The Spinorial Energy Functional: Solutions of the Gradient Flow on Berger Spheres

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We study the negative gradient flow of the spinorial energy functional (introduced by Ammann, Weiß, and Witt) on 3-dimensional Berger spheres. For a certain class of spinors we show that the Berger spheres collapse to a 2-dimensional sphere. Moreover, for special cases, we prove that the volume-normalized standard 3-sphere together with a Killing spinor is a stable critical point of the volume-normalized version of the flow. Our results also include an example of a critical point of the volume-normalized flow on the 3-sphere, which is not a Killing spinor.

1 Introduction

Let $M$ be a compact spin manifold and $\mathcal{N}$ the union of all pairs $(g, \varphi)$ where $g$ is a Riemannian metric on $M$ and $\varphi \in \Gamma(\Sigma(M, g))$ is a spinor of the spin manifold $(M, g)$ whose pointwise norm is constant and equal to 1. The spinorial energy functional $\mathcal{E}$, introduced in [2], is defined by

$$\mathcal{E}: \mathcal{N} \rightarrow [0, \infty), \quad (g, \varphi) \mapsto \frac{1}{2} \int_M |\nabla_{\Sigma(M,g)} \varphi|^2 dv^g,$$

where $dv^g$ is the Riemannian volume form of $(M, g)$ and $|$ is the pointwise norm on $T^*M \otimes \Sigma(M, g)$. If dim$M \geq 3$, then the critical points of $\mathcal{E}$ are precisely the pairs $(g, \varphi)$ consisting of a Ricci-flat Riemannian metric $g$ and a parallel spinor $\varphi$. In the surface case, the spinorial energy functional is related to the Willmore energy of immersions and treated in detail in [3].

On the Fréchet-bundle $\mathcal{N} \rightarrow \mathcal{M}$, $\mathcal{M} := \{\text{Riemannian metrics on } M\}$, there exists a natural connection, which is defined in [2] with the aid of results in [5]. This connection defines a splitting of $T\mathcal{N}$ in horizontal and vertical subbundles, which allows us to define a Riemannian metric on $\mathcal{N}$. The negative gradient flow of $\mathcal{E}$ with respect to this Riemannian metric is called the spinor flow. Short time existence and uniqueness of the spinor flow was shown in [2] with a variant of DeTurck’s trick.
In this paper the spinor flow on 3-dimensional Berger spheres is treated. We view the 3-sphere $S^3$ as a $S^1$-principal bundle over $S^2$ via the Hopf fibration $\pi: S^3 \to S^2$. Rescaling the standard metric $g_{S^3}$ along the fibers of the Hopf fibration by $\varepsilon > 0$ yields the Berger metrics $g^\varepsilon$ on $S^3$. We call $(S^3, g^\varepsilon)$ a Berger sphere.

There is a certain class of spinors on $S^3$, the so-called $S^1$-invariant spinors \cite{1, 11}, which are in one-to-one correspondence to the spinors on the base manifold $S^2$. Our first theorem concerns these spinors.

**Theorem A** (Collapse). Let $M = S^3$ and as initial value $(g_0, \varphi_0)$ choose $g_0 = g^\varepsilon$ and $\varphi_0$ a spinor of unit length that corresponds to an arbitrary Killing spinor on the base $S^2$. Then, if the fibers are sufficiently short (i.e. $\varepsilon$ is small enough), the spinor flow converges to a 2-dimensional sphere in infinite time.

This theorem can be seen as a special case of the conjecture that $S^1$-principal bundles with suitable Riemannian metrics and sufficiently short fibers together with $S^1$-invariant spinors collapse to the base manifold under the spinor flow.

In \cite{2} it was observed that the volume-normalized standard metric on $S^3$ together with a Killing spinor is a critical point of the volume-normalized spinor flow. It is not clear whether this critical point is stable. However, there are such stability results for other geometric flows, see e.g. \cite{3} 1.1 Theorem] in the case of the mean curvature flow. Our second theorem is a first positive result concerning this stability question.

**Theorem B** (Stability). Let $M = S^3$ and as initial value $(g_0, \varphi_0)$ choose $g_0 = c(\varepsilon)g^\varepsilon$ the volume-normalized Berger metric and $\varphi_0$ a spinor that is obtained via parallel transport of an arbitrary Killing spinor of unit length from $(S^3, g_{S^3})$ to $(S^3, g^\varepsilon)$ as described in Remark 3.11. Then, if we are not too far away from $c(1)g_{S^3}$ (i.e. $\varepsilon$ is sufficiently close to 1), the volume-normalized spinor flow converges in infinite time to the volume-normalized standard metric on $S^3$ together with a Killing spinor.

**1.1 Overview of the proof**

First of all, in \cite{2} it was shown that under the splitting of $TN$ the negative gradient of $E$ has an expression

$$-\text{grad} E_{(g,\varphi)} = (Q_1(g,\varphi), Q_2(g,\varphi)) \in \Gamma(\otimes^2 T^*M) \oplus \Gamma(\Sigma(M,g)),$$

where $Q_1(g,\varphi)$ and $Q_2(g,\varphi)$ depend mainly on $\nabla^{(M,g)}\varphi$. This fact is important for us, because it means, essentially, that we can understand $-\text{grad} E_{(g,\varphi)}$ by understanding $\nabla^{(M,g)}\varphi$.

Furthermore, one of the main tools for us to prove the above theorems are generalized cylinders \cite{4}, which provide a way to identify spinors for different metrics. To be more concrete, given a smooth 1-parameter family $(g_t)_{t \in I}$ of Riemannian metrics on a manifold $M$, $I$ an interval, the generalized cylinder is the manifold $Z := I \times M$ together with the Riemannian metric $g_Z := dt^2 + g_t$. If the dimension of $M$ is odd, as in our case, we get an identification $\Sigma^+(Z, g_Z)|_{\{t\} \times M} \cong \Sigma(M, g_t)$. In particular, we can think of
sections \( \varphi \in \Gamma(\Sigma^+(Z, g_Z)) \) as families of sections \((\varphi_t)_{t \in I}\) with \( \varphi_t \in \Gamma(\Sigma(M, g_t)) \) where \( \varphi_t(.) := \varphi(t,.) \).

Denote by \( \pi : S^3 \to S^2 \) the Hopf fibration as above. We write

\[
g_t(X_1 + Y_1, X_2 + Y_2) := g_{S^3}(\alpha(t)X_1 + \beta(t)Y_1, \alpha(t)X_2 + \beta(t)Y_2)
\]

(1.1)

for \( t \in I = [0, b), b \in (0, \infty), X_i \in \ker(d\pi), Y_i \in \ker(d\pi)^\perp, i = 1,2 \), and smooth functions \( \alpha, \beta : I \to (0, \infty) \). We will choose (1.1) as ansatz for the metric part of the solution where we require that \( \alpha(0) \) and \( \beta(0) \) are chosen so that \( g_0 \) is the metric part of our initial value.

Using the generalized cylinder with respect to (1.1) we write

\[
\varphi_t(p) := P_{0,t}(p)(\varphi_0(p))
\]

(1.2)

where \( P_{0,t}(p) \) is the parallel transport in \( \Sigma^+(Z, g_Z) \) with respect to \( \nabla^{\Sigma^+(Z, g_Z)} \) along the curve \( \gamma_p(s) := (s,p) \) from \( \gamma_p(0) \) to \( \gamma_p(t) \). Then we choose (1.2) as ansatz for the spinor part of the solution. In the next step, we derive an expression for \( \nabla^{\Sigma(S^3, g_t)} \varphi_t \) that depends in particular on \( \alpha \) and \( \beta \). To achieve this, we use curvature terms to construct suitable differential equations in \( \Sigma^+(Z, g_Z) \). We use these expressions for \( \nabla^{\Sigma(S^3, g_t)} \varphi_t \) to calculate \( Q_1(g_t, \varphi_t) \) and \( Q_2(g_t, \varphi_t) \). After that we show \( \frac{\partial}{\partial t} \varphi_t = 0 = Q_2(g_t, \varphi_t) \) independent of the choice of \( \alpha \) and \( \beta \). Finally, we will see that \( \frac{\partial}{\partial t} g_t = Q_1(g_t, \varphi_t) \) is equivalent to a system of two non-linear ordinary differential equations for \( \alpha \) and \( \beta \). We solve these systems to get the desired properties of the solutions.

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2 Preliminaries

2.1 Spin geometry

In this section we fix notation and review basics of spin geometry which will be relevant in the following. For more details we refer to e.g. [10], [8], [7] and [12].

