PDM Klein-Gordon particles in Gödel-type Som-Raychaudhuri cosmic string spacetime background

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Abstract: In Gödel-type Som-Raychaudhuri (SR) cosmic string spacetime background, we recycle the Klein-Gordon (KG) oscillators and report their correct exact solutions. We argue that the mathematical collapse of the KG-equation into the 2-dimensional radial Schrödinger-like oscillator does not yield that the parametric characterizations of one is inherited by the other. The angular frequency (positive) in the Schrödinger case is replaced by an irrational frequency-like (positive and negative) in the KG-case. Inheriting the Schrödinger oscillator’s parametric characterizations implies that at least half of the spectra (the negative part) is lost in the process. We also introduce KG-oscillators in pseudo-Gödel SR-type spacetime that admit invariance and isospectrality with those in Gödel SR-type spacetime background. We introduce position-dependent mass (PDM) settings to KG-particles in 4-vector and scalar Lorentz potentials in magnetic field in the Gödel SR-type spacetime background. Four illustrative examples of fundamental nature are discussed and their exact or conditionally exact solvability are reported. Amongst are, a PDM KG-Coulombic particle in 4-vector and scalar Coulombic Lorentz potentials in magnetic field, a PDM KG-Coulombic particle in equally mixed 4-vector and scalar Coulombic Lorentz potentials in magnetic field, and a quasi-free PDM KG-oscillator. We also emphasize that the biconfluent Heun polynomial approach, to the effective oscillator plus Cornell type potential, yields conditionally exact solvability that paralyzes the solution from collapsing into that of pure Coulombic one.

PACS numbers: 05.45.-a, 03.50.Kk, 03.65.-w

Keywords: Klein-Gordon (KG) particles, Gödel-type Som-Raychaudhuri cosmic string spacetime, effective position-dependent mass (PDM), PDM-KG Coulombic and oscillators, quasi-free KG-oscillators.

I. INTRODUCTION

Early universe theories have predicted several topological structural defects in spacetime that have inspired intensive research studies in quantum gravity, such as domain wall [1, 2], cosmic string [3, 4], global monopole [5] and textures [6]. It has been observed that the energy levels of relativistic and non-relativistic quantum systems are infected by such topological defects, not only in general relativity but also in the geometrical theory of topological defects in condensed matter. For example, in general relativity the Gödel spacetime metric [7], with an embedded cosmic string, introduces itself as the first cosmological solution with rotating matter. Its compact form allowed analytical research studies of many physical and mathematical systems in gravitational backgrounds with rotation and causality violation. Rebouças and Tiomno [8] have examined the conditions for spacetime homogeneity of the Riemannian manifold with a Gödel spacetime background and found that the Gödel universe is homogeneous in spacetime (ST-homogeneous). They have shown that all ST-homogeneous Gödel-type metrics characterized by the vorticity \( \Omega \) and a given value of the parameter \( \hat{\mu} (-\infty \leq \hat{\mu}^2 \leq \infty) \) can be transformed in polar coordinates [9–12] to

\[
ds^2 = -(dt + \alpha \Omega \frac{\sinh(\hat{\mu} r)}{\hat{\mu}^2} d\varphi)^2 + \alpha^2 \frac{\sinh(2 \hat{\mu} r)^2}{4 \hat{\mu}^2} d\varphi^2 + dr^2 + dz^2. \tag{1}\n\]

Where for \( \hat{\mu}^2 < 0 \) there is an infinite number of successive causal and noncausal regions, for \( 0 \leq \hat{\mu}^2 < \Omega^2 \) there is one noncausal region for a given \( r > r_c \) \( (r_c \) is the critical radius), and \( \hat{\mu} = \Omega \) corresponds to the Gödel spacetime metric (the first exact Gödel solution of the Einstein equation describing a complete causal, ST-homogeneous rotating universe [8]). Nevertheless, at the limit \( \hat{\mu} \to 0 \) of the Gödel spacetime metric [11] we obtain the ST-homogeneous Som-Raychaudhuri (SR) solution

\[
ds^2 = -(dt + \alpha \Omega r^2 d\varphi)^2 + \alpha^2 r^2 d\varphi^2 + dr^2 + dz^2 \tag{2}\n\]
of the Einstein field equations [13]. Where the disclination parameter $\alpha$ admits the values $0 < \alpha < 1$ in general relativity for cosmic strings with positive curvature, $\alpha > 1$ in the geometric theory of defects in condensed matter for a negative curvature, and $\alpha = 1$ corresponds to Minkowski flat spacetime metric. Moreover, the covariant metric tensor associated with the Som-Raychaudhuri spacetime is given by

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & -\alpha \Omega^2 & 0 \\ 0 & 1 & 0 & 0 \\ -\alpha \Omega^2 & 0 & \alpha^2 r^2 (1 - \Omega^2 r^2) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \iff \quad g^{\mu\nu} = \begin{pmatrix} \Omega^2 r^2 - 1 & 0 & -\Omega \alpha & 0 \\ 0 & 1 & 0 & 0 \\ -\Omega \alpha & 0 & \Omega - \Omega^2 r^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; \quad \det (g) = -\alpha^2 r^2. \quad (3)$$

The Gödel SR-type ST-homogeneous metric [2] is the core of the current methodical proposal.

