) \) and \( (\beta', \alpha) \) red, and the old edge \( (\beta, \beta') \) has already been coloured
red (earlier in the game). If \((\beta, \beta')\) was coloured by \(\exists\) that is \(\exists\) is their owner, then there is no problem. So suppose, for a contradiction, that \((\beta, \beta')\) was coloured by \(\exists\) since \(\exists\) chose red colours for \((\alpha, \beta)\) and \((\alpha, \beta')\), it must be the case that there are cones in \(N^*\) with apexes \(\alpha, \beta, \beta'\) and the same base, \(F\), each inducing the same linear ordering \(\bar{f} = (f_0, \ldots, f_{n-2})\), say, on \(F\). Of course, the tints of these cones may all be different. Clearly, no edge in \(F\) is labelled green, as no cone base can contain green edges. It follows that \(\bar{f}\) must be labeled by some yellow colour, \(y_S\), say. Since \(\Phi \in J\), it obeys its definition, so the tint \(i\) (say) of the cone from \(\alpha\) to \(\bar{f}\) lies in \(S\). Suppose that \(\lambda\) was the last node of \(F \cup \{\beta, \beta'\}\) to be created, as the game proceeded. As \(|F \cup \{\beta, \beta'\}| = n + 1\), we see that \(\forall\) must have chosen the colour of at least one edge in this; say, \((\lambda, \mu)\). Now all edges from \(\beta\) into \(F\) are green, so \(\exists\) is the owner of them as well as of \((\beta, \beta')\).

The same holds for edges from \(\beta'\) to \(F\). Hence \(\lambda, \mu \in F\). We can now see that it was \(\exists\) who chose the colour \(y_S\) of \(\bar{f}\). For \(y_S\) was chosen in the round when \(F\)'s last node, i.e., \(\lambda\) was created. It could only have been chosen by \(\forall\) if he also picked the colour of every edge in \(F\) involving \(\lambda\). This is not so, as the edge \((\lambda, \mu)\) was coloured by \(\exists\) and lies in \(F\). As \(i \in S\), it follows from the definition of \(\forall\)'s strategy that at the time when \(\lambda\) was added, there was already an \(i\)-cone with base \(\bar{f}\), and apex \(N\) say. We claim that \(F \cup \{\alpha\}\) and \(F \cup \{N\}\) are isomorphic over \(F\). For this, note that the only \((n - 1)\)-tuples of either \(F \cup \{\alpha\}\) or \(F \cup \{N\}\) with a yellow colour are in \(F\) (since all others involve a green edge). But this means that \(\exists\) could have taken \(\alpha = N\) in the current round, and not extended the graph. This is contrary to our original assumption, and completes the proof.

atoms for \(n - 1\) tuples in the amalgamation move, like above

4 Blow up and blur

The idea is to blow up a finite structure, replacing each 'colour or atom' by infinitely many, using blurs to represent the resulting term algebra, but the blurs are not enough to blur the structure of the finite structure in the complex algebra. Then, the latter cannot be representable due to a finite-infinite contradiction. This structure can be a finite clique in a graph or a finite relation algebra or a finite cylindric algebra.

We discuss the possibility of obtaining stronger results concerning completions, for example we approach the problem as to whether classes of subneat reducts are closed under completions, and analogous results for infinite dimensions. Partial results in this direction are obtained by Sayed Ahmed, some of which will be mentioned below.

The main idea is to split and blur. Split what? You can split a clique by taking \(\omega\) many disjoint copies of it, you can split a finite relation algebra, by
splitting each atom into \( \omega \) many, you can split a finite cylindric algebra. Generally, the splitting has to do with blowing up a finite structure into infinitely many.

Then blur what? On this split one adds a subset of a set of fixed in advance blurs, usually finite, and then define an infinite atom structure, induced by the properties of the finite structure he originally started with. It is not this atom structure that is blurred but rather the original finite structure. This means that the term algebra built on this new atom structure, that is the algebra generated by the atoms, coincides with a carefully chosen partition of the set of atoms obtained after splitting and bluring up to minimal deviations, so the original finite relation algebra is blurred to the extent that is invisible on this level.

The term algebra will be representable, using all such blurs as colours, But the original algebra structure re-appears in the completion of this term algebra, that is the complex algebra of the atom structure, forcing it to be non representable, due to a finite-infinite discrepancy. However, if the blurs are infinite, then, they will blur also the structure of the small algebra in the complex algebra, and the latter will be representable, inducing a complete representation of the term algebra.

4.1 Main definition and examples

We start by giving rigorous definitions of blowing up and bluring a finite structure. In what follows, by an atom structure, we mean an atom structure of any class of completely additive Boolean algebras.

Let \( N \) be a graph, in our subsequent investigations \( N \) will be finite. But there is no reason to impose restriction on our next definition, which we try keep as general as possible. By induce, we mean ‘define in a natural way’, and we keep natural at this level of ambiguity.

**Definition 4.1.**

1. A splitting of \( N \) is a disjoint union \( N \times I \), where \( I \) is an infinite set.

2. A blur for \( N \) is any set \( J \).

3. An atom structure \( \alpha \) is blown up and blurred if, there exists a subset \( J' \) of a set \( J \) of blurs, possibly empty, such that \( \alpha \) has underlying set \( X = N \times I \times J' \); the latter atom structure is called a blur of \( N \) via \( J \), and is denoted by \( \alpha(N, J) \). Furthermore, every \( j \in J \), induces a non-principal ultrafilter in \( \wp(X) \).

4. An atom structure \( \alpha(N, J) \) reflects \( N \), if \( N \) is faithfully represented in \( \mathfrak{C}m\alpha(N, J) \).
An atom structure $\alpha(N, J)$ is weak if $\mathfrak{T}m_\alpha(N, J)$ is representable.

An atom structure $\alpha(N, J)$ is very weak if $\mathfrak{T}m_\alpha(N, J)$ is not representable.

An atom structure $\alpha(N, J)$ is strong if $\mathfrak{C}m_\alpha(N, J)$ is representable.

We give two examples of weak atom structures. The first construction builds two relativized set algebras based on a certain model that is in turn a Fraisse limit of a class of certain labelled graphs, with the labels coming from $G \cup \{\rho\} \times n$, where $G$ is an arbitrary graph and $\rho$ is a new colour. Under certain conditions on $G$, the first set algebra can be represented on square units, the second, its completion, cannot.

