Wannier functions for quasi-periodic finite-gap potentials

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Abstract

In this paper we consider Wannier functions of quasi-periodic g-gap (g ≥ 1) potentials and investigate their main properties. In particular, we discuss the problem of averaging underlying the definition of Wannier functions for both periodic and quasi-periodic potentials and express Bloch functions and quasi-momenta in terms of hyperelliptic σ functions. Using this approach we derive a power series expansion of the Wannier function for quasi-periodic potentials valid at |x| ≃ 0 and an asymptotic expansion valid at large distance. These functions are important for a number of applied problems.

1 Introduction

The Schrödinger operator with periodic potentials is characterized by two distinguished complete sets of orthogonal functions: the Floquet-Bloch functions, introduced by Floquet (1882) and F.Bloch (1928), and Wannier functions, introduced by G.Wannier (1937). The Schrödinger operator with algebraically integrable potentials allows one to construct Bloch (Floquet-Bloch in one dimension) and Wannier functions and describe their properties with unprecedented completeness. While Bloch functions were studied for a long time and there are many reviews and monographs on the topic, the study of the Wannier functions corresponding to finite-gap potentials has begun only recently [10].

The aim of the paper is to further expand these studies by presenting a definition and a complete description of the properties of Wannier functions for g-gap potentials with g ≥ 1. To this regard, we remark that the usual Wannier functions considered in Condensed Matter Physics are defined for periodic potentials. By considering Wannier functions for g-gap potentials, which are in general quasi-periodic functions, we are generalizing the usual Wannier functions to quasi crystals i.e. to a quasi periodic lattices, this being a first main
result of the paper. This circumstance leads to another interesting problem. As well known, in order to define Wannier functions one must use an averaging procedure which differs essentially for the periodic and quasi-periodic cases. Although the averaging problem has been considered for some algebraically integrable system (see e.g. [21], [20], [31]), to the best of our knowledge, it has not been discussed with sufficient completeness and not linked to Wannier functions, this being indeed a second main result of the paper. The study of the averaging problem in connection with integrable systems, however, goes beyond the frames of this paper. Here we shall only remind that averaging procedures also arise in Krylov-Bogoliubov-Mitropolis perturbation theory of integrable systems, in the study of Seiberg-Witten theory, in multi-matrix models etc. We hope to expand this direction of our study in the future.

The theory of hyperelliptic curves of different genera leads to a rich structure of objects which inherits the rich structure of the moduli spaces of the corresponding hyperelliptic curves. For example, in the case of elliptic curves $g = 1$, periodic and ergodic cases coincide because there exists only one frequency. In the case of the two-gap potential the spectral variety is given by the ultra-elliptic $g = 2$ curve and periodic and ergodic cases are rather different. Moreover, periodic case admits further specialization to the elliptic periodicity. In the cases $g \geq 3$ a new phenomenon appears: the curve admits so-called singular half-periods which leads to certain complications of the theory.

Technically speaking, our development is based on the realization of hyperelliptic functions in terms of multi-variable $\sigma$-function. This realization represents a natural generalization of the Weierstrass theory of elliptic functions to hyperelliptic functions of higher genera. Higher genera theory was developed by K. Weierstrass and F. Klein and its exposition is fixed in classical monograph of H. Baker [2]; recent results in the area are given in [12, 13, 14, 15] whilst various applications see e.g. in [18, 36, 34, 35, 19, 5, 6].

The restricted length of the paper does not permit us to give proofs of the theorems for which we plan a separate publication.

The paper is organized as follows. In the Section 2 we describe quasi-periodic potentials from the pure spectral point of view. We show that the basic objects like finite-gap potential, Weyl and Bloch functions, which are defined initially on the complex plane can be reasonably lift to hyperelliptic Riemann surfaces whose genus coincides with the number of gaps in the spectrum. In the Section 3 we collect necessary results from the theory of hyperelliptic $\sigma$-function such as hyperelliptic $\wp$ and $\zeta$-functions and differential relations between them, as well as, addition theorems. We also describe certain sub-varieties of the $\theta$-divisor. In Section 4 we show how to evaluate Bloch function and quasi-momentum in terms of multidimensional $\sigma$ functions. In section 5 we discuss periodic and quasi-periodic potentials and show how to compute averages of squared Bloch functions in both cases. In Section 6 we discuss Wannier functions for periodic and quasi-periodic potentials and derive a power series expansion of the Wannier function for quasi-periodic potentials valid at $|x| \approx 0$ and an asymptotic expansion valid at large distance. Finally in Section 7 the main conclusion and perspective of future work are presented.
The Schrödinger operator with quasi-periodic and finite-gap potential

2.1 Quasi-periodic potential

The function \( u(x) \) from a Banach space \( C_b(\mathbb{R}) \) of continuous bounded functions is called almost periodic if the set \( \{ T_x u(\cdot) \mid x \in \mathbb{R} \} \), where \( T_x u(\cdot) = u(\cdot + x) \), is relatively compact in \( C_b(\mathbb{R}) \). A closure \( \Gamma \) of this set is a compact in metrizable Abelian group. On the set \( \Gamma \) there exists a normalized Haar measure \( \mu \) which turns out to be \( T_x \)-invariant and ergodic. Thus, each almost–periodic function generates a probability space \( (\Gamma,\mu,T_x) \). The operation of averaging on this space is given by

\[
\langle f(u) \rangle = \lim_{x \to \infty} \frac{1}{x} \int_0^x f(T_x u) dx = \int_{\Gamma} f(u) \mu(du).
\]

By means of the differential expression

\[
L(u) = -\partial_{xx} + u(x)
\]

we define for \( u \in \Gamma \) a Schrödinger operator in \( L^2(\mathbb{R}) \), which is essentially self–adjoint. Let \( \lambda \in \mathbb{C} \) and \( c(x,\lambda), s(x,\lambda) \) denote solutions of the equation \( L\varphi = \lambda\varphi \) with the initial data \( c(0,\lambda) = 1, \ c'(0,\lambda) = 0, \ s(0,\lambda) = 0, \ s'(0,\lambda) = 1 \). The functions \( c(x,\lambda), s(x,\lambda) \) are integral functions of order \( 1/2 \) with respect to \( \lambda \) and continuous with respect to all variables \( (x,\lambda) \). The limits

\[
w_{\pm}(\lambda) \equiv w_{\pm}(\lambda, u(\cdot)) = \mp \lim_{x \to \infty} \frac{c(x,\lambda)}{s(x,\lambda)}
\]

exist and are called the Weyl functions. Besides these, we shall also employ the Weyl functions \( w_{\pm}(x,\lambda) \equiv w_{\pm}(\lambda, T_x u(\cdot)) \). It is well known that all Weyl functions are holomorphic in \( \mathbb{C}_+ = \{ \lambda \in \mathbb{C} \mid \text{Im} \lambda > 0 \} \), map \( \mathbb{C}_+ \to \mathbb{C}_+ \), and if they have zeros or poles on the real axis, these could be simple only \( [16] \). By means of the Weyl functions we can define the functions

\[
\psi(x,\lambda) = c(x,\lambda) \pm w_{\pm}(x,\lambda)s(x,\lambda) = \exp \left( \pm \int_0^x w_{\pm}(y,\lambda) dy \right),
\]

which, for each \( \lambda \in \mathbb{C}_+ \), belong to \( L^2(\mathbb{R}_\pm) \), where \( \mathbb{R}_+ = [0,\infty) \), and \( \mathbb{R}_- = (-\infty,0) \), respectively. By definition, the functions \( \psi_{\pm}(x,\lambda) \) satisfy the equation

\[
-\partial_{xx} \psi_{\pm}(x,\lambda) + (u(x) - \lambda) \psi_{\pm}(x,\lambda) = 0.
\]

