Determinants of the Calabi-Yau Metrics on K3 Surfaces, Discriminants, Theta Lifts and Counting Problems in the A and B Models

To Serge Lang with deep respect

Andrey Todorov
UC Santa Cruz
Department of Mathematics
Santa Cruz, CA 95064
Institute of Mathematics
Bulgarian Academy of Sciences
Sofia, Bulgaria

March 8, 2022

Abstract

The Dedekind eta functions plays important role in different branches of Mathematics and Theoretical Physics. One way to construct Dedekind Eta function to use the explicit formula (Kroncker limit formula) for the regularized determinants of the Laplacian of the flat metric acting of (0,1) forms on elliptic curves. The holomorphic part of the regularized determinant is the Dedekind eta functions. In this paper we generalized the above approach to the case of K3 surfaces. We give an explicit formula of the regularized determinants of the Laplacians of Calabi Yau metrics on K3 Surfaces, following suggestions by R. Borcherds. The holomorphic part of the regularized determinants will be the higher dimensional analogue of Dedekind Eta function.

We give explicit formulas for the number of non singular rational curves with a fixed volume with respect to a Hodge metric in the case of K3 surfaces with Picard group unimodular even lattice by using the holomorphic part exp Φ_{3,19} of the regularized determinants det Δ_{0,1}.

We gave the combinatorial interpretation of the restriction of the automorphic form exp Φ_{3,19} on the moduli of K3 surfaces with unimodular Picard lattices in the A and B models. The results obtained in this paper are related to some results of Bershadsky, Cecotti, Ouguri and Vafa. See [7].
## Contents

1 Introduction .......................... 3
  1.1 Acknowledgements ..................... 6

2 Symmetric Spaces $h_{p,q} := SO_0(p,q)/SO(p) \times SO(q)$
  2.1 Global Flat Coordinates on the Symmetric Space $h_{p,q}$ ........... 6
  2.2 Decomposition of $h_{p,q}$ ................... 8
  2.3 Definition of the Standard Metric on $h_{p,q}$ ..................... 9

3 Discriminants in $h_{p,q}$
  3.1 Definition and Basic Properties of the Discriminant ............... 9
  3.2 The Irreducibility of the Discriminant .......................... 10

4 Moduli of K3 Surfaces ......... 14
  4.1 Moduli of Marked, Algebraic and Polarized K3 surfaces ............ 14
  4.2 Moduli Space of K3 Surfaces with a B-Field ....................... 15
  4.3 Discriminant of Pseudo-Polarized K3 Surfaces ..................... 15

5 Mirror Symmetry ......... 16
  5.1 Mirror Symmetry for K3 Surfaces .......................... 16
  5.2 Mirror Symmetry and Algebraic K3 Surfaces ........................ 17
  5.3 The Mirror Map for Marked M-K3 ............................... 18

6 Automorphic Forms on $\Gamma \backslash h_{p,q}$, Theta Lifts and Regularized Determinants of CY metrics on K3 Surfaces ... 19
  6.1 General Facts about Regularized Determinants .................... 19
  6.2 Special Automorphic Form of Weight -2 on $\Gamma \backslash h_{p,q}$ ........ 20
  6.3 Theta Lift and Automorphic Form with a Zero Set Supported by the Discriminant Locus on $\Gamma \backslash h_{3,19}$ .................... 21
  6.4 The Analogue of the Kronecker Limit Formula for the Regularized Determinants on K3 Surfaces .......................... 22

7 Harvey-Moore-Borcherds Products and Counting Problems in A and B Models .... 25
  7.1 Counting Problems on K3 .......................... 25
  7.2 A and B Models ............................... 27

8 The Canonical Class of the Moduli Space of Polarized Algebraic K3 ... 29
  8.1 The Projection Formula .......................... 29
  8.2 The Divisor of the Restriction of the Automorphic Form on $\mathcal{M}_{K3,n}$ 33
1 Introduction

The Dedekind eta function

\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \]

where \( q = e^{2\pi i \tau} \) plays a very important role in different branches of mathematics. It is closely related to the study of the moduli of elliptic curves. One way to construct Dedekind Eta function in case of elliptic curves is to use the explicit formula for the regularized determinants of the Laplacian of the flat metric acting of (0,1) forms. See [6]. The holomorphic part of the regularized determinant is the Dedekind eta functions.

In this paper we generalized the above approach to the case of K3 surfaces. We give an explicit formula of the regularized determinants of the Laplacians of Calabi Yau metrics on the moduli space of Calabi-Yau metrics on the K3 surface, following suggestions by R. Borcherds. The holomorphic part of the regularized determinants will be the higher dimensional analogue of Dedekind Eta function.

Next we will review the moduli theory of K3 surfaces. It was A. Weil who outline the main problems in the study of the moduli of K3 surfaces. See [29]. The first main result in the study of moduli of K3 surfaces is due to Shafarevich and Piatetski-Shapiro. See [25]. They proved the global Torelli Theorem for polarized algebraic K3 surfaces. Combining the Theorem of Shafarevich and Piatetski Shapiro with the description of the mapping class group of K3 surface one obtain that the moduli space \( \mathcal{M}_{K3,n} \) of polarized algebraic K3 surfaced with a polarization class \( e \) such that \( \langle e, e \rangle = 2n > 0 \) is a Zariski open set in

\[ \Gamma_{K3,n}^+ \backslash SO(2,19)/SO(2) \times SO(19), \]

where \( \Gamma_{K3,n}^+ \) is an index two subgroup in the group of the automorphisms \( O_{\Lambda_{K3,n}}^*(\mathbb{Z}) \) of the lattice \( H^2(M,\mathbb{Z}) \) which is isomorphic to

\[ \Lambda_{K3,n} := -2n\mathbb{Z} \oplus U^2 \oplus E_8(-1) \oplus E_8(-1). \]

In [27] it was proved that every point of \( SO(3,19)/SO(2) \times SO(1,19) \) corresponds to a marked K3 surface. Based on this result in [22] it was proved that the moduli space of Ricci flat metrics on K3 surfaces with a fixed volume is isomorphic to

\[ \mathcal{M}_{KE} := \Gamma^+ \backslash (SO_0(3,19)/SO(3) \times SO(19) - \mathcal{D}_{KE}), \]

where \( \Gamma^+ \) is a subgroup of index 2 in the group of automorphisms of the group of the automorphisms of the Euclidean lattice \( \Lambda_{K3} = U^3 \oplus E_8(-1) \oplus E_8(-1), \) where

\[ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

is the hyperbolic lattice and \( E_8(-1) \) is the standard lattice and \( \mathcal{D}_{KE} \) is the subspace whose points correspond to Ricci flat metrics on orbifolds. Donaldson
proved in [12] that the mapping class group $\Gamma$ of a K3 surface is a subgroup of index 2 in the group of the automorphisms of the Euclidean lattice $\Lambda_{K3}$.

Mirror Symmetry is based on the observation that there are two different models A and B in string theory which define one and the same partition function. The A model is related to the deformation of the Kähler-Einstein metrics. The B-model is related to the deformations of complex structures. To study mirror symmetric on K3 surfaces we need to define a B-field on a K3 surface $X$ such that

$$\int_X \text{Im} \omega \wedge \text{Im} \omega > 0.$$ 

The moduli space of marked K3 surfaces with a B-field is isomorphic to $h_{4,20} := \mathcal{O}(\Lambda_{K3})/SO(3) \times SO(20)$. Aspinwall and Morrison proved that the moduli space of Super Conformal Field Theories with supersymmetry $(4,4)$ is described by $\Gamma^+ \setminus h_{4,20}$, where $\Gamma^+$ is a subgroup of index two in $\mathcal{O}(\Lambda_{K3})$. It is well known that $h_{4,20}$ parametrizes the four-dimensional oriented subspaces in $\mathbb{R}^{4,20}$ on which the bilinear form is strictly positive. See [1]. To a pair $(X, \omega_X(1,1))$ of a marked K3 surface with a B-field $\omega_X(1,1)$ we assign a oriented four dimensional subspace $E_{X,\omega_X(1,1)}$ in

$$H^*(X,\mathbb{Z}) \otimes \mathbb{R} = (H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})) \otimes \mathbb{R}$$

on which the bilinear form defined by the cup product is positive. We will assume that $(H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})) = U_0$ and the B-field $\omega_X(1,1)$ we will be identified with

$$\left(1, -\frac{1}{2} (\omega_X(1,1) \wedge \omega_X(1,1))\right) \in H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}). \quad (1)$$

From now on we will consider the B-field $\omega_X(1,1)$ as defined by (1). The four dimensional subspace $E_{X,\omega_X(1,1)}$ contains the two dimensional subspace $E_{\omega_X}$ spanned by $\text{Re} \omega_X$ and $\text{Im} \omega_X$, where $\omega_X$ is the holomorphic two form on $X$ defined up to a constant and the two dimensional subspace $E_{\omega_X(1,1)}$ spanned by $\text{Re} \omega_X(1,1)$ and $\text{Im} \omega_X(1,1)$, where $\omega_X(1,1)$ is defined by (1). $E_{\omega_X}$ will the orthogonal to $E_{\omega_X(1,1)}$ in $E_{X,\omega_X(1,1)}$.

Mirror Symmetry is pretty well understood in the case of K3 surfaces. See [1], [13] and [28]. The mirror symmetry is exchanging $E_{\omega_X}$ with $E_{\omega_X(1,1)}$. Special case of mirror symmetry of algebraic K3 surfaces was studied in details in [13].

In this paper we will consider the moduli space of K3 surfaces with $B$-fields. We prove the existence of an automorphic form exp ($\Phi_{4,20}$) which vanishes on the totally geodesic subspaces that are orthogonal to $-2$ vectors form following [11].

The regularized determinants of the Laplacian of Ricci flat metrics $\det(\Delta_{KE})$ acting on $(0,1)$ forms will be a function on on the moduli space of Einstein metric

$$\mathcal{M}_{KE} = \mathcal{O}_{\Lambda_{K3}}^+ \setminus \mathcal{O}_0(3,19)/\mathcal{O}(3) \times \mathcal{O}_0(19).$$
R. Borcherds suggested that one can compute the determinants of the Laplacians of Ricci flat metrics explicitly by using the method of the theta lifts. See [11]. In this paper we will give an explicit expression of the regularized determinants of the Laplacians of CY metrics $\text{det}$ as a function on the moduli space of Einstein metrics $M_{KE}$.

The restriction of $\exp(\Phi_{4,20})$ on the moduli space of elliptic K3 surfaces with a section

$$M_{\text{ell}} := \Gamma_{\text{ell}} \setminus \mathfrak{h}_{2,10}$$

vanishes on the discriminant locus

$$D_{\text{ell}} \subset M_{\text{ell}} \subset \Gamma_{\text{ell}}^+ \setminus \mathfrak{h}_{4,20}$$

which is defined by the points orthogonal to $-2$ vectors. The mirror $Y$ of the elliptic K3 $X$ with the section has Picard group $\text{Pic}(Y) = \mathbb{U} \oplus \mathbb{E}(-1) \oplus \mathbb{E}(-1)$. $\exp(\Phi_{4,20})$ restricted on a line $tL$ in the Kähler cone $K(Y)$ spanned by the imaginary part $L$ of a Hodge metric, has a Fourier expansion. The Fourier coefficients $a_n$ of

$$\frac{d}{dt} \log \Phi(it)$$

in front of $\frac{\exp(-int)}{1-\exp(-int)}$ are positive integers and they count the number of rational curves of fixed volume.

In the study of moduli of elliptic curves the Dedekind eta function

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

where $q = e^{2\pi i \tau}$ plays a very important role. We will point out the three main properties of $\eta$.

1. It is well known fact that $\eta^{24}$ is a automorphic form which vanishes at the cusp. In fact $\eta^{24}$ is the discriminant of the elliptic curve.

2. The Kronecker limit formula gives the explicit relations between the regularized determinant of the flat metric on the elliptic and $\eta$.

3. The Fourier expansion of $\frac{d}{dt} \log \eta(it)$ are positive integers which give the number of elliptic curve that are covering of the elliptic curve $E_\tau$ of degree $n$.

By using the results obtained in [6] we prove the analogues of the above properties of the Dedekind eta functions in case of K3 surface for the restriction of the function $\exp(\Phi_{3,19})$ on the moduli space of K3 surfaces with a unimodular Picard lattice. Thus we establish that $\exp(\Phi_{3,19})$ is the analogue of the Dedekind eta function for K3 surfaces.

We also give the combinatorial interpretation of the restriction of the function $\exp(\Phi_{3,19})$ on the moduli space of K3 surfaces with a unimodular Picard
lattice in the A-model and for the first time in the B-model. In the B-model the holomorphic part of the regularized determinant of CY metric counts invariant vanishing calibrated 2-cycles related to finite monodromy operators with a given volume when $\text{Im} \omega_Y$ has integer periods. By invariant vanishing cycles we mean vanishing invariant cycles under the monodromy that appeared in a families $\pi : \mathcal{X} \to D$ over the unit disk such that $\pi^{-1}(0) = X_0$ has singularities.

We hope that the combinatorial properties of the holomorphic part of the regularized determinant of CY metric for CY threefolds also holds in the B-model. It counts the number of invariant calibrated invariant 3-cycles of infinite monodromy.

There are some relations of this paper with the papers [9] and [10].

1.1 Acknowledgements

The author want to acknowledge the help and suggestions of Greg Zuckerman. He proposed to study the behavior of the regularized determinants on the moduli space of K3 surfaces fibred by elliptic curves with sections.

Special thanks to Jay Jorgenson for his help and comments. I am grateful to him for introducing me to the ideas of regularized determinants.

I want to thank Jun Li for his interest in this paper and help. Special thanks to the Center of Mathematical Sciences of Zhe Jiang University and National Center for Theoretical Sciences, Mathematical Division, National Tsing Hua University for the financial support during the preparation of the paper.

2 Symmetric Spaces $\mathfrak{h}_{p,q} := \mathbb{SO}_0(p,q)/\mathbb{SO}(p) \times \mathbb{SO}(q)$

2.1 Global Flat Coordinates on the Symmetric Space $\mathfrak{h}_{p,q}$

We will need some basic facts about the symmetric space

$$\mathfrak{h}_{p,q} := \mathbb{SO}_0(p,q)/\mathbb{SO}(p) \times \mathbb{SO}(q).$$

The following Theorem is standard.

**Theorem 1** Let $\mathbb{R}^{p,q}$ be a $p+q$ dimensional real vector space with a metric with signature $(p,q)$. There is a one to one correspondence between points $\tau$ in $\mathfrak{h}_{p,q}$ and all oriented $p$-dimensional $E_\tau$ subspaces in $\mathbb{R}^{p,q}$ on which the intersection form on $\mathbb{R}^{p,q}$ is strictly positive.