4.2 First example

Let $G$ be a graph. One can define a family of labelled graphs $F$ such that every edge of each graph $\Gamma \in F$, is labelled by a unique label from $G \cup \{\rho\} \times n$, $\rho \notin G$, in a carefully chosen way. The colour of $(\rho, i)$ is defined to be $i$. The colour of $(a, i)$ for $a \in G$ is $i$. $F$ consists of all complete labelled graphs $\Gamma$ (possibly the empty graph) such that for all distinct $x, y, z \in \Gamma$, writing $(a, i) = \Gamma(y, x)$, $(b, j) = \Gamma(y, z)$, $(c, l) = \Gamma(x, z)$, we have:

1. $|\{i, j, l\}| > 1$, or
2. $a, b, c \in G$ and $\{a, b, c\}$ has at least one edge of $G$, or
3. exactly one of $a, b, c$ — say, $a$ — is $\rho$, and $bc$ is an edge of $G$, or
4. two or more of $a, b, c$ are $\rho$.

One forms a labelled graph $M$ which can be viewed as model of a natural signature, namely, the one with relation symbols $R_{(a, i)}$, for each $a \in G \cup \{\rho\}$, $i < n$ and

Then one takes a subset $W \subseteq {}^nM$, by roughly dropping assignments that do not satisfy $(\rho, l)$ for every $l < n$. Formally, $W = \{\bar{a} \in {}^nM : M \models (\bigwedge_{i<j<n, l<n} - (\rho, l)(x_i, x_j))(\bar{a})\}$. Basically, we are throwing away assignments $\bar{a}$ whose edges between two of its elements are labelled by $\rho$, and keeping those whose edges of its elements are not. All this can be done with an arbitrary graph.

Now for particular choices of $G$; for example if $G$ is a certain rainbow graph, or more simply a countable infinite collection of pairwise union of disjoint $N$ cliques with $N \geq n(n-1)/2$, or is the graph whose nodes are the natural numbers, and the edge relation is defined by $iEj$ iff $0 < |i - j| < N$, for same
Here, the choice of $N$ is not haphazard, but it a bound of edges of complete graphs having $n$ nodes.

The relativized set algebras based on $M$, but permitting as assignments satisfying formulas only $n$ sequences in $W$ will be an atomic representable algebra.

This algebra, call it $\mathfrak{A}$, has universe $\{\phi^M : \phi \in L^n\}$ where $\phi^M = \{s \in W : M \models \phi[s]\}$. (This is not representable by its definition because its unit is not a square.) Here $\phi^M$ denotes the permitted assignments satisfying $\phi$ in $M$. Its completion is the relativized set algebra $\mathfrak{C}$ with universe the larger $\{\phi^M : \phi \in L^n_{\infty,\omega}\}$, which turns out not representable. (All logics are taken in the above signature). The isomorphism from $\mathfrak{CmAtA} \to \mathfrak{C}$ is given by $X \mapsto \bigcup X$.

Let us formulate this construction in the context of split and blur. Take the $n$ disjoint copies of $N \times \omega = \mathfrak{G}$. Let $a \in \mathfrak{G} \times n$. Then $a \in N \times \omega \times n$.

Then for every $(a,i)$ where $a \in N \times \omega$, and $i < n$, we have an atom $R^M_{a,i} \in \mathfrak{A}$.

The term algebra of $\mathfrak{A}$ is generated by those.

Hence $\mathfrak{G} \times \omega \times n$ is the atom structure of $\mathfrak{A}$ which can be weakly represented using the $n$ blurs, namely the set $\{(\rho,i) : i < n\}$. The clique $N$ appears on the complex algebra level, forcing a finite $N$ colouring, so that the complex algebra cannot be representable.

We note that if $N$ is infinite, then the complex algebra (which is the completion of the algebra constructed as above) will be representable and so $\mathfrak{A}$, together the term algebra will be completely representable.

### 4.3 The relation algebra

We use the graph $N \times \omega$ of countably many disjoint $N$ cliques. We define a relation algebra atom structure $\alpha(\mathfrak{G})$ of the form $\{(1') \cup (\mathfrak{G} \times n), R_V, \bar{R}, R_r\}$. The only identity atom is $1'$. All atoms are self converse, so $\bar{R} = \{(a,a) : a \text{ an atom}\}$. The colour of an atom $(a,i) \in \mathfrak{G} \times n$ is $i$. The identity $1'$ has no colour. A triple $(a,b,c)$ of atoms in $\alpha(\mathfrak{G})$ is consistent if $R_r; (a,b,c)$ holds. Then the consistent triples are $(a,b,c)$ where

- one of $a, b, c$ is $1'$ and the other two are equal, or
- none of $a, b, c$ is $1'$ and they do not all have the same colour, or
- $a = (a',i), b = (b',i)$ and $c = (c',i)$ for some $i < n$ and $a', b', c' \in \mathfrak{G}$, and there exists at least one graph edge of $G$ in $\{a', b', c'\}$.

$\alpha(\mathfrak{G})$ can be checked to be a relation atom structure. It is exactly the same as that used by Hirsch and Hodkinson, except that we use $n$ colours, instead of just 3. This allows the relation algebra to have an $n$ dimensional cylindric basis and, in fact, the atom structure of $\mathfrak{A}$ is isomorphic (as a cylindric algebra
atom structure) to the atom structure \( \mathcal{M}_n \) of all \( n \)-dimensional basic matrices over the relation algebra atom structure \( \alpha(G) \).

Indeed, for each \( m \in \mathcal{M}_n \), let \( \alpha_n = \bigwedge_{i,j<n} \alpha_{ij} \). Here \( \alpha_{ij} \) is \( x_i = x_j \) if \( m_{ij} = 1 \) and \( R(x_i, x_j) \) otherwise, where \( R = m_{ij} \in L \). Then the map \( (m \mapsto \alpha^W_m)_{m \in \mathcal{M}_n} \) is a well-defined isomorphism of \( n \)-dimensional cylindric algebra atom structures.

It can be shown that the complex algebras of this atom structure is not representable, because its chromatic number is finite; indeed it is exactly \( N \). (This will be demonstrated below.)

