CASIMIR EFFECT FOR BIAXIAL ANISOTROPIC PLATES WITH SURFACE CONDUCTIVITY

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The Casimir energy is constructed for a system consisting of two semi-infinite slabs of anisotropic material. Each of them is characterized by bulk complex dielectric permittivity tensor and surface conductivity on the free boundary. We found general form of the scattering matrix and Fresnel coefficients for each part of the system by solving Maxwell equations in the anisotropic media.

Keywords: Casimir energy; biaxial anisotropy.

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1. Introduction

Last decade, the great interest was connected with 2D systems due to discovering graphene[1] (see, for example last reviews[2,3]). In the same time, many interesting and non-trivial 3D materials appear like metamaterials, three-dimensional topological and Chern insulators, Dirac and Weyl semi-metals, all highly anisotropic.

Different aspects of Casimir effect involving these anisotropic media of different complexity were previously studied. In particular, interaction of passive uniaxial and biaxial media with one of the optical axes being perpendicular to the interface[4,6] anisotropic single-negative metamaterials[7] were among the subjects of research, to

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have 8 amplitudes, \(a_{ij}\) where the subscript \(i\) is to be defined in its turn through boundary (matching) conditions.

\begin{align*}
\text{Fig. 1. Two dielectric semi-spaces} \quad z < 0 \quad \text{and} \quad z > a \quad \text{with dielectric tensors} \quad \varepsilon_{\alpha\beta} \quad \text{and} \quad \varepsilon_{\alpha'}\beta' \quad \text{and boundaries with surface conductivities} \quad \sigma_{ij} \quad \text{and} \quad \sigma'_{ij}. \quad \text{Magnetic properties we assume to be trivial,} \quad \mu^{(r)} = 1.
\end{align*}

mention just a few. However, bianisotropic optically active media with arbitrary orientation of the axes has not yet been considered, while such materials (e.g. TaAs, Na\(_3\)Sb, MoTe\(_2\), etc.) are now under active research.\(^9\) In this paper we consider the Casimir energy for two anisotropic objects with planar symmetry and surface conductivity. The formulas obtained may be applied for the above noted 3D materials.

Using a scattering matrix approach\(^9\)\(^10\) the Casimir energy can be given as

\[
\mathcal{E}_C = \int_\mathbb{R} d^2 k \sum_{k=1}^{4} d\xi \ln \det \left( 1 - e^{-2\pi i k \cdot \mathbf{R}^{(l)} \mathbf{R}^{(r)}(\xi)} \right),
\]

where, \(\kappa = \sqrt{\xi^2 + k_{\perp}^2}, \quad k_3 = i\varepsilon \text{sign} \, \xi, \) and \(\mathbf{R}^{(l)}_{1}(i\xi, k_{\perp}), \mathbf{R}^{(l)}_{2}(i\xi, k_{\perp})\) are the Fresnel reflection matrices. Prime means the opposite direction of scattering.\(^10\)

2. Scattering Problem

Let us consider the Casimir energy for the system plotted in Fig. 1. We consider first a general scattering problem with matter described by hermitian tensor\(^9\)\(\varepsilon_{\alpha\beta}\) in the left (index \(l\)) of the boundary \(z = 0\) and vacuum, \(\varepsilon_{\alpha\beta} = \delta_{\alpha\beta}, \) in the right (index \(r\)), which corresponds to the left part of the system in Fig. 1.

Presence of imaginary part of \(\varepsilon \) (corresponding to optically active media) makes it impossible to find an orthogonal coordinate system where dielectric permittivity tensor would be diagonal. However, it is still perfectly possible to solve Maxwell equation. Generally speaking, Maxwell equations in anisotropic media give a dispersion relation which has 4 distinctive roots, \(\kappa_n = k_3(k, \omega)\) and 4 corresponding distinct eigenvectors \(\mathcal{E}_n\) and \(\mathcal{H}_n\) \(\left(n = 1, 2, 3, 4\right)\). We choose numeration of roots such that in the vacuum limit \(\kappa_{1,2} \rightarrow +k_z\) and \(\kappa_{3,4} \rightarrow -k_z\) \(\left(k_z = \sqrt{\omega^2 - k^2}\right)\).

