TWICE-PUNCTURED HYPERBOLIC SPHERE WITH A CONICAL SINGULARITY AND GENERALIZED ELLIPTIC INTEGRAL

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ABSTRACT. We describe, in terms of generalized elliptic integrals, the hyperbolic metric of the twice-punctured sphere with one conical singularity of prescribed order. We also give several monotonicity properties of the metric and a couple of applications.

1. INTRODUCTION

The hyperbolic metric $\rho(z)|dz|$ on the thrice-punctured sphere $\hat{\mathbb{C}}\setminus\{0, 1, \infty\}$ is one of the fundamental tools in complex analysis. Indeed, for instance, the big Picard theorem can be derived by a careful look at the metric $\rho(z)|dz|$ and the distance induced by it. It is known that the density function $\rho(z)$ can be expressed explicitly as

$$\rho(z) = \frac{\pi}{8|z(1-z)|\Re\{K(z)K(1-z)\}},$$

where $K(z)$ is the complete elliptic integral of the first kind given in (3.1) (see [2] or [15]). On the other hand, it has been recognized that generalized elliptic integrals $K_a(z)$ and $E_a(z)$, defined in (3.2) and (3.3) respectively, share many properties with the original complete elliptic integrals (cf. [4]).

In the present paper, it is shown that the hyperbolic metric of a twice-punctured sphere with one conical singularity of prescribed angle can be expressed in terms of these generalized complete elliptic integrals.

2. HYPERBOLIC METRIC WITH CONICAL SINGULARITIES

A hyperbolic metric of a compact Riemann surface $R$ with conical singularities of angle $2\pi\theta_j$, $\theta_j \in [0, +\infty) \setminus \{1\}$, at points $p_j \in R$, $j = 1, \ldots, N$, is a conformal metric on $R \setminus \{p_1, \ldots, p_N\}$ of the form $ds = e^{\varphi(z)}|dz|$, where $\varphi$ is a smooth function satisfying the Liouville equation

$$\Delta \varphi = 4e^{2\varphi}$$

on $R \setminus \{p_1, \ldots, p_N\}$ and possessing the asymptotic behavior

$$\varphi(z) = \begin{cases} -(1-\theta_j)\log|z-z_j| + O(1) & \text{if } \theta_j > 0, \\ -\log|z-z_j| - \log(-\log|z-z_j|) + O(1) & \text{if } \theta_j = 0 \end{cases}$$

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as \( z \to z_j = z(p_j) \), where \( z \) is a local coordinate of \( R \) around \( p_j \). Note that a conical singularity of angle 0 is called a puncture or a cusp.

The remainder term \( O(1) \) in the above is known to be continuous at \( z = z_j \) by a detailed study of the local behavior of solutions to the Liouville equation at the isolated singularities by Nitsche [12] (see also [9]).

Heins [8, Chap. II] proved that for a compact Riemann surface \( R \) of genus \( g \) and finite points in it with given angles as above, a hyperbolic metric on \( R \) with the behavior described in (2.2) exists uniquely as long as the condition

\[
2(1 - g) - \sum_{j=1}^{N} (1 - \theta_j) < 0
\]

is satisfied. This constraint comes from the Gauss-Bonnet formula. This result was previously known by Picard [13] when \( g = 0 \). Practically, this unique metric as above is called the (complete) hyperbolic metric of the Riemann surface \( R \) \{ \( p_1, \ldots, p_N \} \) with conical singularities of angle \( 2\pi \theta_j \) at \( p_j \) \( (j = n + 1, \ldots, N) \), where \( \theta_1 = \cdots = \theta_n = 0 < \theta_j \neq 1 \) \( (j = n + 1, \ldots, N) \).

The hyperbolic metric treated in the present paper corresponds to the case when \( R = \hat{\mathbb{C}}, g = 0, N = 3, (p_1, p_2, p_3) = (0, 1, \infty) \) and \( (\theta_1, \theta_2, \theta_3) = (0, 0, \alpha) \), where \( 0 \leq \alpha < 1 \). Note here that this case always satisfies the condition (2.3). We denote this metric by \( \rho_\alpha(z)|dz| \).

When \( \alpha = 0 \), the metric \( \rho_0 \) is simply the usual hyperbolic metric \( \rho \) of \( \hat{\mathbb{C}} \setminus \{ 0, 1, \infty \} = \mathbb{C} \setminus \{ 0, 1 \} \) (without conical singularities). By uniqueness of the hyperbolic metric with conical singularities, the metric admits the obvious symmetry \( \rho_\alpha(z) = \rho_\alpha(1 - z) = \rho_\alpha(\bar{z}) \).