But we want more. Is it possible, that the constructed relation algebras not in \( S\text{RaCA}_{n+2} \) which is strictly smaller that \( \text{RRA} \). The idea that could work here, is to use relativized representations. Algebras in \( S\text{RaCA}_{n+2} \) do posses representations that are only locally square. So is the blurring, using \( n \) colours, based on \( N \), namely \( (\rho, i) \ i < n \), enough to prohibit the complex algebra to be representable in a weaker sense, which means that we have to strengthen our conditions, involving the superscrit 2 in the equation with \( N \) and \( n \). We have \( N \geq n(n-1)/2 \) but we need a further combinatorial property relating the triple \((2, N, n)\).

In any event, there is a finite-infinite discrepancy here, as well, no matter what kind of representation we consider, the base has to be infinite. A representation maps the complex algebra into the powerset of a set of ordered pairs, with base \( X \), the latter has to be infinite. At the same time the graph has an \( N \) coloring, and this can be used to partition the complex algebra into \((N \times n) + 1\) blocks.

But this is not enough; the idea in the classical case, works because one member of the partition induced by the finite colouring will be monochromatic, and will satisfy \( (P; P) : P \neq 0 \), which is a contradiction.

The last condition is not guaranteed when we have only relativized representations, because if \( h \) is such a representation, it is not really a faithful one, in the sense that it can happen that there are \( x_0, x_1, x_2 \in X \), and \( (x_0, x_1) \in h(a) \), \( (x_1, x_2) \in h(b) \), \( (x_0, x_2) \in h(c) \), and \( a, b, c \in \text{EmG} \), but \( h((a; b).c) = 0 \) if the node \( x_1 \) witnessing composition, lies outside the \( n \) clique determined by \( x_0, x_2 \). This cannot happen in case of classical representation. Finite clique is the measure of squareness. It will be defined shortly.

But we are also certain that the complex algebra is not in \( S\text{RaCA}_{n+k} \) for some \( k \in \omega \), by the neat embedding theorem for relation algebras, namey, \( \text{RRA} = \bigcap_{k \in \omega} \text{S\text{RaCA}}_{n+k} \).

Now, accordingly, let us keep \( k \) loose, for the time being. We want to determine the least such \( k \). Remember that we required that \( N \geq n(n-1)/2 \), this was necessary to show that permutations of \( \omega \cap \{\rho\} \) induces \( n \) back and forth systems of partial isomorphisms of size less than \( n \) in our limiting labelled graph \( M \), showing that is strongly \( n \) homogeneous, when viewed as a model
for the language $L$. This in turn enabled us to show that the term algebra is representable.

The plan is to go on with the proof and see what other combinatorial properties one should impose on the relationship between $N$, $n$ and $k$ to prohibit even a relativized representation. Obviously one should keep the condition $N \geq n(n - 1)/2$ not to tamper with the first part of the proof.

Let $\mathfrak{A} = \mathfrak{mG}$, and assume that $V \subseteq X \times X$ is a relativized representation. An arbitrary relativized representation, that is if we take any set of ordered pairs, is useless, its not what we want.

We need locally square representations that are like representations only on finite cliques of the base. But what does locally square mean? A clique $C$ of $X$ is a subset of the domain $X$, that can indeed be viewed as a complete graph, in the sense that any two points in it we have $X \models 1(x, y)$, equivalently $(x, y) \in V$, where $V$ is the unit of the relativization. The property of $n + k$ squareness means, then for all cliques $C$ of $X$ with $|C| < n + k$, can always be extended to another clique having at most one more element witnessing composition, so that composition can be preserved in the representation, but only locally. It is easier to build such representations; from the game theoretic point of view because $\forall$ moves are restricted by the size of cliques, which means that the chance that exists provide a node witnessing composition is higher.

Now lets getting starting with our plan.

Assume for contradiction that $\mathfrak{om}(G) \in SR\mathcal{CA}_{n+k}$, and $k \geq 2$. Then $\mathfrak{om}(G)$ has an $n + k - 2$-flat representation $X$ [23] 13.46, which is $n + k - 2$ square [23] 13.10.

In particular, there is a set $X$, $V \subseteq X \times X$ and $g : \mathfrak{om}(G) \to \wp(V)$ such that $h(a)$ ($a \in \mathfrak{om}(G)$) is a binary relation on $X$, and $h$ respects the relation algebra operations. Here $V = \{(x, y) \in X \times X : (x, y) \in h(1)\}$, where 1 is the greatest element of $\mathfrak{om}(G)$. We write $1(x, y)$ for $(x, y) \in h(1)$.

For any $m < \omega$, let $C_{m}(X) = \{a \in mX : Range(a) \text{ is an } m \text{ clique}\}$, then $n + k - 2$ squareness means that that if $a \in C_{n+k-2}(X)$, $r, s \in \mathfrak{mG}$, $i, j, k < n, k \neq i, j$, and $X \models (r; s)(a_{i}, a_{j})$ then there is $b \in C_{n+k-2}(X)$ with $b$ agreeing with $a$ except possibly at $k$ such that $X \models r(b_{i}, b_{k})$ and $X \models s(b_{k}, b_{j})$.

This is the definition. But it is not hard to show that this is equivalent to the simpler condition that for all cliques $C$ of $X$ with $|C| < n + k$, all $x, y \in C$ and $a, b \in \mathfrak{om}(G)$, $X \models (a; b)(x, y)$ there exists $z \in X$ such that $C \cup \{z\}$ is a clique and $X \models a(x, z) \land b(z, y)$.

Now $G$ has a finite colouring using $N$ colours. Indeed, the map $f : N \times \omega$ defined by $f(l, i) = l$ is a finite colouring using $N$ colours. For $Y \subseteq N \times \omega$ and $l < n$ define $(Y, k) = \{(a, i, l) : (a, i) \in Y\}$, regarded as a subset of $\mathfrak{mG}$.

The nodes of $N \times \omega$ can be partitioned into sets $\{C_{j} : j < n\}$ such that there are no edges within $C_{j}$. Let $J = \{1', (C_{j}, k) : j < n, k < n\}$ Then clearly, $\sum J = 1$ in $\mathfrak{om}(G)$, so that $J$ is partition of $\mathfrak{om}(G)$ into $N \times n + 1$ blocks.
As \( J \) is finite, we have for any \( x, y \in X \) there is a \( P \in J \) with \((x, y) \in h(P)\). Since \( \mathcal{C}m\alpha(G) \) is infinite then \( X \) is infinite. Ramsy’s theorem applies in this context, to allow us to infer, that there are distinct \( x_i \in X \) \((i < \omega)\), \( J \subseteq \omega \times \omega \) infinite and \( P \in J \) such that \((x_i, x_j) \in h(P)\) for \((i, j) \in J, i \neq j\). Then \( P \neq 1'\).

