2D COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DEGENERATE VISCOSITIES AND FAR FIELD VACUUM

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Abstract. In this paper, the 2-D isentropic Navier-Stokes systems for compressible fluids with density-dependent viscosity coefficients are considered. In particular, we assume that the viscosity coefficients are constant multiple of density. These equations, including several models in 2-D shallow water theory, are degenerate when vacuum appears. We introduce the notion of regular solutions and prove the local existence of solutions in this class allowing the initial vacuum in the far field. This solution is further shown to be stable with respect to initial data in $H^2$ sense. A Beal-Kato-Majda type blow-up criterion is also established.

1. Introduction

In this paper, we aim at proving the local-in-time well-posedness and blow-up criterion of classical solutions to the compressible Navier-Stokes equations for the isentropic flow when viscosity coefficients, shear and bulk, are degenerate and the initial data are arbitrarily large with vacuum appearing in the far field. For this purpose, we consider the following compressible isentropic Navier-Stokes equations (CINS) in $\mathbb{R}^2$:

\begin{equation}
\begin{aligned}
&\rho_t + \text{div}(\rho u) = 0, \\
&((\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \text{div} T.
\end{aligned}
\end{equation}

We look for local classical solution with initial data

\begin{equation}
(\rho, u)|_{t=0} = (\rho_0(x), u_0(x)), \quad x \in \mathbb{R}^2,
\end{equation}

and far field behavior

\begin{equation}
(\rho, u) \to (0, 0) \quad \text{as} \quad |x| \to +\infty, \quad t > 0.
\end{equation}

Such far field behavior occurs naturally when the solutions have finite total mass and energy.

In system (1.1), $x = (x_1, x_2) \in \mathbb{R}^2$, $t \geq 0$ are space and time variables, respectively. $\rho$ is the density, and $u = (u^{(1)}, u^{(2)})^\top \in \mathbb{R}^2$ is the velocity of the fluid. We assume that the pressure $P$ satisfies

\begin{equation}
P = A\rho^\gamma, \quad \gamma > 1,
\end{equation}

Date: July 3, 2014.

1991 Mathematics Subject Classification. Primary: 35B40, 35A05, 76Y05; Secondary: 35B35, 35L65, 85A05.

Key words and phrases. Compressible Navier-Stokes equations, regular solutions, vacuum, degenerate viscosity, blow-up criterion, shallow water, Local wellposedness.
where $A$ is a positive constant, $\gamma$ is the adiabatic exponent. $T$ denotes the viscosity stress tensor with the following form

$$T = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\text{div}uI_2.$$  

Here $I_2$ is the $2 \times 2$ identity matrix, $\mu(\rho) = \alpha\rho$ is the shear viscosity coefficient, $\lambda(\rho) = \beta\rho$, and $\mu(\rho) + \lambda(\rho)$ is the bulk viscosity coefficient, $\alpha$ and $\beta$ are both constants satisfying

$$\alpha > 0, \quad \alpha + \beta > 0.$$  

System (1.1), along with the structural conditions (1.4)–(1.6), occurs in many important applications. For instance, in geophysical flows, when we replace density $\rho$ with the free surface height $h$, the viscous shallow water equations take the form (1.1), with $\gamma = 2$. Furthermore, conditions (1.5)–(1.6) were satisfied by several important models, including the viscous Saint-Venant system, cf. [3] and [5]. For more detailed discussion on this aspect, we refer to [3] and [21], and the later part of this section. On the other hand, in the theory of gas dynamics, in the derivation of the Navier-Stokes equations from the Boltzmann equations through the Chapman-Enskog expansion to the second order, cf. [6] and [18], the viscosity coefficients are shown to be functions of absolute temperature in a power law. If we restrict the gas flow to be isentropic, such dependence is inherited through the laws of Boyle and Gay-Lussac, and one finds that the viscosity coefficients are proportional to powers of density, see [18]. In this paper, due to some technical difficulty, and in view of the application to shallow water equations, we focus on the case when viscosity coefficients are constant multiple of density.

When initial density has positive lower bound, the local existence of classical solutions for (1.1)–(1.2) follows from a standard Banach fixed point argument due to the contraction property of the solution operators of the linearized problem, cf. [28]. This approach is not applicable to the case when $\inf \rho_0(x) = 0$, which occurs when some physical requirements are imposed, such as finite total initial mass and finite total initial energy. When density function connects to vacuum continuously locally or in the far field, there are two major issues in our situation. The first one lies at the degeneracy in time evolution in momentum equations. We notice that the leading coefficient of $u_t$ in momentum equation vanishes at vacuum, and this leads to infinitely many ways to define velocity (if it exits) when vacuum appears. Mathematically, this degeneracy leads to an essential difficulty in determining the velocity when vacuum occurs, since it is difficult to find a reasonable way to extend the definition of velocity into vacuum region. Physically, it is not clear how to define the fluid velocity when there is no fluid at vacuum. When viscosity coefficients are constants, a remedy was suggested in a series of papers by Cho et. al. (see [7], [8], [9]), where they imposed initially a compatibility condition

$$-\text{div}T_0 + \nabla P(\rho_0) = \sqrt{\rho_0}g$$
for some $g \in L^2$. Under this initial layer compatibility condition, a local theory was established successfully; see also [24]. Such a local solution was further extended globally by [13], when initial energy is small. Some similar results can also be seen in [14] and [15].

In addition to this issue, there is another essential difficulty due to vacuum in our case. We note that under assumption (1.5), viscosity coefficients vanish as density function connects to vacuum continuously in the far field. This degeneracy gives rise to some difficulties in our analysis because of the less regularizing effect of the viscosity on the solutions. This is one of the major obstacles preventing us from utilizing a similar remedy proposed by Cho et. al. for the case of constant viscosity coefficients. In order to overcome these difficulties, we observe that, assuming $\rho > 0$, the momentum equations can be rewritten as

$$u_t + u \cdot \nabla u + \frac{2A\gamma}{\gamma - 1} \rho^{\frac{\gamma - 1}{2}} \nabla \rho^{\frac{\gamma - 1}{2}} + Lu = (\frac{\nabla \rho}{\rho})_t \cdot Q(u),$$

where the so-called Lamé operator $L$ and operator $Q$ are

$$Lu = -\alpha \Delta u - (\alpha + \beta)\nabla \text{div} u, \quad Q(u) = \alpha(\nabla u + (\nabla u)^T) + \beta \text{div} u I_{2},$$

Therefore, the two quantities $\rho^{\frac{\gamma - 1}{2}}$ (local sound speed when multiplied by constant $\sqrt{A\gamma}$) and $\nabla \rho$ play significant roles in our analysis on velocity. With the help of this observation, we introduce a proper class of solutions called regular solutions for our problem.

**Definition 1.1 (Regular solution to Cauchy problem (1.1)-(1.3)).**

Let $T > 0$ be a finite constant. A solution $(\rho, u)$ to Cauchy problem (1.1)-(1.3) is called a regular solution in $[0, T] \times \mathbb{R}^2$ if $(\rho, u)$ satisfies

(A) $\rho > 0, \quad \rho \in C^1([0, T] \times \mathbb{R}^2), \quad \rho^{\frac{\gamma - 1}{2}} \in C([0, T]; H^3), \quad (\rho^{\frac{\gamma - 1}{2}})_t \in C([0, T]; H^2)$;

(B) $\nabla \rho/\rho \in C([0, T]; L^6 \cap D^1 \cap D^2), \quad (\nabla \rho/\rho)_t \in C([0, T]; H^1)$;

(C) $u \in C([0, T]; H^3) \cap L^2([0, T]; H^4), \quad u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2)$;

(D) $u_t + u \cdot \nabla u + Lu = (\nabla \rho/\rho) \cdot Q(u), \quad \text{for } |x| \to +\infty$.

Here and throughout this paper, we adopt the following simplified notations for the standard homogeneous and inhomogeneous Sobolev space:

$$D^{k,r} = \{ f \in L^1_{\text{loc}}(\mathbb{R}^2) : |f|_{D^{k,r}} = |\nabla^k f|_{L^r} < +\infty \}, \quad D^k = D^{k,2}.$$

$$|f|_{D^{k,r}} = \| f \|_{D^{k,r}(\mathbb{R}^2)}, \quad \| f \|_{s} = \| f \|_{H^s(\mathbb{R}^2)}, \quad \| f \|_p = \| f \|_{L^p(\mathbb{R}^2)}.$$

A detailed study of homogeneous Sobolev space can be found in [11].

**Remark 1.1.** From Definition 1.1 we know that

$$\rho^{\frac{\gamma - 1}{2}} \in C([0, T]; H^3), \quad \nabla \rho/\rho \in C([0, T]; L^6 \cap D^1 \cap D^2),$$

which imply that $\nabla \rho/\rho \in L^\infty$. Therefore, our regular solution does not contain local vacuum, but vacuum occurs in the far field.

This notion of regular solutions is motivated from Makino, Ukai and Kawashima [26] where the existence of local classical solutions to compressible Euler equations was established. Similar notion was used also in [19], [22], [27], [35] and [10]. Here, the regular solutions select velocity in a physically reasonable way when density approaches to vacuum. With the help of this notion of solutions, the momentum equations can be reformulated.
into a standard quasi-linear parabolic system with some special source terms. Thus the problem becomes trackable through a successful linearization and approximation process.

In this paper, we impose the following regularity conditions on the initial data:

\[ \rho_0 > 0, \quad (\rho_0^{\frac{1}{3}}, u_0) \in H^3, \quad \nabla \rho_0/\rho_0 \in L^6 \cap D^1 \cap D^2. \]  

\[ (1.9) \]

**Remark 1.2.** We remark that (1.9) identifies a class of admissible initial data that provides unique solvability to our problem (1.1)–(1.3). On the other hand, this set of initial data contains a large class of functions, for example, \[ \rho_0(x) = \frac{1}{1 + |x|^\sigma}, \quad u_0(x) \in C^3_0(\mathbb{R}^2), \quad x \in \mathbb{R}^2, \]

where \( \sigma > \max \{1, \frac{1}{\gamma - 1}\} \).

Now we are ready to state our main results.

**Theorem 1.1.** If the initial data \((\rho_0, u_0)\) satisfy the regularity conditions in (1.9), then there exists a time \( T^* > 0 \) and a unique regular solution \((\rho, u)\) to the Cauchy problem (1.1)–(1.3), satisfying

\[ \rho^{\frac{1}{3}} \in C([0, T^*]; H^3), \quad (\rho^{\frac{1}{3}})_t \in C([0, T^*]; H^2), \]

\[ \nabla \rho/\rho \in C([0, T^*]; L^6 \cap D^1 \cap D^2), \quad (\nabla \rho/\rho)_t \in C([0, T^*]; H^1), \]

\[ u \in C([0, T^*]; H^3) \cap L^2([0, T^*]; H^4), \quad u_t \in C([0, T^*]; H^1) \cap L^2([0, T^*]; D^2), \]

\[ u_{tt} \in L^2([0, T^*]; L^2), \quad t^\frac{1}{2} u \in L^\infty([0, T^*]; D^1), \]

\[ t^\frac{1}{2} u_{tt} \in L^\infty([0, T^*]; L^2) \cap L^2([0, T^*]; D^1). \]  

\[ (1.10) \]

Moreover, if \( 1 < \gamma \leq 3, \rho(t, x) \in C^4([0, T^*] \times \mathbb{R}^2) \) (see [9]).

**Remark 1.3.** The smoothing effect of the velocity \( u \) in positive time \( t \in [\tau, T^*], \forall \tau \in (0, T^*), \) tells us that the regular solution obtained in Theorem 1.1 is indeed a classical one in \((0, T^*) \times \mathbb{R}^2\) (see [3]).

As a direct consequence of Theorem 1.1 and the standard theory of quasilinear hyperbolic equations, we have

**Corollary 1.1.** Let \( 1 < \gamma \leq \frac{5}{3} \) or \( \gamma = 2, 3 \). If the initial data \((\rho_0, u_0)\) satisfy (1.9), then there exists a time \( T^* > 0 \) and a unique regular solution \((\rho, u)\) to the Cauchy problem (1.1)–(1.3), satisfying (1.10) and

\[ \rho \in C([0, T^*]; H^3), \quad \rho_t \in C([0, T^*]; H^2), \quad \rho_{tt} \in C([0, T^*]; L^2). \]  

\[ (1.11) \]

We remark that the local existence time \( T^* \) and all the estimates for the regularity of regular solutions in Theorem 1.1 depend only on norms of \((\rho_0, u_0)\) stated in (1.9). The following theorem is about the continuous dependence of the solution on the initial data at least for a small time interval.

**Theorem 1.2.** Let conditions in Theorem 1.1 hold. For each \( i = 1, 2 \), let \((\rho_i, u_i)\) be the local regular solutions to Cauchy problem (1.1)–(1.3) on \([0, T^*]\) with the initial data \((\rho_{0i}, u_{0i})\) satisfying (1.9). Let \( K > 0 \) be a constant such that

\[ \left\| \frac{\gamma - 1}{\rho_{0i}} \right\|_2 + \left\| \frac{\nabla \rho_{0i}}{\rho_{0i}} \right\|_{L^6 \cap D^1} + \|u_{0i}\|_2 \leq K. \]
Then for \(0 \leq t \leq T_*\), there exists a positive constant \(C(T_*, K)\) such that

\[
\left\| \frac{\rho_{1}^{2}}{\rho_{2}^{2}} - \frac{\rho_{1}^{2}}{\rho_{2}^{2}} \right\|_2 + \left\| \rho_{1} \frac{\rho_{1}}{\rho_{2}} - \rho_{2} \frac{\rho_{1}}{\rho_{2}} \right\|_{L^6 \cap D^1} + \left\| u_1 - u_2 \right\|_2 + \int_{0}^{T_*} \left\| \nabla u_1 - \nabla u_2 \right\|_2^2 dt \leq C \left( \left\| \frac{\rho_{1}^{2}}{\rho_{01}^{2}} - \frac{\rho_{1}^{2}}{\rho_{02}^{2}} \right\|_2 + \left\| \rho_{01} \frac{\rho_{01}}{\rho_{01}} - \rho_{02} \frac{\rho_{01}}{\rho_{01}} \right\|_{L^6 \cap D^1} + \left\| u_{01} - u_{02} \right\|_2 \right).
\]

(1.12)

Finally, we establish some blow-up criterion for classical solutions in terms of \(\nabla \rho/\rho\) and the deformation tensor \(D(u)\) defined by

\[
D(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^\top \right),
\]

which is analogous to the Beal-Kato-Majda criterion for the ideal incompressible flow [29].

**Theorem 1.3.** Let \((\rho(x, t), u(x, t))\) be a regular solution obtained in Theorem 1.1. If \(T < +\infty\) is the maximal existence time, then both

\[
\lim_{T \to T^*} \left( \sup_{0 \leq t \leq T} \left\| \frac{\nabla \rho}{\rho}(\cdot, t) \right\|_{L^6(\mathbb{R}^2)} + \int_{0}^{T} \left\| D(u)(\cdot, t) \right\|_{L^\infty(\mathbb{R}^2)} dt \right) = +\infty,
\]

(1.13)

and

\[
\lim_{T \to T^*} \int_{0}^{T} \left\| D(u)(\cdot, t) \right\|_{L^\infty \cap D^1 \cap H^6(\mathbb{R}^2)} dt = +\infty.
\]

(1.14)

As explained before, one of the main purposes of this article is to establish the theories applicable to various models in shallow water. We now consider the Cauchy problem of shallow water equations of the following form:

\[
\begin{cases}
  h_t + \text{div}(hU) = 0, \\
  (hU)_t + \text{div}(hU \otimes U) + \nabla h^2 = \mathcal{V}(h, U), \\
  (h, U)|_{t=0} = (h_0(x), U_0(x)), \quad x \in \mathbb{R}^2, \\
  (h, U) \to (0, 0) \quad \text{as} \quad |x| \to \infty, \quad t > 0,
\end{cases}
\]

(1.15)

where \(h\) denotes the height of the free surface; \(U = (U^{(1)}, U^{(2)})^\top \in \mathbb{R}^2\) is the mean horizontal velocity of fluids. \(\mathcal{V}(h, U)\) is the viscous term. There are several different viscous terms imposed, such as \(\text{div}(hD(U))\), \(\text{div}(h\nabla U)\), \(h\Delta U\), \(\Delta(hU)\) (see [3] and [21]).

Our framework for system (1.1) is applicable with minor modifications to the viscous terms of the forms \(\text{div}(hD(U))\), \(\text{div}(h\nabla U)\), \(h\Delta U\). More precisely, when \(\mathcal{V} = \text{div}(hD(U))\), this is just a special case of system (1.1) with \(\beta = 0\), \(\alpha = 1/2\), and \(\gamma = 2\). For viscous Saint-Venant model, \(\mathcal{V} = \text{div}(h\nabla U)\), we let \(\gamma = 2\) in condition (A) and \(Q(U) = \nabla U\) in condition (D) of Definition 1.1. Then we have the following theorem.

**Theorem 1.4.** If \(\mathcal{V} = \text{div}(h\nabla U)\), and the initial data \((h_0, U_0)\) satisfy the regularity condition

\[
h_0 > 0, \quad (h_0, U_0) \in H^2, \quad \nabla h_0/h_0 \in L^6 \cap D^1 \cap D^2,
\]

(1.16)
new entropy offers a nice estimate λ on water equation, there is a remarkable discovery of a new mathematical entropy function states remains an interesting open problem. In the development of theories on shallow

then there exists a time \( T_* > 0 \) and a unique regular solution \((h, U)\) to Cauchy problem (1.13), satisfying

\[
\begin{align*}
    h &\in C([0, T_*]; H^3), \quad h_t \in C([0, T_*]; H^2), \\
    \nabla h/h &\in C([0, T_*]; L^6 \cap D^1 \cap D^2), \quad (\nabla h/h)_t \in C([0, T_*]; H^1), \\
    U &\in C([0, T_*]; H^3) \cap L^2([0, T_*]; H^4), \quad U_t \in C([0, T_*]; H^1) \cap L^2([0, T_*]; D^2), \\
    U_{tt} &\in L^2([0, T_*]; L^2), \quad t^\frac{1}{2} U \in L^\infty([0, T_*]; D^1), \\
    t^\frac{1}{2} U_t &\in L^\infty([0, T_*]; D^2) \cap L^2([0, T_*]; D^3), \quad t^\frac{1}{2} U_{tt} \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^1).
\end{align*}
\]

Moreover, if \( T < +\infty \) is the maximal existence time of the regular solution \((h, U)\), then

\[
\lim_{T \to T} \left( \sup_{0 \leq t \leq T} \left\| \frac{\nabla h}{h}(\cdot, t) \right\|_{L^6(\mathbb{R}^2)} + \int_0^T \| \nabla U(\cdot, t) \|_{L^\infty(\mathbb{R}^2)} dt \right) = +\infty,
\]

and

\[
\lim_{T \to T} \left( \int_0^T \| \nabla U(\cdot, t) \|_{L^\infty(\mathbb{R}^2)} dt \right) = +\infty.
\]

Finally, when \( \mathcal{V} = h\Delta U \), let \( \gamma = 2 \) and \( Q(U) = 0 \). Then we have the following result.

**Theorem 1.5.** If \( \mathcal{V} = h\Delta U \), and the initial data \((h_0, U_0)\) satisfy the regularity condition

\[
h_0 \geq 0, \quad (h_0, U_0) \in H^3,
\]

then there exists a time \( T_* \) and a unique regular solution \((h, U)\) to (1.13), satisfying

\[
\begin{align*}
    h &\in C([0, T_*]; H^3), \quad h_t \in C([0, T_*]; H^2), \\
    U &\in C([0, T_*]; H^3) \cap L^2([0, T_*]; H^4), \quad U_t \in C([0, T_*]; H^1) \cap L^2([0, T_*]; D^2), \\
    U_{tt} &\in L^2([0, T_*]; L^2), \quad t^\frac{1}{2} U \in L^\infty([0, T_*]; D^1), \\
    t^\frac{1}{2} U_t &\in L^\infty([0, T_*]; D^2) \cap L^2([0, T_*]; D^3), \quad t^\frac{1}{2} U_{tt} \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^1).
\end{align*}
\]

**Remark 1.4.** For this model, we note that the previous requirement on the regularity of \( \nabla h_0/h_0 \) is not needed due to the good structure of the viscous term \( \mathcal{V} \). Current theorem allows initial data containing local vacuum.

