AN INFINITE FAMILY OF GROMOLL-MEYER SPHERES

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Abstract. We construct a new infinite family of models of exotic 7-spheres. These models are direct generalizations of the Gromoll-Meyer sphere. From their symmetries, geodesics and submanifolds half of them are closer to the standard 7-sphere than any other known model for an exotic 7-sphere.

1. Introduction

This paper provides a new geometric way to construct all exotic 7-spheres. Exotic spheres are differentiable manifolds that are homeomorphic but not diffeomorphic to standard spheres. The first examples were found by Milnor [Mi1] in 1956 among the $S^3$-bundles over $S^4$. It turned out that 7 is the smallest dimension where exotic spheres can occur except possibly in the special dimension 4. In any dimension $n > 4$ the exotic spheres and the standard sphere form a finite abelian group: the group $\Theta_n$ of (orientation preserving diffeomorphism classes of) homotopy spheres [KM]. The inverse element in $\Theta_n$ can be obtained by a change of orientation. In dimension 7 we have $\Theta_7 \approx \mathbb{Z}_{28}$. Hence, ignoring orientation there are 14 exotic 7-spheres. From these 14 exotic 7-spheres four (corresponding to 2, 5, 9, 12, 16, 19, 23, 26 $\in \mathbb{Z}_{28}$) are not diffeomorphic to an $S^3$-bundle over $S^4$ [EK].

In 1974 Gromoll and Meyer [GM] constructed an exotic 7-sphere, $\Sigma_{GM}$, as quotient of the compact group $\text{Sp}(2)$ by a two-sided $S^3$-action. This construction provided $\Sigma_{GM}$ automatically with a metric of nonnegative sectional curvature ($K \geq 0$). The Gromoll-Meyer sphere $\Sigma_{GM}$ was the only exotic sphere known to admit such a metric until 1999 when Grove and Ziller [GZ] constructed metrics with $K \geq 0$ on all Milnor spheres, i.e., on all exotic 7-spheres that are $S^3$-bundles over $S^4$. In 2002 Totaro [To] and independently Kapovitch and Ziller [KZ] showed that $\Sigma_{GM}$ is the only exotic sphere that can be modeled by a biquotient of a compact group and thus underlined the singular status of the Gromoll-Meyer sphere among all models for exotic spheres.

We nevertheless provide an elementary and direct generalization of the Gromoll-Meyer construction. The essential components in this construction are natural self-maps of $S^7$, namely, the $n$-powers of unit octonions, $n \in \mathbb{Z}$. In terms of quaternions these maps are defined by

$$\rho_n : S^7 \to S^7, \quad (\cos t + p \sin t) \mapsto (\cos nt + p \sin nt) \quad (\cos nt + p \sin nt)$$

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where \( p \in \text{Im} \mathbb{H} \) and \( w \in \mathbb{H} \) with \( |p|^2 + |w|^2 = 1 \). Let \( \langle u, v \rangle := \bar{u}v \) denote the standard Hermitian product on \( \mathbb{H}^2 \). The submanifolds
\[
E_{n}^{10} := \{ (u, v) \in S^7 \times S^7 \mid \langle \rho_n(u), v \rangle = 0 \}
\]
come equipped with a free action of the unit quaternions:
\[
S^3 \times E_{n}^{10} \to E_{n}^{10}, \quad q \ast (u, v) = (qu, qv).
\]
Here, \( quq \) means that the two quaternionic components of \( u \) are simultaneously conjugated by \( q \in S^3 \). The quotient of \( E_{n}^{10} \) by the free \( \ast \)-action is a smooth manifold
\[
\Sigma_{n}^{7} := E_{n}^{10}/S^3.
\]
For \( n = 1 \) we have \( E_{1}^{10} = \text{Sp}(2) \) (the group of quaternionic \( 2 \times 2 \) matrices \( A \) with \( \bar{A}A = I \)) and the \( \ast \)-action is the original Gromoll-Meyer action. Hence, \( \Sigma_{1}^{7} = \Sigma_{GM}^{7} \).

Theorem 1. The differentiable manifold \( \Sigma_{n}^{7} \) is a homotopy sphere and represents the \( (n \mod 28) \)-th element in \( \Theta_7 \approx \mathbb{Z}_{28} \).

Let \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) denote the diagonal matrices of \( O(2) \subset \text{Sp}(2) \). All \( E_{n}^{10} \) admit a smooth action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times S^3 \) that commutes with the free \( \ast \)-action:
\[
\mathbb{Z}_2 \times \mathbb{Z}_2 \times E_{n}^{10} \to E_{n}^{10}, \quad B \ast (u, v) = (Bu, Bv),
\]
\[
S^3 \times E_{n}^{10} \to E_{n}^{10}, \quad q \ast (u, v) = (u, qv).
\]
The induced effective action on \( \Sigma_{n}^{7} \) is an action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times SO(3) \) where \( SO(3) = S^3/\{ \pm 1 \} \). On \( \Sigma_{n}^{7} \) this action can be identified with the linear action
\[
(B, \pm q) \cdot (x, u) = (Bx, Bqu)
\]
on \( S^7 \subset \mathbb{R}^2 \times (\text{Im} \mathbb{H})^2 \). On \( \Sigma_{1}^{7} = \Sigma_{GM}^{7} \) the action coincides with the subaction of the \( O(2) \times SO(3) \)-action given in \( \text{GM} \).

The surprising fact is the following even/odd grading of the \( \Sigma_{n}^{7} \):

Theorem 2. All \( \Sigma_{n}^{7} \) with even \( n \) are equivariantly homeomorphic to \( S^7 \) with the linear \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times SO(3) \)-action given above. All \( \Sigma_{n}^{7} \) with odd \( n \) are equivariantly homeomorphic to the Gromoll-Meyer sphere \( \Sigma_{GM}^{7} \) with the above \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times SO(3) \)-action. If \( n \) is even all fixed point sets in \( \Sigma_{n}^{7} \) are spheres while if \( n \) is odd there are also 3-dimensional fixed point sets with fundamental groups \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \).

