FINITUDE OF PHYSICAL MEASURES FOR RANDOM MAPS

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Abstract. For random compositions of independent and identically distributed measurable maps on a Polish space, we study the existence and finitude of absolutely continuous ergodic stationary probability measures (which are, in particular, physical measures) whose basins of attraction cover the whole space almost everywhere. We characterize and hierarchize such random maps in terms of their associated Markov operators, as well as show the difference between classes in the hierarchy by plenty of examples, including additive noise, multiplicative noise, and iterated function systems. We also provide sufficient practical conditions for a random map to belong to these classes. For instance, we establish that any continuous random map on a compact Riemannian manifold with absolutely continuous transition probability has finitely many physical measures whose basins of attraction cover Lebesgue almost all the manifold.

1. Introduction

Uniform hyperbolicity was originally intended to encompass a residual, or at least a dense subset of all smooth dynamical systems [5, 83], although it was soon realized that this is not true [1, 72]. Uniformly hyperbolic systems are structurally stable [5] and admit a very precise topological description of their behavior: there are finitely many compact transitive invariant subsets such that every forward orbit of the system accumulates on one of them [83]. The dynamics near such attractors may be quite chaotic and thus essentially unpredictable after a long period of time. However, Sinai, Ruelle and Bowen demonstrated that these attractors are the supports of physical measures and thus behave well from a statistical point of view [28, 80, 82]. Kifer further showed that such systems are stochastically stable (i.e. the physical measures continuously vary) under small random perturbations [50]. Building on this background, in a conference in honor of Douady in 1995 (refer to [75, 76]), Palis developed a global picture recovering, in a more probabilistic formulation, much of the paradigm of uniform hyperbolic systems. Namely, Palis conjectured that every smooth dynamical system can be approximated by systems having only finitely many attractors, which support physical measures that describe the time averages

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for Lebesgue almost all points and are stochastically stable under small random perturbations.

An important contribution to the stochastic part of the global Palis conjecture was provided by Brin–Kifer [30] and Araújo [6]. Under some natural nondegenerate assumptions on noise, they found finitely many absolutely continuous ergodic stationary measures (in particular physical measures) with pairwise disjoint supports and whose statistical basins of attraction cover the whole ambient space almost everywhere. From now on, we refer to this property as finitude of physical measures or (FPM) for short\(^1\). See Definition 1.1 for a more precise description. To be more specific on the nondegenerate assumptions, Brin and Kifer assumed that the transition probability has a continuous density, while Araújo assumed that the transition probability has a density whose support includes a ball of which the diameter is uniformly bounded from below. In the last two decades, their nondegenerate conditions appeared as the main assumption of many works on stochastic stability around the Palis conjecture (especially of systems without uniform hyperbolicity); see e.g. [7, 9, 10, 16, 22].

In this paper, we will refine the Brin–Kifer and Araújo conditions by showing that (FPM) follows merely if one assumes that the transition probability is absolutely continuous. This is a significant improvement for applications. For instance, random dynamical systems generated by additive noise (the most common noise in applications; see Remark 1.4 for details) are some examples that satisfy these conditions. Our proof is quite different from [6, 30]. It is based on Markov operators theory, which enables us to obtain much stronger properties than (FPM), such as the exponential decay of the annealed correlation functions of each physical measure for some iterate of the random map. Moreover, we give a necessary and sufficient condition for (FPM) in terms of Markov operators, extending a previous work by Inoue and Ishitani [45] for Perron–Frobenius operators. That is, we introduce the notion of mean constrictivity for Markov operators and show its equivalence with both (FPM) and the property of asymptotic periodicity in mean introduced by Inoue and Ishitani. This is also a generalization of the result by Lasota, Lü and Yorke in [62] for the equivalence between constrictivity and asymptotic periodicity of Markov operators, which served as a stepping-stone for several later papers studying the existence of absolutely continuous invariant density for stochastic operators (see e.g., [63]). This functional approach allows us to give a hierarchy of classes of Markov operators, which implies, together with plenty of examples indicating the difference between the classes, that (FPM) are much weaker than the Brin–Kifer and Araújo conditions, refer to Figures 1 and 2 (and Remark 6.3). For instance, we include some random dynamical systems generated by finitely many continuous maps (so-called iterated

\(^1\)All the measures considered in this paper will be probabilities. Also, the context of this paper is the study of absolutely continuous ergodic stationary probabilities (which are, in particular, physical). For that reason, the property is simply called “finitude of physical measures”.

function systems) and by multiplicative noise with a common fixed point (another important class of noise in applications). These systems never have absolutely continuous transition probabilities.

1.1. Finitude of physical measures (FPM). Let $X$ be a Polish space equipped with a probability measure $m$ on the Borel $\sigma$-field $\mathcal{B}$ of $X$. Let $(T, \mathcal{A}, p)$ be a probability space and consider the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (T^\mathbb{N}, \mathcal{A}_{\mathbb{N}}, p^\mathbb{N})$. In this paper, we will deal with a measurable map $f : T \times X \to X$ where we denote $f_t = f(t, \cdot)$ for $t \in T$ and consider the following nonautonomous iterations

$$f_0^\omega = \text{id} \quad \text{and} \quad f_n^\omega = f_{\omega_n} \circ \cdots \circ f_{\omega_1} \text{ for } n \in \mathbb{N} \text{ and } \omega = (\omega_1, \omega_2, \ldots) \in \Omega.$$ 

Since we consider the Bernoulli probability $\mathbb{P} = p^\mathbb{N}$ on $\Omega$, the sequence $\{\omega = (\omega_1, \omega_2, \ldots) \mapsto \omega_n\}_{n \geq 1}$ of noises at each step is an independent and identically distributed random process. Thus, the sequence $\{f_n^\omega(x_0)\}_{n \geq 0}$ can be viewed as a (time-homogeneous) discrete Markov chain $\{X_n\}_{n \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with initial distribution $X_0(\omega) = x_0$ and transition probability given by

$$P(x, A) = p(\{t \in T : f_t(x) \in A\}) = \mathbb{P}(\{\omega \in \Omega : f_\omega(x) \in A\}) \quad \text{for } x \in X, \ A \in \mathcal{B}. \quad (1.1)$$

Recall that a nonnegative function $Q(x, A)$ defined for $x \in X$ and $A \in \mathcal{B}$ is called a (Markov) transition probability if

(i) $Q(x, \cdot)$ is a probability measure for every fixed $x \in X$,
(ii) $Q(\cdot, A)$ is a $\mathcal{B}$-measurable function for every fixed $A \in \mathcal{B}$.

We shall also use the $n$-th transition probability $P_n(x, A)$ of the process $\{X_n\}_{n \geq 0}$ given by

$$P_n(x, A) = \mathbb{P}(\{\omega \in \Omega : f_n^\omega(x) \in A\})$$

which is also a Markov transition probability. We refer to [11] and [69] for general theories of random dynamical systems and Markov processes, respectively. A probability measure $\mu$ on $X$ is said to be a stationary measure of $f$ if

$$\mu(A) = \int (f_\omega)_\# \mu(\omega) \, dp(\omega) \quad \text{for all } A \in \mathcal{B}. \quad (1.2)$$

Here, $g, v$ denotes the pushforward of a measure $v$ by a measurable map $g$, that is, $g_*v(A) = v(g^{-1}A)$ for $A \in \mathcal{B}$. It is well known that $\mu$ is a stationary measure of $f$ if and only if $\mathbb{P} \times \mu$ is an invariant measure for the skew-product

$$F : \Omega \times X \to \Omega \times X, \quad F(\omega, x) = (\sigma \omega, f_\omega(x)) \quad (1.3)$$

where $\sigma$ is the shift operator defined on $\Omega$ (see [74]). Moreover, we say that $\mu$ is ergodic if $\mathbb{P} \times \mu$ is an ergodic probability measure of $F$. This is equivalent to asking that any $A \in \mathcal{B}$ such that $P(x, A) \geq 1_A(x)$ where $1_A$ is the indicator function has $\mu$-measure 0 or 1 (see [51, Appendix A.1] and Appendix C). Finally, recall that a
\( \nu \)-null set is a measurable set that has \( \nu \)-measure zero, and a measure \( \nu \) is said to be absolutely continuous with respect to a measure \( \lambda \) if any \( \lambda \)-null set is \( \nu \)-null.

Our goal is to find a condition on \( f \) under which (FPM) holds.

**Definition 1.1.** We say that \( f \) satisfies (FPM) if there exist finitely many ergodic stationary probability measures \( \mu_1, \ldots, \mu_r \) of \( f \) such that
1) they are absolutely continuous with respect to \( m \);
2) they have pairwise disjoint supports (up to an \( m \)-null set);
3) the union of the fiberwise statistical basins of attraction of these measures \( \mathcal{P} \)-almost surely covers \( X \) up to a set of null \( m \)-measure. That is,
\[
m(X \setminus (B_\omega(\mu_1) \cup \cdots \cup B_\omega(\mu_r))) = 0 \quad \text{for } \mathcal{P} \text{-almost every } \omega \in \Omega
\]
where
\[
B_\omega(\mu_i) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_\omega^j(x)} = \mu_i \right\} \quad \text{for } i = 1, \ldots, r.
\]

Here, the limits of measures are taken in the weak*-topology.

It is not difficult to show (see Lemma 5.1) that, for the measure \( \mu_i \) above, \( B_\omega(\mu_i) \) has full \( \mu_i \)-measure for \( \mathcal{P} \)-almost all \( \omega \in \Omega \) and \( i = 1, \ldots, r \). Moreover, since \( \mu_i \) is absolutely continuous with respect to \( m \), it holds that \( m(B_\omega(\mu_i)) > 0 \) for \( \mathcal{P} \)-almost every \( \omega \in \Omega \). Therefore, \( \mu_i \) is a **physical measure** with respect to the reference measure \( m \) in the sense that \( \mathcal{P} \)-almost surely the fiberwise statistical basin of attraction \( B_\omega(\mu_i) \) has a positive \( m \)-measure for each \( i = 1, \ldots, r \). See more details on the notion of physical measures for random maps in Remark 5.3.

In the deterministic case, that is, when \( f_t = g \) for all \( t \in T \) with some \( g : X \to X \), (FPM) means that the dynamics of \( g \) can be statistically understood by finitely many ergodic invariant physical probability measures that describe the time average of almost all points in \( X \). When and where this property holds attracted great interest in dynamical systems theory, and we refer e.g. to \([27, 75]\). Notice that there are some obstacles to the finitude of physical ergodic invariant probability measures, such as the so-called Newhouse domains \([32, 39, 64, 73, 77]\) in which generically infinitely many attractors coexist. Moreover, recently Berger has realized that the coexistence of infinitely many attractors is locally Kolmogorov typical in parametric families of endomorphisms of surfaces and diffeomorphisms of higher dimensional manifolds, see \([18, 19, 23, 24]\). In contrast to the Palis conjecture, the so-called Takens’ last problem asked if it is possible to construct a persistent class of dynamics of a compact manifold where time averages do not exist (named as historic behavior; cf. \([81, 87]\)) on a Lebesgue positive measure set. Some important advances in this question have been obtained in \([53]\) where the authors constructed a locally dense
class of $C^r$-surface diffeomorphisms ($r \geq 2$) with historic behavior on a positive Lebesgue measure set. This result has been extended to $C^\infty$ and real analytic surface diffeomorphisms in [25] and to $C^r$-diffeomorphisms with $r \geq 1$ in dimension three and higher in [17] (see also [52] for a specific three-dimensional example).

Araújo [6] gave a quite useful sufficient condition (see (1) below) for (FPM) for the following class of random maps. First of all, here $X$ is a compact Riemannian manifold and $T$ is the unit ball of a Euclidean space with $m$ and $p$ being the normalized Lebesgue measures on $X$ and $T$, respectively. Now,

(1) $f : T \times X \to X$ is a continuous map and $f_t$ is a $C^1$-diffeomorphism for every $t \in T$; there are $n_0 \in \mathbb{N}$ and a positive number $\xi_0$ such that for all $n \geq n_0$ and $x \in X$

(A) \{f^n_\omega(x) : \omega \in \Omega\} contains the ball of radius $\xi_0$ and centered at $f^n_0(x)$;

(B) $P^n(x, \cdot)$ is absolutely continuous with respect to $m$.

Here, we wrote $f^n_0$ for the usual $n$-th iteration of the single map $f_0 : X \to X$. Condition (A) above is a topological requirement. On the other hand, since the $n$-th and $(n-1)$-th transition probabilities are related by the recurrence

$$P^n(x, A) = \int P^{n-1}(y, A)P(x, dy) \quad (1.4)$$

we have condition (B) by only requiring that $P^n(x, \cdot)$ be absolutely continuous with respect to $m$ for all $x \in X$. A greater requirement is the continuity of $f$ in item (1). For instance, such a requirement is not necessary for the approach by Brin and Kifer [30, Section 2] to get (FPM). These authors dealt with abstract Markov chains $\{X_n\}_{n \geq 0}$ on a compact Riemannian manifold $X$ with transition probability $P(x, A)$ having a continuous density in the following sense:

(2) there are an integer $n_0 \geq 1$ and a nonnegative function $p(x, y)$ that is continuous in both variables such that for any $x \in X$ and $A \in \mathcal{B}$,

$$P^{n_0}(x, A) = \int_A p(x, y)dm(y).$$

Thus, $P^{n_0}(x, \cdot)$ is absolutely continuous with respect to the normalized Lebesgue measure $m$ for all $x \in X$ as in the Araújo’s condition (B). Although this abstract approach seems more general, this is not the case because any Markov chain in a Polish space can be represented by a random map as in (1.1) (cf. [49, 51]).

We will prove that condition (1) and condition (2), respectively, imply that there are $n_0 \in \mathbb{N}$ and a nonnegative function $p(x, y)$ such that $P^{n_0}(x, dy) = p(x, y)dm(y)$ and $x \mapsto p(x, \cdot)$ is a continuous map from $X$ to $L^1(m)$. See Remarks 1.2 and 1.3. Here $L^1(m) = L^1(X, \mathcal{B}, m)$ denotes, as usual, the Banach space of all real-valued $\mathcal{B}$-measurable functions $\varphi$ on $X$ whose $L^1$-norm $||\varphi|| := \int_X |\varphi|dm$ is bounded, where two functions that coincide with each other $m$-almost everywhere are identified. In view
of this, the following theorem is a notable improvement of Brin–Kifer’s and Araújo’s sufficient conditions to get (FPM):

**Theorem A.** Let \((X, \mathcal{B}, m)\) and \((T, \mathcal{A}, p)\) be a compact Polish probability space and a probabilistic space, respectively. Consider a measurable map \(f : T \times X \to X\) and let \(P^n(x, A)\) be the \(n\)-th transition probabilities for the associated Markov chain induced by \(f\). Assume that for some \(n_0 \in \mathbb{N}\),
\[
P^{n_0}(x, dy) = p(x, y) \, dm(y) \quad \text{such that} \quad x \in X \mapsto p(x, \cdot) \in L^1(m) \quad \text{is continuous.}
\]
Then, \(f\) satisfies (FPM).

In the following series of remarks, we will show some easily checkable assumptions from which Theorem A is applicable, allowing us to compare with the previous sufficient conditions from the literature.

We say that \(f : T \times X \to X\) is a continuous random map if \(f\) is measurable and \(f_t = f(t, \cdot) : X \to X\) is continuous for \(p\)-almost every \(t \in T\).

**Remark 1.2.** Firstly, (FPM) follows from the following assumption.

(i) Let \((X, \mathcal{B}, m)\) and \((T, \mathcal{A}, p)\) be a compact Polish probability space and a probabilistic space, respectively, such that for some \(n_0 \in \mathbb{N}\), it holds that
\[
(a) \quad f : T \times X \to X \quad \text{is a continuous random map,}
\]
\[
(b) \quad P^{n_0}(x, \cdot) \quad \text{is absolutely continuous with respect to} \quad m \quad \text{for all} \quad x \in X.
\]
This generalizes Araújo’s result in [6]. The fact that (i) implies (FPM) follows immediately from Corollary 2.11 and Theorem A. We will prove in Proposition 1.13 that in the context of (i), neither condition (a) nor condition (b) is sufficient to get (FPM). Moreover, in Proposition 1.13 we also show that these conditions are not necessary to obtain (FPM). Notice also that (i) generalizes Araújo’s result by removing condition (A) in his result, and we will give in Theorem 6.4 (1) another practically sufficient condition for (FPM) which utilizes condition (A).

**Remark 1.3.** Secondly, (FPM) holds under the following assumption:

(ii) Let \((X, \mathcal{B}, m)\) be a compact probability Polish space and \(P^{n_0}(x, dy) = p(x, y) \, dm(y)\) for some \(n_0 \in \mathbb{N}\), where \(p(\cdot, y)\) is a continuous function on \(X\) for \(m\)-almost every \(y \in X\).

This clearly generalizes Brin–Kifer’s result in [30]. The proof of this observation follows from the Schaffé–Riez theorem, cf. [61]. Indeed, if \(\{x_n\}_{n \in \mathbb{N}}\) is a converging sequence to \(x\), then by the continuity of \(p(\cdot, y)\) we get \(p(x_n, y) \to p(x, y)\) and thus the Schaffé–Riez theorem implies that \(\|p(x_n, \cdot) - p(x, \cdot)\| \to 0\) as \(n \to \infty\). This means that \(p(x, \cdot)\) varies continuously on \(L^1(m)\) with respect to \(x\). Now, Theorem A implies (FPM). See also Theorem 6.4 (2) for another weakening of Brin–Kifer’s condition.
Remark 1.4. One of the most important noises in real applications that satisfy the assumption in Theorem A is the random dynamical system generated by additive noise with absolutely continuous distribution. Here, $X = \mathbb{T}$ is a compact Lie group where the algebraic multiplication is denoted by "$\cdot$", $m$ is the Haar measure, and $p$ is an absolutely continuous probability with respect to $m$. The torus $\mathbb{R}^d/\mathbb{Z}^d$ is often considered in the literature as an example. The random map $f$ is defined by $f_t(x) = f_0(x) + t$ for some continuous map $f_0 : X \to X$. As a slight abuse of notation, we denote the density function of $p$ by $p(x)$. Then, $P(x, A)$ can be written as

$$P(x, A) = \int_A p(y - f_0(x)) \, dm(y) \quad \text{for } x \in X \text{ and } A \in \mathcal{B}$$

(cf. [63, Equation (10.5.5)]). Hence, $P(x, \cdot)$ is absolutely continuous with respect to $m$ for all $x \in X$. Notice that if the support of $p(x)$ does not include any open ball centered at 0 nor $p(x)$ is not continuous, then both Araújo’s condition (A) and Brin–Kifer’s condition (2) are violated in general. Furthermore, it seems difficult to have the condition in Theorem A with $n_0 = 1$ because $x \in X \mapsto p(y - f_0(x)) \in \mathbb{R}$ is not continuous for some $y \in X$ in general. However, surprisingly, by virtue of Remark 1.2 we know that the condition in Theorem A is always satisfied (and thus (FPM) holds). In fact, by Corollary 2.11, the condition in Theorem A holds with $n_0 = 2$.

Remark 1.5. Another important class of noise that appears in real applications is multiplicative noise with absolutely continuous distribution (cf. [12, 86]). For instance, consider the case when $X = \mathbb{T} = [0, 1]$, $m$ is the Lebesgue measure and $p$ is an absolutely continuous probability with respect to $m$ with density function $p(x)$. Define $f_t(x) = tg(x)$, where $g$ is some continuous map on $X$. It is straightforward to see that $P(x, A)$ is of the form

$$P(x, A) = \int_A p\left(\frac{y}{g(x)}\right) \, dm(y) \quad \text{for } x \in X \text{ and } A \in \mathcal{B}$$

(cf. [63, Equation (10.7.5)]). Therefore, if $g$ is bounded away from zero, then $P(x, \cdot)$ is absolutely continuous with respect to $m$, and by Remark 1.2, (FPM) holds. Actually, the condition in Theorem A holds (with $n_0 = 2$). In fact, this (together with Theorem B) generalizes [63, Theorem 10.7.1]. On the other hand, in contrast to additive noise, if $g(x) = 0$ for some $x \in X$, then the absolute continuity of $P(x, \cdot)$, the condition in Theorem A and (FPM) can fail to hold. See Section 1.4.2 for details.

Remark 1.6. Finally, we remark that the important class of random dynamical systems generated by iterated function systems with probabilities (see Section 1.4 for its formal definition) does not meet the assumption in Theorem A. Indeed, in this
case, we have that \( T \) is a finite set \( \{1, \ldots, k\} \) and thus
\[
P(x, \cdot) = \sum_{i=1}^{k} p_i \delta_{f(i)} \quad \text{where } p_i = p((i)) > 0
\]
which cannot be absolutely continuous with respect to \( m \) in general. However, some of them will be in the range of Section 1.2 to get (FP\( \pi \)), in which several weaker versions of the condition in Theorem A are given.

The proof of Theorem A is based on the analysis of the annealed Perron–Frobenius operator associated with the random dynamical system generated by \( f \). We will show that this operator belongs to the class of constrictive Markov operators, which has been extensively studied in the literature. This observation allows us to generalize Theorem A in terms of Markov operators. In particular, we can obtain new practical sufficient conditions implying (FP\( \pi \)), as indicated in Theorem 6.4. As we will explain in Remark 6.5, such conditions generalize the sufficient condition studied in the work by Araújo and Aytaç [8] to get uniform ergodicity (a stronger property than (FP\( \pi \))).

1.2. Markov operators. In order to provide a general definition of Markov operator, assume that \((X, \mathcal{B}, m)\) is any abstract probability space (not necessarily a Polish space as in the previous subsection). Let \( D(m) = D(X, \mathcal{B}, m) \) be the space of density functions, that is,
\[
D(m) = \{ h \in L^1(m) : h \geq 0 \text{ m-almost everywhere and } \|h\| = 1 \}.
\]
An operator \( P : L^1(m) \to L^1(m) \) is called a Markov operator if \( P \) is linear, positive (i.e. \( P \varphi \geq 0 \) m-almost everywhere if \( \varphi \geq 0 \) m-almost everywhere) and
\[
\int P \varphi \, dm = \int \varphi \, dm \quad \text{for all } \varphi \in L^1(m). \tag{1.5}
\]
Note that a positive linear operator \( P \) on \( L^1(m) \) is a Markov operator\(^2\) if and only if \( P(D(m)) \subset D(m) \). It is not difficult to see that this is equivalent to \( P^* 1_X = 1_X \), where \( P^* \) is the adjoint operator of \( P \), that is, \( P^* \) is a bounded linear operator on \( L^\infty(m) \cong (L^1(m))^* \) given by \( \int P^* \psi \cdot \varphi \, dm = \int \psi \cdot P \varphi \, dm \) for \( \psi \in L^\infty(m) \), \( \varphi \in L^1(m) \). Recall that \( L^\infty(m) = L^\infty(X, \mathcal{B}, m) \) is the Banach space of bounded real-valued \( \mathcal{B} \)-measurable \( m \)-essentially bounded functions defined on \( X \). As usual, two functions that coincide with each other \( m \)-almost everywhere, are identified.

\(^2\)Any Markov operator \( P \) is a bounded operator: Given \( \varphi \in L^1(m) \), consider \( \varphi_+ := \max(\varphi, 0), \varphi_- := \max(-\varphi, 0) \). Then, \( P\varphi_+, P\varphi_- \geq 0 \), and thus \( \|P\varphi\| \leq \int (P\varphi_+ + P\varphi_-) \, dm = \int \varphi_+ \, dm + \int \varphi_- \, dm = \|\varphi\| \).
A key property of Markov operators for the purpose of this paper is constrictivity.

Definition 1.7. A sequence \((Q_n)_{n \geq 1}\) of Markov operators on \(L^1(m)\) is called

1. **constrictive** if there exists a compact set \(F\) of \(L^1(m)\) such that for any \(h \in D(m)\),
   \[
   \lim_{n \to \infty} d(Q_nh, F) = 0;
   \]
2. **uniformly constrictive** if there exists a compact set \(F\) of \(L^1(m)\) such that
   \[
   \lim_{n \to \infty} \sup_{h \in D(m)} d(Q_nh, F) = 0.
   \]

Here \(d(\varphi, F) = \inf_{\psi \in F} \| \varphi - \psi \|\). In particular, a Markov operator \(P\) on \(L^1(m)\) is called

3. **(uniformly) constrictive** if the sequence \((P^n)_{n \geq 1}\) is (uniformly) constrictive.
4. **mean constrictive** if the sequence \((A_n)_{n \geq 1}\) is constrictive, where \(A_n\) is given by
   \[
   A_n \varphi = \frac{1}{n} \sum_{i=0}^{n-1} P_i \varphi \quad \text{for} \; \varphi \in L^1(m).
   \]

The compact set \(F\) above is called a **constrictor**. These conditions appeared in the context of mean ergodic theorems, see [36, 63]. Notice that, by definition, we have

uniformly constrictive \(\Rightarrow\) constrictive \(\Rightarrow\) mean constrictive.

See also Figure 1 for a global picture. Furthermore, the uniform constrictivity of \(P\) is known to be equivalent to the quasi-compactness of \(P\) in \(L^1(m)\), cf. [20, Theorem 2].

Recall that an operator \(Q\) is called **quasi-compact** if there is a compact linear operator \(R\) such that \(\|Q^n - R\|_{op} < 1\) for some \(n \in \mathbb{N}\). Hence, the class of quasi-compact operators contains the subclass of **eventually compact operators**, that is, the linear operators \(Q\) such that \(Q^n\) is compact for some \(n \in \mathbb{N}\).

1.2.1. **Perron–Frobenius operators.** One of the most important examples of Markov operators is the **Perron–Frobenius operator** induced by a nonsingular transformation \(g : X \to X\). Recall that \(g\) is nonsingular (with respect to the reference measure \(m\) on \(X\)) if the preimage of any \(m\)-null set by \(g\) is \(m\)-null. The Perron–Frobenius operator \(\mathcal{L}_g : L^1(m) \to L^1(m)\) of \(g\) is defined by the formula

\[
g.m_\varphi(A) = \int_A \mathcal{L}_g \varphi \, dm \quad \text{for all} \; \varphi \in L^1(m) \; \text{and} \; A \in \mathcal{B}
\]
where \( m_\varphi \) is the finite signed measure given by
\[
m_\varphi(A) = \int_A \varphi \, dm \quad \text{for} \quad A \in \mathcal{B}.
\] (1.6)

It is easy to see that \( \mathcal{L}_g \) is a Markov operator and \( m_{\mathcal{L}_g \varphi} = g_* m_\varphi \). As in this example, Markov operators \( P \) naturally appear in the study of (random) dynamical systems, and \( (P^n \varphi)_{n \geq 0} \) is interpreted as the evolution of density functions driven by the system. We refer to \([13, 14, 36, 37, 63]\).

Let \( f : T \times X \to X \) be as in Section 1.1. We simply write \( \mathcal{L}_t \) for the Perron–Frobenius operator of \( f_t = f(t, \cdot) \) for \( t \in T \) and define the annealed Perron–Frobenius operator \( \mathcal{L}_f : L^1(m) \to L^1(m) \) by
\[
\mathcal{L}_f \varphi(x) = \int \mathcal{L}_t \varphi(x) \, dp(t).
\]

Then, it is straightforward to see that \( \mathcal{L}_f \) is also a Markov operator. Now we are in a position to state two of our main results.

**Theorem B.** Under the assumption of Theorem A, \( \mathcal{L}_f \) is eventually compact, in particular, uniformly constrictive.

We will obtain the above result by proving an equivalent but apparently more general theorem stated in terms of Markov operators and Markov processes. See Theorem 2.3 together with Proposition 2.2 and Remark 2.5.

In the setting of Theorem B, since \( \mathcal{L}_f \) is quasi-compact, we can obtain exponential decay of annealed correlation functions, in the sense that there are finitely many absolutely continuous ergodic stationary measures \( \mu_1, \ldots, \mu_r \) with pairwise disjoint supports and constants \( k \in \mathbb{N}, C > 0, \rho \in (0, 1) \) such that for any \( \varphi, \psi \in L^\infty(m) \), \( n \in \mathbb{N} \) and \( j = 1, \ldots, r \), we have
\[
\left| \int \psi \circ f_n^k \cdot \varphi \, d(P \times \mu_j) - \int \psi \, d\mu_j \int \varphi \, d\mu_j \right| \leq C \rho^k \| \varphi \psi \|_\infty.
\] (1.7)

Refer to, for example, \([31, 67]\) for background and, \([15]\) for proof\(^3\). Actually, this claim with \( r = k = 1 \) was shown by Araújo and Aytaç in \([8]\) under a bit stronger assumption than Araújo’s conditions (A) and (B) in a different manner (using a purely probability-theoretic technique); see also Remark 6.5. Furthermore, the quasi-compactness of \( \mathcal{L}_f \) may lead to other several limit theorems (such as central limit theorem, large

---

\(^3\)Although the paper \([15]\) only dealt with deterministic maps, one can show the claim by literally repeating the argument in \([15, \text{Appendix B}]\) with \( \mathcal{L}_f \) instead of a Perron–Frobenius operator \( \mathcal{L}_g \) of a nonsingular map \( g : X \to X \), after realizing the duality \( \int \psi \circ f_n^k \cdot \varphi \, d(P \times m) = \int \psi \cdot \mathcal{L}_g^k \varphi \, dm \) corresponding to \( \int \psi \circ g^n \cdot \varphi \, dm = \int \psi \cdot \mathcal{L}_g^n \varphi \, dm \).
deviation principle, local limit theorem and almost sure invariance principle) via the so-called Nagaev–Guivarc'h perturbative spectral method, refer to [2, 40, 44].

**Theorem C.** Let \((X, \mathcal{B}, m)\) and \((\Omega, \mathcal{F}, \mathbb{P}) = (T^N, \mathcal{N}, p^N)\) be a locally compact Polish probability space and the infinite product space of a probability space \((T, \mathcal{A}, \mathbb{P})\), respectively. Consider a measurable map \(f : T \times X \to X\). Then, the followings are equivalent:

(i) \(\mathcal{L}_f\) is mean constrictive;

(ii) \(f\) satisfies (FPM).

Moreover, if \(f\) satisfies any of the above equivalent conditions and \(\mu_1, \ldots, \mu_r\) denote the measures that appear in the property (FPM), then

(1) \((\mathbb{P} \times m) (B(\mu_1) \cup \cdots \cup B(\mu_r)) = 1\);

(2) \(\mathbb{P} (B_x(\mu_1) \cup \cdots \cup B_x(\mu_r)) = 1\) for \(m\)-almost every \(x \in X\);

where for each \(i = 1, \ldots, r\),

\[
B(\mu_i) = \left\{ (\omega, x) \in \Omega \times X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu_i \right\}
\]

and

\[
B_x(\mu_i) = \left\{ \omega \in \Omega : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} = \mu_i \right\}.
\]

We note that since uniform constrictivity implies mean constrictivity, Theorem A is just a consequence of Theorems B and C. In fact, conclusions (1) and (2) in Theorem C also hold under the assumptions of Theorem A.