By substituting (2.3) into this equation, we have the following equation for the Weyl functions:

\[
\pm \partial_x w_{\pm}(x,\lambda) + w_{\pm}^2(\lambda, x) + \lambda - u(x) = 0.
\]
Following [28], we define the Floquet function

\[ f(x, \lambda) = \frac{1}{2}(w_+(x, \lambda) + w_-(x, \lambda)). \]  

Let \( \lambda = \xi + i\eta \). The Floquet function has a finite limit almost everywhere at \( \xi \in \mathbb{R}, \eta \downarrow 0 \)

\[ f(\xi + i0, x) = -l(x, \xi) + i\pi n(x, \xi), \]  

(2.7)

where \( l(\xi) \) is the Lyapunov exponent, and \( n(\xi) \) is the number of states.

The number of states, \( n(x, \xi) \), determines the spectrum \( \Sigma(u) \) of the operator \( L(u) \): for a.e. \( u \in \Gamma \) the spectrum is a set of growth points of the number of states [38],

\[ \Sigma(u) = \text{supp}(dn). \]

The Lyapunov exponent \( l(x, \xi) \) determines an absolutely continuous spectrum i.e. a closed set of spectral points with zero Lyapunov exponent,

\[ \Sigma_{a.c.}(u) = \{ \xi \in \mathbb{R} | l(x, \xi) = 0 \}, \]

where the closure is with respect to Lebesgue measure [38, 29]. We now proceed to discuss a special subset of quasi-periodic potentials, the so called finite–gap potentials (more details see in [42], [7], [23]).

2.2 The finite-gap potential

Theory of finite-gap potentials of the Schrödinger operator has long history which goes up to Hermite, Halphen, Darboux. Recent development was stimulated by the soliton theory and achieved in the works made at the middle of seventieth by Novikov, Dubrovin and Novikov, Its and Matveev and Krichever (see e.g. [42] and references therein).

Definition 2.1 The almost–periodic function \( u(x) \) is called a finite–gap potential if the spectrum of the Schrödinger operator \( L(u) = -\partial_x^2 + u(x) \) is the union of a finite set of segments of Lebesgue (double absolutely continuous) spectrum \([E_1, E_2] \cup [E_3, E_4] \cup \cdots \cup [E_{2g+1}, \infty] \), where boundaries of the bands are supposed to be real and ordered as: \( E_1 < E_2 \ldots, < E_{2g+1} < +\infty \).

Starting from this definition, we can derive explicit expressions and basic properties of finite–gap potentials, Weyl and Bloch functions. We shall show that the adequate language to describe these object is the theory of hyperelliptic Riemann surfaces. To this end we fix the Riemann surface \( X \) of genus \( g \) of the algebraic curve given by the equation

\[ \mu^2 = R_{2g+1}(\lambda), \quad R_{2g+1} = 4 \prod_{k=1}^{2g+1} (\lambda - E_k). \]  

(2.8)

We denote the running coordinate of the curve \( X \) as \( P = (\lambda, \mu) \) and coordinate of branch points as \( (E_k, 0), k = 1, \ldots, 2g + 1 \).
Theorem 2.1 The Weyl function of the finite-gap potential \( u(x) \) is defined on the Riemann surface \( X \) of the hyperelliptic curve

\[
\mu^2 = R_{2g+1}(\lambda), \quad R_{2g+1}(\lambda) = 4 \prod_{i=1}^{2g+1} (\lambda - E_i) \tag{2.9}
\]

by the formula

\[
w(x, P) = \frac{1}{2} \left( \frac{\partial_x S(x, P)}{S(x, P)} + \frac{i\mu}{S(x, P)} \right), \tag{2.10}
\]

where

\[
S(x, \lambda) = \prod_{k=1}^{g} (\lambda - \lambda_k(x)) \tag{2.11}
\]

and points \((\lambda_k(x), \mu_k(x))\), \(k = 1, \ldots, g\), are distinct points of the curve \( X \) evaluated accordingly with changing of \( x \).

For our further discussion we need the expression of the number of states, \( n(\xi) \), in terms of the function \( w_+(x, \xi) \),

\[
n(\xi) = \frac{1}{\pi} \langle w_+(x, \xi) \rangle = \frac{1}{2\pi} \langle \text{Im} \ w_+(x, \xi) \rangle = \frac{1}{2\pi} \int_\xi^{\xi^*} \frac{1}{\text{Im} \ w_+(x, \xi)} \ d\xi.
\]

For a finite-gap potential \( u(x) \) we have

\[
\text{Im} \ w_+(x, \xi) = \frac{\sqrt{R_{2g+1}(\xi)}}{2S(x, \xi)}.
\]

Therefore in this case the number of states is

\[
n(\xi) = \frac{1}{\pi} \left\langle \frac{\sqrt{R_{2g+1}(\xi)}}{2S(x, \xi)} \right\rangle = \frac{1}{\pi} \int_\xi^{\xi^*} \left\langle \frac{S(x, \xi)}{\sqrt{R_{2g+1}(\xi)}} \right\rangle \ d\xi.
\]

The number of states and the wave number for all finite-gap potentials are connected as follows

\[
n(\xi) = \frac{1}{\pi} k(\xi).
\]

A direct consequence of the above discussion is the following theorem.

Theorem 2.2 The quasi-momentum of eigenfunction can be defined by the expression

\[
k(P) = \langle w(x, P) \rangle = \left\langle \frac{\mu}{S(x, P)} \right\rangle = \int_{P_0}^{P} \frac{\langle S(x, P) \rangle}{\mu} \ d\lambda, \tag{2.12}
\]

where \( P = (\lambda, \mu) \) and \( P_0 = (\lambda_0, \mu_0) \) are points of the hyperelliptic curve \( \mu^2 = R_{2g+1}(\lambda) \).
The last integral in Eq. (2.12) is the Schwarz-Cristoffel integral which maps the l.h.s. half-plane of complex $\lambda$-plane, to a rectangle in the complex $k$-plane [33, 26]. We shall further interpret this integral as a second kind Abelian integral on $X$ over meromorphic differential,
\[ dX = \frac{\langle S(x, P) \rangle}{\mu} d\lambda, \] (2.13)
which we shall call differential of quasi-momentum.

**Theorem 2.3** The eigenfunction of the $g$-gap potential, which is normalized by the condition
\[ \langle |\psi(x, P)|^2 \rangle = 1, \] (2.14)
is of the form
\[ \psi(x, P) = \sqrt{\frac{S(x, P)}{\langle S(x, P) \rangle}} \exp \left\{ \frac{i\mu}{2} \int_{x_0}^{x} \frac{dy}{S(y, P)} \right\}. \] (2.15)

We see that in order to define the quasi-momentum and normalize the eigenfunction we must fulfill the averaging. As a consequence we should say some words about the averaging for quasi- and almost-periodic functions since the potentials with finite number of gaps are quasi-periodic functions and the potentials with infinite number of gaps are almost-periodic functions.

