Observational Viability and Stability of Nonlocal Cosmology

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ABSTRACT

We show that the nonlocal gravity models, proposed to explain current cosmic acceleration without dark energy, pass two major tests: First, they can be defined so as not to alter the, observationally correct, general relativity predictions for gravitationally bound systems. Second, they are stable, ghost-free, with no additional excitations beyond those of general relativity. In this they differ from their, ghostful, localized versions. The systems’ initial value constraints are the same as in general relativity, and our nonlocal modifications never convert the original gravitons into ghosts.

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1 Introduction

Explaining the current phase of cosmic acceleration is an ongoing challenge \[1\]. The data are consistent with general relativity operating on a critical energy density whose current composition is about 70% cosmological constant plus about 30% nonrelativistic, and small amounts of relativistic, matter \[2, \, 3\]. However, there is no good explanation for why the cosmological constant should be so small, nor why it should recently have come into dominance \[4\]. Scalar potential models \[5, \, 6\] can be devised to reproduce the observed expansion history \[7, \, 8\] but they must be fine tuned and are difficult to motivate. Quantum effects from a very light scalar have also been suggested \[9\].

Various modifications of general relativity that generalize its Lagrangian from $R$ to $f(R)$ \[10, \, 11\] represent the only local, metric-based, generally coordinate invariant and stable modification of gravity \[12\]. But the sole model within this class that exactly reproduces the $\Lambda$CDM expansion history is general relativity with $f(R) = R - 2\Lambda$ \[13\].

More modification freedom is available if locality is abandoned \[14\], but this novel territory raises the worry of new degrees of freedom (DoF), possibly of instability-negative energy \[15\]. While we do not believe such models to be fundamental, even if observationally viable in some regime of validity, they must still face the above problems of principle, as well as more phenomenological ones. Their origin would be the gravitational corrections that grew non-perturbatively during the primordial inflation epoch \[7\], a conjecture that, while plausible \[16, \, 17\], is as yet unverified \[18\]. Independent of their ultimate origin, these models have been proposed and studied purely phenomenologically. Ours \[19\] adds the nonlocal piece

$$\Delta \mathcal{L} \equiv \frac{1}{16\pi G} R \sqrt{-g} \times f \left( \frac{1}{\Box} R \right),$$

(1)

to the Einstein term $R \sqrt{-g}/16\pi G$. Our signature is $(- + + +)$, with the convention $R_{\mu\nu} \sim +\partial_\rho \Gamma^\rho_{\mu\nu}$. The inverse of the (scalar) d’Alembertian $\Box \equiv (-g)^{-\frac{1}{2}} \partial_\mu [\sqrt{-g} g^{\mu\nu} \partial_\nu]$ is the retarded one, with vanishing 0th and 1st time derivatives at the initial time \[19\]. In addition to simplicity, the great advantage of this class of models is to provide a natural delay for the onset of cosmic acceleration: because the Ricci scalar $\tilde{R}$ vanishes during radiation dominance, $\Box^{-1} \tilde{R}$ cannot begin to grow until after the onset of matter dominance; thereafter, because of the propagator, its growth becomes logarithmic.
The model’s defining equations \[19\] take the form, \(G_{\mu\nu} + \Delta G_{\mu\nu} = 8\pi GT_{\mu\nu}\), with

\[
\Delta G_{\mu\nu} = \left[ G_{\mu\nu} + g_{\mu\nu} \nabla - D_{\mu} D_{\nu} \right] \left\{ f \left( \frac{1}{\Box} R \right) + \frac{1}{\Box} \left[ R f' \left( \frac{1}{\Box} R \right) \right] \right\} \\
+ \left[ \delta_{\mu}^{(\rho} \delta_{\nu)}^{\sigma} - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \right] \partial_{\rho} \left( \frac{1}{\Box} R \right) \partial_{\sigma} \left( \frac{1}{\Box} \left[ R f' \left( \frac{1}{\Box} R \right) \right] \right) .
\]

(2)

