Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two parameters

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Abstract

In the article, we present the best possible parameters \( \lambda = \lambda(p) \) and \( \mu = \mu(p) \) on the interval \([0, 1/2]\) such that the double inequality

\[
G_p\left[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a\right] A^{1-p}(a, b) < E(a, b) < G_p\left[\mu a + (1 - \mu)b, \mu b + (1 - \mu)a\right] A^{1-p}(a, b)
\]

holds for any \( p \in [1, \infty) \) and all \( a, b > 0 \) with \( a \neq b \), where \( A(a, b) = (a + b)/2 \), \( G(a, b) = \sqrt{ab} \) and \( E(a, b) = [2 \int_0^{\pi/2} \sqrt{a \cos^2 \theta + b \sin^2 \theta} \, d\theta / \pi]^2 \) are the arithmetic, geometric and special quasi-arithmetic means of \( a \) and \( b \), respectively.

MSC: 26E60; 33E05

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1 Introduction

Let \( r \in (0, 1) \). Then the Legendre complete elliptic integrals \( K(r) \) and \( E(r) \) \([1, 2]\) of the first and second kinds are defined as

\[
K(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - r^2 \sin^2(t)}}, \quad E(r) = \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2(t)} \, dt,
\]

respectively. It is well known that the function \( r \to K(r) \) is strictly increasing from \((0, 1)\) onto \((\pi/2, \infty)\) and the function \( r \to E(r) \) is strictly decreasing from \((0, 1)\) onto \((1, \pi/2)\), and they satisfy the formulas (see [3, Appendix E, pp. 474,475])

\[
\frac{dK(r)}{dr} = \frac{E(r) - r^2 K(r)}{rr'}, \quad \frac{dE(r)}{dr} = \frac{E(r) - K(r)}{r},
\]

\[
K\left(\frac{2\sqrt{r}}{1 + r}\right) = (1 + r)K(r), \quad E\left(\frac{2\sqrt{r}}{1 + r}\right) = 2E(r) - r^2 K(1 + r),
\]

where \( r' = \sqrt{1 - r^2} \).
The complete elliptic integrals $K(r)$ and $E(r)$ are the particular cases of the Gaussian hypergeometric function [4–10]

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n n!} \quad (-1 < x < 1),$$

where $(a)_0 = 1$ for $a \neq 0$, $(a)_n = a(a+1)(a+2) \cdots (a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the shifted factorial function and $\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t} \, dt$ $(x > 0)$ is the gamma function [11–18]. Indeed,

$$K(r) = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; r^2 \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^2}{(n^2)!} r^{2n},$$

$$E(r) = \frac{\pi}{2} F \left( -\frac{1}{2}, \frac{1}{2}; 1; r^2 \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{-\left(\frac{1}{2}\right)_n}{(n!)^2} r^{2n}.$$

Recently, the bounds for the complete elliptic integrals have attracted the attention of many researchers. In particular, many remarkable inequalities and properties for $K(r)$, $E(r)$ and $F(a, b; c; x)$ can be found in the literature [19–52].

In 1998, a class of quasi-arithmetic mean was introduced by Toader [53] which is defined by

$$M_{p,n}(a, b) = p^{-1} \left( \frac{1}{\pi} \int_0^\pi p(r_n(\theta) \, d\theta) \right) = p^{-1} \left( \frac{2}{\pi} \int_0^{\pi/2} p(r_n(\theta) \, d\theta) \right),$$

where $r_n(\theta) = (a^n \cos^2 \theta + b^n \sin^2 \theta)^{1/n}$ for $n \neq 0$, $r_0(\theta) = a \cos^2 \theta + b \sin^2 \theta$, and $p$ is a strictly monotonic function. It is well known that many important means are the special cases of the quasi-arithmetic mean. For example,

$$M_{1/2,2}(a, b) = \frac{\pi}{2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \begin{cases} \pi a/[2K(\sqrt{1-(b/a)^2})], & a \geq b, \\ \pi b/[2K(\sqrt{1-(a/b)^2})], & a < b, \end{cases}$$