### 2.3 Averaging for the finite-gap potentials: periodic and ergodic cases

Any almost-periodic function has a mean value
\[ \langle f \rangle = \lim_{L \to \infty} \frac{1}{L} \int_{0}^{L} f(x) dx. \]
This allows for any almost-periodic function to build a Fourier series
\[ f(x) \simeq \sum_{n} A_n \exp(i\lambda_n x), \quad A_n = \langle f(x) \exp(i\lambda_n x) \rangle. \]
The numbers $\lambda_n$ are designated as the Fourier frequencies and the numbers $A_n$ as the Fourier coefficients of the function $f(x)$.

We remind that a countable set of real numbers $\lambda_1, \ldots, \lambda_n, \ldots$ has a rational basis $\alpha_1, \alpha_2, \ldots$ if the numbers $\alpha_1, \alpha_2, \ldots$ are linearly independent and any number $\lambda_n$ can be presented as their finite linear combination with rational coefficients, i.e.
\[ \lambda_n = \sum_{k=1}^{S_n} r^n_k \alpha_k, \quad r^n_k \in \mathbb{Q}. \]
We say that the basis is finite if it is finite set, we say that the basis is integer if all numbers $r^n_k$ are integer numbers. If the Fourier frequencies of almost-periodic function have a finite and integer basis we designate the appropriate almost-periodic function as the quasi-periodic one. A quasi-periodic function with one period is pure periodic one.

Function $F(x_1, x_2, \ldots)$ of finite or countable set of variables, each of which admits all real values, is called limiting periodic if it is a uniform limit of periodic ones, i.e. if for any positive real number $\varepsilon$ we can point out such an integer positive number $n(\varepsilon)$ and such a periodic function $F_\varepsilon(x_1, x_2, \ldots, x_n)$ that

$$\sup_{-\infty < x_k, k = 1, 2, \ldots < +\infty} |F(x_1, x_2, \ldots) - F_\varepsilon(x_1, x_2, \ldots, x_n)| < \varepsilon.$$ 

It appears that for any almost-periodic function $f(x)$ there exists such a limiting periodic function $F(x_1, x_2, \ldots)$ of finite or countable set of variables such that

$$f(x) = F(x, x, \ldots).$$

In other words any almost-periodic function is a diagonal restriction of some limiting periodic function. The properties of the limiting periodic function $F(x_1, x_2, \ldots)$ depend essentially on the basis of the Fourier frequencies of the function $f(x)$. If the basis $\alpha_1, \alpha_2, \ldots$ of the almost-periodic function $f(x)$ is integer then the limiting periodic function $F(x_1, x_2, \ldots)$ is periodic with periods $2\pi/\alpha_1, 2\pi/\alpha_2, \ldots$. If the basis $\alpha_1, \alpha_2, \ldots$ of the almost-periodic function $f(x)$ is finite then the limiting periodic function $F(x_1, x_2, \ldots)$ depends on finite set of variables. If the basis $\alpha_1, \alpha_2, \ldots$ of the almost-periodic function $f(x)$ is finite and integer then the limiting periodic function $F(x_1, x_2, \ldots)$ is a periodic function on finite set of variables. Obviously, the mean value of the function $f(x)$ depends essentially on whether this function is periodic, quasi-periodic or almost-periodic. If the function $f(x)$ is periodic then the mean value of the function is just the average over the period. If the function $f(x)$ is quasi-periodic or almost-periodic then the mean value of the function is the average over the phase space, i.e. ergodicity takes place. This is illustrated with the following classical statement.

**Theorem 2.4 (Kronecker–Weyl)** Let $\lambda_k, k = 1, \ldots, n$ be real linearly independent numbers, $\theta_k, k = 1, \ldots, n$ be arbitrary real numbers, $\delta_k, k = 1, \ldots, n$ be arbitrary positive numbers. Let $\chi(x_1, x_2, \ldots, x_n)$ be a characteristic function of parallelepiped in $\mathbb{R}^n$ defined by inequalities

$$\theta_k - \delta_k < x_k < \theta_k + \delta_k, \quad k = 1, \ldots, n.$$ 

Then, uniformly in $L$ we have

$$\lim_{L \to \infty} \int_0^L \chi(\lambda_1 x - \theta_1, \ldots, \lambda_n x - \theta_n)dx = \pi^{-n} \delta_1 \ldots \delta_n,$$

where the function $\chi(x_1, x_2, \ldots, x_n)$ is the periodic continuation to the whole $\mathbb{R}^n$ in all variables $x_k, k = 1, \ldots, n$, with periods $2\pi$. 

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In the problem under consideration a finite-gap potential appears as a restriction to a straight line of some meromorphic function defined on a Jacobian. Of course on this line the potential must be real and non-singular. Depending on a direction of the line with respect to periods of the Jacobian this finite-gap potential can be periodic or quasi-periodic and this circumstance influences the averaging essentially. The averaging of quasi-periodic functions which are restrictions of the θ—functions was previously described in the literature (see e.g. [41]), and the averaging of solutions of integrable evolution equations (see e.g. [21], [20]). We shall use these results when discussing the averaging for the finite-gap potentials. We first shall formulate our problem in terms of the σ—functions.

We remark that it is convenient to use the language of σ—function in our foregoing discussions since all expressions in σ—functions are form-invariant with respect to the number of gaps of potential (or the genus of curve) and this fact is very useful for exposition. Keeping this in mind, we give in the next section a brief account of necessary facts from the theory of hyperelliptic σ—functions.

3 Hyperelliptic σ—functions

The theory of hyperelliptic σ—functions represents a many-dimensional generalization of the theory of elliptic functions and provides suitable language of our development.

3.1 Curve and differentials

Let \( X \) be the hyperelliptic curve given by the equation

\[
\mu^2 = 4 \lambda^{2g+1} + \sum_{i=0}^{2g} \alpha_i \lambda^i = 4 \prod_{k=1}^{2g+1} (\lambda - E_k) = R(\lambda),
\]

realized as a two sheeted covering over the Riemann sphere branched in the real points \((E_k, 0), k \in \mathbb{Z} = \{1, \ldots, 2g + 1\}\), with \(E_j \neq E_k\) for \(j \neq k\), and at infinity, \(E_{2g+2} = \infty\). The order of branch points is according to \(E_1 < E_2 < \ldots < E_{2g+1}\) (see Fig. 1).

A set of \(2g\) canonic holomorphic differentials \(dh = (dh_1, \ldots, dh_g)^T\) and meromorphic differentials \(dr = (dr_1, \ldots, dr_g)^T\) is defined by the following expressions,

\[
dh_j = \frac{\lambda^{j-1}}{\mu} d\lambda, \quad j = 1, \ldots, g,
\]

\[
dr_j = \sum_{k=j}^{2g-j+1} (k+1-j) \alpha_{k+1+j} \frac{\lambda^k}{4\mu} d\lambda, \quad j = 1, \ldots, g.
\]
Figure 1: A homology basis on a Riemann surface of the hyperelliptic curve of genus \( g \) with real branching points \( E_1, \ldots, E_{2g+2} = \infty \) (upper sheet). The cuts are drawn from \( E_{2i-1} \) to \( E_{2i} \) for \( i = 1, \ldots, g + 1 \). The \( b \)-cycles are completed on the lower sheet (the picture on lower sheet is just flipped horizontally).

