On a generalized Brauer group in mixed characteristic cases

Makoto Sakagaito

Department of Mathematical Sciences IISER Mohali,
Knowledge city, Sector 81, SAS Nagar, Manauli P.O. 140306, India
E-mail address: makoto@iisermohali.ac.in

Abstract

We define a generalization of the Brauer group $H^n_B(X)$ for an equi-dimensional scheme $X$ and $n > 0$. In the case where $X$ is the spectrum of a local ring of a smooth algebra over a discrete valuation ring, $H^n_B(X)$ agrees with the étale motivic cohomology $H^{n+1}_{\text{ét}}(X, \mathbb{Z}(n-1))$. We prove (a part of) the Gersten-type conjecture for the generalized Brauer group for a local ring of a smooth algebra over a mixed characteristic discrete valuation ring and an isomorphism $H^n_B(R) \simeq H^n_B(k)$ for a henselian local ring $R$ of a smooth algebra over a mixed characteristic discrete valuation ring and the residue field $k$.

1 Introduction

Let $A$ be a Dedekind ring or field, $X$ a smooth scheme over $\text{Spec}(A)$, $\mathbb{Z}(n)_{\text{ét}}$ the Bloch’s cycle complex for étale topology and $\mathbb{Z}/m\mathbb{Z}(n)_{\text{ét}} = \mathbb{Z}(n)_{\text{ét}} \otimes \mathbb{Z}/m$ for a positive integer $m$.

Then

(i) If $l \in \mathbb{N}$ is invertible in $A$, there is a quasi-isomorphism

$$\mathbb{Z}/l(n)_{\text{ét}} \xrightarrow{\sim} \mu_l^{\otimes n}[0]$$

by [2, p.774, Theorem 1.2.4]. Here $\mu_l$ is the sheaf of $l$-th roots of unity.

(ii) If $\text{char}(A) = p > 0$, there is a quasi-isomorphism

$$\mathbb{Z}/p^r(n)_{\text{ét}} \xrightarrow{\sim} W \Omega^n_{r, \log}[-n]$$

for any positive integer $r$ by [3, Theorem 8.5] and [9, Lemma 4.1]. Here $W \Omega^n_{r, \log}$ is the logarithmic de Rham-Witt sheaf.

Assume that a positive integer $m$ equals $l$ in (i) or $p^r$ in (ii). Let $R$ be a local ring of a smooth algebra over $A$, $k(R)$ its fraction field and $\kappa(p)$ the residue field of $p \in \text{Spec}(R)$.
Then the sequence of étale cohomology groups

\[ 0 \to H^n_{\text{ét}}(R, \mathbb{Z}/m(n)) \to H^n_{\text{ét}}(k(R), \mathbb{Z}/m(n))' \to \bigoplus_{p \in \text{Spec}(R)} H^n_{\text{ét}}(\kappa(p), \mathbb{Z}/m(n - 1)) \]

is exact by ([13], [2, p.774, Theorem 1.2.2,4,5]) and ([9, Proposition 4.5], [12, p.600, Theorem 4.1]) where

\[ H^n_{\text{ét}}(k(R), \mathbb{Z}/m(n))' = \text{Ker} \left( H^n_{\text{ét}}(k(R), \mathbb{Z}/m(n)) \to \prod_{p \in \text{Spec}(R), \text{ht}(p)=1} H^n_{\text{ét}}(k(R_p), \mathbb{Z}/m(n)) \right) \]

and \( R_p \) is the strictly henselization of \( R_p \).

Since \( H^i_{\text{Zar}}(R, \mathbb{Z}(n)) = 0 \) for \( i > n \) by ([2, p.779, Theorem 3.2 b]) and ([2, p.786, Corollary 4.4]), we have

\[ H_i^{n+1}(R, \mathbb{Z}(n)) = H_i^{n+1}(R, \mathbb{Q}(n)) = 0 \]

for \( i > n \) by ([2, p.774, Theorem 1.2.2] and [2, p.781, Proposition 3.6]. Hence the étale motivic cohomology \( H^{n+2}_{\text{ét}}(R, \mathbb{Z}(n)) \) is torsion group and

\[ H^{n+1}_{\text{ét}}(R, \mathbb{Z}/m(n)) = \text{Ker} \left( H^{n+2}_{\text{ét}}(R, \mathbb{Z}(n)) \xrightarrow{x \cdot m} H^{n+2}_{\text{ét}}(R, \mathbb{Z}(n)) \right) \]

for any positive integer \( m \). Therefore we can regard the sequence (1) as (a part of) the Gersten type resolution for the étale motivic cohomology.

In this article, we prove results which relate to the above facts.

First we define a generalization of the Brauer group in §3. This definition (Definition 3.1) is a revised version of ([9, §3, Definition 1]). We show the reason why we can regard it as a generalization of the Brauer group (Proposition 3.1) and a relation between it and étale motivic cohomology (Proposition 3.2).

Second we prove the following main results in §4 and §5.

**Theorem 1.1.** (Proposition 3.2 and Theorem 4.2) Let \( A \) be a discrete valuation ring of mixed-characteristic \((0, p)\) and \( R \) a local ring of a smooth algebra over \( A \).

Then the sequence (1) is exact for \( m = p^r \).

Moreover,

**Theorem 1.2.** (Theorem 5.2) Let \( R \) be a henselian local ring of a smooth algebra over a mixed-characteristic discrete valuation ring \( A \) and \( k \) the residue field of \( R \).
Then the canonical map
\[ H^{n+1}_\text{ét}(R, \mathbb{Z}/m(n)) \xrightarrow{\sim} H^{n+1}_\text{ét}(k, \mathbb{Z}/m(n)) \]
is an isomorphism for any positive integer \( m \).

Theorem 1.2 is proved in the case where \( m \) is invertible in \( A \) by [2, p.774, Theorem 1.2.3].

Finally we prove the following local-global principle in §6 by applying the above results.

