On a transformation of Bohl and its discrete analogue

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Abstract. Fritz Gesztesy’s varied and prolific career has produced many
transformational contributions to the spectral theory of one-dimensional Schrö-
deringer equations. He has often done this by revisiting the insights of great
mathematical analysts of the past, connecting them in new ways, and rein-
venting them in a thoroughly modern context.

In this short note we recall and relate some classic transformations that
figure among Fritz Gesztesy’s favorite tools of spectral theory, and indeed
thereby make connections among some of his favorite scholars of the past,
Bohl, Darboux, and Green. After doing this in the context of one-dimensional
Schrödinger equations on the line, we obtain some novel analogues for discrete
one-dimensional Schrödinger equations.

Dem einzigartigen Fritz gewidmet.

1. Introduction

In 1906 [3], Bohl introduced a nonlinear transformation for solutions of Sturm-
Liouville equations, which is an exact, albeit implicit, counterpart to the Liouville-
Green approximation [23]. Bohl used the transformation as a tool in oscillation
theory, and this has continued to be the main use of the Bohl transformation in the
hands of later authors. Notably, Ráb [24] used the Bohl transformation to prove
necessary and sufficient conditions for oscillation of solutions, and showed that it
is an effective foundation for Sturm-Liouville oscillation theory. Willett’s lecture in
[29] provides a clear description in English of the Bohl transformation in oscillation
theory, including the contributions of Ráb, while Reid’s monograph [27] compares
and contrasts it with the Prüfer transformation. See also [26, 13, 16, 17, 18].

In [10] §4, Davies and Harrell introduced a non-oscillatory variant of the Bohl
transformation to connect the notions of Liouville-Green approximation, Green
functions, and Agmon metrics for exponential decay of solutions. Some spectral
bounds were derived as consequences. This analysis was extended in a series of
articles by Chernyavskaya and Shuster (e.g., [4, 5, 8]), to address questions of
solvability, regularity, estimates of Green functions, and asymptotics in Sturm-
Liouville theory.

In this note we begin with a largely expository treatment of the classic Bohl
transformation, concentrating for simplicity on the situation where all coefficients
are real and regular and the Sturm-Liouville equation is in the standard form of
the one-dimensional Schrödinger equation. Then in the last section we show how
the technique can be adapted to the case of a discrete Schrödinger equation on the
integers.

2. The interplay of the Bohl and Green functions

Let $V$ be real-valued and continuous, and consider a solution basis for the
Sturm-Liouville equation

\[ -u'' + V(x)u = 0; \]

we may normalize the basis $u_{1,2}(x)$ so that $W[u_1, u_2] := u_1 u_2' - u_2 u_1' = 1$. (There
is no assumption of an eigenvalue 0. For our purposes a possible nonzero spectral
parameter has simply been incorporated into $V$.) The Bohl transformation maps
this solution basis onto a second solution basis with remarkable prope rties, some of
which are collected in a nutshell version in this section.

**Definition 2.1.** Given a solution basis $\{u_{1,2}(x)\}$ of (2.1), chosen so that the
Wronskian $W[u_1, u_2] := u_1 u_2' - u_2 u_1' = 1$, we define the
diagonal function by

\[ Z[u_1, u_2](x) := \left( u_1(x) u_2(x) \right)^{1/2}. \]

The Bohl transformation of $\{u_{1,2}(x)\}$ is an equivalent solution basis of (2.1), defined
in terms of $\{u_{1,2}(x)\}$ by

\[ B : \{u_1(x), u_2(x)\} \rightarrow \left\{ \phi^\pm(x) := Z(x) \exp \left( \pm \int_{x_1}^x \frac{1}{2Z^2(t)} dt \right) \right\}. \]

**Remark 1.** a) The choice of the complex phase of the square root in (2.2)
is unimportant, but should be continuous in $x$. For brevity we write $Z(x)$ for $Z[u_1, u_2](x)$ when the dependence on $u_{1,2}$ is clear. The reason for calling it the
diagonal function is that, as will be seen below,

\[ Z^2(x) = G_0(x, x), \]

where $G_0(x, x)$ is the diagonal of a certain Green function $G_0(x, y)$ for (2.1). Among
the useful properties of the function $Z$ is that it solves the diagonal differential
equation

\[ J[Z] := -Z'' + V(x)Z - \frac{1}{4Z^3} = 0, \]