We denote periods

\[
2\omega = \left( \oint_{a_k} dh_i \right)_{i,k=1,\ldots,g}, \quad 2\omega' = \left( \oint_{b_k} dh_i \right)_{i,k=1,\ldots,g},
\]

\[
2\eta = \left( -\oint_{a_k} dr_i \right)_{i,k=1,\ldots,g}, \quad 2\eta' = \left( -\oint_{b_k} dr_i \right)_{i,k=1,\ldots,g}.
\]

The half-periods \( \omega, \omega', \eta, \eta' \) satisfy the generalized Legendre relation

\[
\mathfrak{M} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \mathfrak{M}^T = \frac{i\pi}{2} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix},
\]

where \( 1_g \) is the unit \( g \times g \) matrix and \( \mathfrak{M} \) is the \( 2g \times 2g \)-matrix

\[
\mathfrak{M} = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}.
\]

Denote \( \tau = \omega^{-1} \omega' \)-matrix belonging to the Siegel upper half-space \( \mathcal{S}_g = \{ \tau | \tau^T = \tau, \text{Im} \tau > 0 \} \) and necessarily symmetric matrix \( \varkappa = \eta(2\omega)^{-1} \). Jacobi variety of the curve is defines as the factor over the normalized period lattice, \( \text{Jac}(X) = \mathbb{C}/1_g \oplus \tau \). Abel map \( \mathfrak{A} : \{ P \in X \} \rightarrow \text{Jac}(X) \) maps the point \( P \) of the curve to Jacobian by the rule

\[
\mathfrak{A}(P) = \int_{P_0}^P dv,
\]

where \( P_0 \) is the base point of the Abel map. It maps a positive divisor \( D \) of degree \( n \), \( D = P_1 + \ldots + P_n \) as follows

\[
\mathfrak{A}(D) = \mathfrak{A}(P_1) + \ldots + \mathfrak{A}(P_n).
\]
In what follows we deal with non-special divisors (we remind that a special divisor $D$ is a divisor for which it exists a meromorphic function on $X$ with poles at most in $D$). The base point of the Abel map will be taken as $P_0 = (\infty, \infty)$ unless a contrary is stated.

### 3.2 $\theta$-functions and $\sigma$-functions

Introduce canonic $\theta$-function, $\theta(z; \tau)$, by the formula

$$\theta(z; \tau) = \sum_{m \in \mathbb{Z}} \exp \{i \pi m^T \tau m + 2i \pi z^T n\}. \quad (3.6)$$

Fundamental Riemann vanishing theorem says that if for a vector $z \in \text{Jac}(X)$ one has that $\theta(z; \tau) \not\equiv 0$, then there exists a non-special divisor $D$ of degree $g$ and a vector $K_\infty$ (vector of Riemann constant with the base point $P_0 = (\infty, \infty)$) such that

$$z = \mathcal{A}(D) - K_\infty.$$ 

Moreover Riemann $\theta$-function

$$\theta(P) = \theta(\mathcal{A}(P) - \mathcal{A}(D) + K_\infty; \tau)$$

vanishes precisely in $g$ points of the divisor $D$. Let $(\lambda_1, \mu_1), \ldots, (\lambda_g, \mu_g)$ be a non-special divisor. Denote

$$h = \sum_{k=1}^{g} \int_{\infty}^{\lambda_k} du_k - K_\infty. \quad (3.7)$$

Then the $\sigma$-function is defined by the formula

$$\sigma(h) = \sqrt{\frac{\pi^g}{\text{det}(2\omega)}} \sqrt{\prod_{1 \leq i < j \leq 2g+1} (E_i - E_j)} \times \theta((2\omega)^{-1} h|\tau) \exp(h^T \eta(2\omega)^{-1} h; \tau). \quad (3.8)$$

The function $\sigma$ represents a $g$-dimensional generalization of the elliptic $\sigma$-function. It is invariant with respect to the action of certain subgroups of the symplectic group. It also has the following transformation property with respect to shift of the argument by a period,

$$\sigma(h + 2\omega n + 2\omega' m) = \sigma(h) \exp \{(2\eta m + 2\eta' m)^T (h + \omega n + \omega' m)\}, \quad (3.9)$$

where $n, m$ are integer $g$-vectors.

Variety of zeros of the $\sigma$-function called $\sigma$-divisor $\Theta$, coincides with the $\theta$-divisor $(\theta)$. The $\sigma$-divisor in rational limit is given as zeros of a Schur-Weierstrass polynomial analyzed in [14].

Denote Abelian image $\Theta_1$ of the curve $X$,

$$\Theta_1 = \left\{ v \in \text{Jac}(X) \mid v = \int_{\infty}^{P} dh, \quad P \in X \right\}.$$
At genera $g > 1$ the variety $\Theta_1$ belongs to the $\sigma$-divisor, $\Theta_1 \subset \Theta$. We shall need the following result from [37] describing vanishing of certain partial derivatives of the $\sigma$-function.

**Lemma 3.1 (Ônishi, 2004)** Let $\sigma_3$ be partial derivative

$$\sigma_3(h) = \prod_{j \in I} \frac{\partial}{\partial h_j} \sigma(h),$$

where $I$ be multi-index given for different genera $g$ of hyperelliptic curve $X$ in the Table 1

| $g$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
|-----|---|---|---|---|---|---|---|---|---|-----|
| 2   | 2 | 2 | 24| 24| 246|246|2468|2468|...|

*Table 1. Multi-indices corresponding to different genera $g*$

Then in the vicinity of the point $h = (0, \ldots, 0)^T \in \Theta_1$ the following expansions are valid for the $\lambda$-coordinate of the curve $X$ and partial derivative $\sigma_3$

$$\lambda = \frac{1}{\xi^2} + O(1), \quad (3.10)$$

$$\sigma_3(h)^2 = \xi^{2g} + O(\xi^{2g+2}). \quad (3.11)$$

Moreover all derivatives $\sigma_3(h)$ vanish on $h \in \Theta_1$ for all multi-indices $J \subset I, |J| < |I|$.

The proof of the lemma is based on the detailed analysis of the Schur polynomials associated with hyperelliptic curves and certain results of [14].

### 3.3 Kleinian $\varphi$ and $\zeta$-functions

Introduce $g$-dimensional $\zeta$ and $\varphi$-functions as logarithmic derivatives of the $\sigma$-function as follows

$$\zeta_i(h) = \frac{\partial}{\partial u_i} \log \sigma(h), \quad i = 1 \ldots, g,$$

$$\varphi_{i,j}(h) = -\frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(h), \quad i,j = 1 \ldots, g,$$

$$\varphi_{i,j,k}(h) = -\frac{\partial^3}{\partial u_i \partial u_j \partial u_k} \log \sigma(h), \quad i,j,k = 1 \ldots, g,$$

etc.