Let \( M \) be an oriented \( n \)-dimensional manifold and denote by \( GL^+(n, \mathbb{R}) \)-principal bundle of oriented frames for \( M \). Moreover, we denote by \( \theta : \widetilde{GL}^+(n, \mathbb{R}) \to GL^+(n, \mathbb{R}) \) the universal covering for \( n \geq 3 \) and the connected twofold covering for \( n = 2 \). A topological spin structure on \( M \) is a \( \theta \)-reduction of \( GL^+ M \), i.e. a topological spin structure on \( M \) is a \( \widetilde{GL}^+(n, \mathbb{R}) \)-principal bundle \( GL^+ M \) over \( M \) together with a
twofold covering $\Theta : \widetilde{\text{GL}}^+ M \to \text{GL}^+ M$ such that the following diagram commutes

\[
\begin{array}{ccc}
\text{GL}^+ M \times \text{GL}^+ (n, \mathbb{R}) & \longrightarrow & \text{GL}^+ M \\
\downarrow \Theta \times \theta & & \downarrow \Theta \\
\text{GL}^+ M \times \text{GL}^+ (n, \mathbb{R}) & \longrightarrow & \text{GL}^+ M \\
\end{array}
\]

where the horizontal arrows denote the group actions of the principal bundles. Now let $(M, g)$ be an oriented Riemannian manifold and $\text{SO}(M, g)$ the $\text{SO}(n, \mathbb{R})$-principal bundle of oriented orthonormal frames for $M$. Restricting $\theta$ to the spin group given by $\text{Spin}(n) := \theta^{-1}(\text{SO}(n, \mathbb{R}))$, we define a metric spin structure on $M$ to be a $\theta|_{\text{Spin}(n)}$-reduction of $\text{SO}(M, g)$. Again, this means that a metric spin structure on $M$ is a $\text{Spin}(n)$-principal bundle $\text{Spin}(M, g)$ over $M$ together with a twofold covering $\Theta : \text{Spin}(M, g) \to \text{SO}(M, g)$ such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Spin}(M, g) \times \text{Spin}(n) & \longrightarrow & \text{Spin}(M, g) \\
\downarrow \Theta \times \theta & & \downarrow \Theta \\
\text{SO}(M, g) \times \text{SO}(n, \mathbb{R}) & \longrightarrow & \text{SO}(M, g) \\
\end{array}
\]

Given a topological spin structure $\widetilde{\text{GL}}^+ M$ on an oriented manifold $M$, every Riemannian metric $g$ on $M$ defines a metric spin structure on $(M, g)$ by $\text{Spin}(M, g) := \widetilde{\text{GL}}^+ M|_{\text{SO}(M, g)}$. In the following, the term spin structure refers to a topological or metric spin structure and it should always be clear from the context which one we mean.

In order to introduce (complex) spinors, we consider representations of $\text{Spin}(n)$. We first note that the spin group can be realized as a subgroup of the group of invertible elements in $\mathbb{C}l_n$ where $\mathbb{C}l_n$ is the Clifford algebra of $\mathbb{C}^n$ with inner product given by the complex bilinear extension of the standard inner product of $\mathbb{R}^n$, namely $\text{Spin}(n) \cong \{x_1 \cdot \ldots \cdot x_{2k} \mid x_i \in S^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}l_n, \ k \in \mathbb{N}\}$. If $n$ is even, then there exists exactly one equivalence class of irreducible complex representations of $\mathbb{C}l_n$ and every such representation is of dimension $2^n$. If $n$ is odd, then there exist exactly two equivalence classes of irreducible complex representations of $\mathbb{C}l_n$ and every such representation is of dimension $2^n$. Introducing the complex volume element $\omega_n := i^{\frac{n-1}{2}} e_1 \cdot \ldots \cdot e_n \in \mathbb{C}l_n$ where $a = [b]$ is the largest integer $a \leq b$ and $(e_1, \ldots, e_n)$ is the standard basis of $\mathbb{C}^n$, we can distinguish the two different equivalence classes for $n$ odd by the action of $\omega_n$, i.e. $\omega_n$ acts as the identity $id$ on one equivalence class and as $-id$ on the other. The complex spinor representation $\rho : \text{Spin}(n) \to \text{Aut}(\Sigma_n)$ is the restriction of an irreducible complex representation $\rho : \mathbb{C}l_n \to \text{End}(\Sigma_n)$ of $\mathbb{C}l_n$ to $\text{Spin}(n)$ where for $n$ odd we require $\rho(\omega_n) = id_{\Sigma_n}$. For
n odd, the complex spinor representation is irreducible. For n even, it splits into two irreducible representations \( \rho = \rho^+ \oplus \rho^- \) where \( \rho^\pm : \text{Spin}(n) \to \text{Aut}(\Sigma_n) \) have dimension \( 2^{\frac{n}{2}} \) and \( \Sigma_n^\pm \) are the \( \pm 1 \)-eigenspaces of \( \rho(\omega_n) \).

Let \( \text{Spin}(M,g) \) be a spin structure on \( (M,g) \). The (complex) spinor bundle \( \Sigma(M,g) \) is the complex vector bundle associated to the spin structure and the complex spinor representation, i.e. \( \Sigma(M,g) := \text{Spin}(M,g) \times \rho \Sigma_n \). For n even, we have an isomorphism \( \Sigma(M,g) \cong \Sigma^+(M,g) \oplus \Sigma^-(M,g) \) where \( \Sigma^\pm(M,g) := \text{Spin}(M,g) \times \rho^\pm \Sigma_n^\pm \). Next we introduce the so-called Clifford multiplication, which allows to multiply spinors and tangent vectors. To that end, notice that \( TM \cong \text{Spin}(M,g) \times \tau_{\omega(n)} \mathbb{R}^n \), where \( \tau \) is the standard representation of \( SO(n,\mathbb{R}) \) on \( \mathbb{R}^n \). Given \( \varphi = [p,\sigma] \in \Sigma_x(M,g) \) and \( X = [p,v] \in T_xM \) we define the Clifford multiplication (on \( \Sigma(M,g) \)) by \( X \cdot \varphi := [p,\rho(v)(\sigma)] \).

From the relations of the Clifford algebra \( \mathbb{C}l_n \), it follows that

\[
X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X,Y)\varphi,
\]

for all \( X,Y \in T_xM \) and \( \varphi \in \Sigma_x(M,g) \). For \( n \) even, Clifford multiplication interchanges the factors \( \Sigma^\pm(M,g) \). Moreover, given an oriented orthonormal basis \( (e_1, \ldots, e_n) \) of \( T_xM \) and \( \varphi \in \Sigma_x(M,g) \) for \( n \) odd, respectively \( \varphi \in \Sigma^+_x(M,g) \) for \( n \) even, we have

\[
\imath(\frac{n+1}{2})e_1 \cdot \ldots \cdot e_n \cdot \varphi = \varphi.
\]

To measure the length of spinors, we introduce a natural bundle metric on \( \Sigma(M,g) \). First, given an irreducible representation \( \rho : \mathbb{C}l_n \to \text{End}(\Sigma_n) \) of \( \mathbb{C}l_n \), there exists a hermitian inner product \( \langle , \rangle_{\Sigma_n} \) on \( \Sigma_n \) such that \( \langle \rho(x)(\psi), \varphi \rangle_{\Sigma_n} = \langle \psi, \rho(x)(\varphi) \rangle_{\Sigma_n} \) for all \( x \in \mathbb{R}^n, \varphi, \psi \in \Sigma_n \). In particular, the inner product \( \langle , \rangle_{\Sigma_n} \) is \( \text{Spin}(n) \)-invariant and therefore induces a bundle metric on \( \Sigma(M,g) \), which we denote by \( \langle , \rangle \). It holds that

\[
\langle X \cdot \psi, \varphi \rangle = -\langle \psi, X \cdot \varphi \rangle,
\]

for all \( X \in T_xM, \varphi, \psi \in \Sigma_x(M,g) \). For \( \varphi \in \Gamma(\Sigma(M,g)) \) we set \( |\varphi| := \sqrt{\langle \varphi, \varphi \rangle} \). In order to differentiate spinors we note that the Levi-Civita connection \( \nabla \) on \( (M,g) \) can be lifted to a metric connection \( \nabla^{\Sigma(M,g)} \) on \( \Sigma(M,g) \), the spinorial Levi-Civita connection. For all \( X, Y \in \Gamma(TM) \) and \( \varphi \in \Gamma(\Sigma(M,g)) \) we have

\[
\nabla^{\Sigma(M,g)}_X(Y \cdot \varphi) = (\nabla_XY) \cdot \varphi + Y \cdot \nabla^{\Sigma(M,g)}_X \varphi.
\]

For \( n \) even, the factors \( \Sigma^\pm(M,g) \) are invariant under \( \nabla^{\Sigma(M,g)} \). In particular, we get connections \( \nabla^{\Sigma^\pm(M,g)} \) on \( \Sigma^\pm(M,g) \). Denote by \( R^{\Sigma(M,g)} \) the curvature of \( \Sigma(M,g) \). Let \( (e_1, \ldots, e_n) \) be a local orthonormal frame for \( (M,g) \). It holds that

\[
R^{\Sigma(M,g)}(X,Y)\varphi = \frac{1}{2} \sum_{1 \leq i < j \leq n} g(R^M(X,Y)e_i, e_j)e_i \cdot (e_j \cdot \varphi),
\]

for all \( X, Y \in \Gamma(TM) \), \( \varphi \in \Gamma(\Sigma(M,g)) \) where \( R^M \) is the curvature of \( (M,g) \).
2.2 Generalized cylinders

Details concerning this section can be found in [3]. Let \( M \) be a manifold, \( I \subset \mathbb{R} \) an interval and \( (g_t)_{t \in I} \) a smooth 1-parameter family of Riemannian metrics on \( M \). The \textit{generalized cylinder} is the Riemannian manifold \((Z, g_Z)\), where \( Z := I \times M \) and \( g_Z := dt^2 + g_t \). The Riemannian hypersurface \( \{t\} \times M \) is isometric to \((M, g_t)\) and we denote both by \( M_t \). Moreover, the vector field \( \nu := \frac{\partial}{\partial t} \in \Gamma(TZ) \) is of unit length and \( \nu|M_t \) is orthogonal to \( M_t \). We write \( W = W_t \) for the Weingarten map of \( M_t \) with respect to \( \nu|\mathcal{M}_t \).

