Characterization of the $n$-dimensional Sierpiński carpet as an inverse limit of closed balls

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Abstract

In this paper we generalize, for any dimension, a theorem of Tshishiku and Walsh that characterizes the Sierpiński carpet as a limit of a set of maps from the disc to the sphere.

Keywords Sierpiński carpet · inverse limit

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Introduction

In [19], Whyburn did two different topological characterizations of the 1-dimensional Sierpiński carpet. After that, Cannon [6] generalized one of these characterizations for any dimension different than 3. It says that if we take an \( n \)-dimensional sphere and remove a countable set of tame \( n \)-balls such that their union is dense and their diameter tends to 0, then the space that we get does not depend on the choice of the removed balls. Such space is the \( n-1 \)-dimensional Sierpiński carpet.

Recently, Tshishiku and Walsh did another characterization of the 1-dimensional Sierpiński carpet as an inverse limit of closed discs (Lemma 3.1 of [17] and its correction in [18]) for the purpose of using it to have a topological characterization of some boundaries of groups. Roughly speaking, it says that if we take a sphere, choose a countable dense subset of it and for each of these points we remove it and replace it nicely with a circle, then we get a 1-dimensional Sierpiński carpet. With a similar purpose, this paper is devoted to generalizing Tshishiku and Walsh result for any dimension. It consists of the following:

**Theorem 2.8** Let \( P \) be a countable dense subset of the sphere \( S^n \). Let, for every \( p \in P \), \( \pi_p : D^n \to S^n \) be the map that collapses the boundary of the \( n \)-ball to the point \( p \). Then the limit space \( \lim_{\leftarrow} \{ D^n, \pi_p \}_{p \in P} \) is homeomorphic to the \( n-1 \)-dimensional Sierpiński carpet.

Constructions of the \( n-1 \)-dimensional Sierpiński carpet as inverse limits of \( n \)-balls are well known in the literature, as seen in [1] (proof of Theorem 1 (1)), [3] (proof of Theorem 3) and [4] (proof of Theorem 1.2). However, as far as I know, all of these constructions are special cases of the theorem above, since the inverse limits described there depend on extra structures, like well behaved homeomorphisms or group actions.

In [16] we will use this characterization to construct some Bowditch boundaries of relatively hyperbolic groups that are homeomorphic to the \( n \)-dimensional Sierpiński carpet. For instance, if \( (G, \mathcal{P}) \) is a relatively hyperbolic pair with its Bowditch boundary homeomorphic to \( S^3 \), \( \mathcal{Q} \) is a proper subset of \( \mathcal{P} \) such that every group in \( \mathcal{P} - \mathcal{Q} \) is virtually torsion-free and hyperbolic, then the Bowditch boundary of \( (G, \mathcal{Q}) \) is homeomorphic to the 2-dimensional Sierpiński carpet (Theorem 2.8 of [16]).
Acknowledgements. This paper contains part of my PhD thesis. It was written under the advisorship of Victor Gerasimov, to whom I am grateful. I would also like to thank to Jan Boronski (who showed me the references [1], [3] and [4]) and Kazuhiro Kawamura (who showed me that Freedman’s theorem would solve the problem on the restriction of Cannon’s theorem in dimension 4) for the nice conversations that I had with each of them about this paper.

1 Preliminaries

Here we list some well known topological results that are used in the next section.

Some notation that we use in this paper:

1. If $X$ is is a topological space and $Y \subseteq X$, then the closure of $Y$ is denoted by $\overline{Y}$ or $Cl(Y)$ and the interior of $Y$ is denoted by $int(Y)$.

2. If $X$ is a metric space, $Y \subseteq X$ and $\epsilon > 0$, then the $\epsilon$-neighborhood of $Y$ is denoted by $B(Y, \epsilon)$.

1.1 Homogeneity

Definition 1.1. A topological space $X$ is strongly locally homogeneous if $\forall x \in X$, $\forall U$ neighbourhoud of $x$, there exists an open set $V$ such that $x \in V \subseteq U$ and $\forall y \in V$, there exists a homeomorphism $f : X \to X$ such that $f(x) = y$ and $f|_{X-U} = id_{X-U}$.

Definition 1.2. A topological space $X$ is countable dense homogeneous if for every $A, B$ countable dense subsets of $X$, there exists a homeomorphism $f : X \to X$ such that $f(A) = B$.

Proposition 1.3. Let $X$ be a metrizable locally compact strongly locally homogeneous space and $(\hat{X}, \hat{d})$ a metric compactification of $X$. If $A$ and $B$ are two countable dense subsets of $X$, there exists a homeomorphism $f : \hat{X} \to \hat{X}$ such that $f(A) = B$ and $f|_{\hat{X}-X} = id_{\hat{X}-X}$.

Remark. This proposition slightly generalizes Bennett’s Theorem (Theorem 3 of [2]) and the proof is essentially the same. We write it here for the sake of completeness.

Proof. Let’s construct a family of homeomorphisms $\{f_n\}_{n \in \mathbb{N}}$, using a back and forth argument, that converges to the homeomorphism that we need.
Let $A = \{a_i\}_{i \in \mathbb{N}}$ and $B = \{b_i\}_{i \in \mathbb{N}}$. Let $i_1 \in \mathbb{N}$ such that $Cl_X(\mathcal{B}(a_1, \frac{1}{2^{i_1}})) \subseteq X$. Since $X$ is strongly locally homogeneous, there exists $V_1 \subseteq \mathcal{B}(a_1, \frac{1}{2^{i_1+2}})$ an open set and $b \in V_1 \cap B$ such that there is a homeomorphism $h_1 : X \to X$ with $h_1(a_1) = b$ and $h_{1} |_{X - \mathcal{B}(a_1, \frac{1}{2^{i_1+1}})} = id_{X - \mathcal{B}(a_1, \frac{1}{2^{i_1+1}})}$. Since $Cl_X(\mathcal{B}(a_1, \frac{1}{2^{i_1}}))$ do not intersect $\bar{X} - X$, the map $h_1$ extends to a homeomorphism $f_1 : X \to \bar{X}$ such that $f_1 |_{\bar{X} - X} = id_{\bar{X} - X}$.

Suppose we have a homeomorphism $f_{2n} : \bar{X} \to \bar{X}$ such that $f_{2n} |_{\bar{X} - X} = id_{\bar{X} - X}$, $f_{2n}(\{a_1, \ldots, a_n\}) \subseteq B$ and $f_{2n}^{-1}(\{b_1, \ldots, b_n\}) \subseteq A$. If $f_{2n}(a_{n+1}) \notin B$, take $f_{2n+1} = f_{2n}$. Suppose that $f_{2n}(a_{n+1}) \notin B$. There is an index $i_{n+1} \geq 2n + 1$ such that $Cl_X(\mathcal{B}(f_{2n}(a_{n+1}), \frac{1}{2^{i_{n+1}}}))$ does not intersect the set $\{f_{2n}(a_1), \ldots, f_{2n}(a_n), b_1, \ldots, b_n\} \cup (X - X)$. Then there exists an open set $V_{n+1}$ such that $f_{2n}(a_{n+1}) \in V_{n+1} \subseteq \mathcal{B}(f_{2n}(a_{n+1}), \frac{1}{2^{i_{n+1}}})$ and a homeomorphism $h_{2n+1} : X \to X$ such that $h_{2n+1}(f_{2n}(a_{n+1})) \in V \cap B$ and $h_{2n+1} |_{X - \mathcal{B}(f_{2n}(a_{n+1}), \frac{1}{2^{i_{n+1}}})} = id_{X - \mathcal{B}(f_{2n}(a_{n+1}), \frac{1}{2^{i_{n+1}}})}$. Take $f_{2n+1} = h_{2n+1} \circ f_{2n}$.

