On Linearizability and the Termination of Randomized Algorithms

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Abstract

We study the question of whether the “termination with probability 1” property of a randomized algorithm is preserved when one replaces the atomic registers that the algorithm uses with linearizable (implementations of) registers. We show that in general this is not so: roughly speaking, every randomized algorithm \( A \) has a corresponding algorithm \( A' \) that solves the same problem if the registers that it uses are atomic or strongly-linearizable, but does not terminate if these registers are replaced with “merely” linearizable ones. Together with a previous result shown in [15], this implies that one cannot use the well-known ABD implementation of registers in message-passing systems to automatically transform any randomized algorithm that works in shared-memory systems into a randomized algorithm that works in message-passing systems: with a strong adversary the resulting algorithm may not terminate.

1 Introduction

A well-known property of shared object implementations is linearizability [15]. Intuitively, with a linearizable object (implementation) each operation must appear as if it takes effect instantaneously at some point during the time interval that it actually spans. As pointed out by the pioneering work of Golab et al. [14], however, linearizable objects are not as strong as atomic objects in the following sense: a randomized algorithm that works with atomic objects may lose some of its properties if we replace the atomic objects that it uses with objects that are only linearizable. In particular, they present a shared-memory randomized algorithm that guarantees that some random variable has expected value 1, but if we replace the algorithm’s atomic registers with linearizable registers, a strong adversary can manipulate the schedule to ensure that this random variable has expected value \( \frac{1}{2} \). To avoid this weakness of linearizability, and “limit the additional power that a strong adversary may gain when atomic objects are replaced with implemented objects”, Golab et al. introduced the concept of strong linearizability [14].

A natural question is whether this additional power of a strong adversary also applies to termination properties, more precisely: is there a randomized algorithm that (a) terminates with probability 1 against a strong adversary when the objects that it uses are atomic, but (b) when these objects are replaced with linearizable objects (of the same type), a strong adversary can ensure that the algorithm never terminates? To the best of our knowledge, the question whether the “termination with probability 1” property can be lost when atomic objects are replaced with linearizable ones is not answered by the results in [15], or in subsequent papers on this subject [9, 12, 16].

This question is particularly interesting because one of the main uses of randomized algorithms in distributed computing is to achieve termination with probability 1 [1, 2, 3, 4, 5, 6, 8, 10, 11] (e.g., to “circumvent” the famous FLP impossibility result [13]). For example, consider the well-known ABD algorithm that implements linearizable shared registers in message-passing systems [7]. One important use of this algorithm is to relate message-passing and shared-memory systems as follows: any algorithm

\[\text{This implementation works under the assumption that fewer than half of the processes may crash.}\]
that works with atomic shared registers can automatically be transformed into an algorithm for message-passing systems by replacing its atomic registers with the ABD register implementation. But can we use the ABD algorithm to automatically transform any shared-memory randomized algorithm that terminates with probability 1 (e.g., a randomized algorithm that solves consensus) into an algorithm that works in message-passing systems?

In this paper, we show that replacing atomic registers with linearizable registers can indeed affect the termination property of randomized algorithms: termination with probability 1 can be lost. In fact we prove that this loss of termination is general in the following sense: every randomized algorithm $A$ has a corresponding algorithm $A'$ that solves the same problem if the registers that it uses are atomic or strongly-linearizable, but does not terminate if these registers are replaced with “merely” linearizable ones. More precisely, we show that for every randomized algorithm $A$ that solves a task $T$ (e.g., consensus) and terminates with probability 1 against a strong adversary, there is a corresponding randomized algorithm $A'$ that also solves $T$ such that: (1) $A'$ uses only a set of shared registers in addition to the set of base objects of $A$; (2) if these registers are atomic or strongly-linearizable, then $A'$ terminates with probability 1 against a strong adversary, and its expected running time is only a small constant more than the expected running time of $A$; but (3) if the registers are only linearizable, then a strong adversary can prevent the termination of $A'$.

It is worth noting that this result allows us to answer our previous question about the ABD register implementation, namely, whether we can use it to automatically transform any randomized algorithm that works in shared-memory systems into a randomized algorithm that works in message-passing systems. In another paper, we proved that, although the registers implemented by the ABD algorithm are linearizable, they are not strongly linearizable [15]. Combining this result with the result of this paper proves that, in general, using the ABD register implementation instead of atomic registers in a randomized algorithm may result in an algorithm that does not terminate.

