Bridging fermionic and bosonic short range entangled states

Yi-Zhuang You¹, Zhen Bi¹, Alex Rasmussen¹, Meng Cheng² and Cenke Xu¹

1 Department of physics, University of California, Santa Barbara, CA 93106, USA
2 Station Q, Microsoft Research, Santa Barbara, CA 93106, USA

Abstract
In this paper we construct bosonic short range entangled (SRE) states in all spatial dimensions by coupling a $\mathbb{Z}_2$ gauge field to fermionic SRE states with the same symmetries, and driving the $\mathbb{Z}_2$ gauge field to its confined phase. We demonstrate that this approach allows us to construct many examples of bosonic SRE states, and we demonstrate that the previous descriptions of bosonic SRE states such as the semiclassical nonlinear sigma model field theory and the Chern–Simons field theory can all be derived using the fermionic SRE states.

1. Introduction

A short range entangled (SRE) state is the ground state of a quantum many-body system that does not have bulk ground state degeneracy or topological entanglement entropy. However, these states can still have stable nontrivial edge states. Some of the SRE states need certain symmetry to protect the edge states, and these SRE states are also called symmetry protected topological (SPT) states. The most well-known SPT states include the Haldane phase of spin-1 chain [1, 2], quantum spin Hall (QSH) insulator [3, 4], topological insulator [5–7], and topological superconductor such as Helium$^3$-B phase [8, 9]. All the free fermion SPT states have been well understood and classified in [10–12], and recent studies suggest that interaction may not lead to new SRE states, but it can reduce the classification of fermionic SRE states [13–20]. Unlike fermionic systems, bosonic SPT states do need strong interaction. Most bosonic SRE states can be classified by symmetry group cohomology [21, 22], Chern–Simons theory [23] and semiclassical nonlinear sigma model [24].

In this work we demonstrate that there is a close relation between fermionic and bosonic SRE states, more precisely many bosonic SRE states can be constructed from fermionic SRE states with the same symmetry. All fermion systems have at least a $\mathbb{Z}_2$ symmetry $c_i \rightarrow -c_i$, where $c_i$ is a local fermion annihilation operator, thus we can couple all fermion Hamiltonians to a dynamical $\mathbb{Z}_2$ gauge field, and microscopically this $\mathbb{Z}_2$ gauge field commutes with the actual physical symmetry of the fermion system. Once the $\mathbb{Z}_2$ gauge field is in its confined phase, the fermionic degree of freedom no longer exists in the spectrum of the Hamiltonian, and the system becomes a bosonic system. However, in many cases, confinement of a gauge field necessarily breaks certain symmetry of the system, thus we have to be very careful. In both two-dimensional (2D) and three-dimensional (3D), a $\mathbb{Z}_2$ gauge field has a confined phase and a deconfined phase. The deconfined phase is characterized by topological excitations of the $\mathbb{Z}_2$ gauge field. In 2D, the $\mathbb{Z}_2$ gauge field has a ‘vison’ excitation, which corresponds to a $\pi$–flux seen by the matter fields. In 3D, the topological excitation is a ‘vison loop’, which is a closed ring of $\pi$–flux. In 2D/3D, when the visons/vison loops proliferate (condense), the system enters the confined phase, i.e. fermions carrying $\mathbb{Z}_2$ gauge charge cannot propagate freely in the bulk due to the phase fluctuations induced by the vison/vison loop condensation.

However, when the $\mathbb{Z}_2$ gauge field is coupled to a fermionic SRE state, the vison and vison loop often carry nontrivial quantum numbers, or degenerate low-energy spectrum. In these cases, when visons and vison loops condense, the condensate would not be a fully gapped nondegenerate state that does not break any symmetry. Also, sometimes visons in 2D would have a nontrivial statistics, thus it cannot trivially condense. Thus only in certain specific cases can we confine the fermionic SRE states and obtain a fully gapped and symmetric bosonic state. Thus analysis of spectrum and quantum number carried by the vison and vison loop is the key of our study.
Our approach can also be viewed as a slave fermion construction of bosonic SRE states, which has been considered in [25–29]. However, in all of these previous studies the gauge group associated with the slave fermion is bigger than $\mathbb{Z}_2$, which means that when the gauge fluctuation is ignored, at the mean field level the slave fermion has a much larger symmetry than the boson system, and the analysis of gauge confined phase is much more complicated. In our case the gauge group is $\mathbb{Z}_2$, and since any fermion system has this $\mathbb{Z}_2$ symmetry, the fermion SRE states would have the same symmetry as the bosonic states after gauge confinement. Thus in our case the nature of the confined phase can be analyzed reliably, and it only depends on the properties of visons and vison loops.

