On $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function II

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Abstract. A representation of a specialization of a $q$-deformed class one lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function in terms of cohomology groups of line bundles on the space $\mathcal{Q}\mathcal{M}_d(\mathbb{P}^\ell)$ of quasi-maps $\mathbb{P}^1 \to \mathbb{P}^\ell$ of degree $d$ is proposed. For $\ell = 1$, this provides an interpretation of non-specialized $q$-deformed $\mathfrak{gl}_2$-Whittaker function in terms of $\mathcal{Q}\mathcal{M}_d(\mathbb{P}^1)$. In particular the ($q$-version of) Mellin-Barnes representation of $\mathfrak{gl}_2$-Whittaker function is realized as a semi-infinite period map. The explicit form of the period map manifests an important role of $q$-version of $\Gamma$-function as a substitute of topological genus in semi-infinite geometry. A relation with Givental-Lee universal solution ($J$-function) of $q$-deformed $\mathfrak{gl}_2$-Toda chain is also discussed.

Introduction

In the first part [GLO1] of this series of papers we have proposed an explicit representation of a $q$-deformed class one lattice $\mathfrak{gl}_{\ell+1}$-Whittaker function defined as a common eigenfunction of a complete set of commuting quantum Hamiltonians of $q$-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain. Here “class one” means that Whittaker function is non-zero only in the dominant domain. On $q$-deformed Toda chains see e.g. [Et]. The case $\ell = 1$ was discussed previously in [GLO3] (for related results in this direction see [KLS, GiL, GKL1, BF, FFJMM]). A special feature of the proposed representation is that $q$-deformed class one $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi_{\ell+1}(\mathfrak{z})$ with $\mathfrak{z} = (z_1, \ldots, z_{\ell+1})$ and $\mathfrak{z} = (p_1, \ldots, p_{\ell+1}) \in \mathbb{Z}^{\ell+1}$, is given by a character of a $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{\mathfrak{z}}$. The expression in terms of a character can be considered as a $q$-analog of the Givental-Casselman-Shalika representation of class one $p$-adic Whittaker functions [Sh], [CS]. Indeed our representation of $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function reduces, in a certain limit, to the Shintani-Casselman-Shalika representation of $p$-adic Whittaker function. Note that the representation $q$-deformed Whittaker function as a character is a $q$-analog of the Givental integral representation [Gi2], [GKL2] of the classical $\mathfrak{gl}_{\ell+1}$-Whittaker function.

The main objective of this paper is a better understanding of the representation of $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a character. Below we will consider a specialization of the $q$-deformed Whittaker function given by the trace over $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{n,k}$ (in the case $\ell = 1$ there is actually no specialization). Our main result is presented in Theorem 3.1. We provide a description of $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{n,k}$ as a zero degree cohomology group of a line bundle on an algebraic version $\mathcal{L}_{\ell+1}$ of a semi-infinite cycle $\widetilde{\mathcal{L}}_{\ell+1}$ in a universal covering $\widetilde{\mathcal{L}}$ of the space of loops in $\mathbb{P}^\ell$. We define $\mathcal{L}$ as an appropriate limit $d \to \infty$ of the space $\mathcal{Q}\mathcal{M}_d(\mathbb{P}^\ell)$ of degree $d$ quasi-maps of $\mathbb{P}^1$ to $\mathbb{P}^\ell$ [GI], [CJS]. In particular for $\ell = 1$ this provides a description of a $q$-deformed $\mathfrak{gl}_2$-Whittaker function in terms of cohomology of line bundles over $\mathbb{P}^1$. A universal solution of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain [GiL] was given in terms of cohomology groups of line bundles over $\mathcal{Q}\mathcal{M}_d(X), X = G/B$ for finite $d$. We demonstrate how our interpretation of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is reconciled with the results of [GiL].
Using Theorems 3.1 we interpret a $q$-version of the Mellin-Barnes integral representation of the specialized $q$-Whittaker function as a semi-infinite analog of the Riemann-Roch-Hirzebruch theorem. The corresponding Todd class is expressed in terms of a $q$-version of $\Gamma$-function. Analogously, the classical $\Gamma$-function appears in a description of the fundamental class of semi-infinite homology theory and enters the Mellin-Barnes integral representation of the classical Whittaker function. We briefly consider an analog of the elliptic genus arising in the $\mathbb{C}^*$-localization on $\mathbb{L}^{p\ell}_+$. We demonstrate that proliferation of fixed points of $\mathbb{C}^*$-action obstructs identification of the result as a topological genus of an extraordinary cohomology theory. Note also that the ($q$-version of) $\Gamma$-function which appears in our calculations of a semi-infinite analog of the Todd class was considered as a candidate for a topological genus by Kontsevich [K] (see also [L3], [Ho]).

Let us stress that the $\mathbb{C}^* \times GL_{\ell+1}(\mathbb{C})$-module $\mathcal{V}_{n,k}$ arising in the description of $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function is not irreducible. It would be natural to look for an interpretation of $\mathcal{V}_{n,k}$ as an irreducible module of a quantum affine Lie group. A relation of the geometry of semi-infinite flags to representation theory of affine Lie algebras was proposed in [FF]. The semi-infinite flag space is defined as $X_{\mathbb{C}^*} = G(K)/H(O)N(K)$ where $K = \mathbb{C}((t))$, $O = \mathbb{C}[[t]]$, $B = HN$ is a Borel subgroup of $G$, $N$ is its unipotent radical and $H$ is the associated Cartan subgroup. The semi-infinite flag spaces are not easy to deal with. An interesting approach to the semi-infinite geometry was proposed by Drinfeld. He introduced a space of quasi-maps $\mathcal{Q}M_{d}(\mathbb{P}^1, G/B)$ that should be considered as a finite-dimensional substitute of the semi-infinite flag space $X_{\mathbb{C}^*}$ (see e.g. [FM], [FFM], [Bra]). Thus, taking into account constructions proposed in this paper one can expect that ($q$-deformed) $\mathfrak{gl}_{\ell+1}$-Whittaker functions (encoding Gromov-Witten invariants and their $K$-theory generalizations) can be expressed in terms of representation theory of affine Lie algebras (see [GiL] for a related conjecture and [FF, JMM] for a recent progress in this direction). The paper [GLO2] deals with a relation of our results with the representation theory of (quantum) affine Lie groups.

The paper is organized as follows. In Section 1, explicit solutions of $q$-deformed $\mathfrak{gl}_{\ell+1}$-Toda chain ($q$-versions of Whittaker functions) are recalled. In Section 2, we derive integral expressions for the counting of holomorphic sections of line bundles in the space of quasi-maps. In Section 3 we derive a representation of specialized $q$-Whittaker functions in terms of cohomology of holomorphic line bundles on the space of quasi-maps of $\mathbb{P}^1$ to $\mathbb{P}^\ell$. We propose an interpretation of $q$-Whittaker functions as semi-infinite periods. In Section 4 the analogous interpretation of the classical Whittaker functions is discussed. In Section 5, we clarify the connection of our interpretation of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function with the results of [GiL]. Finally, in Section 6 we consider an analog of elliptic genus arising after $\mathbb{C}^*$-localization on $\mathbb{L}^{p\ell}_+$ and its possible relation with extraordinary cohomology theories.

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1 $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function

In this section we recall a construction [GLO1] of the $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function $\Psi^\mathfrak{gl}_{\ell+1}(\mathbf{p}_{\ell+1})$ defined on the lattice $\mathbb{Z}^\ell = (p_{\ell+1,1}, \ldots, p_{\ell+1,\ell+1}) \in \mathbb{Z}^{\ell+1}$. We will consider only class one Whittaker functions satisfying the condition

$$\Psi^\mathfrak{gl}_{\ell+1}(\mathbf{p}_{\ell+1}) = 0$$
outside dominant domain \( p_{\ell+1,1} \geq \ldots \geq p_{\ell+1,\ell+1} \).

The \( q \)-deformed \( \mathfrak{g}_{\ell+1} \)-Whittaker functions are common eigenfunctions of \( q \)-deformed \( \mathfrak{g}_{\ell+1} \)-Toda chain Hamiltonians:

\[
\mathcal{H}^p_{\ell+1}(P_{\ell+1}) = \sum_{I_r} (\tilde{X}_1^{-1-\delta_2-i_1} \cdots \tilde{X}_{i_r-1}^{-1-\delta_{i_r-i_r-1}} \cdot \tilde{X}_{i_r}^{-1-\delta_{i_r+1-i_r-1}}) T_{i_1} \cdots T_{i_r},
\]

where the sum is over ordered subsets \( I_r = \{ i_1 < i_2 < \ldots < i_r \} \subset \{ 1, 2, \ldots, \ell + 1 \} \) and we assume \( i_{r+1} = \ell + 2 \). In (1.1) we use the following notations:

\[
T_if(P_{\ell+1}) = f(\tilde{P}_{\ell+1}), \quad \tilde{p}_{\ell+1,k} = p_{\ell+1,k} + \delta_{k,i}, \quad i, k = 1, \ldots, \ell + 1,
\]

and \( \tilde{X}_{\ell+1} = 1 \). We assume \( q \in \mathbb{C}^*, |q| < 1 \). For example, the first nontrivial Hamiltonian has the following form:

\[
\mathcal{H}^p_{1}(P_{\ell+1}) = \sum_{i=1}^{\ell} (1 - q^{p_{\ell+1,i} - p_{\ell+1,i+1} + 1}) T_{i} + T_{\ell+1}.
\]

The main result of [GL01] is a construction of common eigenfunctions of quantum Hamiltonians (1.1):

\[
\mathcal{H}^p_{r}(P_{\ell+1}) \Psi_{z_1,\ldots,z_{\ell+1}}^p(P_{\ell+1}) = \left( \sum_{I_r} \prod_{i \in I_r} z_i \right) \Psi_{z_1,\ldots,z_{\ell+1}}^p(P_{\ell+1}).
\]

Denote by \( P^{(\ell+1)} \subset \mathbb{Z}^{(\ell+1)/2} \) a subset of integers \( p_{n,i}, n = 1, \ldots, \ell + 1, i = 1, \ldots, n \) satisfying the Gelfand-Zetlin conditions \( p_{k+1,i} \geq p_{k,i} \geq p_{k+1,i+1} \) for \( k = 1, \ldots, \ell \). In the following we use the standard notation \( (n)_q! = (1 - q)(1 - q^2) \cdots (1 - q^n) \).

**Theorem 1.1** Let \( \Psi_{z_1,\ldots,z_{\ell+1}}^p(P_{\ell+1}) \) be a function given in the dominant domain \( p_{\ell+1,1} \geq \ldots \geq p_{\ell+1,\ell+1} \) by

\[
\Psi_{z_1,\ldots,z_{\ell+1}}^p(P_{\ell+1}) = \sum_{P_{\ell+1} \subset P^{(\ell+1)}} \prod_{k=1}^{\ell+1} z_k^{p_{k,i} - \sum_{i}^k p_{k,i}} \prod_{k=2}^\ell \prod_{i=1}^{k-1} (p_{k,i} - p_{k,i+1} q^i) q! \\
\times \prod_{k=1}^\ell \prod_{i=1}^k (p_{k+1,i} - p_{k,i}) q! \big( p_{k,i} - p_{k+1,i+1} \big)_q q! 
\]

and zero otherwise. Then, \( \Psi_{z_1,\ldots,z_{\ell+1}}^p(P_{\ell+1}) \) is a common solution of the eigenvalue problem (1.3).

