I briefly review the description of inclusive heavy hadron decays based on the inverse heavy quark mass expansion. In particular, I consider the transition $B \to X_u \ell \nu$ and the problem of extracting $V_{ub}$, mainly as far as the separation from the background represented by $b \to c$ modes is concerned. I also discuss how to obtain complementary information from the decay mode $B \to X_s \gamma$.

1 General Framework

Inclusive decay widths of hadrons $H_Q$ with a single heavy quark $Q$ can be computed using an expansion in powers of $m_Q^{-1}$, starting from the optical theorem that allows to write $\Gamma(H_Q \to X_f) = 2Im\langle H_Q|\hat{T}|H_Q \rangle/2M_{H_Q}$ with $\hat{T} = i \int d^4x T[H_w(x)H_w^\dagger(0)]$ the transition operator describing the heavy quark $Q$ with the same momentum in the initial and final state, and $H_w$ the effective hamiltonian governing the decay $Q \to X_f$. An operator product expansion (OPE) of $\hat{T}$ in the inverse mass of the heavy quark: $\hat{T} = \sum_i C_i \mathcal{O}_i$ can be given in terms of local operators $\mathcal{O}_i$ ordered by increasing dimension, and coefficients $C_i$ proportional to increasing powers of $m_Q^{-1}$. As a result, for a beauty hadron $H_b$ the general expression of the inclusive width $\Gamma(H_b \to X_f)$ is:

$$\Gamma(H_b \to X_f) = \Gamma_0 \left[ c_f^2 \langle \bar{b}b \rangle H_b + \frac{c_f^2}{m_b^2} \langle \bar{b}g_{sb} \sigma \cdot Gb \rangle H_b + \mathcal{O}\left(\frac{1}{m_b^3}\right) \right], \quad (1)$$

with $\langle O \rangle_{H_b} = \langle H_b|O|H_b \rangle$, $\Gamma_0 = \frac{G_F^2 m_b^5}{192\pi^3} |V_{qb}|^2$ and $V_{qb}$ the relevant CKM matrix element.
The first operator in (1) is $\bar{b}b$, with dimension $D = 3$; the chromomagnetic operator $O_G = \bar{b}b \sigma^{\mu\nu}G_{\mu\nu}$, responsible of the heavy quark-spin symmetry breaking, has $D = 5$. In the limit $m_b \to \infty$, the heavy quark equation of motion allows to write:

$$\langle \bar{b}b \rangle_{Hb} = 1 + \frac{\langle O_G \rangle_{Hb}}{2m_b^2} - \frac{\langle O_\pi \rangle_{Hb}}{2m_b^2} + \mathcal{O}\left(\frac{1}{m_b^2}\right),$$

(2)

with $O_\pi = \bar{b}(i\bar{D})^2b$ the heavy quark kinetic energy operator. Inserting (2) in (1), the leading term reproduces the spectator model result. $\mathcal{O}(m_b^{-1})$ terms are absent [2] since $D = 4$ operators are reducible to $\bar{b}b$ by the equation of motion. Finally, the operators $O_G$ and $O_\pi$ are spectator blind, not sensitive to light flavour. Their matrix elements $\mu^2_G(H_b) = \langle O_G \rangle_{H_b}$, $\mu^2_\pi(H_b) = \langle O_\pi \rangle_{H_b}$ enter in the mass formula

$$M_{H_b} = m_b + \Lambda + \frac{\mu^2_\pi - \mu^2_G}{2m_b} + \mathcal{O}(m_b^{-2})$$

(3)

and in principle can be determined experimentally. $\mu^2_G$ can be obtained from the mass splitting; for $B$ mesons: $\mu^2_G(B) = \frac{3(M^2_B - M^2_{B_\pi})}{4}$. $\mathcal{O}(m_b^{-3})$ terms in (3) come from four-quark operators and are responsible for lifetime differences between baryons and mesons.

Each term in (3) can be further expanded in powers of $\alpha_s$, therefore one envisages in this approach a model independent framework to compute inclusive decay widths and to reliably determine $V_{ub}$ or $V_{cb}$. The drawback is that predictions are really reliable only if truly inclusive quantities are considered, while it is often necessary to put cuts on the relevant kinematical variables. A further uncertainty is represented by quantities such as $m_b$ or $\Lambda$, which enter as input parameters, although one aims at determining them from data, as well.

Finally, an open question remains the problem of the lifetime ratio $\tau(\Lambda_b)/\tau(B_d)$, where it seems still unlikely that the inclusion of $\mathcal{O}(m_b^{-3})$ terms allows to reproduce the current experimental datum [3].

