Algorithms for Generating Ordered Solutions for Explicit AND/OR Structures : Extended Abstract*

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Abstract

We present algorithms for generating alternative solutions for explicit acyclic AND/OR structures in non-decreasing order of cost. Our algorithms use a best first search technique and report the solutions using an implicit representation ordered by cost. Experiments on randomly constructed AND/OR DAGs and problem domains including matrix chain multiplication, finding the secondary structure of RNA, etc. show that the proposed algorithms perform favorably to the existing approach in terms of time and space.

1 Introduction

Traditionally AND/OR graphs [Nilsson, 1980] have been used in problem reduction search for solving complex problems, logical reasoning, theorem proving, etc. Over the last decade there has been a renewed research interest towards applying AND/OR structures in a wide variety of other areas like planning with uncertainty [Russell and Norvig, 2003], graphical models [Dechter and Mateescu, 2007], web service composition [Lang et al., 2005], solution sampling and counting [Gogate and Dechter, 2008], model based programming [Elliott, 2007], etc. However, the problem of generating alternative solutions which has been studied for ordinary graphs extensively [Nilsson, 1998; Eppstein, 1998; 1990; Chegireddy and Hamacher, 1987], has not been deeply studied for AND/OR graphs and the approaches related to these topics are often not obvious.

An ordered set of solutions of an explicit AND/OR DAG can be used to develop useful variants of the AO* algorithm. Currently in AO*, only the minimum cost solution is computed whereas several variants of the A* algorithm exist, where solutions are often sought within a factor of cost of the optimal solution. These approaches were developed to adapt the A* algorithm for using inadmissible heuristics, leveraging multiple heuristics, generating solutions quickly within bounded sub-optimality. [Ehendt and Drechsler, 2009; Chakrabarti et al., 1989; Pearl, 1984], etc. Similar techniques can be developed for AO* search if ordered set of potential solutions are made available. An initial algorithm for generating k-best solutions for AND/OR DAGs has been proposed in [Elliott, 2007] in the context of model based programming. For graphical models, which can be expressed using AND/OR structures, ordered sets of solutions are useful [Flerova and Dechter, 2011; 2010]. In the domain of service composition, the primary motivation behind providing a set of alternative solutions ordered by cost is to offer more choices, while trading off the specified cost criterion (to a limited extent) in favor of other ‘unspecified’ criteria (primarily from the standpoint of quality) and algorithms have been developed for generating a ranked set of solutions for the service composition problem [Shiaa et al., 2008].

The existing method [Elliott, 2007] (will be referred to as BU henceforth) works bottom-up by computing k-best solutions for the current node. We present a best first search algorithm, named Alternative Solution Generation (ASG) for generating an ordered set of solutions. The proposed algorithm maintains a list of candidate solutions, initially containing only the optimal solution, and iteratively generates the next solution in non-decreasing order of cost by selecting the minimum cost solution from the list. In each iteration, this minimum cost solution is used to construct another set of candidate solutions, which is again added to the current list. We present two versions of the algorithm, namely, (i) Basic ASG (will be referred to as ASG henceforth) : This version of the algorithm may construct a particular candidate solution more than once; (ii) Lazy ASG or LASG : A variant of ASG that constructs every candidate solution only once.

We use a compact representation, named signature, for storing the solutions. From the signature of a solution, the actual explicit form of that solution can be constructed through a top-down traversal of the given DAG. This representation allows the proposed algorithms to work in a top-down fashion starting from the initial optimal solution. Unlike the existing approach, our proposed algorithms are interruptible in nature, i.e., our algorithms can be stopped at any point of time during the execution and the set of ordered solutions obtained so far can be observed and subsequent solutions will be generated when the algorithms are resumed again. Moreover, if an upper limit estimate on the number of solutions required is known a priori, our algorithms can be further optimized using that estimate. We have used randomly constructed trees and DAGs as well as domains including the 5-peg Tower of...
Hanoi problem, the matrix-chain multiplication problem and the problem of finding the secondary structure of RNA as test domains. Experimental results show that our proposed algorithms outperform the existing approach by a large margin in terms of the memory used and time required for generating a specific number of ordered solutions for different domains.