Formula (1.4) can be written also in the recursive form.

**Corollary 1.1** Let \( P_{\ell+1,\ell} \) be a set of \( P_{\ell} = (p_{\ell,1}, \ldots, p_{\ell,\ell}) \) satisfying the conditions \( p_{\ell+1,i} \geq p_{\ell,i} \geq p_{\ell+1,i+1} \). The following recursive relation holds:

\[
\Psi_{z_1,\ldots,z_{\ell+1}}^p(P_{\ell+1}) = \sum_{P_{\ell} \subset P_{\ell+1,\ell}} \Delta(p_{\ell}) \sum_{k=1}^{p_{\ell+1,i} - \sum_{i} p_{\ell,i}} Q_{\ell+1,\ell}(P_{\ell+1}, P_{\ell} | q) \Psi_{z_1,\ldots,z_{\ell}}^p(P_{\ell}).
\]
where

\[
Q_{\ell+1}(p_{\ell+1}, p_j) |q\rangle = \frac{1}{\prod_{i=1}^{\ell} (p_{\ell+1,i} - p_{\ell,i})! (p_{\ell,i} - p_{\ell+1,i+1})!},
\]

\[
\Delta(p_\ell) = \prod_{i=1}^{\ell-1} (p_{\ell,i} - p_{\ell,i+1})!.
\]

Remark 1.1 The representation \((1.4)\) is a q-analog of Givental’s integral representation of the classical \(gl_{\ell+1}\)-Whittaker function \([Gi2], [JK]\):

\[
\Psi_{\Delta}^{gl_{\ell+1}}(x_1, \ldots, x_{\ell+1}) = \int_{\mathbb{R}^{\ell+1}} \prod_{i=1}^{\ell} \prod_{p_{i}} dt_{k,i} e^{t_{k,i}F^{\ell+1}_0(t)},
\]

where

\[
F^{\ell+1}_0(t) = t \sum_{k=1}^{\ell+1} \lambda_k (\sum_{i=1}^{k} t_{k,i} - \sum_{i=1}^{k-1} t_{k-1,i}) - \sum_{k=1}^{\ell} \sum_{i=1}^{k} (e_{t_{k+1,i}} - t_{k,i}) + e_{t_{k,i}} - t_{k+1,i+1},
\]

\[\Delta = (\lambda_1, \ldots, \lambda_{\ell+1}),
\]

\[
x_i := t_{\ell+1,i},
\]

\[
i = 1, \ldots, \ell + 1
\]

\[
z_i = q^{i},
\]

\[
\lambda_i = \gamma_i \log q.
\]

For the representation theory derivation of this integral representation of \(gl_{\ell+1}\)-Whittaker function see \([GKLO]\). The representation \((1.4)\) of the q-Whittaker function turns into representation \((1.6)\) the classical Whittaker function in appropriate limit.

As an example consider \(g = gl_2\). Let \(p_1 := p_{2,1} \in \mathbb{Z}, p_2 := p_{2,2} \in \mathbb{Z}\) and \(p := p_{1,1} \in \mathbb{Z}\). Then the function

\[
\Psi_{\Delta}^{gl_{2}}(p_1, p_2) = \sum_{p_2 \leq p_1} \frac{z_{p_1} z_{p_2} - p_1 - p_2}{(p_1 - p)q!(p - p_2)q!},
\]

\[
\Psi_{\Delta}^{gl_{2}}(p_1, p_2) = 0,
\]

\[p_1 < p_2,
\]

is a solution of the system of equations:

\[
\begin{cases}
(1 - q^{p_1 - p_2 + 1})T_1 + T_2 \Psi_{\Delta}^{gl_{2}}(p_1, p_2) = (z_1 + z_2) \Psi_{\Delta}^{gl_{2}}(p_1, p_2), \\
T_1 T_2 \Psi_{\Delta}^{gl_{2}}(p_1, p_2) = z_1 z_2 \Psi_{\Delta}^{gl_{2}}(p_1, p_2).
\end{cases}
\]

Let us consider the following specialization of the \(q\)-deformed \(gl_{\ell+1}\)-Whittaker function

\[
\Psi_{\Delta}^{gl_{\ell+1}}(n, k) := \Psi_{\Delta}^{gl_{\ell+1}}(n + k, k, \ldots, k).
\]

Theorem 1.2 \(\Psi_{\Delta}^{gl_{\ell+1}}(n, k)\) satisfies following difference equation:

\[
\left\{ \prod_{i=1}^{\ell+1} (1 - z_i T^{-1}) \right\} \Psi_{\Delta}^{gl_{\ell+1}}(n, k) = q^n \Psi_{\Delta}^{gl_{\ell+1}}(n, k),
\]

where \(T \cdot f(n) = f(n + 1)\).
Proof: The proof is based on the explicit expression \((1.4)\). Let \(\mathcal{P}_{n,k}\) be a Gelfand-Zetlin pattern such that \((p_{\ell+1,1}, \ldots, p_{\ell+1,\ell+1}) = (n + k, k, \ldots, k)\). Then, the relations \(p_{\ell+1,i} \geq p_{\ell,i} \geq p_{\ell+1,i+1}\) for the elements of a Gelfand-Zetlin pattern imply \(p_{k,i} \neq 1\) and we have that

\[
\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(n, k) = \left(\prod_{i=1}^{\ell+1} z_i^k\right) \sum_{\mathcal{P}_{n,k}} \frac{z_{\ell+1}^{n+\ldots+k-p_{\ell,1}}}{(n+k-p_{\ell,1})q!} \frac{z_{\ell+1}^{p_{\ell,1}-p_{\ell-1,1}}}{p_{\ell-1,1}q!} \cdots \frac{z_1^{p_{1,1}-k}}{(p_{1,1}-k)q!}.
\]

(1.11)

Introduce the generating function

\[
\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(t, k) = \sum_{n \in \mathbb{Z}} t^n \Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(n, k) = \left(\prod_{i=1}^{\ell+1} z_i^k\right) \sum_{n_1 + \ldots + n_{\ell+1} = n} \frac{n_{\ell+1}}{(n_{\ell+1})q!} \cdots \frac{n_1}{(n_1)q!}.
\]

where we use the identity

\[
\prod_{m=0}^{\infty} \frac{1}{(1 - x q^m)} = \sum_{n=0}^{\infty} \frac{x^n}{(n)!}.
\]

Due to the fact that \(\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(n, k) = 0\) for \(n < 0\), the generating function \(\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(t, k)\) is regular at \(t = 0\). It is easy to check now the following identity

\[
\prod_{j=1}^{\ell+1} (1 - tz_j) \Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(t, k) = \Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(qt, k).
\]

Expanding the latter relation in powers of \(t\), we obtain (1.10) for the coefficients of \(\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(t, k)\). \(\square\)

**Remark 1.2** The difference equation (1.10) for the specialized \(q\)-Whittaker function \(\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(n, k)\) can be derived directly from the system of equations (1.3) for the non-specialized \(q\)-deformed Whittaker function \(\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(p_1, p_2, \ldots, p_{\ell+1})\) and the condition

\[
\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(p_1, p_2, \ldots, p_{\ell+1}) = 0
\]

outside dominant domain \(p_1 \geq \ldots \geq p_{\ell+1}\).

**Lemma 1.1** The following integral representation for the specialized \(q\)-deformed \(\mathfrak{g}_{\ell+1}\)-Whittaker functions holds

\[
\Psi_{z_1, \ldots, z_{\ell+1}}^{\ell+1}(n, k) = \left(\prod_{i=1}^{\ell+1} z_i^k\right) \oint_{t=0} dt \frac{dt}{2\pi i} t^{-n} \prod_{i=1}^{\ell+1} \Gamma_q(z_i t),
\]

(1.12)

where

\[
\Gamma_q(x) = \prod_{n=0}^{\infty} \frac{1}{1 - q^n x}.
\]
Proof: Using the identity
\[ \prod_{n=0}^{\infty} \frac{1}{1-xq^n} = \sum_{m=0}^{\infty} \frac{x^m}{(m)_q^q}, \]
one obtains, for \( n \geq 0 \), that
\[ \Psi_{\lambda+1}^{\text{gl}_{\ell+1}}(n,k) = \left( \prod_{i=1}^{\ell+1} z_i^k \right) \sum_{n_1 + \cdots + n_{\ell+1} = n} \frac{z_1^{n_1}}{(n_1)_q^q} \cdots \frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q^q}, \quad n \geq 0, \]
(1.13)
\[ \Psi_{\lambda+1}^{\text{gl}_{\ell+1}}(n,k) = 0, \quad n < 0 \]
For \( n < 0 \), we obviously have that \( \Psi_{\lambda+1}^{\text{gl}_{\ell+1}}(n,k) = 0 \) \( \Box \)

The corresponding integral representation for the classical \( \text{gl}_2 \)-Whittaker function is given by the Mellin-Barnes representation for the \( \text{gl}_2 \)-Whittaker function
\[ \psi_{\lambda_1,\lambda_2}^{\text{gl}_2}(x_1, x_2) = e^{\frac{1}{\hbar}}(\lambda_1 + \lambda_2)x_2 \int_{i\sigma - \infty}^{i\sigma + \infty} d\lambda e^{\lambda(x_1 - x_2)} \Gamma\left( \frac{\lambda - \lambda_1}{\hbar} \right) \Gamma\left( \frac{\lambda - \lambda_2}{\hbar} \right), \]
where \( \sigma > \max\{\text{Im}\lambda_j, j = 1, \ldots, \ell + 1\} \).

Remark 1.3 The expression
\[ \Psi_{\lambda_1,\ldots,\lambda_{\ell+1}}^{\text{gl}_{\ell+1}+1}(n,k) = \left( \prod_{i=1}^{\ell+1} z_i^k \right) \sum_{n_1 + \cdots + n_{\ell+1} = n} \frac{z_1^{n_1}}{(n_1)_q^q} \cdots \frac{z_{\ell+1}^{n_{\ell+1}}}{(n_{\ell+1})_q^q}, \quad n \geq 0, \]
(1.15)
\[ \Psi_{\lambda_1,\ldots,\lambda_{\ell+1}}^{\text{gl}_{\ell+1}+1}(n,k) = 0, \quad n < 0 \]
is a \( q \)-analog of the Givental integral representation for the equivariant Gromov-Witten invariants of \( X = \mathbb{P}^{\ell+1} \)
\[ f_{\lambda}(T) = \int_{\mathbb{R}^{\ell}} \prod_{k=1}^{\ell} dt_{k,1} e^{\frac{1}{\hbar} F(t)}, \]
(1.16)
where \( \lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) \), \( T := t_{\ell+1,1} \) and
\[ F(t) = i\lambda_1 t_{11} + \sum_{k=1}^{\ell} i\lambda_{k+1}(t_{k+1,1} - t_{k,1}) - e^{t_{11}} - \sum_{k=1}^{\ell} e^{t_{k+1,1} - t_{k,1}}. \]
The representation (1.15) for specialized \( q \)-Whittaker function turns into (1.16) in appropriate limit.

