Some Notions of (Open) Dynamical System on Polynomial Interfaces

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We define indexed categories of (open) dynamical system and random dynamical system over polynomial interfaces, where time is given by an arbitrary monoid $T$. We consider the case of open random dynamical systems over both open and closed noise sources, and the case where the interface of the random system is ‘nested’ over the interface of its noise source. We show that, in discrete time, our categories of dynamical systems over polynomial interfaces $p$ are equivalent to Spivak’s categories $p$-$\text{Coalg}$ of $p$-coalgebras. We then define a notion of generalized $pT$-coalgebra for a monad $T$, thereby extending the coalgebraic notion of dynamical system to general time, and show that this construction bestows a notion of open Markov process when the monad $T$ is a probability monad. Finally, we speculate on some further connections and open questions.

1 Polynomials for embodiment and interaction

Each system in our universe inhabits some interface or boundary. It receives signals from its environments through this boundary, and can act by changing its shape (and, as we will see later, its position). As a system changes its shape, the set of possible immanent signals might change accordingly: consider a hedgehog rolling itself into a ball, thereby protecting its soft underbelly from harm (amongst other immanent signals). A system may also change its shape by coupling itself to some other system, such as when we pick up chalk to work through a problem. And shapes can be abstract: we change our ‘shapes’ when we enter an online video conference, or move within a virtual reality. We describe all of these interactions formally using polynomial functors, drawing on the work of Spivak [1].

Definition 1.1. Let $\mathcal{E}$ be a locally Cartesian closed category, and denote by $y^A$ the representable copresheaf $y^A := \mathcal{E}(A, -) : \mathcal{E} \to \mathcal{E}$. A polynomial functor $p$ is a coproduct of representable functors, written $p := \sum_{i:p(1)} y^{p_i}$, where $p(1) : \mathcal{E}$ is the indexing object. The category of polynomial functors in $\mathcal{E}$ is the full subcategory $\text{Poly}_\mathcal{E} \hookrightarrow [\mathcal{E}, \mathcal{E}]$ of the $\mathcal{E}$-copresheaf category spanned by coproducts of representables. A morphism of polynomials is therefore a natural transformation.

∗Early draft of work in progress. Comments and suggestions welcome.
Remark 1.2. Every copresheaf $P : \mathcal{E} \to \mathcal{E}$ corresponds to a bundle $p : E \to B$ in $\mathcal{E}$, for which $B = P(1)$ and for each $i : P(1)$, the fibre $p_i$ is $P(i)$. We will henceforth elide the distinction between a copresheaf $P$ and its corresponding bundle $p$, writing $p(1) := B$ and $p[i] := p_i$, where $E = \sum_i p[i]$. A natural transformation $f : p \to q$ between copresheaves therefore corresponds to a map of bundles. In the case of polynomials, by the Yoneda lemma, this map is given by a ‘forwards’ map $f_1 : p(1) \to q(1)$ and a family of ‘backwards’ maps $f^# : q[f_1(-)] \to p[-]$ indexed by $p(1)$, as in the left diagram below. Given $f : p \to q$ and $g : q \to r$, their composite $g \circ f : p \to r$ is as in the right diagram below.

\[
\begin{array}{ccc}
E & \overset{f^#}{\longleftarrow} & f^* F \\
| & \downarrow & \downarrow \delta \\
B & \overset{f_1}{\longrightarrow} & C
\end{array}
\quad
\begin{array}{ccc}
E & \overset{(gf)^#}{\longleftarrow} & f^* G \\
| & \downarrow & \downarrow \delta \\
B & \overset{g \circ f_1}{\longrightarrow} & D
\end{array}
\]

where $(gf)^#$ is given by the $p(1)$-indexed family of composite maps $r[g_1(f_1(-))] \overset{f^#}{\longrightarrow} q[f_1(-)] \overset{f^#}{\longrightarrow} p[-]$.

In our morphological semantics, we will call a polynomial $p$ a phenotype, its base type $p(1)$ its morphology and the total space $\sum_i p[i]$ its sensorium. We will call elements of the morphology shapes or configurations, and elements of the sensorium immanent signals.

Proposition 1.3 (Spivak [1]). There is a monoidal structure $(\text{Poly}_\mathcal{E}, \otimes, y)$ that we interpret as “putting systems in parallel”. Given $p : \sum_i p[i] \to p(1)$ and $q : \sum_j q[j] \to q(1)$, we have $p \otimes q = \sum_i \sum_j p[i] \times q[j] \to p(1) \times q(1)$. $y : 1 \to 1$ is then clearly unital. \qed

Proposition 1.4 (Spivak [1]). The monoidal structure $(\text{Poly}_\mathcal{E}, \otimes, y)$ is closed, with corresponding internal hom denoted $[-, -]$.

We interpret morphisms $(f_1, f^#)$ of polynomials as encoding interaction patterns; in particular, such morphisms encode how composite systems act as unities. For example, a morphism $f : p \otimes q \to r$ specifies how the systems $p$ and $q$ come together to form a system $r$: the map $f_1$ encodes how $r$-configurations are constructed from configurations of $p$ and $q$; and the map $f^#$ encodes how immanent signals on $p$ and $q$ result from signals on $r$ or from the interaction of $p$ and $q$. For intuition, consider two people engaging in a handshake, or an enzyme acting on a protein to form a complex. The internal hom $[o, p]$ encodes all the possible ways that an $o$-phenotype system can “plug into” a $p$-phenotype system.

Remark 1.5. In the literature on active inference and the free energy principle, there is much debate about the concept of ‘Markov blanket’, an informal notion conceived to represent the boundary of an adaptive system. We believe that the algebra of polynomials is sufficient to formalize this concept precisely, and clear up much of the confusion in the literature.

2 Nested systems and dependent polynomials

The polynomial formalism as presented in the previous section suffices to describe systems’ shapes, and behaviours of those shapes that depend on their sensoria. But in our world, a system has a position as well as a shape! Indeed, one might want to consider systems nested within systems, such that the outer systems constitute the ‘universes’ of the inner systems; in this way, inner shapes may depend on outer shapes, and inner sensoria on outer sensoria.\footnote{We might even consider the outer shapes explicitly as positions in some world-space, and the outer sensorium as determined by possible paths between positions, in agreement with the perspective of Spivak [1] on polynomials.} We can model this situation polynomially.
Recall that an object in $\text{Poly}_E$ corresponds to a bundle $E \to B$, equivalently a diagram $1 \leftarrow E \xrightarrow{p} B \to 1$, and note that the unit polynomial $y$ corresponds to a bundle $1 \to 1$. We can then think of $\text{Poly}_E$ as the category of “polynomials in one variable”, or “polynomials over $y$”. This presents a natural generalization, to polynomials in many variables, corresponding to diagrams $J \leftarrow E \to B \to I$; these diagrams form the objects of a category $\text{Poly}_E(J, I)$. When $J$ is a (polynomial) bundle $\beta$ over $I$, then we can take the subcategory of $\text{Poly}_E(J, I)$ whose objects are commuting squares and whose morphisms are prisms as follows; the commutativity ensures that inner and outer sensoria are compatible.

**Proposition 2.1.** There is an indexed category of nested polynomials which by abuse of notation we will call $\text{Poly}_E(-) : \text{Poly}_E \to \text{Cat}$. Given $\beta : J \to I$, the category $\text{Poly}_E(\beta)$ has commuting squares as on the left below as objects and prisms as on the right as morphisms. Its action on polynomial morphisms $\beta \to \gamma$ is given by composition.

\[
\begin{array}{ccc}
E & \xleftarrow{f^*F} & J \\
\downarrow & & \downarrow \\
B & \xleftarrow{F} & I
\end{array}
\]

**Proposition 2.2.** The base category of polynomials $\text{Poly}_E$ is isomorphic to its image over the trivial polynomial $y$: $\text{Poly}_E \cong \text{Poly}_E(y)$.

**Remark 2.3.** This construction can be repeatedly iterated, modelling systems within systems within systems. We leave the consideration of the structure of this iteration to future work, though we expect it to have a “mutually coinductive” type and an opetopic shape, perhaps equivalent to that obtained by iterating the $\text{Para}$ construction.

### 3 Dynamical systems with polynomial interfaces

We describe categories of dynamical systems. We begin with classical ‘closed’ deterministic systems, before defining closed measure-preserving systems and closed random dynamical systems: we can think of random dynamical systems as deterministic systems parameterized by some dynamical noise source. More precisely, a random dynamical system will be a bundle of a dynamical system over a measure-preserving dynamical system on some probability space.

We then progressively open these various system types up to polynomial interaction: first, general (deterministic) open dynamical systems; proceeded by open random dynamical systems over closed bases, open measure-preserving systems, and then “fully open” random dynamical systems (with open measure-preserving bases). Finally, we consider the case where an open random dynamical system is defined on a nested polynomial interface: that is, where the interface of the total random system is nested in the interface of the base measure-preserving system. This formalizes the idea that randomness comes from some noise source that is “really external” to the random system, in such a way that it pervades the random system’s environment: think for instance of the cosmic microwave background in our own universe. In order for this randomness to be compatible with any change-of-scale implied by the nesting of open systems, we also require that the dynamics be accordingly compatible. At the end of the section, we consider the connection between our categories of dynamical systems and $p$-coalgebras defined internally to $\text{Poly}_E$. 

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Definition 3.1. Let \((\mathbb{T}, +, 0)\) be a monoid, representing time. Let \(X : \mathcal{E}\) be some space, called the state space. Then a closed dynamical system \(\vartheta\) with state space \(X\) and time \(\mathbb{T}\) is an action of \(\mathbb{T}\) on \(X\) satisfying a flow condition. When \(\mathbb{T}\) is also an object of \(\mathcal{E}\), then this amounts to a morphism \(\vartheta : \mathbb{T} \times X \to X\) (or equivalently, a time-indexed family of \(X\)-endomorphisms, \(\vartheta(t) : X \to X\)), such that \(\vartheta(0) = \text{id}_X\) and \(\vartheta(s + t) = \vartheta(s) \circ \vartheta(t)\).

We will call these criteria the flow condition for \(\vartheta\).

Proposition 3.2. When time is discrete, as in the case \(\mathbb{T} = \mathbb{N}\), any dynamical system \(\vartheta\) is entirely determined by its action at \(1 : \mathbb{T}\). That is, letting the state space be \(X\), we have \(\vartheta(t) = \vartheta(1)^{\circ t}\) where \(\vartheta(1)^{\circ t}\) means "compose \(\vartheta(1) : X \to X\) with itself \(t\) times".

Proof. The proof is by (co)induction on \(t : \mathbb{T}\). We must have \(\vartheta(0) = \text{id}_X\) and \(\vartheta(t + s) = \vartheta(t) \circ \vartheta(s)\). So for any \(t\), we must have \(\vartheta(t + 1) = \vartheta(t) \circ \vartheta(1)\). The result follows immediately; note for example that \(\vartheta(2) = \vartheta(1 + 1) = \vartheta(1) \circ \vartheta(1)\).

Example 3.3. Suppose \(X : M \to TM\) is a vector field on \(M\), with a corresponding solution (integral curve) \(\chi_x : \mathbb{R} \to M\) for all \(x : M\); that is, \(\chi'(t) = X(\chi_x(t))\) and \(\chi_x(0) = x\). Then letting the point \(x\) vary, we obtain a map \(\chi : \mathbb{R} \times M \to M\). This \(\chi\) is a closed dynamical system with state space \(M\) and time \(\mathbb{R}\).

Proposition 3.4. Closed dynamical systems with state spaces in \(\mathcal{E}\) and time \(\mathbb{T}\) are the objects of the functor category \(\text{Cat}(\mathcal{B}\mathbb{T}, \mathcal{E})\), where \(\mathcal{B}\mathbb{T}\) denotes the delooping of the monoid \(\mathbb{T}\). Morphisms of dynamical systems are therefore natural transformations.

Proof. The category \(\mathcal{B}\mathbb{T}\) has a single object \(*\) and morphisms \(t : * \to *\) for each point \(t : \mathbb{T}\); the identity is the monoidal unit \(0 : \mathbb{T}\) and composition is given by \(+\). A functor \(\vartheta : \mathcal{B}\mathbb{T} \to \mathcal{E}\) therefore picks out an object \(\vartheta(*) : \mathcal{E}\), and, for each \(t : \mathbb{T}\), a morphism \(\vartheta(t) : \vartheta(*) \to \vartheta(*)\), such that the functoriality condition is satisfied. Functoriality requires that identities map to identities and composition is preserved, so we require that \(\vartheta(0) = \text{id}_{\vartheta(*)}\) and that \(\vartheta(s + t) = \vartheta(s) \circ \vartheta(t)\). Hence the data for a functor \(\vartheta : \mathcal{B}\mathbb{T} \to \mathcal{E}\) amount to the data for a closed dynamical system in \(\mathcal{E}\) with time \(\mathbb{T}\), and the functoriality condition amounts precisely to the flow condition. A morphism of closed dynamical systems \(f : \vartheta \to \psi\) is a map on the state spaces \(f : \vartheta(*) \to \psi(*)\) that commutes with the flow, meaning that \(f\) satisfies \(f \circ \vartheta(t) = \psi(t) \circ f\) for all times \(t : \mathbb{T}\); this is precisely the definition of a natural transformation \(f : \vartheta \to \psi\) between the corresponding functors.