**Theorem 2** Let $\mathbb{R}^{p,q}$ be a $p+q$ dimensional real vector space with a metric with signature $(p,q)$. Let $E_{\tau_0}$ be a $p$-dimensional subspace in $\mathbb{R}^{p,q}$ such the restriction of the quadratic form on $E_{\tau_0}$ is strictly positive. Let $e_1,\ldots,e_p$ be an orthonormal basis of $E_{\tau_0}$. Let $e_{p+1},\ldots,e_{p+q}$ be orthogonal vectors to $E_{\tau_0}$ such that $\langle e_i,e_j \rangle = -\delta_{ij}$ for $p+1 \leq i,j \leq p+q$. Let $E_\tau$ be any $p$-dimensional subspace in $\mathbb{R}^{p,q}$ such that the restriction of the quadratic form in $E_\tau$ is strictly
positive. Then there exists a basis \( \{ g_1(\tau), \ldots, g_p(\tau) \} \) in \( E_\tau \) such that

\[
g_j(\tau) = e_j + \sum_{i=p+1}^{p+q} \tau_i^j e_i.
\] (2)

**Proof:** Let

\[
f_i = \sum_{j=1}^{p} \mu_i^j e_j + \sum_{j=p+1}^{p+q} \lambda_i^j e_j
\] (3)

be an orthonormal basis of \( E_\tau \) where \( 1 \leq i \leq p \) and \( 1 \leq j \leq p + q \). Let \( (A_{ij}(\mu)) \) be the \( p \times p \) matrix \( \left( \mu_i^j \right) \) whose elements \( \mu_i^j \) are defined by the expression (3).

**Lemma 3** \( \det(A_{ij}(\mu)) \neq 0 \).

**Proof:** Suppose that \( \det(A_{ij}(\mu)) = 0 \). This implies that \( \text{rk}(A_{ij}(\mu)) < p \). Thus the rows vectors of the matrix \( A_{ij}(\mu) \) are linearly independent. So we can find constants \( a_i \) for \( i = 1, \ldots, q \) such that at least one of them is non zero and

\[
\sum_{i=1}^{p} a_i \left( \sum_{j=1}^{p} \mu_i^j e_j \right) = 0.
\] (4)

Let us consider the vector

\[
g(\tau) = \sum_{i=1}^{p} a_i g_i.
\] (5)

Combining (4) and (5) we obtain that

\[
g(\tau) = \sum_{j=p+1}^{p+q} \mu_j e_j.
\] (6)

(6) implies that

\[
\langle g(\tau), g(\tau) \rangle = -2 \sum_{j=p+1}^{p+q} |\mu_j|^2 < 0.
\] (7)

The definition of the vectors \( g_i(\tau) \) and (6) imply that \( g(\tau) \) is a non zero vector in \( E_\tau \). Since on \( E_\tau \) the restriction of the metric is strictly positive we get

\[
\langle g(\tau), g(\tau) \rangle > 0.
\]

Thus we get a contradiction with (7). Lemma 3 is proved. ■.

**Theorem 2** follows directly from Lemma 3. ■.

**Corollary 4** There is one to one correspondence between the set of all \( p \times q \) matrices \( (\tau_i^j) \) for \( 1 \leq i \leq p \) and \( p + 1 \leq j \leq p + q \) such that the vectors \( g_i(\tau) \) for \( i = 1, \ldots, p \) defined by (6) spanned a \( p \)-dimensional subspace \( E_\tau \) in \( \mathbb{R}^{p+q} \) on which the restriction of the quadratic form \( \langle u, v \rangle \) is strictly positive and the set of points in \( h_{3,19} \). Thus \( (\tau_i^j) \) define global coordinates on \( h_{3,19} \).
2.2 Decomposition of $\mathfrak{h}_{p,q}$

The following two facts are well known:

**Theorem 5** We have the following decomposition of $\mathfrak{h}_{2,p} = \mathbb{R}^{1,p-1} + \sqrt{-1}\mathfrak{h}_{1,p-1}$.

**Proof:** It is a well known fact that $\mathfrak{h}_{1,p-1}$ is one of the components $V^+$ of the cone $V := \{v \in \mathbb{R}^{1,p-1} \mid \langle v, v \rangle > 0\}$. Let us consider $\mathbb{R}^{2,p} = \mathbb{R}^{1,p-1} \oplus \mathbb{R}^{1,1}$.

Let us consider the map:

$$\Psi : \mathbb{R}^{1,p-1} + \sqrt{-1}\mathfrak{h}_{1,p-1} \to \mathbb{P}((\mathbb{R}^{1,p-1} \oplus \mathbb{R}^{1,1}) \otimes \mathbb{C})$$

defined as follows

$$\Psi : w = (w_1, ..., w_p) \to \left(\frac{w_1}{\sqrt{\langle w, w \rangle}}, ..., \frac{w_p}{\sqrt{\langle w, w \rangle}}, 1\right).$$

It is easy to check that in $\mathbb{P}(\mathbb{R}^{2,p} \otimes \mathbb{C})$ we have

$$\langle \Psi(w), \Psi(w) \rangle = 0 \& \left(\Psi(w), \overline{\Psi(w)}\right) > 0.$$

Thus the image of $\mathbb{R}^{1,p-1} + \sqrt{-1}\mathfrak{h}_{1,p-1}$ under the map $\Psi$ will be $\mathfrak{h}_{2,p}$, since $\mathfrak{h}_{2,p}$ in $\mathbb{P}(\mathbb{R}^{2,p} \otimes \mathbb{C})$ is given by one of the components of the open set in the quadratic $\langle w, w \rangle = 0$ defined by $\langle w, \overline{w} \rangle > 0$. It is very easy to prove that $\Psi$ is one to one map. 

**Theorem 6** Suppose that $p \geq 3$, and $q \geq 2$. Then we have the following decomposition $\mathfrak{h}_{p,q} = \mathfrak{h}_{p-1,q-1} \times \mathbb{R}^{p-1,q-1} \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of real positive numbers.

**Proof:** Let us consider in the space $\mathbb{R}^{p,q}$ two vectors $e_{p+q-1}$ and $e_{p+q}$ such that

$$\langle e_{p+q}, e_{p+q} \rangle = \langle e_{p+q-1}, e_{p+q-1} \rangle = 0 \text{ and } \langle e_{p+q-1}, e_{p+q} \rangle = 1.$$ 

Clearly the orthogonal complement to the subspace $\{e_{p+q}, e_{p+q}\}$ will be isometric to $\mathbb{R}^{p-1,q-1}$. Let us consider a basis $\{e_1, ..., e_{p+1}\}$ of $\mathbb{R}^{p,q}$, where $e_1, ..., e_{p+q-2}$ is a basis of $\mathbb{R}^{p-1,q-1}$.

There is one to one correspondence between the points $\tau \in \mathfrak{h}_{p,q}$ and the oriented $p$-dimensional subspaces $E_\tau$ in $\mathbb{R}^{p,q}$ on which the bilinear form is strictly positive. The intersection $E_\tau \cap \mathbb{R}^{p-1,q-1}$ will be $(p-1)$-dimensional subspace in $\mathbb{R}^{p-1,q-1}$ on which the bilinear form is strictly positive. Let $f_1$ be a vector in $\mathbb{R}^{p,q}$ orthogonal to $\mathbb{R}^{p-1,q-1} \cap E_\tau$. It is easy to see that the coordinates of $f_1$ can be normalized in such a way that its coordinates in $\mathbb{R}^{p,q}$ are such that

$$f_1 = (\mu_1, ..., \mu_{p+q-2}, 1, \lambda),$$

where $\mu = (\mu_1, ..., \mu_{p+q-2})$ is any vector in $\mathbb{R}^{p-1,q-1}$ and $\lambda > 0$ and $\lambda > \langle \mu, \mu \rangle$. Thus the correspondence $E_\tau \to (f_1, E_\tau \cap \mathbb{R}^{p-1,q-1})$ establishes the decomposition (??).
2.3 Definition of the Standard Metric on \( h_{p,q} \)

Since \( h_{p,q} \subset \text{Grass}(p, p + q) \) then the tangent space \( T_{\tau_0, h_{p,q}} \) at a point \( \tau_0 \in h_{p,q} \) can be identified with \( \text{Hom}(E_{\tau_0}, E_{\tau_0}^\perp) \). Thus any tangent vector \( A \in T_{\tau_0, h_{p,q}} \) can be written in the form

\[
A = \sum_{i=1}^{p} \sum_{j=p+1}^{p+q} \tau_i^j (e_i^* \otimes e_j),
\]

where \( e_i \) for \( i = 1, \ldots, q \) is an orthonormal basis of \( E_{\tau_0} \) and \( e_j \) for \( j = p+1, \ldots, p+q \) is an orthonormal basis of \( E_{\tau_0}^\perp \). Then we define the metric on \( T_{\tau_0, h_{p,q}} \) given by

\[
\|A^2\| = \text{Tr}(A \times A^t) = \sum_{i,j} |\tau_i^j|^2,
\]

where \( \tau_i^j \) are defined by (8). We will call this metric the Bergman metric on \( h_{3,19} \).

**Lemma 7** The Bergman metric \( ds_B^2 \) is invariant metric on \( h_{p,q} \). It is given in the flat coordinate system \( (\tau_i^j) \) by

\[
 ds_B^2 = \sum_{1 \leq j \leq 3, 1 \leq i \leq 19} (d\tau_i^j)^2 + O(2). \tag{10}
\]

**Proof:** The proof of Lemma 7 follows directly from the definition of the Bergman metric. 

3 Discriminants in \( h_{p,q} \)

3.1 Definition and Basic Properties of the Discriminant

From now on we will consider the symmetric spaces \( h_{p,q} \) for which \( p - q \equiv 0 \mod 8 \). In this paper \( \Lambda_{p,q} \) will be unimodular even lattice of signature \( (p = q + 8k, q) \). We have the following description all \( \Lambda_{p,q} \) :

**Theorem 8** Suppose that \( \Lambda_{p,q} \) the unimodular even lattice of signature \( (p, q) \) for \( p - q \equiv 0 \mod 8 \). Then

\[
\Lambda_{p,q} \cong U \oplus \ldots \oplus U \oplus E_8(-1) \oplus \ldots \oplus E_8(-1).
\]

**Definition 9** Define the set \( \Delta_{p,q}(e) := \{ \delta \in \Lambda_{p,q} \mid \langle \delta, \delta \rangle = -2 \} \). Let us define by \( O_{p,q} \) the group of the automorphisms of the lattice \( \Lambda_{p,q} \). Let \( O^+_{p,q} \) be the subgroup of \( O_{p,q} \) which preserve the orientation of the positive subspaces of dimension \( p \) in \( \Lambda_{p,q} \otimes \mathbb{R} \). Then \( O^+_{p,q} \) has index two in \( O_{p,q} \).
Definition 10 We know that \( h_{p,q} \) can be realized as an open set in the Grassmanian \( \text{Grass}(p, p + q) \). Let us denote by \( h_{p,q-1}(\delta) \) the set of all \( p \)-dimensional subspaces in the orthogonal complement of the vector \( \delta \) in \( \Lambda_{p,q} \otimes \mathbb{R} \). We will define the discriminant locus \( D_{p,q} \) in \( O_{p,q}^+ \setminus h_{p,q} \) as follows:

\[
D_{p,q} := O_{p,q}^+ \setminus \left( \bigcup_{\delta \in \Delta(e)} (h_{p,q-1}(\delta)) \right).
\]

This definition is motivated by the definition of the discriminant locus in the moduli of algebraic K3 surfaces.

3.2 The Irreducibility of the Discriminant

Theorem 11 The discriminant locus \( D_{p,q} \) is an irreducible real analytic sub-

space in \( O_{p,q}^+ \setminus h_{p,q} \), where \( \Lambda_{p,q} \) is an even unimodular lattice.

Proof: The proof of Theorem 11 will follow if we prove that on the set of

vectors \( \Delta_{\Lambda_{p,q}} \) the group \( O_{\Lambda_{p,q}}^+ \) acts transitively. Thus they form one orbit and therefore the discriminant locus \( D_{p,q} \) in \( O_{p,q}^+ \setminus h_{p,q} \) is an irreducible divisor.

The proof that on the set of vectors \( \Delta_{\Lambda_{p,q}} \) the group \( O_{\Lambda_{p,q}}^+ \) acts transitively

will be based on ideas used in [8] to prove the irreducibility of the discriminant

locus in the moduli space of Enriques surfaces.

We will proceed by induction on \( p \) to prove that the action of \( O_{\Lambda_{p,q}}^+ \) on

the set \( \Delta_{\Lambda_{p,q}} \) is transitive. For \( p = 0 \) the Theorem 11 is obvious. Suppose that

Theorem 11 is true for \( p > 0 \). We will denote by \( L \) the lattice

\[
\bigoplus_p U \oplus \bigoplus_q E_8(-1)
\]

and by \( M \) the lattice \( M = L \oplus U \).

The plan of the proof is the following. We will denote by \( R_0 \) and \( R_1 \) the set of

norm \(-2\) vectors of \( M \) which have inter product respectively 0 or 1 with the vector

\[
e = (0, 0, 1) \in L \oplus U = M.
\]

Let \( \Gamma_1 \) be the group generated by reflections of elements of the set \( R_1 \) and \( \Gamma_2 \) be

the group generated by reflections of elements of \( R_0 \cup R_1 \) and \(-id\). We will show

first that any \(-2\) vector of \( M \) is conjugate to an element of the set \( R_0 \cup R_1 \).

Then we will show that the group \( O_{\Lambda_{p,q}}^+(\mathbb{Z}) \) interchange the sets \( R_0 \) and \( R_1 \).

Lemma 12 Any norm \(-2\) vector \( \delta \) of \( M \) is conjugate to an element of \( R_0 \cup R_1 \)

under the group \( \Gamma_1 \).

Proof: The proof of Lemma 12 is based on the following Propositions 13 and

14.