is the arithmetic-geometric mean of Gauss [54–60],

$$M_{1/2,2}(a, b) = \frac{2}{\pi} \int_0^{\pi/2} \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \, d\theta = \begin{cases} 2aE(\sqrt{1-(b/a)^2})/\pi, & a \geq b, \\ 2bE(\sqrt{1-(a/b)^2})/\pi, & a < b, \end{cases}$$

is the Toader mean [61–70], and

$$M_{1/2,0}(a, b) = \frac{2}{\pi} \int_0^{\pi/2} a^{\cos^2 \theta} b^{\sin^2 \theta} \, d\theta$$

is the Toader-Qi mean [71–74].

Let $p = \sqrt{x}$ and $n = 1$. Then $M_{p,n}(a, b)$ reduces to a special quasi-arithmetic mean

$$E(a, b) = M_{\sqrt{x},2}(a, b) = \begin{cases} 4a[\mathcal{E}(1-a/b)^2]/\pi^2, & a \geq b, \\ 4b[\mathcal{E}(1-a/b)^2]/\pi^2, & a < b. \end{cases}$$ (1.1)
Let
\[ A(a, b) = \frac{a + b}{2}, \quad G(a, b) = \sqrt{ab}, \]
\[ M_p(a, b) = \left( \frac{a^p + b^p}{2} \right)^{1/p} \quad (p \neq 0), \quad M_0(a, b) = \sqrt{ab}, \]
be the arithmetic, geometric and pth power means of \( a \) and \( b \), respectively. Then it is well known that the inequality
\[ G(a, b) = M_0(a, b) < A(a, b) = M_1(a, b) \]
holds for all \( a, b > 0 \) with \( a \neq b \), and the double inequality
\[ \frac{\pi}{2} M_{3/2}(1, r') < E(r) < \frac{\pi}{2} M_2(1, r') \]
holds for all \( r \in (0, 1) \) (see [75, 19.9.4]).

From (1.1)-(1.3) we clearly see that
\[ G(a, b) < E(a, b) < A(a, b) \]
for all \( a, b > 0 \) with \( a \neq b \).

Let \( p \in [1, \infty) \) and
\[ f(x; p; a, b) = G^p[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a] A^{1-p}(a, b). \]

Then it is not difficult to verify that the function \( x \to f(x; p; a, b) \) is strictly increasing on \([0, 1/2]\) for fixed \( p \in [1, \infty) \) and \( a, b > 0 \) with \( a \neq b \). Note that
\[ f(0; p; a, b) = G^p(a, b) A^{1-p}(a, b) \leq G(a, b) \]
\[ < E(a, b) < A(a, b) = f(1/2; p; a, b) \]
for all \( p \in [1, \infty) \) and \( a, b > 0 \) with \( a \neq b \).

Motivated by inequalities (1.4) and the monotonicity of the function \( x \to f(x; p; a, b) \) on the interval \([0, 1/2]\), in the article, we shall find the best possible parameters \( \lambda = \lambda(p), \mu = \mu(p) \) on the interval \([0, 1/2]\) such that the double inequality
\[ G^p[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a] A^{1-p}(a, b) \]
\[ < E(a, b) < G^p[\mu a + (1 - \mu)b, \mu b + (1 - \mu)a] A^{1-p}(a, b) \]
holds for any \( p \in [1, \infty) \) and all \( a, b > 0 \) with \( a \neq b \).

2 Lemmas

Lemma 2.1 (see [3, Theorem 1.25]) Let \(-\infty < a < b < +\infty, f, g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) and differentiable on \((a, b)\), and \( g'(x) \neq 0 \) on \((a, b)\). If \( f'(x)/g'(x) \) is increasing
(decreasing) on \((a, b)\), then so are the functions
\[
\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

If \(f'(x)/g'(x)\) is strictly monotone, then the monotonicity in the conclusion is also strict.