The following lemma will be used later.

\textbf{Lemma 2.1.} For all \( U, X, Y \in T_pM, p \in M, \) and \( t \in I \), it holds that
\[
\nabla^Z_\nu \nu = 0,
\]
\[
g_t(W_t(X), Y) = -\frac{1}{2} \hat{g}_t(X, Y),
\]
\[
g_Z(R^Z(X, Y)U, \nu) = \frac{1}{2}((\nabla^M_Y \hat{g}_t)(X, U) - (\nabla^M_X \hat{g}_t)(Y, U)),
\]
\[
g_Z(R^Z(X, \nu)U, Y) = -\frac{1}{2}(\hat{g}_t(X, Y) + \hat{g}_t(W_t(X), Y)).
\]

If \( \tilde{Z} \in \Gamma(TZ) \) with \( \tilde{Z}(t, p) = (0, Z_t(p)) \in T_tI \times T_pM \) for all \( t \in I, p \in M \), then
\[
[\nu, \tilde{Z}](t, p) = \left(0, \frac{d}{ds}_{s=t} Z_s(p)\right) \in T_tI \times T_pM,
\]
\[
(\nabla^Z_\nu \tilde{Z})(t, p) = \left(0, \frac{d}{ds}_{s=t} Z_s(p) - W_t(Z_t(p))\right)
\]
for all \( t \in I, p \in M \) where \( \nabla^Z \) is the Levi-Civita connection of \((Z, g_Z)\).

The next lemma describes how we can identify spinors of different spinor bundles with the help of generalized cylinders.

\textbf{Lemma 2.2.} Let \( M \) be an oriented manifold together with a topological spin structure \( GL_+^+ M \). The topological spin structure on \( M \) induces a metric spin structure on \((Z, g_Z)\) and metric spin structures \( \text{Spin}(M, g_t) := \overline{GL_+^+ M|_{SO(M, g_t)}} \) on \((M, g_t)\). For the respective spinor bundles we have the following isomorphisms of vector bundles: If \( n \) is even, then \( \Sigma(Z, g_Z)|_{M_t} \cong \Sigma M_t \). If \( n \) is odd, then \( \Sigma^+(Z, g_Z)|_{M_t} \cong \Sigma M_t \). The bundle metrics \( (\cdot, \cdot) \) are preserved by these isomorphisms. Moreover, if \( \cdot \) and \( ^* \) denote the Clifford multiplications in \( \Sigma^+(Z, g_Z) \) and \( \Sigma M_t \), then it holds that
\[
\nu \cdot (X \cdot \varphi) = X^* \nu \varphi
\]
for all \( X \in TM, \varphi \in \Sigma M_t \). If we write \( \nabla^t = \nabla^\Sigma M_t \), then we have
\[
\nabla^\Sigma(Z, g_Z) X \varphi = \nabla^t X \varphi - \frac{1}{2} W_t(X) \nu \varphi
\]
for all \( \varphi \in \Gamma(\Sigma M_t) \).
2.3 The spinorial energy functional and its gradient flow

In the following we work with the real part of \( \langle \ldots \rangle \) and we write \((\ldots) := \text{Re} \langle \ldots \rangle\). It will be useful that \((X \cdot \varphi, Y \cdot \varphi) = g(X, Y) (\varphi, \varphi)\) and \((X \cdot \varphi, \varphi) = 0\) hold for all \(X, Y \in T_xM\) and \(\varphi \in \Sigma_x(M, g)\). These identities follow directly from (2.1) and (2.3).

Let \(M\) be a connected, compact, oriented manifold with a fixed topological spin structure \(\tilde{\text{GL}}^+ M\) and \(\dim M \geq 2\). As stated before, every choice of Riemannian metric \(g\) on \(M\) defines a metric spin structure \(\text{Spin}(M, g) := \tilde{\text{GL}}^+ M \mid_{SO(M, g)}\) and so we have the corresponding spinor bundles \(\Sigma(M, g)\). We set \(N_g := \{ \varphi \in \Gamma(\Sigma(M, g)) \mid |\varphi| = 1 \}\), and \(N := \bigsqcup_{g \in M} N_g\). The **spinorial energy functional** \(E\) is defined by

\[
E : N \to [0, \infty), \quad (g, \varphi) \mapsto \frac{1}{2} \int_M |\nabla^{\Sigma(M, g)} \varphi|_g^2 dv_g,
\]

As mentioned in the introduction there exists a natural connection on the Fréchet-bundle \(N \to M\). For details we refer to [2]. From that connection we get horizontal tangent spaces \(H_{(g, \varphi)}N \cong \Gamma(\otimes^2 T^*M)\) and a splitting

\[
T_{(g, \varphi)}N \cong \Gamma(\otimes^2 T^*M) \oplus V_{(g, \varphi)} \tag{2.9}
\]

where \(V_{(g, \varphi)} = \{ \psi \in \Gamma(\Sigma(M, g)) \mid (\varphi, \psi) = 0 \}\).

On the first factor, we choose the inner product which we get by integrating the natural inner product on \((2,0)\)-tensors. On the second factor, we choose the \(L^2\)-inner product defined by

\[
(\psi_1, \psi_2)_{L^2} := \int_M (\psi_1, \psi_2) dv_g,
\]

for \(\psi_1, \psi_2 \in V_{(g, \varphi)}\). The negative gradient flow of \(E\),

\[
\frac{\partial}{\partial t} (g_t, \varphi_t) = -\text{grad} E_{(g_t, \varphi_t)},
\]

is called the **spinor flow**. Under the splitting (2.9) we have

\[
-\text{grad} E_{(g, \varphi)} = (Q_1(g, \varphi), Q_2(g, \varphi)),
\]

with

\[
Q_1(g, \varphi) = -\frac{1}{4} |\nabla^{\Sigma(M, g)} \varphi|_g^2 g - \frac{1}{4} \text{div}_g T_{g, \varphi} + \frac{1}{2} \left( \nabla^{\Sigma(M, g)} \varphi \otimes \nabla^{\Sigma(M, g)} \varphi \right),
\]

\[
Q_2(g, \varphi) = - \left( \left( \nabla^{\Sigma(M, g)} \varphi \right) \nabla^{\Sigma(M, g)} \varphi + |\nabla^{\Sigma(M, g)} \varphi|_g^2 \varphi, \right.
\]

for all \((g, \varphi) \in N\).

Here, \(\left( \nabla^{\Sigma(M, g)} \varphi \otimes \nabla^{\Sigma(M, g)} \varphi \right) (X, Y) := \left( \nabla^{\Sigma(M, g)}_X \varphi, \nabla^{\Sigma(M, g)}_Y \varphi \right)\) for all \(X, Y \in \Gamma(TM)\),

and \(T_{g, \varphi}\) is the symmetrization of \(\left( X \cdot Y \cdot \varphi + g(X, Y) \varphi, \nabla^{\Sigma(M, g)}_Z \varphi \right)\) in the second and third component where \(X, Y, Z \in \Gamma(TM)\).
The \textit{(volume) normalized spinor flow} is the negative gradient flow of $\mathcal{E}|_{\mathcal{N}_1}$ for $\mathcal{N}_1 := \{(g, \varphi) \in \mathcal{N} \mid \text{vol}(M, g) = 1\}$. We have $-\text{grad}(\mathcal{E}|_{\mathcal{N}_1}) = (\tilde{Q}_1, Q_2|_{\mathcal{N}_1})$ with

$$
\tilde{Q}_1(g, \varphi) = Q_1(g, \varphi) + \frac{n - 2}{2n} \frac{1}{\text{vol}(M, g)} \mathcal{E}(g, \varphi) g
$$

for all $(g, \varphi) \in \mathcal{N}_1$. Note that in the above identity for $\tilde{Q}_1$ the \textquote{\(\frac{1}{\text{vol}(M, g)}\)} is equal to 1. However, it is important in our strategy of solving the normalized spinor flow. More concretely, we will construct $(g_t, \varphi_t)$ with $|\varphi_t| = 1$ for all $t$ and $\frac{d}{dt}(g_t, \varphi_t) = (\tilde{Q}_1(g_t, \varphi_t), Q_2(g_t, \varphi_t))$ where for the initial value we have $\text{vol}(M, g_0) = 1$. To make sure that $(g_t, \varphi_t)$ is a solution of the normalized spinor flow, we then need to verify that $\text{vol}(M, g_t) = 1$ for all $t$. To that end, we calculate

$$
\frac{d}{dt}\bigg|_{t=s} \text{vol}(M, g_t) = \frac{d}{dt}\bigg|_{t=s} \int_M dv^{g_t} = \int_M \frac{d}{dt}\bigg|_{t=s} dv^{g_t}
$$

$$
= \int_M \frac{1}{2} \text{tr}_g \tilde{Q}_1(g_s, \varphi_s) dv^{g_s}
$$

$$
= \int_M \frac{1}{2} \text{tr}_g Q_1(g_s, \varphi_s) dv^{g_s} + \frac{1}{2} \frac{n - 2}{2n} \frac{1}{\text{vol}(M, g_s)} \mathcal{E}(g_s, \varphi_s) \int_M \text{tr}_g g_s dv^{g_s}
$$

$$
= \frac{1}{2} \int_M -\frac{n - 2}{4} |\nabla \Sigma(M, g_s)|^2 dv^{g_s} + \frac{1}{2} \frac{n - 2}{2n} \frac{1}{\text{vol}(M, g_s)} \mathcal{E}(g_s, \varphi_s) \text{vol}(M, g_s)
$$

$$
= -\frac{1}{2} \frac{n - 2}{2} \mathcal{E}(g_s, \varphi_s) + \frac{1}{2} \frac{n - 2}{2} \mathcal{E}(g_s, \varphi_s)
$$

$$
= 0,
$$

so $\text{vol}(M, g_t) = 1$ for every $t$ and $(g_t, \varphi_t)$ is in fact a solution of the normalized spinor flow.

