Make Schubert’s Calculus Calculable

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Abstract

Hilbert’s 15th problem asked for a rigorous foundation of Schubert’s enumerative calculus, in which a long standing and challenging part is Schubert’s problem of characteristics. In the course to secure the foundation of intersection theory André Weil attributed the problem to the determination of the intersection rings of flag manifolds.

This article surveys the background, content, and resolution of the problem of characteristics. Our main results are a unified formula for the characteristics, and a systematic description on the intersection rings of flag manifolds. We illustrate the effectiveness of the formula and the algorithm with explicit examples.

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1 Introduction

Hilbert’s 15th problem [25] is a far-reaching one in contemporary mathematics. It promotes the enumerative geometry of the 19th century growing into the algebraic geometry founded by Van der Waerden and André Weil, and makes Schubert calculus integrated deeply into many branches of mathematics. However, despite great many achievements in the 20th century the part of the problem of finding an effective rule performing the calculus has been stagnant for a long time, notably, the Schubert’s problem of characteristics [44 §8], or the Weil’s problem [45, p.331] on the intersection rings of flag manifolds $G/P$, where $G$ is a compact connected Lie group and $P \subset G$ a parabolic subgroup.

In the series of works [10, 12, 17, 19] we have addressed both of the problem of characteristics and the Weil’s problem for all flag manifolds, implying that the 15th problem has been solved satisfactorily [20, Remark 6.3]. The purpose of this article is to give an overview of the background, content and the resolution of the problem of characteristics. In §2 we bring a linkage from the Apollonius’s Theorem to the homology theory of Lefschetz [31, 1926] that reflects the historic evolution of the ideas from enumerative geometry to the intersection theory of projective geometry. The subject of §3 is the basis theorem of Schubert calculus, by which the problem of characteristics, considered by Schubert as the fundamental problem of enumerative geometry, admits a concise expression.
Our main results are introduced in §4, where we present a formula expressing the characteristics of a flag manifold $G/P$ as a polynomial in the Cartan numbers of the Lie group $G$ (Theorem 4.5), and develop a systematic description on the intersection ring of flag manifolds (Theorem 4.8). In particular, since our algorithmic approach uses the Cartan matrices of Lie groups as the main input, the solutions are implemented successfully by computer programs, so that the intersection theory of flag manifolds becomes accessible by readers familiar with computing science.

2 An invitation to intersection theory

In the 2nd century B.C. the Greek mathematician Apollonius obtained the following result in the paper “Tangencies”.

**Apollonius’s Theorem.** The number of circles tangent to three circles in plane is 8. □

The original proof of Apollonius was lost, but a record of the theorem by Pappus dated in the 4th century survived. During the Renaissance three different proofs were founded respectively by Adriaan van Roomen, Joseph Diaz Gergonne and Isaac Newton. For a pictorial illustration of this theorem, see the cover-page story of the book “3264 and all that”[22].

Descartes’s discovery of the Euclidean coordinates makes it possible for geometers (e.g. Maclaurin, Euler, Bezout) to exploit polynomial system to characterize geometric figures that satisfy a system of incidence conditions. Consequently, many enumerative problems admit the following algebraic formulation.

**Problem 2.1.** Given a system of $n$ polynomials in $n$ variables

\[
\begin{align*}
  f_1(x_1, \ldots, x_n) &= 0 \\
  \vdots \\
  f_n(x_1, \ldots, x_n) &= 0
\end{align*}
\]  

find the number of solutions to the system. □

In line with nowadays complex projective geometry we have assumed that the coefficients of the polynomials $f_i$’s lie in the field $\mathbb{C}$ of complex numbers. Problem 2.1 is a fundamental one in algebra. In the case $n = 1$ Gauss proved in the 1820’s that the number of zero’s of a polynomial in a single variable is the degree of that polynomial, well-known as the **fundamental theorem of algebra**.

Let $g_i$ be the homogenization of the polynomial $f_i$ in (2.1) we get in the $n$-dimensional complex projective space $\mathbb{C}P^n$ a hypersurface $N_i := g_i^{-1}(0)$. In general the zero locus of a homogeneous system on a complex projective space is called a **projective variety**. Naturally, the study of problem 2.1 leads to the fundamental problem of intersection theory.
Problem 2.2. Given $k$ projective subvarieties $N_1, \ldots, N_k$ in a smooth projective manifold $M$ that satisfy the dimension constraint $\dim N_1 + \cdots + \dim N_k = (k-1) \dim M$, find the number $|N_1 \cap \cdots \cap N_k|$ of the common intersection points when the $N_i$’s are in general position.

In the course of studying Problem 2.2 Lefschetz developed the homology theory for the cellular complexes [31, 1926]. In the perspective of this theory let $\alpha_i \in H^{\dim M - \dim N_i}(M)$ be the Poincaré dual of the cycle class represented by the subvariety $N_i \subset M$.

Problem 2.3. Given $k$ projective subvarieties $N_1, \ldots, N_k$ in a smooth projective manifold $M$ that satisfy the dimension constraint $\dim N_1 + \cdots + \dim N_k = (k-1) \dim M$, compute the Kronecker pairing

$$\langle \alpha_1 \cup \cdots \cup \alpha_k, [M] \rangle = ?$$

where $\cup$ means the cup product on the cohomology ring $H^*(M)$, and where $[M]$ denotes the fundamental class of $M$.

Through problems 2.1 to 2.3 we have reviewed three seemingly different approaches to the problems of enumerative geometry. A natural inquiry is: which is the mostly calculable one? The development of the intersection theory shall tell the answer.

