On the basin of attraction of McKean-Vlasov paths

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Abstract

In this paper we provide short proofs and mild extensions of some statements about the ergodicity and the basins of attraction of the McKean-Vlasov evolution. The proofs are based on the representation of these as Wasserstein gradient flows.

Key words and phrases. Wasserstein gradient flows, McKean-Vlasov evolution, ergodicity, basin of attraction.

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1 Introduction

In this paper we study the ergodicity and the energy landscape of the flow \((\mu_t)_{t \in [0, \infty)}\) of the marginal laws of the stochastic differential equation given by

\[
dx_t = -\Psi'(x_t) \, dt + \frac{J}{2} \int R z \, d\mu_t \, dt + \sqrt{2} \, dB_t. \tag{1.1}
\]

Here, the single-site potential \(\Psi : \mathbb{R} \to \mathbb{R}\) and the interaction strength \(J > 0\) satisfy Assumption 1.1 below, and \(B\) is a one-dimensional Brownian motion. This flow \((\mu_t)_{t \in [0, \infty)}\) is often called McKean-Vlasov evolution in the literature.

In order to understand the main motivation for this paper, we recall five well-known facts.

(i) Let \((\mathcal{P}_2(\mathbb{R}), W_2)\) be the Wasserstein space; see Section 2.1 below. Then, we know from [1, Chapter 11] that \((\mu_t)_{t \in [0, \infty)}\) can be represented as a so-called Wasserstein gradient flow (again see Section 2.1) for the functional \(F : \mathcal{P}_2(\mathbb{R}) \to (-\infty, \infty]\), which is defined by

\[
F(\mu) = \int_{\mathbb{R}} \log(\rho) d\mu + \int_{\mathbb{R}} \Psi d\mu - \frac{J}{2} \left( \int_{\mathbb{R}} z d\mu(z) \right)^2 \tag{1.2}
\]

if \(\mu \in \mathcal{P}_2(\mathbb{R})\) has a Lebesgue density \(\rho\), and \(F(\mu) = \infty\) otherwise. Moreover, in [1, 11.2.8] it is shown that for all \(\mu \in \overline{D(F)} = \mathcal{P}_2(\mathbb{R})\), there exists a unique Wasserstein gradient flow for \(F\) with initial value \(\mu\). In this paper, we denote this gradient flow by \((S[\mu](t))_{t \in (0, \infty)}\).

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(ii) Consider the system of $N \in \mathbb{N}$ mean-field interacting diffusions given by

$$dx_i^N(t) = -\Psi'(x_i^N(t)) dt + \frac{J}{N} \sum_{j=0}^{N-1} x_j^N(t) dt + \sqrt{2} dB_i(t) \quad \text{for } 0 \leq i \leq N-1,$$

where $B^N = (B_i)_{i=0,\ldots,N-1}$ is an $N$-dimensional Brownian motion. Let $(L^N(t))_{t \in [0,\infty)}$ denote the corresponding empirical distribution process, i.e.,

$$L^N(t) = \frac{1}{N} \sum_{i=0}^{N-1} \delta_{x_i^N(t)} \quad \text{for all } t \in [0,\infty).$$

Then, ever since the classic papers [7] and [8], it is known that, in the limit as $N \to \infty$, the process $(L^N(t))_{t \in [0,\infty)}$ converges weakly to the deterministic McKean-Vlasov evolution.

(iii) Already in the paper [6] it was conjectured that the process $(L^N(t))_{t \in [0,\infty)}$ exhibits metastable behaviour. It is a long outstanding problem to verify this conjecture rigorously. Although some progress in this direction was established in the paper [4], there are still many open and challenging questions.

(iv) It is well-known that, in order to analyse the metastable behaviour of a stochastic system, it is essential to have deep knowledge on the underlying energy landscape of the system and its ergodicity, i.e., its possible convergence towards stationary measures. We refer the reader with no background in metastability to the monumental monographs in this subject given by [5] and [12].

(v) In order to study curves and other objects that belong to the infinite-dimensional space of probability measures, the Wasserstein formalism provides a natural and convenient framework. Indeed, ever since the seminal papers [10] and [13], it is known that the Wasserstein formalism provides the structure of a Riemannian manifold on the space of probability measures. We refer to [2, p. 421] or [4, Section 1.4] for more arguments that speak in favour of the Wasserstein formalism.

We now formulate the main motivation for this paper. Combining the facts (ii), (iii) and (iv), we see that, in order to understand the metastable behaviour of $(L^N(t))_{t \in [0,\infty)}$, it is essential to study the ergodicity and the energy landscape of the McKean-Vlasov evolution. Moreover, from fact (i) we see that the energy landscape associated to $(\mu_t)_{t \in [0,\infty)}$ is connected to the basins of attraction of $\mathcal{F}$; see Proposition 1.5 for the precise definition of the latter. This is the main motivation why we study the ergodicity of $(\mu_t)_{t \in [0,\infty)}$ and the basins of attraction of $\mathcal{F}$. Finally, fact (v) explains why we use the Wasserstein setting as the framework for this paper.

We make the following assumptions throughout this paper.

**Assumption 1.1**

(i) $\ell \in \mathbb{N} \cap [2,\infty)$. We suppose that $\Psi : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree $2\ell$, and we suppose that the coefficient of degree $2\ell$ is positive.

(ii) $\Psi(z) = \Psi(-z)$ for all $z \in \mathbb{R}$.