\section{Solutions of the spinor flow on Berger spheres}

In this section we state and prove our main results, Theorem 3.5 and Theorem 3.10. First we collect necessary technical ingredients. Then we define $S^1$-invariant spinors, which are part of the initial values of theorem 3.5. After that we prove our main results with the strategy explained in the introduction.

As stated in the introduction we view $S^3$ as a $S^1$-principal bundle over $S^2$ via the Hopf fibration $\pi: S^3 \to S^2$. If we equip $S^2 = \mathbb{C}P^1$ with the Fubini-Study metric $g_{FS}$, then the Hopf fibration $\pi: (S^3, g_{S^3}) \to (S^2, g_{FS})$ turns into a Riemannian submersion. The action of $S^1$ on $S^3$ induces a global flow whose infinitesimal generator we denote by $K \in \Gamma(TS^3)$. We have that $K(p) \in \ker(d\pi_p)$ for all $p \in S^3$ and $K$ is of unit length with respect to $g_{S^3}$. On $S^3$ we choose the connection which is given by

$$
p \mapsto \ker(d\pi_p)\perp.
$$
This connection induces a connection form $\tilde{\omega} : TS^3 \to i\mathbb{R}$. It holds that $\tilde{\omega}(K) = i$. We write $\omega := \frac{1}{i} \tilde{\omega}$ and denote by $d\omega$ the differential of $\omega$, i.e.

$$d\omega(X,Y) = L_X(\omega(Y)) - L_Y(\omega(X)) - \omega([X,Y]).$$

With $X^*$ we denote the horizontal lift of $X$ (with respect to the above connection).

**Remark 3.1** (Orientation convention for $S^3$). For the rest of this paper we fix an orientation on $S^2$. On $S^3$ we fix the orientation that satisfies the following: If $(f_1, f_2)$ is any oriented local orthonormal frame for $(S^2, g_{FS})$, then $(K, f_1^*, f_2^*)$ is an oriented local orthonormal frame for $(S^3, g_{S^3})$.

In Remark 3.9 we explain how our results change if we choose the other orientation on $S^3$ (i.e. $(-K, f_1^*, f_2^*)$ is oriented).

**Remark 3.2** (Notation for frames). For the rest of this paper we use the following notion: $(f_1, f_2)$ denotes an arbitrary oriented local orthonormal frame for $(S^2, g_{FS})$. Moreover, we set

$$(f_1(t), f_2(t)) := (\frac{1}{\beta(t)} f_1 - \frac{1}{\beta(t)} f_2),$$

$$(e_0(t), e_1(t), e_2(t)) := (\frac{1}{\alpha(t)} K, f_1(t)^*, f_2(t)^*).$$

Then $(e_0(t), e_1(t), e_2(t))$ is an oriented local orthonormal frame for $(S^3, g_t)$ where $g_t$, $\alpha(t)$, and $\beta(t)$ are defined by (1.1).

We set

$$a := d\omega(f_1^*, f_2^*) = \pm 2.$$ 

Note that $a$ is a constant that does not depend on the choice of the oriented local orthonormal frame $(f_1, f_2)$.

**Lemma 3.3.** If $\nabla^t$ denotes the Levi-Civita connection on $(S^3, g_t)$ and $(S^2, \beta(t)^2 g_{FS})$ respectively, we have

$$\nabla^t_{e_0(t)} e_0(t) = 0, \quad \nabla^t_{e_1(t)} e_1(t) = \frac{1}{2\beta(t)^2} a \varepsilon_2(t),$$

$$\nabla^t_{e_0(t)} e_1(t) = \frac{1}{2\beta(t)^2} a \varepsilon_1(t), \quad \nabla^t_{e_1(t)} e_0(t) = \frac{1}{2\beta(t)^2} a \varepsilon_1(t),$$

$$\nabla^t_{e_1(t)} e_1(t) = \left(\nabla^t_{f_1(t)} f_1(t)^*\right)^*.$$

$$\nabla^t_{e_2(t)} e_1(t) = \frac{1}{2\beta(t)^2} a \varepsilon_0(t) + \left(\nabla^t_{f_1(t)} f_2(t)^*\right)^*,$$

$$\nabla^t_{e_2(t)} e_2(t) = \frac{1}{2\beta(t)^2} a \varepsilon_0(t) + \left(\nabla^t_{f_1(t)} f_2(t)^*\right)^*.$$
Proof. Since horizontal lifts are right invariant, it follows that
\[ [e_0(t), e_j(t)] = 0 \]
on \( S^3 \) for \( j = 1, 2 \). Using the Koszul formula we then compute the Christoffel symbols \( \Gamma^k_{ij} \) of \((e_0(t), e_1(t), e_2(t))\) with respect to \( \nabla^t \):
\[
-\tilde{\Gamma}^0_{ij} = \tilde{\Gamma}^j_{i0} = \tilde{\Gamma}^0_{0i} = \frac{1}{2} \frac{\alpha(t)}{\beta(t)^2} a,
\]
\[
\tilde{\Gamma}^i_{00} = \tilde{\Gamma}^0_{i0} = \tilde{\Gamma}^0_{0i} = 0,
\]
\[
\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} \circ \pi,
\]
for \( i, j, k \in \{1, 2\} \) where \( \Gamma^k_{ij} \) are the Christoffel symbols of \((f_1(t), f_2(t))\) with respect to \( \nabla^t \). The lemma now follows from an easy computation. \( \square \)

Lemma 3.4. For all \( \varphi \in \Gamma(\Sigma^+(Z, g_Z)) \) and all horizontal vector fields \( Y \in \Gamma(TS^3) \) it holds that
\[
R^{\Sigma+(Z,g_Z)}(\nu, K)\varphi = \frac{1}{2} \left( \frac{\alpha''(t)}{\alpha(t)} + \frac{\alpha'(t)}{\beta(t)^2} a - \frac{\alpha(t)\beta'(t)}{\beta(t)^3} a \right) \nu \cdot K \cdot \varphi,
\]
\[
R^{\Sigma+(Z,g_Z)}(\nu, Y)\varphi = \frac{1}{2} \left( \frac{\beta''(t)}{\beta(t)} - \frac{\alpha'(t)}{2 \beta(t)^2} a + \frac{1}{2} \frac{\alpha(t)\beta'(t)}{\beta(t)^3} a \right) \nu \cdot Y \cdot \varphi,
\]
where \((Z, g_Z)\) is with respect to \((1.1)\).

Proof. We use \((2.1)\). As local orthonormal frame for \( Z \) we choose \((\nu, e_0(.), e_1(.), e_2(.)\)). Furthermore, we use the notation
\[
R_{X,Y,Z,W} := g_Z(R^Z(X,Y)Z,W)
\]
With the aid of Lemma \((3.3)\) and \((2.5)-(2.6)\) it follows from straight forward calculations (for details we refer to [13], Lemma 6.9) that
\[
R_{\nu,e_0,e_0,e_1} = R_{\nu,e_0,e_0,e_2} = R_{\nu,e_0,e_1,a} = R_{\nu,e_0,e_2} = 0,
\]
\[
R_{\nu,e_0,\nu,e_0} = \frac{\alpha''(t)}{\alpha(t)},
\]
\[
R_{\nu,e_0,\nu,e_1} = \frac{\alpha'(t)}{\beta(t)^2} a - \frac{\alpha(t)\beta'(t)}{\beta(t)^3} a.
\]
Plugging this into \((2.1)\) and using \((2.2)\), we get
\[
R^{\Sigma+}(\nu,e_0)\varphi = \frac{1}{2} \left( R_{\nu,e_0,\nu,e_0} \nu \cdot e_0 \cdot \varphi + R_{\nu,e_0,\nu,e_1} e_1 \cdot e_2 \cdot \varphi \right)
\]
\[
= \frac{1}{2} \left( \frac{\alpha''(t)}{\alpha(t)} \nu \cdot e_0 \cdot \varphi + \left( \frac{\alpha'(t)}{\beta(t)^2} a - \frac{\alpha(t)\beta'(t)}{\beta(t)^3} a \right) e_1 \cdot e_2 \cdot \varphi \right)
\]
\[
= \frac{1}{2} \left( \frac{\alpha''(t)}{\alpha(t)} + \frac{\alpha'(t)}{\beta(t)^2} a - \frac{\alpha(t)\beta'(t)}{\beta(t)^3} a \right) \nu \cdot e_0 \cdot \varphi.
\]
From this, the first equation in Lemma 3.4 directly follows. The second equation follows from
\[ R^{S^3} \mathcal{Z}(\nu, e_i) \varphi = \frac{1}{2} \left( \frac{\beta''(t)}{\beta(t)} - \frac{1}{2} \frac{\alpha'(t)}{\beta(t)^2} \nu + \frac{1}{2} \frac{\alpha(t) \beta'(t)}{\beta(t)^3} a \right) \nu \cdot e_i \cdot \varphi, \]
i = 1, 2, which is shown with the same method as above.

