A CRITERIUM FOR NON-UNIQUENESS OF \( g \) MEASURES

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Abstract. We introduce a general criterium for non-uniqueness of \( g \)-measures. We show that the existence of multiple \( g \)-measures compatible with a single \( g \)-function can be proved by estimating the \( \bar{d} \)-distance between some suitable Markov chains. Our method is optimal for the important class of binary attractive \( g \)-functions, which includes for example the Bramson-Kalikow model \cite{Bramson1993}. We illustrate our approach by explicitly computing parameters for which the Bramson-Kalikow model is non-unique.

1. Introduction

Let \( A \) be a finite set we call alphabet and \( \mathcal{X} = A^{\mathbb{Z}_-} \). We denote by \( x_i \) the \( i \)-th coordinate of \( x \in \mathcal{X} \) and for \( i \leq j \) we write \( x_{-j}^i := (x_{-i} \ldots x_{-j}) \). For \( x, y \in \mathcal{X} \), a concatenation \( x_0^i y \) is a new sequence \( z \in \mathcal{X} \) with \( z_0^i = x_0^i \) and \( z_{-\infty}^{-i-1} = y \). We introduce in \( \mathcal{X} \) the metric \( \rho(x, y) := \min\{1/j+1: x_{-j}^0 = y_{0}^0\} \), which turns \( \mathcal{X} \) into a compact metric space. Denote by \( \mathcal{B} \) the Borel \( \sigma \)-algebra on \( \mathcal{X} \). Let \( T: \mathcal{X} \to \mathcal{X} \) be the shift operator such that for \( x \in \mathcal{X} \) we have \( (Tx)_i = x_{i-1} \). A Borel measurable function \( g: \mathcal{X} \to [0, 1] \) satisfying \( \sum_{a \in A} g(ax) = 1 \) for any \( x \in \mathcal{X} \) is called \( g \)-function in \( \mathcal{X} \). We denote by \( \mathcal{C}(\mathcal{X}) \) the set of continuous functions with the norm \( \|f\| := \sup_{x \in \mathcal{X}} |f(x)| \) and by \( \mathcal{G} \) the set of all continuous and strictly positive \( g \)-functions in \( \mathcal{X} \). If \( A \) is well ordered and \( \mathcal{X} \) is endowed with partial order \( x \geq y \Leftrightarrow x_i \geq y_i \) for all \( i \in \mathbb{Z}_- \), a function \( g \in \mathcal{G} \) is attractive if, for all \( a \in A \), \( \sum_{b \geq a} g(bx) \) is an increasing function of \( x \in \mathcal{X} \).

Let \( g \in \mathcal{G} \), a probability measure \( \mu \) on \( \mathcal{X} \) is compatible with \( g \) if it is \( T \)-invariant and, for all \( a \in A \) and \( x \in \mathcal{X} \), \( \mu\{|x \in \mathcal{X} : x_0 = a\}|T^{-1}\mathcal{B})(x) = g(ax_{-\infty}^{-1}) \) or

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equivalently,
\[ \int_X f \, d\mu = \int_X \sum_{a \in A} g(ax) f(ax) \, d\mu, \]
for all \( f \in C(X) \) (Walters, 1975). The set of measures compatible with \( g \) is called \( g \)-measures. In this article, we are interested on conditions for non-uniqueness of \( g \)-measures, i.e., sufficient conditions for the existence of multiple \( g \)-measures compatible with the same \( g \)-function.

The set \( \mathcal{G} \) and the associated \( g \)-measures constitute a rich class of stochastic models, which includes, for example, Markov chains, stochastic models that exhibit non-uniqueness (Hulse, 2006; Berger et al., 2005; Bramson & Kalikow, 1993), and models that are not Gibbsian (Fernández et al., 2011). The question of uniqueness of \( g \)-measures was extensively studied and important progresses have been obtained (Johansson et al., 2012; Fernández & Maillard, 2005). Notwithstanding, the problem of non-uniqueness is much less understood and the literature is still based on few examples of \( g \)-functions (Hulse, 2006; Berger et al., 2005; Bramson & Kalikow, 1993) and general criteria for non-uniqueness have only been obtained for the class of attractive \( g \)-functions (Gallo & Takahashi, 2012; Hulse, 1991).

Our main result (Theorem 1) is a simple criterium for non-uniqueness of \( g \)-measures that, in principle, can be applied to general \( g \)-functions. The idea of the result goes along the following informal lines. Let \( d \) be a distance function between two measures in \( X \). Let \( \mu \) and \( \mu' \) be two \( g \)-measures, a priori not necessarily distinct. Assume \( (\mu_j)_{j \geq 0} \) and \( (\mu'_j)_{j \geq 0} \) be two sequences of measures converging in \( d \) to \( \mu \) and \( \mu' \) respectively. If there exists an integer \( k \geq 0 \), such that
\[ d(\mu_k, \mu) + d(\mu'_k, \mu') < d(\mu_k, \mu'_k), \]
then, by triangular inequality, we conclude that \( \mu \neq \mu' \). Now, we can show by successive applications of the triangular inequality and taking the limit that
\[ d(\mu_k, \mu) \leq \sum_{j \geq k} d(\mu_j, \mu_{j+1}). \]
Therefore, to prove the existence of phase transition it is enough to show that there exists some \( k \geq 0 \) such that
\[ \sum_{j \geq k} d(\mu_j, \mu_{j+1}) + \sum_{j \geq k} d(\mu'_j, \mu'_{j+1}) < d(\mu_k, \mu'_k). \]
(2)

This amounts to say that the \( g \)-measures \( \mu \) and \( \mu' \) are different if we can show that they are slightly perturbed versions of \( \mu_k \) and \( \mu'_k \) with \( \mu_k \neq \mu'_k \). The question is then which distance and sequences of measures are adequate to prove (2). We show that the Orstein’s \( \bar{d} \)-distance and sequences of Markov chains are relatively easy to
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manipulate and suited for applications. Furthermore, as a consequence of use of $d$-distance, we show that our result can also be used to establish that both $g$-measures $\mu$ and $\mu'$ have Bernoulli natural extensions, i.e., isomorphic to a Bernoulli shift.

The idea of studying perturbation of the chains to prove non-uniqueness is not entirely new and is already present in the seminal work of Bramson & Kalikow (1993) and in other related works (Hulse, 2006, Lacroix, 2000, Friedli, 2010). But explicit mention to (2) and consideration of $d$-distance seem to be novel, and, most importantly, our method allow us to obtain stronger results. For example, Theorem 2 states that our criterium (Theorem 1) is optimal in the important class of binary attractive $g$-functions, giving a necessary and sufficient criterium for non-uniqueness in this class. Moreover, Theorem 3 which is an application of Theorem 1 to the Bramson-Kalikow (BK) model, elucidates the intricate relationship between the parameters of the model and allow us explicit compute the parameters for the BK model with non-unique $g$-measures. Finally, Corollaries 1 and 2 give explicit examples of choice for the parameters of BK model with multiple compatible stationary measures. To our knowledge, this is the first time that any explicit numerical parameters are computed for non-unique BK model (see Friedli (2010) for related discussion).

The article is organized as follows. We state the main results and relevant definitions in Section 2. In Section 3 we prove Theorem 1 and Theorem 2. Then, in Section 4 we introduce the couplings used to prove Theorem 3. Finally in Section 5 we prove Theorem 3 and Corollaries 1 and 2.

