On certain semigroups of transformations that preserve a partition

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ABSTRACT

Let \( X \) be a nonempty set, and let \( T_X \) be the full transformation semigroup on \( X \). For a partition \( \mathcal{P} = \{X_i | i \in I\} \) of \( X \), we consider the semigroup \( T(X, \mathcal{P}) = \{f \in T_X | \forall X_i \in \mathcal{P}, \exists X_j \subseteq X_i, \exists X_i \cap X_j \neq \emptyset \} \), the subsemigroup \( \Sigma(X, \mathcal{P}) = \{f \in T(X, \mathcal{P}) | \forall X_i, X_j, X_i \cap X_j \neq \emptyset \} \), and the group of units \( S(X, \mathcal{P}) \) of \( T(X, \mathcal{P}) \). In this paper, we first characterize the elements of \( \Sigma(X, \mathcal{P}) \). For a permutation \( f \) of finite \( X \), we next observe whether there exists a nontrivial partition \( \mathcal{P} \) of \( X \) such that \( f \in S(X, \mathcal{P}) \). We then characterize and enumerate the idempotents in the semigroup \( \Sigma(X, \mathcal{P}) \) for arbitrary and finite \( X \), respectively. We also characterize the elements of \( S(X, \mathcal{P}) \). For finite \( X \), we finally calculate the cardinality of \( T(X, \mathcal{P}), \Sigma(X, \mathcal{P}), \) and \( S(X, \mathcal{P}) \).

1. Introduction

We assume that the reader is familiar with elementary concepts of combinatorics and semigroup theory. Throughout the paper, let \( X \) denote a set with more than two elements, and let \( \mathcal{P} \) denote a partition of \( X \). The symbols \( T_X \) and \( S_X \) will be used to denote the full transformation semigroup and the symmetric group on \( X \), respectively. For a subset \( A \subseteq X \), we denote by \( A \) the image of \( A \) under \( f \in T_X \). A map \( f \in T_X \) preserves a partition \( \mathcal{P} \) if for every \( X_i \in \mathcal{P} \), there exists \( X_j \in \mathcal{P} \) such that \( X_i \subseteq X_j \).

For a partition \( \mathcal{P} \) of a set \( X \), Pei [18] introduced and studied the subsemigroup

\[
T(X, \mathcal{P}) = \{f \in T_X | \forall X_i \in \mathcal{P}, \exists X_j \in \mathcal{P}, X_i \subseteq X_j \}
\]

of \( T_X \). Moreover, Pei proved in [18, Theorem 2.8] that \( T(X, \mathcal{P}) \) is exactly the semigroup of all continuous selfmaps on \( X \) endowed with the topology having \( \mathcal{P} \) as a basis. Since then, \( T(X, \mathcal{P}) \) and its subsemigroups have received considerable attention and their several fascinating algebraic and combinatorial aspects have been investigated (see for example [2–4, 10–12, 14, 19–21, 26, 27]). There have also been a number of interesting works on certain generalizations of the semigroup \( T(X, \mathcal{P}) \) (see for example [8, 9, 22–24]).

Pei [21] studied the regularity and Green’s relations in the semigroup \( T(X, \mathcal{P}) \). When \( \mathcal{P} \) is a uniform partition of finite \( X \), Pei [20] gave an upper bound for the rank of \( T(X, \mathcal{P}) \). Later, Araújo et al. [4] calculated the rank of \( T(X, \mathcal{P}) \) and thus settled a conjecture on the rank of \( T(X, \mathcal{P}) \) posed by Pei in [20]. Araújo et al. [2] also calculated the rank of \( T(X, \mathcal{P}) \) for an arbitrary partition \( \mathcal{P} \) of finite \( X \). Dolinka et al. characterized as well as enumerated the idempotents of the
semigroup \( T(X, \mathcal{P}) \) for finite set \( X \) in [10] and [11] for the uniform and non-uniform cases, respectively. The cardinality of particular classes of subsemigroups of the semigroup \( T(X, \mathcal{P}) \) has also been calculated (see for example [13, 25]).

Pei [20] also considered the group of units \( S(X, \mathcal{P}) \) of the semigroup \( T(X, \mathcal{P}) \) and observed that \( S(X, \mathcal{P}) \) is exactly the subgroup of all homeomorphisms on \( X \) endowed with the topology having \( \mathcal{P} \) as a basis, and called it the homeomorphism group. For a uniform partition \( \mathcal{P} \) of finite \( X \), Pei [20] further deduced an upper bound for the rank of \( S(X, \mathcal{P}) \). Later, Araújo et al. [4] calculated the rank of \( S(X, \mathcal{P}) \) when \( \mathcal{P} \) is a uniform partition of finite \( X \). The homeomorphism group \( S(X, \mathcal{P}) \) is also studied by Araújo et al. in [2].

For a partition \( \mathcal{P} = \{X_i \mid i \in I\} \) of a set \( X \), let
\[
\Sigma(X, \mathcal{P}) = \{ f \in T(X, \mathcal{P}) \mid Xf \cap X_i \neq \emptyset \ \forall X_i \in \mathcal{P} \}.
\]

It is clear that \( \Sigma(X, \mathcal{P}) \) is a subsemigroup of \( T(X, \mathcal{P}) \). When \( \mathcal{P} \) is a uniform partition of finite \( X \), Araújo et al. [4] calculated the rank of the semigroup \( \Sigma(X, \mathcal{P}) \). For a finite \( X \), some interesting properties of \( \Sigma(X, \mathcal{P}) \) are also investigated in [2].

The rest of the paper is organized as follows. In Section 2, we recall necessary concepts from semigroup theory and combinatorics and introduce notation used within the paper. In Section 3, we characterize the elements of the semigroup \( \Sigma(X, \mathcal{P}) \). For a permutation \( f \) of finite \( X \), we observe whether there exists a nontrivial partition \( \mathcal{P} \) of \( X \) such that \( f \in S(X, \mathcal{P}) \) in Section 4. In Section 5, we characterize and enumerate the idempotents in the semigroup \( \Sigma(X, \mathcal{P}) \) for arbitrary and finite \( X \), respectively. Moreover, we characterize the elements of \( S(X, \mathcal{P}) \). In Section 6, we finally calculate the cardinality of \( T(X, \mathcal{P}), \Sigma(X, \mathcal{P}), \) and \( S(X, \mathcal{P}) \) when \( X \) is a finite set.