[10, 17, 18, 29].

b). In fact, with the oscillatory situation in mind Bohl originally wrote the solution
basis in the Liouville-Green form

\[ \frac{1}{\sqrt{R}} \sin \left( \int_{x_1}^x R(t) dt \right) \]

and

\[ \frac{1}{\sqrt{R}} \cos \left( \int_{x_1}^x R(t) dt \right), \]

which is equivalent to (2.3) under the identification $2Z^2 \rightarrow i/R$ and some harmless
linear combinations.

c). We recall that Gesztesy and Simon [12] have made connections between the
Krein spectral shift function, the related Xi function, and the diagonal of the Green
function.
Calling upon §4, we collect some facts, which are verifiable directly:

**Theorem 2.1.**

1. If \( u_1(x) \) and \( u_2(x) \) are solutions to (2.1) such that \( W[u_1, u_2] = 1 \), and \( u_1(x)u_2(x) \) does not vanish on the interval \((a, b)\), then \( Z[u_1, u_2](x) \) satisfies (2.3) on \((a, b)\).

2. If \( Z \) is a nonvanishing solution of (2.3) on \((a, b)\), then \( \phi^\pm(x) \) as defined in (2.3) provide a pair of independent solutions of (2.1) on \((a, b)\). In particular, each \( \phi^\pm(x) \) is a linear combination of \( \{u_1, u_2\} \) and vice versa.

3. If \( x_{>,<} := \max(x, y) \), resp. \( \min(x, y) \), then

\[
G_0(x, y) := Z(x)Z(y)\exp\left(-\int_{x_{<}}^{x_{>}} \frac{1}{2Z^2(t)} dt\right)
\]

is a Green function for (2.1), in the sense that

\[
\left(-\frac{\partial^2}{\partial x^2} + V(x)\right) G_0(x, y) = \delta(x - y).
\]

As an integral kernel, \( G_0 \) defines the inverse of a particular realization of \(-\frac{d^2}{dx^2} + V\), but not a priori one for which the domain of definition includes \( u_1 \) or \( u_2 \), because of a possible mismatch of boundary conditions at finite points. This issue is not important for questions of oscillation or asymptotic behavior at infinity, but another concern remains, namely the possibility that \( Z \) vanishes, which would invalidate the transformation. Because of this we recall that in the absence of imposed finite boundary conditions, complex solutions can always be used to prevent \( Z \) from vanishing:

**Lemma 2.2.** Suppose that \( u \) is a solution of (2.1) on a finite or infinite interval \((a, b)\), and that at some \( x_0 \in (a, b) \), \( \text{Re}(u(x_0))\text{Im}(u(x_0)) \neq 0 \) and \( u'(x_0)/u(x_0) \notin \mathbb{R} \). Then \( u \) does not vanish on \((a, b)\).

**Proof.** Because \( V \) is real-valued, \( \text{Re} u \) and \( \text{Im} u \) each satisfy (2.1), and it therefore suffices to show that they are independent.

Letting \( \alpha = u'(x_0)/u(x_0) \), a calculation shows that

\[
\frac{\text{Re} u'(x_0)}{\text{Re} u(x_0)} = \text{Re} \alpha - \text{Im} \alpha \frac{\text{Im} u(x_0)}{\text{Re} u(x_0)},
\]

\[
\frac{\text{Im} u'(x_0)}{\text{Im} u(x_0)} = \text{Re} \alpha + \text{Im} \alpha \frac{\text{Re} u(x_0)}{\text{Im} u(x_0)}.
\]

It follows that

\[
\frac{\text{Re} u'(x_0)}{\text{Re} u(x_0)} - \frac{\text{Im} u'(x_0)}{\text{Im} u(x_0)} = -\text{Im} \alpha \left( \frac{\text{Im} u(x_0)}{\text{Re} u(x_0)} + \frac{\text{Re} u(x_0)}{\text{Im} u(x_0)} \right),
\]

and therefore

\[
W[\text{Im} u, \text{Re} u] = -\text{Im} \alpha \left( (\text{Re} u(x_0))^2 + (\text{Im} u(x_0))^2 \right) \neq 0.
\]

□

This standard lemma implies that given any two linearly independent solutions of (2.1), it is always possible to find a pair of complex-valued linearly independent combinations that are non-vanishing on \((a, b)\). The Wronskian of the new pair may be set to 1 by multiplying one solution by an appropriate constant, justifying the
conclusions of Theorem 2.1. It is of some use to consider a particular $Z$ determined as follows.

