A note on the boundary spin $s$ XXZ chain

Anastasia Doikou

University of Bologna, Physics Department, INFN Section
Via Irnerio 46, 40126 Bologna, Italy

Abstract

The open spin $s$ XXZ model with non-diagonal boundaries is considered. Within the algebraic Bethe ansatz framework and in the spirit of earlier works we derive suitable reference states. The derivation of the reference state is the crucial point in this investigation, and it involves the solution of sets of difference equations. For the spin $s$ representation, expressed in terms of difference operators, the pseudo-vacuum is identified in terms of $q$-hypergeometric series. Having specified such states we then build the Bethe states and also identify the spectrum of the model for generic values of the anisotropy parameter $q$. 

\footnote{e-mail: doikou@bo.infn.it}
1 Introduction

Much interest has been focused during the past years on integrable models with boundaries [1, 2], further invigorated by advances on problems related to condensed matter physics and statistical mechanics, with immediate applications and possible experimental realizations (see e.g. [3, 4]), and also by studies on the understanding of D-branes by means of the boundary conformal field theory [5, 6].

The main objective of the present work is the study of the open XXZ model with integrable non-diagonal boundaries for the spin $s$ representation. The spin $s$ XXZ model [7] with periodic boundary conditions was investigated in [8], whereas studies concerning the model with toroidal and generic twisted boundary conditions were presented in [9, 10]. Until recently, the formulation of the Bethe ansatz equations for open spin chains with non-diagonal boundaries was an open problem. The main difficulty arising when trying to obtain the transfer matrix eigenvalues and the corresponding Bethe ansatz equations, by standard algebraic or analytical Bethe ansatz techniques, is the lack of an obvious reference state ‘pseudo–vacuum’. However, the problem was solved rather recently for the spin $\frac{1}{2}$ XXZ chain by two different methods [11, 12, 13]. More precisely, in the approach described in [11, 12] such reference state is not a necessary requirement anymore as long as $q$ is a root of unity. On the other hand in the method developed in [13], which is valid for all values of $q$, suitable local gauge transformations are implemented, along the lines described in [14, 15], rendering the derivation of a suitable reference state possible.

Note that the spectrum of the spin $s$ XXZ model in the presence of non-diagonal boundaries was derived in [16] using the method of [11], however only for $q$ root of unity (see also [17] for spin-1 type chains with diagonal boundaries). Moreover, in [18] the spectrum, Bethe ansatz equations and Bethe states were derived for the so called cyclic representation of $U_q(sl_2)$, and for the $q$ harmonic oscillator. The generalized spin $s$ XXX ($q = 1$) model with non diagonal boundaries was also investigated within the Bethe ansatz framework in [19, 20]. Here using the algebraic Bethe ansatz framework [2] [13, 15, 21] we are able to specify the spectrum of the spin $s$ XXZ model with non-diagonal boundaries for generic values of $q$, and more importantly deduce the corresponding eigenstates. It should be stressed that the derivation of the corresponding eigenstates is of great significance allowing for instance the calculation of exact correlation functions [22, 23]. The diagonalization process rests on the identification of an appropriate reference state upon which all the Bethe states are built. It is also worth remarking that in the case where the spin $s$ representation is expressed in terms of difference operators, the reference state may be derived by solving sets of difference equations, and it is expressed in terms of $q$-hypergeometric series.
2 The open spin s XXZ model

