AN ANALOGUE OF LADDER REPRESENTATIONS FOR CLASSICAL GROUPS

HIRAKU ATOBE

Abstract. In this paper, we introduce a notion of ladder representations for split odd special orthogonal groups and symplectic groups over a non-archimedean local field of characteristic zero. This is a natural class in the admissible dual which contains both strongly positive discrete series representations and irreducible representations with irreducible $A$-parameters. We compute Jacquet modules and the Aubert duals of ladder representations, and we establish a formula to describing ladder representations in terms of linear combinations of standard modules.

1. Introduction

Let $F$ be a non-archimedean local field of characteristic zero with the normalized absolute value $|\cdot|$. For $n \geq 0$, we set $G_n$ to be a split special orthogonal group $SO_{2n+1}(F)$ or a symplectic group $Sp_{2n}(F)$. Denote by $\mathcal{R}(GL_n(F))$ (resp. $\mathcal{R}(G_n)$) the Grothendieck group of the category of smooth representations of $GL_n(F)$ (resp. $G_n$) of finite length. For an object $\pi$ of this category, we denote by $[\pi]$ the corresponding element of the Grothendieck group. It is known that $\mathcal{R}(GL_n(F))$ (resp. $\mathcal{R}(G_n)$) has two natural bases. One consists of the irreducible representations and the other consists of the standard modules. The change of basis matrix is triangular and unipotent (in an appropriate sense). At least for the general linear groups case, as predicted by Zelevinsky [20, 21] and proven by Chriss–Ginzburg [8], this matrix is written using Kazhdan–Lusztig polynomials. It is very complicated to compute this matrix in general, but in a certain special class of irreducible representations $\pi$, it may be relatively simple to describing $[\pi]$ in terms of a linear combination of standard modules.

As an example, let us recall the ladder representations of $GL_n(F)$. In Introduction, we only consider unipotent representations. Let

$$[x, y] := \{ |\cdot|^x, |\cdot|^x - 1, \ldots, |\cdot|^y \}$$

be a segment with $x, y \in \mathbb{R}$ such that $x \geq y$ and $x - y \in \mathbb{Z}$. The parabolically induced representation

$$|\cdot|^x \times |\cdot|^{x-1} \times \cdots \times |\cdot|^y$$

of $GL_{x-y+1}(F)$ has a unique irreducible subrepresentation, which is denoted by $\Delta[x, y]$ and is called a Steinberg representation. For the notation of parabolically induced representations of $GL_n(F)$, see [2.2] below. We also write $\Delta[x, x+1] := 1_{GL_n(F)}$ and $\Delta[x, y] := 0$ for $y > x + 1$.

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Theorem 1.1
(Lapid–Mínguez [11, Theorem 1 (ii)]) following is called a determinantal formula ladder representation of \( GL_n \) determinantal formula. For example:\n
and its proof was simplified by Chenevier–Renard [7]. There are several applications for the holds.

Speh representation if \( \pi \) should be irreducible representations with irreducible representations of \( GL_n \) and can be realized as local components of discrete spectrum of automorphic forms. For more precision, see [1]. However, as is the same as Speh representations, the class of \( GL_n \) is called a standard module.

Let \( \mathfrak{S}_t \) be the symmetric group on \( \{1, \ldots, t\} \) with the sign character \( \text{sgn}: \mathfrak{S}_t \to \{\pm 1\} \). The following is called a determinantal formula.

\[ [\pi] = \sum_{\sigma \in \mathfrak{S}_t} \text{sgn}(\sigma) \left[ \Delta[x_1, y_{\sigma(1)}] \times \cdots \times \Delta[x_t, y_{\sigma(t)}] \right] \]

holds.

When \( \pi \) is a Speh representation, the determinantal formula was early proven by Tadić [16] and its proof was simplified by Chenevier–Renard [7]. There are several applications for the determinantal formula. For example:

- The determinantal formula for Speh representations was used in the critical final step of the global classification of Arthur’s endoscopic classification (see [11] §7.5, §8.2).
- Using Theorem 1.1, Tadić computed the full derivative (in the sense of Zelevinsky) of ladder representations (see [11] Theorem 14).
- Mínguez and Sécherre established another determinantal formula to extend the mod \( \ell \) Jacquet–Langlands correspondence to all irreducible representations (see [13] Proposition 1.15, Théorème 1.16]).
- Theorem 1.1 was one of the key ingredients of the theory of local newforms for non-generic representations of \( GL_n(F) \) established by the author together with Kondo and Yasuda (see [1] §6).

Let us turn our attention to the classical group \( G_n \). An analogue of Speh representations should be irreducible representations with irreducible \( A \)-parameters. Such representations are unitary, and can be realized as local components of discrete spectrum of automorphic forms. For more precision, see [1]. However, as is the same as Speh representations, the class of irreducible representations of \( G_n \) with (irreducible) \( A \)-parameters for all \( n \geq 0 \) is not preserved by derivatives (in the sense of Jantzen–Mínguez). Note that every ladder representation of \( GL_n(F) \) can be obtained from a Speh representation of \( GL_{n'}(F) \) for \( n' \geq n \) by derivatives. Inspired by Mœglin’s construction of irreducible representations with \( A \)-parameters (see [18]), we will define a notion of ladder representations of \( G_n \) in Definition 3.1. Here, we only state the unipotent ladder representations of \( \text{Sp}_{2n}(F) \) for simplicity.

Definition 1.2. A unipotent ladder datum for \( G_n = \text{Sp}_{2n}(F) \) is a triple \( \mathcal{L} = (X, l, \eta) \) satisfying the following conditions.
(1) \( X = \{ |x_1|, |x_2|, \ldots, |x_t| \} \) with
- \( x_i \in \mathbb{Z} \);
- \( x_1 < x_2 < \cdots < x_t \);
- \( x_1 + x_2 + \cdots + x_t + \frac{t-1}{2} = n \) (so that \( t \) is odd).
(2) \( l \in \mathbb{Z} \) such that
- \( 0 \leq 2l \leq t \);
- \( x_j + x_{t-j+1} \geq 0 \) for \( 1 \leq j \leq l \);
- \( x_i \geq 0 \) for \( l + 1 \leq i \leq t-l \).
(3) \( \eta \in \{ \pm 1 \} \) such that \((-1)^{l+1+\eta} = 1\).

We define \( \pi(\mathcal{L}) \in \text{Irr}(G_n) \) by the unique irreducible subrepresentation of the standard module
\[
\Delta[x_1, -x_1] \times \Delta[x_2, -x_{t-1}] \times \cdots \times \Delta[x_t, -x_{t-l+1}] \times \pi(\phi, \epsilon),
\]
where
\[
\phi := \bigoplus_{i=l+1}^{t-l} S_{2x_i+1}
\]
and \( \epsilon(S_{2x_i+1}) := (-1)^{i-1-\eta} \) for \( l + 1 \leq i \leq t - l \). We call \( \pi(\mathcal{L}) \) a unipotent ladder representation of \( G_n = \text{Sp}_{2n}(F) \).

For the notation of parabolically induced representations and standard modules of \( G_n \), see \( \S 2.3 \) and \( \S 2.4 \), respectively. The class of ladder representations of \( G_n \) contains
- strongly positive discrete series (see, e.g., [12] for this notion); and
- irreducible representations with irreducible \( A \)-parameters.

As an analogue of [11] \( \S 3 \), for a unipotent ladder datum \( \mathcal{L} \), we will define a directed 3-colorable graph \( \mathcal{E}(\mathcal{L}) \) in Definition 3.3. As in Proposition 3.6 this graph knows derivatives of \( \pi(\mathcal{L}) \). In particular, one can see that the class of ladder representations of \( G_n \) for all \( n \geq 0 \) is preserved by derivatives. More generally, we can compute all Jacquet modules of \( \pi(\mathcal{L}) \) by Theorem 3.8 together with [2, Lemma 2.6]. It would help in the unitary dual problem for \( G_n \). As another application of the graph \( \mathcal{E}(\mathcal{L}) \), one can compute the Aubert dual of \( \pi(\mathcal{L}) \) by Theorem 3.12.

Among irreducible representations of the form \( L(\Delta[x_1, y_1], \ldots, \Delta[x_t, y_t]) \) with \( x_1, \ldots, x_t \in \mathbb{Z} \), the ladder representations of \( \text{GL}_n(F) \) are characterized so that all Jacquet modules are completely reduced. This fact was proven by Ram [14] Theorem 4.1 (c) in the context of affine Hecke algebras, and by Gurevich [9] Corollary 4.11 in general. In Corollary 3.10 we will see that all Jacquet modules of \( \pi(\mathcal{L}) \) are completely reduced as well. However, the converse in an appropriate sense does not hold (see Examples 3.11 and 4.7).

As a main result of this paper, we give an analogue of the determinantal formula in Theorem 4.2. For unipotent ladder representations of \( G_n = \text{Sp}_{2n}(F) \), it is stated as follows.

**Theorem 1.3.** Let \( \mathcal{L} = (X, l, \eta) \) be a unipotent ladder datum for \( G_n = \text{Sp}_{2n}(F) \) with \( X = \{ |x_1|, \ldots, |x_t| \} \) such that \( x_1 < \cdots < x_t \). Let \( \mathcal{E}(\mathcal{L}) \) be the subset of \( \mathcal{E}_t \) consisting of \( \sigma \) such that
- for \( 1 \leq i < j \leq l \), we have \( \sigma(i) < \sigma(j) \);
- for \( l + 1 \leq i < j \leq t - l \), we have \( \sigma(i) < \sigma(j) \);
• if $x_i \leq -1$, then $\sigma^{-1}(i) \leq l$.

For $\sigma \in \mathfrak{S}(\mathcal{L})$, set

$$J^+_\sigma := \{1 \leq j \leq l \mid \sigma(j) < \sigma(t - j + 1)\},$$
$$J^-_\sigma := \{1 \leq j \leq l \mid \sigma(j) > \sigma(t - j + 1)\},$$

and define a direct sum of standard modules $I_\sigma(\mathcal{L})$ by

$$I_\sigma(\mathcal{L}) := \left( \prod_{j \in J^+_\sigma} \Delta[x_{\sigma(j)}, -x_{\sigma(t-j+1)}] \right) \times \left( \bigoplus_{\delta: J^-_\sigma \to \{\pm 1\}} \pi(\phi_\sigma, \varepsilon_{\sigma, \delta}) \right),$$

where $(\phi_\sigma, \varepsilon_{\sigma, \delta})$ is given by

$$\phi_\sigma := \left( \bigoplus_{i=l+1}^{t-1} S_{2x_{\sigma(i)}+1} \right) \oplus \left( \bigoplus_{j \in J^-_\sigma} S_{2x_{\sigma(j)}+1} \oplus S_{2x_{\sigma(t-j+1)}+1} \right)$$

and

$$\varepsilon_{\sigma, \delta}(S_{2x_{\sigma(i)}+1}) := (-1)^{j-1-1} \eta,$$
$$\varepsilon_{\sigma, \delta}(S_{2x_{\sigma(j)}+1}) = \varepsilon_{\sigma, \delta}(S_{2x_{\sigma(t-j+1)}+1}) := \delta(j)$$

for $l + 1 \leq i \leq t - l$ and $j \in J^-_\sigma$. Then in the Grothendieck group $\mathcal{R}(G_n)$, an equality

$$[\pi(\mathcal{L})] = p_5 \left( \sum_{\sigma \in \mathfrak{S}(\mathcal{L})} \text{sgn}(\sigma)[I_\sigma(\mathcal{L})] \right)$$

holds, where $p_5: \mathcal{R}(G_n) \to \mathcal{R}(G_n)$ is the $\mathbb{Z}$-linear extension of

$$[\pi] \mapsto \begin{cases} [\pi] & \text{if } \pi \text{ has the same cuspidal support as } \pi(\mathcal{L}), \\ 0 & \text{otherwise} \end{cases}$$

for each irreducible representation $\pi$ of $G_n$.