We have that $f_{2n+1} |_{\bar{X} - X} = id_{\bar{X} - X}$, $f_{2n+1}(\{a_1, \ldots, a_n\}) = f_{2n}(\{a_1, \ldots, a_n\})$, $\forall i \in \{1, \ldots, n + 1\}$, $f_{2n+1}(a_i) \in B$, $f_{2n+1}^{-1}(\{b_1, \ldots, b_n\}) = f_{2n}^{-1}(\{b_1, \ldots, b_n\})$ and $\forall i \in \{1, \ldots, n\}$, $f_{2n+1}^{-1}(b_i) \in A$.

Suppose we have a homeomorphism $f_{2n+2} : \bar{X} \to X$ such that $f_{2n+2} |_{\bar{X} - X} = id_{\bar{X} - X}$, $f_{2n+2}(\{a_1, \ldots, a_{n+1}\}) \subseteq B$ and $f_{2n+2}^{-1}(\{b_1, \ldots, b_n\}) \subseteq A$. If $f_{2n+2}^{-1}(b_{n+1}) \notin A$, take $f_{2n+2} = f_{2n+1}$. Suppose that $f_{2n+2}^{-1}(b_{n+1}) \subseteq A$. There exists $i_{n+2} \in \mathbb{N}$ such that $2n + 2 \leq i_{n+2}$ and the set $Cl_X(\mathcal{B}(f_{2n+2}^{-1}(b_{n+1}), \frac{1}{2^{i_{n+2}}}))$ do not intersect $\{f_{2n+2}^{-1}(b_1), \ldots, f_{2n+2}^{-1}(b_n), a_1, \ldots, a_{n+1}\} \cup (X - X)$. Then there exists an open set $V_{n+2}$ such that $f_{2n+2}^{-1}(b_{n+1}) \in V_{n+2} \subseteq \mathcal{B}(f_{2n+2}^{-1}(b_{n+1}), \frac{1}{2^{i_{n+2}}})$ and a homeomorphism $h_{2n+2} : X \to X$ such that $h_{2n+2}^{-1}(f_{2n+2}^{-1}(b_{n+1})) \subseteq V \cap A$ and $h_{2n+2} |_{X - \mathcal{B}(f_{2n+2}^{-1}(b_{n+1}), \frac{1}{2^{i_{n+2}}})} = id_{X - \mathcal{B}(f_{2n+2}^{-1}(b_{n+1}), \frac{1}{2^{i_{n+2}}})}$. Take $f_{2n+2} = f_{2n+1} \circ h_{2n+2}$.

We have that $f_{2n+2} |_{\bar{X} - X} = id_{\bar{X} - X}$, $f_{2n+2}(\{a_1, \ldots, a_{n+1}\}) = f_{2n+1}(\{a_1, \ldots, a_{n+1}\})$, $\forall i \in \{1, \ldots, n + 1\}$, $f_{2n+2}(a_i) \in B$, $f_{2n+2}^{-1}(\{b_1, \ldots, b_n\}) = f_{2n+1}^{-1}(\{b_1, \ldots, b_n\})$ and $\forall i \in \{1, \ldots, n + 1\}$, $f_{2n+2}^{-1}(b_i) \in A$.

We have that, if $n \leq m$, $\rho(f_m, f_n) \leq \sum_{j=n}^{m} \frac{1}{2^j}$, where $\rho$ is the uniform distance. Since $\{\sum_{j=n}^{\infty} \frac{1}{2^j}\}_{n \in \mathbb{N}}$ converges to 0, $\{f_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, which implies that it converges uniformly to a continuous map $f : X \to \bar{X}$. Analogously, $\{f_n\}_{n \in \mathbb{N}}$ converges uniformly to a continuous map $g : X \to \bar{X}$.

Since $\forall n \in \mathbb{N}$, $f_n |_{\bar{X} - X} = id_{\bar{X} - X}$, we have that $f |_{\bar{X} - X} = id_{\bar{X} - X}$. We have also that $\forall i \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\forall n' > n$, $f_{n'}(a_i) = f_n(a_i) \in B$ and $f_{n'}^{-1}(b_i) = f_n^{-1}(b_i) \in A$, which implies that $f(a_i) = f_n(a_i)$ and $g(b_i) = f_n^{-1}(b_i)$. So $f(A) = B$, $g(B) = A$, $g \circ f |_A = id_A$ and $f \circ g |_B = id_B$. Since $A$ and $B$ are dense on $\bar{X}$, then the maps $f$ and $g$ are inverses. Thus $f$ is the homeomorphism that we need.

**Remark.** We have that manifolds with no boundary are strongly homoge-
neous, which implies that a manifold $M$ with boundary have the property that for every $A, B$ countable dense subsets of $M$ that do not intersect the boundary, there exists a homeomorphism $f : M \to M$ such that $f(A) = B$. In particular, Bennett’s Theorem implies that manifolds without boundary are countable dense homogeneous.

**Corollary 1.4.** Let $X$ be a metrizable locally compact strongly locally homogeneous space and $(\bar{X}, d)$ a metric compactification of $X$. If $A$ and $B$ are two subsets of $X$ such that $A$ is countable dense and $B$ has empty interior, then there exists a homeomorphism $f : \bar{X} \to \bar{X}$ such that $f(B) \cap A = \emptyset$ and $f|_{\bar{X} - X} = id_{\bar{X} - X}$.

**Proof.** We have that $X - B$ has a countable basis, which implies that it is separable. Let $C$ be a countable dense subset of $X - B$. Since $X - B$ is dense on $X$, then $C$ is dense on $X$. By the last proposition, there is a homeomorphism $f : \bar{X} \to \bar{X}$ such that $f(C) = A$ and $f|_{\bar{X} - X} = id_{\bar{X} - X}$. This homeomorphism satisfies the property $f(B) \cap A = \emptyset$.

1.2 Approximation of maps

**Proposition 1.5.** Let $M$ and $N$ be two compact metric spaces and $\{F_i\}_{i \in \mathbb{N}}$ be a family of maps $F_i : M \to \text{Closed}(N)$ satisfying:

1. $\forall x \in M, \forall i \in \mathbb{N}, \forall U$ neighbourhood of $F_i(x)$, there exists a neighbourhood $V$ of $x$ such that $\forall x' \in V$, there exists $j > i$ such that $F_j(x') \subseteq U$.

2. $\forall i \in \mathbb{N}, \forall x \in M$, $F_i(x) \supseteq F_{i+1}(x)$.

3. $\forall x \in M$, $\lim_{i \in \mathbb{N}} \text{diam}(F_i(x)) = 0$.

Then the map $f : M \to N$ defined by $\{f(x)\} = \bigcap_{i \in \mathbb{N}} F_i(x)$ is continuous.

