Semi-classical approximations based on Bohmian mechanics

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Abstract
Semi-classical theories are approximations to quantum theory that treat some degrees of freedom classically and others quantum mechanically. In the usual approach, the quantum degrees of freedom are described by a wave function which evolves according to some Schrödinger equation with a Hamiltonian that depends on the classical degrees of freedom. The classical degrees of freedom satisfy classical equations that depend on the expectation values of quantum operators. In this paper, we study an alternative approach based on Bohmian mechanics. In this approach the quantum system is not only described by the wave function, but with additional variables such as particle positions or fields. By letting the classical equations of motion depend on these variables, rather than the quantum expectation values, a semi-classical approximation is obtained that is closer to the exact quantum results than the usual approach. We discuss the Bohmian semi-classical approximation in various context, such as non-relativistic quantum mechanics, quantum electrodynamics and quantum gravity. The main motivation comes from quantum gravity. The quest for a quantum theory is still going on. Therefore a semi-classical approach where gravity is treated classically may be an approximation that already captures some quantum gravitational aspects.

1 Introduction
Quantum gravity is often considered to be the holy grail of theoretical physics. One approach is canonical quantum gravity, which concerns the Wheeler-DeWitt equation and which is obtained by applying the usual quantization methods (which were so successful in the case of high energy physics) to Einstein’s field equations. However, this approach suffers from a host of problems, some of technical and some of conceptual nature (such as finding solutions to the Wheeler-DeWitt equation, the problem of time, . . .). For this reason one often resorts to a semi-classical approximation where gravity is treated classically and matter quantum mechanically [1, 2]. The hope is that such an approximation is easier to analyze and yet reveals some effects of quantum gravitational nature.

In the usual approach to semi-classical gravity, matter is described by quantum field theory on curved space-time. For example, in the case the matter is described by a
quantized scalar field, the state vector can be considered to be a function $\Psi(\phi)$ on the space of fields, which satisfies a particular Schrödinger equation

$$i\partial_t \Psi(\phi, t) = \hat{H}(\phi, g) \Psi(\phi, t),$$

where the Hamiltonian operator $\hat{H}$ depends on the space-time metric $g$. This metric satisfies Einstein’s field equations

$$G_{\mu\nu}(g) = 8\pi G \langle \Psi | \hat{T}_{\mu\nu}(\phi, g) | \Psi \rangle,$$

where the source term is given by the expectation value of the energy-momentum tensor operator.

This semi-classical approximation of course has limited validity. For example, it will form a good approximation when the matter state approximately corresponds to a classical state (i.e., a coherent state), but will fail to be so when the state is a macroscopic superposition of such states. Namely, for such a superposition $\Psi = (\Psi_1 + \Psi_2)/\sqrt{2}$, we have that $\langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle \approx \left( \langle \Psi_1 | \hat{T}_{\mu\nu} | \Psi_1 \rangle + \langle \Psi_2 | \hat{T}_{\mu\nu} | \Psi_2 \rangle \right)/2$, so that the gravitational field is affected by two matter sources, one coming from each term in the superposition. However, one expects that according to a full theory for quantum gravity, the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ each have their own gravitational field and that the total state is a superposition of those. And, indeed, Page and Geilker showed with an experiment of this type that this semi-classical theory is not adequate [2, 3].

Of course, as already noted by Page and Geilker, it could be that this problem is not due to fact gravity is treated classically, but due to the choice of the version of quantum theory. Namely, Page and Geilker adopted the Many Worlds point of view, according to which the wave function never collapses. However, according to standard quantum theory the wave function is supposed to collapse upon measurement. Which physical processes act as measurements is of course rather vague and this is the source of the measurement problem. But it could be that such collapses explain the outcome of their experiment. If an explanation of this type is sought, one should consider so-called spontaneous collapse theories, where collapses are objective, random processes that do not in a fundamental way depend on the notion of measurement. (See [4] and [5] for actual proposals combining such a spontaneous collapse approach with respectively (2) and its non-relativistic version.)

In this paper, we consider another approach to quantum mechanics, namely Bohmian mechanics [6–9]. Bohmian mechanics solves the measurement problem by introducing an actual configuration (particle positions in the non-relativistic domain, particle positions or fields in the relativistic domain [10]) that evolves under the influence of the wave function. According to this approach, instead of coupling classical gravity to the wave function it is natural to couple it to the actual matter configuration. For example, in the case of a scalar field there is an actual field $\phi_B$ whose time evolution is determined by the wave functional $\Psi$. There is an energy-momentum tensor $T_{\mu\nu}(\phi_B, g)$ corresponding to this scalar field and this tensor can be introduced as the source term in Einstein’s
field equations:
\[ G_{\mu\nu}(g) = T_{\mu\nu}(\phi_B, g). \] (3)

This approach solves the problem with the macroscopic superposition, since the energy-momentum tensor will correspond to just one of the macroscopic matter distributions.

However, there is an immediate problem with this approach, namely that equation (3) is not consistent. The Einstein tensor \( G_{\mu\nu} \) is identically conserved, i.e., \( \nabla^\mu G_{\mu\nu} \equiv 0. \) So the Bohmian energy-momentum tensor \( T_{\mu\nu}(\phi_B, g) \) must be conserved as well. However, the equations of motion for the scalar field do not guarantee this. (Similarly, in the Bohmian approach to non-relativistic systems, the energy is generically not conserved.)

We will explain that the root of the problem seems to be the gauge invariance, which in this case is the invariance under spatial diffeomorphisms. Because the scalar field and the space-time metric are connected by spatial diffeomorphisms, it seems that one cannot just assume the metric to be classical without also assuming the scalar field \( \phi_B \) to be classical (in which case the energy-momentum tensor is conserved).

We will see that a similar problem arises when we consider a Bohmian semi-classical approach to scalar electrodynamics, which describes a scalar field interacting with an electromagnetic field. In this case, the wave equation for the scalar field is of the form
\[
i\partial_t \Psi(\phi, t) = \hat{H}(\phi, A)\Psi(\phi, t),
\] (4)

where \( A \) is the vector potential. There is also a Bohmian scalar field \( \phi_B \) and a charge current \( j^\nu(\phi_B, A) \) that could act as the source term in Maxwell’s equations
\[
\partial_\mu F^{\mu\nu}(A) = j^\nu(\phi_B, A),
\] (5)

where \( F^{\mu\nu} \) is the electromagnetic field tensor. In this case, we have \( \partial_\nu \partial_\mu F^{\mu\nu} \equiv 0 \) due to the anti-symmetry of \( F^{\mu\nu} \). As such, the charge current must be conserved. However, the Bohmian equation of motion for the scalar field does not imply conservation. Hence, just as in the case of gravity, a consistency problem arises. We will find that this problem can be overcome by eliminating the gauge invariance, either by assuming some gauge or (equivalently) by working with gauge-independent degrees of freedom. In this way, we can straightforwardly derive a semi-classical approximation starting from the full Bohmian approach to scalar electrodynamics. For example, in the Coulomb gauge, the result is that there is an extra current \( j^\nu_Q \) which appears in addition to the usual charge current and which depends on the quantum potential, so that Maxwell’s equations read
\[
\partial_\mu F^{\mu\nu}(A) = j^\nu(\phi_B, A) + j^\nu_Q(\phi_B, A).
\] (6)

While it is easy to eliminate the gauge invariance in the case of electrodynamics, this is notoriously difficult in the case of general relativity. One can formulate a Bohmian theory for the Wheeler-DeWitt approach to quantum gravity, but the usual formulation does not explicitly eliminate the gauge freedom arising from spatial diffeomorphism invariance. Our expectation is that one could find a semi-classical approximation given
such a formulation. At least we find our expectation confirmed in simplified models, called mini-superspace models, where this invariance is eliminated. We will illustrate this for the model described by the homogeneous and isotropic Friedman-Lemaître-Robertson-Walker metric and a uniform scalar field.

In this paper, we are merely concerned with the formulation of Bohmian semi-classical approximations. Practical applications will be studied elsewhere. Such applications have already been studied for non-relativistic systems in the context of quantum chemistry [11–16]. It appears that Bohmian semi-classical approximations yield better or equivalent results compared to the usual semi-classical approximation. (They are better in the sense that they are closer to the exact quantum results.) This provides good hope that also in other contexts, such as quantum gravity, the Bohmian approach also gives better results. Potential applications might be found in inflation theory, where the back-reaction from the quantum fluctuations onto the classical background can be studied, or in black hole physics, to study the back-reaction from the Hawking radiation onto space-time.

Other semi-classical approximations have been proposed, see for example [17–20], and in particular [21, 22] where also Bohmian ideas are used. We will not make a comparison with these proposals here.

The semi-classical approximation is just one practical application of Bohmian mechanics. In recent years, others have been explored. See [23, 24] for overviews. For example, one may use Bohmian approximation schemes to solve problems in many-body systems [25, 26] or one may use Bohmian trajectories as a calculational tool to simulate wave function evolution [27]. So even though Bohmian mechanics yields the same predictions as standard quantum theory (insofar the latter are unambiguous), it leads to new practical tools and ideas.

The outline of the paper is as follows. We will start with an introduction to Bohmian mechanics in section 2. In section 3, we present the Bohmian semi-classical approximation to non-relativistic quantum theory and its derivation from the full Bohmian theory. Then we will respectively discuss the Bohmian semi-classical approximation to the quantized Abraham model in section 4, scalar quantum electrodynamics in section 5 and quantum gravity in section 6. In the latter section, we will work out a simple concrete example to compare the usual semi-classical approximation to the Bohmian one and find that the latter gives better results.

2 Bohmian mechanics

Non-relativistic Bohmian mechanics (also called pilot-wave theory or de Broglie-Bohm theory) is a theory about point-particles in physical space moving under the influence of the wave function [6–9]. The equation of motion for the configuration $X = (X_1, \ldots, X_n)$
of the particles is given by\(^1\)

\[
\dot{X}(t) = v^\psi(X(t), t),
\]

(7)

where \(v^\psi = (v_1^\psi, \ldots, v_n^\psi)\) with

\[
v_k^\psi = \frac{1}{m_k} \text{Im} \left( \frac{\nabla_k \psi}{\psi} \right) = \frac{1}{m_k} \nabla_k S
\]

(8)

and \(\psi = |\psi| e^{iS}\). The wave function \(\psi(x, t) = \psi(x_1, \ldots, x_n)\) itself satisfies the non-relativistic Schrödinger equation

\[
i\partial_t \psi(x, t) = \left( -\sum_{k=1}^n \frac{1}{2m_k} \nabla_k^2 + V(x) \right) \psi(x, t).
\]

(9)

For an ensemble of systems all with the same wave function \(\psi\), there is a distinguished distribution given by \(|\psi|^2\), which is called the quantum equilibrium distribution. This distribution is equivariant. That is, it is preserved by the particles dynamics (7) in the sense that if the particle distribution is given by \(|\psi(x, t_0)|^2\) at some time \(t_0\), then it is given by \(|\psi(x, t)|^2\) at all times \(t\). This follows from the fact that any distribution \(\rho\) that is transported by the particle motion satisfies the continuity equation

\[
\partial_t \rho + \sum_{k=1}^n \nabla_k \cdot (v_k^\psi \rho) = 0
\]

(10)

and that \(|\psi|^2\) satisfies the same equation, i.e.,

\[
\partial_t |\psi|^2 + \sum_{k=1}^n \nabla_k \cdot (v_k^\psi |\psi|^2) = 0,
\]

(11)

as a consequence of the Schrödinger equation. It can be shown that for a typical initial configuration of the universe, the (empirical) particle distribution for an actual ensemble of subsystems within the universe will be given by the quantum equilibrium distribution \([28, 29]\). Therefore for such a configuration Bohmian mechanics reproduces the standard quantum predictions.

Note that the velocity field is of the form \(j^\psi/|\psi|^2\), where \(j^\psi = (j_1^\psi, \ldots, j_n^\psi)\) with \(j_k^\psi = \text{Im}(\psi^* \nabla_k \psi)/m_k\) is the usual quantum current. In other quantum theories, such as for example quantum field theories, the velocity can be defined in a similar way by dividing the appropriate current by the density. In this way equivariance of the density will be ensured. (See \([30]\) for a treatment of arbitrary Hamiltonians.)

