A NOTE ON HARRIS MORRISON SWEEPING FAMILIES OF MAXIMAL GONALITY

BEORCHIA VALENTINA AND ZUCCONI FRANCESCO

ABSTRACT. In [HMo, theorem 2.5] Harris and Morrison construct semistable families \( f : F \to Y \) of \( k \)-gonal curves of genus \( g \) such that for every \( k \) the corresponding modular curves give a sweeping family in the \( k \)-gonal locus \( \mathcal{M}_g^k \). Their construction depends on the choice of a smooth curve \( X \). We show that if the genus \( g(X) \) is sufficiently high with respect to \( g \) then the ratio \( \frac{K_F^2}{\chi(O_F)} \) is 8 asymptotically with respect to \( g(X) \). Moreover, if the conjectured estimates given in [HMo, p. 351-352] hold, we show that if \( g \) is big enough, then \( F \) is a surface of positive index.

CONTENTS

1. Introduction. 1
2. The basic construction. 2
  2.1. Harris-Morrison families. 2
  2.2. Numerical invariants. 5
  2.3. The proof of the Theorem 9
3. Maximal gonality and surfaces of positive index. 10
References 11

1. INTRODUCTION.

Let \( \mathcal{M}_g \) be the moduli space of stable curves of genus \( g \). A family \( \mathcal{B} \) of curves \( Z \subset \mathcal{M}_g \) is said to be a sweeping family if \( \cup \mathcal{B} \) is a Zariski dense subset of \( \mathcal{M}_g \). The invariants of sweeping families can furnish important information on the geometry of \( \mathcal{M}_g \); see: [CFM].

In this paper we focus on the geometry of the families of curves constructed in [HMo, Theorem 2.5]. We revise this construction in Section 2; here we point out only that it depends on choosing a smooth curve \( X \) of genus \( g(X) \). For the aim of [HMo] \( g(X) \) can be taken to be zero, but Harris Morrison construction applies then to any curve \( X \).

We compute the invariants of the smooth surface \( F \) supporting the Harris Morrison family. We denote by \( e(F) \), \( \chi(O_F) \), \( K_F^2 \) respectively the topological characteristic of \( F \), its cohomological characteristic and the self-intersection of the canonical divisor. We also denote by \( q(F) \) the irregularity of \( F \). We have:

**Theorem** Let \( g, k \in \mathbb{N} \) such that \( k \) can occur as the gonality of a smooth curve of genus \( g \). There exist Harris Morrison sweeping \( k \)-gonal semistable genus-\( g \) fibrations \( f : F \to Y \) such that the ratio \( \frac{K_F^2}{\chi(O_F)} \) is 8 asymptotically with respect to \( g(X) > 0 \).

1991 *Mathematics Subject Classification.* Primary 14D22, 14H51; Secondary 14H10.

*Key words and phrases.* moduli of curves, fibration, gonality.
See Theorem 2.18 for a refined statement. We stress that our Theorem is not obtained by a base change over a family as in [HMo] Theorem 2.5 starting from $\mathbb{P}^1$. We remark that the above result applies to any gonality $k$.

If we consider only those families with maximal gonality, then we show that (see Proposition 3.2):

**Proposition** Let $f : F \to Y$ be an Harris Morrison genus-$g$ fibration starting from any plane curve $X$ of genus $g(X) \gg 0$. Assume that the gonality $k$ is maximal and that the conjectured estimates in [HMo] p. 351-352 are true. If $g$ is big enough, then $F$ is a surface of positive index i.e. $K_F^2 > 8\chi(O_F)$. Moreover if $f : F \to Y$ is general in its class then the irregularity of $F$ is $g(Y)$. In particular, $f : F \to Y$ is the Albanese morphism of $F$.

We recall that any surface of general type satisfies the Miyaoka-Yau inequality $K_F^2 \leq 9\chi(O_F)$, and equality holds if and only if $F$ is a ball quotient. Surfaces of positive index satisfy the inequalities $8\chi(O_F) < K_F^2 < 9\chi(O_F)$. These surfaces are still quite mysterious objects [Re], [MT], [My].

The statement of the Proposition above is close to the recent results of Urzúa. He has found in [Ur1] simply connected projective surfaces of general type with $K_F^2$ arbitrarily close to $\frac{26}{71}$; this improves previous estimates by Persson-Peters-Xiao [PPX]. By his method, based on a generalisation of the method of line arrangements on $\mathbb{P}^2$, he finds interesting surfaces of positive index; see: [Ur2]. Thanks to the Kapranov construction of the moduli space $\overline{M}_{0,d+1}$ of rational curves with $d + 1$ marked points, see [Ka1], [Ka2], line arrangements correspond to curves inside $\overline{M}_{0,d+1}$. In this sense Urzúa construction is related to the Harris Morrison one. Hence the fact that our results on Harris Morrison construction of $k$-gonal sweeping families match those obtained in [Ur2] is an evidence for the conjectural results stated in [HM].

**Acknowledgment.** The authors would like to thank D. Chen for his comments on the first version of the paper.

This research is supported by MIUR funds, PRIN project Geometria delle varietà algebriche (2010), coordinator A. Verra. The first author is also supported by funds of the Università degli Studi di Trieste - Finanziamento di Ateneo per progetti di ricerca scientifica - FRA 2011.

2. **The basic construction.**

2.1. **Harris-Morrison families.** We review and we explain some features of the basic construction of [HMo]. We could construct a slight refinement of the basic construction of [HMo] Section 2] using not a product surface $X \times \mathbb{P}^1$ but a ruled surface $\mathbb{P}(\mathcal{E}) \to X$ over $X$, but we have decided to follow [HMo] Section 2] for simplicity.