We remark that for a Möbius transformation \( M \), \( \rho_\alpha(M(0)), \rho_\alpha(M(1)) \) gives the density of the hyperbolic metric of \( \hat{\mathbb{C}} \setminus \{ M(0), M(1) \} \) with a conical singularity of angle \( 2\pi \alpha \) at \( M(\infty) \). For instance, the hyperbolic metric \( \tilde{\rho}_\alpha(z)|dz| \) of the twice-punctured sphere \( \hat{\mathbb{C}} \setminus \{ 1, \infty \} = \mathbb{C} \setminus \{ 1 \} \) with a conical singularity of angle \( 2\pi \alpha \) at 0 can be obtained by \( \tilde{\rho}_\alpha(z) = \rho_\alpha(1/z)|dz|/z^2 \).

### 3. Generalized elliptic integrals

The complete elliptic integrals of the first and the second kind are defined, respectively, by

\[
(3.1) \quad K(z) = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - zt^2)}} \quad \text{and} \quad E(z) = \int_0^1 \sqrt{\frac{1 - zt^2}{1 - t^2}} dt.
\]

Note that these functions can be expressed also by the hypergeometric function:

\[
F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \quad |z| < 1,
\]
where \((a, 0) = 1\) and \((a, n) = a(a+1)(a+2)\cdots(a+n-1)\) for \(n \geq 1\), and \(c \neq 0, -1, -2, \ldots\). Indeed,

\[
K(z) = \frac{\pi}{2^a} F\left(\frac{1}{2}, \frac{1}{2}; z\right) \quad \text{and} \quad E(z) = \frac{\pi}{2^a} F\left(-\frac{1}{2}, \frac{1}{2}; z\right).
\]

Let \(0 < a < 1\). The generalized complete elliptic integrals of the first and the second kind with signature \(1/a\) are defined, respectively, by

\[
K_a(z) = \frac{\pi}{2} F\left(a, 1-a; 1\right) = \sin(\pi a) \int_0^1 \frac{t^{1-2a}dt}{(1-t^2)^{1-a}(1-z t^2)^a}
\]

and

\[
E_a(z) = \frac{\pi}{2} F\left(a, 1-a; 1\right) = \sin(\pi a) \int_0^1 \left(\frac{1-z t^2}{1-t^2}\right)^{1-a} t^{1-2a}dt.
\]

Here, note that \(K_a(z)\) and \(E_a(z)\) are defined as (single-valued) analytic functions in \(z \in \mathbb{C} \setminus [1, +\infty)\).

We remark that the above definition is slightly different from the usual one. The (traditional) generalized complete elliptic integrals of the first and the second kind usually refer to \(K_a(x^2)\) and \(E_a(x^2)\) for \(0 < x < 1\) in our notation.

We mean by \(K'_a\) and \(E'_a\) the derivatives of \(K_a\) and \(E_a\), though these are often used to mean the complementary functions. For the complementary functions, we adopt the notation \(K_a^*(z) = K_a(1-z)\) and \(E_a^*(z) = E_a(1-z)\) in the present paper.

The following formula, which is a special case of Elliott’s identity (see [4]), will be used at a crucial step in the computation of the hyperbolic metric:

\[
K_a^*(z)E_a(z) + E_a^*(z)K_a(z) = K_a(z)K_a^*(z) - K_a^*(z)K_a(z) = \frac{\pi \sin(\pi a)}{4(1-a)}.
\]

Some information about the behavior of the hypergeometric function near \(z = x = 1\) will be needed below. The following result can be found in Chapter 15 of the book [1].

\[
F(a, b; c; 1-) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad c > a + b,
\]

\[
F(a, b; a+b; x) = \frac{1}{B(a, b)} \log \frac{1}{1-x} + O(1), \quad \text{as } x \to 1-,
\]

\[
F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x), \quad c < a + b.
\]

Here \(B(a, b)\) denotes the beta function.
4. Computation of $\rho_\alpha(z)$

A relation between conformal mappings and the generalized complete elliptic integral $K_\alpha(z)$ of the first kind is given by \cite{[4]} Theorem 2.2 when the argument $z$ is real and between 0 and 1. We will now give another aspect of $K_\alpha(z)$ for the complex argument $z$.

As is stated in \cite{[16]} Lemma 2, the hyperbolic metric of the sphere with given conical singularities can be described in terms of solutions to a second-order Fuchsian differential equation with regular singularities at the cone points. In our case, the metric is described explicitly in terms of generalized elliptic integrals.

We begin with a general case. Let $a$, $b$, and $c$ be real numbers. It is a classical fact that the function $f(z) = i K_a(1/z)/K_a(z)$ maps the upper half plane $\mathbb{H}$ onto a curvilinear triangle bounded by three circular arcs and having the interior angles $(1-c)\pi$ at $f(0)$, $(c-a-b)\pi$ at $f(1)$, and $(b-a)\pi$ at $f(\infty)$, provided that these angles are all nonnegative and the sum is less than $\pi$ (see, for instance, \cite{[4]} pp. 206, 207). Note that the segment $(0,1)$ of the real axis is mapped by $f$ to a part of the imaginary axis and that $f$ maps $\mathbb{C} \setminus ((-\infty,0] \cup [1,\infty))$ conformally onto the domain which is the union of $f((0,1))$, $f(\mathbb{H})$, and its reflection in the imaginary axis.

