Extension of Lipschitz Functions Defined on Metric Subspaces of Homogeneous Type

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Abstract
If a metric subspace $M^o$ of an arbitrary metric space $M$ carries a doubling measure $\mu$, then there is a simultaneous linear extension of all Lipschitz functions on $M^o$ ranged in a Banach space to those on $M$. Moreover, the norm of this linear operator is controlled by logarithm of the doubling constant of $\mu$.

1 Formulation of the Main Result
Let $(M, d)$ be a metric space and $X$ be a Banach space. The space $Lip(M, X)$ consists of all $X$-valued Lipschitz functions on $M$. The Lipschitz constant

$$L(f) := \sup_{m \neq m'} \left\{ \frac{||f(m) - f(m')||}{d(m, m')} : m, m' \in M \right\}$$

(1.1)

of a function $f$ from this space is therefore finite and the function $f \mapsto L(f)$ is a Banach seminorm on $Lip(M, X)$.

Let $M^o$ be a metric subspace of $M$, i.e., $M^o \subset M$ is a metric space endowed with the induced metric $d|_{M^o \times M^o}$.

Convention. We mark all objects related to the subspace $M^o$ by the upper "o".

*Research supported in part by NSERC.
2000 Mathematics Subject Classification. Primary 26B35, Secondary 54E35, 46B15.
Key words and phrases. Metric space of homogeneous type, Lipschitz function, linear extension.
A linear operator $E : \text{Lip}(M^o, X) \to \text{Lip}(M, X)$ is called a simultaneous extension if for all $f \in \text{Lip}(M^o, X)$

$$Ef|_{M^o} = f$$

and, moreover, the norm

$$||E|| := \sup \left\{ \frac{L(Ef)}{L(f)} : f \in \text{Lip}(M^o, X) \right\}$$

is finite.

To formulate the main result we also need

**Definition 1.1** A Borel measure $\mu$ on a metric space $(M, d)$ is said to be doubling if the $\mu$-measure of every open ball

$$B_R(m) := \{m' \in M : d(m, m') < R\}$$

is strictly positive and finite and the doubling constant

$$D(\mu) := \sup \left\{ \frac{\mu(B_{2R}(m))}{\mu(B_R(m))} : m \in M, \ R > 0 \right\}$$

(1.2)

is finite.

A metric space carrying a fixed doubling measure is called of homogeneous type.

Our main result is

**Theorem 1.2** Let $M^o$ be a metric subspace of an arbitrary metric space $(M, d)$. Assume that $(M^o, d^o)$ is of homogeneous type and $\mu^o$ is the corresponding doubling measure. Then there exists a simultaneous extension $E : \text{Lip}(M^o, X) \to \text{Lip}(M, X)$ satisfying

$$||E|| \leq c(\log_2 D(\mu^o) + 1)$$

(1.3)

with some numerical constant $c > 1$.

Let us discuss relations of this theorem to some known results. First, a similar result holds for an arbitrary subspace $M^o$ provided that the ambient space $M$ is of pointwise homogeneous type, see [BB1, Theorem 2.21] and [BB2, Theorem 1.14]. The class of metric spaces of pointwise homogeneous type contains, in particular, all metric spaces of homogeneous type, Riemannian manifolds $M_\omega \cong \mathbb{R}^n \times \mathbb{R}_+$ with the path metric defined by the Riemannian metric

$$ds^2 := \omega(x_{n+1})(dx_1^2 + \ldots + dx_{n+1}^2), \quad (x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}_+,$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nonincreasing function (e.g., the hyperbolic spaces $\mathbb{H}^n$ are in this class), and finite direct products of these objects.

The following problem is of a considerable interest.

**Problem 1.3** Is it true that Theorem 1.2 is valid for $M^o (\subset M)$ isometric to a subspace of a metric space $(\hat{M}, \hat{d})$ of pointwise homogeneous type with $||E|| \leq c(\hat{M})$?
(Here \( c(\hat{M}) \) depends on some characteristics of \( \hat{M} \) only.)

