Heegaard Splittings and Virtually Haken Dehn Filling

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Abstract. We use Heegaard splittings to give some examples of virtually Haken 3-manifolds.

A compact connected 3-manifold is said to be virtually Haken if it has a finite sheeted covering space which is Haken. The virtual Haken conjecture states that every compact, connected, orientable, irreducible 3-manifold with infinite fundamental group is virtually Haken. Since virtually Haken 3-manifolds and Haken 3-manifolds possess similar properties, such as geometric decompositions and, in the closed case, topological rigidity, the resolution of this conjecture would provide solutions to several fundamental problems about compact 3-manifolds with infinite fundamental groups.

Some recent results in attacking the conjecture can be found in [CL] [BZ] [M] [DT]. A summary of earlier results can be found in [K, Problem 3.2]. For connections between the virtual Haken conjecture, Heegaard splittings, and the Property \(\tau\) conjecture, see [L].

Motivated by the work of Casson and Gordon ([CG]), we shall show that lifted Heegaard surfaces can often be compressed to become essential. Our techniques can be used to produce many families of non-Haken but virtually Haken 3-manifolds, a few of which are given here to illustrate the method. A more general result will be proved in a forthcoming paper.

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We proceed to give the examples. Let \(K_{2n+1}\) be the twist knot in \(S^3\) as shown in Figure 1. Let \(M_n\) be the exterior of \(K_{2n+1}\), with standard meridian-longitude framing on \(\partial M_n\). Recall that a connected, compact, orientable 3-manifold whose boundary is a torus is called small if every closed, orientable, embedded, incompressible surface is parallel to the boundary, and called large otherwise.

**Theorem 1** The 3-fold cyclic cover of \(M_n\) is large for every \(n > 0\). Every Dehn filling of \(M_n\) with slope \(3p/q\), \((3p,q) = 1\), \(|p| > 1\), yields a virtually Haken 3-manifold.

Note that by [HT], \(M_n\) is hyperbolic, small, and has exactly three boundary slopes, for every \(n > 0\). It follows (combining with [CGLS, Theorem 2.0.3]) that all but exactly three

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Dehn fillings of $M_n$ give irreducible non-Haken 3-manifolds. Also note that each $K_{2n+1}$, $n > 0$, is a non-fibered knot with a genus one Seifert surface, and thus by [CL] it was known that every $m$-fold cyclic cover of $M_n$, $m \geq 4$, is large and every Dehn filling of $M_n$ with slope $p/q$, $(p, q) = 1$, $|p| \geq 8$, is virtually Haken.

**Proof.** Let $\tilde{M}_n$ be the 3-fold cyclic cover of $M_n$ with induced meridian-longitude framing on $\partial \tilde{M}_n$. We shall show that $\tilde{M}_n$ contains a connected, essential (i.e. orientable, incompressible, non-boundary-parallel) genus two closed surface which has an essential simple closed curve isotopic to a longitude curve of the cover. It follows from [CGLS, Theorem 2.4.3] that the surface remains incompressible in every Dehn filling of $\tilde{M}_n$ with slope $p/q$, $(p, q) = 1$, $|p| > 1$. As every Dehn filling of $M_n$ with slope $3p/q$, $(3p, q) = 1$, $|p| > 1$, is free covered by Dehn filling of $\tilde{M}_n$ with slope $p/q$, $(p, q) = 1$, $|p| > 1$, the second conclusion of the theorem will follow.

To make the illustration simple, we first prove the theorem with all details in case $n = 1$, i.e. for the $5_2$ knot $K = K_3$. The knot $K$ is tunnel number one, and Figure 2 shows an unknotting tunnel. Also pictured in Figure 2 is a longitude $\lambda$ of $K$. Let $N$ be a regular neighborhood of $K$ in $S^3$, $M = M_1 = S^3 - N$, $B$ a regular neighborhood of the unknotting tunnel in $M$, and $H = M - B$. Then $H$ is a handle body of genus two. Let $D$ be a meridian disk of the 1-handle $B$ whose boundary is shown in Figure 2. We deform the handle body $H' = N \cup B$ by an isotopy in $S^3$ so that its exterior $H$ can be recognized as a standard handle body in $S^3$ and at the same time we trace the corresponding deformation of $\partial D$ and $\lambda$ under the isotopy. The process is shown through Figures 3-6.

