On sampling discretization in $L_2$

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Abstract

A sampling discretization theorem for the square norm of functions from a finite dimensional subspace satisfying Nikol’skii’s inequality is proved. The obtained upper bound on the number of sampling points is of the order of the dimension of the subspace.

Keywords and phrases: real and complex sampling discretization, submatrices of orthogonal matrices.

1 Introduction

Let $\Omega$ be a compact subset of $\mathbb{R}^d$ with the probability measure $\mu$. By $L_q$, $1 \leq q < \infty$, norm we understand

$$\|f\|_q := \|f\|_{L_q(\Omega, \mu)} := \left(\int_{\Omega} |f|^q d\mu\right)^{1/q}.$$ 

By discretization of the $L_q$-norm we understand a replacement of the measure $\mu$ by a discrete measure $\mu_m$ with support on a set $\xi = \{\xi_j\}_{j=1}^m \subset \Omega$. This means that integration with respect to measure $\mu$ is replaced by an appropriate cubature formula. Thus, integration is replaced by evaluation of a function $f$ at a finite set $\xi$ of points. This is why this way of discretization is called sampling discretization. Discretization is a very important step in making a

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continuous problem computationally feasible. The reader can find a corresponding discussion in a recent survey [3]. The first results in sampling discretization were obtained by Marcinkiewicz and by Marcinkiewicz-Zygmund (see [19]) for discretization of the $L_q$-norms of the univariate trigonometric polynomials in 1930s. Therefore, we also call sampling discretization the Marcinkiewicz-type theorems (see [17], [18], [3]). Recently, a substantial progress in sampling discretization has been made in [17], [18], [9], [3], [4], [5], [10]. In this paper we present results on sampling discretization in the case $q = 2$. We proceed to a detailed discussion.

**Condition E.** We say that the orthonormal system $\{u_i(x)\}_{i=1}^N$ defined on $\Omega$ satisfies Condition E with a constant $t$ if for all $x \in \Omega$

$$\sum_{i=1}^N |u_i(x)|^2 \leq Nt^2.$$  

We begin with the formulation of Rudelson’s result from [15]. In the paper [15] it is formulated in terms of submatrices of an orthogonal matrix. We reformulate it in our notations. Note that Theorem 1.1 can be derived from the original result of Rudelson in the same way as we derive Theorem 3.1 from Lemma 2.2 (see Section 3 below).

**Theorem 1.1.** Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \ldots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real orthonormal on $\Omega_M$ system satisfying Condition E.

Then for every $\epsilon > 0$ there exists a set $J \subset \{1, \ldots, M\}$ of indices with cardinality

$$m := |J| \leq C\frac{t^2}{\epsilon^2}N \log \frac{Nt^2}{\epsilon^2}$$  

(1.1)

such that for any $f = \sum_{i=1}^N c_i u_i$ we have

$$(1 - \epsilon)^2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j \in J} f(x^j)^2 \leq (1 + \epsilon)^2 \|f\|_2^2.$$  

In [18] it was demonstrated how the Bernstein-type concentration inequalities for random matrices can be used to prove an analog of the above Rudelson’s result for a general $\Omega$. The proof in [18] is based on a different idea than the Rudelson’s proof. Here is the corresponding result from [18] (see Theorem 6.6 there).
Theorem 1.2. Let \( \{u_i(x)\}_{i=1}^N \) be a real orthonormal in \( L_2(\Omega, \mu) \) system satisfying Condition E. Then for every \( \epsilon > 0 \) there exists a set \( \{\xi_j\}_{j=1}^m \subset \Omega \) with

\[
m \leq C \frac{t^2}{\epsilon^2} N \log N
\]
such that for any \( f = \sum_{i=1}^N c_i u_i \) we have

\[
(1 - \epsilon) \|f\|_2^2 \leq \frac{1}{m} \sum_{j=1}^m f(\xi_j)^2 \leq (1 + \epsilon) \|f\|_2^2.
\]

We note that Theorem 1.2 is more general and slightly stronger than Theorem 1.1. Theorem 1.2 provides the Marcinkiewicz-type discretization theorem for a general domain \( \Omega \) instead of a discrete set \( \Omega_M \). Also, in Theorem 1.2 we have an extra factor \( \log N \) instead of \( \log N t^2 / \epsilon^2 \) in (1.1). The necessary condition for the Marcinkiewicz-type discretization theorem to hold for an \( N \)-dimensional subspace \( X_N \) is \( m \geq N \). Both Theorem 1.1 and Theorem 1.2 provide sufficient conditions on \( m \) (the upper bound) for existence of a good set of cardinality \( m \) for sampling discretization. These sufficient conditions are close to the necessary condition, which is \( m \geq N \), but still have an extra \( \log N \) factor in the bound for \( m \). The main goal of this paper is to prove a sufficient condition on \( m \) without an extra \( \log N \) factor in the upper bound, which guarantees the Marcinkiewicz-type discretization theorem in \( L_2 \). The first result in that direction was obtained in [17] (see Theorem 4.7 there) under a condition stronger than Condition E.

