CURIOUS CONJECTURES ON THE DISTRIBUTION OF PRIMES
AMONG THE SUMS OF THE FIRST $2n$ PRIMES

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ABSTRACT. Let $p_n$ be $n$th prime, and let $(S_n)_{n=1}^\infty := (S_n)$ be the sequence of the sums of the first $2n$ consecutive primes, that is, $S_n = \sum_{k=1}^{2n} p_k$ with $n = 1, 2, \ldots$. Heuristic arguments supported by the corresponding computational results suggest that the primes are distributed among sequence $(S_n)$ in the same way that they are distributed among positive integers. In other words, taking into account the Prime Number Theorem, this assertion is equivalent to

$$\# \left\{ p : p \text{ is a prime and } p = S_k \text{ for some } k \text{ with } 1 \leq k \leq n \right\} \sim \# \left\{ p : p \text{ is a prime and } p = k \text{ for some } k \text{ with } 1 \leq k \leq n \right\} \sim \frac{\log n}{n} \text{ as } n \to \infty,$$

where $|S|$ denotes the cardinality of a set $S$. Under the assumption that this assertion is true (Conjecture 3.3), we say that $(S_n)$ satisfies the Restricted Prime Number Theorem. Motivated by this, in Sections 1 and 2 we give some definitions, results and examples concerning the generalization of the prime counting function $\pi(x)$ to increasing positive integer sequences.

The remainder of the paper (Sections 3-7) is devoted to the study of mentioned sequence $(S_n)$. Namely, we propose several conjectures and we prove their consequences concerning the distribution of primes in the sequence $(S_n)$. These conjectures are mainly motivated by the Prime Number Theorem, some heuristic arguments and related computational results. Several consequences of these conjectures are also established.

1. INTRODUCTION, MOTIVATION AND PRELIMINARIES

An extremely difficult problem in number theory is the distribution of the primes among the natural numbers. This problem involves the study of the asymptotic behavior of the counting function $\pi(x)$ which is one of the more intriguing functions in number theory. The function $\pi(x)$ is defined as the number of primes $\leq x$. For elementary methods in the study of the distribution of prime numbers, see [12].

Although questions in number theory were not always mathematically en vogue, by the middle of the nineteenth century the problem of counting primes had attracted the attention of well-respected mathematicians such as Legendre, Tchébychev, and the prodigious Gauss.

A query about the frequency with which primes occur elicited the following response: I pondered this problem as a boy, and determined that, at around $x$, the primes occur with density $1/\log x$–C. F. Gauss (letter to Encke, 24 December 1849). Gauss wrote:

This remark of Gauss can be interpreted as predicting that

$$\# \{ \text{primes } \leq x \} \approx \sum_{n=2}^{[x]} \frac{1}{\log n} \approx \int_2^x \frac{dt}{\log t} = \text{Li}(x).$$

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Studying tables of primes, C. F. Gauss in the late 1700s and A.-M. Legendre in the early 1800s conjectured the celebrated **Prime Number Theorem**:

$$\pi(x) = \{p \leq x : \text{p prime}\} \sim \frac{x}{\log x}$$

(here, as always in the sequel, $|S|$ denotes the cardinality of a set $S$).

This theorem was proved much later (8 p. 10, Theorem 1.1.4); for its simple analytic proof see [31] and [46], and for its history see [3], [21], [22] and [28, p. 21]. Briefly, $\pi(x) \sim x/\log x$ as $x \to \infty$, or in other words, the density of primes $p \leq x$ is $1/\log x$; that is, the ratio $\pi(x) : (x/\log x)$ converges to 1 as $x$ grows without bound. Using L’Hôpital’s rule, Gauss showed that the logarithmic integral $Li(x)$, denoted by $Li(x)$, is asymptotically equivalent to $x/\log x$. Recall that Gauss felt that $Li(x)$ gave better approximations to $\pi(x)$ than $x/\log x$ for large values of $x$.

Though unable to prove the Prime Number Theorem, several significant contributions to the proof of Prime Number Theorem were given by P. L. Chebyshev in his two important 1851–1852 papers ([6] and [7]). Chebyshev proved that there exist positive constants $c_1$ and $c_2$ and a real number $x_0$ such that $c_1 x/\log x \leq \pi(x) \leq c_2 x/\log x$ for $x > x_0$. In other words, $\pi(x)$ increases as $x \log x$. Using methods of complex analysis and the ingenious ideas of Riemann (forty years prior), this theorem was first proved in 1896, independently by J. Hadamard and C. de la Vallée-Poussin (see e.g., [33, Section 4.1]).

A **generalized prime system** (or $g$-prime system) $G$ is a sequence of positive real numbers $q_1, q_2, q_3, \ldots$ satisfying $1 < q_1 \leq q_2 \leq \cdots \leq q_n \leq q_{n+1} \leq \cdots$ and $q_n \to \infty$ as $n \to \infty$. From these can be formed the system $N$ of **generalized integers** or Beurling integers; that is, the numbers of the form $q_1^{k_1} q_2^{k_2} \cdots q_l^{k_l}$, where $l \in \mathbb{N}$ and $k_1, k_2, \ldots, k_l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Notice that $N$ denotes the **multiplicative semigroup** generated by $G$, and it consists of the unit 1 together with all finite power-products of $g$-primes, arranged in increasing order and counted with multiplicity.

Clearly, this system generalizes the notion of primes and positive integers obtained from them. Such systems were first introduced by A. Beurling [5] and have been studied by many authors since then (see in particular [4], [2], [11], [13], [25], [32] and [47]). In particular, Nyman [32] and Malliavin [25] sharpened Beurling’s results in various ways.

Much of the theory concerns connecting the asymptotic behaviour of $g$-prime counting function and $g$-counting function $\pi_G(x)$ and $N_G(x)$, defined on $[1, \infty)$ respectively by

$$\pi_G(x) = \sum_{q \in G,q \leq x} 1$$

and

$$N_G(x) = \sum_{n \in N,n\leq x} 1,$$

where in the first sum the summation is taken over all $g$-primes, counting multiplicities. Similarly, for the second sum $\sum_{n \in N,n\leq x} 1$. Accordingly, we have

$$\pi_G(x) = \#\{i : q_i \in G \text{ and } q_i \leq x\}$$

and

$$N_G(x) = \#\{i : n_i \in N \text{ and } n_i \leq x\}.$$

If $G = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} = (a_n)_{n=1}^\infty$ is a sequence such that $a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$, then obviously, we have $\pi_G(a_n) = n$ for each $n \in \mathbb{N}$.

In 1937 Beurling proved [5 Théorème IV] that if $N_G$ satisfies the asymptotic relation $N_G(x) = Ax + O(x/\log^\gamma x)$ with some constants $A > 0$ and $\gamma > 3/2$, then the number of $q_n$’s such that $q_n \leq x$ is equal to $x/\log x + o(x/\log x)$, i.e.,

$$\pi_G(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right).$$
In other words, the conclusion of the Prime Number Theorem (in the sequel, shortly written as PNT) is valid for such a system $G$ (from this reason often called a Beurling prime number system). Beurling also gave an example in which $N_G(x) = Ax + O(x/\log^{3/2} x)$ but PNT is not valid. This result was refined in 1969 by H. G. Diamond [9, Theorem (B)]. In 1970 Diamond [10] also proved that Beurling’s condition is sharp, namely, the PNT does not necessarily hold if $\gamma = 3/2$.

In particular, if $G$ is a set $\mathcal{P} := \{p_1, p_2, p_3, \ldots\}$ of all primes $2 = p_1 < p_2 < p_3 < \cdots$ with the associated multiplicative semigroup $N = \mathbb{N} = \{1, 2, 3, \ldots\}$, then PNT states that

$$\pi(x) \sim \frac{x}{\log x}, \quad \text{as } x \to \infty,$$

where $\pi(x)$ is the usual prime counting function, that is,

$$\pi(x) = \sum_{p \text{ prime}, \ p < x} 1.$$

As observed in [4, Introduction], the additive structure of the positive integers is not particularly relevant to the distribution of primes. Therefore, for a given $g$-prime system $G$ defined above, it can be of interest to consider the distribution of $g$-primes (the elements in $G$) with respect to certain associated system of generalized integers without any algebraic (multiplicative) structure. This means that the associated system $N$ to $G$ defined above may be some subset of $[1, +\infty)$ which is not a multiplicative semigroup (generated by $G$).

In particular, here we mainly consider the case when $G$ is an infinite set of primes and the associated system $N$ to $G$ is an increasing integer sequence $(a_n)_{n=1}^{\infty}$. We focus our attention when $G$ is a set of all primes whose associated system $N$ is the sequence $(a_n)_{n=1}^{\infty} := (\sum_{i=1}^{2n} p_i)_{n=1}^{\infty}$ where $2 = p_1 < p_2 < \cdots < p_n < \cdots$ are all the primes.

Let $(G, N := (a_k)_{k=1}^{\infty})$ be a pair defined above. Then we define its counting function $N_{G,(a_k)}(x)$ as

$$N_{G,(a_k)}(x) = \# \{ i : i \in \mathbb{N} \text{ and } a_i \leq x \}.$$

Furthermore, the prime counting function for $(G, N := (a_k)_{k=1}^{\infty})$ is the function $x \mapsto \pi_{G,(a_k)}(x)$ defined on $[1, \infty)$ as

$$\pi_{G,(a_k)}(x) = \# \{ q : q \in G \text{ and } q = a_i \text{ for some } i \text{ with } a_i \leq x \}.$$

Heuristic and computational results show that for many “natural pairs” $(G, N := (a_k)_{k=1}^{\infty})$ the associated counting function $N_{G,(a_k)}(x)$ has certain asymptotic growth as $x \to \infty$. Notice that for each $n \in \mathbb{N}$ we have

$$\pi_{G,(a_k)}(a_n) = \# \{ q : q \in G \text{ and } q = a_i \text{ for some } i \text{ with } 1 \leq i \leq n \}.$$

The normalizable prime counting function for $(G, N := (a_k)_{k=1}^{\infty})$ is the function $(n, x) \mapsto p_{G,(a_k)}(n, x)$ defined for $(n, x) \in \mathbb{N} \times [1, +\infty)$ as

$$p_{G,(a_k)}(n, x) = \frac{\log a_n}{a_n} \pi_{G,(a_k)}(x).$$

The above expression induces the companion sequence $(b_n)_{n=1}^{\infty}$ of $(a_k)_{k=1}^{\infty}$ defined as

$$b_n = p_{G,(a_k)}(a_n) = \frac{\log a_n}{a_n} \pi_{G,(a_k)}(a_n), \quad n = 1, 2, \ldots.$$
We also define another “companion” sequence \((c_n)_{n=1}^{\infty}\) of \((a_k)_{k=1}^{\infty}\) defined as
\[
(5) \quad c_n = \frac{a_n}{(\log n)(\log a_n)} = \frac{\pi_{G, (a_k)}(a_n)}{b_n \log n}, \quad n = 1, 2, \ldots.
\]

Here as always in the sequel, we will suppose that \(\mathcal{G}\) is a set of all primes whose associated system \(\mathcal{N}\) is a sequence \((a_k)_{k=1}^{\infty}\). For brevity, in the sequel the functions defined by (1), (2), (3) and (4) will be denoted by \(p_{(a_k)}(x), \pi_{(a_k)}(a_n), p_{(a_k)}(n, x)\) and \(p_{(a_k)}(a_n)\), respectively.

**Definition 1.1.** Let \(\Omega\) be a set of all nonnegative continuous real functions defined on \((1, +\infty)\) and let \((a_k)_{k=1}^{\infty} := (a_k)\) be an increasing sequence of positive integers. We say that \((a_k)\) is a prime-like sequence if there exists the function \(\omega_{(a_k)} = \omega \in \Omega\) such that the function \(n \mapsto \pi_{(a_k)}(a_n)\) defined by (2) is asymptotically equivalent to \(\omega(n)\) as \(n \to \infty\). Then we say that a sequence \((a_k)\) satisfies the \(\omega\)-Restricted Prime Number Theorem.

In particular, if \(\omega(x) \sim x/\log x\) as \(x \to \infty\), then we say that a sequence \((a_k)\) satisfies the Restricted Prime Number Theorem (RPNT).

**Proposition 1.2.** Let \((a_k)_{k=1}^{\infty}\) be a positive integer sequence, and let \((b_n)_{n=1}^{\infty}\) be its companion sequence defined by (4). Then
\[
(6) \quad \limsup_{n \to \infty} b_n \leq 1.
\]

**Proof.** Taking the obvious inequality \(\pi_{(a_k)}(a_n) \leq \pi(a_n)\) with \(n = 1, 2, \ldots\) into (4) we get
\[
b_n \leq \frac{\pi(a_n) \log a_n}{a_n}, \quad n = 1, 2, \ldots,
\]
which by the Prime Number Theorem immediately yields
\[
\limsup_{n \to \infty} b_n \leq \lim_{n \to \infty} \frac{\pi(a_n) \log a_n}{a_n} = 1,
\]
as desired. \(\Box\)

**Proposition 1.3.** Let \((a_n)\) be a prime-like sequence with the associated function \(\omega(x)\). Then
\[
(7) \quad \limsup_{n \to \infty} \frac{\omega(n)}{\pi(a_n)} \leq 1.
\]
This means that \(\omega(n)\) grows slowly than \(\pi(a_n)\) as \(n \to \infty\).

