q-graded Heisenberg algebras and deformed supersymmetries

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Abstract

The notion of q-grading on the enveloping algebra generated by products of q-deformed Heisenberg algebras is introduced for q complex number in the unit disc. Within this formulation, we consider the extension of the notion of supersymmetry in the enveloping algebra. We recover the ordinary $\mathbb{Z}_2$ grading or Grassmann parity for associative superalgebra, and a modified version of the usual supersymmetry. As a specific problem, we focus on the interesting limit $q \to -1$ for which the Arik and Coon deformation of the Heisenberg algebra allows to map fermionic modes to bosonic ones in a modified sense. Different algebraic consequences are discussed.

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1 Introduction

Deformations of the bosonic Heisenberg algebra by parameters have known successful achievements in mathematical physics [1]-[11] and in nonlinear physics (e.g. nonlinear quantum optics) [12]-[17]. One of the simplest deformation of the bosonic algebra, a one parameter $q$-deformation, was introduced by Arik and Coon [1] and is defined by

$$aa^\dagger - qa^\dagger a = I, \quad 0 < q \leq 1.$$  \hspace{1cm} (1)

Clearly, one recovers the ordinary Fock algebra of the harmonic oscillator at the limit $q \to 1$, with then $[a, a^\dagger] = I$.

The most of studies pertaining to such deformations are made with the parameter $q \in ]0, 1]$. However, in [8], a study was performed even for complex values of $q$. Concerning the issue of convergence, infinite products and deformed exponential series require at least that the modulus $|q| \leq 1$. This leads to the consideration that $q \in \mathbb{D}^*$, with $\mathbb{D}^*$ the unit complex disc but the zero. Keeping in mind these last remarks, nothing prevents to perform the following limit

$$\lim_{q \to -1} aa^\dagger - qa^\dagger a = aa^\dagger + a^\dagger a = \{a, a^\dagger\} = I,$$ \hspace{1cm} (2)

reminiscent of a fermionic algebra [18]-[24]. It then raises many natural questions. Is it possible to understand the generators associated to this limit as fermions? Then, in the case of a positive answer, is there a mapping from the bosonic operators (defined for $q = 1$) to fermionic ones (defined for $q = -1$), i.e. a kind of supersymmetry?

Recent years, many investigations on $q$-deformed algebras and supersymmetry have been undertaken dealing with $q$-deformed supersymmetric factorization [23] or differential representation, intertwining properties and coherent states [24] (and more references therein). Nevertheless, as far as we can establish, none of them focuses on the complete study of the product of these deformed algebras for different parameters $q$. So doing, one will immediately generate a full deformed universal algebra of all different deformed generators acting on a unique representation Hilbert space. We propose to investigate how the notions of $\mathbb{Z}_2$ Grassmann grading and supersymmetry can be extended to this multi-deformed enveloping algebra.

In this paper, we introduce the notion of $q$-deformed grading on the enveloping algebra generated by all products in different deformed Heisenberg algebras. This notion generalizes the ordinary Grassmann grading and, moreover, by defining a generalized $q$-graded bracket, one is
able to recover, in each subalgebra, the correct structure for bosonic, fermionic, $\mathbb{Z}_2$ graded and basic deformed bosonic algebras. The extension of Grassmann parity affords us to understand ordinary associative superalgebras and their $\mathbb{Z}_2$ graded structure (the usual framework of supersymmetry) as limit algebras when the parameter $q \to \pm 1$. We then determine the modified supersymmetric Hamiltonian and its deformed supercharges mapping some deformed fermions on deformed bosons.

The paper’s outline is as follows. The following section is dedicated to the definition and basic properties of the $q$-deformation of the Heisenberg algebra in the sense of Arik and Coon, for complex parameter $q$, and its representation. The limit $q \to -1$ is clarified. Afterwards, Section 3 addresses the algebraic settlement of the deformed structure producing the general $q$-deformed grading. The particular case of ordinary Grassmann parity is discussed. Section 4 investigates the extended notion of supersymmetry on the enveloping algebra. The specific limit $q \to \pm 1$, producing a modified version of the ordinary supersymmetry, is also discussed. The paper ends by some remarks in Section 5 and an appendix provides useful identities and illustrations.

2 Complex $q$-deformed Heisenberg algebras

Let us consider the Arik and Coon deformation of the Heisenberg algebra [1]

$$a_q a_q^\dagger - qa_q^\dagger a_q = \mathbb{I},$$

(3)

with parameter a complex number $q$. If we regard $a_q^\dagger$ as the adjoint of $a_q$, it follows that, by Hermitian conjugation of (3), $a_q a_q^\dagger - q a_q^\dagger a_q = \mathbb{I}$. By simple substraction of these equations, one ends with $\bar{q} = q$ from the positivity of $a_q^\dagger a_q$. Hence, $q$ should be a real parameter. However, introducing a new operator $a_q^\natural$, let us reconsider the same kind of deformed structure, namely

$$a_q a_q^\natural - q a_q^\natural a_q = \mathbb{I},$$

(4)
and relax the previous condition of adjoint property between $a_q$ and $a_q^\natural$. Then, nothing can be said, \textit{a priori}, on the parameter $q$. We will place ourself in this general situation such that

$$q \in \mathbb{D}^* = \{z \in \mathbb{C}, \ |z| \leq 1, \ z \neq 0\},$$
$$q = r_q e^{i\varphi_q}, \ r_q \in ]0, 1[, \ \varphi_q \in [0, 2\pi]. \quad (5)$$

In order to define the power function of $q$, namely $q^x$, one uses the complex form of the exponential function $e^{x \Ln q}$, where $\Ln(\cdot)$ stands for the principal branch of complex logarithm.

