Free Fermions and Extended Conformal Algebras

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ABSTRACT

A class of algebras is constructed using free fermions and the invariant anti-symmetric tensors associated with irreducible holonomy groups.

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Given a set of free fermions \( \{ \lambda^a; a = 1, \ldots, n \} \) in two dimensions, one can construct a set of conserved currents \( \{ J^{(p)}; p = 1, \ldots, n \} \) where \( J^{(p)} \) is the normal-ordered product of \( p \) fermion fields (with uncontracted indices). Thus each current is an antisymmetric tensor in an associated \( n \)-dimensional vector space \( V \) (\( \dim V = n \)). The set of currents which are quadratic in the fermion fields \( (p = 2) \) form an \( SO(n) \) Kac-Moody current. As such, it is natural to ask whether there are other current algebras constructed by taking higher order polynomials of this currents. Indeed it was observed [1] that one could generate the \( N = 1 \) superconformal algebra by constructing the supercharge from the three fermion current contracted with the structure constant of a Lie algebra.

In this note we will show that certain subalgebras of this larger algebra may be extracted using antisymmetric tensors of \( V \) that are invariant under of a subgroup \( G \) of \( O(n) \) acting on \( V \) with some representation \( R \). Invariant forms can exist in every irreducible representation of a compact Lie group. To be more precise, we shall use the invariant forms which occur in the fundamental representation of irreducible holonomy groups of non-symmetric Riemannian manifolds. We will also use the invariant forms of the adjoint representation of these groups. Typically we find algebras with additional spin \( 3/2 \) and spin \( 2 \) generators. For \( SO(n) \) and \( SU(n/2) \) higher spin currents arise for large enough \( n \). In these cases, one does not generate a closed algebra using only the invariant tensors (and \( T \)) although it might be possible to achieve closure by including a finite number of additional currents.

We recall that the possible irreducible holonomy groups of \( n \)-dimensional non-symmetric Riemannian manifolds are given by Berger’s list [2]: \( SO(n), U(n/2), SU(n/2), Sp(n/4), Sp(n/4) \cdot Sp(1) \) and two exceptional cases, \( G_2 \) (\( n = 7 \)) and \( \text{Spin}(7) \) (\( n=8 \)). The corresponding invariant forms are: the \( \epsilon \)-tensor (\( SO(n) \)); the Kahler 2-form (\( U(n/2) \)), the holomorphic \( \epsilon \)-tensor (\( SU(n/2) \)), the three Kahler 2-forms (\( Sp(n/4) \)), and for \( Sp(n/4) \cdot Sp(1) \), a four-form which is locally the sum of the wedge product of each of the three local Kahler forms with itself. In \( G_2 \), there is a 3-form and its 4-form dual, and in \( \text{Spin}(7) \) a self-dual 4-form. It has been noted
that (two-dimensional) supersymmetric sigma models with these manifolds as
target spaces have additional symmetries associated with these form; and that
the corresponding currents belong, at least classically, to extended superconformal
algebras which close in a non-linear fashion, i.e. they are of the W-symmetry type.
The $SU(3)$, $G_2$ and $Spin(7)$ cases are relevant in string theory compactifications
to 4, 3 and 2 dimensions respectively (see for example [5]).

Let us now turn to a free fermion theory with $n$ fermions $\{\lambda^a\}$. In the general
case, we can introduce the currents

$$J_{a_1\ldots a_p} = :\lambda_{a_1}\cdots\lambda_{a_p}:$$ (1)

The OPE of two such currents (with $q \leq p$) is

$$J_{a_1\ldots a_p}(z)J_{b_1\ldots b_q}(w) = \sum_{m=1}^{p} \frac{(-1)^{pm + \frac{m(m-1)}{2}}}{(z-w)^m} \delta^{[b_1\ldots b_m}_{[a_1\ldots a_m} \lambda^{a_{m+1}\ldots a_p]}(z)\lambda^{b_{m+1}\ldots b_q]}(w)$$ (2)

The antisymmetry symbols in the above equation are with weight one. Clearly,
this O.P.E algebra closes in the sense that all the terms on the right-hand-side can
be arranged into products of $J$’s and their derivatives.