2 Counting holomorphic sections

In this Section we are going to provide an interpretation of the explicit expressions for \( q \)-deformed specialized class one \( \text{gl}_{\ell+1} \)-Whittaker functions in terms of traces of operators acting on the spaces of holomorphic sections of line bundles on infinite-dimensional manifolds. For this aim, we first consider an auxiliary problem of counting holomorphic sections on finite-dimensional manifolds approximating the infinite-dimensional ones. The relevant finite-dimensional manifolds are spaces of the quasi-maps of \( \mathbb{P}^1 \) to \( GL_{\ell+1}(\mathbb{C}) \)-homogeneous spaces.
2.1 Space of quasi-maps

Let us start with recalling the general construction of the quasi-map compactification of the space of holomorphic maps of $\mathbb{P}^1$ to the partial flag spaces of complex Lie group $GL_{\ell+1}$ due to Drinfeld. Let $\alpha_i$, $i = 1, \ldots, \ell$, be a set of simple roots of the complex Lie algebra $\mathfrak{gl}_{\ell+1}$. To any ordered subset of simple roots $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$ indexed by an ordered subset $I^P = \{i_1 < \ldots < i_r\} \subset \{1, \ldots, \ell\}$ one can associate a parabolic subgroup $P \subset GL_{\ell+1}$. Namely, let $B \subset GL_{\ell+1}$ be the subgroup of upper-triangular matrices generated by Cartan torus and one-parameter unipotent subgroups corresponding to positive simple roots. Then a parabolic subgroup $P$ is generated by $B$ and one-parameter unipotent subgroups corresponding to negative roots $-\alpha_i$ such that $i \notin I^P$. In particular, when $r = \ell$ one gets $P = B$, and the corresponding homogeneous space $GL_{\ell+1}/B$ coincides with the full flag space. On the other hand for a parabolic subgroup $P_0 \subset GL_{\ell+1}$ associated to the first simple root (i.e. $I^{P_0} = \{1\} \subset \{1, 2, \ldots, \ell\}$), the corresponding homogeneous space $GL_{\ell+1}/P_0$ is isomorphic to the projective space $\mathbb{P}^\ell$. Partial flag spaces $GL_{\ell+1}/P$ possess canonical projective embeddings

$$\pi : GL_{\ell+1}/P \to \Pi = \prod_{j \in I^P} \mathbb{P}^{n_j-1}, \quad n_j = (\ell + 1)!/j! (\ell + 1 - j)!.$$  \hfill (2.1)

The group $H^2(GL_{\ell+1}/P, \mathbb{Z}) = \mathbb{Z}^r$ is naturally isomorphic to a sublattice of the weight lattice of $\mathfrak{sl}_{\ell+1}$ and is spanned by the weights $\omega_i$ indexed by $I^P$. Let $\mathcal{L}_j$, $j = 1, \ldots, r$, be the line bundles on $GL_{\ell+1}/P$ obtained as pull backs of $O(1)$ form the direct factors $\mathbb{P}^{n_j-1}$ in the right hand side (r.h.s.) of (2.1). The lattice $H^2(GL_{\ell+1}/P, \mathbb{Z}) = \mathbb{Z}^r$ is generated by the first Chern classes $c_1(\mathcal{L}_i)$.

Let $\mathcal{M}_d(GL_{\ell+1}/P)$ be a non-compact space of holomorphic maps of $\mathbb{P}^1$ of multi-degree $d \in H^2(GL_{\ell+1}/P, \mathbb{Z})$ to the flag space $GL_{\ell+1}/P$. Due to (2.1), $\mathcal{M}_d(GL_{\ell+1}/P)$ is a subspace of the product of space $\mathcal{M}_{d_j}(\mathbb{P}^{n_j-1})$. Explicitly, each $\mathcal{M}_{d_j}(\mathbb{P}^{n_j-1})$ can be described as a set of collections of $n$, relatively prime polynomials of degree $d$, up to a common constant factor. The space $\mathcal{M}_{d_j}(\mathbb{P}^{n_j-1})$ allows for a compactification by the space of quasi-maps $\mathcal{QM}_{d_j}(\mathbb{P}^{n_j-1})$ defined as a set of collections of $n$, polynomials of degree $d$, up to a common constant factor. The space of quasi-maps $\mathcal{QM}_d(GL_{\ell+1}/P)$ is then constructed as a closure of $\mathcal{M}_d(GL_{\ell+1}/P)$ in $\prod_j \mathcal{QM}_{d_j}(\mathbb{P}^{n_j})$. Thus defined $\mathcal{QM}_d(GL_{\ell+1}/P)$ is (in general singular) irreducible projective variety. A small resolution of this space is known due to [La], [Ku].

On the space of holomorphic maps $\mathcal{M}_d(GL_{\ell+1}/P)$ of $\mathbb{P}^1$ to $GL_{\ell+1}/P$, there is a natural action of the group $\mathbb{C}^* \times GL_{\ell+1}$ (and, thus, of its maximal compact subgroup $S^1 \times U_{\ell+1}$). Here, the action of $GL_{\ell+1}$ is induced by the standard action on flag spaces and the action of $\mathbb{C}^*$ is induced by the action of $\mathbb{C}^*$ on $\mathbb{P}^1$ given by $(y_1, y_2) \to (\xi y_1, y_2)$ in homogeneous coordinates $(y_1, y_2)$ on $\mathbb{P}^1$. This action of $\mathbb{C}^* \times GL_{\ell+1}$ can be extended to an action on the space $\mathcal{QM}_d(GL_{\ell+1}/P)$ of quasi-maps.

In the following we consider a parabolic subgroup $P_0 \subset GL_{\ell+1}$ associated to the first simple root (and thus $I^{P_0} = \{1\} \subset \{1, 2, \ldots, \ell\}$). The corresponding homogeneous space $GL_{\ell+1}/P_0$ is a projective space $\mathbb{P}^\ell$. The space of quasi-maps $\mathcal{QM}_d(\mathbb{P}^\ell)$ is a non-singular projective variety $\mathbb{P}^{(\ell+1)(d+1)-1}$. A quasi-map $\phi \in \mathcal{QM}_d(\mathbb{P}^\ell)$ is given by a collection

$$(a_0(y) : a_1(y) : \ldots : a_\ell(y)).$$

\footnote{The compactification of $\mathcal{M}_d(\mathbb{P}^\ell)$ by the space $\mathcal{QM}_d(\mathbb{P}^\ell)$ of quasi-maps arises naturally in the linear sigma-model description of Gromov-Witten invariants of projective spaces [W].}
of homogeneous polynomials $a_i(y)$ in variables $y = (y_1, y_2)$ of degree $d$

$$a_k(y) = \sum_{i=0}^{d} a_{k,i} y_1^i y_2^{d-i}, \quad k = 0, \ldots, \ell.$$ considered up to the multiplication of all $a_i(y)$'s by a nonzero complex number. The action of $(\xi, g) \in \mathbb{C}^\ast \times GL_{\ell+1}$ on $\mathcal{O}_{M_d}(\mathbb{P}^{\ell})$ is given by

$$\xi : (a_0(y) : a_1(y) : \ldots : a_{\ell}(y)) \mapsto (a_0(y^\xi) : a_1(y^\xi) : \ldots : a_{\ell}(y^\xi)),$$

$$g : (a_0(y) : a_1(y) : \ldots : a_{\ell}(y)) \mapsto \left( \sum_{k=1}^{\ell+1} g_{1,k} a_{k-1}(y) : \ldots : \sum_{k=1}^{\ell+1} g_{\ell+1,k} a_{k-1}(y) \right), \quad (2.2)$$

where $g = \|g_{ij}\|$ and $y^\xi = (\xi y_1, y_2)$.

### 2.2 Generating functions of holomorphic sections

Let $\mathcal{O}(1)$ be a standard line bundle on $\mathbb{P}^{(\ell+1)(d+1)-1}$. The space of sections of the line bundle $\mathcal{O}(n) := \mathcal{O}(1)^{\otimes n}$ on $\mathcal{O}_{M_d}(\mathbb{P}^{\ell})$ is naturally a $\mathbb{C}^\ast \times GL_{\ell+1}$-module. We are interested in calculating the corresponding character.

Let $T \in GL_{\ell+1}$ be a Cartan torus, $H_1, \ldots, H_{\ell+1}$ be a basis in $\text{Lie}(T)$, and $L_0$ be a generator of $\text{Lie}(\mathbb{C}^\ast)$. The equivariant cohomology of a point with respect to the maximal compact subgroup $G = S^1 \times U_{\ell+1}$ of $\mathbb{C}^\ast \times GL_{\ell+1}$ can be described as

$$H^*_G(\text{pt}, \mathbb{C}) = \mathbb{C}[\lambda_1, \ldots, \lambda_{\ell+1}]^{S_{\ell+1}} \otimes \mathbb{C}[h],$$

where $\lambda_1, \ldots, \lambda_{\ell+1}$ and $h$ are associated with the generators $H_1, \ldots, H_{\ell+1}$ and $L_0$ respectively.

Let $\mathcal{L}_k$ be a one-dimensional $GL_{\ell+1}$-module such that $H_i \mathcal{L}_k = k \mathcal{L}_k$, for $i = 1, \ldots, \ell + 1$. Cohomology groups $H^*(\mathcal{O}_{M_d}(\mathbb{P}^{\ell}), \mathcal{O}(n)) \otimes \mathcal{L}_k$ have a natural structure of $\mathbb{C}^\ast \times GL_{\ell+1}(\mathbb{C})$-module. We denote by $\mathcal{L}_k(n) = \mathcal{O}(n) \otimes \mathcal{L}_k$ the line bundles $\mathcal{O}(n)$ twisted by one-dimensional $GL_{\ell+1}$-module $\mathcal{L}_k$.