## 2 Model sketch

We consider a standard asynchronous shared-memory system with atomic registers [19, 17] where processes are subject to crash failures. We consider register implementations that are linearizable [18] or strongly linearizable [14]. For brevity, in this paper a “linearizable [strongly-linearizable] register” refers to an “implemented register whose implementation is linearizable [strongly-linearizable]”.

The precise definition of strong linearizability of [14] is reproduced here for convenience:

**Definition 1.** A set of histories $H$ over a set of shared objects is strongly linearizable if there exists a function $f$ mapping histories in close($H$) to sequential histories, such that:

- (L) for any $H \in \text{close}(H)$, $f(H)$ is a linearization of $H$, and
- (P) for any $G, H \in \text{close}(H)$, if $G$ is a prefix of $H$, then $f(G)$ is a prefix of $f(H)$.

The function $f$ is called a strong linearization function for $H$.

## 3 Result

Consider Algorithm 1 for $n \geq 3$ processes $p_0, p_1, p_2, \ldots, p_{n-1}$. This algorithm uses linearizable registers $R_1[j], R_2[j],$ and $C_1[j]$ for $j \geq 0$. We first show that if these registers are not strongly linearizable, then a strong adversary $S$ can construct an execution of Algorithm 1 in which all the processes execute infinitely many rounds (and therefore never return in line 15 or 28) (Theorem 2). We then show that if these registers are strongly linearizable, then all the correct processes return from the algorithm with probability 1 (within 2 rounds in expectation) (Theorem 3).

**Theorem 2.** If the registers of Algorithm 1 are linearizable but not strongly linearizable, a strong adversary $S$ can construct a run where all the processes execute infinitely many rounds (and therefore never return in line 15 or 28).

**Theorem 3.** If the registers of Algorithm 1 are strong linearizable, a strong adversary $S$ can construct a run where all the processes execute infinitely many rounds.
Algorithm 1 Weakener algorithm

For $j = 0, 1, 2, \ldots$
- $R_1[j]$: MWMR register initialized to $\bot$
- $C_1[j]$: SWMR register initialized to $-1$
- $R_2[j]$: SWMR register initialized to $\text{false}$

1: CODE OF PROCESS $p_i$, $i \in \{0, 1\}$:
2: for rounds $j = 0, 1, 2, \ldots$ do
3:   {\textbf{* Phase 1: writing $R_1[j] \ast$}}
4:     $R_1[j] \leftarrow i$
5:   if $i = 0$ then
6:     {\textbf{* code executed only by $p_0 \ast$}}
7:     $C_1[j] \leftarrow \text{flip coin}$
8:     end if
9:   {\textbf{* Phase 2: reading $R_2[j] \ast$}}
10:     $v_1 \leftarrow R_2[j]$
11:   if $v_1 = \text{false}$ then
12:     exit for loop
13:   end if
14: end for
15: return

16: CODE OF PROCESS $p_i$, $i \in \{2, 3, \ldots, n-1\}$:
17: for rounds $j = 0, 1, 2, \ldots$ do
18:   {\textbf{* Phase 1: reading $R_1[j]$ and $C_1[j] \ast$}}
19:     $u_1 \leftarrow R_1[j]$
20:     $u_2 \leftarrow R_1[j]$
21:     $c_1 \leftarrow C_1[j]$
22:     if $(u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1)$ then
23:         exit for loop
24:     end if
25:   {\textbf{* Phase 2: writing $R_2[j] \ast$}}
26:     $R_2[j] \leftarrow \text{true}$
27: end for
28: return

\textbf{Proof.} Assume that $R_1[j]$, $R_2[j]$, and $C_1[j]$ (for all $j \geq 0$) are linearizable but not strongly linearizable. A strong adversary $S$ can construct an infinite execution of Algorithm 1 as follows (Figure 1):

![Figure 1: Phase 1 in a single round of an infinite execution](image-url)

1. \textbf{Phase 1:} At time $t_0$, process $p_0$ starts writing 0 into $R_1[0]$ in line 4 process $p_1$ starts writing 1 into $R_1[0]$ in line 4 and processes $p_2, p_3, \ldots, p_{n-1}$ start reading $R_1[0]$ in line 19.

