Asymptotic distribution of Brownian Excursions into an Interval.

B. Rajeev
Indian Statistical Institute, 8th Mile,
Mysore Road, Bangalore 560 059, India
email:brajeev@isibang.ac.in

May 25, 2022

Abstract

In this paper, following earlier results in [2] we derive the asymptotic distribution, as \( t \to \infty \), of the excursion of Brownian motion straddling \( t \), into an interval \((a, b)\), conditional on the event that there is such an excursion.

Key words and phrases: Brownian motion, excursions, last exit/entrance times, asymptotic distribution of excursions.

1 Introduction:

In this paper we consider the asymptotic distribution of the excursions \((\zeta_t)\) of a one dimensional standard Brownian motion \((W_t)\), straddling the time \( t \), into an interval \((a, b)\) as \( t \to \infty \). Although, excursions straddling a given time are well studied in the literature for general Markov processes (for a
sample, see [6, 7, 11, 12]), the study of their asymptotics as \( t \to \infty \) seems to be new.

To describe our results in more detail let for each \( t > 0 \), \( \sigma_t, d_t \) be the last entrance before \( t \) into \((a, b)\) and first exit after \( t \) from \((a, b)\) respectively for a sample path \( W \) such that at time \( t, W_t \in (a, b) \). The excursion straddling time \( t \) is the portion of the trajectory \( \zeta_t(s) := W_{\sigma_t+s\wedge d_t-\sigma_t}, s \geq 0 \). We view the process \( (\zeta_t) \) as a process with values in the space \( C([0, \infty), [a, b]) \), the space of continuous functions with values in \([a, b]\) so that the convergence in question reduces to weak convergence in this space. The crucial step in the proof is to express the expected value \( E[f(\zeta_t)|W_t \in (a, b)] \), where \( f : C([0, \infty), [a, b]) \to \mathbb{R} \) a bounded and continuous function as \( E[q(W_{\sigma_t}, t-\sigma_t, f)|W_t \in (a, b)] \) where the kernel \( q(x, s, A) \) is a bounded continuous function of \((x, s), s > 0, x = a \) or \( b \) for a Borel set \( A \in C([0, \infty), [a, b]) \) and then use the weak convergence of the pair \((W_{\sigma_t}, t-\sigma_t)\). It is known (see [2]) that the latter pair converges to \((X, Y)\), the exit place and time respectively of a Brownian motion started uniformly in the interval \((a, b)\). Together with the explicit form of \( q(x, s, A) \), this gives the limiting distribution of \( \zeta := \lim_{t \to \infty} \zeta_t \) as follows: Starting from \( X = a \) or \( b \) with probability \( \frac{1}{2} \), the conditional distribution of \( \zeta \in A \) given that the lifetime \( Y \) of \( \zeta \) is at least \( s \), is given by \( q(x, s, A) \). Together with the known distribution of the lifetime of excursions into \((a, b)\), starting from \( a \) or \( b \) this gives a complete description of the limiting distribution.

In Section 2, we set up the necessary machinery from excursion theory. Rather than use the extension of Itô’s excursion theory, due to B.Maisonneuve ([6]), in cases where the boundary of the excursion set involves more than one point, we give a more intuitive, ‘bare hands’ construction, using Itô’s original result (as presented in [11]) (Theorem 2.3). Although we are dealing with a very basic example as far as excursion theory is concerned, our approach maybe of interest in constructing non trivial ‘exit systems’ starting with excursions from a single point. See Remark 2.4 for the connection with the exit system formalism of [6]. As mentioned above, the crucial point in the proof is to express the expected value of functionals of the excursions into \((a, b)\) at time \( t \), in terms of the kernel \( q(x, s, A) \) describing the conditional excursion measures and then the continuity in \((x, s)\) of these kernels. This is done via a ‘conditional excursion formula’ for the excursion straddling \( t \), proved in Section 3. Such formulas are well known for general Markov processes (see [1, 6]) but at specific time instances and not for the excursion process as a...
whole, as in [11] and as required in our situation.

In section 4, we describe the limiting distribution of $\zeta_t$ and prove the convergence to this distribution. As mentioned above the proof involves the convergence, as $t \to \infty$ of the pair of variables $(W_{\sigma_t}, t - \sigma_t)$ to the pair $(X, Y)$ described above. This latter result is a consequence of a more general result, proved in [2], about the convergence as $t \to \infty$ of the time reversal $(W_{t-s}, 0 \leq s \leq t)$ of Brownian motion $W_t$ from a point $t$ with $W_t \in (a, b)$. Finally, we give an application to the evaluation of asymptotic distribution of functionals of the excursion straddling $t$ as $t \to \infty$.

2 Excursions into an interval:

Let $S$ be an open subset of $\mathbb{R}$. In this paper, $S$ will be either $(0, \infty) \cup (-\infty, 0)$ or the finite interval $(a, b)$ or the set $(-\infty, a) \cup (a, b) \cup (b, \infty)$. $\partial S$ will denote the boundary of $S$. We denote the closure of a set $S$ by $\bar{S}$. Let $C([0, \infty), \bar{S})$ denote the space of continuous functions from $[0, \infty)$ into $\bar{S}$, equipped with the topology of uniform convergence on compact subsets of $[0, \infty)$. We will denote by $U = U(S)$ the space of excursions from $\partial S$ into $S$. In other words,

$$U := \{u \in C([0, \infty), \bar{S}) : u(0) \in \partial S, \text{ and } \exists R(u) > 0 \text{ such that } u(t) \in S, 0 < t < R(u) \text{ and } u(t) \in \partial S \forall t \geq R(u)\}.$$ 