A realization of the algebra (4) is also well known. To construct it, one starts with the ordinary bosonic operators $a$ and $a^\dagger$, fulfilling $[a,a^\dagger] = \mathbb{I}$ with the number $N = a^\dagger a$, generating the ordinary number operator in the Fock Hilbert space $\mathcal{H} = \text{span}\{|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle\}$. Then we define

$$a_q|n\rangle = \sqrt{[N+1]/N+1}a|n\rangle, \quad a_q^\natural|n\rangle = \sqrt{[N]/N}a^\dagger|n\rangle, \quad [N]_q = \frac{1-q^N}{1-q}, \quad (6)$$

where one refers to $[N]_q$ as the $q$-basic number of the theory. Note that $[N]_q$ is not necessarily self-adjoint. Indeed, $([N]_q)^\dagger = [N]_q$, which is not $[N]_q$ unless $q$ is real. The adjoint of the operator $a_q$ can be expressed as

$$a_q^\dagger = \sqrt{[N]_q}a_q \quad (7)$$

from which it appears possible to define naturally a self-adjoint deformed number operator as

$${N}^q_{q,q} := a_q^\dagger a_q = \sqrt{[N]_q[N]_q}. \quad \text{From (7), a relation between } a_q^\dagger \text{ and } a_q^\natural \text{ can be inferred:}$$

$$a_q^\natural = a_q^\dagger \quad \Leftrightarrow \quad a_q^\natural = (a_q^\dagger)^\natural = a_q^\natural.$$

We are then in position to define properly the unary operation $^\natural$ which is the adjoint operation composed with the complex conjugation. The operator $a_q^\natural = (a_q^\dagger)^\natural = a_q^\dagger$, viewed as a matrix, can be understood as the transpose of $a_q$. Moreover, it can be checked that $(a_q^\natural)^\natural = a_q$, therefore $^\natural$ is an involution; we also have $(a_q^\natural a_q)^\natural = a_q^\natural a_q$. For a real parameter $q$, the definitions of $a_q^\dagger$ and $a_q^\natural$ coincide.

\footnote{In fact, this condition may not be imposed because nothing prevents to do an extension $q \in \mathbb{C}, \ \varphi_q \in [0, 2\pi[, \ q \neq 0$. All the following main equations are again valid outside the unit disc. Only the notion of convergence of functions series and infinite products involved in deformed special function theory has to be reconsidered. We are not dealing with these ideas here, but we want, as much as possible, to have a theory with interesting properties for the theoretician community.}
Let us briefly mention the limit \( q \to 0 \). The corresponding basic number \([N]_0\) proves to be the constant operator \( \mathbb{I} \). This implies that \( a_0 \) and \( a_0^\dagger \) are mutually inverse in the Fock space without the vacuum \(|0\rangle\). As a result of the triviality of the \((q = 0)\)-commutator, the \((q = 0)\)-deformed algebra is again a Lie algebra. Then the enveloping algebra over \( \{a, a^{-1}, \mathbb{I}\} \) becomes a division algebra (other relations concerning division algebras built over the Heisenberg generators are available in [17]).

Let us focus now on the limit \( q \to -1 \) of the algebra (4). This limit can be written as

\[
\lim_{q \to -1} a_q a_q^\dagger - qa_q^\dagger a_q = a_{-1} a_{-1}^\dagger + a_{-1}^\dagger a_{-1} =: \{a_{-1}, a_{-1}^\dagger\} = \mathbb{I}.
\]

A prime remark would be that, recalling that \( q \neq 0 \), the above limit could be performed only by avoiding the forbidden value \( q = 0 \); this can be done by varying continuously \( q \) along a straight line if \( q \) does not belong to the segment \([0, 1]\). In the case \( q \in ]0, 1[ \), then the same limit can be only made by choosing a contour through the complex plane.

Noting that, in any state \(|n\rangle\),

\[
[N]_{-1}|n\rangle = \lim_{q \to -1} [N]_q |n\rangle = \frac{1 - (-1)^N}{2} |n\rangle = \begin{cases} 0, & \text{if } n = 2p \\ |1\rangle, & \text{if } n = 2p + 1 \end{cases}
\]

then we infer the following representation for the operators

\[
\lim_{q \to -1} a_q |n\rangle = \sqrt{\frac{1 - (-1)^{N+1}}{2(N+1)}} |n\rangle = \sqrt{\frac{1 - (-1)^{n}}{2}} |n - 1\rangle = \begin{cases} 0, & \text{if } n = 2p \\ |1\rangle, & \text{if } n = 2p + 1 \end{cases}
\]

\[
\lim_{q \to -1} a_q^\dagger |n\rangle = \sqrt{\frac{1 - (-1)^N}{2N}} a_q |n\rangle = \sqrt{\frac{1 - (-1)^n}{2}} |n + 1\rangle = \begin{cases} 0, & \text{if } n = 2p \\ |1\rangle, & \text{if } n = 2p + 1 \end{cases}
\]

Let us recall that a fermionic algebra is usually defined by a set of algebraic relations

\[
cc^\dagger + c^\dagger c = \mathbb{I}, \quad c^2 = 0 = (c^\dagger)^2.
\]

The anticommutation rule is already satisfied by the pair \((a_{-1}, a_{-1}^\dagger)\). Checking, that for any \( q \in \mathbb{D}^* \), \( a_q^2 |n\rangle = \sqrt{|n - 1\rangle_q |n\rangle_q |n - 2\rangle_q} \), \( (a_q^\dagger)^2 |n\rangle = \sqrt{|n + 1\rangle_q |n + 2\rangle_q} |n + 2\rangle_q \), one infers from (10) that, indeed, for any state, \( a_{-1}^2 |n\rangle = 0 \) and \( (a_{-1}^\dagger)^2 |n\rangle = 0 \). Thus, the pair \((a_{-1}, a_{-1}^\dagger)\) is close to what one usually refers to as a fermionic algebra. For this reason, we will refer henceforth to these operators to fermions and to their algebra, to a fermionic algebra. Here, more rigorously, the operators \((a_{-1}, a_{-1}^\dagger)\) are fermionic operators with an infinite dimensional representation space which is a direct sum of ordinary two dimensional fermionic representation spaces.
3 \( q \)-grading of deformed Heisenberg algebras

The notion of \( \mathbb{Z}_2 \) Grassmann grading for associative complex superalgebras [25] will find, in the next lines, an extension according to the present \( q \)-deformed study. But before, for the sake of rigor, let us put in algebraic terms the definition of the deformation of the Heisenberg algebra [4].

**Building a \( q \)-grading on the enveloping algebra.** For all \( q \in \mathbb{D}^* \), we introduce the deformed complex Heisenberg algebra with its three generators and deformed commutator as the pair

\[
\left( H_q = \text{span}_\mathbb{C}\{a_q, a^\dagger_q, \mathbb{I}\} \ ; \ [\cdot, \cdot]_q \right). \tag{14}
\]

Some remarks are in order at this stage. First, the deformed Heisenberg algebra \( H_q \) is not a Lie algebra unless that one considers the limit points \( q \in \{0, \pm 1\} \). The Jacobi identity fails to be satisfied in the general situation when \( q \notin \{0, \pm 1\} \). Note also that these algebras are not disjoint since \( \mathbb{I} \in H_q \cap H_{q'} \), for \( q \neq q' \). The data of the pair \( (14) \) are equivalent to the data of a complex vector space \( H_q \) and a constraint (equivalence relation) \([a_q, a^\dagger_q]_q = \mathbb{I}\) on the tensor algebra built out of its generators.