(iii) $z \mapsto \Psi'(z)$ is convex on $[0,\infty)$.

(iv) $1/J < \int_{\mathbb{R}} z^2 e^{-\Psi(z)} \, dz / (\int e^{-\Psi(z)} \, dz)$. 

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Remark 1.2 As an immediate consequence of Assumption 1.1, we have that there exists $c > 0$ such that
\[ F(\mu) \geq c \left( \int_{\mathbb{R}} |x|^4 \, d\mu(x) - 1 \right) \quad \text{for all } \mu \in \mathcal{P}_2(\mathbb{R}), \]
and there exists $\lambda < 0$ such that $F$ is $\lambda$-convex along generalized geodesics in the sense of [1, 4.0.1]. For more details on this, we refer to [1, Section 9.3] or [4, 3.34 and 3.35].

An important observation in Lemma 2.4 is that, as an immediate consequence of Assumption 1.1, the system (1.1) admits exactly three stationary points at some measures $\mu^-, \mu^0, \mu^+ \in \mathcal{P}_2(\mathbb{R})$, which are defined in (2.13); see Lemma 2.4 for more details. We also mention here that, as we will see in Lemma 2.3, the measures $\mu^-$ and $\mu^+$ are the global minimizers of the functional $F$.

We now formulate the main result of this paper in the following theorem.

**Theorem 1.3** Suppose Assumption 1.1. Let $\mu \in \mathcal{P}_2(\mathbb{R})$. Then, there exists $\mu^* \in \{\mu^-, \mu^0, \mu^+\}$ such that
\[ \lim_{t \to \infty} W_2(S[\mu](t), \mu^*) = 0 \quad \text{and} \quad \lim_{t \to \infty} F(S[\mu](t)) = F(\mu^*). \]

**Proof.** The proof is postponed to Section 5. $\square$

As a by-product of the proof of Theorem 1.3, we obtain the following two propositions, which are interesting on their own. The first one shows that inside the valleys of the set $\{\mu \in \mathcal{P}_2(\mathbb{R}) \mid F(\mu) \leq F(\mu^0)\}$ the convergence of the gradient flows for $F$ is determined by the sign of the mean of the initial value.

**Proposition 1.4** Suppose Assumption 1.1. Let $\mu \in \mathcal{P}_2(\mathbb{R})$ be such that $\int_{\mathbb{R}} z \, d\mu(z) \neq 0$ and $F(\mu) \leq F(\mu^0)$. Then,
\[ \lim_{t \to \infty} F(S[\mu](t)) = F(\mu^-) = F(\mu^+), \]
and
\[ \lim_{t \to \infty} W_2(S[\mu](t), \mu^-) = 0 \quad \text{if } \int_{\mathbb{R}} z \, d\mu(z) < 0 \quad \text{and} \]
\[ \lim_{t \to \infty} W_2(S[\mu](t), \mu^+) = 0 \quad \text{if } \int_{\mathbb{R}} z \, d\mu(z) > 0. \]

**Proof.** The proof is postponed to Section 3. $\square$

The second by-product is the following proposition, which provides useful informations on the energy landscape determined by $F$. This is an important ingredient for the proof of Theorem 1.3.

**Proposition 1.5** Suppose Assumption 1.1. Let $\mathcal{B}^-, \mathcal{B}^0$ and $\mathcal{B}^+$ be the basins of attraction of the stationary measures $\mu^-, \mu^0$ and $\mu^+$, respectively. That is,
\[ \mathcal{B}^- = \{\mu \in \mathcal{P}_2(\mathbb{R}) \mid \lim_{t \to \infty} S[\mu](t) = \mu^-\}, \]
\[ \mathcal{B}^+ = \{\mu \in \mathcal{P}_2(\mathbb{R}) \mid \lim_{t \to \infty} S[\mu](t) = \mu^+\}, \quad \text{and} \]
\[ \mathcal{B}^0 = \{\mu \in \mathcal{P}_2(\mathbb{R}) \mid \lim_{t \to \infty} S[\mu](t) = \mu^0\}. \]

Then, $\mathcal{B}^-$ and $\mathcal{B}^+$ are open subsets of $\mathcal{P}_2(\mathbb{R})$, and $\mathcal{B}^0$ is a closed subset of $\mathcal{P}_2(\mathbb{R})$. 

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Proof. The claim follows from Proposition 4.1 and Corollary 5.1 below. □

The results of this paper are not completely new. Indeed, Theorem 1.3 and Proposition 1.4 are mild extensions of the results that have already been obtained in the paper [14]. The proofs in [14] are based on methods from the theory of partial differential equations. The main contributions of this paper are that we use the Wasserstein framework to prove these results (which provides shorter proofs than in [14]), and that the results hold in the stronger topology of the Wasserstein distance (whereas the results in [14] are formulated in terms of the weak topology). However, to our knowledge, Proposition 1.5 is a new result. It is expected that this proposition will become useful in the study of the metastable behaviour of the system (1.3) via the Wasserstein framework. The latter is left for future research.

This paper is organized as follows. First, we recall some elements of the construction of Wasserstein gradient flows in Section 2.1. Then, in Section 2.2, we compare \( F \) with the functional \( \bar{H} \), which appeared in [4]. In Section 2.3 we characterize the stationary measures, and in Section 2.4 we show a useful symmetry property of the McKean-Vlasov evolution. In Chapter 3 we first show some compactness property of the gradient flows for \( F \), and then use this property to prove Proposition 1.4. In Chapter 4 we prove the main part of Proposition 1.5. In Chapter 5 we provide the proof of Theorem 1.3. We conclude this paper with some immediate consequences of this theorem for the set \( B^0 \).