2. Preliminaries and notation

In this section, we introduce relevant notation and recall concepts from combinatorics and semigroup theory that are requisite to the paper. We refer the reader to the standard books [5, 17] for more detailed information from combinatorics and semigroup theory, respectively.

Unless stated otherwise, we will use capital letter to denote nonempty subset, calligraphic letter to denote collection of subsets, and small letter to denote set element, map, or positive integer. The letter \( I \) will be reserved for an arbitrary indexing set. The set of all positive integers is denoted by \( \mathbb{N} \). We will always presume that \( m \) and \( n \) are positive integers. The symbol \( I_m \) denote the subset \( \{1, \ldots, m\} \). The number of elements of a set \( A \) is denoted by \( |A| \) and is called the cardinality or size of \( A \). A set of cardinality \( n \) is called an \( n \)-element set. We write \( A \setminus B \) to denote the set of all elements \( x \in A \) such that \( x \notin B \). We denote by \([a]\) the equivalence class of an element \( a \) of a set \( A \) under an equivalence relation on \( A \). We denote by \( \binom{n}{r} \) the number of \( r \)-element subsets of an \( n \)-element set. We will use \( A = \{n_1 \cdot a_1, \ldots, n_k \cdot a_k\} \) to denote the multiset \( A \) with \( n_i \) copies of \( a_i \) for each \( i \in I_k \).

Let \( X \) be a nonempty set. A partition of \( X \) is a collection of nonempty disjoint subsets of \( X \), called blocks, whose union is \( X \). A partition is called trivial if it has only singleton blocks or a single block. A partition is called uniform if all its blocks have the same size. An \( m \)-partition is a partition that has exactly \( m \) blocks. For \( m, k \in \mathbb{N} \) with \( m \geq k \), an \((m, k)\)-partition is an \( m \)-partition that has exactly \( k \) different size blocks. It is well known that any partition of \( X \) induces naturally an equivalence relation on \( X \), and vice versa (cf. [17, Proposition 1.4.6]).

The composition of maps will be denoted by juxtaposition. A selfmap on a set \( A \) is a map from \( A \) to \( A \). Let \( f, g \in T_X \). For \( x \in X \), we will use \( xf \) to denote the image of \( x \) under \( f \) and compose maps from left to right: \( x(fg) = (xf)g \). The symbols \( \text{dom}(f) \) and \( \text{codom}(f) \) will be used to denote respectively the domain and the codomain of \( f \). The pre-image of a subset \( B \subseteq X \) under \( f \) is denoted by \( Bf^{-1} = \{x \in X \mid xf \in B\} \). If \( A, B \subseteq X \) such that \( Af \subseteq B \), then there is a map \( g : \)}
Let $S$ be a semigroup. An element $a \in S$ is called an idempotent provided that $a^2 = a$. The set of idempotents of $S$ is denoted by $E(S)$. It is well known that if $f \in T_X$ is an idempotent if and only if $f$ acts as the identity map on its image set (cf. [6, p. 6]). The group of units of $S$ is the subgroup of all invertible elements of $S$. An equivalence relation $\rho$ on $S$ is called a congruence if for all $x, y, z, w \in S, (x, y) \in \rho$ and $(z, w) \in \rho$ implies $(xz, yw) \in \rho$. If $\rho$ is a congruence on $S$, then the factor set $S/\rho$ is a semigroup, called a quotient semigroup, equipped with the multiplication defined by $[x][y] = [xy]$ for all $[x], [y] \in S/\rho$. We will write $S \cong T$ to mean that there is an isomorphism between two semigroups $S$ and $T$.

\section*{3. The semigroup \( \Sigma(X, \mathcal{P}) \)}

In this section, we first characterize the elements of the semigroup $\Sigma(X, \mathcal{P})$. We then prove two simple but important lemmas on $S(X, \mathcal{P})$ and $\Sigma(X, \mathcal{P})$, respectively. We begin by recalling a definition.

\begin{definition} \textbf{(cf. [24])} Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of an arbitrary set $X$, and let $f \in T(X, \mathcal{P})$. The character of $f$ is a selfmap $\chi(f) : I \to I$ defined by $i\chi(f) = j$ whenever $X_i f \subseteq X_j$.

When $X$ is a finite set, the selfmap $\chi(f)$ has been discussed, and also denoted by $\tilde{f}$ in [2, 10, 11].

Denote by $S_{\mathcal{P}}(X)$ the semigroup of all continuous selfmaps on $X$ endowed with the topology having $\mathcal{P}$ as a basis. By [18, Theorem 2.8], we know that $S_{\mathcal{P}}(X) = T(X, \mathcal{P})$. We now have the following.

\begin{theorem} \label{thm:characterization}
Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of an arbitrary set $X$, and let $f \in T(X, \mathcal{P})$. Then the following statements are equivalent:

(i) $f \in \Sigma(X, \mathcal{P})$.

(ii) $\chi(f)$ is a surjective map.

(iii) $f \in S_{\mathcal{P}}(X)$ such that $Af^{-1} \neq \emptyset$ for all nonempty open set $A$.

\end{theorem}

\begin{proof}
(i) $\implies$ (ii). Let $j \in I$. Since $f \in \Sigma(X, \mathcal{P})$, we have $Xf \cap X_j \neq \emptyset$. Then, there exists $i \in I$ such that $Xf \subseteq X_i$. It follows that $i\chi(f) = j$ by definition of $\chi(f)$. Hence, $\chi(f)$ is surjective.