Theorem 1.3. Let \( \Omega_M = \{x^j\}_{j=1}^M \) be a discrete set with the probability measure \( \mu(x^j) = 1/M, j = 1, \ldots, M \). Assume that \( \{u_i(x)\}_{i=1}^N \) is an orthonormal on \( \Omega_M \) system (real or complex). Assume in addition that this system has the following property: for all \( j = 1, \ldots, M \) we have

\[
\sum_{i=1}^N |u_i(x^j)|^2 = N. \tag{1.2}
\]

Then there is an absolute constant \( C_1 \) such that there exists a subset \( J \subset \{1, 2, \ldots, M\} \) with the property: \( m := |J| \leq C_1 N \) and for any \( f = \sum_{i=1}^N c_i u_i \) we have

\[
C_2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j \in J} |f(x^j)|^2 \leq C_3 \|f\|_2^2,
\]

where \( C_2 \) and \( C_3 \) are absolute positive constants.
The following Theorem 1.4 is the main result of the paper.

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^d$ be a compact set with the probability measure $\mu$. Assume that $\{u_i(x)\}_{i=1}^N$ is a real (or complex) orthonormal system in $L_2(\Omega, \mu)$ satisfying Condition E. Then there is an absolute constant $C_1$ such that there exists a set $\{\xi_j\}_{j=1}^m \subset \Omega$ of $m \leq C_1 t^2 N$ points with the property:

For any $f = \sum_{i=1}^N c_i u_i$ we have

$$C_2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^2 \leq C_3 t^2 \|f\|_2^2,$$

where $C_2$ and $C_3$ are absolute positive constants.

Let us make a remark on weighted discretization. We begin with known results on weighted discretization for the reader’s convenience. In the case of weighted discretization, namely, when instead of $\frac{1}{m} \sum_{j=1}^m |f(\xi_j)|^2$ we use the weighted sum $\sum_{j=1}^m \lambda_j |f(\xi_j)|^2$, the problem of discretization is solved in the sense of order in the case of real subspaces $X_N$. It is pointed out in [18] that the paper by J. Batson, D.A. Spielman, and N. Srivastava [1] (see Theorem 3.1 there) basically solves the discretization problem with weights. We present an explicit formulation of this important result in our notations.

**Theorem 1.5.** Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \ldots, M$, and let $X_N$ be an $N$-dimensional subspace of real functions defined on $\Omega_M$. Then for any number $b > 1$ there exists a set of weights $\lambda_j \geq 0$ such that $|\{j : \lambda_j \neq 0\}| \leq [bN]$ so that for any $f \in X_N$ we have

$$\|f\|_2^2 \leq \sum_{j=1}^M \lambda_j f(x^j)^2 \leq \frac{b + 1 + 2 \sqrt{b}}{b + 1 - 2 \sqrt{b}} \|f\|_2^2.$$

As was observed in [13, Theorem 2.13], this last theorem with a general probability space $(\Omega, \mu)$ in place of the discrete space $(\Omega_M, \mu)$ remains true (with other constant in the right hand side) if $X_N \subset L_4(\Omega, \mu)$. It was proved in [3] (see Theorem 6.3 there) that the additional assumption $X_N \subset L_4(\Omega, \mu)$ can be dropped as well.

**Theorem 1.6.** If $X_N$ is an $N$-dimensional subspace of the real $L_2(\Omega, \mu)$, then for any $b \in (1, 2]$, there exist a set of $m \leq [bN]$ points $\xi^1, \ldots, \xi^m \in \Omega$
and a set of nonnegative weights $\lambda_j$, $j = 1, \ldots, m$, such that
\[
\|f\|_2^2 \leq \sum_{j=1}^m \lambda_j f(\xi_j)^2 \leq \frac{C}{(b - 1)^2} \|f\|_2^2, \quad \forall f \in X_N,
\]
where $C > 1$ is an absolute constant.