**3.1 S^1-invariant spinors**

For details concerning this section we refer to [1] and also [11]. Define \((g_t)_{t \in I}\) by \((1.1)\). The \(S^1\)-action on \(S^3\) induces an \(S^1\)-action on \(SO(S^3, g_t)\) which lifts uniquely to an \(S^1\)-action on \(\text{Spin}(S^3, g_t)\) as follows: We use the fact that for every Riemannian metric on \(S^3\) (respectively, \(S^2\)) there exists, up to equivalence of reductions, exactly one metric spin structure. Pulling back \(\text{Spin}(S^2, \beta(t)^2 g_{FS})\) along the Hopf fibration and enlarging the structure group to \(\text{Spin}(3)\) we get the spin structure on \((S^3, g_t)\).

\[
\text{Spin}(S^3, g_t) = \pi^* (\text{Spin}(S^2, \beta(t)^2 g_{FS})) \times \text{Spin}(2) \text{ Spin}(3).
\]

Now we define the \(S^1\)-action by
\[
[(x, \sigma), g] \cdot e^{i \phi} := [(x \cdot e^{i \phi}, \sigma), g]
\]
for \((x, \sigma) \in \pi^* (\text{Spin}(S^2, \beta(t)^2 g_{FS})) \subset S^3 \times \text{Spin}(S^2, \beta(t)^2 g_{FS}), g \in \text{Spin}(3), \) and \(e^{i \phi} \in S^1 \subset \mathbb{C}\). This action is the desired lift of the \(S^1\)-action on \(SO(S^3, g_t)\). Uniqueness follows from the fact that \(\text{Spin}(S^3, g_t) \cong S^3 \times \text{Spin}(3)\) and \(SO(S^3, g_t) \cong S^3 \times SO(3, \mathbb{R})\) are connected.

This yields a \(S^1\)-action on \(\Sigma(S^3, g_t)\). Spinors which are invariant under this action are called \(S^1\)-invariant. Denote by \(V(t) \subset \Gamma(\Sigma(S^3, g_t))\) the vector space of \(S^1\)-invariant spinors. For every \(\varphi \in V(t)\) we have
\[
\nabla^{t}_{K} \varphi = \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a K \cdot t \varphi, \tag{3.1}
\]
see [1, Lemma 4.3.].

The \(S^1\)-invariant spinors are in one-to-one correspondence to the spinors on the base manifold. To be more precise, by [1, Lemma 4.4.] there is an isomorphism of vector spaces
\[
Q = Q(t) : \Gamma(\Sigma(S^3, \beta(t)^2 g_{FS})) \rightarrow V(t).
\]

The following identities will be used later: For every vector field \(X \in \Gamma(TS^2)\) and every spinor \(\sigma \in \Gamma(\Sigma(S^2, \beta(t)^2 g_{FS}))\)
\[
\nabla^{t}_{X} \cdot Q(\sigma) = Q(\nabla^{t}_{X} \sigma) - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a X^* \cdot t \sigma, \tag{3.2}
\]
\[
Q(X \cdot t \sigma) = X^* \cdot t \sigma. \tag{3.3}
\]

In (3.1)-(3.3) we denote by \(\nabla^{t}\) the spinorial Levi-Civita connection on \(\Sigma(S^3, g_t)\) and \(\Sigma(S^2, \beta(t)^2 g_{FS})\) respectively and \(\cdot t\) is the Clifford multiplication in the respective spinor bundles.
3.2 A collapsing theorem

Our first main result is the following theorem.

**Theorem 3.5 (Collapse).** Let \( \varepsilon > 0 \) and \( \lambda \in \{ \pm 1 \} \). Write \((g_0, \varphi_0) := (g^\varepsilon, Q(\sigma))\) for \( \sigma \) a \( \lambda \)-Killing spinor on \((S^2, g_{FS} = \frac{1}{4}g_{S^2})\) such that \( |\varphi_0| = 1 \). Then the solution of the spinor flow on \( M = S^3 \) with initial value \((g_0, \varphi_0)\) is given by \((1.1), (1.2)\) where \( b = t_{\text{max}}, \quad t_{\text{max}} \in (0, \infty) \) maximum time of existence (to the right), such that:

If \( a\lambda = 2 \), then:
- For \( 0 < \varepsilon < \frac{2}{3} \) we have \( t_{\text{max}} = \infty, \lim_{t \to \infty} \alpha(t) = 0, \text{ and } \lim_{t \to \infty} \beta(t) =: \beta_\infty > 0 \).
- For \( \varepsilon \geq \frac{2}{3} \) we have \( \lim_{t \to t_{\text{max}}} \alpha(t) = \lim_{t \to t_{\text{max}}} \beta(t) = 0 \).
- For \( \varepsilon = \frac{2}{3} \) we have \( t_{\text{max}} = 12, \alpha(t) = \frac{2}{3} \beta(t), \text{ and } \beta(t) = \frac{1}{6} \sqrt{36 - 3t} \).
- For \( \varepsilon = 1 \) we have \( t_{\text{max}} = 16 \) and \( \alpha(t) = \beta(t) = \frac{1}{4} \sqrt{16 - t} \).

If \( a\lambda = -2 \), then:
- For every \( \varepsilon > 0 \) we have \( t_{\text{max}} = \infty, \lim_{t \to \infty} \alpha(t) = 0, \text{ and } \lim_{t \to \infty} \beta(t) =: \beta_\infty > 0 \).

Moreover, in any of the above cases the spinor flow preserves the class of \( S^1 \)-invariant spinors which correspond to Killing spinors on \( S^2 \). More precisely,

\[ \varphi_t = Q(\sigma_t) \]

for every \( t \in I \) where \( \sigma_t \) is a \( \frac{\lambda}{\beta(t)} \)-Killing spinor on \((S^2, \beta(t)^2 g_{FS})\).

**Remark 3.6.** In the case \( a\lambda = 2 \) the result can be interpreted as follows: If we start with fibers that are sufficiently short (\( \varepsilon < \frac{2}{3} \)), then the \( S^1 \)-fiber converges to a point under the spinor flow (\( \alpha \to 0 \)), but the complement does not (\( \beta \to \beta_\infty > 0 \)). In that sense the \( S^1 \)-principal bundle \( S^3 \) collapses against its base \( S^2 \). If we start with fibers that are too long (\( \varepsilon > \frac{2}{3} \)), then \( S^3 \) converges to a point under the spinor flow.

In the case \( a\lambda = -2 \) the collapse is independent of the length of the fibers.

Now we carry out the steps mentioned in the introduction to prove Theorem 3.5.

**Lemma 3.7.** Choose \((g_0, \varphi_0)\) as in Theorem 3.5 and define \((g_t, \varphi_t)_{t \in I}\) by \((1.1), (1.2)\). Then, for every \( t \in I \) and every horizontal vector field \( Y \in \Gamma(TS^3) \), it holds that

\[ \nabla^K_t \varphi_t = \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} g K \cdot \varphi_t, \]

\[ \nabla^Y_t \varphi_t = \left( \frac{1}{\beta(t)} \lambda - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} g \right) Y \cdot \varphi_t \]

where \( \nabla^t = \nabla^{\Sigma(S^3, g_0)} \) and \( \cdot \) is the Clifford multiplication in \( \Sigma(S^3, g_t) \).

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Proof. First of all, from (3.1)-(3.3) we get
\[
\nabla^0_K \varphi_0 = \frac{1}{4} \epsilon a K \cdot \varphi_0, \quad \\
\nabla^0_Y \varphi_0 = (\lambda - \frac{1}{4} \epsilon a) Y \cdot \varphi_0,
\]
for every horizontal vector field \( Y \in \Gamma(TS^3) \). From (2.8) we get \( \nabla^2_\nu e_0 = 0 \) and (2.7) yields \([\nu, e_0](t, p) = -\frac{\alpha'(t)}{\alpha(t)} e_0(t, x)\). In the following we write \( \nabla^{Z+}_\nu = \nabla^{\Sigma+}_{e_0} \). Using Lemma \( \text{2.2} \) and Lemma \( \text{3.3} \) it follows that
\[
\nabla^{\Sigma^+}_\nu \left( \nabla^t_{e_0(t)} \varphi_t - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a e_0(t \cdot \varphi_t) \right)
\]
\[
= \nabla^{\Sigma^+}_\nu \left( \nabla^t_{e_0(t)} \varphi_t + \frac{1}{2} \nu \cdot W_t(e_0(t)) \cdot \varphi_t \right) - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= \nabla^{\Sigma^+}_\nu \nabla^t_{e_0(t)} \varphi_t - \frac{1}{2} L_\nu (\alpha'(t) \nu \cdot e_0 - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= \nabla^t_{e_0(t)} \varphi_t + \nabla^{\Sigma^+}_{\nu, e_0} \varphi_t - \frac{1}{2} L_\nu (\alpha'(t) \nu \cdot e_0 + \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= -\frac{\alpha'(t)}{\alpha(t)} \nabla^t_{e_0(t)} \varphi_t + \frac{1}{2} \nu \left( \frac{\alpha''(t)}{\alpha(t)} + \frac{\alpha'(t)}{\beta(t)^2} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= -\frac{\alpha'(t)}{\alpha(t)} \nabla^t_{e_0(t)} \varphi_t + \frac{1}{2} \nu \left( \frac{\alpha''(t)}{\alpha(t)} + \frac{\alpha'(t)}{\beta(t)^2} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= -\frac{\alpha'(t)}{\alpha(t)} \nabla^t_{e_0(t)} \varphi_t + \frac{1}{2} \nu \left( \frac{\alpha''(t)}{\alpha(t)} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= -\frac{\alpha'(t)}{\alpha(t)} \nabla^t_{e_0(t)} \varphi_t + \frac{1}{2} \nu \left( \frac{\alpha''(t)}{\alpha(t)} \nu \cdot e_0 \cdot \varphi_t
\]
\[
= -\frac{\alpha'(t)}{\alpha(t)} \nabla^t_{e_0(t)} \varphi_t - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a e_0(t \cdot \varphi_t).
\]