2. Construction of 3D bosonic SPT phases

Let us take the 3D topological superconductor (TSC) phase with time-reversal symmetry as an example. One example of such TSC is the $^3$He B phase. Here instead of focusing on the real $^3$He system, we are discussing a more general family of TSC phases defined on a lattice that are topologically equivalent to $^3$He-B. One typical Hamiltonian of such TSC defined on the cubic lattice reads

$$H = \sum_k \chi_k \left[ \sum_{i=1}^3 \Gamma_i \sin k_i - \Gamma^0 \left( 3 - m - \sum_{i=1}^3 \cos k_i \right) \right] \chi_k.$$ 

(1)

Here $m$ plays the same role as the chemical potential in real $^3$He system; $m = 0$ is the trivial-TSC transition critical point. The time-reversal symmetry acts as $\chi_k \rightarrow i\Gamma^0 \chi_k$. Close to the trivial-TSC phase transition, in the continuum limit this TSC phase can be described by the following universal real space Hamiltonian:

$$H_0 = \int dx \sum_{a=1}^n \chi_a^T \left( i\Gamma^1 \partial_x + i\Gamma^2 \partial_y + i\Gamma^3 \partial_z + m\Gamma^4 \right) \chi_a,$$

$$\Gamma^1 = \sigma^{10}, \Gamma^2 = \sigma^{22}, \Gamma^3 = \sigma^{33}, \Gamma^4 = \sigma^{23},$$

(2)

where $\sigma^a = \sigma^i \otimes \sigma^j$ denotes the tensor product of Pauli matrices, and $a = 1 \cdots n$ is the flavor index. This is a widely used approximate form for this class of TSC (for example, [30, 31]). For each flavor index $a$, $\chi_a$ is a four-component Majorana fermion. In this Hamiltonian $m > 0$ and $m < 0$ correspond to the TSC phase and the trivial phase respectively. The time-reversal-symmetry acts as $\mathbb{Z}_2^s : \chi \rightarrow i\Gamma^5 \chi$. Our conclusion is that, when we couple $n$-copies of this TSC to the same $\mathbb{Z}_2$ gauge field, the $\mathbb{Z}_2$ gauge field can have a fully gapped nondegenerate confined phase when and only when $n$ is an integer multiple of 8. And when $n = 8$, the confined phase is the 3D bosonic SPT state with time-reversal symmetry first characterized in [32].

First of all, when $n = 1$, the vison loop must be gapless, and the gaplessness is protected by time-reversal symmetry [9]. On a vison line along x direction, there will be a pair of counter-propagating Majorana modes, so the effective 1D Hamiltonian along the vison line reads (see appendix B for derivation):

$$H_{1D,x} = \int dx \chi^T i\sigma^3 \partial_x \chi.$$ 

(3)

In this reduced 1D theory, time-reversal symmetry acts as $\mathbb{Z}_2^s : \chi \rightarrow i\sigma^3 \chi$. The only mass term $\chi^T i\sigma^3 \chi$ in this vison line would break time-reversal symmetry, thus as long as time-reversal is preserved, the vison line is always gapless. This implies that when $n = 1$ the vison line definitely cannot drive the system into a fully gapped state by proliferation without breaking time-reversal.

For $n > 1$, the effective theory along the vison line becomes

$$H_{1D,x} = \int dx \sum_{a=1}^n \chi_a^T i\sigma^3 \partial_x \chi_a.$$ 

(4)

Then for even integer $n$, it appears that there is a time-reversal symmetric mass term $\chi_a^T \sigma^i A_{ab} \chi_b$, where $A$ is an antisymmetric matrix in the flavor space. In the bulk theory equation (2), this mass term can correspond to several terms such as $\chi_a^T \sigma^i \alpha_{ab} \chi_b$ (see appendix B). However, none of these terms can gap out vison lines along all directions. For example, for vison loops along y direction, the modes moving along $+y$ is an eigenstate of $\Gamma^2$ with $\Gamma^2 = +1$, and modes moving along $-y$ direction have eigenvalue $\Gamma^2 = -1$. Because $\sigma^i$ commutes with $\Gamma^2 = \sigma^{10}$, $\chi_a^T \sigma^i \alpha_{ab} \chi_b$ can never back-scatter modes in the y vison line. In fact no flavor mixing time-reversal invariant fermion bilinear terms in the bulk would gap out the vison lines along all directions, while a $\mathbb{Z}_2$ gauge confined phase requires dynamically condensing vison lines in all directions. Therefore the fermion bilinear flavor mixing terms in the bulk do not allow us to condense the vison lines in order to generate a fully gapped symmetric bosonic state.

Since no fermion bilinear term can gap out all the vison loops, we need to consider interaction effects. In [13, 14], the authors studied the interaction effect on equation (4), and the conclusion is that for $n = 8$ there is an
SO(7) invariant interaction term $H_{\text{1D,int}} = \int dx \, V_{\text{abcd}} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta$ that can gap out the 1D theory equation (4) without generating nonzero expectation value of any fermion bilinear operator, where $V_{\text{abcd}}$ is some coefficient tensor specified in [13, 14]. The same field theory analysis applies here: the effective interaction $H_{\text{1D,int}}$ can gap out the 1D theory equation (4) along the vison loop without degeneracy. $H_{\text{1D,int}}$ corresponds\(^3\) to the following term in the bulk:

$$H_{\text{int}} = \int d^d x \, V_{\text{abcd}} \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\delta.$$  

Since this term is rotationally invariant, it will gap out vison lines along all directions. Thus with $n = 8$, and with the interaction term $H_{\text{int}}$ in the bulk, all vison loops can be gapped out without breaking time-reversal symmetry, thus we can safely condense the vison loops and drive the system into a fully gapped, time-reversal invariant bosonic state. But this is only possible when $n$ is an integral multiple of 8. In the following paragraphs we will argue that when $n = 8$ the confined bosonic state is a bosonic SPT state.