Remark that the projection operator $p_5$ can be computed by using [2, Theorem 4.2]. In particular, all terms in the right hand side of our formula can be write down explicitly.

In the following example, when $\phi = \bigoplus_{i=1}^r S_{2x_i+1}$ and $\varepsilon_i = \varepsilon(S_{2x_i+1})$ for $1 \leq i \leq r$, we write $
\pi(\phi, \varepsilon) = \pi(x_1^{r_1}, \ldots, x_r^{r_r}).$

**Example 1.4.** Let us consider $\mathcal{L} := (\{ | \cdot |^0, | \cdot |^1, | \cdot |^2 \}, 1, +1)$. Then $\pi(\mathcal{L})$ is the unique irreducible subrepresentation of $\Delta[0, -2] \times \pi(1^+)$. We give the list of $\sigma \in \mathfrak{S}(\mathcal{L}) = \mathfrak{S}_3$, $\text{sgn}(\sigma)$ and $I_\sigma(\mathcal{L})$ in Table 1.

| $\sigma$ | $\text{sgn}(\sigma)$ | $I_\sigma(\mathcal{L})$ |
|----------|----------------------|------------------------|
|          |                      |                        |

Table 1: List of $\sigma$, $\text{sgn}(\sigma)$, $I_\sigma(\mathcal{L})$ in Example 1.4.
Since \( p_n(\{\pi(0^-,1^+,2^-)\}) = 0 \), Theorem 1.3 asserts that
\[
[\pi(\mathcal{L})] = [\Delta[0,-2] \times \pi(1^+)] - [\Delta[0,-1] \times \pi(2^+)] - [\Delta[1,-2] \times \pi(0^+)] + [\pi(0^+,1^+,2^+)] + [\pi(0^-,1^-,2^+)] + [\pi(0^+,1^-,2^-)] + [\pi(0^-,1^+,2^-)].
\]

This paper is organized as follows. First of all, we recall some general results on representation theory of p-adic classical groups in §2. In §3 we define ladder representations and establish several properties. Finally, in §4 we state and prove our determinantal formula (Theorem 4.2).

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2. Preliminaries

In this section, we introduce our notation.

2.1. Notation. Throughout this paper, we fix a non-archimedean local field \( F \) of characteristic zero with normalized absolute value \(| \cdot |\).

For the group \( G \) of \( F \)-points of a connected reductive group defined over \( F \), representations of \( G \) mean that smooth representations over \( \mathbb{C} \), that is, the stabilizer of every vector is an open subgroup of \( G \). We denote by Rep\((G)\) the category of smooth representations of \( G \) of finite length, and by Irr\((G)\) the set of equivalence classes of irreducible objects of Rep\((G)\). Let \( \mathcal{R}(G) \) be the Grothendieck group of Rep\((G)\). The canonical map from the objects of Rep\((G)\) to \( \mathcal{R}(G) \) will be denoted by \( \pi \mapsto [\pi] \).

Fix a minimal \( F \)-parabolic subgroup \( P_0 \) of \( G \). In this paper, every parabolic subgroup of \( G \) is assumed to be standard, i.e., it contains \( P_0 \). Let \( P = MN \) be a (standard) parabolic subgroup of \( G \). For a representation \( \tau \) of \( M \), we denote by Ind\(_P^G(\tau)\) the normalized parabolically induced representation of \( \tau \). We view Ind\(_P^G\) as a functor. Its left adjoint, the Jacquet functor with respect to \( P \), will be denoted by Jac\(_P\).

We say that an irreducible representation \( \pi \) of \( G \) is supercuspidal if it is not a composition factor of any representation of the form Ind\(_P^G(\tau)\), where \( P = MN \) is a proper parabolic subgroup of \( G \) and \( \tau \) is a representation of \( M \). We write \( \mathcal{C}(G) \) for the subset of Irr\((G)\) consisting of supercuspidal representations. For any \( \pi \in \text{Rep}(G) \), we denote by \( \pi^\vee \) the contragredient of \( \pi \).
2.2. Representations of general linear groups. Set
\[ \text{Irr}^{\text{GL}} := \bigcup_{n \geq 0} \text{Irr}(\text{GL}_n(F)), \quad \mathcal{C}^{\text{GL}} := \bigcup_{n \geq 0} \mathcal{C}(\text{GL}_n(F)). \]

For \( \rho \in \mathcal{C}^{\text{GL}} \), let \( d_\rho \geq 0 \) be such that \( \rho \in \mathcal{C}(\text{GL}_{d_\rho}(F)) \). We also write
\[ \mathcal{R}^{\text{GL}} := \bigoplus_{n \geq 0} \mathcal{R}(\text{GL}_n(F)). \]

For \( \tau_i \in \text{Rep}(\text{GL}_{n_i}(F)) \) for \( i = 1, \ldots, r \), we denote the normalized parabolically induced representation by
\[ \tau_1 \times \cdots \times \tau_r := \text{Ind}_{P(n_1, \ldots, n_r)}^{\text{GL}_1 \times \cdots \times \text{GL}_n}(\tau_1 \boxtimes \cdots \boxtimes \tau_r), \]
where \( P(n_1, \ldots, n_r) \) is the parabolic subgroup of \( \text{GL}_{n_1 + \cdots + n_r}(F) \) with Levi subgroup \( \text{GL}_{n_1}(F) \times \cdots \times \text{GL}_{n_r}(F) \). We define a map \( m: \mathcal{R}^{\text{GL}} \times \mathcal{R}^{\text{GL}} \rightarrow \mathcal{R}^{\text{GL}} \) by the \( \mathbb{Z} \)-linear extension of
\[ [\tau_1] \times [\tau_2] \mapsto [\tau_1 \times \tau_2] \]
for \( \tau_i \in \text{Irr}(\text{GL}_{n_i}(F)) \). It gives a \( \mathbb{Z} \)-graded ring structure on \( \mathcal{R}^{\text{GL}} \).

The Jacquet functor for \( \text{GL}_{n_1 + \cdots + n_r}(F) \) with respect to \( P(n_1, \ldots, n_r) \) is denoted by \( \text{Jac}_{(n_1, \ldots, n_r)} = \text{Jac}_{P(n_1, \ldots, n_r)} \). We define a map \( m^*: \mathcal{R}^{\text{GL}} \rightarrow \mathcal{R}^{\text{GL}} \otimes \mathcal{R}^{\text{GL}} \) by the \( \mathbb{Z} \)-linear extension of
\[ [\tau] \mapsto \sum_{k=0}^{n} [\text{Jac}_{(k, n-k)}(\tau)] \]
for \( \tau \in \text{Irr}(\text{GL}_n(F)) \). It gives a co-multiplication of \( \mathcal{R}^{\text{GL}} \), i.e., \( m^* \) is a ring homomorphism.

Finally, we set
\[ M^*: \mathcal{R}^{\text{GL}} \rightarrow \mathcal{R}^{\text{GL}} \otimes \mathcal{R}^{\text{GL}} \]
to be the composition \( M^* := (m \otimes \text{id}) \circ (\vee \otimes m^*) \circ s \circ m^* \), where \( s: \mathcal{R}^{\text{GL}} \otimes \mathcal{R}^{\text{GL}} \rightarrow \mathcal{R}^{\text{GL}} \otimes \mathcal{R}^{\text{GL}} \) denotes the transposition \( s(\sum_i \tau_i \otimes \tau_i') := \sum_i \tau_i' \otimes \tau_i \).

2.3. Representations of classical groups. In this paper, we let \( G_n \) be either the split special orthogonal group \( \text{SO}_{2n+1}(F) \) or the symplectic group \( \text{Sp}_{2n}(F) \) of rank \( n \). Set
\[ \text{Irr}^G := \bigcup_{n \geq 0} \text{Irr}(G_n), \quad \mathcal{C}^G := \bigcup_{n \geq 0} \mathcal{C}(G_n) \]
and
\[ \mathcal{R}^G := \bigoplus_{n \geq 0} \mathcal{R}(G_n), \]
where the unions and the direct sum are taken over groups of the same type.

Let \( P \) be the standard parabolic subgroup of \( G_n \) with Levi subgroup isomorphic to \( \text{GL}_{d_1}(F) \times \cdots \times \text{GL}_{d_r}(F) \times G_{n_0} \). For \( \pi_0 \in \text{Rep}(G_{n_0}) \) and \( \tau_i \in \text{Rep}(\text{GL}_{d_i}(F)) \) with \( 1 \leq i \leq r \), we denote the normalized parabolically induced representation by
\[ \tau_1 \times \cdots \times \tau_r \times \pi_0 := \text{Ind}_P^{G_n}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0). \]
It gives an \( \mathcal{R}^{\text{GL}} \)-module structure on \( \mathcal{R}^G \) by
\[ \times: \mathcal{R}^{\text{GL}} \otimes \mathcal{R}^G \rightarrow \mathcal{R}^G, \quad [\tau] \otimes [\pi] \mapsto [\tau \times \pi] \]
for \( \tau \in \text{Irr}(\text{GL}_{d}(F)) \) and \( \pi \in \text{Irr}(G_n) \).
On the other hand, Jacquet functors give a comodule structure
\[ \mu^*: \mathcal{R}^G \to \mathcal{R}^{GL} \otimes \mathcal{R}^G \]
by the \( \mathbb{Z} \)-linear extension of
\[ [\pi] \mapsto \sum_{k=0}^n \left[ \text{Jac}_{P_k}^{G_n}(\pi) \right] \]
for \( \pi \in \text{Irr}(G_n) \), where \( P_k \) is the standard parabolic subgroup of \( G_n \) with Levi subgroup isomorphic to \( GL_k(F) \times G_{n-k} \). The Geometric Lemma at the level of Grothendieck groups is commonly known in this case as Tadić’s formula.

**Proposition 2.1** (Tadić’s formula [15]). For \( [\tau] \in \mathcal{R}^{GL} \) and \( [\pi] \in \mathcal{R}^G \), we have
\[ \mu^*([\tau] \times [\pi]) = M^*([\tau]) \rtimes \mu^*([\pi]). \]

For \( \pi \in \text{Irr}(G_n) \), there exist \( \rho_1, \ldots, \rho_r, \sigma \in \mathcal{C}^G \) such that \( \pi \) is a subquotient of \( \rho_1 \times \cdots \times \rho_r \rtimes \sigma \). The multi-set
\[ \text{supp}(\pi) := \{ \rho_1, \ldots, \rho_r, \rho_1^\vee, \ldots, \rho_r^\vee \} \cup \{ \sigma \} \]
is uniquely determined by \( \pi \). We call \( \text{supp}(\pi) \) the *cuspidal support* of \( \pi \). Similarly, when \( \rho_1 \times \cdots \times \rho_r \rtimes \sigma \) is a representation of \( G_n \), we say that the multi-set \( \{ \rho_1, \ldots, \rho_r, \rho_1^\vee, \ldots, \rho_r^\vee \} \cup \{ \sigma \} \) is a *cuspidal support* of \( G_n \). For a cuspidal support \( s \) of \( G_n \), we define a map \( p_s: \mathcal{R}(G_n) \to \mathcal{R}(G_n) \) by the \( \mathbb{Z} \)-linear extension of
\[ [\pi] \mapsto \begin{cases} [\pi] & \text{if } \text{supp}(\pi) = s, \\ 0 & \text{otherwise} \end{cases} \]
for \( \pi \in \text{Irr}(G_n) \). By the Langlands classification explained in §2.4 below, the computation of \( p_s([\pi]) \) for \( \pi \in \text{Irr}(G_n) \) is reduced to the case where \( \pi \) is tempered. Since the cuspidal supports of irreducible tempered representations can be computed by [2, Theorem 4.2], one can easily determine \( p_s([\pi]) \) for any \( \pi \in \text{Irr}(G_n) \).

