Coherent population trapping in the stochastic limit

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Abstract

A 2–level atom with degenerate ground state interacting with a quantum field is investigated. We show, that the field drives the state of the atom to a stationary state, which is non–unique, but depends on the initial state of the system through some conserved quantities. This non–uniqueness follows from the degeneracy of the ground state of the atom, and when the ground subspace is two–dimensional, the family of stationary states will depend on a one–dimensional parameter. Only one of the stationary states in this family is a pure state, and this state coincides with the known population trapped state (zero population in the excited level $|\text{NC}\rangle$). Another one stationary state corresponds to an equal weight mixture of the excited level $|3\rangle$ and of the coupled state $|C\rangle$.

1 Introduction

In the present paper we consider a 2–level atom with a degenerate ground state (or, equivalently, a 3–level atom with equal energies of the two lower levels). We prove that the interaction with radiation drives this atom to a family of stationary states, depending linearly on a one–dimensional parameter, which varies in explicitly determined interval. For a particular (extremal) value of this parameter the stationary state coincides with the coherent population trapped state, described in [1]–[5].

Our starting point are the papers [2], [4], which discuss coherent population trapping (CPT) in a 3–level Λ–system (i.e. a 3-level atom where only two transitions between the the lower levels 1 and 2 and the higher level 3 are allowed, and the transition between 1 and 2 is forbidden). CPT is based on the preparation of atoms in a special coherent superposition of the two lower states. In [4] it was argued that the CPT process may be described in the basis of coupled and non–coupled states defined by

$$|C\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle)$$

$$|\text{NC}\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)$$

where the two levels $|1\rangle$, $|2\rangle$, correspond to the hyperfine split $3S_{1/2}$ sodium ground state, coupled by the laser fields to a common excited level $|3\rangle$ within $3P_{1/2}$ levels.

In [2], [5] it is explained that this scheme, although highly simplified with respect to the real situation, nevertheless captures the main physical features of the phenomenon of CPT.
Coherent population trapping (CPT) consists in driving the atomic state to the superposition wave function $|NC\rangle$, often called population trapped or dark state cf. [1]–[5]. An atom driven in this state is transparent, i.e. it does not absorb the incoming laser radiation, in the sense that transitions from the trapped state to the excited $|3\rangle$ are forbidden [3]. The idea is that, at resonance and in the stationary state, all the atomic population is pumped into the non–coupled state, which is not excited by the laser radiation, so that the excited state population, hence the emitted fluorescent intensity reaches the minimum.

In [6] trapped states were used to propose a scheme to utilize photons for ideal quantum transmission between atoms located at spatially separated nodes of a quantum network.

Now, the process of driving the atom to the non coupled state is dynamical and therefore it is natural to try and apply the stochastic limit technique [7] in order to deduce a (non phenomenological) master equation for which the $|NC\rangle$ state is an attractor. The present note essentially confirms this intuition, but also shows that the situation is more complex than this. In fact the $|NC\rangle$ state is indeed an attractor for the reduced dynamics, but not a global one: there exists a one dimensional parameter family of stationary states and the precise interval of this parameter is determined in Theorem 4 below. This means that the set of atomic states is split into nonintersecting sets, one for each value of the parameter, and each of these sets is a domain of attraction for the state corresponding to the given value of the parameter. These stationary states generally are mixed. Only one of these states is pure and coincides with the known non–coupled, or population trapped state (2), investigated in [1]–[5].

This extends the effect of population trapping and predicts a dependence of the trapped state on the preparation procedure. This also extends our ability to control the quantum states of the atom. In fact, by preparing the initial state, switching the interaction with the field and (possibly) filtering the excited state, one can realize a switch between the $|NC\rangle$ and the $|C\rangle$ states.

In the present note the role of the velocity of the atoms has not been investigated. It is natural to conjecture, from the analysis of [2], [5], that the selection parameter of the stationary states depends on this velocity. The explicit form of this dependence is now under investigation. Also the dependence on the initial state of the field is investigated. It is shown that the above scenario can be realized only in equilibrium or non equilibrium, but non vacuum, states. In the vacuum state new phenomena, such as quantum beats, may arise (cf. Remark 11 below).

## 2 The master equation

For the investigation of the dynamics of a 3–level Λ–system interacting with radiation we use the stochastic limit approach, [7]. In this approach one introduces a slow time scale $t/\lambda^2$, where $\lambda$ is a coupling constant for the interaction of the system with radiation. In the limit $\lambda \to 0$ the dynamics is given by Langevin and master equations, cf. [7], [8], which are unambiguously derived from the original Hamiltonian. For the mathematical discussion of Langevin and master equations see also [9], [10]. Evolution of the slow degrees of freedom of the filed in the stochastic limit approach was considered in [11].