We denote by $[N]$ and $\widetilde{N}$ a finite set and its order respectively. If $[M]$ is a subset of $[N]$ and is invariant under the action of $\mathfrak{S}_k$ we will denote the quotient $[M]/\mathfrak{S}_k$ by $[M]$.

Let $\overline{M}_{0,b}$ be the compactification of the moduli space of $b$-pointed stable curves of genus 0 and let $\beta : \overline{H}_{k,b} \to \overline{M}_{0,b}$ be the natural morphism on the Hurwitz scheme of admissible $k$-covers of stable $b$-pointed curves of genus 0 constructed in [HM]. Then the morphism $\beta : H_{k,b} \to \overline{M}_{0,b}$ is a $\widetilde{N}(k,b)$ sheeted unramified covering, where $\widetilde{N}(k,b)$ counts the $k$-sheeted connected covers of $\mathbb{P}^1$ simply branched at $b$ fixed points. From now on we set $N := N(k,b)$, where $N(k,b)$ is defined in [HMo] page 334. Concerning the subset $[N]$ we recall that $\mathfrak{S}_k$ acts on $[N]$ and by [HMo] Lemma 1.24 this action is free if $k \geq 3$ and trivial for $k = 2$, so $N = k! \widetilde{N}$ if $k \geq 3$ and $N = \widetilde{N}$ if $k = 2$.

Let $X$ be any smooth complete curve and let $\pi_X : X \times \mathbb{P}^1 \to X$ be the projection. By [Ha] $\text{NS}(X \times \mathbb{P}^1) = [s] \mathbb{Z} \oplus [f] \mathbb{Z}$ where $[s]$ is the numerical class of a $\pi_X$-section $s$ and $[f]$ is the numerical
class of a $\pi_X$-fiber $f$. In the sequel we will not distinguish between $s$ and its class $[s]$ if no danger of confusion can arise.

Let $s$ be a $\pi_X$-section with $s^2 = 0$. Let $b \in \mathbb{N}$ and let $C_1, \ldots, C_b$ be effective divisors on $X$ of degrees $c_i > 0$, such that $O_{X, \mathbb{P}^1}(s + \pi_X^* C_i)$ is very ample, $i = 1, \ldots, b$. Let $\sigma_i \in [s + \pi_X^* C_i]$ be a smooth curve for every $i = 1, \ldots, b$, such that the sections $\sigma_i$ meet transversely everywhere and such that each $\pi_X$-fiber contains at most one of these intersections.

We set

$$\sigma := \sum \sigma_i$$

and we point out that $[\sigma] = b[s] + [\pi_X^* C] = b[s] + c[f]$ where $C = \sum C_i$ and $c = \sum c_i$. The reader is warned that if $X = \mathbb{P}^1$ the above construction easily works. If $X$ is a curve of positive genus then it is sufficient to assume that for every $i = 1, \ldots, b$ the number $c_i$ is at least equal to the degree of a very ample divisor on $X$.

Let $[I_X]$ be the set of nodes of $\sigma$ or equivalently the set of intersections of the $\sigma_i$’s. Then

$$I_X = (b - 1)c;$$

see: [HMo] Formula 2.1 on page 338.

Let $B_X$ be the blow-up of $X \times \mathbb{P}^1$ at $[I_X]$ and denote by $B_x$ its fiber over $x \in X$ and $\tilde{\sigma}_i$ the strict transform of $\sigma_i$ on $B_X$. Let $\alpha: X \to \overline{M}_{0,b}$ be the map which sends a point $x \in X$ to the class of the fiber $B_x$ marked by its $b$ points of intersections with the $\tilde{\sigma}_i$ where $i = 1, \ldots, b$. Let $Y := X \times \overline{M}_{0,b} \tilde{\mathcal{H}}_{K,b}$. By the same argument of [HMo] pages 338-339 the induced map $\mu: Y \to X$ is a covering of degree $\tilde{N}$. Let $\pi_Y: Y \times \mathbb{P}^1 \to Y$ be natural projection. If we set $\nu := \mu \times \text{id}: Y \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ we see that $[I_Y] = \nu^*([I_X])$ is the scheme of singularities of the divisor

$$\tau := \nu^* \sigma$$
on $Y \times \mathbb{P}^1$. Let $[J_Y] \subset Y$ be the $\pi_Y$-projection of $[I_Y]$ and let $[J_X] \subset X$ be the $\pi_X$-projection of $[I_X]$. By construction we have a map from the complement of $[J_Y]$ to $\overline{M}_{0,b}$, and we extend it to a morphism $\rho: Y \to \overline{M}_{0,b}$ following the argument of [HMo] Theorem 2.15.

Here we recall first that $\mu: Y \setminus [J_Y] \to X \setminus [J_X]$ is unramified and that $[J_Y]$ is partitioned into three classes:

$$[J_Y] = [J_Y,(1)] \cup [J_Y,(2,2)] \cup [J_Y,(3)].$$

The topological description given in [HMo] Page 340 yields that $\mu$ is not ramified on $[J_Y,(1)] \cup [J_Y,(2,2)]$ and it is triply branched at points of $[J_Y,(3)]$. In particular

$$I_Y = \left( \tilde{N}_1 + \tilde{N}_2 + \frac{\tilde{N}_3}{3} \right) (b - 1)c;$$

where $\tilde{N} = \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3$ and a combinatorical definition of $\tilde{N}_1, \tilde{N}_2, \tilde{N}_3$ is given in [HMo] Proposition 2.9.