Let $U$ be the trace of the Borel $\sigma$-field $B$ of $C([0, \infty), \bar{S})$ on $U$ i.e. $U = U \cap B$. Let $U_\delta := U \cup \{\delta\}$, with $\delta$ attached as an isolated point. Let $U_\delta$ be the sigma field generated by $U$ and $\{\delta\}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t)$ a filtration on it satisfying the usual conditions i.e. $\mathcal{F}_0$ contains $P$ null sets and the filtration is right continuous. Let $(W_t)$ be a 1-dimensional standard $\mathcal{F}_t$-Brownian motion. $\mathcal{P}$ will denote the previsible $\sigma$-field over $(\mathcal{F}_t)$. Let $(L_t^x)_{t \geq 0}$ denote the local time process of $(W_t)_{t \geq 0}$ at $x \in \mathbb{R}$. For a continuous, non decreasing, $\mathcal{F}_t$-adapted process $(L_t)$, we define its right continuous inverse $(\tau_t)$ in the usual way by

$$\tau_t := \inf\{s > 0 : L_s > t\}.$$
We note that $(\tau_t)$ is a non decreasing, right continuous process such that for each $t, \tau_t$ is an $\mathcal{F}_t$-stopping time. Let $Z$ be the random closed set $Z := \{ t : W_t \in S^c \}$. We will assume that a.s. (P), $L_t = \int_0^t I_Z(s) \, dL_s$. For $t > 0$ let $e_t : \Omega \to U_\delta$ be defined by

$$e_t(\omega)(s) = W_{\tau_t + s \wedge (\tau_t - \tau_t)}(\omega), \quad s \geq 0, \quad \tau_t(\omega) - \tau_t - (\omega) \neq 0,$$

$$\tau_t(\omega) - \tau_t(\omega) = 0.$$

Then $e(\cdot)$ defines a sigma finite point process on $\Omega$ with state space $U_\delta$ and time domain $D(\omega) = \{ t > 0 : \tau_t(\omega) \neq \tau_t(\omega) \}$. For $\Gamma \in U_\delta$ we define

$$N_t^\Gamma(\omega) := \#\{ s \leq t : s \in D(\omega), e_s(\omega) \in \Gamma \}.$$

With $Z$ as defined above we can write $Z(\omega)^c = \bigcup_{i=1}^\infty (\alpha_i(\omega), \beta_i(\omega))$. Let $G(\omega) := \{ \alpha_i(\omega) : i = 1, \ldots, \}$ be the set of left end points of the excursion intervals , $d_t(\omega) := \inf\{ s > t : W_s(\omega) \notin S \}$ and $\sigma_t = \sup\{ s \leq t : W_s \notin S \}$. We use the convention that $\inf\{ \phi \} = \infty$ and $\sup\{ \phi \} = 0$. For $\omega \in \Omega$, and $t > 0$, we define the portion of the excursion straddling $t$, from $t$ onwards viz. $i_t(\omega)$ as follows:

$$i_t(\omega)(s) := W_{t + s \wedge (d_t(\omega) - t)}(\omega), \quad 0 \leq s \leq d_t(\omega) - t. \tag{2.1}$$

Note that if $t \in G(\omega)$, then $i_t(\omega) \in U(S)$.

Now we specialise to the case $S = (0, \infty) \cup (-\infty, 0)$ and recall a few well known facts. We take $L_t := L_0^\delta$, the local time at 0 of Brownian motion. Then it is well known that $e$ is a $\sigma$- finite $(\mathcal{F}_\tau)$ Poisson point process with characteristic measure $n(\Gamma), \Gamma \in U_\delta$ given by

$$n(\Gamma) := \frac{1}{t} E \left( N_t^\Gamma \right).$$

Note that $n(\{ \delta \}) = 0$.

**Remark 2.1** Let $t > 0$. Then it is known (see [11], Propn.2.8, Chap.XII) that $n\{ u \in U : R(u) > t \} = \left( \frac{2}{\pi t} \right)^{1/2}$. In particular $N_t^\Gamma < \infty$ a.s. with $\Gamma = \{ u : R(u) > t \}$. 

4
We also note that if $\Gamma$ is such that $EN_{1}^{\Gamma} < \infty$ then $n(\Gamma) < \infty$ and
$
\{N_{t}^{\Gamma} - tn(\Gamma); t \geq 0\}
$
is an $\mathcal{F}_{\tau_{t}}$-martingale.

Suppose $H : [0, \infty) \times \Omega \times U_{\delta} \rightarrow [0, \infty]$, with $H(t, \omega, \delta) \equiv 0$ for all $(t, \omega)$. Suppose $H$ is $\mathcal{P} \otimes U_{\delta}$ measurable. We have the following well known Theorem of K.Ito\^{(2)}.

**Theorem 2.2** Let $S = (-\infty, 0) \cup (0, \infty)$, $L_{t} := L_{t}^{0}$, and $H, G, i_{t}, \tau_{t}$ and $n$ be as above. Then,

$$
E \left[ \sum_{t \in G(\omega)} H(t, \omega, i_{t}(\omega)) \right] = E \left[ \sum_{s \in D(\omega)} H(\tau_{s-}(\omega), \omega, e_{s}(\omega)) \right]
$$

$$
= E \int_{0}^{\infty} ds \int_{U_{\delta}} H(\tau_{s}(\omega), \omega, u)n(du)
$$

$$
= E \int_{0}^{\infty} dL_{s}(\omega) \int_{U_{\delta}} H(s, \omega, u)n(du). \quad (2.2)
$$

**Proof:** The proof follows as in Proposition 2.6, using Proposition 1.10 and Theorem 2.4 of Chapter XII of [11](see also [3], Chap. III, Thm. 3.18 and Thm 3.24). The only difference in our case is that $\mathcal{F}_{t}$ is not the canonical filtration. □