By construction any solution $u$ as set forth in Lemma 2.2 will be linearly independent of its complex conjugate. We may therefore choose a complex number $\alpha$ so that

$$W[u, \alpha^2 \pi] = 1,$$

which ensures that

$$Z(x) = \alpha |u(x)|.$$

Indeed, $\arg(\alpha)$ is restricted to the values $k\pi/4$ for integer $k$, as can be seen for example from (2.4), which implies that

$$-|u'' + V|u| = \frac{1}{4\alpha^2 |u|^3} \in \mathbb{R}.$$

(In passing we note the implication that the expression on the left does not change sign.) In fact, there are only two truly distinct cases for $\arg(\alpha)$, viz., 0 and $\pi/4$, due to the simple scalings in (2.4) and Theorem 2.1 when $Z$ is replaced by $iZ$. If $\alpha > 0$, i.e., $Z(x) > 0$, then the solutions $\phi^{\pm}$ do not change sign, which corresponds to the case of disconjugacy for the ODE (2.1) in the classical theory [20, 26]. This situation was the focus of [10], in which some spectral bounds were derived and it was argued that $1/2Z^2$ defines an Agmon metric.

Otherwise, it may be assumed without of loss of generality that $\arg(\alpha) = \pi/4$, for which solutions may oscillate, in that their arguments increase or decrease by $n\pi$ for $n > 1$ as $x \to \infty$. A central question of Sturmian theory is whether solutions oscillate infinitely often, or only finitely often. When the increase in the argument of a solution is infinite, the equation (2.1) is said to be oscillatory. An approach to oscillation theory, equivalent to that of [24, 29] but bringing out the role of Green functions, can be based on the following version of a result of Gagliardo, as cited in [29]:

**Corollary 2.3** (Cf. [29], Corollary 3.2.). Suppose that (2.1) holds on an infinite interval $(a, \infty)$, and let $G_B(x, y)$ be the Green function constructed according to the prescription leading to (2.7). Then either $G_B(x, y) \in \mathbb{R}$ for all $x, y$, in which case the solution basis $\phi^{\pm}$ is nonoscillatory, or else: The phase of $\phi^{\pm}$ has only finite increase on $(a, \infty)$ iff $1/G(x, x) \in L^1(a, \infty)$.

As has been known since the work of Wigner and von Neumann and Wigner, it is possible for eigenvalues to be embedded in the continuous spectrum of Sturm-Liouville equations [22, 25]. This phenomenon requires oscillatory solutions to be square-integrable, and can thus be related to the Bohl transformation as follows:

**Corollary 2.4** (Ráb). An oscillatory solution of (2.1), written in the form

$$Z(x) \exp \left( \pm \frac{i}{2} \int_a^x \frac{1}{|Z(t)|^2} dt \right),$$

exists and is square integrable if and only if the nonlinear equation

$$-w''(x) + V(x)w(x) = -\frac{1}{4w^3(x)}$$

has a square-integrable solution.

We close this section with a Darboux-type factorization [9, 14], the novel feature of which is the role played by the diagonal function and the Bohl solution
basis (2.3). For any complex valued, nonvanishing function $Z(x) \in AC^1([a,b])$, define

$$D^\pm[Z] := \frac{d}{dx} - \frac{Z'}{Z} \mp \frac{1}{2Z^2}.$$  

It is immediate to see that

$$D^\pm[Z] \phi^\pm = 0,$$

where $\phi^\pm$ are defined in terms of $Z$ by (2.3). A further calculation reveals that

$$(2.9) \quad \left( D^\pm[Z] - 2\frac{d}{dx} \right) D^\pm[Z] = -\frac{d^2}{dx^2} + \frac{Z''}{Z} + \frac{1}{4Z^4} = -\frac{d^2}{dx^2} + V(x),$$

provided that $Z$ satisfies the diagonal differential equation (2.4). An alternative way to express the two factorizations in (2.9) is that

$$-\frac{d^2}{dx^2} + V(x) = \left( D[Z] \right)^* D[Z],$$

where $^*$ designates the formal adjoint operation.