Following [HMo] we blow-up $\epsilon: A \to Y \times \mathbb{P}^1$ at the points of the set $[I_Y,(1)]$ and we construct a finite cover $\pi: G \to A$ and a semistable fibration $f: F \to Y$ see [HMo] Diagram (2.2), page 338] such that over the complement $Y'$ of $[J_Y]$ in $Y$ the surface $G$ is obtainable by the pull-back of the smooth universal admissible cover $\theta: \mathcal{U}_{H_{b, k}} \subset \mathbb{P}^1 \times \mathcal{H}_{k,b} \to \mathcal{H}_{k,b}$ whose existence is shown in [HM]. We recall that over $Y'$, $G$ is a stable fibration $G_{Y'} \to Y' \subset Y$, it coincides with $f: F \setminus f^{-1}([J_Y]) = F_{Y'} \to Y'$. Moreover $A$ coincides with $Y \times \mathbb{P}^1$ over $Y'$ and there exists a finite morphism $G_{Y'} = F_{Y'} \to A_{Y'} = \mathbb{P}(O_Y \oplus O_Y)$.

By the very explicit local analytic description of the finite cover $\pi: G \to A$ and of the blow-up $\epsilon: A \to Y \times \mathbb{P}^1$ done in [HMo] from page 343 to the top of page 347 we obtain an explicit description
of the smooth surface $F$ and of the semistable fibration $f: F \to Y$ inducing the desired moduli map $\rho: Y \to \mathcal{M}_g$. We set $Z := \rho(Y)$. We sum up the above construction in the following diagram:

\[
\begin{array}{c}
S \\
\downarrow u \\
G \\
\downarrow \pi \\
\uparrow k:1 \\
A \\
\downarrow \nu \\
Y \times \mathbb{P}^1 \pi_Y \\
\downarrow \mu \\
\bar{N}_1 \\
\downarrow \pi \\
X \times \mathbb{P}^1 \pi_X \\
\downarrow \alpha \\
\bar{M}_{0,b}
\end{array}
\]

What makes this families very interesting is that thanks to the work done in [HMo] the dominant morphism $\zeta: G \to F$ is very explicit and hence the nature of the fibers of $f: F \to Y$ over $[J_Y]$ is evident.

If $y \in Y$ we denote by $F_y$ and $G_y$ respectively the $f$-fiber over $y$ and the $(f \circ \zeta)$-fiber over $y$.

The analysis of fibers of type (1), that is $y \in [J_{Y,(1)}]$, splits into four cases: $(1_0)$, $(1_{j,0})$ $(1_{j,g})$, $(1_{j,i})$ where $j > 0$ and $1 \leq i \leq g/2$; see [HMo] Diagram (2.12)] and with obvious notation we write $y \in [J_{Y,1}]$ if $y \in [J_{Y,(1)}]$ and $y$ is of type II.

If $y \in [J_{Y,(1_0)}]$ then $F_y = F'_y \cup E$ where $F'_y$ is a genus $g - 1$ curve and $E$ is a rational $(-2)$ curve; see the top of diagram [HMo] Diagram (2.12)]. The corresponding fiber $G_y$ on $G$ is

\[G_y = G''_y \cup E' \cup \bigcup_{j=1}^{k-2} E_{j,y}\]

where $G''_y$ is a genus $g - 1$ curve, $E_{j,y} = -1$ and $E'$ is a $-2$ curve. Over a neighbourhood of $y$ the morphism $\zeta: G \to F$ consists on the contraction of the $k - 2$ rational curves $E_{j,y}$.

If $y \in [J_{Y,(1_{j,0})}]$ or $y \in [J_{Y,(1_{j,g})}]$ then $F_y$ is a smooth semistable fiber and over a neighbourhood of $y$ the morphism $\zeta: G \to F$ consists on an ordered contraction of $k$ rational curves. More precisely if $y \in [J_{Y,(1_{j,0})}] \cup [J_{Y,(1_{j,g})}]$ then

\[G_y = G'_y \cup E'_y \cup G''_y \cup \bigcup_{i=1}^{k-2} E_{i,y}\]

where $G'_y$ is a rational curve, $E'_y$ is a $(-2)$-rational curve, $E'_y \cdot G'_y = 1$, $G''_y$ is a smooth curve of genus $g$, $E'_y \cdot G''_y = 1$, $G''_y$ is a smooth curve of genus $g$ and $E_{i,y} = -1$. The map $\zeta$ first contracts $E_{i,y}$ where $i = 1, \ldots, k - 2$. After this contraction process the image of $G'_y$ is a $-1$ curve which intersects the image of $E'_y$ in a unique point. Hence to get the semistable reduction $F$ we need first to contract the image of $G_y$ and then the image of $E'_y$.

If $y \in [J_{Y,(1_{j,i})}]$ where $j > 0$ and $1 \leq i \leq [g/2]$ then $F_y = F_{y,i} \cup E_y \cup F_{y,g-i}$ where $F_{y,i} \cap F_{y,g-i} = \emptyset$, $E_y^2 = -2$, $F_{y,i} \cdot E_y = F_{y,g-i} \cdot E_y = 1$ and

\[G_y = G'_y \cup E'_y \cup G''_y \cup \bigcup_{i=1}^{k-2} E_{i,y},\]
where $E_{i,y}$ are $-1$-rational curves, $E_y$ is a $(−2)$ curve. Moreover $F_{y,i}$ is smooth of genus $i$ and admits a $j$ covering over the fiber $\mathbb{P}^1_y \subset Y \times \mathbb{P}^1$ and $F_{y,g-i}$ is smooth of genus $g-i$ and admits a $k−j$ covering over $\mathbb{P}^1_y$. Note that if $e_y \subset A$ is the exceptional curve over $y$ arising in the blow-up $A \rightarrow Y \times \mathbb{P}^1$ then the restriction of $\pi \colon G \rightarrow A$ to $E'_y \cup \left(\bigcup_{i=1}^{k-2} E_{i,y}\right)$ gives the $k$-cover of $e_y$ contained in $G$. Over a neighborhood of $y$ the morphism $\zeta : G \rightarrow F$ consists on the contraction of the $k-2$ rational curves $E_{i,y}$.

