Various Abelian Projections of $SU(2)$ Lattice Gluodynamics and Aharonov-Bohm Effect in the Field Theory *

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Abstract

We show that in general abelian projection of lattice gluodynamics it is not only monopoles but also strings are present. Both these topological excitations may be responsible for the confinement of color. We illustrate our ideas by some explicit results in the Abelian Higgs model with the Villain action.

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1 Introduction

The idea of the abelian projection [1] is to reduce the gluodynamics to an abelian system, in which we can understand all physical effects including confinement. The last year’s review by Suzuki [2] and many other talks given at the present symposium (sections “QCD vacuum”) contain various numerical and analytical results confirming the monopole confinement mechanism in $SU(2)$ lattice gluodynamics. All these results are obtained for the so-called maximal abelian projection [3, 4], in which the gauge transformation makes the link matrices diagonal ”as much as possible”. Formally, the matrices of the gauge transformation $\Omega_x$ maximize the quantity

$$R(U') = \sum_{x,\mu} Tr(U'_x \sigma_3 U'_{x+\mu} \sigma_3),$$

(1)

$$U'_x = \Omega_x U_{x+\hat{\mu}} \Omega_{x+\hat{\mu}}.$$ 

(2)

The other abelian projections (such as the diagonalization of the plaquette matrix $U_{x,12}$) do not give evidence that vacuum behaves as the dual superconductor. Below we give two examples. First, it occurs [5] that the fractal dimensionality of the monopole currents extracted from the lattice vacuum by means of the maximal abelian projection is strongly correlated with the string tension. If monopoles are extracted by means of the other projections (diagonalization of $U_{x,12}$) this correlation is absent [5]. The other example is the temperature dependence of the monopole condensate $C$ measured on the basis of the percolation properties of the clusters of the monopole currents [6]. For the maximal abelian projection the condensate is non-zero below the critical temperature $T_c$ and vanishes above it. For the projection which corresponds to the diagonalization of $U_{x,12}$, the condensate is nonzero at $T > T_c$. The last result was obtained by the authors of [6], but has not been published.

In Sect. 2 we discuss why the maximal abelian projection may be the unique projection in which only the monopoles are responsible for the confinement. In Sect. 3 we argue that for a general projection not only monopoles but also strings are present in the abelian vacuum. In Sect. 4 we discuss a simplified abelian model in which confinement of quarks is due to the interaction with strings. It occurs that this interaction is the field theoretical analogue of the Aharonov-Bohm effect.
2 Confinement in the Maximal Abelian Gauge.

There is a small parameter $\varepsilon$ in lattice gluodynamics in the maximal abelian gauge. This parameter is the natural measure of closeness between the diagonal matrices and the link matrices after the gauge projection. For the $SU(2)$ gluodynamics, we have

$$\varepsilon^2 = \frac{\sum_{x,\mu} |U_{x\mu}^{12}|^2}{\sum_{x,\mu} |U_{x\mu}^{11}|^2},$$

where $U_{x\mu}^{ij}$ are the matrix elements of the link matrix $U_{x\mu}$.

If we neglect the fluctuations of $|U_{x\mu}^{11}|$, as well as the gauge fixing term and the Faddeev-Popov determinant, the $SU(2)$ action in the minimal abelian gauge becomes:

$$S_P = \beta \frac{1}{2} \text{Tr} U_P = \bar{\beta} \cos \theta_P + O(\varepsilon^2) + ...,$$

(4)

where $\bar{\beta} = \beta |U_{11}|^4$, $\theta_P$ is the plaquette angle constructed in the usual way from the link angles $\theta_{x\mu}$, $U_{x\mu}^{11} = |U_{x\mu}^{11}| e^{i\theta_{x\mu}}$.

Thus, in the naive approximation, we get the action of the compact electrodynamics. Since in the compact electrodynamics the confinement is due to the monopole condensation, it is not surprising that, in the maximal abelian gauge, the vacuum of gluodynamics is an analogue of the dual superconductor.

Of course, this is only an intuitive argument. The confinement in the $U(1)$ theory exists in the strong coupling region. Therefore, in order to explain the confinement at large values of $\beta$ in $SU(2)$ gluodynamics, we have to study in detail the special features of the gauge fixing procedure (such as Faddeev-Popov–determinant, gauge fixing term, fluctuations of $|U_{x\mu}^{11}|$ etc.).