The condition we need on \( k \), is that if \((x_0, x_1) \in h(a), (x_1, x_2) \in h(b) \) and \((x_0, x_2) \in h(c)\), then \( a; b; c \neq 0\).

So this prompts:

Find a combinatorial relation between \( n, k, N \) with \( N \geq n(n - 1)/2 \) that forces \((P; P) \cdot P \neq 0\). What is the least such \( k \)? This is formulated for any \( P \), but maybe the condition would also force Ramsey’s theorem to give the right block.

A non-zero element \( a \) of \( \mathcal{C}m\alpha(G) \) is monochromatic, if \( a \leq 1' \), or \( a \leq (\Gamma, s) \) for some \( s < n \). Now \( P \) is monochromatic, and the \( C_j \) s are independent, it follows also from the definition of \( \alpha \) that \((P; P) \cdot P = 0\).

\( \mathcal{C}m\alpha(G) \) is not in \( SR\alpha CA_{n+m} \), and from this, it will follow that \( \mathcal{C}m\mathcal{M}_n \notin SR\alpha n\mathcal{C}A_{n+m} \) for al \( m \geq k \). Showing that the latter two cases are not closed under completions.

For a relation algebra \( R \) having an \( n \) dimensional cylindric basis, let \( \text{Mat}_n R \) be the term cylindric algebra of dimension \( n \) generated by the basic matrices.

**Theorem 4.2.** Let \( G \) be a graph that is a disjoint union of cliques having size \( n \). Then there is a strongly \( n \) homogenous labelled graph \( M \), every edge is labelled by an element from \( G \cup \{\rho\} \times n, W \subseteq ^n M \), such that the set algebra based on \( W \) is an atomic \( \mathfrak{A} \in \text{RCA}_n \), and there is an atomic \( R \in \text{RRA} \) the latter with an \( n \) dimensional cylindric basis, such that \( \mathfrak{A} \cong \text{Mat}_n R \), and the completions of \( \mathfrak{A} \) and \( R \) are not representable, hence they are not completely representable.

### 4.4 Second Example

Here we turn to our second split and blur construction. It is a simplified version of the proof of Andréka and Németi, except that for a set of blinds \( J \), they defined infinitely many ternary relations on \( \omega \) with suffixes from \( J \), to synchronize the composition operation. This was necessary to show that the required algebras are generated by a single element; here we use one uniform relation, and we sacrifice with this part of the result, which is worthwhile, due to the reduction of the complexity of the proof. We think that our simplified version captures the essence of the blow up and blur construction of Andréka and Németi.

Let \( I \) and \( J \) be sets, for the time being assume they are finite. We will define two partitions \((H^P : P \in I)\) and \((E^W : W \in J)\) of a given infinite set \( H \), using atoms from a finite relation algebra for the first superscripts, and ”blurs” (literally) for the second superscript.
The blurs do two things. They are just enough to distort the structure of $M$ in the term algebra, but not in its completions, but at the same time they are colors that are necessary for representing the term algebra.

Indeed, we use the first partition to show that the complex algebra of our atom structure is not representable, while we use the second to show that the term algebra is representable.

Let us start getting more concrete. Let $I$ be a finite set with $|I| \geq 6$. Let $J$ be the set of all 2 element subsets of $I$, and let

$$H = \{ a_{iP,W} : i \in \omega, P \in I, W \in J, P \in W \}. $$

In a minute we will get even more concrete by choosing a specific finite relation relation $M$ with certain properties, namely, it cannot be represented on infinite sets. The atoms of $M$ will be $I$. This algebra is finite, so it cannot do what we want. A completion of a finite algebra is itself.

The index $i$ here says that we will replace each atom of this relation algebra by infinitely many atoms, that will define an atom structure of a new infinite relation algebra, the desired algebra. (This is an instance of a technique called splitting, which involves splitting an atom into smaller atoms. Invented by Andreka, it is very useful in proving non representability results).

The structure of $M$ will be blown up by splitting the atoms, then 'blurred' in the term algebra, but it will not be blurred in the completion of the term algebra. More precisely, $M$ will be a subalgebra of the completion, but it may (and will not be) a subalgebra of the term algebra.

The best way to visualize the partitions we will define is to imagine that the atoms of the new algebras, form a partition of an infinite rectangle with finite base $I$ and side $\omega$ reflecting the infinite splitting of $I$. Or to view it as an infinite tenary matrix, with each entry indexed by $(i, P, W) \in \omega \times I \times J$, $P \in W$.

We now define two finite partitions of the rectangle, namely $H$. For $P \in I$, let

$$H^P = \{ a_{iP,W} : i \in \omega, W \in J, P \in W \}. $$

The finite relation algebra will be embedable in the completion via $P \mapsto H^P$, no distortion involved. $M$ will still be up there on the global level.

The $J$s are the blurs, for $W \in J$, let

$$E^W = \{ a_{iP,W} : i \in \omega, P \in W \}. $$

The singletons will generate this partition up to a 'finite blurring'. That is the term algebra will consist of all those $X$ such that $X$ intersects $E^W$ finitely or cofinitely. For each $W \in J$, we have $W \subseteq I$, and so $E^W$ will be the subrectangle of $H$ on the base $W$. 

44
To implement our plan we further need a tenary relation, which synchronizes composition; it will tell us which rows in the rectangle, allow composition like \( M \).

For \( i, j, k \in \omega \) \( e(i, j, k) \) abbreviates that \( i, j, k \) are \textit{evenly distributed}, i.e.

\[
e(i, j, k) \text{ iff } (\exists p, q, r)\{p, q, r\} = \{i, j, k\}, r - q = q - p
\]

For example 3,5,7 are evenly distributed, but 3,5,8 are not. All atoms are self-converse. This always makes life easier. We define the consistent triples as follows (Involving identity are as usual \( (a, b, Id) : a \neq b \)).

Let \( i, j, k \in \omega \), \( P, Q, R \in I \) and \( S, Z, W \in J \) such that \( P \in S, Q \in Z \) and \( R \in W \). Then the triple \( (a_i^{PS}, a_j^{QZ}, a_k^{RW}) \) is consistent iff either 

(i) \( S \cap Z \cap W = \emptyset \), or

(ii) \( e(i, j, k) \& P \leq Q; R \).

