Searching for gravitational-wave signals emitted by eccentric compact binaries using a non-eccentric template bank: implications for ground-based detectors

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Abstract

Most of the inspiralling compact binaries are expected to be circularized by the time their gravitational-wave signals enter the frequency band of ground-based detectors such as LIGO or VIRGO. However, it is not excluded that some of these binaries might still possess a significant eccentricity at a few tens of hertz. Despite this possibility, current search pipelines—based on matched filtering techniques—consider only non-eccentric templates. The effect of such an approximation on the loss of signal-to-noise ratio (SNR) has been investigated by Martel and Poisson (1999 Phys. Rev. D 60 124008) in the context of initial LIGO detector. They ascertained that non-eccentric templates will be successful at detecting eccentric signals. We revisit their work by incorporating current and future ground-based detectors and precisely quantify the exact loss of SNR. In order to be more faithful to an actual search, we maximized the SNR over a template bank, whose minimal match is set to 95%. For initial LIGO detector, we claim that the initial eccentricity does not need to be taken into account in our searches for any system with total mass $M \in [2–45]M_\odot$ if $e_0 \lesssim 0.05$ because the loss of SNR (about 5%) is consistent with the discreteness of the template bank. Similarly, this statement is also true for systems with $M \in [6–35]M_\odot$ and $e_0 \lesssim 0.10$. However, by neglecting the eccentricity in our searches, significant loss of detection (larger than 10%) may arise as soon as $e_0 \gtrsim 0.05$ for neutron-star binaries. We also provide exhaustive results for VIRGO, Advanced LIGO and Einstein Telescope detectors. It is worth noting that for Einstein Telescope, neutron star binaries with $e_0 \gtrsim 0.02$ lead to a 10% loss of detection.

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(Some figures in this article are in colour only in the electronic version)
1. Introduction

Inspiralling compact binaries are one of the most promising sources of gravitational-wave signals for interferometric ground-based detectors such as LIGO [1] and VIRGO [2]. The phase and amplitude of the signal strongly depend on the masses of the two compact objects, the spin effects (neglected in this paper) and eccentricity. However, it is known that most of the inspiralling compact binaries would be circularized by the time they enter the lower cut-off frequency of the detectors ($F_L$ hereafter, which is about 20 Hz to 40 Hz). Indeed, the eccentricity of an isolated compact binary evolving under the effect of gravitational radiation reaction is reduced by a factor of three when the semi-major axis is halved [3]. For instance, even though the orbital eccentricity of the Hulse and Taylor binary pulsar is $e_0 = 0.617$ (from the last measurements published in [4]), it would decrease to $10^{-6}$ by the time the orbital period of the binary reaches 0.02 s [5]. Therefore, any isolated binary star formed by stellar evolution would be circularized by the time its signal becomes measurable by ground-based detectors.

Nevertheless, in addition to the usual route to binary star formation, there exists a different formation scenario in densely populated regions (e.g., in globular clusters) where binary–binary interactions can produce hierarchical triplets of black holes. The inner binaries of these triplets (undergoing Kozai oscillations) can merge under gravitational radiation reaction with large eccentricity [6]. Consequently, even at the lower cut-off frequency of ground-based detectors, there are compact binaries with significant eccentricity. As predicted in [7], about 30% of the binaries will possess eccentricities larger than 0.10 when their emitted gravitational-wave signals reach a frequency of 10 Hz. There is also semi-analytical description of the final stages of mergers of black-hole neutron-star (BHNS) binaries showing that they may transit to inspiralling compact binaries in eccentric orbits [8].

More recently, in [9], the authors study the case of stellar mass black holes (BHs) that segregate and form a steep density cusp around supermassive black holes in galactic nuclei. They found that BH binaries that form this way in galactic nuclei have an expected rate of coalescence events detectable by Advanced LIGO about 1–100 per year, depending on the initial mass function of stars in galactic nuclei and the mass of the massive BHs. The authors also claimed that such BH binaries have significant eccentricities as they enter the LIGO band (90% with $e > 0.9$), and hence they are distinguishable from other circularized binaries. They also show that mergers of eccentric binaries can be detected to larger distances and greater BH masses than circular mergers, up to ($\sim 700M_\odot$).

Despite the fact that binaries may have non-negligible eccentricities, current searches for inspiralling compact binaries neglect this parameter. For instance, the matched filtering technique that is deployed to analyze scientific runs of LIGO data considers templates with circularized orbits only [10]. Yet, matched filtering is sub-optimal if the template is not a faithful representation of the signal that is searched for. In a previous work, Martel and Poisson [11] studied the loss of SNR when signals from eccentric binaries are filtered with templates based on circularized binaries. They performed this study in the context of initial LIGO. For instance, in the case of a $(1.4 + 1.4)M_\odot$ binary with an initial eccentricity of 0.05, they found that only a 2.4% loss of SNR is expected. This result combined with the idea that eccentricities are small when a gravitational-wave signal enters a detector’s band, led to searches where eccentricity is systematically neglected.

In [11], the authors considered a continuous space of templates to compute the loss of SNR. In this paper, we consider a more realistic situation by using a discrete grid of templates, which is identical to that used in LIGO searches (where eccentricity is neglected). We also extend our target sources not only to binary neutron star (BNS) and low mass binary black hole
(BBH) but also to large mass BBH and BHNS. Our goal is to provide a precise description of the loss of SNR when waveforms from eccentric inspiralling compact binaries are searched with waveforms from circularized binaries.

