LYAPUNOV INEQUALITIES FOR PARTIAL DIFFERENTIAL EQUATIONS AT RADIAL HIGHER EIGENVALUES

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To our dear friend and colleague Jean Mawhin on the occasion of his seventieth birthday

Abstract. This paper is devoted to the study of $L_p$ Lyapunov-type inequalities ($1 \leq p \leq +\infty$) for linear partial differential equations at radial higher eigenvalues. More precisely, we treat the case of Neumann boundary conditions on balls in $\mathbb{R}^N$. It is proved that the relation between the quantities $p$ and $N/2$ plays a crucial role to obtain nontrivial and optimal Lyapunov inequalities. By using appropriate minimizing sequences and a detailed analysis about the number and distribution of zeros of radial nontrivial solutions, we show significant qualitative differences according to the studied case is subcritical, supercritical or critical.

1. Introduction

Let us consider the linear problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (0, L), \quad u'(0) = u'(L) = 0$$

where $a \in \Lambda$ and $\Lambda$ is defined by

$$\Lambda = \{ a \in L^1(0, L) \setminus \{0\} : \int_0^L a(x) \, dx \geq 0 \text{ and (1.1) has nontrivial solutions \} }$$

The well known $L_1$ Lyapunov inequality states that if $a \in \Lambda$, then $\int_0^L a^+(x) \, dx > 4/L$. Moreover, the constant $4/L$ is optimal since $4/L = \inf_{a \in \Lambda} \| a^+ \|_{L^1(0,L)}$ and this infimum is not attained (see [1], [7] and [8]). This result is as a particular case of the so called $L_p$ Lyapunov inequalities, $1 \leq p \leq \infty$. In fact, if for each $p$ with $1 \leq p \leq \infty$, we define the quantity

$$\beta_p = \inf_{a \in \Lambda \cap L^p(0,L)} I_p(a)$$

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where

\[ I_p(a) = \|a^+\|_{L^p(0,L)} = \left( \int_0^L (a^+(x))^p \, dx \right)^{1/p}, \forall \, a \in \Lambda \cap L^p(0,L), \, 1 \leq p < \infty, \]

\[ I_\infty(a) = \text{sup ess } a^+, \forall \, a \in \Lambda \cap L^\infty(0,L), \]

then \( \beta_1 = \frac{4}{L} \) and for each \( p \) with \( 1 \leq p < \infty \), it is possible to obtain an explicit expression for \( \beta_p \) as a function of \( p \) and \( L \) (11, 10).

Let us observe that the real number zero is the first eigenvalue of the eigenvalue problem

\[ u''(x) + \rho u(x) = 0, \, x \in (0,L), \, u'(0) = u'(L) = 0 \]

and that for Neumann boundary conditions the restriction on the function \( a \) in the definition of the set \( \Lambda \),

\[ a \in L^1(0,L) \setminus \{0\}, \int_0^L a(x) \, dx \geq 0, \]

or the more restrictive pointwise condition

\[ a \in L^1(0,L), \, 0 \prec a, \]

are natural if we want to obtain nontrivial optimal Lyapunov inequalities (see Remark 4 in 11). Here, for \( c, d \in L^1(0,L) \), we write \( c \prec d \) if \( c(x) \leq d(x) \) for a.e. \( x \in [0,L] \) and \( c(x) < d(x) \) on a set of positive measure.

In fact, it can be easily proved that if

\[ \Lambda_0 = \{a \in L^1(0,L) : 0 \prec a \text{ and (1.1) has nontrivial solutions } \} \]

then the constant \( \beta_p \) defined in (1.3) satisfies

\[ \beta_p = \inf_{a \in \Lambda_0 \cap L^p(0,L)} I_p(a) \]

Since zero is the first eigenvalue of (1.5), it is coherent to affirm that \( \beta_p \) is the \( L^p \) Lyapunov constant for the Neumann problem at the first eigenvalue.

On the other hand, the set of eigenvalues of (1.5) is given by \( \rho_k = k^2 \pi^2/L^2, \, k \in \mathbb{N} \cup \{0\} \) and if for each \( k \in \mathbb{N} \cup \{0\} \), we consider the set

\[ \Lambda_k = \{a \in L^1(0,L) : \rho_k \prec a \text{ and (1.1) has nontrivial solutions } \} \]

then for each \( p \) with \( 1 \leq p \leq \infty \), we can define the constant

\[ \beta_{p,k} = \inf_{a \in \Lambda_k \cap L^p(0,L)} I_p(a - \rho_k) \]

An explicit value for \( \beta_{1,k} \) has been obtained by the authors in 3. The case \( p = \infty \) is trivial (\( \beta_{\infty,k} = \rho_{k+1} - \rho_k \)) and, to the best of our knowledge, an explicit value of \( \beta_{p,k} \) as a function of \( p, k \) and \( L \) is not known when \( 1 < p < \infty \). Nevertheless, since \( \beta_{1,k} > 0 \), we trivially deduce \( \beta_{p,k} > 0 \), for each \( p \) with \( 1 \leq p \leq \infty \).
With regard to Partial Differential Equations, the linear problem
\[
\begin{align*}
\Delta u(x) + a(x)u(x) &= 0, & x \in \Omega \\
\frac{\partial u}{\partial n}(x) &= 0, & x \in \partial \Omega 
\end{align*}
\]
has been studied in [2], where \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\) is a bounded and regular domain, \(\frac{\partial}{\partial n}\) is the outer normal derivative on \(\partial \Omega\) and the function \(a : \Omega \to \mathbb{R}\) belongs to the set \(\Gamma\) defined as
\[
\Gamma = \{ a \in L^{\frac{N}{2}}(\Omega) : \int_{\Omega} a(x) \, dx \geq 0 \text{ and } (1.12) \text{ has nontrivial solutions} \}
\]
if \(N \geq 3\) and
\[
\Gamma = \{ a : \Omega \to \mathbb{R} : \exists q \in (1, \infty] \text{ with } a \in L^q(\Omega) \setminus \{0\}, \int_{\Omega} a(x) \, dx \geq 0 \text{ and } (1.12) \text{ has nontrivial solutions} \}
\]
if \(N = 2\).
Obviously, the quantity
\[
\gamma_p \equiv \inf_{a \in \Gamma \cap L^p(\Omega)} \| a^+ \|_{L^p(\Omega)}, \quad 1 \leq p \leq \infty
\]
is well defined and it is a nonnegative real number. A remarkable novelty (see [2]) with respect to the ordinary case is that \(\gamma_1 = 0\) for each \(N \geq 2\). Moreover, if \(N = 2\), then \(\gamma_p > 0\), \(\forall p \in (1, \infty]\) and if \(N \geq 3\), then \(\gamma_p > 0\) if and only if \(p \geq N/2\). In contrast to the ordinary case, it seems difficult to obtain an explicit expressions for \(\gamma_p\), as a function of \(p, \Omega\) and \(N\), at least for general domains.