2 Preliminaries

2.1 Wasserstein gradient flows

In this section, we briefly recall some elements of the construction of Wasserstein gradient flows. For simplicity, we restrict all definitions to the functional \( F \) from (1.2). For more general functionals and for the details, we refer to [1].

Let \( P_2(\mathbb{R}) \) denote the space of all probability measures on \( \mathbb{R} \), whose second moment is finite, and let \( W_2 \) denote the Wasserstein distance on \( P_2(\mathbb{R}) \), i.e.,

\[
W_2(\mu, \nu)^2 := \inf_{\gamma \in \text{Cpl}(\mu,\nu)} \int_{\mathbb{R}^2} |y - y'|^2 \, d\gamma(y, y'),
\]

where \( \mu, \nu \in P_2(\mathbb{R}) \) and \( \text{Cpl}(\mu,\nu) \) denotes the space of all probability measures on \( \mathbb{R}^2 \) that have \( \mu \) and \( \nu \) as marginals.

We say that a curve \((\mu_t)_{t \in [0,\infty)} \subset P_2(\mathbb{R}^d)\) is absolutely continuous if there exists \( m \in L^2_{\text{loc}}((0,\infty)) \) such that

\[
W_2(\mu_s, \mu_t) \leq \int_s^t m(r) \, dr \quad \text{for all } 0 < s < t < \infty.
\]

We denote the set of all absolutely continuous curves in \((P_2(\mathbb{R}), W_2)\) by \( \text{AC}((0,\infty); P_2(\mathbb{R})) \).

It is shown in [1, 1.1.2] that for all \((\mu_t)_{t \in [0,\infty)} \in \text{AC}((0,\infty); P_2(\mathbb{R}))\), there exists \( |\mu'| \in L^2_{\text{loc}}((0,\infty)) \), called the metric derivative of \((\mu_t)_{t \in [0,\infty)}\), such that

\[
|\mu'|(t) = \lim_{s \to t} \frac{W_2(\mu_s, \mu_t)}{|s - t|} \quad \text{for almost every } t \in (0,\infty).
\]
Another important object is the metric slope (cf. [1, 1.2.4]) of $\mathcal{F}$, which is defined by

$$|\partial \mathcal{F}|(\mu) = \limsup_{\nu \to \mu} \left( \frac{\mathcal{F}(\mu) - \mathcal{F}(\nu)}{W_2(\mu, \nu)} \right)^+ \quad \text{for } \mu \in D(\mathcal{F}).$$  \hspace{1cm} (2.4)

We are now in the position to define the notion of Wasserstein gradient flows for $\mathcal{F}$. There are four different and equivalent ways to do this, which are all listed in [1, Chapter 11]. In this paper, we choose the definition as a curve of maximal slope (cf. [1, 1.3.2]).

**Definition 2.1** We say that a curve $(S[\mu](t))_{t \in (0, \infty)} \in AC((0, \infty); \mathcal{P}(\mathbb{R}))$ is a Wasserstein gradient flow for $\mathcal{F}$ with initial value $\mu \in \mathcal{P}(\mathbb{R})$ if $\lim_{t \to 0} W_2(S[\mu](t), \mu) = 0$, and if

$$0 = \mathcal{F}(S[\mu](t)) - \mathcal{F}(\mu) + \frac{1}{2} \int_0^t \left( |\partial \mathcal{F}|^2(S[\mu](r)) + |(S[\mu])'|^2(r) \right) dr \quad \text{for all } t \in (0, \infty).$$ \hspace{1cm} (2.5)

### 2.2 Macroscopic Hamiltonians

In this section we first introduce and recall some facts about the function $\bar{H} : \mathbb{R} \to \mathbb{R}$, which was the object of investigation in the paper [4]. Then, in Lemma 2.3, we show the relation between $\mathcal{F}$ and $\bar{H}$, and infer from that useful analytic facts about $\mathcal{F}$.

Let the function $\varphi^* : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi^*(\sigma) = \log \int_{\mathbb{R}} e^{\sigma z - \Psi(z)} dz \quad \text{for } \sigma \in \mathbb{R}. \hspace{1cm} (2.6)$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the Legendre transform of $\varphi^*$, i.e.,

$$\varphi(m) = \sup_{\sigma \in \mathbb{R}} (\sigma m - \varphi^*(\sigma)) \quad \text{for } m \in \mathbb{R}. \hspace{1cm} (2.7)$$

It is then well-known from standard properties of Legendre transforms that for all $m, \sigma \in \mathbb{R}$,

$$\varphi'(m)m - \varphi^*(\varphi'(m)) = \varphi(m), \quad (\varphi^*)'(\varphi'(m)) = m \quad \text{and} \quad (\varphi^*)'(\sigma) = \int_{\mathbb{R}} z d\mu^\sigma, \hspace{1cm} (2.8)$$

where, for $\sigma \in \mathbb{R}$, the probability measure $\mu^\sigma \in \mathcal{M}_1(\mathbb{R})$ is defined by

$$d\mu^\sigma(z) = e^{-\varphi^*(\sigma) + \sigma z - \Psi(z)} dz = \frac{e^{\sigma z - \Psi(z)}}{\int_{\mathbb{R}} e^{\sigma z - \Psi(z)} dz} dz. \hspace{1cm} (2.9)$$

Finally, we define the function $\bar{H} : \mathbb{R} \to \mathbb{R}$ by

$$\bar{H}(z) = \varphi(z) - \frac{1}{2} z^2 \quad \text{for } z \in \mathbb{R}. \hspace{1cm} (2.10)$$

**Remark 2.2** The function $\bar{H}$ played the role of the macroscopic Hamiltonian in [4], where the metastable behaviour of the system (1.3) was studied. The crucial difference in [4] was that the macroscopic order parameter was chosen to be the empirical mean. Recall from fact (i) and (ii) of the introduction that the functional $\mathcal{F}$ appears as the macroscopic Hamiltonian of the system (1.3) by choosing the empirical distribution as the macroscopic order parameter; see [4, Section 1.4] for more details on this.