**Theorem 1.3.** (Theorem 6.1) Let \( R \) be a henselian local ring of a smooth algebra over a mixed-characteristic discrete valuation ring and \( k \) the residue field of \( k \). Assume that \( \text{char}(k) = p > 0 \).

Then the local-global map
\[ H^{n+1}_\text{ét}(k(R), \mu_p \otimes n) \to \prod_{p \in \text{Spec}(R), \text{ht}(p) = 1} H^{n+1}_\text{ét}(k(\tilde{R}_p), \mu_p \otimes n) \]
is an injective where \( \tilde{R}_p \) is the henselization of \( R \) at \( p \).

**2 Notations**

For a scheme \( X \), \( X_\text{ét} \) and \( X_\text{Zar} \) denote the category of étale schemes over \( X \) equipped with the étale and Zariski topology, respectively. \( X^{(i)} \) denotes the set of points of codimension \( i \) and \( X_{(i)} \) denotes the set of points of dimension \( i \). \( k(X) \) denotes the ring of rational functions on \( X \) and \( \kappa(x) \) denotes the residue field of \( x \in X \).

For \( t \in \{ \text{ét}, \text{Zar} \}, S_X \), denotes the category of sheaves on \( X_t \).

**3 Definition of a generalized Brauer group**

In this section, we define a generalization of the Brauer group. This definition is a revised version of [9, §3, Definition 1].

Let \( D_i = \mathbb{Z}[t_0, \cdots, t_i]/(\sum_i t_i - 1) \), and \( \Delta^i = \text{Spec} D_i \) be the algebraic \( i \)-simplex. For an equi-dimensional scheme \( X \), let \( z^n(X, i) \) be the free abelian group on closed integral subschemes of codimension \( n \) of \( X \times \Delta^i \), which intersect all faces property. Intersecting with faces defines the structure of a simplicial abelian group, and hence gives a (homological) complex \( z^n(X, *) \).

The complex of sheaves \( \mathbb{Z}(n)_i \) on the site \( X_t \), where \( t \in \{ \text{ét}, \text{Zar} \} \), is defined as the cohomological complex with \( z^n(-, 2n - i) \) in degree \( i \).

Assume that \( X \) is a smooth scheme of finite type over a field or a Dedekind ring, then there is a quasi-isomorphism \( \mathbb{Z}(1) \simeq \mathbb{G}_m[-1] \).
For an abelian group $A$ we define $A(n)$ to be $\mathbb{Z}(n) \otimes A$.

Let $\epsilon : X_{\text{et}} \to X_{\text{Zar}}$ be the canonical map of sites. Then we define a generalization of the Brauer group.

**Definition 3.1.** Let $X$ be an equi-dimensional scheme. Then we define $H^0_B(X)$ as

$$H^0_B(X) = \Gamma \left( X, \mathbb{R}^{n+1} \epsilon_* \mathbb{Z}(n-1) \right).$$

This cohomology group relates to Kato homology (cf. [3] p.29, (5.3)).

For the following reason we can regard $H^0_B$ as a generalization of the Brauer group.

**Proposition 3.1.** Let $X$ be an essentially smooth scheme over a Dedekind domain. Then

$$H^1_B(X) = H^1_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}) \text{ and } H^2_B(X) = Br(X).$$

where $Br(X)$ is the cohomological Brauer group $H^2(\mathbb{X}_{\text{et}}, \mathbb{G}_m)$.

**Proof.** We prove

$$H^2_B(X) = Br(X). \quad (2)$$

We have the morphism

$$H^3_{\text{et}}(X, \mathbb{Z}(1)) \to \Gamma \left( X, \mathbb{R}^3 \epsilon_* \mathbb{Z}(1) \right) \quad (3)$$

which is induced by the morphism

$$\tau_{\leq 3} (\mathbb{R} \epsilon_* \mathbb{Z}(1)) \to \mathbb{R}^3 \epsilon_* \mathbb{Z}(1).$$

Let $x \in X_{(0)}$ and $i_x : x \to X$ the closed immersion. Then the morphism

$$\Gamma \left( X, \mathbb{R}^3 \epsilon_* \mathbb{Z}(1) \right) \to \prod_{x \in X_{(0)}} \Gamma \left( x, (i_x)^* \mathbb{R}^3 \epsilon_* \mathbb{Z}(1) \right)$$

is an injective and

$$\Gamma \left( x, (i_x)^* \mathbb{R}^3 \epsilon_* \mathbb{Z}(1) \right) = H^3_{\text{et}}(\text{Spec}(\mathcal{O}_{X,x}), \mathbb{Z}(1))$$

by [5] p.88, III, Proposition 1.13. Hence

$$\Gamma \left( X, \mathbb{R}^3 \epsilon_* \mathbb{Z}(1) \right) \subset \bigcap_{x \in X_{(0)}} H^3_{\text{et}}(\text{Spec}(\mathcal{O}_{X,x}), \mathbb{Z}(1)).$$

On the other hand,

$$H^3_{\text{et}}(X, \mathbb{Z}(1)) = \bigcap_{x \in X_{(0)}} H^3_{\text{et}}(\text{Spec}(\mathcal{O}_{X,x}), \mathbb{Z}(1))$$

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by \([9, \S 1]\), the equation (2)]. Therefore
\[
\Gamma \left( X, R^j\epsilon_*Z(1) \right) = \bigcap_{x \in X_{(0)}} H^3_{\text{et}}(\text{Spec}(O_{X,x}), Z(1))
\]
and the morphism \([9]\) is an isomorphism. Hence the equation (2) follows.

Moreover we can show that
\[
H^1_{\text{et}}(X) = H^1_{\text{et}}(X, \mathbb{Q}/\mathbb{Z})
\]
as above. Therefore the statement follows. \(\square\)

In the following case, \(H^n_B\) is expressed by étale motivic cohomology.

**Proposition 3.2.** Let \(X\) be an essentially smooth scheme over a Dedekind domain and \(H^i_{\text{Zar}}(X, Z(n - 1)) = 0\) for \(i \geq n + 1\).