On the other hand, effective position-dependent mass (PDM) settings in the von Roos Schrödinger Hamiltonian [14] have attracted research interest over the last few decades both in quantum and classical mechanics (e.g., [15–26]). Hereby, it has been asserted that the PDM notion is a metaphoric manifestation of coordinate transformation/deformation [27–32]. This would, in turn, effectively change the canonical momentum for PDM classical systems and the momentum operator for PDM quantum systems. Namely, the PDM canonical momentum reads $p (x) = m (x) \dot{x}$ and consequently negative the gradient of the potential force field is no longer given by the time derivative of the canonical momentum but rather related to the time derivative of the pseudo-momentum (i.e., Noether momentum) $\pi (x) = \sqrt{m (x)} \dot{x}$ [18]. Yet, in quantum mechanics the PDM-momentum operator is shown to be given by $\hat{p}_j (\mathbf{r}) = -i \left( \partial_j - \partial_j m (\mathbf{r}) / 4m (\mathbf{r}) \right) ; \quad j = 1, 2, 3$, and consequently yields, in its most simplistic one-dimensional form, the von Roos PDM kinetic energy operator $\hat{T} = \left( \hat{p}_x (x) / \sqrt{m (x)} \right)^2 = -m (x)^{-1/4} \partial_x m (x)^{-1/2} \partial_x m (x)^{-1/4}$, which is known in the literature as Mustafa and Mazharimousavi’s [20] ordering. Basically, in short, a quantum mechanical PDM particle should be associated with the PDM-momentum operator both for relativistic and non-relativistic PDM-quantum particles. Therefore, in the relativistic Dirac and/or Klein-Gordon (KG) equations, the assumption that the rest mass energy term $m \rightarrow m + m (x) + S (x)$ does not introduce PDM relativistic particles but rather introduces an amendment to the already existing Lorentz scalar potential, i.e., $\tilde{S} (x) = m (x) + S (x)$, (e.g., [33, 34] and references cited therein). Very recently, however, we have introduced and discussed PDM relativistic particles [29] in Gödel spacetime background [38], in cosmic string spacetime in Kaluza-Klein theory [31] and in a cosmic spacetime with a cosmic space-like dislocation [30]. It would be, therefore, interesting to study PDM KG-particles in the Gödel-type Som-Raychaudhuri (SR) spacetime background [11, 32, 44].

The organization of our paper is in order. In section 2, we recyle and discuss a KG-oscillators in the Gödel SR-type cosmic string spacetime background. Hereby, we pinpoint and emphasis (along with the analysis carried out by Fernandez [45] on the related references cited therein, some of which are cited here) that the KG-oscillators and the Schrödinger-oscillator are two different quantum mechanical systems and should not be confused with each other. The mathematical collapse of the KG equation into Schrödinger-like oscillator does not mean that the parametric characterizations are copied from one into the other. By recycling this problem, we provide solid/correct interpretation grounds to be used in the subsequent sections. We report, in section 3, a set of KG-oscillators in pseudo-Gödel SR-type cosmic string spacetime that admit invariance and isospectrality with the KG-oscillator discussed in section 2. We introduce, in section 4, PDM-settings (metaphorically speaking) for KG-particles in 4-vector and/or scalar Lorentz potentials in Gödel SR-type cosmic string spacetime background and subjected to a magnetic field. Therein, we discuss four illustrative examples of fundamental nature: (i) PDM-KG particles in a 4-vector and scalar Lorentz potentials and a magnetic field in the Gödel SR-type cosmic string spacetime background, (ii) a PDM-KG Coulombic particle in 4-vector and scalar coulombic potentials and a magnetic field in Gödel SR-type spacetime at zero vorticity, (iii) a PDM-KG Coulombic particle in equally mixed 4-vector and scalar Coulombic Lorentz potentials and a magnetic field in Gödel SR-type spacetime, and (iv) a quasi-free PDM KG-oscillator in Gödel SR-type cosmic string spacetime background. Our concluding remarks are given in section 5.

II. KLEIN-GORDON OSCILLATORS IN THE GÖDEL SR-TYPE COSMIC STRING SPACETIME BACKGROUND

The KG-equation describing a spin-0 particle of rest mass energy $m$ (denoting $mc^2$ in $c = \hbar = 1$ units) in the Gödel SR-type ST-homogeneous metric [2] is given by

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-gg^{\mu\nu}} \partial_\nu \Psi \right) = m^2 \Psi. \quad (4)$$
Which, in a straightforward manner, would result

\[ \left\{ -\partial_t^2 + \left( \partial_r^2 + \frac{1}{r} \partial_r \right) + \left( \Omega r \partial_t - \frac{1}{\alpha r} \partial_\phi \right)^2 + \partial_z^2 - m^2 \right\} \Psi = 0. \]

(5)