In this paper we obtain analogs of Theorems 1.5 and 1.6 in the case of complex subspaces $X_N$ (see Theorems 3.2 and 3.3 in Section 3). We note that there are related results on the Banach–Mazur distance between two finite dimensional spaces of the same dimension (see, for instance, [2], [16], [7]).

2 Main lemma

Results of this section are based on the following result by A. Marcus, D.A. Spielman and N. Srivastava from [12] (see Corollary 1.5 with $r = 2$ there).

**Theorem 2.1.** Let a system of vectors $v_1, \ldots, v_M$ from $\mathbb{C}^N$ have the following properties: for all $w \in \mathbb{C}^N$
\[
\sum_{j=1}^M |\langle w, v_j \rangle|^2 = \|w\|_2^2
\]
and for some $\epsilon > 0$
\[
\|v_j\|_2^2 \leq \epsilon, \quad j = 1, \ldots, M.
\]
Then there is a partition of $\{1, 2, \ldots, M\}$ into two sets $S_1$ and $S_2$ such that for all $w \in \mathbb{C}^N$ and for each $i = 1, 2$
\[
\sum_{j \in S_i} |\langle w, v_j \rangle|^2 \leq \frac{(1 + \sqrt{2}\epsilon)^2}{2} \|w\|_2^2.
\]

The following Lemma 2.1 was derived from Theorem 2.1 in [13] (see Lemma 2 there and also see [14], Lemma 10.22, p.105).
Lemma 2.1. Let a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_M$ from $\mathbb{C}^N$ satisfy (2.1) for all $\mathbf{w} \in \mathbb{C}^N$ and

$$
\|\mathbf{v}_j\|_2^2 = N/M, \quad j = 1, \ldots, M.
$$

Then there is a subset $J \subset \{1, 2, \ldots, M\}$ such that for all $\mathbf{w} \in \mathbb{C}^N$

$$
c_0 \|\mathbf{w}\|_2^2 \leq \frac{M}{N} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \leq C_0 \|\mathbf{w}\|_2^2,
$$

where $c_0$ and $C_0$ are some absolute positive constants.

A simple Remark 2.1 is from [17].

Remark 2.1. For the cardinality of the subset $J$ from Lemma 2.1 we have

$$
c_0 N \leq |J| \leq C_0 N.
$$

In this section we prove the following generalization of Lemma 2.1.

Lemma 2.2. Let a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_M$ from $\mathbb{C}^N$ satisfy (2.1) for all $\mathbf{w} \in \mathbb{C}^N$ and

$$
\|\mathbf{v}_j\|_2^2 \leq \theta N/M, \quad \theta \leq M/N, \quad j = 1, \ldots, M.
$$

Then there is a subset $J \subset \{1, 2, \ldots, M\}$ such that for all $\mathbf{w} \in \mathbb{C}^N$

$$
c_0 \theta \|\mathbf{w}\|_2^2 \leq \frac{M}{N} \sum_{j \in J} |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \leq C_0 \theta \|\mathbf{w}\|_2^2, \quad |J| \leq C_1 \theta N,
$$

where $c_0$, $C_0$, and $C_1$ are some absolute positive constants.

To obtain an analog of Theorem 1.5 in the complex case we need the following corollary.

Corollary 2.1. Let a system of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_M$ from $\mathbb{C}^N$ satisfy (2.1) for all $\mathbf{w} \in \mathbb{C}^N$. Then there exists a set of weights $\lambda_j \geq 0$, $j = 1, \ldots, M$, such that $|\{j : \lambda_j \neq 0\}| \leq 2C_1 N$ and for all $\mathbf{w} \in \mathbb{C}^N$ we have

$$
c_0 \|\mathbf{w}\|_2^2 \leq \sum_{j=1}^M \lambda_j |\langle \mathbf{w}, \mathbf{v}_j \rangle|^2 \leq C_0 \|\mathbf{w}\|_2^2.
$$

where $c_0$, $C_0$, and $C_1$ are absolute positive constants from Lemma 2.2.
Proof. Without loss of generality we assume that $\|v_1\|_2 = \min_{j=1,\ldots,M} \|v_j\|_2$. Let $n_1, \ldots, n_M$ be natural numbers such that for every $j$, $1 \leq j \leq M$