**Proof.** Notice that the inequality (7) is equivalent to
\[
(8) \quad \limsup_{n \to \infty} \frac{\log a_n}{a_n} \pi_{(a_k)}(a_n) \leq 1.
\]
Since by the assumption, \(\omega(n) \sim \pi_{(a_k)}(a_n)\), the inequality (8) yields
\[
\limsup_{n \to \infty} \frac{\log a_n}{a_n} \omega(n) \leq 1,
\]
whence, in view of the fact that \(\log a_n/a_n \sim 1/\pi(a_n)\), immediately follows (7). \(\square\)

**Remark 1.4.** The inequality (6) is sharp since by the Prime Number Theorem (see Example 2.1), equality in (6) holds for the sequences \(a_k = k\) with \(k = 1, 2, \ldots\).
Remark 1.5. If a sequence \((a_k)\) satisfies the \(\omega\)-Restricted Prime Number Theorem, then by (4) we have

\[
\omega(n) \sim \pi(a_k)(a_n) = \frac{a_n b_n}{\log a_n} \quad \text{as } n \to \infty.
\]

Remark 1.6. Let \((a_k)\) be a sequence satisfying the \(\omega\)-Restricted Prime Number Theorem. Then clearly, \(\omega(a_k)(n) \leq n\) for all sufficiently large \(n\). Moreover, \(\omega(a_k)(n) \sim n\) as \(n \to \infty\) if and only if the density of primes in a sequence \((a_k)\) is equal to 1.

Notice also that by the Prime Number Theorem, \(\omega_n(x) = x/\log x\) for the sequence of all positive integers \(\mathbb{N} = \{1, 2, \ldots n, \ldots\}\), that is, \(\mathbb{N}\) satisfies the Restricted Prime Number Theorem (cf. Conjecture 3.3).

Here, as always in the sequel, \(\mathcal{P} = (p_n) := \{p_1, p_2, \ldots p_n, \ldots\}\) will denote the set of all primes, where \(2 = p_1 < 3 = p_2 < p_3 < \cdots < p_n < \cdots\). Moreover, \((a_n)\) will always denote an infinite strictly increasing sequence of positive integers. Hence, for such a sequence must be \(a_n \geq n\) for each \(n \in \mathbb{N}\).

The remainder of the paper is organized as follows. In Section 2 we present five examples concerning the determination of the function \(\omega(a_k)(x)\) and a sequence \((b_n)\) associated to a given sequence \((a_k)\). In particular, we consider the sequence \((a_k)_{k=1}^{\infty}\) with \(a_k = a + (k - 1)d\), where \(a \geq 1\) and \(d > 1\) are relatively prime integers.

In Section 3 we consider the distribution of primes in the sequence \((S_n)_{n=1}^{\infty}\) whose terms are given by \(S_n = \sum_{i=1}^{2n} p_i\), where \(p_i\) is the \(i\)th prime. Heuristic arguments supported by related computational results suggest the curious conjecture that the sequence \((S_n)\) satisfies the Restricted Prime Number Theorem (Conjecture 3.3). In other words, this means that the primes are distributed amongst all the terms of the sequence \((S_n)\) in the same way that they are distributed amongst all the positive integers. Under this conjecture, we prove that if \(q_k\) is the \(k\)th prime in \((S_n)_{n=1}^{\infty}\), then \(q_k \sim 2k^2 \log^3 k \sim 2p_k^2 \log k\) as \(k \to \infty\) (Corollaries 3.6 and 3.7).

Assuming that Conjecture 3.3 is true, in Section 4 we give the asymptotic expression for the \(k\)th prime in the sequence \((S_n)\) (Corollary 4.2); namely, \(q_k \sim 2k^2 \log^3 k\) as \(k \to \infty\). This result is refined by Theorem 4.4. We also conjecture that \([k \log k] + 1 \leq m\) for each pair \((k, m)\) of positive integers with \(k \geq 1\) and \(q_k = S_m\) (Conjecture 4.6). Some consequences of Conjectures 3.3 and 4.6 are also presented.

Section 5 is devoted to the estimations of the values \(M_k\) \((k = 1, 2, \ldots)\) involving in the expression for \(q_k\) from Theorem 4.4. We also propose some other conjectures concerning the sequences \((S_k)\) and \((M_k)\). Related consequences are also established.

The conjectures presented in this paper, as well as some their consequences, are mainly supported by some computational results given in Section 6. In particular, the number \(\pi_n := k\) of primes in the set \(S_n := \{S_1, S_2, \ldots, S_n\}\) for 38 values of \(n\) up to \(10^9 + 5 \cdot 10^8\) are presented in Table 1. For such values \(k\) and the associated indices \(m\) such that \(q_k = S_m\), the corresponding approximate values of \(q_k, M_k\) (together with lower and upper bounds of \(M_k\)), \((k \log k)/m\) and \(S_m \sqrt{k \log k} / (2m^{5/2} \log m)\) are also given in this table. Under the previous notations, related numerical results for \(q_k / (2k^2 \log^3 k)\), \(q_k / (2m^2 \log m)\) and two estimates involving \(q_k\) which are discussed in Section 4, are given in Table 3. Some additional computational results, the conjectures and their consequences are also given in Section 6.
In the last Section 7 we propose the stronger (asymptotic) version of Conjecture 3.3 which coincides with well known form of Prime Number Theorem involving the function \( \text{li}(x) \).

Notice that similar considerations to those for the sequence \((S_n)\) concerning alternating sums of consecutive primes are given in [29].

2. Examples

Example 2.1. For the sequence \((a_k)\) with \(a_k = k\) \((k = 1, 2, \ldots)\), we clearly have \(\pi(k)(n) = \pi(n)\), and hence

\[
(10) \quad b_n = \frac{\pi(n) \log n}{n}, \quad n = 1, 2, \ldots.
\]

By the Prime Number Theorem, from (10) we find that

\[
(11) \quad \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{\pi(n) \log n}{n} = 1.
\]

Example 2.2. Let \((a_k)\) be a sequence of all primes, that is, \(a_k = p_k\) with \(k \in \mathbb{N} := \{1, 2, \ldots\}\), where \(p_k\) is the \(k\)th prime. Since by (1), \(\pi(p_k)(p_n) = \pi(p_n) = n\), substituting this into (4) yields

\[
(12) \quad b_n = \frac{n \log p_n}{p_n}, \quad n = 1, 2, \ldots.
\]

Now applying to (12) the well known fact that \(p_n \sim n \log n\) as \(n \to \infty\) (see, e.g., [30]), we find that

\[
(13) \quad \lim_{n \to \infty} b_n = 1.
\]

Notice also that the known inequality \(p_n > n \log n\) with \(n \geq 1\) (see, e.g., [38], (3.10) in Theorem 3) implies that \(b_n < 1\) for all \(n \geq 1\).

Example 2.3. Suppose that \(a\) and \(d\) are relatively prime positive integers. Then concerning Dirichlet’s theorem de la Vallée Poussin established (see, e.g., [35, p. 205]) that the number of primes \(p < x\) with \(p \equiv a (\text{mod } d)\) is approximately

\[
(14) \quad \frac{\pi(x)}{\varphi(d)} \sim \frac{1}{\varphi(d)} \cdot \frac{x}{\log x}.
\]

Here \(\varphi(n)\) is the Euler totient function defined as the number of positive integers not exceeding \(n\) and relatively prime to \(n\). Note that the right hand side of (14) is the same for any \(a\) such that \(\gcd(a, d) = 1\). This shows that primes are in a certain sense uniformly distributed in reduced residue classes with respect to a fixed modulus. Notice that for a sequence \((a_k)_{k=1}^{\infty}\) given by \(a_k = a + (k - 1)d\), (14) can be written as

\[
(15) \quad \pi(a_k)(a_n) \sim \frac{\pi(a_n)}{\varphi(d)} \text{ as } n \to \infty.
\]

Inserting (15) together with \(\pi(a_k) \sim a_k/ \log a_k\) into (4) immediately gives

\[
(16) \quad \lim_{n \to \infty} b_n = \frac{1}{\varphi(d)} \lim_{n \to \infty} \frac{\pi(a_n) \log a_n}{a_n} = \frac{1}{\varphi(d)}.
\]
Then substituting (16) into (9), we obtain that for the associated function \( \omega_{a,d}(x) := \omega(a_k) \) of the sequence \((a_k)\) there holds
\[
\omega_{a,d}(n) \sim \pi(a_n) \frac{an b_n}{\varphi(d) \log a_n} = \frac{a + (n - 1)d}{\varphi(d) \log(a + (n - 1)d)} \sim \frac{dn}{\varphi(d) \log n} \text{ as } n \to \infty.
\]

It follows that \( \omega_{a,d}(x) = dx/(\varphi(d) \log x) \) for \( x \in (1, +\infty) \).

**Example 2.4.** Let \((a_n)\) be a sequence defined as \(a_n = 2^{p_n} - 1\), where \(p_n\) is the \(n\)th prime. The numbers \(a_n\) are called **Mersenne numbers**. A prime that appears in the sequence \((a_n)\) is called **Mersenne prime**. Namely, it is easy to show (see, e.g., [36, p. 28]) that if \(2^n - 1\) is prime, then so is \(n\). The greatest known Mersenne prime is \(2^{43112609} - 1\) with the exponent 43112609 (12978169 digit number), and it is discovered in August 2008. This is in fact one between 45 known Mersenne primes, and so \(a_{45} \leq 2^{43112609} - 1\).

In 1980 H. Lenstra and C. Pomerance, working independently, came the conclusion that the probability that a Mersenne number \(2^p - 1\) is prime is \(e^\gamma \log(ap)/(p \log 2)\) with \(\gamma = 0.577216\ldots\) (the Euler-Mascheroni constant), where \(a = 2\) if \(p \equiv 3 \pmod{4}\) and \(a = 6\) if \(p \equiv 1 \pmod{4}\). Recall that the constant \(e^\gamma = 1.781072\ldots\) is important in number theory; namely, \(e^\gamma = \lim_{n \to \infty} \frac{\log \prod_{k=1}^{n} \frac{p_k}{p_k - 1}}{\log p_n}\) which restates the third of Mertens’ theorems ([27], also see [23, pp. 351–353, Theorem 428]). Then notice that the distribution of the log of the Mersenne primes is a **Poisson Process** (see [45]).

Accordingly to the above assumption given by Lenstra and Pomerance, if \(a_k = 2^{q_k} - 1\), where \((q_k)_{k=1}^\infty\) is a sequence of all primes \(\equiv 3 \pmod{4}\) \((q_1 = 3, q_2 = 7, q_3 = 11, \ldots\)\), for the associated function \(\omega^{(3,4)}(x)\) to \((a_k)\) we have that “the expected number” of primes between the first \(n\) terms of the sequence \((q_k)\) is
\[
\omega^{(3,4)}(n) \sim \sum_{k=1}^{n} e^\gamma \log(2q_k)/q_k \log 2 \text{ as } n \to \infty.
\]

Since \(q_k \sim p_{2k} \sim 2k \log k\), substituting this into (18) and using the well known asymptotic formula \(\sum_{k=1}^{n} 1/k \sim \gamma + \log n\) as \(n \to \infty\), we get
\[
\omega^{(3,4)}(n) \sim \frac{e^\gamma}{2 \log 2} \sum_{k=2}^{n} \frac{\log(4k \log k)}{k \log k} = \frac{e^\gamma}{2 \log 2} \sum_{k=2}^{n} \frac{\log k + \log 4 + \log \log k}{k \log k}
\]
\[
\sim \frac{e^\gamma}{2 \log 2} \left( \sum_{k=2}^{n} \frac{1}{k} + \sum_{k=2}^{n} \frac{\log 4}{k \log k} + \sum_{k=2}^{n} \frac{\log \log k}{k \log k} \right)
\]
\[
\sim \frac{e^\gamma}{2 \log 2} \left( (\gamma + \log n) + \log 4 \int_{2}^{n} \frac{dx}{x \log x} + \int_{2}^{n} \frac{\log \log x}{x \log x} \, dx \right)
\]
\[
\text{(the changes } \log x = s \text{ and } \log \log x = t \text{)}
\]
\[
= \frac{e^\gamma}{2 \log 2} \left( \log n + \int_{\log 2}^{\log n} \frac{ds}{s} + \int_{\log \log 2}^{\log \log n} \frac{t \, dt}{t} \right)
\]
\[
\sim \frac{e^\gamma}{2 \log 2} \left( \log n + \log \log n + \frac{(\log \log n)^2}{2} \right) \text{ as } n \to \infty.
\]

This shows that \(\omega^{(3,4)}(x) = e^\gamma (\log x + \log \log x + (\log \log x)^2/2)/(2 \log 2)\), and hence \(\pi(a_k)(a_n) \sim e^\gamma/(2 \log 2) (\log n + \log \log n + (\log \log n)^2/2)\). Substituting this in (4),
where \((b_n)\) is the companion’ sequence of \((a_k)\), and using the fact that \(q_n \sim 2n \log n\), we find that

\[
(20) \quad b_n \sim \frac{e^\gamma n (\log n)^2}{4n \log n} \text{ as } n \to \infty.
\]

Similarly, under the above assumptions attributed by Lenstra and Pomerance, if \(a'_k = 2^{r_k} - 1\), where \((r_k)_{k=1}^\infty\) is a sequence of all primes \(\equiv 1 \pmod{4}\) (\(r_1 = 5, r_2 = 13, r_3 = 17, \ldots\)), then for the associated function \(\omega^{(1,4)}(x)\) to \((a'_k)\) and the companion sequence \((b'_n)\) of \((a'_k)\) the same relations \((18)–(20)\) are satisfied.