Before we proceed with the derivation of the spectrum and eigenstates of the spin s XXZ chain with non diagonal boundaries we shall first provide a brief review of the model. Consider the Lax operator \( L(\lambda) \in \mathbb{C}^2 \otimes U_q(\widehat{sl}_2) \) being a solution of the fundamental algebraic relation

\[
R_{12}(\lambda_1 - \lambda_2) \; L_1(\lambda_1) \; L_2(\lambda_2) = L_2(\lambda_2) \; L_1(\lambda_1) \; R_{12}(\lambda_1 - \lambda_2),
\]

where \( R \) is the spin \( \frac{1}{2} \) XXZ matrix, solution of the Yang-Baxter equation \[14, 22\]. The \( L \) operator may be written, with the help of the evaluation homomorphism \[24\], which maps \( U_q(\widehat{sl}_2) \rightarrow U_q(sl_2) \), as a \( 2 \times 2 \) matrix

\[
L(\lambda) = \frac{1}{2} \begin{pmatrix} e^{\mu\lambda} q^{\frac{1}{2}} A - e^{-\mu\lambda} q^{-\frac{1}{2}} D & (q - q^{-1}) B \\ (q - q^{-1}) C & e^{\mu\lambda} q^{\frac{1}{2}} D - e^{-\mu\lambda} q^{-\frac{1}{2}} A \end{pmatrix}
\]

(2.2)

A, B, C, D are the generators of \( U_q(sl_2) \) and they satisfy the well known defining relations:

\[
AD = DA = I, \quad AC = qCA, \quad AB = q^{-1}BA, \quad DC = q^{-1}CD, \quad DB = qBD \quad [C, B] = \frac{A^2 - D^2}{q - q^{-1}}.
\]

(2.3)

This algebra admits various realizations, such as the known spin integer or half integer \( s \) representation. The \( 2s + 1 \) dimensional representation may be also realized by difference operators acting on the space of polynomials of degree \( 2s \) \( [27] \) (see also e.g. \[25\]). For the particular case where the generators admit the spin \( \frac{1}{2} \) representation the \( L \) operator reduces to the spin \( \frac{1}{2} \) XXZ \( R \) matrix. More precisely, \( A = D^{-1} \mapsto q^{\sigma^z}, \; B \mapsto \sigma^-, \; C \mapsto \sigma^+ \) with \( \sigma^{\pm,\pm} \) being the usual \( 2 \times 2 \) Pauli matrices.

(a) Let us recall the spin \( s \) representation of \( U_q(sl_2) \) in more detail. This is a \( n = 2s + 1 \) dimensional representation, and may be expressed in terms of \( n \times n \) matrices as

\[
A = \sum_{k=1}^{n} q^{\alpha_k} e_{kk}, \quad C = \sum_{k=1}^{n-1} \tilde{C}_k e_{k+1, k}, \quad B = \sum_{k=1}^{n-1} \tilde{C}_k e_{k+1, k}
\]

(2.4)

where we define the matrix elements \((e_{ij})_{kl} = \delta_{ik} \delta_{jl}\) and

\[
\alpha_k = \frac{n + 1}{2} - k, \quad \tilde{C}_k = \sqrt{[k]_q[n - k]_q}, \quad [k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}.
\]

(2.5)

(b) Another realization of the spin \( s \) representation, equivalent to the latter one \((2.4)\), may
be given in terms of the Heisenberg algebra—essentially in terms of difference operators. Let $X$ and $Z$ be operators satisfying the following commutation relation

$$XZ = q\, Z\, X, \quad q = e^{i\mu}. \quad (2.6)$$

It is clear that $X$ and $Z$ can be realized in terms of the Heisenberg algebra elements i.e.,

$$X = \pm e^{\hat{p}}, \quad Z = \pm e^{\hat{q}}, \quad \text{and} \quad [\hat{p}, \hat{q}] = i\mu, \quad (2.7)$$

then the generators $A, B, C, D$ may be expressed as:

$$A = q^{-s}X, \quad D = q^{s}X^{-1}, \quad B = -\frac{Z^{-1}}{q - q^{-1}}(X^{-1} - X), \quad C = \frac{Z}{q - q^{-1}}(q^{2s}X^{-1} - q^{-2s}X). \quad (2.8)$$