Next, let us give the definition of the \( q \)-grading of generators of any \( H_q \) and find an extension for any element of the enveloping algebra spanned by all \( H_q \)'s, \( q \in \mathbb{D}^* \). This concept will be introduced by the data of two attributes related to the parameters \( q \): the “degree”, denoted by \( |(\cdot)| \), and the “radius”, denoted by \( \ell(\cdot) \).

By convention, elements of \( h_0 = \text{span}_\mathbb{C}\{\mathbb{I}\} \) are of degree 0 and we define the degree of the generators of \( H_q \) as

\[
|a_q| = \sqrt{\varphi_q / \pi} = |a^\dagger_q|, \quad |\mathbb{I}| = 0. \tag{15}
\]

Given a generator of \( H_q \), its degree becomes a real parameter in the segment \([0, 2]\) which can be viewed as the normalized phase of the deformation parameter \( q \). For instance, the degree of an ordinary (Heisenberg) boson is \( |a_{q=1}| = 0 = |a^\dagger_{q=1}| \), while the degree of the operators \( |a_{q=-1}| = 1 = |a^\dagger_{q=-1}| \) reproducing a well defined notion of \( \mathbb{Z}_2 \) Grassmann parity for these limit.

We will characterize the generators of \( H_q \), by another quantity that we will refer to as its “radius” or “length” which is nothing but

\[
\ell(a_q) = \sqrt{r_q} = \ell(a^\dagger_q), \quad \ell(\mathbb{I}) := 1, \tag{16}
\]

given the modulus \( r_q \) of the deformation parameter \( q \).
At this stage, the following deformed bracket for elementary generators can be defined

\[ [x_q, y_{q'}]_{q,q'} := x_q y_{q'} - g(q, q') y_{q'} x_q, \quad g(q, q') := e^{i\pi|y_{q'}|\ell(x_q)\ell(y_{q'})}. \quad (17) \]

A quick verification, using (15) and (16), yields the following limits

\[ \begin{align*}
(\text{boson}) \quad & g(1, 1) = 1 : \quad [a_1, a\uparrow_1]_{1,1} = [a_1, a\uparrow_1]_{q=1} = a_1 a\uparrow_1 - a\uparrow_1 a_1 = \mathbb{I}, \quad (18) \\
(\text{fermion}) \quad & g(-1, -1) = -1 : \quad [a_{-1}, a\uparrow_{-1}]_{-1,-1} = [a_{-1}, a\uparrow_{-1}]_{q=-1} = a_{-1} a\uparrow_{-1} + a\uparrow_{-1} a_{-1} = \mathbb{I}, \quad (19) \\
(\text{q-def.}) \quad & g(q, q) = q : \quad [a_q, a\uparrow_q]_{q,q} = [a_q, a\uparrow_q]_{q} = a_q a\uparrow_q - qa\uparrow_q a_q = \mathbb{I}. \quad (20)
\end{align*} \]

Another interesting property of the deformed bracket (17) is that it reproduces the \(\mathbb{Z}_2\) graded bracket between fermion and bosons. In other words, in addition to (18) and (19), the bracket of a fermion and a boson is a commutator, because of \(g(1, -1) = 1\).

Having properly defined the notion of \(q\)-grading of basic generators, let us go further by defining similar ideas for more complex structures.

The (noncommutative) product of elements of two algebras \(H_q\) and \(H_{q'}\) lies in the complex vector space \(H_q H_{q'}\). By iteration, one can build monomials in basic generators living in a product of deformed algebras \(H_q, i = 1, 2, \ldots\). Taking the complex span of these monomials, one forms a complex vector space. Negative integer powers of generators can be defined algebraically as \(x_{q}^{-n} := (x_{q}^{-1})^{n}\), \(n \in \mathbb{N}\), where the inverse of \(x_{q}\), i.e. \(x_{q}^{-1}\), acts by representation such that \(x_{q} x_{q}^{-1} = \mathbb{I}\) or \(x_{q}^{-1} x_{q} = \mathbb{I}\) (right or left inverse). Further precisions on the division algebra generated by the Heisenberg operators \(a_q\) and \(a\uparrow_q\) can be found in [17]. The overall algebra spanned by any linear combination of any kind of products of generators (including inverse integer powers) will be called the deformed enveloping algebra denoted by \(U_q(H)\).

We would like to give a sense to the notion of grading for any elements of the enveloping algebra \(U_q(H)\). We start by the degree and radius of bilinear products which can be defined as

\[ |x_q y_{q'}| = |x_q| + |y_{q'}|, \quad \ell(x_q y_{q'}) = \ell(x_q)\ell(y_{q'}), \quad (21) \]

where \(x_q\) and \(y_{q'}\) are generators of \(H_q\) and \(H_{q'}\), respectively. It is remarkable that \(|x_q y_{q'}| = |y_{q'} x_q|\) and \(\ell(x_q y_{q'}) = \ell(y_{q'} x_q)\). Integer powers of elementary generators of \(H_q\) belonging to \((H_q)^{\alpha} \subset U_q(H)\) can be also assigned with a degree and a radius as

\[ |(x_q)^{\alpha}| = \alpha |x_q|, \quad \ell((x_q)^{\alpha}) = (\ell(x_q))^{\alpha}, \quad \alpha \in \mathbb{Z}. \quad (22) \]
More generally, the following relations, valid for finite products of integer powers of elementary
generators, stand for definition:
\[
| \prod_{i=1}^{n} (x_{q_i})^{\alpha_i} | := \sum_{i=1}^{n} \alpha_i | x_{q_i} |, \quad \ell \left( \prod_{i=1}^{n} (x_{q_i})^{\alpha_i} \right) := \prod_{i=1}^{n} (\ell(x_{q_i}))^{\alpha_i}.
\] (23)

