One-BEC-species coherent oscillations with frequency controlled by a second species atom number

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Abstract

Controlling the tunneling of atoms of one species using a different atom species is a fundamental step in the development of a new class of atom quantum devices, where detection, motion control, and other functions over the atoms, can be achieved by exploiting the interaction between two different atomic species. Here, we theoretically study coherent oscillations of a non-self-interacting Bose–Einstein condensate (BEC) species in a triple-well potential controlled by a self-interacting species self-trapped in the central well of the potential. In this system, a blockade, due to the interspecies interaction, prevents atoms of the non-self-interacting species from populating the central well. Thus, for an initial population imbalance between the left- and right-hand wells of the non-self-interacting species, coherent BEC oscillations are induced between these two wells, resembling those of Rabi-like BEC oscillations in a double-well potential. The oscillation period is found to scale linearly with the number of self-trapped atoms as well as with the interspecies interaction strength. This behavior is corroborated by the quantum many-particle and the mean-field models of the system. We show that BEC oscillations can be described by using an effective bosonic Josephson junction with a tunneling amplitude that depends on the number of the self-trapped atoms in the central well. We also consider the effect of the self-trapped atom losses on the coherent oscillations. We show, by using quantum trajectories, that this type of losses leads to a dynamical change in the oscillation period of the non-self-interacting species, which in turn allows the number of self-trapped atoms lost from the system to be estimated.

1. Introduction

A fascinating coherent quantum phenomenon in Bose–Einstein condensates (BECs) is the Josephson-like oscillations [1–10], which are based on the tunneling of atoms between two weakly coupled condensates. A straightforward implementation of a weak coupling can be realized with a double-well potential [7, 8], where a small overlap of the condensate wavefunctions in the inter-well barrier enables the transfer of atoms between the wells. Unlike the standard Josephson effect [11], the tunneling dynamics of a BEC can be strongly affected by the interaction between atoms. The most striking effect is the macroscopic quantum self-trapping [5–7, 12], which inhibits the tunneling of atoms. In a double-well potential, this phenomenon occurs when the initial atom population imbalance between the wells exceeds a critical value. This is manifested by a self-locked atomic distribution in a single well. On the other hand, in the limit where the interaction of atoms is negligible, an oscillatory exchange of atoms between the wells takes place [1]. This oscillating behavior, equivalent to single-atom dynamics, is usually referred to as a Rabi oscillation [6].

A relevant aspect in these small systems is the ability to control them, which has encouraged further studies of BECs in two [13–19] and three-well [20–27] setups. In particular, it has been shown that ultracold atoms trapped in a triple-well potential can operate as a transistor-like device [27]. In this type of transistor, a flow of atoms across the central or gate well is achieved by applying chemical potential gradients. In this context, however, it has not yet been revealed to what extent phase coherence of the BEC...
can be important for the motion of atoms. This raises the basic but important question of whether in a triple-well potential, coherent BEC oscillations can be observed as they have been in two-well potentials. Furthermore, can coherent BEC oscillations be induced and controlled by a gate well containing a second BEC species?

Two bosonic species in optical lattices allow for the study of interesting many-body physics such as the emergence of novel quantum phases [28–37]. The interplay between the interactions of atoms from distinct species and the interactions of the species itself has driven the search for new effects related to coherent quantum phenomena [38, 39]. In particular, extensive research has been carried out in small systems of two wells [40–52] and three-well rings [36, 53]. Very importantly, the tuning of the interaction strength between atoms of distinct species, which is an accessible experimental parameter [54, 55], has been suggested as a tool of control of quantum systems [35, 56–59].

In the case of the triple-well transistor-like potential with two BEC species, the interaction between species can play a regulating role in the functioning of the transistor-like device. Here, in order to control the tunneling process in a triple-well potential of one BEC species, a second BEC species is loaded into the central well, becoming self-trapped due to the atom interaction. Additionally, the BEC species to be controlled is loaded into the left-hand well for which a null interaction between atoms is assumed, whereas an interaction with atoms of the second species is enabled. In this configuration, the species in the central well plays a role similar to that of an ion placed in the center of a bosonic Josephson junction (BJJ) [9, 10].

Because of the interaction blockade resulting from the two interacting BEC species in the central well, the atoms of the non-self-interacting species are prevented from populating this well. So, when atoms of the non-self-interacting species are placed in the left-hand well, they are induced to perform coherent oscillations between the left- and right-hand wells. This is similar to Rabi oscillations for noninteracting atoms in a double-well potential [6]. A striking result of this engineered system is that the period of the coherent BEC oscillations scales linearly with the number of atoms of the species trapped in the central well. This opens up the possibility to use coherent oscillation processes to accurately estimate the number of atoms in a trap.

As for the control of the coherent BEC oscillations is concerned, dissipative effects have been used to steer the dynamics of quantum systems [60]. In particular, the loss of particles, which resembles the emission of photons in quantum optics [61], have drawn large attention because of the possibility to change the tunneling properties of some quantum systems [62, 63]. In the same spirit, here we simulate the removal of atoms from the central well during the time evolution. We find that dynamical changes of the number of atoms in this well lead to variations in the period of the coherent oscillations of the BEC species subject to control. This could be used as a tool for experimentally sensing atoms in the central well or for controlling the BEC oscillations between the outer wells.

The paper is organized as follows. In section 2, we introduce the model and discuss its properties. In section 3, we present our main results organized in several subsections. We start discussing the phenomenon of coherent BEC oscillations between left- and right-hand wells of the triple-well potential and the importance of the interaction between BEC species to create a blockade that prevents the population of the central well. In subsection 3.1, we investigate the conditions for the emergence of coherent oscillations by analyzing the energy spectrum and properties of the eigenstates that contribute to it. Next, we analytically show, that the period of the oscillations of the non-self-interacting species scales linearly with the number of self-trapped atoms and the interspecies interaction strength, which is corroborated by numerical simulations. In subsection 3.2, we study the dynamics of the coherent BEC oscillations using a mean-field theory. We show that BEC oscillations can be described by using an effective BJJ with a tunneling amplitude that depends on the number of the self-trapped atoms in the central well. Importantly, we find that the relation between the oscillation period with the self-trapped atoms and the interspecies interaction strength is the same in both descriptions. In last subsection 3.3, we consider the presence of atom losses in the central well. For that purpose, we use the formalism of quantum trajectories to analyze the effects of atom losses on the tunneling speed of atoms when performing coherent oscillations between the opposite edge wells of the triple-well potential. We also perform a time-frequency analysis in order to track down the decay of the number of atoms during the time evolution. In section 4 we summarize our results. Finally, some of our technical details are deferred to the appendices A, B, C and D.

2. The model

We consider a one-dimensional triple-well potential similar to that in reference [24] (see figure 1) containing two different atomic species A and B and model the dynamics of the condensates within the weak coupling regime [8]. The two atomic species can be either different atoms [54, 64] or identical atoms in
different internal states [18]. Additionally, we assume negligible intraspecies interaction of the species A, which can be achieved with Feshbach resonances [55, 65, 66].

Taking into account the above assumptions, the dynamics of the weakly-coupled condensates can be described by the quantum many-particle Hamiltonian (QMPH)

\[
H_0 = -J_A \left[ \hat{A}_L \hat{A}_C^\dagger + \hat{A}_C \hat{A}_L^\dagger + \hat{A}_C \hat{A}_R^\dagger + \hat{A}_R \hat{A}_C^\dagger \right] - J_B \left[ \hat{B}_L \hat{B}_C^\dagger + \hat{B}_C \hat{B}_L^\dagger + \hat{B}_C \hat{B}_R^\dagger + \hat{B}_R \hat{B}_C^\dagger \right]
+ \frac{U_B}{2} \left[ \hat{B}_L \hat{B}_C^\dagger \hat{B}_C^\dagger \hat{B}_L^\dagger + \hat{B}_L^\dagger \hat{B}_C \hat{B}_C^\dagger \hat{B}_L^\dagger - 1 \right] + \hat{B}_C \hat{B}_L^\dagger \hat{B}_B^\dagger \hat{B}_C^\dagger - 1 \right]
+ U_{AB} \left[ \hat{A}_L^\dagger \hat{A}_L \hat{B}_L^\dagger \hat{B}_L + \hat{A}_C^\dagger \hat{A}_C \hat{B}_C^\dagger \hat{B}_C + \hat{A}_R^\dagger \hat{A}_R \hat{B}_R^\dagger \hat{B}_R \right].
\]

In equation (1) \( \hat{O}_O^\dagger \) (\( O = \hat{A}, \hat{B} \)) is the creation (annihilation) operator for the atoms at the well \( Y(Y = L, C, R) \). \( J_A \) and \( J_B \) are the tunneling strengths between adjacent wells in the triple-well potential for the species A and B, respectively. \( U_B \) is the intraspecies interaction strength for the atoms of species B, whereas \( U_{AB} \) is the strength of the inter-species interaction. Since we consider identical wells, the local onsite energies are the same at each well and therefore have been omitted as they do not contribute to the dynamics of the coherent oscillations.