Notice that \(B(\mu_i)\) is basically the statistical basin of attraction of the measure \(\mathbb{P} \times \mu_i\) for the skew-product map \(F\) in (1.3). Thus, the above theorem is actually a characterization of an absolutely continuous version of the Palis conjecture for the class of deterministic systems of the form (1.3), in terms of the annealed Perron–Frobenius operator of the random map \(f\). In particular, when \(T\) (or \(\Omega\)) is a singleton, Theorem C provides a characterization of the existence of finitely many \(m\)-absolutely continuous invariant probability measures for deterministic dynamics such that the union of their basins has full \(m\)-measure, in terms of the Perron–Frobenius operator.

### 1.2.2. General Markov operators.

We next generalize the equivalence in Theorem C to general Markov operators. However, to do this, we need some preliminaries.

Let \(P\) be a Markov operator on \(L^1(m)\) and consider the adjoint operator \(P^*\) on \(L^\infty(m)\). We define the support of a real-valued function \(h\) (up to an \(m\)-null set) by \(\text{supp } h = \{ x \in X : h(x) \neq 0 \}\). We say that \(h\) is an invariant density of \(P\) (or a \(P\)-invariant density) if \(h \in D(m)\) and \(Ph = h\). As usual, \(P^n\) and \((P^*)^n\) denote the \(n\)-th iterated of
Let \((P^*P)^*\) be the adjoint operator of \(P^*\). Using recursively the duality relation, it is not difficult to see that \((P^*P)^* = (P^*)^*\). For simplicity of notation, we will simply write \(P^*P\) when no confusion can arise. Having this notation in mind, we say that a \(P\)-invariant density \(h\) has the maximal support if
\[
\lim_{n \to \infty} P^n 1_{\text{supp}\, h}(x) = 1 \quad \text{for } m\text{-almost all } x \in X.
\] (1.8)

A probability measure \(\mu\) is called ergodic if any \(A \in \mathcal{B}\) such that \(P^*A \geq A\) satisfies \(\mu(A) \in \{0, 1\}\). We will say that a \(P\)-invariant density \(h \in D(m)\) is ergodic if the probability measure \(m_h\) given in (1.6) is ergodic. See Appendix C for more details on equivalent definitions.

**Theorem D.** Let \((X, \mathcal{B}, m)\) be an abstract probability space and consider a Markov operator \(P : L^1(m) \to L^1(m)\). Then, the following conditions are equivalent:

1. (MC) \(P\) is mean constrictive;
2. (FED) \(P\) admits finitely many ergodic \(P\)-invariant densities \(h_1, \ldots, h_r\) with mutually disjoint supports (up to an \(m\)-null set) and the invariant density function \(h = \frac{1}{r}(h_1 + \cdots + h_r)\) has the maximal support;
3. (APM) There exist finitely many ergodic \(P\)-invariant densities \(h_1, \ldots, h_r\) with mutually disjoint supports (up to an \(m\)-null set) and positive bounded linear functionals \(\lambda_1, \ldots, \lambda_r\) on \(L^1(m)\) such that
   \[
   \lim_{n \to \infty} \left\| A_n \varphi - \sum_{i=1}^r \lambda_i(\varphi) h_i \right\| = 0 \quad \text{for any } \varphi \in L^1(m).
   \]

The condition (APM), named as asymptotic periodicity in mean, was introduced by Inoue and Ishitani for Perron–Frobenius operators in [45] as a weaker version of the classic property asymptotic periodicity. In turn, asymptotic periodicity was introduced and shown to be equivalent to constrictivity in [62]. The equivalence between the conditions (FED) and (APM) is the generalization of Inoue–Ishitani [45] to the case of general Markov operators. Similarly, the equivalence between (APM) and (MC) is a generalization of the spectral decomposition theorem (on \(L^1(m)\)) that has been developed by Lasota, Li, Yorke, Komorník, Bartoszek during the eighties and nineties, and by Storozhuk, Toyokawa more recently. Refer to [21, 88] and references therein. Actually, we will prove in Theorem 4.1 a more complete version of Theorem D where we will provide a sequence of equivalences between (MC) and, a priori, weaker conditions.

**Remark 1.8.** If \(1_X\) is an ergodic \(P\)-invariant density, then \(P\) is (MC). Indeed, since any two ergodic \(P\)-invariant densities are equal or have disjoint support up to an \(m\)-null set, see Proposition C.10 in Appendix C, \(1_X\) is actually the unique invariant ergodic density of \(P\). Then \(P\) satisfies (FEM) and consequently, by Theorem D, \(P\) is (MC).
1.3. **Hierarchy of classes of Markov operators.** We have considered the classes of uniform constrictive (UC), constrictive (C) and mean constrictive (MC) Markov operators introduced in Definition 1.7. In this subsection, we introduce in Definition 1.10 a new class between them that we call asymptotic constrictivity (AC). In the next subsection, we will provide examples of random maps that show the difference between these classes. First, to show the global picture, we characterize the class of Markov operators in $L^1(m)$ for which there is an invariant density, which we denote by $(S)$.

1.3.1. **Straube class.** In [85] Straube studied the existence of invariant densities for the Perron–Frobenius operator associated with a nonsingular transformation $g : X \rightarrow X$. Namely, Straube showed that there exists a $g$-invariant probability measure which is absolutely continuous with respect to $m$ if and only if there exist $\delta > 0$ and $0 < \alpha < 1$ such that $m(A) < \delta$ implies $m(g^{-k}(A)) < \alpha$ for all $k \geq 0$. Recall that a $g$-invariant absolutely continuous probability measure corresponds to an invariant density of the Perron–Frobenius operator $\mathcal{L}_g$. Moreover,

$$m\left(g^{-k}(A)\right) = \int_A 1 \circ g^k(x) \, dm = \int_A \mathcal{L}_g^k 1_X \, dm.$$

More recently, Islam, Góra and Boyarsky in [46] proved also similar necessary and sufficient conditions in terms of Markov operator for the existence of absolutely continuous stationary measures for a certain class of iterated function systems with probabilities. The following theorem finally provides a characterization of the class (S) in the spirit of (MC), generalizing the previous results in [46, 85]. A more complete version will be given in Theorem 3.4.

**Theorem E.** For a Markov operator $P : L^1(m) \rightarrow L^1(m)$, the following assertions are equivalent:

1. There exists an invariant density for $P$;

2. There exist $\alpha \in (0, 1)$ and $\delta > 0$ such that

$$\sup_{n \geq 0} \int_A P^n 1_X \, dm < \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta;$$

3. There exist $\alpha \in (0, 1)$ and $\delta > 0$ such that

$$\sup_{n \geq 0} \int_A A_n 1_X \, dm < \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta;$$

Moreover, the equivalence also holds by taking $\alpha = 1$ in all the above items.
1.3.2. Weakly almost periodic class. Consider the class of Markov operators on $L^1(m)$ having an invariant density with the maximal support as in (1.8). This class was characterized in [88] as the well-known class of Markov operators in $L^1(m)$ called weakly almost periodic.

**Definition 1.9.** A Markov operator $P$ on $L^1(m)$ is weakly almost periodic if $(P^n \varphi)_{n \geq 1}$ is weakly precompact for any $\varphi \in L^1(m)$, that is, if any sequence $(P^n \varphi)_{k \geq 1}$ contains a further subsequence $(P^{n_j} \varphi)_{j \geq 1}$ that weakly converges to a function in $L^1(m)$.

By the Dunford–Pettis theorem\(^4\), a Markov operator $P$ is weakly almost periodic if and only if

$$\text{(WAP) for any } \varepsilon > 0 \text{ and } \varphi \in L^1(m), \text{ there is } \delta > 0 \text{ such that }$$

$$\int_A P^n \varphi \, dm < \varepsilon \quad \text{for any } n \in \mathbb{N} \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta.$$

As mentioned, due to [88, Theorem 3.1], we have that (WAP) is equivalent to the existence of a $P$-invariant density with the maximal support. Therefore, (WAP) implies (S), and moreover, (MC) implies (WAP) from Theorem D.

1.3.3. Asymptotically constrictive class. An equivalent formulation for the class of constrictive Markov operators $P : L^1(m) \to L^1(m)$ is

$$(C) \quad \text{for any } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that for any } \varphi \in D(m), \text{ there is } n_0 \in \mathbb{N} \text{ satisfying}$$

$$\int_A P^n \varphi \, dm < \varepsilon \quad \text{for any } n \geq n_0 \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta.$$ (1.9)

One of the implications follows easily from Dunford–Pettis theorem. Indeed, if $P$ is constrictive with a constrictor $F$, since $F$ is compact (in particular, weakly precompact) in $L^1(m)$ and $\int_A P^n \varphi \, dm \leq d(P^n \varphi, F) + \sup_{\varphi \in F} \int_A \varphi \, dm$ for each $\varphi \in D(m)$, $n \in \mathbb{N}$ and $A \in \mathcal{B}$, we immediately obtain (C). One can find the proof of the converse e.g. in [56].

Having in mind these equivalent formulations of (C) and (WAP), we introduce the following middle property between constrictivity and weak almost periodicity, which we could come up with while looking for non-constrictive examples.

**Definition 1.10.** A Markov operator $P$ on $L^1(m)$ is said to be asymptotically constrictive (AC) if for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\limsup_{n \to \infty} \int_A P^n \varphi \, dm < \varepsilon \quad \text{for any } \varphi \in D(m) \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta.$$ (1.9)

---

\(^4\)The theorem states that $F \subset L^1(m)$ is weakly precompact (i.e. any sequence in $F$ contains a weakly converging subsequence in $L^1(m)$) if and only if $F$ is bounded (i.e. there is $M > 0$ such that $\|\varphi\| < M$ for all $\varphi \in F$) and uniformly integrable (i.e. for any $\varepsilon > 0$, there is $\delta > 0$ such that $\int_A \varphi \, dm < \varepsilon$ for any $\varphi \in F$ and $A \in \mathcal{B}$ with $m(A) < \delta$). Refer to e.g. [26, Theorem 4.7.18].
It is straightforward to see that (C) implies (AC). We prove in Theorem 4.9 that (AC) implies (MC). We summarize the hierarchy between classes in the following figure.

\[
\begin{array}{cccc}
\text{(UC)} & \text{Obvious} & \text{(C)} & \text{Obvious} \\
\text{Prop. 1.15} & \phantom{\text{Thm. 4.9}} & \text{(AC)} & \phantom{\text{Thm. 4.9}} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \text{Obvious} & \phantom{\text{Thm. 4.9}} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \text{Prop. 1.16} & \phantom{\text{Thm. 4.9}} \\
\text{(AC)} & \phantom{\text{Thm. 4.9}} & \text{(MC)} & \text{Obvious} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \text{Prop. 1.19} & \phantom{\text{Thm. 4.9}} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \phantom{\text{Prop. 1.19}} & \text{(WAP)} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \phantom{\text{Prop. 1.19}} & \phantom{\text{Thm. 4.9}} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \phantom{\text{Prop. 1.19}} & \phantom{\text{Thm. 4.9}} \\
\phantom{\text{Prop. 1.15}} & \phantom{\text{Thm. 4.9}} & \phantom{\text{Prop. 1.19}} & \text{(S)} \\
\end{array}
\]

**Figure 1.** The hierarchy of the classes between (UC) and (S).

1.3.4. **Sub-hierarchy in (UC).** A transition probability \( P(x, A) \) on \( X \times \mathcal{B} \) is said to be \textit{m-nonsingular} if \( P(x, A) = 0 \) for \( m \)-almost every \( x \in X \) whenever \( m(A) = 0 \). It is well-known that the theories of Markov operators and Markov processes (transition probabilities) are intimately related. Indeed, given an \( m \)-nonsingular transition probability \( P(x, A) \) for \( x \in X \) and \( A \in \mathcal{B} \), we can induce a Markov operator \( P \) on \( L^1(m) \) such that \( P \varphi \) is the Radon–Nikodým derivative with respect to \( m \) of the finite signed measure

\[
\mu_\varphi(A) = \int \varphi(x) P(x, A) \, dm, \quad \text{for } A \in \mathcal{B}.
\]

See [47] or [70, Proposition V.4.2] for more details on the construction. Conversely, given a Markov operator \( P \) on \( L^1(m) \) one can define

\[
P^n(\cdot, A) = P^n 1_A \quad \text{for all } A \in \mathcal{B} \text{ and } n \geq 1.
\]

Notice that \( P^n(\cdot, A) \) is an equivalent class and thus as a real-valued function is only defined up to \( m \)-null sets. It is not hard to see that one may choose in each equivalent class \( P^n(\cdot, A) \) a \( \mathcal{B} \)-measurable function \( P^n(x, A) \) on \( X \) for every fixed \( A \in \mathcal{B} \) such that it differs from an \( m \)-nonsingular transition probability only on a negligible set of points. Nevertheless, if \( X \) is a Polish space, Neveu proved in [70, Proposition V.4.4] that these representations can be chosen appropriately to get that \( P^n(x, A) \) is a transition probability which induces the Markov operator \( P^n \) on \( L^1(m) \) as explained above. Taking into account this relation between Markov operators and Markov processes, in what follows we will introduce two subclasses in (UC).

Let \((X, \mathcal{B}, m)\) be a Polish probability space and consider a Markov operator \( P \) on \( L^1(m) \). Let \( P^n(x, A) \) be an \( n \)-transition probability that induces \( P^n \) as explained above. We first introduce the following classical condition from the theory of Markov chain adapted to Markov operators in \( L^1(m) \):

**(D)** there exist \( n_0 \geq 1, \, 0 < \varepsilon < 1, \, \delta > 0 \) and a probability \( \mu \) absolutely continuous with respect to \( m \) such that \( P^{n_0}(x, A) < \varepsilon \) for all \( x \in X \) and \( A \in \mathcal{B} \) with \( \mu(A) < \delta \).

\[\text{Recall that } P(x, A) \text{ is said to satisfy the Doeblin condition if there exist } n_0 \geq 1, \, \varepsilon > 0, \, \delta < 1 \text{ and a probability } \mu \text{ such that } P^n(x, A) > \varepsilon \text{ for all } x \in X \text{ and } A \in \mathcal{B} \text{ with } \mu(A) > \delta \text{ (cf. [69]). If } \mu \text{ is in addition required to be absolutely continuous with respect to } m, \text{ then this condition is equivalent to (D).} \]
The second class that we introduce is also (an equivalent condition to) a classical property widely discussed and studied in the literature of Markov processes under the name of uniform ergodicity that we adapt to Markov operators on $L^1(m)$:

$$(D^*) \text{ there exist } n_0 \geq 1, 0 < \varepsilon < 1, \delta > \frac{1}{2} \text{ and a probability } \mu \text{ absolutely continuous with respect to } m \text{ such that } P^{n_0}(x,A) < \varepsilon \text{ for all } x \in X \text{ and } A \in \mathcal{B} \text{ with } \mu(A) < \delta.$$

Dorea and Pereira [33] introduced $(D^*)$ without the absolute continuity of $\mu$ as an equivalent condition to uniform ergodicity. In Proposition 6.6 we will show equivalent conditions to $(D^*)$ including uniform ergodicity adapted to Markov operators in $L^1(m)$. Furthermore, we will show that the condition $(D^*)$ implies that $P$ only admits a unique invariant density, see Remark 6.7.

In Proposition 6.1 we will prove that $(UC)$ is the class of Markov operators $P$ on $L^1(m)$ that satisfy the following condition:

$$(UC) \text{ there are } n_0 \geq 1, 0 < \varepsilon < 1, \delta > 0 \text{ and a probability } \mu \text{ absolutely continuous with respect to } m \text{ such that } P^{n_0}(x,A) < \varepsilon \text{ for all } A \in \mathcal{B} \text{ with } \mu(A) < \delta \text{ and } m\text{-almost every } x \in X (\text{depending on } A).$$

In view of these characterizations, one immediately obtains the following relations:

\begin{align*}
(D^*) & \xleftrightarrow{Prop \ 1.12} (D) & \xleftrightarrow{Prop \ 1.11} (UC).
\end{align*}

**Figure 2.** The subhierarchy in $(UC)$.

In the next subsection of examples, we will show that neither of the converse implications above holds. However, if we restrict ourselves to the class of strong Feller operators, then $(UC)$ implies $(D)$. See Proposition 6.2. Due to this result, the annealed Perron–Frobenius operator $L_f$ satisfies $(D)$ when $f$ satisfies the Araújo or Brin–Kifer conditions (see Remark 6.3). Furthermore, $L_f$ satisfies $(D^*)$ for the conditions in Araújo–Aytaç [8]. In fact, we will generalize the argument in Araújo–Aytaç to show that a version of Araújo’s condition (other than the condition in Remark 1.2) is sufficient to obtain $(UC)$. See Remark 6.5 for details.

1.4. **Examples and counterexamples.** In this subsection, to complete Remark 1.2 and the list that reverse implications in Figures 1 and 2 fail, we will consider several examples coming from the three aforementioned important classes of random dynamical systems, i.e. additive noise, multiplicative noise and iterated function systems. We will also explain that some examples can be easily modified to be deterministic systems.
1.4.1. Additive type noise. First, we consider some perturbed systems with additive type noise, which will indicate the reverse implications in Figures 2. Let $X$ and $T$ be the closed interval $[0, 1]$ equipped with the Lebesgue measure, denoted by $m$ and $p$, respectively. Consider a measurable map $f_0 : X \to X$ and consider the random map $f : T \times X \to X$ given by $f(t, x) = f_t(x)$,

$$
    f_t(x) = \begin{cases} 
      0 & \text{for } x = 0, \\
      f_0(x) + t \pmod{1} & \text{for } x \neq 0.
    \end{cases}
$$

Then the following proposition holds for this $f$.

**Proposition 1.11.** $\mathcal{L}_f$ satisfies (UC), but does not satisfy (D).

Next, let us consider $X = X_- \cup X_+$ with $X_- = (-1, 0]$, $X_+ = (0, 1]$ and $T = X_+$. We equip $X$ and $T$ with the Lebesgue measures $m$ and $p$, respectively. Let $\iota : X \to X$ be the involution given by $\iota(x) = x + 1$ for $x \in X_-$ and $\iota(x) = x - 1$ for $x \in X_+$, so that $\iota(X_-) = X_+$ and $\iota(X_+) = X_-$. Let $f_0 : X \to X$ be a measurable map satisfying that $f_0(X_-) \subset X_+$, $f_0(X_+) \subset X_-$ and $f_0 \circ \iota = \iota \circ f_0$ on $X_+$. Define the random map $\tilde{f} : T \times X_+ \to X_+$ by

$$
    \tilde{f}_t(x) = \iota \circ f_0(x) + t \pmod{1}
$$

and let $f : T \times X \to X$ be the random map given by

$$
    f_t(x) = \begin{cases} 
      \iota \circ \tilde{f_t}(x) & \text{for } x \in X_+, \\
      \tilde{f_t} \circ \iota(x) & \text{for } x \in X_-.
    \end{cases}
$$

(1.10)

See Figure 5. Then, the following proposition holds for this $f$.

**Proposition 1.12.** $\mathcal{L}_f$ satisfies (D), but does not satisfy ($D^*$).

1.4.2. Multiplicative noise. We secondly focus on a family of perturbed systems with multiplicative noise. As announced, these examples prove the assertions in Remark 1.2. Let us consider a random map $f : T \times X \to X$ given by the following multiplicative noise,

$$
    f_t(x) = (1 - \varepsilon t) f_0(x), \quad x \in [0, 1], \quad t \in [0, 1] \quad \text{and} \quad 0 < \varepsilon < 1.
$$

(1.11)

Here $f_0 : [0, 1] \to [0, 1]$ is a measurable map and $X = T = [0, 1]$ is equipped with the Lebesgue measure which, as before, we denote by $m$ and $p$ respectively. The next three examples in Proposition 1.13 show that both conditions (a) and (b) in Remark 1.2 are neither sufficient nor necessary to get $(FPM)$. Realize that the first example gives an important warning: the natural relaxation of the condition (b) from “for all $x \in X$” to “for $m$-almost every $x \in X$” (denoted by almost-(b)) is not useful to get $(FPM)$.
Proposition 1.13. Consider the random map given in (1.11).

(1) Let \( f_0(x) = \frac{x}{2} \). Then, \( f \) does not satisfy (b), but satisfies (a) and almost-(b). Moreover, (S) does not hold. In particular, (FPM) does not hold. However, we have that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_0^j(x)} = \delta_0 \quad \text{for all} \ x \in X \text{ and } \omega \in \Omega.
\]

(2) Let \( f_0(x) = \frac{x}{2} \) if \( x \neq 0 \) and \( f_0(0) = \frac{1}{2} \). Then, \( f \) does not satisfy (a), but satisfies (b). Moreover, (S) does not hold. In particular, (FPM) does not hold.

(3) Let \( f_0(x) = 2x \mod 1 \). Then, \( f \) does not satisfy (a) and (b), but \( L \) satisfies (C). In particular, (FPM) holds.

Proposition 1.13 (1) provides a class of examples that does not satisfy (FPM) but almost-(b) holds. However, this example still has a unique non-absolutely continuous physical measure. The next example shows that there is a drastic gap between (b) and almost-(b): finitude versus infinitude.

Consider a random map \( f \) under a multiplicative type noise, given by
\[
f_t(x) = tx + (1-t)f_0(x) \quad x \in X = [0,1], \quad t \in T = [0,1]. \tag{1.12}
\]

Now, we consider a concrete \( C^1 \) map \( f_0 : X \to X \) with infinitely many sinks, which was essentially given by Araújo in \cite[Example 1]{Araujo}. Let \( \phi : X \to \mathbb{R} \) be a \( C^1 \) function given by
\[
\phi(x) = X^4 \sin \frac{1}{X} \quad \text{where} \ X = \frac{2}{\pi} \left( x - \frac{1}{2} \right). \tag{1.13}
\]

Note that \( \phi \) can be seen as a \( C^1 \) function on the Lie group \( S^1 = \mathbb{R}/\mathbb{Z} \) under the identification of \( S^1 \) with \( X = [0,1] \). Notice also that \( \phi \) has (countably) infinitely many local maxima and minima, which accumulates to \( \frac{1}{2} \). Therefore, the one-time map \( f_0 \) of the gradient flow given by \( \dot{x} = \nabla \phi(x) \) has infinitely many sinks that accumulate to \( \frac{1}{2} \) and whose basin covers \( X \) except the sources of \( f_0 \).

Proposition 1.14. Consider the random map given in (1.12) with the time-one map \( f_0 \) of the gradient flow induced by the potential function (1.13). Then, \( f \) does not satisfy (b), but satisfies (a) and almost-(b). Moreover, there are infinitely many points \( (s_k)_{k \geq 1} \subset X \) such that
\[
G(\delta_{s_k}) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_0^j(x)} = \delta_{s_k} \quad \text{for all} \ \omega \in \Omega \right\}
\]
is a non-empty open set (in particular, has an \( m \)-positive measure) for all \( k \geq 1 \), and \( \bigcup_{k=1}^{\infty} G(\delta_{s_k}) = X \) up to an \( m \)-zero measure set. In particular, (FPM) does not hold.
1.4.3. **Iterated function systems.** Finally, we consider some random maps generated by iterated function systems (IFS). These examples will disprove the reverse implications in Figures 1. As explained in Remark 1.6, an IFS with probabilities is a random map \( f : T \times X \to X \) on a finite set \( T = \{1, 2, \ldots, k\} \) with a probability measure \( p \) where \( p(||i||) = p_i > 0 \) for \( i = 1, \ldots, k \). Notice that this setting allows the deterministic case, that is, the case \( k = 1 \). As before, set \( \Omega = T^\mathbb{N} \) and \( \mathbb{P} = p^\mathbb{N} \). Then, \( \mathbb{P} \) is the Bernoulli probability on \( \Omega \), and the corresponding annealed Perron–Frobenius operator is of the form

\[
\mathcal{L}_f \varphi = \int \mathcal{L}_t \varphi \, dp(t) = \sum_{i=1}^k p_i \mathcal{L}_i \varphi \quad \text{for } \varphi \in L^1(m)
\]

where \( \mathcal{L}_i \) is the Perron–Frobenius operator of \( f_i = f(i, \cdot) \) for \( i = 1, \ldots, k \). Throughout this subsection, we keep the setting and notations.

(a) **Random expanding maps.** Let \( X \) be the unit interval \([0, 1] \) equipped with the Borel \( \sigma \)-field \( \mathcal{B} \) and the normalized Lebesgue measure \( m \). Let \( f_i \) be a piecewise \( C^2 \) nonsingular transformation with a finite partition for each \( i = 1, \ldots, k \). Assume the following expanding (on average) condition:

\[
\sum_{i=1}^k \frac{p_i}{|f'_i(x)|} < 1 \quad \text{for all } x \in X.
\]  

(1.14)

When \( x \) is a discontinuity point of some \( f_i \), the left and right limits of \( f'_i(x) \) are considered and (1.14) with these limits instead of \( f'_i(x) \) are required. Then, the following proposition holds for this \( f \).

**Proposition 1.15.** \( \mathcal{L}_f \) satisfies (C), but does not satisfy (UC).

(b) **Random contracting maps.** Let \((X, \mathcal{B}, m)\) be as in the previous example. Consider the case \( k = 2 \) and

\[
f_1(x) = \frac{x}{2}, \quad f_2(x) = \frac{x}{2} + \frac{1}{2} \quad (x \in X) \quad \text{with} \quad p_1 = p_2 = \frac{1}{2}
\]

known as a special case of Bernoulli convolution [78]. Then the following proposition holds for this \( f \).

**Proposition 1.16.** \( \mathcal{L}_f \) satisfies (AC), but does not satisfy (C).

**Remark 1.17.** Note that each \( f_i \) does not satisfy even (S): the Dirac measure at 0 (resp. 1) is the only invariant probability measure of \( f_1 \) (resp. \( f_2 \)), but it is not absolutely continuous with respect to Lebesgue measure. Namely, the random map \( f \) has a property that each deterministic map \( f_\omega \) does not have (such behaviors are called *noise-induced phenomena*, and have been attracting attention among physicists, cf. [38]). This is contrastive to other examples in Section 1.4.3, which causes us to work a bit harder in Section 1.4.4.
Remark 1.18. A slightly weaker condition than (AC) (i.e., “some $\varepsilon > 0$” instead of “any $\varepsilon > 0$”) appeared in [55, 58] with the name of smoothing property, and was later called almost constrictivity in [56]. Note that the definition of smoothing in [57] is stronger than ones in [55, 58], whence the one in [57] is indeed equivalent to constrictivity. The author of [55] asserted that a Markov operator $P$ is asymptotically periodic (or equivalently constrictive) when $P$ is smoothing. However, this claim cannot be true because the example in Proposition 1.16 satisfies the smoothing property, but does not satisfy (C). We also remark that it was proven in [56] that the almost constrictivity implies (WAP). Figure 1 shows that the results in this paper largely improved it.

(c) Random rotations. Now let $X = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ equipped with the Borel $\sigma$-field $\mathcal{B}$ and the normalized Lebesgue measure $m$. Let $k = 2$ and consider two irrational rotations $f_1$ and $f_2$ with angles $\alpha$ and $\beta$, that is,

$$f_1(x) = x + \alpha \pmod{1} \quad \text{and} \quad f_2(x) = x + \beta \pmod{1}$$

where $\alpha, \beta \in [0, 1]$. Let $p_1 = p_2 = \frac{1}{2}$.

Proposition 1.19. The following hold.

1. If $\alpha - \beta$ is irrational, then $\mathcal{L}_f$ satisfies (C).
2. If $\alpha$ and $\beta$ are irrational numbers such that $\alpha - \beta$ is rational, then $\mathcal{L}_f$ satisfies (MC) but does not satisfy (AC).
3. If $\alpha$ and $\beta$ are rational numbers, then $\mathcal{L}_f$ satisfies (WAP) but does not satisfy (MC).

(d) Direct sums of random contraction and expanding map. Let $k = 2$ and consider the direct sum of random transformations, where one satisfies (S) and the other does not. That is, let $X = X_- \cup X_+$ and equip $X$ with a probability measure $m$ for which both $X_-$ and $X_+$ have a positive measure. With $p_1 = p_2 = \frac{1}{2}$, define

$$f_1(x) = \begin{cases} \tau_1^{-}(x) & \text{for } x \in X_- \\ \tau_1^{+}(x) & \text{for } x \in X_+ \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} \tau_2^{-}(x) & \text{for } x \in X_- \\ \tau_2^{+}(x) & \text{for } x \in X_+ \end{cases}$$

where the random dynamics generated by $\tau_1^{+}, \tau_2^{+} : X_+ \to X_+$ (with equal probabilities) satisfies (S), and the random dynamics generated by $\tau_1^{-}, \tau_2^{-} : X_- \to X_-$ does not satisfy (S). A typical example is the case when $\tau_1^{+}, \tau_2^{+}$ are one-dimensional piecewise $C^2$ nonsingular transformations with finite partitions satisfying the expanding condition (1.14) (with $\tau_1^{+}$ instead of $f_1$), and $\tau_1^{-}(x) = \tau_2^{-}(x) = \frac{x}{2}$ on $X_- = [0, 1]$. Then, the following proposition holds for this $f$.

Proposition 1.20. $\mathcal{L}_f$ satisfies (S), but does not satisfy (WAP).
1.4.4. **Deterministic systems.** By slightly modifying examples in Section 1.4.3, one can easily make examples of deterministic dynamical systems that disprove the reverse implications in Figures 1, as follows. Recall that the baker’s transformation is a map $g : X \to X$ on $X = [0, 1]^2$ (equipped with the Lebesgue measure $m$) defined by $g(x, y) = (2x, \frac{y}{2})$ when $0 \leq x < \frac{1}{2}$ and $g(x, y) = (2x - 2, \frac{y}{2} + \frac{1}{2})$ when $\frac{1}{2} \leq x \leq 1$.

**Proposition 1.21.** The following hold.

1. Any one-dimensional piecewise $C^2$ nonsingular transformation with a finite partition satisfying (1.14) is (C) but not (UC).
2. The baker’s transformation is (AC) but not (C).
3. Any irrational rotations are (MC) but not (AC).
4. Any rational rotations are (WAP) but not (MC).
5. Define $g : X \to X$ as a measurable map preserving a splitting of $X$ by two positive measure sets $X_-, X_+$ such that the restriction of $g$ on $X_+$ satisfies (S) and the restriction of $g$ on $X_-$ does not. Then, $g$ is (S) but not (WAP).

1.5. **Questions.** We would like to leave questions on the notion of physical noise, a possible generalization of Theorem C for more general probabilities, and statistical properties of Markov operators satisfying (AC).

1.5.1. **On the physical noise.** We call noise satisfying condition (b) in Remark 1.2 as physical noise. This kind of noise is frequently used in the literature as indicated in Remarks 1.4 and 1.5. As an important consequence of Remark 1.2, any continuous random perturbations of a dynamics on a compact space $X$ by a physical noise satisfies (FPM). Recall that (FPM) asks the existence of finitely many ergodic measures with disjoint supports which are absolutely continuous with respect to $m$ and the union of its fiberwise statistical basin of attraction covers almost surely $X$. Proposition 1.13 provides a class of examples of continuous random perturbation of a dynamics by an almost physical noise (that is, condition almost-(b)) which does not satisfy (FPM). However, this example still has a unique non-absolutely continuous physical measure where any fiberwise statistical basin of attraction covers the full space up to a null set. Then, a natural question is whether (FPM) could be weakened by asking finitely many physical ergodic stationary measures instead of finitely many absolutely continuous ergodic stationary measures. Call this property (fpm) for short. On the other hand, Proposition 1.14 shows that considering random perturbations by almost physical noise is still insufficient to get (fpm).