**Lemma 2.2** The inequality
\[
\frac{1}{4p} + \left(\frac{2\sqrt{\frac{2}{\pi}}}{p}\right)^{4/p} < 1
\]
holds for all \(p \in [1, \infty)\).

**Proof** Let
\[
f(p) = \frac{1}{4p} + \left(\frac{2\sqrt{\frac{2}{\pi}}}{p}\right)^{4/p}.
\]

Then simple computations lead to
\[
\lim_{p \to \infty} f(p) = 1, \quad (2.2)
\]
\[
f'(p) = \frac{4}{p^2} \log \left(\frac{\sqrt{\frac{2}{\pi}}}{4}\right) \left[\left(\frac{2\sqrt{\frac{2}{\pi}}}{p}\right)^{4/p} - \frac{1}{16 \log(\frac{\sqrt{2\pi}}{4})}\right]
\]
\[
\geq \frac{4}{p^2} \log \left(\frac{\sqrt{\frac{2}{\pi}}}{4}\right) \left[\left(\frac{2\sqrt{\frac{2}{\pi}}}{p}\right)^{4} - \frac{1}{16 \log(\frac{\sqrt{2\pi}}{4})}\right]
\]
\[
= \frac{1024 \log(\frac{\sqrt{2\pi}}{4}) - \pi^4}{4 \pi^4 p^2} > 0 \quad (2.3)
\]
for \(p \in [1, \infty)\).

Therefore, Lemma 2.2 follows easily from (2.1)-(2.3). \(\square\)

**Lemma 2.3** The following statements are true:

1. The function \(r \mapsto [E(r) - (1 - r^2)K(r)]/r^2\) is strictly increasing from \((0, 1)\) onto \((\pi/4, 1)\).
2. The function \(r \mapsto [K(r) - E(r)]/r^2\) is strictly increasing from \((0, 1)\) onto \((\pi/4, \infty)\).
3. The function \(r \mapsto [E(r) + (1 - r^2)K(r)]/(1 - r^2)\) is strictly increasing from \((0, 1)\) onto \((\pi, \infty)\).
4. The function \(r \mapsto [2E(r) - (1 - r^2)K(r)]/(1 + r^2)\) is strictly decreasing from \((0, 1)\) onto \((1, \pi/2)\).
5. The function \(r \mapsto r^2[2E(r) - (1 - r^2)K(r)]/[(1 + r^2)^2(K(r) - E(r))]\) is strictly decreasing from \((0, 1)\) onto \((0, 2)\).

**Proof** Parts (1) and (2) can be found in the literature [3, Theorem 3.21(1) and Exercise 3.43(11)].
For part (3), let \( f_1(r) = [E(r) + (1 - r^2)K(r)]/(1 - r^2) \). Then simple computations lead to

\[
\begin{align*}
  f_1(0^+) &= \pi, & f_1(1^-) &= \infty, \\
  f_1'(r) &= \frac{r}{1 - r^2} \left[ \frac{2}{r^2} \left( \frac{E(r) - (1 - r^2)K(r)}{r^2} + (1 - r^2)K(r) \right) \right].
\end{align*}
\]

(2.4)

(2.5)

It follows from part (1) and (2.5) that

\[
f_1'(r) > 0
\]

(2.6)

for all \( r \in (0,1) \). Therefore, part (3) follows from (2.4) and (2.6).

For part (4), let \( f_2(r) = [2E(r) - (1 - r^2)K(r)]/(1 + r^2) \), then one has

\[
\begin{align*}
  f_2(0^+) &= \frac{\pi}{2}, & f_2(1^-) &= 1, \\
  f_2'(r) &= \frac{r}{1 + r^2} \left[ \frac{1}{r^2} \left( \frac{E(r) - (1 - r^2)K(r)}{r^2} - 2E(r) \right) \right].
\end{align*}
\]

(2.7)

(2.8)

From part (1) and (2.8) we clearly see that

\[
f_2'(r) < -\frac{r}{1 + r^2} < 0
\]

(2.9)

for all \( r \in (0,1) \). Therefore, part (4) follows from (2.7) and (2.9).

