**Exceptional Super Yang-Mills in 27 + 3 and Worldvolume M-Theory**

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Bars and Sezgin have proposed a super Yang-Mills theory in $D = 11 + 3$ space-time dimensions with an electric 3-brane that generalizes the 2-brane of M-theory. More recently, the authors found an infinite family of exceptional super Yang-Mills theories in $D = (8n+3)+3$ via the study of exceptional periodicity (EP). A particularly interesting case occurs in signature $D = 27 + 3$, where the superalgebra supports an electric 11-brane and its 15-brane magnetic dual. The worldvolume symmetry of the 11-brane has signature $D = 11 + 3$ and can reproduce super Yang-Mills theory in $D = 11 + 3$. Upon reduction to $D = 26 + 2$, the 11-brane reduces to a 10-brane with 10 + 2 worldvolume signature. A single time projection gives a 10 + 1 worldvolume signature and can serve as a model for $D = 10 + 1$ M-theory as a reduction from the $D = 26 + 1$ signature of bosonic M-theory.

Extending previous results of Dijkgraaf, Verlinde and Verlinde, we also put forward the realization of spinor generators of EP algebras as total cohomologies of (the largest spatially extended) branes which centrally extend the $(1,0)$ superalgebra underlying the corresponding exceptional super Yang-Mills theory.

**Keywords**: Super Yang-Mills, exceptional periodicity, 11-brane, M-theory.

I. INTRODUCTION

After Witten’s introduction of M-theory \cite{Witten95} in $D = s + t = 10 + 1$ space-time dimensions, Vafa proposed F-theory in $11 + 1$ \cite{Vafa96}. Bars went further studying S-theory in $11 + 2$ \cite{Bars03, Bars04} and super Yang-Mills (SYM) theory in $11 + 3$, with subsequent further investigations by Sezgin et al. \cite{Sezgin96, Sezgin97}. Pushing beyond, Nishino defined SYM’s in signature $D = s + t = (9 + m) + (1 + m)$, for arbitrary $m \in \mathbb{N} \cup \{0\}$ \cite{Nishino99}. Using the algebraic structure of exceptional periodicity (EP) \cite{Dijkgraaf96, Dijkgraaf97, Dijkgraaf98}, in \cite{Rios09} the authors defined infinite families of exceptional $(1,0)$ SYM theories; among these, a family in $D = s + t = (8n+3)+3$ that directly generalizes Sezgin’s SYM in $11 + 3$. It is here worth recalling that the structure of SYM in $11 + 3$, with a 64-dimensional Majorana-Weyl (MW) semispinor\footnote{It is amusing to observe that the semispinor 64 of Spin(14) recently appeared in the $X_1$ algebraic structure of the Vogel plane in \cite{Bianchi94}.}, interestingly arises in a certain 5-grading of “extended Poincaré type” of $e_8(-24)$ and has found use in unification models (see e.g. \cite{Bianchi94}).

In this work, we ascend to $D = 27 + 3$ space-time dimensions, in which an electric 11-brane and its 15-brane magnetic dual arise as central extensions of the $(1,0)$ global supersymmetry algebra. In particular, the 11-brane gives rise to a worldvolume theory with 11 + 3 signature, thus providing a worldvolume embedding for the chiral SYM in 11 + 3 of Bars and Sezgin \cite{Bars03, Bars04}.

Following the 11-brane in the reduction from $27 + 3 \rightarrow 26 + 2 \rightarrow 26 + 1$ leads to the reduction of the $D = 11 + 3$ worldvolume of the 11-brane to a $D = 10 + 1$ worldvolume of a 10-brane, suggesting that M-theory may be a worldvolume theory. This chain of reductions along the worldvolume of the electric 11-brane yields a natural map of the conjectured “bosonic M-theory” of Horowitz and Susskind \cite{Horowitz97} in $26 + 1$ down to M-theory in $10 + 1$. Moreover, the electric 10-brane in $D = 26 + 2$ has a 14-brane magnetic dual (both centrally extending the corresponding $(1,0)$ global superalgebra). Upon reduction to $D = 26 + 1$, this implies the existence of a “dual” (worldvolume-realized) M-theory in $D = s + t = 13 + 1$.
II. (1, 0) SYM IN 27 + 3 AND M-THEORY