2 Background and Proposed Algorithms

In this paper we use $G_{\alpha\beta} = (V, E)$ to denote an AND/OR directed acyclic graph where $V$ is the set of nodes, and $E$ is the set of edges. The nodes of $G_{\alpha\beta}$ with no successors are called terminal nodes. The non-terminal nodes of $G_{\alpha\beta}$ are of two types – i) OR nodes and ii) AND nodes. The start (or root) node of $G_{\alpha\beta}$ is denoted by $v_R$. OR edges and AND edges are the edges that emanate from OR nodes and AND nodes respectively. We use the standard notion of solution graph, $S(v_q)$, rooted at any node $v_q \in V$ [Martelli and Montanari, 1973]. By a solution graph $S$ of $G_{\alpha\beta}$ we mean a solution graph with root $v_R$.

We present our algorithms using the notion of alternating AND/OR trees which offer a succinct representation. Also, the correctness proofs are much simpler for alternating AND/OR trees. Formally, an alternating AND/OR tree, $T_{\alpha\beta} = (V, E)$, is an AND/OR tree with the restriction that there is an alternation between the AND nodes and the OR nodes. Every child of an AND node is either an OR node or a terminal node, and every children of an OR node is either an AND node or a terminal node. We have shown in [Ghosh et al., 2012] that every AND/OR tree can be converted to an equivalent alternating AND/OR tree with respect to the solution space. The notion of cost is as follows. In $G_{\alpha\beta}$, every edge $e_{qr} \in E$ from node $v_q$ to node $v_r$ has a finite non-negative cost $c_e((v_q, v_r))$ or $c_e(e_{qr})$. Similarly every node $v_q$ has a finite non-negative cost denoted by $c_v(v_q)$. We use the notion of additive costs for given solution graph $S$ [Martelli and Montanari, 1973].

Definition 2.2a [Cost of a Solution Graph] For every node $v_q$ in $S$, $C(S, v_q)$ denotes the cost of the solution graph, $S(v_q)$. The cost of a solution $S$ is $C(S, v_R)$ which is also denoted by $C(S)$. We denote the optimal solution below every node $v_q$ as $opt(v_q)$. Therefore, the optimal solution of the entire AND/OR DAG $G_{\alpha\beta}$, denoted by $S_{opt}$, is $opt(v_R)$. The cost of the optimal solution rooted at every node $v_q$ in $G_{\alpha\beta}$ is $C_{opt}(v_q)$. The cost of the optimal solution $S_{opt}$ of $G_{\alpha\beta}$ is denoted by $C_{opt}(v_R)$ or, alternatively, by $C_{opt}(S_{opt})$. □

The existing method, BU [Elliott, 2007], works with the input parameter $k$, i.e., the number of solutions to be generated have to be known a priori. Also this method is not inherently incremental in nature, thus does not perform efficiently when the solutions are needed on demand, e.g., first initial 20 solutions are needed, then the next 10 are needed. In this case the first 20 solutions will have to be recomputed again while computing next 10 solutions, i.e., from the 21st solution to the 30th solution. We address these limitations in our proposed approach. In this paper, we present our proposed algorithms for alternating AND/OR trees for brevity. The details of the changes required for handling AND/OR DAGs are presented in detail in [Ghosh et al., 2012].

2.1 Core Concepts and Definitions

Our proposed ASG algorithm works top-down using edge markings, to generate the next best solutions from the previously generated solutions. ASG works iteratively — it generates the next best solution based on the previously generated solutions. Thus, we first compute the optimal solution, and subsequently generate other solutions from the already generated solutions. The following terminology and notions are used to describe the ASG algorithm. In the context of AND/OR trees, we use $e_q$ to denote the edge that points to the vertex $v_q$. Our algorithm has been developed on the notion of

- (a) marking of an OR edge, (b) swap option and swap operation and (b) the implicit representation of a solution. Next, we briefly describe the concepts.