This theory solves the measurement problem. Notions such as measurement or observer play no fundamental role. Instead measurement can be treated as any other physical process.

There are two aspects of the theory that are important for deriving the semi-classical approximation. Firstly, Bohmian mechanics allows for an unambiguous analysis of the

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\(^1\)Throughout the paper we assume units in which \(\hbar = c = 1\).
classical limit. Namely, the classical limit is obtained whenever the particles (or at least the relevant macroscopic variables, such as the center of mass) move classically, i.e., satisfy Newton’s equation. By taking the time derivative of (7), we find that

\[ m_k \ddot{X}_k(t) = -\nabla_k (V(x) + Q^\psi(x,t)) \bigg|_{x=X(t)}, \]

where

\[ Q^\psi = -\sum_{k=1}^{n} \frac{1}{2m_k} \frac{\nabla^2 |\psi|^2}{|\psi|^2} \]

is the quantum potential. Hence, if the quantum force \(-\nabla_k Q^\psi\) is negligible compared to the classical force \(-\nabla_k V\), then the \(k\)-th particle approximately moves along a classical trajectory.

Another aspect of the theory is that it allows for a simple and natural definition for the wave function of a subsystem [8, 28]. Namely, consider a system with wave function \(\psi(x,y)\) where \(x\) is the configuration variable of the subsystem and \(y\) is the configuration variable of its environment. The actual configuration is \((X,Y)\), where \(X\) is the configuration of the subsystem and \(Y\) is the configuration of the other particles. The wave function of the subsystem \(\chi(x,t)\), called the conditional wave function, is then defined as

\[ \chi(x,t) = \psi(x,Y(t),t). \]

This is a natural definition since the trajectory \(X(t)\) of the subsystem satisfies

\[ \dot{X}(t) = v^\psi(X(t),Y(t),t) = v^X(X(t),t). \]

That is, for the evolution of the subsystem’s configuration we can either consider the conditional wave function or the total wave function (keeping the initial positions fixed). (The conditional wave function is also the wave function that would be found by a natural operationalist method for defining the wave function of a quantum mechanical subsystem [31].) The time evolution of the conditional wave function is completely determined by the time evolution of \(\psi\) and that of \(Y\). This makes that the conditional wave function does not necessarily satisfy a Schrödinger equation, although in many cases it does. This wave function collapses according to the usual text book rules when an actual measurement is performed.

We will also consider semi-classical approximations to quantum field theories. More specifically, we will consider bosonic quantum field theories. In Bohmian approaches to such theories it is most easy to introduce actual field variables rather than particle positions [10, 32]. To illustrate how this works, let us consider the free massless real scalar field (for the treatment of other bosonic field theories see [32]). Working in the functional Schrödinger picture, the quantum state vector is a wave functional \(\Psi(\phi)\) defined on a space of scalar fields in 3-space and it satisfies the functional Schrödinger equation

\[ i\partial_t \Psi(\phi,t) = \frac{1}{2} \int d^3x \left( -\frac{\delta^2}{\delta \phi(x)^2} + \nabla \phi(x) \cdot \nabla \phi(x) \right) \Psi(\phi,t). \]
The associated continuity equation is
\[ \partial_t |\Psi(\phi, t)|^2 + \int d^3x \left( \frac{\delta S(\phi, t)}{\delta \phi(x)} |\Psi(\phi, t)|^2 \right) = 0, \] (17)
where \( \Psi = |\Psi|e^{iS} \). This suggests the guidance equation
\[ \dot{\phi}(x, t) = \frac{\delta S(\phi, t)}{\delta \phi(x)} \bigg|_{\phi(x) = \phi(x, t)}. \] (18)
(Note that in this case we did not notationally distinguish the actual field variable from the argument of the wave functional.) Taking the time derivative of this equation results in
\[ \Box \phi(x, t) = -\frac{\delta Q_\Psi(\phi, t)}{\delta \phi(x)} \bigg|_{\phi(x) = \phi(x, t)}, \] (19)
where
\[ Q_\Psi = -\frac{1}{2|\Psi|^2} \int d^3x \frac{\delta^2 |\Psi|}{\delta \phi(x)^2}, \] (20)
where \( Q_\Psi \) is the quantum potential. The classical limit is obtained whenever the quantum force, i.e., the right-hand side of equation (19), is negligible. Then the field approximately satisfies the classical field equation \( \Box \phi = 0 \).

One can also consider the conditional wave functional of a subsystem. A subsystem can in this case be regarded as a system confined to a certain region in space. The conditional wave functional for the field confined to that region is then obtained from the total wave functional by conditioning over the actual field value on the complement of that region. However, in this paper we will not consider this kind of conditional wave functional. Rather, there will be other degrees of freedom, like for example other fields, which will be conditioned over.

This Bohmian approach is not Lorentz invariant. The guidance equation (18) is formulated with respect to a preferred reference frame and as such violates Lorentz invariance. This violation does not show up in the statistical predictions given quantum equilibrium, since the theory makes the same predictions as standard quantum theory which are Lorentz invariant.\(^2\) The difficulty in finding a Lorentz invariant theory resides in the fact that any adequate formulation of quantum theory must be non-local [33]. One approach to make the Bohmian theory Lorentz invariant is by introducing a foliation which is determined by the wave function in a covariant way [34]. In this paper, we will not attempt to maintain Lorentz invariance. As such, the Bohmian semi-classical approximations will not be Lorentz invariant, (very likely) not even concerning the statistical predictions. This is in contrast with the usual approach like the one for gravity given by (1) and (2) which is Lorentz invariant. However, this does not take away the expectation that the Bohmian semi-classical approximation will give better or at least equivalent results compared to the usual approach.

\(^2\)Actually, this statement needs some qualifications since regulators need to be introduced to make the theory and its statistical predictions well defined [32].
3 Non-relativistic quantum mechanics

3.1 Usual versus Bohmian semi-classical approximation

Consider a composite system of just two particles. The usual semi-classical approach (also called the mean-field approach) goes as follows. Particle 1 is described quantum mechanically, by a wave function $\chi(x_1, t)$, which satisfies the Schrödinger equation

$$i\hbar \frac{\partial \chi(x_1, t)}{\partial t} = \left[-\frac{1}{2m_1} \nabla^2_1 + V(x_1, X_2(t))\right] \chi(x_1, t),$$

where the potential is evaluated for the position of the second particle $X_2$, which satisfies Newton’s equation

$$m_2 \ddot{X}_2(t) = -\left\langle \chi \left| \nabla_2 V(x_1, x_2) \right|_{x_2=\dot{X}_2(t)} \right\rangle = \int d^3x_1 |\chi(x_1, t)|^2 \left[-\nabla_2 V(x_1, x_2)\right] \bigg|_{x_2=\dot{X}_2(t)}. \tag{22}$$

So the force on the right-hand-side is averaged over the quantum particle.

An alternative semi-classical approach based on Bohmian mechanics was proposed independently by Gindensperger et al. [11] and Prezhdo and Brookby [12]. In this approach there is also an actual position for particle 1, denoted by $X_1$, which satisfies the equation

$$\dot{X}_1(t) = v^\chi(X_1(t), t), \tag{23}$$

where

$$v^\chi = \frac{1}{m_1} \text{Im} \frac{\nabla \chi}{\chi}, \tag{24}$$

and where $\chi$ satisfies the Schrödinger equation (21). But instead of equation (22), the second particle now satisfies

$$m_2 \ddot{X}_2(t) = -\nabla_2 V(X_1(t), x_2) \bigg|_{x_2=\dot{X}_2(t)}, \tag{25}$$

where the force depends on the position of the first particle. So in this approximation the second particle is not acted upon by some average force, but rather by the actual particle of the quantum system. This approximation is therefore expected to yield a better approach than the usual approach, in the sense that it yields predictions closer to those predicted by full quantum theory, especially in the case where the wave function evolves into a superposition of non-overlapping packets. This is indeed confirmed by a number of studies, as we will discuss below.

Let us first mention some properties of this approximation and compare them to the usual approach. In the mean field approach, the specification of an initial wave function $\chi(x, t_0)$, an initial position $X_2(t_0)$ and velocity $\dot{X}_2(t_0)$ determines a unique solution for the wave function and the trajectory of the classical particle. In the Bohmian approach also the initial position $X_1(t_0)$ of the particle of the quantum system needs to be specified in order to uniquely determine a solution. Different initial positions $X_1(t_0)$ yield different
evolutions for the wave function and the classical particle. This is because the evolution of each of the variables $X_1, X_2, \chi$ depends on the others. Namely, the evolution of $\chi$ depends on $X_2$ via (21), whose evolution in turn depends on $X_1$ via (25), whose evolution in turn depends on $\chi$ via (23). (This should be contrasted with the full Bohmian theory, where the wave function acts on the particles, but there is no back-reaction from the particles onto the wave function.)

The initial configuration $X_1(t_0)$ should be considered random with distribution $|\chi(x, t_0)|^2$. However, this does not imply that $X_1(t)$ is random with distribution $|\chi(x, t)|^2$ for later times $t$. It is not even clear what the latter statement should mean, since different initial positions $X_1(t_0)$ lead to different wave function evolution; so which wave function should $\chi(x, t)$ be?

This semi-classical approximation has been applied to a number of systems. Prezhdo and Brookby studied the case of a light particle scattering off a heavy particle [12]. They considered the scattering probability over time and found that the Bohmian semi-classical approximation was in better agreement with the exact quantum mechanical prediction than the usual approximation. The Bohmian semi-classical approximation gives probability one for the scattering to have happened after some time, in agreement with the exact result, whereas the probability predicted by the usual approach does not reach one. The reported reason for the better results is that the wave function of the quantum particle evolves into a superposition of non-overlapping packets, which yields bad results for the usual approach (since the force on the classical particle contains contributions from both packets), but not for the Bohmian approach. These results were confirmed and further expanded by Gindensperger et al. [13]. Other examples have been considered in [11, 14, 15]. In those cases, the Bohmian semi-classical approximation gave very good agreement with the exact quantum or experimental results. It was always either better or comparable to the usual approach. These results give good hope that the Bohmian semi-classical approximation will also give better results than the usual approximation in other domains such as quantum gravity.

3.2 Derivation of the Bohmian semi-classical approximation

The Bohmian semi-classical approach can easily be derived from the full Bohmian theory. Consider a system of two particles. In the Bohmian description of this system, we have a wave function $\psi(x_1, x_2, t)$ and positions $X_1(t), X_2(t)$, which respectively satisfy

\footnote{The derivation is very close to the one followed by Gindensperger et al. [11]. A difference is that they also let the wave function of the quantum system depend parametrically on the position of the classical particle. This leads to a quantum force term in the equation (25) for particle 2. However, this does not seem to lead to a useful set of equations. In particular, they can not be numerically integrated by simply specifying the initial wave function and particle positions. In any case, Gindensperger et al. drop this quantum force when considering examples [11, 13, 14], so that the resulting equations correspond to the ones presented above.}
the Schrödinger equation
\[ i \partial_t \psi = \left[ -\frac{1}{2m_1} \nabla_1^2 - \frac{1}{2m_2} \nabla_2^2 + V(x_1, x_2) \right] \psi \] (26)
and the guidance equations
\[ \dot{X}_1(t) = v_1^\psi(X_1(t), X_2(t), t), \quad \dot{X}_2(t) = v_2^\psi(X_1(t), X_2(t), t). \] (27)

The conditional wave function \( \chi(x_1, t) = \psi(x_1, X_2(t), t) \) for particle 1 satisfies the equation
\[ i \partial_t \chi(x_1, t) = \left( -\frac{\nabla_1^2}{2m_1} + V(x_1, X_2(t)) \right) \chi(x_1, t) + I(x_1, t), \] (28)
where
\[ I(x_1, t) = \left( -\frac{\nabla_2^2}{2m_2} \psi(x_1, x_2, t) \right) \bigg|_{x_2=X_2(t)} + i \nabla_2 \psi(x_1, x_2, t) \bigg|_{x_2=X_2(t)} \cdot v_2^\psi(X_1(t), X_2(t), t). \] (29)

So in case \( I \) is negligible in (28), up to a time-dependent factor times \( \chi \), we are led to the Schrödinger equation (21). This will for example be the case if \( m_2 \) is much larger than \( m_1 \) (\( I \) is inversely proportional to \( m_2 \)) and if the wave function slowly varies as a function of \( x_2 \). We also have that
\[ m_2 \ddot{X}_2(t) = -\nabla_2 \left[ V(X_1(t), x_2) + Q^\psi(X_1(t), x_2, t) \right] \bigg|_{x_2=X_2(t)}, \] (30)

with \( Q^\psi \) the quantum potential. We obtain the classical equation (25), if the quantum force is negligible compared to the classical force.