**Question 1.** Find a reasonable sufficient condition to get (fpm) generalizing the physical noise.
1.5.2. Generalization of Theorem C. Let \( f : T \times X \to X \) be a \( \mathbb{P} \)-random map as defined in Appendix A where \( \mathbb{P} \) is a shift-invariant probability measure on \((\Omega, \mathcal{F}) = (\mathbb{T}^N, \mathcal{A}^N)\) but not necessarily a Bernoulli probability as before. Consider the annealed Perron–Frobenius operator \( \mathcal{L}_f \) given by

\[
\mathcal{L}_f \varphi = \int \mathcal{L}_\omega \varphi \, d\mathbb{P}(\omega) \quad \text{for } \varphi \in L^1(m)
\]

where \( \mathcal{L}_\omega \) is the Perron–Frobenius of \( f_\omega = f_t \) for \((\omega_i)_{i \geq 0} \in \Omega\) with \( \omega_0 = t \). Then Theorem D applies to \( \mathcal{L}_f \). However, we cannot conclude Theorem C from this because an invariant density of \( \mathcal{L}_f \) does not correspond in general with a stationary measure with respect to \( \mathbb{P} \). To be more clear, an invariant density \( h \) for \( \mathcal{L}_f \) defines a probability measure \( \mu \) by

\[
d\mu = h \, dm
\]

which satisfies that

\[
\mu(A) = \int (f_\omega)_* \mu(A) \, d\mathbb{P}(\omega) \quad \text{for all } A \in \mathcal{B}.
\]

(1.15)

However, a stationary measure \( \mu \) with respect to \( \mathbb{P} \) is not necessarily a measure that satisfies (1.15). See, for instance, [68] where stationary measures for general Markov probabilities are characterized. This stationary is necessary to obtain invariant measures for the skew-product \( F \) in (1.3) associated with \( f \). One can see that if \( \mu \) satisfies (1.15) and \( \mathbb{P} \) is not a Bernoulli measure, \( \mathbb{P} \times \mu \) is not in general an invariant measure of \( F \) (cf. [11, Example 1.4.7]). Since the \( F \)-invariance of \( \mathbb{P} \times \mu \) is essential for us to prove Theorem C, we do not know how to obtain this result in this case.

**Question 2.** Is it possible to obtain a similar result to Theorem C for a random iteration driving by a general probability \( \mathbb{P} \) on \( \Omega \) of a measurable map \( f \) as above?

1.5.3. Equivalent conditions to (AC) and (UC). We next consider the counterpart for (AC) in the position of (APM) for (MC). First, we introduce some definitions.

An invariant density \( h \) of a Markov operator \( P : L^1(m) \to L^1(m) \) is called mixing (resp. exact) if

\[
\lim_{n \to \infty} P^n \varphi = h \int \varphi \, dm \quad \text{weakly (resp. strongly) in } L^1(m)
\]

for any \( \varphi \in L^1(m) \) whose support is included in \( \text{supp } h \). If \( h \) is exact and the above converge occurs exponentially fast (i.e. there are constant \( C > 0, 0 < \rho < 1 \) independently of \( \varphi \) such that \( \|P^n \varphi - h \int \varphi \, dm\| \leq C \rho^n \|\varphi\| \) for each \( n \)), we say that \( P \) is exponentially exact. In particular, given a non-singular measurable map \( g : X \to X \) and an \( \mathcal{L}_g \)-invariant density \( h \), a probability measure \( m_h \) given in (1.6) is mixing (resp. exact) for \( g \) if and only if \( h \) is mixing (resp. exact) for \( \mathcal{L}_g \). See Appendix D for more details.

We also recall the well-known fact that a Markov operator \( P \) is (C) if and only if
(AP) \( P \) is asymptotically periodic, that is, there exist finitely many densities \( g_1, \ldots, g_r \) with mutually disjoint supports (up to an \( m \)-null set), positive bounded linear functionals \( \lambda_1, \ldots, \lambda_r \) on \( L^1(m) \) and a permutation \( \rho \) of \( \{1, \ldots, r\} \) such that \( Pg_i = g_{\rho(i)} \) for each \( i = 1, \ldots, r \) and for \( \varphi \in L^1(m) \),

\[
\lim_{n \to \infty} P^m(\varphi - \sum_{i=1}^r \lambda_i(\varphi)g_i) \to 0 \quad \text{strongly in } L^1(m).
\]  

(1.16)

It is also known that if \( P \) satisfies (AP) with \( r = 1 \), then \( g_1 \) is exact. Refer to e.g. [63].

Mimicking the above asymptotically periodic condition, we introduce the following class of Markov operators:

(\( APW \)) \( P \) is asymptotically periodic weakly, that is, (AP) holds with “weakly” instead of “strongly” in (1.16).

**Question 3.** Is (AC) equivalent to (APW)? Moreover, is \( g_1 \) mixing if \( r = 1 \)?

The different classes considered in this paper can be classified into three categories according to the type of conditions involved in the definition of the class: conditions on constrictor, conditions à la Dunford–Pettis or conditions on periodicity in the limit. Since many of these conditions are ultimately equivalent, we have avoided as far as possible introducing many names to indicate the different equivalent definitions. In Table 1 we organize this classification of classes, but first we introduce two more classes:

(EAP) \( P \) is exponentially asymptotically periodic, that is, (AP) holds with

\[
\left\| P^n(\varphi - \sum_{j=1}^r \lambda_j(\varphi)g_j) \right\| \leq C \rho^n \left\| \varphi \right\| \quad \text{for any } \varphi \in L^1(m)
\]

for some \( C > 0 \), \( 0 < \rho < 1 \) taken independently of \( n, \varphi \), instead of (1.16).

(CW) \( P \) is constrictive weakly\(^6\), that is, there exists a weakly compact set \( F \) of \( D(m) \) such that for any \( h \in D(m) \) there exists \( (\psi_n)_{n \in \mathbb{N}} \subset F \) satisfying that

\[
\lim_{n \to \infty} (P^n h - \psi_n) = 0 \quad \text{weakly in } L^1(m)
\]

In Proposition 4.15 we will show that (APW) \( \Rightarrow \) (CW) \( \Rightarrow \) (AC). As we see below, these implications have some consequences. The first obvious consequence is that Question 3 is actually reduced to prove (AC) \( \Rightarrow \) (APW). Moreover,

\[
(C) \Rightarrow (CW) \Rightarrow (MC) \quad \text{and} \quad (AP) \Rightarrow (APW) \Rightarrow (APM).
\]  

(1.17)

\(^6\)There exists a quite similar name in literature, called weak constrictivity. \( P \) is said to be weakly constrictive if there is a weakly compact set \( F \) of \( \lim_{n \to \infty} d(P^n h, F) = 0 \) for any \( h \in D(m) \). It is known that weak constrictivity is an equivalent condition to (C) (cf. [56]).
Indeed, notice first that “\(\lim_{n \to \infty} d(P^nh, F) = 0\)” in the notion of constrictive Markov operator given in Definition 1.7 can be rephrased as there is \((\psi_n)_{n \in \mathbb{N}} \in F\) such that \(P^nh - \psi_n \to 0\) strongly in \(L^1(m)\). Thus, clearly (C) implies (CW). Also, since (AC) implies (MC), from Proposition 4.15 and Theorem D, we have (CW) \(\Rightarrow\) (MC). Similarly, we have (AP) \(\Rightarrow\) (APW) \(\Rightarrow\) (APM). Finally, we mention that neither of the converses of the implications in (1.17) holds. To see this, we will prove in Proposition 7.3 that in example (b) of Section 1.4.3, \(1_X\) is a \(P\)-invariant density (called the trivial density) which is mixing, but not exact. In particular, \(P\) satisfies (APW) with \(r = 1\) and consequently (CW) from Proposition 4.15. However, since any Markov operator with the mixing and non-exact trivial density is not constrictive (cf. [63]), \(P\) in the example does not satisfy (C). Also, (MC) (resp. (APM)) does not imply (CW) (resp. (APW)) since (AC) is not equivalent to (MC) (and thus (APM)) from Propositions 1.19 (2) and 1.21 (3).

In Theorem 4.1 we prove the equivalence between the different definitions for the class of constrictive in mean Markov operators. In Proposition 6.1, we prove the equivalence between the definitions of uniformly constrictive à la Danford–Pettis and the version on the constrictor. Finally, to complete Table 1 it remains to solve (together with (AC) \(\Rightarrow\) (APW)) the following question:

**Question 4.** Is (UC) equivalent with (EAP)? Moreover, if (EAP) holds with \(r = 1\), then is \(g_1\) exponentially exact?

1.6. **Organization of the paper.** In Section 2 we will prove Theorem B and show the generalization of Araújo’s result mentioned after Theorem A. The proof of the most general versions of Theorems D and E is carried on in Section 4 and 3 respectively. Theorem C is proved in Section 5. In Section 6 we study the sub-hierarchy in (UC). Finally, in Section 7 we provide the proof of the propositions of the examples discussed in the introduction. We also include four appendices that may be of independent interest. In Appendix A we briefly mention in a general framework some properties of the annealed Perron–Frobenius operator that we will use. Appendix B studies and generalizes the restriction of a Markov operator to the support of a density. In Appendix C, we study the ergodicity of invariant densities. Finally, in Appendix D we relate the definition of mixing and exactness in this section with the classical definition of mixing and exactness for deterministic maps.
2. Feller continuity and quasi-compactness: proof of Theorem B

A Markov transition probability \( P(x, A) \) is said to be Feller continuous, strong Feller continuous and ultra Feller continuous if the family of probabilities \( P(x, \cdot) \) varies continuously with respect to the weak* topology, setwise convergence and total variation distance on the space of probabilities respectively. Namely, for any \( x \in X \) and every sequence \( \{x_n\}_{n \geq 1} \) with \( x_n \to x \),

1. \( P(x, A) \) is Feller if \( P(x_n, \cdot) \to P(x, \cdot) \) in the weak* topology, i.e., if
   \[
   \int \varphi(y) P(x_n, dy) \to \int \varphi(y) P(x, dy)
   \]
   for all bounded continuous real-valued function \( \varphi \) on \( X \).

2. \( P(x, A) \) is strong Feller if \( P(x_n, \cdot) \to P(x, \cdot) \) in setwise convergence, i.e., if
   \[
   P(x_n, A) \to P(x, A) \quad \text{for all} \, A \in \mathcal{B}.
   \]

3. \( P(x, A) \) is ultra Feller if \( P(x_n, \cdot) \to P(x, \cdot) \) in total variation distance. i.e., if
   \[
   ||P(x_n, A) - P(x, A)||_{TV} \to 0
   \]
   where the total variation distance of two Borel probability measures \( \mu \) and \( \nu \) on \( X \) is given by
   \[
   ||\mu - \nu||_{TV} = 2 \cdot \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|.
   \]

It is clear by definition that ultra Feller continuity implies strong Feller continuity. Moreover, since \( X \) is a Polish space, an equivalent way of describing the setwise convergence of a sequence of measures \( (\mu_n)_{n \geq 1} \) to \( \mu \) is the following:

\[
\lim_{n \to \infty} \int \varphi \, d\mu_n = \int \varphi \, d\mu,
\]

for all bounded Borel measurable real-valued function \( \varphi \) on \( X \). This is because the simple functions are dense among the bounded Borel measurable real-valued functions on \( X \) under the supremum norm. Thus, as a consequence, strong Feller continuity implies Feller continuity. The converse of these implications is not true in general. However, although ultra Feller continuity seems, at first sight, to be stronger than the strong Feller continuity, it turns out that the two are almost equivalent. More precisely, according to [42, Theorem. 3.37], if two Markov transition probabilities \( Q(x, A) \) and \( R(x, A) \) are strong Feller, then the convolution \( QR \) given by

\[
QR(x, A) = \int R(y, A) Q(x, dy)
\]
is an ultra Feller continuous Markov transition probability. In view of Chapman–Kolmogorov relation,

\[ P^{n+k}(x, A) = \int P^n(y, A)P^k(x, dy) \quad \text{for all } n, k \in \mathbb{N}, \]

we have that \( P^{n+k}(x, A) \) is the convolution of \( P^n(x, A) \) and \( P^k(x, A) \). In particular, we get the following remark:

**Remark 2.1.** If \( P^{n_0}(x, A) \) is strong Feller continuous for some \( n_0 \geq 1 \), then \( P^{2n_0}(x, A) \) is ultra Feller continuous.

To prove Theorem B we need the following proposition that shows some equivalent formulation of the assumption of Theorem A. First, recall that a Markov transition probability \( P(x, A) \) is said to be \( m \)-nonsingular if \( m(A) = 0 \) implies that \( P(x, A) = 0 \) for \( m \)-almost every \( x \in X \). We also address the reader to recall how a transition probability can be associated with a Markov operator explained at the beginning of Section 1.3.4, see also [70, Chapter V.4].

**Proposition 2.2.** Let \((X, \mathcal{B}, m)\) be a Polish probability space. Consider a Markov transition probability \( P(x, A) \) with \( x \in X \) and \( A \in \mathcal{B} \). Then, the following conditions are equivalent:

(a) there exists \( n_0 \geq 1 \) such that
   1. \( P^{n_0}(x, A) \) is strong Feller continuous, and
   2. \( P^{n_0}(x, \cdot) \) is \( m \)-nonsingular;

(b) there exists \( n_0 \geq 1 \) such that
   1. \( P^{n_0}(x, A) \) is ultra Feller continuous, and
   2. \( P^{n_0}(x, \cdot) \) is absolutely continuous with respect to \( m \) for all \( x \in X \);

(c) there exists \( n_0 \geq 1 \) such that

\[ P^{n_0}(x, dy) = p(x, y) \, dm(y) \quad \text{with } x \in X \mapsto p(x, \cdot) \in L^1(m) \text{ continuous.} \]

Moreover, if \( P^n(x, A) \) is an \( n \)-th Markov transition probability associated to a Markov operator \( P : L^1(m) \to L^1(m) \), then \( P^n(x, A) \) is \( m \)-nonsingular for each \( n \geq 1 \).

**Proof.** Assume the condition (a), and show (b). According to Remark 2.1, the condition (1) implies that \( P^{2n_0}(x, A) \) is ultra Feller continuous. Moreover, since \( P^n(x, A) \) is the convolution of \( P \) and \( P^{n-1} \) (see (1.4)) we get from (2) that \( P^n(x, \cdot) \) is \( m \)-nonsingular for all \( n \geq n_0 \) and \( m \)-almost every \( x \in X \). In particular, \( P^{2n_0}(x, \cdot) \) is \( m \)-nonsingular. In fact, the continuity with respect to the total variation distance implies that \( P^{2n_0}(x, \cdot) \) is actually absolutely continuous with respect to \( m \) for all \( x \in X \). Indeed, take \( A \in \mathcal{B} \) with \( m(A) = 0 \), \( x \in X \) and consider \( x_n \to x \) with \( x_n \in X \) such that \( P^{2n_0}(x_n, A) = 0 \) for all \( n \geq 1 \). By the ultra Feller continuity of \( P^{2n_0}(\cdot, A) \) we have that \( P^{2n_0}(x_n, A) \to P^{2n_0}(x, A) \).
as \( n \to \infty \). Then \( P^{2n0}(x, A) = 0 \) as required. Consequently, we get (i) and (ii) for the positive integer \( 2n_0 \).

Conversely, (i) and (ii) clearly imply (1) and (2). We now prove that (i) and (ii) are equivalent to \( L^1(m) \)-continuity of the Radon–Nikodým derivative \( p(x, \cdot) \) of \( P^{n0}(x, \cdot) \) with respect to \( m \). But this follows immediately from the well-known fact that

\[
\|P^{n0}(x, \cdot) - P^{n0}(x', \cdot)\|_{TV} = \|p(x, \cdot) - p(x', \cdot)\| \quad \text{for all } x, x' \in X.
\]

See the equation before Lemma 2 of [61]. This completes the proof of the equivalences.

The last assertion follows immediately from the fact that \( P^n(x, A) = P^{n1} 1_A(x) \) for \( m \)-almost every \( x \in X \). Indeed, by duality

\[
\int P^n(x, A) \, dm = \int_A P^{n1} 1_X \, dm = 0 \quad \text{whenever } m(A) = 0.
\]

Consequently, \( P^n(x, A) = 0 \) for \( m \)-almost every \( x \in X \) concluding that \( P^n(x, A) \) is \( m \)-singular.

The following result is Theorem B stated in terms of Markov processes and Markov operators.

**Theorem 2.3.** Let \((X, \mathcal{B}, m)\) be a compact Polish probability space. Consider a Markov operator \( P : L^1(m) \to L^1(m) \) and let \( P^n(x, A) \) be an associated \( n \)-th transition probability. Assume that there exists \( n_0 \in \mathbb{N} \) such that \( P^{n0}(x, A) \) is strong Feller continuous. Then \( P \) is eventually compact.

**Proof.** According to Proposition 2.2, we can assume that \( P^{n0}(x, A) \) satisfies conditions (i) and (ii) in that proposition. Now, from [43, Lemma 1], we have that the absolute continuity of \( P^{n0}(x, \cdot) \) with respect to \( m \) is equivalent to the uniformly absolutely continuity of such measure with respect to \( m \) for all \( x \in X \). That is, for any \( x \in X \) and \( \varepsilon > 0 \), there is \( \delta_x = \delta_x(\varepsilon) > 0 \) such that

\[
P^{n0}(x, A) < \varepsilon \quad \text{for all } A \in \mathcal{B} \text{ with } m(A) < \delta_x.
\]

**Claim 2.4.** For each \( \varepsilon > 0 \), there is \( \delta = \delta(\varepsilon) > 0 \) such that

\[
P^{n0}(x, A) < \varepsilon \quad \text{for all } A \in \mathcal{B} \text{ with } m(A) < \delta \text{ and } x \in X.
\]

**Proof.** By the continuity of the family of probability measures \( P^{n0}(x, \cdot) \) with respect to the total variation, for each \( x \in X \), there exists a neighborhood \( V(x, \varepsilon) \) of \( x \) such that \( P^{n0}(x', A) < \varepsilon \) for all \( x' \in V(x, \varepsilon) \) and \( A \in \mathcal{B} \) with \( m(A) < \delta_x \). Since the union of the open sets \( V(x, \varepsilon) \) for \( x \in X \) covers the compact set \( X \), we can extract a finite subcover \( V(x_1, \varepsilon), \ldots, V(x_k, \varepsilon) \). Thus, the claim follows by taking \( \delta = \min\{\delta_{x_1}, \ldots, \delta_{x_k}\} \). \( \square \)
Finally, let us conclude the proof. First, recall that a bounded family \( F \subset L^1(m) \) is said to be uniform integrable (in \( L^1(m) \)) if for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that
\[
\int_A |g| \, dm < \varepsilon \quad \text{for all} \quad g \in F \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta.
\]
Writing \( P^{n_0}(x, dy) = p(x, y) \, dm(y) \) and take \( F = \{p(x, \cdot) : x \in X\} \subset D(m) \). Claim 2.4 implies that \( F \) is uniform integrable (in \( L^1(m) \)). Then, according to [91, Corollary 2.5 (b)], the operator \( \pi^2 \) is compact where \( \pi : L^\infty(m) \to L^\infty(m) \) is given by
\[
\pi \psi(x) = \int \psi(y) p(x, y) \, dm(y) = \int \psi(y) P^{n_0}(x, dy) = P^{n_0} \psi(x), \quad \psi \in L^\infty(m).
\]
That is, \( \pi \) is the adjoint operator \( P^{n_0} \) of \( P^{n_0} \). Thus, \( P^{2n_0} = (P^{n_0})^2 = \pi^2 \) is a compact operator. Hence, by Schauder’s theorem (cf. [79, Theorem 4.19]), \( P^{2n_0} \) is also compact, concluding the proof. \( \square \)

We just indicate that Theorem B follows immediately from Proposition 2.2 and Theorem 2.3. Actually, Theorem B and Theorem 2.3 are equivalent from the following observation.

**Remark 2.5.** As indicated in the introduction, it is well known that the theory of Markov operators and Markov processes are intimately related. Less known perhaps is that the general theory of Markov operators is actually equivalent to the particular theory of annealed Perron–Frobenius operator associated with random maps on Polish probability space \((X, \mathcal{B}, m)\). Indeed, clearly given a random map \( f \) we define a Markov operator by means of the annealed Perron–Frobenius operator \( \mathcal{L}_f \). Conversely, given a Markov operator \( P : L^1(m) \to L^1(m) \), we consider a transition probability \( P(x, A) \) which induces \( P \). See [70, Proposition V.4.4] and Section 1.3.4 for more details. Notice that
\[
P(x, A) = P^* 1_A(x) \quad \text{for } m\text{-almost every } x \in X
\]
where \( P^* \) is the adjoint operator of \( P \). Now, Kifer proved in [51, Theorem 1.1] that any transition probability in a Polish\(^7\) space \( X \) can be represented by a random product of independent and identically distributed measurable maps. That is, there exists a probability space \((T, \mathcal{A}, p)\) and a measurable map \( f : T \times X \to X \) such that
\[
P(x, A) = p(\{t \in T : f_t(x) \in A\})
\]
where \( f_t = f(t, \cdot) \). This implies that \( P(x, A) = \mathcal{L}^*_f 1_A(x) \) for \( m\)-almost every \( x \in X \) where \( \mathcal{L}^*_f \) is the adjoint operator of the annealed Perron–Frobenius operator \( \mathcal{L}_f \) associated with \( f \). Hence, \( P^* 1_A = \mathcal{L}^*_f 1_A \) and therefore \( P = \mathcal{L}^*_f \). To see this final assertion,

\(^7\)Actually the only requirement is that \((X, \mathcal{B})\) will be countably generated measurable space, i.e., that the \( \sigma \)-algebra \( \mathcal{B} = \sigma(A) \) for some countable subset \( A \) of \( \mathcal{B} \).
observe first that any \( g \in L^\infty(m) \) can be approximated uniformly by simple functions \( g_n = \sum_{i=1}^N a_i 1_{A_i} \) where \( a_i = a_i(n) \in \mathbb{R}, A_i = A_i(n) \in \mathcal{B} \) and \( N = N(n) \in \mathbb{N} \). Then, \( g_n \) converges to \( g \) in \( L^\infty(m) \)-norm and hence

\[
P^*g = \lim_{n \to \infty} P^* g_n = \lim_{n \to \infty} \sum_{i=1}^N a_i P^* 1_{A_i} = \lim_{n \to \infty} \sum_{i=1}^N a_i \mathcal{L}^* 1_{A_i} = \lim_{n \to \infty} \mathcal{L}^* g_n = \mathcal{L}^* f
\]

This implies that \( P^* = \mathcal{L}^* f \) as well as \( P = \mathcal{L} f \) as required.

2.1. Generalization of Araújo’s result. We will prove that under the condition (i) mentioned in Remark 1.2, the assumptions of Theorems A and B hold. To do this, we need some preliminaries.

The following proposition provides a well-known sufficient condition to obtain that a given Markov transition probability is Feller continuous. The proof is straightforward and could be founded in [41, Theorem 4.22]. Recall that a continuous random map is a measurable map \( f : T \times X \to X \) such that \( f_t = f(t, \cdot) \) is continuous for \( p \)-almost every \( t \in T \).

**Proposition 2.6.** If \( P(x, A) \) is the transition probability associated to a continuous random map, then \( P(x, A) \) is Feller continuous.

The following result extends [60, Theorem 1.2 in Section 4] to the case of compact Polish probability spaces.

**Theorem 2.7.** Let \((X, \mathcal{B}, m)\) be a compact Polish probability space. Consider a Markov transition probability \( P(x, A) \) with \( x \in X \) and \( A \in \mathcal{B} \). Assume that

1. \( P(x, A) \) is Feller continuous, and
2. \( P(x, \cdot) \) is absolutely continuous with respect to \( m \) for all \( x \in X \).

Then \( P(x, A) \) is a strong Feller continuous transition probability.

We will prove the above result by modifying the argument in [60] where Theorem 2.7 was shown with \( \mathbb{R}^d \) instead of \( X \). For this modification, we need the following.

**Lemma 2.8.** Let \((X, \mathcal{B}, m)\) be a Polish probability space. Then, for any \( A \in \mathcal{B} \) and \( \delta > 0 \), there exists \( B \in \mathcal{B} \) such that

\[
m(\partial B) = 0 \quad \text{and} \quad m(A \Delta B) < \delta
\]

where \( \partial B \) is the boundary of \( B \) and \( A \Delta B \) is the symmetric difference of \( A \) and \( B \).

**Proof.** Consider the subalgebra \( \mathcal{A} \) in \( \mathcal{B} \) of \( m \)-continuity sets, i.e., \( B \in \mathcal{A} \) if and only if \( B \in \mathcal{B} \) and \( m(\partial B) = 0 \). Let us prove that the \( \sigma \)-algebra generated by the \( \mathcal{A} \) is
To see this, fix any metric \( d \) compatible with the topology of \( X \). Note that for \( x \in X, \partial B_r(x) \subset \{ y \in X : d(x, y) = r \} \) where \( B_r(x) \) denotes the ball centered at \( x \) and of radius \( r > 0 \). In particular, \( \partial B_r(x) \cap \partial B_s(s) = \emptyset \) for \( r \neq s \). But in a space of finite measure, there can only be countably many pairwise disjoint sets of positive measure. Hence, \( B_r(x) \in \mathcal{A} \) for all but at most countably many \( r \). In particular, \( \mathcal{A} \) contains a neighborhood base of \( x \). This shows that \( \sigma(\mathcal{A}) = \mathcal{B} \) as desired. Now, since the subalgebra \( \mathcal{A} \subset \mathcal{B} \) generates \( \mathcal{B} \), according to the well-known result on approximation generating subalgebras (cf. [66, Theorem 1.1]), for every \( A \in \mathcal{B} \) and \( \delta > 0 \), there is \( B \in \mathcal{A} \) such that \( m(AB) < \delta \). This concludes the proof of the lemma.

Now, we are in a position to prove Theorem 2.7.

**Proof of Theorem 2.7.** Let us fix \( A \in \mathcal{B} \) and \( x \in X \), and consider a sequence \( \{x_n\}_{n \geq 1} \) converging to \( x \). We will prove that \( P(x_n, A) \to P(x, A) \) as \( n \to \infty \) concluding the strong Feller continuity of \( P(x, A) \). It follows from Portemanteau’s theorem (cf. [54, Theorem 13.16]) that \( P(x_n, \cdot) \) converges to \( P(x, \cdot) \) in the weak topology if and only if \( P(x_n, B) \) converges to \( P(x, B) \) for any continuity set \( B \) of \( P(x, \cdot) \), i.e., any \( B \in \mathcal{B} \) satisfying \( P(x, \partial B) = 0 \). The former condition holds because we assumed that \( P(x, A) \) is Feller continuous, and thus, recalling that \( P(x, \cdot) \) is absolutely continuous with respect to \( m \) by assumption,

\[
P(x_n, B) \to P(x, B) \quad \text{for any } B \in \mathcal{B} \text{ satisfying } m(\partial B) = 0. \tag{2.1}
\]

Now, according to [43, Lemma 1], the absolute continuity of a probability measure \( \nu \) with respect to \( m \) is equivalent to the uniform absolute continuity of \( \nu \) with respect to \( m \). That is, \( \nu(B) \to 0 \) as \( m(B) \to 0 \). Thus, for each \( \varepsilon > 0 \) and \( n \geq 1 \), by Lemma 2.8 and the absolute continuity of \( P(x_n, \cdot) \) with respect to \( m \), one can find \( B_{\varepsilon,n} \in \mathcal{B} \) such that

\[
m(\partial B_{\varepsilon,n}) = 0 \quad \text{and} \quad P(x_n, A \Delta B_{\varepsilon,n}) < \frac{\varepsilon}{2^n}.
\]

Set

\[
C_{\varepsilon} = \bigcup_{n \geq 1} B_{\varepsilon,n} \quad \text{and} \quad D_{\varepsilon} = \bigcap_{n \geq 1} B_{\varepsilon,n}.
\]

**Claim 2.9.** It holds that

\[
P(x, C_{\varepsilon}) - 2\varepsilon \leq \liminf_{n \to \infty} P(x_n, A) \leq \limsup_{n \to \infty} P(x_n, A) \leq P(x, C_{\varepsilon}).
\]

**Proof.** Since

\[
m(\partial(C_{\varepsilon} \cup D_{\varepsilon})) \leq m(C_{\varepsilon} \cup \partial D_{\varepsilon}) \leq 2 \sum_{n=1}^{\infty} m(\partial B_{\varepsilon,n}) = 0
\]
it follows from (2.1) that
\[ \limsup_{n \to \infty} P(x_n, C_\varepsilon \setminus B_{\varepsilon,n}) \leq \limsup_{n \to \infty} P(x_n, C_\varepsilon \setminus D_\varepsilon) = P(x, C_\varepsilon \setminus D_\varepsilon) \leq P(x, (A\Delta C_\varepsilon) \cup (A\Delta D_\varepsilon)) \leq 2 \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = 2\varepsilon. \]

Therefore, since
\[ P(x_n, C_\varepsilon) \leq P(x_n, C_\varepsilon \setminus B_{\varepsilon,n}) + P(x_n, A \cup B_{\varepsilon,n}) \leq P(x_n, C_\varepsilon \setminus B_{\varepsilon,n}) + P(x_n, A\Delta B_{\varepsilon,n}) + P(x_n, A) \]
by applying (2.1) again (note that \( m(\partial C_\varepsilon) \leq \sum_{n=1}^{\infty} m(\partial B_{\varepsilon,n}) = 0 \)) we have
\[ \liminf_{n \to \infty} P(x_n, A) \geq \liminf_{n \to \infty} P(x_n, C_\varepsilon) - 2\varepsilon - \lim_{n \to \infty} \frac{\varepsilon}{2^n} = P(x, C_\varepsilon) - 2\varepsilon. \]

On the other hand, since
\[ P(x_n, A) \leq P(x_n, B_{\varepsilon,n}) + P(x_n, A\Delta B_{\varepsilon,n}) \leq P(x_n, C_\varepsilon) + \frac{\varepsilon}{2^n} \]
we have
\[ \limsup_{n \to \infty} P(x_n, A) \leq P(x, C_\varepsilon) \]
concluding the proof of the claim. \( \square \)

Finally, observe that
\[ |P(x, C_\varepsilon) - P(x, A)| \leq P(x, A\Delta C_\varepsilon) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon. \]
Thus, \( P(x, C_\varepsilon) \to P(x, A) \) as \( \varepsilon \to 0 \). Hence, by Claim 2.9, we get that
\[ \lim_{n \to \infty} P(x_n, A) = \lim_{\varepsilon \to 0} P(x, C_\varepsilon) = P(x, A). \]
This concludes the proof of the theorem. \( \square \)

Remark 2.10. A converse of Theorem 2.7 follows from [34, Proposition 12.1.7]. Also, from [47, Remark 2], by arguing as in the implication of (b) from (a) in Proposition 2.2, we immediately get the following more specific converse: if \( P(x, A) \) is a strong Feller continuous Markov transition probability on a Polish probability space \( (X, \mathscr{B}, m) \), then there exists a probability measure \( m^* \) on \( (X, \mathscr{B}) \) such that \( m \) is absolutely continuous with respect to \( m^* \) and \( P(x, A) \) satisfies (1) and (2) with \( m^* \) instead of \( m \).