The (1, 0) superalgebra in \( D = 27 + 3 \) space-time dimensions (corresponding to the level \( n = 3 \) of EP) [8,10] takes the form [11]

\[
27 + 3 : \ \{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu\rho})_{\alpha\beta} Z_{\mu\nu\rho} + (\gamma^{\mu_1...\mu_7})_{\alpha\beta} Z_{\mu_1...\mu_7} + (\gamma^{\mu_1...\mu_{11}})_{\alpha\beta} Z_{\mu_1...\mu_{11}} + (\gamma^{\mu_1...\mu_{15}})_{\alpha\beta} Z_{\mu_1...\mu_{15}}. \tag{2.1}
\]

Namely, the central extensions are given by a 3-brane, a 7-brane, an electric 11-brane and its dual, a magnetic 15-brane. Note that the magnetic duals of the 3-brane and 7-brane, \textit{i.e.} the 23-brane resp. 19-brane, do \textit{not} centrally extend the algebra (2.1); however, they can be found as the largest spatially extended central charges at \( n = 5 \) resp. \( n = 4 \) levels of EP [11].

In \( D = 27 + 3 \), the electric 11-brane has a multi-time worldvolume, with signature \( 11 + 3 \), which can be used to provide a worldvolume realization for the \( 11 + 3 \) SYM of Bars and Sezgin [3–5]. In other words, the multi-time worldvolume of the electric 11-brane in \( 27 + 3 \) can support a corresponding \((1, 0)\) superalgebra in \( 11 + 3 \):

\[
11 + 3 : \ \{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (\gamma^{\mu_1...\mu_7})_{\alpha\beta} Z_{\mu_1...\mu_7}, \tag{2.2}
\]

which Sezgin \textit{et al.} have shown to unify the superalgebras of \( D = 10 + 1 \) (\( N = 1 \)) M-theory and \( D = 9 + 1 \) IIB ((2, 0)) superstring theory [6].

Hence, one can consider a reduction \( 27 + 3 \rightarrow 26 + 1 \), and focus on the corresponding reduction \( 11 + 3 \rightarrow 10 + 1 \) of the 11-brane (multi-time) worldvolume down to the (single-time) 10-brane worldvolume (in \( 26 + 1 \)); in turn, this latter can be used to provide a worldvolume realization of \( 10 + 1 \) M-theory.

This simple reasoning yields the following consequences:

- it puts forward the realization of \( 10 + 1 \) M-theory as a worldvolume theory of an electric 10-brane in a higher \( 26 + 1 \) space-time (pertaining to bosonic M-theory of Horowitz and Susskind [14]);
- as such, it provides a map from the bosonic M-theory in \( D = 26 + 1 \) to M-theory in \( D = 10 + 1 \);
- we observe that bosonic M-theory can be completed to a two-time theory in \( D = 26 + 2 \), in which a \((1, 0)\) exceptional SYM can be defined, with central extensions given by [11]

\[
26 + 2 : \ \{Q_\alpha, Q_\beta\} = (\gamma^{\mu\nu})_{\alpha\beta} Z_{\mu\nu} + (\gamma^{\mu_1...\mu_6})_{\alpha\beta} Z_{\mu_1...\mu_6} + (\gamma^{\mu_1...\mu_{10}})_{\alpha\beta} Z_{\mu_1...\mu_{10}} + (\gamma^{\mu_1...\mu_{14}})_{\alpha\beta} Z_{\mu_1...\mu_{14}}, \tag{2.3}
\]

i.e. by a 2-brane, a 6-brane, and by an electric 10-brane and its dual, a magnetic 14-brane. This implies that there exists a theory which is \textit{dual} to the worldvolume-realized \( 10 + 1 \) M-theory embedded in \( 26 + 2 \), namely the worldvolume-realized \( 14 + 2 \) theory reduced to \( 13 + 1 \) of the magnetic 13-brane embedded in \( 26 + 1 \); we dub such a \( 13 + 1 \) theory the \textit{“dual M-theory”}, and we leave its study for future work.