We consider a 3–level system with degenerate (for example, hyperfine split) ground states $|1\rangle$, $|2\rangle$ and the excited state $|3\rangle$.

The interaction of the system with the radiation field is described by the Hamiltonian

$$H = H_S + H_R + \lambda H_I$$

(3)
where the system degrees of freedom are described by the Hamiltonian $H_S$:

$$H_S = \varepsilon_1 |1\rangle\langle 1| + \varepsilon_2 |2\rangle\langle 2| + \varepsilon_3 |3\rangle\langle 3|$$

where $\varepsilon_i$ is the energy of the level $|i\rangle$ (note that $\varepsilon_1 = \varepsilon_2$).

The radiation degrees of freedom are described by the Hamiltonian

$$H_R = \sum_i \int \omega(k) a_i^*(k) a_i(k) dk$$

where $a_i(k)$ is a boson field with a mean zero gauge invariant Gaussian state characterized by the pair correlations

$$\langle a_i^*(k) a_j(k') \rangle = N_i(k) \delta_{ij} \delta(k - k')$$

and $i, j = 1, 2$ are the polarization indices.

The interaction Hamiltonian $H_I$ is defined as follows

$$H_I = \int \sum_{i\alpha} g_{i\alpha}(k) a_i(k) D_{\alpha}^* dk + \text{h.c.}$$

where $\alpha$ takes two values 1 and 2 and

$$D_1 = |1\rangle\langle 3|, \quad D_2 = |1\rangle\langle 2|$$

The free evolution of the interaction is equivalent to an effective free evolution of the boson field of the form

$$e^{-it(\omega(k) - \omega)} a_i(k)$$

where $\omega = \varepsilon_3 - \varepsilon_1$ is the Bohr frequency, which is equal to the difference of energies of the two energy levels.

By the stochastic golden rule [7] the rescaled free evolution of the field above, in the stochastic limit, becomes a quantum white noise $b_{i\omega}(t, k)$, or master field satisfying the commutation relations

$$[b_{i\omega}(t, k), b_{j\omega'}(t', k')] = 2\pi \delta_{\omega, \omega'} \delta_{ij} \delta(t - t') \delta(\omega(k) - \omega) \delta(k - k')$$

and with the mean zero gauge invariant Gaussian state with correlations:

$$\langle b_{i\omega}^*(t, k) b_{j\omega'}(t', k') \rangle = 2\pi \delta_{\omega, \omega'} \delta_{ij} \delta(t - t') \delta(\omega(k) - \omega) \delta(k - k') N_i(k)$$

$$\langle b_{i\omega}^*(t, k) b_{j\omega'}^*(t', k') \rangle = 2\pi \delta_{\omega, \omega'} \delta_{ij} \delta(t - t') \delta(\omega(k) - \omega) \delta(k - k') (N_i(k) + 1)$$

The Schrödinger equation becomes a white noise Hamiltonian equation, cf. [7], [8] which when put in normal order is equivalent to the quantum stochastic differential equation (QSDE)

$$dU_t = (-i dH(t) - Gdt)U_t \quad ; \quad t > 0$$

with initial condition $U_0 = 1$ and where

(i) $h(t)$ is the white noise Hamiltonian and $dH(t)$, called the martingale term, is the stochastic differential:

$$dH(t) = \int_t^{t+dt} h(s) ds = \sum_{i\alpha\omega} (D_{\alpha}^* dB_{i\omega}(t) + D_{\alpha} dB_{i\omega}^*(t))$$
driven by the quantum Brownian motions

\[ dB_{i\omega}(t) := \int_{t}^{t+dt} \int dk \overline{g_{ia}(k)} b_{i\omega}(\tau, k) d\tau =: \int_{t}^{t+dt} b_{i\omega}(\tau, g_{i\alpha}) d\tau \]  \hspace{1cm} (12)

(ii) The operator \( G \), called the drift, is given by

\[ G = \sum_{i\alpha, i\beta, \omega} \left( (g_{i\alpha}|g_{i\beta})_\omega \right) \overline{D_\alpha D_\beta} + \left( g_{i\alpha}|g_{i\beta}\right)^\dagger_\omega \overline{D_\alpha D_\beta} \]  \hspace{1cm} (13)

where the explicit form of the constants \( (g_{i\alpha}|g_{i\beta})_\omega^\pm \), called the generalized susceptivities, is:

\[ (g_{i\alpha}|g_{i\beta})_\omega^- = -i \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} \frac{N_i(k) + 1}{\omega(k) - \omega - i0} \]  \hspace{1cm} (14)

\[ = \pi \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} (N_i(k) + 1)\delta(\omega(k) - \omega) - i \text{P.P.} \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} \frac{N_i(k) + 1}{\omega(k) - \omega} \]  \hspace{1cm} (15)

\[ (g_{i\alpha}|g_{i\beta})_\omega^+ = -i \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} \frac{N_i(k)}{\omega(k) - \omega - i0} \]

\[ = \pi \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} N_i(k)\delta(\omega(k) - \omega) - i \text{P.P.} \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} \frac{N_i(k)}{\omega(k) - \omega} \]