Corollary 2.11. Let \( (X, \mathscr{B}, m) \) be a compact Polish probability space. Consider a continuous random map \( f : T \times X \to X \) where \( (T, \mathcal{A}, p) \) is a probability space and let \( P^n(x, A) \) be the associated \( n \)-th transition probability. Assume that there is \( n_0 \geq 1 \) such that
\[ P^{n_0}(x, \cdot) \text{ is absolutely continuous with respect to } m \text{ for all } x \in X. \]
Then, for each \( x \in X \), there exists \( p(x, \cdot) \in D(m) \) such that
\[
P^{2n_0}(x, A) = p(x, y) \, dm(y) \quad \text{and} \quad x \in X \mapsto p(x, \cdot) \in L^1(m) \text{ is continuous.}
\]
Moreover, the annealed Perron–Frobenius operator \( L_f \) associated with \( f \) is eventually compact.

Proof. According to Proposition 2.6, \( P^{n_0}(x, A) \) is a Feller continuous Markov transition probability. Hence, by Theorem 2.7, we have that \( P^{n_0}(x, A) \) is actually strong Feller continuous. Therefore, Remark 2.1, Proposition 2.2 and Theorem 2.3 immediately imply the conclusion of the corollary. \( \square \)

3. Existence of invariant measures: proof of Theorem E

In what follows, \( (X, \mathcal{B}, m) \) denotes any abstract probability space. We will prove in this section a generalization of Theorem E which will be used to prove Theorem D in the next section. But first, to prove these results we will need some preliminaries that can be interesting by themselves.

3.1. Fundamental lemma. If \( \phi \) belongs to \( L^1(m) \), then the support of \( \phi \), denoted by \( \text{supp} \phi \), is not defined in a unique way, since \( \phi \) can be represented by two functions whose values are different in a zero \( m \)-measuring set. However, since it is common to simplify terminology, we consider \( \text{supp} \phi \) as a set defined by the relation \( \phi \neq 0 \). Moreover, if we want to emphasize that a relationship between sets holds except a set of zero \( m \)-measure, we say that it holds up to an \( m \)-null set. For instance, \( A \subseteq B \) up to an \( m \)-null set if \( m(A \setminus B) = 0 \).

Lemma 3.1. Let \( E \) be either \( L^1(m) \) or \( L^\infty(m) \). Let \( P : E \to E \) be a positive linear bounded operator and consider \( \phi, \psi \in E \) with \( \psi \geq 0 \). It holds that
\[
\begin{align*}
(1) & \quad \text{supp} \, P\phi \subseteq \text{supp} \, P1_{\text{supp} \phi} \text{ and } \text{supp} \, P\psi = \text{supp} \, P1_{\text{supp} \psi}, \\
(2) & \quad \text{if } \text{supp} \phi \subseteq \text{supp} \psi, \text{ then } \text{supp} \, P\phi \subseteq \text{supp} \, P\psi.
\end{align*}
\]

Proof. It is easy to see that (1) implies (2). Indeed, if \( \text{supp} \phi \subseteq \text{supp} \psi \), we have that \( 1_{\text{supp} \phi} \leq 1_{\text{supp} \psi} \) and since \( P \) is positive linear operator, \( P1_{\text{supp} \phi} \leq P1_{\text{supp} \psi} \). This implies that \( \text{supp} \, P1_{\text{supp} \phi} \subseteq \text{supp} \, P1_{\text{supp} \psi} \). Now, (1) immediately implies the conclusion of (2). But, actually, if (2) holds for any pair \( \phi \) and \( \psi \) in the assumptions of the lemma, we also get (1) as follows. Since \( \text{supp} \phi = \text{supp} \, P1_{\text{supp} \phi} \), by (2) we get that \( \text{supp} \, P\phi \subseteq \text{supp} \, P1_{\text{supp} \phi} \). Same argument and inclusion hold for \( \psi \). But, in this case, taking into account that \( \psi \geq 0 \), we can also apply (2) to \( \text{supp} \, P1_{\text{supp} \psi} = \text{supp} \psi \) getting the other inclusion and proving (1).
In view of the above observation, the proof of the lemma is reduced to show (2). To do this, let \( \phi \) and \( \psi \) be as in the statement. From the approximation by simple functions, \( \phi \) and \( \psi \) can be written as the pointwise limit of a functions

\[
\phi_n = \sum_{k=1}^{N} a_k 1_{F_k} \quad \text{and} \quad \psi_n = \sum_{k=1}^{M} b_k 1_{G_k}
\]

respectively, where \( N = N(n), M = M(n) \in \mathbb{N}, a_k = a_k(n) \neq 0, b_k = b_k(n) > 0 \) and \( F_k = F_k(n), G_k = G_k(n) \in \mathcal{B} \) such that \( F_\ell \cap F_j = \emptyset \) and \( G_\ell \cap G_j = \emptyset \) if \( \ell \neq j \). Moreover, \( \phi_n \) and \( \psi_n \) can be chosen\(^8\) so that

(i) \( \text{supp } \phi_n = \text{supp } \phi \subset \text{supp } \psi = \text{supp } \psi_n \), and

(ii) \( \phi_n \) and \( \psi_n \) converges in norm of \( E \) to \( \phi \) and \( \psi \), respectively.

Since \( P \) is a bounded operator, (ii) implies that \( P\phi_n \to P\phi \) and \( P\psi_n \to P\psi \). Then

(iii) \( m((\text{supp } P\phi_n) \Delta (\text{supp } P\phi)) \to 0 \) and \( m((\text{supp } P\psi_n) \Delta (\text{supp } P\psi)) \to 0 \) as \( n \to \infty \).

Note that by (i), \( \phi_n \) and \( \psi_n \) satisfy the assumption in (2). It is easy to see that if (2) holds for \( \phi_n \) and \( \psi_n \), then (iii) implies that \( \text{supp } P\phi \subset \text{supp } P\psi \) up to an \( m \)-null set.

Hence, we reduce the proof to check the conclusion of (2) for \( \phi_n \) and \( \psi_n \).

Taking into account \( a_k \neq 0, P1_{F_k} \geq 0 \) and the linearity of \( P \), we see that

\[
\text{supp } P\phi_n = \text{supp } \sum_{k=1}^{N} a_k P1_{F_k} \subset \bigcup_{k=1}^{N} \text{supp } P1_{F_k}
\]

\[
= \text{supp } \sum_{k=1}^{N} P1_{F_k} = \text{supp } P1_{\cup_{k=1}^{N} F_k} = \text{supp } P1_{\text{supp } \phi_n}. \tag{3.1}
\]

Now, using that \( b_k > 0, P1_{G_k} \geq 0 \) and the linearity of \( P \),

\[
\text{supp } P1_{\text{supp } \psi_n} = \text{supp } P1_{\cup_{k=1}^{M} G_k} = \text{supp } \sum_{k=1}^{M} P1_{G_k} = \text{supp } \sum_{k=1}^{M} b_k P1_{G_k} = \text{supp } P\psi_n. \tag{3.2}
\]

Observe that (3.1) and (3.2) implies (1) for \( \phi_n \) and \( \psi_n \). As we see at the beginning of the proof, (1) implies (2) and we conclude the lemma. \( \square \)

\(^8\) For a non-negative measurable function \( g \), we consider the mesh with width \( 2^{-n} \) from the level \( 2^{-n} \) up to \( 2^n \), i.e.,

\[
g_n = \sum_{k=0}^{2^n-1} (k+1)2^{-n}1_{g^{-1}(k2^{-n},(k+1)2^{-n}]} + 2^n1_{g^{-1}(2^n,\infty)}.
\]

Then we can see that \( g_n \) pointwise converges to \( g \) since \( 0 \leq g_n(x) - g(x) < 2^{-n} \) on the set where \( g \leq 2^n \). Moreover, \( \text{supp } g = \text{supp } g_n \). For a general \( g \), we consider \( g = g^+ - g^- \) where \( g^+ \) and \( g^- \) are the positive and negative parts of \( g \) which we approximate by simple functions \( g_n^+ \) and \( g_n^- \) as before. Finally, it is simple to verify that \( g_n = g_n^+ - g_n^- \) converges in the norm \( L^1(m) \) or \( L^{\infty}(m) \) where appropriate, to \( g \) and \( \text{supp } g_n = \text{supp } g \).
3.2. Restrictions of Markov operators. Fix $S \in \mathcal{B}$ with $m(S) > 0$. Let us consider the probability space $(S, \mathcal{B}_S, m_S)$ where

$$\mathcal{B}_S = \{A \in \mathcal{B} : A \subset S\} \quad \text{and} \quad m_S(A) = \frac{m(A)}{m(S)}, \quad A \in \mathcal{B}_S \quad (\text{i.e., } dm_S = \frac{1_s}{m(S)} dm).$$

Denote $L^1(S, \mathcal{B}_S, m_S)$ by $L^1(m_S)$. Observe that $L^1(m_S) \hookrightarrow L^1(m)$ by means of the inclusion

$$1_S : \phi \in L^1(m_S) \mapsto 1_S \phi \in L^1(m) \quad \text{given by} \quad 1_S \phi(x) = \begin{cases} \phi(x) & x \in S, \\ 0 & x \in X \setminus S. \end{cases} \quad (3.3)$$

Although we are using the same notation $1_S$ to designate the characteristic function of $S$ and the inclusion by the characteristic function, it should cause no confusion. Now, given a Markov operator $P : L^1(m) \to L^1(m)$, we define an operator

$$P_S : L^1(m_S) \to L^1(m_S), \quad P_S \phi = 1_S \cdot P(1_S \phi) \quad \text{for } \phi \in L^1(m_S). \quad (3.4)$$

It is not difficult to see that $P_S$ is a positive contraction of $L^1(m_S)$, that is, $P_S \phi \geq 0$ if $\phi \geq 0$ and $\|P_S\| \leq 1$. Taking advantage of the abuse of notation, we can extend $P_S$ to $L^1(m)$ as follows:

$$P_S \phi = 1_S \cdot P(1_S \phi) \quad \text{for } \phi \in L^1(m).$$

When no confusion can arise, we will omit the dot in the above expression and in (3.4). We also identify $L^1(m_S)$ with

$$1_S(L^1(m_S)) = \{\phi \in L^1(m) : \text{supp } \phi \subset S\} \subset L^1(m).$$

**Lemma 3.2.** Let $P : L^1(m) \to L^1(m)$ be a Markov operator and consider $S \in \mathcal{B}$ with $m(S) > 0$. If $\text{supp } P1_S \subset S$ up to an $m$-null set, then $P_S : L^1(m_S) \to L^1(m_S)$ defined in (3.4) is a Markov operator and

$$P_S \phi = P(1_S \phi) \quad \text{for all } \phi \in L^1(m).$$

**Proof.** Let $\phi \in L^1(m)$. Note that $\text{supp } 1_S \phi \subset S = \text{supp } 1_S$. Thus, by Lemma 3.1, $\text{supp } P(1_S \phi) \subset \text{supp } P1_S \subset S$ up to an $m$-null set. In particular, $1_S P(1_S \phi) = P(1_S \phi)$. Then, for any $\phi \in L^1(m)$,

$$\int P_S \phi \, dm_S = \frac{1}{m(S)} \int_S P(1_S \phi) \, dm = \frac{1}{m(S)} \int_S P(1_S \phi) \, dm = \frac{1}{m(S)} \int_S \phi \, dm = \int_S \phi \, dm_S.$$

Therefore, $P_S$ is a Markov operator. \qed

The implication in the previous lemma is actually an equivalence. To see this observation and other interesting equivalent conditions see Proposition B.1 in the Appendix B where the theory of the restriction of a Markov operator is extended and clarified. For the following result, recall the notion of weak almost periodicity (WAP)
in Definition 1.9. As mentioned, according to [88, Theorem 3.1], (WAP) is equivalent to the existence of an invariant density with the maximal support.

**Proposition 3.3.** Let $P : L^1(m) \to L^1(m)$ be a Markov operator and consider $h \in D(m)$ such that $Ph = h$ and set $S = \text{supp} h$. Then, $P_S : L^1(m_S) \to L^1(m_S)$ is a weakly almost periodic Markov operator. Moreover, $P_S h = h$ and $P_S \phi = P(1_S \phi)$ for all $\phi \in L^1(m)$.

**Proof.** Note that $\text{supp} h = S = \text{supp} 1_S$ and by Lemma 3.1 and the $P$-invariance of $h$, it follows that $S = \text{supp} h = \text{supp} Ph = \text{supp} P 1_S$ up to an $m$-null set as well as $m(S) > 0$. Hence, Lemma 3.2 implies that $P_S$ is a Markov operator and $P_S \phi = P(1_S \phi)$ for all $\phi \in L^1(m)$. To prove that $P_S$ is also weakly almost periodic, it suffices to see that $P_S h = h$ since $P_S 1_S = 1_S$ by the Markov property and thus $h$ has trivially the maximal support (as a fixed point of $P_S$). But this is proved as follows: $P_S h = P(1_S h) = Ph = h$. □

3.3. **Proof of Theorem E.** In this subsection, we prove a more general version of Theorem E. In particular, we remark that the new item (5) below was shown in [84] to be a sufficient condition for the existence of a $P$-invariant density, and below we show it is indeed also a necessary condition.

**Theorem 3.4.** Let $(X, \mathcal{B}, m)$ be an abstract probability space and consider a Markov operator $P : L^1(m) \to L^1(m)$. Then, the following conditions are equivalent:

1. There exists an invariant density for $P$;
2. There exist $\alpha \in (0, 1)$ and $\delta > 0$ such that
   \[ \sup_{n \geq 0} \int_A P^n 1_X \, dm < \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta; \]
3. There exist $\alpha \in (0, 1)$ and $\delta > 0$ such that
   \[ \sup_{n \geq 0} \int_A A^n 1_X \, dm < \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta; \]
4. There exist $\alpha \in (0, 1)$ and $\delta \geq 0$ such that
   \[ \inf_{n \geq 1} \int_A A^n 1_X \, dm > 1 - \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) > \delta. \]
5. There exist $\alpha \in (0, 1)$ and $\delta > 0$ such that
   \[ \limsup_{n \to \infty} \int_A P^n 1_X \, dm < \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta; \]
6. There exist $\alpha \in (0, 1)$ and $\delta > 0$ such that
   \[ \limsup_{n \to \infty} \int_A A^n 1_X \, dm < \alpha \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta; \]

The function $1_X$ in (5) and (6) could be substituted by any arbitrary density $h \in D(m)$. Moreover, the equivalence also holds by taking $\alpha = 1$ in all the above items.
Proof. We first prove (1) implies (2). Let $g \in D(m)$ be an invariant density of $P$, that is, $Pg = g$. Hence, Proposition 3.3 implies that $P_S : L^1(m_S) \to L^1(m_S)$ is a weak almost periodic Markov operator where $S = \text{supp } g$. According to [88, Theorem 3.1], this is equivalent to weak compactness of $(P^n_{S1_S})_{n \geq 0}$ in $L^1(m_S)$. Hence, by Dunford–Pettis characterization, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{n \geq 0} \int_A P^n_{S1_S} dm_S < \varepsilon \quad \text{for any } A \in \mathcal{B}_S \text{ with } m_S(A) < \delta. \quad (3.5)$$

Note that clearly $m(S) > 0$ since $g$ is a density function. Thus, $m(X \setminus S) < 1$ and hence, we can take $\varepsilon > 0$ small enough such that $\alpha := \varepsilon \cdot m(S) + m(X \setminus S) < 1$. Then, by Dunford–Pettis characterization, there is $\delta > 0$ such that (3.5) holds. Let $\delta' = \delta \cdot m(S) > 0$. Observe that, according to Proposition 3.3, $P_S1_S = P1_S$ and then, $P^n_{S1_S} = P^n1_S$ and supp $P^n1_S \subset S$ up to an $m$-null set for all $n \geq 0$. Finally, for any $A \in \mathcal{B}$ with $m(A) < \delta'$ it holds that $m_S(A \cap S) < \delta$ and then,

$$\sup_{n \geq 0} \int_A P^n1_X dm = \sup_{n \geq 0} \left( \int_A P^n1_S dm + \int_A P^n1_{X \setminus S} dm \right)$$

$$= \sup_{n \geq 0} \left( m(S) \cdot \int_{A \cap S} P^n1_S dm_S + \int_A P^n1_{X \setminus S} dm \right)$$

$$< m(S) \cdot \varepsilon + \sup_{n \geq 0} \int_{X \setminus S} P^n1_X dm$$

$$< \varepsilon \cdot m(S) + m(X \setminus S) = \alpha.$$

The implication from the condition (2) to (3) follows immediately taking into account that

$$\int_A A_n1_X dm = \frac{1}{n} \sum_{i=0}^{n-1} \int_A P^i1_X dm < \frac{1}{n} \sum_{i=0}^{n-1} \alpha = \alpha$$

for all $n \geq 1$ and $A \in \mathcal{B}$ with $m(A) < \delta$.

We show the implication from (3) to (4). Observe that since $P$ is a Markov operator, then $P^11_X = 1_X$ and thus,

$$\int A_n1_X dm = \int A^n1_X dm = \frac{1}{n} \sum_{k=0}^{n-1} \int P^k1_X dm = 1.$$

From this and assumption (3), we have

$$\inf_{n \geq 1} \int_{X \setminus E} A_n1_X dm = \inf_{n \geq 1} \int A_n1_X dm - \sup_{n \geq 1} \int_{E} A_n1_X dm > 1 - \alpha$$

for all $E \in \mathcal{B}$ with $m(E) < \delta$. Setting $\delta$ in (4) as $1 - \delta$ in (3), we get the desired implication.
Next we show the implication from (4) to (1). Suppose contrarily that there is no invariant density for $P$. Then, according to [59, Theorem 4.6 and Lemma 4.5], there exists $\psi \in L^\infty(m)$ with $\psi \geq 0$, fully supported on $X$ (i.e., supp $\psi = X$) and $\|A_n^*\psi\|_\infty \to 0$ as $n \to 0$. Now, since $h$ is fully supported on $X$, for $\delta \geq 0$ as in the condition (4), we can find $\eta > 0$ such that the set $E = \{h \geq \eta\}$ satisfies $m(E) > \delta$. Since $\psi \geq \eta$ on $E$, $\eta^{-1}\psi \geq 1_E$. Hence,

$$\int_E A_n 1_X \, dm = \int 1_E A_n 1_X \, dm \leq \frac{\psi}{\eta} \int A_n 1_X \, dm = \frac{1}{\eta} \int A_n^* \psi \, dm \leq \frac{1}{\eta} \|A_n^* \psi\|_\infty \to 0$$

as $n \to \infty$. This contradicts (4) and the proof is done. Finally, observe that the previous arguments work assuming $\alpha = 1$ in all of the items.

To complete the proof we will see now the equivalence with (5), (6). Clearly (2) implies (5). Also, (5) immediately implies (6). On the other hand, from [84, Theorem 1], we have that (6) implies (1). Moreover, a slight modification of the argument of [84, Theorem 1] also shows that (6) implies (1). Indeed, assume (6) instead of (5) (which is called (C) in [84] and where $1_X$ could be substituted by any density in $D(m)$).

Define

$$\lambda(\varphi) = \text{Lim} \int A_n 1_X \varphi \, dm \quad \text{for } \varphi \in L^\infty(m)$$

where Lim is a Banach limit (i.e. replace $P^n$ in the definition of $\lambda(h)$ of [84] with $A_n$).

The rest of the proof in [84, Theorem 1] literally works to conclude (1) in our case, except proving $\lambda(P^n \varphi) = \lambda(\varphi)$ for each $\varphi \in L^\infty(m)$, that is the unique calculation that one needs to do. To see it,

$$\lambda(P^n \varphi) = \text{Lim} \int A_n 1_X P^n \varphi \, dm = \text{Lim} \int \frac{1}{n} \sum_{i=0}^{n-1} P^{i+1} 1_X \varphi \, dm$$

$$= \text{Lim} \int \left( A_n 1_X + \frac{1}{n} P^n 1_X - \frac{1}{n} 1_X \right) \varphi \, dm = \lambda(\varphi) + \text{Lim} \frac{1}{n} \int (P^n 1_X - 1_X) \varphi \, dm.$$  

In the last equality we used the linearity of Banach limits. On the other hand,

$$\left| \int (P^n 1_X - 1_X) \varphi \, dm \right| \leq \|\varphi\|_\infty \int (P^n 1_X + 1_X) \, dm = 2\|\varphi\|_\infty$$

since $P$ is a Markov operator. Hence we get $\frac{1}{n} \int (P^n 1_X - 1_X) \varphi \, dm \to 0$ as $n \to \infty$, and obtain the desired equality $\lambda(P^n \varphi) = \lambda(\varphi)$. The version for $\alpha = 1$ follows since, actually, the assumption in [84, Theorem 1] is (5) for $\alpha = 1$ and where $1_X$ could be substituted by any density in $D(m)$. This completes the proof. $\Box$.

The implications between (1)–(4) in Theorem 3.4 require strongly that $P$ is a Markov operator. However, the equivalence between (1) and (4) for $\alpha = 1$ holds even under the weaker assumption that the linear positive operator $P$ is just a contraction (i.e.,
∥P∥_{op} ≤ 1). As a final result of this section, we will prove it as follows. Compare with [59, Theorem 4.2 and Corollary 4.7 in Chapter 3] and [71, Theorem 2].

**Theorem 3.5.** If $P : L^1(m) \to L^1(m)$ is a linear positive contraction, then the following are equivalent:

(i) there exists a $P$-invariant density;
(ii) there exists $δ ≥ 0$ such that
\[
\inf_{n≥0} \int_E P^n 1_X dm > 0 \quad \text{for all } E ∈ ℬ \text{ with } m(E) > δ;
\]
(iii) there exists $δ ≥ 0$ such that
\[
\inf_{n≥0} \int_E A^n 1_X dm > 0 \quad \text{for all } E ∈ ℬ \text{ with } m(E) > δ.
\]

**Proof.** Observe that the proof of the implication of (1) from (4) in Theorem 3.4 does not require that \( \int P φ \, dm = \int φ \, dm \) for all $φ \in L^1(m)$. This shows that (iii) implies (i). The equivalence between (i) and (ii) follows from [59, Theorem 4.2] as follows. This result says that if $h$ is a $P$-invariant density, then
\[
\inf_{n≥0} \int E P^n 1_X dm > 0 \quad \text{for all } E \subset \text{supp } h \text{ with } m(E) > δ.
\]
Taking $δ = 1 - m(\text{supp } h) ≥ 0$, we have that if $E \subset X$ with $m(E) > δ$, then $m(E) > 0$ where $E = E \cap \text{supp } h$. Hence, from the above inequality it holds that
\[
\inf_{n≥0} \int E P^n 1_X dm ≥ \inf_{n≥0} \int E P^n 1_X dm > 0.
\]
Finally, since (ii) implies (iii) immediately, we conclude the cycle of implications and thus the proof of the theorem. \( □ \)

### 4. Mean constrictivity: proof of Theorem D

In the sequel, we will prove the following result which is a more general version of Theorem D.

**Theorem 4.1.** Let $(X, ℬ, m)$ be an abstract probability space and consider a Markov operator $P : L^1(m) \to L^1(m)$. Then the following conditions are equivalent:

(ℳ) $P$ is mean constrictive;
(ℳ2) There is a compact set $F \subset L^1(m)$ and $κ < 1$ such that
\[
\limsup_{n→∞} d(A_n φ, F) ≤ κ \quad \text{for any } φ \in D(m);
\]
(WMC) \( P \) is weakly mean constrictive, i.e., there is a weakly compact set \( F \subset L^1(m) \) such that
\[
\lim_{n \to \infty} d(A_n \varphi, F) = 0 \quad \text{for any } \varphi \in D(m);
\]

(WMC2) There is a weakly compact set \( F \subset L^1(m) \) and \( \kappa < 1 \) such that
\[
\limsup_{n \to \infty} d(A_n \varphi, F) \leq \kappa \quad \text{for any } \varphi \in D(m);
\]

(MCDP) \( P \) is mean constrictive à la Dunford–Pettis, i.e., for every \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that for any \( \varphi \in D(m) \), there is \( n_0 = n_0(\varepsilon, \varphi) \geq 1 \) satisfying that
\[
\int_E A_n \varphi \ dm < \varepsilon \quad \text{for any } n \geq n_0 \text{ and } E \in \mathcal{B} \text{ with } m(E) < \delta;
\]

(MCDP2) There exist \( \kappa < 1 \) and \( \delta > 0 \) such that for any \( \varphi \in D(m) \), there is \( n_0 = n_0(\varphi) \geq 1 \) satisfying that
\[
\int_E A_n \varphi \ dm \leq \kappa \quad \text{for any } n \geq n_0 \text{ and } E \in \mathcal{B} \text{ with } m(E) < \delta;
\]

(AMC) \( P \) is asymptotically mean constrictive, i.e., for every \( \varepsilon > 0 \), there is \( \delta > 0 \) such that
\[
\limsup_{n \to \infty} \int_E A_n \varphi \ dm < \varepsilon \quad \text{for any } \varphi \in D(m) \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta;
\]

(AMC2) There exist \( \kappa < 1 \) and \( \delta > 0 \) such that
\[
\limsup_{n \to \infty} \int_E A_n \varphi \ dm < \kappa \quad \text{for any } \varphi \in D(m) \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta;
\]

(FED) \( P \) admits finitely many ergodic invariant densities \( h_1, \ldots, h_r \) with mutually disjoint supports (up to an \( m \)-null set) and the invariant density function \( h = \frac{1}{r} (h_1 + \cdots + h_r) \) has the maximal support;

(APM) \( P \) is asymptotically periodic in mean, i.e., there exist finitely many ergodic invariant densities \( h_1, \ldots, h_r \) with mutually disjoint supports (up to an \( m \)-null set) and positive bounded linear functionals \( \lambda_1, \ldots, \lambda_r \) on \( L^1(m) \) such that
\[
\lim_{n \to \infty} \left\| A_n \varphi - \sum_{i=1}^r \lambda_i(\varphi) h_i \right\| = 0 \quad \text{for any } \varphi \in L^1(m).
\]

We will prove Theorem 4.1 following the implications described in Figure 3. To organize the proof, we divide the proof into two subsections.

4.1. Proof of the lemmas.

Lemma 4.2. Condition (WMC) (resp. (WMC2)) implies condition (MCDP) (resp. (MCDP2)).
Proof. Let $F$ be as in the condition (WMC). Fix $\varepsilon > 0$. The condition (WMC) guarantees that for any $\varphi \in D(m)$ there exists $n_0 = n_0(\varepsilon, \varphi) \geq 1$ such that
\[
\inf_{\phi \in F} \| A_n \varphi - \phi \| < \frac{\varepsilon}{2} \quad \text{for any } n \geq n_0.
\]
Since $F$ is weakly compact, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $\phi \in F$ and $E \in \mathcal{B}$ with $m(E) < \delta$, we have $\int_E |\phi| \, dm < \frac{\varepsilon}{2}$. Thus, for any set $E$ with $m(E) < \delta$
\[
\int_E A_n \varphi \, dm \leq \inf_{\phi \in F} \| A_n \varphi - \phi \| + \int_E |\phi| \, dm < \varepsilon \quad \text{for any } n \geq n_0.
\]
This proves (MCDP).

The implication (WMC2) implies (MCDP2) follows analogously. Namely, we put $\kappa$ in (MCDP2) as the sum of $\kappa$ in (WMC2) and $\varepsilon = 1 - \kappa^2$ from the weak compactness of $F$. \qed

Lemma 4.3. The condition (FED) implies the condition (APM).

Proof. Suppose the condition (FED). In particular, the existence of an invariant density with the maximal support implies that $P$ is (WAP), cf. [88, Theorem 3.1]. Yosida and Kakutani proved in [92] that (WAP) implies mean ergodicity.\footnote{The converse of this implication was recently proved in [88, Proposition 3.9].} Recall that a Markov operator $P : L^1(m) \to L^1(m)$ is called mean ergodic if $A_n \varphi$ converges in $L^1(m)$-norm for any $\varphi \in L^1(m)$. For each density $h \in D(m)$, denote by $h^*$ the limit in $L^1(m)$-norm of $A_n h$. Observe that $h^*$ is an invariant density of $P$. Hence, from the finitude of ergodic invariant densities $h_1, \ldots, h_r$, or equivalently extremal points in the space of invariant densities (see Proposition C.10 in Appendix C), we can find $\lambda_1(h), \ldots, \lambda_r(h)$ with $\lambda_i(h) \geq 0, \lambda_1(h) + \cdots + \lambda_r(h) = 1$ such $h^* = \lambda_1(h) h_1 + \cdots + \lambda_r(h) h_r$. We have the equation in (APM) for any $h \in D(m)$ and also for $\varphi \in L^1(m)$ with $\varphi \geq 0$.\footnote{The converse of this implication was recently proved in [88, Proposition 3.9].}
Then, considering the positive part and negative part of any function in $L^1(m)$, we can extend the above argument to $L^1(m)$. Obviously $\lambda_i : L^1(m) \to \mathbb{R}$ is bounded and linear so that the proof is done.

□

**Lemma 4.4.** The condition (APM) implies the condition (MC).

**Proof.** The proof is straightforward, taking the compact constrictor for $A_n$ by

$$F = \{ \varphi \in L^1(m) : \varphi \text{ is a convex combination of } h_1, \ldots, h_r \}.$$ 

Therefore, the proof is done.

□

4.2. **Proof of the theorem.** In this subsection, we will show that (AMC2) implies (FED).

**Remark 4.5.** Observe that from Proposition 3.3, the restriction $P_S$ of a Markov operator $P$ to the support $S$ of a $P$-invariant density is a Markov operator. Hence, Lemma B.1 in the Appendix B implies that $P^*1_S \geq 1_S$. In particular the sequence $(P^*1_S)_{n \geq 1}$ is increasing.

**Proposition 4.6.** (AMC2) implies that the set of ergodic invariant densities of $P$ is nonempty and finite. Moreover, these ergodic invariant densities have mutually disjoint support.

**Proof.** Observe first that by virtue of Theorem 3.4 (6), it immediately follows from (AMC2) that there exists a $P$-invariant density. Thus, by Corollary C.10,

$$E = \{ g \in L^1(m) : g \text{ is an ergodic } P\text{-invariant density} \} \neq \emptyset.$$ 

The following claim is the key to obtaining the finitude of ergodic $P$-invariant densities.

**Claim 4.7.** Consider $E \in \mathcal{B}$ such that $m(E) > 0$ and $P^*1_E \geq 1_E$. Let $\delta > 0$ be the constant given in (AMC2). Then, $m(E) \geq \delta$.

**Proof.** Assume that $m(E) < \delta$. Then, applying (AMC2) to $h = \frac{1_E}{m(E)} \in D(m)$, we have

$$\limsup_{n \to \infty} \int_E A_n \frac{1_E}{m(E)} dm \leq \kappa < 1.$$ 

On the other hand, for any $n \geq 1$,

$$\int_E A_n \frac{1_E}{m(E)} dm = \frac{1}{m(E)} \int_E A_n^*1_E dm = \frac{1}{m(E)} \int_E \frac{1}{n} \sum_{i=0}^{n-1} P^i1_E dm \geq \frac{1}{m(E)} \int_E 1_E dm = 1.$$ 

This is a contradiction, and thus, we have $m(E) \geq \delta$. 

