

$t$-perfection in $P_5$-free graphs

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Abstract

A graph is called $t$-perfect if its stable set polytope is fully described by non-negativity, edge and odd-cycle constraints. We characterise $P_5$-free $t$-perfect graphs in terms of forbidden $t$-minors. Moreover, we show that $P_5$-free $t$-perfect graphs can always be coloured with three colours, and that they can be recognised in polynomial time.

1 Introduction

There are three quite different views on perfect graphs, a view in terms of colouring, a polyhedral and a structural view. Perfect graphs can be seen as:

- the graphs for which the chromatic number $\chi(H)$ always equals the clique number $\omega(H)$, and that in any induced subgraph $H$;
- the graphs for which the stable set polytope, the convex hull of stable sets, is fully described by non-negativity and clique constraints; and
- the graphs that do not contain any odd hole (an induced cycle of odd length at least 5) or their complements, odd antiholes.

(The polyhedral characterisation is due to Fulkerson [15] and Chvátal [9], while the third item, the strong perfect graph theorem, was proved by Chudnovsky, Robertson, Seymour and Thomas [6].)

In this article, we work towards a similar threefold view on $t$-perfect graphs. These are graphs that, similar to perfect graphs, have a particularly simple stable set polytope. For a graph to be $t$-perfect its stable set polytope needs to be given by non-negativity, edge and odd-cycle constraints; for precise definitions we defer to the next section. The concept of $t$-perfection, due to Chvátal [9], thus takes its motivation from the polyhedral aspect of perfect graphs. The corresponding colouring and structural view, however, is still missing. For some graph classes, though, claw-free graphs for instance [5], the list of minimal obstructions for $t$-perfection is known. We extend this list to $P_5$-free graphs. (A graph is $P_5$-free if it does not contain the path on five vertices as an induced subgraph.)

Perfection is preserved under vertex deletion, and the same is true for $t$-perfection. There is a second simple operation that maintains $t$-perfection: a $t$-contraction, which is only allowed at a vertex with stable neighbourhood, contracts all the incident edges. Any graph obtained by a sequence of vertex deletions and $t$-contractions is a $t$-minor. The concept of $t$-minors makes it more convenient to characterise $t$-perfection in certain graph classes as it allows for more succinct lists of obstructions.
For that characterisation denote by $C^k_n$ the $k$th power of the $n$-cycle $C_n$, that is, the graph obtained from $C_n$ by adding an edge between any two vertices of distance at most $k$ in $C_n$. We, moreover, write $\overline{G}$ for the complement of a graph $G$, and $K_n$ for the complete graph on $n$ vertices and $W_n$ for the wheel on $n + 1$ vertices.

**Theorem 1.** Let $G$ be a $P_5$-free graph. Then $G$ is $t$-perfect if and only if it does not contain any of $K_4$, $W_5$, $C^7_7$, $C^2_{10}$ or $C^{13}_3$ as a $t$-minor.

This answers a question of Benchetrit [2, p. 76].

![Forbidden $t$-minors in $P_5$-free graphs](image)

The forbidden graphs of the theorem are minimally $t$-imperfect, in the sense that they are $t$-imperfect but any of their proper $t$-minors are $t$-perfect. Odd wheels, even Möbius ladders (see Section 3), the cycle power $C^7_7$ and the graph $C^2_{10}$ are known to be minimally $t$-imperfect. The graph $C^{13}_3$ appears here for the first time as a minimally $t$-imperfect graph. We prove this in Section 4 where we also present two more minimally $t$-imperfect graphs.

A starting point for Theorem 1 was the observation of Benchetrit [2, p. 75] that $t$-minors of $P_5$-free graphs are again $P_5$-free. Thus, any occurring minimally $t$-imperfect graph will be $P_5$-free, too. This helped to whittle down the list of prospective forbidden $t$-minors. We prove Theorem 1 in Sections 5 and 6.

A graph class in which $t$-perfection is quite well understood is the class of near-bipartite graphs; these are the graphs that become bipartite whenever the neighbourhood of any vertex is deleted. In the course of the proof of Theorem 1 we make use of results of Shepherd [26] and of Holm, Torres and Wagler [20]; together they yield a description of $t$-perfect near-bipartite graphs in terms of forbidden induced subgraphs. We discuss this in Section 3.

As a by-product of the proof of Theorem 1 we also obtain a polynomial-time algorithm to check for $t$-perfection in $P_5$-free graphs (Theorem 20).

Finally, in Section 7 we turn to the third defining aspect of perfect graphs: colouring. Shepherd and Sebő conjectured that every $t$-perfect graph can be coloured with four colours, which would be tight. For $t$-perfect $P_5$-graphs we show (Theorem 23) that already three colours suffice. We, furthermore, offer a conjecture that would, if true, characterise $t$-perfect graphs in terms of (fractional) colouring, in a way that is quite similar as for perfect graphs.
We end the introduction with a brief discussion of the literature on \( t \)-perfect graphs. A general treatment may be found in Grötschel, Lovász and Schrijver [19, Ch. 9.1] as well as in Schrijver [25, Ch. 68]. The most comprehensive source of literature references is surely the PhD thesis of Bencherit [2]. A part of the literature is devoted to proving \( t \)-perfection for certain graph classes. For instance, Boulala and Uhry [3] established the \( t \)-perfection of series-parallel graphs. Gerards [16] extended this to graphs that do not contain an odd-\( K_4 \) as a subgraph (an odd-\( K_4 \) is a subdivision of \( K_4 \) in which every triangle becomes an odd circuit). Gerards and Shepherd [17] characterised the graphs with all subgraphs \( t \)-perfect, while Barahona and Mahjoub [1] described the \( t \)-imperfect subdivisions of \( K_4 \). Wagler [29] gave a complete description of the stable set polytope of antiwebs, the complements of cycle powers. These are near-bipartite graphs that also play a prominent role in the proof of Theorem 1. See also Wagler [30] for an extension to a more general class of near-bipartite graphs. The complements of near-bipartite graphs are the quasi-line graphs. Chudnovsky and Seymour [8], and Eisenbrand, Oriolo, Stauffer and Ventura [12] determined the precise structure of the stable set polytope of quasi-line graphs. Previously, this was a conjecture of Ben Rebea [24].

Algorithmic aspects of \( t \)-perfection were also studied: Grötschel, Lovász and Schrijver [18] showed that the max-weight stable set problem can be solved in polynomial-time in \( t \)-perfect graphs. Eisenbrand et al. [11] found a combinatorial algorithm for the unweighted case.

\section{Definitions}

All the graphs in this article are finite, simple and do not have parallel edges or loops. In general, we follow the notation of Diestel [10], where also any missing elementary facts about graphs may be found.

Let \( G = (V, E) \) be a graph. The stable set polytope \( \text{SSP}(G) \subseteq \mathbb{R}^V \) of \( G \) is defined as the convex hull of the characteristic vectors of stable, i.e. independent, subsets of \( V \). The characteristic vector of a subset \( S \) of the set \( V \) is the vector \( \chi_S \in \{0, 1\}^V \) with \( \chi_S(v) = 1 \) if \( v \in S \) and 0 otherwise. We define a second polytope \( \text{TSTAB}(G) \subseteq \mathbb{R}^V \) for \( G \), given by

\[
\begin{align*}
&x \geq 0, \\
x_u + x_v &\leq 1 \text{ for every edge } uv \in E, \\
\sum_{v \in V(C)} x_v &\leq \left\lfloor \frac{|C|}{2} \right\rfloor \text{ for every induced odd cycle } C \text{ in } G.
\end{align*}
\]

These inequalities are respectively known as non-negativity, edge and odd-cycle inequalities. Clearly, \( \text{SSP}(G) \subseteq \text{TSTAB}(G) \).