System (1.1) has received extensive attention recently. For some interesting development, we refer the readers to [2], [3], [20], [17], [24], [35], [39]. We remark that, in spite of these significant achievements, a lot of questions remain open, including the local well-posedness of classical solutions in multiple dimensions when \( \inf \rho_0 = 0 \). Our results in this paper is a very first step toward this direction. In fact, our conditions in (1.9) rules out local vacuum. Therefore, a theory for more general initial data allowing local vacuum states remains an interesting open problem. In the development of theories on shallow water equation, there is a remarkable discovery of a new mathematical entropy function by Bresch and Desjardins [4] for \( \lambda(\rho) \) and \( \mu(\rho) \) satisfying \( \lambda(\rho) = 2(\mu'(\rho)\rho - \mu(\rho)) \). This new entropy offers a nice estimate

\[
\mu'(\rho) \frac{\nabla \rho}{\sqrt{\rho}} \in L^\infty([0, T]; L^2(\mathbb{R}^2))
\]
provided that $\mu'(\rho_0)\sqrt{\rho_0} \in L^2(\mathbb{R}^2)$. For our problem, such an entropy exists when \( \beta = 0 \). For technical reason, we assume a stronger condition $\frac{\rho_0}{\sqrt{\rho_0}} \in L^6 \cap D^1 \cap D^2$ on the initial data to obtain a better regularity $\frac{\rho_0}{\rho} \in C([0,T];L^6 \cap D^1 \cap D^2)$. It is interesting to find out if it is possible to establish the local existence of regular solutions with initial data allowing local vacuum, with the aid of the entropy of Bresch and Desjardins.

We now outline the organization of the rest of paper. In Section 2, we list some important lemmas that will be used frequently in our proof. In Section 3, we first reformulate our problem into a simpler form. Next we give the proof of the local existence of classical solutions to this reformulated problem. This is achieved in four steps: 1) we construct approximate solutions for the linearized problem when initial density has positive lower bound; 2) we establish the a priori estimates independent of the lower bound of density for the linearized problem; 3) we then pass to the limit to recover the solution of this linearized problem allowing vacuum in the far field; 4) we prove the unique solvability of the reformulated problem through a standard iteration process. Section 4 is devoted to proving the $H^2$ stability with respect to the initial data, i.e. Theorem 1.2. The proof of Theorem 1.3 is given in Section 5.

Finally, we remark that our framework provided in this paper is applicable to the case of three dimensions, with some minor modifications. We will not pursue this in this article.

2. Preliminaries

In this section, we present some important lemmas that appear frequently in our proof. The first one is the following well-known Gagliardo-Nirenberg inequality, which can be found in [16].

**Lemma 2.1.** [16] Let $r \in (1, +\infty)$ and $h \in W^{1,p}(\mathbb{R}^2) \cap L^r(\mathbb{R}^2)$. Then

$$|h|_q \leq C|\nabla h|_p^\theta|h|_r^{1-\theta}$$

where $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{r} - \frac{1}{p} + \frac{1}{2}\right)^{-1}$. If $p < 2$, then $q \in [r, \frac{2p}{2-p}]$ when $r < \frac{2p}{2-p}$; and $q \in [\frac{2p}{2-p}, r]$ when $r \geq \frac{2p}{2-p}$. If $p = 2$, then $q \in [r, +\infty)$. If $p > 2$, then $q \in [r, +\infty]$.

Some common versions of this inequality can be written as

$$|f|_3 \leq C|f|_2^\frac{3}{2}|\nabla f|_2^\frac{3}{2}, \quad |f|_6 \leq C|f|_2^\frac{3}{4}|\nabla f|_2^\frac{3}{4}, \quad |f|_\infty \leq C|f|_2^\frac{3}{2}|\nabla f|_2^\frac{3}{2};$$

(2.1)

which will be used frequently in our following proof.

The second one can be found in Majda [25], and we omit its proof.

**Lemma 2.2.** [25] Let positive constants $r$, $a$ and $b$ satisfy the relation $\frac{1}{r} = \frac{1}{a} + \frac{1}{b}$ and $1 \leq a, b, r \leq +\infty$. \( \forall s \geq 1 \), if $f, g \in W^{s,a}(\mathbb{R}^2) \cap W^{s,b}(\mathbb{R}^2)$, then we have

$$|D^s(fg) - fD^s g|_r \leq C_s(|\nabla f|_a|D^{s-1}g|_b + |D^s f|_a|g|_b),$$

(2.2)

$$|D^s(fg) - fD^s g|_r \leq C_s(|\nabla f|_a|D^{s-1}g|_b + |D^s f|_a|g|_b),$$

(2.3)

where $C_s > 0$ is a constant depending on $s$ only.

The following lemma is important in the derivation of uniqueness in Section 3, which can be found in Remark 1 of [1].
Lemma 2.3. [1] If \( f(t,x) \in L^2([0,T];L^2) \), then there exists a sequence \( s_k \) such that \( s_k \to 0 \), and \( s_k|f(s_k,x)|^2 \to 0 \), as \( k \to +\infty \).

Due to harmonic analysis, we have the following regularity estimate result for Lamé operator. For problem

\[
-\alpha \Delta u - (\alpha + \beta)\nabla \text{div} u = Lu = F, \quad u \to 0 \quad \text{as} \quad |x| \to +\infty, \quad (2.4)
\]

we have

Lemma 2.4. [33] If \( u \in D^{1,q}(\mathbb{R}^2) \) with \( 1 < q < +\infty \) is a weak solution to problem (2.4), then

\[
|u|_{D^{k+2,q}} \leq C|F|_{D^{k,q}},
\]

where \( C \) depending only on \( \alpha, \beta \) and \( q \).

The proof can be obtained via the classical estimates from harmonic analysis, which can be found in [33] or [34].

Finally, using Aubin-Lions Lemma, one has (c.f. [31]),

Lemma 2.5. [31] Let \( X_0, X \) and \( X_1 \) be three Banach spaces satisfying \( X_0 \subset X \subset X_1 \). Suppose that \( X_0 \) is compactly embedded in \( X \) and that \( X \) is continuously embedded in \( X_1 \).

I) Let \( G \) be bounded in \( L^p(0,T;X_0) \) with \( 1 \leq p < +\infty \), and \( \frac{\partial G}{\partial t} \) be bounded in \( L^1(0,T;X_1) \). Then \( G \) is relatively compact in \( L^p(0,T;X) \).

II) Let \( F \) be bounded in \( L^\infty(0,T;X_0) \) and \( \frac{\partial F}{\partial t} \) be bounded in \( L^q(0,T;X_1) \) with \( q > 1 \). Then \( F \) is relatively compact in \( C(0,T;X) \).

3. Existence of Regular Solutions

In this section, we aim at proving Theorem 1.1. To this end, we first reformulated our main problem (1.1)-(1.3) into a simpler form.

3.1. Reformulation. Let \( \phi = \rho^{\frac{2\gamma-1}{\gamma}} \). System (1.1) can be written as

\[
\begin{aligned}
\phi_t + u \cdot \nabla \phi + \frac{\phi - 1}{2} \phi \text{div} u &= 0, \\
u_t + u \cdot \nabla u + \frac{2\gamma}{\gamma - 1} \phi \nabla \phi + Lu &= \psi \cdot Q(u),
\end{aligned}
\]

where \( \psi = \nabla \rho / \rho = \frac{2}{\gamma - 1} \nabla \phi / \phi = (\psi^{(1)},\psi^{(2)})^T \). The initial data are given by

\[
(\phi,u)|_{t=0} = (\phi_0,u_0), \quad x \in \mathbb{R}^2. \quad (3.2)
\]

To prove Theorem 1.1, our first step is to establish the following existence result for the reformulated problem (3.1)-(3.2).

Theorem 3.1. If the initial data \((\phi_0,u_0)\) satisfy the following regularity conditions:

\[
\phi_0 > 0, \quad (\phi_0,u_0) \in H^3, \quad \psi_0 = \frac{2}{\gamma - 1} \nabla \phi_0 / \phi_0 \in L^6 \cap D^1 \cap D^2, \quad (3.3)
\]
then there exist a time $T_*>0$ and a unique regular solution $(\phi, u)$ to Cauchy problem (3.1)-(3.2), satisfying

$$\phi \in C([0, T_*]; H^3), \quad \phi_t \in C([0, T_*]; H^2),$$
$$\psi \in C([0, T_*]; L^6 \cap D^1 \cap D^2), \quad \psi_t \in C([0, T_*]; H^1),$$
$$V \in C([0, T_*]; H^3) \cap L^2([0, T_*]; H^4), \quad u_t \in C([0, T_*]; H^1) \cap L^2([0, T_*]; D^2),$$
$$u_{tt} \in L^2([0, T_*]; L^2), \quad t^2 u_t \in L^\infty([0, T_*]; D^4),$$
$$t^2 u_{tt} \in L^\infty([0, T_*]; D^2) \cap L^2([0, T_*]; D^3), \quad t^2 u_{tt} \in L^\infty([0, T_*]; L^2) \cap L^2([0, T_*]; D^1).$$

We will prove this theorem in subsequent four subsections, and at the end of this section we will show that this theorem indeed implies Theorem 1.1. For simplicity, in the following sections, we denote $\frac{\gamma}{\gamma - 1} = \theta$.

### 3.2. Linearization

In order to proceed with nonlinear problem, we first need to consider the following linearized problem

$$\begin{cases}
\phi_t + v \cdot \nabla \phi + \frac{\gamma - 1}{2} \phi \text{div} v = 0, \\
u_t + v \cdot \nabla v + 2\theta \phi \nabla \phi + Lu = \psi \cdot Q(v),
\end{cases}$$

(3.5)

where

$$Lu = -\alpha \Delta u - (\alpha + \beta) \nabla \text{div} u, \quad Q(v) = \alpha (\nabla v + (\nabla v)^\top) + \beta \nabla \text{div} u,$$

and $v = (v^{(1)}, v^{(2)}) \in \mathbb{R}^2$ is a known vector satisfying $v(t = 0, x) = u_0(x)$ and

$$v \in C([0, T]; H^3) \cap L^2([0, T]; H^4), \quad v_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2),$$
$$v_{tt} \in L^2([0, T]; L^2), \quad t^2 v \in L^\infty([0, T]; D^4),$$
$$t^2 v_t \in L^\infty([0, T]; D^2) \cap L^2([0, T]; D^3), \quad t^2 v_{tt} \in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^1).$$

We assume that

$$\phi_0 > 0, \quad (\phi_0 - \phi^\infty, u_0) \in H^3, \quad \psi_0 = \frac{2}{\gamma - 1} \nabla \phi_0 / \phi_0 \in L^6 \cap D^1 \cap D^2,$$

(3.7)

where $\phi^\infty \geq 0$ is a constant.

In the following two subsections, we first solve this linearized problem when the initial density is away from vacuum, then we establish the uniform estimates with respect to the lower bound of the density which enable us to pass to the limit of the case when $\inf \phi_0 = 0$.

### 3.3. A priori estimate with uniformly positive density ($\inf \phi_0 = \delta > 0$)

Now we want to get some local (in time) a priori estimate which is independent of the lower bound of $\phi$ for the classical solution $(\phi, u)$ to (3.5). First we have the following existence of classical solutions to (3.5) by the standard hyperbolic theory.
Lemma 3.1. Assume that the initial data \((\phi_0, u_0)\) satisfy (3.8) and \(\phi_0 \geq \delta\) for some constant \(\delta > 0\). Then there exists a unique classical solution \((\phi, u)\) to (3.5) such that

\[
\begin{align*}
\phi &\geq \delta, \; \phi - \phi^\infty \in C([0, T]; H^3), \; \phi_t \in C([0, T]; H^2), \\
\psi &\in C([0, T]; L^6 \cap D^1 \cap D^2), \; \psi_t \in C([0, T]; H^1), \; \psi_{tt} \in L^2([0, T]; L^2), \\
u &\in C([0, T]; H^3) \cap L^2([0, T]; H^4), \; u_t \in C([0, T]; H^1) \cap L^2([0, T]; D^2), \\
u_{tt} &\in L^2([0, T]; L^2), \; t^\frac{1}{2}u_t \in L^\infty([0, T]; D^1), \\
t^\frac{1}{2}u_{tt} &\in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^2), \; t^\frac{1}{2}u_{tt} \in L^\infty([0, T]; L^2) \cap L^2([0, T]; D^1),
\end{align*}
\]

for any \(T > 0\), where \(\delta > 0\) is a constant depending on \(\delta\) and \(T\).

Proof. The existence and regularity of a unique solution \(\phi\) to the first equation of (3.5) can be obtained essentially according to Lemma 6 in [8] via the standard theory of transport equation, and \(\phi\) can be written as

\[
\phi(t, x) = \phi_0(W(0, t, x)) \exp\left(-\frac{\gamma - 1}{2} \int_0^t \text{div}(v, W(s, t, x)) ds\right),
\]

where \(W \in C^1([0, T] \times [0, T] \times \mathbb{R}^2)\) is the solution to the initial value problem

\[
\begin{align*}
\frac{d}{dt}W(t, s, x) &= v(t, W(t, s, x)), \quad 0 \leq t \leq T, \\
W(s, s, x) &= x, \quad 0 \leq s \leq T, \; x \in \mathbb{R}^2.
\end{align*}
\]

So we easily know that there exists a positive constant \(\delta\) such that \(\phi \geq \delta\).

Now, it is easy to show that \(\psi\) satisfies

\[
\psi_t + \nabla(v \cdot \psi) + \nabla \text{div}v = 0.
\]

A direct calculation shows that \(\partial_t \psi^{(j)} = \partial_j \psi^{(i)}\) in the sense of distribution, then the above system can be rewritten as

\[
\psi_t + \sum_{l=1}^2 A_l \partial_l \psi + B \psi + \nabla \text{div}v = 0,
\]

where \(A_l = (a_{ij}^{(l)})_{2 \times 2}\) \((i, j, l = 1, 2)\) are symmetric with \(a_{ij}^{(l)} = v^{(l)}\) when \(i = j\) and \(a_{ij}^{(l)} = 0\) otherwise, \(B = (\nabla v)^\top\). Therefore, system (3.12) is a positive symmetric system. Then the assertion of the regularity on \(\psi\) follows.

Finally, with the regularity properties of \(\phi\) and \(\psi\), it is not difficult to solve \(u\) from the linear parabolic equations

\[
u_t + Lu = -v \cdot \nabla v - 2\theta \phi \nabla \phi + \psi \cdot Q(v),
\]

to complete the proof of this lemma. Here we omit the details. \(\square\)

Now we are going to establish the uniform estimates on the solutions obtained in Lemma 3.1. For this purpose, we fix \(T > 0\) and a positive constant \(c_0\) large enough such that

\[
2 + \phi^\infty + |\phi_0|_\infty + ||\phi_0 - \phi^\infty||_3 + ||\psi_0||_{L^\infty \cap D^1 \cap D^2} + ||u_0||_3 \leq c_0,
\]
The constants \(c \leq T\) and the fixed constants \(\alpha, \beta, \gamma\) for some time constant depending only on fixed constants \(\varphi\).

We start with the estimates for \(\phi\). For \(\phi = \min(1, v, s)\), since

\[
\sup_{0 \leq t \leq T^*} (|v(t)|^2_{D_2} + |v_t(t)|^2_{D_2}) + \int_0^{T^*} (|v(s)|^2_{D_4} + |v_t(s)|^2_{D_2} + |v_{tt}(s)|^2_{D_2}) \, ds \leq c_2^2,
\]

for some time \(T^* \in (0, T)\) and constants \(c_i (i = 1, 2, 3, 4)\) such that

\[
1 < c_0 \leq c_1 \leq c_2 \leq c_3 \leq c_4.
\]

The constants \(c_i (i = 1, 2, 3, 4)\) and \(T^*\) will be determined later and depend only on \(c_0\) and the fixed constants \(\alpha, \beta, \gamma\) and \(T\).

Let \((\phi, u)\) be the unique classical solution to (3.5) on \([0, T] \times \mathbb{R}^2\). In the following we are going to establish a series of uniform local (in time) estimates listed as Lemmas 3.2, 3.3.

We start with the estimates for \(\phi\). Hereinafter, we use \(C \geq 1\) to denote a generic positive constant depending only on fixed constants \(\alpha, \beta, \gamma\) and \(T\).

**Lemma 3.2.** Let \((\phi, u)\) be the unique classical solution to (3.5) on \([0, T] \times \mathbb{R}^2\). Then

\[
|\phi(t)|^2_{\infty} + |\phi(t) - \phi^\infty|^3_{\infty} \leq Cc_0^2,
\]

\[
|\phi_t(t)\|_{D_4} \leq Cc_0c_1, \quad |\phi_{tt}(s)|^2_{D_1} \leq Cc_0c_2,
\]

\[
|\phi_t(t)|^2_{D_2} + |\phi_{tt}(s)|^2_{D_1} + \int_0^t |\phi_{tt}(s)|^2_{D_1} \, ds \leq Cc_3^2
\]

for \(0 \leq t \leq T_1 = \min(T^*, (1 + c_3)^{-2})\).

**Proof.** From stand energy estimates (see, for instance, [8]) and (2.1), we easily have

\[
|\phi(t) - \phi^\infty|^3_{\infty} \leq \left(\|\phi_0 - \phi^\infty\|_{\infty}^2 + \phi^\infty \int_0^t \|\nabla v(s)\|_{\infty}^2 \, ds\right) \exp\left(C \int_0^t \|v(s)\|_{\infty}^4 \, ds\right).
\]

Therefore, observing that

\[
\int_0^t \|v(s)\|_{\infty}^4 \, ds \leq t^{\frac{1}{2}} \left(\int_0^t \|v(s)\|_{\infty}^2 \, ds\right)^{\frac{3}{2}} \leq C(c_3 t + c_3 t^{\frac{1}{2}}),
\]

we get

\[
|\phi(t) - \phi^\infty|^3_{\infty} \leq Cc_0 \quad \text{for} \quad 0 \leq t \leq T_1 = \min(T^*, (1 + c_3)^{-2}).
\]

For \(\phi_t\), since

\[
\phi_t = -v \cdot \nabla \phi - \frac{\gamma - 1}{2} \phi \text{div} v,
\]

then for \(0 \leq t \leq T_1\), it holds that

\[
|\phi_t(t)|_{D_2} \leq C(|v \cdot \nabla \phi|_{D_2} + |\phi \text{div} v|_{D_2})(t) \leq Cc_0c_1,
\]

\[
|\phi_{tt}(s)|^2_{D_1} \leq C(|v \cdot \nabla \phi|_{D_1} + |\phi \text{div} v|_{D_1})(t) \leq Cc_0c_2,
\]

\[
|\phi_t(t)|_{D_2} \leq C(|v \cdot \nabla \phi|_{D_2} + |\phi \text{div} v|_{D_2})(t) \leq Cc_0c_3,
\]
where we used (2.1) and Hölder’s inequality.

Using the equation

\[\phi_{tt} = -v_t \cdot \nabla \phi - v \cdot \nabla \phi_t - \frac{\gamma - 1}{2} \phi_t \text{div} v - \frac{\gamma - 1}{2} \phi \text{div} v_t\]

and the assumption (3.4), we have, for \(0 \leq t \leq T_1\),

\[
|\phi_{tt}(t)|_2 \leq C(|v_t \cdot \nabla \phi(t)|_2 + |v \cdot \nabla \phi_t(t)|_2 + |\phi_t \text{div} v(t)|_2 + |\phi \text{div} v_t(t)|_2)
\]

\[\leq C(\|v_t(t)\|_1\|\phi(t) - \phi^\infty\|_3 + \|\phi_t(t)\|_1\|v(t)\|_3) \leq Cc_3^4.\]  \hspace{1cm} (3.18)

Similarly, for \(0 \leq t \leq T_1\), we also have

\[
\int_0^t \|\phi_{tt}\|^2_2ds \leq C \int_0^t (|v_t \cdot \nabla \phi|^2_1 + |v \cdot \nabla \phi_t|^2_1 + \|\phi_t \text{div} v|^2_1 + \|\phi \text{div} v_t|^2_1)ds
\]

\[\leq C \int_0^t (\|v_t\|^2_2\|\phi - \phi^\infty\|^2_3 + \|v\|^2_3\|\phi_t\|^2_2)ds \leq Cc_3^6.\]  \hspace{1cm} (3.19)

This completes the proof of the lemma. \(\square\)

Now we establish the estimates for \(\psi\) by the stand energy estimates for positive symmetric hyperbolic system.