Then
\[
H^n_B(X) = H^{n+1}_{\text{et}}(X, Z(n - 1)).
\]
Especially, if \(A\) is a local ring of smooth algebra over a Dedekind domain and \(X = \text{Spec}(A)\), then the equation (4) holds and
\[
H^n_B(X)_m \overset{\text{def}}{=} \text{Ker} \left( H^n_B(X) \xrightarrow{m} H^n_B(X) \right)
\]
\[
= H^n_{\text{et}}(X, Z/m(n - 1))
\]
for any positive integer \(m\).

**Proof.** Since the canonical map induces a quasi-isomorphism
\[
\mathbb{Z}(n - 1)_{\text{Zar}} \simeq \tau_{\leq n} R\epsilon_* Z(n - 1)_{\text{et}}
\]
([2, p.774, Theorem 1.2.2]), we have a distinguished triangle
\[
\cdots \rightarrow \mathbb{Z}(n - 1)_{\text{Zar}} \rightarrow \tau_{\leq n+1} R\epsilon_* Z(n - 1)_{\text{et}} \rightarrow R^{n+1} \epsilon_* Z(n - 1)_{\text{et}} [- (n + 1)] \rightarrow \cdots
\]
Hence the sequence
\[
0 \rightarrow H^{n+1}_{\text{Zar}}(X, Z(n - 1)) \rightarrow H^{n+1}_{\text{et}}(X, Z(n - 1)) \rightarrow \Gamma \left( X, R^{n+1} \epsilon_* Z(n - 1)_{\text{et}} \right)
\]
\[
\rightarrow H^{n+2}_{\text{Zar}}(X, Z(n - 1)) \rightarrow H^{n+2}_{\text{et}}(X, Z(n - 1))
\]
is exact. Therefore the equation (4) holds.

Assume that \(A\) is a local ring of a smooth algebra over a Dedekind domain and \(X = \text{Spec}(A)\). Then
\[
H^i_{\text{Zar}}(X, Z(n - 1)) = 0
\]
for \(i \geq n\) by [2, p.786, Corollary 4.4] and
\[
H^n_{\text{et}}(X, Z(n - 1)) = H^n_{\text{Zar}}(X, Z(n - 1)) = 0
\]
by the equation (6). Therefore the equations (4) and (5) hold. This completes the proof. \(\square\)
Remark 1. In general,
\[ H^{n+1}_{\text{ét}}(X, \mathbb{Z}(n-1))_{\text{tor}} \neq \Gamma \left( X, R^{n+1}_\ast \epsilon_\ast \mathbb{Z}(n-1) \right)_{\text{tor}}. \]

Let \( K \) be a field and \( l \) a positive integer. Suppose that \( \mu_l \subset K \).
Then we have
\[ H^5_{\text{Zar}}(\text{Spec} \, P^m_K, \mathbb{Z}(3)) = H^1_{\text{Zar}}(\text{Spec} \, K, \mathbb{Z}(1)) = K^* \]
for an integer \( m \geq 2 \) by the relation
\[ H^i_{\text{Zar}}(\text{Spec} \, P^m_K, \mathbb{Z}(n)) = \bigoplus_{j=0}^m H^i-2j_{\text{Zar}}(\text{Spec} \, K, \mathbb{Z}(n-j)). \]

Hence
\[ H^5_{\text{Zar}}(\text{Spec} \, P^m_K, \mathbb{Z}(3))_{\text{tor}} \neq 0. \]

Since the sequence
\[ 0 \to H^5_{\text{Zar}}(\text{Spec} \, P^m_K, \mathbb{Z}(3))_{\text{tor}} \to H^5_{\text{ét}}(\text{Spec} \, P^m_K, \mathbb{Z}(3))_{\text{tor}} \to \Gamma \left( X, R^5_\ast \epsilon_\ast \mathbb{Z}(3) \right)_{\text{tor}} \]
is exact,
\[ H^5_{\text{ét}}(\text{Spec} \, P^m_K, \mathbb{Z}(3))_{\text{tor}} \neq \Gamma \left( X, R^5_\ast \epsilon_\ast \mathbb{Z}(3) \right)_{\text{tor}}. \]

Proposition 3.3. Let \( X \) be an essentially smooth scheme over the spectrum of a Dedekind domain. Let \( \alpha : X_{\text{ét}} \to X_{\text{Nis}} \) be the canonical map of sites. Then
\[ H^{n+1}_{\text{ét}}(X) = \Gamma \left( X, R^{n+2}_\ast \alpha_\ast \mathbb{Z}(n)_{\text{ét}} \right). \]

Proof. Let \( \beta : X_{\text{Nis}} \to X_{\text{Zar}} \) be the canonical map of sites. Since \( \beta^* \) is exact and
\[ \beta^* \mathbb{Z}(n-1)_{\text{Zar}} = \mathbb{Z}(n-1)_{\text{Nis}}, \]
we have an quasi-isomorphism
\[ \mathbb{Z}(n-1)_{\text{Nis}} \simeq \tau_{\leq n} R_{\alpha_\ast} \mathbb{Z}(n-1)_{\text{ét}} \]
by [2] p.774, Theorem 1.2.2 and the sequence
\[ 0 \to H^{n+1}_{\text{Nis}}(X, \mathbb{Z}(n-1)) \to H^{n+1}_{\text{ét}}(X, \mathbb{Z}(n-1)) \to \Gamma \left( X, R^{n+1}_\ast \alpha_\ast \mathbb{Z}(n-1)_{\text{ét}} \right) \]
\[ \to H^{n+2}_{\text{Nis}}(X, \mathbb{Z}(n-1)) \to H^{n+2}_{\text{ét}}(X, \mathbb{Z}(n-1)) \]
is exact. Moreover the sequence (8) is exact and
\[ H^2_{\text{Zar}}(X, \mathbb{Z}(n-1)) = H^1_{\text{Nis}}(X, \mathbb{Z}(n-1)) \]
for any \( i \) by [2] p.781, Proposition 3.6. Therefore the statement follows from the Five lemma. \( \square \)
Proposition 3.4. Let $X$ be an essentially smooth scheme over the spectrum of a Dedekind domain. Then

$$H_{\mathcal{B}}^{n+1}(X) = \Gamma(X, R^{n+1}\epsilon_* \mathbb{Q}/\mathbb{Z}_{\text{et}}) = \Gamma(X, R^{n+1}\alpha_* \mathbb{Q}/\mathbb{Z}_{\text{et}}).$$