References [24, 32] pointed out that this 3D bosonic SPT state can be described by a $O(5)$ NLSM field theory with a topological $\Theta$-term. Let us couple the eight copies of $^3$He B to a five-component unit vector $\mathbf{n}$:

$$H = H_0 + \int d^d x \sum_{n=1}^5 n_i \gamma^i \rho_{\text{ab}} \rho_{\text{cd}}$$

where $\gamma^i$ are five $8 \times 8$ symmetric matrices in the flavor space that satisfy $[\gamma^i, \gamma^j] = 2\delta_{ij}$ (e.g. a particular choice could be $\gamma^i = \sigma_1^{100}, \sigma_1^{310}, \sigma_1^{331}, \sigma_1^{312}$). Under time-reversal transformation, $\mathbf{n} \rightarrow -\mathbf{n}$. Following the calculation in [33], we can show that for the $^3$He B phase with $m > 0$, after integrating out the fermions, the effective field theory for the vector $\mathbf{n}$ contains a topological $\Theta$-term at $\Theta = 2\pi$:

$$S = \int d^d x d\tau \left( \frac{1}{g} \left( \partial_\mu n^\mu \right)^2 + \frac{i\Theta}{\Omega_4} \epsilon_{\text{abcd}} n^a \partial_\mu n^b \partial_\nu n^c \partial_\sigma n^d \partial_\tau n^e, \right)$$

where $\Omega_4$ is the volume of a four-dimensional sphere with unit radius. Equation (7) is precisely the field theory introduced in [24, 32] to describe the 3D bosonic topological SC with time-reversal symmetry.

Using the field theory equation (7), we can demonstrate that the 2D boundary of this 3D bosonic SPT state could be a 2D $\mathcal{Z}_2$ topological order, whose mutually semionic excitations $e$ and $m$ are both Kramers’ doublet [32] (the so called $\epsilon T m T$ state). [19, 20, 34] argued that the boundary of eight copies of $^3$He B is the (fermionized) $\epsilon T m T$ state. For the sake of completeness, we will repeat this argument. Based on the field theory equation (7), the $e$ and $m$ excitations at the 2D boundary of the 3D bosonic SPT phase correspond to the vortex of boson field $b_1 \sim n_1 + i n_2$, and vortex of $b_2 \sim n_3 + i n_4$ respectively, which can be considered as surface terminations of bulk vortex lines. By solving the Bogoliubov–de Gennes equation with a vortex at the boundary, we can demonstrate that there are four Majorana fermion zero modes located at each vortex core. These four Majorana fermion zero modes can in total generate four different states. Under interaction, time-reversal symmetry\(^4\) guarantees that these four states split into two degenerate doublets with opposite fermion number parity. Thus in the bulk each vortex line is effectively four copies of 1D Kitaev’s Majorana chain. Since we are in a $\mathcal{Z}_2$ gauge confined phase, we are only allowed to consider states with even number of fermions, thus after gauge projection, only one of the two doublets survives, which according to the appendix and [20] is a Kramers doublet. Also the vortex of $b_1$ carries charge $\pm 1/2$ of $b_2$, and vortex of $b_2$ carries $\pm 1/2$ charge of $b_1$, thus these two vortices are both Kramers doublet, and they have mutual semion statistics. This means that boundary of the confined phase is really the $\epsilon T m T$ state.

Combining all the results together, we conclude that the $\mathcal{Z}_2$ confined phase of eight copies of $^3$He B is really the bosonic SPT phase with time-reversal symmetry. Furthermore, since this bosonic SPT state has $\mathcal{Z}_2$ classification, it implies that two copies of the bosonic state is trivial (which can be shown in our NLSM field theory by directly coupling two copies of equation (7) together [24]), which then implies that 16 multiples of the $^3$He-B TSC is trivial under interaction. This conclusions is consistent with the well-known $\mathcal{Z}_{16}$ classification of DIII class fermionic SPT states [19, 20, 35].

We can also give the eight copies of $^3$He B phase various flavor symmetries, and we can construct many 3D bosonic SPT phases with symmetry that contains $\mathcal{Z}_2$ as a normal subgroup by confining the bulk $\mathcal{Z}_2$ gauge field.

\(^3\) The fact that $\sigma^i$ on the vison line is extended to $\Gamma^i$ in the bulk is explained in appendix B.

\(^4\) The $\mathcal{Z}_2$ topological order at the 2D boundary has nothing to do with the bulk $\mathcal{Z}_2$ gauge field that we will confine by proliferating the vison loops.

\(^5\) Assuming tentatively an enlarged $U(1) \times U(1)$ symmetry for rotation of $(n_1, n_2)$ and $(n_3, n_4)$, respectively.

\(^6\) Here the time-reversal symmetry is a modified time-reversal symmetry defined in [20], which is a product of ordinary time-reversal and a $\pi$-rotation of boson field $b_1$ or $b_2$. 

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3
Since all the free fermion SPT states in 3D require the time-reversal symmetry, thus so far our approach does not allow us to construct 3D bosonic SPT phases without $\mathbb{Z}_2^T$.