**Lemma 2.1.** If $y \in [J_Y,(2,2)] \cup [J_Y,(3)]$ then the fiber $F_y$ of $f : F \rightarrow Y$ over $y$ is a smooth semistable curve. The rational map $F \dashrightarrow Y \times \mathbb{P}^1$ is finite in a neighborhood of $F_y$. Moreover the surfaces $G$ and $F$ coincide over an analytic neighborhood of $y \in Y$.

**Proof.** See [HMo] Pages 343-346. □

**Proposition 2.2.** Over each of the points of $[I_X] = \pi_X([J_X]) \subset X$ there are $\tilde{N}_{\text{sing}}$ points $y \in Y$ such that the fiber $F_y \subset F$ is singular. Any singular fiber of $f : F \rightarrow Y$ contains a unique $(−2)$ rational curve.

**Proof.** This is a restatement of [HMo] Theorem 3.1 and it easily follows by the above description of the singularities of the fibration $f : F \rightarrow Y$.

**Definition 2.3.** A semistable fibration $f : F \rightarrow Y$ as the one of Proposition 2.2 is called a Harris-Morrison family.

Note that by Harris Morrison construction it follows

**Proposition 2.4.** Let $X$ be any smooth complete curve of genus $g(X)$. Let $g \geq 3$ be any natural number. If $g$ is odd let $k = \frac{g+3}{2}$ and set $k = \frac{g+2}{2}$ if $g$ is even. Then the family of curves $Z \subset \overline{\mathcal{M}}_g$ constructed by Harris and Morrison varying the curve $X$ forms a sweeping family.

**Proof.** The chosen number $k$ is the maximal gonality for a curve of genus $g$. Hence the claim follows by a standard result of Brill-Noether theory. See also [HMo] page 350]. □

**Remark.** Notice that, even without the assumption of maximal gonality, the families of curves $Z \subset \overline{\mathcal{M}}_g$ constructed by Harris and Morrison varying the curve $X$ depend freely on $g(X)$. In particular we will study those families such that the parameter $g(X) >> 0$.

### 2.2. Numerical invariants.

We recall that $\epsilon : A \rightarrow Y \times \mathbb{P}^1$ is the blow-up of $[I_Y,(1)]$ which is a reduced scheme of length $(b-1)c\tilde{N}_1$. Let $E_A$ be the exceptional divisor of $\epsilon : A \rightarrow Y \times \mathbb{P}^1$. Hence

\begin{equation}
E_A^2 = -I_{Y,(1)} = -(b-1)c\tilde{N}_1.
\end{equation}

Let $\tau$ be the strict transform of $\tau$, that is $\tau = \epsilon^*_A(\tau) - 2E_A$. Since $\tau = \nu^*(\sigma) = \nu^*(bs + cf)$ then

\begin{equation}
(\tau)^2 = 2bc\tilde{N} - 4(b-1)c\tilde{N}_1.
\end{equation}

Let $R$ be the ramification divisor of $\pi : G \rightarrow A$. The following relation holds:

\begin{equation}
\pi_* R = \tau.
\end{equation}

Since we have simple branch on the fibers the local analysis shows that

\begin{equation}
\pi^* \pi_* R = 2R + \bar{R}
\end{equation}

and that:
\[ R \cdot \widetilde{R} = (b-1)c \left( 2\widetilde{N}_{2,2} + \frac{4}{3}\widetilde{N}_3 \right). \]

**Proposition 2.5.** The ramification divisor \( R \) of the finite cover \( \pi: G \to A \) satisfies:

\[ R^2 = \left( \widetilde{N} + (b-1) \left( \frac{4}{3}\widetilde{N}_3 - \widetilde{N}_1 \right) \right) c \]

**Proof.** It is as in [HMo, Theorem 2.15]. \( \square \)

By construction we have seen that over any point \( y \in [J_{Y,(1)}] \) there exist \( k - 2 \) rational curves \( E_{i,y} \subset G_y \) such that \( E_{i,y}^2 = -1 \). In particular there are \((k-2)\widetilde{N}_1(b-1)c\) exceptional curves contained inside fibers of \( f_G: G \to Y \) over \( J_{Y,(1)} \).

**Lemma 2.6.** Let \( u: G \to S \) be the contraction of these \((k-2)\widetilde{N}_1(b-1)c\) rational curves. Then \( S \) is a smooth surface. Moreover \( K^2_S = K^2_G + (k-2)\widetilde{N}_1(b-1)c \).

**Proof.** It follows straightly from Castelnuovo contraction theorem. \( \square \)

Now we want to compute the invariants of \( S \).

**Proposition 2.7.** Let \( E \) be any \(-1\) rational curve which is contracted by \( u: G \to S \). Then \( R \cdot E = 0 \).

**Proof.** We use the ramification divisor formula: \( K_{G|Y} = \pi^*K_{A|Y} + R \). Let \( E \) be any \(-1\) rational curve which is contracted by \( u: G \to S \) then the finite morphism \( \pi: G \to A \) restricts to an isomorphism \( \pi_{|E}: E \to L \subset A \) where \( L \) is an exceptional curve produced via the blow-up \( \epsilon: A \to Y \times \mathbb{P}^1 \). In particular \( \pi_*E = L \). Then \( R \cdot E = -1 - \pi^*(K_{A|Y})E = -1 - K_{A|Y} \cdot L = -1 + 1 = 0 \). \( \square \)

**Corollary 2.8.** Let \( E \) be any \(-1\) rational curve which is contracted by \( u: G \to S \). Then \( E \cdot \widetilde{R} = 2 \).

**Proof.** By Equation 2.5 and by Proposition 2.7 we have: \( 0 = E \cdot R = E \cdot \frac{1}{2}(\pi^*\pi_*R - \widetilde{R}) = \frac{1}{2}(L \cdot \tau - E \cdot \widetilde{R}) \).