Proof. We prove the equation (9). The sequence

$$R^{n+1}\epsilon_* \mathbb{Q}(n)_{\text{et}} \to R^{n+1}\epsilon_* \mathbb{Q}/\mathbb{Z}(n)_{\text{et}} \to R^{n+2}\epsilon_* \mathbb{Z}(n)_{\text{et}} \to R^{n+2}\epsilon_* \mathbb{Q}(n)_{\text{et}}$$

is exact. Thus, the canonical map

$$\mathbb{Q}(n)_{\text{Zar}} \sim R\epsilon_* \mathbb{Q}(n)_{\text{et}}$$

is quasi-isomorphism by [2, p.781, Proposition 3.6], hence

$$R^{n+1}\epsilon_* \mathbb{Q}(n)_{\text{et}} = R^{n+2}\epsilon_* \mathbb{Q}(n)_{\text{et}} = 0$$

by [2, p.786, Corollary 4.4]. Therefore we have the equation (9). We can also prove the equation (10) as above. \qed

4 Purity

At first, we show the exactness of the following sequence in equi-characteristic cases.

Proposition 4.1. Let $A$ be a field or an equi-characteristic Dedekind domain. Let $\mathfrak{X}$ be an essentially smooth scheme over Spec($A$). Suppose that $\mathfrak{X}$ is an integral quasi-compact scheme.

Then the sequence

$$0 \to H_{\mathcal{B}}^{n+1}(\mathfrak{X}) \to \text{Ker} \left( H^{n+2}_{\text{et}}(k(\mathfrak{X}), \mathbb{Z}(n)) \to \prod_{x \in \mathfrak{X}(\mathbb{Q})} H^{n+2}_{\text{et}}(k(O_{\mathfrak{X}, x}), \mathbb{Z}(n)) \right)$$

$$\to \bigoplus_{x \in \mathfrak{X}(\mathbb{Q})} H^{n+1}_{\text{et}}(\kappa(x), \mathbb{Z}(n-1))$$

is exact.

Proof. Let $g : \text{Spec}(k(\mathfrak{X})) \to \mathfrak{X}$ be the generic point.

Since $R^{n+1}\epsilon_* \mathbb{Z}/m(n)$ is the Zariski sheaf on $\mathfrak{X}$ associated to the presheaf

$$U \mapsto H^{n+1}_{\text{et}}(U, \mathbb{Z}/m(n))$$

for any positive integer $m$, we have homomorphisms

$$R^{n+1}\epsilon_* \mathbb{Z}/m(n) \to g_* \left( H^{n+1}_{\text{et}}(k(\mathfrak{X}), \mathbb{Z}/m(n)) \right)$$
\[ \text{and} \]
\[ g_\ast \left( H^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n)) \right)' \rightarrow \bigoplus_{x \in X^{(1)}} (i_x)_\ast \left( H^{n+1}_{\text{ét}}(\kappa(x), \mathbb{Z}/m(n)) \right). \]

Here
\[ g_\ast \left( H^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n)) \right)' = \text{Ker} \left( g_\ast \left( H^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n)) \right) \rightarrow \prod_{x \in X^{(1)}} (i_x)_\ast \left( H^{n+1}_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/m(n)) \right) \right). \]

Then it is sufficient to show that the sequence
\[ 0 \rightarrow R^{n+1} \epsilon_* \mathbb{Z}/m(n) \rightarrow g_\ast \left( H^{n+1}_{\text{ét}}(k(\mathfrak{X}), \mathbb{Z}/m(n)) \right)' \rightarrow \prod_{x \in X^{(1)}} (i_x)_\ast \left( H^{n+1}_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{Z}/m(n)) \right) \]
is exact. Hence it is sufficient to show that the sequence \([11]\) is exact in the case where \( \mathfrak{X} \) is the spectrum of a local ring of a smooth scheme over \( A \). Therefore the statement follows from \([8, 5.2.\text{Theorem C}]\) and \([9, \text{Proposition 4.5}], [12, \text{p.600, Theorem 4.1}]\). \(\square\)

In the following, we consider \( H^n_{\text{ét}}(\mathfrak{X}) \) in the case where \( \mathfrak{X} \) is mixed-characteristic cases.

**Theorem 4.1.** Let \( R \) be a local ring of a smooth algebra over a discrete valuation ring with quotient field \( K \).

Then the homomorphism
\[ H^{n+2}_{\text{ét}}(R, \mathbb{Z}(n)) \rightarrow H^{n+2}_{\text{ét}}(K, \mathbb{Z}(n)) \]
is an injective.

**Proof.** Let \( Y \rightarrow \text{Spec}(R) \) be the inclusion of the closed fiber with open complement \( j : U \rightarrow \text{Spec}(R) \). Then \( Y \) is the spectrum of a local ring of smooth scheme over a field and
\[ H^n_{\text{ét}}(Y, \mathbb{Z}(n-1)) = H^n_{\text{Zar}}(Y, \mathbb{Z}(n-1)) \]
by \([2, \text{p.774, Theorem 1.2.2}], [13]\). Moreover
\[ H^n_{\text{Zar}}(Y, \mathbb{Z}(n-1)) = 0 \]
by \([2, \text{p.779, Theorem 3.2}]\) and \([2, \text{p.786, Corollary 4.4}]\). Hence
\[ H^{n+2}_{\text{ét}}(R_{\text{ét}}, \mathbb{Z}(n)) = H^n_{\text{ét}}(Y, \mathbb{Z}(n-1)) = 0 \]
by \([9, \text{Proposition 4.3}]\). Therefore the homomorphism
\[ H^{n+2}_{\text{ét}}(R, \mathbb{Z}(n)) \rightarrow H^{n+2}_{\text{ét}}(U, \mathbb{Z}(n)) \]
is an injective. On the other hand,
\[ H_{\text{Zar}}^i(U, \mathbb{Z}(n)) = 0 \]
for \( i \geq n + 2 \) by [2] p.779, Theorem 3.2 and [2] p.786, Corollary 4.4.