We have shown
\[
\begin{align*}
\nabla^{\Sigma^+}_\nu \left( \nabla^t_{e_0(t)} \varphi_t - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a e_0(t \cdot \varphi_t) \right) &= -\frac{\alpha'(t)}{\alpha(t)} \left( \nabla^t_{e_0(t)} \varphi_t - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a e_0(t \cdot \varphi_t) \right), \\
\nabla^t_{e_0(t)} \varphi_t - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a e_0(t \cdot \varphi_t) &= 0.
\end{align*}
\]

This differential equation with initial value has zero as unique solution, so
\[
\nabla^t_{e_0(t)} \varphi_t - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} a e_0(t \cdot \varphi_t) = 0
\]
on $Z$.

To prove the second equation, we define $\tilde{Y}_t \in \Gamma(TS^3)$ by $\tilde{Y}_t(x) := \frac{1}{\beta(t)} Y(x)$. Using the same ideas as above, we get

$$\nabla^t_{\tilde{Y}_t} \varphi_t - \left( \frac{1}{\beta(t)} \lambda - \frac{1}{4 \beta(t)^2} a \right) \tilde{Y}_t \cdot \varphi_t = 0.$$  

\[\square\]

Using the previous lemma, a straightforward calculation yields the following lemma. (Details can be found in [13, Lemma 6.14].)

**Lemma 3.8.** Let $(g_t, \varphi_t)_{t \in I}$ as in Lemma 3.7. For every $t \in I$ it holds that

$$Q_1(g_t, \varphi_t)(e_0(t), e_0(t)) = -\frac{9}{64} \frac{\alpha(t)^2}{\beta(t)^4} a^2 + \frac{1}{2} \frac{\alpha(t)}{\beta(t)^3} a \lambda - \frac{1}{2} \frac{1}{\beta(t)^2} \lambda^2,$$

$$Q_1(g_t, \varphi_t)(e_1(t), e_1(t)) = \frac{3}{64} \frac{\alpha(t)^2}{\beta(t)^4} a^2 - \frac{1}{8} \frac{\alpha(t)}{\beta(t)^3} a \lambda,$$

$$Q_1(g_t, \varphi_t)(e_2(t), e_2(t)) = Q_1(g_t, \varphi_t)(e_1(t), e_1(t)),$$

$$Q_1(g_t, \varphi_t)(e_i(t), e_j(t)) = 0 \text{ for } i \neq j,$$

$$Q_2(g_t, \varphi_t) = 0.$$

**Proof of Theorem 3.5.** First of all, we have

$$\frac{\partial}{\partial t} \varphi_t = \nabla_{\frac{\partial}{\partial t}} \varphi = 0 = Q_2(g_t, \varphi_t).$$

Moreover, $|\varphi_t| = 1$ for all $t \in I$ follows from the fact that $\nabla^{\Sigma^+(Z,g,\varphi)}$ is a metric connection. From Lemma 3.8 we deduce that $\frac{\partial}{\partial t} g_t = Q_1(g_t, \varphi_t)$ with $g_0 = g^\varepsilon$ holds iff $(\alpha, \beta)$ is the solution of the following system of two non-linear ordinary differential equations:

$$\alpha'(t) = -\frac{9}{128} \frac{\alpha(t)^3}{\beta(t)^4} a^2 + \frac{1}{4} \frac{\alpha(t)^2}{\beta(t)^3} a \lambda - \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} \lambda^2,$$

$$\beta'(t) = \frac{3}{128} \frac{\alpha(t)^2}{\beta(t)^3} a^2 - \frac{1}{16} \frac{\alpha(t)}{\beta(t)^2} a \lambda,$$

$$\alpha(0) = \varepsilon,$$

$$\beta(0) = 1.$$

Let $F: U := \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \mid y = 0\} \rightarrow \mathbb{R}^2$ be the vector field associated to that system, i.e.

$$F \left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} -\frac{9}{128} \frac{x^3}{y^4} a^2 + \frac{1}{4} \frac{x^2}{y^3} a \lambda - \frac{1}{4} \frac{x}{y^2} \lambda^2 \\ \frac{3}{128} \frac{x^2}{y^3} a^2 - \frac{1}{16} \frac{x}{y^2} a \lambda \end{array} \right),$$

see figure 1.
Let $c = (x, y) : J \to U$, $J \subset \mathbb{R}$ interval, be an integral curve of $F$. If there exists $t \in J$ such that $x(t) \neq 0$, then $c(J)$ lies in one quadrant of $\mathbb{R}^2$. Using

$$x'(t) = \frac{x(t)}{y(t)^2} \left( -\frac{9}{128} \left( \frac{x(t)}{y(t)} a \lambda \right)^2 + \frac{1}{4} \left( \frac{x(t)}{y(t)} a \lambda \right) - \frac{1}{4} \right)$$

and $-\frac{9}{128} z^2 + \frac{1}{4} z - \frac{1}{4} < 0$ for all $z \in \mathbb{R}$, we get that $x(t)$ is either strictly decreasing or strictly increasing (depending on in which quadrant $c(J)$ lies). It follows that the critical points of $F$ are precisely the points $(0, k)$ for $k \neq 0$.

Let us now prove the case $a \lambda = -2$. First we show that the integral curves of $F$ remain in certain compact subsets of $U$. To that end, let

$$K(v, w) := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq v, \ w \leq y \leq w + v - x\}.$$

If $c$ is an integral curve of $F$ as above with $x(l), y(l) > 0$ for some $l \in J$, then $c(t) \in K(c(l))$ for all $t \in J$ with $t \geq l$. The idea to prove that is as follows: For every boundary point $(v, w) \in \partial K(c(l))$ with $v > 0$ the vector $F(v, w)$ points inside $K(c(l))$. Then the integral curve $c$ can’t leave $K(c(l))$ since its movement is prescribed by $F$.

Let $c : J \to U$ be a maximal integral curve of $F$ with $c(0) = (\varepsilon, 1)$, $\varepsilon > 0$, and $J = [0, t_{\max})$. Then $c(J) \subset K(c(0)) \subset U$. Therefore, we have $t_{\max} = \infty$. Using the Poincaré-Bendixson theorem (we use the version from [6]) and observing that $F$ has no periodic orbits, we get a sequence $(t_n)_{n \in \mathbb{N}}$ such that $t_n \geq 0$, $t_n \to \infty$, and $c(t_n) \to p$ for $n \to \infty$ where $p$ is a critical point of $F$. So we have $p = (0, k)$ for some $k > 0$. It follows that

$$\lim_{t \to \infty} c(t) = (0, k),$$

Figure 1: Plots of the vector field $F$ of Theorem 3.5.

(a) $F$ for $a\lambda = 2$ with integral curves. The integral curves with starting point $(1, 1)$ and $(\frac{1}{2}, 1)$ are highlighted.

(b) $F$ for $a\lambda = -2$ with integral curves.
Figure 2: The sets $K_1(v, w)$, $K_2(v)$, and $K_3(v, w)$ together with the integral curves of $F$ through $(1, 1)$ and $(\frac{2}{3}, 1)$ for $a\lambda = 2$.

since for every $n \in \mathbb{N}$ we have $c([t_n, \infty)) \subset K(c(t_n))$. This proves Theorem 3.5 in the case $a\lambda = -2$.

The case $a\lambda = 2$ can be treated with the same methods, i.e. by showing that integral curves remain in certain compact sets and using the Poincaré-Bendixson theorem. This time, however, we have to consider three different cases depending on the value of $\varepsilon > 0$.

We briefly outline the proof in this case. Define

$K_1(v, w) := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq v, \frac{3}{2}x + w - \frac{3}{2}v \leq y \leq w\}$ for $0 < v < \frac{2}{3}w$,

$K_2(v) := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq v, x \leq y \leq \frac{3}{2}x\}$ for $v > 0$,

$K_3(v, w) := \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq v, \frac{w}{v}x \leq y \leq x\}$ for $0 < w < v$,

see figure 2.

Proof for $a\lambda = 2$ and $0 < \varepsilon < \frac{2}{3}$: We show as before: If $c = (x, y) : J \to U$ is an integral curve of $F$ with $0 < x(l) < \frac{2}{3}y(l)$ and $y(l) > 0$ for some $l \in J$, then $c(t) \in K_1(c(l))$ for all $t \in J$ with $t \geq l$. Using the Poincaré-Bendixson theorem as above proves the theorem in this case.