These functions have the following periodicity properties. Let $n, m \in \mathbb{Z}^g$ two arbitrary integer vectors. Then

$$\zeta_k(u + 2\omega n + 2\omega' m) = \zeta_k(u) + 2\eta m + 2\eta' m, \quad k = 1 \ldots, g, \quad (3.12)$$

$$\varphi(u + 2\omega n + 2\omega' m) = \varphi(u), \quad (3.13)$$
where $I$ be arbitrary set of two, three, etc., indices.

Consider the principal object of the hyperelliptic theory the \textit{master polynomial}

$$F(\lambda; \lambda_1, \ldots, \lambda_g) = \prod_{k=1}^{g}(\lambda - \lambda_k),$$

where $\lambda_1, \ldots, \lambda_g$ are first coordinated of non-special divisor of degree $g$. $P_1 = (\lambda_1, \mu_1), \ldots, P_g = (\lambda_g, \mu_g)$. O. Bolza \cite{11} found expression for the master polynomial in terms of $\wp$-functions (see also \cite{2}, \cite{3}, \cite{12} for derivation and applications) in the form

$$F(\lambda; \lambda_1, \ldots, \lambda_g) \equiv \mathcal{P}(\lambda, h) = \lambda^g - \wp_{g,g}(h)\lambda^{g-1} - \cdots - \wp_{g,1}(h). \quad (3.14)$$

In what follows we shall refer to this parametrization of the master polynomial as to Bolza polynomial.

Zeros of Bolza polynomial $\mathcal{P}(\lambda, h)$ are known to solve the Jacobi inversion problem,

$$\lambda_1 + \ldots + \lambda_g = \wp_{g,g}(h),$$

$$\vdots$$

$$\lambda_1 \cdots \lambda_g = (-1)^{g+1}\wp_{1,g}(h),$$

whilst the $\mu$-coordinates of the divisor $P_1 = (\mu_1, \lambda_1), \ldots, P_g = (\mu_g, \lambda_g)$ are given by

$$\mu_k = -\frac{\partial}{\partial u_k}\mathcal{P}(\lambda, h) \bigg|_{\lambda = \lambda_k}, \quad k = 1, \ldots, g. \quad (3.16)$$

Among various relations of the theory we shall use, for our derivation the following formula for $\zeta$-function,

$$-\int_{E_{g+1}} \frac{\lambda^g}{\mu} d\lambda + \frac{1}{2} \frac{\mu}{\mathcal{P}(\lambda, h)} + \frac{1}{2} \frac{\partial}{\partial u_g}\mathcal{P}(\lambda, h) = \zeta_g \left( h + \int_{\mathcal{P}(\lambda, h)} d\lambda \right) - \zeta_g(h). \quad (3.17)$$

The derivation of this formula is documented in \cite{2} \cite{3} and \cite{12} \cite{13} and based on the finding of explicit expression for the fundamental bi-differential - so called Bergman kernel in modern terminology.

Another set of formulae describe Jacobi variety $\text{Jac}(X)$ as algebraic variety in $\mathbb{C}^{g+\binom{g+1}{2}}$. The functions $\wp_{gg}$ and $\wp_{ik}$ are related by

$$\wp_{gg} \wp_{gk} = 4\wp_{gg} \wp_{gi} \wp_{gk} - 2(\wp_{gi} \wp_{g-1,k} + \wp_{g,k} \wp_{g-1,i})$$

$$+ 4(\wp_{gk} \wp_{g,i-1} + \wp_{gi} \wp_{g,k-1}) + 4\wp_{k-1,i-1} - 2(\wp_{k,i-2} + \wp_{i,k-2}) +$$

$$+ \alpha_{2g-1} \frac{1}{2}(\delta_{ig} \wp_{kg} + \delta_{kg} \wp_{ig}) + c_{i,k}, \quad (3.18)$$

where

$$c_{i,k} = \alpha_{2i-2} \delta_{ik} + \frac{1}{2}(\alpha_{2i-1} \delta_{k,i+1} + \alpha_{2k-1} \delta_{i,k+1}). \quad (3.19)$$
We shall also use derivative of these formulae,

\[
\varphi_{gggi} = (6\varphi_{gg} + \alpha_2 g)\varphi_{gi} + 6\varphi_{g,i-1} - 2\varphi_{g-1,i} + \frac{1}{2} \delta_{g,i} \alpha_{2g-1},
\]  

(3.20)

which describe according to [12] the KdV hierarchy. We remark that the above formulae were derived for the case ultraelliptic curves \( g = 2 \) in [4] where relation to integrable equations like KdV or sine-Gordon had not been noticed. The comprehensive generalization to hyperelliptic curves of higher genera was obtained to the first time in [12, 13].

We complete our brief description of \( \sigma \)-functions by the addition theorem for hyperelliptic \( \sigma \)-functions.

**Theorem 3.2** Let

\[
v = \int_{(\infty, \infty)}^{(\lambda, \mu)} dh \in \Theta_1
\]

be Abelian image of the curve \( X \) and let \( h \) be a vector in general position in \( \text{Jac}(X) \). Then the following formula is valid

\[
\frac{\sigma(h - v)\sigma(h + v)}{\sigma_2(v)^2\sigma(h)^2} = P(\lambda, h),
\]

(3.21)

where \( P(\lambda, h) \) is the Bolza polynomial (3.14) and the multi-index \( I \) is given in the Table of Lemma 3.1.

To the best knowledge of the authors the formula (3.21) as well the foregoing relation (3.22) are new. We shall publish detailed proof elsewhere and give here only its short version. This proof is based on the Riemann vanishing theorem and solution of the Jacobi inversion problem in the form (3.15) what leads to the Bolza polynomial in the right hand side. Lemma 3.1 permits to check the leading terms of the expansions in \( v \) near the point \( v = (0, \ldots, 0)^T \in \Theta_1 \) in both sides of the equality.

**Corollary 3.3** Let \( \Omega = \sum_{k \in I} \Omega_k \), where \( |I| = g \) be non-singular even half-period represented in the form \( \Omega = 2\omega n + 2\omega' n' \) with \( n, n' \in \mathbb{Z}^g \). Then the following formula is valid

\[
\frac{\sigma(v - \Omega)^2}{\sigma_2(v)^2} = Q_0(\lambda) \prod_{i \in I} (\lambda - E_i), \quad v \in \Theta_1, \quad \Omega \in \text{Jac}(X),
\]

(3.22)

where \( Q_0(\lambda) = \sigma(\Omega)^2 \exp\left\{ \Delta^T \int_{\infty}^{P} dh \right\} \) with \( \Delta = 2\eta n + 2\eta' n' \), \( E_i \) are branch points of the curve \( X \) and \( I \) is the set of multi-indices given in the Lemma 3.1.

### 4 \( \sigma \)-functional realization of the basic functions

Consider \( h \)-vectors of the form

\[
h = i e_g - \Omega, \quad e_g = (0, \ldots, 0, 1)^T,
\]

(4.1)
where $\Omega = (\Omega_1, \ldots, \Omega_g)^T$ be non pure imaginary even non-singular half-period supported by $g$ branch points

$$\Omega = \sum_{i \in I} A_i, \quad |I| = g. \quad (4.2)$$

Then the Bolza polynomial is identified as the $S(x, \lambda)$ function introduced in Section 2,

$$P(\lambda; h) = S(x, \lambda). \quad (4.3)$$

4.1 Its-Matveev theorem

The following theorem describes algebro-geometric solutions of the Schrödinger equation with finite-gap potentials and was proved by Its and Matveev in 1975 [27]. Different variants of proof the Its-Matveev theorem are known see e.g. [7], [17], [23].