It is proved in \([BB2]\) that as such \( M^o \) one can take, e.g., finite direct products of Gromov hyperbolic spaces of bounded geometry and that the answer in Problem 1.3 is positive in this case.

Second, as a consequence of Theorem 1.2 we obtain a deep extension result due to Lee and Naor, see \([LN, \text{Theorem 1.6}]\). The latter asserts that a simultaneous extension \( \hat{E} : Lip(\hat{M}^o, X) \to Lip(\hat{M}, X) \) exists whenever the subspace \((M^o, d^o)\) of \((M, d)\) has the finite doubling constant \( \delta(M^o) \) and, moreover,

\[
||E|| \leq c \log_2 \delta(M^o) \tag{1.4}
\]

with some numerical constant \( c > 1 \).

Let us recall that the doubling constant \( \delta(M) \) of a metric space \((M, d)\) is the infimum of integers \( N \) such that every closed ball of \( M \) of radius \( R \) can be covered by \( N \) closed balls of radius \( R/2 \). The space \( M \) is said to be doubling if \( \delta(M) < \infty \).

To derive the Lee-Naor theorem from our main result we first note that without loss of generality one may assume that \((M^o, d^o)\) is complete. By the Koniagin-Vol’berg theorem \([KV]\) (see also \([LS]\)) a complete doubling space \( M \) carries a doubling measure \( \mu \) such that

\[
\log_2 D(\mu) \leq c \log_2 \delta(M) \tag{1.5}
\]

where \( c \geq 1 \) is a numerical constant. Together with (1.3) this implies the Naor-Lee result.

On the other hand, it was noted in \([CW]\) that if \( M \) carries a doubling measure \( \mu \), then this space is doubling and

\[
\log_2 \delta(M) \leq c \log_2 D(\mu) \tag{1.6}
\]

with some numerical constant \( c > 1 \). Hence, Theorem 1.2 is, in turn, a consequence of (1.6) and the Lee-Naor theorem. However, the rather elaborated proof of the latter result is nonconstructive (it exploits an appropriate stochastic metric decomposition of \( M \setminus M^o \)). In contrast, our proof is constructive and is based on a simple average procedure. Therefore our proof can be also seen as a streamlining constructive method of the proof of the Lee-Naor theorem.

2. Proof of Theorem 1.2.

We begin with the following remark reducing the required result to a special case.

Let \( M \) and \( M^o \) be isometric to subspaces of a new metric space \( \hat{M} \) and its subspace \( \hat{M}^o \), respectively. Assume that there exists a simultaneous extension \( \hat{E} : Lip(\hat{M}^o, X) \to Lip(\hat{M}, X) \). Then, after identification of \( M^o \) and \( M \) with the corresponding isometric subspaces of \( \hat{M} \), the operator \( \hat{E} \) gives rise to a simultaneous extension \( E : Lip(M^o, X) \to Lip(M, X) \) satisfying

\[
||E|| \leq ||\hat{E}||. \tag{2.1}
\]
If, in addition, $||\hat{E}||$ is bounded by the right-hand side of (1.3), then the desired result immediately follows.

We choose as the above pair $\hat{M}^o \subset \hat{M}$ metric spaces denoted by $M_N^0$ and $M_N$ where $N \geq 1$ is a fixed integer and defined as follows.

The underlying sets of these spaces are

$$M_N := M \times \mathbb{R}^N, \quad M_N^0 := M^o \times \mathbb{R}^N; \quad \text{(2.2)}$$

a metric $d_N$ on $M_N$ is given by

$$d_N((m, x), (m', x')) := d(m, m') + |x - x'|_1 \quad \text{(2.3)}$$

where $m, m' \in M$ and $x, x' \in \mathbb{R}^N$, and $|x|_1 := \sum_{i=1}^N |x_i|$ is the $l_1$-metric of $x \in \mathbb{R}^N$. Further, $d_N^o$ denotes the metric on $M_N^0$ induced by $d_N$.