**Example 2.5.** Let \((a_k)_{k=1}^\infty\) be an increasing sequence of positive integers satisfying

\[
(21) \quad \frac{\log a_k}{a_k} = o(k^{-1}).
\]

Then from \((4)\) and the obvious fact that \(\pi(a_k)(a_n) \leq n\) for each \(n \in \mathbb{N}\), we find that

\[
(22) \quad \lim_{n \to \infty} b_n = 0.
\]

In particular, \((22)\) holds for any sequence \((a_k)\) satisfying one of the following asymptotics: \(a_n \sim a^n\) with a fixed \(a > 1\); \(a_n \sim n^\alpha \log \alpha n\) with \(\alpha > 1\); \(a_n \sim n^\alpha\) with \(\alpha > 1\); or \(a_n \sim n^\alpha \log^\beta n\) with \(\alpha \geq 1\) and \(\beta > 1\).

Accordingly, we ask the following question.

**Question 2.6.** For what real numbers \(\alpha \in (0, 1)\) there exists a sequence \((a_k)\) whose companion sequence \((b_n)\) defined by \((4)\) satisfies the limit relation

\[
\limsup_{n \to \infty} b_n = \alpha?
\]

3. **Distribution of Primes in the Sequence \((S_n)\) with \(S_n = \sum_{i=1}^{2n} p_i\)**

Here, as always in the sequel, we consider the distribution of primes in the sequence \((S_n)_{n=1}^\infty\), whose terms are given by \(S_n = \sum_{i=1}^{2n} p_i\), where \(p_i\) is the \(i\)th prime. Recall that the prime counting function \(\pi(x)\) is defined as the number of primes \(\leq x\).

**Proposition 3.1.** Let \((S_n)\) be the sequence defined as \(S_n = \sum_{i=1}^{2n} p_i\). Then as \(n \to \infty\),

\[
(23) \quad S_n \sim 2n^2 \log n
\]

and

\[
(24) \quad \pi(S_n) \sim n^2.
\]

Furthermore, if \(x\) is a real number such that \(S_n \leq x < S_{n+1}\), then

\[
(25) \quad n \sim \sqrt{\frac{x}{\log x}} \text{ as } n \to \infty.
\]

**Proof.** Let \((S'_n)\) be the sequence defined as \(S'_n = \sum_{i=1}^{n} p_i\) (this is Sloane’s sequence A007504 in [42]). By the Prime Number Theorem, we have (see, e.g., [43] page 5),

\[
(26) \quad S'_n := \sum_{i=1}^{n} p_i \sim \sum_{k=1}^{n} k \log k \sim \int_1^n x \log x \, dx = \frac{x^2}{2} \log x \bigg|_1^n - \int_1^n \frac{x^2}{2} (\log x)' \, dx
\]

\[
\sim \frac{n^2 \log n}{2} \text{ as } n \to \infty.
\]
It follows from (26) that
\begin{equation}
S_n = S_{2n}^\prime \sim 2n^2 \log n,
\end{equation}
which implies (23). By the Prime Number Theorem, from (27) we have
\begin{equation}
\pi(S_n) \sim \frac{2n^2 \log n}{\log(2n^2 \log n)} \sim \frac{2n^2 \log n}{\log 2 + 2 \log n + \log \log n} \sim n^2.
\end{equation}
Finally, (25) immediately follows from (23).

Remark 3.2. For refinements of the estimate (23), see [15], [39] and [41] Theorem 2.3). We see from (23) that there are \( \sim n^2 \) primes less than \( S_n \). Using this fact, Z.-W. Sun [43, Remark 1.6] conjectured that the number of primes in the interval \((\sum_{i=1}^{n} p_i, \sum_{i=1}^{n+1} p_i)\) is asymptotically equivalent to \( n/2 \) as \( n \to \infty \). Under the validity of this conjecture, in particular it follows that the number of primes in the interval \((S_n, S_{n+1})\) is asymptotically equivalent to \( n \) as \( n \to \infty \). Moreover, we also believe that the “probability” that \( \sum_{i=1}^{2n} p_i \) is a prime is \( 2n/p_{2n} \), which is \( \sim 1/\log n \) because of \( p_{2n} \sim 2n \log 2n \). Notice that the “probability” of a large integer \( n \) being a prime is also asymptotically equal to \( 1/\log n \).

Furthermore, some computational results and heuristic arguments show that between these \( \sim n^2 \) primes which are less than \( S_n \) there are \( \sim 2n/\log S_n \sim n/\log n \) primes that belong to the set \( S_n := \{S_1, S_2, \ldots, S_n\} \). For example, if \( n = 10^8 \) then \( n/\log n = 10^8/\log 10^8 = 5428681.02 \), while from the second column of Table 1 of Section 6 we see that there are 5212720 primes in the set \( S_{10^8} \) (cf. Table 2 of Section 6). Accordingly, we propose the following curious conjecture which is basic in this paper.

Conjecture 3.3. The sequence \((S_n)\) with \( S_n = \sum_{i=1}^{2n} p_i \) satisfies the Restricted Prime Number Theorem. In other words,
\begin{equation}
\pi_n := \pi(S_n) = \#\{p : p \text{ is a prime and } p = S_i \text{ for some } i \text{ with } 1 \leq i \leq n\}
\sim \frac{n}{\log n} \quad \text{as } n \to \infty.
\end{equation}

Let us recall that in all results of this section (Corollaries 3.4, 3.6, 3.7, 3.8 and 3.13) we assume the truth of Conjecture 3.3. In particular, Conjecture 3.3 implies Euclid’s theorem (on the infinitude of primes) for \((S_n)\) as follows.

Corollary 3.4 (Euclid’s theorem for the sequence \((S_n)\)). The sequence \((S_n)\) contains infinitely many primes.

Remark 3.5. Notice that the sequence \((S_n)\) is closely related to the Sloane’s sequence A013918 [42] containing all primes (in increasing order) equal to the sum of the first \( m \) primes for some \( m \in \mathbb{N} \) (A013918 is in fact the intersection of A000040-the sequence of all primes and A007504-sum of first \( n \) primes). The first few terms of the sequence A013918 are: 2, 5, 17, 41, 197, 281, 7699, 8893, 22039; see the related link by T. D. Noe [42, A013918] which gives the table of the first 10000 terms of this sequence (10000th term is 402638678093). Notice also that the Sloane’s sequence A013916 in [42] associated to the sequence A013918 gives numbers \( n \) such that the sum of the first \( n \) primes is prime. The first few terms of this sequence are: 1, 2, 4, 6, 12, 14, 60, 64, 96 (see the related link by D. W. Wilson [42, A013918] which gives table of the first 10000 terms of this sequence (10000th term is 244906). Similarly, the second Sloane’s sequence A013917 (\((a_n)\)) associated to A013918, is defined as: \( a_n \) is prime and sum of all primes \( \leq a_n \) is prime. The first few terms of this sequence are: 2, 3, 7, 13, 37, 43, 281.
As a further application of Conjecture 3.3, here we obtain the asymptotic expression for the $k$th prime in the sequence $(S_n)$.

**Corollary 3.6** (The asymptotic expression for the $k$th prime in the sequence $(S_n)$). Let $q_k$ $(k = 1, 2, \ldots)$ be the $k$th prime in the sequence $(S_n)$. Then

$$q_k \sim 2k^2 \log^3 k \text{ as } k \to \infty.$$ \hspace{1cm} (30)

*Proof.* If for a pair $(k, n)$ there holds $q_k = S_n$, then by Conjecture 3.3, we have

$$k \sim \frac{n}{\log n} \text{ as } n \to \infty,$$ \hspace{1cm} (31)

so that $n \sim k \log n$, and hence $\log n \sim \log k$ as $n \to \infty$. Inserting this into (23), we find that

$$q_k = S_n \sim 2n^2 \log n \sim 2(k \log n)^2 \log n = 2k^2 \log^3 n \sim 2k^2 \log^3 k,$$

as desired. \hfill \Box

**Corollary 3.7.** Let $q_k$ $(k = 1, 2, \ldots)$ be the $k$th prime in the sequence $(S_n)$. Then

$$q_k \sim 2p_k^2 \log k \text{ as } k \to \infty$$ \hspace{1cm} (32)

and

$$q_k \sim p_k^2 \log^2 k \text{ as } k \to \infty.$$ \hspace{1cm} (33)

*Proof.* From (30) and the fact that $p_k \sim k \log k$ we find that

$$q_k \sim 2(k \log k)^2 \log k \sim 2p_k^2 \log k,$$

which proves (32).

Similarly, from (30) and $p_k^2 \sim k^2 \log^2 k = 2k^2 \log k$ we find that

$$q_k \sim (k^2 \log k)^2 \log^2 k \sim p_k^2 \log^2 k,$$

which implies (33). \hfill \Box

Furthermore, we have the following result.

**Corollary 3.8.** Let $q_k$ be the $k$th prime in the sequence $(S_n)$ with $q_k = S_n$. Then

$$\lim_{k \to \infty} \frac{k \log k}{n} = 1.$$ \hspace{1cm} (34)

*Proof.* The asymptotic relation (31) implies that $\log n / \log k \sim 1$, which substituting in (31) immediately gives (34).

Motivated by some heuristic arguments and computations for some small integer values $d$, we propose the following generalization of Conjecture 3.3.

**Conjecture 3.9.** For any fixed nonnegative integer $d$ the sequence $(S_n^{(d)})_{n=1}^{\infty}$ defined as

$$S_n^{(d)} = 2d + S_n = 2d + \sum_{i=1}^{2n} p_i, \quad n = 1, 2, \ldots$$

satisfies the Restricted Prime Number Theorem. In other words, as $n \to \infty$,

$$\pi_n^{(d)} := \pi(2d + S_n)(2d + S_n) = \#\{ p : p \text{ is a prime and } p = 2d + S_i \}
\sim \frac{n}{\log n}.$$ \hspace{1cm} (35)

for some $i$ with $1 \leq i \leq n$. 

For \( d = 0 \) this conjecture is in fact Conjecture 3.3 (cf. Sloane’s sequence A013918 mentioned above).

**Remark 3.10.** Conjecture 3.3 and the fact that by (23) \( S_n \sim 2n^2 \log n \) imply that the average difference between consecutive primes in the sequence \( (S_n) \) near to \( 2n^2 \) is approximately \( \log(2n^2) \sim 2 \log n \).

**Remark 3.11.** Numerous computational results concerning the sums of the first \( n \) primes (partial sums of consecutive primes) given by the Sloane’s sequence A007504 (here denoted as \( S'_n \)), and certain their curious arithmetical properties are presented in the following Sloane’s sequences in OEIS [42]: A051838, A116536, A067110, A067111, A045345, A114216, A024011, A077023, A033997, A071089, A083186, A166448, A196527, A065595, A165906, A061568, A066039, A077022, A110997, A112997, A156778, A167214, A038346, A038347, A054972, A072476, A076570, A076873, A077354, A110996, A123119, A189072, A196528, A022094, A024447, A121756, A143121, A117842, A118219, A131740, A143215, A161436, A161490, A013918 etc.

Since the sequence \( (S_n) \) is a subsequence of the sequence \( (S'_n) \) with \( S'_n = \sum_{k=1}^{n} p_k \) whose all terms with odd indices \( n \) are even integers, it follows that in accordance to Definition 1.1, Conjecture 3.3 is equivalent to

\[
\omega(S'_n(n)) \sim n^2 \log n.
\]

Therefore, Conjecture 3.3 is equivalent with the following one.

