Functional integral with $\varphi^4$ term in the action beyond standard perturbative methods II

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To avoid problems with infinite measure, the functional integral for harmonic oscillator can be calculated by time-slicing method with continuum limit procedure proposed Gelfand and Yaglom. In previous article we proved by nonperturbative calculation the generalized Gelfand-Yaglom equation for anharmonic oscillator with positive or negative mass term. In this article we prove by step-by-step the calculation of the correction function to the Gelfand-Yaglom equation for anharmonic oscillator.

Introduction

Let us demonstrate the meaning of continuum limit procedure proposed by Gel’fand-Yaglom on the example of the harmonic oscillator. In Euclidean variant of the theory the continuum functional integral for harmonic oscillator can be read:

$$Z = \int [D\varphi(x)] \exp(-S)$$

Where the euclidean action is:

$$S = \int_0^\beta d\tau \left[ c/2 \left( \frac{\partial \varphi(\tau)}{\partial \tau} \right)^2 + b\varphi(\tau)^2 \right]$$ (1)

In time-slicing approximation of Wiener unconditional measure functional integral we calculate the $N-$dimensional integral:

$$Z_N^0 = \int \prod_{i=1}^{N} \left( \frac{d\varphi_i}{\sqrt{2\pi\triangle}} \right) \exp \left\{ -\sum_{i=1}^{N} \triangle \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\triangle} \right)^2 + b\varphi_i^2 \right] \right\}$$ (2)

where $\triangle = \beta/N$, and $b, c$ are the parameters of the model. The unconditional measure $N$ dimensional approximation (fixed $\varphi_0$ and integration over $\varphi_N$) can be evaluated explicitly:

$$Z_N^0 = \left[ \prod_{i=0}^{N} 2(1 + b\triangle^2/c)\omega_i \right]^{\frac{1}{\triangle}}$$ (3)

where $\omega_i$ is defined by recursion

$$\omega_i = 1 - \frac{A^2}{\omega_{i-1}}$$

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with the first term
\[ \omega_0 = 1/2 + \frac{b\Delta^2/c}{2(1 + b\Delta^2/c)}, \]

where
\[ A = \frac{1}{2(1 + b\Delta^2/c)}. \]

The recurrence relation for functional integral is introduced by time – slicing procedure. Explicitly this recurrence is represented by factor \( \omega_i \). Following this recurrence, we can prove the difference equation for inverse square root of the finite dimensional integral. Based on this procedure the continuum limit \( N \rightarrow \infty \) of the \( Z_0^N \) can be defined by:
\[ Z_0^{(\beta)} = \frac{1}{\sqrt{F(\beta)}}. \]

It was shown by Gelfand and Yaglom that \( F(\beta) \) is the solution of the equation:
\[ \frac{\partial^2}{\partial \tau^2} F(\tau) = \frac{2b}{c} F(\tau), \quad \tau \in (0, \beta). \] (4)

For the harmonic oscillator the equation (4) can be calculated by taking the continuum limit of the difference equation extracted from the recurrence relation for \( Z_0^N \).

The general solution of the above equation is:
\[ F(\tau) = C_1 \cosh \left( \sqrt{\frac{2b}{c}} \tau \right) + C_2 \sinh \left( \sqrt{\frac{2b}{c}} \tau \right) \] (5)

where \( C_1, C_2 \) are the constants fixed by initial conditions of the unconditional (when \( \varphi_0 \) is fixed and \( \varphi_N \) is free), or conditional (when both \( \varphi_0, \varphi_N \) are fixed) measure functional integral. From the analytical form of the result for functional integral follows the clear interpretations of the calculated quantities as energy levels and others.

In this article we report on the attempt to evaluate non-perturbatively the functional integral for an-harmonic oscillator with positive as well as negative mass squared term. Our aim is to calculate the \( N \)-dimensional integral by another method as well-known conventional perturbative calculation. We find the result suitable to evaluate by the recurrence procedure the difference equation. Following the idea of Gelfand - Yaglom we define the differential, Gelfand – Yaglom type equation for the quantity \( y(\tau) \) related (similarly as \( F(\tau) \)) to the unconditional measure functional integral. The differential equations for \( y(\tau) \) reads:
\[ \frac{\partial^2}{\partial \tau^2} y(\tau) + 4 \frac{\partial}{\partial \tau} y(\tau) \frac{\partial}{\partial \tau} \ln S(\tau) = y(\tau) \left( \frac{2b}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right)^2 \right), \] (6)

We shall evaluate the function \( S(\tau) \) analytically in this article.

As we explain in the text, our result possesses the form of an asymptotic expansion power series. Nevertheless we evaluate precisely all finite difference mathematical objects. In the article [2] quoted as "article I" and in this article we explain and prove all analytical calculations.

For an-harmonic oscillator the correction term is \(-2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right)^2 \) to the equation for the harmonic oscillator. It would be desirable to recover it using more general arguments and to see its form for some more general classes of potentials.

This article is organized as follows. In the next section we resume the article I. In the third section we evaluate the function \( S(\tau) \) by recurrence relation from the result of \( N \)-dimensional integral. In the fourth section we discuss some preliminary conclusions of our calculation.
Resume of the article I.

Our aim is to solve the problem of evaluation of the continuum unconditional measure Wiener functional integral:

\[ Z = \int |D\varphi(x)| \exp(-S) , \]

where continuum action possesses the fourth order term:

\[ S = \int_0^\beta d\tau \left[ c/2 \left( \frac{\partial\varphi(\tau)}{\partial\tau} \right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4 \right] . \]  

(7)

The functional integral \( Z \) is defined by limiting procedure of the finite dimensional integral \( Z_N \):

\[ Z_N = \prod_{i=1}^{-\infty} \left( \frac{d\varphi_i}{\sqrt{2\pi\Delta}} \right) \exp \left\{ -\sum_{i=1}^N \Delta \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\Delta} \right)^2 + b\varphi_i^2 + a\varphi_i^4 \right] \right\} , \]

(8)

where \( \Delta = \beta/N \). Then, the continuum Wiener unconditional measure functional integral is defined by the formal limit:

\[ Z = \lim_{N \to \infty} Z_N . \]

The first important task is to calculate the one dimensional integral

\[ I_1 = \int_{-\infty}^{+\infty} dx \exp\{-(\alpha x^4 + \beta x^2 + \gamma x)\} \]

for \( \text{Re} \alpha > 0 \).

Standard perturbative procedure rely on Taylor’s decomposition of \( \exp(-\alpha x^4) \) term with consecutive replacements of the integration and summation order. The integrals can be calculated, but the sum is divergent.

We propose the power expansion in \( \gamma \):

\[ I_1 = \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{n!} \int_{-\infty}^{+\infty} dx x^n \exp\{-(\alpha x^4 + \beta x^2)\} . \]

(10)

The integral is given in terms of the parabolic cylinder functions:

\[ D_{-m-1/2}(z) = \frac{e^{-z^2/4}}{\Gamma(m + 1/2)} \int_0^{+\infty} dx x^{m-1/2} \exp\{-\frac{1}{2}x^2 - zx\} . \]

(11)

The integral \( I_1 \) then can be read:

\[ I_1 = \frac{\Gamma(1/2)}{\sqrt{\beta}} \sum_{m=0}^{\infty} \frac{\xi^m}{m!} D_{-m-1/2}(z) , \]

(12)

where

\[ \xi = \frac{\gamma^2}{4\beta} , \quad z = \frac{\beta}{\sqrt{2\alpha}} , \]

and we have used the abbreviation:

\[ D_{-m-1/2}(z) = z^{m+1/2} e^{z \frac{\beta}{2}} D_{-m-1/2}(z) . \]
It was shown, that sum in Eq. (12) is convergent and for finite values of the parameters of the model this sum converges uniformly.