The second says that if \( i, j, k \) are \( e \) related then the composition of \( P, Q \) and \( R \) existing on those three rows, is defined like \( M \).

Let \( F \) denote this atom structure, \( F = H \cup \{Id\} \)

Now, as promised, we choose a (finite) relation algebra \( M \) with atoms \( I \cup \{1d\} \) such that for all \( P, Q \in I \), \( P \neq Q \) we have

\[
P; P = \{Q \in I : Q \neq P\} \cup \{Id\} \text{ and } P; Q = H
\]

Such an \( M \) exists It is also known that \( M \), if representable, can be only represented on finite sets. Now using the above partitions we show:

**Theorem 4.3.**

(1) \( \mathfrak{CmF} \) is a relation algebra that is not representable.

(2) \( \mathcal{R} \) the term algebra over \( F \) is representable.

**Proof.**

(1) Non representability uses the first partition of \( H \). Note that \( ; \) is defined on \( \mathfrak{Cm(F)} \) so that

\[
H^P; H^Q = \bigcup\{H^Z : Z \leq P; Q \in M\}.
\]

So \( M \) is isomorphic to a subalgebra of \( \mathfrak{CmF} \). But \( \mathfrak{CmF} \) can only be represented on infinite sets, while \( \mathfrak{M} \) only on finite ones, hence we are done.

(2) The representability of the term algebra uses the second partition. The blow up and blur algebra is \( \mathcal{R} = \{X \subseteq F : X \cap E^W \in Cof(E^W), \forall W \in J\} \). For any \( a \in F \) and \( W \in J \), let

\[
U^a = \{X \in R : a \in X\}
\]
and

\[ U^W = \{ X \in R : |Z \cap E^W| \geq \omega \} \]

Let

\[ \text{Uf} = \{ U^a : a \in F \} \cup \{ U^W : W \in J : |E^W| \geq \omega \}. \]

\text{Uf} denotes the set of ultrafilters of \( \mathcal{R} \), that include at least one non-principal ultrafilter, that is an element of the form \( U^W \).

Let \( F, G, K \) be boolean ultrafilters in a relation algebra and let \( ; \) denote composition. Then

\[ F; G = \{ X; Y : X \in F, Y \in G \}. \]

The triple \( (F, G, K) \) is \emph{consistent} if the following holds:

\[ F; G \subseteq K, F \subseteq G \text{ and } G \subseteq F. \]

So to represent \( \mathcal{R} \) using \( \text{Uf} \) as colours, we want to achieve (i) -(iii) below:

(i) \((U^a, U^b, U^W)\) is consistent whenever \( a, b \in H \) and \( a; b \in U^W \).

(ii) \((F, G, K)\) is consistent whenever at least two of \( F, G, K \) are non-principal and \( F, G, K \in \text{Uf} - \{ U^{Id} \} \).

(iii) For any \( a, b, c, d \in H \), there is \( W \in J' \) such that \( a; b \cap c; d \in U^W \).

Let us see how to represent this algebra. We call \((G, l)\) a \emph{consistent coloured graph} if \( G \) is a set, \( l : G \times G \rightarrow \text{Uf} \) such that for all \( x, y, z \in G \), the following hold:

(i) \( l(x, y) = U^{Id} \) iff \( x = y \),

(ii) \( l(x, y) = l(y, x) \)

(iii) The triple \((l(x, y), l(x, z), l(y, z))\) is consistent.

We say that a consistent coloured graph \((G, l)\) is complete if for all \( x, y \in G \), and \( F, K \in \text{Uf} \), whenever \((l(x, y), F, K)\) is consistent, then there is a node \( z \) such that \( l(z, x) = F \) and \( l(z, y) = K \). We will build a complete consistent graph step-by-step. So assume (inductively) that \((G, l)\) is a consistent coloured graph and \((l(x, y), F, K)\) is a consistent triple. We shall extend \((G, l)\) with a new point \( z \) such that \((l(x, y), l(z, x), l(z, y)) = (l(x, y), G, K)\). Let \( z \notin G \). We define \( l(z, p) \) for \( p \in G \) as follows:

\[ l(z, x) = F \]

\[ l(z, y) = K, \text{ and if } p \in G \setminus \{x, y\}, \text{ then} \]

46
\[
l(z, p) = U^W \text{ for some } W \in J' \text{ such that both (}U^W, F, l(x, p)) \text{ and } (U^W, K, l(y, p)) \text{ are consistent}.
\]

Such a \( W \) exists by our assumptions (i)-(iii). Conditions (i)-(ii) guarantee that this extension is again a consistent coloured graph.

We now show that any non-empty complete coloured graph \((G, l)\) gives a representation for \( R \). For any \( X \in R \) define

\[
\text{rep}(X) = \{(u, v) \in G \times G \mid X \in l(u, v)\}
\]

We show that

\[
\text{rep} : R \rightarrow R(G)
\]

is an embedding. \( \text{rep} \) is a boolean homomorphism because all the labels are ultrafilters.

\[
\text{rep}(\text{Id}) = \{(u, u) \mid u \in G\},
\]

and for all \( X \in R \),

\[
\text{rep}(X)^{-1} = \text{rep}(X).
\]

The latter follows from the first condition in the definition of a consistent coloured graph. From the second condition in the definition of a consistent coloured graph, we have:

\[
\text{rep}(X); \text{rep}(Y) \subseteq \text{rep}(X; Y).
\]

Indeed, let \((u, v) \in \text{rep}(X), (v, w) \in \text{rep}(Y)\) i.e. \( X \in l(u, v), Y \in l(v, w) \). Since \((l(u, v), l(v, w), l(u, w))\) is consistent, then \( X; Y \in l(u, w) \), i.e. \((u, w) \in \text{rep}(X; Y)\). On the other hand, since \((G, l)\) is complete and because (i)-(ii) hold, we have:

\[
\text{rep}(X; Y) \subseteq \text{rep}(X); \text{rep}(Y),
\]

because \((G, l)\) is complete and because (i) and (ii) hold. Indeed, let \((u, v) \in \text{rep}(X; Y)\). Then \( X; Y \in l(u, v) \). We show that there are \( F, K \in \text{Uf} \) such that

\[
X \in F, Y \in K \text{ and } (l(u, v), F, K) \text{ is consistent}.
\]

We distinguish between two cases:

**Case 1.** \( l(u, v) = U^a \) for some \( a \in F \). By \( X; Y \in U^a \) we have \( a \in X; Y \). Then there are \( b \in X, c \in Y \) with \( a \leq b; c \). Then \((U^a, U^b, U^c)\) is consistent.