We now take $S = (a, b), -\infty < a < b < \infty$ and $L_{t} := L_{t}^{a} + L_{t}^{b}$, $t \geq 0$. Let $(\tau_{t})_{t \geq 0}, D$ be as defined above. We note that for $t \in D(\omega)$ the excursion $e_{t}(\omega)$ defined earlier, may not be in $U$ since $D(\omega)$ includes excursions below $a$ and above $b$. Consequently $G(\omega) \subseteq D(\omega)$. Since we are only interested in excursions into $(a, b)$ we proceed as follows: For $t \in D(\omega)$, define $e_{t}(\omega) \in U$ as

$$
e_{t}(\omega)(s) := W_{\tau_{s}(\omega) + a}(\omega) \quad 0 \leq s < \tau_{t}(\omega) - \tau_{s-}(\omega)
$$

$$
W_{\tau_{s}(\omega)}(\omega) \quad s \geq \tau_{t}(\omega) - \tau_{s-}(\omega).
$$

If $t \notin D(\omega)$, put $e_{t}(\omega) \equiv \delta$. Define

$$
e_{t}(\omega) := I_{U}(e_{t}(\omega))e_{t}(\omega) + \delta I_{U^{c}}(e_{t}(\omega)).
$$
We shall take this as the definition of our excursion process when $S = (a, b)$.

We define the kernel $n(x, \cdot), x \in \mathbb{R}$ on $U = U(S), S = (a, b)$ as follows: We denote by $U_0 := U(S)$, when $S = (-\infty, 0) \cup (0, \infty)$, the space of excursions from 0. Let $U_0^+ := \{u \in U_0, u(t) > 0, 0 < t < R(u)\}$. $U_0^-$ is defined similarly. $U_0 = U_0^+ \cup U_0^-$ and $U_{0, \delta} := U_0 \cup \{\delta\}$. The sigma field $U_{0, \delta}$ is the sigma field generated by $U_0$ and $\{\delta\}$, where $U_0 = U_0 \cap B$ and $B$ the Borel sigma field of $C([0, \infty), \mathbb{R})$. For $c \in \mathbb{R}$, let $T_c := \inf\{s > 0 : W_s = c\}$. For $u \in U_0$, let $u^c$ denote the path in $C([0, \infty), \mathbb{R})$ which is given by $u(\cdot)$ stopped when it reaches level $c$ viz. $u(T_c \wedge \cdot)$. Define maps $\lambda^a : U_{0, \delta} \rightarrow U_{\delta}$, $\lambda^b : U_{0, \delta} \rightarrow U_{\delta}$ as follows: For $u \in U_0^+$, $\lambda^a(u) := a + u^{b-a}$ and for $u \in U_0^-$, $\lambda^b(u) := b + u^{a-b}$.

We extend $\lambda^a, \lambda^b$ to the whole of $U_{0, \delta}$ by setting $\lambda^a = \delta (= \lambda^b)$ on $U_0^+ \cup \{\delta\}$ (respectively $U_0^- \cup \{\delta\}$).

Let $n$ be the Itô excursion measure and $n^+ := n \mid_{U_0^+}, n^- := n \mid_{U_0^-}$, the restrictions of $n$ to $U_0^+$ and $U_0^-$ respectively.

Define $n_a := n^+ \circ (\lambda^a)^{-1}$; $n_b := n^- \circ (\lambda^b)^{-1}$. For $\Gamma \in \mathcal{U}, x \in \mathbb{R}$, define

$$n(x, \Gamma) := 1_{\{a\}}(x)n_a(\Gamma) + 1_{\{b\}}(x)n_b(\Gamma).$$

We extend $n(x, \cdot)$ to the whole of $U_{\delta}$ by setting $n(x, \{\delta\}) = 0$ for every $x$. Let $H : [0, \infty) \times \Omega \times U_{\delta} \rightarrow [0, \infty]$ be $\mathcal{P} \times U_{\delta}$ measurable and such that $H(t, \omega, \delta) = 0$ for all $(t, \omega)$.

**Theorem 2.3** Let $S = (a, b)$. Let, $G, H, i_t, \tau_t, L_t$ and $n(x, \cdot)$ be as above, corresponding to the interval $(a, b)$. Then,

$$E \left[ \sum_{t \in G(\omega)} H(t, \omega, i_t(\omega)) \right] = E \left[ \sum_{s \in D(\omega)} H(\tau_s-(\omega), \omega, e_s(\omega)) \right]$$

$$= E \int_0^\infty dL_s(\omega) \int_{U_{\delta}} H(s, \omega, u)n(W_s, du)$$

$$= E \int_0^\infty ds \int_{U_{\delta}} H(\tau_s, \omega, u)n(W_{\tau_s}, du). \tag{2.3}$$
**Proof:** The first equality follows from the inclusion \( G(\omega) \subseteq D(\omega) \) and the fact that for \( s \in D(\omega) - G(\omega) \), \( H(\tau_{s-}, \omega, e_s(\omega)) = 0 \). The third equality follows by time change. Thus it suffices to prove the 2nd equality in the statement. Let \( D^+(\omega) := \{ t \in G(\omega) : W_{\tau_{t-}} = a \} \) and \( D^-(\omega) := \{ t \in G(\omega) : W_{\tau_{t-}} = b \} \). Then

\[
E \sum_{t \in D} H(\tau_{t-}, \omega, e_t) = E \sum_{t \in D^+} H(\tau_{t-}, \omega, e_t) + E \sum_{t \in D^-} H(\tau_{t-}, \omega, e_t) =: S_1 + S_2.
\]

To analyse \( S_1 \) introduce the standard Brownian motion \((\tilde{W}_t)\) where

\[
\tilde{W}_t := (W_{t+T_a} - W_{T_a}) I_{\{T_a < \infty\}} = (W_{t+T_a} - a) I_{\{T_a < \infty\}}.
\]