3 Schubert’s problem of characteristics

The fundamental problem which occupies Schubert is to express the product of two of these symbols in terms of others linearly. He succeeds in part. -Coolidge J.L. [5, Chapt. IV]

Hermann Schubert (1848-1911) received his Ph.D. from the University of Halle, Germany in 1870. His doctoral thesis “The theory of characteristics” [35] is about enumerative geometry. Prior to this he had published several relevant works. In particular, he had shown that there are 16 spheres tangent to 4 general spheres in space, a direct extension of the Apollonius theorem.

In 1879 Schubert published the celebrated book “Calculus of Enumerative Geometry” [36] that represents the summit of intersection theory in the late 19th
While developing M. Chasles’s work on conics he demonstrated amazing applications of intersection theory to enumerative geometry, such as

i) The number of conics tangent to 8 quadrics in space is 4,407,296;

ii) The number of quadrics tangent to 9 quadrics in space is 666,841,088.

iii) The number of twisted cubic curves tangent to 12 quadrics in space is 5,819,539,783,680.

However, in addition to the extensive use of the controversial principle of conservation of numbers, Schubert’s exposition was so sketching that gave “no definition of intersection multiplicity, no way to find it nor to calculate it”. In the 15th problem Hilbert called for a rigorous foundation of Schubert calculus, while he praised the advantage of the calculus to foresee the final degree of a polynomial system before carrying out the actual process of elimination.

To illustrate the central part of Schubert’s approach to those spectacular enumerative numbers we resort to a table of computation from his book:

| $\mu^3\nu^5$ | $\mu^2\nu^6$ | $\mu^5\nu$ | $\nu^7$ | $\rho^8$ |
|--------------|--------------|------------|---------|---------|
| 1            | 8            | 34         | 92      | 4       |
| 2            | 14           | 52         | 116     | 16      |
| 4            | 24           | 76         | 128     | 24      |
| 4            | 24           | 76         | 128     | 24      |
| 2            | 16           | 48         | 64      | 32      |
| 1            | 12           | 48         | 64      | 32      |
| 6            | 6            | 12         | 16      | 4       |
| 8            | 8            | 16         | 4       |         |

It consists of the equations that evaluate the monomials $\mu^m\nu^n\rho^{8-m-n}$ in the symbols $\mu, \nu, \rho$ by integers, which were called characteristics by Schubert, and the Schubert’s symbolic equations by earlier researchers. Schubert emphasized that the problem of characteristics is the fundamental one of enumerative geometry. However, to state the problem in its natural simplicity and generality, one has to wait until 1950’s for the celebrated “basis theorem of Schubert calculus”. Let us recall the story.

The study of the characteristics began with the Italian school headed by Segre, Enriques and Severi. Two representing papers of the school are “The principle of conservation of numbers” and “The foundation of enumerative geometry and the theory of characteristics” due to Severi. Regarding these works Van de Wareden commented that they “erected an admirable structure, but its logical foundation was shaky. The notions were not well-defined, and the proofs were insufficient”.

In the pioneer work “Topological foundation of enumerative geometry” Van der Waerden first interpreted the characteristics in the perspective of the homology theory developed by Lefschetz (e.g. Problem 2.3), where he had the following crucial observations that guided the course of the later studies on the 15th problem:
1) Each Schubert’s symbolic equation is a relation on the homology of a projective manifold;

2) The solvability of Schubert’s characteristic problem relies on a finite basis of the homology of the relevant manifold;

3) The determination of the intersection products in homology is the goal of all enumerative methods.

C. Ehresmann [23, 1934] went two important steps further, who discovered that

4) The parameter spaces of the geometric figures of Schubert are in principle certain types of flag manifolds \( G/P \) (Remark 4.6);

5) For the Grassmannian \( G_{n,k} \) of \( k \)-planes on the \( n \)-space \( \mathbb{C}^n \) the Schubert symbols form exactly a basis of the homology \( H_*(G_{n,k}) \).

In what follows we denote by \( W(P, G) \) the set of left cosets of the Weyl group \( W(G) \) of \( G \) by the Weyl group \( W(P) \) of \( P \), and let \( l : W(P, G) \to \mathbb{Z} \) be the associated length function [3]. With the in-depth research on the structures of Lie groups the vague term “Schubert symbols” in the early literature was gradually replaced by such rigorous defined geometric objects as “Schubert cells” or “Schubert varieties”. In particular, Chevalley [8, 1958] and Bernstein-Gel’fand-Gel’fand [3, 1973] obtained the following result.

**Theorem 3.1.** Every flag manifold \( G/P \) admits an canonical decomposition into cells indexed by the elements of \( W(P, G) \)

\[
G/P = \bigcup_{w \in W(P, G)} X_w, \quad \dim X_w = 2l(w),
\]

where each cell \( X_w \) the closure of an algebraic affine space, called the Schubert variety on \( G/P \) associated to \( w \).

Since only even dimensional cells are involved in the partition (3.1) the set \( \{[X_w], w \in W(P, G)\} \) of fundamental classes forms an additive basis of the homology \( H_*(G/P) \). The co-cycle classes \( s_w \in H^*(G/P) \) Kronecker dual to the basis (i.e. \( \langle s_w, [X_u] \rangle = \delta_{w,u}, w, u \in W(P, G) \)) gives rise to the Schubert class associated to \( w \in W(P, G) \). Theorem 3.1 implies the following result predicted by Van der Waerden [44, §8], well-known as the basis theorem of Schubert calculus.