We now consider closed random dynamical systems.

Definition 3.5. Suppose \(\mathcal{E}\) is a category equipped with a probability monad \(\mathcal{P} : \mathcal{E} \to \mathcal{E}\) and a terminal object \(1 : \mathcal{E}\). A probability space in \(\mathcal{E}\) is an object of the slice \(1/\mathcal{K}(\mathcal{P})\) of the Kleisli category of the probability monad under \(1\). Explicitly, a probability space is therefore equivalently a pair \((B, \beta)\) where \(B : \mathcal{E}\) is an object and \(\beta : 1 \to \mathcal{P}B\) is a measure over \(B\). Morphisms \(f : (A, \alpha) \to (B, \beta)\) between probability spaces are stochastic channels \(f : A \to \mathcal{P}B\) that preserve the measure; that is, they satisfy \(f \bullet \alpha = \beta\).

Proposition 3.6. There is a forgetful functor \(1/\mathcal{K}(\mathcal{P}) \to \mathcal{E}\) taking probability spaces \((B, \beta)\) to the underlying spaces \(B\) and their morphisms \(f : (A, \alpha) \to (B, \beta)\) to the underlying maps \(f : A \to \mathcal{P}B\). We will write \(B\) to refer to the space in \(\mathcal{E}\) underlying a probability space \((B, \beta)\), in the image of this forgetful functor.

Definition 3.7. Let \((B, \beta)\) be a probability space in \(\mathcal{E}\). A closed metric or measure-preserving dynamical system \((\vartheta, \beta)\) on \((B, \beta)\) with time \(\mathbb{T}\) is a closed dynamical system \(\vartheta\) with state space \(B : \mathcal{E}\) such that, for all \(t : \mathbb{T}\), \(\mathcal{P} \vartheta(t) \circ \beta = \beta\); that is, each \(\vartheta(t)\) is a \((B, \beta)\)-endomorphism in \(1/\mathcal{K}(\mathcal{P})\).
Proposition 3.8. Closed measure-preserving dynamical systems in $\mathcal{E}$ with time $\mathbb{T}$ form the objects of a category $\text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})_\mathcal{P}$ whose morphisms $f : (\vartheta, \alpha) \to (\psi, \beta)$ are maps $f : \vartheta(\ast) \to \psi(\ast)$ in $\mathcal{E}$ between the state spaces that preserve both flow and measure, as in the following commutative diagram, which also indicates their composition:

![Diagram]

Proof. The identity morphism on a closed measure-preserving dynamical system is the identity map on its state space. It is easy to check that composition as in the diagram above is thus both associative and unital with respect to these identities.

Proposition 3.9. There is a forgetful functor $U : \text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})_\mathcal{P} \to \text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})$ which simply forgets the probability space structures. Given a closed measure-preserving dynamical system $\vartheta, \beta$, we will write $\vartheta$ for the closed dynamical system in the image of this forgetful functor.

Definition 3.10. Let $(\vartheta, \beta)$ be a closed measure-preserving dynamical system. A closed random dynamical system over $(\vartheta, \beta)$ is an object of the slice category $\text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})/\vartheta$; it is therefore a bundle of the corresponding functors.

Proposition 3.11. The indexing of categories of closed random dynamical systems $\text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})/\vartheta$ by their base measure-preserving systems $(\vartheta, \beta)$ constitutes an indexed category $\text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})_\mathcal{P}^{\text{op}} \to \text{Cat}$ taking closed measure-preserving systems to the categories of closed random dynamical systems (bundles of functors) above them, and morphisms of closed measure-preserving systems to the corresponding base-change functors. This indexed category is therefore defined as $\text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})/U^{\text{op}}(-)$ where $U^{\text{op}} : \text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})_\mathcal{P}^{\text{op}} \to \text{Cat}(\mathbb{B}\mathcal{T}, \mathcal{E})^{\text{op}}$ is the opposite of the forgetful functor in Proposition 3.9.

Example 3.12. The solutions $X(t, \omega; x_0) : \mathbb{R}_+ \times \Omega \times M \to M$ to a stochastic differential equation $dX_t = f(t, X_t)dt + \sigma(t, X_t)dW_t$, where $W : \mathbb{R}_+ \times \Omega \to M$ is a Wiener process in $M$, define a random dynamical system $\mathbb{R}_+ \times \Omega \times M \to M : (t, \omega, x) \mapsto X(t, \omega; x_0)$ over the Wiener base flow $\theta : \mathbb{R}_+ \times \Omega \to \Omega : (t, \omega) \mapsto W(s + t, \omega) - W(t, \omega)$ for any $s : \mathbb{R}_+$. We can alternatively represent this system as a bundle system over $(\theta, \gamma)$, where $\gamma$ is the Wiener measure on the Wiener space $\Omega$, by writing $\vartheta : \mathbb{R}_+ \times \Omega \times M \to \Omega \times M : (t, \omega, x) \mapsto (\theta(t, \omega), X(t, \omega; x_0))$. This gives a closed random dynamical system in $\text{Cat}(\mathbb{B}\mathbb{R}_+, \mathcal{E})/\theta$.

Example 3.13. In discrete time ($\mathbb{T} = \mathbb{N}$), closed random dynamical systems (with independent-increment noise) correspond to Markov chains, for the same reason that general closed discrete-time dynamical systems correspond to transition functions (Proposition 3.2). Let $\vartheta$ be a closed discrete-time random dynamical system over $(\theta, \gamma)$, and suppose the state space of $\vartheta$ corresponds to a trivial bundle $\pi : \Omega \times M \to \Omega$. Since the systems $\vartheta$ and $\theta$ correspond to transition functions, we have $\vartheta^\ast : \Omega \times M \to \Omega \times M$ and $\theta^\ast : \Omega \to \Omega$. By the universal property of the product, the map $\vartheta^\ast : \Omega \times M \to \Omega \times M$ corresponds to a pair of maps $\vartheta^\Omega : \Omega \times M \to \Omega$ and $\vartheta^M : \Omega \times M \to M$, but the former component must coincide with $\theta^\ast$, since $\vartheta$ is a bundle over $\theta$. At each time step $n : \mathbb{N}$, the noise is distributed according to $\gamma : 1 \to \mathcal{P}\Omega$. Pushing $\gamma$...
forward along \( \psi^M \) induces a \( \mathcal{P} \)-coalgebra, \( \psi^\gamma : M \to \mathcal{P} M \), which is precisely a Markov chain. Conversely, note that, by randomness pushback [2, Def. 11.19], any such map \( \tau : M \to \mathcal{P} M \) canonically induces a pair \( (\tau^\gamma : \Omega \times M \to M, \gamma : 1 \to \mathcal{P} \Omega) \) for some probability space \((\Omega, \gamma)\), and one can construct a random dynamical system accordingly; see Arnold [3, Theorem 2.1.6].

**Definition 3.14.** Let \( p : \text{Poly}_E \) be a polynomial in \( E \), to be called the **interface**. Let \( S : E \) be an object, to be called the **state space**. Let \((T, +, 0)\) be a monoid, representing time. An open dynamical system on the interface \( p \) with state space \( S \) and time \( T \) consists in a pair of morphisms \( \vartheta^o : T \times S \to p(1) \) and \( \vartheta^u : \sum_{t \in T} \sum_{s \in S} p[\vartheta^o(-, s)] \to S \), such that, for any global section \( \sigma : p(1) \to \sum_{i : p(1)} p[i] \) of \( p \), the maps \( \vartheta^\sigma : T \times S \to S \) given by

\[
\sum_{t \in T} \sum_{s \in S} p[\vartheta^o(-, s)] \to S
\]

constitute a closed dynamical system, i.e., an object in \( \text{Cat}(BT, E) \). That is, the maps \( \vartheta^\sigma \) must satisfy the flow conditions, that \( \vartheta^o(0) = id_S \) and \( \vartheta^o(s + t) = \vartheta^o(s) \circ \vartheta^\sigma(t) \). We collect the data of such a dynamical system over \( p \) into a tuple \( \vartheta = (S, \vartheta^o, \vartheta^u) \). We call the closed system \( \vartheta^\sigma \), induced by a section \( \sigma \) of \( p \), the **closure** of \( \vartheta \) by \( \sigma \).

**Proposition 3.15.** Open dynamical systems over \( p \) with time \( T \) form a category, denoted \( \text{Dyn}^T(p) \). Its morphisms are defined as follows. Let \( \vartheta := (X, \vartheta^o, \vartheta^u) \) and \( \psi := (Y, \psi^o, \psi^u) \) be two dynamical systems over \( p \). A morphism \( f : \vartheta \to \psi \) consists in a morphism \( f : X \to Y \) such that, for any time \( t : T \) and global section \( \sigma : p(1) \to \sum_{i : p(1)} p[i] \) of \( p \), the following naturality squares commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\vartheta^o(t) \cdot \sigma} & \sum_{x : X} p[\vartheta^o(t, x)] & \xrightarrow{\vartheta^o(t)} & X \\
\downarrow{f} & & \downarrow{f} & & \\
Y & \xrightarrow{\psi^o(t) \cdot \sigma} & \sum_{y : Y} p[\psi^o(t, y)] & \xrightarrow{\psi^o(t)} & Y
\end{array}
\]

The identity morphism \( id_\vartheta \) on the dynamical system \( \vartheta \) is given by the identity morphism \( id_X \) on its state space \( X \). Composition of morphisms of dynamical systems is given by composition of the morphisms of the state spaces.

**Proof.** We need to check unitality and associativity of composition. This amounts to checking that the composite naturality squares commute. But this follows immediately, since the composite of two commutative diagrams along a common edge is again a commutative diagram. \( \square \)

**Proposition 3.16.** \( \text{Dyn}^T(p) \) extends to a polynomially-indexed category, \( \text{Dyn}^T : \text{Poly}_E \to \text{Cat} \). Suppose \( \varphi : p \to q \) is a morphism of polynomials. We define a corresponding functor \( \text{Dyn}^T(\varphi) : \text{Dyn}^T(p) \to \text{Dyn}^T(q) \) as follows. Suppose \((X, \vartheta^o, \vartheta^u) : \text{Dyn}^T(p) \) is an object (dynamical system) in \( \text{Dyn}^T(p) \). Then \( \text{Dyn}^T(\varphi)(X, \vartheta^o, \vartheta^u) \) is defined as the triple \((X, \vartheta^o \circ \vartheta^o, \vartheta^o \circ \vartheta^o \circ \vartheta^o) : \text{Dyn}^T(q) \), where the two maps are explicitly the following composites:

\[
T \times X \xrightarrow{\vartheta^o} p(1) \xrightarrow{\varphi_1} q(1), \quad \sum_{t \in T} \sum_{x : X} q[\varphi_1 \circ \vartheta^o(t, x)] \xrightarrow{\vartheta^o \circ \vartheta^o} \sum_{t \in T} \sum_{x : X} p[\vartheta^o(t, x)] \xrightarrow{\vartheta^u} X.
\]

On morphisms, \( \text{Dyn}^T(\varphi)(f) : \text{Dyn}^T(\varphi)(X, \vartheta^o, \vartheta^u) \to \text{Dyn}^T(\varphi)(Y, \psi^o, \psi^u) \) is given by the same underlying map \( f : X \to Y \) of state spaces.
Proof. We need to check that $\text{Dyn}^T(\varphi)(X, \vartheta^o, \vartheta^u)$ satisfies the flow conditions of Definition 3.14, that $\text{Dyn}^T(\varphi)(f)$ satisfies the naturality condition of Proposition 3.15, and that $\text{Dyn}^T$ is functorial with respect to polynomials. We begin with the flow condition. Given a section $\tau : q(1) \to \sum_{j:q(1)} q[j]$ of $q$, we require the closures $\text{Dyn}^T(\varphi)(\vartheta)^\tau : T \times X \to X$ given by

$$
\sum_{t:T} X \xrightarrow{\vartheta^o(-)^\tau} t:T \sum_{x:X} q[\varphi_1 \circ \vartheta^o(t, x)] \xrightarrow{\vartheta^o^* \varphi^#} \sum_{t:T} p[\vartheta^o(t, x)] \xrightarrow{\vartheta^u} X
$$

to satisfy $\text{Dyn}^T(\varphi)(\vartheta)^\tau(0) = \text{id}_X$ and $\text{Dyn}^T(\varphi)(\vartheta)^\tau(s + t) = \text{Dyn}^T(\varphi)(\vartheta)^\tau(s) \circ \text{Dyn}^T(\varphi)(\vartheta)^\tau(t)$. Note that the following diagram commutes, by the definition of $\varphi^#$,

$$
\begin{array}{ccc}
\sum_{i:p(1)} p[i] & \xrightarrow{\varphi^#} & \sum_{i:p(1)} q[\varphi_1(i)] & \xrightarrow{\varphi^1_\tau} & p(1) \\
\downarrow & & \downarrow & & \\
p & \xrightarrow{\varphi^1_q} & p(1) & \xrightarrow{\varphi^1_q} & p(1)
\end{array}
$$

so that $\varphi^# \circ \varphi^1_\tau$ is a section of $p$. Therefore, letting $\sigma := \varphi^# \circ \varphi^1_\tau$, for $\text{Dyn}^T(\varphi)(\vartheta)^\tau$ to satisfy the flow condition for $\tau$ reduces to $\vartheta^o$ satisfying the flow condition for $\sigma$. But this is given ex hypothesi by Definition 3.14, for any such section $\sigma$, so $\text{Dyn}^T(\varphi)(\vartheta)^\tau$ satisfies the flow condition for $\tau$. And since $\tau$ was any section, we see that $\text{Dyn}^T(\varphi)(\vartheta)$ satisfies the flow condition generally.

