Generalized Misner-Sharp energy in the generalized Rastall theory

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Employing the unified first law of thermodynamics and the field equations of the generalized Rastall theory, we get the generalized Misner-Sharp mass which differs from those of the Einstein and Rastall theories. Moreover, using the first law of thermodynamics, the obtained generalized Misner-Sharp mass and the field equations, the entropy of the static spherically symmetric horizons is also addressed in the framework of the generalized Rastall theory. In addition, by generalizing the study to the flat FRW universe, the apparent horizon entropy is also calculated. Considering the effects of applying the Newtonian limit to the field equations on the coupling coefficients of the generalized Rastall theory, our study indicates i) the obtained entropy-area relation is the same as that of the Rastall theory, and ii) the Bekenstein entropy is recovered when the generalized Rastall theory reduces to the Einstein theory. The validity of the second law of thermodynamics is also investigated in the flat FRW universe.

I. INTRODUCTION

Based on the curvature-matter non-minimal coupling theories, the ordinary energy-momentum conservation law is not valid \[ \mathcal{L} \neq 0 \], a hypothesis that returns to Rastall [4]. Recently, introducing a generalized version for the Rastall theory, it has been shown that such non-minimal coupling, and indeed, the geometry ability to couple to the matter fields in a non-minimal way, can theoretically describe inflation and the current accelerated universe without need for considering the dark energy candidates and inflationary fields [3]. The profound connection between thermodynamics and gravitational theories [4–17] motivates physicists to look for the Misner-Sharp mass in the various gravitational theories for studying the thermodynamic aspects of the theories [18–25]. The same analysis has been done in the Rastall framework [26] indicating that only when this theory reduces to the Einstein theory, the Bekenstein entropy is recovered in the static spherically symmetric spacetimes [26], a result in agreement with those of the dynamic studies [27, 28].

In Ref. [29], studying some cosmological consequences of the generalized Rastall theory [3], authors address some similarities between the generalized Rastall theory and other cosmological frameworks such as the Einstein gravity with particle creation mechanism [30] such that the Rastall parameter is related to the particle creation parameter. They also used the Misner-Sharp mass of the Einstein theory [18] as well as the Cal-Kim temperature [16] in order to obtain the entropy of the apparent horizon of FRW spacetime. Based on their results, the generalized Rastall theory preserves the Bekenstein bound of entropy [29].

Here, we are interested in finding the generalized Misner-Sharp mass in the generalized Rastall theory by using the thermodynamic laws and the corresponding field equations. Our final goals are also to find the entropy corresponding to the horizons of the static spherically symmetric spacetime and the flat FRW universe. We present our analysis and results in the next section, where the requirements for the validity of the second law of thermodynamics will also been addressed. The last section is devoted to a summary. The unit of \( c = \hbar = 1 \) is used throughout this paper.

II. THERMODYNAMICS OF THE GENERALIZED RASTALL THEORY

Based on the generalized Rastall theory [3]

\[ T^{\mu \nu} - \mu = (\lambda R)^{\mu \nu}, \]

(1)

which finally leads to

\[ G_{\mu \nu} + \kappa \lambda g_{\mu \nu} R = \kappa T_{\mu \nu}. \]

(2)

where \( \kappa \) is an unknown constant, called the Rastall gravitational coupling constant, and \( \lambda \) is the Rastall parameter. The field equations look very similar to those of the Rastall theory, and indeed, only one difference exists. Unlike the Rastall theory, the Rastall parameter is variable here. Applying the Newtonian limit to this field equations, one easily reaches

\[ \frac{\kappa}{4\kappa \lambda - 1}(3\kappa \lambda - \frac{1}{2}) = \kappa_G, \]

(3)

in which \( \kappa_G \equiv 4\pi G \), meaning that this generalized Rastall theory let \( G \) change, because \( \lambda \) is not generally constant. As a result, one easily obtains that if we presume \( \kappa \equiv 8\pi G \), then the Newtonian limit automatically indicates that \( \lambda = 0 \) meaning that we are in the Einstein framework, and thus the Einstein results should be recovered. Eq. (3) is similar to that of the Rastall theory [4, 26], a result due to the fact that both theories include

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the same additional geometrical term (the Ricci scalar) compared with the Einstein theory.

The unified first law of thermodynamics is written as

\[ dE = A \Psi_a dx^a + W dV. \]  \hspace{1cm} (4)

Here, \( \Psi_a = T^a_b \partial_b r + W \partial_a r \) and \( W = -\frac{h^{ab} \partial_b r}{2} \) denote the energy supply vector and the work density, respectively, where \( h^{ab} \) is the metric on the hypersurface \((t,r)\). Additionally, \( A = 4\pi r^2 \) is the area of the system boundary located at radius \( r \). For the FRW and spherically symmetric spacetimes, the apparent and event horizons are proper causal boundary, respectively [6–29].