2. Main results

Before stating Theorem 1 we need to introduce Orstein’s $d$-distance (Shields, 1996). We say that a measure $\nu$ on $\mathcal{X} \times \mathcal{X}$ is a coupling between $\mu$ and $\mu'$ if for all measurable subset $\Gamma$ of $\mathcal{X}$ we have $\nu(\Gamma \times \mathcal{X}) = \mu(\Gamma)$ and $\nu(\mathcal{X} \times \Gamma) = \mu'(\Gamma)$. The set of all stationary couplings between $\mu$ and $\mu'$ is denoted by $C(\mu, \mu')$ and the $d$-distance between $\mu$ and $\mu'$ is defined by

$$d(\mu, \mu') = \inf_{\nu \in C(\mu, \mu')} \nu(\{(x, x') \in \mathcal{X} \times \mathcal{X} : x_0 \neq x_0'\}).$$

The $g$-measure $\mu$ on $\mathcal{X}$ is $T$-invariant, this implies that we can extend $\mu$ and $T$ to a measure $\tilde{\mu}$ and shift $\tilde{T}$ on $A^\mathbb{Z}$ in a unique way. We say that this natural extension $(\tilde{\mu}, \tilde{T})$ is Bernoulli if it is isomorphic to a Bernoulli shift.

Finally, we denote the set of $k$-th order ergodic Markov kernels on $\mathcal{X}$ by $\mathcal{M}_k \subset C(\mathcal{X})$ and the set of all Markov kernels by $\mathcal{M} = \bigcup_{k \geq 0} \mathcal{M}_k$. 


Theorem 1. Let \((g_j)_{j \geq 0}\) and \((g'_j)_{j \geq 0}\) be two sequences of functions in \(\mathcal{M}\) both converging to \(g \in \mathcal{G}\) in \(\mathcal{C}(\mathcal{X})\). Let \(\mu_j\) and \(\mu'_j\) be the associated \(g_j\) and \(g'_j\)-measures. If there exists an integer \(k \geq 0\) such that
\[
\sum_{j \geq k} \bar{d}(\mu_j, \mu_{j+1}) + \sum_{j \geq k} \bar{d}(\mu'_j, \mu'_{j+1}) < \bar{d}(\mu_k, \mu'_k),
\]
then there exist at least two distinct \(g\)-measures \(\mu\) and \(\mu'\). Moreover, the natural extension of both \(g\)-measures are Bernoulli.

The main advantage of Theorem 1 is that we need to know very little about the measures compatible with \(g\). In particular, we don’t need to know a priori how to construct a good coupling between the Markov processes and a \(g\)-measure \(\mu\) to upper bound \(\bar{d}(\mu_k, \mu')\). Even less, we don’t need to establish \textit{a priori} the weak convergence of the sequence of \(g_j\)-measures to a \(g\)-measure. The only requirement is a good control of the coupling between its Markov approximations.

An interesting question is whether the converse of Theorem 1 holds, \textit{i.e.}, if non-uniqueness of \(g\)-measure implies the existence of sequences \((g_j)_{j \geq 0}\) and \((g'_j)_{j \geq 0}\) that satisfy (3). We show below that the converse of Theorem 1 is indeed true for the important class of binary attractive \(g\)-functions, which includes for example the Bramson-Kalikow model (Bramson & Kalikow, 1993).

Theorem 2. Let \(A = \{-1, +1\}\). If \(g \in \mathcal{G}\) is attractive, then there exist multiple \(g\)-measures if and only if there exist two sequences \((g_j)_{j \geq 0}\) and \((g'_j)_{j \geq 0}\) of functions in \(\mathcal{M}\) both converging to \(g \in \mathcal{G}\) in \(\mathcal{C}(\mathcal{X})\) and the associated \(g_j\) and \(g'_j\)-measures \(\mu_j\) and \(\mu'_j\) satisfy, for some \(k \geq 0\), the inequality (3).

Now we apply Theorem 1 to the model studied in Bramson & Kalikow (1993) and we define below. Let \(A = \{-1, +1\}\), \(\epsilon \in (0, 1/2)\), and \((m_j)_{j \geq 1}\) be an increasing sequence of positive odd numbers. Let \(x \in \mathcal{X}\), we denote by \(p_{[m_j]} \in \mathcal{M}_{m_j}\) the function
\[
p_{[m_j]}(x) = 1 \left\{ x_0 \sum_{l=1}^{m_j} x_{-l} > 0 \right\} (1 - \epsilon) + 1 \left\{ x_0 \sum_{l=1}^{m_j} x_{-l} < 0 \right\} \epsilon.
\]
Let \((\lambda_j)_{j \geq 1}\) be a sequence of positive numbers such that \(\sum_{j=1}^{\infty} \lambda_j = 1\). Given \((m_j)_{j \geq 1}\) and \((\lambda_j)_{j \geq 1}\), the BK-model is defined by function \(p \in \mathcal{G}\) such that, for all \(x \in \mathcal{X}\),
\[
p(x) = \sum_{j=1}^{\infty} \lambda_j p_{[m_j]}(x).
\]
It is immediate that BK-model is attractive and regular. Bramson & Kalikow (1993) showed that if \(\lambda_j = (1 - r)^{j - 1}\) for \(r \in (2/3, 1)\), there exist sequences \((m_j)_{j \geq 1}\) for
which the BK model is not unique, i.e., there is multiple $p$-measures. However, it is not known how the sequence $(m_j)_{j \geq 1}$ should be explicitly chosen for fixed sequences $(\lambda_j)_{j \geq 1}$. Theorem 3 below exhibit an explicit relationship between sequences $(\lambda_j)_{j \geq 1}$ and $(m_j)_{j \geq 1}$ for which the BK model is non-unique.

**Theorem 3.** Let $(\lambda_j)_{j \geq 1}$ and $(m_j)_{j \geq 1}$ be the sequences that define the BK model $p$ in (5). Let $m_0 = 0$, $r : \{1, 2, \ldots\} \rightarrow \mathbb{Z}_+$ be a function such that $r(k) < k$, and $0 < \alpha < \frac{1}{2} - \epsilon$. If for all $k \geq 1$ we have

$$m_{k+1} \geq \left( \sum_{j \geq k+2} \lambda_j - \sum_{j=r(k+1)+1}^{k+1} \lambda_j \right)^2,$$

(6)

where

$$A_k := 8 \left( 1 - 2\epsilon \right)^{-2} (1 + m_{r(k+1)}(2\epsilon)^{-m_{r(k+1)}})^2 \ln \left( 2^{k+2}(1 + m_k(2\epsilon)^{-m_k})\alpha^{-1} \right),$$

(7)

then the corresponding BK model $p$ has multiple compatible stationary measures.

Let us give two numerical examples of sequences $(\lambda_j)_{j \geq 1}$ and $(m_j)_{k \geq 1}$ for which the BK model is non-unique illustrating the relationship between the sequences $(\lambda_j)_{j \geq 1}$ and $(m_j)_{k \geq 1}$ in Theorem 3.

**Corollary 1.** Let $\epsilon = 1/4$ and for $j \geq 1$, $\lambda_j = \frac{1}{2} \left( \frac{2}{3} \right)^j$. Let $m_1 = 217$, $c$ be and odd positive integer, and for $j \geq 1$, $m_{j+1} = c^{m_j}$. If $c \geq 577$, then the associated BK model has multiple compatible stationary measures.

