LICKORISH TYPE CONSTRUCTION OF MANIFOLDS OVER SIMPLE POLYTOPES

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ABSTRACT. This paper is a survey on the Lickorish type construction of some kind of closed manifolds over simple convex polytopes. Inspired by Lickorish’s theorem, we propose a method to describe certain families of manifolds over simple convex polytopes with torus action. Under this construction, many classical classification results of these families of manifolds could be interpreted by this construction and some further problems will be discussed.

1. WHAT IS LICKORISH TYPE CONSTRUCTION?

This paper is a survey on the Lickorish type construction of some kind of closed manifolds over simple convex polytopes. We first explain what is “Lickorish type construction”.

In algebra, it is natural to describe algebraic systems, such as rings and algebras, by generators and relations. In geometry and topology, it is often convenient to construct spaces from some very special examples by certain type of operations. We write this construction in terms of algebraic system by

\[ \text{AS} \{ \text{generators} \mid \text{some operations} \}, \]

where AS is the abbreviation for “algebraic system”. One typical example is the following theorem obtained by Lickorish in 1962.

**Theorem 1.1** (Lickorish [8]). Any orientable closed connected 3-manifold can be obtained from \( S^3 \) by a finite number of Dehn surgeries on knots.

This theorem provides a global viewpoint of the construction of orientable closed connected 3-manifolds under algebraic system with generators and operations. We call this kind of construction or description **Lickorish type construction**. Under this point of view, we can rewrite the above theorem as:

\[ \{ \text{All orientable closed connected 3-manifolds} \} = \text{AS} \{ S^3 \mid \text{Dehn surgeries on knots} \}. \]

It is natural to ask: *is there any other examples of this construction?* This is the main motivation of this survey paper. Under the viewpoint of Lickorish’s

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construction, we survey some known results of constructing some families of closed manifolds arising in toric topology. We hope that such constructions would be further studied with more applications. The reader is referred to [3] and [2] for the backgrounds of toric topology.

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2. Closed manifolds over simple polytopes

Let \( P^n \) be an \( n \)-dimensional simple convex polytope in an Euclidean space. Suppose the number of facets (codimension-one faces) of \( P^n \) is \( m \). According to [3], one can construct a \( T^m \)-manifold \( \mathbb{Z}_{P^n} \) and a \( \mathbb{Z}_2^m \)-manifold \( \mathbb{R}\mathbb{Z}_{P^n} \) whose orbit spaces are both \( P^n \). Indeed, let \( \{F_1, \ldots, F_m\} \) be the set of facets of \( P^n \). Let \( \{e_1, \ldots, e_m\} \) be an unimodular basis of \( \mathbb{Z}^m \). Define a function \( \lambda_0: \{F_1, \ldots, F_m\} \to \mathbb{Z}^m \) by

\[
\lambda_0(F_i) = e_i, \; 1 \leq i \leq m.
\]

For a proper face \( f \) of \( P^n \), let \( G_f \) denote the subtorus of the \( m \)-dimensional real torus \( T^m \) determined by the set \( \{\lambda_0(F_i) | f \subseteq F_i\} \subseteq \mathbb{Z}^m \) under the exponential map \( \mathbb{Z}^m \to \mathbb{R}^m \to T^m \). For any point \( p \in P^n \), let \( f(p) \) denote the unique face of \( P^n \) that contains \( p \) in its relative interior. Then by [3] Construction 4.1, the moment-angle manifold \( \mathbb{Z}_{P^n} \) of \( P^n \) is defined to be the following quotient space

\[
\mathbb{Z}_{P^n} := P^n \times T^m / \sim
\]

where \( (p, g) \sim (p', g') \) if and only if \( p = p' \) and \( g^{-1}g' \in G_{f(p)} \). In addition, let

\[
\Theta_{P^n}: P^n \times T^m \to \mathbb{Z}_{P^n}
\]

be the quotient map. There is a canonical \( T^m \)-action on \( \mathbb{Z}_{P^n} \) by

\[
g' \cdot \Theta_{P^n}(p, g) = \Theta_{P^n}(p, g'g), \; p \in P^n, g, g' \in T^m.
\]

If we replace \( T^m \) by the \( \mathbb{Z}_2\)-torus \( (\mathbb{Z}_2)^m \) and replace \( \lambda_0 \) by a function that maps \( \{F_1, \ldots, F_m\} \to \) a basis of \( (\mathbb{Z}_2)^m \), we can similarly define \( \mathbb{R}\mathbb{Z}_{P^n} = P^n \times (\mathbb{Z}_2)^m / \sim \) as \( \Theta_{P^n} \) and a canonical \( (\mathbb{Z}_2)^m \)-action on \( \mathbb{R}\mathbb{Z}_{P^n} \) as \( \Theta_{P^n} \).