For part (5), let \( f_3(r) = r^2[2E(r) - (1 - r^2)K(r)]/[(1 + r^2)^2(K(r) - E(r))] \), then \( f_3(r) \) can be rewritten as

\[
f_3(r) = \frac{2E(r) - (1 - r^2)K(r)}{1 + r^2} \times \frac{1}{K(r) - E(r)} \times \frac{1}{1 + r^2}.
\]

(2.10)

Therefore, part (5) follows easily from parts (2) and (4) together with (2.10).

\[\Box\]

**Lemma 2.4** The function

\[
g(r) = \frac{r^2K(r)}{(1 + r^2)[K(r) - E(r)]}
\]

is strictly decreasing from \((0,1)\) onto \((1/2, 2)\).

**Proof** Let \( g_1(r) = r^2K(r) \) and \( g_2(r) = (1 + r^2)[K(r) - E(r)] \). Then we clearly see that

\[
\begin{align*}
  g_1(0^+) &= g_2(0^+) = 0, & g(r) &= \frac{g_1(r)}{g_2(r)}, \\
  g(1^-) &= \frac{1}{2}, \\
  g_1'(r) &= \frac{1}{2} - \frac{3E(r)}{2(1 - r^2)}, \\
  g_2'(r) &= 2 - \frac{3E(r)(1 - r^2)}{2(1 - r^2)}.
\end{align*}
\]

(2.11)

(2.12)

(2.13)
From Lemma 2.3(3), (2.11) and (2.13) we know that
\[ g(0^+) = \lim_{r \to 0^+} \frac{g_1'(r)}{g_2'(r)} = 2 \]  
(2.14)
and the function \( g_1'(r)/g_2'(r) \) is strictly decreasing on (0, 1).

Therefore, Lemma 2.4 follows easily from Lemma 2.1, (2.11), (2.12) and (2.14) together with the monotonicity of the function \( g_1'(r)/g_2'(r) \). \( \square \)

**Lemma 2.5** Let \( u \in [0, 1], \ r \in (0, 1), \ p \in [1, \infty) \) and
\[
h(u, p; r) = \frac{1}{2} p \log \left[ 1 - \frac{4ur^2}{(1 + r^2)^2} \right] - \log \left[ \frac{4(2\mathcal{E}(r) - (1 - r^2)\mathcal{K}(r))^2}{\pi^2(1 + r^2)} \right].
\]  
(2.15)
Then one has
(1) \( h(u, p; r) > 0 \) for all \( r \in (0, 1) \) if and only if \( u \leq 1/4p \);
(2) \( h(u, p; r) < 0 \) for all \( r \in (0, 1) \) if and only if \( u \geq 1 - (2\sqrt{2/\pi})^{4/p} \).

**Proof** It follows from (2.15) that
\[
h(u, p; 0^+) = 0,
\]  
(2.16)
\[
h(u, p; 1^-) = \frac{P}{2} \log(1 - u) + \log \left( \frac{\pi^2}{8} \right),
\]  
(2.17)
\[
\frac{\partial h(u, p; r)}{\partial r} = \frac{2(1 - r^2)[K(r) - \mathcal{E}(r)]}{r(1 + r^2)[2\mathcal{E}(r) - (1 - r^2)K(r)]} - \frac{4pur(1 - r^2)}{(1 + r^2)^2 - 4ur^2}
\]  
(2.18)
\[
= \frac{2(1 - r^2)[2(K(r) - \mathcal{E}(r)) + p(2\mathcal{E}(r) - (1 - r^2)K(r))]}{(1 + r^2)^2 - 4ur^2}[2\mathcal{E}(r) - (1 - r^2)K(r)] \left[ h_1(p; r) - 2u \right],
\]  
(2.18)
where
\[
h_1(p; r) = \frac{(1 + r^2)^2[K(r) - \mathcal{E}(r)]}{r^2[2(K(r) - \mathcal{E}(r)) + p(2\mathcal{E}(r) - (1 - r^2)K(r))]}
\]  
(2.19)
\[
= \frac{1}{g(r) + (p - 1)f_5(r)},
\]  
(2.19)
where \( f_5(r) \) and \( g(r) \) are defined by (2.10) and Lemma 2.4, respectively.