**Corollary 2.** Let $\epsilon = 1/4$ and for $j \geq 1$, $m_j = 2^{c^2} - 1$. Let $b_1 = 1$, $c \geq 0$, and for $l \geq 2$, $b_l = 2^{(c\sum_{k=1}^{l-1} b_k)^2}$. For $l \geq 1$ and $j \in \{\sum_{k=1}^{l-1} b_k + 1, \ldots, \sum_{k=1}^l b_k\}$ we set $\lambda_j = (3/4)^{l-1}/(4b_l)$. If $c \geq 7$, then the associated BK model has multiple compatible stationary measures.

Finally, to prove that Theorem 3 is tight, we need a criterium for uniqueness with a comparable condition on the parameters. Straightforward but tedious calculations show that known criteria for uniqueness (Johansson et al. 2012; Fernández & Maillard, 2005) don’t give such conditions and, therefore, the explicit computation of parameters that give a sharp transition from uniqueness to non-uniqueness regime for the BK model is still an open problem.

3. Proof of Theorems 1 and 2

*Proof of Theorem 7*

We proceed in three main steps. First, we prove the existence of a subsequence
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Let $(\mu_{v_j})_{j \geq 0}$ be a sequence of measures that converges in $\bar{d}$ to a measure $\mu$ compatible with $g$. The same naturally holds for a subsequence $(\mu'_{u_j})_{j \geq 0}$ and some measure $\mu'$ compatible with $g$. Then we prove that $\mu$ and $\mu'$ are actually distinct, and finally, that their natural extensions are Bernoulli.

For the first step, let us prove that there exist a subsequence $\mu_{v_j}$ converging weakly and in entropy to a measure $\mu$ compatible with $g$. Because ergodic Markov processes are finitely determined, and for this class of processes, the weak convergence and the convergence of the entropy together imply convergence in $\bar{d}$, we conclude that $\mu_{v_j}$ converges to $\mu$ in $\bar{d}$-distance (see definition in p.221 and Theorem IV.2.9 of Shields (1996)).

We consider processes on finite alphabet, therefore the space of respective probability measures endowed with the weak topology is compact. Hence, for any sequence $(\mu_j)_{j \geq 0}$ there exists a convergent subsequence $(\mu_{v_j})_{j \geq 0}$. Let $\mu$ be its weak limit. It is immediate that $\mu$ is $T$-invariant. We will now show that it satisfies (1) and therefore it is a $g$-measure. For this, observe that for $f \in C(X)$

$$\left| \int_X \sum_{a \in A} g(ax) f(ax) d\mu - \int_X f d\mu \right| \leq \sum_{a \in A} \left[ \int_X g(ax) f(ax) d\mu - \int_X g(ax) f(ax) d\mu_{v_j} \right]$$

$$+ \left| \sum_{a \in A} \int_X (g(ax) - g_{v_j}(ax)) f(ax) d\mu_{v_j} \right| + \left| \int_X f d\mu_{v_j} - \int_X f d\mu \right|.$$

The first and third terms in the right hand side of the above inequality goes to zero because $\mu_{v_j} \to \mu$ weakly. The second term also goes to zero because

$$\lim_{j \to \infty} \| g_{v_j} - g \| = \lim_{j \to \infty} \sup_{x \in X} | g_{v_j}(x) - g(x) | = 0.$$

Now, we will show that existence of the weak limit $\mu$ and the convergence of $(g_{v_j})$ to a $g$ in the norm $\| \cdot \|$ implies that the entropies of $(\mu_{v_j})_{j \geq 1}$ also converges to the entropy of $\mu$.

For ergodic Markov processes $\mu_{v_j}$, we can write their entropies $H(\mu_{v_j})$ as

$$H(\mu_{v_j}) = -\int_X \log g_{v_j} d\mu_{v_j}.$$

Also, a simple computation shows that the entropy $H(\mu)$ of $\mu$ is given by

$$H(\mu) = -\int_X \log g d\mu.$$
Furthermore,
\[ \left| \int_X \log g d\mu - \int_X \log g_{v_j} d\mu_{v_j} \right| \leq \left| \int_X \log g d\mu - \int_X \log g_{v_j} d\mu_{v_j} \right| + \left| \int_X \log g_{v_j} d\mu_{v_j} - \int_X \log g_{v_j} d\mu_{v_j} \right|. \]

We observe that \( g \in G \) and therefore \( \log g \in C(X) \). Now, the first term of the right hand side of the above equation goes to zero because \( \mu_{v_j} \to \mu \) weakly. To show that the second term also converges to zero, we note that, by assumption, we have
\[ \lim_{j \to \infty} \sup_{x \in X} |\log g_{v_j}(x) - \log g(x)| = 0. \]

Thus, we conclude that \( \mu_{v_j} \) converges in \( \bar{d} \) to \( \mu \).

We now come to the second step, and prove that the limits \( \mu \) and \( \mu' \) are distinct.

Taking \( v_0 = k \), we have
\[ \lim_{j \to \infty} \bar{d}(\mu_k, \mu_{v_j}) = \bar{d}(\mu_k, \mu). \]

By triangular inequality, we also have
\[ \lim_{j \to \infty} \bar{d}(\mu_k, \mu_{v_j}) \leq \sum_{j=0}^{\infty} \bar{d}(\mu_{v_j}, \mu_{v_{j+1}}) \]

and, therefore,
\[ \bar{d}(\mu_k, \mu) \leq \sum_{j=0}^{\infty} \bar{d}(\mu_{v_j}, \mu_{v_{j+1}}). \]

Again, by triangular inequality we also have,
\[ \sum_{j=0}^{\infty} \bar{d}(\mu_{v_j}, \mu_{v_{j+1}}) \leq \sum_{j=k}^{\infty} \bar{d}(\mu_j, \mu_{j+1}). \]

Thus, if (3) is satisfied, we have
\[ \bar{d}(\mu_k, \mu) + \bar{d}(\mu'_k, \mu') < \bar{d}(\mu_k, \mu'_k) \]

showing that there exist two distinct \( g \)-measures \( \mu \) and \( \mu' \).

To conclude the proof, let \( (\widehat{\mu}, \widehat{T}) \) and \( (\widehat{\mu}', \widehat{T}) \) be respectively the natural extensions of \( \mu \) and \( \mu' \). Ornstein (1974) proved that the set of \( \widehat{T} \)-invariant measure on \( A^\mathbb{Z} \) having Bernoulli property is closed in the topology induced by the \( \bar{d} \)-metric. Because ergodic Markov chains are Bernoulli, this proves that \( \mu \) and \( \mu' \) have Bernoulli natural extensions.

\[ \square \]
Proof of Theorem 2

We define a sequence of functions $g_j, g'_j \in \mathcal{M}$ for each $j \geq 1$ and $x \in \mathcal{X}$ by

$$g_j(x^{-1}_- y) = \sup_{y \in \mathcal{X}} g(1x^{-1}_- y).$$

and

$$g'_j(x^{-1}_- y) = \inf_{y \in \mathcal{X}} g(1x^{-1}_- y).$$

For $j = 0$ we define $g_0(x^{-1}_- y) = \sup_{y \in \mathcal{X}} g(1y)$ and $g'_0(x^{-1}_- y) = \inf_{y \in \mathcal{X}} g(1y)$. Observe that if $g$ is attractive, $g_j$ and $g'_j$ are also attractive.