**Conjecture 3.3’**. Let \( (S'_n) \) be a sequence defined as \( S'_n = \sum_{k=1}^{n} p_k, n = 1, 2, \ldots \). Then

\[
\omega(S'_n)(x) = \frac{x}{2 \log x} \quad \text{for} \quad x \in (1, \infty).
\]

**Proposition 3.12.** For each \( n \geq 3 \) we have

\[
1 \leq \frac{S_n}{2n^2 \log n} < 1 + \frac{\log 2}{\log n} + \frac{\log \log(2n)}{\log n}.
\]

**Proof.** By Mandl’s inequality (see, e.g., [39], [15]), for each \( n \geq 9 \) there holds

\[
S'_n < \frac{n}{2} p_n
\]

(for a refinement of (38), see [17, the inequality 2.4]). Mandl’s inequality (38) with \( 2n \) instead of \( n \) becomes \( S_n < np_{2n} \) with \( n \geq 5 \). This inequality together with the known inequality (see, e.g., [38, p. 69])

\[
p_{2n} < 2n(\log n + \log 2 + \log \log(2n)) \quad \text{for all} \quad n \geq 3
\]

immediately yields

\[
S_n < 2n^2(\log n + \log 2 + \log \log(2n)) \quad \text{for all} \quad n \geq 5.
\]

On the other hand, a lower bound for \( S'_n \) can be obtained by using Robin’s inequality (see, e.g., [15, p. 51]) which asserts that for every \( n \geq 2 \)

\[
np_{\lfloor n/2 \rfloor} \leq S'_n.
\]

The inequality (40) with \( 2n \) instead of \( n \) and the inequality \( n \log n \leq p_n \) with \( n \geq 3 \) (see, e.g., [38, p. 69]) yield

\[
2n^2 \log n \leq S_n \quad \text{for} \quad n \geq 3.
\]
The inequalities (39) and (41) immediately yield
\[ \log n \leq \frac{S_n}{2n^2} < \log n + \log 2 + \log \log(2n) \quad \text{for all } n \geq 5, \]
or equivalently,
\[ 1 \leq \frac{S_n}{2n^2 \log n} < 1 + \frac{\log 2}{\log n} + \frac{\log \log(2n)}{\log n} \quad \text{for all } n \geq 5. \tag{42} \]
The inequalities given by (42) coincide with these of (37) for \( n \geq 5 \). A direct calculation shows that (37) is also satisfied for \( n = 3 \) and \( n = 4 \). This completes the proof. \( \square \)

**Corollary 3.13.** Let \( q_k = S_m \) be the \( k \)th prime in the sequence \( (S_n)_{n=1}^{\infty} \). Then for all \( k \geq 3 \) there holds
\[ 2m^2 \log m < q_k < 2m^2(\log m + \log(2m) + \log 2). \tag{43} \]

**Proof.** The above inequalities coincide with (37) of Proposition 3.12 with \( n = m \) and \( q_k = S_m \). \( \square \)

**Remark 3.14.** Z.-W. Sun \([44]\), the case \( \alpha = 1 \) in Lemma 3.1] showed that for all \( n \geq 2 \)
\[ S'_n > 2 + \frac{n^2 \log n}{2} \left( 1 - \frac{1}{2 \log n} \right), \]
which with \( 2n \) instead of \( n \) becomes
\[ S_n > 2 + 2n^2 \left( \log n + \log \frac{2}{\sqrt{e}} \right) \approx 2 + 2n^2(\log n + 0.193147), \]
whence it follows that
\[ \frac{S_n}{2n^2 \log n} > 1 + \frac{0.193147}{n}. \]
The above inequality is stronger than the left hand side of the inequality (37). Accordingly, if \( q_k = S_m \), then the first inequality of (43) can be refined in the form
\[ q_k > 2 + 2m^2(\log m + 0.193147) \quad \text{for all } k \geq 3. \]

On the other hand, combining the inequalities (46) and (47) from the next section with the inequalities \( S_n > 2np_n \) and \( S_n < np_{2n} \) (given in proof of Proposition 3.12), respectively, we immediately obtain the following refinement of Proposition 3.12.

**Proposition 3.15.** For each \( n \geq 3 \) there holds
\[ \frac{S_n}{2n^2 \log n} \geq 1 + \frac{\log \log n - 1}{\log n} + \frac{\log \log n - 2.2}{\log^2 n}, \]
and for each \( n \geq 344192 \), we have
\[ \frac{S_n}{2n^2 \log n} \leq 1 + \frac{\log \log(2n) + \log 2 - 1}{\log n} + \frac{\log \log(2n) - 2}{(\log n \log(2n)).} \]

**Remark 3.16.** If \( q_k = S_m \), then in view of the first inequality of Proposition 3.15, the first inequality of (43) may be replaced by the following one:
\[ q_k > 2m^2 \left( \log m + \log \log m - 1 + \frac{\log \log m - 2.2}{\log m} \right) \quad \text{for all } k \geq 3. \]
Remark 3.17. The inequalities (38) and (40) and the asymptotic expression \( p_n \sim n \log n \) show that the average of the first \( n \) primes is asymptotically equal to \( (n \log n)/2 \) (cf. Sloane’s sequence A060620 in [42]), that is,

\[
\frac{S'_n}{n} \sim \frac{n \log n}{2} \quad \text{as} \quad n \to \infty.
\]

Conjecture 3.3 suggests the fact that for the sequence \( (S_n) \) would be valid the analogues of some other classical results and conjectures closely related to the Prime Number Theorem and Riemann Hypothesis. In particular, if \( Q = \{q_1, q_2, \ldots, q_k, \ldots\} \) is a set of all primes \( q_1 < q_2 < \ldots < q_k < \cdots \) in the sequence \( (S_n) \), it can be of interest to establish the asymptotic expression for \( q_k \) as \( k \to \infty \).

Finally, heuristic arguments, some computational results and Conjecture 3.3 lead to the following its generalization (cf. Sloane’s sequence A143121 - triangle read by rows, \( T(n, k) = \sum_{j=1}^{n} p_{i+k} \), \( 1 \leq k \leq n \); see the columns in Example of this sequence).

**Conjecture 3.18.** For any fixed positive integer \( k \), let \( (S^{(k)}_n) := (S^{(k)}_n)_{n=1}^{\infty} \) be the sequence whose \( n \)th term is defined as

\[
S^{(k)}_n = \sum_{i=1}^{2n+1} p_{i+k}, \quad n \in \mathbb{N}.
\]

Then the sequence \( (S^{(k)}_n) \) satisfies the Restricted Prime Number Theorem.

For example, there are 78498 (resp. 664579) primes less than \( 10^6 \) (resp. \( 10^7 \)), while the computations show that among the first \( 10^6 \) (resp. \( 10^7 \)) terms of the sequences \( (S_n) \), \( (S^{(k)}_n) \) with \( k = 1, 2, \ldots, 12 \) there are 69251 (resp. 594851), 69581 (resp. 594377), 68844 (resp. 593632), 68883 (resp. 593773), 69602 (resp. 596609), 69540 (resp. 596558), 69414 (resp. 595539), 69317 (resp. 594626), 69455 (resp. 595474), 69268 (resp. 594542), 68891 (resp. 593807), 69251 (resp. 594383), 69564 (resp. 595270) primes, respectively.

4. The Asymptotic Expression for the \( k \)th Prime in the Sequence \( (S_n) \)

As an easy consequence of the Prime Number Theorem, it can be deduced that \( p_n \sim n \log n \) as \( n \to \infty \) (see, e.g., [26]). Furthermore, a particular asymptotic expansion for \( p_n \) (see [34] or [43] the equality (66) of Section 6; also see Sloane’s sequence A200265) yields

\[
p_n = n \left( \log n + \log \log n + O \left( \frac{\log \log n}{\log n} \right) \right).
\]

It is also known that (see [16] and [38, p. 69])

\[
n(\log n + \log \log n - 1) < p_n < n(\log n + \log \log n).
\]

A more precise work about this can be found in [37] and [40] where related results are as follows:

\[
n \left( \log n + \log \log n - 1 + \frac{\log \log n - 2.2}{\log n} \right) \leq p_n \quad \text{for} \quad n \geq 3
\]

and
Corollary 4.1. For each \( n \geq 688383 \) the interval
\[
\left[ n \left( \log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} \right), n \left( \log n + \log \log n - 1 + \frac{\log \log n - 2}{\log n} \right) \right]
\]
contains at least one prime. Furthermore, the length \( l_n \) of this interval is
\[
l_n = \frac{0.2n}{\log n} \sim \frac{0.2p_n}{\log^2 n}.
\]

As an application of Conjecture 3.3, we obtain the following result.

Corollary 4.2. Let \( q_k \) be the \( k \)th prime in the sequence \( (S_n) \) with \( S_n = \sum_{k=1}^{2n} p_k \). If \( q_k = S_m \), then under Conjecture 3.3 there holds
\[
q_k \sim 2k^2 \log^3 k \sim 2m^2 \log m \sim p_{[k^2 \log^2 k]} \quad \text{as} \quad k \to \infty,
\]
and
\[
\lim_{k \to \infty} \frac{k \log k}{m} = 1.
\]

Proof. The first asymptotic relation of (50) coincides with (30) of Corollary 3.6. Further, by (37) of Proposition 3.12, we have
\[
q_k = S_m \sim 2m^2 \log m.
\]

Moreover, we have
\[
p_{[k^2 \log^2 k]} \sim k^2 (\log^2 k) \log(k^2 \log^2 k) \sim 2k^2 \log^3 k \quad \text{as} \quad k \to \infty.
\]
The last two asymptotic expressions of (50) follow from (52) and (53).

It remains to prove (51). If we suppose that (51) is not satisfied, then there exists \( \varepsilon > 0 \) and an infinite subsequence \( (k_j, m_j)_{j=1}^{\infty} \) of the sequence \( (k, m)_{k=1}^{\infty} \) such that \( k_j \log k_j \geq 1 + \varepsilon \) for all \( j \in \mathbb{N} \) or \( k_j \log k_j \leq 1 - \varepsilon \) for all \( j \in \mathbb{N} \). In the first case, using (50) for all sufficiently large \( j \), we find that
\[
m_j^2 \log m_j \sim k_j^2 \log^3 k_j \geq (1 + \varepsilon)^2 m_j^2 \log k_j,
\]
whence we immediately get \( \log m_j \geq (1 + \varepsilon)^2 \log k_j \), or equivalently, \( m_j \geq k_j^t \) with \( t = (1 + \varepsilon)^2 > 1 \). From the previous inequality and the fact that \( t > 2 \) we have
\[
m_j^2 \log m_j > m_j^2 \geq k_j^{2t} \gg k_j^2 \log^3 k_j,
\]
which contradicts the fact that by (50) \( 2k^2 \log^3 k \sim 2m^2 \log m \). In a similar way as in the first case, in the second case we find that \( m_j \leq k_j^s \) for a constant \( s = (1 - \varepsilon)^2 < 1 \). Then choosing a sufficiently large \( j_0 \) such that \( \log m_j < m_j^{1/s - 1} \) for all \( j > j_0 \), in view of the fact that \( s + 1 < 2 \) we get
\[
m_j^2 \log m_j < m_j^{1+1/s} \leq k_j^{s+1} \ll k_j^2 \log^3 k_j.
\]
A contradiction, and therefore, (51) is true. \( \square \)
Remark 4.3. From (50) and \( p_k \sim k \log k \) we see that
\[
\frac{q_k}{2k \log^2 k} \sim k \log k \sim p_k \quad \text{as} \quad k \to \infty.
\]
The above asymptotic expression together with the assumption that Conjecture 3.3 is true suggests the fact that for the sequence \( \left( \frac{q_k}{k \log^2 k} \right) \) would be satisfied the asymptotic expression similar to (44), i.e.,
\[
\frac{q_k}{2k \log^2 k} = k(\log k + \log \log k + Q_k),
\]
with some sequence \((Q_k)\). Motivated by (55), we establish the following asymptotic expression for the \(k\)th prime in the sequence \((S_n)\).

Theorem 4.4 (The asymptotic expression for the \(k\)th prime in the sequence \((S_n)\)). Let \( q_k \) be the \(k\)th prime in the sequence \((S_n)\) \((k = 2, 3, \ldots)\). Then under Conjecture 3.3 there exists a sequence \((M_k)\) of positive real numbers such that
\[
limit_{k \to \infty} M_k = 1
\]
and
\[
q_k = 2M_k^5k^2 \log^2 k(\log k + \log \log k + 2 \log M_k).
\]

For the proof of Theorem 4.4 we will need the following result.

Lemma 4.5. Let \( S_m = q_k \) be the \(k\)th prime in the sequence \((S_n)\). Then under Conjecture (3,3) we have
\[
q_k \sim \frac{2m^2 \sqrt{m \log m}}{\sqrt{k \log k}} \quad \text{as} \quad k \to \infty.
\]

Proof. First notice that, under notations of Lemma 4.5, Conjecture 3.3 yields (cf. (51) of Corollary 4.2)
\[
k \sim \frac{m}{\log m} \quad \text{as} \quad k \to \infty.
\]
Using (58), we find that
\[
\frac{2m^2 \sqrt{m \log m}}{\sqrt{k \log k}} \sim \frac{2m^2 \sqrt{m \log m}}{\sqrt{m \left(1 - \frac{\log \log m}{\log m}\right)}} \sim 2m^2 \log m \quad \text{as} \quad k \to \infty.
\]
The asymptotic relation (59) and the fact that by (50) of Corollary 4.2, \( q_k = S_m \sim 2m^2 \log m \) immediately yield (57).