As in the ordinary case, the real number zero is the first eigenvalue of the eigenvalue problem
\[
\begin{align*}
\Delta u(x) + \rho u(x) &= 0, & x \in \Omega \\
\frac{\partial u}{\partial n}(x) &= 0, & x \in \partial \Omega 
\end{align*}
\]
so that it is natural to say that the constant \(\gamma_p\) defined in (1.14) is the \(L^p\) Lyapunov constant at the first eigenvalue for the Neumann problem (1.12).

To our knowledge, there are no significant results concerning to \(L^p\) Lyapunov inequalities for PDE at higher eigenvalues and this is the main subject of this paper where we provide some new qualitative results which extend to higher eigenvalues those obtained in [2] for the case of the first eigenvalue. We carry out a complete qualitative study of the question pointing out the important role played by the dimension of the problem.

Since in the case of ODE our proof are mainly based on an exact knowledge about the number and distribution of the zeros of the corresponding solutions ([3]), in the PDE case we are able to study \(L^p\) Lyapunov inequalities if \(\Omega\) is a ball and for radial higher eigenvalues. It is not restrictive to assume that \(\Omega = B_{\mathbb{R}^N}(0; 1) \equiv B_1\), the open ball in \(\mathbb{R}^N\) of center zero and radius one.
In Section 2 we describe the problem in a precise way and we present the main results of this paper. In Section 3 we study the subcritical case, i.e. $1 \leq p < \frac{N}{2}$, if $N \geq 3$, and $p = 1$ if $N = 2$. To prove the results in this section we will construct some explicit and appropriate sequences of problems like (1.12) where Dirichlet type problems play an essential role. In this subcritical case we prove that the optimal Lyapunov constants are trivial, i.e., zero.

In Section 4, we treat with the supercritical case: $p > \frac{N}{2}$, if $N \geq 2$. By using some previous results of Section 2, about the number and distribution of the zeros of nontrivial and radial solutions, together with some compact Sobolev inclusions, we use a reasoning by contradiction to prove that the optimal Lyapunov constants are strictly positive and they are attained. In Section 5 we consider the critical case, i.e. $p = \frac{N}{2}$, if $N \geq 3$. Because in this case the Sobolev inclusions are continuous but no compact, we demonstrate that the optimal Lyapunov constants are strictly positive but we do not know if they are attained or not.

Finally, we study the case of Neumann boundary conditions but similar results can be obtained in the case of Dirichlet type problems.

2. MAIN RESULTS

From now on, $\Omega = B_1$, the open ball in $\mathbb{R}^N$ of center zero and radius one. It is very well known ([4]) that the operator $-\Delta$ exhibits an infinite increasing sequence of radial Neumann eigenvalues $0 = \mu_0 < \mu_1 < \ldots < \mu_k < \ldots$ with $\mu_k \to +\infty$, all of them simple and with associated eigenfunctions $\varphi_k \in C^1[0,1]$ solving

$$-(r^{N-1}\varphi')' = \mu_k r^{N-1}\varphi, \quad 0 < r < 1,$$

$$\varphi'(0) = \varphi'(1) = 0.$$  

Moreover, each eigenfunction $\varphi_k$ has exactly $k$ simple zeros $r_k < r_{k-1} < \ldots < r_1$ in the interval $(0,1)$.

For each integer $k \geq 0$ and number $p$, $1 \leq p \leq \infty$, we can define the set

$$\Gamma_k = \{ a \in L^{N/2}(B_1) : a \text{ is a radial function, } \mu_k < a \text{ and } (1.12) \text{ has radial and nontrivial solutions} \}$$

if $N \geq 3$ and

$$\Gamma_k = \{ a : B_1 \to \mathbb{R} \text{ s. t. } \exists q \in (1,\infty) \text{ with } a \in L^q(B_1) : a \text{ is a radial function, } \mu_k < a \text{ and } (1.12) \text{ has radial and nontrivial solutions} \}$$

if $N = 2$.

We also define the quantity

$$\gamma_{p,k} = \inf_{a \in \Gamma_k \cap L^p(B_1)} \| a - \mu_k \|_{L^p(B_1)}$$

The main result of this paper is the following.

**Theorem 2.1.** Let $k \geq 0$, $N \geq 2$, $1 \leq p \leq \infty$. The following statements hold:
(1) If $N = 2$ then $\gamma_{p,k} > 0 \iff 1 < p \leq \infty$.
   If $N \geq 3$ then $\gamma_{p,k} > 0 \iff \frac{N}{2} \leq p \leq \infty$.

(2) If $N \geq 2$ and $\frac{N}{2} < p \leq \infty$ then $\gamma_{p,k}$ is attained.

A key ingredient to prove this theorem is the following proposition on the number and distribution of zeros of nontrivial radial solutions of (1.12) when $a \in \Gamma_k$.

**Proposition 2.2.** Let $\Omega = B_1$, $k \geq 0$, $a \in \Gamma_k$ and $u$ any nontrivial radial solution of (1.12). Then $u$ has, at least, $k + 1$ zeros in $(0,1)$. Moreover, if $k \geq 1$ and we denote by $x_k < x_{k-1} < \ldots < x_1$ the last $k$ zeros of $u$, we have that

$$r_i \leq x_i, \forall 1 \leq i \leq k,$$

where $r_i$ denotes the zeros of the eigenfunction $\varphi_k$ of (2.1).

