General decay of the solution for a viscoelastic wave equation with a time-varying delay term in the internal feedback

Wenjun Liu
College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
E-mail: wjliu@nuist.edu.cn

Abstract: In this paper we consider a viscoelastic wave equation with a time-varying delay term, the coefficient of which is not necessarily positive. By introducing suitable energy and Lyapunov functionals, under suitable assumptions, we establish a general energy decay result from which the exponential and polynomial types of decay are only special cases.

Keywords: viscoelastic wave equation; time-varying delay; internal feedback; general energy decay

AMS Subject Classification (2000): 35L05; 35L15; 35L70; 93D15

1 Introduction

In this work, we investigate the following viscoelastic wave equation with a linear damping and a time-varying delay term in the internal feedback

\[
\begin{aligned}
& u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s) \Delta u(x,s) \, ds \\
& \quad + a_0 u_t(x,t) + a_1 u_t(x,t - \tau(t)) = 0, \quad (x,t) \in \Omega \times (0,\infty), \\
& u(x,t) = 0, \quad (x,t) \in \partial \Omega \times [0,\infty), \\
& u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
& u_t(x,t) = f_0(x,t) \quad (x,t) \in \Omega \times [-\tau(0),0),
\end{aligned}
\]

(1.1)

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n (n \geq 2) \) with a boundary \( \partial \Omega \) of class \( C^2 \), \( a_0 \) and \( a_1 \) are real numbers with \( a_0 > 0, \tau(t) > 0 \) represents the time-varying delay, and the initial datum \( u_0, u_1, f_0 \) are given functions belonging to suitable spaces.

The viscoelastic wave equation without delay (i.e., \( a_1 = 0 \)), has been considered by many authors during the past decades. Cavalcanti et al. [7] studied

\[ u_{tt} - \Delta u + \int_0^t g(t-\tau) \Delta u(\tau) \, d\tau + a(x) u_t + |u|^\gamma u = 0, \quad (x,t) \in \Omega \times (0,\infty), \]

for \( a : \Omega \to \mathbb{R}^+ \), a function, which may be null on a part of the domain \( \Omega \). Under the conditions that \( a(x) \geq a_0 > 0 \) on \( \omega \subset \Omega \), with \( \omega \) satisfying some geometry restrictions and

\[-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,\]

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the authors established an exponential rate of decay. Berri mi and Messaoudi [3] improved Cavalcanti’s result by introducing a different functional which allowed to weaken the conditions on both $a$ and $g$. In [8], Cavalcanti et al. considered

$$u_{tt} - k_0 \Delta u + \int_0^t \text{div}[a(x)g(t-\tau)\nabla u(\tau)]d\tau + b(x)h(u_t) + f(u) = 0,$$

under similar conditions on the relaxation function $g$ and $a(x) + b(x) \geq \rho > 0$, for all $x \in \Omega$. They improved the result of [7] by establishing exponential stability for $g$ decaying exponentially and $h$ linear and polynomial stability for $g$ decaying polynomially and $h$ nonlinear. Berrimi and Messaoudi [4] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = |u|^{p-2}u, \quad p > 2$$

in a bounded domain. They showed, under weaker conditions than those in [8], that the solution is global and decay in a polynomial or exponential fashion when the initial data is small enough. Then Messaoudi [19] improved this result by establishing a general decay of energy which is similar to the relaxation function. For other related works, we refer the readers to [6, 15, 17, 18, 26, 20, 29, 31, 32, 33] and the references therein.

In recent years, the control of PDEs with time delay effects has become an active area of research, see for instance [1, 16, 27, 28] and the references therein. The presence of delay may be a source of instability. For example, it was proved in [9, 10, 21, 22, 30] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. In [21], Nicaise and Pignotti examined (1.1) with $g \equiv 0$, $a_0 > 0$, $a_1 > 0$ and $\tau(t) \equiv \tau$ be a constant delay in the case of mixed homogeneous Dirichlet-Neumann boundary conditions, under a geometric condition on the Neumann part of the boundary. Assuming that $0 \leq a_1 < a_0$, a stabilization result is given, by using a suitable observability estimate and inequalities obtained from Carleman estimates for the wave equation due to Lasiecka et al. in [14]. However, for the opposite case $a_1 \geq a_0$, they were able to construct a sequence of delays for which the corresponding solution is unstable. The same results were obtained for the case when both the damping and the delay act on the boundary, see also [2] for the treatment of this problem in more general abstract form. Kirane and Said-Houari [13] considered (1.1) with $a_0 > 0$, $a_1 > 0$ and $\tau(t) \equiv \tau$ be a constant delay in the case of initial and Dirichlet boundary conditions. They established general energy decay results under the condition that $0 \leq a_1 \leq a_0$.