$$\|v_1\|_2^2 \leq \frac{\|v_j\|_2^2}{n_j} < 2\|v_1\|_2^2. \quad (2.4)$$

Denote $M' = \sum_{j=1}^M n_j$. We build a system $V$ of vectors $v_1', \ldots, v_{M'}'$ from $\mathbb{C}^N$ in the following way: for every $j$, $1 \leq j \leq M$, we include in $V$ $n_j$ copies of the vector $v_j/\sqrt{n_j}$. Let us check that $V$ satisfies (2.1) and (2.2) with $\theta = 2$. By construction and by our assumption that the system of vectors $v_1, \ldots, v_M$ satisfies (2.1), we have

$$\sum_{j=1}^{M'} |\langle w, v_j' \rangle|^2 = \sum_{j=1}^M n_j |\langle w, v_j/\sqrt{n_j} \rangle|^2 = \|w\|_2^2. \quad (2.5)$$

By construction of the system $V$ we obtain from (2.5) applied to the canonical basis of $\mathbb{C}^N$ and from (2.4)

$$\|v_1\|_2^2 M' \leq \sum_{j=1}^{M'} \|v_j'\|_2^2 = N.$$

By construction for each $j = 1, \ldots, M'$, we have a number $k(j) \in \{1, \ldots, M\}$ such that $v_j' = v_{k(j)}/\sqrt{n_{k(j)}}$. Therefore, by (2.4) we get

$$\|v_j'\|_2^2 = \frac{\|v_{k(j)}\|_2^2}{n_{k(j)}} < 2\|v_1\|_2^2 \leq 2\frac{N}{M'}, \quad j = 1, \ldots, M'.$$

We apply Lemma 2.2 to the system $V$ and obtain a subset $J \subset \{1, \ldots, M'\}$ with $|J| \leq 2C_1 N$ such that for all $w \in \mathbb{C}^N$

$$c_0\|w\|_2^2 \leq \frac{M'}{2N} \sum_{j \in J} |\langle w, v_j' \rangle|^2 \leq C_0\|w\|_2^2.$$

It is clear that

$$\frac{M'}{2N} \sum_{j \in J} |\langle w, v_j' \rangle|^2 = \sum_{j=1}^M \lambda_j |\langle w, v_j \rangle|^2$$

for some nonnegative $\lambda_j, j = 1, \ldots, M$, so that $|\{j : \lambda_j \neq 0\}| \leq 2C_1 N$. \qed
Note that condition (2.1) implies that $M \geq N$. Lemma 2.2 in some sense improves the celebrated result of M. Rudelson [15] where a similar to Lemma 2.2 result was proved with $|J| \leq C_1(t)N \log N$ and with bounds depending on $\epsilon$ (see Theorem 1.1 in Introduction). Proof of Lemma 2.2 uses the iteration method suggested by A. Lunin [11]. We also refer the reader to the paper [8] for a discussion of recent outstanding progress in the area of submatrices of orthogonal matrices.

Proof. We use the following known results (for Proposition 2.1 see Corollary B from [13], Corollary 10.19 from [14], p.104, or [6], and for Lemma 2.3 see Lemma 1 in [13] or Lemma 10.20 in [14], p.104).

**Proposition 2.1.** Let $v_1, \ldots, v_M \in \mathbb{C}^N$ be such that $\|v_j\|^2 \leq \delta$ for all $j = 1, \ldots, M$. If

$$\alpha \|w\|^2 \leq \sum_{j=1}^M |\langle w, v_j \rangle|^2 \leq \beta \|w\|^2, \quad \forall w \in \mathbb{C}^N,$$

with some numbers $\beta \geq \alpha > \delta$, then there exists a partition of $\{1, \ldots, M\}$ into $S_1$ and $S_2$ such that for each $i = 1, 2$,

$$\frac{1 - 5\sqrt{\delta/\alpha}}{2} \alpha \|w\|^2 \leq \sum_{j \in S_i} |\langle w, v_j \rangle|^2 \leq \frac{1 + 5\sqrt{\delta/\alpha}}{2} \beta \|w\|^2, \quad \forall w \in \mathbb{C}^N.$$

**Lemma 2.3.** Let $0 < \delta < 1/100$, and let $\alpha_j, \beta_j, j = 0, 1, \ldots$, be defined inductively

$$\alpha_0 = \beta_0 = 1, \quad \alpha_{j+1} := \alpha_j \frac{1 - 5\sqrt{\delta/\alpha_j}}{2}, \quad \beta_{j+1} := \beta_j \frac{1 + 5\sqrt{\delta/\alpha_j}}{2}.$$