**Theorem 3.2.** The set \( \{s_w, w \in W(P, G)\} \) of Schubert classes forms a basis of the cohomology \( H^*(G/P) \).

An immediate consequence of the basis theorem is that any product \( s_{u_1} \cdots s_{u_k} \) in the Schubert classes can be uniquely expressed into a linear combination of the basis elements

\[
s_{u_1} \cdots s_{u_k} = \sum_{w \in W(P, G), l(w) = l(u_1) + \cdots + l(u_k)} c_{u_1, \ldots, u_k}^w \cdot s_w, \quad c_{u_1, \ldots, u_k}^w, a_{u_1, \ldots, u_k}^w \in \mathbb{Z},
\]
where the coefficients \( c_{w_1, \ldots, w_k} \) are the characteristics of Schubert \([23, 37, 38, 44]\). Granted with the basis theorem “the problem of characteristics” admits the following concise statement.

**Problem 3.3.** Given a product \( s_{u_1} \cdots s_{u_k} \) in the Schubert classes determine the characteristics numbers \( c_{w_1, \ldots, w_k} \) in (3.2).

In the momentous treaties “Foundations of Algebraic Geometry” \([45, 1962]\) A. Weil completed the definition of intersection multiplicities of subvarieties, and summarized the task of the classical Schubert calculus into the following accessible form. \( \square \)

**Problem 3.4 (Weil).** Determine the cohomology rings of all flag manifolds \( G/P \). \( \square \)

Weil commented his problem as “the modern form taken by the topic formerly known as enumerative geometry” \([45, \text{p.331}]\). We show that

**Theorem 3.5.** For the flag manifolds \( G/P \) the Weil’s problem is equivalent to the Schubert’s one.

**Proof.** A ring is an abelian group \( R \) that is furnished with a multiplication \( R \times R \rightarrow R \). By the basis theorem the cohomology \( H^*(G/P) \) has a canonical basis consisting of Schubert classes. Therefore, the multiplication on \( H^*(G/P) \) is uniquely determined by the product among the basis elements, which is handled by the problem of characteristics. \( \square \)

**Remark 3.6.** For the case \( k = 2 \) the characteristics \( c_{w_1, \ldots, w_2} \) admit various interpretations. They are called the structure constants of the flag manifold \( G/P \) in topology; and the Littlewood-Richardson coefficients in representation theory \([32]\).

In certain cases the parameter spaces of the geometric figures concerned by Schubert \([36]\) fail to be flag manifolds, but can be constructed by performing finite steps of blow-ups on flag manifolds, see examples in Fulton \([24, \text{Section 10.4}]\), Eisenbud-Harries \([22, \text{Chap.13}]\), or in \([13]\) for the constructions of the parameter spaces of the complete conics and quadrics on \( \mathbb{CP}^3 \). As results the relevant characteristics can be computed from those of flag manifolds via strict transformations (e.g. \([13, \text{Examples 5.11; 5.12}]\)). \( \square \)

### 4 A formula for Schubert’s characteristics

A calculus, or science of calculation, is one which has organized processes by which a passage is made, mechanically, from one result to another. – De Morgan.
To secure the foundation of a “calculus” it suffices to decide the objects to be calculated, and to determine accordingly the rules of the calculation (e.g. [1, Chap.2], [33]). As for the Schubert’s calculus the objects to be calculated are the Schubert symbols, or Schubert varieties. In this section we accomplish the rule of the calculus by a unified formula computing the characteristics.

4.1 Observation and expectation

The major difficulties that one encounters with the characteristics are fairly easy to observe:

i) The simply-connected simple Lie groups $G$ consist of the three infinite families of classical Lie groups $Spin(n), Sp(n), SU(n)$, as well as the five exceptional ones $G_2, F_4, E_6, E_7, E_8$;

ii) for a simple Lie group $G$ with rank $n$ there are precisely $2^n - 1$ parabolic subgroups $P$ on $G$.

That is, there exist plenty of flag manifolds $G/P$ whose geometries and topologies vary considerably with respect to different choices of $G$ and $P$. In addition

iii) the number of basis elements of $H^\ast(G/P)$ agrees with the Euler characteristic $\chi(G/P)$, which is normally very large,

not to mention the number of the relevant characteristics. For instance, for an exceptional Lie group $G$ with a maximal torus $T$ the Euler characteristics $\chi(G/T)$ is given in the table below

| $G$  | $G_2$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|------|-------|-------|-------|-------|-------|
| $\chi(G/T)$ | 12    | 1152  | $2^4 \cdot 3^4 \cdot 5$ | $2^{10} \cdot 3^4 \cdot 5^2 \cdot 7$ | $2^7 \cdot 3^3 \cdot 5^2 \cdot 7$ |

Summarizing, studies case by case can never reach a complete solution to the problem.

On the other hand, according to Cartan’s beautiful classification on compact Lie groups, associate to each simple Lie group $G$ there is a Cartan matrix $C$, that can serve as “the cosmological constants” to classify all flag manifolds $G/P$, see discussions in the coming section. As examples, for the five exceptional Lie groups those matrices are

$G_2 : \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$, $F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$, $E_6 : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$.
This motivates the inquiry whether one can formulate the characteristics $c_{w_1, \ldots, w_k}$, as well as the intersection ring of $G/P$, merely from the Cartan matrix of the Lie group $G$? In this section we fulfill this expectation.

