Exotic compact objects with soft hair

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Motivated by the lack of a general parametrization for exotic compact objects, we construct a class of perturbative solutions valid for small (but otherwise generic) multipolar deviations from a Schwarzschild metric in general relativity. We introduce two classes of exotic compact objects, with “soft” and “hard” hair, for which the curvature at the surface is respectively comparable to or much larger than that at the corresponding black-hole horizon. We extend the Hartle-Thorne formalism to relax the assumption of equatorial symmetry and to include deformations induced by multipole moments higher than the spin, thus constructing the most general, axisymmetric quasi-Schwarzschild solution to Einstein’s vacuum equations. We explicitly construct several particular solutions of objects with soft hair, which might be useful for tests of quasi-black-hole metrics, and also to study deformed neutron stars. We show that the more compact a soft exotic object is, the less hairy it will be. All its multipole moments can approach their corresponding Kerr values only in two ways as their compactness increases: either logarithmically (or faster) if the moments are spin-induced, or linearly (or faster) otherwise. Our results suggest that it is challenging (but possibly feasible with next-generation gravitational-wave detectors) to distinguish Kerr black holes from a large class of ultracompact exotic objects on the basis of their different multipolar structure.

I. INTRODUCTION

As a by-product of the black-hole (BH) uniqueness and no-hair theorems [1,2] (see also [3,5]), the multipole moments of any stationary BH in isolation can be written as [6,7],

\[ M_\ell^\text{BH} + iS_\ell^\text{BH} = M^{\ell+1}(i\chi) \ell, \tag{1} \]

where \( M_\ell \) (\( S_\ell \)) are the Geroch-Hansen mass (current) multipole moments [6,7], the suffix “BH” refers to the Kerr metric, and

\[ \chi \equiv \frac{S_1}{M_0} \tag{2} \]

is the dimensionless spin. Equation (1) implies that \( M_\ell^\text{BH} \) and \( S_\ell^\text{BH} \) vanish when \( \ell \) is odd (even), and that all moments with \( \ell \geq 2 \) can be written only in terms of the mass \( M_0 = M \) and angular momentum \( S_1 = J \) (or, equivalently, \( \chi \)) of the BH. Therefore, any independent measurement of three multipole moments (e.g. the mass, the spin and the mass quadrupole \( M_2 \)) provides a null-hypothesis test of the Kerr metric and, in turn, it might serve as a genuine strong-gravity confirmation of general relativity [8-13].

Motivated by some scenarios inspired by semiclassical and quantum gravity which predict exotic ultracompact objects without a horizon (see, e.g., [14-19]) or new physics at the horizon scale (see [20]-[22] for some overview), the aim of this paper is to identify some generic features in the multipolar structure of exotic compact objects (ECOs) and to construct explicit quasi-BH solutions to Einstein’s equations in vacuum.

The vacuum region outside a spinning object is not generically described by the Kerr geometry, due to the absence of an analog to the Birkhoff’s theorem in axisymmetry. Thus, the multipole moments of an axisymmetric ECO (spinning or not) will generically satisfy relations of the form

\[ M_\ell^{\text{ECO}} = M_\ell^{\text{BH}} + \delta M_\ell, \tag{3} \]
\[ S_\ell^{\text{ECO}} = S_\ell^{\text{BH}} + \delta S_\ell, \tag{4} \]

where \( \delta M_\ell \) and \( \delta S_\ell \) are model-dependent corrections, whose precise value can be obtained by matching the metric describing the interior of the object to that of the exterior. We shall assume that matter fields are confined in the interior and that the exterior is governed by Einstein’s equations. This is a nontrivial assumption on the physics of the ECO, namely that the exotism resides entirely in the matter and not in the gravity sector. It allows us to perform an analysis of great generality independently of the details of the exotic interior: different models of ECOs are defined by the boundary conditions at the object’s surface, which uniquely define \( \delta M_\ell \) and \( \delta S_\ell \).

We introduce two classes of ECO models:

- “soft” ECOs: for which the boundary conditions are such that the curvature at the surface is comparable to that at the horizon of the corresponding BH, i.e. \( K^{1/2} \sim 1/M^2 \) (we adopt \( G = c = 1 \) units throughout and use the Kretschmann scalar \( K \) as a measure of the compactness). This corresponds

\[^1\text{Although our main motivation is to study ECO spacetimes, we shall construct a general perturbative solution of vacuum Einstein’s equations, which might be useful also to study deformed neutron stars within general relativity.}\]
to models in which there is no further length scale in the exterior other than $\mathcal{M}$ or in which the new scale is parametrically close to $\mathcal{M}$. An example of the latter case is the "scrambling time" $\tau \sim -\mathcal{M} \log(\mathcal{L}/\mathcal{M}) \gtrsim \mathcal{M}$, where even if there is a new length scale $\mathcal{L} \ll \mathcal{M}$, its effect on the curvature at the surface is logarithmically suppressed. In other words, these models the near-surface geometry is similar to that of a BH and smoothly approaches the horizon in the BH limit (hence their "softness").

- "hard" ECOs: for which the curvature at the surface can be much larger than that at the horizon of the corresponding BH, presumably because the underlying theory involves a new length scale, $\mathcal{L}$, such that $\mathcal{L} \ll \mathcal{M}$ so the ECO can support large curvatures on its surface without collapsing. The curvature in this case typically behaves like a power of $\mathcal{M}/\mathcal{L}$, hence $K^{1/2} \gg 1/\mathcal{M}^2$. In other words, in these models high-energy effects drastically modify the near-surface geometry (hence their "hardness").

In this work with focus only on soft ECO solutions (which we refer to also as "ECOs with soft hair") relative to a Kerr BH, as we shall show. Soft ECOs are characterized by multipolar hair that do not by itself require physics on a short-distance scale, but this does not mean that such ECOs are possible in the absence of any very high-energy physics. Indeed, an ECO with a surface just above the BH limit requires large internal stresses in order to prevent its collapse, even if the exterior is exactly the Schwarzschild or Kerr geometry. Our notion of soft ECOs refers to the scale of the physics that is implied by the existence of multipolar hair. This is a feature distinguishable purely from the ECO exterior, and therefore, to the extent that this exterior is governed by Einstein's theory, the characterization of the softness of an ECO is very much model-independent. Likewise, the exterior of hard ECOs might be described by soft ECO solutions far from the surface, where the curvature is perturbatively close to that of a BH.

In this paper, we shall construct perturbative solutions obtained by solving the vacuum Einstein equations order by order in a small multipole moment expansions. We shall classify the solutions in terms of the type of independent multipole moments that they possess to leading order. In this scheme, each solution can possess an infinite tower of multipole moments which are sourced by the leading-order ones. For example, a solution possessing only $\mathcal{S}_1 = J$ to leading order will contain an infinite number of multipole moments at higher orders. We shall refer to the latter as spin-induced moments since they vanish as $J \to 0$. On the other hand, we shall refer to the multipole moments that remain nonzero as $J \to 0$ as nonspin-induced moments.

We shall provide evidence for the following conjecture. In the BH limit, the deviations from the Kerr multipole moments (with $\ell \geq 2$) vanish as

$${\delta \mathcal{M}_\ell \over \mathcal{M}^{\ell+1}} \to a_\ell {\chi^\ell \over \log \delta} + b_\ell \delta + \ldots, \quad (5)$$

$${\delta \mathcal{S}_\ell \over \mathcal{M}^{\ell+1}} \to c_\ell {\chi^\ell \over \log \delta} + d_\ell \delta + \ldots, \quad (6)$$

or faster. Here $a_\ell$, $b_\ell$, $c_\ell$, and $d_\ell$ are numbers of order unity or smaller, the ellipsis stand for terms which are subleading in our expansion, whereas $\delta \ll 1$ is a dimensionless number that can be expressed in terms of coordinate-independent geometrical quantities and measures the compactness of the object in a way to be specified below. The BH limit corresponds to $\delta \to 0$. The coefficients $a_\ell$, $b_\ell$, $c_\ell$, and $d_\ell$ are related to the spin-induced contribution to the multipole moments; selection and $\mathbb{Z}_2$ rules presented below imply that $a_\ell$ and $c_\ell$ be identically zero when $\ell$ is odd and even, respectively. On the other hand, the coefficients $b_\ell$ and $d_\ell$ are related to the nonspin-induced contributions up to order $\ell$. Note that the selection rules imposed by the equatorial symmetry of the Kerr solution do not necessarily apply to ECOs, so $\mathcal{S}_\ell^{\mathrm{ECO}} (\mathcal{S}_\ell^{\mathrm{K}})$ can be nonzero also when $\ell$ is odd (even) and the solution can break the equatorial symmetry through the terms proportional to $b_\ell$ and $d_\ell$ in Eqs. (5) and (6).

In other words, in this perturbative scheme the deviations from the Kerr multipole moments must die sufficiently fast as the compactness of the object approaches that of a BH, or otherwise the curvature at the surface will grow and the models cannot classify as soft ECOs. As the ECO approach the BH limit, spin-induced moments are less strongly suppressed than other moments and are therefore easier to detect.