From Lemma 2.3(5) and Lemma 2.4 together with (2.19) we clearly see that the function \( r \to h_1(p; r) \) is strictly increasing on (0, 1) and
\[
h_1(p; 0^+) = \frac{1}{2p},
\]  
(2.20)
\[
h_1(p; 1^-) = 2.
\]  
(2.21)
From Lemma 2.2 we know that \( 1 - (2\sqrt{2/\pi})^{4/p} > 1/(4p) \). Therefore, we only need to divide the proof into three cases as follows.

**Case 1** \( u \leq 1/(4p) \). Then Lemma 2.3(4), (2.18), (2.20) and the monotonicity of the function \( r \to h_1(p; r) \) on the interval (0, 1) lead to the conclusion that the function \( r \to h(u, p; r) \)
is strictly increasing on (0, 1). Therefore, \( h(u, p; r) > 0 \) for all \( r \in (0, 1) \) follows from (2.16) and the monotonicity of the function \( r \rightarrow h(u, p; r) \).

**Case 2** \( u \geq 1 - (2\sqrt{2}/\pi)^{4/p} \). Then from Lemma 2.2, Lemma 2.3(5), (2.17), (2.18), (2.20), (2.21) and the monotonicity of the function \( r \rightarrow h(u, p; r) \) on the interval (0, 1) we clearly see that there exists \( r_0 \in (0, 1) \) such that the function \( r \rightarrow h(u, p; r) \) is strictly decreasing on \((0, r_0)\) and strictly increasing on \((r_0, 1)\), and

\[
h(u, p; 1^-) \leq 0. \tag{2.22}
\]

Therefore, \( h(u, p; r) < 0 \) for all \( r \in (0, 1) \) follows from (2.16) and (2.22) together with the piecewise monotonicity of the function \( r \rightarrow h(u, p; r) \) on the interval (0, 1).

**Case 3** \( 1/(4p) < u < 1 - (2\sqrt{2}/\pi)^{4/p} \). Then (2.17) leads to

\[
h(u, p; 1^-) > 0. \tag{2.23}
\]

It follows from Lemma 2.3(5), (2.18), (2.20), (2.21) and the monotonicity of the function \( r \rightarrow h_1(p; r) \) on the interval (0, 1) that there exists \( r^* \in (0, 1) \) such that the function \( r \rightarrow h(u, p; r) \) is strictly decreasing on \((0, r^*)\) and strictly increasing on \((r^*, 1)\). Therefore, there exists \( \lambda \in (0, 1) \) such that \( h(u, p; r) < 0 \) for \( r \in (0, \lambda) \) and \( h(u, p; r) > 0 \) for \( r \in (\lambda, 1) \). \( \square \)

### 3 Main result

**Theorem 3.1** Let \( \lambda, \mu \in [0, 1/2] \). Then the double inequality

\[
G^p[\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a]A^{1-p}(a, b) < E(a, b) < G^p[\mu a + (1 - \mu)b, \mu b + (1 - \mu)a]A^{1-p}(a, b)
\]

holds for any \( p \in [1, \infty) \) and all \( a, b > 0 \) with \( a \neq b \) if and only if \( \lambda \leq 1/2 - \sqrt{1 - (2\sqrt{2}/\pi)^{4/p}/2} \) and \( \mu \geq 1/2 - \sqrt{4/(4p)} \).