Moreover, as it is shown in [4, Chapter 3], under Assumption 1.1, the function $\bar{H}$ admits exactly three critical points, which are located at $-m^*$, $0$ and $m^*$ for some $m^* > 0$. Furthermore, $\bar{H}''(0) < 0$, $\bar{H}''(m^*) = \bar{H}''(-m^*) > 0$, and $\bar{H}(0) > \bar{H}(m^*) = \bar{H}(-m^*)$. That is, $\bar{H}$ has a local maximum at $0$, and the two global minima of $\bar{H}$ are located at $\pm m^*$.  

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In the following let \( m[\mu] = \int_{\mathbb{R}} z d\mu(z) \) denote the mean of a probability measure \( \mu \in \mathcal{P}_2(\mathbb{R}) \). We have the following relation between the macroscopic Hamiltonians \( F \) and \( \bar{H} \).

**Lemma 2.3** Suppose Assumption 1.1. Then, for all \( m \in \mathbb{R} \), we have that

\[
F(\mu) > F(\mu^{\varphi'(m)}) \quad \text{for all } \mu \in \mathcal{P}_2(\mathbb{R}) \text{ such that } m[\mu] = m \text{ and } \mu \neq \mu^{\varphi'(m)},
\]

and,

\[
\bar{H}(m) = \min_{\mu \in \mathcal{P}_2(\mathbb{R}), m[\mu] = m} F(\mu) = F(\mu^{\varphi'(m)}). \tag{2.12}
\]

Moreover, let

\[
\mu^- := \mu^{\varphi'(-m^*)}, \quad \mu^0 := \mu^{\varphi'(0)} \quad \text{and} \quad \mu^+ := \mu^{\varphi'(m^*)}. \tag{2.13}
\]

Then, \( F \) admits exactly two global minima, one at \( \mu^- \) and one at \( \mu^+ \), and we have that \( F(\mu^-) = F(\mu^+) < F(\mu^0) \).

**Proof.** If \( F(\mu) = \infty \), then (2.11) is trivially satisfied. So we assume that \( F(\mu) < \infty \). In the following let \( \mathcal{H}(\cdot | \cdot) \) denote the relative entropy functional (see e.g., [1, 9.4.1]). Then, by using (2.8) and by denoting the Lebesgue density of \( \mu \) by \( \rho \),

\[
F(\mu) = \int_{\mathbb{R}} \log(\rho e^{\psi}) d\mu - \frac{J}{2} \frac{m^2}{m^2} = \mathcal{H}(\mu | \mu^{\varphi'(m)}) + \varphi'(m) m - \varphi^*(\varphi'(m)) - \frac{J}{2} m^2 \tag{2.14}
\]

since \( \mathcal{H}(\mu | \mu^{\varphi'(m)}) > 0 \) if \( \mu \neq \mu^{\varphi'(m)} \). This shows (2.11). The claim (2.12) follows from (2.8), (2.14) and the fact that \( \mathcal{H}(\mu^{\varphi'(m)} | \mu^{\varphi'(m)}) = 0 \). Finally, (2.12) and Remark 2.2 imply the last two claims. \( \square \)

### 2.3 Stationary points of the McKean-Vlasov evolution

In this section we characterize the *stationary points* of the McKean-Vlasov evolution\(^2\), where we say that \( \mu \in \mathcal{P}_2(\mathbb{R}) \) is stationary if

\[
S[\mu](t) = \mu \quad \text{for all } t \in (0, \infty), \tag{2.15}
\]

or equivalently,

\[
\| (S[\mu])' \| (t) = 0 \quad \text{for almost every } t \in (0, \infty). \tag{2.16}
\]

**Lemma 2.4** Suppose Assumption 1.1. Let \( \mu \in \mathcal{P}_2(\mathbb{R}) \). Then, the following statements are equivalent.

(i) \( \mu \) is stationary.

(ii) \( |\partial F(\mu)| = 0 \).

(iii) \( \mu \in \{ \mu^-, \mu^0, \mu^+ \} \).

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\(^1\)See also [11, Section IV.2] for a more general result.

\(^2\)See also [9] for similar results.
Proof. (i) ⇒ (ii). Suppose that μ is stationary. Recall from [1, 2.4.15] that $|(S[μ])'|(t) = |∂F(S[μ](t))|$ for almost every $t \in (0, \infty)$. Then, (2.16) implies part (ii).