Proof of Theorem 4.4. Let \((C_k)_{k=2}^\infty\) be a sequence of positive real numbers such that \( m(k) = m = C_k k \log k \) with \( k \geq 2 \) and \( q_k = S_m \). Then by (51) of Corollary 4.2, we have \( C_k \to 1 \) as \( k \to \infty \). Taking \( m = C_k k \log k \) into (57) of Lemma 4.5, as \( k \to \infty \) we obtain that
\[
q_k \sim \frac{2 \sqrt{C_k^5 k^5 \log^5 k(\log k + \log \log k + \log C_k)}}{\sqrt{k \log k}}
\]
\[
= 2C_k^2 \sqrt{C_k^5 k^2 \log^2 k(\log k + \log \log k + \log C_k)} =: f(k, C_k).
\]
Let \((\delta_k)\) be a positive real sequence such that
\[
q_k = \delta_k f(k, C_k) \quad \text{for each} \quad k \geq 2.
\]
Then from (60) we see that $\delta_k \to 1$ as $k \to \infty$. For a fixed $k \geq 2$ consider the equation $f_k(x) = \delta_k f(k, C_k)$ which can be written in the form

\begin{equation}
\frac{x^2}{\sqrt{x}}(\log k + \log \log k + \log x) = \delta_k C_k^2 \sqrt{C_k}(\log k + \log \log k + \log C_k).
\end{equation}

Notice that for any fixed integer $k \geq 2$, the real function $f_k(x)$ defined as

\begin{equation}
f_k(x) = 2 \sqrt{x} (\log k + \log \log k + \log x), \quad x > 0,
\end{equation}

satisfies the limit relations $\lim_{x \to +\infty} f_k(x) = +\infty$ and $\lim_{x \to +0} f_k(x) = 0$. From this it can be easily shown that for each integer $k \geq 2$, the equation (62) has a positive real solution $x_k$. Using the facts that $\lim_{x \to +\infty} C_k = \lim_{x \to +\infty} \delta_k = 1$, it can be easily show that $\lim_{k \to \infty} x_k = 1$. Then taking $x_k = M_k^2$ ($k = 2, 3, \ldots$), then we find that $f_k(M_k^2) = \delta_k f(k, C_k) = q_k$, whence it follows that

\begin{equation}
q_k = 2 M_k^2 \log^2 k (\log k + \log \log k + 2 \log M_k).
\end{equation}

This proves (56) and the proof is completed. □

Computational results (cf. the eighth column of Table 1 of Section 6) suggest the additional relationship between $k$’s and $m$’s as follows.

**Conjecture 4.6.** For each pair $(k, m)$ with $k \geq 1$ and $q_k = S_m$ we have

\begin{equation}
\lfloor k \log k \rfloor + 1 \leq m,
\end{equation}

or equivalently,

\begin{equation}
q_k \geq S_{\lfloor k \log k \rfloor + 1}.
\end{equation}

Furthermore, for each $k \geq 10^4$,

\begin{equation}
m \leq \lfloor 1.4 k \log k \rfloor,
\end{equation}

or equivalently,

\begin{equation}
q_k \leq S_{\lfloor 1.4 k \log k \rfloor}.
\end{equation}

**Corollary 4.7.** If the inequality (63) of Conjecture 4.6 is true, then for each $k \geq 1$ there holds

\begin{equation}
q_k > 2k^2 (\log^2 k) (\log k + \log \log k).
\end{equation}

**Proof.** Combining the inequality (63) with the inequality on the left hand side of (37) of Proposition 3.12, we find that

\begin{equation}
q_k = S_m \geq S_{\lfloor k \log k \rfloor + 1} \geq 2 (\lfloor k \log k \rfloor + 1)^2 \log (\lfloor k \log k \rfloor + 1)
\end{equation}

\begin{equation}
> 2k^2 (\log^2 k) \log (k \log k) = 2k^2 (\log^2 k) (\log k + \log \log k),
\end{equation}

as desired. □

**Corollary 4.8.** If the inequality (63) of Conjecture 4.6 is true, then $M_k > 1$ for each $k \geq 1$, where $(M_k)$ is the sequence defined by (56) of Theorem 4.4.

**Proof.** The assertion follows immediately from the inequality (67) and the expression (56) for $q_k$ given by Theorem 4.4. □

Finally, in view of the data of the last column in Table 1 of Section 6 and some considerations presented above, we propose the following conjecture which is stronger than Corollary 4.7.
**Conjecture 4.9.** For every $k \geq 252028$ with $q_k = S_m$ there holds

\[
q_k > \frac{2m^2\sqrt{m\log m}}{k\log k}.
\]

In view of the well known inequality $p_k > k(\log k + \log \log k)$, the following conjecture is also stronger than Corollary 4.7.

**Conjecture 4.10.** There exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there holds

$$q_k > 2kp_k \log^2 k,$$

where $p_k$ is the $k$th prime.

**Remark 4.11.** The last column of Table 1 presented in Section 6 shows that

$$q_k \approx \frac{2m^2\sqrt{m\log m}}{k\log k}$$

is a “good” approximation for the $k$th prime sum $q_k$. Notice that this approximation can be written as

$$q_k \approx 2m^2 \log m \cdot \sqrt{\frac{m}{k\log k}},$$

where the values $\sqrt{m/(k\log k)}$ slowly tend to 1 as $k$ grows. In particular, from the last row of Table 1 of Section 6 we see that for $m = 10^9 - 2$ (i.e., for $k = 46388006$) we have $\sqrt{m/(k\log k)} \approx 1.105079$. Hence, in view of the above approximation, we believe that for all values $m$ up to $10^9$ there holds

$$q_k > 2.2m^2 \log m.$$

Notice that some values of the sequences $(Q'_k)$ such that

$$Q'_k = \frac{q_k - 2k^2(\log^2 k)(\log k + \log \log k)}{2k^2 \log^2 k \log \log k}$$

and the sequence $(Q''_k)$ such that

$$Q''_k = \frac{q_k - 2(p_k)^2 \log k}{2k^2 \log^2 k \log \log k}$$

are given in Table 3 of Section 6. Table 3 also shows that for almost all values $m$ up to $10^9$ (i.e., for $k \leq 46388006$) there holds

$$kp_k > 1.1m^2 \log m.$$

Finally, Table 2 of Section 6 leads to the following conjecture whose both parts are obviously stronger than Conjecture 4.6.

**Conjecture 4.12.** Let $\pi(x)$ be the prime counting function, and let $\pi_n$ be the number of primes in the set $\{S_1, S_2, \ldots, S_n\}$. Then

$$\pi_n < \pi(n)$$

for each $n \geq 10^4$ and

$$\pi_n < \frac{n}{\log n}$$

for each $n \geq 10^5$. 
5. Estimations of values $M_k$ from Theorem 4.4

The computational results related to the search of primes in the sequence $(S_n)$ given in the following section (Table 1) and some heuristic arguments suggest the fact that the sequence $(S_n/(2n^2 \log n))_{n=2}^\infty$ plays an important role for estimating the values $M_k$ $(k = 1, 2, \ldots)$ in the expression (56) for the $k$th prime $q_k$ in the sequence $(S_n)$.

Here we first consider the sequence $(S_n/(2n^2))$.

**Proposition 5.1.** The sequence $(v_n)$ defined as

$$v_n = \frac{S_n}{2n^2}, \quad n \in \mathbb{N},$$

is increasing for $n \geq 2$.

**Proof.** Since $S_{n+1} = S_n + p_{2n+1} + p_{2n+2}$, an easy calculation shows that $r_n < r_{n+1}$ is equivalent with

$$\frac{S_n}{2n^2} < \frac{p_{2n+1} + p_{2n+2}}{2(2n + 1)},$$

which can be written as

$$\frac{p_{2n+1} + p_{2n+2}}{2} > S_n \left(\frac{1}{n} + \frac{1}{2n^2}\right).$$

By a refinement of Mandl’s inequality due to Hassani [17], for every $n \geq 10$ we have

$$\frac{n}{2}p_n - \sum_{i=1}^{n} p_i > 0.01659n^2.$$  

Replacing $n$ by $2n$ into (72) it becomes

$$p_{2n} - \frac{S_n}{n} > 0.06636n^2 \quad \text{for all } n \geq 5.$$  

Further, by the inequality (37) of Proposition 3.12 we have

$$\log(2n) + \log \log(2n) > \frac{S_n}{2n^2} \quad \text{for all } n \geq 5.$$  

By using Mathematica 8, it is easy to to prove the inequality

$$0.06636n^2 > \log(2n) + \log \log(2n) \quad \text{for all } n \geq 8.$$  

Finally, combining the inequalities (73), (74), (75) and the obvious inequality $(p_{2n+1} + p_{2n+2})/2 > p_{2n}$ immediately gives (71) for all $n \geq 8$. This together with a direct verification that $v_n < v_{n+1}$ for $2 \leq n \leq 8$ concludes the proof. □

**Remark 5.2.** Notice that the sequence $(v_n)$ defined by (70) is a subsequence of the sequence $(v'_n)$ defined as

$$v'_n = \frac{2S_n'}{n^2} := \frac{2\sum_{i=1}^{n} p_i}{n^2}, \quad n \in \mathbb{N};$$

namely, $v_n = v'_{2n}$ for all $n = 1, 2, \ldots$ Similarly as in the proof of Proposition 5.1, it can be shown that the sequence $(v'_n)$ is increasing for $n \geq 4$.

Contrary to Proposition 5.1, we propose the following conjecture.
Conjecture 5.3. The sequence \((t_n)\) defined as
\[
t_n = \frac{S_n}{2n^2 \log n}, \quad n \in \mathbb{N} \setminus \{1\},
\]
is decreasing on the range \(\{n \in \mathbb{N} : n \geq 1100\}\) \((m = 1099\) is a maximal value between total 40 values up to \(n = 200000\) for which \(t_{m+1} > t_m\)).

Remark 5.4. Notice that the sequence \((t_n)\) defined by (76) is a subsequence of the sequence \((t'_n)\) defined as
\[
t'_n = \frac{2S'_n}{n^2 \log(n/2)} := \frac{2 \sum_{i=1}^{n} p_i}{n^2 \log(n/2)}, \quad n \in \mathbb{N};
\]
namely, \(t_n = t'_2n\) for all \(n = 1, 2, \ldots\). We conjecture that the sequence \((t'_n)\) is decreasing on the range \(\{n \in \mathbb{N} : n \geq 2199\}\) \((m = 2198\) is a maximal value up to \(n = 10^6\) for which \(t'_{m+1} > t'_m\)).

Corollary 5.5. Let \((t_n)\) be the sequence defined in Conjecture 5.3. Then under Conjecture 5.3, for each \(n \geq 1100\) there holds
\[
t_{n+1} < t_n < t_{n+1} \left(1 + \frac{1}{n \log n}\right).
\]

Proof. Proposition 5.1, Conjecture 5.3 and the well known inequality \((1 + 1/n)^n < e\) with \(n \geq 1\) immediately imply that for all \(n \geq 1100\) there holds
\[
0 < t_n - t_{n+1} = \frac{S_n}{2n^2 \log n} - \frac{S_{n+1}}{2(n+1)^2 \log(n+1)} < \frac{S_{n+1}}{2(n+1)^2 \log n} - \frac{S_{n+1}}{2(n+1)^2 \log(n+1)}
\]
\[
= \frac{S_{n+1}}{2(n+1)^2} \cdot \frac{\log (1 + \frac{1}{n})^n}{n \log n} \cdot \frac{1}{n \log n}
\]
\[
< \frac{S_{n+1}}{2(n+1)^2 \log(n+1)} \cdot \frac{1}{n \log n}.
\]
From (79) we immediately get (78). \(\square\)

Corollary 5.6. Let \((t_n)\) be the sequence defined in Conjecture 5.3. Then under Conjecture 5.3, for each \(n \geq 1101\) we have
\[
t_n > \frac{17}{8 \log 2} \left(\left(1 + \frac{1}{2 \log 2}\right) \left(1 + \frac{1}{3 \log 3}\right) \cdots \left(1 + \frac{1}{(n-1) \log(n-1)}\right)\right)^{-1}
\]
and
\[
S_n > \frac{17 n^2 \log n}{4 \log 2} \left(\left(1 + \frac{1}{2 \log 2}\right) \left(1 + \frac{1}{3 \log 3}\right) \cdots \left(1 + \frac{1}{(n-1) \log(n-1)}\right)\right)^{-1}.
\]

Proof. By the right hand side of the inequality (78), we obtain that for each \(n \geq 1101\)
\[
t_n > t_{n-1} \left(1 + \frac{1}{(n-1) \log(n-1)}\right)^{-1}.
\]
By iterating the inequality (82) \((n - 2)\) times and taking \(S_2 = 17\) in \((n - 2)\)th step, we immediately obtain the inequality (80). Substituting \(t_n = \frac{S_n}{2n^2 \log n}\) into (80) gives the inequality (81).