Let $f \in \mathcal{C}(\mathcal{X}), g \in \mathcal{G}$, and $x \in \mathcal{X}$. We define the Ruelle operator $L_g$ by

$$L_g f(x) = \sum_{a \in A} g(ax) f(ax).$$

Let $h \in \mathcal{C}(\mathcal{X})$ be an increasing function and $+1, -1 \in \mathcal{X}$ such that $+1_i = 1$ and $-1_i = -1$ for $i \leq 0$. By Lemma 2.1 in Hulse (1991) we have that

$$\lim_{n \to \infty} L^n_g h(+1) = \int_{\mathcal{X}} h d\mu^+, \quad \lim_{n \to \infty} L^n_g h(-1) = \int_{\mathcal{X}} h d\mu^-,$$

where $\mu^+$ and $\mu^-$ are extremal $g$-measures, not necessarily distinct. Now, let $x, y \in \mathcal{X}$, we define the function $p_j : \mathcal{X} \times \mathcal{X} \to [0, 1]$ by

$$p_j(x, y) = \min \{g_j(x), g(1y)\},$$

$$p_j(x, y) + p_j(-x, 1y) = g_j(x),$$

$$p_j(x, y) + p_j(1x, -1y) = g(1y),$$

$$\sum_{a \in A} \sum_{b \in A} p_j(ax, by) = 1.$$ 

This function $p_j$ was introduced in Hulse (1991) and can be used to define a coupling between $\mu_j$ and $\mu^+$, where $\mu_j$ is the unique $g_j$-measure. To see this, let $f_1, f_2 \in \mathcal{C}(\mathcal{X})$ and $x, y \in \mathcal{X}$. We introduce the Ruelle operator $L_{p_j}$ as

$$L_{p_j}(f_1 \otimes f_2)(x, y) = \sum_{a \in A} \sum_{b \in A} p_j(ax, by) f_1(ax) f_2(by).$$

For any increasing functions $h_1, h_2 \in \mathcal{C}(\mathcal{X})$, we have that $\lim_{n \to \infty} L^n_{p_j} (h_1 \otimes h_2)(+1, +1)$ exists and defines a coupling $\nu_j$ between $\mu_j$ and $\mu^+$ (see p. 442 in Hulse (2000)). By construction and definition of $p_j$, this coupling has the property that $\nu_j(\{(x, y) \in \mathcal{X} \times \mathcal{X} : x_0 < y_0\}) = 0.$
This implies that for any $j \geq 0$

\[
\bar{d}(\mu_j, \mu^+) = \mu_j(\{x \in \mathcal{X} : x_0 = 1\}) - \mu^+(\{x \in \mathcal{X} : x_0 = 1\}). \tag{8}
\]

Now, by definition of $\mu^+$, we have that for any $g$-measure $\mu$ we have

\[
\mu^+(\{x \in \mathcal{X} : x_0 = 1\}) - \mu(\{x \in \mathcal{X} : x_0 = 1\}) \geq 0.
\]

We also note that $\mu_j$ converges weakly to some $g$-measure, therefore we have that

\[
\lim_{j \to \infty} \mu_j(\{x \in \mathcal{X} : x_0 = 1\}) - \mu^+(\{x \in \mathcal{X} : x_0 = 1\}) = 0
\]

Hence by (8)

\[
\lim_{j \to \infty} \bar{d}(\mu_j, \mu^+) = 0.
\]

Thus, we conclude that

\[
\sum_{j=k}^{\infty} \bar{d}(\mu_j, \mu_{j+1}) = \mu_k(\{x \in \mathcal{X} : x_0 = 1\}) - \mu^+(\{x \in \mathcal{X} : x_0 = 1\}). \tag{9}
\]

Now, let $\mu'_j$ be the $g'_j$ measures. Repeating again the argument, we have that

\[
\sum_{j=k}^{\infty} \bar{d}(\mu'_j, \mu'_{j+1}) = \mu^- (\{x \in \mathcal{X} : x_0 = 1\}) - \mu'_k (\{x \in \mathcal{X} : x_0 = 1\}). \tag{10}
\]

Also, using an argument analogous to the one used to prove (8), we have

\[
\bar{d}(\mu_k, \mu'_k) = \mu_k(\{x \in \mathcal{X} : x_0 = 1\}) - \mu'_k (\{x \in \mathcal{X} : x_0 = 1\}). \tag{11}
\]

Combining (9), (10), and (11) we have that inequality

\[
\sum_{j=k}^{\infty} \bar{d}(\mu_j, \mu_{j+1}) + \sum_{j=k}^{\infty} \bar{d}(\mu'_j, \mu'_{j+1}) < \bar{d}(\mu_k, \mu'_k)
\]

is equivalent to

\[
\mu^+(\{x \in \mathcal{X} : x_0 = 1\}) - \mu^- (\{x \in \mathcal{X} : x_0 = 1\}) > 0. \tag{12}
\]

From Theorem 2.2 in [Hulse, 1991], we have that inequality (12) holds if and only if the $g$-measures are non-unique.

\[\square\]

Because our method of proof of non-uniqueness involves the $\bar{d}$-distance, we will need to construct several couplings and stationary measures to prove Theorem [3]. The constructions are conceptually straightforward but tedious to write. Therefore, for convenience of the reader we will first define all the constructions and later use them in the proof of Theorem [3].
4. Couplings and perfect simulations

The aim of the present section is to construct simultaneously all the Markov chains needed in the proof of Theorem 3 using only a single sequence \( U := (U_j)_{j \in \mathbb{Z}} \) of i.i.d. r.v.’s uniformly distributed in \([0,1)\). We let \((\Omega, \mathcal{F}, \mathbb{P})\) denote the probability space corresponding to the i.i.d. sequence \((U_j)_{j \in \mathbb{Z}}\), and \(\mathbb{E}\) the expectation under \(\mathbb{P}\).

The idea of the construction is the following. First, for \(g \in G\), we associate an update function \(F : [0,1) \times A^{\mathbb{Z}} \to A\) that satisfies \(\mathbb{P}(F(U_0, x) = a) = g(ax)\) for any \(x \in \mathcal{X}\) and \(a \in A\). For any pair of integers \(i, j\) such that \(-\infty < i \leq j \leq +\infty\), let \(F_{\{i,j\}}(U_j, x) \in A^{j-i+1}\) be the sample obtained by applying recursively \(F\) on the fixed past \(x\), i.e, let \(F_{\{i,i\}}(U_i, x) := F(U_i, x)\) and for any \(j > i\)

\[
F_{\{j,i\}}(U_j, x) := F(U_j, F_{\{j-1,i\}}(U_{j-1}, x))F_{\{j-1,i\}}(U_{j-1}, x).
\]

Secondly, define \(F_{\{j,i\}}(U_j, x) := F(U_j, x)\) and

\[
F_{\{j,i\}}(U_j, x) = F(U_j, F_{\{j-1,i\}}(U_{j-1}, x)x).
\]

\(F_{\{j,i\}}(U_j, x)\) is the last symbol of the sample \(F_{\{j,i\}}(U_j, x)\).

With these definitions, for all \(x \in \mathcal{X}\) we can construct the sequence \(\left( X^{(x)}_j \right)_{j \geq 1} \) defined by

\[
F_{\{j,1\}}(U_j, x) := X^{(x)}_j,
\]

which is the stochastic process starting with a fixed past \(x \in \mathcal{X}\) and updated according to \(g\).