The substitution of

\[ \Psi (t, r, \phi, z) = \exp \left( i [\ell \varphi + k_z z - Et] \right) \psi (r) = \exp \left( i [\ell \varphi + k_z z - Et] \right) \frac{R (r)}{\sqrt{r}} \]

would yield

\[ R'' (r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - \tilde{\omega}^2 r^2 \right] R (r) = 0, \]

(7)

where

\[ \lambda = E^2 - 2 \ell \Omega E - k_z^2 - m^2 ; \quad \tilde{\ell} = \frac{\ell}{\alpha} ; \quad \tilde{\omega}^2 = \Omega^2 E^2. \]

(8)

It is obvious that this equation represents a KG-oscillator (as a manifestation of the Gödel SR-type spacetime structure) and resembles the one-dimensional form of the two-dimensional radially symmetric Schrödinger-oscillator. However, the parameter \( \tilde{\omega} = \pm [\Omega E] \) should never be confused with the Schrödinger-oscillator frequency for it does not admit the same parametric characterization. Therefore, this equation would only mathematically inherit the textbook exact eigenvalues and eigenfunctions of the Schrödinger-oscillator so that

\[ \lambda = 2 \tilde{\omega} \left( 2n_r + |\tilde{\ell}| + 1 \right) \]

(9)

and

\[ R (r) \sim r^{|\tilde{\ell}|+1/2} \exp \left( -\frac{\tilde{\omega} r^2}{2} \right) L_{n_r}^{|\tilde{\ell}|} (\tilde{\omega} r^2) , \]

(10)

where \( L_{n_r}^{|\tilde{\ell}|} (\tilde{\omega} r^2) \) are the associated Laguerre polynomials. Under such settings, one obtains

\[ E = \Omega \left( 2n_r + |\tilde{\ell}| + \tilde{\ell} + 1 \right) \pm \sqrt{\Omega^2 \left( 2n_r + |\tilde{\ell}| + \tilde{\ell} + 1 \right)^2 + m^2 + k_z^2}. \]

(11)

We clearly observe, hereby, that the vorticity parameter \( \Omega \) plays a magnetic field-like role of lifting the degeneracies associated with the irrational magnetic quantum number \( \tilde{\ell} = \ell/\alpha = \pm |\tilde{\ell}| \) and results in quasi Landau energy levels. Moreover, it should be noted here that for \( \Omega = 0, k_z = 0 \) and \( \alpha = 1 \), the rest mass energies for the one-dimensional KG-particle are retrieved, with \( k_z = 0 \), as a natural tendency of the more general case at hand. However, when this result is compared with that of Carvalho et al. (i.e., equation (14) in (11)) we observe that the negative energies are missed in their results and should be corrected into those reported in (11) above. So should be the case with the results reported on the linear confinement of a scalar particle in Gödel-type space-time by Vitória (their Eq. (23)) and in the related comment by Neto (their Eq. (18)) for zero linear confinement. That would also include the results reported by Ahmed, where not only half of the spectra (the negative energies) is missed but also all nodeless (i.e., states with \( n_r = 0 \)) states.

### III. KG-Oscillators in Pseudo-Gödel SR-Type Cosmic String Spacetime Admitting Invariance and Isospectrality

Let us consider that the Gödel SR-type spacetime metric \( ds^2 \) of be transformed so that

\[ ds^2 \rightarrow d\tilde{s}^2 = - (d\tilde{t} + \alpha \tilde{r}^2 d\tilde{\varphi})^2 + \alpha^2 \tilde{r}^2 d\tilde{\varphi}^2 + d\tilde{r}^2 + d\tilde{z}^2 \]

(12)

where

\[ d\tilde{t} = dt, d\tilde{r} = \sqrt{f (r)} dr, \tilde{r} = \sqrt{Q (r)} r, d\tilde{\varphi} = d\varphi, d\tilde{z} = dz, \]