Proposition 13 Suppose that \( v \notin M \subset M \otimes \mathbb{Q} \). Suppose that \( x \) is some real

number. Then there exists a vector \( \vec{\mu} \in M \) such that

\[
|\langle \vec{\mu} - \vec{\nu}, \vec{\mu} - \vec{\nu} \rangle - x| < 1.
\]
Proof: The proof of Proposition 14 follows the proof of Lemma 2.1 given in [8]. Since \( \mathcal{V} \not\subset M \times \mathbb{Q} \) we can find a primitive isotropic vector \( \overrightarrow{\mu} \) such that \( \langle \overrightarrow{\mu}, \overrightarrow{\nu} \rangle \) is not an integer. This is because primitive isotropic vectors span \( L \). As the group \( \mathcal{O}_M(\mathbb{Z}) \) acts transitively on norm 0 vectors we can assume that

\[
e = (0, 0, 1) \in \bigoplus_{p} U \oplus \bigoplus_{q} E_8(-1) \oplus \mathbb{U}
\]

Then \( \overrightarrow{v} = (\overrightarrow{\lambda}, a, b) \) with \( a \) not an integer. We will find some \( \overrightarrow{\mu} \) of the form \( \overrightarrow{\mu} = (0, m, n) \) with integers \( m \) and \( n \) such that for \( x \) we have

\[
|\langle \overrightarrow{\mu} - \overrightarrow{v}, \overrightarrow{\mu} - \overrightarrow{v} \rangle - x| = |\langle \overrightarrow{v}, \overrightarrow{v} \rangle - 2(a - m)(b - n) - x| < 1.
\]

Since \( a \) is not an integer we can find some integer \( m \) such that \( |a - m| < 1 \). Whenever we add 1 to \( n \), the expression \( 2(a - m)(b - n) \) is changed by a non zero number less than 2, so we can choose some integer \( n \) such that \( 2(a - m)(b - n) \) is at a distance of less then 1 from any given number \( x - \langle \overrightarrow{v}, \overrightarrow{v} \rangle \). This proves Proposition 14.

Proposition 14 Suppose that \( R_1 \) is the set of norm \(-2\) vectors of \( M \) having inner product \( 1 \) with

\[
e = (0, 0, 1) \in M = L \oplus \mathbb{U}.
\]

Suppose that \( \Gamma_1 \) is the subgroup of \( \mathcal{O}_M(\mathbb{Z}) \) generated by reflections of vectors of \( R_1 \) and the automorphism \(-id\). Then any vector \( \overrightarrow{r} \in M \) is conjugate under \( \Gamma_1 \) to a vector of the form \( (\overrightarrow{v}, m, n) \in M \) such that either \( m = 0 \) or \( \frac{\overrightarrow{v}}{m} \in L \) and \( m > 0 \).

Proof: We can assume that \( \overrightarrow{\nu} = (\overrightarrow{v}, m, n) \) has the property that \( |\langle \overrightarrow{\nu}, \overrightarrow{\nu} \rangle| = |m| \) is minimal among all conjugates of \( \overrightarrow{r} \) under \( \Gamma_1 \), where \( \overrightarrow{v} = (0, 0, 1) \). If \( m = 0 \) then we are done. So we can assume that \( m \neq 0 \), and wish to prove that \( \frac{\overrightarrow{v}}{m} \in L \) and \( m > 0 \).

Suppose that \( \frac{\overrightarrow{v}}{m} \notin L \). By Proposition 13 we can find a vector \( \overrightarrow{\mu} \in L \) satisfying

\[
\left| \left\langle \frac{\overrightarrow{\mu} - \overrightarrow{v}}{m}, \frac{\overrightarrow{\mu} - \overrightarrow{v}}{m} \right\rangle + \left( -\frac{2n}{m} - \frac{\langle \overrightarrow{v}, \overrightarrow{v} \rangle}{m^2} \right) \right| < 1. \tag{11}
\]

Let \( \delta = \left( \frac{\overrightarrow{\mu}}{m}, 1, \frac{-\langle \overrightarrow{v}, \overrightarrow{v} \rangle - 2}{2m} \right) \). It is easy to see that \( \langle \delta, \delta \rangle = -2 \). Let

\[
T_3(\overrightarrow{v}) = \overrightarrow{r'}, \overrightarrow{v}, \langle \overrightarrow{v}, \delta \rangle \delta,
\]

i.e. \( r' \) is the reflection of \( r \) with respect to the hyperplane of \( \delta \in R_1 \). Direct computations show that

\[
|\langle \overrightarrow{r'}, \overrightarrow{v} \rangle| = |\langle T_3(\overrightarrow{v}), \overrightarrow{v} \rangle| = |\langle \overrightarrow{v}, T_3(\overrightarrow{v}) \rangle| = |\langle \overrightarrow{v}, \overrightarrow{v} + \langle \overrightarrow{v}, \delta \rangle \delta \rangle| =
\]

11
Combining (11) and (12) we deduce that

$$\left| \langle \vec{r}, \vec{v} \rangle \right| = m \left( \frac{\langle \vec{m}, \vec{v} \rangle}{m} + \left( -\frac{2n}{m} - \frac{\langle \vec{v}, \vec{v} \rangle}{m^2} \right) \right).$$  \hspace{1cm} (12)$$

We have chosen $|\langle \vec{r}, \delta \rangle| = m$ to be minimal. So (13) contradicts $|\langle \vec{r}, \delta \rangle| = m$. Proposition 14 is proved. $\blacksquare$

Proof of Lemma 12 Let $\delta = (\vec{v}, m, n)$. By Proposition 14 we can assume that either $m = 0$ or $\frac{\vec{v}}{m} \in M$. If $m = 0$ then Lemma 12 is proved. Suppose that $\frac{\vec{v}}{m} \in M$ holds. Then $\langle \vec{v}, \frac{\vec{v}}{m} \rangle \in 2\mathbb{Z}$. Thus

$$\langle \delta, \delta \rangle = -2 = m^2 \left( \frac{\vec{v}}{m}, \frac{\vec{v}}{m} \right) + 2mn$$

implies that $-2$ is divisible by $2m$. From here we conclude that $m = 1$. Lemma 13 is proved. $\blacksquare$

Let us define the group $\Gamma_3$ as the group generated by the automorphisms $\mathcal{O}_L(\mathbb{Z})^+$ extended to automorphisms of $M$ by letting them act trivially on $\mathbb{U}$, the group of automorphisms taking

$$(\vec{v}, m, n) \rightarrow (\vec{v} + 2m \vec{\lambda}, m, n - \langle \vec{v}, \vec{\lambda} \rangle - m \langle \vec{\lambda}, \vec{\lambda} \rangle)$$

for $\lambda \in L$, and the group of automorphisms given by reflections of norm $-2$ vectors in $R_1$.

Lemma 15 The group $\Gamma_3$ acts transitively on the set of vectors of norm $-2$ in $M$.

Proof: The proof of Lemma 15 is based on the following Propositions:

Proposition 16 The group $\mathcal{O}_L(\mathbb{Z})^+$ acts transitively on the set of vectors of norm $-2$ in $L$.

Proof: Since by definition

$$L = \mathbb{U} \oplus \ldots \oplus \mathbb{U} \oplus \mathbb{E}_8(-1) \oplus \ldots \oplus \mathbb{E}_8(-1) \oplus \mathbb{U}$$

then Proposition 16 follows from the induction hypothesis. $\blacksquare$

Proposition 17 Let $\vec{\lambda} \in L$. There exists an element $g_{\vec{\lambda}} \in \text{Aut} (M)$ such that if $\delta = (\vec{v}, 0, k)$ and $\delta^2 = -2$, then $g_{\vec{\lambda}}(\delta) = (\mu, 0, 0)$. 

12
Proof: The conditions \( \delta = (\vec{v}, 0, k) \) and \( \delta^2 = -2 \) imply that \( \langle \vec{v}, \vec{v} \rangle = -2 \). Thus \( \vec{v} \) is a primitive element in \( L \). Proposition \ref{prop16} implies that there exists an element \( \sigma \in O_L(\mathbb{Z})^+ \) such that

\[
\sigma(\vec{v}) = \vec{\mu} = f_1 - \mu \vec{f}_2 \in \mathbb{U} \subset L,
\]

where \( \langle f_1, f_2 \rangle = 0 \) and \( \langle f_1, f_2 \rangle = 1 \).

Next we will construct an element \( g_{\vec{\lambda}} \in Aut(M) \) such that

\[
g_{\vec{\lambda}}((\vec{\mu}, 0, k)) = (\vec{v}, \vec{v}),
\]

where \( \vec{\lambda} \in M \) is any element. Direct computations show that \( g_{\vec{\lambda}} \) preserve the scalar product in \( M \). Thus \( g_{\vec{\lambda}} \in Aut(M) \). The definition of \( g_{\vec{\lambda}} \) and since we choose \( \lambda \in L \) such that \( \langle \vec{\lambda}, \vec{v} \rangle = k \) then

\[
g_{\vec{\lambda}}((\vec{\mu}, 0, k)) = (\vec{\mu}, 0, k) - \langle \vec{\mu}, \vec{\lambda} \rangle f_2 = (\vec{v}, 0, 0).
\]

Proposition \ref{prop18} is proved. ■

Proposition 18 Suppose that \( \delta \in R_0 \). Then there exists an element \( \sigma \in \Gamma_3 \) such that \( \sigma(\delta) \in R_1 \).

Proof: Proposition \ref{prop17} implies that without loss of generality we may assume that \( \delta \in L \). Let

\[
\vec{\lambda} \in L, \langle \delta, \vec{\lambda} \rangle \neq 0 \text{ and } \langle \vec{\lambda}, \vec{\lambda} \rangle \neq 0.
\]

Let us consider

\[
\delta_1 = \left( \vec{\lambda}, 1, -\frac{\langle \vec{\lambda}, \vec{\lambda} \rangle}{2} \right) \in L \oplus U = M.
\]

Clearly \( \langle \delta_1, \delta_1 \rangle = -2 \). Then the map \( T_{\delta_1}(\delta) = \delta + \langle \delta_1, \delta \rangle \delta_1 \) is an element of \( \Gamma_3 \) and clearly \( T_{\delta_1}(\delta) \in R_1 \). Proposition \ref{prop18} is proved. ■
4 Moduli of K3 Surfaces

4.1 Moduli of Marked, Algebraic and Polarized K3 surfaces

A K3 surface is a compact, complex two dimensional manifold with the following properties: i. There exists a non-zero holomorphic two form $\omega$ on $X$ without zeroes. ii. $H^1(X,\mathcal{O}_X) = 0$.

In [2] and [5], the following topological properties are proved. The surface $X$ is simply connected, and the homology group $H_2(X,\mathbb{Z})$ is a torsion free abelian group of rank 22. The intersection form $\langle u,v \rangle$ on $H_2(X,\mathbb{Z})$ has the properties:

1. $\langle u,u \rangle = 0 \mod(2)$.
2. $\det (\langle e_i,e_j \rangle) = -1$.
3. The symmetric form $\langle , \rangle$ has a signature $(3,19)$.

Theorem 5 on page 54 of [26] implies that as an Euclidean lattice $H_2(X,\mathbb{Z})$ is isomorphic to the K3 lattice $\Lambda_{K3}$, where $\Lambda_{K3} := \mathbb{U}^3 \oplus (-E_8)^2$. Every K3 surface is also simply connected.

Definition 19 Let $\alpha = \{\alpha_i\}$ be a basis of $H_2(X,\mathbb{Z})$ with intersection matrix $\Lambda_{K3}$. The pair $(X,\alpha)$ is called a marked K3 surface. Let $l \in H^{1,1}(X,\mathbb{R}) \cap H^2(X,\mathbb{Z})$ be the Poincare dual class of a hyperplane section, i.e. an ample divisor. The triple $(X,\alpha,l)$ is called a marked, polarized K3 surface. The degree of the polarization is an integer $2d$ such that $\langle l,l \rangle = 2d > 0$.

Definition 20 The period map $\pi$ for marked K3 surfaces $(X,\alpha)$ is defined by integrating the holomorphic two form $\omega$ along the basis $\alpha$ of $H_2(X,\mathbb{Z})$, meaning

$$\pi(X,\alpha) := (\ldots, \int_{\alpha_i} \omega, \ldots) \in \mathbb{P}^{21}.$$ 

The Riemann bilinear relations hold for $\pi(X,\alpha)$, meaning

$$\langle \pi(X,\alpha), \pi(X,\alpha) \rangle = 0 \text{ and } \left\langle \pi(X,\alpha), \overline{\pi(X,\alpha)} \right\rangle > 0. \quad (15)$$

Choose a primitive vector $l \in \Lambda_{K3}$ such that $\langle l,l \rangle = 2d > 0$. Let us denote

$$\Lambda_{K3,l} := \{v \in \Lambda_{K3} | \langle l,v \rangle = 0 \}.$$ 

Then $\pi(X,\alpha,l) \in \mathbb{P}(\Lambda_{K3,l} \otimes \mathbb{C})$ and it satisfies (15). The set of points in $\mathbb{P}(\Lambda_{K3,l} \otimes \mathbb{C})$ that satisfy (15) consists of two components isomorphic to the symmetric space $h_{2,19}$. In [25] the following Theorem was proved:

**Theorem 21** The moduli space $\mathcal{M}_{K3,mpa}^{2d}$ of marked, polarized, algebraic K3 surfaces of a fixed degree $2d$ exists and it is embedded by the period map into $h_{2,19}$ is an open everywhere dense subset. Let

$$\Gamma_{K3,2d} = \{ \phi \in \text{Aut}^+(\Lambda_{K3}) | \langle \phi(u),\phi(u) \rangle = \langle u,u \rangle \text{ and } \phi(l) = l \},$$ 

14
where \( l \) is a primitive vector such that \( \langle l, l \rangle = 2d > 0 \). Then the moduli space \( \mathcal{M}_{K3, pa}^{2d} \) of polarized, algebraic K3 surfaces of a fixed degree 2d is isomorphic to a Zariski open set in the quasi-projective variety \( \Gamma_{K3, 2d} \backslash \mathfrak{h}_{2,19} \).

By pseudo-polarized algebraic K3 surface we understand a pair \((X, l)\) where \( l \) corresponds to either ample divisor or pseudo ample divisor, which means that for any effective divisor \( D \) in \( X \), we have \( \langle D, l \rangle \geq 0 \). Mayer proved the linear system \( |3l| \) defines a map:

\[
\phi_{|3l|} : X \rightarrow X_1 \subset \mathbb{P}^m
\]

such that: i. \( X_1 \) has singularities only double rational points. ii. \( \phi_{|3l|} \) is a holomorphic birational map. Let us denote by \( \mathcal{M}_{K3, ppa}^{2d} \) the moduli space of pseudo-polarized algebraic K3 surfaces of degree 2d. From the results proved in [12], [23], [27] and [25] the following Theorem follows:

**Theorem 22** The moduli space of \( \mathcal{M}_{K3, ppa}^{2d} \) is isomorphic to the locally symmetric space \( \Gamma_{K3, 2d} \backslash \mathfrak{h}_{2,19} \).

### 4.2 Moduli Space of K3 Surfaces with a B-Field

**Definition 23** Let \( X \) be a K3 surface. Let \( \omega_X(1, 1) \in H^{1,1}(X, \mathbb{C}) \) such that

\[
\int_X \text{Im} \omega_X(1, 1) \wedge \text{Im} \omega_X(1, 1) > 0.
\]

Then \( \omega_X(1, 1) \) will be called a B-field on \( X \).

