THERE EXIST NO MINIMALLY KNOTTED PLANAR SPATIAL GRAPHS ON THE TORUS

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Abstract. We show that all nontrivial embeddings of planar graphs on the torus contain a nontrivial knot or a nonsplit link. This is equivalent to showing that no minimally knotted planar spatial graphs on the torus exist that contain neither a nontrivial knot nor a nonsplit link all of whose components are unknots.

1. Introduction

All considered graphs are undirected finite graphs and we will work in the piecewise linear category. A graph embedding is an embedding $f : G \to S^3$ of a graph $G$ in $S^3$ up to ambient isotopy and the corresponding spatial graph $G$ is the image of this embedding. A graph $G$ is planar if there exists an embedding $f : G \to S^2$. Such an embedding is called trivial and its image is a trivial spatial graph. A spatial graph $G$ is minimally knotted if $G$ is nontrivial but for every edge $e$, $G - e$ is trivial. Some authors call minimally knotted spatial graphs almost trivial, almost unknotted or Brunnian. In this paper, a nontrivial link is a nonsplit link with at least two components.

Previous research on minimally knotted spatial graphs has been undertaken: The first example of a minimally knotted spatial graph was an embedding of a handcuff graph given by Suzuki [1]. Kawauchi [2], Wu [3] and Inaba and Soma [4] showed that every planar graph has a minimally knotted embedding. Ozawa and Tsutsumi [5] proved that minimally knotted embeddings of planar graphs are totally knotted. Especially minimally knotted $\Theta_n$-graphs have generated some interest. Kinoshita [6] gave the first example of a minimally knotted $\Theta_3$-graph (see Figure 1) which Suzuki [7] generalised to give examples of minimally knotted $\Theta_n$-graphs for all $n \geq 3$. Closely related are ravels which are nontrivial embeddings of $\Theta_n$-graphs that contain no nontrivially knotted subgraph; this definition is equivalent to the one given by Farkas, Flapan and Sullivan [8]. The concept of ravels has been introduced by Castle, Evans and Hyde [9] as local entanglements that are not caused by knots or links and may lead to new topological structures in coordination polymers. A ravel in a molecule has been synthesized by Lindoy et al [10]. Castle, Evans and Hyde [11] conjectured the following:

Conjecture. (Castle, Evans, Hyde [11])

All nontrivial embeddings of planar graphs on the torus include a nontrivial knot or a nonsplit link.

With Theorem 1 we prove that their conjecture is true.

Theorem 1. (Knots and links existence)

Let $G$ be a planar graph and $f : G \to S^3$ be an embedding of $G$ with image $G$. If $G$ is contained in the torus $T^2$ and contains no nontrivial knot nor a nonsplit link, then $f$ is trivial.

Since $\Theta_n$-graphs are planar, it follows from Theorem 1 that on the torus there exist no minimally knotted embeddings of $\Theta_n$-graphs with $n > 2$. This gives us the following Corollary:

Corollary. (Ravels do not embed on the torus)

Every nontrivial embedding of $\Theta_n$-graphs on the torus contains a knot.

We conclude by showing that all made assumptions are necessary. Explicit ambient isotopies that transform spatial graphs which fulfil the assumptions of Theorem 1 into the plane $\mathbb{R}^2$, are given in [12]. There is also a consequence of Theorem 1 shown: Nontrivial 3-connected and simple planar spatial graphs that are embedded on a torus are chiral. A graph is simple if it contains no loops and no multi-edges. It is 3-connected if at least three vertices and their incident edges have to be deleted to decompose the graph or to reduce it to a single vertex.

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2.1. Outline of the proof. The proof uses two theorems of Scharlemann, Thompson [13] and Ozawa, Tsutsumi [5]. We assume that the spatial graph $G$ we consider is given by an embedding $f : G \to T^2$ of a planar graph $G$ and furthermore that $G$ contains no nontrivially knotted or linked subgraph. We conclude that $G$ must be trivial. During the proof, we need the following two definitions:

**Definition.** An embedding $f : G \to S^3$ of a graph $G$ is **primitive**, if for each component $G_i$ of $G$ and any spanning tree $T_i$ of $G_i$, the bouquet graph $f(G_i)/f(T_i)$ obtained from $f(G_i)$ by contracting all edges of $f(T_i)$ in $S^3$ is trivial.

**Definition.** An embedding $f : G \to S^3$ of a graph $G$ is **free**, if the fundamental group of $S^3 - f(G)$ is free.