The form of the nonlocal distortion function \(f(X)\) can, unlike the local models \[13\], be chosen to reproduce the ΛCDM background cosmology exactly \[20, 21, 22\]. Indeed, there is a simple analytic form for \(f(X)\), effectively equivalent to the numerical solution \[21\]

\[
f(X) \approx 0.245 \left[ \tanh \left( 0.350Y + 0.032Y^2 + 0.003Y^3 \right) - 1 \right] , \quad Y \equiv X + 16.5 .
\]

(3)

Like all modified gravity theories, nonlocal cosmology can be differentiated from general relativity with dark energy by how it alters results in the solar system and how it affects structure formation \[23\]. Koivisto has argued that there are no conflicts with solar system constraints \[20\]. A recent study of structure formation by Park and Dodelson revealed deviations from general relativity in the 10%-30% range, which are interesting because they are not currently excluded and should be observable by the next generation of large scale structure surveys \[24\]. While we await these observations, it is worth examining the theoretical consistency of nonlocal cosmology in its own right. In particular, how does the model behave for gravitationally bound systems, does it possess extra degrees of freedom and is it stable? Those are the questions we will study in sections \[2, 3\] and \[4\] respectively.

2 Screening: Absence of Effects on Bound Systems

In this section we discuss the issue of screening in modified theories; \(f(R)\) models suffer from the major problem that \(R\) typically has the same sign for cosmology, where we want big effects to explain the acceleration data, and for the solar system, where significant deviations from general relativity are excluded by the data. This has prompted the development of elaborate “chameleon mechanisms” in which the extra scalar degree of freedom present
in $f(R)$ models is light in cosmological settings and heavy inside the solar system [23]. Nonlocal cosmology differs from $f(R)$ models in two crucial ways: there are no extra degrees of freedom to mediate new forces; and the propagator $\Box^{-1}$ acting on $R$ allows us to so define the nonlocal distortion function so that there are no changes at all from general relativity in a gravitationally bound system, yet without affecting the model’s predictions for cosmology. The first point will be demonstrated in section 4; it is the second point which concerns us here.

The key fact is that the scalar d’Alembertian $\Box \equiv (-g)^{-\frac{1}{2}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$ has opposite signs when acting on functions of time than on functions of space. In the background cosmology, and perturbations about it, the time dependence of the Ricci scalar is stronger than its space-dependence. This means that $\Box^{-1}R$ is typically negative for cosmology. Indeed, reproducing the $\Lambda$CDM expansion history fixes the nonlocal distortion function $f(X)$ only for negative $X$ [21].

While gravitationally bound systems are not always static, their space-dependence is generally stronger than that on time. That means $\Box^{-1}R$ is positive inside a gravitationally bound system. Further, reproducing the $\Lambda$CDM expansion history requires $f(0) = 0$ [21]. To completely annul all corrections inside gravitationally bound systems it suffices to define $f(X) = 0$ for all $X > 0$. Hence there is a very simple way for nonlocal models to completely screen inside the solar system, the galaxy, or any other gravitationally bound system, all without affecting the model’s desired behavior for cosmology.

We should comment that small values of $f(X)$ for $X > 0$ are quite reasonable if one accepts our view that nonlocal cosmology is the gravitational vacuum polarization induced by the vast ensemble of infrared gravitons created by primordial inflation. However, from the purely phenomenological perspective of model building, it is worth noting that the actual data from gravitationally bound systems do not require anything like the severe restriction of $f(X) = 0$ for all $X > 0$. If the characteristic mass of a system is $M$ and we observe at distance $r$, then $\Box^{-1}R \sim GM/c^2r$. This is never larger than about $10^{-6}$ for the solar system, where observational constraints are tightest. Under the assumption that $f(X)$ is analytic, Koivisto has shown that the best solar system constraint only fixes the first derivative to the range $-5.8 \times 10^{-6} < f’(0) < 5.7 \times 10^{-6}$, and fixes none of the higher derivatives [20]. This bound is easily met by the simple analytic form (3) which was found to reproduce the $\Lambda$CDM expansion history [21]. Even giant spiral
galaxies do not have larger values of $\square^{-1}R$ — it is about $10^{-7}$ at the Sun’s orbit for our own galaxy — and the data are of course much less restrictive. The largest $\square^{-1}R$ can get for an observable structure is about unity, on the surface of a neutron star. Although high quality observations of pulsar timing have been made, their interpretation as tests of general relativity on the neutron star surface is complicated by continuing uncertainty about the nuclear equation of state. Values of $f(X)$ as large as $1/10$ at $X \approx 1$ are probably consistent, while observation provides no constraint at all on $f(X)$ for larger values of $X$.