In the particular case when $0 < a < 1$, $b = 1-a$, and $c = 1$, the function $f$ can be written in the form $i K_a(1-z)/K_a(z)$, and the image $f(\mathbb{H})$ is a circular triangle with interior angles $0$, $0$, and $|1-2a|\pi$. More specifically, we have the following result.

4.1. Lemma. Let $f_a(z) = i K_a(1-z)/K_a(z)$, $0 < a < 1$. Then the image $f_a(\mathbb{H})$ of the upper half plane $\mathbb{H}$ under $f_a$ is the hyperbolic triangle $\Delta_a$ in $\mathbb{H}$ whose interior angles are $0$, $0$, and $|1-2a|\pi$ at the vertices $0$, $\infty$, and $e^{1-2a\pi i/2}$, respectively. More precisely, $\Delta_a = \{w \in \mathbb{H} : 0 < \Re w < \sin(\pi a), \ |2w \sin(\pi a) - 1| > 1 \}$.

Proof. Since $f_a(\mathbb{H})$ is a Jordan domain, $f_a$ extends to a homeomorphism from the closure of $\mathbb{H}$ onto the closure of $f_a(\mathbb{H})$. First note that $f_a$ maps the interval $(0,1)$ onto the whole positive imaginary axis. Since the interior angles of $f_a(\mathbb{H})$ at $f_a(0)$ and $f_a(1)$ are both $0$, the boundary arcs $f_a((1,\infty))$ and $f_a((-\infty,0))$ are contained in hyperbolic geodesics in $\mathbb{H}$ of the forms $|w - r| = r (0 < r)$ and $\Re w = p (p > 0)$, respectively. In particular, the image $f_a(\mathbb{H})$ is a hyperbolic triangle in $\mathbb{H}$. Since we know that these two geodesics form an angle of $\theta = |1-2a|\pi$, we find that $r(1 + \cos \theta) = p$ by elementary geometry. Thus, it is enough to show that $p = \sin(\pi a)$, which leads to the relation $r = \sin(\pi a)/(1 + \cos((1-2a)\pi)) = 1/(2 \sin(\pi a))$.

In order to make statements precise, we introduce some notation. Let $f$ be an analytic function defined in $\mathbb{C} \setminus \mathbb{R}$. For each $x \in \mathbb{R}$, we denote by $f^\pm(x)$ the limit $\lim_{t \to 0^\pm} f(x \pm it)$ (if it exists). If $f$ extends analytically to a neighborhood $V$ of $x$ as a single-valued function on $(\mathbb{C} \setminus \mathbb{R}) \cup V$, then we write simply $f(x)$ as usual instead of $f^+(x) = f^-(x)$. 

With this notation, using [7 (3), (27), pp. 105, 106], we obtain the transformation formulas

\begin{equation}
\frac{2}{\pi} K_a(-x) = F(a, 1 - a; 1; -x) = (1 + x)^{-a} F(a, a; 1; \frac{x}{1+x}) \tag{4.2}
\end{equation}

and

\begin{equation}
\frac{2}{\pi} K_a^\pm(1 + x) = F^\pm(a, 1 - a; 1; 1 + x) = (1 + x)^{-a} \left[ \frac{\Gamma(a)}{\Gamma(2a)\Gamma(1 - a)} F(a, a; 2a; \frac{1}{1+x}) - e^{\mp \pi a i} F(a, a; 1; \frac{x}{1+x}) \right], \tag{4.3}
\end{equation}

for all \( x > 0 \). Therefore,

\begin{equation}
f_a^+(-x) = \frac{i}{\pi} K_a^+(-x) = \frac{i \Gamma(a)}{\Gamma(2a)\Gamma(1 - a)} \cdot \frac{F(a, a; 2a; \frac{1}{1+x})}{F(a, a; 1; \frac{x}{1+x})} - ie^{\pi a i}, \quad x > 0. \tag{4.4}
\end{equation}

Thus \( \text{Re} f_a^+(-x) = \sin(\pi a) \) for \( x > 0 \), as required.