Then, the graph \( G \) is called \emph{\( t \)-perfect} if \( \text{SSP}(G) \) and \( \text{TSTAB}(G) \) coincide. Equivalently, \( G \) is \( t \)-perfect if and only if \( \text{TSTAB}(G) \) is an integral polytope, i.e. if all its vertices are integral vectors. It is easy to see that bipartite graphs are \( t \)-perfect. The smallest \( t \)-imperfect graph is \( K_4 \). Indeed, the vector \( \frac{1}{2}(1, 1, 1, 1) \) lies in \( \text{TSTAB}(K_4) \) but not in \( \text{SSP}(K_4) \).

It is easy to verify that vertex deletion preserves \( t \)-perfection. Another operation that keeps \( t \)-perfection was found by Gerards and Shepherd [17]: whenever there is a vertex \( v \), so that its neighbourhood is stable, we may contract all
edges incident with $v$ simultaneously. We will call this operation a $t$-contraction at $v$. Any graph that is obtained from $G$ by a sequence of vertex deletions and $t$-contractions is a $t$-minor of $G$. Let us point out that any $t$-minor of a $t$-perfect graph is again $t$-perfect.

3 $t$-perfection in near-bipartite graphs

Part of the proof of Theorem 1 consists in a reduction to near-bipartite graphs. A graph is near-bipartite if it becomes bipartite whenever the neighbourhood of any of its vertices is deleted. We will need a characterisation of $t$-perfect near-bipartite graphs in terms of forbidden induced subgraphs. Fortunately, such a characterisation follows immediately from results of Shepherd [26] and of Holm, Torres and Wagler [20].

We need a bit of notation. Examples of near-bipartite graphs are antwebs: an antiweb $C_n^k$ is the complement of the $k$th power of the $n$-cycle $C_n$. The antiweb is prime if $n \geq 2k + 2$ and $k + 1$, $n$ are relatively prime. We simplify the notation for antwebs $C_n^k$ slightly by writing $A_n^k$ instead. Even Möbius ladders, the graphs $A_{2t+4}$, are prime antwebs; see Figure 2 for the Möbius ladder $C_8^3$. We view $K_4$ alternatively as the smallest odd wheel $W_3$ or as the smallest even Möbius ladder $C_4^1$. Trotter [27] found that prime antwebs give rise to facets in the stable set polytope—we only need that prime antwebs other than odd cycles are $t$-imperfect, a fact that is easier to check.

Figure 2: Two views of the Möbius ladder on 8 vertices

Shepherd proved:

**Theorem 2** (Shepherd [26]). Let $G$ be a near-bipartite graph. Then $G$ is $t$-perfect if and only if

(i) $G$ contains no induced odd wheel; and
(ii) $G$ contains no induced prime antiweb other than possibly an odd hole.

Holm, Torres and Wagler [20] gave a neat characterisation of $t$-perfect antwebs. For us, however, a direct implication of the proof of that characterisation is more interesting: an antiweb is $t$-perfect if and only if it does not contain any even Möbius ladder, or any of $A_1^1$, $A_2^2$, $A_3^3$, $A_4^4$, $A_5^5$ and $A_{10}^1$ as an induced subgraph. We may omit $A_{12}^4$ from that list as it contains an induced $A_7^1$. Combining the theorem of Holm et al. with Theorem 2 one obtains:
Proposition 3. A near-bipartite graph is \( t \)-perfect if and only if it does not contain any odd wheel, any even Möbius ladder, or any of \( A_7^1 \), \( A_{10}^2 \), \( A_{13}^3 \), \( A_{13}^4 \) and \( A_{19}^7 \) as an induced subgraph.

4 Minimally \( t \)-imperfect antiwebs

For any characterisation of \( t \)-perfection in minimally \( t \)-imperfect, that is, all graphs that are \( t \)-imperfect but whose proper \( t \)-minors are \( t \)-perfect. Even Möbius ladders and odd wheels, for instance, are known to be minimally \( t \)-imperfect. This follows from the result of Fonlupt and Uhry [14] that almost bipartite graphs are \( t \)-perfect; a graph is almost bipartite if it contains a vertex whose deletion renders it bipartite. It is easy to check that any proper \( t \)-minor of an even Möbius ladder or an odd wheel is almost bipartite.

All the other forbidden \( t \)-minors in Theorem 1 or Proposition 3 are minimally \( t \)-imperfect, too. That \( C_{10}^2 \) is minimally \( t \)-imperfect is proved in [5]. There, also minimality for \( C_{10}^2 \) is shown, which allows us to verify that \( A_{10}^4 \) is minimally \( t \)-imperfect as well. Indeed, for this we first observe that \( A_{10}^4 \) can be obtained from \( C_{10}^2 \) by adding diagonals of the underlying 10-cycle. The second necessary observation is that any two vertices directly opposite in the 10-cycle form a so-called odd pair: any induced path between them has odd length. Minimality now follows from the result of Fonlupt and Hadjar [13] that adding an edge between the vertices of an odd pair preserves \( t \)-perfection.

In this section, we prove that \( A_{13}^3 \), \( A_{13}^4 \) and \( A_{19}^7 \) are minimally \( t \)-imperfect, which was not observed before. As prime antiwebs these are \( t \)-imperfect. This follows from Theorem 2 but can also be seen directly by observing that the vector \( x \equiv \frac{1}{3} \) lies in TSTAB but not in SSP for any of the three graphs.

To show that the graphs are minimally \( t \)-imperfect, it suffices to consider the \( t \)-minors obtained from a single vertex deletion or from a single \( t \)-contraction. If these are \( t \)-perfect then the antiweb is minimally \( t \)-imperfect.

Trotter gave necessary and sufficient conditions when an antiweb contains another antiweb:

Theorem 4 (Trotter [27]). \( A_{n'}^{k'} \) is an induced subgraph of \( A_n^k \) if and only if
\[
n(k' + 1) \geq n'(k + 1) \quad \text{and} \quad nk' \leq n'k.
\]

We fix the vertex set of any antiweb \( A_n^k \) to be \{0, 1, \ldots, n - 1\}, so that \( ij \) is an edge of \( A_n^k \) if and only if \( |i - j| \mod n > k \).

Proposition 5. The antiweb \( A_{13}^3 \) is minimally \( t \)-imperfect.

Proof. For \( A_{13}^3 \) to be minimally \( t \)-imperfect, every proper \( t \)-minor \( A_{13}^3 \) needs to be \( t \)-perfect. As no vertex of \( A_{13}^3 \) has a stable neighbourhood, any proper \( t \)-minor is a \( t \)-minor of a proper induced subgraph \( H \) of \( A_{13}^3 \). Thus, it suffices to show that any such \( H \) is \( t \)-perfect.

By Proposition 3 \( H \) is \( t \)-perfect unless it contains an odd wheel or one of \( A_7^1 \), \( A_2^2 \) or \( A_{10}^2 \) as an induced subgraph. Since the neighbourhood of every vertex is stable, \( H \) cannot contain any wheel. For the other graphs, we check the inequalities of Theorem 4 and see that none can be contained in \( H \). Thus, \( H \) is \( t \)-perfect and \( A_{13}^3 \) therefore minimally \( t \)-imperfect. 