Applying this idea of integration on the $N$ dimensional integral (8) integral we have the result:

$$Z_N = \left[ \prod_{l=0}^N 2(1 + b\Delta^2/c)\omega_l \right]^{-\frac{1}{2}} S_N$$  \hspace{1cm} (13)

with

$$S_N = \sum_{k_1,\ldots,k_{N-1}=0}^\infty \prod_{i=0}^N \left[ \frac{(\rho)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2) \sqrt{2\omega_i} D_{-k_{i-1} - k_i - 1/2}(z) \right],$$  \hspace{1cm} (14)

where the constants and symbols in the above relation are connected to the constants of the model by the relations:

$$k_0 = k_N = 0, \quad \rho = (1 + b\Delta^2/c)^{-1}, \quad z = c(1 + b\Delta^2/c)/\sqrt{2a\Delta^3}, \quad \omega_i = 1 - A^2/\omega_{i-1}, \quad \omega_0 = 1/2 + b\Delta^2/c,$$

$$A = \frac{1}{2}(1 + b\Delta^2/c).$$

In the formula for $S_N$ is useful to mention that:

- $\rho$ is independent of the coupling constant;
- only the argument $z$ of parabolic cylinder function is coupling constant dependent;
- It was shown, that for finite values of the parameters of the model and one summation index $k_i \to \infty$ the $k_i$-th term of the sum approaches to zero as

$$\frac{k_i^\alpha \beta_k}{k_i! \exp(\sqrt{k_i})},$$

where $\alpha$ and $\beta$ are finite numbers. This asymptotic is sufficient for a proof of the uniform convergence of the series for $S_N$ not only for single $k_i$, but for arbitrary tuple \{${k_i}$\} of indices.

Following the idea of Gelfand and Yaglom the functional integral in the continuum limit is defined by the formal limit

$$\lim_{N \to \infty} Z_N = \frac{1}{\sqrt{F(\beta)}},$$

where $F(\beta)$, for our case, is the solution of the differential equation:

$$\frac{\partial^2}{\partial \tau^2} F(\tau) + 4 \left( \frac{\partial}{\partial \tau} F(\tau) \right) \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right) = F(\tau) \left( \frac{2b}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right)^2 \right), \quad \tau \in (0, \beta)$$  \hspace{1cm} (15)

calculated at the point $\beta$ (the upper limit of the time interval in the continuum action) with the initial conditions: $F(0) = 1$ and $\partial F(\tau)/\partial \tau|_{\tau=0} = 0$. Evaluating the continuum limit of the difference equation we use the convention for definition of the continuum variable $\tau = n \Delta$.

The function $S(\tau)$ is given as the continuum limit of the Eq. (14)

$$S(\tau) = \lim_{N \to \infty} S_N.$$

Equation (15) can be simplified by the substitution:

$$F(\tau) = \frac{y(\tau)}{S(\tau)^2}.$$

For $y(\tau)$ we find the equation:

$$\frac{\partial^2}{\partial \tau^2} y(\tau) = y(\tau) \left( \frac{2b}{c} \right),$$  \hspace{1cm} (16)

accompanied by the initial conditions:

$$y(0) = S(0)^2, \quad \frac{\partial y(\tau)}{\partial \tau} \bigg|_{\tau=0} = \frac{\partial S(\tau)^2}{\partial \tau} \bigg|_{\tau=0}.$$
For harmonic oscillator we have $S = 1$.

To evaluate $S_N$, we must solve the problem how to sum up the product of two parabolic cylinder functions in Eq. (14). The parabolic cylinder functions are the representation of the group of the upper triangular matrices, so we implicitly expect the simplification of the product due to a group principles. This problem was not solved completely yet. We adopt less complex method of summation, namely we use the asymptotic expansion one of parabolic cylinder function, with precise sum over $k_i$ of the rest of the relation, containing the other function. In the algebra such relation for precise summation is available. Surely, the result is degraded to the form of an asymptotic expansion only, but still we shall have an analytical solution of the problem. This procedure is widely discussed in article I, here we repeat the result:

$$Z_N = \left\{ \prod_{i=0}^{N} \left[ 2(1 + b\Delta^2/c)\omega_i \right] \right\}^{-1/2} S_N,$$

(17)

$$S_N = 1$$

$$S_N = \sum_{\mu=0}^{J} \frac{(-1)^{\mu}}{\mu!} \frac{(2z^2\Delta^3)^{\mu}}{(2z^2\Delta^3)^{\mu}} \triangle^{3\mu} (\Lambda)^{2\mu},$$

(18)

where the symbols $(\Lambda)^{2j}$ satisfy the following recurrence relation:

$$(\Lambda)^{2\mu-p} = \sum_{\lambda=0}^{\mu} \left( \begin{array}{c} \mu \\ \lambda \end{array} \right) \frac{1}{\omega^{2\mu-2\lambda}} \sum_{i=\max\{0, 2\lambda-p\}}^{2\lambda} \left( \frac{A^2}{\omega^{2\lambda-2\omega^{-1}}} \right)^i (A - 1)^{2\lambda} a_{2\mu-2\lambda+i}$$

(19)

The recurrence procedure begins from:

$$(1)^{2\lambda}_i = \frac{1}{\omega^0} a_{2\lambda}_i.$$

We repeat also the definition of the value $z$, where the dependence on the coupling constant is hidden:

$$z = c(1 + b\Delta^2/c) \sqrt{2a\Delta^3}.$$

As follows in the calculation the important role plays the objects $\omega_i$ defined by recurrence as

$$\omega_i = 1 - \frac{A^2}{\omega_i^{-1}},$$

with the first term for unconditional measure integral:

$$\omega_0 = 1/2 + \frac{b\Delta^2/c}{2(1 + b\Delta^2/c)}.$$

For the forthcoming calculation we will use the more convenient variables, introduced in Appendix 2 of article I:

$$Q_i = w_1 x^i + w_2 y^i,$$

$$\bar{Q}_i = w_1 x^i - w_2 y^i,$$

where

$$x = \frac{1}{2A} + \sqrt{\frac{1}{4A^2} - 1}, \quad y = \frac{1}{2A} - \sqrt{\frac{1}{4A^2} - 1}, \quad xy = 1,$$

$$w_1 = 1 + \frac{2B}{\sqrt{1 - 4A^2}}, \quad w_2 = 1 - \frac{2B}{\sqrt{1 - 4A^2}}.$$
\[ A = \frac{1}{2(1 + b \triangle^2/c)} , \quad B = \frac{b \triangle^2/c}{2(1 + b \triangle^2/c)}. \]

We also use the fruitful identity following from the definition of the \(n - \text{th}\) convergent of the continued fraction, calculated in article I:

\[ \frac{A}{\omega_{k-1}} = \frac{Q_{k-1}}{Q_k}. \]

The symbols \(a_i^j\) were defined by need to rewrite the Pochhammer symbols \((k + 1/2)_j\) in the form:

\[ (k + 1/2)_j = \sum_{i=0}^{\min(j,k)} a_i^j (k-i)(k-i+1), \]

and by help the recurrence procedure we found:

\[ a_i^j = \binom{j}{i} \binom{1/2}{i}. \]

In this article we explicitly evaluate the recurrence relation for \((\Lambda)_{2\mu-p}^2\) and we calculate the continuum limit of the function \(S_{\Lambda}\).

**Evaluation of the recurrence relation**

We rewrite the recurrence relation (19) into more convenient form for consecutive calculation. We introduce the quantities \(Q_k\) by the identity:

\[ \frac{A}{\omega_{k-1}} = \frac{Q_{k-1}}{Q_k}. \]

We replace the summation index \(i\) by the summation index \(j\) defined by:

\[ i = 2\lambda - j. \]

Finally, we interchange the order of summations over indexes \(j\) and \(\lambda\). We read:

\[(AQ_{\Lambda}Q_{\Lambda-1})^{2d-p} (\Lambda)^{2d-p} =
\]

\[ = \sum_{j=0}^{p} \frac{a_{2d-p}}{(AQ_{\Lambda}Q_{\Lambda-1})^{p-j}} \sum_{\lambda=\lceil \frac{j+1}{2} \rceil}^{d} (AQ_{\Lambda-2}Q_{\Lambda-1})^{2\lambda-j} (\Lambda - 1)^{2\lambda-j} \left( \frac{d}{\lambda} \right) (Q_{\Lambda-1}^{4})^{d-\lambda}, \quad p \in <0,2d> \quad (20)\]

The right hand side of the equation is \((2d, p)\)-th matrix element of the products of three matrices. For fixed \(d\) and \(p\), on the left hand side of the equation, we read only the \(d\)-th column of a matrix, which is recurrently tied to the matrix in the center of the product on left hand side. We can use the notation:

\[ X_{p,\mu}^{d}(\Lambda) = \sum_{j=0}^{p} \sum_{\lambda=\lceil \frac{j+1}{2} \rceil}^{d} A_{p,j}^{d}(\Lambda - 1)C_{j,\lambda}^{d}(\Lambda - 1)M_{\lambda,\mu}^{d}(\Lambda - 1). \]

The definition of the matrices \(A_{p,j}^{d}(\Lambda - 1), C_{j,\lambda}^{d}(\Lambda - 1), M_{\lambda,\mu}^{d}(\Lambda - 1)\) is the following:

1. The \(A_{p,j}^{d}\) is the lower triangular matrix with the zeros over the main diagonal of the dimension \((2\mu+1)(2\mu+1)\).