4.2 Numerical construction in a Weyl group

Therefore, let $C = (c_{i,j})_{n \times n}$ be the Cartan matrix of some compact simple Lie group $G$, and let $\mathbb{R}^n$ be the $n$-dimensional real vector space with basis $\{\omega_1, \ldots, \omega_n\}$. Define in term of $C$ the endomorphisms $\sigma_i \in \text{Hom}(\mathbb{R}^n), 1 \leq i \leq n$, by the formula

$$\sigma_i(\omega_k) = \begin{cases} 
\omega_k & \text{if } i \neq k; \\
\omega_k - (c_{k,1}\omega_1 + c_{k,2}\omega_2 + \cdots + c_{k,n}\omega_n) & \text{if } i \neq k.
\end{cases}$$

By the property of Cartan matrix we have $\sigma_2 = \text{id}$, implying that $\sigma_i \in \text{Aut}(\mathbb{R}^n)$. It can be further shown that

**Lemma 4.1.** The subgroup of Aut($\mathbb{R}^n$) generated by the $\sigma_i$'s is the Weyl group $W(G)$ of $G$.

For each subset $K \subset \{1, \ldots, n\}$ there is a parabolic subgroup $P = P_K$, unique up to the conjugations on $G$, whose Weyl group $W(P)$ is generated by those $\sigma_j$ with $j \notin K$. Resorting to the length function $l$ on $W(G)$ we can furthermore embed the set $W(P;G)$ as the subset of the group $W(G)$ $\mathbb{R}$, by Lemma 5.1 each element $w \in W^m(P;G)$ admits a factorization of the form

$$w = \sigma_{i_1} \circ \cdots \circ \sigma_{i_m} \text{ with } 1 \leq i_1, \ldots, i_m \leq n,$$

hence can be denoted by $w = \sigma_I$, where $I = (i_1, \ldots, i_m)$. Such expressions of $w$ may not be unique, but the ambiguity can be dispelled by employing the following notion. Furnish the set $D(w) := \{I = (i_1, \ldots, i_m) \mid w = \sigma_I\}$ with the lexicographical order $\preceq$ on the multi-indices $I$'s. We call a decomposition $w = \sigma_I$ minimized if $I \in D(w)$ is the minimal one. Clearly,
Lemma 4.2. Every $w \in W(P; G)$ possesses a unique minimized decomposition. □

It follows that the set $W^m(P; G)$ is also ordered by the lexicographical order $\preceq$ on the multi-index $I$'s, hence can be uniquely presented as

\begin{equation}
W^m(P; G) = \{ w_{m,i} \mid 1 \leq i \leq \beta(m) \}, \quad \beta(m) := |W^m(P; G)|,
\end{equation}

where $w_{m,i}$ denotes the $i^{th}$ element in $W^m(P; G)$. In [13] the package “Decomposition” in MATHEMATICA is compiled, whose function is stated below.

Algorithm I. Decomposition.

**Input:** The Cartan matrix $C = (a_{ij})_{n \times n}$ of $G$, and a subset $K \subset \{1, \ldots, n\}$ to specify a parabolic subgroup $P$.

**Output:** The set $W(P; G)$ being presented by the minimized decompositions of its elements, together with the index system (5.1) imposed by the order $\preceq$. □

**Example 4.3.** Let $G = SU(n)$ be the special unitary group, and $k \in \{1, \ldots, n-1\}$. The flag manifold $G/P(k)$ is the well-known Grassmannian manifold $G_{n,k}$. Taking $(n, k) = (9, 4)$ and applying the Decomposition we obtain the following table that presents the elements $w \in W(P; G)$ with $l(w) \leq 9$ in terms of their minimized decompositions, together with the index system given in (4.1).