3 Local versus Nonlocal Formulations

Soon after our nonlocal model [19] appeared, a “localized” version, based on two additional scalars, was proposed [27, 28]. Briefly, it replaced the nonlocal terms in (1) by

$$Rf\left(\frac{1}{\square}R\right)\sqrt{-g} \rightarrow Rf(\Phi)\sqrt{-g} + \Psi(\phi R)\sqrt{-g} .$$  (4)

The local mechanism then, relied on two new scalars: $\Psi$ is a Lagrange multiplier that enforces $\phi = \square^{-1}R$ to recover the original nonlinearity. As long as one is interested in the inhomogeneous response of $\square^{-1}R$ to a given source of stress-energy, and how that ultimately affects gravity, there is no problem employing this localized version of the model. For example, Koivisto’s solar system constraint was derived using it [20]. However, the localized version has severe ghost problems when one considers the homogeneous DoF associated with the initial value data of the two scalars, as we now show.

Consider first just the scalar, two-field sector. After a partial integration, the off-diagonal term $\Psi \square \phi$ is just the difference of two diagonal free scalar Lagrangians, one of which is therefore a ghost:

$$-\partial_{\mu} \Psi \partial_{\nu} \phi g^{\mu\nu} = -\frac{1}{2} \partial_{\mu}(\Psi + \phi)\partial_{\nu}(\Psi + \phi)g^{\mu\nu} + \frac{1}{2} \partial_{\mu}(\Psi - \phi)\partial_{\nu}(\Psi - \phi)g^{\mu\nu} .$$  (5)

With our spacelike metric, the combination $(\Psi - \phi)$ has negative kinetic energy. (We thank G. Esposito-Farese for this observation.)

While we have established the ghost nature of the purely scalar sector, ours is really a three-field system; to include the graviton, one must first perform a conformal metric rescaling to the Einstein frame, as given in (17)
of [26]. As correctly stated in [26], this implies that the necessary condition for ghostlessness is
\[ 6f'(\Phi) > 1 + f(\Phi) - \Psi > 0. \] (6)

No matter what we assume about the nonlocal distortion function \( f \), condition (6) can never be met as long as the scalar \( \Psi \) is allowed to have arbitrary initial value data. The authors of ref. [26] actually concluded that the localized model can be ghost-free for a period of time, but this ignores the virulence of kinetic instabilities. There are so many excitations at large wave number that quantum fluctuations in \( \Psi \) would instantly result in violation of (6), no matter what classical mean was imposed. Note also that even if gravity had stabilized the ghost in (5), it could not have prevented \( (\Phi - \Psi) \) — and a corresponding part of the metric field — from developing rapid and phenomenologically unacceptable time dependence.

We now allay the worry that this disease also infects the original system. Clearly, (1) only yields (1) after discarding precisely the homogeneous scalar’s solutions through requiring that they, and their first time derivatives, vanish at the initial time; this precisely discards their DoF! The next worry that might arise is that perhaps these excitations could somehow appear in the original, nonlocal form. Here general relativity itself offers the prime example of how this danger is averted: the “Newtonian” third mode, after the two \( g_{ij}^{TT} \) gravitons, is indeed dangerous if dynamical — but is saved from propagating by general relativity’s constraint equations. This — identical — salvation of (1) will indeed be demonstrated in the next section.

### 4 Nonlocal Stability

We will proceed for concreteness in a particular, synchronous, gauge. There we will see that the nonlocal equations require the same initial data, subject to exactly the same constraints, as general relativity. We will also show that none of the DoF common to the nonlinear and general relativistic terms is ever converted to ghost stature by the nonlocal corrections, as will also be illustrated by a simple, gauge independent, linearized treatment. The section ends with a discussion of the extent to which our conclusions depend upon assuming retarded boundary conditions for \( \Box^{-1} \), and on the form (3) for the nonlocal distortion function.
4.1 Synchronous gauge

Synchronous gauge is the coordinate frame of a system of timelike, freely falling observers who are released from a spacelike surface with zero initial relative velocities [29]

\[ ds^2 = -dt^2 + h_{ij}(t, \vec{x})dx^i dx^j . \]  

(7)

The basic analysis and conclusions should apply in any gauge, as we will see they do at linearized, kinematical level.