We will use the notations

\[ \text{Re} \left( g_{i\alpha}|g_{i\beta} \right)_\omega^\pm = \pi \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} N_i(k)\delta(\omega(k) - \omega) \]  \hspace{1cm} (16)

\[ \text{Im} \left( g_{i\alpha}|g_{i\beta} \right)_\omega^\pm = -i \text{P.P.} \int dk \overline{g_{i\alpha}(k)g_{i\beta}(k)} \frac{N_i(k)}{\omega(k) - \omega} \]  \hspace{1cm} (17)

\[ (g_{i\alpha}|g_{i\beta})_\omega^+ = \text{Re} \left( g_{i\alpha}|g_{i\beta} \right)_\omega^+ + i \text{Im} \left( g_{i\alpha}|g_{i\beta} \right)_\omega^+ \]  \hspace{1cm} (18)

\[ \overline{(g_{i\alpha}|g_{i\beta})_\omega^+} = \text{Re} \left( g_{i\alpha}|g_{i\beta} \right)_\omega^+ - i \text{Im} \left( g_{i\alpha}|g_{i\beta} \right)_\omega^+ \]  \hspace{1cm} (19)

Note that for \( \alpha = \beta \) the values (16), (17) coincides with real and imaginary part of the generalized susceptivities, but the are not necessarily equal to real and imaginary parts for \( \alpha \neq \beta \). The values in (18), (19) are related in the following way: the complex conjugation of \( (g_{i\alpha}|g_{i\beta})_\omega^+ \) is equal to \( \overline{(g_{i\beta}|g_{i\alpha})_\omega^+} \).

We also use the notation

\[ (g_{i\alpha}|g_{i\beta})_\omega^\pm = \sum_i (g_{i\alpha}|g_{i\beta})_\omega^\pm \]  \hspace{1cm} (20)

**Remark 1.** Typically the \( g_{i\alpha} \) are matrix elements (cf. the description in section 4.9.3 of [7]). Therefore their dependence on the index \( \alpha \) is often unavoidable. Therefore we will develop the theory, as far as possible, keeping this dependence explicit. In some cases, e.g. some particular classes of 3–level atoms, the assumption that the formfactors \( g_{i\alpha} \) do not depend on the index \( \alpha \), is justified. In this case the formulae simplify and are easier to interpret. Thus situation is described in the Section 3 below.
Remark 2. Note that if the expectation of number operators $N_i(k)$ in the reference state depends only on the dispersion $\omega(k)$ (in this case we will denote this value $N(\omega)$), then we have the identity

$$R_\omega = \frac{\text{Re} \left( g|g\rangle_\omega^+ \right)}{\text{Re} \left( g|g\rangle_\omega^+ \right)} = \frac{N(\omega)}{N(\omega) + 1}$$

(21)

which shows that this quotient of generalized susceptivities does not depend on the formfactor $g$ and is a natural non equilibrium generalization of the Einstein emission–absorption coefficient (cf. section 5.9 of [7]).

The master equation for the reduced density matrix $\rho(t)$ of the system for a general discrete system with the dipole interaction in the stochastic limit approach was found in [8]. For the considered degenerate 3–level $\Lambda$–system it takes the form

$$\frac{d\rho(t)}{dt} = \sum_j \left( i \text{ Im} \left( g_{j1}|g_{j1}\rangle_\omega \langle \rho, |3\rangle\langle 3| \right) - i \text{ Im} \left( g_{j1}|g_{j1}\rangle_\omega^+ \langle \rho, |1\rangle\langle 1| \right) + 2\text{Re} \left( g_{j1}|g_{j1}\rangle_\omega \langle \rho, |3\rangle\langle 3| \right) + 2\text{Re} \left( g_{j1}|g_{j1}\rangle_\omega^+ \langle \rho, |1\rangle\langle 1| \right) \right)$$

$$+ \left( i \text{ Im} \left( g_{j2}|g_{j2}\rangle_\omega \langle \rho, |3\rangle\langle 3| \right) - i \text{ Im} \left( g_{j2}|g_{j2}\rangle_\omega^+ \langle \rho, |2\rangle\langle 2| \right) + 2\text{Re} \left( g_{j2}|g_{j2}\rangle_\omega \langle \rho, |3\rangle\langle 3| \right) + 2\text{Re} \left( g_{j2}|g_{j2}\rangle_\omega^+ \langle \rho, |2\rangle\langle 2| \right) \right)$$