(ii) ⇒ (iii). Using [1, 10.4.13], we have that the Lebesgue density, ρ, of μ belongs to the Sobolev space $W^{1,1}_{loc}(\mathbb{R})$. Suppose that ρ is continuous³, and let $m = m[μ]$. Then, by using again [1, 10.4.13],

$$|∂F|(μ) = \int_{\mathbb{R}} \left| \frac{∂z}{ρ(z)} + Ψ'(z) - Jm \right|^2 dμ(z) = \int_{\mathbb{R}} \left| \frac{∂z}{ρ(z)} \left( Ψ(z) - Jm(z) \right) \right|^2 dμ(z). \quad (2.17)$$

Since $|∂F|(μ) = 0$, (2.17) implies that for μ-a.e. $z \in \mathbb{R}$,

$$ρ(z) = ρ(0) e^{Ψ(0)} e^{−Ψ(z)+Jmz} = e^{−Ψ(z)+Jmz−φ^*(Jm)}, \quad (2.18)$$

where we used in the last step the definition of $φ^*$ and that μ is a probability measure. In particular, combining (2.8) and (2.18) yields that $m = (φ^*)'(Jm)$. And by using the second claim in (2.8), we infer that $H'(m) = 0$. However, in Remark 2.2 we have seen that there are only three solutions to this equation. This implies that

$$m \in \{-m^*,0,m^*\}. \quad (2.19)$$

Combining (2.18) and (2.19) yields part (iii).

(iii) ⇒ (ii). Combining the representation (2.17) with the definition of the measures $μ^−, μ^0$ and $μ^+$ yields part (ii).

(ii) ⇒ (i). From [1, 2.4.15], we have that for all $t > 0$,

$$|∂F|(S[μ](t)) ≤ e^{−Lt}|∂F|(μ) = 0. \quad (2.20)$$

Again, using that $|(S[μ])'|(t) = |∂F|(S[μ](t))$ for almost every $t \in (0, \infty)$, (2.20) yields part (i). \qed

2.4 Symmetry property

The following lemma shows that gradient flows for $F$ admit a useful symmetry property.

Lemma 2.5 Let $ς : \mathbb{R} \to \mathbb{R}$ be defined by $ς(z) = −z$, and let $μ ∈ \mathcal{P}_2(\mathbb{R})$. Then,

$$S[ς#μ](t) = ς#S[μ](t) \quad \text{for all } t \in (0, \infty). \quad (2.21)$$

Proof. First note that

$$F(ν) = F(ς#ν) \quad \text{for all } ν ∈ \mathcal{P}_2(\mathbb{R}), \quad (2.22)$$

and therefore,

$$|∂F|(ν) = |∂F|(ς#ν) \quad \text{for all } ν ∈ \mathcal{P}_2(\mathbb{R}). \quad (2.23)$$

Moreover, for all $ν ∈ \mathcal{AC}((0, \infty); \mathcal{P}_2(\mathbb{R}))$ and $0 < s < t < \infty$,

$$W_2(ν_s, ν_t) = W_2(ς#(ς#ν_s), ς#(ς#ν_t)) ≤ W_2(ς#ν_s, ς#ν_t) ≤ W_2(ν_s, ν_t). \quad (2.24)$$

³Recall that there exists a continuous representative for each element in $W^{1,1}_{loc}(\mathbb{R})$. 7
Therefore, \( W_2(\nu_s, \nu_l) = W_2(\zeta_\# \nu_s, \zeta_\# \nu_l) \), and we have that the metric derivatives coincide, i.e.,
\[
|\nu'(t)| = |(\zeta_\# \nu)'(t)| \quad \text{for almost every } t \in (0, \infty) \text{ and for all } \nu \in \mathcal{AC}((0, \infty); \mathcal{P}_2(\mathbb{R})).
\] (2.25)

Suppose first that \( \mu \in D(F) \). Then, by combining (2.5), (2.22), (2.23) and (2.25),
\[
0 = F(S[\mu](T)) - F(\mu) + \frac{1}{2} \int_0^T (|\partial F|^2(S[\mu](t)) + |(S[\mu])'|^2(t)) \, dt
\]
\[
= F(\zeta_\# S[\mu](T)) - F(\zeta_\# \mu) + \frac{1}{2} \int_0^T (|\partial F|^2(\zeta_\# S[\mu](t)) + |(\zeta_\# S[\mu])'|^2(t)) \, dt
\] (2.26)

for all \( T \in (0, \infty) \). Hence, \((\zeta_\# S[\mu](t))_{t \in [0, \infty)}\) is the unique gradient flow for \( F \) with initial value \( \zeta_\# \mu \). This shows (2.21) for all \( \mu \in D(F) \). Combined with the regularization estimate (see [1, 4.3.2]), this also yields (2.21) for all \( \mu \in \mathcal{P}_2(\mathbb{R}) \setminus D(F) \). \(\square\)

3 Convergence in the valleys

In this section we first show some compactness property of the McKean-Vlasov paths in Lemma 3.1. Then, we use this result to prove Proposition 1.4.

Lemma 3.1 Suppose Assumption 1.1. Let \( \mu \in D(F) \). Then, there exist a sequence \((t_k)_k\) and \( \mu^* \in \{\mu^-, \mu^0, \mu^+\} \) such that \( \lim_{k \to \infty} t_k = \infty \),
\[
\lim_{k \to \infty} W_2(S[\mu](t_k), \mu^*) = 0 \quad \text{and} \quad \lim_{t \to \infty} F(S[\mu](t)) = F(\mu^*). \tag{3.1}
\]

Proof. In the following let \( \mu_t = S[\mu](t) \). We prove this lemma in three steps.