The even/odd grading of the \( \Sigma_{n}^{7} \) also transfers to some of the invariant submanifolds. The most important one is \( \Sigma_{5}^{7} \) whose preimage under the map \( E_{n}^{10} \to \Sigma_{n}^{7} \) consists of points \( (u, v) \) where both quaternionic components of \( u \) are purely imaginary.

Proposition 3. \( \Sigma_{5}^{7} \) is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times SO(3) \)-equivariantly diffeomorphic to \( S^5 \subset (\text{Im} \mathbb{H})^2 \) with the linear action \( (B, \pm q) \cdot u = Bqu \) if \( n \) is even and to the Brieskorn sphere \( W_5^{3} \) if \( n \) is odd. The subsphere \( \Sigma_{5}^{7} \) is minimal for every \( \{ \pm I \} \times SO(3) \)-invariant metric on \( \Sigma_{n}^{7} \).
Recall here that the Brieskorn sphere $W_d^7$ with $d \in \mathbb{N}$ is the intersection of the unit sphere in $\mathbb{C}^4 = \mathbb{C} \oplus \mathbb{C}^3$ with the complex hypersurface
\[ z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0 \]
and that there is a natural $O(2) \times SO(3)$-action on $W_d^5$.

\[ O(2) \times SO(3) \times W_d^5 \rightarrow W_d^5, \]

\[
\begin{align*}
&\left( \begin{array}{c}
\cos \theta - \sin \theta \\
\sin \theta \\
\cos \theta 
\end{array} , A \right) \cdot (z_0, z) = (e^{2i\theta} z_0, e^{d\theta} A z), \\
&\left( \begin{array}{c}
1 \\
0 \\
-1
\end{array} , A \right) \cdot (z_0, z) = (\overline{z}_0, A \overline{z}).
\end{align*}
\]

The classification theorems of Jänich and Hsiang-Hsiang imply that for $G = O(2) \times SO(3)$ and even for the smaller group $G = \{\pm 1\} \times SO(3)$ the Brieskorn sphere $W_d^5$ is not $G$-equivariantly homeomorphic to $S^5$ with any linear action, see [HMa]. However, $W_d^5$ is SO(3)-equivariantly diffeomorphic to $S^5$. In the case $d = 3$ an explicit formula for such a diffeomorphism is given in [DP].

The invariant subsphere $\Sigma_n^5$ is dual to the invariant circle $\Sigma_n^1$ whose preimage under the map $E_{n}^{10} \rightarrow \Sigma_n^7$ consists of points $(u, v)$ for which both components of $u$ are real. These two dual submanifolds play a central role for the geodesic geometry of $M_n^7$. We construct a one parameter family of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times SO(3)$-invariant metrics $\langle \cdot, \cdot \rangle_{\nu}$ on each $\Sigma_n^7$ with the following property:

**Theorem 4.** All points $p \in \Sigma_n^1$ have the wiedersehen property, i.e., every unit speed geodesic $\gamma$ in $\Sigma_n^7$ with $\gamma(0) = p$ is length minimizing on $[0, \pi[$ and obeys $\gamma(\pi) = -p$ and $\gamma(2\pi) = p$. Moreover, $\Sigma_n^1 \times \Sigma_n^5 \rightarrow \Sigma_n^7$ that maps $(x, y, t)$ to $\gamma(t)$, where $\gamma : [0, \frac{\pi}{2}] \rightarrow \Sigma^7$ is the unique unit speed geodesic segment from $x$ to $y$, is a homeomorphism.

This invariant geodesic join structure actually is the key to prove Theorem 1 and Theorem 2. In the particular case of exotic 7-spheres our method is an improvement over the general construction that equips all exotic spheres with pointed wiedersehen metrics [Ba].

The even/odd grading of the $\Sigma_n^7$ is in contrast to what happens for the Milnor spheres $M_{k,l}^7$ and the Brieskorn spheres $W_{6n-1,3}^7$.

The Milnor sphere $M_{k,l}^7$ with $k + l = 1$ is defined by gluing two copies of $\mathbb{H} \times S^3$ along $(\mathbb{H} \setminus \{0\}) \times S^3$ by the map

\[
(u, v) \mapsto \left( \frac{u}{|u|}, \frac{v}{|v|} \right)^k \cdot \left( \frac{u}{|u|} \right)^l.
\]

For convenience, we set $M_{k,l}^d = M_{d}^d$ where $d = k - l$ is odd. The Milnor sphere $M_d^7$ represents the $\frac{d+1}{2}$-th element in $\Theta_7$, see [Ek]. There is a natural $SO(3) = S^3/\{\pm 1\}$-action on $M_d^7$ which is in both charts defined by

\[
\pm q \cdot (u, v) = (quq^{-1}, qvq^{-1}).
\]

Davis [Da] has shown that $M_d^7$ is $SO(3)$-equivariantly diffeomorphic to $M_d^7$ if and only if $d' = \pm d$ and that all $M_d^7$ are $SO(3)$-equivariantly homeomorphic to $S^7 \subset \mathbb{H}^2$. 


with the linear SO(3)-action given by \((\pm q, u) \mapsto qu\bar{q}\). We show that the latter situation changes when one extends the SO(3)-action by the commuting involution \((u, v) \mapsto (u, -v)\) in both charts. This involution fixes all points in the base of the bundle \(M^7_d \to S^4\) and induces the antipodal map on all the \(S^3\)-fibers. For consistency, the group generated by SO(3) and the involution is denoted by \(\{\pm 1\} \times SO(3)\).

**Theorem 5.** The fixed point set of the involution \((-1, \pm i)\) on \(M^7_d\) is a 3-dimensional lens space with fundamental group \(\mathbb{Z}_{|d|}\). Hence, \(M^7_d\) is \(\{\pm 1\} \times SO(3)\)-equivariantly homeomorphic to any of the \(\Sigma^n_7\).

This theorem is a consequence of Theorem 5.1 which is the analogue of Proposition 3 for the Milnor spheres.