To describe the evolution of the system, the wavefunction is spanned as

\[
|\psi(t)\rangle = \sum_{s_A=0}^{N_A} \sum_{q_A=0}^{N_A} \sum_{s_B=0}^{N_B} \sum_{q_B=0}^{N_B} C_{q_A,s_A,q_B,s_B}(t) |N_A - q_A - s_A, q_A, s_A\rangle_A |N_B - q_B - s_B, q_B, s_B\rangle_B,
\]

where the amplitudes \( C_{q_A,s_A,q_B,s_B}(t) \) are governed by the Schrödinger equation with the Hamiltonian (1).

Here, \( |p, q, s\rangle_P \) denotes the state with \( p \) atoms in the left well, \( q \) atoms in the central well and \( s \) atoms in the right well for the species \( D(D = A, B) \); \( N_D \) is the total number of atoms of species \( D \). In what follows, we set \( \hbar = 1 \) and consider attractive interaction between A and B atoms, i.e. \( U_{AB} < 0 \).

### 3. Results and discussion

Let us consider the triple-well system prepared so that atoms of species B are loaded into the central well, and all atoms of species A are initially loaded into the left well of the triple-well potential (see figure 1). When the number of atoms of species B in the central well overcomes a critical value, for a given intraspecies interaction strength, a macroscopic self-trapping occurs [6, 12]. This effect inhibits the tunneling of atoms of species B and therefore prevents atoms of this species to populate the other wells. In addition, the interaction between atoms of distinct species in the central well can block atoms of species A to populate this well, which is known as interaction blockade [21, 67, 68]. In the scenario where both, self-trapping of species B and an interaction blockade take place in the central well, atom population of species A can oscillate between the left and right wells with negligible contribution of the central well. Thus,
to analyze the population dynamics of species A in these wells, we define $z(t)$ as the population imbalance between the left and right wells \[^6\], viz
\begin{equation}
z(t) = n^A_L - n^A_R,
\end{equation}
where $n^A_L$ and $n^A_R$ are the populations of atoms of species A in the left and right wells, respectively. It is important to remark that $z(t)$ is a quantity that can be experimentally monitored during the time evolution \[^7\]. For example, it has been used to characterize the nonlinear self-trapping and Josephson oscillations in a single BJJ \[^7\]. In a BJJ, the condensate tunnel through the inter-well barrier of a double-well potential. The theoretical description of the BJJ can be realized using a one-dimensional Bose–Hubbard model with a tunnel coupling estimated from the solution of the Gross–Pitaevskii equation in 3D \[^8\].

In what follows, we present results from simulations of the Schrödinger equation with the Hamiltonian equation (1) and the wavefunction equation (2). Let us first consider the scenario when $N_A = 2$, and $N_B = 6$ whose dynamics is featured in figure 2. In the (left column), $U_{AB}/J_A = -0.1$, so that the interaction between atoms of distinct species is weak. In this case, atoms of species A can populate the central well as shown in figure 2(b) and the fractional population imbalance $z(t)/N_A$ displays a seemingly quasiperiodic oscillation as depicted in figure 2(a). The $z(t)/N_A$ evolution is actually described by a biharmonic function, whose incommensurability depends on the interspecies interaction strength, as will be shown below.

In the (right column) of figure 2, we consider $U_{AB}/J_A = -3$ that corresponds to a strong interspecies interaction. So, a strong interaction blockade for A atoms takes place in the central well and so $n^A_C \ll 1$ as shown in figure 2(e). In this scenario the populations of the left and right wells oscillate in anti-phase with the same frequency (see figure 2(d)). This is similar to Rabi oscillations for noninteracting atoms in a double-well potential \[^6\]. Notice that for $U_{AB}/J_A = -0.1$ and $U_{AB}/J_A = -3$ the population of the species B remains trapped in the central well (see figure 2(c) and (f)).

### 3.1. Coherent BEC oscillations: period $T_A$ vs atom number $N_B$

Let us now analyze the coherent BEC oscillation of species A, i.e. the coherent oscillatory process of the population of species A between the left and right wells of the system. To gain insight into this phenomenon, we first analyze the energy spectrum as a function of the interaction strength $U_B$. In figure 3, we plot the energy spectrum for $N_B = 6, N_A = 2$ and $U_{AB}/J_A = -3$ that follows from the numerical diagonalization of the Hamiltonian equation (1). This figure exhibits several energy bands that grow linearly with $U_B$. In particular, the energy bands that follow straight lines in orange color are described by the expression $E_{nB} \approx N_B(N_B - 1)U_B/2 = \sigma_N U_B$, which is the energy of interaction between atoms of species B. Among the energy bands with $\sigma_N$, there are eigenstates that correspond to the case where all 6 atoms of species B are trapped in the central well. This implies the existence of quasidegenerate eigenstates with close energy values \[^6\], as shown for example in the inset of figure 3. In this inset, the eigenstates depicted in red are linear combinations of the Fock states $\ket{2, 0, 0}_A \ket{0, 6, 0}_B, \ket{1, 0, 1}_A \ket{0, 6, 0}_B$ and $\ket{0, 0, 2}_A \ket{0, 6, 0}_B$, where
all 6 atoms of species B are trapped in the central well and the atoms of species A populate the left and right wells.

Inasmuch as the system is initially prepared with all the atoms of species B in the central well, the wave function $|\psi(t = 0)\rangle$ can be spanned by the eigenstates depicted in red in the inset of figure 3. Therefore, the overlap of $|\psi(0)\rangle$ with these eigenstates enables the atoms of species A, that initially occupy the left well, to populate the right well as well. This raises the questions of how coherent BEC oscillations arise and what is the role of the central well in this process.

To gain understanding of this process, we resort to a reduced exactly solvable problem of one atom of species A, and $N_B$ constant atoms of species B in the central well. This means that the wavefunction (2) is spanned by three basis vectors: $|1, 0, 0\rangle_A|0, N_B, 0\rangle_B$, $|0, 1, 0\rangle_A|0, N_B, 0\rangle_B$, and $|0, 0, 1\rangle_A|0, N_B, 0\rangle_B$. In the following, we define this set of states as the manifold $N_B$ where the states can be written as $|1\rangle_A \otimes |0, N_B, 0\rangle_B$. In this basis, one can find three eigenvectors that correspond to the oscillating modes with the eigenvalues $\lambda_1 = 0$, $\lambda_{2,3} = \frac{U_{AB} N_B + \sqrt{(U_{AB} N_B)^2 + U_{AB}^2}}{2}$ (see appendix A).

We focus on the last two modes that exhibit oscillating dynamics. In the limit $N_B|U_{AB}| \gg J_A$, the frequencies can be approximated as $\omega_2 = N_B|U_{AB}|$ and $\omega_3 = 2J_A^2/(|U_{AB}|N_B|)$, whose eigenmodes can be approximated to $\psi_2 = \frac{1}{\sqrt{1 + 2(\frac{U_{AB} N_B}{2})}} \left( \frac{U_{AB} N_B}{2} |1, 0, 0\rangle_A - |0, 1, 0\rangle_A + \frac{U_{AB} N_B}{2} |0, 0, 1\rangle_A \right) |0, N_B, 0\rangle_B$ and $\psi_3 = \frac{1}{\sqrt{2 + (\frac{U_{AB} N_B}{2})}} \left( |1, 0, 0\rangle_A + \frac{U_{AB} N_B}{2} |0, 1, 0\rangle_A + |0, 0, 1\rangle_A \right) |0, N_B, 0\rangle_B$. Hence, in this limit, the eigenmode $\psi_2$ is characterized by an atom populating the central well. In contrast, the eigenmode $\psi_3$ is mainly characterized by a superposition of the states $|1, 0, 0\rangle_A$ and $|0, 0, 1\rangle_A$, thus enabling an oscillation of the A atom between the left and right wells with the frequency $\omega_3$.