3. Construction of 2D bosonic SPT phases

Now let us look at 2D examples. In 2D the simplest fermionic SRE state is the $p + ip$ TSC that does not require any symmetry, and the simplest bosonic SRE state is the so called ‘$E_8$’ state with chiral central charge $c_\ast = 8$ at its boundary [36, 37]. In the following we will prove that if we couple $n$ copies of $p + ip$ TSC to a $\mathbb{Z}_2$ gauge field, the $\mathbb{Z}_2$ gauge field can confine to a gapped bosonic state when and only when $n$ is a integral multiple of 16. And when $n = 16$, the confined phase is precisely the bosonic $E_8$ SRE state [38]. First of all, when $n = 1$, the vison of the $\mathbb{Z}_2$ gauge field carries a Majorana fermion zero mode, which grants the vison a nonabelian statistics, thus when $n = 1$ (and generally for odd integer $n$) the $\mathbb{Z}_2$ gauge field cannot enter its confined phase by condensing the vison.

When $n$ is even, $n$-copies of $p + ip$ TSC is equivalent to an integer quantum Hall (IQH) state with Hall conductivity $\nu = n/2$, thus a vison (half flux quantum) would carry charge $n/4$, and has statistics angle $\pi n/8$ under exchange. Thus the smallest $n$ that makes vison a boson is 16, and when $n = 16$, the $\mathbb{Z}_2$ gauge field can enter a confined phase by condensing the bosonic vison.

The vison condensation can be formulated by the Chern–Simons theory [39]. Let us start from the Chern–Simons description for $n$-copies of $p + ip$ TSC with even $n = 2\nu$ (i.e. $\nu$ layers of IQH), and couple the fermion currents $d a^I$ ($I = 1, \cdots, \nu$) to the $\mathbb{Z}_2$ gauge field. The Lagrangian density can be written as

$$\mathcal{L} = \sum_I \frac{1}{4\pi} d a^I \wedge d a^I + \sum_I \frac{1}{2} A^I \wedge d a^I + \frac{1}{4\pi} A \wedge d A. \quad (8)$$

Here the $\mathbb{Z}_2$ gauge theory is described by the mutual Chern–Simon theory of two gauge fields $A$ and $\tilde{A}$ with the $K$-matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

which is a standard representation of $\mathbb{Z}_2$ gauge theory, for instance see [40, 41]. $\tilde{A}$ can be considered as a Higgs field that Higgs the $U(1)$ gauge structure of $A$ down to $\mathbb{Z}_2$. The field $A$ couples to the fermion current $j^I = \star d a^I$ with equal charge, and the field $\tilde{A}$ couples to the vison current in the $\mathbb{Z}_2$ gauge theory. The field $A$ can be treated as a Lagrangian multiplier and integrated out first, which leads to the constraint $\sum_I n a^I + 2\tilde{A} = 0$. This constraint can be solved by the following reparameterization

$$a^I = \tilde{a}^I, \quad \tilde{a}^{I-1} = \tilde{a}^I + \tilde{a}^{I-1} - \tilde{a}^{-I}, \quad a^I = \tilde{a}^I - \tilde{a}^{I-1},$$

$$a^I = \tilde{a}^I - \tilde{a}^{I-1} \quad \text{for } I = 2, \cdots, \nu, \quad \tilde{A} = -\tilde{a}^1. \quad (9)$$

Substituting equation (9) into equation (8), we arrive at a bosonic theory in terms of the new set of gauge fields $\tilde{a}^I$, as $\mathcal{L} = \sum_i \frac{1}{4\pi} K^\text{SO}(n)_{ij} \tilde{a}^i \wedge d \tilde{a}^j$, where $K^\text{SO}(n)$ is the Cartan matrix of the $\text{SO}(n)$ Lie algebra (for even $n > 2$).

For $n = 16$, the K-matrix reads

$$K^\text{SO}(16) = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad (10)$$

which gives the $\text{SO}_1(16)$ Chern–Simons theory. We now extend $K^\text{SO}(16)$ by a block of trivial boson, given by the $K$-matrix $\sigma^I$, and define $K^{\text{ext}} = K^\text{SO}(16) \oplus \sigma^I$. One finds a transform $W$, with $\det W = 1$, given by

$$W^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 4 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 5 & -5 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 4 & -4 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}. \quad (11)$$
such that

\[
W^T K^{\text{ext}} W = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}.
\]

(12)

The last 2 × 2 block describes a \(Z_2\) topological order. The fermion excitations of this \(K\)-matrix corresponds to the original fermion in the \(p + ip\) TSC. The vison couples to the last gauge field, i.e. it corresponds to the charge vector (0, 0, 0, 0, 0, 0, 0, 1), and is a boson ready to condense. Thus after the vison condensation, the \(Z_2\) topological order is destroyed and the original fermion is confined. The \(K\)-matrix is left with the upper 8 × 8 block, which is exactly the Cartan matrix of the \(E_8\) Lie algebra. Since all the charge vectors of the upper 8 × 8 block are self-bosons, and they are bosons relative to the vison, these charge vectors are unaffected by the vison condensate. Thus we have shown by explicit calculation that confining the fermions in 16-copies of \(p + ip\) TSC leads to the \(E_8\) bosonic SRE state.