Definition 2.2b [Aggregated Cost] In an AND/OR DAG $G_{\alpha\beta}$, the aggregated cost, $c_{\alpha\beta}$, for an edge $e_{ij}$ from node $v_i$ to node $v_j$ is defined as: $c_{\alpha\beta}(e_{ij}) = c_e(e_{ij}) + C_{opt}(v_j)$. □

Definition 2.2c [Vopt and Eopt] For any solution graph $S_m$ of an AND/OR DAG $G_{\alpha\beta}$, we define a set of nodes, $V_{opt}(S_m)$, and a set of OR edges, $E_{opt}(S_m)$, as:

1. $V_{opt}(S_m) = \{v_q \mid v_q \in S_m \text{ and solution graph } S_m(v_q) \text{ is identical to the solution graph } opt(v_q)\}$

2. $E_{opt}(S_m) = \{e_{pr} \mid \text{OR edge } e_{pr} \in S_m, \text{ and } v_r \in V_{opt}(S_m)\}$

Clearly, for any node $v_q \in V_{opt}(S_m)$, if $v_q$ is present in $S_{opt}$, then the solution graph $S_m(v_q)$ is identical to the solution graph $S_{opt}(v_q)$. Also $C(S_m, v_q) = C_{opt}(v_q)$ □

Marking of an OR edge. The notion of marking an OR edge is as follows. For an OR node $v_q$, $L(v_q)$ is the list of OR edges of $v_q$ sorted in non-decreasing order of the aggregated cost of the edges. We define $\delta_{i,i+1}$ as the difference of the cost of OR edge $e_i$ and $e_{i+1}$, where $e_i$ and $e_{i+1}$ emanate from the same OR node $v_q$, and $e_{i+1}$ is the edge next to $e_i$ in $L(v_q)$. Consider a solution, $S_{curr}$, containing the edge $e_i = (v_q, v_{pr})$, where $e_i \in E_{opt}(S_{curr})$. We mark $e_i$ with the cost increment which will be incurred to construct the next best solution from $S_{curr}$ by choosing another child of $v_q$.

Definition 2.2d [Swap Option] A swap option $\sigma_{ij}$ is defined as a three-tuple $(e_i, e_j, \delta_{ij})$ where $e_i$ and $e_j$ emanate from the same OR node $v_q$, $e_j$ is the edge next to $e_i$ in $L(v_q)$, and $\delta_{ij} = c_e(e_j) - c_e(e_i)$. Also, we say that the swap option $\sigma_{ij}$ belongs to the OR node $v_q$. □

Consider the OR node $v_q$ and the sorted list $L(v_q)$. It may be observed that in $L(v_q)$ every consecutive pair of edges forms a swap option. Therefore, if there are $k$ edges in $L(v_q)$, $k-1$ swap options will be formed. At node $v_q$, these swap options are ranked according to the rank of their original edges in $L(v_q)$. We present an example of an alternating AND/OR tree in Figure 1. The terminal nodes are represented by a circle with thin outline. AND nodes are shown in the figures with their outgoing edges connected by a semi-circular curve in all the examples. The edge costs are shown by the side of each edge within an angled bracket. The cost of the terminal nodes are shown inside a box. For every non-terminal node $v_q$, the pair of costs, $c_v(v_q)$ and $C_{opt}(v_q)$, is shown inside a rectangle. The optimal solution below every node is shown
using by thick dashed edges with an arrow head. The marks corresponding to OR edges $e_2$, $e_3$, $e_6$, $e_{11}$, and $e_{13}$ are $[e_2 : 5]$, $[e_3 : 1]$, $[e_6 : 3]$, $[e_{11} : 4]$, and $[e_{13} : 3]$. Similarly, the corresponding swap options are: $\sigma_{(2,3)} = \langle e_2, e_3, 5 \rangle$, $\sigma_{(3,4)} = \langle e_3, e_4, 4 \rangle$, $\sigma_{(9,10)} = \langle e_9, e_{10}, 3 \rangle$, $\sigma_{(11,12)} = \langle e_{11}, e_{12}, 4 \rangle$, and $\sigma_{(13,14)} = \langle e_{13}, e_{14}, 3 \rangle$.

**Figure 1**: Example of OR-edge marking and swap option

**Definition 2.1 [Order Relation $\mathcal{R}$]** We define an order relation, namely $\mathcal{R}$, between a pair of swap options as follows.