2. At time $t_1 > t_0$, process $p_0$ completes its writing of 0 into $R_1[0]$ in line 4.
3. After time \( t_1 \), process \( p_0 \) flips a coin and writes the result into \( C_1[0] \) in line 7. Let \( t_c > t_1 \) be the time when \( p_0 \) completes this write. Depending on the result of \( p_0 \)’s coin flip (and therefore the content of \( C_1[0] \)), the adversary \( S \) continues the run it is constructing in one of the following two ways:

**Case 1: \( C_1[0] = 0 \) at time \( t_c \).**

The continuation of the run in this case is shown at the top of Figure 1.

(a) At time \( t_2 > t_c \), \( p_1 \) completes its writing of 1 into \( R_1[0] \) (line 4).

Note that both \( p_0 \) and \( p_1 \) have now completed Phase 1 of round \( j = 0 \).

(b) The adversary \( S \) linearizes the write of 1 into \( R_1[0] \) by \( p_1 \) after the write of 0 into \( R_1[0] \) by \( p_0 \).

(c) Note that \( p_2, p_3, \ldots, p_{n-1} \) are still reading \( R_1[0] \) in line 19. Now the adversary linearizes these read operations between the above write of 0 by \( p_0 \) and the write of 1 by \( p_1 \).

(d) At time \( t_3 > t_2 \), processes \( p_2, p_3, \ldots, p_{n-1} \) complete their read of \( R_1[0] \) in line 19. By the above linearization, they read 0, and so they set (their local variable) \( u_1 = 0 \) in that line.

(e) Then processes \( p_2, p_3, \ldots, p_{n-1} \) start and complete their read of \( R_1[0] \) in line 20. Since (1) these reads start after the time \( t_2 \) when \( p_1 \) completed its write of 1 into \( R_1[0] \), and (2) this write is linearized after the write of \( p_0 \) into \( R_1[0] \), processes \( p_2, p_3, \ldots, p_{n-1} \) read 1. So they all set (their local variable) \( u_2 = 1 \) in line 20. Let \( t_4 > t_3 \) be the time when every process \( p_2, p_3, \ldots, p_{n-1} \) has set \( u_2 = 1 \).

(f) After time \( t_4 \), processes \( p_2, p_3, \ldots, p_{n-1} \) start reading \( C_1[0] \) in line 21. Since \( C_1[0] = 0 \) at time \( t_c \) and it is not modified thereafter, \( p_2, p_3, \ldots, p_{n-1} \) read 0 and set (their local variable) \( c_1 = 0 \) in line 21.

(g) Then \( p_2, p_3, \ldots, p_{n-1} \) execute line 22 and find that the condition of this line is *not* satisfied because they have \( u_1 = c_1 = 0 \) and \( u_2 = 1 - c_1 = 1 \).

So \( p_2, p_3, \ldots, p_{n-1} \) complete Phase 1 of round \( j = 0 \) without exiting in line 23. Recall that both \( p_0 \) and \( p_1 \) also completed Phase 1 of round \( j = 0 \) without exiting.

**Case 2: \( C_1[0] = 1 \) at time \( t_c \).**

The continuation of the run in this case is shown at the top of Figure 1. This continuation is essentially symmetric to the one for Case 1: the key difference is that the adversary \( S \) now linearizes the write of 1 before the write of \( p_0 \), as we describe in detail below.

(a) At time \( t_2 > t_c \), \( p_1 \) completes its writing of 1 into \( R_1[0] \) (line 4).

Note that both \( p_0 \) and \( p_1 \) have now completed Phase 1 of round \( j = 0 \).

(b) \( S \) linearizes the write of 1 into \( R_1[0] \) by \( p_1 \) before the write of 0 into \( R_1[0] \) by \( p_0 \).

(c) Note that \( p_2, p_3, \ldots, p_{n-1} \) are still reading \( R_1[0] \) in line 19. Now the adversary linearizes these read operations between the above write of 1 by \( p_1 \) and the write of 0 by \( p_0 \).