Notice that under Conjecture 5.3, the sequence \((t_n)\) defined by (76) as

\[ t_n = \frac{S_n}{2n^2 \log n}, \quad n \in \mathbb{N} \setminus \{1\}, \]

is decreasing on the range \(\{n \in \mathbb{N} : n \geq 1100\}\). As noticed above, the computational results for “prime sums” given in Table 1 of Section 6 suggest the fact that the sequence \((t_n)\) plays an important role for estimating the values \(M_k\) for all \(k \geq 2\).

Reference to Conjectures 3.3 and 5.7 for all \(k \geq 2\) there holds

\[ q_k < 2k^2 \log^3 k \left(1 + \frac{\log 2 + 2 \log \log(2k)}{\log k}\right)^5 \left(1 + \frac{\log \log k}{\log k} + \frac{2 \log 2 + 2 \log \log(2k)}{\log^2 k}\right). \]

Proof. Applying the inequality \(\log(1 + x) < x\) with \(x > 0\) to (84), we find that

\[ \log M_k \leq \frac{\log 2 + \log \log(2k)}{\log k} \quad \text{for all } k \geq 2. \]

Inserting the inequalities (84) and (86) into the expression (56) of Theorem 4.4 for \(q_k\), we immediately obtain (85).

Corollary 5.10. Let \(q_k\) be the \(k\)th prime in the sequence \((S_n)\) \((k = 1, 2, \ldots)\). Then under Conjectures 3.3 and 5.7 there holds

\[ q_k = 2k^2 \log^2 k (\log k + O(\log \log k)), \]

or equivalently,

\[ \frac{q_k}{2k^2 \log^2 k} = 1 + O\left(\frac{\log \log k}{\log k}\right). \]

Proof. The inequality (85) immediately yields the asymptotic expression (87).
Corollary 5.11. Let \( q_k \) be the \( k \)-th prime in the sequence \((S_n)_n \) \((k = 1, 2, \ldots)\). Then under Conjectures 3.3 and 5.7 there exists an absolute positive constant \( C \) with \( 1 \leq C \leq 6 \) such that
\[
q_k = 2k^2 \log^2 k (\log k + C \log \log k + o(\log \log k)).
\]

Proof. Using the binomial expansion, we find that
\[
\left(1 + \frac{\log 2 + \log \log(2k)}{\log k}\right)^5 = 1 + \frac{5 \log \log k}{\log k} + o\left(\frac{\log \log k}{\log k}\right),
\]
which substituting in (85) immediately yields the estimation from Corollary 5.11. \( \Box \)

Remark 5.12. The determination of a constant \( C \) from Corollary 5.11 is closely related to the sequence \((Q_k)_k \) with \( Q_k = (q_k - 2k^2 \log^3 k)/(2k^2 (\log^2 k) \log \log k) \), whose values are presented in Table 3 of Section 6. Related data from Table 3 and the additional computations suggest that
\[
q_k < 2k^2 \log^2 k (\log k + 6 \log \log k) \quad \text{for all } k \geq 5 \cdot 10^6.
\]

Conjecture 5.13 (A refined upper bound of the sequence \((M_k)_k \)). Let \((M_k)_{k=1}^\infty \) be the sequence defined by the expression (56) of Theorem 4.4. Then
\[
M_k \leq t_{[k \log k]} = \frac{S_{[k \log k]}}{2 ([k \log k]^2 \log [k \log k]} := M_{(t)} \quad \text{for all } k \geq 5 \times 10^7,
\]
where \([k \log k]\) is the greatest integer not exceeding \( k \log k \).

Corollary 5.14. Let \((t_n)_n \) be the sequence defined by (76). Then under the inequality (63) of Conjecture 4.6 and Conjecture 5.13, for all \( k \geq 2 \) the interval
\[
\left[2k^2 (\log^2 k)(\log k + \log \log k), 2k^2 (\log^2 k) \left(1 + \frac{\log(2 \log(2k))}{\log k}\right)^5 \times \right.
\]
\[
\left.\log k + \log \log k + 2 \log \left(1 + \frac{\log(2 \log(2k))}{\log k}\right)\right]
\]
contains at least one prime that belongs to the sequence \((S_n)_n \). In particular, the prime \( q_k \) belongs to the interval given by (90).

Furthermore, for all \( k \geq 2 \) the length \( l_k \) of the interval (90) satisfies the inequality
\[
l_k < 62k^2 (\log k) \log(k \log k) \log(2 \log(2k)) + 4k^2 (\log k + 31 \log(2 \log(2k)) \log(2 \log(2k))).
\]

Proof. The first assertion immediately follows from the inequality (67) of Corollary 4.7 and the inequality (83) of Conjecture 5.7. Notice that by the inequality on the right hand side of (37) from Proposition 3.12, we find that
\[
t_k < 1 + \frac{\log(2 \log(2k))}{\log k} \quad \text{for all } k \geq 5.
\]

Then the inequality (83) of Conjecture 5.7 and the inequality (92) immediately yield
\[
q_k = 2M_k^5 k^2 (\log^2 k)(\log k + \log \log k + 2 \log M_k)
\leq 2t_k^5 k^2 (\log^2 k)(\log k + \log \log k + 2 \log t_k)
\leq 2k^2 (\log^2 k) \left(1 + \frac{\log(2 \log(2k))}{\log k}\right)^5 (\log k + \log \log k + 2 \log \left(1 + \frac{\log(2 \log(2k))}{\log k}\right)).
\]
The inequalities (93) and (67) of Corollary 4.7 show that the interval defined by (90) contains the prime sum $q_k$.

Further, using the inequality $(1 + x)^5 \leq 1 + 31x$ for $0 \leq x : = \log(2 \log(2k))/\log k \leq 1$ and the inequality $\log(1 + x) < x$ for $x : = \log(2 \log(2k))/\log k > 0$, the length $l_k$ of interval defined by (90) can be estimated as follows.

(94) $$l_k \leq \left(\left(1 + \frac{\log(2 \log(2k))}{\log k}\right)^5 - 1\right) 2k^2(\log^2 k)(\log k + \log \log k)$$

$$+ 4 \left(1 + \frac{\log(2 \log(2k))}{\log k}\right)^5 k^2(\log^2 k) \log \left(1 + \frac{\log(2 \log(2k))}{\log k}\right)$$

$$\leq \frac{31 \log(2 \log(2k))}{\log k} 2k^2(\log^2 k)(\log k + \log \log k)$$

$$+ 4 \left(1 + 31 \cdot \frac{\log(2 \log(2k))}{\log k}\right) k^2(\log^2 k) \frac{\log(2 \log(2k))}{\log k}$$

$$= 62k^2(\log k)(\log k \log k)(\log(2 \log(2k)) + 4k^2(\log k + 31 \log(2 \log(2k)))(\log(2 \log(2k))).$$

This completes the proof. $\square$

Nevertheless the fact that $M_k^{(l)}$ is probably the upper bound of $M_k$ for all $n > 5 \cdot 10^7$ (see Table 1), we propose the following conjecture.

**Conjecture 5.15.** Let $(t_n)$ be the sequence defined by (76). Then for all $k \geq 2$ the interval

(95) $$[2t_{\lfloor k \log k \rfloor}^5 k^2(\log^2 k)(\log k + \log \log k + 2 \log t_{\lfloor k \log k \rfloor}), 2t_k^5 k^2(\log^2 k)(\log k + \log \log k + 2 \log t_k)]$$

contains at least one prime sum $q_i$ from the sequence $(S_n)$.

As an application of Corollary 5.14, we obtain the following $(S_n)$-analogue of the well known fact that the series $\sum_{n=1}^{\infty} 1/p_n$ diverges.

**Corollary 5.16.** The series

(96) $$\sum_{k=1}^{\infty} \frac{k \log^2 k}{q_k}$$

diverges.

**Proof.** It is easy to see that for each $k \geq 2$ the right bound of the interval given by (90) is less than $288k^2 \log^3 k$, and hence, by Corollary 5.14, $q_k < 288k^2 \log^3 k$ for each $k \geq 2$. Therefore, $k \log^2 k/q_k > 1/(288k \log k)$, and hence,

$$\sum_{k=1}^{n} \frac{k \log^2 k}{q_k} > \frac{1}{288} \sum_{k=2}^{n} \frac{1}{k \log k} \sim \int_{2}^{n} \frac{dx}{x \log x}$$

$$= \log \log x \bigg|_{2}^{n} = \log \log n - \log \log 2 \to \infty \text{ as } n \to \infty.$$

Therefore, the series (96) diverges. $\square$

On the other hand, we have the following consequence of Corollary 5.14.
Corollary 5.17. For every $\varepsilon > 0$ the series
\[ \sum_{k=1}^{\infty} \frac{k \log^{2-\varepsilon} k}{q_k} \]
converges.

Proof. By Corollary 5.14 (see the interval (90)), $q_k > 2k^2 \log^3 k$ for each $k \geq 2$. Therefore, $k \log^{2-\varepsilon}/q_k < 1/(2k \log^{1+\varepsilon} k)$, and hence,
\[ \sum_{k=1}^{n} \frac{k \log^{2-\varepsilon} k}{q_k} < \frac{1}{2} \sum_{k=2}^{n} k \log^{1+\varepsilon} k \sim \int_{2}^{n} \frac{dx}{x \log^{1+\varepsilon} x} = -\frac{1}{\varepsilon \log^{\varepsilon} 2} = \frac{1}{\varepsilon \log^{\varepsilon} n} \rightarrow \frac{1}{\varepsilon \log^{\varepsilon} 2} \text{ as } n \to \infty. \]

Therefore, the series (97) converges. \qed

6. COMPUTATIONAL RESULTS

By using Mathematica 8, here we present our computational results concerning the number of expression “prime sums” $q_k$ (under Conjecture 3.3) and related expression (the equality (56) of Theorem 4.4). The notion $\pi_n := k$ in the second column of Tables 1, 3 and 4 presents the number of primes in a set $S_n := \{S_1, S_2, \ldots, S_n\}$, where $n$ is a related value given in the first column of this table. Hence, under notations of Section 1 and Conjecture 3.3,
\[ k := \pi_n := \pi_{(S_k)}(S_n) = \# \{p : p \text{ is a prime and } p = S_i \text{ for some } i \text{ with } 1 \leq i \leq n \}. \]

Accordingly, the value $k$ in the second column of Table 1 presents the number of primes in a set $S_n$, where $n$ is a related value given in the first column of this table. The appropriate rounded value of the greatest prime $q_k$ in $S_n$ is given in the third column (related exact values are given in Table 3), while in the next column it is written the values $n - m$, where $m$ are related indices such that $q_k = S_m$. In the fifth column of Table 1 we present the corresponding values of $M_k$ obtained as solutions of the equation (56) in Theorem 4.4. The refined upper bound $M_k^{(1)}$ and the upper bound $M_k^{(u)}$ of $M_k$ given in Conjectures 5.13 and 5.7, respectively, are given in the next two columns of Table 1. Notice that the data from the last two columns of this table are closely related to Conjecture 4.6 and Conjecture 4.9, respectively.

For example, a computation gives the following exact values:
\[ q_{59129} = S_{849995} = 22420773979207, q_{62297} = S_{899999} = 25235697805141, q_{2707378} = 99262810294692679, q_{5212720} = S_{10^8} = 411680592327546713. \]
Recall that $\pi(x)$ denotes the number of primes less or equal to $x$. Then Table 2 present the quotients $\frac{\pi(n)}{\log n}$ and $\frac{\pi(n)}{\log(\pi(n))}$.

Table 2. Distribution of primes in the sequence $(S_n)$ in the range $1 \leq n \leq 10^9 + 5 \cdot 10^8$

```
| $n$  | $\pi(n)$ | $\frac{\pi(n)}{\log n}$ | $\frac{\pi(n)}{\log(\pi(n))}$ |
|------|----------|--------------------------|-------------------------------|
| $10^2$ | 1059190  | 0.920990                | 0.958787                      |
| $10^3$ | 0.973993 | 1.011300                | 0.959903                      |
| $10^4$ | 0.961329 | 0.983409                | 0.960219                      |
| $10^5$ | 0.958673 | 0.934617                | 0.961311                      |
| $10^6$ | 0.958673 | 0.934617                | 0.961311                      |
| $5 \cdot 10^6$ | 0.958673 | 0.934617                | 0.961311                      |
```

Notice that from Table 1 we see that $M_k^{(l)}$ is probably the upper bound of $M_k$ for $n > 5 \cdot 10^7$ (Conjecture 5.13) which is better estimate than $M_k^{(u)}$ (Conjecture 5.7).