In this way we obtain the equations for a semi-classical formulation. In addition, we also have the conditions under which they will be valid. For other quantum theories, such as quantum gravity, we can follow a similar path to find a Bohmian semi-classical approximation.

### 4 The quantized Abraham model

The next theory we consider is the Abraham model, which on the classical level describes extended, rigid, charged particles, which interact with an electromagnetic field [35]. We will consider a semi-classical approximation, first considered by Kiessling [36], where the charges are treated quantum mechanically, while the electromagnetic field is treated classically. Just as in the case of non-relativistic particles, it is straightforward to find a consistent set of equations, i.e., no difficulties such as the one mentioned in the introduction are met. We will again start from the full Bohmian theory to guide us to the semi-classical approximation.

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4If \( I \) contains a term of the form \( f(t) \chi \), then it can be eliminated by changing the phase of \( \chi \) by a time-dependent term.
In the classical Abraham model, the charge distribution of the $i$-th particle is centered around a position $\mathbf{q}_i$ and is given by $\rho_i(\mathbf{x}) = e\varphi(\mathbf{x} - \mathbf{q}_i)$, where the function $\varphi$ is assumed to be smooth, radial, with compact support and normalized to 1. The particles move under the Lorentz force law

$$m\mathbf{\ddot{q}}_i = e\left[\mathbf{E}_\varphi(\mathbf{q}_i) + \mathbf{q}_i \times \mathbf{B}_\varphi(\mathbf{q}_i)\right], \quad (31)$$

where $\mathbf{E}_\varphi(\mathbf{x})$ is shorthand for the convolution $(\mathbf{E} * \varphi)(\mathbf{x})$ and similarly for $\mathbf{B}_\varphi(\mathbf{x})$. $\mathbf{E}$ and $\mathbf{B}$ are respectively the electric and magnetic field. In terms of the electromagnetic potential $A^\mu = (A_0, \mathbf{A})$, they are given by $\mathbf{E} = -\partial_t \mathbf{A} - \nabla A_0$ and $\mathbf{B} = \nabla \times \mathbf{A}$. The electromagnetic field satisfies Maxwell’s equations

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad (32)$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and charge current $j^\nu$ given by

$$j^0(\mathbf{x}, t) = \rho(\mathbf{x}, t) = \sum_{i=1}^n e\varphi(\mathbf{x} - \mathbf{q}_i(t)), \quad j^i(\mathbf{x}, t) = \sum_{i=1}^n e\mathbf{q}_i(t) \varphi(\mathbf{x} - \mathbf{q}_i(t)). \quad (33)$$

Note that although we have employed a covariant notation this model is not covariant, due to the rigidity of the particles.

The quantization of the classical model is straightforward [35]. In the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the Schrödinger equation reads

$$i\hbar \partial_t \Psi = \left[-\frac{1}{2m} \sum_{i=1}^n \left(\nabla_i - ie\mathbf{A}_\varphi^T(\mathbf{q}_i)\right)^2 + V_c(q) + \int d^3x \left(-\frac{1}{2} \frac{\delta^2}{(\delta^2 + 1/2)(\nabla \times \mathbf{A}^T(\mathbf{x}))^2}\right)\right] \Psi \quad (34)$$

for the wave function $\Psi(q, \mathbf{A}^T)$, where $q = (\mathbf{q}_1, \ldots, \mathbf{q}_n)$, $\mathbf{A}^T$ is the transverse part of the vector potential$^5$ and $V_c$ is the Coulomb potential$^6$.

$$V_c = -\frac{1}{2} \int d^3x \rho(\mathbf{x}) \frac{\rho(\mathbf{x})}{(\nabla \times \mathbf{A}^T)^2} = \frac{1}{8\pi} \int d^3xd^3y \frac{\rho(\mathbf{x}) \rho(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \quad (35)$$

$$= \frac{e^2}{8\pi} \sum_{i,j=1}^n \int d^3xd^3y \varphi(\mathbf{x}) \varphi(\mathbf{y}) \left|\mathbf{x} - \mathbf{y} + \mathbf{q}_i - \mathbf{q}_j\right| \quad (36)$$

So for the electromagnetic field we have used the functional Schrödinger representation (see e.g. [32] for more details).

In the Bohmian approach we have the following guidance equations

$$\mathbf{\dot{q}}_i(t) = \frac{1}{m} \left[\nabla_i S(q, \mathbf{A}^T, t) - e\mathbf{A}_\varphi^T(\mathbf{q}_i)\right]_{q(t), \mathbf{A}^T(\mathbf{x}, t)}, \quad (38)$$

$^5$We can write $\mathbf{A} = \mathbf{A}^T + \mathbf{A}^L$, with $\mathbf{A}^T = \mathbf{A} - \nabla \varphi \mathbf{\cdot} \mathbf{A}$ and $\mathbf{A}^L = \nabla \varphi \mathbf{\cdot} \mathbf{A}$ respectively the transverse and longitudinal part of the vector potential. The Coulomb gauge corresponds to $\mathbf{A}^L = 0$.

$^6$We have used the notation $\nabla_j f(\mathbf{x}) = -\frac{1}{i\hbar} \int d^3y f(\mathbf{y})$. 

11
\[ \dot{A}^T(x, t) = \frac{\delta S(q, A^T, t)}{\delta A^T(x)} \bigg|_{q(t), A^T(x, t)}, \]  

(39)

where \( \Psi = |\Psi|e^{iS} \). Taking the time derivative, we find

\[ m\ddot{q}_i = e \left[ E_\varphi(q_i) + \dot{q}_i \times B_\varphi(q_i) \right] - \nabla_i Q, \]  

(40)

\[ \partial_\mu F^{\mu\nu} = j^\nu + j_Q^\nu, \]  

(41)

\[ Q = -\frac{1}{2m} \sum_{i=1}^n \frac{\nabla_i^2 |\psi|}{|\psi|} - \frac{1}{2|\psi|} \int d^3x \frac{\delta^2 |\psi|}{\delta A^T^2}, \quad j_Q^\nu = \left(0, -\frac{\delta Q}{\delta A^T} \right) \]  

(42)

are respectively the quantum potential and what can be called the quantum charge current, which enters Maxwell’s equation in addition to the usual current. Both currents are conserved, i.e., \( \partial_\mu j_\mu = \partial_\mu j_Q^\mu = 0 \). These equations are written in terms of \( A^\mu \) and it is assumed that the Coulomb gauge \( \nabla \cdot A = 0 \) holds. So the equations (41) are equivalent to

\[ \Box A^T = j^T + j_Q, \quad A^L = 0, \quad \nabla^2 A_0 + j^0 = 0, \quad \nabla \dot{A}_0 = j^L, \]  

(43)

where \( A^L \) is the longitudinal part of the vector potential. The first equation follows from differentiating (39) with respect to time, the second one is implied by the Coulomb gauge, and the third (and the fourth) defines \( A_0 \) in terms of the charge density.

We can now easily consider a semi-classical limit where either the charges or the electromagnetic field approximately behave classically. Let us consider the latter case. In that case, we assume the quantum current \( j_Q^\nu \) to be negligible in (41).

Consider further the conditional wave function for the charges \( \chi(q, t) = \psi(q, A^T(x, t)) \), where \( (q(t), A^T(x, t)) \) is a particular solution to the guidance equations. It satisfies the equation

\[ i\partial_t \chi = \left[ -\frac{1}{2m} \sum_{i=1}^n \left( \nabla_i - ieA^T_\varphi(q_i, t) \right)^2 + V_c(q) \right] \chi + I, \]  

(44)

where

\[ I = \int d^3x \left( -\frac{1}{2m} \sum_{i=1}^n \left( \nabla_i - ieA^T_\varphi(q_i, t) \right)^2 + \frac{1}{2} \left( \nabla \times A^T(x) \right)^2 \right) \psi + i\partial_t A^T(x, t) \cdot \frac{\delta \psi}{\delta A^T(x)} \bigg|_{A^T(x) = A^T(x, t)} \]  

(45)

Whenever \( I \) is negligible, up to a time-dependent factor times \( \chi \) (cf. footnote 4), the wave equation reduces to

\[ i\partial_t \chi = \left[ -\frac{1}{2m} \sum_{i=1}^n \left( \nabla_i - ieA^T_\varphi(q_i, t) \right)^2 + V_c(q) \right] \chi. \]  

(46)

Together with

\[ \dot{q}_i = \frac{1}{m} \left( \nabla_i S - eA^T_\varphi(q_i) \right), \quad \partial_\mu F^{\mu\nu} = j^\nu \]  

(47)
and the Coulomb gauge, this defines a consistent semi-classical approximation. The second-order equation for the charges is still of the form (40), but now with quantum potential

\[ Q = -\frac{1}{2m} \sum_{i=1}^{n} \nabla_i^2 |x| \]  \hspace{1cm} (48)

This semi-classical theory was studied by Kiessling [36]. Kiessling also considers charges with spin and possible alternative guidance equations. Our analysis shows how it may be derived from Bohmian approach to the quantized Abraham model.

5 Scalar electrodynamics

So far the semi-classical approximations could have easily been guessed without considering the full quantum theory. In this section, we consider semi-classical approximations to scalar electrodynamics, i.e., a scalar field interacting with the electromagnetic field, for which this would have been much harder. The approximation will be obtained by starting from different but equivalent formulations of the Bohmian approach to quantum electrodynamics. These formulations can be found either by considering different gauges or by working with different choices of gauge-independent variables (which more or less amounts to the same thing [32, 37]). The gauges we will consider here are the Coulomb gauge, the unitary gauge and the temporal gauge. However, while the Coulomb and the unitary gauge completely fix the gauge freedom, the temporal gauge does not. We will find that in order to a (consistent) semi-classical approximation, it will appear necessary to completely eliminate the gauge freedom (or at least separate gauge degrees of freedom from gauge-independent degrees of freedom). As such, the Bohmian formulation in the temporal gauge does not immediately seem to lead to a semi-classical approximation.

5.1 Coulomb gauge

5.1.1 Classical theory

We start by formulating the classical theory of scalar electrodynamics. The Lagrangian is

\[ L = \int d^3x \left( (D_\mu \phi)^* D^\mu \phi - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) , \]  \hspace{1cm} (49)

where \( D_\mu = \partial_\mu + ieA_\mu \) is the covariant derivative, with \( A^\mu = (A_0, A) \) and \( F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \). The corresponding field equations are

\[ D_\mu D^\mu \phi + m^2 \phi = 0 , \hspace{1cm} \partial_\mu F^{\mu\nu} = j^\nu , \]  \hspace{1cm} (50)

where

\[ j^\nu = ie (\phi^* D^\nu \phi - \phi D^\nu \phi^*) \]  \hspace{1cm} (51)

is the charge current. The theory has a local gauge symmetry

\[ \phi \rightarrow e^{i e \alpha} \phi , \hspace{1cm} A^\mu \rightarrow A^\mu - \partial^\mu \alpha . \]  \hspace{1cm} (52)
In this section we will focus on the Coulomb gauge $\nabla \cdot A = 0$. Writing $A = A^T + A^L$ (cf. footnote 5), this gauge corresponds to $A^L = 0$ and the equations of motion reduce to
\[
\left( \Box + 2ieA_0 \partial_t + ie\dot{A}_0 - e^2A_0^2 + 2ieA^T \cdot \nabla + e^2A^T^2 + m^2 \right) \phi = 0 , \tag{53}
\]

\[
\Box A^T + \nabla \dot{A}_0 = j , \tag{54}
\]

\[
-\nabla^2 A_0 = j_0 , \tag{55}
\]

with $j = ie(\phi \nabla \phi^* - \phi^* \nabla \phi) - 2e^2A^T|\phi|^2$ and $j_0 = ie(\phi^* \dot{\phi} - \phi \dot{\phi}^*) - 2e^2A_0^2|\phi|^2$. Equation (55) can be written as
\[
(\nabla^2 - 2e^2|\phi|^2) A_0 = -ie(\phi^* \dot{\phi} - \phi \dot{\phi}^*) . \tag{56}
\]

This is not a dynamical equation but rather a constraint on $A_0$; it could be used to solve for $A_0$ in terms of $\phi$.