\begin{remark}
Since \( f_a^+(x) = f_{1-a}^-(x) \), as a by-product of (4.4), we obtain the relation

\[ \frac{\Gamma(a)}{\Gamma(2a)\Gamma(1 - a)} \cdot \frac{F(a, a; 2a; \frac{1}{1+x})}{F(a, a; 1; \frac{x}{1+x})} - \cos(\pi a) = \frac{\Gamma(1 - a)}{\Gamma(2 - 2a)\Gamma(a)} \cdot \frac{F(1 - a, 1 - a; 2 - 2a; \frac{1}{1+x})}{F(1 - a, 1 - a; 1; \frac{x}{1+x})} + \cos(\pi a) \]

for \( 0 < a < 1 \) and \( x > 0 \). This is equivalent to the identity

\begin{align*}
&\frac{\Gamma(a)}{\Gamma(2a)\Gamma(1 - a)} \cdot F(a, a; 2a; 1 - x)F(1 - a, 1 - a; 1; x) \\
&\quad - \frac{\Gamma(1 - a)}{\Gamma(2 - 2a)\Gamma(a)} \cdot F(1 - a, 1 - a; 2 - 2a; 1 - x)F(a, a; 1; x) \\
&\quad - 2 \cos \pi a \cdot F(a, a; 1; x)F(1 - a, 1 - a; 1; x) = 0.
\end{align*}

As far as we know, this is a new identity for hypergeometric functions.

Let \( \alpha = |1 - 2a| \). Since \( f_a \) maps each of the intervals \((-\infty, 0), (0, 1)\), and \((1, +\infty)\) onto a hyperbolic geodesic segment in \( \mathbb{H} \), the pull-back \( f_a^* \rho_\mathbb{H} = \rho_\mathbb{H}(f_a(z))|f'_a(z)||dz| \) of the hyperbolic (or Poincaré) metric \( \rho_\mathbb{H}(z)|dz| = |dz|/(2 \text{Im} (z)) \) of the upper half plane \( \mathbb{H} \), together with its reflection \( f_a^* \rho_\mathbb{H}(\bar{z})|dz| \) defines a smooth conformal metric on \( \mathbb{C} \setminus \{0, 1\} \). This is the hyperbolic metric \( \rho_\alpha(z)|dz| \) of the twice-punctured sphere \( \mathbb{C} \setminus \{0, 1\} \) with a conical singularity of angle \( 2\pi \alpha \) at \( \infty \) (cf. [16, Lemma 2]). We emphasize that the curvature equation, which is equivalent to (2.1),

\begin{equation}
\Delta \log \rho_\alpha = 4\rho_\alpha^2 \tag{4.6}
\end{equation}

plays an important role in investigation of the metric.
Agard [2] gave a formula for $\rho_{C \setminus \{0,1\}} = \rho_0$ in terms of complete elliptic integrals. In the same way, we can compute $\rho_{\alpha}$ for $0 \leq \alpha < 1$ with the help of the above construction.

4.7. **Theorem.** Let $0 \leq \alpha < 1$ and choose $0 < a < 1$ so that $\alpha = |1 - 2a|$. The hyperbolic metric $\rho_{\alpha}(z)|dz|$ of the twice-punctured sphere $\mathbb{C} \setminus \{0,1\}$ with conical singularity of angle $2\pi\alpha$ at $\infty$ is given by

$$
\rho_{\alpha}(z) = \frac{\pi \cos(\pi\alpha/2)}{8|z(1-z)| \text{Re} (K_a(z)K_{\alpha}(1 - \bar{z})).}
$$

**Proof.** By Gauss’ contiguous relations (see (2.5.8) of [5]), one obtains

$$
z(1-z)K_a'(z) = (1-a)[E_a(z) - K_a(z)].
$$

Using this identity, we derive

$$
f_{a}'(z) = -i \frac{K_a'(1-z)K_a(z) - K_{\alpha}(1-z)K_{\alpha}'(z)}{(K_a(z))^2}$$

$$
= -i \frac{1-a}{z(1-z)} \cdot \frac{E_a^*(z)K_a(z) + K_{\alpha}^*(z)E_a(z) - K_{\alpha}^*(z)K_a(z)}{(K_a(z))^2}$$

$$
= -i \frac{\pi \sin(\pi a)}{4z(1-z)(K_a(z))^2},
$$

where we have used (3.4). Hence, using the relation $K_a(z) = K_{\alpha}(\bar{z})$, we obtain

$$
\rho_{\alpha}(z) = \frac{|f_{a}'(z)|}{2 \text{Im} f_{a}(z)} = \frac{\pi \sin(\pi a)}{8|z(1-z)| (K_a(z))^2 \text{Re} (K_a(1 - \bar{z})/K_a(\bar{z})),}
$$

from which the required formula follows. \qed

By the representation formula for $\rho_{\alpha}$, we have the following.

4.9. **Corollary.** The quantity $\rho_{\alpha}(z)$ is jointly continuous in $\alpha$ and $z$.

Because the formula

$$
K_a\left(\frac{1}{2}\right) = \frac{\Gamma(\frac{1-a}{2})\Gamma(\frac{a}{2}) \sin(\pi a)}{4\sqrt{\pi}}
$$

is known (see, for instance, [4, (4.5)]), we have the following consequence.