Since \( \tau = \epsilon^*(\tau) - 2E_A \) and \( E_A \cdot L = -1 \) then the claim follows. \( \square \)

We now consider the \((-2)\) rational curves of \( G \). We notice that if \( y \in [J_{Y,1(2,0)}] \cup [J_{Y,1(2k-2,0)}] \) then there is a \((-2)\) rational curve \( G'_y \subset G_y \) which is mapped to a fiber of the canonical projection \( \pi_{Y}: Y \times \mathbb{P}^1 \to Y \); that is: \( \epsilon \circ \pi(G'_y) = \pi_{Y}^{-1}(y) \subset Y \times \mathbb{P}^1 \). Next lemma deals with the \((-2)\) curves which are mapped to the exceptional curves of the morphism \( \epsilon: A \to Y \times \mathbb{P}^1 \).

**Lemma 2.9.** Let \( E \) be any \((-2)\) rational curve which is contained in a fiber of \( f_G: G \to Y \) and which is mapped 2-to-1 to a curve \( L \subset A \) such that \( \epsilon(L) \in [I_Y] \). Then \( R \cdot E = 2 \). Moreover \( E \cdot \widetilde{R} = 0 \)

**Proof.** If \( E \) is a \((-2)\) curve contained in a fiber of \( f_G: G \to Y \) which is mapped to an exceptional curve \( L \) of the morphism \( \epsilon: A \to Y \times \mathbb{P}^1 \) then \( \pi_{|E}: E \to L \) is a 2-to-1 covering. Hence \( \pi_*E = 2L \). Then \( 0 = E \cdot K_{G|Y} = E \cdot (\pi^*K_{A|Y} + R) = 2L^2 + E \cdot R \). This implies \( R \cdot E = 2 \). Then \( E \cdot \widetilde{R} = E \cdot \pi^*\pi_*R - 4 = 2L \cdot \tau - 4 = 0 \). \( \square \)

By the local description of \( f: S \to Y \) we know that there are \((b-1)c \left[ \widetilde{N}_{\text{sing}} + \sum_{j=0}^{12} (\widetilde{M}_{j,0} + \widetilde{M}_{j,9}) \right] \) mutually disjoint \((-2)\) rational curves, where the \([M_{j,i}]\) are given in [HMo] Formula 1.35.

**Lemma 2.10.** \( \sum_{j=0}^{12} (\widetilde{M}_{j,0} + \widetilde{M}_{j,9}) \leq \widetilde{N}_1 \).
Proof. By [HMo] Formula 1.30 $[N_j] = \bigcup_{i=0}^{1} [M_j]$ where $[M_j]$ is defined in [HMo] Formula 1.28. Since by [HMo] Formula 1.36 $[M_j] = \bigcup_{i=1}^{\theta} [M_j,i]$ the claim follows. $\square$

Moreover let

\[(2.7)\]

\[e := \sum_{j=0}^{\frac{\theta}{2}} (\bar{M}_{j,0} + \bar{M}_{j,\theta}).\]

Obviously $(b - 1)ce = \sum_{j=0}^{\frac{\theta}{2}} (J_{Y,1,j,0} + J_{Y,1,j,\theta})$. We set

\[\bigcup_{j=0}^{\frac{\theta}{2}} ([J_{Y,1,j,0}] \cup [J_{Y,1,j,\theta}]) =: \{a_1, \ldots, a_{(b - 1)ce}\} \subset Y.\]

We recall that the morphism $h: S \to F$ contracts two rational curves of any fiber of $f_S: S \to Y$ over $\{a_1, \ldots, a_{(b - 1)ce}\} \subset Y$. Moreover the fiber $F_{a_i}$, $i = 1, \ldots, (b - 1)ce$ of $f: F \to Y$ is smooth, but over a neighbourhood $U_i$ of $a_i \in Y$ the map $\rho_F: F_U \to U \times \mathbb{P}^1$ is only a rational one. We know that the fiber $f_S(a_i) := S_{a_i} = F'_{a_i} + E_{a_i} + E'_{a_i} \subset S$ where $F'_{a_i}$ is a smooth curve of genus $g$, $E_{a_i}$ is a $(-2)$ rational curve such that $E_{a_i} \cdot F'_{a_i} = 1$. Moreover we know that $E'_{a_i}$ is a $-1$-rational curve and $E_{a_i} \cdot E'_{a_i} = 1$.

Corollary 2.11. Let $h: S \to F$ be the morphism which factorises the morphism $\zeta: G \to F$. Then $K_F^2 = K_S^2 + 2(b - 1)ce$.

Proof. By Harris Morrison construction the contraction of $k - 2$ rational curves in each one of the $(b - 1)ce$ singular fibers of $f_G: G \to Y$ gives the morphism $u: G \to S$. To reach $F$ we need to contract first $E_{a_i}$ and then the image of $E_{a_i}$ for every $i = 1, \ldots, (b - 1)ce$. Hence $\zeta: G \to F$ is given by $h\zeta = h \circ u$ where $h: S \to F$ is a composition of $2(b - 1)ce$ simple contractions. Hence $K_F^2 = K_S^2 + 2(b - 1)ce$. $\square$

By Lemma 2.1 the points of type $(2, 2)$ and the points of type $(3)$ give a smooth semistable fiber of $f: F \to Y$. By Proposition 2.2 there are $\bar{N}_{\text{sing}}(b - 1)ce$ singular fibers of $f: F \to Y$. Set

\[r := \bar{N}_{\text{sing}}(b - 1)ce.\]

Let $\{y_i\}_{i=1}^{r} = \text{Sing}(f)$ be the subset of $Y$ given by the points $y \in Y$ such that $f^{-1}(y) = F_y$ is a singular fiber. By the local description of the singular fibers of $f: F \to Y$ it follows that for any $l = 1, \ldots, r$, $f^{-1}(y_l) = F_{y_l} = F_{y_l,i} \cup E_{y_l} \cup F_{y_l,g-i}$ where $E_{y_l}$ is a $(-2)$ curve, $F_{y_l,i} \cdot E_{y_l} = F_{y_l,g-i} \cdot E_{y_l} = 1$ and obviously $F_{y_l,i} \cdot F_{y_l,g-i} = 0$, or $F_{y_l} = F'_{y_l} \cup E$ where $F'_{y_l}$ is a genus $g - 1$ curve and $E$ is a rational $(-2)$ curve.