A. Thermodynamic analysis of the static spherically symmetric spacetimes

Now, consider the spherically symmetric static spacetime

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \]  \hspace{1cm} (5)

while it is supported by an energy-momentum source with energy density \( \rho \) and pressure \( p \) (or equally the energy-momentum tensor \( T^\nu_\nu = diag(-\rho, p, p, p) \)), and its horizon is located at \( r_h \) (leading to \( f(r) = 0 \)). In this manner, Eq. (11) easily leads to

\[ dE = 4\pi r^2 \rho dr. \]  \hspace{1cm} (6)

Using the \( t-t \) component of (2), one can get \( \rho \) placed in the above equation to reach at

\[ dE = \frac{4\pi}{\kappa} [1 - \frac{d(rf(r))}{dr}] + \kappa \lambda \left( \frac{dr^2 f'(r)}{dr} - 2(1 - \frac{d(rf(r))}{dr}) \right) dr. \]  \hspace{1cm} (7)

This is in fact the differential of the Misner-Sharp mass content of the generalized Rastall theory introduced in \[3\], and its integral leads to

\[ E = \frac{4\pi}{\kappa} \left[ r(1 - f(r)) + \kappa \left( \frac{dr^2 f'(r)}{dr} - 2(1 - \frac{d(rf(r))}{dr}) \right) \right] dr. \]  \hspace{1cm} (8)

In general, one can find \( E \) by knowing the exact from of \( \lambda \). For the Misner-Sharp mass confined to the radius \( r_h \), this equation leads to

\[ E_h = \frac{4\pi}{\kappa} \left[ r_h + \kappa \left( \int \frac{dr^2 f'(r)}{dr} - 2(1 - \frac{d(rf(r))}{dr}) \right) dr \right] r = r_h, \]  \hspace{1cm} (9)

and therefore, the energy changes due to the hypothetical displacement of the horizon radius from \( r_h \) to \( r_h + dr_h \) is evaluated as

\[ dE_h = E_{r_h + dr_h} - E_{r_h} = \frac{dE_h}{dr_h} dr_h = \frac{4\pi}{\kappa} [dr_h + \kappa \lambda \left( \frac{dr^2 f'(r)}{dr} - 2(1 - \frac{d(rf(r))}{dr}) \right) dr_h]. \]  \hspace{1cm} (10)

At the \( \lambda = 0 \) limit, the above equations reduce to

\[ dE = \frac{4\pi}{2\kappa G} [1 - \frac{d(rf(r))}{dr}] dr \Rightarrow E = \frac{r}{2G} [1 - f(r)], \]  \hspace{1cm} (11)

nothing but the Misner-Sharp mass in the Einstein theory [13, 23]. As a proper result, it leads to \( E_h = \frac{4\pi r_h}{2G} \) for the Misner-Sharp mass confined to the horizon located at \( r_h \). Thus, the emergence of the Einstein result is parallel to assume \( \kappa = 2\kappa G \) or equally \( \lambda = 0 \). It is also easy to check the \( \lambda = constant \) case recovers the result of the Rastall theory [20].

Now, let us use the \( r-r \) component of (2) to find pressure at radius \( r \) as

\[ P(r) = \frac{1}{\kappa} \left( \frac{1}{r^2} [rf'(r) - 1 + f(r)] - \frac{\kappa \lambda}{r^2} \left[ r^2 f''(r) + 4rf'(r) - 2 + 2f(r) \right] \right), \]  \hspace{1cm} (12)

leading to

\[ P(r_h) = \frac{1}{\kappa r_h^2} \left( r_h f'(r_h) - 1 \right) - \frac{\kappa \lambda}{r_h^2} \left[ r_h f''(r_h) + 4rf'(r_h) - 2 \right], \]  \hspace{1cm} (13)

on the event horizon where \( f(r_h) = 0 \). Assuming \( dV = 4\pi r^2 dr \) (in accordance with Eq. (3) compatible with the \( dE = \rho dV \) relation), and using Eq. (13), we can reach

\[ P(r_h) dV = \frac{2f'(r_h)}{\kappa} d\left( \frac{A}{4} \right) - \frac{4\pi}{\kappa} [1 + \kappa \lambda (r_h^2 f''(r_h) + 4rf'(r_h) - 2)] dr, \]  \hspace{1cm} (14)

on the event horizon \( (f(r_h) = 0) \). Additionally, since \( f(r_h) = 0 \), one writes

\[ [1 + \kappa \lambda (r_h^2 f''(r_h) + 4rf'(r_h) - 2)] dr = (1 - 2\kappa \lambda) dr_h + \kappa \lambda [d(r^2 f'(r))_{r = r_h}] + 2\kappa \lambda r_h f'(r) + dr_h + \kappa \lambda [d(r^2 f'(r))_{r = r_h}] = 2dr_h - d(rf(r))_{r = r_h}, \]  \hspace{1cm} (15)
nothing but $\frac{d}{dt}dE_h$ compared with Eq. (10). Thus, Eq. (14) reduces to

$$P(r_h)dV = \frac{2f'(r_h)}{\kappa}d(A^4) - dE_h,$$

(16)

compared with the first law of thermodynamics ($PdV = TdS - dE$) \cite{19,20} to get

$$TdS_h = \frac{2f'(r_h)}{\kappa}d\left(\frac{A^4}{4}\right),$$

(17)

where $T$ and $S_h$ denote the horizon temperature and entropy, respectively. Now, bearing the horizon temperature ($T = \frac{f'(r_h)}{4\pi}$) in mind \cite{10,22}, the horizon entropy is obtained as

$$dS_h = \frac{8\pi}{\kappa}d\left(\frac{A^2}{4}\right) \Rightarrow S_h = \frac{8\pi A}{\kappa}.$$  