**Theorem 24** Let \((X, \omega_X(1, 1), \gamma_1, \ldots, \gamma_22)\) be a marked K3 surface with a B-field. Then the moduli space \( \mathcal{M}_{m,B} \) of marked K3 surfaces with a B-field is isomorphic to \( SO_0(4, 20)/SO(4) \times SO(20) \).

**Proof:** See [1] or [28].

### 4.3 Discriminant of Pseudo-Polarized K3 Surfaces

The complement of \( \mathcal{M}_{K3, mpa}^{2d} \) in \( \mathfrak{h}_{2,19} \) can be described as follow. Given a polarization class \( e \in \Lambda_{K3} \), set \( T_e \) to be the orthogonal complement to \( e \) in \( \Lambda_{K3} \), i.e. \( T_e \) is the transcendental lattice. Then we have the realization of \( \mathfrak{h}_{2,19} \) as one of the components of

\[
\mathfrak{h}_{2,19} \cong \{ u \in \mathbb{P}(T_e \otimes \mathbb{C}) | \langle u, u \rangle = 0 \text{and} \langle u, \overline{\nu} \rangle > 0 \}.
\]

For each \( \delta \in \Delta(e) \), define the hyperplane

\[
H(\delta) = \{ u \in \mathbb{P}(T_e \otimes \mathbb{C}) | \langle u, \delta \rangle = 0 \}.
\]

Let \( \mathcal{H}_{K3, 2d} = \bigcup_{\delta \in \Delta(e)} (H(\delta) \cap \mathfrak{h}_{2,19}) \). Let us define the discriminant \( \mathcal{D}_{K3}^{2d} := \Gamma_{K3, 2d} \backslash \mathcal{H}_{K3, 2d} \). Results from [24], [25], [27] and [23] imply that \( \mathcal{D}_{K3}^{2d} \) is the
complement of the moduli space of algebraic polarized K3 surfaces \( \mathcal{M}_{K3,pa}^{2d} \) in the locally symmetric space \( \Gamma_{K3,2d} \backslash b_{K3,2d} \), i.e.
\[
\mathcal{D}_{K3}^{2d} = (\Gamma_{K3,2d} \backslash b_{K3,2d}) - \mathcal{M}_{K3,pa}^{2d}.
\]

5 Mirror Symmetry

5.1 Mirror Symmetry for K3 Surfaces

Let \((X, \alpha, \omega_X(1,1))\) be a marked K3 surface with a B-field \(\omega_X(1,1)\). To define the mirror of \((X, \alpha, \omega_X(1,1))\) we need to fix an unimodular hyperbolic lattice \(U\) in \(H_2(X,\mathbb{Z})\) with generators \(\{\gamma_0, \gamma_1\}\) such that for the holomorphic two form \(\omega_X\) we have
\[
\int_{\gamma_0} \omega_X \neq 0 \text{ and } \int_{\gamma_1} \omega_X \neq 0.
\]
Thus we can normalize \(\omega_X\) in the following manner
\[
\int_{\gamma_0} \omega_X = 1 \text{ and } \int_{\gamma_1} \omega_X \neq 0.
\]
From now on we will consider the set \((X, \alpha, \omega_X(1,1), U, \omega_X)\), where \(\alpha\) is a marking, \(U\) is a fixed sublattice in \(H^2(X,\mathbb{Z})\) such that the holomorphic two form satisfies (16).

Let \(U_0\) the unimodular hyperbolic sublattice \(H_0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})\) in the cohomology ring \(H^*(X,\mathbb{Z})\). We will assign to the B-field \(\omega_X(1,1)\) the vector
\[
\hat{\omega}_X := \left( \omega_X(1,1), 1, -\frac{\omega_X(1,1) \wedge \omega_X(1,1)}{2} \right)
\]
in \(H^2(X,\mathbb{Z}) \oplus U_0 = H^*(X,\mathbb{Z})\).

We will need the following Theorem:

**Theorem 25** Let \((X, \alpha, \omega_X(1,1), U, \omega_X)\), where \(\alpha\) is a marking, \(U\) is a fixed sublattice in \(H^2(X,\mathbb{Z})\) such that the holomorphic two form satisfies (16). Then there exists a marked K3 surface \((Y, \alpha)\) with a B-field \(\omega_Y(1,1)\) such that if we identify \(H^2(Y,\mathbb{Z})\) with \(U^\perp \oplus U_0\), then the class of the cohomology \([\omega_Y]\) of the K3 surfaces \(Y\) is such that \([\omega_Y] = \omega_X(1,1) \in (U^\perp \oplus U_0) \otimes \mathbb{C}\) and \(\omega_Y(1,1) = [\omega_X] \in (H^2(Y,\mathbb{Z}) \oplus U) \otimes \mathbb{C}\).

**Proof:** Let us consider \(\omega_X(1,1) \in (U^\perp \oplus U_0) \otimes \mathbb{C} = \Lambda_{K3} \otimes \mathbb{C}\). Then direct computations show that we have \(\langle \hat{\omega}_X, \hat{\omega}_X \rangle = 0\) and \(\langle \hat{\omega}_X, \overline{\omega_X} \rangle > 0\). From the epimorphism of the period map for K3 surfaces proved in [27] it follows that there exists a marked K3 surface \((Y, \alpha)\) with a holomorphic two form \(\omega_Y\) such that the class of cohomology \([\omega_Y]\) is the same as the class of cohomology of
Next we will prove that the class of cohomology $\omega_X \in H^{1,1}(Y, \mathbb{C})$ satisfies
\[
\int_Y \text{Im} \omega_X \wedge \text{Im} \omega_X = \langle \text{Im} \omega_X, \text{Im} \omega_X \rangle > 0.
\]
Indeed on $X$ we have
\[
\langle \omega_X, \omega_X(1, 1) \rangle = \langle \omega_X, \overline{\omega_X(1, 1)} \rangle = 0 \quad (17)
\]
since $\omega_X(1, 1)$ is a form of type $(1, 1)$ and $\omega_X$ is a form of type $(2, 0)$. On the other hand the form $\omega_X(1, 1)$ with respect to the new complex structure $Y$ on $X$ it is a form of type $(2, 0)$. So $(17)$ means that on $Y$ $\omega_X$ is a form of type $(1, 1)$. On the other hand we have
\[
\int_X \omega_X \wedge \overline{\omega_X} = 2 \int_X \text{Im} \omega_X \wedge \text{Im} \omega_X = 2 \langle \text{Im} \omega_X, \text{Im} \omega_X \rangle > 0. \quad (18)
\]
Thus $(18)$ proves that $\omega_X$ is a B-field on $Y$. Theorem 25 is proved. ■

Now we are ready to define the mirror symmetry:

**Definition 26** We will define the marked surface $(Y, \alpha, \omega_Y(1, 1), U, \omega_Y)$ constructed in Theorem 25 will be the mirror of $(X, \alpha, \omega_X(1, 1), U_0, \omega_X)$.

### 5.2 Mirror Symmetry and Algebraic K3 Surfaces

Let us consider the Neron-Severi group
\[ M = \text{Pic}(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R}). \]

We can characterize in another way $NS(X)$. It is the dual group in $H^2(X, \mathbb{Z})$ of the kernel of the functional:
\[ (\omega_X) : H_2(X, \mathbb{Z}) \to \mathbb{C} \]
defined by $\gamma \to \int_X \omega_X$. We define the transcendental classes of homologies $T(X) \subset H^2(X, \mathbb{Z})$ on $X$ as follows: $T(X) := \ker (\omega_X)^\perp$.

**Definition 27** We will say that pairs $(X, M)$ $M$–marked K3 surface if $M$ is the Picard lattice of some algebraic K3 surface together with a primitive imbedding of $M$ into $H^2(X, \mathbb{Z})$.

The following Theorem was proved in [28] or [13]

**Theorem 28** The moduli space $\mathcal{M}_M$ of marked pairs $(X, M)$ exists and $\mathcal{M}_M \cong \Gamma_M \setminus \mathfrak{h}_{2,20 - \rho}$, where $\rho = \text{rk} M$ and $\Gamma_M = \{ \phi \in \text{Aut} \Lambda_{K3} | \phi_M = \text{id} \}$. 




Suppose that we consider $M$ such that $U$ can be embedded into $M^\perp$. According to a Theorem of Nikulin this is always possible if $rkM = \rho \geq 9$. The construction of mirror symmetry for $M$ marked K3 surfaces $(X, \alpha, M, U, \omega_X)$, where $U \subset M^\perp$ was described in [28] and [13] as follows: Let $(X, \alpha, M, U, \omega_X)$ be an algebraic polarized K3 surface. Then Theorem 25 implies the following Corollary:

**Corollary 29** Let $(X, \alpha, M, U, \omega_X)$ be $M$-marked K3 surface such that $U \subset T_X$ and the B-field $\omega_X(1,1) \in \mathbb{R}^T$ satisfies $\omega_X(1,1)|_{U^\perp \subset T_X} = 0$. Then the mirror $(Y, M_1, U, \omega_Y(1,1), \omega_Y)$ satisfies the following conditions: i. $Pic(Y) = M_1 = U^\perp \subset T_X$. ii. $T_Y = M \oplus U \cong Pic(X) \oplus U$.

**Proof:** Corollary 29 follows directly from Theorem 25.

**Remark 30** Some interesting examples and applications of Corollary 29 were discussed in [13].

### 5.3 The Mirror Map for Marked M-K3

Part of the mirror conjecture states that the

**Definition 31** Let $X$ be a K3 surface. We will define the Kähler cone of $K(X)$ of $X$ as follows:

$$K(X) := \{ \omega \in H^{1,1}(X, \mathbb{R}) | \omega = \text{Im} g, \text{ and } g \text{ is a Kähler metric on } X \}.$$  

We will need the characterization of the Kähler cone that is given bellow. Denote by 

$$\Delta(X) := \{ \delta \in NS(X) | \langle \delta, \delta \rangle = -2 \}.$$  

We will need the following Lemma from [25]:

**Lemma 32** Let $\delta \in \Delta(X)$. Then $\delta$ or $-\delta$ can be realized as an effective curve on $X$.

We will denote by 

$$\Delta^+(X) := \{ \delta \in \Delta(X) | \delta \text{ can be realized as an effective curve} \}.$$  

Let us denote by $V := \{ v \in H^{1,1}(X, \mathbb{R}) | \langle v, v \rangle > 0 \}$. Since the restriction of the bilinear form on $H^{1,1}(X, \mathbb{R})$ has a signature $(1,19)$, then $V$ will consists of two components. Let us denote by $V^+$ the component of $V$ which contains a Kähler class.

Each $\delta \in \Delta^+(\Delta)$ generates a reflection $s_\delta$ of $V^+$, where $s_\delta(v) = v + \langle v, \delta \rangle \delta$. Let us denote by $\Gamma(\Delta)$ the subgroup of $O_{\Delta_K3}^+$ generated by $s_\delta$. In [27] the following Theorem was proved:

**Theorem 33** The Kähler cone $K(X)$ coincides with the fundamental domain of the group $\Gamma(\Delta)$ in $V^+$ which contains a Kähler class.
Proof: See [27]. □

Remark 34 According to Theorem 28 $\mathcal{M}_{K3,M} = \Gamma_M \setminus \mathfrak{h}_{2,20-\rho}$ is the moduli space of $M$-marked $K3$ surfaces. Suppose that $U \subset T_X$ is fixed and $M_1 \subset T_X$ is the orthogonal complement of $U$ in $M$. Let $(Y,M_1)$ be some $M_1$ marked $K3$ surface defined by the primitive embedding $M_1 \subset T_X \subset \Lambda_{K3}$. Let $\mathfrak{h}_{M_1} = M_1 \otimes \mathbb{R} + iK(Y)$, where $K(Y)$ is the Kähler cone of $Y$. Then according to Theorem 28 $\mathcal{M}_{K3,M} \cong \Gamma_M \setminus \mathfrak{h}_{M_1}$. Thus we have a complex analytic covering map:

$$\psi_M : \mathfrak{h}_{M_1} \to \mathcal{M}_{K3,M} = \Gamma_M \setminus \mathfrak{h}_{M_1}.$$ 

The map $\psi^{-1}_M$ which is multivalued is called the mirror map. It identifies in the case described in this Remark the moduli space of $M$-marked $K3$ surfaces with the complexified Kähler cone of its mirror.

6 Automorphic Forms on $\Gamma \setminus \mathfrak{h}_{\rho,q}$, Theta Lifts and Regularized Determinants of CY metrics on K3 Surfaces

6.1 General Facts about Regularized Determinants

Definition 35 Let $M$ be a compact $C^\infty$ manifold. Let $g$ be a Riemannian metric on $M$. Let

$$\Delta_{g,q} = d \circ d^* + d^* \circ d$$

be the Laplacian associated with the metric $g$ acting on the space of $C^\infty$ infinity $q$-forms $C^\infty(M,\Omega^0_M)$. It is a well known fact that the spectrum of $\Delta_{g,q}$ is nonnegative, i.e. $0 \leq \lambda_1 \leq ... \lambda_k \leq ...$ and

$$\lim_{k \to \infty} \frac{\lambda_k}{k^n} = c > 0, \quad (19)$$

where $n = \dim_M M$. We define the zeta function $\zeta_q(s)$ of $\Delta_{g,q}$ as follows:

$$\zeta_q(s) = \sum_{\lambda_k > 0} \lambda_k^{-s}.$$ 

Then $\zeta_q(s)$ is a well defined function for $s \in \mathbb{C}$, where Re $s$ large enough. One can prove that $\zeta_q(s)$ has a meromorphic continuation in $\mathbb{C}$ and $\zeta_q(0)$ is well defined. Then we define the regularized determinant of $\Delta_{g,q}$ as follows: $\det \Delta_{g,q} = \exp \left(-\zeta'_q(0)\right)$.

In [6] the following Theorem was proved:

Theorem 36 Let $M$ be a CY manifold with a polarization class $L \in H^2(M,\mathbb{Z}) \cap H^{1,1}(M,\mathbb{R})$. Let $\det \Delta_{(0,1)}$ be the regularized determinant of the Laplacian corresponding to the Calabi Yau metric corresponding to the polarization class $L$ and acting on the space of $(0,1)$ forms. Then $d^c \log \det \Delta_{(0,1)} = -\text{Im W.P.}$.
6.2 Special Automorphic Form of Weight -2 on $\Gamma \backslash h_{p,q}$

In this paper the group $\Gamma$ will be the group of automorphisms of $\Lambda_{K3}$ which preserve the spinor norm, i.e. $\Gamma = \mathcal{O}_{K3}(\mathbb{Z})$ is a subgroup of index 2 in the group of automorphism $\mathcal{O}_{K3}(\mathbb{Z})$ of the lattice $\Lambda_{K3}$. Donaldson proved in [12] that the mapping class group of a K3 surface is isomorphic to $\Gamma$.