V. Buchstaber (cf. [2]) defines \( s_{\mathbb{C}} = s_{\mathbb{C}}(P^n) \) (or \( s_{\mathbb{R}} = s_{\mathbb{R}}(P^n) \)) to be the maximal dimension of the subtorus of \( T^n \) (or sub-\( \mathbb{Z}_2 \)-torus of \( (\mathbb{Z}_2)^m \)) that can act freely on \( \mathbb{Z}_{P} \) (or \( \mathbb{R}\mathbb{Z}_{P} \)) through the canonical action. It is easy to see that

\[
s_{\mathbb{C}} \leq m - n, \; s_{\mathbb{R}} \leq m - n.
\]

Note that \( \mathbb{Z}_{P^n} \) and \( \mathbb{R}\mathbb{Z}_{P^n} \) are the “highest level” manifolds over \( P^n \). If we have a subtorus \( H_{\mathbb{C}} \) of \( T^n \) (or a sub-\( \mathbb{Z}_2 \)-torus \( H_{\mathbb{R}} \subset \mathbb{Z}_2^n \)) that acts freely on \( \mathbb{Z}_{P^n} \) (or \( \mathbb{R}\mathbb{Z}_{P^n} \)) where \( \text{rank}(H_{\mathbb{C}}) \leq s_{\mathbb{C}} \) (\( \text{rank}(H_{\mathbb{R}}) \leq s_{\mathbb{R}} \)), we can obtain a \( T^n / H_{\mathbb{C}} \)-manifold \( \mathbb{Z}_{P^n} / H_{\mathbb{C}} \) (or \( \mathbb{Z}_2^n / H_{\mathbb{R}} \)-manifold \( \mathbb{R}\mathbb{Z}_{P^n} / H_{\mathbb{R}} \)) with orbit space \( P^n \), called a partial quotient of \( \mathbb{Z}_{P^n} \) (or \( \mathbb{R}\mathbb{Z}_{P^n} \)). Therefore, one can construct a series of manifolds
with real torus (or $\mathbb{Z}_2$-torus) actions whose orbit spaces are all $P^n$ as shown in the following picture.

Note that any partial quotient of $\mathbb{Z}P^n$ (or $\mathbb{R}ZP^n$) can be described by a non-degenerate $\mathbb{Z}^r$-coloring (or $(\mathbb{Z}_2)^r$-coloring) on $P^n$ with $n \leq r \leq m$ in the similar fashion as $\mathbb{Z}P^n$ (or $\mathbb{R}ZP^n$). A nondegenerate $\mathbb{Z}^r$-coloring (or $(\mathbb{Z}_2)^r$-coloring) on $P^n$ is a function $\mu : \{F_1, \cdots, F_m\} \to \mathbb{Z}^r$ (or $(\mathbb{Z}_2)^r$) such that at any vertex $v = F_{i_1} \cap \cdots \cap F_{i_n}$ of $P^n$, the set $\{\mu(F_{i_1}), \cdots, \mu(F_{i_n})\}$ is part of a unimodular basis of $\mathbb{Z}^r$ (or $(\mathbb{Z}_2)^r$). Given any nondegenerate $\mathbb{Z}^r$-coloring $\mu$ on $P^n$, we can obtain an $(n+r)$-dimensional manifold $M(P^n, \mu)$ defined by:

$$M(P^n, \mu) = P^n \times T^r / \sim$$

where $(p, g) \sim (p', g')$ if and only if $p = p'$ and $g^{-1}g'$ is in the subtorus of $T^r$ determined by the set $\{\mu(F_i) | F_i \subset F_1\} \subset \mathbb{Z}^r$. There is a canonical action of $T^r$ on $M(P^n, \mu)$ whose orbit space is $P^n$. Similarly, given a nondegenerate $(\mathbb{Z}_2)^r$-coloring $\mu'$ on $P^n$, we can construct an $n$-dimensional manifold $M(P^n, \mu')$ with a canonical $(\mathbb{Z}_2)^r$-action whose orbit space is $P^n$. It is not hard to see that any partial quotient of $\mathbb{Z}P^n$ (or $\mathbb{R}ZP^n$) can be realized as $M(P^n, \mu)$ by some nondegenerate $\mathbb{Z}^r$-coloring (or $(\mathbb{Z}_2)^r$-coloring) $\mu$ on $P^n$.

Inspired by Lickorish’s theorem, we propose the following problem.

**Problem 2.1.** Give a Lickorish type construction for all the partial quotients of (real) moment-angle manifolds over simple polytopes described above. In other words, find some generators and operations that can produce all such kind of closed manifolds.
a small cover) is also called its characteristic function. These manifolds are introduced by Davis-Januszkiewicz [3] as analogues of smooth projective toric variety in the category of closed manifolds with real torus and \( \mathbb{Z}_2 \)-torus actions. In this case, it is also interesting to study the Lickorish type construction of quasitoric manifolds and small covers.

Roughly speaking, these manifolds over simple convex polytopes can be determined by some informations of their polytopes. If these polytopes admits some Lickorish type constructions, then these corresponding manifolds will also admit the induced Lickorish type construction. Therefore, we will discuss the Lickorish type construction of simple convex polytopes in the next section first.

3. Operations on polytopes

3.1. Simple polytopes and flips. One has a suitable “algebraic system” to describe all simple polytopes in any dimension as follows. First of all, let us recall the definition of flips on a simple polytope.

Let \( P \) be an \( n \)-dimensional simple polytope with \( m \) facets. Assume there exists a face \( f \) that is a simplex of dimension \( a - 1 \) and let \( b = n + 1 - a \). One can define the flip on \( P \) at \( f \) as follows (called a flip of type \((a,b)\)).