When $\omega_2 \gg \omega_3$, which is fulfilled in the limit $|U_{AB}| \gg \sqrt{2J_A}/N_B$, the frequency $\omega_2$ corresponds to a very fast oscillating mode, whose amplitude can be neglected within the rotating wave approximation. Therefore, in the region that $|U_{AB}| \gg \sqrt{2J_A}/N_B$, the coherent oscillations becomes determined by $\omega_3 = 2J_A^2/(|U_{AB}|N_B)$. It is important to note that atoms of species A are not self-interacting, therefore this mode frequency can characterize the coherent oscillation of a condensate with different number of atoms of species A, as shown below. The expression for $\omega_3$ implies that the period of the oscillation of atoms of species A is given by

$$T_A = \frac{\pi|U_{AB}|}{J_A^2 N_B}.$$  

This expression is independent of the sign of $U_{AB}$, meaning that it is applicable to attractive and repulsive interspecies interactions. Let us consider again the parameter values associated with figures 2(a)–(c), namely $U_{AB}/J_A = -0.1$ with $N_B = 6$. These values correspond to a point in the parameter space which is not included in the region $|U_{AB}| \gg \sqrt{2J_A}/N_B$. According to the reduced eigenvalue problem proposed above, the oscillating mode with frequency $\omega_2$ plays a relevant role in the dynamics. Therefore atoms of
species $A$ populate the central well, which is in agreement with the behavior exhibited by the dynamics of $n_B^2(t)$ in figure 2(b). On the other hand, we have shown in figure 2(a) for these parameters, that the function $z(t)/N_A$ exhibits a seemingly quasi-periodic dynamics. This behavior of the $z(t)/N_A$ function corresponds, in fact, to a biharmonic function with frequencies provided by the eigenvalues $\lambda_2$ and $\lambda_3$ of the reduced eigenvalue problem. This statement will be later revisited in a mean-field description.

The oscillation period in equation (4) relies on the self-trapping of $N_B$ atoms in the central well, which in turn depends on the overlap of the wavefunction with the energy eigenstates. Thus, assuming that $|\psi(0)\rangle$ overlaps with eigenstates of the manifold $N_B$, it will remain so, if they are separated by large energy gaps from eigenstates of the manifold $N_B - 1$ as shown, for example, in the energy spectrum in figure 3. Hence, eigenstates of the manifolds $N_B - p$ with $p = 1, 2, ..., N_B - 1$ can be adiabatically eliminated (see e.g. references [40, 70]). An estimation of the energy gaps in the spectrum of the Hamiltonian (1) allows showing that, in the case of $U_{AB} < 0$, a full adiabatic elimination of the manifold $N_B - 1$ is possible for $|U_{AB}| \lesssim U_B/3$. This is independent of the number of atoms of species $A$ provided that $N_A$ is not very large compared to $N_B$ (see appendix A).

To numerically test the above bound estimation of $U_{AB}$, we look into the evolution of the function $z(t)/N_A$ for different values of the intraspecies interaction strength $U_B/J_B$ and the same values $U_{AB}/J_A = -10$ and $J_A = J_B$. In figure 4(a), we observe for $U_B/J_B = 30$, i.e. $|U_{AB}| = U_B/3$, that the $z(t)/N_A$ function exhibits a coherent oscillation. While for $U_B/J_B = 15, 10, 5$, the oscillatory behavior deviates from the coherent process predicted by the theory.

Thus, whenever interspecies interaction strength value is in the parameter region $\sqrt{2}J_A/N_B \ll |U_{AB}| \lesssim U_B/3$, $z(t)$ oscillates coherently. This means that a well-defined oscillation period can be measured in this parameter region, opening the opportunity to investigate the linear dependency on the interspecies interaction strength predicted in equation (4). In figure 4(b) is shown the period $T_A$ is units of $(1/J_A)$ as a

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Figure 4. (a) $z(t)/N_A$ vs time for three different interaction strengths of the species $B$: $U_B/J_B = 5$ with $N_A = 2$ (orange dotted-dashed line), $U_B/J_B = 10$ with $N_A = 2$ (red dashed line), $U_B/J_B = 15$ with $N_A = 2$ (black solid line), $U_B/J_B = 30$ with $N_A = 2$ (blue solid line) and $N_A = 6$ (green dashed line). The other parameters are $U_{AB}/J_A = -10$ and $N_B = 6$. (b) Period of the coherent oscillations $T_A$ in units of $1/J_A$ as a function of the interspecies interaction strength $|U_{AB}|$ for different number of atoms $N_B$. $N_B = 6 \quad N_B = 8 \quad N_B = 10 \quad N_B = 15$. The other parameters are $U_B/J_B = 60, N_A = 3$ and $J_A = J_B$. (c) Period of the coherent oscillations $T_A$ in units of $1/J_A$ as a function of the atom number $N_A$ for $N_A = 1 \quad N_A = 2 \quad N_A = 3 \quad N_A = 8$. The inset shows the same dependence of the period for a larger number of atoms $N_B$ with $N_A = 2$. The dashed straight lines have a slope of $9.42(1/J_A)$ that follows from equation (4). The other parameters are $U_B/J_B = 10, U_{AB}/J_A = -3$ and $J_A = J_B$. 

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function of the interspecies interaction strength \(|U_{AB}|/J_A\) for different number of atoms of the species B. From this picture, we can observe a linear dependence of the period \(T_A\) with the interaction strength between the species \(|U_{AB}|\), which holds regardless of the sign. One can also notice in this figure an increase in the slope in the linear behavior when the number of atoms \(N_B\) is increased, in agreement with equation (4).

For a complete check of equation (4), we plot in figure 4(c) the oscillations period as a function of the number \(N_B\). In the main panel are plotted period data vs \(N_B\) for different values of the number of atoms \(N_A\), i.e., \(N_A = 1, 2, 3\) and \(8\), whose behavior appears extended for a relatively large \(N_B\) in the inset of figure 4(c). In general, the period exhibits a linear dependence on the number \(N_B\) of atoms of species B, where the relation is independent on the number of atoms of species A. The linear dependence is highlighted by a dashed straight line, whose slope, in both the main and inset panels, is analytically obtained with equation (4).

### 3.2. Mean-field description

Condensates with a number of atoms of the order of one hundred have been successfully studied in the mean-field regime [71]. In the mean-field approach the two condensates are represented by the macroscopic wavefunctions \(\Phi_D(r, t)\) with \(D = A, B\), whose mixture dynamics can be described by coupled Gross–Pitaevskii equations [45, 72]

\[
i \Phi_A = -\frac{\nabla^2 \Phi_A}{2m_A} + V_A(r)\Phi_A + \tilde{U}_A|\Phi_A|^2\Phi_A + \tilde{U}_{AB}|\Phi_B|^2\Phi_A,
\]

\[
i \Phi_B = -\frac{\nabla^2 \Phi_B}{2m_B} + V_B(r)\Phi_B + \tilde{U}_B|\Phi_B|^2\Phi_B + \tilde{U}_{AB}|\Phi_A|^2\Phi_B,
\]

where \(m_D\) and \(V_D(r)\) denote the mass and trapping potential for each species. \(\tilde{U}_A, \tilde{U}_B\) are the intra-species interactions for the species A and B, respectively. \(\tilde{U}_{AB}\) is the strength of the interaction between distinct species. Here, as before, we set \(\hbar = 1\).

Since we are interested in the dynamics of three weakly linked BECs, we write the wave-function as

\[
\begin{pmatrix}
\Phi_A \\
\Phi_B
\end{pmatrix}
= \begin{pmatrix}
\phi_A(t)u_A^0(r) \\
\phi_B(t)u_B^0(r)
\end{pmatrix}
+ \begin{pmatrix}
\phi_A(t)u_A^1(r) \\
\phi_B(t)u_B^1(r)
\end{pmatrix}
+ \begin{pmatrix}
\phi_A(t)u_A^2(r) \\
\phi_B(t)u_B^2(r)
\end{pmatrix},
\]

where \(\int u_Q^0(r)u_Q^1(r)dr = \delta_{Q,Y}\) with \(Q, P = A, B\) and \(Y, \bar{Y} = L, C, R\).

Let us now consider a lattice of identical wells with vanishing intraspecies interaction for the species A. Thus, inserting equation (7) into equations (5) and (6) and taking into account the orthogonality of the functions \(u_Q^0(r)\), with \(D = A, B\) we find, for the description of the two species dynamics, the following equations

\[
i \dot{\phi}_A^P = -\Omega_A\phi_C^P + g_{AB}|\phi_C^P|^2\phi_A^P,
\]

\[
i \dot{\phi}_A^C = -\Omega_A(\phi_C^A + \phi_C^B) + g_{AB}|\phi_C^B|^2\phi_A^C,
\]

\[
i \dot{\phi}_A^R = -\Omega_A\phi_C^R + g_{AB}|\phi_C^R|^2\phi_A^R,
\]

and,

\[
i \dot{\phi}_B^P = -\Omega_B\phi_C^P + g_B|\phi_C^P|^2\phi_B^P + g_{AB}|\phi_C^A|^2\phi_B^P,
\]

\[
i \dot{\phi}_B^C = -\Omega_B(\phi_C^A + \phi_C^B) + g_B|\phi_C^A|^2\phi_B^C + g_{AB}|\phi_C^B|^2\phi_B^C,
\]

\[
i \dot{\phi}_B^R = -\Omega_B\phi_C^R + g_B|\phi_C^R|^2\phi_B^R + g_{AB}|\phi_C^A|^2\phi_B^R,
\]

where \(\Omega_D\) and \(g_D\) are the respective effective coupling strengths between the condensates in neighboring wells and the strengths of the atoms interaction for each species. \(g_{AB}\) is the effective strength for the interaction between atoms of distinct species, i.e. A and B.