A **meridian disk system** of a handlebody of genus $g$ is a set of $g$ properly embedded
Figure 2: An unknotting tunnel, its co-core $\partial D$ and a standard longitude of $K$

Figure 3: The deformation of $H'$, $\partial D$ and $\lambda$ (part a)

mutually disjoint disks in the handle body such that cutting the handlebody along these disks results in a 3-ball. Let $\{X, Y\}$ be a meridian disk system of $H$ whose boundary are shown in Figure 6. Following $\partial D$ in the given orientation, we get a geometric presentation
Figure 4: The deformation of $H'$, $\partial D$ and $\lambda$ (part b)

Figure 5: The deformation of $H'$, $\partial D$ and $\lambda$ (part c)
of the fundamental group $\pi_1(M)$ of $M$:

$$\pi_1(M) = \langle x, y; x^{-1}y^{-1}x^{-1}yxyx^{-1}y^{-1}x^{-1}xyy >,$$

where $x$ is chosen such that it has a representative curve which is a simple closed curve in $\partial H$ which is disjoint from $\partial Y$ and intersects $\partial X$ exactly once and $y$ is also chosen similarly. (We shall call such generators dual to the disk system.) Also we can read off the longitude in terms of these two generators:

$$\lambda = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^2.$$

Cutting $H$ along $X$ and $Y$, we get a 3-ball. Figure 7 shows the boundary 2-sphere of the 3-ball, which records $X^+, X^-, Y^+, Y^-$ and $\partial D$. Figure 8 shows $H$ in a standard position, and $\partial D$ in $\partial H$.

The exterior of $H$ in $M$ is a compression body which we denote by $C$. Topologically, $C$ is $\partial M \times [0,1]$ with a 1-handle attached on $\partial M \times \{1\}$. It has two boundary components: one is $\partial M = \partial M \times \{0\}$ and the other is the genus two surface $\partial H$. We have that $H \cup_{\partial H} C$ is a Heegaard splitting of $M$.

Let $\tilde{M} = \tilde{M}_1$ be the 3-fold cyclic cover of $M = M_1$. Note that each of $x$ and $y$ is a generator of $H_1(M;\mathbb{Z}) = \mathbb{Z}$. Let $\tilde{M}$ have the induced Heegaard splitting from that of $M$. 

Figure 6: The deformation of $H'$, $\partial D$ and $\lambda$ (part d)
Figure 7: $\partial D$ on the sphere $\partial(\mathcal{H} - \{X \times I \cup Y \times I\})$

Figure 8: $\mathcal{H}$ and $\partial D$ in standard position

We can easily give the Heegaard diagram of $\tilde{M}$, as shown in Figure 9. The genus four handle body $\tilde{H}$ in Figure 9 is the corresponding cover of $\mathcal{H}$. The corresponding cover $\tilde{C}$ of $C$ is a compression body obtained by attaching three 1-handles to $\partial \tilde{M} \times [0, 1]$ on the side $\partial \tilde{M} \times \{1\}$. The disk $X$ lifts to three disks $X_1, X_2, X_3$; and the disk $Y$ lifts to three disks $Y_1, Y_2, Y_3$, as shown in Figure 9. Pick the meridian disk $X_4$ of $\tilde{H}$ as shown in Figure 9. Then $\{X_1, X_2, X_3, X_4\}$ forms a disk system of $\tilde{H}$. The disk $D$ lifts to three disks $\{W_1, W_2, W_3\}$ whose boundary $\{\partial W_1, \partial W_2, \partial W_3\}$ is shown in Figure 9. Figure 9 also shows the longitude $\tilde{\lambda}$ of $\tilde{M}$, which is a lift of $\lambda$. 

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This Heegaard splitting of $\tilde{M}$ is weakly reducible: $\partial X_4$ is disjoint from $\partial W_3$. We now show that the closed, genus 2 surface $S$ obtained by compressing the Heegaard surface $\partial \tilde{H}$ using the disks $W_3$ and $X_4$ is essential in $\tilde{M}$. It is enough to show that the surface $S$ is incompressible in $\tilde{M}(2)$, which is the manifold obtained by Dehn filling $M$ with the slope 2. $\tilde{M}(2)$ has the induced Heegaard splitting $\tilde{H} \cup \tilde{C}(2)$. Note that $\tilde{M}(2)$ is the free 3-fold cyclic cover of $M(6)$, extending the cover $\tilde{M} \rightarrow M$, and that $\tilde{C}(2)$ is a handle body of genus four covering the handle body $C(6)$ of genus two, extending the cover $\tilde{C} \rightarrow C$. Let $\tilde{V}$ be the filling solid torus in $\tilde{M}(2)$ and let $W_4$ be a meridian disk of $\tilde{V}$. Then $\{W_1, W_2, W_3, W_4\}$ is a disk system of the handle body $\tilde{C}(2)$.