**Theorem 4.1 (Its-Matveev, 1975)** Let $X$ be non-degenerate hyperelliptic curve of genus $g$. Let $\Omega = (\Omega_1, \ldots, \Omega_g)^T$ be non pure imaginary even non-singular half-period supported by $g$ branch points. Then the smooth and real potential $u(x)$ and one-valued on the curve $X$ Bloch function $\psi(x, P)$, are given by the formulae

$$u(x) = \varphi_g(ixe_g - \Omega), \quad (4.4)$$

$$\psi(x, P) = C(P) \frac{\sigma \left( \int P^1 dh - ixe_g + \Omega \right)}{\sigma_g \left( \int P^1 dh \right) \sigma(ixe_g - \Omega)} \times \exp \left\{ ix \int_0^P dh - \Omega^T \int_0^P dr \right\}, \quad (4.5)$$

where $\varphi$ is the set of indices given in the Table of the Lemma 3.1, $C(P)$ is a normalization constant which depends on the point $P \in X$, $e_g = (0, \ldots, 0, 1)^T$.

4.2 $\sigma$-functional description of the Weyl function and the wave number

**Theorem 4.2** Let conditions of the Theorem 4.1 are satisfied. Then Weyl function $w(x, P)$ and wave number $k(P)$ are given by the formulae

$$w(x, P) = \zeta_g \left( ixe_g - \Omega + \int_0^P dh \right) - \zeta_g(ixe_g - \Omega) - \int_0^P dr_g. \quad (4.6)$$
The wave number or quasi-momentum is

\[ k(P) = \left\langle \zeta_g \left( ixe_g - \Omega + \int \limits_{\infty}^{P} dh \right) - \zeta_g(ixe_g - \Omega) - \int \limits_{(E_{2g+1},0)}^{P} dr_g \right\rangle. \quad (4.7) \]

Remark that due to the formula (3.17) and mentioned identification \( S(x,\lambda) \) as the Bolza polynomial the quasi-momentum is given as the second kind Abelian integral (2.12). Denote \( a \)-periods of \( k(P) \) as

\[ \mathcal{X}_i = \oint_{a_i} d\mathcal{X}, \quad i = 1, \ldots, g. \quad (4.8) \]

### 4.3 Normalization of the Bloch function

Different normalizations of the Bloch function are accepted in various problems of physics. In our case the normalizing constant \( C(P) \) is computed from the condition (2.14). We shall do that on the basis of the relation (3.21). As the result we obtain the following expression for the normalizing constant

\[ C(P) = \frac{1}{\sqrt{\langle S(x;\lambda) \rangle}}. \quad (4.9) \]

The proof in general case is based on the addition theorem [12]. Expression (4.9) was already given in [7], Chapter 8. We emphasize that it is valid both for ergodic and periodic cases.

Further we shall fix the value (4.9) for the normalization constant and consider only normalized Bloch functions \( \psi(x,P) \).

### 5 Periodic and ergodic finite-gap potentials

We shall now consider two classes of finite-gap potentials: periodic and ergodic.

**Definition 5.1** We shall call the vector \( U = (U_1, \ldots, U_g)^T \),

\[ U = (2\omega')^{-1} e_g \quad (5.1) \]

as the winding vector

**Definition 5.2** The finite-gap potential is called periodic if the curve \( X \) admits existence of an integer vector \( n \in \mathbb{Z}^g \) and real number \( U \in \mathbb{R} \) \( U \neq 0 \) such that

\[ U = nU, \quad (5.2) \]

otherwise the finite-gap potential is called ergodic.

We remark that in generic case the finite-gap potential is ergodic, it is periodic only under special conditions on the curve presented above.
5.1 Condition of periodicity of the finite-gap potential

The direct consequence of the above definition is that the finite-gap potential is periodic if and only if period matrix \( \omega' \) satisfy to the equation for certain vector \( n \in \mathbb{Z}^g \) and real number \( U \neq 0 \),

\[
\omega' n = \frac{1}{2U} e_g.
\]  

(5.3)

One can easily see that under condition (5.3) the functions in \( x \) \( g_{kl}(x) = \varphi_{kl}(ixe_g - \Omega) \) are periodic in \( x \) with period \( 1/U \). Indeed,

\[
\varphi_{kl} \left( i \left( x + \frac{i}{U} \right) e_g - \Omega \right) = \varphi_{kl} (ixe_g - \Omega - 2\omega' n) = \varphi_{kl} (ixe_g - \Omega).
\]

Conditions of periodicity (5.3) represent \( g \) transcendental conditions on the moduli of the curve. But these conditions are weaker then conditions of double periodicity in \( x \) of the finite-gap potentials. The last conditions are formulated in terms of period matrix of the curve \( \tau \) as follows

**Theorem 5.1** ([7]) *The finite gap potential \( u(x) \) is elliptic function of \( x \) if and only if*

- there exist such homology basis \( B = (a_1, \ldots, a_g; b_1, \ldots, b_g) \) that the period matrix \( \tau \) has the form

\[
\tau = \begin{pmatrix}
\tau_{11} & k/N & 0 & \ldots & 0 \\
k/N & * & * & * & * \\
0 & * & * & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
k \in \{1, \ldots, N - 1\}, \quad N > 1, N \in \mathbb{N}.
\end{pmatrix}
\]  

(5.4)

- in the homology basis \( B \) the winding vector \( U \) is of the form

\[
U = (*, 0, \ldots, 0)^T.
\]  

(5.5)

Conditions (5.4), (5.5) represent \( 2g - 2 \) equations and therefore the associated double periodic potentials can be included as particular cases into the set of periodic potentials. These more particular potentials permit explicit analytic description and we shall use this circumstance in what follows. We remark that the reduction theory of Abelian functions to elliptic functions goes back to K.Weierstrass and A.Poincaré and is exposed in the classical A.Krazer monograph [31]; the modern exposition and applications to integrable system are given, in particular, in [8], [9]; the Theorem 5.1 was recently considered in [22], where an alternative proof is given.
5.2 Calculation of averages

The average

\[ \langle S(x, \lambda) \rangle = \lambda^g + \sum_{j=1}^{g} \lambda^{j-1} s_j \]  

(5.6)

is sensible to commensurability or non-commensurability of the frequencies (components of the winding vector \( U \)). In what follows we shall specify coefficients of the polynomial (5.6) as \( s_j^{(p)} \) or \( s_j^{(e)} \) with superscript \((p)\) taken for periodic case and \((e)\) – for ergodic, we shall also provide \( \langle S(x, \lambda) \rangle \) with subscripts, \( \langle S(x, \lambda) \rangle_p \) and \( \langle S(x, \lambda) \rangle_e \).