Now we can define the notion of perfect simulation by coupling from the past (CFTP). Let \(\theta\) be the coalescence time defined by

\[
\theta := \min \{ i \geq 0 : F_{\{0,-i\}}(U_0, x) = F_{\{0,-i\}}(U_0, y) \text{ for all } x, y \in \mathcal{X} \}.
\]

It can be proved (see Propp & Wilson (1996); Comets et al. (2002); De Santis & Piccioni (2012) for instance) that if \(\theta\) is \(\mathbb{P}\)-a.s. finite then there is a unique process \((X_j)_{j \in \mathbb{Z}}\), compatible with \(g\), such that,

\[
F_{\{0,-\theta\}}(U_{-\theta}, x) \overset{\mathbb{P}}{=} X_0 \quad \forall x \in \mathcal{X}.
\]

Therefore, when an update function \(F\) and a \(\mathbb{P}\)-a.s. finite \(\theta\) exists, we say that there exists a CFTP algorithm that perfectly simulates \((X_j)_{j \in \mathbb{Z}}\). Observe that we are considering the bi-infinite stationary process on \(\mathbb{Z}\) rather then the process restricted on \(\mathbb{Z}_-\), as this is more convenient for the proof of Theorem 3.
Update function for the truncated Markov kernels. We will consider different Markov kernels in the proof of Theorem 3. They are truncations of order $m_k$ of the Bramson-Kalikow’s function $p \in \mathcal{C}(\mathcal{X})$ defined in [5]. Let $x \in \mathcal{X}$, $p_{[0]} \in \mathcal{M}_0$ be defined by $p_{[0]}(x) = (1 - \epsilon)1\{x_0 > 0\} + \epsilon 1\{x_0 < 0\}$, and $p_{[m_j]} \in \mathcal{M}_{m_j}$ be defined as in [4]. For $l > k \geq 0$, consider the following $m_k$-th order Markov kernels

$$p_k(x) = \sum_{j=1}^{k} \lambda_j p_{[m_j]}(x) + \sum_{j=k+1}^{\infty} \lambda_j p_{[0]}(x),$$

(15)

$$p_k'(x) = \sum_{j=1}^{k} \lambda_j p_{[m_j]}(x) + \sum_{j=k+1}^{\infty} \lambda_j (1 - p_{[0]}(x)),$$

(16)

$$q_{k,l}(x) = \sum_{j=1}^{k} \lambda_j p_{[m_j]}(x) + \sum_{j=k+1}^{l} \lambda_j (1 - p_{[0]}(x)) + \sum_{j=l+1}^{\infty} \lambda_j p_{[0]}(x)$$

(17)

$$q_{k,l}'(x) = \sum_{j=1}^{k} \lambda_j p_{[m_j]}(x) + \sum_{j=k+1}^{l} \lambda_j p_{[0]}(x) + \sum_{j=l+1}^{\infty} \lambda_j (1 - p_{[0]}(x))$$

(18)

where $\sum_{j=1}^{0} x_j$ means that the summand is zero.

Defining $\bar{\lambda}_0 := 2\epsilon$ and $\bar{\lambda}_j := \lambda_j(1 - 2\epsilon)$ for $j \geq 1$, we can respectively rewrite (15) and (17) as

$$p_k(x) = \bar{\lambda}_0 \frac{1}{2} + \sum_{j=1}^{k} \bar{\lambda}_j 1\{x_0 \sum_{i=1}^{m_j} x_{-i} > 0\} + \sum_{j=k+1}^{\infty} \bar{\lambda}_j \left(1 + \frac{x_0}{2}\right),$$

$$q_{k,l}(x) = \bar{\lambda}_0 \frac{1}{2} + \sum_{j=1}^{k} \bar{\lambda}_j 1\{x_0 \sum_{i=1}^{m_j} x_{-i} > 0\} + \sum_{j=k+1}^{l} \bar{\lambda}_j \left(1 + \frac{x_0}{2}\right) + \sum_{j=l+1}^{\infty} \bar{\lambda}_j \left(1 - \frac{x_0}{2}\right).$$

Similar equations hold for (16) and (18).

Now, for any past $x \in \mathcal{X}$, consider the intervals

$$I_0(-1) := [0, \epsilon[, \ I_0(+1) = [\epsilon, 2\epsilon[ \text{ and } I_j = \left[\sum_{i=0}^{j-1} \bar{\lambda}_i, \sum_{i=0}^{j} \bar{\lambda}_i\right], \ j \geq 1.$$  

(19)

We observe that the lengths $|I_0(-1)| = |I_0(+1)| = \epsilon$ and for $k \geq 1$, $|I_k| = \bar{\lambda}_k$.

It is natural to consider the following update functions for the Markov kernels $p_k$ and $q_{k,l}$ respectively.

$$F^{p_k}(U_0, x) = \sum_{a \in A} a 1\{U_0 \in I_0(a)\} + \sum_{a \in A} \sum_{j=1}^{k} a 1\{U_0 \in I_j\} \left(1 + \frac{a}{2}\right)\left(1 + \frac{x_0}{2}\right)$$

$$+ \sum_{a \in A} \sum_{j=k+1}^{l} a 1\{U_0 \in I_j\} \left(1 + \frac{a}{2}\right),$$

(18)
and
\[
F^{q_{k,l}}(U_0, x) = \sum_{a \in A} a \mathbf{1}\{U_0 \in I_0(a)\} + \sum_{a \in A} \sum_{j=1}^{k} a \mathbf{1}\{U_0 \in I_j\} \left\{ a \sum_{i=0}^{m_j-1} x_{-i} > 0 \right\}
\]
\[
+ \sum_{a \in A} \sum_{j=k+1}^{i} a \mathbf{1}\{U_0 \in I_j\} \left( \frac{1-a}{2} \right) + \sum_{a \in A} \sum_{j \geq k+1} a \mathbf{1}\{U_0 \in I_j\} \left( \frac{1+a}{2} \right).
\]

We can define analogous update functions for \( p'_k \) and \( q'_{k,l} \).

Let \( +1, -1 \in \mathcal{X} \) be defined by \( +1_j = 1 \) and \( -1_j = -1 \) for \( j \leq 0 \). We define the coalescence time
\[
\theta^{p_k} := \min \left\{ i \geq 0 : F^{p_k}_{[0,-i]}(U_{-i}, x) = F^{p_k}_{[0,-i]}(U_{-i}, y) \text{ for all } x, y \in \mathcal{X} \right\}
\]
\[
= \min \left\{ i \geq 0 : F^{p_k}_{[0,-i]}(U_{-i}, +1) = F^{p_k}_{[0,-i]}(U_{-i}, -1) \right\},
\]
where the last equality is a direct consequence of the attractivity of \( p_k \). We substitute in the above definitions \( p_k \) by \( p'_k \), \( q_{k,l} \), or \( q'_{k,l} \) to define \( \theta^{p'_k} \), \( \theta^{q_{k,l}} \), and \( \theta^{q'_{k,l}} \).

We also define, for any \( i \in \mathbb{Z} \) and \( k \geq 1 \), the regeneration time of order \( k \)
\[
\eta_k := \min \{ i \geq m_k - 1 : U_{-j} \in I_0(-1) \cup I_0(+1), j = i - m_k + 1, \ldots, i \}.
\]

**Couplings between the chains and an upper bound for \( \theta^{p_k} \) and \( \eta_k \).** We couple all the chains together constructing them simultaneously using the CFTP algorithm with same sequence \( U \) and the respective update functions. Consequently, the coupling law is always \( \mathbb{P} \), i.e., the product law of \( U \). We also use the same symbol to indicate the marginal process and coupled process, when there is no ambiguity.

