Emergence of the laws of thermodynamics for autonomous, arbitrary quantum systems

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Originally formulated for macroscopic machines, the laws of thermodynamics were recently shown to hold for quantum systems coupled to ideal sources of work (external classical fields) and heat (systems at equilibrium). Ongoing efforts have been focusing on extending the validity of thermodynamic laws to more realistic, non-ideal energy sources. Here, we go beyond these extensions and show that energy exchanges between arbitrary quantum systems are structured by the laws of thermodynamics. We first generalize the second law and identify the associated work and heat exchanges. After recovering known results from ideal work and heat sources, we analyze some consequences of hybrid work and heat sources. We illustrate our general laws with microscopic machines realizing thermodynamic tasks in which the roles of heat and work sources are simultaneously played by elementary quantum systems. Our results open perspectives to understand and optimize the energetic performances of realistic quantum devices, at any scale.

The laws of thermodynamics have been formalized centuries ago to predict the performances of macroscopic machines, composed of large systems exchanging work and heat. While the first law of thermodynamics defines the splitting of the energy exchanges between heat and work, the second law states their different nature by expressing the fundamental constraints they obey due to the arrow of time. In the last two decades, the application domain of these concepts has been dramatically expanded, as similar laws were found to rule the average energy flows received by a single nanoscopic classical $^{11}$–$^{14}$, or quantum system $^{15}$–$^{18}$, far from equilibrium and from the so-called thermodynamic limit. Importantly, the second law was shown to emerge universally from the Schrödinger equation ruling the dynamics of a quantum system of interest coupled on the one hand to a heat bath – a quantum system initially at thermal equilibrium and identified with a pure source of heat – and on the other hand to a pure source of work $^8$ – modeled as a time-dependence in the system’s Hamiltonian. Recent advances in the analysis of quantum effects in heat engines has however motivated to look further into full quantum description of sources of heat and work $^{12}$–$^{17}$. Indeed, time-dependent Hamiltonians can be considered as effective semi-classical models. One could expect that work exchanges should emerge from a fully autonomous scenario, where the system, the sources of work, and the sources of heat are described by a global time-independent Hamiltonian. In such scenarios, the division between heat and work becomes fuzzy, rising doubt whether a consistent thermodynamic framework can be built at this level. Some noticeable progresses in that direction include the introduction of the notion of ergotropy $^{18}$–$^{19}$ as work contribution in fully autonomous systems $^{20}$–$^{25}$ and extensions of the traditional frameworks to non-thermal baths $^{26}$–$^{28}$. In parallel to these developments, it has been noticed that energy exchanges with a large quantum system initially at thermal equilibrium, which would traditionally be regarded as heat source, can also contain coherent, deterministic contributions, exhibiting the properties of work $^{25}$–$^{29}$. As a matter of fact, in several implementations of quantum heat engines, the very same physical devices can often be used to provide either work or heat (e.g. a microwave transmission line can both induce thermalization or convey a driving field performing work on a superconducting qubit $^{31}$), reinforcing the need to describe hybrid quantum energy sources providing both work-like and heat-like energy. All the above efforts contributed to build fundamental definitions of work and heat in the quantum regime which do not presuppose the role of work or heat sources for each device, but relate directly to the properties of the exchanged energy $^{32}$–$^{34}$.

Here, building on the above studies, we introduce an expression of the second law of thermodynamics valid for arbitrary set of interacting quantum systems, which leads us to identify consistent definitions for heat and work. We show that the emerging notion of work corresponds to a generalization of the concept of ergotropy, the latter being in general insufficient to fully quantify energetic resources that can be consumed to decrease entropy, as we show on a counterexample. On the other hand, the notion of heat which naturally emerges is the variation of complete passivity $^{15}$–$^{18}$, which, interestingly, sets an intrinsic notion of temperature as developed in $^{15}$–$^{16}$.

As an illustrative application of the suggested framework, we show that sets of quantum systems as small as two interacting qubits can behave as autonomous thermal machines, where work is consumed to cool down one of the qubit (refrigerator), or conversely, where work is produced out of initial temperature gradient between the two qubits (engine). We show that the efficiencies of both operations are limited by the same upper bounds as classical macroscopic thermal machines, namely the Carnot bounds. Before stating our results, we recall one...
Second law for a quantum system coupled to ideal heat and work sources—We consider a quantum system $A$ that is externally driven and interacts with a heat bath, modeled by a global Hamiltonian of the form $H(t) = H_A(t) + V(t) + H_B$, where $V(t)$ contains the system-bath interaction terms. The action of the drive is captured semi-classically by the time-variation of $H(t)$. We assume that at time $t = 0$, the system $A$ and the bath $B$ are in a factorized state $\rho_S(0) \otimes \rho_B[\beta]$, where $\rho_B[\beta] = e^{-\beta H_B}/Z_B$ denotes the thermal equilibrium state at inverse temperature $\beta$, $Z_B$ being the partition function. At times $t > 0$, the joint dynamics of the two systems generates a correlated state of both systems $\rho_{AB}(t)$, exhibiting in general a non-zero mutual information, which can be associated with a production of entropy. More precisely, the following equality has been derived $[3]$:

$$
\sigma_0(t) = \Delta S_A(t) + \beta_2 \Delta E_B(t) = I_{SB}(t) + D(\rho_B(t)||\rho_B^\beta(t)).
$$

(1)

Here, $\Delta X(t) = X(t) - X(0)$ for any quantity $X$. Moreover, $S_A = S[\rho_A(t)]$ denotes the Von Neumann entropy of system $A$, with $S[\rho] = -k_B \text{Tr} (\rho \log \rho)$, $k_B$ being Boltzmann’s constant, and $D(\rho_1||\rho_2) = k_B \text{Tr} (\rho_1 (\log \rho_1 - \log \rho_2))$ denotes the relative entropy of states $\rho_1$ and $\rho_2$. We have also introduced the partial states of the system $\rho_A = \text{Tr}_B(\rho_{AB}(t))$ and the bath $\rho_B = \text{Tr}_A(\rho_{AB}(t))$. In addition $I_{AB}(t) = D(\rho_{AB}(t)||\rho_A(t) \otimes \rho_B(t))$ is the mutual information of $A$ and $B$ that built up during their joint evolution ($I_{AB}(0) = 0$). As the mutual information and the relative entropy are positive quantities, so is the right-hand side of Eq. (1). As a consequence, $\sigma_0$ is often interpreted as the second law of thermodynamics for quantum systems $[3, 37]$ and the energy change of the reservoir is usually categorized as heat (interpreting $-\beta \Delta E_B$ as the entropy exchanged with the reservoir). Introducing the work performed by the external drive as the change of energy of the total system $A + B$, namely $W_{dr}(t) = \int_0^t dt \text{Tr}\{\dot{H}(t')\rho_{AB}(t')\}$, and defining the nonequilibrium free energy of system $A$, namely $F_A(t) = E_A(t) - S_A(t)/k_B \beta$, one can also rewrite Eq. (1) as:

$$
W_{dr}(t) \geq \Delta F_A + \Delta E_{int},
$$

(2)

with $E_{int}(t) = \langle V(t) \rangle$ the energy stored in the coupling. We recover in this way a usual statement of the second law for an isothermal transformation leading to a change of free energy $\Delta F_A$, which takes into account the role of the interaction energy when the latter cannot be neglected (as it is often the case for nanoscale systems). Finally, we stress that in this paradigm, the irreversibility of an evolution, as quantified by the entropy production, is ultimately related to lack or loss of information occurring when tracing out the bath $B$.