### 5.1.2 Bohmian approach

The classical theory can easily be quantized using the Coulomb gauge. The same quantum field theory can be obtained by eliminating the gauge degrees of freedom on the classical level and then quantizing the remaining gauge-independent degrees of freedom [32]. In the resulting Bohmian approach there are actual fields $\phi$ and $A^T$ that are guided by a wave functional $\Psi(\phi, A^T, t)$ which satisfies the functional Schrödinger equation\(^7\)
\[
i\partial_t \Psi = \int d^3x \left( -\frac{\delta^2}{\delta \phi^* \delta \phi} + |(\nabla - ieA^T)\phi|^2 + m^2|\phi|^2 - \frac{1}{2} C \frac{1}{\nabla^2} C - \frac{1}{2} \frac{\delta^2}{\delta A^T \delta A^T} + \frac{1}{2} (\nabla \times A^T)^2 \right) \Psi , \tag{57}
\]

where
\[
C(x) = e \left( \phi^*(x) \frac{\delta}{\delta \phi^*(x)} - \phi(x) \frac{\delta}{\delta \phi(x)} \right) \tag{58}
\]
is the charge density operator in the functional Schrödinger picture. The first three terms in the Hamiltonian correspond to the Hamiltonian of a scalar field minimally coupled to a transverse vector potential. The fourth term corresponds to the Coulomb potential and the remaining terms to the Hamiltonian of a free electromagnetic field.

The guidance equations are
\[
\dot{\phi} = \frac{\delta S}{\delta \phi^*} - e\phi \frac{1}{\nabla^2} CS , \quad \dot{A}^T = \frac{\delta S}{\delta A^T} , \tag{59}
\]

with $\Psi = |\Psi|e^{iS}$. Defining
\[
A_0 = -i \frac{1}{\nabla^2} CS , \tag{60}
\]

---

\(^7\)The wave functional should be understood as a functional of the real and imaginary part of $\phi$. In addition, writing $\phi = (\phi_\text{r} + i\phi_\text{i})/\sqrt{2}$, we have that the functional derivatives are given by Wirtinger derivatives $\delta/\delta \phi = (\delta/\delta \phi_\text{r} - i\delta/\delta \phi_\text{i})/\sqrt{2}$ and $\delta/\delta \phi = (\delta/\delta \phi_\text{r} + i\delta/\delta \phi_\text{i})/\sqrt{2}$.
we can rewrite the guidance equation for the scalar field as
\[ D_0 \phi = \frac{\delta S}{\delta \phi^*}. \] (61)

With this definition of \( A_0 \), the classical equation (56) is satisfied.

Taking the time derivative of the guidance equations we obtain, after some calculation, that
\[ D_\mu D^\mu \phi - m^2 \phi = -\frac{\delta Q}{\delta \phi^*}, \] (62)
\[ \partial_\mu F^{\mu \nu} = j^\nu + j_Q^\nu, \] (63)
where the Coulomb gauge is assumed. The quantum potential is given by
\[ Q = -\frac{1}{|\Psi|} \int d^3x \left( \frac{\delta^2}{\delta \phi^* \delta \phi} + \frac{1}{2} \frac{\delta}{\delta \nabla} \right) |\Psi|^2. \] (64)
The current \( j^\mu \) is given by the classical expression (51) and \( j_Q^\nu = (0, j_Q^\nu) \), with
\[ j_Q = i\nabla \frac{1}{\nabla^2} CQ - \frac{\delta Q}{\delta A^T} \] (65)
can be considered an additional quantum current. The total current \( j^\nu + j_Q^\nu \) is conserved, i.e., \( \partial_\nu (j^\nu + j_Q^\nu) = 0 \), as a consequence of (62). This is required for consistency of (63), since \( \partial_\nu \partial_\mu F^{\mu \nu} \equiv 0 \) (due to the anti-symmetry of \( F^{\mu \nu} \)). The current \( j^\nu \) satisfies \( \partial_\nu j^\nu = -iCQ \) and is hence not necessarily conserved.

### 5.1.3 Semi-classical approximation: classical electromagnetic field

Let us first consider a semi-classical approximation by treating the electromagnetic field classically and the scalar field quantum mechanically. Similarly as before, it can be found by considering the conditional wave functional \( \chi(\phi, t) = \Psi(\phi, A^T(t), t) \) for the scalar field and conditions under which the electromagnetic field approximately behaves classically.

We will not carry through this procedure, but just give the resulting equations.

The wave functional \( \chi(\phi, t) \) satisfies the functional Schrödinger equation for a quantized scalar field moving in an external classical transverse vector potential and Coulomb potential:
\[ i\partial_t \chi = \int d^3x \left( -\frac{\delta^2}{\delta \phi^* \delta \phi} + |\nabla - ieA^T| \phi|^2 + m^2 |\phi|^2 - \frac{1}{2} C \frac{\delta}{\delta \nabla} \right) \chi, \] (66)
where \( C \) is defined as before. The actual scalar field satisfies
\[ D_0 \phi = \frac{\delta S}{\delta \phi^*}, \] (67)
where \( A_0 \) is defined as before, with \( S \) now the phase of \( \chi \). The vector potential \( A^\mu = (A_0, A^T) \) satisfies Maxwell’s equations
\[ \partial_\mu F^{\mu \nu} = j^\nu + j_Q^\nu, \] (68)
where \( j_\nu^Q = (0, j_\phi^Q) \) is an additional quantum current, with
\[
j_Q = i \nabla \frac{1}{\sqrt{2}C} Q \tag{69}
\]
and
\[
Q = -\frac{1}{|\chi|} \int d^3 x \left( \frac{\delta^2}{\delta \phi^* \delta \phi} + \frac{1}{2} C \frac{1}{\sqrt{2}C} \right) |\chi| \tag{70}
\]
So the equations (66)-(68) define the semi-classical approximation.

We still have that \( \partial_\nu (j^\mu + j_\phi^Q) = 0 \), so that the Maxwell equations (68) are consistent. The correct source term in Maxwell’s equations was found by considering the Bohmian approach to the full quantum theory. Without this it would have been hard to guess the right current. Just using the current \( j^\mu \) would yield an inconsistent set of equations since \( \partial_\nu j_\nu = -iCQ \). Adding an extra current to \( j^\mu \) so that the total one would be conserved would still leave an ambiguity since one could always add a vector that is conserved.

Another essential ingredient in our derivation was that the gauge freedom was eliminated. We will see in section 5.3 that without such an elimination it does not seem possible to derive a semi-classical approximation.

### 5.1.4 Semi-classical approximation: classical scalar field

We can also consider a semi-classical approximation where the scalar field is treated classically and the electromagnetic field quantum mechanically. In this case, the functional Schrödinger equation for the wave function \( \chi(A_T, t) \) is
\[
i \partial_t \chi = \int d^3 x \left( -\frac{1}{2} \frac{\delta^2}{\delta A_T^2} + \frac{1}{2} (\nabla \times A_T)^2 + ieA_T \cdot (\phi^* \nabla \phi - \phi \nabla \phi^*) + e^2 A_T^2 |\phi|^2 \right) \chi. \tag{71}
\]
The actual potential \( A_T \) satisfies the guidance equation
\[
\dot{A}_T = \frac{\delta S}{\delta A_T} \tag{72}
\]
and the scalar field satisfies the classical equation
\[
D_\mu D^\mu \phi - m^2 \phi = 0, \tag{73}
\]
where again \( A^\mu = (A_0, A_T) \) with \( A_0 \) defined by (56).

### 5.2 Unitary gauge

There are other possible semi-classical approximations one can consider. We have 4 independent field degrees of freedom, namely the two transverse degrees of freedom of the vector potential and the two (real) degrees of freedom of the scalar field, and any of these or combinations thereof could in principle be assumed classical. Here, we will consider two different semi-classical approximations. One where the amplitude of the scalar field is assumed classical and another one where all degrees of freedom but the
amplitude are assumed classical. While we could consider these approximations using the Bohmian formulation in terms of the Coulomb gauge, it is more elegant to consider it in a different yet equivalent formulation which can be obtained by imposing the unitary gauge.

### 5.2.1 Classical theory

The unitary gauge is given by $\phi = \phi^*$. Writing

$$\phi = \eta e^{i\theta}/\sqrt{2},$$

with $\eta = \sqrt{2}\phi$ (which can be done where $\phi \neq 0$), this gauge amounts to $\theta = 0$. The classical equations become

$$\Box \eta + m^2 \eta - e^2 A^\mu A_\mu \eta = 0, \quad \partial_\mu F^{\mu\nu} = j^\nu,$$

where now $j^\nu = -e^2 \eta^2 A^\nu$. Maxwell’s equations become

$$\Box A + \nabla \nabla \cdot A + e^2 \eta^2 A + \nabla \dot{A}_0 = 0,$$

$$\left(\nabla^2 - e^2 \eta^2\right) A_0 + \nabla \cdot \dot{A} = 0.$$

The last equation is again a constraint rather than a dynamical equation; it can be used to express $A_0$ in terms of $A$.

### 5.2.2 Bohmian approach

Quantization of the classical theory leads to the following functional Schrödinger equation for $\Psi(\eta, A)$:

$$i \partial_t \Psi = \frac{1}{2} \int d^3x \left( -\frac{1}{\eta} \frac{\delta}{\delta \eta} \left( \frac{\delta}{\delta \eta} \right) + (\nabla \eta)^2 + m^2 \eta^2 + e^2 A^2 \eta^2 - \frac{\delta^2}{\delta A^2} - \frac{1}{e^2 \eta^2} \left( \nabla \cdot \frac{\delta}{\delta A} \right)^2 + (\nabla \times A)^2 \right) \Psi.$$

The guidance equations are

$$\dot{\eta} = \frac{\delta S}{\delta \eta}, \quad \dot{\mathbf{A}} = \frac{\delta S}{\delta \mathbf{A}} - \nabla \left( \frac{1}{e^2 \eta^2} \nabla \cdot \frac{\delta S}{\delta \mathbf{A}} \right).$$

Defining

$$A_0 = \frac{1}{e^2 \eta^2} \nabla \cdot \frac{\delta S}{\delta \mathbf{A}},$$

the latter equation can be written as $\dot{\mathbf{A}} = \delta S/\delta \mathbf{A} - \nabla A_0$. With this definition the classical equation (77) also holds for the Bohmian dynamics.

The second order equations are

$$\Box \eta + m^2 \eta - e^2 A^\mu A_\mu \eta = -\frac{\delta Q}{\delta \eta}, \quad \partial_\mu F^{\mu\nu} = j^\nu + j^\nu_Q,$$
where $j^\nu_Q = (0,j_Q)$, with
\[ j_Q = \frac{\delta Q}{\delta A}, \]
and
\[ Q = -\frac{1}{2|\Psi|^2} \int d^3x \left( \frac{1}{\eta} \frac{\delta}{\delta \eta} \left( \eta \frac{\delta}{\delta \eta} \right) + \frac{\delta^2}{\delta A^2} + \frac{1}{e^2} \left( \nabla \cdot \frac{\delta}{\delta A} \right)^2 \right) |\Psi|. \]

This Bohmian approach is equivalent to the approach considered in the previous section. This can easily be checked by using the field transformation $(\phi, A^T) \rightarrow (\eta, A)$ which is obtained by using the polar decomposition (74) for $\phi$ and $A = A^T + \frac{1}{e} \nabla \theta$ (with inverse transformation $\theta = \frac{1}{e^2} \nabla \cdot A$).

