A kinetic theory for electromagnetic ion waves in a cold relativistic plasma is derived. The kinetic equation for the broadband electromagnetic ion waves is coupled to the slow density response via an acoustic equation driven by ponderomotive force like term linear in the electromagnetic field amplitude. The modulational instability growth rate is derived for an arbitrary spectrum of waves. The monochromatic and random phase cases are studied.

The interaction of intense radiation with plasmas is of fundamental importance in a wide variety of applications, such as inertial confinement fusion and pulsar emissions, and such interactions can lead to a range of instabilities, e.g. Brillouin and Raman scattering as well as modulational instabilities. In some of the instabilities, relativistic effects play an important role. In Refs. it has been shown that a novel type of ion electromagnetic wave may propagate through a cold magnetized relativistic plasma. Such waves may be of relevance in high-energy astrophysical environments, e.g. pulsars, as well as in intense laser interactions with high density targets. However, the coherence length of intense electromagnetic (EM) waves could often be short. Moreover, it is well-known that effects of partial coherence may be used to stabilize the propagation of EM pulses in
nonlinear dispersive media \([12, 13]\). Thus, it is of interest to analyze the effects of partial coherence of EM ion wave propagation.

In this paper, the effects of partial coherence ion EM waves in a magnetized relativistic plasma is analyzed. In particular, a kinetic equation describing the evolution of the EM quasi-particles is derived and the general nonlinear dispersion relation is obtained. We study the effects of partial coherence using a random phase background of EM ion waves, leading to a Lorentz distribution for the quasi-particles. It is shown that the spatial spectral broadening induced by the random phase gives rise to a reduction of the modulational instability growth rate, in addition to the appearance of a new short wavelength instability region.

Let us consider large amplitude circularly polarized electromagnetic ion waves, with the electric field amplitude \(E_0\), propagating parallel to a constant magnetic field \(\vec{B}_0 = B_0 \hat{z}\) in a cold electron–ion plasma. Stenflo and Tsintsadze \([11]\) have shown that there are situations in which the ions are strongly relativistic. For such electromagnetic waves, the linear dispersion relation is of the form

\[
1 \approx \frac{k^2 c^2}{\omega^2} \pm \frac{\omega_{\text{pi}}^2}{\omega \omega_{\text{Ei}}},
\]

where \(k\) is the wavenumber, \(\omega\) is the wave frequency, \(\omega_{\text{pi}} = \left(\frac{e^2 n_0}{\epsilon_0 m_i}\right)^{1/2}\) is the ion plasma frequency, \(\omega_{\text{Ei}} = eE_0/cm_i\), \(e\) is the magnitude of the electron charge, \(n_0\) is the unperturbed ion density, \(\epsilon_0\) is the vacuum dielectric constant, \(m_i\) is the ion rest mass, and \(c\) is the speed of light in vacuum. The \(+\) \((-\) in (1) correspond to \(|\vec{p}_i|/m_i c > \omega_{\text{ci}}/\omega\) \(|\vec{p}_i|/m_i c < \omega_{\text{ci}}/\omega\), where \(\vec{p}_i\) is the ion particle momentum and \(\omega_{\text{ci}} = eB_0/m_i\) is the ion cyclotron frequency. If the plus sign is used in the dispersion relation (1), we obtain the so called second ion-cyclotron wave, whereas the minus sign gives the ion helicon wave.

The nonlinear evolution of electromagnetic ion waves, with amplitude \(E\), in a cold quasi-neutral relativistic plasma is given by \([11]\)

\[
i \left( \frac{\partial}{\partial t} + v_g \frac{\partial}{\partial z} \right) E + v_g' \frac{\partial^2 E}{\partial z^2} \pm \omega \left( \frac{|E| - E_0}{E_0} - \frac{\delta n}{n_0} \right) E = 0,
\]

where the \(+\) \((-\) refers to the \(+\) \((-\) in the linear dispersion relation (1), and the slow density
response is obtained from
\[
\left( \frac{\partial^2}{\partial t^2} - v_s^2 \frac{\partial^2}{\partial z^2} \right) \frac{\delta n}{n_0} = \pm \beta \frac{\partial^2}{\partial z^2} \left( \frac{|E| - E_0}{E_0} \right).
\]
(3)

Here \( v_g = \partial \omega / \partial k \) is the group velocity, \( v'_g = \partial^2 \omega / \partial k^2 \) is the group velocity dispersion, \( \delta n \) is the ion density perturbation, \( n_0 \) is the constant background density, \( v_s = (T_e/m_i)^{1/2} \) is the ion sound speed, and \( \beta = (c^2/2)(E_0/cB_0) \). Thus, the above system of equations is effectively a quadratic nonlinear Schrödinger equation, in contrast to the regular cubic nonlinear Schrödinger equation.

In the quasi-stationary limit, we may integrate Eq. (3) to obtain the quadratic nonlinear Schrödinger equation
\[
i \frac{\partial E}{\partial \tau} + \frac{v'_g}{2} \frac{\partial^2 E}{\partial \zeta^2} \pm \omega \left( 1 \pm \frac{\beta}{v_s} \right) \frac{|E|}{E_0} E = 0,
\]
(4)
where we have transformed to a co-moving frame \( \tau = t \) and \( \zeta = z - v_g t \).