Hence
\[ H_{\text{et}}^{n+2}(U, \mathbb{Z}(n)) = H_B^{n+1}(U) \]
by Proposition 3.2 and the homomorphism
\[ H_B^{n+1}(U) \to H_{\text{et}}^{n+2}(K, \mathbb{Z}(n)) \]
is an injective by Proposition 4.1. Therefore the statement follows.

**Corollary 4.1.** Let \( A \) be a Dedekind domain and \( \mathfrak{X} \) an essentially smooth scheme over \( \text{Spec}(A) \).

Then
\[ H_B^{n+1}(\mathfrak{X}) = \bigcap_{x \in \mathfrak{X}(0)} H_B^{n+1}(\text{Spec} \mathcal{O}_{\mathfrak{X}, x}). \]

**Proof.** Let \( j : \mathfrak{X}_K \to \mathfrak{X} \) be the inclusion of the generic fiber. Then we have a distinguished triangle
\[ \cdots \to \bigoplus_{a \in \text{Spec}(A)^{(1)}} i_{ax} R_{i_{ax}}^1 (R^{n+1} \epsilon_* \mathbb{Z}/m(n)) \to R^{n+1} \epsilon_* \mathbb{Z}/m(n) \]
\[ \to R j_* j^* (R^{n+1} \epsilon_* \mathbb{Z}/m(n)) \to \cdots \]
by [2] p.778, Lemma 2.4. On the other hand,
\[ (j_* j^* (R^{n+1} \epsilon_* \mathbb{Z}/m(n)))_x = \Gamma((\text{Spec} \mathcal{O}_{\mathfrak{X}, x})_K, R^{n+1} \epsilon_* \mathbb{Z}/m(n)) \]
by [6] p.88, III, Proposition 1.13 and the homomorphism
\[ R^{n+1} \epsilon_* \mathbb{Z}/m(n) \to j_* j^* (R^{n+1} \epsilon_* \mathbb{Z}/m(n)) \]
is an injective by Theorem 4.1. Moreover, the homomorphism
\[ \Gamma (\mathfrak{X}, i_{ax} R_{i_{ax}}^1 (R^{n+1} \epsilon_* \mathbb{Z}/m(n))) \to \prod_{x \in \mathfrak{X}(0)} H_{(\mathcal{O}_{\mathfrak{X}, x})_a}^1 (\mathcal{O}_{\mathfrak{X}, x}, R^{n+1} \epsilon_* \mathbb{Z}/m(n)) \]
is an in injective. Hence the sequence
\[ 0 \to H_B^{n+1}(\mathfrak{X}) \to H_B^{n+1}(\mathfrak{X}_K) \to \prod_{x \in \mathfrak{X}(0)} H_{(\mathcal{O}_{\mathfrak{X}, x})_a}^1 (\mathcal{O}_{\mathfrak{X}, x}, R^{n+1} \epsilon_* \mathbb{Z}/m(n)) \]
is exact and
\[ H_B^{n+1}(\mathfrak{X}_K) = \bigcap_{x \in (\mathfrak{X}_K)(0)} H_B^{n+1}(\text{Spec} \mathcal{O}_{\mathfrak{X}_K, x}) \]
by Proposition 4.1. Therefore the statement follows. \( \square \)
Lemma 4.1. Let $R$ be a local ring of a smooth algebra over a discrete valuation ring $A$ and $X = \text{Spec}(R)$. Let $i : Z \to X$ be a regular closed subscheme of codimension 2 with open complement $j : U \to X$. Suppose that $\text{char}(Z) = p > 0$.

Then

$$H^i_{\text{Zar}}(U, Z(n)) = 0$$

for $i \geq n + 2$ and

$$H^n_B(U) = H^{n+1}_{\text{et}}(U, \mathbb{Z}(n - 1))$$  \hspace{1cm} (12)

Proof. Let $Z = \text{Spec}(R')$. Then $R'$ is a local ring of a regular ring of finite type over a field. By Quillen’s method (cf. [7, The proof of Theorem 5.11]),

$$R' = \lim_{\rightarrow} R'_i$$

where $R'_i$ is a local ring of a smooth algebra over $\mathbb{F}_p$ and the maps $R'_i \to R'_j$ are flat. Hence

$$H^i_{\text{Zar}}(Z, \mathbb{Z}(n)) = 0$$  \hspace{1cm} (13)

for $i \geq n + 1$ by [2, p.786, Corollary 4.4]. Therefore

$$H^i_{\text{Zar}}(U, \mathbb{Z}(n)) = 0$$

by [2, p.779, Theorem 3.2] and we have the equation (12) by Proposition 3.2.

This completes the proof. \qed

Proposition 4.2. With the notations of Lemma 4.1, we have

$$H^n_B(X) = H^n_B(U).$$

Proof. We have

$$H^n_B(U) = \bigcap_{x \in U^{(0)}} H^n_B(\mathcal{O}_{U,x})$$

by Corollary 4.1.

Let $l$ be a positive integer which is prime to $\text{char}(Z) = p > 0$. Then

$$H^n_B(\mathcal{O}_{U,x})_l \overset{\text{def}}{=} \text{Ker} \left( H^n_B(\mathcal{O}_{U,x}) \xrightarrow{x_l} H^n_B(\mathcal{O}_{U,x}) \right)$$

$$= \bigcap_{y \in U^{(1)}} H^n_B(\mathcal{O}_{U,y})_l \left( = \bigcap_{y \in U^{(1)}} \text{Ker} \left( H^n_B(\mathcal{O}_{U,y}) \xrightarrow{x_l} H^n_B(\mathcal{O}_{U,y}) \right) \right)$$

for $x \in U^{(0)}$ by [2, p.774, Theorem 1.2] and [13].