(ii) $\implies$ (iii). Let $A$ be a nonempty open set. Then $A = \bigcup_{j \in J} X_j$ for some $J \subseteq I$ (cf. [7, Definition 2.2.1]). Since $\chi(f) : I \to I$ is surjective, for each $j \in J$ there exists $i \in I$ such that $i\chi(f) = j$ and subsequently $X_j f \subseteq X_i$. That means $X_j f^{-1} \neq \emptyset$ for all $j \in J$. Hence we obtain

$$Af^{-1} = \left( \bigcup_{j \in J} X_j \right) f^{-1} = \bigcup_{j \in J} (X_j f^{-1}) \neq \emptyset.$$ 

(iii) $\implies$ (i). Let $X_i \in \mathcal{P}$. By (iii), we have $X_i f^{-1} \neq \emptyset$. It follows that $Xf \cap X_i \neq \emptyset$ and hence $f \in \Sigma(X, \mathcal{P})$. \hfill \Box

\end{proof}

\begin{definition} \textbf{(cf. [21])} Let $E$ be an equivalence relation on a set $X$. A selfmap $f : X \to X$ is said to be $E^*$-preserving if $f$ satisfies the following.

\end{definition}
\((x,y) \in E\) if and only if \((xf, yf) \in E\).

The next result characterizes the \(E^*\)-preserving maps.

**Theorem 3.4.** Let \(\mathcal{P} = \{X_i \mid i \in I\}\) be the partition associated with an equivalence relation \(E\) on an arbitrary set \(X\), and let \(f \in T(X, \mathcal{P})\). Then the following statements are equivalent:

(i) \(\chi^{(f)}\) is an injective map.

(ii) \(f\) is an \(E^*\)-preserving map.

**Proof.** (i) \(\Rightarrow\) (ii). Let \(x, y \in X\). Since \(f \in T(X, \mathcal{P})\), it is clear that if \((x, y) \in E\), then \((xf, yf) \in E\).

On the contrary, suppose that \(x \in X_i\) and \(y \in X_j\) for two distinct blocks \(X_i, X_j \in \mathcal{P}\). Since \(xf, yf \in X_t\), and \(f \in T(X, \mathcal{P})\), we have \(X_tf \subseteq X_t\) and \(X_lf \subseteq X_l\). Then, \(s\chi^{(f)}(x) = t\chi^{(f)}(y)\) for some \(s, t \in I\). Since \(X_i \cap X_j = \emptyset\), we have \((x, y) \not\in E\).

Recall that \(X_tf \subseteq X_r\) and \(x \in X_i\). Also, \(X_lf \subseteq X_r\) and \(y \in X_i\), it follows that \(yf \in X_r\). Thus, \((xf, yf) \in E\) which contradicts the hypothesis that \(f\) is an \(E^*\)-preserving map. Hence, \(\chi^{(f)}\) is an injective map.

Note that any selfmap on a finite set that is injective or surjective must be bijective (cf. [16, Proposition 1.1.3]). Combining this fact with Theorem 3.2 and Theorem 3.4, we have the following immediate corollary.

**Corollary 3.5.** Let \(\mathcal{P} = \{X_1, ..., X_m\}\) be an \(m\)-partition associated with an equivalence relation \(E\) on an arbitrary set \(X\), and let \(f \in T(X, \mathcal{P})\). Then the following four statements are equivalent:

(i) \(f \in \Sigma(X, \mathcal{P})\).

(ii) \(\chi^{(f)}\) is a bijective map on \(I\).

(iii) \(f\) is an \(E^*\)-preserving map.

(iv) \(f \in S_{\mathcal{P}}(X)\) such that \(Af^{-1} \neq \emptyset\) for all nonempty open set \(A\).

In the following lemma, we prove that every bijection in \(T(X, \mathcal{P})\) maps each block to a block of the same cardinality.

**Lemma 3.6.** Let \(\mathcal{P} = \{X_i \mid i \in I\}\) be a partition of an arbitrary set \(X\), and let \(f \in T(X, \mathcal{P})\). If \(f \in S(X, \mathcal{P})\), then

(i) \(X_i \in \mathcal{P}\) for all \(i \in I\).

(ii) \(|X_i| = |X_j|\) if \(i\chi^{(f)}(j) = j\).

**Proof.** Note that any two blocks of a partition are either equal or disjoint.

(i) On the contrary, suppose that \(X_i \notin \mathcal{P}\). Then, there exists some \(X_j \in \mathcal{P}\) such that \(X_i \subseteq X_j\). Since \(f \in S(X, \mathcal{P})\), it follows that the inverse map \(f^{-1}\) of the bijection \(f\) belongs to \(S(X, \mathcal{P})\). Then, \(X_jf^{-1} \subseteq X_k\) for some \(X_k \in \mathcal{P}\). Thus, we obtain \(X_i = (X_i)f^{-1} \subseteq X_j f^{-1} \subseteq X_k\), which contradicts the fact that any two blocks are either equal or disjoint. Hence, \(X_i \in \mathcal{P}\).
(ii) If $i \chi^{(i)} = j$, then $X_f \subseteq X_j$ by definition of $\chi^{(i)}$. By (i), we have $X_f \in \mathcal{P}$. Recall that any two blocks are either equal or disjoint. Since $X_i \in \mathcal{P}$, we have $X_f = X_j$ whence $|X_i| = |X_j|$ since $f$ is a bijection.

We next recall the following congruence on the semigroup $T(X, \mathcal{P})$ which is induced by $\chi^{(i)}$.

Remark 3.7. (cf. [24]) Let $\mathcal{P} = \{X_i \mid i \in I\}$ be a partition of an arbitrary set $X$. Then, the relation $\chi$ on $T(X, \mathcal{P})$ defined by

$$(f, g) \in \chi \iff \chi^{(f)} = \chi^{(g)}$$

is a congruence on the semigroup $T(X, \mathcal{P})$.

Note that the congruence $\chi$ defined above on the semigroup $T(X, \mathcal{P})$ is also a congruence on the semigroup $\Sigma(X, \mathcal{P})$, and the quotient semigroup $\Sigma(X, \mathcal{P})/\chi$ is a subsemigroup of $T(X, \mathcal{P})/\chi$. We now prove the following lemma.

Lemma 3.8. Let $\mathcal{P} = \{X_i \mid i \in I\}$ be a partition of a set $X$. Then the semigroup $\Sigma(X, \mathcal{P})/\chi$ is isomorphic to the semigroup of all surjective selfmaps on $I$.

