Pointwise convergence problem of one dimensional Schrödinger equation in Fourier-Lebesgue spaces

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Abstract. In this paper, we consider the pointwise convergence problem of one dimensional Schrödinger equation. We show the almost everywhere pointwise convergence of one dimensional Schrödinger equation in Fourier-Lebesgue spaces $\hat{H}^{s,r}(\mathbb{R})$ with $s \geq \frac{1}{2r}, 2 \leq r < \infty$ with rough data. We also present counterexamples showing that the maximal function estimate related to one Schrödinger equation can fail with data in $\hat{H}^{s,r}(\mathbb{R})(s < \frac{1}{2r})$. The key ingredients are maximal function estimate related to $\hat{H}^{s,r}(\mathbb{R}), 2 \leq r < \infty$ and the density theorem in $\hat{H}^{s,r}(\mathbb{R}), 2 \leq r < \infty$.

Keywords: Pointwise convergence; One dimensional Schrödinger equation; Fourier-Lebesgue spaces

AMS Subject Classification: 42B25; 42B15; 35Q53

1. Introduction

In this paper, we investigate the pointwise convergence problem of one dimensional Schrödinger equation

\begin{equation}
    iu_t + \partial_x^2 u = 0,
\end{equation}
\begin{equation}
    u(x, 0) = f(x).
\end{equation}

Carleson [5] showed pointwise convergence problem of the one dimensional Schrödinger equation in $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$. Dahlberg and Kenig [8] showed that the pointwise convergence of the Schrödinger equation is invalid in $H^s(\mathbb{R}^n)$ with $s < \frac{1}{4}$. Some authors have studied the pointwise convergence problem of Schrödinger equations in higher dimension [1, 2, 6, 7, 9, 13, 15, 18–27]. Bourgain [3] presented counterexamples showing that when $s < \frac{n}{2(n+1)}, n \geq 2$ the pointwise convergence problem of $n$ dimensional Schrödinger equation does not hold. Du et al. [11] proved that the pointwise convergence problem of two
dimensional Schrödinger equation in $H^s(\mathbb{R}^2)$ with $s > \frac{1}{3}$. Du and Zhang [12] showed that the pointwise convergence problem of $n$ dimensional Schrödinger equation is valid for data in $H^s(\mathbb{R}^n)(s > \frac{n}{2(n+1)}, n \geq 3)$.

In this paper, motivated by [10, 17], we use the Fourier-Lebesgue space $\hat{H}^{s,r}(\mathbb{R})$ which is used in [4, 14, 16] to study the pointwise convergence problem of Schrödinger equation. We show the almost everywhere pointwise convergence of one dimensional Schrödinger equation in Fourier-Lebesgue spaces $\hat{H}^{s,r}(\mathbb{R})$ with $s \geq \frac{1}{2} r, 2 \leq r < \infty$ with rough data. We also present counterexamples showing that the maximal function estimate related to one Schrödinger equation can fail with data in $\hat{H}^{s,r}(\mathbb{R})(s < \frac{1}{2} r)$.

We present some notations before stating the main results. $|E|$ denotes by the Lebesgue measure of set $E$.

\[
\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(x)dx, \quad \mathcal{F}^{-1}_x f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(x)dx,
\]

\[
U(t)u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi-it\xi^2} \hat{u}_0(\xi)d\xi, D^\alpha_u U(t)u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^\alpha e^{ix\xi-it\xi^2} \hat{u}_0(\xi)d\xi,
\]

\[
\|f\|_{L^q_t L^p_x} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x,t)|^p dt \right)^\frac{q}{p} dx \right)^\frac{1}{q}, \quad \frac{1}{r} + \frac{1}{r'} = 1.
\]

\[
\hat{H}^{s,r}(\mathbb{R}) = \left\{ f \in \hat{H}^{s,r}(\mathbb{R}) | f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{\hat{H}^{s,r}(\mathbb{R})} = \|\langle \xi \rangle^s \hat{f}\|_{L^{r'}(\mathbb{R})} < \infty \right\}, \text{ where } \langle \xi \rangle^s = (1 + \xi^2)^{s/2} \text{ for any } \xi \in \mathbb{R}.
\]

**Theorem 1.1.** Let $f \in \hat{H}^{s,r}(\mathbb{R})$ with $s \geq \frac{1}{2r}, 2 \leq r < \infty$. Then, we have

\[
\lim_{t \to 0} U(t)f(x) = f(x) \quad (1.3)
\]

almost everywhere.

**Remark 1.** When $r = 2$, from Theorem 1.1, we obtain the same result as its of Carleson [5]. Thus, we extend the result of [5].