Second law for two arbitrary quantum systems—We now present our framework allowing to extend the second law to two arbitrary quantum systems $A$ and $B$ in arbitrary local states (arbitrary uncorrelated state $\rho_A(0) \otimes \rho_B(0)$), which can each be sources of both work and heat. The difference with the setup above is therefore that we now allow the state of $B$ to be not thermal initially and finally. For the sake of completeness and pedagogy, we still allow for the Hamiltonians of the systems to be time-dependant, but our final goal is to consider time-independent Hamiltonians, and show the autonomous emergence of the notion of work. Moreover, we wish to separate the energy exchanges between the systems into heat-like contributions that will be related to entropy and work-like part that can be considered as a resource for instance to decrease entropy, consistently with the expression of the second law. This splitting can be introduced by comparing the state $\rho_B(t)$ of system $B$ at any time $t$ to the thermal state $\rho_B[\beta_B(t)] = e^{-\beta_B(t) H_B}/Z_B(t)$ which is chosen to have

FIG. 1. a: Recent formulations of the laws of thermodynamics apply to a system $A$ (which can be microscopic and/or quantum), coupled to ideal (macroscopic) sources of heat $Q$ and of work $W$. b: We demonstrate the laws of thermodynamics for two or more quantum systems, that can be of any scale and initialized in any arbitrary uncorrelated state. Each system can in general behave as both a source of work and heat to the other system. Note that the heat (respectively work) provided by system $A$ does not necessarily equals the heat (work) received by system $B$ as some energy $E_{int}$ can be stored in the coupling between the systems and as one kind of energy can be consumed to the profit of the other one.
the same entropy as $\rho_B(t)$. Here $Z_B(t) = \text{Tr}\{e^{-\beta_B(t)H_B}\}$ is the associated partition function. Namely, the inverse temperature $\beta_B(t)$ is defined via the equation:

$$S[w_B[\beta_B(t)]] = S[\rho_B(t)],$$

where the left-hand term is a function of $\beta_B(t)$ only $S[w_B[\beta_B(t)]] = \beta_B(t)\text{Tr}\{H_B w_B[\beta_B(t)]\} + \log Z_B(t)$. As $S[w_B[\beta_B(t)]]$ is a monotonously decreasing function of $\beta_B(t)$ spanning the interval $[0, k_B \log d_B]$ of all the possible values taken by $S[\rho_B(t)]$ ($d_B \in [2; +\infty)$ is the dimension of the Hilbert space of $B$), this equation always admits a unique solution, defining uniquely the thermal state $w_B[\beta_B(t)]$. Note that this appealing intrinsic notion of temperature was extensively analyzed in [15, 36].

We now introduce the quantity

$$E_B^\text{th}(t) = \text{Tr}\{H_B w_B[\beta_B(t)]\},$$

that we refer to in the following as the thermal energy of $B$, so as to state our central result:

$$\sigma_A \equiv \Delta S_A(t) - \beta_B(0)Q_B(t) = I_{AB}(t) + D(w_B[\beta_B(t)] \v| w_B[\beta_B(0)]) \geq 0,$$  \hspace{1cm} (5)

with

$$Q_B(t) = -\Delta E_B^\text{th}(t).$$

Proof — Using that the unitary evolution of $A$ and $B$ preserves the total Von Neumann entropy, we can write:

$$\Delta S_A = I_{AB}(t) - \Delta S_B.$$ \hspace{1cm} (7)

Adding on both sides $\beta_B(0)\Delta E_B^\text{th}$ and using that

$$\beta_B(0)\Delta E_B^\text{th} - \Delta S_B$$

$$= -\text{Tr}\{[w_B[\beta_B(t)] - w_B[\beta_B(0)] \log w_B[\beta_B(0)]]\}$$

$$= -S[w_B[\beta_B(t)] + S[w_B[\beta_B(0)]]$$

$$= D(w_B[\beta_B(t)] \v| w_B[\beta_B(0)]),$$ \hspace{1cm} (8)

where to go to the second line we have used that by definition $S[w_B[\beta_B(t)]] = S_B(t)$, we finally obtain Eq. (5) \hspace{1cm} ■

Discussion — We interpret Eq. (5) as the second law of thermodynamics for a transformation of system $A$ caused by the interaction with system $B$. By comparing with Eq. (1), we identify $Q_B(t)$ as the heat provided by system $B$ during the transformation of $A$ (with positive sign when the heat exits system $B$). In addition, we observe the emergence of an effective inverse temperature $\beta_B(0)$ associated with the initial entropy of system $B$, which sets constraints on this heat flow via Eq (5). We stress that system $B$ itself is not in thermal equilibrium (and a consequence will be that it can also provide work), but the definition of heat compatible with Eq. (5) is the energy difference between the two “thermal backgrounds” $w_B[\beta_B(t)]$ and $w_B[\beta_B(0)]$ associated with $\rho_B(t)$ and $\rho_B(0)$. Thus, $Q_B(t)$ corresponds to an energy exchange intrinsically linked to entropy change, verifying notably

$$\dot{S}_B(t) = -\beta_B(t)\dot{Q}_B(t).$$ \hspace{1cm} (9)

Simultaneously, system $B$ also behaves as a source of work for $A$. The quantity,

$$W_B = -\Delta E_B - Q_B,$$ \hspace{1cm} (10)

plays the role of the work performed by system $B$. This can be illustrated by introducing the nonequilibrium free energy of system $A$ at temperature $\beta_B(0)$, namely $F_A[\beta_B(0)] = E_A(t) - S_A/k_B \beta_B(0)$, to obtain from Eq. (5):

$$W_{\text{dr}}(t) + W_B \geq \Delta F_A[\beta_B(0)] + \Delta E_{\text{int}}(t).$$ \hspace{1cm} (11)

We see that $W_B$ and the external driving work $W_{\text{dr}}(t)$ appear on an equal footing in Eq. (11) as resources that can be consumed to vary the system’s free energy. Moreover, in the absence of external driving (i.e. a fully autonomous machine described by a time-independent Hamiltonian), one can identify the work $W_B$ spent by system $B$ without resorting to a semi-classical description. In what follows, we will always assume a time-independent total Hamiltonian such that $W_{\text{dr}} = 0$. We also stress that during a transformation where $\Delta S_B = 0$, $W_B$ corresponds to the whole energy variation of system $B$, as expected. This justifies the intuition that work is the iso-entropic part of the energy exchange.

It is interesting to note that the definition of work $W_B$ which emerges from our formalism is connected to the concept of ergotropy, often considered as the adequate notion of work in autonomous scenarios [18][21][23][28]. The ergotropy $\mathcal{E}_B$ of system $B$ with Hamiltonian $H_B$ is defined as the maximum amount of energy that can be extracted via a unitary transformation on $B$. There is an extended notion of ergotropy, defined as $\mathcal{E}^\infty_B = E_B - E^\text{th}_B$, which was introduced as the maximal amount of energy which can be extracted from an infinite number of copies of the system via global unitary operations on all the copies [35] (see [35] for alternative extraction protocol). The notion of work emerging from the second law Eq. (5) is precisely the variation of $\mathcal{E}^\infty_B$. This quantity $\mathcal{E}^\infty_B$ also bears another important physical interpretation: it is the minimum amount of work one needs to pay to prepare the nonequilibrium state of $B$ from thermal equilibrium, with the help of a thermal bath (see Supplementary Material as well as [15]). The inequality $\mathcal{E}^\infty_B \geq \mathcal{E}_B$ implies that there exist other resources than the ergotropy which can be consumed to obtain effects similar to a work expense (that is, e.g. inducing a heat flow from a cold to a hot system, or reducing a system’s entropy). Furthermore,
using the variation of ergotropy instead of the variation of $\mathcal{E}^\infty_B$ would lead to breaking-down the positivity of the entropy production in some cases. In the Supplementary Material, we show a concrete example based of two qubits, demonstrating the need to identify the variation of $\mathcal{E}^\infty_B$ as work rather than the variation of $\mathcal{E}_B$. A necessary condition to obtain such a positivity violation is to choose an initial state verifying $\mathcal{E}^\infty_B \neq \mathcal{E}_B$, which is possible only in dimension greater than 2.