Proof. Denote by $O(I)$ the semigroup of all surjective selfmaps on $I$. Define a map $\phi : \Sigma(X, \mathcal{P})/\chi \to O(I)$ by setting $[f] \phi = \chi^{(f)}$ for all $[f] \in \Sigma(X, \mathcal{P})/\chi$. By Theorem 3.2, it is clear that $\phi$ is well-defined.

Let $[f], [g] \in \Sigma(X, \mathcal{P})/\chi$. Note that $\chi^{(fg)} = \chi^{(f)} \chi^{(g)}$ (cf. [24, Lemma 2.3]). Therefore, we obtain

$$([fg]) \phi = \chi^{(fg)} = \chi^{(f)} \chi^{(g)} = ([f] \phi)([g] \phi)$$

whence $\phi$ is a homomorphism. To see $\phi$ is injective, suppose that $[f] \phi = [g] \phi$. Then

$$\chi^{(f)} = \chi^{(g)} \Rightarrow (f, g) \in \chi \Rightarrow [f] = [g]$$

and $\phi$ is injective.

To show that $\phi$ is surjective, let $\psi \in O(I)$. For each $j \in I$, we arbitrarily choose an element $x_j \in X_j$. Define a map $f : X \to X$ by setting $xf = x_j$ whenever $x \in X_i$ and $i \psi = j$. Since $\psi$ is surjective, one can verify, in a routine manner, that $f \in \Sigma(X, \mathcal{P})$ and $\chi^{(f)} = \psi$. Therefore, $[f] \phi = \chi^{(f)} = \psi$ whence $\phi$ is surjective. This completes the proof.

4. Permutations that preserve nontrivial partitions

When $\mathcal{P}$ is a trivial partition of $X$, it is clear that $T(X, \mathcal{P}) = T_X$ and $S(X, \mathcal{P}) = S_X$. Observe that $T(X, \mathcal{P}) \subseteq T_X$ and $S(X, \mathcal{P}) \subseteq S_X$ for all nontrivial partition $\mathcal{P}$ of $X$. We will now naturally be concerned of the converse problem: For a map $f \in T_X \ (f \in S_X)$, does there exist a nontrivial partition $\mathcal{P}$ of $X$ such that $f \in T(X, \mathcal{P}) \ (f \in S(X, \mathcal{P}))$? This section deals with this interesting problem when $X$ is finite. In the rest of this section, let $X = \{1, \ldots, n\}$.

Let $f \in T_n \setminus S_n$. If $f$ is a constant map, it is clear that $f \in T(X, \mathcal{P})$ for each nontrivial partition $\mathcal{P}$ of $X$. Otherwise, if $f$ is a nonconstant map, we see that $f \in T(X, \mathcal{P})$ where $\mathcal{P}$ is the nontrivial partition of $X$ induced by the equivalence relation $\ker(f)$ on $X$ defined as $\ker(f) = \{(x, y) \mid xf = yf\}$. Thus,

Remark 4.1. If $f \in T_n \setminus S_n$, then there exists a nontrivial partition $\mathcal{P}$ of $X$ such that $f \in T(X, \mathcal{P})$.

Proposition 4.2. Let $f \in S_n$. If $f$ is not an $n$-cycle, then there exists a nontrivial partition $\mathcal{P}$ of $X$ such that $f \in S(X, \mathcal{P})$.

Proof. The result is obvious for the identity permutation since the identity permutation preserves each partition of $X$. Therefore, suppose that $f$ is a nonidentity permutation. If $f$ is not an $n$-cycle, we can write
\[ f = \beta_1 \beta_2 \cdots \beta_s, \]

where \( \beta_i \)'s are cycles of length less than \( n \) (cf. [15, Theorem 5.1]). Let \( X_i = \{ x \in X \mid x \beta_1 \neq x \} \). It is clear that \( X_1f = X_1 \) and \( (X \setminus X_1)f = (X \setminus X_1) \). Moreover, one can verify, in a routine manner, that \( \mathcal{P} = \{ X_1, X \setminus X_1 \} \) is a nontrivial partition of \( X \). Hence, \( f \in S(X, \mathcal{P}) \) where \( \mathcal{P} \) is a nontrivial partition of \( X \).

**Theorem 4.3.** Let \( f \in S_n \) be an \( n \)-cycle, and let \( m \) be an integer such that \( 1 < m < n \). Then \( m \) divides \( n \) if and only if there exists a nontrivial \( m \)-partition \( \mathcal{P} \) of \( X \) such that \( f \in S(X, \mathcal{P}) \).

**Proof.** We first assume that \( n = km \) for some integer \( k \) with \( 1 < k < n \). Without loss of generality, let \( f = (1, \ldots, n) \) be an \( n \)-cycle. Consider, for each \( 1 \leq i \leq m \), the subset

\[ X_i = \{ i, i + m, i + 2m, \ldots, i + (k - 1)m \}. \]

of \( X \). One can verify, in a routine manner, that \( X_i \cap X_j = \emptyset \) for \( i \neq j \), and \( X = \bigcup_{i \in I_m} X_i \). This implies \( \mathcal{P} = \{ X_1, \ldots, X_m \} \) is a nontrivial \( m \)-partition of \( X \). Since \( f = (1,2,\ldots,n) \), we next see that \( X_if = X_i \) for some \( s \in I_m \). Then, \( (X \setminus X_i)f = X \setminus X_i \). It follows that \( f \) is not an \( n \)-cycle, which is a contradiction. Hence, \( X_if \neq X_i \) for all \( i \in I_m \).

Let \( i,j \in I_m \) such that \( i \neq j \). If \( X_if = X_j \), then we claim that \( X_if \neq X_i \). If possible, let \( X_if = X_i \). Then, \( f \) is not an \( n \)-cycle, which is a contradiction. Thus, without loss of generality, assume that \( X_if = X_i \) for each \( 1 \leq i \leq m - 1 \) and \( X_mf = X_1 \). Since \( f \in S(X, \mathcal{P}) \), by Lemma 3.6(ii), we have \( |X_1| = \cdots = |X_m| \). Then letting \( |X_1| = t \) for each \( i \in I_m \), we obtain

\[ |X_1| + \cdots + |X_m| = n \Rightarrow tm = n \]

whence \( m \) divides \( n \). This completes the proof.