**Lemma 3.3.** Let \((\phi, u)\) be the unique classical solution to (3.3) on \([0, T] \times \mathbb{R}^2\). Then

\[
|\psi(t)|_\infty^2 + |\psi(t)|_D^2 \leq Cc_0^2, \quad |\psi|_2 \leq Cc_2^2,
\]

\[
|\psi(t)|_{D_1}^2 + \int_0^t |\psi_{tt}(s)|_2^2ds \leq Cc_3^4, \quad \text{for} \quad 0 \leq t \leq T_1.
\]  \hspace{1cm} (3.20)

**Proof.** According to the proof of Lemma 3.1, we know that \(\psi\) satisfies the hyperbolic system (3.12). First, multiplying (3.12) by \(6|\psi|^4\psi\) and then integrating over \(\mathbb{R}^2\), we easily deduce that

\[
\frac{d}{dt}\|\psi\|_6^6 \leq C \left( \sum_{l=1}^{2} \|\partial_l A_l\|_\infty + \|B\|_\infty \right) \|\psi\|_6^6 + C\|\nabla^2 v\|_6 \|\psi\|_6^5.\]  \hspace{1cm} (3.21)

Noting that

\[
\|\nabla v\|_\infty \leq C\|v\|_3, \quad \|\nabla^2 v\|_6 \leq C\|\nabla^2 v\|_2^\frac{7}{2}\|\nabla^3 v\|_2^\frac{5}{2} \leq C\|\nabla^2 v\|_1,
\]  \hspace{1cm} (3.22)

(3.21) implies that

\[
\frac{d}{dt}\|\psi\|_6 \leq C\|\nabla v\|_3\|\psi\|_6 + C\|\nabla^2 v\|_1.
\]  \hspace{1cm} (3.23)

Second, let \(\varsigma = (\varsigma_1, \varsigma_2)^T\) \((1 \leq |\varsigma| \leq 2 \text{ and } \varsigma_i = 0, 1, 2)\). Taking derivative \(\partial_\varsigma\) to (3.12), we have

\[
(\partial_\varsigma \psi)_t + \sum_{l=1}^{2} A_l \partial_l \partial_\varsigma \psi + B \partial_\varsigma \psi + \partial_\varsigma \text{div} v
\]

\[= \left( - \partial_\varsigma^2 (B\psi) + B \partial_\varsigma \psi \right) + \sum_{l=1}^{2} \left( - \partial_\varsigma^2 (A_l \partial_l \psi) + A_l \partial_l \partial_\varsigma \psi \right) = \Theta_1 + \Theta_2.\]  \hspace{1cm} (3.24)
Using the fact that
\[ (3.22) \]
and
\[ (3.23) \]
for 0 \leq t \leq T, we have
\[ (3.24) \]
Similarly, using
\[ (3.25) \]
while the choice
\[ (3.26) \]
for 0 \leq t \leq T, it holds that
\[ (3.27) \]
Using the fact that
\[ (3.28) \]
formulas (3.22)-(3.30) and Gagliardo-Nirenberg inequality lead to
\[ (3.29) \]
Then the Gronwall’s inequality implies that
\[ (3.30) \]
For 0 \leq t \leq T_1, this estimate gives the part of \( \| \psi(t) \|_{L^6 \cap D^1 \cap D^2} \) in this lemma. Noting that
\[ (3.31) \]
for 0 \leq t \leq T_1, it holds that
\[ (3.32) \]
Similarly, using
\[ (3.33) \]
for 0 \leq t \leq T_1, we have
\[ (3.34) \]
This concludes the proof of the lemma. \( \square \)

Now we turn to the estimate of the velocity \( u \).
Lemma 3.4. Let \((\phi, u)\) be the unique classical solution to (3.3) on \([0, T] \times \mathbb{R}^2\). Then
\[
\|u(t)\|_{L^1}^2 + \int_0^T (\|\nabla u(s)\|_{L^2}^2 + |u_t(s)|_{L^2}^2) \, ds \leq Cc_0^2,
\]
\[
|u(t)|_{L^2}^2 + |u_t(t)|_{L^2}^2 + \int_0^T (|u(s)|_{L^3}^2 + |u_t(s)|_{L^3}^2) \, ds \leq Cc_1^3 c_2^3,
\]
for \(0 \leq t \leq T = \min(T^*, (1 + c_3)^{-6})\).

Proof. We divide the proof into three steps.

Step 1 (Estimate of \(|u|_{L^2}\)). Multiplying (3.3) by \(u\) and integrating over \(\mathbb{R}^2\), we have
\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \alpha |\nabla u|^2 + (\alpha + \beta) |\text{div} u|^2 \leq \int_{\mathbb{R}^2} \left( -v \cdot \nabla v - u - 2\theta \phi \nabla \phi \cdot u + \psi \cdot Q(v) \cdot u \right) \, dx \equiv \sum_{i=1}^3 I_i.
\]
Due to Gagliardo-Nirenberg inequality, Hölder's inequality and Young’s inequality, we have
\[
I_1 = -\int_{\mathbb{R}^2} v \cdot \nabla v \cdot u \, dx \leq C|v|_\infty |\nabla v|_{L^2} |u|_{L^2} \leq C|\nabla v|_{L^2}^2 + C|u|_{L^2}^2,
\]
\[
I_2 = -\int_{\mathbb{R}^2} 2\theta \phi \nabla \phi \cdot u \, dx \leq C |\nabla \phi|_{L^2} |\phi|_{L^\infty} |u|_{L^2} \leq C |\nabla \phi|_{L^2}^2 + C |\phi|_{L^\infty}^2,
\]
\[
I_3 = \int_{\mathbb{R}^2} \psi \cdot Q(v) \cdot u \, dx \leq C |\psi|_6 |\nabla v|_{L^3} |u|_{L^2} \leq C |\nabla v|_{L^2}^2 + C |\psi|_6^2 |\nabla v|_{L^2}^2.
\]
Then we have
\[
\frac{1}{2} \frac{d}{dt} |u|^2 + \alpha |\nabla u|^2 \leq C |u|^2 + C|\nabla v|^2 |\nabla v|_{L^2}^2 + C |\nabla \phi|_{L^2}^2 |\phi|_{L^\infty}^2 + C |\psi|_6^2 |\nabla v|_{L^2}^2.
\]
Integrating (3.38) over \((0, t)\), for \(0 \leq t \leq T_1\), it gives
\[
|u(t)|_{L^2}^2 + \int_0^t \alpha |\nabla u(s)|_{L^2}^2 \, ds \leq C \int_0^t |u(s)|_{L^2}^2 \, ds + C|u_0|_{L^2}^2 + Cc_3^4 t.
\]
Then Gronwall’s inequality implies
\[
|u(t)|_{L^2}^2 + \int_0^t \alpha |\nabla u(s)|_{L^2}^2 \, ds \leq C \left( |u_0|_{L^2}^2 + c_3^4 t \right) \exp(Ct) \leq Cc_0^2
\]
for \(0 \leq t \leq T_2 = \min(T^*, (1 + c_3)^{-6})\).

Step 2 (Estimate of \(|\nabla u|_{L^2}\)). Multiplying (3.3) by \(u_t\) and integrating over \(\mathbb{R}^2\), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \alpha |\nabla u|^2 + (\alpha + \beta) |\text{div} u|^2 \right) + |u|^2 \leq \int_{\mathbb{R}^2} \left( -v \cdot \nabla v \cdot u_t - 2\theta \phi \nabla \phi \cdot u_t + \psi \cdot Q(v) \cdot u_t \right) \, dx \equiv \sum_{i=4}^6 I_i.
\]
Using Gagliardo-Nirenberg inequality, Hölder’s inequality and Young’s inequality, we have

\[ I_4 = -\int_{\mathbb{R}^2} v \cdot \nabla v \cdot u_t \, dx \leq C|v|_\infty |\nabla v|_2 |u_t|_2 \leq C|v|^2 |\nabla v|^2_2 + \frac{1}{10} |u_t|^2, \]
\[ I_5 = -\int_{\mathbb{R}^2} 2\theta \phi \nabla \phi \cdot u_t \, dx \leq C|\phi|_\infty |\nabla \phi|_2 |u_t|_2 \leq \frac{1}{10} |u_t|^2 + C|\nabla \phi|^2 |\phi|_\infty^2, \quad (3.41) \]
\[ I_6 = \int_{\mathbb{R}^2} \psi \cdot Q(v) \cdot u_t \, dx \leq C|\psi|_6 |\nabla v|_3 |u_t|_2 \leq \frac{1}{10} |u_t|^2 + C|\psi|_6^2 \|\nabla v\|_1^2. \]

Then

\[
\frac{d}{dt} \left( \alpha |\nabla u|^2 + (\alpha + \beta)|\text{div}u|^2 \right) + |u_t|^2 \\
\leq C|v|^2 |\nabla v|^2_2 + C|\text{div}u|^2 |\phi|^2_\infty + C|\psi|_6^2 \|\nabla v\|^2_1. \quad (3.42)
\]

Integrating (3.42) over \((0, t)\), we get

\[
|\nabla u(t)|^2 + \int_0^t |u_t(s)|^2 \, ds \leq C|\nabla u_0|^2 + Cc_3^2 t \leq Cc_0^2 \quad (3.43)
\]

for \(0 \leq t \leq T_2 = \min(T^*, (1 + c_3)^{-6})\).

From the classical estimates for elliptic system in Lemma 2.4, and

\[ Lu = -u_t - v \cdot \nabla v - 2\theta \phi \nabla \phi + \psi \cdot Q(v), \quad (3.44) \]

we easily have, for \(0 \leq t \leq T_2\),

\[
|u(t)|_{D^2} \leq C \left( |u_t(t)|_2 + |v \cdot \nabla v(t)|_2 + |\phi \nabla \phi(t)|_2 + |\psi \cdot Q(v(t))|^2 \right) \\
\leq C \left( |u_t(t)|_2 + c_1^2 c_2^2 + c_0^2 + c_0 |\nabla v|^2 \right), \quad (3.45)
\]

where we have used the fact that

\[
|v \cdot \nabla v(t)|_2 \leq C|v|_6 |\nabla v|_3 \leq C|v|_2^\frac{2}{3} |\nabla v|^\frac{2}{3} \leq C |v|_2^\frac{2}{3} |\nabla v|^\frac{2}{3}. \]

Then (3.45) implies that

\[
\int_0^{T_2} |u(t)|_{D^2}^2 \, dt \leq C \int_0^{T_2} \left( |u_t|^2 + c_1^2 c_2^2 + c_0^2 + c_0^2 |\nabla v|^2 \right)(t) \, dt \leq Cc_0^2. \quad (3.46)
\]

**Step 3 (Estimate of \( |u|_{D^2} \)).** First we differentiate (3.3) with respect to \(t\):

\[ u_{tt} + Lu_t = -(v \cdot \nabla v)_t - 2\theta (\phi \nabla \phi)_t + (\psi \cdot Q(v))_t. \quad (3.47) \]

Multiplying (3.47) by \(u_t\) and integrating over \(\mathbb{R}^2\), we have

\[
\frac{1}{2} \frac{d}{dt} |u_t|^2 + \alpha |\nabla u_t|^2 + (\alpha + \beta)|\text{div}u_t|^2 \\
= \int_{\mathbb{R}^2} \left( -(v \cdot \nabla v)_t \cdot u_t - 2\theta (\phi \nabla \phi)_t \cdot u_t + (\psi \cdot Q(v))_t \cdot u_t \right) \, dx \equiv \sum_{i=7}^9 I_i, \quad (3.48)
\]
where the right-hand side terms can be estimated as follows,

\[
I_7 = -\int_{\mathbb{R}^2} (v \cdot \nabla v) t \cdot u_t dx \leq C|v|_{\infty} |\nabla v_t|_2 |u_t|_2 + C|v_t|_2 |\nabla v|_{\infty} |u_t|_2
\]

\[
\leq C||v||_3^2 |v_t|_1^2 + C|u_t|_2^2,
\]

\[
I_8 = -\int_{\mathbb{R}^2} 2\theta(\phi \nabla \phi) t \cdot u_t dx = \theta \int_{\mathbb{R}^2} (\phi^2) t \text{div} u_t dx
\]

\[
\leq C|\phi_t|_2 |\phi|_\infty |\nabla u_t|_2 \leq \frac{\alpha}{10} |\nabla u_t|_2^2 + C|\phi_t|_2^2 |\phi|_\infty^2,
\]

\[
I_9 = \int_{\mathbb{R}^2} (\psi \cdot Q(u)) t \cdot u_t dx \leq C|\psi|_{\infty} |\nabla v_t|_2 |u_t|_2 + C|\psi_t|_2 |\nabla v|_{\infty} |u_t|_2
\]

\[
\leq C|u_t|_2^2 + C|\psi|_{L^6 \cap D^1 \cap D^2}^2 |\nabla v_t|_2^2 + C|\psi_t|_2^2 |\nabla v|_2^2.
\]

Then we have

\[
\frac{d}{dt} |u_t|^2 + |\nabla u_t|^2 \leq C|u_t|^2 + C|v|^3_3 |v_t|^2_1 + C|\phi_t|^2_2 |\phi|_\infty^2 + C|\psi|_{L^6 \cap D^1 \cap D^2}^2 |\nabla v_t|^2_2 + C|\psi_t|^2_2 |\nabla v|_2^2.
\]

(3.50)

Integrating (3.50) over \((\tau, t) \ (\tau \in (0, t))\), we have

\[
|u_t(t)|_2^2 + \int_\tau^t |\nabla u_t(s)|_2^2 ds
\]

\[
\leq |u_t(\tau)|_2^2 + C \int_\tau^t \left( |u_t|^2 + |v_t|^2_3 |v_t|^2 + |\phi_t|^2_2 |\phi|^2_\infty \right) (s) ds
\]

\[
+ C \int_\tau^t |\psi|_{L^6 \cap D^1 \cap D^2}^2 |\nabla v_t|^2_2 + |\psi_t|^2_2 |\nabla v|^2_2 |\nabla v|_2^2 (s) ds
\]

\[
\leq |u_t(\tau)|_2^2 + C \int_0^t |u_t(s)|_2^2 ds + Cc_2^4 t, \quad \text{for} \ 0 \leq t \leq T_2.
\]

(3.51)

From the momentum equations (3.5) we have

\[
|u_t(\tau)|_2 \leq C(|v|_{\infty} |\nabla v_t|_2 + |\phi|_{\infty} |\nabla \phi|_2 + |u|_{D^2} + |\psi|_{\infty} |\nabla v|_2) (\tau).
\]

(3.52)

Then it is clear from the assumption (3.7) and Lemma 3.1 that

\[
\lim_{\tau \to 0} |u_t(\tau)|_2 \leq C(|v_0|_{\infty} |\nabla v_0|_2 + |\phi_0|_{\infty} |\nabla \phi_0|_2 + |u_0|_{D^2} + |v_0|_{\infty} |\nabla v_0|_2) \leq Cc_2^4.
\]

(3.53)

Letting \(\tau \to 0\) in (3.51), it follows from Gronwall's inequality that

\[
|u_t(t)|_2^2 + \int_0^t \left( \alpha |\nabla u_t|^2 + (\alpha + \beta) |\text{div} u_t|^2 \right) (s) ds \leq c_0^4 \exp(Ct) \leq Cc_2^4
\]

(3.54)

for \(0 \leq t \leq T_2\). So from (3.45), we easily have for \(0 \leq t \leq T_2\),

\[
|u(t)|_{D^2} \leq C(|u(t)|_2 + c_1^2 c_2^2 + c_0^2 + c_0 |\nabla v|_2) \leq Cc_1^3 c_2^4.
\]
From the classical estimates for elliptic system in Lemma 2.3, we have
\[ |u(s)|_{D^3} \leq C \left( |u_t|_{D^1} + |v \cdot \nabla v|_{D^1} + |\phi \nabla \phi|_{D^1} + |\psi \cdot Q(v)|_{D^1} \right) \]
\[ \leq C \left( |u_t|_{D^1} + |v|_6 |\nabla^2 v|_3 + |\nabla v|_6 |\nabla v|_3 + |\phi|_\infty |\nabla^2 \phi|_2 \right) \]
\[ + C \left( |\nabla \phi|_6 |\nabla \phi|_3 + |\psi|_\infty |\nabla^2 v|_2 + |\nabla \psi|_6 |\nabla v|_3 \right) \]
\[ \leq C \left( |u_t|_{D^1} + c_1 c_2 + c_1 c_2 c_3^2 \right), \tag{3.55} \]
where we have used the face that
\[ |v|_6 |\nabla^2 v|_3 \leq C |v|_2^2 |\nabla v|_2^2 |\nabla^2 v|_2^2, \quad |\nabla v|_6 |\nabla v|_3 \leq C |\nabla v|_2^2 |\nabla^2 v|_2^2 |\nabla v|_2 |\nabla^2 v|_2. \]
Then (3.55) implies that
\[ \int_0^{T_2} |u(s)|^2_{D^3} \, dt \leq C \int_0^{T_2} \left( |u_t|^2_{D^1} + c_3^2 \right) \, dt \leq C c_0^4. \]

Now we will give some estimates for the higher order terms of the velocity \( u \) in the following Lemma.

**Lemma 3.5.** Let \( (\phi, u) \) be the unique classical solution to (3.3) on \([0, T] \times \mathbb{R}^2\). Then,
\[ |u_t(t)|_{D^1}^2 + |u(t)|_{D^3}^2 + \int_0^t \left( |u_t(s)|_{D^2}^2 + |u_t(s)|_{D^1}^2 + |u(s)|_{D^1}^2 \right) \, ds \leq C c_2^4 c_3^4, \tag{3.56} \]
\[ t|u_t(t)|_{D^2}^2 + t|u_{tt}(t)|_{D^1}^2 + t|u(t)|_{D^3}^2 + \int_0^t \left( s|u_{tt}|_{D^1}^2 + s|u_t|_{D^3}^2 \right) \, ds \leq C c_3^4 \]
for \( 0 \leq t \leq T_3 = \min(T^*, (1 + c_3)^{-8}) \).

**Proof.** We divide the proof into two steps.

**Step 1** (Estimate of \( |u|_{D^3} \)). Multiplying (3.47) by \( u_{tt} \) and integrating over \( \mathbb{R}^2 \), we have
\[ \frac{1}{2} \frac{d}{dt} \left( \alpha |\nabla u_t|_2^2 + (\alpha + \beta) |\text{div} u|_2^2 \right) + |u_{tt}|_2^2 \]
\[ = \int_{\mathbb{R}^2} \left( -(v \cdot \nabla v)_t \cdot u_{tt} - 2\theta (\phi \nabla \phi)_t \cdot u_{tt} + (\psi \cdot Q(v))_t \cdot u_{tt} \right) \, dx \equiv \sum_{i=10}^{12} I_i. \tag{3.57} \]

Applying Gagliardo-Nirenberg inequality, Hölder’s inequality and Young’s inequality, we get
\[ I_{10} = - \int_{\mathbb{R}^2} (v \cdot \nabla v)_t \cdot u_{tt} \, dx \leq C |v_t|_2 |\nabla v|_\infty |u_{tt}|_2 + C |v|_\infty |\nabla v_t|_2 |u_{tt}|_2 \]
\[ \leq C |v_t|_2^2 |\nabla v|_2^2 + C |v_t|_2^2 |\nabla v|_2^2 + \frac{1}{10} |u_{tt}|_2^2, \]
\[ I_{11} = - \int_{\mathbb{R}^2} 2\theta (\phi \nabla \phi)_t \cdot u_{tt} \, dx \leq C |\phi_t|_2 |\nabla \phi|_\infty |u_{tt}|_2 + C |\phi|_\infty |\nabla \phi_t|_2 |u_{tt}|_2 \]
\[ \leq C |\phi_t|_2^2 |\nabla \phi|_2^2 + C |\phi|_\infty^2 |\nabla \phi_t|_2^2 + \frac{1}{10} |u_{tt}|_2^2, \tag{3.58} \]
\[ I_{12} = \int_{\mathbb{R}^2} (\psi \cdot Q(v))_t \cdot u_{tt} \, dx \leq C|\psi_t|_2 |\nabla v|_\infty |u_{tt}|_2 + C|\psi|_\infty |\nabla v_t|_2 |u_{tt}|_2 \]
\[ \leq C|\psi_t|_2^2 |\nabla v|_2^2 + C|\psi|_{L^6 \cap D_1 \cap D_2}^2 |\nabla v_t|_2^2 + \frac{1}{10} |u_{tt}|_2^2. \]

Then
\[ \frac{d}{dt} \left( \alpha |\nabla u_t|^2 + (\alpha + \beta) |\text{div} u_t|^2 \right) + |u_{tt}|_2^2 \]
\[ \leq C|\psi_t|_2^2 |\nabla v|_2^2 + C||v||_2^2 |\nabla v_t|_2^2 + C|\phi_t|_2^2 |\nabla \phi|_2^2 + C|\phi|_\infty^2 |\nabla \phi_t|_2^2 \]
\[ + C|\psi|_2^2 |\nabla v|_2^2 + C|\psi|_{L^6 \cap D_1 \cap D_2}^2 |\nabla v_t|_2^2. \]

Integrating (3.60) over \((\tau, t)\), we have
\[ |\nabla u_t(t)|_2^2 + \int_\tau^t |u_{tt}(s)|_2^2 \, ds \leq C |\nabla u_t(\tau)|_2^2 + C_c_0 \tau, \] for \(0 \leq \tau \leq T_3 = \min(T^*, (1 + c_3)^{-8})\).

