Deconfinement and chiral restoration in nonlocal SU(3) chiral quark models

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We study the features of nonlocal SU(3) chiral quark models with wave function renormalization. Model parameters are determined from meson phenomenology, considering different nonlocal form factor shapes. In this context we analyze the characteristics of the deconfinement and chiral restoration transitions at finite temperature, introducing the couplings of fermions to the Polyakov loop. We analyze the results obtained for various thermodynamical quantities considering different Polyakov loop potentials and nonlocal form factors, in comparison with data obtained from lattice QCD calculations.

I. INTRODUCTION

The detailed understanding of the behavior of strongly interacting matter under extreme conditions of temperature and/or density has become an issue of great interest in recent years. It is widely believed that as the temperature and/or density increase, one finds a transition from a hadronic phase, in which chiral symmetry is broken and quarks are confined, to a partonic phase in which chiral symmetry is restored and/or quarks are deconfined. From the theoretical point of view, one way to address this problem is through lattice QCD calculations\cite{1-3}, which have been significantly improved in the last years. However, this ab initio approach is not yet able to provide a full understanding of the QCD phase diagram and the related hadron properties, owing to the well-known difficulties of dealing with small current quark masses and finite chemical potentials. Thus it is worth to develop effective models that show consistency with lattice results, and can be extrapolated into regions not accessible by lattice calculation techniques. Here we will concentrate on one particular class of effective theories, namely the so-called nonlocal Polyakov–Nambu–Jona-Lasinio (nlPNJL) models\cite{4-8}, in which quarks move in a background color field and interact through covariant nonlocal chirally symmetric four-point couplings. These approaches, which can
be considered as an improvement over the (local) PNJL model [9–15], offer a common framework to study both the chiral restoration and deconfinement transitions. In fact, the nonlocal character of the interactions arises naturally in the context of several successful approaches to low-energy quark dynamics [16, 17], and leads to a momentum dependence in the quark propagator that can be made consistent with lattice results [19–21]. Moreover, it has been found that, under certain conditions, it is possible to derive the main features of nlPNJL models starting directly from QCD [22].

Some previous works have addressed the study of nlPNJL models for the case of two dynamical quarks, showing that the presence of nonlocal form factors in the current-current quark interactions leads to a momentum dependent mass and wave function renormalization (WFR) in the quark propagator [23–25]. As stated, it is possible to choose the model parameters and form factors so as to fit these momentum dependences to those obtained in lattice QCD [18]. The aim of this work is to extend those works to three flavors, including flavor mixing through a nonlocal ’t Hooft-like six-fermion interaction. The case of a three-flavor nlPNJL model with simple Gaussian form factors and no WFR in the quark propagator has been previously addressed in Refs. [5, 6], where the phenomenology of light scalar and pseudoscalar mesons is analyzed. In addition, the introduction of a Gaussian form factor to account for the WFR in the three flavor case has been considered in Ref. [26]. For comparison, we analyze here both the case of a model in which the form factors are Gaussian functions (which ensure a fast ultraviolet convergence of loop integrals), and a model in which these are given by the mentioned lattice QCD-inspired functions [see Eqs. (29,30) below] of the momentum. In this framework we determine several properties of light mesons (masses, mixing angles, decay constants), analyzing the compatibility with the corresponding phenomenological values. Then we study the deconfinement and chiral restoration phase transitions that occur at finite temperature, and we determine the corresponding critical temperatures. Our analyses are carried out at the mean field level, considering the above mentioned form factor shapes and different parameters and functional forms for the Polyakov potential. We also analyze the behavior of thermodynamical quantities: specific heat, interaction energy, entropy and energy densities. The results arising from our model are discussed in comparison with lattice QCD estimations and with the results obtained using the parameterization of Ref. [26].

This article is organized as follows. In Sect. II we present the general formalism, including analytical results for the scalar and pseudoscalar meson properties. We discuss the model parameterization and compare our predictions with phenomenological expectations. In Sect. III we extend our analysis to nonzero temperature. The Polyakov loop potential is introduced and the
deconfinement and chiral restoration phase transitions are analyzed. In Sect. IV we summarize our results and conclusions. A final Appendix includes some of our analytical expressions.

II. NONLOCAL SU(3) CHIRAL QUARK MODEL - ZERO TEMPERATURE

We start by considering the Euclidean effective action

\[
S_E = \int d^4 x \left\{ \overline{\psi}(x)(-i\not{\partial} + \hat{m})\psi(x) - \frac{G}{2} \left[ j_5^S(x) j_5^S(x) + j_5^P(x) j_5^P(x) + j_5^r(x) j_5^r(x) \right] 
- \frac{H}{4} A_{abc} \left[ j_5^S(x) j_5^S(x) j_5^S(x) - 3 j_5^S(x) j_5^P(x) j_5^P(x) \right] + U[A(x)] \right\},
\]

where \( \psi(x) \) is the \( N_f = 3 \) fermion triplet \( \psi = (u \ d \ s)^T \), and \( \hat{m} = \text{diag}(m_u, m_d, m_s) \) is the current quark mass matrix. We will work in the isospin symmetry limit, assuming \( m_u = m_d \). The fermion currents are given by

\[
\begin{align*}
    j_5^S(x) &= \int d^4 z \ g(z) \overline{\psi}(x + \frac{z}{2}) \lambda_\alpha \psi(x - \frac{z}{2}), \\
    j_5^P(x) &= \int d^4 z \ g(z) \overline{\psi}(x + \frac{z}{2}) i\lambda_\alpha \gamma_5 \psi(x - \frac{z}{2}), \\
    j_5^r(x) &= \int d^4 z \ f(z) \overline{\psi}(x + \frac{z}{2}) \frac{i}{2\kappa} \theta \psi(x - \frac{z}{2}),
\end{align*}
\]

where \( f(z) \) and \( g(z) \) are covariant form factors responsible for the nonlocal character of the interactions, and the matrices \( \lambda_\alpha, \ a = 0, ..., 8 \), are the standard eight Gell-Mann matrices, plus \( \lambda_0 = \sqrt{2/3} \ 1_{3\times3} \). The relative weight of the interaction driven by \( j_5^r(x) \), responsible for the quark wave function renormalization, is controlled by the parameter \( \kappa \). The model includes flavor mixing through a \( \ 't \) Hooft-like term, in which the SU(3) symmetric constants \( A_{abc} \) are defined by

\[
A_{abc} = \frac{1}{3!} \epsilon_{ijk} \epsilon_{mnl} (\lambda_\alpha)_{im} (\lambda_\beta)_{jn} (\lambda_\gamma)_{kl}.
\]

The interaction between fermions and color gauge fields \( G_\mu^a \) takes place through the covariant derivative in the fermion kinetic term, \( D_\mu \equiv \partial_\mu - iA_\mu \), where \( A_\mu = g G_\mu^a \lambda^a / 2 \). Finally, the action includes an effective potential \( U \) that accounts for gauge field self-interactions. At the mean field level we will assume that fermions move on a uniform background gauge field, which for zero temperature decouples from matter (finite temperature effects will be discussed in the next sections).