**Remark.** This proposition slightly generalizes Fort’s Theorem about upper semi-continuous maps.

**Proof.** By the properties 2 and 3 it is clear that $f$ is well defined.

Let $x \in M$ and $U$ a neighbourhood of $f(x)$. Since $\{F_i(x)\}_{i \in \mathbb{N}}$ is a nested family of compact spaces with intersection $\{f(x)\}$, there is $i_0 \in \mathbb{N}$ such that $F_{i_0}(x) \subseteq U$ (Proposition 1.7 of [13]). By the first condition, there exists a neighbourhood $V$ of $x$ such that $\forall x' \in V$, there exists $i_{x'} > i_0$ such that $F_{i_{x'}}(x') \subseteq U$. But $f(x') \in F_{i_{x'}}(x')$, which implies that $f(x') \in U$. Thus, $f$ is continuous at $x$, which implies that $f$ is continuous. \qed
### 1.3 Topological quasiconvexity

**Definition 1.6.** Let \( X \) be a compact metric space and \( \sim \) an equivalence relation on \( X \). We say that \( \sim \) is topologically quasiconvex if \( \forall q \in X, [q] \) is closed and \( \forall \epsilon > 0, \#\{x \in X : \text{diam} [x] > \epsilon\} < \aleph_0 \), with \([x]\) the equivalence class of \( x \). The partition of \( X \) by such equivalence relation is called a null family.

Let \( X, Y \) be compact metric spaces. A quotient map \( f : X \to Y \) is topologically quasiconvex if the relation \( \sim = \Delta X \cup \bigcup_{y \in Y} f^{-1}(y)^2 \) is topologically quasiconvex.

**Remark.** This definition does not depend on the choice of the metric compatible with the topology of \( X \).

**Proposition 1.7.** *(Propositions 1 and 3 of Section 2 of [7] and Proposition 1.15 and Corollary 1.16 of [15]*) Let \( f : X \to Y \) be a continuous map, where \( X \) and \( Y \) are Hausdorff compact spaces. For \( A \subseteq Y \), let \( \sim_A = \Delta X \cup \bigcup_{q \in A} f^{-1}(q)^2 \) and \( X_A = X / \sim_A \). For \( p \in Y \), let \( \sim_p = \sim_{\{p\}} \), \( X_p = X_{\{p\}} \) and \( \pi_p : X_p \to Z \) the quotient map. The following are equivalent:

1. \( f \) is topologically quasiconvex.
2. \( \forall A \subseteq Y, X_A \text{ is Hausdorff} \).
3. \( \forall p \in Y, X_p \text{ is Hausdorff} \).
4. The induced map \( X \to \lim_{\leftarrow} \{X_p, \pi_p\}_{p \in Y} \) is a homeomorphism.

### 1.4 Spheres

Let \( D \) be an open or closed subset of \( S^n \). We say that \( D \) is a tame ball if \( \text{Cl}(D) \) and \( S^n - \text{int}(D) \) are homeomorphic to closed balls. If \( n = 2 \) then every interior of a closed disc is tame by Schoenflies Theorem.

However this is false for \( n > 2 \) (a counterexample is the Alexander horned sphere), so we need to be careful with it. We say that a subspace \( Y \) of \( S^n \) that is homeomorphic to \( S^{n-1} \) is tame in \( S^n \) if it bounds two closed balls. The Generalized Schoenflies Theorem, says that if the embedding of \( Y \) on \( S^n \) is well behaved, then \( Y \) is tame:

**Proposition 1.8.** *(Generalized Schoenflies Theorem - Brown [5]*) A subset \( Y \) of \( S^n \) that is homeomorphic to \( S^{n-1} \) is tame if and only if the inclusion map \( Y \to S^n \) extends to an embedding \( Y \times [-1, 1] \to S^n \) (where we identify \( Y \) with \( Y \times \{0\} \)).
If a space $X$ is homeomorphic to $S^n$ minus a finite number of tame open balls, we say that a subspace $Y$ of $X$ that is homeomorphic to $S^{n-1}$ is tame in $X$ if it is tame in $X'$, where $X'$ is the space homeomorphic to $S^n$ constructed from $X$ by adjoint the balls that are missing.

We have also a theorem that characterizes a sphere minus two tame discs:

**Proposition 1.9.** (Annulus Theorem - Kirby [10] for dimension $\neq 4$ and Quinn [14] for dimension 4) The space $S^n - (A \cup B)$, where $A$ and $B$ are disjoint tame copies of $S^{n-1}$, has three connected components, which two of them are balls and the other one has the closure homeomorphic to $S^{n-1} \times [0, 1]$.

The following theorem is well known. It is proved in [8] (Theorem 15) for dimension 2 and 3. For the other dimensions, it comes as a consequence of the Stable Homeomorphism Theorem ([10], [14]) and the second corollary of Theorem 2 of [10].

**Proposition 1.10.** (Isotopy Theorem for Spheres) Choose an orientation of $S^n$ and let $f, g : S^n \to S^n$ be homeomorphisms such that both preserve or reverse orientation. Then there exists an isotopy between $f$ and $g$.

**Proposition 1.11.** (Decomposition Theorem - Meyer, Theorem 2 of [11]) Let $C_1, ..., C_n, ...$ be a (possibly finite) family of tame open balls in $S^n$ which their closures are pairwise disjoint and such that the equivalence relation $\Delta S^n \cup \bigcup_i \text{Cl}_{S^n} C_i^2$ is topologically quasiconvex. If $U$ is an open set of $S^n$ that contains $\bigcup_i \text{Cl}_{S^n} C_i$, then there exists a continuous map $f : S^n \to S^n$ that is surjective, it is the identity outside $U$, each $\text{Cl}_{S^n} C_i$ is saturated (i.e. it is the preimage of some point) and $\forall x \in S^n - f(\bigcup_i \text{Cl}_{S^n} C_i)$, $f^{-1}(x)$ is a single point.

**Remark.** The version that we stated is in [6]. This result is due to Moore (Theorem 25 of [12]) when $n = 2$.

As a consequence of all these theorems, it is possible to characterize spheres with a finite set of tame balls removed. The following are easy and probably well known:

**Lemma 1.12.** Let $A_1, A_2, ..., A_m$ be disjoint closed tame balls of $S^n$, with $n > 1$, and $F$ a closed set of $S^n$ such that $S^n - F$ is connected and $\forall i \in \{1, ..., m\}, A_i \cap F = \emptyset$. Then there is a closed tame ball $C$ that contains $F$ and $A_i \cap C = \emptyset$, $\forall i \in \{1, ..., m\}$.

**Proof.** By Decomposition Theorem, there is a continuous surjective map $\pi : S^n \to S^n$ that collapses only the tame balls $A_1, ..., A_m$. Let $X$ be...
$S^n \setminus \{p\}$, where $p \notin \pi(A_1 \cup \ldots \cup A_n \cup F)$. We have that $X$ is homeomorphic to $\mathbb{R}^n$ and we choose a metric $d$ on $X$ induced by the euclidean metric on $\mathbb{R}^n$. Let $k = \text{diam } \pi(F)$. Since $X$ is homeomorphic to $\mathbb{R}^n$ and $X - \pi(F)$ is connected, for every pair of distinct $m$-tuples that do not intersect $\pi(F)$, there is a homeomorphism $f : X \to X$ that send one $m$-tuple to the other and it is the identity map on $\pi(F)$. So we choose the first $m$-tuple as $(\pi(A_1), \ldots, \pi(A_m))$ and the second one any $m$-tuple $(x_1, \ldots, x_m)$ such that $\forall i \in \{1, \ldots, m\}, d(x_i, \pi(F)) > k$. If $y \in \pi(F)$, then the closed ball $Cl(\mathcal{B}(y, k))$ is a tame ball that contains $\pi(F)$ and do not intersect $\{x_1, \ldots, x_m\}$. Since $\pi|_{S^n - (A_1 \cup \ldots \cup A_m)} : S^n - (A_1 \cup \ldots \cup A_m) \to S^n - \pi(A_1 \cup \ldots \cup A_m)$ is a homeomorphism, then $D = \pi^{-1}(f^{-1}(Cl(\mathcal{B}(y, k))))$ is the tame ball that we are looking for.