On the other hand, we have

$$H^n_B(X)_l = \bigcap_{x \in X^{(1)}} H^n_B(\mathcal{O}_{X,x})_l$$

by [2, p.774, Theorem 1.2] and [13].

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Since \( X^{(1)} = U^{(1)} \), we have
\[
H^n_B(X)_t = H^n_B(U)_t.
\]
If \( \text{char}(X) = p > 0 \), we have also
\[
H^n_B(X)_p = H^n_B(U)_p
\]
by Corollary 4.1 and [9, Theorem 4.1] as above.

Therefore it is sufficient to prove that
\[
H^n_B(X)_p = H^n_B(U)_p
\]
in the case where \( A \) is mixed-characteristic \((0,p)\).

Assume that \( A \) is a discrete valuation of mixed-characteristic \((0,p)\). Then we have quasi-isomorphisms
\[
\tau_{\leq n+1} \left( \mathbb{Z}(n-3)_{\text{et}}[-4] \right) \xrightarrow{\sim} \tau_{\leq n+1} \mathbb{R}^! \mathbb{Z}(n-1)_{\text{et}}^X
\]
by [9, Proposition 4.3] and
\[
\mathbb{Z}/p(n-3)_{\text{et}}[-4] \xrightarrow{\sim} \tau_{\leq n+1} \mathbb{R}^! \mathbb{Z}/p(n-1)_{\text{et}}^X
\]
by [10, p.540, Theorem 4.4.7] and [11, p.187, Remark 3.7].

Since the sequence
\[
H^{n-3}_{\text{et}}(\mathbb{Z}, \mathbb{Z}(n-3)) \xrightarrow{x^p} H^{n-3}_{\text{et}}(\mathbb{Z}, \mathbb{Z}(n-3)) \rightarrow H^{n-3}_{\text{et}}(\mathbb{Z}, \mathbb{Z}/p(n-3)) \rightarrow 0
\]
is exact by ([2, p.774, Theorem 1.2.2], [13]) and the equation (13), the sequence
\[
H^{n+1}_{\text{et}}(X, \mathbb{Z}(n-1)) \xrightarrow{x^p} H^{n+1}_{\text{et}}(X, \mathbb{Z}(n-1)) \rightarrow H^{n+1}_{\text{et}}(X, \mathbb{Z}/p(n-1)) \rightarrow 0
\]
is exact and we have
\[
H^{n+2}_{\text{et}}(X, \mathbb{Z}(n-1))_p = 0.
\]
Therefore we have
\[
H^{n+1}_{\text{et}}(X, \mathbb{Z}(n-1))_p = H^{n+1}_{\text{et}}(U, \mathbb{Z}(n-1))_p
\]
and the statement follows from Lemma 4.1 and Proposition 3.2.

**Theorem 4.2.** Let \( \mathcal{X} \) be an essentially smooth scheme over the spectrum of a Dedekind domain. Then
\[
H^n_B(\mathcal{X}) = \bigcap_{x \in \mathcal{X}^{(1)}} H^n_B(O_{\mathcal{X},x})
\]
and the sequence
\[
0 \rightarrow H^n_B(\mathcal{X}) \rightarrow \text{Ker} \left( H^n_B(\mathcal{X}) \rightarrow \prod_{x \in \mathcal{X}^{(1)}} H^n_B(k(O_{\mathcal{X},x})) \right) \rightarrow \bigoplus_{x \in \mathcal{X}^{(1)}} H^{n-1}_B(k(\kappa(x)))
\]
is exact.

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Proof. By [9, Proposition 4.5], it is sufficient to prove the equation (14). We shall show the equation (14) by induction on dim \( X \). In the case where dim \( X = 1 \), we can prove the equations (14) and (15) by [9, Proposition 4.5].

Let \( x \in X(0) \) and \( i_x : Z_x \to \text{Spec}(\mathcal{O}_{X,x}) \) a regular closed subscheme of codimension 2 with open complement \( j_x : U_x \to \text{Spec}(\mathcal{O}_{X,x}) \). Then we have

\[
H^n_B(\mathcal{O}_{X,x}) = H^n_B(U_x)
\]

by Proposition 4.2. Since \( \text{Spec}(\mathcal{O}_{X,x})^{(1)} = U_x^{(1)} \) and dim \( (U_x) = \text{dim}(X) - 1 \), we have

\[
H^n_B(U_x) = \bigcap_{y \in X^{(1)}} H^n_B(\mathcal{O}_{X,y})
\]

by induction hypothesis. Therefore the equation (14) follows by Corollary 4.1 and the sequence (15) is exact by [9, Proposition 4.5]. This completes the proof.

5 Étale motivic cohomology of Henselian regular rings

Let \( A \) be a mixed-characteristic henselian discrete valuation ring, \( K \) its fraction field and \( \pi \) a prime element of \( A \).

We consider the following diagram of schemes.

\[
\begin{array}{ccc}
\mathcal{X} \otimes_A K & \xrightarrow{j} & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec}(K) & \longrightarrow & \text{Spec}(A)
\end{array}
\]

where the vertical arrows are smooth.

Let \( \text{char}(A/(\pi)) = p > 0 \). Then the filtration \( U^n \mathcal{M}_r^n \) of

\[
\mathcal{M}_r^n = i^* R^n j_* \mu_p^{\otimes n}
\]

is defined and the structure of \( \mathcal{M}_r^n \) is studied in [1].

Let \( \Omega_Y^1 = \Omega_Y^{1,\mathbb{Z}} \) be the exterior algebra over \( \mathcal{O}_Y \) of the sheaf \( \Omega_Y^{1,\mathbb{Z}} \) of absolute differentials on \( Y \) and \( \Omega_{Y,\log} \) the part of \( \Omega_Y^1 \) generated étale locally by local sections of the forms

\[
\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}.
\]

Then the structure of \( \mathcal{M}_r^n \) is as follows.