Recently, the stability of PDEs with time-varying delays was studied in [5, 11, 23, 24, 25]. In [24], Nicaise et al. analyzed the exponential stability of the heat and wave equations with time-varying boundary delay in one space dimension, under the condition $0 \leq a_1 < \sqrt{1 - da_0}$, where $d$ is a constant such that $\tau'(t) \leq d < 1, \forall t > 0$. In [23], Nicaise and Pignotti studied the stabilization problem by interior damping of the wave equation with internal time-varying delay feedback and obtained exponential stability estimates by introducing suitable Lyapunov
functionals, under the condition \( |a_1| < \sqrt{1-d} a_0 \) in which the positivity of the coefficient \( a_1 \) is not necessary.

Motivated by these results, we investigate in this paper system (1.1) under suitable assumptions and prove a general decay result from which the exponential and polynomial types of decay are only special cases. Our main novel contribution is an extension of previous results from [13, 21] to time-varying delays with not necessarily positive coefficient \( a_1 \) of the delay term. This extension is not straightforward due to the loss of translation-invariance. For our purpose, we introduce new energy and Lyapunov functionals, which take into account the dependence of the delay with respect to time.

The paper is organized as follows. In Section 2 we present some assumptions and state the main result. The general decay result is proved in Sections 3.

\section{Preliminaries and main result}

In this section, we present some assumptions and state the main result. We use the standard Lebesgue space \( L^2(\Omega) \) and the Sobolev space \( H^1_0(\Omega) \) with their usual scalar products and norms. Throughout this paper, \( C_i \) is used to denote a generic positive constant.

For the relaxation function \( g \), we assume the following (see [18, 19]):

\( \text{(G1)} \) \( g(t) : [0, \infty) \to (0, \infty) \) is a non-increasing \( C^1 \) function such that
\[
1 - \int_0^\infty g(s) \, ds = l > 0.
\]

\( \text{(G2)} \) There exists a positive non-increasing differentiable function \( \xi(t) \) such that
\[
g'(t) \leq -\xi(t)g(t), \quad t \geq 0,
\]
and
\[
\int_0^{+\infty} \xi(t) \, dt = \infty.
\]

For the time-varying delay, we assume as in [23] that there exist positive constants \( \tau_0, \overline{\tau} \) such that
\[
0 < \tau_0 \leq \tau(t) \leq \overline{\tau}, \quad \forall \ t > 0.
\]

Moreover, we assume that the speed of the delay satisfies
\[
\tau'(t) \leq d < 1, \quad \forall \ t > 0,
\]
that
\[
\tau \in W^{2,\infty}([0, T]), \quad \forall \ T > 0
\]
and that $a_0, a_1$ satisfy

$$|a_1| < \sqrt{1 - d} a_0. \quad (2.4)$$

As in [23], let us introduce the function

$$z(x, \rho, t) = u_t(x, t - \tau(t) \rho), \quad x \in \Omega, \ \rho \in (0, 1), \ t > 0. \quad (2.5)$$

Then, problem (1.1) is equivalent to

$$\begin{cases}
  u_{tt}(x, t) - \Delta u(x, t) + \int_0^t g(t - s) \Delta u(x, s) \, ds \\
  + a_0 u_t(x, t) + a_1 z(x, 1, t) = 0, & (x, t) \in \Omega \times (0, \infty), \\
  \tau(t) z_t(x, \rho, t) + (1 - \tau'(t) \rho) z_\rho(x, \rho, t) = 0, & (x, \rho, t) \in \Omega \times (0, 1) \times (0, \infty), \\
  u(x, t) = 0, & (x, t) \in \partial \Omega \times [0, \infty), \\
  z(x, 0, t) = u_t(x, t), & (x, t) \in \Omega \times (0, \infty), \\
  u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), & x \in \Omega, \\
  z(x, \rho, 0) = f_0(x, -\rho \tau(0)), & (x, \rho) \in \Omega \times (0, 1),
\end{cases} \quad (2.6)$$

We now state, without a proof, a well-posedness result, which can be established by combining the arguments of [12, 13].