Finally, we define a Borel measure $\mu^o_N$ on $M_N^0$ as the tensor product of the measure $\mu^o$ and the Lebesgue measure $\lambda_N$ on $\mathbb{R}^N$:

$$\mu^o_N := \mu^o \otimes \lambda_N. \quad \text{(2.4)}$$

We extend this measure to the $\sigma$-algebra consisting of subsets $S \subset M_N$ such that $S \cap M_N^0$ is a Borel subset of $M_N^0$. Namely, we set for these $S$

$$\pi_N(S) := \mu^o_N(S \cap M_N^0).$$

It is important for the subsequent part of the proof that every open ball $B_R((m, x)) \subset M_N$ belongs to this $\sigma$-algebra. In fact, its intersection with $M_N^0$ is a Borel subset of this space, since the function $(m', x') \mapsto d_N((m, x), (m', x'))$ is continuous on $M_N^0$. Hence,

$$\pi_N(B_R((m, x))) = \mu^o_N(B_R((m, x)) \cap M_N^0). \quad \text{(2.5)}$$

**Auxiliary results**

The measure $\mu^o_N$ is clearly doubling. Therefore its dilation function given for $l \geq 1$ by

$$D^o_N(l) := \sup \left\{ \frac{\mu^o_N(B_R^o(\hat{m}))}{\mu^o_N(B_R^o(\hat{m}))} : \hat{m} \in M_N^o \text{ and } R > 0 \right\}$$

is finite.

Hereafter we denote by $\hat{m}$ the pair $(m, x)$ with $m \in M$ and $x \in \mathbb{R}^N$, and by $B_R^o(\hat{m})$ the open ball in $M_N^0$ centered at $\hat{m} \in M_N^0$ and of radius $R$. The open ball $B_R(\hat{m})$ of $M_N$ relates to that by

$$B_R^o(\hat{m}) = B_R(\hat{m}) \cap M_N^o$$

provided $\hat{m} \in M_N^o$.

In [BB1] the value $D^o_N(1 + 1/N)$ is proved to be bounded by some numerical constant for all sufficiently large $N$. In the argument presented below we require a similar estimate for a (modified) dilation function $D_N$ for the extended measure $\pi_N$. This is given for $l \geq 1$ by

$$D_N(l) := \sup \left\{ \frac{\pi_N(B_R(\hat{m}))}{\pi_N(B_R(\hat{m}))} \right\}. \quad \text{(2.6)}$$
where the supremum is taken over all $R$ satisfying
\[ R > 4d(\tilde{m}, M^o_N) := 4 \inf \{ d_N(\tilde{m}, \tilde{m}') : \tilde{m}' \in M^o_N \} \quad (2.7) \]
and then over all $\tilde{m} \in M_N$.

Due to (2.5) and (2.7) the denominator in (2.6) is not zero and $D_N(l)$ is well defined.

Comparison of the above dilation functions shows that $D_N^o(l) \leq D_N(l)$. Nevertheless, the converse is also true for $l$ close to 1.

**Lemma 2.1** Assume that $N$ and the doubling constant $D := D(\mu^o)$, see (1.2), are related by
\[ N \geq [3 \log_2 D] + 5. \quad (2.8) \]
Then the following is true:
\[ D_N(1 + 1/N) \leq \frac{6}{5} e^4. \]

**Proof.** In accordance with the definition of $D_N$, see (2.6), we must estimate the function
\[ \frac{\overline{\mu}_N(B_{R_N}(\tilde{m}))}{\overline{\mu}_N(B_{R}(\tilde{m}))} \quad \text{where} \quad R_N := \left(1 + \frac{1}{N}\right) R. \quad (2.9) \]
Since the points $\tilde{m}'$ of the ball $B_{R_N}(\tilde{m})$ of $M_N$ satisfy the inequality
\[ d(m, m') + |x - x'|_1 < R_N, \]
the Fubini theorem and (2.5) yield
\[ \overline{\mu}_N(B_{R_N}(\tilde{m})) = \gamma_N \int_{M^o \cap B_{R_N}(m)} (R_N - d(m, m'))^N d\mu^o(m'); \quad (2.10) \]
here $\gamma_N$ is the volume of the unit $t^N_1$-ball.