In section 2, we describe the relevant features of the inspiralling compact binaries. We also emphasize the differences between our waveforms and those obtained in [11]. In section 3.2, we describe the simulation protocol and present exhaustive results showing the loss of SNR for different detectors, namely initial LIGO, Advanced LIGO, Einstein Telescope and VIRGO detectors. Finally, in section 4, we discuss the implications of our results for ground-based detectors and future directions for this work.

2. The eccentric model

2.1. Eccentric waveform calculation

The evolution of the orbital elements of an eccentric binary was first calculated by Peters and Mathews [3, 12]. The explicit expressions of the gravitational-wave signals were first derived in [13], in the context of spacecraft Doppler detection of gravitational waves. Although eccentric waveforms have been derived to a higher post-Newtonian order [14], we restrict ourselves to the case where the binary’s orbital motion is governed by Newtonian gravity and orbital elements by the quadrupole formula. We use the same notation as in [11]. Let us start from the orbital radius \( r \) that is given by

\[
r = \frac{pM}{1 + e \cos \phi},
\]

(1)

where \( p \) is the semi-latus rectum, \( M \) is the total mass, \( e \) is the eccentricity and \( \phi \) is the orbital phase. The semi-major axis is related to \( p \) and \( e \) by the relation

\[
a = \frac{pM}{1 - e^2}.
\]

(2)

Note that the orbital period is defined by

\[
P = 2\pi M \left( \frac{p}{1 - e^2} \right)^{\frac{1}{2}}.
\]

(3)

The decay of orbital elements \( p \) and \( e \) is calculated in the quadrupole approximation [3]. Together with the orbital phase evolution, we have three ordinary differential equations (ODE) that can be used to describe the evolution of the system and form the starting point to determine the eccentric waveforms

\[
\frac{d\phi}{dt} = \frac{(1 + e \cos \phi)^2}{p^2 M},
\]

(4)

\[
\frac{dp}{dt} = -\frac{64}{5} \frac{\mu}{M^2} (1 - e^2)^{3/2} \frac{1 + 7/8 e^2}{p^3},
\]

(5)

\[
\frac{de}{dt} = -\frac{304}{15} \frac{\mu}{M^2} (1 - e^2)^{3/2} \frac{1 + 121/304 e^2}{p^4}.
\]

(6)

In these equations \( \mu = m_1 m_2 / (m_1 + m_2) \) is the reduced mass and, \( m_1 \) and \( m_2 \) are the two component masses. We can convert equations (4), (5) and (6) into the same notation as in [3], by replacing \( p \) by \( a \) using equation (2).

In order to compute eccentric waveforms, we need to solve the system of three ODEs as defined by equations (4), (5) and (6). The method employed is described in details in section 2.2.
Once the evolution of the three parameters \( e, p \) and \( \phi \) are known, we can generate the eccentric waveforms. The source and the gravitational-wave detector are separated by a distance, \( R \). The detector is placed in a direction defined by the polar angles \( \iota \) and \( \beta \) relative to the Cartesian frame (following standard convention [16]). The unit vectors defined by \( \hat{\iota} \) and \( \hat{\beta} \) are chosen as polarization axes. The metric perturbation is given by

\[
h = F_+ s_+ + F_\times s_\times,
\]

where \( F_+ \), \( F_\times \) are the detector’s beam factor [16] and the two fundamental polarizations of the gravitational waves [13] are defined as follows:

\[
s_+ = -\frac{\mu_p}{p R} \left[ 2 \cos(2\phi - 2\beta) + \frac{5e}{2} \cos(\phi - 2\beta) + \frac{e^2}{2} \cos(3\phi - 2\beta) + e^2 \cos(2\beta) \right] \times (1 + \cos^2 \iota) + [e \cos(\phi) + e^2] \sin^2 \iota.
\]

\[
s_\times = -\frac{\mu_p}{p R} [4 \sin(2\phi - 2\beta) + 5e \sin(\phi - 2\beta) + e \sin(3\phi - 2\beta) - 2e^2 \sin(2\beta)] \cos \iota.
\]

In section 3.2, we will see how \( \iota \) and \( \beta \) affect the waveform shape. It is important to note that the signal can be decomposed into components that oscillate at once, twice and three times the orbital frequency; we consider only the first three harmonics. The presence of the first and third harmonics depends on the presence of eccentricity. Indeed if we set \( e_0 = 0 \), the remaining terms in equations (7) and (8) contain components that oscillate at twice the orbital frequency only (second harmonic).

We also need the initial conditions for the semi-latus rectum, \( p_0 \), the orbital phase, \( \phi_0 \) and the eccentricity, \( e_0 \). The initial conditions on \( e_0 \) and \( \phi_0 \) are set arbitrarily and are specified in section 3.2. The remaining initial condition, \( p_0 \), is derived from the initial eccentricity, \( e_0 \). Using equation (3), we have

\[
p_0 = \frac{1 - e_0^2}{(2\pi M f_0)^{2/3}},
\]

where \( f_0 \) depends on the lower cut-off frequency, \( F_L \), at which the signal enters a detector’s sensitivity band. If we want the waveform to be valid from \( F_L \) onward, we must have the third harmonic to start at \( F_L \). The consequence is that the first harmonic, which contributes to the third, must start at \( F_L / 3 \). So, in equation (9), we must set \( f_0 = F_L / 3 \).