Now let us investigate the \(p ± ip\) TSC with a \(Z_2\) symmetry discussed in [17]. In this system the fermions with zero \(Z_2\) charge form a \(p + ip\) TSC, while fermions carrying \(Z_2\) charge form a \(p − ip\) TSC. This \(Z_2\) global symmetry is different from the \(Z_2\) gauge symmetry, since all the fermions in our system carry \(p\) gauge charge. For one copy of the \(p ± ip\) TSC coupled to the \(Z_2\) gauge field, the vison carries two independent Majorana fermion zero modes \(\chi\) and \(\chi^\dagger\), and the global \(Z_2\) symmetry acts \(Z_2 : \chi \rightarrow e^{i\pi} \chi\). There is no nontrivial Hamiltonian for these two Majorana fermion modes that preserves the \(Z_2\) symmetry, thus the spectrum of the vison is always two fold degenerate, and hence condensing the vison will not lead to a nondegenerate state.

Two copies of the \(p ± ip\) TSC is formally equivalent to a QSH insulator: fermions that carry global \(Z_2\) charge 0 and 1 form \(p ± ip\) TSC states respectively. Then after coupling to the \(Z_2\) gauge field, the vison would carry two complex localized fermion modes \(c_1\) and \(c_2\), and a vison would carry charge \(±1/2\) of the \(Z_2\) global symmetry, which corresponds to \(n_2 = n_2^2 = 1, 0\) respectively. (The \(Z_2\) symmetry in our system is just the \(Z_2\) subgroup of the \(U(1)\) symmetry of the \(\nu = −1\) IQH state, and it is known that a vison, or a \(p − ip\) flux in a \(\nu = −1\) IQH state carries \(±1/2\) charge, as was shown in [43]). Thus the condensate of the vison always spontaneously breaks the \(Z_2\) symmetry. This situation is very similar to the case discussed in [43]. The universality class of the confinement transition is the so-called 3D XY transition, namely at the quantum critical point the \(Z_2\) symmetry order parameter has an anomalous dimension \(\eta \sim 1.49\) [44, 45].

Eventually for four copies of this \(p ± ip\) TSC, a vison carries four complex fermion modes \(c_1, c_2, c_1^\dagger, c_2^\dagger\). The vison now can be a boson that does not carry any \(Z_2\) global charge, for example the state with \(n_1{\pi} = 1\) and \(n_2{\pi} = 0\) is a \(Z_2\) charge neutral boson. Thus condensing this vison would lead to a fully gapped nondegenerate bosonic state that preserves the global \(Z_2\) symmetry.

Now let us couple four copies of the \(p ± ip\) TSC to a four-component unit vector \(n\):

\[
H = \int d^3x \chi^T (i\sigma^{1000} \partial_x + i\sigma^{2000} \partial_y + m\sigma^{3000})\chi \\
+ \sum_{j=1}^4 n^j \chi^j \chi^j, \tag{13}
\]

with \(\chi^1 = \sigma^{2100}, \chi^2 = \sigma^{2211}, \chi^3 = \sigma^{2223}, \chi^4 = \sigma^{2202}\). The global \(Z_2\) symmetry acts as \(Z_2 : \chi \rightarrow \sigma^{3000}\chi\), and \(n \rightarrow n\). After integrating out the fermions, the resulting theory is a \((2 + 1)d\) O(4) NLSM with a topological \(\Theta\)-term at \(\Theta = 2\pi\):

\[
S = \int d^3x dt \frac{1}{\Omega_3} (\partial_t n)^2 + \frac{i\Theta}{\Omega_3} \epsilon_{abcd} n^a \partial_x n^b \partial_y n^c n^d, \tag{14}
\]

where \(\Omega_3 = 2\pi^2\) is the volume of a 3D sphere with unit radius, and this is precisely the field theory describing the 2D bosonic SPT phase with \(Z_2\) symmetry, which was first studied in [46]. This field theory was studied in [24, 47].

Finally we condense the vison in this system to confine the fermions. Similar to our previous \(K\)-matrix calculation, we couple the four copies of \(p ± ip\) TSC to the \(Z_2\) gauge field, as described by the Lagrangian density
\[ L = \sum_{i,j} K_{ij}^{\text{QSH}} a_i^\dagger a_j + \sum_j \frac{1}{\pi} a_i^\dagger A a_i + \frac{1}{\pi} A^\dagger A, \]

where the matrix \( K_{ij}^{\text{QSH}} \) is diagonal with the diagonal elements \((1, 1, -1, -1)\). In the theory, the global \( \mathbb{Z}_2 \) symmetry charge is given by the charge vector \( q_{\mathbb{Z}_2} = (0, 0, 1, 1) \). Integrating out \( A \) leads to the constraint \( \sum_j a_j^\dagger + 2A = 0 \), which can be solved by

\[
\begin{bmatrix}
a_1^\dagger \\
a_2^\dagger \\
a_3^\dagger \\
a_4^\dagger \\
A
\end{bmatrix} =
\begin{bmatrix}
1 & 1 & -1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & -1 & 1 \\
0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix}.
\]

Substituting equation (16) into equation (15) yields a Chern–Simons theory \( L = \sum_{i,j} \frac{1}{4\pi} K_{ij}^{\text{SPT}^*} a_i^\dagger a_j^\dagger \) with

\[
K_{ij}^{\text{SPT}^*} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 2 & 0
\end{bmatrix}.
\]