The following corollary is an immediate consequence of Theorem 4.3.

**Corollary 4.4.** Let \( f \in S_n \) be an \( n \)-cycle. If \( n \) is a prime, then \( f \notin S(X, \mathcal{P}) \) for any nontrivial partition \( \mathcal{P} \) of \( X \).

**Proposition 4.5.** Let \( \mathcal{P} = \{ X_1, \ldots, X_m \} \) be an \( m \)-partition of \( X \), and let \( f \in S(X, \mathcal{P}) \). If \( f \) is an \( n \)-cycle, then

(i) \( \chi^{(f)} \) is an \( m \)-cycle on \( I_m \)

(ii) \( \mathcal{P} \) is a uniform partition of \( X \).

**Proof.**

(i) Note that \( S(X, \mathcal{P}) \subseteq \Sigma(X, \mathcal{P}) \). In view of Corollary 3.5, \( \chi^{(f)} \) is a permutation of \( I_m \). If \( f \) is an \( n \)-cycle, then \( m \) divides \( n \) by Theorem 4.3. If \( m \) is an improper divisor of \( n \), one can easily verify, in a routine manner, that \( \chi^{(f)} \) is an \( m \)-cycle. Otherwise, assume that \( m \) is a proper divisor of \( n \).

On the contrary, suppose that \( \chi^{(f)} \) is not an \( m \)-cycle. Then, \( \chi^{(f)} \) can be written as a product of disjoint cycles (cf. [15, Theorem 5.1]). Let \( (i_1, \ldots, i_t) \) be a \( t \)-cycle, \( 1 < t < m \), in the cycle decomposition of \( \chi^{(f)} \). Then, \( i_1 \chi^{(f)} = i_{t+1} \) for \( r = 1, \ldots, t - 1 \) and \( i_t \chi^{(f)} = i_1 \).

Since \( f \in S(X, \mathcal{P}) \), by Lemma 3.6(i) we thus obtain \( X_{i_r}f = X_{i_{r+1}} \) for all \( r = 1, \ldots, t - 1 \) and \( X_{i_t}f = X_{i_1} \). This means that \( f \) is a cycle of length at most \( tk \), which is obviously less than \( n \),
where $k = |X_i|$ for each $j = 1, \ldots, t$. This gives a contradiction of the hypothesis that $f$ is an $n$-cycle. Hence, $\chi^{(f)}$ is an $m$-cycle on $I_m$.

(ii) Without loss of generality, assume that $\chi^{(f)} = (1, \ldots, m)$ by (i). This means $i\chi^{(f)} = i + 1$ for all $i = 1, \ldots, m - 1$ and $m\chi^{(f)} = 1$. Since $f \in S(X, \mathcal{P})$, by Lemma 3.6(ii) we have $|X_i| = \cdots = |X_m|$ whence $\mathcal{P}$ is a uniform partition of $X$. \hfill $\square$

5. Idempotents in $\Sigma(X, \mathcal{P})$

In this section, we first define a notion of block maps associated with a set partition. We then proceed with lemmas and a corollary. By using these lemmas, we next characterize the idempotents of the semigroup $\Sigma(X, \mathcal{P})$. We also prove a necessary and sufficient condition for a map in $T(X, \mathcal{P})$ to be in $S(X, \mathcal{P})$. We finally count the idempotents in the semigroup $\Sigma(X, \mathcal{P})$ when $X$ is finite.

**Definition 5.1.** Let $\mathcal{P}$ be a partition of an arbitrary set. A *block map* is a map whose both domain and codomain are blocks of $\mathcal{P}$.

We now prove the following simple lemmas.

**Lemma 5.2.** Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of a set $X$, and let $f \in T_X$. Then $f \in T(X, \mathcal{P})$ if and only if there exists a unique indexed family $B(f, I)$ of block maps induced by $f$, where $B(f, I) = \{f_i \mid f_i \text{ is induced by } f \text{ and } \text{dom}(f_i) = X_i \text{ for each } i \in I\}$.

**Proof.** We first assume that $f \in T(X, \mathcal{P})$, and let $X_i \in \mathcal{P}$. Then, there exists $X_j \in \mathcal{P}$ such that $X_jf \subseteq X_j$. We subsequently have a block map from $X_j$ to $X_j$ induced by $f$. Denote this induced block map by $f_j$. Since $X_i \in \mathcal{P}$ is an arbitrary block, we therefore obtain a unique indexed family $B(f, I)$ of block maps induced by $f$, where $B(f, I) = \{f_i \mid f_i \text{ is induced by } f \text{ and } \text{dom}(f_i) = X_i \text{ for each } i \in I\}$.

Conversely, suppose that the condition holds. Let $X_i \in \mathcal{P}$. Then, $X_if = X_if_i$. Since $\text{codom}(f_i) \in \mathcal{P}$, we have $X_if \subseteq X_i$, where $X_i = \text{codom}(f_i)$. Hence, $f \in T(X, \mathcal{P})$. This completes the proof. \hfill $\square$

**Lemma 5.3.** Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of a set $X$, and let $f \in T(X, \mathcal{P})$. If each block map of $B(f, I)$ is an idempotent, then $f$ is an idempotent.

**Proof.** Let $x \in X$. Then, $x \in X_i$ for some $i \in I$. Since $f_i \in B(f, I)$ is an idempotent, it follows that $f_i : X_i \to X_i$. Therefore

$$x(f^2) = (xf)f = (xf_i)f_i = x(f_i)^2 = xf_i = xf$$

whence $f$ is an idempotent. \hfill $\square$

**Lemma 5.4.** Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of a set $X$, and let $f \in E(T(X, \mathcal{P}))$. If $i \in 1\chi^{(f)}$, then the block map $f_i \in B(f, I)$ is an idempotent.