To work with mesonic degrees of freedom we proceed to perform a standard bosonization of the fermionic theory, introducing scalar fields \( \sigma_\alpha(x) \), \( \zeta(x) \) and pseudoscalar fields \( \pi_\alpha(x) \), together with
auxiliary fields $S_a(x)$, $P_a(x)$ and $R(x)$, with $a = 0, \ldots, 8$. After integrating out the fermion fields we obtain a partition function

\[
Z = \int \mathcal{D}\sigma \mathcal{D}\pi \mathcal{D}\zeta A(\sigma, \pi, \zeta) \\
\times \int \mathcal{D}S_a \mathcal{D}P_a \mathcal{D}R \exp \left[ \int d^4x \left\{ \sigma_a S_a + \pi_a P_a + \zeta R + \frac{G}{2}(S_a S_a + P_a P_a + R^2) \right. \right.
\]
\[+ \left. \left. \frac{H}{4} A_{abc}(S_a S_b S_c - 3S_a P_b P_c) \right\} \right],
\]

where the operator $A(p, p')$ (in momentum space) is given by

\[
A(p, p') = (2\pi)^4 \delta^{(4)}(p - p')(-\not{p} + m_c) + g \left( \frac{p + p'}{2} \right) [\sigma_a (p' - p) + i\gamma_5 \pi_a (p' - p)] \lambda_a + \\
+ \frac{1}{2\kappa} f \left( \frac{p + p'}{2} \right) (\not{p} + \not{p'}) \zeta (p' - p).
\]

Now we follow the stationary phase approximation, replacing the path integrals over the auxiliary fields by the corresponding argument evaluated at the minimizing values $\tilde{S}_a$, $\tilde{P}_a$, and $\tilde{R}$. The procedure is similar to that carried out in Ref. [27], where more details can be found.

**A. Mean Field Approximation**

We consider the mean field approximation (MFA), in which the meson fields are expanded around their vacuum expectation values. One has thus

\[
\sigma_a(x) = \bar{\sigma}_a + \delta \sigma_a(x), \\
\pi_a(x) = \delta \pi_a(x), \\
\zeta(x) = \bar{\zeta} + \delta \zeta(x),
\]

where we have assumed that pseudoscalar mean field values vanish, owing to parity conservation. Moreover, for the scalar fields only $\bar{\sigma}_{0,8}$ and $\bar{\zeta}$ can be different from zero due to charge and isospin symmetries. Thus the Euclidean action reduces to

\[
\frac{S_{E}^{MFA}}{V^{(4)}} = -2 \text{Tr} \int \frac{d^4p}{(2\pi)^4} \log \left[ \frac{M^2(p) + p^2}{Z^2(p)} \right] - \bar{\sigma}_a \tilde{S}_a - \bar{\zeta} \tilde{R} - \frac{G}{2}(S_a S_a + R^2) - \frac{H}{4} A_{abc} S_a S_b S_c,
\]

where $\tilde{S}_a$, $\tilde{P}_a$ and $\tilde{R}$ stand for the values of $\tilde{S}_a$, $\tilde{P}_a$ and $\tilde{R}$ within the MFA.

For the neutral fields ($a = 0, 3, 8$) it is convenient to change to a flavour basis, $\phi_a \rightarrow \phi_i$, where $i = u, d, s$, or equivalently $i = 1, 2, 3$. In this basis, by minimizing the mean field action in Eq. [7]
we obtain the gap equations given in Ref. [27],

\[
\begin{align*}
\bar{\sigma}_u + G\bar{S}_u + \frac{H}{2} \bar{S}_d \bar{S}_s &= 0 , \\
\bar{\sigma}_d + G\bar{S}_d + \frac{H}{2} \bar{S}_s \bar{S}_u &= 0 , \\
\bar{\sigma}_s + G\bar{S}_s + \frac{H}{2} \bar{S}_u \bar{S}_d &= 0 , 
\end{align*}
\]

(8)

plus an extra equation arising from the \( j^r(x) \) current-current interaction,

\[
\bar{\zeta} + G\bar{R} = 0 ,
\]

(9)

where the mean field values \( \bar{S}_i \) and \( \bar{R} \) are given by

\[
\begin{align*}
\bar{S}_i &= -8N_c \int \frac{d^4p}{(2\pi)^4} \frac{g(p) Z(p) M_i(p)}{p^2 + M_i^2(p)} , \ i = u, d, s , \\
\bar{R} &= \frac{4N_c}{\kappa} \int \frac{d^4p}{(2\pi)^4} p^2 f(p) \sum_{i=1}^{3} \frac{Z(p)}{p^2 + M_i^2(p)} .
\end{align*}
\]

(10)

The functions \( M_i(p) \) and \( Z(p) \) correspond to momentum-dependent effective masses and WFR of the quark propagators. In terms of the model parameters and form factors, these are given by

\[
\begin{align*}
M_i(p) &= Z(p) \left[ m_i + \bar{\sigma}_i g(p) \right] , \\
Z(p)^{-1} &= 1 - \frac{\bar{\zeta}}{\kappa} f(p) .
\end{align*}
\]

(11)

Thus, for a given set of model parameters and form factors, from Eqs. (8-11) one can numerically obtain the mean field values \( \bar{\sigma}_{u,s} \) and \( \bar{\zeta} \).