Some products of basic generators admit a spectral decomposition of the form \( A_q = \prod_{i=1}^{n} x_{q_i} = \sum_{n=0}^{\infty} A_q([n]) | n \rangle \langle n | \), where \( A_q([n]) \in \mathbb{C} \), any function of number operators being a typical example. For this kind of operators, rational and real powers also have a rigorous definition. For instance, one sets \( A_q^\alpha := \sum_{n=0}^{\infty} (A_q([n]))^\alpha | n \rangle \langle n |, \alpha \in \mathbb{R} \). In this situation of a possible diagonal decomposition of an operator being a product of elementary generators, the formulas (24) can be extended to real powers:
\[
\left| \left( \prod_{i=1}^{n} (x_{q_i})^{\alpha_i} \right)^{\beta} \right| := \beta \sum_{i=1}^{n} \alpha_i | x_{q_i} |, \quad \ell \left( \left( \prod_{i=1}^{n} (x_{q_i})^{\alpha_i} \right)^{\beta} \right) := \prod_{i=1}^{n} (\ell(x_{q_i}))^{\beta\alpha_i}, \quad \beta \in \mathbb{R}.
\] (24)

In order to compute the deformed bracket of composite elements in \( U_q(H) \), one has to perform
first the decomposition in sum of monomials in the elementary generators before the computation.
Finally, the general bracket for any monomials \( A_{q_i} = \prod_{i_k} x_{q_{i_k}}^{\alpha_{i_k}} \) and \( B_{q_j} = \prod_{j_k} x_{q_{j_k}}^{\alpha_{j_k}} \) can be expressed as follows (an explicit example is provided in the appendix)
\[
[A_{q_i}, B_{q_j}]_{G(q_i, q_j)} := A_{q_i} B_{q_j} - G(q_i, q_j) B_{q_j} A_{q_i}, \quad G(q_i, q_j) := e^{i \pi | a_{-1} a_{-1}^\dagger |} = e^{i \pi A_{q_i} |B_{q_j}| \ell(A_{q_i} B_{q_j})}.
\] (25)

In conformity with the above definition of degree, the degree of a product of two fermions is 2
(for instance \( | a_{-1} a_{-1}^\dagger | = 1 + 1 = 2 \) and not 0 as it is customarily the case in the context of
\( \mathbb{Z}_2 \) superalgebras. This does not lead to any contradiction and the anticommutator
(19) is still valid and based on the product of degrees. Although one is tempted to take for definition of the
degree a kind of number modulo 2 (or \( 2\pi \) from (15)), the study can be pursued in this general
context proving that there is no need to make further assumptions in the definition (15).

\( (q = \pm 1) \)-grading and matrix representation. As a prime interesting feature with implications in supersymmetry, we discuss the matrix representation. Let us recall that the ordinary
notion of \( \mathbb{Z}_2 \) grading applied to matrix algebras \( \mathbb{K} \) can be introduced by the data of two integers \( n \) and \( m \), and the decomposition of any element of \( M_{n+m}(\mathbb{K}) \), the set of square matrices of
dimension \((n+m)^2\) with coefficients in the field \( \mathbb{K} \), into four submatrices of dimensions \( n \times n \),
\( n \times m \), \( m \times n \) and \( m \times m \) \( [25] \), i.e.
\[
M(n+m, n+m) = \begin{pmatrix}
A(n,n) & B(n,m) \\
C(m,n) & D(m,m) 
\end{pmatrix}.
\] (26)
A matrix is said to be “even” if its entries belong either to $A(n, n)$ or to $D(m, m)$ and “odd” if its entries belong to the matrices $B(n, m)$ or $C(m, n)$. One can check that any product of matrices obeys to the law $even \cdot even = even$, $even \cdot odd = odd$, $odd \cdot odd = even$ such that the matrix product is stable under this grading. The same idea can be simply illustrated in ordinary matrix formulation of supersymmetry where the supersymmetric Hamiltonian is diagonal (even quantity) and supercharges consist in off diagonal matrices (odd elements). One thing remains to be clarified: if the ordinary bosonic modes $a_1 = a$ and $a_1^\dagger = a^\dagger$ or usual fermionic operators $c$ and $c^\dagger$ admit a matrix representation onto the Fock Hilbert space basis, it can be suggested to find the equivalent feature such that the notion of $\mathbb{Z}_2$ grading as previously discussed can be readily read from their matrix representation.

At first, using the conventional matrix representation of bosonic modes, nothing can be said. However, if one organizes the states differently such that we write the Fock basis in the following form

$$\{\{|0\rangle, |2\rangle, |4\rangle, \ldots, |2p\rangle, \ldots\}, \{|1\rangle, |3\rangle, |5\rangle, \ldots, |2p+1\rangle, \ldots\}\}$$  \hspace{1cm} (27)

then the ordinary boson $a_1$ and fermion $a_{-1}$ have the following matrices with respect to the order (27)

$$a_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & \sqrt{5} & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \cdots & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \sqrt{5} & \cdots & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}, \quad a_{-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}. \hspace{1cm} (28)$$

The associated adjoint operators can be easily inferred. In this context, the notion of “odd” matrix can be affected to either the pair $(a_1, a_1^\dagger)$ or to the pair $(a_{-1}, a_{-1}^\dagger)$. Then, legitimately in this context, the fermions $(a_{-1}, a_{-1}^\dagger)$ can be seen as “odd” elements while the fact that the operators $(a_1, a_1^\dagger)$ can be also seen as “odd” quantities becomes confusing. Nevertheless, another interesting feature emerges: multiplying “odd” matrices, for instance $a_1^\dagger a_1$ (of degree 0), or $a_{-1}^\dagger a_{-1}$ (of degree 2) will produce “even” elements (as they should be) which are the bosonic and fermionic numbers, respectively.
4 Deformed supersymmetry

This section aims at defining a general notion of supersymmetry on the enveloping algebra $U_q(H)$. The technical difficulty comes from the fact that operators are noncommuting objects in contrast to the situation of ordinary supersymmetric quantum theory. In addition, for different $q$’s, all $q$-deformed operators act on an identical Hilbert space (the Fock Hilbert space).

$(1, -1)$-Supersymmetry. In this paragraph, we define, as a guiding model to next discussions, a supersymmetric theory on the enveloping subalgebra spanned only by products in $H_{-1}$ and $H_1$. Supersymmetry is realized through a set of charges commuting with a Hamiltonian. Operators mapping in a deformed way fermions $a_{-1}$ and $a_{-1}^\dagger$ to bosons $a_1$ and $a_1^\dagger$ are identified. The converse is essentially not true due to the deformation.

Simple properties allow us to investigate which kind of operators can generate a supersymmetry. We will restrict the study to the situation of a supersymmetry generated by only quadratic products of operators which is, in fact, the closest possible to the ordinary notion of supersymmetry where a Hamiltonian appears as a (supersymmetrically) factorized by bilinear operators: the supercharges [26]. These latter operators generically are of the form of a product of bosons and fermions. These supercharges via a graded structure maps bosonic to fermionic degrees of freedom and vice versa.