**Case 2.** \( l(u, v) = U^W \) for some \( W \in J' \). Then \(|X; Y \cap E^W| \geq \omega\) by \( X; Y \in U^W \). Now if both \( X \) and \( Y \) are finite, then there are \( a \in X, b \in Y \)
with \( |a;b \cap E^W| \geq \omega \). Then \((U^W, U^a, U^b)\) is consistent by (i). Assume that one of \(X, Y\), say \(X\) is infinite. Let \(S \in J'\) such that \(|X \cap E^S| \geq \omega\) and let \(a \in Y\) be arbitrary. Then \((U^W, U^S, U^a)\) is consistent by (ii) and \(X \in U^S, Y \in U^a\).

Finally, \(\text{rep}\) is one to one because \(\text{rep}(a) \neq \emptyset\) for all \(a \in A\). Indeed \((u, v) \in \text{rep}(\text{Id})\) for any \(u \in G\). Let \(a \in H\). Then \((U^{\text{Id}}, U^a, U^a)\) is consistent, so there is a \(v \in G\) with \(l(u, v) = U^a\). Then \((u, v) \in \text{rep}(a)\).

\[\square\]

4.5 The cylindric algebra

We define the atom structure like we did before. The basic matrices of the atom structure above form a 3 dimensional cylindric algebra. We want an \(n\) dimensional one. Our previous construction of the atom structure satisfied (*) satisfies \((\forall a_1 \ldots a_3 b_1 \ldots b_3 \in I)(\exists W \in J)(a_1; b_1) \cap \ldots \cap (a_3; b_3) \in U^W.\)

We strengthen this condition to (**)

\[(\forall a_1 \ldots a_n b_1 \ldots b_n \in I)(\exists W \in J)W \cap (a_1; b_1) \cap \ldots \cap (a_n; b_n) \neq \emptyset.\]

(This is referred to in [9] as an \(n\) complex blur for \(M\), our first construction was a 3 complex blur).

This condition will entail that the set of all \(n\) by \(n\) matrices is a cylindric basis on the new relation algebra \(R_n\) defined as before, with minor modifications.

Now \(R_n\) is defined by taking \(I\) be a finite set with \(|I| \geq 2n + 2\), \(J\) be the set of all 2 (See the proof) element subsets of \(I\). And then define everything as before. The resulting cylindric algebra is also called the blow up and blur cylindric algebra of dimension \(n\), which actually blows up and blurs the \(n\) dimensional finite cylindric algebra consisting of \(n\) basic matrices of \(M\), which is representable, so such an algebra exists for every \(n\).

The new condition (**) guarantess the amalgamation property of matrices (corresponding to commutativity of cylindrifiers) which is the essential property of basis.

We know that the term algebra is a subneat reduct of an algebra in \(\omega\) extra dimensions. But we need a final tick so that the the term cylindric algebra is a full neat reduct. This requires a yet another strengthenig of (**) by replacing \(\exists\) by \(\forall\).

Now under this stronger condition, let \(\mathcal{B}_n\) be the set of basic matrices of our blown up and blurred \(R_n\). In the first order language \(L\) of \((\omega, <)\), which has quantifier elimination, diagrams are defined for each \(K \subseteq n\) and \(\phi \in L\), via maps \(e: K \times K \to R_n\). For an atom let \(v(a)\) be its ith co-ordinate, or its \(i\)th level in the rectangle.
The pair $e$ and $\phi$ defines an element in $Cm\mathcal{B}_n$, called a diagram, that is a set of matrices, defined by

$$M(e, \phi) = \{ m \in B_n, i, j \in K, m_{ij} \leq \phi(e_{ij}, v(m_{ij})) \}.$$  

A normal diagram is one whose entries are either atoms or finitely many blurs (by (J)), that is elements of the form $E^W$, in addition to the condition that $\phi$ implies $\phi_e$. Any diagram can be approximated by normal ones; and actually it is a finite union of normal diagrams. The term algebra turns, denoted by $\mathcal{B}_n(M, J, e)$, consists of those diagrams, and finally we get that that for $t < n$

$$\mathfrak{N}_t\mathcal{B}_n(M, J, e) \cong \mathcal{B}_t(M, J, e).$$

Here actually we are also blowing and bluring the finite dimensional cylindric algebra atom consisting of matrices on $M$, we blow up every $n$ dimensional matrix to infinitely many, where each entry is either an atom of the relation algebra or a blur; these are exactly the diagrams.

### 4.6 The analogy, first informaly, then formally in a map

This construction actually has a lot of affinity with the first model theoretic construction. First they both prove the same thing; the Andreka et all construction proves that in addition the term algebra is a $k$ neat reduct. Now here we are comparing a relation algebra construction with a cylindric algebra one, but the analogy is worthwhile pointing out.

Replace the clique $N$ in Hodkinson’s construction by $M$, in this case the term algebra, $\mathcal{R}$ is also obtained by replacing every atom by infinitely many ones, and $M$ appears on the global level as a subalgebra of the complex algebra.

To this construction we can also associate a finite graph with finite chromatic number, namely the complete graph on $M$. The blurs are the colours, that correspond to the colours $(\rho, i)$ in Sayed Ahmed’s construction.

In the first case the splitting of the clique $N$, uses just one index, in the second we use two indices, the atoms of $M$ and the blurs. The first partition replaces the use of Ramseys theorem, the second partition, is a division of the whole splitting into finitely many rectangles, one for each blur. The homogeneous model $M$ in the second construction correspond to the second partition, in the sense that it is not the base of the representable term algebra, but $W$ is, which is basically obtained by removing the blurs, that are the same time essential in representing it, $W$ thus corresponds to the term algebra of co-finite finite intersections with the second partition, which in turn is representable.

In short, we start up with a finite structure, blow it up, on the term algebra level, using blurs to represent it, but it will not be blurred enough to disappear on the complex algebra level, forcing the latter to be non-representable (due to
incompatibility of “a finiteness condition”) with the inevitability of representing the complex algebra on an infinite set.