4.11. **Corollary.**

$$
\rho_{\alpha}\left(\frac{1}{2}\right) = \frac{8\pi^2}{(\Gamma \left(\frac{1-a}{4}\right))^2 (\Gamma \left(\frac{1-a}{4}\right))^2 \cos(\frac{\pi a}{2})}.
$$

The explicit formula in (4.8) of $\rho_{\alpha}$ can be used to determine the constant terms of asymptotic expansions of $\rho_{\alpha}$ around singularities.
4.12. Theorem. For $0 < \alpha < 1$, the metric $\rho_\alpha$ satisfies

$$\log \rho_\alpha(z) = \begin{cases} 
\log \frac{1}{|z|} - \log \log \frac{1}{|z|} - \log 2 + o(1) & \text{as } z \to 0, \\
\log \frac{1}{|z-1|} - \log \log \frac{1}{|z-1|} - \log 2 + o(1) & \text{as } z \to 1, \\
-(1 + \alpha) \log |z| + \log \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(1-\alpha)}{(\Gamma(\frac{\alpha}{2}))^2\Gamma(\alpha)} + o(1) & \text{as } z \to \infty.
\end{cases}$$

Proof. Choose $a \in (0, 1/2]$ so that $1 - 2a = \alpha$. First we investigate $\rho_\alpha(z)$ around $z = 0$. Since the $O(1)$ term, say $w(z)$, is known to be continuous at $z = 0$ (see [12, Satz 1] or [9, Theorem 1.1]), it suffices to show that $w(0) = \log 2$. By (3.5), for $x > 0$ we have $K_a(x) = \pi + O(x)$ and $K_a(1 - x) = \sin(\pi a) x + O(1)$ as $x \to 0 +$. Substitution of these formulas into (4.8) yields $w(0) = \log 2$ as required. The corresponding result for $z = 1$ follows from the previous one by the symmetry $\rho_\alpha(1 - z) = \rho_\alpha(z)$.

Finally, we consider the case $z \to \infty$. By the general property of conical singularities, one has the expression $\log \rho_\alpha(z) = (2a - 2) \log |z| + v(z)$, where $v(z)$ is a continuous function near $z = \infty$ (see [12, Satz 1] or [9, Theorem 1.1]). For $x > 0$, by (4.2), (4.3), and (3.5), we have

$$\frac{2}{\pi} K_a(-x) = \frac{\Gamma(1 - 2a)}{(\Gamma(1 - a))^2} x^{-a}(1 + o(1))$$

and

$$\frac{2}{\pi} \text{Re} K_a^\pm(1 + x) = (1 + x)^{-a} \left[ \frac{\Gamma(a)}{\Gamma(2a)\Gamma(1 - a)} F(a, a; 2a; \frac{1}{1+x}) - \cos(\pi a) F(a, a; 1; \frac{1}{1+x}) \right]$$

as $x \to +\infty$. Combining (4.13) and (4.14) with (4.8), we see that

$$\rho_\alpha(-x) = \frac{(\Gamma(1 - a))^2\Gamma(2a)}{(\Gamma(a))^2\Gamma(1 - 2a)} x^{2a-2}(1 + o(1)), \quad x \to +\infty,$$

which implies that $v(\infty) = \log(\Gamma(1 - a))^2\Gamma(2a)/\Gamma(a)^2\Gamma(1 - 2a))$. This is equal to the required constant term. □
Lehto, Virtanen and Väisälä \cite{10} proved the useful inequality $\rho_0(|z|) \leq \rho_0(z)$ for all $z \in \mathbb{C} \setminus \{0, 1\}$. Later on, Weitsman \cite{17} proved a monotonicity property of the hyperbolic metric on a circularly symmetric domain, which means that $\rho_0(re^{i\theta})$ is a non-increasing function of $\theta$ in $0 < \theta < \pi$ for a fixed $r > 0$ for the particular domain $\mathbb{C} \setminus \{0, 1\}$. We can deduce the same result for $\rho_\alpha$ by employing the method developed in \cite{10}.

4.15. \textbf{Theorem.} For $0 \leq \alpha < 1$ and fixed $r > 0$, $\rho_\alpha(re^{i\theta})$ is a non-increasing (non-decreasing) function of $\theta$ in $0 < \theta < \pi$ ($-\pi < \theta < 0$). In particular, the inequalities $\rho_\alpha(|z|) \leq \rho_\alpha(z) \leq \rho_\alpha(|z|)$ hold for each $z \in \mathbb{C} \setminus \{0, 1\}$.