With the ansatz \( E = (E_0 + \delta E) \exp(i\phi) \), where \( \delta E \ll E_0 \) and \( \phi \ll 1 \), we linearize Eqs. (2) and (3). For a harmonic dependence, i.e. \( \delta E, \phi \propto \exp(iKz - i\Omega t) \), we obtain the dispersion relation
\[
(\Omega^2 - v_s^2 K^2) \left[ (\Omega - v_g K)^2 \pm \frac{1}{2} \omega v'_g K^2 - \frac{1}{4} v'_g^2 K^4 \right] = \frac{1}{2} \omega \beta v'_g K^4,
\]
(5)
consistent with the results in Refs. [11] and [14]. In the quasi-stationary limit (cf. Eq. (4)), we can solve for \( \Omega \) to obtain the growth rate \( \Gamma = \text{Im}(\Omega) \) according to
\[
\Gamma = K \left[ \frac{1}{2} \omega v'_g \left( \frac{\beta}{v_s^2} \pm 1 \right) - \frac{1}{4} v'_g^2 K^2 \right]^{1/2}.
\]
(6)

For positive group velocity dispersion, we see that the second order dispersive term competes with the nonlinear term, giving the characteristic modulational instability growth rate curve (cf. Figs. 1 and 2).

We are now interested in analyzing the effects of partial coherence of the electromagnetic ion waves. For this purpose, we introduce the Wigner function \[\rho(t, z, p) = \frac{1}{2\pi} \int d\zeta e^{ip\zeta} E^*(t, z + \zeta/2) E(t, z - \zeta/2),\]
(7)
such that

\[ |E| = \left( \int dp \rho(t, z, p) \right)^{1/2}. \]  

(8)

We note that the Wigner method [19], as well as the equivalent mutual coherence method [20], have been used to analyze the modulational instability of the cubic nonlinear Schrödinger equation relevant for e.g. nonlinear optics. Moreover, in plasma applications the Zakharov equations have been analyzed using the above method [21] in order to obtain the statistical dynamics and Landau like damping of Langmuir waves.

Applying the time derivative to the definition (7) and using (2), we obtain the kinetic equation

\[
\frac{\partial \rho}{\partial t} + \left( v_g + v'_g p \right) \frac{\partial \rho}{\partial z} \pm 2\omega \left( \frac{|E|}{E_0} - \frac{\delta n}{n_0} \right) \sin \left( \frac{1}{2} \frac{\partial}{\partial z} \frac{\partial}{\partial p} \right) \rho = 0,
\]  

(9)

where the sin-operator is determined in terms of its Taylor expansion and the arrows denotes direction of operation. The kinetic equation (9), together with (3) and (8), determines the evolution of broad band electromagnetic ion waves in cold relativistic plasmas.

In order to analyze the modulational instability of the system (3), (8), and (9), we let

\[ \rho(t, z, p) = \rho_0(p) + \delta \rho \exp(iKz - i\Omega t) \text{ and } \delta n \propto \exp(iKz - i\Omega t), \]

where \( \delta \rho \ll \rho_0 \). Linearizing Eqs. (3), (8), and (9), we then obtain the nonlinear dispersion relation

\[
E_0^2 = \pm \omega \left( 1 \mp \frac{K^2}{\Omega^2 - v_s^2 K^2} \right) \int dp \frac{\rho_0(p + K/2) - \rho_0(p - K/2)}{\Omega - (v_g + v'_g p) K},
\]  

(10)

where \( E_0 = (\int dp \rho_0)^{1/2} \). This dispersion relation generalizes (5) to the case of arbitrary spatial spectral background distributions \( \rho_0 \).

In the case of a monochromatic spectral distribution, \( \rho_0(p) = E_0^2 \delta(p) \), we retrieve the dispersion relation (5), as expected. However, if the background distribution \( \rho_0 \) has a finite spectral width, the dispersion relation is altered. Next, we look at the case of a random phased background electromagnetic ion wave. This will give rise to the spectral distribution in the form of the Lorentzian

\[
\rho_0(p) = \frac{E_0^2}{\pi} \frac{\Delta}{p^2 + \Delta^2},
\]  

(11)
where \( \Delta \) is the spectral width of the distribution. With this, the nonlinear dispersion relation (10) becomes

\[
(\Omega^2 - v_s^2 K^2) \left[ (\Omega - (v_g - i \nu'_g K) K)^2 \pm \frac{1}{2} \omega \nu'_g K^2 - \frac{1}{4} \nu'_g^2 K^4 \right] = \frac{1}{2} \omega \beta \nu'_g K^4. \tag{12}
\]

As \( \Delta \to 0 \), we obtain from above the monochromatic dispersion relation (5). Moreover, the effect of the spectral broadening is to introduce a damping of the perturbation modes.