Correspondingly, the global \( \mathbb{Z}_2 \) charge is transformed to \( \tilde{q}_{\mathbb{Z}_2} = W^T q_{\mathbb{Z}_2} = (1, -1, 0, 0) \), with the transformation matrix \( W \) taken from the first 4 rows of the matrix in equation (16). In \( K^{\text{SPT}^*} \), the lower 2 × 2 block describes the \( \mathbb{Z}_2 \) topological order, which contains the bosonic vison with neutral global \( \mathbb{Z}_2 \) charge (as seen from \( q_{\mathbb{Z}_2} \)). As the vison condenses, the \( \mathbb{Z}_2 \) topological order is removed, leaving the upper 2 × 2 block, i.e. the \( \sigma^i \) matrix, as the \( K \)-matrix describing a SRE bosonic state, with the global \( \mathbb{Z}_2 \) charge \( q = (1, -1) \) (as taken from \( \tilde{q}_{\mathbb{Z}_2} \)). Such a \( K \)-matrix equipped with the \( \mathbb{Z}_2 \) symmetry matches [23] the Chern–Simons description of the \( \mathbb{Z}_2 \) SPT state.

Therefore after confining the fermions in four copies of \( p \pm ip \) TSC, we obtain the bosonic SPT state with \( \mathbb{Z}_2 \) global symmetry. This bosonic SPT state has \( \mathbb{Z}_2 \) classification [21, 24, 46], which implies that eight copies of the \( p \pm ip \) TSC with \( \mathbb{Z}_2 \) symmetry is a trivial state, which is consistent with the well-known \( \mathbb{Z}_2 \) classification of such \( p \pm ip \) TSC under interaction [15–18, 35].

Extra symmetries can be added to the four copies of \( p \pm ip \) TSC discussed above, and other 2D bosonic TSC can be constructed in the same way. Construction of 1D bosonic SPT phases is much more obvious, which will be discussed in the appendix.

4. Summary

In this paper we demonstrate that many bosonic SRE phases can be constructed by fermionic SRE phases with the same symmetry. The fermionic SRE states and the \( \mathbb{Z}_2 \) gauge field can all be defined on a lattice, thus our method has provided a projective construction of the lattice wave function of these bosonic SRE states. Also, our method provides a full lattice regularization of the CS field theory [23] and semiclassical NLSM field theory [24] description of bosonic SPT phases. However, some bosonic SPT phases cannot be constructed using the method discussed in the current paper, for example, there is one bosonic SPT phase with \( U(1) \rtimes \mathbb{Z}_2 \) symmetry in 3D, while there is no free fermion SPT phase with the same symmetry. We will leave the construction of these bosonic SPT phases to future study.

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Appendix A. Construction of 1D bosonic SPT

In this appendix, we construct the 1D Haldane phase using four copies of Kitaev’s chains with the time-reversal symmetry \( \mathbb{Z}_2^4 \). Let us start from the fermionic SPT phase composed of four copies of Kitaev’s chains coupled to a fluctuating three-component unit vector \( n \):

\[ H = \chi^\dagger (i \sigma^{100} \partial_\xi + \sigma^{200}) \chi + \sum_{j=1}^{3} n^\dagger \gamma j^j \chi, \]

with \( \gamma^1 = \sigma^{332}, \gamma^2 = \sigma^{320}, \gamma^3 = \sigma^{312} \). The time reversal symmetry acts as \( \mathbb{Z}_2^4 : \chi \to \sigma^{500} \chi \) and \( n \to -n \) followed by the complex conjugation (denoted \( \mathcal{K} \)). Note that the time reversal operator \( T = \mathcal{K} \sigma^{500} \) behaves as \( T^2 = 1 \) on the Majorana fermions \( \chi \). After integrating out the fermions, the resulting theory is a \((1 + 1)d\) O(3) NLSM with a topological \( \theta \)-term at \( \theta = 2\pi \)
\[ S = \int \! dx \! dz \frac{1}{g} \left( \frac{\partial_{\mu} \mathbf{n}}{g} \right)^2 + \frac{i \Theta}{\Omega_2} e_{abc} n^a \partial_{\mu} n^b \partial_{\nu} n^c, \]  

where \( \Omega_2 = 4\pi \) is the volume of a 2D sphere with unit radius, and this is precisely the field theory describing the 1D bosonic SPT phase with \( \mathbb{Z}_2 \) symmetry, i.e. the Haldane phase of 1D spin chain [1, 2].

Then we can couple the fermions to a \( \mathbb{Z}_2 \) gauge field, namely we impose the following gauge constraint on every site: \( \chi_{0} \chi_1 \chi_2 \chi_3 = 1 \). The same gauge constraint is imposed on the edge Majorana fermion zero modes. The edge Majorana fermion zero modes may be arranged in a matrix as [48]

\[ F = \frac{1}{2} \left( \chi_0 \sigma^0 + i \chi_1 \sigma^1 + i \chi_2 \sigma^2 + i \chi_3 \sigma^3 \right). \]

Under time-reversal transformation, \( \mathbb{Z}_2^n \): \( F \to F^* = (i \sigma^2) F (-i \sigma^2) \).