We must estimate the integral in (2.10) from above under the condition
\[ d_N(\tilde{m}, M^o_N) < R/4. \quad (2.11) \]
To this end split the integral into one over $B_{3R/4}(m) \cap M^o$ and one over the remaining part $(B_{R_N}(m) \setminus B_{3R/4}(m)) \cap M^o$. Denote these integrals by $I_1$ and $I_2$. For $I_2$ we get
\[ I_2 \leq \gamma_N (R_N - 3R/4)^N \mu^o(B_{R_N}(m) \cap M^o). \quad (2.12) \]
Further, from (2.11) we clearly have
\[ d(m, M^o) < R/4. \]
Pick a point $\tilde{m} \in M^o$ so that
\[ d(m, M^o) \leq d(m, \tilde{m}) < R/4. \]
Then we have the following embeddings

\[ B_{R_{N/4}}(\hat{m}) \subset B_{R_{N/2}}(m) \cap M^o \subset B_{R_N}(m) \cap M^o \subset B_{5R_{N/4}}(\hat{m}). \]

Applying the doubling inequality for the measure \( \mu^o \), see (1.2), we then obtain

\[ \mu^o(\partial B_{R_N}(m) \cap M^o) \leq D^3 \mu^o(\partial B_{R_{N/2}}(m) \cap M^o). \]

Moreover, due to (2.8)

\[ D^3 < 2^{[\log_2 D]+1} \leq 2^{N-4}. \]

Combining the last two inequalities with (2.12) we have

\[ I_2 \leq \gamma_N 2^{-N+4} \left(1 + \frac{4}{N}\right)^N \int_{\partial B_{R_N}(m) \cap M^o} (R - d(m, m'))^N d\mu^o(m'). \]

(2.13)

To estimate the integral \( I_1 \) we rewrite its integrand as follows:

\[ (R_N - d(m, m'))^N = \left(1 + \frac{1}{N}\right)^N (R - d(m, m'))^N \left(1 + \frac{d(m, m')}{(N+1)(R - d(m, m'))}\right)^N. \]

Since \( m' \in B_{3/4R}(m) \), the last factor is at most \( \left(1 + \frac{3R/4}{(N+1)R/4}\right)^N = \left(1 + \frac{3}{N+1}\right)^N \).

This yields

\[ I_1 \leq \gamma_N \left(1 + \frac{1}{N}\right)^N \left(1 + \frac{3}{N+1}\right)^N \int_{\partial B_{3R/4}(m) \cap M^o} (R - d(m, m'))^N d\mu^o(m') \leq \]

\[ e^4 \overline{\kappa}_N(B_R(\hat{m})). \]

Hence for the part of fraction (2.9) related to \( I_1 \) we have

\[ \overline{I}_1 := \frac{I_1}{\overline{\kappa}_N(B_R(\hat{m}))} \leq e^4. \]

(2.14)

To estimate the remaining part \( \overline{I}_2 := \frac{I_2}{\overline{\kappa}_N(B_R(\hat{m}))} \) we note that its denominator is greater than

\[ \gamma_N \int_{M^o \cap \partial B_{R_{N/2}}(m)} (R - d(m, m'))^N d\mu^o(m'). \]

Since here \( d(m, m') \leq R_N/2 \), this, in turn, is bounded from below by

\[ \gamma_N 2^{-N} \left(1 - \frac{1}{N}\right)^N \int_{\partial B_{R_N/2}(m) \cap M^o} (R - d(m, m'))^N d\mu^o(m'). \]

Combining this with (2.13) and noting that \( N \geq 5 \) we get

\[ \overline{I}_2 \leq 2^{-4} \left(1 - \frac{1}{N}\right)^{-N} \left(1 + \frac{4}{N}\right)^N \leq \frac{1}{5} e^4. \]