The chiral condensates \( \langle \bar{q}q \rangle \), order parameters of the chiral restoration transition, can be obtained by varying the MFA partition function with respect to the current quark masses. These quantities are in general divergent, and can be regularized by subtracting the free quark contributions. One has

\[
\langle \bar{q}q \rangle = -4N_c \int \frac{d^4p}{(2\pi)^4} \left[ \frac{Z(p) M_q(p)}{p^2 + M_q^2(p)} - \frac{m_q}{p^2 + m_q^2} \right] , \ q = u, d, s .
\]

(12)

\section{Quadratic Fluctuations - Meson masses and weak decay constants}

In order to analyze the properties of meson fields it is necessary to go beyond the MFA, considering quadratic fluctuations in the Euclidean action:

\[
S_E^{\text{quad}} = \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} \sum_M r_M G_M(p^2) \phi_M(p) \bar{\phi}_M(-p) ,
\]

(13)
where meson fluctuations $\delta \sigma_a, \delta \pi_a$ have been translated to a charge basis $\phi_M$, $M$ being the scalar and pseudoscalar mesons in the lowest mass nonets ($\sigma, \pi^0$, etc.), plus the $\zeta$ field. The coefficient $r_M$ is 1 for charge eigenstates $M = a_0^0, \sigma, f_0, \zeta, \eta, \eta'$, and 2 for $M = a_0^+, K_0^{*+}, K_0^{*0}, \pi^+, K^+, K^0$. Meson masses are then given by the equations

$$G_M(-m_M^2) = 0. \quad (14)$$

In addition, physical states have to be normalized through

$$\tilde{\phi}_M(p) = Z_{M^{-1/2}}^{-1} \phi_M(p), \quad (15)$$

where

$$Z_M^{-1} = \frac{dG_M(p)}{dp^2} \bigg|_{p^2 = -m_M^2}. \quad (16)$$

The full expressions for the one-loop functions $G_M(q)$ are quoted in the Appendix. They can be written in terms of the coupling constants $G$ and $H$, the mean field values $\bar{S}_{u,s}$ and quark loop functions that prove to be ultraviolet convergent owing to the asymptotic behavior of the nonlocal form factors. For the pseudoscalar meson sector, the $\pi$ and $K$ mesons decouple, while the $I = 0$ states get mixed. Since the corresponding mixing angles are momentum dependent functions, it is necessary to introduce two mixing angles $\theta_\eta$ and $\theta_{\eta'}$, defined at $p^2 = -m_\eta^2$ and $p^2 = -m_{\eta'}^2$ respectively, see Eq. (47). In the case of the scalar meson sector, the $a_0$ and $K_0^*$ mesons decouple, while the $\zeta, \sigma_0$ and $\sigma_8$ fields get mixed by a $3 \times 3$ matrix, see Eq. (48).

One can also calculate the weak decay constants of pseudoscalar mesons. These are given by the matrix elements of the axial currents $A_\mu^a$ between the vacuum and the physical meson states,

$$i f_{ab}(p^2) p_\mu = \langle 0 | A_\mu^a(0) | \delta \pi_b(p) \rangle. \quad (17)$$

The matrix elements can be calculated from the expansion of the Euclidean effective action in the presence of external axial currents,

$$\langle 0 | A_\mu^a(0) | \delta \pi_b(p) \rangle = \frac{\delta^2 S_E}{\delta A_\mu^a \delta \pi_b(p)} \bigg|_{A_\mu^a = \delta \pi_b = 0}. \quad (18)$$

It is important to notice that, owing to nonlocality, the axial currents have to be introduced not only into the covariant derivative in the Euclidean action, but also in the fermion fields entering the nonlocal currents, through the replacements [27, 28]

$$\psi \left( x - \frac{z}{2} \right) \rightarrow W_A \left( x, x - \frac{z}{2} \right) \psi \left( x - \frac{z}{2} \right)$$

$$\psi^\dagger \left( x + \frac{z}{2} \right) \rightarrow \psi^\dagger \left( x + \frac{z}{2} \right) W_A \left( x + \frac{z}{2}, x \right). \quad (19)$$
Here the transport function \( W_A(x, y) \) is given by

\[
W_A(x, y) = P \exp \left\{ \frac{i}{2} \int_x^y ds \mu \gamma_5 \lambda_\alpha A_\mu^\alpha(s) \right\},
\]

where \( s \) runs over an arbitrary path connecting \( x \) with \( y \).

After a rather lengthy calculation, we find that the relevant term in the expansion of the Euclidean action can be written as

\[
S_E^{[A, \phi]} = \int \frac{d^4p}{(2\pi)^4} \frac{d^4p'}{(2\pi)^4} \sum_{i,j=1}^3 A_{\mu ij}(p) \delta_{pij}(p') \ G_{ij}^\mu(p, p') ,
\]

where we have defined \( A_\mu = \lambda_\alpha A_\mu^\alpha / \sqrt{2} \), \( \delta_\pi = \lambda_\alpha \delta_\alpha / \sqrt{2} \). The functions \( G_{ij}^\mu(p, p') \) are found to satisfy the relation

\[
p_\mu G_{ij}^\mu(p, p') = -i \delta^{(4)}(p + p') F_{ij}(p^2) ,
\]

where

\[
F_{ij}(p^2) = 2N_c \int \frac{d^4q}{(2\pi)^4} \left[ g(q^+) + g(q^-) - 2g(q) \right] Z(q) \left[ \frac{M_i(q)}{q^2 + M_i^2(q)} + \frac{M_j(q)}{q^2 + M_j^2(q)} \right] Z(q^+ + q^2 + q^2 - M_i^2(q) - M_j^2(q))
\]

\[
\times \left[ \frac{Z(q^+)}{M_i^2(q)} + q^2 - M_i^2(q) \right] \left[ \frac{Z(q^-)}{M_j^2(q)} + q^2 - M_j^2(q) \right] \left[ (q^+ \cdot q^-) + M_i(q^+) M_j(q^-) \right]
\]

\[
+ 4N_c \int \frac{d^4q}{(2\pi)^4} g(q) \left[ M_i(q^+) q^- - M_j(q^-) q^+ \right] \left[ Z(q^-) q^+ - Z(q^+) q^- \right]
\]

\[
\left[ q^2 + M_i^2(q) \right] \left[ q^2 + M_j^2(q) \right] ,
\]

with \( q^\pm = q \pm p/2 \). It is worth to point out that the functions \( F_{ij} \) (and, therefore, the weak decay constants) are given by the longitudinal component of \( G_{ij}^\mu(p, p') \), which does not depend on the arbitrary path chosen in the transport functions \( W_A(x, y) \).