Finally, let us have a short digression on the waveform duration, which may lead to difficulties from the point of view of data analysis. In the case of an eccentric waveform implementation that depends on equation (9), the second harmonic starts at \( 2F_L / 3 \) even though \( e_0 = 0 \) and the first and third harmonic are null. In the case of non-eccentric waveform implementation, starting the waveform at \( F_L \) is sufficient. Therefore, the durations of those two waveforms differ even when \( e_0 = 0 \). The ratio between the two durations is proportional to \( (2/3)^{-8/3} \). In the case of a standard non-eccentric waveform implementation \( (f_0 = F_L) \), a \( (1.4+1.4)M_\odot \) binary lasts 25 s if \( F_L = 40 \) Hz, and 158 s if \( F_L = 20 \) Hz. These durations have to be compared to an eccentric implementation where \( f_0 = F_L / 3 : 74 \) and 465 s, respectively. Dealing with such long waveforms may lead to technical issues from a computational point of view (e.g., memory allocation).

2.2. Eccentric waveforms: validation and investigations

2.2.1. ODE integrator and waveform validation. The main difficulty in generating eccentric waveforms from equations (7) and (8) resides in the resolution of the system of differential equations. This is done by using a numerical code based on the GNU Scientific Library [19].
Table 1. Duration of the gravitational-wave signal as a function of initial eccentricity and component masses. The lower cut-off frequency is $F_L = 40$ Hz. The numbers provided in this table show a systematic difference of about 5% compared to numbers from table I of [11]; we believe that our results are more accurate than those from [11] (see the text for an explanation).

Because the duration decreases when eccentricity and total mass increase, there is a combination for which the ending frequency of the waveform is below $F_L$. In such a case, no waveform can be generated, which is represented by the / symbol. In brackets, we also provide the ratio of the duration of an eccentric waveform with respect to the first line, where $e_0 = 0$. This ratio is independent of the component masses (see figure 2 and the text for more details).

| $e_0$ | $1.4 + 1.4$ | $1.4 + 10$ | $5 + 5$ | $10 + 10$ | $20 + 20$ |
|-------|-------------|------------|--------|-----------|-----------|
| 0     | 73.30(1.00) | 16.37(1.00)| 8.77(1.00)| 2.75(1.00)| 0.85(1.00)|
| 0.1   | 70.66(0.96) | 15.78(0.96)| 8.46(0.96)| 2.65(0.96)| 0.82(0.96)|
| 0.2   | 63.18(0.86) | 14.11(0.86)| 7.56(0.86)| 2.37(0.86)| 0.73(0.86)|
| 0.3   | 52.09(0.71) | 11.63(0.71)| 6.23(0.71)| 1.95(0.71)| 0.60(0.71)|
| 0.4   | 39.13(0.53) | 8.73(0.53) | 4.68(0.53)| 1.46(0.53)| 0.43(0.50)|
| 0.5   | 26.22(0.36) | 5.84(0.36) | 3.13(0.36)| 0.97(0.35)| 0.27(0.31)|
| 0.6   | 15.09(0.21) | 3.35(0.20) | 1.80(0.20)| 0.55(0.20)| 0.06(0.07)|
| 0.7   | 6.92(0.09)  | 1.52(0.09) | 0.82(0.09)| 0.23(0.08)| /         |
| 0.8   | 2.12(0.03)  | 0.40(0.02) | 0.23(0.02)| /         | /         |

The ODE integrator is based on a Runge–Kutta–Fehlberg (4, 5) method. The step size is taken to be the inverse of the sampling frequency ($4 kHz$). The stopping condition is the same as in [11], which is $p = 6$. Using this method, we compute the waveform durations as a function of initial eccentricity and component masses (see table 1). Our results do not agree with those obtained in [11]: durations generated by our code are consistently 5% lower than those computed in [11]. This effect has been seen independently [27]. We performed another sanity check in the case where $e_0 = 0$: we compare our waveform with a non-eccentric waveform (at Newtonian order) that is generated with the LIGO Algorithm Library (LAL) [18]. We found a perfect agreement\(^1\) between the two waveforms. So, we think that the results in [11] may be biased due to a lack of accuracy in the evolution of the ODEs. We are now able to extend the waveform generation to higher masses and larger eccentricities as compared to [11], whose study was limited to $(8.0 + 8.0)M_\odot$.

The discrepancy that has been noted and the ability to extend investigations to higher mass range is also one of the motivations for revisiting their work.