Cutting $\tilde{H}$ along $X_4$, we get a handle body $H_\#$ of genus three, and $\{X_1, X_2, X_3\}$ is a disk system of $H_\#$. Using the Whitehead algorithm [S], we see that $\partial H_\# - \partial W_3$ is incompressible in $H_\#$. In fact, from Figure 9, we can read off the Whitehead graph of $\partial W_3$ with respect to the disk system $\{X_1, X_2, X_3\}$ of $H_\#$, which is given as Figure 10. The graph is connected with no cut vertex, which means, by the Whitehead algorithm, that $\partial W_3$ must intersect every essential disk of $H_\#$. Now by the Handle Addition Lemma due to Przytycki [P] and Jaco [J], the manifold $H_\# \cup W_3 \times I$, obtained by attaching the 2-handle $W_3 \times I$ to $H_\#$, has incompressible boundary.

On the other hand, cutting the handle body $\tilde{C}(2)$ along the disk $W_2$, we get a handle body $H_\ast$, which is homeomorphic to $\tilde{V}$ with the two 1-handles $W_1 \times I$ and $W_2 \times I$ attached on $\partial \tilde{V}$. The genus of $H_\ast$ is three, and $\{W_1, W_2, W_4\}$ gives a disk system. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope 6. We can easily see that with respect to the generators $x, y$ of $\pi_1(M)$, 

$$\alpha = \lambda x^6 = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^8.$$ 

Let $\tilde{\alpha} \subset \partial \tilde{M}$ be a lift of $\alpha$. Then $\tilde{\alpha}$ has slope 2 in $\partial \tilde{M}$ which can be considered as the boundary of the disk $W_4$. Figure 11 shows $\tilde{\alpha} = \partial W_4, \partial W_1$ and $\partial W_2$ in $\partial \tilde{H}$.

Again using the Whitehead algorithm, we see that $\partial H_\ast - \partial X_4$ is incompressible in $\tilde{H}_\ast$. In fact, from Figure 11, we can read off the Whitehead graph of $\partial X_4$ with respect to the disk system $\{W_1, W_2, W_4\}$, which is given as Figure 12. The graph is connected with no cut vertex, which means, by the Whitehead algorithm, that $\partial H_\ast - \partial X_4$ is incompressible in $H_\ast$. Again by the Handle Addition Lemma, the manifold $H_\ast \cup X_4 \times I$ has incompressible boundary of genus two. Note that $\partial (H_\ast \cup X_4 \times I) = \partial (\tilde{H}_\# \cup Y_3 \times I) = S$ (up to a small isotopy), and thus $S$ is incompressible in $\tilde{M}(2)$. But the surface $S$ is contained $\tilde{M}$, and thus it is an essential surface in $\tilde{M}$.

Obviously the longitude $\tilde{\lambda}$ in $\partial \tilde{M}$ is isotopic to an essential simple closed curve in the
Figure 9: the Heegaard diagram of the 3-fold cyclic cover $\tilde{M}$ and the longitude $\tilde{\lambda}$

surface $S$, as shown in Figure 9. The proof of Theorem 1 is complete for $n = 1$.

The proof for general $K_{2n+1}$, $n > 0$, is similar. The knot $K_{2n+1}$ is tunnel number one, with an unknotted tunnel shown in Figure 2 (replacing the bottom three crossings by $2n + 1$ crossings). Let $M_n$ be the exterior of $K_{2n+1}$, $H'$ the handlebody which is a regular neighborhood of the knot and its unknotted tunnel, $H = M_n - H'$, and $D$ a meridian disk of the unknotted tunnel. There is a corresponding Heegaard splitting $M_n = H \cup_{\partial H} C$, where $C$ is a compression body. We let $\lambda$ be a standard longitude, and again we deform the
handlebody $H'$ by an isotopy in $S^3$ so that its exterior $H$ can be recognized as a standard handlebody in $S^3$, while tracing the corresponding deformations of $\partial D$ and $\lambda$ under the isotopy. We thus get two essential disks $X$ and $Y$ which form a disk system of $H$. From $\partial D$, we get a geometric presentation of the fundamental group $\pi_1(M)$ of $M$ with respect to the disk system $\{X, Y\}$:

$$\pi_1(M) = \langle x, y; (x^{-1}y^{-1})^{2n-1}x^{-1}(yx)^n+1y^{-1}x^{-1}y^{-1})^{2n-1}(xy)^n+1 \rangle.$$ 