The argument of the proof is as follows: We start showing that the statement is true for non-standardly embedded tori in Lemma 1. With Lemma 2 we argue that it is sufficient to consider connected graphs. Then we show in Lemma 3 that a bouquet graph on $T^2$ either contains a nontrivial knot or is trivial. Since any connected spatial graph $G$ on $T^2$ contracts to a bouquet graph on $T^2$, it follows that $G$ is primitive if it contains no nontrivial knot. By Theorem 2 we know that the restriction $f_{|G'}$ is free for all connected subgraphs $G'$ of $G$. By Lemma 2 together with Theorem 3 we conclude that $G$ is trivial.

2.2. Preparations for the proof.

**Lemma 1.** (Nonstandardly embedded torus) Let $\mathbb{T}^2$ be a torus that is not standardly embedded. Any spatial graph $G$ that is embedded on $\mathbb{T}^2$ and that contains no nontrivial knot is trivial.

**Proof.** If the spatial graph $G$ contains a cycle that follows a longitude of the torus $\mathbb{T}^2$, this cycle is knotted since $\mathbb{T}^2$ itself is knotted. Therefore, no such subgraph of $G$ can exist and we find a meridian $m$ of $\mathbb{T}^2$ that has no intersection with $G$. This shows that $G$ in embedded in the twice punctured sphere $\mathbb{T}^2 - m \cong S^2 - \{p_1, p_2\}$. Therefore, $G$ is trivial.

It follows from Lemma 1 that the statement of Theorem 1 is true for nonstandardly embedded tori. Therefore, we consider the standardly embedded torus $T^2$ from now on which saves us from considering case studies.

**Lemma 2.** (Connectivity Lemma) The image $G$ of an embedding $f : G \to T^2 \subset S^3$ of a graph $G$ with $n > 1$ connected components on the standard torus $T^2$ either contains a nonsplit link, or no nonsplit linked subgraph and decomposes into $n$ disjoint components of which at least $n - 1$ components are trivial.

**Proof.** Take any connected component $f(G_i)$ of the embedding $f(G)$ on the torus $T^2$. The complement of $f(G_i)$ in the torus (without considering the rest of the spatial graph $f(G - G_i)$) is a collection of pieces that can be the punctured torus, discs and essential annuli. (An essential annulus contains a simple closed curve that does not bound a disc in the torus.)

In the case that the complement of $f(G_i)$ in $T^2$ includes the punctured torus, $f(G_i)$ is trivial and splits from the other components.

If the complement of $f(G_i)$ in $T^2$ is only a collection of discs, then all other components of $f(G)$ lie in one of those discs and therefore the graph is split. ($f(G_i)$ might or might not contain a nonsplit link.)

In the case that the complement of $f(G_i)$ in $T^2$ includes an essential annulus $A$, it is possible that other components of $G$ are embedded in this annulus. A component $G_j$ might be embedded in the annulus in two ways: Either the complement of $f(G_j)$ in $A$ is a punctured annulus and therefore $f(G_j)$ is trivial and splits from the rest of the spatial graph $f(G - G_j)$. Or $f(G_j)$ splits the annulus into two annuli. The annulus $A$ has one type of an essential curve $c$ running inside it; $c$ is parallel to the boundary curves of $A$. If $f(G_j)$ splits $A$ into two annuli, a subgraph of $f(G_j)$ must be deformable to be parallel to $c$. If $c$ is a meridian or a preferred longitude of $T^2$, both components $f(G_i)$ and $f(G_j)$ are split and trivial since the torus is a standard torus. If $c$ is neither a meridian nor a longitude of $T^2$, $f(G_i)$ and $f(G_j)$ form a nonsplit link.

**Lemma 3.** (Bouquet Lemma) The image $B$ of an embedding $f : B \to T^2 \subset S^3$ of a connected bouquet graph $B$ on the torus $T^2$ either contains a nontrivial knot or is trivial.
Every nontrivial embedding of planar graphs in the torus cannot be planar by assumption, it follows from Theorem 3 that \( f \) is not a nontrivial knot. By Lemma 3, an unknotted bouquet graph \( B \) on the torus without self-intersections:

- (1) \( T(0, 0) \) loops that bound a disc in \( T^2 \) (trivial elements in \( \pi_1(T^2) \)),
- (2) \( T(0, 1) \) meridional loops,
- (3) \( T(1, 0) \) longitudinal loops,
- (4) \( T(1, n) \) loops or alternatively \( T(n, 1) \) loops, \( n \geq 1 \)

Loops of type (1) do not contribute to nontriviality of \( B \).