We will denote the excursions from 0 of \((\tilde{W}_t)\) with a tilde. Thus, \( \tilde{L}_0^t \) is the local time at 0 of \((\tilde{W}_t)\) with right continuous inverse \((\tilde{\tau}_t^0)\); \( \tilde{e}_t^0 \) is the excursion process for \( t \in \tilde{D}_0^0 = \{ s : \tilde{\tau}_s^0 \neq \tilde{\tau}_s^0 \} \). Let \( \tilde{D}_0^{0,+} = \{ s \in \tilde{D}_0^0 : \tilde{e}_s^0 \in U_0^+ \} \). Then note that, almost surely, there is a 1-1 correspondence between \( t \in D^+(\omega) \) and \( s \in \tilde{D}_0^{0,+} \) in the sense that \( \tau_{t-}(\omega) = \tilde{\tau}_s^0(\omega) + T_a \). This follows from two facts: Firstly the local time at 0 for \((\tilde{W}_t)\) at time \( t \) is precisely the local time at \( a \) for \((W_t)\) at time \( t + T_a \) on \( T_a < \infty \). Secondly, the positive excursions of \((\tilde{W}_t)\) from 0 until the hitting time of \( b - a \) are exactly the excursions of \((W_t)\) from \( a \) until the hitting time of \( b \). Further for such \( t \) and \( s, e_t(\omega) = e_{s+T_a}^0 + a = \lambda^a(e_{s}^0) \). Hence

\[
S_1 = E \sum_{s \in \tilde{D}_0^{0,+}} H(\tilde{\tau}_s^0, T_a, \omega, \lambda^a(e_{s}^0)) I_{\{T_a < \infty\}} = E \sum_{s \in \tilde{D}_0^0} \tilde{H}(\tilde{\tau}_s^0, \omega, e_{s}^0)
\]

where \( \tilde{H}(t, \omega, u) := H(t + T_a, \omega, \lambda^a(u)) I_{\{T_a < \infty\}} \) for \( t \geq 0, \omega \in \Omega, u \in U_{0,\delta} \). Note that \( \tilde{H}(t, \omega, u) \) is \( \tilde{P} \otimes U_{0,\delta} \) measurable, where \( \tilde{P} \) is the previsible sigma field with respect to the filtration \((\mathcal{F}_{t+T_a})\). Since \((\tilde{W}_t)\) is an \((\mathcal{F}_{t+T_a})\) Brownian motion, we get using basic Brownian excursion theory, the definition of the
map $\lambda_a$ and a change of variable, that

$$S_1 = E \int_0^\infty d\tilde{L}_s \int_0^U \tilde{H}(s, \omega, u) n(du)$$

$$= E \int_0^\infty d\tilde{L}_s \int_{U_0^+} H(s + T_a, \omega, \lambda^a(u)) I_{\{T_a < \infty\}} n(du)$$

$$= E \int_0^\infty dL_{s+T_a} \int_U H(s + T_a, \omega, u) 1_{\{a\}}(W_{s+T_a}) n_a(du)$$

where we have used the fact that $\tilde{L}_s^0 = L_{s+T_a}^a$, and that the latter process is, almost surely, supported on the set $\{s : W_{s+T_a} = a\}$ to obtain the last equality. Hence

$$S_1 = E \int_0^\infty dL_{s+T_a} \int_U H(s, \omega, u) 1_{\{a\}}(W_s) n_a(du).$$

Similarly,

$$S_2 = E \int_0^\infty dL_{s+T_a} \int_U H(s, \omega, u) 1_{\{b\}}(W_s) n_b(du).$$
Hence

\[
S_1 + S_2 = E \int_0^\infty dL_s \int_U H(s, \omega, u) \, n(W_s, du) \\
+ E \int_0^\infty dL_s^b \int U H(s, \omega, u) \, n(W_s, du) \\
= E \int_0^\infty d(L_s^a + L_s^b) \int_U H(s, \omega, u) \, n(W_s, du) \\
= E \int_0^\infty dL_s \int_U H(s, \omega, u) \, n(W_s, du) \\
= E \int_0^\infty ds \int_U H(\tau_s, \omega, u) \, n(W_{\tau_s}, du).
\]

This completes the proof of the Theorem. \[ \square \]

**Remark 2.4** We can arrive at the above result, using the results in [6]. Consider the closed homogenous set \( M = \{ t : W_t = a \text{ or } b \} \) and the corresponding exit system \( \{(\tilde{L}, \tilde{n}(x, .)); x \in \mathbb{R}\} \). Then \( \tilde{L}_t = L_t = L_s^a + L_s^b \) and \( \tilde{n}(x, .) \) is a measure on \( \tilde{U} := U(S) \cup U(S_1) \cup U(S_2) \) where \( S := (a, b), S_1 := (-\infty, a), S_2 := (b, \infty) \) and \( \tilde{n}(a, .)|_{U(S)} = n(a, .), \tilde{n}(b, .)|_{U(S)} = n(b, .) \).