\textbf{Proof.} It is enough to show the assertion by assuming that $0 < a < \frac{1}{2}$. (The case $a = \frac{1}{2}$ can be treated similarly with the special relation $\rho_0(1/z) = \rho_0(z)|z|^2$ being taken into account.) By the obvious symmetry $\rho_\alpha(z) = \rho_\alpha(a(z)$, it is enough to prove the inequality $\rho_\alpha(re^{i\theta_1}) \geq \rho_\alpha(re^{i\theta_2})$ for $0 \leq \theta_1 < \theta_2 \leq \pi$. Let $\lambda_1(z) = \rho_\alpha(e^{-i\theta_0}z)$ and $\lambda_2(z) = \rho_\alpha(e^{i\theta_0}z)$, where $\theta_0 = (\theta_1 + \theta_2)/2$. Consider now the function $h(z) = \log \lambda_1(z) - \log \lambda_2(z)$. Then $h$ is smooth in $\mathbb{C} \setminus \{0, e^{i\theta_0}, e^{-i\theta_0}\}$ and, by the above symmetry, $h = 0$ on $\mathbb{R} \setminus \{0\}$.

We will show that $h(z) \geq 0$ for $z \in \mathbb{H}$. To this end, we first observe the asymptotic behavior of $h(z)$. It is easy to see that $h(z) \to +\infty$ as $z \to e^{i\theta_0}$. By Theorem 4.12, we also have $h(z) \to 0$ as $z \to \infty$ or $z \to 0$. Therefore, the set $\{z \in \mathbb{H} \setminus \{e^{i\theta_0}\} : h(z) \leq -\varepsilon\}$ is compact for each $\varepsilon > 0$. Suppose now that $h < 0$ somewhere in $\mathbb{H}$. Then, there would be a minimum point $z_0$ for $h$ in $\mathbb{H} \setminus \{e^{i\theta_0}\}$. Then $\Delta h(z_0) \geq 0$ by minimality. On the other hand, the inequality $h(z_0) < 0$ would imply $\lambda_1(z_0) < \lambda_2(z_0)$. Hence, by (4.6),

$$\Delta h(z_0) = 4\lambda_1(z_0)^2 - 4\lambda_2(z_0)^2 < 0,$$

which would be impossible. Thus, we have shown that $h(z) \geq 0$ for $z \in \mathbb{H}$.

We now take the point $z_0 = re^{i(\theta_2 - \theta_1)/2}$. Then $0 \leq h(z_0) = \log \rho_\alpha(re^{i\theta_1}) - \log \rho_\alpha(re^{i\theta_2}) = \log \rho_\alpha(re^{i\theta_1}) - \log \rho_\alpha(re^{i\theta_2})$, and thus, $\rho_\alpha(re^{i\theta_1}) \geq \rho_\alpha(re^{i\theta_2})$. \hfill \square

The hyperbolic distance on $\hat{\mathbb{C}} \setminus \{0, 1\}$ induced by $\rho_\alpha$ is defined, as usual, by

$$d_\alpha(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \rho_\alpha(z)|dz|,$$

where $\gamma$ runs over all the rectifiable paths $\gamma$ connecting $z_1$ and $z_2$ in $\hat{\mathbb{C}} \setminus \{0, 1\}$.

As a corollary of Theorem 4.15 we derive a lower estimate for the hyperbolic distance.

4.16. \textbf{Corollary.} For $0 < a < 1$ and $z_1, z_2 \in \hat{\mathbb{C}} \setminus \{0, 1\}$ with $|z_1| \leq |z_2|$, the following inequality holds:

\begin{equation}
(4.17) \quad d_\alpha(z_1, z_2) \geq d_\alpha(-|z_1|, -|z_2|) = \int_{|z_1|}^{|z_2|} \rho_\alpha(-t)dt.
\end{equation}
We can compute the last integral by the following result.

4.18. **Theorem.** Let $\alpha = |1 - 2a|$ for $0 < a < 1$. The formula

$$\int_x^y \rho_\alpha(-t) dt = \Phi_\alpha(y) - \Phi_\alpha(x)$$

holds for $0 < x < y$, where

$$\Phi_\alpha(x) = -\frac{1}{2} \log \left( \frac{\Gamma(a) \ F(a, a; 2a; \frac{1}{1+x}) - \cos(\pi a)}{\Gamma(2a) \Gamma(1-a) \ F(a, a; 1; \frac{1}{1+x})} \right).$$

**Proof.** One can proceed almost as in the proof of [15, Lemma 5.1]. We can write $f_a^+(-t)$ in the form $i u(x) + \sin(\pi a)$ for $x > 0$ by (4.4). Since $\rho_\alpha(-t) = |(f_a^+)'(-t)|/2 \Im f_a^+(-t) = -u'(t)/2u(t)$, we obtain

$$\int_x^y \rho_\alpha(-t) dt = -\int_x^y \frac{u'(t)}{2u(t)} dt = \frac{1}{2} \log \frac{u(x)}{u(y)} = \Phi_\alpha(y) - \Phi_\alpha(x).$$

$\square$

Note that when $a \neq \frac{1}{2}$,

$$\Phi_\alpha(\infty) = -\frac{1}{2} \log \cos(\pi a)$$

is positive and finite, whereas $\Phi_{1/2}(\infty) = \infty$.