In what follows, we collect some lemmas that will be used in the proof Theorem \[3\]. Let us give an upper bounds on the expectation of the coalescence and regeneration times that hold for \( p_k \), \( p'_k \), \( q_k \), and \( q'_k \). First, observe that by construction,
\[
F^{p_k}_{[0,-\eta_k]}(U_{-\eta_k}, +1) = F^{p_k}_{[0,-\eta_k]}(U_{-\eta_k}, -1)
\]
\( \mathbb{P} \)-a.s. and, therefore,
\[
\mathbb{P}(\eta_k \geq \theta^{p_k}) = 1.
\]
The same holds for \( p'_k \), \( q_{k,l} \), and \( q'_{k,l} \). We have the following lemma.

**Lemma 1.** Let \( \eta_k \) be the regeneration time of order \( k \). We have that
\[
\mathbb{E}[\theta^{p_k}] \leq \mathbb{E}[\eta_k] \leq \frac{m_k}{(2\epsilon)^{m_k}}.
\]
The same bound holds for \( \theta^{p'_k} \), \( \theta^{q_{k,l}} \), and \( \theta^{q'_{k,l}} \).
Proof. By the definition of \( \eta_k \) we have
\[
\mathbb{P}(\eta_k \geq n.m_k) \leq \prod_{i=1}^{n} \mathbb{P}\left(\bigcap_{j=-(i-1)m_k}^{i m_k + j - 1} \left\{ U_j \in \bigcup_{l=0}^{i m_k + j - 1} I_l \right\} \right)^c.
\]
Using the stationarity and independence of \( U \), we have for \( i = 1, \ldots, n \)
\[
\mathbb{P}\left(\bigcap_{j=-(i-1)m_k}^{i m_k + j - 1} \left\{ U_j \in \bigcup_{l=0}^{i m_k + j - 1} I_l \right\} \right) = \prod_{j=1}^{m_k} \mathbb{P}\left( U_j \in \bigcup_{l=0}^{j} I_l \right) = \prod_{j=1}^{m_k} \mathbb{P}\left( U_0 \in \bigcup_{l=0}^{j} I_l \right).
\]
A simple upper bound is \( \prod_{j=1}^{m_k} \mathbb{P}(U_0 \in \bigcup_{l=0}^{j} I_l) \leq (2\varepsilon)^{m_k} \). This yields
\[
\mathbb{E}[\eta_k] \leq m_k \sum_{n \geq 1} \mathbb{P}(\eta_k \geq n.m_k) \leq m_k \sum_{n \geq 1} (1 - (2\varepsilon)^{m_k})^n \leq \frac{m_k}{(2\varepsilon)^{m_k}}.
\]

\[\Box\]

Lemma 2. Let \((Y_{r,k+1}^r)_{j \in \mathbb{Z}}\) be the stationary process compatible with \( q_{r,k+1} \). If \( \sum_{j>k+1} \lambda_j > \sum_{j=r+1}^{k+1} \lambda_j \) then
\[
\mathbb{E}[Y_{0,r,k+1}] \geq \left( 1 - 2\varepsilon \right) \left( \sum_{j>k+1} \lambda_j - \sum_{j=r+1}^{k+1} \lambda_j \right) > 0. \tag{20}
\]

Proof. Let \((Z_{r,k+1}^r)_{j \in \mathbb{Z}}\) be the stationary process compatible with \( q_{r,k+1}' \) we observe that
\[
\mathbb{E}[Y_{0,r,k+1}^r] = \mathbb{P}(Y_{0,r,k+1}^r = 1) - \mathbb{P}(Y_{0,r,k+1}^r = -1) = \mathbb{P}(Y_{0,r,k+1}^r = 1) - \mathbb{P}(Z_{0,r,k+1}^r = 1).
\]
Now, we want to construct a maximal coupling between \((Y_{r,k+1}^r)_{j \in \mathbb{Z}}\) and \((Z_{r,k+1}^r)_{j \in \mathbb{Z}}\). For this we define an update function for \( q_{r,k+1}' \) using a set of intervals slightly different from the intervals defined in (23). We have
\[
I_0'(-1) := [0, \varepsilon[, \quad I_0'(1) = [\varepsilon, 2\varepsilon[ \text{ and } I_j' = \left[ \sum_{i=0}^{j-1} \bar{\lambda}_i, \sum_{i=0}^{j} \bar{\lambda}_i \right], \text{ for } r \geq j \geq 1; \tag{21}
\]
\[
I_j' = \left[ \sum_{i=0}^{j-1} \bar{\lambda}_i, \sum_{i=k+1}^{j} \bar{\lambda}_i, \sum_{i=0}^{j} \bar{\lambda}_i, \sum_{i=k+1}^{j} \bar{\lambda}_i \right], \text{ for } k + 1 \geq j \geq r + 1, \tag{22}
\]
and
\[
I_j' = \left[ \sum_{i=0}^{j} \bar{\lambda}_i, \sum_{i=k+1}^{j} \bar{\lambda}_i, \sum_{i=0}^{j} \bar{\lambda}_i, \sum_{i=k+1}^{j} \bar{\lambda}_i \right], \text{ for } j \geq k + 2. \tag{23}
\]
The update function for \( q'_{r,k+1} \) is then defined by

\[
H_{r,k+1}(U_0, x) = \sum_{a \in A} a1\{U_0 \in I_0'(a)\} + \sum_{a \in A} \sum_{j=1}^r a1\{U_0 \in I'_j\} 1\left\{ a \sum_{i=0}^{m_j-1} x_i > 0 \right\} + \sum_{a \in A} \sum_{j=r+1}^{k+1} a1\{U_0 \in I'_j\} \left( \frac{1-a}{2} \right) + \sum_{a \in A} \sum_{j=k+2}^{r} a1\{U_0 \in I'_j\} \left( \frac{1+a}{2} \right).
\]

Observe that this update function is different from \( F_{r,k+1} \), which uses the intervals defined in \( \text{(23)} \).

By construction

\[
\mathbb{P}(Y_{r,v_k+1}^r = 1) = \mathbb{P}(Z_0^{r,v_k+1} = 1) = \mathbb{P}(Y_{r,v_k+1}^r \neq Z_0^{r,v_k+1}).
\]

Now, the following lower bound is a straightforward consequence of the construction of the coupling.

\[
\mathbb{P}(Y_{r,v_k+1}^r \neq Z_0^{r,v_k+1}) \geq \left(1 - 2\epsilon\right) \left( \sum_{j>v_k+1} \lambda_j - \sum_{j=r+1}^{v_k+1} \lambda_j \right).
\]

\[\square\]

**Lemma 3.** Let \((Y_{j}^{r,k+1})_{j \in \mathbb{Z}}\) be the stationary process compatible with \( q_{r,k+1} \). For all \( k > 0 \) and \( j \geq 1 \), we have

\[
\mathbb{P}\left( \left| \frac{1}{m_{k+1}} \sum_{l=1}^{m_{k+1}} Y_{l}^{r,k+1} - \mathbb{E}[Y_{0}^{r,k+1}] \right| \geq \frac{\mathbb{E}[Y_{0}^{r,k+1}] + \epsilon}{2} \right) \leq 2 \exp\left( \frac{-m_{k+1} \mathbb{E}[Y_{0}^{r,k+1}]^2}{8 \left(1 + \mathbb{E}[\theta_{\theta^r}]\right)^2} \right).
\]

\[\text{(24)}\]

**Proof.** We will use Theorem 1 of [Chazottes et al. (2007)] to obtain a concentration upper bound. Let \((X_{j}^{[+1\sigma])}_{j \geq 1}\) and \((X_{j}^{[-1\sigma])}_{j \geq 1}\) be respectively the process with measure defined by the conditional distributions \(\mathbb{P}((X_{j}^{k+1})_{j \geq 1} = \cdot \mid X_0 = 1, X_{-1} = \sigma_{-2}, \ldots, X_{-i+1} = \sigma_{-i})\) and \(\mathbb{P}((X_{j}^{k+1})_{j \geq 1} = \cdot \mid X_0 = -1, X_{-1} = \sigma_{-2}, \ldots, X_{-i+1} = \sigma_{-i})\). We denote by \( \mathcal{Q}_i^\sigma \) the maximal coupling between the conditional distributions.