5
Proposition 6. The antiweb $A_{13}^4$ is minimally $t$-imperfect.

Proof. By Proposition 3, any proper induced subgraph of $A_{13}^4$ that is $t$-imperfect contains one of $A_1^1$, $A_2^2$, or $A_{10}^4$ as an induced subgraph; note that $A_{13}^4$ does not contain odd wheels. However, routine calculation and Theorem 4 show that $A_{13}^4$ contains neither of these. Therefore, deleting any vertex in $A_{13}^4$ always results in a $t$-perfect graph.

It remains to consider the graphs obtained from $A_{13}^4$ by a single $t$-contraction. By symmetry, it suffices check whether the graph $H$ obtained by $t$-contraction at 0 is $t$-perfect; see Figure 3. Denote by $\tilde{0}$ the new vertex that resulted from the contraction.

The graph $H$ is still near-bipartite and still devoid of odd wheels. Thus, by Proposition 3, it is $t$-perfect unless it contains $A_1^1$ and $A_2^2$ as an induced subgraph—all the $t$-imperfect antiwebs of Proposition 3 are too large for the nine-vertex graph $H$.

Now, $A_1^1$ is 4-regular but $H$ only contains five vertices of degree at least 4. Similarly, $A_2^2$ is 3-regular but two of the nine vertices of $H$, namely 1 and 12, have degree 2. We see that neither of the two antiwebs can be contained in $H$, so that $H$ is $t$-perfect and, thus, $A_{13}^4$ minimally $t$-imperfect.

Proposition 7. The antiweb $A_{19}^7$ is minimally $t$-imperfect.

Proof. We claim that any proper induced subgraph of $A_{19}^7$ is $t$-perfect. Indeed, as $A_{19}^7$ does not contain any induced odd wheel, this follows from Proposition 3 unless $A_{19}^7$ contains one of $A_1^1$, $A_2^2$, $A_{10}^4$, $A_{12}^4$, $A_{13}^3$, $A_{13}^4$, or $A_{16}^6$ as an induced subgraph. We can easily verify with Theorem 4 that this is not the case.

It remains to check that any $t$-contraction in $A_{19}^7$ yields a $t$-perfect graph, too. By symmetry, we may restrict ourselves to a $t$-contraction at the vertex 0. Let $H$ be the resulting graph, and let $\tilde{0}$ be the new vertex; see Figure 4.

The graph $H$ is a near-bipartite graph on 15 vertices. It does not contain any odd wheel as an induced subgraph. Thus, by Proposition 3, $H$ is $t$-perfect unless it has an induced subgraph $A$ that is isomorphic to a graph in

$$A := \{A_1^1, A_8^2, A_{10}^4, A_{12}^4, A_{13}^3, A_{13}^4 \}.$$
Figure 4: Antiweb $A^1_{19}$, and its $t$-minor obtained by $t$-contraction at 0

Since this is not the case for $A^1_{19}$, we may assume that $\hat{0} \in V(A)$.

Note that the graphs $A^1_7, A^1_{10}, A^1_{13}$ and $A^1_{13}$ have minimum degree at least 4. Yet, $\hat{0}$ has only two neighbours of degree 4 or more (namely, 3 and 16). Thus, neither of these four antiwebs can occur as an induced subgraph in $H$.

It remains to consider the case when $H$ contains an induced subgraph $A$ that is isomorphic to $A^2_2$ or to $A^2_{12}$, both of which are 3-regular graphs. In particular, $A$ is then contained in $H' = H - \{1, 18\}$ as the vertices 1 and 18 have degree 2.

As $H'$ has only 13 vertices, $A$ cannot be isomorphic to $A^2_{12}$ since deleting any single vertex of $H'$ never yields a 3-regular graph. That leaves only $A = A^2_2$.

Since $A^2_2$ is 3-regular, we need to delete exactly one of the four neighbours of $\hat{0}$ in $H'$. Suppose this is the vertex 3. Then, 12 has degree 2 and thus cannot be part of $A$. Deleting 12 as well leads to vertex 2 having degree 2, which thereby is also excluded from $A$. This, however, is impossible as 2 is one of the three remaining neighbours of $\hat{0}$.

By symmetry, we may therefore assume that the neighbours of $\hat{0}$ in $A$ are precisely 2, 3, 16. That 17 is not part of $A$ entails that the vertex 7 has degree 2 and thus cannot lie in $A$ either. Then, however, 16 $\in V(A)$ has degree 2 as well, which is impossible.

5 Harmonious cutsets

We investigate the structure of minimally $t$-imperfect graphs, whether they are $P_5$-free or not. We hope this more general setting might prove useful in subsequent research.

A structural feature that may never appear in a minimally $t$-imperfect graph $G$ is a clique separator: any clique $K$ of $G$ so that $G - K$ is not connected.

Lemma 8 (Chvátal [2]; Gerards [16]). No minimally $t$-imperfect graph contains a clique separator.

A generalisation of clique separators was introduced by Chudnovsky et al. [7] in the context of colouring $K_4$-free graphs without odd holes. A tuple $(X_1, \ldots, X_s)$ of disjoint subsets of the vertex set of a graph $G$ is $G$-harmonious if
• any induced path with one endvertex in \( X_i \) and the other in \( X_j \) has even length if and only if \( i = j \); and

• if \( s \geq 3 \) then \( X_1, \ldots, X_s \) are pairwise complete to each other.

A pair of subgraphs \( \{ G_1, G_2 \} \) of \( G = (V, E) \) is a separation of \( G \) if \( V(G_1) \cup V(G_2) = V \) and \( G \) has no edge between \( V(G_1) \setminus V(G_2) \) and \( V(G_2) \setminus V(G_1) \). If both \( V(G_1) \setminus V(G_2) \) and \( V(G_2) \setminus V(G_1) \) are non-empty, the separation is proper.

A vertex set \( X \) is called a harmonious cutset if there is a proper separation \( (G_1, G_2) \) of \( G \) so that \( X = V(G_1) \cap V(G_2) \) and if there exists a partition \( X = (X_1, \ldots, X_s) \) so that \( (X_1, \ldots, X_s) \) is \( G \)-harmonious.

We prove:

**Lemma 9.** If a \( t \)-imperfect graph contains a harmonious cutset then it also contains a proper induced subgraph that is \( t \)-imperfect. In particular, no minimally \( t \)-imperfect graph admits a harmonious cutset.

For the proof we need a bit of preparation.

**Lemma 10.** Let \( S_1 \subseteq \cdots \subseteq S_k \) and \( T_1 \subseteq \cdots \subseteq T_l \) be nested subsets of a finite set \( V \). Let \( \sigma := \sum_{i=1}^{k} \lambda_i \chi_{S_i} \), and \( \tau := \sum_{j=1}^{l} \mu_j \chi_{T_j} \) be two convex combinations in \( \mathbb{R}^V \) with non-zero coefficients. If \( \sigma = \tau \) then \( k = \ell \), \( \lambda_i = \mu_i \) and \( S_i = T_i \) for all \( i = 1, \ldots, k \).

The lemma is not new. It appears in the context of submodular functions, where it may be seen to assert that the Lovász extension of a set-function is well-defined; see Lovász [21]. For the sake of completeness, we give a proof here.