Let us now investigate the dependence of the coherent BEC oscillations with the number of atoms using the mean-field equations (8)–(13). In figure 5(a), we show the mean-field behavior of the oscillation period as a function of the number \(N_B\). Here, as in the quantum many-body problem counterpart, the period exhibits a linear dependence on the number of self-trapped atoms, namely \(T_A = T_0N_B\). While the simulations are considered for \(N_A = N_B\), further additional computation demonstrates the independence of the period from the number \(N_A\) of atoms of species A. An important question that arises here is whether the slope of the straight line, \(T_0\), can be modified by the parameter \(g_{AB}\).
In the QMPD, we have seen that coherent BEC oscillations are determined by the strength of the interaction between species. To find out what occurs on the mean-field level, we first analyze the behavior of the period versus the interspecies interaction. In the inset of figure 5(a) we display the behavior of the dimensionless quantity $T_A/\Omega_A$ as a function of the absolute value of the interaction strength between the species, $|g_{AB}|$, for $N_A = N_B = 15$. It shows a linear scaling of the period $T_A$ with $|g_{AB}|$, which holds regardless of the sign of $g_{AB}$. This is in agreement with the results from QMPD (see section 3.1). However, in contrast to the QMPD results, the interspecies interaction strength $|g_{AB}|$ does not require to be smaller than $g_0$. On the other hand, for small interaction values, it is possible to observe the presence of more than a single harmonic, which is discussed next.

To gain further insight into the relation of the period with the interspecies interaction, we analyze the behavior of the dimensionless quantity $T_A\Omega_A^2/(N_B|g_{AB}|)$ as a function of $|g_{AB}|/\Omega_A$ with $g_{AB} < 0$. The quantity $T_A\Omega_A^2/(N_B|g_{AB}|)$ can provide relevant information associated with the values of $g_{AB}$ in which the coherent BEC oscillations are controlled by the self-trapped atoms. In figure 5(b) it is plotted the behavior of $T_A\Omega_A^2/(N_B|g_{AB}|)$ for two different values of the number $N_B$ of atoms of species B, namely $N_B = 5$ (blue curve) and $N_B = 7$ (orange curve). In this figure, both curves converge to the same constant $\pi$ value for values $|g_{AB}|/\Omega_A \gtrsim 3$. This convergence indicates a definite relation between the period $T_A$, the interspecies interaction strength $|g_{AB}|$ and the number $N_B$ of atoms of species B, as predicted in equation (4) which in turn follows from the QMPD. On the other hand, for small $|g_{AB}|$ values (e.g. $|g_{AB}| / \Omega_A \lesssim 3$ for $N_B = 5$), the curves separate from each other and from the horizontal line that determines the coherent oscillation, as the value $|g_{AB}|/\Omega_A$ decreases. We have checked that such separations are associated with the emergence of more than one harmonic and depend on the number $N_B$ of atoms of species B considered. This behavior is in line with the predictions shown below.

To derive an analytical expression of the coherent BEC oscillation period within the mean-field description, we resort to equations (8)–(13). Assuming $|\phi_C(t)|^2 = N_B$, implies that the interaction between species takes place in the central well only. Thus, the system of equations (8)–(10) reduces to the form

$$i\dot{\phi}_L^A = -\Omega_A\phi_C^A,$$
$$i\dot{\phi}_C^A = -\Omega_A(\phi_L^A + \phi_R^A) + g_{AB}N_B\phi_C^A,$$
$$i\dot{\phi}_R^A = -\Omega_A\phi_C^A.$$

Moreover, in the limit $|g_{AB}|N_B \gg \Omega_A$, the system of equations (14)–(16) can be further reduced to the coupled equations (see appendix B)

$$i\dot{\phi}_L^A = -\tilde{\Omega}_A\tilde{\phi}_R^A,$$
$$i\dot{\tilde{\phi}}_R^A = -\tilde{\Omega}_A\tilde{\phi}_L^A.$$

\[Figure 5.\] (a) Period of the coherent oscillations $T_A$ in units of $1/\Omega_A$ as a function of the atoms $N_A$. Numerical results of the set of equations (8)–(10) and (11)–(13) (blue squares) $N_A = N_B, g_{AB}/\Omega_A = -3$. The slope of the straight line is given by $T_0 = 9.425(1/\Omega_A)$. The inset shows a linear dependence of the period $T_A$ as a function of $|g_{AB}|$ for $N_A = N_B = 15$. (b) Dimensionless quantity $T_A\Omega_A^2/(N_B|g_{AB}|) \propto |g_{AB}|/\Omega_A$ ($g_{AB} < 0$) for two different set of parameters. (Blue solid line) $N_A = 7$; (orange solid line) $N_A = 5$. The dashed horizontal line corresponds to the constant $\pi$ value. The other parameters are $N_A = 5$ and $g_0/\Omega_A = 10$. 

\[In the QMPD,\] we have seen that coherent BEC oscillations are determined by the strength of the interaction between species. To find out what occurs on the mean-field level, we first analyze the behavior of the period versus the interspecies interaction. In the inset of figure 5(a) we display the behavior of the period as a function of the absolute value of the interaction strength between the species, $|g_{AB}|$, for $N_A = N_B = 15$. It shows a linear scaling of the period $T_A$ with $|g_{AB}|$, which holds regardless of the sign of $g_{AB}$. This is in agreement with the results from QMPD (see section 3.1). However, in contrast to the QMPD results, the interspecies interaction strength $|g_{AB}|$ does not require to be smaller than $g_0$. On the other hand, for small interaction values, it is possible to observe the presence of more than a single harmonic, which is discussed next.

To gain further insight into the relation of the period with the interspecies interaction, we analyze the behavior of the dimensionless quantity $T_A\Omega_A^2/(N_B|g_{AB}|)$ as a function of $|g_{AB}|/\Omega_A$ with $g_{AB} < 0$. The quantity $T_A\Omega_A^2/(N_B|g_{AB}|)$ can provide relevant information associated with the values of $g_{AB}$ in which the coherent BEC oscillations are controlled by the self-trapped atoms. In figure 5(b) it is plotted the behavior of $T_A\Omega_A^2/(N_B|g_{AB}|)$ for two different values of the number $N_B$ of atoms of species B, namely $N_B = 5$ (blue curve) and $N_B = 7$ (orange curve). In this figure, both curves converge to the same constant $\pi$ value for values $|g_{AB}|/\Omega_A \gtrsim 3$. This convergence indicates a definite relation between the period $T_A$, the interspecies interaction strength $|g_{AB}|$ and the number $N_B$ of atoms of species B, as predicted in equation (4) which in turn follows from the QMPD. On the other hand, for small $|g_{AB}|$ values (e.g. $|g_{AB}| / \Omega_A \lesssim 3$ for $N_B = 5$), the curves separate from each other and from the horizontal line that determines the coherent oscillation, as the value $|g_{AB}|/\Omega_A$ decreases. We have checked that such separations are associated with the emergence of more than one harmonic and depend on the number $N_B$ of atoms of species B considered. This behavior is in line with the predictions shown below.

To derive an analytical expression of the coherent BEC oscillation period within the mean-field description, we resort to equations (8)–(13). Assuming $|\phi_C(t)|^2 = N_B$, implies that the interaction between species takes place in the central well only. Thus, the system of equations (8)–(10) reduces to the form

$$i\dot{\phi}_L^A = -\Omega_A\phi_C^A,$$  \hfill (14)
$$i\dot{\phi}_C^A = -\Omega_A(\phi_L^A + \phi_R^A) + g_{AB}N_B\phi_C^A,$$  \hfill (15)
$$i\dot{\phi}_R^A = -\Omega_A\phi_C^A.$$  \hfill (16)

Moreover, in the limit $|g_{AB}|N_B \gg \Omega_A$, the system of equations (14)–(16) can be further reduced to the coupled equations (see appendix B)

$$i\dot{\phi}_L^A = -\tilde{\Omega}_A\tilde{\phi}_R^A,$$  \hfill (17)
$$i\dot{\tilde{\phi}}_R^A = -\tilde{\Omega}_A\tilde{\phi}_L^A.$$  \hfill (18)
where \( \langle \tilde{\phi}_A^L, \tilde{\phi}_B^L \rangle = \langle \phi_A^L, \phi_B^L \rangle \exp[-i \tilde{\Omega}_A t] \) and \( \tilde{\Omega}_A = \frac{\Omega^2_{AB,N_B}}{g_{AB}} \) is the tunneling amplitude. Equations (17) and (18) allow describing the atom dynamics of species A between the left and right wells, as a tunneling process in a BJJ with null particle interaction but with a tunneling amplitude that depends on the number of atoms of species B, \( N_B \).