For the proof of this proposition we will need the following lemma. Some of the results of this lemma can be proved in a different way, by using the version of the Sturm Comparison Lemma proved in [4], Lemma 4.1, for the $p$-laplacian operator (see also [7]). Other results are new.

**Lemma 2.3.** Let $k \geq 1$. Under the hypothesis of Proposition 2.2 we have that

i) $u$ vanishes in the interval $(0,r_k]$. If $r_k$ is the only zero of $u$ in this interval then $a(r) \equiv \mu_k$ in $(0,r_k]$.

ii) $u$ vanishes in the interval $[r_{i+1},r_i)$, for $1 \leq i \leq k - 1$. If $r_{i+1}$ is the only zero of $u$ in this interval then $u(r_i) = 0$ and $a(r) \equiv \mu_k$ in $[r_{i+1},r_i]$.

ii) $u$ vanishes in the interval $[r_1,1)$. If $r_1$ is the only zero of $u$ in this interval then $a(r) \equiv \mu_k$ in $[r_1,1]$.

**Proof.** To prove i), multiplying (1.12) by $\varphi_k$ and integrating by parts in $B_{r_k}$ (the ball centered in the origin of radius $r_k$), we obtain

$$\int_{B_{r_k}} \nabla u \nabla \varphi_k = \int_{B_{r_k}} au \varphi_k.$$

On the other hand, multiplying (2.1) by $u$ and integrating by parts in $B_{r_k}$, we have

$$\int_{B_{r_k}} \nabla \varphi_k \nabla u = \mu_k \int_{B_{r_k}} \varphi_k u + \int_{\partial B_{r_k}} u \frac{\partial \varphi_k}{\partial n}.$$

Subtracting these equalities yields

(2.3) $$\int_{B_{r_k}} (a - \mu_k) u \varphi_k = \omega_N r_k^{N-1} u(r_k) \varphi_k'(r_k),$$
where $\omega_N$ denotes de measure of the $N$-dimensional unit sphere. Assume, by contradiction, that $u$ does not vanish in $(0, r_k]$. We can suppose, without loss of generality, that $u > 0$ in this interval. We can also assume that $\varphi_k > 0$ in $(0, r_k)$. Since $r_k$ is a simple zero of $\varphi_k$, we have $\varphi'_k(r_k) < 0$ and since $a \geq \mu_k$ in $(0, r_k)$ we obtain a contradiction.

Finally, if $r_k$ is the only zero of $u$ in $(0, r_k]$, equation 2.3 yields

$$\int_{B_{r_k}} (a - \mu_k) u \varphi_k = 0,$$

which gives $a(r) \equiv \mu_k$ in $(0, r_k]$.

To deduce ii), we proceed similarly to the proof of part i), substituting $B_{r_k}$ by $A(r_{i+1}, r_i)$ (the annulus centered in the origin of radii $r_{i+1}$ and $r_i$) and obtaining

$$\int_{A(r_{i+1}, r_i)} (a - \mu_k) u \varphi_k = \omega_N r_{i+1}^{N-1} u(r_i) \varphi'_k(r_i) - \omega_N r_i^{N-1} u(r_{i+1}) \varphi'_k(r_{i+1})$$

and ii) follows easily by arguments on the sign of these quantities, as in the proof of part i).

To obtain iii), a similar analysis to that in the previous cases shows that

$$\int_{A(r_1, 1)} (a - \mu_k) u \varphi_k = -\omega_N r_1^{N-1} u(r_1) \varphi'_k(r_1),$$

and the lemma follows easily as previously.

Proof of Proposition 2.2. Let $k = 0$. If we suppose that $u$ has no zeros in $(0, 1]$ and we integrate the equation $-\Delta u = a u$ in $B_1$, we obtain $\int_{B_1} a u = 0$, a contradiction. Hence, for the rest of the proof we will consider $k \geq 1$.

Let $1 \leq i \leq k$. By the previous lemma $u$ vanishes in the $i$ disjoint intervals $[r_i, r_{i-1}), ..., [r_2, r_1), [r_1, 1)$. Therefore $u$ has, at least, $i$ zeros in the interval $[r_i, 1)$ which implies that $r_i \leq x_i$.

Finally, let us prove that $u$ has, at least, $k + 1$ zeros. From the previous part, taking $i = k$, $u$ has at least $k$ zeros in the interval $[r_k, 1]$, one in each of the $k$ disjoint intervals $[r_k, r_{k-1}), ..., [r_2, r_1), [r_1, 1)$. Suppose, by contradiction, that these are the only zeros of $u$. Then $u$ does not vanish in $(0, r_k)$ and applying part i) of Lemma 2.3 we obtain $u(r_k) = 0$ and $a \equiv \mu_k$ in $(0, r_k]$. Applying now part ii) of this lemma, we deduce $u(r_{k-1}) = 0$ and $a \equiv \mu_k$ in $[r_k, r_{k-1})$. Repeating this argument and using part iii) of the previous lemma we conclude $u(r_1) = 0$, for all $1 \leq i \leq k$ and $a \equiv \mu_k$ in $(0, 1]$, which contradicts $a \in \Gamma_k$.

For the proof of Theorem 2.1 we will distinguish three cases: the subcritical case ($1 \leq p < \frac{N}{N-1}$ if $N \geq 3$, and $p = 1$ if $N = 2$), the supercritical case ($p > \frac{N}{N-1}$ if $N \geq 2$), and the critical case ($p = \frac{N}{N-1}$ if $N \geq 3$).
3. The subcritical case

In this section, we study the subcritical case, i.e. $1 \leq p < \frac{N}{2}$, if $N \geq 3$, and $p = 1$ if $N = 2$. In all those cases we will prove that $\gamma_{p,k} = 0$.