## 3 Excursions Straddling a fixed time:

We next look at excursions into \( S \), straddling a given time \( t > 0 \). We work with the filtration generated by the Brownian motion \( (W_t) \). In other words, \( \mathcal{F}_t \) is the same as \( \mathcal{F}_t^W := \sigma\{W_s, s \leq t\} \) augmented by all \( P \) null sets. Recall that \( \sigma_t := \sigma\{s \leq t : W_s \notin S\} \). Let \( \mathcal{F}_{\sigma_t} := \sigma\{H(\sigma_t) : H(t, \omega) \text{ an } \mathcal{F}_t \text{ optional process}\} \). Recall that \( i_t(\omega) \) denotes the portion of the excursion straddling \( t \), from \( t \) upto its lifetime \( d_t(\omega) \) and that for \( u \in U \),
$R(u) := \inf\{s > 0 : u_s \notin S\}$ denotes the lifetime of the excursion $u$. Define for $s > 0$ and $F : U_{0,\delta} \to [0, \infty]$ measurable, with $F(\delta) = 0$,

$$q(s, F) := \frac{1}{n\{R > s\}} \int_{\{R > s\}} F(u) n(du). \quad (3.4)$$

Note that $n\{R > s\} > 0$ (see Remark 2.1). We then have the following proposition from Chap.XII, Propn.3.3, [11]:

**Proposition 3.1** Let $S = (-\infty, 0) \cup (0, \infty)$ and $\sigma_t$ the associated last entrance time before $t$ for $S$. Then for every $t > 0$,

$$E[F(i_{\sigma_t}) | \mathcal{F}_{\sigma_t}] = q(t - \sigma_t, F) \text{ a.s.} \quad (3.5)$$

We now wish to generalise this proposition to the case of excursions into $(a, b)$ straddling $t > 0$. For $s > 0, x \in \mathbb{R}$, let

$$q(x, s, F) = \frac{1}{n(x, \{R > s\})} \int_{\{R > s\}} F(u)n(x, du). \quad (3.6)$$

**Theorem 3.2** Let $S = (a, b)$ and $U_{\delta}, \sigma_t, \mathcal{F}_{\sigma_t}$ be associated with $S$ as above. Let $F : U_{\delta} \to [0, \infty]$ be measurable with $F(\delta) = 0$. Then for every $t > 0$, we have

$$E[F(i_{\sigma_t}(\omega)) | \mathcal{F}_{\sigma_t}] = q(W_{\sigma_t}, t - \sigma_t, F) \quad (3.7)$$

almost surely on $(\sigma_t < t)$.

The proof of the Theorem 3.2 depends on an extension of Proposition 3.1 which we now formulate. Let $(W_s) \equiv (\{W_{s+T_a} - W_{T_a}\}I_{(T_a<\infty)})$ be the standard Brownian motion introduced in the proof of Theorem 2.3 and $\tilde{\sigma}_t = \sup\{s \leq t : \tilde{W}_s = 0\}$. Let

$$\tilde{T}_{b-a} := \inf\{s > 0 : \tilde{W}_{s+\tilde{\sigma}_t} = b - a\} \text{ on } \{\tilde{\sigma}_t < t\}$$

$$= \infty \text{ on } \{\tilde{\sigma}_t = t\}.$$
Similarly let \( \hat{W}_s \equiv ((W_{s+T_b} - W_{T_b}) I_{(T_b < \infty)}), \hat{\sigma}_t := \sup\{s \leq t : \hat{W}_s = 0\} \) and 
\[
\hat{T}_{-\sigma}^{t-a} := \inf\{s > 0 : \hat{W}_{s+\hat{\sigma}_t} = -(b-a)\} \text{ on } \{\hat{\sigma}_t < t\} = \infty \text{ on } \{\hat{\sigma}_t = t\}.
\]

In what follows, we will abuse notation to refer to \( \hat{\sigma}_t \) as the excursion of \( \hat{W}_s \) from zero, starting at time \( \hat{\sigma}_t < t \) and a similar reference to \( \hat{\sigma}_t \) will mean the excursion of \( \hat{W}_s \) from zero starting at time \( \hat{\sigma}_t < t \). For \( u \in U_0 \), again by abusing notation we will denote by \( T_{b-a}(u) \) the hitting time of level \( b-a \) by the excursion \( u \) with a similar convention for \( T_{-\sigma}^{t-a}(u) \). The following Proposition relates the excursions (straddling \( t \)) of \( \hat{W}_s \) and \( \tilde{W}_s \) below and above zero with state spaces \( U_0^+, U_0^- \) respectively to the excursions (straddling \( t \)) of \( W_s \) into \( S = (a, b) \) with state space \( U = U(S) = U(a, b) \).

**Proposition 3.3** Let \( F : U_\delta \rightarrow [0, \infty] \) be measurable with \( F(\delta) = 0 \). Let \( t > 0 \) be fixed.

a) \[ E[1_{(0,t)}(\hat{\sigma}_t)1_{(t-\hat{\sigma}_t,\infty)}(T_{b-a}(\hat{\sigma}_t))F \circ \lambda^a(i_{\hat{\sigma}_t})|\mathcal{F}_\sigma] = q(W_{\sigma_t}, t - \sigma_t, F) \]
a.s. on the set \( \{\hat{\sigma}_t < t, \hat{T}_{b-a}^{t} > t - \hat{\sigma}_t\} \) and 

b) \[ E[1_{(0,t)}(\hat{\sigma}_t)1_{(t-\hat{\sigma}_t,\infty)}(T_{-\sigma}^{t-a})(i_{\hat{\sigma}_t}))F \circ \lambda^b(i_{\hat{\sigma}_t})|\mathcal{F}_\sigma] = q(W_{\sigma_t}, t - \sigma_t, F) \]
a.s. on the set \( \{\hat{\sigma}_t < t, \hat{T}_{-\sigma}^{t-a} > t - \hat{\sigma}_t\} \).