1. If there is a path from $v_i$ to $v_r$ in $T_{\alpha \beta}$, where $e_i$ and $e_r$ are OR edges, $\sigma_{qi}$ and $\sigma_{rj}$ are swap options, then $(\sigma_{qi}, \sigma_{rj}) \in \mathcal{R}$.
2. If $\sigma_{pq} = \langle e_p, e_q, \delta_{pq} \rangle$ and $\sigma_{rt} = \langle e_r, e_t, \delta_{rt} \rangle$ are two swap options such that $v_q = v_r$, then $(\sigma_{pq}, \sigma_{rt}) \in \mathcal{R}$.

**Implicit Representation of the Solutions.** We use an implicit representation for storing every solution other than the optimal one. These other solutions can be constructed from the optimal solution by applying a set of swap options to the optimal solution in the following way. If $(\sigma_i, \sigma_j) \in \mathcal{R}$, $\sigma_i$ has to be applied before $\sigma_j$. Therefore, every solution is represented as a sequence $\Sigma$ of swap options, where $\sigma_i$ appears before $\sigma_j$ in $\Sigma$ if $(\sigma_i, \sigma_j) \in \mathcal{R}$. Intuitively the application of every swap option specifies that the swapped edge will be the part of the solution.

**Definition 2.2 [Signature of a Solution]** The minimal sequence of swap options corresponding to a solution, $S_m$, is defined as the signature, $\text{Sig}(S_m)$, of that solution. It may be noted that for the optimal solution $S_{\text{opt}}$ of any alternating AND/OR tree $T_{\alpha \beta}$, $\text{Sig}(S_{\text{opt}}) = \{\}$, i.e., an empty sequence. It is possible to construct more than one signature for a solution, as $\mathcal{R}$ is a partial order. It is important to observe that all different signatures for a particular solution are of equal length and the sets of swap options corresponding to these different signatures are also equal. Therefore the set of swap options corresponding to a signature is a canonical representation of the signature. Henceforth we will use the set notation for describing the signature of a solution.

**Definition 2.3 [Swap List]** The swap list corresponding to a solution $S_m$, $L(S_m)$, is the list of swap options that are applicable to $S_m$. Let $\text{Sig}(S_m) = \{\sigma_1, \cdots, \sigma_m\}$ and $\forall i, \ 1 \leq i \leq m$, each swap option $\sigma_i$ belongs to node $v_{p_i}$. The application of all other swap options that belong to the OR edges in the path from the root node to $v_{p_i}$, is invalidated in the solution $S_m$ for the same reason. This condition restricts the set of swap options that can be applied on a particular solution.

**Definition 2.4 [Successors and Predecessors of a Solution]** The set of successors and predecessors of a solution $S_m$ is:
1. \( \text{Succ}(S_m) = \{ S'_m \mid S'_m \text{ can be constructed from } S_m \text{ by applying a swap option that belongs to the swap list of } S_m \} \)

2. \( \text{Pred}(S_m) = \{ S''_m \mid S_m \in \text{Succ}(S''_m) \} \)

### 2.2 ASG algorithm

In our ASG algorithm, we maintain a list, Open, which initially contains only the optimal solution \( S_{opt} \). At any point of time Open contains a set of candidate solutions from which the next best solution in the non-decreasing order of cost is selected. At each iteration the minimum cost solution (\( S_{min} \)) in Open is removed from Open and added to another list, named, Closed. The Closed list contains the set of ordered solutions generated so far. Then the successor set of \( S_{min} \) is constructed and any successor solution which is not currently present in Open as well as is not already added to Closed is inserted to Open. However as a further optimization, we use a sublist of Closed, named TList, to store the relevant portion of Closed such that checking with respect to the solutions in TList is sufficient to figure out whether the successor solution is already added to Closed. It is interesting to observe that this algorithm can be interrupted at any time and the set of ordered solutions computed so far can be obtained. Also, the algorithm can be resumed if some more solutions are needed.

We briefly mention the key lemma and theorems.

**Definition 2.1** [Solution Space DAG (SSDAG)] The solution space DAG of an AND/OR tree \( T_{\alpha \beta} \) is a directed acyclic graph (DAG), \( G^* = (V, E) \), where \( V \) is the set of all possible solutions of the AND/OR tree \( T_{\alpha \beta} \), and \( E \) is the set of edges which is defined as:

\[
E = \left\{ e^a_{pm} \mid S_p, S_m \in V, \text{ and } e^a_{pm} \text{ is a directed edge from } S_p \text{ to } S_m, \text{ and } S_m \in \text{Succ}(S_p) \right\}
\]

Clearly \( S_{opt} \) is the root node of \( G^* \).