Finally in the 'space' representation $|x\rangle \in V(R)$ of the Heisenberg algebra one defines

$$\hat{p}|x\rangle = i\mu \frac{\partial}{\partial x}|x\rangle, \quad \hat{q}|x\rangle = x|x\rangle. \quad (2.9)$$

then it is clear that for any function of $z = e^X$

$$X^{\pm 1} F(z) = F(q^{\pm 1}z). \quad (2.10)$$

Recall that our objective here is to study the spin $s$ XXZ chain with non diagonal boundaries. As is well known to build the model one first needs to construct the open transfer matrix, defined as \[t(\lambda) = \text{tr}_0 \left\{ K^+(\lambda) \ T(\lambda) \ K^-(\lambda) \ \hat{T}(\lambda) \right\} = \text{tr}_0 \left\{ K^+(\lambda) \ T(\lambda) \right\} \quad \text{where} \]

$$T(\lambda) = L_{01}(\lambda) \ldots L_{0N}(\lambda), \quad \hat{T}(\lambda) = L_{0N}(\lambda) \ldots L_{01}(\lambda) \quad (2.11)$$

$L$ is given by \[2.2\] and also satisfies $L(\lambda) \ L(-\lambda) = 2 \sinh \mu(\lambda + is + \frac{i}{2}) \sinh \mu(-\lambda + is + \frac{i}{2})$. $K^\pm$ are solutions of the reflection equation \[\text{[1]}\]:

$$R_{12}(\lambda_1 - \lambda_2) \ K_1(\lambda_1) \ R_{21}(\lambda_1 + \lambda_2) \ K_2(\lambda_2) = K_2(\lambda_2) \ R_{12}(\lambda_1 + \lambda_2) \ K_1(\lambda_1) \ R_{21}(\lambda_1 - \lambda_2) \quad (2.12)$$

The general solution is a $2 \times 2$ matrix with entries \[26, 27\] :

$$K_{11}(\lambda) = \sinh \mu(-\lambda + i\xi), \quad K_{22}(\lambda) = \sinh \mu(\lambda + i\xi)$$

$$K_{12}(\lambda) = \kappa q^\theta \sinh(2\mu\lambda), \quad K_{21}(\lambda) = \kappa q^{-\theta} \sinh(2\mu\lambda). \quad (2.13)$$

Here we shall consider $K^+ = K(-\lambda - i; \xi^+, \kappa^+, \theta^+)$, and $K^-(\lambda) = K(\lambda; \xi^-, \kappa^-, \theta^-)$. Using the fact that $T$ satisfies the reflection equation one can show the transfer matrix \[2.11\] provides a family of commuting operators \[2\]: $[t(\lambda), t(\lambda')] = 0$.  

3
3 Bethe states and spectrum

We can now come to our main aim which is the derivation of the spectrum and Bethe states for the generic open transfer matrix (2.11). We shall apply in what follows the approach developed in [13] in order to deduce the eigenstates, the spectrum, and the Bethe ansatz equations of the spin \( s \) XXZ chain with non-diagonal boundaries. Define the gauge transformed \( L \)-operators

\[
\tilde{L}_n(m|\lambda) = \tilde{M}^{-1}_{m_n-1} (\lambda) L_n(\lambda) \tilde{M}_{m_n}(\lambda) \equiv \begin{pmatrix} \tilde{\alpha}_n & \tilde{\beta}_n \\ \tilde{\gamma}_n & \tilde{\delta}_n \end{pmatrix},
\]

\[
S_n(m|\lambda) = M_{m_n}^{-1} (-\lambda) L_n^{-1} (-\lambda) M_{m_n-1} (-\lambda) \equiv \begin{pmatrix} \tilde{\alpha}'_n & \tilde{\beta}'_n \\ \tilde{\gamma}'_n & \tilde{\delta}'_n \end{pmatrix}, \quad m_n = m + \sum_{k=1}^{n} g_k (3.1)
\]

where the local gauge transformations are defined as (for a more detailed description of the method see [13])

\[
M_n(\lambda) = \begin{pmatrix} e^{-\mu \lambda x_n} & e^{-\mu \lambda y_n} \\ 1 & 1 \end{pmatrix}, \quad \tilde{M}_n(\lambda) = \begin{pmatrix} e^{-\mu \lambda x_{n+1}} & e^{-\mu \lambda y_{n-1}} \\ 1 & 1 \end{pmatrix}
\]

