Superstable groups acting on trees

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Groups acting on trees

Definition

A simplicial tree is a connected graph without circuits.

A group $G$ acts on a simplicial tree $T$ if it acts by automorphisms on $T$.

A real tree $X$ is a geodesic metric space such that any two points are joined by an unique arc.

This equivalent to saying that $X$ is a 0-hyperbolic geodesic space.

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Groups acting on trees

Definition (Classification of elements)

Let $G$ be a group acting on a simplicial or a real tree $T$. An element $g$ is said an inversion if $ge = \bar{e}$ for some edge $e$ (when $T$ is simplicial). An element $g$ is said elliptic if $gx = x$ for some $x$ of $T$. An element $g$ is said hyperbolic if it is neither an inversion nor elliptic. A group $G$ acts freely if every nontrivial element of $G$ is hyperbolic.
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- A group $G$ acts *freely* if every nontrivial element of $G$ is hyperbolic.
Motivation

Theorem 1 (Sela)
A free group is stable.

Reformulation:
A group acting freely on a simplicial tree is stable.

Remark:
A superstable group acting freely on a real (or simplicial) tree is abelian.
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The action of $G$ is trivial if there is no hyperbolic elements.

Question

What can be said about the model theory of groups acting nontrivially on simplicial trees? Is it possible for such groups to be $\omega$-stable or superstable?
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A group acts without inversions and nontrivially on a simplicial tree if and only if either $G$ splits as a free product with amalgamation or $G$ has an infinite cyclic quotient.
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Free products

What about the superstability of free products?

Theorem 3 (Poizat, 1984)

A nontrivial free product $G_1 \ast G_2$ is superstable if and only if $G_1 = G_2 = \mathbb{Z}_2$. 
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An action of an $\omega$-stable group on a simplicial tree is trivial.

Using Bass-Serre theorem:

Corollary 1

A free product with amalgamation or an HNN-extension is not $\omega$-stable.
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Superstable groups

Examples:

$\mathbb{Z}$ is superstable and acts freely on a simplicial tree.

If $G$ is superstable then $G \oplus \mathbb{Z}$ is superstable and acts nontrivially on a simplicial tree.

If $G$ is superstable then $G \oplus (\mathbb{Z}_2 \ast \mathbb{Z}_2)$ is superstable and acts nontrivially on a simplicial tree.

Moreover $G \oplus (\mathbb{Z}_2 \ast \mathbb{Z}_2) = (G \oplus \mathbb{Z}_2) \ast G (G \oplus \mathbb{Z}_2)$. 
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Let $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Let $G$ be a group acting on a $\Lambda$-tree $T$. We suppose that if $T$ is simplicial then the action is without inversions.

**Definition**
The hyperbolic length function is defined by:

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\ell(g) = \inf \{ d(x, gx) | x \in T \}.
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**Fact**
$g$ is hyperbolic if and only if $\ell(g) > 0$. 
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Classifications of actions

One of the following cases holds:

1. **Abelian actions**
   - The hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \Lambda$ is a homomorphism.

2. **Dihedral actions**
   - The hyperbolic length function is given by $\ell(g) = |\rho(g)|$ for $g \in G$, where $\rho : G \to \text{Isom}(\Lambda)$ is a homomorphism whose image contains a reflection and a nontrivial translation, and the absolute value signs denote hyperbolic length for the action of Isom(\Lambda).

3. **Irreducible actions**
   - $G$ contains a free subgroup of rank 2 which acts freely, without inversions and properly discontinuously on $T$. 
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(3) (Irreducible actions) \( G \) contains a free subgroup of rank 2 which acts freely, without inversions and properly discontinuously on \( \mathcal{T} \).
Theorem 5

Let $G$ be a superstable group acting nontrivially on a $\Lambda$-tree, where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. If $G$ is $\alpha$-connected and $\Lambda = \mathbb{Z}$, or if the action is irreducible, then $G$ interprets a simple group having a nontrivial action on a $\Lambda$-tree.

Corollary 2

If $G$ is superstable and splits as $G = G_1 \ast A G_2$, with the index of $A$ in $G_1$ different from 2, then $G$ interprets a simple superstable non-$\omega$-stable group acting nontrivially on a simplicial tree.
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Minimal Superstable groups

Definition

Let $G$ be a group and $B$ be a family of definable subgroups of $G$. We say that $B$ is a Borel family, if for any $B \in B$, $N_G(B)/B$ is finite and $B$ is generous, for any $g \in G$, $B^g \in B$, and any two elements of $B$ are conjugate to each other.
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Theorem 6

Let $G$ be a superstable group of finite Lascar rank acting nontrivially on a $\Lambda$-tree where $\Lambda = \mathbb{Z}$ or $\Lambda = \mathbb{R}$. Suppose that, if $H$ is a definable subgroup such that $U(H) < U(G)$, and having a nontrivial action on a $\Lambda$-tree, then $H$ is nilpotent-by-finite. Then there are definable subgroups $H_1 \triangleleft H_2 \triangleleft G$ such that $H_2$ is of finite index in $G$, and one of the following cases holds:

1. $H_1$ is connected, any action of $H_1$ on a $\Lambda$-tree is trivial, $H_2/H_1$ is soluble and has a nontrivial action on a $\Lambda$-tree.

2. $H_2/H_1$ is simple and acts nontrivially on a $\Lambda$-tree, $H_2/H_1$ has a Borel family of equationally-definable nilpotent subgroups such that there exists $m \in \mathbb{N}$ such that for every hyperbolic element $g$ in $H_2/H_1$, there is $1 \leq n \leq m$, such that $g^n$ is in some $B \in B$.

If $\Lambda = \mathbb{Z}$ then $H_2/H_1 = G_1 \ast A G_2$ with the biindex of $A$ is 2 in both $G_1$ and $G_2$. 
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