Finally, we emphasize that the quantities $Q_B$ and $W_B$ only depend on the initial and final states of system $B$, not on its whole trajectory. Nevertheless, these quantities depend on the trajectory of $A$. Indeed, in an autonomous scenario where the total Hamiltonian is time-independent, the trajectory of $A$ can be varied only by selecting different parameters for $B$ (varying $H_B$, $V$ or $\rho_B(0)$), which in general will affect the final state of $B$ and therefore the value of $Q_B$ and $W_B$. We therefore retrieve the expected properties from standard non-autonomous machines, namely that the heat and work provided by the sources depend on the trajectory of $A$.

An infamous consequence is that less work can be extracted in a stroke if the transformation is performed faster. This point is illustrated within our formalism in Supplementary Material.

**General consequences** – First, we consider the case where system $B$ is initially in a thermal state $w_B[\beta_B(0)]$ (that is when Eq. (1) applies). In this case, $\sigma_A$ still differs from $\sigma_0$ as our approach takes into account the possibility that the final state of $B$ is out-of-equilibrium. As $E_B(t) - E_B^\text{th}(t) = -W_B \geq 0$, we have $\sigma_A \leq \sigma_0$, meaning that Eqs. (5) and (11) correspond to tighter constraints than Eqs. (1)-(2) on the entropy variation of $A$ and on the work cost required to perform a given transformation (associated with a given free energy variation).

A second important consequence of our formulation is that system $A$ and $B$ can be treated on an equal footing. We define the entropy production $\sigma_B$ from the point of view of system $B$ by swapping the roles of $A$ and $B$ in Eq. (6). In general $\sigma_A \neq \sigma_B$, reflecting the fact that tracing over one or the other system does not lead to the same information loss. We can yet obtain an expression emphasizing the symmetric roles of the two systems by rewriting in Eq. (6) the entropy variation of system $A$ as $\Delta S_A/k_B = -\beta_A(0)Q_A(t) - D(w_A[\beta_A(t)]||w_A[\beta_A(0)])$, leading to (see Supplementary material):

$$-\beta_B(0)Q_B(t) - \beta_A(0)Q_A(t) \geq 0. \quad (12)$$

Eq. (12) can be interpreted as a universally valid expression of Clausius’s law, in which system $i = A, B$ plays the role of a heat reservoir at inverse temperature $\beta_i(0)$ and providing heat $Q_i = -\Delta E_i^\text{th}(t)$. Meanwhile, each system also receives an amount of work $W_i(t) = -\Delta E_i(t) + \Delta E_i^\text{th}(t)$. Eq. (12) bear similarities with the results presented in [28], obtained by focusing on the dynamics of the system only, and for the special case where $B$ is a bath initially in a state unitarily related to a thermal state. By including explicitly the system playing the role of heat and work source, we were able to reach a more general law applicable to any scale.

We now turn to more specific scenarios. If we first consider that we put in contact two systems which have initially the same effective temperature, i.e. $\beta_A(0) = \beta_B(0) = \beta$. Then, by energy conservation $\Delta E_A + \Delta E_B + \Delta E_{\text{int}} = 0$, Eq. (12) implies:

$$W_A + W_B + \Delta E_{\text{int}}(t) \leq 0, \quad (13)$$

namely, the total amount of non-thermal energy in the systems, which as seen earlier can be treated as a reserve of work, tends to decrease (up to the energy $-\Delta E_{\text{int}}$ which is paid when decoupling the two systems), preventing us, as expected, from building a heat engine working with a single temperature. On the other hand, if $\beta_A \neq \beta_B$, one can examine the conditions to build an autonomous, microscopic engine. Let us focus on the case of a refrigerator aiming at cooling down system $A$, i.e. $\beta_A > \beta_B$ and $Q_A \geq 0$. Eq. (12) implies:

$$0 \leq (\beta_A(0) - \beta_B(0))Q_A \leq \beta_B(0) (W_A + W_B - \Delta E_{\text{int}}(t)). \quad (14)$$

In the line of macroscopic thermodynamics, we see that such process is only possible if some resource is consumed: either system $A$ or $B$ provides work by reducing their energy of non-thermal nature, either energy is provided by a decrease of the coupling energy, which in turn will require an amount of work $-\Delta E_{\text{int}}(t) \geq 0$ to uncouple the systems, and for instance operate cyclically. The efficiency of conversion of those resources into cooling power $Q_A$ is upper bounded by the Carnot coefficient of performance $\eta_{\text{Carnot}} = \beta_B(0)/(\beta_A(0) - \beta_B(0))$. The striking novelty is that the macroscopic principles ruling the performances of a refrigerator are here extended for two arbitrary quantum systems, in the absence of macroscopic thermal bath and of external work source. This is exemplified below on the case of two qubits, where it is shown that the role of the hot bath (the entropy sink) and the work source can be both played by one of the two qubits so as to cool down the other. Conversely, one can also consider elementary engines, where work is generated within $A$ or $B$ out of a temperature gradient between the thermal backgrounds of $A$ and $B$. The efficiency of the engine is upper bounded by the Carnot efficiency $(\beta_A(0) - \beta_B(0))/\beta_A(0) \leq 1$ for $\beta_A(0) \geq \beta_B(0)$ (see plots in Supplementary Material).

**Example: a refrigerator composed of two qubits** – We consider two qubits of Hamiltonians $H_j = (\hbar / 2) \sigma_j^z$, with $j = A, B$ and $\sigma_j^{x,y,z}$ are the Pauli matrices in the Hilbert space of $j$. Initially, the qubit $A$ is in a thermal state $\rho_A(0) = w_A[\beta_A(0)]$ at smaller temperature than $B$, i.e. $\beta_A(0) \geq \beta_B(0)$. The goal of the refrigerator is
to extract heat from qubit $A$ and reject it into qubit $B$. While in a traditional refrigerator this would be obtained by spending work from an external drive, it can be seen from Eq. (14) that one can also consume non-thermal energy initially present in qubit $A$. The latter then plays the role of both the hot bath and the work source needed to build a generic heat engine.

To be specific, we assume that the initial state of qubit $B$ is $\rho_B(0) = e^{-i\pi \sigma_z^B/2} w_B(0) e^{i\pi \sigma_z^B/2}$, i.e. a population-inverted state without coherences in the energy eigenbasis, and that the two qubits are coupled via Hamiltonian $V = \hbar g \sigma_z^A \otimes \sigma_z^B/2$. From the solution of the Schrödinger equation for the two qubits, we computed analytically the heat flow $Q_A(t)$ provided by the qubit $A$, the work $W_B(t)$ provided by the qubit $B$, and the variation of the internal energy. Their expressions are given in the Supplementary Material and are plotted in Fig. S1, showing that refrigeration indeed occurs if the systems interact for a time $t \simeq 2\pi/\sqrt{g^2 + (\omega_B - \omega_A)^2}$. We also plot on panel b the coefficient of performance of the refrigerator $\eta(t)$ together with Carnot’s bound. The relative low performance of the refrigerator observed in Fig. S1 is due to the generation of correlations between $A$ and $B$ as well as the increase of the thermal distances $D(w_B[\beta_B(t)][w_B[\beta_B(0)]]$ and $D(w_A[\beta_A(t)][w_A[\beta_A(0)]]$, that is a consequence of the extreme smallness of the heat sources.