Two three-component vector operators can be conveniently constructed with these edge Majorana operators \( (a = 1, 2, 3) \):

\[ S^a = \frac{1}{2} \text{Tr} F^a F^a, \quad K^a = \frac{1}{2} \text{Tr} F \sigma^a F^\dagger. \]

In fact, the boundary Majorana fermions have an emergent SO(4) symmetry, and the two vectors correspond to the two independent SU(2) subgroups of the SO(4). The full SO(4) rotational symmetry among the four flavors of Majorana fermions is decomposed to \( \text{SU}(2)_{\text{spin}} \times \text{SU}(2)_{\text{gauge}} \) generated by \( S \) and \( K \) respectively. For the fermions in \( F \), the \( \text{SU}(2)_{\text{spin}} \) rotation corresponds to a left rotation \( F \to U F \) with \( U \in \text{SU}(2)_{\text{spin}} \), while the \( \text{SU}(2)_{\text{gauge}} \) rotation corresponds to a right rotation \( F \to FG \) with \( G \in \text{SU}(2)_{\text{gauge}} \).

Under the constraint \( \chi_{0} \chi_1 \chi_2 \chi_3 = 1 \), which is equivalent to the requirement of gauge neutrality, i.e. \( K = 0 \). Therefore under the gauge constraint, the physical state of the boundary is only two fold degenerate, and these states are invariant under \( \text{SU}(2)_{\text{gauge}} \). This means that we are free to combine time-reversal symmetry with a \( \text{SU}(2)_{\text{gauge}} \) transformation. For example, we can define a new time-reversal transformation \( T: F \to F^* (i \sigma^2) = -i \sigma^2 F \), this new time-reversal transformation satisfies \( T^2 = -1 \), and it is exactly the same time-reversal transformation for spin-1/2 object. Thus we conclude that under gauge constraint, four copies of Kitaev’s chain is equivalent to the Haldane’s phase.

### Appendix B. Vison loops in \( ^3 \)He B TSC

In this appendix, we derive the effective theory along the vison loop in the \( ^3 \)He B TSC. Let us start with equation (2), and first consider a straight vison line along the \( x \)-direction. The vison line can be considered as a thin hollow cylinder through the bulk of the TSC with a \( \mathbb{Z}_2 \) flux (\( \pi \)-flux) threading through the hole of the tube. For this configuration, it could be convenient to use the cylindrical coordinate defined as \( (x, y, z) = (x, \rho \cos \theta, \rho \sin \theta) \). Applying the coordinate transform to equation (2), the Schrödinger equation reads

\[ \left( i \Gamma^x \partial_x + \frac{m^2}{2} \right) |x, \rho \rangle = E |x, \rho \rangle, \]  

where \( m \Omega_0 = \Gamma^z / 2 \) is the spin connection that corresponds to threading the \( \pi \)-flux (as \( e^{i \Omega_0 \omega \phi} = -1 \)). The low-energy fermion modes around the vison line are given by the following ansatz in the asymptotic limit

\[ \chi_0 (x, \rho, \theta) \approx e^{-im\rho} e^{-i \Gamma^y \theta} \chi_0 (x). \]

Substitute equation (23) to (22), one can see \( \chi_0 (x) \) must satisfy \( i \Gamma^y \chi_0 (x) = \chi_0 (x) \) in order to obtain the low-energy modes (whose energy \( E \to 0 \) as the \( x \)-direction momentum \( i \partial_x \to 0 \)). The matrix \( i \Gamma^y \Gamma^z = \sigma^1 \) has two eigenvectors of the +1 eigenvalue:

\[ \chi_1 = \frac{1}{\sqrt{2}} |1, 0, 0, 0\rangle^T, \quad \chi_2 = \frac{1}{\sqrt{2}} |0, 0, 1, -1\rangle^T, \]  

corresponding to the two counter-propagating Majorana modes along the vison line. It is straightforward to see that the 4 \( \times \) 4 matrix \( \Gamma^y = \sigma^0 \) represented on the basis \( (\chi_1, \chi_2) \) becomes the 2 \( \times \) 2 matrix \( \sigma^3 \), so the effective 1D Hamiltonian should be \( H_{1D, \chi} = \int d\chi \chi^T (x) i \sigma^3 \partial_x \chi(x) \) as shown in equation (3).

In general, any operator \( \hat{O} \) (as a 4 \( \times \) 4 matrix) defined in the 3D bulk can be thus projected to the subspace of the fermion modes along the vison line, as the corresponding 2 \( \times \) 2 matrix \( \hat{O} \) by \( (a, b = 1, 2) \).
Table 1. Projection of bulk operators to the vison line (x-direction).

| \( \mathcal{O} \) | \( \rightarrow \mathcal{O} \) |
|------------------|------------------|
| \( \sigma^{00} \) | \( \int \frac{d\theta}{2\pi} \frac{d\phi}{2\pi} \chi_{a}^{T}(x, \rho, \theta) \mathcal{O}_{i,j}^{a} (x, \rho, \theta) \) |
| \( \sigma^{21} \) | \( \int \frac{d\theta}{2\pi} \frac{d\phi}{2\pi} e^{iF^{a} b^{a} \frac{\pi}{2}} e^{-iF^{a} b^{a} \frac{\pi}{2}} \mathcal{O}_{i,j}^{a} (x, \rho, \theta) \) |

in Table 1, we conclude the projection of all 4 \( \times 4 \) Hermitian matrices (16 complete basis) to the 2D subspace of counter propagating Majorana modes along the vison line. This establishes the correspondence between the operators in the bulk and that on the vison line. One can see \( \Gamma^{2} \) in the bulk would correspond to \( \sigma^{2} \) on the vison line. So the action of the time-reversal symmetry \( T \) is reduced to \( \chi \rightarrow i\sigma^{z} \chi \) on the vison line.