**Lemma 2.1** Let (2.1)–(2.3) be satisfied and $g$ satisfies (G1). Then given $u_0 \in H^1_0(\Omega)$, $u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0, 1))$ and $T > 0$, there exists a unique weak solution $(u, z)$ of the problem (2.6) on $(0, T)$ such that

$$u \in C(0, T; H^1_0(\Omega)) \cap C^1(0, T; L^2(\Omega)), \quad u_t \in L^2(0, T; H^1_0(\Omega)) \cap L^2((0, T) \times \Omega).$$

Inspired by [19, 23], we define the new energy functional as

$$E(t) := \frac{1}{2} \int_\Omega \left[ u_t^2 + \left( 1 - \int_0^t g(s) \, ds \right) |\nabla u|^2 \right] \, dx + \frac{1}{2} (g \circ \nabla u)(t) + \frac{\xi}{2} \int_0^t \int_\Omega e^{\lambda(s-t)} u^2_t(x, s) \, dx \, ds, \quad (2.7)$$

where $\xi, \lambda$ are suitable positive constants, and

$$(g \circ v)(t) = \int_\Omega \int_0^t g(t - s) |v(t) - v(s)|^2 \, ds \, dx, \quad \forall \ v \in L^2(\Omega).$$

We will fix $\xi$ such that

$$2a_0 - \frac{|a_1|}{\sqrt{1 - d}} - \xi > 0 \quad \text{and} \quad \xi - \frac{|a_1|}{\sqrt{1 - d}} > 0, \quad (2.8)$$

and

$$\lambda < \frac{1}{\bar{\tau}} \log \frac{|a_1|}{\xi \sqrt{1 - d}}. \quad (2.9)$$

In fact, the existence of such a constant $\xi$ is guaranteed by the assumption (2.4).

Our main result is the following.
Theorem 2.2 Let (2.1) - (2.4) be satisfied and $g$ satisfies (G1) and (G2). Then there exist two positive constants $K, k$ such that, for any solution of problem (1.1), the energy satisfies

$$E(t) \leq Ke^{-k \int_0^t \xi(s) \, ds}, \quad \forall \ t \geq t_0. \quad (2.10)$$

Remark 1 Note that the exponential or polynomial decay estimate is only a particular case of (2.10). More precisely, we can obtain exponential decay for $\xi(t) \equiv a$ and polynomial decay for $\xi(t) \equiv a(1 + t)^{-1}$, where $a > 0$ is a constant.

Remark 2 Estimate (2.10) is also true for $t \in [0, t_0]$ by virtue of the continuity and boundedness of $E(t)$ and $\xi(t)$.

3 General decay of the solution

As mentioned earlier, the proof of the general decay result is given in this section. We have the following lemmas.

Lemma 3.1 Let (2.1) - (2.4) be satisfied and $g$ satisfies (G1). Then for all regular solution of problem (1.1), the energy functional defined by (2.7) is non-increasing and satisfies

$$E'(t) \leq \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} g(t) \int_\Omega |\nabla u|^2 \, dx - C_1 \int_\Omega \left[ u_t^2(x, t) + u_t^2(x, t - \tau(t)) \right] \, dx$$

$$- \frac{\lambda \xi}{2} \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x, s) \, dx \, ds \leq 0$$

for some positive constant $C_1$.

Proof. Differentiating (2.7) and by (1.1), we obtain

$$E'(t) = \int_\Omega \left[ u_t u_{tt} + \left( 1 - \int_0^t g(s) \, ds \right) \nabla u \cdot \nabla u_t - \frac{1}{2} g(t) |\nabla u|^2 \right] \, dx$$

$$+ \int_0^t g(t-s) \int_\Omega \nabla u_t(t) \cdot [\nabla u(t) - \nabla u(s)] \, dx \, ds + \frac{1}{2} \int_0^t g'(t-s) \int_\Omega |\nabla u(t) - \nabla u(s)|^2 \, dx \, ds$$