In synchronous gauge the covariant scalar d’Alembertian takes the form

\[ \Box = -\partial_t^2 - \frac{1}{2}h^{ij}\dot{h}_{ij} + \frac{1}{\sqrt{h}}\partial_t\left(\sqrt{h} h^{ij}\partial_j\right) . \]  

(8)

Here and henceforth, \( h^{ij} \) denotes the inverse of the spatial metric \( h_{ij} \), \( h \) stands for the determinant of \( h_{ij} \), and an overdot represents differentiation with respect to time. The various curvatures we require are

\[ R_{00} = -\frac{1}{2}h^{kl}\dddot{h}_{kl} + \frac{1}{4}h^{ik}h^{j\ell}\dot{h}_{ij}\dot{h}_{k\ell} , \]  

(9)

\[ R_{ij} = \frac{1}{2}\dddot{h}_{ij} + \frac{1}{4}h^{kl}\dddot{h}_{ij}\dot{h}_{k\ell} - \frac{1}{2}h^{kl}\dddot{h}_{ik}\dot{h}_{j\ell} + \frac{3}{2}R_{ij} , \]  

(10)

\[ R = h^{kl}\dddot{h}_{kl} + \frac{1}{4}h^{ij}h^{kl}\dddot{h}_{ij}\dot{h}_{k\ell} - \frac{3}{4}h^{ik}h^{j\ell}\dot{h}_{ij}\dot{h}_{k\ell} + 3R \]  

(11)

where \( 3R \) means, as usual, the intrinsic spatial curvature.

4.2 Initial value data and constraints

Let us first see that the nonlocal field equations (2) require the same initial value data as general relativity, namely, the values of the 3-metric and its first time derivative at \( t = 0 \): \( h_{ij}(0, \vec{x}) \) and \( \dot{h}_{ij}(0, \vec{x}) \). The retarded Green’s function associated with \( \Box^{-1} \) is defined by the differential equation

\[ \sqrt{h} \Box G[h](t, \vec{x}; t', \vec{x}') = \delta(t-t')\delta^3(\vec{x} - \vec{x}') , \]  

(12)

subject to retarded boundary conditions

\[ G[h](t, \vec{x}; t', \vec{x}') = 0 \quad \forall \ t' > t . \]  

(13)

\footnote{While this gauge has well-known problems with caustics, they are not relevant to our treatment.}
Even though we cannot solve equations (12-13) for an arbitrary 3-metric, their form clearly defines the Green’s function \( G[h] \) at time \( t \) using only the values of \( h_{ij} \) and its first time derivative for times less than or equal to \( t \).

Because \( \Box^{-1}R \) is the integral \( \int d^4x'G[h](t,\vec{x};t',\vec{x}')R(x') \), we need only consider the second time derivatives of the metric in \( R \); the first time derivatives and all spatial derivatives are shielded by the inverse differential operator. From expression (11) we see that these second time derivatives can be written in form

\[
R = \partial_t^2 \ln(h) + \frac{1}{4}(h^{ij}h^{k\ell} + h^{ik}h^{j\ell})\dot{h}_{ij}\dot{h}_{k\ell} + (3)R. \tag{14}
\]

Now use relation (8) to express second time derivatives in terms of the scalar d’Alembertian

\[
\partial_t^2 = -\Box - \frac{1}{2}h^{ij}\dot{h}_{ij}\partial_t + \frac{1}{\sqrt{h}}\partial_t\left(\sqrt{h}h^{ij}\partial_j\right). \tag{15}
\]