Proof for $a\lambda = 2$ and $\frac{2}{3} < \varepsilon < 1$: Again we have: If $c = (x, y) : J \to U$ is an integral curve of $F$ with $0 < x(l) < y(l) < \frac{3}{2}x(l)$ for some $l \in J$, then $c(t) \in K_2(x(l))$ for all $t \in J$ with $t \geq l$. Let $c = (x, y) : J \to U$ be a maximal integral curve of $F$ with $c(0) = (\varepsilon, 1)$, $\frac{2}{3} < \varepsilon < 1$, and $J = [0, t_{\text{max}})$. We show $c(t) \overset{t\to t_{\text{max}}}{\to} 0$. It holds that $c(J) \subset K_2(x(0))$. Because of the Poincaré-Bendixson theorem, there exists no $\delta > 0$ such that $c(J) \subset K_2(x(0)) \cap \{(x, y) \in \mathbb{R}^2 \mid x \geq \delta\}$. Together with the fact that $x(t)$ is
strictly decreasing we get \( x(t) \xrightarrow{t \to t_{\max}} 0 \) and therefore \( c(t) \xrightarrow{t \to t_{\max}} 0 \). This completes the proof in that case.

**Proof for** \( a\lambda = 2 \) and \( \varepsilon > 1 \): If \( c = (x, y): J \to U \) is an integral curve of \( F \) with \( 0 < y(l) < x(l) \) for some \( l \in J \), then \( c(t) \in K_3(c(l)) \) for all \( t \in J \) with \( t \geq l \). Now proceed as in the case \( \frac{2}{3} < \varepsilon < 1 \).

It remains to prove the last statement of the theorem. Let \( GL^+ S^3 \) be the topological spin structure on \( S^3 \). For fixed \( e^{is} \in S^1 \) we construct a spin-diffeomorphism \( F: GL^+ S^3 \to GL^+ S^3 \) which restricts to the action of \( e^{is} \) on \( Spin(S^3, g_t) \) defined in Section 3.1 for every \( t \in I \). (We use the definition of “spin-diffeomorphism” which is given in [2] Section 4.1.) Then we use the diffeomorphism invariance of the spinor flow (see [2] Corollary 4.5. (ii)) together with the uniqueness of the solution finish the proof.

The \( S^1 \)-action on \( S^3 \) induces an \( S^1 \)-action on \( GL^+ S^3 \) which lifts uniquely to a \( S^1 \)-action on \( GL^+ S^3 \). This can be shown as in the case of metric spin structures (see Section 3.1) using topological spin structures instead. From the action of \( e^{is} \) on \( S^3 \), \( GL^+ S^3 \) and on \( Spin(S^3, g_t) \) (the latter is defined in Section 3.1) we then get maps

\[
F: GL^+ S^3 \to GL^+ S^3,
F_t: Spin(S^3, g_t) \to Spin(S^3, g_t),
f: S^3 \to S^3
\]
where \( F \) is a spin-diffeomorphism. The \( S^1 \)-actions on \( GL^+ S^3 \) and \( SO(S^3, g_t) \) coincide on \( SO(S^3, g_t) \subset GL^+ S^3 \). Combining that with the uniqueness of the \( S^1 \)-action on \( Spin(S^3, g_t) \) we get

\[
F|_{Spin(S^3, g_t)} = F_t
\]
for every \( t \in I \). Using [2] Section 4.1 we get a map

\[
F_*: \mathcal{N} \to \mathcal{N}, \quad \mathcal{N}_g \ni \varphi \mapsto F_*\varphi \in \mathcal{N}_{(f^{-1})\circ g},
\]
defined by: If locally \( \varphi = [\tilde{s}, \tilde{\varphi}] \), then \( F_*\varphi = [F \circ \tilde{s} \circ f^{-1}, \tilde{\varphi} \circ f^{-1}] \). From the definitions it follows that for every \( t \in I \) and every spinor \( \varphi \in \mathcal{N}_{g_t} \)

\[
(F_*\varphi)(x) = \varphi(x \cdot e^{-is}) \cdot e^{is}. \tag{3.4}
\]

Now let \((g_t, \varphi_t)_{t \in I}\) be the solution of the spinor flow with initial value as in Theorem 3.3. By the diffeomorphism invariance of the spinor flow, \((f^{-1})^* g_t, F_*\varphi_t)_{t \in I}\) is also a solution. We have \((f^{-1})^* g_0 = (f^{-1})^* g^\circ g^c = g^c \) and from (3.4) we get \( F_*\varphi_0 = \varphi_0 \). Because of the uniqueness of the solution of the spinor flow it follows that

\[
F_*\varphi_t = \varphi_t
\]
for every \( t \in I \). Using again (3.4) and noting that \( e^{is} \in S^1 \) was arbitrary, we see that \( \varphi_t \) is \( S^1 \)-invariant, i.e. \( \varphi_t \in V(t) \). Define \( \sigma_t \in \Gamma(\Sigma(S^2, \beta(t)^2 g_{FS}) \) by \( Q(\sigma_t) = \varphi_t \). Combining the second equation in Lemma 3.7 with (3.2) and (3.3) yields that \( \sigma_t \) is a \( \frac{\lambda}{\beta(t)} \)-Killing spinor. This finishes the proof of the theorem. \( \square \)
Remark 3.9 (Change of orientation convention). If we choose the other orientation on $S^3$ (i.e. the orientation that satisfies: If $(f_1, f_2)$ is any oriented local orthonormal frame on $(S^2, g_{S^2})$, then $(-K, f_1^*, f_2^*)$ is an oriented local orthonormal frame on $(S^3, g_{S^3})$), then Theorem 3.5 still holds if we switch the results for the cases “$a\lambda = 2$” and “$a\lambda = -2$”. This can be seen as follows: In Lemma 3.4 one has to replace “$a$” by “$-a$”. The additional sign enters because in the proof we used (2.2). For the same reason we have to replace “$a$” by “$-a$” in (3.1)–(3.2). Then we also need to replace “$a$” by “$-a$” in Lemma 3.7 and Lemma 3.8 and therefore also in Theorem 3.5.

3.3 A stability theorem

For $\varepsilon > 0$ we define $c(\varepsilon) > 0$ by $\text{vol}(S^3, c(\varepsilon)g^s) = 1$, i.e. $c(\varepsilon) = \left(\frac{1}{2\varepsilon^2}\right)^{\frac{3}{2}}$.

Our second main result is the following theorem.

Theorem 3.10. (Stability) Let $\varepsilon > 0$ and $\mu \in \{\pm \frac{1}{2}\}$. Moreover, let $g_0 := c(\varepsilon)g^s$ and $\varphi_0$ a spinor which is obtained via parallel transport of a $\mu$-Killing spinor from $(S^3, g_{S^3})$ to $(S^3, g^s)$ such that $|\varphi_0| = 1$. (In Remark 3.11 we will explain how $\varphi_0$ is defined in a more formal way.) Then the solution of the normalized spinor flow on $M = S^3$ with initial value $(g_0, \varphi_0)$ is given by (1.1)–(1.2) where $b = t_{\text{max}}, t_{\text{max}} \in (0, \infty]$ maximum time of existence (to the right), such that:

If $a\mu = 1$, then:

- For $0 < \varepsilon < \frac{2}{3}$ we have $\lim_{t \to t_{\text{max}}} \alpha(t) = 0$ and $\lim_{t \to t_{\text{max}}} \beta(t) = \infty$.
- For $\varepsilon = \frac{2}{3}$ we have $t_{\text{max}} = \infty$, $\alpha(t) \equiv \frac{2}{3}\sqrt{c(\frac{3}{2})}$, and $\beta(t) \equiv \sqrt{c(\frac{3}{2})}$. In particular, $(c(\frac{2}{3})g^s, \varphi_0)$ is a critical point of the normalized spinor flow.
- For $\varepsilon > \frac{2}{3}$ we have $t_{\text{max}} = \infty$ and $\lim_{t \to \infty} \alpha(t) = \lim_{t \to \infty} \beta(t) = \sqrt{c(1)}$. Moreover, there exist smooth functions $f, g : I \to (0, \infty)$ with $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} g(t) = \frac{\mu}{\sqrt{c(1)}}$ and

$$\nabla^I_Y \varphi_t = f(t)K \cdot \varphi_t, \quad \nabla^I_K \varphi_t = g(t)Y \cdot \varphi_t,$$

for every horizontal vector field $Y \in \Gamma(TS^3)$ and every $t \in I$.
- For $\varepsilon = 1$ we have $t_{\text{max}} = \infty$ and $\alpha(t) \equiv \beta(t) \equiv \sqrt{c(1)}$. In particular, $(c(1)g_{S^3}, \varphi_0)$ is a critical point of the normalized spinor flow.

If $a\mu = -1$, then:

- For every $\varepsilon > 0$ we have $t_{\text{max}} = \infty$ and $\lim_{t \to \infty} \alpha(t) = \lim_{t \to \infty} \beta(t) = \sqrt{c(1)}$. Moreover, there exist smooth functions $f, g : I \to (0, \infty)$ with $\lim_{t \to \infty} f(t) = \lim_{t \to \infty} g(t) = \frac{\mu}{\sqrt{c(1)}}$ and

$$\nabla^I_Y \varphi_t = f(t)K \cdot \varphi_t, \quad \nabla^I_K \varphi_t = g(t)Y \cdot \varphi_t,$$
\[ \nabla_t^Y \varphi_t = g(t) Y^\iota \varphi_t, \]

for every horizontal vector field \( Y \in \Gamma(TS^3) \) and every \( t \in I \).