On the other hand, from the momentum equations (3.5) we have
\[ |\nabla u_t(\tau)|_2 \leq C (1 + \|v\|_2^2 + \|\phi\|_2^2 + |\psi|_{L^6 \cap D_1 \cap D_2}^2 \|v\|_2) (\tau). \]

Then from the assumption (3.7) and Lemma 3.1 one has
\[ \limsup_{\tau \to 0} |\nabla u_t(\tau)|_2 \leq C (1 + \|v_0\|_2^2 + \|\phi_0\|_2^2 + |\psi_0|_{L^6 \cap D_1 \cap D_2}^2 \|v_0\|_2) \leq C_c_0. \]

Letting \(\tau \to 0\) in (3.61), it reads that
\[ |\nabla u_t(t)|_2^2 + \int_0^t |u_{tt}(s)|_2^2 \, ds \leq C_c_0, \quad \text{for} \; 0 \leq t \leq T_3. \]

For the higher order terms, from (3.55) and (5.57), it is easy to show that
\[ |u(t)|_{D^3} \leq C \left( |u_t(t)|_{D^1} + c_1c_2 + c_1c_2^2c_3^2 \right) \leq C_c_2^4c_3^4. \]

From (3.41) we have
\[ Lu_t = -u_{tt} - (v \cdot \nabla v)_t - 2\theta (\phi \nabla \phi)_t + (\psi \cdot Q(v))_t. \]

We apply Lemma 2.4 to (3.65) to show that, for \(0 \leq t \leq T_3\),
\[ |u_t(t)|_{D^2} \leq C (|u_{tt}|_2 + |(v \cdot \nabla v)_t|_2 + |(\phi \nabla \phi)_t|_2 + |(\psi \cdot Q(v))_t|_2) (t) \]
\[ \leq C \left( |u_{tt}|_2 + c_2 |\nabla v_t|^2 + c_2^2 |\nabla v|^2 + c_2^3 |\nabla^3 v|_2 \right) (t), \]
\[ |u(t)|_{D^4} \leq C (|u_t|_{D^2} + |v \cdot \nabla v|_{D^2} + |\phi \nabla \phi|_{D^2} + |\psi \cdot Q(v)|_{D^2}) (t) \]
\[ \leq C (|u_t|_{D^2} + c_2^2 + c_2 \|\nabla v\|_2) (t), \]

which quickly implies that
\[ \int_0^{T_3} \left( |u_t(t)|_{D^2}^2 + |u(t)|_{D^4}^2 \right) \, dt \leq C_c_0, \quad \text{for} \; 0 \leq t \leq T_3. \]

**Step 2 (Estimate of \(t^{\frac{1}{2}}|u|_{D^4}\)).** Now we differentiate (3.41) with respect to \(t\):
\[ u_{ttt} + Lu_{tt} = -2v_t \cdot \nabla v_t - u_{tt} \cdot \nabla v - v \cdot \nabla v_t \]
\[ - \theta (\nabla \phi_t)^2 + 2\psi_t \cdot (Q(v))_t + \psi_{tt} \cdot Q(v) + \psi \cdot (Q(v))_{tt}. \]
Multiplying \( (3.67) \) by \( u_{tt} \) and integrating over \( \mathbb{R}^2 \), we have
\[
\frac{1}{2} \frac{d}{dt} |u_{tt}|^2 + \alpha |v_{tt}|^2 + (\alpha + \beta) |\text{div} u_{tt}|^2
\]
\[
= \int_{\mathbb{R}^2} \left( -2v_t \cdot \nabla v_t - v_{tt} \cdot \nabla v - v \cdot \nabla v_{tt} - \theta (\nabla \phi)^2_{tt} + 2\psi_t \cdot (Q(v))_t \right) \cdot u_{tt} \, dx
\]
\[
+ \int_{\mathbb{R}^2} \left( \psi_t \cdot Q(v) + \psi \cdot (Q(v))_t \right) \cdot u_{tt} \, dx \equiv \sum_{i=13}^{19} I_i.
\] (3.68)

Similarly, we can estimate the right-hand side term by term as follows.
\[
I_{13} = - \int_{\mathbb{R}^2} 2v_t \cdot \nabla v_t \cdot u_{tt} \, dx \leq C |u_{tt}|_2 |\nabla v_t|_3 |v_t|_6 \leq C |u_{tt}|_2^2 + C |v_t|_1^2 |\nabla v_t|_1^2.
\]
\[
I_{14} = - \int_{\mathbb{R}^2} v_{tt} \cdot \nabla v \cdot u_{tt} \, dx \leq C |\nabla v|_{\infty} |v_{tt}|_2 |u_{tt}|_2 \leq C |u_{tt}|_2^2 + C |\nabla v|_2^2 |u_{tt}|_2^2.
\] (3.69)

For the term \( I_{15} \), via the integration by parts, we have
\[
I_{15} = - \int_{\mathbb{R}^2} v \cdot \nabla u_{tt} \cdot u_{tt} \, dx \leq C |v|_{\infty} |v_{tt}|_2 |\nabla u_{tt}|_2 + C |\nabla v|_{\infty} |v_{tt}|_2 |u_{tt}|_2
\]
\[
\leq \frac{\alpha}{20} |\nabla u_{tt}|_2^2 + C |u_{tt}|_2^2 + C |v|_3^3 |u_{tt}|_2^2.
\] (3.70)

Similarly, for the terms \( I_{16}-I_{18} \), we have
\[
I_{16} = \int_{\mathbb{R}^2} \theta (\phi^2)_{tt} \cdot \text{div} u_{tt} \, dx \leq C |\phi_{tt}|_2 |\phi|_{\infty} |\nabla u_{tt}|_2 + |\phi_t|_6 |\phi_t|_3 |\nabla u_{tt}|_2
\]
\[
\leq \frac{\alpha}{10} |\nabla u_{tt}|_2^2 + C |\phi_{tt}|_2^2 |\phi|_{\infty}^2 + C |\phi_t|_4^4,
\]
\[
I_{17} = \int_{\mathbb{R}^2} 2\psi_t \cdot (Q(v))_t \cdot u_{tt} \, dx \leq C |\psi_t|_6 |\nabla v_t|_3 |u_{tt}|_2 \leq C |u_{tt}|_2^2 + C |\psi_t|_2^2 |\nabla v_t|_2^2,
\]
\[
I_{18} = \int_{\mathbb{R}^2} \psi_t \cdot Q(v) \cdot u_{tt} \, dx \leq C |\psi_t|_2 |\nabla v|_{\infty} |u_{tt}|_2 \leq C |u_{tt}|_2^2 + C |\psi_t|_2^2 |\nabla v|_2^2.
\] (3.71)

For the last term, via the integration by parts, we have
\[
I_{19} = \int_{\mathbb{R}^2} \psi \cdot (Q(v))_{tt} \cdot u_{tt} \, dx \leq C |\psi|_{\infty} |v_{tt}|_2 |\nabla u_{tt}|_2 + C |\psi|_3 |v_{tt}|_2 |u_{tt}|_6
\]
\[
\leq \frac{\alpha}{20} |\nabla u_{tt}|_2^2 + C |\psi|_6^2 |\nabla v_t|_2^2 |v_{tt}|_2 + C |u_{tt}|_2^2.
\] (3.72)

These estimates, together with Lemmas 3.2, 3.3, and Step 1 lead to
\[
\frac{d}{dt} |u_{tt}|_2^2 + |\nabla u_{tt}|_2^2
\]
\[
\leq C |u_{tt}|_2^2 + C |v_t|_6^2 |\nabla v_t|_2^2 + C |\phi_t|_2^2 |\phi|_{\infty}^2 + C |\phi_t|_4^4
\]
\[
+ C (|v|_3^2 + |\psi|_6^2 |\nabla v_t|_2^2) |v_{tt}|_2^2 + C |\psi_t|_2^2 |\nabla v|_2^2 + C |\psi_t|_2^2 |\nabla v|_2^2
\]
\[
\leq C |u_{tt}|_2^2 + C \sigma_3^2 |u_{tt}|_2^2 + C \sigma_3^2 |v_t|_{D_2}^2 + C \sigma_3^2 |\psi_t|_2^2 + C \sigma_3^3
\] (3.73)

for \( 0 \leq t \leq T_3 \).
Multiplying both sides of (3.73) by \( t \) and integrating over \((\tau, t)\), we have

\[
t \| u_t(t) \|_2^2 + \int_{\tau}^{t} s |\nabla u_t(s)|_2^2 \, ds \\
\leq \tau |u_t(\tau)|_2^2 + \int_{\tau}^{t} |u_t(s)|_2^2 \, ds + C \int_{\tau}^{t} s \left( |u_t(s)|_2^2 + C_3^2 |v_t(s)|_2^2 \right) \, ds \\
+ C \int_{\tau}^{t} \left( C_3^4 |v_t|_{D^2} + C_3^3 |\psi_t|_2^2 \right) \, ds + C C_3^3 t \\
\leq \tau |u_t(\tau)|_2^2 + C C_3^4 + C C_3^3 t.
\]

(3.74)

We know from (3.56) and Lemma 2.3 that, there exists a sequence \( s_k \) such that

\[
s_k \to 0, \quad \text{and} \quad s_k |u_t(s_k, x)|_2^2 \to 0, \quad \text{as} \quad k \to +\infty.
\]

Taking \( \tau = s_k \) and letting \( k \to +\infty \) in (3.74), we arrive at

\[
t |u_t(t)|_2^2 + \int_{\tau}^{t} s \left( |\nabla u_t(s)|_2^2 + (\alpha + \beta) |\text{div} u_t(s)|_2^2 \right) \, ds \leq C C_3^4.
\]

(3.75)

Now combining Lemmas 3.2, 3.4, 3.5, 3.6 and (3.66), from Lemma 2.3, we obtain

\[
t s |u_t(s)|_{D^2} + t s |u(s)|_{D^2} \leq C C_3^3, \\
t s |u_t(s)|_{D^3} \leq C t s \left( C_3^4 + |u_t(t)|_{D^1} + C_3^1 |v_t(t)|_{D^2} \right)
\]

for \( 0 \leq t \leq T_3 \), which, according to (3.75), proves (3.56). \( \square \)

Then from Lemmas 3.2, 3.4 for \( 0 \leq t \leq T_3 \), we have

\[
|u(t)|_{D^2}^2 + |u_t(t)|_{D^2}^2 + \int_{0}^{t} \left( |u(s)|_{D^2}^2 + |u_t(s)|_{D^2}^2 \right) \, ds \leq C C_3^{10} C_3^2,
\]

\[
|u(t)|_{D^3}^2 + |u_t(t)|_{D^3}^2 + \int_{0}^{t} \left( |u(s)|_{D^3}^2 + |u_t|_{D^3}^2 \right) \, ds \leq C C_3^{10} C_3^2,
\]

\[
t |u_t(t)|_{D^2} + t |u(t)|_{D^2} + t |u_t(t)|_{D^2} + \int_{0}^{t} \left( |u(s)|_{D^2}^2 + |u_t(s)|_{D^2}^2 \right) \, ds \leq C C_3^4,
\]

\[
|\phi(t)|_\infty^2 + |\phi(t) - \phi_0|_3^2 + |\phi_t(t)|_2^2 + |\phi_{tt}(t)|_2^2 + \int_{0}^{t} |\phi_{tt}|_1^2 \, ds \leq C C_3^6,
\]

\[
|\psi(t)|_\infty^2 + |\psi(t) - \psi_0|_3^2 + |\psi_t(t)|_1^2 + \int_{0}^{t} |\psi_{tt}|_2^2 \, ds \leq C C_3^4.
\]

(3.76)

Therefore, if we define the constants \( c_i \) \((i = 1, 2, 3, 4, 5)\) and \( T^* \) by

\[
c_1 = C C_3^2 c_0, \quad c_2 = C C_3^2 c_1^2 = C^2 C_3^2 c_0^2, \quad c_3 = C C_3^4 c_2^2 = C C_3^2 c_0^2 C_3^2,
\]

\[
c_4 = C C_3^2 c_3^2 = C C_3^2 c_0^2, \quad \text{and} \quad T^* = \min(T, (1 + c_3)^{-8}),
\]

(3.77)

then we deduce that
Proof. For \( (\phi_0, u_0) \) such that \( \phi_0 > 0 \), we have the following existence result under the assumption that \( \phi_0 > 0 \).

**Lemma 3.6.** Assume that the initial data \( (\phi_0, u_0) \) satisfy (3.3). Then there exists a unique classical solution \( (\phi, u) \) to (3.7) such that

\[
\phi \geq 0, \quad \phi \in C([0, T^*]; H^3), \quad \dot{\phi} \in C([0, T^*]; H^2), \quad \dot{\psi} \in C([0, T^*]; L^6 \cap D^1 \cap D^2),
\]

\[
\partial_t \psi = \partial_j \psi_i (i, j = 1, 2), \quad \psi_t \in C([0, T^*]; H^1), \quad \psi_{tt} \in L^2([0, T^*]; L^2),
\]

\[
u \in C([0, T^*]; H^3) \cap L^2([0, T^*]; H^4), \quad u_t \in C([0, T^*]; H^1) \cap L^2([0, T^*]; D^2),
\]

\[
u_{tt} \in L^2([0, T^*]; L^2), \quad \partial_t^4 u_t \in L^\infty([0, T^*]; D^4),
\]

\[
\partial_t^4 u_t \in L^\infty([0, T^*]; D^4) \cap L^2([0, T^*]; D^2), \quad \partial_t^4 u_{tt} \in L^\infty([0, T^*]; L^2) \cap L^2([0, T^*]; D^1).
\]

Moreover, the solution \( (\phi, u) \) also satisfies the estimates in (3.7).

**Proof.** For \( \delta \in (0, 1) \), we define

\[
\phi_{\delta 0} = \phi_0 + \delta, \quad \psi_{\delta 0} = \frac{2}{\gamma - 1} \nabla \phi_0 / (\phi_0 + \delta).
\]

From (3.8) we know that \( \psi_0 \in L^6 \cap D^1 \cap D^2(\mathbb{R}^2) \), then Gagliardo-Nirenberg inequality implies that, there exists a finite and positive constant \( C \) such that

\[
|\nabla \phi_0(x)| \leq C \phi_0(x), \quad \text{for} \quad x \in \mathbb{R}^2.
\]

So if \( \phi_0(x) = 0 \), we immediately have \( \nabla \phi_0(x) = 0 \), which means that

\[
\psi_{\delta 0} = 0, \quad \text{if} \quad \phi_0(x) = 0; \quad \psi_{\delta 0} - \psi_0 = \frac{\delta}{\phi_0} \psi_0, \quad \text{if} \quad \phi_0(x) > 0.
\]
From the assumption (3.13), there exists a \( \delta_1 > 0 \) such that if \( 0 < \delta < \delta_1 \), then

\[
1 + |\phi_0|_\infty + ||\phi_0 - \delta||_3 + |\psi_0|_{L^6 \cap D^1 \cap D^2} + ||u_0||_3 \leq C_0^2 = \overline{c}_0.
\]

Therefore, taking \((\phi_0, u_0)\) as the initial data, problem (3.5) admits a unique classical solution \((\phi^\delta, u^\delta)\) satisfying the local estimates in (3.78). We note that the estimates in (3.78) are independent of \( \delta \), then there exists a subsequence (still denoted by \((\phi^\delta, u^\delta)\)) converges to a limit \((\phi, \psi, u)\) in weak or weak* sense:

\[
\begin{align*}
(\phi^\delta - \delta, u^\delta) &\rightharpoonup (\phi, u) \quad \text{weak* in } L^\infty([0, T^*]; H^3(R^2)), \\
\psi^\delta &\rightharpoonup \psi \quad \text{weak* in } L^\infty([0, T^*]; L^6 \cap D^1 \cap D^2(R^2)), \\
\phi_t^\delta &\rightharpoonup \phi_t \quad \text{weak* in } L^\infty([0, T^*]; H^2(R^2)), \\
(\psi_t^\delta, u_t^\delta) &\rightharpoonup (\psi_t, u_t) \quad \text{weak* in } L^\infty([0, T^*]; H^1(R^2)).
\end{align*}
\]

(3.80)

In addition, for any \( R > 0 \), due to the Aubin-Lions Lemma (see [31]) (i.e., Lemma 2.5), there exists a subsequence (still denoted by \((\phi^\delta, \psi^\delta, u^\delta)\)) satisfying

\[
(\phi^\delta, \psi^\delta, u^\delta) \rightharpoonup (\phi, \psi, u) \quad \text{in } C([0, T^*]; H^1(B_R)),
\]

(3.81)

where \( B_R \) is a ball centered at origin with radius \( R \). It is clear that \((\phi, \psi, u)\) also satisfies the local estimates in (3.78). So it is easy to show that \((\phi, u)\) is a weak solution of problem (3.5) satisfying the regularity:

\[
\begin{align*}
\phi &> 0, \; \phi \in L^\infty([0, T^*]; H^3), \; \phi_t \in L^\infty([0, T^*]; H^2), \\
\psi &\in L^\infty([0, T^*]; L^6 \cap D^1 \cap D^2), \; \psi_t \in L^\infty([0, T^*]; H^1), \; \psi_{tt} \in L^2([0, T^*]; L^2), \\
u &\in L^\infty([0, T^*]; H^3) \cap L^2([0, T^*]; H^1), \; u_t \in L^\infty([0, T^*]; H^1) \cap L^2([0, T^*]; D^2), \\
\psi_t &\in L^2([0, T^*]; L^2), \; t^2 \psi_t \in L^\infty([0, T^*]; D^4), \\
\psi_{tt} &\in L^\infty([0, T^*]; D^2) \cap L^2([0, T^*]; D^3), \; t^2 u_{tt} \in L^\infty([0, T^*]; L^2) \cap L^2([0, T^*]; D^4).
\end{align*}
\]

We remark that, in this step, even though vacuum appears in the far field, \( \psi \) satisfies \( \partial_i \psi^{(j)} = \partial_j \psi^{(i)} \; (i, j = 1, 2) \) and solves the following positive symmetric hyperbolic system in the sense of distribution:

\[
\psi_t + \nabla(v \cdot \psi) + \nabla \text{div}v = 0.
\]

(3.82)

The uniqueness and time continuity for \((\phi, u)\) can be obtained by standard procedure and we omit the details here. \( \square \)

3.5. Proof of Theorem 3.1

Our proof is based on the classical iteration scheme and the existence results for the linearized problem in Sections 3.2-3.4. Like in Section 3.3, we define constants \( c_0, c_1, c_2, c_3, c_4 \) and assume that

\[
2 + |\phi_0|_\infty + ||(\phi_0, u_0)||_3 + ||\psi_0||_{L^6 \cap D^1 \cap D^2} \leq c_0.
\]
Denote by $u^0 \in C([0, T^*]; H^3) \cap L^2([0, T^*]; H^4)$ the solution of the following Cauchy problem of heat equation:
\[
\begin{cases}
h_t - \Delta h = 0 & \text{in } (0, +\infty) \times \mathbb{R}^2, \\
h(0) = u_0 & \text{in } \mathbb{R}^2.
\end{cases}
\]

Taking a time $T_{**} \in (0, T^*)$ such that
\[
\sup_{0 \leq t \leq T_{**}} \|u^0(t)\|_1^2 + \int_0^{T_{**}} \left( \|\nabla u^0(s)\|_1^2 + |u_t^0(s)|_2^2 \right) ds \leq c_1^2,
\]
\[
\sup_{0 \leq t \leq T_{**}} (|u^0(t)|_{D^2}^2 + |u_t^0(t)|_{D^3}^2) + \int_0^{T_{**}} \left( |u^0(s)|_{D^4}^2 + |u_t^0(s)|_{D^2}^2 + |u_{tt}^0(s)|_2^2 \right) ds \leq c_2^2,
\]
\[
\text{ess sup}_{0 \leq t \leq T_{**}} (t |u_t^0(t)|_{D^2}^2 + t |u^0(t)|_{D^3}^2) + \int_0^{T_{**}} \left( s |u_t^0|_{D^4}^2 + s |u^0|_{D^2}^2 \right) ds \leq c_4^2.
\]

We first prove the existence of regular solutions. Let $v = u^0$, we can get a classical solution $(\phi^1, u^1)$ of problem (3.3) as well as function $\psi^1$. Inductively, we construct approximate sequences $(\phi^{k+1}, \psi^{k+1}, u^{k+1})$ as follows: given $(\phi^k, \psi^k, u^k)$ for $k \geq 1$, define $(\phi^{k+1}, \psi^{k+1}, u^{k+1})$ by solving the following problem:
\[
\begin{cases}
\phi_t^{k+1} + u^k \cdot \nabla \phi^{k+1} + \gamma - 1 \gamma 2 \phi^{k+1} \text{div} u^k = 0, \\
\psi_t^{k+1} + \sum_{l=1}^{2} A_l(u^k) \partial_l \psi^{k+1} + B(u^k) \psi^{k+1} + \nabla \text{div} u^k = 0, \\
u_t^{k+1} + u^k \cdot \nabla u^{k+1} + 2 \theta \phi^{k+1} \nabla \phi^{k+1} = -Lu^{k+1} + \psi^{k+1} \cdot Q(u^k), \\
(\phi^{k+1}, \psi^{k+1}, u^{k+1})|_{t=0} = (\phi_0, \psi_0, u_0).
\end{cases}
\]

This problem was obtained from (3.5) by replacing $v$ with $u^k$. Then we know that $(\phi^k, \psi^k, u^k)$ $(k = 1, 2, \ldots)$ satisfy the estimates in (3.78).