**Proof.** By allowing \( \lambda_1 \) and \( \mu_1 \) to be 0, we may clearly assume that \( S_1 = \emptyset = T_1 \). Moreover, if two elements \( u, v \in V \) always appear together in the sets \( S_i, T_j \) then we may omit one of \( u, v \) from all the sets. So, in particular, we may assume \( S_2 \) and \( T_2 \) to be singleton-sets.

Let \( s \) be the unique element of \( S_2 \). Then \( \sum_{i=2}^{k} \lambda_i = \sigma_s = \tau_s \leq \sum_{j=2}^{l} \mu_j \). By symmetry, we also get \( \sum_{i=2}^{k} \lambda_i \geq \sum_{j=2}^{l} \mu_j \), and thus we have equality. We deduce that \( T_2 = \{ s \} \), and that \( \lambda_1 = \mu_1 \) as \( \lambda_1 = 1 - \sum_{i=2}^{k} \lambda_i = 1 - \sum_{j=2}^{l} \mu_j = \mu_1 \).

Then

\[
(\lambda_1 + \lambda_2)\chi_{S_1} + \sum_{i=3}^{k} \lambda_i \chi_{S_i \setminus \{s\}} = (\mu_1 + \mu_2)\chi_{T_1} + \sum_{j=3}^{l} \mu_j \chi_{T_j \setminus \{s\}}
\]

are two convex combinations. Induction on \( |S_k| \) now finishes the proof, where we also use that \( \lambda_1 = \mu_1 \).

**Lemma 11.** Let \( G \) be a graph, and let \( (X, Y) \) be a \( G \)-harmonious tuple (with possibly \( X = \emptyset \) or \( Y = \emptyset \)). If \( S_1, \ldots, S_k \) are stable sets then there are stable sets \( S'_1, \ldots, S'_k \) so that

(i) \( S'_1 \cap X \subseteq \cdots \subseteq S'_k \cap X \);

(ii) \( S'_1 \cap Y \supseteq \cdots \supseteq S'_k \cap Y \); and

(iii) \( \sum_{i=1}^{k} \chi_{S'_i} = \sum_{i=1}^{k} \chi_{S_i} \).
Proof. We start with two easy claims. First:

For any two stable sets $S, T$ there are stable sets $S'$ and $T'$ such that $\chi_S + \chi_T = \chi_{S'} + \chi_{T'}$ and $S' \cap X \subseteq T' \cap X$. \hfill (1)

Indeed, assume there is an $x \in (S \cap X) \setminus T$. Denote by $K$ the component of the induced graph $G[S \cup T]$ that contains $x$, and consider the symmetric differences $\tilde{S} = S \triangle K$ and $\tilde{T} = T \triangle K$. Clearly, $\chi_S + \chi_T = \chi_{\tilde{S}} + \chi_{\tilde{T}}$. Moreover, $K$ meets $X$ only in $S$ as otherwise $K$ would contain an induced $x$–$(T \cap X)$ path, which then has necessarily odd length. This, however, is impossible as $(X, Y)$ is $G$-harmonious. Therefore, $x \notin \tilde{S} \cap X \subseteq S \cap X$. By repeating this exchange argument for any remaining $x' \in (S \cap X) \setminus T$, we arrive at the desired stable sets $S'$ and $T'$. This proves (1).

We need a second, similar assertion:

For any two stable sets $S, T$ with $S \cap X \subseteq T \cap X$ there are stable sets $S'$ and $T'$ such that $\chi_S + \chi_T = \chi_{S'} + \chi_{T'}$, $S' \cap X = S \cap X$ \hfill (2)
and $S' \cap Y \supseteq T' \cap Y$.

To see this, assume there is a $y \in (T \cap Y) \setminus S$, and let $K$ be the component of $G[S \cup T]$ containing $y$, and set $\tilde{S} = S \triangle K$ and $\tilde{T} = T \triangle K$. The component $K$ may not meet $T \cap X$, as then it would contain an induced $y$–$(T \cap X)$ path. This path would have even length, contradicting the definition of a $G$-harmonious tuple. As above, we see, moreover, that $K$ meets $Y$ only in $T$; otherwise there would be an induced odd $y$–$(S \cap Y)$ path, which is impossible. Thus, $\tilde{S}, \tilde{T}$ satisfy the first two conditions we want to have for $S', T'$, while $(T \cap Y) \setminus \tilde{S}$ is smaller than $(T \cap Y) \setminus S$. Again repeating the argument yields $S', T'$ as desired. This proves (2).

We now apply (1) iteratively to $S_1$ (as $S$) and each of $S_2, \ldots, S_k$ (as $T$) in order to obtain stable sets $R_1, \ldots, R_k$ with $R_i \cap X \subseteq R_i \cap X$ for every $i = 2, \ldots, k$ and $\sum_{i=1}^k \chi_{R_i} = \sum_{i=1}^k \chi_{R_i}$. We continue applying (1), first to $R_2$ and each of $R_3, \ldots, R_k$, then to the resulting $R'_3$ and each of $R'_4, \ldots, R'_k$ and so on, until we arrive at stable sets $T_1, \ldots, T_k$ with $\sum_{i=1}^k \chi_{T_i} = \sum_{i=1}^k \chi_{T_i}$ that are nested on $X$: $T_1 \cap X \subseteq \ldots \subseteq T_k \cap X$.

In a similar way, we use (2) to force the stable sets to become nested on $Y$ as well. First, we apply (2) to $T_1$ (as $S$) and to each of $T_2, \ldots, T_k$ (as $T$), then to the resulting $T'_3$ and each of $T'_4, \ldots, T'_k$, and so on. Proceeding in this manner, we obtain the desired stable sets $S'_1, \ldots, S'_k$. \hfill $\square$

Lemma 12. Let $(G_1, G_2)$ be a proper separation of a graph $G$ so that $X = V(G_1) \cap V(G_2)$ is a harmonious cutset. Let $z \in \mathbb{Q}^V(G)$ be so that $z|_{G_1} \in SSP(G_1)$ and $z|_{G_2} \in SSP(G_2)$. Then $z \in SSP(G)$.

The lemma generalises the result by Chudnovsky et al. \cite{Chudnovsky} that $G = G_1 \cup G_2$ is 4-colourable if $G_1$ and $G_2$ are 4-colourable.

Proof of Lemma 12. Let $(X_1, \ldots, X_s)$ be a $G$-harmonious partition of $X$. As $z|_{G_j} \in SSP(G_j)$, for $j = 1, 2$, we can express $z|_{G_1}$ as a convex combination of stable sets $S_1, \ldots, S_m$ of $G_1$, and $z|_{G_2}$ as a convex combination of stable sets $T_1, \ldots, T_m$ of $G_2$. Since $z$ is a rational vector, we may even assume that

$$z|_{G_1} = \frac{1}{m} \sum_{i=1}^m \chi_{S_i} \text{ and } z|_{G_2} = \frac{1}{m} \sum_{i=1}^m \chi_{T_i}.$$
Indeed, this can be achieved by repeating stable sets.

We first treat the case when \( s \leq 2 \). If \( s = 1 \), then set \( X_2 = \emptyset \), so that whenever \( s \leq 2 \), we have \( X = X_1 \cup X_2 \).