More formally, we define a function that maps the ingredients of the first construction to that of the second:

\[ N \mapsto M \]

\[ N \times \omega \times n \mapsto \omega \times P \times J \]

In the former case \( J' = \emptyset \), the blurs do not appear on this level, in the second splitting all blurs are used.

\[ \{(\rho, i) : i < n\} \mapsto \{W : W \in J\}. \]

Here in the first case \( n \) blurs are needed to represent the new term algebra. In the latter it is the number of two elements subsets of \( I \). For \( \phi \in L^+ \), let \( \phi^M = \{s \in {}^n M : M \models \phi[s]\} \). Here we are not relativizing semantics, in particular, tuples whose edges can be labelled \( \rho \) are there, but then the representable algebra is \( \{\phi^M \cap W : \phi \in L^+\} \). In the second case we have a finite partition of the rectangle \( H \), via \( (E^W : W \in J) \).

\[ \{\phi^M \cap W : \phi \in L^+\} \mapsto \{X \subseteq F : X \cap E^W \in \text{Cof}(E^W) : \text{for all } W \in J\}. \]

\[ \mathfrak{A} \mapsto \mathcal{R} \]

\[ \text{CmAt}\mathfrak{A} \rightarrow \text{CmAt}\mathcal{R} \]

Here we include more examples.

**Example 4.4.** Let \( l \in \omega, l \geq 2 \), and let \( \mu \) be a non-zero cardinal. Let \( I \) be a finite set, \( |I| \geq 3l \). Let

\[ J = \{(X, n) : X \subseteq I, |X| = l, n < \mu\}. \]

Let \( H \) be as before, i.e.

\[ H = \{a_i^{P,W} : i \in \omega, P \in I, W \in J\}. \]

Define \( (a_i^{P.S,P}, a_j^{Q.Z,q}, a_k^{R.W,r}) \) is consistent if

\[ S \cap Z \cap W = \emptyset \text{ or } c(i, j, k) \text{ and } |\{P, Q, R\}| \neq 1. \]

Pending on \( l \) and \( \mu \), let us call these atom structures \( \mathcal{F}(l, \mu) \). Then our first example in is just \( \mathcal{F}(2, 1) \).

If \( \mu \geq \omega \), then \( J \) as defined above would be infinite, and \( \text{Uf} \) will be a proper subset of the ultrafilters. It is not difficult to show that if \( l \geq \omega \) (and we relax the condition that \( I \) be finite), then \( \text{Cm}\mathcal{F}(l, \mu) \) is completely representable,
and if $l < \omega$ then $\mathfrak{Cm}\mathcal{F}(l, \mu)$ is not representable. In the former case we have infinitely many colours, so that the chromatic number of the graph is infinite, while in the second case the chromatic number is finite. Informally, if the blurs get arbitrarily large, then in the limit, the resulting algebra will be completely representable, and so its complex algebra will be representable. If we take a sequence of blurs, each finite, but increasing in size we get a sequence of algebras that are not completely representable, and the sequence of their complex algebras will not be representable. The limit of the former, will be completely representable (with an infinite set of blurs); its completion will be the limit of the second sequence of non representable algebras, will be representable. Either construction can be used to achieve this.

This phenomena has many reincarnations in the literature. One is the following: It is is nothing more than Monk’s classical non finite axiomatizability result; it gives a sequence of non representable algebras whose ultraproduct is completely representable.

Using such examples, we now prove:

**Corollary 4.5.**  
1. The classes $\text{RRA}$ is not finitely axiomatizable.  
2. The elementary closure of the class $\text{CRA}$ is not finitely axiomatizable.

**Proof.** For the second we use the second construction. Let $D$ be a non-trivial ultraproduct of the atom structures $\mathcal{F}(i, 1), i \in \omega$. Then $\mathfrak{Cm}D$ is completely representable. Thus $\mathfrak{Cm}\mathcal{F}(i, 1)$ are $\text{RRA}$’s without a complete representation while their ultraproduct has a complete representation.

Also $\mathfrak{Cm}\mathcal{F}(i, 1), i \in \omega$ are non representable with a completely representable ultraproduct. This yields the desired result.

We prove the cylindric case. Take $G_i$ to be the disjoint union of cliques of size $n(n - 1)/2 + i$. Let $\alpha_i$ be the corresponding atom astructure of $\mathfrak{A}_i$, as constructed above. Then $\mathfrak{Cm}\alpha_i$ is not representable, but $\prod_{i \in \omega} \mathfrak{Cm}\alpha_i = \mathfrak{Cm}(\prod_{i \in \omega} \mathfrak{A}_i)$. Then the latter is based on the disjoint union of the cliques which is arbitrarily large, hence is representable.

The first construction also works, by using relation algebra atom structures with $n$ dimensional cylindric bases, this will yield the analogous result for cylindric algebras.

The second re-incarnation is due to Hirsch and Hodkinson, it also works for relation and cylindric algebras, and this is the essence. For each graph $\Gamma$, they associate a cylindric algebra atom structure of dimension $n$, $\mathfrak{M}(\Gamma)$ such that $\mathfrak{Cm}\mathfrak{M}(\Gamma)$ is representable if and only if the chromatic number of $\Gamma$, in symbols $\chi(\Gamma)$, which is the least number of colours needed, $\chi(\Gamma)$ is infinite. Using a famous theorem of Erdos, they construct a sequence $\Gamma_r$ with infinite chromatic number and finite girth, whose limit is just 2 colourable, they show
that the class of strongly representable algebras is not elementary. Notice that this is a reverse process of Monk-like constructions, given above, which gives a sequence of graphs of finite chromatic number whose limit (ultaproduct) has infinite chromatic number.

And indeed, the construction also, is a reverse to Monk’s construction in the following sense: Some statement fail in $\mathcal{A}$ iff $\mathcal{A}\mathcal{T}\mathcal{A}$ be partitioned into finitely many $\mathcal{A}$-definable sets with certain ‘bad’ properties. Call this a bad partition. A bad partition of a graph is a finite colouring. So Monks result finds a sequence of badly partitioned atom structures, converging to one that is not. As we did above, this boils down, to finding graphs of finite chromatic numbers $\Gamma_i$, having an ultraproduct $\Gamma$ with infinite chromatic number.

An atom structure is strongly representable iff it has no bad partition using any sets at all. So, here, the idea find atom structures, with no bad partitions, with an ultraproduct that does have a bad partition. From a graph Hirsch and Hodkinson constructed an atom structure that is strongly representable iff the graph has no finite colouring. So the problem that remains is to find a sequence of graphs with no finite colouring, with an ultraproduct that does have a finite colouring, that is, graphs of infinite chromatic numbers, having an ultraproduct with finite chromatic number.