Fix \( \rho \in \mathcal{C}(\text{GL}_d(F)) \). For \( \pi \in \text{Rep}(G_n) \) with \( n \geq d \), we define the \( \rho \)-derivative \( D_\rho(\pi) \) of \( \pi \) by the semisimple representation of \( G_{n-d} \) satisfying that
\[ [\text{Jac}_{P_\rho}(\pi)] = [\rho] \boxtimes [D_\rho(\pi)] + \sum_i [\tau_i] \otimes [\pi_i], \]
where \( \tau_i \in \text{Irr}(\text{GL}_d(F)) \) and \( \pi_i \in \text{Irr}(G_{n-d}) \) such that \( \tau_i \not\cong \rho \). When \( n < d \), we understand that \( D_\rho(\pi) = 0 \) for any \( \pi \in \text{Rep}(G_n) \). By the \( \mathbb{Z} \)-linear extension, we regard \( D_\rho \) as a map \( \mathcal{R}^G \to \mathcal{R}^G \). We note that for a cuspidal support \( s \) of \( G_n \), we have
\[ D_\rho \circ p_s = \begin{cases} p_{s'} \circ D_\rho & \text{if } \rho \in s, \\ 0 & \text{otherwise}, \end{cases} \]
where \( s' \) is the cuspidal support of \( G_{n-d} \) such that \( s = s' \cup \{ \rho, \rho^\vee \} \).

We say that \( \rho \in \mathcal{C}^GL \) is *good* if
- \( \rho^\vee \cong \rho| \cdot |^a \) for some \( a \in \mathbb{Z} \); and
- \( \rho| \cdot |^m \rtimes 1_{G_0} \) is reducible for some \( m \in \mathbb{Z} \).

Note that for any \( \rho \in \mathcal{C}^GL \) satisfying that \( \rho \cong \rho^\vee \), exactly one of \( \rho \) or \( \rho| \cdot |^{1/2} \) is good. An irreducible representation \( \pi \) of \( G_n \) is said to be *of good parity* if every \( \rho \in \text{supp}(\pi) \cap \mathcal{C}^GL \) is good.
2.4. Langlands classification. For \( \tau \in \text{Rep}(\text{GL}_n(F)) \) and a character \( \chi \) of \( F^\times \), we denote by \( \tau \chi \) the representation obtained from \( \tau \) by twisting by the character \( \chi \circ \det \). A segment \([x, y]_\rho \) is a set of supercuspidal representations of the form

\[
\{ \rho \cdot |^\tau \cdot \rho| \cdot |^{x-1} \ldots, \rho| \cdot |^y \},
\]

where \( \rho \in \mathcal{C}^{\text{GL}} \) and \( x, y \in \mathbb{R} \) with \( x - y \in \mathbb{Z} \) and \( x \geq y \). For a segment \([x, y]_\rho \), we define the Steinberg representation \( \Delta_\rho[x, y] \) of \( \text{GL}_{d_\rho(x-y+1)}(F) \) by the unique irreducible subrepresentation of

\[
\rho \cdot |^x \times \rho| \cdot |^{x-1} \times \ldots \times \rho| \cdot |^y .
\]

It is an essentially discrete series and all essentially discrete series are of this form [19, Theorem 9.3]. By convention, we set \( \Delta_\rho \) to be the trivial representation of the trivial group \( \text{GL}_0(F) \), and \( \Delta_\rho[x, y] := 0 \) for \( y > x + 1 \).

Let \([x_1, y_1]_{\rho_1}, \ldots, [x_r, y_r]_{\rho_r} \) be segments, where \( \rho_i \in \mathcal{C}^{\text{GL}} \) is assumed to be unitary for \( 1 \leq i \leq r \), and let \( \pi_{\text{temp}} \) be an irreducible tempered representation of \( G_{n_0} \). When \( x_1 + y_1 \leq \cdots \leq x_r + y_r < 0 \), the parabolically induced representation

\[
\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \times \pi_{\text{temp}}
\]

is called a standard module. It has a unique irreducible subrepresentation, denoted by

\[
L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi_{\text{temp}}).
\]

Conversely, the Langlands classification says that any irreducible representation \( \pi \) of \( G_n \) is of this form for some standard module. It is known that the isomorphism classes of standard modules of \( G_n \) give a basis of \( \mathcal{B}(G_n) \).

By the local Langlands correspondence established by Arthur [1], every irreducible tempered representation \( \pi_{\text{temp}} \) of \( G_n \) is written as \( \pi_{\text{temp}} = \pi(\phi, \epsilon) \), where \( \phi \) is a tempered \( L \)-parameter for \( G_n \) and \( \epsilon \) is a character of the component group of \( \phi \). When \( \pi_{\text{temp}} = \pi(\phi, \epsilon) \) is of good parity, they are described as follows:

- \( \phi \) is decomposed into a (formal) sum

\[
\phi = \bigoplus_{i=1}^{r} \rho_i \otimes S_{a_i},
\]

where \( \rho_i \in \mathcal{C}^{\text{GL}} \) is assumed to be unitary, and \( S_{a_i} \) is a unique irreducible algebraic representation of \( \text{SL}_2(\mathbb{C}) \) of dimension \( a_i \) such that

1. \( \rho_i \cdot \frac{a_{i+1}}{2} \) is good for \( i = 1, \ldots, r \);
2. \( \sum_{i=1}^{r} d_{\rho_i} a_i = 2n \) if \( G_n = \text{SO}_{2n+1}(F) \) (resp. \( \sum_{i=1}^{r} d_{\rho_i} a_i = 2n+1 \) if \( G_n = \text{Sp}_{2n}(F) \));
3. \( \prod_{i=1}^{r} \omega_{\rho_i} = 1 \), where \( \omega_i \) is the central character of \( \rho_i \);

- \( \epsilon \) is characterized by a tuple of sign \( (\epsilon(\rho_1 \otimes S_{a_1}), \ldots, \epsilon(\rho_r \otimes S_{a_r})) \in \{ \pm 1 \}^r \) such that

4. if \( \rho_i \cong \rho_j \) and \( a_i = a_j \), then \( \epsilon(\rho_i \otimes S_{a_i}) = \epsilon(\rho_j \otimes S_{a_j}) \);
5. \( \prod_{i=1}^{r} \epsilon(\rho_i \otimes S_{a_i}) = 1 \).

By convention, let \( S_0 \) be the zero representation of \( \text{SL}_2(\mathbb{C}) \). When \( \rho \cdot |^{1/2} \) is good, we formally allow the situation \( \phi \preceq \rho \otimes S_0 \). In this case, we set

\[
\pi(\phi, \epsilon) := \begin{cases} 
\pi(\phi - \rho \otimes S_0, \epsilon) & \text{if } \epsilon(\rho \otimes S_0) = 1, \\
0 & \text{if } \epsilon(\rho \otimes S_0) = -1.
\end{cases}
\]
3. Ladder representations

In this section, we define ladder representations for $G_n$, and establish several properties.

3.1. Definition. Recall that ladder representations of $\text{GL}_n(F)$ are obtained by appropriate Jacquet modules of Speh representations. Inspired by Mœglin’s construction of irreducible representations with $A$-parameters, we introduce the following notion.

**Definition 3.1.** We say that $\mathcal{L} = (X, l, \eta)$ is a ladder datum for $G_n$ if the following conditions hold.

1. $X = (X_\rho)_{\rho}$, $l = (l_\rho)$ and $\eta = (\eta_\rho)$, where $\rho$ runs over $\mathcal{C}^{\text{GL}}$ such that $\rho' \cong \rho$.
2. $X_\rho$ is a finite subset of $\mathcal{C}^{\text{GL}}$ of the form
   \[ X_\rho = \{ \rho \cdot \vert x_1, \ldots, \rho \cdot \vert x_\nu \} \]
   such that
   - all but finitely many $\rho$, the cardinality $t_\rho$ is zero;
   - $\rho \cdot \vert x_j$ is good (so that $2x_i \in \mathbb{Z}$) for $1 \leq j \leq t_\rho$;
   - $x_1 < \cdots < x_{t_\rho}$, which depend on $\rho$;
   - the sum
     \[ \sum_\rho (2x_1 + \cdots + 2x_{t_\rho} + t_\rho)d_\rho \]
     is equal to $2n$ if $G_n = \text{SO}_{2n+1}(F)$ (resp. $2n + 1$ if $G_n = \text{Sp}_{2n}(F)$).
3. $l_\rho \in \mathbb{Z}$ such that
   - $0 \leq 2l_\rho \leq t_\rho$;
   - $x_j + x_{t_\rho-j+1} \geq 0$ for $1 \leq j \leq l_\rho$ (especially $x_{t_\rho-l_\rho+1} > 0$);
   - $x_i \geq -1/2$ for $l_\rho + 1 \leq i \leq t_\rho - l_\rho$.
4. $\eta_\rho \in \{ \pm 1 \}$ such that
   - $\eta_\rho = 1$ if $X_\rho = \emptyset$ or if $x_{t_\rho+1} = -1/2$;
   - $\eta_\rho = -1$ if $2l_\rho = t_\rho$;
   - the equation
     \[ \prod_\rho (-1)^{t_\rho + l_\rho} \eta_\rho^{t_\rho} = 1 \]
     holds.

Define $\pi(\mathcal{L}) \in \text{Irr}(G_n)$ by the unique irreducible subrepresentation of the standard module
\[ I(\mathcal{L}) := \bigotimes_\rho (\Delta_\rho[x_1, -x_{t_\rho}] \times \Delta_\rho[x_2, -x_{t_\rho-1}] \times \cdots \times \Delta_\rho[x_{t_\rho}, -x_{t_\rho-l_\rho+1}]) \rtimes \pi(\phi, \varepsilon), \]
where
\[ \phi := \bigoplus_{\rho \in \mathbb{Z}} t_{\rho} - l_{\rho} \rho \boxtimes S_{2x_1+1} \]
and $\varepsilon(\rho \boxtimes S_{2x_i+1}) := (-1)^{i-l_\rho-1} \eta_\rho$ for $l_\rho+1 \leq i \leq t_\rho - l_\rho$. We call $\pi(\mathcal{L})$ a ladder representation of $G_n$.

Note that $\pi(\mathcal{L})$ is an irreducible representation of good parity of $G_n$. Moreover, the map $\mathcal{L} \mapsto \pi(\mathcal{L})$ is injective.

**Remark 3.2.** Let $\mathcal{L} = (X, l, \eta)$ be a ladder datum with $X = (X_\rho)_\rho$ and $l = (l_\rho)_\rho$. 
(1) If \( l_\rho = 0 \) for all \( \rho \), then \( \pi(\mathcal{L}) \) is strongly positive discrete series. For this notion, see \cite{[12]} for example. Conversely, every strongly positive discrete series representation can be written as \( \pi(\mathcal{L}) \) for some ladder datum \( \mathcal{L} = (X, L, \eta) \) with \( l_\rho = 0 \) for all \( \rho \).