Grove and Ziller [GZ] constructed SO(3)-actions on \(M^7_d\) that are entirely different from the SO(3)-actions on \(M^7_d\) and \(\Sigma^n_7\) above. The SO(3)-actions on \(M^7_d\) and \(\Sigma^n_7\) fix a circle pointwise while the Grove-Ziller actions are almost free.

The Brieskorn sphere \(W^7_{6n-1, 3}\) is defined by the intersection of the the unit sphere \(S^9 \subset \mathbb{C}^5 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^3\) with the complex hypersurface

\[w^{6n-1} + z_0^3 + z_1^2 + z_2^2 + z_3^2 = 0.\]

It represents the \((n \text{ mod } 28)\)-th homotopy sphere in \(\Theta_7\) (see [BK]) and admits the natural SO(3)-action \((A, (w, z_0, z)) \mapsto (w, z_0, Az)\).

**Theorem 6.** None of the \(W^7_{6n-1, 3}\) are SO(3)-equivariantly diffeomorphic to any of the \(\Sigma^n_7\) or to any of the \(M^7_{k,l}\).

In particular, \(W^7_{6n-1, 3}\) is not SO(3)-equivariantly homeomorphic to the join of a circle and \(W^3_3\). Thus, the equivariant topology of the \(\Sigma^n_7\) with odd \(n\) is much more determined by the equivariant topology of \(W^3_3\) than the equivariant topology of \(W^7_{6n-1, 3}\) although the latter contain \(W^3_3\) in a much more obvious way (just by setting \(w = 0\)).

Many of the constructions in this paper generalize the constructions given in [DP] for the original Gromoll-Meyer sphere \(\Sigma^n_{7\text{GM}}\).

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2. A CONSTRUCTION OF THE \(S^3\)-PRINCIPAL BUNDLES OVER \(S^7\)

Recall from the introduction the definition of \(E^{10}_n \subset S^7 \times S^7\):

\[E^{10}_n = \{(u, v) \in S^7 \times S^7 \mid \langle \rho_n(u), v \rangle = 0\}.\]

For \(n = 1\) the space \(E^{10}_1\) can be equivalently seen as the group Sp(2) of \(2 \times 2\) quaternionic matrices \(A\) such that \(A^tA = \mathbb{I}\). The standard projection \(Sp(2) \to S^7\), \(A = (u, v) \mapsto u\) turns Sp(2) into an \(S^3\)-principal bundle over \(S^7\).
Lemma 2.1. $E_{n}^{10}$ is the pull-back of $\text{Sp}(2)$ by the map $\rho_{n} : S^{7} \to S^{n}$.

Proof. By the usual explicit construction, the total space of the pull-back bundle $\rho_{n}(\text{Sp}(2))$ is the submanifold of $S^{7} \times \text{Sp}(2)$ consisting of all pairs $(u, A)$ such that $\rho_{n}(u)$ is the first column of $A$. It is evident, however, that in this construction we log the first column of $A$ twice. Eliminating this redundancy leads to the definition of $E_{n}^{10}$ above. This in particular shows that $E_{n}^{10}$ is a submanifold of $S^{7} \times S^{7}$. □

Corollary 2.2. $E_{n}^{10}$ is an $S^{3}$-principal bundle over $S^{7}$ classified by $n \mod 12$.

Proof. The $S^{3}$-principal bundles over $S^{7}$ are classified by $\pi_{6}(S^{3}) \approx \mathbb{Z}_{12}$ and the characteristic map of the bundle $\text{Sp}(2) \to S^{7}$ generates $\pi_{6}(S^{3})$ (see [Hu] or [DMR] for a more explicit reference). The map $\rho_{n}$ has degree $n$. □

The principal bundle map $E_{n}^{10} \to S^{7}$ is given by the projection to the first column. The corresponding free $S^{3}$-action on $E_{n}^{10}$ is given by

$$S^{3} \times E_{n}^{10} \to E_{n}^{10}, \quad (q \star (u, v)) = (u, v \bar{q}).$$

The map $\tilde{\rho}_{n}$ in the pull-back diagram

$$E_{n}^{10} \xrightarrow{\tilde{\rho}_{n}} \text{Sp}(2) \quad \downarrow \quad \downarrow$$

$$S^{7} \xrightarrow{\rho_{n}} S^{7}$$

takes the explicit form

$$\tilde{\rho}_{n} : E_{n}^{10} \to \text{Sp}(2), \quad (u, v) \mapsto (\rho_{n}(u), v).$$

Recall from the introduction that there is a free $S^{3}$-action $q \star (u, v) = (qu, qv)$ on $E_{n}^{10}$ that commutes with the $\cdot$-action and whose orbit space is the smooth manifold $\Sigma_{n}^{7}$. The pull-back diagram above extends to the following commutative diagram:

$$E_{n}^{10} \xrightarrow{\tilde{\rho}_{n}} \text{Sp}(2) \quad \downarrow \quad \downarrow$$

$$\Sigma_{n}^{7} \xrightarrow{\rho_{n}} \Sigma_{GM}^{7}$$

The degree of the induced map $\rho_{n}^{\prime} : \Sigma_{n}^{7} \to \Sigma_{GM}^{7}$ is $n$. The proof that $\Sigma_{n}^{7}$ represents the $(n \mod 28)$-th element of $\Theta_{7}$ requires several geometric constructions and is postponed until section 4.

Each principal bundle $E_{n}^{10}$ admits a natural action of $Z_{2} \times Z_{2} \times S^{3} \times S^{3} \times S^{3}$, where $Z_{2} \times Z_{2}$ denotes the diagonal matrices in $O(2) \subset \text{Sp}(2)$:

(3) \hspace{1cm} $Z_{2} \times Z_{2} \times E_{n}^{10} \to E_{n}^{10}, \quad B \cdot (u, v) = (Bu, Bv),$

(4) \hspace{1cm} $S^{3} \times S^{3} \times E_{n}^{10} \to E_{n}^{10}, \quad (q_{1}, q_{2}, q_{3}) \cdot (u_{1}, v_{1}) = q_{1}u_{1}q_{1}^{-1}q_{2}u_{2}q_{2}^{-1}q_{3}v_{3}q_{3}^{-1}$.