It is not obvious, a priori, that such graphs actually exist. And here is where Erdos’ methods offer solace. Indeed, graphs like this can be found using the probabilistic methods of Erdos, for those methods render finite graphs of arbitrarily large chromatic number and girth. By taking disjoint unions, one can get graphs of infinite chromatic number (no bad partitions) and arbitrarily large girth. A non principal ultraproduct of these has no cycles, so has chromatic number 2 (bad partition).

Monk proved non finite axiomatizability of the representable algebras using a lifting argument. Here we do the same thing with anti-Monk algebras, in the hope of getting a weakly representable $\omega$ dimensional atom structure that is not strongly representable.

Let $\Gamma_r$ be a sequence of Erdos graphs. Let $\mathfrak{A}_n = \mathfrak{A}(n, \Delta_n)$ be the representable atomic algebra of dimension $n$, based on $\Delta_n$, the disjoint union of $\Gamma_r$, $r > n$. Then we know that $\mathcal{A}\mathcal{T}\mathcal{A}_n$ is a strongly representable atom structure of dimension $n$. Let $\mathfrak{A}_n^+$ be an $\omega$ dimensional algebra such that $\mathcal{R}\mathcal{D}_n\mathfrak{A}_n^+ = \mathfrak{A}_n$; we can assume that for $n < m$, there is an $x \in \mathfrak{A}_m$, such that $\mathfrak{A}_n \cong \mathcal{R}\mathcal{D}_l\mathfrak{A}_m$. Now let $\mathfrak{A} = \prod_{n \in F} \mathfrak{A}_n^+$, be any non trivial ultraproduct of the $\mathfrak{A}_n^+$s, then $\mathfrak{A}$ is an atomic $\mathcal{R}\mathcal{C}_\omega$ that has an atom structure that is based on the graph $\Delta = \prod \Delta_n$, with chromatic number 2, hence it is only weakly representable.
References

[1] Andréka, Ferenczi, Németi (Editors) Cylindric-like Algebras and Algebraic Logic, Andréka, Ferenczi, Németi (Editors) Bolyai Society Mathematical Studies p.205-222 (2013).

[2] L. Henkin, J.D. Monk and A.Tarski, Cylindric Algebras Part I. North Holland, 1971.

[3] L. Henkin, J.D. Monk and A.Tarski, Cylindric Algebras Part II. North Holland, 1985.

[4] R. Hirsch, Relation algebra reducts of cylindric algebras and complete representations, The Journal of Symbolic Logic, Vol. 72, Number 2, June 2007.

[5] T. Sayed Ahmed, Martin’s axiom, Omiting types and Complete representations in algebraic logic Studai Logica, 72 (2002), pp. 285-309.

[6] T. Sayed Ahmed, The class of neat reducts of polyadic algebras is not elementary, Fundementa Mathematica, 172 (2002), pp. 61-81.

[7] T. Sayed Ahmed, A model-theoretic solution to a problem of Tarski, Math Logic Quarterly, 48 (2002), pp. 343-355.

[8] T. Sayed Ahmed, Neat embedding is not sufficient for complete representability, Bulletin of the Section of Logic, Volume 36:1/2 (2007), pp. 21-27.

[9] T. Sayed Ahmed, Completions, Complete representations and Omiting types in. In: Cylindric-like Algebras and Algebraic Logic. Editors: H. Andréka, M. Ferenczi, I. Németi, Bolyai Society Mathematical Studies, pp.205-222 (2013).

[10] Hirsch and Hodkinson Completion and complete representations in algebraic logic in [17]

[11] T. Sayed Ahmed The class of neat reducts is not elementary Logic Journal of IGPL, 2002.

[12] Nemeti The class of neat reducts is not closed under forming subalgebras Notre dame Journal of Formal Logic.

[13] Sayed Ahmed Neat reducts and neat embedings in cylindric algebras in [17]

[14] Hirsch R. Relation algebra reducts of cylindric algebras and complete representations. Journal of Symbolic Logic, 72(2) (2007), p.673-703.
[15] Sayed Ahmed, A note on neat reducts, Studia Logica 85 (2007) 139-151

[16] Sayed Ahmed, The class of quasipolyadic algebras of infinite dimensions is not axiomatizable. Mathematical Logic quarterly.

[17] Andréka, Ferenczi, Németi (Editors) Cylindric-like Algebras and Algebraic Logic, Andréka, Ferenczi, Németi (Editors) Bolyai Society Mathematical Studies p.205-222 (2013).

[18]

[19] L. Henkin, J.D. Monk and A. Tarski, Cylindric Algebras Part I. North Holland, 1971.

[20] L. Henkin, J.D. Monk and A. Tarski, Cylindric Algebras Part II. North Holland, 1985.

[21] R. Hirsch, Relation algebra reducts of cylindric algebras and complete representations, The Journal of Symbolic Logic, Vol. 72, Number 2, June 2007.

[22] Hirsch R. Relation algebra reducts of cylindric algebras and complete representations. Journal of Symbolic Logic, 72(2) (2007), p.673-703.

[23] Hirsch and Hodkinson complete representations in algebraic logic Journal of Symbolic Logic (1996).

[24] Hirsch and Hodkison Completions and complete representations in algebraic logic In [17].

[25] Hirsch R., Hodkinson I., Relation algebras by games. Studies in Logic and the Foundations of Mathematics. Volume 147. (2002)

[26] Sayed Ahmed, T., Neat reducts and neat embeddings in cylindric algebras In 'Cylindric-like algebras and Algebraic Logic', Bolyai Society of Mathematical Studies, vol 22, Editors: Andréka H, Ferenczi, M. and Németi. I (2013).

[27] H. Andréka, I. Németi, T. Sayed Ahmed, Omitting types for finite variable fragments and complete representations of algebras, Journal of Symbolic Logic, 73(1) (2008), p.65-89.

[28] L. Henkin, J.D. Monk and A. Tarski, Cylindric Algebras Part I. North Holland, 1971.

[29] L. Henkin, J.D. Monk and A. Tarski, Cylindric Algebras Part II. North Holland, 1985.
[30] R. Hirsch, *Relation algebra reducts of cylindric algebras and complete representations*, The Journal of Symbolic Logic, Vol. 72, Number 2, June 2007.

[31] I. Hodkinson, *A construction of cylindric and polyadic algebras from atomic relation algebras*, Algebra Universalis, 68 (2012), pp. 25