The field has the following structure at the left of the boundary (inside matter):

\[
\begin{align*}
\mathbf{E}_l, \mathbf{H}_l &= e^{i k_z z} \{ A_{\delta}^{(l)}(\mathcal{E}_1^{(l)}, \mathcal{H}_1^{(l)}) + e^{i k_{\perp} z} B_{\delta}^{(l)}(\mathcal{E}_2^{(l)}, \mathcal{H}_2^{(l)}) \} + e^{i k_{\perp} z} A_{\delta}^{(l)}(\mathcal{E}_3^{(l)}, \mathcal{H}_3^{(l)}) + e^{i k_z z} B_{\delta}^{(l)}(\mathcal{E}_4^{(l)}, \mathcal{H}_4^{(l)}),
\end{align*}
\]

and on its right (in vacuum)

\[
\begin{align*}
\mathbf{E}_r, \mathbf{H}_r &= e^{i k_z z} \{ A_{\delta}^{(r)}(\mathcal{E}_1^{(r)}, \mathcal{H}_1^{(r)}) + B_{\delta}^{(r)}(\mathcal{E}_2^{(r)}, \mathcal{H}_2^{(r)}) \} + e^{-i k_{\perp} z} \{ A_{\delta}^{(r)}(\mathcal{E}_3^{(r)}, \mathcal{H}_3^{(r)}) + B_{\delta}^{(r)}(\mathcal{E}_4^{(r)}, \mathcal{H}_4^{(r)}) \},
\end{align*}
\]

where the subscript \(i,o\) denotes incoming (outgoing) wave on the boundary. We have 8 amplitudes, \(A_{\delta}^{(l,o)}, B_{\delta}^{(l,o)}\) to be defined. They are related by the scattering matrix which is to be defined in its turn through boundary (matching) conditions.

\(^a\)The Greek indexes \(\alpha, \beta\) run from 1 to 3 and the Latin ones run from 1 to 2.
The in and out states and S-matrix have the following form: \( \mathbf{E}^{\text{out}} = [A'_0, B'_o, A'_r, B'_r]^T, \mathbf{E}^{\text{in}} = [A'_r, B'_r, A'_i, B'_i]^T, \mathbf{E}^{\text{out}} = \mathbf{S} \cdot \mathbf{E}^{\text{in}}, \) where
\[
\mathbf{S} = \begin{bmatrix} R & T' \\ T & R' \end{bmatrix}, \quad R = \begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}, \quad T = \begin{bmatrix} t_{xx} & t_{xy} \\ t_{yx} & t_{yy} \end{bmatrix}.
\]
\( \mathbf{S} \) matrix, we shall use explicit expression for \( \mathbf{E}_{r,t}, \mathbf{H}_{r,t}, \) obtained in the next Section, and impose on them the following boundary conditions:
\[
(\mathbf{E}^l - \mathbf{E}^r) \times n_{t \rightarrow r} \big|_{z = 0} = 0, \quad (\mathbf{H}^l - \mathbf{H}^r) \times n_{t \rightarrow r} \big|_{z = 0} = 4\pi \sigma \mathbf{E}^r \big|_{z = 0}.
\]

3. Maxwell Equation in Media and the S-matrix

Let us seek the solutions to the Maxwell equations in the plane waves form \( (\mathbf{E}, \mathbf{H}) = e^{ikxz + ik_2y + ik_3z - \omega t} (\mathbf{E}, \mathbf{H}), \) with constant amplitudes \( \mathbf{E} \) and \( \mathbf{H}. \)

The equations can be represented in the form of an eigenproblem \( \mathbf{M} \cdot \mathbf{v} = k_3 \mathbf{v}, \)
where the matrix \( \mathbf{M} \) and \( \mathbf{v} \) are given by \( (\varepsilon_{\alpha \beta} \text{ is minor of element } (\alpha, \beta) \text{ in } \varepsilon) \)
\[
\mathbf{M} = \begin{bmatrix}
  -k_1 \frac{e_{31}}{\varepsilon_{33}} & -k_1 \frac{e_{23}}{\varepsilon_{33}} & k_1 k_2 \frac{e_{33}}{\varepsilon_{33}} & \omega - k_2^2 \frac{e_{33}}{\varepsilon_{33}} \\
  -k_1 \frac{e_{21}}{\varepsilon_{33}} & -k_1 \frac{e_{13}}{\varepsilon_{33}} & -\omega + k_2^2 \frac{e_{33}}{\varepsilon_{33}} & k_1 k_2 \frac{e_{33}}{\varepsilon_{33}} \\
  -k_1 k_2 \frac{e_{12}}{\varepsilon_{33}} & -k_1 k_2 \frac{e_{21}}{\varepsilon_{33}} & k_1^2 \frac{e_{33}}{\varepsilon_{33}} & k_1 \frac{e_{33}}{\varepsilon_{33}} \\
  -\omega k_z + \frac{k_2^2}{\varepsilon_{33}} & \omega k_z + \frac{k_2^2}{\varepsilon_{33}} & \omega k_z - \frac{k_2^2}{\varepsilon_{33}} & -k_1 k_2 \frac{e_{33}}{\varepsilon_{33}}
\end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{E}_x \\ \mathbf{E}_y \\ \mathbf{H}_x \\ \mathbf{H}_y \end{bmatrix}.
\]