Lemma 2.3. This action on $E_{n}^{10}$ is of cohomogeneity 2.
Proof. The third $S^3$-factor yields the principal action related to the bundle $E_7^{10} \rightarrow S^7$, $(u,v) \mapsto u$, i.e., this $S^3$-factor acts simply transitively on the fiber over any $u \in S^7$. The action of the first two $S^3$-factors on $S^7$ has kernel $\{\pm(1,1)\}$ and induces a standard linear $SO(4)$-action on $S^7$. By applying all three $S^3$-factors one can transform an arbitrary point in $E_7^{10}$ to a point of the form

$$\begin{pmatrix} \cos t + i \cos s \sin t & -\sin s \sin nt \\ \sin s \sin t & \cos nt - i \cos s \sin nt \end{pmatrix}.$$  

The diagonal in the first two $S^3$-factors gives the Gromoll-Meyer action $\star$ corresponding to the principal bundle $E_7^{10} \rightarrow \Sigma_{7}$, The third $S^3$-factor and the $Z_2 \times Z_2$-factor yield the effective $Z_2 \times Z_2 \times SO(3)$-action $\bullet$ on $\Sigma_{7}$ from the introduction. It is an interesting question for which $n$ this $\bullet$-action can be extended. The maximum dimension of any compact differentiable transformation group of an exotic 7-sphere is four $[St]$. On the original Gromoll-Meyer sphere $\Sigma_{GM} = \Sigma_{1}$ there is a natural $O(2) \times SO(3)$-action. It is induced by the action

$$O(2) \times SO(3) \times Sp(2) \rightarrow Sp(2), \quad (A,q) \bullet (u,v) \mapsto (Au,Avq)$$

on $Sp(2) = E_1^{10}$ and extends the $\bullet$-action naturally. A corresponding $O(2) \times SO(3)$-action exists of course on $\Sigma_{7-1}$. On $\Sigma_{3}$ an $O(2) \times SO(3)$-action is induced by the action

$$O(2) \times SO(3) \times E_6^{10} \rightarrow E_6^{10}, \quad (A,q) \bullet (u,v) \mapsto (Au,vq).$$

On the other $\Sigma_{n}$ with $n \neq -1,0,1$, however, it seems likely that the $Z_2 \times Z_2 \times SO(3)$-action cannot be extended to any larger group, see Remark 4.4.

Question 2.4. Which $E_n^{10}$ admit Riemannian metrics with $K \geq 0$ that are invariant under the cohomogeneity 2 action above? If some $E_n^{10}$ admits such a metric then by the O’Neill formulas the induced metric on $\Sigma_{n}$ also has $K \geq 0$. This would be particularly interesting for those $\Sigma_{n}$ that are not diffeomorphic to $S^3$-bundles over $S^4$ since on such exotic spheres no metrics with $K \geq 0$ are known so far.

Remark 2.5. While there are twelve $S^3$-principal bundles over $S^7$ there are 28 homotopy 7-spheres. This means in particular that some $\Sigma_{n}$ are quotients of trivial bundles $E_n^{10}$. This phenomenon is well-known from surgery theory (see [Wa]) in an inexplicit way.

Remark 2.6. Grove and Ziller [GZ] constructed cohomogeneity one metrics with $K \geq 0$ on all $S^3 \times S^3$-principal bundles over $S^4$. It is known that the $(n \mod 12)$-th $S^3$-principal bundle over $S^4$ is diffeomorphic to an $S^3 \times S^3$-principal bundle over $S^4$ if and only if $n \mod 12 \in \{0,1,3,4,6,7,9,10\}$. It is easy to see that the set of all integers $n$ with $n \mod 12 \in \{0,1,3,4,6,7,9,10\}$ maps surjectively on $Z_{28}$. Thus, every element in $\Theta_7$ can be represented by some $\Sigma_{n}$ such that $E_n^{10}$ admits a cohomogeneity one metric with $K \geq 0$. However, this does not mean that $\Sigma_{n}$ admits a metric with $K \geq 0$ since the Gromoll-Meyer action $E_n^{10}$ is not isometric with respect to the Grove-Ziller metric.
3. Invariant submanifolds and parity

In this section we will see that the even/odd grading of the generalized Gromoll-Meyer spheres $\Sigma_n^7$ is based on an elementary property of the maps $\rho_n$.

Consider the subsets
\[ E_n^0 := \{(u, v) \in E_n^{10} \mid u \in \text{Im} \mathbb{H} \times \mathbb{H} \}, \]
\[ E_n^8 := \{(u, v) \in E_n^{10} \mid u \in \text{Im} \mathbb{H} \times \text{Im} \mathbb{H} \} \]
of $E_n^{10} \subset S^7 \times S^7$. These are the preimages of the subspaces
\[ S^6 = \{ (\frac{p}{w}, w) \mid p \in \text{Im} \mathbb{H}, w \in \mathbb{H}, |p|^2 + |w|^2 = 1 \}, \]
\[ S^5 = \{ (\frac{p_1}{p_2}, p_1, p_2) \in \text{Im} \mathbb{H}, |p_1|^2 + |p_2|^2 = 1 \} \]
of $S^7 \subset \mathbb{H} \times \mathbb{H}$ under the principal bundle projection $E_n^{10} \to S^7$.

**Lemma 3.1.** $E_n^0$ and $E_n^8$ are submanifolds of $E_n^{10}$ diffeomorphic to $S^6 \times S^3$ and $S^5 \times S^3$, respectively.

**Proof.** $E_n^0 \to S^6$ is a proper subbundle of $E_n^{10} \to S^7$ and hence trivial. \(\square\)

**Lemma 3.2.** $E_n^0$ and $E_n^8$ are invariant under the free $*$-action of $S^3$ and under the $\bullet$-action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S^3$. Hence, the $*$-quotients $\Sigma_n^6$ and $\Sigma_n^5$ are submanifolds of $\Sigma_n^7$ with a natural $\bullet$-action of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3)$.