We can obviously combine relation (15) with (14) to conclude that

\[
R = -\Box \ln(h) + \frac{1}{4}(h^{ik}h^{j\ell} - h^{ij}h^{k\ell})\dot{h}_{ij}\dot{h}_{k\ell} + \frac{1}{2}h^{k\ell}(\dot{h}_{\ell i,j} + \dot{h}_{j\ell,i} - \dot{h}_{ij,\ell}) + \frac{1}{2}h^{k\ell}(\Gamma^k_{i,j,k} + \Gamma^k_{k,i,j} - \Gamma^k_{k\ell i}\Gamma^\ell_{ij} - \Gamma^k_{k\ell j} - \Gamma^k_{ki}\Gamma^\ell_{ij}). \tag{16}
\]

Here \( \Gamma^k_{i,j} \equiv \frac{1}{2}h^{k\ell}(h_{\ell i,j} + h_{j\ell,i} - h_{ij,\ell}) \) is the 3-space affinity, and commas denote partial differentiation.

With relations (12,13), equation (16) shows that \( \Box^{-1}R \) involves only the usual initial value data, \( h_{ij}(0,\vec{x}) \), and \( \dot{h}_{ij}(0,\vec{x}) \) of general relativity. That these initial value data are apportioned, also as in general relativity, between constrained fields and gravitational radiation modes is seen by examining the nonlocal corrections \( \Delta G_{00} \) and \( \Delta G_{0i} \) to the constraint equations. Note first from (13) that \( \Box^{-1} \) and its first time derivative both vanish at \( t = 0 \). Further, the nonlocal distortion function vanishes at \( t = 0 \). So we need only examine the two terms of (2) in which two covariant derivatives act upon \( f(\Box^{-1}R) + \Box^{-1}[Rf'(|\Box^{-1}R|)] \). It is easy to see that neither of the two combinations in the constraint equations contains a second time derivative:

\[
g_{00}\Box - D_0D_0 = \frac{1}{2}h^{k\ell}\dot{h}_{k\ell}\partial_t - \frac{1}{\sqrt{h}}\partial_t\left(\sqrt{h}h^{k\ell}\partial_\ell\right), \tag{17}
\]

\[
g_{0i}\Box - D_0D_i = -\partial_t\partial_i + \frac{1}{2}h^{k\ell}\dot{h}_{ik}\partial_\ell. \tag{18}
\]
Hence we conclude that the nonlocal corrections to the constraint equations
\[ t = 0 \implies \Delta G_{00} = 0 = \Delta G_{0i} \] (19)
vanish at \( t = 0 \). This completes the verification that the nonlocal model and
general relativity share the same initial data and constraints.

4.3 No ghosts

To see that there are no ghosts it suffices to examine the second derivative terms (still in synchronous gauge) of the dynamical equations, \( G_{ij} + \Delta G_{ij} = 8\pi GT_{ij} \). The second derivatives of \( h_{ij}(t, \vec{x}) \) in the Einstein tensor are, from (10-11),
\[ G_{ij} = \frac{1}{2} \ddot{h}_{ij} - \frac{1}{2} h_{ij} h^{kl} \ddot{h}_{kl} + O(\partial_t) . \] (20)
Of course it is only the first term, \( \frac{1}{2} \ddot{h}_{ij} \), that involves unconstrained fields; the second term represents completely constrained ones. Because general relativity has no ghosts, we need only check that the nonlocal corrections in (2) don’t change the sign of the \( \frac{1}{2} \ddot{h}_{ij} \) term in (20).

The work of the previous subsection shows that local second time derivatives can only come from the parts of \( \Delta G_{ij} \) which either multiply \( G_{ij} \) or have two covariant derivatives acting on \( f(\Box^{-1}R) + \Box^{-1}[Rf'(\Box^{-1}R)] \). The latter terms,
\[ g_{ij} \Box - D_i D_j = h_{ij} \Box + O(\partial_t) \] (21)
are simple to analyze. The local second derivative terms are therefore,
\[ G_{ij} + \Delta G_{ij} = \frac{1}{2} \ddot{h}_{ij} \times \left[ 1 + f\left( \frac{1}{\Box} R \right) + \frac{1}{\Box} \left[ Rf'(\frac{1}{\Box} R) \right] \right] \]
\[ - \frac{1}{2} h_{ij} h^{kl} \ddot{h}_{kl} \times \left[ 1 + f\left( \frac{1}{\Box} R \right) + \frac{1}{\Box} \left[ Rf'(\frac{1}{\Box} R) \right] - 4f'(\frac{1}{\Box} R) \right] + O(\partial_t) . \] (22)
Only the first line of expression (22) represents the unconstrained, dynamical part of \( h_{ij} \). By comparing with the approximate analytic form (3) of the nonlocal distortion function \( f(X) \) we see that the coefficient of the dynamical term is reduced at late times, but never by enough to make it change sign. We therefore conclude that no dynamical graviton mode ever becomes a ghost.
4.4 No linearized ghosts