The next lemma is related to the continuous domain dependence of the eigenvalues of the Dirichlet Laplacian. In fact, the result is valid under much more general hypothesis (see [6]). Here we show a very simple proof for this special case.

**Lemma 3.1.** Let $N \geq 2$ and $R > 0$. Then

$$\lim_{\varepsilon \to 0} \lambda_1(A(\varepsilon,R)) = \lambda_1(B_R),$$

where $\lambda_1(A(\varepsilon,R))$ and $\lambda_1(B_R)$ denotes, respectively, the first eigenvalues of the Laplacian operator with Dirichlet boundary conditions of the annulus $A(\varepsilon,R)$ and the ball $B_R$.

**Proof.** For $N \geq 3$ and $\varepsilon \in (0, R/2)$ define the following radial function $u_\varepsilon \in H^1_0(A(\varepsilon,R))$:

$$u_\varepsilon(x) = \begin{cases} 
\phi_1(x), & \text{if } 2\varepsilon < |x| < R, \\
\frac{|x| - \varepsilon}{\varepsilon} \phi_1(2\varepsilon), & \text{if } \varepsilon < |x| < 2\varepsilon,
\end{cases}$$

where $\phi_1$ denotes the first eigenfunction with Dirichlet boundary conditions of the ball $B_R$. It is easy to check that

$$\lim_{\varepsilon \to 0} \int_{A(\varepsilon,2\varepsilon)} |\nabla u_\varepsilon|^2 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{2\varepsilon} \omega_N r^{N-1} \frac{\phi_1(2\varepsilon)^2}{\varepsilon^2} dr = 0.$$

In the same way it is obtained $\lim_{\varepsilon \to 0} \int_{A(\varepsilon,2\varepsilon)} u_\varepsilon^2 = 0$. In addition, from the variational characterization of the first eigenvalue it follows that $\lambda_1(A(\varepsilon,R)) \leq \frac{\int_{A(\varepsilon,R)} |\nabla u_\varepsilon|^2}{\int_{A(\varepsilon,R)} u_\varepsilon^2}$. Therefore

$$\limsup_{\varepsilon \to 0} \lambda_1(A(\varepsilon,R)) \leq \limsup_{\varepsilon \to 0} \frac{\int_{A(\varepsilon,R)} |\nabla u_\varepsilon|^2}{\int_{A(\varepsilon,R)} u_\varepsilon^2} = \frac{\int_{B_R} |\nabla \phi_1|^2}{\int_{B_R} \phi_1^2} = \lambda_1(B_R).$$

On the other hand, using that the first Dirichlet eigenvalue $\lambda_1(\Omega)$ is strictly decreasing with respect to the the domain $\Omega$, it follows that $\lambda_1(A(\varepsilon,R)) > \lambda_1(B_R)$. Thus

$$\liminf_{\varepsilon \to 0} \lambda_1(A(\varepsilon,R)) \geq \lambda_1(B_R)$$

and the lemma follows for $N \geq 3$.

The same proof works for $N = 2$ if we consider, for every $\varepsilon \in \left(0, \min\{1, R^2\}\right)$, the radial function $u_\varepsilon \in H^1_0(A(\varepsilon,R))$:
\begin{equation}
\begin{aligned}
u_\varepsilon(x) &= \begin{cases}
\phi_1(x), & \text{if } \sqrt{\varepsilon} \leq |x| < R, \\
\log |x| - \log \varepsilon - \frac{1}{\varepsilon} \sqrt{\varepsilon} \phi_1(\sqrt{\varepsilon}), & \text{if } \varepsilon < |x| < \sqrt{\varepsilon}.
\end{cases}
\end{aligned}
\end{equation}

Lemma 3.2. Let \( k \geq 0, N \geq 3 \) and \( 1 \leq p < N/2 \). Then \( \gamma_{p,k} = 0 \).

Proof. If \( k = 0 \), this lemma follows from [2 Lem. 3.1]. In this lemma a family of bounded, positive and radial solutions were used. Hence, for the rest of the proof we will consider \( k \geq 1 \).

To prove this lemma we will construct an explicit family \( a_\varepsilon \in \Gamma_k \) such that \( \lim_{\varepsilon \to 0} \| a_\varepsilon - \mu_k \|_{L^p(B_1)} = 0 \). To this end, for every \( \varepsilon \in (0, r_k) \), define \( u_\varepsilon : B_1 \to \mathbb{R} \) as the radial function

\begin{equation}
u_\varepsilon(x) = \begin{cases}
\varphi_k, & \text{if } r_k \leq |x| < 1, \\
\phi_1 (A(\varepsilon, r_k)), & \text{if } \varepsilon \leq |x| < r_k, \\
\phi_1 (B_\varepsilon), & \text{if } |x| < \varepsilon.
\end{cases}
\end{equation}

where \( \phi_1 (A(\varepsilon, r_k)) \) and \( \phi_1 (B_\varepsilon) \) denotes, respectively, the first eigenfunctions with Dirichlet boundary conditions of the annulus \( A(\varepsilon, r_k) \) and the ball \( B_\varepsilon \).

Moreover these eigenfunctions are chosen such that \( u_\varepsilon \in C^1(B_1) \).

Then, it is easy to check that \( u_\varepsilon \) is a solution of (1.12), being \( a_\varepsilon \in L^\infty(B_1) \) the radial function

\begin{equation}
a_\varepsilon(x) = \begin{cases}
\mu_k, & \text{if } r_k < |x| < 1, \\
\lambda_1 (A(\varepsilon, r_k)), & \text{if } \varepsilon < |x| < r_k, \\
\lambda_1 (B_\varepsilon), & \text{if } |x| < \varepsilon,
\end{cases}
\end{equation}

where \( \lambda_1 (A(\varepsilon, r_k)) \) and \( \lambda_1 (B_\varepsilon) \) denotes, respectively, the first eigenvalues with Dirichlet boundary conditions of the annulus \( A(\varepsilon, r_k) \) and the ball \( B_\varepsilon \).