A natural question that arises here is whether this effective one-dimensional BJJ can emulate coherent BEC oscillations with the timing and number of atoms of a real BJJ [8]. According to the amplitude \( \tilde{\Omega}_A \), an increase in the oscillation period due to a larger number \( N_B \) of atoms can be compensated by reducing the interspecies interaction strength. Thus, considering, e.g. species populations \( N_A \sim N_B \sim 1500 \) atoms with a small interspecies interaction strength \( |g_{AB}|/\Omega_A \sim 0.1 \) and the value \( \Omega_A/h \sim 1.5k\omega \) that follows from reference [8], the effective one-dimensional BJJ predicts the appearance of coherent BEC oscillations with a timing similar to those reported in BJJ experiments [7, 8].

As for the population imbalance, by solving equations (17) and (18) with initial conditions \( \phi_A^L(0) = \sqrt{N_A} \) and \( \phi_B^L(0) = 0 \), one finds that (see appendix B)

\[
z(t) = |\phi_A^L|^2 - |\phi_B^L|^2 = N_A \cos \left( \frac{\omega_0}{N_B} t \right),
\]

where \( \omega_0 = \frac{\pi \Omega^2_{AB,N_B}}{g_{AB}} \). It follows then that \( T_0 = \pi |g_{AB}|/\Omega_A^2 \) which fully agrees with the numerical predictions shown in figure 5(a).

The convergence of the quantity \( T_A \Omega_A^2/(N_B|g_{AB}|) \) to the constant \( \pi \) value for the two different sets of parameters shown in figure 5(b) underlines the generality of the equation (19). Also, as in the QMPD, we find a similar condition \( |g_{AB}|/\Omega_A \gg \sqrt{2}/N_B \) for which equation (19) is fulfilled (see appendix B). In other words, the lower bound of the interaction strength \( |g_{AB}|/\Omega_A \), required for \( z(t) \) to oscillate with the single frequency \( \omega_0/N_B \), becomes smaller as the number of atoms \( N_B \) increases. This explains the dependence on \( N_B \) shown in figure 5(b). On the other hand, when the above condition is not fulfilled, the \( z(t) \) dynamics is described by a biharmonic function, whose frequencies are obtained by solving equations (14)–(16) (see appendix B).

Finally, we note that equations (8)–(13) straightforwardly follow from the mean-field counterpart, equation (C1), of the Hamiltonian equation (1) (see appendix C). So, a one-to-one correspondence between the parameters of both descriptions, equations (1) and (C1) can be found. This demonstrates the equivalence between the coherent-BEC oscillation period, \( T_A = \pi |g_{AB}|N_B/\Omega_A^2 \), from the mean-field regime and equation (4) from the QMPD.

According to the relation for \( T_A \), variations in the number of atoms of species B change the tunneling time between the left and right wells of the atoms of species A. Consequently, changing the number of atoms of species B trapped in the central well during the time evolution may contribute to control the speed of the tunneling process of atoms of species A. The physical realization of this latter scenario is discussed and simulated in the next subsection.

### 3.3. Coherent BEC oscillations in the presence of atom losses in the central well

In previous subsections, we have analyzed the dynamics of the coherent BEC oscillations where the number of self-trapped atoms in the central well is conserved, i.e. \( n_B^C(t) \approx N_B = \text{const} \). Now we address the situation when the number of atoms in the central well changes in time. For that purpose, we consider the scenario in which atoms are removed from the central well. Localized single-particle losses is a dissipative process that can help in the dynamical control of quantum systems [60, 62, 63, 73, 74].

In the presence of dissipation, the dynamics is described by a master equation [61],

\[
\dot{\rho} = -i[H, \rho] + \mathcal{L}\rho,
\]

where \( \rho = |\psi\rangle\langle\psi| \) is the density operator and \( \mathcal{L}\rho \) is a dissipation term. We are especially interested in the effects of localized atoms loss, which can be experimentally implemented by using an electron beam [75, 76] or a resonant laser. Other dissipative effects such as phase noise [77, 78] induced in the atoms of species B are neglected, since the population imbalance dynamics of A atoms is mainly determined by the number of atoms in the central well \( n_B^C \).

Atom losses of species B in the central well can be described by the Liouvillian

\[
\mathcal{L}_{\text{loss}}\rho = -\frac{\gamma}{2}(B_C^\dagger B_C\rho + \rho B_C^\dagger B_C - 2B_C^\dagger B_C\rho),
\]

where \( \gamma \) denotes the loss rate.
Here, rather than solving equations (20) and (21), we simulate the evolution of a stochastic wavefunction whose propagation in time without atom losses is given by

$$|\psi(t + \Delta t)\rangle = e^{-i\hat{H}t\Delta t}|\psi(t)\rangle.$$  

(22)

In equation (22), $\hat{H}$ is an effective Hamiltonian that can be written as

$$\hat{H} = \hat{H}_0 - i\frac{\gamma}{2}\hat{B}_C^\dagger\hat{B}_C,$$

(23)

where $\hat{H}_0$ is the Hermitian Hamiltonian given by equation (1) while the second term is a non-Hermitian contribution due to atom losses.

To mimic the dynamical modification of the system, due to the $B$-species atom losses, we use the quantum trajectory formalism [61, 79, 80]. In particular, we are interested in the observation of quantum jumps that would account for atom losses.

To take into account the quantum jumps in the evolution, the wave function is evaluated using the following algorithm [61, 79]: (i) compute $\delta p = \delta t(\psi(t)|\hat{B}_C^\dagger\hat{B}_C|\psi(t))$. This is the probability of a quantum jump to occur to the state $\hat{B}_C|\psi(t)\rangle$. (ii) Define a random number $\epsilon$ distributed uniformly between zero and one, namely $\epsilon \in (0, 1)$ and compare it to $\delta p$.

If $\epsilon \leq \delta p$, a quantum jump takes place, so

$$|\psi(t + \Delta t)\rangle = \frac{\hat{B}_C|\psi(t)\rangle}{\sqrt{\delta p/\delta t}}.$$  

(24)

If $\epsilon \geq \delta p$ then

$$|\psi(t + \Delta t)\rangle = \frac{e^{-i\hat{H}t|\psi(t)\rangle}}{\sqrt{\langle\psi(t)|e^{-i\hat{H}t|\psi(t)\rangle}}}.$$  

(25)

Let us recall that the wavefunction $|\psi(t)\rangle$ is spanned in the product of the basis of the two species. So, by performing exact diagonalization this set a limit to the initial number $N_B$ of atoms in the central well. Thus, for the sake of convenience, we consider a number of atoms of species $B$ much larger than the atoms of species $A$. We refer to appendix D for information on the exact diagonalization method in the presence of atom losses.

In figure 6, we analyze a quantum trajectory starting with $N_B = 8$. Figure 6(a) shows the stepwise evolution of the number of atoms of species $B$ in the central well. This stepwise behavior is reflected in the frequency of oscillations of the population imbalance of species $A$, as shown in figure 6(b). As expected, the frequency of the oscillations increases as the number of atoms in the central well decreases. A characterization of the changing frequency behavior during the evolution of the coherent oscillation signal can be achieved through a time-frequency analysis [81, 82]. In figure 6(c) is displayed the intensity distribution of a Continuous Wavelet Transform (CWT) as a function of time and $T_{\text{CWT}}$ of the signal plotted in figure 6(b). $T_{\text{CWT}}$ is the inverse of the frequency in the CWT. The jumps in $T_{\text{CWT}}$ are clearly associated with the jumps in $n_B^C$ and thus could be used to detect atom losses in the central well. In the inset of figure 6(c) is plotted the maximal values of the CWT intensity for the entire time domain as a function of $T_{\text{CWT}}$. In the plot we observe three peaks separated by the same $T_0$, namely the proportionality factor between $T_0$ and $N_B$ predicted by the theory.