Let $A_{n,k}^{(d)}(z, q)$, be the character of the $\mathbb{C}^\ast \times GL_{\ell+1}$-module $\mathcal{V}_{n,k,d} = H^0(\mathcal{O}_{M_d}(\mathbb{P}^{\ell}), \mathcal{L}_k(n))$, $n \geq 0$,

$$A_{n,k}^{(d)}(z, q) = \text{Tr} \mathcal{V}_{n,k,d} q^{L_0} e^{\sum \lambda_i H_i},$$

where we assume that $q \in \mathbb{C}^\ast, |q| < 1$. This character can be straightforwardly calculated as follows. The space $\mathcal{V}_{n,k,d} = H^0(\mathcal{O}_{M_d}(\mathbb{P}^{\ell}), \mathcal{L}_k(n))$ can be identified with the space of degree $n$ homogeneous polynomials in $(\ell + 1)(d + 1)$ variables $a_{k,i}$, for $k = 0, \ldots, \ell$ and $i = 0, \ldots, d$. Define

$$\mathcal{V}_{k,d} = \oplus_{n=0}^{\infty} \mathcal{V}_{n,k,d},$$

and the grading on $\mathcal{V}_{k,d}$ is defined by the eigenvalue decomposition with respect to the action of an operator $D$

$$t^D : \mathcal{V}_{n,k,d} \to t^n \mathcal{V}_{n,k,d}, \quad t \in \mathbb{C}^\ast.$$ The action of the subgroup $(\mathbb{C}^\ast \times T) \subset G(\mathbb{C}) = \mathbb{C}^\ast \times GL_{\ell+1}$ is given by

$$e^{\sum \lambda_i H_i} : (a_0(y) : a_1(y) : \ldots : a_{\ell}(y)) \mapsto (e^{\lambda_1} a_0(y) : e^{\lambda_2} a_1(y) : \ldots : e^{\lambda_{\ell+1}} a_{\ell}(y)), \quad (2.3)$$

where

$$a_k(y) = \sum_{j=0}^{d} a_{k,j} y_1^j y_2^{d-j}, \quad k = 0, \ldots, \ell.$$
The action of the generator $L_0$ of $\mathbb{C}^*$ is as follows

$$q^{L_0} : a_{k,j} \mapsto q^i a_{k,j}.$$  \hspace{1cm} (2.4)

**Proposition 2.1** For the $\mathbb{C}^* \times GL_{\ell+1}$-character of the module $\mathcal{V}_{n,k,d}$, the following integral representation holds

$$A^{(d)}_{n,k}(\underline{z}, q) = \text{Tr}_{\mathcal{V}_{n,k,d}} q^{L_0} e^{\sum \lambda_i H_i} = \left( \prod_{i=1}^{\ell+1} z_i^k \right) \prod_{m=1}^{\ell+1} \prod_{j=0}^{d} \frac{1}{(1 - t q^j z_m)},$$  \hspace{1cm} (2.5)

where $\underline{z} = (z_1, \ldots, z_{\ell+1})$ and $z_m = e^{\lambda_m}$. \hspace{1cm} $\Box$

**Proof:** A simple calculation gives us that

$$A^{(d)}(\underline{z}, t, q) = \text{Tr}_{\mathcal{V}_{k,d}} t^{D} q^{L_0} e^{\sum \lambda_i H_i} = \left( \prod_{i=1}^{\ell+1} z_i^k \right) \prod_{m=1}^{\ell+1} \prod_{j=0}^{d} \frac{1}{(1 - t q^j z_m)},$$  \hspace{1cm} (2.6)

The projection on the subspace of $\mathcal{V}_{k,d}$ of the grading $n$ with respect to $D$ can be realized by taking a residue,

$$\text{Tr}_{\mathcal{V}_{n,k,d}} q^{L_0} e^{\sum \lambda_i H_i} = \oint_{t=0} dt \frac{1}{2\pi i t^{n+1}} \text{Tr}_{\mathcal{V}_{k,d}} t^{D} q^{L_0} e^{\sum \lambda_i H_i}.$$  \hspace{1cm} (2.7)

This gives us the integral expression (2.5) \hspace{1cm} $\Box$

**2.3 Equivariant Euler characteristic of line bundles on $\mathcal{Q}\mathcal{M}_d(\mathbb{P}^d)$**

Characters (2.5) of the space of holomorphic sections can be related to equivariant holomorphic Euler characteristics of line bundles on $\mathcal{Q}\mathcal{M}_d(\mathbb{P}^d)$. First we recall the standard facts about line bundles on projective spaces. Line bundles $\mathcal{O}(n)$ on projective spaces $\mathbb{P}^N$ are equivariant with respect to the standard action of $U_{N+1}$ on $\mathbb{P}^N$. The $U_{N+1}$-equivariant Euler characteristic of $\mathcal{O}(n)$ is given by the character

$$\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{O}(n)) = \sum_{m=0}^{N} (-1)^m \text{Tr}_{\mathcal{O}(n)} e^{\sum \lambda_i H_i}, \quad e^{\sum \lambda_i H_i} \in U_{N+1}. \hspace{1cm} (2.8)$$

Cohomology groups of $\mathcal{O}(n)$ on projective space $\mathbb{P}^N$ have the following properties (see e.g. [OSS])

$$\dim H^m(\mathbb{P}^N, \mathcal{O}(n)) = 0, \quad m \neq 0, N,$$

$$\dim H^N(\mathbb{P}^N, \mathcal{O}(n)) = 0, \quad n \geq 0,$$

$$\dim H^0(\mathbb{P}^N, \mathcal{O}(n)) = 0, \quad n < 0. \hspace{1cm} (2.9)$$

Taking into account (2.9) the expression (2.8) reduces to

$$\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{O}(n)) = \text{Tr}_{\mathcal{O}(n)} e^{\sum \lambda_i H_i} + (-1)^N \text{Tr}_{\mathcal{O}(n)} e^{\sum \lambda_i H_i}.$$  \hspace{1cm} (2.10)

We have $\mathcal{Q}\mathcal{M}_d(\mathbb{P}^d) = \mathbb{P}^{(d+1)(d+1)-1}$ and, thus, for $n \geq 0$, we can identify $A^{(d)}_{n,k}(\underline{z}, q)$ with equivariant holomorphic Euler characteristic of $\mathcal{L}_k(n)$

$$A^{(d)}_{n,k}(\underline{z}, q) = \text{Tr}_{\mathcal{Q}\mathcal{M}_d(\mathbb{P}^d), \mathcal{L}_k(n)} e^{H_0 + \sum \lambda_i H_i} = \chi_{G(\mathcal{Q}\mathcal{M}_d(\mathbb{P}^d), \mathcal{L}_k(n))}, \quad n \geq 0,$$
where $G = S^1 \times U_{t+1}$. The equivariant Euler characteristic of a holomorphic vector bundle on the projective space possesses a canonical holomorphic integral representation. According to the Riemann-Roch-Hirzebruch (RRH) theorem, one can express the $U_{N+1}$-equivariant holomorphic Euler characteristic of a $U_{N+1}$-equivariant vector bundle $\mathcal{E}$ on $\mathbb{P}^N$ as follows
\begin{equation}
\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{E}) = \sum_{m=0}^{N} (-1)^m \text{Tr}_{H^m(\mathbb{P}^N, \mathcal{E})} e^\sum \lambda_i H_i = \langle \text{Ch}_{U_{N+1}}(\mathcal{E}) \text{Td}_{U_{N+1}}(T\mathbb{P}^N), [\mathbb{P}^N] \rangle,
\end{equation}
where $H_i$ are generators of the Lie algebra $\mathfrak{g}_{N+1}$, $T\mathbb{P}^N$ is the tangent bundle to $\mathbb{P}^N$, $\text{Ch}_{U_{N+1}}(\mathcal{E})$ is a $U_{N+1}$-equivariant Chern character of $\mathcal{E}$ and $\text{Td}_{U_{N+1}}(\mathcal{E})$ is a $U_{N+1}$-equivariant Todd genus of $\mathcal{E}$ 

The tangent bundle $T\mathbb{P}^N$ to the projective space $\mathbb{P}^N$ is $U_{N+1}$-equivariantly stable-equivalent to $O(1)^\oplus(N+1)$ as the following lemma shows.

**Lemma 2.1** The following relation holds in $U_{N+1}$-equivariant topological $K$-theory $K_{U_{N+1}}(\mathbb{P}^N)$
\begin{equation}[T\mathbb{P}^N] \oplus [O] = [O(1)]^\oplus(N+1),
\end{equation}
where $[\mathcal{E}]$ is a class of a vector bundle $\mathcal{E}$ in $K_{U_{N+1}}(\mathbb{P}^N)$.

**Proof:** For a tangent sheaf to the complex projective space $\mathbb{P}^N$ we have the Euler exact sequence (see e.g. [GH])
\begin{equation}0 \longrightarrow O \longrightarrow O(1)^\oplus(N+1) \longrightarrow T\mathbb{P}^N \longrightarrow 0 \tag{2.13}
\end{equation}
The maps (2.13) are explicitly $U_{N+1}$-equivariant and, thus, we obtain the relation (2.12) in $U_{N+1}$-equivariant $K$-groups of $\mathbb{P}^N$.

Lemma 2.1 and the fact that the Todd class depends only on stable equivalence class of a vector bundle allows us to rewrite RRH-theory on projective spaces as follows
\begin{equation}\chi_{U_{N+1}}(\mathbb{P}^N, \mathcal{E}) = \langle \text{Ch}_{U_{N+1}}(\mathcal{E}) \text{Td}_{U_{N+1}}(O(1)^\oplus(N+1), [\mathbb{P}^N]) \rangle. \tag{2.14}
\end{equation}
In the following we will consider only the case of line bundles and thus we take $\mathcal{E} = O(n)$, $n \in \mathbb{Z}$. The pairing of the cohomology classes with the fundamental class entering the formulation of RRH-theorem can be expressed explicitly using a particular model for the cohomology ring $H^*(\mathbb{P}^N, \mathbb{C})$. The cohomology ring $H^*(\mathbb{P}^N, \mathbb{C})$ is generated by an element $x \in H^2(\mathbb{P}^N, \mathbb{C})$ with a single relation $x^{N+1} = 0$
\begin{equation}H^*(\mathbb{P}^N, \mathbb{C}) = \mathbb{C}[x]/x^{N+1}. \tag{2.15}
\end{equation}
The $U_{N+1}$-equivariant analog of (2.15) is given by
\begin{equation}H^*_{U_{N+1}}(\mathbb{P}^N, \mathbb{C}) = \mathbb{C}[x] \otimes \mathbb{C}[\lambda_1, \cdots, \lambda_{N+1}]/\left(\prod_{j=1}^{N+1} (x - \lambda_j)\right),
\end{equation}
which is naturally a module over $H^*_{U_{N+1}}(\text{pt}, \mathbb{C}) = \mathbb{C}[\lambda_1, \cdots, \lambda_{N+1}]$ where $\mathfrak{S}_{N+1}$ is the permutation group of a set of $N+1$ elements. The pairing of an element of $H^*_{U_{N+1}}(\mathbb{P}^N, \mathbb{C})$ represented by $P(x, \lambda)$ with a $U_{N+1}$-equivariant fundamental cycle $[\mathbb{P}^N]$ can be expressed in terms of the integral
\begin{equation}\langle P(\lambda), [\mathbb{P}^N] \rangle = \frac{1}{2\pi i} \oint_{C_0} dx \frac{P(x, \lambda)}{\prod_{j=1}^{N+1} (x - \lambda_j)},
\end{equation}
where the integration contour $C_0$ encircles the poles $x = \lambda_j, j = 1, \ldots, (N + 1)$. The pairing for $H^*(\mathbb{P}^N, \mathbb{C})$ is obtained by a specialization $\lambda_j = 0, j = 1, \ldots, (N + 1)$. The equivariant Chern character and Todd class can be written in terms of this model of characteristic as

$$Ch_{U_{N+1}}(\mathcal{O}(n)) = e^{nx}, \quad Td_{U_{N+1}}(\mathcal{O}(1)^{\oplus(N+1)}) = \prod_{j=1}^{N+1} \frac{(x - \lambda_j)}{1 - e^{-(x-\lambda_j)}}. $$