(3.2)

\[
x_n = -ie^{-i\mu(\beta+\gamma)} e^{-i\mu m}, \quad y_n = -ie^{-i\mu(\beta-\gamma)} e^{i\mu m}, \text{ and } \beta, \gamma \text{ are } \lambda \text{ independent } \mathbb{C} \text{ numbers.}
\]

Note that \( g_n \) depends on the choice of representation and it will be specified later on. For the spin \( \frac{1}{2} \) representation in particular \( g = 1 \) [13] (for a relevant discussion see also [18]). The transfer matrix (2.11) can be rewritten with the help of the aforementioned gauge transformations (3.1) as (see also Appendix and [13] [18])

\[
t(\lambda) = tr_0 \{ \tilde{K}^+ (\lambda) \tilde{T}(\lambda) \},
\]

(3.3)

where \( \tilde{K}^\pm, \tilde{T} \) are the gauge transformed matrices (see Appendix).

To diagonalize the transfer matrix (2.11) one first needs a suitable pseudo-vacuum state of the general form:

\[
\Omega^{(m)} = \otimes_{n=1}^{N} \varpi_n^{(m)}
\]

(3.4)

\( \varpi_n^{(m)} \) is the local pseudo-vacuum annihilated by the action of the lower left elements of the transformed \( L \) matrices (3.1), i.e.

\[
\tilde{\gamma}_n, \tilde{\gamma}'_n \varpi_n^{(m)} = 0.
\]

(3.5)

From the action of \( \tilde{\gamma}_n \) on the local pseudo-vacuum the following constraint is obtained

\[
\left[ -x_{m_n+1} (e^{\mu \lambda} q^{\frac{1}{2}} A_n - e^{-\mu \lambda} q^{\frac{1}{2}} D_n) + x_{m_n-1+1} (e^{\mu \lambda} q^{\frac{1}{2}} D_n - e^{-\mu \lambda} q^{\frac{1}{2}} A_n) + e^{-\mu \lambda} x_{m_n+1} x_{m_n-1+1} (q - q^{-1}) C_n - e^{\mu \lambda} (q - q^{-1}) B_n \right] \varpi_n^{(m)} = 0,
\]

(3.6)
a similar constraint arises from the action of $\tilde{\gamma}'_n$ on the local pseudo-vacuum. Notice that the transformed non-diagonal elements of $[A.3], [A.7]$ acting on the pseudo-vacuum state $(3.4)$ must satisfy:

$$\tilde{K}^+ (m^0 | \lambda) = \tilde{K}^+ (m^0 | \lambda) = 0, \quad \tilde{K}^- (m_0 | \lambda) = 0$$

(3.7)

where the integers $m^0$ and $m_0$ are associated to the left and right boundaries respectively. Solving the latter constraints (3.7) one can fix the relations between the left and right boundary parameters (see also [11, 13, 18]). Note however that the problem was solved in [12] in the more general case with no constraints between left and right boundary parameters, however only for $q$ root of unity.

We may now specify the exact reference state for the spin $s$ XXZ model in both realizations (2.4) and (2.8).

(a) We associate each site of the chain with the spin $s_n$ representation and we define the local pseudo-vacuum state as

$$\mathcal{v}_n^{(m)} = \sum_{i=1}^{j_n} w^{(m,n)}_i f^{(n)}_l$$

(3.8)

$f^{(n)}_l$ is the $j_n = 2s_n + 1$ column vector with zeroes everywhere apart from the $l^{th}$ position.