| $w_{i,j}$ | decomposition | $w_{i,j}$ | decomposition | $w_{i,j}$ | decomposition |
|-----------|---------------|-----------|---------------|-----------|---------------|
| $w_{1,1}$ | [4]           | $w_{2,1}$ | [3, 4]        | $w_{2,2}$ | [5, 4]        |
| $w_{3,1}$ | [2, 3, 4]     | $w_{3,2}$ | [3, 5, 4]     | $w_{3,3}$ | [6, 5, 4]     |
| $w_{4,1}$ | [1, 2, 3, 4]  | $w_{4,2}$ | [2, 3, 5, 4]  | $w_{4,3}$ | [3, 6, 5, 4]  |
| $w_{4,4}$ | [4, 3, 5, 4]  | $w_{4,5}$ | [7, 6, 5, 4]  | $w_{5,1}$ | [1, 2, 3, 5, 4]|
| $w_{5,2}$ | [2, 3, 6, 5, 4]| $w_{5,3}$ | [2, 4, 3, 5, 4]| $w_{5,4}$ | [3, 7, 6, 5, 4]|
| $w_{5,5}$ | [4, 3, 6, 5, 4]| $w_{5,6}$ | [8, 7, 6, 5, 4]| $w_{6,1}$ | [1, 2, 3, 6, 5, 4]|
| $w_{6,2}$ | [1, 2, 4, 3, 5, 4]| $w_{6,3}$ | [2, 3, 7, 6, 5, 4]| $w_{6,4}$ | [2, 4, 3, 6, 5, 4]|
| $w_{6,5}$ | [3, 2, 4, 3, 5, 4]| $w_{6,6}$ | [3, 8, 7, 6, 5, 4]| $w_{6,7}$ | [4, 3, 7, 6, 5, 4]|
| $w_{6,8}$ | [5, 4, 3, 6, 5, 4]| $w_{6,9}$ | [9, 8, 7, 6, 5, 4]| $w_{7,1}$ | [1, 2, 3, 7, 6, 5, 4]|
| $w_{7,2}$ | [1, 2, 4, 3, 6, 5, 4]| $w_{7,3}$ | [1, 3, 2, 4, 3, 5, 4]| $w_{7,4}$ | [2, 3, 8, 7, 6, 5, 4]|
| $w_{7,5}$ | [2, 4, 3, 7, 6, 5, 4]| $w_{7,6}$ | [2, 5, 4, 3, 6, 5, 4]| $w_{7,7}$ | [3, 2, 4, 3, 6, 5, 4]|
| $w_{7,8}$ | [3, 9, 8, 7, 6, 5, 4]| $w_{7,9}$ | [4, 3, 8, 7, 6, 5, 4]| $w_{7,10}$ | [5, 4, 3, 7, 6, 5, 4]|
| $w_{8,1}$ | [1, 2, 3, 8, 7, 6, 5, 4]| $w_{8,2}$ | [1, 2, 4, 3, 7, 6, 5, 4]| $w_{8,3}$ | [1, 2, 5, 4, 3, 6, 5, 4]|
| $w_{8,4}$ | [1, 3, 2, 4, 3, 6, 5, 4]| $w_{8,5}$ | [2, 1, 3, 2, 4, 3, 5, 4]| $w_{8,6}$ | [2, 3, 9, 8, 7, 6, 5, 4]|
| $w_{8,7}$ | [2, 4, 3, 8, 7, 6, 5, 4]| $w_{8,8}$ | [2, 5, 4, 3, 7, 6, 5, 4]| $w_{8,9}$ | [3, 2, 4, 3, 7, 6, 5, 4]|
| $w_{8,10}$ | [3, 2, 5, 4, 3, 6, 5, 4]| $w_{8,11}$ | [4, 3, 9, 8, 7, 6, 5, 4]| $w_{8,12}$ | [5, 4, 3, 8, 7, 6, 5, 4]|

For more examples of the results produced by Decomposition we refer to [18] Sections 1.1–7.1. □
4.3 The formula for Schubert’s characteristics

Given an element \( w \in W(P_K; G) \) with minimized decomposition
\[
w = \sigma_{i_1} \circ \sigma_{i_2} \circ \cdots \circ \sigma_{i_m}, \quad 1 \leq i_1, i_2, \cdots, i_m \leq n,
\]
the structure matrix of \( w \) is the strictly upper triangular matrix \( A_w = (a_{s,t})_{m \times m} \)
defined by the Cartan matrix \( C = (c_{i,j})_{n \times n} \) of \( G \) as
\[
a_{s,t} = 0 \text{ if } s \geq t, \quad -c_{i,s,i} \text{ if } s < t.
\]

As examples recall that the Cartan matrix of the exceptional Lie group \( G_2 \) is
\[
C = \begin{pmatrix}
2 & -1 \\
-3 & 2
\end{pmatrix}.
\]

By Lemma 4.1 the Weyl group \( W(G_2) \) has two generators \( \sigma_1, \sigma_2 \). Consider the
following elements with length 4:
\[
u = \sigma_1 \circ \sigma_2 \circ \sigma_1 \circ \sigma_2 \quad \text{and} \quad v = \sigma_2 \circ \sigma_1 \circ \sigma_2 \circ \sigma_1.
\]

From the Cartan matrix \( C \) one reads
\[
A_u = \begin{pmatrix}
0 & 1 & -2 & 1 \\
0 & 0 & 3 & -2 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
A_v = \begin{pmatrix}
0 & 3 & -2 & 3 \\
0 & 0 & 1 & -2 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \( \mathbb{Z}[x_1, \ldots, x_m] \) be the ring of polynomials in \( x_1, \ldots, x_m \) that is graded by \( \text{deg } x_i = 1 \), and let \( \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \) be its subgroup spanned by the homogeneous monomials with degree \( m \). Given a \( m \times m \) strictly upper triangular integer matrix \( A = (a_{i,j}) \) the triangular operator \( T_A \) associated to \( A \) is the linear map
\[
T_A : \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \rightarrow \mathbb{Z}
\]
defined recursively by the following elimination rules:

i) If \( m = 1 \) (i.e. \( A = (0) \)) then \( T_A(x_1) = 1 \);
ii) If \( h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m)} \) then \( T_A(h) = 0 \);
iii) If \( h \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m-r)} \) with \( r \geq 1 \) then
\[
T_A(h \cdot x_m^r) = T_{A_1}(h \cdot (a_{1,m}x_1 + \cdots + a_{m-1,m}x_{m-1})^{r-1}),
\]
where \( A_1 \) is the \((m-1) \times (m-1)\) (strictly upper triangular matrix) obtained from \( A \) by deleting both of the \( m^{th} \) column and row. Since every polynomial \( h \in \mathbb{Z}[x_1, \ldots, x_m]^{(m)} \) admits the unique expansion
\[
h = \sum_{0 \leq r \leq m} h_r \cdot x_m^r \quad \text{with} \quad h_r \in \mathbb{Z}[x_1, \ldots, x_{m-1}]^{(m-r)},
\]
the operator $T_A$ is well-defined by the rules i), ii) and iii). It follows that

**Lemma 4.4.** For any polynomial $h \in \mathbb{Z}[x_1, \ldots, x_m]^{(m)}$ the number $T_A(h)$ is a polynomial in the entries of the matrix $A$ with degree $m$. □

Extending the main results of [9,11,12] we have shown in [20] Theorem 2.4 the following formula that expresses the Schubert’s characteristics of a flag manifold $G/P$ as polynomials in the Cartan numbers of the group $G$.