If \( B \) has loops of the types (1), (2) and (3) only, it is trivial.

If \( B \) has loops of type (4), there are – beside the loops \( T(0, 0) \) – only three types of loops simultaneously embeddable on the torus without self-intersections: \( T(1, 0), T(1, n) \) and \( T(1, n + 1) \) (respectively \( T(0, 1), T(n, 1) \) and \( T(n + 1, 1) \)). This can easily be confirmed by applying the formula of Rolfsen’s exercise 2.7 [14]: If two torus knots \( T(p, q) \) and \( T(p', q') \) intersect in one point transversally, then \( pq' - qp' = \pm 1 \). Such a bouquet is trivial. \( \square \)

**Theorem 2.** (Ozawa and Tsutsumi’s freeness criterion [5])

An embedding \( f \) of a graph \( G \) in \( S^3 \) is primitive if and only if the restriction \( f|_{G'} \) is free for all connected subgraphs \( G' \) of \( G \).

**Theorem 3.** (Scharlemann and Thompson’s planarity criterion [13])

An embedding \( f(G) \to S^3 \) of a graph \( G \) is trivial if and only if

(a) \( G \) is planar and
(b) for every subgraph \( G' \subset G \), the restriction \( f|_{G'} \) is free.

2.3. **The proof.** We are now ready to prove Theorem 1:

Proof. It follows from Lemma 1 that the statement of Theorem 1 is true for nonstandardly embedded tori. Therefore, we assume that \( G \) is embedded in the standard torus \( T^2 \). By the connectivity Lemma 2 we can assume that \( G \) is connected. Any connected spatial graph contracts to a spatial bouquet graph \( B \) if a spanning tree \( T \) is contracted in \( S^3 \). If the spatial graph is embedded in a surface, edge contractions can be realised in the surface. It follows that a connected spatial graph \( G \) which is embedded in the torus \( T^2 \), contracts to a bouquet graph \( B \) which also is embedded in \( T^2 \) if a spanning tree is contracted. Since \( G \) contains no nontrivial knot by assumption, \( B \) also contains no nontrivial knot. By Lemma 3, an unknotted bouquet graph \( B \) on the torus \( T^2 \) is trivial. Therefore, any bouquet graph \( B = f(G)/f(T) \) which is obtained from \( f(G) \) by contracting all edges of \( f(T) \) in \( S^3 \) is trivial and \( f \) is primitive by definition. By Theorem 2, the restriction \( f|_{G'} \) is free for all connected subgraphs \( G' \) of \( G \). Then Lemma 2 ensures that the restriction \( f|_{G'} \) is free for all subgraphs \( G' \subset G \) since \( G \) contains no nonsplit link by assumption. As \( G \) is planar by assumption, it follows from Theorem 3 that \( f(G) \) is trivial. \( \square \)

**Corollary.** (Ravels do not embed on the torus)

Every nontrivial embedding of \( \Theta_n \)-graphs on the torus contains a knot.

Proof. As there exist no pair of disjoint cycles in a \( \Theta_n \)-graph, such a graph does not contain a nonsplit link. Since \( \Theta_n \)-graphs are planar, the corollary follows directly from Theorem 1. \( \square \)

**Remark.** Simple 3-connected nontrivial embeddings of planar graphs in the torus are chiral as shown in [15].

2.4. **All assumptions that have been made are necessary.** This can be seen by considering the following examples:

- There exist nontrivial embeddedings on \( T^2 \) that contain neither a nontrivial knot nor a nonsplit link. Those are embeddings of graphs which are not planar.
  
  **Examples:** \( K_{3,3} \) and \( K_4 \) embedded as shown left in the figure below.

- There exist nontrivial embeddings of planar graphs that contain neither a nontrivial knot nor a nonsplit link. Those are not embedded in the torus.
  
  **Examples:** Kinoshita-theta curve (middle in the figure below) and every ravel.

- There exist nontrivial embeddings of planar graphs on \( T^2 \).
  
  **Examples:** Spatial graphs that are subdivisions of nontrivial torus knots with \( n > 0 \) vertices and \( n \) edges.
  (Right in the figure below.)
Figure 1. All assumptions are necessary.

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