Pricing of basket options II

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Abstract

We consider the problem of approximation of density functions which is important in the theory of pricing of basket options. Our method is well adopted to the multidimensional case. Observe that implementations of polynomial and spline approximation in this situation are connected with difficulties of fundamental nature. A simple approximation formula for European call options is presented. It is shown that this approximation formula has exponential rate of convergence.

1 Approximation of density functions

We shall use notations adopted in [7]. Remind that the pricing formula has the form $V = e^{-rT}E^Q[\phi]$. Since the reward function $\phi$ has a simple structure the main problem is to approximate the respective risk-neutral density function. For our model the characteristic function $\Phi^Q(v,T)$ has the form

$$\Phi^Q(v,T) = \prod_{s=1}^{n} \exp \left( -T \psi^Q_s (v_s) \right) \cdot \prod_{m=1}^{n} \exp \left( -T \phi^Q_m \left( \sum_{k=1}^{n} a_{k,m} v_k \right) \right), \quad (1)$$
and the respective characteristic exponent

\[ \tau^Q(v, T) := -T \left( \sum_{s=1}^{n} \psi_s^Q(v_s) + \sum_{m=1}^{n} \phi_m^Q \left( \sum_{k=1}^{n} a_{k,m} v_k \right) \right) \]

satisfies the condition

\[ \tau^Q(-i e_s, T) = -r, 1 \leq s \leq n. \]  \hfill (2)

In particular, in the case of KoBoL family

\[ \psi_s^Q(\xi_s) = -i \mu_s \xi_s + c_{s,+} \Gamma (-\nu_s) ( ( -\lambda_s,- )^{\nu_s} - ( -\lambda_s,- - i \xi_s )^{\nu_s} ) + c_{s,-} \Gamma (-\nu_s) ( \lambda_s,+^{\nu_s} - ( \lambda_s,+ + i \xi_s )^{\nu_s} ), \]

\[ \lambda_{s,-} < 0 < \lambda_{s,+}, \nu_s \in (0, 1), \mu_s \in \mathbb{R}, c_{s,+} > 0, c_{s,-} > 0, 1 \leq s \leq n. \]  \hfill (3)

It is possible to show \([1]\) that for any \(\nu \in (0, 1) \cup (1, 2)\),

\[ \psi^Q(\xi) = -i \mu \xi + c_+ \Gamma (-\nu) ( ( -\lambda_- )^{\nu} - ( -\lambda_- - i \xi )^{\nu} ) + c_- \Gamma (-\nu) ( \lambda_-^{\nu} - ( \lambda_- + i \xi )^{\nu} ) \]

and

\[ \psi^Q(\xi) = -i \mu \xi + c_+ ( \ln ( -\lambda_- - i \xi ) - \ln ( -\lambda_- ) ) + c_- ( \ln ( \lambda_- - i \xi ) - \ln \lambda_- ) , \nu = 0, \]

\[ \psi^Q(\xi) = -i \mu \xi + c_+ ( ( -\lambda_- ) \ln ( -\lambda_- ) - ( -\lambda_- - i \xi ) \ln ( -\lambda_- - i \xi ) ) + c_- ( \lambda_- \ln \lambda_- - ( \lambda_- - i \xi ) \ln ( \lambda_- - i \xi ) ) , \nu = 1. \]

Assume that all characteristic exponents \(\psi_s^Q, 1 \leq s \leq n\) and \(\phi_m^Q, 1 \leq m \leq n\) correspond a KoBoL process and hence analytically extendable into the strips \(\lambda_{s,-} < \kappa_{s,-} < 0 < \kappa_{s,+} < \lambda_{s,+}, 1 \leq s \leq n\) and \(\lambda_{s,-}' < \kappa_{s,-}' < 0 < \kappa_{s,+}' < \lambda_{s,+}', 1 \leq s \leq n\) respectively. In this case \(\Phi^Q(v, t) = \Phi^Q(v_1, \ldots, v_n, t)\) admits an analytic extension into the tube \(T_n \subset \mathbb{C}^n\),

\[ T_n := \left( \prod_{s=1}^{n} \{ \Im v_s \in [\kappa_{s,-}, \kappa_{s,+}] \} \right) \cap \left( \{ \Im \left( \sum_{k=1}^{n} a_{k,m} v_k \right) \in [\kappa_{m,-}', \kappa_{m,+}'], 1 \leq m \leq n \} \right). \]
Let
\[ a_+ := \min_{1 \leq s, m \leq n} \left\{ \kappa_{s,+}, \kappa_{m,+} \min \left\{ 1, \left( \sum_{k=1}^{n} a_{k,m} \right)^{-1} \right\} \right\} \]
and
\[ a_- := \max_{1 \leq s, m \leq n} \left\{ \kappa_{s,-}, \kappa_{m,-} \min \left\{ 1, \left( \sum_{k=1}^{n} a_{k,m} \right)^{-1} \right\} \right\} \]
then
\[ T'_n := \prod_{s=1}^{n} \{ a_{-} \in [a_{-}, a_{+}] \} \subset T_n. \]

Let \( a_- := (a_+, \ldots, a_+) \) and \( a_+ := (a_-, \ldots, a_-). \)

**Theorem 1.** Let \( \psi_Q^s, 1 \leq s \leq n \) and \( \phi_Q^m, 1 \leq m \leq n \) defined by (3) then the respective density function \( p_Q^s (\cdot) \) can be represented as
\[ p_Q^s (\cdot) = \frac{(2\pi)^{-n}}{\exp (\langle \cdot, a_+ \rangle) + \exp (\langle \cdot, a_- \rangle)} \times \int_{\mathbb{R}^n} \exp (-i \langle \cdot, v \rangle) \left( \Phi_Q^s (v - ia_+, T) + \Phi_Q^s (v - ia_-, T) \right) dv. \]
In particular, if \(-a_- = a_+ := a\) then
\[ p_Q^s (\cdot) = \frac{1}{2 (2\pi)^n} (\cosh (\langle \cdot, a \rangle))^{-1} \times \int_{\mathbb{R}^n} \exp (-i \langle \cdot, v \rangle) \left( \Phi_Q^s (v + ia, T) + \Phi_Q^s (v - ia, T) \right) dv. \]