Let us now consider initial larger number of atoms in the central well, for which the dissipative term in equation (23) predicts an increase in the probability of atom losses. Figure 7 shows the evolution of the population for two quantum trajectories with different initial values of $N_B$. The red curves show the evolution of the period of the maximal value of the intensity of the CWT ($T_{\text{CWT}}^{\text{max}}$) normalized by $T_0$ for each quantum trajectory. Thus, the ratio $T_{\text{CWT}}^{\text{max}}/T_0$ obtained from the CWT of the coherent oscillations, provides an stepwise estimation of the atom population in the central well. To check the pertinence of the stepwise estimation $T_{\text{CWT}}^{\text{max}}/T_0$ in figure 7, we plot also the population of atoms of species $B$ in the central well, $n_B^C(t)$. By comparing the $T_{\text{CWT}}^{\text{max}}/T_0$ and $n_B^C(t)$ behavior in each quantum trajectory, we observe that, in fact, the curve, $T_{\text{CWT}}^{\text{max}}/T_0$, correctly follows the stepwise $n_B^C(t)$ evolution. Notice that both behaviors are stepwise.

Jumps in the function $T_{\text{CWT}}^{\text{max}}/T_0$, however, are described by smooth decays, which are due to the time-frequency uncertainty in the wavelet transform [82]. Thus, estimating the quantities $T_{\text{CWT}}^{\text{max}}(t)$ and $T_0$ enables determining the number of atoms of species $B$ at each moment in time, i.e. $n_B^C(t)_{\text{CWT}} = T_{\text{CWT}}^{\text{max}}(t)/T_0$.

In figure 7, fast atom losses produce a rapid change in the oscillation period which continuously change the tunneling speed of the atoms of species $A$. An example of a rapid changing oscillation period of $z(t)$ is shown in the inset of figure 7. This behavior is characterized by a steep drop of the function $T_{\text{CWT}}^{\text{max}}/T_0$.
Figure 6. Quantum trajectory in the presence of atom losses in the central well with initial number of atoms $N_B = 8$; (a) population of the atoms of species $B$ in the central well as a function of time, $n_B(t)$; (b) Fractional population imbalance $z(t)/N_A$ vs time; (c) intensity plot of the CWT vs time vs period $T_{CWT}$ (inverse of the CWT frequency) for the curve shown in panel (b). Inset shows the maximal values of CWT for the entire time domain vs the period $T_{CWT} \in (50, 83)$. The spectral peaks appear separated by the same time interval $9.42(1/J_A)$. The other parameters are $\gamma/J_A = 10^{-4}$, $U_B/J_B = 10$, $U_{AB}/J_A = -3$, and $J_A = J_B$.

Figure 7. Period of the maximal value of the CWT, $T_{CWT}^{\text{max}}$, divided by the characteristic period $T_0$ as a function of time for two quantum trajectories with initial $N_B = 20$ and $N_B = 25$ atoms in the central well (red solid lines). Superimposed appear the curves of the populations of the species $B$ in the central well as a function of time, $n_B(t)$. The inset shows the evolution of the fractional population imbalance for a time interval of the curve in blue in the main panel. The other parameters are the same as in figure 6.

associated with the $n_B^0(t)$ evolution depicted in blue in the main panel. These results not only confirm the connection between $T_A$ and $N_B$ predicted in equation (4), but also allow proposing a new mechanism for the control of the tunneling speed of atoms in atom triple-well potentials.

4. Conclusions and outlook

We have investigated coherent oscillations of one BEC species emerging from the interaction with a second species in a triple-well potential. It was shown that one BEC species, self-trapped in the central well, enables
controlling the coherent oscillations of a non-self-interacting species between the left and right wells of the triple-well potential. Moreover, it was found, in this engineered setup, that the period of the coherent BEC oscillations scales linearly with both, the number of atoms trapped in the central well and the interspecies interaction strength. This behavior was demonstrated by the quantum many-particle dynamics and further corroborated by the mean-field dynamics.

We have also investigated the presence of atom losses from the central well and their effect on the coherent BEC oscillation period. By using quantum trajectories, we showed that fluctuations in the atom number from the central well can be associated with variations of the oscillation period, thus making possible to measure the atoms lost from the system.

In this regard, coherent BEC oscillations studied here can potentially be used as a method for accurately estimate the number of atoms in a trap [83]. Very importantly, it provides a mechanism to detect fluctuations in the number of atoms, which can be used for sensing purposes. A peculiar feature of this mechanism is that, unlike other atom detection techniques [84, 85], this one relies on atoms of a second species to achieve it.

With the current proposal, new avenues are opened up for the implementation of coherent BEC oscillations that are controlled by another BEC species. On one side, the implementation of a gate-like BEC species in atomic transistor-like systems [86] could significantly improve the control over the tunneling of atoms in future devices. On the other side, the atom detection mechanism proposed here, could find applications in the fields of ultracold atoms and quantum information processing, where reliable methods of atom detection are essential [87, 88]. Likewise, similar mechanism could be applicable in atomtronics circuits [89–93] in the detection of matter wave motion.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Oscillatory modes in the quantum many-body description

To gain insight in the frequency of the coherent BEC oscillations, we analyze the exactly solvable case of a constant number $N_B$ of atoms of species $B$ in the central well and $N_A = 1$. In this case the basis of the wavefunction equation (2) consists in Fock states $|1, 0, 0\rangle_A |0, N_B, 0\rangle_B, |0, 1, 0\rangle_A |0, N_B, 0\rangle_B$ and $|0, 0, 1\rangle_A |0, N_B, 0\rangle_B$ of amplitudes $C_j$ with $j = 1, 2, 3$.

In this reduced basis, the Schrödinger equation with the Hamiltonian (1) can be written as

$$i\dot{C}_j = H_{j,s} C_s \quad (A1)$$

where $j, s = 1, 2, 3$. Here, $H_{j,s}$ are the matrix elements of the matrix

$$H = \begin{pmatrix} 0 & -J_A & 0 \\ -J_A & U_{AB}N_B & -J_A \\ 0 & -J_A & 0 \end{pmatrix}, \quad (A2)$$

where a diagonal term $U_{AB}N_B(N_B - 1)$ is absorbed by the phase of the vector $\tilde{C}$. From solving the secular equation for the matrix $H$, we obtain the eigenvalues equation

$$\lambda(\lambda^2 - \lambda U_{AB}N_B - 2J_A^2) = 0 \quad (A3)$$

with three roots, $\lambda_1 = 0, \lambda_{2,3} = \frac{U_{AB}N_B \pm \sqrt{U_{AB}^2N_B^2 + 8J_A^2}}{2}$. The respective eigenvectors are $\{-1, 0, 1\}$, $\{1, -\frac{1 - \sqrt{1 + 8x^2}}{2x}, 1\}$ and $\{1, -\frac{1 + \sqrt{1 + 8x^2}}{2x}, 1\}$, where $x = J_A/N_B U_{AB}$. 

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In the case $|U_{AB}|N_B \gg I_A$, i.e. $|x| \ll 1$, the roots can be approximated to $\lambda_2 = U_{AB}N_B$ and $\lambda_3 = -\frac{2U}{U_{AB}N_B}$, while the eigenvectors after normalization can be approximated to $\sqrt{\frac{1}{2+\left(\frac{U}{U_{AB}N_B}\right)^2}} \left\{ \frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}}, \frac{\lambda_2}{\sqrt{\lambda_1^2+\lambda_2^2}} \right\}$, and

$$\frac{1}{\sqrt{2+\left(\frac{U}{U_{AB}N_B}\right)^2}} \left\{ 1, \frac{-2U}{U_{AB}N_B} \right\}.$$

In the above scenario, we have considered a constant number of atoms $N_B$ in the central well. However, tunneling of atoms of species B toward the neighboring wells is possible due to the hopping term with amplitude $J_B$ in the Hamiltonian (1). In this case, Fock states of the manifold $N_B-1$ may contribute to change the dynamics of the population imbalance $z(t)$.