As a complement to our detailed treatment of DoF in the full nonlinear theory, the present subsection is devoted to the linearized (about flat space $g_{\mu\nu} = \eta_{\mu\nu} + k_{\mu\nu}$) treatment of the problem\footnote{For an analysis of perturbations from a related nonlocal model [30] in a general cosmological background see [31].} This has several advantages: First, it is of course simpler, yet it retains the main point of the DoF analysis, since their content resides. Second, it allows us to treat the desired results gauge invariantly. [Of course, the full nonlinear treatment is needed to make sure no higher order failure of the critical constraint equations occurs, as notoriously happen in generic massive gravity models [32].]

We first derive the relevant field equation; varying $R_{\text{lin}} \partial^{-2} R_{\text{lin}}$ yields

$$\Delta G_{\mu\nu}^{\text{lin}} = \left( \eta_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu} \right) \frac{1}{\partial^2} R_{\text{lin}} \equiv \Pi_{\mu\nu} \frac{1}{\partial^2} R_{\text{lin}} , \quad R_{\text{lin}} = \Pi^{\rho\sigma} k_{\rho\sigma} . \quad (23)$$

The transverse projector’s $0\mu$ components are respectively of zero and first order in time derivatives: $\Pi_{00} = -\nabla^2$, $\Pi_{0i} = -\partial_0 \partial_i$, already showing these are constraint components, as in (17-18). For orientation, we revert to synchronous gauge (here $k_{0\mu} = 0$), for which $R_{\text{lin}} = \partial^2 k^T - \nabla^2 k^L$. Here $k^T$ and $k^L$ are components of the usual ADM “TT” decomposition of a symmetric spatial tensor $k_{ij} = k_{ij}^{TT} + \frac{1}{2}(\delta_{ij} - \partial_i \partial_j / \nabla^2) k^T + \frac{1}{2}(\partial_i k_j^T + \partial_j k_i^T) + \partial_i \partial_j k^L$; indeed, $k^T$ is precisely the Newtonian metric of concern, while $k^L$ is the doubly longitudinal, pure gauge, term [33]. The rest of the story is of course just the linearization of the results of subsections 4.2 and 4.3.

Now we go to the linearized, but gauge invariant $R_{\text{lin}}$:

$$R_{\text{lin}} = \partial^2 k^T - C , \quad C \equiv \nabla^2 k_{00} + \nabla^2 k^L - 2 k_{0i,0i} . \quad (24)$$

The first, gauge invariant, Newtonian term is unchanged, while the additional (also gauge invariant) combination $C$ differs from its synchronous gauge value only by lower time derivative terms, so the justifications previously exhibited for that gauge simply carry over unchanged to any frame.

4.5 Generalized models

Here we consider how generalizations of the model would affect our conclusions. We begin with the initial time, which can be any instant during the
epoch of radiation dominance [21]. Unless the nonlocal distortion function $f(X)$ is changed from the form (3), the initial time could not be taken during the epoch of primordial inflation because $^{-1}R$ behaves there like $-4$ times the number of e-foldings [7]. Nor would it make any physical sense to assume such an early time because our physical picture of the nonlocal modification is the gravitational vacuum polarization that was built up during primordial inflation by the continual production of infrared gravitons.