**Proof** In our notations the density function can be represented as
\[ p_Q^s (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle \cdot, v \rangle) \Phi_Q^s (v, T) dv = (2\pi)^{-n} F \Phi_Q^s (v, T) (\cdot). \]

It is easy to check that \( \psi_Q^s (\xi_s), 1 \leq s \leq n \) admits an analytic extension into the strip \( \Im \xi_s \in [\kappa_{s,-}, \kappa_{s,+}] \), where \( \lambda_{s,-} < \kappa_{s,-} < 0 < \kappa_{s,+} < \lambda_{s,+}, 1 \leq s \leq n \) and
\[ \lim_{\Im \xi_s \in [\kappa_{s,-}, \kappa_{s,+}], |\xi_s| \to \infty} \frac{\psi_Q^s (\xi_s)}{\eta_s (|\Re \xi_s|)} = 1, \]
\( \eta_s(x) = \begin{cases} 
-i \mu_s x - \Gamma (-\nu_s) (c_{s,+} \exp \left( \frac{-i \mu_s x}{2} \right) + c_{s,-} \exp \left( \frac{i \mu_s x}{2} \right) ) x^\nu, & \nu_s \in (0, 1) \cup (1, 2), \\
-i (\mu_s x + c_{s,+} x \ln x - c_{s,-} x \ln x) + \frac{\pi (c_{s,+} + c_{s,-}) x}{2}, & \nu_s = 1, \\
-i \mu_s x + (c_{s,+} + c_{s,-}) \ln x, & \nu_s = 0. 
\end{cases} \) 

(4)

In particular, if \( c_{s,+} = c_{s,-} = c \) then \( \eta_s(x) \) simplifies as

$$
\eta_s(x) = \begin{cases} 
-i \mu_s x - 2c \Gamma (-\nu_s) \cos \left( \frac{\nu_s \pi}{2} \right) x^\nu, & \nu_s \in (0, 1) \cup (1, 2), \\
-i \mu_s x + \pi c x, & \nu = 1, \\
-i \mu_s x + 2 \ln x, & \nu = 0. 
\end{cases}
$$

(5)

The same asymptotic are valid for \( \phi_m^Q, 1 \leq m \leq n \). Observe that

$$
\text{sign} \left( -\Gamma (-\nu) \cos \left( \frac{\nu \pi}{2} \right) \right) > 0.
$$

Then from (4) and (5) it follows that

$$
\lim_{\langle v, v \rangle \to \infty, v \in T_n'} \Phi^Q (v, T) \, dv = 0.
$$

Hence, applying Cauchy’s theorem \( n \) times in the tube \( T_n' \), which is justified by (4), we get

$$
p_T^Q (\cdot) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, v \rangle \right) \Phi^Q (v, T) \, dv
$$

$$
= (2\pi)^{-n} \int_{\mathbb{R}^n + ia_+} \exp \left( -i \langle \cdot, v \rangle \right) \Phi^Q (v, T) \, dv
$$

$$
= (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, x - ia_+ \rangle \right) \Phi^Q (x - ia_+, T) \, dx
$$

$$
= \exp \left( -\langle \cdot, a_+ \rangle \right) (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, x \rangle \right) \Phi^Q (x - ia_+, T) \, dx,
$$

or

$$
p_T^Q (\cdot) \exp \left( \langle \cdot, a_+ \rangle \right) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, x \rangle \right) \Phi^Q (x - ia_+, T) \, dx. \quad (6)
$$

Similarly,

$$
p_T^Q (\cdot) \exp \left( \langle \cdot, a_- \rangle \right) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp \left( -i \langle \cdot, x \rangle \right) \Phi^Q (x - ia_-, T) \, dx. \quad (7)
$$
Comparing (6) and (7) we get the proof.

We will need the following result which is known as the Poisson summation formula.

**Theorem 2.** ([3] p.252) Suppose that for some \(A > 0\) and \(\delta > 0\) we have

\[
\max \{ f(x), Ff(x) \} \leq A (1 + |x|)^{-n-\delta}
\]

then

\[
\sum_{m \in \mathbb{Z}^n} f(x + Pm) = \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} Ff \left( \frac{m}{P} \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right)
\]

for any \(P > 0\). The series converges absolutely.

Put

\[
MT = MT (\Phi^Q, a)
\]

\[
:= \frac{1}{2 (2\pi)^n} \left\| \int_{\mathbb{R}^n} \exp \left( -i \left\langle \cdot, v \right\rangle \right) \left( \Phi^Q (v + ia, T) + \Phi^Q (v - ia, T) \right) dv \right\|_{\infty},
\]

fix \(T > 0, \epsilon > 0\) and select such \(P \in \mathbb{N}\) that

\[
MT \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \left( \cosh \left( \frac{2P - 1}{2} m, a \right) \right)^{-1} \leq \epsilon. \quad (8)
\]

**Theorem 3.** Let \(-a_- = a_+ := a\) then in our notations

\[
\left\| p_T^Q (x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( -\frac{m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_p \left( \frac{Q_n}{2} \right)} \leq \epsilon P^{n/p},
\]

where \(1 \leq p \leq \infty\).