(2) If \( X_\rho \) is a segment \([x, y]\) for all \( \rho \), then \( \pi(\mathcal{L}) \) is an irreducible representation with an \( A \)-parameter. In this case, the definition of \( \pi(\mathcal{L}) \) is exactly the same as Mœglin’s construction. For more precision, see \cite{[18]}. Conversely, if \( \pi \) is an irreducible representation with an irreducible \( A \)-parameter, then \( \pi \) can be written as \( \pi(\mathcal{L}) \) for some ladder datum \( \mathcal{L} = (X, L, \eta) \) such that \( X_\rho \) is a segment for some \( \rho \) and \( X_{\rho'} = \emptyset \) for \( \rho' \not\equiv \rho \).

3.2. The graph. Fix \( \rho \in \mathcal{C}^{GL} \) such that \( \rho' \cong \rho \). For a ladder datum \( \mathcal{L} = (X, L, \eta) \), we write \( \mathcal{L}_\rho = (X_\rho, l_\rho, \eta_\rho) \). As an analogue of \cite{[14]} §3, we attach a directed 3-colorable graph to \( \mathcal{L}_\rho \).

**Definition 3.3.** Write \( X_\rho = \{ \rho \cdot |x_1, \ldots, \rho \cdot |x_t \} \) with \( x_1 < \cdots < x_t \) and \((l, \eta) = (l_\rho, \eta_\rho)\). We define a directed 3-colorable graph \( \mathcal{E}(\mathcal{L}_\rho) = (\mathcal{V}(\mathcal{L}_\rho), \prec, f_\rho) \) as follows.

1. The vertex set of \( \mathcal{E}(\mathcal{L}_\rho) \) is

\[
\mathcal{V}(\mathcal{L}_\rho) := \{(x_{t+1+i}, -i), (x_{t+1+i} - 1, -i), \ldots, (-x_{t-l-i} - 1, -i) \mid -l \leq i \leq t - l - 1\}.
\]

2. The edges of \( \mathcal{E}(\mathcal{L}_\rho) \) consist of the horizontal arrows and the diagonal arrows assigned as follows:

- If \( x_1, \ldots, x_t \in \mathbb{Z} \), then the horizontal arrows are

\[
(j, -i) \prec (j - 1, -i)
\]

for \(-l \leq i \leq t - l - 1 \) and \(-x_{t-l-i} + 1 \leq j \leq x_{t+1+i} \), and the diagonal arrows are

\[
(j, -i) \prec (j + 1, -(i + 1))
\]

for \(-l \leq i < t - l - 1 \) and \(-x_{t-l-(i+1)} - 1 \leq j \leq x_{t+1+i} \).

- If \( x_1, \ldots, x_t \in (1/2)\mathbb{Z} \setminus \mathbb{Z} \), then the horizontal arrows are

\[
(j + 1/2, -i) \prec (j - 1/2, -(i + 1))
\]

for \(-l \leq i \leq t - l - 1 \) and \(-x_{t-l-i} + 1 \leq j + 1/2 \leq x_{t+1+i} \), and the diagonal arrows are

\[
(j - 1/2, -i) \prec (j + 1/2, -(i + 1))
\]

for \(-l \leq i < t - l - 1 \) and \(-x_{t-l-(i+1)} - 1 \leq j - 1/2 \leq x_{t+1+i} \).

3. The coloring map \( f_\rho : \mathcal{V}(\mathcal{L}_\rho) \rightarrow \{-1, 0, 1\} \) is given as follows:

- If \( x_1, \ldots, x_t \in \mathbb{Z} \), then

\[
f_\rho(j, -i) := \begin{cases} (-1)^i \eta & \text{if } 0 \leq i \leq t - 2l - 1, \ i - (t - 2l - 1) \leq j \leq i, \\ 0 & \text{otherwise}. \end{cases}
\]

Here, we note that \( x_{t+1+i} \geq i \) for \( 0 \leq i \leq t - 2l - 1 \).

- If \( x_1, \ldots, x_t \in (1/2)\mathbb{Z} \setminus \mathbb{Z} \), then

\[
f_\rho\left(j - \frac{1}{2} \eta, -i\right) := \begin{cases} (-1)^j \eta & \text{if } 0 \leq i \leq t - 2l - 1, \ i - (t - 2l - 1) + \eta \leq j \leq i, \\ 0 & \text{otherwise}. \end{cases}
\]

Here, we note that \( x_{t+1+i} \geq i - \frac{1}{2} \eta \) for \( 0 \leq i \leq t - 2l - 1 \).
Corollary 3.7.  

Note that $\rho \in \mathcal{L}$. Thus, the map $\rho \mapsto \mathcal{E}(\mathcal{L}_\rho) = (\mathcal{V}(\mathcal{L}_\rho), \prec, f_\rho)$ is injective. In particular, one can specify $\mathcal{E}(\mathcal{L}_\rho)$ to assign $\mathcal{L}_\rho$.

Proposition 3.6 immediately implies $\mathcal{L}_\rho$. Now the proposition follows from [5, Theorem 7.1].

Remark 3.4.  

(1) The map $\mathcal{L}_\rho \mapsto \mathcal{E}(\mathcal{L}_\rho) = (\mathcal{V}(\mathcal{L}_\rho), \prec, f_\rho)$ is injective. In particular, one can specify $\mathcal{E}(\mathcal{L}_\rho)$ to assign $\mathcal{L}_\rho$.

(2) Note that $f_\rho(x, y) = 0 \iff f_\rho(-x, -(t - 2l - 1) - y) = 0$. Moreover, if $(x, y) = (-x, -(t - 2l - 1) - y)$, then $t$ is odd and $(x, y) = (0, -(\frac{t - 2l - 1}{2}))$ so that $f_\rho(x, y) \neq 0$. Hence $|f_\rho^{-1}(0)|$ is even. We will write $2m_\rho = |f_\rho^{-1}(0)|$.

(3) For each $x \in \mathbb{R}$, the graph $\mathcal{E}(\mathcal{L}_\rho)$ has at most one minimal vertex of the form $(x, y)$ for $y \in \mathbb{R}$.

Example 3.5.  

(1) Consider $\mathcal{L}_\rho = (\{|1/2, \rho| \cdot |5/2, \rho| \cdot |7/2, \rho|\}, 0, -1)$. Then the graph $\mathcal{E}(\mathcal{L}_\rho)$ is as follows:

Here, for a vertex $(x, y) \in \mathcal{V}(\mathcal{L}_\rho)$, we put a white circle if $f_\rho(x, y) = 0$, whereas a block circle if $f_\rho(x, y) \neq 0$.

(2) Consider $\mathcal{L}_\rho = (\{4, 0\}, 1, -1)$. Then the graph $\mathcal{E}(\mathcal{L}_\rho)$ is as follows:

3.3. Jacquet modules. The graph $\mathcal{E}(\mathcal{L}_\rho) = (\mathcal{V}(\mathcal{L}_\rho), \prec, f_\rho)$ defined in the previous subsection knows Jacquet modules of $\pi(\mathcal{L})$. First, we consider derivatives of $\pi(\mathcal{L})$.

Proposition 3.6. For $x \in \mathbb{R}$, the derivative $\mathcal{D}_{\rho | x}(\pi(\mathcal{L}))$ is nonzero if and only if $\mathcal{E}(\mathcal{L}_\rho)$ has a minimal vertex of the form $(x, y)$ for some $y$ such that $f_\rho(x, y) = 0$. In this case, $\mathcal{D}_{\rho | x}(\pi(\mathcal{L}))$ is also a ladder representation. If we write $\mathcal{D}_{\rho | x}(\pi(\mathcal{L})) = \pi(\mathcal{L}')$, then $\mathcal{L}_\rho' = \mathcal{L}_\rho$ for all $\rho' \neq \rho$, and the graph $\mathcal{E}(\mathcal{L}_\rho')$ is given from $\mathcal{E}(\mathcal{L}_\rho)$ by removing two vertices $(x, y)$ and $(-x, -(t_\rho - 2l_\rho - 1) - y)$ from $\mathcal{V}(\mathcal{L}_\rho)$, where $t_\rho = |X_\rho|$.

Proof. Note that $\mathcal{D}_{\rho | x} \circ \mathcal{D}_{\rho | x}(\pi(\mathcal{L})) = 0$ for any $x \in \mathbb{R}$ by Tadić’s formula (Proposition [2,1] and [17, Lemma 7.3]. Now the proposition follows from [5, Theorem 7.1].

As a consequence, the class of ladder representations is preserved by derivatives, which is the same as in the general linear groups case (see [10]). Proposition 3.6 immediately implies the following corollary.

Corollary 3.7. Let $\mathcal{L}$ be a ladder datum for $G_n$. 


(1) The ladder representation $\pi(L)$ is supercuspidal if and only if $m_\rho = 0$ for all $\rho \in \mathcal{C}^{\text{GL}}$ with $\rho^\vee \cong \rho$.

(2) More generally, the unique representation $\sigma \in \text{supp}(\pi(L)) \cap \mathcal{C}^G$ is a ladder representation with the ladder datum $L_\sigma$ such that for any $\rho \in \mathcal{C}^{\text{GL}}$ with $\rho^\vee \cong \rho$, the graph $\mathcal{E}(L_{\sigma,\rho})$ is given from $\mathcal{E}(L_\rho)$ by removing all vertices $(x, y)$ such that $f_\rho(x, y) = 0$.

Recall that we have a $\mathbb{Z}$-linear map $\mu^*: \mathcal{B}^G \to \mathcal{B}^{\text{GL}} \otimes \mathcal{B}^G$ in §2.3. For $\pi \in \text{Irr}^G$, when

$$\mu^*([\pi]) = \sum_{i \in I} [\tau_i] \otimes [\pi_i]$$

with $\tau_i \in \text{Irr}^{\text{GL}}$ and $\pi_i \in \text{Irr}^G$, for fixed $\rho \in \mathcal{C}^{\text{GL}}$ with $\rho \cong \rho^\vee$, set

$$\mu^\rho_\rho([\pi]) := \sum_{i \in I_\rho} [\tau_i] \otimes [\pi_i],$$

where $I_\rho$ is the subset of $I$ consisting of indices $i$ such that $\tau_i$ is a subquotient of $\rho| \cdot |x_1 \times \cdots \times \rho| \cdot |x_r$ for some $x_1, \ldots, x_r \in \mathbb{R}$. By the same argument as [2, Lemma 2.6], when $\pi \in \text{Irr}^G$ is of good parity, $\mu^*([\pi])$ is recovered from $\mu^\rho_\rho([\pi])$ for all $\rho \in \mathcal{C}^{\text{GL}}$ with $\rho \cong \rho^\vee$.

When $\pi$ is a ladder representation of $G_n$, one can compute $\mu^\rho_\rho([\pi])$.

**Theorem 3.8.** Let $L = (X, l, \eta)$ be a ladder datum for $G_n$. Write $X_\rho = \{\rho| \cdot |x_1 \times \cdots \times \rho| \cdot |x_t\}$ with $x_1 < \cdots < x_t$ and $(l, \eta) := (l_\rho, \eta_\rho)$. Then

$$\mu^\rho_\rho([\pi(L)]) = \sum_{\mathbf{y}=(y_1, \ldots, y_t)} \left[ L(\Delta_\rho[x_1, y_1 + 1], \ldots, \Delta_\rho[x_t, y_t + 1]) \right] \otimes \left[ \pi(L_y) \right],$$

where $\mathbf{y} = (y_1, \ldots, y_t)$ runs over the subset of $\mathbb{R}^l$ such that

- $-x_{t-i} - 1 \leq y_i \leq x_i$ and $y_i \equiv x_i \text{ mod } \mathbb{Z}$ for $1 \leq i \leq t$;
- $y_1 < \cdots < y_t$;
- $y_i + y_t-i+1 \geq -1$ for $1 \leq i \leq l$;
- if $l+1 \leq i \leq t-l$, then

$$y_i \geq \begin{cases} i - l - 1 & \text{if } x_1, \ldots, x_t \in \mathbb{Z}, \\ i - l - 1 - \frac{1}{2} \eta & \text{if } x_1, \ldots, x_t \in \mathbb{Z} \setminus \mathbb{Z}, \end{cases}$$

and $L_y = (X, l', \eta)$ is defined by

- $X_\rho := X_{\rho'}$ and $l_\rho' := l_\rho$ for $\rho' \not\cong \rho$;
- $Y_\rho := \{\rho| |y_i| \leq t, y_i + y_t-i+1 \geq 0\}$;
- $l_\rho' := l_\rho - \#\{1 \leq i \leq l | y_i + y_t-i+1 = -1\}$.