The proof that $\text{Dyn}^T(\varphi)(f)$ satisfies the naturality condition of Proposition 3.15 proceeds similarly. Supposing again that $\tau$ is any section of $q$, we require the following diagram to commute for any time $t : T$:

$$
\begin{array}{ccc}
X & \xrightarrow{\vartheta^o(t)^* \varphi^1_\tau} & \sum_{x:X} q[\varphi_1 \circ \vartheta^o(t, x)] \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{\psi^o(t)^* \varphi^1_\tau} & \sum_{y:Y} q[\varphi_1 \circ \psi^o(t, x)]
\end{array}
$$

Again letting $\sigma := \varphi^# \circ \varphi^1_\tau$, we see that this diagram reduces exactly to the diagram in Proposition 3.15 by the functoriality of pullback, and since $f$ makes that diagram commute, it must also make this diagram commute.

Finally, to show that $\text{Dyn}^T$ is functorial with respect to polynomials amounts to checking that composition and pullback are functorial; but this is a basic result of category theory. \qed

To confirm that our definition of open dynamical system subsumes the classical case of ‘closed’ dynamical systems, we now consider the case of dynamical systems on the trivial interface $y$: such a system has a trivial shape, exposing no configuration to any environment nor receiving any signals from it.

**Proposition 3.17.** $\text{Dyn}^T(y)$ is equivalent to the classical category $\text{Cat}(B^T, \mathcal{E})$ of closed dynamical systems in $\mathcal{E}$ with time $T$ (cf. Proposition 3.4).

**Proof.** The trivial interface $y$ corresponds to the trivial bundle $\text{id}_1 : 1 \to 1$. Therefore, a dynamical system over $y$ consists of a choice of state space $S$ along with a trivial output map $\vartheta^u : \text{id}_1 : T \times S \to 1$ and a time-indexed update map $\vartheta^o : T \times S \to S$. This therefore has the form of a classical closed dynamical system,
so it remains to check the flow condition. There is only one section of $id_1$, which is again $id_1$. Pulling this back along the unique map $\theta^o(t) : S \to 1$ gives $\theta^o(t)^*id_1 = id_S$. Therefore the requirement that, given any section $\sigma$ of $y$, each map $\theta^o \circ \theta^o(t)^*\sigma$ satisfies the flow condition reduces to the classical requirement that $\theta^o : \mathbb{T} \times S \to S$ satisfies the flow condition. Since the pullback of the unique section $id_1$ along the trivial output map $\theta^o(t) = \mathbb{T} : S \to 1$ of any dynamical system in $\text{Dyn}^\mathbb{T}(y)$ is the identity of the corresponding state space $id_S$, a morphism $f : (\theta^o, \theta^u, \mathbb{T}) \to (\psi^o, \psi^u, \mathbb{T})$ in $\text{Dyn}^\mathbb{T}(y)$ amounts precisely to a map $f : \theta^o \to \psi^o$ on the state spaces in $\mathcal{E}$ such that the naturality condition $f \circ \theta^o(t) = \psi^o(t) \circ f$ of Proposition 3.4 is satisfied, and every morphism in $\text{Cat}(\mathcal{B\mathbb{T}}, \mathcal{E})$ corresponds to a morphism in $\text{Dyn}^\mathbb{T}(y)$ in this way.

Example 3.18. We now consider the case of a dynamical system $(S, \theta^o, \theta^u)$ with outputs but no inputs. Such a system lives over a polynomial bundle $p : p(1) \longrightarrow p(1)$ that is an isomorphism. A section of this bundle must therefore be its inverse $p^{-1}$, and so, $\theta^o(t)^*p^{-1} = id_{p(1)}$. Once again, the update map corresponds to a dynamical system in $\text{Cat}(\mathcal{B\mathbb{T}}, \mathcal{E})$; just now we have outputs $\theta^o : \mathbb{T} \times S \to p(1)$ exposed to the environment.

**Proposition 3.19.** When time is discrete, as with $\mathbb{T} = \mathbb{N}$, any open dynamical system $(X, \theta^o, \theta^u)$ over $p$ is entirely determined by its components at $1 : \mathbb{T}$. That is, we have $\theta^o(t) = \theta^o(1) : X \to p(1)$ and $\theta^o(t) = \theta^o(1) : \sum_{x,X} p[\theta^o(x)] \to X$. A discrete-time open dynamical system is therefore a triple $(X, \theta^o, \theta^u)$, where the two maps have types $\theta^o : X \to p(1)$ and $\theta^u : \sum_{x,X} p[\theta^o(x)] \to X$.

**Proof.** Suppose $\sigma$ is a section of $p$. We require each closure $\theta^o$ to satisfy the flow conditions, that $\theta^o(0) = id_X$ and $\theta^o(t+s) = \theta^o(t) \circ \theta^o(s)$. In particular, we must have $\theta^o(t+1) = \theta^o(t) \circ \theta^o(1)$. By induction, this means that we must have $\theta^o(t) = \theta^o(1)^{ot}$ (compare Proposition 3.2). Therefore we must in general have $\theta^o(t) = \theta^o(1)$ and $\theta^u(t) = \theta^u(1)$.

Example 3.20. Suppose $\dot{x} = f(x, a)$ and $b = g(x)$, with $f : X \times A \to TX$ and $g : X \to B$. Then, as for the ‘closed’ vector fields of Example 3.3, this induces an open dynamical system $(X, f, g) : \text{Dyn}^\mathbb{R}(By^A)$, where $\int f : \mathbb{R} \times X \times A \to X$ returns the $(X, A)$-indexed solutions of $f$.

Example 3.21. The preceding example is easily extended to the case of a general polynomial interface. Suppose similarly that $\dot{x} = f(x, a, \theta)$ and $b = g(x)$, now with $f : \sum_{x,X} p[g(x)] \to TX$ and $g : X \to p(1)$. Then we obtain an open dynamical system $(X, f, g) : \text{Dyn}^\mathbb{R}(p)$, where now $\int f : \mathbb{R} \times \sum_{x,X} p[g(x)] \to X$ is the ‘update’ and $g : X \to p(1)$ the ‘output’ map.

We now move on to define open random dynamical systems. We do so in stages, starting with defining open random dynamical systems over closed base measure-preserving systems, and then opening up the base systems to form “fully open” random dynamical systems. We then ask for the base and total open systems to be compatible with appropriately nested corresponding polynomial interfaces, resulting in a categorical notion of random nested dynamical system.

**Definition 3.22.** Let $(\theta, \beta)$ be a closed measure-preserving dynamical system in $\mathcal{E}$ with time $\mathbb{T}$, and let $p : \text{Poly}_\mathcal{E}$ be a polynomial in $\mathcal{E}$. Write $\Omega := \theta(s)$ for the state space of $\theta$, and let $\pi : S \to \Omega$ be an object (bundle) in $\mathcal{E}/\Omega$. An open random dynamical system over $(\theta, \beta)$ on the interface $p$ with state space $\pi : S \to \Omega$ and time $\mathbb{T}$ consists in a pair of morphisms $\theta^o : \mathbb{T} \times S \to p(1)$ and $\theta^u : \sum_{s:S} p[\theta^o(t, s)] \to S$, such that, for any global section $\sigma : p(1) \to \sum_{i:p(1)} p[i]$ of $p$, the maps $\theta^o : \mathbb{T} \times S \to S$ defined as

$$\sum_{t: \mathbb{T}} S \frac{\theta^o(t)^* \sigma}{t: \mathbb{T}} \sum_{s:S} p[\theta^o(t, s)] \to S \frac{\theta^o}{\mathbb{T}} \to S$$
form a closed random dynamical system in \( \text{Cat}(\mathbb{B}_T, \mathcal{E})/\theta \). That is to say, for all \( t : T \) and sections \( \sigma \), the following naturality square commutes:

\[
\begin{array}{ccc}
S & \xrightarrow{\vartheta^o(t) \sigma} & \sum_{s:S} p[\vartheta^o(t, s)] \\
\pi \downarrow & & \downarrow \vartheta^o(t) \\
\Omega & \xrightarrow{\vartheta(t)} & \Omega
\end{array}
\]

We collect the data of such a dynamical system into a tuple \( \vartheta = (\pi, \vartheta^o, \vartheta^u) \). Given a section \( \sigma \) of \( p \), the induced closed system \( \vartheta^\sigma \) will again be called the \textbf{closure} of \( \vartheta \) by \( \sigma \).

**Proposition 3.23.** Let \( (\theta, \beta) \) be a closed measure-preserving dynamical system in \( \mathcal{E} \) with time \( T \), and let \( p : \text{Poly}_\mathcal{E} \) be a polynomial in \( \mathcal{E} \). Open random dynamical systems over \( (\theta, \beta) \) on the interface \( p \) form the objects of a category \( \text{RDyn}^T(p, \theta) \). Writing \( \vartheta := (\pi_X, \vartheta^o, \vartheta^u) \) and \( \psi := (\pi_Y, \psi^o, \psi^u) \), a morphism \( \vartheta \to \psi \) is a map \( f : X \to Y \) in \( \mathcal{E} \) making the following diagram commute for all times \( t : T \) and sections \( \sigma \) of \( p \):

\[
\begin{array}{ccc}
X & \xrightarrow{\vartheta^o(t) \sigma} & \sum_{x:X} p[\vartheta^o(t, x)] \\
\pi \downarrow & & \downarrow \vartheta^u(t) \\
\Omega & \xrightarrow{\vartheta(t)} & \Omega
\end{array}
\]

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi^o(t) \sigma} & \sum_{y:Y} p[\psi^o(t, y)] \\
\pi \downarrow & & \downarrow \psi^u(t) \\
\Omega & \xrightarrow{\psi(t)} & \Omega
\end{array}
\]

Identities are given by the identity maps on state-spaces. Composition is given by pasting of diagrams.

**Proposition 3.24.** The categories \( \text{RDyn}^T(p, \theta) \) collect into a doubly-indexed category of the form \( \text{RDyn}^T : \text{Poly}_\mathcal{E} \times \text{Cat}(\mathbb{B}_T, \mathcal{E}) \to \text{Cat} \). By the universal property of the product \( \times \) in \( \text{Cat} \), it suffices to define the actions of \( \text{RDyn}^T \) separately on morphisms of polynomials and on morphisms of closed measure-preserving systems.

Suppose therefore that \( \varphi : p \to q \) is a morphism of polynomials. Then, for each measure-preserving system \( (\theta, \beta) : \text{Cat}(\mathbb{B}_T, \mathcal{E}) \), we define the functor \( \text{RDyn}^T(\varphi, \theta) : \text{RDyn}^T(p, \theta) \to \text{RDyn}^T(q, \theta) \) as follows. Let \( \vartheta := (\pi_X : X \to \Omega, \vartheta^o, \vartheta^u) : \text{RDyn}^T(p, \theta) \) be an object (open random dynamical system) in \( \text{RDyn}^T(p, \theta) \). Then, as in Proposition 3.16, \( \text{RDyn}^T(\varphi, \theta)(\vartheta) \) is defined as the triple \( (\pi_X, \varphi_1 \circ \vartheta^o, \vartheta^u \circ \varphi^o \circ \varphi^#) : \text{RDyn}^T(q, \theta) \), where the two maps are explicitly the following composites:

\[
T \times X \xrightarrow{\vartheta^o} q(1) \xrightarrow{\varphi_1} q(1) , \quad \sum_{t:T} \sum_{x:X} q[\varphi_1 \circ \vartheta^o(t, x)] \xrightarrow{\varphi^o \circ \varphi^#} \sum_{t:T} \sum_{x:X} p[\vartheta^o(t, x)] \xrightarrow{\vartheta^u} X .
\]

Again as in Proposition 3.16, on morphisms \( f : (\pi_X : X \to \Omega, \vartheta^o, \vartheta^u) \to (\pi_Y : Y \to \Omega, \psi^o, \psi^u) \), the image \( \text{RDyn}^T(\varphi, \theta)(f) : \text{RDyn}^T(\varphi, \theta)(\pi_X, \vartheta^o, \vartheta^u) \to \text{RDyn}^T(\varphi, \theta)(\pi_Y, \psi^o, \psi^u) \) is given by the same underlying map \( f : X \to Y \) of state spaces.