$$- \frac{\xi}{2} \int_\Omega u_t^2(x, t) \, dx - \frac{\xi}{2} \int_\Omega e^{-\lambda t(t-s)} u_t^2(x, t-(t-s)) \, dx \, ds$$

$$- \frac{\lambda \xi}{2} \int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)} u_t^2(x, s) \, dx \, ds$$

$$= \int_\Omega \left[ u_t u_{tt} + \nabla u \cdot \nabla u_t - \int_0^t g(t-s) \nabla u(s) \cdot \nabla u_t(t) \, ds \right] \, dx$$

$$- \frac{1}{2} g(t) \int_\Omega |\nabla u|^2 \, dx + \frac{1}{2} g'(t) \circ \nabla u(t) + \frac{\xi}{2} \int_\Omega u_t^2(x, t) \, dx$$

$$- \frac{\xi}{2} \int_\Omega e^{-\lambda t(t-s)} u_t^2(x, t-(t-s)) \, dx - \frac{\lambda \xi}{2} \int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)} u_t^2(x, s) \, dx \, ds,$$
and then, using integration by parts, the assumptions (2.1)–(2.2) and some manipulations as in [23],

\[ E'(t) = -a_0 \int_\Omega u_t^2(x,t)dx - a_1 \int_\Omega u_t(t) \int_\Omega u_t(t - \tau(t))dx - \frac{1}{2}g(t) \int_\Omega |\nabla u|^2dx \\
+ \frac{1}{2}(g' \circ \nabla u)(t) + \frac{\xi}{2} \int_\Omega u_t^2(x,t)dx - \frac{\xi}{2} \int_\Omega e^{-\lambda \tau(t)}u_t^2(x,t - \tau(t))(1 - \tau'(t))dx \\
- \frac{\lambda \xi}{2} \int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds \\
\leq -a_0 \int_\Omega u_t^2(x,t)dx - a_1 \int_\Omega u_t(t) \int_\Omega u_t(t - \tau(t))dx - \frac{1}{2}g(t) \int_\Omega |\nabla u|^2dx \\
+ \frac{1}{2}(g' \circ \nabla u)(t) + \frac{\xi}{2} \int_\Omega u_t^2(x,t)dx - \frac{\xi}{2}(1 - d)e^{-\lambda \tau} \int_\Omega u_t^2(x,t - \tau(t))dx \\
- \frac{\lambda \xi}{2} \int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds \\
\leq \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t) \int_\Omega |\nabla u|^2dx - \left(a_0 - \frac{|a_1|}{2\sqrt{1-d}} - \frac{\xi}{2}\right) \int_\Omega u_t^2(x,t)dx \\
- \left(e^{-\lambda \tau} \frac{\xi}{2}(1 - d) - \frac{|a_1| \sqrt{1-d}}{2}\right) \int_\Omega u_t^2(x,t - \tau(t))dx \\
- \frac{\lambda \xi}{2} \int_{t-\tau(t)}^t \int_\Omega e^{-\lambda(t-s)}u_t^2(x,s)dxds. \tag{3.2} \]

Combining (2.8)–(2.9), (3.2) and hypothesis (G1), (3.1) is established. \( \square \)

Now we are going to construct a Lyapunov functional \( L \) equivalent to \( E \). For this purpose, we define the following functionals:

\[ I(t) := \int_\Omega uu_t dx, \tag{3.3} \]

\[ K(t) := -\int_\Omega u_t \int_0^t g(t - s)(u(t) - u(s))ds dx, \tag{3.4} \]

Set

\[ L(t) = NE(t) + \varepsilon I(t) + K(t) \tag{3.5} \]

where \( N \) and \( \varepsilon \) are suitable positive constants to be determined later. Similar as in [19], we can prove that, for \( \varepsilon \) small enough while \( N \) large enough, there exist two positive constants \( \beta_1, \beta_2 \) such that

\[ \beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \forall \ t \geq 0. \tag{3.6} \]

The following estimates hold true.