### 5.2.3 Semi-classical approximation: classical electromagnetic field

In the case the vector potential $A$ evolves approximately classically, we have the following semi-classical approximation. The wave functional $\chi(\eta,t)$ satisfies
\[ i\partial_t \chi = \frac{1}{2} \int d^3x \left( -\frac{1}{\eta} \frac{\delta}{\delta \eta} \left( \eta \frac{\delta}{\delta \eta} \right) + (\nabla \eta)^2 + m^2 \eta^2 - e^2 A^\mu A^\mu \eta^2 \right) \chi \]
and the guidance equation
\[ \dot{\eta} = \frac{\delta S}{\delta \eta}. \]

The electromagnetic field satisfies
\[ \partial_\mu F^{\mu\nu} = -e^2 \eta^2 A^\nu. \]

The appearance of the term containing $A_0$ in the Schrödinger equation requires some explanation. As before, the equation can be obtained by assuming certain terms negligible when considering the time derivative of the conditional wave functional $\chi(\eta,t) = \Psi(\eta, A(t), t)$. In this case, $i\partial_t \chi$ contains the term
\[ \int d^3x \frac{e^2 \eta^2(x)}{2} \left[ -A_0^2(x,t) + \left( A_0^2(x,t) - \frac{1}{e^4} \frac{\delta S}{\delta A(x)} \right)^2 \right] \chi. \]

The term within the big curly bracket on the right hand side would be zero if we evaluated it for the actual field $\eta(x,t)$ because of (80). If we do not evaluate it then it is not necessarily zero. In our semi-classical approximation we assume this term negligible. As such, the resulting Schrödinger equation (84) corresponds to the one of a quantized scalar field minimally coupled to a classical electromagnetic field.

Note that there is no consistency issue in this case. Just as in the classical or the full Bohmian case, the equation $\partial_\mu j^\mu = \partial_\mu (-e^2 \eta^2 A^\mu) = 0$ does not follow from the equation of $\eta$, but from the Maxwell equations themselves.

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8The equivalence of the Schrödinger picture for the Coulomb gauge and the unitary gauge was considered before in [38] (but seems to require a small correction for the kinetic term for $\eta$).
5.2.4 Semiclassical approximation: classical scalar field

In the case we assume the scalar field classical, we have the following semiclassical approximation. The wave functional $\chi(A,t)$ satisfies

$$i \partial_t \chi = \frac{1}{2} \int d^3 x \left( - \frac{\delta^2}{\delta A^2} - \frac{1}{e^2 \eta^2} \left( \nabla \cdot \delta A \right)^2 + (\nabla \times A)^2 + e^2 \eta^2 A^2 \right) \chi$$  \hspace{1cm} (88)

and guides the field $A$ through the guidance equation

$$\dot{A} = \delta S/\delta A - \nabla A_0,$$  \hspace{1cm} (89)

where $A_0$ is defined as in (80). The field $\eta$ satisfies the classical equation

$$\Box \eta + m^2 \eta - e^2 A^2 \eta = 0.$$  \hspace{1cm} (90)

The Schrödinger equation corresponds to a spin-1 field with mass squared $e^2 \eta^2$ [32].

5.3 Scalar electrodynamics: temporal gauge

In this section, we consider a Bohmian approach to scalar electrodynamics where not all gauge freedom is eliminated. It will appear that we need to deal with the remaining gauge freedom in order to get an adequate Bohmian semi-classical approximation.

5.3.1 Classical theory

The temporal gauge is given by $A_0 = 0$. It does not completely fix the gauge. There is still a residual gauge symmetry given by the time-independent transformations:

$$\phi \rightarrow e^{ie\theta} \phi, \hspace{0.5cm} A \rightarrow A + \nabla \theta,$$  \hspace{1cm} (91)

with $\dot{\theta} = 0$.

In this gauge, the classical equations of motion read

$$\ddot{\phi} - D^2 \phi + m^2 \phi = 0, \hspace{0.5cm} \Box A + \nabla \nabla \cdot A = j, \hspace{0.5cm} -\nabla \cdot \dot{A} = j_0,$$  \hspace{1cm} (92)

where $D = \nabla - ieA$, $j = ie(\phi D^* \phi^* - \phi^* D \phi)$ and $j_0 = ie(\phi^* \dot{\phi} - \phi \dot{\phi}^*)$.

5.3.2 Bohmian approach

Quantization of the classical theory leads to the Schrödinger equation\(^9\)

$$i \partial_t \Psi = \int d^3 x \left( - \frac{\delta^2}{\delta \phi^* \delta \phi} + |D \phi|^2 + m^2 |\phi|^2 - \frac{1}{2} \frac{\delta^2}{\delta A^2} + \frac{1}{2} (\nabla \times A)^2 \right) \Psi,$$  \hspace{1cm} (93)

\(^9\)In [39] this equation is considered to study the semiclassical approximation in the context of standard quantum theory.
together with the constraint
\[ \nabla \cdot \frac{\delta \Psi}{\delta A} + ie \left( \phi \frac{\delta \Psi}{\delta \phi^*} - \phi \frac{\delta \Psi}{\delta \phi} \right) = 0, \] (94)
for the wave functional \( \Psi(\phi, A, t) \). The constraint expresses the fact that the wave functional is invariant under time-independent gauge transformations, i.e., \( \Psi(\phi, A) = \Psi(e^{ie\theta} \phi, A + \nabla \theta) \), with \( \theta \) time-independent. The constraint is compatible with the Schrödinger equation: if it is satisfied at one time, it is satisfied at all times.

In the Bohmian approach [40], there are actual configurations \( \phi \) and \( A \) that satisfy
\[ \dot{\phi} = \frac{\delta S}{\delta \phi}, \quad \dot{A} = \frac{\delta S}{\delta A}, \] (95)
These equations are invariant under the time-independent gauge transformations (91) because of the constraint (94).

The corresponding second-order equations are
\[ \ddot{\phi} - D^2 \phi + m^2 \phi = -\frac{\delta Q}{\delta \phi^*}, \quad \Box A + \nabla \nabla \cdot A = j + j_Q, \] (96)
where
\[ Q = -\frac{1}{|\Psi|} \int d^3x \left( \frac{\delta^2}{\delta \phi^* \delta \phi} + \frac{1}{2} \frac{\delta^2}{\delta A^2} \right) |\Psi| \] (97)
and
\[ j_Q = -\frac{\delta Q}{\delta A}. \] (98)
The constraint (94) further implies that
\[ -\nabla \cdot \dot{A} = j_0, \] (99)
so that, assuming the gauge \( A_0 = 0 \), (92) and (99) can be written as
\[ D_\mu D^\mu \phi - m^2 \phi = -\frac{\delta Q}{\delta \phi^*}, \quad \partial_\mu F^{\mu \nu} = j^\nu + j_Q^\nu, \] (100)
where \( j_Q^\nu = (0, j_Q) \).

This Bohmian approach is equivalent to the one formulated using the Coulomb gauge (and hence also to the one formulated using the unitary gauge). To see this, consider the field transformation \( \phi, A \rightarrow \phi', A^T, A^L \) defined by \( \phi' = \phi \exp(-ie \frac{1}{\sqrt{2}} \nabla \cdot A) \) and the usual decomposition of \( A \) into transverse and longitudinal part. In terms of the new variables, the constraint (94) reads \( \delta \Psi/\delta A^L = 0 \), i.e., the wave functional does not depend on \( A^L \), just on \( \phi \) and \( A^T \). The Schrödinger equation (93) reduces to the one in the Coulomb gauge given in (57). (The latter equivalence is discussed in detail in [38].) The guidance equations (95) also reduce to the ones in the Coulomb gauge. They yield the extra equation \( \dot{A}^L = 0 \), which implies that the field \( A^L \) is static. But \( A^L \) is the gauge degree of freedom that remains after imposing the temporal gauge. So one could gauge-fix it to be zero or just discard it. Therefore, these Bohmian formulations are equivalent. (See [32] for a similar comparison in the case of the free electromagnetic field.)
5.3.3 Usual semi-classical approximation

In the framework of standard quantum theory, there is a natural semi-classical approximation that treats the vector potential classically and the scalar field quantum mechanically. The scalar field is described by a wave functional $\chi(\phi,t)$ which satisfies

$$i\partial_t \chi = \int d^3x \left( -\frac{\delta^2}{\delta\phi^* \delta\phi} + |D\phi|^2 + m^2|\phi|^2 \right) \chi$$

and the electromagnetic field satisfies the classical Maxwell equations

$$\partial_\mu F^{\mu\nu} = \langle \chi | \hat{j}^\nu | \chi \rangle,$$

where

$$\langle \chi | \hat{j}^\nu | \chi \rangle = \int D\phi \Psi^* C \Psi = e \int D\phi \Psi^* \left( \phi^* \frac{\delta \Psi}{\delta\phi^*} - \phi \frac{\delta \Psi}{\delta\phi} \right),$$

$$\langle \chi | \hat{j} | \chi \rangle = ie \int D\phi |\Psi|^2 \left( \phi D^* \phi^* - \phi^* D \phi \right),$$

and where of course the temporal gauge is assumed.

This theory is consistent since $\partial_\mu \langle \chi | \hat{j}^\mu | \chi \rangle = 0$, as a consequence of the Schrödinger equation (101). It is also invariant under the time-independent gauge transformations $A \rightarrow A' = A + \nabla \theta$, $\Psi(\phi) \rightarrow \Psi'(\phi) = \Psi(e^{-ie\theta} \phi)$.

5.3.4 Bohmian semi-classical approximation

A natural guess for a Bohmian semi-classical approximation similar to the usual one is the following. An actual field $\phi$ is introduced that satisfies $\dot{\phi} = \delta S/\delta\phi^*$, where the wave functional satisfies (101), and the Maxwell equations are $\partial_\mu F^{\mu\nu} = j^\nu$, where $j^\mu$ is the classical expression for the charge current. However, the second-order equation for the Bohmian field is

$$\ddot{\phi} - D^2 \phi + m^2 \phi = -\frac{\delta Q}{\delta\phi^*},$$

where $Q = -\frac{1}{|\chi|} \int d^3x \left( \frac{\delta^2 |\chi|}{\delta\phi^* \delta\phi} \right)$. As a consequence, we have that $\partial_\mu j^\mu = -iCQ$ and hence Maxwell’s equations imply that $CQ = 0$ or $Q = Q(|\phi|^2)$. This is a constraint on the wave functional that was absent in the usual semi-classical theory. It also seems to be a rather strong condition. It will for example be satisfied if the scalar field evolves classically (i.e., when the right-hand side of (104) is zero) but it is unclear whether there are other solutions.

We arrive to a similar conclusion from a more careful approach trying to derive the semi-classical approximation from the full Bohmian theory. If we want a semi-classical approximation with $A$ classical, then we should require that $j_Q = 0$ (since we want the wave functional to depend solely on the scalar field and no longer on the vector potential). However, due to the constraint (94) (or again using the equation of motion (100) for the scalar field), this implies that $CQ = 0$. 
So the conclusion seems to be that if we assume A classical, then φ should also behave classically. This is not surprising since the gauge symmetry implies that the physical (i.e., gauge invariant) degrees of freedom are some combination of the fields A and φ. So one can not just assume A classical and keep φ fully quantum.