Retrieving ideal sources of work and heat—Our formalism defines work and heat exchanges between two arbitrary systems, and allows us to derive consistent expressions of the laws of thermodynamics. We finally show which limits leads to retrieve the usual cases where one of the systems is an ideal heat or work source. We first assume that system $B$ is at any time in a thermal state, namely $\rho_B(t) = w_B[\beta_B(t)]$ (with a possibly time-dependent temperature). Then, by our definitions $W_B = 0$, meaning that $B$ is a source of heat only. Note that this condition implies $E_B(t) = -\beta(t) \dot{S}(t)$, which is consistent with a definition of ideal heat source emerging in the context of repeated interaction with units [41]. In this case, the heat can be computed from the energy variation of $B$, namely $Q_B = -\Delta E_B$ as proposed in [8]. Conversely, a perfect work source is obtained when assuming that system $B$ does not become correlated with system $A$, i.e. $I_{AB}(t) = 0$ at any time $t$ or equivalently $\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t)$. It is then easy to show that system $B$ follows a unitary evolution, along which neither its entropy nor its effective temperature $\beta_B(t)$ vary, leading to $Q_B(t) = 0$ and $W_B(t) = -\Delta E_B(t)$ (see Supplementary material). Moreover, system $A$ evolves under the action of effective Hamiltonian $H_A^{\text{eff}}(t) = H_A + \text{Tr}\{\rho_B(t)V\}$, and the work can be computed according to the broadly used formula $W_B = \text{Tr}\left\{\rho_A(t) \frac{\partial}{\partial t} H_A^{\text{eff}}(t)\right\}$. The internal dynamics of $B$ is then responsible for the emergence of a time-dependent Hamiltonian for $A$, often considered to model a work source. One motivation of our formalism is precisely to interpolate between these two extremes and consider deviations from ideality that naturally emerge in realistic setups.

Generalizations—Finally, we mention that our approach can be used to probe more general scenarios. Our main results Eq. (5) and (12) can be straightforwardly extended to more than two systems put in contact at a
time \( t = 0 \) (see Supplementary material), leading to:

\[
\Delta S_j - \sum_{i \neq j} \beta_j(0)Q_j(t) \geq 0 \tag{15}
\]

and

\[
- \sum_j \beta_j(0)Q_j(t) \geq 0, \tag{16}
\]

where \( Q_i(t) = -\Delta E_i(t) \) and \( \beta(t) \) are defined analogously as before for each system \( i \). Eqs. (15)-(16) can be used to investigate realistic setups of quantum heat engines, where some systems play the role of non-ideal hybrid work and heat sources, while others play the role of working media.

We also mention that the violation of Eqs. (15)-(16) means (i) that the systems \( S_i \) are not evolving unitarily, which can be useful for instance to detect the presence of noise in the setup [24]; or (ii) the systems are initially correlated. Building up on this last possibility, it is a known fact that the consumption of initial correlations is a resource to extract work or invert the sign of natural heat flows [42] [43] (e.g. it captures the case of Maxwell demons). We can therefore refine the inequalities to include such situations, obtaining

\[
- \sum_j \beta_j(0)Q_j(t) \geq \Delta I_{\text{tot}}(t), \tag{17}
\]

where \( I_{\text{tot}}(t) = D(\rho_{\text{tot}}(t)\| \bigotimes_{j=1}^N \rho_j(t)) \) quantifies correlations between the \( N \) systems. Another approach to take into account these initial correlations was proposed in [24] [25].

Finally, it can be noticed that as the effective temperature \( \beta(t) \) of the systems generally depend on time, their thermal part can be seen as finite-size heat baths. In particular, from Eqs. (15)-(16) one can derive tighter inequalities using the instantaneous temperatures to quantify the entropy flows:

\[
\Delta S_j - \sum_{i \neq j} \int_0^t dt' \beta_i(t')\dot{Q}_i(t') \geq 0,
\]

and

\[
- \int_0^t dt' \sum_j \beta_j(t')\dot{Q}_j(t') \geq 0, \tag{18}
\]

where \( \dot{Q}_j(t) = -\dot{E}_j(t) \) is the heat current associated to \( Q_j(t) \). These inequalities resemble the one derived in [12] for finite-size thermal baths, but are valid for arbitrary systems, initially in arbitrary uncorrelated states, and interacting at arbitrary strength.

**Conclusion** — We have demonstrated a version of the second law of thermodynamics valid for several arbitrary quantum systems, initially in arbitrary uncorrelated states, and evolving autonomously under their joint unitary evolution. The initial entropy of each system leads to the emergence of an effective temperature and an amount of thermal energy, the variation of which plays the role of heat. The energy beyond the thermal energy can be instead used as a resource to decrease entropy, and its variation is therefore assimilated to work. Moreover, this quantity is reminiscent of the notion of ergotropy, already considered as a promising candidate to extend the notion of work to the quantum domain. We also showed that these notions of heat and work become equivalent to earlier well-accepted expressions in the case of systems behaving as ideal work and heat sources. We illustrated the reach of our results by demonstrating the possibility to design refrigerators and heat engines based on microscopic quantum systems each able to combine simultaneously the role of work and heat sources. With our universally valid notions of work and heat, these microscopic engines then follow laws similar to macroscopic thermodynamics, with efficiencies complying with Carnot’s bounds. The consequences of our results are multifaceted and yet to be explored. On one hand, they ground recent attempts to refine the splitting between work and heat in a fully quantum description. On the other hand, they allow one to understand the performances of realistic quantum devices, in which no such things as perfect heat sources or work sources exist, and determine which microscopic properties lead to devices which are close to, or reciprocally deviate from, these ideal models. Finally, they open an avenue towards the design of extremely compact heat engines where the role of heat and work sources can be played by elementary systems.

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[1] C. Jarzynski, Nonequilibrium Equality for Free Energy Differences, Phys. Rev. Lett. 78, 2690 (1997)
[2] G. E. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences, Phys. Rev. E 60, 2721 (1999)
[3] C. Jarzynski, Equalities and Inequalities: Irreversibility and the Second Law of Thermodynamics at the Nanoscale, Annu. Rev. Condens. Matter Phys. 2, 329 (2011)
[4] U. Seifert, Stochastic thermodynamics, fluctuation theorems and molecular machines, [Rep. Prog. Phys. 75, 126001 (2012)]
[5] H. Spohn, Entropy production for quantum dynamical semigroups, J. Math. Phys. 19, 1227 (1978)
[6] R. Alicki, The quantum open system as a model of the heat engine, J. Phys. A: Math. Gen. 12, L103 (1979)
F. Binder, L. A. Correa, C. Gogolin, J. Anders, and H.-H. Hasegawa, J. Ishikawa, K. Takara, and D. J. Driebe, J. Kurchan, A Quantum Fluctuation Theorem, B. Fernando, H. Michael, N. Nelly, O. Jonathan, and P. Strasberg and A. Winter, First and Second Law

S. L. Jacob, M. Esposito, J. M. R. Parrondo, and M. N. Bera, A. Riera, M. Lewenstein, Z. B. Khanian, and M. P. Muller, Correlating Thermal Machines and the Second Law at the Nanoscale, Phys. Rev. X 10, 021064 (2018).

D. Gelbwaser-Klimovsky and G. Kurizki, Heat-machine control by quantum-state preparation: From quantum engines to refrigerators, Phys. Rev. E 90, 022102 (2014).

D. Gelbwaser-Klimovsky and G. Kurizki, Catalysis of heat-to-work conversion in quantum machines, Proc. Natl. Acad. Sci. U.S.A. 114, 12156 (2017).

S. Seah, S. Nimmrichter, and V. Scarani, Work production of quantum rotor engines, New J. Phys. 20, 043045 (2018).

R. Uzdin and S. Rahav, Global Passivity in Microscopic Thermodynamics, Phys. Rev. X 8, 021064 (2018).

R. Uzdin, The Second Law and Beyond in Microscopic Quantum Setups, in Thermodynamics in the Quantum Regime (Springer, Cham, Switzerland, 2019) pp. 681–712.