The principal minor of the dimension \((2d+1)(2d+1)\) is non-zero only with the elements:

\[ \{A_{p,j}^{d}(\Lambda - 1)\}_{p,j} = \frac{a_{2d-p}}{\left( AQ_{\Lambda}Q_{\Lambda-1} \right)^{p-j}}. \]
2. The $\mathbb{C}^d_{p,\lambda}(\Lambda)$ is the upper triangular matrix with the zeros under the main diagonal of the dimension $(2d+1)(\mu+1)$. The nonzero elements form the main minor of the dimension $(2d+1)(d+1)$ with $\lambda^{th}$ column:

$$\mathbb{C}^d_{p,\lambda}(\Lambda) = (AQ_{\lambda}^2 Q_{\lambda-1}^{2\lambda-p}) (\Lambda)^{2\lambda}$$

where $p = 0, 1, ..., 2\lambda$ and $\lambda = 0, 1, ..., d$.

3. The $\mathbb{M}^d_{\lambda,k}$ is the upper triangular matrix with the zeros under the main diagonal of the dimension $(\mu+1)(\mu+1)$. The nonzero elements form the main minor of the dimension $(d+1)(d+1)$:

$$\mathbb{M}^d_{\lambda,k}(\Lambda - 1) = \left( \begin{array}{c} k \\ \lambda \end{array} \right) (Q_{\lambda-1}^d)^{k-\lambda}, \quad d \geq k \geq \lambda \geq 0.$$

To evaluate the matrix $\mathbb{C}(\Lambda)$, we must calculate $\mathbb{X}(\Lambda)$ for all dimensions up to $\mu$, for each dimension to extract the last column of matrix $\mathbb{X}(\Lambda)$ and from these columns to compose the matrix $\mathbb{C}(\Lambda)$. We define such linear operation as follows:

1. Let $\mathbb{A}^d$ and $\mathbb{M}^d$ are the matrices of the dimensions $(2\mu+1)(2\mu+1)$ and $(\mu+1)\times(\mu+1)$ respectively. Let $\mathbb{C}^d$ is the matrix of dimensions $(2d+1)(2d+1)$, $(d+1)(d+1)$, and $(2d+1)(d+1)$ respectively.

2. Let $\sup\mathbb{M}$ is the supermatrix possessing on the $d-th$ place of the main diagonal the matrix $\mathbb{M}^d$ the same is defined for $\sup\mathbb{A}$.

3. The $\sup\mathbb{C}$ is the supermatrix with $d-th$ diagonal element of the form:

$$\mathbb{C}^d(\Lambda) = \sum_{i_1=0}^{d} \mathbb{X}^{d-i\lambda}(\Lambda).$$

4. Matrix $\tilde{\mathbb{X}}^d$ is the one column matrix defined by the relation:

$$\tilde{\mathbb{X}}^d(\Lambda) = \mathbb{X}^d(\Lambda) * \mathbb{P}^d,$$

where $\mathbb{P}^d$ is the projector of the $d$-th column of the matrix $\mathbb{X}^d(\Lambda)$ into $d$-th column of the matrix $\tilde{\mathbb{X}}^d(\Lambda)$. $\mathbb{P}^d$ is matrix with only nonzero term

$$\{\mathbb{P}^d\}_{d,k} = \delta_{d,k}.$$

5. The matrix $\mathbb{X}^d(\Lambda)$ is defined by relation:

$$\mathbb{X}^d(\Lambda) = \mathbb{A}^d(\Lambda - 1) * \mathbb{C}^d(\Lambda - 1) * \mathbb{M}^d(\Lambda - 1).$$

6. Then, for $\mathbb{C}^d(\Lambda)$ we have the result:

$$\mathbb{C}^d(\Lambda) = \sum_{i_1=0}^{d} \mathbb{A}^{d-i\lambda}(\Lambda - 1) * \mathbb{C}^{d-i\lambda}(\Lambda - 1) * \mathbb{M}^{d-i\lambda}(\Lambda - 1).$$

7. After evaluation of the full recurrence we find:

$$\mathbb{C}^d(\Lambda) = \sum_{i_1=0}^{d} \sum_{i_2=0}^{d-i_1} \sum_{i_3=0}^{d-i_1-i_2-\cdots-i_3} \{ \mathbb{A}^{d-i\lambda}(\Lambda - 1) * \mathbb{A}^{d-i_1-i_2-\cdots-i_3}(\Lambda - 2) * \cdots * \mathbb{A}^{d-i_1-i_2-\cdots-i_3}(1) \} * \mathbb{C}^{d-i\lambda-i_1-i_2-\cdots-i_3}(1) *$$

$$\left\{ \mathbb{M}^{d-i\lambda-i_1-i_2-\cdots-i_3}(1) * \cdots * \mathbb{M}^{d-i\lambda-i_2-\cdots-i_3}(\Lambda - 2) * \mathbb{M}^{d-i\lambda}(\Lambda - 1) \right\}.$$

To evaluate the product of two consecutive matrices from multi-product:

$$\{ \mathbb{A}^{d-i\lambda}(\Lambda - 1) * \mathbb{A}^{d-i_1-i_2-\cdots-i_3}(\Lambda - 2) * \cdots * \mathbb{A}^{d-i_1-i_2-\cdots-i_3}(1) \}$$

we use the two identities for the summation over the index $j$: 
Then, for product of two lower-triangular matrices we have:

\[ a_{2I_3-p}^{12} \cdot a_{2I_2-j}^{12} = 2^{-2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!(p-\lambda)!} \left( \frac{p-\lambda}{I_3 - 2j} \right) \left( \frac{4I_3 - 2j}{I_2 - 2j} \right)! \]

and

\[ \frac{(4I_3 - 2j)!}{(4I_2 - 2j)!} = \delta_{\epsilon}^{I_3 - 4I_2} (\epsilon^{4I_2 - 2j}) |_{\epsilon=1} \]

We introduced the abbreviation

\[ I_j = d - (i_j + i_{j+1} + \cdots + i_{\lambda}) \]

Then, for product of two lower-triangular matrices we have:

\[
\sum_{j=\lambda}^{p} \left\{ A^{I_3}(2) \right\}_{p,j} \left\{ A^{I_2}(1) \right\}_{j,\lambda} = \sum_{j=\lambda}^{p} \frac{a_{2I_3-p}^{12}}{(AQ_3Q_2)^{p-j}} \cdot \frac{a_{2I_2-j}^{12}}{(AQ_2Q_1)^{j-\lambda}} \]

\[ = 2^{-2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!(p-\lambda)!} \partial_{4I_2}^{(4)} \left\{ \left( \epsilon^{I_3 - 2p} \right) \sum_{j=\lambda}^{p} \left( \frac{p-\lambda}{I_3 - 2j} \right) \left( \frac{\epsilon^2}{AQ_3Q_2} \right)^{p-j} \left( \frac{1}{AQ_2Q_1} \right)^{j-\lambda} \right\} |_{\epsilon=1} \]

In the above relation the summation over index \( j \) can be performed explicitly. By the substitution:

\[ \epsilon^2 = \xi, \]

we find:

\[
\sum_{j=\lambda}^{p} \left\{ A^{I_3}(2) \right\}_{p,j} \left\{ A^{I_2}(1) \right\}_{j,\lambda} = 2^{-2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!(p-\lambda)!} 2^{4I_2} D_{\xi} \left\{ \frac{1}{AQ_3Q_2} \left( \frac{\xi}{AQ_3Q_2} + \frac{1}{AQ_2Q_1} \right)^{p-\lambda} \right\} |_{\xi=1} \]

where \( D_{\xi} \) is the differential operator calculated from \( \partial_{4I_2}^{(4)} \) given as:

\[ D_{\xi} = 3/4 \partial_{\xi}^2 + 3\xi \partial_{\xi}^3 + \xi^2 \partial_{\xi}^4 \]