Now, we introduce the upper-triangular matrix \( D^\sigma \) defined for \( 1 \leq i < j \leq n \) by

\[
D^\sigma_{i,i} := 1
\]

\[
D^\sigma_{i,j} := \mathcal{Q}_i^\sigma (X_{j}^{[+1\sigma])} \neq X_{j}^{[-1\sigma])}
\]

\[\text{(25)}\]

Then, we define the matrix \( \bar{D} \) as \( \bar{D}_{i,j} := \sup_{\sigma \in \{-1,1\}^n} D^\sigma_{i,j} \). For a given function \( f : \{-1,1\}^n \rightarrow \mathbb{R} \), we define the variation of \( f \) at site \( i \) with \( 1 \leq i \leq n \) by

\[
\delta_i f := \sup_{\sigma_j = \sigma'_j, i \neq j} |f(\sigma) - f(\sigma')|.
\]
Thus we have the following result: let \( n > 1 \) be arbitrary. Assume that \( \| \bar{D} \|_2 < \infty \) and \( \| \delta f \|_2 < \infty \). Then, Theorem 1 in [Chazottes et al. (2007)] states that, for all functions \( f : \{-1, 1\}^n \to \mathbb{R} \) and all \( t > 0 \), we have

\[
P(|f - \mathbb{E}[f]| \geq t) \leq 2 \exp \left( -\frac{2t^2}{\| \bar{D} \|_2^2 \| \delta f \|_2^2} \right). \tag{26}
\]

In our case, observe that the elements of matrix \( \bar{D}_{i,j} \) are bounded from above by the probabilities \( P(\theta^q > j) \) for all \( i, j \in \mathbb{N}, i \geq 1 \). To see this we note that

\[
P(\theta^q > j) = P \left( F_{q,0}^q(U_{0}^{j}, \pm 1) \neq F_{q,0}^q(U_{0}^{j}, -1) \right)
\]

where the last equality is a consequence of attractivity of \( q_r \). Now, by the stationarity of \( U \) we have

\[
P(\theta^q > j) = P \left( X_{j}^{(+1)} \neq X_{j}^{(-1)} \right)
\]

for all \( j \). Taking \( f = f(x_1, \ldots, x_{m_k+1}) = \frac{1}{m_{k+1}} \sum_{i=1}^{m_{k+1}} x_i \) we have \( \delta_i f = \frac{2}{m_{k+1}} \) if \( i \in \{1, \ldots, m_{k+1}\} \). Thus, we obtain

\[
\| \delta g \|_2 = \sum_{i=1}^{m_{k+1}} \left( \frac{2}{m_{k+1}} \right) = \frac{4}{m_{k+1}}. \tag{29}
\]
Applying (26) and using (28) and (29), we obtain

\[ P\left(\left|\sum_{j=1}^{m_{k+1}} Y_{r,k+1} + \sum_{j=1}^{m_{k+1}} \mathbb{E}[Y_{r,k+1}^+ - \mathbb{E}[Y_{r,k+1}^-]] \right| \geq \mathbb{E}[Y_{r,k+1}^0]\right) \leq 2 \exp\left(-\frac{m_{k+1}\mathbb{E}[Y_{r,k+1}^0]^2}{8\left(1 + \mathbb{E}[\theta^r_\eta]\right)^2}\right). \]

\[ \square \]

5. PROOF OF THEOREM 3 AND PROPOSITION 1

Proof of Theorem 3

We fix a sequence \((\lambda_j)_{j \geq 1}\) of positive real numbers such that \(\sum_{j \geq 1} \lambda_j = 1\). Let \(r : \{1, 2, \ldots\} \to \mathbb{Z}_+\) such that \(r(k) < k\) and \(\sum_{j \geq k+1} \lambda_j > \sum_{j=r(k)+1}^k \lambda_j, \forall k \geq 1\). The sequence of odd positive integer numbers \((m_j)_{j \geq 1}\) will be chosen afterwards.

Clearly \((p_j)_{j \geq 1}\) and \((p'_j)_{j \geq 1}\) defined in (15) and (16) converge to the Bramson-Kalikow’s \(p\) in \(\mathcal{C}(X)\). For all \(k \geq 0\), let \(P_k\) (resp. \(P'_k\)) be the unique stationary measure compatible with \(p_k\) (resp. \(p'_k\)). Observe that for \(k = 0\), \(P_0\) (resp. \(P'_0\)) is just a Bernoulli process of parameter \(1 - \epsilon\) (resp. \(\epsilon\)).

We will apply Theorem 1 with \(k = 0\), \(\mu_j = p_j\) and \(\mu'_j = p'_j\) (here \(k\) and \(j\) are from Theorem 1). Since \(\bar{d}(P_0, P'_0) = 1 - 2\epsilon\), we need to find an explicit sequence \((m_j)_{j \geq 1}\) such that

\[ \sum_{k \geq 0} \bar{d}(P_k, P_{k+1}) + \sum_{k \geq 0} \bar{d}(P'_k, P'_{k+1}) < 1 - 2\epsilon. \] (30)

By symmetry of the kernels \(p_k\) and \(p'_k\), (30) is equivalent to

\[ 2 \sum_{k \geq 0} \bar{d}(P_k, P_{k+1}) < 1 - 2\epsilon. \] (31)

Now, our task is to upper bound \(\bar{d}(P_k, P_{k+1})\). For all \(k \geq 0\), let \((X_j^k)_{j \in \mathbb{Z}}\) be the stationary process compatible with the measure \(P_k\).

By definition of the \(\bar{d}\)-distance we have that

\[ \bar{d}(P_k, P_{k+1}) \leq \mathbb{P}\left(X_0^k \neq X_0^{k+1}\right). \] (32)

Define for all \(i \in \mathbb{Z}_-\), the interval \(I_i := [i - m_{k+1}, i - 1]\) and the events

\[ S_i := \left\{ \sum_{j \in I_i} X_j^{k+1} > 0 \right\}. \]

As \(\eta_k\) (defined as in Lemma 11) is a stopping time for the filtration \((\mathcal{F}_i)_{i \geq 0} = (\sigma(U_0, U_{-1}, \ldots, U_{-i}))_{i \geq 0}\) and the events \(S_i^{k+1}\) are independent of \(\mathcal{F}_i\) for all \(i \geq 0\),
we have by construction of the coupling $\mathbb{P}$ and Wald’s equality
\[
\mathbb{P}(X_0^k \neq X_0^{k+1}) = \mathbb{P}\left(\bigcup_{i=0}^{\eta_k} S_i^0\right) \leq \mathbb{E}\left[\sum_{i=0}^{\eta_k} 1_{\{S_i^0\}}\right] = (\mathbb{E}[\eta_k] + 1)\mathbb{P}(S_0^0).
\] (33)

Combining (32) and (33) we obtain
\[
\bar{d}(P_k, P_{k+1}) \leq (\mathbb{E}[\eta_k] + 1)\mathbb{P}(S_0^0)
\] (34)
for all $k \geq 0$.