3. The discrete form of the Bohl $J[Z]$

In this section we show that most of the transformations and relationships presented in the first section have counterparts for discrete one-dimensional Schrödinger equations. (Part of the material in this section has appeared in a preprint [19], which has been expanded and divided for publication as two articles.) Some details are rather different from the continuous case, making it uncertain how far the analogy goes, especially in the oscillatory case. A full-fledged oscillation theory for discrete problems based on an analogue of the Bohl transformation and its connection to Green functions would be an interesting next project.

Let $\Delta$ denote the discrete second-difference operator on the positive integers. We standardize the Laplacian such that $(\Delta f)_n := f_{n+1} + f_{n-1} - 2f_n$ for $f = (f_n) \in \ell^2(\mathbb{N})$, and consider equations of the form

$$(3.1) \quad (-\Delta + V)u = 0,$$

where the potential-energy function $V$ is a diagonal operator with real values $V_n$.

Eq. (3.1) and its solutions share many of the properties of classical Sturm-Liouville equations, as is laid out for example in [11]. For our purposes we recall that: The solution space is two-dimensional, and the Wronskian of any two solutions

$$(3.2) \quad W[u^{(1)}, u^{(2)}] := u_n^{(1)} u_n^{(2)} - u_{n+1}^{(1)} u_{n+1}^{(2)}$$

is constant. A Green matrix as a solution of

$$(3.3) \quad (-\Delta + V)G = I,$$

where $I$ is the identity matrix, and every Green matrix can be written as the sum of a vector in the null space of $(-\Delta + V)$ and the particular Green matrix

$$(3.4) \quad G_{m,n}^{(p)} := \frac{u_n^{(1)} u_n^{(2)}}{W[u^{(1)}, u^{(2)}]}.$$

provided that $\{u^{(1)}, u^{(2)}\}$ are linearly independent.
A feature of the discrete Schrödinger equation (3.1) that is not shared by (2.1) is an invariance under the transformation
\[ u_n \rightarrow (-1)^n u_n \]
\[ V_n \rightarrow -4 - V_n, \]
(3.5)
as can be easily checked. Among other things, this implies that any fact proved under the assumption, for example, that \( V_n > 0 \) has a counterpart for \( V_n < -4 \), with systematic sign changes.

Our goal in this section is to present an analogue of the Bohl transformation for the discrete Schrödinger equation (3.1). In particular, we offer a discrete version of some of the results of [10], §4, and show in particular that the diagonal elements \( G_{nn} \) of the Green matrix allow the full solution space to be recovered formulaically. We build on some earlier steps in this direction by Chernyavskaya and Shuster [6, 7]. As in [10] we furthermore point out connections between the diagonal of the Green matrix and an Agmon distance for (3.1).

In the discrete situation the use of exponentials of integrals is not the most natural, so we instead seek to represent a pair of solutions in the forms
\[ \varphi^+_n = z_n \prod_{\ell=1}^{n} S_{\ell}, \quad \varphi^-_n = z_n \left( \prod_{\ell=1}^{n} S_{\ell} \right)^{-1}. \]
(3.6)
Since the product of these two solutions is the diagonal of a Green matrix, up to a constant multiple, this suggests that if we begin by selecting a Green matrix such that \( G_{nn} \) is nonvanishing, then we can directly define
\[ z_n := (G_{nn})^{1/2}. \]

It remains to work out the most convenient form of \( S_{\ell}^{\pm 1} \). If the Wronskian is scaled so that \( W[\varphi^-, \varphi^+] = 1 \), then substitution of the ansatz (3.6) leads after a calculation to
\[ S_n - \frac{1}{S_n} = \frac{1}{z_n z_{n-1}}. \]
(3.7)
Here we pause to observe two ambiguities in relating \( \varphi^+_n \) to the potential \( V \). The first is that, due to the invariance (3.5), if
\[ G_{mn} = \psi^+_{\min(m,n)} \psi^-_{\max(m,n)} \]
is the Green matrix for some potential function \( V_n \), then the same diagonal elements \( G_{nn} \) also belong to the Green matrix for an equation of type (3.1) but with potential function \( \tilde{V}_n = -4 - V_n \). Secondly, (3.7) is equivalent to a quadratic expression for \( S_n \), and therefore the solution is generally nonunique. These ambiguities are avoided when the Schrödinger operator \( H = -\Delta + V_n \) in (3.1) is positive, so \( G_{nn} > 0 \) and by convention \( z_n > 0 \). We can then fix \( S_n \) as the larger root of (3.7).