- For \( \varepsilon = 1 \) we have \( t_{\text{max}} = \infty \) and \( \alpha(t) \equiv \beta(t) \equiv \sqrt{c(1)} \). In particular, \((c(1)g_{S^3}, \varphi_0)\) is a critical point of the normalized spinor flow.

**Remark 3.11.**

1. In the case \( a\mu = 1 \) we can interpret the result as follows: If we are not too far away (\( \varepsilon > \frac{2}{3} \)) from the normalized standard metric \( c(1)g_{S^3} \) together with a Killing spinor, then the metric part of the solution flows back to the normalized standard metric \((\alpha, \beta \to \sqrt{c(1)})\) and the spinor part of the solution flows back to a Killing spinor \((f, g \to \frac{\mu}{\sqrt{c(1)}})\). However, if we are too far away \((\varepsilon \leq \frac{2}{3})\), then the solution no longer flows back \((\alpha \to 0, \beta \to \infty)\).

2. If we choose \( g_0 = g^\varepsilon \) and \( \varphi_0 \) as in Theorem 3.10 then \((g_0, \varphi_0)\) converges to a point under the unnormalized spinor flow (for \( a\mu = -1 \) or \( a\mu = 1 \) and \( \varepsilon \geq \frac{2}{3} \)), see \([13, \text{Theorem 6.17}]\). In that sense the interesting behavior is only captured in the normalized flow.

3. In the following we make precise what we mean by \( \varphi_0 \) in Theorem 3.10. To that end, let \( \sigma_0 \in \Gamma(\Sigma(S^3, g_{S^3})) \) be a \( \mu \)-Killing spinor with \( |\sigma_0| = 1 \).

   **Case** \( \varepsilon > 1 \): Define \( \sigma_t \) by (1.2) where \( \alpha(t) = 1 + t, \beta(t) \equiv 1 \) and \( b = \varepsilon - 1 \). Set \( \varphi_0 := \sigma_{\varepsilon - 1} \in \Gamma(\Sigma(S^3, g^\varepsilon)) \).

   **Case** \( \varepsilon < 1 \): Define \( \sigma_t \) by (1.2) where \( \alpha(t) = 1 - t, \beta(t) \equiv 1 \), and \( b = 1 - \varepsilon \). Set \( \varphi_0 := \sigma_{1-\varepsilon} \in \Gamma(\Sigma(S^3, g^\varepsilon)) \).

   **Case** \( \varepsilon = 1 \): Simply set \( \varphi_0 := \sigma_0 \).

   Note that \( \Sigma(S^3, g^\varepsilon) = \Sigma(S^3, c(\varepsilon)g^\varepsilon) \), so \( \varphi_0 \in \Gamma(\Sigma(S^3, c(\varepsilon)g^\varepsilon)) \).

**Lemma 3.12.** Choose \((g_0, \varphi_0)\) as in Theorem 3.10 and define \((g_t, \varphi_t)_{t \in I}\) by (1.1)–(1.2). Then, for every \( t \in I \) and every horizontal vector field \( Y \in \Gamma(TS^3) \), it holds that

\[ \nabla_{K^t}^t \varphi_t = \left( \frac{\mu - \frac{1}{4}a}{\alpha(t)} + \frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} \right) K^t \varphi_t, \]

\[ \nabla_{Y^t}^t \varphi_t = \frac{1}{\beta(t)} \left( -\frac{1}{4} \frac{\alpha(t)}{\beta(t)^2} - 1 \right) + \mu \right) Y^t \varphi_t. \]

**Proof.** First, these equations hold for \( t = 0 \). (This can be shown as follows: Let \( \sigma_t \) be as in Remark 3.11. We can deduce equations for \( \nabla^t \sigma_t \) similar to the proof of Lemma 3.7. Evaluating the equations for \( \nabla^t \sigma_t \) at \( t = \varepsilon - 1 \) and \( t = 1 - \varepsilon \), respectively, yields the equations of Lemma 3.12 for \( t = 0 \), see also [13, Lemma 6.21].) Using the same method as in the proof ofLemma 3.7 yields the equations for all \( t \in I \).
Using Lemma 3.12 one easily proves the following lemma.

**Lemma 3.13.** Let \((g_t, \varphi_t)_{t \in I}\) as in Lemma 3.12. For every \(t \in I\) it holds that

\[
Q_1(g_t, \varphi_t)(e_0(t), e_0(t)) = \frac{1}{4} \frac{1}{\alpha(t)^2} \left( \mu - \frac{1}{4} a \right)^2 + \frac{1}{\beta(t)^2} \left( -\frac{1}{2} \mu^2 - \frac{3}{8} a \mu \right) + \frac{\alpha(t)}{\beta(t)^3} \left( \frac{1}{8} a^2 + \frac{1}{2} a \mu \right) - \frac{9}{64} \frac{\alpha(t)^2}{\beta(t)^3} a^2,
\]

\[
Q_1(g_t, \varphi_t)(e_1(t), e_1(t)) = -\frac{1}{4} \frac{1}{\alpha(t)^2} \left( \mu - \frac{1}{4} a \right)^2 + \frac{1}{\beta(t)^2} \left( -\frac{1}{32} a^2 - \frac{1}{8} a \mu \right) + \frac{3}{64} \frac{\alpha(t)^2}{\beta(t)^3} a^2,
\]

\[
Q_1(g_t, \varphi_t)(e_2(t), e_2(t)) = Q_1(g_t, \varphi_t)(e_1(t), e_1(t)),
\]

\[
Q_1(g_t, \varphi_t)(e_j(t), e_j(t)) = 0 \text{ for } i \neq j,
\]

\[
Q_2(g_t, \varphi_t) = 0,
\]

\[
\frac{1}{6 \ \text{vol}(S^3, g_t)} E(g_t, \varphi_t) = \frac{1}{12} \left( \mu - \frac{1}{4} a \right)^2 \frac{1}{\alpha(t)^2} + \frac{1}{64} \frac{\alpha(t)^2}{\beta(t)^4} + \frac{1}{24} \frac{\alpha(\mu - 1 - a)}{\beta(t)^2} + \frac{1}{6} \frac{\alpha(\mu + a)}{\beta(t)^2} - \frac{1}{12} \frac{\alpha(t)}{\beta(t)^2}.
\]

**Proof of Theorem 3.14.** From Lemma 3.13 we get that \(\frac{\partial}{\partial t} g_t = \tilde{Q}_1(g_t, \varphi_t)\) with \(g_0 = c(\varepsilon)g^c\) is equivalent to the following systems of two non-linear ordinary differential equations for \((\alpha, \beta)\) with initial values \(\alpha(0) = \sqrt{c(\varepsilon)} \varepsilon, \beta(0) = \sqrt{c(\varepsilon)}\): For \(a \mu = 1\):

\[
\alpha'(t) = -1 \frac{\alpha(t)^3}{4 \beta(t)^4} + 5 \frac{\alpha(t)^2}{12 \beta(t)^3} - \frac{1}{6} \frac{\alpha(t)}{\beta(t)^2},
\]

\[
\beta'(t) = 1 \frac{\alpha(t)^2}{8 \beta(t)^3} - 5 \frac{\alpha(t)}{24 \beta(t)^2} + \frac{1}{12} \frac{1}{\beta(t)},
\]

and for \(a \mu = -1\):

\[
\alpha'(t) = -1 \frac{\alpha(t)^3}{4 \beta(t)^4} + 1 \frac{\alpha(t)}{12 \beta(t)^2} + \frac{1}{6} \frac{1}{\alpha(t)},
\]

\[
\beta'(t) = 1 \frac{\alpha(t)^2}{8 \beta(t)^3} - 1 \frac{\beta(t)}{12 \alpha(t)^2} - \frac{1}{24} \frac{1}{\beta(t)}.\]

Denote by \(F = F(x, y)\) the vector field associated to these differential equations as in the proof of Theorem 3.15. First, we show that it suffices to restrict \(F\) to a certain 1-dimensional submanifold of \(\mathbb{R}^2\). To that end, define

\[
u: (0, \infty) \to \mathbb{R}^2, \quad t \mapsto \left( \sqrt{c(t)} t, \sqrt{c(t)} \right) = \left( (2\pi)^{-\frac{1}{4}} t^\frac{3}{4}, (2\pi)^{-\frac{3}{4}} t^{-\frac{1}{4}} \right).
\]

The image of \(u\) is an embedded submanifold of \(\mathbb{R}^2\) and precisely the set of the initial values we are interested in. Noting that

\[
F(u(t)) = k(t) u'(t),
\]

20
with

\[ k(t) = \begin{cases} \frac{1}{5}(2\pi^2)\frac{3}{5}t^3(-3t^2 + 5t - 2), & \text{for } a\mu = 1, \\ \frac{1}{3}(2\pi^2)\frac{3}{5}t^{-\frac{4}{3}}(-3t^4 + t^2 + 2), & \text{for } a\mu = -1, \end{cases} \]

we see that \( F \) is tangent to the image of \( u \). So we only have to understand the integral curves of \( F|_{\text{Im}(u)} \). One way to do this is to look at the corresponding vector field on \((0, \infty)\) which is given by \( d(u^{-1})(V \circ u) \). Since \( d(u^{-1})(V \circ u) = k(t) \) the claims about \( \alpha \), \( \beta \) and \( t_{\text{max}} \) of Theorem 3.10 follow easily. The claimed convergence of the functions \( f \) and \( g \) follows from Lemma 3.12 and the shown convergence of \( \alpha \) and \( \beta \). This finishes the proof of the theorem. \( \Box \)
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