In the rest of this paper we shall construct solutions obtained by solving the vacuum Einstein equations order by order in a small multipole moment expansions. We shall classify the solutions in terms of the type of independent multipole moments that they possess to leading order. In this scheme, each solution can possess an infinite tower of multipole moments which are sourced by

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\[ \text{multipole moments of the Kerr solution.}
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Several interesting particular solutions are explicitly discussed below and their metric is publicly available in closed form in an online repository containing supplemental Mathematica® notebooks [29]. As an anticipation of our main result, Fig. 1 presents the embedding diagrams for some representative solutions.

Henceforth we adopt the Geroch-Hansen definition of the multipole moments [6] [7]: the latter are equivalent [20] to the multipole moments defined by Thorne [31] using asymptotically mass-centered Cartesian coordinates.

II. ECOS WITH SOFT HAIR: A GENERAL SMALL-MULTIPOLE EXPANSION

In this section we implement our method for characterizing the soft hair of an ECO. Our assumptions are that

1. the exterior of the ECO is governed by Einstein’s vacuum equations,
2. its geometry deviates by a small amount from the Schwarzschild metric.

We consider generic stationary axisymmetric deviations which include not only the soft hair but also angular momentum, so these ECOS can rotate slowly. By inserting these perturbations in the vacuum Einstein equations and decomposing them into spherical harmonics of degree \( \ell \), we obtain a set of ordinary differential equations for each multipolar mode, which we solve analytically. After requiring asymptotic flatness at infinity, we must still give boundary conditions at the ECO surface. These can be completely specified, at each perturbative order, by a pair of constants \( M_\ell \) and \( S_\ell \) for each \( \ell \geq 2 \). These constants are directly connected to the physical mass and current multipoles, \( M_\ell \) and \( S_\ell \), and therefore provide a general, physical parametrization of the soft hair.

A. Setup

We consider a metric of the form

\[
\begin{align*}
g_{\mu\nu} = g^{(0)}_{\mu\nu} + \sum_{n=1}^{\infty} \epsilon^n h^{(n)}_{\mu\nu},
\end{align*}
\]

where \( g^{(0)}_{\mu\nu} \) is the background Schwarzschild metric in Schwarzschild coordinates, \( \epsilon \) is a small book-keeping parameter, and \( h^{(n)}_{\mu\nu} \) is the perturbation entering at order \( O(\epsilon^n) \). Note that, although we introduced a single book-keeping parameter \( \epsilon \), there might exist several physical expansion parameters, one for each multipole moment introduced at the leading order.\(^4\)

We focus on stationary and axisymmetric perturbations, expand them in a complete basis of spherical harmonics (reducing to Legendre polynomials \( P_\ell(\cos \theta) \) in axisymmetry) and in the Regge-Wheeler gauge [32].

\[
h^{(n)}_{\mu\nu} = \begin{pmatrix} -fH_0^{\ell} P_\ell & 0 & 0 & h_0^{\ell} P'_\ell \\ 0 & f^{-1} H_2^{\ell} P_\ell & 0 & 0 \\ 0 & 0 & r^2 K^{\ell} P_\ell & 0 \\ h_0^{\ell} P'_\ell & 0 & 0 & r^2 \sin^2 \theta K^{\ell} P_\ell \end{pmatrix},
\]

with \( P'_\ell = \frac{dP_\ell(\cos \theta)}{d\cos \theta} \) and \( f(r) = 1 - 2M/r \). The parameter \( \ell \) is related to the multipolar series in the Legendre polynomials (a sum over \( \ell \) is implicit), whereas \( n \) denotes the order of the perturbative scheme. It is convenient to separate the perturbations in two sets, according to how they transform under parity. The odd (or axial) sector contains only the function \( h_0^{\ell}(r) \), whereas the even (or polar) sector contains the functions \( H_0^{\ell}(r) \), \( H_2^{\ell}(r) \), and \( K^{\ell}(r) \). Owing to the harmonic decomposition, all these functions depend only on the radial coordinate \( r \).

B. Separation of variables

By inserting metric (7) into the vacuum equations, \( R_{\mu\nu} = 0 \), we obtain a set of ordinary differential equations for the perturbation functions \( h^{(n)}_{\mu\nu}(r) \). We solve them analytically by using an extension of the perturbative scheme of Ref. [33].

The odd parity sector is entirely characterized by the functions \( h_0^{\ell} \). At each given order \( n \geq 1 \) the differential equations can be obtained from \( R_{tt} = 0 \) by using the orthogonality of the axisymmetric vector spherical harmonics,

\[
\int_0^\pi d\theta \sin \theta P_\ell'(\cos \theta) R_{t\varphi} = 0,
\]

with \( \ell = 1, 2, 3, \ldots \). Due to the symmetry of the background and of the Einstein’s field equations, this procedure gives a set of purely radial, ordinary differential equations for \( h_0^{\ell} \). These equations are inhomogeneous (for \( n \geq 2 \)) with source terms given by the lower-order functions.

The even parity sector is characterized by the functions \( H_0^{\ell} \), \( H_2^{\ell} \) and \( K^{\ell} \). Similarly to the odd parity case, at each given order \( n \geq 1 \) the differential equations can be obtained from \( E^i \equiv (R_{tt}, R_{t\varphi}, R_{\varphi\varphi}) = 0 \) and by using the orthogonality of the Legendre polynomials \( P_\ell \),

\[
\int_0^\pi \sin \theta P_\ell(\cos \theta) E^i d\theta = 0,
\]

are the dimensionless quantities \( \epsilon J/M^2 \) and \( \epsilon M_2/M^3 \); the latter are independent from one another.

\(^4\) For example, if the spin \( J \) and the mass quadrupole \( M_2 \) are present at the leading order, the physical expansion parameters
FIG. 1. Embedding diagrams for some representative soft ECO solutions presented in the text. We show the embedding of $g_{00}d\Omega^2 = r^2(1 + \sum_{n, \ell} e^{i} K^{nt} P_{\ell}) d\Omega^2$ for surfaces of constant $r_0$ and $t$. The colors are weighted according to $g_{\ell\phi}$ to represent the current multipole moments. The first row contains equatorially-symmetric solutions, as evident by the symmetry between the North and South hemispheres in each diagram. The second line contains nonequatorially symmetric solutions: the differences between the North and South hemispheres are either in the colors (for $S_{\ell}$ with even $\ell$), or in the shape (for $M_{\ell}$ with odd $\ell$). The shape of the $[S_2]$-solution is equatorially symmetric since it does not contain mass multipole moments of odd order. For illustrative purposes the multipolar deviations $\delta M_{\ell}$ and $\delta S_{\ell}$ are chosen artificially large to magnify the effect.

with $\ell = 0, 1, 2, 3, \ldots$. This procedure leads to a set of purely radial, coupled, inhomogeneous differential equations for $H_{0}^{\ell t}$, $H_{2}^{\ell t}$ and $K^{nt}$, where again the source terms are given by the lower order functions.

C. Solutions and multipole moments

The set of ordinary differential equations provided in (9) and (10) can be solved analytically for the metric functions to any given order. As we shall discuss, the structure of the solution is very similar order by order, so we expect that the following results will hold to any order in the perturbative scheme. This is due to the fact that each solution is a polynomial in $M/r$ and also contains terms such as $\log(1 - 2M/r)$ and powers thereof [see Eq. (20) below]. When appearing in the source of the differential equation for the higher-order corrections, these terms give rise to the same polynomial and logarithmic terms. The procedure continues iteratively, although the higher-order solutions are cumbersome. Table I summarizes the notation used in this section. In the following we briefly outline how to obtain the solution and extract the multipole moments.

1. $O(\epsilon)$ terms and $\ell \geq 2$

At linear order, from Eq. (9) we obtain a set of differential equations for the functions $h_{0}^{\ell t}$ and, from Eq. (10), a set of differential equations for the functions $H_{0}^{\ell t}$, $H_{2}^{\ell t}$, $K^{nt}$. For each value of $\ell \geq 2$, these systems can be reduced to a single equation for $H_{0}^{\ell t}$ and to a single equation for $h_{0}^{\ell t}$.

\begin{align*}
\mathcal{D}_1 H_{0}^{\ell t} &= 0, \\
\mathcal{D}_2 h_{0}^{\ell t} &= 0,
\end{align*}
The above equations can be solved analytically, each constant associated with the homogeneous solution of the perturbations given by

\begin{align}
D_1 &\equiv \frac{d^2}{dr_*^2} + \frac{2f}{r} \frac{d}{dr_*} - \left(\ell(\ell+1) + \frac{4M^2}{r^4}\right), \\
D_2 &\equiv \frac{d^2}{dr_*^2} + \frac{4M - \ell(\ell+1)r}{r^2(r - 2M)},
\end{align}

and \(r_*\) is the tortoise coordinate defined by \(dr/dr_* = f\). The above equations can be solved analytically, each function being defined by two arbitrary constants. The first constant can be fixed by imposing asymptotic flatness, whereas the second constant is related to the corresponding multipole moment of order \(\ell\). The large-distance behaviour of the solutions reads

\begin{align}
H_0^{\ell} &\rightarrow -\frac{2M_1^{(1)}}{r^{\ell+1}} + O\left(r^{-(\ell+2)}\right), \\
h_0^{\ell} &\rightarrow \frac{2S_1^{(1)}}{r^{\ell+1}} + O\left(r^{-(\ell+1)}\right),
\end{align}

where \(M_1^{(1)} (S_1^{(1)})\) can be identified with the mass (current) \(\ell\)-pole moment to linear order in the perturbation scheme.