Given the effective Hamiltonian equation (3) and the above \( \mathbb{Z}_{2} \) symmetry on the vison line, it seems that if we make even copies of the system, the vison line can be gapped out by a bilinear mass term of the form \( \chi^{T} \sigma^{1} \otimes \chi \) (with \( A = -A^{T} \)) which does not breaks the time-reversal symmetry. However, this is only true for our analysis of the straight vison line along the x-direction. Because according to 1, the mass term \( \chi^{T} \sigma^{1} \otimes \chi \) would extend to the bulk as \( \chi^{T} \sigma^{13} \otimes \chi \), which can not gap out the vison lines along any other directions, as \( \sigma^{13} \) commutes with both \( \Gamma^{2} = \sigma^{10} \) and \( \Gamma^{3} = \sigma^{22} \). Therefore it is impossible to fully gap out the vison loop by any fermion bilinear term.

References

[1] Haldane F D M 1983 Phys. Lett. A 93 464
[2] Haldane F D M 1983 Phys. Rev. Lett. 50 1153
[3] Kane C L and Mele E J 2005 Phys. Rev. Lett. 95 226801
[4] Kane C L and Mele E J 2005 Phys. Rev. Lett. 95 146802
[5] Fu L, Kane C L and Mele E J 2008 Phys. Rev. Lett. 98 106801
[6] Moore J E and Balents L 2007 Phys. Rev. B 75 121306(R)
[7] Roy R 2009 Phys. Rev. B 79 195322
[8] Roy R 2008 arXiv:0803.2868
[9] Qi X-L, Hughes T L, Raghu S and Zhang S-C 2009 Phys. Rev. Lett. 102 187001
[10] Schnyder A P, Ryu S, Furusaki A and Ludwig A W W 2009 AIP Conf. Proc. 1134 10
[11] Ryu S, Schnyder A, Furusaki A and Ludwig A 2010 New J. Phys. 12 065010
[12] Kitaev A 2009 AIP Conf. Proc. 1134 22
[13] Fidkowski L and Kitaev A 2010 Phys. Rev. B 81 134509
[14] Fidkowski L and Kitaev A 2011 Phys. Rev. B 83 075103
[15] Qi X-L 2013 New J. Phys. 15 065002
[16] Ryu S and Zhang S-C 2012 Phys. Rev. B 85 245132
[17] Gu Z-C and Levin M 2014 Phys. Rev. B 89 201113(R)
[18] Yao H and Ryu S 2013 Phys. Rev. B 88 064507
[19] Fidkowski L, Chen X and Vishwanath A 2013 Phys. Rev. X 3 041016
[20] Wang C and Senthil T 2014 Phys. Rev. B 89 195124
[21] Chen X, Gu Z-C, Liu Z-X and Wen X-G 2013 Phys. Rev. B 87 155114
[22] Chen X, Gu Z-C, Liu Z-X and Wen X-G 2012 Science 338 1604
[23] Lu Y-M and Vishwanath A 2012 Phys. Rev. B 86 125119
[24] Bi Z, Rasmussen A, Slagle K and Xu C 2013 Phys. Rev. B 91 134404
[25] Ye P and Wen X-G 2013 Phys. Rev. B 87 195128
[26] Ye P and Wen X-G 2014 Phys. Rev. B 89 045127
[27] Lu Y-M and Lee D-H 2014 Phys. Rev. B 89 195143
[28] Grover T and Vishwanath A 2013 Phys. Rev. B 87 045129
[29] Oon J, Cho G Y and Xu C 2013 Phys. Rev. B 88 014425
[30] Wang Z, Qi X-L and Zhang S-C 2011 Phys. Rev. B 84 195136
[31] Ryu S, Moore J E and Ludwig A W W 2012 Phys. Rev. B 85 045104
[32] Vishwanath A and Senthil T 2013 Phys. Rev. X 3 011016
[33] Abanov A G and Wiegmann P B 2000 Nucl. Phys. B 570 685
[34] Metlitski M A, Fidkowski L, Chen X and Vishwanath A 2013 arXiv:1406.3032
[35] You Y-Z and Xu C 2014 Phys. Rev. B 90 115112
[36] Kitaev A Y 2006 Ann. Phys. 321 2
[37] Kitaev A (http://online.kitp.ucsb.edu/online/topomat11/kitaev)
[38] Plamadeala E, Mulligan M and Nayak C 2013 Phys. Rev. B 88 045131
[39] Cheng M and Gu Z-C 2014 Phys. Rev. Lett. 112 141602
[40] Kou S-P, Levin M and Wen X-G 2008 Phys. Rev. B 78 155134
[41] Xu C and Sachdev S 2009 Phys. Rev. B 79 064405
[42] Cano J, Cheng M, Mulligan M, Nayak C, Plamadeala E and Yard J 2014 Phys. Rev. B 89 115116
[43] Ran Y, Vishwanath A and Lee D-H 2008 Phys. Rev. Lett. 101 066801
[44] Isakov S V, Hastings M B and Melko R G 2011 Nat. Phys. 7 772
[45] Isakov S V, Hastings M B and Melko R G 2012 Science 335 193
[46] Levin M and Gu Z-C 2012 Phys. Rev. B 86 115109
[47] Xu C and Senthil T 2013 Phys. Rev. B 87 174412
[48] Hermele M 2007 Phys. Rev. B 76 035125