For the resulting product of all matrices \( A^{I} (k) \) we find:

\[
\left\{ A^{d-i_{\lambda}} (\Lambda - 1) \cdot A^{d-i_{\lambda} - i_{\lambda - 1}} (\Lambda - 2) \cdot \cdots \cdot A^{d-i_{\lambda} - i_{\lambda - 1} - \cdots - i_2} (1) \right\}_{p,\lambda} =
\]

\[ = 2^{-2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!(p-\lambda)!} 2^{4I_2} D_{\xi} \left\{ \prod_{m=2}^{\Lambda} \left[ \frac{1}{AQ_3Q_2} + \frac{\xi_2}{AQ_5Q_2} + \cdots + \frac{\xi_2 \cdots \xi_{\lambda-1}}{AQ_\lambda Q_{\lambda-1}} \right]^{p-\lambda} \right\} |_{(a_d\xi_{\lambda-1})} \]

Evaluating the product of matrices:

\[
\left\{ \tilde{M}^{d-i_{\lambda} - i_{\lambda - 1} - \cdots - i_2} (1) \cdot \cdots \cdot \tilde{M}^{d-i_{\lambda} - i_{\lambda - 1}} (\Lambda - 2) \cdot \tilde{M}^{d-i_{\lambda}} (\Lambda - 1) \right\}
\]

we use that:

\( \tilde{M}_{j+1}^{I_{j+1}} \) is one-column matrix with \( I_{j+1} \) non-zero elements in \( j + 1 \) column:
\[ \left\{ \tilde{M}^i_{I_{j+1}}(j) \right\}_{\lambda,j+1} = \begin{pmatrix} I_{j+1} \\ \lambda \end{pmatrix} Q^{4(I_{j+1}-\lambda)}, \]

where \( \lambda = 0, 1, \ldots, I_{j+1} \). Product of such matrices is one-column matrix with the elements:

\[ \left\{ \tilde{M}^{d-i_{j-1}}(1) \ast \ldots \ast \tilde{M}^{d-i_{j-1}}(A-2) \ast \tilde{M}^{d-i_{j-1}}(A-1) \right\}_{\lambda,j} = \begin{pmatrix} I_2 \\ \lambda \end{pmatrix} \begin{pmatrix} I_3 \\ \lambda \end{pmatrix} \cdots \begin{pmatrix} I_{\lambda} \\ \lambda-1 \end{pmatrix} Q^{4(I_{j-2}-\lambda)}Q^{4(I_{j-2}-\lambda)} \cdots Q^{4(I_{j-2}-\lambda)} \]

From definition (19) of the recurrence steps we have for the matrix \( C^{I_2}(1) \) the nonzero elements:

\[ \{ C^{I_2}(1) \}_{j,\lambda} = \frac{Q^{4\lambda}_{0}}{(AQ_{1}Q_{0})^{2}} a_{2\lambda-j}^{2\lambda} \]

with the conditions for indices:

\[ 0 \leq j \leq 2\lambda \leq 2I_{2} \leq 2\mu \]

Collecting all partial results together, inserting them into Eq. (21) and remember that for function \( S_{\Lambda} \) defined in Eq. (18) only matrix elements \( \{ C(\Lambda)_{2\mu}^{2\mu} \}_{2\mu,2\mu} \) are important, we find the result:

\[ \{ C(\Lambda)_{2\mu}^{2\mu} \}_{2\mu,2\mu} = \sum_{\mu=0}^{\mu} \sum_{j=0}^{j} \frac{(-1)^{\mu}}{\sum_{\lambda=0}^{\lambda-1} \cdots} \left( \begin{pmatrix} I_{j} \\ \lambda \end{pmatrix} Q^{4(I_{j}-\lambda)} \cdots Q^{4(I_{j}-\lambda)} \right) \]

\[ \times \frac{1}{(AQ_{0}Q_{1})^{2}} \right]\{ \begin{pmatrix} I_{2} \\ \lambda \end{pmatrix} a_{2\lambda-j}^{2\lambda} Q^{4(I_{2}-\lambda)} \}
\]

The asymptotic function \( S_{\Lambda} \) can be read:

\[ S_{\Lambda} = \sum_{\mu=0}^{\mu} \frac{(-1)^{\mu}}{\sum_{\lambda=0}^{\lambda-1} \cdots} \left( \begin{pmatrix} I_{j} \\ \lambda \end{pmatrix} Q^{4(I_{j}-\lambda)} \cdots Q^{4(I_{j}-\lambda)} \right)
\]

because \( \{ C(\Lambda)_{2\mu}^{2\mu} \}_{2\mu,2\mu} = (\Lambda)^{2\mu}_{0} \)

Due to the analytic form for \( \{ C(\Lambda)_{2\mu}^{2\mu} \}_{2\mu,2\mu} \) we can express \( S_{\Lambda} \) in the continuum \( \triangle \rightarrow 0 \) limit as well as in asymptotic \( \mu \rightarrow \infty \) limit.

The continuum limit

In continuum limit we must take into account that:
- in Eq. (24) the \( \mu - \text{th} \) term \( \{ C(\Lambda)_{2\mu}^{2\mu} \}_{2\mu,2\mu} \) is multiplied by \( \triangle^{3\mu} \).
- the sum

\[ \left( \frac{1}{AQ_{2}Q_{1}} + \frac{\xi_2}{AQ_{3}Q_{2}} + \cdots + \frac{\xi_2 \cdots \xi_{\Lambda-1}}{AQ_{\Lambda}Q_{\Lambda-1}} \right)_{(all\xi_{m-1})} \]
is given by exact formula:
\[
\frac{1}{2Aw_1w_2(x-y)} \left( \frac{Q_\Lambda}{Q_\Lambda} - \frac{\tilde{Q}_0}{\tilde{Q}_0} \right).
\]

From the relation
\[
\frac{1}{2Aw_1w_2(x-y)} \sim \frac{1}{\sqrt{2b/c}} \Delta,
\]
we deduce that to the leading term of Eq. (23) contribute the terms with summation index \(j = 0\) with the contribution proportional to \( \left( \frac{1}{\Delta} \right)^{2\mu} \).

- to obtain the additional necessary factor \( \left( \frac{1}{\Delta} \right)^{\mu} \) the leading term must be composed from the contributions where \( \mu \) summation indices of the \( \{i_2, i_3, \ldots, i_{\Lambda-1}\} \) are equal to 1. Let it be the combination, say, \( \{i_{n_1}, i_{n_2}, \ldots, i_{n_{\mu}}\} \). The sum of these indices is equal to \( \mu \), and in such case also \( I_2 = 0 \), because of definition \( I_2 = \mu - i_{n_1} - i_{n_2} - \cdots - i_{n_{\mu}} \).

The contribution to the body of principal formula (23) then can be read:
\[
\left( I_{n_2} \right) \left( I_{n_3} \right) \cdots \left( I_{n_{\mu}} \right) Q_{n_2}^4 \cdots Q_{n_{\mu}}^4
\]
\[
\frac{1}{(2\mu)!} \left\{ \prod_{m=1}^{\mu} D^{\xi_m}_{\Lambda^m} \frac{1}{\xi_{n_m}^{2\mu-2I_{n_{m+1}}} + I_{n_{m+1}}} \right\} \left( \frac{1}{AQ_2Q_1} + \frac{\xi_2}{AQ_3Q_2} + \cdots + \frac{\xi_2 \cdots \xi_{\Lambda-1}}{AQ_{\Lambda}Q_{\Lambda-1}} \right)^{2\mu}(\text{all} \xi \rightarrow 1)
\]

We must sum over all such combinations, this can be done by \( \mu \) summations. Every summation is proportional to \( \Delta^{-1} \). Therefore leading term proportional to \( \Delta^{-3\mu} \) can be achieved only if \( \mu \) different indices \( i_n = 1 \). Taking into account, that the difference of two consecutive \( I_{n_i} \) is one, we can rewrite the dominant contribution in the continuum limit into the form:
\[
\left\{ C(\Lambda)^{2\mu} \right\}_{2\mu,2\mu} = \sum_{n_1=2}^{\Lambda-\mu} \sum_{n_2=n_1+1}^{\Lambda-\mu+1} \cdots \sum_{n_{\mu}=n_{\mu-1}+1}^{\Lambda} \mu!Q_{n_1}^4 \cdots Q_{n_{\mu}}^4
\]
\[
\frac{1}{(2\mu)!} \left\{ \prod_{m=1}^{\mu} D^{\xi_m}_{\Lambda^m} \frac{1}{\xi_{n_m}^{2\mu-2I_{n_{m+1}}} + I_{n_{m+1}}} \right\} \left( \frac{1}{AQ_2Q_1} + \frac{\xi_2}{AQ_3Q_2} + \cdots + \frac{\xi_2 \cdots \xi_{\Lambda-1}}{AQ_{\Lambda}Q_{\Lambda-1}} \right)^{2\mu}(\text{all} \xi \rightarrow 1)
\]

The effect of the operator \( D_{\xi} \).