2. \(O(\epsilon^n)\) terms and \(\ell \geq 2\)

To \(O(\epsilon^n)\) the functions that appear in the metric are \(H_0^{\ell}, H_2^{\ell}, K^{n\ell}\) and \(h_0^{n\ell}\). As we discussed above, the equations for these solutions are sourced by the multipoles in the lower order terms. The differential equations can be written as

\[D_1 H_0^{\ell} = T_1^{n\ell}, \quad D_2 h_0^{\ell} = T_2^{n\ell},\]

where \(T_1^{n\ell}\) and \(T_2^{n\ell}\) are the source terms generated by the lower order moments.

To any order the new solution is defined by a free constant associated with the homogeneous solution of Eq. \(17\), which is related to the corresponding multipole moments. In the asymptotic limit, the inhomogeneous solutions read

\begin{align}
H_0^{n\ell} &\rightarrow -\frac{2M_1^{(n)}}{r^{\ell+1}} + O\left(r^{-(\ell+2)}\right), \\
h_0^{n\ell} &\rightarrow -\frac{2S_1^{(n)}}{r^{\ell+1}} + O\left(r^{-(\ell+1)}\right),
\end{align}

where \(M_1^{(n)} (S_1^{(n)})\) is an \(n\)-th order correction to the object’s mass (current) \(\ell\)-pole moment. These quantities are in general free constants that are proportional to the multipole moments which source them.

The analytical expressions of the metric functions are too cumbersome to present them explicitly here but they are provided in an online repository \cite{29}. For any \(n\), their schematic form reads

\[x_{\ell}^{(n)} = \sum_{i=0}^{n} a_i(r) \log(1 - 2M/r)^i,\]

where \(x_{\ell}^{(n)} = (H_0^{\ell}, H_2^{n\ell}, K^{n\ell}, h_0^{n\ell})\) collectively represents all variables, and \(a_i(r)\) are generic polynomials in \(M/r\) which are regular at \(r = 2M\) and depend on the multipole moments of the object.

3. \(\ell = 0, 1\)

For the axial sector of the perturbations, the structure of the solutions described above is still valid even when \(\ell = 1\). However, in the polar sector the case of \(\ell = 0, 1\) is different. For \(\ell = 0\) we can set \(K^{n0} = 0\) for any \(n\) without loss of generality, since the metric is invariant under a transformation \(r \rightarrow A(r) r\). On the other hand, for \(\ell = 1\) the system of ordinary differential equations is underdetermined. This is a consequence of the fact that the \(\ell = 1\) polar perturbations include a displacement of the center of mass, which can be compensated by an appropriate choice of coordinates. To fix this remaining freedom we choose the constant \(K^{n1} = 0\) for any \(n\). With this choice one can solve the system of Eqs. \([9] and \([10]\). The equations for the \(\ell = 0\) and \(\ell = 1\) components of the polar sector can be written schematically as

\begin{align}
\frac{d^2 H_0^{n0}}{dr^2} + \frac{2H_0^{n0}}{r - 2M} = T_1^{n0}, \\
\frac{d^2 H_0^{n1}}{dr^2} + \frac{2H_0^{n1}}{r - 2M} = T_1^{n1},
\end{align}

where the source terms are zero when \(n = 1\). The functions \(H_2^{n0}\) and \(H_2^{n1}\) can be written in terms of \(H_0^{n0}\) and \(H_0^{n1}\).

4. Truncating the multipolar orders

Although our approach does not require to restrict to any given multipolar order, the multipolar expansion

\[H_{\ell}(r) = \sum_{n=0}^{\infty} H_{\ell}^{n}(r),\]

where \(H_{\ell}^{n}(r)\) are the multipole moments of a function.
needs to be truncated in order to obtain a finite set of equations. The order of the truncation is in general arbitrary and depends on the particular solution. For example, in Sec. [IVA 1] below we shall restrict to spin-induced multipole moments and truncate the multipolar order by imposing that only \( \ell = 1 \) perturbations are nonvanishing at \( \mathcal{O}(\epsilon) \), whereas \( \ell > 1 \) perturbations are sourced at higher order.

More in general, the couplings between multipoles follow the standard addition rules for angular momenta in quantum mechanics, so that if one mode with \( \ell_1 \) is present at \( \mathcal{O}(\epsilon) \) and another mode with \( \ell_2 > \ell_1 \) is present at \( \mathcal{O}(\epsilon^2) \), they will source a multipole moment to \( \mathcal{O}(\epsilon^{n+1}) \) with \( \ell \) such that \( \ell_2 - \ell_1 \leq \ell \leq \ell_2 + \ell_1 \), provided some order is not forbidden by the selection rules described in Sec. [II] (For a related discussion, see Ref. [23].)

For example, if only \( \ell = 1 \) is present to \( \mathcal{O}(\epsilon) \), it will source \( \ell = 0, 1, 2 \) to \( \mathcal{O}(\epsilon^2) \), and \( \ell = 0, 1, 2, 3 \) to \( \mathcal{O}(\epsilon^3) \), and so on (this particular case is simply the Hartle-Thorne small-spin expansion \( \mathcal{O}(\epsilon) \), cf. Sec. [IVA 1]). On the other hand, if only \( \ell = 1, 2 \) are present to \( \mathcal{O}(\epsilon) \), they will source \( \ell = 0, 1, 2, 3, 4 \) to \( \mathcal{O}(\epsilon^2) \), and \( \ell = 0, 1, 2, 3, 4, 5, 6 \) to \( \mathcal{O}(\epsilon^3) \), and so on. In general, if the solution contains moments up to multipolar index \( \ell = L \) to the leading order, then moments up to index \( \ell = n \times L \) will be induced to \( \mathcal{O}(\epsilon^n) \).

Thus, building a consistent solution requires to include at each given order \( n \) in \( \epsilon \ll 1 \) a sufficiently high number of terms in the multipolar expansion, depending on initial assumptions for the system under consideration. As we shall discuss, for some solutions the number of terms in the ansatz can be significantly reduced by making use of some parity rules.

### III. PROPERTIES OF THE QUASI-SCHWARZSCHILD SOFT ECO METRIC

A useful graphical representation of the multipolar structure of the general solution is given in Table [II] Each row represents the multipole moments of order \( \ell \) while each column represents the \( \mathcal{O}(\epsilon^n) \) correction to a given multipole. We can identify each entry of Table [II] with a set \( (n, \ell) \) corresponding to its parameters. Thus, each cell \( (n, \ell) \) represents a \( n \)th-order correction to a multipole moment of order \( \ell \). Furthermore, each entry is divided into two subcells. The upper right part represents the mass multipole moment contribution, whereas the bottom left part represents the current multipole moment contribution. The solution given by ansatz [17] contains all multipolar contributions represented in Table [II] Each line of Table [II] can be reduced to two physical parameters, \( M_\ell \) and \( S_\ell \), describing the physical mass and current multipoles of that order.

In addition to the sum rules for the multipolar index \( \ell \) at each perturbative order, our framework also enjoys two sets of symmetries, which are related to the properties of each term under a parity transformation and under a reflection on the equatorial plane, respectively:

- **Parity rule**: Since mass (current) multipole moments are related with even (odd) functions in the metric decomposition [cf. Eqs (18)–(19)], terms associated to mass multipole moments are parity-even, whereas those associated to current multipole moments are parity-odd. The coupling between two parity-even (-odd) terms gives rise to a parity even term, whereas the coupling between a parity-even and a parity-odd term gives rise to a parity-odd term.

- **\( \mathbb{Z}_2 \) rule**: Terms associated to \( M_\ell \ (S_\ell) \) are symmetric under a reflection on the equatorial plane when \( \ell \) is even (odd), whereas they change sign when \( \ell \) is odd (even). Thus, as previously mentioned, equatorial symmetry of the full metric implies \( M_\ell = 0 \ (S_\ell = 0) \) when \( \ell \) is odd (even). In the general case, these restrictions do not hold and the solution enjoys a simple “\( \mathbb{Z}_2 \) rule”: the coupling between two terms that are both even or odd under this transformation gives rise to a \( \mathbb{Z}_2 \)-even term, whereas the coupling between an even and an odd term gives rise to a \( \mathbb{Z}_2 \)-odd term. In other words, the coupling between two (non)equatorially-symmetric moments gives rise to equatorially-symmetric moments, whereas the coupling between an equatorially-symmetric moment and a nonequatorial one gives rise to nonequatorially-symmetric moments.