Using Lemma \text{[11]} we find stable sets \( S'_1, \ldots, S'_m \) of \( G_1 \) so that \( z|_{G_1} = \frac{1}{m} \sum_{i=1}^{m} \chi_{S'_i} \) and

\[
S'_1 \cap X_1 \subset \ldots \subset S'_m \cap X_1, \text{ and } S'_i \cap X_2 \supseteq \ldots \supseteq S'_m \cap X_2
\]

holds. Analogously, we obtain a convex combination \( z|_{G_2} = \frac{1}{m} \sum_{i=1}^{m} \chi_{T'_i} \) of stable sets \( T'_1, \ldots, T'_m \) of \( G_2 \) that are increasingly nested on \( X_1 \) and decreasingly nested on \( X_2 \).

Define \( \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_k \) to be the distinct restrictions of the sets \( S'_i \) to \( X_1 \).

More formally, let \( 1 = i_1 < \ldots < i_k < i_{k+1} = m + 1 \) be so that

\[
\mathcal{S}_i = S'_i \cap X_1 \text{ for all } i \leq i < i_{i+1}
\]

We set, moreover, \( \lambda_i = \frac{1}{m}(i_{i+1} - i_i) \). Equivalently, \( m \lambda_i \) is the number of \( S'_i \) with \( \mathcal{S}_i = S'_i \cap X_1 \). Then \( z|_{X_1} = \sum_{i=1}^{k} \lambda_i \chi_{\mathcal{S}_i} \) is a convex combination.

We do exactly the same in \( G_2 \) in order to obtain \( z|_{X_1} = \sum_{i=1}^{k} \mu_i \chi_{T'_i} \), where the sets \( \mathcal{T}_i \) are the distinct restrictions of the \( T'_i \) to \( X_1 \). With Lemma \text{[11]} we deduce first that \( \mathcal{T}_i = T'_i \) and \( \lambda_i = \mu_i \) for all \( i \), from which we get that

\[
S'_i \cap X_1 = T'_i \cap X_1 \text{ for all } 1 \leq i \leq m.
\]

The same argument, only applied to the restrictions of \( S'_i \) and of \( T'_i \) to \( X_2 \), yields that also

\[
S'_i \cap X_2 = T'_i \cap X_2 \text{ for all } 1 \leq i \leq m.
\]

Thus, \( R_i = S'_i \cup T'_i \) is, for every \( i = 1, \ldots, m \), a stable set of \( G \). Consequently, \( z = \frac{1}{m} \sum_{i=1}^{m} \chi_{R_i} \) is a convex combination of stable sets and thus a point of \( \text{SSP}(G) \).

It remains to treat the case when the harmonious cutset has at least three parts, that is, when \( s \geq 3 \). We claim that there are sets \( S_0, S_1, \ldots, S_s \) of stable sets of \( G_1 \) so that

(a) \( z|_{G_1} = \frac{1}{m} \sum_{j=0}^{s} \sum_{S \in S_j} \chi_S \) and \( \sum_{j=0}^{s} |S_j| = m \);

(b) for \( j = 1, \ldots, s \) if \( S \in S_j \) then \( X_j \cap S \) is non-empty; and

(c) for \( j = 0, \ldots, s \) if \( S, T \in S_j \) then \( X_j \cap S \subseteq X_j \cap T \) or \( X_j \cap S \supseteq X_j \cap T \).

Moreover, there are analogous sets \( T_0, T_1, \ldots, T_s \) for \( G_2 \).

To prove the claim note first that each \( S_i \) meets at most one of the sets \( X_1 \) as each two induce a complete bipartite graph. Therefore, we can partition \( \{ S_1, \ldots, S_m \} \) into sets \( S'_1, \ldots, S'_s \) so that \( a \) and \( b \) are satisfied. Next, we apply Lemma \text{[11]} to each \( S'_j \) and \( (X_j, \emptyset) \) in order to obtain sets \( S''_j \) that satisfy \( a \) and \( b \) but not necessarily \( c \); property \( d \) still holds as Lemma \text{[11]} guarantees \( \sum_{S \in S'_j} \chi_S = \sum_{S \in S''_j} \chi_S \) for each \( j \). If \( f \) is violated, then only because for some \( j \neq 0 \) there is \( S \in S''_j \) that is not only disjoint from \( X_1 \) but also from all other \( X_{j'} \). In order to repair \( h \) we remove all stable sets \( S \) in \( \bigcup_{j=1}^{s} S''_j \) that
are disjoint from $\bigcup_{j=1}^s X_j$ from their respective sets and add them to $S''$. The resulting sets $S_0, S_1, \ldots, S_s$ then satisfy (2) and (3). The proof for the $T_j$ is the same.

As a consequence of (2) and (3) it follows for $j = 0, 1, \ldots, s$ that

$$\sum_{S \in S_j} \lambda_{S \cap X_j} = m \cdot z(v) = \sum_{T \in T_j} \chi_{T \cap X_j} \quad (3)$$

Now, consider $j \neq 0$. Then, by (1) and (2), there is a vertex $v \in X_j$ that lies in every $S \in S_j$. Thus, we have $\sum_{S \in S_j} \lambda_S(v) = |S_j|$. Evaluating (3) at $v \in X_j$, we obtain

$$|S_j| = m \cdot z(v) = \sum_{T \in T_j} \chi_T(v) \leq |T_j|.$$  

Reversing the roles of $S_j$ and $T_j$, we also get $|T_j| \leq |S_j|$, and thus that $|T_j| = |S_j|$, as long as $j \neq 0$. That this also holds for $j = 0$ follows from $m = \sum_{j=0}^s |S_j| = \sum_{j=0}^s |T_j|$, so that we get $m_j := |S_j| = |T_j|$ for every $j = 0, 1, \ldots, s$. Together with (3) this implies, in particular, that

$$\frac{1}{m_j} \sum_{S \in S_j} \lambda_{S \cap X_j} = \frac{1}{m_j} \sum_{T \in T_j} \chi_{T \cap X_j}.$$  

We may, therefore, define a vector $y^j$ on $V(G)$ by setting

$$y^j|_{G_1} := \frac{1}{m_j} \sum_{S \in S_j} \lambda_S \text{ and } y^j|_{G_2} := \frac{1}{m_j} \sum_{T \in T_j} \chi_T \quad (4)$$

For any $j = 0, \ldots, s$, define $G^j = G - \bigcup_{r \neq j} X_r$, and observe that $X_j$ is a harmonious cutset of $G^j$ consisting of only one part. (That is, $X_j$ is $G^j$-harmonious.) Moreover, as (2) shows, the restriction of $y^j$ to $G_1 \cap G^j$ lies in $\text{SSP}(G_1 \cap G^j)$, while the restriction to $G_2 \cap G^j$ lies in $\text{SSP}(G_2 \cap G^j)$. Thus, we can apply the first part of this proof, when $s \leq 2$, in order to deduce that $y^j \in \text{SSP}(G^j) \subseteq \text{SSP}(G)$.

To finish the proof we observe, with (2) and (3), that

$$z = \sum_{j=0}^s m_j y^j.$$

As, by (2), $\sum_{j=0}^s m_j = m$, this means that $z$ is a convex combination of points in $\text{SSP}(G)$, and thus itself an element of $\text{SSP}(G)$.