**Lemma 3.2** Under the assumption (G1), the functional \( I \) satisfies, along the solution, the estimate

\[ I'(t) \leq -\frac{l}{2} \int_\Omega |\nabla u|^2 dx + C_2 \int_\Omega [u_t^2(x,t) + u_t^2(x,t - \tau(t))]dx + C_3(g \circ \nabla u)(t). \tag{3.7} \]
Proof. Differentiating and integrating by parts

\[
I'(t) = \int_{\Omega} u_t^2 \, dx + \int_{\Omega} u \left( \Delta u - \int_0^t g(t-s) \Delta u(s) \, ds - a_0 u_t(t) - a_1 u_t(t - \tau(t)) \right) \, dx
\]

\[
= \int_{\Omega} u_t^2 \, dx - l \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \nabla u \cdot \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) \, ds \, dx - a_0 \int_{\Omega} u(t) u_t(t) \, dx - a_1 \int_{\Omega} u(t) u_t(t - \tau(t)) \, dx.
\] (3.8)

Now, using Young’s inequality and (G1), we obtain (see [19])

\[
\int_{\Omega} \nabla u \cdot \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) \, ds \, dx
\]

\[
\leq \delta \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{4\delta} \int_{\Omega} \left( \int_0^t g(t-s)|\nabla u(s) - \nabla u(t)| \, ds \right)^2 \, dx
\]

\[
\leq \delta \int_{\Omega} |\nabla u|^2 \, dx + \frac{1 - l}{4\delta}(g \circ \nabla u)(t), \quad \forall \delta > 0.
\] (3.9)

Also, using Young’s and Poincaré’s inequalities gives

\[
-a_0 \int_{\Omega} u(t) u_t(t) \, dx \leq \delta \int_{\Omega} |\nabla u|^2 \, dx + C(\delta) \int_{\Omega} u_t^2 \, dx,
\] (3.10)

\[
-a_1 \int_{\Omega} u(t) u_t(t - \tau(t)) \, dx \leq \delta \int_{\Omega} |\nabla u|^2 \, dx + C(\delta) \int_{\Omega} u_t^2(t - \tau(t)) \, dx.
\] (3.11)

Combining (3.8)–(3.11) and choosing \( \delta \) small enough, we obtain (3.7). \( \square \)

Lemma 3.3 Under the assumption (G1), the functional \( K \) satisfies, along the solution, the estimate

\[
K'(t) \leq - \left( \int_0^t g(s) \, ds - 2\delta \right) \int_{\Omega} u_t^2 \, dx + \delta \int_{\Omega} |\nabla u|^2 \, dx + \frac{C_4}{\delta}(g \circ \nabla u)(t)
\]

\[
- \frac{C_5}{\delta}(g \circ \nabla u)(t) + \delta \int_{\Omega} u_t^2(t - \tau(t)) \, dx.
\] (3.12)

Proof. By exploiting (1.1) and integrating by parts, we have

\[
K'(t) = \left( 1 - \int_0^t g(s) \, ds \right) \int_{\Omega} \nabla u \cdot \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx
\]

\[
+ \int_{\Omega} \left( \int_0^t g(t-s)(\nabla u(s) - \nabla u(t)) \, ds \right)^2 \, dx - \int_{\Omega} u_t \int_0^t g'(t-s)(u(t) - u(s)) \, ds \, dx
\]

\[
- \int_0^t g(s) \, ds \int_{\Omega} u_t^2 \, dx + \int_{\Omega} \left( \int_0^t g(t-s)(u(t) - u(s)) \, ds \right) [a_0 u_t(t) + a_1 u_t(t - \tau(t))] \, dx.
\]

Using Young’s and Poincaré’s inequalities, we obtain (see [19] [18])

\[
\left( 1 - \int_0^t g(s) \, ds \right) \int_{\Omega} \nabla u \cdot \int_0^t g(t-s)(\nabla u(t) - \nabla u(s)) \, ds \, dx
\]

\[
\leq \delta \int_{\Omega} |\nabla u|^2 \, dx + \frac{C}{\delta}(g \circ \nabla u)(t),
\]

\[7\]
By using (3.1), (3.5), (3.7) and (3.12), a series of computations yields, for \( \varepsilon \) small such that
\[
\varepsilon \ll 1
\]
Thus, it follows from (G2) and (3.13) that
\[\text{Combining all above estimates, (3.12) is established.} \] □

Now, we are ready to prove the general decay result.