**Theorem 4.5.** Let $w \in W(P;G)$ be an element with minimized decomposition $\sigma_{i_1} \circ \cdots \circ \sigma_{i_m}$ and structure matrix $A_w$. For any monomial $s_{u_1} \cdots s_{u_k}$ in the Schubert classes with total degree $m$ one has

\[
(5.2) \quad c_{u_1,\ldots,u_k}^w = T_{A_w} \left( \prod_{i=1}^{m} \left( \sum_{\sigma_I = u_i, |I| = \{u_i\}, I \subseteq \{1, \ldots, m\}} x_I \right) \right),
\]

where for a multi-index $I = \{j_1, \ldots, j_t\}$ we have set $|I| := t$ and

\[
\sigma_I := \sigma_{j_1} \circ \cdots \circ \sigma_{j_t} \in W(G), \quad x_I := x_{i_{j_1}} \cdots x_{i_{j_t}} \in \mathbb{Z}[x_1, \ldots, x_m]. \quad \square
\]

Since the matrix $A_w$ is constructed from the Cartan matrix of the group $G$ in term of the minimized decomposition $w$, while the operator $T_{A_w}$ is evaluated easily by the elimination rules i)-iii) stated above, the formula (4.2) indicates an effective algorithm evaluating $c_{u_1,\ldots,u_k}^w$. Combining these ideas the package “Characteristics” in MATHEMATICA has been compiled (e.g. [13]) whose function is described as follows.

**Algorithm II:** Characteristics.

**Input:** The Cartan matrix $C = (a_{ij})_{n \times n}$ of $G$, and a subset $K \subset \{1, \ldots, n\}$ to specify a parabolic subgroup $P$.

**Output:** The characteristics $c_{u_1,\ldots,u_k}^w$ of $G/P$. □

**Example 4.6.** Traditionally, the characteristics $c_{u_1,u_2}^w$ of the Grassmannian $G_{n,k}$ are given by the combinatorial Littlewood-Richardson rule [32], rather than a closed formula. In comparison the formula (4.2) is practical for explicit computation. Taking the cases $(n,k) = (9,4)$ and $l(w) = 8$ as example the values of $c_{u_1,u_2}^w$ produced by Characteristics are tabulated below.

| $u_1$ | $u_2$ | $c_{u_1,u_2}^w$, $l(w) = 8$ |
|-------|-------|-----------------------------|
| $u_{1,1}$ | $w_{7,1}$ | 1 1 | 0 0 0 0 0 0 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,2}$ | 0 1 | 1 1 0 0 0 0 0 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,3}$ | 0 0 | 0 1 1 0 0 0 0 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,4}$ | 1 0 | 0 0 0 1 1 0 0 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,5}$ | 0 1 | 0 0 0 0 0 1 1 1 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,6}$ | 0 0 | 1 0 0 0 0 1 0 1 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,7}$ | 0 0 | 0 1 0 0 0 1 1 0 0 0 0 0 0 0 |
| $u_{1,1}$ | $w_{7,8}$ | 0 0 | 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 |
| $u_{1,1}$ | $w_{7,9}$ | 0 0 | 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 |
| $u_{1,1}$ | $w_{7,10}$ | 0 0 | 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0 |
The Characteristics works equally well for other types of flag manifolds. As example the characteristics for the flag manifold $E_7/P_{(2)}$ ($P_{(2)} = S^1 \cdot SU(7)$) with $k = 2, l(w) = 9$ are given below.

| $u_1$ | $u_2$ | $c_{u_1,u_2}^w$ | $l(w) = 9$ |
|-------|-------|------------------|-------------|
| 0     | 0     | 1                | 0           |
| 0     | 0     | 1                | 1           |
| 0     | 0     | 0                | 0           |
| 0     | 1     | 1                | 1           |
| 0     | 1     | 0                | 0           |
| 0     | 1     | 1                | 1           |
| 0     | 0     | 1                | 0           |
| 0     | 0     | 0                | 1           |
| 0     | 1     | 0                | 0           |

The characteristics are given in the following table.
4.4 The solution to Weil’s problem

For the Grassmann manifold $G_{n,k}$ let $c_i \in H^{2i}(G_{n,k})$ be the $i^{th}$ special Schubert class (i.e. the $i^{th}$ universal Chern class) on $G_{n,k}$. Borel [3] has shown that

$$H^\ast(G_{n,k}) = \mathbb{Z}[c_1, \ldots, c_k] / \langle c_{n-k+1}^{-1}, \ldots, c_n^{-1} \rangle,$$

where $c_j^{-1}$ denotes the component of the formal inverse of $1 + c_1 + \cdots + c_k$ in degree $j$, and where $\langle \cdots \rangle$ denotes the ideal generated by the enclosed polynomials. Comparing formula (4.3) with the characteristics of $G_{3,4}$ given in Example 4.7 reveals the following phenomena: the characteristics are essential for enumerative geometry, but fail to be a concise way to characterize the structure of the ring $H^\ast(G_{n,k})$. It is the Weil’s problem that motivates us the following extension of Borel’s formula (4.3) to all flag manifolds.