**Proof.** Write $d := d_\rho$. Let $P_{dk}$ be the standard parabolic subgroup of $G_n$ with Levi subgroup isomorphic to $\text{GL}_{dk}(F) \times G_{n-dk}$. Write

$$[\text{Jac}_{P_{dk}}(\pi(L))] = \sum_{i \in I} [\tau_i] \otimes [\pi_i]$$

with $\tau_i \in \text{Irr}(\text{GL}_{dk}(F))$ and $\pi_i \in \text{Irr}(G_{n-dk})$. Set $I_\rho$ to be the subset of $I$ consisting of indices $i$ such that $\tau_i$ is a subquotient of $\rho| \cdot |x_1 \times \cdots \times \rho| \cdot |x_k$ for some $x_1, \ldots, x_k \in \mathbb{R}$. Note that by Proposition 3.6 we see that $\pi_i$ is a ladder representation of $G_{n-dk}$.
We claim that for a fixed ladder datum \( \mathcal{L}_0 \) of \( G_{n-dk} \) such that \( \mathcal{L}_{0, \rho'} = \mathcal{L}_{\rho'} \) for \( \rho' \neq \rho \), the equation
\[
\sum_{\pi \in \Pi(\mathcal{L}_{0, \rho})} [\tau_\pi] = \sum_{\pi \in \Pi(\mathcal{L}_0)} \left[ L(\Delta_\rho[x_1, y_1 + 1], \ldots, \Delta_\rho[x_t, y_t + 1]) \right]
\]
holds. For simplicity, we write \( A \) (resp. \( B \)) for the left (resp. the right) hand side of this equation. Let \( Q \) be the standard parabolic subgroup of \( GL_{dk}(F) \) with Levi subgroup isomorphic to \( GL_d(F) \times \cdots \times GL_d(F) \). To prove \( A = B \), it is enough to show that \( [\text{Jac}_Q(A)] = [\text{Jac}_Q(B)] \).

By applying Proposition 3.6 repeatedly, we see that
\[
[\text{Jac}_Q(A)] = \sum_{\Delta = (\lambda_1, \ldots, \lambda_k)} \rho \cdot |\lambda_1 \otimes \cdots \otimes \lambda_k|,
\]
where \( \Delta = (\lambda_1, \ldots, \lambda_k) \) runs over the subset of \( \mathbb{R}^k \) such that

- the graph \( \mathcal{E}_0 := \mathcal{E}((\lambda_1, \mu_1)) \) has a minimal vertex of the form \( (\lambda_1, \mu_1) \) for some \( \mu_1 \) such that \( f_\rho(\lambda_1, \mu_1) = 0 \);
- for \( 1 \leq i \leq k - 1 \), the graph
  \[
  \mathcal{E}_i := \mathcal{E}_{i-1} \setminus \{ (\lambda_i, \mu_i), (-\lambda_i, -(t - 2l - 1) - \mu_i) \}
  \]
  has a minimal vertex of the form \( (\lambda_{i+1}, \mu_{i+1}) \) for some \( \mu_{i+1} \) such that \( f_\rho(\lambda_{i+1}, \mu_{i+1}) = 0 \);
- the graph
  \[
  \mathcal{E}_k := \mathcal{E}_{k-1} \setminus \{ (\lambda_k, \mu_k), (-\lambda_k, -(t - 2l - 1) - \mu_k) \}
  \]
is equal to \( \mathcal{E}(\mathcal{L}_0, \rho) \).

By [11 Theorem 7 (2)], one can compute \( [\text{Jac}_Q(B)] \) similarly. Then we see that \( [\text{Jac}_Q(A)] = [\text{Jac}_Q(B)] \), as desired. This completes the proof. \( \square \)

Note that even if \( y \neq y' \), one might have \( \mathcal{L}_y = \mathcal{L}_{y'} \).

**Example 3.9.** Set \( \rho = 1_{GL_1(F)} \). Consider \( \mathcal{L} = (X, \eta, \eta) \) with \( X' = \emptyset \) for \( \rho' \neq \rho \) and

\[
\mathcal{L}_\rho = (\{ |\rho| \}, 1, 1, 1, +1).
\]

Then \( \pi(\mathcal{L}) = L(\Delta_\rho[0, -2]; \pi(\rho \boxtimes S_3, +)) \in \text{Irr}(Sp_8(F)) \). If \( P \) is the standard Siegel parabolic subgroup, then Theorem 3.8 says that

\[
[\text{Jac}_P(\pi(\mathcal{L}))] = [L(\Delta_\rho[0, -2], |\rho| \cdot |\cdot^1|)] \otimes [1_{Sp_0(F)}] + [L(\Delta_\rho[0, -1], |\rho| \cdot |\cdot^1| \cdot |\cdot^2|)] \otimes [1_{Sp_0(F)}].
\]

By Theorem 3.8, we see that all irreducible subquotients of \( \text{Jac}_P(\pi(\mathcal{L})) \) have distinct cuspidal supports. Hence we have:

**Corollary 3.10.** Let \( \pi(\mathcal{L}) \) be a ladder representation of \( G_n \). Then for any parabolic subgroup \( P \) of \( G_n \), the Jacquet module \( \text{Jac}_P(\pi(\mathcal{L})) \) is completely reducible.

As an analogue of the general groups case [11 Corollary 4.11], one might expect that the converse of Corollary 3.10 would hold in an appropriate sense. However, we have a counterexample as follows.
Example 3.11. Consider $\pi = \pi(\phi, \varepsilon) \in \text{Irr}(\text{SO}_7(F))$ with $\phi = S_2 \oplus S_4$ and $\varepsilon(S_2) = \varepsilon(S_4) = -1$. It is a discrete series representation but not ladder. For $k \in \{1, 2, 3\}$, let $P_k$ be the standard parabolic parabolic subgroup of $\text{SO}_7(F)$ with Levi subgroup isomorphic to $\text{GL}_k(F) \times \text{SO}_{7-2k}(F)$. Then by [2] Theorems 4.2, 4.3, we have

$$\text{Jac}_{P_7}(\pi) = | \cdot |^{7/2} \otimes \pi(\phi', \varepsilon'),$$
$$\text{Jac}_{P_2}(\pi) = \Delta[3, -1] \otimes L(| \cdot |^{-1/2}; 1_{\text{SO}_1(F)}),$$
$$\text{Jac}_{P_3}(\pi) = \Delta[3, -1/2] \otimes 1_{\text{SO}_4(F)}.$$ 

Combining with [10] 9.5 Proposition, we see that $\text{Jac}_P(\pi)$ is irreducible for every parabolic subgroup $P$ of $\text{SO}_7(F)$.

3.4. Aubert duality. Recall from [5] that for $\pi \in \text{Irr}(G_n)$, there exists a sign $\varepsilon \in \{\pm 1\}$ such that the virtual representation

$$\hat{\pi} := \varepsilon \sum_{P=MN} (-1)^{\dim A_M} \text{Ind}_{P^*}^{G^*}(\text{Jac}_P(\pi))$$

is in fact an irreducible representation, where $P = MN$ runs over all standard parabolic subgroups of $G_n$, and $A_M$ is the maximal split torus of the center of $M$. We call $\hat{\pi}$ the Aubert dual of $\pi$.

An explicit formula for the Aubert duals of strongly positive discrete series representations (resp. irreducible representations with (irreducible) $A$-parameters) was given in [12] Theorem 3.6] (resp. [3] Theorem 6.2]). Using the graph $E(L_\rho)$, we can extend these explicit formulas to the case of ladder representations.

Theorem 3.12. For a ladder datum $L$, we define another ladder datum $\hat{L}$ so that for $\rho \in \mathcal{E}^{\text{GL}}$ with $\rho^\vee \cong \rho$, the graphs $E(L_\rho) = (\mathcal{V}(L_\rho), \prec, f_\rho)$ and $E(L_\hat{\rho}) = (\mathcal{V}(\hat{L}_\rho), \prec, \hat{f}_\rho)$ are related as follows.

1. $(\mathcal{V}(\hat{L}_\rho), \prec)$ is given from $(\mathcal{V}(L_\rho), \prec)$ by changing the diagonal arrows and the horizontal arrows. More precisely,
   - if $\rho$ is good, then for $k \in \mathbb{Z}$, the line $x + y = k$ (resp. $y = k$ in $\mathcal{V}(L_\rho)$ corresponds to the line $y = k$ (resp. $x + y = k$) in $\mathcal{V}(\hat{L}_\rho)$;
   - if $|\rho| \cdot \frac{1}{2}$ is good, then for $k \in \mathbb{Z}$, the line $x + y = k - \frac{1}{2} \eta$ (resp. $y = k$ in $\mathcal{V}(L_\rho)$ corresponds to the line $y = k$ (resp. $x + y = k + \frac{1}{2} \eta$) in $\mathcal{V}(\hat{L}_\rho)$.

2. $\hat{f}_\rho : \mathcal{V}(\hat{L}_\rho) \to \{-1, 0, 1\}$ is given by

$$\hat{f}_\rho(\hat{x}, \hat{y}) = \begin{cases} 
  f_\rho(x, y) & \text{if } \rho \text{ is good}, \\
  -f_\rho(x, y) & \text{if } |\rho| \cdot \frac{1}{2} \text{ is good},
\end{cases}$$

where $(\hat{x}, \hat{y}) \in \mathcal{V}(\hat{L}_\rho)$ is the vertex corresponding to $(x, y) \in \mathcal{V}(L_\rho)$ by (1).

Then the Aubert dual of $\pi(L_\rho)$ is isomorphic to $\pi(\hat{L}_\rho)$.

Proof. This follows from [5] Algorithm 4.1] together with Proposition 3.6. \qed

Example 3.13. Consider $E(L_\rho)$ as follows:
Then $E(\hat{L}_\rho)$ is as follows:

In particular, if $X_\rho' = \emptyset$ for any $\rho' \not\sim \rho$, then

$$\pi(L) = L(\Delta_\rho[0,-4]; \pi(\rho \boxtimes S_3,+)),$$

$$\pi(\hat{L}) = L(\rho \mid \cdot \mid^{-4}, \rho \mid \cdot \mid^{-3}, \Delta_\rho[0,-2]; \pi(\rho \boxtimes S_3,+)).$$

One can check that $\hat{\pi}(L) \cong \pi(\hat{L})$ by [5, Algorithm 4.1].

4. Determinantal formula

In this section, we describe ladder representations of $G_n$ in terms of linear combinations of standard modules in $R^G$.

4.1. Statement. Let $\mathcal{L} = (X, l, \eta)$ be a ladder datum of $G_n$. For $\rho \in C^{GL}$ with $\rho^\vee \cong \rho$, we write $X_\rho = \{\rho \mid |x_1|, \ldots, \rho \mid |x_{t_\rho}\}$ with $x_1 < \cdots < x_{t_{\rho}}$. For a non-negative integer $t$, we denote by $\mathcal{S}_t$ the symmetric group of order $t!$. In particular, $\mathcal{S}_0 = \mathcal{S}_1$ is the trivial group.