Then there exist a positive absolute constant $C$ and a number $L \in \mathbb{N}$ such that

$$\alpha_j \geq 100\delta, \quad j \leq L, \quad 25\delta \leq \alpha_{L+1} < 100\delta, \quad \beta_{L+1} < C\alpha_{L+1}.$$

If $\delta := \theta N/M \geq 1/100$, then (2.3) holds with $J = \{1, 2, \ldots, M\}$ and $C_1 = 1/\delta \leq 100, c_0 = 1, C_0 = 100$. Assume $\delta < 1/100$. Let $\alpha_j, \beta_j$ be as defined in Lemma 2.3, then the vectors $v_1, \ldots, v_M$ satisfy the assumptions of Proposition 2.1 with $\alpha = \beta = 1$. We apply Proposition 2.1 and choose a
subset of the obtained partition with a smaller cardinality. We obtain a set $J_1 \subset \{1, 2, \ldots, M\}$ with $|J_1| \leq M/2$ such that $\forall w \in \mathbb{C}^N$

$$\alpha_1 \|w\|^2 \leq \sum_{i \in J_1} |\langle w, v_i \rangle|^2 \leq \beta_1 \|w\|^2.$$ 

Since $\alpha_1 > 25\delta$ we can apply Proposition 2.1 again and obtain $J_2 \subset J_1$ with $|J_2| \leq M/2^2$, for which we have two-sided inequalities with $\alpha_2 > 0$ and $\beta_2$. Let $L$ be the number from Lemma 2.3. We consecutively apply Proposition 2.1 (choosing at each step the subset $S_i$ with the smallest cardinality) and find $J_1 \supset J_2 \supset \cdots \supset J_{L+1}$ with the property

$$1 - \frac{5\sqrt{\delta/\alpha_L}}{2} \|w\|_2^2 \leq \sum_{j \in J_{L+1}} |\langle w, v_j \rangle|^2 \leq \frac{1 + 5\sqrt{\delta/\alpha_L}}{2} \beta_L \|w\|_2^2, \ \forall w \in \mathbb{C}^N.$$

By Lemma 2.3 we obtain

$$1 - \frac{5\sqrt{\delta/\alpha_L}}{2} \alpha_L = \alpha_{L+1} \geq 25\delta,$$

$$1 + \frac{5\sqrt{\delta/\alpha_L}}{2} \beta_L = \beta_{L+1} \leq C\alpha_{L+1} < 100C\delta.$$

Thus, for $J := J_{L+1}$ we have

$$25\theta \frac{N}{M} \|w\|_2^2 \leq \sum_{i \in J} |\langle w, v_i \rangle|^2 \leq 100C\theta \frac{N}{M} \|w\|_2^2.$$

Note that $2^{-L-1} \leq \beta_{L+1} < 100C\delta$, therefore $|J_{L+1}| \leq M/2^{L+1} \leq 100CM\delta = 100C\theta N$ as required. 

### 3 Application to discretization

The following corollary of Lemma 2.2 is a generalization of Theorem 4.7 from [17] (see Theorem 1.3 in Introduction). In [17] instead of condition (3.1) a stronger assumption (1.2) was imposed.

**Theorem 3.1.** Let $\Omega_M = \{x^j\}_{j=1}^M$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \ldots, M$. Assume that $\{u_i(x)\}_{i=1}^N$ is an orthonormal
on $\Omega_M$ system (real or complex). Assume in addition that this system has the following property: for all $j = 1, \ldots, M$ we have

$$\sum_{i=1}^{N} |u_i(x^j)|^2 \leq Nt^2. \quad (3.1)$$

Then there is an absolute constant $C_1$ such that there exists a subset $J \subset \{1, 2, \ldots, M\}$ with the property: $m := |J| \leq C_1 t^2 N$ and for any $f = \sum_{i=1}^{N} c_i u_i$ we have

$$C_2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j \in J} |f(x^j)|^2 \leq C_3 t^2 \|f\|_2^2, \quad (3.2)$$

where $C_2$ and $C_3$ are absolute positive constants.