Let us evaluate the operation of the operator \( D_{\xi} \) accompanied by operator’s variable term:
\[
\lim_{x_j \rightarrow 1} D_{x_j} \frac{1}{x_j^{2(i_j+1)+i_\Lambda}}
\]

\( i_j \) times on the \( n-th \) power of the function

\[ f_j = a_{j-1} + (x_2 \cdots x_j)b_j \]

where:

\[ a_{j-1} = \frac{1}{Q_2Q_1} + \frac{x_2}{Q_3Q_2} + \cdots + \frac{x_2 \cdots x_{j-1}}{Q_{j}Q_{j-1}} \]
The proof of this formula is given in Appendix A.

We meet very important feature of the application of the operator \( D_x \).
1. Applying \( D_x \) the first time for first nonzero \( i_j \), all \( i_{j+1}, \ldots, i_{\Lambda} \) are zero, then factor \( \frac{1}{x^{c_{j+1}+\cdots+c_{\Lambda}}} = 1 \).

2. We find in the evaluation, that there always appears the terms, killing the variables in denominator of (27) in the next steps of calculation.

From the practical reasons, our aim is to express the resulting formula in the form where the dependence on the next derivative variable, \( x_{j-1} \) is in the function \( f_{j-1} \) only. We find:

\begin{equation}
\lim_{x_{j-1} \to 1} D_{x_{j-1}}^{i_j} \frac{1}{x^2 \cdots (x + 1) \cdots x} f_j^n = (x_2 \cdots x_j b_j)^{2i_j} \sum_{l=0}^{MIN} \binom{2i_j}{l} (a_{j-2})^l (f_{j-1})^{n-2i_j-l} J(l, MIN; n, i_j; b_j/b_{j-1})
\end{equation}

where

\( J(l, MIN; n, i_j; b_j/b_{j-1}) = \sum_{p=l}^{MIN} \binom{2i_j - l}{p - l} \binom{2i_j - l/2}{2i_j - p} n!(2i_j - p)! \frac{(b_j)}{b_{j-1}}^p \),

\( MIN = \min(2i_j, n - 2i_j) \).

The proof of this formula is given in Appendix A.

We see, that in the term \( [(x_2 \cdots x_{j-1})b_j]^{2i_j} \) is the variable power \( x_k^{2i_j} \) canceling the same power of the variable in the denominator of the operator effecting over variable \( x_k \), to left the evaluation simpler. It can be shown, that function \( J \) is proportional to the Gegenbauer orthogonal polynomial following the relation for the Jacobi orthogonal polynomial \( P_n^{(\alpha, \beta)} \) (see e.g. Prudnikov [3]):

\begin{equation}
\sum_k^n \binom{n + \alpha}{k} \binom{n + \beta}{n - k} \frac{x + 1}{x - 1}^k = \frac{2^n}{(x - 1)^n} P_n^{(\alpha, \beta)}(x)
\end{equation}

If the \( \beta \) is a half number, the Jacobi polynomial can be expressed by help of the Gegenbauer polynomial.

The result of two consecutive application of the operator \( D_x \) and \( D_{x_{j-1}} \) is:

\begin{equation}
[(x_2 \cdots x_{j-2})b_{j-1}]^{2i_{j-1}} [(x_2 \cdots x_{j-2})b_j]^{2i_j} \sum_{l=0}^{MIN1} \binom{2i_j}{l} (a_{j-2})^l \sum_{l=0}^{MIN2} \binom{2i_j}{l} (a_{j-3})^l (f_{j-2})^{n-2i_j-2i_{j-1}-l} J(l, MIN1; n, i_j; b_j/b_{j-1}) J(l, MIN2; n - 2i_j - l, i_{j-1}; b_{j-1}/b_{j-2})
\end{equation}
where

\[ MIN1 = \min (2i_j, n - 2i_j), \]
\[ MIN2 = \min (2i_{j-1}, n - 2i_j - 2i_{j-1} - l). \]

On the result of application of three operations the nonlinear character of the our result is clearly visible:

\[
\left\{ \lim_{x_{j-1} \to 1} \frac{D^{ij-2}_{x_{j-2}} - 1}{x_{j-2}^{2(i_{j-1} + \ldots + i_{j-1})}} \right\} \left\{ \lim_{x_{j-1} \to 1} \frac{D^{ij-1}_{x_{j-2}} - 1}{x_{j-2}^{2(i_j + \ldots + i_{j-1})}} \right\} \left\{ \lim_{x_{j-1} \to 1} \frac{D^{ij}_{x_{j}} - 1}{x_{j}^{2(i_{j+1} + \ldots + i_{j})}} \right\} (f_j)^n = \tag{31}
\]

\[
[(x_2 \cdots x_{j-3}) b_{j-2}]^{2i_j-2} [(x_2 \cdots x_{j-3}) b_{j-3}]^{2i_{j-1} - 2} [(x_2 \cdots x_{j-3}) b_{j}]^{2} \]

\[
\sum_{m=0}^{MIN1} \binom{2i_j}{m} \binom{b_{j-2}}{b_{j-1}}^m \sum_{l=0}^{MIN1} \binom{2i_j - m}{l - m} \left(1 - \frac{b_{j-1}}{b_{j-2}}\right)^{l-m} \sum_{l=0}^{MIN2} \binom{2i_{j-1}}{l} (a_{j-3})^{l+m} \sum_{\nu=0}^{MIN3} \binom{2i_{j-2}}{\nu} (f_{j-3})^{n - 2i_j - 2i_{j-1} - 2i_j - 2 - l - m - \nu} J(l, MIN1; n, i_j; b_j/b_{j-1}) \]

\[ J(l, MIN2; n - 2i_j - l, i_{j-1}; b_{j-1}/b_{j-2}) J(\nu, MIN3; n - 2i_j - 2i_{j-1} - m - l, i_{j-2}; b_{j-2}/b_{j-3}) \]

where

\[ MIN3 = \min (2i_{j-2}, n - 2i_j - 2i_{j-1} - 2i_j - 2 - m - l). \]

The above relations give first three terms of the asymptotic expansion for function \( S_\Lambda \). We find:

For \( \mu = 1 \) only one summation index \( i_j \) is nonzero, then

\[
\{ C(\Lambda)^2 \}_{2,2} = \frac{\mu^1}{(2\mu)!} \sum_{k=2}^{\Lambda} Q_k^l b_k^2 J(0, 0; 2, 1; b_k/b_{k-1})
\]

For \( \mu = 2 \) two summation index \( i_j \) are nonzero, then

\[
\{ C(\Lambda)^4 \}_{4,4} = \frac{\mu^1}{(2\mu)!} \sum_{k=2}^{\Lambda} \sum_{p=k+1}^{\Lambda} Q_k^l Q_p^l b_k^2 b_p^2 J(0, 0; 2, 4, 1; b_k/b_p) J(0, 0; 2, 1; b_p/b_{p-1})
\]

For \( \mu = 3 \) three summation index \( i_j \) are nonzero, then

\[
\{ C(\Lambda)^6 \}_{6,6} = \frac{\mu^1}{(2\mu)!} \sum_{k=2}^{\Lambda} \sum_{p=k+1}^{\Lambda} \sum_{q=p+1}^{\Lambda} Q_k^l Q_p^l Q_q^l b_k^2 b_p^2 b_q^2 \sum_{l=0}^{2} \binom{2}{l} \left(1 - \frac{b_p}{b_q}\right)^l J(l, 2; 6, 1; b_k/b_p) J(0, 2 - l; 4 - l, 1; b_p/b_q) J(0, 0; 2, 1; b_q/b_{q-1})
\]

The continuum limit is introduced by prescription:

\[ \triangle k \to x, \triangle \Lambda \to \tau, \]
In continuum limit we obtain:

\[ Q_k \rightarrow 2 \cosh(\gamma x) \]

\[ b_k \rightarrow \frac{1}{\Delta \gamma} (\tanh(\gamma \tau) - \tanh(\gamma x)) \]

where \( \gamma = \sqrt{2b/c} \), \( b \) and \( c \) are the parameters of the model. The continuum limit of the relation (24) we will call \( S(a, b, c, \tau) \). To illustrate the analytical form of the result, we show the first nontrivial term (\( \mu = 1 \)):

\[
\{ C^2(a, b, c, \tau) \}_2,2 = \frac{3}{\gamma^2} \left[ 3\gamma \tau \tanh^2(\gamma \tau) + \tanh(\gamma \tau) - \gamma \tau \right]
\] (33)

For the higher we have the analytical formulas also as the results of algebraic evaluation by Mathematica. The continuum function \( S(a, b, c, \tau) \) for the first three nontrivial contributions is shown in the Fig. 1.