$$+ \left( -i \text{ Im} \left( g_{j1}|g_{j1}\rangle_\omega \langle \rho, |1\rangle\langle 2| \right) + 2\text{Re} \left( g_{j1}|g_{j1}\rangle_\omega \langle \rho, |3\rangle\langle 3| \right) + 2\text{Re} \left( g_{j1}|g_{j1}\rangle_\omega^+ \langle \rho, |1\rangle\langle 2| \right) \right)$$

$$+ \left( -i \text{ Im} \left( g_{j2}|g_{j2}\rangle_\omega \langle \rho, |1\rangle\langle 2| \right) + 2\text{Re} \left( g_{j2}|g_{j2}\rangle_\omega \langle \rho, |3\rangle\langle 3| \right) + 2\text{Re} \left( g_{j2}|g_{j2}\rangle_\omega^+ \langle \rho, |1\rangle\langle 2| \right) \right)$$

(22)

where, as usual $[a, b] = ab - ba$ and $\{a, b\} = ab + ba$.

One of our main results is the following separation of the density matrix into parts corresponding to invariant subspaces of the evolution.

Lemma 1. The vector space $H(3)$ of the Hermitian $3 \times 3$ matrices is the direct sum of two subspaces, $V_0$, $V_1$, which are invariant under the evolution, defined by (22):

$$H(3) = V_0 \oplus V_1$$

A linear basis of $V_0$ is given by $\{|2\rangle\langle 3|, |3\rangle\langle 2|, |3\rangle\langle 1|, |1\rangle\langle 3|\}$. Any matrix in this space decays exponentially to zero under the reduced evolution if the real parts of generalized susceptivities (16) (for the indices $\alpha = 1, 2$ and $\beta = 3$ and vice versa) are non zero.

A linear basis of $V_1$ is given by $\{|2\rangle\langle 1|, |1\rangle\langle 2|, |3\rangle\langle 3|, |1\rangle\langle 1|, |2\rangle\langle 2|\}$. This space contains all the stationary states for the evolution.

Proof Direct verification from the right hand side of (master3).

Remark 3. Notice that the space $V_1 = \mathbb{C}|3\rangle\langle 3| \oplus M$, where $M$ is the $2 \times 2$ matrix algebra generated by $|1\rangle\langle 2|$, is itself a $\ast$–algebra.
Remark 4. From Lemma 1 we deduce that the evolution of the density matrix $\rho(t)$ can be split into the sum of two evolutions $\rho_0(t)$ and $\rho_1(t)$, where $\rho_0(t)$ is an off diagonal matrix and $\rho_1(t)$ is a density matrix.

$$\rho(t) = \rho_0(t) + \rho_1(t) = \begin{pmatrix} 0 & \rho_{32}(t) & \rho_{33}(t) \\ \rho_{23}(t) & 0 & 0 \\ \rho_{13}(t) & 0 & 0 \end{pmatrix} + \begin{pmatrix} \rho_{33}(t) & 0 & 0 \\ 0 & \rho_{22}(t) & \rho_{21}(t) \\ 0 & \rho_{12}(t) & \rho_{11}(t) \end{pmatrix}$$

Moreover, $\|\rho_0(t)\| \leq e^{-ct}$, where $2c = \min \text{Re} (g_{j\alpha}|g_{j\alpha})^\dagger$, $j = 1, 2$, $\alpha = 1, 2$, i.e. the off–diagonal part of $\rho(t)$ (in $V_0$) decays exponentially whenever $c > 0$.

3 Stationary states for the master equation

In the present section we will describe the set of stationary states for the evolution, generated by the master equation (22). By lemma 1 the invariant states of (22) belong to space $V_1$. On this subspace equation (22) reduces to the following system of three differential equations

$$\frac{d\rho_{22}(t)}{dt} = 2\text{Re} (g_2|g_2)_\omega \rho_{33} - 2\text{Re} (g_2|g_2)^\dagger \rho_{22} - (g_1|g_2)_\omega \rho_{21} - (g_1|g_2)^\dagger \rho_{12}$$  \hspace{1cm} (23)

$$\frac{d\rho_{11}(t)}{dt} = 2\text{Re} (g_1|g_1)_\omega \rho_{33} - 2\text{Re} (g_1|g_1)^\dagger \rho_{11} - (g_2|g_1)_\omega \rho_{21} - (g_2|g_1)^\dagger \rho_{12}$$  \hspace{1cm} (24)

$$\frac{d\rho_{12}(t)}{dt} = -((g_1|g_1)^\dagger + (g_2|g_2)_\omega) \rho_{12} - (g_1|g_2)_\omega \rho_{11} - (g_2|g_1)_\omega \rho_{22} + 2\text{Re} (g_2|g_1)^\dagger \rho_{33}$$  \hspace{1cm} (25)

which together with the normalization condition

$$\rho_{11} + \rho_{22} + \rho_{33} = 1$$

the conjugation rule

$$\rho_{12}^* = \rho_{21}, \quad \rho_{11}, \rho_{22}, \rho_{33} \in \mathbb{R}$$

and the conditions of positivity of the density matrix discussed in the following Lemma, form the set of equations determining the evolution of density matrix.