The above rules, together with the addition rules of angular momenta, strongly constrain the types of multipole moments that can be induced for each perturbative solution. For example, to order \( \mathcal{O}(\epsilon^2) \) a \( M_4 \) moment cannot be induced by the coupling between \( M_2 \) and \( S_2 \), even if the angular-momentum sum rules would allow for it. Indeed, both \( M_2 \) and \( M_4 \) are \( \mathbb{Z}_2 \) even while \( S_2 \) is \( \mathbb{Z}_2 \) odd. Likewise, a \( S_3 \) moment (which is parity odd) cannot be induced by \( M_2 \) (which is parity even) to order \( \mathcal{O}(\epsilon^2) \), even if both the angular-momentum sum and the \( \mathbb{Z}_2 \) rules would allow for it. Explicit solutions in which all these rules are at play are discussed in Sec. [IV] below.

As we shall discuss in the next section, our solution can be reduced to the Hartle-Thorne small-spin expansion if we consider all multipole moments to be spin induced and take the spin to be \( \mathcal{O}(\epsilon) \). In this particular case the only nonzero entries in Table [II] are those identified in blue. Due to the aforementioned \( \mathbb{Z}_2 \) rule, these terms are all equatorially symmetric since are sourced by powers of the spin \( \mathcal{J} \equiv S_1 \), which is \( \mathbb{Z}_2 \) even.

It is also informative to compare our quasi-Schwarzschild solution to known exact vacuum solutions to Einstein’s equation, for example with the Munko-

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5 The row corresponding to \( \ell = 1 \) is characterized by one parameter only, since the mass dipole can be set zero without loss of generality.
Novikov metric \[35\] with arbitrarily large multipole moments \[20\]. The Manko-Novikov solution has a multipolar structure that in Table II can be identified by the red cells. As clear from the table, this solution is not a particular case of the Hartle-Thorne approximation – even in the limit of small multipole moments – but it is actually orthogonal to it. This is consistent with the fact that, in the small-multipole moment limit, the Manko-Novikov solution contains quadratic spin terms in the multipole moments equal to or higher than \(\ell = 4\) (see, e.g., Ref. \[36\]), which is not the case for the Hartle-Thorne metric, since in that case \(M_\ell \sim \chi^\ell\) and \(S_\ell \sim \chi^\ell\) (at least) to the leading order \[27\] \[28\] \[33\] \[37\] (see discussion below). Also in this case, the selection and \(\mathbb{Z}_2\) rules discussed above enforce the solution to be equatorially symmetric.

Besides these particular cases, the solution resulting from ansatz \(\Theta\) is generically not equatorially symmetric, as can be seen by the presence of even (odd) current (mass) multipole moments in Table II. Some specific nonequatorially symmetric solutions are discussed in Sec. \[IVB\].

### IV. PARTICULAR SOLUTIONS

We have explicitly computed six different solutions, which we divide in two sets. The first set (Sec. \[IVA\]) are three equatorially-symmetric solutions: (i) a small-multipole solution up to \(O(\epsilon^3)\) assuming only nonvanishing \(J\) at linear order (cf. Sec. \[IVA1\]); (ii) a solution up to \(O(\epsilon^3)\) built by assuming that only \(M_2\) is present at linear order (cf. Sec. \[IVA2\]); and (iii) a more general solution up to \(O(\epsilon^2)\) which assumes that both \(J\) and \(M_2\) are present at linear order (cf. Sec. \[IVA3\]). The second set (Sec. \[IVB\]) comprises three nonequatorially symmetric solutions: (iv) a solution up to \(O(\epsilon^3)\) built with the assumption that at linear order only the \(S_2\) multipole moment is present (cf. Sec \[IVB1\]); (v) a solution up to \(O(\epsilon^2)\) assuming both \(J\) and \(S_2\) at linear order (cf. Sec. \[IVB3\]); (vi) a more general solution up to \(O(\epsilon^2)\) constructed with the combination of \(J\), \(M_2\) and

| \(\ell = 0\) | \(\epsilon^1\) | \(\epsilon^2\) | \(\epsilon^3\) | \(\epsilon^4\) | \(\epsilon^5\) |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| \(\ell = 1\) | | | | | |
| \(S_1^{(1)}\) | 0 | | | | |
| \(M_1^{(1)}\) | | | | | |
| \(M_1^{(2)}\) | | | | | |
| \(S_1^{(2)}\) | | | | | |
| \(S_1^{(3)}\) | | | | | |
| \(M_1^{(3)}\) | | | | | |
| \(M_1^{(4)}\) | | | | | |
| \(M_1^{(5)}\) | | | | | |
| \(\ell = 2\) | | | | | |
| \(S_2^{(1)}\) | | | | | |
| \(M_2^{(1)}\) | | | | | |
| \(M_2^{(2)}\) | | | | | |
| \(S_2^{(2)}\) | | | | | |
| \(M_2^{(3)}\) | | | | | |
| \(M_2^{(4)}\) | | | | | |
| \(M_2^{(5)}\) | | | | | |
| \(\ell = 3\) | | | | | |
| \(S_3^{(1)}\) | | | | | |
| \(M_3^{(1)}\) | | | | | |
| \(M_3^{(2)}\) | | | | | |
| \(S_3^{(2)}\) | | | | | |
| \(M_3^{(3)}\) | | | | | |
| \(M_3^{(4)}\) | | | | | |
| \(M_3^{(5)}\) | | | | | |
| \(\ell = 4\) | | | | | |
| \(S_4^{(1)}\) | | | | | |
| \(M_4^{(1)}\) | | | | | |
| \(M_4^{(2)}\) | | | | | |
| \(S_4^{(2)}\) | | | | | |
| \(M_4^{(3)}\) | | | | | |
| \(M_4^{(4)}\) | | | | | |
| \(M_4^{(5)}\) | | | | | |
| \(\ell = 5\) | | | | | |
| \(S_5^{(1)}\) | | | | | |
| \(M_5^{(1)}\) | | | | | |
| \(M_5^{(2)}\) | | | | | |
| \(S_5^{(2)}\) | | | | | |
| \(M_5^{(3)}\) | | | | | |
| \(M_5^{(4)}\) | | | | | |
| \(M_5^{(5)}\) | | | | | |

TABLE II. Representation of the multipolar structure of a soft ECO (described by the metric \(\Theta\)) up to \(O(\epsilon^5)\) and \(\ell \leq 5\). Each column corresponds to the different multipole moments at a given perturbation order, while each line contains the different higher-order corrections to a given \(\ell\)-pole. Each cell entry \((n, \ell)\) is divided into two: the upper-right (lower-left) entry corresponds to the \(n\)-th order coefficient of the mass (current) \(\ell\)-pole, \(M_\ell^{(n)}\) \((S_\ell^{(n)})\). Since our ansatz for metric \(\Theta\) does not assume equatorial symmetry, all the entries in this table are present in the solution, with the exception of \(M_1^{(n)}\), which can always be set to zero without loss of generality. The blue (red) cells correspond to the entries present in the Hartle-Thorne (Manko-Novikov) solution, whereas cells which are half blue and half red represent entries present in both of these particular cases.
the $\mathcal{S}_2$ at linear order (cf. Sec. [V B 3]). The explicit form of these metrics is provided in an online notebook [29], whereas some illustrative embedding diagrams are given in Fig. [1].

Since in this perturbative scheme each solution is completely characterized by its leading-order corrections only, we shall denote it by the multipole moments that exist at the linear level. Schematically, we shall denote as $[\mathcal{M}_1, \mathcal{M}_2, \mathcal{S}_1, \mathcal{S}_2, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4, \mathcal{J}_5, \mathcal{J}_6, \ldots]$ a solution which has $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{S}_1$, $\mathcal{S}_2$, etc nonvanishing multipole moments (besides the mass) at the linear order. With this nomenclature, the six solutions described above are denoted as $[\mathcal{J}], [\mathcal{M}_2], [\mathcal{J}\mathcal{M}_2], [\mathcal{S}_2], [\mathcal{J}\mathcal{S}_2]$ and $[\mathcal{J}\mathcal{M}_2\mathcal{S}_2]$, respectively. We stress that the quantities within brackets are the multipole moments entering at the leading order, but the full solution will contain more moments that are induced by the fundamental ones.

We also remark that, beyond $\mathcal{O}(\epsilon)$, a linear combination of two solutions is not a solution of the field equations, e.g., we cannot simply combine the $[\mathcal{J}]$-solution and the $[\mathcal{M}_2]$-solution to obtain the $[\mathcal{J}\mathcal{M}_2]$-solution. This is a consequence of the nonlinearity of Einstein’s equations. For example, the $[\mathcal{J}\mathcal{M}_2]$-solution contains terms proportional to $\mathcal{J}\mathcal{M}_2$ at $\mathcal{O}(\epsilon^2)$, which do not appear in a linear combination of the $[\mathcal{J}]$ and $[\mathcal{M}_2]$ solutions.