Let us analyze, as above, the oscillation dynamics of one atom of species A. For the sake of understanding, we rewrite the tensor product in equation (2) in the basis of the three energy eigenstates in the energy spectrum of the Hamiltonian equation (1). So, to numerically test the associated with the state the state may still show some overlap with the initial wavefunction. The interspecies interaction energy $|\Delta\rangle$ diagonal terms of the states interaction should fulfill the relation

$$\tilde{\Delta}_i, \tilde{\Delta}_s = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2}} \frac{\lambda_1}{\sqrt{\lambda_1^2+\lambda_2^2}} 0 0 0 \lambda_1 \lambda_2 \begin{pmatrix}
Q_1 & -J_A & 0 & -J_B \sqrt{N_B} & 0 & 0 & 0 & 0 \\
-J_A & Q_2 & -J_A & 0 & -J_B \sqrt{N_B} & 0 & 0 & 0 \\
0 & -J_A & Q_3 & 0 & 0 & -J_B \sqrt{N_B} & 0 & 0 \\
-J_B \sqrt{N_B} & 0 & 0 & -J_A & U_{AB}N_B & -J_A & 0 & -J_B \sqrt{N_B} \\
0 & -J_B \sqrt{N_B} & 0 & -J_A & U_{AB}N_B & -J_A & 0 & -J_B \sqrt{N_B} \\
0 & 0 & -J_B \sqrt{N_B} & 0 & -J_A & 0 & 0 & Q_3 & -J_A \\
0 & 0 & 0 & -J_B \sqrt{N_B} & 0 & -J_A & 0 & 0 & Q_2 \\
0 & 0 & 0 & 0 & -J_B \sqrt{N_B} & 0 & -J_A & Q_2 & -J_A
\end{pmatrix},$$

where $Q_1 = -U_B(N_B-1) + U_{AB}$, $Q_2 = -(U_B - U_{AB})(N_B - 1)$ and $Q_3 = -U_B(N_B - 1)$. Here as well, the diagonal term $U_{AB}N_B$ is absorbed by the phase of the vector $C$.

Let us consider $|U_{AB}|N_B \gg I_A$ and $U_{AB}N_B \gg J_B$ with $I_A = J_B$. In this scenario, the diagonal terms in the $H$ matrix, related to the states $|0, N_B-1, 0\rangle_B \otimes |1\rangle_A$ and $|1, N_B-1, 0\rangle_B \otimes |1\rangle_A$, are largely shifted from the diagonal terms of the states $|0, N_B, 0\rangle_B \otimes |1\rangle_A$. Here as well, the energy gap between the manifolds $N_B$ and $N_B-1$ is mainly provided by the interspecies interaction $\Delta E_B = U_B(N_B-1)$. This large energy gap implies that states $|0, N_B-1, 1\rangle_B \otimes |1\rangle_A$ and $|1, N_B-1, 0\rangle_B \otimes |1\rangle_A$ are hardly occupied at any time and any transition to these states must quickly return to the states $|0, N_B, 0\rangle_B \otimes |1\rangle_A$. Therefore, it enables performing adiabatic elimination of the manifold $N_B-1$. By doing so, one realize that for $U_{AB} \sim U_B$, some diagonal matrix elements such as $Q_3$ in the $H$ matrix may vanish, leading to a break down of the adiabatic approximation for some states. In the case, $N_A > 1$, a similar diagonal element as above $Q_3 = -(U_B - N_A U_{AB})(N_B - 1)$ vanishes for $U_{AB} = U_B/N_A$. So, in order to eliminate adiabatically all the states of the manifold $N_B-1$, the strength of the interspecies interaction should fulfill the relation $U_{AB} \lesssim U_B/2 = U_B/(2N_A)$.

In the case $U_{AB} < 0$, one can also find a region of parameters where the adiabatic process fails. To gain insight let us define the characteristic state of the manifold $N_B$, $|\psi\rangle_A = |N_A - s, s, 0\rangle_A \otimes |0, N_B, 0\rangle_B$, where $s = 0, 1, 2, \ldots, N_A - 1$. For $s = 0$, the state $|\psi\rangle_B$ is the same initial wavefunction. However, for small $s > 0$, the state may show some overlap with the initial wavefunction. The interspecies interaction energy associated with the state $|\psi\rangle_A$ is $-sU_{AB}|N_B|$. Thus, the energy gap between the state $|\psi\rangle_A$ and the manifold $N_B-1$ becomes $\Delta_1 = U_B(N_B - 1) - sU_{AB}|N_B|$. Here, we have taken into account that the largest energy of the manifold $N_B-1$ is $U_B(N_B - 1)/(N_B - 2)/2$, by neglecting any contribution coming from the kinetic energy. For the case $N_A = 1$, the energy gap $\Delta_1 = U_B(N_B - 1)$ is constant for any $U_{AB} < 0$. Therefore, all the states related to the manifold $N_B-1$ can be adiabatically eliminated. In the case, $N_A = 2$, we find that for $s = 1$, the energy gap $\Delta_1$ vanishes for $U_{AB} = -U_B(N_B - 1)/N_B$. This implies that, when $|U_{AB}| \lesssim U_B$, the adiabatic approach breaks down for some states of the manifold $N_B-1$, which affects the dynamics of the coherent BEC oscillations. According to this analysis, when $N_A > 2$, the energy gap $\Delta_1$ between the manifolds $N_B$ and $N_B-1$ is further reduced. However, in general, for values $s > 1$, the overlap of the initial wavefunction with $|\psi\rangle_A$ becomes negligible regardless of $N_A$. On the other hand, we should note that the state $|\psi\rangle_B$ as well as the states $N_B-1$ and $N_B$, share the same Fock states and energies of some of the eigenstates in the energy spectrum of the Hamiltonian equation (1). So, to numerically test the aforementioned energy gap between the manifolds $N_B-1$ and $N_B$, we look into the overlap of the initial wavefunction with the eigenstates of the system. In figure 8 are displayed the probabilities $p_i = |\langle \psi(0)|\phi_i\rangle|^2$ of finding the initial wavefunction in the eigenstates $|\phi_i\rangle$ of the Hamiltonian equation (1) for two A atoms.
Figure 8. Probabilities $p_i = |\langle \psi(0) | \phi_i \rangle|^2$ of finding the initial wavefunction in an eigenstate $|\phi_i \rangle$ of the Hamiltonian equation (1) for two cases $N_A = 2$ (red dots) and $N_A = 6$ (green dots). The initial wavefunction is $|\psi(0) \rangle = |\psi_0 \rangle = |N_A, 0, 0 \rangle \otimes |0, N_B, 0 \rangle$, which consists in $N_B$ atoms in the central well and $N_A$ atoms in the first well. The parameters are $U_{AB}/J_A = -3, U_B/J_B = 10$ and $N_B = 6$.

number $N_A = 2$ and $N_A = 6$ with parameters $U_{AB}/J_A = -3, U_B/J_B = 10$ and $N_B = 6$. In this picture appear indicated the energy values due to the intraspecies interaction of the manifold $N_B, E_{N_B} = 150J_B$, and of the manifold $N_B - 1, E_{N_B-1} = 100J_B$. Around these two energy values, appear a bunch of eigenstates, whose spreading is characterized by the kinetic energy of the atoms of species A. Very importantly, since the energy difference $E_{N_B} - E_{N_B-1} = 50J_B$ is approximately three times $|U_{AB}|N_B = 18J_B$, it allows the adiabatic elimination of the states of the manifold $N_B - 1$ in both cases $N_A = 2$ and $N_A = 6$. Here, one should bear in mind that increasing the number of A atoms leads to a spreading in energy, as e.g. shown in the case of $N_A = 6$ around the energies $E_{N_B}$ and $E_{N_B-1}$ in figure 8. Thus, increasing $N_A$ further can reduce the gap between the energy bands corresponding to the manifolds $N_B - 1$ and $N_B$ to a critical value. So, to prevent a failure of the adiabatic approximation, the number of atoms of species A cannot largely exceed the number of atoms of species B and the interspecies interaction must meet the condition $|U_{AB}| \lesssim U_B/3$.

Appendix B. Frequency dependence on the number of atoms: mean-field description

In this section we derive the analytical expression of the population imbalance and its frequency for the oscillations of the atoms of the species A using the mean-field framework. To this end, we analyze the system of equations (14)–(16)

$$i \dot{\phi}_A = -\Omega_A \phi_C,$$
$$i \dot{\phi}_C = -\Omega_A(\phi_A + \phi_B) + g_{AB}N_B \phi_A,$$
$$i \dot{\phi}_B = -\Omega_A \phi_C.$$

By making the substitution $\Psi = \phi_A + \phi_B$ into the equations above, and doing some algebra one finds that $\Psi$ can be described by the second-order dynamical equation

$$\ddot{\Psi} + i g_{AB}N_B \dot{\Psi} + 2\Omega_A^2 \Psi = 0.$$ (B4)

Proposing $\Psi = \Psi_0 e^{i\lambda t}$ leads to the secular equation

$$\lambda^2 + g_{AB}N_B \lambda - 2\Omega_A^2 = 0,$$ (B5)

where the roots are

$$\lambda_{\pm} = -M \pm \left(M^2 + 8\Omega_A^2\right)^{1/2}.$$ (B6)

being $M = g_{AB}N_B$. In this latter equation, taking into account $|g_{AB}|N_B \gg \Omega_A$, the roots become

$$\lambda_+ \approx \omega_+ = \frac{2\Omega_A^2}{g_{AB}N_B} \quad \text{and} \quad \lambda_- \approx \omega_- = -g_{AB}N_B.$$
The solution for $\Psi$ can then be expressed as
\[ \Psi = a_+ e^{i \omega_+ t} + a_- e^{-i \omega_- t} \approx a_+ \exp(i \omega_+ t), \] (B7)
where the second oscillatory term with the root $\omega_- = -g_{AB} N_B$ is neglected because of its fast oscillating contribution. That is equivalent to say that $|\omega_+| \ll |\omega_-|$, which yields the condition $|g_{AB}| \gg \sqrt{\Omega_A/N_B}$.