**Proof**: Let \( \alpha(s, \omega) \) be \( (\mathcal{F}_t) \)-optional. To prove a) we need to show that 
\[
E[\alpha(\hat{\sigma}_t)1_{(0,t)}(\hat{\sigma}_t)1_{(t-\hat{\sigma}_t,\infty)}(T_{b-a}(\hat{\sigma}_t))F(\lambda^a(i_{\hat{\sigma}_t}))] = E[\alpha(\hat{\sigma}_t)q(W_{\sigma_t}, t - \sigma_t, F)1_{(0,t)}(\hat{\sigma}_t)1_{(t-\hat{\sigma}_t,\infty)}(T_{b-a}^{t})]
\]

For \( (s, \omega, u) \in [0, \infty) \times \Omega \times U_{0,\delta} \) define 
\[
H(s, \omega, u) := \alpha(s)F \circ \lambda^a(u)I_{(0,t)}(s)I_{\{R > t-s\}}(u)I_{\{T_{b-a} > t-s\}}(u).
\]

Let \( \tilde{G}(\omega) \subset [0, \infty) \) be the left end points of excursion intervals of \( \hat{W}_t \) from zero. Let \( L_t := \tilde{L}_t^0 \), the local time of zero of \( \hat{W}_s \). Recall that \( \tau_t \) is the
right continuous inverse of \((L_t)\). Then we may write as in Proposition 3.3, Chapter XII, [11],

\[
E \left[ \alpha(\tilde{\sigma}_t) I_{(0,t)}(\tilde{\sigma}_t) I_{(t-\tilde{\sigma}_t,\infty)} (T_{b-a}(i_{\tilde{\sigma}_t})) F \circ \lambda^a(i_{\tilde{\sigma}_t}) \right]
\]

\[
= E \sum_{s \in \tilde{G}} H(s, \omega, i_s(\omega))
\]

\[
= E \int ds \int_{U_{0,\delta}} H(\tau_s, \omega, u) n(du)
\]

\[
= E \int ds I_{(0,t)}(\tau_s) \alpha(\tau_s) \int_{\{R > t - \tau_s \cap \{T_{b-a} > t - \tau_s\}} F \circ \lambda^a(u) n(du)
\]

\[
= E \int ds I_{(0,t)}(\tau_s) q(a, t - \tau_s, F) n^a(R > t - \tau_s)
\]

\[
= E \sum_{s \in \tilde{G}} \alpha(s) q(a, t - s, F) I_{(0,t)}(s) G(s, i_s)
\]

\[
= E \alpha(\tilde{\sigma}_t) I_{(0,t)}(\tilde{\sigma}_t) q(a, t - \tilde{\sigma}_t, F) G(\tilde{\sigma}_t, i_{\tilde{\sigma}_t})
\]

where for \((s, u) \in [0, \infty) \times U_{0,\delta}\) we define

\[
G(s, u) := I_{U_0^+ \cap \{R > t - s, T_{b-a} > t - s\}}(u);
\]

and where in the 4th equality above we have used the fact that for \(0 < s < t\),

\[(\lambda^a)^{-1}\{u \in U : R(u) > t - s\} = U_0^+ \cap \{R > t - s, T_{b-a} > t - s\}.
\]

Finally we note that when \(G(\tilde{\sigma}_t, i_{\tilde{\sigma}_t}) = 1\), \(\tilde{\sigma}_t = \sigma_t\) and \(W_{\sigma_t} = a\). This completes the proof of a). The proof of b) is similar. \(\Box\)

**Proof of Theorem 3.2:** Let \(\alpha(s, \omega)\) be \((\mathcal{F}_t)\)-optional and \(t > 0\) be given. We need to show

\[
E \alpha(\sigma_t) 1_{(\sigma_t < t)} F(i_{\sigma_t}) = E \alpha(\sigma_t) 1_{(\sigma_t < t)} q(W_{\sigma_t}, t - \sigma_t, F).
\]
Recalling the notation $i_{\tilde{\sigma}_t}$ and $\tilde{\sigma}_t$ introduced before the statement of Proposition(3.3), we note that

$$(\sigma_t < t) = (\tilde{\sigma}_t < t, \bar{T}_{b-a}^t > t - \tilde{\sigma}_t) \cup (\tilde{\sigma}_t < t, \hat{T}_{b-a}^t > t - \tilde{\sigma}_t)$$

where the sets in the right hand side are disjoint. Further we note that $\sigma_t = \tilde{\sigma}_t$ on the set ($\tilde{\sigma}_t < t, \hat{T}_{b-a}^t > t - \tilde{\sigma}_t$) and $\sigma_t = \tilde{\sigma}_t$ on the set ($\tilde{\sigma}_t < t, \bar{T}_{b-a}^t > t - \tilde{\sigma}_t$). We then have

$$E\alpha(\sigma_t) 1_{(\sigma_t < t)} F(\delta_{i_{\sigma_t}}) = E[\alpha(\tilde{\sigma}_t) 1_{(0,t)}(\bar{\sigma}_t) 1_{(t-\tilde{\sigma}_t,\infty)}(\bar{T}_{b-a}^t) F(\lambda^a(i_{\tilde{\sigma}_t}))]$$

$$+ E[\alpha(\tilde{\sigma}_t) 1_{(0,t)}(\hat{\sigma}_t) 1_{(t-\hat{\sigma}_t,\infty)}(\hat{T}_{b-a}^t) F(\lambda^b(i_{\hat{\sigma}_t}))]$$

$$= E[\alpha(\tilde{\sigma}_t) q(W_{\sigma_t}, t - \sigma_t, F) 1_{(0,t)}(\tilde{\sigma}_t) 1_{(t-\tilde{\sigma}_t,\infty)}(\bar{T}_{b-a}^t)]$$

$$+ E[\alpha(\tilde{\sigma}_t) q(W_{\sigma_t}, t - \sigma_t, F) 1_{(0,t)}(\hat{\sigma}_t) 1_{(t-\hat{\sigma}_t,\infty)}(\hat{T}_{b-a}^t)]$$

$$= E\alpha(\sigma_t) 1_{(\sigma_t < t)} q(W_{\sigma_t}, t - \sigma_t, F),$$

where to obtain the second equality we have used the result of Proposition(3.3). This completes the proof of Theorem(3.2). \( \square \)