Lemma 2. The Hermitian matrix

$$\rho = \begin{pmatrix} \rho_{33} & 0 & 0 \\ 0 & \rho_{22} & \rho_{21} \\ 0 & \rho_{12} & \rho_{11} \end{pmatrix}$$

is a density matrix iff the diagonal elements satisfy

$$\rho_{11} + \rho_{22} + \rho_{33} = 1, \quad \rho_{11}, \rho_{22}, \rho_{33} \geq 0$$  \hspace{1cm} (26)

and the off-diagonal elements satisfy

$$\rho_{12}^* = \rho_{21}, \quad |\rho_{12}|^2 \leq \rho_{11}\rho_{22}$$  \hspace{1cm} (27)
Lemma 3. When the susceptivities \((g_\alpha | g_\beta)_{\omega}^\pm\) do not depend on \(\alpha, \beta\) (in this case we denote them \((g|g)^\pm_{\omega}\)) the system (23)–(25) of linear equations determining the evolution of the atom has the conservation law
\[
\rho_{11}(t) + \rho_{22}(t) = \rho_{12}(t) + \rho_{21}(t) + C; \quad \forall t
\] (28)
Moreover, if Re\((g|g)_{\omega}^-\) > 0, then \(\rho_{12}(t) + \rho_{21}(t)\) converges exponentially in time to the stationary value
\[
\rho_{12} + \rho_{21} = \frac{4 \text{Re}(g|g)_{\omega}^- (1 - C) - 2 \text{Re}(g|g)_{\omega}^+ C}{4(\text{Re}(g|g)_{\omega}^- + \text{Re}(g|g)_{\omega}^+)} = \frac{1 - C - CR_{\omega}/2}{1 + R_{\omega}}
\] (29)
where \(C\) is a real constant.

Proof. The conservation law (28) follows from the identity
\[
\frac{d\rho_{11}(t)}{dt} + \frac{d\rho_{22}(t)}{dt} = \frac{d\rho_{12}(t)}{dt} + \frac{d\rho_{21}(t)}{dt}
\]
It implies that for fixed \(C\) the evolution of the system is characterized by the real function of time \(\rho_{12}(t) + \rho_{21}(t)\) which we denote \(2s(t)\):
\[
s(t) = \frac{1}{2}(\rho_{12}(t) + \rho_{21}(t))
\]
With this notation the system (23)–(25) implies that
\[
\frac{ds(t)}{dt} = -4 \left(\text{Re}(g|g)_{\omega}^- + \text{Re}(g|g)_{\omega}^+\right) s(t) + 2\text{Re}(g|g)_{\omega}^-(1 - C) - \text{Re}(g|g)_{\omega}^+ C
\] (30)
If Re\((g|g)_{\omega}^-\) > 0, then equation (30) implies the exponential decay of \(s(t)\) to the stationary value (29) and this proves the lemma.

Remark 5. Note that the condition Re\((g|g)_{\omega}^-\) > 0 means that
\[
\int \overline{g(k)g(k)} \delta(\omega(k) - \omega) dk \neq 0
\]
which is automatically satisfied when the support of the formfactor \(g(k)\) intersects with the resonant surface \(\omega(k) = \omega\).

The stationary solution of the system (23)–(25) is determined by the system of equations
\[
2\text{Re}(g_2|g_2)_{\omega}^- (\rho_{11} + \rho_{22}) + 2\text{Re}(g_2|g_2)_{\omega}^+ \rho_{22} + (g_1|g_2)_{\omega}^+ \rho_{12} + (g_1|g_2)_{\omega}^- \rho_{12} = 2\text{Re}(g_2|g_2)_{\omega}^-
\] (31)
\[
2\text{Re}(g_1|g_1)_{\omega}^- (\rho_{11} + \rho_{22}) + 2\text{Re}(g_1|g_1)_{\omega}^+ \rho_{11} + (g_2|g_1)_{\omega}^+ \rho_{12} + (g_2|g_1)_{\omega}^- \rho_{12} = 2\text{Re}(g_1|g_1)_{\omega}^-
\] (32)
\[
\left((g_1|g_1)_{\omega}^+ + (g_2|g_2)_{\omega}^+\right) \rho_{12} = -(g_1|g_2)_{\omega}^+ \rho_{11} - (g_2|g_1)_{\omega}^+ \rho_{22} + 2\text{Re}(g_2|g_1)_{\omega}^- \rho_{33}
\] (33)