**Theorem 1.2.** The maximal function inequality

\[
\|U(t)f\|_{L^r_t L^\infty_x} \leq C\|f\|_{\hat{H}^{s,r}(\mathbb{R})} \quad (1.4)
\]

does not hold if $s < \frac{1}{2r}$.

2. Preliminaries

In this section, we present some preliminaries.
Lemma 2.1. Let $f \in L^2(\mathbb{R})$. Then, we have

$$\| U(t)f \|_{L^4_t L^\infty_x} \leq \| D_x^{\frac{1}{2}}f \|_{L^2(\mathbb{R})} = \| |\xi|^{\frac{1}{2}} \hat{f} \|_{L^2(\mathbb{R})}. \quad (2.1)$$

For the proof of Lemma 2.1, we refer the readers to [17].

Lemma 2.2. (Maximal function estimate related to $\dot{H}^{\frac{1}{p}, \frac{p}{2}}$, $p \geq 4$) Let $f \in \dot{H}^{\frac{1}{p}, \frac{p}{2}}$, $p \geq 4$. Then, we have

$$\| U(t)f \|_{L^p_t L^\infty_x} \leq C \| |\xi|^{\frac{1}{p}} \hat{f} \|_{L^p(\mathbb{R})}. \quad (2.2)$$

Proof. Obviously, we have

$$\| U(t)f \|_{L^\infty_t L^\infty_x} \leq \| \hat{f} \|_{L^1}. \quad (2.3)$$

Interpolating (2.1) with (2.3), we have that (2.2) is valid.

This completes the proof of Lemma 2.2.

Lemma 2.3. (Density Theorem in $\dot{H}^{s,r}$) Let $f \in \dot{H}^{s,r}$, $s \in \mathbb{R}$, $2 \leq r < \infty$. Then, $\forall \epsilon > 0$, there exists a rapidly decreasing function $g$ and a function $h$ with $\| h \|_{\dot{H}^{s,r}} < \epsilon$ such that

$$f = g + h. \quad (2.4)$$

Proof. From $f \in \dot{H}^{s,r}$, we have that $\langle \xi \rangle^s \hat{f} \in L^r$, according to the density theorem in $L^r$, we know that there exists rapidly decreasing function $g_1$ and a function $h_1$ such that $\langle \xi \rangle^s \hat{f} = g_1 + h_1$ with $\| h_1 \|_{L^r} < \epsilon$. Thus, we have that

$$f = \mathcal{F}_x^{-1} (\langle \xi \rangle^{-s} g_1) + \mathcal{F}_x^{-1} (\langle \xi \rangle^{-s} h_1). \quad (2.5)$$

Let $g = \mathcal{F}_x^{-1} (\langle \xi \rangle^{-s} g_1), h = \mathcal{F}_x^{-1} (\langle \xi \rangle^{-s} h_1)$. Since $g_1$ is a decreasing rapidly function, thus, $g$ is a decreasing rapidly function. Obviously,

$$\| h \|_{\dot{H}^{s,r}} = \| h_1 \|_{L^r}. \quad (2.6)$$

Thus, we have $f = g + h$.

This completes the proof of Lemma 2.3.

Lemma 2.4. Let $f$ be a rapidly decreasing function. Then, we have

$$| U(t)f - f | \leq C |t|. \quad (2.7)$$
Proof. By using a direct computation, since $f$ is a decreasing rapidly function, we have

$$|U(t)f - f| = C \left| \int_{\mathbb{R}} e^{ix\xi} \left[ e^{-it\xi^2} - 1 \right] \hat{f}(\xi) d\xi \right| \leq C|t| \int_{\mathbb{R}} |\xi|^2 |\hat{f}(\xi)| d\xi$$

$$\leq C|t| \int_{\mathbb{R}} (\xi)^2 |\hat{f}(\xi)| d\xi \leq C|t| \int_{\mathbb{R}} (\xi)^{-2} d\xi \leq C|t|. \quad (2.8)$$

This completes the proof of Lemma 2.4.

3. Proof of Theorem 1.1

In this section, we apply Lemmas 2.1-2.4 to establish Theorem 1.1.

Proof of Theorem 1.1. If $f$ is rapidly decreasing function, from Lemma 2.4, we have

$$|U(t)f - f| \leq C|t|. \quad (3.1)$$

From (3.1), we know that Theorem 1.1 is valid.