(13)
and hence
\[ \frac{d\tilde{r}}{dr} \implies \sqrt{f(r)} = \sqrt{Q(r)} \left[ 1 + \frac{Q'(r)}{2Q(r)} r \right], \] (14)
to govern the correlation between \( f(r) \) and \( Q(r) \). Under such transformation settings, a pseudo-Gödel SR-type spacetime metric is manifestly introduced as
\[ d\bar{s}^2 = -(dt + \alpha \Omega g(r) d\varphi)^2 + \alpha^2 g(r) d\varphi^2 + f(r) dr^2 + dz^2 \] (15)
Then the covariant metric tensor associated to such a pseudo-Gödel SR-type spacetime is given by
\[ \tilde{g}_{\mu\nu} = \begin{pmatrix} -1 & 0 & -\alpha \Omega g(r) & 0 \\ 0 & f(r) & 0 & 0 \\ -\alpha \Omega g(r) & 0 & \alpha^2 g(r) (1 - \Omega^2 g(r)) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \implies \tilde{g}^{\mu\nu} = \begin{pmatrix} (\Omega^2 g(r) - 1) & 0 & -\Omega^2 g(r) & 0 \\ 0 & f(r) & 0 & 0 \\ -\Omega^2 g(r) & 0 & \alpha^2 g(r) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (16)
where \( g(r) = Q(r) r^2 \) and \( \text{det}(g) = -\alpha^2 f(r) g(r) \). We may now use similar assumption to that in (6) and define
\[ \Psi(t, \tilde{r}, \varphi, z) = \exp(i [\ell \varphi + k z - Et]) \frac{R(\tilde{r})}{\sqrt{f(r)}}; \quad \tilde{r} = \sqrt{g(r)} = \sqrt{Q(r)r}. \] (17)
Consequently, the KG-equation (4) would read
\[ \left[ \frac{1}{\sqrt{g(r) f(r)}} \frac{\partial}{\partial r} \left( \sqrt{g(r)} f(r) \partial_r \right) + \lambda - \frac{\tilde{r}^2}{g(r)} - \tilde{\omega}^2 g(r) \right] \frac{R(\tilde{r})}{\sqrt{r}} = 0, \] (18)
to imply, with \( \tilde{r} = \sqrt{g(r)} = \sqrt{Q(r)r} \), that
\[ \frac{d^2 R(\tilde{r})}{d\tilde{r}^2} + \left[ \lambda - \frac{\left( \tilde{r}^2 - 1/4 \right)}{\tilde{r}^2} \right] R(\tilde{r}) = 0. \] (19)
Obviously, this equation is invariant with (7) and shares the same eigenvalues. We may now conclude that the two equations (7) and (19) are invariant and isospectral. Moreover, our transformed metric in (15) is a pseudo-Gödel SR-type spacetime metric under our transformation recipe in (13) and (14). This would in effect introduce a new set of the Gödel SR-type spacetime metrics that may very well be classified as pseudo-Gödel SR-type spacetime metrics that can be transformed back into the Gödel SR-type spacetime metrics through the transformation in (14).

IV. PDM-KG PARTICLES IN A 4-VECTOR AND SCALAR LORENTZ POTENTIALS AND A MAGNETIC FIELD IN GÖDEL SR-TYPE COSMIC STRING SPACETIME BACKGROUND

The KG-equation for a spin-0 particle in a 4-vector potential \( A_\mu \) in the Gödel SR-type spacetime background (2) is given by
\[ \frac{1}{\sqrt{-g}} D_\mu \left( \sqrt{-g} g^{\mu\nu} D_\nu \Psi \right) = m^2 \Psi, \] (20)
where the gauge-covariant derivative is given by \( D_\mu = \partial_\mu - ie A_\mu \). Moreover, in a recent paper, we have asserted that the PDM-setting is a manifestation of coordinate deformation/transformation (27) (31) that yields to the introduction of the PDM-momentum operator
\[ \hat{p}_j(r) = -i \frac{\nabla f(r)}{4f(r)} \iff p_j = -i \left( \partial_j - \frac{\partial_j f(r)}{4f(r)} \right). \] (21)
This would, in effect, suggest that a PDM-quantum particle (metaphorically speaking) is described by the PDM-momentum operator in (21) and may be subjected to some interaction potential force fields. However, should one be interested in invariant and isospectral systems, then our recipe in the preceding section shall be followed (c.f., e.g., [28]). In the current section, however, we shall assume that we have only a deformation/defect in the radial coordinate so that

$$\partial_\mu \rightarrow \partial_\mu + F_\mu; F_\mu = (0, F_r, 0, 0), F_r = \frac{f'(r)}{4f(r)}$$  \hspace{1cm} (22)

(e.g., [29] 31). This would in turn allow us to cast equation (20) as

$$\frac{1}{\sqrt{-g}} \hat{D}_\mu \left( \sqrt{-g} g^{\mu\nu} \hat{D}_\nu \Psi \right) = (m + S(r))^2 \Psi; \hat{D}_\mu = \partial_\mu + F_\mu - ieA_\mu.$$  \hspace{1cm} (23)

Consequently, one obtains

$$\left\{ -D_t^2 + \frac{1}{r} D_r (r [D_r - F_r]) + \left( \Omega r D_t - \frac{1}{ar} D_\varphi \right)^2 + D_z^2 + F_r (D_r - F_r) - (m + S(r))^2 \right\} \Psi = 0. \hspace{1cm} (24)$$

At this point, we may now define the corresponding gauge-covariant derivatives so that

$$D_t = \partial_t - ieA_t = \partial_t - iV(r), \hspace{0.5cm} D_r = \partial_r, \hspace{0.5cm} D_\varphi = \partial_\varphi - ieA_\varphi, \hspace{0.5cm} D_z = \partial_z,$$

where, $V(r) = eA_t$ is the Lorentz 4-vector potential (i.e., transforms within the 4-vector potential $A_\mu$), $S(r)$ is the Lorentz scalar potential (i.e., transforms like the rest mass energy $m \rightarrow m + S(r)$), and $eA_\varphi$ may include both magnetic and Aharonov-Bohm flux fields effects [31] 32. We may now use the assumption of (8) in (24) and obtain

$$\left\{ \partial_r^2 + \frac{1}{4r^2} + (E + V(r))^2 - \left[ \Omega r (E + V(r)) + \frac{(\ell - eA_\varphi)}{ar} \right] \right\} M(r) - k_z^2 - (m + S(r))^2 \right\} R(r) = 0, \hspace{1cm} (26)$$

where

$$M(r) = - \frac{3}{16} \left( \frac{f'(r)}{f(r)} \right)^2 + \frac{1}{4} \frac{f''(r)}{f(r)} + \frac{f'(r)}{4rf(r)}.$$  \hspace{1cm} (27)