The corresponding term for the Gelfand-Yaglom equation, \(-2\partial^2_\mu \ln(S(a, b, c, \tau)) - 4(\partial_\mu \ln(S(a, b, c, \tau)))^2\) is shown in the Fig. 2.

**The leading divergent term in the limit \( \mu \rightarrow \infty \)**

In this limit the terms (23) are divergent. We are going to evaluate the leading divergent term for each \( \mu \). To provide this, we must to evaluate the double sums in Eq. (23):

\[
\sum_{j=0}^{I_2} \sum_{\lambda=1}^{I_2} \frac{(4I_2 - 2j)!}{(2\mu - j)!} \frac{2^{4(I_2-\lambda+j)}}{\xi^{(4\mu-2j)}} \left\{ \prod_{m=2}^{\Lambda} D_{\xi_m} \left[ \frac{1}{\xi_{m-2I_2+1}} + \cdots + \frac{\xi_2 \cdots \xi_{\Lambda-1}}{AQ_\Lambda Q_{\Lambda-1}} \right]^{2\mu-j} \right\} \Bigg|_{(\xi_m \rightarrow 1)}
\]

\[
\left( \frac{1}{AQ_0 Q_1} \right)^{\frac{I_2}{\lambda}} \left\{ \left( \frac{I_2}{\lambda} \right) Q_{2\lambda-j} Q_0 Q_1^{4(I_2-\lambda)} \right\}
\] (34)

The details of the calculation are explained in the Appendix B, we obtained the result:
Fig. 2: $\tau$ dependence of the continuum function $-2\partial_{\tau}^2 \ln(S(a, b, c, \tau)) - 4(\partial_{\mu} \ln(S(a, b, c, \tau)))^2$ for fixed $a, b, c$. The first three nontrivial terms of the asymptotic series \[24\] were used.

\[
\{C(\Lambda)^{2\mu}\}_{2\mu, 2\mu} = (1/2)^{2\mu} \left( \frac{1}{AQ_2 Q_1} + \cdots + \frac{1}{AQ_\Lambda Q_{\Lambda-1}} \right)^{2\mu} (Q_0^4 + Q_1^4)^\mu \sim 2^\mu (1/2)^{2\mu} \left( \frac{\tanh(\gamma \tau)}{\Delta \gamma} \right)^{2\mu}
\]

For the leading divergent term $\mu \to \infty$ of the asymptotic series for $S_{\Lambda} \[24\]$ we finally have:

\[
\frac{(-1)^\mu a^\mu}{\mu! c^{2\mu}} \Delta^\mu 2\mu (1/2)^{2\mu} \left( \frac{\tanh(\gamma \tau)}{\gamma} \right)^{2\mu}
\]  \(35\)

The series of this form is an asymptotic expansion of the parabolic cylinder function of the index $-1/2$ and the argument

\[
z^{-1} = 2a\Delta \left( \frac{\tanh(\gamma \tau)}{c\gamma} \right)^2.
\]

**Gelfand - Yaglom equation for anharmonic oscillator with mass zero**

As a test of our calculation we evaluate the energy levels of anharmonic oscillator with zero mass. The Gelfand-Yaglom equation for this case will be obtained by limit $b \to 0$, and we have:

\[
y''(\tau) + 4g_0(\tau)y'(\tau) = f_0(\tau)y(\tau)
\]  \(36\)

For the functions $f_0(\tau)$ and $g_0(\tau)$ we evaluate from Eq. \[24\] for first three non-trivial terms:

\[
f_0(\tau) = -2 \left[ \frac{d^2}{d\tau^2} \ln \left( 1 - \frac{a\tau^3}{c^2} + \frac{43}{30} \frac{(a\tau^3)^2}{c^2} - \frac{8111}{1890} \frac{(a\tau^3)^3}{c^2} + \cdots \right) \right]
\]

\[
- 4 \left[ \frac{d}{d\tau} \ln \left( 1 - \frac{a\tau^3}{c^2} + \frac{43}{30} \frac{(a\tau^3)^2}{c^2} - \frac{8111}{1890} \frac{(a\tau^3)^3}{c^2} + \cdots \right) \right]^2,
\]

\[
g_0(\tau) = \left[ \frac{d}{d\tau} \ln \left( 1 - \frac{a\tau^3}{c^2} + \frac{43}{30} \frac{(a\tau^3)^2}{c^2} - \frac{8111}{1890} \frac{(a\tau^3)^3}{c^2} + \cdots \right) \right].
\]

For first order in the coupling constant $a$, we have the equation:
\[ y''(\tau) - \frac{3.4a\tau^2}{c^2} y'(\tau) = \frac{3.4a}{c^2} y(\tau) \] (38)

This equation can be solved analytically, (see Kamke [3], Eq. 2.60). By substitution \( y(\tau) = u(\tau) \exp\left(\frac{2a\tau^3}{c^2}\right) \) we find the equation:

\[ u''(\tau) - \left(\frac{6a}{c^2}\right)^2 \tau^4 u(\tau) . \]

The general solution is expressed as the linear combination of the Bessel functions:

\[ y(\tau) = \sqrt{\tau} \left( C_1 J_{1/6}(i\frac{2a\tau^3}{c^2}) + C_2 Y_{1/6}(i\frac{2a\tau^3}{c^2}) \right) \exp\left(\frac{2a\tau^3}{c^2}\right) . \] (39)

The constants \( C_1 \) and \( C_2 \) will be fixed from boundary conditions. For \( z \sim 0 \) we follow the identities [6]:

\[
J_{\nu}(z) = \left( \frac{z}{2} \right)^{\nu} \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+\nu+1)} \left( \frac{z}{2} \right)^{2m},
\]

and

\[
Y_{\nu}(z) = \frac{1}{\sin(\nu\pi)} \left[ J_{\nu}(z) \cos(\nu\pi) - J_{-\nu}(z) \right].
\]

Inserting to the equations \( y(0) = 1 \) and \( y'(0) = 0 \) we find:

\[
C_1 = - \left( \frac{ai}{c^2} \right)^{1/6} \Gamma(1-1/6) \sin(\pi/6)
\]

\[
C_2 = \left( \frac{ai}{c^2} \right)^{1/6} \Gamma(1-1/6) \cos(\pi/6)
\]

For evaluation of the energy of the ground state we need \( y(\tau) \) for \( \tau \to \infty \). For this limit we can use the relations [6]:

\[
J_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{2\nu + 1}{4} \pi \right),
\]

\[
Y_{\nu}(z) = \sqrt{\frac{2}{\pi z}} \sin \left( z - \frac{2\nu + 1}{4} \pi \right),
\]

and finally we have for the functional integral in this limit:

\[
F(\beta) = \left[ \left( \frac{ai}{c^2} \right)^{1/6} \Gamma(1-1/6) \left( \frac{c^2}{i\alpha\beta^3} \right)^{1/2} \sin \left( \frac{2i\alpha\beta^3}{c^2} - \frac{\pi}{6} \right) \exp \left( \frac{2i\alpha\beta^3}{c^2} \right) \right]^{-1/2}
\]

The unconditional measure functional integral is the partition function for the model solved. for the harmonic oscillator there is the simple relation for the energy of the ground state:

\[
E_0 = - \lim_{\beta \to \infty} \left( \frac{1}{\beta^3} \ln F(\beta) \right).
\]

By the direct application of this relation we obtain zero for the ground state energy of the an-harmonic oscillator with zero mass. We find the nonzero result for the definition:

\[
E_0 = - \lim_{\beta \to \infty} \left( \frac{1}{\beta^3} \ln F(\beta) \right) = \frac{2a}{c^2}.
\]
Conclusions

In this article we calculated step-by-step the correlation function for differential Gelfand-Yaglom equation anharmonic oscillator with positive or negative mass squared term. We stress, that generalized Gelfand-Yaglom equation is the non-perturbative equation, the correlation function is evaluated in the form of the asymptotic series. The analytical form of the correlation function enables us to evaluate the continuum limit as well as asymptotic limit. The continuum limit can be used for evaluation such physical quantities as energy levels. The asymptotic limit can be used for attempts to sum the series by Borel’s method.