**Corollary 13.** Let $(G_1, G_2)$ be a proper separation of $G$ so that $X = V(G_1) \cap V(G_2)$ is a harmonious cutset. Then $G$ is $t$-perfect if and only if $G_1$ and $G_2$ are $t$-perfect.

**Proof.** Assume that $G_1$ and $G_2$ are $t$-perfect, and consider a rational point $z \in \text{TSTAB}(G)$. Then $z|G_1 \in \text{SSP}(G_1)$ and $z|G_2 \in \text{SSP}(G_2)$, which means that Lemma 12 yields $z \in \text{SSP}(G)$. Since this is true for all rational $z$ it extends to real $z$ as well.

The corollary directly implies Lemma 9.
6 $P_5$-free graphs

Let $\mathcal{F}$ be the set of graphs consisting of $P_5$, $K_4$, $W_5$, $C_7^2$, $A_2^3$, and $A_3^3$ together with the three graphs in Figure 5. Note that the latter three graphs all contain $K_4$ as a $t$-minor: for (a) and (b) $K_4$ is obtained by a $t$-contraction at any vertex of degree 2, while for (c) both vertices of degree 2 need to be $t$-contracted. In particular, every graph in $\mathcal{F}$ besides $P_5$ is $t$-imperfect. We say that a graph is $\mathcal{F}$-free if it contains none of the graphs in $\mathcal{F}$ as an induced subgraph.

![Figure 5: Three graphs that $t$-contract to $K_4$](image)

We prove a lemma that implies directly Theorem 1:

**Lemma 14.** Any $\mathcal{F}$-free graph is $t$-perfect.

We first examine how a vertex may position itself relative to a 5-cycle in an $\mathcal{F}$-free graph.

**Lemma 15.** Let $G$ be an $\mathcal{F}$-free graph. If $v$ is a neighbour of a 5-hole $C$ in $G$ then $v$ has either exactly two neighbours in $C$, and these are non-consecutive in $C$; or $v$ has exactly three neighbours in $C$, and these are not all consecutive.

![Figure 6: The types of neighbours of a 5-hole](image)

**Proof.** See Figure 6 for the possible types of neighbours (up to isomorphy). Of these, (b) and (c) contain an induced $P_5$; (e) and (g) are the same as (a) and (b) in Figure 5 and thus in $\mathcal{F}$; (h) is $W_5$. Only (d) and (f) remain.

**Lemma 16.** Let $G$ be an $\mathcal{F}$-free graph, and let $u$ and $v$ be two non-adjacent vertices such that both of them have precisely three neighbours in a 5-hole $C$. 

12
Then $u$ and $v$ have either all three or exactly two non-consecutive neighbours in $C$ in common.

![Diagram](image)

**Figure 7:** The possible configurations of Lemma 16

*Proof.* By Lemma 15, both of $u$ and $v$ have to be as in (f) of Figure 6. Figure 7 shows the possible configurations of $u$ and $v$ (up to isomorphism). Of these, (b) is impossible as there is an induced $P_5$—the other two configurations (a) and (c) may occur.

A subgraph $H$ of a graph $G$ is *dominating* if every vertex in $G - H$ has a neighbour in $H$.

**Lemma 17.** Let $G$ be an $F$-free graph. Then, either any 5-hole of $G$ is dominating or $G$ contains a harmonious cutset.

*Proof.* Assume that there is a 5-hole $C = c_1 \ldots c_5 c_1$ that fails to dominate $G$. Our task consists in finding a harmonious cutset. We first observe:

Let $u \in N(C)$ be a neighbour of some $x \notin N(C)$. Then $u$ has exactly three neighbours in $C$, not all of which are consecutive.  

(5)

So, such a $u$ is as in (f) of Figure 6. Indeed, by Lemma 15, only (d) or (f) in Figure 6 are possible. In the former case, we may assume that the neighbours of $u$ in $C$ are $c_1$ and $c_3$. Then, however, $xuc_1c_4c_5$ is an induced $P_5$. This proves (5).

![Diagram](image)

**Figure 8:** $x$ in solid black.

Consider two adjacent vertices $y, z \notin N(C)$, and assume that there is a $u \in N(y) \cap N(C)$ that is not adjacent to $z$. We may assume that $N(u) \cap C = \{c_1, c_2, c_4\}$ by (5). Then, $zyuc_2c_3$ is an induced $P_5$, which is impossible. Thus:

$$N(y) \cap N(C) = N(z) \cap N(C) \text{ for any adjacent } y, z \notin N(C).$$

(6)
Next, fix some vertex \( x \) that is not dominated by \( C \) (and, by assumption, there is such a vertex). As a consequence of (6), \( N(x) \cap N(C) \) separates \( x \) from \( C \). In particular, \[ X := N(x) \cap N(C) \text{ is a separator.} \] (7)

Consider two vertices \( u, v \in X \). Then, by (5), each of \( u \) and \( v \) have exactly three neighbours in \( C \), not all of which are consecutive. We may assume that \( N(u) \cap V(C) = \{c_1, c_2, c_4\} \).

First, assume that \( uv \in E(G) \), and suppose that the neighbourhoods of \( u \) and \( v \) in \( C \) are the same. This, however, is impossible as then \( u, v, c_1, c_2 \) form a \( K_4 \). Therefore, \( uv \in E(G) \) implies \( N(u) \cap V(C) \neq N(v) \cap V(C) \).

Now assume \( uv \notin E(G) \). By Lemma 13, there are only two possible configurations (up to isomorphism) for the neighbours of \( v \) in \( C \); these are (a) and (c) in Figure 8. The first of these, (a) in Figure 8, is impossible, as this is a graph of \( F_7 \); see Figure 5(c). Thus, we see that \( u, v \) are as in (b) of Figure 8 that is, that \( u \) and \( v \) have the same neighbours in \( C \).

To sum up, we have proved that:

\[ uv \in E(G) \iff N(u) \cap V(C) \neq N(v) \cap V(C) \quad \text{for any two } u, v \in X \] (8)

An immediate consequence is that the neighbourhoods in \( C \) partition \( X \) into stable sets \( X_1, \ldots, X_k \) such that \( X_i \) is complete to \( X_j \) whenever \( i \neq j \). As \( X \) cannot contain any triangle—together with \( x \) this would result in a \( K_4 \)—it follows that \( k \leq 2 \). If \( k = 1 \), we put \( X_2 = \emptyset \) so that always \( X = X_1 \cup X_2 \).

We claim that \( X \) is a harmonious cutset. As \( X \) is a separator, by (7), we only need to prove that \( (X_1, X_2) \) is \( G \)-harmonious. For this, we have to check the parities of induced \( X_1 \)-paths and of \( X_2 \)-paths; since \( X_1 \) is complete to \( X_2 \) any induced \( X_1-X_2 \) path is a single edge and has therefore odd length.

Suppose there is an odd induced \( X_1 \)-path or \( X_2 \)-path. Clearly, we may assume there is such a path \( P \) that starts in \( u \in X_1 \) and ends in \( v \in X_1 \). As \( X_1 \) is stable, and as \( G \) is \( P_3 \)-free, it follows that \( P \) has length 3. So, let \( P = upxv \).

Let us consider the position of \( p \) and \( q \) relative to \( C \). We observe that neither \( p \) nor \( q \) can be in \( C \). Indeed, if, for instance, \( p \) was in \( C \) then \( p \) would also be a neighbour of \( v \) since \( N(u) \cap V(C) = N(v) \cap V(C) \), by (8). This, however, is impossible as \( P \) is induced.