**Proof** Using Theorem [1] we get

\[
p_T^Q (\cdot) \leq MT (\cosh (\cdot, a))^{-1}. \quad (9)
\]

Applying (9) we can check that the conditions of Theorem [1] are satisfied. Hence using condition (8) we get

\[
\epsilon \geq \left\| \sum_{m \in \mathbb{Z}^n \setminus \{0\}} p_T^Q (x + Pm) \right\|_{L_\infty \left( \frac{Q_n}{2} \right)}
\]
\[
\begin{aligned}
&= \left\| p_T^Q (x) - \sum_{m \in \mathbb{Z}^n} p_T^Q (x + Pm) \right\|_{L_\infty (\mathbb{R}^n)} \\
&= \left\| p_T^Q (x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \mathbf{F} p_T^Q \left( \frac{m}{P} \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_\infty (\mathbb{R}^n)} ,
\end{aligned}
\]

where \( Q_n := \{ x \mid x = (x_1, \ldots, x_n) \in \mathbb{R}^n, |x_k| \leq 1, 1 \leq k \leq n \} \) is the unit cube in \( \mathbb{R}^n \). Observe that

\[
\Phi^Q (-x, T) = (2\pi)^n \mathbf{F}^{-1} p_T^Q (-x)
\]

\[
= (2\pi)^n \left( \frac{1}{(2\pi)^n} \right) \int_{\mathbb{R}^n} \exp \left( i \left\langle x, y \right\rangle \right) p_T^Q (-y) \, dy \\
= \int_{\mathbb{R}^n} \exp \left( i \left\langle -x, y \right\rangle \right) p_T^Q (y) \, dy \\
= \mathbf{F} p_T^Q (x).
\]

Consequently,

\[
\left\| p_T^Q (x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( \frac{-m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_\infty (\mathbb{R}^n)} \leq \varepsilon
\]

and

\[
\left\| p_T^Q (x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( \frac{-m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_1 (\mathbb{R}^n)} \leq \varepsilon P^n.
\]

Finally, applying Riesz-Thorin interpolation theorem we obtain

\[
\left\| p_T^Q (x) - \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( \frac{-m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_p (\mathbb{R}^n)} \leq \varepsilon P^{n/p}.
\]

Observe that the function \(|\Phi^Q (-m, T)|\) exponentially decays as \(|m| \to \infty\). Hence the series

\[
\frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( \frac{-m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right)
\]
converges absolutely and represents an infinitely differentiable function. The next statements deal with approximation of this function using \( M \)-term exponential sums.

**Theorem 4.** Let
\[
\Omega \left( \Phi^Q (v, T), \varrho \right) := \{ v \in \mathbb{R}^n, |\Phi^Q (v, T)| \geq \varrho \},
\]
\[
\kappa_n = 2^n \Gamma^{-1} \left( 1 + \sum_{k=1}^{n} \nu_k \right) \left( \prod_{s=1}^{n} \Gamma (1 + \nu_s) \right) \left( \prod_{s=1}^{n} (d_s T)^{-\nu_s - 1} \right)
\]
and
\[
d_s := -\Gamma (-\nu_s) \cos \left( \frac{\nu_s \pi}{2} \right) (c_{s,+} + c_{s,-}), 1 \leq s \leq n.
\]
Then
\[
\text{Card} \left( \Omega \left( \Phi^Q (v, T), \varrho \right) \cap \mathbb{Z}^n \right) \lesssim \kappa_n \left( \ln \varrho^{-1} \right) \sum_{k=1}^{n} \nu_k^{-1}, \varrho \to 0.
\]

**Proof** Observe that
\[
p^Q_T (\cdot) = (2\pi)^{-n} F \left( \Phi^Q (v, T) \right) (\cdot).
\]
From (1) it follows that \( \Phi^Q \in C(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \). Hence by Plancherel’s theorem
\[
F^{-1} \left( p^Q_T (\cdot) \right) (\cdot) = (2\pi)^{-n} \Phi^Q (v, T),
\]
or
\[
\Phi^Q (v, T) = \int_{\mathbb{R}^n} \exp \left( i \langle v, x \rangle \right) p^Q_T (x) dx.
\]
Since
\[
p^Q_T (x) \geq 0, \int_{\mathbb{R}^n} p^Q_T (x) dx = 1
\]
then
\[
\max \{ v \in \mathbb{R}^n \mid |\Phi^Q (v, T)| \} = \Phi^Q (0, T) = 1. \tag{10}
\]
Let \( \nu_s \in (0, 1), 1 \leq s \leq n \). From (1), (4) and (10) it follows that \( |\Phi^Q (v, T)| \) can be asymptotically majorated as \( |v_s| \to \infty, 1 \leq s \leq n, v \in \mathbb{T}^n \),
\[
|\Phi^Q (v, T)| \lesssim \prod_{s=1}^{n} \left| \exp \left( -T \psi^Q_s (v_s) \right) \right|
\]
\[ \simeq \prod_{s=1}^{n} \left| \exp \left( -T \left( -i\mu_s x_s - \Gamma (-\nu_s) \left( c_{s,+} \exp \left( \frac{-i\nu_s \pi}{2} \right) + c_{s,-} \exp \left( \frac{i\nu_s \pi}{2} \right) \right) \right) |x_s|^\nu \right) \right| \]
\[ \lesssim \prod_{s=1}^{n} \exp \left( -T \left( -\Gamma (-\nu_s) \right) \cos \left( \frac{\nu_s \pi}{2} \right) \right) \left( c_{s,+} + c_{s,-} \right) |x_s|^\nu \]
\[ := \prod_{s=1}^{n} \exp \left( -T d_s |x_s|^\nu \right) = \exp \left( -\sum_{s=1}^{n} \left| \left( d_s T \right)^{\nu_s^{-1}} x_s \right|^\nu \right) \quad (11) \]

where
\[ d_s := -\Gamma (-\nu_s) \cos \left( \frac{\nu_s \pi}{2} \right) \left( c_{s,+} + c_{s,-} \right), \]
\[ x_s = \Re q\nu_s \text{ and } d_s > 0, 1 \leq s \leq n. \]

For a fixed \( 0 < \varrho < 1 \) consider the set
\[ \Omega \left( \Phi^Q (v,t), \varrho \right) := \{ v \in \mathbb{R}^n, |\Phi^Q (v,t)| \geq \varrho \} . \]

From (11) it follows that
\[ \Omega \left( \Phi^Q (v,T), \varrho \right) \subset \Omega' := \left\{ x \in \mathbb{R}^n, \exp \left( -\sum_{s=1}^{n} \left| \left( d_s T \right)^{\nu_s^{-1}} x_s \right|^\nu \right) \geq \varrho \right\} \]
or
\[ \Omega' := \left\{ x \in \mathbb{R}^n, \sum_{s=1}^{n} \left| \frac{d_s T}{(\ln \varrho)^{-1}} \right|^{\nu_s^{-1}} x_s \right|^\nu \leq 1 \right\} . \]