This result would describe KG-particles in Gödel SR-type cosmic string spacetime background within effective PDM introduced as a topological defect in the coordinate system. In the following subsections, we discuss some illustrative examples of fundamental nature.

A. PDM-KG Coulomb particle in 4-vector and scalar Coulombic potentials and a magnetic field in Gödel SR-type cosmic string spacetime background

Let us consider a PDM-KG particle with an effective PDM function (a dimensionless scalar multiplier) given by

$$f(r) = \exp (4\beta r),$$  \hspace{1cm} (28)

so that equations (22) and (27) yield

$$F_r = \frac{f'(r)}{4f(r)} = \beta \iff M(r) = \beta^2 + \frac{\beta}{r}. \hspace{1cm} (29)$$

Moreover, for $V(r) = \beta_1/r$, $S(r) = \beta_2/r$, and $A_\varphi = B_\varphi r/2$ equation (20) would read

$$\left\{ \partial_r^2 - \frac{(\ell^2 - 1/4)}{r^2} - \tilde{\lambda}^2 r^2 - 2E \Omega \left( \beta_1 + \tilde{B} \right) r + \frac{\beta}{r} + \tilde{\lambda} \right\} R(r) = 0, \hspace{1cm} (30)$$

where

$$\tilde{\lambda} = E^2 - k_z^2 - m^2 - \tilde{B}^2 - \Omega^2 \beta_1^2 - \beta^2 - 2 \frac{\ell}{\alpha} \Omega E + 2 \Omega \tilde{B} \beta_1; \hspace{0.5cm} \frac{\ell}{\alpha} \tilde{B} = \frac{eB_\varphi}{2\alpha},$$  \hspace{1cm} (31)


\[
\beta = -\beta - 2m\beta_2 + 2E\beta_1 + 2\ell \tilde{B} - 2\ell \Omega \beta_1, \quad \mathcal{L}^2 = \ell^2 - \beta_1^2 + \beta_2^2, \quad \bar{\omega}^2 = \Omega^2 E^2.
\]  

(32)

The differential equation (30) at hand admits a solution in the form of biconfluent Heun functions so that

\[
\psi(r) = \frac{R(r)}{\sqrt{r}} = N \cdot r^{|\mathcal{L}|} e^{-(\frac{i\bar{\omega}}{2} + \Omega \beta_1 r + \bar{B} r)} \cdot H_B \left( \alpha', \beta', \gamma', \delta', \sqrt{\bar{\omega} r} \right),
\]  

where

\[
\alpha' = 2|\mathcal{L}|, \quad \beta' = \frac{2(\Omega \beta_1 + \bar{B})}{\sqrt{\bar{\omega}}}, \quad \gamma' = \frac{\lambda + (\Omega \beta_1 + \bar{B})^2}{\bar{\omega}}, \quad \delta' = -\frac{2\beta}{\bar{\omega}}.
\]  

(34)

This would result in a biconfluent Heun polynomial of degree \(n' = 2n_r \geq 0\) (where \(n_r\) denotes the radial quantum number) when \(\gamma' = 2(2n_r + 1) + \alpha'\) and consequently yields \(E = \Omega (2n_r + |\mathcal{L}| + \ell + 1) + \sqrt{\Omega^2 (2n_r + |\mathcal{L}| + \ell + 1)^2 + m^2 + k^2 + \beta^2 - 4\Omega \beta_1 \bar{B}}\).  

(35)

This condition would in turn classify the solution as a conditionally exact solution, therefore (c.f., e.g., [29, 31, 45] for more details). Nevertheless, the choice, \(n' = 2n_r \geq 0\), is manifested by the requirement that the eigenvalues as well as the eigenfunctions should naturally retrieve those of the KG-oscillator when \(\beta = \beta_1 = \beta_2 = \bar{B} = 0\) (e.g., [29]). However, one would notice that for the case where the vorticity \(\Omega = 0\), the differential equation in (30) effectively reduces to that of the radial Schrödinger Coulomb problem. However, neither the reported biconfluent Heun polynomial in (33) nor the eigenvalues in (35) collapse into the corresponding known Schrödinger-like KG-Coulombic solution. One has therefore to restart from the beginning and work out the related solution as in the sequel subsection.