3 Topological Excitations in the General Abelian Projection.

Now we consider the general abelian projection of the $SU(2)$ lattice gluodynamics. In this case there is no small parameter, similar to $\varepsilon$ and not only diagonal elements $U_{x\mu}^{11}$ but also non-diagonal elements $U_{x\mu}^{12}$ contributes to the effective action. It is convenient to consider the following parametrization of the link matrix:

$$U_{x\mu} = \exp\{\frac{i}{2}(\rho_{x\mu}^1\sigma_1 + \rho_{x\mu}^2\sigma_2)\} \exp\{\frac{i}{2}\theta_{x\mu}\sigma_3\}.$$
After the abelian projection, the fields $\theta_{x\mu}$ and $\rho_{x\mu} = \rho_{1x\mu} + i\rho_{2x\mu}$ transform under the abelian gauge transformation as follows:

$$\theta_{x\mu} \rightarrow \theta_{x\mu} + \alpha_x - \alpha_{x+\mu}, \quad (6)$$

$$\rho_{x\mu} \rightarrow e^{2i\alpha_x} \rho_{x\mu}, \quad (7)$$

so the diagonal gluons become the gauge field (photon) $\theta$, and the non-diagonal gluons play the role of the charge 2 matter vector field $\rho$, [1]. For the general abelian projection we have

$$< \sum_{\mu} | \rho_{x\mu} |^2 > \neq 0, \quad (8)$$

therefore we have the condensed matter field and in this system the counterpart of the Nielsen–Olesen strings exists. These topological excitations, as shown later, may play as important role in the confinement mechanism as the monopoles in the maximal abelian projection. Using standard algorithm, we can extract the monopoles from a given configuration of the compact abelian gauge fields on the lattice. The Nielsen–Olesen strings can be extracted in a similar way from a given configuration of the compact matter field and gauge field. Consider, for simplicity, a condensed scalar matter field, for instance, the composite field

$$\Phi_x = \sum_{\mu} \rho_{x\mu} = | \Phi_x | e^{i\phi_x}. \quad (9)$$

Below we use the notations of the calculus of differential forms on the lattice, which are briefly described in Appendix A. Consider the gauge invariant link variable $L = d\varphi + 2\theta + 2\pi l$ and the plaquette variable $P = d\theta + 2\pi n$, where $l$ and $n$ are integer numbers such that $-\pi < L \leq \pi$, and $-\pi < P \leq \pi$. The world sheets $\sigma$ of the Nielsen–Olesen strings and world trajectories $j$ of the monopoles are defined on the dual lattice by:

$$\sigma = \frac{1}{2\pi} (*dL - 2*P) = *dl - 2*n, \quad (10)$$

$$j = \frac{1}{2\pi} *dP = *dn. \quad (11)$$

Since $\delta \sigma = j$, the strings may be opened, and may carry monopole and antimonopole at the ends; the monopole currents form closed lines: $\delta j = 0$. Thus, in general abelian projection gluodynamics is reduced to an abelian Higgs model in which the topological excitations are monopoles and strings.

3
4 Abelian Higgs Theory and Aharonov–Bohm Effect in the Field Theory

We consider now the Abelian Higgs model with noncompact gauge field, in order to study the contribution of the strings into the confinement mechanism. The contribution of monopoles, which exist due to the compactness of the gauge field, is widely discussed in the literature. The partition function of the 2D XY-model is equivalent to the partition function of the Coulomb gas, the partition function of the 4D compact electrodynamics can be represented as a sum over the monopole–antimonopole world lines. Now we show that the partition function of the four–dimensional Abelian Higgs theory can be represented as a sum over closed surfaces which are the world sheets of the Abrikosov–Nielsen–Olesen strings.