**Theorem 4.8.** For each flag manifold $G/P$ there exist Schubert classes $x_1, \ldots, x_n$ such that
\[(4.4) \quad H^\ast(G/P) = \mathbb{Z}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m \rangle,\]

where \(f_i \in \mathbb{Z}[x_1, \ldots, x_n], 1 \leq i \leq m, \) and where the numbers \(n\) and \(m\) are minimum subject to the presentation.

**Proof.** Let \(D(H^\ast(G/P))\) be the ideal of the decomposable elements of the ring \(\subset H^\ast(G/P)\). Since the cohomology \(H^\ast(G/P)\) is torsion free and has a basis consisting of Schubert classes, there exists Schubert classes \(x_1, \ldots, x_n\) on \(G/P\) that correspond to a basis of the quotient group \(H^\ast(G/P)/D(H^\ast(G/P))\).

It follows that the inclusion \(\{x_1, \ldots, x_n\} \subset H^\ast(G/P)\) induces a ring epimorphism

\[h : \mathbb{Z}[x_1, \ldots, x_n] \rightarrow H^\ast(G/P)\]

By the Hilbert’s basis theorem there exist polynomials \(f_1, \ldots, f_m \in \mathbb{Z}[x_1, \ldots, x_n]\) such that \(\ker h = \langle f_1, \ldots, f_m \rangle\). We can of course assume that the number \(m\) is minimum with respect to the formula (4.4).

As the cardinality of a basis of the quotient group \(H^\ast(G/P)/D(H^\ast(G/P))\) the number \(n\) is an invariant of \(G/P\). In addition, if one changes the generators \(x_1, \ldots, x_n\) to \(x'_1, \ldots, x'_n\), then each old generator \(x_i\) can be expressed as a polynomial \(g_i\) in the new ones \(x'_1, \ldots, x'_n\). The invariance of the number \(m\) is shown by the presentation

\[H^\ast(G/P) = \mathbb{Z}[x'_1, \ldots, x'_n]/\langle f'_1, \ldots, f'_m \rangle,\]

where \(f'_j\) is obtained from \(f_j\) by substituting \(g_i\) for \(x_i, 1 \leq j \leq m. \) \(\square\)

The proof of Theorem 4.8 serves the purpose to clarify two crucial steps in solving the Weil’s problem:

i) Find a minimal set of Schubert classes \(\{x_1, \ldots, x_n\}\) on \(G/P\) that generates the quotient group \(H^+(G/P)/D(H^+(G/P))\);

ii) Seek a Hilbert basis \(\{f_1, \ldots, f_m\}\) of the ideal \(\ker h\).

Since both of the tasks can be implemented by the *Characteristics* (e.g. [20 Section 4.4]) we obtain therefore the package "Chow-ring" in MATHEMATICA [17, 20] whose function is stated below.

**Algorithm III: Chow-ring.**

**Input:** The Cartan matrix \(C = (a_{ij})_{n \times n}\) of \(G\), and a subset \(K \subset \{1, \ldots, n\}\) to specify a parabolic subgroup \(P\).

**Output:** A presentation (4.4) of the cohomology \(H^\ast(G/P)\). \(\square\)

\(^2\)For a flag manifold \(G/P\) Chow has shown that the Chow ring \(A^\ast(G/P)\) is canonically isomorphic to the cohomology \(H^\ast(G/P)\).
Example 4.9. To economize notation we write \( s_{r,i} \) to denote the Schubert class associated to the coset \( w_{r,i} \in W^r(P;G) \) (see (4.1)), and called it the \( i \)th Schubert class in degree \( r \).

If \( G \) is a simple Lie group of rank \( n \) and if \( K = \{1, \cdots, n \} \), then the parabolic subgroup \( P_K \) is a maximal torus \( T \) on \( G \), and the flag manifold \( G/T \) is called the complete flag manifold of the group \( G \). As applications of the Chow-ring the cohomologies \( H^*(G/T) \) for all exceptional Lie groups \( G \) have been determined in terms of a minimal system of generators and relations in the Schubert classes (e.g. [19]). We present below the results for the cases \( G = F_4, E_6, E_7 \).

(4.5) \[ H^*(F_4/T) = \mathbb{Z}[\omega_1, \cdots, \omega_6, y_3, y_4]/ \langle \rho_2, \rho_4, r_3, r_6, r_8, r_{12} \rangle, \]
where
\[
\begin{align*}
\rho_2 &= c_2 - 4\omega_1^2; \\
\rho_4 &= 3y_4 + 2\omega_1y_3 - c_4; \\
r_3 &= 2y_3 - \omega_1^3; \\
r_6 &= y_3^2 + 2c_6 - 3\omega_1^2y_4; \\
r_8 &= 3y_4^2 - \omega_1^2c_6; \\
r_{12} &= y_4^3 - c_6^2. \\
\end{align*}
\]

(4.6) \[ H^*(E_6/T) = \mathbb{Z}[\omega_1, \cdots, \omega_7, y_3, y_4, y_5, y_6]/ \langle \rho_2, \rho_3, \rho_4, \rho_5, r_6, r_8, r_9, r_{12} \rangle, \]
where
\[
\begin{align*}
\rho_2 &= 4\omega_2^2 - c_2; \\
\rho_3 &= 2y_3 + 2\omega_2^3 - c_3; \\
\rho_4 &= 3y_4 + \omega_2^4 - c_4; \\
\rho_5 &= 2\omega_2^2y_3 - \omega_2c_4 + c_5; \\
r_6 &= y_3^2 - \omega_2c_5 + 2c_6; \\
r_8 &= 3y_4^2 - 2c_5y_3 - \omega_2^2c_6 + \omega_2^3c_5; \\
r_9 &= 2y_3c_6 - \omega_2^3c_6; \\
r_{12} &= y_4^3 - c_6^2. \\
\end{align*}
\]