Remark 6. For different formfactors \(g_\alpha(k)\) the system (23)–(25) may have different behaviors. In the generic case for \(g_1 \neq g_2\) the stationary solution is unique. For instance when \(g_1\) is orthogonal
to \(g_2\) (in the sense of the bilinear form \((g_1|g_2)^\dagger\), then the determinant of the system (31), (32) reduces to

\[
-\left(2 \text{Re} (g_2|g_2)^\dagger 2 \text{Re} (g_1|g_1) + 2 \text{Re} (g_2|g_2)^\dagger 2 \text{Re} (g_1|g_1)^\dagger + 2 \text{Re} (g_2|g_2)^\dagger 2 \text{Re} (g_1|g_1)^\dagger\right)
\]

and whenever this determinant is non-zero, the solution is unique.

When \(g_1 = g_2\) the solution is non–unique due to Lemma 3.

Now we are ready to formulate the following theorem describing the structure of the stationary density matrices.

**Remark 7.** If for \(\alpha, \beta = 1, 2\)

\[(g_{\alpha}|g_{\beta})^\dagger = 0\]

in particular, in the Fock case, the stationary solutions of (23)–(25) (neglecting the trivial case when also \((g_{\alpha}|g_{\beta}) = 0\)) is characterized by the single condition

\[\rho_{11} + \rho_{22} = 1, \quad \rho_{11}, \rho_{22} \geq 0\]

so that \(\rho_{33} = 0\) and \(\rho_{12}\) is arbitrary and subject only to the constraints (27).

**Theorem 4.** For \((g_{\alpha}|g_{\beta})^\dagger\) not depending on \(\alpha, \beta\) and when

\[\text{Re} (g|g)^\dagger > 0\]  

the system (31)–(33) of linear equations determining the stationary state of the atom possesses a family of solutions parameterized by the one–dimensional parameter:

\[\rho = \begin{pmatrix} \rho_e & 0 & 0 \\ 0 & \rho_g & s \\ 0 & s & \rho_g \end{pmatrix}\]  

where, in the notations (21)

\[\rho_e = \frac{2 \text{Re} (g|g)^\dagger (1 + 2s)}{4 \text{Re} (g|g)^\dagger + 2 \text{Re} (g|g)^\dagger} = \frac{(1 + 2s)R_{\omega}}{2 + R_{\omega}}\]  

\[\rho_g = \frac{2 \text{Re} (g|g)^\dagger - 2 \text{Re} (g|g)^\dagger s}{4 \text{Re} (g|g)^\dagger + 2 \text{Re} (g|g)^\dagger} = \frac{1 - sR_{\omega}}{2 + R_{\omega}}\]

The admissible values of the parameter \(s\) are precisely those for which

\[\frac{1}{2(1 + R_{\omega})} = \frac{1}{2} \left(1 + \frac{\text{Re} (g|g)^\dagger}{\text{Re} (g|g)^\dagger}\right)^{-1} \geq s \geq -\frac{1}{2}\]

Moreover, if (34) is satisfied, the solution of the system (23)–(25) converges, as \(t \to \infty\), to the stationary state (35).

**Proof** If \(g_1 = g_2 = g\), then (31)–(33) take respectively the form:

\[2 \text{Re} (g|g)^\dagger (\rho_{11} + \rho_{22}) + 2 \text{Re} (g|g)^\dagger \rho_{22} + (g|g)^\dagger \rho_{21} + (g|g)^\dagger \rho_{12} = 2 \text{Re} (g|g)^\dagger\]
implies

\[ 2\Re (g|g) \rho_{11} + 2\Re (g|g)^\dagger \rho_{22} = 2\Re (g|g) \rho_{12} = 2\Re (g|g) \rho_{21} = 2\Re (g|g) \rho_{12} \]

Equations (42) and (43) imply that any stationary density matrix must satisfy the following condition:

\[ 2\Re (g|g)^\dagger (\rho_{22} - \rho_{21}) + 2i (\Im (g|g)^\dagger) (\rho_{21} - \rho_{12}) = 0 \]

Taking the sum of two equations above and dividing by two, we get

\[ (g|g)^\dagger (\rho_{22} - \rho_{11} + \rho_{21} - \rho_{12}) = 0 \]

If \( (g|g)^\dagger \neq 0 \), then since \( \rho_{22} - \rho_{11} \) is real, and \( \rho_{12} - \rho_{21} \) is imaginary, we obtain

\[ \rho_{22} = \rho_{11}, \quad \rho_{21} = \rho_{12} \]

Then, the sum of (39) and (40) takes the form

\[ 2\Re (g|g)^\dagger (\rho_{11} + \rho_{22} + \rho_{12} + \rho_{21}) = 4\Re (g|g) \rho_{33} \]