We consider the model describing interaction of the noncompact gauge field $A_{\mu}$ with the scalar field $\Phi = |\Phi| e^{i\varphi}$, whose charge is $Ne$ (to be in agreement with the previous section, we have to set $N = 2$, since the charge of the nondiagonal gluon is $2e$). The selfinteraction of the scalar field is described by the potential $V(\Phi) = \lambda(|\Phi|^2 - \eta^2)^2$. For simplicity, we consider the limit as $\lambda \to \infty$, so that the radial part of the scalar field is frozen, and the dynamical variable is compact: $\varphi \in (-\pi, \pi]$. The partition function for the Villain form of the action is given by:

$$Z = \int_{-\infty}^{+\infty} DA \int_{-\pi}^{+\pi} D\varphi \sum_{l(\epsilon_1) \in \mathbb{Z}} \exp\{-S_l(A, d\varphi)\}, \quad (12)$$

where

$$S_l(A, d\varphi) = \frac{1}{2e^2}||dA||^2 + \frac{\kappa}{2}||d\varphi + 2\pi l - NA||^2. \quad (13)$$

We use the notations of the calculus of differential forms on the lattice described in Appendix A. The symbol $\int D\varphi$ ($\int DA$) denotes the integral over all site (link) variables $\varphi$ ($A$). Fixing the gauge $d\varphi = 0$, we get the following expression for the action (13): $S_l = \frac{1}{2e^2}[A, (\delta d + N^2 \kappa e^2)A] + \text{(terms linear in } A); \text{ therefore, due to the Higgs mechanism, the gauge field acquires the mass } m = N\kappa^2 e; \text{ there are also soliton sectors of the Hilbert space which contain Abrikosov–Nielsen–Olesen strings hidden in the summation variable } l \text{ in (12).}$

The partition function of the compact electrodynamics can be represented as a sum over closed world lines of monopoles. In the same way the partition
function (14) can be rewritten as a sum over closed world sheets of the Nielsen–Olesen strings:

\[ Z_{BKT} = \text{const.} \sum_{\sigma^*(c_2) \in \mathbb{Z}} \exp \left\{ -2\pi^2 \kappa (\sigma^*, (\Delta + m^2)^{-1} \sigma) \right\}, \]  

(14)

A derivation of this representation is given in [12]. The sum here is over the integer variables \( \sigma^* \) attached to the plaquettes \( c_2 \) of the dual lattice. The condition \( \delta^* \sigma = 0 \) means that for each link of the dual lattice the “conservation law” is satisfied: \( \sigma_1 + \sigma_2 + \sigma_3 = \sigma_4 + \sigma_5 + \sigma_6 \), where \( \sigma_i \) are integers corresponding to plaquettes connected to the considered link. The signs of \( \sigma_i \)’s in this “conservation law” are dictated by the definition of \( \delta \) (by the orientation of the plaquettes 1,...,6). If \( \sigma = 0,1 \), then the condition \( \delta^* \sigma = 0 \) means that we consider closed surfaces made of plaquettes with \( \sigma = 1 \). In (14) we have \( \sigma \in \mathbb{Z} \), which means that a single plaquette may “decay” into several ones, but still the surfaces made of plaquettes with \( \sigma \neq 0 \) are closed. It follows from (14) that the strings interact with each other via the Yukawa forces \( (\Delta + m^2)^{-1} \).

Now we calculate the quantum average of the Wilson loop for the charge \( Me \), \( W_M(C) = \exp \{ iM(A,jC) \} \), in the BKT representation.

Repeating all steps which transform (12) into (14), we get:

\[ < W_M(C) > = \frac{1}{Z_{BKT}} \sum_{\sigma^*(c_2) \in \mathbb{Z}} \exp \left\{ -2\pi^2 \kappa (\sigma^*, (\Delta + m^2)^{-1} \sigma) - \frac{M^2 e^2}{2} (jC, (\Delta + m^2)^{-1} jC) - 2\pi i \frac{M}{N} (jC, (\Delta + m^2)^{-1} \delta \sigma) + 2\pi i \frac{M}{N} \mathcal{L}(\sigma, jC) \right\} \]  

(15)

The first three terms in the exponent describe the short–range (Yukawa) interactions: surface – surface, current – current and current – surface. In spite of the gauge field acquiring the mass \( m = N\kappa \frac{4}{e} \), there a is long–range interaction

\footnote{We use the superscript BKT, since a similar transformation of the partition function was first found by Berezinskii [7], Kostlitz and Thouless [8], who showed that the XY model is equivalent to the Coulomb gas in two dimensions.}