(4.7) \[ H^*(E_7/T) = \mathbb{Z}[\omega_1, \cdots, \omega_7, y_3, y_4, y_5, y_6]/ \langle \rho_i, r_j \rangle, \]
where
\[
\begin{align*}
\rho_2 &= 4\omega_2^2 - c_2; \\
\rho_3 &= 2y_3 + 2\omega_2^3 - c_3; \\
\rho_4 &= 3y_4 + \omega_2^4 - c_4; \\
\rho_5 &= 2y_5 - 2\omega_2^3y_3 + \omega_2c_4 - c_5; \\
r_6 &= y_3^2 - \omega_2c_5 + 2c_6; \\
r_8 &= 3y_4^2 + 2y_5y_5 - 2y_3c_5 + 2\omega_2c_7 - \omega_2^2c_6 + \omega_2^3c_5; \\
r_9 &= 2y_9 + 2y_4y_5 - 2y_3c_6 - \omega_2^2c_7 + \omega_2^3c_6; \\
r_{10} &= y_9^2 - 2y_3c_7 + \omega_2^3c_7; \\
r_{12} &= y_4^3 - 4y_5c_7 - c_6^2 - 2y_3y_9 - 2y_3y_4y_5 + 2\omega_2y_5c_6 + 3\omega_2y_4c_7 + c_5c_7; \\
r_{14} &= c_7^2 - 2y_5y_9 + 2y_3y_4c_7 - \omega_2^3y_4c_7; \\
r_{18} &= y_3^2 + 2y_5c_6c_7 - y_4c_7^2 - 2y_4y_5y_9 + 2y_3y_9^2 - 5\omega_2y_5^2c_7.
\end{align*}
\]
where $\omega_i = s_{1,i}$, the $c_i$'s are the polynomials in $\omega_1, \ldots, \omega_n$ defined in \cite{20} (5.17), and where the $y_i$'s are the Schubert classes on $G/T$ specified by the minimized decompositions of their corresponding elements of the Weyl group $W(G)$ in the table below:

| $y_i$ | $y_3$ | $y_4$ | $y_5$ | $y_9$ |
|-------|-------|-------|-------|-------|
| $F_4/T$ | [3, 2, 1] | [4, 3, 2, 1] |       |       |
| $E_6/T$ | [5, 4, 2] | [6, 5, 4, 2] |       |       |
| $E_7/T$ | [5, 4, 2] | [6, 5, 4, 2] | [7, 6, 5, 4, 2] | [1, 5, 4, 3, 7, 6, 5, 4, 2] |

For more examples of the applications of Chow-ring to computing with partial flag manifolds $G/P$ we refer to \cite{17} Theorems 1-7.

4.5 Applications to the topology of homogeneous spaces

For a compact Lie group $G$ with a closed subgroup $H$ the quotient space $G/H$ is called a homogeneous space of $G$. In contrast to the flag manifolds the cohomology of a homogeneous space may be nontrivial in odd degrees, and may contain torsion elements.

A classical problem of topology is to express the cohomology of a Lie group $G$, or a homogeneous space $G/H$, by a minimal system of explicit generators and relations. The traditional approaches due to H.Cartan, A.Borel, P.Baum and H.Toda utilize various spectral sequence techniques \cite{2, 4, 30, 41, 46}, and the calculation encounters the same difficulties when applied to a Lie group $G$ whose integral cohomology has torsion elements, in particular when $G$ is one of the exceptional Lie groups.

Schubert calculus makes the cohomology theory of homogeneous spaces appearing in a new light. For examples, inputting the formulae (4.5) – (4.7) into the second page of the Serre-spectral sequence of the fibration $G \to G/T$

$$E_2^{2*}(G) = H^*(G/T) \otimes H^*(T),$$

the integral cohomology $H^*(G)$, as well as the Hopf algebra structure on the mod $p$ cohomology $H^*(G; \mathbb{Z}_p)$, has been determined by computing with the Schubert classes on $G/T$ \cite{16} \cite{21}. For more examples of the extension of Schubert calculus to computing with the homogeneous spaces, see in \cite{17} Sect.5.

5 Concluding remarks

Throughout the ages a common hope of geometers is to find calculable mechanisms among the geometric entities they are caring of (e.g. algebraic varieties, cellular complexes, vector bundles, or the cobordism classes of smooth manifolds). The emergence of Schubert’s calculus, or the birth of the intersection theory, had catered to this demand. Today, Schubert’s calculus has widely integrated into many branches of mathematics, and has profoundly affected the
trajectories of the development of those fields, such as the theory of characteristic classes and the string theory \[26, 34\]. All of these vigorously witnessed Hilbert’s broad vision and foresight, and put forward the essential request to explore effective rules performing the computation. In this sense among the 23 problems of Hilbert the 15th one is relevant to computing science.

In this paper we have recalled the earlier studies on the characteristics before 1960’s, presented a formula computing the characteristics, and illustrated a passage from the Cartan matrices of Lie groups to the intersection rings of flag manifolds in which the Schubert’s characteristics play a central role. For the rigorous treatments of the Schubert’s enumerative examples mentioned in Section 3 we refer to the beautiful survey articles S. Kleiman \[27, 29\], or the relevant sections of the excellent textbooks Fulton \[24, 1998\] and Eisenbud-Harries \[22, 2016\] on intersection theory.

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