□
Now, by Remark 4.5, the support $S$ of any invariant density satisfies $P^*1_S \geq 1_S$ and by Corollary C.10 and Remark C.11 two distinct ergodic $P$-invariant densities have disjoint support. Hence, it follows from Claim 4.7 that the cardinality of $\mathcal{E}$ is less than $\delta^{-1}$. Set $r$ be this cardinality (notice that $r > 0$ since $\mathcal{E} \neq \emptyset$). Thus, we get that $P$ admits finitely many ergodic invariant densities $h_1, \ldots, h_r$ in $D(m)$ with mutually disjoint supports. □

**Proposition 4.8.** (WAP) holds if (AMC2) holds.

We postpone the proof of the above proposition and show how to use it to conclude the following main result of this subsection:

**Theorem 4.9.** (AMC2) implies (FED). In particular, (AC) implies (MC).

**Proof.** Let $h_1, \ldots, h_r$ be the ergodic invariant densities of $P$ obtained in Proposition 4.6. To conclude (FED) we need to prove that the $P$-invariant density $h = \frac{1}{r}(h_1 + \cdots + h_r)$ has the maximal support. According to Proposition 4.8, $P$ is (WAP). Then there is a $P$-invariant density $g$ with the maximal support, i.e., such that $\lim_{n \to \infty} P^*1_{\text{supp } g} = 1_X$. Since the set of $P$-invariant densities is convex and the ergodic densities are its extremal points, it follows that $g = \lambda_1 h_1 + \cdots + \lambda_r h_r$ with $\lambda_1 \geq 0$ and $\lambda_1 + \cdots + \lambda_r = 1$. Then, $\text{supp } g \subset \text{supp } h$ and thus we also have $\lim_{n \to \infty} P^*1_{\text{supp } h} = 1_X$. This shows that $h$ has the maximal support concluding the proof.

The final implication follows immediately from the first part of the theorem since, trivially, (AC) $\Rightarrow$ (AMC2) and (FED) $\Rightarrow$ (MC) by Lemmas 4.3 and 4.4. □

4.2.1. **Proof of Proposition 4.8.** Let $h$ be a $P$-invariant density. Set $S = \text{supp } h$. Define

$$\varphi \overset{\text{def}}{=} \lim_{n \to \infty} P^n 1_S.$$

This limit exists since the sequence $(P^n 1_S)_{n \geq 1}$ is increasing. See Remark 4.5. The first observation is that $0 \leq \varphi \leq 1$. Also, by the monotone continuity property of $P^*$ (see [70, Proposition V.4.1]), $P^* \varphi = \varphi$. Moreover, since $P^*1_S \geq 1_S$, one has that $\varphi = 1$ on $S$. Let us split

$$X = E_0 \cup E \cup E_1, \quad E = \{x \in X : 0 < \varphi < 1\} \quad \text{and} \quad E_i = \{x \in X : \varphi = i\} \quad \text{for } i = 0, 1.$$

**Lemma 4.10.** We have $P^*1_E \leq 1_E$ and $P^*1_{E_0} \geq 1_{E_0}$.

**Proof.** Note that $\text{supp } 1_E \subset \text{supp } \varphi$ and $\text{supp } 1_E \subset \text{supp } (1_X - \varphi)$. Then, by Lemma 3.1, it follows that

$$\text{supp } P^*1_E \subset \text{supp } P^*\varphi \cap \text{supp } P^*(1_X - \varphi) = \text{supp } \varphi \cap \text{supp } (1_X - \varphi) = E.$$
Then, Proposition B.1 in Appendix B implies that $P^1_E \leq 1_E$. On the other hand, since \( \lim_{n \to \infty} \min\{1_X, n\varphi\} = 1_{\supp \varphi} \), it holds that
\[
P^1_{\supp \varphi} = \lim_{n \to \infty} P^n \min\{1_X, n\varphi\} \leq \lim_{n \to \infty} \min\{1_X, n\varphi\} = 1_{\supp \varphi}.
\]
Equivalently, $P^1_{E_0} \geq 1_{E_0}$ since $E_0 = X \setminus \supp \varphi$. \hfill \( \Box \)

**Lemma 4.11.** For every $A \subset \supp \varphi \setminus S$, \( \lim_{n \to \infty} P^n(\varphi 1_A) = 0 \).

**Proof.** By assumption, $1_A \leq 1_{\supp \varphi \setminus S}$. Hence, $0 \leq \varphi 1_A \leq \varphi 1_{\supp \varphi \setminus S}$. Then, since
\[
\lim_{n \to \infty} P^n(\varphi 1_{\supp \varphi \setminus S}) = \lim_{n \to \infty} P^n(\varphi 1_{\supp \varphi} - \varphi 1_S) = \lim_{n \to \infty} P^n\varphi - \lim_{n \to \infty} P^n 1_S = \varphi - \varphi = 0,
\]
it also follows that \( \lim_{n \to \infty} P^n(\varphi 1_A) = 0 \). \hfill \( \Box \)

Let us define
\[
\psi^{\text{inj}} = \lim_{n \to \infty} P^n 1_E.
\]
As above, notice that this limit exists since, from Lemma 4.10, the sequence \( (P^n 1_E)_{n \geq 1} \) is decreasing. Moreover, again, by the monotone continuity property of $P^n$, we have that $P^n \psi = \psi$ and $\supp \psi \subset E$.

**Lemma 4.12.** If $m(E_0) = 0$, then $\varphi = 1_X - \psi$.

**Proof.** Since $m(E_0) = 0$,
\[
1_X = \lim_{n \to \infty} P^n 1_X = \lim_{n \to \infty} P^n (1_S + 1_{E_1 \setminus S} + 1_E) = \varphi + \lim_{n \to \infty} P^n 1_{E_1 \setminus S} + \psi.
\]
Having it in mind that $1_{E_1 \setminus S} = \varphi 1_{E_1 \setminus S}$ and $E_1 \setminus S \subset \supp \varphi \setminus S$, Lemma 4.11 implies that \( \lim_{n \to \infty} P^n 1_{E_1 \setminus S} = 0 \). Substituting above, we get $1_X = \varphi + \psi$ concluding the proof. \hfill \( \Box \)

For each $\varepsilon > 0$, let us denote $B_\varepsilon = \{ x \in X : 1 - \varepsilon \leq \psi(x) \}$.

**Lemma 4.13.** If $\varphi = 1_X - \psi$, then $\lim_{n \to \infty} P^n(\psi 1_{B_\varepsilon}) = \psi$ for every $\varepsilon > 0$. Moreover, if $m(B_\varepsilon) = 0$ for some $\varepsilon > 0$, then $\psi = 0$ (and hence, $\varphi = 1$) $m$-almost everywhere.

**Proof.** For each $k \geq 1$, define $C_k = \{ x \in X : \frac{1}{k + 1} \leq \psi(x) < \frac{1}{k} \}$. Since $\varphi = 1_X - \psi$, we have
\[
C_k = \{ x \in X : 1 - \frac{1}{k} < \psi(x) \leq 1 - \frac{1}{k + 1} \}.
\]
Hence,
\[
1_{C_k} \psi \leq 1_{C_k} \left( 1 - \frac{1}{k + 1} \right) = 1_{C_k} \cdot \frac{k}{k + 1} \leq k1_{C_k} \varphi.
\]
Moreover, from Lemma 4.11, since $C_k \subset E \subset \supp \varphi \setminus S$, we have that
\[
0 \leq \lim_{n \to \infty} P^n (1_{C_k} \psi) \leq k \lim_{n \to \infty} P^n (1_{C_k} \varphi) = 0
\]
and thus, for each $k \geq 1$,
\[
\lim_{n \to \infty} P^n (1_{C_k} \psi) = 0. \tag{4.1}
\]
On the other hand, for each $\varepsilon > 0$, there exists $k_0 \geq 1$ such that

$$E \setminus B_\varepsilon = \{x \in X : 0 < \psi(x) < 1 - \varepsilon \} \subset \bigcup_{k=1}^{k_0} C_k.$$  

Then, from (4.1) we get

$$\lim_{n \to \infty} P^n(x) = 0 \quad \text{for all } x > 0. \quad (4.2)$$

Now, observe that $\psi = \psi_1 E = \psi_1 B_\varepsilon + \psi_1 E \setminus B_\varepsilon$. Hence, by (4.2), we conclude that

$$\lim_{n \to \infty} P^n(\psi_1 E) = \lim_{n \to \infty} P^n(\psi) = \psi.$$  

Finally, note that if $m(B_\varepsilon) = 0$ for some $\varepsilon > 0$ then $\psi < 1 - \varepsilon$ $m$-almost everywhere. From this, it follows that

$$\psi = P^n(\psi) = P^n(\psi_1 E) < (1 - \varepsilon)P^n(1 - \varepsilon) \psi \quad \text{as } n \to \infty.$$  

Then $\psi = 0$ $m$-almost everywhere. \hfill \Box

Now we set

$$\mathcal{F} = \{S \subset X : S \text{ is the support of a } P\text{-invariant density}\}.$$  

Then, according to Proposition 4.6 there is at least one $P$-invariant density and thus $\mathcal{F} \neq \emptyset$. The inclusion up to an $m$-null set induces a partial order in $\mathcal{F}$. Given a chain, by the Zorn lemma, there exists a maximal element $S \in \mathcal{F}$ of the chain in the sense that if $S \subset S'$ and $S' \in \mathcal{F}$ then $S' = S$ up to an $m$-null set. Let $h \in D(m)$ be a $P$-invariant density such that $S = \text{supp } h$ and consider for this density the sets $E_0$, $E$ and $E_1$ defined above.

**Lemma 4.14.** (AMC2) implies that $m(E_0) = 0$ and $m(B_\varepsilon) = 0$ for some $\varepsilon > 0$.

**Proof.** By Lemma 4.10, $P^1 E_0 \geq 1 E_0$. If $m(E_0) > 0$, then Proposition B.1 implies that $P_{E_0} : L^1(m_0) \to L^1(m_0)$ is a Markov operator. Therefore, by virtue of Theorem 3.4 (6), it immediately follows from (AMC2) that there exists a $P_{E_0}$-invariant density $g \in D(m_{E_0})$. In particular, it satisfies that $P(1_{E_0} g) = 1_{E_0} g$ and $\int 1_{E_0} g dm = m(E_0)$. Hence, $\phi = \frac{1}{h} (h + \frac{1_{E_0} g}{m(E_0)})$ is a $P$-invariant density and thus $S' = \text{supp } \phi \in \mathcal{F}$ and $S = \text{supp } h \subseteq S'$. This contradicts the maximality of $S$, concluding that $m(E_0) = 0$.

Next we will see that there exists $\varepsilon > 0$ such that $m(B_\varepsilon) = 0$. From the assumption (AMC2), there are $0 < \kappa < 1$ and $\delta > 0$ so that

$$\limsup_{n \to \infty} \int_B A_n \phi dm \leq \kappa \quad \text{for each } \phi \in D(m) \text{ and } B \in \mathcal{B} \text{ with } m(B) < \delta. \quad (4.3)$$

On the other hand, one can find $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$ it holds $m(B_\varepsilon) < \delta$ since $m(B_\varepsilon) \to 0$ as $\varepsilon \to 0$. We fix $0 < \varepsilon < \min\{\varepsilon_0, 1 - \kappa\}$. We will see that $m(B_\varepsilon) = 0$. 


Arguing by contradiction, we assume that \( m(B_\epsilon) > 0 \). Then for a function \( \phi = \frac{1_{B_\epsilon}}{m(B_\epsilon)} \), it follows from (4.3) that
\[
\kappa \geq \limsup_{n \to \infty} \int_{B_\epsilon} A_n \phi \, dm
= \limsup_{n \to \infty} \int_X \phi \cdot A_n 1_{B_\epsilon} \, dm \geq \limsup_{n \to \infty} \int_X \phi \cdot A_n (\psi 1_{B_\epsilon}) \, dm = \int_X \phi \psi \, dm
\]
by the Lebesgue dominated convergence theorem and Lemma 4.13 (see \( \varphi = 1_X - \psi \) due to \( m(E_0) = 0 \) and Lemma 4.12). But we also have
\[
\int_X \phi \psi \, dm = \frac{1}{m(B_\epsilon)} \int_{B_\epsilon} \psi \, dm \geq \frac{1}{m(B_\epsilon)} \int_{B_\epsilon} (1 - \epsilon) \, dm > \kappa.
\]
Therefore, we conclude \( m(B_\epsilon) = 0 \). □

Proof of Proposition 4.8. The result follows from Lemmas 4.12, 4.13 and 4.14. □

4.3. On the classes \((AC)\), \((CW)\) and \((APW)\). In Theorem 4.9 we have proved that \((AC) \Rightarrow (MC)\) and, in particular from Theorem D, \((AC)\) implies \((APM)\). The following proposition concludes that conditions \((APW)\) and \((CW)\) introduced in Section 1.5.3 are sufficient for \((AC)\). Thus, as a consequence, \((APW) \Rightarrow (APM)\) and \((CW) \Rightarrow (MC)\).

Proposition 4.15. It holds that \((CW)\) implies \((AC)\). Furthermore, \((APW)\) implies \((CW)\).

Proof. Assume that \((CW)\) holds. Hence, given \( h \in D(m) \), there is \( (\psi_n)_{n \geq 1} \subset F \) such that \( P^n h - \psi_n \to 0 \) weakly in \( L^1(m) \) as \( n \to \infty \). Therefore,
\[
\limsup_{n \to \infty} \int_A P^n h \, dm = \limsup_{n \to \infty} \left( \int_A \psi_n \, dm + \int_A (P^n h - \psi_n) \, dm \right)
= \limsup_{n \to \infty} \int_A \psi_n \, dm \leq \sup_{\psi \in F} \int_A \psi \, dm.
\]
Since \( F \) is a weakly compact set, it follows from Dunford–Pettis theorem that for any \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( \sup_{\psi \in F} \int_A \psi \, dm < \epsilon \) for any \( A \in \mathcal{B} \) with \( m(A) < \delta \). This concludes that \( P \) satisfies \((AC)\).

Next we assume that \((APW)\) holds. Then, for \( h \in D(m) \), since \( P \) is a Markov operator, due to the weak convergence in \((APW)\),
\[
0 = \lim_{n \to \infty} \int P^n (h - \sum_{i=1}^r \lambda_i(h) g_i) \, dm = \int (\varphi - \sum_{i=1}^r \lambda_i(h) g_i) \, dm = 1 - \sum_{i=1}^r \lambda_i(h). \quad (4.4)
\]
Set
\[
F := \left\{ \sum_{i=1}^r a_i g_i : a_i \in \mathbb{R}, \sum_{i=1}^r a_i = 1 \right\} \subset D(m).
\]
Notice that $F$ is a weakly compact set and also $P$-invariant since $P g_i = g_{\rho(i)}$ where $\rho$ is the permutation of $\{1, \ldots, r\}$ in the definition of $(\text{APW})$. Moreover, in view of (4.4), the weak convergence in $(\text{APW})$ means that for any $h \in D(m)$, there exists

$$
\psi_n := \sum_{i=1}^r \lambda_i(h) P^n_i = \sum_{i=1}^r \lambda_i(h) g_{\rho(i)} \in F
$$

such that $P^n h - \psi_n \to 0$ weakly as $n \to \infty$. This concludes that $P$ satisfies $(\text{CW})$. \hfill \Box

5. Characterization of finitude of physical measures: proof of Theorem C

Let $(X, \mathcal{B}, m)$ and $(\Omega, \mathcal{F}, P) = (T^N, \mathcal{A}^N, p^N)$ be a locally compact Polish probability space and the infinite product space of a probability space $(T, \mathcal{A}, p)$, respectively. Consider a measurable map $f : T \times X \to X$ and $f_0^\omega = \text{id}$ and $f_n^\omega = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}$ for $n > 0$ and $\omega = (\omega_0, \omega_1, \ldots) \in \Omega$, where we denoted $f_t = f(t, \cdot)$ for $t \in T$. Recall the notion of stationary measure of $f$ in (1.2). Also, recall that the convergence of measures that we are considering is in the weak* topology. That is, $\mu_n \to \mu$ if and only if

$$
\int \varphi \ d\mu_n \to \int \varphi \ d\mu \quad \text{for all } \varphi \in C_b(X) \quad (5.1)
$$

where $C_b(X)$ denotes the set of bounded real-valued continuous functions of $X$. However, since $X$ is a locally compact Polish space and the measures $\mu_n$ and $\mu$ are assumed to be probabilities, according to Portemanteau’s theorem (cf. [54, Theorem 13.16]), (5.1) is equivalent to

$$
\int \varphi \ d\mu_n \to \int \varphi \ d\mu \quad \text{for all } \varphi \in C_c(X).
$$

Here $C_c(X) \subset C_b(X)$ denotes the set of compactly supported continuous functions.

**Lemma 5.1.** Let $\mu$ be an ergodic stationary measure of $f$. Then,

(i) $\bar{\mu}(B(\mu)) = 1$ where $\bar{\mu} = P \times \mu$;

(ii) $\mu(B_w(\mu)) = 1$ for $P$-almost every $\omega \in \Omega$;

(iii) $P(B_x(\mu)) = 1$ for $\mu$-almost every $x \in X$.

Here,

$$
B_w(\mu) = \{ x \in X : (\omega, x) \in B(\mu) \} \quad \text{and} \quad B_x(\mu) = \{ \omega \in \Omega : (\omega, x) \in B(\mu) \} \quad (5.2)
$$

and

$$
B(\mu) = \left\{ (\omega, x) \in \Omega \times X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_j^\omega(x)} = \mu \right\}.
$$
Proof. Since $\mu$ is an ergodic stationary measure of $f$, the measure $\bar{\mu} = \mathbb{P} \times \mu$ is an ergodic invariant probability measure of the corresponding skew-product transformation $F$. We now consider the set $B(\mu)$ of points $(\omega, x) \in \Omega \times X$ such that for any $\varphi \in C_c(X)$, it holds

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_j^\omega(x) = \int_X \varphi \, d\mu.$$ 

Similarly we define the set $B(\mu, \varphi)$ of points $(\omega, x) \in \Omega \times X$ for which the above limit holds for a fixed continuous function $\varphi : X \to \mathbb{R}$. Taking into account that $\varphi \circ f_n^\omega(x) = \bar{\varphi} \circ F_n^\omega(\omega, x)$ and $\int_X \varphi \, d\mu = \int_X \bar{\varphi} \, d\bar{\mu}$ where $\bar{\varphi}(\omega, x) = \varphi(x)$ for $(\omega, x) \in \Omega \times X$ ($\bar{\varphi} \in C_c(\Omega \times X)$), it follows from the Birkhoff Ergodic Theorem that $B(\mu, \varphi)$ has full $\bar{\mu}$-measure. Now, since we assumed that $X$ is locally compact Polish space, $C_c(X)$ is separable (cf. [90, Section 42]). Thus, taking a countable dense subset $S$ of $C_c(X)$, it is not difficult to see that

$$B(\mu) = \bigcap_{\varphi \in S} B(\mu, \varphi)$$

and therefore $B(\mu)$ has also full $\bar{\mu}$-measure. Finally, we observe that $B_\omega(\mu)$ and $B_x(\mu)$ are the $\omega$-section and $x$-section of $B(\mu)$ respectively, i.e., it holds (5.2). Hence, by Fubini’s theorem, we have that $\mu(B_\omega(\mu)) = 1$ and $\mathbb{P}(B_x(\mu)) = 1$ for $\mathbb{P}$-almost every $\omega \in \Omega$ and $\mu$-almost every $x \in X$ respectively. \qed

We highlight the following lemma for future reference. This lemma follows immediately by the Fubini theorem and the definition of $B_\omega(\mu)$ and $B_x(\mu)$ in (5.2).

**Lemma 5.2.** The following are equivalent:

1. $\bar{\mu}(B(\mu)) > 0$ where $\bar{\mu} = \mathbb{P} \times \mu$;
2. $\mu(B_\omega(\mu)) = 1$ for $\mathbb{P}$-almost every $\omega \in \Omega$;
3. $\mathbb{P}(B_x(\mu)) = 1$ for $m$-almost every $x \in X$.

**Remark 5.3.** The equivalence in Lemma 5.2 also holds asking ”$> 0$" instead of ”$= 1$". Thus, an ergodic stationary probability $\mu$ of $f$ is said to be a physical measure if one of the following equivalent conditions holds:

1. $\bar{\mu}(B(\mu)) > 0$ where $\bar{\mu} = \mathbb{P} \times \mu$;
2. $\mu(B_\omega(\mu)) > 0$ for $\mathbb{P}$-almost every $\omega \in \Omega$;
3. $\mathbb{P}(B_x(\mu)) > 0$ for $m$-almost every $x \in X$. 

This definition agrees with the classical notion of physical measure in the deterministic case (that is when $\Omega$ is a singleton). However, this new definition differs from previous notions of physical measure for random maps introduced in the literature (see for instance [4, Equation (2)] or [6, page 313]). This concept was introduced by asking $m(G(\mu)) > 0$ where

$$G(\mu) = \left\{ x \in X : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_j^{\omega}(x)} = \mu \right\}$$

for $\mathbb{P}$-almost every $\omega \in \Omega$.

Since $\bar{m}(B(\mu)) \geq m(G(\mu))$, one has that $m(G(\mu)) > 0$ implies $\bar{m}(B(\mu)) > 0$. However, a priori, it is not expected the converse. Also, defining $G(\mu)$ as the statistical basin of attraction does not seem appropriate, as the following example shows.

**Example 5.4.** Let $X = T = [-1, 1]$ be equipped with the normalized Lebesgue measure and let $X_- = [-1, 0)$, $X_+ = (0, 1]$. Consider a continuous map $f_0 : X \to X$ which has exactly two sinks $p_- = -\frac{3}{4}$, $p_+ = \frac{1}{4}$ whose basin of attraction is $X_-, X_+$, respectively, and $0$ is the fixed point of $f_0$. With the notation $B(x, r)$ for the ball of radius $r$ centered at $x$, we also assume that $f_0(X) \subset B(0, \frac{3}{4})$ and $f_0(B(p_+\frac{1}{4})) \subset B(p_+\frac{3}{4})$. Then, for the random map $f : T \times X \to X$ with additive noise given by $f_t(x) = f_0(x) + \frac{3}{4}t$, it holds that $f_t(B(p_+\frac{1}{4})) \subset B(p_+\frac{3}{4})$ for any $t \in T$. See Figure 4.

![Figure 4](image_url)

**(a) The map $f_0$ on $[-1, 1]$.

(b) The map $f_t$ on $[-1, 1]$**

**Figure 4. Illustrations of the random map $f$ in Example 5.4.**

By the argument in Remark 1.4, the annealed Perron–Frobenius operators associated with the restrictions of $f$ on both $B(p_-, \frac{1}{4})$ and $B(p_+, \frac{1}{4})$ satisfy (FPM). Therefore, $f$ admits at least two absolutely continuous ergodic stationary measures $\mu_-, \mu_+$ whose
supports are included in $B(p_-, \frac{1}{4}), B(p_+, \frac{1}{4})$, respectively. Since the annealed Perron–Frobenius operator associated with $f$ itself also satisfies (FPM), $f$ admits finitely many absolutely continuous ergodic stationary measures $\mu_1, \ldots, \mu_r$, two of which are $\mu_-, \mu_+$, such that

$$m(B_\omega(\mu_1) \cup \cdots \cup B_\omega(\mu_j)) = 1 \quad \text{for } \mathbb{P}\text{-almost every } \omega.$$

On the other hand, by the continuity of $f_0$, there is a neighborhood $U$ of 0 such that $f_0(U) \subset B(0, \frac{1}{16})$, so that for each $x \in U$, both $\{f_t(x) : t \in T\} \cap X_+$ and $\{f_t(x) : t \in T\} \cap X_-$ have positive Lebesgue measure. Consequently, one can find positive $\mathbb{P}$-measure sets $\Gamma_-, \Gamma_+$ and $n_0 \geq 1$ such that if $\omega \in \Gamma_\pm$ then $f_n^\omega(x) \in B(p_\pm, \frac{1}{4})$ for any $n \geq n_0$ and $x \in U$. This concludes that $U \not\subset \bigcup_{j=1}^r G(\mu_j)$. Therefore,

$$0 < m(G(\mu_1) \cup \cdots \cup G(\mu_j)) < 1.$$

In conclusion, if one expects finitely many physical measures where the union of their basins of attraction covers the whole space almost everywhere, then the good notion of the basin should be the fiberwise statistical basin.

The following lemma will be essential to prove Theorem C. In this lemma, the support $\text{supp } \eta$ of a measure $\eta$ on $X$ (which is not necessarily absolutely continuous with respect to $m$) is defined as the set of all points $x$ in $X$ for which every open neighborhood $V$ of $x$ has positive measure.

**Lemma 5.5.** Let $\mu$ be a stationary measure of $f$. Then for any probability measure $\eta$ with $\text{supp } \eta \subset B_\omega(\mu)$,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} (f_j^\omega)_* \eta = \mu.$$

In particular, if $x \in B_\omega(\mu)$, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_j^\omega(x)) = \int \varphi \, d\mu \quad \text{for all } \varphi \in C_b(X). \quad (5.3)$$

**Proof.** Take any probability measure $\eta$ with support contained in $B_\omega(\mu)$. Given any $\varphi \in C_c(X)$, by definition of $B_\omega(\mu)$, we have that for each $x \in \text{supp } \eta$ it holds

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f_j^\omega(x)) = \int \varphi \, d\mu.$$

Taking integrals over $\text{supp } \eta$ with respect to the probability measure $\eta$ on both sides of the equality, the dominated convergence theorem gives

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \varphi(f_j^\omega(x)) \, d\eta = \int \varphi \, d\mu.$$
Since
\[ \int \varphi d(f^j_\omega) \eta = \int \varphi \circ f^j_\omega \, d\eta \quad \text{for every } j \geq 0 \]
we get the first part of the lemma. To prove (5.3), it suffices to apply the first part of this lemma to \( \eta = \delta_x \) for \( x \in B_\omega(\mu) \).

Let us show that (ii) implies (i) in Theorem C. To do this, according to the equivalences shown in Theorem D, it suffices to show the following proposition.

**Proposition 5.6.** If \( f \) satisfies (FPM), then \( \mathcal{L}_f \) satisfies the condition (FED).

**Proof.** Since \( f \) satisfies (FPM) we have only finitely many absolutely continuous (with respect to \( m \)) ergodic stationary probability measures \( \mu_1, \ldots, \mu_r \) which have pairwise disjoint supports and \( m(B_\omega(\mu_1) \cup \cdots \cup B_\omega(\mu_r)) = 1 \) for \( P \)-almost every \( \omega \in \Omega \). Let \( h_i \) be the Radon–Nikodým derivative of the measure \( \mu_i \) with respect to \( m \) for \( i = 1, \ldots, r \). Observe that \( h_i \) is an invariant density of \( \mathcal{L}_f \) because \( \mu_i \) is a stationary probability measure of \( f \). Moreover, since the supports of such measures are pairwise disjoints, we also get that the densities \( h_1, \ldots, h_r \) have mutually disjoint supports. To conclude (FED) for \( \mathcal{L}_f \) we need to prove that the density \( h = \sum_{i=1}^r (h_1 + \cdots + h_i) \) has the maximal support. This means that

\[ \lim_{n \to \infty} \mathcal{L}^n h(\text{supp } h)(x) = 1 \quad \text{for } m\text{-almost every } x \in X \quad (5.4) \]

where \( \mathcal{L}^*_f \) is the adjoint operator of \( \mathcal{L}_f \).

Given \( x \in X \) and \( \omega \in B_x(\mu_i) \) for some \( i \in \{1, \ldots, r\} \), we have that \( x \in B_\omega(\mu_i) \). Therefore, by Lemma 5.5,

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{\text{supp } h \circ f^j_\omega}(x) = \mu_i(\text{supp } h) = 1. \]

That is, for every \( x \in X \),

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{\text{supp } h \circ f^j_\omega}(x) = 1 \quad \text{for all } \omega \in B_x(\mu_1) \cup \cdots \cup B_x(\mu_r). \quad (5.5) \]

Taking into account Lemma 5.2, we have that \( P(B_x(\mu_1) \cup \cdots \cup B_x(\mu_r)) = 1 \) for \( m \)-almost every \( x \in X \). Then, using the notation introduced in Appendix A, by Lemma A.4 (4),
the dominated convergence theorem, Lemma A.4 (2) and (5.5) imply that, for m-
amingly every \( x \in X \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathcal{L}_f^j 1_{\text{supp } h}(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \mathcal{L}_f^j 1_{\text{supp } h}(x) d\mathbb{P}(\omega) = \int_{\bigcup \mu_i B_\infty(\mu_i)} \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{\text{supp } h} \circ f^j_\omega(x) d\mathbb{P}(\omega) = 1.
\]

Further, according to Remark 4.5, we have that \((\mathcal{L}_f^j 1_{\text{supp } h})_{j \geq 1}\) is monotonic increasing and thus, mean converges imply sequence converges (this follows as a consequence of the Stolz–Cesàro theorem). That is, we obtain (5.4) concluding the proof. \( \square \)

Now we will prove that (i) implies (ii). According to Theorem D, if \( \mathcal{L}_f \) is mean constrictive, then \( \mathcal{L}_f \) satisfies (APM). That is, \( \mathcal{L}_f \) admits finitely many ergodic invariant densities \( h_1, \ldots, h_r \) in \( D(m) \) with mutually disjoint supports and \( \lambda_1, \ldots, \lambda_r \) bounded linear functionals such that for any \( \varphi \in L^1(m) \),

\[
\lim_{n \to \infty} \left\| A_\omega \varphi - \sum_{i=1}^{r} \lambda_i(\varphi) h_i \right\| = 0. \tag{5.6}
\]

Since \( h_i \) is an ergodic invariant density \( h_i \) for \( \mathcal{L}_f \), the measure \( \mu_i \) is an absolutely continuous (with respect to \( m \)) ergodic stationary measures of \( f \) where \( d\mu_i = h_i dm \).

Moreover, since the support \( \text{supp } \nu \) of a measure \( \nu \) absolutely continuous with respect to \( m \) is defined as the support of the Radon–Nikodým derivative \( \frac{d\nu}{dm} \) of \( \nu \) with respect to \( m \), we also obtain that \( \text{supp } \mu_1, \ldots, \text{supp } \mu_r \) are pairwise disjoints.

To prove that \( f \) satisfies (FPM) it remains to show that the union of fiberwise basins of attraction of these measures is \( X \) modulus a set of zero \( m \)-measure. This will be obtained in the following proposition.