2 Inclusive $B \to X_u \ell \nu$ transition

In the application of the previous formalism to the decay $B \to X_u \ell \nu$ for the extraction of $V_{ub}$ the main problem is represented by the background due to $b \to c$ transitions. One should consider differential distributions in the relevant kinematical variables and put suitable cuts in order to achieve an effective discrimination. However, in some of the resulting regions of the phase space the OPE is no more reliable, and singular terms appear that signal the inadequacy of the approach.

Let us consider the distribution in the lepton energy $E_\ell$; one can subtract the charm background imposing the cut $E_\ell > (m^2_B - \Lambda)$.
Since the largest energy available is $E_{\text{max}}^\ell = \left( m_B^2 - s_H \right) / 2m_B$, where $s_H$ is the hadronic invariant mass, the range which can be experimentally useful is $(\Delta E_{\ell})_{\text{end-point}} = m_D^2 / 2m_B \approx 0.33 \text{ GeV}$, too small to guarantee a significant statistics. In this region the OPE is also no more reliable, since the true expansion parameter is $\bar{\Lambda} / (m_b - 2E_{\ell})$, which is large in the end point.

Actually, the calculation of the lepton energy spectrum shows the appearance of singular distributions $\delta^{(n)}(y - 1)$, where $y = 2E_{\ell}/m_b$. The inadequacy of the approach is also evident from the fact that although $m_b / 2$ is the largest energy available for a free quark decay, the true end point corresponds to $E_{\ell} = m_B / 2$. In this window bound state effects, due to the Fermi motion of the heavy quark, become important. They can be taken into account introducing a non-perturbative form factor, known as shape function, representing a resummation of all the singular terms \[4\]. The physical spectra are then obtained through a convolution of the differential distributions with such a function. The knowledge of the first terms of the expansion of the shape function, gives us information about its first few moments. However, the reconstruction procedure is not unique and the uncertainty linked to the shape function hampers the extraction of $V_{ub}$.

The hadronic invariant mass distribution is more promising. To subtract the charm background one should impose $s_H < m_D^2$ and, in order to assess the effectiveness of considering such a distribution, it is necessary to evaluate how many events are left after the cut is imposed:

$$\Gamma(m^2) = \int_{\bar{\Lambda}^2}^{m^2} ds_H \frac{d\Gamma}{ds_H} \quad \bar{\Lambda}^2 \leq m^2 \leq m_B^2. \quad (4)$$

In \[5\] the fully differential distribution relative to the decay $B \to X_u \ell \nu$ has been calculated including $O(\alpha_s)$ corrections, reducing the theoretical uncertainty linked to such corrections. The results for the hadronic invariant mass spectrum show that the cut $s_H < m_D^2$ reduces only slightly the number of events, while eliminating completely the charm background. In order to consider the realistic situation, in \[6\] the spectrum at $O(\alpha_s)$ has been convoluted with a form of the shape function \[6\], obtaining that the number of events below the cut $s_H < m_D^2$ is $80 \pm 10\%$. Therefore, the analysis of the hadronic invariant mass distribution should be effective for the determination of $V_{ub}$.

An alternative procedure is to consider the spectrum in the leptonic invariant mass $q^2$ \[7\]. The cut $q^2 > (m_B - m_D)^2$ discriminates the charm signal. The range where the OPE is no more reliable can roughly be estimated as the window between the parton end-point $q^2 = m_b^2$, and the true end-point $q^2 = m_B^2$. However, the range that can be exploited experimentally is much larger: $\Delta q^2 = m_B^2 - (m_B - m_D)^2$, \[7\].
indicating that non perturbative effects should have a minor impact in this case. Nevertheless, the fraction of events surviving to the cut is only \( \simeq 20\% \). Probably, from the experimental side, the most suitable procedure is the analysis of a double differential distribution, e.g. both in \( s_H \) and \( q^2 \).

An important step towards the reduction of the theoretical uncertainty in the extraction of \( V_{ub} \) from \( B \to X_u \ell \nu \) comes from the analysis of the photon energy spectrum in \( B \to X_s \gamma \). In the quark transition \( b \to s \gamma \) such a spectrum would be monochromatic. However, real gluon emission as well as the Fermi motion modify the spectrum. The effect of the Fermi motion can be included introducing the same form factor as in \( B \to X_u \ell \nu \). Since real gluon emission is computable in perturbation theory, one could try to extract information about the shape function from this mode. This is what the CLEO Collaboration is doing, and, from the combined analysis of the modes \( B \to X_s \gamma \) and \( B \to X_c \ell \nu \), the first moments of the shape function have been determined, leading to the results (5):

\[
\mu^2 = 0.236 \pm 0.071 \pm 0.078 \text{ GeV}^2 \quad \bar{\Lambda} = 0.35 \pm 0.08 \pm 0.10 \text{ GeV}.
\]

The experimental analysis is ongoing, as well as the theoretical improvements, reinforcing our hope to get soon more precise information.

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