We will define the one cocycle $\mu(\gamma, \tau)$ of the group $\Gamma$ with coefficients the non singular $3 \times 3$ matrices with coefficients functions on $h_{3,19}$. Let an element $\gamma \in \Gamma$ be represented by a $(22 \times 22)$ matrix $(\gamma_{k,l})$. We proved that any point $\tau \in h_{3,19}$ can be represented by the $3 \times 22$ matrix $\tau = (E_3, \tau_j)$, where $E_3$ is the identity $3 \times 3$ matrix. The action of $\gamma = (\gamma_{k,l}) \in \Gamma$ on $h_{3,19}$ is described as follow:

$$\gamma(\tau) = (E_3, \tau_i,j) \times (\gamma_{k,l}) = (\mu(\gamma, \tau), \sigma_{\gamma,i,j}(\tau)),$$

where $\mu(\gamma, \tau)$ is $3 \times 3$ matrix defined by the first three columns of the matrix $[20]$ and $\sigma_{\gamma,i,j}(\tau)$ is some $3 \times 19$ matrix. Theorem $[2]$ implies that the $3 \times 3$ matrix $\mu(\gamma, \tau)$ has rank 3, i.e. $\det(\mu(\gamma, \tau)) \neq 0$. It is easy to see that we have:

$$\mu(\gamma_1\gamma_2, \tau) = \mu(\gamma_1, \tau) \times \mu(\gamma_2, \gamma_1(\tau)).$$

**Definition 37** Let $\Phi(\tau)$ be a function on $h_{3,19}$ such that it satisfies the following functional equation:

$$\Phi(\tau \gamma) = (\det(\mu(\gamma, \tau)))^k \Phi(\tau).$$

Then we will call $\Phi(\tau)$ an automorphic form of weight $k$.

**Definition 38** Let us recall that according to Theorem $[2]$ to each point $\tau = (\tau^j_i) \in h_{3,19}$, $1 \leq j \leq 3$ and $1 \leq i \leq 19$ we assigned the row vectors $g_i$ of the matrix $(E_3, \tau^j_i)$. We will define the function $g(\tau)$ on $h_{3,19}$ as follows

$$g(\tau) := \det((g_i(\tau), g_j(\tau))).$$

**Theorem 39** The function $g(\tau)$ defined in Definition $[38]$ is an automorphic form of weight $-2$.

**Proof:** We need to compute

$$g((\gamma(\tau)) = \det(\langle (\mu(\gamma, \tau)^{-1} \times \gamma_i(\tau)), (\mu(\gamma, \tau)^{-1} \times \gamma_j(\tau)) \rangle) = ?,$$

where $\gamma_i(\tau)$ is the $i^{th}$ row of the $(3 \times 22)$ matrix $(\gamma^i_j) \times \gamma$. Theorem $[2]$ and the expression of the matrix $\mu(\gamma, \tau)$ given by $[20]$ imply

$$g((\gamma(\tau)) = \det(\langle (\mu(\gamma, \tau)^{-1} \times \gamma_i(\tau)), (\mu(\gamma, \tau)^{-1} \times \gamma_j(\tau)) \rangle) =$$

$$= (\det(\mu(\gamma, \tau)))^{-2} \det(\langle g_i(\tau), g_j(\tau) \rangle) = \det(\mu(\gamma, \tau))^{-2} \times g(\tau).$$

Thus we get $g(\gamma(\tau)) = \det(\mu(\gamma, \tau))^{-2} g(\tau)$. So Theorem $[39]$ is proved. ■

20
6.3 Theta Lifts and Automorphic Form with a Zero Set Supported by the Discriminant Locus on $\Gamma \backslash \mathfrak{h}_{3,19}$

Suppose that $\Lambda_{p,q}$ is a unimodular even lattice. We will define the Siegel kernel $\Theta_{\Lambda_{p,q}}(\tau)$ as follows:

$$\Theta_{\Lambda_{p,q}}(\tau) := \sum_{\lambda \in \Lambda_{p,q}} \exp \left( 2\pi \sqrt{-1} \langle \text{Pr} E_{\tau} \lambda, \text{Pr} E_{\tau} \lambda \rangle \rho - \langle \text{Pr} E_{\tau} \lambda, \text{Pr} E_{\tau} \lambda \rangle \rho \right),$$

where $E_{\tau}$ is a $p$-dimensional real vector subspace in $\Lambda_{p,q} \otimes \mathbb{R}$ on which the quadratic form is positive definite, $E_{\tau}^\perp$ is the $q$-dimensional vector subspace in $\Lambda_{p,q} \otimes \mathbb{R}$ orthogonal to $E_{\tau}$, $\text{Pr} E_{\tau} \lambda$ is the orthogonal projection of $\lambda$ on $E_{\tau}$, and $\rho = x + iy, y > 0$.

The following result follows directly from the results proved in [11].

**Theorem 40** Let $\Lambda_{p,q}$ be an even unimodular lattice of signature $(p,q)$. Then there exists a non zero automorphic form $\exp \left( \Phi_{\Lambda_{p,q}}(\tau) \right)$ such that the zero set of $\exp \left( \Phi_{\Lambda_{p,q}}(\tau) \right)$ coincide with the discriminant

$$D_{\Lambda_{p,q}} \subset O_{\Lambda_{p,q}} \backslash \mathfrak{h}_{p,q}. $$

Moreover let $\Lambda_{p_1,q_1}$ be an even unimodular sublattice in $\Lambda_{p,q}$ such that $p - q = p_1 - q_1$. Then

$$\exp \left( \Phi_{\Lambda_{p,q}}(\tau) \right) |_{O_{\Lambda_{p_1,q_1}} \backslash \mathfrak{h}_{p_1,q_1}} = \exp \left( \Phi_{\Lambda_{p_1,q_1}}(\tau) \right).$$

**Proof:** Let us consider the regularized integral as described in [11] or in [20]

$$\Phi_{\Lambda_{p,q}}(\tau) = \int_{\mathcal{H}} \Theta_{\Lambda_{p,q}}(\tau) y^2 \frac{E(\rho) \, d\rho \wedge \overline{d\rho}}{\Delta(\rho) y^2},$$

where $\mathcal{H}$ is the fundamental domain of the group $\text{PSL}_2(\mathbb{Z})$, $\rho = x + iy, y > 0$ and $E(\rho) / \Delta(\rho)$ is a meromorphic automorphic form of weight $q - p$ with a pole of order one at $\infty$. Thus we have

$$\frac{E(\rho)}{\Delta(\rho)} = \frac{1}{\exp \left( 2\pi \sqrt{-1} \rho \right)} + a_0 + a_1 \exp \left( 2\pi \sqrt{-1} \rho \right) + ...$$

It was proved in [11] and in [20] that (22) implies that $\exp \left( \Phi_{\Lambda_{p,q}}(\tau) \right)$ will vanish on the discriminant of $O^*(\Lambda_{p,q}) \backslash \mathfrak{h}_{p,q}$.

The relation (21) follows from the condition $p - q = p_1 - q_1 = 8k$ and the definition of $\Phi_{\Lambda_{p,q}}(\tau)$. Theorem 40 is proved. ■

We will consider the case of K3 surfaces. We know that $\Lambda_{K3} = \Lambda_{3,19}$. We will study the relations between the non zero automorphic form $\exp \left( \Phi_{\Lambda_{K3}}(\tau) \right)$ and the regularized determinants.

**Theorem 41** $\Delta_B \Phi_{\Lambda_{K3}}(\tau,\sigma) = 0$, where $\Delta_B$ is the Laplacian of the Bergman metric on $\Lambda_{K3} = \Lambda_{3,19}$. 

21
Proof: Any choice of an embedding of the hyperbolic lattice $U \subset \Lambda_{K3}$ defines a totally geodesic subspace $h_{2,18} \subset h_{3,19}$. This follows from Theorem 6. According to the construction of the automorphic form $\exp(\Phi_{\Lambda_{18}}(\tau))$ given in [11] it follows that $\Phi_{\Lambda_{18}}$ is a holomorphic function on $h_{2,18}$. Thus we have $\Delta_B \Phi_{\Lambda_{18}} = 0$. All the embeddings $h_{2,18} \subset h_{3,19}$ corresponding to primitive embeddings $U \subset \Lambda_{K3}$ form an everywhere dense subset of totally geodesic submanifolds in $h_{3,19}$. Since $h_{2,18}$ is a totally geodesic subspace in $h_{3,19}$ we get that

$$\Delta_B (\Phi_{\Lambda_{K3}}|_{h_{2,18}}) = \Delta_B \Phi_{\Lambda_{18}}.$$  

Thus the restriction of the Bergman Laplacian applied to on $\Phi_{\Lambda_{18}}$ is zero on an everywhere dense subset in $h_{3,19}$. Thus the continuous function $\Delta_B \Phi_{\Lambda_{K3}}$ is zero on everywhere dense subset in $h_{3,19}$. From here we deduce that $\Delta_B \Phi_{\Lambda_{K3}} = 0$. Theorem 41 is proved. ■

6.4 The Analogue of the Kronecker Limit Formula for the Regularized Determinants on K3 Surfaces

Theorem 42 The function $\log \frac{\det \Delta_{KE}}{\det(\langle g_i(\tau), g_j(\tau) \rangle)}$ is a harmonic function on the moduli space $M_{KE}$ of Einstein metrics of the K3 surface with respect to the Laplacian corresponding to the Bergman metric.

Proof: The proof of Theorem 42 is based on the following Lemmas:

Lemma 43 Let $\tau_0 \in h_{3,19}$. Then there exists $L \in \Lambda_{K3} \otimes \mathbb{R}$ and totally geodesic subspace $h_{2,19}$ passing through $\tau_0 \in h_{3,19}$ such its points correspond to polarized marked K3 surfaces with class of polarization $L$.

Proof: We know that each point $\tau = (\tau_i^j) \in h_{3,19}$ corresponds to a three dimensional subspace $E_\tau \subset \Lambda_{K3} \otimes \mathbb{R}$ on which the cup product is strictly positive. Let $L \in E_\tau$ be a non zero vector. Then $\langle L, L \rangle > 0$. Let us consider the following set:

$$h_L := \{ E \subset \Lambda_{K3} \otimes \mathbb{R} | L \in E, \ dim \mathbb{C} E = 3 \land \langle , , \rangle |_E > 0 \} .$$

It is easy to see that there is one to one correspondence between the two dimensional oriented positive subspaces in the orthogonal complement $L^\perp = \mathbb{R}^{2,19}$ and $h_L$. Thus we get that

$$h_L = h_{2,19} = SO_0(2,19)/SO(2) \times SO(19).$$

Lemma 43 is proved. ■

Let us choose an orthonormal basis $e_1, e_2$ and $e_3 = L$ of the three dimensional subspace $E_{\tau_0} \in h_L$. Lemma 43 and Corollary 3 imply that the three dimensional subspaces $E_\tau$ that correspond $\tau \in h_L = h_{2,19} \subset h_{3,19}$ are spanned by the orthonormal vectors:

$$g_1(\tau) = e_1 + \sum_{i=1}^{19} \tau_i^1 e_i, \quad g_2(\tau) = e_2 + \sum_{i=1}^{19} \tau_i^2 e_i \quad \text{and} \quad g_3(\tau) = L = e_3. \quad (23)$$
Lemma 44 The subspace given by the equations \( \tau_i^3 = 0 \) for \( i = 1, \ldots, 19 \), where \( \tau_i^j \) are coordinates defined by \( \tau_i^3 = 0 \) is the totally geodesic subspace in \( \mathfrak{h}_L = \mathfrak{h}_{2,19} \) in \( \mathfrak{h}_{3,19} \).

Proof: The proof follows directly from (23). \( \blacksquare \)

We know that \( \mathfrak{h}_{2,19} \) is a complex manifold of dimension 19. The complex coordinates on \( \mathfrak{h}_{2,19} \) are defined as follows: \( \rho^i = \tau_i^1 + \sqrt{-1} \tau_i^2, \) \( 1 \leq i \leq 19. \) From the epimorphism of the period map we know that \( \tau_0 \) corresponds to a K3 surface \( X_{\tau_0} \) and the class of cohomology of the complex two form \( \omega_{\tau_0} := e_1 + \sqrt{-1} e_2 \in \Lambda_{K3} \otimes \mathbb{C} \) can be identified with the class of cohomology of the holomorphic two form \( \omega_{\tau_0}(2,0) \) on a marked K3 surface \( X_{\tau_0} \) such that the vector \( e_3 = L \) can be identified with the class of cohomology of the imaginary part of a Kähler metric on \( X_{\tau_0} \). The subspace in \( \Lambda_{K3} \otimes \mathbb{R} \) spanned by \( e_4, \ldots, e_{22} \) can be identified with the primitive class of cohomology of type \( (1,1) \), i.e. with \( H^{1,1}(X_{\tau_0}, \mathbb{R}) = E_{\tau_0}^2. \) See [27].

Definition 45 We will define the Weil-Petersson metric on the totally geodesic subspace \( \mathfrak{h}_{2,19} \) as the restriction of the metric on \( \mathfrak{h}_{3,19} \) defined by (9).

Lemma 46 Let \( \tau_0 \in \mathfrak{h}_{3,19} \). Let \( \mathfrak{h}_L = \mathfrak{h}_{2,19} \) be the totally geodesic subspace passing through \( \tau_0 \in \mathfrak{h}_{3,19} \) and defined by the \( L \in E_{\tau_0} \) as in Lemma 43. Let \( g_i(\tau) \) be vectors defined by (23). Then the function

\[
\log \det (\langle g_i(\tau), g_j(\tau) \rangle) |_{\mathfrak{h}_{2,19}}
\]

is a potential of the Weil-Petersson metric on \( \mathfrak{h}_{2,19} \).