Next, suppose that \( \phi : (\theta, \beta) \to (\theta', \beta') \) is a morphism of closed measure-preserving dynamical systems, and let \( \Omega' := \theta'^*(\theta) \) be the state space of the system \( \theta' \). By Proposition 3.8, the morphism \( \phi \) corresponds to a map \( \phi : \Omega \to \Omega' \) on the state spaces that preserves both flow and measure. Therefore, for each polynomial \( p : \text{Poly}_\mathcal{E} \), we define the functor \( \text{RDyn}^T(p, \phi) : \text{RDyn}^T(p, \theta) \to \text{RDyn}^T(p, \theta') \) by post-composition. That
is, suppose given open random dynamical systems and morphisms over \((p,\theta)\) as in the diagram of Proposition 3.23. Then \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)\) returns the following diagram:

That is, \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)(\theta) := (\phi \circ \pi_X, \vartheta^o, \vartheta^u)\) and \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)(f)\) is given by the same underlying map \(f : X \to Y\) on state spaces.

**Proof.** We need to check: the naturality condition of Definition 3.22 for both \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(\vartheta)(\theta)\) and \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)(\theta)\); functoriality of \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)\) and \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)\); and (pseudo)functoriality of \(\mathsf{RDyn}_T^\mathcal{T}\) with respect to both morphisms of polynomials and of closed measure-preserving systems.

We begin by checking that the conditions of Definition 3.22 and Proposition 3.16 are satisfied by the objects \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(\pi_X,\vartheta^o, \vartheta^u) : \mathsf{RDyn}_T^\mathcal{T}(q,\theta)\) and morphisms \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(f) : \mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(\pi_X,\vartheta^o, \vartheta^u) \to \mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(\pi_Y,\psi^o, \psi^u)\) in the image of \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)\). We proceed similarly to the proof of Proposition 3.16. Therefore, given a section \(\tau : q(1) \to \sum_{j,q(1)} q[j]\) of \(q\), we need to check that the closure \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(\vartheta)(\theta)^\tau\) forms a closed random dynamical system in \(\mathsf{Cat}(\mathcal{B}\mathcal{T},\mathcal{E})/\theta\). That is to say, for all \(t : T\) and sections \(\tau\), we need to check that the following naturality square commutes:

As before, we find that \(\varphi^\# \circ \varphi^\tau\) is a section of \(p\), so that commutativity of the diagram above reduces to commutativity of the diagram in Definition 3.22. Similarly, given a morphism \(f : (\pi_X,\vartheta^o, \vartheta^u) \to (\pi_Y,\psi^o, \psi^u)\), we need to check that the diagram in Proposition 3.23 induced for \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(f)\) commutes for all times \(t : T\) and sections \(\tau\) of \(q\). But as in the proof of Proposition 3.16, given such a section \(\tau\), the diagram for \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)(f)\) reduces to that for \(f\) and the section \(\varphi^\# \circ \varphi^\tau\) of \(p\), which commutes ex hypothesi; and functoriality of \(\mathsf{RDyn}_T^\mathcal{T}(\varphi,\theta)\) follows immediately.

Next, we check that the conditions of Definition 3.22 and Proposition 3.16 are satisfied in the image of \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)\). It is clear by the definition of the action of \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)\) that the condition that the diagram in Proposition 3.16 commutes is satisfied, from which it follows by pasting that \(\mathsf{RDyn}_T^\mathcal{T}(p,\phi)\) is functorial. We therefore just have to check the induced diagram in Definition 3.22 commutes. Consider the following
case (Proposition 3.17), the category of random dynamical system over a closed measure-preserving dynamical system and a polynomial makes its type functorial with respect to morphisms of polynomials and morphisms of closed measure-preserving dynamical systems (Proposition 3.8), and the outer square is the induced diagram we need to check, which therefore commutes by the pasting of commuting squares.

Finally, we check that $\mathcal{RDyn}^T$ is functorial with respect to morphisms of polynomials and morphisms of closed measure-preserving dynamical systems. As in the proof of Proposition 3.16, these reduce to checking that pullback and composition are functorial, which we again leave to the dedicated reader.

In applications, it is often desirable to connect together systems with different noise sources, which means collecting together the indexing over closed measure-preserving systems into a single total category. Formally, this amounts to constructing an opfibration using the Grothendieck construction.

**Proposition 3.25.** The indexing of $\mathcal{RDyn}^T$ by closed measure-preserving systems generates, for each polynomial $p : \text{Poly}_E$, an opfibration over $\text{Cat}(\mathcal{B}^T, E)_p$, denoted $\int \mathcal{RDyn}^T(p)$; retaining the indexing by polynomials makes its type $\int \mathcal{RDyn}^T : \text{Poly}_E \to \text{Fib}(\text{Cat}(\mathcal{B}^T, E)_p^{op})$.

Explicitly, an object of $\int \mathcal{RDyn}^T(p)$ is a pair $(\vartheta, \theta)$ where $\theta := (\vartheta^0, \beta)$ is an object of $\text{Cat}(\mathcal{B}^T, E)_p$ (i.e., a closed measure-preserving dynamical system) and $\vartheta := (\pi, \vartheta^0, \vartheta^\mu)$ is an object of $\mathcal{RDyn}^T(p, \theta)$ (i.e., an open random dynamical system over $\theta$ on the interface $p$). A morphism $(\vartheta, \theta) \to (\vartheta', \theta')$ consists in a pair $(f, \phi)$, where $\phi : \theta \to \theta'$ is a morphism of closed measure-preserving systems and $f : \mathcal{RDyn}^T(p, \vartheta, \theta) \to \mathcal{RDyn}^T(p, \vartheta', \theta')$ is a morphism in $\mathcal{RDyn}^T(p, \vartheta')$ of open random dynamical systems. The identity morphism on an object of $\int \mathcal{RDyn}^T$ is given by the corresponding pair of identities. Given $(f, \phi) : (\vartheta, \theta) \to (\vartheta', \theta')$ and $(f', \phi') : (\vartheta''', \theta''') \to (\vartheta''', \theta''')$, their composite is given by the pair $(f' \circ \mathcal{RDyn}^T(p, \vartheta'), \phi' \circ \phi)$.

**Proof.** We start with a doubly-indexed category $\mathcal{RDyn}^T : \text{Poly}_E \times \text{Cat}(\mathcal{B}^T, E)_p \to \text{Cat}$. By the Cartesian closure of $\text{Cat}$, this induces an indexed category of indexed categories $\text{Poly}_E \to \text{Cat}(\text{Cat}(\mathcal{B}^T, E)_p, \text{Cat})$. Applying the covariant Grothendieck construction to each indexed category in the codomain generates an indexed opfibration $\int \mathcal{RDyn}^T : \text{Poly}_E \to \text{Fib}(\text{Cat}(\mathcal{B}^T, E)_p^{op})$. Unpacking the structure generated by this indexed Grothendieck construction gives the explicit form presented in the proposition.

**Proposition 3.26.** Suppose $(\theta, \beta)$ is a closed measure-preserving dynamical system. As in the deterministic case (Proposition 3.17), the category $\mathcal{RDyn}^T(y, \theta)$ of open random dynamical systems on the trivial interface $y$ with base system $\theta$ is equivalent to the category of closed random dynamical systems $\text{Cat}(\mathcal{B}^T, E)/\theta$.

**Proof.** The proof is directly analogous to the proof of Proposition 3.17, except that we now need to check that the flow condition induced by Definition 3.22 in the case of the trivial interface $y$ corresponds to the definition of a bundle in $\text{Cat}(\mathcal{B}^T, E)/\theta$, and that the morphisms of systems coincide similarly. Both of these verifications are straightforward and we leave them to the reader.
To be satisfactorily 'open', we should want the noise sources themselves to be open systems, preserving measure in an appropriately generalized sense. Fortunately, we can follow the theme of the developments above to define an indexed category of open measure-preserving systems.

**Definition 3.27.** Let $p : \text{Poly}_E$ be a polynomial and let $(S, \nu) : 1/\mathcal{K}(P)$ be a probability space, both in $\mathcal{E}$. An open measure-preserving dynamical system on the interface $p$ with state space $(S, \nu)$ and time $T$ consists in a pair of morphisms $\vartheta^p : T \times S \to p(1)$ and $\vartheta^{u} : \sum_{i:p(1)} p[i] s \to S$, such that, for any global section $\sigma : p(1) \to \sum_{i:p(1)} p[i]$ of $p$, the maps $\vartheta^\sigma : T \times S \to S$ given by

$$\sum_{t:T} \sum_{s:S} p[\vartheta^p(t, s)] \xrightarrow{\vartheta^u} S$$

constitute a closed measure-preserving dynamical system—again called the closure of $\vartheta$—with respect to $\nu$, i.e., an object in $\text{Cat}(\mathcal{B}_T, \mathcal{E})_p$; see Definition 3.7 for the explicit conditions. We collect the data of such an open dynamical system into a tuple $\vartheta = (S, \nu, \vartheta^p, \vartheta^u)$.

**Proposition 3.28.** Open measure-preserving dynamical systems $p$ with time $T$ form the objects of a category, denoted $\text{Dyn}^T(p)_p$. Its morphisms $f : (X, \mu, \vartheta^p, \vartheta^u) \to (Y, \nu, \psi^p, \psi^u)$ are maps between the state spaces $f : X \to Y$ in $\mathcal{E}$ such that, for all times $t : T$ and sections $\sigma : p(1) \to \sum_{i:p(1)} p[i]$ of $p$, the map $f$ lifts to a morphism $f^\sigma : \vartheta^\sigma \to \psi^\sigma$ in $\text{Cat}(\mathcal{B}_T, \mathcal{E})_p$ between the closures. More explicitly, this condition corresponds to an extension of the condition on morphisms of plain open dynamical systems (Proposition 3.15) so that they preserve measure (Proposition 3.8).

**Proof.** The proof amounts to the straightforward proof of Proposition 3.15, extended so that morphisms and their composition preserve measure. Just as checking compositionality in the plain case is straightforward, so is checking that measure is preserved: the relevant composite triangles must commute by pasting. □

**Proposition 3.29.** The categories $\text{Dyn}^T(p)_p$ collect into a polynomially-indexed category, $\text{Dyn}^T_p : \text{Poly}_E \to \text{Cat}$. The action of $\text{Dyn}^T_p$ on morphisms $\varphi : p \to q$ of polynomials is defined as for plain open dynamical systems in Proposition 3.16.

**Proof.** The proof proceeds as for the proof of Proposition 3.16. We also need to check that the structures in the image of $\text{Dyn}^T_p(\varphi)$ for each polynomial morphism $\varphi : p \to q$ satisfy the relevant measure-preservation property. But as in the other parts of the proof, this property follows from the facts that the structures in $\text{Dyn}^T(p)_p$ satisfy the property ex hypothesi for any section of $p$, and that any section $\tau$ of $q$ can be pulled back along $\varphi$ to a section of $p$ accordingly. □

**Proposition 3.30.** Just as the category $\text{Cat}(\mathcal{B}_T, \mathcal{E})$ of general closed dynamical systems is equivalent to $\text{Dyn}^T(y)_p$ (Proposition 3.17), there is an analogous equivalence in the measure-preserving case; that is, $\text{Cat}(\mathcal{B}_T, \mathcal{E})_p \cong \text{Dyn}^T(y)_p$.

**Proof.** Given Proposition 3.17, we only need to check that the measure-preservation condition coincides for objects and morphisms; but this is immediate from the definitions. □

**Proposition 3.31.** There is a forgetful indexed functor $U : \text{Dyn}^T_p \to \text{Dyn}^T$ which simply forgets the probability space structures.