**Proof.** Straightforward. \(\square\)

**Lemma 3.3.** As submanifolds of $S^7 \times S^7$ we have
\[ \ldots = E_{-3}^0 = E_{-2}^0 = E_{-1}^0 = E_1^0 = E_2^0 = E_3^0 = \ldots, \]
\[ \ldots = E_{-4}^0 = E_{-2}^8 = E_0^8 = E_2^8 = E_4^8 = \ldots \]
and the same identities also hold for $E_n^8 \subset E_n^0$ and for the quotients $\Sigma_n^6$ and $\Sigma_n^5$.

**Proof.** This is an immediate consequence of the two basic identities
\[ \rho_{2m+1}( (\frac{p}{w}) ) = (-1)^m (\frac{p}{w}) \quad \text{and} \quad \rho_{2m}( (\frac{p}{w}) ) = (-1)^m (\frac{0}{1}) \]
for $(\frac{p}{w}) \in S^6 \subset \text{Im} \mathbb{H} \times \mathbb{H}$. \(\square\)

**Corollary 3.4.** If $n$ is odd, $\Sigma_n^5$ is equivariantly diffeomorphic to the Brieskorn sphere $W_3^5$ with its natural $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3)$-action. If $n$ is even, $\Sigma_n^5$ is equivariantly diffeomorphic to the Euclidean sphere $S^5 \subset \mathbb{R}^3 \times \mathbb{R}^3$ where $\text{SO}(3)$-acts diagonally on both $\mathbb{R}^3$-factors and each $\mathbb{Z}_2$-factor acts on one of the $\mathbb{R}^3$-factors.

**Proof.** From Lemma 7.4 of [DF] we know that $\Sigma_5^7$ is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3)$-equivariantly diffeomorphic to $W_3^5$. For $\Sigma_0^5$ we observe that
\[ E_0^8 = \{ (\frac{p_1}{p_2}, q) \mid p_1, p_2 \in \text{Im} \mathbb{H}, q \in S^3 \} \]
The natural embedding $S^5 \to E_0^8$, $(\frac{p_1}{p_2}) \mapsto (\frac{p_1}{p_2}, 1)$ identifies the $*$-quotient of $E_0^8$ with $S^5$. \(\square\)

**Lemma 3.5.** The subsphere $\Sigma_n^5$ is minimal in $\Sigma_n^6$ and $\Sigma_n^7$ for all $\{\pm 1\} \times \text{SO}(3)$-invariant Riemannian metrics on $\Sigma_n^7$. 
Proof. Analogous to the proof of Corollary 3.4 in [DP] this follows from the fact that $\Sigma^5_n$ is the union of orbits whose isotropy groups contain elements of the form $(-1, \pm q)$. \hfill \Box

4. The geodesic join structure of $\Sigma^7_n$

We will now study the geometry of a one parameter family of Riemannian metrics on $E^{10}_n$ and $\Sigma^7_n$ and use the results to prove Theorem 1, Theorem 2 and Theorem 4. The one parameter family of metrics is defined in such a way that the constructions of [Du] and [DP] for $\Sigma^7_{GM}$ can be extended to all $\Sigma^7_n$.

We equip the total space of the principal bundle $E^{10}_n \to S^7$ with the Riemannian metric $\langle \cdot, \cdot \rangle_{\nu}$ with $\nu > 0$ defined by the following properties:

- The $S^3$-fibers have constant curvature $\frac{1}{\nu}$.
- The horizontal distribution is given by the pull-back of the horizontal distribution of $Sp(2)$ via the map $\rho_n$, i.e., we pull-back the principal bundle connection of $Sp(2)$.
- The metric $\langle \cdot, \cdot \rangle_{\nu}$ induces on $S^7$ the metric with constant curvature 1 by Riemannian submersion.

Such metrics are called connection metrics or Kaluza-Klein metrics.

The $Z_2 \times Z_2 \times S^3 \times S^3 \times S^3$-action given in (3) and (4) is isometric with respect to the metric $\langle \cdot, \cdot \rangle_{\nu}$. In particular, the Gromoll-Meyer action $\star$ is isometric and $\Sigma^7_n$ inherits a Riemannian metric by Riemannian submersion, which will again be denoted by $\langle \cdot, \cdot \rangle_{\nu}$. The $\bullet$-action of $Z_2 \times Z_2 \times S^3$ on $E^{10}_n$ is also isometric.

Since the $\bullet$-action commutes with the $\star$-action, it induces an effective isometric $Z_2 \times Z_2$-action on $(\Sigma^7_n, \langle \cdot, \cdot \rangle_{\nu})$.

**Lemma 4.1.** The common fixed point set of $SO(3)$ in $\Sigma^7_n$ is the circle

$$\Sigma^1_n := \left\{ \pi_{\Sigma^7_n} \left( \begin{pmatrix} \cos t & -\sin nt \\ \sin t & \cos nt \end{pmatrix} \right) \mid t \in \mathbb{R} \right\}.$$  

Hence, for any $SO(3)$-invariant Riemannian metric on $\Sigma^7_n$, this circle $\Sigma^1_n$ is a simple closed geodesic.

**Proof.** $\pi_{\Sigma^7_n}(u, v)$ is a fixed point of $SO(3)$ if and only if for every $q \in S^3$ there is a $q' \in S^3$ such that $(q'uv^2, q'vq) = (u, v)$. It is easy to see from the second column of this equation that all elements of $S^3$ occur for $q'$. Therefore, $u$ must have two real components. \hfill \Box

Note that the $Z_2 \times Z_2$-action on $\Sigma^1_n$ is equivalent to the standard $Z_2 \times Z_2$-action on $S^1$. In particular, for each point $p \in \Sigma^1_n$ there is a natural antipode $-p$.

**Theorem 4.2.** Every unit speed geodesic $\gamma$ in $(\Sigma^7_n, \langle \cdot, \cdot \rangle_{\nu})$ with $\gamma(0) = p \in \Sigma^1_n$ is length minimizing on $[0, \pi]$ and we have $\gamma(\pi) = -p$ and $\gamma(2\pi) = p$.