To extract the sub-algebras of interest we shall contract some of the above $J$’s
with the invariant tensors of the holonomy groups $G$ which we have listed above. Let

$$g = g_{a_1\ldots a_k}e^{a_1} \wedge \ldots \wedge e^{a_k}$$

be a (constant) k-form of the vector space $V$ and $\{e^a\}$, $a = 1, \ldots, \dim V$ a basis of
$V$. The currents that we will study are

$$X = \frac{1}{k!} g_{a_1\ldots a_k} :\lambda_{a_1}\cdots\lambda_{a_k}:$$ (3)

All these currents are primary with conformal spin $\frac{k}{2}$ with respect to the energy
momentum

\[ T = \frac{1}{2} \lambda^a \partial \lambda_a \]

of the free fermions, i.e. the O.P.E of \( T \) and \( X \) is

\[ T(z)X(w) = \frac{\partial X}{z-w} + \frac{k}{2} \frac{X}{(z-w)^2}. \quad (4) \]

1. \( G = SO(4) \)

We define

\[ X = \frac{1}{4!} \epsilon_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \quad (5) \]

we then find

\[ X(z)X(w) = \frac{1}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T}{(z-w)} \quad (6) \]

The O.P.E of \( T \) with \( X \) closes as in equation (4). Thus, we have an algebra with two spin 2 currents. It might be thought that this algebra should factorise, but it does not as one may verify by considering all possible linear combinations of \( X \) and \( T \).

If one goes to higher \( n \), one finds that the OPE algebra does not close on \( T \) and the spin \( n/2 \) current

\[ X = \frac{1}{n!} \epsilon_{abcde...} : \lambda^a \lambda^b \lambda^c \lambda^d \lambda^e \ldots : \quad (7) \]

Indeed, if one considers the OPE of \( X \) with itself, the two most singular terms, are

\[ \frac{(-1)^{\frac{n(n-1)}{2}}}{(z-w)^n} - \frac{(-1)^{\frac{n(n+1)}{2}}}{(z-w)^{n-2}} \frac{1}{2} \lambda^{ab}(z) \lambda_{ab}(w). \quad (8) \]

Up to a sign we can identify the coefficient of the latter term as \( 2T^2 \). However, were the algebra to close on \( X \) and \( T \) alone then the coefficient of the above \( T^2 \)
term is fixed [4], by conformal invariance, to be
\[ 2 \frac{5s+1}{2s+10c} \]
in magnitude, where, for us, \( c = s = \frac{3}{2} \). Clearly, the above O.P.E. can not close on \( X \) and \( T \) alone. Explicit calculation of the full O.P.E. for \( \text{SO}(5) \) bears out this conclusion. This does not mean that the O.P.E. does not close if one includes more currents or a different choice for energy-momentum tensor.

2. \( G = SU(3) \)

In \( SU(3) \) we have \( n = 6 \) \( \lambda \)'s and replace them by 3 complex ones, i.e. \( \lambda^a \rightarrow \{\lambda^\alpha, \bar{\lambda}^\alpha\}, \alpha = 1, 2, 3 \). There is complex spin 3/2 current

\[ G = \frac{1}{3!} \varepsilon_{\alpha\beta\gamma} : \lambda^\alpha \lambda^\beta \lambda^\gamma : \]  

(9)

and its conjugate

\[ \bar{G} = \frac{1}{3!} \varepsilon_{\alpha\beta\gamma} : \bar{\lambda}^\alpha \bar{\lambda}^\beta \bar{\lambda}^\gamma : \]

(10)

One finds

\[ G(z)\bar{G}(w) = \frac{1}{(z - w)^2} - \frac{J}{(z - w)^2} + \frac{\{-\frac{1}{2}\partial J + T'\}}{(z - w)} \]

(11)

where

\[ J = :\lambda^\alpha \bar{\lambda}^\alpha : \quad T' = T + \frac{1}{2} :\lambda^\beta \bar{\lambda}_\beta \lambda^\gamma \bar{\lambda}_\gamma : \]

(12)

While \( J \) is a \( U(1) \) current, one can show by re-normal ordering the \( \lambda^4 \) term that \( T' = \frac{1}{2} : J^2 : \) and so is the Sugawara energy-momentum tensor associated with the \( U(1) \) current. Evaluating the other O.P.E’s and, making a suitable rescaling, one finds that \( T', G, \bar{G} \) and \( J \) define an \( N = 2 \) superconformal algebra with central charge \( c = 1 \). The field \( T - \frac{T'}{3} \) commutes with this algebra. This algebra occurs as a subalgebra of the extended \( N = 2 \) superconformal algebra which arises in the context of Calabi-Yau compactifications [6].
For the $SU(n)$ case, we have the spin $\frac{n}{2}$ current

$$X = \frac{1}{n!} \epsilon_{\alpha \beta \gamma \delta \ldots} : \lambda^\alpha \lambda^\beta \lambda^\gamma \lambda^\delta \ldots :$$