Random dynamical systems form a subcategory of bundles of dynamical systems, and so, to define a general category of "fully open" random dynamical systems, it makes sense to start by defining open bundles of dynamical systems.
**Definition 3.32.** Let \( p, b : \text{Poly}_E \) be polynomials in \( E \), and let \( \theta := (\theta(\cdot), \theta^\ell, \theta^u) : \text{Dyn}^T (b) \) be an open dynamical system over \( b \). An open bundle dynamical system over \((p, b, \theta)\) is a pair \((\pi_{\theta^\ell}, \vartheta)\) where \( \vartheta := (\vartheta(\cdot), \vartheta^\ell, \vartheta^u) : \text{Dyn}^T (p) \) is an open dynamical system over \( p \) and \( \pi_{\theta^\ell} : \vartheta(\cdot) \to \theta(\cdot) \) is a bundle in \( E \), such that, for all time \( t : T \) and sections \( \sigma \) of \( p \) and \( \varsigma \) of \( b \), the following diagrams commute, thereby inducing a bundle of closed dynamical systems \( \pi_{\theta^\ell} : \vartheta^\sigma \to \theta^\sigma \) in \( \text{Cat}(B_T, E) \):

\[
\begin{array}{ccc}
\vartheta(\cdot) & \xrightarrow{\vartheta^\ell(\cdot)^\sigma \pi_{\theta^\ell}} & \sum_{w : \vartheta(\cdot)} p[\vartheta^\ell(t, w)] \\
& \downarrow \pi_{\theta^\ell} & \downarrow \vartheta^\ell(t) \\
\theta(\cdot) & \xrightarrow{\theta^\ell(\cdot)^\varsigma \pi_{\theta^\ell}} & \sum_{x : \theta(\cdot)} b[\theta^\ell(t, x)]
\end{array}
\]

**Proposition 3.33.** Let \( p, b : \text{Poly}_E \) be polynomials in \( E \), and let \( \theta := (\theta(\cdot), \theta^\ell, \theta^u) : \text{Dyn}^T (b) \) be an open dynamical system over \( b \). Open bundle dynamical systems over \((p, b, \theta)\) form the objects of a category \( \text{BunDyn}^T (p, b, \theta) \). Morphisms \( f : (\pi_{\theta^\ell}, \vartheta) \to (\pi_{\vartheta}, \vartheta) \) are maps \( f : \vartheta(\cdot) \to \vartheta(\cdot) \) in \( E \) making the following diagram commute for all times \( t : T \) and sections \( \sigma \) of \( p \) and \( \varsigma \) of \( b \):

\[
\begin{array}{ccc}
\vartheta(\cdot) & \xrightarrow{\vartheta^\ell(\cdot)^\sigma \pi_{\theta^\ell}} & \sum_{w : \vartheta(\cdot)} p[\vartheta^\ell(t, w)] \\
& \downarrow \pi_{\theta^\ell} & \downarrow \vartheta^\ell(t) \\
\theta(\cdot) & \xrightarrow{\theta^\ell(\cdot)^\varsigma \pi_{\theta^\ell}} & \sum_{x : \theta(\cdot)} b[\theta^\ell(t, x)]
\end{array}
\]

That is, \( f \) is a map on the state spaces that induces a morphism \((\pi_{\vartheta}, \vartheta^\ell) \to (\pi_{\theta^\ell}, \vartheta^\ell) \) in \( \text{Cat}(B_T, E)/\theta^c \) of bundles of the closures. Identity morphisms are the corresponding identity maps, and composition is by pasting.

**Proposition 3.34.** Varying the polynomials \( p \) in \( \text{BunDyn}^T (p, b, \theta) \) induces a polynomially indexed category \( \text{BunDyn}^T (-, b, \theta) : \text{Poly}_E \to \text{Cat} \). On polynomials \( p \), it returns the categories \( \text{BunDyn}^T (p, b, \theta) \) of Proposition 3.33. On morphisms \( \varphi : p \to q \) of polynomials, define the functors \( \text{BunDyn}^T (\varphi, b, \theta) : \text{BunDyn}^T (p, b, \theta) \to \text{BunDyn}^T (q, b, \theta) \) as in Propositions 3.16 and 3.24. That is, suppose \((\pi_{\theta^\ell}, \vartheta) : \text{BunDyn}^T (p, b, \theta) \) is object (open bundle dynamical system) in \( \text{BunDyn}^T (p, b, \theta) \), where \( \vartheta := (\vartheta(\cdot), \vartheta^\ell, \vartheta^u) \). Then its image \( \text{BunDyn}^T (\varphi, b, \theta)(\pi_{\theta^\ell}, \vartheta) \) is defined as the pair \((\pi_{\theta^\ell}, \varphi \vartheta)\), where \( \varphi \vartheta := (\vartheta(\cdot), \varphi \vartheta \circ \vartheta^\ell, \varphi \vartheta \circ \vartheta^u) \). On morphisms \( f : (\pi_{\theta^\ell}, \vartheta) \to (\pi_{\vartheta}, \vartheta) \), \( \text{BunDyn}^T (\varphi, b, \theta)(f) \) is again given by the same underlying map \( f : \vartheta(\cdot) \to \vartheta(\cdot) \) of state spaces.

**Proof.** The proof amounts to the proof for Proposition 3.24 that \( \text{RDyn}^T (\varphi, \theta) \) constitutes an indexed category, except that the closed base dynamical system \( \theta \) of that Proposition is here replaced, for any section \( \varsigma \) of \( b \), by the closure \( \theta^\varsigma \) by \( \varsigma \) of the open dynamical system \( \theta : \text{Dyn}^T (b) \) of the present Proposition. The proof goes through accordingly, since the relevant diagrams are guaranteed to commute for any such \( \varsigma \) by the conditions in Definition 3.32 and Proposition 3.33.
Proposition 3.35. Letting the base system $\theta$ also vary induces a doubly-indexed category $\text{BunDyn}^T(-, b, \cdot) : \text{Poly}_E \times \text{Dyn}^T(b) \to \text{Cat}$. Given a polynomial $p : \text{Poly}_E$ and morphism $\phi : \theta \to \rho$ in $\text{Dyn}^T(b)$, the functor $\text{BunDyn}^T(p, b, \phi) : \text{BunDyn}^T(p, b, \theta) \to \text{BunDyn}^T(p, b, \rho)$ is defined by post-composition, as in Proposition 3.24 for the action of $\text{RDyn}^T$ on morphisms of the base systems there. More explicitly, such a morphism $\phi$ corresponds to a map $\phi : \theta(*) \to \rho(*)$ of state spaces in $E$. Given an object $(\pi_{\theta\theta}, \vartheta)$ of $\text{BunDyn}^T(p, b, \theta)$, we define $\text{BunDyn}^T(p, b, \phi)(\pi_{\theta\theta}, \vartheta) := (\phi \circ \pi_{\theta\theta}, \vartheta)$. Given a morphism $f : (\pi_{\theta\theta}, \vartheta) \to (\pi_{\theta\theta}, \varrho)$ in $\text{BunDyn}^T(p, b, \theta)$, its image $\text{BunDyn}^T(p, b, \phi)(f) : (\phi \circ \pi_{\theta\theta}, \vartheta) \to (\phi \circ \pi_{\theta\theta}, \varrho)$ is given by the same underlying map $f : \theta(*) \to \rho(*)$ of state spaces.

Proof. As for Proposition 3.34, the proof here amounts to the proof for Proposition 3.24 that $\text{RDyn}^T(p, \phi)$ constitutes an indexed category, except again the closed systems are replaced by (the appropriate closures of) open ones, and the measure-preserving structure is forgotten.

Proposition 3.36. There is an indexed opfibration $\int \text{BunDyn}^T(-, b) : \text{Poly}_E \to \text{Fib}(\text{Dyn}^T(b))$ generated from $\text{BunDyn}^T(-, b, \cdot)$ by the Grothendieck construction.

Explicitly, given a polynomial $p : \text{Poly}_E$, the objects of $\int \text{BunDyn}^T(p, b)$ are triples $(\pi_{\theta\theta}, \vartheta, \theta)$, where $\theta : \text{Dyn}^T(b)$ is an open dynamical system over $b$ and $(\pi_{\theta\theta}, \vartheta) : \text{BunDyn}^T(p, b, \theta)$ is an open bundle dynamical system over $(p, b, \theta)$. Morphisms $f : (\pi_{\theta\theta}, \vartheta, \theta) \to (\pi_{\theta\theta}, \varrho, \rho)$ are pairs $(f_p, f_\theta)$ of a morphism $f_p : \theta \to \rho$ in $\text{Dyn}^T(b)$ and a morphism $f_\theta : (f_\theta \circ \pi_{\theta\theta}, \vartheta) \to (\pi_{\theta\theta}, \varrho)$ in $\text{BunDyn}^T(p, b, \rho)$ making the following diagram commute for all sections $\sigma$ of $p$ and $\varsigma$ of $b$:

\[
\begin{array}{ccc}
\theta(*) & \xrightarrow{\phi^*(t)\cdot \sigma} & \sum_{w : \theta(*)} p[\phi^*(t, w)] & \xrightarrow{\phi^*(t)} & \theta(*) \\
\downarrow & & \downarrow & & \downarrow \\
\rho(*) & \xrightarrow{\rho^*(t)\cdot \varsigma} & \sum_{y : \rho(*)} p[\rho^*(t, y)] & \xrightarrow{\rho^*(t)} & \rho(*)
\end{array}
\]

Identity morphisms are the pairs of the corresponding identities, and composition is again by pasting.

Proof. Compare Proposition 3.25.

Proposition 3.37. Varying the base polynomial $b$ extends $\text{BunDyn}^T(-, b, \cdot)$ to a triply indexed category, $\text{BunDyn}^T : \text{Poly}_E \times \sum_{b : \text{Poly}_E} \text{Dyn}^T(b) \to \text{Cat}$ and $\int \text{BunDyn}^T(-, b)$ to a doubly indexed fibration, $\int \text{BunDyn}^T : \text{Poly}_E \to \prod_{b : \text{Poly}_E} \text{Fib}(\text{Dyn}^T(b))$; these are equivalent by the Grothendieck construction.

Let $p, b, c : \text{Poly}_E$ be polynomials, and let $\chi : b \to c$ be a morphism accordingly. Let $\theta : \text{Dyn}^T(b)$ range over open dynamical systems over $b$. We define the (dependent) functor $\text{BunDyn}^T(p, \chi, \theta) : \text{BunDyn}^T(p, b, \theta) \to \text{BunDyn}^T(p, c, \text{Dyn}^T(\chi)(\theta))$ as follows. This functor is equivalent to its image under the Grothendieck
construction, \( \int \text{BunDyn}^\tau(p, \chi) : \int \text{BunDyn}^\tau(p, b) \to \int \text{BunDyn}^\tau(p, c) \), which is easier to describe. Therefore, let \((\pi_{\theta \theta}, \vartheta, \theta)\) be an object of \( \int \text{BunDyn}^\tau(p, b) \). Its image under \( \int \text{BunDyn}^\tau(p, \chi) \) is the object \((\pi_{\theta \theta}, \vartheta, \text{Dyn}^\tau(\chi)(\theta)) : \int \text{BunDyn}^\tau(p, c) \). Given a morphism \((f_p, f_b) : (\pi_{\theta \theta}, \vartheta, \theta) \to (\pi_{\theta \theta}, \vartheta, \varrho, \rho)\) in \( \int \text{BunDyn}^\tau(p, b) \), its image \( \int \text{BunDyn}^\tau(p, \chi)(f) : (\pi_{\theta \theta}, \vartheta, \text{Dyn}^\tau(\chi)(\theta)) \to (\pi_{\theta \theta}, \vartheta, \text{Dyn}^\tau(\chi)(\rho)) \) is given by the same maps \( f_p : \vartheta(\cdot) \to \varrho(\cdot) \) and \( f_b : \theta(\cdot) \to \rho(\cdot) \) of state spaces.