The spectrum of the problem, \( k_3 = k_3(k, \omega), \) is solution of the solvability condition of \( (\mathbf{M} - k_3 \mathbf{I}), \) which is a 4th degree equation in \( k_3: \det(\mathbf{M} - k_3 \mathbf{I}) = 0. \) This equation has 4 solutions, \( \kappa_n. \) For vacuum case, \( \varepsilon_{\alpha \beta} = \delta_{\alpha \beta}, \) we obtain two double-degenerate roots \( \kappa_{1,2} = +k_z \) and \( \kappa_{3,4} = -k_z. \) Corresponding eigenvectors read
\[
\mathbf{v}^0_1 = \begin{bmatrix} 1, 0, -k_1 k_2, k_2^2 + k_2 \frac{e_{33}}{\varepsilon_{33}} \end{bmatrix}^T, \quad \mathbf{v}^0_2 = \begin{bmatrix} 0, 1, -k_1 k_2, \omega k_z \end{bmatrix}^T, \quad \mathbf{v}^0_{3,4} = \mathbf{v}^0_{1,2} \big|_{k_z \rightarrow -k_z}.
\]

Then, the general form of the field in vacuum case is a linear combination of these solutions \( \mathbf{v}^0 = e^{ik_z z} (\mathbf{v}^0_1 \mathbf{v}^0_1 + \mathbf{v}^0_2 \mathbf{v}^0_2) + e^{-ik_z z} (\mathbf{v}^0_3 \mathbf{v}^0_3 + \mathbf{v}^0_4 \mathbf{v}^0_4), \) with constants \( \mathbf{v}^0_1. \)

In the non-vacuum case the amplitudes read
\[
\mathbf{v}_1 = \begin{bmatrix}
  \frac{1}{k_3(k_3 k_3 - k_3 k_1) + \omega^2 (k_3 e_{31} - k_1 e_{31})} \\
  \frac{k_3(k_3 k_3 - k_3 k_1) + \omega^2 (k_3 e_{31} - k_1 e_{31}) - k_1 k_2 (k_3 k_3 + k_1 k_2 e_{31} - k_3 e_{31})}{\omega (k_1 k_3 k_3 + k_1 (k_3 - k_3 e_{31}) + \omega^2 (k_1 e_{31} - k_3 e_{31})))} \\
  \frac{k_3 k_3 k_3 - k_3 k_3 + \omega^2 (k_3 e_{31} + k_1 k_2 e_{31} + k_3 e_{31}) - k_1 k_2 (k_3 e_{31} + k_1 k_3 e_{31})}{\omega (k_1 k_3 k_3 + k_1 (k_3 - k_3 e_{31}) + \omega^2 (k_1 e_{31} - k_3 e_{31}))} \\
  \frac{k_3 k_3 k_3 + \omega^2 (k_3 e_{31} + k_1 k_2 e_{31} + k_3 e_{31}) - k_1 k_2 (k_3 e_{31} + k_1 k_3 e_{31})}{\omega (k_1 k_3 k_3 + k_1 (k_3 - k_3 e_{31}) + \omega^2 (k_1 e_{31} - k_3 e_{31}))}
\end{bmatrix},
\]
\[
\mathbf{v}_2 = \begin{bmatrix}
  \frac{1}{k_3(k_3 k_3 - k_3 k_1) + \omega^2 (k_3 e_{31} - k_1 e_{31})} \\
  \frac{-k_1 k_2 (k_3 k_3 + k_1 k_2 e_{31} - k_3 e_{31})}{\omega (k_1 k_3 k_3 + k_1 (k_3 - k_3 e_{31}) + \omega^2 (k_1 e_{31} - k_3 e_{31})))} \\
  \frac{1}{k_3(k_3 k_3 - k_3 k_1) + \omega^2 (k_3 e_{31} + k_1 k_2 e_{31} + k_3 e_{31}) - k_1 k_2 (k_3 e_{31} + k_1 k_3 e_{31})}{\omega (k_1 k_3 k_3 + k_1 (k_3 - k_3 e_{31}) + \omega^2 (k_1 e_{31} - k_3 e_{31})))} \\
  \frac{-k_1 k_2 (k_3 k_3 + k_1 k_2 e_{31} - k_3 e_{31})}{\omega (k_1 k_3 k_3 + k_1 (k_3 - k_3 e_{31}) + \omega^2 (k_1 e_{31} - k_3 e_{31})))}
\end{bmatrix},
\]
general solution is again a linear combination of these 4 solutions $v_{\xi_4}$ where $\xi_4 = k^\alpha \varepsilon_{\alpha \beta}$ and $(k \varepsilon) = k^\alpha k^\beta \varepsilon_{\alpha \beta}$. Also $v_3 = v_1|_{k_3 \to -k_3}$, $v_4 = v_2|_{k_3 \to -k_3}$. General solution is again a linear combination of these 4 solutions $v = \sum_{n=1}^4 e^{i k_n z} v_n v_n$.