2.2.2. Eccentricity evolution. Even though we fix the initial eccentricity, $e_0$, in order to solve the ODEs, it is clear from equation (6) that the eccentricity, $e$, evolves with time and depends on the orbital element $p$. In figure 1 (left panel), we plot the evolution of three typical systems in a $(p, e)$ plane, where $e_0 = 0.4$; $p_0$ is given by equation (9). For instance, a BNS system $(1.4 - 1.4)M_\odot$ starts at $(e = e_0 = 0.4, \ p = p_0 \sim 80)$ and stops at $(e \sim 0, \ p \sim 6)$. So, the orbit has circularized by the time the two objects merge. More generally, in the right panel, we represent in a contour plot the ratio between the initial and final eccentricity for any system with a total mass between 2 and $60M_\odot$ and an initial eccentricity between 0 and 0.4. If we look at an extreme case where $M \sim 60M_\odot$ and $e_0 = 0.4$, we can see that the system ends up with a final eccentricity twice as less as the initial eccentricity. Most of the systems with $M < 20M_\odot$ end up with an eccentricity 10 times smaller than $e_0$.

\(^1\) To obtain the same waveform duration, we need to set the lower cut-off frequency of the non-eccentric waveform to $2F_L/3$ Hz.
Figure 1. Eccentricity evolution. In the left panel, we show the trajectory of three typical systems in the $(p, e)$ plane. Those systems are made of a BNS $(1.4 - 1.4)M_\odot$, BHNS $(1.4 - 10)M_\odot$ and a BBH $(10 - 10)M_\odot$. The initial eccentricity is fixed to $e_0 = 0.4$. The eccentricity, $e$, decreases as $p$ decreases and the evolution is faster for more massive systems. In the right panel, we plot a contour of the quantity $e_0/e_{\text{final}}$, where $e_{\text{final}}$ is the final eccentricity of the system (when $p \sim 6$). The initial eccentricity $e_0$ is set to $2sL/3$ (e.g., initial LIGO).

2.2.3. Waveform duration. In table 1, we provide the durations of the eccentric waveforms, denoted as $T(e_0)$, for various component masses and eccentricities. Table 1 shows that for a given total mass, the duration of the signal decreases with increasing initial eccentricity. If a binary is highly eccentric initially, then its orbit shrinks faster by losing energy at a faster rate via radiation of gravitational waves. Moreover, for a given initial eccentricity, the waveform duration decreases as the total mass increases. This behavior can be anticipated by looking at equation (5), for which a larger initial value is expected for larger masses and larger eccentricity.

It would be useful to know the duration of the eccentric waveform (e.g., optimization of the vector’s length for memory storage and data analysis purposes). In table 1, we also provide the ratio $T_e(e_0)/T_e(e_0 = 0)$ (in brackets) to emphasize the large differences in duration between waveforms set with $e_0 = 0$ and $e_0 \neq 0$. Ideally, we would like to have an analytical expression to anticipate the eccentric waveform duration. In [3], the author started from equations (5) and (6) and derived the ratio $T_e(e)/T_e(e_0 = 0)$ in the approximation of either small $e_0$ or $e_0 \sim 1$. Considering the latter approximation, we found that the following expression (the last equation in [3] without the constant factor)

$$T(e_0) = T_e(e_0 = 0)(1 - e_0^2)^{7/2}$$

is a perfect fit to our data for any value of $e_0$ in the range $[0, 1]$ (not only $e \sim 1$). In figure 2, we plot the ratio $T_e(e_0)/T_e(e_0 = 0)$ from our experimental data in the case of two lower cut-off frequencies together with the fit given by equation (10). The duration of the eccentric waveforms depends only on the initial eccentricity and duration of the non-eccentric waveform.

2.2.4. $\iota$ and $\beta$ parameters. Let us consider a $(5.0 + 5.0)M_\odot$ system with $e_0 = 0.5$. Duration of such a waveform is $3.13$ s. We want to emphasize that the parameters $\beta$ and $\iota$ are of no importance for the simulations we present in section 3. In figure 3, we plot the first $0.5$ s of
Figure 2. The ratio of the duration of an eccentric binary to that of a quasi-circularized waveform as a function of eccentricity. The durations of the eccentric ($T$) and circularized ($T_c$) systems are extracted from the integration of the ODEs. The ratio as a function of eccentricity can be accurately fitted with $(1 - e_0^2)^{7/2}$, which depends only on the initial eccentricity $e_0$. The crosses and circles show the ratio $T/T_c$ for two different lower cut-off frequencies $F_L = 20$ Hz and $F_L = 40$ Hz. The fit is independent of $F_L$ and the total mass (see also table 1).

Figure 3. Effect of $\iota$ and $\beta$ on the waveform and matches. In the left panel, variation of a $(5.0+5.0)M_\odot$ eccentric waveform with an initial eccentricity $e_0 = 0.5$ as a function of different $\beta$ and $\iota$ values. In the right panel, matches between a waveform with $\iota = 0$ and $\beta = 0$ and templates for a wide range of $\iota$ and $\beta$. The matches are always larger than 0.999; we can set the pair $(\iota, \beta)$ to arbitrary value without loss of generality.

this waveform when $\beta = 0$ and $\iota = \pi/4$. Then, we keep $\beta = 0$ and set $\iota = 0$. We see that only the amplitude of the waveform changes, the period remaining unchanged. Finally, we set $\beta = \pi/4$ and $\iota = \pi/4$. Now, the amplitude changes as well as the phase. However, the
phase is only shifted. These results are expected if we look at equations (7) and (8), where we see that \( \beta \) contributes to the phase, \( \phi \), in the same manner in all the three harmonics, and \( \iota \) comes as a factor for the amplitude only. So, neither \( \beta \) nor \( \iota \) affect the overall frequency behavior of the waveform. In the right panel of the figure, we also plot the matches between a \((5+5)M_\odot\) with \( e_0 = 0.5 \) and a set of templates (same masses and eccentricity) that span the \((\iota, \beta)\) plane. All matches are above 0.999, which means that we can neglect the effect of these two parameters in the following simulations and fix them to constant values.