Accordingly, in this situation we simply define
\[ S_n^{[\pm]} := \frac{1 + \sqrt{1 + 4 z_n^2 z_{n-1}^2}}{2 z_n z_{n-1}}. \]
(3.9)
A pair of functions $\varphi^\pm_n$ can now be defined by the ansatz (3.6), i.e., when expressed in terms of $z_n$,

\begin{align}
\varphi^+_n &= z_n \prod_{k=m+1}^n \left( 1 + \frac{1 + 4z_n^2}{2z_n z_{n-1}} \right)^{-\frac{1}{2}}, \\
\varphi^-_n &= z_n \prod_{k=m+1}^n \left( 1 + \frac{1 + 4z_n^2}{2z_n z_{n-1}} \right)^{-1}.
\end{align}

(3.10)

Remarkably, with this definition, both $\varphi^+$ and $\varphi^-$ solve a single equation of the form (3.1), where the potential function $V_n$ is determined from $z_n$ via

\begin{align}
V_n[z] := \frac{\Delta \varphi^+_n}{\varphi^+_n} &= \frac{1 + \sqrt{1 + 4z_n^2}}{2z_n} + \frac{2z_n^2}{1 + \sqrt{1 + 4z_n^2}} - 2 \\
&= \frac{z_{n+1}}{z_n} S[z_{n+1}] + \frac{z_{n-1}}{z_n} S[z_n] - 2,
\end{align}

(3.11)

provided that $V_n > -2$. (Else a different root must be chosen in (3.9).) To see that $\varphi^\pm_n$ solve the same discrete Schrödinger equation, let us separately calculate

\begin{align}
\Delta \varphi^-_n = \frac{z_{n+1}}{z_n} S[z_{n+1}] + \frac{z_{n-1}}{z_n} S[z_n] - 2,
\end{align}

(3.12)

and note that since $S[z_i]$ has been chosen to satisfy (3.11), the difference between these last two expressions is

$$\frac{1}{z_n^2} - \frac{1}{z_n^2} = 0.$$

This leads to a theorem in the spirit of [10].

**Theorem 3.1.** Suppose that (3.1) has two independent positive solutions for $m \leq n \leq N$, with $N \geq M + 2$, and denote the associated Green matrix $G_{mn}$. Since $G_{nn} > 0$ for $m \leq n \leq N$, we may define $z_n := \sqrt{G_{nn}} > 0$. In terms of $z_n$, determine $S_n[z]$ and $\varphi^\pm_n$ according to (3.9) and (3.10). Then

1. $\varphi^\pm_n$ is an independent pair of solutions of (3.1) for $m < n \leq N$.
2. $G_{nm} = z_n z_m \prod_{k=m+1}^n \frac{1}{S_k[z]}$, $M < m < n \leq N$.
3. The potential function is determined from $G_{nn}$ by a nonlinear difference equation,

\begin{align}
\frac{1}{2} \left( \sqrt{1 + 4G_{nn} G_{n+1} n+1} + \sqrt{1 + 4G_{nn} G_{n-1} n-1} \right) = (V_n + 2)G_{nn}.
\end{align}

(3.13)

**Remark 2.** In what follows we are mainly concerned with what happens when $N \to \infty$. In that case the assumption that there are two positive solutions is a question of disconjugacy in the theory of ordinary differential equations, cf. [20]. If, for example, $V_n > 0$ for $n \geq N_0$, then it is not difficult to show that no solution can change sign more than once, and that therefore the positivity assumption is satisfied for $n$ sufficiently large. As will be seen in the proof, a necessary condition for the assumption is that $V_n > -2$. 