Now, to obtain an upper bound for $\mathbb{E}[\eta_k]$ we use Lemma 1 and for $\mathbb{P}(S_0^0)$ we use a concentration of measure result, which we describe below. Let $r := r(k + 1)$ and $(Y_{r,k}^{r,k+1})_{j \in \mathbb{Z}}$ be the process compatible with $q_{r,k+1}$. Observe that for all $k \geq r \geq 0$ we have $q_{r,k+1} \in \mathcal{M}_r$. Also note that, for any $n \geq 1$, and integers $l_1, \ldots, l_n$, we have by construction that
\[
\mathbb{P}\left(\bigcap_{j=1}^n \left\{X_j^{k+1} < Y_j^{r,k+1}\right\}\right) = 0,
\]
and therefore
\[
\mathbb{P}\left(\sum_{j=1}^{m_{k+1}} X_j^{k+1} < 0\right) \leq \mathbb{P}\left(\sum_{j=1}^{m_{k+1}} Y_j^{r,k+1} < 0\right). \] (35)

Furthermore, we have
\[
\mathbb{P}\left(\sum_{j=1}^{m_{k+1}} Y_j^{r,k+1} < 0\right) \leq \mathbb{P}\left(\left|\frac{1}{m_{k+1}} \sum_{j=1}^{m_{k+1}} Y_j^{r,k+1} - \mathbb{E}[Y_j^{r,k+1}]\right| \geq \frac{\mathbb{E}[Y_0^{r,k+1}]}{2}\right), \] (36)
and therefore we can upper bound $\mathbb{P}(S_0^0)$ using a concentration of measure for a Markov chain of order $r < k + 1$.

Combining (34), (35), (36), and Lemmas 2, 3, we deduce that, for all $k \geq 0$,
\[
\bar{d}(P_k, P_{k+1}) \leq 2(\mathbb{E}[\eta_k] + 1)\exp\left(-\frac{m_{k+1} \left(\sum_{j \geq k+2} \lambda_j - \sum_{j=r+1}^{k+2} \lambda_j\right)^2 \left(1 - 2\epsilon\right)^2}{8\left(1 + \mathbb{E}[\theta^{r,k}]\right)^2}\right).
\]

Let $\alpha > 0$ such that $\alpha < \frac{1}{2} - \epsilon$. Define
\[
A_0 := 8\left(1 - 2\epsilon\right)^{-2} \ln\left(4\alpha^{-1}\right)
\]
and for all $k \geq 1$,
\[
A_k := 8\left(1 - 2\epsilon\right)^{-2} (1 + m_r(2\epsilon)^{-m_r})^2 \ln\left(2^{k+2}(1 + m_k(2\epsilon)^{-m_k})\alpha^{-1}\right). \] (37)

Then, for all $k \geq 0$ choose $m_{k+1}$ as the first odd integer such that
\[
m_{k+1} \geq \frac{A_k}{\left(\sum_{j \geq k+2} \lambda_j - \sum_{j=r+1}^{k+1} \lambda_j\right)^2}.
\] (38)
With these choices, using Lemma 1 we obtain
\[
\bar{d}(P_k, P_{k+1}) \leq \frac{\alpha}{2^{k+1}}
\] (39)
for all \( k \geq 0 \). Since \( \alpha < \frac{1}{2} - \epsilon \) we obtain (31), which proves the theorem. \( \square \)

Proof of Corollary 1

If, for \( j \geq 1 \), we choose \( \lambda_j = \frac{1}{2} \left( \frac{2}{3} \right)^j \), we have for \( k \geq 1 \), \( \sum_{j \geq k+1} \lambda_j - \lambda_k \geq 0 \), i.e., we have a function \( r \) in (6) defined by \( r(k) = k - 1 \). Let \( \epsilon = 1/4 \) and \( \alpha = 1/8 \), then \( A_0 = 160 \ln 2 \) and, by (38), \( m_1 \) must be chosen greater than \( 320 \left( \frac{3}{2} \right)^2 \ln 2 \approx 216,74 \). Let us take \( m_1 = 217 \). Now, from (37), we can see that in this case, the sequence \( (m_k)_{k \geq 1} \) must satisfy \( m_k \geq k + 1 \) for all \( k \geq 1 \). Therefore, for \( k \geq 1 \), we have
\[
\frac{A_k}{\left( \sum_{j \geq k+2} \lambda_j - \lambda_{k+1} \right)^2} \leq 512 \left( \frac{9}{2} \right)^{k+1} (1 + m_k 2^{m_k})^3 \leq 512 \left( \frac{9}{2} \right)^{k+1} (64)^{m_k} \leq (577)^{m_k}.
\]

Proof of Corollary 2

Let \( b_1 = 1 \) and \( c \) a positive constant to be fixed afterwards. For \( l \geq 2 \), we define
\( b_l = \lceil 2^c \sum_{j=1}^{l-1} b_j \rceil \), where \( \lceil \cdot \rceil \) is the ceiling function. Let \( s = 3/4 \), for \( l \geq 1 \) and \( j \in \{ \sum_{k=1}^{l-1} b_k + 1, \ldots, \sum_{k=1}^{l} b_k \} \) we define
\[
\lambda_j = \frac{s^{l-1} - s^l}{b_l}.
\]
It is straightforward to verify that \( \sum_{j \geq 1} \lambda_j = 1 \). Let \( r(k) = \lceil \sqrt{\log(k)/c} \rceil \) where \( \log \) is base 2 logarithm and \( \lceil \cdot \rceil \) is the floor function. We observe that by construction, for \( l \geq 1 \), we have
\[
\sum_{j \geq k+1} \lambda_j - \sum_{j = r(k)+1} \lambda_j \geq s^l - (s^{l-2} \wedge 1 - s^l) = \frac{1}{8} \left( \frac{3}{4} \right)^{l-2}.
\]
We set \( m_j = \lfloor 2^{c_j^2} \rfloor \) if \( \lfloor 2^{c_j^2} \rfloor \) is odd, otherwise \( m_j = \lfloor 2^{c_j^2} \rfloor - 1 \). We want to obtain a sequence \((m_j)_{j \geq 1}\) that satisfies (6) and (7). Let

\[
B_k = 4(k + 1)2^{2(k+1)(\log(2\epsilon)^{-1})}.
\]

We have the following upper bound for (7).

\[
A_k \leq \frac{8(k + 2)\ln 2}{(1 - 2\epsilon)^2} B_k + \frac{8\ln \left( 1 + 2^{ck^2}(2\epsilon)^{-2ck^2} \right)}{(1 - 2\epsilon)^2} B_k.
\]

Now, let \( \epsilon = 1/4 \) and \( \alpha = 1/8 \), we have,

\[
A_k \leq 128\ln 2B_k(k + 2) + 32\ln 2B_k(ck^2 + 1) + 32B_k2^{ck^2}
\]

\[
\leq 81B_k2^{ck^2}.
\]

Also, we observe that for \( c \geq 7 \), \( B_k \leq 2^{2+\log(k+1)+2(k+1)} \leq 2^{ck} \), and, therefore,

\[
A_k \leq 81.2^{c(k^2+k)}.
\]

Also for \( c \geq 2 \), we have \( \frac{1}{8} \left( \frac{3}{7} \right)^{l-2} \geq \left( \frac{1}{7} \right)^{l+1} \geq 2^{-ck} \). Finally, to satisfy the conditions in Theorem \( \text{(1)} \) it is enough that

\[
2^{c(k+1)^2} \geq 81.2^{c(k^2+2k)}.
\]

The above inequality is satisfied if \( c \geq 7 \).

\[\square\]

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