Also we get

$$\lambda = y(xy)^n(x^{-1}y^{-1})^nx^{-1}y^{-2}(x^{-1}y^{-1})^nx^{-1}(yx)^n+1x.$$ 

Let $\tilde{M}_n$ be the 3-fold cyclic cover of $M_n$ and let $\tilde{M}_n = \tilde{H} \cup_{\partial \tilde{H}} \tilde{C}$ have the induced Heegaard splitting from that of $M_n$, where $\tilde{H}$ is a genus four handle body which is the corresponding 3-fold cyclic cover of $H$ and $\tilde{C}$ a compression body which covers $C$. Again the disk $X$ lifts to three disks $X_1, X_2, X_3$; and the disk $Y$ lifts to three disks $Y_1, Y_2, Y_3$, as shown in Figure 9 (ignore the $\partial W_i$ and $\lambda$ part), and we pick the meridian disk $X_4$ of $\tilde{H}$ as shown in Figure 9. Then $\{X_1, X_2, X_3, X_4\}$ forms a disk system of $\tilde{H}$. The disk $D$ lifts to three disks $\{W_1, W_2, W_3\}$ which form a disk system of $\tilde{C}$. Again exactly one of the disks $\{W_1, W_2, W_3\}$, say $W_3$, is disjoint from $X_4$, which shows that the Heegaard splitting of $\tilde{M}_n$ is weekly reducible. Again one can show that the surface $S$ obtained by compressing the Heegaard surface $\partial \tilde{H}$ using the disks $W_3$ and $X_4$ is an essential closed genus two surface in $\tilde{M}_n$. In fact, cutting $\tilde{H}$ along $X_4$, we get a handle body $H_\#$ of genus three and $\{X_1, X_2, X_3\}$ is a disk system of $H_\#$. The Whitehead graph of $\partial W_3$ with respect to the disk system $\{X_1, X_2, X_3\}$ of $H_\#$ is given as Figure 13. The graph is connected with no
cut vertex, which means that $\partial H_# - \partial W_3$ is incompressible. Thus by the handle addition lemma, the manifold $H_# \cup W_3 \times I$, obtained by attaching the 2-handle $W_3 \times I$ to $H_#$, has incompressible boundary.

On the other hand, letting $\tilde{C}(2)$ be the handle body obtained by Dehn filling $\tilde{C}$ with slope 2 and letting $W_4$ be a meridian disk of the filling solid torus, then $\{W_1, W_2, W_3, W_4\}$ forms a disk system of $\tilde{C}(2)$. Cutting $\tilde{C}(2)$ along the disk $W_3$, we get a handlebody $H_*$ with disk system $\{W_1, W_2, W_4\}$. Let $\alpha \subset \partial M$ be an essential simple closed curve of slope
6. Then with respect to the generators $x, y$ of $\pi_1(M)$,

$$\alpha = \lambda x^6 = y(xy)^n(x^{-1}y^{-1})^n x^{-1}y^{-2}(x^{-1}y^{-1})^n x^{-1}(yx)^{n+1} x^6.$$ 

We may consider $\partial W_4$ as a lift of $\alpha$. From the word $\alpha$, we can draw $\partial W_4$ on $\partial \tilde{\mathcal{H}}$. Consequently we can read off the Whitehead graph of $\partial X_4$ with respect to the disk system $\{W_1, W_2, W_4\}$ and see that the graph is the same as that shown in Figure 12, showing that $\partial H_* - \partial X_4$ is incompressible in $H_*$. Thus the manifold $H_* \cup X_4 \times I$ has incompressible boundary of genus two. We thus have justified the incompressibility of the surface $S$ in $\tilde{M}_n(2)$ and thus in $\tilde{M}_n$.