Let us then list the possible minimal bilinears, built from products of the fermions $a_{-1}$ and $a_{-1}^\dagger$ by bosons $a_1$ and $a_1^\dagger$. Bearing in mind that the order of operators is important, 6 monomials are of interest

\[ q_1 = a_{-1}a_1 = \sqrt{\frac{N+1}{N+1}} a_2^2, \quad \tilde{q}_1 = a_1a_{-1} = \sqrt{\frac{N+2}{N+2}} a_2^2, \quad (29) \]
\[ \tilde{q}_1^\dagger = a_{-1}^\dagger a_1^\dagger = \sqrt{\frac{N-1}{N}} (a_1^\dagger)^2, \quad q_1^\dagger = a_1^\dagger a_{-1}^\dagger = \sqrt{\frac{N-1}{N-1}} (a_1^\dagger)^2, \quad (30) \]
\[ q_2 = a_1^\dagger a_{-1} = \sqrt{\frac{N-1}{N}} N = a_{-1}^\dagger a_1, \quad \tilde{q}_2 = a_{-1}a_1^\dagger = \sqrt{\frac{N+1}{N+1}} (N+1) = a_1a_{-1}^\dagger, \quad (31) \]

all of degree $^21$ and radius 1. Note that $q_2$ and $\tilde{q}_2$ are self-adjoint. A set of Hermitian Hamiltonian

\[ ^2 \text{By consistency, for instance, we can check that for } |q_1| = |a_{-1}a_1| = 1 + 0 = |\sqrt{N}a_1| + |a_2| + |N^{-1/2}| = (1/2 + 1/2) + 2 \cdot 0 + (-1/2) \cdot 0 = 1, \text{ obtained from [26]. The degrees of the other operators can be derived in a similar way.} \]
operators can be readily obtained from these operators

\[
\mathfrak{h}_1 = [\mathfrak{a}_1, \mathfrak{a}_1^\dagger]_G = \mathfrak{a}_1^\dagger \mathfrak{a}_1^\dagger + \mathfrak{q}_1 \mathfrak{q}_1 = 2[\mathcal{N} + 1]_{-1}(\mathcal{N} + 1),
\]

\[
\tilde{\mathfrak{h}}_1 = [\tilde{\mathfrak{q}}_1, \tilde{\mathfrak{q}}_1^\dagger]_G = \tilde{\mathfrak{q}}_1^\dagger \tilde{\mathfrak{q}}_1^\dagger + \tilde{\mathfrak{q}}_1 \tilde{\mathfrak{q}}_1 = 2[\mathcal{N}]_{-1} \mathcal{N},
\]

\[
\mathfrak{h}_2 = (\mathfrak{q}_2)^2 = [\mathcal{N}]_{-1} \mathcal{N},
\]

\[
\tilde{\mathfrak{h}}_2 = (\tilde{\mathfrak{q}}_2)^2 = [\mathcal{N} + 1]_{-1}(\mathcal{N} + 1).
\]

We are now in position to define the basic Hermitian supersymmetric Hamiltonian (up to some energy scale \(\hbar \omega\) that we omit)

\[
\mathfrak{h}_{ss} = [\mathcal{N}]_{-1} \mathcal{N} = \mathfrak{a}_1^\dagger \mathfrak{a}_1 \mathfrak{a}_1^\dagger \mathfrak{a}_1 = 2|\mathcal{N}]_{-1} + |\mathcal{N}| = 2, \quad \ell(\mathfrak{h}_{ss}) = 1,
\]

and the following supersymmetric algebra can be verified

\[
[\mathcal{Q}, \mathfrak{h}_{ss}]_G = 0 = [\mathfrak{Q}^\dagger, \mathfrak{h}_{ss}]_G, \quad [\mathcal{Q}, \mathfrak{Q}^\dagger]_G = \mathfrak{h}_{ss}, \quad \mathcal{Q} := \frac{1}{\sqrt{2}} \mathfrak{q}_1,
\]

\[
[q_2, \mathfrak{h}_{ss}]_G = 0, \quad \mathfrak{h}_{ss} = (q_2)^2 \quad (1/2)[\tilde{\mathfrak{q}}_1, \tilde{\mathfrak{q}}_1^\dagger]_G.
\]

Thus the formulation allows to generate a \(\mathcal{N} = 3\) supersymmetry (with three different symmetries). The other operators \(\tilde{\mathfrak{q}}_1\) and \(\tilde{\mathfrak{q}}_2\) have a simple meaning, in the present context. They define the partner charges allowing the construction of another supersymmetric Hamiltonian \(\tilde{\mathfrak{h}}_{ss}\), the so-called superpartner of \(\mathfrak{h}_{ss}\), obtained by reversing the order of operators, namely

\[
\tilde{\mathfrak{h}}_{ss} = (\tilde{\mathfrak{q}}_2)^2 = (1/2)[\tilde{\mathfrak{q}}_1, \tilde{\mathfrak{q}}_1^\dagger]_G.
\]

To the question “is this notion of \((1, -1)\)-supersymmetry equivalent to the ordinary one?” the answer is no. A hint to recognize this fact is the form of the ordinary supersymmetric Hamiltonian \(h_{ss} = \mathcal{N}_B + \mathcal{N}_F\), where \(\mathcal{N}_B\) and \(\mathcal{N}_F\) are the bosonic and fermionic number operators, respectively. Here the supersymmetric Hamiltonian \(\mathfrak{h}_{ss}\) is clearly not of this form. However, the operator \(h_{ss}\) can be rebuilt as \(h_{ss} = \mathfrak{h}_{ss} + \tilde{\mathfrak{h}}_{ss} = [\mathcal{N}]_{-1} \mathcal{N} + [\mathcal{N} + 1]_{-1}(\mathcal{N} + 1) = \mathcal{N} + [\mathcal{N} + 1]_{-1}\).

From this point of view, the supersymmetries (37) or (38) appear therefore as the basic ones even though it is not true that all properties of the ordinary supersymmetry can be recovered. In the following, we focus on the evidence of the deformation of the ordinary supersymmetry if the symmetry is realized as (37) or (38).