To simplify the notation, in the following we will use the dimensionless multipole moments

$$\hat{\mathcal{M}}_\ell = \frac{\mathcal{M}_\ell}{\mathcal{M}^{\ell+1}}, \quad \hat{\mathcal{S}}_\ell = \frac{\mathcal{S}_\ell}{\mathcal{M}^{\ell+1}}. \quad (23)$$

A. Equatorially symmetric case

1. $[\mathcal{J}]$-solution

Let us assume that the multipole moments are sourced only by the object’s rotation. Thus, to linear order in the spin, corresponding to $\mathcal{O}(\epsilon)$, the multipolar structure of the object is modified only by means of the body’s angular momentum, and all multipole moments with $\ell \geq 2$ are spin induced. This allows us to truncate the multipole moments of $\mathcal{O}(\epsilon)$ at $\ell = 1$, i.e. $H_0^{1\ell} = H_2^{1\ell} = K_0^{\ell\ell} = h_0^{\ell\ell} = 0$ for any $\ell > 1$. This solution corresponds to the Hartle-Thorne approximation for the external spacetime of slowly-rotating objects in general relativity [27, 28, 37].

The properties of this solution are well-known but for the sake of clarity we will summarize them here. Due to the $\mathbb{Z}_2$ rules presented in Sec. [III], this solution is equatorially symmetric. Furthermore, due to the addition rules of angular momenta, to second order in the spin, the object’s angular momentum ($\ell = 1$) will source a mass quadrupole ($\ell = 2$) and a correction to the object’s mass ($\ell = 0$). Similarly, to third order in the spin, it will source a current octupole ($\ell = 3$) and a correction to the object’s angular momentum ($\ell = 1$). The corrections to the lower-order moments (mass and angular momentum) can be reabsorbed by defining the physical quantities $M = M + \epsilon^2 M_0^{(2)} + \mathcal{O}(\epsilon^4)$ and $J = J + \epsilon^3 J_1^{(3)} + \mathcal{O}(\epsilon^5)$. We can summarize the structure of the equations in the following form: to each even (odd) order $n$, the spin will source a mass (current) multipole moment of order $\ell = n$, and will also source corrections to the lower mass (current) multipole moments. Schematically, the spin-induced multipole moments of this solution read, to the leading order in the spin expansion:

$$\hat{\mathcal{M}}_\ell \propto \chi^\ell + \ldots, \quad \text{even } \ell \geq 2 \quad (24)$$

$$\hat{\mathcal{S}}_\ell \propto \chi^\ell + \ldots, \quad \text{odd } \ell \geq 2 \quad (25)$$

where the prefactors are model-dependent constants that depend only on the compactness, $\mathcal{M}_\ell = 0$ ($\mathcal{S}_\ell = 0$) for odd (even) $\ell$, and we have defined the dimensionless moments as in Eq. (23). This multipolar structure corresponds to the blue cells in Table [1]. We note that the exterior solution of a slowly-spinning object in general relativity has been computed up to $\mathcal{O}(\chi^4)$ in Ref. [37]; our $[\mathcal{J}]$ metric extends that result to $\mathcal{O}(\chi^5)$ and might therefore be useful to construct more precise models of slowly-spinning neutron stars.

2. $[\mathcal{M}_2]$-solution

One of the simplest extensions to the previous model consists in assuming that the multipole moments are induced not by the spin but by the mass quadrupole moment. Similarly to the role of the spin in the case above (Sec. [IV A 1]), the quadrupole moment $\mathcal{M}_2$ will appear in the source terms of the higher-order equations. For example, at second order in the expansion, the coupling between two mass quadrupole terms will source an $\mathcal{M}_4$ multipole moment and subleading corrections to the quadrupole and the mass. In a similar way, at third order the coupling between the multipoles will source an $\mathcal{M}_6$ multipole moment and corrections to all the existing lower order multipole moments. As discussed previously, the subleading corrections to the multipole moments existing to the leading order (mass monopole and mass quadrupole) can be reabsorbed in the definition of the physical multipole moments. In contrast to the $[\mathcal{J}]$-solution, this case does not contain current multipole moments. At each order $n$ only mass multipole moments up to order $\ell = 2n$ will be sourced. Schematically, the quadrupole-induced multipole moments of this solution read:

$$\hat{\mathcal{M}}_\ell \propto (\mathcal{M}_2)^{\ell/2} + \ldots, \quad \text{even } \ell \geq 4 \quad (26)$$

$$\hat{\mathcal{S}}_\ell = 0, \quad \text{any } \ell, \quad (27)$$

---

6 Generically, the spin-induced moments $\hat{\mathcal{M}}_\ell$ and $\hat{\mathcal{S}}_\ell$ might also get corrections at order $\chi^{\ell+2n}$ ($n = 1, 2, 3, \ldots$), but these are subleading in the small-spin expansion. Henceforth we shall denote the possible subleading corrections with ellipsis.
whereas \( M_\ell = 0 \) for odd \( \ell \). Again, the ellipsis in the above equation refer to the subleading corrections entering at higher order in the \( M_2 \ll 1 \) expansion, and the prefactors are model-dependent constants that depend only on the compactness.

3. \([JM_2]\)-solution

Another interesting situation comes from the combination of the two cases above, namely when both the spin and the quadrupole moment act as source terms. Obviously this solution must reduce to the \([J]\)-solution (\([M_2]\)-solution) when we take \( M_2 = 0 \) (\( J = 0 \)). However, it is not a simple linear combination of \([J]\) and \([M_2]\), since it also contains mixed coupling terms at \( \mathcal{O}(\epsilon^2) \), i.e. terms proportional to \( JM_2 \). For example, at second order in the expansion, this coupling will source a current octupole moment, \( S_3 \), and a higher-order correction to the object’s angular momentum, which can be reabsorbed as in the \([J]\) solution. For each order \( n \) this solution contains mass multipole moments for even \( \ell \) and current multipole moments for odd \( \ell \) up to \( \ell = 2n \).

The induced multipole moments of this solution can be written as a combination of terms sourced by the spin and by the quadrupole:

\[
\mathcal{M}_\ell := \sum_{p=0}^{\ell/2} \alpha_p \chi^{\ell-2p} (M_2)^p + \ldots, \quad \text{even } \ell \geq 4
\]

\[
\bar{S}_\ell := \sum_{p=0}^{(\ell-1)/2} \beta_p \chi^{\ell-2p} (M_2)^p + \ldots, \quad \text{odd } \ell \geq 3,
\]

whereas \( M_\ell = 0 \) (\( \bar{S}_\ell = 0 \)) for odd (even) \( \ell \). The prefactors \( \alpha_p \) and \( \beta_p \) are model-dependent, dimensionless constants that depend only on the compactness, and the ellipsis refers to subleading terms.

The above formulas can be understood as follows. The angular-momentum addition rules imply that, to the leading order, \( M_\ell \sim \chi^\ell (M_2)^p \), with \( p + 2p = \ell \) since \( \chi \) is a \( \ell = 1 \) moment and \( M_2 \) is a \( \ell = 2 \) moment. By replacing \( q = \ell - 2p \) for each order \( p \), we obtain Eq. (28).

A similar argument applies to the derivation of Eq. (29).

For example, in this solution the (dimensionless) induced \( M_4 \) up to \( \mathcal{O}(\epsilon^4) \) reads

\[
M_4 = \alpha_2 (M_2)^2 + \alpha_1 \chi^2 \bar{M}_2 + \alpha_0 \chi^4,
\]

where the second term on the right-hand side is an example of the aforementioned coupling terms between two different multipole moments existing at the leading order.

B. Nonequatorially symmetric case

1. \([S_2]\)-solution

The simplest nonequatorially symmetric solution can be obtained by including only a current quadrupole moment at the leading order; i.e., higher-order multipoles will be sourced exclusively by \( S_2 \). Due to the angular momentum and the \( \mathbb{Z}_2 \) selection rules described in Sec. III, the current quadrupole moment will source, to second order in the perturbation expansion, the (equatorial-symmetric) moments \( M_4 \) and \( M_2 \), and also a subleading correction to \( M \). Using the same selection rules we find that third-order couplings between these multipole moments will source the nonequatorially symmetric terms \( S_4, S_6 \) and a subleading correction to \( S_2 \). As in the above cases, the subleading corrections to the fundamental moments (mass monopole and current quadrupole) can be consistently reabsorbed in the definition of the physical multipole moments \( M \) and \( S_2 \). Although we computed the explicit solution up to third order in \( S_2 \ll 1 \), from the structure of the solution and the selection rules it is easy to find a peculiar general pattern, namely that the current multipole moments appear only at odd orders in the expansion while the mass multipole moments are sourced only at even orders in the expansion. Furthermore, it can also be shown that odd order terms are nonequatorially symmetric while even order terms are equatorially symmetric.