3. Searching for eccentric binaries with templates from non-eccentric binaries

In this section, we briefly describe the different tools that we have used in our simulations. Then, we present the simulation protocol that we have followed. Finally, we present the results of our simulations in the context of different detectors and mass ranges.

3.1. Match and template bank

We denote the Fourier transform of two functions \( x(t) \) and \( h(t) \) by \( \hat{x}(f) \) and \( \hat{h}(f) \), respectively. The matched-filtering inner product is defined by

\[
(x, h) = 4 \int_0^\infty \frac{\hat{x}^*(f) \hat{h}(f) + \hat{x}(f) \hat{h}^*(f)}{S_n(f)} \, df,
\]

where \( S_n(f) \) is the noise power spectral density (PSD) of the detector. A normalized signal is given by

\[
\hat{h} = \frac{h}{\sqrt{(h, h)}},
\]

where a hat denotes normalized signal. The SNR is defined by

\[
\rho(t) = \frac{(x, h)}{\sqrt{(h, h)}} = (x, \hat{h}).
\]

We can now imagine that \( x(t) = n(t) + s(t, \vartheta_\mu) \), where \( n(t) \) is Gaussian noise and \( s(t, \vartheta_\mu) \) is a gravitational-wave signal characterized by a set of \( p \) parameters \( \vartheta_\mu, \mu = 0, 1, \ldots, p - 1 \). These parameters may be separated into intrinsic and extrinsic parameters. The intrinsic parameters are represented by the component masses \( m_1 \) and \( m_2 \) and the initial eccentricity \( e_0 \), while the extrinsic parameters are the initial orbital phase \( \varphi_0 \) and the time of arrival, \( t_0 \). When the noise fluctuations are neglected, we can define the match between a signal \( s(t, \vartheta_\mu) \) and a template \( h(t, \vartheta_\nu) \) as follows [25]:

\[
M(s(t, \vartheta_\mu), h(t, \vartheta_\nu)) = \max_{\vartheta_\mu, \vartheta_\nu} M(\hat{s}(t, \vartheta_\mu), \hat{h}(t, \vartheta_\nu)),
\]

where the extrinsic parameters are maximized over automatically (using Fourier transform).

Using a continuous template space, we can maximize the matches over the intrinsic parameters to obtain the fitting factor, \( FF \), between the signal and the template family [28], which is defined by

\[
FF(s(t, \vartheta_\mu), h(t, \vartheta_\nu)) = \max_{\vartheta_\nu} M(\hat{s}(t, \vartheta_\mu), \hat{h}(t, \vartheta_\nu)).
\]

The fitting factor can be interpreted as the maximum fraction of SNR that can be obtained by filtering a signal with an approximate template family. The signal and template families can be different, like in this paper, where the signal parameters are \( \vartheta_\mu = m_1, m_2, e_0 \) and the template parameters are \( \vartheta_\nu = m_1, m_2 \). The fitting factor was used in [11] to estimate the loss of SNR when an eccentric signal is filtered with a non-eccentric signal.
In practice, the continuous template space is replaced by a discrete one, which is called a template bank \cite{22, 23, 25}. It is represented by \( \{ h(t, \vartheta^i) \} \), where \( i = 0, 1, \ldots, N_b - 1 \), and \( N_b \) is the number of templates. A template bank is optimally designed if \( N_b \) is the smallest such that for any signal there always exists at least one template in the bank that gives

\[
\min_{\vartheta^\mu} \max_{i} M(s(t, \vartheta^\mu), h(t, \vartheta^i)) \geq MM,
\]

where \( MM \) is the minimal match defined by the user. Usually, \( MM \) is set to 95\%, or 97\%, which corresponds to a loss in the detection rate of about 14\% and 9\%, respectively \((1 - MM^2)\).

So, the minimal match is a measure of how well a discrete template bank covers the parameter space defined by the intrinsic parameters.

Let us define a new notation that will be useful to quantify our results. We define the match over the bank, \( M_B \), as follows:

\[
M_B(s(t, \vartheta^\mu), h(t, \vartheta^i)) = \max_i M(s(t, \vartheta^\mu), h(t, \vartheta^i)).
\]

The quantity \( 1 - M_B \) can be interpreted as the average fractional loss of SNR due to modeling the signal with a simplified or approximate template.

Searches for inspiralling compact binaries performed in LIGO data use a template bank constructed so that signal from any circularized binary, \( s(t) \), is found with a match greater than the minimal match, \( MM \). Exhaustive simulations were performed in \cite{26, 20} to test the template bank placement in various cases: mass parameters related to BNS, BBH and BHNS systems and various design sensitivity curves of ground-based detectors.

In the rest of this paper, we will study the distribution of the quantity \( M_B \) when eccentric binaries are searched with the same template bank placement as that used in LIGO analysis, where templates are generated with a non-eccentric model.