**Theorem 5.1.** ([1] p.112, Corollary (1.4.1)) Let \( e \) be the absolute ramification index of \( K \) and \( e' = \frac{ep}{p-1} \).

Then the sheaf \( \mathcal{M}_r^n \) has the following structure.
(i) \[ M_1^n / U^1 M_1^n \simeq \Omega_{V, \log}^n \oplus \Omega_{Y, \log}^{n-1}. \]

(ii) If \( 1 \leq m < e' \) and \( m \) is prime to \( p \),
\[ U^m M_1^n / U^{m+1} M_1^n \simeq \Omega_{Y, \log}^{n-1}. \]

(iii) If \( 1 \leq m < e' \) and \( p \mid m \),
\[ U^m M_1^n / U^{m+1} M_1^n \simeq B_{i-1}^n \oplus B_i^{n-2} \]
where \[ B_i^q = \text{Image} \left( d : \Omega_{Y}^{q-1} \rightarrow \Omega_{Y}^q \right) \]
for an integer \( q \).

(iv) For \( m \geq e' \),
\[ U^m M_1^n = 0. \]

[4, p.548, Corollary 1.7] and [11, p.184, Theorem 3.3] are the improved versions of Theorem 5.1.

As an application, we have the following lemma.

**Lemma 5.1.** Let \( R \) be a henselian local ring of a smooth scheme over a mixed-characteristic discrete valuation ring \( A \) and \( \pi \) a prime element of \( A \).

Then we have
\[ H^q_{\text{ét}} \left( R/(\pi), U^1 M_1^n \right) = 0 \]
for \( q \geq 1 \).

**Proof.** We may assume that \( A \) is a henselian discrete valuation ring. Let \( q \geq 1 \). Since \( R/(\pi) \) is a henselian local ring and \( \text{char}(R/(\pi)) > 0 \), we have
\[ H^q_{\text{ét}} \left( R/(\pi), B_{i-1}^n \right) = H^q_{\text{ét}} \left( R/(\pi), B_i^{n-2} \right) = 0 \]
by [9, The proof of Lemma 5.2]. Moreover
\[ H^q_{\text{ét}} \left( R/(\pi), \Omega_{R/(\pi)}^{n-1} \right) = 0 \]
by [6, p.103, III, Lemma 2.15] and [6, p.114, III, Proposition 3.7]. Therefore the statement follows from Theorem 5.1.

We prove Theorem 1.2 by computing the cohomology group
\[ H^{n+1}_{\text{ét}} \left( R/(\pi), i_\ast \tau_{\leq n} R_{j_{\ast} \mu_p^{\otimes n}} \right) \]
as follows.
Theorem 5.2. Let $R$ be a henselian local ring of a smooth scheme over a mixed-characteristic discrete valuation ring $A$ and $k$ the residue field of $R$.

Then the canonical map
\[ H^{n+1}_{\text{ét}}(R, \mathbb{Z}/m(n)) \xrightarrow{\sim} H^{n+1}_{\text{ét}}(k, \mathbb{Z}/m(n)) \]
is an isomorphism for any positive integer $m$.

Proof. Let $\pi$ be a prime element of $A$ and $i : \text{Spec}(R/(\pi)) \to \text{Spec}(R)$ the closed subscheme with open complement $j : \text{Spec}(R[\pi^{-1}]) \to \text{Spec}(R)$.

Then the restriction map
\[ H^{n+1}_{\text{ét}}(R, \mathbb{Z}/m(n)) \xrightarrow{\sim} H^{n+1}_{\text{ét}}(R/(\pi), i^*\mathbb{Z}/m(n)) \]
is an isomorphism by [2, p.777, The proof of Proposition 2.2.b)]. Therefore it is sufficient to show that the homomorphism
\[ H^{n+1}_{\text{ét}}(R/(\pi), i^*\mathbb{Z}/m(n)) \to H^{n+1}_{\text{ét}}(R/(\pi), \mathbb{Z}/m(n)) \]
is an isomorphism in the case where $m = \text{char}(k) = p > 0$ by [9, Theorem 5.1].

We consider the cohomology group $H^{n+1}_{\text{ét}}(R/(\pi), i_*\tau_{\leq n} R_j \mu^{\otimes n}_p)$.

We have the spectral sequence
\[ H^{n+1}_{\text{ét}}(R/(\pi), i^* R^f_j \mu^{\otimes n}_p) \Rightarrow R^{s+1} \Gamma_{\text{ét}}(i^* \tau_{\leq n} R_j \mu^{\otimes n}_p) \tag{17} \]
and
\[ \mathbb{R}^{n+1} \Gamma_{\text{ét}}(i^* \tau_{\leq n} R_j \mu^{\otimes n}_p) = H^{n+1}_{\text{ét}}(R/(\pi), i^* \tau_{\leq n} R_j \mu^{\otimes n}_p) \]
where $\mathbb{R}^s \Gamma_{\text{ét}}$ is the right hyper-derived functor of the global sections functor $\Gamma_{\text{ét}}$ from $\mathcal{S}_{\text{Spec}(R/(\pi))_{\text{ét}}}$.