Finally the longitude $\tilde{\lambda}$ in $\partial \tilde{\mathcal{M}}$ is isotopic to an essential simple closed curve in the
Let $J_{2n+1}$, $n > 0$, be the family of two bridge knots shown in Figure 14. Note that these knots are hyperbolic, small and non-fibered with genus two Seifert surfaces.

**Theorem 2** The 5-fold cyclic cover of the exterior of $J_{2n+1}$ is large and every Dehn filling of the exterior of $J_{2n+1}$ with slope $5p/q$, $(5p, q) = 1$, $|p| > 1$, yields a virtually Haken 3-manifold, for every $n > 0$.

This theorem gives another family of non-Haken, virtually Haken 3-manifolds to which the results of [CL] do not apply. As the proof of Theorem 2 is very similar to that of Theorem 1, we omit the details and indicate only the steps. In fact the exterior of $J_{2n+1}$ is tunnel number one and a genus two Heegaard splitting of it can be explicitly given as in the case for the exterior of the twist knot $K_{2n+1}$. In the 5-fold cyclic cover of the exterior of $J_{2n+1}$, the lifted Heegaard surface is of genus 6 and can be compressed along two reducing disks, one on each side of the Heegaard surface, to a closed incompressible surface of genus 4. Also a lift of the longitude can be isotoped into the resulting incompressible surface.

We now go back to the twist knots $K_{2n+1}$ and prove the following Theorem 3. Although the result of the theorem is covered by [CL], we have included it primarily because its proof illustrates two complications which arise in more general settings. First, we have to deal with multi 2-handle additions, which requires the multi 2-handle addition theorem of Lei [L]. Also, one of the Whitehead graphs contains a cut vertex, and must be simplified using Whitehead moves.
Figure 15: The Heegaard splitting of the 5-fold cover of $M$
Figure 16: The Whitehead graph of \( \{ \partial W_4, \partial W_5 \} \) with respect to the disk system \( \{ X_1, X_2, X_3, X_4 \} \) of the handle body \( H_\# \)

**Theorem 3** The 5-fold cyclic cover of the exterior \( M_n \) of \( K_{2n+1} \) is large for every \( n > 0 \). Every Dehn filling of \( M_n \) with slope \( 5p/q, (5p,q) = 1, \vert p \vert > 1 \), yields a virtually Haken 3-manifold.

**Proof.** Again we give details only for the \( n = 1 \) case. We continue to use the Heegaard splitting of \( M = M_1 = H \cup C \) as given in the proof of Theorem 1. Let \( \tilde{M} \) be the 5-fold cyclic cover of \( M \) with the induced Heegaard splitting from that of \( M \). The Heegaard diagram of \( \tilde{M} \) is shown in Figure 15. The genus six handle body of Figure 14 is \( \tilde{H} \) which covers \( H \). The disks \( X \) and \( Y \) of \( H \) lift to disks \( X_1, ..., X_5 \) and \( Y_1, ..., Y_5 \), as shown in Figure 15. Pick the meridian disk \( X_6 \) of \( \tilde{H} \) as shown in Figure 15. Then \( \{ X_1, X_2, X_3, X_4, Y_5, X_6 \} \) forms a disk system of \( \tilde{H} \). The disk \( D \) lifts to five disks \( \{ W_1, W_2, W_3, W_4, W_5 \} \) whose boundaries are shown in Figure 15. Figure 15 also shows a longitude \( \tilde{\lambda} \) of \( \tilde{M} \), which is a lift of the longitude \( \lambda \) of \( M \).

This Heegaard splitting of \( \tilde{M} \) is weakly reducible: \( \{ \partial Y_5, \partial X_6 \} \) is disjoint from \( \{ \partial W_4, \partial W_5 \} \). We now show that the surface \( S \) obtained by compressing the Heegaard surface \( \partial \tilde{H} \) using these four disks is an essential closed genus two surface in \( \tilde{M} \). It is enough to show that the surface \( S \) is incompressible in \( \tilde{M}(2) \), which is the free 5-fold cyclic cover of \( M(10) = H \cup C(10) \), and has the induced Heegaard splitting \( \tilde{H} \cup \tilde{C}(2) \). Let \( \tilde{V} \) be the filling solid torus in \( \tilde{M}(2) \) and let \( W_6 \) be a meridian disk of \( \tilde{V} \). Then \( \{ W_1, ..., W_5, W_6 \} \) is a disk system of the handle body \( \tilde{C}(2) \).