Per the symmetry remarked upon in (3.5), an alternative to positivity is the assumption that there are two solutions $\psi_{n}^{\pm}$ such that $(-1)^{n}\psi_{n}^{\pm} > 0$. A sufficient condition for this is that $V_n < -4$ and a necessary condition is that $V_n < -2$.

**PROOF.** The essential calculation was provided in the discussion before the statement of the theorem. Given that the Wronskian of $\varphi^{-}$ and $\varphi^{+}$ is 1, these two functions are linearly independent and therefore a basis for the solution space of 

$$(-\Delta + V_n^{[z]})\varphi = 0,$$

$V_n^{[z]}$ being defined by (3.11). Moreover,

$$G_{mn} = \varphi_{\min(m,n)}^{+} \varphi_{\max(m,n)}^{-}$$

is a Green function for $-\Delta + V_n^{[z]}$.

Hence the crux is to show that $V_n^{[z]}$ is the same as the original $V_n$ of (3.1).

Because $S_n^{[z]}$ was defined such that

$$S_n^{[z]} - \frac{1}{S_n^{[z]}} = \frac{1}{z_n z_{n-1}},$$

we may rewrite (3.11) as

(3.14)

$$V_n^{[z]} + 2 = \frac{1}{2z_n^2} \left( \sqrt{1 + 4z_n^2} - \sqrt{1 + 4z_n^2 + 2} \right).$$

From the definition of $z_n$ and the assumptions of the theorem, we know that for some independent set of positive solutions $\psi_{n}^{\pm}$ of (3.1), with Wronskian 1, $z_n^2 = \psi_{n}^{+}\psi_{n}^{-}$.

Therefore

$$4z_n^2 z_n^{\pm} = 4(\psi_{n}^{+}\psi_{n-1}^{+})(\psi_{n}^{-}\psi_{n+1}^{+})$$

$$= (\psi_{n}^{+}\psi_{n+1}^{-} + \psi_{n-1}^{+}\psi_{n}^{-})^2 - (\psi_{n}^{+}\psi_{n-1}^{+} - \psi_{n}^{-}\psi_{n+1}^{+})^2$$

$$= (\psi_{n}^{+}\psi_{n+1}^{-} + \psi_{n-1}^{+}\psi_{n}^{-})^2 - 1.$$  

Hence (3.14) yields

$$V_n^{[z]} + 2 = \frac{1}{2\psi_n^{+}\psi_n^{-}} \left( \psi_n^{+}\psi_{n+1}^{-} + \psi_n^{-}\psi_{n+1}^{+} + \psi_{n-1}^{+}\psi_{n}^{-} + \psi_n^{+}\psi_{n-1}^{-} \right)$$

$$= \frac{1}{2\psi_n^{+}\psi_n^{-}} \left( \psi_n^{+} V_n^{-} + \psi_n^{-} V_n^{+} \right)$$

$$= V_n + 2,$$

as claimed, and establishes (3.13). \qed

It may well be asked at this stage why we have restricted ourselves to the situation where $G_{nn} > 0$, for at the formal level the calculations given above remain valid without assuming positivity. In the discrete setting, continuity is not available to connect the values of a solution $\varphi_n$ as $n$ varies, and hence without an assumption such as positivity, there is a degree of indeterminateness in defining solutions by a prescription such as (3.9). For some choice of phases in (3.14), it will still be true that $V_n^{[z]}$ as defined there coincides with $V_n$, but the implicit nature of these choices of square root is problematic. Possibly a suitable canonical choice of phase or ideas from Teschl’s oscillation theory for Jacobi operators [28] could help avoid implicit definitions, and we hope to elaborate this point in future work.
Returning to the case where $G_{nn} > 0$, Formula (3.10) suggests that $S_n$ can be related to an Agmon distance $[2, 21]$, that is, a metric $d_A(m, n)$ on the positive integer lattice such that every $\ell^2$ solution $\phi^-$ of (3.1) satisfies a bound of the form

$$e^{d_A(0, n)}\phi^- \in \ell^\infty,$$

and that as a consequence $\phi^-$ decays rapidly as $n \to \infty$. Thus if $z_n$ is bounded we expect an Agmon distance to be something like $\sum_{\ell = m+1}^{n} \ln S^{[z]}_{\ell}$, assuming $n > m$.