Now we have
\[ \text{Card} \left( \Omega' \cap \mathbb{Z}^n \right) \simeq \text{Vol}_n \left( \Omega' \right), \]
as \( \varrho \to 0 \) and
\[ \text{Vol}_n \left( \Omega' \right) = \prod_{s=1}^{n} \left( \frac{(\ln \varrho)^{-1}}{d_s T} \right)^{\nu_s^{-1}} B \left( \nu_1, \cdots, \nu_n \right) , \]
where
\[ B \left( \nu_1, \cdots, \nu_n \right) := \left\{ x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \sum_{s=1}^{n} |x_s|^\nu_s \leq 1 \right\} \]
and \( \nu_1 > 0, \cdots, \nu_n > 0 \). It is known that
\[ \text{Vol}_n B \left( \nu_1, \cdots, \nu_n \right) = 2^n \prod_{s=1}^{n} \frac{\Gamma (1 + \nu_s)}{\Gamma (1 + \sum_{s=1}^{n} \nu_s)} . \]
Hence
\[
\text{Card} \left( \Phi^n(\Omega, T), \Omega \cap \mathbb{Z}^n \right) \lesssim \kappa_n (\ln \varrho^{-1})^{\frac{1}{p'}} \nu_{p'}^{-1}, \varrho \to 0.
\]

**Theorem 5.** (F. Riesz, [12] p. 102) Let \( \omega_n(x), n \in \mathbb{N} \) be othonormal system of functions which is uniformly boned over \( \Omega \subset \mathbb{R}^n \),
\[
\sup_{x \in \mathbb{R}^n} |\omega_n(x)| \leq L.
\]
If \( 1 \leq p \leq 2 \) then there is such \( f \in L_p \) such that
\[
\|f\|_{p'} \leq L^{2/p-1} \|c\|_p,
\]
where \( 1/p + 1/p' = 1 \) and
\[
\|c\|_p := \left( \sum_{k=1}^{\infty} |c_k|^{1/p} \right)^{1/p},
\]
\[
c_k = \int_\Omega f \omega_n dx.
\]

**Theorem 6.** Let \( 2 \leq p \leq \infty, 1/p + 1/p' = 1 \),
\[
e_n := n^{-n/p'} \left( \frac{\kappa_n}{p'} \sum_{s=1}^{n} \nu_{s-1}^{-1} \right)^{1/p'},
\]
\[
\eta_n := e_n \kappa_n \left( \sum_{s=1}^{n} \nu_{s-1}^{-1} \right)^{1/p'}^{-1}
\]
and
\[
M := \kappa_n (\ln R)^{\sum_{s=1}^{n} \nu_{s-1}^{-1}}.
\]
Then in our notations
\[
E(M) := \left\| \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi(-m/P, T) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_p}(P Q_n)
\]
\[
- \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n \cap \Omega_{1/R}} \Phi(-m/P, T) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_p}(P Q_n)
\]
\[ \eta_n \exp \left( -2 \kappa \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} M \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right) \left( \sum_{s=1}^{n} (\nu_s - 1) \left( p' \right)^{-1} \right), \]

as \( M \to \infty. \)

**Proof** Observe that the system of functions

\[
\chi_m (x) := \frac{P^{-n/2}}{\nu} \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right), m \in \mathbb{Z}^n, x \in \frac{P}{2} Q_n
\]

is uniformly bounded

\[ |\chi_m (x)| \leq \frac{P^{-n/2}}{\nu}, \forall m \in \mathbb{Z}^n \]

and orthonormal in \( L_2 \left( \frac{P}{2} Q_n \right). \) Let \( \rho \to \infty \) then in our notations

\[
\text{Vol} \left( \Omega_{1/\rho}' \right) \simeq \kappa_n \left( \ln \rho \right) \sum_{s=1}^{n} \nu_s^{-1}.
\]

Applying Theorem 1 we get

\[
\left\| \frac{1}{P^n} \sum_{m \in \mathbb{R}^n \setminus \Omega_{1/\rho}' \cap \mathbb{Z}^n} \Phi^Q \left( -\frac{m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_p \left( \frac{P}{2} Q_n \right)} \leq \frac{1}{P^{n/2}} \left( \int_{R}^{\infty} \rho^{-p'} \, d\text{Vol} \left( \Omega_{1/\rho}' \right) \right)^{1/p'}
\]

\[
\simeq \frac{1}{P^{n/2}} \left( \frac{\kappa_n}{p} \sum_{s=1}^{n} \nu_s^{-1} \right)^{1/p'} R^{-1} \left( \ln R \right)^{\left( \sum_{s=1}^{n} \nu_s^{-1} - 1 \right) \left( p' \right)^{-1}, R \to \infty.}
\]

It means that using

\[ M := \kappa_n \left( \ln R \right) \sum_{s=1}^{n} \nu_s^{-1} \]

harmonics from \( \Omega_{1/\rho}' \) we get the error \( E (M) \) of approximation

\[
E (M) \lesssim \frac{L^{2/p' - 1}}{P^{n/2}} \left( \frac{\kappa_n}{p} \sum_{s=1}^{n} \nu_s^{-1} \right)^{1/p'} R^{-1} \left( \ln R \right)^{\left( \sum_{s=1}^{n} \nu_s^{-1} - 1 \right) \left( p' \right)^{-1}}
\]
\[ P^{−n/p'} \left( \frac{\xi_n}{p'} \sum_{s=1}^n \nu_s^{-1} \right)^{1/p'} \left( \ln R \right)^{\left( \sum_{s=1}^n \nu_s^{-1} - 1 \right) \left( p' \right)^{-1}} \]

\[ := e_n R^{-1} \left( \ln R \right)^{\left( \sum_{s=1}^n \nu_s^{-1} - 1 \right) \left( p' \right)^{-1}}, R \to \infty, \]

or

\[ E(M) \lesssim \eta_n \exp \left( -\xi_n \left( \sum_{s=1}^n \nu_s^{-1} \right)^{-1} M \left( \sum_{s=1}^n \nu_s^{-1} \right)^{-1} \left( 1 - \sum_{s=1}^n \nu_s^{-1} \right)^{-1} \left( p' \right)^{-1} \right) \]

as \( M \to \infty. \)