Next we are going to prove that the whole sequence $(\phi^k, \psi^k, u^k)$ converges strongly to a limit $(\phi, \psi, u)$ which satisfies the regularity (3.4). Let
\[
\overline{\phi}^{k+1} = \phi^{k+1} - \phi^k, \quad \overline{\psi}^{k+1} = \psi^{k+1} - \psi^k, \quad \overline{u}^{k+1} = u^{k+1} - u^k.
\]

From (3.84) we have
\[
\begin{cases}
\overline{\phi}_t^{k+1} + u^k \cdot \nabla \phi^k + u^k \cdot \nabla \phi^k + \gamma - 1 \gamma 2 (\phi^{k+1} \text{div} u^k + \phi^k \text{div} u^k) = 0, \\
\overline{\psi}_t^{k+1} + \sum_{l=1}^{2} A_l(u^k) \partial_l \psi^{k+1} + B(u^k) \psi^{k+1} + \nabla \text{div} u^k = \Upsilon_1^k + \Upsilon_2^k, \\
\overline{u}_t^{k+1} + u^k \cdot \nabla \overline{u}^{k+1} + u^k \cdot \nabla u^{k+1} + \theta \nabla ((\phi^{k+1})^2 - (\phi^k)^2) = -Lu^{k+1} + \psi^{k+1} \cdot Q(u^k) + \overline{\psi}^{k+1} \cdot Q(u^{k-1}),
\end{cases}
\]
(3.85)
where $\Upsilon^k_1$ and $\Upsilon^k_2$ are defined by

$$\Upsilon^k_1 = -\sum_{l=1}^{2} (A_l(u^k)\partial_l\psi^k - A_l(u^{k-1})\partial_l\psi^k), \quad \Upsilon^k_2 = -(B(u^k)\psi^k - B(u^{k-1})\psi^k).$$

We first estimate $\|\phi^{k+1}\|_1$. Multiplying (3.85) by $2\phi^{k+1}$ and integrating over $\mathbb{R}^2$, we have

$$\frac{d}{dt}\|\phi^{k+1}\|_2^2 = -2\int_{\mathbb{R}^2} (u^k \cdot \nabla \phi^{k+1} + \nabla \phi^k + \frac{\gamma - 1}{2} (\phi^{k+1} \text{div} u^k + \phi^k \text{div} u^k)) \phi^{k+1} dx,$$

$$\leq C\|\nabla u^k\|_{\infty}\|\phi^{k+1}\|_2^2 + C\|\phi^{k+1}\|_2\|\nabla\psi^k\|_{\infty} + C\|\phi^{k+1}\|_2\|\nabla\phi^k\|_{\infty},$$

which means that $(0 < \eta \leq \frac{1}{10}$ is a constant)

$$\frac{d}{dt}\|\phi^{k+1}(t)\|_2^2 \leq A_k(t)\|\phi^{k+1}(t)\|_2^2 + \eta\|\nabla\phi(t)\|_1^2,$$

$$A_k(t) = C\left(\|\nabla u^k\|_2 + \frac{1}{\eta}(1 + \|\phi^k\|_3^2)\right), \text{ and } \int_0^t A_k(s) ds \leq \tilde{C} + \tilde{C}_\eta t$$

for $t \in [0, T_\ast]$, where $\tilde{C}_\eta$ is a positive constant depending on $\eta$ and constant $\tilde{C}$.

Now taking derivative $\partial_x^k (|\xi| = 1)$ to (3.85), multiplying by $2\partial_x^k \phi^{k+1}$ and integrating over $\mathbb{R}^2$, we have

$$\frac{d}{dt}\|\phi^{k+1}\|_2^2 = -2\int_{\mathbb{R}^2} \partial_x^k (u^k \cdot \nabla \phi^{k+1} + \nabla \phi^k) \partial_x^k \phi^{k+1} dx,$$

$$\leq C\|\nabla u^k\|_{\infty}\|\phi^{k+1}\|_2^2 + C\|\phi^{k+1}\|_2\|\nabla\psi^k\|_{\infty} + C\|\phi^{k+1}\|_2\|\nabla\phi^k\|_{\infty} + C\|\psi^k\|_{\infty}\|\nabla\div\phi^k\|_2\|\phi^{k+1}\|_2,$$

which means that

$$\frac{d}{dt}\|\phi^{k+1}(t)\|_2^2 \leq B_k(t)\|\phi^{k+1}(t)\|_2^2 + \eta\|\nabla\phi(t)\|_2^2,$$

$$B_k(t) = C\left(\|u^k\|_4 + \frac{1}{\eta}\|\phi^k\|_3^2\right), \text{ and } \int_0^t B_k(s) ds \leq \tilde{C} + \tilde{C}_\eta t$$

for $t \in [0, T_\ast]$. Combining (3.86-8.87), it is easy to show that, for $t \in [0, T_\ast]$,

$$\begin{cases}
\frac{d}{dt}\|\phi^{k+1}(t)\|_1^2 \leq \Phi_k(t)\|\phi^{k+1}(t)\|_1^2 + \eta\|\nabla\phi(t)\|_2^2, \\
\Phi_k(t) = C(A_k(t) + B_k(t)), \text{ and } \int_0^t \Phi_k(s) ds \leq \tilde{C} + \tilde{C}_\eta t.
\end{cases}$$
Now we estimate $|\psi^{k+1}|_2$. Multiplying $[3.85]_2$ by $2\psi^{k+1}$ and integrating over $\mathbb{R}^2$, we have

$$
\frac{d}{dt}|\psi^{k+1}|_2^2 \leq \left( \sum_{l=1}^{2} |\partial_l A_l(u^k)|_\infty + |B(u^k)|_\infty \right) |\psi^{k+1}|_2^2 + (|\Upsilon^k_1|_2 + |\Upsilon^k_2|_2 + |\nabla^2 \pi^k|_2) |\psi^{k+1}|_2.
$$

From Hölder’s inequality, it is easy to deduce that

$$
|\Upsilon^k_1|_2 \leq C |\nabla \psi^k|_3 |\pi^k|_6, \quad |\Upsilon^k_2|_2 \leq C |\psi^k|_\infty |\nabla \pi^k|_2.
$$

From $[3.89]$-$[3.90]$, for $t \in [0, T_\ast]$, we have

$$
\begin{aligned}
\left\{ 
\frac{d}{dt}|\psi^{k+1}|_2^2 &\leq \Psi^k_\eta(t) |\psi^{k+1}|_2^2 + \eta |\pi^k|_2^2, \\
\Psi^k_\eta(t) &= C \left( \|\nabla u^k\|_2 + \frac{1}{\eta} |\psi^k|_\infty^2 + \frac{1}{\eta} |\nabla \psi^k|_3^2 + \frac{1}{\eta} \right), \text{ and } \int_0^t \Psi^k_\eta(s) ds \leq \hat{C} + \hat{C}_\eta t.
\end{aligned}
$$

For $||\pi^k+1||_1$, multiplying $[3.85]_3$ by $2\pi^{k+1}$ and integrating over $\mathbb{R}^2$, we have

$$
\begin{aligned}
\frac{d}{dt}|\pi^{k+1}|_2^2 + 2\alpha |\nabla \pi^{k+1}|_2^2 + 2(\alpha + \beta) |\nabla \pi^{k+1}|_2^2 \\
&= -2 \int_{\mathbb{R}^3} \left( u^k \cdot \nabla \pi^k + \psi^k \cdot \nabla u^k \right) \cdot \pi^{k+1} dx \\
&- 2 \int_{\mathbb{R}^3} \left( \theta \nabla ((\phi^k)^2 - (\phi^k)^2) - \psi^{k+1} \cdot Q(\pi^k) + \psi^k \cdot Q(u^k) \right) \cdot \pi^{k+1} dx \\
&\leq C |u^k|_\infty |\nabla \pi^k|_2 |\pi^{k+1}|_2 + C |u^k|_6 |\nabla \pi^k|_2 |\pi^{k+1}|_2 + C |\psi^k|_\infty |\nabla \pi^k|_2 |\pi^{k+1}|_2 \\
&+ C \left( |\phi^{k+1}|_\infty + |\phi^k|_\infty \right) |\nabla \pi^k|_2 |\pi^{k+1}|_2 + C |\psi^{k+1}|_2 |\nabla \pi^k|_2 |\pi^{k+1}|_2,
\end{aligned}
$$

which implies that

$$
\frac{d}{dt}|\pi^{k+1}|_2^2 + \alpha |\nabla \pi^{k+1}|_2^2 \leq E^k_\eta(t) |\pi^{k+1}|_2^2 + E^k_2(t) |\pi^{k+1}|_2^2 + E^k_3(t) |\pi^{k+1}|_2^2 + \eta |\pi^k|_2^2,
$$

where

$$
E^k_\eta(t) = C \left( 1 + \frac{1}{\eta} |u^k|_\infty^2 + \frac{1}{\eta} |\nabla u^{k-1}|_3^2 + \frac{1}{\eta} |\psi^{k+1}|_\infty^2 \right),
$$

$$
E^k_2(t) = C \left( |\phi^{k+1}|_\infty^2 + |\phi^k|_\infty \right), \quad E^k_3(t) = C |\nabla u^{k-1}|_2^2,
$$

and

$$
\int_0^t \left( E^k_\eta(s) + E^k_2(s) + E^k_3(s) \right) ds \leq \hat{C} + \hat{C}_\eta t.
$$
Now taking $\partial_x^k$ to (3.85)$_3$ ($|\zeta|=1$), multiplying by $\partial_x^k u^{k+1}$ and integrating over $\mathbb{R}^2$, we have

$$
\frac{d}{dt}|\nabla u^{k+1}|_2^2 + \alpha |\nabla \partial_x^k u^{k+1}|^2_2 + (\alpha + \beta) |\partial_x^k \text{div} u^{k+1}|^2_2
= \int_{\mathbb{R}^2} \left( - \partial_x^k (u^k \cdot \nabla u^k) - \partial_x^k (\overline{u}^k \cdot \nabla u^{k-1}) - \theta \partial_x^k (\nabla ((\phi^{k+1})^2 - (\phi^k)^2)) \right) \cdot \partial_x^k u^{k+1} dx
+ \int_{\mathbb{R}^2} \left( \partial_x^k (\nabla^2 (\overline{\psi}^{k+1} \cdot Q(u^{k-1}))) + \partial_x^k (\nabla^2 (\overline{\psi}^{k+1} \cdot Q(u^{k+1}))) \right) \cdot \partial_x^k u^{k+1} dx = \sum_{i=1}^5 J_i,
$$

where

$$
J_1 = \int_{\mathbb{R}^2} -\partial_x^k (u^k \cdot \nabla u^k) \cdot \partial_x^k u^{k+1} dx
\leq C|\nabla u^k|_\infty |\nabla u^{k+1}|_2 + C|u^k|_\infty |\overline{u}^k|_D |\nabla u^{k+1}|_2,
$$

$$
J_2 = \int_{\mathbb{R}^2} -\partial_x^k (\overline{u}^k \cdot \nabla u^{k-1}) \cdot \partial_x^k u^{k+1} dx
\leq C|\nabla \overline{u}^k|_2 |\nabla u^{k+1}|_2 |\nabla u^{k-1}|_\infty + C |\overline{u}^k|_6 |\nabla u^{k+1}|_2 |\nabla^2 u^{k-1}|_3,
$$

$$
J_3 = \int_{\mathbb{R}^2} -\theta \partial_x^k (\nabla ((\phi^{k+1})^2 - (\phi^k)^2)) \cdot \partial_x^k u^{k+1} dx
\leq C|((\phi^{k+1})^2 + (\phi^k)^2)|_\infty |\nabla^2 (\overline{\psi}^{k+1} \cdot u^{k+1})|_2 \cdot (\overline{\phi}^{k+1})_2 + C|(\phi^{k+1} + \phi^k)|_\infty |\nabla u^{k+1}|_2 |\nabla (\phi^{k+1})|_2,
$$

$$
J_4 = \int_{\mathbb{R}^2} \partial_x^k (\nabla^2 (\overline{\psi}^{k+1} \cdot Q(u^{k-1}))) \cdot \partial_x^k u^{k+1} dx
\leq C|\nabla u^{k+1}|_\infty |\nabla (\overline{\psi}^{k+1})|_2 |\nabla u^{k-1}|_\infty + C|\overline{u}^k|_6 |\nabla u^{k+1}|_2 |\nabla^2 (\overline{\psi}^{k+1} \cdot u^{k-1})|_2,
$$

Using Young’s inequality and (3.93), we have

$$
\frac{d}{dt}|\nabla u^{k+1}|_2^2 + \alpha |\nabla \partial_x^k u^{k+1}|^2_2
\leq F^{k}_\eta (t) |\nabla u^{k+1}|_2^2 + F^{2}_\eta (t) |\nabla u^{k+1}|_1^2 + F^{3}_\eta (t) |\overline{\psi}^{k+1}|_2^2 + \eta |\overline{u}^k|_2^2,
$$

where

$$
F^{k}_\eta (t) = C \left( 1 + \frac{1}{\eta} |u^k|_3^3 + \frac{1}{\eta} |\nabla u^{k-1}|_2^2 + \frac{1}{\eta} |\psi^{k+1}|_L^2 |D^1 \cap D^2,\right),
F^{2}_\eta (t) = C |(\phi^{k+1})_3 + |(\phi^k)_3|_3^2,\ F^{3}_\eta (t) = C |\nabla u^{k-1}|_2^2,
$$

and

$$
\int_0^t (F^{k}_\eta(s) + F^{2}_\eta(s) + F^{3}_\eta(s)) ds \leq \tilde{C} + \tilde{C}_\eta t,
$$

for $t \in (0, T_{ss}]$. 

Combining (3.92) and (3.94), we easily get
\[
\frac{d}{dt} \|\overline{u}^{k+1}\|_1^2 + \alpha \|\nabla \overline{u}^{k+1}\|_1^2 \leq \Theta_{\eta}^k(t) \|\overline{u}^{k+1}\|_1^2 + \Theta_2^k(t) \|\phi^{k+1}\|_1^2 + \Theta_3^k(t) \|\psi^{k+1}\|_2^2 + \eta \|\overline{u}^k\|_2^2,
\]  
(3.95)
and
\[
\int_0^t (\Theta_{\eta}^k(s) + \Theta_2^k(s) + \Theta_3^k(s))ds \leq \hat{C} + \hat{C}_\eta t,
\]
for \( t \in (0, T_\ast] \).

Finally, let
\[
\Gamma^{k+1}(t) = \sup_{0 \leq s \leq t} \|\phi^{k+1}(s)\|_1^2 + \sup_{0 \leq s \leq t} \|\psi^{k+1}(s)\|_2^2 + \sup_{0 \leq s \leq t} \|\overline{u}^{k+1}(s)\|_1^2.
\]

According to (3.88), (3.91), (3.95) and Gronwall’s inequality, we have
\[
\Gamma^{k+1}(t) + \int_0^t \mu \|\nabla \overline{u}^{k+1}\|_1^2 ds \leq \left( C\eta \int_0^t \|\nabla \overline{u}\|_1^2 ds + C\eta t \sup_{0 \leq s \leq t} \|\overline{u}^k(s)\|_2^2 \right) \exp (\hat{C} + \hat{C}_\eta t).
\]

We choose \( \eta > 0 \) and \( T_\ast \in (0, \min(1, T_\ast)) \) small enough such that
\[
C\eta \exp \hat{C} \leq \min \left( \frac{1}{4}, \frac{\mu}{4} \right), \quad \text{and} \quad \exp(\hat{C}_\eta T_\ast) \leq 2.
\]

Then we easily have
\[
\sum_{k=1}^{\infty} \left( \Gamma^{k+1}(T_\ast) + \int_0^{T_\ast} \mu \|\nabla \overline{u}^{k+1}\|_1^2 ds \right) \leq C < +\infty.
\]

Thanks to
\[
\lim_{k \to +\infty} \|\psi^{k+1}\|_6 \leq C \lim_{k \to +\infty} (\|\psi^{k+1}\|_6 \|\psi^{k+1}\|_2^2) \leq C \lim_{k \to +\infty} \|\psi^{k+1}\|_2^2 = 0,
\]
we easily know that the whole sequence \((\phi^k, \psi^k, u^k)\) converges to a limit \((\phi, \psi, u)\) in the following strong sense:
\[
\phi^k \to \phi \quad \text{in} \quad L^\infty([0, T_\ast]; H^1(\mathbb{R}^2)),
\]
\[
\psi^k \to \psi \quad \text{in} \quad L^\infty([0, T_\ast]; L^6(\mathbb{R}^2)),
\]
\[
u^k \to u \quad \text{in} \quad L^\infty([0, T_\ast]; H^1(\mathbb{R}^2)) \cap L^2([0, T_\ast]; D^2(\mathbb{R}^2)).
\]  
(3.96)

It is clear that \((\phi, \psi, u)\) satisfies the estimates in (3.78). Thanks to (3.96), \((\phi, u)\) is a weak solution of problem (3.31, 3.2) with the following regularities:
\[
\phi \in L^\infty([0, T_\ast]; H^3), \quad \phi_t \in L^\infty([0, T_\ast]; H^2), \quad \psi \in L^\infty([0, T_\ast]; L^6 \cap D^1 \cap D^2),
\]
\[
\partial_t \psi^{(i)} = \partial_j \psi^{(j)} \quad (i, j = 1, 2), \quad \psi \in L^\infty([0, T_\ast]; H^1), \quad \psi_{tt} \in L^2([0, T_\ast]; L^2),
\]
\[
u \in L^\infty([0, T_\ast]; H^3) \cap L^2([0, T_\ast]; H^4), \quad \nu_t \in L^\infty([0, T_\ast]; H^1) \cap L^2([0, T_\ast]; D^2),
\]
\[
u_{tt} \in L^2([0, T_\ast]; L^2), \quad t^{\frac{1}{2}} \nu \in L^\infty([0, T_\ast]; D^1),
\]
\[
u t^\frac{1}{2} \in L^\infty([0, T_\ast]; D^2), \quad t^\frac{1}{2} \nu_t \in L^\infty([0, T_\ast]; L^2) \cap L^2([0, T_\ast]; D^1).
\]  
(3.97)
The time-continuity of the above solution can be obtained by standard procedure (see [8]). Therefore, it is a regular solution.