**Proof.** We just need to check that the diagrams of Definition 3.32 and Proposition 3.36 induced in the image of each \( \int \text{BunDyn}^\tau(p, \chi) \) commute, and that \( \int \text{BunDyn}^\tau(p, -) \) is functorial with respect to morphisms of polynomials. To check the first diagram, we note that the following commutes for the usual reason, that \( \rho^\# \circ \chi^1 \tau \) is a section of \( b \):

\[
\begin{array}{cccccccccc}
\vartheta(\cdot) & \xrightarrow{\vartheta^\#(t) \sigma} & \sum_{w : \vartheta(\cdot)} p[\vartheta^\#(t, w)] & \xrightarrow{\vartheta^\#(t)} & \vartheta(\cdot) \\
\pi_{\theta \theta} & & & & & & & & \pi_{\theta \theta} \\
\theta(\cdot) & \xrightarrow{\vartheta^\#(t) \chi^1 \tau} & \sum_{x : \theta(\cdot)} c[\chi_1 \circ \vartheta^\#(t, x)] & \xrightarrow{\vartheta^\#(t) \chi^\#} & \sum_{x : \theta(\cdot)} b[\vartheta^\#(t, x)] & \xrightarrow{\vartheta^\#(t)} & \theta(\cdot) \\
\end{array}
\]

To check the second diagram, we observe that the same property makes the following commute, given a morphism \((f_p, f_b) : (\pi_{\theta \theta}, \vartheta, \theta) \to (\pi_{\theta \theta}, \vartheta, \varrho, \rho)\) in \( \int \text{BunDyn}^\tau(p, b) \):

\[
\begin{array}{cccccccccccc}
\vartheta(\cdot) & \xrightarrow{\vartheta^\#(t) \sigma} & \sum_{w : \vartheta(\cdot)} p[\vartheta^\#(t, w)] & \xrightarrow{\vartheta^\#(t)} & \vartheta(\cdot) \\
\pi_{\theta \theta} & & & & & & & & \pi_{\theta \theta} \\
\theta(\cdot) & \xrightarrow{\vartheta^\#(t) \chi^1 \tau} & \sum_{x : \theta(\cdot)} c[\chi_1 \circ \vartheta^\#(t, x)] & \xrightarrow{\vartheta^\#(t) \chi^\#} & \sum_{x : \theta(\cdot)} b[\vartheta^\#(t, x)] & \xrightarrow{\vartheta^\#(t)} & \theta(\cdot) \\
\end{array}
\]

To check the second diagram, we observe that the same property makes the following commute, given a morphism \((f_p, f_b) : (\pi_{\theta \theta}, \vartheta, \theta) \to (\pi_{\theta \theta}, \vartheta, \varrho, \rho)\) in \( \int \text{BunDyn}^\tau(p, b) \):

\[
\begin{array}{cccccccccccc}
\vartheta(\cdot) & \xrightarrow{\vartheta^\#(t) \sigma} & \sum_{w : \vartheta(\cdot)} p[\vartheta^\#(t, w)] & \xrightarrow{\vartheta^\#(t)} & \vartheta(\cdot) \\
\pi_{\theta \theta} & & & & & & & & \pi_{\theta \theta} \\
\theta(\cdot) & \xrightarrow{\vartheta^\#(t) \chi^1 \tau} & \sum_{x : \theta(\cdot)} c[\chi_1 \circ \vartheta^\#(t, x)] & \xrightarrow{\vartheta^\#(t) \chi^\#} & \sum_{x : \theta(\cdot)} b[\vartheta^\#(t, x)] & \xrightarrow{\vartheta^\#(t)} & \theta(\cdot) \\
\end{array}
\]

Finally, we note that, as in the proof of Proposition 3.16, functoriality on morphisms \( b \to c \to d \) of polynomials follows from the functoriality of pullback and composition.

\[\square\]

**Proposition 3.38.** By restricting \( \text{BunDyn}^\tau \) and \( \int \text{BunDyn}^\tau \) to those systems which preserve measure in the base, we obtain the indexed categories \( \text{RBDyn}^\tau : \text{Poly}_\mathcal{E} \times \sum_b \text{Poly}_\mathcal{E} \text{Dyn}^\tau(b)p \to \text{Cat} \) and \( \int \text{RBDyn}^\tau : \text{Poly}_\mathcal{E} \to \prod_b \text{Poly}_\mathcal{E} \text{Fib}(\text{Dyn}^\tau(b)p) \) of open random bundle dynamical systems, or alternatively, fully open random dynamical systems.

**Proof.** The only extra check required is that the measure-preservation property is retained by the action of \( \text{RBDyn}^\tau \) on morphisms of polynomials \( p \to q \) above and below \( b \to c \) and on morphisms of open measure-preserving systems in the base. On the latter, the result is immediate, since such morphisms are defined to
As usual, we also require an open dynamical system \( \mathcal{E} \). We call the above condition the nesting condition. That is, \( \text{RDyn}^\mathbb{T} = \text{RBDyn}^\mathbb{T}(y) : \text{Poly}_\mathcal{E} \times \text{Dyn}^\mathbb{T}(y)_p \rightarrow \text{Cat} \).

**Proposition 3.39.** Open random dynamical systems embed into fully open random dynamical systems with trivial base polynomial. That is, \( \text{RDyn}^\mathbb{T} \cong \text{RBDyn}^\mathbb{T}(y) : \text{Poly}_\mathcal{E} \times \text{Dyn}^\mathbb{T}(y)_p \rightarrow \text{Cat} \).

**Proof.** This follows immediately from the facts in Proposition 3.30 that \( \text{Dyn}^\mathbb{T}(y)_p \) is equivalent to the category of closed measure-preserving systems, and that open random dynamical systems are fully open random dynamical systems with closed bases.

When the polynomial \( p \) is nested over \( b \), meaning that \( p : \text{Poly}_\mathcal{E}(b) \) in the sense of Proposition 2.1, we should expect the dynamics on the two interfaces to be compatible with this nesting. We can formalize this with the following structure.

**Proposition 3.40.** There is an indexed category \( \text{NDyn}^\mathbb{T} : \sum_b \text{Poly}_\mathcal{E}(y) \text{Poly}_\mathcal{E}(b) \times \text{Dyn}^\mathbb{T}(b) \rightarrow \text{Cat} \) of open dynamical systems over nested polynomials, and a corresponding dependently-indexed fibration \( \int \text{NDyn}^\mathbb{T} : \prod_b \text{Poly}_\mathcal{E}(y) \text{Poly}_\mathcal{E}(b) \rightarrow \text{Fib}(\text{Dyn}^\mathbb{T}(b)) \). These are defined as \( \text{BunDyn}^\mathbb{T} \) and \( \int \text{BunDyn}^\mathbb{T} \) with the extra condition that the dynamics are compatible with the nesting. More explicitly, suppose that \( p \) is a polynomial nested over \( b \). Then there are morphisms \( m : \sum_i p[i] \rightarrow \sum_k b[k] \) and \( n : p(1) \rightarrow b(1) \) in \( \mathcal{E} \), such that \( n \circ p = b \circ m \). An open dynamical system over \( (m, n) : p \rightarrow b \), i.e. an object of \( \int \text{NDyn}^\mathbb{T}(m, n) : \text{Fib}(\text{Dyn}^\mathbb{T}(b)) \), is a triple \( (\pi_{\vartheta \theta}, \vartheta, \theta) \) where \( \theta : \text{Dyn}^\mathbb{T}(b) \) is an open dynamical system over \( b \), \( \vartheta : \text{Dyn}^\mathbb{T}(p) \) is an open dynamical system over \( p \), and \( \pi_{\vartheta \theta} : \vartheta(*) \rightarrow \theta(*) \) is a map between the state spaces which lifts uniquely to make the front face of the following cube, and therefore the whole cube, commute:

![Diagram](image.png)

We call the above condition the nesting condition; the unlabelled edges of this cube are the obvious projections. As usual, we also require an open dynamical system \( (\pi_{\vartheta \theta}, \vartheta, \theta) \) to satisfy a flow condition, such that for all times \( t : \mathbb{T} \) and sections \( \sigma \) of \( p \) and \( \varsigma \) of \( b \), the following diagrams commute, where the dashed arrow below is the same dashed lift above:
Morphisms in $\int \text{NDyn}^T(m, n)$ are as for $\int \text{BunDyn}^T$ (Proposition 3.36), with the addition of the dashed arrows on the top and bottom of the defining cube (in 3.36). The actions of $\text{NDyn}^T$ and of $\int \text{NDyn}^T$ on morphisms of polynomials, nested polynomials, and base systems are defined as for $\text{BunDyn}^T$; given a morphism of polynomials, the dashed lifts are transformed by pullback. Because the pasting of two pullback squares is again a pullback square, it is easy to check that this also constitutes an indexed category.

The existence of the dashed lift asserts that the bundle of state spaces $\vartheta(*) \to \theta(*)$ is compatible with the nesting of polynomials $p \to b$, in the sense that each section $\sigma$ of $p$ projects onto a compatible section $\varsigma$ of $b$.

**Proposition 3.41.** By restricting $\text{NDyn}^T$ and $\int \text{NDyn}^T$ to those systems which preserve measure in the base, we obtain the indexed categories $\text{RNDyn}^T : \prod_{b: \text{Poly}_E(b)} \text{Poly}_E(b) \times \text{Dyn}^T(b)p \to \text{Cat}$ and $\int \text{RNDyn}^T : \text{Poly}_E(b) \to \text{Fib}(\text{Dyn}^T(b)p)$ of random nested dynamical systems.

**Proof.** The construction and proof are analogous to those of Proposition 3.38, with consideration for the nesting condition of Proposition 3.40.

**Proposition 3.42.** Open random dynamical systems embed into random nested dynamical systems with trivial base polynomial. That is, $\text{RDyn}^T \cong \text{RNDyn}^T(y) : \text{Poly}_E(y) \times \text{Dyn}^T(y)p \to \text{Cat}$.

**Proof.** The result is the same as Proposition 3.39, with the additional nesting condition. We therefore need to check that there exists a unique dashed lift as in Proposition 3.40. Since we have $b = y$ ex hypothesi, we have $\sum_{x: \vartheta(*)} b[\theta^p(t, x)] \cong \sum_{x: \theta(*)} 1 \cong \theta(*)$. Then, letting $\lambda$ denote the projection $\sum_{w: \vartheta(*)} p[\theta^p(t, w)] \to \vartheta(*)$, the dashed lift must be equal to $\pi \theta \circ \lambda$.

**Example 3.43.** Let $p$ be a polynomial, $M : E$ an object, and $g : M \to p(1)$ a map. Then suppose $dx_t = f(t, x_t, a_{x_t})dt + \sigma(t, x_t)dW_t$ is a stochastic differential equation, with $f : \mathbb{R}_+ \times \sum_{x: M} p[g(x)] \to TM$. Its solutions $\chi : \mathbb{R}_+ \times \Omega \times \sum_{x: M} p[g(x)] \to M$ induce an open random dynamical system $(\pi_\Omega : \Omega \times M \to M, g, \chi)$ on the interface $p$ with Wiener base flow $(\theta, \gamma)$, following the recipe in Example 3.12.

### 3.1 Internalizing dynamics in Poly

**Proposition 3.44.** When $\mathbb{T} = \mathbb{N}$, the category $\text{Dyn}^N(p)$ of open dynamical systems over $p$ with time $\mathbb{N}$ is equivalent to the topos $p\text{-Coalg}$ of $p$-coalgebras.

**Proof.** $p\text{-Coalg}$ has as objects pairs $(S, \beta)$ where $S : E$ is an object in $E$, $\beta : S \to p \triangleleft S$ is a morphism of polynomials (interpreting $S$ as the constant copresheaf on the set $S$), and $\triangleleft$ denotes the composition monoidal product in $\text{Poly}_E$ (i.e., composing the corresponding copresheaves $E \to \mathcal{E}$). A straightforward computation shows that, interpreted as an object in $E$, $p \triangleleft S$ corresponds to $\sum_{i: p(1)} S^{p[i]}$. By the universal property of the dependent sum, a morphism $\beta : S \to \sum_{i: p(1)} S^{p[i]}$ therefore corresponds bijectively to a pair of maps $\beta^p : S \to p(1)$ and $\beta^u : \sum_{i: S} p[\beta^p(i)] \to X$. By Proposition 3.19, such a pair is equivalently a discrete-time open dynamical system over $p$ with state space $S$: that is, the objects of $p\text{-Coalg}$ are in bijection with those of $\text{Dyn}^N(p)$.

Next, we show that the hom-sets $p\text{-Coalg}((S, \beta), (S', \beta'))$ and $\text{Dyn}^N(p)((S, \beta^p, \beta^u), (S', \beta'^p, \beta'^u))$ are in bijection. A morphism $f : (S, \beta) \to (S', \beta')$ of $p$-coalgebras is a morphism $f : S \to S'$ between the state spaces such that $\beta' \circ f = (p \triangleleft f) \circ \beta$. Unpacking this, we find that this means the following diagram in $E$:...
must commute for any section \( \sigma \) of \( p \):  

Pulling the arbitrary section \( \sigma \) back along the 'output' maps \( \beta^o \) and \( \beta'^o \) means that the following commutes:
Forgetting the vertical projections out of the pullbacks gives:

\[
S \xleftarrow{\beta^u} \sum_{s:S} p[\beta^o(s)] \xrightarrow{\beta^{o*\sigma}} S
\]

\[
S' \xleftarrow{\beta'^u} \sum_{s':S'} p[\beta'^o(s')] \xrightarrow{\beta'^{o*\sigma}} S'
\]

Finally, by collapsing the identity maps and reflecting the diagram horizontally, we obtain

\[
S \xrightarrow{\beta^{o*\sigma}} \sum_{s:S} p[\beta^o(s)] \xrightarrow{\beta^u} S
\]

\[
S' \xrightarrow{\beta'^{o*\sigma}} \sum_{s':S'} p[\beta'^o(s')] \xrightarrow{\beta'^u} S'
\]

which we recognize from Proposition 3.15 as the defining characteristic of a morphism in \( \text{Dyn}^\mathbb{N}(p) \). Finally, we note that each of these steps is bijective, and so we have the desired bijection of hom-sets. 

**Question 3.45.** Some questions:

1. Is there a correspondence between \( \text{Dyn}^\mathbb{T}(p) \) and \( \text{Cat}^\#(\text{Cofree}_p, y^\mathbb{T}) \)?
2. Are either of \( \text{Dyn}^\mathbb{T}(p) \) or \( \text{Cat}^\#(\text{Cofree}_p, y^\mathbb{T}) \) a topos, perhaps similarly to \( p\text{-Coalg} \)?
3. How do these topoi relate to behaviour topoi?
4. How do the internal languages of these topoi relate to coalgebraic logic?