**Corollary 1.**

\[ \left\| p_t^Q (x) - \frac{1}{Pn} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( -\frac{m}{P}, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, x \right\rangle \right) \right\|_{L_p (\mathbb{Z}^n)} \]

\[ \leq \epsilon P^{n/p} + \eta_n \exp \left( -\xi_n \left( \sum_{s=1}^n \nu_s^{-1} \right)^{-1} M \left( \sum_{s=1}^n \nu_s^{-1} \right)^{-1} \left( 1 - \sum_{s=1}^n \nu_s^{-1} \right)^{-1} \left( p' \right)^{-1} \right), \]

as \( M \to \infty. \)

### 2 The problem of optimal approximation of density functions and \( n \)-widths

In this section we discuss the problem of optimal approximation of density functions using a wide range of approximation methods. In problems of optimal recovery arise quantities which are known as cowidths. Let \((X, d)\) be a given metric (Banach) space, \(Y\) a certain set (coding set), \(A \subset X\), \(\Phi\) a family of mappings \(\phi : A \to Y\), then the respective cowidth can be defined as

\[ \text{co}^\Phi (A, X) = \inf_{\phi \in \Phi} \sup_{y \in \phi(A)} \text{diam} \{ \phi^{-1} (y) \cap A \}, \]

where

\[ \phi^{-1} (y) = \{ x \mid x \in X, \phi (x) = \phi (y) \}. \]
In particular, let $Y$ be $\mathbb{R}^m$ and $\Phi : \text{lin} (A) \to \mathbb{R}^m$ be a linear application, $\Phi = L (\text{lin} (A), \mathbb{R}^m)$, then we get a linear cowidth $\lambda^m (A, X)$. It is easy to check that $\lambda^m = 2d^m$, where $d^m$ is the Gelfand’s $m$-width defined by

$$d^m (A, X) = \inf \{ L_m \subset X | \sup \{ x | x \in A \cap L_m \} \},$$

where inf is taken over all subspaces $L_m$ of codimension $m$. Letting $Y$ be the set of all $m$-dimensional complexes in $X$ and $\Phi = C (A, Y)$ be the set of all continuous mappings $\phi : A \to Y$, then we get Alexandrov’s cowidths $a^m (A, X)$.

Let $P \in \mathbb{N}$ and $\{ \varrho_k (x), k \in \mathbb{N} \}$ be a set of continuous orthonormal functions on the $n$-dimensional torus, $P\mathbb{T}^n := \mathbb{R}^n / P\mathbb{Z}^n$. Let $\Lambda := \{ \lambda_k, k \in \mathbb{N} \}$ be a fixed sequence of complex numbers such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_m| \geq \cdots$. For any $f \in L_1 (P\mathbb{T}^n)$ we can construct a formal Fourier series

$$s [f] = \sum_{k=1}^{\infty} c_k (f) \varrho_k, c_k (f) := \int_{P\mathbb{T}^n} f \varrho_k dx.$$ 

Consider set of functions

$$\Lambda := \{ f | |c_k (f)| \leq |\lambda_k|, k \in \mathbb{N} \}.$$ 

It is easy to check that $\Lambda$ is a convex and symmetric set. Also, $\Lambda$ is compact in $C (P\mathbb{T}^n)$ if

$$\sum_{k=1}^{\infty} |\lambda_k| < \infty.$$ 

**Theorem 7.** Let $\sum_{k=1}^{\infty} \lambda_k < \infty$ then

$$a^m (\Lambda, L_1 (P\mathbb{T}^n)) \geq 2^{-1} |\lambda_{m+1}|.$$ 

**Proof** Let us remind some basic definitions. Let $X$ be a Banach space with the norm $\| \cdot \|$ and the unit ball $B$ and $A$ be a convex, compact, centrally symmetric subset of $X$. Let $L_{m+1}$ be an $(m+1)$-dimensional subspace in $X$. Bernstein’s $m$-width is defined as

$$b_m (A, X) = \sup \{ L_{m+1} \subset X | \sup \{ \epsilon > 0 | \epsilon B \cap L_{m+1} \subset A \} \}.$$ 

The Alexandrov $m$-width is the value

$$a_m (A, X) = \inf_{\Theta_m \subset X} \inf_{\sigma : A \to \Theta_m} \sup \{ x \in A, \| x - \sigma (x) \| \},$$

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where the infimum is taken over all \( m \)-dimensional complexes \( \Theta_m \), lying in \( X \) and all continuous mappings \( \sigma : A \to \Theta_m \).

The Uryson width \( u_m(A, X) \) is the infimum of those \( \epsilon > 0 \) for which there exists a covering of \( A \) by open sets (in the sense of topology induced by the norm \( \| \cdot \| \) in \( X \)) of diameter < \( \epsilon \) in \( X \) of multiplicity \( m + 1 \) (i.e., such that each point is covered by \( \leq m + 1 \) sets and some point is covered by exactly \( m + 1 \) sets). Observe that the width \( u_m(A, X) \) was introduced by Uryson \[11\] and inspired by the Lebesgue-Brouwer definition of dimension.