G. Manzano, F. Galve, R. Zambrini, and J. M. R. Parrondo, Entropy production and thermodynamic power of the squeezed thermal reservoir, Phys. Rev. E 93, 052120 (2016).

J. P. Santos, G. T. Landi, and M. Paternosrro, Wigner Entropy Production Rate, Phys. Rev. Lett. 118, 220601 (2017).

W. Niedenzu, V. Mukherjee, A. Ghosh, A. G. Kofman, and G. Kurizki, Quantum engine efficiency bound beyond the second law of thermodynamics, Nat. Commun. 9, 1 (2018).

J. Monsel, M. Fellous-Asiani, B. Huard, and A. Auffeves, The Energetic Cost of Work Extraction, Phys. Rev. Lett. 124, 130601 (2020).

E. Bernhardt, C. Eloard, and K. Le Hur A topologically protected quantum dynamo effect in a driven spin-boson model, arXiv:2208.01707 (2022).

C. Nathanaël, J. Sébastien, B. Landry, C.-I. Philippe, F. Quentin, A. Janet, A. Alexia, A. Rémi, R. Pierre, and H. Benjamin, Observing a quantum Maxwell demon at work, Proc. Natl. Acad. Sci. U.S.A. 114, 7561 (2017).

H. Weimer, M. J. Henrich, F. Rempp, H. Schröder, and G. Mahler, Local effective dynamics of quantum systems: A generalized approach to work and heat, Europhys. Lett. 83, 30008 (2008).

H. Hossein-Nejad, E. J. O’Reilly, and A. Olaya-Castro, Work, heat and entropy production in bipartite quantum systems, arXiv 10.48550/arXiv.1507.00441 (2015), 1307.0044.

M. Maffei, P. A. Camati, and A. Auffeves, Probing non-classical light fields with energetic witnesses in waveguide quantum electrodynamics, Phys. Rev. Res. 3, L032073 (2021).

R. Alicki and M. Fannes, Dissipation, correlation and entropy production rate, Phys. Rev. Lett. 118, 220601 (2017).

S. Defner and E. Lutz, Nonequilibrium Entropy Production for Open Quantum Systems, Phys. Rev. Lett. 107, 140404 (2011).

J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics—a topical review, J. Phys. A: Math. Theor. 49, 143001 (2016).

M. Campisi and R. Fazio, Dissipation, correlation and lags in heat engines, J. Phys. A: Math. Theor. 49, 345002 (2016).

D. J. Bedingham and O. J. E. Maroney, The thermodynamic cost of quantum operations, New J. Phys. 18, 113050 (2016).

P. Strasberg, G. Schaller, T. Brandes, and M. Esposito, Quantum and Information Thermodynamics: A Unifying Framework Based on Repeated Interactions, Phys. Rev. X 7, 021003 (2017).

T. Sagawa and M. Ueda, Fluctuation Theorem with Information Exchange: Role of Correlations in Stochastic Thermodynamics, Phys. Rev. Lett. 109, 180602 (2012).

K. Micadei, J. P. S. Peterson, A. M. Souza, R. S. Sarthour, I. S. Oliveira, G. T. Landi, T. B. Batalhão, R. M. Serra, and E. Lutz, Reversing the direction of heat flow using quantum correlations, Nat. Commun. 10, 1 (2019).
Supplemental Materials: An autonomous formulation of the second law of thermodynamics valid for arbitrary quantum systems

Interpretation of $E^\infty$ as the preparation work cost

We are interested in this section in the minimum amount of work one needs to prepare an arbitrary non-equilibrium state $\rho_A^0$ of a system $A$. For reference, the same conclusion as the one derived below was obtained in [S1].

We assume that the system $A$ of Hamiltonian $H_A$ is initially in an equilibrium state $w_A[\beta, H_A]$ with a bath at inverse temperature $\beta$. Assuming we can perform arbitrary driving and quenches, one reversible protocol to reach $\rho_A^0$ is as follows.

- Perform an isothermal reversible quasi-static driving $H_A \rightarrow \tilde{H}_A$ such that the final state is our target state, $w_A[\beta, \tilde{H}_A] = \rho_A^0$. This can always be done, at least in theory, for arbitrary initial state $\rho_A^0$. It can be shown as follows. One expresses $\rho_A^0$ in its diagonal form,

$$\rho_A^0 := \sum_{l=1}^{L} r_l |r_l\rangle\langle r_l|,$$  \hspace{1cm} (S1)

and

$$H_A = \sum_{k=1}^{K} e_k |e_k\rangle\langle e_k|,$$  \hspace{1cm} (S2)

with $1 \leq L \leq K \leq +\infty$. For $l \in [1; L]$, we introduce pseudo-energies $E_l$ as

$$E_l := \frac{1}{\beta} (\ln r_l + c),$$ \hspace{1cm} (S3)

where $c$ is a free constant one can use to choose the energy origin. Then, if $\rho_A^0$ is full rank, meaning if $L = K$, we define $\tilde{H}_A$ as

$$\tilde{H}_A := \sum_{l=1}^{L} E_l |r_l\rangle\langle r_l|.$$ \hspace{1cm} (S4)

However, if $\rho_A^0$ is not full rank $L < K$, we define $\tilde{H}_A$ as

$$\tilde{H}_A := \sum_{l=1}^{L} E_l |r_l\rangle\langle r_l| + \sum_{l=L+1}^{K} E_l |e_l\rangle\langle e_l|,$$ \hspace{1cm} (S5)

where $E_l = E_{l'} \gg 1/\beta$ for all $l, l' \geq L + 1$. With this choice, one can verify that

$$w_A[\beta, \tilde{H}_A] := e^{-\beta \tilde{H}_A} / \text{Tr}[e^{-\beta \tilde{H}_A}] = \rho_A^0,$$ \hspace{1cm} (S6)

for the full-rank situation. This identity becomes only approximate in the non-full-rank situation, but the approximation is exponentially good for large $E_l$, $l > L$. The work involved in this reversible quasi-static driving is given by the variation of equilibrium free energy $F[w_A[\beta, H]] := \text{Tr}[\rho H] - \frac{1}{\beta} S[\rho]$,

$$W_{\text{quasi-static}} = F[w_A[\beta, \tilde{H}_A]] - F[w_A[\beta, H_A]].$$ \hspace{1cm} (S7)

- Switch off the bath interaction.
• Perform a quench $\tilde{H}_A \rightarrow H_A$ to come back to the initial Hamiltonian. The work involved in the quench is

$$W_{\text{quench}} = \text{Tr}[\rho_A^0(H_A - \tilde{H}_A)],$$ \hspace{1cm} (S8)

since the quench does not change the state of $A$ for occurring on a timescale much smaller than the system evolution time.

Then, the total work invested to prepare $\rho_A^0$ is

$$W_\beta = W_{\text{quasi-static}} + W_{\text{quench}} = F_{\beta,H_A}[\rho_A^0] - F[w_A[\beta,H_A]]$$ \hspace{1cm} (S9)

where $F_{\beta,H_A}[\rho_A^0] = \text{Tr}[\rho_A^0 H_A] - \frac{1}{2}S[\rho_A]$ denotes the non-equilibrium free-energy. Interestingly, one can show that the overall work $W$ is related to the relative entropy between $\rho_A^0$ and $w_A[\beta,H_A]$,

$$W_\beta = \frac{1}{\beta}D[\rho_A^0|w_A[\beta,H_A]],$$ \hspace{1cm} (S10)

which is always positive, as expected: one always has to spend work to prepare a non-equilibrium state. Since the above protocol is reversible, it guarantees to be the one with the smallest amount of work to invest, for fixed $\beta$. Then, what is the value of $\beta$ which minimizes $W_\beta$? The answer is given by taking the derivative of $W_\beta$ with respect to $\beta$.