4.19. **Remark.** More generally, the $\rho_\alpha$-distance between $z_1$ and $z_2$ in $\mathbb{H} \setminus \{0, 1, \infty\}$ can be expressed by

$$d_\alpha(z_1, z_2) = \arctanh \left( \frac{f_a(z_2) - f_a(z_1)}{f_a(z_2) - f_a(z_1)} \right),$$

where $0 < a < 1$ is chosen so that $\alpha = |1 - 2a|$ and $f_a$ is given in Lemma 4.1. Indeed, by construction, $f_a$ is an isometric embedding of $(\mathbb{H}, \rho_\alpha)$ into $(\mathbb{H}, \rho_\xi)$ and its image $\Delta_a$ is (hyperbolically) convex in $\mathbb{H}$. Therefore, the geodesic segment joining $z_1$ and $z_2$ in $\hat{\mathbb{C}} \setminus \{0, 1\}$ with respect to $\rho_\alpha$ is contained in the closure of $\mathbb{H}$ and its image under $f_a$ is the hyperbolic geodesic joining $f_a(z_1)$ and $f_a(z_2)$. It is well known that the hyperbolic distance between two points $w_1$ and $w_2$ in $\mathbb{H}$ is given by $\arctanh |(w_2 - w_1)/(w_2 - \overline{w_1})|$, and the above formula follows.

Finally, we mention monotonicity of $\rho_\alpha(z)$ with respect to the parameter $\alpha$.

4.20. **Proposition.** The density $\rho_\alpha(z)$ is non-increasing in $0 \leq \alpha < 1$ for a fixed $z \in \mathbb{C} \setminus \{0, 1\}$.
Though this result is contained in [14, Prop. 2.4] as a special case, we
give a proof for convenience of the reader. The assertion is established by
a simple application of the Schwarz-Pick-Ahlfors lemma (cf. [3]). Here, we
employ the same technique as in Theorem 4.15.

Proof. For a given pair $\alpha, \alpha'$ with $0 \leq \alpha < \alpha' < 1$, we
consider the function
$$h = \log \rho_{\alpha'} - \log \rho_{\alpha}$$
in $\mathbb{C} \setminus \{0, 1\}$. By Theorem 4.12 the function $h$ extends
continuously to 0 and 1 if we set $h(0) = h(1) = 0$, and has the asymptotic
behavior $h(z) = (\alpha - \alpha' + o(1)) \log |z|$ as $z \to \infty$. Therefore, if $h$
takes a positive value, there is a point $z_0 \in \mathbb{C} \setminus \{0, 1\}$ at which
$h$ attains its (positive) maximum. Then
$$\Delta h(z_0) \leq 0.$$ On the other hand, by (4.6),
$$\Delta h(z_0) = 4\rho_{\alpha'}(z_0)^2 - 4\rho_{\alpha}(z_0)^2 > 0,$$
which is a contradiction. Hence, we conclude that $h(z) \leq 0$, in other words,
$\rho_{\alpha}(z) \geq \rho_{\alpha'}(z)$ for $z \in \mathbb{C} \setminus \{0, 1\}$. □

4.21. Remark. The expression $\rho(a, z) \equiv \rho_{|1-2a|}(z)$ as in (4.8) can be viewed
as a smooth function in $(a, z) \in (0, 1) \times (\mathbb{C} \setminus \{0, 1\})$. Then it has the obvious
symmetry $\rho(1-a, z) = \rho(a, z)$. By the above theorem, $\rho(a, z)$ attains its
maximum at $a = \frac{1}{2}$ for a fixed $z$. In particular, by this observation we
obtain $(\partial \rho/\partial a)(\frac{1}{2}, z) = 0$. We also see that $\rho_{\alpha}(z) \to 0$ as $\alpha \to 1$ from (4.8).
This corresponds to the well-known fact that the twice-punctured sphere
\( \hat{\mathbb{C}} \setminus \{0, 1\} \) does not carry a hyperbolic metric.

5. Applications

We conclude the present note with a few applications of our metric $\rho_{\alpha}$.
Since no concrete estimates for $\rho_{\alpha}$ are given so far, we will give only general
principles to refine classical results.

If a meromorphic function $f$ on the unit disk $\mathbb{D}$ does not assume the
three points 0, 1 and $\infty$, then the principle of hyperbolic metric gives us the
inequality $f^* \rho \leq \rho_{\mathbb{D}}$, namely;
$$\rho(f(z)|f'(z)| \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$ The classical theorems of Picard and Schottky follow essentially from the
above inequality (see, for example, [3, §1-9]). We can now relax the assumption
about the omitted values as in the following.