Solving the constraint (3.6) on the local pseudo-vacuum one can specify the value of $g_n = j_n - 1$ appearing in the local gauge transformations, and we also acquire recursion relations among the coefficients $w^{(m,n)}_i$, which read as:

$$w^{(m,n)}_i = \frac{q^{j_{n-1}} (q^{j_{n-l+1}} - q^{-j_{n-l+1}})}{(q - q^{-1}) \tilde{C}_{l-1} x_{m_{n-1}+1}} w^{(m,n)}_{l-1}$$

(3.9)

with normalization $w^{(m,n)}_1 = 1$. The solution of the later recursion formula provides the exact form of the coefficients:

$$w^{(m,n)}_i = \left( \frac{q^{j_{n-1}}}{x_{m_{n-1}+1}} \right)^{l-1} \prod_{k=1}^{l-1} \frac{|j_n - k| q}{\tilde{C}_k}.$$  

(3.10)

(b) Let us now consider the difference realization of the spin $s$ representation (2.8). In this case the pseudo-vacuum will be expressed as a polynomial of the local spin parameter associated to the $n^{th}$ site of the chain $z^{(n)}$

$$\mathcal{v}_n^{(m)} = \phi^{(m)}(z^{(n)}) = \sum_{l=1}^{j_n} w^{(m,n)}_i (z^{(n)})^{l-1}.$$  

(3.11)
As in the previous case the aim is to determine the coefficients of the polynomial. The solution of the constraint (3.6) this time leads to sets of difference equations for the spin variables $z^{(n)}$, namely

$$
(z^{(n)}x_{m_n+1} q^{-\frac{ln}{2}+1} + 1) \phi^{(m)}(qz^{(n)}) - (z^{(n)}x_{m_{n-1}+1} q^{\frac{ln}{2}} + 1) \phi^{(m)}(q^{-1}z^{(n)}) = 0. \tag{3.12}
$$

It is worth stressing that difference equations of the type (3.12) occur also in the study of the Azbel-Hofstadter problem (see e.g. [28, 29]). Although for our purposes here relations (3.12) are sufficient we derive, for the sake of completeness, the explicit form of the polynomial. In any case, the derivation of the pseudo-vacuum is of great significance per se leading to the explicit form of the Bethe states as we shall see. In addition we wish to stress the connection of these polynomials with the $q$-hypergeometric series. Let us illuminate this point a bit further. Equation (3.12) also provides recursion relations among $w^{(m,n)}$, whose solution gives the coefficients:

$$
w^{(m,n)} = (x_{m_{n-1}+1} q^{-\frac{ln}{2}+1})^{l-1} \prod_{k=1}^{l-1} \frac{[jn - k]_q}{[k]_q}. \tag{3.13}
$$

It will be instructive at this stage to express the polynomial $\phi^{(m)}$ in terms of $q$-hypergeometric series (see e.g. [30] and references therein). For that purpose it is convenient to rescale the local spin variables as: $\bar{z}^{(n)} = -q^{\frac{ln}{2}+1}x_{m_{n-1}+1}z^{(n)}$, then it follows from (3.13) that the new coefficients of the polynomial, written in terms of $\bar{z}^{(n)}$, are given by

$$
w^{(m,n)} = \prod_{k=0}^{l-2} \frac{1 - q^{-2jn+2} q^{2k}}{1 - q^{2} q^{2k}}. \tag{3.14}
$$

Let us also introduce some useful notation. Set $\bar{q} = q^2$, and also define the $q$-shifted factorials as,

$$(a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad \text{and} \quad (a_1, a_2, \ldots, a_k; q)_n = \prod_{j=1}^{k} (a_j; q)_n. \tag{3.15}
$$

Now we may define the so called $q$-hypergeometric functions in the following manner

$$
s+1 \Phi_s(a_1, a_2, \ldots, a_s+1; b_1, \ldots, b_s; q, z) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_s+1; q)_k}{(b_1, \ldots, b_s, q; q)_k} z^k, \tag{3.16}
$$

it is thus clear that the polynomial (3.13) may be written as a $q$-hypergeometric series

$$
\phi^{(m)}(\bar{z}^{(n)}) = \sum_{k=0}^{jn-1} \frac{(\bar{q}^{-jn+1}; q)_k}{(\bar{q}; q)_k} (\bar{z}^{(n)})^k = \Phi_0(\bar{q}^{-jn+1}; \bar{q}, \bar{z}^{(n)}). \tag{3.17}
$$
This series obviously terminates because of the presence of the $q^{-j_{n+1}}$ term in the $q$-shifted factorial appearing in the numerator of (3.13).