Equations (42) and (43) imply that any stationary density matrix must satisfy the following condition:

\[ 2\Re (g|g)^\dagger (\rho_{11} + \rho_{12}) = 2\Re (g|g) \rho_{33} \]

Since under the condition (42) the equations (39) and (40) coincide, equations (42), (44) describe the general stationary solution for (23)–(25). Using (42), (44) and (26), we obtain

\[ \frac{\rho_{11} - \rho_{22}}{4\Re (g|g) + 2\Re (g|g)^\dagger}, \quad \rho_{33} = \frac{2\Re (g|g)^\dagger + 4\Re (g|g) \rho_{12} - 2\Re (g|g) \rho_{21}}{4\Re (g|g) + 2\Re (g|g)^\dagger} \]

In particular, \( \rho_{12} \) must be a real number. From (45) and (27) one sees that the positivity of the density matrix is equivalent to inequalities

\[ \frac{1}{2} \left( 1 + \frac{\Re (g|g)^\dagger}{\Re (g|g)} \right)^{-1} \geq \rho_{12} \geq -\frac{1}{2} \]

Conversely, taking any real value of \( \rho_{12} \) satisfying (46) and determining \( \rho_{11} \) and \( \rho_{33} \) by (45), one obtains a stationary state for the master equation (23)–(25).

Let us now prove the convergence of the system to a stationary state. The system (23)–(25) implies

\[ \frac{d}{dt} (\rho_{22} - \rho_{11}) = -2\Re (g|g)^\dagger (\rho_{22} - \rho_{11}) + 2i \Im (g|g)^\dagger (\rho_{12} - \rho_{21}) \]

\[ \frac{d}{dt} (\rho_{12} - \rho_{21}) = -2\Re (g|g)^\dagger (\rho_{12} - \rho_{21}) + 2i \Im (g|g)^\dagger (\rho_{22} - \rho_{11}) \]

Adding these two equations we see that

\[ \rho_{22} - \rho_{11} + \rho_{12} - \rho_{21} = \text{const} e^{i(-2\Re (g|g)^\dagger + 2i \Im (g|g)^\dagger)} \]
For the case \( \text{Re}(g|g)_{\omega}^+ > 0 \) the linear combination (49) converges exponentially to zero. Since \( \rho_{22} - \rho_{11} \) is real and \( \rho_{12} - \rho_{21} \) is imaginary, we obtain that (49) converges to the state where \( \rho_{22} = \rho_{11} \) and \( \rho_{12} = \rho_{21} \) (and therefore real).

Then, applying Lemma 3, we get that \( \rho_{12} = \rho_{21}, \rho_{22} = \rho_{11} \) and \( \rho_{33} \) converge to stationary values, which are controlled by the stationary value \( s = \frac{1}{2}(\rho_{12} + \rho_{21}) \).

This finishes the proof of the theorem.

**Remark 8.** Note that if \( \text{Re}(g|g)_{\omega}^- = 0 \) and \( \text{Im}(g|g)_{\omega}^- \neq 0 \) then (49) implies that the system does not converge to a stationary state but has an oscillatory behavior.

Since the generalized susceptivities are given by the expression

\[
\text{Re}(g_i|g_i)_{\omega}^+ = \pi \int |g_i(k)|^2 N_i(k) \delta(\omega(k) - \omega) dk
\]

\[
\text{Re}(g_i|g_i)_{\omega}^- = \pi \int |g_i(k)|^2 (N_i(k) + 1) \delta(\omega(k) - \omega) dk
\]

\[
\text{Re}(g|g)_{\omega}^- = \sum_i \text{Re}(g_i|g_i)_{\omega}^-
\]

It follows that one has inequality

\[
\text{Re}(g|g)_{\omega}^- > \text{Re}(g|g)_{\omega}^+
\]

One can see that for high intensity of radiation, i.e. when \( N_i(k) >> 1 \), one can put \( \frac{N_i(k)}{N_i(k) + 1} = 1 \). In this case the solution (35), (38) will be simplified as follows

\[
\rho = \left( \begin{array}{ccc}
\frac{1+2s}{3} & 0 & 0 \\
0 & \frac{1-s}{3} & s \\
0 & s & \frac{1-s}{3}
\end{array} \right), \quad \frac{1}{4} \geq s \geq -\frac{1}{2}
\]

The most interesting states correspond to the extremal values of the parameter \( \rho_{12} \). The minimal value of \( \rho_{12} \) is \( -\frac{1}{2} \), which correspond to the density matrix for the pure state \(|NC\rangle\): 

\[
\rho_{\text{min}} = \frac{1}{2} \left( \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{array} \right) = |NC\rangle\langle NC| = \frac{1}{2} (|1\rangle\langle 1| + |2\rangle\langle 2| - |1\rangle\langle 2| - |2\rangle\langle 1|)
\]

where the vector

\[
|NC\rangle = \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle)
\]

is exactly the coherent population trapped state (2), discussed in the literature [1]–[4]. In the same approximation the maximal value \( \rho_{12} = \frac{1}{4} \) corresponds to the density matrix

\[
\rho_{\text{max}} = \frac{1}{4} \left( \begin{array}{ccc}
2 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{array} \right) = \frac{1}{2} |3\rangle\langle 3| + \frac{1}{2} |C\rangle\langle C|
\]
This state is mixed, but the state of the reduced system corresponding to levels $|1\rangle$ and $|2\rangle$ is pure with the state vector

$$|C\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle)$$

which coincides with the coupled state (1).