Proof: The \( 2 \times 2 \) matrix \( (\langle g_i(\tau), g_j(\tau) \rangle) |_{\mathfrak{h}_{2,19}} \) is symmetric. Since

\[
\langle g_i(0), g_j(0) \rangle = \delta_{ij}
\]

it can be represented as follows:

\[
(\langle g_i(\tau), g_j(\tau) \rangle) |_{\mathfrak{h}_{2,19}} = I_2 + (h_{ij}(\tau)).
\]

Then we have:

\[
\log \det (\langle g_i(\tau), g_j(\tau) \rangle) |_{\mathfrak{h}_{2,19}} = \sum_{i=1}^{2} \log(1 + \lambda_i(\tau)), \quad (24)
\]

where \( \lambda_i(\tau) \) are the eigen values of the matrix \( (h_{ij}(\tau)) \). Thus we get

\[
\sum_{i=1}^{2} \lambda_i(\tau) = h_{11}(\tau) + h_{22}(\tau). \quad (25)
\]

From the definition of the matrix \( (\langle g_i(\tau), g_j(\tau) \rangle) |_{\mathfrak{h}_{2,19}} \) we get that

\[
h_{11} = \sum_{i=4}^{22} (\tau_i^1)^2 \quad \text{and} \quad h_{22} = \sum_{i=4}^{22} (\tau_i^2)^2. \quad (26)
\]
Combining (24), (25) and (26) we get that

$$\log \det \left( \langle g_i(\tau), g_j(\tau) \rangle \right) \bigg|_{h_{2,19}} = \frac{1}{4} \sum_{i=4}^{22} |\rho_i|^2 + O(3). \quad (27)$$

Thus we get from (27) that

$$dd^c \log \det \left( \langle g_i(\tau), g_j(\tau) \rangle \right) \bigg|_{h_{2,19}} = \sqrt{-1} \sum_{i=4}^{22} \partial \rho_i \wedge \overline{\partial \rho_i} + O(2). \quad (28)$$

From (28) we conclude the proof of Lemma 46. ■

**Lemma 47** Let $\Delta_B$ be the Laplacian of the Bergman metric on $\mathfrak{h}_{3,19}$. Then the restriction of the function

$$\log \frac{\det \left( \Delta_{KE} \right)}{\det \left( \langle g_i(\tau), g_j(\tau) \rangle \right)}$$

on each totally geodesic subspace $h_{2,19} \subset \mathfrak{h}_{3,19}$ is a harmonic function with respect of the Laplacian of the Weil-Petersson metric.

**Proof:** Combining Theorem 36 with Lemma 46 we deduce Lemma 47. ■

It is an obvious fact that the set of three dimensional positive subspaces in $\Lambda_{K3} \otimes \mathbb{R}$ which contain a vector in $\Lambda_{K3} \otimes \mathbb{Q}$ form an everywhere dense subset in $\mathfrak{h}_{3,19}$. From here it follows that we can find an everywhere dense subset of totally geodesic subspaces $h_{2,19}$ in $\mathfrak{h}_{3,19}$ on which the continuous function

$$\Delta_B \left( \log \det \Delta_{KE} - \log \det \left( \langle g_i(\tau), g_j(\tau) \rangle \right) \right)$$

is zero. Therefore it is zero on $\mathfrak{h}_{3,19}$. Theorem 42 is proved. ■

**Theorem 48** The following formula holds for the regularized determinant of the Laplacian of the Einstein metrics

$$\det(\Delta_{KE})(\tau) = \det \left( \langle g_i(\tau), g_j(\tau) \rangle \right) \times |\exp(\Phi_{K3}(\tau))|^2.$$

**Proof:** According to Theorem 12 the function

$$\log \det \Delta_{KE} - \log \det \left( \langle g_i(\tau), g_j(\tau) \rangle \right)$$

is a harmonic function with respect to the Laplacian of the Bergman metric on $\mathfrak{h}_{3,19}$. Let us consider the function:

$$\frac{\det \Delta_{KE}}{\det \left( \langle g_i(\tau), g_j(\tau) \rangle \right)} = \phi(\tau)$$

on $\mathfrak{h}_{3,19}$. According to Theorem 39 the function $\det \left( \langle g_i(\tau), g_j(\tau) \rangle \right)$ is an automorphic form of weight $-2$. Therefore the function $\phi$ is an automorphic function of weight 2. In 17 we proved that $\det \Delta_{KE}$ is a bounded non negative function.
Therefore the only zeroes of $\det \Delta_{KE}$ can be located on the discriminant locus $D_{KE}$. We know that $|\exp(\Phi_{A_{K3}}(\tau))|$ is an automorphic function with a zero set on the discriminant locus $D_{KE}$. Since $D_{KE}$ is an irreducible in $M_{KE}$, by taking suitable powers of $\phi$ and $|\exp(\Phi_{A_{K3}}(\tau))|$, we may assume that the function

$$\left(\frac{|\exp(\Phi_{A_{K3}}(\tau))|}{\phi^m}\right)^n = \psi$$

is a non zero function such $\Delta_B \log \psi = 0$. Thus we get a harmonic non zero function on $M_{KE}$.

**Lemma 49** $\psi|_{\mathfrak{M}_{\ell}} = \text{const.}$

**Proof:** Since

$$dd^c \left( \log \frac{\det(\Delta_{KE}(\tau))}{\det(g_i(\tau), g_j(\tau))} |\mathfrak{M}_{\ell} \right) = 0$$

we can conclude that

$$\frac{\det(\Delta_{KE}(\tau))}{\det(g_i(\tau), g_j(\tau))} |\mathfrak{M}_{\ell} = |\eta|,$$

where $\eta$ is a holomorphic automorphic form defined up to a character $\chi \in \Gamma_{\ell} \backslash [\Gamma_{\ell}, \Gamma_{\ell}]$ and with a zero set $D_{\ell}$. Since $D_{\ell}$ is an irreducible divisor, we can conclude that $\eta = \exp(\Phi_{A_{\ell}}(\tau))$. Thus since

$$\exp(\Phi_{A_{K3}}(\tau)) |\mathfrak{M}_{\ell} = \exp(\Phi_{A_{\ell}}(\tau)),$$

we get that $\psi|_{\mathfrak{M}_{\ell}} = \text{const.}$ Since any two $\mathfrak{M}_{\ell,1}$ and $\mathfrak{M}_{\ell,2}$ intersect. So the continuous function $\psi$ is a constant on an everywhere dense subset in $M_{KE}$. Thus $\psi$ is a constant. Lemma 49 is proved. ■

Lemma 49 imply Theorem 48. ■

### 7 Harvey-Moore-Borcherds Products and Counting Problems in A and B Models

#### 7.1 Counting Problems on K3

**Theorem 50** Let $X$ be an algebraic K3 surface such that $\text{Pic}(X)$ is an unimodular lattice. Then we have either $NS(X) = U \oplus E_8(-1)$ or $NS(X) = U \oplus E_8(-1) \oplus E_8(-1)$. Let $l \in NS(X)$ be the polarization class. Let us consider the components $V^+_{\text{Enr}}$ and $V^+_{\ell}$ of the positive cones in $(U \oplus E_8(-1)) \otimes \mathbb{R}$ and in $(U \oplus E_8(-1) \oplus E_8(-1) \otimes \mathbb{R}$ which contain the polarization vector $l$. Let us consider the discriminant automorphic forms $\exp(\Phi_{\text{Enr}}(\tau))$ and $\exp(\Phi_{\ell}(\tau))$ on $(U \oplus E_8(-1)) \otimes \mathbb{R} \oplus \sqrt{-1} V^+_{\text{Enr}}$ and on $((U \oplus E_8(-1)) \otimes \mathbb{R}) \oplus \sqrt{-1} V^+_{\ell}$. Then the
restriction of the functions \( \exp (\Phi_{ren}(\tau)) \) and \( \exp (\Phi_{ell}(\tau)) \) on the lines \( \sqrt{-1}\ell t \) are periodic. The Fourier expansions

\[
\frac{d}{dt} \left( \Phi_{ren}(\sqrt{-1}\ell t) \right) = -\sum_n a_n \frac{e^{-nt}}{1 - e^{-nt}}
\]

and

\[
\frac{d}{dt} \left( \Phi_{ell}(\sqrt{-1}\ell t) \right) = -\sum_n b_n \frac{e^{-nt}}{1 - e^{-nt}}
\]

(29)

have integer coefficients \( a_n \) and \( b_n \). \( a_n \) and \( b_n \) are equal to the number of non-singular rational curves of degree \( n \) on a K3 surface \( X \) with \( \text{NS}(X) = U \oplus E_8(-1) \) or \( \text{NS}(X) = U \oplus E_8(-1) \oplus E_8(-1) \).

**Proof:** Let us fix a bases \( \{\gamma_i\} \) and \( \{\epsilon_j\} \) of \( U \oplus E_8(-1) \) and \( U \oplus E_8(-1) \oplus E_8(-1) \) respectively. Then we fix the flat coordinates \( \{\tau_1^{\ast}, \ldots, \tau_{10}^{\ast}\} \) and \( \{\tau_1^\ast, ..., \tau_{18}^\ast\} \) in the symmetric spaces \( \mathfrak{h}_{2,10} \) and \( \mathfrak{h}_{2,18} \) respectfully represented as tube domains since we have

\[
\mathfrak{h}_{2,10} = (U \oplus E_8(-1)) \otimes \mathbb{R} + iV^+ \subset (U \oplus E_8(-1)) \otimes \mathbb{C}
\]

and

\[
\mathfrak{h}_{2,18} = (U \oplus E_8(-1) \oplus E_8(-1)) \otimes \mathbb{R} + iV^+ \subset (U \oplus E_8(-1) \oplus E_8(-1)) \otimes \mathbb{C},
\]

where \( V^+ \) is one of the components of the positive cone in \( (U \oplus E_8(-1)) \otimes \mathbb{R} \) or \( (U \oplus E_8(-1) \oplus E_8(-1)) \otimes \mathbb{R} \).

We will denote by \( \langle \delta, \tau \rangle \) the following expressions:

\[
\langle \delta, \tau \rangle = \sum_{i=1}^{10} \langle \delta, \gamma_i \rangle \, \tau^i \quad \text{and} \quad \langle \delta, \tau \rangle = \sum_{i=1}^{18} \langle \delta, \epsilon_i \rangle \, \tau^i.
\]

Then Harvey-Moore-Borcherds product formula states that there exist automorphic forms on \( \Gamma_{2,10} \backslash \mathfrak{h}_{2,10} \) or on \( \Gamma \backslash \mathfrak{h}_{2,18} \) which can be represented for some large \( \text{Im} \, \tau^i \) as the following products.

\[
\exp (\Phi_{ren}(\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{E_{ren}}^+} \left( 1 - \exp \left( 2\pi i \sum_{i=1}^{10} \langle \delta, \gamma_i \rangle \, \tau^i \right) \right)
\]

and

\[
\exp (\Phi_{ell}(\tau)) = \exp(2\pi i \langle \tau, w \rangle) \prod_{\delta \in \Delta_{E_{ren}}^+} \left( 1 - \exp \left( 2\pi i \sum_{i=1}^{18} \langle \delta, \epsilon_i \rangle \, \tau^i \right) \right).
\]

(30)
It was proved that \( \exp (\Phi_{Enr}(\tau)) \) and \( \exp (\Phi_{ell}(\tau)) \) have an analytic continuation in \( h_{2,10} \) and \( h_{2,18} \) and the zeroes remain the same. Substituting

\[
\sum_{i=1}^{10} \gamma_i \tau^i = i\ell t \quad \text{and} \quad \sum_{i=1}^{18} \varepsilon_i \tau^i = i\ell t
\]

in (30) we get

\[
\exp (\Phi_{Enr}(\tau)) = \exp(2\pi i \langle \tau, w \rangle \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t)))
\]

and

\[
\exp (\Phi_{ell}(\tau)) = \exp(2\pi i \langle \tau, w \rangle \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t))) . \tag{31}
\]

Let us split the irreducible non singular on disjoint finite sets \( A_n \), where \( A_n = \{ \delta \in \Delta^+ | \langle \delta, l \rangle = n \} \). Suppose that \( \# A_n = a_n \) in the case of \( \Lambda_{Enr} \) and \( \# A_n = b_n \) in the case \( \Lambda_{ell} \). We can rewrite (31) as follows

\[
\exp (\Phi_{Enr}(\tau)) = \exp(2\pi i \langle \tau, w \rangle \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t))) = \exp(2\pi i \langle \tau, w \rangle \prod_{n=1}^{\infty} \left( \prod_{\delta \in A_n} (1 - \exp (-2\pi nt)) \right) = \exp(2\pi i \langle \tau, w \rangle \prod_{n=1}^{\infty} ((1 - \exp (-2\pi nt))^{a_n})) . \tag{32}
\]

In the same way we will get that

\[
\exp (\Phi_{ell}(\tau)) = \exp(2\pi i \langle \tau, w \rangle \prod_{\delta \in \Delta_{Enr}^+} (1 - \exp (-2\pi \langle \delta, l \rangle t))) = \exp(2\pi i \langle \tau, w \rangle \prod_{n=1}^{\infty} \left( (1 - \exp (-2\pi nt))^{b_n} \right) . \tag{33}
\]

From (32) and (33) we derive (29) and thus Theorem 50. \( \blacksquare \)

### 7.2 A and B Models

**Remark 51** In the A model the automorphic function \( \exp (\Phi_{3,19}(\tau)) \) which is the holomorphic part of the regularized determinant when restricted on the line \( \mathbb{R}l \) in the Kähler cone of a K3 surface with \( \text{Pic}(X) \) unimodular lattice counts rational curves with a given volume according to Theorem 27.

We will consider the B model of \( M \)-marked K3 surfaces where \( M \) is an unimodular lattice and \( M = \text{Pic}(Y) \). The moduli space of \( \text{Pic}(Y) \)-marked K3 surfaces \( \mathfrak{M}_{\text{Pic}(Y)} \), where \( \text{Pic}(Y) \) is a unimodular lattice can be represented as a tube domain \( \mathbb{R}^k + i\mathbb{V}^+ \) modulo action of an arithmetic group \( \Gamma_{\text{Pic}(Y)} \). Now we will study the combinatorial properties of the restriction of the automorphic function \( \exp (\Phi_{4,20}(\tau)) \) on \( \mathfrak{M}_{\text{Pic}(Y)} \).
Lemma 52 Let $Y$ be a K3 surface. Let $g$ be a Calabi-Yau metric on $Y$. Let $\gamma \in H_2(Y,\mathbb{Z})$. We will call $\gamma$ a calibrated cycle if the restriction of $\alpha \Re \omega_Y + \beta \Im \omega_Y$ on $\gamma$ is the volume form of the restriction of the CY metric on $\gamma$.

Theorem 53 Suppose that $Y$ is a K3 surface such that $\Im \omega_Y \in H^2(Y,\mathbb{Z}) \cap H^{1,1}(Y,\mathbb{R})$. Let $g$ be a CY metric on $Y$. Then any $\delta \in T(Y) := Pic(Y)^{\perp} \subset H^2(Y,\mathbb{Z})$ such that $\langle \delta, \delta \rangle = -2$ can be realized as calibrated cycle. Then the restriction of the automorphic function $\exp(\Phi_{3,19}(\tau))$ on the line $\mathbb{R} \Im \omega_Y \subset \mathbb{R}^k + iV^+ \subset Pic(Y) \otimes \mathbb{C}$ is a periodic function such that the coefficients $a_n$ in front of $\frac{\exp(-int)}{1-\exp(-int)}$ are integer such that $a_n$ is equal to calibrated cycles $\delta$ such that $\operatorname{vol}(\delta) = n$.