**Theorem 5.2 (Averaging in ergodic case)** Let the components of the winding vector \( U \) be all non-commensurable. Then the average \( \langle S(x, \lambda) \rangle_e \) is ergodic and given by

\[ \langle S(x, \lambda) \rangle_e = \lambda^g + \sum_{j=1}^{g} \lambda^{j-1} s_j^{(e)}, \]  

(5.7)

where

\[ s_j^{(e)} = \frac{1}{\det 2\omega'} \det \begin{pmatrix} \omega_{1,1}' \ldots \omega_{1,g}' \\ \vdots \ldots \vdots \\ \omega_{j-1,1}' \ldots \omega_{j-1,g}' \\ \eta_{1,1} \ldots \eta_{g,1} \\ \omega_{j+1,1}' \ldots \omega_{j+1,g}' \\ \vdots \ldots \vdots \\ \omega_{g,1} \ldots \omega_{g,g}' \end{pmatrix}, \quad j = 1, \ldots, g. \]

In particular, for \( g = 1 \)

\[ \langle S(x, \lambda) \rangle_e = \lambda + \frac{\eta'}{\omega'} \]

and for \( g = 2 \)

\[ \langle S(x, \lambda) \rangle_e = \lambda^2 + \lambda \frac{\det \begin{pmatrix} \omega_{11} & \omega_{12} \\ \eta_{11} & \eta_{12} \end{pmatrix}}{\det \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}} + \frac{\det \begin{pmatrix} \eta_{11} & \eta_{12} \\ \omega_{11} & \omega_{12} \end{pmatrix}}{\det \begin{pmatrix} \eta_{11} & \eta_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}}. \]

In the case of commensurability we obtain another result for the average.

**Theorem 5.3 (Averaging in periodic case)** Let the components of the winding vector \( U \) be all commensurable. Then the average \( \langle S(x, \lambda) \rangle_p \) is periodic and given by

\[ \langle S(x, \lambda) \rangle_p = z^g + \sum_{j=1}^{g} z^{j-1} s_j^{(p)}, \]  

(5.8)
where

\[ s_j^{(p)} = \frac{1}{\det 2\omega} \det \begin{pmatrix} \omega_{1,1}^1 & \cdots & \omega_{1,g}^1 \\ \vdots & \cdots & \vdots \\ \omega_{g-1,1}^1 & \cdots & \omega_{g-1,g}^1 \\ \eta_{j,1}^1 & \cdots & \eta_{j,g}^1 \end{pmatrix}, \quad j = 1, \ldots, g. \]  

(5.9)

In particular, for \( g = 1 \)

\[ \langle S(x, \lambda) \rangle_p = \lambda + \frac{\eta_j'}{\omega'} \]

and for \( g = 2 \)

\[ \langle S(x, \lambda) \rangle_p = \lambda^2 + \lambda \frac{\det \begin{pmatrix} \omega_{11}^1 & \omega_{12}^1 \\ \eta_{21}^1 & \eta_{22}^1 \end{pmatrix}}{\det \begin{pmatrix} \omega_{11}^1 & \omega_{12}^1 \\ \omega_{21}^1 & \omega_{22}^1 \end{pmatrix}} + \frac{\det \begin{pmatrix} \omega_{21}^2 & \omega_{22}^2 \\ \eta_{11}^2 & \eta_{12}^2 \end{pmatrix}}{\det \begin{pmatrix} \omega_{21}^2 & \omega_{22}^2 \\ \omega_{11}^2 & \omega_{12}^2 \end{pmatrix}}. \]

We see that for the case of elliptic curve the averages in ergodic and periodic case coincide, but for genera bigger than one \( \langle S(x, \lambda) \rangle_p \) and \( \langle S(x, \lambda) \rangle_e \) are given by different expressions. The comprehensive proof of these formulae will be given elsewhere.

6 Wannier function

**Definition 6.1** The Wannier function \( W_n(x) \) in the \( n \)-th spectral band, \( n = 1, \ldots, g \), is defined as the following integral of normalized Bloch function,

\[ W_n(x) = \frac{1}{\sqrt{\mathcal{K}_n}} \int_{a_n} \psi(x, \lambda) d\lambda, \quad n = 1, \ldots, g, \]  

(6.1)

where \( \mathcal{K}_n \) are \( a_n \)-periods of the differential of quasi-momentum (2.13).

Using translation operator one then construct a countable set of Wannier functions,

\[ W_n^{(l)}(x) = \frac{1}{\sqrt{\mathcal{K}_n}} \int_{a_n} \psi_l(x, \lambda) d\lambda, \quad n = 1, \ldots, g, l \in \mathbb{Z} \]  

(6.2)

with

\[ \psi_l(x, P) = \frac{1}{\sqrt{\mathcal{K}(x; \lambda)}} \frac{\sigma \left( \int_{\Omega}^P d\hbar - ixe_g + \Omega + 2l \Omega' \right)}{\sigma \left( \int_{\Omega}^P d\hbar \right)} \sigma(ixe_g - \Omega - 2l \Omega') \times \exp \left\{ ix \int_{(E_{2s+1,0})}^P dr_g - (2l \Omega' + \Omega)^T \int_{(E_{2s+1,0})}^P dr \right\}, \]  

(6.3)
where $\Omega'$ is a pure imaginary non-singular even half-period. In the case of genus $g = 1$ the above definitions are those given in [10].

To complete this definition we must compute periods (2.13). To this regard the following lemma apply.

**Lemma 6.1** Denote and $s_k$ are coefficients of the polynomial

$$\langle S(x, \lambda) \rangle = \lambda^g + s_g \lambda^{g-1} + \ldots + s_1.$$  

Then

$$\begin{pmatrix} \mathcal{K}_1 \\ \vdots \\ \mathcal{K}_g \end{pmatrix} = 2\omega^T \begin{pmatrix} 2\kappa_{g,1} - s_1. \\ \vdots \\ 2\kappa_{g,g} - s_g \end{pmatrix}. \quad (6.4)$$

In particular in the case $g = 1$, $s_1 = \eta'/\omega'$, $\kappa_{11} = \kappa = \eta/2\omega$ and

$$\mathcal{K}_1 = \frac{i\pi}{\omega'}. \quad (6.5)$$

Note that in the case $g = 1$, the Wannier function (6.1) coincides with the classical one derived in [10],

$$W_1(x) = \left( \frac{T}{2\pi} \right)^{1/2} \int_{-\pi/T}^{\pi/T} \psi(x, P)dk, \quad T = -2i\omega'.$$

The principal expected properties for the set of Wannier functions, $W_k(x)$, $k = 1, \ldots, g$ is orthogonality

$$\int_{-\infty}^{\infty} W_k^{(k')} (x) W_l^{(l')} (x) dx = \delta_{kl} \delta_{k'l'}, \quad k, l = 1, \ldots, g, \quad k', l' \in \mathbb{Z} \quad (6.6)$$

and completeness in space of eigenfunctions of the Schrödinger operator with finite-gap quasi-periodic potential. We shall postpone the discussion of these questions to a future publication.

Since the normalization of Bloch function is different for periodic and ergodic cases we shall distinguish two kind of Wannier functions - the classical Wannier functions associated with periodic potentials and the quasi-periodic Wannier functions associated with quasi-periodic potentials. Both kinds of Wannier functions are defined by the same formula involving Bolza polynomial in the expression of the normalization constant. Normalizing constants, however, are computed in different ways for periodic and ergodic cases.