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Appendix A

We are going to apply the derivative operator

$$D_{x_j} = \frac{3}{2}\partial^2_{x_j} + 3x_j\partial^3_{x_j} + x_j^2\partial^4_{x_j}$$

\(i\) times on the \(n-th\) power of the function

$$f_j = a_{j-1} + (x_2\cdots x_j)b_j$$

where:

$$a_{j-1} = \frac{1}{Q_2Q_1} + \frac{x_2}{Q_3Q_2} + \cdots + \frac{x_2\cdots x_{j-1}}{Q_jQ_{j-1}}$$

$$b_j = \frac{1}{Q_{j+1}Q_j} + \cdots + \frac{1}{Q_{\Lambda}Q_{\Lambda} - 1}$$

We see, that \(b_j\) is a constant from point of derivative operator \(D_{x_j}\) and \(a_{j-1}\) also. Therefore for \((D_{x_j})^i f_j^n\) we find:

\[(D_{x_j})^i f_j^n = \left((x_2x_3\cdots x_{j-1})b_j\right)^{2i} \sum_{m=0}^{\min(2i,n-2i)} (-1)^m \binom{2i}{m} a_{j-1}^m \frac{n!}{(n-2i-m)!} (n-2i+1/2)_{2i-m} f_{j-1}^{n-2i-p} \] (A.41)

The above result is nonzero only if \(2i \leq n\). The variable \(x_{j-1}\) is in the term \(f_j\) and \(a_{j-1}\) as well. By the identity

$$a_{j-1} = f_j - (x_2x_3\cdots x_{j-1})b_j$$

we introduce summation due to binomial expansion of \(a_{j-1}\) into Eq. (A.41) over index \(p = 0, 1, \cdots, m\). We exchange the order of summations and apply the identity:

$$\binom{2i}{m} \binom{m}{p} = \binom{2i}{p} \binom{2i-p}{m-p}$$

Finally, performing the limit \(x_j \to 1\) we find:

\[(D_{x_j})^i f_j^n |_{x_j=1} = \left((x_2x_3\cdots x_{j-1})b_j\right)^{2i} \sum_{p=0}^{\min(2i,n-2i)} \binom{2i}{p} \left((x_2x_3\cdots x_{j-1})b_j\right)^p f_{j-1}^{n-2i-p} \]

\[\sum_{m=p}^{\min(2i,n-2i)} (-1)^{p-m} \binom{2i-p}{m-p} \frac{n!}{(n-2i-m)!} (n-2i+1/2)_{2i-m} \] (A.42)
Now, we can perform the summation over index \(m\). For the case \(\min(2i, n-2i) = 2i\) we use the identities:

\[
\frac{n!}{(n-2i-m)!} = \lim_{x \to 1} \partial_x^{2i+m} (x^n)
\]  
(A.43)

and

\[
(n-2i+1/2)_{2i-m} = \lim_{x \to 1} \partial_x^{2i-m} \left( \frac{(-1)^{2i-m}}{x^{n-2i+1/2}} \right)
\]  
(A.44)

Inserting this into sum over \(m\) in Eq \((A.42)\), replacing the order of limit and summation, as well as the summation index \(m\) by \(l = m - p\) we have:

\[
\lim_{x \to 1} \sum_{l=0}^{2i-p} (-1)^l \left( \frac{2i-1}{l} \right) \partial_x^l \left( \partial_x^{2i+p} (x^n) \right) \partial_x^{2i-p-l} \left( \frac{(-1)^{2i-p-l}}{x^{n-2i+1/2}} \right) = \frac{n!}{(n-2i-p)!} (p + 1/2)_{2i-p} \]  
(A.45)

When \(\min(2i, n-2i) = n - 2i\) we find the identical result. We proved that:

\[
(D_{x_j})^i f_j^n |_{x_j=1} =

((x_2 x_3 \cdots x_{j-1}) b_j)^{2i} \sum_{p=0}^{\min(2i, n-2i)} \left( \frac{2i}{p} \right) ((x_2 x_3 \cdots x_{j-1}) b_j)^p f_j^{n-2i-p} \frac{n!}{(n-2i-p)!} (p + 1/2)_{2i-p}
\]  
(A.46)

In the sum over index \(p\) we recognize the formula for the Jacobi orthogonal polynomial. By help of the identity:

\[
\left( \frac{2i}{p} \right) \frac{n!}{(n-2i-p)!} (p + 1/2)_{2i-p} = \frac{(2i)!}{(n-2i)!} \left( \frac{n - 2i}{p} \right) \left( \frac{2i - 1}{2i - p} \right)
\]

for the case \(\min(2i, n-2i) = 2i\) we find the similarity with identity (see e.g. Prudnikov [3]):

\[
\sum_{k=0}^{n} \left( \frac{n + \alpha}{k} \right) \left( \frac{n + \beta}{n - k} \right) \left( \frac{x + 1}{x - 1} \right)^k = \frac{2^n}{(x - 1)^n} P_n^{(\alpha,\beta)}(x)
\]  
(A.47)

Where \(P_n^{(\alpha,\beta)}(x)\) is Jacobi orthogonal polynomial.

The expression in Eq. \((A.46)\) we simplify further for application of the operator \((D_{x_{j-1}})\). In Eq. \((A.46)\) the variable \(x_{j-1}\) is in two terms, therefore we simplify that equation by the identities:

\[
(x_2 x_3 \cdots x_{j-1}) b_j = \frac{b_j}{b_{j-1}} (x_2 x_3 \cdots x_{j-1}) b_{j-1},
\]

and

\[
(x_2 x_3 \cdots x_{j-1}) b_{j-1} = f_{j-1} - a_{j-2}.
\]

Now, the variable \(x_{j-1}\) will be in \(f_{j-1}\) only and for application of the operator \((D_{x_{j-1}})\) we prepare the relation:

\[
(D_{x_j})^i f_j^n |_{x_j=1} =

\left( \frac{2i}{p} \right) \sum_{p=0}^{\min(2i, n-2i)} \left( \frac{b_j}{b_{j-1}} \right)^p \sum_{\nu=0}^{p} \left( \frac{p}{\nu} \right) (-a_{j-2})^\nu f_{j-1}^{n-2i-\nu} \frac{n!}{(n-2i-p)!} (p + 1/2)_{2i-p}
\]

By exchange of the order of summation we finally red:

\[
(D_{x_j})^i f_j^n |_{x_j=1} =

\left( \frac{2i}{p} \right) \left( -a_{j-2} \right)^\nu f_{j-1}^{n-2i-\nu} \sum_{\nu=0}^{p} \left( \frac{2i - \nu}{p - \nu} \right) \left( \frac{b_j}{b_{j-1}} \right)^p \frac{n!}{(n-2i-p)!} (p + 1/2)_{2i-p}
\]
In this form is result of application of operator $(D_x)^i$ suitable for the next evaluations.

Appendix B

We are going to evaluate the leading divergent term. To provide this, we must to evaluate the double sum in Eq. (23):

$$
\sum_{j=0}^{2I_2} \sum_{\lambda=|\lambda|+1}^{I_2} \frac{(4I_2 - 2j)!}{(2\mu - j)!} \frac{2^{4(i_2 + i_3 + \cdots + i_\lambda)}}{2(4\mu - 2j)} \left\{ \prod_{m=2}^{\Lambda} D_{\xi_m}^{\lambda_m} \left[ \frac{1}{\epsilon_{2m-2I_m+1}^2} \left( \frac{1}{AQ_2 Q_1} + \cdots + \frac{\xi_2 \cdots \xi_{\Lambda-1}}{AQ_2 Q_{\Lambda-1}} \right) \right] \right\}_{(\xi_{m-1})}
$$

$$
\left( \frac{1}{AQ_0 Q_1} \right)^{j} \left\{ \left( \frac{I_2}{\lambda} \right) a_{2\lambda - j}^{2\lambda} Q_0^{2\lambda} Q_1^{4(I_2 - \lambda)} \right\}
$$

(A.48)