### 3.2 \( p\mathbb{T}\text{-coalgebras and open Markov processes} \)

In the preceding sections, we noted connections between (deterministic) discrete-time dynamical systems over a polynomial interface \( p \) and \( p \)-coalgebras, with Proposition 3.44 showing their equivalence, as well as connections between random dynamical systems and Markov chains and Markov processes (REF). In this section, we connect these connections by generalizing the notion of coalgebra to systems evolving in arbitrary time.

Recall therefore that Markov chains are coalgebras for a probability monad \( \mathcal{P} : \mathcal{E} \to \mathcal{E} \), and that open Markov chains over a polynomial interface \( p \) are coalgebras for the composite functor \( p \mathcal{P} \), where here we interpret \( p \) as an endofunctor on \( \mathcal{E} \). Recall also that a Markov process in general time is given by a time-indexed family of Markov kernels—i.e., morphisms in \( K(\mathcal{P}) \)—satisfying a familiar flow condition. We can therefore use the recipes above to define a notion of \( p\mathbb{T}\text{-coalgebra} \) in general time for arbitrary monads \( T \). Instantiating this notion with \( T = \mathcal{P} \) will then give us a notion of open Markov process over a polynomial
interface with general time, and we can then extend the results above to exhibit these as random dynamical systems.

**Definition 3.46.** Let $T : \mathcal{E} \to \mathcal{E}$ be a monad on the category $\mathcal{E}$, and let $p : \text{Poly}_\mathcal{E}$ be a polynomial in $\mathcal{E}$. Let $(\mathbb{T}, +, 0)$ be a monoid in $\mathcal{E}$, representing time. Then a $pT$-coalgebra with time $\mathbb{T}$ consists in a triple $(S, \vartheta^p, \vartheta^u)$ of a state space $S : \mathcal{E}$ and two morphisms $\vartheta^p : \mathbb{T} \times S \to p(1)$ and $\vartheta^u : \sum_{t \in \mathbb{T}} \sum_{s \in S} p[\vartheta^p(t, s)] \to TS$, such that, for any section $\sigma : p(1) \to \sum_{i \in p(1)} p[i]$ of $p$, the maps $\vartheta^p : \mathbb{T} \times S \to TS$ given by

$$\sum_{t \in \mathbb{T}} S \overset{\vartheta^p(-, \sigma)}{\to} \sum_{t \in \mathbb{T}} \sum_{s \in S} p[\vartheta^p(-, s)] \overset{\vartheta^u}{\to} TS$$

constitute an object in the functor category $\text{Cat}(\mathbb{B}\mathbb{T}, \mathcal{K}(T))$, where $\mathbb{B}\mathbb{T}$ is the delooping of $\mathbb{T}$ and $\mathcal{K}(T)$ is the Kleisli category of $T$. We call $\vartheta^p$ the closure of $\vartheta$ by $\sigma$.

We now recall the development after Definition 3.14 of the indexed category $\text{Dyn}^\mathbb{T}$, in order to define the analogous indexed category $(-)T\text{-Coalg}^\mathbb{T}$.

**Proposition 3.47.** $pT$-coalgebras with time $\mathbb{T}$ form a category, denoted $pT\text{-Coalg}^\mathbb{T}$. Its morphisms are defined as follows. Let $\vartheta := (X, \vartheta^p, \vartheta^u)$ and $\psi := (Y, \psi^p, \psi^u)$ be two $pT$-coalgebras. A morphism $f : \vartheta \to \psi$ consists in a morphism $f : X \to Y$ such that, for any time $t : \mathbb{T}$ and global section $\sigma : p(1) \to \sum_{i \in p(1)} p[i]$ of $p$, the following naturality squares commute:

$$
\begin{array}{ccc}
X & \overset{\vartheta^p(t, \sigma)}{\xrightarrow{}} & \sum_{x \in X} p[\vartheta^p(t, x)] \overset{\vartheta^u(t)}{\xrightarrow{}} TX \\
\downarrow f & & \downarrow Tf \\
Y & \overset{\psi^p(t, \sigma)}{\xrightarrow{}} & \sum_{y \in Y} p[\psi^p(t, y)] \overset{\psi^u(t)}{\xrightarrow{}} TY
\end{array}
$$

The identity morphism $\text{id}_\vartheta$ on the $pT$-coalgebra $\vartheta$ is given by the identity morphism $\text{id}_X$ on its state space $X$. Composition of morphisms of $pT$-coalgebras is given by composition of the morphisms of the state spaces.

**Proof.** As for Proposition 3.15, the proof follows immediately by pasting. $\square$

**Proposition 3.48.** $pT\text{-Coalg}^\mathbb{T}$ extends to a polynomially-indexed category, $(-)T\text{-Coalg}^\mathbb{T} : \text{Poly}_\mathcal{E} \to \text{Cat}$. Suppose $\varphi : p \to q$ is a morphism of polynomials. We define a corresponding functor $\varphi T\text{-Coalg}^\mathbb{T} : pT\text{-Coalg}^\mathbb{T} \to qT\text{-Coalg}^\mathbb{T}$ as follows. Suppose $(X, \vartheta^p, \vartheta^u) : pT\text{-Coalg}^\mathbb{T}$ is an object ($pT$-coalgebra) in $pT\text{-Coalg}^\mathbb{T}$. Then $\varphi T\text{-Coalg}^\mathbb{T}(X, \vartheta^p, \vartheta^u)$ is defined as the triple $(X, \varphi_1 \circ \vartheta^p, \vartheta^u \circ \vartheta^u \circ \vartheta^u \circ \vartheta^u) : qT\text{-Coalg}^\mathbb{T}$, where the two maps are explicitly the following composites:

$$\mathbb{T} \times X \overset{\vartheta^p}{\xrightarrow{}} p(1) \overset{\varphi_1}{\xrightarrow{}} q(1), \quad \sum_{t \in \mathbb{T}} \sum_{x \in X} q[\varphi_1 \circ \vartheta^p(t, x)] \overset{\vartheta^u \circ \vartheta^u \circ \vartheta^u}{\xrightarrow{}} \sum_{t \in \mathbb{T}} \sum_{x \in X} p[\vartheta^u(t, x)] \overset{\vartheta^u}{\xrightarrow{}} TX .$$

On morphisms, $\varphi T\text{-Coalg}^\mathbb{T}(f) : \varphi T\text{-Coalg}^\mathbb{T}(X, \vartheta^p, \vartheta^u) \to \varphi T\text{-Coalg}^\mathbb{T}(Y, \psi^p, \psi^u)$ is given by the same underlying map $f : X \to Y$ of state spaces.

**Proof.** The proof is directly analogous to that of Proposition 3.16. $\square$

**Proposition 3.49.** $pT\text{-Coalg}^\mathbb{T}$ is equivalent to $\text{Dyn}^\mathbb{T}(p)$ when $T = \text{id}_\mathcal{E}$.

**Proof.** This is easy to see by noting that $\text{id}_\mathcal{E} X = X$ for all objects $X : \mathcal{E}$. $\square$
Corollary 3.50. $p \text{id}_\mathcal{C} \text{-Coalg}^N$ is equivalent to $p \text{-Coalg}$.

Proof. This follows directly from Propositions 3.44 and 3.49. □

Remark 3.51. Note that $pT\text{-coalg}$ as defined above really are coalgebras $X \rightarrow pTX$ in the traditional sense when $T = \mathbb{N}$. We can see this by observing that an analogue of Proposition 3.2 holds for $pT\text{-Coalg}^N$, so that, following our definition, a $pT\text{-coalgebra } \vartheta$ with state space $X$ is determined by two morphisms $\vartheta^o : X \rightarrow p(1)$ and $\vartheta^u : \sum_{x \in X} p[\vartheta^u(x)] \rightarrow TX$. Then, note that a ‘classical’ $pT\text{-coalgebra } \vartheta' : X \rightarrow pTX$ is equivalently a morphism $\vartheta : X \rightarrow \sum_{i \in p(1)} X^{i[1]}$, by the definition of the polynomial functor $p$. But, by the universal property of the dependent sum (and as in the proof of Proposition 3.44), such a morphism corresponds bijectively to such a pair of maps $(\vartheta^o, \vartheta^u)$ as determines our earlier $pT\text{-coalgebra } \vartheta'$. And as in Proposition 3.44, our definition of $pT\text{-coalgebra morphism corresponds to the classical notion of coalgebra homomorphism, so that our category } pT\text{-Coalg}^N \text{ is equivalent to the classical category } \text{Coalg}(pT) \text{ of } pT\text{-coalgebras and coalgebra homomorphisms. This justifies our thinking of the categories } pT\text{-Coalg}^T \text{ as generalized categories of coalgebras.}$

Proposition 3.52. $yT\text{-Coalg}^T$ is equivalent to $\text{Cat}(\mathbf{B}_T, \mathcal{K}(T))$.

Proof. Analogous to the proof of Proposition 3.17 for the equivalence $\text{Dyn}^T(y) \cong \text{Cat}(\mathbf{B}_T, \mathcal{E})$. □

Remark 3.53 (Closed Markov chains and Markov processes). A closed Markov chain is given by a map $X \rightarrow \mathcal{P}X$, where $\mathcal{P} : \mathcal{E} \rightarrow \mathcal{E}$ is a probability monad on $\mathcal{E}$; this is equivalently a $y \mathcal{P}\text{-coalgebra with state space } \mathbb{N}$, and an object in $\text{Cat}(\mathbf{B}^\mathbb{N}, \mathcal{K}(\mathcal{P}))$. With more general time $\mathbb{T}$, one obtains closed Markov processes: objects in $\text{Cat}(\mathbf{B}_\mathbb{T}, \mathcal{K}(\mathcal{P}))$. More explicitly, a closed Markov process is a time-indexed family of Markov kernels; that is, a morphism $\vartheta : \mathbb{T} \times X \rightarrow \mathcal{P}X$ such that, for all times $s, t : \mathbb{T}$, $\vartheta_{s+t} = \vartheta_s \circ \vartheta_t$ as a morphism in $\mathcal{K}(\mathcal{P})$.

Note that composition $\bullet$ in $\mathcal{K}(\mathcal{P})$ is given by the Chapman-Kolmogorov equation, so this means that

$$\vartheta_{s+t}(y|x) = \int_{x' \in X} \vartheta_s(y|x') \vartheta_t(dx'|x).$$

Open Markov processes over a polynomial interface $p$ are therefore the objects of the category $p\mathcal{P}\text{-Coalg}^\mathbb{T}$ for a given time monoid $\mathbb{T}$; the generalized flow condition corresponds to the satisfaction of an analogous Chapman-Kolmogorov equation by the closures of the systems by any section of $p$.

We now translate the development of bundle and nested open dynamical systems (after Definition 3.32) to the coalgebraic setting. Our principal aim is to define a notion of “nested Markov process” and corresponding generalizations for arbitrary monads $T$.

Definition 3.54. Let $p, b : \text{Poly}_\mathcal{E}$ be polynomials in $\mathcal{E}$, and let $\theta := (\theta(\ast), \theta^o, \theta^u) : bT\text{-Coalg}^\mathbb{T}$ be an $bT\text{-coalgebra}. A pT\text{-coalgebra bundle over } \theta$ is a pair $(\pi_{\theta b}, \vartheta)$ where $\vartheta := (\vartheta(\ast), \vartheta^o, \vartheta^u) : pT\text{-Coalg}^\mathbb{T}$ is a $pT\text{-coalgebra and } \pi_{\theta b} : \theta(\ast) \rightarrow \theta(\ast)$ is a bundle in $\mathcal{E}$, such that, for all time $t : \mathbb{T}$ and sections $\sigma$ of $p$ and $\varsigma$ of $b$, the following diagrams commute, thereby inducing a bundle of closed dynamical systems $\pi^\sigma_{\theta b} : \theta^\sigma \rightarrow \theta^\varsigma$ in $\text{Cat}(\mathbf{B}_\mathbb{T}, \mathcal{K}(\mathbb{T}))$:

$$\begin{array}{c c}
\vartheta(\ast) & \xrightarrow{\vartheta^o(t)^\ast \sigma} & \sum_{w: \vartheta(\ast)} \pi[\vartheta^o(t, w)] & \xrightarrow{\vartheta^u(t)} & T\vartheta(\ast) \\
\pi_{\theta b} & & & & T\pi_{\theta b} \\
\theta(\ast) & \xrightarrow{\vartheta^o(t)^\ast \varsigma} & \sum_{x: \theta(\ast)} b[\vartheta^o(t, x)] & \xrightarrow{\vartheta^u(t)} & T\theta(\ast)
\end{array}$$
Proposition 3.55. Let \( p, b : \text{Poly}_E \) be polynomials in \( E \), and let \( \theta := (\theta, \theta^o, \theta^a) : bT\text{-Coalg}^T \) be a \( bT \)-coalgebra. \( pT \)-coalgebra bundles over \( \theta \) form the objects of a category \( (p, b)T\text{-Coalg}^T / \theta \). Morphisms \( f : (p, b) \to (q, \rho) \) are maps \( f : \vartheta(*) \to \varrho(*) \) in \( E \) making the following diagram commute for all times \( t : T \) and sections \( \sigma \) of \( p \) and \( \varsigma \) of \( b \):

\[
\begin{array}{cccc}
\vartheta(*) & \rightarrow & \sum_{w : \vartheta(*)} p[\vartheta^o(t, w)] & \rightarrow T\vartheta(*) \\
\downarrow \vartheta^o(t) & & \downarrow T\vartheta^o(t) \\
\varrho(*) & \rightarrow & \sum_{x : \varrho(*)} p[\varrho^o(t, x)] & \rightarrow T\varrho(*) \\
\downarrow \varrho^o(t) & & \downarrow T\varrho^o(t) \\
\end{array}
\]

That is, \( f \) is a map on the state spaces that induces a morphism \( (\pi_{\theta^o} \circ \vartheta^o, \vartheta^o) \rightarrow (\pi_{\theta^o} \circ \varrho^o, \varrho^o) \) in \( \text{Cat}(bT, K(T))/\theta \) of bundles of the closures. Identity morphisms are the corresponding identity maps, and composition is by pasting.