5.1. Theorem. Let $f$ be a meromorphic function on the unit disk omitting
the two values 0 and 1. Suppose that every pole of $f$ is of order at least
$k \geq 2$. Then the following inequality holds:
$$\rho_{1/k}(f(z)|f'(z)| \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}.$$
5.2. Let $\alpha > 1/k$. We may assume that $f$ is not constant. Let $\lambda$ be the pull-back metric $f^*\rho_\alpha$ of $\rho_\alpha$ under $f$. Then it is easily verified that the Gaussian curvature of $\lambda$ is $-4$ off the set of poles and branch points of $f$. Let $z_0$ be a pole of $f$. Then the order $m$ of the pole at $z_0$ is at least $k$ by assumption. In view of \cite{[5,5]}, we have

$$\log \lambda(z) = -(1 + \alpha) \log |f(z)| + \log |f'(z)| + O(1)$$

$$= [(1 + \alpha)m - (m + 1)] \log |z - z_0| + O(1)$$

$$= \alpha m - 1 \log |z - z_0| + O(1)$$

as $z \to z_0$. Since $\alpha m > 1$, we see that $\lambda(z_0) = 0$. Thus $\lambda$ is an ultrahyperbolic metric on $D$ in the sense of Ahlfors \cite{[3]}. Thus, Ahlfors’ lemma now yields $f^*\rho_\alpha \leq \rho_D$. Taking the limit as $\alpha \to 1/k$, we obtain the required inequality.

Knowledge about the hyperbolic metric $\rho = \rho_0$ of the thrice-punctured sphere $\mathbb{C} \setminus \{0, 1\}$ has led to various useful estimates for the hyperbolic metric of a general plane domain (see, for instance, \cite{[3]} or \cite{[15]}). We now use $\rho_\alpha$ instead of $\rho_0$ to obtain similar estimates for the hyperbolic metric with conical singularities.

5.2. **Theorem.** Let $\Omega$ be a subdomain of the Riemann sphere $\hat{\mathbb{C}}$ with $\infty \in \Omega$ such that $\hat{\mathbb{C}} \setminus \Omega$ contains at least two points. Let $\lambda$ be a conformal metric on $\Omega$ with conical singularities of angle less than $2\pi$. Suppose that $\lambda$ has a conical singularity of angle $2\pi \alpha > 0$ and that for each $w_0 \in \partial\Omega$, $|z - w_0| \log(1/|z - w_0|)\lambda(z)$ is bounded away from $0$ in $V \cap \Omega$ for a neighborhood $V$ of $w_0$ if $w_0$ is isolated in $\partial\Omega$ and $|z - w_0| \log(1/|z - w_0|)\lambda(z) \to +\infty$ as $z \to w_0$ in $\Omega$ otherwise. Then

$$\lambda(z) \geq \sup_{w_0, w_1 \in \partial\Omega} \frac{1}{|w_1 - w_0|} \rho_\alpha \left( \frac{z - w_0}{w_1 - w_0} \right), \quad z \in \Omega \setminus \{\infty\}.$$

**Proof.** We follow the argument used by Heins \cite{[3]} \S 20. First note that the set $S$ of conical singularities of $\lambda$ can be characterized as $\{z \in \Omega \setminus \{\infty\} : \lambda(z) = \infty\} \cup \{\infty\}$. Pick $\alpha' \in (\alpha, 1)$ and fix a pair of distinct points $w_0$ and $w_1$ in $\partial\Omega$. Set $\mu(z) = \rho_{\alpha'}((z - w_0)/(w_1 - w_0))/(w_1 - w_0)$ and let

$$v = \max\{\log \mu - \log \lambda, \; 0\}.$$

Then $v$ is subharmonic on $\Omega \setminus S$ and vanishes in a neighborhood of $S$. Moreover, $v = 0$ near every boundary point of $\Omega$ except possibly for $w_0$ and $w_1$. If $w_j$ is not isolated, then this is still valid. Otherwise, by the local behavior of solutions to the Liouville equation around an isolated singularity due to Nitsche \cite{[12]}, we see that $v$ can be extended continuously to the point $w_j$. Recall now the following fact: Suppose that $u$ is a continuous function on an open neighborhood $D$ of a point $a$ and subharmonic on $D \setminus \{a\}$. Then $u$ is subharmonic on $D$. Thus $v$ is subharmonic on $\Omega' = \Omega \cup \{w_j : w_j$ is isolated$\}$ and vanishes in a neighborhood of $\partial\Omega' \cup S$. We now appeal to the maximum
principle to conclude that $v = 0$ in $\Omega$, which means $\mu \leq \lambda$. The proof is now complete. □

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