From the above expressions, the weak decay constants for \( \pi \) and \( K \) mesons in the isospin limit are given by

\[
f_{\pi} = \frac{Z_{\pi}^{1/2}}{m_{\pi}^2} F_{uu}(p^2) \bigg|_{p^2 = -m_{\pi}^2} ,
\]

\[
f_K = \frac{Z_K^{1/2}}{m_K^2} F_{us}(p^2) \bigg|_{p^2 = -m_K^2} .
\]

For the \( \eta - \eta' \) sector, the functions \( f_{ab}(p^2) \) defined in Eq. \([17]\) are related to \( F_{ij}(p^2) \) through

\[
f_{00}(p^2) = \frac{1}{3} \left[ 2F_{uu}(p^2) + F_{ss}(p^2) \right] ,
\]

\[
f_{ss}(p^2) = \frac{1}{3} \left[ F_{uu}(p^2) + 2F_{ss}(p^2) \right] ,
\]

\[
f_{08}(p^2) = \frac{\sqrt{2}}{3} \left[ F_{uu}(p^2) - F_{ss}(p^2) \right] .
\]
These can be translated to the mass eigenstate basis through the mixing angles in Eq. (47). Thus one defines

$$f^a_\eta = \frac{Z_{\eta}^{1/2}}{m_\eta} \left[ f_{a8}(p^2) \cos \theta_\eta - f_{a0}(p^2) \sin \theta_\eta \right] \bigg|_{p^2 = -m_\eta^2} , \quad a = 0, 8 ,$$

$$f^a_{\eta'} = \frac{Z_{\eta'}^{1/2}}{m_{\eta'}^2} \left[ f_{a8}(p^2) \sin \theta_{\eta'} + f_{a0}(p^2) \cos \theta_{\eta'} \right] \bigg|_{p^2 = -m_{\eta'}^2} , \quad a = 0, 8 . \quad (26)$$

In order to compare with phenomenological determinations, it is convenient to consider an alternative parametrization in terms of two decay constants $f_0$, $f_8$ and two mixing angles $\theta_0$, $\theta_8$ [29, 30]. Both parametrizations are related by

$$\begin{pmatrix} f^8_{\eta} & f^0_\eta \\ f^8_{\eta'} & f^0_{\eta'} \end{pmatrix} = \begin{pmatrix} f_8 \cos \theta_8 & -f_0 \sin \theta_0 \\ f_8 \sin \theta_8 & f_0 \cos \theta_0 \end{pmatrix}. \quad (27)$$

C. Model parameters and form factors

The model includes five parameters, namely the current quark masses $m_{u,s}$ and the coupling constants $G$, $H$ and $\kappa$. In addition, one has to specify the form factors $f(z)$ and $g(z)$ entering the nonlocal fermion currents. Here, following Ref. [18], we will consider two parameter sets, corresponding to two different functional forms for $f(z)$ and $g(z)$. The first one corresponds to the often used exponential forms

$$g(p) = \exp \left( -\frac{p^2}{\Lambda_0^2} \right) , \quad f(p) = \exp \left( -\frac{p^2}{\Lambda_1^2} \right) , \quad (28)$$

which guarantee a fast ultraviolet convergence of the loop integrals. Note that the range (in momentum space) of the nonlocality in each channel is determined by the parameters $\Lambda_0$ and $\Lambda_1$, respectively. The second set of form factors considered here is

$$g(p) = \frac{1 + \alpha_z}{1 + \alpha_z} \frac{\alpha_m f_m(p) - m \alpha_z f_z(p)}{\alpha_m - m \alpha_z} , \quad f(p) = \frac{1 + \alpha_z}{1 + \alpha_z} \frac{f_z(p)}{f_z(p)} , \quad (29)$$

where

$$f_m(p) = \left[ 1 + \left( \frac{p^2}{\Lambda_0^2} \right)^{3/2} \right]^{-1} , \quad f_z(p) = \left[ 1 + \left( \frac{p^2}{\Lambda_1^2} \right) \right]^{-5/2}. \quad (30)$$

As shown in Ref. [18], for the SU(2) version of the model these functional forms can very well reproduce the momentum dependence of mass and wave function renormalization obtained in lattice calculations.

Given the form factor functions, one can fix the model parameters so as to reproduce the observed meson phenomenology. To the above mentioned parameters $m_{u,s}$, $G$, $H$ and $\kappa$ one has
to add the cutoffs $\Lambda_0$ and $\Lambda_1$, introduced through the form factors. Here we have chosen to take as input value the light quark mass $m_u$, while the remaining six parameters are determined by fixing the value of the quark WFR at momentum zero, $Z(0) = 0.7$ (as dictated by lattice QCD estimations), and by requiring that the model reproduces the empirical values of five physical quantities. These are the masses of the pseudoscalar mesons $\pi$, $K$ and $\eta'$, the pion weak decay constant $f_\pi$ and the light quark condensate $\langle \overline{u}u \rangle$. In Table I we quote the numerical results for the model parameters that we have obtained for the above described form factor functions. In what follows, the parameter sets corresponding to the form factors in Eqs. (28) and (29-30) will be referred to as Set I and Set II, respectively. As expected from the ansatz chosen for the form factors, for Set II the momentum dependent mass and WFR in the light quark propagators are able to fit adequately the results obtained in lattice QCD calculations. This is shown in Fig. 1 where we plot the curves obtained for the functions $M(p)$ and $Z(p)$, together with $N_f = 2 + 1$ lattice data taken from Ref. [20]. For comparison we also quote the results corresponding to Set I.

|               | Set I  | Set II |
|---------------|--------|--------|
| $m_u$ [MeV]   | 5.7    | 2.5    |
| $m_s$ [MeV]   | 136    | 63.9   |
| $GA^2_0$      | 23.64  | 15.55  |
| $-HA^5_0$     | 526    | 241    |
| $\kappa$ [GeV]| 4.36   | 8.08   |
| $\Lambda_0$ [GeV] | 0.814  | 0.824  |
| $\Lambda_1$ [GeV] | 1.032  | 1.550  |

Table I: Model parameters for the form factors in Eqs. (28) (Set I) and (29,30) (Set II).

D. MESON PHENOMENOLOGY

Once the parameters have been determined, we can calculate the values of several meson properties for the scalar and pseudoscalar sector. Our numerical results for Sets I and II are summarized in Table II, together with the corresponding phenomenological estimates. The quantities marked with an asterisk are those that have been chosen as input values. In general, it is seen that the meson masses, mixing angles and weak decay constants predicted by the model are in a reasonable agreement with phenomenological expectations. Moreover, the results for Set I do not differ significantly from those obtained in Ref. [6] for a nlPNJL model with a Gaussian form factor $g(p)$.
and no WFR. Regarding the scalar meson sector, a new ingredient with respect to the model with no WFR is the presence of the additional field $\zeta$, which mixes with the $I = 0$ fields $\sigma_0$ and $\sigma_8$. The mass of the physical particles can be obtained by determining the zeroes of the functions $G_{\zeta,\sigma,f_0}(p^2)$ arising from the diagonalization of the $3 \times 3$ matrix in Eq. (48) (see Appendix). From the corresponding numerical calculation it is seen that one of these functions is positive definite for the momentum range described by our models, which reflects that the eigenstate associated with $\zeta$ does not correspond to a physical particle. For the remaining two states, which can be interpreted as the $\sigma$ and $f_0$ scalar mesons, we obtain masses of about 550 and 1200 MeV. In fact, in the case of the $f_0$ meson it happens that the loop integrals in Eq. (49) become divergent, and need some regularization prescription. This occurs since $p^2$ exceeds a threshold above which both effective quarks are simultaneously on shell, thus it can be interpreted as the possibility of a decay of the meson into two massive quarks. The integrals can be properly defined e.g. following the prescription in Ref. [27]. Since the threshold lies at about 1 GeV, we have estimated the mass value for the $f_0$ meson by an extrapolation from the momentum region in which the integrals are well defined. The same procedure has been used in the case of the $K^*_0$ meson.