A possible way to develop a semi-classical approximation is thus to separate gauge degrees of freedom from gauge-independent ones and assume only some of the latter to be classical. One way to do this is as follows. Writing $A = A_T + A_L$, we can assume $A_T$ to behave classically and not $A_L$, since $A_T$ does not change under a gauge transformation, whereas $A_L$ does. We could fully eliminate $A_L$ by introducing the field variable $\phi' = \phi \exp(-ie \frac{1}{\sqrt{2}} \nabla \cdot A)$, which would lead to the full Bohmian approach and the semi-classical approximation of section 5.1.3. However, we can also formulate a semi-classical approximation by keeping the variable $A_L$, but which is still equivalent to the one of section 5.1.3. The functional Schrödinger equation is

$$i \partial_t \Psi = \int d^3x \left( - \frac{\delta^2}{\delta \phi^* \delta \phi} + |D \phi|^2 + m^2 |\phi|^2 - \frac{1}{2} \frac{\delta^2}{\delta A_L^2} \right) \Psi \quad (105)$$

and the constraint

$$\nabla \cdot \frac{\delta \Psi}{\delta A_L} + ie \left( \phi^* \frac{\delta \Psi}{\delta \phi^*} - \phi \frac{\delta \Psi}{\delta \phi} \right) = 0. \quad (106)$$

The guidance equations are

$$\dot{\phi} = \frac{\delta S}{\delta \phi^*}, \quad \dot{A}_L = \frac{\delta S}{\delta A_L}. \quad (107)$$

The equation of motion for $A_T$ is

$$\Box A_T = j_T. \quad (108)$$

Together these equations imply

$$D_{\mu} D^{\mu} \phi - m^2 \phi = -\frac{\delta Q}{\delta \phi^*}, \quad \partial_{\mu} F^{\mu\nu} = j^{\nu} + j_Q^{\nu} \quad (109)$$

in the temporal gauge, where $j^\nu_Q = (0, j_Q)$, with

$$j_Q = -\frac{\delta Q}{\delta A_L} \quad (110)$$

and quantum potential

$$Q = -\frac{1}{|\Psi|} \int d^3x \left( \frac{\delta^2}{\delta \phi^* \delta \phi} + \frac{1}{2} \frac{\delta^2}{\delta A_L^2} \right) |\Psi|. \quad (111)$$

We still have gauge invariance under time independent gauge transformations. To show that this semi-classical approximation is equivalent to the one of section 5.1.3, one just has to apply the field transformation $(\phi, A_L) \rightarrow (\phi', A_L)$ with $\phi' = \phi \exp(-ie \frac{1}{\sqrt{2}} \nabla \cdot A_L)$ (cf. the last paragraph of section 5.1.3).
6 Quantum gravity

6.1 Canonical quantum gravity

In canonical quantum gravity, the state vector is a functional of a spatial metric $h_{ij}(x)$ on a 3-dimensional manifold and the matter degrees of freedom, say a scalar field $\phi(x)$. The wave functional is static and merely satisfies the constraints [2]:

$$\mathcal{H}\Psi(h, \phi) = 0, \quad (112)$$
$$\mathcal{H}_i\Psi(h, \phi) = 0. \quad (113)$$

Their explicit forms are not important here. The latter constraint expresses the fact that the wave functional is invariant under infinitesimal diffeomorphisms of 3-space. The former equation is the Wheeler-DeWitt equation. It is believed that this equation contains the dynamical content of the theory. However, it is as yet not clear how this dynamical content should be extracted. This is the problem of time [2, 41].

In the Bohmian approach, there is an actual 3-metric and a scalar field, whose dynamics depends on the wave function [42–44]. The dynamics expresses how the Bohmian configuration changes along a succession of 3-dimensional space-like surfaces.\(^{10}\) Although the wave function is stationary, the Bohmian configuration will change along these surfaces for generic wave functions. This is how the Bohmian approach solves the problem of time.

The structure of the theory is similar to that of scalar electrodynamics in the temporal gauge, which was discussed in section 5.3. Namely, in both cases there is a constraint on the wave functional which expresses invariance under infinitesimal gauge transformations: spatial diffeomorphisms in the case of gravity and phase transformations in the case of scalar electrodynamics. Therefore we may encounter similar complications in developing a consistent Bohmian semi-classical approximation. Indeed, as we will see below, the Bohmian energy-momentum tensor will not be covariantly conserved and hence can not be used in the Einstein equations. This feature is analogous to what we saw in the case of scalar electrodynamics. In that case, the Bohmian charge current was not conserved so that it could not enter as the source in Maxwell’s equations. The possible solution to the problem is presumably similar to that in the case of electrodynamics, namely the gauge invariance should be eliminated by working with gauge invariant degrees of freedom or by choosing a gauge. However, this is a notoriously hard problem in the case of gravity. It can be solved in the case of simplified models of quantum gravity called mini-superspace models. We will discuss such a model in section 6.4.

First, we will consider the problem in formulating a Bohmian semi-classical approximation in more detail. Instead of starting from the Bohmian formulation for canonical

\(^{10}\)The succession of the surfaces is determined by the lapse function and different choices of lapse function lead to a different Bohmian dynamics. This is analogous to the fact that in Minkowski space-time, the Bohmian dynamics (at least in the usual formulation) depends on the choice of a preferred reference frame or foliation.
quantum gravity, we will start from a quantized scalar field on an classical curved space-time.

6.2 Semi-classical gravity

The formulation of quantum field theory on a classical curved space-time in the functional Schrödinger picture was detailed in [45, 46]. We assume that the space-time manifold $\mathcal{M}$ is globally hyperbolic so that it can be foliated into space-like hypersurfaces. $\mathcal{M}$ is then diffeomorphic to $\mathbb{R} \times \Sigma$, with $\Sigma$ a 3-surface. We choose coordinates $x^\mu = (t, \mathbf{x})$ such that the time coordinate $t$ labels the leaves of the foliation and $\mathbf{x}$ are coordinates on $\Sigma$. In terms of these coordinates the space-time metric and its inverse can be written as

$$g_{\mu\nu} = \left( N^2 - N_i N^i - N_i \right), \quad g^{\mu\nu} = \left( \frac{1}{N^2} \frac{-N^i}{N^2} N^i - h_{ij} \right), \quad (114)$$

where $N$ is the lapse function and $N_i = h_{ij} N^j$ are the shift functions. $h_{ij}$ is the induced Riemannian metric on the leaves of the foliation. The unit vector field normal to the leaves is $n_\mu = (N, 0, 0, 0)$.

For a mass-less scalar field, the Schrödinger equation for this space-time background reads

$$i \frac{\partial \Psi}{\partial t} = \int_\Sigma d^3x \left( \hat{\mathcal{H}} + N_i \hat{\mathcal{H}}_i \right) \Psi, \quad (115)$$

where $\Psi$ is a functional on the space of fields $\phi$ on $\Sigma$ and

$$\hat{\mathcal{H}} = \frac{1}{2} \sqrt{h} \left( - \frac{1}{h} \frac{\delta^2}{\delta \phi^2} + h^{ij} \partial_i \phi \partial_j \phi \right), \quad (116)$$

$$\hat{\mathcal{H}}_i = - \frac{i}{2} \sqrt{h} \left( \partial_i \phi \frac{\delta}{\delta \phi} + \frac{\delta}{\delta \phi} \partial_i \phi \right). \quad (117)$$

This equation describes the action of the classical metric onto the quantum field. The usual way to introduce a back-reaction is by using the expectation value of the energy-momentum tensor operator as the source term in Einstein’s field equations, i.e.,

$$G_{\mu\nu}(g) = \langle \Psi | \hat{T}_{\mu\nu}(\phi, g) | \Psi \rangle. \quad (118)$$

This equation is consistent since $\nabla^\mu \langle \Psi | \hat{T}_{\mu\nu}(\phi, g) | \Psi \rangle = 0$.

In the Bohmian approach, there is an actual scalar field which satisfies the guidance equation

$$\dot{\phi} = \frac{N}{\sqrt{h}} \frac{\delta S}{\delta \phi} + N_i \partial_i \phi \quad (119)$$

or, equivalently,

$$n^\mu \partial_\mu \phi = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta \phi}. \quad (120)$$

This dynamics depends on the foliation (unlike the wave function dynamics). That is, different foliations will lead to different evolutions of the scalar field.
Taking the time derivative of this equation, we obtain
\[ \nabla_\mu \nabla^\mu \phi = -\frac{1}{\sqrt{-g}} \frac{\delta Q}{\delta \phi}, \] (121)
with
\[ Q = -\frac{1}{2|\Psi|} \int d^3x \left[ \frac{N}{\sqrt{h}} \frac{\delta^2|\Psi|^2}{\delta \phi^2} \right] \] (122)
the quantum potential.

To find the energy-momentum tensor for the Bohmian field, we consider the Lagrangian
\[ L = \frac{1}{2} \int d^3x \sqrt{-g} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right) - Q \] (123)
from which the equation of motion (121) can be derived. We can then use the usual definition of the energy-momentum tensor
\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta L}{\delta g^{\mu\nu}} \] (124)
to obtain
\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - \frac{2}{\sqrt{-g}} \frac{\delta Q}{\delta g^{\mu\nu}}. \] (125)
This seems to be the natural expression for the Bohmian energy-momentum tensor, which could be used in the classical Einstein field equations.\(^\text{11}\) However, by taking the covariant derivative, we find
\[ \nabla_\mu T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \left( \frac{\delta Q}{\delta \phi} \partial_\nu \phi + 2\nabla_\nu \frac{\delta Q}{\delta g^{\mu\nu}} \right), \] (126)
which is generically different from zero, so that Einstein’s equations would not be consistent with this energy-momentum tensor as matter source.

### 6.3 Bohmian analogue of the Schrödinger-Newton equation

In the non-relativistic domain, one often describes the coupling between quantum matter and classical Newtonian gravity by the Schrödinger-Newton equation (see e.g. [48]). For one particle, this equation, which can be derived from the equations (115) and (118) [49], is given by
\[ i\partial_t \psi(x,t) = \left[ -\frac{1}{2m} \nabla^2 + m\Phi(x,t) \right] \psi(x,t), \] (127)
where \( \Phi \) is the gravitational potential, which satisfies
\[ \nabla^2 \Phi = 4\pi Gm|\psi|^2. \] (128)
\(^{11}\)In a more precise derivation, we should start from the Bohmian theory of quantum gravity, deduce the modified Einstein field equations (along the lines of [47]), and find the energy-momentum tensor for the matter field.
So the mass distribution that generates the gravitational potential is given by \( m|\psi|^2 \). This leads to the non-linear Schrödinger equation

\[
i \partial_t \psi(x, t) = \left[ -\frac{1}{2m} \nabla^2 - \frac{Gm^2}{|x - y|^3} \right] \psi(x, t).
\] (129)

One could consider a Bohmian analogue of the Schrödinger-Newton equation, where in addition to (127) we have the guidance equation

\[
\dot{X}(t) = v^\psi(X(t), t)
\] (130)

and the Poisson equation

\[
\nabla^2 \Phi(x, t) = 4\pi Gm\delta(x - X(t)).
\] (131)

So in this case the gravitational potential is generated by the Bohmian particle rather than the wave function. This leads to the modified Schrödinger equation

\[
i \partial_t \psi(x, t) = \left[ -\frac{1}{2m} \nabla^2 - \frac{Gm^2}{|x - X(t)|} \right] \psi(x, t).
\] (132)

The generalization to many particles is straightforward. In this case, the gravitational potential generated by the Bohmian particles is

\[
\Phi(x, t) = -G \sum_k m_k \frac{1}{|x - X_k(t)|},
\] (133)

so that the potential in the many-particle Schrödinger equation is given by

\[
V(x) = \sum_k m_k \Phi(x_k, t).
\] (134)

Such a Bohmian version of the Schrödinger-Newton equation does not immediately seem to follow from a more fundamental Bohmian theory. Of course, to possibly obtain such an equation, we should introduce positions rather than a field configuration for the matter, which could be naturally done for the Dirac field. (It seems that Bohmian field ontologies are well-suited for bosonic field theories, while particle ontologies are well-suited for fermionic field theories [10]). However, the reason why we think (132) does not follow from such a theory is that we obtained nothing of this sort for the Bohmian semi-classical Abraham model considered in section 4. In the classical Abraham model, the Poisson equation expresses the electric potential in terms of the charges. This may suggest a Bohmian analogue in which the electric potential is generated by Bohmian point charges. However, this is not what we find in the Bohmian semi-classical approach, cf. (46). This being said, it might still be worth investigating the Bohmian analogue of the Schrödinger-Newton equation, perhaps as a possible alternative to collapse models. (A model similar in spirit was studied in [50].)
6.4 Mini-superspace model

In this section, we consider a symmetry-reduced model of quantum gravity where homogeneity and isotropy are assumed. In this model, the spatial diffeomorphism invariance is eliminated and we can straightforwardly develop a Bohmian semi-classical approximation.

In the classical mini-superspace model, the universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric

$$ds^2 = N(t)^2 dt^2 - a(t)^2 d\Omega_3^2,$$  \hspace{1cm} (135)

where $N$ is the lapse function, $a = e^\alpha$ is the scale factor\(^{12}\) and $d\Omega_3^2$ is the metric on 3-space with constant curvature $k$. Assuming matter that is described by a scalar field $\phi$, the Lagrangian is $[51, 52]$

$$L = Ne^{3\alpha} \left[ -\kappa \left( \frac{\dot{\alpha}^2}{2N^2} + V_G \right) + \frac{\dot{\phi}^2}{2N^2} - V_M \right],$$  \hspace{1cm} (136)

where $\kappa = 3/4\pi G$.

$$V_G = -\frac{1}{2} k e^{-2\alpha} + \frac{1}{6} \Lambda$$ \hspace{1cm} (137)

is the gravitational potential, with $\Lambda$ the cosmological constant, and $V_M$ is the potential for the matter field. The corresponding equations of motion are, after imposing the gauge $N = 1$:\(^{13}\)

$$\frac{1}{2} \ddot{\alpha} = \frac{1}{2\kappa} \left( \frac{1}{2} \dot{\phi}^2 + V_M \right) + V_G,$$  \hspace{1cm} (138)

$$\ddot{\phi} + 3\dot{\alpha} \dot{\phi} + \partial_\phi V_M = 0.$$ \hspace{1cm} (139)

(The second-order equation for $\alpha$ which arises from variation with respect to $\alpha$ is redundant since it can be derived from the other two equations.)