In fact, Fernández [43] has provided through and comprehensive details on the condition exact and/or numerical exact solvability of the corresponding one-dimensional Schrödinger model

\[
\begin{align*}
\left\{ \partial_r^2 - \frac{(\mathcal{L}^2 - 1/4)}{r^2} - A r^2 - B r + \frac{C}{r} + E \right\} R(r) &= 0.
\end{align*}
\]  

(36)

Fernández work, along with the arguments reported in the model above, provide a brut-force evidence on not only the conditional exactness of the biconfluent Heun polynomials solution but also on the lack of the correct convergence of these polynomials into the pure Coulombic (when \(A = B = 0\) in (30)) spectral problem.

### B. A PDM-KG Coulombic particle in 4-vector and scalar Coulombic potentials and magnetic field in Gödel SR-type spacetime at zero vorticity

For such a model, we choose to work with \(\Omega = 0\) (no vorticity) so that equation (30) reads

\[
\left\{ \partial_r^2 - \frac{(\mathcal{L}^2 - 1/4)}{r^2} + \frac{i\beta}{r} + \lambda \right\} R(r) = 0.
\]  

(37)

Which resembles the two-dimensional (2D) Coulombic problem that admits exact solution in the form of Whittaker and confluent hypergeometric functions as

\[
R(r) \sim W_M \left( -\frac{i\beta}{2\sqrt{\lambda}}, |\mathcal{L}|, 2i\sqrt{\lambda} r \right) \sim r^{|\mathcal{L}|+1/2} \exp \left( i\sqrt{\lambda} r \right) \cdot 1_F \left( \left[ \frac{1}{2} + |\mathcal{L}| + \frac{i\beta}{2\sqrt{\lambda}} \right], \left[ 1 + 2|\mathcal{L}| \right], 2i\sqrt{\lambda} r \right). \tag{38}
\]

Consequently, a finite confluent hypergeometric polynomial of order \(n_r\) is obtained when

\[
\frac{1}{2} + |\mathcal{L}| + \frac{i\beta}{2\sqrt{\lambda}} = -n_r \iff \sqrt{\lambda} = -\frac{i\beta}{2n_r + 2|\mathcal{L}| + 1} \iff \lambda = -\frac{\beta^2}{(2n_r + 2|\mathcal{L}| + 1)^2}.
\]  

(39)
In fact, $\tilde{\lambda}$ in (39) represents the exact eigenvalues of the 2D-Schrödinger Coulomb problem and the exact eigenfunctions are given by

$$\psi (r) = \frac{R(r)}{\sqrt{r}} = N r^{|L|} \exp \left( i \sqrt{\tilde{\lambda}} r \right) |_{L} \left( -n_r, [1 + 2 |L|], \Lambda r \right); \Lambda = \frac{\tilde{\beta}}{2n_r + 2 |L| + 1}. \tag{40}$$

where $n_r = 0, 1, 2, \cdots$ is the radial quantum number.

Under such settings, equation (31) along with (39) would imply, with $n = n_r + |L| + 1/2$, that

$$E^2 - k_z^2 - m^2 - \tilde{B}^2 - \beta^2 = -\left( -\beta - 2m\beta_2 + 2E\beta_1 + 2\tilde{B} \right)^2 \left( 2n \right)^2, \tag{41}$$

where $\tilde{\ell} = \ell/\alpha, \tilde{B} = eB_o/2\alpha$, and $L^2 = \tilde{\ell}^2 - \beta_1^2 - \beta_2^2$. This result can be simplified into

$$E = -\beta_1 \Lambda_2 \pm \sqrt{4n^2 \Lambda_1 \left( \tilde{n}^2 + \beta_1^2 \right) - \tilde{n}^2 \Lambda_2^2} \tag{42}$$

where

$$\Lambda_1 = k_z^2 + m^2 + \tilde{B}^2 + \beta^2, \quad \Lambda_2 = -\beta - 2m\beta_2 + 2\tilde{B}. \tag{43}$$

Obviously, the term $2\tilde{\ell}\tilde{B}$ (i.e., the magnetic field effect) in $\Lambda_2$ removes the degeneracies associated with $|L|$ in the new irrational quantum number $\tilde{n} = n_r + |L| + 1/2$ of (42). Moreover, one should notice that the PDM settings of (28) and (20) have provided a Coulomb-like interaction as a byproduct of its own.

C. A PDM-KG Coulombic particle in equally mixed 4-vector and scalar Lorentz potentials and a magnetic field in Gödel SR-type spacetime

We now consider a PDM particle described by (28) in Gödel SR-type spacetime in a magnetic field along with equally mixed 4-vector and scalar Lorentz potentials and a magnetic field, where $V(r) = S(r) = \beta_o/r : \beta_1 = \beta_2 = \beta_o$. In this case, equation (39) reads

$$\left\{ \partial_r^2 - \left( \frac{\tilde{\ell}^2 - 1/4}{r^2} \right) - \omega^2 r^2 - 2E\Omega \left( \Omega \beta_o + \tilde{B} \right) r + \frac{\tilde{\beta}}{r} + \tilde{\lambda} \right\} R(r) = 0, \tag{44}$$

where

$$\tilde{\lambda} = E^2 - k_z^2 - m^2 - \tilde{B}^2 - \Omega^2 \beta^2 - \beta^2 - 2\tilde{\ell} \Omega E + 2\Omega \beta \beta_o; \quad \tilde{\ell} = \frac{\ell}{\alpha}, \tilde{B} = \frac{eB_o}{2\alpha}, \tag{45}$$

and

$$\tilde{\beta} = -\beta - 2m\beta_o + 2E\beta_o + 2\tilde{B} - 2\tilde{\ell} \Omega \beta_o, \quad \omega^2 = \Omega^2 E^2. \tag{46}$$