**Proof.** If $i \in 1\chi^{(f)}$, then we see that $Xf \cap X_i \neq \emptyset$. Let $y \in Xf \cap X_i$. Since $f \in E(T(X, \mathcal{P}))$, we have $yf = y$ (cf. [6, p. 6]). If follows that $f_i : X_i \to X_i$. Let $x \in X_i$. Then

$$x(f_i^2) = (xf_i)f_i = (xf_i)f_i = x(f_i)^2 = xf_i = xf$$

whence the block map $f_i$ is an idempotent. \hfill $\square$

**Corollary 5.5.** Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of a set $X$, and let $f \in T(X, \mathcal{P})$. If $f$ is an idempotent, then $\chi^{(f)}$ is an idempotent.
Proof. Let \( j \in I \chi(I) \). It is sufficient to show that \( j \chi(I) = j \) (cf. [6, p. 6]). Since \( j \in I \chi(I) \), the block map \( f_j \in B(f, I) \) is an idempotent by Lemma 5.4. It follows that \( f_j : X_j \to X_j \) and subsequently \( j \chi(I) = j \). Hence, \( \chi(I) \) is an idempotent.

The following theorem that characterizes the idempotents of the semigroup \( \Sigma(X, \mathcal{P}) \) is a direct consequence of Lemma 5.3 and Lemma 5.4.

**Theorem 5.6.** Let \( \mathcal{P} = \{X_i | i \in I\} \) be a partition of a set \( X \), and let \( f \in \Sigma(X, \mathcal{P}) \). Then \( f \) is an idempotent if and only if each block map of \( B(f, I) \) is an idempotent.

Theorem 5.6 yields the following.

**Corollary 5.7.** Let \( \mathcal{P} = \{X_i | i \in I\} \) be a partition of a set \( X \), and let \( f \in \Sigma(X, \mathcal{P}) \). If \( f \) is an idempotent, then \( \chi(I) \) is the identity map.

Proof. Let \( i \in I \). If \( f \in \Sigma(X, \mathcal{P}) \) is an idempotent, then \( f_i : X_i \to X_i \) by Theorem 5.6. It follows that \( X_i f = X_i f_i \subseteq X_i \). By definition of \( \chi(I) \), we then have \( i \chi(I) = i \). Hence, \( \chi(I) \) is the identity map on \( I \).

The next theorem provides a necessary and sufficient condition for a map in \( T(X, \mathcal{P}) \) to be in \( S(X, \mathcal{P}) \).

**Theorem 5.8.** Let \( \mathcal{P} = \{X_i | i \in I\} \) be a partition of an arbitrary set \( X \), and let \( f \in T(X, \mathcal{P}) \). Then \( f \in S(X, \mathcal{P}) \) if and only if

(i) each block map of \( B(f, I) \) is a bijective map; and
(ii) \( \chi(I) \) is a bijective map.

Proof. Assume that \( f \in S(X, \mathcal{P}) \). Let \( f_i \in B(f, I) \). It is clear that the block map \( f_i \) is injective. If \( \text{codom}(f_i) = X_j \), then \( X_j f = X_j f_i \subseteq X_j \). Since \( f \in S(X, \mathcal{P}) \), we have \( X_j f_i \in \mathcal{P} \) by Lemma 3.6(i). We then have \( X_i f_i = X_i \) and so \( f_i \) is surjective. Since \( f_i \in B(f, I) \) is arbitrary, this proves (i).

We next prove (ii). Note that \( S(X, \mathcal{P}) \subseteq \Sigma(X, \mathcal{P}) \). Since \( f \in S(X, \mathcal{P}) \), by Theorem 3.2, the map \( \chi(I) \) is surjective. On the contrary, suppose that there exist two distinct elements \( i, j \in I \) such that \( i \chi(I) = j \chi(I) \), say equal to \( k \) for some \( k \in I \). Then, \( X_k f \subseteq X_k \) and \( X_k f \subseteq X_k \) by definition of \( \chi(I) \). Since \( f \in S(X, \mathcal{P}) \), by Lemma 3.6(i), we have \( X_k f = X_k \) and \( X_k f = X_k \) and subsequently \( X_i = X_j \). This is a contradiction since \( i \neq j \). Hence, the map \( \chi(I) \) is bijective.

Conversely, suppose that the conditions hold. It is sufficient to prove that \( f \) is a bijection. We first prove that \( f \) is injective. Let \( x, y \in X \) and suppose that \( x \neq y \). Then, \( x \neq y \). Therefore, \( f \) is injective. We thus obtain \( x = y \). We next prove (ii). Note that \( S(X, \mathcal{P}) \subseteq \Sigma(X, \mathcal{P}) \). Since \( f \in S(X, \mathcal{P}) \), by Theorem 3.2, the map \( \chi(I) \) is surjective. On the contrary, suppose that there exist two distinct elements \( i, j \in I \) such that \( i \chi(I) = j \chi(I) \), say equal to \( k \) for some \( k \in I \). Then, \( X_k f \subseteq X_k \) and \( X_k f \subseteq X_k \) by definition of \( \chi(I) \). Since \( f \in S(X, \mathcal{P}) \), by Lemma 3.6(i), we have \( X_k f = X_k \) and \( X_k f = X_k \) and subsequently \( X_i = X_j \). This is a contradiction since \( i \neq j \). Hence, the map \( \chi(I) \) is bijective.

Conversely, suppose that the conditions hold. It is sufficient to prove that \( f \) is a bijection. We first prove that \( f \) is injective. Let \( x, y \in X \) and suppose that \( x \neq y \). Then, \( x \neq y \). Therefore, \( f \) is injective. We thus obtain \( x = y \).

We now prove that \( f \) is surjective. Let \( y \in X \). Then, \( y \in X_i \) for some \( i \in I \). By (ii), there exists \( i \in I \) such that \( x, y \in X_i \). By (i), we know that \( f_i \) is injective. We thus obtain \( x = y \). Therefore, \( f \) is surjective. This completes the proof.

When \( X \) is a finite set, the following theorem counts the number of the idempotents in the semigroup \( \Sigma(X, \mathcal{P}) \).