We obtain

$$\frac{1}{\beta^2} W_\beta = \frac{1}{\beta^2} F[w_A[\beta,\tilde{H}_A]] - \frac{1}{\beta^2} F[w_A[\beta,H_A]]$$

$$= \frac{1}{\beta^2} \{S[\rho_A^0] - S[w_A[\beta,H_A]]\}. \hspace{1cm} (S11)$$

Since $S[w_A[\beta,H_A]]$ is a monotonic decreasing function of $\beta$, we deduce that $W_\beta$ is a monotonic decreasing function of $\beta$ on $[0,\beta_0]$ and a monotonic increasing function of $[\beta_0;+\infty[$, where $\beta_0$ denotes the inverse temperature such that $S[w_A[\beta_0,H_A]] = S[\rho_A^0]$. Consequently, the minimum amount of work to prepare $\rho_A^0$ is $W_{\beta_0}$. By noticing that $W_{\beta_0} = \text{Tr}[\rho_A^0 H_A] - \text{Tr}[w_A[\beta_0,H_A]H_A] = \mathcal{E}_A^\infty$, the "asymptotic" ergotropy S35 introduced in the main text, one concludes that $\mathcal{E}_A^\infty$ represents the minimal amount of work needed to prepare the state $\rho_A^0$ out of a thermal bath.

**Negativity of the entropy production defined from the splitting ergotropy - passive energy**

One natural choice widely used for autonomous systems is to identify the variation of ergotropy of a system as work. If one assumes that the variation of ergotropy corresponds to the whole exchange of work, the second law of thermodynamics should take the form,

$$\sigma_{A,\text{erg}}(t) := \Delta S_A + \beta B(0) (\Delta E_B - \Delta E_B) \geq 0.$$ \hspace{1cm} (S12)

From Eq. [1], one finds that:

$$\sigma_{A,\text{erg}}(t) = I_{AB}(t) + D[\pi_B(t)|w_B[\beta_B(0)]] - D[\pi_B(0)|w_B[\beta_B(0)]]. \hspace{1cm} (S13)$$

We now show that the quantity on the right-hand side can become negative, or equivalently that the free energy of system $A$ can be increased by more than the entropy consumed. We first note that in the case where system $B$ is a qubit, $E_B - W_B = E_B^{\text{th}}$, such that $\sigma_{A,\text{erg}} = \sigma_A \geq 0$. Our counterexample must therefore involve a system of dimension at least 3. We consider the case where systems $A$ and $B$ are two identical qutrits of Hamiltonians

$$H_j = \sum_{k=0}^2 \omega_k |k\rangle_j \langle k|, \hspace{1cm} j = A, B$$ \hspace{1cm} (S14)

initialized in state $\rho_{AB}(0) = \rho_A(0) \otimes \rho_B(0)$, with $\rho_A(0) = w_A[\beta_B(0)]$ a thermal state at temperature $\beta(0)$ and $\rho_B(0) = \sum_{k=0}^2 \omega_k |k\rangle_B \langle k|$. We further assume that $p_{B,2} \leq p_{B,1} \leq p_{B,0}$ (that is $\rho_B(0)$ is a passive state and $W_B(0) = 0$), but non-thermal, that is, there is no positive real number $\beta$ such that $p_{B,i}/p_{B,j} = e^{-\beta (\omega_i - \omega_j)}$ for all couples $(i,j) \in \{0,2\}^2$. Such a non-thermal passive state can be simply built by choosing $p_{B,1} \propto e^{-\beta_1 \omega_1} p_{B,0}$ and $p_{B,2} \propto e^{-\beta_2 \omega_2} p_{B,0}$ with two different positive numbers $\beta_1 \neq \beta_2$. These conditions imply that $E_B(0) - E_B^{\text{th}}(0) \neq 0$. As
before, we denote $\beta_B(0)$ the inverse temperature of the thermal state which has the same entropy as $\rho_B(0)$. Finally, we choose a coupling Hamiltonian implementing a swap of the two qutrit states, namely:

$$V = g \sum_{k \neq l} |kl\rangle_A \langle lk|.$$  \hspace{1cm} (S15)

It is straightforward to show that for $t = t_{\text{SWAP}}$ verifying $gt_{\text{SWAP}} = \pi/2$, we have $\rho_{AB}(t_{\text{SWAP}}) = \rho_B(0) \otimes \rho_A(0)$, that is the states of qutrits $A$ and $B$ are swapped. This means that the state of qutrit $B$ is replaced with a thermal state which has the same entropy, and therefore still no ergotropy $W_B(0)$, however its energy has decreased by $E_B(t_{\text{SWAP}}) = E_B(0) = E_B^{\text{th}}(0) - E_B(0) < 0$. Moreover, the entropy of qubit $A$ did not change:

$$S_A(t) = S[\rho_A(t_{\text{SWAP}})] = S[\rho_B(0)] = S[w_B[\beta_B(0)]] = S_A(0).$$  \hspace{1cm} (S16)

Finally, we have $\sigma_A < 0$. The qutrit swap allows for a spontaneous diminution of the passive amount of energy $E_B - W_B$ in qubit $B$. Similarly, the variation of the free energy of system $A$ is equal to its variation of internal energy as $\Delta S_A = 0$, that is $\Delta E_A(t_{\text{SWAP}}) = -\Delta E_B(t_{\text{SWAP}}) \geq 0$: the free energy of qubit $A$ was increased without consuming any ergotropy.

Consequently, identifying work with the variation of ergotropy may lead to underestimation of entropy production. More generally, as the present example allows for a spontaneous decrease of the passive amount of energy $E_B - W_B$ of $B$ without any entropy variation of the entropy of $A$, it also demonstrates that there is no other choice of effective temperature that one could inject in Eq. (S12) to ensure its positivity: this demonstrates that the notion of ergotropy is insufficient to capture all the energy that can be assimilated as work. One can note however that under special assumptions, e.g. restricting the possible initial state of system $B$ to unitary-transformed thermal states, i.e. $\rho_B(0) = U_0 w_B[\beta_B(0)] U_0^\dagger$, one obtains

$$\Delta S_A(t) + \beta_B(0) \Delta E_A(t) \geq 0,$$

in agreement with the results of [S2, S3]. Note that that inequality is less tight than Eq. (5) in general as system $B$ can still end up in a state that contains more non-thermal energy than ergotropy.

**Clausius inequality**

To go from Eq. (5) to inequality (12) of main text, we use the definitions of $w_A[\beta_A(t)]$:

$$\Delta S_A = S[w_A[\beta_A(t)]] - S[w_A[\beta_A(0)]]$$

$$= S[w_A[\beta_A(t)]] + \text{Tr}(w_A[\beta_A(t)] \log w_A[\beta_A(0)]) - \text{Tr}(w_A[\beta_A(t)] \log w_A[\beta_A(0)])$$

$$- S[w_A[\beta_A(0)]] - \text{Tr}(w_A[\beta_A(0)] \log w_A[\beta_A(0)]) + \text{Tr}(w_A[\beta_A(0)] \log w_A[\beta_A(0)])$$

$$= -D(w_A[\beta_A(t)] || w_A[\beta_A(0)]) - \text{Tr}(w_A[\beta_A(t)] \log w_A[\beta_A(0)])$$

$$+ \text{Tr}(w_A[\beta_A(0)] \log w_A[\beta_A(0)]).$$  \hspace{1cm} (S18)

This finally leads to the identity:

$$\Delta S_A = -D(w_A[\beta_A(t)] || w_A[\beta_A(0)]) + \beta_A(0) Q_A(t),$$  \hspace{1cm} (S19)

valid for any quantum system.