(d) At time \( t_3 > t_2 \), processes \( p_2, p_3, \ldots, p_{n-1} \) complete their read of \( R_1[0] \) in line 19. By the above linearization, they read 1, and so they set (their local variable) \( u_1 = 1 \) in that line.

(e) Then processes \( p_2, p_3, \ldots, p_{n-1} \) start and complete their read of \( R_1[0] \) in line 20. Since (1) these reads start after the time \( t_3 \) when \( p_0 \) completed its write of 0 into \( R_1[0] \), and (2) this write is linearized after the write of \( p_1 \) into \( R_1[0] \), processes \( p_2, p_3, \ldots, p_{n-1} \) read 1. So they all set (their local variable) \( u_2 = 1 \) in line 20. Let \( t_4 > t_3 \) be the time when every process \( p_2, p_3, \ldots, p_{n-1} \) has set \( u_2 = 0 \).

(f) After time \( t_4 \), processes \( p_2, p_3, \ldots, p_{n-1} \) start reading \( C_1[0] \) in line 21. Since \( C_1[0] = 1 \) at time \( t_c \) and it is not modified thereafter, \( p_2, p_3, \ldots, p_{n-1} \) read 1 and set (their local variable) \( c_1 = 1 \) in line 21.

(g) Then \( p_2, p_3, \ldots, p_{n-1} \) execute line 22 and find that the condition of this line is *not* satisfied because they have \( u_1 = c_1 = 1 \) and \( u_2 = 1 - c_1 = 0 \).

So \( p_2, p_3, \ldots, p_{n-1} \) complete Phase 1 of round \( j = 0 \) without exiting in line 23. Recall that both \( p_0 \) and \( p_1 \) also completed Phase 1 of round \( j = 0 \) without exiting.
Suppose no process reaches line 26 in round $j = 0$ while executing Algorithm 1. (Note that $p_2$ has now completed Phase 2 of round 0.)

4. **Phase 2**: Process $p_2$ writes true into $R_2[0]$ in line 25, let $t'_0$ be the time when this write operation completes. (Note that $p_2$ has now completed Phase 2 of round 0.)

5. After time $t'_0$, processes $p_0$ and $p_1$ read $R_2[0]$ in line 10. Since $p_2$ completes its write of true into $R_2[0]$ before $p_0$ and $p_1$ start to read this register, $p_0$ and $p_1$ set $v_1 = \text{true}$ in line 10.

6. Then $p_0$ and $p_1$ execute line 11 and find that the condition "$v_1 = \text{false}$" of this line is not satisfied. So $p_0$ and $p_1$ complete Phase 2 of round $j = 0$ without exiting in line 12.

7. Processes $p_3, \ldots, p_n$ execute line 26 and so they also complete Phase 2 of round $j = 0$.

So all the $n$ processes $p_0, p_1, \ldots, p_{n-1}$ have completed Phase 2 of round 0 without exiting; they are now poised to execute round $j = 1$.

The adversary $S$ continues to build the run by repeating the above scheduling of $p_0, p_1, \ldots, p_{n-1}$ for rounds $j = 1, 2, \ldots$. This gives a non-terminating run of Algorithm 1 with probability 1: in this run, all processes are correct, i.e., each takes an infinite number of steps, but loops forever in a for loop and never reaches the return statement that follows this loop (in line 15 or 28).

We now prove that if the registers $R_i[j]$ for $j = 1, 2, \ldots$ are strongly linearizable, then Algorithm 1 terminates with probability 1, even against a strong adversary. Roughly speaking, this is because if $R_i[j]$ is strongly linearizable, then the order in which 0 and 1 are written into $R_i[j]$ in line 4 is already fixed before the adversary $S$ can see result of the coin flip in line 7 of round $j$. So for every round $j \geq 0$, the adversary cannot "retroactively" decide on this linearization order according to the coin flip result (as it does in the proof of Theorem 2 where $R_i[j]$ is merely linearizable) to ensure that processes $p_i, i \geq 2$ do not exit by the condition of line 22. Thus, with probability $1/2$, all these processes will exit in line 23. And if they all exit there, then no process will write true in register $R_2[j]$ in line 26 and so $p_0$ and $p_1$ will also exit in line 12 of round $j$.