### 4 Asymptotic Distribution of Excursions

**Stradling a fixed time:**

Let $C := C([0, \infty), \bar{S}) \cup \{\delta\}$ with the $\sigma$-field $\mathcal{C}$ generated by the Borel sigma field of $C([0, \infty), \bar{S})$ and the singleton $\{\delta\}$. We now consider only the case $S = (a, b)$. All excursion related objects are considered with respect to this $S$. We consider the excursions $i_{\sigma_t}$ as a stochastic process with values in $C$ and accordingly use a new notation. We define the $C$ valued stochastic process $(\zeta_t)$, measurable in $(t, \omega)$ as follows:

$$\zeta_t(\cdot) := W_{\sigma_t + \wedge (d_t - \sigma_t)} I(\sigma_t < t) + \delta I(\sigma_t = t)$$

where on $(\sigma_t < t)$, we note that the function $s \rightarrow W_{\sigma_t + \wedge (d_t - \sigma_t)}$ belongs to $C$. Let $E := \{a, b\} \times [0, \infty) \times C$ and $\mathcal{E} := \{a, b, \phi, \{a, b\}\} \times \mathcal{B}[0, \infty) \times \mathcal{C}$ the product sigma field on $E$. For $A \in \mathcal{E}, x = a, b, \text{ and } s > 0$, let $A(x, s) :=$
\( \{ \omega : (x, s, \omega) \in A \} \). For \( A \in \mathcal{E}, s > 0, x \in \mathbb{R} \), we recall from eqn.(3.6) the kernel
\[ q(x, s, A(x, s)) := \frac{n(x, A(x, s)) \cap \{ R > s \}}{n(x, \{ R > s \})}. \]
We then define the probability measure \( P^0 \) on \((E, \mathcal{E})\) as follows:
\[ P^0(A) := \int_0^\infty \left( \frac{q(a, s, A(a, s)) + q(b, s, A(b, s))}{2} \right) dF(s) \quad (4.8) \]
where \( F(\cdot) \) is a distribution function on \([0, \infty)\) defined as follows: Let \( P_x, x \in \mathbb{R} \), denote the distribution of \((W_s + x)\) on \( C([0, \infty), \mathbb{R})\). Then,
\[ F(s) := \frac{1}{b - a} \int_a^b (1 - \psi(x, s)) \, dx \]
where
\[ \psi(x, s) := P_x \left( a < \inf_{0 \leq r \leq s} W_r < \sup_{0 \leq r \leq s} W_r < b \right). \]
We note that \( F \) is the asymptotic distribution of \( t - \sigma_t \), conditional on \( \{ W_t \in (a, b) \} \) as \( t \to \infty \) (see [1], Thm.(4.2)). Further it is clear that \( P^0(A) = EQ(X, Y, A(X, Y)) \) where \( X, Y \) are independent, \( Y \sim F \) and \( P(X = a) = P(X = b) = \frac{1}{2} \).

We then have the following theorem.

**Theorem 4.1** Let \((\zeta_t), P^0, X, Y\) be as above. Then, conditional on \((\sigma_t < t), (W_{\sigma_t}, t - \sigma_t, \zeta_t)\) converges weakly to \( P^0 \) on \((E, \mathcal{E})\) as \( t \to \infty \).

**Proof:** Let \( f : E \to \mathbb{R} \) be a bounded and continuous function. It suffices to show that
\[ \lim_{t \to \infty} E[f(W_{\sigma_t}, t - \sigma_t, \zeta_t) \mid \sigma_t < t] = \int_E f \, dP^0. \]
We have
\[ E[f(W_{\sigma_t}, t - \sigma_t, \zeta_t) \mid \sigma_t < t] = \frac{E I_{(\sigma_t < t)} f(W_{\sigma_t}, t - \sigma_t, \zeta_t)}{P(\sigma_t < t)} = \frac{E \left[ I_{(\sigma_t < t)} E[f(W_{\sigma_t}, t - \sigma_t, \zeta_t) \mid F_{\sigma_t}] \right]}{P(\sigma_t < t)}. \]
From Theorem 3.2, we have

\[ E[f(W_{\sigma_t}, t - \sigma_t, \zeta_t) \mid \mathcal{F}_{\sigma_t}] = q(W_{\sigma_t}, t - \sigma_t, f(W_{\sigma_t}, t - \sigma_t, \cdot)) \]

almost surely on \( \{\sigma_t < t\} \). On the other hand we know from the results in [2] that \( (W_{\sigma_t}, t - \sigma_t) \) converges weakly to \( (X,Y) \) conditional on \( \{\sigma_t < t\} \). Using Remark 2.1 it can be shown that \( q(x, s, f(x, s, \cdot)) \) is a bounded continuous function on \( \{a, b\} \times (0, \infty) \). The result follows. □

**Remark 4.2** The limiting distribution \( P_0 \) can be described in terms of the measure \( Q_{ex} \) introduced in [7]. We recall that the measure \( Q_{ex} \) was introduced on the space of excursions as an 'equilibrium measure' or more specifically as the 'Palm measure' corresponding to a stationary point process. It is natural to interpret our results in Sec.4 as \( t \to \infty \) in terms of this equilibrium measure. Let \( M, \{(\tilde{L}, \tilde{n}(x, \cdot)); x = a \text{ or } b\} \) be as in Remark 2.4, with \( \tilde{L} = L = L^a + L^b \). Then it was shown in [7] that \( Q_{ex} \) was given as \( Q_{ex}(\cdot) = \int \alpha(dx)\tilde{n}(x, \cdot) \) where \( \alpha(A) := \int dxE^x \int_0^1 I_A(W_s) dL_s \). It follows from the occupation density formula that \( \alpha = \delta_a + \delta_b \) and consequently from Remark 2.4 that, \( Q_{ex}\mid U(S) = n(a, \cdot) + n(b, \cdot) \). We than have for \( x = a \text{ or } b, s > 0, A \in \mathcal{U}(S) \)

\[ q(x, s, A) := \frac{Q_{ex}(A \cap \{R(u) > s, u(0) = x\})}{Q_{ex}(\{R(u) > s, u(0) = x\})}. \]