Which admits conditionally exact solution in the form of biconfluent Heun polynomials so that

$$\psi(r) = \frac{R(r)}{\sqrt{r}} = N r^{|L|} e^{-\left( \omega^2 + \Omega \beta_o r + \tilde{B} r \right)} H_B \left( \alpha', \beta', \gamma', \delta', \sqrt{\omega} r \right), \tag{47}$$

where

$$\alpha' = 2|\tilde{\ell}|, \quad \beta' = \frac{2}{\sqrt{\omega}} \left( \Omega \beta_o + \tilde{B} \right), \quad \gamma' = \frac{\tilde{\lambda} + \left( \Omega \beta_o + \tilde{B} \right)^2}{\omega}, \quad \delta' = \frac{2\tilde{\beta}}{\sqrt{\omega}}, \tag{48}$$

with the condition $\gamma' = 2(2n_r + 1) + \alpha'$ to consequently yield

$$E = \Omega \left( 2n_r + |\tilde{\ell}| + \tilde{\ell} + 1 \right) \pm \sqrt{\Omega^2 \left( 2n_r + |\tilde{\ell}| + \tilde{\ell} + 1 \right)^2 + m^2 + k_z^2 + \beta^2 - 4\Omega \beta_o \tilde{B}}. \tag{49}$$

In each of the three PDM-KG Coulombic particles above, we notice that the first term of the energy eigenvalues (i.e., equations (35), (42), and (49)) lifts the degeneracies associated with the magnetic quantum number $\ell = \pm |\tilde{\ell}|$. 

D. A quasi-free PDM KG-oscillator in Gödel SR-type cosmic string spacetime background

We now consider a free PDM KG-particle with not only position-dependent but also energy-dependent scalar multiplier deformation function in the form of

$$f(r) = \exp\left(2\xi Er^2\right),$$

which implies that

$$M(r) = E^2\xi^2 r^2 + 2E\xi.$$  \hfill(51)

Let us assume that this PDM KG-particle is free from the Lorentz potentials (i.e., $V(r) = S(r) = 0$ and no magnetic field effect $eA_\phi = 0$, hence quasi-free). In this case, equation (26) reads

$$\begin{cases}
\partial_r^2 r + \left(\tilde{\ell}^2 - \frac{1}{4}\right) \frac{1}{r^2} - \tilde{\Omega}^2 r^2 + \tilde{\epsilon} \right) R(r) = 0, \\
\end{cases}$$

which resembles the 2D radial Schrödinger harmonic oscillator problem with

$$\tilde{\epsilon} = E^2 - 2\tilde{\Omega} E - 2E\xi - (m^2 + k^2_z), \quad \tilde{\Omega}^2 = E^2 (\Omega^2 + \xi^2).$$  \hfill(53)

Under such settings, the textbook solution is therefore given by

$$\tilde{\epsilon} = 2\tilde{\Omega} \left(2n_r + |\tilde{\ell}| + 1\right)$$

and

$$R(r) \sim r^{|\tilde{\ell}| + 1/2} \exp\left(-\frac{\tilde{\Omega}^2 r^2}{2}\right) L_{n_r}^{\tilde{\ell}} \left(\tilde{\Omega} r^2\right),$$

where $L_{n_r}^{\tilde{\ell}} \left(\tilde{\Omega} r^2\right)$ are the associated Laguerre polynomials. One should notice that the choice of the deformation in (50) is just to facilitate the calculation to be able to get

$$E = \tilde{\gamma} \pm \sqrt{\gamma^2 + m^2 + k^2_z}; \quad \tilde{\gamma} = \tilde{\ell}\Omega + \xi + (\Omega^2 + \xi^2) \left(2n_r + |\tilde{\ell}| + 1\right).$$  \hfill(56)

This particular example shows that PDM-KG particles (PDM-particles in general) may very well create their own byproducted effective interactions that may lead to bound states. Hence the notion of quasi-free particles is unavoidable in the process. Moreover, we observe that quasi-Landau type energy levels manifestly introduced by the vorticity parameter $\Omega$ in $\tilde{\gamma}$. Hence, the vorticity parameter have similar effect as that of the magnetic field..