Since the first Dirichlet eigenvalue \( \lambda_1 (\Omega) \) is strictly decreasing with respect to the domain \( \Omega \), it follows that

\[ \lambda_1 (A(\varepsilon, r_k)) > \lambda_1 (B_\varepsilon) = \mu_k, \]

which gives \( a_\varepsilon \in \Gamma_k \). (The equality \( \lambda_1 (B_\varepsilon) = \mu_k \) follows from the fact that \( \varphi_k \) is a positive solution of \( -\Delta \varphi = \mu_k \varphi \) in \( B_\varepsilon \) which vanishes on \( \partial B_\varepsilon \)).

Let us estimate the \( L^p \)-norm of \( a_\varepsilon - \mu_k \):
\[ \|a_\varepsilon - \mu_k\|_{L^p(B_1)} = \left( \int_{B_\varepsilon} (\lambda_1(B_\varepsilon) - \mu_k)^p + \int_{A(\varepsilon, r_k)} (\lambda_1(A(\varepsilon, r_k)) - \mu_k)^p \right)^{\frac{1}{p}} = \\
\left( (\lambda_1(B_\varepsilon) - \mu_k)^p \frac{\omega_N \varepsilon^N}{N} + (\lambda_1(A(\varepsilon, r_k)) - \mu_k)^p \frac{\omega_N (r_k^N - \varepsilon^N)}{N} \right)^{\frac{1}{p}}. \]

Taking into account that \( \lambda_1(B_\varepsilon) = \lambda_1(B_1)/\varepsilon^2 \), \( \lambda_1(B_{r_k}) = \mu_k \), using \( N > 2p \), and applying Lemma 3.1, we conclude

\[ \lim_{\varepsilon \to 0} \|a_\varepsilon - \mu_k\|_{L^p(B_1)} \leq \lim_{\varepsilon \to 0} \left( (\lambda_1(B_1))^p \frac{\omega_N \varepsilon^N}{N} + (\lambda_1(A(\varepsilon, r_k)) - \mu_k)^p \frac{\omega_N (r_k^N - \varepsilon^N)}{N} \right)^{\frac{1}{p}} = 0, \]

and the proof is complete. \( \square \)

**Lemma 3.3.** Let \( k \geq 0, N = 2 \) and \( p = 1 \). Then \( \gamma_{1,k} = 0 \).

**Proof.** If \( k = 0 \), this lemma follows from [2, Lem. 3.2]. In this lemma a family of bounded, positive and radial solutions were used. Hence, for the rest of the proof we will consider \( k \geq 1 \).

Similarly to the proof of the previous lemma, we will construct some explicit sequences in \( \Gamma_k \). In this case, this construction will be slightly more complicated. First, for every \( \alpha \in (0, 1) \), define \( v_\alpha, A_\alpha : B_1 \to \mathbb{R} \) as the radial functions:

\[ v_\alpha(r) = \begin{cases} 
\alpha(1 - r^2)(3 - r^2) - \log r, & \text{if } \alpha \leq r < 1, \\
\alpha(1 - r^2)(3 - r^2) - \log \alpha + \frac{\alpha^2 - r^2}{2\alpha^2}, & \text{if } r < \alpha,
\end{cases} \]

\[ A_\alpha(r) = \begin{cases} 
\frac{16\alpha(1 - r^2)}{\alpha(1 - r^2)(3 - r^2) - \log r}, & \text{if } \alpha < r < 1, \\
\frac{16\alpha(1 - r^2) + 2}{\alpha}, & \text{if } r < \alpha,
\end{cases} \]

where \( r = |x| \). It is easily seen that \( v_\alpha \in C^1(B_1), A_\alpha \in L^\infty(B_1) \), and

\[ \left\{ \begin{array}{ll}
\Delta v_\alpha(x) + A_\alpha(x)v_\alpha(x) = 0, & x \in B_1 \\
v_\alpha(x) = 0, & x \in \partial B_1
\end{array} \right. \]

Now, for every \( \alpha \in (0, 1) \) and \( \varepsilon \in (0, r_k) \), define \( u_{\alpha, \varepsilon} : B_1 \to \mathbb{R} \) as the radial function:
where the eigenfunctions $\varphi_k$ and $\phi_1(A(\varepsilon, r_k))$ are chosen such that $u_{\alpha, \varepsilon} \in C^1(B_1)$.

An easy computation shows that $u_{\alpha, \varepsilon}$ is a solution of (1.12), being $a_{\alpha, \varepsilon} \in L^\infty(B_1)$ the radial function

\begin{equation}
(3.10) \quad a_{\alpha, \varepsilon}(x) = \begin{cases} 
\mu_k, & \text{if } r_k < |x| < 1, \\
\lambda_1(A(\varepsilon, r_k)), & \text{if } \varepsilon < |x| < r_k, \\
\frac{1}{\varepsilon^2} A_{\alpha}(\frac{x}{\varepsilon}), & \text{if } |x| < \varepsilon.
\end{cases}
\end{equation}

Again, using that the first Dirichlet eigenvalue $\lambda_1(\Omega)$ is strictly decreasing with respect to the domain $\Omega$, it follows that

$$
\lambda_1(A(\varepsilon, r_k)) > \lambda_1(B_{r_k}) = \mu_k.
$$

Moreover, \(\inf_{|x|<\varepsilon} a_{\alpha, \varepsilon}(x) = \left( \inf_{x \in B_1} A_{\alpha}(x) \right) / \varepsilon^2 := m_\alpha / \varepsilon^2\). We see at once that $m_\alpha > 0$ for every $\alpha \in (0, 1)$. Hence, if we fix $\alpha$ and choose $\varepsilon \in (0, 1)$ such that $m_\alpha / \varepsilon^2 \geq \mu_k$, it is deduced that $a_{\alpha, \varepsilon} \in \Gamma_k$.