Lemma 1.13. Consider the cylinder $S^{n-1} \times [0, 1]$ and a pair of homeomorphisms $f_i : S^{n-1} \times \{i\} \to S^{n-1} \times \{i\}$, for $i \in \{0, 1\}$, such that both preserve or reverse orientation (for some orientation of $S^{n-1} \times [0, 1]$). Then, there exists a homeomorphism $f : S^{n-1} \times [0, 1] \to S^{n-1} \times [0, 1]$ such that $f|_{S^{n-1} \times \{i\}} = f_i$.

Proof. Let $\phi_i : S^{n-1} \times \{i\} \to S^{n-1}$ be the projection maps, for $i \in \{0, 1\}$. So the maps $\phi_1 \circ f_1 \circ \phi_1^{-1}$ and $\phi_2 \circ f_2 \circ \phi_2^{-1}$ both preserve or reverse orientation. Then, by the Isotopy Classification Theorem, there is an isotopy between the maps $\phi_1 \circ f_1 \circ \phi_1^{-1}$ and $\phi_2 \circ f_2 \circ \phi_2^{-1}$: $\eta : S^{n-1} \times [0, 1] \to S^{n-1}$. Then the map $f : S^{n-1} \times [0, 1] \to S^{n-1} \times [0, 1]$ defined as $f(x, t) = (\eta(x, t), t)$ is a homeomorphism. If $x \in S^{n-1}$, then $f((x, i)) = (\phi_i \circ f_i \circ \phi_i^{-1}(x), i) = (\phi_i \circ f_i((x, i)), i) = \phi_i^{-1} \circ \phi_i \circ f_i((x, i)) = f_i((x, i))$. Thus, $f$ is the homeomorphism that we want.

Lemma 1.14. Let $A_1, A_2$ be disjoint closed tame balls in $S^n$ and $f_i : A_i \to A_i$ be homeomorphisms that both preserve or reverse orientation (for some orientation of $S^n$). Then there exists a homeomorphism $f : S^n \to S^n$ such that $f|_{A_i} = f_i$.

Proof. Immediate from the **Annulus Theorem** and the last lemma.

Lemma 1.15. Let $A, B, C$ be closed tame balls in $S^n$ such that $A, B \subseteq C$ and $g : A \to B$ be a orientation preserving map (for some orientation of $S^n$). Then there exists a homeomorphism $f : C \to C$ such that $f|_{\partial C} = id_{\partial C}$ and $f|_A = g$.

Proof. Let $Y_A$ be the connected manifold whose boundary is $\partial A \cup \partial C$ and $Y_B$ the connected manifold whose boundary is $\partial B \cup \partial C$. By the Annulus Theorem, the spaces $Y_A$ and $Y_B$ are both homeomorphic to $S^{n-1} \times [0, 1]$. Since $g$ is an orientation preserving map, there is a homeomorphism $h$:
Let \( Y_A \to Y_B \) such that \( h|_{\partial A} = g|_{\partial A} \) and \( h|_{\partial C} = id_{\partial C} \). Thus the map \( f : C \to C \) defined by \( f(x) = g(x) \) if \( x \in A \) and \( f(x) = h(x) \) if \( x \in Y_A \) is the homeomorphism that we want.

**Proposition 1.16.** Consider the spaces \( X = S^n - (A_1 \cup A_2 \cup \ldots \cup A_n) \) and \( Y = S^n - (B_1 \cup B_2 \cup \ldots \cup B_n) \), where \( \{A_1, A_2, \ldots, A_n\} \) and \( \{B_1, B_2, \ldots, B_n\} \) are families of disjoint closed tame balls. Let, for every \( i \in \{1, \ldots, n\} \), a homeomorphism \( f_i : A_i \to B_i \) that preserves the orientation of the spheres (induced by a fixed orientation of \( S^n \)). Then there exists a homeomorphism \( f : S^n \to S^n \) such that \( \forall i \in \{1, \ldots, n\}, f|_{A_i} = f_i \).

**Proof.** By Lemma [1.12] there exists \( C_1 \) a closed tame ball that contains \( A_1 \) and \( B_1 \) and does not intersect the tame balls \( A_2, \ldots, A_n, B_2, \ldots, B_n \). We also construct, recursively, tame balls \( C_2, \ldots, C_n \) such that \( C_i \) contains \( A_i \) and \( B_i \) and does not intersect the tame balls \( C_1, \ldots, C_{i-1}, A_{i+1}, \ldots, A_n, B_{i+1}, \ldots, B_n \). By the last lemma, there exists \( f_i' : C_i \to C_i \) a homeomorphism such that \( f_i'|_{\partial C_i} = id_{\partial C_i} \) and \( f_i'|_{A_i} = f_i \). Then the map \( f : S^n \to S^n \) such that \( f(x) = f_i'(x) \) if \( x \in C_i \) and \( f(x) = x \) if \( x \in X - \bigcup_{i=1}^n C_i \) is a homeomorphism that restricts to a homeomorphism between \( X \) and \( Y \).

**Remark.** In particular, it shows that the homeomorphism class of the space \( S^n \) minus \( n \) open tame balls does not depend on the choice of these balls removed.

### 1.5 Cellular complexes

**Proposition 1.17.** Let \( \{\tilde{T}_i\}_{i \in \mathbb{N}} \), \( \{\tilde{Q}_i\}_{i \in \mathbb{N}} \) be two families of cellular structures of a compact \( n \)-manifold \( M \) (possibly with boundary) such that \( \tilde{T}_{i+1} \) is a subdivision of \( \tilde{T}_i \), \( \tilde{Q}_{i+1} \) is a subdivision of \( \tilde{Q}_i \), every \( n \)-cell of \( \tilde{T}_i \) and \( \tilde{Q}_i \) has diameter less than \( \frac{1}{2} \). Let \( T_i, Q_i \) be the \( n-1 \)-skeletons of \( \tilde{T}_i, \tilde{Q}_i \), respectively. Let \( \{f_i\}_{i \in \mathbb{N}} \), where \( f_i : (M, \tilde{T}_i) \to (M, \tilde{Q}_i) \) is a cellular isomorphism which satisfies that \( \forall i \in \mathbb{N}, \forall j < i, f_i|_{T_j} = f_j|_{T_j} \). Then there is a homeomorphism \( f : M \to M \) such that \( \forall i \in \mathbb{N}, f|_{T_i} = f_i|_{T_i} \).