**Lemma 2.1.** For any alternating AND/OR tree \( T_{\alpha \beta} \), for every node of the SSDAG of \( T_{\alpha \beta} \), ASG algorithm generates the solution corresponding to that node.

**Theorem 2.1.** \( \forall S_j \in V, S_j \) is generated by ASG algorithm only once and in the non-decreasing order of costs while ties among the solutions having same costs are resolved arbitrarily in the favour of predecessor solutions.

### 2.3 LASG algorithm

Although ASG algorithm adds a solution to Closed only once, a particular solution may be present in the successor set of more than one solutions. Thus it is necessary to check whether a particular solution \( S_i \) is already present in Open or TList before adding \( S_i \) to Open. We present a lazy version of ASG, LASG, which avoids this checking for duplicates using the notion of native swap options. Conceptually, the native swap options for a solution \( S_q \) are the swap options that become available immediately after applying \( \sigma_{ij} \), but were not available in the predecessor solution of \( S_q \).

We use the notion of solution space tree which is conceptually a sub-tree of a spanning tree of the solution space DAG and the spanning trees of the solution space DAGs are complete solution space trees. In LASG, instead of using the entire swap list of a solution to construct all successors at once and then add those solutions to Open, the native swap options are used for constructing a subset of the successor set. This subset consists of only those solutions that are currently not present in Open and thus can be added to Open without comparing with the existing entries in Open. The construction of each remaining successor solution \( S'_m \) of \( S_m \) and subsequently the insertion of \( S'_m \) to Open is delayed until every other predecessor solution of \( S'_m \) is added to Closed. In LASG, the solution space tree \( T^* \) is maintained to determine when every other predecessor of \( S'_m \) is added to Closed.

**Theorem 2.2.** The solution space tree constructed by LASG algorithm is complete.

**Complexity**: We briefly compare the complexities of our proposed algorithms, ASG and LASG with the existing approach BU. For BU algorithm, the time complexity of generating the \( c \) best solutions for an AND/OR tree is \( O(n_{\alpha \beta} \cdot c \cdot \log_e c) \) and the space complexity is \( O(n_{\alpha \beta} \cdot c) \), where \( n_{\alpha \beta} \) denotes the number of nodes in the AND/OR tree. Whereas, the space complexity of both ASG and LASG algorithm is \( O(n_{\alpha \beta} \cdot c) \). The time complexity of LASG is \( O(c \cdot n_{\alpha \beta}) \) which is log \( c \) factor better than BU. When an additional hash-map is used to reduce the time overhead of duplicate checking, the time complexity of ASG becomes \( O(n_{\alpha \beta} + \sqrt{n_{\alpha \beta} \cdot (c \cdot \log_e c + c \cdot \log_e n_{\alpha \beta})}) \), which is asymptotically lower than both LASG and BU. It is important to note that, using the additional hash map does not change asymptotic space complexity of ASG, although the exact space requirement is doubled.

### 3 Summary of Experimental Results

We have used randomly constructed trees and DAGs as well as some well-known problem domains including the 5-peg Tower of Hanoi problem, the matrix-chain multiplication problem and the problem of finding the secondary structure of RNA as test domain. For the randomly constructed trees, LASG runs up to 20 times faster than the existing approach BU. For randomly constructed DAGs, the running time of LASG is up to 100 times lower than the running time of BU and between ASG and BU, the running times are comparable. Also, for randomly constructed DAGs, the space requirement is up to 5 times lower on average for ASG compared to BU; LASG requires around 50% of the space required by ASG. We have also empirically shown that, if the number of solutions to be generated is known a priori, ASG and LASG can be further optimized in terms of running time.

For 5-peg tower of Hanoi, matrix-chain multiplication and secondary structure of RNA, the performances of ASG and LASG are similar. For 5-peg tower of Hanoi problem LASG runs around two times faster on average than BU. Whereas, for matrix-chain multiplication and secondary structure of RNA the running time of LASG is two orders of magnitude better on average than that of BU. Details of the experimental results as well as the analysis based on the size of Open are presented in [Ghosh et al., 2012].
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