(We write the Agmon distance in this way because the triangle inequality is an equality on the integer lattice, which implies that any metric takes the form of a sum of quantities defined at values of $\ell$ from $m+1$ to $n$.) In Agmon’s theory, however, it is desirable that the distance function be a quantity that can be calculated directly from the potential alone (or at least dominated by some such expression). As we shall now see, understanding the diagonal of the Green matrix allows the derivation of Agmonish bounds. We begin by showing that $G_{nn}$ is comparable to $(V_n + 2)^{-1}$ in a precise sense.

**Lemma 3.2.** Suppose that $\liminf_{n \to \infty} V_n > C > 0$, and let $G_{mn}$ be any positive Green matrix for (3.1) on the positive integers. Define

$$K_A := \sqrt{1 + \left(\frac{2}{C(C + 2)}\right)^2 + \frac{2}{C(C + 2)}}.$$

Then for $n$ sufficiently large,

$$\frac{1}{V_n + 2} \leq G_{nn} \leq \frac{K_A}{V_n + 2}.$$  \hspace{1cm} (3.15)

Consequently,

$$\frac{\sqrt{(V_n + 2)(V_{n-1} + 2)} + \sqrt{4 + (V_n + 2)(V_{n-1} + 2)}}{2K_A} \leq S^{[z]}_n \leq \frac{\sqrt{(V_n + 2)(V_{n-1} + 2)} + \sqrt{4 + (V_n + 2)(V_{n-1} + 2)}}{2}.$$  \hspace{1cm} (3.16)

**Remark 3.** The upper bound is of the same form as a semiclassical upper bound proved in [19]. To simplify it, $K_A$ could be replaced in these inequalities by

$$\sqrt{1 + \frac{4}{C^2}} > K_A$$

(see proof).

**Proof.** The lower bound on $G_{nn}$ is immediate from Statement (3) of Theorem 3.1 the left member of which is larger than 1.

The upper bound in (3.15) requires a spectral estimate. The Green matrix $G_{mn}$ is the kernel of the resolvent operator of a self-adjoint realization of $-\Delta + V$ on $\ell^2([N, \infty))$ for some $N$, where the boundary condition at $n = N, N + 1$ is that satisfied by $\varphi^+_n$. Since $-\Delta > 0$ on this space (as an operator), $\inf \text{sp}(-\Delta + V) > C$, and hence, by the spectral mapping theorem, $\|(-\Delta + V)^{-1}\|_{\text{op}} < C^{-1}$. Since $G_{nn} = \langle e_n, (-\Delta + V)^{-1} e_n \rangle$, where $\{e_n\}$ designate the standard unit vectors in $\ell^2$, it follows that $G_{nn} < C^{-1}$. Inserting this into (3.11) would already imply (3.15).
with $K_A$ replaced by $\sqrt{1 + 4/C^2}$. To improve the constant, replace only the terms $G_{n+1} \pm 1$ in (3.14) by $1/C$, getting

$$(3.17) \quad (V_n + 2) \leq \sqrt{1 + \frac{4G_{n+\pm1}}{C^2}}.$$ 

Since $\sqrt{1 + xy}$ is a decreasing function of $x$ when $x, y > 0$, an upper bound on $G_{nn}$ is the larger root of the case of equality in (3.17) (which is effectively a quadratic). The claimed upper bound with the constant $K_A$ results by keeping one factor $V_n + 2$ in the solution of the quadratic, replacing the others by $C + 2$.

The bounds on $S_n^{[z]}$ result from inserting the bounds on $G_{nn}$ into (3.9) and collecting terms. □

We can now state some Agmonish bounds.

COROLLARY 3.3. Suppose that $\liminf_{n \to \infty} V_n > C > 0$ and fix a positive integer $m$. Then the subdominant (i.e., eventually decreasing) solution $\phi^-$ of (3.1) satisfies

(a)  \[ \left( \prod_{\ell=m}^{n} \frac{V_{\ell} + 2}{K_A} \right) \phi^\ell_n \in \ell^\infty. \]

(b) If, in addition, $n(V_{n+1} - V_n) \in \ell^1$, then

\[ \left( \prod_{\ell=m}^{n} \frac{V_{\ell} + 2 + \sqrt{V_{\ell}(V_{\ell} + 4)}}{2} \right) \phi^\ell_n \in \ell^\infty. \]

PROOF. The ansatz (3.9) allows an identification of $\phi^-$ with a constant multiple of $\varphi^-$, in the representation (3.10). Because $z_n$ is bounded, so is

\[ \left( \prod_{\ell} S_{\ell}^{[z]} \right) \varphi^- n. \]

We then use the lower bound on $S_{\ell}^{[z]}$ from the lemma, but simplify by dropping the 4, which allows the product to telescope in a pleasing way, producing (a).