Using the canonical momenta

$$\pi_N = 0, \quad \pi_\alpha = -\kappa e^{3\alpha} \frac{\dot{\alpha}}{N}, \quad \pi_\phi = e^{3\alpha} \frac{\dot{\phi}}{N},$$  \hspace{1cm} (140)

we can pass to the Hamiltonian formulation. This leads to the Hamiltonian constraint (which is just eq. (138))

$$-\frac{1}{2\kappa e^{3\alpha}} \pi_\alpha^2 + \frac{1}{2e^{3\alpha}} \pi_\phi^2 + e^{3\alpha} (\kappa V_G + V_M) = 0.$$ \hspace{1cm} (141)

\(^{12}\)The reason for introducing the variable $\alpha$ is that it is unbounded, unlike the scale factor, which satisfies $a \geq 0$.

\(^{13}\)The theory is time-reparamaterization invariant. Solutions that differ only by a time-reparameterization are considered physically equivalent. Choosing the gauge $N = 1$ corresponds to a particular time-parameterization.
Quantization yields the Wheeler-DeWitt equation:

\[(\hat{H}_G + \hat{H}_M)\psi = 0,\] (142)

where

\[
\hat{H}_G = \frac{1}{2\kappa e^{3\alpha}} \partial^2_\alpha + \kappa e^{3\alpha} V_G, \quad \hat{H}_M = -\frac{1}{2e^{3\alpha}} \partial^2_\phi + e^{3\alpha} V_M. \] (143)

In the corresponding Bohmian approach [52], there is an actual FLRW metric of the form (135) and scalar field, whose time evolutions are determined by the guidance equations

\[
\dot{\alpha} = -\frac{N}{\kappa e^{3\alpha}} \partial_\alpha S, \quad \dot{\phi} = \frac{N}{e^{3\alpha}} \partial_\phi S, \] (144)

where \(N\) is an arbitrary lapse function.\(^{14}\) In the gauge \(N = 1\), these equations imply

\[
\frac{1}{2} \dot{\alpha}^2 = \frac{1}{\kappa} \left( \frac{1}{2} \dot{\phi}^2 + V_M + Q_M^\psi \right) + V_G + Q_G^\psi, \] (145)

\[
\dot{\phi} + 3\dot{\alpha} \dot{\phi} + \partial_\phi (V_M + Q_M^\psi + \kappa Q_G^\psi) = 0, \] (146)

where

\[
Q_M^\psi = \frac{1}{2\kappa^2 e^{6\alpha}} \frac{\partial^2_\alpha |\psi|^2}{|\psi|}, \quad Q_M^\psi = -\frac{1}{2e^{6\alpha}} \frac{\partial^2_\phi |\psi|^2}{|\psi|}. \] (147)

We will now look for a semi-classical approximation where the scale factor behaves approximately classical. In order to do so, we assume again the gauge \(N = 1\) and we consider the conditional wave function \(\chi(\phi, t) = \psi(\phi, \alpha(t))\), given a set of trajectories \((\alpha(t), \phi(t))\). Using

\[
\partial_t \chi(\phi, t) = \partial_\alpha \psi(\phi, \alpha)|_{\alpha = \alpha(t)} \dot{\alpha}(t), \] (148)

we can write

\[
\frac{i}{\dot{\alpha}} \partial_t \chi = \hat{H}_M \chi + I, \] (149)

where\(^{15}\)

\[
I = \frac{i}{\dot{\alpha}} \partial_t \chi \left( \dot{\alpha} + \frac{1}{\kappa e^{3\alpha}} \partial_\alpha S |_{\alpha(t)} \right) + \frac{1}{2\kappa e^{3\alpha}} \left[ (\partial_\alpha S)^2 + i \partial^2_\alpha S \right] |_{\alpha(t)} \chi + \kappa e^{3\alpha} (V_G + Q_G^\psi) |_{\alpha = \alpha(t)} \chi. \] (150)

When \(I\) is negligible (up to a real time-dependent function times \(\chi\)), (149) becomes the Schrödinger equation for a homogeneous matter field in an external FLRW metric. We

\(^{14}\)Just as the classical theory, the Bohmian approach is time-reparameterization invariant. This is a special feature of mini-superspace models [53, 54]. As mentioned before, for the usual formulation of the Bohmian dynamics for the Wheeler-DeWitt theory of quantum gravity, a particular space-like foliation of space-time or, equivalently, a particular choice of “initial” space-like hypersurface and lapse function, needs to be introduced. Different foliations (or lapse functions) yield different Bohmian theories.

\(^{15}\)To obtain this equation, note that \(\partial^2_\alpha \psi|_{\alpha = \alpha(t)} = [(\partial_\alpha S)^2 + i \partial^2_\alpha S + \partial^2_\alpha |\psi|/|\psi| |\psi| + 2i \partial_\alpha S \partial_\alpha \psi \), so that \(\partial^2_\alpha \psi|_{\alpha = \alpha(t)} = [(\partial_\alpha S)^2 + i \partial^2_\alpha S + \partial^2_\alpha |\psi|/|\psi| |\psi|]_{\alpha = \alpha(t)} \chi + 2i \partial_\alpha S \partial_\alpha \chi/\dot{\alpha}. \) Using this equation together with (142) we obtain (149).
can further assume the quantum potential $Q_G^\psi$ to be negligible compared to other terms in eq. (145). As such, we are led to the semi-classical theory:

\begin{align}
\hat{H}_M \chi, \\
\dot{\phi} &= \frac{1}{e^{3\alpha}} \partial_\phi S, \\
\frac{1}{2} \dot{\alpha}^2 &= \frac{1}{\kappa} \left( \frac{1}{2} \dot{\phi}^2 + V_M + Q_M^\chi \right) + V_G \equiv -\frac{1}{\kappa e^{3\alpha}} \partial_t S + V_G. 
\end{align}

Let us now consider when the term $I$ will be negligible. The quantity in brackets in the first term would be zero when evaluated for the actual trajectory $\phi(t)$ (because of the guidance equation for $\alpha$). As such, the first term will be negligible if the actual scale factor evolves approximately independently of the scalar field. The second term will be negligible if $S$ varies slowly with respect to $\alpha$ or if the term in square brackets is approximately independent of $\phi$. In the latter case, the second term becomes a time-dependent function times $\chi$, which can be eliminated by changing the phase of $\chi$. Similarly, if $Q_G^\psi \ll V_G$ then the third term also becomes a time-dependent function times $\chi$.

In the usual semi-classical approximation, one has (151) and

\begin{equation}
\frac{1}{2} \dot{\alpha}^2 = \frac{1}{\kappa e^{3\alpha}} \langle \chi | \hat{H}_M | \chi \rangle + V_G, 
\end{equation}

with $\chi$ normalized to one. These equations follow from (1) and (2). In the next section, we will compare this approximation with the Bohmian one for a particular example. It will appear that the latter gives better results than the usual approximation. (Note that Vink himself, in his seminal paper [52] on applying the Bohmian approach to quantum gravity, considers a derivation of the usual semi-classical approximation, rather than the Bohmian one. But he hinted on the Bohmian semi-classical approximation in [55].)

### 6.5 Example in mini-superspace

In this section, we will work out a simple example to compare the Bohmian and usual semi-classical approximations to the full Bohmian result.

We put $\kappa = 1$ and assume $V_G = V_M = 0$. It will also be useful to introduce the time parameter $\tau$, defined by $d\tau e^{3\alpha} = dt$ (which corresponds to choosing $N = e^{3\alpha}$ instead of $N = 1$). Derivatives with respect to $\tau$ will be denoted by primes. In this way, the classical equations (138) and (139) reduce to

\begin{equation}
\alpha'' = \phi'^2, \quad \phi'' = 0. 
\end{equation}

The possible solutions are

\begin{equation}
\alpha = c_1 \tau + c_2, \quad \phi = \pm c_1 \tau + c_3, 
\end{equation}

where $c_i, \ i = 1, 2, 3,$ are constants. In the case $c_1 = 0$, the scale factor is constant and we have Minkowski space-time. If $c_1 > 0$ the universe starts from a big bang and keeps
expanding forever. If \( c_1 < 0 \) the universe contracts until a big crunch. If \( c_1 \neq 0 \), the corresponding paths in \((\phi, \alpha)\)-space are given by

\[
\alpha = \pm \phi + c, \tag{157}
\]

with \( c \) constant.

### 6.5.1 Full Bohmian analysis

In the full quantum case, we have the Wheeler-DeWitt equation

\[
(\partial_{\alpha}^2 - \partial_{\phi}^2)\psi = 0 \tag{158}
\]

and guidance equations

\[
\alpha' = -\partial_{\alpha}S, \quad \phi' = \partial_{\phi}S. \tag{159}
\]

For the state

\[
\psi_R(\phi, \alpha) = \exp \left[ iu(\phi - \alpha) - \frac{(\phi - \alpha)^2}{4\sigma^2} \right], \tag{160}
\]

the guidance equations read \( \alpha' = \phi' = u \), so that we have only classical solutions

\[
\alpha = u\tau + c_1, \quad \phi = u\tau + c_2 \tag{161}
\]

or

\[
\alpha = \phi + c. \tag{162}
\]

See fig. 1 for some trajectories.

Similarly, for the state

\[
\psi_L(\phi, \alpha) = \exp \left[ -iv(\phi + \alpha) - \frac{(\phi + \alpha)^2}{4\sigma^2} \right] \tag{163}
\]

the solutions are also classical:

\[
\alpha = v\tau + c_1, \quad \phi = -v\tau + c_2, \tag{164}
\]

or

\[
\alpha = -\phi + c, \tag{165}
\]

see fig. 2.

Consider now the superposition \( \psi = \psi_R + \psi_L \) and assume \( v > u \gg 0 \). For \( \alpha \to \pm \infty \) the wave functions \( \psi_R \) and \( \psi_L \) are non-overlapping functions of \( \phi \) so that, asymptotically, the Bohmian dynamics is either determined by \( \psi_R \) or \( \psi_L \). This means that asymptotically, i.e., for \( \alpha \to \pm \infty \), we have classical motion, given either by (162) or (165). Some trajectories are plotted in figs. 3 and 4. (Trajectories for the case \( u = -v \) were plotted in [56].) We see that trajectories starting from the “left”, i.e. \( \phi < 0 \), for \( \alpha \to -\infty \) will end up on the left, while trajectories that start from the “right”, i.e. \( \phi > 0 \), might either end up moving to the left or to the right. (If \( u = v \), then trajectories on starting on
the left stay on the left and trajectories starting on the right stay on the right.) Some trajectories are closed. They correspond to cyclic universes (which oscillate between a minimum and maximum scale factor), see fig. 4.

For trajectories with classical asymptotic behavior, there is possible non-classical behavior in the region of overlap, where \( \alpha \approx 0 \). For trajectories starting on the left there is a transition from \( \alpha = u\phi \) to \( \alpha = -v\phi \) (which is impossible classically). For some trajectories starting on the right, there is the opposite transition.

Note that there is no natural measure on the set of trajectories [57], so we can not make any probabilistic statements like about the probability for a trajectory to move from left to right.

6.5.2 Usual semi-classical approximation

Let us first consider the usual semi-classical approximation and compare it to the full Bohmian approach:

\[
\begin{align*}
\text{i} \partial_\tau \chi &= -\frac{1}{2} \partial_\phi^2 \chi, \\
\alpha' &= 2\langle \chi | \hat{H}_M | \chi \rangle = -\langle \chi | \partial_\phi^2 | \chi \rangle.
\end{align*}
\]

The wave equation corresponds to that of a single particle of unit mass in one dimension. Hence, \( \langle \chi | \partial_\phi^2 | \chi \rangle \) is time-independent and we have that \( \alpha' \) is constant.