**Example: two qubit autonomous refrigerator.**

For a qubit system characterized by state $\rho_j(t)$, the Von Neumann entropy depends only of the parameter $r_j(t) = \sqrt{\text{Tr}(\sigma^z_j \rho_j(t))^2 + \text{Tr}(\sigma^x_j \rho_j(t))^2 + \text{Tr}(\sigma^y_j \rho_j(t))^2}$, namely:

$$S[\rho_j(t)] = -\frac{1 + r_j(t)}{2} \log \left(\frac{1 + r_j(t)}{2}\right) - \frac{1 - r_j(t)}{2} \log \left(\frac{1 - r_j(t)}{2}\right).$$  \hspace{1cm} (S20)

Comparison with the entropy of a qubit thermal state $w_j[\beta_j] = e^{-\beta_j \omega_j \sigma^z_j/2}/Z_j$ allows one to identify the effective temperature $\beta_j(t) = \log((1 + r_j(t))/(1 - r_j(t))/\hbar \omega_j$. Consequently, we can compute the thermal energy $E_j^{\text{th}}(t) = \text{Tr}(\sigma^z_j \rho_j(t))\beta_j(t)$.
\[-r_j(t)\hbar \omega_j/2\text{, and therefore the work and heat provided by qubit } i \text{ between } t = 0 \text{ and } t:\]

\[Q_j = \hbar \omega_i \Delta r_j(t)/2\quad (S21)\]

\[W_j = -\hbar \omega_i (\Delta r_j(t) + \Delta z_j(t))/2,\quad (S22)\]

where \(z_j(t) = \text{Tr}\{\sigma^z_j \rho_j(t)\}\). Assuming the initial states \(\rho_A(0) = \omega_A|\beta_A(0)\rangle\) and \(\rho_B = e^{-i\pi \sigma^z_B / 2} w_B|\beta_B(0)\rangle e^{i\pi \sigma^z_B / 2} = e^{i\pi \sigma^z_B / 2} w_B|\beta_B(0)\rangle Z_B(0)\); we can propagate the two-qubit state according to \(\rho_{AB}(t) = e^{-iH_{AB}t}\rho_A(0) \otimes \rho_B(0) e^{iH_{AB}t}\) and compute:

\[z_A(t) = \frac{g^2}{2\Omega^2 \lambda^2} \left[ \lambda^2 (\tanh(\beta_A \omega_A/2) + \tanh(\beta_B(0) \omega_B/2)) \cos(\Omega t) + \Omega^2 (\tanh(\beta_A \omega_A/2) - \tanh(\beta_B(0) \omega_B/2)) \cos(\Omega t) \right]
\]

\[+ \frac{1}{\lambda^2 \Omega^2} \left[ (g^2(\omega_B^2 + \omega_A^2) + (\omega_B^2 - \omega_A^2)^2) \tanh(\beta_A \omega_A/2) - 2g^2 \omega_B \omega_B \tanh(\beta_B \omega_B/2) \right] \quad (S23)\]

\[z_B(t) = \frac{g^2}{2\Omega^2 \lambda^2} \left[ \lambda^2 (\tanh(\beta_A \omega_A/2) + \tanh(\beta_B(0) \omega_B/2)) \cos(\Omega t) - \Omega^2 (\tanh(\beta_A \omega_A/2) - \tanh(\beta_B(0) \omega_B/2)) \cos(\Omega t) \right]
\]

\[+ \frac{1}{\lambda^2 \Omega^2} \left[ (g^2(\omega_B^2 + \omega_A^2) + (\omega_B^2 - \omega_A^2)^2) \tanh(\beta_B \omega_B/2) - 2g^2 \omega_B \omega_B \tanh(\beta_A \omega_A/2) \right] \quad (S24)\]

and \(r_A(t) = |z_A(t)|, r_B(t) = |z_B(t)|\). Moreover:

\[\Delta E_{\text{int}}(t) = -\frac{g^2}{4\Omega^2 \lambda^2} \left[ \lambda^2 (\omega_B - \omega_A) (\tanh(\beta_B(0) \omega_B/2) + \tanh(\beta_B(0) \omega_B/2)) \cos(\Omega t) \right]
\]

\[+ \Omega^2 (\omega_B + \omega_A) (\tanh(\beta_A \omega_A/2) - \tanh(\beta_B(0) \omega_B/2)) \cos(\Omega t) \]

\[+ \frac{g^2}{2\Omega^2 \lambda^2} \left[ e^{\beta_A(0) \omega_B} + e^{\beta_B(0) \omega_B} - 1 \right] \frac{\lambda^2 (\omega_B^2 - \omega_A^2) + e^{\beta_A(0) \omega_B} - e^{\beta_B(0) \omega_B}}{(1 + e^{\beta_A(0) \omega_B})(1 + e^{\beta_B(0) \omega_B})} \Omega^2 (\omega_B + \omega_A) \quad (S25)\]

We have introduced the parameters:

\[\lambda = \sqrt{g^2 + (\omega_A + \omega_B)^2}, \quad \Omega = \sqrt{g^2 + (\omega_A - \omega_B)^2}.\quad (S26)\]

**Autonomous two-qubits engine**

Here we present some plots corresponding to the regime of work generation by the two qubits, thanks to the temperature gradient of the qubits’ thermal background. As in the example of the two-qubit refrigerator, the local Hamiltonians of \(A\) and \(B\) are \(H_j = (\hbar \omega_j/2) \sigma^z_j\), but we now consider a coupling along the \(x\) and \(y\) directions, namely \(g_x \sigma^x_A \sigma^x_B + g_y \sigma^y_A \sigma^y_B\). The plot of Fig. S1(a) represents the work stored (in the form of population inversion) in \(A\) (purple curve) and in the coupling energy (green curve), in function of the time. Note that for the parameters and time interval considered here the work stored in \(B\) is null. Starting with a thermal background of \(B\) hotter than \(A\)’s one, we define the efficiency of work generation as

\[\eta = \frac{-W_A - W_B + \Delta E_{\text{int}} \Theta(-W_A - W_B + \Delta E_{\text{int}})}{Q_B}\quad (S27)\]

where the Heaviside function \(\Theta(x)\) ensure that efficiency is non-zero only when the device operates in engine mode, i.e. converts heat from \(B\) into work in \(A, B\) or increase of the coupling energy. We include the variation of the energy coupling \(\Delta E_{\text{int}}\) because, if negative, it means one will have to pay at least a work cost equal to \(-\Delta E_{\text{int}}\) to reinitialize the engine (or equivalently to separate \(A\) from \(B\)). Then, it is a work expenditure that should be subtracted from the generated work. Conversely, if \(\Delta E_{\text{int}}\) is positive, it means one can extract (at most) an amount of work equal to \(\Delta E_{\text{int}}\), and thus it is fair to add it to the generated work. The plots of the efficiency is presented in Fig.S1(b). Thanks to the expression of the second law introduced in the main text, one can show that the efficiency \(\eta\) is upper bounded by the Carnot efficiency, namely \(1 - \beta_B(0)/\beta_A(0)\) (black dashed line). As for the refrigerator, the loss of efficiency compared to the Carnot efficiency is due to generation of \(A-B\) correlations and the increase of the thermal distances \(D(w_B[\beta_B(t)]||w_B[\beta_B(0)])\) and \(D(w_A[\beta_A(t)]||w_A[\beta_A(0)])\).
FIG. S1. **Performances of the two-qubit engine.** a: Work $W_A(t)$ provided by $A$ (purple curve), heat $Q_B(t)$ provided by $B$ (red curve), and variation of the coupling energy $\Delta E_{\text{int}}(t)$ (green curve) in units of $\hbar \omega_A$ after the qubits interacted during a time $t$. Time is given in unit of the frequency $\Omega = \sqrt{g_x^2 + g_y^2 + (\omega_B - \omega_A)^2}$. Note that $W_B(t)$ is equal to 0 for the considered time interval, and thus is not appearing in the plot. b: Efficiency of the engine $\eta(t)$ (blue curve) and Carnot’s efficiency $1 - \beta_B(0)/\beta_A(0)$ (brown straight line). Parameters: $\beta_A(0)\hbar \omega_A = 2$, $\beta_B(0)\hbar \omega_A = 0.1$, $\omega_B/\omega_A = 1.63$, $g_x/\omega_A = 2$, $g_y/\omega_A = 0.8$.