- Let \( \Delta^{a-1} = [v_1, \cdots, v_a] \) be a simplex of dimension \( a - 1 \) in an \( n \)-simplex \( \Delta^n = [v_1, \cdots, v_{n+1}] \) and \( \Delta^{b-1} = [v_{a+1}, \cdots, v_{n+1}] \) be opposite face of \( \Delta^{a-1} \) in \( \Delta^n \). The set of points \( \{ \frac{1}{2}v_i + \frac{1}{2}v_j \ | \ 1 \leq i \leq a, a+1 \leq j \leq n+1 \} \) spans a hyperplane \( L \) which intersect \( \Delta^n \) transversely. Let \( H_{a,b} = L \cap \Delta^n \). It is easy to see that \( H_{a,b} \) is a simple \((n-1)\)-polytope combinatorially equivalent to the product \( \Delta^{a-1} \times \Delta^{b-1} \). We have \( \Delta^n = (\Delta^{a-1} \cup H_{a,b}) \cup (\Delta^{b-1} \cup H_{a,b}) \) where \( A \cup B \) is the convex hull of two sets \( A \) and \( B \).
- Choose a hyperplane to cut off a small neighborhood \( N(f) \) of \( f \) in \( P \) which is combinatorially equivalent to \( \Delta^{a-1} \cup H_{a,b} \).
- Define the flip on \( P \) at \( f \) be

\[
\text{flip}_f(P) = (P - N(f)) \cup (\Delta^n - \Delta^{a-1} \cup H_{a,b})
\]

\[
= (P - N(f)) \cup (\Delta^{b-1} \cup H_{a,b})
\]

It is shown in [1, Corollary 2.7] that the combinatorial type of \( \text{flip}_f(P) \) is uniquely determined by \( P \) and \( f \). We also use \( F_{(a,b)} \) to refer to a general flip of type \((a,b)\) on a simple polytope.

**Example 3.1.** Doing a flip on a simple \( n \)-polytope \( P \) at a vertex \( v \) (i.e. a flip of type \((1,n)\)) is equivalent to “cutting off” \( v \) from \( P \). In addition, the flip of the simple 3-polytope \( P \) in Figure 1 at the 1-simplex \( f \) gives us a polytope combinatorially equivalent to the 3-cube.
It is shown in [1, Lemma 2.3] that up to combinatorial equivalence any simple $n$-polytope can be obtained from the $n$-simplex by a finite number of flips. We can restate their theorem as follows:

**Theorem 3.2** (Bosio–Meersseman [1]).

\[
\{ \text{All simple } n\text{-polytopes} \}/\sim_C \subseteq \text{AS}\{ \Delta^n | \text{flips } F(a,b), a + b = n + 1, 1 \leq a, b \leq n \},
\]

where the equivalent relation $\sim_C$ is up to combinatorial equivalence.

We warn that the class of simple polytopes is not closed under “combinatorial flips”. Indeed, the result of cutting off a neighborhood of a simplicial face $f$ of a simple polytope $P$ and gluing the neighborhood of another simplex in its place may not be a convex polytope (see [1, Example 2.10]).

### 3.2. Polytopal spheres and bistellar moves.

Recall that a polytopal sphere is defined to be the boundary of a simplicial polytope. A PL sphere is a simplicial sphere $K$ which is PL homeomorphic to the boundary of a simplex (i.e. there is a subdivision of $K$ isomorphic to a subdivision of the boundary of a simplex). Polytopal spheres are all PL spheres. But there exist non-polytopal PL spheres in dimension $\geq 3$ (e.g. Brückner sphere). The following operations called “bisetellar moves” can help us to obtain new PL spheres from any given one.

**Definition 3.3.** Let $K$ be a simplicial $q$-manifold (or any pure $q$-dimensional simplicial complex) and $\sigma \in K$ be a $(q - i)$-simplex ($0 \leq i \leq q$) such that $\text{link}_K \sigma$ is the boundary $\partial \tau$ of an $i$-simplex $\tau$ that is not a face of $K$. Then the operation $\chi_\sigma$ on $K$ defined by

\[
\chi_\sigma(K) := K \setminus (\sigma \ast \partial \tau) \cup (\partial \sigma \ast \tau)
\]

is called a bistellar $i$-move.

A $q$-dimensional PL manifold is a simplicial complex $K$ such that the link of every nonempty simplex $\sigma$ in $K$ is a PL sphere of dimension $q - \text{dim}(\sigma) - 1$. Two PL manifolds $K$ and $K'$ are called bistellarly equivalent if we can obtain $K'$ from $K$ by a finite sequence of bistellar moves. It is easy to see that two bistellarly equivalent PL manifolds are PL homeomorphic. The following remarkable result shows that the converse is also true.

**Theorem 3.4** (Pachner [14 15]). Two PL manifolds are bistellarly equivalent if and only if they are PL homeomorphic.
Since a simple polytope \( P \) and its dual simplicial polytope \( P^* \) determine each other, we have the following one-to-one correspondence:

\[
\{ \text{simple polytopes } P \} \leftrightarrow \{ \text{polytopal spheres } \partial P^* \}.
\]

It is easy to see that for any \( 1 \leq a \leq n \), a \((a,b)\)-type flip on \( P \) corresponds to a bistellar \((a-1)\)-move on \( \partial P^* \). Indeed, any proper face \( f \) of \( P \) determines a unique simplex of \( \partial P^* \) denoted by \( \sigma_f \), where \( \dim(\sigma_f) = n - \dim(f) - 1 \). Then for any simplicial face \( f \) of \( P \), \( \chi_{\sigma_f}(\partial P^*) \) corresponds to flip \( f(\partial P^*) \). So we have the correspondence:

\[
\{ \text{flips on simple } n\text{-polytopes} \} \leftrightarrow \{ \text{bistellar moves on polytopal } (n-1)\text{-spheres} \}.
\]

By Pachner’s theorem, all the simplicial complexes obtained from bistellar moves on \( \partial \Delta^n \) are exactly PL \((n-1)\)-spheres. So we have the following algebraic system of Lickorish type construction of all polytopal simplicial spheres.

\[
\{ \text{All polytopal } (n-1)\text{-spheres} \} \subseteq \text{PL } (n-1)\text{-spheres } = \text{AS}\{\partial \Delta^n | \text{bistellar moves}\}
\]

Moreover, the following theorem implies that we do not need flips of type \((n,1)\) to obtain a simple \( n\)-polytope \( P \) from the \( n\)-simplex \( \Delta^n \).