Our conclusions about the number of DoF and the initial value constraints are independent of any assumption about the nonlocal distortion function $f(X)$. However, our no-ghost conclusion does require an $f(X)$ which maintains the positivity of the coefficient $h_{ij}$ in equation (22). The choice (3) which reproduces the $\Lambda$CDM expansion history will do this. Because that $f(X)$ does not come near to changing the sign at the current time, any excursion from (3) which is still consistent with constraints on the expansion history should be acceptable. However, our no-ghost conclusion would be endangered by an $f(X)$ which can become smaller than $-1$, or whose slope can become too positive.

Finally we come to the question of modifying our assumption about the use of retarded boundary conditions to define $^{-1}$. Retarded boundary conditions seem very natural from our perspective of viewing nonlocal cosmology as the gravitational vacuum polarization that was built up from nothing during primordial inflation. We cannot consider promoting the boundary condition to a new DoF because this would recover the localized model, with its fatal ghost. However, one might consider how the model looks with some other, but definite boundary condition.

Suppose we fix the initial values of $^{-1}R$ and its first time derivatives as $\Phi_0(\vec{x})$ and $\dot{\Phi}_0(\vec{x})$. Green’s Second Identity allows us to express $^{-1}R$ in terms of the Green’s function defined by relations (12-13),

$$
\left[ \frac{1}{\Box} R \right](t, \vec{x}) = \int_{t'>0} d^4x' \sqrt{h(t', \vec{x}')} R(x')G[h](t, \vec{x}; t', \vec{x}') \\
+ \int d^3x' \sqrt{h(0, \vec{x}')} \left[ \Phi_0(\vec{x}') \partial_t G[h](t, \vec{x}; 0, \vec{x}') - G[h](t, \vec{x}; 0, \vec{x}') \dot{\Phi}_0(\vec{x}') \right].
$$

(25)

Of course the Green’s function is not known for arbitrary $h_{ij}(t, \vec{x})$ but the cosmological background $h_{ij}(t, \vec{x}) = \delta_{ij}a^2(t)$ is simple enough to analyze: spatially homogeneous contributions to $\Phi_0$ add constants, whereas spatially homogeneous contributions to $\dot{\Phi}_0$ behave as $\int dt / a^3(t)$.

\(^3^{See [30, 31] for a related nonlocal model which describes primordial inflation.}\)
Permitting nonzero $\Phi_0$ and $\dot{\Phi}_0$ would obviously change our result that the initial value constraints of nonlocal cosmology agree with those of general relativity. Nonzero values for $\Phi_0$ and $\dot{\Phi}_0$ also change the numerical value — although not the functional form — of the crucial coefficient of the $\ddot{h}_{ij}$ term in equation (22). Although small changes of this type pose no essential problem, neither change is particularly desirable. So it is just as well to stick with the original model with $\Phi_0(\vec{x}) = 0 = \dot{\Phi}_0(\vec{x})$, which is also what the putative physical origin of the nonlocal correction would suggest.

5 Discussion

Our nonlocal model (1-2) exactly reproduces the $\Lambda$CDM expansion history with zero cosmological constant [21]. The model has no current problem either with solar system tests [20] nor with existing data on structure formation [24]. The small deviations from general relativity it predicts for structure formation should be resolvable with the next generation of large scale structure surveys [24]. In anticipation, we have considered the theoretical issues of screening and stability.

Our first result is that screening inside any gravitationally bound system can be made 100% effective by simply defining the nonlocal distortion function to vanish for positive argument, which has no effect on the (desired) cosmological behavior. Our second result is that the localized model [27] is inequivalent to ours in that it has extra (scalar) excitations, one of which is unavoidably a ghost. Instead, we saw that the nonlocal model has the same DoF as general relativity; the variables of both separate into identical sets of constrained and radiation excitation modes, subject to the identical initial value constraints. Further, despite the (also desired!) difference in the evolution equations, explicit and nonperturbative examination of the highest time derivatives shows that no graviton degree of freedom ever becomes a ghost. That ensures the absence of kinetic energy instabilities. The more difficult issue, ruling out instabilities due to possible negative potential energy excitations is, if anything, more difficult than proving the positive energy theorem in general relativity; while these bad modes seem unlikely on physical grounds, we have not attempted to exclude them.

Note added in proof: A very recent study [34] now finds that our model deviates quite significantly from observed structure formation data. Whatever the outcome, this of course in no way affects our nonlocal models’ inner
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