### 3.2. Simulation parameters

As mentioned in the introduction, we want to identify the parameter space (in total mass, \( M \), and initial eccentricity, \( e_0 \)) where searching for eccentric binaries with templates corresponding to circularized systems suffices to obtain a negligible loss of SNR.

Our signals are based on eccentric waveforms whose phase is at Newtonian order only, we will therefore also consider Newtonian order for our templates. We will use the so-called TaylorT3 approximant (see, e.g., \cite{18, 20} for a precise definition). The lower cut-off frequency of the template is chosen to be \( F_L \). Since our templates do not have any eccentricity, the match with the signal can be close to one only in the regime where \( e_0 \sim 0 \).

In all the simulations that follows, we use an hexagonal template bank with a minimal match \( MM = 95\% \). It has been shown that for VIRGO, LIGO, Einstein Telescope and Advanced LIGO design sensitivity curves (see, e.g., \cite{20} for analytical expressions), \( M_B \) is guaranteed to be larger than the minimal match (95\%) for any system with \( 2M_\odot \leq M \leq 60M_\odot \) \cite{20}. Yet, in some cases, as described later in section 3.3, we will extend the search to \( 80M_\odot \).

Note that the template bank is actually optimized for 2PN order in phase and might therefore not be optimal for the Newtonian order considered here. In principle, since the phase of the signal and template are based on a Newtonian order only, we could use a one-dimensional template bank (e.g., the chirp mass as defined later). So, using the template bank described in \cite{20} and used in LIGO searches, which is a two-dimensional grid, our simulations may provide slightly overestimated values of matches.

In all our simulations, we use the following common parameters. The number of waveforms in each simulation is 10 000. The sampling frequency is 4096 Hz. The total mass is uniformly distributed in the range considered (see section 3.3). The initial starting
Table 2. Number of templates, $N_B$, of each template bank used in our simulations. The template bank size of initial LIGO is the smallest because its lower cut-off frequency, $F_L$, equals 40 Hz whereas other detectors have $F_L = 20$ Hz. The parameter space of the first row is $m_1, m_2 \in [1, 30]M_\odot$ for initial LIGO and $m_1, m_2 \in [1, 40]M_\odot$ for the others. The second row gives $N_B$ for a BNS parameter space, where $m_1, m_2 \in [1, 3]M_\odot$ (see the appendix).

|        | Initial LIGO | Advanced LIGO | ET  | VIRGO |
|--------|--------------|---------------|-----|-------|
| Bank size | 6792         | 25194         | 65104 | 52455 |
| Bank size BNS | 2319         | 6969          | 16046 | 12955 |

phase is randomized between 0 and $2\pi$. Although the choice of the initial phase should be of no consequence, we randomize it in all our simulations. The initial eccentricity is uniformly distributed so that $e_0 \in [0.0, 0.4]$. The maximum value was chosen to maintain the final frequency of the heaviest injected waveform in band (i.e, above the lower cut-off frequency of the detector so as to consider waveforms that are physically meaningful only). For instance, in the initial LIGO case, a $(30 + 30)M_\odot$ system with $e_0 = 0.4$ has a final frequency of about 60 Hz. It is quite close to $F_L = 40$ Hz but the waveform still has three cycles. The case of Advanced LIGO and Einstein detectors is less critical: the shortest waveform corresponds to a $(40 + 40)M_\odot$ and lasts 0.9 s, it has a final frequency of about 55 Hz and possesses about 10 cycles. If we consider those extreme cases to be physically reasonable, then our choice of mass–eccentricity plane is also reasonable because lighter and less eccentric systems necessarily have longer durations. Concerning the $\iota$ and $\beta$ parameters, we fix them to $\iota = \pi/4$ and $\beta = 0$. As explained in section 2.2.4, fixing $\beta$ and $\iota$ to arbitrary values does not affect the waveform significantly. However, we performed tests where $\beta$ and $\iota$ were randomized. As expected, we did not see any significant effects on the matches. Finally, let us note that if $\beta$ varies, then the ending frequency may slightly change and the matches as well.

3.3. Search for eccentric waveforms in ground-based detectors

We consider four design sensitivity curves: initial LIGO, Advanced LIGO, Einstein Telescope and VIRGO. In this section we summarize the results we obtained when searching for eccentric waveforms with circular waveforms. As mentioned above, we limited the maximum value of eccentricity to $e_0 = 0.4$; when $e_0 > 0.4$, the loss of SNR is always larger than 50% whatever is the total mass. Finally, let us note that for all the results presented below, the initial eccentricity is defined by equation (3), which is defined at $f_{\text{min}}$; in other words, when the main harmonic frequency equals $2F_L/3$.

3.3.1. Initial LIGO. In the case of initial LIGO, the lower cut-off frequency is $F_L = 40$ Hz and the total mass range $M \in [2, 60]M_\odot$. The template bank size is provided in table 2. In figure 4, we show the distribution of the match over the bank, $M_B$, in the $(M, e_0)$ plane. As expected, around $e_0 = 0$, $M_B$ is close to 1. However, in most of the parameter space, $M_B$ is much smaller than unity. It is convenient to represent $M_B$ with isocontours set to [96.5, 95, 90, 80, 50]%. Using a value of 96.5% is useful because the loss of detection associated with it is about 10%. Using a value of 95% is also useful because it corresponds to the minimal match of the template bank.

