THE (abc, pqr)-PROBLEM FOR APPROXIMATE SCHAUDER FRAMES FOR BANACH SPACES
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Abstract: Motivated from the complete solution of important abc-problem for Gabor system for the Hilbert space $L^2(\mathbb{R})$ by Dai and Sun [Memoirs of Amer. Math. Soc., 2016] and from the existential result of approximate Schauder frames for $L^p(\mathbb{R})$ using translation operators on $L^p(\mathbb{R})$ by Freeman, Odell, Schlumprecht, and Zsak [Israel. J. Math, 2014], we formulate (abc, pqr)-problem for approximate Schauder frames for Banach spaces $L^p(\mathbb{R})$, $1 < p < \infty$.

Keywords: abc-problem for Gabor system, Approximate Schauder Frame.

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1. Introduction

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. Recall that [7, 16, 29] a sequence $\{\tau_n\}$ in $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist $a, b > 0$ such that

$$a\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \tau_n \rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$ 

Among various examples of frames for Hilbert spaces, one of the most studied classes of frames for the Hilbert space $L^2(\mathbb{R})$ is the frames arising from the Gabor systems $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$, where $g \in L^2(\mathbb{R})$ and for $c, d \in \mathbb{R}$,

$$T_c : L^2(\mathbb{R}) \ni f \mapsto T_c f \in L^2(\mathbb{R}); \quad T_c f : \mathbb{R} \ni x \mapsto (T_c f)(x) := f(x - c) \in \mathbb{C},$$
$$E_d : L^2(\mathbb{R}) \ni f \mapsto E_d f \in L^2(\mathbb{R}); \quad E_d f : \mathbb{R} \ni x \mapsto (E_d f)(x) := e^{2\pi idx} f(x) \in \mathbb{C}.$$ 

A detailed analysis of Gabor frames for Hilbert spaces can be found in [13, 19, 21, 23, 25, 55]. A fundamental problem in Gabor analysis is to characterize parameters $a > 0$, $b > 0$ and the function $g$ such that $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. The problem has been solved fully or partially for certain classes of functions in [3, 6, 12, 14, 15, 18, 24, 26, 28, 30, 41, 47, 49, 51, 53, 56, 58]. Let $c > 0$. When $g = \chi_{[0,c]}$, the characteristic function on $[0, c]$, the problem of characterizing parameters $a > 0$, $b > 0$ and $c > 0$ such...
that
$$\{E_m T_n \chi_{[0,c]}\}_{m,n \in \mathbb{Z}}$$
is a frame for $L^2(\mathbb{R})$ is known as $abc$-problem for Gabor system. A list of partial answers for this are obtained in [6, 20, 25, 28, 30, 31, 48, 49, 51]. In 2013, Dai and Sun fully resolved this problem in their Memoirs [10, 11]. In this paper, we formulate a similar problem for approximate Schauder frames for function spaces.

2. THE $(abc, pqr)$-PROBLEM FOR APPROXIMATE SCHAUDEER FRAMES

We are motivated from the following two results.

(i) The complete solution of $abc$-problem for Gabor system by Dai and Sun [11].

(ii) For every $2 < p < \infty$, there is an approximate Schauder frame for $L^p(\mathbb{R})$ obtained using translation operators on $L^p(\mathbb{R})$ [22].

Generalized Fourier expansion which resulted from the theory of frames for Hilbert spaces led the notion of framing in [5] which is generalized to Schauder frames in [4] which is further generalized in [22, 59] to approximate Schauder frames defined as follows.

**Definition 2.1.** [22, 59] Let $X$ be a separable Banach space and $X^*$ be its dual. Let $\{\tau_n\}_{n}$ be a sequence in $X$ and $\{f_n\}_{n}$ be a sequence in $X^*$. The pair $(\{f_n\}_{n}, \{\tau_n\}_{n})$ is said to be an approximate Schauder frame (ASF) for $X$ if the map
$$S_{f,\tau} : X \ni x \mapsto S_{f,\tau} x := \sum_{n=1}^{\infty} f_n(x) \tau_n \in X$$
is a well-defined bounded linear, invertible operator.

We now consider the Banach space $X = L^p(\mathbb{R})$, for $1 < p < \infty$. Let $q$ be the conjugate index of $p$. Since dual of $L^p(\mathbb{R})$ is isometrically isomorphic to $L^q(\mathbb{R})$, given $\phi \in X^*$, there exists a unique $\omega \in L^q(\mathbb{R})$ such that
$$\phi(u) = \int_{\mathbb{R}} u(\alpha) \omega(\alpha) d\alpha := [u, \omega], \quad \forall u \in L^p(\mathbb{R}).$$
In particular, if $\{f_n\}_n$ is a sequence in $X^*$, then there exists a unique sequence $\{\omega_n\}_n$ in $X$ such that
$$f_n(x) = [x, \omega_n], \quad \forall x \in X, \forall n \in \mathbb{N}.$$ Thus asking whether a pair $(\{f_n\}_n, \{\tau_n\}_n)$ is an ASF for $X$ is same as asking whether the map
$$S_{\omega,\tau} : X \ni x \mapsto S_{\omega,\tau} x := \sum_{n=1}^{\infty} [x, \omega_n] \tau_n \in X$$
is a well-defined bounded linear, invertible operator. We then have the following problem.