Let us check if the ordinary properties of supercharges are satisfied. First, the square of non Hermitian supercharges are usually vanishing quantities, here

\[
\mathcal{Q}_1^2 \neq 0 \quad \text{and} \quad (\mathcal{Q}_1^\dagger)^2 \neq 0.
\]
This shows that the supersymmetry is actually realized in a deformed way. Second, supercharges used to map bosons onto fermions and conversely. In the present situation, any $q$-commutation relation does not lead to interesting results. Nevertheless, after scrupulous analysis (see Appendix), one can reach the following interesting algebras

\begin{align}
[q_1, a_{-1}]_G &= [N]_{-1} a_1, \\ (40) \\
[q_1^+, a_{-1}]_G &= [N - 1]_{-1} a_1^+, \\ (41) \\
[q_2, a_{-1}]_G &= [N + 1]_{-1} a_1, \\ (42) \\
[q_2^+, a_{-1}]_G &= [N]_{-1} a_1^+, \\ (43)
\end{align}

revealing that fermions are actually mapped on deformed bosons (up to a function of the fermionic number operator). The converse is not true (see Appendix) pointing out the peculiar aspects of this deformed supersymmetry.

(q, \bar{q})-Supersymmetry. In this last paragraph, we define in a more general context of $q$-deformation, the notion of deformed supersymmetry.

We start by the simple remark that, as shown above, elements of the Heisenberg algebra $H_{q=-1}$ can be seen as deformed supersymmetric partners of elements of $H_{q=1}$. The complex number $-1$ is obtained after a rotation by $\pi$ from the complex number $1$. The notion of supersymmetry could find an equivalence in $U_q(H)$, mapping generators of $H_q$ onto a generator of $H_{\bar{q}}$, with $\bar{q}$ a transformation of $q$:

\[ \bar{q} = S_{f,k}(q) = f(r_q, \varphi)e^{ik(r_q,\varphi)}, \quad 0 < f(r_q, \varphi) \leq 1, \quad 0 \leq k(r_q, \varphi) < 2\pi, \]

where $f$ and $k$ are real functions. The above case of $(1, -1)$-supersymmetry corresponds to $S_{1\text{d},\pi}$, a simple rotation $R_{\pi}(\cdot) = e^{i\pi}(\cdot)$ in the complex plane by an angle $\pi$. However, we will assume that $f \equiv f(r_q)$ and $k \equiv k(\varphi_q)$; so doing, we write $\bar{q} = f(r_q)e^{ik(\varphi_q)}$, thereby defining $r_{\bar{q}} := f(r_q)$ and $\varphi_{\bar{q}} := k(\varphi_q)$.

The second step is to define again bilinears which are of interest. It does not take long to find the following operators (the same notation as in the previous paragraph is used but
operators now refer to different quantities)

\[ q_1 = a_q a_q = \sqrt{\frac{[N+1]_q [N + 2]_q}{(N + 1)(N + 2)}} (a_1)^2, \quad \bar{q}_1 = a_q a_q = \sqrt{\frac{[N+1]_q [N + 2]_q}{(N + 1)(N + 2)}} (a_1)^2, \tag{45} \]

\[ \bar{q}_1^* = a_q^\dagger a_q^\dagger = \sqrt{\frac{[N]_q [N - 1]_q}{N(N - 1)}} (a_1^\dagger)^2, \quad q_1^* = a_q^\dagger a_q^\dagger = \sqrt{\frac{[N]_q [N - 1]_q}{N(N - 1)}} (a_1^\dagger)^2, \tag{46} \]

\[ q_2 = a_q^\dagger a_q = \sqrt{[N]_q [N]_q} a_q^2, \quad \bar{q}_2 = a_q a_q = \sqrt{[N+1]_q [N + 1]_q} a_q^2. \tag{47} \]

They all share the same degree \( \sqrt{\varphi_q/\pi} + \sqrt{\varphi_{\bar{q}}/\pi} \) and the same radius \( \sqrt{r_q f(r_q)} \). We also notice that \( q_2 \) and \( \bar{q}_2 \) remain Hermitian, while \( q_1 \) and \( \bar{q}_1^* \) become adjoint of one another.

In order to obtain a Hermitian Hamiltonian operator, we consider the operator generated by \( q_2 \)

\[ h_2 = (q_2)^2 = [N]_q [N]_q = (\{N\}_q q_q)^2, \quad |h_2| = 2 \left( \sqrt{\frac{\varphi_q}{\pi}} + \sqrt{\frac{\varphi_{\bar{q}}}{\pi}} \right), \quad \ell(h_2) = r_q f(r_q) \tag{48} \]

and require that \( q_2 \) is a symmetry of \( h_2 \) with respect to the deformed bracket. We are led to the following

\[ [q_2, h_2]_G = \sqrt{[N]_q [N]_q [N]_q [N]_q} \left[ 1 - e^{2i\pi \left( \sqrt{\frac{\varphi_q}{\pi}} + \sqrt{\frac{\varphi_{\bar{q}}}{\pi}} \right)^2 \left( r_q f(r_q) \right)^3} \right] \tag{49} \]

A set of necessary and sufficient conditions for \([q_2, h_2]_G = 0\) to hold is

\[ \left( \sqrt{\frac{\varphi_q}{\pi}} + \sqrt{\frac{\varphi_{\bar{q}}}{\pi}} \right)^2 = p \in \mathbb{N}, \quad r_q f(r_q) = 1, \tag{50} \]

which can be easily solved by

\[ p = 0, \quad \varphi_q = 0 \quad \text{and} \quad \varphi_{\bar{q}} = k_0(\varphi_q) = 0; \quad f(r_q) = \frac{1}{r_q}; \tag{51} \]

\[ p \geq 1, \quad k_p(\varphi_q) = \varphi_{\bar{q}} = (\sqrt{\varphi_q} - \sqrt{p \pi})^2 \in [0, 2\pi]; \quad f(r_q) = \frac{1}{r_q}. \tag{52} \]

Before regarding the phase \( \varphi_q \) problems, let us focus on the equation \( f(r_q) = 1/r_q \) having a drastic consequence. Indeed, if \( r_q \in [0, 1] \), then \( f(r_q) \geq 1 \). This is to say that if we impose \( \bar{q} \in \mathbb{D}^* \), then obligatory we will restrict to the situation where \( r_q = f(r_q) = 1 \) in order to get relevant solutions. Enlarging the scope of value of \( q \in \mathbb{C} \), another interesting feature emerges: superpartners of operators with deformation parameter \( q \) strictly inside of the unit disc \( \mathbb{D}^* \) are operators with parameter \( \bar{q} \) lying strictly outside of the unit disc (and vice-versa); superpartners

\[ \text{From the beginning, we never assume this condition which is without any consequence on the present formulation.} \]
of operators labeled by a $q$ belonging to the unit circle $S^1 = \{ |z| = 1, z \in \mathbb{C} \}$ are operators with parameter still on the circle.