We have specified the exact form of the reference state for both realizations of the spin $s$ XXZ model (2.4), (2.8). To derive the spectrum of the transfer matrix it is also necessary to consider the actions of the transformed diagonal elements on the pseudo-vacuum. Indeed it is relatively easy to show by simply using the transformations (3.1), and the recursion relations (3.13) that the action of the diagonal elements on the pseudo-vacua, derived above, takes the general form:

$$\tilde{\delta}_n^{(m)} \omega_n^{(m)} = a_n(\lambda) \omega_n^{(m+1)},$$

$$\tilde{\delta}_n^{(m)} \omega_n^{(m)} = \frac{x_{m+1} - y_{m+1}}{x_{m-1} - y_{m-1}}b_n(\lambda) \omega_n^{(m-1)},$$

$$\tilde{\delta}_n^{(m)} \omega_n^{(m)} = a'_n(\lambda) \omega_n^{(m-1)},$$

$$\tilde{\delta}_n^{(m)} \omega_n^{(m)} = \frac{x_{m-1} - y_{m-1}}{x_{m} - y_{m}}b'_n(\lambda) \omega_n^{(m+1)}$$

where the values of $a_n$, $b_n$, $a'_n$, $b'_n$ for each realization are given below

(a) \quad a_n(\lambda) = q^{-j_{n+1}} \sinh \mu(\lambda + j_n \frac{i}{2}), \quad b_n(\lambda) = q^{j_{n-1}} \sinh \mu(\lambda - j_n \frac{i}{2} + i)$

\quad $a'_n(\lambda) = q^{j_{n-1}} \sinh \mu(\lambda + j_n \frac{i}{2}), \quad b'_n(\lambda) = q^{-j_{n+1}} \sinh \mu(\lambda - j_n \frac{i}{2} + i)$

(b) \quad $a_n(\lambda) = \sinh \mu(\lambda + j_n \frac{i}{2}), \quad b_n(\lambda) = \sinh \mu(\lambda - j_n \frac{i}{2} + i)$

\quad $a'_n(\lambda) = \sinh \mu(\lambda + j_n \frac{i}{2}), \quad b'_n(\lambda) = \sinh \mu(\lambda - j_n \frac{i}{2} + i).$ (3.19)

Our aim now is to solve the following eigenvalue problem

$$t(\lambda) \Psi = \Lambda(\lambda) \Psi \quad \text{where} \quad \Psi = \mathcal{B}_{m^0-2}(\lambda_1) \cdots \mathcal{B}_{m^0-2M}(\lambda_M) \Omega^{(m)}$$ (3.20)

$\Psi$ is of course the general Bethe ansatz state.

The transfer matrix eigenvalues may be deduced by virtue of algebraic relations emerging from the reflections equation, which omitted here for brevity (see e.g [13, 18]). The exchange relations, arising from the reflection equation, are identical to the ones presented in [13]. The action of the diagonal elements on the pseudo-vacuum is naturally modified, depending explicitly on the choice of the representation. It may be shown that the state $\Psi$ is indeed an eigenstate of the transfer matrix if we impose $m \equiv m^0 - 2M$, and then it follows that $m_0 = Ng + m^0 - 2M$. Let us also define

$$f_n(\lambda) = a_n(\lambda)a'_n(\lambda), \quad h_n(\lambda) = b_n(\lambda) b'_n(\lambda)$$

$$K_1^+(m|\lambda) = \tilde{K}_1^+(m|\lambda) + \frac{\sinh \mu(i\gamma + m + 2\lambda + i) \sinh i\mu}{\sinh i\mu(1 + m + \gamma) \sinh (2\lambda + i)} \tilde{K}_1^+(m|\lambda),$$