It is known \[9\] p.190 that for any compact set \( A \) in a Banach space \( X \)

\[
b_m(A, X) \leq 2a_m(A, X)\]

\[10\] p.222,

\[
a_m(A, X) \leq u_n(A, X)\]

and \[9\] p.190,

\[
u_m(A, X) \leq a^n(A, X)\]

Hence (see \[3\] for more details)

\[
b_m(A, X) \leq 2a^m(A, X) .\] (12)

Let us fix

\[
L_{m+1} = \text{lin} \{ \varrho_k, 1 \leq k \leq m + 1 \}\]

and consider the set

\[
QL_{m+1} := \left\{ t_{m+1} = \sum_{k=1}^{m+1} c_k \varrho_k, |c_k| \leq |\lambda_{m+1}| \right\}.
\]

Clearly

\[
QL_{m+1} \subset \Lambda
\]

and

\[
|\lambda_{m+1}| \ B_1(P^n) \cap L_{m+1} \subset QL_{m+1},
\]

where

\[
B_1(P^n) := \{ f : \| f \|_1 \leq 1 \}.
\]

It means that

\[
b_m(\Lambda, L_1(P^n)) \geq |\lambda_{m+1}| .\] (13)
Finally, comparing (13) and (12) we get
\[ a^m (\Lambda, L_1 (P T^n)) \geq 2^{-1} |\lambda_{m+1}|. \]

For simplicity assume that \( A = 0 \). In this case
\[ \Phi^Q (v, t) = \prod_{s=1}^{n} \exp \left(-T \psi^Q_s (v_s)\right). \]

Consider function class
\[ \Lambda := \left\{ f (x) = \sum_{m \in \mathbb{Z}^n} c_m \varrho_m (x) \right\}, \]
where \( |c_m| \leq \lambda_m \),
\[ \varrho_m (x) := \exp \left(i \left\langle \frac{2\pi m}{P}, x \right\rangle \right), m \in \mathbb{Z}^n \]
and
\[ \lambda_m = \frac{1}{P^{n/2}} \left| \Phi^Q \left(-\frac{m}{P}, T\right) \right|. \]

Using Theorem 2 and Theorem 1 we get the following statement.

**Corollary 1.** In our notations
\[ a^M (\Lambda, L_1 (P T^n)) \gtrsim \frac{1}{2} \exp \left(-\left(\kappa_n^{-1} M \right) \left(\sum_{s=1}^{n} \nu_s^{-1}\right)^{-1}\right), M \to \infty. \]

### 3 Pricing of basket options

In applications it is important to construct such pricing theory which includes a wide range of reward functions \( \varphi \). For instance, the European call reward function which is given by
\[ \varphi = \varphi (x_1, \cdots, x_n) = \left( S_{0,1} \exp (x_1) - \sum_{j=2}^{n} S_{0,j} \exp (x_j) - K \right)_+, \]

admits an exponential grows with respect to \( x_1 \) as \( x_1 \to \infty \). Hence we need to introduce the following definition.
Definition 1. We say that the reward function \( \varphi \) is adopted to the model process \( U_t = \{ U_t, t \in \mathbb{R}_+ \} \) if \( \mathbb{E}^Q [\varphi] < \infty \).

Clearly, if \( \mathbb{E}^Q [\varphi] = \infty \) then the option can not be priced. Remind that the operator of expectation is taken with respect to the density function \( p_t^Q \) which satisfies the equivalent martingale measure condition \( [2] \).

The next statement reduces European call reward function to the canonic form.

Lemma 1. In our notations

\[
V = K \exp (-rT) \int_{\mathbb{R}^n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) p_T^Q (y - b) \, dy,
\]

where

\[
b := (a_1, \cdots, a_n), b_j = \ln \left( \frac{S_{0,j}}{K} \right), 1 \leq j \leq n.
\]

Proof Remind that \( V = \exp (-rT) \mathbb{E}^Q [\varphi] \). In our case

\[
\varphi = \left( S_{1,T} - \sum_{j=2}^{n} S_{j,T} - K \right) _+,
\]

where

\[
S_{j,T} = S_{0,j} \exp (U_{j,T}), 1 \leq j \leq n.
\]

It means that

\[
V = \exp (-rT) \int_{\mathbb{R}^n} \left( S_{0,1} \exp (x_1) - \sum_{j=2}^{n} S_{0,j} \exp (x_j) - K \right) p_T^Q (x) \, dx,
\]

\[
= K \exp (-rT)
\]

\[
\times \int_{\mathbb{R}^n} \left( \exp \left( x_1 + \ln \left( \frac{S_{0,1}}{K} \right) \right) - \sum_{j=2}^{n} \exp \left( x_j + \ln \left( \frac{S_{0,j}}{K} \right) \right) - 1 \right) p_T^Q (x) \, dx,
\]

where \( S_{0,j}, 1 \leq j \leq n \) are the respective spot prices. Making change of variables

\[
y_j = x_j + \ln \left( \frac{S_{0,j}}{K} \right), 1 \leq j \leq n
\]

\(15\)
we get
\[
V = K \exp(-rT) \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) p_T^Q(y - b) \, dy,
\]
where
\[
b := (b_1, \cdots, b_n), \quad b_j = \ln \left( \frac{S_{0,j}}{K} \right), \quad 1 \leq j \leq n.
\]

We will need the following result \[2\].

**Theorem 8.** Let \( n \geq 2 \). For any real numbers \( \epsilon = (\epsilon_1, \cdots, \epsilon_n) \) with \( \epsilon_m > 0 \) for \( 2 \leq m \leq n \) and \( \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m \),
\[
\left( \exp(x_1) - \sum_{m=2}^{n} \exp(x_m) - 1 \right) = (2\pi)^{-n} \int_{\mathbb{R}^n + i\epsilon} \exp(i \langle u, x \rangle) \mathbf{FS} (u) \, du,
\]
where \( x = (x_1, \cdots, x_n) \) and for \( u = (u_1, \cdots, u_n) \in \mathbb{C}^n \)
\[
\mathbf{FS} (u) = \frac{\Gamma (i (u_1 + \sum_{m=2}^{n} u_m) - 1) \prod_{m=2}^{n} \Gamma (-iu_m)}{\Gamma (iu_1 + 1)}.
\]

The next statement gives a general approximation formula for the European call options which is important in various applications. Observe that it does not show the rate of convergence. This problem will be discussed later, it explains just how to construct the approximation formula.