**Theorem 3.** If the registers of Algorithm 1 are strongly linearizable, then the algorithm terminates, even against a strong adversary: with probability 1, all the correct processes reach the return statement in line 15 or 28. Furthermore, they do so within $2$ expected rounds.

To prove the above theorem, we first show the following two lemmas.

**Lemma 4.** For all rounds $j \geq 0$, if no process reaches line 26 in round $j$, then neither $p_0$ nor $p_1$ enters round $j + 1$.

**Proof.** Suppose no process reaches line 26 in round $j$. Then no process writes into $R_2[j]$, and so $R_2[j] = \text{false}$ (the initial value of $R_2[j]$) at all times. Assume, for contradiction, that some process $p_i$ with $i \in \{0, 1\}$ enters round $j + 1$. So $p_i$ did not exit in line 12 of round $j$. Thus, when $p_i$ evaluated the exit condition "$v_1 = \text{false}$" in line 11 of round $j$, it found that $v_1 = \text{true}$. But $v_1$ is the value that $p_i$ read from $R_2[j]$ in line 10 of that round, and so $v_1$ can only be false — a contradiction.

**Lemma 5.** For all rounds $j \geq 0$, with probability at least $1/2$, no process enters round $j + 1$.

**Proof.** Consider any round $j \geq 0$. There are two cases:

1. **Process $p_0$ does not complete its write of register $R_0[j]$ in line 4 in round $j$.**

   Thus, $p_0$ never reaches line 7 (where it writes $C_1[j]$) in round $j$. So $C_1[j] = -1$ (the initial value of $C_1[j]$) at all times.

   **Claim 5.1** No process enters round $j + 1$.

   **Proof.** We first show that no process reaches line 26 in round $j$. To see why, suppose, for contradiction, some process $p_i$ reaches line 26 in round $j$. So $p_i$ did not exit in line 23 of round $j$. Thus, when $p_i$ evaluated the exit condition ($u_1 \neq c_1$ or $u_2 \neq -1 - c_1$) in line 22, it found the condition to be false, i.e., it found that $u_1 = c_1$ and $u_2 = 1 - c_1$. Note that $c_1$ is the value that $p_i$ read from $C_1[j]$.
in line 21 and so \( c_1 = -1 \). Thus, \( p_1 \) found that \( u_1 = -1 \) and \( u_2 = 2 \) in line 22. But \( u_1 \) is the value that \( p_i \) read from \( R_i[j] \) in line 19, and so \( u_1 \) can only be \( \bot \) (the initial value of \( R_i[j] \)), or 0 or 1 (the values written into it by \( p_0 \) and \( p_1 \), respectively). So \( u_1 \neq -1 \) in line 22 — a contradiction. So no process reaches line 26 in round \( j \). This implies that: (i) processes \( p_2, \ldots, p_{n-1} \) do not enter round \( j+1 \), and (ii) by Lemma 1 neither \( p_0 \) nor \( p_1 \) enters round \( j+1 \). □

(II) Process \( p_0 \) completes its write of register \( R_i[j] \) in line 4 in round \( j \).

**Claim 5.2** With probability at least \( 1/2 \), no process enters round \( j+1 \).

Proof. Consider the set of histories \( \mathcal{H} \) of Algorithm 1; this is a set of histories over the registers \( R_i[j], R_2[j], C_1[j] \) for \( j \geq 0 \). Since these registers are strongly linearizable, by Lemma 4.8 of [13], \( \mathcal{H} \) is strongly linearizable, i.e., it has at least one strong linearization function that satisfies properties (L) and (P) of Definition 1. Let \( f \) be the strong linearization function that the adversary \( S \) uses.

Let \( G \) be an arbitrary history of the algorithm up to and including the completion of the write of 0 into \( R_i[j] \) by \( p_0 \) in line 4 in round \( j \). Since \( p_0 \) completes its write of 0 into \( R_i[j] \) in \( G \), this write operation appears in the strong linearization \( f(G) \). Now there are two cases:

- **Case A:** In \( f(G) \), the write of 1 into \( R_i[j] \) by \( p_1 \) in line 4 in round \( j \) occurs before the write of 0 into \( R_i[j] \) by \( p_0 \) in line 4 in round \( j \).