**Step 1.** [There exists a subsequence \((t_n)_n\) such that \( \lim_{n \to \infty} |\partial F|'(\mu_{t_n}) = 0 \).]

Note that the sequence \((\mathcal{F}(\mu_t))_{t \in [0, \infty)}\) is a continuous, monotone and bounded sequence of real numbers by (1.5) and [1, 2.4.15 and 10.3.18]. Therefore, it converges, as \( t \to \infty \), to a number \( L^* \in \mathbb{R} \). In particular, by [1, 2.4.15 and 10.3.18],
\[
\int_0^\infty |\partial F|'(\mu_r) \, dr = - \int_0^\infty \frac{d}{dr} \mathcal{F}(\mu_r) \, dr = -L^* + \mathcal{F}(\mu) < \infty. \tag{3.2}
\]

This implies the claim of Step 1.

**Step 2.** [\( \lim_{k \to \infty} W_2(\mu_{t_{n_k}}, \mu^*) \) for some \( \mu^* \in \{\mu^-, \mu^0, \mu^+\} \) and a subsequence \((t_{n_k})_k\).]

By (1.5), the monotonicity of \( t \mapsto \mathcal{F}(\mu_t) \) and the fact that \( \mu_0 = \mu \in D(F) \), we have that
\[
\sup_{n \in \mathbb{N}} \int_{\mathbb{R}} |x|^4 \, d\mu_{t_n}(x) \leq \sup_{n \in \mathbb{N}} \left( \frac{1}{c} \mathcal{F}(\mu_{t_n}) + 1 \right) \leq \frac{1}{c} \mathcal{F}(\mu) + 1 < \infty. \tag{3.3}
\]

Using [15, 6.8 (iii)], this implies that there exist a further subsequence \((t_{n_k})_k\) and \( \mu^* \in \mathcal{P}_2(\mathbb{R}) \) such that \( \lim_{k \to \infty} W_2(\mu_{t_{n_k}}, \mu^*) \). It remains to show that \( \mu^* \in \{\mu^-, \mu^0, \mu^+\} \). In order to do this, we use the lower semi-continuity of \( |\partial F| \) ([1, 4.3.2]) and Step 1 to observe that
\[
|\partial F|(\mu^*) \leq \liminf_{k \to \infty} |\partial F|(\mu_{t_{n_k}}) = 0. \tag{3.4}
\]

Combining this with Lemma 2.4 yields the claim of Step 2.
Step 3. \( \lim_{t \to \infty} \mathcal{F}(\mu_t) = \mathcal{F}(\mu^*) \).\\First note that by the lower semi-continuity of \( \mathcal{F} \) (see [3, 3.35] or [1, Section 9.3]), we have that
\[
L^* = \lim_{t \to \infty} \mathcal{F}(\mu_t) = \lim_{k \to \infty} \mathcal{F}(\mu_{t_{nk}}) \geq \mathcal{F}(\mu^*). 
\] (3.5)
To show the other inequality, we use [1, 2.4.9], and observe that for all \( k \in \mathbb{N} \),
\[
|\partial \mathcal{F}|(\mu_{t_{nk}}) \geq \left( \frac{\mathcal{F}(\mu_{t_{nk}}) - \mathcal{F}(\mu^*)}{W_2(\mu_{t_{nk}}, \mu^*)} + \frac{\lambda}{2} W_2(\mu_{t_{nk}}, \mu^*) \right)^+,
\] (3.6)
which is equivalent to
\[
W_2(\mu_{t_{nk}}, \mu^*) |\partial \mathcal{F}|(\mu_{t_{nk}}) \geq \left( \mathcal{F}(\mu_{t_{nk}}) - \mathcal{F}(\mu^*) + \frac{\lambda}{2} W_2^2(\mu_{t_{nk}}, \mu^*) \right)^+.
\] (3.7)
Taking the limit as \( k \to \infty \) on both sides, and using Step 1 and Step 2, this implies that
\[
0 \geq (L^* - \mathcal{F}(\mu^*))^+.
\] (3.8)
We conclude that \( L^* \leq \mathcal{F}(\mu^*) \).

With this compactness result in hand, we are able to prove Proposition 1.4.

Proof of Proposition 1.4. In the following let \( \mu_t = S[\mu](t) \). It suffices to consider only the case that \( m[\mu] < 0 \). We know from Lemma 3.1 that there exists a subsequence \( (\mu_{t_k})_k \) such that
\[
\lim_{k \to \infty} W_2(\mu_{t_k}, \mu^*) = 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{F}(\mu_t) = \mathcal{F}(\mu^*) \quad \text{for some} \quad \mu^* \in \{\mu^-, \mu^0, \mu^+\}.
\] (3.9)
We first show that \( \mu^* = \mu^- \) (which implies (1.7)), and then show that \( \lim_{t \to \infty} W_2(\mu_t, \mu^-) = 0 \) (which implies (1.8)).

Step 1. \( \mu^* = \mu^- \).