Definition 4.1. (1) Let $\mathcal{S}(\mathcal{L})$ be the subset of the direct sum of symmetric groups $\oplus_\rho \mathcal{S}_{t_\rho}$ consisting of $\sigma = (\sigma_\rho)_\rho$ such that

- for $1 \leq i < j \leq l_\rho$, we have $\sigma_\rho(i) < \sigma_\rho(j)$;
- for $l_\rho + 1 \leq i < j \leq t_\rho - l_\rho$, we have $\sigma_\rho(i) < \sigma_\rho(j)$;
- if $x_i \leq -1$, or if $x_i = -1/2$ and $\eta_\rho = -1$, then $\sigma_\rho^{-1}(i) \leq l_\rho$.

(2) The restriction of the product of the sign characters

$$\bigoplus_\rho \mathcal{S}_{t_\rho} \xrightarrow{\oplus \text{sgn}} \bigoplus_\rho \{\pm 1\} \xrightarrow{\text{product}} \{\pm 1\}.$$
gives a sign map
\[ \text{sgn}: \mathfrak{S}(L) \to \{ \pm 1 \}. \]

(3) For \( \sigma = (\sigma_\rho)_\rho \in \mathfrak{S}(L) \), set
\[
J_{\sigma_\rho}^+ := \{ 1 \leq j \leq l_\rho \mid \sigma_\rho(j) < \sigma_\rho(t_\rho - j + 1) \},
\]
\[
J_{\sigma_\rho}^- := \{ 1 \leq j \leq l_\rho \mid \sigma_\rho(j) > \sigma_\rho(t_\rho - j + 1) \}.
\]

We define a direct sum of standard modules \( I_\sigma(L) \) by
\[
I_\sigma(L) := \bigoplus_{\rho} \bigoplus_{j \in J_{\sigma_\rho}^+} \Delta_\rho[x_{\sigma_\rho(j)}, -x_{\sigma_\rho(t_\rho - j + 1)}] \otimes \left( \bigoplus_{\delta} \pi(\phi_\sigma, \varepsilon_{\sigma_\delta}) \right),
\]
where \( \delta = (\delta_\rho)_\rho \) runs over the set of tuples of maps
\[ \delta_\rho: J_{\sigma_\rho}^- \to \{ \pm 1 \}, \]
and \( (\phi_\sigma, \varepsilon_{\sigma_\delta}) \) is given by
\[
\phi_\sigma := \bigoplus_{\rho} \bigoplus_{i = t_\rho + 1}^{t_\rho - l_\rho} \rho \boxtimes S_{2x_{\sigma_\rho}(i)} + 1
\]
\[
\oplus \bigoplus_{\rho} \bigoplus_{j \in J_{\sigma_\rho}^-} \rho \boxtimes (S_{2x_{\sigma_\rho}(j)} + 1 \oplus S_{2x_{\sigma_\rho}(t_\rho - j + 1) + 1})
\]
and
\[
\varepsilon_{\sigma_\delta} (\rho \boxtimes S_{2x_{\sigma_\rho}(i + 1)}) := (-1)^{i - l_\rho - 1} \eta_\rho,
\]
\[
\varepsilon_{\sigma_\delta} (\rho \boxtimes S_{2x_{\sigma_\rho}(i + 1)}) = \varepsilon_{\sigma_\delta} (\rho \boxtimes S_{2x_{\sigma_\rho}(t_\rho - j + 1) + 1}) := \delta_\rho(j)
\]
for \( l_\rho + 1 \leq i \leq t_\rho - l_\rho \) and \( j \in J_{\sigma_\rho}^- \). Here, if \( \phi_\sigma \supset \rho \boxtimes S_0 \) and \( \varepsilon_{\sigma_\delta} (\rho \boxtimes S_0) = -1 \), then we interpret \( \pi(\phi_\sigma, \varepsilon_{\sigma_\delta}) \) to be 0.

For a ladder datum \( \mathcal{L} \) for \( G_n \), we have a ladder representation \( \pi(\mathcal{L}) \in \text{Irr}(G_n) \). Recall in \( \mathbb{1} \mathbb{3} \) for a cuspidal support \( \mathcal{S} \) of \( G_n \), we have a projection operator \( p_\mathcal{S}: \mathcal{R}(G_n) \to \mathcal{R}(G_n) \). If \( \Pi \) is a standard module of \( G_n \), since all irreducible subquotients of \( \Pi \) have the same cuspidal support, one has \( p_\mathcal{S}([\Pi]) = [\Pi] \) or \( p_\mathcal{S}([\Pi]) = 0 \). By \( \mathbb{2} \) Theorem 4.2, one can determine \( p_\mathcal{S}([\Pi]) \).

Now we can state our main result, which is an analogue of Tadić’s determinantal formula \( \mathbb{1} \mathbb{4} \) Theorem 1 (ii).

**Theorem 4.2.** Let \( \mathcal{L} \) be a ladder datum for \( G_n \). Then
\[
[\pi(\mathcal{L})] = p_{\mathbb{S}} \left( \sum_{\sigma \in \mathfrak{S}(\mathcal{L})} \text{sgn}(\sigma)[I_\sigma(L)] \right),
\]
where \( \mathbb{S} = \text{supp}(\pi(\mathcal{L})) \).
4.2. Examples. In this subsection, except for Example 4.7 below, we assume that \( L = (X, L, \eta) \) satisfies that \( X_\rho = \emptyset \) unless \( \rho = 1_{\text{GL}_1(F)} \). We identify \( L \) with \( 1_{\text{GL}_1(F)}(F) \) and drop \( 1_{\text{GL}_1(F)}(F) \) from the subscript. For example, \( [x, y] = \{ \cdot | x, | x^{-1}, \ldots, \cdot | y \} \) is a segment, and \( \Delta[x, y] \) is the associated Steinberg representation.

Once \( \rho \) is fixed, for \( \phi = \bigoplus_{i=1}^{r} \rho \otimes S_{2x_i+1} \) and \( \epsilon_i = \epsilon(\rho \otimes S_{2x_i+1}) \), we write \( \pi(x_1^r, \ldots, x_r^\epsilon) := \pi(\phi, \epsilon) \).

**Example 4.3.** Let us consider the trivial representation \( 1_{SO_5(F)} \) of \( SO_5(F) \). Note that \( 1_{SO_5(F)} = \pi(L) \) with \( L := ([3/2, -3/2], 2, -1) \). By Definition 4.1 (1), we have
\[
\mathcal{G}(L) = \{ \sigma \in \mathcal{G}_5 \mid \sigma(1) = 1 \}.
\]

We give the list of \( \sigma \in \mathcal{G}(L) \), \( \text{sgn}(\sigma) \) and \( I_\sigma(L) \) in Table 4. Here, we recall that \( \Delta[-3/2, -1/2] = 1_{\text{GL}_0(F)}, \Delta[-3/2, 1/2] = 0 \) and \( \pi((-\frac{1}{2})^-, (\frac{3}{2})^-) = \pi((-\frac{1}{2})^-, (\frac{3}{2})^-) = 0 \).

**Table 4:** List of \( \sigma \), \( \text{sgn}(\sigma) \), \( I_\sigma(L) \) in Example 4.3

| \( \sigma \) | \( \text{sgn}(\sigma) \) | \( I_\sigma(L) \) |
|---|---|---|
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4}) \) | 1 | \( \cdot | -\frac{2}{3} \times \cdot \frac{1}{2} \times 1_{SO_1(F)} \) |
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4}) \) | -1 | \( \Delta[-\frac{1}{2}, -\frac{3}{2}] \times 1_{SO_1(F)} \) |
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4}) \) | -1 | \( \cdot \frac{2}{3} \times \pi((\frac{3}{2})^+) \) |
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4}) \) | 1 | 0 |
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4}) \) | -1 | \( \pi((\frac{3}{2})^+) \) |
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4}) \) | -1 | 0 |

Then Theorem 4.2 says that
\[
\left[ 1_{SO_5(F)} \right] = \left[ \cdot | -\frac{2}{3} \times \cdot \frac{1}{2} \times 1_{SO_1(F)} \right] - \left[ \Delta[-\frac{1}{2}, -\frac{3}{2}] \times 1_{SO_1(F)} \right] - \left[ \cdot | -\frac{2}{3} \times \pi((\frac{3}{2})^+) \right] + \left[ \pi((\frac{3}{2})^+) \right].
\]

**Example 4.4.** Let us consider \( L := ([3, -1], 2, +1) \). Then
\[
\pi(L) = L(\Delta[-1, -3], \Delta[0, -2], \pi(1^+))
\]
is an irreducible representation of \( \text{Sp}_{14}(F) \). By Definition 4.1 (1), we have
\[
\mathcal{G}(L) = \{ \sigma \in \mathcal{G}_5 \mid \sigma(1) = 1 \}.
\]

The list of \( \sigma \in \mathcal{G}(L) \), \( \text{sgn}(\sigma) \) and \( I_\sigma(L) \) is given in Table 5. Here, we recall that \( \Delta[-1, 0] = 1_{\text{GL}_0(F)} \).

**Table 5:** List of \( \sigma \), \( \text{sgn}(\sigma) \), \( I_\sigma(L) \) in Example 4.4

| \( \sigma \) | \( \text{sgn}(\sigma) \) | \( I_\sigma(L) \) |
|---|---|---|
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{5}{2}) \) | 1 | \( \Delta[-1, -3] \times \Delta[0, -2] \times \pi(1^+) \) |
| \( (\frac{1}{2} \frac{2}{3} \frac{3}{4} \frac{5}{2}) \) | -1 | \( \Delta[-1, -2] \times \Delta[0, -3] \times \pi(1^+) \) |
(\begin{array}{c|c}
1 & 2 & 3 & 4 & 5 \\
\hline
-1 & \Delta[-1,-3] \times \Delta[0,-1] \rtimes \pi(2^+) \\
1 & \Delta[-1,-2] \times \Delta[0,-3] \rtimes \pi(2^+) \\
1 & \Delta[-1,-2] \times \Delta[0,-1] \rtimes \pi(3^+) \\
-1 & \Delta[-1,-2] \rtimes \pi(3^+) \\
-1 & \Delta[-1,-3] \times \Delta[1,-2] \rtimes \pi(0^+) \\
1 & \Delta[-1,-2] \times \Delta[1,-3] \rtimes \pi(0^+) \\
1 & \Delta[-1,-3] \rtimes (\pi(0^+,1^+,2^+) \oplus \pi(0^-,1^-,2^+)) \\
-1 & \Delta[1,-3] \rtimes \pi(2^+) \\
-1 & \Delta[-1,-2] \rtimes (\pi(0^+,1^+,3^+) \oplus \pi(0^-,1^-,3^+)) \\
1 & \Delta[1,-2] \rtimes \pi(3^+) \\
1 & \Delta[-1,-3] \rtimes (\pi(0^+,1^+,2^+) \oplus \pi(0^-,1^+,2^-)) \\
-1 & \Delta[-1,-2] \rtimes \pi(0^+,-1^+,3^-) \\
1 & \Delta[-1,-3] \rtimes (\pi(0^+,1^+,2^+) \oplus \pi(0^-,1^+,2^-)) \\
1 & \Delta[2,-3] \rtimes \pi(1^+) \\
1 & \pi(1^+,2^+,3^+) \oplus \pi(1^-,3^+,3^-) \\
-1 & \Delta[-1,-2] \rtimes (\pi(0^+,1^+,3^+) \oplus \pi(0^+,1^-,3^-)) \\
1 & \pi(1^+,2^+,3^+) \oplus \pi(1^+,2^-,3^-) \\
1 & \Delta[-1,-2] \rtimes (\pi(0^+,1^+,3^+) \oplus \pi(0^+,1^+,3^-)) \\
1 & \pi(1^+,2^+,3^+) \oplus \pi(1^+,2^-,3^-) \\
-1 & \pi(0^+,2^+,3^+) \oplus \pi(0^-,2^+,3^-) \\
1 & \pi(1^+,2^+,3^+) \oplus \pi(1^+,2^+,3^-)
\end{array})

Among the standard modules appearing in this list, exactly 4 representations

\[ \Delta[-1,-3] \rtimes \pi(0^+,1^+,2^-), \Delta[-1,-2] \rtimes \pi(0^-,1^+,3^-) \]

are killed by \( p_s([\cdot]) \) with \( s = \text{supp}(\pi(L)) \). In addition, the 4 representations

\[ \Delta[-1,-3] \rtimes \pi(0^+,1^+,2^+), \Delta[-1,-2] \rtimes \pi(0^+,1^+,3^+) \]

appear three times, but two of them are cancelled, respectively. Therefore, by Theorem 4.2, we conclude that \([\pi(L)]\) is a linear combination of exactly 24 standard modules with coefficients in \( \{\pm 1\} \).