Proof: We will prove the following Lemma:

Lemma 54 The 2-cycle $\delta \in T(Y) = Pic(Y)^{\perp} \subset H^2(Y,\mathbb{Z})$ on the K3 surface $Y$ such that $\langle \delta, \delta \rangle = -2$, and $\langle \delta, \Im \omega_Y \rangle > 0$ can be realized as calibrated cycle.

Proof: We know that $\Im \omega_Y \in H^{2.0}(Y,\mathbb{C}) \oplus H^{0.2}(Y,\mathbb{C}) \subset T(Y) \otimes \mathbb{R}$. Let us choose a CY metric $g$ on $Y$ such that

$$\langle \Im g, \delta \rangle = 0 \quad \text{and} \quad \langle \Im g, \omega_Y \rangle = 0.$$

Let us consider isometric deformation of $Y$ with respect to the CY metric $g$. From the properties of the isometric deformation of CY metrics on K3 surfaces studied in [27] we can change the complex structure on $Y$ in such a way that

1. $\langle \alpha \Re \omega_Y + \beta \Im \omega_Y, \delta \rangle > 0$ for some real numbers $\alpha$ and $\beta$,
2. the vector $\gamma \Re \omega_Y + \mu \Im \omega_Y$ in the three dimensional subspace in $H^2(Y,\mathbb{R})$ spanned by $\Re \omega_Y$, $\Im \omega_Y$ and $\Im g$ perpendicular to $\alpha \Re \omega_Y + \beta \Im \omega_Y$ is such that

$$\langle \gamma \Re \omega_Y + \mu \Im \omega_Y, \delta \rangle = 0$$

and

3. $\Im g$ and $\alpha \Re \omega_Y + \beta \Im \omega_Y$ will be realized as the imaginary part of a CY metric with respect to the new complex structure on $Y$. It is easy to see that the Poincare dual class of cohomology of $\delta$ can be realized as a form of type $(1,1)$ with respect to the new isometric complex structure on $Y$. Thus as it was proved in [25] $\delta$ can be realized as a rational non singular curve on the new K3 surface. Then the volume form of the restriction of CY metric with imaginary part $\alpha \Re \omega_Y + \beta \Im \omega_Y$ on the rational curve with class of homology $\delta$ will be $\Im \omega_Y$. Lemma 54 is proved. \hfill \blacksquare

Lemma 55 The restriction of the automorphic function $\exp(\Phi_{3,19}(\tau))$ on the line $\mathbb{R} \Im \omega_Y \subset \mathbb{R}^k + iV^+ \subset Pic(Y) \otimes \mathbb{C}$ is a periodic function such that the coefficients $a_n$ in front of $\frac{\exp(-int)}{1-\exp(-int)}$ are integer such that $a_n$ is equal to calibrated cycles $\delta$ such that $\operatorname{vol}(\delta) = n$.

Proof: The proof of Lemma 55 is exactly the same as the proof of Theorem 50. \hfill \blacksquare

Theorem 53 is proved. \hfill \blacksquare
there exists an automorphism $\sigma$ Let

Lemma 61

an isolated singularity of type $r_3$ with a monodromy group $r_3(v) = v + \langle v, \delta \rangle \delta$ for any $v \in \Lambda_{K3}$. This means that there exists a family of $K3$ surfaces $\pi : X \to D$ such that $X_0 = \pi^{-1}(0)$ has an isolated singularity of type $A_n$, $D_n$, or $E_6$, $E_7$ and $E_8$ and the monodromy operator acting on $H_2(X_t, \mathbb{Z})$ by the reflection $r_3$ described above. Thus in the B-model the partition function count invariant calibrated cycles with a given volume when we choose the complex structure such on the $K3$ surface with unimodular Picard group such that $\text{Im} \omega_Y \in H_2(X_t, \mathbb{Z})$.

Conjecture 57 The analogue of Theorem 53 holds for the B-model of CY threefolds, i.e. the partition function counts invariant vanishing calibrated cycles in the B-model when the monodromy operator is of infinite order.

8 The Canonical Class of the Moduli Space of Polarized Algebraic $K3$

8.1 The Projection Formula

Theorem 58 Let $l \in \Lambda_{K3}$ be a primitive vector such that $\langle l, l \rangle = 2n > 0$. Let $(l) \perp$ be the sublattice in $\Lambda_{K3}$ orthogonal to $\mathbb{Z}l$. Then we have $(l) \perp \cong \mathbb{Z}l^\ast \oplus \mathbb{U}^2 \oplus (-E_8)^2$, where $l^\ast$ is a primitive vector in $\Lambda_{K3}$ such that $\langle l^\ast, l^\ast \rangle = -2n < 0$.

Proof: According to [25] the subgroup $O^+_{\Lambda_{K3}}$ of index two that preserve the spinor norm acts transitively on the primitive vectors with a fixed positive self intersection. Let us fix $U$ in $\Lambda_{K3}$ with a basis $e_0$ and $e_1$ such that $\langle e_i, e_i \rangle = 0$ and $\langle e_1, e_2 \rangle = 1$. Then $l = e_1 + ne_2 \in U$ is a primitive vector such that $\langle l, l \rangle = 2n > 0$. Let $l^\ast = e_1 - ne_2 \in U$. Clearly $l^\ast$ is a primitive vector such that $\langle l, l^\ast \rangle = 0$ and $\langle l^\ast, l^\ast \rangle = -\langle l, l \rangle = -2n$. Then we have

$$(l) \perp \cong \mathbb{Z}l^\ast \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

(34)

Theorem 58 is proved. $\blacksquare$

Notation 59 Let $\Lambda_{K3,n} := \mathbb{Z}l^\ast \oplus \mathbb{U}^2 \oplus (-E_8)^2$ where $\langle l^\ast, l^\ast \rangle = -2n$. Let $\{e_1, e_2, f_1, f_2, g_1 \text{ and } g_2\}$ be a basis of $U \oplus U \oplus U \oplus \mathbb{Z}l$ such that $\langle e_i, e_i \rangle = \langle f_i, f_i \rangle = \langle g_i, g_i \rangle = 0$ and $\langle e_1, e_2 \rangle = \langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle = 1$.

Theorem 60 The orthogonal projection of the discriminant $\mathcal{D}_{3,19}$ on $\mathcal{M}_{K3,pa}^{2d}$ is $\mathcal{D}_n$, where $\mathcal{D}_n$ is the divisor in $\mathcal{M}_{K3,pa}^{2d}$ defined by the hyperplanes in $\mathcal{H}_{2,n}$ orthogonal to $l^\ast$ and all vectors $\delta \in \Lambda_n$ such that $\langle \delta, \delta \rangle = -2$.

Proof: The proof of Theorem 60 follows directly from the following Lemma:

Lemma 61 Let $\delta \in \Lambda_{K3}$ be such that $\langle \delta, \delta \rangle = -2$. Suppose that $\delta \notin \Lambda_{K3,n}$. Then there exists an automorphism $\sigma$ of the lattice $\Lambda_{K3,n}$ such that $\text{Pr}_U \sigma(\delta) = l^\ast$. 

29
Proposition 62 Let $\Lambda_{K3} = U \oplus L$ and let $e_1$ and $e_2$ be the isotropic generators of $U$. Let $l = e_1 + ne_2 \in U$ and $n > 0$. Suppose that $\langle \delta, \delta \rangle = -2$. Then there exists an element $\sigma \in \Gamma_n$ such that in the representation

$$\sigma(\delta) = n_1 e_1 + n_2 e_2 + \mu_{\sigma(\delta)}, \mu_{\sigma(\delta)}$$

satisfies

$$(Pr_U(\sigma(\delta)), Pr_U(\sigma(\delta))) < 0 \iff \langle \mu_{\sigma(\delta)}, \mu_{\sigma(\delta)} \rangle > 0.$$ \hfill (35)

Proof: Let $\delta = m_1 e_1 + m_2 e_2 + \mu$. Let us consider

$$\delta_1 = k_3 l^* + \mu_\delta \in \Lambda_{K3,n}, \langle \delta_1, \delta_1 \rangle = -2.$$ \hfill (36)

Then $\mu_\delta \in \langle l^* \rangle = L = U \oplus U \oplus \mathbb{E}(1) \oplus \mathbb{E}(1)$. Since $\langle l^*, l^* \rangle < 0$, \hfill (36) and $\langle \delta_1, \delta_1 \rangle = -2$ then $\langle \mu_\delta, \mu_\delta \rangle > 0$. Let us consider the reflection map

$$\sigma(\delta) = r_{\delta_1}(\delta) = \delta + \langle \delta, \delta_1 \rangle \delta_1,$$

where $\nu \in \Lambda_{K3,n}$. Let us compute the projection $Pr_U(\sigma(\delta))$ of $\sigma(\delta)$ on $U$ spanned by $e_1$ and $e_2$. Direct computations show that

$$Pr_U(2n\delta) = 2nm_1 e_1 + 2nm_2 e_2 = nm_1 (l + l^*) + m_2 (l - l^*) = (nm_1 + m_2) l + (nm_1 - m_2) l^*.$$ \hfill (37)

Direct computations show that

$$Pr_U(2n\sigma(\delta)) = (nm_1 + m_2) l - 2n (k_3, \langle \delta, \delta_1 \rangle + (nm_1 - m_2)) l^*.$$

Suppose that $nm_1 - m_2 \neq 0$. So

$$\langle Pr_U(2n\sigma(\delta)), Pr_U(2n\sigma(\delta)) \rangle = (nm_1 + m_2)^2 \langle l, l \rangle + (2n (k_3, \langle \delta, \delta_1 \rangle + (nm_1 - m_2)))^2 \langle l^*, l^* \rangle = n (nm_1 + m_2)^2 - n (2n (k_3, \langle \delta, \delta_1 \rangle + (nm_1 - m_2)))^2.$$ \hfill (38)

We can choose $\delta_1$ such that $|k_3|$ is big enough. Thus \hfill (38) will imply \hfill (35)

$$\langle Pr_U(2n\sigma(\delta)), Pr_U(2n\sigma(\delta)) \rangle = n (nm_1 + m_2)^2 - n (2n (k_3, \langle \delta, \delta_1 \rangle + (nm_1 - m_2)))^2 < 0.$$

Suppose that $nm_1 - m_2 = 0$. Then \hfill (37) implies

$$\delta = (m_1 n + m_2) l + \mu.$$
Thus $\Pr_U(\delta) = (m_1n + m_2)l$. Let us choose $\delta_1 = k_\delta l^* + \mu_\delta$, such that $\langle \delta_1, \delta_1 \rangle = -2$, and $\langle \delta, \delta_1 \rangle \neq 0$. Let us compute
\[
\langle r_\delta, (2n\delta) \rangle = 2n\delta + 2n \langle \delta, \delta_1 \rangle = (m_1n + m_2)l + \langle \delta, \delta_1 \rangle (k_\delta l^* + \mu_\delta). \tag{39}
\]
Thus (39) implies that
\[
\langle \Pr_U(2n\delta), \Pr_U(2nr_\delta) \rangle = (m_1n + m_2)^2 l + (m_1n + m_2)l + (\langle \delta, \delta_1 \rangle k_\delta l^*)^2 = (m_1n + m_2)^2 n - n (\langle \delta, \delta_1 \rangle k_\delta)^2. \tag{40}
\]
If we choose $\delta_1$ such that $\langle \delta, \delta_1 \rangle \neq 0$ and $|k_\delta|$ big is enough then (40) implies
\[
\langle \Pr_U(r_\delta), \Pr_U(2n\delta) \rangle = (m_1n + m_2)^2 n - n (\langle \delta, \delta_1 \rangle k_\delta)^2 < 0.
\]
Proposition 62 is proved. ■

**Proposition 63** Let $\Gamma_n$ be the generated by the reflections
\[
r_\delta : v \rightarrow \langle v, \delta \rangle \delta
\]
for
\[
\delta \in \mathbb{Z}l^* \oplus U \oplus U \oplus E_\delta(-1) \oplus E_\delta(-1).
\]
Let $\delta_{\text{min}} \in \{ \Gamma_n \delta \}$ be such that
\[
\langle \mu_{\delta_{\text{min}}}, \mu_{\delta_{\text{min}}} \rangle = \min_{\sigma \in G_n} \langle \mu_{\sigma(\delta)}, \mu_{\sigma(\delta)} \rangle \geq 0. \tag{41}
\]
Then $\langle \mu_{\delta_{\text{min}}}, \mu_{\delta_{\text{min}}} \rangle = 0$.

**Proof:** Let $\delta_{\text{min}} = pe_1 + qe_2 + \mu_{\delta_{\text{min}}}$. Then according to (??) we have
\[
\Pr_U(2n\delta_{\text{min}}) = (pn + q)l + (pn - q)l^*, \tag{42}
\]
where $l = e_1 + ne_2$ and $l^* = e_1 - ne_2$. So (41) implies
\[
\langle \Pr_U(2n\delta_{\text{min}}), \Pr_U(2n\delta_{\text{min}}) \rangle < 0
\]
which is equivalent to
\[
(pn + q)^2 - (pn - q)^2 < 0. \tag{43}
\]
Let us choose
\[
\langle \delta, \delta \rangle = -2 \quad \text{and} \quad \delta = k_\delta l^* + \mu_\delta. \tag{44}
\]
Let us consider $r_\delta(\delta_{\text{min}}) = \delta_{\text{min}} + \langle \delta, \delta_{\text{min}} \rangle \delta$. Direct computations using (41) and (43) show that
\[
r_\delta(2n\delta_{\text{min}}) = (pn + q)l + ((pn - q) + 2nk_\delta \langle \delta_{\text{min}}, \mu_\delta \rangle)l^* + \mu_{r_\delta(\delta)}. \tag{45}
\]
Remark 64 Let \( \delta = k_\delta l^* + \mu_\delta \) and \( \delta_1 = -k_\delta l^* + \mu_\delta \), satisfy \( \langle \delta, \delta \rangle = \langle \delta_1, \delta_1 \rangle = -2 \), then we can choose \( \mu_\delta \) to be such that the sign of \( \langle \delta_{\min}, \delta \rangle \) to be the same as that of \( \langle \delta_{\min}, \delta_1 \rangle \).