Now we prove the uniqueness of regular solutions. Let \((\phi_1, u_1)\) and \((\phi_2, u_2)\) be two regular solutions to Cauchy problem (3.1)-(3.2) satisfying the uniform estimates in (3.78). We denote that

\[
\phi = \phi_1 - \phi_2, \quad \psi = u_1 - u_2,
\]

and

\[
\psi = \psi_1 - \psi_2 = \frac{2}{\gamma - 1} \left( \nabla \phi_1 / \phi_1 - \nabla \phi_2 / \phi_2 \right).
\]

Then \((\phi, \psi, u)\) satisfies the system

\[
\begin{align*}
\phi_t + u_1 \cdot \nabla \phi + u_2 \cdot \nabla \phi + \frac{\gamma - 1}{2} (\phi \text{div} u + \phi_1 \text{div} u_2) &= 0, \\
\psi_t + \sum_{l=1}^{2} A_l(u_1) \partial_l \psi + B(u_1) \psi + \nabla \text{div} u &= \Upsilon_1 + \Upsilon_2, \\
u_t + u_1 \cdot \nabla u + u_2 \cdot \nabla u_2 + \theta \nabla ((\phi_1)^2 - (\phi_2)^2) &= -L \pi + \psi_1 \cdot Q(\pi) + \psi_2 \cdot Q(u_2),
\end{align*}
\]

with \(\Upsilon_1\) and \(\Upsilon_2\) defined by

\[
\Upsilon_1 = -\sum_{l=1}^{2} (A_l(u_1) \partial_l \psi_2 - A_l(u_2) \partial_l \psi_2), \quad \Upsilon_2 = -(B(u_1) \psi_2 - B(u_2) \psi_2).
\]

Let

\[
\Phi(t) = \|\phi(t)\|^2_1 + \|\psi(t)\|^2_2 + \|\pi(t)\|^2_2.
\]

Similarly to the derivation of (3.86)-(3.92), we can show that

\[
\frac{d}{dt} \Phi(t) + C \|\nabla \pi(t)\|^2_1 \leq G(t) \Phi(t),
\]

where \(\int_0^t G(s) ds \leq \hat{C}\), for \(0 \leq t \leq T_\ast\). From the Gronwall’s inequality, we conclude that

\[
\phi = \psi = \pi = 0,
\]

then the uniqueness is obtained.

3.6. Proof of Theorem 1.1 and Corollary 1.1

Based on Theorem 3.1, we are now ready to prove the local existence of regular solution to the original Cauchy problem (1.1)-(1.3). Moreover, we will show that the regular solutions that we obtained satisfy system (1.1) classically.

Proof of Theorem 1.1

Proof. For initial data (1.9), we know from Theorem 3.1 that there exists a time \(T_\ast > 0\) such that the problem (3.1)-(3.2) has a unique regular solution \((\phi, u)\) satisfying the regularity (3.4), which means that

\[
(\rho^{\frac{\gamma - 1}{2}}, u) \in C^1([0, T_\ast] \times \mathbb{R}^2), \quad (\nabla \rho / \rho, \partial_x u) \in C([0, T_\ast] \times \mathbb{R}^2),
\]

(3.100)
where \( \xi \in \mathbb{R}^2 \) with \(|\xi| = 2\). Since

\[
\rho(t, x) = \phi^{\frac{2}{\gamma - 1}}(t, x),
\]

and \( \frac{2}{\gamma - 1} \geq 1 \) for \( 1 < \gamma \leq 3 \), it is easy to show that

\[
\rho(t, x) \in C^1([0, T_\ast] \times \mathbb{R}^2).
\]

Multiplying (3.1) by \( \frac{\partial \rho}{\partial \phi}(t, x) = \frac{2}{\gamma - 2} \phi^{\frac{3}{\gamma - 1}}(t, x) \in C([0, T_\ast] \times \mathbb{R}^2) \), we get the continuity equation in (1.1):

\[
\rho_t + u \cdot \nabla \rho + \rho \text{div} u = 0.
\]

(3.101)

Multiplying (3.2) by \( \phi^{\frac{2}{\gamma - 1}} = \rho(t, x) \in C^1([0, T_\ast] \times \mathbb{R}^2) \), we get the momentum equations in (1.1):

\[
\rho u_t + \rho u \cdot \nabla u + \nabla P = \text{div} \left( \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\text{div} I_2 \right).
\]

(3.102)

That is to say, \((\rho, u)\) satisfies problem (1.1)-(1.3) in classical sense with regularity (1.10).

Proof of Corollary 1.1.

Proof. When \( 1 < \gamma \leq \frac{5}{3} \), \( \frac{2}{\gamma - 1} \geq 3 \). Since \( \phi \in C([0, T_\ast], H^3) \cap C^1([0, T_\ast], H^2) \), we read

\[
\rho(t, x) = \phi^{\frac{2}{\gamma - 1}}(t, x),
\]

that

\[
\rho \in C([0, T_\ast], H^3).
\]

With the help of the continuity equation

\[
\rho_t + u \cdot \nabla \rho + \rho \text{div} u = 0,
\]

and the fact that \( u(t, x) \in C([0, T_\ast], H^3) \cap C^1([0, T_\ast], H^2) \), it is clear that

\[
\rho \in C([0, T_\ast], H^3) \cap C^1([0, T_\ast], H^2),
\]

and the regularity on \( \rho_t \) in Corollary 1.1 follows.

Furthermore, when \( \gamma = 2 \), or 3, \( \rho(t, x) = \phi^2(t, x) \) and \( \rho(t, x) = \phi(t, x) \), respectively. By the same token, the regularity of \( \rho \) in these cases can be achieved. \( \square \)
4. Stability in $H^2$ sense

Now we prove the stability in $H^2$, i.e., Theorem 1.2. For $i = 1, 2$, let $(\phi_i, u_i)$ be the regular solution to Cauchy problem (3.1) with initial data $(\phi_{0i}, u_{0i})$ satisfying (1.9). Let $K > 0$ be a constant such that

$$\|\phi_{0i}\|_2 + \left| \frac{\nabla \phi_{0i}}{\phi_{0i}} \right|_{L^q(D^1)} + \|u_{0i}\|_2 \leq K.$$ 

Denote

$$\tilde{\phi} = \phi_1 - \phi_2, \quad \tilde{u} = u_1 - u_2,$$

and

$$\tilde{\psi} = \psi_1 - \psi_2 = \frac{2}{\gamma - 1} \left( \frac{\nabla \phi_1}{\phi_1} - \frac{\nabla \phi_2}{\phi_2} \right).$$

Then $(\tilde{\phi}, \tilde{\psi}, \tilde{u})$ satisfies the system

$$\begin{cases}
\tilde{\phi}_t + u_1 \cdot \nabla \tilde{\phi} + \tilde{u} \cdot \nabla \phi_2 + \frac{\gamma - 1}{2} (\tilde{\phi} \text{div} u_2 + \phi_1 \text{div} \tilde{u}) = 0, \\
\tilde{\psi}_t + \sum_{l=1}^2 A_l(u_1) \partial_t \tilde{\psi} + B(u_1) \tilde{\psi} + \nabla \text{div} \tilde{u} = \tilde{Y}_1 + \tilde{Y}_2, \\
\tilde{u}_t + u_1 \cdot \nabla \tilde{u} + \tilde{u} \cdot \nabla u_2 + \theta \nabla ((\phi_1)^2 - (\phi_2)^2) \\
\quad = -L \tilde{u} + \psi_1 \cdot Q(\tilde{u}) + \tilde{\psi} \cdot Q(u_2),
\end{cases} \tag{4.1}$$

where $\tilde{Y}_1$ and $\tilde{Y}_2$ are defined by

$$\tilde{Y}_1 = -\sum_{l=1}^2 (A_l(u_1) \partial_t \psi_2 - A_l(u_2) \partial_t \psi_2), \quad \tilde{Y}_2 = -(B(u_1) \psi_2 - B(u_2) \psi_2).$$

Similarly to the derivation of (3.88), we have

$$\begin{cases}
\frac{d}{dt} \|\tilde{\phi}(t)\|_2^2 \leq A_\eta(t) \|\tilde{\phi}(t)\|_2^2 + C \|\tilde{u}(t)\|_2^2 + \eta \|\nabla \tilde{u}(t)\|_2^2, \\
\int_0^t A_\eta(s) ds \leq \tilde{C} + \tilde{C}_\eta t \quad \text{for} \quad t \in [0, T_*]. \tag{4.2}
\end{cases}$$

Then taking $\partial_x^\xi$ to (4.1) $(\xi = 2)$, multiplying by $2\partial_x^\xi \tilde{\phi}$ and integrating over $\mathbb{R}^2$, we have

$$\frac{d}{dt} \|\tilde{\phi}(t)\|_2^2 = -2 \int_{\mathbb{R}^2} \partial_x^\xi \left( u_1 \cdot \nabla \tilde{\phi} + \tilde{u} \cdot \nabla \phi_2 + \frac{\gamma - 1}{2} (\tilde{\phi} \text{div} u_2 + \phi_1 \text{div} \tilde{u}) \right) \partial_x^\xi \tilde{\phi} dx \leq C(\|\nabla u_1\|_2 + \|\nabla u_2\|_2 + \|\phi_1\|_3 + \|\nabla \phi_2\|_2^2) \|\tilde{\phi}\|_2^2 + C \|\tilde{u}\|_2^2 + \eta \|\nabla \tilde{u}\|_2^2. \tag{4.3}$$

(4.2) and (4.3) yield

$$\begin{cases}
\frac{d}{dt} \|\tilde{\phi}(t)\|_2^2 \leq \Phi_\eta(t) \|\tilde{\phi}(t)\|_2^2 + C \|\tilde{u}(t)\|_2^2 + \eta \|\nabla \tilde{u}(t)\|_2^2, \\
\int_0^t \Phi_\eta(s) ds \leq \tilde{C} + \tilde{C}_\eta t \quad \text{for} \quad t \in [0, T_*]. \tag{4.4}
\end{cases}$$
From Hölder’s inequality and Lemma 2.2, it is easy to deduce that
\[
\frac{d}{dt} |\tilde{\psi}|_6^6 \leq C \left( \sum_{l=1}^{2} |\partial_l A_l(u_1)|_{\infty} + |B(u_1)|_{\infty} \right) |\tilde{\psi}|_6^6 + |\nabla^2 \tilde{\psi}|_6^6 + C(|\tilde{\Psi}|_6 + |\tilde{\Psi}|^6_6). \tag{4.5}
\]
From Hölder’s inequality, we easily have
\[
|\tilde{\Psi}|_6 + |\tilde{\Psi}|^6_6 \leq C(\|\tilde{\psi}\|_{\infty}\|\nabla\tilde{\psi}\|_6 + \|\tilde{\psi}\|_6 \|\nabla \tilde{\psi}\|_6) \leq C\|\tilde{\psi}\|_{L^6_0\cap D^1_\delta^2}\|\tilde{\psi}\|_2, \tag{4.6}
\]
and
\[
\frac{d}{dt} |\tilde{\psi}|_6^6 \leq B_\eta(t)|\tilde{\psi}(t)|_6^6 + C|\tilde{\eta}(t)|^2_2 + \eta\|\nabla \tilde{\psi}(t)\|_2^2, \tag{4.7}
\]
where
\[
\tilde{\Psi} = \tilde{\psi} + \sum_{l=1}^{2} A_l(u_1) \partial_l \tilde{\psi} + B(u_1) \partial^2 \tilde{\psi} + \partial^2 \nabla \tilde{\psi}^k
\]
\[
= \tilde{\Psi}_1 + \tilde{\Psi}_2 + \Psi_1 + \Psi_2,
\]
where
\[
\Psi_1 = \sum_{l=1}^{2} \left( A_l(u_1) \partial_l \tilde{\psi} - \partial^2_l (A_l(u_1) \partial_l \tilde{\psi}) \right), \quad \Psi_2 = B(u_1) \partial^2 \tilde{\psi} - \partial^2_l (B(u_1) \tilde{\psi}).
\]
Multiplying (4.8) by \(2\partial^2 \tilde{\psi}\) and integrating over \(\mathbb{R}^2\), we have
\[
\frac{d}{dt} |\partial^2 \tilde{\psi}|_2^2 \leq C \left( \sum_{l=1}^{2} |\partial_l A_l(u_1)|_{\infty} + |B(u_1)|_{\infty} \right) |\partial^2 \tilde{\psi}|_2^2 + |\nabla^3 \tilde{\psi}|_2^2 + C(|\partial^2 \tilde{\Psi}|_2^2 + |\tilde{\Psi}|_2^2) \tag{4.9}
\]
From Hölder’s inequality and Lemma 2.2, it is easy to deduce that
\[
|\partial^2 \tilde{\Psi}|_2^2 + |\partial^2 \tilde{\Psi}|_2^2 \leq C\|\tilde{\psi}\|_{L^6_0\cap D^1_\delta^2}\|\nabla \tilde{\psi}\|_1, \quad |\Psi_1|_2 + |\Psi_2|_2 \leq C|\nabla \tilde{\psi}|_2 \|\nabla \tilde{u}\|_3. \tag{4.10}
\]
From (4.7) - (4.10), we have
\[
\frac{d}{dt} |\tilde{\psi}(t)|_{L^6_0\cap D^1}^2 \leq |\Psi_\eta(t)||\tilde{\psi}(t)|_{L^6_0\cap D^1}^2 + C|\tilde{\rho}(t)|^2_2 + \eta\|\nabla \tilde{\psi}(t)\|_2^2,
\]
\[
\int_0^t \Psi_\eta(s)ds \leq \tilde{C} + \tilde{C}_\eta t \quad \text{for} \quad t \in [0, T_*]. \tag{4.11}
\]
Similarly to the derivation of (3.95), we easily have
\[
\frac{d}{dt} \|\tilde{\psi}(t)\|^2_1 + \|\nabla \tilde{\psi}(t)\|_1^2
\]
\[
\leq E_\eta(t)\|\tilde{\psi}(t)\|^2_1 + E_2(t)\|\tilde{\psi}(t)\|^2_1 + E_3(t)\|\tilde{\psi}(t)\|^2_1 + \eta\|\nabla \tilde{\psi}(t)\|_2^2,
\]
where we have \(\int_0^t (E_\eta(s) + E_2(s) + E_3(s))ds \leq \tilde{C} + \tilde{C}_\eta t \) for \(t \in (0, T_*)\).
Taking $\partial_2^\xi$ to (4.13) ($|\xi| = 2$), multiplying by $\partial_2^\xi \tilde{u}$ and integrating over $\mathbb{R}^2$, we have
\[
\frac{1}{2} \frac{d}{dt} |\partial_2^\xi \tilde{u}|^2 + \alpha |\nabla \partial_2^\xi \tilde{u}|^2 + (\alpha + \beta)|\partial_2^\xi \text{div}\tilde{u}|^2
\]
\[
= \int_{\mathbb{R}^2} \left( -\partial_2^\xi (u_1 \cdot \nabla \tilde{u}) - \partial_2^\xi (\tilde{u} \cdot \nabla u_2) - \theta \partial_2^\xi (\nabla (\phi_1^2 - \phi_2^2)) \right) \cdot \partial_2^\xi \tilde{u} dx
\]
\[
+ \int_{\mathbb{R}^2} \left( \partial_2^\xi (\psi_1 \cdot Q(\tilde{u})) + \partial_2^\xi (\tilde{\psi} \cdot Q(u_2)) \right) \cdot \partial_2^\xi \tilde{u} dx = \sum_{i=6}^{10} J_i.
\]
The right-hand side can be estimated term by term as follows.

- $J_6 = \int_{\mathbb{R}^2} -\partial_2^\xi (u_1 \cdot \nabla \tilde{u}) \cdot \partial_2^\xi \tilde{u} dx$

- $J_7 = \int_{\mathbb{R}^2} -\partial_2^\xi (\tilde{u} \cdot \nabla u_2) \cdot \partial_2^\xi \tilde{u} dx$

- $J_8 = \int_{\mathbb{R}^2} -\theta \partial_2^\xi (\nabla (\phi_1^2 - \phi_2^2)) \cdot \partial_2^\xi \tilde{u} dx$

\[
\leq C(\nabla^2 u_1|\nabla^2 \tilde{u}|_2 + C|u_1|_{\infty} |\nabla^3 \tilde{u}|_2 |\nabla^2 \tilde{u}|_2 + C|\nabla u_1|_{\infty} |\nabla^2 \tilde{u}|_2^2,
\]

- $J_9 = \int_{\mathbb{R}^2} -\partial_2^\xi (\psi_1 \cdot Q(\tilde{u})) \cdot \partial_2^\xi \tilde{u} dx$

\[
\leq C(\nabla(\phi_1 + \phi_2)|_{\infty} |\nabla^2 \tilde{u}|_2 + |(\phi_1 + \phi_2)|_{\infty} |\nabla^2 \tilde{u}|_2 + |\nabla^2 \phi_1 + \phi_2|_{\infty} |\nabla^3 \tilde{u}|_2,
\]

- $J_{10} = \int_{\mathbb{R}^2} -\partial_2^\xi (\tilde{\psi} \cdot Q(u_2)) \cdot \partial_2^\xi \tilde{u} dx$

\[
\leq C(\nabla^3 \tilde{u}|_2 |\nabla^2 \tilde{u}|_2 + C|\nabla^2 \psi_1|_{\infty} |\nabla^3 \tilde{u}|_2 + C|\nabla^2 \psi|_{\infty} |\nabla^3 \tilde{u}|_2 |\nabla^2 \tilde{u}|_2,
\]

According to Young’s inequality and (4.13), we have
\[
\left\{ \begin{array}{l}
\frac{d}{dt} |\tilde{u}(t)|^2_{D^2} + \alpha |\tilde{u}(t)|^2_{D^3} \leq F_\eta(t) ||\tilde{u}(t)||^2_{D^2} + F_2(t) ||\tilde{\phi}(t)||^2_{D^3} + F_3(t) ||\tilde{\psi}(t)||^2_{L^6 \cap D^3}, \\
\int_0^t (F_\eta(s) + F_2(s) + F_3(s)) ds \leq \tilde{C} + \tilde{C}_\eta t \quad \text{for} \quad t \in (0, T_*).
\end{array} \right.
\]

Then combining (4.12) and (4.14), we easily have
\[
\left\{ \begin{array}{l}
\frac{d}{dt} ||\tilde{u}(t)||^2_{D^2} + \alpha ||\nabla \tilde{u}(t)||^2_{D^3} \leq \Theta_\eta(t) ||\tilde{u}(t)||^2_{D^2} + \Theta_2(t) ||\tilde{\phi}(t)||^2_{D^3} + \Theta_3(t) ||\tilde{\psi}(t)||^2_{L^6 \cap D^3}, \\
\int_0^t (\Theta_\eta(s) + \Theta_2(s) + \Theta_3(s)) ds \leq \tilde{C} + \tilde{C}_\eta t \quad \text{for} \quad t \in (0, T_*).
\end{array} \right.
\]

Finally, let
\[
\Gamma(t) = ||\tilde{\phi}(t)||^2_2 + ||\tilde{\psi}(t)||^2_{L^6 \cap D^3} + ||\tilde{u}(t)||^2_2.
\]

Then we have
\[
\frac{d}{dt} \Gamma(t) + \mu ||\nabla \tilde{u}(t)||^2_2 \leq \Pi_\eta(t) \Gamma(t)
\]
for some $\Pi'_n$ such that $\int_0^t \Pi'_n(s)ds \leq \bar{C} + \bar{C}_\eta t$ for $t \in (0, T)$. Then our stability result follows from Gronwall’s inequality.

5. Blow-up criterion

In this section, we give the proof to Theorem 1.3. In order to prove (1.13), we use a contradiction argument. Let $(\rho, u)$ be the unique regular solution to the Cauchy problem (1.1)-(1.3) with the maximal existence time $T$. We assume that $T < T$ and $\lim_{T \to T} \sup_{0 \leq t \leq T} |\nabla \rho(x)| \leq C_0 < +\infty$. (5.1)

We will show that under assumption (5.1), $T$ is actually not the maximal existence time for the regular solution.