**Example 4.5.** Let us consider \( L := ([4,0],1,-1) \). Then

\[ \pi(L) = L(\Delta[0,-4]; \pi(1^-,2^+,3^-)) \]
is an irreducible representation of $\text{Sp}_{2n}(F)$. Any element $\sigma \in \mathcal{G}(\mathcal{L}) \subset \mathcal{G}_5$ is determined by the pair $(\sigma(1), \sigma(5))$ since $\sigma(2) < \sigma(3) < \sigma(4)$. Hence $|\mathcal{G}(\mathcal{L})| = 20$. We list $\sigma \in \mathcal{G}(\mathcal{L})$, $\text{sgn}(\sigma)$ and $I_\sigma(\mathcal{L})$ in Table 4.

Table 4: List of $\sigma$, $\text{sgn}(\sigma)$, $I_\sigma(\mathcal{L})$ in Example 4.5

| $\sigma$ | $\text{sgn}(\sigma)$ | $I_\sigma(\mathcal{L})$ |
|----------|----------------------|------------------------|
| $\{1\ 2\ 3\ 4\ 5\}$ | 1 | $\Delta[0,-4] \times \pi(1^-,2^+,3^-)$ |
| $\{1\ 2\ 3\ 4\ 5\}$ | $-1$ | $\Delta[0,-3] \times \pi(1^-,2^+,4^-)$ |
| $\{1\ 2\ 3\ 4\ 5\}$ | 1 | $\Delta[0,-2] \times \pi(1^-,3^+,4^-)$ |
| $\{1\ 2\ 3\ 4\ 5\}$ | $-1$ | $\Delta[0,-1] \times \pi(2^-,3^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\Delta[1,-4] \times \pi(0^-,2^+,3^-)$ |
| $\{1\ 2\ 1\ 4\ 5\}$ | 1 | $\Delta[1,-3] \times \pi(0^-,2^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\Delta[1,-2] \times \pi(0^-,3^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | 1 | $\pi(0^+,-1^+,2^-,3^+,4^-) \oplus \pi(0^-,1^-,2^-,3^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | 1 | $\Delta[2,-4] \times \pi(0^-,1^+,3^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\Delta[2,-3] \times \pi(0^-,1^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | 1 | $\pi(0^-,1^+,2^+,3^+,4^-) \oplus \pi(0^-,1^-,2^-,3^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\pi(0^+,1^-,2^+,3^+,4^-) \oplus \pi(0^-,1^-,2^-,3^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\Delta[3,-4] \times \pi(0^-,1^+,2^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | 1 | $\pi(0^-,1^+,2^+,3^+,4^-) \oplus \pi(0^-,1^+,2^-,3^+,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\pi(0^-,1^+,2^-,3^+,4^-) \oplus \pi(0^-,1^-,2^-,3^-,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | 1 | $\pi(0^-,1^+,2^-,3^-,4^-) \oplus \pi(0^-,1^-,2^-,3^-,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | 1 | $\pi(0^-,1^+,2^-,3^-,4^-) \oplus \pi(0^-,1^-,2^-,3^-,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\pi(0^-,1^+,2^+,3^-,4^-) \oplus \pi(0^-,1^-,2^+,3^-,4^-)$ |
| $\{1\ 2\ 1\ 4\ 3\}$ | $-1$ | $\pi(0^-,1^+,2^+,3^-,4^-) \oplus \pi(0^-,1^-,2^+,3^-,4^-)$ |

If we put $s = \text{supp}(\pi(\mathcal{L}))$, we note that

\[
\begin{align*}
    p_s([\pi(0^+,1^-,2^+,3^+,4^-) \oplus \pi(0^-,1^-,2^+,3^-,4^-)]) &= 0, \\
    p_s([\pi(0^+,1^-,2^+,3^-,4^-) \oplus \pi(0^-,1^-,2^+,3^-,4^-)]) &= 0, \\
    p_s([\pi(0^+,1^-,2^+,3^-,4^-) \oplus \pi(0^-,1^-,2^+,3^-,4^-)]) &= 0,
\end{align*}
\]

and

\[
\begin{align*}
    p_s([\pi(0^-,1^+,2^+,3^+,4^-) \oplus \pi(0^-,1^-,2^-,3^+,4^-)]) \\
    - p_s([\pi(0^+,1^-,2^+,3^-,4^-) \oplus \pi(0^-,1^-,2^-,3^-,4^-)]) \\
    + p_s([\pi(0^-,1^+,2^+,3^-,4^-) \oplus \pi(0^-,1^+,2^-,3^-,4^-)]) \\
    - p_s([\pi(0^-,1^+,2^-,3^+,4^-) \oplus \pi(0^-,1^+,2^-,3^-,4^-)])
\end{align*}
\]
By taking 

\[ -p_s([\pi(0^-, 1^+, 2^+, 3^+, 4^-) \oplus \pi(0^-, 1^-, 2^+, 3^-, 4^-)]) = [\pi(0^-, 1^+, 2^+, 3^+, 4^-)]. \]

Here, in the left hand side of the last equation, we note that \([\pi(0^-, 1^+, 2^+, 3^+, 4^-)]\) appears three times. In conclusion, Theorem 4.2 tells us that \([\pi(\mathcal{L})]\) is an alternating sum of exactly 15 standard modules.

**Remark 4.6.** As we have seen these examples, one might expect that \([\pi(\mathcal{L})]\) would be a linear combination of standard modules with coefficients in \([\pm 1]\).

As in Example 3.11, there are irreducible representations \(\pi\) which are not ladder but all Jacquet modules of \(\pi\) are completely reducible. For such representations, our determinantal formula may fail.

**Example 4.7.** In this example, we fix an irreducible cuspidal symplectic representation \(\rho\) of \(GL_2(F)\). Consider

\[
\pi := L(\Delta_\rho[-1, -2]; \pi(0^+, 1^+)) \in \text{Irr}(SO_{17}(F)).
\]

One can compute all Jacquet modules of \(\pi\) by the argument of [2, Theorem 4.3] together with [17, Section 6], [2, Theorem 4.2] and [5, Theorem 7.1]. To be precise, if \(P_{2k}\) is the standard parabolic subgroup of \(SO_{2k}(F)\) with Levi subgroup isomorphic to \(GL_k(F) \times SO_{17-2k}(F)\), then

\[
\text{Jac}_{P_2}(\pi) = \left( \rho \cdot |^{-1} \otimes L(\rho \cdot |^{-2}; \pi(0^+, 1^+)) \right)
\]

\[
\oplus \left( \rho \cdot |^1 \otimes L(\Delta_\rho[-1, -2]; \pi(0^+, 0^+)) \right),
\]

\[
\text{Jac}_{P_2}(\pi) = \left( \Delta_\rho[-1, -2] \otimes \pi(0^+, 1^+) \right)
\]

\[
\oplus \left( L(\rho \cdot |^{-1}, \rho \cdot |^1) \otimes L(\rho \cdot |^{-2}; \pi(0^+, 0^+)) \right),
\]

\[
\oplus \left( L(\rho \cdot |^1, \rho \cdot |^2) \otimes L(\rho \cdot |^{-1}; \pi(0^+, 0^+)) \right),
\]

\[
\text{Jac}_{P_h}(\pi) = \left( \Delta_\rho[-1, -2], | \cdot |^1 \otimes \pi(0^+, 0^+) \right)
\]

\[
\oplus \left( L(\rho \cdot |^{-1}, \rho \cdot |^1, 0]; \pi(0^+, 0^+)) \right),
\]

\[
\oplus \left( L(\rho \cdot |^{-1}, \rho \cdot |^1, \rho \cdot |^2) \otimes \pi(0^+, 0^+) \right),
\]

\[
\text{Jac}_{P_h}(\pi) = \left( L(\Delta_\rho[-1, -2], \Delta_\rho[1, 0]) \otimes 1_{SO_1(F)} \right)
\]

\[
\oplus \left( L(\rho \cdot |^{-1}, \Delta_\rho[1, 0], \rho \cdot |^2) \otimes 1_{SO_1(F)} \right).
\]

Since the representations of \(GL_{2k}(F)\) appearing above are all ladder, we see that \(\text{Jac}_P(\pi)\) is completely reducible for any parabolic subgroup \(P\) of \(SO_{17}(F)\).

On the other hand, if we try to apply Theorem 4.2 to \(\pi\) formally, the right hand side should be equal to

\[
(*) \quad [\Delta_\rho[-1, -2] \times \pi(0^+, 1^+)] - [\rho \cdot |^{-1} \times \pi(0^+, 2^+)] + [\pi(1^+, 2^+)].
\]

By taking \(D_{\rho \cdot |^2}\), it becomes \([\pi(1^+, 1^+)]\), whereas \(D_{\rho \cdot |^2}(\pi) = 0\) by [5, Theorem 7.1]. Hence \([\pi]\) is not equal to \((*)\).
4.3. Proof of Theorem 4.2. In this subsection, we prove Theorem 4.2.

Let \( \mathcal{L} = (X, \xi, \eta) \) be a ladder datum for \( G_n \) and \( \pi(\mathcal{L}) \in \text{Irr}(G_n) \) be the associated ladder representation. For \( \rho \in \mathcal{E}(\text{GL}_d(F)) \) such that \( \rho^\vee \cong \rho \), we write \( X_\rho = \{ \rho \cdot |x_1|, \ldots, \rho \cdot |x_{t_\rho}| \} \) with \( x_1 < \cdots < x_{t_\rho} \). We defined a directed 3-colorable graph \( \mathcal{E}(\mathcal{L}_\rho) = (\mathcal{V}(\mathcal{L}_\rho), \prec, f_\rho) \) in Definition 3.3. Set \( 2m_\rho = |f_\rho^{-1}(0)| \), which is even.

We will prove Theorem 4.2 by induction on \( n \). If \( m_\rho = 0 \) for all \( \rho \), then by Corollary 3.7 (1), \( \pi(\mathcal{L}) \) is supercuspidal. In this case, since \( |\mathcal{G}(\mathcal{L})| = 1 \), Theorem 4.2 is trivial.

From now, assume the assertion of Theorem 4.2 for \( G_{n'} \) with \( n' < n \). Furthermore, we fix \( \rho \in \mathcal{E}(\text{GL}_d(F)) \) with \( \rho^\vee \cong \rho \) such that \( m_\rho > 0 \). Write \( t := t_\rho \), \( (l, \eta) := (l_\rho, \eta_\rho) \) and \( m := m_\rho \) for simplicity.