  Since \( f \) is a strong linearization function, for every extension \( H \) of the history \( G \) (i.e., for every history \( H \) such that \( G \) is a prefix of \( H \) ), the write of 1 into \( R_i[j] \) occurs before the write of 0 into \( R_i[j] \) in the linearization \( f(H) \). Thus, in \( G \) and every extension \( H \) of \( G \), no process can first read 0 from \( R_i[j] \) and then read 1 from \( R_i[j] \) (∗).

  Let \( \mathcal{P} \) be the set of processes in \( \{p_2, p_3, \ldots, p_{n-1}\} \) that reach line 22 in round \( j \) and evaluate the condition \( (u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1) \) of that line. Note that \( u_1 \) and \( u_2 \) are the values that the processes in \( \mathcal{P} \) read from \( R_i[j] \) consecutively in lines 19 and 20. So \( u_1 \) and \( u_2 \) are in \( \{0,1,\bot\} \), and, by (∗), no process can have both \( u_1 = 0 \) and \( u_2 = 1 \) (∗∗). Moreover, \( c_1 \) is the value that the processes in \( \mathcal{P} \) read from \( C_1[j] \) in line 21 and so \( c_1 \) is in \( \{0,1,\bot\} \).

  Let \( \mathcal{P}' \subseteq \mathcal{P} \) be the subset of processes in \( \mathcal{P} \) that have \( c_1 = 1 \) or \( u_1 = 0 \) when they evaluate the condition \( (u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1) \) in line 22 in round \( j \).

**Claim 5.2.1**

(a) No process in \( \mathcal{P}' \) reaches line 26 in round \( j \).

(b) If \( \mathcal{P}' = \mathcal{P} \) then neither \( p_0 \) nor \( p_1 \) enters round \( j+1 \).

Proof. To see why (a) holds, note that: (i) every process \( p_i \) in \( \mathcal{P}' \) that has \( c_1 = 1 \) evaluates the condition \( (u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1) \) in line 22 to true because \( u_1 \neq 1 \); and (ii) every process \( p_i \), in \( \mathcal{P}' \) that has \( c_1 = 0 \), also evaluates the condition \( (u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1) \) in line 22 to true (otherwise \( p_i \) would have both \( u_1 = c_1 = 0 \) and \( u_2 = 1 - c_1 = 1 \), which is not possible by (∗∗)). Thus, no process \( p_i \) in \( \mathcal{P}' \) reaches line 26 in round \( j \) (it would exit in line 23 before reaching that line).

To see why (b) holds, suppose \( \mathcal{P}' = \mathcal{P} \) and consider an arbitrary process \( p \). If \( p \notin \mathcal{P} \) then \( p \) does not evaluate the condition \( (u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1) \) in line 22 in round \( j \); and if \( p \in \mathcal{P} \), then \( p \in \mathcal{P}' \), and so from part (a), \( p \) does not reach line 26 in round \( j \). So in both cases, \( p \) does not reach line 26 in round \( j \). Thus no process reaches line 26 in round \( j \), and so, by Lemma 4 neither \( p_0 \) nor \( p_1 \) enters round \( j+1 \). □

Now recall that \( G \) is the history of the algorithm up to and including the completion of the write of 0 into \( R_i[j] \) by \( p_0 \) in line 4 in round \( j \). After this write, i.e., in any extension \( H \) of \( G \), \( p_0 \) is supposed to flip a coin and write the result into \( C_1[j] \) in line 7. Thus, with probability at least \( 1/2 \), \( p_0 \) will not invoke the operation to write 1 into \( C_1[j] \). So with probability at least \( 1/2 \), processes never read 1 from \( C_1[j] \). Thus with probability at least \( 1/2 \), no process in \( \mathcal{P} \) has \( c_1 = 1 \) when it evaluates the condition \( (u_1 \neq c_1 \text{ or } u_2 \neq 1 - c_1) \) in line 22 in round \( j \). Since \( c_1 \in \{0,1,\bot\} \), this implies that with probability at least \( 1/2 \), every process in \( \mathcal{P} \) has \( c_1 = -1 \) or \( c_1 = 0 \) when it evaluates this condition in line 22 in round \( j \); in other words, with probability at least \( 1/2 \), \( \mathcal{P}' = \mathcal{P} \). Therefore, from Claim (II), with probability at least \( 1/2 \):

(a) No process in \( \mathcal{P} \) reaches line 26 in round \( j \).
(b) Neither \( p_0 \) nor \( p_1 \) enters round \( j + 1 \).