The values of first three columns of Table 3 are defined in the same way as these of Table 1 (with exact values of $q_k$), and the related values of ratios $q_k/(2k^2 \log^3 k)$ are given in fourth column of this table. The asymptotic relation (87) of Corollary 5.10 shows that
it can be of interest to analyze the sequence \((Q_k)_{k=2}^{\infty}\) whose \(k\)th term is defined by the equality
\[
q_k = 2k^2 \log^3 k + 2Q_k k^2 (\log^2 k)(\log \log k), \quad k = 2, 3, \ldots ,
\]
or equivalently,
\[
\frac{q_k}{2k^2 \log^3 k} = 1 + Q_k \cdot \frac{\log \log k}{\log k}, \quad k = 2, 3, \ldots ,
\]
We also consider two similar sequences \((Q'_k)\) and \((Q''_k)\) which are closely related to Corollary 4.7 and Theorem 4.4, respectively (cf. Remark 4.11), and they are defined as
\[
Q'_k = \frac{q_k - 2k^2 (\log^2 k)(\log k + \log \log k)}{2k^2 (\log^2 k) \log k}, \quad k = 2, 3, \ldots ,
\]
and
\[
Q''_k = \frac{q_k - 2(p_k)^2 \log k}{2k^2 (\log^2 k) \log k}, \quad k = 2, 3, \ldots ,
\]
Some values of these sequences are given in the last three columns of Table 3.

Table 3. Some “prime sums” \(q_k\)’s in the sequence \((S_n)\) with \(n \leq 10^9 + 5 \cdot 10^8\) and related values \(q_k/(2k^2 \log^3 k), q_k/(2m^2 \log m), Q_k, Q'_k\) and \(Q''_k\)

| \(n\) | \(k := \pi_n\) | \(q_k\) | \(\frac{q_k}{2k^2 \log^3 k}\) | \(\frac{q_k}{2m^2 \log m}\) | \(Q_k\) | \(Q'_k\) | \(Q''_k\) |
|------|----------------|-------|---------------------|---------------------|------|------|------|
| 10   | 5              | 281   | 1.34807             | 1.47352             | 2.35436 | 0.17772 | -1.76013 |
| 100  | 23             | 107934| 3.30944             | 1.19829             | 6.36347 | 5.33647 | 5.44583 |
| 1000 | 141            | 15501706 | 3.21679            | 1.17689             | 6.86016 | 5.86016 | 5.77437 |
| 10000| 8350           | 264074170741 | 2.58273  | 1.14710             | 6.49405 | 5.49405 | 5.28293 |
| 20000| 15504          | 116374522657 | 2.56968  | 1.14347             | 6.68238 | 5.68238 | 5.44027 |
| 30000| 22595          | 2591079720139 | 2.61956  | 1.14145             | 7.03697 | 6.03697 | 5.79201 |
| 40000| 29495          | 470419172003 | 2.56741  | 1.14004             | 6.91365 | 5.91365 | 5.66455 |
| 50000| 36302          | 747253338077 | 2.51877  | 1.13867             | 6.77736 | 5.77736 | 5.51158 |
| 60000| 43119          | 10901967324637 | 2.46956  | 1.13810             | 6.62021 | 5.62021 | 5.35079 |
| 70000| 49834          | 1500269948023 | 2.43548  | 1.13737             | 6.51781 | 5.51781 | 5.24772 |
| 80000| 56419          | 1977612132971 | 2.41801  | 1.13675             | 6.48149 | 5.48149 | 5.2025 |
| 90000| 62770          | 25235697805141 | 2.41648  | 1.13621             | 6.51155 | 5.51155 | 5.22977 |
| 10^6 | 69251          | 31380813002879 | 2.40495  | 1.13572             | 6.30126 | 5.30126 | 5.02105 |
| 4 \cdot 10^6| 252028 | 459524557423241 | 2.24844  | 1.12966             | 6.15989 | 5.15989 | 4.84045 |
| 5 \cdot 10^6| 310756 | 870522520170287 | 2.22830  | 1.12873             | 6.12200 | 5.12200 | 4.79728 |
| 10^7 | 594851         | 3296595208696199 | 2.181921 | 1.12593             | 6.07346 | 5.07346 | 4.7374 |
| 5 \cdot 10^7| 2707378 | 99262810294692679 | 2.083831 | 1.11987             | 5.95575 | 4.95575 | 4.89396 |
| 7 \cdot 10^7| 3720648 | 198036666738658321 | 2.065440 | 1.11868             | 5.93361 | 4.93361 | 4.85144 |
| 8 \cdot 10^7| 4220531 | 26046364887226043 | 2.059225 | 1.11821             | 5.93009 | 4.93009 | 4.85643 |
| 10^8 | 5212720 | 41168059237546713 | 2.047463 | 1.11744             | 5.91551 | 4.91551 | 4.53769 |
| 2 \cdot 10^8| 10047823 | 1705122556732581169 | 2.014906 | 1.11511             | 5.88554 | 4.88554 | 4.64973 |
| 3 \cdot 10^8| 14763858 | 39132741026057161 | 1.995499 | 1.11379             | 5.86106 | 4.86106 | 4.64806 |
| 4 \cdot 10^8| 19404439 | 7053651472078073839 | 1.982108 | 1.11287             | 5.84373 | 4.84373 | 4.44675 |
| 5 \cdot 10^8| 23985388 | 1113847945180255153 | 1.972857 | 1.11217             | 5.83583 | 4.83583 | 4.43625 |
| 7 \cdot 10^8| 33031264 | 2217740160508699829 | 1.958466 | 1.11113             | 5.81944 | 4.81944 | 4.41565 |
| 10^9 | 46388006 | 4600776423412508181 | 1.943427 | 1.11005             | 5.80097 | 4.80097 | 4.39275 |

\[10^9 + 5 \cdot 10^8\] 68259534 105428905479616558423 1.927428 | 1.10885             | 5.78377 | 4.78377 | 4.37116 |

It is easy to prove the following result.

**Proposition 6.1.** Let \((Q_k), (Q'_k)\) and \((Q''_k)\) be the sequences defined by (99), (100) and (101), respectively. Then
\[
\lim_{k \to \infty} (Q_k - Q'_k) = \lim_{k \to \infty} (Q'_k - Q''_k) = 0.
\]

We also propose the following conjecture.
Conjecture 6.2. The all sequences \((Q_k), (Q'_k)\) and \((Q''_k)\) converge to 1.

Remark 6.3. In view of the above conjecture, it can be of interest to consider the sequence \((Q''_k)\) defined as

\[
Q''_k = \frac{q_k - 2k^2 (\log^2 k)(\log k + \log \log k)}{2k^2 (\log^2 k) \log \log k}, \quad k = 2, 3, \ldots ,
\]

The values of \(Q''_k\) for \(s = 3, 5, 6, 7, 8, 9\) are equals to 19.961, 15.329, 14.524, 13.809, 13.621, 13.069, respectively.

Remark 6.4. The values \(V_k := q_k/(2m^2 \log m) = S_m/(2m^2 \log m)\) presented in the fourth column of Table 3 are in fact terms of the sequence \(t_n := S_n/(2n^2 \log n)\) with \(n = 2, 3, \ldots\), which is decreasing under Conjecture 5.3 on the range \(\{n \in \mathbb{N} : n \geq 1100\}\). Accordingly, under Conjectures 4.6 and 5.3 and the fact that \(q_{151} = S_{1100} = 19949537\), we immediately get

\[
V_k = t_m \leq t_{\lfloor k \log k \rfloor + 1} < t_{\lfloor k \log k \rfloor} := M_k^{(l)} \quad \text{for all} \ k \geq 151,
\]

that is,

\[
(102) \quad V_k < M_k^{(l)} \quad \text{for all} \ k \geq 151,
\]

where \(M_k^{(l)}\) are approximative values for \(M_k\) given by (89) and presented in Table 1.

Moreover, the comparison of values of \(M_k\) with those of \(V_k\) from Tables 1 and 2, respectively, leads to the following conjecture.

Conjecture 6.5. Let \((M_k)_{k=2}^{\infty}\) be the sequence defined by (56) of Theorem 4.4, and let \(m(k) = m\) be defined as \(S_m = q_k\). Then

\[
(103) \quad M_k < \frac{q_k}{2m^2 \log m} \quad \text{for all} \ k \geq 4 \times 10^6.
\]

Consequently, we obtain the following “weak version” of Conjecture 6.5.

Corollary 6.6. Let \((M_k)_{k=2}^{\infty}\) be the sequence defined by (56) of Theorem 4.4, and let \(m(k) = m\) be defined as \(S_m = q_k\). Then under Conjectures 4.6, 5.3 and 6.5 we have

\[
M_k < t_{\lfloor k \log k \rfloor + 1} := \frac{S_{\lfloor k \log k \rfloor + 1}}{2(\lfloor k \log k \rfloor + 1)^2 \log(\lfloor k \log k \rfloor + 1)} \quad \text{for all} \ k \geq 4 \times 10^6.
\]

Proof. Combining Conjectures 4.6, 5.3 and 6.5, we find that for all \(k \geq 4 \times 10^6\) with \(q_k = S_m\)

\[
M_k < \frac{q_k}{2m^2 \log m} = \frac{S_m}{2m^2 \log m} = t_m \leq t_{\lfloor k \log k \rfloor + 1} = \frac{S_{\lfloor k \log k \rfloor + 1}}{2(\lfloor k \log k \rfloor + 1)^2 \log(\lfloor k \log k \rfloor + 1)},
\]

as desired. \qed

Remark 6.7. The ratios \(L_k := q_k/(2k^2 \log^2 k(\log k + \log \log k))\) are closely related to Corollary 4.7. Of course, the values \(L_k\) are small than the related values \(q_k/(2k^2 \log^3 k)\) presented in the fourth column of Table 3. For example, \(L_k\) is equal to 1.762999, 1.696920 for \(k = 2707378, 19404439\), respectively. However, the sequence \((L_k)\) slowly tends to 1 as \(k\) grows. This is directly connected with the fact that the sequence \((k \log k/m(k))\) converges very slowly to 1 as \(k\) grows (see the eighth column of Table 1).
Remark 6.8. A good approximation from Remark 4.11 arising from the last column of Table 1 can be written as

\[
\sqrt{k \log k} \approx \frac{2m^2 \sqrt{m \log m}}{S_m},
\]

where \(q_k = S_m\). The approximation (104) allows us for given \(n\) to determine the index \(k = k(n)\) such that the prime sum \(q_k\) is “very close” to \(S_n\), especially, for each \(n \geq 4 \times 10^6\) (i.e., for \(k \geq 252028\)), assuming that Conjecture 4.9 is true, then \(q_{k(n)} < S_n\). Accordingly, for given \(n\) we assume that \(k_0(n) = \lfloor x_0 \rfloor\), where \(x_0 = x_0(n)\) is a root of the equation

\[
\sqrt{x \log x} = \frac{2n^2 \sqrt{n \log n}}{S_n}.
\]

For some values \(n\) from Table 1 Table 4 presents the exact largest values \(k(n)\) such that \(q_{k(n)} \leq S_n\) (these values are in fact, given in the second column of Table 1) and related differences \(k(n) - k_0(n)\).

**Table 4.** The values \(k = k_0(n)\) and \(k(n) - k_0(n) = k - k_0\) for some values \(n \leq 10^9\)

| \(n\)  | \(k := \pi_n\) | \(k - k_0\) | \(n\)  | \(k\) | \(k - k_0\) | \(n\)  | \(k\) | \(k - k_0\) | \(n\)  | \(k\) | \(k - k_0\) |
|-------|----------------|-----------|-------|------|-----------|-------|------|-----------|-------|------|-----------|
| 10^4  | 1098           | -33       | 10^6  | 8350 | -59       | 10^9  | 141  | -3        | 10^7  | 59485 | 1469      |
| 3 \cdot 10^5 | 22595       | -338     | 4 \cdot 10^5 | 29495 | -371     | 5 \cdot 10^5 | 36302 | -812    | 6 \cdot 10^5 | 49834 | -177     | 8 \cdot 10^5 | 56419 | -627    |
| 9 \cdot 10^5 | 62770       | -308     | 10^{6} | 69251 | -283     | 2 \cdot 10^6 | 131841 | -368   | 3 \cdot 10^{6} | 252028 | 5       | 5 \cdot 10^6 | 310756 | 404     |
| 10^{7} | 192655        | -110     | 4 \cdot 10^{6} | 2707378 | 14644   | 6 \cdot 10^7 | 3216515 | 18621  | 7 \cdot 10^{7} | 1141478 | 4638    | 3 \cdot 10^7 | 1671839 | 7462    |
| 10^{8} | 5212720       | 32606    | 10^{8} | 5073536 | 35030   | 10^{8} + 2 \cdot 10^7 | 6191655 | 37303    | 10^{8} + 3 \cdot 10^7 | 6679364 | 41059   | 10^{8} + 4 \cdot 10^7 | 7165567 | 45196    |
| 10^{9} | 8132623       | 53221    | 2 \cdot 10^{8} | 10047823 | 67743   | 3 \cdot 10^8 | 14763858 | 107704  | 4 \cdot 10^8 | 19404439 | 149159  | 5 \cdot 10^8 | 23985388 | 186542   |
| 8 \cdot 10^8 | 37508452   | 309262   | 9 \cdot 10^8 | 41960555 | 3517799 | 10^9 | 46388006 | 392660 |

In view of the above considerations and computational results given in Table 4, we propose the following conjecture.