**Proof of Theorem 2.2.** Since \( g \) is positive, we have, for any \( t_0 > 0 \),
\[
\int_0^t g(s) \, ds \geq \int_0^{t_0} g(s) \, ds := g_0 > 0, \quad t \geq t_0.
\]
By using (3.1), (3.5), (3.7) and (3.12), a series of computations yields, for \( t \geq t_0 \),
\[
L'(t) \leq \frac{N}{2} (g' \circ \nabla u)(t) - \frac{N}{2} g(t) \int_\Omega |\nabla u|^2 \, dx - NC_1 \int_\Omega [u_t^2(x, t) + u_t^2(x, t - \tau(t))] \, dx
- \frac{\lambda \xi N}{2} \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x, s) \, dx \, ds + \varepsilon C_2 \int_\Omega [u_t^2(x, t) + u_t^2(x, t - \tau(t))] \, dx
- \frac{\varepsilon l}{\Omega} \int_\Omega |\nabla u|^2 \, dx + \varepsilon C_3 (g \circ \nabla u)(t) - \left( \int_0^t g(s) \, ds - 2\delta \right) \int_\Omega u_t^2 \, dx + \delta \int_\Omega |\nabla u|^2 \, dx
+ \frac{C_4}{\delta} (g \circ \nabla u)(t) - \frac{C_5}{\delta} (g' \circ \nabla u)(t) + \delta \int_\Omega u_t^2(t - \tau(t)) \, dx
= - [(NC_1 + g_0) - 2\delta - \varepsilon C_2] \int_\Omega u_t^2 \, dx + \left( \varepsilon C_3 + \frac{1}{\delta} C_4 \right) (g \circ \nabla u)(t)
+ \left( \frac{N}{2} - \frac{C_5}{\delta} \right) (g' \circ \nabla u)(t) - \left( \frac{\varepsilon l}{2} - \delta \right) \int_\Omega |\nabla u|^2 \, dx
- (NC_1 - \delta - \varepsilon C_2) \int_\Omega u_t^2(x, t - \tau(t)) \, dx
- \frac{\lambda \xi N}{2} \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x, s) \, dx \, ds.
\]
At this point, we choose \( \varepsilon \) small enough such that \( \varepsilon < \frac{g_0}{\lambda} \) and (3.6) hold, and \( \delta \) sufficiently small such that
\[
\alpha_1 = \frac{\varepsilon l}{2} - \delta > 0.
\]
As long as \( \varepsilon \) and \( \delta \) are fixed, we choose \( N \) large enough such that
\[
NC_1 - 2\delta > 0, \quad \alpha_2 = NC_1 - \delta - \varepsilon C_2 > 0 \quad \text{and} \quad \alpha_3 = \frac{N}{2} - \frac{C_5}{\delta} > 0.
\]
Thus, it follows from (G2) and (3.13) that
\[
L'(t) \leq \frac{g_0}{2} \int_\Omega u_t^2 \, dx - \alpha_1 \int_\Omega |\nabla u|^2 \, dx - \alpha_2 \int_\Omega u_t^2(x, t - \tau(t)) \, dx
- \frac{\lambda \xi N}{2} \int_{t-\tau(t)}^t \int_\Omega e^{\lambda(s-t)} u_t^2(x, s) \, dx \, ds + \alpha_4 (g \circ \nabla u)(t)
\leq - C_6 E(t) + \alpha_4 (g \circ \nabla u)(t), \quad \forall t \geq t_0,
\] (3.14)
where \( \alpha_4 = \varepsilon C_3 + \frac{1}{\beta} C_4 > 0 \). It follows from (3.14), (G1) and (3.1) that

\[
\xi(t) L'(t) \leq - C_6 \xi(t) E(t) + \alpha_4 \xi(t) (g \circ \nabla u)(t)
\leq - C_6 \xi(t) E(t) - \alpha_4 (g' \circ \nabla u)(t)
\leq - C_6 \xi(t) E(t) - C_7 E'(t), \quad \forall \, t \geq t_0,
\]

That is

\[
F'(t) \leq - C_8 \xi(t) E(t) \leq - k \xi(t) F(t), \quad \forall \, t \geq t_0,
\]

where

\[
F(t) = \xi(t) L(t) + C_7 E(t)
\]

is clearly equivalent to \( E(t) \) and \( k \) is a positive constant.

Consequently, (2.10) can be obtained by (3.6) and (3.16). \( \square \)

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