Since this solution sources two new multipole moments at each perturbation order, a schematic form for current quadrupole-induced multipole moments is more involved than in the previous cases. Schematically, we obtain

\[
\mathcal{M}_\ell \propto (S_2)^{\ell/2+1,2} + \ldots \quad \text{even } \ell \geq 2
\]

\[
\bar{S}_\ell \propto (S_2)^{\ell/2,2} + \ldots \quad \text{even } \ell \geq 4
\]

whereas \( M_\ell = S_\ell = 0 \) for odd values of \( \ell \). In the above expressions \( [x,2] := \max\{2n \in \mathbb{Z} \mid 2n \leq x \} \). More explicitly, the above relations imply the multipole moments are sourced in pairs: \( M_2 \propto (S_2)^2, M_4 \propto (S_2)^2, M_6 \propto (S_2)^4, M_8 \propto (S_2)^4, \) and likewise \( \bar{S}_4 \propto (S_2)^3, \bar{S}_6 \propto (S_2)^3, \bar{S}_8 \propto (S_2)^5, \bar{S}_{10} \propto (S_2)^5, \) and so on.

2. \([JS_2]\)-solution

In this case, the quadrupole moment \( S_2 \) plays the same role of the mass quadrupole moment in the equatorially-symmetric case \([JM_2]\) (Sec. IV A 3), and will appear in the source terms of the higher-order equations. For example, at second order in the perturbation expansion, the coupling between the (nonequatorially-symmetric) current quadrupole moment \( S_2 \) and the (equatorial-symmetric) angular momentum \( J \) will source a (nonequatorially-symmetric) mass octupole moment \( M_3 \).

\footnote{In general also \( M_1 \) would be sourced, however, choosing specific values for the integration constants of the homogeneous part of the equation we can set \( M_1 = 0 \), as expected since the mass dipole can be gauged away in general relativity.}
With this result in mind we can easily generalize the structure of the \([\mathcal{J}\mathcal{S}_2]\)-solution in the following way. It describes a quasi-spherical symmetric body without equatorial symmetry with just two independent parameters (besides the mass): its spin \(\mathcal{J}\), and its current quadrupole \(\mathcal{S}_2\). To each even (odd) order \(n > 2\), these two multipole moments will source a mass (current) multipole moment of order \(\ell = n\), and will also source corrections to all lower mass (current) multipole moments which can be consistently reabsorbed. Up to \(O(\epsilon^3)\), the first induced multipole moments of this solution can be written as a combination of terms sourced by spin and by the current quadrupole:

\[
\mathcal{M}_3 = a_3 \chi (\tilde{S}_2), \quad (33)
\]
\[
\mathcal{M}_4 = c_4 (\tilde{S}_2)^2 + b_4 \chi^4, \quad (34)
\]
\[
\tilde{S}_4 = d_4 \chi^3, \quad (35)
\]
\[
\tilde{S}_4 = f_4 (\tilde{S}_2)^3 + g_4 \chi^2 (\tilde{S}_2), \quad (36)
\]
where the prefactors \(a_\ell, b_\ell, c_\ell\) and \(d_\ell, f_\ell, g_\ell\) are again model-dependent constants that depend only on the compactness.

3. \([\mathcal{J}\mathcal{M}_2\mathcal{S}_2]\)-solution

The most general solution which contains all multipole moments with \(\ell \leq 2\) at the leading order is the \([\mathcal{J}\mathcal{M}_2\mathcal{S}_2]\)-solution. This solution is characterized by three independent “hairs” (in addition to the mass) – the angular momentum, the mass quadrupole moment, and the current quadrupole moment – the combination of which will source higher-order moments accordingly to the aforementioned selection rules. Similarly to the two previous cases, the presence of a nonvanishing current quadrupole moment breaks the \(\mathbb{Z}_2\) symmetry of the solution. Indeed, all previous cases discussed so far are included in this solution and can be obtained by setting some of the independent parameters to zero (e.g. the \([\mathcal{J}\mathcal{M}_2\mathcal{S}_2]\)-solution reduces to \([\mathcal{J}\mathcal{M}_2]\) if we set \(\mathcal{S}_2 = 0\)). However, as mentioned before, a linear combination of the previous models is not sufficient to describe this solution nor will it be in general a solution of Einstein’s field equations.

The structure of the solution is similar to the previous ones with the exception of new coupling terms that appear at higher order, as a consequence of the nonlinearity of Einstein’s equations. In addition to the couplings described in the former cases, this solution will contain a mixed coupling between \(\mathcal{S}_2\) and \(\mathcal{M}_2\) which will source higher-order multipole moments. Due to the selection rules described in Sec. III, the coupling between the former (nonequatorial-symmetric) and the latter (equatorial-symmetric) will induce two nonequatorially symmetric terms with \(\ell = 2\) and \(\ell = 4\) at second order in the perturbative expansion. Thus, at second order the coupling \(\mathcal{S}_2\mathcal{M}_2\) sources an \(\mathcal{S}_4\) and a subleading (and re-absorbable) correction to \(\mathcal{S}_2\).

Up to \(O(\epsilon^4)\), the first induced multipole moments of this solution can be written as a combination of terms sourced by spin, the mass quadrupole, and the current quadrupole:

\[
\mathcal{M}_3 = a_{110} \chi \tilde{S}_2 + a_{111} \chi \tilde{S}_2 \mathcal{M}_2 + a_{310} \chi^3 \tilde{S}_2
\]
\[
+ a_{130} \chi (\tilde{S}_2)^3 + a_{112} \chi \tilde{S}_2 (\mathcal{M}_2)^2, \quad (37)
\]
\[
\mathcal{M}_4 = a_{020} (\tilde{S}_2)^2 + a_{002} (\mathcal{M}_2)^2 + a_{021} (\tilde{S}_2)^2 \mathcal{M}_2
\]
\[
+ a_{201} \chi^2 \mathcal{M}_2 + a_{400} \chi^4 + a_{040} (\tilde{S}_2)^4 + a_{003} (\mathcal{M}_2)^4 + a_{220} \chi^2 (\tilde{S}_2)^2 + a_{202} (\mathcal{M}_2)^2 + a_{022} (\tilde{S}_2)^2 (\mathcal{M}_2)^2, \quad (38)
\]
\[
\tilde{S}_3 = a_{101} \chi \mathcal{M}_2 + a_{300} \chi^3 + a_{102} (\mathcal{M}_2)^2 + a_{120} (\tilde{S}_2)^2
\]
\[
+ a_{301} \chi^2 \mathcal{M}_2 + a_{103} (\mathcal{M}_2)^3 + a_{121} (\tilde{S}_2)^2 \mathcal{M}_2, \quad (39)
\]
\[
\tilde{S}_4 = a_{011} \tilde{S}_2 \mathcal{M}_2 + a_{030} (\tilde{S}_2)^3 + a_{230} \chi^2 \tilde{S}_2
\]
\[
+ a_{012} (\tilde{S}_2)^2 \mathcal{M}_2 + a_{031} (\tilde{S}_2)^3 \mathcal{M}_2 + a_{211} \chi^2 \tilde{S}_2 \mathcal{M}_2, \quad (40)
\]
where the prefactors \(a_{ijk}\) are again model-dependent constants that depend only on the compactness, and the subscripts are assigned such that the terms are of the form \(a_{ijk} \chi^i (\tilde{S}_2)^j (\mathcal{M}_2)^k\).

V. HAIR CONDITIONER FOR ECO: SOFT HAIR IN THE BH LIMIT

With the explicit form of the soft ECO solutions at hand we can now study the magnitude of the curvature on the surface of the ECO, and thus explore under what conditions the multipole moments constitute soft hair. To this end, we look at two curvature invariants, namely the Kretschmann scalar \(K \equiv R_{abcd} R^{abcd}\) and the Pontryagin scalar, \(\ast RR \equiv \frac{1}{2} R_{abcd} \ast R^{abcd}\). These are the leading-order nonvanishing curvature scalars, since our solutions satisfy \(R_{\mu
u} = 0\). The general expression of these invariants is cumbersome; however, they can be schematically decomposed into terms that are regular at \(r = 2\mathcal{M}\) and terms that are divergent. The terms containing a curvature singularity at \(r = 2\mathcal{M}\) must vanish identically in the BH case, yielding the unique Kerr solution.

However, the singularity is avoided if the object is just slightly less compact than a BH, i.e. if it has a radius \(r_0 > 2\mathcal{M}\). This is the case (for instance) of perfect-fluid stars, whose deformations can also be described by our solution. An interesting question is therefore what happens for an object with \(r_0 = 2\mathcal{M}(1 + \delta)\) in the \(\delta \to 0\) limit. Here, \(r_0\) is the proper circumferential radius of a circle around the axis of symmetry \([26]\), so

\[
\delta = \frac{r_0}{2\mathcal{M}} - 1 \quad (41)
\]
is defined in terms of geometrical quantities. Likewise, one could also introduce \([35]\) the proper radial distance
\[ \Delta = \int_{2M}^{r_0} dr \sqrt{g_{rr}} \approx 4M \sqrt{\delta} + \mathcal{O}(\delta), \] which is directly related to \( \delta \) in a model-independent way in the \( \epsilon \to 0 \) and \( \delta \to 0 \) limits.