Hence the fraction (2.9) is bounded by \( \overline{I}_1 + \overline{I}_2 \leq \frac{6}{5} e^4 \), see (2.14), and this immediately implies the required estimate of \( D_N(1 + 1/N) \). \( \square \)

In the next lemma we estimate \( \overline{\kappa}_N \)-measure of the spherical layer \( B_{R_2}(\hat{m}) - B_{R_1}(\hat{m}) \), \( R_2 \geq R_1 \), by a kind of a surface measure. For its formulation we set

\[ A_N := \frac{12}{5} e^4 N. \]

(2.15)
Lemma 2.2 Assume that

\[ N \geq [3 \log_2 D] + 6. \]

Then for all \( \tilde{m} \in M_N \) and \( R_1, R_2 > 0 \) satisfying

\[ R_2 \geq \max\{R_1, 8d_N(\tilde{m}, M^o_N)\} \]

the following is true

\[ \bar{\mu}_N(B_{R_2}(\tilde{m}) \setminus B_{R_1}(\tilde{m})) \leq A_N \frac{\bar{\mu}_N(B_{R_2}(\tilde{m}))}{R_2} (R_2 - R_1). \]

Proof. By definition \( M_N = M_{N-1} \times \mathbb{R} \) and \( \bar{\mu}_N = \bar{\mu}_{N-1} \otimes \lambda_1 \). Then by the Fubini theorem we have for \( R_1 \leq R_2 \) with \( \tilde{m} = (\tilde{m}, t) \)

\[ \bar{\mu}_N(B_{R_2}(\tilde{m})) - \bar{\mu}_N(B_{R_1}(\tilde{m})) = 2 \int_{R_1}^{R_2} \bar{\mu}_{N-1}(B_s(\tilde{m})) ds \leq \frac{2R_2 \bar{\mu}_{N-1}(B_{R_2}(\tilde{m}))}{R_2} (R_2 - R_1). \]

We claim that for arbitrary \( l > 1 \) and \( R \geq 8d_N(\tilde{m}, M^o_N) : = 8d_{N-1}(\tilde{m}, M^o_{N-1}) \)

\[ R \bar{\mu}_{N-1}(B_R(\tilde{m})) \leq \frac{l D_{N-1}(l)}{l - 1} \bar{\mu}_N(B_R(\tilde{m})). \quad (2.16) \]

Together with the previous inequality this will yield

\[ \bar{\mu}_N(B_{R_2}(\tilde{m})) - \bar{\mu}_N(B_{R_1}(\tilde{m})) \leq \frac{2l D_{N-1}(l)}{l - 1} \frac{\bar{\mu}_N(B_{R_2}(\tilde{m}))}{R_2} (R_2 - R_1). \]

Finally choose here \( l = 1 + \frac{1}{N-1} \) and use Lemma 2.1. This will give the required inequality.

Hence, it remains to establish (2.16). By the definition of \( D_{N-1}(l) \) we have for \( l > 1 \) using the previous lemma

\[ \bar{\mu}_N(B_{lR}(\tilde{m})) = 2l \int_0^R \bar{\mu}_{N-1}(B_{ls}(\tilde{m})) ds \leq 4l \int_{R/2}^R \bar{\mu}_{N-1}(B_{ls}(\tilde{m})) ds \leq 4l D_{N-1}(l) \int_{R/2}^R \bar{\mu}_{N-1}(B_s(\tilde{m})) ds \leq 2l D_{N-1}(l) \bar{\mu}_N(B_R(\tilde{m})). \]

On the other hand, replacing \([0, R]\) by \([l^{-1}R, R]\) we also have

\[ \bar{\mu}_N(B_{lR}(\tilde{m})) \geq 2l \bar{\mu}_{N-1}(B_R(\tilde{m}))(R - l^{-1}R) = 2(l - 1)R \bar{\mu}_{N-1}(B_R(\tilde{m})). \]