**Theorem 9.** Let
\[
b := (b_1, \cdots, b_n), \quad b_j = \ln \left( \frac{S_{0,j}}{K} \right), \quad 1 \leq j \leq n
\]
and \( \epsilon = (\epsilon_1, \cdots, \epsilon_n), \quad -i\epsilon \in T', \quad 2 \leq m \leq n, \epsilon_1 < -1 - \sum_{m=2}^{n} \epsilon_m \). Then formal approximation formula can be written as
\[
V \approx \frac{K \exp(-rT - \langle b, \epsilon \rangle)}{P^n} \sum_{m \in \Omega_{1/R}} \Phi^Q \left( \frac{m}{P} + i\epsilon, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, b \right\rangle \right)
\]
\[
\times \frac{\Gamma \left( \frac{-2\pi i}{P} (m_1 + \sum_{s=2}^{n} m_s) + 1 \right) \prod_{s=2}^{n} \Gamma \left( \frac{2\pi i}{P} m_s \right)}{\Gamma \left( 1 - \frac{2\pi i m_1}{P} \right)},
\]
16
where \( R \to \infty, P \to \infty \) and

\[
\Omega'_{1/R} = \left\{ x \in \mathbb{R}^n, \sum_{s=1}^n \left( \frac{d_s T}{\ln R} \right)^{\nu_s} x_s \leq 1 \right\}
\]

and

\[
d_s = -\Gamma (-\nu_s) \cos \left( \frac{\nu_s \pi}{2} \right) (c_{s,+} + c_{s,-}), \quad 0 < \nu_s < 1.
\]

**Proof** Applying Lemma 3 we get

\[
V = \exp (-rT) \mathbb{E}^Q [\varphi]
\]

\[
= \exp (-rT) \int_{\mathbb{R}^n} \left( S_{0,1} \exp (x_1) - \sum_{j=2}^n S_{0,j} \exp (x_j) - K \right) p_T^Q (x) \, dx,
\]

\[
= K \exp (-rT) \int_{\mathbb{R}^n} \left( \exp (y_1) - \sum_{j=2}^n \exp (y_j) - 1 \right) p_T^Q (y - b) \, dy,
\]

where

\[
b := (b_1, \ldots, b_n), \quad b_j = \ln \left( \frac{S_{0,j}}{K} \right), \quad 1 \leq j \leq n.
\]

Assume that \(-i\epsilon \in T'_n\). Applying Cauchy’s theorem \( n \) times in the tube \( T'_n \), which is justified by (4), we get

\[
p_T^Q (y) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle y, x \rangle) \Phi^Q (x, T) \, dx
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n - i\epsilon} \exp (-i \langle y, x \rangle) \Phi^Q (x, T) \, dx
\]

\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle y, x + i\epsilon \rangle) \Phi^Q (x + i\epsilon, T) \, dx
\]

\[
= \exp (\langle y, \epsilon \rangle) (2\pi)^{-n} \int_{\mathbb{R}^n} \exp (-i \langle y, x \rangle) \Phi^Q (x + i\epsilon, T) \, dx
\]

Let \( y \in \mathbb{P}^Q \) then from Corollary 1 we get

\[
p_T^Q (y) \approx \exp (\langle y, \epsilon \rangle) \left( \frac{1}{P^n} \sum_{m \in \mathbb{Z}^n} \Phi^Q \left( \frac{-m}{P} + i\epsilon, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, y \right\rangle \right) \right)
\]
and

\[ p_T^Q(y - b) \approx \exp(\langle y - b, \epsilon \rangle) \]

\[ \times \frac{1}{P^n} \sum_{m \in \Omega_{1/R}'} \left( \Phi_Q \left( -\frac{m}{P} + i\epsilon, T \right) \exp \left( i \left\langle -\frac{2\pi m}{P}, b \right\rangle \right) \right) \exp \left( i \left\langle \frac{2\pi m}{P}, y \right\rangle \right) \]

Finally, assuming \( \epsilon = (\epsilon_1, \ldots, \epsilon_n), 2 \leq j \leq n, \epsilon_1 < -1 - \sum_{j=2}^{n} \epsilon_j, \) using Theorem 3 and the fact that the domain \( \Omega_{1/R}' \) is centrally symmetric we obtain

\[ V = K \exp(-rT) \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) p_T^Q(y - b) \, dy \]

\[ \approx \frac{K \exp(-rT)}{P^n} \sum_{m \in \Omega_{1/R}'} \left( \Phi_Q \left( -\frac{m}{P} + i\epsilon, T \right) \exp \left( i \left\langle -\frac{2\pi m}{P}, b \right\rangle \right) \right) \]

\[ \times \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \exp(\langle y - b, \epsilon \rangle) \exp \left( i \left\langle \frac{2\pi m}{P}, y \right\rangle \right) \, dy \]

\[ = \frac{K \exp(-rT - \langle b, \epsilon \rangle)}{P^n} \sum_{m \in \Omega_{1/R}'} \left( \Phi_Q \left( -\frac{m}{P} + i\epsilon, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, b \right\rangle \right) \right) \]

\[ \times \int_{\mathbb{R}^n} \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \exp(\langle y, \epsilon \rangle) \exp \left( i \left\langle \frac{2\pi m}{P}, y \right\rangle \right) \, dy \]

\[ = \frac{K \exp(-rT - \langle b, \epsilon \rangle)}{P^n} \sum_{m \in \Omega_{1/R}'} \left( \Phi_Q \left( \frac{m}{P} + i\epsilon, T \right) \exp \left( i \left\langle \frac{2\pi m}{P}, b \right\rangle \right) \right) \]

\[ \times \Gamma\left(-\frac{2\pi}{P} (m_1 + \sum_{s=2}^{n} m_s) + 1\right) \prod_{s=2}^{n} \Gamma\left(\frac{2\pi}{P} m_s\right) \]

\[ \times \frac{1}{\Gamma\left(1 - \frac{2\pi m_1}{P}\right)} . \]

The following statement is trivial.