**Proposition 5.7.** Assume that \( \mathcal{L}_f \) satisfies (APM) as above. Then

\[
m(X \setminus (B_\omega(\mu_1) \cup \cdots \cup B_\omega(\mu_r))) = 0 \quad \text{for } \mathbb{P}-\text{almost every } \omega \in \Omega.
\]

**Proof.** Let \( \Omega_0 = \{ \omega \in \Omega : \mu_i(B_\omega(\mu_i)) = 1, \text{ for } i = 1, \ldots, r \} \). From item (i) in Lemma 5.1 we get \( \mathbb{P}(\Omega_0) = 1 \). Set

\[
A_\omega = X \setminus (B_\omega(\mu_1) \cup \cdots \cup B_\omega(\mu_r)).
\]

Then, for any \( \omega \in \Omega_0 \) it holds that \( \mu_i(A_\omega) = 0 \) for all \( i = 1, \ldots, r \) because \( A_\omega \subset X \setminus B_\omega(\mu_i) \). Consequently, since \( d\mu_i = h_i dm \)

\[
\int_{A_\omega} h_i dm = 0 \quad \text{for all } i = 1, \ldots, r \text{ and } \omega \in \Omega_0. \tag{5.7}
\]
Set
\[ \Omega_1 = \bigcap_{n=0}^{\infty} \sigma^{-n}(\Omega_0), \]
which is also a full \( \mathbb{P} \)-measure set. Notice that \( \omega \in \Omega_1 \) implies \( \sigma^n \omega \in \Omega_0 \) for any \( n \geq 0 \). Thus, from (5.7)
\[ \int_{A_{\sigma^n \omega}} h_i \, dm = 0 \quad \text{for each } i = 1, \ldots, r \text{ and } n \geq 1 \text{ and } \omega \in \Omega_1. \quad (5.8) \]

For simplicity of notation, we write \( \bar{m} = \mathbb{P} \times m \).

**Claim 5.8.** For \( \bar{m} \)-almost every \( (\omega, x) \in \Omega \times X \), there is \( k \geq 1 \) such that \( f_k \omega(x) \in X \setminus A_{\sigma^k \omega} \).

**Proof.** By contradiction, assume that the claim does not hold. Therefore,
\[ \bar{m}(B_0) > 0, \quad B_0 = \{(\omega, x) \in \Omega \times X : f^n_\omega(x) \in A_{\sigma^n \omega} \text{ for all } n \geq 1\}. \]
Set \( B = B_0 \cap (\Omega_1 \times X) \), which is still a positive \( \bar{m} \)-measure set because \( \Omega_1 \times X \) is a full measure set. Denoting by \( B_\omega = \{x \in X : (\omega, x) \in B\} \) the \( \omega \)-section of \( B \), Fubini’s theorem implies that for every \( n \geq 1 \),
\[ \int_{\Omega} \int_{B_\omega} 1_{A_{\sigma^n \omega}} \circ f^n_\omega \, dm \, d\mathbb{P} = \int_{\Omega} \int_{B_\omega} dm \, d\mathbb{P} = \bar{m}(B) > 0. \]
Hence, with the notation from Appendix A, see Lemma A.4 (2),
\[ \frac{1}{n} \sum_{i=0}^{n-1} \int_{\Omega} \int_{B_\omega} L_i^{n\omega} 1_{A_{\sigma^n \omega}} \, dm \, d\mathbb{P} = \bar{m}(B) > 0. \quad (5.9) \]

Given \( \varphi \in L^1(m) \) and \( \omega \in \Omega \), let us write
\[ L_n^\omega \varphi = \sum_{j=1}^{r} \lambda_j(\varphi) h_j + Q_n^\omega \varphi \quad \text{and} \quad A_n \varphi = \sum_{j=1}^{r} \lambda_j(\varphi) h_j + Q_n \varphi \]
where we recall that (see Lemma A.4 (4)),
\[ A_n \varphi = \frac{1}{n} \sum_{i=0}^{n-1} L_i^\omega \varphi = \frac{1}{n} \sum_{i=0}^{n-1} \int L_i^\omega \varphi \, d\mathbb{P}. \]
Then,
\[ \frac{1}{n} \sum_{i=0}^{n-1} \int_{X} Q_i^\omega 1_X \, dm \, d\mathbb{P} = \int_{X} (A_n 1_X - \sum_{j=1}^{r} \lambda_j(1_X) h_j) \, dm = \int_{X} Q_n 1_X \, dm \leq ||Q_n 1_X||. \]
On the other hand, by Lemma A.4 (3), Equation (5.8) and the above inequality,

\[
\frac{1}{n} \sum_{i=0}^{n-1} \int_{\Omega} \int_{B_{\omega}} \mathcal{L}_{i,\omega}^i A_{i,\omega} d\mu d\mathbb{P} \leq \frac{1}{n} \sum_{i=0}^{n-1} \int_{X} \int_{\tilde{A}_{\omega}} Q_{i,\omega}^i d\mu \mathbb{P} \leq \frac{1}{n} \sum_{i=0}^{n-1} \int_{X} Q_{i,\omega}^i d\mu \mathbb{P} \leq \|Q_{i,\omega}^i\| \to 0.
\]

The last limit (as \( n \to \infty \)) follows from the assumption (5.6). But this limit provides a contradiction with (5.9). \( \square \)

Fix \((\omega, x) \in \Omega \times M\) (up to zero measure sets) and let \( k \geq 1 \) be the integer of Claim 5.8. Set \( y = f_k^\sigma(x) \in X \setminus A_{\sigma^k,\omega} \). By definition of \( A_{\sigma^k,\omega} \), there is \( i \) such that \( y \in G_{\sigma^k,\omega}(\mu_i) \). Then, in view of (5.2), \((\sigma^k \omega, y) \in B(\mu_i)\) and hence

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_j^\sigma(x)}(y) = \mu_i.
\]

On the other hand,

\[
\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_j^\sigma(x)} = \frac{n-k}{n} \left( \frac{1}{n-k} \sum_{j=0}^{n-k-1} \delta_{f_j^\sigma(x)}(y) + \frac{1}{n} \sum_{j=0}^{k-1} \delta_{f_j^\sigma(x)} \right) \to \mu_i
\]

as \( n \to \infty \). This shows that \( x \in B_{\omega}(\mu_i) \) concluding the proof. \( \square \)

Finally, we summarize and complete the last details of the proof of Theorem C:

**Proof of Theorem C.** If \( f \) satisfies (FPM), then \( \mathcal{L}_f \) is (FED) by Proposition 5.6 and hence Theorem D implies that \( \mathcal{L}_f \) is (MC). This proves (ii) implies (i) in Theorem C. On the other hand, if \( \mathcal{L}_f \) is (MC), then it is also (APM) by Theorem D and hence, one concludes that \( f \) satisfies (FPM) from Proposition 5.7. This shows (i) implies (ii). Finally, (1) and (2) follow from Lemma 5.2. \( \square \)

### 6. Sub-hierarchy in (UC)

In this section, we complete the proof of the implications in the hierarchies of Figures 2, as well as provide practical sufficient conditions for (FPM) other than conditions in Section 1.1. In the sequel, \((X, \mathcal{B}, \mu)\) denotes a Polish probability space, \( P : L^1(m) \to L^1(m) \) is a Markov operator and \( P^n(x, A) \) is an \( n \)-transition probability that induces \( P^n \). We first give equivalent conditions for (UC).

**Proposition 6.1.** The following assertions are equivalent:
(1) $P$ satisfies (UC);
(2) $P : L^1(m) \to L^1(m)$ is a quasi-compact operator;
(3) for any $\epsilon > 0$, there are $\delta > 0$ and $n_0 \geq 1$ such that
\[
\sup_{\varphi \in D(m)} \int_A P^{n_0} \varphi < \epsilon \quad \text{for all } A \in \mathcal{B} \text{ with } m(A) < \delta;
\]
(4) there are $n_0 \geq 1$, $0 < \epsilon < 1$ and $\delta > 0$ such that
\[
\sup_{\varphi \in D(m)} \int_A P^{n_0} \varphi < \epsilon \quad \text{for all } A \in \mathcal{B} \text{ with } m(A) < \delta;
\]
(5) for any $\epsilon > 0$, there are $\delta > 0$ and $n_0 \geq 1$ such that $P^{n_0}(x, A) < \epsilon$ for all $A \in \mathcal{B}$ with $m(A) < \delta$ and $m$-almost every $x \in X$ (depending on $A$);
(6) there are $n_0 \geq 1$, $0 < \epsilon < 1$ and $\delta > 0$ such that $P^{n_0}(x, A) < \epsilon$ for all $A \in \mathcal{B}$ with $m(A) < \delta$ and $m$-almost every $x \in X$ (depending on $A$);
(7) for any $\epsilon > 0$, there are $\delta > 0$, $n_0 \geq 1$ and a probability $\mu$ absolutely continuous with respect to $m$ such that $P^{n_0}(x, A) < \epsilon$ for all $A \in \mathcal{B}$ with $\mu(A) < \delta$ and $m$-almost every $x \in X$ (depending on $A$);
(8) there are $n_0 \geq 1$, $0 < \epsilon < 1$, $\delta > 0$ and a probability $\mu$ absolutely continuous with respect to $m$ such that $P^{n_0}(x, A) < \epsilon$ for all $A \in \mathcal{B}$ with $\mu(A) < \delta$ and $m$-almost every $x \in X$ (depending on $A$).

**Proof.** The equivalence between (1)–(4) follows from [20, Theorem 2]. On the other hand, observe that
\[
\int_A P^{n_0} \varphi dm = \int \varphi(x) P^{n_0} 1_A(x) dm = \int \varphi(x) P^{n_0}(x, A) dm
\]  
for all $\varphi \in L^1(m)$, $A \in \mathcal{B}$ and $n \geq 1$. Taking into account that if $\varphi \in D(m)$ then $\int \varphi dm = 1$, from (6.1) one immediately gets that (5) implies (3) and (6) implies (4).

We are going to see now that (3) implies (5). By contradiction, assume that there exists $\epsilon > 0$ satisfying that for any $\delta > 0$ and $n \geq 1$ there are $A \in \mathcal{B}$ with $m(A) < \delta$ and set $E \subset X$ with $m(E) > 0$ such that $P^n(x, A) \geq \epsilon$ for all $x \in E$. Take $\varphi = \frac{1}{m(E)} 1_E \in D(m)$. Hence, (3) implies that $\int_A P^n \varphi dm < \epsilon$. However, by (6.1)
\[
\int_A P^n \varphi dm = \int \varphi(x) P^n(x, A) dm = \frac{1}{m(E)} \int_E P^n(x, A) dm \geq \epsilon
\]
we get a contradiction.

The implication of (6) from (4) follows analogously as the implication of (5) from (3). This concludes the equivalence from (1) to (6). To complete the proof, we will show that (5) is equivalent to (7) and (6) is equivalent to (8).

Observe that clearly, (5) implies (7) and (6) implies (8) by taking $\mu = m$. As before, the converse implications are analogous and thus we only prove (7) implies (5). To
do this, by (7) we have that \( \mu \) is absolutely continuous with respect to \( m \). In particular, according to [43, Lemma 1], \( \mu \) is uniform absolutely continuous with respect to \( m \). That is, for any \( \eta > 0 \) there is \( \alpha > 0 \) such that \( \mu(A) < \eta \) for all \( A \in \mathcal{B} \) with \( m(A) < \alpha \). Then taking \( \eta = \delta \) in (7), we get that for any \( \epsilon > 0 \), there are \( \alpha > 0 \) and \( n \geq 1 \) such that \( P^n(x, A) < \epsilon \) for all \( A \in \mathcal{B} \) with \( m(A) < \alpha \) and \( m \)-almost every \( x \in X \) (depending on \( A \)). This concludes (5). \( \square \)

As already mentioned in Section 1.3.4, it follows from ((8) of) Proposition 6.1 that (D) implies (UC). The following proposition shows that the converse is also true if \( P \) is strong Feller.

**Proposition 6.2.** Let \( P : L^1(m) \to L^1(m) \) be a Markov operator. Assume that \( P \) is strong Feller, that is, there exists \( k \in \mathbb{N} \) such that an associated \( k \)-th transition probability \( P^k(x, A) \) is strong Feller continuous. Then \( P \) satisfies (UC) if and only if \( P \) satisfies (D).

**Proof.** In view of Proposition 6.1, (D) implies (UC) even if we do not assume that \( P \) is strong Feller. We will prove now the converse.

Assume first that \( P(x, A) \) is strong Feller continuous and (UC). That is, \( k = 1 \) in the statement of the proposition. The first observation in this case is that \( P^n(x, A) \) is strong Feller continuous for all \( n \geq 1 \). To see this, recall that \( P^{n+1}(x, A) = \int P(y, A) P^n(x, dy) \) for \( n \geq 1 \) where \( P^1(x, A) = P(x, A) \) is strong Feller continuous by assumption. Arguing by induction, if \( P^n(x, A) \) is strong Feller continuous, it is Feller continuous and thus \( P^n(x, \cdot) \) varies continuously in the weak* topology. Moreover, according to Proposition 2.2, since \( P(x, A) \) is strong Feller continuous, \( y \mapsto P(y, A) \) is a bounded continuous function for all \( A \in \mathcal{B} \). Thus,

\[
P^{n+1}(x', A) = \int P(y, A) P^n(x, dy) \to \int P(y, A) P^n(x, dy) = P^{n+1}(x, A) \quad \text{if } x' \to x.
\]

Therefore, \( P^{n+1}(x, A) \) is strong Feller continuous. Now, having the continuity of \( x \mapsto P^{n^0}(x, A) \) in mind, we get that (8) in Proposition 6.1 holds for all \( x \in X \). This concludes (D).

Let us prove the proposition for \( k > 1 \), i.e. assume now that \( P \) satisfies (UC) and \( P^k(x, A) \) is strong Feller continuous for \( k > 1 \). To see the conclusion, consider the operator \( Q = P^k \). By definition, \( Q \) satisfies (UC). Moreover, the associated transition probability of \( Q \) is \( Q(x, A) = P^k(x, A) \). Thus, we can apply the case with \( k = 1 \) for \( Q \), concluding that \( Q \) satisfies (D). In particular, \( P \) satisfies (D). \( \square \)

**Remark 6.3.** If a random map \( f \) satisfies the Araújo or Brin–Kifer conditions, Proposition 2.6, Theorem 2.7 and Proposition 6.2 implies that the Perron–Frobenius operator \( \mathcal{L}_f \) satisfies (D).
Another important consequence of Proposition 6.1 is the following practical sufficient conditions for \((\text{FPM})\).

**Theorem 6.4.** Assume that \(P^{\mu_0}(x,dy) = p(x,y)\,dm(y)\) and one of the following holds:

1. there are \(\gamma > 0\) and \(\alpha > 0\) such that
   
   (a) \(m(\text{supp} \, p(x,\cdot)) > \gamma\) for \(m\)-almost every \(x \in X\), and
   
   (b) \(\alpha \leq p(x,y)\) for \(m\)-almost every \(x \in X\) and \(m\)-almost every \(y \in \text{supp} \, p(x,\cdot)\);

2. there is \(\beta > 0\) such that \(p(x,y) \leq \beta\) for \(m\)-almost every \(x \in X\) and \(m\)-almost every \(y \in \text{supp} \, p(x,\cdot)\).

Then, \(\mathcal{L}_f\) satisfies \((\text{UC})\). In particular, if \(X\) is locally compact, \(f\) satisfies \((\text{FPM})\).

**Proof.** By Proposition 6.1 (6), the condition (2) implies that \(\mathcal{L}_f\) satisfies \((\text{UC})\). Therefore, it follows from the implications in Figure 1 and Theorem C that \(f\) satisfies \((\text{FPM})\).

Assume the condition (1). Then, for any \(A \in \mathcal{B}\) with \(m(A) > 1 - \frac{\gamma}{2}\), it holds

\[
P^{\mu_0}(x,A) = \int_A p(x,y)\,dm(y) = \int_{A \cap \text{supp} \, p(x,\cdot)} p(x,y)\,dm(y)
\]

\[
\geq \alpha m(A \cap \text{supp} \, p(x,\cdot)) > \frac{\alpha \gamma}{2}.
\]

In the last inequality, we used that \(m(A \cap \text{supp} \, p(x,\cdot)) = m(A) + m(\text{supp} \, p(x,\cdot)) - m(A \cup \text{supp} \, p(x,\cdot)) > (1 - \frac{\gamma}{2}) + \frac{\gamma}{2} - \frac{\gamma}{2} = \frac{\gamma}{2}\). Hence, by considering complements and using Proposition 6.1 (6) again, we get \((\text{UC})\) for \(\mathcal{L}_f\), and \(f\) satisfies \((\text{FPM})\).

**Remark 6.5.** As previously mentioned, Theorem 6.4 (2) is another weakening of Brin–Kifer’s condition (see (2) in page 5). Moreover, we find Araújo–Aytaç [8] for Theorem 6.4 (1). Indeed, they considered the case when \(X\) is a compact manifold equipped with the normalized Lebesgue measure \(m\), and assumed that there exist \(\alpha, \beta, \gamma > 0\) and \(t_* \in T\) such that for every \(x \in X\),

1. \(\text{supp} \, p(x,\cdot)\) includes the ball of radius \(\gamma\) centered at \(f_{t_*}(x)\), and
2. \(\alpha \leq p(x,y) \leq \beta\) for \(m\)-almost every \(y \in \text{supp} \, p(x,\cdot)\).

Obviously, the conditions in Theorem 6.4 relaxed Araújo–Aytaç’s condition to get that \((\text{UC})\). Moreover, in view of the following section, under the assumptions in [8, Thm. A], one gets \((\text{D}^*)\).

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\(^{10}\)They also assumed an aperiodicity condition to ensure uniform ergodicity, meaning that \(X\) cannot be decomposed into \(\ell\) subsets \(X = X_1 \cup \cdots \cup X_\ell\) (\(\ell \geq 2\)) such that \(p(x,X_{i+1 \mod \ell})\) for all \(x \in X_i\) and \(1 \leq i \leq \ell\). We do not assume it, as indicated in the second example in Section 1.4.1, which satisfies the conditions in Theorem 6.4 but violates the aperiodicity condition.
6.1. \((D^*)\) and uniform ergodicity. Let us also give equivalent conditions for \((D^*)\).

**Proposition 6.6.** The following assertions are equivalent:

1. \(P\) satisfies \((D^*)\);
2. there is a probability measure \(\pi\) absolutely continuous with respect to \(m\) such that
   \[
   \lim_{n\to\infty} \sup_{x \in X} \|P^n(x, \cdot) - \pi\|_{TV} = 0;
   \]
3. there are a probability measure \(\pi\) absolutely continuous with respect to \(m\) and constants \(C > 0, 0 < \lambda < 1\) such that
   \[
   \|P^n(x, \cdot) - \pi\|_{TV} \leq C\lambda^n \quad \text{for all } x \in X.
   \]

**Proof.** The equivalence between (2) and (3) follows from [69, Theorem 16.0.2]. It is also clear that (2) implies (1) by taking the measure \(\mu\) in (1) as the measure \(\pi\) in (2). Finally, the implication of (2) from (1) follows basically from [33, Theorem 2]. Namely, Dorea and Pereira proved that (1) implies the convergence in (2) but they do not conclude that \(\pi\) is absolutely continuous with respect to \(m\). To prove this, observe that the convergence in (2) implies that \(\pi\) is the unique measure such that
   \[
   \pi(A) = \int P(x, A) \, d\pi(x) \quad \text{for all } A \in \mathcal{B}.
   \]
See Remark 6.7 for more details. On the other hand, (1) implies condition \((D)\) adapted to Markov operators in \(L^1(m)\) (see Section 1.3.4). In view of Theorem E (together with the observation that \(P(x, A) = \int_A P^n 1_X \, dm\)), one has that \((D)\) implies \((S)\). That is, there is \(g \in D(m)\) such that \(P^*g = g\). Hence,
   \[
   m_\delta(A) \overset{def}{=} \int_A g \, dm = \int_A P^*g \, dm = \int g \, P^*1_A \, dm = \int P(x, A) \, dm_\delta \quad \text{for all } A \in \mathcal{B}. \quad (6.2)
   \]
Therefore, by the uniqueness, \(m_\delta = \pi\), proving that \(\pi\) is absolutely continuous. \(\Box\)

**Remark 6.7.** In Markov processes theory, condition (2) in Proposition 6.6 without the absolute continuity of \(\pi\) with respect to \(m\) is called uniform ergodicity (cf. [69]). Apart from the background, it would still make sense to call any of the equivalent conditions in Proposition 6.6 uniform ergodicity (for Markov operators on \(L^1(m)\)) because the conditions imply that there exists a unique invariant density of \(P\). To see this, observe first that (2) in Proposition 6.6 implies that \(\pi\) is the unique invariant probability in the sense that
   \[
   \pi(A) = \int P(x, A) \, d\pi(x) \quad \text{for all } A \in \mathcal{B}.
   \]
Indeed, if \(\nu\) is a probability satisfying also the above relation, then
   \[
   \nu(A) = \int P(x, A) \, d\nu(x) = \int P(x, A)P(y, dx) \, d\nu(y) = \cdots = \int P^n(x, A) \, d\nu.
   \]
Since $P^n(x, A)$ converges uniformly to $\pi(A)$ on $x$ as $n \to \infty$, the right-hand side of the above expression converges to $\pi(A)$, and therefore, $v = \pi$. Since $\pi$ is absolutely continuous with respect to $m$, let us write $d\pi = h \, dm$ with $h \in D(m)$. Then,

\[
\int_A h(x) \, dm = \pi(A) = \int P(x, A) \, d\pi = \int h(x)P(x, A) \, dm = \int h(x)P^1_A(x) \, dm = \int_A Ph(x) \, dm.
\]

This implies that $Ph = h$. Moreover, by the uniqueness shown above, arguing as in (6.2), we conclude that $h$ is the unique invariant density of $P$.

7. Examples and counterexamples: proof of propositions

In this section, we prove all propositions given in Section 1.4.

7.1. Proof of Propositions 1.11 and 1.12: Additive type noise. We first prove Proposition 1.11. Recall that our random map is given by (1.10).

To see (UC), notice that if $x \neq 0$ then

\[
\{t \in T : f_t(x) \in A\} = \{t \in T : f_0(x) + t \in A \pmod{1}\}
\]

is the translated set of $A$ by $f_0(x)$. Hence, by the invariance of the Lebesgue measure for translations,

\[
P(x, A) = p(\{t \in T : f_t(x) \in A\}) = p(A) = m(A).
\]

Now, (UC) immediately follows from the interpretation of (UC) in Section 1.3.4 (e.g. for $n_0 = 1$, $\delta = \varepsilon = \frac{1}{2}$ and $\mu = m$).

On the other hand, since $\{t \in T : f_t(0) = 0\} = T$, we have $P(0, \{0\}) = 1$, namely $P(0, \cdot) = \delta_0$. Thus it follows from (1.4) that $P^n(0, \{0\}) = 1$ for any $n \geq 1$. This means that (D) is violated with $x = 0$ and $A = \{0\}$.

We next prove Proposition 1.12. Recall the random map displayed in Figure 5. Notice that $p(A) = 2m(A)$ for any Borel set $A \subset X_+$. Furthermore, by the argument obtaining (7.1),

\[
p(\{t : \tilde{f}_t(y) \in B\}) = 2m(B) \quad \text{for any } y \in X_+ \text{ and Borel set } B \subset X_+
\]

and thus

\[
P(x, A) = \begin{cases} 
p(\{t : \tilde{f}_t(x) \in \iota(A)\}) = 2m(\iota(A)) = 2m(A) & \text{for } x \in X_+ \\
p(\{t : f_t(x) \in A\}) = 2m(A) & \text{for } x \in X.
\end{cases}
\]

for each $A \in \mathcal{B}$. By (1.4), $P^n(x, A) = 2m(A)$ for each $n \geq 1$, $x \in X$ and $A \in \mathcal{B}$. Therefore, obviously (D) holds (e.g. for $n_0 = 1$, $\delta = \varepsilon = \frac{1}{2}$ and $\mu = m$). On the other hand, given $n_0 \geq 1$, $0 < \varepsilon < 1$, $\delta > \frac{1}{2}$ and a probability measure $\mu$, it holds that $\mu(X_+) \leq \frac{1}{2}$ or
The maps \( f \) and \( f_0 \) on \([-1, 1]\).

Figure 5. Illustrations of the random map \( f \) in Proposition 1.11.

\[ \mu(X_+ \cap Z) = 0 \] implies almost-(b).

\[ \mu(X_-) \leq \frac{1}{2} \]. For simplicity assume that \( \mu(X_+) \leq \frac{1}{2} \), and set \( A = X_+ \). Then, \( \mu(A) < \delta \) but \( P_n(x, A) = 2m(A) = 1 > \varepsilon \). This means that \((D*)\) is violated.

7.2. Proof of Propositions 1.13 and 1.14: Multiplicative noise. Recall condition (b) in Remark 1.2. Also recall that condition almost-(b) is the natural relaxation of the condition (b) from “for all \( x \in X \)” to “for \( m \)-almost every \( x \in X \)”. Let \( f \) be the random map given in (1.11), i.e. random perturbation of a measurable map \( f_0 : [0, 1] \rightarrow [0, 1] \) by multiplicative noise. To prove Propositions 1.13, we need the following lemma.

Lemma 7.1. Denote the set of fixed points and zeros of \( f_0 \) by \( F \) and \( Z \), respectively, that is, \( F = \{ x : f_0(x) = x \} \) and \( Z = \{ x : f_0(x) = 0 \} \). Then, the following holds:

1. If \( 0 \in F \cap Z \), then \( f \) does not satisfy (b).
2. If \( Z = \emptyset \), then \( f \) satisfies (b).
3. If \( m(Z) = 0 \), then \( f \) satisfies almost-(b).

Proof. Assume that \( 0 \in F \cap Z \). Then, \( \{ f_\omega^n(0) : \omega \in \Omega \} = \{ 0 \} \) for all \( n \geq 1 \), implying that \( P_n(0, \cdot) = \delta_0 \) for all \( n \geq 1 \). Since \( \delta_0 \) is not absolutely continuous with respect to the Lebesgue measure \( m \) of \([0, 1]\), (b) does not hold.

Take a point \( x \) such that \( x \notin Z \). Then \( f_0(x) > 0 \), so \( \{ f_\omega(x) : \omega \in \Omega \} \) is a closed interval which is not a point set, and \( P(x, \cdot) \) is the normalized Lebesgue measure on the closed interval, implying that \( P(x, \cdot) \) is absolutely continuous for all \( x \notin Z \). Hence, \( Z = \emptyset \) implies (b), and \( m(Z) = 0 \) implies almost-(b). \( \square \)
Proof of Proposition 1.13. Let \( f_0 \) be the measurable map in (1) or (2) of Proposition 1.13, that is, \( f_0(x) = \frac{x}{2} \) for \( x \in (0, 1] \) and \( f_0(0) = 0 \) or \( \frac{1}{2} \). To see \( \mathcal{L}_f \) does not satisfy (S) for this \( f \), recall that for a given Markov operator \( P : L^1(m) \to L^1(m) \), according to Theorem E, the existence of a \( P \)-invariant density is equivalent to the following condition: there is \( \delta > 0 \) such that

\[
\sup_{n \geq 1} \int_A P^n 1_A \, dm < 1 \quad \text{for any } A \in \mathcal{B} \text{ with } m(A) < \delta.
\]

By duality, this implies that

\[
\int P^n 1_A \, dm < 1 \quad \text{for all } n \geq 1 \text{ and } A \in \mathcal{B} \text{ with } m(A) < \delta.
\]

Since \( P^n 1_A(x) = P^n(x, A) \leq 1 \) for any \( x \in X \), we conclude the following necessary condition for the existence of a \( P \)-invariant density:

\( \star \) there is \( \delta > 0 \) such that for any \( n \geq 1 \) and \( A \in \mathcal{B} \) with \( m(A) < \delta \), there exists \( E \in \mathcal{B} \) with \( m(E) > 0 \) satisfying that \( P^n(x, A) < 1 \) for all \( x \in E \).

We will prove that \( \mathcal{L}_f \) does not satisfy \( \star \). Notice that, for that purpose, it suffices to show that for any \( \delta > 0 \) there are \( n_0 \geq 1 \) and \( A_0 \in \mathcal{B} \) such that \( m(A_0) < \delta \) but \( P^{n_0}(x, A_0) = 1 \) for \( m \)-almost every \( x \). Observe also that since \( 0 < f_t(x) \leq f_0(x) \) for all \( t \in [0, 1] \) and \( x \neq 0 \), we have that, if \( x \neq 0 \) then

\[
0 < f_t(x) \leq f_0(x) = \frac{1}{2^n} \quad \text{for any } n \geq 1 \text{ and } \omega \in \Omega.
\]

Hence, the support of \( P^n(x, \cdot) = \mathbb{P}(\omega : f_t^n(x) \in \cdot) \) is contained in the interval \( (0, 1/2^n) \) of length \( 1/2^n \) if \( x \neq 0 \). Therefore, for any \( \delta > 0 \), by taking \( n_0 \geq \frac{\log \delta}{\log 2} \) and \( A_0 = \left(0, \frac{1}{2^{n_0}}\right) \), we have that \( m(A_0) < \delta \) and \( P^{n_0}(x, A_0) = 1 \) whenever \( x \neq 0 \). From these observations, \( \star \) does not hold, and consequently, neither does (S) for \( \mathcal{L}_f \). In particular, due to the implications in Figure 1, (FPM) does not hold for \( \mathcal{L}_f \).

Let us complete the proof of items (1) and (2) of Proposition 1.13. When \( f_0 \) is the continuous map in (1), obviously (a) in Remark 1.2 holds and \( Z = \{0\} \). So, by Lemma 7.1, \( f \) does not satisfy (b) but satisfies almost-(b). When \( f_0 \) is the map in (2) having discontinuity at 0, (a) does not hold and \( Z = \emptyset \). Therefore, by using Lemma 7.1 again, we conclude that \( f \) satisfies (b).

We next prove item (3) of Proposition 1.13. Let \( f_0(x) = 2x \pmod{1} \) for \( x \in [0, 1] \). Then, since \( 0 \in F \cap Z \), it follows from Lemma 7.1 that \( f \) does not satisfy (b). On the other hand, by [48, Theorem 3.1], we find that \( \mathcal{L}_f \) satisfies (C). Therefore, \( f \) satisfies (FPM) by Theorem C together with the implications in Figure 1. \( \square \)
Now, we will prove Proposition 1.14. Hence, let \( f \) be the random map given in (1.12), i.e. random perturbation of a measurable map \( f_0 : [0, 1] \to [0, 1] \) by multiplicative type noise. First, we need the following lemma.

**Lemma 7.2.** Denote the set of fixed points of \( f_0 \) by \( F \). The following hold:

1. \( F = \emptyset \) if and only if then \( f \) satisfies (b);
2. If \( m(F) = 0 \), then \( f \) satisfies almost-(b).

**Proof.** Take a point \( x \notin F \). Then \( f_0(x) - x \neq 0 \) and thus \( I_x = \{ f_t(x) : t \in T \} \) is a closed interval which is not a point set. Hence, \( P(x, \cdot) \) is the normalized Lebesgue measure on the closed interval \( I_x \). This implies that \( P(x, \cdot) \) is absolutely continuous for all \( x \notin F \). On the other hand, if \( x \in F \), then \( \{ f^n_\omega(x) : \omega \in \Omega \} = \{ x \} \) for all \( n \geq 1 \). Consequently \( P^n(x, \cdot) = \delta_x \) for all \( n \geq 1 \). Since \( \delta_x \) is not absolutely continuous with respect to \( m \), (b) does not hold. The claim immediately follows from these observations.