We study the invariants of the surface $F$.

Proposition 2.12. Let $f: F \to Y$ be a Harris Morrison family. Then the topological Euler characteristic of $F$ is:

\[e(F) = 4(g - 1)(g(Y) - 1) + 2r.\]

Proof. By a standard topological argument it follows that $e(F) = 4(g - 1)(g(Y) - 1) + \sum_{i=1}^{r} (e(F_{y_i}) - (2 - 2g))$. By the analysis of singular fibers, see Proposition 2.2, it follows that for every $i = 1, \ldots, r$ $e(F_{y_i}) = (2 - 2g) = 2$. Then $e(F) = 4(g - 1)(g(Y) - 1) + 2r$. $\square$
We point out the reader the very important relation:

\[ \tilde{N}_1 = e + \tilde{N}_{\text{sing}} \]

where \( e \) is defined in Equation 2.7.

**Proposition 2.13.** Let \( f : F \to Y \) be an Harris Morrison semistable fibration. Then

\[ K_F^2 = c \left[ (b - 1) \left[ 2e + \tilde{N}_1 + (8g - 7) \frac{\tilde{N}_3}{3} \right] - 3\tilde{N} \right] + 8\tilde{N}(g(X) - 1)(g - 1) \]

**Proof.** We notice that by Corollary 2.11 \( K_F^2 = 2c(b - 1)c + K_S^2 \). Then by Lemma 2.6 \( K_F^2 = 2c(b - 1)c + (k - 2)(b - 1)c\tilde{N}_1 + K_G^2 \). By definition of \( R \) we have \( K_G^2 = R^2 + 2R \cdot \pi^*K_A + (\pi^*K_A)^2 \). By Equation 2.2 we have

\[ (\pi^*K_A)^2 = cK_A^2 = -k \left[ 8(g(Y) - 1) + (b - 1)c\tilde{N}_1 \right] . \]

By projection formula and by Equation 2.4 we have:

\[ 2R \cdot \pi^*K_A = 2\tau K_A = 2b(g(X) - 1) - 2e \tilde{N} + b(b - 1)c\frac{2\tilde{N}_3}{3} . \]

Now the claim follows by Proposition 2.5 taking into account that \( b = 2g + 2k - 2 \) and

\[ 2g(Y) - 2 = \tilde{N}(2g(X) - 2) + \frac{2}{3}(b - 1)c\tilde{N}_3 . \]

since Riemann-Hurwitz formula.

We put

\[ \alpha := (b - 1) \left[ 2e + \tilde{N}_1 + (8g - 7) \frac{\tilde{N}_3}{3} \right] - 3\tilde{N} \]

**Corollary 2.14.** For every fibration constructed as in [HM, Theorem 2.5] we can write by Proposition 2.13 and by Equation 2.10

\[ K_F^2 = c\alpha + 8\tilde{N}(g(X) - 1)(g - 1) \]

where the coefficient \( \alpha \) does not depend on \( g(X) \).

**Proposition 2.15.** If \( f : F \to Y \) is the fibration constructed in [HM, Theorem 2.5] then

\[ \chi(O_F) = \frac{c}{12} \left[ (b - 1) \left[ +3\tilde{N}_1 + (12g - 11) \frac{\tilde{N}_3}{3} \right] - 3\tilde{N} \right] + \tilde{N}(g(X) - 1)(g - 1) . \]

**Proof.** By Noether Identity and by Proposition 2.12 we have

\[ 12\chi(O_F) = K_F^2 + 4(g - 1)(g(Y) - 1) + 2\tilde{N}_{\text{sing}}(b - 1)c . \]

By Proposition 2.13 we can write

\[ 12\chi(O_F) = c \left[ (b - 1) \left( 2e + \tilde{N}_1 + (8g - 7) \frac{\tilde{N}_3}{3} \right) - 3\tilde{N} \right] + 8\tilde{N}(g(X) - 1)(g - 1) + 4(g - 1)(g(Y) - 1) + 2\tilde{N}_{\text{sing}}(b - 1)c . \]
By Equation 2.8 and by Equation 2.9 we next obtain:
\[
12\chi(O_F) = c \left[ (b - 1) \left( 3\tilde{N}_1 + (12g - 11)\frac{\tilde{N}_3}{3} \right) - 3\tilde{N} \right] + 12\tilde{N}(g(X) - 1)(g - 1)
\]
and the claim follows. \(\square\)

We put
\[
(2.12) \quad \alpha' := \frac{1}{12} \left[ (b - 1) \left( 3\tilde{N}_1 + (12g - 11)\frac{\tilde{N}_3}{3} \right) - 3\tilde{N} \right].
\]

In particular we have:

**Corollary 2.16.** For every fibration constructed as in [HMo, Theorem 2.5] we can write by Proposition 2.15 and by Equation 2.12:
\[
(2.13) \quad \chi(O_F) = \alpha'c + \tilde{N}(g(X) - 1)(g - 1).
\]

where the coefficient \(\alpha'\) does not depend on \(g(X)\).