We show that the cases \( \mu^* = \mu^+ \) or \( \mu^* = \mu^0 \) lead to contradictions. First suppose that \( \mu^* = \mu^+ \). Since the map \( t \mapsto m[\mu_t] \) is continuous and since \( m[\mu_0] = m[\mu] < 0 \), we have that there exists \( t' \in (0, \infty) \) such that \( m[\mu_{t'}] = 0 \). Then, by the monotonicity of \( t \mapsto \mathcal{F}(\mu_t) \) and by Lemma 2.3,
\[
\mathcal{F}(\mu^0) \geq \mathcal{F}(\mu) \geq \mathcal{F}(\mu_{t'}) \geq \mathcal{F}(\mu^0).
\] (3.10)
Hence, \( \mathcal{F}(\mu_{t'}) = \mathcal{F}(\mu) \), which implies that \( |\partial \mathcal{F}|(\mu) = 0 \), since \( |\partial \mathcal{F}|(\mu_{t'}) = -\frac{d}{dt} \mathcal{F}(\mu_{t'}) \). Moreover, (3.10) yields that \( \mathcal{F}(\mu^0) = \mathcal{F}(\mu) \). By Lemma 2.3, we infer that \( \mu = \mu^0 \). This contradicts the fact that \( m[\mu] < 0 \). The case \( \mu^* = \mu^0 \) is treated analogously.

Step 2. \( \lim_{t \to \infty} W_2(\mu_t, \mu^-) = 0 \).

Let \( (\mu_{s_n})_{n \in \mathbb{N}} \) be any subsequence of \( (\mu_t)_{t \in [0, \infty)} \). Using the same compactness argument from Step 2 of the proof of Lemma 3.1, we know that there exists a further subsequence \( (\mu_{s_{nk}})_{k \in \mathbb{N}} \) such that \( \lim_{k \to \infty} W_2(\mu_{s_{nk}}, \mu') = 0 \) for some \( \mu' \in \mathcal{P}_2(\mathbb{R}) \). In order to show the claim of Step 2, it remains to show that \( \mu' = \mu^- \). If \( m[\mu'] \geq 0 \), we infer a contradiction by repeating the same arguments from Step 1. So we have that \( m[\mu'] < 0 \). Moreover, we have that
\[
\mathcal{F}(\mu') \leq \liminf_{k \to \infty} \mathcal{F}(\mu_{s_{nk}}) = \lim_{t \to \infty} \mathcal{F}(\mu_t) = \mathcal{F}(\mu^-).
\] (3.11)
In view of Lemma 2.3, this implies that \( \mu' = \mu^- \) or \( \mu' = \mu^+ \). The latter case is not possible, since \( m[\mu'] < 0 \). This yields the claim of Step 2.

\[ \square \]
Proposition 4.1 Suppose Assumption 1.1. Recall the definition of $\mathcal{B}^-$ and $\mathcal{B}^+$ from (1.10). Then, $\mathcal{B}^-$ and $\mathcal{B}^+$ are open subsets of $\mathcal{P}_2(\mathbb{R})$.

Proof. In view of Lemma 2.5, it suffices to show the claim only for $\mathcal{B}^-$. Let $\nu \in \mathcal{B}^-$. We abbreviate $\Delta := \mathcal{F}(\mu^0) - \mathcal{F}(\mu^-)$. Let $t' > 0$ be such that for all $t \geq t'$,

- $W_2(S[\nu](t), \mu^-) \leq \frac{1}{4}m^*$,
- $\mathcal{F}(S[\nu](t)) \leq \mathcal{F}(\mu^-) + \frac{1}{4}\Delta$, and
- $e^{\lambda t} = e^{-|\lambda|t} \leq \frac{1}{2}$.

It is easy to see that the number $t'$ exists by using Lemma 3.1 and the fact that $\nu \in \mathcal{B}^-$. Set

$$\delta := \min \left\{ e^{2\lambda t'} \frac{m^*}{4}, \sqrt{e^{2\lambda t'} \frac{1}{4} \Delta} \right\}. \quad (4.1)$$

We now show that $B_\delta(\nu) = \{ \mu \in \mathcal{P}_2(\mathbb{R}) | W_2(\mu, \nu) < \delta \} \subset \mathcal{B}^-$. Let $\mu \in B_\delta(\nu)$. We have to show that

1. $m[S[\mu](2t')] < 0$, and that
2. $\mathcal{F}(S[\mu](2t')) \leq \mathcal{F}(\mu^0)$.

In order to show (i), note that by the contraction property (see [1, 11.2.1]) and the definition of $t'$ and $\delta$,

$$W_2(S[\mu](2t'), \mu^-) \leq W_2(S[\nu](2t'), \mu^-) + e^{-2\lambda t'}\delta \leq \frac{m^*}{2}. \quad (4.2)$$

This implies claim (i). To show claim (ii), we use the regularization estimate ([1, 4.3.2]4), and obtain that

$$\mathcal{F}(S[\mu](2t')) \leq \mathcal{F}(S[\nu](t')) + |\lambda|W_2(S[\nu](t'), S[\mu](t'))^2 \leq \mathcal{F}(\mu^-) + \frac{1}{2}\Delta < \mathcal{F}(\mu^0). \quad (4.3)$$

This concludes the proof of claim (ii). \halmos

5 Proof of Theorem 1.3

Proof of Theorem 1.3. Applying the semigroup property of the McKean-Vlasov path and the regularization estimate ([1, 4.3.2]), we can assume without restriction that $\mu \in D(\mathcal{F})$.

We know from Lemma 3.1 that there exists a subsequence $(\mu_{k})_k$ and $\mu^* \in \{ \mu^-, \mu^0, \mu^+ \}$ such that

$$\lim_{k \to \infty} W_2(\mu_k, \mu^*) = 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{F}(\mu_t) = \mathcal{F}(\mu^*). \quad (5.1)$$

4Recall that there is a typo in [1, (4.3.2)]: It must be $e^{\lambda t - 1}$ instead of $e^{\lambda t - 1}$. 