The isocontours follow more or less the same structure. If we focus on the 95% isocontour, we can see that BNSs are found with matches above 95% if initial eccentricity is below 0.05. While the total mass increases up to $20M_\odot$, the initial eccentricity for which the match is
greater than 95% increases up to a maximum value of 0.15. As the total mass increases for a match of 95% or greater, from \(20M_\odot\) to about \(40M_\odot\), the initial eccentricity again goes down to 0.05. Above \(40M_\odot\), matches can go down to 90% even for negligible eccentricities but this is related to the template bank design. Indeed, the template bank is optimally designed for signals that have a significant part of their power in a frequency band that resides where the detector has the best sensitivity. This assumption breaks down for signals with \(M > 40M_\odot\) because the signal’s ending frequency is already as low as 110 Hz whereas the detector best sensitivity is about 200 Hz. Consequently the bank is sub-optimal for these systems.

In the appendix, in figure A1, we provide for convenience the same results as in figure 4 where \(M\) has been replaced by the chirp mass, \(M_c\), which is defined by \(M = M\eta^{3/5}\) and \(\eta = \frac{m_1 m_2}{M^2}\). We also provide, in figure A2, the same results as in figure 4 where \(M_B\) has been replaced by the loss of detection \((1 - M_B^3)\).

Finally, since the parameter space related to BNSs is rather small in figure 4, we performed another simulation where we focus on the range \(M \in [2, 6]M_\odot\). As we can see in figure A3, using non-eccentric waveforms is sufficient to detect a (1.4 + 1.4)\(M_\odot\) eccentric waveforms if \(e_0 \leq 0.05\).

### 3.3.2. Advanced LIGO

The simulation in the case of Advanced LIGO differs from that of initial LIGO in two ways: the lower cut-off frequency is set to \(F_L = 20\) Hz, and the maximum total mass is extended to \(80M_\odot\). Advanced LIGO will use signal recycling which allows us to tune the frequency response of the detector. This flexibility will allow a wide range of responses that should be determined by the astrophysics of the objects that are targeted. In this paper and simulation, we consider the case of a NS–NS tuning.

Results are shown in figure 5 and are very similar to the initial LIGO case. The main difference being that the high mass range is extended because the lower cut-off frequency is smaller and that \(e_0\) is fixed at \(2F_L/3 = 13.33\) Hz.
In the left panel of figure 5, we can see that BNSs are found with a match larger than 95% if initial eccentricity is below 0.05. Then, keeping the same match, the initial eccentricity increases up to a maximum value of 0.15 as the total mass increases up to $25M_\odot$. Finally, when the total mass increases from $25M_\odot$ to about $70M_\odot$, the initial eccentricity remains constant around 0.15 for the same match of 95%. Above $70M_\odot$, matches go down to 90% even for negligible eccentricities.

Since the lower cut-off frequency, $F_L$, is different between initial LIGO and Advanced LIGO detectors (40 Hz and 20 Hz respectively), it is not convenient to compare the results provided in the left panel with those from figures 4 and 5. Indeed, the initial eccentricity is taken at two different frequencies. For convenience we will also provide the results of the simulation (masses and matches) as a function of the eccentricity at 26.66 Hz, which will be our frequency.

In the appendix (figure A1), we provide for convenience the same results as in figure 4 where $M$ has been replaced by the chirp mass. In figure A2, we also provide the same results as in figure 4 where $M_B$ has been replaced by the loss of detection $(1 - M_B^3)$. We performed another simulation so as to focus on the BNS region. Results are shown in figure A3: using the non-eccentric model is sufficient to detect $(1.4 + 1.4)M_\odot$ eccentric systems with $e_0 \leq 0.05$.

### 3.3.3. Einstein Telescope.

The simulation in the case of the Einstein Telescope is similar to that of Advanced LIGO (same $F_L$, same mass range). Results are shown in figure 6.

In the left panel of figure 6, if we focus on the 95% isocontour, we can see that BNSs are found with matches above 95% if $e \sim 0$. Then, the initial eccentricity increases up to a maximum value of 0.10 while total mass increases up to $30M_\odot$ while obtaining a match $\geq 95\%$. Finally, when the total mass increases from $30M_\odot$ to about $60M_\odot$, the initial eccentricity decreases to $e_0 \sim 0$. Among the four design sensitivity curves investigated in this paper, the Einstein Telescope case gives the least satisfactory results.

The right panel as explained in previous section shows the same results but when $e = 26.66$ Hz.
In the appendix, in figure A1, we replace $M$ by $M$ using results from figure 6. In figure A2, we replace $M_B$ by the loss of detection $(1 - M_B^3)$. Finally, we focus on the area $M \in [2, 6]M_\odot$ and provide the results in figure A3: using the non-eccentric model is sufficient to detect a $(1.4 + 1.4)M_\odot$ eccentric system with $e_0 \leq 0.02$.

3.3.4. VIRGO. The simulation in the case of VIRGO is similar to that of Advanced LIGO (same $f_L$, same mass range). The results are shown in figure 7.