**Theorem 3.5** (see [4]). Let \( P \) be simple polytope of dimension \( n \geq 3 \). Then there is a sequence of simple polytopes \( P_1, \ldots, P_m \) such that \( P_1 = \Delta^n \), \( P_m = P \) and for \( i = 1, \cdots, m-1 \), \( \partial P_{i+1} \) is obtained from \( \partial P_i \) by a bistellar \( k \)-move with \( 0 \leq k \leq n-2 \).

So we have the following corollary, up to combinatorial equivalence

**Corollary 3.6.** \( \{ \text{All simple } n\text{-polytopes} \}/\sim_C \subseteq \text{AS}\{\Delta^n | \text{flips } F_{(a,b)}, a + b = n + 1, 1 \leq a \leq n-1, 2 \leq b \leq n\} \).

From the point of view of surgery, flips and bistellar moves are some sort of combinatorial surgeries. Analogy to Dehn surgeries in Lickorish’s Theorem [1.1], these two kinds of operations are concrete and constructive. From finite concrete generators, one can use these two kind of operations to construct all objects in the above sets of combinatorial classes.

**Remark 3.7.** It is known that for a compact PL manifold \( M \), the differential structure on \( M \) is determined by the homotopy set \([M, PL/O]\). Since \( \pi_n(PL/O) = 0, n < 7 \), it follows that if \( M^n \) is a PL \( n \)-sphere with \( n < 7 \), there is a one-to-one correspondence between PL structures on \( M^n \) and smooth structures on \( M^n \). The most interesting case is in dimension 4, in this case the classification of PL structure on \( S^4 \) is equivalent to the classification of smooth structure on \( S^4 \).

On the other hand, bistellar move doesn’t change PL structure. So it may be interesting to find some kind of “bistellar move” invariants on \( S^4 \).
4. LIFTING OPERATIONS ON $P$ TO $\mathbb{Z}_P$ AND $\mathbb{R}\mathbb{Z}_P$

In section 2, we introduce two families of manifolds over a simple polytope $P$ equipped with some special actions of real torus and $\mathbb{Z}_2$-torus. It is natural to consider the question of lifting the operations on $P$ to these manifolds. First, let’s consider the surgery on “the highest level” $\mathbb{Z}_P$ and $\mathbb{R}\mathbb{Z}_P$.

Suppose $P$ is a simple $n$-polytope with $m$ facets. Bosio-Meersseman [1] describes the equivariant surgery $\tilde F^C_{(a,b)}$ on $\mathbb{Z}_P$ corresponding to a $(a,b)$-type flip $\tilde F_{(a,b)}$ on $P$ at a simplicial face $f$ of $P$ which have no intersection with $f$. So when removing $N(f)$ from $P$ and glue back $\Delta^{b-1} \circ H_{a,b} \subset \Delta^n$ in the flip, the corresponding equivariant surgery on $\mathbb{Z}_P$ is given by:

$$\tilde F^C_{(a,b)}(\mathbb{Z}_P) = \begin{cases} (\mathbb{Z}_P \setminus (S^{2a-1} \times D^{2b} \times T^{m-n-1})) \\ \cup (D^{2a} \times S^{2b-1} \times T^{m-n-1}), & \text{if } a \neq 1, n; \\ ((\mathbb{Z}_P \times S^1) \setminus (S^1 \times D^{2n} \times T^{m-n})), & \text{if } a = 1; \\ (\mathbb{Z}_P \setminus (D^2 \times S^{2n-1} \times T^{m-n})) \cup \\ (S^1 \times D^{2n} \times T^{m-n})/S^1, & \text{if } a = n. \end{cases}$$

Note that the $a = n$ case is the converse operation of the $a = 1$ case, and the quotient /$S^1$ corresponds to the fact that a simplicial facet is shrunk to a vertex. These operations on $\mathbb{Z}_P$ are also given in Buchstaber-Panov [2 §6.4].

Similarly, let $\pi_\mathbb{R} : \mathbb{R}\mathbb{Z}_P \to P$ be the orbit map of the canonical $(\mathbb{Z}_2)^m$-action on $\mathbb{R}\mathbb{Z}_P$. For a small neighborhood $N(f)$ of $f$ in $P$, $\pi_\mathbb{R}^{-1}(N(f)) \cong S^{a-1} \times D^b \times (S^0)^{m-n-1}$, the equivariant surgery $\tilde F^\mathbb{R}_{(a,b)}$ on $\mathbb{R}\mathbb{Z}_P$ corresponding to the flip of $P$ at $f$ is given by

$$\tilde F^\mathbb{R}_{(a,b)}(\mathbb{R}\mathbb{Z}_P) = \begin{cases} (\mathbb{R}\mathbb{Z}_P \setminus (S^{a-1} \times D^b \times (\mathbb{Z}_2)^{m-n-1})) \\ \cup (D^a \times S^{b-1} \times (\mathbb{Z}_2)^{m-n-1}), & \text{if } a \neq 1, n; \\ ((\mathbb{R}\mathbb{Z}_P \times \mathbb{Z}_2) \setminus (S^0 \times D^n \times (\mathbb{Z}_2)^{m-n})), & \text{if } a = 1; \\ (\mathbb{R}\mathbb{Z}_P \setminus (D^1 \times S^{n-1} \times (\mathbb{Z}_2)^{m-n})) \cup \\ (S^0 \times D^n \times (\mathbb{Z}_2)^{m-n})/\mathbb{Z}_2, & \text{if } a = n. \end{cases}$$