$$K_4^+(m|\lambda) = \frac{\sinh i\mu(m + \gamma) \sinh i\mu}{\sinh i\mu(1 + m + \gamma) \sinh (2\lambda + i)} \tilde{K}_4^+(m|\lambda),$$ (3.21)
\( \tilde{K}_{1,4}^± \) are given in the Appendix. Finally the spectrum may be written as

\[
\Lambda(\lambda) = \left( K^+_1(m^0|\lambda)K^-_1(m^0|\lambda) \prod_{n=1}^N f_n(\lambda) \prod_{j=1}^M \frac{\sinh \mu(\lambda + \lambda_j) \sinh \mu(\lambda - \lambda_j - i)}{\sinh \mu(\lambda + \lambda_j + i) \sinh \mu(\lambda - \lambda_j)} \right) + \left( K^+_4(m^0|\lambda)K^-_4(m^0|\lambda) \prod_{n=1}^N h_n(\lambda) \prod_{j=1}^M \frac{\sinh \mu(\lambda + \lambda_j + 2i) \sinh \mu(\lambda - \lambda_j + i)}{\sinh \mu(\lambda + \lambda_j + i) \sinh \mu(\lambda - \lambda_j)} \right) \tag{3.22}
\]

provided that certain unwanted terms appearing in the eigenvalue expression are vanishing. This is achieved as long as \( \lambda_i \)'s satisfy the Bethe ansatz equations, which are written below in the familiar form:

\[
\frac{K^+_1(m^0|\lambda_i - i\frac{\pi}{2}) K^-_1(m^0|\lambda_i - i\frac{\pi}{2})}{K^+_4(m^0|\lambda_i - i\frac{\pi}{2}) K^-_4(m^0|\lambda_i - i\frac{\pi}{2})} \prod_{n=1}^N f_n(\lambda_i - i\frac{\pi}{2}) \prod_{j=1}^M \frac{\sinh \mu(\lambda_i - \lambda_j + i) \sinh \mu(\lambda_i + \lambda_j + i)}{\sinh \mu(\lambda_i - \lambda_j - i) \sinh \mu(\lambda_i + \lambda_j - i)} = \prod_{j=1}^M \frac{\sinh \mu(\lambda_i + \lambda_j + i)}{\sinh \mu(\lambda_i - \lambda_j - i)} \sinh \mu(\lambda_i + \lambda_j - i) \tag{3.23}
\]

\( K^±_{1,4}, f_n, h_n \) are defined in (3.21), and this completes the derivation of the spectrum and Bethe states of the spin \( s \) XXZ chain with non-diagonal boundaries.

## 4 Discussion

Let us briefly recall the main findings of the present investigation. We have been able to derive the spectrum and Bethe ansatz for the open spin \( s \) XXZ chain with special non-diagonal boundaries and for generic values of the anisotropy parameter. Note that the spectrum (3.22), and Bethe ansatz equations (3.23) for the spin \( s \) XXZ model with non-diagonal boundaries are similar with the ones found in [16], although the investigation in [16] was restricted for \( q \) root of unity. More importantly in the present study we were able to specify the Bethe states of the model by means of suitable local gauge transformations. The first important step was the derivation of an appropriate reference arising as a solution of certain difference equations (3.9), (3.12). Then within the algebraic Bethe ansatz framework we built the corresponding Bethe states and derived the spectrum and Bethe ansatz equations.