**Proof.** The proof is similar to the proofs of Theorem I in [Du] and Theorem 2.1 in [DP]. We lift $\gamma$ horizontally to a geodesic $\tilde{\gamma}$ in $E^{10}_n$ with

$$\tilde{\gamma}(0) = \alpha(t) := \begin{pmatrix} \cos t & -\sin nt \\ \sin t & \cos nt \end{pmatrix} \in E^1_n.$$
That \( \hat{\gamma} \) is horizontal with respect to \( E_{10}^n \to \Sigma_n^7 \) means that the geodesic \( \hat{\gamma} \) passes perpendicularly through all \( \star \)-orbits. It is straightforward to check that
\[
S^3 \star \alpha(t) = S^3 \star \alpha(t).
\]
Thus, \( \hat{\gamma} \) passes perpendicularly through \( S^3 \star \hat{\gamma}(0) \). A geodesic that passes perpendicularly through one orbit passes perpendicularly through all orbits. Hence, \( \hat{\gamma} \) passes perpendicularly through all \( S^3 \)-orbits of the \( \bullet \)-action. In other words, \( \hat{\gamma} \) is horizontal to the principal fibration \( E_{10}^n \to \Sigma_n^7 \). Hence, \( \hat{\gamma} \) projects to a geodesic \( \beta \) in \( \Sigma_n^7 \). By definition of \( \langle \cdot, \gamma \rangle \nu \) the sphere \( S^7 \) inherits the metric with constant curvature 1 from \( E_{10}^n \) by Riemannian submersion. Since all unit speed geodesics of \( S^7 \) that start at \( \beta(0) = \pi_{S^7}(\alpha(t)) \) pass through \( \beta(\pi) = -\beta(0) \) at time \( \pi \) we have \( \beta(\pi) = \pi_{S^7}(\alpha(t + \pi)) \). Thus, \( \hat{\gamma}(\pi) \) is contained in \( S^3 \star \alpha(t + \pi) = S^3 \star \alpha(t + \pi) \) and \( \hat{\gamma}(2\pi) \) is contained in \( S^3 \star \alpha(t + 2\pi) = S^3 \star \hat{\gamma}(0) \). This shows \( \gamma(\pi) = -\gamma(0) \) and \( \gamma(2\pi) = \gamma(0) \). Now let \( \gamma \) be a unit speed geodesic in \( \Sigma_n^7 \) with \( \gamma(0) = p \) and \( \gamma_1(t) = -p \). By the construction above \( \beta \) is a unit speed geodesic in \( S^7 \) with \( \beta(l) = -\beta(0) \). Hence, \( l \) cannot be less than \( \pi \).

Recall that the join \( X \star Y \) of two spaces \( X \) and \( Y \) is the quotient of \( X \times Y \times [0, 1] / \sim \) where \((x, y, 0) \sim (x, y', 0) \) and \((x, y, 1) \sim (x', y, 1) \) for all \( x \in X \) and all \( y \in Y \). For our purposes it is convenient to substitute \([0, 1] \) by \([0, \frac{\pi}{2}] \).

**Corollary 4.3.** \( \Sigma_n^1 \) and \( \Sigma_n^5 \) have constant distance \( \frac{\pi}{2} \). Moreover, the map \( \Sigma_n^1 \star \Sigma_n^5 \to \Sigma_n^7 \) that maps \((x, y, t) \) to \( \gamma(t) \), where \( \gamma : [0, \frac{\pi}{2}] \to S^7 \) is the unique unit speed geodesic segment from \( x \) to \( y \), is an equivariant homeomorphism.

**Proof.** This follows from the construction in the proof of Theorem 4.2 if one recalls that the submanifolds \( E_{10}^n \) and \( E_{10}^n \) project to the submanifolds
\[
S^1 = \{ (\cos t, \sin t) \mid t \in \mathbb{R} \},
\]
\[
S^5 = \{ (p_1, p_2) \mid p_1, p_2 \in \text{Im } \mathbb{H}, |p_1|^2 + |p_2|^2 = 1 \}
\]
of \( S^7 \subset \mathbb{H}^2 \) under the principal fibration \( E_{10}^n \to S^7 \) and to the submanifolds \( \Sigma_n^1 \) and \( \Sigma_n^5 \) of \( \Sigma_n^7 \) under the principal fibration \( E_{10}^n \to \Sigma_n^7 \).

Theorem 4.2 and Corollary 4.3 together yield Theorem 4 from the introduction.

**Corollary 4.4.** \( \Sigma_n^7 \) is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3) \)-equivariantly homeomorphic to \( S^1 \star S^5 \) if \( n \) is even and to \( S^1 \star W_3^5 \) if \( n \) is odd. Here, the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) acts on \( S^1 \) in the standard way.

**Proof.** This is evident from Corollary 4.4 and Corollary 4.3.

In particular, all \( \Sigma_n^7 \) with even \( n \) are mutually equivariantly homeomorphic and that all \( \Sigma_n^7 \) with odd \( n \) are mutually equivariantly homeomorphic. This proves Theorem 2 from the introduction.

**Proof of Theorem 1.** Consider the unit speed geodesic
\[
\beta(t) = \left( \frac{\cos t + p \sin t}{\sqrt{\sin t}} \right).
\]
in $S^7 \subset \mathbb{H}^2$ that emanates from the north pole with initial velocity $(p, w) \in S^6 \subset \text{Im } \mathbb{H} \times \mathbb{H}$. A lift $\tilde{\gamma}_n$ of this curve to $E^{10}_n$ with $\tilde{\gamma}_n(0) = (1, 0, 0)$ is given by

\begin{equation}
\tilde{\gamma}_n(t) = \left( \cos t + p \sin t \frac{w}{|w|} e^{-n|p| \sin(nt)} - e^{n|p| \cos(nt)} \frac{w}{|w|} t \right).
\end{equation}