We are ready now to solve (2) taking into account boundary conditions (3). With a somewhat cumbersome calculation we obtain the Fresnel reflection matrices

$$R = -\frac{1}{\Delta} \begin{bmatrix} \bar{v}_1 \bar{v}_1 \bar{v}_2 \bar{v}_2 \bar{v}_3 \bar{v}_3 \bar{v}_4 \bar{v}_4 \end{bmatrix} \begin{bmatrix} v_1 v_1 v_2 v_2 v_3 v_3 v_4 v_4 \end{bmatrix}$$

$$R' = -\frac{1}{\Delta'} \begin{bmatrix} \bar{v}_1' \bar{v}_1' \bar{v}_2' \bar{v}_2' \bar{v}_3' \bar{v}_3' \bar{v}_4' \bar{v}_4' \end{bmatrix} \begin{bmatrix} v_1' v_1' v_2' v_2' v_3' v_3' v_4' v_4' \end{bmatrix},$$

where $\Delta = |v_3|^2 v_1^2 \hat{v}_1^2 \hat{v}_2^2|$, $\hat{v}_n' = v_n' + v_n$, $v_n' = (0, -4 \pi i \sigma_2 \sigma, \varepsilon_n')^T$, $v_n' = v_n$ and $\sigma_2$ is Pauli matrix. $\sigma_n$ is the surface conductivity on the interface.

If the matter is on the right and vacuum is on the left, we have the same formulas for scattering matrix (5), where for $v'$ we have to use vacuum case (5), and for $v'$ – expressions for matter (3). With these changes the conductivity appears within vacuum vectors, only. Therefore, the Fresnel reflection matrices read

$$R = -\frac{1}{\Delta} \begin{bmatrix} \bar{v}_1 \bar{v}_1 \bar{v}_2 \bar{v}_2 v_3' v_3' v_4' v_4' \end{bmatrix} \begin{bmatrix} v_1 v_1 v_2 v_2 v_3' v_3' v_4' v_4' \end{bmatrix}$$

$$R' = -\frac{1}{\Delta'} \begin{bmatrix} \bar{v}_1' \bar{v}_1' \bar{v}_2' \bar{v}_2' v_3'^* v_3'^* v_4'^* v_4'^* \end{bmatrix} \begin{bmatrix} v_1' v_1' v_2' v_2' v_3'^* v_3'^* v_4'^* v_4'^* \end{bmatrix},$$

where $\Delta' = |v_3'|^2 v_1'^2 \hat{v}_1'^2 \hat{v}_2'^2|$, $\hat{v}_n'^* = v_n'^* + v_n^*$ and $v_n'^* = v_n$. Also, in general, we have to change $\sigma \to \sigma'$ and $\varepsilon \to \varepsilon'$.

4. Conclusion

We considered the scattering problem for 3D extended system (see Fig. 1). The system consists of two parts characterized by surface conductivities $\sigma$, $\sigma'$ and the bulk dielectric permittivities $\varepsilon$, $\varepsilon'$. The Casimir energy for this system has the form given by Eq. (1), where the Fresnel matrices are given by Eqs. (6) and (7). The limiting cases of uniaxial materials can be shown to coincide with known results [8].

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