**Theorem 5.9.** Let \( \mathcal{P} = \{X_1, ..., X_m\} \) be an \((m, k)\)-partition of a finite set \( X \) such that \( \mathcal{P} \) has exactly \( m_i \geq 1 \) blocks of size \( n_i \geq 1 \) for each \( i \in I_k \). Then

\[
| E(\Sigma(X, \mathcal{P})) | = \prod_{i=1}^{k} \left( \sum_{j=1}^{n_i} \binom{n_i}{j} \right)^{m_i}.
\]
Theorem 6.1. Let \( \mathcal{P} \) be an \((m, k)\)-partition of a finite set \( X \) such that \( \mathcal{P} \) has exactly \( m_i \geq 1 \) blocks of size \( n_i \geq 1 \) for each \( i \in I_k \). Then

\[
|T(X, \mathcal{P})| = \prod_{i=1}^{k} \left( \sum_{j=1}^{k} m_j n_i^{n_j} \right)^{m_i}.
\]

Proof. Note that each map \( f \in \Sigma(X, \mathcal{P}) \) is uniquely determined by the \( m \)-family \( B(f, I_m) \) of block maps (cf. Lemma 5.2). From Theorem 5.6, we know that a map \( f \in \Sigma(X, \mathcal{P}) \) is an idempotent if and only if each block map \( f_i \in B(f, I_m) \) is an idempotent. It is therefore sufficient to count the total number of such \( m \)-families \( B(f, I_m) \) of idempotent block maps. To count it, we break up the problem into \( k \) subfamilies depending on the domain sizes of block maps.

Let \( i \in I_k \). Since \( \mathcal{P} \) has \( m_i \) blocks of size \( n_i \), we begin by counting the number of possible \( m_i \)-subfamilies of idempotent block maps from \( m_i \) distinct blocks of size \( n_i \). Note that the total number of idempotents in the full transformation semigroup on an \( n \)-element set is \( \sum_{j=1}^{n} \left( \binom{n}{j} \right) n^{-j} \) (cf. [16, Corollary 2.7.4]). Moreover, any map which is an idempotent must be selfmap. Therefore, the number of possible idempotent block maps from a block of size \( n_i \) is \( \sum_{j=1}^{n_i} \left( \binom{n_i}{j} \right) n_i^{-j} \).

Recall that \( \mathcal{P} \) has \( m_i \) blocks of size \( n_i \) by the multiplication principle, the total number of possible \( m_i \)-subfamilies of idempotent block maps from \( m_i \) distinct blocks of size \( n_i \) is \( \left( \sum_{j=1}^{n_i} \left( \binom{n_i}{j} \right) n_i^{-j} \right)^{m_i} \). Since \( \mathcal{P} \) has exactly \( k \) blocks of different size and \( i \in I_k \) is an arbitrarily chosen element, the total number of possible \( m \)-families of idempotent block maps is now followed by applying the multiplication principle. This completes the proof.

\[
|T(X, \mathcal{P})| = \prod_{i=1}^{k} \left( \sum_{j=1}^{k} m_j n_i^{n_j} \right)^{m_i}.
\]

6. The cardinality of \( T(X, \mathcal{P}), \Sigma(X, \mathcal{P}), \) and \( S(X, \mathcal{P}) \)

This section calculates the size of \( T(X, \mathcal{P}), S(X, \mathcal{P}), \) and \( \Sigma(X, \mathcal{P}) \), respectively, when \( X \) is a finite set. We begin by calculating the cardinality of the semigroup \( T(X, \mathcal{P}) \).

Theorem 6.1. Let \( \mathcal{P} = \{ X_1, \ldots, X_m \} \) be an \((m, k)\)-partition of a finite set \( X \) such that \( \mathcal{P} \) has exactly \( m_i \geq 1 \) blocks of size \( n_i \geq 1 \) for each \( i \in I_k \). Then

\[
|T(X, \mathcal{P})| = \prod_{i=1}^{k} \left( \sum_{j=1}^{k} m_j n_i^{n_j} \right)^{m_i}.
\]

Proof. Note that each map \( f \in T(X, \mathcal{P}) \) is uniquely determined by the \( m \)-family \( B(f, I_m) \) of block maps (cf. Lemma 5.2). Therefore, it is sufficient to count the total number of such \( m \)-families \( B(f, I_m) \) of block maps. To count it, we break up the problem into \( k \) subfamilies depending on the domain sizes of block maps.

Let \( i \in I_k \). Since \( \mathcal{P} \) has \( m_i \) blocks of size \( n_i \), we begin by counting the number of possible \( m_i \)-subfamilies of block maps from \( m_i \) distinct blocks of size \( n_i \). Clearly, the codomain of a block map from a block of size \( n_i \) can be any block of \( \mathcal{P} \). Note that the number of maps from an \( n \)-element set into an \( t \)-element set is \( t^n \). Therefore, the number of possible block maps from a block of size \( n_i \) is \( \sum_{j=1}^{k} m_j n_i^{n_j} \) by the addition principle.

Recall that \( \mathcal{P} \) has \( m_i \) blocks of size \( n_i \) by the multiplication principle, the total number of possible \( m_i \)-subfamilies of block maps from \( m_i \) distinct blocks of size \( n_i \) is \( \left( \sum_{j=1}^{k} m_j n_i^{n_j} \right)^{m_i} \). Since \( \mathcal{P} \) has exactly \( k \) blocks of different size and \( i \in I_k \) is an arbitrarily chosen element, the total number of possible \( m \)-families of block maps is now followed by applying the multiplication principle. This completes the proof.

The next theorem calculate the cardinality of the group of units \( S(X, \mathcal{P}) \) of the semi-group \( T(X, \mathcal{P}) \).
Theorem 6.2. Let \( P = \{X_1, \ldots, X_m\} \) be an \((m, k)\)-partition of a finite set \( X \) such that \( P \) has exactly \( m_i \geq 1 \) blocks of size \( n_i \geq 1 \) for each \( i \in I_k \). Then

\[
|S(X, P)| = \prod_{i=1}^{k} (m_i!(n_i!)^{m_i}).
\]

Proof. Consider the equivalence relation \( \sim \) on the partition \( P \) defined by \( P \sim Q \) if and only if \( |P| = |Q| \). Clearly, there are exactly \( k \) equivalence classes under the equivalence \( \sim \). Let \( [X_1], \ldots, [X_k] \) be the equivalence classes under \( \sim \), where \( |X_i| = n_i \). Note that \( |[X_i]| = m_i \) for each \( i = 1, \ldots, k \).