Proof: Let \( f_1 \) and \( g_i \) be the generators of \( U \oplus U \), where \( \langle f_1, f_i \rangle = \langle g_i, g_i \rangle = 0 \) and \( \langle f_1, f_2 \rangle = \langle g_1, g_2 \rangle = 1 \). Then we can choose

\[
\mu_{\delta_{\min}} = f_1 + \frac{\langle \mu_{\delta_{\min}}, \mu_{\delta_{\min}} \rangle}{2} f_2, \quad \mu_{\delta_1} = g_1 + \frac{\langle \mu_{\delta_1}, \mu_{\delta_1} \rangle}{2} g_2 + m f_2. \tag{46}
\]

Then it is clear that

\[
\langle \delta_{\min}, \delta \rangle = \langle pe_1 + qe_2, k_\delta (e_1 + ne_2) \rangle + \langle \mu_{\delta_{\min}}, \nu \rangle = k_\delta q + \langle \mu_{\delta_{\min}}, \nu \rangle.
\]

On the other hand we derive from (46)

\[
\langle \delta_{\min}, \delta_1 \rangle = -k_\delta q + \langle \mu_{\delta_{\min}}, \nu \rangle = -k_\delta q + m. \tag{47}
\]

It is clear that we can choose \( m \) such that the sign of \( \langle \delta_{\min}, \delta \rangle \) to be the same as the sign of \( \langle \delta_{\min}, \delta_1 \rangle \). Remark 64 is proved.

Thus Remark 64 implies that we can choose the sign of \( k_\delta \) in the expression of \( \delta \) such that the sign of \( k_\delta \langle \delta_{\min}, \delta \rangle \) to be the opposite of the sign of \( (pn - q) \). So

\[
((pn - q) + 2nk_\delta \langle \delta_{\min}, \delta \rangle)^2 < (pn - q)^2. \tag{48}
\]

Thus (43) and (48) imply that

\[
\langle \Pr_U (2nr_\delta (\delta_{\min})), \Pr_U (2nr_\delta (\delta_{\min})) \rangle = 4n^2 \left( (pn + q)^2 - ((pn - q) + 2nk_\delta \langle \delta_{\min}, \delta \rangle)^2 \right) < 0. \tag{49}
\]

So (49) implies that

\[
\langle \mu_{r_{\delta}} (\delta_{\min}), \mu_{r_{\delta}} (\delta_{\min}) \rangle \geq 0. \tag{50}
\]

Since

\[
2nr_\delta (\delta_{\min}) = \Pr_U (2nr_\delta (\delta_{\min})) + \mu_{r_{\delta}} (\delta_{\min})
\]

then (50), (48) and (49) show that

\[
\langle 2n\mu_{\min}, 2n\mu_{\min} \rangle = -8n^2 + 4n^2 \left( (pn - q)^2 - (pn + q)^2 \right) >
\]

\[
-8n^2 + 4n^2 \left( ((pn - q) + 2nk_\delta \langle \delta_{\min}, \delta \rangle)^2 - (pn + q)^2 \right) =
\]

\[
\langle 2n\mu_{r_{\delta}} (\delta_{\min}), 2n\mu_{r_{\delta}} (\delta_{\min}) \rangle > 0.
\]

So we get that

\[
\langle \mu_{\min}, \mu_{\min} \rangle > \langle \mu_{r_{\delta}} (\delta_{\min}), \mu_{r_{\delta}} (\delta_{\min}) \rangle \geq 0. \tag{51}
\]

Thus we get a contradiction with \( \| \mu_{\delta_{\min}} \|^2 > 0 \) being the minimal value. Proposition 63 is proved. ■ Proposition 63 implies Lemma 61 ■ Lemma 61 implies Theorem 60 ■
8.2 The Divisor of the Restriction of the Automorphic Form on \( \mathcal{M}_{K3,n} \)

Let us consider the moduli space \( \mathcal{M}_{K3,n} \) of pseudo polarized algebraic K3 surfaces with a polarization class \( l \in \Lambda_{K3} \), where \( l \) is a primitive vector in \( \Lambda_{K3} \) such that \( \langle l, l \rangle = 2n \). Then according to [24] and [12] we have \( \mathcal{M}_{K3,n} = \Gamma_n \setminus \mathfrak{h}_{2,19} \), where \( \Gamma_n := \{ \phi \in \mathcal{O}^+_{\Lambda_{K3}} \mid \phi(l) = l \} \). According to [27] we can define \( \mathfrak{h}_{2,19} \) as one of the open components of the quadric \( Q \subset \mathbb{P}(\Lambda_{K3,n} \otimes \mathbb{C}) \) defined as follows

\[
Q := \{ u \in \mathbb{P}(\Lambda_{K3,n} \otimes \mathbb{C}) \mid \langle u, u \rangle = 0 \text{ and } \langle u, \overline{\pi} \rangle > 0. \}
\]

Let us define \( \mathcal{D}_n \) in \( \mathcal{M}_{K3,n} \) as follows: Let \( \lambda \in \Lambda_{K3,n} \), then

\[
\mathcal{H}_\lambda := \{ u \in \mathbb{P}(\Lambda_{K3,n} \otimes \mathbb{C}) \mid \langle u, \lambda \rangle = 0. \}
\]

Let

\[
\mathcal{D}_n := \left( \bigcup_{\langle \delta, \delta \rangle = -2 \text{ and } \delta \in \Lambda_{K3,n}} (\mathfrak{h}_{2,19} \cap \mathcal{H}_\delta) \right) \cup \left( \bigcup_{\phi \in \Gamma_n} (\mathfrak{h}_{2,19} \cap \mathcal{H}_{\phi(l^*)}) \right). \tag{52}
\]

Then \( \mathcal{D}_n := \Gamma_n \setminus \mathcal{D}_n \).

**Theorem 65** There exists an automorphic form \( \Psi_{19,n} \) on \( \mathcal{M}_{K3,n} = \Gamma_n \setminus \mathfrak{h}_{2,19} \) such that the zero set of \( \Psi_{19,n} \) is \( \mathcal{D}_n \).

**Proof:** According to the results of Harvey, Moore and Borcherds on we can find an automorphic form \( |\Psi_{\Lambda_{K3}}|^2 \) on the moduli space of Einstein metrics \( \mathcal{O}^+_{\Lambda_{K3}} \setminus \mathfrak{h}_{3,19} \) such that its zeros are exactly on the discriminant locus of \( \mathcal{O}^+_{\Lambda_{K3}} \setminus \mathfrak{h}_{3,19} \). Recall that the discriminant locus on \( \mathcal{O}^+_{\Lambda_{K3}} \setminus \mathfrak{h}_{3,19} \) is defined as the set of three dimensional positive vector subspaces in \( \Lambda_{K3} \otimes \mathbb{R} \) perpendicular to \( \delta \) such that \( \langle \delta, \delta \rangle = -2 \) modulo the action of the arithmetic group \( \mathcal{O}^+_{\Lambda_{K3}} \). The moduli space \( \mathcal{M}_{K3,n} = \Gamma_n \setminus \mathfrak{h}_{2,19} \) can be embedded in \( \mathcal{O}^+_{\Lambda_{K3}} \setminus \mathfrak{h}_{3,19} \) as the set of all three dimensional oriented subspaces in \( \Lambda_{K3} \otimes \mathbb{R} \) containing the polarization vector \( l \) modulo the action of \( \mathcal{O}^+_{\Lambda_{K3}} \). The restriction of some power of \( \Psi_{\Lambda_{K3}} \) on \( \mathcal{M}_{K3,n} \) will give us an automorphic form \( \Psi_{19,n} \) on \( \mathcal{M}_{K3,n} \). Thus we have the following obvious fact:

**Remark 66** The zero set of the restriction of \( \Psi_{\Lambda_{K3}} = \exp(\Phi_{\Lambda_{K3}}(\tau)) \) on \( \mathcal{M}_{K3,n} \) is the projection of the zero set of \( \Psi_{\Lambda_{K3}} = \exp(\Phi_{\Lambda_{K3}}(\tau)) \) on \( \mathcal{M}_{K3,n} \).

Thus we need to compute the projection of the zero set of \( \exp(\Phi_{\Lambda_{K3}}) \) on \( \Gamma^+ \setminus \mathfrak{h}_{3,19} \) to \( \mathcal{M}_{K3,n} = \Gamma_n \setminus \mathfrak{h}_{2,19} \). Theorem 65 will follow from the following Lemma:

**Lemma 67** The zero set of \( \Psi_{19,n} \) on \( \mathcal{M}_{K3,n} \) is \( \mathcal{D}_n \).
Proof: Let $\delta \in \Lambda_{K_3}$ be such that $\langle \delta, \delta \rangle = -2$. Let $Pr_{l,n}(\delta) \in \Lambda_{K_3,n}$ be the orthogonal projection of $\delta$ on $\Lambda_{K_3,n}$. If

$$Pr_{l,n}(\delta) = \delta \iff \langle l, \delta \rangle = 0,$$

then it implies that the component

$$\bigcup_{(\delta, \delta) = -2 \& \delta \in \Lambda_{K_3,n}} (h_{2,19} \cap H_{\delta})$$

defines the components of $\mathcal{D}_n := \Gamma_n \setminus \mathcal{D}_n$ corresponding to the vectors with $-2$ norm in $\Lambda_{K_3,n}$.

Suppose that $\delta \in \Lambda_{K_3}$, $\langle \delta, \delta \rangle = -2$ and $Pr_{l,n}(\delta) \neq \delta$. Theorem 60 implies that we can find $\sigma \in \Gamma_n$ such that $\sigma(\delta) = m_1 e_1 + m_2 e_2$. Thus $Pr_{l,n}(\delta) = k_{l,l^*}$.

Then

$$\pi(H_{\delta} \cap h_{2,19}) = \pi(H_{l^*} \cap h_{2,19}) \quad (53)$$

where $\pi : h_{2,19} \to \Gamma_n \setminus h_{2,19}$. Thus (53) implies Lemma 67.

Theorem 65 is proved.

Corollary 68 The zero set of the restriction of the automorphic form $\Psi_{\Lambda_{K_3}} = \exp(\Phi_{\Lambda_{3,19}}(\tau))$ on $\mathcal{M}_{K_3,n}$ is a divisor $\mathcal{D}_n := \Gamma_n \setminus \mathcal{D}_n$ which consists of two components $\pi(H_{l^*} \cap h_{2,19})$ and $\pi(H_{\delta} \cap h_{2,19})$, where $\delta \in (l^*)^\perp = U \oplus U \oplus E_8(-1) \oplus E_8(-1)$ and $\pi : h_{2,19} \to \Gamma_n \setminus h_{2,19} = \mathcal{M}_{K_3,n}$.

Proof: Corollary 68 follows from Theorem 11 which implies that the divisor $\pi(H_{\delta} \cap h_{2,19})$ is an irreducible since we assumed that $\delta \in (l^*)^\perp = U \oplus U \oplus E_8(-1) \oplus E_8(-1)$. The irreducibility of $\pi(H_{l^*} \cap h_{2,19})$ follows from Theorem 60.

Corollary 68 is generalization of the results obtained in [18, 17] and [19].

References

[1] P. Aspinwall and D. Morrison, "String Theory on K3 Surfaces", Mirror Symmetry II, (B. Greene and S.-T. Yau ed.), International Press, Cambridge, 1997, 703-716.

[2] "Géométrie des Surfaces K3: modules and périodes." Astérisque 126 Paris: Société Mathématique de France (1985).

[3] W. Baily and Borel, "On Compactification of Arithmetically Defined Quotients of Bounded Symmetric Domains", Bull. Amer. Math. Society 70(1964) 588-593.

[4] W. Baily and Borel, On Compactification of Arithmetic Quotients of Bounded Symmetric Domains", Ann. Math. (2) 84(1966) 442-528.

[5] W. Barth, C. Peters and A. Van de Ven, "Compact Complex Surfaces", Ergebnisse der Math. 4 New York, Springer-Verlag (1984).

[6] J. Bass, A. Todorov, "The Analogue of the DedekindEta Function for CY Manifolds I.", to appear in Crelle.
[7] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, "Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitude", Comm. Math. Phys. 165 (1994), 311-428.

[8] R. Borcherds, "The Moduli of Enriques Surfaces and the Fake Monster Lie Superalgebra", Topology 35(1996), 699-710.

[9] A. Clinger and Ch. Doran, "On K3 Surfaces with Large Complex Structure", math.AG/0508249.

[10] A. Clinger and J. Morgan, "Mathematics Underlying F-Theory/Heterotic String Duality in Eight Dimensions, math.AG/0308106.

[11] R. Borcherds, "Automorphic Forms with Singularities on Grassmanians", Inv. Math.132 (1998), 491-562

[12] S. Donaldson, "Polynomial Invariants for Smooth 4-Manifolds", Topology 29(1990), No. 3, 257-315.

[13] I. Dolgachev, "Mirror Symmetry for Lattice Polarized K3 Surfaces", math.AG/9502005.

[14] Ph. Griffiths and J. Harris, "Principle of Algebraic Geometry",

[15] Horikawa, "Surjectivity of the Period Map of K3-surfaces of Degree 2." Math. Ann. 104 (1978), 113-146.

[16] G. Harvey and G. Moore, "On the Algebra of BPS States", Com. Math. Physics, 197 (1998) 489-519.

[17] J. Jorgenson and A. Todorov, "Analytic Discriminant for Manifolds with Zero Canonical Class", Manifolds and Geometry, ed. P. de Bartolomeis, F. Tircerri and E. Vesantini, Symposia Mathematica 36, (1996) 223-260.

[18] J. Jorgenson and A. Todorov, "Analytic Discriminant for Polarized Algebraic K3 Surfaces", Mirror Symmetry III, ed. S.-T. Yau and Phong, AMS(1998), p. 211-261.

[19] J. Jorgenson and A. Todorov, "A Conjectural Analogue of Dedekind Eta Function for K3 Surfaces", Math. Research Lett. 2(1995) 359-360.

[20] M. Kontsevich, "Product Formulas for Modular Forms on O(2,n)", Sémairie Bourbaki, 49ème anné, 1996-97, n° 821.

[21] K. Kodaira, "On the Structure of Compact Analytic Surfaces I", Amer. J. Math. 86 (1964), 751-798.

[22] R. Kobayashi and A. Todorov, "Polarized Period Map for Generalized K3 Surfaces and the Moduli of Einstein Metrics.", Tohoku Journal of Math. vol. 39, No 3(1987).
[23] V. Kulikov, "Degeneration of K3 Surfaces and Enriques Surfaces", USSR Izv. Ser. Math. 11 (1977) 957-989.

[24] A. Mayer, "Families of K3 Surfaces", Nagoya Math. J. 48 (1972), 1-17.

[25] I. I. Piatetski-Shapiro and I. R. Shafarevich, "A Torelli Theorem for Algebraic Surfaces of Type K3", USSR Izv. Ser. Math. 5 (1971) 547-588.

[26] J. P. Serre, "A course in Arithmetic", Graduate Text in Mathematics 7, Springer Verlag (1973).

[27] A. N. Todorov, "Applications of Kähler-Einstein-Calabi-Yau metric to Moduli of K3 Surfaces", Inv. Math. 61 (1980) 251-265.

[28] A. N. Todorov, "Applications of Some Ideas of Mirror Symmetry to Moduli of K3 Surfaces", preprint, 1993.

[29] A. Weil, "Collected Work" Springer-Verlag, vol. 3.