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A Appendix

The local gauge transformations may be expressed as:

\[ M_n(\lambda) = \left( X_n(\lambda), \ Y_n(\lambda) \right), \quad \tilde{M}_n(\lambda) = \left( X_{n+1}(\lambda), \ Y_{n-1}(\lambda) \right) \]  

(A.1)

and

\[ X_n(\lambda) = \begin{pmatrix} e^{-\mu \lambda} x_n \\ 1 \end{pmatrix}, \quad Y_n(\lambda) = \begin{pmatrix} e^{-\mu \lambda} y_n \\ 1 \end{pmatrix}, \quad x_n = x_0 e^{-i\mu n}, \quad y_n = y_0 e^{i\mu n}. \]  

(A.2)

It is also convenient to introduce the matrices

\[ M_n^{-1}(\lambda) = \begin{pmatrix} \tilde{Y}_n(\lambda) \\ \tilde{X}_n(\lambda) \end{pmatrix}, \quad \tilde{M}_n^{-1}(\lambda) = \begin{pmatrix} \tilde{Y}_{n-1}(\lambda) \\ \tilde{X}_{n+1}(\lambda) \end{pmatrix}, \]  

(A.3)

with

\[ \tilde{X}_n(\lambda) = \frac{1}{x_n - y_n} \begin{pmatrix} 1 & -e^{\mu \lambda} \\ -e^{-\mu \lambda} & x_n \end{pmatrix}, \quad \tilde{Y}_n(\lambda) = \frac{1}{x_n - y_n} \begin{pmatrix} e^{\mu \lambda} & -1 \\ 1 & -y_n \end{pmatrix}, \]

\[ \tilde{X}_{n+1}(\lambda) = \frac{1}{x_{n+2} - y_n} \begin{pmatrix} 1 & -e^{\mu \lambda} \\ -e^{-\mu \lambda} & x_n \end{pmatrix}, \quad \tilde{Y}_{n+1}(\lambda) = \frac{1}{x_{n+2} - y_n} \begin{pmatrix} e^{\mu \lambda} & -1 \\ 1 & -y_n \end{pmatrix}. \]  

(A.4)

The transformed \( K^\pm \) and \( T \) matrices are expressed as:

\[ \tilde{K}^+(m|\lambda) = \begin{pmatrix} \tilde{K}_1^+(m|\lambda) & \tilde{K}_2^+(m|\lambda) \\ \tilde{K}_3^+(m|\lambda) & \tilde{K}_4^+(m|\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{Y}_m(-\lambda) K^+(\lambda) X_m(\lambda) & \tilde{Y}_m(-\lambda) K^+(\lambda) Y_{m-2}(\lambda) \\ \tilde{X}_m(-\lambda) K^+(\lambda) X_{m+2}(\lambda) & \tilde{X}_m(-\lambda) K^+(\lambda) Y_m(\lambda) \end{pmatrix} \]  

(A.5)

and

\[ \tilde{T}(\lambda) = \begin{pmatrix} A_m(\lambda) & B_m(\lambda) \\ C_m(\lambda) & D_m(\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{Y}_{m-2}(\lambda) T(\lambda) X_m(-\lambda) & \tilde{Y}_m(\lambda) T(\lambda) Y_m(-\lambda) \\ \tilde{X}_m(\lambda) T(\lambda) X_m(-\lambda) & \tilde{X}_{m+2}(\lambda) T(\lambda) Y_m(-\lambda) \end{pmatrix}. \]  

(A.6)

Similarly to [13], one defines the transformed \( K^- \) matrix as

\[ \tilde{K}^-(l|\lambda) = \begin{pmatrix} \tilde{K}_1^-(l|\lambda) & \tilde{K}_2^-(l|\lambda) \\ \tilde{K}_3^-(l|\lambda) & \tilde{K}_4^-(l|\lambda) \end{pmatrix} = \begin{pmatrix} \tilde{Y}_{l-2}(\lambda) K^-(\lambda) X_l(-\lambda) & \tilde{Y}_l(\lambda) K^-(\lambda) Y_{l-2}(\lambda) \\ \tilde{X}_l(\lambda) K^-(\lambda) X_l(-\lambda) & \tilde{X}_{l+2}(\lambda) K^-(\lambda) Y_l(-\lambda) \end{pmatrix} \]  

(A.7)

with \( l = m + \sum_{n=1}^N g_n \).
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