III. (1, 0) SYM IN 27 + 3, EP, AND SPINORS AS BRANE COHOMOLOGIES

Through the algebraic structure of EP, let us consider the generalization of the split form \( \mathfrak{e}_{8(-24)} \) of the largest finite-dimensional exceptional Lie algebra \( \mathfrak{e}_8 \) provided by \( \mathfrak{e}_8^{(3)} \) of the corresponding \textit{EP algebra} at level \( n = 3 \) [8,10]:

\[
\mathfrak{e}_8^{(3)} := \mathfrak{so}_{28,4} \oplus 32768 \tag{3.1}
= 30_{-2} \oplus (16384)'_{-1} \oplus (\mathfrak{so}_{27,3} \oplus \mathbb{R})_0 \oplus 16384_{+1} \oplus 30_{+2}, \tag{3.2}
\]

where \( 32768 = 2^{15} \) is the MW semispinor in \( 28 + 4 \), while \( 16384 = 2^{14} \) and \( (16384)' = (2^{14})' \) denote the MW spinor and its conjugate in \( D = 27 + 3 \). \( \mathfrak{e}_8^{(3)} \) is the \( n = 3 \) element of the countably, Bott-periodized infinite sequence of \textit{EP-generalizations} of \( \mathfrak{e}_{8(-24)} \). By denoting with \( 2^{N-1} \) and \((2^{N-1})'\) the chiral semispinor representations of \( \mathfrak{so}_{2N} \), as well as with \( \wedge^i \mathbb{N} \) the rank-\( i \) antisymmetric (\( i \)-form) representation of \( \mathfrak{so}_N \), we recall that, as a consequence of the well known cohomological structure of Clifford algebra \( Cl(N) \) in \( N \) dimensions (see e.g. [15], and Refs. therein), it holds that

\[
\dim_{\mathbb{R}} Cl(N) = 2^N = \dim_{\mathbb{R}} \left( 2^{N-1} \oplus (2^{N-1})' \right) = \sum_{i=0}^{N} \binom{N}{i} = \dim_{\mathbb{R}} \left( \bigoplus_{i=0}^{N} \wedge^i \mathbb{N} \right). \tag{3.3}
\]
FIG. 1: The “Magic Star” of EP [8] allows for \( \epsilon^{(n)}_8 \) to be found as a finite-dimensional, Jacobi-violating generalization of \( \epsilon_8 \) \[8–10\]. \( T_3^{n,n} \) denotes a T-algebra of rank-3 of special type [17] parametrized by the octonions \( \mathbb{O} \) with \( n \in \mathbb{N} \) [9, 17].

Thus, in the case under consideration, the 2^{15}-dimensional MW semispinor 32768 of \( \mathfrak{so}_{28,4} \), which branches as \( 16384 \oplus 16384' \) under \( \mathfrak{so}_{28,4} \to \mathfrak{so}_{27,3} \), can be regarded as the total cohomology of a 15-brane, which in turn can be identified with the maximally spatially extended central charge of \( \mathcal{N} = (1, 0) \text{ SYM} \) (2.1) in 27 + 3 space-time dimensions [11].

This is nothing but the \( n = 3 \) case of a general fact, namely that the (chiral) spinor component \( 2^{4n+3} \) of the EP algebra

\[
\epsilon^{(n)}_{8(-24)} = \mathfrak{so}_{8n,4,4} \oplus 2^{4n+3}
\]

\[
= (8n + 6)_{-2} \oplus (2^{4n+2})_{-1} \oplus (\mathfrak{so}_{8n+3,3} \oplus \mathbb{R})_0 \oplus 2^{4n+2} \oplus (8n + 6)_{+2},
\]

can be realized as the total cohomology of a \((4n+3)\)-brane, which in turn can be identified with the largest spatially extended central extension of the \((1, 0)\) supersymmetry algebra in \((8n + 3) + 3\) space-time dimensions [11]

\[
(8n + 3) + 3: \{Q_\alpha, Q_\beta\} = (\gamma^{\mu_1...\mu_7})_{\alpha\beta} Z_{\mu_1...\mu_7} + (\gamma^{\mu_1...\mu_{n+3}})_{\alpha\beta} Z_{\mu_1...\mu_{n+3}}.
\]

Therefore, the spinor generators of the EP algebra \( \epsilon^{(n)}_{8(-24)} \) are realized, exploiting the central extensions of the \((1, 0)\) supersymmetry algebra in \((8n + 3) + 3\), in terms of brane cohomology. We stress that this realization extends the results found by Dijkgraaf, Verlinde and Verlinde [16] which, in the BPS quantization of the 5-brane, realized the 16 components of the central charge as fluxes through the odd homology cycles on the five-brane itself. Since the spinor generators are the very ones responsible for the violation of the Jacobi identity in EP algebras [8–10], this is a further hint that the Lie subalgebra of EP yields a purely bosonic sector. Moreover, it is worth here reminding that the \( n = 1 \) (trivial) level of EP boils down to the fact the spinor component \( 128 \) of \( \epsilon_{8(-24)}^{(1)} \equiv \mathfrak{e}_{8(-24)} \) can be realized as the total cohomology of the 7-brane which centrally extends the \((1, 0)\) superalgebra in 11 + 3 [3,6].