Lemma 4.8. If \( \lambda \in \mathbb{R} \) satisfies that \( D_{\rho^\mid \setminus \lambda}([\pi(\mathcal{L})]) = 0 \), then

\[
D_{\rho^\mid \setminus \lambda} \circ p_\delta \left( \sum_{\sigma \in \mathcal{G}(\mathcal{L})} \operatorname{sgn}(\sigma)[I_\sigma(\mathcal{L})] \right) = 0.
\]

Proof. If \( \rho^\mid \setminus \lambda \notin \mathfrak{s} \), then \( D_{\rho^\mid \setminus \lambda} \circ p_\delta = 0 \) so that the assertion is trivial. Hence we may assume that \( \rho^\mid \setminus \lambda \in \mathfrak{s} \). In this case, since \( D_{\rho^\mid \setminus \lambda} \circ p_\delta = p_{\lambda'} \circ D_{\rho^\mid \setminus \lambda} \) with \( \mathfrak{s}' = \mathfrak{s} \setminus \{ \rho^\mid \setminus \lambda, \rho \cdot |\lambda| \} \), it suffices to show that \( D_{\rho^\mid \setminus \lambda}(\sum_{\sigma \in \mathcal{G}(\mathcal{L})} \operatorname{sgn}(\sigma)[I_\sigma(\mathcal{L})]) = 0 \).

To show this, we may assume that \( D_{\rho^\mid \setminus \lambda}([I_\sigma(\mathcal{L})]) \neq 0 \) for some \( \sigma \in \mathcal{G}(\mathcal{L}) \). In this case, we have \( \lambda \in \{ x_1, \ldots, x_t \} \) by Tadić’s formula (Proposition 2.1) and [17 Lemma 7.3]. Since \( D_{\rho^\mid \setminus \lambda}([\pi(\mathcal{L})]) = 0 \), by Proposition 3.6 one of the following holds:

- \( \lambda = x_1 \) and \( l = 0 \); or
- \( \lambda \) is \( 2 \leq j \leq t \) such that \( x_j = \lambda \) and \( x_{j-1} = \lambda - 1 = x_j - 1 \).

However, in the former case, we have \( \sigma_\rho = \text{id} \) for any \( \sigma \in \mathcal{G}(\mathcal{L}) \). This implies that \( D_{\rho^\mid \setminus \lambda}([I_\sigma(\mathcal{L})]) = 0 \), which is a contradiction. Hence the latter case must occur.

Suppose that \( \sigma = (\sigma_{\rho'})_{\rho'} \in \mathcal{G}(\mathcal{L}) \) satisfies that \( D_{\rho^\mid \setminus \lambda}([I_\sigma(\mathcal{L})]) \neq 0 \). Then we note that one of \( j \) or \( j - 1 \) does not belong to \( \sigma(\{ t+1, \ldots, t-l \}) \). We will define \( \sigma' = (\sigma_{\rho'})_{\rho'} \) as follows. For \( \rho' \neq \rho \), set \( \sigma'_{\rho'} := \sigma_{\rho'} \). To define \( \sigma'_{\rho} \), we consider two cases.

1. If \( j, j - 1 \in \sigma(\{ 1, \ldots, l \}) \), writing \( j - 1 = \sigma(i) \) and \( j = \sigma(i') \) with \( 1 \leq i < i' \leq l \), we set \( \sigma'_{\rho} := \sigma_{\rho'} \cdot (t - i + 1, t - i' + 1) \). Then \( \sigma' \in \mathcal{G}(\mathcal{L}) \). Moreover, there exists a direct sum of standard modules \( \Pi \) satisfying \( D_{\rho^\mid \setminus \lambda}(\Pi) = 0 \) such that

\[
I_{\sigma}(\mathcal{L}) = \Delta_{\rho}[\lambda - 1, x_{\sigma(t-i+1)}] \times \Delta_{\rho}[\lambda, x_{\sigma(t-i'+1)}] \times \Pi,
\]

\[
I_{\sigma'}(\mathcal{L}) = \Delta_{\rho}[\lambda - 1, x_{\sigma(t-i'+1)}] \times \Delta_{\rho}[\lambda, x_{\sigma(t-i+1)}] \times \Pi.
\]

Hence

\[
D_{\rho^\mid \setminus \lambda}([I_\sigma(\mathcal{L})]) = \Delta_{\rho}[\lambda - 1, x_{\sigma(t-i+1)}] \times \Delta_{\rho}[\lambda - 1, x_{\sigma(t-i'+1)}] \times \Pi = D_{\rho^\mid \setminus \lambda}([I_{\sigma'}(\mathcal{L})]).
\]

2. If one of \( j \) or \( j - 1 \) does not belong to \( \sigma(\{ 1, \ldots, l \}) \), set \( \sigma'_{\rho} := (j, j - 1) \cdot \sigma_{\rho} \). In this case, \( \sigma' \in \mathcal{G}(\mathcal{L}) \). We claim that

\[
D_{\rho^\mid \setminus \lambda}([I_\sigma(\mathcal{L})]) = D_{\rho^\mid \setminus \lambda}([I_{\sigma'}(\mathcal{L})]).
\]

This equation can be proven similarly to the first case unless there exists \( 1 \leq i \leq l \) such that \( (j - 1, j) = (\sigma(i), \sigma(t-i+1)) \) or \( (j - 1, j) = (\sigma(t-i+1), \sigma(i)) \). By symmetry, we consider the former case so that \( i \in J_{\sigma, \rho} \) and \( i \in J_{\sigma', \rho} \). Hence we can find a
product of Steinberg representations \(\tau\) and a sequence of tempered representations \(\pi(\phi, \varepsilon_1), \ldots, \pi(\phi, \varepsilon_r)\) such that

\[
I_\sigma(\mathcal{L}) = \Delta[\lambda, -(\lambda - 1)] \times \tau \times \bigoplus_{i=1}^{r} \pi(\phi, \varepsilon_i),
\]

\[
I_{\sigma'}(\mathcal{L}) = \tau \times \bigoplus_{i=1}^{r} (\pi(\phi', \varepsilon'_{r,+}) \oplus \pi(\phi', \varepsilon'_{r,-})),
\]

where \(\phi' = \phi \oplus (\rho \boxtimes S_{2\lambda-1} \oplus \rho \boxtimes S_{2\lambda+1})\) and

\[
\varepsilon'_{r,\pm}(\rho \boxtimes S_{2\lambda-1}) = \varepsilon'_{r,\pm}(\rho \boxtimes S_{2\lambda+1}) = \pm 1.
\]

Then by [17, Lemma 7.3], with \(\phi_0 := \phi \oplus (\rho \boxtimes S_{2\lambda-1})^{\oplus 2}\), we have

\[
D_{\rho^{|\lambda}}([I_{\sigma'}(\mathcal{L})]) = \left[\tau \times \bigoplus_{i=1}^{r} (\pi(\phi_0, \varepsilon'_{r,+}) \oplus \pi(\phi_0, \varepsilon'_{r,-}))\right]
\]

\[
= \left[\tau \times \bigoplus_{i=1}^{r} (\Delta_{\rho}[\lambda - 1, -(\lambda - 1)] \times \pi(\phi, \varepsilon_i))\right]
\]

\[
= \left[\Delta_{\rho}[\lambda - 1, -(\lambda - 1)] \times \tau \times \bigoplus_{i=1}^{r} \pi(\phi, \varepsilon_i)\right]
\]

\[
= D_{\rho^{|\lambda}}([I_{\sigma}(\mathcal{L})]),
\]

as desired.

Since the map \(\sigma \mapsto \sigma'\) is an involution on

\[
\{\sigma \in \mathcal{G}(\mathcal{L}) \mid D_{\rho^{|\lambda}}([I_{\sigma}(\mathcal{L})]) \neq 0\}
\]

satisfying \(\text{sgn}(\sigma') = -\text{sgn}(\sigma)\) and \(D_{\rho^{|\lambda}}([I_{\sigma}(\mathcal{L})]) = D_{\rho^{|\lambda}}([I_{\sigma'}(\mathcal{L})])\), we deduce that

\[
\sum_{\sigma \in \mathcal{G}(\mathcal{L})} \text{sgn}(\sigma) D_{\rho^{|\lambda}}([I_{\sigma}(\mathcal{L})]) = 0.
\]

This completes the proof. \(\Box\)

**Lemma 4.9.** If \(\lambda \in \mathbb{R}\) satisfies that \(D_{\rho^{|\lambda}}([\pi(\mathcal{L})]) \neq 0\), then

\[
D_{\rho^{|\lambda}}([\pi(\mathcal{L})]) = D_{\rho^{|\lambda}} \circ p_s \left(\sum_{\sigma \in \mathcal{G}(\mathcal{L})} \text{sgn}(\sigma)[I_{\sigma}(\mathcal{L})]\right),
\]

where \(s = \text{supp}(\pi(\mathcal{L}))\).

**Proof.** By Proposition 3.6, we have \(\lambda \in \{x_1, \ldots, x_t\}\) and \(\lambda - 1 \not\in \{x_1, \ldots, x_t\}\). Moreover, if we define \(\mathcal{L}' = (X', \underline{1}, \eta)\) from \(\mathcal{L}\) by replacing \(X_{\rho}\) with

\[
X'_{\rho} := (X_{\rho} \setminus \{\rho| \cdot |^1\}) \cup \{\rho| \cdot |^{-1}\},
\]

then \(D_{\rho^{|\lambda}}([\pi(\mathcal{L})]) = [\pi(\mathcal{L}')]\). By Definition 4.1, we can see that \(\mathcal{G}(\mathcal{L}') = \mathcal{G}(\mathcal{L})\) and \(D_{\rho^{|\lambda}}([I_{\sigma}(\mathcal{L})]) = [I_{\sigma}(\mathcal{L}')]\). Finally, if we set \(s' = \text{supp}(\pi(\mathcal{L}'))\), then \(D_{\rho^{|\lambda}} \circ p_s = p_{s'} \circ D_{\rho^{|\lambda}}\).
Since we know the assertion of Theorem 4.2 for $\pi(L')$ by the induction hypothesis, we have

$$D_{\rho|\lambda}([\pi(L)]) = \left[\pi(L')\right]$$

$$= p_{\rho'} \left( \sum_{\sigma \in G(L')} \text{sgn}(\sigma)[I_{\sigma}(L')] \right)$$

$$= p_{\rho'} \circ D_{\rho|\lambda} \left( \sum_{\sigma \in G(L)} \text{sgn}(\sigma)[I_{\sigma}(L)] \right)$$

$$= D_{\rho|\lambda} \circ p_{\rho} \left( \sum_{\sigma \in G(L)} \text{sgn}(\sigma)[I_{\sigma}(L)] \right),$$

as desired. \[\square\]

Let $P = MN$ be the standard parabolic subgroup of $G_n$ such that $\pi(L)$ is an irreducible subquotient of $\text{Ind}_{G_n}^G(\pi_M)$ for some $\pi_M \in \mathcal{C}(M)$. Then by applying Lemmas 4.8 and 4.9 repeatedly, we see that

$$\text{Jac}_P([\pi(L)]) = \text{Jac}_P \left( p_{\rho} \left( \sum_{\sigma \in G(L)} \text{sgn}(\sigma)[I_{\sigma}(L)] \right) \right).$$

Since $[\pi(L)]$ can be written as a linear combination of standard modules with the same cuspidal support as $\pi(L)$, this equation implies that

$$[\pi(L)] = p_{\rho} \left( \sum_{\sigma \in G(L)} \text{sgn}(\sigma)[I_{\sigma}(L)] \right).$$

This completes the proof of Theorem 4.2.

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**Department of Mathematics, Hokkaido University, Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan**

*Email address: atobe@math.sci.hokudai.ac.jp*