Therefore we have the following integral representation of the equivariant holomorphic Euler characteristic $( t = e^{-x}, z_i = e^{\lambda_i}, i = 1, \ldots, N + 1)$:

$$\chi_{U_{N+1}}(z) = \langle Ch_{U_{N+1}}(\mathcal{O}(n)) Td_{U_{N+1}}(\mathcal{O}(1)^{\oplus(N+1)}), [\mathbb{P}^N] \rangle = \frac{1}{2\pi i} \int_{C_0} \frac{dx}{\prod_{i=1}^{N+1} (x - \lambda_i)} e^{nx} \prod_{i=1}^{N+1} \frac{(x - \lambda_i)}{1 - e^{-(x-\lambda_i)}} $$

$$= -\frac{1}{2\pi i} \int_{C_0} \frac{dt}{t^{n+1}} \prod_{i=1}^{N+1} \frac{1}{1 - tz_i}, \quad (2.16)$$

where in the last expression the integration contour $C_0$ encircles the poles $t = z_j^{-1}, j = 1, \ldots, \ell + 1$. The integral representation $(2.17)$ can be obtained directly using a particular realization of $(U_{N+1}$-equivariant) $K$-theory on $\mathbb{P}^N$ (see e.g. [A]). The $K$-group $K(\mathbb{P}^N)$ is generated by a class $t$ of the line bundle $\mathcal{O}(1)$ satisfying the relation $(1 - t)^{N+1} = 0$. We have the following isomorphisms for $(U_{N+1}$-equivariant) $K$-groups of projective spaces

$$K(\mathbb{P}^N) = \mathbb{C}[t, t^{-1}]/(1 - t)^{N+1}, \quad K_{U_{N+1}}(\mathbb{P}^N) = \mathbb{C}[t, t^{-1}, z, z^{-1}] \bigg/ \prod_{j=1}^{N+1} (1 - tz_j). \quad (2.18)$$

The equivariant analog of the pairing with the fundamental class of $\mathbb{P}^N$ in $K$-theory is given by

$$\langle R, [\mathbb{P}^N] \rangle_K = -\frac{1}{2\pi i} \int_{C_0} \frac{dt}{t} \frac{R(t)}{\prod_{j=1}^{N+1} (1 - tz_j)}, \quad (2.19)$$

where $R(t)$ is a rational function representing an element of $K_{U_{N+1}}(\mathbb{P}^N)$ and the integration contour $C_0$ encircles the poles $t = z_j^{-1}, j = 1, \ldots, (N + 1)$.

Using the representation of the pairing $(2.19)$ one can represent RRH expression for the Euler characteristic as

$$\chi_{U_{N+1}}(z) = \langle [\mathcal{O}(n)], [\mathbb{P}^N] \rangle_K = -\frac{1}{2\pi i} \int_{C_0} \frac{dt}{t^{n+1}} \prod_{i=1}^{N+1} \frac{1}{1 - tz_i}. \quad (2.20)$$

This reproduces the representation $(2.17)$. Now we would like to apply the integral representation for equivariant Euler characteristic to the $S^1 \times U_{\ell+1}$-equivariant line bundle $\mathcal{L}_k(n)$ on $\mathcal{M}_d(\mathbb{P}^\ell)$.

Consider the $S^1 \times U_{\ell+1}$-equivariant cohomology of the projective space $\mathbb{P}(V_{(\ell+1)(d+1)})$ where the vector space $V_{(\ell+1)(d+1)} = \oplus_{j=0}^{\ell+1} \oplus_{m=0}^{d} V_{j,m}$ has the structure of an $S^1$-module with an action given by

$$e^{i\theta} : V_{j,m} \to e^{im\theta} V_{j,m}, \quad \dim V_{j,m} = 1, \quad \theta \in S^1,$$
and each $V_m = V_{1,m} \oplus V_{2,m} \oplus \ldots \oplus V_{\ell+1,m}$ is standard $U_{\ell+1}$-module. Then for the $G = S^1 \times U_{\ell+1}$-equivariant cohomology of $\mathbb{P}(V(\ell + 1)(d + 1))$ we have an isomorphism

$$H^{\ast}_{S^1 \times U_{\ell+1}}(\mathbb{P}(V(\ell + 1)(d + 1))) = \mathbb{C}[x, \lambda, \hbar] \left/ \prod_{j=1}^{\ell+1} \prod_{m=0}^{d} (x - \lambda_j - mh) \right.,$$

(2.21)

where $x$ is a generator of $H^{\ast}(\mathbb{P}(V(\ell + 1)(d + 1)), \mathbb{C})$. The pairing with the $S^1 \times U_{\ell+1}$-equivariant fundamental cycle $[\mathbb{P}(V(\ell + 1)(d + 1))]$ can be expressed in the form of the contour integral

$$\langle P(\lambda, \hbar), [\mathbb{P}(V(\ell + 1)(d + 1))] \rangle = \frac{1}{\pi i} \oint_{C} \frac{P(x, \lambda, \hbar) \, dx}{\prod_{j=1}^{\ell+1} \prod_{m=0}^{d} (x - \lambda_j - mh)},$$

(2.22)

where the integration contour $C$ encircles the poles $x = \lambda_j + mh$, $j = 1, \ldots, \ell + 1$, $m = 0, 1, \ldots, d$.

Specializing to the action of $G = S^1 \times U_{\ell+1}$ on $QM_d(\mathbb{P}^\ell) \simeq \mathbb{P}(\ell + 1)(d + 1)$ described in Section 2.1 we obtain

$$\text{Ch}_G(L_k(n)) = e^{nx + k(1 + \ldots + \lambda_{\ell+1})}, \quad \text{Td}_G(T^\mathbb{P}(\ell + 1)(d + 1)) = \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{x - mh - \lambda_i}{1 - e^{\lambda_i + mh - x}}.$$

Let $q = e^h$, $t = e^{-x}$, and $z_i = e^{\lambda_i}$, $1 \leq i \leq \ell + 1$, then

$$\chi_G(QM_d(\mathbb{P}^\ell), L_k(n)) = \langle \text{Ch}_G(L_k(n)) \rangle \text{Td}_G(T^\mathbb{P}(\ell + 1)(d + 1)) \big|_{[\mathbb{P}^\ell]} = \oint_{C} \frac{dx}{\prod_{i=1}^{\ell+1} \prod_{m=0}^{d} (x - \lambda_i - mh)} e^{nx + k(1 + \ldots + \lambda_{\ell+1})} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{(x - \lambda_i - mh)}{1 - e^{-(x - \lambda_i - mh)}}$$

(2.23)

For $n \geq 0$ one has the identity

$$\chi_G(QM_d(\mathbb{P}^\ell), L_k(n)) = \text{Tr} \langle_{V_{n, k, d}} \rangle q^{L_0} e^\sum \lambda_i H_i.$$

Deforming the contour for $n \geq 0$ we obtain the following integral representation for the character

$$\text{Tr} \langle_{V_{n, k, d}} \rangle q^{L_0} e^\sum \lambda_i H_i = \left( \prod_{i=1}^{\ell+1} \frac{dt}{2\pi i t_0 + 1} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{1}{1 - tz_i q^m} \right)$$

which coincides with (2.25).

**Remark 2.1** Without a restriction $n \geq 0$, the integral representation for the equivariant Euler characteristic can be represented as a difference of two terms

$$\chi_G(QM_d(\mathbb{P}^\ell), L_k(n)) =$$

$$= \left( \prod_{i=1}^{\ell+1} \frac{dt}{2\pi i t_0 + 1} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{1}{1 - tz_i q^m} \right) + \left( \prod_{i=1}^{\ell+1} \frac{dt}{2\pi i t_0 + 1} \prod_{i=1}^{\ell+1} \prod_{m=0}^{d} \frac{1}{1 - tz_i q^m} \right)$$

This decomposition corresponds to the decomposition (2.10)

$$\chi_G(QM_d(\mathbb{P}^\ell), L_k(n)) =$$

$$= \text{Tr} H^0(QM_d(\mathbb{P}^\ell), L_k(n)) q^{L_0} e^\sum \lambda_i H_i + (-1)^{(\ell + 1)(d + 1) - 1} \text{Tr} H^{(\ell + 1)(d + 1) - 1}(QM_d(\mathbb{P}^\ell), L_k(n)) q^{L_0} e^\sum \lambda_i H_i.$$
Remark 2.2 In the limit $q \to 0$ one has an integral representation for a character $\chi^{(0)}_{n,k}$ of an irreducible finite-dimensional representation $\mathcal{V}_{n,k,0} = \text{Sym}^n \mathbb{C}^{\ell+1} \otimes \mathcal{L}_k$ of $\mathfrak{gl}_{\ell+1}$:

$$\chi^{(0)}_{n,k}(z) = \text{tr} \mathcal{V}_{n,k,0} e^{\lambda_1 H_1 + \cdots + \lambda_{\ell+1} H_{\ell+1}} = \left( \prod_{i=1}^{\ell+1} \frac{\lambda_i}{\ell+1} \right) \int_{t=0}^{\ell+1} \prod_{i=1}^{\ell+1} \frac{1}{1 - tz_i} \, dt$$

where $z_i = \exp \lambda_i$ and the $GL(\ell+1)$-module $\mathcal{V}_{n,k,0}$, $n \geq 0$ is realized as a zero cohomology space $H^0(\mathbb{P}^\ell, \mathcal{L}_k(n))$.

3 K-theory of $\mathcal{L}\mathbb{P}^\ell_+$ and $q$-Whittaker functions

In this section we establish a direct connection between $q$-deformed class one specialized $\mathfrak{gl}_{\ell+1}$-Whittaker functions and geometry of the space $\mathcal{L}\mathbb{P}^\ell_+$ defined as an appropriate limit of $\mathcal{Q} \mathcal{M}_d(\mathbb{P}^\ell)$ when $d \to +\infty$. Geometrically the $\mathcal{L}\mathbb{P}^\ell_+$ should be considered as a space of algebraic disks in $\mathbb{P}^\ell$ (see [Gi1] for details). In general, let $LX = \text{Map}(S^1, X)$ be the space of free contractible loops in a compact Kähler manifold $X$. There is a natural action of $S^1$ on $LX$ by loop rotations. The universal covering $\tilde{LX}$ can be defined as a space of maps $D \to X$ of the disk $D$ considered up to a homotopy of the map preserving the image of the boundary loop $S^1 \subset D$. The group of covering transformations of the universal cover $\tilde{LX} \to LX$ is isomorphic to $\pi_2(X)$. Let $\tilde{LX}_+ \subset \tilde{LX}$ be a semi-infinite cycle of loops that are boundaries of holomorphic maps $D \to X$. For $X = \mathbb{P}^\ell$ define an algebraic version $\mathcal{L}\mathbb{P}^\ell_+$ of $\tilde{L\mathbb{P}^\ell}_+$ as a set of collections of regular series