Concerning the quark masses and condensates, it is found that in the case of Set II we obtain relatively low values for $m_u$ and $m_s$, and a somewhat large value for the light quark condensate. Similar results have been previously obtained in Refs. [18] and [26], within two- and three-flavor parameterizations respectively. As discussed in those articles, this can be in part attributed to the fact that our fit to lattice data for the function $Z(p)$ is based on the calculations in Ref. [20], which correspond to a rather large renormalization scale $\mu = 3$ GeV. On the other hand, for both Sets I
and II we find that the quark mass ratio is $m_s/m_u \simeq 25$, which is phenomenologically adequate. Something similar happens with the product $-\langle \bar{u}u \rangle m_u$, which gives $7.9 \times 10^{-5}$ GeV$^4$ for Set I and $8.2 \times 10^{-5}$ GeV$^4$ for Set II: these values are in agreement with the scale-independent result obtained from the Gell-Mann-Oakes-Renner relation at the leading order in the chiral expansion, namely $-\langle \bar{u}u \rangle m_u = f_\pi^2 m_\pi^2/2 \simeq 8.3 \times 10^{-5}$ GeV$^4$.

|                           | Set I | Set II | Empirical |
|---------------------------|-------|--------|-----------|
| $\bar{\sigma}_u$ [MeV]   | 529   | 454    | -         |
| $\bar{\sigma}_s$ [MeV]   | 702   | 663    | -         |
| $\bar{\zeta}/\kappa$     | -0.429| -0.429 | -         |
| $-\langle \bar{u}u \rangle^{-1/3}$ [MeV] * | 240 | 320 | - |
| $-\langle \bar{s}s \rangle^{-1/3}$ [MeV] | 198 | 343 | - |
| $m_\pi$ [MeV] *           | 139   | 139    | 139       |
| $m_K$ [MeV] *             | 495   | 495    | 495       |
| $m_\eta$ [MeV]            | 527   | 537    | 547       |
| $m_{\eta'}$ [MeV] *       | 958   | 958    | 958       |
| $m_{\omega}$ [MeV]        | 936   | 916    | 980       |
| $m_{K^0}$ [MeV]           | 1300  | 1300   | 1430      |
| $m_\sigma$ [MeV]          | 599   | 537    | 400 - 550 |
| $m_{f_0}$ [MeV]           | 1300  | 1200   | 980       |
| $f_\pi$ [MeV] *           | 92.4  | 92.4   | 92.4      |
| $f_K/f_\pi$               | 1.17  | 1.16   | 1.22      |
| $f_{\eta}/f_\pi$         | 0.17  | 0.14   | (0.11 - 0.507) |
| $f_{\eta'}/f_\pi$        | 1.12  | 1.12   | (1.17 - 1.22) |
| $f_{\eta'}/f_\pi$        | 1.09  | 1.43   | (0.98 - 1.16) |
| $f_{\eta'}/f_\pi$        | -0.48 | -0.42  | -(0.42 - 0.46) |
| $\theta_\eta$            | -2.95°| -1.01° | -         |
| $\theta_{\eta'}$         | -41.62°| -30.79°| -         |
| $\theta_0$               | -8.63°| -5.53° | -(0° - 10°) |
| $\theta_8$               | -22.94°| -20.67°| -(19° - 22°) |

Table II: Numerical results for various phenomenological quantities. Input values are marked with an asterisk.
III. NONZERO TEMPERATURE

A. Polyakov Loop

As stated in the previous section, the effective action of the model includes the interaction of quarks with color gauge fields through the covariant derivative in the fermion kinetic term. This coupling will be treated at the mean field level, considering that quarks move on a constant background field $\phi = A_4 = i A_0 = ig \delta_{\mu 0} G_{a}^{\mu} \lambda^a / 2$, where $G_{a}^{\mu}$ are the SU(3) color gauge fields. Then the traced Polyakov loop, which in the infinite quark mass limit can be taken as order parameter of confinement, is given by $\Phi = \frac{1}{3} \text{Tr} \exp(i\phi/T)$. We will work in the so-called Polyakov gauge, in which the matrix $\phi$ is given a diagonal representation $\phi = \phi_3 \lambda_3 + \phi_8 \lambda_8$. Owing to the charge conjugation properties of the QCD Lagrangian [31], the mean field traced Polyakov loop field $\Phi$ is expected to be a real quantity. Assuming that $\phi_3$ and $\phi_8$ are real-valued [13], this implies $\phi_8 = 0$, $\Phi = \left[2 \cos(\phi_3/T) + 1 \right]/3$.

The effective gauge field self-interactions are given by the Polyakov-loop potential $U[A(x)]$. At finite temperature $T$, it is usual to take an ansatz for the the functional form of this potential based on the properties of pure gauge QCD. We consider here two alternative forms, commonly used in the literature. The first one is a polynomic function, based on a Ginzburg-Landau ansatz. It reads [12]

$$U_{\text{poly}}(\Phi, T) = T^4 \left[ - \frac{b_2(T)}{2} \Phi^2 - \frac{b_3}{3} \Phi^3 + \frac{b_4}{4} \Phi^4 \right] ,$$  \hspace{1cm} (31)

where

$$b_2(T) = a_0 + a_1 \left( \frac{T_0}{T} \right) + a_2 \left( \frac{T_0}{T} \right)^2 + a_3 \left( \frac{T_0}{T} \right)^3 .$$  \hspace{1cm} (32)

The potential parameters can be fitted to pure gauge lattice QCD data so as to properly reproduce the corresponding equation of state and Polyakov loop behavior. This yields [12]

$$a_0 = 6.75 , \quad a_1 = -1.95 , \quad a_2 = 2.625 ,$$

$$a_3 = -7.44 , \quad b_3 = 0.75 , \quad b_4 = 7.5 .$$  \hspace{1cm} (33)