When $f \in H^{s,r}(\mathbb{R})(s \geq \frac{1}{2r})$, by using Lemma 2.3, there exists a rapidly decreasing function $g$ such that $f = g + h$, where $\|h\|_{\hat{H}^{s,r}(\mathbb{R})} < \epsilon(s \geq \frac{1}{2r})$. Thus, we have

$$\lim_{t \to 0} |U(t)f - f| \leq \lim_{t \to 0} |U(t)g - g| + \lim_{t \to 0} |U(t)h - h|. \quad (3.2)$$

We define

$$E_{\alpha} = \left\{ x \in \mathbb{R} : \lim_{t \to 0} |U(t)f - f| > \alpha \right\}. \quad (3.3)$$

Obviously, $E_{\alpha} \subset E_{1\alpha} \cup E_{2\alpha}$.

$$E_{1\alpha} = \left\{ x \in \mathbb{R} : \lim_{t \to 0} |U(t)g - g| > \frac{\alpha}{2} \right\}, \quad (3.4)$$

$$E_{2\alpha} = \left\{ x \in \mathbb{R} : \lim_{t \to 0} |U(t)h - h| > \frac{\alpha}{2} \right\}. \quad (3.5)$$

Obviously,

$$E_{\alpha} \subset E_{1\alpha} \cup E_{2\alpha}. \quad (3.6)$$

From Lemmas 2.4, we have

$$|E_{1\alpha}| = 0. \quad (3.7)$$

Obviously,

$$E_{2\alpha} \subset E_{21\alpha} \cup E_{22\alpha}. \quad (3.8)$$

where

$$E_{21\alpha} = \left\{ x \in \mathbb{R} : \sup_{t > 0} |U(t)h| > \frac{\alpha}{4} \right\}, \quad (3.9)$$

$$E_{22\alpha} = \left\{ x \in \mathbb{R} : |h| > \frac{\alpha}{4} \right\}. \quad (3.10)$$
Thus, from Lemma 2.2, we have
\[ |E_{21\alpha}| = \int_{E_{21\alpha}} dx \leq \int_{E_{21\alpha}} \left[ \sup_{t>0} |U(t)h| \right]^{2r} dx \leq \frac{\|U(t)h\|_{L^2_rL^\infty_t}^{2r}}{\alpha^{2r}} \leq C \frac{\|h\|_{\dot{H}^{\frac{1}{r}}}^{2r}}{\alpha^{2r}} \leq C \frac{\epsilon^{2r}}{\alpha^{2r}}. \] (3.11)

By using the Hausdorff-Young inequality, we have
\[ |E_{22\alpha}| = \int_{E_{22\alpha}} dx \leq \int_{E_{22\alpha}} \frac{|h|^r}{\alpha^r} dx \leq \frac{\|h\|_{L^r}^r}{\alpha^r} \leq C \frac{\|h\|_{L^r}^r}{\alpha^r} \leq C \frac{\epsilon^r}{\alpha^r}. \] (3.12)

From (3.7), (3.11) and (3.12), we have
\[ |E_{\alpha}| \leq |E_{1\alpha}| + |E_{2\alpha}| \leq |E_{1\alpha}| + |E_{21\alpha}| + |E_{22\alpha}| \leq C \epsilon^{2r} + C \frac{\epsilon^r}{\alpha^r}. \] (3.13)

Thus, since \( \epsilon \) is arbitrary, from (3.13), we have
\[ |E_{\alpha}| = 0. \] (3.14)

Thus, from (3.3) and (3.14), we have \( U(t)f - f \to 0 (t \to 0) \) almost everywhere.

This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we present the counterexamples showing that \( s \geq \frac{1}{2r} \) is the necessary condition for the maximal function estimate.

**Proof of Theorem 1.2.** We choose \( f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} 2^{-k(s+\frac{1}{r})} \chi_{2^k \leq |\xi| \leq 2^{k+1}}(\xi) d\xi \), obviously,
\[ \|f\|_{\dot{H}^{s,r}} \sim 1. \] (4.1)

We choose \( t \leq \frac{\delta}{100} 2^{-2k} \), where \( \delta \) will be chosen later. For \( |x| \leq 2^{-k} \) and sufficiently small \( \delta \), we have
\[ \|U(t)f\|_{L^2_rL^\infty_t} \sim 2^{-k(s+\frac{1}{r})}. \] (4.2)

We choose \( t \leq \frac{\delta}{100} 2^{-2k} \), where \( \delta \) will be chosen later. For \( |x| \leq 2^{-k} \) and sufficiently small \( \delta \), we have
\[ \|U(t)f\|_{L^2_rL^\infty_t} \leq C \|f\|_{\dot{H}^{s,r}(\mathbb{R})} \] and (4.1)-(4.2), we have
\[ 2^{-k(s+\frac{1}{r})} \leq C. \] (4.3)

We know that for sufficiently large \( k \), when \( s < \frac{1}{2r} \), (4.3) is invalid.

This completes the proof of Theorem 1.2.

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