$$a_i(z) = a_{i,0} + a_{i,1}z + a_{i,2}z^2 + \cdots, \quad 0 \leq i \leq \ell,$$

modulo the action of $\mathbb{C}^*$. The topology on this space should be defined by considering $\mathcal{L}\mathbb{P}^\ell_+$ as a limit of $\mathcal{Q} \mathcal{M}_d(\mathbb{P}^\ell)$ when $d \to +\infty$. This space inherits the action of $G = S^1 \times U(\ell+1)$ defined previously on $\mathcal{Q} \mathcal{M}_d(\mathbb{P}^\ell)$. In the following we do not define appropriate topology rigorously leaving this for another occasion. Instead we define the limit $d \to +\infty$ on the level of cohomology algebra $H^*(\mathcal{Q} \mathcal{M}_d(\mathbb{P}^\ell), \mathbb{C})$ and the space of holomorphic sections of line bundles on $\mathcal{Q} \mathcal{M}_d(\mathbb{P}^\ell)$. Let us take the limit $d \to +\infty$ of the character $A_{n,k}^{(d)}(\mathbb{P}^\ell, q)$ given by the integral expression (2.5). The limit of $A_{n,k}^{(d)}(\mathbb{P}^\ell, q)$ can be interpreted as a character of a $\mathbb{C}^* \times GL_{\ell+1}$-module $\mathcal{V}_{n,k,\infty}$ defined as follows. Let $\mathcal{V}_{k,\infty}$ be a linear space of polynomials of infinite number of variables $a_{i,m}$, $i = 0, \ldots, \ell$, $m \in \mathbb{Z}_{\geq 0}$. Let $L_0$ be a generator of $\text{Lie}(\mathbb{C}^*)$, $T \in \mathfrak{gl}_{\ell+1}$ be a Cartan torus and $H_1, \ldots, H_{\ell+1}$ be a basis in $\text{Lie}(T)$. Define the action of $L_0$ and $H_j$ on the generators $a_{i,m}$ as follows

$$L_0 : a_{i,m} \rightarrow m \, a_{i,m};$$

$$e^{\sum_j \lambda_j H_j} : a_{i,m} \rightarrow e^{\lambda_i} a_{i,m}$$

This supplies $\mathcal{V}_{k,\infty}$ with the structure of a $\mathbb{C}^* \times GL_{\ell+1}$-module. Now the linear subspace $\mathcal{V}_{n,k,\infty} \subset \mathcal{V}_{k,\infty}$ is defined as a subspace of polynomials of the variables $a_{i,m}$, $i = 0, \ldots, \ell$, $m \in \mathbb{Z}_{\geq 0}$ of the total degree $n$.

**Theorem 3.1** Let $\Psi^{(\ell+1)}_\mathbb{P}(n, k)$ be a specialization (1.9) of the solution of $q$-deformed Toda lattice defined in the Theorem 1.7. Then the following holds.

$$\Psi^{(\ell+1)}_\mathbb{P}(n, k) = \chi_{n,k}(z),$$

where

$$\chi_{n,k}(z) = \lim_{d \to \infty} A_{n,k}^{(d)}(\mathbb{P}^\ell, q) = \text{Tr} \mathcal{V}_{n,k,\infty} q^{L_0} e^{\sum \lambda_i H_i}, \quad z_i = e^{\lambda_i}.$$
Proof: For the function $\chi_{n,k}(z) = \lim_{d \to \infty} A_{n,k}^{(d)}(z, q)$ the following integral representation holds:

$$
\chi_{n,k}(z) = \left( \prod_{i=1}^{\ell+1} z_i^k \right) \oint_{t=0} \frac{dt}{2\pi i t^{n+1}} \prod_{m=1}^{\ell+1} \prod_{j=0}^{\infty} \frac{1}{1 - t q^j z_m} = \left( \prod_{i=1}^{\ell+1} z_i^k \right) \oint_{C} \frac{dt}{2\pi i t^{n+1}} \prod_{m=1}^{\ell+1} \Gamma_q(t z_m).
$$

The relations (3.1) follows directly from the explicit integral expression (3.2) and Lemma 1.1. The representation in terms of the trace over $V_{n,k,\infty}$ follows from the statement of Proposition 2.1 with obvious modifications for $d \to +\infty$.

For $n \geq 0$ we can identify the character $A_{n,k}^{(d)}$ with the equivariant Euler characteristic expressed through Riemann-Roch-Hirzebruch formula

$$
\chi_G(\mathcal{Q}M_d(\mathbb{P}^\ell), \mathcal{L}_k(n)) = \int_{\mathcal{Q}M_d(\mathbb{P}^\ell)} \text{Ch}_G(\mathcal{L}_k(n)) \text{Td}_G(T \mathcal{Q}M_d(\mathbb{P}^\ell)).
$$

Taking the limit $d \to +\infty$ we obtain formal Riemann-Roch-Hirzebruch formula for $\chi_G(\mathcal{L}^\ell_+, \mathcal{L}_k(n))$.

Using the description (2.13), (2.19) of the equivariant K-groups of projective spaces and taking the limit $d \to +\infty$ in the integral representation of the Euler characteristic (2.23) one obtains the following integral representation for the equivariant Euler characteristic of line bundles on $\mathcal{L}^\ell_+$

$$
\chi_G(\mathcal{L}^\ell_+, \mathcal{L}_k(n)) = \left( \prod_{j=1}^{\ell+1} z_j^k \right) \oint_{C} \frac{dt}{2\pi i t^{n+1}} \prod_{i=1}^{\ell+1} \prod_{j=0}^{\infty} \frac{1}{1 - t q^j z_i} = \left( \prod_{j=1}^{\ell+1} z_j^k \right) \oint_{C} \frac{dt}{2\pi i t^{n+1}} \prod_{i=1}^{\ell+1} \Gamma_q(t z_i),
$$

where the integration contour $C$ encircles all poles except $t = 0$ and

$$
\Gamma_q(y) = \prod_{n=0}^{\infty} \frac{1}{1 - y q^n}.
$$

Problem 3.1 Define an equivariant (co)homology theory for $\mathcal{L}^\ell_+$ in such a way that Chern and Todd classes $\text{Ch}_G(\mathcal{L}_k(n))$, $\text{Td}_G(T \mathcal{L}^\ell_+)$ make sense and the expression

$$
\int_{\mathcal{L}^\ell_+} \text{Ch}_G(\mathcal{L}_k(n)) \text{Td}_G(T \mathcal{L}^\ell_+)
$$

is well-defined and is equal to

$$
\Psi_{\ell+1}^{(n,k)}(n,k) = \left( \prod_{j=1}^{\ell+1} z_j^k \right) \oint_{C} \frac{dt}{2\pi i t^{n+1}} \prod_{i=1}^{\ell+1} \Gamma_q(t z_i).
$$

Remark 3.1 The conjectural relation above provides a description of the specialized $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function as a semi-infinite period

$$
\Psi_{\ell+1}^{(n,k)}(n,k) = \int_{\mathcal{L}^\ell_+} \text{Ch}_G(\mathcal{L}_k(n)) \text{Td}_G(T \mathcal{L}^\ell_+), \quad n \geq 0.
$$
The $K$-theory of the semi-infinite spaces $\mathcal{L} \mathbb{P}^\ell$ is closely connected with a quantum version of $K$-theory of projective spaces proposed in [GiL]. The generating function $F(n, z, q)$ of the correlation functions in $K$-theory version of Gromov-Witten theory with the target space $\mathbb{P}^\ell$ obeys the following difference equation [GiL]

\[
\left\{ \prod_{i=1}^{\ell+1} (1 - z_i T^{-1}) \right\} \cdot F(n, z, q) = q^n F(n, z, q),
\]

where $T \cdot f(n) = f(n+1)$. The specialized $q$-deformed $\mathfrak{gl}_{\ell+1}$-Whittaker function satisfies the same equation (3.7) (see Lemma [11] and relation [11.10]). Therefore the Whittaker function can be considered as a correlation function of some special operator singled out by the class one condition (i.e. the condition $\Psi^{gl}_{k,n}(n,k) = 0$ for $n < 0$). We provide some information on this operator in the Section 5.

4 Quantum cohomology and Whittaker function

In the previous Section we proposed a description of $q$-deformed class one specialized $\mathfrak{gl}_{\ell+1}$-Whittaker function in terms of a semi-infinite version of Riemann-Roch-Hirzebruch theorem. This expresses the $q$-Whittaker function as a semi-infinite period. Its classical (i.e. non-deformed) counterpart can be also expressed in terms of a semi-infinite period. In this Section we provide this conjectural representation.

We start from recalling the notion of quantum cohomology. The quantum cohomology $QH^*(X)$ of a compact symplectic manifold $X$ can be defined in terms of semi-infinite geometry of a universal cover $\tilde{L}X$ of the loop space $LX$. One of the descriptions is given by a Morse-Smale-Bott-Novikov-Floer complex constructed in terms of critical points of an area functional on $\tilde{L}X$. Its cohomology groups (interpreted as Floer cohomology groups $FH^*(\tilde{L}X)$ of $\tilde{L}X$) are isomorphic to the semi-infinite cohomology $H^\infty/2^+(LX)$ arising naturally in the Hamiltonian formalism of a topological two-dimensional sigma model with the target space $X$. In the following we will use an equivariant version of quantum cohomology $QH^*(\mathbb{P}^\ell)$ of projective spaces considered in [Gi1] (see also [CJS] for a non-equivariant version).

We have defined the universal covering $\tilde{L}X$ of the loop space $LX$ as a space of maps $D \to X$ of the disk $D$ considered up to a homotopy preserving the image of the boundary loop $S^1 \subset D$. The group of covering transformations of the universal cover $\tilde{L}X \to LX$ is isomorphic to the image $\Gamma \subset H_2(X)$ of the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$ where $H_2(X)$ denotes integral homology modulo torsion. Let $r$ be the rank of $\Gamma$ and $\mathbb{C}[\Gamma] \simeq \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$ be its group algebra.

As a vector space the quantum cohomology $QH^*(X)$ of $X$ as a vector space is isomorphic to the ordinary cohomology $H^*(X, \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}])$, over the group algebra $\mathbb{C}[\Gamma] \simeq \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}]$. Let $S^1$ act on the loop space $LX$ by loop rotations. For the corresponding $S^1$-equivariant quantum cohomology we have the following isomorphism of vector spaces:

$QH^*_{S^1}(X) = H^*(X, \mathbb{C}[u_1^{\pm 1}, \ldots, u_r^{\pm 1}](\hbar))$.

\footnote{Whittaker functions naturally arise in the description of Gromov-Witten invariants of flag spaces. In the mirror dual description they expressed in terms of periods of top-dimensional holomorphic forms on non-compact Calabi-Yau spaces [Gi2]. Thus, the possibility to express Whittaker functions as semi-infinite periods leads to a formulation of the mirror symmetry as an identification of two period maps - semi-infinite and finite ones.}
where we use the identification

\[ H^*_S(\text{pt}, \mathbb{C}) = H^*(BS^1, \mathbb{C}) = \mathbb{C}[h], \]

and the standard localization of the equivariant cohomology \( H^*_S(\text{pt}) \) with respect to the maximal ideal generated by \( h \) is implied.