From the definition of regular solutions, we know that, for $\phi = \rho^{\gamma - \frac{1}{2}}$, $(\phi, u)$ satisfies

$$\begin{cases}
\phi_t + u \cdot \nabla \phi + \frac{\gamma - 1}{2} \rho \div \mathbf{u} = 0, \\
\psi_t + \nabla (u \cdot \psi) + \nabla \div \mathbf{u} = 0, \\
u_t + u \cdot \nabla u + 2\theta \phi \nabla \phi + Lu = \psi \cdot Q(u).
\end{cases}$$

(5.2)

For (5.2), we have the equivalent form

$$\psi_t + \sum_{i=1}^2 A_i \partial_i \psi + B \psi + \nabla \div \mathbf{u} = 0.$$  

(5.3)

Here $A_i = (a^{(i)}_{ij})_{2 \times 2}$ $(i, j, l = 1, 2)$ are symmetric with $a^{(i)}_{ij} = u^{(i)}$ when $i = j$; and $a^{(i)}_{ij} = 0$, otherwise. $B = (\nabla \mathbf{u})^\top$, so (5.3) is a positive symmetric hyperbolic system.

By assumptions (5.1) and (5.2), we first show that density $\rho$ is uniformly bounded.

**Lemma 5.1.** Let $(\rho, u)$ be the unique regular solution to the Cauchy problem (1.1)-(1.3) on $[0, T]$ satisfying (5.1). Then

$$\|\rho\|_{L^\infty([0, T] \times \mathbb{R}^2)} + \|\phi\|_{L^\infty([0, T]; L^q(\mathbb{R}^2))} \leq C, \quad 0 \leq T < T,$$

where $C > 0$ depends on $C_0$, constant $q \in [2, +\infty]$ and $T$.

**Proof.** First, it is obvious that $\phi$ can be represented by

$$\phi(t, x) = \phi_0(W(0, t, x)) \exp \left( -\frac{\gamma - 1}{2} \int_0^t \div (s, W(s, t, x))ds \right),$$

(5.4)

where $W \in C^1([0, T] \times [0, T] \times \mathbb{R}^2)$ is the solution to the initial value problem

$$\begin{cases}
\frac{d}{dt} W(t, s, x) = u(t, W(t, s, x)), \quad 0 \leq t \leq T, \\
W(s, s, x) = x, \quad 0 \leq s \leq T, \quad x \in \mathbb{R}^2.
\end{cases}$$

(5.5)

Then it is clear that $\|\phi\|_{L^\infty([0, T] \times \mathbb{R}^2)} \leq |\phi_0|_\infty \exp(CC_0)$. 
Next, multiplying (5.2)_1 by 2φ and integrating over \( \mathbb{R}^2 \), we get
\[
\frac{d}{dt}|\phi|^2_2 \leq C|\text{div}u|_\infty |\phi|^2_2,
\] (5.6)
from (5.1) and Gronwall’s inequality, we immediately obtain the desired conclusions. \( \square \)

Now we give the basic energy estimates.

**Lemma 5.2.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem (1.1)-(1.3) on \([0, T)\) satisfying (5.1). Then
\[
\sup_{0 \leq t \leq T} |u(t)|^2_2 + \int_0^T |\nabla u(t)|^2_2 \, dt \leq C,
\]
where \( C \) only depends on \( C_0 \) and \( T \).

**Proof.** Multiplying (5.2)_3 by 2u and integrating over \( \mathbb{R}^2 \), we have
\[
\frac{d}{dt}|u|^2_2 + \alpha|\nabla u|^2_2 + (\alpha + \beta)|\text{div}u|^2_2
= \int_{\mathbb{R}^2} 2\left(-u \cdot \nabla u \cdot u - \theta \nabla \phi \cdot u + \psi \cdot Q(u) \cdot u\right) \, dx \equiv: L_1 + L_2 + L_3.
\] (5.7)
The right-hand side terms can be estimated as follows.
\[
\begin{align*}
L_1 &= -\int_{\mathbb{R}^2} 2u \cdot \nabla u \cdot u \, dx \leq C|\text{div}u|_\infty |u|^2_2, \\
L_2 &= \int_{\mathbb{R}^2} \theta \phi^2 |\text{div}u|^2_2 \leq C|\phi|^2_2 |\text{div}u|_\infty \leq C|\text{div}u|_\infty, \\
L_3 &= \int_{\mathbb{R}^2} 2\psi \cdot Q(u) \cdot u \, dx \leq C|\psi|_6 |\nabla u|^2_2 |u|^3_3 \\
&\leq \frac{\alpha}{4}|\nabla u|^2_2 + C|\psi|_6 |u|^2_2 |\nabla u|^2_2 \leq \frac{\alpha}{2}|\nabla u|^2_2 + C|\psi|_6 |u|^2_2,
\end{align*}
\] (5.8)
where we have used the fact \(|u|^3_3 \leq C|u|^2_2 |\nabla u|^\frac{1}{2}_2\). (5.7) and (5.8) yield
\[
\frac{d}{dt}|u|^2_2 + \frac{\alpha}{2}|\nabla u|^2_2 \leq C(|\text{div}u|_\infty + 1)|u|^2_2 + C|\text{div}u|_\infty.
\] (5.9)
According to Gronwall’s inequality, we have
\[
|u(t)|^2_2 + \int_0^t |\nabla u(s)|^2_2 \, ds \leq C, \quad 0 \leq t \leq T.
\] (5.10) \( \square \)

The next lemma is a key estimate on \( \nabla \phi \) and \( \nabla u \). We denote \( \omega = \partial_{x_1} u^{(2)} - \partial_{x_2} u^{(1)} \).

**Lemma 5.3.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem (1.1)-(1.3) on \([0, T)\) satisfying (5.1). Then
\[
\sup_{0 \leq t \leq T} |\nabla u(t)|^2_2 + \sup_{0 \leq t \leq T} |\nabla \phi(t)|^2_2 + \int_0^T (|\nabla^2 u|^2_2 + |u_t|^2_2) \, dt \leq C,
\]
where \( C \) only depends on \( C_0 \) and \( T \).
Proof. First, multiplying (5.2) by $-Lu - \theta\nabla\phi^2$ and integrating over $\mathbb{R}^2$, we have

$$\frac{1}{2} \frac{d}{dt} \left( \alpha |\nabla u|^2_{L^2} + (\alpha + \beta) |\operatorname{div} u|^2_{L^2} \right) + \int_{\mathbb{R}^2} (-Lu - \theta\nabla\phi^2)^2 dx = -\alpha \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \omega' dx + (2\alpha + \beta) \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \nabla \operatorname{div}u dx$$

$$- \theta \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \nabla \phi^2 dx - \theta \int_{\mathbb{R}^2} u \cdot \nabla \phi^2 dx + \theta \int_{\mathbb{R}^2} (\psi \cdot Q(u)) \cdot \nabla \phi^2 dx$$

$$- \int_{\mathbb{R}^2} (\psi \cdot Q(u)) \cdot (\alpha \Delta u + (\alpha + \beta) \nabla \operatorname{div}u) dx \equiv \sum_{i=4}^{9} L_i,$$

where we have used the fact that $-\Delta u + \nabla \operatorname{div}u = (\partial_{x_2} \omega, -\partial_{x_1} \omega)^T = \omega'$.

From the standard elliptic estimate shown in Lemma 2.4, we have

$$|\nabla^2 u|^2_{L^2} \leq C |\theta \nabla\phi^2|^2_{L^2} \leq C |\alpha \Delta u + (\alpha + \beta) \nabla \operatorname{div}u|^2_{L^2} - C |\theta \nabla\phi^2|^2_{L^2},$$

(5.12)

Now we estimate the right-hand side of (5.11) term by term. According to

$$\frac{1}{2} \nabla(|u|^2) - u \cdot \nabla u = (u^{(2)} \omega, -u^{(1)} \omega)^T = \omega'',$$

and Hölder’s inequality, Gagliardo-Nirenberg inequality and Young’s inequality, we obtain

$$|L_4| = \alpha \left| \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \omega' dx \right| = \alpha \left| \int_{\mathbb{R}^2} \left( \frac{1}{2} \nabla(|u|^2) - \omega'' \right) \cdot \omega' dx \right|$$

$$= \alpha \left| \int_{\mathbb{R}^2} -\omega'' \cdot \omega' dx \right| = \alpha \left| \int_{\mathbb{R}^2} u^{(2)} \partial_{x_2} \omega^2 + u^{(1)} \partial_{x_1} \omega^2 dx \right|$$

$$= \alpha \left| \int_{\mathbb{R}^2} \omega^2 \operatorname{div}u dx \right| \leq C |\operatorname{div}u|_{L^\infty} |\nabla u|^2_{L^2},$$

$$|L_6| = \theta \left| \int_{\mathbb{R}^2} (u \cdot \nabla u) \cdot \nabla \phi^2 dx \right|$$

$$= \theta \left| \int_{\mathbb{R}^2} \nabla u : (\nabla u)^\top \phi^2 dx - \int_{\mathbb{R}^2} \phi^2 u \cdot \nabla (\operatorname{div}u) dx \right|$$

$$= \theta \left| \int_{\mathbb{R}^2} \nabla u : (\nabla u)^\top \phi^2 dx + \int_{\mathbb{R}^2} (\operatorname{div}u)^2 \phi^2 dx + \int_{\mathbb{R}^2} u \cdot \nabla \phi^2 \operatorname{div}u dx \right|$$

$$\leq C |\nabla u|^2_{L^2} + C |\operatorname{div}u|_{L^\infty} |u|^2_{L^2} |\nabla \phi^2|^2_{L^2}$$

$$\leq C (|\nabla u|^2_{L^2} + |\operatorname{div}u|_{L^\infty} + |\operatorname{div}u|_{L^\infty} |\nabla \phi|^2_{L^2}),$$

$$\leq C (|\nabla u|^2_{L^2} + |\operatorname{div}u|_{L^\infty} + |\operatorname{div}u|_{L^\infty} |\nabla \phi|^2_{L^2}),$$
\[ L_7 = -\theta \int_{\mathbb{R}^2} u_t \cdot \nabla \phi^2 dx = \theta \frac{d}{dt} \int_{\mathbb{R}^2} \phi^2 dx - \theta \int_{\mathbb{R}^2} (\phi^2)_t dx \]

\[ = \theta \frac{d}{dt} \int_{\mathbb{R}^2} \phi^2 dx + \theta \int_{\mathbb{R}^2} u \cdot \nabla \phi^2 dx + \theta(\gamma - 1) \int_{\mathbb{R}^2} \phi^2 dx \]

\[ \leq \theta \frac{d}{dt} \int_{\mathbb{R}^2} \phi^2 dx + C(|\nabla u|^2_2 + |\text{div} u|_\infty + |\text{div} u|_\infty |\nabla \phi|^2_2), \quad (5.14) \]

\[ L_0 = -\int_{\mathbb{R}^2} (\psi \cdot Q(u)) \cdot (\alpha \Delta u + (\alpha + \beta) \nabla \text{div} u) dx \]

\[ \leq C |\psi|_6 |\nabla^2 u|^2_2 |\nabla u|^2_2 \leq C(\varepsilon) |\nabla u|^2_2 + \varepsilon |\nabla^2 u|^2_2, \]

where \( \varepsilon > 0 \) is a sufficiently small constant. Combining \((5.11)-(5.14)\), we have

\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (\alpha |\nabla u|^2 + (\alpha + \beta) |\text{div} u|^2 - \theta \phi^2 |\text{div} u|^2) dx + C |\nabla^2 u|^2_2 \]

\[ \leq C(|\nabla u|^2_2 + |\nabla \phi|^2_2)(|\text{div} u|_\infty + 1) + |\text{div} u|_\infty). \quad (5.15) \]

Second, applying \( \nabla \) to \((5.21)\) and multiplying by \( (\nabla \phi)^T \), we have

\[ (|\nabla \phi|^2)_t + \text{div}(|\nabla \phi|^2 u) + (\gamma - 2) |\nabla \phi|^2 \text{div} u \]

\[ = -2(\nabla \phi)^T \nabla u(\nabla \phi) - (\gamma - 1) \phi \nabla \phi \cdot \nabla \text{div} u \]

\[ = -2(\nabla \phi)^T D(u)(\nabla \phi) - (\gamma - 1) \phi \nabla \phi \cdot \nabla \text{div} u. \quad (5.16) \]

Integrating \((5.16)\) over \( \mathbb{R}^2 \), we get

\[ \frac{d}{dt} |\nabla \phi|^2_2 \leq C(\varepsilon)(|D(u)|_\infty + 1)|\nabla \phi|^2_2 + \varepsilon |\nabla^2 u|^2_2. \quad (5.17) \]

Adding \((5.17)\) to \((5.15)\), from Gronwall’s inequality we immediately obtain

\[ |\nabla u(t)|^2_2 + |\nabla \phi(t)|^2_2 + \int_0^t |\nabla^2 u(s)|^2_2 dt \leq C, \quad 0 \leq t \leq T. \]

Finally, due to \( u_t = -Lu - u \cdot \nabla u - 2\theta \phi \nabla \phi + \psi \cdot Q(u) \), we deduce that

\[ \int_0^t |u_t|^2_2 dt \leq C \int_0^t \left( |\nabla^2 u|^2_2 + |\nabla u|^2_2 |u|^2_0 + |\phi|^2 |\nabla \phi|^2_2 + |\nabla u|^2_2 |\psi|^2_0 \right) dt \leq C. \]

Next, we proceed to improve the regularity of \( \phi, \psi \) and \( u \). To this end, we first drive some bounds on derivatives of \( u \) based on the above estimates.

**Lemma 5.4.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem \((1.1)-(1.3)\) on \([0, T)\) satisfying \((5.1)\). Then

\[ \sup_{0 \leq t \leq T} |u(t)|^2_2 + \sup_{0 \leq t \leq T} |u(t)|_{D^2} + \int_0^T |u(t)|^2_2 dt \leq C, \quad 0 \leq T < \overline{T}, \quad (5.18) \]

where \( C \) only depends on \( C_0 \) and \( T \).
Proof. Using \( Lu = -u_t - u \cdot \nabla u - 2\theta \psi \nabla \phi + \psi \cdot Q(u) \) and Lemma 2.4, we have

\[
|u|_{D^2} \leq C(|u_t|_2 + |u_0| \nabla u_2^{\frac{2}{3}} |\nabla^2 u|^{\frac{1}{3}} + |\phi|_{\infty} |\nabla \phi|_2 + |\psi|_6 \nabla u_2^{\frac{2}{3}} |\nabla^2 u|^{\frac{1}{3}}),
\]

which implies, from Young’s inequality with appropriate weights, that

\[
|u|_{D^2} \leq C(|u_t|_2 + |u_0| \nabla u_2^{\frac{3}{4}} |\nabla \phi|_2 + |\nabla u|_2) \leq C(1 + |u_t|_2).
\]

Next, differentiating (5.21) with respect to \( t \), it reads

\[
u_{tt} + Lu_t = -(u \cdot \nabla u)_t - 2\theta (\phi \nabla \phi)_t + (\psi \cdot Q(u))_t.
\]

Multiplying (5.21) by \( u_t \) and integrating over \( \mathbb{R}^2 \), one has

\[
\frac{1}{2} \frac{d}{dt} |u_t|^2 + \alpha |\nabla u_t|^2 + (\alpha + \beta) |\text{div} u_t|^2
\]

\[
= \int_{\mathbb{R}^2} \left(- (u \cdot \nabla u)_t \cdot u_t - 2\theta (\phi \nabla \phi)_t \cdot u_t + (\psi \cdot Q(u))_t \cdot u_t \right) dx \equiv \sum_{i=10}^{12} L_i.
\]

We estimate the right-hand side of (5.22) term by term as follows.

\[
L_{10} = - \int_{\mathbb{R}^2} (u \cdot \nabla u)_t \cdot u_t dx = - \int_{\mathbb{R}^2} \left((u_t \cdot \nabla u) \cdot u_t + (u \cdot \nabla u_t) \cdot u_t \right) dx
\]

\[
= - \int_{\mathbb{R}^2} (u_t \cdot D(u) \cdot u_t - (u_t)^2 \text{div} u_t) dx \leq C|D(u)|_{\infty} |u_t|^2,
\]

\[
L_{11} = - \int_{\mathbb{R}^2} 2\theta (\phi \nabla \phi)_t \cdot u_t dx = \theta \int_{\mathbb{R}^2} (\phi^2)_t \text{div} u_t dx
\]

\[
= - \frac{\theta(\gamma - 1)}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \phi^2 (\text{div} u)^2 dx - \frac{\theta(\gamma - 1)}{2} \int_{\mathbb{R}^2} u \phi^2 \nabla (\text{div} u)^2 dx
\]

\[
- \frac{\theta(\gamma - 1)^2}{2} \int_{\mathbb{R}^2} \phi^2 (\text{div} u)^3 dx - \theta \int_{\mathbb{R}^2} u \cdot \nabla \phi^2 \text{div} u_t dx
\]

\[
\leq - \frac{\theta(\gamma - 1)}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \phi^2 (\text{div} u)^2 dx + C|u|_{\infty} |\phi|_{\infty}^2 |\nabla u_2| |\nabla^2 u|_2
\]

\[
+ C|\phi|_{\infty}^2 |D(u)|_{\infty} |\nabla u_2|^2 + C|\phi|_{\infty} |\nabla \phi|_2 |u|_{\infty} |\nabla u_t|_2
\]

\[
\leq - \frac{\theta(\gamma - 1)}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \phi^2 (\text{div} u)^2 dx + \frac{\alpha}{4} |u_t|_2^2 + C(1 + |D(u)|_{\infty} + |u_2|_{D^2}),
\]

\[
L_{12} = \int_{\mathbb{R}^2} (\psi \cdot Q(u))_t \cdot u_t dx = \int_{\mathbb{R}^2} \psi \cdot Q(u)_t \cdot u_t dx
\]

\[
- \int_{\mathbb{R}^2} \nabla \text{div} u \cdot Q(u) \cdot u_t dx + \int_{\mathbb{R}^2} \psi \cdot u \text{div} (Q(u) \cdot u_t) dx
\]

\[
\leq C|\psi|_6 |\nabla u_2| |u_t|_2^{\frac{4}{3}} |\nabla u_2|^{\frac{1}{3}} + C|\nabla^2 u_2| |Q(u)|_{\infty} |u_t|_2
\]

\[
+ C|\psi|_6 |u_6| |\nabla^2 u_2| |u_t|_6 + C|\psi|_6 |u_6| |Q(u)|_6 |\nabla u_t|_2
\]

\[
\leq \frac{\alpha}{10} |\nabla u_t|_2^2 + C(1 + |D(u)|_{\infty}) (|u_t|_2^2 + |u_2|_{D^2}).
\]
It is clear from (5.22)-(5.23) and (5.1) that
\[
\frac{d}{dt}(\|u_t\|^2 + |\phi|^2) + |\nabla u_t|^2 \\
\leq C(1 + |D(u)|_\infty)\|u_t\|^2 + C(1 + |D(u)|_\infty).
\] (5.24)

Integrating (5.24) over \((\tau, t)\) \((\tau \in (0, t))\), we have
\[
\|u_t(t)\|^2 + |\phi u_t|^2 + \int_\tau^t |\nabla u_t(s)|^2 ds \\
\leq \|u_t(\tau)\|^2 + |\phi u_t|^2 + C \int_\tau^t (|D(u)|_\infty + 1)\|u_t\|^2 (s) ds + C.
\] (5.25)

From the momentum equations (5.23), we obtain
\[
\|u_t(t)\|^2 \leq C(|u_\infty|\|\nabla u_0\|_2 + |\phi_\infty|\|\nabla \phi_0\|_2 + |u_0|D^2 + |\psi_0|\|\nabla u_0\|_2)(\tau),
\] (5.26)
which, together with (5.10), gives
\[
\limsup_{\tau \to 0} \|u_t(\tau)\|_2 \leq C(|u_\infty|\|\nabla u_0\|_2 + |\phi_\infty|\|\nabla \phi_0\|_2 + |u_0|D^2 + |\psi_0|\|\nabla u_0\|_2) \leq C_0.
\] (5.27)

Letting \(\tau \to 0\) in (5.25), applying Gronwall’s inequality, we arrive at
\[
\|u_t(t)\|^2 + \int_0^t |\nabla u_t(s)|^2 ds + |u(t)|D^2 \leq C, \quad 0 \leq t \leq T.
\]
This completes the proof of this lemma. \qed

The following lemma gives bounds of \(\nabla \phi\) and \(\nabla^2 u\).

**Lemma 5.5.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem (1.1)-(1.3) on \([0, T]\) satisfying (5.1). Then
\[
\sup_{0 \leq t \leq T} \|\phi(t)\|_{W^{1,6}} + \sup_{0 \leq t \leq T} |\phi_t(t)|_6 + \int_0^T |u(t)|_{D^{2,6}} dt \leq C, \quad 0 \leq T < \bar{T},
\] (5.28)
where \(C\) only depends on \(C_0\) and \(T\).