Concerning the phase, the case $p = 0, \varphi_q = 0 = k_0(\varphi_q)$, for $r_q = 1 = f(r_q)$ refers to the trivial point $q = 1 = \bar{q}$. We will focus only on points $\bar{q} \neq q$ encoding values where partners differ from one another. It turns out that for $\varphi_q \in \{ 0, \pi/4, \pi/2, 3\pi/4, \ldots, 7\pi/4 \}$, there always exists a value $\bar{q} = q$ since one can show that $p = (4/\pi)\varphi_q \in [0, 7]$. More generally, solving the first condition of (52), for $p \geq 1$, one infers the following constraints on $\varphi_q$

$$\forall \varphi_q \in [0, 2\pi[, \quad p = 1, 2, \quad k_p(\varphi_q) = (\sqrt{p\pi} - \sqrt{\varphi_q})^2 \in [0, 2\pi[, \quad \text{for } p \geq 1, \quad (53)$$

These solutions imply that, regardless of their modulus which can be fixed to be equal $r_q = 1$, given $q$ fixing once for all $r_q$ and $\varphi_q$, there are at least 2 and at most 7 parameters $\bar{q}_p, p \in [1, 7]$, providing good parameter candidates for different supersymmetries $q_{2,p}$ for at least 2 and at most 7 different models defined by $h_{2,p}$, such that $[q_{2,p}, h_{2,p}] G = 0$ (see Figure 1 and Figure 2). In the particular situation of the so-called $(1, -1)$-supersymmetry as built in the previous paragraph, given $(q = 1, r_1 = 1, \varphi_1 = 0)$, the above solutions are consistent and reduce to the unique possibility of $p = 1$ in (53) such that $k_1(0) = \pi$, thus implying a unique choice for $\bar{q} = e^{i\pi}$.

Finally, the constraint such that $k(\varphi_q) \in [0, 2\pi]$ is actually too strong and more solutions can
be determined by relaxing this condition. In fact, it seems that an infinite set of inequivalent (modulo $2\pi$) solutions of the phase equation \([52]\) are available due to the fact that \(k(\varphi_q)\) is a nonlinear function of \(\varphi_q\). A careful analysis and the meaning of these solutions will be treated in a subsequent work.

The meaning of supersymmetry is not clear when considering the other operators \(q_1\) and \(\tilde{q}_1\). A Hermitian Hamiltonian can be easily identified using the deformed bracket \(h_1 = [q_1, \tilde{q}_1]_G\). However, the set of conditions in order to impose \([q_1, h_1] = 0 = [\tilde{q}_1, h_1]\) is more involved. Moreover, the resulting operator \(h_1\) is not equal to \(h_2\) \([48]\). Other more complicated issues arise when one reverses the order of the basic generators. The former \(\mathcal{N} = 3\) \((1, -1)\)-deformed supersymmetry is therefore explicitly broken in the general deformation theory, with a reduced number of supercharges equal to \(\mathcal{N} = 1\).

Let us turn to the properties of the mapping boson-fermions. The following relations can be obtained:

\[
[q_2, a_{\tilde{q}}] = \left[\sqrt{\frac{[N]_q[N+1]_q}{[N+1]_q}} - e^{i\sqrt{\varphi_q(\varphi_q + \varphi_q)} f(r_q)\sqrt{[N+1]_q}}\right] a_q, \tag{55}
\]

\[
[q_2, a_{\tilde{q}}^2] = \left[[N]_q - e^{i\sqrt{\varphi_q(\varphi_q + \varphi_q)} f(r_q)\sqrt{[N+1]_q}}\right] a_q, \tag{56}
\]

which at the limit \(q = 1\) and \(\tilde{q} = -1\), characterized by \(f(r_1) = 1 = r_1, \varphi_0 = 0\) and \(\varphi_1 = \pi\), reproduce correctly \([42]\) and \([43]\). Hence, one draws the conclusion that \(a_{\tilde{q}}\) and \(a_{\tilde{q}}^2\) are deformed partners of \(a_q\) and \(a_{\tilde{q}}^2\), respectively.

### 5 Conclusion

We have succeeded in setting a notion of \(q\)-grading onto the deformed enveloping algebra built from all possible products of \(q\)-deformed Heisenberg algebras for different parameters \(q \in \mathbb{D}^*, \mathbb{D}^*\) being the complex disc of radius one without 0. This notion of \(q\)-grading encompasses in specific limit the ordinary notion of \(\mathbb{Z}_2\) grading of ordinary associative superalgebra. A generalized bracket is defined on the enveloping algebra which reproduces the ordinary bosonic, fermionic, \(\mathbb{Z}_2\) graded and \(q\)-deformed commutators for corresponding subspaces in the total algebra. The formalism is then used to show that the notion of supersymmetry can be extended and, even, realized in the present situation where the fermions do not commute with bosons. In the specific instance such that \(q = \pm 1\), a supersymmetric Hamiltonian has been defined and its \(\mathcal{N} = 3\) supersymmetry properly identified with new deformed features. Finally, in the full \(q\)-deformed
theory, we have identified many kinds of operators (restricting the deformed phase parameter in $[0, 2\pi]$) which are able to define different supersymmetric models. In the general context of deformation, the $\mathcal{N} = 3 \, (1, -1)$-supersymmetry is explicitly broken and at least $\mathcal{N} = 1$ supercharge can be defined in any $(q, \bar{q})$-deformed supersymmetric model.

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**Appendix**

This appendix provides useful relations and explicit illustrations to the text.