Next, we analyze the modulational instability properties of the dispersion relation (12). We introduce the dimensionless variables \( \bar{\Omega} = \Omega / \omega \), \( \bar{K} = v_g K / \omega \), \( \bar{\Delta} = v_g \Delta / \omega \), \( \bar{v} = v_s / v_g \), \( \bar{v}' = \omega v'_g / v_g^2 \), and \( \bar{\beta} = \beta / v_g^2 \). In Figs. 1 (+ in Eq. (2)) and 2 (− in Eq. (2)) we have displayed the growth rate \( \Gamma = \text{Im}(\bar{\Omega}) \) for some typical parameter values. Using \( \bar{v} = 0.33 \), \( \bar{\beta} = 1 \), and \( \bar{v}' = 0.5 \), it can be seen that a finite spectral width \( \bar{\Delta} \) gives rise to a reduced growth rate. On the other hand, the \( \bar{K} \)-region, where the instability occurs, is enlarged. We also note that the + mode has a larger growth rate as compared to the − mode in Eq. (9).

To summarize, we have analyzed the effects of a partial coherence of circularly polarized electromagnetic ion waves in relativistic plasmas. In particular, the modulational instability growth rate was found for both coherent waves and for waves with a random phase. It was shown that the effect of partial coherence is to stabilize the propagation of the aforementioned waves, while broadening the possible instability wavenumber region. The latter could
FIG. 2: The growth rate $\Gamma = \text{Im}(\bar{\Omega})$ plotted as a function of the normalized wavenumber $\bar{K}$ for the parameter values $\bar{v} = 0.33$, $\bar{\beta} = 1$, and $\bar{v}' = 0.5$, with $-\Delta$ in Eq. (2). In (a) we have $\bar{\Delta} = 0$, while (b) has $\bar{\Delta} = 0.2$. The reduction in growth rate can be seen from (a) to (b).

lead to the formation of short wavelength nonlinear structures due to partial coherence.

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[1] G. A. Mourou, T. Tajima, and S. V. Bulanov, Rev. Mod. Phys. 78, 309 (2006).
[2] M. Marklund and P. K. Shukla, Rev. Mod. Phys. 78, 591 (2006).
[3] V. S. Beskin, A. V. Gurevich, and Ya. N. Istomin, Physics of the Pulsar Magnetosphere (Cambridge, 1993).
[4] M. Y. Yu, K. H. Spatschek, and P. K. Shukla, Zh. Naturforsch. A 29, 1736 (1974).
[5] P. K. Shukla, M. Y. Yu and K.H. Spatschek, Phys. Fluids 18, 265 (1975).
[6] R. P. Sharma and P. K. Shukla, Phys. Fluids 26, 87 (1983); P. K. Shukla and L. Stenflo, ibid. 28, 1576 (1985); G. Murtaza and P. K. Shukla, J. Plasma Phys. 31, 423 (1984); P. K. Shukla and L. Stenflo, Phys. Rev. A 30, 2110 (1984).
[7] P. K. Shukla, N. N. Rao, M. Y. Yu, and N. L. Tsintsadze, Phys. Rep. 135, 1 (1986).
[8] N. L. Tsintsadze and L. Stenflo, Phys. Lett. A 48, 399 (1974).
[9] C. E. Max, J. Arons, and A. B. Langdon, Phys. Rev. Lett. 33, 209 (1974).
[10] N. Y. Kotsarenko, Soviet J. Plasma Phys. 3, 197 (1977).
[11] L. Stenflo and N. L. Tsintsadze, Astrophys. Space Sci. 64, 513 (1979).
[12] Y. Kato, K. Mima, N. Miyanaga, S. Arinaga, Y. Kitagawa, M. Nakatsuka, and C. Yamanaka, Phys. Rev. Lett. 53, 1057 (1984).
[13] M. Koenig, B. Faral, J. M. Boudenne, D. Batani, A. Benuzzi, and S. Bossi, Phys. Rev. E 50, R3314 (1994).
[14] N. L. Tsintsadze, N. A. Papuashvili, E. C. Tsikarishvili, and L. Stenflo, Physica Scripta 21, 183 (1980).
[15] E. P. Wigner, Phys. Rev. 40, 749 (1932);
[16] J. E. Moyal, Proc. Cambridge Philos. Soc. 45, 99 (1949).
[17] Yu. L. Klimontovich, The Statistical Theory of Non-Equilibrium Processes in a Plasma (Pergamon Press, Oxford, 1967).
[18] J. T. Mendonça, Theory of Photon Acceleration (IOP Publishing, Bristol, 2001).
[19] D. Anderson, B. Hall, M. Lisak and M. Marklund, Phys. Rev. E 65, 046417 (2002); B. Hall, M. Lisak, D. Anderson, R. Fedele, and V. E. Semenov, ibid. 65, 035602 (2002).
[20] M. Soljacic, M. Segev, T. Coskun, D. N. Christodoulides, and A. Vishwanath, Phys. Rev. Lett. 84, 467 (2000).
[21] R. Fedele, P.K. Shukla, M. Onorato, D. Anderson, and M. Lisak, Phys. Lett. A 303, 61 (2002).