**Proof.** First, using Lemma 2.4, we read from \(Lu = -u_t - u \cdot \nabla u - 2\theta \phi \nabla \phi + \psi \cdot Q(u)\) that
\[
|\nabla^2 u|_6 \leq C(|u_6| + |u \cdot \nabla u|_6 + |\phi \nabla \phi|_6 + |\psi \cdot Q(u)|_6)
\leq C(1 + |\nabla u|_2 + |\nabla \phi|_6 + |D(u)|^{\frac{8}{5}} |\nabla D(u)|^{\frac{8}{5}}),
\] (5.29)
where we have used the fact that \(|D(u)|_\infty \leq C|D(u)|^{\frac{8}{5}} |\nabla D(u)|^{\frac{8}{5}}\). Now, Young’s inequality implies that
\[
|\nabla^2 u|_6 \leq C(1 + |\nabla u|_2 + |\nabla \phi|_6).
\] (5.30)

Next, applying \(\nabla\) to (5.21), multiplying the result equations by \(6|\nabla \phi|^4 \nabla \phi\), we have
\[
(|\nabla \phi|^4)_t + \text{div}(|\nabla \phi|^6 u) + (3\gamma - 4) |\nabla \phi|^6 \text{div} u
\]
\[
= -6|\nabla \phi|^4 (\nabla \phi)^T \nabla u (\nabla \phi) - (3\gamma - 3) \phi |\nabla \phi|^4 \nabla \phi \cdot \nabla \text{div} u
\]
\[
= -6|\nabla \phi|^4 (\nabla \phi)^T D(u)(\nabla \phi) - (3\gamma - 3) \phi |\nabla \phi|^4 \nabla \phi \cdot \nabla \text{div} u,
\] (5.31)
which implies, upon integrating over \( \mathbb{R}^2 \), that

\[
\frac{d}{dt} \| \nabla \phi \|_6 \leq C(\| D(u) \|_\infty) \| \nabla \phi \|_6 + C \| \nabla^2 u \|_6. \tag{5.32}
\]

Therefore, we obtain from \((5.30)\) that

\[
\frac{d}{dt} \| \nabla \phi \|_6 \leq C(1 + \| D(u) \|_\infty) \| \nabla \phi \|_6 + C(1 + \| \nabla u_t \|_2^2). \tag{5.33}
\]

In view of Lemma 5.4 and \((5.1)\), we apply Gronwall’s inequality to conclude that

\[
\| \nabla \phi(t) \|_6 \leq C(1 + \| \nabla \phi_0 \|_6) \exp \left( \int_0^t (1 + \| D(u) \|_\infty) ds \right) \leq C, \quad 0 \leq t \leq T.
\]

Finally, we infer from \((5.30)\) and Lemma 5.4 that

\[
\int_0^t \| u(s) \|_{D^2_x}^2 ds \leq C \int_0^t (1 + \| \nabla \phi(s) \|_6^2 + \| \nabla u_t(s) \|_2^2) ds \leq C, \quad 0 \leq t \leq T. \tag{5.34}
\]

The proof of the lemma is finished. \( \square \)

Lemma 5.5 implies that

\[
\int_0^t \| \nabla u(\cdot, s) \|_\infty ds \leq C, \tag{5.35}
\]

for any \( t \in [0, T) \) with \( C > 0 \) a finite number. Noting that \((5.2)\) is essentially a parabolic-hyperbolic system, it is then standard to derive other higher order estimates for the regularity of the regular solutions. We will show this fact in the following 4 lemmas.

**Lemma 5.6.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem \((1.1)-(1.3)\) on \([0, T)\) satisfying \((5.1)\). Then

\[
\sup_{0 \leq t \leq T} \| \phi(t) \|_{D^2}^2 + \sup_{0 \leq t \leq T} \| \psi(t) \|_{D^1}^2 + \sup_{0 \leq t \leq T} \| \phi_t(t) \|_{1}^2 + \sup_{0 \leq t \leq T} \| \psi_t(t) \|_{2}^2
\]

\[+\int_0^T \left( \| u(t) \|_{D^3}^2 + \| \phi_{tt}(t) \|_{2}^2 \right) dt \leq C, \quad 0 \leq T < T, \]

where \( C \) only depends on \( C_0 \) and \( T \).

**Proof.** Using \( Lu = -u_t - u \cdot \nabla u - 2 \theta \phi \nabla \phi + \psi \cdot Q(u) \) and Lemma 2.4, we have

\[
\| u \|_{D^3} \leq C(\| u_t \|_{D^1} + \| u \cdot \nabla u \|_{D^1} + \| \phi \nabla \phi \|_{D^1} + \| \psi \cdot Q(u) \|_{D^1})
\]

\[\leq C(1 + \| u_t \|_{D^1} + \| \phi \|_{D^2} + \| \psi \|_6 \| \nabla^2 u \|_3 + \| \nabla \psi \|_2 \| D(u) \|_\infty)
\]

\[\leq C(1 + \| u_t \|_{D^1} + \| \phi \|_{D^2} + \| \psi \|_6 \| \nabla^2 u \|_2^\frac{2}{3} \| \nabla^3 u \|_2^\frac{1}{3} + \| \nabla \psi \|_2 \| D(u) \|_6^\frac{2}{3} \| \nabla D(u) \|_6^\frac{1}{3}), \tag{5.36}
\]

where we have used the fact that \( \| D(u) \|_\infty \leq C \| D(u) \|_6^\frac{2}{3} \| \nabla D(u) \|_6^\frac{1}{3} \). With the help of Young’s inequality, \((5.36)\) offers that

\[
\| u \|_{D^3} \leq C(1 + \| u_t \|_{D^1} + \| \phi \|_{D^2} + \| D(u) \|_\infty \| \nabla \psi \|_2). \tag{5.37}
\]
Next, applying $\partial_i$ $(i = 1, 2)$ to $(5.2)_2$ with respect to $x$, we obtain

$$
(\partial_i \psi)_t + \sum_{l=1}^{2} A_l \partial_l \partial_i \psi + B \partial_i \psi + \partial_i \nabla \text{div} u
$$

$$
= \left( - \partial_i (B \psi) + B \partial_i \psi \right) + \sum_{l=1}^{2} \left( - \partial_i (A_l) \partial_l \psi \right) = \Theta'_1 + \Theta'_2.
$$

(5.38)

Multiplying $(5.38)$ by $2(\partial_i \psi)^T$, integrating over $\mathbb{R}^2$, and then summing over $i$, noting that $A_l$ ($l = 1, 2$) are symmetric, it is not difficult to show that

$$
\frac{d}{dt} |\nabla \psi|_2^2 \leq C|\nabla A|_\infty |\nabla \psi|_2^2 + \int_{\mathbb{R}^2} 2 \sum_{i=1}^{2} (\partial_i \psi)^T (\nabla u)^T (\partial_i \psi) + C|\nabla^3 u|_2 |\nabla \psi|_2
$$

$$
+ C|\Theta'_1|_2 |\nabla \psi|_2 + 2 \int_{\mathbb{R}^2} \sum_{i=1}^{2} (\partial_i \psi)^T \Theta'_2 dx,
$$

(5.39)

where $\nabla A = \sum_{l=1}^{2} \partial_l A_l$. We treat each term on the right-hand side of the above inequality as follows. From the definition of matrices $A_l$ and $B$, it is clear that

$$
|\nabla A|_\infty |\nabla \psi|_2^2 + \int_{\mathbb{R}^2} 2 \sum_{i=1}^{2} (\partial_i \psi)^T (\nabla u)^T (\partial_i \psi) \leq C|\nabla u|_\infty |\nabla \psi|_2^2.
$$

(5.40)

When $|\zeta| = 1$, choosing $r = 2$, $a = 3$, $b = 6$ in $(2.3)$, we have

$$
|\Theta'_1|_2 = |D^\zeta (B \psi) - BD^\zeta \psi|_2 \leq C|\nabla^2 u|_3 |\psi|_6.
$$

(5.41)

For the last term on the right-hand side of $(5.39)$, we have

$$
2 \int_{\mathbb{R}^2} \sum_{i=1}^{2} (\partial_i \psi)^T \Theta'_2 dx \leq C|\nabla u|_\infty |\nabla \psi|_2^2.
$$

(5.42)

Combining $(5.37)$, $(5.39)$-$(5.42)$, and using Gagliardo-Nirenberg inequality, we have

$$
\frac{d}{dt} |\psi|_{D^1}^2 \leq C(1 + |\nabla u|_\infty) |\psi|_{D^1}^2 + C(1 + |\nabla^3 u|_2) |\psi|_{D^1} + C
$$

$$
\leq C(1 + |\nabla u|_\infty) |\psi|_{D^1}^2 + C(1 + |\phi|_{D^2} + |\nabla u|_2^2).
$$

(5.43)

On the other hand, let $\nabla \phi = G = (G^{(1)}, G^{(2)})^T$. Applying $\nabla^2$ to $(5.2)_1$, we have

$$
0 = (\nabla G)_t + \nabla((\nabla u)^T \cdot G) + \nabla((\nabla G)^T \cdot u) + \frac{\gamma-1}{2} \nabla(G \text{div} u + \phi \nabla \text{div} u)
$$

$$
= (\nabla G)_t + \sum_{i=1}^{2} (G^{(i)} \nabla^2 u^{(i)} + \nabla u^{(i)} \otimes \nabla G^{(i)}) + \sum_{i=1}^{2} (u^{(i)} \nabla^2 G^{(i)} + \nabla G^{(i)} \otimes \nabla u^{(i)})
$$

$$
+ \frac{\gamma-1}{2} (\nabla G \text{div} u + G \otimes \nabla \text{div} u) + \frac{\gamma-1}{2} (\phi \nabla^2 \text{div} u + \nabla \text{div} u \otimes G).
$$

(5.44)

Similarly to the previous step, we multiply $(5.44)$ by $2\nabla G$ and integrate it over $\mathbb{R}^2$ to derive
This estimate, together with (5.43), gives that
\[
\frac{d}{dt}|G|^2_{D^1} \leq C|\nabla u|_\infty|G|^2_{D^1} + C|G|_6|\nabla G|_2|\nabla^2 u|_3 + C|\phi|_\infty|\nabla G|_2|\nabla^3 u|_2
\]
\[
\leq C(1 + |\nabla u|_\infty)(|G|_2^2 + |\psi|^2_{D^1}) + C(1 + |\nabla u_t|^2_{D^1}).
\] (5.45)

This estimate, together with (5.43), gives that
\[
\frac{d}{dt}(|G|^2_{D^1} + |\psi|^2_{D^1}) \leq C(1 + |\nabla u|_\infty)(|G|^2_{D^1} + |\psi|^2_{D^1}) + C(1 + |\nabla u|^2_{D^1}).
\] (5.46)

Then the Gronwall’s inequality and (5.35) imply
\[
|\phi(t)|^2_{D^2} + |\psi(t)|^2_{D^1} + \int_0^t |u(s)|^2_{D^3} \, dt \leq C, \quad 0 \leq t \leq T.
\]

Finally, using the following relations
\[
\psi_t = -\nabla(u \cdot \psi) - \nabla \text{div}u, \quad \phi_t = -u \cdot \nabla \phi - \frac{\gamma - 1}{2} \phi \text{div}u,
\]
\[
\phi_{tt} = -u_t \cdot \nabla \phi - u \cdot \nabla \phi_t - \frac{\gamma - 1}{2} \phi_t \text{div}u - \frac{\gamma - 1}{2} \phi \text{div}u_t,
\] (5.47)

we conclude the proof of this lemma. \(\square\)

In order to obtain higher order regularity, we need the following improved estimate.

**Lemma 5.7.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem (1.1)-(1.3) on \([0, T]\) satisfying (5.1). Then
\[
\sup_{0 \leq t \leq T} |u_t(t)|^2_{D^2} + \sup_{0 \leq t \leq T} |u(t)|^2_{D^1} + \int_0^T |u_{tt}(t)|^2_{D^1} \, dt \leq C, \quad 0 \leq T < T,
\]
where \(C\) only depends on \(C_0\) and \(T\).

**Proof.** First, \(Lu_t = -u_{tt} - (u \cdot \nabla u)_t - 2\theta(\phi \nabla \phi)_t + (\psi \cdot Q(u))_t\) and Lemma 2.4 yield
\[
|u_t|_{D^2} \leq C(|u_{tt}|_2 + |u \cdot \nabla u|_2 + |\nabla (\phi^2)|_2 + |(\psi \cdot Q(u))|_2)
\]
\[
\leq C(|u_{tt}|_2 + |u|_\infty|\nabla u|_2 + |\nabla u|_3|u_t|_2^\frac{1}{2}|\nabla u_t|_2^\frac{3}{2} + |\phi|_\infty|\nabla \phi_t|_2)
\]
\[
+ C(|\nabla \phi|_3|\phi_t|_6 + |\psi|_6|\nabla u_t|_2^\frac{4}{3}|\nabla^2 u_t|_2^\frac{1}{3} + |\psi_t|_2|Q(u)|_\infty)
\]
\[
\leq C(1 + |u_{tt}|_2 + |\nabla u_t|_2 + |u|_{D^2, \infty}),
\] (5.48)

which implies, with the help of Young’s inequality, that
\[
|u_t|_{D^2} \leq C(1 + |u_{tt}|_2 + |\nabla u_t|_2 + |u|_{D^2, \infty}).
\] (5.49)

Now, multiplying (5.21) by \(u_{tt}\) and integrating over \(\mathbb{R}^2\), we have
\[
\frac{1}{2} \frac{d}{dt} \left( \frac{\alpha}{2} |u_{tt}|^2 + (\alpha + \beta)|\text{div}u_t|^2 \right) + |u_{tt}|^2
\]
\[
= \int_{\mathbb{R}^2} \left( -(u \cdot \nabla u_t) \cdot u_{tt} - 2\theta(\phi \nabla \phi)_t \cdot u_{tt} + (\psi \cdot Q(u))_t \cdot u_{tt} \right) \, dx \equiv \sum_{i=1}^{15} L_i.
\] (5.50)
For the terms $L_{13} - L_{15}$, we perform the following estimates:

$$L_{13} = -\int_{\mathbb{R}^2} (u \cdot \nabla u) \partial_t u_t dx \leq C|u_t|_6 |\nabla u|_3 |u_t|_2 + C|u|_\infty |\nabla u_t|_2 |u_t|_2$$
$$\leq C((\nabla u)_3^2 + |u|_\infty^2) |\nabla u_t|_2^2 + \frac{1}{10} |u_t|_2^2,$$

$$L_{14} = -\int_{\mathbb{R}^2} 2\theta(\phi \nabla \phi) \cdot u_t dx = \theta \frac{d}{dt} \int_{\mathbb{R}^2} (\phi^2)_{\partial_t} \cdot \nabla u_t dx - \int_{\mathbb{R}^2} \theta (\phi^2)_{\partial_t} \cdot \nabla u_t dx$$
$$\leq \theta \frac{d}{dt} \int_{\mathbb{R}^2} (\phi^2)_{\partial_t} \cdot \nabla u_t dx + C|\nabla u_t|^2_2 + C|\nabla u_t|^2_2,$$

$$L_{15} = \int_{\mathbb{R}^2} (\psi \cdot Q(u))_{\partial_t} u_t dx = \int_{\mathbb{R}^2} \psi \cdot Q(u)_{\partial_t} u_t dx + \int_{\mathbb{R}^2} \psi_t \cdot Q(u) \cdot u_t dx$$
$$\leq C|\psi|_6 |\nabla u|_3 |u|_2 + |\psi_t|_2 |\nabla u|_\infty |u_t|_2$$
$$\leq C|\psi|_6^2 |\nabla u|_2^2 |\nabla^2 u|_2^2 + C|\psi_t|_2 |\nabla u|_2^4 |\nabla^2 u|_6^2 + \frac{1}{10} |u_t|_2^2$$
$$\leq C(1 + |\nabla u_t|^2_2 + |\nabla^2 u|^2_6) + \frac{1}{5} |u_t|_2^2.$$ 

Therefore, (5.50) and (5.51) imply that

$$\frac{1}{2} \frac{d}{dt} \left( (\nabla u_t)^2_2 + (\alpha + \beta)|\nabla u_t|^2_2 - \theta \int_{\mathbb{R}^2} (\phi^2)_{\partial_t} \cdot \nabla u_t dx \right) + |u_t|_2^2$$

$$\leq C(1 + |\nabla u_t|^2_2 + |\nabla^2 u|^2_6),$$

which, upon integrating over $(\tau, t)$, yields

$$|\nabla u_t(t)|^2_2 + \int_{\tau}^t |u_t(s)|^2_2 ds \leq C + |\nabla u_t(\tau)|^2_2 + \int_{\tau}^t |\nabla u_t|^2_2 ds, \quad 0 \leq t \leq T,$$ (5.53)

where we used the fact that for any $\epsilon > 0$

$$\int_{\mathbb{R}^3} (\phi^2)_{\partial_t} \cdot \nabla u_t dx \leq \epsilon |\nabla u_t|^2_2 + C.$$

(5.54)

From the momentum equations (5.2)3, we have

$$|\nabla u_t(\tau)|^2 \leq C(1 + |u|^2_3 + |\phi|^2_3 + |\psi|_{L^6(\cap D^1 \cap D^2)} |u|^2_2) (\tau).$$

(5.55)

Using the regularity in (1.10), we find

$$\limsup_{\tau \to 0} |\nabla u_t(\tau)|^2_2 \leq C(1 + |u_0|^2_3 + |\phi_0|^2_3 + |\psi_0|_{L^6(\cap D^1 \cap D^2)} |u_0|^2_2) \leq C_0.$$ (5.56)

Letting $\tau \to 0$ in (5.53), we finally proved that

$$|\nabla u_t(t)|^2_2 + \int_{0}^{t} |u_t(s)|^2_2 ds \leq C, \quad 0 \leq t \leq T.$$ (5.57)

In order to complete the proof of this lemma, we observe from (5.36) and (5.49) that

$$|u(t)|_{D^1} \leq C(1 + |u_t|_2)(t) \leq C, \quad 0 \leq t \leq T.$$

$$\int_{0}^{t} |u_t|^2_2 ds \leq \int_{0}^{t} C(1 + |u_t|^2_2 + |\nabla u_t|^2_2 + |u|^2_{D^2, u}) ds \leq C, \quad 0 \leq t \leq T.$
It remains to prove the following lemma for the required regularity estimate.

**Lemma 5.8.** Let \((\rho, u)\) be the unique regular solution to the Cauchy problem \((1.1)-(1.3)\) on \([0, T]\) satisfying (5.1). Then

\[
\sup_{0 \leq t \leq T} \left( |\phi(t)|^2_{D^3} + |\psi(t)|^2_{D^2} + |\phi_t(t)|^2_{D^3} + |\psi_t(t)|^2_{D^3} \right) + \int_0^T |u|^2_{D^4} dt \leq C,
\]

\[
\sup_{0 \leq t \leq T} \left( |u_t(t)|^2_{D^2} + t|u_tt(t)|^2_{D^4} + t|u(t)|^2_{D^4} \right) + \int_0^T (t|u_{tt}(t)|^2_{D^3} + t|u(t)|^2_{D^3}) dt \leq C,
\]

where \(0 \leq T < T\), and \(C\) only depends on \(C_0\) and \(T\).

**Proof.** The first assertion in this lemma follows in the similar lines of proof as that for Lemma 5.5, while the second assertion can be proved by the same method used in Lemma 5.7. We omit the details for simplicity of presentation. \(\square\)

Now we know from Lemmas 5.1-5.8 that, if the regular solution \((\rho, u)(x, t)\) exists up to the time \(T > 0\), with the maximal time \(T < +\infty\) such that the assumption (5.1) holds, then \((\rho^{\frac{\gamma-1}{2}}, \nabla \rho/\rho, u)\) satisfy the conditions imposed on the initial data (1.9). If we solve the system (1.1) with the initial time \(T\), then Theorem 1.1 ensures that \((\rho, u)(x, t)\) extends beyond \(T\) as the unique regular solution. This contradicts to the fact that \(T\) is the maximal existence time. We thus complete the proof of Theorem 1.3.

**Acknowledgement:** The research of Y. Li and S. Zhu were supported in part by National Natural Science Foundation of China under grant 11231006 and Natural Science Foundation of Shanghai under grant 14ZR1423100. S. Zhu was also supported by China Scholarship Council. The research of R. Pan was partially supported by National Science Foundation under grants DMS-0807406 and DMS-1108994.

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