This implies that in Case A, with probability (at least) 1/2, no process enters round \( j + 1 \).

- **Case B:** In \( f(G) \), the write of 1 into \( R_1[j] \) by \( p_1 \) in line 4 in round \( j \) does not occur before the write of 0 into \( R_1[j] \) by \( p_0 \) in line 4 in round \( j \). This case is essentially symmetric to the one for Case A, we include it below for completeness.

Since \( J \) is a strong linearization function, for every extension \( H \) of the history \( G \), the write of 1 into \( R_1[j] \) does not occur before the write of 0 into \( R_1[j] \) in the linearization \( f(H) \). Thus, in \( G \) and every extension \( H \) of \( G \), no process can first read 1 from \( R_1[j] \) and then read 0 from \( R_1[j] \).

Let \( \mathcal{P} \) be the set of processes in \( \{p_2, p_3, \ldots, p_{n-1}\} \) that reach line 22 in round \( j \) and evaluate the condition \((u_1 \neq c_1 \lor u_2 \neq 1 - c_1)\) of that line. Note that \( u_1 \) and \( u_2 \) are the values that the processes in \( \mathcal{P} \) read from \( R_1[j] \) consecutively in lines 19 and 20. So \( u_1 \) and \( u_2 \) are in \( \{0, 1, \bot\} \), and, by \((\dagger)\), no process can have both \( u_1 = 1 \) and \( u_2 = 0 \) \((\dagger\dagger)\). Moreover, \( c_1 \) is the value that the processes in \( \mathcal{P} \) read from \( C_1[j] \) in line 21, and so \( c_1 \) is in \( \{0, 1, -1\} \).

Let \( \mathcal{P}' \subseteq \mathcal{P} \) be the subset of processes in \( \mathcal{P} \) that have \( c_1 = -1 \) or \( c_1 = 1 \) when they evaluate the condition \((u_1 \neq c_1 \lor u_2 \neq 1 - c_1)\) in line 22 in round \( j \).

**Claim 5.2.2**

(a) No process in \( \mathcal{P}' \) reaches line 26 in round \( j \).

(b) If \( \mathcal{P}' = \mathcal{P} \) then neither \( p_0 \) nor \( p_1 \) enters round \( j + 1 \).

**Proof.** To see why (a) holds, note that: (i) every process \( p_i \) in \( \mathcal{P}' \) that has \( c_1 = -1 \) evaluates the condition \((u_1 \neq c_1 \lor u_2 \neq 1 - c_1)\) in line 22 to true because \( u_1 \neq -1 \); and (ii) every process \( p_i \) in \( \mathcal{P}' \) that has \( c_1 = 1 \), also evaluates the condition \((u_1 \neq c_1 \lor u_2 \neq 1 - c_1)\) in line 22 to true (otherwise \( p_i \) would have both \( u_1 = c_1 = 1 \) and \( u_2 = 1 - c_1 = 0 \), which is not possible by \((\dagger\dagger)\)). Thus, no process \( p_i \) in \( \mathcal{P}' \) reaches line 26 in round \( j \) (it would exit in line 23 before reaching that line).

To see why (b) holds, suppose \( \mathcal{P}' = \mathcal{P} \) and consider an arbitrary process \( p \). If \( p \notin \mathcal{P} \) then \( p \) does not evaluate the condition \((u_1 \neq c_1 \lor u_2 \neq 1 - c_1)\) in line 22 in round \( j \); and if \( p \in \mathcal{P} \), then \( p \in \mathcal{P}' \), and so from part (a), \( p \) does not reach line 26 in round \( j \). So in both cases, \( p \) does not reach line 26 in round \( j \). Thus no process reaches line 26 in round \( j \), and so, by Lemma 3, neither \( p_0 \) nor \( p_1 \) enters round \( j + 1 \).