Thus the application of the stochastic limit approach allows to generalize the coherent population trapping phenomenon. We see that the family of stationary density matrices realizes a continuous interpolation between the coupled and the non–coupled state.

**Remark 9.** To distinguish experimentally different population trapped stationary states, one can measure the following observable

$$A = |1\rangle\langle 2| + |2\rangle\langle 1|$$

which for instance may describe the interaction of the hyperfine split levels with a magnetic field.

In fact the measurement of $A$ in the generic stationary state gives

$$\text{tr} \rho A = \rho_{12} + \rho_{21} = 2s$$

and the different trapped states give different mean values of $A$.

**Remark 10.** Equations (37), (38) imply that the minimum ground state population is achieved when the parameter $s$ is maximum, i.e.

$$\rho_{g}^{\text{min}} = \frac{1}{2 + R_\omega} \left( 1 - \frac{R_\omega}{2 + 2R_\omega} \right) = \frac{1}{2} \frac{1}{1 + R_\omega} = \frac{1}{2} \frac{N(\omega) + 1}{2N(\omega) + 1} > \frac{1}{4}$$

Since the ground level population is $2 \rho_g$, it follows that in any stationary state at least 1/2 of the population is in the ground level. On the other hand the above chain of identities shows, in that the region of high radiation intensity $N(\omega) >> 1$, the estimate $\rho_{g}^{\text{min}} = \frac{1}{4}$ is almost exact and therefore we can conclude that, in this region, for any stationary state, the population of the excited state is about 1/2.

**Remark 11.** Consider the regime when there is no decay to the stationary state and we have the oscillations. This regime is possible when $\text{Re} (g_{j\alpha}|g_{j\beta})^+ = 0$. We consider again the case when the susceptivities $(g_{\alpha}|g_{\beta})^+_{\omega}$ do not depend on $\alpha$, $\beta$ and all $\text{Re} (g_{j\alpha}|g_{j\beta})^+_{\omega} = 0$.

By Remark 5 after Lemma 3 it is natural to assume that $\text{Re} (g|g)^-_{\omega} > 0$ and there is a convergence of $s(t) = \frac{1}{2} (\rho_{12}(t) + \rho_{21}(t))$ to its stationary value $-\frac{1}{2} \leq s \leq \frac{1}{2}$. Analyzing the system equations for the density matrix, one can check that in the considered case the dynamics in the invariant subspace $V_1$ is described by equation (49), which takes the form

$$\rho_{22} - \rho_{11} + \rho_{12} - \rho_{21} = \text{const} e^{2it} \text{Im} (g|g)^+_{\omega}$$

where $\rho_{ij}$ are complex numbers satisfying Lemma 2.

This kind of pure oscillatory behavior without damping is related to the quantum beating. When $\text{Re} (g|g)^-_{\omega} > 0$, the off–diagonal matrix elements $\rho_{13}$, $\rho_{23}$ decay exponentially by Remark 4, cf. [8]. We see that in the regime $\text{Re} (g|g)^-_{\omega} > 0$, $\text{Re} (g|g)^+_{\omega} = 0$ (which is satisfied, for instance
in the Fock (vacuum) state) the behavior of the 3–level degenerate Λ–system for large times is described by the oscillations (50), when \( \rho_{13} = \rho_{23} = \rho_{31} = \rho_{32} = 0 \) and \( s(t) = \frac{1}{2} (\rho_{12}(t) + \rho_{21}(t)) = \text{const}, \ -\frac{1}{2} \leq s \leq \frac{1}{2}. \)

In conclusion: in the present paper we investigated the interaction of an atom with a degenerate ground state with a quantum field. We find (under natural conditions for the formfactors), that the evolution drives the atom exponentially to a stationary state. This stationary state is not unique, and the family of stationary states may be parameterized by a one–dimensional parameter. For a special (minimal) value of this parameter the obtained stationary state is pure and coincides with the population trapped state, known in the literature [1]–[4]. The obtained results show the possibility of emergence of mixed stationary states, which continuously interpolate between the coupled and the non–coupled states. This difference can be experimentally detected.

In the case of special states (the Fock state) also the oscillatory behavior (50) is possible.

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