Combining the last two inequalities we get (2.16). \qed

Extension operator

We define the required simultaneous extension \( E : Lip(M^o_N, X) \to Lip(M_N, X) \) using the standard average operator \( Ave \) defined on continuous and locally bounded functions \( g : M^o_N \to X \) by

\[ Ave(g; \tilde{m}, R) := \frac{1}{\bar{\mu}_N(B_R(\tilde{m}))} \int_{B_R(\tilde{m})} g d\bar{\mu}_N. \]
To be well-defined the domain of integration $B_R(\hat{m}) \cap M_N^o$ should be of strictly positive $\mu_N$-measure (i.e., $\mu_N$-measure). This condition is fulfilled in the case presented now. Namely, we define the simultaneous extension $E$ on functions $f \in Lip(M_N^o, X)$ by

$$(Ef)(\hat{m}) := \begin{cases} f(\hat{m}) & \text{if } \hat{m} \in M_N^o \\ \text{Ave}(f; m, R(\hat{m})) & \text{if } \hat{m} \notin M_N^o \end{cases}$$ \tag{2.17}$$

where we set

$$R(\hat{m}) := 8d_N(\hat{m}, M_N^o).$$

The required estimate of $\|E\|$ is presented below. To formulate the result we set

$$K_N(l) := A_N D_N(l)(4l + 1)$$ \tag{2.18}$$

where the first of two factors are defined by (2.15) and (2.6).

**Proposition 2.3** The following inequality

$$\|E\| \leq 20A_N + \max \left( \frac{4l + 1}{2(l - 1)}, K_N(l) \right)$$

is true provided $l := 1 + 1/N$.

Before we begin the proof let us derive from here the desired result. Namely, choose

$$N := \lfloor 3 \log_2 D \rfloor + 6$$

and use Lemma 2.1 and (2.15) to estimate $D_N(1 + 1/N)$ and $A_N$. Then we get

$$\|E\| \leq C(\log_2 D + 2)$$

with some numerical constant $C$. This clearly gives (1.3).

**Proof.** We have to show that for every $\tilde{m}_1, \tilde{m}_2 \in M_N$

$$\|(Ef)(\tilde{m}_1) - (Ef)(\tilde{m}_2)\|_X \leq K\|f\|_{Lip(M_N^o, X)} d_N(\tilde{m}_1, \tilde{m}_2)$$ \tag{2.19}$$

where $K$ is the constant in the inequality of the proposition.

It suffices to consider only two cases:

(a) $\tilde{m}_1 \in M_N^o$ and $\tilde{m}_2 \notin M_N^o$;

(b) $\tilde{m}_1, \tilde{m}_2 \notin M_N^o$.

We assume without loss of generality that

$$\|f\|_{Lip(M_N^o, X)} = 1$$ \tag{2.20}$$

and simplify the computations by introducing the following notations:

$$R_i := d_N(\hat{m}_i, M_N^o), \ B_{ij} := B_{8R_i}(%(\hat{m}_i)) \ , \ v_{ij} := \mu_N(B_{ij}) , \ 1 \leq i, j \leq 2 .$$ \tag{2.21}$$
We assume also for definiteness that
\[ 0 < R_1 \leq R_2. \] (2.22)

By the triangle inequality we then have
\[ 0 \leq R_2 - R_1 \leq d_N(\hat{m}_1, \hat{m}_2). \] (2.23)

Further, by Lemma 2.2 the quantities introduced satisfy the following inequality:
\[ v_{i2} - v_{i1} \leq \frac{A_N v_{i2}}{R_2} (R_2 - R_1), \] (2.24)

Let now \( \hat{m}^* \) be such that \( d_N(\hat{m}_1, \hat{m}^*) < 2R_1 \). Set
\[ \hat{f}(\hat{m}) := f(\hat{m}) - f(\hat{m}^*). \] (2.25)

From the triangle inequality we then obtain
\[ \max \{ ||\hat{f}(\hat{m})||_X : \hat{m} \in B_{i2} \cap M_N^o \} \leq 10R_2 + (i - 1)d_N(\hat{m}_1, \hat{m}_2); \] (2.26)

here \( i = 1, 2 \).