(18)

At first sight, it looks like the Bekenstein entropy, but in fact, it reduces to the Bekenstein entropy whenever $G$ is constant and $\lambda = 0$ parallel to the $\kappa = 8\pi G = constant$ constraint. Indeed, this is the same as the entropy of the Rastall theory, despite the fact that $\lambda = constant \neq 0$ (or equally $G$ is constant) in the Rastall theory \cite{26,27}. The latter may be due to that i) both the Rastall theory and its generalized version modify the Einstein field equations with the same geometrical term ($R$), and ii) in both theories, unlike the Rastall parameter, the Rastall gravitational coupling is constant.

**B. Entropy of the apparent horizon of FRW universe**

Now, consider a flat FRW universe with scale factor $a(t)$ and line element

$$ds^2 = -dt^2 + a^2(t) \left[dr^2 + r^2d\Omega^2\right],$$

(19)

filled by an energy-momentum source specified by $T^\mu_\nu = diag(-\rho, p, p, p)$. Its apparent horizon, as the proper causal boundary \cite{12,17}, is located at \cite{16,17}

$$\tilde{r}_A = \frac{1}{H},$$

(20)

where $H = \frac{\dot{a}}{a}$ is called the Hubble parameter, and has temperature $T = \frac{H}{2\pi}$ \cite{15,17}. During the time interval $dt$, the energy flux crossing ($\delta Q$) the apparent horizon is evaluated as

$$\delta Q = A\Psi_adx^a,$$

(21)

in which $A$ denotes the area of the apparent horizon \cite{15,16}. Bearing the definition of $\Psi_a$ in mind, one can reach

$$\delta Q = -\frac{3V(\rho + p)H}{2}dt + \frac{A(\rho + p)}{2}(d\tilde{r} - \tilde{r}Hdt),$$

(22)

combined with the Clausius relation ($TdS_A = \delta Q$) and $V = \frac{4\pi}{3}r^3$ (the volume confined by the apparent horizon) to get

$$dS_A = -\frac{\delta Q}{T} = 6\pi V(\rho + p)dt,$$

(23)

where $S_A$ is the entropy corresponding to the apparent horizon (the causal boundary). Before proceeding, we use the field equations (2) for obtaining

$$\frac{dH}{dt} = \dot{H} = -\frac{\kappa}{2}(\rho + p),$$

(24)

inserted into Eq. (23) to find

$$dS_A = -\frac{12\pi V}{\kappa}dH.$$  

(25)

The integral of this equation is straightforward leading to

$$S_A = \frac{8\pi A}{\kappa}.$$  

(26)

for the apparent horizon entropy in the generalized Rastall theory \cite{3}. The same as the static spherically symmetric spacetimes, the obtained relation i) reduces to that of the Einstein theory at the appropriate limit $\kappa = 8\pi G$ (or equally $\lambda = 0$), and ii) is the same as that of the Rastall theory \cite{26,27}. Finally, one can use Eqs. (23) and (24) to see that, during the cosmic evolution, for which $\dot{H} < 0$, the second law of thermodynamics ($\frac{dS_A}{dt} \geq 0$) is met whenever the $\rho + p > 0$ condition is satisfied which yields $\kappa > 0$.

**III. SUMMARY**

Using the $t - t$ component of the field equations and the unified first law of thermodynamics, we could obtain the generalized Misner-Sharp mass, confined to the event horizon of the static spherically symmetric spacetimes, in the generalized Rastall theory. Thereinafter, combining this result with the $r - r$ component of the field equations, the entropy corresponding to the horizon has been calculated. We also generalized our investigation to the flat FRW universe, and got the apparent horizon entropy. Applying the Newtonian limit to the Rastall field equations, relation between the Newtonian gravitational coupling, $\lambda$ and $\kappa$ has also been established which is similar...
to that of the Rastall theory \[4, 26\] due to the fact that both theories add the same geometrical term \((R)\) to the Einstein field equations.

Our study shows that the horizon entropy in the generalized Rastall theory is the same as that of the Rastall theory \[26, 27\] which may have two reasons including
\(i\) both theories modify the Einstein theory with the same geometrical term, and
\(ii\) unlike the Rastall parameter, the Rastall gravitational coupling is constant in both theories. It is worthwhile mentioning that, the same as the Rastall theory and also other works \[3, 26–28\], the results indicate that the \(S \propto A\) relation is valid in the generalized Rastall theory, a property preserved by the Bekenstein entropy. In both static and dynamic cases, the obtained entropy relation reduces to the Bekenstein entropy at the appropriate limit \(\kappa = 8\pi G\) (or equally \(\lambda = 0\)), a desired result. Finally, we saw that if \(\rho + p > 0\), then the second law of thermodynamics is satisfied in this theory.

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