(13)

and its complex conjugate

$$\bar{X} = \frac{1}{n!} \epsilon_{\alpha \beta \gamma \delta \ldots} : \bar{\lambda}^\alpha \bar{\lambda}^\beta \bar{\lambda}^\gamma \bar{\lambda}^\delta \ldots :$$

(14)

The OPE of $X$ with its complex conjugate does not close on $T$ for the reasons given earlier, however one may hope that it will close using Sugawara-like currents. This possibility remains under study.

3. $Sp(k) \cdot Sp(1)$

This is the holonomy group of quaternionic Kahler manifolds in which case there are three (locally defined) complex structures $I_r$ that obey the algebra of imaginary unit quaternions ($I_r I_s = -\delta_{rs} + \sum_t \epsilon_{rst} I_t$) and three associated Kahler forms $\omega_r$. From these one can define an $Sp(k) \cdot Sp(1)$ invariant 4-form

$$\Omega = \sum_{r=1}^{3} \omega_r \wedge \omega_r$$

(15)

The associated spin 2 current current is

$$X = \frac{1}{4!} \Omega_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d$$

(16)

One finds that the OPE algebra is

$$X(z)X(w) = \frac{1}{4!} \frac{n(n+2)}{(z-w)^4} + \frac{(n+2)}{3} T(w) \frac{T(w)}{(z-w)^2} + \frac{(n+2)}{6} \frac{\partial T(w)}{z-w} - \frac{n-4}{6} \frac{\partial X(w)}{z-w} - \frac{n-4}{3} \frac{X(w)}{(z-w)^2}.$$  

(17)

where $n = 4k$. 
4. **$G_2$ and Spin(7)**

The final examples are provided by the exceptional holonomy groups. In $G_2$ there is a 3-form ($e^a$ a basis in $\mathbb{R}^7$)

$$\phi = f_{abc}e^a \wedge e^b \wedge e^c$$

(18)

and a 4-form which is the dual of $\phi$

$$^*\phi = f_{abcd}e^a \wedge e^b \wedge e^c \wedge e^d$$

(19)

Using these we can form the currents

$$G = \frac{1}{3!} f_{abc} : \lambda^a \lambda^b \lambda^c :$$

(20)

and

$$X = \frac{1}{4!} f_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d :$$

(21)

The OPE algebra is

$$G(z)G(w) = \frac{7}{(z-w)^3} + \frac{6T}{(z-w)^2} - \frac{6X}{(z-w)}$$

$$G(z)X(w) = -\frac{6G(y)}{(z-w)^2} - \frac{2\partial G}{z-w}$$

$$X(z)X(w) = \frac{7}{(z-w)^4} + \frac{8T}{(z-w)^3} + \frac{4\partial T}{z-w} - \frac{6X}{(z-w)^2} - \frac{3\partial X}{z-w}$$

(22)

In the Spin(7) case ($n = 8$), one has an invariant 4-form which is closely related to the forms of $G_2$;

$$\phi = g_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d :$$

(23)

with $g_{0abc} = f_{abc}$ (for $a, b, c = 1, \ldots, 7$) and $g_{abcd} = f_{abcd}$ (for $a, b, c, d = 1, \ldots, 7$).
The associated spin 2 current is

\[ X = \frac{1}{4!} g_{abcd} : \lambda^a \lambda^b \lambda^c \lambda^d : \]  

(24)

The O.P.E is

\[ X(z)X(w) = \frac{14}{(z-w)^4} + \frac{14T}{(z-w)^2} + \frac{7\partial T}{z-w} - \frac{12X}{(z-w)^2} - \frac{6\partial X}{z-w} \]  

(25)

**Note Added**

This paper was based on work carried out a number of years ago, we were motivated to publish it by the recent appearance of the paper "Superstrings and Manifolds of Exceptional Holonomy " by S. Shatashvili and C. Vafa in which the \(G_2\) and spin(7) examples are discussed in a free superfield realisation.

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