We now prove (2.19) for \( \hat{m}_1 \in M_N^o \) and \( \hat{m}_2 \not\in M_N^o \). We begin with the evident inequality
\[ ||(E f)(\hat{m}_2) - (E f)(\hat{m}_1)||_X = \frac{1}{v_{i2}} \left| \int_{B_{i2}} \hat{f}(\hat{m}) d\mu_N \right|_X \leq \max_{B_{i2} \cap M_N^o} ||\hat{f}||_X, \]
see (2.21) and (2.25). Applying (2.26) with \( i = 2 \) we then bound this maximum by
\[ 10R_2 + d_N(\hat{m}_1, \hat{m}_2). \]
But \( \hat{m}_1 \in M_N^o \) and so
\[ R_2 = d_N(\hat{m}_2, M_N^o) \leq d_N(\hat{m}_1, \hat{m}_2); \]
therefore (2.19) holds in this case with \( K = 11 \).

The remaining case \( \hat{m}_1, \hat{m}_2 \not\in M_N^o \) requires some additional auxiliary results. For their formulations we first write
\[ (Ef)(\hat{m}_1) - (Ef)(\hat{m}_2) := D_1 + D_2 \] (2.27)

where
\[ D_1 := \text{Ave}(\hat{f}; \hat{m}_1, 8R_1) - \text{Ave}(\hat{f}; \hat{m}_1, 8R_2) \]
\[ D_2 := \text{Ave}(\hat{f}; \hat{m}_1, 8R_2) - \text{Ave}(\hat{f}; \hat{m}_2, 8R_2), \] (2.28)

see (2.17) and (2.25).

**Lemma 2.4** We have
\[ ||D_i||_X \leq 20A_N d_N(\hat{m}_1, \hat{m}_2). \]
Recall that \( A_N \) is the constant defined by (2.15).
Proof. By (2.28), (2.25) and (2.21),
\[ D_1 = \frac{1}{v_{11}} \int_{B_{11}} \hat{f} \overline{\mu}_N - \frac{1}{v_{12}} \int_{B_{12}} \hat{f} \overline{\mu}_N = \left( \frac{1}{v_{11}} - \frac{1}{v_{12}} \right) \int_{B_{11}} \hat{f} \overline{\mu}_N - \frac{1}{v_{12}} \int_{B_{12} \setminus B_{11}} \hat{f} \overline{\mu}_N. \]

This immediately implies that
\[ \|D_1\|_X \leq 2 \cdot \frac{v_{12} - v_{11}}{v_{12}} \cdot \max_{B_{12} \cap M_N^o} \|\hat{f}\|_X. \]

Applying now (2.24) and (2.23), and then (2.26) with \( i = 1 \) we get the desired estimate. \( \square \)

To obtain a similar estimate for \( D_2 \) we will use the following two facts.

**Lemma 2.5** Assume that for a given \( l > 1 \)
\[ d_N(\hat{m}_1, \hat{m}_2) \leq 8(l - 1)R_2. \] (2.29)

Let for definiteness \( v_{22} \leq v_{12} \). (2.30)

Then we have
\[ \overline{\mu}_N(B_{12} \Delta B_{22}) \leq A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2) \] (2.31)

(*here \( \Delta \) denotes symmetric difference of sets*).