V. CONCLUDING REMARKS

The harmonic oscillator and Coulomb problems form the most fundamental systems in non-relativistic and relativistic quantum mechanics in different spacetime backgrounds. Their exact solvability and impressive/superb pedagogical/research implementation makes them systems of great physical relevance. In the current methodical proposal, we considered KG-particles (harmonic oscillator and Coulombic types) in Gödel SR-type cosmic string spacetime backgrounds and under the influence of different 4-vector and/or scalar Lorentz potentials settings [49]. We started with the KG-oscillators in Gödel SR-type cosmic string spacetime backgrounds and reported their exact solutions. Hereby, we have emphasized that the mathematical collapse of the KG-equation into the two dimensional radially symmetric Schrödinger-oscillator (hence the metaphoric notion of KG-oscillator emerged in the process) does not mean that the parametric characterizations are copied from one to the other. The parameter $\tilde{\omega}$ of [8] is not related to the angular frequencies of the harmonic oscillator (e.g., $46, 48, 51$) but rather admits values given by $\tilde{\omega} = \pm |\Omega E|$ (e.g., [45]). The usage of $\tilde{\omega}$ as the angular frequency of the harmonic oscillator had the consequences of losing at least half of the spectra (only positive energies were reported and anti-particle solutions were dismissed from the relativistic theory). The recycling of the KG-oscillators in Gödel SR-type cosmic string spacetime backgrounds in section 2 is unavoidable, therefore. We have, in the same section, pinpointed that the results reported by Carvalho et al [10] on the
quantum influence of topological defects in Gödel spacetime, by Vitória et al [46] on the linear confinement of a scalar particle in Gödel-type spacetime (including the related comment by Neto et al [47]), and those reported by Ahmed [48] on KG-Coulombic type particles should be redirected to reflect the results of the current proposal, therefore. Not only they have lost all negative energies but also they have lost all nodeless states form the spectra (excluding Carvalho et al [10], where nodeless states, with $n_r = 0$, are reported therein). Here, we have just mentioned few of so many references that are not cited herein but lie far beyond the scope of the current study. Subsequently, we have reported (in section 3) the KG-oscillators in pseudo-Gödel SR-type cosmic string spacetime that admit invariance and isospectrality with the KG-oscillator.

On the PDM settings side of the current methodical proposal, we have introduced a general recipe for PDM-KG particles in a 4-vector and scalar Lorentz potentials and a magnetic field in Gödel SR-type cosmic string spacetime background (in section 4). Therein, we have considered four illustrative examples of fundamental nature: (i) a PDM-KG Coulomb particle in 4-vector and scalar Coulombic potentials and a magnetic field in Gödel SR-type cosmic string spacetime background, (ii) a PDM-KG Coulombic particle in 4-vector and scalar Coulombic potentials and magnetic field in Gödel SR-type spacetime at zero vorticity, (iii) a PDM-KG Coulombic particle in equally mixed 4-vector and scalar Lorentz potentials and a magnetic field in Gödel SR-type spacetime, and (iv) a quasi-free PDM KG-oscillator in Gödel SR-type cosmic string spacetime background.

In the light of our experience on the PDM KG-particles (this would also include constant mass settings with $f(r) = 1$, in general) in Gödel SR-type cosmic string spacetime background, the following notes are inevitably unavoidable:

1. For the PDM-KG Coulomb particle in 4-vector and scalar Coulombic potentials and a magnetic field, the corresponding one-dimensional Schrödinger equation (30), indulges within a harmonic oscillator-like in a Cornel type potentials [49], that admits a biconfluent Heun polynomial type solution (33) which consequently yields some conditionally exact energies reported in (35). However, it is obvious that neither the biconfluent Heun polynomials in (33) nor the eigenvalues in (35) collapse into the corresponding known Schrödinger-like KG-Coulombic solution, when the linear plus oscillator like potential terms are switched off.

2. We have deliberately considered a PDM-KG Coulombic particle in Gödel SR-type spacetime at zero vorticity to provide a brut-force evidence on the drawback of the use of the biconfluent Heun polynomials approach. We have observed that the more general and conditionally exact solution for system (39) does not collapse into that for the pure Coulombic one of (37).

3. The equally mixed 4-vector and scalar Lorentz potentials is a common practice in quarkonium spectroscopy [49, 56]. We have therefore introduced such feasible mixture/model of interaction potentials. Again, this system suffers from the drawback of the conditional exact solvability of the biconfluent Heun polynomial approach, as in point 1 above.

4. PDM-KG particles (PDM-particles in general) may generate their own byproducted effective interactions that support bound states. They should, therefore, be labelled as quasi-free PDM-particles. Yet, quasi-Landau type energy levels could be introduced by the vorticity $\Omega$ of spacetime that has similar effect of the magnetic field.

The drawback in the biconfluent Heun polynomial approach (a power series manifested solution) inspires the search for a more sophisticated/reliable approach. We foresee that an alternative could be sought in the well known shifted $1/N$ or shifted-$\ell$ expansion quasi-perturbation method (e.g., [51–58]). Such quasi-perturbation methods result the exact energies for the harmonic oscillator like and Coulomb like interactions (especial cases of 33) from the zeroth (leading) order correction, where all higher order correction identically vanish. This method is also reported to provide highly accurate results for the Coulomb plus oscillator [53, 54], a close form of potential to the one used in section 4-A and 4-C. Work in this direction is in progress.

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