**Conjecture 6.9.** Let \(n \geq 4 \times 10^6\) be a positive integer, and let \(x_0(n)\) be a real root of the equation

\[
\sqrt{x \log x} = \frac{2n^2 \sqrt{n \log n}}{S_n}.
\]

Then the set \(\{S_1, S_2, \ldots, S_n\}\) contains at least \(\lfloor x_0(n) \rfloor\) primes.

The inequality on right hand side of (37) of Proposition 3.12 immediately gives the following weak version of Conjecture 6.9.

**Conjecture 6.10.** Let \(n \geq 4 \times 10^6\) be a positive integer, and let \(y_0(n)\) be a real root of the equation

\[
\left(1 + \frac{\log 2 + \log \log(2n)}{\log n}\right) \sqrt{y \log y} = \sqrt{n}.
\]

Then the set \(\{S_1, S_2, \ldots, S_n\}\) contains at least \(\lfloor y_0(n) \rfloor\) primes.
It can be also of interest to compare the values \( k_0(n) \) and \( k_1(n) := \lfloor y_0(n) \rfloor \) with the values \( k_2(n) := \lfloor z_0(n) \rfloor \), where \( z_0(n) \) is a real root of the equation

\[
x \log x = n.
\]

**Corollary 6.11.** Let \( n \geq 4 \times 10^6 \) be a positive integer. Then under Conjecture 6.10 and its notations, the sequence \( (S_i)_{i=1}^{\infty} \) contains at least \( k_0(n) := \lfloor y_0(n) \rfloor \) primes which are less than \( 2n^2(\log n + \log \log(2n) + \log 2) \). In other words,

\[
q_{k_0(n)} < 2n^2(\log n + \log \log(2n) + \log 2).
\]

**Proof.** The assertion immediately follows from Conjecture 6.10 and the right hand side of the inequalities (37) from Proposition 3.12. \( \square \)

The values \( k_0(n) \) (derived from Table 4 as the differences \( k_0(n) = k(n) - (k(n) - k_0(n)) \)), \( k_1(n) \) and \( k_2(n) \) concerning the values of \( n \) from Table 4, are presented in Table 5.

**Table 5.** The values \( k_i(n), i = 0, 1, 2 \), the ratios \( \delta_j(n) := (k(n) - k_j(n))/k(n) \) with \( j = 1, 2 \), the ratios \( \eta(n) := k_0(n)/\sqrt{k_1(n)k_2(n)} \) and \( \xi(n) := k(n)/\sqrt{k_1(n)k_2(n)} \)

| \( n \) | \( k_0(n) \) | \( \delta_0(n) \) | \( k_1(n) \) | \( \delta_1(n) \) | \( k_2(n) \) | \( \delta_2(n) \) | \( \eta(n) \) | \( \xi(n) \) |
|---|---|---|---|---|---|---|---|---|
| 10 | 2 | 0.60000 | 2 | 0.60000 | 5 | 1.00000 | 0.63246 | 1.58114 |
| \( 10^2 \) | 22 | 0.04348 | 15 | 0.34783 | 29 | -0.26087 | 1.05482 | 1.10277 |
| \( 10^3 \) | 144 | -0.00213 | 109 | 0.22695 | 190 | -0.34755 | 1.00063 | 0.97978 |
| \( 10^4 \) | 1151 | -0.03005 | 846 | 0.22951 | 1382 | -0.25865 | 1.04598 | 1.01546 |
| \( 10^5 \) | 8409 | -0.00771 | 6928 | 0.17030 | 10770 | -0.28982 | 0.97345 | 0.96666 |
| \( 5 \times 10^5 \) | 37114 | -0.02237 | 30816 | 0.15112 | 46521 | -0.28150 | 0.98022 | 0.95878 |
| \( 10^6 \) | 69534 | -0.00409 | 58857 | 0.15009 | 87845 | -0.26850 | 0.96703 | 0.96309 |
| \( 4 \times 10^6 \) | 252032 | 0.00002 | 216103 | 0.14254 | 315878 | -0.25334 | 0.96461 | 0.96463 |
| \( 5 \times 10^6 \) | 310352 | 0.00120 | 266622 | 0.14202 | 388499 | -0.25017 | 0.96430 | 0.96555 |
| \( 10^7 \) | 593382 | 0.00247 | 512630 | 0.13822 | 739955 | -0.24393 | 0.96345 | 0.96584 |
| \( 5 \times 10^7 \) | 2692734 | 0.00541 | 2353142 | 0.13084 | 3329279 | -0.22971 | 0.96204 | 0.96728 |
| \( 10^8 \) | 5180024 | 0.0063 | 4546674 | 0.12778 | 6382029 | -0.22432 | 0.96162 | 0.96769 |
| \( 5 \times 10^8 \) | 23798846 | 0.00778 | 21080809 | 0.12110 | 29093410 | -0.21296 | 0.96098 | 0.96851 |
| \( 10^9 \) | 45995346 | 0.00846 | 40886757 | 0.11859 | 56048389 | -0.20825 | 0.96082 | 0.96902 |

The last column of Table 5 suggests \( \xi(n) < 1 \) for all \( n \geq 10^5 \), which is obviously equivalent with the following conjecture.

**Conjecture 6.12.** Let \( n \geq 10^5 \) be a positive integer, and let \( k_1(n) = \lfloor y_0(n) \rfloor \) and \( k_2(n) = \lfloor z_0(n) \rfloor \), where \( y_0(n) \) and \( z_0(n) \) are real roots of the equations (107) and (108), respectively. Then the set \( \{S_1, S_2, \ldots, S_n\} \) contains less than \( \lfloor \sqrt{k_1(n) \cdot k_2(n)} \rfloor \) primes.

As an immediate consequence, we obtain the following statement.

**Corollary 6.13.** Let \( n \geq 10^5 \) be a positive integer. Then under Conjecture 6.12 and its notations the sequence \( (S_i)_{i=1}^{\infty} \) contains at most \( \lfloor \sqrt{k_1(n) \cdot k_2(n)} \rfloor \) primes which are less than \( 2n^2 \log n \). In other words,

\[
q_{\lfloor \sqrt{k_1(n) \cdot k_2(n)} \rfloor} > 2n^2 \log n.
\]

**Proof.** The assertion immediately follows from Conjecture 6.12 and left hand side of the inequalities of (37) from Proposition 3.12. \( \square \)

**Remark 6.14.** Conjecture 6.6 may be considered as the “prime sums analogue” of the well known fact that \( p_k \geq k \log k \) for all \( k \geq 3 \), where \( p_k \) is the \( k \)th prime (see e.g., [38], p. 69)}
Remark 6.15. The approximation (105) can be written as

\[
\sqrt{\frac{m}{k \log k}} \approx \frac{S_m}{2m^2 \log m},
\]

which in view of Conjecture 5.3 asserts that the sequence \((m/(k \log k))_{k=1}^\infty\) is decreasing for a fixed large integer \(k_1\).

On the other hand, if we write the estimate (111) in the form

\[
\sqrt{\frac{m \log^2 m}{k \log k}} \approx \frac{S_m}{2m^2},
\]

then Proposition 5.1 suggests that the sequence \((m \log^2 m/(k \log k))_{k=1}^\infty\) is increasing for some fixed large integer \(k_2\).

7. THE STRONGER FORM OF CONJECTURE 3.3

Around 1800, young C. F. Gauss conjectured that for large \(n\) the number of primes not exceeding \(n\) is nearly

\[\text{li}(n) := \int_2^n \frac{dt}{\log t}.\]

Heuristic and computational arguments give the impression that Restricted Prime Number Theorem (RPNT) for the sequence \((S_n)\) (i.e., Conjecture 3.3) probably holds in its stronger form which in fact presents the well known form of Prime Number Theorem (PNT) for primes (see e.g., [18, Chapter 12]). Accordingly, we propose the following conjecture.

Conjecture 7.1. Let \(\pi_{(S_k)}(S_n) = \pi_n\) be the number of primes \(p\) in the sequence \((S_k)\) such that \(p = S_i\) for some \(i\) with \(1 \leq i \leq n\). Then

\[
\pi_n = \text{li}(n) + R(n),
\]

where

\[
\text{li}(n) := \int_2^n \frac{dt}{\log t} = \frac{n}{\log n} + O\left(\frac{1}{\log^2 n}\right) \text{ as } n \to \infty.
\]

is the logarithmic integral and

\[
R(n) \ll ne^{-C\delta(n)} \text{ with } \delta(n) := (\log n)^{3/5}(\log \log n)^{-1/5}.
\]

Assuming the above conjecture, and following related “PNT result” of A. Ivić and J.-M. De Koninck [20, Theorem 9.1] (see also [19, Theorem]), it can be proved the following result.

Corollary 7.2. Under the truth and notations of Conjecture 7.1, we have

\[
\sum_{i=1}^n \frac{1}{\pi_i} = \frac{1}{2} \log^2 n + O(\log n) \text{ as } n \to \infty.
\]

Similarly, using Conjecture 7.1 it can be proved the following result.

Corollary 7.3. Let \(q_k\) be the \(k\)th prime in \((S_n)\). Then under Conjecture 7.1,

\[
\sum_{k=1}^n \frac{k \log^2 k}{q_k} = \log \log n + o\left(\frac{1}{\log n}\right).
\]
Finally, we propose the following conjecture.

**Conjecture 7.4** (Chebyshev inequalities for \((S_n)\)). There exist positive constants \(c_1\), \(c_2\) and a positive positive integer \(n_0\) such that

\[
\frac{c_1 n}{\log n} \leq \pi_S(n)(S_n) \leq \frac{c_2 n}{\log n} \quad \text{for all } n > n_0.
\]

**Remark 7.5.** We also believe that for the sequence \((S_n)\) are valid the analogues of some other classical results and conjectures closely related to the Prime Number Theorem.

**Remark 7.6.** Numerous computational results involving sums of the first \(n\) primes (the Sloane’s sequence A007504 sequence here denoted as \(S_n^\prime\)) and certain their curious arithmetical properties are presented in Sloane’s sequences A051838 (numbers \(n\) such that sum of first \(n\) primes divides product of first \(n\) primes), A116536, A067110, A067111, A045345, A114216 (sum of first \(n\) primes divided by maximal power of 2), A024011 (numbers \(n\) such that \(n\)th prime divides sum of first \(n\) primes), A036439 \(a(n) = 2+\)

- the sum of the first \(n - 1\) primes), A014284 (partial sums of primes, if \(1\) is regarded as a prime; \(1, 3, 6, 11, 18, 29, \ldots\)), A134125 (integral quotients of partial sums of primes divided by the number of summations; \(5, 5, 7, 11, 16, 107, \ldots\)), A134126 (indices \(k\) such that the \((k+1)\)th partial sum of primes divided by \(k\) is integer; \(1, 2, 4, 7, 10, 50, 130, \ldots\)), A134127 (largest prime in the partial sums of primes in A134125 which have integer averages), A134129 (prime partial sums \(A007504(k+1)\) such that \(A007504(k+1)/k\) is integer; \(5, 10, 28, 77, 160, \ldots\)), A077023, A033997, A071089, A083186 (sum of first \(n\) primes whose indices are primes), A166448 (sum of first \(n\) primes minus next prime), A196527, A065595 (square of first \(n\) primes minus sum of squares of first \(n\) primes), A165906 (sum of first \(n\) primes divided by the \(n\)th prime), A061568 (number of primes \(\leq\) sum of first \(n\) primes), A066039 (largest prime less than or equal to the sum of first \(n\) primes), A077022, A110997, A112997 (sum of first \(n\) primes minus sum of their indices), A156778 (sum of first \(n\) primes multiplied by \(n/2\)), A167214 (sum of first \(n\) primes multiplied by \(n\)), A038346 (sum of first \(n\) primes \(\equiv 1(\mod 4)\)), A038347 (sum of first \(n\) primes \(\equiv 3(\mod 4)\)), A054972 (product of sum of first \(n\) primes and product of first \(n\) primes), A072476, A076570 (greatest prime divisor of sum of first \(n\) primes), A076873 (smallest prime not less than sum of first \(n\) primes), A077354 (sum of second string of \(n\) primes-sum of first \(n\) primes, or \(2n\)th partial sum of primes; this is in fact our sequence \((S_n)\)), A110996, A123119 (number of digits in sum of first \(n\) primes), A189072 (semiprimes in the sum of first \(n\) primes), A196528, A022094 (sum of first \(p_n\) primes, where \(p_n\) is the \(n\)th prime), A024447, A121756, A143121 (triangle read by rows, \(T(n, k) = \sum_{j=k}^n p_j\), \(1 \leq k \leq n\)), A171842 (partial sum of smallest prime \(\geq n\)), A118219, A131740 (sum of \(n\) successive primes after \(n\)th prime), A143215 (the sequence whose \(n\)th term is \(p_n \cdot S_n' = p_n \cdot \sum_{i=1}^n p_i\)), A161436 (sum of all primes from \(n\)th prime to \((2n-1)\)th prime), A161490, A013918 (numbers \(n\) such that \(n\) is prime and is equal to the sum of the first \(k\) primes for some \(k\); \(2, 5, 17, 41, 197, 281, 7699, 8893, \ldots\)) etc.

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