**Proof.** Define the column vectors

$$v_j := M^{-1/2}(u_1(x^j), \ldots, u_N(x^j))^T, \quad j = 1, \ldots, M.$$ 

Then our assumption (3.1) implies that the system $v_1, \ldots, v_M$ satisfies (2.2) with $\theta = t^2$. For any $w = (w_1, \ldots, w_N)^T \in \mathbb{C}^N$ we have

$$\sum_{j=1}^{M} |\langle w, v_j \rangle|^2 = \frac{1}{M} \sum_{j=1}^{M} \sum_{i,k=1}^{N} w_i \bar{w}_k u_i(x^j) \bar{u}_k(x^j) = \sum_{i=1}^{N} |w_i|^2$$

by the orthonormality assumption. This implies that the system $v_1, \ldots, v_M$ satisfies (2.1).

Note that the necessary condition for (3.2) to hold is $m \geq N$. Applying Lemma 2.2 we complete the proof of Theorem 3.1. \qed

The following Theorem 3.2, which is a complex analog of Theorem 1.5, can be derived from Corollary 2.1 in the same way as we have derived Theorem 3.1 from Lemma 2.2 above.

**Theorem 3.2.** Let $\Omega_M = \{x^j\}_{j=1}^{M}$ be a discrete set with the probability measure $\mu(x^j) = 1/M$, $j = 1, \ldots, M$. Assume that $\{u_i(x)\}_{i=1}^{N}$ is an orthonormal on $\Omega_M$ system (real or complex). Then there is an absolute constant $C_1$ such that there exists a set of weights $\lambda_j \geq 0$, $j = 1, \ldots, M$, with the property: $m := |\{j : \lambda_j \neq 0\}| \leq C_1 N$ and for any $f = \sum_{i=1}^{N} c_i u_i$ we have

$$c_0 \|f\|_2^2 \leq \sum_{j=1}^{M} \lambda_j |f(x^j)|^2 \leq C_0 \|f\|_2^2,$$

where $c_0$ and $C_0$ are from Lemma 2.2.
Further, using Theorem 3.2 and repeating the argument in the proof of Theorem 6.3 from [5] (with natural modifications from the real case to the complex case), which was used to derive Theorem 1.6 from Theorems 1.5 and 1.2, we obtain the following complex analog of Theorem 1.6. Note that the complex version of Theorem 1.2 can be proved in the same way as Theorem 1.2 was proved in [18].

**Theorem 3.3.** If $X_N$ is an $N$-dimensional subspace of the complex $L_2(\Omega, \mu)$, then there exist three absolute positive constants $C'_1$, $c'_0$, $C'_0$, a set of $m \leq C'_1 N$ points $\xi^1, \ldots, \xi^m \in \Omega$, and a set of nonnegative weights $\lambda_j$, $j = 1, \ldots, m$, such that

$$c'_0 \|f\|_2^2 \leq \sum_{j=1}^m \lambda_j |f(\xi^j)|^2 \leq C'_0 \|f\|_2^2, \quad \forall f \in X_N.$$  

**Remark 3.1.** A combination of the proof of Theorem 6.3 from [5] with Theorem 3.4 (in the proof of Theorem 6.3 from [5] we use Theorem 3.4 instead of Theorem 1.2) gives Theorem 3.3 with $C'_1 = C'_2$, $c'_0 = C'_2$, and $C'_0 = C'_3$, where $C'_i$, $i = 1, 2, 3$, are from Theorem 3.4.

It is important to emphasize that in the proof of Theorems 3.2 and 3.3, which are complex companions of Theorems 1.5 and 1.6, we did not use Theorems 1.5 and 1.6. Thus, our arguments give other proofs of analogs of Theorems 1.5 and 1.6. Note that constants in Theorems 3.2 and 3.3 are not as good as constants in Theorems 1.5 and 1.6.

Let $\Omega$ be a compact set in $\mathbb{R}^d$ and let $X_N$ be an $N$-dimensional subspace of real (or complex) space of continuous functions $\mathcal{C}(\Omega)$. Let $\mu$ be a probability measure on $\Omega$ and let $\{u_i(x)\}_{i=1}^N$ be an orthonormal basis for $X_N$.

**Nikol’skii inequality.** We say that $X_N$ satisfies the Nikol’skii inequality for the pair $(2, \infty)$ if there exists a constant $t$ such that

$$\|f\|_\infty \leq tN^{\frac{1}{2}} \|f\|_2, \quad \forall f \in X_N.$$  

We point out that condition (3.3) with $X_N = \text{span}\{u_i\}_{i=1}^N$ is equivalent to Condition E. This can be seen from the following well-known result.