Since flips on an $n$-simplex $\Delta^n$ can produce all simple polytopes (see Theorem 3.2 and Corollary 3.6), we can use the above equivariant surgeries to produce all (real) moment-angle manifolds from $S^{2n+1} = \mathbb{Z}_\Delta^n$ (or $S^n = \mathbb{R}\mathbb{Z}_\Delta^n$). Moreover, for any PL sphere $K$ we can define moment-angle complex $\mathbb{Z}_K$ and real moment-angle
complex $\mathbb{R}Z_K$; which generalizes the constructions for simple polytopes. Indeed, $Z_K$ and $\mathbb{R}Z_K$ are still topological manifolds for any PL sphere $K$.

Furthermore, two moment-angle manifolds $Z_{P_1}$ and $Z_{P_2}$ are equivariantly homeomorphic if and only if $P_1$ and $P_2$ are combinatorially equivalent. Moreover, in $[1]$ equivariant homeomorphism can be strengthened to equivariantly diffeomorphism.

Combing the results of Theorem 3.2 and Buchstaber-Panov $[2, \S 6.4]$, we have the following.

**Theorem 4.1.** Lickorish type constructions for moment-angle manifolds and real moment-angle manifolds.

1. **$\mathbb{C}$-case:**

   \[
   \{ \text{All moment-angle manifolds over simple $n$-polytopes with $m$ facets} \} / \sim_H \\
   \not\subseteq \{ \text{All moment-angle complexes over PL $(n-1)$-spheres with $m$ vertices} \} / \sim_H
   
   = AS\{ \mathbb{Z}_{\Delta^n} = S^{2n+1} | \tilde{F}_{(a,b)}^\mathbb{C}, a + b = n + 1, 1 \leq a, b \leq n \};
   \]

2. **$\mathbb{R}$-case:**

   \[
   \{ \text{All real moment-angle manifolds over simple $n$-polytopes with $m$ facets} \} / \sim_H
   \not\subseteq \{ \text{All real moment-angle complexes over PL $(n-1)$-spheres with $m$ vertices} \} / \sim_H
   
   = AS\{ \mathbb{R}Z_{\Delta^n} = S^n | \widetilde{F}_{(a,b)}^\mathbb{R}, a + b = n + 1, 1 \leq a, b \leq n \},
   \]

where $\sim_H$ denotes the equivalence relation of equivariant homeomorphism.

Since $\tilde{F}_{(a,b)}^\mathbb{C}$ and $\tilde{F}_{(a,b)}^\mathbb{R}$ both preserve the equivariant cobordism classes of the corresponding manifolds, we can deduce the following from Corollary 3.6.

**Corollary 4.2.** The moment-angle manifold and real moment-angle manifold of any simple polytope are equivariantly cobordant to zero in the category of compact manifolds with effective real torus or $\mathbb{Z}_2$-torus actions.

**Proof.** Let $P$ be a simple $n$-polytope with $m$ facets. By Corollary 3.6 and the definition of $\tilde{F}_{(a,b)}^\mathbb{C}$ and $\tilde{F}_{(a,b)}^\mathbb{R}$, the manifolds $Z_P$ and $\mathbb{R}Z_P$ are equivariantly cobordant to $S^{2n+1} \times T^{m-n-1}$ and $S^n \times (\mathbb{Z}_2)^{m-n-1}$, respectively. There is natural extension of the canonical action of $T^m = T^{n+1} \times T^{m-n-1}$ on $S^{2n+1} \times T^{m-n-1}$ to $D^{2n+2} \times T^{m-n-1}$. So $Z_P$ is equivariantly cobordant to zero. The similar argument works for $\mathbb{R}Z_P$. \[\square\]

Next, we consider some “lower level” classes $Z_P/H_\mathbb{C}$ and $\mathbb{R}Z_P/H_\mathbb{R}$. For the diagonal action, it is well-known that: $D_\mathbb{C} = \{(g, g, \cdots, g)\} \subseteq T^m \rhd \sim Z_P$ is free, and $D_\mathbb{R} = \{(g, g, \cdots, g)\} \subseteq (\mathbb{Z}_2)^m \rhd \sim \mathbb{R}Z_P$ is free. We have the following result parallel to Theorem 4.1.