Multipole moments that would have to vanish in the Kerr limit are ECO hair. We will determine what their size can be, as a function of \( \delta \), in order for the curvature of the solution not to be large (on the scale of \( M \)). Under these conditions the ECO hair will remain soft.

1. Multipole moments sourced by \( \ell = 1 \) moments

As a starting point we focus on the case of spin-induced multipole moments. For this purpose we will use the \([J]\)-solution described in Sec. [VAR] and study its curvature invariants. Up to the linear order, the only multipole moments existing in the solution are the mass and the spin of the object and thus the curvature invariants are regular everywhere. At second order a spin-induced quadrupole moment is sourced. In the \( r_0 \to 2M \) limit, the second-order correction to the Kretschmann scalar at the surface reads

\[ \mathcal{K}^{(2)} \sim -\frac{45 \chi^2}{4M} (1 + A_2(\delta)) \log \left(1 - \frac{2M}{r_0}\right) P_2(\theta), \]

where we have explicitly factored out the spin dependence by defining

\[ \mathcal{M}_2 = A_2(\delta) \chi^2 + \mathcal{O}(\chi^4), \]

with \( A_2(\delta) \) being a function of \( \delta \) only. The logarithmic term above forces the curvature to blow up when \( r_0 \to 2M \) unless \( 1 + A_2(\delta) \to 0 \) logarithmically, or faster. That is, \( 1 + A_2(\delta) \to a_2/\log(\delta) \) or faster, where \( a_2 \) is a model-dependent constant at most of order unity. In other words, requiring that the curvature does not blow up in the BH limit imposes

\[ \mathcal{M}_2 \to \mathcal{M}_2^{\text{BH}} + a_2 \frac{\chi^2}{\log \delta}, \]

or faster as \( \delta \to 0 \), for a purely spin-induced solution.

Here

\[ \mathcal{M}_2^{\text{BH}} = -\chi^2 \]

is the mass quadrupole moment of a Kerr BH [see Eq. [1]]. Thus, as the ECO’s surface approaches the horizon, the ECO’s multipole moments must approach the BH multipole moments as \( 1/\log \delta \) or faster if the curvature is to remain of the same order as the curvature at the horizon of the corresponding BH (i.e., for a soft ECO).

By extending this analysis to higher-order terms we find that this behaviour occurs for all spin-induced multipole moments. As an example, to third order in the perturbation scheme the corrections to the Pontryagin scalar take the form

\[ ^*RR^{(3)} \sim -\frac{315 \chi^3}{4M^2} (1 + C_3(\delta)) \log \left(1 - \frac{2M}{r_0}\right) P_3(\theta), \]

where similarly to the previous case we have defined

\[ \mathcal{S}_3 = C_3(\delta) \chi^3 + \mathcal{O}(\chi^5). \]

Thus, the Pontryagin scalar blows up in the BH limit unless \( 1 + C_3(\delta) \to c_3/\log(\delta) \) or faster, as \( \delta \to 0 \). In other words, the spin-induced current octupole moment of a “soft” ECO must vanish as

\[ \mathcal{S}_3 \to \mathcal{S}_3^{\text{BH}} + c_3 \frac{\chi^3}{\log \delta}, \]

or faster, as \( \delta \to 0 \). Again, \( c_3 \) is a new model-dependent constant of order unity or smaller, and

\[ \mathcal{S}_3^{\text{BH}} = -\chi^3 \]

corresponds to the Kerr current octupole moment.

These results lead to the conclusion that if an ECO is “soft” its spin-induced multipole moments must vanish logarithmically (or faster) as a function of \( \delta \to 0 \), as anticipated by the first terms in Eqs. [5] and [6].

2. Multipole moments sourced by \( \ell > 1 \) moments

An interesting question is how do the induced multipoles vanish when they are sourced by multipole moments higher than the spin, e.g. by a leading-order quadrupole moment. Let us consider the nonequatorially symmetric \([S_2]\)-solution as a simple representative example. In this case the curvature invariants have pathologies at \( r = 2M \) already at the linear order in the perturbative expansion,

\[ ^*RR^{(1)} \sim \frac{90 S_2}{M^4 r_0^3} \left[ \log \left(1 - \frac{2M}{r_0}\right) \right] P_2(\theta). \]

Using the same arguments as in the previous case, the current quadrupole moment of a soft ECO must go to zero as

\[ S_2 \sim 1/\log \delta, \]

or faster, as \( \delta \to 0 \). This result contrasts with the previous scenario where at linear order there was no restriction on the \( \ell = 1 \) current multipole moment (i.e., the spin). By analyzing the second-order corrections to the Kretschmann scalar, we find that the curvature diverges in the BH limit as \( \delta^{-2} \).
\[
\mathcal{K}^{(2)} \sim \frac{225}{256 \mathcal{M}^2 (r_0 - 2 \mathcal{M})^2} \mathcal{S}_2^2 \sin \theta + \ldots - \frac{\mathcal{S}_2^2}{\mathcal{M}^2} \left( \frac{45}{28} (7B_2(\delta) + 5) P_2(\theta) + \frac{135}{56} (147B_4(\delta) + 202) P_4(\theta) \right) \log \delta, \tag{52}
\]

where the ellipsis accounts for other regular second-order terms. Similarly to the previous cases, in the above equation we defined

\[
\mathcal{M}_2 = B_2(\delta) \mathcal{S}_2^2 + O(\mathcal{S}_2^4), \quad \mathcal{M}_4 = B_4(\delta) \mathcal{S}_2^2 + O(\mathcal{S}_2^4).
\tag{53}
\]

Therefore, in this case we find that the saturation of Eq. (51) is not enough to prevent the curvature to blow up in the BH limit: the current quadrupole \( \mathcal{S}_2 \) must vanish as

\[
\mathcal{S}_2 \to d_2 \delta, \quad \tag{54}
\]
or faster, as \( r_0 \to 2 \mathcal{M} \). A similar argument can be presented for any multipole moment with \( \ell > 1 \) entering at the leading order.

The last term on the right hand side of Eq. (52), together with Eq. (54) and the factorization (53), would impose that the multipole moments \( \mathcal{M}_2 \) and \( \mathcal{M}_4 \) should vanish at least logarithmically in the \( \delta \to 0 \) limit. Nonetheless, by extending this analysis to third order it is easy to show that they also must go to zero linearly,

\[
\mathcal{M}_2 \to b_2 \delta, \quad \mathcal{M}_4 \to b_4 \delta, \quad \tag{55}
\]
or faster, similarly to the behaviour found for \( \mathcal{S}_2 \). (In the above equation \( b_\ell \) are again dimensionless constants which depend on the model.) Therefore, in this case all multipole moments must vanish linearly or faster.

The above argument applies to any solution sourced by a single multipole moment with \( \ell > 1 \), showing that multipolar deviations from Kerr sourced by multipole moments other than the spin must vanish linearly or faster as \( \delta \to 0 \). This striking difference relative to the spin-induced case is due to the fact that, already to first order, the solution contains irregular terms that must be suppressed as in Eq. (50). Since the leading-order moments with \( \ell > 1 \) should already vanish logarithmically to the leading order, they induce power-law scalings in the curvature to next-to-the-leading order. The net result is that all moments should vanish at least linearly in this case. This does not happen in the spin-induced case, where the first-order solution is everywhere regular, \( g_{\theta \phi} = 2 J \sin^2 \theta / r \), and therefore induces only logarithmic corrections at higher orders.

The discussion above was inferred from particular solutions where only one multipole moment was present to the leading order. However, as previously discussed, when two or more multipole moments are present at the leading order, the higher-order multipole moments can also be induced by mixing terms. One particular case of interest is the vanishing of a multipole moment induced simultaneously by spin-spin couplings and by the couplings with higher multipole moments. A representative case of this scenario is the \( \mathcal{M}_4 \) moment in the \([J,M_2]\)-solution, see Eq. (39). Although we did not compute the solution up to sufficiently high orders in the perturbative expansion to check explicitly the coupling terms in the curvature, results from the lower orders and the selection rules imply that the first two terms in Eq. (30) must vanish linearly (or faster) as \( \delta \to 0 \), whereas the last term must vanish logarithmically, \( \alpha_0 \sim 1 / \log \delta \). The net result is

\[
\mathcal{M}_4 \to a_4 \frac{\chi^4}{\log \delta} + b_4 \delta, \quad \tag{56}
\]
or faster, in agreement with Eq. (5).

Finally, for all solutions we have checked that if the nonspin-induced multipole moments vanish linearly (or faster) as \( \delta \to 0 \) and the spin-induced multipole moments vanish logarithmically (or faster), then the whole solution is regular in the BH limit also at higher order in the perturbative expansion.