**Problem 2.2.** Let $1 < p < \infty$ and $q$ be the conjugate index of $p$. Characterize parameters $a > 0$, $b > 0$, $p > 0$, $q > 0$ and functions $g \in L^p(\mathbb{R})$, $h \in L^q(\mathbb{R})$ such that the pair
$$\{E_m q \tilde{T}_{np} h\}_{m,n \in \mathbb{Z}}, \{E_m b T_{na} g\}_{m,n \in \mathbb{Z}}.$$


Problem 2.3. (The $(abc,pqr)$-problem for approximate Schauder frames) Let $1 < p < \infty$ and $q$ be the conjugate index of $p$. Characterize parameters $a > 0$, $b > 0$, $c > 0$, $p > 0$, $q > 0$, $r > 0$ such that the pair

$$\left(\{E_{mq}T_{np}\chi[0,r]\}_{m,n\in\mathbb{Z}}, \{E_{mb}T_{na}\chi[0,c]\}_{m,n\in\mathbb{Z}}\right)$$

is an ASF for $\mathcal{L}^p(\mathbb{R})$.

We remark that we can formulate problems similar to Problems 2.2, 2.4 for $p$-approximate Schauder frames, $p$-approximate Bessel sequences, $p$-approximate orthonormal bases and $p$-approximate Riesz bases \cite{43,46}. Following is one such in the discrete case. Our problem is motivated from the discrete Gabor analysis in $\ell^2(\mathbb{Z})$ \cite{1,2,8,9,50,54}.

Problem 2.4. (The $(MN,PQ)$-problem for $p$-approximate Schauder frames) Let $1 < p < \infty$ and $q$ be the conjugate index of $p$. Characterize natural numbers $N,M,P,Q$ and sequences $\{x_j\}_{j\in\mathbb{Z}} \in \mathcal{L}^p(\mathbb{Z})$, $\{y_j\}_{j\in\mathbb{Z}} \in \mathcal{L}^q(\mathbb{Z})$ such that the pair

$$\left(\{\tilde{E}_{m/p}\tilde{T}_{nQ}\{y_j\}_{j\in\mathbb{Z}}\}_{0\leq m\leq p-1,n\in\mathbb{Z}}, \{E_{m/M}\tilde{T}_{nN}\{x_j\}_{j\in\mathbb{Z}}\}_{0\leq m\leq M-1,n\in\mathbb{Z}}\right)$$

is a

(i) $p$-orthonormal basis for $\ell^p(\mathbb{Z})$.
(ii) $p$-approximate Riesz basis for $\ell^p(\mathbb{Z})$.
(iii) $p$-approximate Schauder frame for $\ell^p(\mathbb{Z})$.
(iv) $p$-approximate Bessel sequence for $\ell^p(\mathbb{Z})$.

where

$$T_c : \mathcal{L}^p(\mathbb{R}) \ni f \mapsto T_c f \in \mathcal{L}^p(\mathbb{R}); \quad T_c f : \mathbb{R} \ni x \mapsto (T_c f)(x) := f(x-c), \quad \text{for } c \in \mathbb{R},
$$

$$E_d : \mathcal{L}^p(\mathbb{R}) \ni f \mapsto E_d f \in \mathcal{L}^p(\mathbb{R}); \quad E_d f : \mathbb{R} \ni x \mapsto (E_d f)(x) := e^{2\pi i dx} f(x), \quad \text{for } d \in \mathbb{R}$$

and

$$\tilde{T}_s : \mathcal{L}^q(\mathbb{R}) \ni f \mapsto \tilde{T}_s f \in \mathcal{L}^q(\mathbb{R}); \quad \tilde{T}_s f : \mathbb{R} \ni x \mapsto (\tilde{T}_s f)(x) := f(x-s), \quad \text{for } s \in \mathbb{R},
$$

$$\tilde{E}_t : \mathcal{L}^q(\mathbb{R}) \ni f \mapsto \tilde{E}_t f \in \mathcal{L}^q(\mathbb{R}); \quad \tilde{E}_t f : \mathbb{R} \ni x \mapsto (\tilde{E}_t f)(x) := e^{2\pi itx} f(x), \quad \text{for } t \in \mathbb{R}.$$
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