The quantum cohomology space \( QH^*_S(X) \) has a natural structure of a module over an algebra \( D \) generated by \( u_i = \exp \tau_i, v_i = -h\partial/\partial \tau_i, i = 1, \ldots, r \). More precisely, \( QH^*_S(X) \) as a linear space over \( \mathbb{C}(h) \) is generated by solutions of the system of linear differential equations

\[ \nabla_i f(\zeta) = 0, \quad f(\zeta) = (f_1(\zeta), f_2(\zeta), \ldots, f_n(\zeta)), \quad n = \dim H^*(X), \quad (4.1) \]

where the flat connection \( \nabla = \sum_{i=1}^r d\tau_i \nabla_i \) provides an action of \( v_i \) on \( QH^*_S(X) \).

The \( D \)-module \( QH^*_S(\mathbb{P}^\ell) \) is of rank one. It is generated over \( D \) by an element \( f_* \) satisfying the relation \( (v^{\ell+1} - u)f_* = 0 \) i.e. the quantum cohomology can be represented as \( QH^*(\mathbb{P}^\ell) \simeq D/(v^{\ell+1} - u) \). Explicitly we have the differential equation for the generator \( f_*(\tau) \)

\[ \left\{ \left( -h \frac{\partial}{\partial \tau} \right)^{\ell+1} - e^\tau \right\} f_*(\tau, h) = 0. \quad (4.2) \]

The representation (4.1) arises after transformation of (4.2) to the matrix differential equation of the first order.

The \( (S^1 \times U_{\ell+1}) \)-equivariant analog of quantum cohomology \( QH^*_S(\mathbb{P}^\ell) \) allows for a similar representation with the differential equation (4.2) replaced by

\[ \left\{ \prod_{k=1}^{\ell+1} \left( -h \frac{\partial}{\partial \tau} - \lambda_k \right) - e^\tau \right\} f_*(\tau, \lambda, h) = 0. \quad (4.3) \]

**Lemma 4.1** The general solution of (4.3) is given by a linear combination of the integrals

\[ f^{(a)}(\tau, \lambda, h) = \int_{\gamma_a} d\lambda \, e^{-\frac{\lambda}{h}} \prod_{k=1}^{\ell+1} \frac{\lambda - \lambda_k}{h} \Gamma \left( \frac{\lambda - \lambda_k}{h} \right), \quad a = 1, \ldots, n \quad (4.4) \]

with a suitable choice of integration contours \( \gamma_a \).

**Proof:** Note that the function

\[ Q(\lambda, \lambda) = \prod_{k=1}^{\ell+1} \frac{\lambda - \lambda_k}{h} \Gamma \left( \frac{\lambda - \lambda_k}{h} \right). \quad (4.5) \]

obeys the difference equation

\[ \prod_{k=1}^{\ell+1} (\lambda - \lambda_k) Q(\lambda, \lambda) = Q(\lambda + h, \lambda). \quad (4.6) \]

Therefore, the function

\[ f(\tau, \lambda) = \int_{\gamma} d\lambda \, e^{-\frac{\lambda}{h}} Q(\lambda, \lambda, h) \quad (4.7) \]
satisfies (4.3) provided the choice of the contour $\gamma$ allows for an integration by parts. The contours can be chosen in such a way that the total derivatives do not give a contribution into the integral (4.4).

A particular choice of $\gamma$ in (4.7) gives us a special solution of the equation (4.3)

$$f^*_{\tau, \lambda}(\tau, \lambda, h) = \sigma + \int_{\sigma - \infty}^{\sigma + \infty} \prod_{k=1}^{\ell+1} \lambda^{\lambda - \lambda_k} \Gamma\left(\frac{\lambda - \lambda_k}{h}\right),$$

where $\sigma$ is such that $\sigma < \min\{\Re \lambda_j, j = 1, \ldots, \ell + 1\}$. This is a unique solution of (4.3) exponentially decaying when $\tau \to +\infty$.

**Remark 4.1** In the case of $\ell = 1$, the differential equation (4.2) is equivalent to an eigenvalue problem for the quadratic Hamiltonian of the $\mathfrak{sl}_2$-Toda chain. The solution given in the integral form (4.8) coincides in this case with the Mellin-Barnes representation of the $\mathfrak{gl}_2$-Whittaker function (1.14) for $x_2 = 0$ and $x_1 = \tau$.

Replacing formally $\Gamma$-functions by infinite products over their poles one has for $f^*_{\tau, \lambda}(\tau, \lambda, h)$ the following expression

$$\int dxe^{\tau x/h} \prod_{j=1}^{\ell+1} \prod_{n=0}^{\infty} \frac{1}{x - \lambda_j - hn}. \quad (4.9)$$

This formal representation can be interpreted using the model for cohomology of $\mathcal{Q} \mathcal{M}_d(P^1)$ discussed in Section 2.3. Naively (4.9) can be considered as an integral over $\mathcal{L}^d_1$ of $\exp(\tau \omega/h)$ where $\omega$ is an element of the second $S_1 \times U_{\ell+1}$-equivariant cohomology of $\mathcal{L}^d_1$. Recall that we define $\mathcal{L}^d_1$ on the level of cohomology as a limit of $\mathcal{Q} \mathcal{M}_d(P^1)$ when $d \to +\infty$. However a correct regularization for (4.9) is given by (4.8) and, thus, a geometric interpretation of (4.8) implies some modification of $\mathcal{L}^d_1$. We attribute the difference between (4.9) and (4.8) to the fact that the proper interpretation of $\mathcal{L}^d_1$ as $d \to +\infty$ limit deserves more care in this case and does not coincide with a straightforward limit $d \to +\infty$ on the level of cohomology. Let us denote the corresponding hypothetically modified limit by $\mathcal{L}^d_1$.

**Problem 4.1** Find the space $\mathcal{L}^d_1$ and construct equivariant (co)homology for $\mathcal{L}^d_1$ in such a way that the integral

$$\int_{\mathcal{L}^d_1} e^{\tau \omega/h}, \quad \omega \in H^2_{S_1 \times U_{\ell+1}}(\mathcal{L}^d_1, \mathbb{C})$$

is well-defined and is equal to $f^*_{\tau, \lambda}(\tau, \lambda, h)$ given by (4.8).

## 5 $S^1$-localization

In this Section we calculate the equivariant Euler characteristic $\chi_G(\mathcal{Q} \mathcal{M}_d(P^1), L_k(n))$ for $G = S^1 \times U_{\ell+1}$ using Borel localization for $S^1$-action. This yields a direct relation between our construction of $q$-Whittaker functions and the results of [GIL].

The character (2.23) can be calculated using an equivariant localization as follows. We have a compact Lie group $G = S^1 \times U_{\ell+1}$ acting on a projective space $X = \mathcal{Q} \mathcal{M}_d(P^1) \equiv P^{(\ell+1)(d+1)-1}$. 


Recall that $\mathcal{QM}_d(\mathbb{P}^d)$ is defined as a set of $(\ell + 1)$ polynomials each of degree $d$ considered up to common constant factor

$$(a_0(y), a_1(y), \ldots, a_\ell(y)) \sim (\rho a_0(y), \rho a_1(y), \ldots, \rho a_\ell(y)), \quad \rho \in \mathbb{C}^*,$$

where

$$a_k(y) = \sum_{i=0}^{d} a_{k,i} y_1^i y_2^{d-i}, \quad k = 0, \ldots, \ell.$$ 

The action of an element $q = e^{i\theta}$ of $S^1$ on $a_{k,j}$ is given by

$$q : a_{k,j} \rightarrow q^j a_{k,j}.$$ 

The line bundles $\mathcal{L}_k(n)$ are equivariant with respect to the action of $S^1$. Let $X^{S^1} \subset X$ be a set of $S^1$-fixed points. It is a union of smooth components $Y_i$. The Bott localization formula gives the following expression for the equivariant Euler class (2.23) (see e.g. [BGV])

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^d), \mathcal{L}_k(n)) = \sum_{Y_i \in X^{S^1}} \int_{Y_i} \frac{\text{Ch}_G(\mathcal{L}_k(n)|_{Y_i}) \text{Td}_G(TY_i)}{E_G(N_{Y_i})},$$

(5.1)

where the sum runs over all components $Y_i$ in $X^{S^1}$, $N_{Y_i}$ is the normal bundle of $Y_i$ in $X$, $\text{Ch}_G(\mathcal{L}_k(n))$, $\text{Td}_G(Y_i)$ are equivariant Chern character and Todd class, and $E_G(N_{Y_i})$ is the equivariant Euler class of $N_{Y_i}$.

It is easy to infer that the subvarieties $Y_i$, $i = 0, \ldots, \ell$ in $\mathcal{QM}_d(\mathbb{P}^d)$ are isomorphic to the projective spaces $\mathbb{P}^d$ and are defined by the equations

$$Y_i = \{a_{k,j} = 0, \ j \neq i\}, \quad i = 0, \ldots, \ell.$$ 

To calculate the action of $S^1$ on the normal bundle to $Y_i$ we consider the intersection of $Y_i$ with open subsets $U_{a_{k,i}} = \{a_{k,i} \neq 0\}$ of $\mathcal{QM}_d(\mathbb{P}^d)$. Natural coordinates on $U_{a_{k,i}}$ are

$$\xi_{r,j} = a_{r,j}/a_{k,i}, \quad (r, j) \neq (k, i),$$

and the intersections $Y_i \cap U_{a_{k,i}} = \{a_{k,i} \neq 0\}$ are defined by the equations

$$\xi_{r,i} = 0, \quad r \neq k.$$ 

Thus one can take a collection of coordinates $\xi_{r,j}$, $j \neq i$ as a local section of the dual to the normal bundle $N_{Y_i}$. The action of $S^1$ on $N_{Y_i}$ can then be found by considering the action on section $\xi_{r,j}$:

$$\xi_{r,j} \rightarrow q^{j-i} \xi_{r,j}.$$ 

Similarly, one can show that $q \in S^1$ acts on the restriction of the line bundle $\mathcal{O}(k)$ on $Y_i$ by multiplication on $q^{ni}$. The fixed point formula (5.1) reduces to the following explicit expression

$$\chi_G(\mathcal{QM}_d(\mathbb{P}^d), \mathcal{L}_k(n)) =$$

$$= -\left(\prod_{j=1}^{\ell+1} \frac{z_j}{z_j^k}\right) \sum_{i=0}^{d} \int_{C_0} \frac{dt}{2\pi i} \frac{q^{ni}}{\prod_{j=1}^{\ell+1} \prod_{m=0, m \neq i}^{d} (1 - t z_j q^{m-i}) \prod_{j=1}^{\ell+1} (1 - t z_j)},$$

where the integration contour $C_0$ encircles the $(\ell + 1)$ poles defined by the equations $t = z_j^{-1}$, $j = 1, \ldots, \ell + 1$. 18
Lemma 5.1 The following identity holds for \( n \geq 0 \)

\[
\oint_C \frac{dt}{2\pi i t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1} \prod_{m=0, m \neq i}^d (1 - tz_j q^m)} = \sum_{i=0}^d \int_{C_0} \frac{dt}{2\pi i t^{n+1}} \frac{q^{ni}}{\prod_{j=1}^{\ell+1} \prod_{m=0}^d (1 - tz_j q^{m-i})},
\]

where the integration contour \( C \) encircles poles defined by the equations \( t = z_j^{-1} q^{-i}, j = 1, \ldots, \ell + 1, \)
\( i = 0, \ldots, d \) and the integration contour \( C_0 \) encircles \((\ell + 1)\) poles defined by the equations \( t = z_j^{-1}, j = 1, \ldots, \ell + 1. \)