The initial wave function which we will use in the semi-classical approximation is given by the conditional wave function \( \chi(\phi, \tau_0) = N\psi(\phi, \alpha(\tau_0)) \) at some time \( \tau_0 \), with \( N \) a normalization constant. The conditional wave functions corresponding to \( \psi_R \) and

Figure 1: Some trajectories for \( \Psi_R, \sigma = 1 \).

Figure 2: Some trajectories for \( \Psi_L, \sigma = 1 \).
\( \psi_L \) are

\[
\chi_R(\phi, \tau_0) = N_R \psi_R(\phi, \alpha_0) = (2\pi \sigma^2)^{-1/4} \exp \left[ iu(\phi - \alpha_0) - \frac{(\phi - \alpha_0)^2}{4\sigma^2} \right],
\]

\[
\chi_L(\phi, \tau_0) = N_L \psi_L(\phi, \alpha_0) = (2\pi \sigma^2)^{-1/4} \exp \left[ -iv(\phi + \alpha_0) - \frac{(\phi + \alpha_0)^2}{4\sigma^2} \right],
\] (168)

where \( \alpha_0 = \alpha(\tau_0) \). \( \chi_R \) is a Gaussian packet centered around \( \alpha_0 \) and average momentum \( u \), while \( \chi_L \) is a Gaussian packet centered around \(-\alpha_0\) and average momentum \( v \). The solutions to the Schrödinger equation (166), with these conditional wave functions as initial conditions, are given by [7]:

\[
\chi_R(\phi, \tau) = [2\pi s^2(\bar{\tau})]^{-1/4} \exp \left[ iu(\phi_R - u\bar{\tau}/2) - (\phi_R - u\bar{\tau})^2/4s(\bar{\tau})\sigma \right],
\]

\[
\chi_L(\phi, \tau) = [2\pi s^2(\bar{\tau})]^{-1/4} \exp \left[ -iv(\phi_L + v\bar{\tau}/2) - (\phi_L + v\bar{\tau})^2/4s(\bar{\tau})\sigma \right],
\] (169)

where

\[
\phi_R = \phi - \alpha_0, \quad \phi_L = \phi + \alpha_0, \quad \bar{\tau} = \tau - \tau_0, \quad s(\bar{\tau}) = \sigma(1 + i\bar{\tau}/2\sigma^2).
\] (170)

In the following, we assume that

- \( \tau_0 \) is small enough, so that \( \alpha_0 < 0 \) and \( |\alpha_0| \gg \sigma \),
- \( \alpha'(\tau_0) > 0 \) ((167) determines \( \alpha' \) only up to a sign),
- \( v > u \gg 0 \),
• \( 1 \ll \sigma u \).

The expectation value \( \langle \chi | \hat{H}_M | \chi \rangle \) is time-independent. So we can calculate it at time \( \tau_0 \). We have that
\[
2 \langle \chi R | \hat{H}_M | \chi R \rangle = u^2 + \frac{1}{4\sigma^2}.
\] (171)

Hence \( \alpha' = \sqrt{u^2 + \frac{1}{4\sigma^2}} \). Since \( 1 \ll \sigma u \), we have that \( \alpha' \approx u \), so that we approximately obtain the classical solution for the scale factor.

Similarly, for \( \chi_L \), we have that
\[
2 \langle \chi L | \hat{H}_M | \chi L \rangle = v^2 + \frac{1}{4\sigma^2}.
\] (172)

and since \( 1 \ll \sigma u < \sigma v \), we have \( \alpha' \approx v \).

Now consider the superposition \( \chi = (\chi_R + \chi_L)/\sqrt{2} \). Since \( \chi_R \) and \( \chi_L \) have approximately negligible overlap initially, because \( |\alpha_0| \gg \sigma \), this state is approximately normalized to one. For the same reason we have that
\[
2 \langle \chi | \hat{H}_M | \chi \rangle \approx \langle \chi R | \hat{H}_M | \chi R \rangle + \langle \chi L | \hat{H}_M | \chi L \rangle = \frac{1}{2}(u^2 + v^2) + \frac{1}{4\sigma^2}.
\] (173)

Hence, for \( 1 \ll \sigma u \), we have that \( \alpha' \approx \sqrt{(u^2 + v^2)/2} \). As such, the semi-classical approximation is very close to the exact result, given that \( u \approx v \). But if \( u \) is very different from \( v \), the semi-classical approximation is not so good. In particular, we do not get the asymptotic behavior that \( \alpha' = u \) or \( \alpha' = v \) for early or late times (i.e., \( \tau \to \tau_0 \) or \( \tau \to \infty \)).

### 6.5.3 Bohmian semi-classical approximation

The equations of motion in the Bohmian semi-classical approximation are
\[
i \partial_\tau \chi = -\frac{1}{2} \partial^2_\phi \chi,
\] (174)
\[
\phi' = \partial_\phi S,
\] (175)
\[
\alpha'^2 = \phi'^2 - \frac{\partial^2_\phi |\chi|}{|\chi|} \equiv -2 \partial_\tau S.
\] (176)

For the state \( \psi_R \), the solutions of the guidance equation are [7]:
\[
\phi - \alpha_0 = u(\tau - \tau_0) + (\phi_0 - \alpha_0) \frac{|s(\tau - \tau_0)|}{\sigma},
\] (177)

where \( \phi_0 = \phi(\tau_0) \). Or using the notation (170):
\[
\phi_R = u\bar{\tau} + \phi_{R,0} \frac{|s(\bar{\tau})|}{\sigma},
\] (178)

where \( \phi_{R,0} = \phi_R(\tau_0) = \phi_0 - \alpha_0 \). With the same assumptions about the constants as in the previous section and taking \( |\phi_{R,0}| \lesssim \sigma \), i.e., that the initial value \( \phi_{R,0} \) does not lie
too far outside the bulk of the Gaussian packet, we have that $|\phi_{R,0}|\tilde{\tau}/2\sigma^2 \ll u\tilde{\tau}$. Using in addition that $|\frac{s(\tau)}{\sigma}| = \sqrt{1 + \tilde{\tau}^2/4\sigma^2} \lesssim 1 + \tilde{\tau}/2\sigma^2$, we find that

$$\phi_R \approx u\tilde{\tau} + \phi_{R,0}\tilde{\tau}/2\sigma^2.$$  

(179)

So we approximately get classical motion and hence the full Bohmian result. 

The classical equation for the scale factor becomes

$$\alpha'^2 = u^2 + \frac{3}{2|s(\tau)|^2} + \frac{\phi_{R,0}u\tilde{\tau}}{2\sigma^3|s(\tau)|} + \frac{\phi_{R,0}^2}{4\sigma^4}\left(\frac{\tilde{\tau}^2}{4\sigma^2|s(\tau)|^2} - 1\right).$$  

(180)

We have that

$$\left|\frac{3}{2|s(\tau)|^2} + \frac{\phi_{R,0}u\tilde{\tau}}{2\sigma^3|s(\tau)|} + \frac{\phi_{R,0}^2}{4\sigma^4}\left(\frac{\tilde{\tau}^2}{4\sigma^2|s(\tau)|^2} - 1\right)\right| \leq \frac{3}{2|s(\tau)|^2} + \frac{u|\phi_{R,0}\tilde{\tau}|}{2\sigma^3|s(\tau)|} + \frac{\phi_{R,0}^2}{4\sigma^4}\left(\frac{\tilde{\tau}^2}{4\sigma^2|s(\tau)|^2} - 1\right)$$

$$\leq \frac{3}{2\sigma^2} + \frac{u|\phi_{R,0}|}{\sigma^2} + \frac{\phi_{R,0}^2}{4\sigma^4},$$  

(181)

where we used $|\tilde{\tau}|/2\sigma|s(\tau)| \lesssim 1$ for the last two terms in order to obtain the last inequality. Using the assumptions that $1 \ll \sigma u$ and $|\phi_{R,0}| \lesssim \sigma$, we find that

$$\alpha'^2 \approx u^2.$$  

(182)

So similarly as in the case of the usual semi-classical approximation (just with the extra condition on the initial value $\phi_{R,0}$), we obtain the full quantum results. For the wave function $\psi_L$, similar results hold. 

Consider now the superposition $\chi = (\chi_R + \chi_L)/\sqrt{2}$. This superposition corresponds to two Gaussian packets that move across each other. Initially and finally they are approximately non-overlapping.\(^\text{16}\) This means that before and after the wave packets cross, the motions of $\phi$ and $\alpha$ are approximately classically, in agreement with the full Bohmian results, since they will be determined by either $\chi_R$ or $\chi_L$. This is unlike the usual semi-classical approximation. 

A trajectory for the scalar field starting from the left (i.e., $\phi < 0$) will end up on the left, while a trajectory starting on the right will end up on the right. The reason is that because of equivariance the probability to start from the left equals the probability to end up on the left (which equals 1/2). Since trajectories do not cross (since the dynamics is given by a first-order differential equation in time), the probability for trajectories to start on the left (right) and end up on the right (left) must be zero. This means that for all trajectories starting on the left, we have a transition from $\alpha = u\phi$ to $\alpha = -v\phi$ and the opposite transition for trajectories starting on the right. This is unlike the full Bohmian analysis where trajectories exist where such a transition does not occur.

\(^{16}\)The latter statement follows from the fact that the spread of the wave function, which equals $|s| = \sigma\sqrt{1 + \tilde{\tau}^2/4\sigma^2}$, goes like $\tilde{\tau}/2\sigma$ for large time, which is much smaller than the distance between the centers of the Gaussians, which equals $(u + v)\tilde{\tau}$.
In conclusion, we see that the Bohmian semi-classical approximation is in better agreement with the exact Bohmian results than the usual semi-classical approximation. We came to this conclusion by making the comparison on the level of the actual trajectories for $\alpha$ and $\phi$. We did not attempt to make a comparison in the context of standard quantum theory, since the standard quantum interpretation of the Wheeler-DeWitt equation is problematic due to the problem of time. But it is clear that for approaches to quantum theory that would associate (approximately) the classical evolutions (162) and (165) to the superposition $\Psi_R + \Psi_L$ (like perhaps the consistent histories or many worlds approach) the usual semi-classical approximation would fare worse than the Bohmian semi-classical approximation.

7 Conclusion

We have shown how semi-classical approximations can be developed using Bohmian mechanics. We have obtained these approximations from the full Bohmian theory by assuming certain degrees of freedom to evolve approximately classically. This was illustrated for non-relativistic systems. If there is a gauge symmetry, like in electrodynamics or gravity, then extra care is required in order to obtain a consistent semi-classical theory. By eliminating the gauge symmetry (either by imposing a gauge or by working with gauge-independent degrees of freedom), we were able to find a semi-classical approximation in the case of scalar quantum electrodynamics. For quantum gravity, eliminating the gauge symmetry (more precisely the spatial diffeomorphism invariance) is notoriously hard. We have only considered the simplified mini-superspace approach to quantum gravity, which describes an isotropic and homogeneous universe, and where the diffeomorphism invariance is explicitly eliminated. More general cases in quantum gravity still need to be studied. For example, for the case of inflation theory, where one usually considers quantum fluctuations on a classical isotropic and homogeneous universe, it should not be too difficult to develop a Bohmian semi-classical approximation.

Apart from possible applications in quantum cosmology, such as inflation theory, it might also be interesting to consider potential applications in quantum electrodynamics or quantum optics. In particular, since the results may be compared to the predictions of full quantum theory, this could give us a handle on where to expect better results for the Bohmian semi-classical approximation compared to the usual one in the case of quantum gravity where the full quantum theory is not known. That is, it might give us better insight in which effects are truly quantum and which effect are merely artifacts of the approximation.

Further developments may include higher order corrections to the semi-classical approximation. One way of doing this might be by following the ideas presented in [58, 59]. As explained there, one might introduce extra wave functions for a subsystem in addition to the conditional wave function. These wave functions interact with each other and the Bohmian configuration. By including more of those wave functions one presumably obtains better approximations to the full quantum result.
Finally, although we regard the Bohmian semi-classical approximation for quantum gravity as an approximation to some deeper quantum theory for gravity, one could also entertain the possibility that it is a fundamental theory on its own. At least, there is presumably as yet no experimental evidence against it.

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