**Corollary 2.17.** Let \(X\) be any smooth complete curve with genus \(g(X)\). Let \(f: F \rightarrow Y\) be the fibration constructed in [HMo, Theorem 2.5] then
\[
K_F^2 - 8\chi(O_F) = c \left[ (b - 1) \left( 2e - \tilde{N}_1 + \frac{\tilde{N}_3}{3} \right) - \tilde{N} \right].
\]

**Proof.** It follows by Corollary 2.15 and by Proposition 2.13. \(\square\)

### 2.3. The proof of the Theorem

Now we want to analyse the relation between the fiber degrees \(c_i\) for the divisors \(\sigma_i \in |s + \pi_XC_i|\) and the genus \(g(X)\) in order that a family \(f: F \rightarrow Y\) as in [HMo, Theorem 2.5] exists if we start from a curve \(X\). Since we are working over a product surface \(X \times \mathbb{P}^1\), a divisor \(L\) which is numerically equivalent to \(s + c_i f\) is very ample iff \(L|_{s=0}\) is very ample; hence a necessary condition is that \(c_i\) is the degree of a very ample divisor on \(X\). On the other hand if \(O_X(l)\) is a very ample sheaf on \(X\) then \(s + \pi_X(l)\) is a very ample divisor on \(X \times \mathbb{P}^1\). For a general \(X\) if \(O_X(l)\) is a nonspecial very ample divisor then by Halphen theorem [Ha, Proposition 6.1] it follows that \(c_i \geq g(X) + 3\). Then if we want to construct families \(f: F \rightarrow Y\) as in [HMo, Theorem 2.5] starting from a curve \(X\) where \(c\) is small with respect to \(g(X)\), definitely we need to consider curves \(X\) with very ample special divisors.

**Theorem 2.18.** Let \(g, k \in \mathbb{N}\) with \(3 \leq k \leq \left\lfloor \frac{(g+3)}{2} \right\rfloor\). For every real number \(\epsilon > 0\), there exists a real number \(\Delta(\epsilon) \geq 0\) such that there are families \(f: F \rightarrow Y\) obtained by the Harris Morrison construction starting from any plane curve \(X\) of genus \(g(X) \geq \Delta(\epsilon)\) such that the following holds:
\[
8 - \epsilon \leq \frac{K_F^2}{\chi(O_F)} \leq 8 + \epsilon.
\]

**Proof.** Since \(X\) is a plane curve of genus \(g(X)\) then by Clebsh formula its degree is \(d(X) = \frac{3 + \sqrt{8g(X) + 1}}{2}\). We can consider \(c_i = d(X)\) where \(i = 1, \ldots, b\) where \(b = 2(g + k - 1)\). Then \(c\) can be taken equal to \(bd(X)\). For every Harris-Morrison family we can write:
Theorem 2.18 can be extended easily to subcanonical curves

Proof. By Equation 2.11, by Equation 2.13 and by Corollary 2.17 we have\[\text{[HMo, p. 351-352]}
\]
conjectured estimate in \(N\)

\[\text{Since the parameters } \alpha, \alpha' \text{ do not depend on } g(X) \text{ since } g, k \text{ are fixed and since } d(X) = \frac{3+\sqrt{8g(X)+1}}{2}\]
then we can find\[\frac{2g(X)-2}{g+\sqrt{8g(X)+1}} \geq \frac{b(\alpha-8\alpha')-\epsilon\alpha'}{\epsilon(g-1)}\]
to obtain \(|\frac{K^2_F}{\chi(O_F)} - 8| \leq \epsilon. \]

\[\square\]

Remark. Theorem 2.18 can be extended easily to subcanonical curves \(X\) where the subcanonical degree is sufficiently small with respect to \(g(X)\).

\[\text{Remark. We point out the reader that the invariants of surfaces of general type which supports fibrations as those of Proposition 2.3 are strongly influenced by the base } Y \text{ of the fibration, in a way which is quite new for the theory of surfaces of general type, as far as we know.} \]

We have shown Theorem stated in the Introduction.

3. Maximal gonality and surfaces of positive index.

In \([HMo]\) the genus \(g(X)\) plays no role, see: \([HMo, Corollary 3.15]\). In this work it plays an essential role due to the Equations 2.11 and Equation 2.13.

We consider the expressions of \(K^2_F\) and \(\chi(O_F)\) as (linear) polynomials in the variable \(g(X)\). In this section we consider the Harris Morrison families obtained in \([HM]\) and with maximal gonality.

In particular in this section we have:

\[g = 2n + 1, k = n + 2, b = 6n + 4, m = (6n + 3)c\]

or

\[g = 2n, k = n + 1, b = 6n, m = (6n - 1)c\]

Let us see how \(k\) influences \(K^2_F\) and \(8\chi(O_F)\). We will use the following

Lemma 3.1. Let \(f: F \rightarrow Y\) be an Harris Morrison genus-\(g\) fibration as in \([HMo, Theorem 2.5]\). Assume that the gonality \(k\) is maximal, that is \(k = \frac{g+3}{2}\) if \(g\) is odd and \(k = \frac{g+2}{2}\) if \(g \geq 4\) is even. Assume that the conjectured estimate in \([HMo, p. 351-352]\) is true. Then if \(g \gg 0\), we have \(\alpha > 8\alpha'\).