**Proof.** Let \( F_i : M \to \text{Closed}(M) \) defined by \( F_i(x) = \{f_i(x)\} \) if \( x \in T_i \) and if \( x \notin T_i \), \( F_i(x) \) is the \( n \)-cell of \( \tilde{Q}_i \) that contains \( f_i(x) \) (it is unique since \( f_i \) is a cellular isomorphism, which implies that \( f_i(x) \) is not on the boundary of a \( n \)-cell). By the hypothesis on \( \{Q_i\}_{i \in \mathbb{N}} \), we have that \( \forall x \in M, \lim_{i \to \infty} \text{diam}(F_i(x)) = 0 \). Since \( \forall i \in \mathbb{N}, f_{i+1} \) extends \( f_i|_{T_i} \) and \( \tilde{T}_{i+1} \) is a subdivision of \( \tilde{T}_i \), it follows that \( \forall i \in \mathbb{N}, \forall x \in M, \lim_{i \to \infty} f_{i+1}(x) \subset F_i(x) \). Let \( x \in M, i \in \mathbb{N} \) and \( U \) be a neighbourhood of \( F_i(x) \) (we can suppose that \( U \) is open). Let \( i' > i \) such that every \( n \)-cell in \( \tilde{Q}_{i'} \) that intersects \( F_i(x) \) is contained in \( U \).
Let $V = \bigcup\{D \in \mathring{T}_{\nu} : f_\nu(D) \subseteq U\}$. Let $D$ be a $n$-cell of $\mathring{T}_{\nu}$ such that $x \in D$. Then $f_\nu(D) \subseteq U$ (by the choice of $\nu$), which implies that $D \subseteq V$. Since $D$ is arbitrary, we get that $V$ is a neighbourhood of $x$. Let $x' \in V$. By the construction of $V$ we have that there exists a $n$-cell $D'$ in $\mathring{T}_{\nu}$ such that $x' \in D'$ and $f_\nu(D') \subseteq U$. In any case we have that $F'_\nu(x') \subseteq f_\nu(D')$, which implies that $F'_\nu(x') \subseteq U$. Then, by Proposition 1.5 there is a continuous map $f : M \to M$ defined by $\{f(x) = \bigcap_{i \in \mathbb{N}} F_i(x)\}$. Since $\forall i \in \mathbb{N}$, $\forall x \in T_i$, $F_i(x) = \{f_i(x)\}$, we have that $\forall i \in \mathbb{N}$, $f$ extends $f_i|_{T_i}$.

Let $T = \bigcup_{i \in \mathbb{N}} T_i$ and $Q = \bigcup_{i \in \mathbb{N}} Q_i$. Analogously to the last paragraph, the sequence $\{f_{i-1}\}_{i \in \mathbb{N}}$ converges uniformly to a homeomorphism $g : M \to M$ such that $\forall i \in \mathbb{N}$, $g|_{Q_i} = f_{i-1}|_{Q_i}$. It is clear that $g|_Q \circ f|_T = id_T$ and $f|_T \circ g|_Q = id_Q$. Since $T$ and $Q$ are dense in $M$, we have that $g \circ f = id_M$ and $f \circ g = id_M$. Thus $f$ is a homeomorphism. □

2 Sierpiński carpet

Consider the Menger space $M_{n-1}^n$. It is the Sierpiński carpet of dimension $n - 1$. The space $M_{n-1}^n$ is constructed by taking $S^n$ and removing a family of disjoint open tame balls $\{C_i\}_{i \in \mathbb{N}}$ satisfying:

1. $\bigcup_{i \in \mathbb{N}} \mathring{C}_i$ is dense in $S^n$.
2. $\forall \epsilon > 0$, the set $\{i \in \mathbb{N} : diam \mathring{C}_i > \epsilon\}$ is finite.

From now on we are always regarding the Sierpiński carpet of dimension $n - 1$ as a subset of $S^n$ such as described above.

For $n = 1$ this space is clearly a Cantor set. Whyburn showed (Theorem 3 of [19]) that $M_{1}^1$ doesn’t depend of the choice of the family $\{C_i\}_{i \in \mathbb{N}}$, and Cannon showed it for arbitrary $n \neq 4$ (Theorem 1 of [6]). Cannon’s proof also works in dimension 4, as we discuss on the Appendix.

Consider the equivalence relation $\sim = \Delta M_{n-1}^n \cup \bigcup_{i \in \mathbb{N}} \partial C_i^2$. The second property of the construction of $M_{n-1}^n$ is equivalent to say that the quotient map $\varpi : M_{n-1}^n \to M_{n-1}^n/\sim$ is topologically quasiconvex.

**Proposition 2.1.** $M_{n-1}^n/\sim$ is homeomorphic to $S^n$.

**Proof.** This is a special case of the Decomposition Theorem. □

So we can identify $M_{n-1}^n/\sim$ with the $n$-sphere.

For $A \subseteq S^n$, let $\sim_A = \Delta M_{n-1}^n \cup \bigcup_{i \in \mathbb{N}} \partial C_i^2$. Since $\varpi$ is a topologically quasiconvex map, it follows that $\forall A \subseteq S^n$, $M_{n-1}^n/\sim_A$ is Hausdorff. Let $A \subseteq A'$. Consider also the quotient maps given by $\varpi_A : M_{n-1}^n \to M_{n-1}^n/\sim_A$,
\( \varpi_{A',A} : M^n_{n-1}/\sim_{A'} \rightarrow M^n_{n-1}/\sim_A \) and \( \varpi'_A : M^n_{n-1}/\sim_A \rightarrow S^n \). If \( a \in S^n \) and \( A \subseteq S^n \), we use \( \varpi_a \), \( \varpi_{A,a} \) and \( \varpi'_a \) instead of \( \varpi_{[a]} \), \( \varpi_{A,[a]} \) and \( \varpi'_{[a]} \).

There is also a quotient map \( \tilde{\varpi} : S^n \rightarrow S^n \) such that \( \tilde{\varpi}|_{M^n_{n-1}} = \varpi \) and \( \forall i \in \mathbb{N}, \tilde{\varpi}(C_i) = \varpi(\partial C_i) \). Analogously, if \( A \subseteq S^n \), then there is also a natural quotient map \( \tilde{\varpi}_A : S^n - \bigcup_{\varpi(C_i) \notin A} C_i \rightarrow M^n_{n-1}/\sim_A \).

**Proposition 2.2.** For every finite set \( A \), \( M^n_{n-1}/\sim_A \) is homeomorphic to \( S^n \) minus \( \#A \) disjoint open tame balls.

**Remark.** By Proposition 1.16 the space \( S^n \) minus \( \#A \) disjoint open tame balls is well defined, up to homeomorphisms.

**Proof.** Let \( Z = M^n_{n-1} \cup \bigcup_{i \in A} C_i \). By the Decomposition Theorem \( Z/\approx_A \) is homeomorphic to \( S^n \), where \( \approx_A = \sim_A \cup \Delta Z \). So \( M^n_{n-1}/\sim_A = (Z-\bigcup_{i \in A} C_i)/\approx_A \) is homeomorphic to \( S^n \) minus \( \#A \) disjoint open tame balls.

Note that the last proposition and Proposition 1.7 implies that the space \( M^n_{n-1} \) is homeomorphic to an inverse limit of \( n \)-balls that quotient to the \( n \)-sphere collapsing their boundaries and such that the set of points of the \( n \)-sphere that are the image of some collapsing boundary is countable dense. It remains to prove that such inverse limit is unique, up to homeomorphisms.

**Definition 2.3.** We say that a triangulation \( \tilde{T} \) of \( S^n \) with \( n-1 \)-skeleton \( T \) is compatible with the Sierpiński carpet that we fixed, if:

1. \( T \subseteq M^n_{n-1} \)
2. \( \forall i \in \mathbb{N}, T \cap \partial C_i = \emptyset \) or \( \partial C_i \subseteq T \).
3. \( T \) intersects only a finite number of spheres of \( \{\partial C_i\}_{i \in \mathbb{N}} \).
4. \( \tilde{T} \) induces a triangulation of \( S^n - \bigcup\{C_i : \partial C_i \subseteq T\} \)

**Lemma 2.4.** Let \( \tilde{T} \) be a compatible triangulation of \( S^n \) and \( T \) its \( n-1 \)-skeleton. So \( \forall A \subseteq S^n \), finite, \( \tilde{\varpi}_A(\tilde{T}) = \{\tilde{\varpi}_A(B) : B \in \tilde{T}, B \subseteq S^n - \bigcup_{i \in A} C_i\} \) is a CW structure of \( M^n_{n-1}/\sim_A \) with its \( n-1 \)-skeleton given by \( \varpi_A(T) \).