Since $p$-cohomological dimension of $k$ is at most 1,
\[ H^{s}_{\text{ét}}(R/(\pi), i^* R^f_j \mu^{\otimes n}_p) = 0 \tag{18} \]
for $s \geq 2$ by [2, p.777, The proof of Proposition 2.2.b)]. Hence we have
\[ H^{n+1}_{\text{ét}}(R/(\pi), \tau_{\leq n} (i^* R^f_j \mu^{\otimes n}_p)) = H^{n}_{\text{ét}}(R/(\pi), i^* R^n_j \mu^{\otimes n}_p) \]
by the spectral sequence (17) and
\[ H^1_{\text{ét}}(R/(\pi), i^* R^n_j \mu^{\otimes n}_p) = H^1_{\text{ét}}(R/(\pi), M^n_j / U^n 1^n_j) \]
by Lemma [5.1]. Therefore we have
\[ H^{n+1}_{\text{ét}}(R/(\pi), \tau_{\leq n} (i^* R^f_j \mu^{\otimes n}_p)) \]
\[ = H^{n+1}_{\text{ét}}(R/(\pi), \mathbb{Z}/p(n)) \oplus H^n_{\text{ét}}(R/(\pi), \mathbb{Z}/p(n - 1)) \tag{19} \]
by Theorem [5.1] (i). On the other hand, the homomorphism
\[ H^n_{\text{ét}}(R/(\pi), i^* \tau_{\leq n} R_j \mu^{\otimes n}_p) \to H^n_{\text{ét}}(R/(\pi), i^* R^n_j \mu^{\otimes n}_p) \]
is a surjective by the spectral sequence (17) and the equation (18). Moreover the homomorphism
\[ H^0_{\text{ét}}(R/(\pi), i^* R^n j_* \mu_p^{\otimes n}) \to H^0_{\text{ét}}(R/(\pi), \mathcal{M}_1^n / \mathcal{U}^1 \mathcal{M}_1^n) \]
is a surjective by Lemma 5.1. Therefore the homomorphism
\[ H^n_{\text{ét}}(R/(\pi), i^* \tau_{\leq n} R j_* \mu_p^{\otimes n}) \to H^{n-1}_{\text{ét}}(R/(\pi), \mathbb{Z}/p(n-1)) \]
is a surjective and the sequence
\[ 0 \to H^{n+1}_{\text{ét}}(R/(\pi), i^* \mathbb{Z}/p(n)) \to H^{n+1}_{\text{ét}}(R/(\pi), i^* \tau_{\leq n} R j_* \mu_p^{\otimes n}) \]
\[ \to H^n_{\text{ét}}(R/(\pi), \mathbb{Z}/p(n-1)) \to 0 \] (20)
is exact by the distinguished triangle
\[ \cdots \to i^* \mathbb{Z}/p(n)_{\text{ét}} \to i^* \tau_{\leq n} R j_* \mathbb{Z}/p(n)_{\text{ét}} \to \mathbb{Z}/p(n)_{\text{ét}} \to \cdots . \]
Therefore the homomorphism (16) is an isomorphism by (19) and (20). This completes the proof. \[ \square \]

### 6 An Application

In this section, we apply results of §4 and §5.

**Theorem 6.1.** Let \( R \) be a henselian local ring of a smooth algebra over a mixed-characteristic discrete valuation ring \( A \) and \( k \) the residue field of \( R \). Assume that \( \text{char}(k) = p > 0 \).

Then the local-global map
\[ H^{n+1}_{\text{ét}}(k(R), \mu_p^{\otimes n}) \to \prod_{\mathfrak{p} \in \text{Spec}(R)^{(1)}} H^{n+1}_{\text{ét}}(k(\overline{R}_p), \mu_p^{\otimes n}) \] (21)
is an injective where \( \overline{R}_p \) is the henselization of \( R \) at \( \mathfrak{p} \).

**Proof.** Let \( a \) be an element of the kernel of the local-global map (21). Let
\[ \text{Ker} \left( H^{n+1}_{\text{ét}}(k(R), \mu_p^{\otimes n}) \to \prod_{\mathfrak{p} \in \text{Spec}(R)^{(1)}} H^{n+1}_{\text{ét}}(k(R_p), \mu_p^{\otimes n}) \right) \]
where \( R_p \) is the strictly henselization of \( R_p \). Then
\[ a \in H^{n+1}_{\text{ét}}(k(R), \mathbb{Z}/p^{e(n)})' \]
Since the diagram
\[
\begin{array}{ccc}
H^{n+1}_{et}(k(R), \mathbb{Z}/p^r(n))' & \longrightarrow & H^{n+1}_{et}(\kappa(p), \mathbb{Z}/p^r(n-1)) \\
\downarrow & & \downarrow \\
H^{n+1}_{et}(k(\tilde{R}_p), \mathbb{Z}/p^r(n))' & \longrightarrow & H^{n+1}_{et}(\kappa(p), \mathbb{Z}/p^r(n-1))
\end{array}
\]
is commutative for \( p \in \text{Spec}(R) \),
\[
a \in H^{n+1}_{et}(R, \mathbb{Z}/p^r(n))
\]by Theorem 4.2. Hence it is sufficient to show that the homomorphism
\[
H^{n+1}_{et}(R, \mathbb{Z}/p^r(n)) \rightarrow H^{n+1}_{et}(\tilde{R}(\pi), \mathbb{Z}/p^r(n)) \tag{22}
\]is an injective where \( \pi \) is a prime element of \( A \).

We consider the commutative diagram
\[
\begin{array}{ccc}
H^{n+1}_{et}(R, \mathbb{Z}/p^r(n)) & \rightarrow & H^{n+1}_{et}(\tilde{R}(\pi), \mathbb{Z}/p^r(n)) \\
\downarrow & & \downarrow \\
H^{n+1}_{et}(R/(\pi), \mathbb{Z}/p^r(n)) & \rightarrow & H^{n+1}_{et}(\tilde{R}(\pi)/(\pi), \mathbb{Z}/p^r(n)) \tag{23}
\end{array}
\]
Then the left map in the diagram (22) is an isomorphism by Theorem 5.2. Since the homomorphism
\[
H^{n+1}_{et}(R/(\pi), \mathbb{Z}/p^r(n)) \rightarrow H^{n+1}_{et}(\tilde{R}(\pi)/(\pi), \mathbb{Z}/p^r(n))
\]is an injective by Theorem 4.1 and \( R(\pi)/(\pi) = \tilde{R}(\pi)/(\pi) \), the lower map in the diagram (23) is an injective. Therefore the homomorphism (22) is an injective. This completes the proof.

\[\square\]

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