In the left panel of figure 7, if we focus on the 95% isocontour, we can see that BNSs are found with matches above 95% if initial eccentricity is below 0.03. Then, the initial...
eccentricity can be increased up to a maximum value of 0.15 while total mass increases up to $40M_\odot$ which still achieve matches $\geq 95\%$. Finally, when the total mass increases from $40M_\odot$ to about $80M_\odot$, the initial eccentricity decreases to $e_0 = 0.15$ for matches $\geq 95\%$.

The right panel as explained in previous section shows the same results when $e = 26.66$ Hz.

In the appendix, in figure A1, we replace $M$ by $M$ using results from figure 7. In figure A2, we replace $M_B$ by the loss of detection $(1 - M_B^3)$. Finally, we focus on the area $M \in [2, 6]M_\odot$ and provide the results in figure A3: using the non-eccentric model is sufficient to detect a $(1.4 + 1.4)M_\odot$ eccentric binary with $e_0 \leq 0.03$.

### 4. Conclusion and perspectives

In this paper the main difference with previous works is that we maximize the matches over a discrete template bank, $M_B$, instead of computing the so-called fitting factor, $FF$, which maximizes the matches over a continuous parameter space. The template bank we used is identical to the template bank used in LIGO searches for binaries in quasi-circular orbits. For this reason, and since our waveform generator is slightly more accurate, our results are also slightly different from [11] in the case of initial LIGO detector.

The statements hereafter assume that binary systems are detectable in the sense that they have a SNR large enough. They also consider different lower cut-off frequencies depending on the detector.

If we search for gravitational-wave signal emitted by eccentric inspiralling compact binaries with standard non-eccentric template bank, our main results can be summarized as follows. In initial LIGO, (1) the binary neutron star system can be detected with $M_B \gtrsim 95\%$ if the initial eccentricity $e_0 \lesssim 0.05$ (or $M_B \gtrsim 90\%$ if $e_0 \lesssim 0.10$) and (2) the binary system with a total mass, $6 \leq M \leq 35M_\odot$ can be detected with $M_B \gtrsim 95\%$ if $e_0 \lesssim 0.10$. Similar results have been obtained for the Advanced LIGO case. Concerning the VIRGO detector, the binary neutron star system can be detected with $M_B \gtrsim 95\%$ if $e_0 \lesssim 0.04$, or $M_B \gtrsim 90\%$ if $e_0 \lesssim 0.06$, and (2) a system with total mass larger than $6M_\odot$ can be detected with $M_B \gtrsim 95\%$ if $e_0 \lesssim 0.10$. Similar results have been obtained in the case of the Einstein Telescope. In conclusion, the initial eccentricity does not need to be taken into account in our searches for gravitational-wave signals if $e_0 \lesssim 0.05$, because the loss of SNR (about 5%) is consistent with the discreteness of the template bank that is being used. However, by neglecting the eccentricity in our searches, significant loss of detection may arise as soon as $e_0 \geq 0.05$, especially for binary neutron stars and systems with high masses ($M \gtrsim 35M_\odot$).

Future directions for this work are the inclusion of higher PN order, the development of a template bank placement for eccentric systems and parameter estimation. Theoretical studies related to eccentric waveforms with higher PN order have been developed (e.g, see [29]). It would be interesting to extend our studies (exhaustive Monte-Carlo simulations) to higher PN order, which should be straightforward given the infrastructure which has been developed. Concerning the template bank placement, as we have explained above $e_0 = 0.05$ the loss of SNR may be quite large in some region of the parameter space. A solution may be to use a dedicated template bank to take into account the eccentricity. For instance, using the template bank used in this paper, we can have several layers of templates, each having different values of initial eccentricity. More investigations are needed but preliminary studies show that only a few layers are required to significantly improve the results (the TaylorT3 family being replaced by the eccentric waveform presented here). Finally, using Monte-Carlo simulations it would be interesting to estimate the error bound on the variance of the measured eccentricity.
Figure A1. Matches between the simulated eccentric waveform and a template bank made of non-eccentric waveforms. The eccentricity is uniformly distributed in the range $[0, 0.4]$. The $y$-axis represents the chirp mass parameter. The data used are identical to those of figures 4, 5, 6 and 7, where the design sensitivity curves are from top left to bottom right: initial LIGO, Advanced LIGO, Einstein Telescope and VIRGO.
Figure A2. Loss of detection. Simulated waveforms have eccentricity in the range $[0, 0.4]$. Template bank is made of non-eccentric waveforms. From top left to bottom right, we used initial LIGO, Advanced LIGO, Einstein Telescope and VIRGO design sensitivity curves. The data used are identical to those of figure A1.
Figure A3. Matches between the simulated eccentric waveform (BNS) and a template bank made of non-eccentric waveforms. The initial eccentricity is uniformly distributed in the range [0, 0.1]. From top left to bottom right, we used initial LIGO, Advanced LIGO, Einstein Telescope and VIRGO design sensitivity curves.
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Appendix. Additional results

In figure A1 we provide the results shown in section 3.2 in terms of chirp mass rather than the total mass. See the text for a full description of the simulation parameters. We also transform the results into loss of detection in figure A2. Finally, we further explore the BNS region by carrying out a simulation restricted to the mass range $M \in [2, 6]M_\odot$, and the results of which are shown in figure A3.

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