FIG. S2. **Trade-off velocity of the work production versus efficiency.** Plots of the efficiency of the autonomous two-qubit engine against the time (in unit of $\omega_A$), for coupling strength varying from $g_x/\omega_A = 2$ (lighter curve) to $g_x/\omega_A = 3$ (darker curve). The other parameters are the same as in the previous figure, namely $\beta_A(0)\hbar \omega_A = 2$, $\beta_B(0)\hbar \omega_A = 0.1$, $\omega_B/\omega_A = 1.63$, and $g_y/\omega_A = 0.8$.

**Trade-off velocity versus efficiency**

In this section we aim to briefly illustrate that the heat (and potentially the work) provided by the system $B$ does depend on the “trajectory” of $A$. One general consequence is that faster processes are usually more dissipative/irreversible and less efficient. In Fig. (S2), we plot the efficiency of the autonomous two-qubit engine considered in the previous section against the time evolution (in unit of $\omega_A$) for increasing coupling strength. The lighter curve corresponds to $g_x/\omega_A = 2$, while the darker curve corresponds to $g_x/\omega_A = 3$. Each curve in-between corresponds to an increment of $\delta g_x/\omega_A = 0.1$ with respect to its lighter neighbor. We observe the expected behavior: faster processes are less efficient. This illustrates, in particular, that the heat provided by $B$ does depend on the trajectory of $A$, as announced.

**Ideal work source**

We assume that the unitary evolution of $A$ and $B$ is such that at any time, $I_{AB}(t) = 0$ and $\rho_{AB}(t) = \rho_A(t) \otimes \rho_B(t)$. The Liouville-Von-Neumann evolution equation reads:

$$\dot{\rho}_{AB}(t) = -i[H_A + V + H_B, \rho_A(t) \otimes \rho_B(t)].$$

(S28)
Taking partial trace over one or the other system, we obtain:

$$\dot{\rho}_A(t) = -i[H^\text{eff}_A(t), \rho_A(t)]$$
$$\dot{\rho}_B(t) = -i[H^\text{eff}_B(t), \rho_B(t)],$$

where $H^\text{eff}_A(t) = H_A + \text{Tr}_B\{V \rho_B(t)\}$ (respectively $H^\text{eff}_B(t) = H_B + \text{Tr}_A\{V \rho_A(t)\}$) is an effective Hamiltonian acting on system $A$ (resp. $B$), where $\text{Tr}_i$ denotes the partial trace over the operator space of system $i$. We can see that both systems undergo a unitary evolution due to their interaction. As a consequence, the entropy of $B$, and therefore its effective inverse temperatures $\beta^\text{eff}_B(t)$ are conserved. This leads to $Q_B(t) = 0$ and $W_B(t) = -\Delta E_B(t)$. We now use Eq. (S29) to express the work provided by system $B$. We have

$$W_B(t) = \int_0^t dt' \text{Tr}\{H_B (-i[H^\text{eff}_B(t), \rho_B(t)])\}$$
$$= -i \int_0^t dt' \text{Tr}\{[H_B, H_B + \text{Tr}_A\{V \rho_A(t)\}] \rho_B(t)\}$$
$$= -i \int_0^t dt' \text{Tr}\{[H_B, V] \rho_A(t) \otimes \rho_B(t)\}$$
$$= i \int_0^t dt' \text{Tr}\{\text{Tr}_B\{V[H_B, \rho_B(t)]\} \rho_A(t)\}$$
$$= - \int_0^t dt' \text{Tr}\{\frac{\partial}{\partial t} H^\text{eff}_A(t) \rho_A(t)\}.\quad (S30)$$

We finally note that in the case of only two subsystems, the condition that $B$ is an ideal work source for $A$ implies automatically that $A$ is also an ideal work source for $B$, such that the entropy of both systems is constant. As soon as three or more systems are coupled however, the entropy of system $A$ may vary due to interactions with other energy sources which are not ideal work sources, even though the entropy of $B$ remain constant.

**Autonomous second law for $N$ systems**

We here extend the formulation of the second law to the case of $N$ quantum systems initially in a factorized state $\rho_{\text{tot}}(0) = \bigotimes_{i=1}^N \rho_i(0)$. The total Hamiltonian reads $H_{\text{tot}} = \sum_i H_i + V$, where $V$ contains all the coupling terms. We introduce:

$$I_{\text{tot}}(t) = D(\rho_{\text{tot}}(t)\| \bigotimes_{i=1}^N \rho_i(t)) \geq 0,\quad (S31)$$

which verifies $I_{\text{tot}}(0) = 0$. We can then express the variation of the Von Neumann entropy of system $j \in [1, N]$ between time $t = 0$ and $t$:

$$\Delta S_j = I_{\text{tot}}(t) - \sum_{i \neq j} \Delta S_i.\quad (S32)$$

Analogously to the case of two systems, we define for each system $i$:

$$w_i[\beta_i(t)] = \frac{e^{-\beta_i(t)H_i}}{Z_i(t)},\quad (S33)$$
$$Z_i(t) = \text{Tr}\{e^{-\beta_i(t)H_i}\},\quad (S34)$$
$$E_i^\text{th}(t) = \text{Tr}\{H_i w_i[\beta_i(t)]\},$$
$$Q_i(t) = -E_i^\text{th}(t) + E_i^\text{th}(0),\quad (S35)$$

where $\beta_i(0)$ is the solution of

$$S[w_i[\beta_i(t)]] = S[\rho_i(0)].\quad (S36)$$

We then add on both side of Eq. (S32) the quantity $-\sum_{i \neq j} \beta_i(0)Q_i(t)$ and use that
\[-\beta_i(0)Q_i(t) - (S[\rho_i(t)] - S[\rho_i(0)]) = -\text{Tr}\{(w_i[\beta_i(t)] - w_i[\beta_i(0)]) \log w_i[\beta_i(0)]\} - S[w_i[\beta_i(t)] + S[w_i[\beta_i(0)]] \]
\[= D(w_i[\beta_i(t)] \| w_i[\beta_i(0)]), \tag{S37}\]
to finally get
\[\Delta S_j - \sum_{i \neq j} \beta_i(0) Q_i(t) = I_{\text{tot}}(t) + \sum_{i \neq j} D(w_i[\beta_i(t)] \| w_i[\beta_i(0)]), \tag{S38}\]
which implies inequality (15) of main text. To demonstrate inequality (16) of main text, we rewrite the variation the entropy of system $j$ using the identity Eq. (S19) and inject it into Eq. (S38) to get:
\[-\sum_i \beta_i(0) Q_i(t) = I_{\text{tot}} + \sum_i D(w_i[\beta_i(t)] \| w_i[\beta_i(0)]), \tag{S39}\]
which in turn implies inequality (16) of main text.

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[S1] M. N. Bera, A. Riera, M. Lewenstein, Z. B. Khanian, and A. Winter, Thermodynamics as a Consequence of Information Conservation, Quantum 3, 121 (2019) 1707.01750v3.
[S2] R. Uzdin and S. Rahav, Global Passivity in Microscopic Thermodynamics, Phys. Rev. X 8, 021064 (2018).
[S3] W. Niedenzu, V. Mukherjee, A. Ghosh, A. G. Kofman, and G. Kurizki, Quantum engine efficiency bound beyond the second law of thermodynamics, Nat. Commun. 9, 1 (2018).