**Theorem 4.3.** Lickorish type construction for quotient spaces induced by the diagonal action on (real) moment-angle manifolds.
\[ \tilde{C}\text{-case: } \{ \text{All quotient spaces } \mathbb{Z}_p/D_{\mathbb{C}} \} / \sim_H \subseteq \text{AS}\{ \mathbb{C}P^n = \mathbb{Z}_{\Delta}^n/D_{\mathbb{C}} \mid \tilde{F}_{(a,b)}^{D_{\mathbb{C}}}, a + b = n + 1, 1 \leq a, b \leq n \}; \]
\[ \tilde{R}\text{-case: } \{ \text{All quotient spaces } \mathbb{R}Z_p/D_{\mathbb{R}} \} / \sim_H \subseteq \text{AS}\{ \mathbb{R}P^n = \mathbb{R}Z_{\Delta}^n/D_{\mathbb{R}} \mid \tilde{F}_{(a,b)}^{D_{\mathbb{R}}}, a + b = n + 1, 1 \leq a, b \leq n \} \]
where \( P \) runs over all possible \( n \)-dimensional simple polytopes.

The operations \( \tilde{F}_{(a,b)}^{D_{\mathbb{C}}} \) and \( \tilde{F}_{(a,b)}^{D_{\mathbb{R}}} \) are defined as follows.

\[ \tilde{F}_{(a,b)}^{D_{\mathbb{C}}} (\mathbb{Z}_p) = \left\{ \begin{array}{ll}
((\mathbb{Z}_p/D_{\mathbb{C}}) \setminus (S^{2a-1} \times D^{2b} \times T^{m-n-2})) & \text{if } a \neq 1, n; \\
\cup (D^{2a} \times S^{2b-1} \times T^{m-n-2}), & \\
((\mathbb{Z}_p/D_{\mathbb{C}}) \setminus (S^1 \times D^{2n} \times T^{m-n-1})), & \text{if } a = 1; \\
\cup (D^2 \times S^{2n-1} \times T^{m-n-1}), & \text{converse operation of } a = 1 \text{ case, if } a = n.
\end{array} \right. \]

\[ \tilde{F}_{(a,b)}^{D_{\mathbb{R}}} (\mathbb{R}Z_p/D_{\mathbb{R}}) = \left\{ \begin{array}{ll}
((\mathbb{R}Z_p/D_{\mathbb{R}}) \setminus (S^{a-1} \times D^b \times (\mathbb{Z}_2)^{m-n-2})), & \text{if } a \neq 1, n; \\
\cup (D^n \times S^{b-1} \times (\mathbb{Z}_2)^{m-n-2})), & \\
((\mathbb{R}Z_p/D_{\mathbb{R}} \setminus (S^0 \times D^n \times (\mathbb{Z}_2)^{m-n-1}))), & \text{if } a = 1; \\
\cup (D^1 \times S^{n-1} \times (\mathbb{Z}_2)^{m-n-1}), & \text{converse operation of } a = 1 \text{ case, if } a = n.
\end{array} \right. \]

**Remark 4.4.** In this section, we discuss the liftings from the combinatorial type surgeries to the equivariant surgeries on the highest level. These liftings are one-to-one correspondence between equivariant homeomorphism classes and combinatorial equivalent classes. Hence we could say these surgeries are still constructible, since one can produce these “highest level” objects by concrete date from these concrete combinatorial operations.

### 5. Construction of Quasitoric Manifolds and Small Covers

#### 5.1. Low dimensional cases I: quasitoric manifolds

In the case of 2-dimensional simple polytopes, P. Orlik and F. Raymond’s work [13] implies the following (also see [3, p.427]).

**Theorem 5.1 (Orlik-Raymond [13]).**

\{ All 4-dim quasitoric manifolds \} / \sim_D = \text{AS}\{ \mathbb{C}P^2, \mathbb{C}P^2, S^2 \times S^2, \mathbb{C}P^2 \times \overline{\mathbb{C}P}^2 | \tilde{\varphi} \} \]

where \( \sim_D \) denotes the equivalence relation of \( T^2 \)-diffeomorphism and \( \tilde{\varphi} \) is the equivariant connected sum of two manifolds.

In the case of 3-dimensional small covers, Izmestiev [5] studied a special class of 3-dimensional small covers whose characteristic functions take values in a basis of \( (\mathbb{Z}_2)^3 \) (i.e. three linearly independent elements of \( (\mathbb{Z}_2)^3 \)). Izmestiev [5] gave a Lickorish type construction of such 3-dimensional small covers as follows.
Theorem 5.2 (Izmiestiev [5]).

- Combinatorial case:
  
  \[ C = \{ (P^3, \lambda) \mid |\text{Im} \lambda| = 3 \} = \text{AS}\{ (I^3, \lambda_0) \text{ with } |\text{Im} \lambda_0| = 3 \mid \# \} \]

  where \# is the connected sum of two simple polytopes at some vertices and \# is the operation on a 3-polytope shown in Figure 2.

- Topological case:
  
  \[ \{ M(P^3, \lambda) \mid (P^3, \lambda) \in C \} / = \text{AS}\{ M(I^3, \lambda_0) = T^3 \mid \# \} \]

  where \# and \# are the equivariant connected sum and the equivariant 0-type Dehn surgery, respectively.

In dimension 6, Shintarô Kuroki discussed the equivariant diffeomorphism classification question of all 6-dimensional torus manifolds with vanishing odd-degree cohomology, which is a wider class of manifolds including quasitoric manifolds. Similar to Theorem 5.1

Theorem 5.3 (Kuroki [7]). Up to \( T^3 \)-diffeomorphism, we have one-to-one correspondence:

\[ \{ \text{All 6-dimensional 1-connected torus manifolds with vanishing odd-degree cohomology} \} / \sim_D = \text{AS}\{ S^6, S^4\text{-bundles over } S^2, \text{quasitoric 6-manifolds} \mid \# \}, \]

where \# is the equivariant connected sum of two manifolds and \( \sim_D \) denotes the equivalence relation of equivariant \( T^k \)-diffeomorphism.