### 6.1 Power series for the Wannier function at $|x| \simeq 0$

**Theorem 6.2** The Wannier function of the first lower energy band $W_1(x)$ admits the following expansion

$$W_1(x) = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p)!} W_1^{(2p)} x^{2p}. \quad (6.7)$$
Here
\[ W_1^{(2p)} = \sum_{k=1} M_k q_{pk}, \]  
(6.8)
the quantities \( M_k \) are given as \( a \)-cycles of second kind Abelian integrals,
\[ M_k = \frac{1}{\sqrt{K_k}} \oint_{a_1} \lambda^k \sqrt{\prod_{i \in \{1, \ldots, 2g+1\}-I} (\lambda - E_i)} \ d\lambda, \quad k = 0, \ldots \]  
(6.9)
The quantities \( q_{kl} \) are coefficients of polynomials in \( z \), \( Q_p = \sum_{l=0}^p q_{p,l} z^l \), defined by recurrence
\[ Q_p = \sum_{m=0}^{p-1} \left( \frac{2p}{2m - p} \right) \phi_{m-p-1}(z) Q_m \]
with conditions
\[ \psi_0 = \wp_{gg}(\Omega) + z, \quad \phi_p = \wp \underbrace{g \ldots g}_{2p+2}(\Omega). \]

The first few coefficients are
\[ W_1^{(0)} = M_0, \]
\[ W_1^{(2)} = M_1 + 2\wp_{gg}(\Omega) M_0, \]
\[ W_1^{(4)} = M_2 + 4\wp_{gg}(\Omega) M_1 + (4\wp_{gg}(\Omega)2 + 2\wp_{gggg}(\Omega)) M_0, \]
\[ W_1^{(6)} = M_3 + 6\wp_{gg}(\Omega) M_2 + (12\wp_{gg}(\Omega)2 + 14\wp_{gggg}(\Omega)) M_1 + (2\wp_{gggg}(\Omega) + 28\wp_{gggg}(\Omega)\wp_{gg}(\Omega) + 8\wp_{gg}(\Omega)3) M_0. \]

Here in its turn
\[ \wp_{gg}(\Omega) = \sum_{i \in I} E_i, \]
\[ \wp_{gggg}(\Omega) = (6 \sum_{i \in I} E_i + \alpha_{2k}) \sum_{i \in I} E_i + 4 \sum_{i,k \in I} E_i E_k, \]
where we used (3.20).

We remark that the proof of this theorem requires an expansion of the exponential in \( x \) which is a generating function for Schur polynomials [32] and subsequent integration of these polynomials (details will be given elsewhere). The integrand can be expressed recursively in terms of second kind Abelian differentials described below. In the case of genus one curve and elliptic periodic potentials these integrals are given as standard hypergeometric functions [10].

Another remark is that in the course of the proof a reciprocal hyperelliptic curve \( X^* \) of the same genus and dual to the initial hyperelliptic curve \( X \), naturally arises.
Definition 6.2 We shall call reciprocal curves \(X^\#_k\), \(k = 1, \ldots, g\) associated to the curve \(X\) the set of hyperelliptic curves of genus \(g\) given by the formulae

\[
\mu^\# = 4(\lambda^g + s_g \lambda^{g-1} + \ldots + s_1) \prod_{i \in \{1, \ldots, 2g+1\} - I_k} (\lambda - E_i) = 4(\lambda^{2g+1} + \alpha^\#_{2g}\lambda^{2g} + \ldots + \alpha_0^\#),
\]

where \(I_k\) is the set of \(g\) indices, \(I_k \subset \{1, \ldots, 2g + 1\}\).

We remark that the amplitudes of the Wannier pulses \(W_n(0), n = 1, \ldots, g\) are given as periods of meromorphic hyperelliptic integral of corresponding reciprocal curves.

To describe canonical differentials on \(X^\#_1\) it will be convenient to introduce the set of second kind differentials, which we shall call the Wannier differentials,

\[
dW_1^{(n)} = \lambda^n \sqrt{\frac{\lambda^g + s_g \lambda^{g-1} + \ldots + s_1}{\prod_{i \in \{1, \ldots, 2g+1\} - I_k} (\lambda - E_i)}} d\lambda, \quad n = 0, \ldots.
\]

Canonical holomorphic differentials \(dh^\# = (dh_1^\#, \ldots, dh_g^\#)^T\) of the curve are given as

\[
dh_k^\# = \frac{\partial}{\partial s_k} dW^{(0)} = \frac{\lambda^{k-1}}{\mu^\#} d\lambda, \quad k = 1, \ldots, g.
\]

The fact that holomorphic differentials of the reciprocal curve are given as derivatives in moduli of meromorphic differentials resembles principle feature of the algebraic curves appearing in Seiberg-Witten theory [39], [40], see also [24] where various forms of Seiberg-Witten curves were discussed.

Introduce also canonical second kind differentials \(dr^\# = (dr_1^\#, \ldots, dr_g^\#)^T\) as in (3.3)

\[
dr_j = \sum_{k=j}^{2g+1-j} (k + 1 - j) c_{k+1+j}^\# \frac{\lambda^k}{4\mu^\#} d\lambda \quad j = 1, \ldots, g.
\]

The first \(g\) differentials \(dW^{(k)}, k = 0, \ldots, g - 1\) are expressible linearly in terms of 2\(g\) differentials \(dh^\#\) and \(dh^\#.\) Periods of the differentials \(dW^{(n)}\) with higher \(n\) are also expressible in terms of aforementioned 2\(g\) differentials due to the de Rham theorem [25].

The last point which we wish to discuss here is the asymptotic expansion of the Wannier function at \(x \to +\infty\). To evaluate this expansion one can use a variant of the steepest descents method applicable to the case when the integrand contains a multiplier vanishing in the saddle point. The following theorem apply.

Theorem 6.3 At \(x \to \infty\) the Wannier function of the lower energy band \([E_1, E_2]\) for the one gap potential has the following asymptotic expression

\[
W_1(x) \simeq \text{Re} \left\{ \sqrt{\frac{Q_0(\lambda_0)}{K_n}} \prod_{i \in 1} (\lambda_0 - E_i) \psi(x, \lambda_0) \left[ \frac{i}{2\langle S'(x, \lambda_0) \rangle} \right]^{1/4} \frac{\Gamma \left( \frac{4}{x^2} \right)}{x^2} \right\},
\]

(6.14)
where $\lambda_0$ is a solution of the equation

$$\langle S(x, \lambda) \rangle = 0,$$

(6.15)

and $Q_0(\lambda)$ is given in the Lemma.

The formula for asymptotic expansion naturally generalizes the result for genus one derived in [10] and shows the universal character of the decreasing low at infinity of the Wannier function: $\exp\{c_0 x\} x^{-3/4}$ with phase $c_0 x$ in the saddle point.

7 Conclusion

In this paper we have defined Wannier functions associated with permitted zones of quasi-periodic finite gap potentials. A number of interesting questions remained beyond our considerations. One question is whether Wannier function introduced above generate an orthogonal and complete set of in the space of eigenfunctions of the Schrödinger operator. Moreover, a discussion of the first non-trivial examples, as well as a comparison between our analytical expressions for amplitudes and asymptotic behaviors of Wannier functions with numerical calculations as done in [10] for the one gap case, would also be desirable. We plan to present this material and other related problems in a forthcoming detailed publication.

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