Next, assume that \( p, q \notin N(C) \) holds. Since \( p \) and \( q \) are adjacent, we can apply (6) to \( p \) and \( q \), which results in \( N(p) \cap N(C) = N(q) \cap N(C) \). However, as \( u \) lies in \( N(p) \cap N(C) \) it then also is a neighbour of \( q \), which contradicts that \( upqv \) is induced.

It remains to consider the case when one of \( p \) and \( q \) say, lies in \( N(C) \). As \( p \) is adjacent to \( u \) but not to \( v \), both of which lie in \( X_1 \) and are therefore non-neighbours, it follows from (5) that \( p \notin X \). In particular, \( p \) is not a neighbour of \( x \), which means that \( pxuv \) is an induced path.

Suppose there is a neighbour \( c \in V(C) \) of \( p \) that is not adjacent to \( u \). By (5), \( c \) is not adjacent to \( v \) either, so that \( cpxuv \) forms an induced \( P_3 \), a contradiction. Thus, \( N(p) \cap V(C) \subseteq N(u) \) has to hold. By (5), we may assume that the neighbours of \( u \) in \( C \) are precisely \( c_1, c_2, c_4 \). As \( u \) and \( p \) are adjacent, \( p \) cannot be neighbours with both of \( c_1 \) and \( c_2 \), as this would result in a \( K_4 \). Thus, we may assume that \( N(p) \cap V(C) = \{c_2, c_4\} \). (Note, that \( p \) has at least two neighbours in \( C \), by Lemma 15)
To conclude, we observe that $pc_4c_5c_1c_2p$ forms a 5-hole, in which $v$ has four neighbours, namely $c_1, c_2, c_4, p$. This, however, is in direct contradiction to Lemma 15 which means that our assumption is false, and there is no odd induced $X_1$-path, and no such $X_2$-path either. Consequently, $(X_1, X_2)$ is $G$-harmonious, and $X = X_1 \cup X_2$ therefore a harmonious cutset.

**Proposition 18.** Let $G$ be a $t$-imperfect graph. Then either $G$ contains an odd hole or it contains $K_4$ or $C_5^2$ as an induced subgraph.

**Proof.** Assume that $G$ does not contain any odd hole and neither $K_4$ nor $C_5^2$ as an induced subgraph. Observe that any odd antihole of length $\geq 9$ contains $K_4$. Since the complement of a 5-hole is a 5-hole, and since $C_5^2$ is the odd antihole of length 7, it follows that $G$ cannot contain any odd antihole at all.

Now, by the strong perfect graph theorem it follows that $G$ is perfect. (Note that we do not need the full theorem but only the far easier version for $K_4$-free graphs; see Tucker [28].) Since $G$ does not contain any $K_4$ it is therefore $t$-perfect as well.

**Lemma 19.** Let $G$ be an $F$-free graph. If $G$ contains a 5-hole, and if every 5-hole is dominating then $G$ is near-bipartite.

**Proof.** Let $G$ contain a 5-hole, and assume every 5-hole to be dominating. Suppose that the lemma is false, i.e. that $G$ fails to be near-bipartite. In particular, there is a vertex $v$ such that $G - N(v)$ is not bipartite, and therefore contains an induced odd cycle $T$. As any 5-hole is dominating and any $k$-hole with $k > 5$ contains an induced $P_5$, $T$ has to be a triangle. Let $T = xyz$. We distinguish two cases, both of which will lead to a contradiction.

**Case:** $v$ lies in a 5-hole $C$.

Let $C = c_1 \ldots c_5c_1$, and $v = c_1$. Then $T$ could meet $C$ in 0, 1 or 2 vertices. If $T$ has two vertices with $C$ in common, these have to be $c_3$ and $c_4$ as the others are neighbours of $v$. Then, the third vertex of $T$ has two consecutive neighbours in $C$, which means that by Lemma 15 its third neighbour in $C$ has to be $c_1 = v$, which is impossible.

Next, suppose that $T$ meets $C$ in one vertex, $c_3 = z$, say. By Lemma 15 each of $x, y$ has to have a neighbour opposite of $c_3$ in $C$, that is, either $c_1$ or $c_5$. As $c_1 = v$, both of $x, y$ are adjacent with $c_5$. The vertices $x, y$ could have a third neighbour in $C$; this would necessarily be $c_2$. However, not both can be adjacent to $c_2$ as then $x, y, c_2, c_3$ would induce a $K_4$. Thus, assume $x$ to have exactly $c_3$ and $c_5$ as neighbours in $C$. This means that $C' = c_3x_5c_1c_3c_5$ is a 5-hole in which $y$ has at least three consecutive neighbours, $c_3, x, c_5$, which is impossible (again, by Lemma 15).

Finally, suppose that $T$ is disjoint from $C$. Each of $x, y, z$ has at least two neighbours among $c_2, \ldots, c_5$, and no two have $c_3$ or $c_4$ as neighbour; otherwise we would have found a triangle in $G - N(v)$ meeting $C$ in exactly one vertex, and could reduce to the previous subcase. Thus, we may assume that $x$ is adjacent to $c_2$ and $c_5$. Moreover, since no vertex of $x, y, z$ can be adjacent to both $c_3$ and $c_4$ (as then it would also be adjacent to $c_1$, by Lemma 15) and no $c_i \in C$ can be adjacent to all vertices of $T$ (because otherwise $c_i, x, y, z$ would form a $K_4$), it follows that we may assume that $y$ is adjacent to $c_2$ but not to $c_5$, while $z$ is adjacent to $c_5$ but not to $c_2$. Then, $c_1c_2yzc_5c_1$ is a 5-hole in which $x$ has...
four neighbours, in obvious contradiction to Lemma 15. Therefore, this case is impossible.

**Case:** $v$ does not lie in any 5-hole.

Let $C = c_1 \ldots c_5 c_1$ be a 5-hole. Since every 5-hole is dominating, $v$ has a neighbour in $C$, and thus, by Lemma 15 either as in (f) of Figure 6 or as in (d). The latter, however, is impossible since then $v$ would be contained in a 5-hole. Therefore, we may assume that the neighbours of $v$ in $C$ are precisely $\{c_1, c_2, c_4\}$. As a consequence, $T$ can meet $C$ in at most $c_3$ and $c_5$, both not in both as $C$ is induced.

Suppose $T = xyz$ meets $C$ in $x = c_3$. If $y$ is not adjacent to either of $c_1$ and $c_4$, then $c_1 c_4 c_y$ forms an induced $P_5$. If, on the other hand, $y$ is adjacent to $c_4$ then, by Lemma 15 also to $c_1$. Thus, $y$ is either adjacent to $c_1$ or to both $c_3$ and $c_4$. The same holds for $z$. Since $y$ and $z$ are adjacent, they cannot both have three neighbours in $C$ (otherwise $G$ would contain a $K_4$). Suppose $N(y) \cap C = \{x, c_1\}$. But then $xc_4c_5c_1yx$ forms an induced 5-cycle in which $z$ has at least three consecutive neighbours; a contradiction to Lemma 15.