Let us estimate the $L_1$-norm of $a_{\alpha, \varepsilon} - \mu_k$:

\begin{equation}
(3.11) \quad \|a_{\alpha, \varepsilon} - \mu_k\|_{L^1(B_1)} = \int_{B_\varepsilon} \left( \frac{1}{\varepsilon^2} A_{\alpha}(\frac{x}{\varepsilon}) - \mu_k \right) dx + \int_{A(\varepsilon, r_k)} (\lambda_1(A(\varepsilon, r_k)) - \mu_k) dx.
\end{equation}

Doing the change of variables $x = \varepsilon y$ in the first integral and applying Lemma 3.1 in the second one, it is obtained, for fixed $\alpha \in (0, 1)$:

$$
\lim_{\varepsilon \to 0} \|a_{\alpha, \varepsilon} - \mu_k\|_{L^1(B_1)} = \int_{B_1} A_{\alpha}(y) dy.
$$

Thus, from the definition of $\gamma_{1,k}$ we have

\begin{equation}
(3.12) \quad \gamma_{1,k} \leq \int_{B_1} A_{\alpha}(y) dy, \quad \forall \alpha \in (0, 1).
\end{equation}

Now we will take limit when $\alpha$ tends to 0 in this last expression. For this purpose we first deduce easily from the definition of $A_{\alpha}$ that $A_{\alpha}(r) \leq 16\alpha(1-$
Lemma 4.1. Let \( a \) attained in the unique element \( a_u \) by \( \mu \). We begin by studying the case \( p \equiv 0 \). Hence, multiplying the equation \(-\Delta u = au \) in \( B_1 \) we obtain

\[
\int_{B_1} A_\alpha(y)dy = 2\pi \int_0^1 r A_\alpha(r)dr \leq 2\pi \int_0^\alpha r \frac{16\alpha + 2/\alpha^2}{-\log\alpha}dr + 2\pi \int_\alpha^1 r 32\alpha dr
\]

which gives \( \lim_{\alpha \to 0} \int_{B_1} A_\alpha(y)dy = 0 \) and the lemma follows from (3.12). \( \square \)

4. The supercritical case

In this section, we study the supercritical case, i.e. \( p > \frac{N}{2} \), if \( N \geq 2 \). In all those cases we will prove that \( \gamma_{p,k} \) is strictly positive and that it is attained. We begin by studying the case \( p = \infty \).

Lemma 4.1. Let \( k \geq 0, N \geq 2 \) and \( p = \infty \). Then \( \gamma_{\infty,k} = \mu_{k+1} - \mu_k \) is attained in the unique element \( a_0 \equiv \mu_{k+1} \in \Gamma_k \).

Proof. Clearly \( a_0 \equiv \mu_{k+1} \in \Gamma_k \) satisfies \( \|a_0 - \mu_k\|_{L^\infty(B_1)} = \mu_{k+1} - \mu_k \). Suppose, contrary to our claim, that there exists \( \mu_{k+1} \neq a \in \Gamma_k \) such that \( \|a - \mu_k\|_{L^\infty(B_1)} \leq \mu_{k+1} - \mu_k \). Therefore \( \mu_k < a < \mu_{k+1} \), a contradiction with the fact \( a \in \Gamma_k \) (see [5, 9]). \( \square \)

Next we concentrate on the case \( \frac{N}{2} < p < \infty \).

Lemma 4.2. Let \( N \geq 2, p > N/2 \) and \( M > 0 \). Then, there exists \( \epsilon = \epsilon(N, p, M) \) with the following property:

For every \( a \in L^p(B_1) \) satisfying \( \|a\|_{L^p(B_1)} \leq M \) and every \( u \in H^1(B_1) \) radial nontrivial solution of \(-\Delta u = au \) in \( B_1 \) we have

i) \( z > \epsilon \) for every zero \( z \) of \( u \).

ii) \( |z_2 - z_1| > \epsilon \) for every different zeros \( z_1, z_2 \) of \( u \).

Proof. Let \( z \in (0, 1] \) be a zero of \( u \). Hence, multiplying the equation \(-\Delta u = au \) by \( u \), integrating by parts in the ball \( B_z \) and applying Hölder inequality, we obtain

\[
\int_{B_z} |\nabla u|^2 = \int_{B_z} a u^2 \leq \|a\|_{L^p(B_z)} \|u\|_{L^{\frac{2p}{p-1}}(B_z)}^{2p}.
\]

From the above it follows that

\[
M \geq \|a\|_{L^p(B_1)} \geq \|a\|_{L^p(B_z)} \geq \frac{\|\nabla u\|_{L^2(B_z)}^2}{\|u\|_{L^{\frac{2p}{p-1}}(B_z)}^{2p}} \geq \min_{v \in H^1_0(B_z)} \frac{\|\nabla v\|_{L^2(B_z)}^2}{\|v\|_{L^{\frac{2p}{p-1}}(B_z)}^{2p}}.
\]

From the change \( w(x) = v(zx) \), it is easily deduced that
\[
\min_{v \in H^1_0(B_k)} \|\nabla v\|_{L^2(B_k)}^2 = \frac{N}{p} \min_{w \in H^1_0(B_1)} \|\nabla w\|_{L^2(B_1)}^2 := \frac{N}{p} \alpha(N, p),
\]
where we have used the compact embedding \(H^1_0(B_1) \subset L^{\frac{2p}{p-1}}(B_1)\) (since \(p > N/2\), then \(2 < \frac{2p}{p-1} < \frac{2N}{N-2}\), which is the critical Sobolev exponent). Thus, taking \(\varepsilon > 0\) such that \(M < \varepsilon_1 \frac{N}{p-2} \alpha(N, p)\), we conclude part i) of the lemma with \(\varepsilon = \varepsilon_1\).