As resulting from the star-shaped algebraic structure of EP algebras [8,10], we note that the cubic Vinberg’s T-algebra [17] at EP level \( n = 3 \) can be decomposed (in a manifestly \( \mathfrak{so}_{25,1} \)-covariant way) as

\[
T_3^{8,3} = 26 \oplus 4096 \oplus 1,
\]

where 26 and 4096 = \( 2^{12} \) respectively are the vector and MW semispinor irreps. of \( \mathfrak{so}_{25,1} \). In 25 + 1 space-time dimensions, the 24 (light-cone) transverse dimensions can be projectively mapped to the Cayley-Moufang plane \( \mathbb{O} \mathbb{P}^2 \), thus recovering three chart-wise Hopf fibrations \((i = 1, 2, 3)\):

\[
\pi_i: S^7 \hookrightarrow S^{15} \to S^8(i = 1, 2, 3).
\]

In particular, the third orthogonal idempotent \( 1 \) of the T-algebra (3.7) enhances the signature to 26 + 1, which in turn reduces to 10 + 1 along a sixteen-dimensional Cayley-Moufang plane chart map; this provides a simple explanation,
within the $n = 3$ level of EP, of a remarkable observation by Ramond and Sati [18-21], asserting that M-theory with hidden $\text{O}^2^\text{P}$ fibers underlies the origin of the massless multiplet of eleven-dimensional supergravity in terms of representations of $F_4$ decomposed with respect to its maximal compact subgroup $\text{Spin}(9)$.

Also, we observe that, as resulting from [22], the semispinor $4096 = 2^{12}$ can be realized as a module for the double cover of the centralizer of an element in the class 2B of the Monster Group. Wilson has given an explicit construction of the Leech lattice in terms of $\text{O}^4$ elements [23], which in turn can be projectively mapped to the aforementioned Cayley-Moufang plane $\text{O}^2$. As resulting from [8], the spinorial “extended roots” at EP level $n = 3$ have norm $n + 1 = 4$, thus providing norm-four vectors, as expected for the Leech lattice $\Lambda_L$ [9]. In this framework, we put forward the conjecture that at non-perturbative level the 24 transverse directions can be realized as vectors of $\Lambda_L$, which we recall to be the unique even unimodular lattice with no roots in $D = 24$: taken projectively, the map

$$\text{O}^4(\Lambda_L) \to \text{O}^2$$  (3.9)

reduces to a discrete version of the Hopf fibrations (3.8), with $3 \times 65520 = 196560$ vectors, whereas $S^7$ gets discretized in terms of the 240 roots of $\varepsilon_8$, such that $273 \times 240 = \text{65520}$, as observed by Wilson [23]. The 273-dimensional part is recovered from the T-algebra $T_3^{8,2}$ (EP level $n = 2$), with light-cone coordinates removed. This intriguing scenario, deserving a detailed study, is left for future investigation.

IV. CONCLUSION

Using the exceptional SYM theory in $27 + 3$ space-time dimensions, whose $(1,0)$ non-standard global superalgebra can be centrally extended by an electric 11-brane and its 15-brane magnetic dual [11], we considered the (multi-time) worldvolume theory of the 11-brane itself as support for the $(1,0)$ SYM theory in $11 + 3$ space-time dimensions as introduced by Bars and Sezgin some time ago [6-9].

As the $(1,0)$ superalgebra in $11 + 3$ unifies the $10 + 1 (N = 1)$ superalgebra and the $9 + 1$ IIB chiral superalgebra [6], we proposed the reduced (single-time) 10-brane worldvolume theory in $10 + 1$ as a worldvolume realization of M-theory (this also entails the existence of a would-be “dual worldvolume M-theory” realized as a worldvolume theory in $13 + 1$). In this framework, the space-time reduction $27 + 3 \to 26 + 1$ yields a natural map from the conjectured bosonic M-theory of Horowitz and Susskind [13] in $D = 26 + 1$ to M-theory.

Last but not least, extending the results of [16], we have put forward the intriguing brane-cohomological interpretation of spinors, in particular for the spinor generators of EP algebras. This entangles the algebraic structure of exceptional periodicity with the central extensions of exceptional super Yang Mills theories in higher dimensional space-times.
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