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Let \((\mu_{s_n})_{n \in \mathbb{N}}\) be a subsequence of \((\mu_t)_{t \in [0, \infty)}\). As in Step 2 of the proof of Lemma 3.1, we infer the existence of a further subsequence, still denoted by \((\mu_{s_n})_{n \in \mathbb{N}}\), such that

\[
\lim_{n \to \infty} W_2(\mu_{s_n}, \nu^*) = 0 \quad \text{for some } \nu^* \in \mathcal{P}_2(\mathbb{R}). \tag{5.2}
\]

It remains to show that \(\nu^* = \mu^*\). We divide the proof into the three cases \(\mu^* = \mu^-, \mu^* = \mu^0\) and \(\mu^* = \mu^+\).

**Case 1.** \([ \mu^* = \mu^- ]\)

As in (3.11), we infer that \(\mathcal{F}(\nu^*) \leq \mathcal{F}(\mu^-)\). By Lemma 2.3, this implies that either \(\nu^* = \mu^- = \mu^\ast\) or \(\nu^* = \mu^+\). It remains to show that the latter case leads to a contradiction. Note that by (5.1) and (5.2),

- there exists \(T > 0\) such that \(\mathcal{F}(\mu_t) \in [\mathcal{F}(\mu^-), \mathcal{F}(\mu^0))\) for all \(t \geq T\),
- there exists \(N \in \mathbb{N}\) such that \(s_n \geq T\) and \(m[\mu_{s_n}] > 0\) for all \(n \geq N\), and
- there exists \(K \in \mathbb{N}\) such that \(t_k > s_N\) and \(m[\mu_{t_k}] < 0\) for all \(k \geq K\).

In particular, we have that

\[
\mathcal{F}(\mu_t) < \mathcal{F}(\mu^0) \quad \text{for all } t \in [s_N, t_K], \quad m[\mu_{s_N}] > 0, \quad \text{and} \quad m[\mu_{t_K}] < 0. \tag{5.3}
\]

Hence, there exists \(t' \in [s_N, t_K]\) such that \(\mathcal{F}(\mu_{t'}) < \mathcal{F}(\mu^0)\) and \(m[\mu_{t'}] = 0\). This contradicts Lemma 2.3.

**Case 2.** \([ \mu^* = \mu^+ ]\)

This case is treated in the same way as Case 1.

**Case 3.** \([ \mu^* = \mu^0 ]\)

In this case we have that \(\mathcal{F}(\nu^*) \leq \mathcal{F}(\mu^0)\). There are three subcases given by \(m[\nu^*] = 0\), \(m[\nu^*] > 0\) and \(m[\nu^*] < 0\).

**Case 3.1.** \([ m[\nu^*] = 0 ]\)

By Lemma 2.3, the combination of \(\mathcal{F}(\nu^*) \leq \mathcal{F}(\mu^0)\) and \(m[\nu^*] = 0\) yields that \(\nu^* = \mu^0 = \mu^*\).

**Case 3.2.** \([ m[\nu^*] < 0 ]\)

From Proposition 1.4 we know that \(\nu^* \in \mathcal{B}^\ast\). Hence, by Proposition 4.1, there exists \(\delta > 0\) such that \(B_\delta(\nu^*) \subset \mathcal{B}^-\). In particular, by (5.2), there exists \(N \in \mathbb{N}\) such that \(\mu_{s_N} \in \mathcal{B}^-\). This contradicts (5.1). Indeed, the fact that \(\mu_{s_N} \in \mathcal{B}^-\) implies that

\[
\lim_{t \to \infty} \mu_{s_N+t} = \lim_{t \to \infty} S[\mu_{s_N}](t) = \mu^- \quad \text{in } \mathcal{P}_2(\mathbb{R}), \tag{5.4}
\]

which contradicts the fact that \(\lim_{k \to \infty} \mu_{t_k} = \mu^* = \mu^0\) in \(\mathcal{P}_2(\mathbb{R})\).

**Case 3.3.** \([ m[\nu^*] > 0 ]\)

This case is treated in the same way as Case 3.2.

We conclude this paper with some immediate consequences of Theorem 1.3 for the set \(\mathcal{B}^0\).

**Corollary 5.1**

(i) \(\mathcal{B}^0\) is closed,

(ii) \(\mathcal{B}^0 \cap \{ \mu \in \mathcal{P}_2(\mathbb{R}) \mid \mu \text{ is symmetric, i.e. } \zeta_\# \mu = \mu \} \), and

(iii) \(\mu^0 \in \partial \mathcal{B}^0\).

**Proof.** To show part (i), we simply use Proposition 4.1 and that, by Theorem 1.3, \(\mathcal{P}_2(\mathbb{R}) = \mathcal{B}^- \cup \mathcal{B}^0 \cup \mathcal{B}^+\). Part (ii) is a straightforward consequence of Theorem 1.3 and Lemma 2.5. Finally, to show part (iii), we use that by Proposition 1.4, \(\mu^{\zeta(\eta)}(\mu^0) \in \mathcal{B}^-\) and \(\mu^{\zeta(\eta)}(\mu^0) \in \mathcal{B}^+\) for all \(\eta > 0\), and that \(\lim_{\eta \downarrow 0} W_2(\mu^{\zeta(\eta)}(\mu^0), \mu^0) = \lim_{\eta \downarrow 0} W_2(\mu^{\zeta(\eta)}(\mu^0), \mu^0) = 0\).
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