**Proof** Let \( t \in [0, 1/2] \), since \( G^p[ta + (1 - t)b, tb + (1 - t)a]A^{1-p}(a, b) \) and \( E(a, b) \) are symmetric and homogeneous of degree one, without loss of generality, we assume that \( a > b > 0 \). Let \( r \in (0, 1) \) and \( b/a = (1 - r)^2/(1 + r)^2 \). Then (1.1) leads to

\[
E(a, b) = \frac{4(1 + r)^2}{\pi^2(1 + r^2)}A(a, b)\mathcal{E}^2\left(\frac{2\sqrt{r}}{1 + r}\right) = \frac{4}{\pi^2} A(a, b) \frac{[2\mathcal{E}(r) - (1 - r^2)K(r)]^2}{1 + r^2},
\]

\[
\log\left[G^p[ta + (1 - t)b, tb + (1 - t)a]A^{1-p}(a, b)\right] - \log E(a, b)
\]

\[
= \log \left[\frac{G^p[ta + (1 - t)b, tb + (1 - t)a]A^{1-p}(a, b)}{A(a, b)}\right] - \log \left[E(a, b)\right]
\]

\[
= \frac{1}{2} p \log \left[1 - \frac{4(1 - 2r)^2r^2}{(1 + r^2)^2}\right] - \log \left[\frac{4(2\mathcal{E}(r) - (1 - r^2)K(r))^2}{\pi^2(1 + r^2)}\right].
\]

Therefore, Theorem 3.1 follows easily from Lemma 2.5 and (3.1). \( \square \)

Let \( p = 1, 2 \), then Theorem 3.1 leads to Corollary 3.2 immediately.
Corollary 3.2 Let $\lambda_1, \mu_1, \lambda_2, \mu_2 \in [0, 1/2]$. Then the double inequalities

\[
H[\lambda_1 a + (1 - \lambda_1) b, \lambda_1 b + (1 - \lambda_1) a] < E(a, b) < H[\mu_1 a + (1 - \mu_1) b, \mu_1 b + (1 - \mu_1) a],
\]
\[
G[\lambda_2 a + (1 - \lambda_2) b, \lambda_2 b + (1 - \lambda_2) a] < E(a, b) < G[\mu_2 a + (1 - \mu_2) b, \mu_2 b + (1 - \mu_2) a]
\]

hold for all $a, b > 0$ with $a \neq b$ if and only if $\lambda_1 \leq 1/2 - \sqrt{1 - 8/\pi^2}/2 = 0.2823\ldots$, $\mu_1 \geq 1/2 - \sqrt{2}/8 = 0.3232\ldots$, $\lambda_2 \leq 1/2 - \sqrt{1 - 64/\pi^4}/2 = 0.2071\ldots$ and $\mu_2 \geq 1/4$.

Let $p \in [1, \infty)$, $r \in (0, 1)$, $a = r$, $b = 1 - r^2 = r^2$, $\lambda = 1/2 - \sqrt{1 - (2\sqrt{2}/\pi)^4p}/2$ and $\mu = 1/2 - \sqrt{p}/(4p)$. Then (1.1) and Theorem 3.1 lead to Corollary 3.3 immediately.

Corollary 3.3 The double inequality

\[
\sqrt{\frac{2\pi}{4}} \left(1 + r^2\right)^{(1-p)/2} \left[4r^2 + \left(\frac{8}{\pi^2}\right)^{2/p} r^4\right]^{p/4} < \mathcal{E}(r) < \sqrt{\frac{2\pi}{4}} \left(1 + r^2\right)^{(1-p)/2} \left[\left(1 + r^2\right)^2 - r^4\right]^{p/4} \]

holds for all $r \in (0, 1)$ and $p \in [1, \infty)$.

4 Results and discussion

In this paper, we provide the sharp bounds for the special quasi-arithmetic mean $E(a, b)$ in terms of the arithmetic mean $A(a, b)$ and geometric mean $G(a, b)$ with two parameters. As consequences, we present the best possible one-parameter harmonic and geometric means bounds for $E(a, b)$ and find new bounds for the complete elliptic integral of the second kind.

5 Conclusion

In the article, we derive a new bivariate mean $E(a, b)$ from the quasi-arithmetic mean and provide its sharp upper and lower bounds in terms of the concave combination of arithmetic and geometric means.

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Competing interests

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Authors’ contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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