VI. A “SOFT-HAIR THEOREM” CONJECTURE FOR ECOS

Given the general structure of the field equations, we expect the results obtained in the previous section to be valid at any order in the small-multipole expansion and to any order of the multipolar truncation. Thus, it is natural to conjecture that, within general relativity, the multipole moments of any (axisymmetric) ultracompact object whose exterior spacetime is perturbatively close to a Schwarzschild metric (i.e., the multipole moments of a soft ECO) must satisfy Eqs. (3) and (4), with

\[
\delta \mathcal{M}_\ell \to a_\ell \frac{\chi^\ell}{\log \delta} \mathcal{M}^{\ell+1}, \quad \delta \mathcal{S}_\ell \to c_\ell \frac{\chi^\ell}{\log \delta} \mathcal{M}^{\ell+1}, \quad \tag{57}
\]
or faster, as \( \delta \to 0 \), if the multipole moments are spin-induced. For all other types of multipole moments,

\[
\delta \mathcal{M}_\ell \to b_\ell \mathcal{M}^{\ell+1} \delta, \quad \delta \mathcal{S}_\ell \to d_\ell \mathcal{M}^{\ell+1} \delta, \quad \tag{58}
\]
or faster, as \( \delta \to 0 \). In the above expressions, \( a_\ell, b_\ell, c_\ell \) and \( d_\ell \) dimensionless numbers of order unity or smaller.

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8 For the sake of clarity, in Eq. (40) we have assumed \( \Delta \mathcal{M}_2 = 0 \). Nevertheless, the same conclusions can be reached if we assume that the quadrupole deviations vanish logarithmically, as requested by Eq. (13).
the spin and compactness dependence has been completely factored out. Overall, although we cannot prove it to arbitrary order, our results support the conjecture that the multipolar corrections of a soft ECO in the BH limit should behave as in Eqs. [5] and [6].

In other words, the closer a soft ECO is to a BH, the less hairy it will be, in the manner specified by [57] or [58]. This property can be seen as a weaker version of Birkhoff’s theorem: beyond spherical symmetry the external spacetime is different from Kerr, but the multipolar deviations must die off as the BH limit is approached. Furthermore, spin-induced multipolar deviations die off much more slowly than any other. While – simply by continuity arguments – it is not surprising that $\delta M_L, \delta S_L \to 0$ in the BH limit, the logarithmic scaling with $\delta$ in the spin-induced case is important, since it can give rise to potentially detectable effects, as we briefly discuss in the next section.

VII. DISCUSSION AND EXTENSIONS

We have constructed and analyzed a class of axisymmetric solutions to Einstein’s vacuum equations describing small (but otherwise generic) multipolar deformations of a Schwarzschild geometry. Our solution does not require equatorial symmetry and includes deformations induced by moments other than the spin. In fact, it is the most general axisymmetric quasi-Schwarzschild solution to Einstein’s vacuum equations.

We have introduced a classification for ECOs with “soft” and “hard” hair, respectively. The former are characterized by a curvature at the surface which is comparable to that at the corresponding BH horizon, whereas the latter can have much larger curvature due to putative high-energy effects. Self-consistency of the perturbative approach requires that our general solution belongs to the soft ECO class.

For this family of solutions, we showed that all spin-induced deviations from the BH multipole moments have to vanish logarithmically (or faster) as a function of the compactness parameter $\delta$ [see Eq. (41)] in the BH limit. This logarithmic scaling already appeared in a particular case of our generic solution, namely in slowly-rotating gravastars [59], and we showed here that it is more generic. On the other hand, multipolar deviations which are nonspin induced must vanish linearly (or faster) and are therefore more difficult to constrain. Overall, the approach to the BH limit is summarized in Eqs. (5) and (6). This “soft-hair theorem” implies that the more compact a soft ECO is, the less hairy it will be, thus extending Birkhoff’s theorem to the case of small deviations from spherical symmetry. Saturation of this (either logarithmic or linear) decay yields an upper bound on the compactness of quasi-Schwarzschild ECOs. Furthermore, since spin-induced multipolar corrections die off much more slowly than any other in the BH limit, they are the dominant corrections and those that might provide the most stringent bounds on the ECO compactness.

Models of ECOs with soft hair can be constrained with future observations [20–21, 30–35]. In particular, extreme-mass ratio inspirals (EMRIs) detectable by the future space mission LISA can place a constrain $\delta M_2/M^3 < 10^{-4}$ on the quadrupole moment of the central supermassive object [40]. When the aforementioned decay of hair is saturated, the quadrupolar deviation of soft ECOs reads

$$\frac{\delta M_2}{M^3} \approx \left\{ \begin{array}{ll} |\log \delta|^{-1} \sim 10^{-2} \log \left( \frac{10^6 M_\odot}{L_p} \right) \ & |\log \delta|^{-1} \sim 10^{-4} \left( \frac{L}{10^6 M_\odot} \right) \end{array} \right.,$$

(59)

for spin- and nonspin-induced quadrupolar hair, respectively. In the former case we defined $\delta = \ell_p/(2M)$, where $\ell_p$ is the Planck length, whereas in the latter case $\delta = L/(2M)$, where $L \approx 50$ km. In other words, an EMRI detection by LISA has the potential to constrain spin-induced multipolar deviations from Kerr even for objects motivated by quantum-gravity considerations [14–19], whereas multipolar deviations which are not spin induced can be constrained only for objects with compactness smaller than $M/r_0 \approx 0.49995$, nonetheless an impressive constraint. The scaling rules [5] and [6] imply that a quadrupole moment measurement of a soft ECO will always be dominated by the spin-induced contribution, unless

$$\chi \ll \sqrt{|\log \delta|},$$

(60)

where we assumed $b_2/a_2 \sim O(1)$. The above upper bound is very small for realistic values of $\delta$, e.g. $\chi \ll 0.03$ when $\delta \approx 10^{-4}$.

A detailed studied on the phenomenology of EMRIs in the case of ECOs with soft hair and the connection between our solution and existing parametrizations (e.g. with “bumpy” BHs [47, 48]) will appear elsewhere.

Our argument is based on the properties of the external spacetime and is therefore independent of the internal structure of the body. There are, however, some caveats that are worth discussing:

- We assumed vacuum Einstein equations, so strictly speaking our analysis is not valid in modified gravity or in the presence of long-range fields in general relativity. However, we expect our argument to remain qualitatively valid if putative external fields die off sufficiently fast (e.g., for boson stars, in particular the most compact ones, and for BHs in effective-field-theory corrections to general relativity [49, 53], which are suppressed at large distance by high-curvature terms). In particular, our parametrization could approximately describe the external metric of spinning BHs without equatorial symmetry which were recently constructed in extensions of general relativity [52] and with specific matter sources [53]. We also expect that an extension to Einstein-Maxwell should be relatively straightforward and qualitatively similar to the vacuum case.
We assumed that the multipole moments are perturbatively small. While this necessarily restricts the analysis to soft ECO models, it might not always be the case. For example, the angular momentum of a spinning boson star is quantized so its spin-induced multipole moments cannot be made arbitrarily small. However, stable boson stars have a maximum compactness that is not continuously connected to that of a BH, so they are outside the scope of our study.

An important open issue concerns the stability of these geometries. The answer to this problem depends on the internal composition of the object or, equivalently, on the boundary conditions for time-dependent perturbations at the surface, which we left unspecified (also in the axisymmetric, stationary case) for the sake of clarity. As such, the stability issue can be assessed only case by case. Some particular examples of our general solution (namely, gravastars) are linearly (mode) stable under radial [55] and nonradial [56] perturbations for densities below that corresponding to the maximum mass; in this respect they are therefore similar to ordinary neutron stars. On the other hand, linearized gravitational fluctuations of static ultracompact horizonless objects with perfectly reflective boundary conditions are extremely long-lived and decay no faster than logarithmically [57].

The long damping time of these modes has led to the conjecture that these objects might be nonlinearly unstable [57] [58], although most likely this conclusion and the putative endstate of the instability depend on the specific model. Likewise, spinning ultracompact horizonless objects with perfectly reflective boundary conditions are linearly unstable toward the ergoregion instability [59–61] (see [62] for a review). Also in this case the instability depends on the specific model. In particular, partial absorption in the interior might quench the instability completely [63] [64].

A possible extension of our analysis concerns relaxing the assumption of soft hair. In axisymmetry, this can be done, even at the nonlinear level, using the integrability of stationary axisymmetric metrics (see e.g. Refs. [65] [66]). For static (i.e., without angular momentum) deformations, these solutions belong to the Weyl class [67].

A discussion of the no-hair properties of ECO models with hard hair will appear elsewhere [26]. Future work will also include a study of the geodesic properties of the quasi-Schwarzschild solution in the general case and of the detectability of multipolar corrections with current and future observations.

Finally, although our main motivation was to study ECOs, the family of perturbative solutions constructed in this work might also serve to study generic (axisymmetric) deformations of a neutron star, for example those sourced by an intrinsic quadrupole moment.

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