Here $e^p = \cos |p| + \frac{\omega}{|p|} \sin |p|$ denotes the exponential map of $S^3 \subset \mathbb{H}$ at 1. Note that for $w = 0$ equation (5) simply becomes $\tilde{\gamma}_n(t) = (e^{n|p| \cos(t)}, 0)$. Using the identity

$$\tilde{\rho}_n(\tilde{\gamma}_n(t)) = \tilde{\gamma}_n(nt)$$

for the map $\tilde{\rho}_n : E^{10}_n \to \text{Sp}(2)$ defined in section 4 it is straightforward to verify that $\tilde{\gamma}_n$ is the unique horizontal lift of $\gamma$ to $E^{10}_n$ with $\tilde{\gamma}_n(0) = 1$. Since the fibers of $E^{10}_n \to S^7$ and $E^{10}_n \to \Sigma^7_n$ through $\tilde{\gamma}_n(0) = (1, 0, 0)$ are the same (as sets), the geodesic $\tilde{\gamma}_n$ is horizontal with respect to both these fibrations. This shows that $\gamma_n = \pi_{\Sigma^7_n} \circ \tilde{\gamma}_n$ is a geodesic in $\Sigma^7_n$. Now, considering all possible unit initial vectors $(p, w) \in S^6 \subset \text{Im } \mathbb{H} \times \mathbb{H}$ and times $t \in [0, \frac{\pi}{2}]$ the geodesics $\gamma_n$ provide an embedding of a disk $D^7(\frac{\pi}{2})$ into $\Sigma^7_n$ by Theorem 4.2. In the same way, the geodesics $\pi_{\Sigma^7_n} \circ (-\gamma_n) \circ (-\text{id})$ provide another embedding of the same disk. By Theorem 4.2 $\Sigma^7_n$ is the twisted sphere obtained by gluing these two embedded disks along their common boundary. One easily checks that

$$\tilde{\gamma}_n(p, w, \frac{\pi}{2}) = q \ast (-\tilde{\gamma}_n(-p', -w', \frac{\pi}{2}))$$

for some $q \in S^3$ if and only if $(p', w') = \sigma^n(p, w)$ where $\sigma$ is the exotic diffeomorphism of $S^6 \subset \text{Im } \mathbb{H} \times \mathbb{H}$ first described in [Du]. This diffeomorphism $\sigma$ generates $\pi_0(\text{Diff}_+(S^6))$. It is given by the formula

$$\sigma(p, w) := b(p, w)(p, w)b(p, w)$$

where $b(p, w) = \frac{w}{|w|} e^{n|p| \omega} \frac{w}{|w|}$ is an analytic formula for a generator of $\pi_0(S^3)$. Hence, we have obtained $\Sigma^7_n$ by gluing two 7-disks with the $n$-th power of a generator of $\pi_0(\text{Diff}_+(S^6)) \approx \Theta_7 \approx \mathbb{Z}_{28}$. \hfill \Box

Remark 4.5. Let $G$ be a compact group acting smoothly on $\Sigma^7_n$ with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3) \subset G$. Precisely as in [DP], Lemma 3.7, one can show that $G$ leaves $\Sigma^1_n$ and $\Sigma^5_n$ invariant. Let $n \notin \{-1, 0, 1\}$. Comparing for different $p \in \Sigma^1_n$ the closing behaviour of geodesics that start at $p$ perpendicularly to $\Sigma^1_n$, one can see that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the maximal compact group that acts isometrically on $(\Sigma^7_n, \langle \cdot, \cdot \rangle_\nu)$ and effectively on the circle $\Sigma^1_n$. This difference from the cases $n = -1, 0, 1$ suggests that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \text{SO}(3)$ is the full isometry group of $(\Sigma^7_n, \langle \cdot, \cdot \rangle_\nu)$.

Remark 4.6. If we pull back the metric $\langle \cdot, \cdot \rangle_\nu$ on $\text{Sp}(2)$ by the map $\tilde{\rho}_n$ then we obtain a degenerate metric $\langle \cdot, \cdot \rangle_\nu^\ast$ on $E^{10}_n$ that has the same geodesics through the circle $\Sigma^1_n$ as the metric $\langle \cdot, \cdot \rangle_\nu$. For $n \notin \{-1, 0, 1\}$ the metric $\langle \cdot, \cdot \rangle_\nu^\ast$ is degenerate precisely over $|n| - 1$ subspaces in $S^7$ whose first quaternionic components have constant real part. With such a metric $\Sigma^7_n$ looks like $n$ copies of $\Sigma^7_{\text{GM}}$ stacked one on top of the other, i.e., like a degenerate connected sum of $n$ copies of $\Sigma^7_{\text{GM}}$. 
Remark 4.7. The manifolds \((E^n, \langle \cdot, \cdot \rangle_\nu)\) with even \(n\) are not just mutually equal as submanifolds of \(S^7 \times S^7\) but also mutually equal as Riemannian manifolds. Hence, also the manifolds \((\Sigma^n_\nu, \langle \cdot, \cdot \rangle_\nu)\) with even \(n\) are all mutually equal as Riemannian manifolds. The analogous statements hold for odd \(n\).

5. Comparison to the exotic Milnor and Brieskorn 7-spheres

In this section we compare the equivariant topology of the spheres \(\Sigma^7_n\) with the equivariant topology of the Milnor spheres \(M^7_d\) and the Brieskorn spheres \(W^7_{6n-1,3}\) and prove Theorem 5 and Theorem 6 of the introduction.

Recall from the introduction that the Milnor spheres \(M^7_d\) admit natural \({\pm \mathbb{I}}\times SO(3)\)-actions. Davis [Da] has shown that these actions can be extended to \(GL(2, \mathbb{R}) \times SO(3)\)-actions. In the first chart the \(GL(2, \mathbb{R})\)-action is given by

\[
(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}) \cdot (u, v) = (\frac{bu+dv}{|bu+dv|}, \det (\begin{smallmatrix} a & c \\ b & d \end{smallmatrix})^k v (\frac{bu+dv}{|bu+dv|}))
\]

and in the second one by

\[
(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}) \cdot (u, v) = \left(\frac{b+du}{a+cu}, \det (\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}) \left(\frac{a+cu}{|a+cu|}\right)^k \left(\frac{b+du}{a+cu}\right)\right).
\]

Note that our definition of the action differs from the definition given by Davis by the factor \(\det (\begin{smallmatrix} a & c \\ b & d \end{smallmatrix})\). The reason is that with our definition the identification between \(M^7_d\) and the Gromoll-Meyer sphere \(\Sigma^7_{GM}\) given in [GM] becomes an \(O(2) \times SO(3)\)-equivariant diffeomorphism while without the determinant factor the identification is only \(SO(2) \times SO(3)\)-equivariant. Moreover, note that the map \(M^7_d \to \Sigma^7_{GM}\) given by \((u, v) \mapsto (\bar{u}, \bar{v})\) in both charts is an \(GL(2, \mathbb{R}) \times SO(3)\)-equivariant diffeomorphism.