We start by illustrating the kind of computations involved by the generalized $q$-graded bracket. Let us calculate, for the specific instance, the bracket of the monomials of the form $a_{q_1}^\dagger a_{q_1}'$ and $a_{q_2}^\dagger a_{q_2}' a_{q_3} a_{q_3}'$ (as appeared in computing the bracket of the supercharge $q_2 = a_1^\dagger a_{-1}$ and supersymmetric Hamiltonian $h_{ss} = a_1^\dagger a_{-1} a_{-1}$). Given the complex numbers $q_i = r_{q_i} e^{i\varphi_{q_i}}$ and $q_i' = r_{q_i} e^{i\varphi_{q_i}'}$, $i = 1, 2, 3$, we have

$$|a_{q_1}^\dagger a_{q_1}'| = \sqrt{\varphi_{q_1}} + \sqrt{\varphi_{q_1}'}, \quad |a_{q_2}^\dagger a_{q_2}' a_{q_3} a_{q_3}'| = \sqrt{\varphi_{q_2}} + \sqrt{\varphi_{q_2}'} + \sqrt{\varphi_{q_3}} + \sqrt{\varphi_{q_3}'}$$

$$\ell(a_{q_1}^\dagger a_{q_1}') = \sqrt{r_{q_1} r_{q_1}'}, \quad \ell(a_{q_2}^\dagger a_{q_2}' a_{q_3} a_{q_3}') = \sqrt{r_{q_2} r_{q_2}' r_{q_3} r_{q_3}'}.$$

$$\begin{align*}
& [a_{q_1}^\dagger a_{q_1}', a_{q_2}^\dagger a_{q_2}' a_{q_3} a_{q_3}']_G = a_{q_2}^\dagger a_{q_2}' a_{q_2} a_{q_2}' a_{q_3} a_{q_3}'
& \quad + (-1)^{|a_{q_1}^\dagger a_{q_1}'|} a_{q_1}^\dagger a_{q_1}' a_{q_2} a_{q_2}' a_{q_3} a_{q_3}'
& \quad - (-1)^{|a_{q_1}^\dagger a_{q_1}'|} a_{q_1}^\dagger a_{q_1}' a_{q_2} a_{q_2}' a_{q_3} a_{q_3}'
& \quad - (-1)^{|a_{q_1}^\dagger a_{q_1}'|} a_{q_1}^\dagger a_{q_1}' a_{q_2} a_{q_2}' a_{q_3} a_{q_3}'
& \quad - (-1)^{|a_{q_1}^\dagger a_{q_1}'|} a_{q_1}^\dagger a_{q_1}' a_{q_2} a_{q_2}' a_{q_3} a_{q_3}'
& \quad - (-1)^{|a_{q_1}^\dagger a_{q_1}'|} a_{q_1}^\dagger a_{q_1}' a_{q_2} a_{q_2}' a_{q_3} a_{q_3}.
\end{align*}$$

(57)

Now fixing $q_1 = 1$, $q_1' = -1$, $q_2 = 1 = q_2'$ and $q_3 = -1 = q_3'$, the expression (57) reduces to

$$[q_2, h_{ss}] = [a_1^\dagger a_{-1}, a_1^\dagger a_{-1} a_{-1}^\dagger a_{-1}]_G$$
\begin{align}
    \ &= a_1^\dagger a_{-1}^\dagger a_1 a_{-1}^\dagger a_{-1} - (-1)^{(1+0)-(0+0+1+1)} a_1^\dagger a_1 a_{-1}^\dagger a_{-1}^\dagger a_{-1}^\dagger a_{-1}^\dagger a_{-1} \\
    \ &= \sqrt{[N]_{-1} N} [N]_{-1} N - [N]_{-1} N \sqrt{[N]_{-1} N} = 0,
\end{align}

showing that \( q_2 \) is a symmetry of \( \mathfrak{h}_{ss} \). Of course, using an ordinary commutator this statement becomes obvious since both \( q_2 \) and \( \mathfrak{h}_{ss} \) are pure functions of the number operator \( N \). But here, the difficulty resides in the fact that we use a different commutation relation which turns out to simplify in the form of the ordinary commutator.

We give the complete set of commutation relations between the supercharges \( q_1 \), \( q_1^\dagger \) and \( q_2 \) and the basic degrees of freedom \( a_1 \), \( a_1^\dagger \), \( a_{-1} \) and \( a_{-1}^\dagger \). The following relations hold:

\begin{itemize}
    \item \( q_1 = a_{-1} a_1 \)
    \[ [q_1, a_1^\dagger]_G = (N + 2)a_{-1} - \sqrt{[N]_{-1} N} a_1 = \left( N + 2 \right) \sqrt{\frac{[N+1]_{-1}}{N+1} - \sqrt{[N]_{-1} N}} a_1, \]
    \[ [q_1, a_{-1}]_G = [N]_{-1} a_{-1}, \]
    \[ [q_1, a_1]_G = \left[ \sqrt{\frac{[N+1]_{-1}}{N+1} - \sqrt{[N]_{-1} N}} \right] (a_1)^3, \]
    \[ [q_1, a_{-1}^\dagger]_G = \sqrt{\frac{[N+1]_{-1} [N+3]_{-1}}{(N+1)(N+3)}} (a_1)^3 = \frac{[N+1]_{-1}}{\sqrt{(N+1)(N+3)}} (a_1)^3. \]

    \item \( q_1^\dagger = a_1^\dagger a_{-1}^\dagger \)
    \[ [q_1^\dagger, a_1^\dagger]_G = \left[ \frac{\sqrt{N-1}_{-1}}{N-1} - \frac{\sqrt{N-2}_{-1}}{N-2} \right] (a_1^\dagger)^3, \]
    \[ [q_1^\dagger, a_{-1}]_G = \frac{[N]_{-1}}{\sqrt{N(N-2)}} (a_1^\dagger)^3, \]
    \[ [q_1^\dagger, a_1]_G = a_1^\dagger \sqrt{N-1} \sqrt{N} - (N+1)a_{-1}^\dagger = \left[ \sqrt{[N]_{-1} N} - (N+1) \sqrt{[N]_{-1} N} \right] a_1^\dagger, \]
    \[ [q_1^\dagger, a_{-1}^\dagger]_G = [N-1]_{-1} a_1^\dagger. \]

    \item \( q_2 = a_{-1}^\dagger \)
    \[ [q_2, a_{-1}^\dagger]_G = \left[ \sqrt{[N]_{-1} N} - \sqrt{[N-1]_{-1} (N-1)} \right] a_{-1}^\dagger, \]
    \[ [q_2, a_{-1}]_G = [N]_{-1} a_{-1}^\dagger, \]
    \[ [q_2, a_1]_G = \sqrt{[N]_{-1} N} a_1 - (N+1)a_{-1} = \left[ \sqrt{[N]_{-1} N} - \sqrt{[N+1]_{-1} (N+1)} \right] a_1, \]
    \[ [q_2, a_{-1}^\dagger]_G = \sqrt{[N+1]_{-1} (N+1)} a_{-1} = [N+1]_{-1} a_{-1}. \]
\end{itemize}

In summary, these operators map bosons on a mixed combination of bosons and fermions which cannot be reduced to a deformation of a fermion. On the other hand, fermions can be exactly mapped onto bosons up to a deformation function or a nonlinearity appearing as a power of a bosonic operator.
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