**Proof.** Let \( \tilde{W} \) be the CW structure of \( S^n \) given by the triangulation \( \tilde{T} \) and, for each \( k \in \mathbb{N} \), let \( W^k \) be its \( k \)-skeleton. We have that \( W^k \) coincides with the \( k \)-skeleton of \( \tilde{T} \). For \( i \in \mathbb{N} \), let \( Y^k_i = \{B \in W^k : B \subseteq \partial C_i\} \). We have that \( Y^k_i \) is a subcomplex of \( W^k \). If \( k \leq n-1 \), then the quotient \( \tilde{\varpi}_A(W^k) \) has the CW structure given by \( W^k \) quotiented by all subcomplexes \( \{Y^k_i : \varpi(C_i) \notin A\} \) (note that there is just a finite number of sets \( Y^k_i \) that are not singletons). Let \( B \) be a cell in \( W^n \). If \( B \cap C_i \neq \emptyset \) for some \( i \in \mathbb{N} \)
such that \( \varpi(C_i) \notin A \), then \( B \subseteq C_i \) and then we discard it. If \( \forall i \in \mathbb{N} \) such that \( \varpi(C_i) \subseteq A \), \( B \cap C_i = \emptyset \), then \( \varpi_A(\text{int } B) \) is homeomorphic to \( \text{int } B \) by the Decomposition Theorem. So \( \varpi_A(B) \) is a cell that we attach to \( \varpi_A(W^k) \). Doing this for every \( n \)-cell \( B \), we get the cellular structure of the quotient.

\[ \square \]

**Remark.** Note that the induced structure on the quotient may not be a triangulation, since some \( n-1 \)-faces, that are contained in some \( \partial C_i \) such that \( \varpi(C_i) \notin A \), must collapse.

Analogously, we have:

**Lemma 2.5.** Let \( \tilde{T} \) be a compatible triangulation of \( S^n \) and \( T \) its \( n-1 \)-skeleton. Let \( A = \bigcup \{ \varpi(\partial C_i) : \partial C_i \subseteq T \} \). So \( \varpi_A(\tilde{T}) = \{ \varpi_A(B) : B \in \tilde{T}, B \subseteq S^n - \bigcup_{i \in A} C_i \} \) is a triangulation of \( M_{n-1}/\sim_A \) with its \( n-1 \)-skeleton given by \( \varpi_A(T) \).

Let \( X \) be a Hausdorff compact space and \( \pi : X \to S^n \) a topologically quasiconvex map satisfying:

1. The set \( P = \{ x \in S^n : \#\pi^{-1}(x) > 1 \} \) is countable and dense on \( S^n \).
2. \( \forall p \in P \), the space \( X_p = X/\sim_p' \), where \( \sim_p' = \Delta X \cup \bigcup_{x \neq p} \pi^{-1}(x)^2 \), is homeomorphic to a closed ball \( D^n \).

By Proposition 1.7 these properties of \( X \) are equivalent to \( X \) being an inverse limit of \( n \)-balls that quotient to the \( n \)-sphere collapsing their boundaries and such that the set of points of the \( n \)-sphere that are images of some collapsing boundary is countable dense.

Let, for \( S \subseteq S^n, \sim_S' = \Delta X \cup \bigcup_{x \neq s} \pi^{-1}(x)^2, X_S = X/\sim_S' \) and \( \pi_S : X \to X_S \) the quotient map. If \( S \subseteq S' \subseteq S^n \), let \( \pi_{S',S} : X_{S'} \to X_S \) and \( \pi' : X_S \to S^n \) be the quotient maps that commute with \( \pi \). If \( S \subseteq S^n \) and \( S \subseteq S^n \), we use \( \pi_s, \pi_{S,s} \) and \( \pi'_s \) instead of \( \pi_{\{s\}}, \pi_{S,\{s\}} \) and \( \pi'_{\{s\}} \).

**Proposition 2.6.** If \( S \) is finite, then \( X_S \) is homeomorphic to \( S^n \) minus \( \#S \) open tame balls.

**Proof.** Let \( \{B_s\}_{s \in S} \) be a set of open tame balls on \( S^n \) such that their closures are pairwise disjoint and \( \forall s \in S, s \in B_s \). Let \( Z = S^n - \bigcup_{s \in S} B_s \) and \( Z' = \pi_S^{-1}(Z) \). We have that \( \pi_S|_{Z'} : Z' \to Z \) is a homeomorphism.

Let \( s \in S \). We have that \( \pi_{S,s}(Z') \) is homeomorphic to \( Z' \). Let \( A_s \) be the preimage in \( X_S \) of the boundary of \( B_s \). We have that \( \pi_{S,s}(A_s) \) is tame in \( X_S \) since \( \pi'_S(A_s) \) is tame in \( S^n \) and \( \pi'_S(A_s) = \pi'_S(\pi_{S,s}(A_s)) \), where \( \pi'_s \) is a homeomorphism onto its image, outside of any neighbourhood of \( \pi^{-1}_S(s) \).
Since $\pi_{S,s}(A_s)$ is tame in $X_s$, then, by the Annulus Theorem, the connected submanifold $Y$ that has $\pi_{S,s}(A_s) \cup \pi^{-1}_s(s)$ as its boundary is homeomorphic to $S^{n-1} \times [0,1]$, which implies that $\tilde{Y} \cup \pi_{S,s}(Z')$ is homeomorphic to $Z'$. Let $Z'_1 = \pi_{S,s}(Y \cup \pi_{S,s}(Z'))$. We have that $Z'_1$ is homeomorphic to $Z'$. By an induction process (since $S$ is finite), we get that $X_S$ is homeomorphic to $Z'$ and then it is homeomorphic to $Z$, which is $S^n$ minus $\#S$ open tame balls.

**Proposition 2.7. (Cannon - Lemma 1 of [6])** Let $\tilde{T}$ be a compatible triangulation of $S^n$, $T$ be its $n-1$-skeleton and $\epsilon > 0$. Let $i \in \mathbb{N}$. Then there exists a subdivision $\tilde{T}'$ of $\tilde{T}$ such that it is compatible, $T' \cap \bigcup_{j \in \mathbb{N}} C_j = (T \cap \bigcup_{j \in \mathbb{N}} C_j) \cup \partial C_i$, where $T'$ is the $n-1$-skeleton of $\tilde{T}'$, and for every $n$-simplex $D$ in $\tilde{T}'$, $\text{diam } D < \epsilon$.

**Remark.** For $n = 2$, the proposition above is due to Whyburn (Lemma 1 of [19]).

This proposition is stated and proved in [6] with the restriction that $n \neq 4$, since some theorems used in Cannon’s proof had this restriction at that time. However, Cannon’s proof works without this restriction, as we comment on the Appendix.

**Theorem 2.8.** $X$ is homeomorphic to $M^n_{n-1}$.

**Remark.** This theorem is proved for $n = 2$ in [18]. Our methods used in this proof are similar to theirs.

**Proof.** Here we do a back-and-forth argument.

Let $\tilde{T}_0$ be a compatible triangulation of $S^n$ and its $n-1$-skeleton $T_0$.

Since $S^n$ is countable dense homogeneous, we can assume that $M^n_{n-1}/\sim = S^n$ and $P = \{x \in S^n : \#\varpi^{-1}(x) > 1\}$ (remember that $P$ was originally defined for $X$ and not for $M^n_{n-1}$).