Theorem 5.1. In every Milnor sphere \(M^7_d\) there is a unique invariant submanifold \(M^5_d\) which is \(O(2) \times SO(3)\)-equivariantly diffeomorphic to the Brieskorn sphere \(W^5_{5d}\) with the \(O(2) \times SO(3)\)-action given in [H]. This submanifold \(M^5_d\) is minimal for any \({\pm \mathbb{I}}\times SO(3)\)-invariant Riemannian metric on \(M^7_d\).

Proof. It suffices to consider the case \(d > 0\). Let \(M^5_d\) be the submanifold of \(M^7_d\) given by the equations \(\text{Re} v = 0\) and \(\text{Re} w v = 0\) in both charts (it is essential here that \(k + l = 1\)). Hirsch and Milnor [HM] proved that \(M^5_d\) is homeomorphic and hence (because exotic spheres do not exist in dimension 5) diffeomorphic to \(S^5\). It is straightforward to check that \(M^5_d\) is invariant under the \(SO(2) \times SO(3)\)-action. Consider the curve \(\alpha\) in \(M^7_d\) which is given by \(\alpha(s) = (i \tan s, j)\) in the first chart. The isotropy groups along \(\alpha\) are

\[
K_- = \{(\mathbb{I}, \pm e^{j\tau})\} \cup \{(-\mathbb{I}, \pm ie^{j\tau})\} \cup \{([1 0]_{-1}, \pm e^{j\tau})\} \cup \{([-1 0]_1, \pm e^{j\tau})\}
\]

at \(s = 0\),

\[
H = \{(\mathbb{I}, \pm 1), (-\mathbb{I}, \pm i), ([1 0]_{-1}, \pm j), ([-1 0]_1, \pm k)\}
\]

for \(0 < s < \frac{\pi}{4}\), and

\[
K_+ = \{([\cos \theta - \sin \theta]_{\cos \theta}, \pm e^{-\frac{4}{\theta} j\theta})\} \cup \{([\cos \theta - \sin \theta]_{\cos \theta} [1 0]_{-1}, \pm e^{-\frac{4}{\theta} j\theta})\}
\]
at \( s = \frac{\pi}{4} \). Now consider the Brieskorn sphere \( W^5_d \) with the \( O(2) \times SO(3) \)-action given in \( \| \) and the curve

\[
\beta(s) = \left( s, 0, \frac{1}{\sqrt{2}} \sqrt{1 - s^2 - s^d}, -\frac{1}{\sqrt{2}} \sqrt{1 - s^2 + s^d} \right)
\]
on the interval \([s_-, 0]\) where \( s_- < 0 \) is the root of \( 1 - s^2 + s^d \). Straightforward computations show that the isotropy groups along \( \beta \) are the same as the isotropy groups along \( \alpha \). This proves that \( M^5_d \) and \( W^5_d \) are equivariantly diffeomorphic. The uniqueness and minimality of \( M^5_d \) follows from the following fact: The fixed point set of any element of the form \((\pm 1, \pm q)\) is contained in \( M^5_d \) and even more \( M^5_d \) can be seen to be the union of orbits whose isotropy groups contains such elements. \( \square \)

**Proof of Theorem 5.** The involution \((-1, \pm i)\) is contained in \( M^5_d \approx W^5_{|d|} \). The fixed point set of \((-1, \pm i) = (-1, \text{diag}(1, -1, -1)\) in \( W^5_{|d|} \) is the \( W^3_{|d|} \) given by the equation \( z_1 = 0 \) and hence diffeomorphic to a lens space with fundamental group \( \mathbb{Z}_{|d|} \). \( \square \)

The Milnor sphere \( M^7_d \) have direct analogues \( M^{15}_d \) in dimension 15. They are obtained by gluing two copies of \( O \times S^7 \) along \((O \setminus \{0\}) \times S^7 \) by the map \( \| \). Precisely as above each \( M^{15}_d \) admits a smooth action of \( O(2) \times G_2 \) (see \( \| \)).

**Theorem 5.2.** In every \( M^{15}_d \) there is a unique invariant submanifold \( M^{13}_d \) which is \( O(2) \times G_2 \)-equivariantly diffeomorphic to the Brieskorn sphere \( W^{13}_{|d|} \) with the action of \( O(2) \times G_2 \subset O(2) \times SO(7) \) given analogously to \( \| \). This submanifold \( M^{13}_d \) is minimal for any \( \{\pm 1\} \times G_2 \)-invariant Riemannian metric on \( M^{15}_d \).

**Proof.** Analogous to the proof of Theorem 5.1. \( \square \)

Finally, we turn to the Brieskorn spheres \( W^7_{6n-1,3} \) and prove Theorem 6 from the introduction.

**Proof of Theorem 6.** The involution \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in SO(3) \) on \( W^7_{6n-1,3} \) is given by

\[
(w, z_0, z_1, z_2, z_3) \rightarrow (w, z_0, z_1, -z_2, -z_3).
\]

Its fixed point set is thus identical to \( W^3_{6n-1,3,2} \), which is the intersection of the unit sphere \( S^5 \) in \( \mathbb{C}^3 \) with the complex hypersurface

\[
w^{6n-1} + z_0^3 + z_1^2 = 0.
\]

Milnor \( \| \) has shown that \( W^3_{5,3,2} \) is diffeomorphic to Poincare dodecahedral space and that the universal covering space of \( W^3_{6n-1,3,2} \) is non-compact if \( n > 1 \). \( \square \)

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