Proof. By Equation 2.11 by Equation 2.13 and by Corollary 2.17 we have

\[\alpha - 8\alpha' = (b-1) \left[2c - \tilde{N}_1 + \tilde{N}_3 \right] - \tilde{N}.\]

Since \(c \geq 0\) and \(b = 2(g+k-1) > 0\) it is sufficient to show that \((b-1) \left[-\tilde{N}_1 + \frac{\tilde{N}_3}{9}\right] - \tilde{N} \geq 0.\) In the case of maximal gonality up to the first factor by \([HM, bottom of the page 351]\) if \(k \gg 0\) it is conjectured that \(\tilde{N}_3 \simeq (k-2)\tilde{N}_1, \tilde{N} \simeq (k)(k-1)\tilde{N}_1\). Finally assume that \(g = 2n+1.\) By Equation 3.1 we have \(b-1 = 6n+3\) then up to the first order \((b-1) \left[-\tilde{N}_1 + \frac{\tilde{N}_3}{9}\right] - \tilde{N} \simeq \tilde{N}_1 \left[(6n+3)[1] - (n+2)(n+1)\frac{1}{2}\right] = \tilde{N}_1 \frac{11}{6} > 0\) if \(n \geq 44,\) that is \(k \geq 46.\) By the same argument if \(g = 2n\) we have that \(\alpha \geq 8\alpha'\) if \(3n^2 - 131n + 20 \geq 0\) that is if \(n \geq 43\) and then \(k \geq 44\).

\[\square\]
Proposition 3.2. Let \( f : F \rightarrow Y \) be an Harris Morrison genus-\( g \) fibration starting from any plane curve \( X \) of genus \( g(X) \gg 0 \). Assume that the gonality \( k \) is maximal and that the conjectured estimates in \([HMo], p. 351-352]\) are true. If \( k \) is big enough, then \( F \) is a surface of positive index i.e. \( K_F^2 > 8\chi(O_F) \). Moreover if \( f : F \rightarrow Y \) is general in its class then the irregularity of \( F \) is \( g(Y) \). In particular, \( f : F \rightarrow Y \) is the Albanese morphism of \( F \).

Proof. For every Harris Morrison family as in the statment:

\[
\frac{K_F^2}{\chi(O_F)} = \frac{\alpha + 8\tilde{N}(g-1)(g(X) - 1)}{\alpha' + \tilde{N}(g-1)(g(X) - 1)}
\]

since Equation 2.11 and Equation 2.13. Then the first claim is equivalent to show that \( \alpha \geq 8\alpha' \) and under our assumption this follows by Lemma 3.1 Let \( q(F) \) be the irregularity of \( Y \). By contradiction assume that \( q(Y) > g(Y) \). Then by universal property of the Albanese morphism it follows that the Jacobian of the general fiber of \( f : F \rightarrow Y \) contains an Abelian subvariety; but the locus of such curves in \( \overline{\mathcal{M}}_g \) is a proper closed. Hence the family is not a sweeping one: a contradiction to Proposition 2.4.

We have shown the Proposition stated in the Introduction.

We conclude by observing that if the conjectured estimates of Harris Morrison hold, then the families \( F \) of Proposition 3.2 furnish an intriguing example of surfaces with ratio \( \frac{K_F^2}{\chi(O_F)} \) asymptotically 8, which are minimal as semistable models, but with slope of the supported fibration asymptotically equal to 12. We think that this kind of divergence between the two fundamental ratios among the invariants of a fibered surface which is minimal as a semistable model is worthy to be studied in the light of the recent results of Urzúa quoted in the Introduction of this paper.

References

[CFM] D. Chen, G. Farkas, I. Morrison, Effective divisors on moduli spaces of curves, preprint (2012), arXiv:1205.6138v1.

[HMo] J. Harris and I. Morrison, Slopes of effective divisors on the moduli space of stable curves, Invent. Math. 99 (1990), 321-355.

[HM] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, Invent. Math. 67 (1982), 23-86.

[Ha] R. Hartshorne, Algebraic geometry, Grad. Texts Math., Vol. 52, Springer Verlag, New York-Heidelberg-Berlin, (1977).

[Ka1] M. M. Kapranov, Chow quotients of Grassmannians I, Adv. Soviet Math. 16, part 2, A.M.S. (1993) 29-110.

[Ka2] M. M. Kapranov, Veronese curves and Grothendieck-Knudsen moduli space \( \overline{\mathcal{M}}_{0,n} \), J. Algebraic Geom. 2 (1993), no. 2, 335-365.

[MT] B. Moishezon and M. Teicher, Simply-connected algebraic surfaces of positive index, Invent. Math. 89 (1987), no. 3, 601-643.

[My] Y. Miyaoka, Algebraic surfaces with positive indices. Classification of algebraic and analytic manifolds (Katata, 1982), 281-301, Progr. Math., 39, Birkhäuser Boston, Boston, MA, 1983.

[M] D. Mumford, Stability of Projective Varieties, LEns. Math. 23 (1977) 39-110.

[P] R. Pandharipande Descendent bounds for effective divisors on \( \overline{\mathcal{M}}_g \), J. Algebraic Geometry, vol. 21, (2012), 299-303.

[PPX] U. Persson, C. Peters, G. Xiao, Geography of spin surfaces, Topology 35 (1996), no. 4, 845-862.

[Re] I. Reider Geography and the number of moduli of surfaces of general type, Asian J. Math. 9 (2005), no. 3, 407-448.

[Sz] L. Szpiro, Séminaire sur les pinceaux de courbes de genre au moins deux, Astérisque 86 (1981).

[Ur1] G. Urzúa, Arrangements of curves and algebraic surfaces, J. Algebraic Geom. 19 (2010), no. 2, 335-365.

[Ur2] G. Urzúa, Arrangements of rational sections over curves and the varieties they define. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 22 (2011), no. 4, 453-486.

[X1] G. Xiao, Irregularity of surfaces with a linear pencil, Duke Math. J. vol. 55, no. 3 (1987), 597-602.

Dipartimento di Matematica e Geoscienze, Università degli studi di Trieste, via Valerio 12/B, 34127 Trieste, Italy, beorchia@units.it
DIPARTIMENTO DI MATEMATICA E INFORMATICA, VIA DELLE SCIENZE 206, UNIVERSITÀ DEGLI STUDI DI UDINE, UDINE, 33100 ITALY, FRANCESCO.ZUCCONI@UNIUD.IT