For (b) we note that the additional assumption on $V_n$ allows us to conclude that $\varphi$ is well-approximated by a Liouville-Green expression in [19], Theorem 4.1, which is a bounded quantity times the reciprocal of the expression in parentheses. □

Thus when $\liminf_{n \to \infty} V_n > 0$, a suitable Agmon distance $d_A(m, n)$ for (3.1) is given by

\[ \sum_{\ell=m+1}^{n} (\ln(V_{\ell} + 2) - \ln K_A), \]

or by

\[ \sum_{\ell=m+1}^{n} \ln \frac{V_{\ell} + 2 + \sqrt{V_{\ell}(V_{\ell} + 4)}}{2}, \]

provided that $n(V_{n+1} - V_n) \in \ell^1$.

We close with a Darboux-type factorization for a generic discrete Schrödinger equation (3.1). A Darboux-type factorization for general Jacobi operators was
previously considered by Gesztesy and Teschl in [15]. As in [32] the novel feature of this factorization is that it is constructed using the diagonal of the Green matrix.

To this end, choose a Green matrix such that $G_{nn}$ is nonvanishing for a range of values of $n$, and define a solution $\varphi_n^+$ according to (3.9). The phase of the square roots is chosen (if necessary) to ensure that $V_n^z$ from (3.11) equals $V_n$.

**Theorem 3.4.** Given a Green matrix such that the diagonal $G_{kk}$ is nonvanishing for $n - 1 \leq k \leq n + 2$, and choosing the phase of the square roots as described above,

$$-\Delta V_n = R \left[ -\nabla^+ - 1 + \frac{2G_{nn}}{1 + (1 + 4G_{nn}G_{n+1,n+1})^{1/2}} \right] \left[ \nabla^+ + 1 - \frac{1 + (1 + 4G_{nn}G_{n+1,n+1})^{1/2}}{2G_{nn}} \right],$$

where $R$ is the shift operator such that $[Rf]_n = f_{n-1}$ and the right-difference operator is defined by $[\nabla^+ f]_n := f_{n+1} - f_n$.

**Remark 4.** As with (2.9), there is a second factorization, with shifts and differences reversed, and $n + 1$ replaced by $n - 1$.

**Proof.** Writing

$$Q_n := 1 - \frac{1 + (1 + 4G_{nn}G_{n+1,n+1})^{1/2}}{2G_{nn}} = 1 - \frac{z_n + 1 S_{n+1}^z}{z_n},$$

with $S_k^z$ defined in (3.9), we first note that, by a simple calculation,

$$[\nabla^+ + Q_n] \varphi^+ = 0.$$  

This motivates calculating

$$H = \left[ -\nabla^+ + \frac{Q_n}{1 - Q_n} \right] \left[ \nabla^+ + Q_n \right],$$

which is well-defined because $Q_n \neq 1$, owing to (3.9) with $z_k$ nonvanishing. The left factor was chosen to produce a convenient cancellation, ensuring that $H$ has the form of a discrete Schrödinger equation, with a shifted index:

$$(Hf)_n = (-\Delta f)_{n+1} + \left( \frac{Q_n}{1 - Q_n} - Q_{n+1} \right) f_{n+1} + 0 \cdot f_n.$$  

We now verify that when the index is shifted back, the potential term is indeed $V_n$:

$$\left( \frac{Q_{n-1}}{1 - Q_{n-1}} - Q_n \right) = \left( \frac{z_{n-1}}{z_n S_n^z} - 1 \right) - \left( 1 - \frac{z_{n+1} S_n^z}{z_n} \right)$$

$$= -2 + \frac{z_{n-1}}{z_n S_n^z} + \frac{z_{n+1} S_n^z}{z_n},$$

which reduces to $V_n$ according to (3.11).

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