To accomplish this calculation we change the order of the summations and we apply the identities:

$$(4I_2 - 2j)! = 2^{4(I_2 - 2j)} (2I_2 - j)! (1/2)_{2I_2-j}$$

$$
\frac{a_{2\lambda - j}^{2\lambda}}{a_{2\lambda - j}^{2\mu}} = \frac{2\lambda}{2\lambda - j} \frac{(1/2)_{2\lambda}}{(1/2)_{2\lambda-j}}
$$

$$(1/2)_{2I_2-j} = \partial_\varepsilon^{2I_2-2\lambda} \left( \frac{1}{\varepsilon_{2\lambda-j+1/2}^2} \right) \left|_{\varepsilon \rightarrow 1} \right.
$$

$$(2I_2 - j)! = \int d\vartheta^{2\mu-2I_2} \left( \vartheta^{2I_2-j} \right) \left|_{\vartheta \rightarrow 1} \right.
$$

$$
\frac{2^{4(i_2 + i_3 + \cdots + i_\lambda)}}{2(4\mu - 4I_2)} = 1
$$

We find:

$$
\sum_{\lambda=0}^{I_2} \frac{I_2}{\lambda} \frac{(1/2)_{2\lambda}}{a_{2\lambda - j}^{2\lambda}} Q_0^{2\lambda} Q_1^{4(I_2 - \lambda)}
$$

(A.49)

$$
\sum_{j=0}^{2\lambda} \frac{2\lambda}{j} \left\{ \partial_\varepsilon^{2\lambda - 2\lambda} \left( \frac{1}{\varepsilon_{2\lambda-j+1/2}^2} \right) \left|_{\varepsilon \rightarrow 1} \right. \right\} \left\{ \int d\vartheta^{2\mu-2I_2} \left( \vartheta^{2I_2-j} \right) \left|_{\vartheta \rightarrow 1} \right. \right\}
$$

$$
\left( \frac{1}{AQ_0 Q_1} \right)^{j} \left\{ \prod_{m=2}^{\Lambda} D_{\xi_m}^{\lambda_m} \left[ \frac{1}{\epsilon_{2m-2I_m+1}^2} \left( \frac{1}{AQ_2 Q_1} + \frac{\xi_2}{AQ_2 Q_2} + \cdots + \frac{\xi_2 \cdots \xi_{\Lambda-1}}{AQ_2 Q_{\Lambda-1}} \right) \right] \right\}_{(all\xi_m \rightarrow 1)}
$$

The derivative over variable $\varepsilon$ and inverse derivative over variable $\vartheta$ are independent on the summation index $j$ and we can transfer them out of the sum. The summation over index $j$ gives:

$$
\sum_{j=0}^{2\lambda} \left( \frac{2\lambda}{j} \right) \left( \frac{\varepsilon U}{\vartheta} \right)^{j} = \left( 1 + \frac{\varepsilon U}{\vartheta} \right)^{2\lambda}
$$

(A.50)
where

\[
U = \left( \frac{1}{AQ_2 Q_1} + \frac{\xi_2}{AQ_3 Q_2} + \cdots + \frac{\xi_{L-1}}{AQ_{L-1} Q_{L-1}} \right)^{-1} \left( \frac{1}{AQ_0 Q_1} \right).
\]

For summation over index \( j \) in (A.49) we obtain:

\[
\left\{ \prod_{m=2}^L \frac{1}{\xi_m} \left( \frac{1}{\xi_m^{2\mu} - 2im + 1} \left( 1 + \frac{\varepsilon U}{\vartheta} \right)^{2\lambda} \right) \right\}
\]

\[
\left\{ \int d\vartheta^{2\mu} \left( \vartheta^{2\lambda} \right) \left( 1 + \frac{\varepsilon U}{\vartheta} \right)^{2\lambda} \right\} \left| \varepsilon \rightarrow 1 \right.
\]

\[
\sum_{i=0}^{\min(2\lambda,2\mu-2\lambda)} (-1)^i \left( \frac{2I_2 - 2\lambda}{i} \right) (2\lambda + 1/2)_{2I_2 - 2\lambda - i} \left( \frac{2\lambda}{2I_2 - 2\lambda} \right) (2I_2 - 2\lambda)! \left( \frac{2\lambda}{2I_2 - 2\lambda} \right) \left( \frac{2I_2 - 1/2}{i} \right) (i)!
\]

We converted (A.52) to the final form:

\[
\left\{ \vartheta^{2\lambda} \left( 1 + \frac{\varepsilon U}{\vartheta} \right)^{2\lambda} \right\} \left| \varepsilon \rightarrow 1 \right. = \left( \frac{2I_2 - 1/2}{2I_2 - 2\lambda} \right) (2I_2 - 2\lambda)! \left( \frac{2I_2 - 1/2}{i} \right) (i)!
\]

where \( MIN = \min(2\lambda, 2I_2 - 2\lambda) \), and \( MAX = \max(2\lambda, 2I_2 - 2\lambda) \).

The sum over index \( i \) can be expressed by help of Gegenbauer orthogonal polynomials \( C^n_a \) following the relation (see [3], vol. I, page 634):

\[
\sum_{i=0}^n (-1)^i \left( \frac{n}{i} \right) \left( a - n + 1/2 \right) \left( \begin{array}{c} a \cr i \end{array} \right)^{-1} x^i = \left( \begin{array}{c} a \cr 2n \end{array} \right)^{-1} \left( \begin{array}{c} a \cr 2n \end{array} \right)^{-1} C^1_{2n-a-2n}(\sqrt{1-1/x})
\]

We insert (A.54) into the relation (A.49), and using the identity:

\[
(1/2)^{2\lambda} (2I_2 - 2\lambda)! \left( \frac{2I_2 - 1/2}{2I_2 - 2\lambda} \right) = (1/2)^{2I_2}
\]
we find:

\[
(1/2)_{2I_2} \left\{ \prod_{m=2}^{I_2} D_{\xi_m} \left[ \frac{1}{\xi_m^{4\mu-2I_{m+1}}} \left( \frac{1}{AQ_2Q_1} + \frac{\xi_2}{AQ_3Q_2} + \cdots + \frac{\xi_2 \cdots \xi_{A-1}}{AQ_AQ_{A-1}} \right)^2 \right] \right\}
\]

(A.56)

\[
\sum_{\lambda=0}^{I_2} \left( \begin{array}{l} I_2 \\ \lambda \end{array} \right) \left( \frac{2I_2-1/2}{2M\ln} \right)^{-1} Q_0^{\lambda} Q_1^{4(I_2-\lambda)}
\]

\[
\left\{ \int d\theta^{2\mu-2I_2} \left( \theta^{2I_2-2\lambda+M\ln} \right) \right\} (U + \theta)^{2\lambda - M\ln}
\]

Taking into account that \( I_2 = \mu \), the inverse derivative over variable \( \theta \) have no effect and we red:

\[
\left\{ \int d\theta^{2\mu-2I_2} \left( \theta^{2I_2-2\lambda+M\ln} \right) \right\} (U + \theta)^{2\lambda - M\ln} \rightarrow \theta^{2\mu-2\lambda+M\ln} (U + \theta)^{2\lambda - M\ln}
\]

Taking into account that \( U \sim \Delta \) we have:

\[
\{ C(\Lambda)^{2\mu} \}_{2\mu,2\mu} = (1/2)_{2\mu} \left( \frac{1}{AQ_2Q_1} + \frac{1}{AQ_3Q_2} + \cdots + \frac{1}{AQ_AQ_{A-1}} \right)^{2\mu} (Q_0^4 + Q_1^4)^\mu
\]

For finite but \( \Delta \rightarrow 0 \) we approximatively find:

\[
\{ C(\Lambda)^{2\mu} \}_{2\mu,2\mu} \sim 2^{\mu} (1/2)_{2\mu} \left( \frac{\tan (\gamma \tau)}{\Delta \gamma} \right)^{2\mu}
\]

Finally, for the leading divergent terms for \( \mu \rightarrow \infty \) of the asymptotic series for \( S_\Lambda \) in quasi-continuum relation we have:

\[
\frac{(-1)\mu}{\mu!} \Delta^{2\mu} (1/2)_{2\mu} \left( \frac{\tan (\gamma \tau)}{\gamma} \right)^{2\mu}
\]

(A.58)

[1] Chaichian M., Demichev A., Path Integrals in Physics, Vol. I, IOP Publishing Ltd. 2001.
[2] J. Boháčik and P. Prešnajder, Functional integral for $\varphi^4$ potential beyond classical perturbative methods, [hep-th/0503235]

[3] A.P. Prudnikov, J.A. Brytchkov, O.I. Marichev: Integrals and Series, Gordon & Breach, New York, 1986; and in russian language see Nauka, 1981.

[4] Gabor Szegő: Orthogonal Polynomials, American Mathematical Society Colloquium Publications, vol XXIII, 1959.

[5] E. Kamke, Differentialgleichungen Lösungsmethoden und Lösungen I, Gewöhnliche Differentialgleichungen, Leipzig 1959.

[6] Bateman H., Higher Transcendental Functions, Volume II, McGraw-Hill, 1953.