Let \( i \in I_k \). Consider the class \([X_i]\). By Lemma 3.6 and Theorem 5.8(ii), we first note that the images of two distinct blocks in \([X_i]\) under a map of \( S(X, P) \) must be distinct blocks in \([X_i]\). If \( f \in S(X, P) \), then there are \( m_i \) choices for the image of the first block of \([X_i]\) under \( f \), the remaining \( (m_i - 1) \) choices for the image of the second block of \([X_i]\) under \( f \), etc. For the last block of \([X_i]\), there is exactly one choice under \( f \). Therefore, by the multiplication principle, all \( m_i \) blocks of the class \([X_i]\) can be mapped in \( m_i! \) different ways.

Note that the number of bijections between any two \( n \)-element sets is \( n! \). Therefore, among \( m_i! \) different choices, each fixed choice gives \((n_i!)^{m_i}\) bijections that preserve all blocks of \([X_i]\) by the multiplication principle. Hence, the total number of bijections that preserve all blocks of the class \([X_i]\) is \((m_i!(n_i!)^{m_i})\).

Since there are exactly \( k \) equivalence classes and \([X_i]\) is an arbitrarily chosen class, one can obtain the stated formula of \( |S(X, P)| \) by the multiplication principle.

The following theorem provides a formula for the cardinality of the semigroup \( \Sigma(X, P) \).

Theorem 6.3. Let \( P = \{X_1, \ldots, X_m\} \) be an \((m, k)\)-partition of a finite set \( X \) such that \( P \) has exactly \( m_i \geq 1 \) blocks of size \( n_i \geq 1 \) for each \( i \in I_k \). Then

\[
|\Sigma(X, P)| = m_1! \cdots m_k! \sum n_1^{s(m_1)} \cdots n_k^{s(m_k)},
\]

where the sum runs over all \( k \)-tuple of

\[
A := \{(t_{m_1}, \ldots, t_{m_k}) \mid \forall i \in I_k, t_{m_i} = (l_{1_i}, \ldots, l_{t_{m_i}}), l_j \in \{n_{1_j}, \ldots, n_k\}\}
\]

such that the components of all \( t_{m_i} \) of a \( k \)-tuple of \( A \) form \( \{m_1 \cdot n_1, \ldots, m_k \cdot n_k\} \); each \( s(m_i) \) is the sum of all components of \( t_{m_i} \) in a \( k \)-tuple of \( A \).

Proof. Since \( P \) is an \( m \)-partition of a finite set \( X \), by Lemma 3.8, we must have \( \Sigma(X, P)/\chi \cong S_m \). It follows that there are \( m! \) distinct equivalence classes into which \( \Sigma(X, P) \) splits under the equivalence \( \chi \). Therefore, it is sufficient to calculate the cardinality of all \( m! \) distinct equivalence classes under the equivalence \( \chi \).

Let \([f] \in \Sigma(X, P)/\chi \) be an arbitrary class. We now calculate the cardinality of the class \([f] \). By Lemma 5.2, it is sufficient to count the total number of \( m \)-families of block maps induced by maps of \([f] \). Note, for any \( g, h \in [f] \), that \( \chi^g = \chi^h \). From Corollary 3.5, we recall that the map \( \chi^{(f)} \) is a bijection on \( I_m \).

Let \( X_i \in P \). If \( i \chi^{(f)} = i' \), then \( X_i f = X_i f \subseteq X_{i'} \). Therefore, the number of block maps from \( X_i \) to \( X_{i'} \) is \( r_{i'i} \), where \( r_i, r_{i'} \in \{m_1 \cdot n_1, \ldots, m_k \cdot n_k\} \) and \( |X_i| = r_i, |X_{i'}| = r_{i'} \). Since \( P \) has exactly \( m \) distinct blocks, by the multiplication principle, the total number of possible \( m \)-families \( B(f, I_m) \) of block maps is \( r_{i'i}^m \). Thus, \( |[f]| = r_{i'i}^m \), where \( r_i, r_{i'} \in \{m_1 \cdot n_1, \ldots, m_k \cdot n_k\} \) for all \( 1 \leq i, i' \leq m \).
Since \([f] \in \Sigma(X, \mathcal{P})/\chi\) is an arbitrary class, by the addition principle, we thus obtain
\[
|\Sigma(X, \mathcal{P})| = \sum_{[f] \in \Sigma(X, \mathcal{P})/\chi} |[f]| = \sum_{\phi \in S_m} r_{i_1}^{s_{i_1}} r_{i_2}^{s_{i_2}} \ldots r_{i_n}^{s_{i_n}},
\]
where \(\phi \in S_m\) denotes the isomorphic image of the class \([f]\) and \(i\phi = \bar{i}\). Since \(\mathcal{P}\) has exactly \(m_i\) blocks of size \(n_i\) for each \(i \in I_k\), we see that all the \(r_i\)'s form the multiset \(\{m_1 \cdot n_1, \ldots, m_k \cdot n_k\}\), and also all \(r_i\)'s form the multiset \(\{m_1 \cdot n_1, \ldots, m_k \cdot n_k\}\).

Hence, by [5, Theorem 2.4.2], we obtain
\[
|\Sigma(X, \mathcal{P})| = m_1! \ldots m_k! \sum r_{i_1}^{s_{i_1}} \ldots r_{i_n}^{s_{i_n}},
\]
where the sum runs over all \(k\)-tuple of
\[
A := \{(t_{m_1}, \ldots, t_{m_k}) \mid \forall i \in I_k, t_m = (l_1, \ldots, l_{m_i}), l_j \in \{n_1, \ldots, n_k\}\}
\]
such that the components of all \(t_{m}\) of a \(k\)-tuple of \(A\) form \(\{m_1 \cdot n_1, \ldots, m_k \cdot n_k\}\); each \(s(m_i)\) is the sum of all components of \(t_{m_i}\) in a \(k\)-tuple of \(A\).

\[\square\]

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