5.2. Low dimensional case II: small covers. The Four Color Theorem tells us that any simple 3-polytope admits \((\mathbb{Z}_2)^3\)-colorings. Denote by

- \( \mathcal{P} \) := the set of all pairs \((P^3, \lambda)\), where \( P^3 \) is a 3-dimensional simple convex polytope and \( \lambda \) is a nondegenerate \((\mathbb{Z}_2)^3\)-coloring on it.

- \( \mathcal{M} \) := the set of all 3-dimensional small covers.

By Davis-Januszkiewicz [6], there exists a one-to-one correspondence:

\[ \mathcal{P} \leftrightarrow \mathcal{M} \]

\[ (P^3, \lambda) \mapsto M(P^3, \lambda) \]
Zhi Lü and Li Yu studied general 3-dimensional small covers in [10] and showed that any 3-dimensional small cover can be obtained from small covers over $\Delta^3$ and a triangular prism via a sequence of surgeries. Combinatorially, one has:

**Theorem 5.4 (Lü-Yu [10]).** All pairs $(P^3, \lambda)$ of $P$ form an algebraic system with generators $(\Delta^3, \sigma \circ \lambda_0)$, $(P^3(3), \sigma \circ \lambda_1)$, $(P^3(3), \sigma \circ \lambda_2)$, $(P^3(3), \sigma \circ \lambda_3)$, $(P^3(3), \sigma \circ \lambda_4)$, $\sigma \in \text{GL}(3, \mathbb{Z}_2)$ and six operations $\sharp^v$, $\sharp^e$, $\sharp^{eve}$, $\natural$, $\sharp^\triangle$, $\sharp^\circ$, where $\Delta^3$ is a 3-simplex and $P^3(3)$ is a triangular prism, and $\lambda_0, \cdots, \lambda_4$ are shown in Figure 3.

The six operations $\sharp^v$, $\sharp^e$, $\sharp^{eve}$, $\natural$, $\sharp^\triangle$, $\sharp^\circ$ on $P$ are shown below (cf. [10]).

- Operation $\sharp^v$ on $P$:

- Operation $\sharp^e$ on $P$:
• Operation $\|^\text{eve}\$ on $\mathcal{P}$:

Here $P^3(3)$ is obtained by cutting a vertex from the triangular prism $P^3(3)$.

• Operation $\sharp$ on $\mathcal{P}$:

Note that two neighboring facets marked by $e_2$ and $e_3$ are needed to be big.

• Operation $\sharp^\Delta$ on $\mathcal{P}$:

Case (II): 2-independent coloring

$(P^3, \lambda) = (P^3_1, \lambda_1)\triangle(P^3_2, \lambda_2)$
• Operation $\sharp \circledast$ on $\mathcal{P}$:

\[
(P^3, \lambda) \circledast (Q, \tau) = (P^3 \lambda \circledast (Q, \tau))
\]

By the construction of small covers from pairs in $\mathcal{P}$, we have the following operations on $\mathcal{M}$ corresponding to $\sharp v, \sharp e, \sharp eve, \natural, \sharp \triangle, \sharp \circledast$.

- $\sharp v$ is the equivariant connected sum.
- $\natural$ is equivariant 0-type Dehn surgery.
- $\sharp e, \sharp eve, \sharp \triangle, \sharp \circledast$ are some equivariant cut-and-paste operations which can be understood as the generalized equivariant connected sums.

For the generators of $\mathcal{M}$, we take $\mathcal{M}(\Delta^3, \lambda_0)$ and $\mathcal{M}(P^3(3), \sigma \circ \lambda_i)$ ($i = 1, ..., 4$), $\sigma \in \text{GL}(3, \mathbb{Z}_2)$, which give all elementary generators of the algebraic system $\langle \mathcal{M} ; \sharp v, \sharp e, \sharp eve, \natural, \sharp \triangle, \sharp \circledast \rangle$. The topological types of these generators are as follows:

- $M(\Delta^3, \lambda_0) \approx \mathbb{R}P^3$ with the canonical linear $(\mathbb{Z}_2)^3$-action
- $M(P^3(3), \lambda_i)(i = 1, ..., 4) \approx S^1 \times \mathbb{R}P^2$ with four different $(\mathbb{Z}_2)^3$-actions

So we have the Lickorish type construction of all 3-dimensional small covers.

**Theorem 5.5** (Lü-Yu [10]). All the 3-dimensional small covers form an algebraic system with generators $\mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions and six operations $\sharp v, \sharp e, \sharp eve, \natural, \sharp \triangle, \sharp \circledast$.

In addition, Kuroki [6] studied the relations among six operations $\sharp v, \sharp e, \sharp eve, \natural, \sharp \triangle, \sharp \circledast$ on $\mathcal{P}$ and found that $\natural = \natural \circ (\sharp v P^3(3))$ and $\sharp eve = \natural \circ (\sharp e P^3(3))$. Furthermore, Nishimura [11] discovered more relations among the operations in Theorem 5.5 and obtained another algebraic system by the following theorem, which improved Theorem 5.5.

**Theorem 5.6** (Y. Nishimura [11]). All the 3-dimensional small covers form an algebraic system with generators $T^3, \mathbb{R}P^3$ and $S^1 \times \mathbb{R}P^2$ with certain $(\mathbb{Z}_2)^3$-actions and two operations $\natural, \sharp v$.