Proposition 3.56. Varying the polynomials \( p \) in \( (p, b)T\text{-Coalg}^T / \theta \) induces a polynomially indexed category \((- b)T\text{-Coalg}^T / \theta : \text{Poly}_E \rightarrow \text{Cat} \). On polynomials \( p \), it returns the categories \( (p, b)T\text{-Coalg}^T / \theta \) of Proposition 3.55. On morphisms \( \varphi : p \rightarrow q \) of polynomials, define the functors \((- \varphi, b)T\text{-Coalg}^T / \theta : (p, b)T\text{-Coalg}^T / \theta \rightarrow (q, b)T\text{-Coalg}^T / \theta \) as in Proposition 3.48. That is, suppose \((p, b)T\text{-Coalg}^T / \theta \) is an object in \((p, b)T\text{-Coalg}^T / \theta \), where \( \vartheta := (\vartheta(*) \circ \vartheta^o, \vartheta^o) \). Then its image \((- \varphi, b)T\text{-Coalg}^T / \theta \) is defined as the pair \((\pi_{\theta^o} \circ \vartheta^o, \varphi) \), where \( \varphi \vartheta := (\vartheta(*) \circ \vartheta^o, \vartheta^o) \circ \vartheta^o \circ \vartheta^o \circ \vartheta^o \circ \vartheta^o \). On morphisms \( f : (\pi_{\theta^o} \circ \vartheta^o, \vartheta^o) \rightarrow (\pi_{\theta^o} \circ \vartheta^o, \vartheta^o) \) is again given by the same underlying map \( f : \vartheta(*) \rightarrow \varrho(*) \) of state spaces.

Proof. Analogous to the proof of Proposition 3.34. \( \square \)

Proposition 3.57. Letting the base system \( \theta \) also vary induces a doubly-indexed category \((- b)T\text{-Coalg}^T / (=) : \text{Poly}_E \times bT\text{-Coalg}^T \rightarrow \text{Cat} \). Given a polynomial \( p : \text{Poly}_E \) and morphism \( \phi : \theta \rightarrow \rho \) in \( bT\text{-Coalg}^T \), the functor \((p, b)T\text{-Coalg}^T / \phi : (p, b)T\text{-Coalg}^T / \theta \rightarrow (p, b)T\text{-Coalg}^T / \rho \) is defined by post-composition, as in Propositions 3.24 and 3.35. More explicitly, such a morphism \( \phi \) corresponds to a map \( \phi : \theta(*) \rightarrow \rho(*) \) of state spaces in \( E \). Given an object \((\pi_{\theta^o}, \rho) \) of \((p, b)T\text{-Coalg}^T / \theta \), we define \((p, b)T\text{-Coalg}^T / \phi \) : \((\pi_{\theta^o}, \theta) \rightarrow (\phi \circ \pi_{\theta^o}, \vartheta) \). Given a morphism \( f : (\pi_{\theta^o}, \theta) \rightarrow (\pi_{\theta^o}, \vartheta) \) in \((p, b)T\text{-Coalg}^T / \theta \), its image \((p, b)T\text{-Coalg}^T / \phi \) : \((\phi \circ \pi_{\theta^o}, \vartheta) \rightarrow (\phi \circ \pi_{\theta^o}, \vartheta) \) is given by the same underlying map \( f : \vartheta(*) \rightarrow \varrho(*) \) of state spaces.

Proof. Analogous to the proof of Proposition 3.35. \( \square \)

Proposition 3.58. There is an indexed opfibration \((- b)T\text{-Coalg}^T : \text{Poly}_E \rightarrow \text{Fib} (bT\text{-Coalg}^T) \) generated from \((- b)T\text{-Coalg}^T / (=) \) by the Grothendieck construction.

Explicitly, given a polynomial \( p : \text{Poly}_E \), the objects of \((p, b)T\text{-Coalg}^T \) are triples \((\pi_{\theta^o}, \theta, \theta) \), where \( \theta : bT\text{-Coalg}^T \) is a \( bT \)-coalgebra and \((\pi_{\theta^o}, \theta, \theta) : (p, b)T\text{-Coalg}^T / \theta \) is a p\( T \)-coalgebra bundle over \( \theta \). Morphisms \( f : (\pi_{\theta^o}, \theta, \theta) \rightarrow (\pi_{\theta^o}, \vartheta, \theta) \) are pairs \((f_p, f_b) \) of a morphism \( f_p : \theta \rightarrow \rho \) in \( bT\text{-Coalg}^T \) and a morphism \( f_b : (f_b \circ \pi_{\theta^o}, \theta) \rightarrow (\pi_{\theta^o}, \vartheta) \) in \((p, b)T\text{-Coalg}^T / \theta \) making the following diagram commute for all sections \( \sigma \)
of \( p \) and \( \zeta \) of \( b \):

\[
\begin{align*}
\vartheta(\ast) & \xrightarrow{\vartheta^\ast \sigma} \sum_{w: \vartheta(\ast)} p[\vartheta^\ast(t, w)] \xrightarrow{\varrho^\ast(t)} T\vartheta(\ast) \\
\theta(\ast) & \xrightarrow{\theta^\ast \zeta} \sum_{x: \theta(\ast)} b[\theta^\ast(t, x)] \xrightarrow{\rho^\ast(t)} T\theta(\ast) \\
\rho(\ast) & \xrightarrow{\rho^\ast \zeta} \sum_{z: \rho(\ast)} b[\rho^\ast(t, z)] \xrightarrow{\zeta} T\rho(\ast)
\end{align*}
\]

Identity morphisms are the pairs of the corresponding identities, and composition is again by pasting.

**Proof.** Compare Propositions 3.36 and 3.25. \( \square \)

**Proposition 3.59.** Varying the base polynomial \( b \) extends \((- , b)T{-}\text{Coalg}\simeq /\simeq\) to a triply indexed category, \((- , - )T{-}\text{Coalg}^T / \equiv \) \( : \text{Poly}_E \times \sum_b \text{Poly}_E \rightarrow \text{bT{-}\text{Coalg}}^T \rightarrow \text{Cat} \) and \( (- , - )T{-}\text{Coalg}^T \) to a doubly indexed fibration, \( \int (- , - )T{-}\text{Coalg}^T \rightarrow \text{Poly}_E \rightarrow \prod_b \text{Poly}_E \text{Fib} ( bT{-}\text{Coalg}^T ) \); these are equivalent by the Grothendieck construction.

Let \( p, b, c : \text{Poly}_E \) be polynomials, and let \( \chi : b \rightarrow c \) be a morphism accordingly. Let \( \theta : bT{-}\text{Coalg}^T \) range over \( bT{-}\text{coalgebras} \), and let \( \chi_* := \chi T{-}\text{Coalg}^T \). We define the (dependent) functor \( (p, \chi)T{-}\text{Coalg}^T / \theta : (p, b)T{-}\text{Coalg}^T / \theta \rightarrow (p, c)T{-}\text{Coalg}^T / \chi_* \theta \) as follows. This functor is equivalent to its image under the Grothendieck construction, \( \int (p, \chi)T{-}\text{Coalg}^T : \int (p, b)T{-}\text{Coalg}^T \rightarrow \int (p, c)T{-}\text{Coalg}^T \), which is easier to describe. Therefore, let \( \langle \pi_{\theta \theta}, \theta, \theta \rangle \) be an object of \( \int (p, b)T{-}\text{Coalg}^T \). Its image under \( \int (p, \chi)T{-}\text{Coalg}^T \) is the object \( \langle \pi_{\theta \theta}, \theta, \chi_* \theta \rangle : \int (p, c)T{-}\text{Coalg}^T \). Given a morphism \( (f_p, f_b) : (\pi_{\theta \theta}, \theta, \theta) \rightarrow (\pi_{\theta \theta}, \theta, \chi_* \theta) \) in \( \int (p, b)T{-}\text{Coalg}^T \), its image \( \int (p, \chi)T{-}\text{Coalg}^T \) \( \langle f \rangle : (\pi_{\theta \theta}, \theta, \chi_* \theta) \rightarrow (\pi_{\theta \theta}, \theta, \chi_* \theta) \) is given by the same maps \( f_p : \vartheta(\ast) \rightarrow \varrho(\ast) \) and \( f_b : \theta(\ast) \rightarrow \rho(\ast) \) of state spaces.

**Proof.** Compare Proposition 3.37. \( \square \)

And, when \( p \) is nested over \( b \) we of course have a correspondingly nested notion of coalgebra:

**Proposition 3.60.** There is an indexed category \((- , - )T{-}\text{Coalg}^T / \equiv \) \( : \sum_b \text{Poly}_E / bT{-}\text{Coalg}^T \rightarrow \text{Cat} \) of generalized coalgebras over nested polynomials, and a corresponding dependently-indexed fibration \( \int (- , - )T{-}\text{Coalg}^T : \prod_b \text{Poly}_E / b \rightarrow \text{Fib} ( bT{-}\text{Coalg}^T ) \). We have purposefully overloaded the names of these categories, since they are defined as the categories of coalgebra bundles, only with an extra condition of compatibility with the nesting. That is, suppose that \( p \) is a polynomial nested over \( b \). Then there are morphisms \( m : \sum_i p[i] \rightarrow \sum_k b[k] \) and \( n : p(1) \rightarrow b(1) \) in \( E \), such that \( n \circ p = b \circ m \). A generalized \( T{-}\text{coalgebra} \) over \((m, n) : p \rightarrow b \), i.e. an object of \( \int (m, n)T{-}\text{Coalg}^T \), is a triple \( \langle \pi_{\theta \theta}, \theta, \theta \rangle \) where \( \theta : bT{-}\text{Coalg}^T \) is a \( bT{-}\text{coalgebra} \), \( \vartheta : pT{-}\text{Coalg}^T \) is a \( pT{-}\text{coalgebra} \), and \( \pi_{\theta \theta} : \vartheta(\ast) \rightarrow \theta(\ast) \) is a map between
the state spaces which lifts uniquely to make the front face of the following cube, and therefore the whole cube, commute:

As in the dynamical-system case (Proposition 3.40), we call the above condition the nesting condition; the unlabelled edges of this cube are the obvious projections. As usual, we also require a flow condition to be satisfied; that is, for all times \( t \in \mathbb{T} \) and sections \( \sigma \) of \( p \) and \( \varsigma \) of \( b \), the following diagrams commute, where the dashed arrow below is the same dashed lift above:

Morphisms in \( \int (m, n)T\text{-Coalg}^\mathbb{T} \) are as for \( \int (p, b)T\text{-Coalg}^\mathbb{T} \) (Proposition 3.59), with the addition of the dashed arrows on the top and bottom of the defining cube (in 3.59). The actions of \( (\cdot, \cdot)T\text{-Coalg}^\mathbb{T} / (\equiv) \) and of \( \int (\cdot, \cdot)T\text{-Coalg}^\mathbb{T} \) on morphisms of polynomials, nested polynomials, and base coalgebras are defined as in the non-nested case above; given a morphism of polynomials, the dashed lifts are transformed by pullback. Because the pasting of two pullback squares is again a pullback square, it is easy to check that this also constitutes an indexed category.

**Question 3.61.** Some questions:

1. Under what conditions can we say something like, “\( pT\text{-Coalg}^\mathbb{T} \) is equivalent to \( \text{Cat} (\mathit{BT}, \mathit{K}^{\ell}(pT)) \)”?
2. What about coalgebraic logic? Distributive laws? Does that good stuff lift to this regime?
3. What about mapping cospan-algebra processes to processes over polynomials?

**4 References**

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