A second usual form is based on the logarithmic expression of the Haar measure associated with the SU(3) color group integration. The potential reads in this case [13]

$$U_{\text{log}}(\Phi, T) = \left\{ - \frac{1}{2} a(T) \Phi^2 + b(T) \log \left[ 1 - 6 \Phi^2 + 8 \Phi^3 - 3 \Phi^4 \right] \right\} T^4 ,$$  \hspace{1cm} (34)
where the coefficients are parameterized as
\[
a(T) = a_0 + a_1 \left(\frac{T_0}{T}\right) + a_2 \left(\frac{T_0}{T}\right)^2, \quad b(T) = b_3 \left(\frac{T_0}{T}\right)^3.
\]
(35)

Once again the values of the constants can be fitted to pure gauge lattice QCD results. This leads to \([13]\)
\[
a_0 = 3.51, \quad a_1 = -2.47, \quad a_2 = 15.2, \quad b_3 = -1.75.
\]
(36)

In both cases, the values of \(a_i\) and \(b_i\) are constrained by the condition of reaching the Stefan-Boltzmann limit at \(T \to \infty\) and by imposing the presence of a first-order phase transition at \(T_0\), which is a further parameter of the model. In absence of dynamical quarks, from lattice calculations one expects a deconfinement temperature \(T_0 = 270\) MeV. However, it has been argued that in the presence of light dynamical quarks this temperature scale should be adequately reduced \([32]\). The actual value depends on the number of active flavors; the analysis in Ref. \([32]\) leads to \(T_0 = 208\) MeV and \(T_0 = 187\) MeV for two and three dynamical quarks, respectively.

### B. Thermodynamics

To investigate the phase transitions and the temperature dependence of thermodynamical quantities within our model, we consider the thermodynamical potential per unit volume at the mean field level. We will proceed by using the standard Matsubara formalism, following same prescriptions as in previous works, see e.g. Refs. \([27, 33]\). In this way we obtain
\[
\Omega^{\text{MFA}} = \Omega^{\text{reg}} + \Omega^{\text{free}} + \mathcal{U}(\Phi, T) + \Omega_0 ,
\]
(37)

where
\[
\Omega^{\text{reg}} = -2T \sum_{n=-\infty}^{\infty} \sum_{c,f} \int \frac{d^3p}{(2\pi)^3} \log \left[ \frac{p^2_{nc} + M_f^2(p_{nc})}{Z^2(p_{nc})(p^2_{nc} + m_f^2)} \right]
\]
\[
- \left( \tilde{\zeta} \tilde{R} + \frac{G}{2} \tilde{R}^2 + \frac{H}{4} \bar{S}_u S_d S_s \right) - \frac{1}{2} \sum_f \left( \bar{\sigma}_f S_f + \frac{G}{2} S_f^2 \right),
\]
\[
\Omega^{\text{free}} = -2T \sum_{c,f} \int \frac{d^3p}{(2\pi)^3} \text{Re} \log \left[ 1 + \exp \left( -\frac{\epsilon_{fp} + i\phi_c}{T} \right) \right].
\]
(38)

Here we have defined \(p^2_{nc} = [(2n + 1)\pi T + \phi_c]^2 + \vec{x}^2\), \(\epsilon_{fp} = \sqrt{\vec{x}^2 + m_f^2}\). The sums over color and flavor indices run over \(c = r, g, b\) and \(f = u, d, s\), respectively, and the color background fields are \(\phi_r = -\phi_g = \phi_3\), \(\phi_b = 0\). The term \(\Omega_0\) is just a constant that sets the value of the thermodynamical potential at \(T = 0\).
Now, from the thermodynamic potential we can calculate various thermodynamic quantities such as the energy and entropy densities, which are given by

$$\varepsilon = \Omega + Ts, \quad s = -\frac{\partial \Omega}{\partial T}. \quad (39)$$

We are also interested in the behavior of quark condensates and the corresponding chiral susceptibilities, defined by

$$\langle \bar{q}q \rangle = \frac{\partial \Omega}{\partial m_q}, \quad \chi_q = \frac{\partial \langle \bar{q}q \rangle}{\partial m_q} = \frac{\partial^2 \Omega}{\partial m_q^2}, \quad q = u, d, s. \quad (40)$$

For large temperatures, the behavior of the regularized quark condensates is dominated by the free part contribution, which grows with $T$ as $\langle \bar{q}q \rangle \sim -m_q T^2$. Therefore, in order to analyze the chiral restoration transition it is usual to define a subtracted chiral condensate

$$\langle \bar{q}q \rangle_{\text{sub}} = \frac{\langle \bar{u}u \rangle - \frac{m_u}{m_s} \langle ss \rangle}{\langle \bar{u}u \rangle_0 - \frac{m_u}{m_s} \langle ss \rangle_0}, \quad (41)$$

where we have also introduced a normalization factor given by the values of the chiral condensates at zero temperature.

### C. Numerical Results

Let us present our numerical results for the quantities defined in the previous section, considering different form factors and Polyakov loop potentials. In Figs. 2 and 3 we show the behavior of the subtracted chiral condensate, the traced Polyakov loop $\Phi$ and the associated susceptibilities as functions of the temperature. Curves in Figs. 2 and 3 correspond to the parameter Sets I (gaussian form factors) and II (lattice-inspired form factors), respectively. In the upper panels we show the results for the subtracted chiral condensate $\langle \bar{q}q \rangle_{\text{sub}}$ and the traced Polyakov loop $\Phi$, for three different values of the parameter $T_0$. Left and right panels correspond to logarithmic and polynomic Polyakov potentials, given by Eqs. (31) and (34), respectively. As stated, $T_0 = 270$ MeV is the deconfinement transition temperature obtained from lattice calculations in pure gauge QCD, while the values $T_0 = 208$ MeV and 187 MeV arise from the corresponding rescaling in the case of two and three dynamical quarks, respectively [32]. We also include in the graphs lattice QCD data taken from Refs. [34, 35].