**An application :** For \( u > 0, y > 0 \), consider the probabilities

\[ \phi(t) := P_a\{0 < t - \sigma_t < u, 0 < W_t - W_{\sigma_t} < y\} \]

where for each \( a \in \mathbb{R} \), \( P_a\{\omega : W_0 = a\} = 1 \). In [1],Theorem (5.1), it was shown that \( \phi(t) \) satisfies a renewal equation. With the renewal theorem in mind, a natural question is to find the limit of \( \phi(t) \) as \( t \to \infty \). However, it is easy to see that the probabilities that define \( \phi(t) \) converge to zero since the events in question are contained in \( \{W_t \in (a,b)\} \). On the other hand, since these events can be expressed as functionals of the excursion straddling \( t \), we can apply the previous theorem to compute the limit of the conditional probability

\[ P_a\{\omega : 0 < t - \sigma_t < u, 0 < W_t - W_{\sigma_t} < y|W_t \in (a,b)\} \]
as $t \to \infty$. Let

$$A := \{ (x, s, \omega) : 0 < \omega_s - \omega_0 < y, 0 < s < u, x = a \text{ or } b \}.$$ 

Then by Thm.(4.1),

$$P_a\{ 0 < t - \sigma_t < u, 0 < W_t - W_{\sigma_t} < y | W_t \in (a, b) \} = P_a\{(W_{\sigma_t}, t - \sigma_t, \zeta_t) \in A | \sigma_t < t \} \to P^0(A)$$

provided $P^0(\partial A) = 0$. Note that

$$(\partial A) = (\bar{A} - A^\circ) = \{(x, s, \omega) : s = u, 0 \leq \omega_u - \omega_0 \leq y, x = a \text{ or } b \}
\cup \{(x, s, \omega) : 0 < s < u, \omega_s - \omega_0 = 0 \text{ or } y, x = a \text{ or } b \}
=: A_1 \cup A_2$$

Clearly, from the definition of the measure $P^0$ we have,

$$P^0(A_1) = P^0\{(a, s, \omega) : s = u, 0 \leq \omega_u - \omega_0 \leq y, x = a \text{ or } b \} = 0.$$ 

As for the second set $A_2$ in the union, from the definition of $P^0$ (eqn.(4.8)) it suffices to show that for every $0 < s < u, x = a, b, q(x, s, (A_2)(x, s)) = 0$. Again, from the definition of $q(x, s, \cdot)$ (eqn.(3.6)) it suffices to show that for each $x = a, b$ and $0 < s < u, n(x, A_2(x, s) \cap \{ R > s \}) = 0$. From the definition of the kernels $n(x, \cdot)$, we have for $0 < s < u, x = a$,

$$n(a, \{ \omega : \omega_s - \omega_0 = 0 \text{ or } y \} \cap \{ R > s \})
= n^+ \circ \lambda_a^{-1}(\{ \omega : \omega_s - \omega_0 = 0 \text{ or } y \} \cap \{ R > s \})
= n^+(\{ \omega : \omega(s \wedge T_{b-a}) = y \} \cap \{ R \circ \lambda_a > s \})
= n^+(\{ \omega : \omega(s \wedge T_{b-a}) = y \} \cap \{ R_1 \wedge T_{b-a} > s \})
= n^+(\{ \omega : \omega(s) = y \} \cap \{ R_1 \wedge T_{b-a} > s \})
= 0.$$

where $R_1(\omega)$ and $T_{b-a}(\omega)$ are respectively the life time and the hitting time of $b - a$ of the excursion $\omega$ starting at 0 and the last equality follows from the absolute continuity of the map $n \circ \omega_s^{-1}, s > 0$. This proves that $P^0(\partial A) = 0$.

**Acknowledgement**: The author would like to thank Jean Bertoin for pointing out reference [7].
References

[1] Athreya, K.B. and Rajeev, B (2013) : Brownian Crossings via Regeneration times, Sankhya, Vol. 75, Series A, Part 2, p. 194-210.

[2] Athreya, K.B. and Rajeev, B (2014) : Weak Convergence of the past and future of Brownian motion given the present. (Pre-print)

[3] Blumenthal, Robert M. (1992): *Excursions of a Markov Process*, Birkhauser.

[4] Getoor, R and Sharpe, M (1973): Last exit decompositions and distributions, Indiana Univ. Math. J., 23, 377-404.

[5] Itô, K. (1970): Poisson point processes attached to Markov processes, Proc. Sixth Berkeley Symposium, Math. Stat. Prob., vol. 3, University of California, Berkeley, 1970, p. 225-239.

[6] Maisonneuve, B. (1975): Exit Systems, Annals of Probability, 3, p. 399-411.

[7] Pitman, J. (1987): Stationary excursions, Séminaire de Probabilités, Vol. 21, p. 289-302.

[8] Rajeev, B. (1989a): Sojourn times of Martingales, Sankhya, Series A, Vol. 51, Part 1, p. 1-12.

[9] Rajeev, B (1989) : Crossings of Brownian motion: A semi-martingale approach, Series A, vol. 51, Part 3, p. 251-268.

[10] Rajeev, B. (1989b): Crossings of Semi-Martingales, Seminaire de Probabilites XXIV, Lecture Notes in Mathematics 1426, Springer Verlag, p. 107- p. 116.

[11] Revuz, D. and Yor, M. (1991): *Continuous Martingales and Brownian motion*, 3rd edition, Springer Verlag, Berlin.