So by Theorem 5.6 we can obtain any 3-dimensional small cover by connected sum and some special kind of Dehn surgeries.
It should be pointed out that Nishimura in [12] also gave Lickorish type construction for all orientable 3-dimensional small covers as follows: all orientable 3-dimensional small covers are obtained from $\mathbb{R}P^3$ and $T^3$ by using the equivariant connected sum and equivariant $0 \frac{1}{T}$-type Dehn surgery.

**Remark 5.7.** In this section, we see in the ”lowest level”, there are plenty results similar to Theorem 1.1. Many “Equivariant Dehn surgeries” admit more concrete correspond to their combinatorial surgeries.

Furthermore, in many cases, one could reduce the number of generators and the number of specific types of equivariant surgeries as less as possible. In particular, according to [10], we could see these six operations (equivariant-surgeries) are very concrete and easy to handle in the combinatorial level as well as the manifold level (lowest level).

6. LICKORISH TYPE PROBLEMS IN EQUIVARIANT COBORDISMS

Recall that two smooth closed $n$-manifolds $M_1$ and $M_2$ are bordant if their disjoint union are the boundary of some $n+1$ manifold. One knows that if $M_1$ is bordant to $M_2$, $M_2$ can be obtained from $M_1$ by a finite steps of surgeries, which is also a kind of Lickorish type construction. Therefore, in cobordism, the idea of Lickorish type construction may provide a good point of view to discuss the manifolds in the same bordism classes.

In equivariant case, questions become more difficult. Roughly speaking, let $G$ be a compact Lie group, two $G$-manifolds $M_1$ and $M_2$ are $G$-equivariant bordant if there exists a $G$-manifolds $W$ with boundary $M_1 \sqcup M_2$ such that their $G$-structures are equivalent. We are interested in the equivariant cobordism classification problems of $G$-manifolds.

We can apply the preceding discussion of the Lickorish type construction of small covers to study their equivariant cobordism classification. Define $\widehat{\mathcal{M}}$ to be the set consisting of equivariant unoriented cobordism classes of all 3-dimensional small covers. Since $\widehat{\mathcal{M}}$ forms an abelian group under disjoint union, we can think of $\widehat{\mathcal{M}}$ as a vector space over $\mathbb{Z}_2$.

Let $[M(P^3_1, \lambda_1)]$ and $[M(P^3_2, \lambda_2)]$ be two classes in $\widehat{\mathcal{M}}$ where $\lambda_i$ is a characteristic function on a 3-dimensional simple polytope $P^3_i$, $i = 1, 2$. From Lü-Yu [10], we know

$$[M(P^3_1, \lambda_1) \overset{\text{ev}}{\exists} M(P^3_2, \lambda_2)] = [M(P^3_1, \lambda_1)] + [M(P^3_2, \lambda_2)]$$
$$[M(P^3, \lambda) \overset{\text{ev}}{\exists} M(P^3(3), \tau)] = [M(P^3, \lambda)] + [M(P^3(3), \tau)]$$
$$[M(P^3, \lambda) \overset{\text{ev}}{\exists} M(P^3(3), \tau)] = [M(P^3, \lambda)] + [M(P^3(3), \tau)]$$
\[
\begin{align*}
[M(P^3, \lambda) \# M(\varnothing, \tau)] &= [M(P^3, \lambda)] \\
[M(P^3, \lambda) \# M(P^3(i), \tau)] &= [M(P^3, \lambda)] + [M(P^3(i), \tau)], \quad i = 3, 4, 5. \\
[M(P_1^3, \lambda_1)] &\sim [M(P_2^3, \lambda_2)] \\
&= \left\{ \\
&\quad [M(P_1^3, \lambda_1)] + [M(P_2^3, \lambda_2)] \\
&\quad \mathbb{Z}_2 \cdot [M(P_3^3, \lambda_1 \triangle \lambda_2)] \\
&\quad \text{or } [M(P_1^3, \lambda_1)] + [M(P_2^3, \lambda_2)] + [M(P_3^3, \lambda_1 \triangle \lambda_2)].
\right.
\end{align*}
\]

By the above discussion, it is easy to see that the abelian group \( \hat{M} \) is generated by some small covers over \( \Delta^3 \) and \( P^3(3) \).

**Proposition 6.1** (Equivariant cobordism classification of 3-dim small covers). The abelian group \( \hat{M} \) is generated by classes of \( \mathbb{R}P^3 \) and \( S^1 \times \mathbb{R}P^2 \) with certain \( (\mathbb{Z}_2)^3 \)-actions.

It is known in [9] that as a \( \mathbb{Z}_2 \)-vector space, \( \hat{M} \) has dimension 13.

Similarly, for quasitoric manifolds we would like to study the following problem.

**Problem 6.2.** Give a Lickorish type construction for all 6-dimensional quasitoric manifolds with simple generators and simple operators, and compute the equivariant cobordism group.

**Problem 6.3.** Find some simple Lickorish type constructions between two quasitoric manifolds if they are equivariantly bordant.

In this survey, we’ve introduced and restated many theorems in the “Lickorish type” style. In our point of view, Lickorish’s original idea is to use less generators and finite “easy” operators to construct all the objects. We are happy to find many “Lickorish type” combinatorial and geometric objects and we hope that these “Lickorish type” style theorems could provide another point of view to understand combination, geometry and topology.

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