**Lemma 2.**

\[ \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \leq \min \left\{ \exp(y_1), \exp(y_1 - \sum_{j=2}^{n} \exp(y_j)) \right\} . \]
Theorem 10. Let in our notations

\[ M(P, R) = \left\| \frac{1}{P^n} \sum_{\mathbf{m} \in \Omega_{R/1}} \Phi^\mathbf{Q} \left( -\frac{\mathbf{m}}{P} + i\epsilon, T \right) \exp \left( i \left\langle -\frac{2\pi \mathbf{m}}{P}, \mathbf{b} \right\rangle \right) \exp \left( i \left\langle \frac{2\pi \mathbf{m}}{P}, \mathbf{y} \right\rangle \right) \right\|_{L_\infty(\mathbb{R}^n)} \],

\[ L_\epsilon := \left\| \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle \mathbf{y}, \epsilon \rangle) \right\|_{L_1(\mathbb{R}^n)} \],

and \( \tilde{V} \) be the approximant for \( V \) from the Theorem \( \mathbb{E} \) then

\[ \delta := \left| V - \tilde{V} \right| \]

\[ \leq L_\epsilon \left( \epsilon + \eta_n \exp \left( -\mathcal{K}_n \left( \frac{\nu_1}{n} \right)^{-1} M \left( \sum_{i=1}^{n} \nu_i^{-1} \right)^{-1} \right) \right) M \left( 1 - \sum_{i=1}^{n} \nu_i^{-1} \right)^{-1} + M(P, R) \int_{\mathbb{R}^n \setminus \left( \frac{\pi}{2} - \|\mathbf{b}\|_\infty \right) Q_n} \min \left\{ \exp (y_1), \exp \left( y_1 - \sum_{j=2}^{n} \exp (y_j) \right) \right\} \exp (\langle \mathbf{y}, \epsilon \rangle). \]

Proof Let \( \tilde{V} \) be the approximant for \( V \), then assuming that \( -i\epsilon \in T' \) we get

\[ \delta = \left| V - \tilde{V} \right| \]

\[ = \left| \int_{\mathbb{R}^n} \left( \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle \mathbf{y}, \epsilon \rangle) \right) \right| \]

\[ \times \left( \int_{\mathbb{R}^n} \exp (i \langle \mathbf{y} - \mathbf{b}, \mathbf{x} \rangle) \Phi^\mathbf{Q} (-\mathbf{x} + i\epsilon, T) d\mathbf{x} \right) \]

\[ - \frac{1}{P^n} \sum_{\mathbf{m} \in \Omega_{R/1}} \Phi^\mathbf{Q} \left( -\frac{\mathbf{m}}{P} + i\epsilon, T \right) \exp \left( i \left\langle -\frac{2\pi \mathbf{m}}{P}, \mathbf{b} \right\rangle \right) \exp \left( i \left\langle \frac{2\pi \mathbf{m}}{P}, \mathbf{y} \right\rangle \right) d\mathbf{y} \]

\[ := \int_{\mathbb{R}^n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle \mathbf{y}, \epsilon \rangle) \mu (\mathbf{y}) d\mathbf{y}. \]
From the Theorem 1 and Theorem 3 it follows that for a chosen $P > 0$ and $M > 0$ we have

$$
\delta_1 := \left| \int_{\mathbb{R}^n} \exp \left( i \langle y - b, x \rangle \right) \Phi^Q (x + i\varepsilon, T) \, dx \right|
$$

$$
- \frac{1}{P^n} \sum_{m \in \Omega_{1/R}^{\gamma}} \left( \Phi^Q \left( \frac{m}{P} + i\varepsilon, T \right) \exp \left( i \left\langle -\frac{2\pi m}{P}, b \right\rangle \right) \right) \exp \left( i \left\langle \frac{2\pi m}{P}, y \right\rangle \right)
$$

$$
\leq \epsilon + \eta_n \exp \left( -M \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{1} \right) M \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1}
$$

for $\forall y \in \frac{P}{2} Q_n - b$. Observe that $p' = 1$ in our case. Clearly,

$$
\left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle y, \varepsilon \rangle) \in L_1 (\mathbb{R}^n)
$$

for a chosen $\epsilon$. Let

$$
L_\epsilon = \left\| \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle y, \varepsilon \rangle) \right\|_{L_1(\mathbb{R}^n)}.
$$

Hence $\frac{P}{2} Q_n - b \subset \left( \frac{P}{2} - \|b\|_\infty \right) Q_n$, where $\|b\|_\infty := \max \{|b_k|, 1 \leq k \leq n\}$ and

$$
\int_{\left( \frac{P}{2} - \|b\|_\infty \right) Q_n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle y, \varepsilon \rangle) \mu (y) \, dy
$$

$$
\leq L_\epsilon \left( \epsilon + \eta_n \exp \left( -M \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{1} \right) M \left( \sum_{s=1}^{n} \nu_s^{-1} \right)^{-1} \right).
$$

Then we have

$$
\epsilon' := \int_{\mathbb{R}^n \setminus \left( \frac{P}{2} - \|b\|_\infty \right) Q_n} \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle y, \varepsilon \rangle) \mu (y) \, dy
$$

$$
\leq M (P, R) \left\| \left( \exp (y_1) - \sum_{j=2}^{n} \exp (y_j) - 1 \right) \exp (\langle y, \varepsilon \rangle) \right\|_{L_1(\mathbb{R}^n \setminus \left( \frac{P}{2} - \|b\|_\infty \right) Q_n)}
$$

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and by Lemma 3

\[ \epsilon' \leq M(P, R) \int_{\mathbb{R}^n \setminus \left( \frac{e}{2} - \|b\|_\infty \right)} Q_n \left( \exp(y_1) - \sum_{j=2}^{n} \exp(y_j) - 1 \right) \exp(\langle y, \epsilon \rangle) \]

\[ \leq M(P, R) \int_{\mathbb{R}^n \setminus \left( \frac{e}{2} - \|b\|_\infty \right)} \min \left\{ \exp(y_1), \exp \left( y_1 - \sum_{j=2}^{n} \exp(y_j) \right) \right\} \exp(\langle y, \epsilon \rangle). \]

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