Now recall that \( G \) is the history of the algorithm up to and including the completion of the write of 0 into \( R_1[j] \) by \( p_0 \) in line 4 in round \( j \). After this write, i.e., in any extension \( H \) of \( G \), \( p_0 \) is supposed to flip a coin and write a result into \( C_1[j] \) in line 7. Thus, with probability at least 1/2, \( p_0 \) will not invoke the operation to write 0 into \( C_1[j] \). So with probability at least 1/2, processes never read 0 from \( C_1[j] \). Thus with probability at least 1/2, no process in \( \mathcal{P} \) has \( c_1 = 0 \) when it evaluates the condition \((u_1 \neq c_1 \lor u_2 \neq 1 - c_1)\) in line 22 in round \( j \). Since \( c_1 \in \{0, 1, -1\} \), this implies that with probability at least 1/2, every process in \( \mathcal{P} \) has \( c_1 = -1 \) or \( c_1 = 1 \) when it evaluates this condition in line 22 in round \( j \); in other words, with probability at least 1/2, \( \mathcal{P}' = \mathcal{P} \). Therefore, from Claim 5.2.1, with probability at least 1/2:

(a) No process in \( \mathcal{P} \) reaches line 26 in round \( j \).

(b) Neither \( p_0 \) nor \( p_1 \) enters round \( j + 1 \).

This implies that in Case B, with probability at least 1/2, no process enters round \( j + 1 \).

So in both Cases A and B, with probability at least 1/2, no process enters round \( j + 1 \).

We can now complete the proof of Theorem 3 namely, that with strongly linearizable registers, Algorithm 1 terminates with probability 1 in expected 2 rounds, even against a strong adversary. Consider any round \( j \geq 0 \). By Lemma 3, with probability at least 1/2, no process enters round \( j + 1 \). Since this holds for every round \( j \geq 0 \), then it must be that, with probability 1, all the processes that take an infinite number of steps must exit their loop in lines 23 or 13 and reach the return statement that follows this loop; furthermore, they do so within 2 expected iterations of the loop.
Theorem 6. Let $A$ be any randomized algorithm that solves a task $T$ (such as consensus) for $n \geq 3$ processes and terminates with probability 1 against a strong adversary. There is a corresponding randomized algorithm $A'$ that solves $T$ for $n \geq 3$ processes such that:

1. $A'$ uses a set $R$ of shared registers in addition to the set of base objects of $A$.

2. If the registers in $R$ are atomic or strongly linearizable, then $A'$ terminates with probability 1 against a strong adversary. Furthermore, the expected running time of $A'$ is only a constant more than the expected running time of $A$.

3. If the registers in $R$ are only linearizable, then a strong adversary can prevent the termination of $A'$.

Proof. Consider any randomized algorithm $A$ that solves some task $T$ for $n \geq 3$ processes $p_0, p_1, p_2, \ldots, p_{n-1}$, and terminates with probability 1 against a strong adversary. Using $A$, we construct the following randomized algorithm $A'$: every process $p_i$ with $i \in \{0, 1, 2, \ldots, n-1\}$ first executes Algorithm 1 if $p_i$ returns then it executes algorithm $A$. Note that:

1. In addition to the set of base objects that $A$ uses, the algorithm $A'$ uses the set of shared registers $R = \{R_1[j], R_2[j], C_1[j] \mid j \geq 0\}$.

2. Suppose these registers are strongly linearizable. Then, by Theorem 3, Algorithm 1 (that processes execute before executing $A$) terminates with probability 1 in expected 2 rounds against a strong adversary. Since $A$ also terminates with probability 1 against a strong adversary, the algorithm $A'$ also terminates with probability 1 against a strong adversary, and the expected running time of $A'$ is only a constant more than the expected running time of the given algorithm $A$. Since $A$ solves task $T$, it is clear that $A'$ also solves $T$.

3. Suppose these registers are linearizable but not strongly linearizable. Then, by Theorem 2, a strong adversary can construct a run of Algorithm 1 where, with probability 1, all the processes execute infinitely many rounds and never return. Thus, since $A'$ starts by executing Algorithm 1, it is clear that a strong adversary can prevent the termination of $A'$ with probability 1.

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