For the second part of the lemma, consider two zeros \(0 < z_1 < z_2 < 1\) of \(u\). Taking into account that \(z_1 \geq \varepsilon_1\) and arguing in the same manner of part i), we obtain
\[
M \geq \|a\|_{L^p(B_1)} \geq \|a\|_{L^p(A(z_1, z_2))} \geq \|u\|_{L^2(A(z_1, z_2))}^2 = \frac{1}{\varepsilon_1 N} \int_{z_1}^{z_2} \frac{u''(r)^2}{(2p/(p-1))^{(p-1)/p}} dr \geq \omega_N \left( \int_{z_1}^{z_2} \frac{u(r)^2}{(2p/(p-1))^{(p-1)/p}} dr \right) (p-1)/p.
\]

On the other hand, from the one dimensional change of variable \(w(x) = v(z_1 + (z_2 - z_1)x)\), it is immediate that
\[
\min_{w \in H_0^1(1, z_1)} \|w''\|_{L^2(1, z_1)}^2 = (z_2 - z_1)^{\frac{p}{p-2}} \min_{w \in H^1_0(0, 1)} \|w''\|_{L^2(0, 1)}^2 \geq \omega_N \varepsilon_1 N \int_{z_1}^{z_2} \frac{u''(r)^2}{(2p/(p-1))^{(p-1)/p}} dr \geq \omega_N \varepsilon_1 \varepsilon_1 \frac{N}{p-2} \alpha(N, p).
\]

It follows that \(M \geq \omega_N \varepsilon_1^N \varepsilon_1^{N-1} (z_2 - z_1)^{1/p-2} C_p\). From this, taking \(\varepsilon_2\) such that \(M < \omega_N \varepsilon_1 \varepsilon_1^{N-1} \varepsilon_2^{1/p-2} C_p\), we conclude part ii) of the lemma with \(\varepsilon = \varepsilon_2\).

Obviously, taking \(\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}\), the lemma is proved.

**Lemma 4.3.** Let \(k \geq 0\), \(N \geq 2\) and \(N/2 < p < \infty\). Then \(\gamma_{p,k}\) is strictly positive and it is attained in a function \(a_0 \in \Gamma_k\).

**Proof.** Take a sequence \(\{a_n\} \subset \Gamma_k\) such that \(\|a_n - \mu_k\|_{L^p(B_1)} \to \gamma_{p,k}\). Take \(\{u_n\} \subset H^1(B_1)\) such that \(u_n\) is a radial solution of (1.12), for \(a = a_n\), with the normalization \(\|u_n\|_{H^1(B_1)} = \int_{B_1} (|\nabla u_n|^2 + u_n^2) = 1\). Therefore, we can suppose, up to a subsequence, that \(u_n \rightharpoonup u_0\) in \(H^1(B_1)\) and \(u_n \to u_0\) in \(L^{\frac{2p}{p-1}}(B_1)\) (since \(p > N/2\), then \(2 < \frac{2p}{p-1} < \frac{2N}{N-2}\), which is the critical Sobolev exponent). On the other hand, since \(\{a_n\}\) is bounded in \(L^p(B_1)\), and \(1 \leq N/2 < p < \infty\), we can assume, up to a subsequence, that \(a_n \to a_0\) in \(L^p(B_1)\). Taking limits in the equation (1.12), for \(a = a_0\) and \(u = u_n\), we obtain that \(u_0\) is a solution of this equation for \(a = a_0\). Note that \(u_n \to u_0\)
in $L^{2p}(B_1)$ and $a_n \to a_0$ in $L^p(B_1)$ yields $\lim \int_{B_1} |\nabla u_n|^2 = \lim \int_{B_1} a_n u_n^2 = \int_{B_1} a_0 u_0^2 = \int_{B_1} |\nabla u_0|^2$ and consequently $u_n \to u_0 \neq 0$ in $H^1(B_1)$. Therefore, if $a_0 \neq \mu_k$, then $a_0 \in \Gamma_k$ and $\|a_0 - \mu_k\|_p \leq \lim_{n \to \infty} \|a_n - \mu_k\|_p = \gamma_{p,k}$, and the lemma follows.

On the contrary, suppose by contradiction that $a_0 \equiv \mu_k$. Then $u_0 = \varphi_k$ for some nontrivial radial eigenfunction $\varphi_k$. Consider $\varepsilon$ given in Lemma 4.2. Take $\varepsilon_0 = \min \{\varepsilon, 2r_k/3, 2(1 - r_1), r_i - r_{i+1}; 1 \leq i \leq k-1\}$. Thus, from the previous lemma, $u_n$ has no zeros in $(0, \varepsilon_0)$, and has, at most, one zero in each of the $k$ disjoint intervals $(r_i - \varepsilon_0/2, r_i + \varepsilon_0/2), 1 \leq i \leq k$. Therefore, $u_n$ has, at most, $k$ zeros in the set $A := (0, \varepsilon_0) \cup (\bigcup_{1 \leq i \leq k} (r_i - \varepsilon_0/2, r_i + \varepsilon_0/2))$.

On the other hand, taking into account the continuous embedding $H^1_{rad}(A(\varepsilon_0, 1)) \subset C(A(\varepsilon_0, 1))$ and $u_n \to \varphi_k$ in $H^1_0(B_1)$, we can assert $u_n \to \varphi_k$ in $C(A(\varepsilon_0, 1))$. Clearly $\min_{r \in (0,1) \setminus A} |\varphi_k(r)| > 0$. Then, for large $n$ we see that $\min_{r \in (0,1) \setminus A} |u_n(r)| > 0$, which implies that $u_n$ does not vanish in $(0, 1) \setminus A$, for large $n$. Since $u_n$ has, at most, $k$ zeros in $A$, we conclude that $u_n$ has, at most, $k$ zeros in $(0,1)$, for large $n$. This contradicts Proposition 2.2 and the lemma follows.

\[\square\]

5. The critical case

In this section, we study the critical case, i.e. $p = \frac{N}{2}$, if $N \geq 3$. We will prove that $\gamma_{p,k} > 0$.

**Lemma 5.1.** Let $k \geq 0$, $N \geq 3$ and $p = N/2$. Then $\gamma_{p,k} > 0$.

**Proof.** To obtain a contradiction, suppose that $\gamma_{p,k} = 0$. Then we could find a sequence $\{a_n\} \subset \Gamma_k$ such that $a_n \to \mu_k$ in $L^{N/2}(B_1)$. Similarly to the supercritical case, we can take $\{u_n\} \subset H^1(B_1)$ such that $u_n$ is a radial solution of (1.12), for $a = a_n$, with the normalization $\|u_n\|_{H^1(B_1)} = 1$. Again, we can suppose, up to a subsequence, that $u_n \to u_0$ in $H^1(B_1)$ and taking limits in the equation (1.12), for $a = a_n$ and $u = u_n$, we obtain that $u_0$ is a solution of this equation for $a = \mu_k$.