Cutting \( \tilde{H} \) along \( Y_5, X_6 \), we get a handle body \( H_\# \) of genus four and \( \{ X_1, X_2, X_3, X_4 \} \) is a disk system of \( H_\# \). The Whitehead graph of \( \{ \partial W_4, \partial W_5 \} \) with respect to the disk system \( \{ X_1, ..., X_4 \} \) of \( H_\# \) is given in Figure 16. The graph is connected with no cut vertex, which means that the surface \( \partial H_\# - \{ \partial W_4, \partial W_5 \} \) is incompressible in \( H_\# \). Moreover as \( \partial W_4 \) is disjoint from the disk \( X_1 \), and \( \partial W_5 \) is disjoint from the disk \( X_4 \), each of the surfaces \( \partial H_\# - \partial W_4 \) and \( \partial H_\# - \partial W_5 \) is compressible in \( H_\# \). Therefore all the conditions of the
Figure 17: $\partial W_6 = \tilde{\alpha}$, $\partial W_1$, $\partial W_2$, $\partial W_3$ on the Heegaard surface $\partial \tilde{H}$
Figure 18: The Whitehead graph of \( \{ \partial Y_5, \partial X_6 \} \) with respect to the disk system \( \{ W_1, W_2, W_3, W_6 \} \) of the handle body \( H_* \).

The multi-handle addition theorem of [L] are satisfied, and thus the manifold \( H_\# \cup W_4 \times I \cup W_5 \times I \) has incompressible boundary.

On the other hand, cutting the handle body \( \tilde{C}(2) \) along the disks \( W_4 \) and \( W_5 \), we get a handle body \( H_* \), with disk system \( \{ W_1, W_2, W_3, W_6 \} \). Let \( \alpha \subset \partial M \) be an essential simple closed curve of slope 10. Then

\[
\alpha = \lambda x^{10} = yxyx^{-1}y^{-1}x^{-1}y^{-2}x^{-1}y^{-1}x^{-1}yxyx^{-12}.
\]

Let \( \tilde{\alpha} \subset \partial \tilde{M} \) be a lift \( \alpha \). Then \( \tilde{\alpha} \), which can be considered as the boundary of the disk \( W_6 \), has slope 2 in \( \partial \tilde{M} \). Figure 17 shows \( \tilde{\alpha} = \partial W_6, \partial W_1, \partial W_2, \partial W_3 \) in \( \partial \tilde{H} \).

From Figure 17, we can read off the Whitehead graph of \( \{ \partial Y_5, \partial X_6 \} \) with respect to the disk system \( \{ W_1, W_2, W_3, W_6 \} \) of \( H_* \), which is given as Figure 18. The graph is connected but has a cut vertex (the vertex \( W^-_2 \)). Applying Whitehead moves to the graph twice with results shown in Figure 19, we end up with a graph (shown in Figure 19 (b)) which is connected with no cut vertex. This means that the surface \( \partial H_* - \{ \partial Y_5 \cup \partial X_6 \} \) is incompressible in \( H_* \). From Figure 16, we also see that \( \partial Y_5 \) is disjoint from \( \partial W_6 \) and \( \partial X_6 \) is disjoint from \( \partial W_1 \). Thus each of the surfaces \( \partial H_* - \partial Y_5 \) and \( \partial H_* - \partial X_6 \) is compressible in \( H_* \). Again the multi-handle addition theorem of [L] implies that the manifold \( H_* \cup X_6 \times I \cup Y_5 \times I \) has incompressible boundary. Therefore the genus two surface \( S = \partial (H_* \cup X_6 \times I \cup Y_5 \times I) = \partial (H_\# \cup W_4 \times I \cup W_5 \times I) \) is incompressible in \( \tilde{M}(2) \) and thus is essential in \( \tilde{M} \).

Obviously \( \tilde{\lambda} \) can be isotoped into \( S \). The proof of Theorem 3 is complete in case \( n = 1 \). The proof for the general case is similar (cf the proof of Theorem 1 in general case). We leave the details to the reader to verify.

\( \diamond \)
Figure 19: (a) The resulting graph after the Whitehead move with respect to the cut vertex $W_2^-$ of Figure 18. (b) The resulting graph after the Whitehead move with respect to the cut vertex $W_3^-$ of part (a)

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