On the other hand, in the limit $|g_{AB}| N_B \gg \Omega_A$, the amplitude $\phi^L_\theta$ can be adiabatically eliminated from equations (B1)–(B3) by making $\phi^C_\theta = 0$. By doing so, the system of equations (B1)–(B3) is reduced to
\begin{align}
\frac{i}{2} \phi^L_\theta &= \frac{\omega_+}{2} \phi^L_\theta, \\
\frac{i}{2} \phi^R_\theta &= \frac{\omega_-}{2} \phi^R_\theta, \\
\end{align}
where $(\phi^L_\theta, \phi^R_\theta) = (\phi^L_\theta, \phi^R_\theta) \exp[-i \omega_+ t/2]$.

Considering the initial conditions $\phi^L_\theta(0) = \sqrt{N_A}$ and $\phi^R_\theta(0) = 0$, it is straightforward to find
\begin{align}
\phi^L_\theta(t) &= \sqrt{N_A} \cos \left( \frac{\omega_+ t}{2} \right) e^{i \omega_+ t/2}, \\
\phi^R_\theta(t) &= i \sqrt{N_A} \sin \left( \frac{\omega_+ t}{2} \right) e^{i \omega_+ t/2}. \\
\end{align}
(B9)

Notice that $\phi^L_\theta + \phi^R_\theta = \sqrt{N_A} e^{i \omega_+ t}$, which is in agreement with equation (B7).

Having obtained the solutions of the oscillating amplitudes in the left and right wells, we can finally write the population imbalance as
\[ z(t) = |\phi^L_\theta|^2 - |\phi^R_\theta|^2 = N_A \left[ \cos^2 \left( \frac{\omega_+ t}{2} \right) - \sin^2 \left( \frac{\omega_+ t}{2} \right) \right] = N_A \cos \left( \frac{2\Omega^2}{|g_{AB}| N_B} t \right). \] (B12)

**Appendix C. Mean-field Hamiltonian and its quantum many-particle counterpart**

The system of equations (8)–(10) and (11)–(13) stem from the mean-field Hamiltonian
\[ H = -\Omega_A \left[ \phi^L_\theta (\phi^L_\theta)^* + \phi^C_\theta (\phi^C_\theta)^* + \phi^R_\theta (\phi^R_\theta)^* + \phi^C_\theta (\phi^C_\theta)^* \right] - \Omega_B \left[ (\phi^L_\theta (\phi^L_\theta)^*) + (\phi^C_\theta (\phi^C_\theta)^*) + (\phi^R_\theta (\phi^R_\theta)^*) + (\phi^C_\theta (\phi^C_\theta)^*) \right] + g_{AB} \left[ |\phi^L_\theta|^2 |\phi^C_\theta|^2 + |\phi^R_\theta|^2 |\phi^C_\theta|^2 + |\phi^C_\theta|^2 |\phi^R_\theta|^2 \right] + \frac{g_{AB}^2}{2} \left[ |\phi^L_\theta|^4 + |\phi^C_\theta|^4 + |\phi^R_\theta|^4 \right]. \] (C1)

Following a standard quantization procedure [94, 95] by replacing the $(\phi^L_\theta, \phi^R_\theta)$-number amplitudes $(\phi^L_\theta, \phi^C_\theta, \phi^R_\theta, \phi^C_\theta, \phi^R_\theta, \phi^C_\theta)$, and their complex conjugates, by the Bose operators $(\hat{A}_L, \hat{A}_C, \hat{A}_R; \hat{B}_L, \hat{B}_C, \hat{B}_R)$ which annihilate a particle at the respective site, and their adjoint creation operators, subject to the commutation rules
\[ [\hat{D}_Y, \hat{D}^\dagger_Y] = \delta_{YY}, \text{ being } \hat{D} = \hat{A}, \hat{B} \quad \text{and} \quad Y, \tilde{Y} = L, C, R, \] (C2)
leads to the QMPH
\[ \hat{H} = -\Omega_A \left[ A_L A_L + A_C A_L + A_R A_L + A_R A_R \right] - \Omega_B \left[ B_L B_L + B_C B_L + B_R B_L + B_R B_R \right] + g_{AB} \left[ A_L A_L B_L B_L + A_C A_C B_C B_C + A_R A_R B_R B_R \right] + \frac{g_{AB}^2}{2} \left[ B_L B_L B_L B_L + B_C B_C B_C B_C + B_R B_R B_R B_R \right]. \] (C3)

**Appendix D. Some computational aspects**

In the evolution of the quantum many-particle system, we use a second-order Suzuki–Trotter decomposition method with a time step $\Delta t = 0.01$, while for the mean-field dynamics we use a fourth-order Runge–Kutta method with time steps as small as $\Delta t = 10^{-5}$.

Let us discuss some aspects of the exact diagonalization of the Hamiltonian equation (1) and the specifics of the Hilbert space $\mathcal{H}$ for this system. For two interacting species of condensates, labelled $A$ and $B$, the dimension of $\mathcal{H}$ is given by the product of the dimension of the two corresponding subsystems, i.e.
dim_{H} = \dim_{A} \times \dim_{B}. The dimension of the subsystem corresponding to the bosonic species \( D \) with \( N_D \) number of atoms and \( M_D \) wells is \( \dim_{0} = \left( \frac{N_D + M_D - 1}{2} \right)^2 \).

A relevant issue in the exact diagonalization problem is the selection of a proper basis where the Hamiltonian can be written in matrix form. Available algorithms in the literature for diagonalization of Bose–Hubbard systems rely on some sort of ordering \([96, 97]\) of the basis states \( \{|n_1, n_2, \ldots, n_M\} \). These algorithms are quite flexible and can be readily extended to large sizes. However, they provide little or none insight into the block structure of the Hamiltonian matrix of the system.

Here, the wavefunction equation (2) is spanned such that the basis of each species \( \{|n_1, n_C, n_R\} \) is built using the constraint \( n_1 + n_C + n_R = N \), being \( N \) a fixed total number of atoms of the species. In this basis, the Hamiltonian matrix for each subsystem has a tridiagonal block structure, with each diagonal block being a dimer matrix \([98]\). This facilitates the initial preparation of the populations in the numerical experiment. Moreover, in the case of atom losses, this basis represents a big advantage, since the removal and deposition of atoms in the wells can be pinpointed across the integration scheme.

Exact diagonalization in the presence of atoms losses can be implemented by adding an independent well to the otherwise triple-well system, so that the extra well can be used as a depository of atoms. However, the introduction of an extra well can be computationally costly when dealing with two species. For example, in the case of \( N_B = 25 \) and \( M_B = 4 \), one gets for the species \( B, \dim_B = 3276. \) So, considering only \( N_A = 2 \) with three wells, one gets that \( \dim_H = 196 \times 96 \). Therefore, the evolution requires at every time step the diagonalization of a matrix with several hundred of millions of matrix elements.

As shown in the main text, coherent oscillations take place in the parameter region \( \sqrt{\gamma_{12}}/N_B \ll |U_{AB}| \ll U_B/3 \). In this region, interactions between atoms of species \( A \) and \( B \) outside the central well are negligible or irrelevant for the population imbalance dynamics. This is key because it means that the interactions between atoms of distinct species in the left and right wells are not necessary and therefore can be ‘turned off’ in the numerical experiment.

Turning off the interactions between species in the left and right wells can be advantageous since it opens the opportunity to use these wells as a depository of atoms extracted from the central well. This means, that the dynamics of species \( B \) in the presence of losses can, in fact, be reduced to three wells. This artifice works as long as the number of atoms of species \( B \) deposited in the left or right wells do not tunnel to the central well. However, in most of cases this holds, since atoms of species \( B \) in the central well can block the tunneling of atoms to the central well. On the other hand, atoms of species \( B \) deposited in the left or right wells can localize in these wells due to self-trapping effect.

In the case of three wells with \( N_B = 25 \) and \( N_A = 2 \) one gets that \( \dim_H = 2106 \), leading to a matrix with few millions of matrix elements, two order of magnitude lower than in the case that involves four wells for the species \( B \).

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