We claim that $u_n \to u_0$ in $H^1(B_1)$ and consequently, $u_0 = \varphi_k$, for some nontrivial eigenfunction $\varphi_k$. For this purpose, we set

$$\lim \int_{B_1} |\nabla u_n|^2 = \lim \int_{B_1} a_n u_n^2 = \lim \int_{B_1} (a_n - \mu_k) u_n^2 + \lim \int_{B_1} \mu_k u_n^2 = 0 + \mu_k \int_{B_1} u_0^2 = \int_{B_1} |\nabla u_0|^2,$$

where we have used $a_n \to \mu_k$ in $L^{N/2}(B_1)$ and $u_n^2$ is bounded in $L^{N/(N-2)}(B_1)$ (since $u_n$ is bounded in $H^1(B_1) \subset L^{2N/(N-2)}(B_1)$). Thus, from standard arguments, we deduce that $u_n \to u_0 = \varphi_k$ in $H^1(B_1)$.

In the following, we will fix $\varepsilon \in (0, r_k)$. Since $a_n \to \mu_k$ in $L^{N/2}(A(\varepsilon, 1))$ and $u_n \to u_0 = \varphi_k$ in $H^1_{rad}(A(\varepsilon, 1)) \subset C(A(\varepsilon, 1))$, we can assert that
that, for large $n$, the number of zeros of $u_n$ is equal to the number of zeros of $\varphi_k$ in the annulus $A(\varepsilon, 1)$, which is exactly $k$. Applying Proposition 2.2 we can assert that, for large $n$ there exists a zero $\varepsilon_n \in (0, \varepsilon)$ of $u_n$. Hence, multiplying the equation $-\Delta u_n = a_n u_n$ by $u_n$, integrating by parts in the ball $B_{\varepsilon_n}$ and applying Hölder inequality, we deduce

$$
\int_{B_{\varepsilon_n}} |\nabla u_n|^2 = \int_{B_{\varepsilon_n}} a_n u_n^2 \leq \|a_n\|_{L^{N/2}(B_{\varepsilon_n})} \|u_n\|_{L^{2N/(N-2)}(B_{\varepsilon_n})}^2.
$$

From the above it follows that

$$
\|a_n\|_{L^{N/2}(B_{\varepsilon_n})} \geq \frac{\|\nabla u_n\|_{L^2(B_{\varepsilon_n})}^2}{\|u_n\|_{L^{2N/(N-2)}(B_{\varepsilon_n})}^2} \geq \inf_{u \in H^1_0(B_{\varepsilon_n})} \frac{\|\nabla u\|_{L^2(B_{\varepsilon_n})}^2}{\|u\|_{L^{2N/(N-2)}(B_{\varepsilon_n})}^2}.
$$

From the change $v(x) = u(\varepsilon_n x)$, it is easily deduced that

$$
\inf_{u \in H^1_0(B_{\varepsilon_n})} \frac{\|\nabla u\|_{L^2(B_{\varepsilon_n})}^2}{\|u\|_{L^{2N/(N-2)}(B_{\varepsilon_n})}^2} = \inf_{v \in H^1_0(B_1)} \frac{\|\nabla v\|_{L^2(B_1)}^2}{\|v\|_{L^{2N/(N-2)}(B_1)}^2} := C_N > 0.
$$

From the above it follows that, for fixed $\varepsilon \in (0, r_k)$ and large $n$, we obtain

$$
C_N \leq \|a_n\|_{L^{N/2}(B_{\varepsilon_n})} \leq \|a_n - \mu_k\|_{L^{N/2}(B_{\varepsilon_n})} + \|\mu_k\|_{L^{N/2}(B_{\varepsilon_n})} \leq \|a_n - \mu_k\|_{L^{N/2}(B_1)} + \|\mu_k\|_{L^{N/2}(B_1)}.
$$

Taking limits when $n$ tends to $\infty$ in this expression we deduce

$$
C_N \leq \mu_k \left(\frac{\omega_N \varepsilon^N}{N}\right)^{2/N}.
$$

Choosing $\varepsilon > 0$ sufficiently small we obtain a contradiction.

\[\Box\]

References

[1] A. Cañada, J.A. Montero and S. Villegas. Liapunov-type inequalities and Neumann boundary value problems at resonance. Math. Ineq. Appl., 8 (2005), 459-475.
[2] A. Cañada, J.A. Montero and S. Villegas. Lyapunov inequalities for partial differential equations. J. Funct. Anal., 237, (2006), 176-193.
[3] A. Cañada and S. Villegas. Lyapunov inequalities for Neumann boundary conditions at higher eigenvalues. J. Eur. Math. Soc., 12, (2010), 163-178.
[4] M. Del Pino and R. Manásevich. Global bifurcation from the eigenvalues of the p-Laplacian. J. Differential Equations, 92, (1991), 226-251.
[5] C. L. Dolph. Nonlinear equations of Hammerstein type. Trans. Amer. Math. Soc., 66, (1949), 289-307.
[6] B. Fuglede. Continuous domain dependence of the eigenvalues of the Dirichlet Laplacian and related operators in Hilbert Space. J. Funct. Anal., 167, (1999), 183-200.
[7] P. Hartman. *Ordinary Differential Equations*. John Wiley and Sons Inc., New York-London-Sydney, 1964.

[8] W. Huaizhong and L. Yong. *Neumann boundary value problems for second-order ordinary differential equations across resonance*. SIAM J. Control and Optimization, 33, (1995), 1312-1325.

[9] G. Vidosicch. *Existence and uniqueness results for boundary value problems from the comparison of eigenvalues*. ICTP Preprint Archive, 1979015, 1979.

[10] M. Zhang. *Certain classes of potentials for p-Laplacian to be non-degenerate*. Math. Nachr., 278, (2005), 1823-1836.

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