**Proof of Proposition 1.14.** Now let \( f_0 \) be the \( C^1 \) map given by the gradient flow with potential (1.13). As mentioned, \( f_0 \) has infinitely many sinks \( (s_k)_{k \geq 1} \) and sources \( (r_k)_{k \geq 1} \) such that the union of the basins of \( s_k \) for \( f_0 \) coincides with \( X \setminus \{ r_1, r_2, \ldots \} \). In particular, \( F = \{ r_k, s_k : k \geq 1 \} \) is the set of fixed point of \( f_0 \). Hence, it follows from Lemma 7.2 that \( f \) satisfies almost-(b), but not (b).

Furthermore, take \( x \) in the basin of \( s_k \) for \( f_0 \). In view of (1.12), \( f_i(x) \) is a convex combination of \( x \) and \( f_0(x) \). Thus, \( f_i(x) \) is in the closed interval between these points. In particular (from the orientation preserving) in the closed interval whose endpoints are \( x \) and \( s_k \). Arguing recursively, \( f^n_\omega(x) \) is in the closed interval whose endpoints are \( s_k \) and \( f_0^{n-1}(x) \) for any \( \omega \in \Omega \) and \( n \geq 1 \). Since \( f_0^n(x) \to s_k \), this implies that \( f^n_\omega(x) \to s_k \) as \( n \to \infty \) for any \( \omega \in \Omega \). This completes the proof of Proposition 1.14.

**7.3. Proof of Proposition 1.15: Random expanding maps.** The constrictivity of \( \mathcal{L}_f \) under the condition (1.14) is just the consequence of the work by Boyarsky and Levesque [29, Theorem 2]. Furthermore, it is not difficult to see that \( \mathcal{L}_f \) is not uniformly constrictive as follows. Notice that, for any \( n_0 \geq 1 \) and \( x \in X \),

\[
P^{n_0}(x, \cdot) = \sum_{\omega \in \{1, \ldots, k\}^{n_0}} p_\omega \delta_{f^n_{\omega}(x)}
\]

where \( p_\omega = p^{n_0}((\omega)) \) and \( f^{n_0}_\omega = f_{\omega_{n_0-1}} \circ \cdots \circ f_{\omega_0} \) for \( \omega = (\omega_0, \ldots, \omega_{n_0-1}) \). Thus, if we consider the finite set \( A := \{ f^{n_0}_\omega(x) : \omega \in \{1, \ldots, k\}^{n_0} \} \), then \( P^{n_0}(x, A) = 1 \) but \( \mu(A) = 0 \) for any \( m \)-absolutely continuous probability measure \( \mu \). By virtue of the interpretation of (UC) in Section 1.3.4, this concludes the violation of (UC).
7.4. Proof of Proposition 1.16: Random contracting maps. For the proof of Proposition 1.16, the next proposition is important. Recall the mixing property and exactness of an invariant density for a Markov operator given in Section 1.5.

Proposition 7.3. For $f$ in Proposition 1.16, $1_X$ is an $L_f$-invariant density. Moreover, $1_X$ is mixing, but is not exact.

Proof. It is obvious that $1_X$ is an invariant density for $L_f$. Furthermore, we immediately find that

$$L^n_f \varphi(x) = \frac{1}{2} L^n_1 \varphi(x) + \frac{1}{2} L^n_2 \varphi(x) = \frac{1}{2} \varphi \left( x \frac{1}{2} \right) + \frac{1}{2} \varphi \left( x + \frac{1}{2} \right)$$

which is equivalent to the Perron–Frobenius operator $L_g$ of the (deterministic) dyadic map $g(x) = 2x \mod 1$. Then, for any $A, B \in \mathcal{B}$, we have

$$\int_X L^n_f 1_A \cdot 1_B \, dm = \int_X 1_A \cdot L^n_g 1_B \, dm \to m(A)m(B)$$

as $n \to \infty$ since $m$ is mixing for $g$. Therefore, (using a simple function approximation) we conclude that $1_X$ is mixing.

On other other hand, for $\varphi = 1_A - 1_{A^c}$ with $A = [0, \frac{1}{2}]$ and any $n \in \mathbb{N}$, we have

$$\int_X |L^n_f \varphi| \, dm = \sum_{k=0}^{2^n-1} \int_X \left| 1_{\left[ \frac{2k}{2^n} \cdot \frac{2k+1}{2^n} \right]} - 1_{\left[ \frac{2k+1}{2^n} \cdot \frac{2k+2}{2^n} \right]} \right| \, dm = \int_X 1 \, dm = 1.$$

If $1_X$ will be exact then, $L^n_f \varphi \to \int \varphi \, dm = 0$ in $L^1$-norm. Thus, the above computation implies that $1_X$ is not exact. \hfill \Box

Let us move to the proof of Proposition 1.16. It is well-known that, in the class of constrictive Markov operators preserving $1_X$, the mixing property implies the exactness (cf. [63, Remark 5.5.1]). Thus, the above proposition already concludes that $P$ is not constrictive. As another proof, we can also check it directly as follows.

Recall the Dunford–Pettis interpretation of (C) (given in Section 1.3.3): $L_f$ is constrictive if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $h \in D(m)$, there is $n_0 \geq 1$ for which

$$\int_A L^n_f h \, dm < \varepsilon \quad \text{for any } n \geq n_0 \text{ and } A \in \mathcal{B} \text{ with } m(A) \leq \delta.$$

The key point is that the set $A$ can depend on $n$, compare it with the definition of (AC). Fix $\varepsilon > 0$ and $\delta > 0$. Let $A = [0, \delta]$ and $h = \frac{1}{\delta} 1_A \in D(m)$. Then

$$L^n_fh = \frac{1}{\delta} \sum_{k=0}^{2^n-1} 1_{\left[ \frac{k}{\delta}, \frac{k+1}{\delta} \right]}.$$
Thus, by letting \( B_n = \text{supp } L_n^a h \), we have
\[
\int_{B_n} L_n^a h \, dm = 1 \quad \text{and} \quad m(B_n) = \delta \quad \text{for any } n \geq 1,
\]
which implies that \( L_f \) is not constrictive.

On the other hand, since \( L_f \) is mixing with respect to \( m \), we can conclude \( L_f \) is asymptotically constrictive as follows. For any \( \varepsilon > 0 \), set \( \delta = \varepsilon \). Then for any \( h \in D(m) \) and \( B \in \mathcal{B} \) with \( m(B) < \delta \), by the mixing property, we have
\[
\lim_{n \to \infty} \int_B L_n^a h \, dm = \int_X h \, dm \cdot m(B) \to \varepsilon.
\]
This completes the proof of Proposition 1.16.

7.5. Proof of Proposition 1.19: Random rotations. To refrain notations, in this subsection we identify any closed interval in \( S^1 \) with its corresponding set in \([0, 1) \) (a closed interval or a set \([0, a) \cup [b, 1) \) with some \( a < b \)).

7.5.1. Case (1): Irrational \( \alpha - \beta \). It is sufficient to show that \( L_f \) is asymptotically stable, that is, \( \|L_n^a \varphi - 1_X\| \to 0 \) as \( n \to \infty \) for any \( \varphi \in D(m) \), since an asymptotically stable Markov operator is constrictive (see [63]). For \( \varphi \in D(m) \), we can calculate \( L_n^a \varphi \) as
\[
L_n^a \varphi(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \varphi(x - (n-k)\alpha - k\beta) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \varphi(x - n\alpha + k(\alpha - \beta))
\]
\[
= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \mathcal{L}_{\beta - \alpha}^k \varphi(x - n\alpha)
\]
where \( \mathcal{L}_\gamma \) denotes the Perron–Frobenius operator for the irrational rotation with angle \( \gamma \). Since \( L_f 1_X = 1_X \) and \( \alpha - \beta \) is irrational, using the mean ergodic theorem weighted with binomial coefficients (see [35, Theorem 4.1 and Corollary 4.3]),
\[
L_n^a \mathcal{L}_{-\alpha}^n \varphi = L_n^a \varphi(x + n\alpha) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \mathcal{L}_{\beta - \alpha}^k \varphi(x) \to 1_X \text{ in } L^1(m) \text{ as } n \to \infty.
\]
It is clear that \( L_f \mathcal{L}_{-\alpha} = \mathcal{L}_{-\alpha} L_f, \mathcal{L}_{-\alpha} 1_X = 1_X \) and \( \|\mathcal{L}_{-\alpha} \psi\| = \|\psi\| \) for any \( \psi \in L^1(m) \). Therefore, we have
\[
\|L_n^a \varphi - 1_X\| = \|L_n^a(\mathcal{L}_{-\alpha}^n \varphi - 1_X)\| = \|L_{-\alpha} L_n^a \varphi - L_n^a 1_X\| = \|L_n^a \mathcal{L}_{-\alpha}^n \varphi - 1_X\| \to 0
\]
as \( n \to \infty \), which completes the proof.
7.5.2. Case (2): Irrational $\alpha$ and $\beta$ with rational $\alpha - \beta$. Let $\alpha - \beta = \frac{\ell}{N}$ for some integer $\ell$ and $N$. Fix $0 < \delta < 1$. Set $B_i = [\frac{i}{N}, \frac{i+1}{N}]$ for $i = 0, \cdots, N-1$ and $B = B_0 \cup \cdots \cup B_{N-1}$.

Notice that $m(B) = \delta$.

We first show that $\mathcal{L}_f 1_B = 1_{B + \alpha}$. Indeed, observe that for any $i = 0, \ldots, N-1$ there is a unique $j = 0, \ldots, N-1$ such that $B_i + \alpha = B_j + \beta$, where $A + t = [a + t, b + t]$ for $A = [a, b]$ and $t \in (0,1)$. This follows for $j = i - \ell \pmod{N}$, since

$$\left(\frac{i}{N} + \alpha\right) - \left(\frac{j}{N} + \beta\right) = \frac{i}{N} - \frac{j}{N} + \alpha - \beta = \frac{i}{N} - \frac{j}{N} + \ell = 0 \pmod{1},$$

and such $j$ is unique. Then we have

$$\mathcal{L}_f 1_B = \frac{1}{2} \sum_{i=0}^{N-1} (1_{B_i + \alpha} + 1_{B_i + \beta}) = \frac{1}{2} \sum_{i=0}^{N-1} (1_{B_i + \alpha} + 1_{B_i + \beta}) = \sum_{i=0}^{N-1} 1_{B_i + \alpha} = 1_{B + \alpha}.$$

Moreover, we have that $\mathcal{L}_f^n 1_B = 1_{B + n\alpha}$ for any $n \geq 1$.

Take $\varphi := \frac{1}{2} 1_B \in D(m)$ since $m(B) = \delta$. Since $\alpha \notin \mathbb{Q}$, we can consider a sequence $\{n_j\}_{j \geq 1}$ such that $\alpha n_j \to 0 \pmod{1}$ as $j \to \infty$. Hence,

$$\int_B \mathcal{L}_f^n \varphi \, dm \to 1 \quad \text{as} \quad j \to \infty.$$

Therefore, $\mathcal{L}_f$ does not satisfies (AC).

We next prove that $\mathcal{L}_f$ is mean constrictive. It is clear that the function $1_X$ is invariant for $\mathcal{L}_f$. Then, from Remark 1.8, it is sufficient to show that $1_X$ is an ergodic density. Let $\mathcal{L}_f^* 1_A = 1_A$, where $\mathcal{L}_f^*$ is the adjoint operator for $\mathcal{L}_f$. We will prove $m(A) \in [0,1]$. Since obviously $1_A \in L^2(m)$, we have the Fourier series of $1_A$ as follows,

$$1_A(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}.$$

Then $\mathcal{L}_f^* \varphi = \varphi$ and

$$\mathcal{L}_f^* \varphi(x) = \frac{1}{2} \mathcal{L}_1^* \varphi(x) + \frac{1}{2} \mathcal{L}_2^* \varphi(x) = \frac{1}{2} \varphi(x + \alpha) + \frac{1}{2} \varphi(x + \beta)$$

imply

$$\frac{1}{2} \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n (x+\alpha)} + \frac{1}{2} \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n (x+\beta)} = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}.$$

Due to the uniqueness of Fourier series, it must be satisfied that, for any $n \in \mathbb{Z}$,

$$\frac{1}{2} e^{2\pi i n \alpha} + \frac{1}{2} e^{2\pi i n \beta} = 1.$$
It can be rewritten as
\[ e^{2\pi i y} + 1 = 2e^{-2\pi i \beta} \quad \text{with} \quad y := \alpha - \beta. \]
That is,
\[ \cos(2\pi \gamma) + 1 = 2 \cos(-2\pi \beta) \quad \text{and} \quad \sin(2\pi \gamma) = 2 \sin(-2\pi \beta). \]
Hence
\[ (2 \cos(2\pi \beta) - 1)^2 + (-2 \sin(2\pi \beta))^2 = 1, \]
which leads to \( \cos(2\pi \beta) = 1 \). This equality holds only for \( n = 0 \) since \( \beta \) is irrational.

Thus, all coefficient \( a_n \) must be 0 for any \( n \neq 0 \), which shows \( 1_A \) is constant and hence \( m(A) = 0 \) or 1.

7.5.3. Case (3): Rational \( \alpha \) and \( \beta \). By the same arguments, we have \( \mathcal{L}_f 1_B = 1_{B+a} \).
Unlike the case \( \alpha, \beta \in [0,1) \setminus \mathbb{Q} \), we further have \( \mathcal{L}_f 1_B = 1_B \) by taking \( N \) as the least common multiple of the denominator for \( \alpha \) and \( \alpha - \beta \). Then, for any \( \delta > 0 \), taking \( \varphi = \frac{1}{\delta} 1_B \), we get \( m(B) = \delta \) but
\[ \int_B A_n \varphi \, dm = \frac{1}{n} \sum_{j=0}^{n-1} \int_B \mathcal{L}^j_f \varphi \, dm = 1, \]
since the support of \( \mathcal{L}^n_f \varphi \) and \( A_n \varphi \) is always on \( B \). This concludes that \( \mathcal{L}_f \) does not satisfy (MC).

Finally, since \( 1_X \) is clearly invariant for \( \mathcal{L}_f \), by the fact that (WAP) is equivalent to the existence of an invariant density with the maximal support [88, Theorem 3.1], \( \mathcal{L}_f \) satisfies (WAP). This completes the proof.

7.6. Proof of Proposition 1.20: Direct sums of random maps. Notice that the random dynamics \( f_+ \) on \( X_+ \) generated by \( \tau_+^{1}, \tau_+^{2} \) satisfies (S) because \( f \) satisfies (S) and preserves \( X_+ \). That is, \( \mathcal{L}_{f_+} \) admits an invariant density, denoted by \( h_+ \). Let \( h \) be the density function on \( X \) given by \( h(x) = 0 \) on \( X_- \) and \( h(x) = h_+(x) \) on \( X_+ \). Then, \( h \) is obviously an invariant density of \( \mathcal{L}_f \), namely, (S) holds for \( f \).

On the other hand, (WAP) does not hold. If \( \mathcal{L}_f \) has an invariant density \( h \) with the maximal support, then an integral function \( h_- \) on \( X_- \), given by \( h_-(x) = h(x) \), should be invariant for \( \mathcal{L}_{f_-} \), where \( f_- \) is the random dynamics generated by \( \tau_-^{1}, \tau_-^{2} \), due to the invariance of \( X_-, X_+ \) for \( f \). This contradicts the assumption that \( f_- \) is not in (S).

7.7. Proof of Proposition 1.21: Deterministic systems. All statements except one for the baker’s transformation can be proven by literally repeating the argument in the previous subsections. On the other hand, for the statement for the baker’s transformation, it immediately follows from the fact that the baker’s transformation is mixing but not exact with respect to the Lebesgue measure \( m \) (cf. [3]).
APPENDIX A. ANNEALED PERRON–FROBENIUS OPERATORS

Let \((X, \mathcal{B}, m)\) be a Polish probability space and consider \((\Omega, \mathcal{F}) = (\mathbb{T}^N, \mathcal{A}^N)\) an infinite product space of a measure space. Now, we introduce a probability measure \(P\) on \((\Omega, \mathcal{F})\) which is shift invariant but not necessarily a Bernoulli probability. Let \(f : T \times X \to X\) be a measurable map. We consider iterations \(f_\omega^n = f_{\omega_n} \circ \cdots \circ f_{\omega_1}\) for \(n \geq 1\) where the sequence \(\omega = (\omega_i)_{i \geq 1}\) is chosen from \(\Omega\) according to the probability \(P\). To emphasize this random choice, we call \(f\) as a \(P\)-random map. We can introduce the annealed Perron–Frobenius operator \(L_f\) given by

\[
L_f \varphi = \int L_\omega \varphi \, dP(\omega) \quad \text{for } \varphi \in L^1(m) \tag{A.1}
\]

where \(L_\omega\) is the Perron–Frobenius operator of \(f_\omega\). Recall that we introduced \(L_\omega \varphi\) as the Radon–Nikodým derivative of the signed measure \((f_\omega)_* m_\varphi\) with respect to \(m\) where

\[
m_\varphi(B) = \int_B \varphi \, dm \quad \text{for all } B \in \mathcal{B}.
\]

That is, for each \(\varphi \in L^1(m)\),

\[
L_\omega \varphi \in L^1(m) \quad \text{such that } (f_\omega)_* m_\varphi(B) = \int_B L_\omega \varphi \, dm \quad \text{for all } B \in \mathcal{B}. \tag{A.2}
\]

In other words, \((f_\omega)_* m_\varphi = m_{L_\omega \varphi}\), i.e.,

\[
\int_{(f_\omega)^{-1}(B)} \varphi \, dm = \int_B L_\omega \varphi \, dm \quad \text{for all } B \in \mathcal{B}. \tag{A.3}
\]

The following lemma characterizes \(L_f\) in similar terms.

**Lemma A.1.** For each \(\varphi \in L^1(m)\), \(L_f \varphi\) is the Radon–Nikodým derivative of the measure \(m_\varphi\) with respect to \(m\) where

\[
m_\varphi(B) = \int (f_\omega)_* m_\varphi(B) \, dP \quad \text{for all } B \in \mathcal{B}.
\]

That is,

\[
L_f \varphi \in L^1(m) \quad \text{such that } m_\varphi(B) = \int_B L_f \varphi \, dm \quad \text{for all } B \in \mathcal{B}. \tag{A.4}
\]

In particular,

\[
L_f \varphi = \varphi \iff m_\varphi(B) = \int (f_\omega)_* m_\varphi(B) \, dP \quad \text{for all } B \in \mathcal{B}.
\]

**Proof.** According to the definition of the annealed operator \(L_f\) in (A.1) and the Perron–Frobenius operator of \(f_\omega\) in (A.2), it holds

\[
\int_B L_f \varphi \, dm = \int_B \int L_\omega \varphi \, dP \, dm = \int \int_B L_\omega \varphi \, dP \, dm = \int (f_\omega)_* m_\varphi(B) \, dP.
\]
This proves the first part of the lemma. The second part follows immediately by observing that \( \mathcal{L}_f \varphi = \varphi \) if and only if \( \int_B \mathcal{L}_f \varphi \, d\mu = m_\varphi(B) \). Hence, the above equation reads as \( m_\varphi = \bar{m}_\varphi \) and thus (A.4) follows. \( \square \)

**Remark A.2.** Since the Perron–Frobenius operator also acts on the space of finite signed measures as
\[
\mathcal{L}_f \mu(B) = \int 1_B \circ f_\omega(x) \, d\mathbb{P}(\omega) \, d\mu(x) \quad \text{for all } B \in \mathcal{B},
\]
the equivalence (A.4) can be generalized as follows:
\[
\mathcal{L}_f \mu = \mu \iff \mu(B) = \int (f_\omega)_\ast \mu(B) \, d\mathbb{P} \quad \text{for all } B \in \mathcal{B}.
\]

**Remark A.3.** Note that when \( \mathbb{P} = \rho^N \) is a Bernoulli measure on \( \Omega \),
\[
\int (f_\omega)_\ast m_\varphi(B) \, d\mathbb{P} = \int (f_\omega)_\ast m_\varphi(B) \, d\rho(\omega_1).
\]
Thus, from (A.4), \( \varphi \) is an \( \mathcal{L}_f \)-invariant density if and only if \( m_\varphi \) is an \( f \)-stationary measure in the sense introduced in (1.2) that is absolutely continuous with respect to \( m \).

Let us write
\[
\mathcal{L}_\omega^n = \mathcal{L}_{\omega_0} \circ \cdots \circ \mathcal{L}_{\omega_1} \quad \text{for } \psi \in L^\infty(m) \text{ and } \omega \in \Omega
\]
where \( \sigma \) is the left shift on \( \Omega \) preserving \( \mathbb{P} \). As usual, we introduce the adjoint operator of \( \mathcal{L}_f : L^1(m) \to L^1(m) \) as the operator \( \mathcal{L}_f^\ast : L^\infty(m) \to L^\infty(m) \) satisfying
\[
\langle \mathcal{L}_f^\ast \psi, \varphi \rangle = \langle \psi, \mathcal{L}_f \varphi \rangle \quad \text{for } \psi \in L^\infty(m) \text{ and } \varphi \in L^1(m) \text{ where } \langle h, g \rangle = \int hg \, dm.
\]
Recall that since \( (\mathcal{L}_f^n)^\ast = (\mathcal{L}_f^\ast)^n \) where \( \mathcal{L}_f^n \) and \( (\mathcal{L}_f^\ast)^n \) are the \( n \)-th iteration of \( \mathcal{L}_f \) and \( \mathcal{L}_f^\ast \) respectively, we simply write it as \( \mathcal{L}_f^n \). Similarly, we write \( (\mathcal{L}_\omega^n)^\ast = \mathcal{L}_\omega^n \).

**Lemma A.4.** For any \( n \geq 1 \), \( \varphi \in L^1(m) \) and \( \psi \in L^\infty(m) \), it holds
\[
(1) \quad \mathcal{L}_\omega^n = \mathcal{L}_{f_\omega^n};
(2) \quad \mathcal{L}_\omega^n \psi = \psi \circ f_\omega^n;
(3) \quad \langle \mathcal{L}_\omega^n \psi, \varphi \rangle = \langle \psi, \mathcal{L}_\omega^n \varphi \rangle.
\]
Moreover, if \( \mathbb{P} \) is a Bernoulli measure on \( \Omega \) (i.e., an infinite product \( p^\mathbb{N} \) of a probability measure \( p \) on \( T \)), it holds
\[
(4) \quad \mathcal{L}_f^n \varphi = \int \mathcal{L}_\omega^n \varphi \, d\mathbb{P};
(5) \quad \mathcal{L}_f^n \psi = \int \mathcal{L}_\omega^n \psi \, d\mathbb{P}.
\]
Proof. We prove the claim by induction. It is clear that (1) holds for \( n = 1 \). Assume that \((f_{\omega}^{n-1}), m_\psi = m_{\mathcal{L}_{\omega}^{n-1}}\). Then
\[
(f_{\omega}^n), m_\psi = (f_{\omega}^{n-1}, (f_{\omega}^{n-1}), m_\psi) = (f_{\omega}^{n-1}, m_{\mathcal{L}_{\omega}^{n-1}}).
\]
From this, having in mind (A.3), we get
\[
(f_{\omega}^n), m_\psi(B) = \int_{(f_{\omega}^{n-1})^{-1}(B)} \mathcal{L}_{\omega}^{n-1} \varphi \, dm = \int_B \mathcal{L}_{\omega}^{n-1} \circ \mathcal{L}_{\omega}^{n-1} \varphi \, dm = \int_B \mathcal{L}_{\omega}^n \varphi \, dm.
\]
This implies that \( \mathcal{L}_{\omega}^n = \mathcal{L}_{\omega}^n \).

Now, (2) follows immediately from (1) since \( \mathcal{L}_{\omega}^n = (\mathcal{L}_{\omega})^n = (\mathcal{L}_{\omega}^n) \). Hence, as is well-known, \( (\mathcal{L}_{\omega})\psi = \psi \circ f_{\omega}^n \). Finally, observe that (3) is just the duality relation. To conclude the proof, from now on we assume that the probability \( \mathbb{P} \) is Bernoulli.

**Claim A.5.** For any \( n \geq 1 \),
\[
m_{\mathcal{L}_{\omega}^n}(B) = \int (f_{\omega}^n), m_\psi(B) \, d\mathbb{P} \quad \text{for all } B \in \mathcal{B} \text{ and } \varphi \in L^1(m).
\]

Proof. We prove the claim by induction. For \( n = 1 \), we need to prove that \( m_{\mathcal{L}_{\omega}^1}(B) = \int (f_{\omega}), m_\psi(B) \, d\mathbb{P} \). Observe that this is just Lemma A.1. Thus, we can assume that the claim holds for \( n - 1 \). Then, using again Lemma A.1,
\[
m_{\mathcal{L}_{\omega}^n}(B) = \int_B \mathcal{L}_{\omega}^n \varphi \, dm = \int_B \mathcal{L}_{\omega} \mathcal{L}_{\omega}^{n-1} \varphi \, dm = \int_B (f_{\omega}), m_{\mathcal{L}_{\omega}^{n-1}}(B) \, d\mathbb{P}.
\]
By induction we have that
\[
\int_B (f_{\omega}), m_{\mathcal{L}_{\omega}^{n-1}}(B) \, d\mathbb{P} = \int \int (f_{\omega}^{n-1}), m_\psi(f_{\omega}^{n-1}(B)) \, d\mathbb{P}(\bar{\omega}) \, d\mathbb{P}(\omega)
= \int \int (f_{\omega} \circ f_{\omega}^{n-1}), m_\psi(B) \, d\mathbb{P}(\bar{\omega}) \, d\mathbb{P}(\omega).
\]
Taking into account that \( \mathbb{P} \) is a Bernoulli measure and that \( f_{\omega} = f_{\omega_1} \) and \( f_{\omega}^{n-1} = f_{\omega_{t_{1}}} \circ \cdots \circ f_{\omega_1} \) where \( \omega = (\omega_t)_{t \geq 1} \) and \( \bar{\omega} = (\bar{\omega}_t)_{t \geq 1} \), we have
\[
\int \int (f_{\omega} \circ f_{\omega}^{n-1}), m_\psi(B) \, d\mathbb{P}(\bar{\omega}) \, d\mathbb{P}(\omega) = \int (f_{\omega}^n), m_\psi(B) \, d\mathbb{P}.
\]
Putting together all the above equations, we get the claim.

Now we conclude (4). To do this, we use the first item in this lemma, the fact that the Perron–Frobenius operator of \( f_{\omega}^n \) satisfies that \( m_{\mathcal{L}_{\omega}^n} = (f_{\omega}^n), m_\psi \), and finally Claim A.5. Then we get that
\[
\int_B \int \mathcal{L}_{\omega}^n \varphi \, d\mathbb{P} \, dm = \int_B \int \mathcal{L}_{f_{\omega}} \varphi \, d\mathbb{P} \, dm = \int_B (f_{\omega}^n), m_\psi(B) \, d\mathbb{P} = \int_B P^n \varphi \, dm.
\]
From this, since \( B \in \mathcal{B} \) is arbitrary, we obtain that \( \mathcal{L}_{\omega}^n \varphi = \int \mathcal{L}_{\omega}^n \varphi \, d\mathbb{P} \, dm \) as desired.
Finally, we will prove (5). By (3), (4) and the duality, we have
\[
\langle L^n f, \varphi \rangle = \langle \psi, L^n f \varphi \rangle = \langle \psi, \int L^n \omega f d\mathcal{P} \rangle = \int \langle \psi, L^n \omega f \rangle d\mathcal{P}
\]

This implies \( L^n f = \int L^n \omega \psi d\mathcal{P} \) as required.

**Appendix B. Generalized restrictions of Markov operators**

We will extend the theory of restrictions of Markov operators developed in Section 3.2. Roughly speaking, we want to replace the reference measure \( m \) with an absolutely continuous invariant measure and restrict the Markov operator to the support of such measure recovering the Markov property among other ergodic properties.

Let \( P : L^1(m) \to L^1(m) \) be a Markov operator. Take \( h \in D(m) \) and define the probability measure \( m_h \) as usual by \( dm_h = h d\mathcal{P} \). Denote \( S = \text{supp} \, h \) and note that \( S \in \mathcal{B} \) with \( m(S) > 0 \). Let us consider \( L^1(m_h) = L^1(S, \mathcal{B}_S, m_h) \) where \( \mathcal{B}_S \) denotes the trace \( \sigma \)-algebra of \( S \) in \( \mathcal{B} \). As in Section 3.2, \( L^1(m_h) \hookrightarrow L^1(m) \). Abusing notation, we write this inclusion by \( 1_S \) and identify \( L^1(m_h) \) with

\[
1_S(L^1(m_S)) = \{ \phi \in L^1(m) : \text{supp} \, \phi \subset S \} \subset L^1(m).
\]

We define the operator

\[
P_h : L^1(m_h) \to L^1(m_h), \quad P_h \phi = \frac{1_S P(1_S \phi h)}{h} \text{ for } \phi \in L^1(m_h).
\]

Actually, \( P_h \) acts on \( L^1(m) \) by the same formula. Moreover, if \( h = \frac{1_S}{m(S)} \), then \( P_h \phi = 1_S P(1_S \phi) \). That is, \( m_h \) and \( P_h \) coincides with the measure \( m_S \) and the operator \( P_S \) introduced in (3.3) and (3.4) respectively.

It is clear that \( P_h \) is a bounded linear positive operator on \( L^1(m_h) \). Moreover, \( P_h \) is also a contraction on \( L^1(m_h) \) since \( \|P_h\|_{\text{op}} \leq 1 \). On the other hand, \( L^\infty(m_h) \hookrightarrow L^\infty(m) \) by the same canonical inclusion \( 1_S \). As before, we identify \( L^1(m_h) \) with \( 1_S(L^\infty(m_h)) \). We also have that \( P_h^* \), the adjoint operator of \( P_h \), acting on \( L^\infty(m_h) \) coincides with \( 1_S P^* 1_S \). Indeed, for each \( \varphi \in L^1(m_h) \) and \( \psi \in L^\infty(m_h) \),

\[
\int_S \varphi \cdot P_h^* \psi dm_h = \int_S P_h \varphi \cdot \psi dm_h = \int_X 1_S \frac{P(1_S \phi h)}{h} \cdot \psi h dm = \int_X \varphi \cdot 1_S P^* \psi dm_h = \int_S \varphi \cdot P^* 1_S \psi dm_h
\]

as desired.
Proposition B.1. Let $P : L^1(m) \to L^1(m)$ be a Markov operator and consider $S \in \mathcal{B}$ with $m(S) > 0$. The following conditions are equivalent:

1. $P^*1_{X \setminus S} \leq 1_{X \setminus S}$. Equivalently, $P1_S \geq 1_S$;
2. $\text{supp} \, P1_S \subset S$. Equivalently, $\text{supp} \, P^*1_{X \setminus S} \subset X \setminus S$;
3. $1_S P(1_S \phi) = P(1_S \phi)$ for all $\phi \in L^1(m)$. Equivalently, $P \phi = P(1_S \phi)$ for all $\phi \in L^1(m)$ or $P(1_S \phi) \in L^1(m_S)$ for all $\phi \in L^1(m)$;
4. for every $\phi \in L^1(m)$ it holds that
   \[ \int_S P(1_S \phi) \, dm = \int_S \phi \, dm; \]
5. $P : L^1(m_S) \to L^1(m_S)$ is a Markov operator;
6. $P_h : L^1(m_h) \to L^1(m_h)$ is a Markov operator for any $h \in D(m)$ with $S = \text{supp} \, h$.