\footnote{Due to the definition of the integration by parts \( (\varphi, \delta \psi) = (d\varphi, \psi) \), the operator \( (\Delta + m^2)^{-1} \) (and not \( (-\Delta + m^2)^{-1} \)) is positive definite on the Euclidean lattice.
of geometrical nature, described by the last term in the exponent: $IL(\sigma, j_C)$, $IL$ being the four–dimensional analogue of the Gauss linking number for loops in three dimensions, i.e. the linking number of surfaces defined by $\{\sigma\}$ and loop defined by $j_C$. The explicit expression for $IL$ is:

$$IL = (\ast j_C, \Delta^{-1} d \ast \sigma) = (\ast j_C, \ast n),$$

(16)

where $\ast n$ is an integer valued 3-form which is the solution of the equation $\delta \ast n = \ast \sigma$. It is clear now that $IL$ is equal to the number of points at which the loop $j_C$ intersects the three–dimensional volume $\ast n$ bounded by the closed surface defined by $\ast \sigma(\ast c_2)$. The elements of the surface $\ast \sigma$ may carry any integer number, so that any intersection point may contribute an integer into $IL$. Therefore, $IL$ is the linking number of the world sheet of the strings and the current $j_C$ which define Wilson’s loop $W_M(C)$. The reason for the topological interaction is that the charges $e, 2e, \ldots (N-1)e$ cannot be completely screened by the condensate of the field of charge $Ne$; if $M/N$ is integer, then the screening is complete and there is no topological interaction. From the another point of view this, is the four–dimensional analogue [14, 15, 16] of the Aharonov–Bohm effect: strings correspond to solenoids which scatter charged particles.

To apply the model in question to abelian projected gluodynamics, we simply have to set $M = 1, N = 2$, since the charge of “test infinitely heavy” quark is unity and the charge of the condensed nondiagonal gluon, whose phase plays the role of the field $\varphi$, is two. The topological interaction can yield the area low; this fact can be easily proven for completely random surfaces $\sigma$.

Thus, we conclude that the confinement in gluodynamics can be due not only to monopoles but also to strings.

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Appendix A

Here we briefly summarize the main notions from the theory of differential forms on the lattice [13]. The advantages of the calculus of differential forms consists in the general character of the expressions obtained. Most of the transformations depend neither on the space–time dimension, nor on the rank of the fields. With minor modifications, the transformations are valid for lattices of any form (triangular, hypercubic, random, etc.). A differential form of rank $k$ on the lattice is a function $\phi_k$ defined on $k$-dimensional cells $c_k$ of the lattice, e.g. the scalar (gauge) field is a 0–form (1–form). The exterior differential operator $d$ is defined as follows:

$$\left( d\phi \right)(c_{k+1}) = \sum_{c_k \in \partial c_{k+1}} \phi(c_k).$$  \hspace{1cm} (A.1)

Here $\partial c_k$ is the oriented boundary of the $k$-cell $c_k$. Thus the operator $d$ increases the rank of the form by unity; $d\varphi$ is the link variable constructed, as usual, in terms of the site angles $\varphi$, and $dA$ is the plaquette variable constructed from the link variables $A$. The scalar product is defined in the standard way: if $\varphi$ and $\psi$ are $k$-forms, then $(\varphi, \psi) = \sum_{c_k} \varphi(c_k)\psi(c_k)$, where $\sum_{c_k}$ is the sum over all cells $c_k$. To any $k$–form on the $D$-dimensional lattice there corresponds a $(D-k)$–form $\Phi^*(c_k)$ on the dual lattice, $c_k$ being the $(D-k)$–dimensional cell on the dual lattice. The codifferential $\delta = \ast d \ast$ satisfies the partial integration rule: $(\varphi, \delta \psi) = (d \varphi, \psi)$. Note that $\delta \Phi(c_k)$ is a $(k - 1)$–form and $\delta \Phi(c_0) = 0$. The norm is defined by: $\|a\|^2 = (a, a)$; therefore, $\|d\varphi + 2\pi l\|^2$ in (13) implies summation over all links. $\sum_{l(c_1)} = \mathbb{Z}$ denotes the sum over all configurations of the integers $l$ attached to the links $c_1$. The action (13) is invariant under the gauge transformations $A' = A + d\alpha$, $\varphi' = \varphi + \alpha$ due to the well known property $d^2 = \delta^2 = 0$. The lattice Laplacian is defined by: $\Delta = d\delta + \delta d$.

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