Let $S_0 = \varpi(\bigcup_{j \in \mathbb{N}} C_j \cap T_0)$ and $R_0 = S_0$. By Proposition 1.16 there is a homeomorphism $f_0 : M^n_{n-1}/\sim S_0 \to X_{R_0}$. We have that $\forall x \in \varpi S_0(T_0)$, $\#\pi^{-1}_{R_0}(f_0(x)) = 1$. Then $\pi_{R_0}|_{\pi^{-1}_{R_0}(\varpi S_0(T_0))}$ is a homeomorphism onto its image, which implies that there is an embedding $\tilde{f}_0 : T_0 \to X$ that commutes the diagram:

$$
\begin{array}{ccc}
T_0 \overset{\varpi|T_0}{\longrightarrow} M^n_{n-1}/\sim S_0 & \overset{f_0}{\longrightarrow} & X_{R_0} \\
\downarrow \quad \quad \quad \pi \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
Suppose that we have a family of triangulations $\{\tilde{T}_k : 0 \leq k \leq 2i\}$ that are compatible and, together with their respective $n-1$-skeletons $T_k$, satisfies:

1. $\forall k \leq 2i$, $\tilde{T}_k$ a subdivision of $\tilde{T}_{k-1}$.

2. $\bigcup_{j=1}^{i} \partial C_j \subseteq T_{2i}$.

Let $S_k = \tilde{\omega}(\bigcup_{j \in \mathbb{N}} C_j \cap T_k)$. We have that $S_k \supseteq S_{k-1}$.

Suppose also that we have a family of embeddings $\tilde{f}_k : T_k \to X$ and a family of sets $R_k = \pi(\tilde{f}_k(\bigcup_{j \in \mathbb{N}} C_j \cap T_k))$, with $0 \leq k \leq 2i$, satisfying:

3. $\forall k \leq 2i$, $\bigcup\{\pi(\partial C_j) : 2j \leq k\} \subseteq R_k$.

We have that $\forall k \leq 2i$, $R_k \supseteq R_{k-1}$.

Finally, suppose that we have families of homeomorphisms $f_k : M^n_{n-1} / \sim_{S_k} \to X_{R_k}$, $\hat{f}_k : S^n \to S^n$ and, $f_{s,k} : M^n_{n-1} / \sim_s \to X_{f_k(s)}$, with $0 \leq k \leq 2i$, satisfying:

4. The diagram commutes, for every $s \in S_k$:

$$
\begin{array}{cccc}
T_k & \overset{\tau_{s_k}}{\searrow} & M^n_{n-1} / \sim_{S_k} & \overset{\tau'_{s_k}}{\searrow} & M^n_{n-1} / \sim_s & \overset{\tau'}{\searrow} & S^n \\
\tilde{f}_k & \downarrow & f_k & \downarrow & f_{s,k} & \downarrow & \hat{f}_k \\
X & \overset{\pi_{R_k}}{\searrow} & X_{R_k} & \overset{\pi'_{R_k, f_k(s)}}{\searrow} & X_{f_k(s)} & \overset{\pi'_{f_k(s)}}{\searrow} & S^n
\end{array}
$$

5. $\forall k \leq 2i$, the diameter of every simplex of $\tilde{\omega}_s(\tilde{T}_k)$, $f_{s,k-1} \circ \tilde{\omega}_s(\tilde{T}_k)$, for every choice of $s \in S_{k-1}$, and $\tilde{\omega}(\tilde{T}_k)$ and $\hat{f}_{k-1} \circ \tilde{\omega}(\tilde{T}_k)$ is less than $\frac{1}{2^k}$.

Observe that the maps $f_{s,k}$ and $\hat{f}_k$ are uniquely defined by $f_k$ and the commutative diagram.

Let $\tilde{T}_{2i+1}$ be a subdivision of $\tilde{T}_{2i}$ that is compatible and let $T_{2i+1}$ its $n-1$-skeleton, satisfying:

2. $\partial C_{i+1} \subseteq T_{2i+1}$.

5. The diameter of every simplex of $\tilde{\omega}_s(\tilde{T}_{2i+1})$, $f_{s,2i} \circ \tilde{\omega}_s(\tilde{T}_{2i+1})$, for every choice of $s \in S_{2i}$, and $\tilde{\omega}(\tilde{T}_{2i+1})$ and $\hat{f}_{2i} \circ \tilde{\omega}(\tilde{T}_{2i+1})$ is less than $\frac{1}{2^{2i+1}}$. 
We can do that by subdividing $\tilde{T}_{2i}$, using the last proposition, to have simplexes small enough such that the uniform continuity property of these maps gives induced triangulations satisfying (5).

If $\varpi(\partial C_{i+1}) \in S_{2i}$, then we take $S_{2i+1} = S_{2i}$, $R_{2i+1} = R_{2i}$, $\tilde{T}_{2i+1} = \tilde{T}_{2i}$, $f_{2i+1} = f_{2i}$ and $\hat{f}_{2i+1} = \hat{f}_{2i}$. In this case, all six conditions are satisfied.

Suppose that $\varpi(\partial C_{i+1}) \notin S_{2i}$. Let $S_{2i+1} = \varpi(\bigcup_{j \in \mathbb{N}} C_j \cap T_{2i+1})$. We have that $S_{2i+1} \supseteq S_{2i}$ and $\varpi(\partial C_{2i+1}) \in S_{2i+1}$. For every $s \in S_{2i+1} - S_{2i}$, take $E_s$ as the $n$-simplex in $\tilde{T}_{2i}$ such that $s \in \tilde{\varpi}(E_s)$, choose $p_s \in \tilde{\varpi}(E_s) \cap P$ (since $s \in \tilde{\varpi}(E_s)$, then $\text{int}(\tilde{\varpi}(E_s)) \neq \emptyset$) and define $R_{2i+1} = R_{2i} \cup \{p_s : s \in S_{2i+1} - S_{2i}\}$. We have that $R_{2i+1} \supseteq R_{2i}$ and the condition (3) is automatically satisfied. (it is necessary to add something to the set to satisfy this condition only on even steps).

By Lemma 2.5 $\tilde{T}_{2i}$ induces a triangulation in $M_{n-1}/\sim S_{2i+1}$. Let $D$ be a $n$-simplex in $\tilde{T}_{2i}$ and $S_D = P \cap \varpi(\text{int } D)$. We need a homeomorphism $f_D : (D \cap M_{n-1})/\sim S_{2i+1} \to Y_D$, where $Y_D$ is the closure of the connected component of $X_{R_{2i+1}} - \pi_{R_{2i+1}}(\hat{f}_{2i}(T_{2i}))$ that contains $\pi_{R_{2i+1}}^{-1}(\hat{f}_{2i}(S_D))$, such that $f_D|_{\varpi S_{2i+1}(\partial D)} = \pi_{R_{2i+1}} \circ \hat{f}_{2i} \circ \varpi_S^{-1}|_{\varpi S_{2i+1}(\partial D)}$. Every term of this composition is well defined and continuous, when restricted to suitable domains. So $f_D|_{\varpi S_{2i+1}(\partial D)}$ is well defined, continuous and, by the choice of the points $p_s$, the spaces $(D \cap M_{n-1})/\sim S_{2i+1}$ and $Y_D$ are homeomorphic. Then, by Proposition 1.16 there exists a homeomorphism $f_D$ that extends $\pi_{R_{2i+1}} \circ \hat{f}_{2i} \circ \varpi_S^{-1}|_{\varpi S_{2i+1}(\partial D)}$ and such that $\forall s \in S_{2i+1} \cap S_D$, $f_D(\varpi S_{2i+1}(s)) = \pi_{R_{2i+1}}^{-1}(p_s)$. By Proposition 1.3 we can suppose also that $f_D(S_{2i+1}(S_D)) = \pi_{R_{2i+1}}^{-1}(S_D)$. Take $f_{2i+1} : M_{n-1}/\sim S_{2i+1} \to X_{R_{2i+1}}$ as $f_{2i+1}(x) = f_D(x)$ if $x \in (D \cap M_{n-1})/\sim S_{2i+1}$. Since $f_{2i+1}|_{T_{2i}}$ does not depend of the choice of the $n$-simplex $D$ and two $n$-simplexes intersect only in $T_{2i}$, we have that $f_{2i+1}$ is well defined and a homeomorphism. It is immediate that there is a homeomorphism $\hat{f}_{2i+1} : S^n \to S^n$ that commutes the diagram:

\[
\begin{array}{ccc}
M_{n-1}/\sim S_{2i+1} & \xrightarrow{\varpi S_{2i+1}} & S^n \\
\downarrow f_{2i+1} & & \downarrow f_{2i+1} \\
X_{R_{2i+1}} & \xrightarrow{\pi_{R_{2i+1}}^{-1}} & S^n \\
\end{array}
\]

Observe that, since it holds for every $n$-simplex $D$, we have that $\hat{f}_{1}(P) = P$. We have also that $\hat{f}_{2i+1}|_{\varpi(T_{2i})} = \hat{f}_{2i}|_{\varpi(T_{2i})}$.

Define $\hat{f}_{2i+1} : T_{2i+1} \to X$ as $\hat{f}_{2i+1} = \pi_{R_{2i+1}}^{-1} \circ f_{2i+1} \circ \varpi S_{2i+1}|_{T_{2i+1}}$. It is clear that $\hat{f}_{2i+1}$ is an embedding that extends $\hat{f}_{2i}$.
For every choice of $s \in S_{2i+1}$, let $f_{s,2i+1} : M^n_{n-1} / \sim_s \to X_{f_{2i+1}(s)}$ be the homeomorphism that commutes the diagram (i.e., satisfies (4)):

Now, if we have $\tilde{T}_{2i+1}$, $S_{2i+1}$, $R_{2i+1}$, $f_{2i+1}$, $\tilde{f}_{2i+1}$ and $\forall s \in S_{2i+1}$, $f_{s,2i+1}$, then we do an entirely analogous construction, but requiring that $\pi(\partial C_i) \in R_{2i+2}$.

By Proposition 1.17 there are homeomorphisms $\hat{f} : S^n \to S^n$ and, $\forall s \in P$, $f_s : M^n_{n-1} / \sim_s \to X_{f(s)}$ such that $\forall i \in \mathbb{N}$, $\hat{f}|_{\sim_i} = f_i|_{\sim_i}$ and $f_s|_{\sim_i} = f_s|_{\sim_i}$. Let $T = \bigcup_{i \in \mathbb{N}} T_i$. Since it commutes for every $i \in \mathbb{N}$, the following diagram commutes (for every $s \in P$):

Since $T$ contains $\bigcup\{C_i : i \in \mathbb{N}\}$, it is dense on $M^n_{n-1}$ and $\sim_i$ is dense on $M^n_{n-1} / \sim_s$, which implies that the following diagram commutes:

By the back-and-forth construction, $\hat{f}(P) = P$. Then the family of homeomorphisms $\{\hat{f}, f_i, i \in \mathbb{N}\}$ induces an homeomorphism $f : M^n_{n-1} \to X$. □

Appendix

Here we show a brief comment on what works in the Approximation Theorem for Cellular Maps on dimension 4 that should be enough for Cannon’s lemma that we used before (Proposition 2.7).
Definition 2.9. Let $X$ be a topological space, $(Y,d)$ a metric space and $\pi : X \to Y$ a continuous map. We say that $\pi$ is a near homeomorphism if for every $\epsilon > 0$, there exists a homeomorphism $f : X \to Y$ such that for every $x \in X$, $d(\pi(x), f(x)) < \epsilon$.

Proposition 2.10. (Freedman, Theorem 9.1’ of [9]) Let $\pi : S^n \to S^n$ be a topologically quasiconvex map such that the set $\{x \in S^n : \#\pi^{-1}(x) > 1\}$ is nowhere dense in $S^n$. Then $\pi$ is a near homeomorphism (for any metric on $S^n$).

Proposition 2.11. (Corollary 7.1 of [9]) Let $\pi : M \to N$ be a near homeomorphism between compact manifolds. Let $C$ be a compact subset of $M$ such that for all $p \in C$, $\{p\}$ is saturated. Then, for every $\epsilon > 0$, there exists a homeomorphism $f : M \to N$ such that for every $x \in M$, $d(\pi(x), f(y)) < \epsilon$ and $f|_C = \pi|_C$.

Combining the two propositions, we get:

Proposition 2.12. Let $\pi : S^n \to S^n$ be a topologically quasiconvex map such that the set $\{x \in S^n : \#\pi^{-1}(x) > 1\}$ is nowhere dense in $S^n$, $U = \{U_\alpha\}_{\alpha \in \Gamma}$ be an open cover of $S^n$ where all sets are saturated and $C$ be a compact subset of $S^n$ such that for all $p \in C$, $\{p\}$ is saturated. Then there exists a homeomorphism $g : S^n \to S^n$ such that $g \circ \pi|_C = id_C$ and for every $x \in S^n$, there exists $\alpha \in \Gamma$ such that $x, g \circ \pi(x) \in U_\alpha$.

Proof. Fix a metric $d$ on $S^n$. The set $\pi(U) = \{\pi(U_\alpha)\}_{\alpha \in \Gamma}$ is an open cover of $N$. Let $\epsilon > 0$ be the Lebesgue number of the covering $\pi(U)$. By the Proposition 2.10, the map $\pi$ is a near homeomorphism, which implies, by Proposition 2.11, that there exists a homeomorphism $f : M \to N$ such that $f|_C = \pi|_C$ and $\forall x \in S^n$, $d(\pi(x), f(x)) < \epsilon$.

Let $x \in M$. Take $y = f^{-1} \circ \pi(x)$. We have that $d(f(y), \pi(y)) < \epsilon$. But $f(y) = \pi(x)$, which implies that $d(\pi(x), \pi(y)) < \epsilon$. Since $\epsilon$ is the Lebesgue number of the cover $\pi(U)$, there exists $\alpha \in \Gamma$ such that $\pi(x), \pi(y) \in \pi(U_\alpha)$. Since $U_\alpha$ is saturated, we have that $x, y \in U_\alpha$. Then $x, f^{-1} \circ \pi(x) \in U_\alpha$. We have also that $f|_C = \pi|_C$, which implies that $f^{-1} \circ \pi|_C = id_C$. Thus, $f^{-1}$ is the homeomorphism that we want.

Remark. Note that if we take $C$ as a $n - 1$-sphere embedded in $S^n$, then the fact that there exists an homeomorphism $f : S^n \to S^n$ that sends $C$ to $\pi(C)$ implies that $C$ is tame if and only if $\pi(C)$ is tame.

Proposition 2.12 and the Remark are enough to replace the Approximation Theorem for Cellular Maps and its corollary on Cannon’s proof of
Proposition 2.7 Since the Annulus Theorem also works in dimension 4 (Quinn, [14]), then there are no more restrictions on the dimension. Thus, Proposition 2.7 works in dimension 4 as well. For the same reason, Cannon’s theorem that says that the space $M_{n-1}^n$ does not depend on the choices of removed open balls (Theorem 1 of [6]) also works in dimension 4.

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