Proof. Let us prove the equivalence between the above conditions.

First, we will prove (1) $\Rightarrow$ (2). Suppose that (2) does not hold. Then, there is $B \subset \text{supp} \, P1_S$ with $m(B \setminus S) > 0$. From this and the condition (1), it follows
\[ 0 < \int_{B \setminus S} P1_S \, dm = \int 1_S P^*1_B \, dm \leq \int 1_S P^*1_{X \setminus S} \, dm \leq \int 1_S 1_{X \setminus S} \, dm = 0 \]
which is a contradiction. Note that the equivalence indicated in (1) immediately follows by using $1_X = P1_S + P^*1_{X \setminus S}$. Furthermore, the equivalence indicated in (2) follows immediately by the duality
\[ \int_{X \setminus S} P1_S \, dm = \int_S P^*1_{X \setminus S} \, dm. \]

Now, we will show that (2) $\Rightarrow$ (3). Let $\phi \in L^1(m)$. Note that $\text{supp} \, 1_S \phi \subset S = \text{supp} \, 1_S$. Thus, by Lemma 3.1 and condition (2), $\text{supp} \, P(1_S \phi) \subset \text{supp} \, P1_S \subset S$ up to $m$-null set. In particular, $1_S P(1_S \phi) = P(1_S \phi)$. That is, condition (3) holds.

We will see that (3) $\Rightarrow$ (4). Assuming (3) and using that $P$ is a Markov operator, for any $\phi \in L^1(m)$,
\[ \int_S P(1_S \phi) \, dm = \int_S 1_S P(1_S \phi) \, dm = \int_S 1_S P(1_S \phi) \, dm = \int_S 1_S \phi \, dm = \int_S \phi \, dm \]
and thus (4) holds.

Next, we will prove that (4) and (5) are equivalent. Clearly, $P_S$ is a bounded linear positive operator. Note that, by definition, for any $\phi \in L^1(m_S)$,
\[ \int P_S \phi \, dm_S = \frac{1}{m(S)} \int_S P(1_S \phi) \, dm \quad \text{and} \quad \int \phi \, dm_S = \frac{1}{m(S)} \int_S \phi \, dm. \]
Thus, clearly we get that $\int P_S \phi \, dm_S = \int \phi \, dm_S$ and viceversa.
We will obtain the implication (4) to (6). Similar as above, for any \( \phi \in L^1(m) \) and \( h \in D(m) \) with \( S = \text{supp } h \), \( P_h \) is clearly a bounded linear positive operator. Moreover, taking into account that \( \phi h = 1_S \phi h \),
\[
\int P_h \phi \, dm_h = \int S P(1_S \phi h) \, dm \quad \text{and} \quad \int \phi \, dm_h = \int S \phi h \, dm.
\] (B.1)
Hence, applying (4) to \( \phi = \phi h \), we immediately get
\[
\int P_h \phi \, dm = \int \phi \, dm_h.
\]
Thus, \( P_h \) is Markov and we conclude (6).

Since (6) \( \Rightarrow \) (5) is clear, the rest to show is (4) \( \Rightarrow \) (1). Suppose contrarily that there is some \( B \subset S \) such that \( m(B) > 0 \) and \( 1_B P^* 1_S < 1_B \). This implies that
\[
\int S 1_B \, dm > \int S 1_B P^* 1_S \, dm = \int S P(1_B 1_S) \, dm
\]
but clearly contradicts to the assumption (4) with \( \phi = 1_B \).

Appendix C. Ergodicity of invariant densities

Let \( P : L^1(m) \to L^1(m) \) be a Markov operator as introduced in Section 1.2. We associate \( P \) with a new operator, which we will still denote by \( P \), acting on the set of probability measures \( \mu \) on \( (X, \mathcal{B}) \) which are absolutely continuous with respect to \( m \) by
\[
P \mu(A) = \int P^* 1_A(x) \, d\mu(x) \quad \text{for any } A \in \mathcal{B}.
\]
Here \( P^* : L^\infty(m) \to L^\infty(m) \) denotes the adjoint operator of \( P : L^1(m) \to L^1(m) \). Recall that \( D(m) = \{ \phi \in L^1(m) : \phi \geq 0, \|\phi\| = 1 \} \) and for a given \( \phi \in D(m) \), we denote \( m_\phi \), the probability measure given by \( dm_\phi = \phi \, dm \). We say that \( \phi \in D(m) \) is a \( P \)-invariant density if \( P \phi = \phi \). Similarly, a probability measure \( \mu \) on \( (X, \mathcal{B}) \) is said to be \( P \)-invariant if \( P \mu = \mu \). Denote by \( D_p(m) \) and \( I_p(m) \), respectively, the convex sets of \( P \)-invariant densities and \( P \)-invariant probability measures that are absolutely continuous with respect to \( m \).

**Lemma C.1.** Let \( \phi \in D(m) \). Then, \( P \phi = \phi \) if and only if \( P m_\phi = m_\phi \). In particular, \( D_p(m) \) is identified with \( I_m(P) \).

**Proof.** If \( P \phi = \phi \), then for any \( A \in \mathcal{B} \),
\[
P m_\phi(A) = \int P^* 1_A \, dm_\phi = \int P^* 1_A \phi \, dm = \int A P \phi \, dm = \int_A \phi \, dm = m_\phi(A).
\]
Conversely, if $Pm_\phi = m_\phi$, then for any $A \in \mathcal{B}$,
\[ m_\phi(A) = \int P^1_A \phi \, dm = \int_A P\phi \, dm. \]
Thus $P\phi$ is the Radon–Nikodým derivative of $m_\phi$ with respect to $m$. Since this derivative is exactly $\phi$, we have $P\phi = \phi$. \qed

Recall that an invariant probability measure $\nu$ of a transformation $g$ of a measurable space $(Y, \mathcal{A})$, i.e., $\nu = g_\# \nu$, is said to be ergodic if $\nu(A) \in [0, 1]$ for all set $A \in \mathcal{A}$ such that $A \subset g^{-1}(A)$. Equivalently, for all $A \in \mathcal{A}$ such that $A = g^{-1}(A)$. Moreover, it is also well known that one can weaken both previous conditions of invariance of the set $A$ by simply asking that the relation holds up to a $\mu$-null set. That is, if $\mathcal{L}_g^* 1_A \geq 1_A$ or $\mathcal{L}_g^* 1_A = 1_A$, respectively, for $\mu$-almost everywhere where $\mathcal{L}_g^* \psi = \psi \circ g$ for $\psi \in L^\infty(m)$ is the adjoint Perron–Frobenius operator of $g$. The following proposition shows some of these equivalences in a more general setting.

**Proposition C.2.** Let $\mu \in I_p(m)$. Then the following conditions are equivalent:

1. $\mu(A) \in [0, 1]$ for any $A \in \mathcal{B}$ such that $P^1_A \geq 1_A$ for $m$-almost everywhere;
2. $\mu(A) \in [0, 1]$ for any $A \in \mathcal{B}$ such that $P^1_A \geq 1_A$ for $\mu$-almost everywhere;
3. $\mu(A) \in [0, 1]$ for any $A \in \mathcal{B}$ such that $P^1_A = 1_A$ for $\mu$-almost everywhere.

**Proof.** $(2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (2)$: Let us consider $A \in \mathcal{B}$ such that $P^1_A \geq 1_A$ for $\mu$-almost everywhere. Since $\mu \in I_p(m)$, then $\mu(A) = \int P^1_A \, d\mu$. On the other hand, $\mu(A) = \int 1_A \, d\mu$. Thus, since $P^1_A \geq 1_A$ for $\mu$-almost everywhere, we get $0 \leq \int P^1_A - 1_A \, d\mu = 0$. From this follows that $P^1_A = 1_A$ for $\mu$-almost everywhere. Consequently, it immediately follows that $(2)$ implies $(3)$.

$(2) \Rightarrow (1)$: Let us consider $A \in \mathcal{B}$ such that $P^1_A \geq 1_A$ for $m$-almost everywhere. This means that there is $B \in \mathcal{B}$ such that $m(B) = 0$ and $P^1_A(x) \geq 1_A(x)$ for all $x \in X \setminus B$. Since $\mu$ is absolutely continuous with respect to $m$, then $\mu(B) = 0$. Thus, $P^1_A \geq 1_A$ for $\mu$-almost everywhere. From this, it immediately follows that $(3)$ implies $(2)$.

$(1) \Rightarrow (2)$: Let us consider $A \in \mathcal{B}$ such that $P^1_A \geq 1_A$ for $\mu$-almost everywhere. Since $\mu \in I_p(m)$, by Lemma C.1, we have $\phi \in D_p(m)$ such that $\mu = m_\phi$ (i.e., $d\mu = \phi \, dm$). Then, by Proposition 4.5, we have that $P^1_{1 \chi_{\|S}} \leq 1_{\chi\|S}$ for $m$-almost everywhere where $\chi = \text{supp } \phi$. Then,
\[ 1_S P^1 1_A = 1_S P^1 1_{\|S \cap A} + 1_S P^1 1_{(X\|S) \cap A} = 1_S P^1 1_{\|S \cap A} \leq P^1 1_{\|S \cap A} \quad \text{(C.1)} \]
for $m$-almost everywhere. But observe that $\text{supp } 1_S P^1 1_A \subset S$ and $m$ and $\mu$ are equivalent on $S$. Thus, since $P^1_A \geq 1_A$ for $\mu$-almost everywhere we have $1_{\|S \cap A} = 1_S 1_A \leq 1_S P^1 1_A$ $m$-almost everywhere. Putting this together with (C.1), we get that
1_{S \cap A} \leq P^* 1_{S \cap A} \mu\text{-almost everywhere. By assumption, } m(S \cap A)m(X \setminus (S \cap A)) = 0. Consequently, since \( \mu(A) = \mu(S \cap A) \) and \( \mu(X \setminus A) = \mu(X \setminus (S \cap A)) \), one has that \( \mu(A)\mu(X \setminus A) = 0. \) \hfill \Box

Let \( h \in D_p(m) \). From, Lemma C.1, we can apply Proposition C.2 to \( \mu = m_h \).

**Definition C.3.** Let \( h \in D_p(m) \) and set \( \mu = m_h \). If any of the equivalent items in Proposition C.2 holds, we say that \( \mu \) is an ergodic \( P \)-invariant measure, \( h \) is an ergodic \( P \)-invariant density, \( (P, \mu) \) is ergodic or \( (P, h) \) is ergodic.\(^{11}\)

Observe that if \( P 1_X = 1_X \), then according to Lemma C.1, it holds that \( P m = m \), i.e., \( m \in I_p(m) \). Then, Proposition C.2 implies that \( 1_X \) is an ergodic \( P \)-invariant density if and only if \( m(A) \in [0, 1] \) for all \( A \in \mathcal{B} \) with \( P^* 1_A = 1_A \). Motivated by this last condition and following Krengel [59, page 126], we introduce the following definition:

**Definition C.4.** A Markov operator \( P : L^1(m) \to L^1(m) \) is called ergodic in the sense of Krengel if the set of \( P^* \)-invariant sets \( \mathcal{B}_i = \{ A \in \mathcal{B} : P^* 1_A = 1_A \text{ } m\text{-almost everywhere} \} \) is trivial, i.e., \( \mathcal{B}_i = \{ \emptyset, X \} \) up to an \( m \)-null set.

In view of this definition, the above observation can be written as follows:

**Remark C.5.** If \( P 1_X = 1_X \), then the following are equivalent:

1. \( (P, 1_X) \) is ergodic (or in other words, \( (P, m) \) is ergodic);
2. \( P \) is ergodic in the sense of Krengel.

In general, \( P \) could be ergodic in the sense of Krengel and \( 1_X \not\in D_p(m) \). See, for instance, Remark C.9. Note that Definition C.3 requires that \( 1_X \) is a \( P \)-invariant density to be ergodic.

Now, recalling the theory of restriction of Markov operators introduced in Section 3.2 and generalized in Appendix B we have the following:

**Theorem C.6.** Let \( h \in D_p(m) \) and \( S = \text{supp } h \). Then the following conditions are equivalent:

1. \( (P, h) \) is ergodic (or in other words, \( (P, m_h) \) is ergodic);
2. \( (P_h, 1_S) \) is ergodic (or in other words, \( (P_h, m_h) \) is ergodic);
3. \( P_h \) is ergodic in the sense of Krengel;
4. \( P_S \) is ergodic in the sense of Krengel.

**Proof.** Recall that \( P_h^* \psi = 1_S P^*(1_S \psi) \) for all \( \psi \in L^\infty(m_h) \).

1. \( \Rightarrow \) 3: Let us consider \( A \in \mathcal{B}_S \) such that \( P_h^* 1_A = 1_A \) for \( \mu \)-almost everywhere. Then \( 1_S P^* 1_A = 1_A \mu \)-almost everywhere and hence also \( m \)-almost everywhere. In particular, \( P^* 1_A \geq 1_A m \)-almost everywhere and thus, by assumption, \( m_h(A) \in [0, 1] \).

\(^{11}\)The reference measure \( m \) is always implicit when we mention \( P \).
(3) \(\Leftrightarrow\) (2): Notice that \(P_h 1_S = 1_S\). Thus, the equivalence follows from Remark C.5.

(2) \(\Rightarrow\) (1): Let us consider \(A \in \mathcal{B}\) such that \(P^* 1_A \geq 1_A\) for \(\mu\)-almost everywhere. Since \(h \in D_P(m)\), we have \(P^* 1_{X \backslash S} \leq 1_{X \backslash S}\) for \(m\)-almost everywhere. Hence, as argued in the previous theorem (see (C.1)),

\[
1_{S \cap A} \leq 1_S P^* 1_A = 1_S P^* 1_{S \cap A} + 1_S P^* 1_{(X \backslash S) \cap A} = 1_S P^* 1_{S \cap A} = P_h^* 1_{S \cap A}
\]

\(m\)-almost everywhere. Then, by assumption we have that \(m_h(S \cap A) \in \{0,1\}\). But, since \(m_h(A) = m_h(S \cap A)\) we conclude the implication.

(3) \(\Leftrightarrow\) (4): Notice that \(P^*_S\) and \(P^*_h\) are both of the form \(1_S P^* 1_S\) acting, respectively, on \(L^1(m_S)\) and \(L^1(m_h)\). Since both measures \(m_S\) and \(m_h\) are equivalent, we have that \(P^*_S 1_A = 1_A\) \(m_S\)-almost everywhere if and only if \(P^*_h 1_A = 1_A\) \(m_h\)-almost everywhere. From this, it immediately follows the equivalence. \(\square\)

**Definition C.7.** A Markov operator \(P : L^1(m) \to L^1(m)\) is called **conservative** if there is \(\varphi \in L^1(m)\) such that \(\varphi > 0\) on \(X\) and

\[
X = \left\{ x \in X : \sum_{i=0}^{\infty} P^i \varphi(x) = \infty \right\}
\]

up to an \(m\)-null set.

**Proposition C.8.** Let \(h \in D(m)\) be a \(P\)-invariant density and set \(S = \text{supp} \ h\). Then both, \(P_h\) and \(P_S\) are conservative Markov operators. In particular, if \(h \in D_P(m)\) is ergodic then,

1. \(P_S : L^1(m_S) \to L^1(m_S)\) is a conservative and ergodic Markov operator and \(D_{P_S}(m_S) = \{h\}\);
2. \(P_h : L^1(m_h) \to L^1(m_h)\) is a conservative and ergodic Markov operator and \(D_{P_h}(m_h) = \{1_S\}\).

**Proof.** Since \(h\) is \(P\)-invariant, according to Proposition 3.3 and Proposition B.1, \(P_S\) and \(P_h\) are Markov operators and \(P_S h = h\). Also, it is not difficult to check that \(P_h 1_S = 1_S\). On the other hand, \(P_S\) and \(P_h\) are conservative since \(h > 0\) and \(1_S > 0\) and \(\sum_{i=0}^{\infty} P^i h = \infty\) and \(\sum_{i=0}^{\infty} P^i 1_S = \infty\) on \(S\). If in addition \(h\) is ergodic, then from Theorem C.6 we get that \(P_S\) and \(P_h\) are conservative and ergodic Markov operators. Finally, from [37, Theorem A in Chapter VI] it follows that \(D_{P_S}(m_S) = \{h\}\) and \(D_{P_h} = \{1_S\}\). \(\square\)

**Remark C.9.** In view of the above proposition, if \(h\) is an ergodic \(P\)-invariant density with \(h > 0\) on \(X\) and \(1_X \not\in D_P(m)\), then \(P\) is ergodic in the sense of Krengel but \(1_X \not\in D_P(m)\) = \{h\}.

The next proposition is well-known for the case of transformations. One can generalize this to the case for Markov operators in terms of invariant densities.

**Proposition C.10.** Let \(h \in D_P(m)\) and set \(S = \text{supp} \ h\). Then, \(h\) is ergodic if and only if \(h\) is an extremal point of \(D_P(m)\). That is, it cannot be decomposed as

\[
h = th_1 + (1-t)h_2 \text{ with } t \in (0,1) \text{ and } h_1, h_2 \in D_P(m).
\]
Moreover, if $h_1 \in D_p(m)$ is ergodic and $h_2 \in D_p(m)$, then either
\[ h_1 = h_2 \quad \text{or} \quad m(\text{supp} h_1 \cap \text{supp} h_2) = 0. \]

**Proof.** Suppose that $h$ is ergodic. By Proposition C.8, $P_S$ is a Markov operator and $D_{P_S}(m_S) = \{h\}$. In particular, we have $P_{Sg} = P(1_{Sg})$ for all $g \in L^1(m_S)$. See Proposition 3.3 or Proposition B.1. Now, if $h = ah_1 + (1 - a)h_2$ for some $0 < a < 1$ and $h_1, h_2 \in D_p(m)$, then $h = h_1 = h_2$. Indeed, observe that since $h_i$ is a $P$-invariant density, according again to Proposition C.8, $P_{S_i}$ is a Markov operator and $P_{S_i}h_i = h_i$ where $S_i = \text{supp} h_i$ for $i = 1, 2$. Then, since $S_i \subseteq S$, we have $P_{S_i}h_i = P(1_{S_i}h_i) = P(1_S h_i) = P_{S_i}h_i = h_i$ for $i = 1, 2$. Consequently, it follows $h = h_1 = h_2$ since $D_{P_S}(m_S) = \{h\}$. This proves that $h$ is an extremal point of $D_p(m)$.

Now, we will prove the converse. Suppose $h$ is not ergodic and denote by $\mu := m_h$. Take a set $A \in \mathcal{B}$ such that $P^1_A \geq 1_A$ and $0 < \mu(A) < 1$. Hence, since $\mu(S \setminus A) = 1 - \mu(A)$, we can write $h$ as a convex combination as follows
\[ h = \mu(A) \cdot \frac{1_A h}{\mu(A)} + \mu(S \setminus A) \cdot \frac{1_{S \setminus A} h}{\mu(S \setminus A)}. \]
Moreover, since $(1_A h)/\mu(A)$ and $(1_{S \setminus A} h)/\mu(S \setminus A)$ have both $L^1$-norm equals one, to show that $h$ is not an extremal point in $D_p(m)$ it suffices to prove that $h1_A$ and $h1_{S \setminus A}$ are $P$-invariant. To prove this, observe first that since $h = 1_A h + 1_{S \setminus A} h$ and $h = Ph = P(1_A h) + P(1_{S \setminus A} h)$, one can write
\[ 1_A h - P(1_A h) = P(1_{S \setminus A} h) - 1_{S \setminus A} h. \quad (C.2) \]
From Lemma 3.2, it follows that $\text{supp} P^1_A \subseteq A$ up to an $m$-null set. Moreover, since $\text{supp} h = S = \text{supp} 1_S$, by the $P$-invariance of $h$ and Lemma 3.1, it follows that $S = \text{supp} h = \text{supp} Ph = \text{supp} P^1_S$ up to an $m$-null set. Hence, $\text{supp} P^1_{S \setminus A} \subseteq S \setminus A$. Additionally, since $\text{supp}(1_A h) \subseteq \text{supp} 1_A$ and $\text{supp}(1_{S \setminus A} h) \subseteq \text{supp} 1_{S \setminus A}$, Lemma 3.1 implies that $\text{supp} P(1_A h) \subseteq \text{supp} P^1_A \subseteq A$ and $\text{supp} P(1_{S \setminus A} h) \subseteq \text{supp} P^1_{S \setminus A} \subseteq S \setminus A$. Consequently, $\text{supp}(1_A h - P(1_A h)) \subseteq A$ and $\text{supp}(P(1_{S \setminus A} h) - 1_{S \setminus A} h) \subseteq X \setminus A$ and thus it follows from (C.2) that
\[ P(1_A h) = 1_A h \quad \text{and} \quad P(1_{S \setminus A} h) = 1_{S \setminus A} h \]
as desired.

Finally, we will prove the second part of the proposition. Let us assume $h_1$ and $h_2$ as in the statement. Clearly if $h_1 = h_2$ then $m(S) > 0$ where $S = \text{supp} h_1 \cap \text{supp} h_2$. Conversely, suppose that $m(S) > 0$. Since $P^1_S \leq P^1_{\text{supp} h_i}$ by Lemma 3.1, $\text{supp} P^1_S \subseteq \text{supp} P^1_{\text{supp} h_i}$, $\text{supp} Ph_i = \text{supp} h_i$ for $i = 1, 2$ and thus $\text{supp} P^1_S \subseteq S$. According to Proposition B.1, $P_S$ is a Markov operator, $P_S \phi = P(1_S \phi)$ and
\[ \int_S P(1_S \phi) \, dm = \int_S 1_S \phi \, dm \quad (C.3) \]
for all $\phi \in L^1(m)$. This implies that $1_{S}h_i$ is $P$-invariant. Indeed, since $P h_i = h_i$, it follows that $1_{S}P(1_{S}h_i) = P_\omega h_i = P(1_{S}h_i) \leq h_i$. Hence $P(1_{S}h_i) \leq 1_{S}h_i$. On the other hand, if $P(1_{S}h_i) < 1_{S}h_i$ on a set $A \in \mathcal{B}$ of positive $m$-measure, then

$$\int_S P(1_{S}h_i) \, dm < \int_A 1_{S}h_i \, dm + \int_{S \setminus A} 1_{S}h_i \, dm = \int_S 1_{S}h_i \, dm$$

contradicting (C.3). Therefore, we have $m(A) = 0$ and $P(1_{S}h_i) = 1_{S}h_i$. Then, $g_i = \frac{1_{S}h_i}{H_i} \in D_P(m)$ where $H_i = \int_S h_i \, dm$. Moreover, $m_{g_i}$ is absolutely continuous with respect to $m_{h_i}$ since $dm_{g_i} = \frac{1_{S}}{H_i} \, dm_{h_i}$. In particular, since $m_{h_i}$ is ergodic, we also have that $m_{g_i}$ is ergodic. Since $S = \text{supp } g_1$, we obtain from Proposition C.8 that $D_{P_S}(m_S) = \{g_1\}$. But, since $S = \text{supp } g_2$ and $g_2$ is $P$-invariant, we also have that $g_2 \in D_{P_S}(m_S)$. Thus, $g_2 = g_1$. From this, it follows that $h_1 = h_2$ as desired. \hfill \Box

**Remark C.11.** Recall that the support of $h \in L^1(m)$ is only well-defined up to a set of measure zero. Thus, without loss of generality, from the above corollary, we can assume that the supports of any two ergodic $P$-invariant densities $h_1$ and $h_2$ are identical or disjoint.

To conclude, we will relate the previous definition of ergodicity for invariant measures of Markov operators with the classical approach in random dynamical systems and probability theory. First of all, let us consider an annealed Perron–Frobenius operator $\mathcal{L}_f : X \to X$ associated with a random map $f : T \times X \to X$ (with respect to a Bernoulli probability measure $\mathbb{P}$). Recall that, as we showed in Remark 2.5, when $X$ is a Polish space, any Markov operator can be represented as an annealed Perron–Frobenius operator. As explained in Section 1.3.4, associated with this map we have a transition probability $P(x,A)$ which coincides, for each $A \in \mathcal{B}$, with $\mathcal{L}_f 1_A$ on $m$-almost everywhere. Let us consider an absolutely continuous probability measure $\mu$ on $X$ which is $\mathcal{L}_f$-invariant. A classical approach to introducing the ergodicity of $\mu$ is first to leave this measure to an invariant measure of a deterministic dynamical system that represents $f$ in some sense. Then, ergodicity is defined throughout this leaf measure. A deterministic dynamical system that represents $f$ is its associated skew-product given by

$$F : \Omega \times X \to \Omega \times X, \quad F(\omega,x) = (\sigma \omega, f_\omega(x))$$

with $f_\omega = f(t, \cdot)$ if $\omega_0 = t$, $\omega = (\omega_i)_{i \in \mathbb{Z}} \in \Omega = T^N$ and $\sigma : \Omega \to \Omega$ the shift operator. The measure $\mu$ is lifted to the $F$-invariant measure $\mathbb{P} \times \mu$. But we can also consider the shift operator acting on $(X^\mathbb{Z}, \mathcal{B}^\mathbb{Z})$. By Kolmogorov extension theorem, one can construct a shift-invariant probability measure $\mathbb{P}_\mu$ on $X^\mathbb{Z}$ from the consistent sequence of
measures \( \{\mathbb{P}_\mu^n\}_{n \geq 0} \) where \( \mathbb{P}_\mu^n \) is defined on \( \mathbb{X}^{2n+1} \) by
\[
\int \varphi \, d\mathbb{P}_\mu^n = \int \varphi(x_{-n}, \ldots, x_n) P(x_{n-1}, dx_n) \ldots P(x_{-n}, dx_{-n+1}) \mu(dx_{-n}).
\]
The following result follows basically from classical results of Markov chain and random dynamical systems. We refer to [41, Section 5.2] and [66, Proposition 1.3] (see also [51, 74]) for more details.

**Theorem C.12.** Let \( f : T \times X \to X \) be a random map (with respect to a Bernoulli probability measure \( \mathbb{P} \)) and denote by \( \mathcal{L}_f : L^1(m) \to L^1(m) \) the associated annealed Perron–Frobenius operator. Let \( \mu \in I_m(\mathcal{L}_f) \). Then the following conditions are equivalent:

1. \( (\mathcal{L}_f, \mu) \) is ergodic;
2. \( \mathbb{P}_\mu \) is an ergodic shift invariant measure on \( \mathbb{X}^\mathbb{Z} \);
3. \( \mathbb{P} \times \mu \) is an ergodic \( F \)-invariant probability measure on \( \Omega \times X \).

**Appendix D. Mixing and exactness of invariant densities**

In this appendix we recall definition of mixing and exactness for (deterministic) dynamics, and relate them with mixing and exactness for Markov operators defined in Section 1.5. Although the content of this appendix should be a folklore among experts, we include it for convenience of the reader.

Let \( (X, \mathcal{B}, m) \) be a probability space. Let \( g : X \to X \) be an \( m \)-nonsingular measurable map, and consider a \( g \)-invariant probability measure \( \mu \) on \( X \). Then, \( \mu \) is called mixing if \( \mu(g^{-n}A \cap B) \to \mu(A)\mu(B) \) as \( n \to \infty \) for any \( A, B \in \mathcal{B} \) (cf. [89]). Recall also that \( \mu \) is said to be exact if
\[
\bigcap_{n \geq 0} g^{-n} \mathcal{B} = \{\emptyset, X\} \pmod{\mu}.
\]
From Lin’s theorem ([65]), \( \mu \) is exact if and only if
\[
\lim_{n \to \infty} \|\mathcal{L}_g^n \varphi\|_{L^1(\mu)} = 0 \quad \text{for any } \varphi \in L^1(m) := \left\{ \psi \in L^1(\mu) : \int_X \psi \, d\mu = 0 \right\}.
\]
Below as in Appendix B, we identify \( \varphi \in L^1(m_S) \) with \( 1_S \varphi \in L^1(m) \) for an \( m \)-positive measure set \( S \).

**Lemma D.1.** Assume that \( \mu \) is a \( g \)-invariant probability measure absolutely continuous with respect to \( m \) with the density map \( h \). Let \( S := \text{supp} \, h \). Then, \( \mu \) is mixing if and only if
\[
\lim_{n \to \infty} \mathcal{L}_g^n \varphi = h \int_X \varphi \, dm \quad \text{weakly in } L^1(m) \text{ for any } \varphi \in L^1(m_S).}
\]
Furthermore, $\mu$ is exact if and only if
\[
\lim_{n \to \infty} L^n \varphi = h \int_X \varphi \, dm \quad \text{strongly in } L^1(m) \quad \text{for any } \varphi \in L^1(m_S). \tag{D.1}
\]

Proof. Note first that the mixing property of $\mu$ is equivalent to require that
\[
\int \psi \circ g^n \cdot \varphi \, d\mu \to \int \psi \, d\mu \int \varphi \, d\mu \quad \text{as } n \to \infty
\]
for any $\psi \in L^\infty(\mu)$ and $\varphi \in L^1(\mu)$. Due to the duality between $L^g$ and $\psi \mapsto \psi \circ g$, this can be written as
\[
\int \psi \cdot L^n g(\varphi h) \, dm \to \int \psi \left( h \int (\varphi h) \, dm \right) \, dm \quad \text{as } n \to \infty.
\]
Hence, we immediately get the claim for mixing.

We next show the claim for exactness. Since exactness is a hereditary property between absolutely continuous measures, it follows from Lin’s theorem mentioned above that $\mu$ is exact if and only if
\[
\lim_{n \to \infty} \| L^n g \varphi \|_{L^1(m_S)} = 0 \quad \text{for any } \varphi \in L^1_0(m_S). \tag{D.2}
\]
Thus, it suffices to show the equivalence between (D.2) and (D.1).

(D.2)$\Rightarrow$(D.1): Note that $L^n g \varphi - h \int_X \varphi \, dm = L^n g (\varphi - h \int_X \varphi \, dm)$ and
\[
\int_X \left( \varphi - h \int_X \varphi \, dm \right) \, dm = \frac{1}{m(S)} \int_X \varphi \, dm_S - \int_X h \, dm \cdot \frac{1}{m(S)} \int_X \varphi \, dm_S = 0.
\]

(D.1)$\Rightarrow$(D.2): It is straightforward if we take $\varphi \in L^1_0(m_S)$.

We finally remark that, if a map $g : X \to X$ admits an ergodic invariant probability measure $\mu$, then it holds by Birkhoff’s pointwise ergodic theorem
\[
\lim_{n \to \infty} \int \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ g^j \cdot \psi \, d(\mathcal{P} \times \mu) = \int \psi \, d\mu \int \varphi \, d\mu
\]
for any $\varphi \in L^\infty(\mu)$, $\psi \in L^1(\mu)$. Thus, in the case when $\mu = h \, dm$ with some $h \in D(m)$,
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} L_j g \psi = h \int \psi \, d\mu \quad \text{weakly in } L^1(m) \quad \text{for each } \psi \in L^1(m_S).
\]
On the other hand, since $h$ is an ergodic invariant density and the restriction of $L_g$ on $L^1(m_S)$ satisfies (WAP), it follows from Yoshida–Kakutani’s mean ergodic theorem ([92, Theorem 1]) that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} L_j g \psi = h \int \psi \, d\mu \quad \text{strongly in } L^1(m) \quad \text{for each } \psi \in L^1(m_S).
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