In general, it is found that when the temperature is increased one finds both the expected chiral restoration and deconfinement transitions. It is seen that these proceed in all cases as smooth crossovers, the curves getting steeper for lower values of $T_0$ (in fact, first order phase transitions
Figure 2: Subtracted chiral condensate, Polyakov loop, chiral susceptibilities and PL susceptibility \( d\Phi/dT \) as functions of the temperature. Curves correspond to parameter Set I, for logarithmic (left) and polynomial (right) PL potentials. Triangles, circles and squares stand for lattice QCD results from Refs. \[34, 35\].
are found for $T_0 < 187$ MeV). The transition temperatures can be defined through the peaks in the corresponding susceptibilities. In Figs. 2 and 3 (central and lower panels) we show the curves for the Polyakov loop susceptibility —defined as $d \Phi / dT$— and the chiral susceptibilities $\chi_{u,s}$ [given by Eq. (40)] as functions of the temperature. For clarity we have plotted in the graphs the subtracted
susceptibilities \( \tilde{\chi}_q \equiv \chi_q - \chi_q(T = 0) \). In all cases it is seen that both the SU(2) chiral restoration and deconfinement transitions occur essentially at the same critical temperatures. The corresponding values are quoted in Table III. In addition, in the curves for \( \chi_s \) it is possible to identify a second, broad peak that allows to define an approximate critical temperature for the restoration of the full SU(3) chiral symmetry.

| \( T_0 \) (MeV) | Set B | Set C |
|-----------------|-------|-------|
| 187             | 158   | 163   |
| 208             | 169   | 172   |
| 270             | 202   | 200   |

Table III: Critical deconfinement and SU(2) chiral restoration temperatures for different form factors and Polyakov loop potentials. L and P stand for logarithmic and polynomial potentials, respectively. Values are given in MeV.

From the figures it is seen that the transitions are smoother in the case of the polynomial potential, and that the curves for the subtracted chiral condensate get closer to lattice QCD data for \( T_0 \approx 187 \) MeV. On the contrary, the curves showing the dependence of the traced Polyakov loop with the temperature in general deviate significantly from lattice estimations, with the occasional exception of the model with a polynomial potential and \( T_0 \approx 270 \) MeV.

In order to compare the features of parameterizations I and II, it is important to consider other thermodynamical quantities. While the characteristics of the transition curves shown in Figs. 2 and 3 are qualitatively similar for both parameterizations, this is not the case if one looks e.g. at the interaction energy and the entropy. This is shown in Fig. 4, where we plot the normalized interaction energy \( (\varepsilon - 3p)/T^4 \) (left) and the normalized entropy density \( s/s_{SB} \) (right), where \( s_{SB} \) is the corresponding Stefan-Boltzmann limit. Dashed and solid curves correspond to Sets I and II, respectively, and upper, central and lower panels correspond to \( T_0 = 187, 208 \) and 270 MeV, respectively. In all cases we consider here the models with a logarithmic PL potential. We have included for comparison three sets of lattice data, taken from Refs. 35–37. It can be seen that for both the interaction energy and the entropy the curves for Set I show a pronounced dip at about \( T \approx 300 \) MeV, which is not observed in the case of Set II, where the falloff is smooth. In order to trace the source of this effect we have also considered a third parameterization Set III in which the form factor \( g(p) \) has a Gaussian shape as in Set I, but we do not include the coupling driven by the
Figure 4: Normalized interaction energy (left) and entropy density (right) as functions of the temperature, for different model parameterizations. Curves correspond to models with logarithmic PL potentials, with $T_0 = 187, 208$ and $270$ MeV for upper, central and lower panels, respectively. Squares, circles and triangles stand for lattice data from Refs. [35–37].

...currents $j^r(x)$ [i.e. there is no wave function renormalization, $Z(p) = 1$]. This parameterization has been previously considered in Ref. [6], where the values of model parameters can be found (see also Ref. [8]). In Fig. 4 it corresponds to the dotted curve, which does not show the mentioned dip. This indicates that the effect can be attributed to the exponential behavior of the form factor
$f(p)$ in the wave function renormalization for Set I. Moreover, our results can be also compared with those obtained from the parameterization considered in Ref. [26], where the form factors are introduced so as to fit lattice results for the quark propagator (as in our Set II), but $f(p)$ is assumed to have a Gaussian shape. The curves for the interaction energy and the entropy for this model (dashed-dotted lines in Fig. 4) are similar to those obtained for our parameterization Set I. Thus, from the comparison with lattice data, one can conclude that the choice of a power-like behavior for $f(p)$ as that proposed in Eqs. (29, 30) turns out to be more adequate. It is also seen that for $T_0 \sim 200$ MeV one gets a better agreement with lattice results than if one takes $T_0$ close to the pure gauge transition temperature of about 270 MeV. On the other hand, once again the plots in Fig. 4 indicate that the transition predicted by the nonlocal quarks models is too sharp in comparison with lattice estimations. As stated, the curves in Fig. 4 correspond to the logarithmic Polyakov loop potential; we have omitted the graphs for the polynomic potential, for which this analysis is qualitatively similar.

Finally, for completeness in Fig. 5 we replot our results for the behavior of the interaction energy and the entropy density, together with the energy density and the specific heat $c_V = T \partial s/\partial T$ as functions of the temperature. Here we just consider our parameterization Set II, which shows the best agreement with lattice calculations. We include the curves corresponding to $T_0 = 187, 208$ and 270 MeV for both logarithmic (left) and polynomic (right) PL potentials. In general, it is seen that $T_0 \sim 200$ MeV is preferred in order to get a better agreement with lattice data, and that the major discrepancy with lattice results lies in the rapid growth at the critical temperature (less pronounced in the case of the polynomic potential). In this regard, it is worth to take into account that this behavior can be modified after the inclusion of mesonic corrections to the finite temperature euclidean action: light mesons should be excited before the quarks, which should soften the transitions [34, 35, 36]. At the same time, the incorporation of meson fluctuations should not modify the critical temperatures, which for $T_0 \sim 200$ MeV are in agreement with lattice estimations.

IV. SUMMARY AND CONCLUSIONS

We analyze here the features of three-flavor nlPNJL models that include a wave function renormalization in the effective quark propagators. This represents an extension of previous works that consider two-flavor schemes, and three-flavor models with no quark WFR. In this framework, we obtain a parameterization of the model that reproduces lattice QCD results for the momentum
Figure 5: Normalized interaction energy, entropy density, energy density and specific heat as functions of the temperature, for the lattice-inspired parameterization Set II. Solid, dashed-dotted and dashed curves correspond to $T_0 = 187$, 208 and 270 MeV, respectively. Squares, circles and triangles stand for lattice data from Refs. [35–37].
dependence of the effective quark mass and WFR, and at the same time leads to an acceptable phenomenological pattern for particle masses and decay constants in both the scalar and pseudoscalar meson sectors. For comparison we also consider a parameterization based on Gaussian form factors, which leads to a faster convergence of quark loop integrals. Gaussian and lattice-inspired parameterizations are called here Set I and Set II, respectively. It is seen that the predictions for meson properties are qualitatively similar in both cases, and also agree with those obtained previously within three-flavor models with no WFR.