Consequently, $T$ is disjoint from every 5-hole. By Lemma 15 each of $x$, $y$, $z$ has neighbours in $C$ as in (d) or (f) of Figure 6. However, if any of $x$, $y$, $z$ has only two neighbours in $C$ as in (d) then that vertex together with four vertices of $C$ forms a 5-hole that meets $T$—this is precisely the situation of the previous subcase. Thus, we may assume that all vertices of $T$ have three neighbours in $C$ as in (f) of Figure 6. If we consider the possible configurations of two non-adjacent vertices which have three neighbours in $C$ (namely $v$ and a vertex of $T$) as we have done in Lemma 7 we see that only $(a)$ and $(c)$ in Figure 7 are possible. But then each vertex of $T$ has to be adjacent to $c_4$, which means that $T$ together with $c_4$ induces a $K_4$, which is impossible.

**Proof of Lemma 14**. Suppose that $G$ is a $t$-imperfect and but $F$-free. By deleting suitable vertices we may assume that every proper induced subgraph of $G$ is $t$-perfect. In particular, by Lemma 9 $G$ does not admit a harmonious cutset. Since $G$ is $t$-imperfect it contains an odd hole, by Proposition 18 and since $G$ is $P_5$-free, the odd hole is of length 5. From Lemma 17 we deduce that any 5-hole is dominating. Lemma 19 implies that $G$ is near-bipartite.

Noting that both $A_4^{13}$ and $A_4^{16}$, as well as any Möbius ladder or any odd wheel larger than $W_5$, contain an induced $P_5$, we see with Proposition 6 that $G$ is $t$-perfect after all.

By Lemma 13 a $P_5$-free graph is either $t$-perfect or contains one of eight $t$-imperfect graphs as an induced subgraph. Obviously, checking for these forbidden induced subgraphs can be done in polynomial time, so that we get as immediate algorithmic consequence:

**Theorem 20.** $P_5$-free $t$-perfect graphs can be recognised in polynomial time.

We suspect, but cannot currently prove, that $t$-perfection can be recognised as well in polynomial time in near-bipartite graphs.

### 7 Colouring

Can $t$-perfect graphs always be coloured with few colours? This is one of the main open questions about $t$-perfect graphs. A conjecture by Shepherd and
Sebő asserts that four colours are always enough:

**Conjecture 21** (Shepherd; Sebő [23]). *Every t-perfect graph is 4-colourable.*

The conjecture is known to hold in a number of graph classes, for instance in claw-free graphs, where even three colours are already sufficient; see [5]. It is straightforward to verify the conjecture for near-bipartite graphs:

**Proposition 22.** *Every near-bipartite t-perfect graph is 4-colourable.*

**Proof.** Pick any vertex $v$ of a near-bipartite and $t$-perfect graph $G$. Then $G - N(v)$ is bipartite and may be coloured with colours 1, 2. On the other hand, as $G$ is $t$-perfect the neighbourhood $N(v)$ necessarily induces a bipartite graph as well; otherwise $v$ together with a shortest odd cycle in $N(v)$ would form an odd wheel. Thus we can colour the vertices in $N(v)$ with the colours 3, 4.

Near-bipartite $t$-perfect graphs can, in general, not be coloured with fewer colours. Indeed, this is even true if we restrict ourselves further to complements of line graphs, which is a subclass of near-bipartite graphs. Two $t$-perfect graphs in this class that need four colours are: $L(Π)$, the complement of the line graph of the prism, and $L(W_5)$. The former was found by Laurent and Seymour (see [25, p. 1207]), while the latter was discovered by Benchetrit [2]. Moreover, Benchetrit showed that any 4-chromatic $t$-perfect complement of a line graph contains one of $L(Π)$ and $L(W_5)$ as an induced subgraph.

How about $P_5$-free $t$-perfect graphs? Applying insights of Sebő and of Sumner, Benchetrit [2] proved that $P_5$-free $t$-perfect graphs are 4-colourable. This is not tight:

**Theorem 23.** *Every $P_5$-free $t$-perfect graph $G$ is 3-colourable.*

For the proof we use that there is a finite number of obstructions for 3-colourability in $P_5$-free graphs:

**Theorem 24** (Maffray and Morel [22]). *A $P_5$-free graph is 3-colourable if and only if it does not contain $K_4$, $W_5$, $C^2_7$, $A^2_{10}$, $A^1_{13}$ or any of the seven graphs in Figure 9 as an induced subgraph.*

(Maffray and Morel call these graphs $F_1$–$F_{12}$. The graphs $K_4$, $W_5$, $C^2_7$, $A^2_{10}$, $A^1_{13}$ are respectively $F_1$, $F_2$, $F_3$, $F_{11}$ and $F_{12}$.) A similar result was obtained by Bruce, Hoang and Sawada [4], who gave a list of five forbidden (not necessarily induced) subgraphs.

**Proof of Theorem 23.** Any $P_5$-free graph $G$ that cannot be coloured with three colours contains one of the twelve induced subgraphs of Theorem 24. Of these twelve graphs, we already know that $K_4$, $W_5$, $C^2_7$, $A^2_{10}$, $A^1_{13}$ are $t$-imperfect, and thus cannot be induced subgraphs of a $t$-perfect graph. It remains to consider the seven graphs in Figure 9. These graphs are $t$-imperfect, too: each can be turned into $K_4$ by first deleting the grey vertices and then performing a $t$-contraction at the respective black vertex.

We mention that Benchetrit [2] also showed that $P_6$-free $t$-perfect graphs are 4-colourable. This is tight: both $L(Π)$ and $L(W_5)$ (and indeed all complements of line graphs) are $P_6$-free. We do not know whether $P_7$-free $t$-perfect graphs are 4-colourable.
Figure 9: The remaining 4-critical $P_5$-free graphs of Theorem 24 in Maffray and Morel [22] these are called $F_3 - F_8$ and $F_{10}$. In each graph, deleting the grey vertices and then $t$-contracting at the black vertex results in $K_4$.

We turn now to fractional colourings. A motivation for Conjecture 21 was certainly the fact that the fractional chromatic number $\chi_f(G)$ of a $t$-perfect graph $G$ is always bounded by 3. More precisely, if $og(G)$ denotes the odd girth of $G$, that is, the length of the shortest odd cycle, then $\chi_f(G) = 2 - \frac{og(G)}{og(G) - 1}$ as long as $G$ is $t$-perfect (and non-bipartite). This follows from linear programming duality; see for instance Schrijver [25, p. 1206].

Recall that a graph $G$ is perfect if and only if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. As odd cycles seem to play a somewhat similar role for $t$-perfection as cliques play for perfection, one might conjecture that $t$-perfection is characterised in an analogous way:

**Conjecture 25.** A graph $G$ is $t$-perfect if and only if $\chi_f(H) = 2 - \frac{og(H)}{og(H) - 1}$ for every non-bipartite $t$-minor $H$ of $G$.

Note that the conjecture becomes false if, instead of $t$-minors, only induced subgraphs $H$ are considered. Indeed, in the $t$-imperfect graph obtained from $K_4$ by subdividing some edge twice, all induced subgraphs satisfy the condition (but not the $t$-minor $K_4$).

An alternative but equivalent formulation of the conjecture is: $\chi_f(G) > 2 - \frac{og(G)}{og(G) - 1}$ holds for every minimally $t$-imperfect graph $G$. It is straightforward to check that all minimally $t$-imperfect graphs that are known to date satisfy this. In particular, it follows that the conjecture is true for $P_5$-free graphs, for near-bipartite graphs, as well as for claw-free graphs; see [5] for the minimally $t$-imperfect graphs that are claw-free.

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