**Proposition 3.1.** Let $X_N$ be an $N$-dimensional subspace of $\mathcal{C}(\Omega)$. Then for any orthonormal basis $\{u_i\}_{i=1}^N$ of $X_N \subset L_2(\Omega, \mu)$ we have that for $x \in \Omega$

$$\sup_{f \in X_N: f \neq 0} \frac{|f(x)|}{\|f\|_2} = \left(\sum_{i=1}^N |u_i(x)|^2\right)^{1/2}.$$
The following simple result can be found in [3] (see Proposition 2.1 there). Note that only the real case is discussed in [3]. However, the same argument works for the complex case as well.

**Proposition 3.2.** Let \( Y_N := \text{span}\{u_1(x), \ldots, u_N(x)\} \) with \( \{u_i(x)\}_{i=1}^N \) being a real (or complex) orthonormal on \( \Omega \) with respect to a probability measure \( \mu \) basis for \( Y_N \). Assume that \( \|u_i\|_4 := \|u_i\|_{L_4(\Omega, \mu)} < \infty \) for all \( i = 1, \ldots, N \). Then for any \( \delta > 0 \) there exists a set \( \Omega_M = \{x^j\}_{j=1}^M \subset \Omega \) such that for any \( f \in Y_N \)

\[
\|f\|_{L_2(\Omega)}^2 - \|f\|_{L_2(\Omega_M)}^2 \leq \delta \|f\|_{L_2(\Omega)}^2,
\]

where

\[
\|f\|_{L_2(\Omega_M)}^2 := \frac{1}{M} \sum_{j=1}^M |f(x^j)|^2.
\]

The following generalization of Theorem 3.1, which is equivalent to Theorem 1.4, is the main result of the paper.

**Theorem 3.4.** Let \( \Omega \subset \mathbb{R}^d \) be a compact set with the probability measure \( \mu \). Assume that \( X_N \subset C(\Omega) \) satisfies the Nikol’skii inequality (3.3). Then there is an absolute constant \( C'_1 \) such that there exists a set \( \{\xi^j\}_{j=1}^m \subset \Omega \) of \( m \leq C'_1 t^2 N \) points with the property: for any \( f \in X_N \) we have

\[
C'_2 \|f\|_2^2 \leq \frac{1}{m} \sum_{j=1}^m |f(\xi^j)|^2 \leq C'_3 t^2 \|f\|_2^2,
\]

where \( C'_2 \) and \( C'_3 \) are absolute positive constants.

**Proof.** For a given \( \delta \in (0, 1) \), taking into account Proposition 3.2 we find a set \( \Omega_M = \{x^j\}_{j=1}^M \) such that for any \( f \in X_N \)

\[
\|f\|_{L_2(\Omega)}^2 - \|f\|_{L_2(\Omega_M)}^2 \leq \delta \|f\|_{L_2(\Omega)}^2.
\]

Specify \( \delta = 1/2 \). Then, clearly, subspace \( X_N \) restricted to \( \Omega_M \) (denote it by \( Y_l \)) satisfies the Nikol’skii inequality (3.3) with \( t \) replaced by \( 2t \). Let \( u_1, \ldots, u_l \), \( l \leq N \), be an orthonormal basis of \( Y_l \). By Proposition 3.1 inequality (3.3) is equivalent to (3.1). Now applying Theorem 3.1 to \( Y_l \) we find a subset \( J \subset \{1, 2, \ldots, M\} \) with the property: \( m := |J| \leq C_1(2t)^2 N \) and for any \( f \in X_N \) we have

\[
C_2 \|f\|_{L_2(\Omega_M)}^2 \leq \frac{1}{m} \sum_{j \in J} |f(x^j)|^2 \leq C_3 t^2 \|f\|_{L_2(\Omega_M)}^2,
\]

where \( C'_2 \) and \( C'_3 \) are absolute positive constants.
where $C_2$ and $C_3$ are absolute positive constants from Theorem 3.1. From here and (3.5) with $\delta = 1/2$ we obtain (3.4).

**Remark 3.2.** In Theorem 3.4 we assume that $X_N \subset C(\Omega)$. It is done for convenience. The statement of Theorem 3.4 holds if instead of continuity assumption we require that $X_N$ is a subspace of the space $B(\Omega, \mu)$ of functions, which are bounded and measurable with respect to $\mu$ on $\Omega$.

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