**Proof.** Set
\[ R := 8R_2 + d_N(\hat{m}_1, \hat{m}_2). \]

Then \( B_{12} \cup B_{22} \subset B_R(\hat{m}_1) \cap B_R(\hat{m}_2) \), and
\[ \overline{\mu}_N(B_{12} \Delta B_{22}) \leq (\overline{\mu}_N(B_R(\hat{m}_1)) - \overline{\mu}_N(B_{8R_2}(\hat{m}_1))) + \]
\[ (\overline{\mu}_N(B_R(\hat{m}_2)) - \overline{\mu}_N(B_{8R_2}(\hat{m}_2))). \] (2.32)

Estimating the terms on the right-hand side by Lemma 2.2 we bound them by
\[ A_N \frac{\overline{\mu}_N(B_R(\hat{m}_1))}{R} (R - 8R_2) + A_N \frac{\overline{\mu}_N(B_R(\hat{m}_2))}{R} (R - 8R_2). \]

Moreover, \( 8R_2 \leq R \leq 8lR_2 \) and \( R - 8R_2 := d_N(\hat{m}_1, \hat{m}_2) \), see (2.29); taking into account (2.6), (2.21) and (2.30) we therefore have
\[ \overline{\mu}_N(B_{12} \Delta B_{22}) \leq A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \] (2.33)

**Lemma 2.6** Under the assumptions of the previous lemma we have
\[ v_{12} - v_{22} \leq A_N D_N(l) \frac{v_{12}}{4R_2} d_N(\hat{m}_1, \hat{m}_2). \] (2.33)

**Proof.** By (2.21) the left-hand side is bounded by \( \overline{\mu}_N(B_{12} \Delta B_{22}) \). \( \square \)

We now estimate \( D_2 \) from (2.28) beginning with
Lemma 2.7 Under the conditions of Lemma 2.5 we have

\[ ||D_2||_X \leq K_N(l) d_N(\hat{m}_1, \hat{m}_2) \]

where \( K_N(l) := A_N D_N(l)(4l + 1) \).

Proof. By the definition of \( D_2 \) and our notation, see (2.28), (2.25) and (2.21),

\[
||D_2||_X := \left| \left| \frac{1}{v_{12}} \int_{B_{12}} \hat{f} d\mu_N - \frac{1}{v_{22}} \int_{B_{22}} \hat{f} d\mu_N \right| \right|_X \leq \\
\frac{1}{v_{12}} \int_{B_{12}} ||\hat{f}||_X d\mu_N + \left| \left| \frac{1}{v_{12}} - \frac{1}{v_{22}} \right| \right|_X \int_{B_{22}} ||\hat{f}||_X d\mu_N := J_1 + J_2 .
\]

By (2.31), (2.29) and (2.26)

\[
J_1 \leq \frac{1}{v_{12}} \mu_N(B_{12} \cup B_{22}) \sup_{(B_{12} \cup B_{22}) \cap M_N^*} ||\hat{f}||_X \leq \\
\frac{A_N D_N(l)}{4R_2} d_N(\hat{m}_1, \hat{m}_2)(d_N(\hat{m}_1, \hat{m}_2) + 10R_2) \leq A_N D_N(l)(2l + 1/2)d_N(\hat{m}_1, \hat{m}_2).
\]

Also, (2.33), (2.26) and (2.29) yield

\[
J_2 \leq A_N D_N(l)(2l + 1/2)d_N(\hat{m}_1, \hat{m}_2) .
\]

Combining these we get the required estimate. \( \square \)

It remains to consider the case of \( \hat{m}_1, \hat{m}_2 \in M_N \) satisfying the inequality

\[
d_N(\hat{m}_1, \hat{m}_2) > 8(l - 1)R_2
\]

converse to (2.29). Now the definition (2.28) of \( D_2 \) and (2.26) imply that

\[
||D_2||_X \leq 2 \sup_{(B_{12} \cup B_{22}) \cap M_N^*} ||\hat{f}||_X \leq 2 \left( 10R_2 + d_N(\hat{m}_1, \hat{m}_2) \right) \leq \frac{4l + 1}{2(l - 1)} d_N(\hat{m}_1, \hat{m}_2) .
\]

Combining this with the inequalities of Lemmas 2.4 and 2.7 and equality (2.27) we obtain the required estimate of the Lipschitz norm of the extension operator \( E \):

\[
||E|| \leq 20A_N + \max \left( \frac{4l + 1}{2(l - 1)}, K_N(l) \right)
\]

where \( K_N(l) \) is the constant in (2.18). \( \square \)

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