As a second step we analyze the characteristics of the deconfinement and chiral restoration transitions at finite temperature, introducing the couplings of fermions to a background gauge field and taking the traced Polyakov loop as order parameter for the deconfinement. We quote our numerical results for both parameterization Sets I and II, considering logarithmic and polynomic forms for the PL potential and different values for the corresponding scale parameter $T_0$. It is seen that in all cases the transitions proceed as smooth crossovers at a common critical temperature, which for $T_0 \simeq 200$ MeV (i.e. the approximately expected value for a three-flavor model) is in agreement with that found in lattice QCD. However, it is seen that both the subtracted chiral condensate and the traced Polyakov loop show a steeper behavior in comparison with those obtained from lattice calculations. This is particularly remarkable in the case of the Polyakov loop. For the case of a polynomic PL potential the transition is smoother, and the behavior of the PL approximates to that found in lattice QCD but only for larger values of $T_0$. In order to distinguish between the different parameterizations and those proposed in related works, we have also analyzed the temperature dependence of the interaction energy and the normalized entropy and energy densities. For these thermodynamical quantities it is seen that the lattice-inspired power-like parameterization Set II shows indeed the best agreement with lattice QCD results, which supports the consistency of our approach. The main discrepancy is still related with the steep behavior of the various quantities in the vicinity of the SU(2) chiral restoration transition, which should be softened after the inclusion of meson fluctuations. The consistent incorporation of this effect in the context of our models is presently under study.

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APPENDIX: analytic expressions for $G_M(p^2)$ functions

We present here the analytic expressions for the functions $G_M(p^2)$ appearing in the quadratic expansion of the Euclidean action, see Eq. [13]. Our calculations are in agreement with the results reported in Ref. [26]. For the $I \neq 0$ states, $a_0$, $K$ and $\kappa$ we obtain

$$G(\pi)(p) = (G \pm \frac{H}{2} \bar{S}_s)^{-1} + 4C_{uu}^+(p),$$

$$G(\kappa)(p) = (G \pm \frac{H}{2} \bar{S}_u)^{-1} + 4C_{us}^+(p),$$

where the functions $C_{ij}^+(p)$, with $i,j = u$ or $s$ are defined as

$$C_{ij}^+(p) = -2N_c \int \frac{d^4q}{(2\pi)^4} q^2(q) \frac{Z(q^+)}{q^2 + M^2_{ij}(q^+)} \frac{Z(q^-)}{q^2 + M^2_{ij}(q^-)} [(q^+ q^-) \pm M_i(q^+) M_j(q^-)],$$

with $q^\pm = q + p/2$. At the $I = 0$ pseudoscalar sector one has a mixing between the $\eta_0$ and $\eta_8$ fields. The masses of the physical states $\eta$ and $\eta'$ can be obtained from the functions

$$G(\eta)(p) = \frac{G_{88}(p) + G_{00}(p)}{2} \pm \sqrt{G_{80}(p)^2 + \left(\frac{G_{88}(p) - G_{00}(p)}{2}\right)^2},$$

where we use the definitions

$$G_{00}^+(p) = \frac{4}{3} \left[2C_{uu}^+(p) + C_{ss}^+(p) + \frac{6G \mp HS_s \pm 4HS_u}{8G^2 - 4H^2S_u \mp 4HGS_s}\right],$$

$$G_{88}^+(p) = \frac{4}{3} \left[2C_{ss}^+(p) + C_{uu}^+(p) + \frac{6G \mp 2HS_s \pm 4HS_u}{8G^2 - 4H^2S_u \mp 4HGS_s}\right],$$

$$G_{80}^+(p) = \frac{4}{\sqrt{2}} \sqrt{C_{uu}^+(p) - C_{ss}^+(p)} \pm \frac{H(S_s - \bar{S}_u)}{8G^2 - 4H^2S_u \mp 4HGS_s}.\quad (45)$$

The states $\eta$ and $\eta'$ are thus defined as

$$\eta = \eta_8 \cos \theta_\eta - \eta_0 \sin \theta_\eta,$$

$$\eta' = \eta_8 \sin \theta_\eta' + \eta_0 \cos \theta_\eta',$$

where the mixing angles $\theta_\eta$, $\theta_\eta'$ are given by

$$\tan 2\theta_{\eta,\eta'} = \frac{-2G_{80}^-}{G_{88}^- - G_{00}^-} \bigg|_{p^2 = -m_{\eta,\eta'}^2}.\quad (47)$$

Finally, for the $I = 0$ scalar sector, the quadratic terms involving the fields $\zeta$, $\sigma_8$ and $\sigma_0$ are mixed by the $3 \times 3$ matrix

$$\begin{pmatrix}
4C^+(p) + G^{-1} & \sqrt{\frac{2}{3}}[2C_{uu}^+(p) + C_{ss}^+(p)] & \frac{4}{\sqrt{3}}[C_{uu}^+(p) - C_{ss}^+(p)] \\
\sqrt{\frac{2}{3}}[2C_{uu}^+(p) + C_{ss}^+(p)] & G_{00}^+(p) & G_{80}^+(p) \\
\frac{4}{\sqrt{3}}[C_{uu}^+ - C_{ss}^+(p)] & G_{80}^+(p) & G_{88}^+(p)
\end{pmatrix},$$

(48)
where

\[
C^\zeta(p) = \frac{N_c}{\kappa^2} \int \frac{d^4q}{(2\pi)^4} q^2 f^2(q) \sum_{i=1}^{3} \frac{Z(q^+)}{q^{+2} + M_i^2(q^+)} \frac{Z(q^-)}{q^{-2} + M_i^2(q^-)} \\
\times \left[ q^+ q^- + q^{+2} q^{-2} - (q^+ q^-)^2 - M_i(q^+) M_i(q^-) \right]
\]

\[
C_i^{+\zeta}(p) = -\frac{2N_c}{\kappa} \int \frac{d^4q}{(2\pi)^4} g(q) f(q) \frac{Z(q^+)}{q^{+2} + M_i^2(q^+)} \frac{Z(q^-)}{q^{-2} + M_i^2(q^-)} \\
\times q \cdot \left[ q^- M_i(q^+) + q^+ M_i(q^-) \right],
\]

with \(i = u, s\). For a given value of \(p^2\), we denote the eigenvalues of this matrix by \(G_\zeta(p)\), \(G_\sigma(p)\) and \(G_{f_0}(p)\). As stated in Sect. [11] from the functions \(G_\sigma(p)\) and \(G_{f_0}(p)\) one can determine the masses of the \(\sigma, f_0\) physical states (the function \(G_\zeta(p)\) turns out to be positive definite for the allowed values of \(-p^2\)). The corresponding mixing angles can be obtained in a similar way as in the \(\eta\) meson sector, now defining \(SO(3)\) rotation matrices for the \(\sigma\) and \(f_0\) physical states.

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