Proof: We have that

\[
\oint_C \frac{dt}{2\pi i t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1} \prod_{m=0}^d (1 - tz_j q^m)} = \sum_{i=0}^d \int_{C_i} \frac{dt}{2\pi i t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1} \prod_{m=0}^d (1 - tz_j q^m)},
\]

where the integration contour \( C_i \) encircles \((\ell + 1)\) poles defined by the equations \( t = z_j^{-1} q^{-i}, j = 1, \ldots, \ell + 1. \) Making the change of variables \( t \to tz_j^{-1} \) in the r.h.s., we obtain that

\[
\oint_C \frac{dt}{2\pi i t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1} \prod_{m=0}^d (1 - tz_j q^m)} = \sum_{i=0}^d \int_{C_0} \frac{dt}{2\pi i t^{n+1}} \frac{1}{\prod_{j=1}^{\ell+1} \prod_{m=0}^d (1 - tz_j q^{m-i})}
\]

\( \square \)

We are going to consider a continuation of the expression \((5.2)\) to \( q \in \mathbb{C}^*, |q| < 1 \) and the limit \( d \to \infty. \)

Proposition 5.1 The specialization \((1.9), (1.12)\) of the q-deformed \( gl_{\ell+1}-\)Whittaker function can be written in the following form

\[
\Psi_{\ell+1}^{gl}(n, k) = \langle I_{n,k}(z) \tilde{L}(z), [\mathbb{P}] \rangle_K,
\]

where

\[
\tilde{L}(z, t) = \prod_{j=1}^{\ell+1} \prod_{k=1}^\infty \frac{1}{(1 - tz_j q^k)} = \prod_{j=1}^{\ell+1} \Gamma_q(qtz_j),
\]

\[
I_{n,k}(z, t) = \left( \prod_{j=1}^{\ell+1} z_j^k \right) t^{-n} \sum_{i=0}^\infty q^{ni} \prod_{j=1}^{\ell+1} \prod_{m=1}^i \frac{1}{(1 - tz_j q^{-m})},
\]

and the pairing \( \langle , \rangle_K \) is the standard pairing \((2.19)\) on \( K_{U_{\ell+1}}(\mathbb{P}) \) taking values in \( K_{U_{\ell+1}}(pt). \)
The representation of the Whittaker function given in Proposition 5.1 establishes a direct connection with the results of Givental-Lie [GL]. In [GL] the function (5.4) was interpreted as a universal solution of the reduction (1.10) of \( q \)-deformed \( gl_{\ell+1} \)-Toda chain. Indeed, the function \( I_{n,k}(z,t) \) satisfies the eigenvalue problem

\[
\prod_{j=1}^{\ell+1} (1 - z_i T^{-1}) I_{n,k}(z,t) = q^n I_{n,k}(z,t),
\]

modulo the relation \( \prod_{j=1}^{\ell+1} (1 - tz_j) = 0 \) holding in \( K_{U_{\ell+1}}(P^\ell) \) and is uniquely determined by the normalization condition

\[
I_{n,k}(z,t)|_{q=0} = \left( \prod_{j=1}^{\ell+1} z_j^k \right) t^{-n}, \quad n \geq 0.
\]

The solution \( I_{n,k}(z,t) \) is universal in the sense that taking the pairing

\[
\langle I_{n,k}(z), f \rangle_K = -\frac{1}{2\pi i} \int_{C_0} dt \frac{I_{n,k}(z,t) f(t)}{\prod_{j=1}^{\ell+1} (1 - z_j t)}
\]

with arbitrary \( f \in K_{U_{\ell+1}}(P^\ell) \) one obtains a solution (5.6) of the \( q \)-deformed reduced \( gl_{\ell+1} \)-Toda chain (1.10).

### 6 Semi-infinite Todd genus and \( q \)-Gamma function

In the explicit expression for the cohomological pairing on \( LP^\ell_+ \), conjectured in Problem 3.1, the \( q \)-Gamma function \( \Gamma_q \) plays the role similar to the Todd genus in the analogous pairing for underlying finite-dimensional space \( P^\ell \). The \( S^1 \)-localization discussed in the previous section reduces the pairing of the Chern and Todd classes on \( LP^\ell_+ \) to the paring of some cohomology classes on \( P^\ell \). It is an interesting problem to interpret the resulting cohomology classes on \( P^\ell \) in terms of some geometric objects on \( P^\ell \). For example, in an analogous case of \( S^1 \)-localization of \( K \)-theory on the loop space \( LX \), the elliptic genus of \( X \) arises. The corresponding elliptic cohomology is an instance of an extraordinary cohomology theory. In this Section we discuss the result of \( S^1 \)-localization on \( LP^\ell_+ \) from the extraordinary cohomology perspective. We demonstrate that an intrinsic non-local nature of the \( C^* \)-localization on \( LP^\ell_+ \) obstructs a straightforward relation with the formalism of extraordinary cohomology theories and corresponding multiplicative genera. Let us remark that classical \( \Gamma \)-function was considered as a candidate for a topological genus by Kontsevich in [K]. Also a kind of \( \Gamma \)-genus also appeared in obviously related context in [Le], [Ho] (see also [CGi1], [CGi2] for a discussion of formal groups in a quantum version of cobordism theory).

We first recall standard facts on multiplicative topological genera and formal group laws corresponding to complex oriented cohomology theories (see e.g. [BMN]). The Hirzerbruch multiplicative genus is a homomorphism \( \varphi : \Omega^* \to R \) of the ring of complex cobordisms \( \Omega^* = \Omega^*(pt) \) to a ring of coefficients \( R \). One has a Thom isomorphisms \( \Omega^* \otimes \mathbb{Q} = \mathbb{Q}[x_1, x_2, \ldots], \deg(x_i) = -2i \) and the topological genus \( \varphi \) is characterized by its values on generators \( x_i \) that can be represented by Pontryagin-Thom duals to complex projective spaces \( \mathbb{P}^i \). Equivalently \( \varphi \) defined over \( \mathbb{Q} \) can be described in terms of a one-dimensional commutative formal group law

\[
f_\varphi(z, w) = e_\varphi(\log_\varphi(z) + \log_\varphi(w)),
\]

where
expressed through the logarithm function
\[
\log_\varphi(z) = z + \sum_{n=1}^\infty \frac{\varphi([\mathbb{P}^n])}{n+1} z^{n+1},
\]
and its inverse \( e_\varphi(u) \). For instance rational cohomology and \( K \)-theory correspond to additive and multiplicative group laws
\[
f_H(z, w) = z + w, \quad f_K(z, w) = z + w - zw.
\]
To a genus \( \varphi \) one associates a multiplicative sequence \( \{ \Phi_n(c_i) \} \), \( \deg(\Phi_n) = n \) of cohomology classes
\[
P_\varphi = \sum_{n=0}^\infty \Phi_n(c_i) = \prod_{j=1}^N \frac{x_j}{e_{\varphi}(x_j)}.
\]
and a map
\[
X \rightarrow \varphi(X) = \langle \Phi(TX), [X] \rangle, \quad \dim \mathbb{C} X = N.
\]
Here \( TX \) is the tangent bundle to a manifold \( X \), \( [X] \) is the fundamental class in the homology of \( X \) and \( \langle \cdot, \cdot \rangle \) is a standard pairing. The classes \( x_i \) are defined in terms of Chern classes \( c_i \) of \( TX \) using a splitting of \( TX \)
\[
c(X) = 1 + \sum_{i=1}^N c_i(X) = \prod_{j=1}^N (1 + x_j).
\]
In the case of additive and multiplicative group laws we have respectively that
\[
P_{\varphi_H}(x) = 1, \quad \log_{\varphi_H}(z) = z, \quad e_\varphi(u) = u;
\]
\[
P_{\varphi_K}(x) = \prod_{j=1}^n \frac{x_j}{1 - e^{-x_j}}, \quad \log_{\varphi_K}(z) = -\ln(1 - z), \quad e_{\varphi_K}(u) = 1 - e^{-u}.
\]
Note that \( P_{\varphi_K}(x) \) defines the Todd class of \( TX \). For example, the equivariant Riemann-Roch-Hirzebruch theorem for a line bundle \( \mathcal{L}_k(n) \) on \( \mathbb{P}^\ell \) can be represented in the following form
\[
\chi_{U_{\ell+1}}(\mathbb{P}^\ell, \mathcal{L}_k(n)) = \langle \text{Ch}_{U_{\ell+1}}(\mathcal{L}_k(n)) \text{Td}_{U_{\ell+1}}(T\mathbb{P}^\ell), [\mathbb{P}^\ell] \rangle
\]
\[
= \frac{1}{2\pi i} \oint_{C_0} dx e^{nx+k(\lambda_1+\ldots+\lambda_{\ell+1})} \prod_{i=1}^{\ell+1} \frac{1}{1 - e^{\lambda_i-x}}
\]
\[
= \frac{1}{2\pi i} \oint_{C_0} dx e^{nx+k(\lambda_1+\ldots+\lambda_{\ell+1})} \prod_{i=1}^{\ell+1} \frac{1}{e_{\varphi_K}(x - \lambda_i)}.
\]
Here \( e_{\varphi_K}(x) \) is the exponent corresponding to \( K \)-theory (see the Remark 2.2).

Using the conjectural relation in Problem 3.1, the \( S^1 \times U_{\ell+1} \)-equivariant Riemann-Roch-Hirzebruch theorem for a trivial line bundle on \( \mathcal{L}_0(0) \) can be represented in the form similar to (6.2) for \( k = n = 0 \)
\[
\chi_{S^1 \times U_{\ell+1}}(\mathcal{L}_0(0)) = \frac{1}{2\pi i} \oint_{C} dx \prod_{j=1}^{\ell+1} \prod_{m=0}^{\infty} \frac{1}{1 - e^{\lambda_j + mh-x}} = \]
\[
(6.3)
\]
\[ = \frac{1}{2\pi i} \oint_C dx \prod_{i=1}^{\ell+1} \frac{1}{e^{\varphi_i(x - \lambda_i)}}, \]

where \( C \) encircles all the poles \( x = \lambda_j + mh; \ j = 1, \ldots, (\ell + 1) \), \( m \in \mathbb{Z}_{\geq 0} \) and we used a notation

\[ e_{\varphi_i}(u; h) = \frac{1}{\Gamma_q(e^{-u})} = \prod_{n=0}^{\infty} (1 - e^{nh-u}). \] (6.4)

However, despite a similarity of (6.2) and (6.3) the difference in integration contours does not allow directly to interpret \( e_{\varphi_i}(u; h) \) as a topological genus corresponding to a extraordinary cohomology theory on \( \mathbb{P}^\ell \). The way to transform (6.3) into an integral over the contour \( C_0 \) was discussed in Section 5. As a result the integration contour in (6.3) can be replaces by \( C_0 \) at the expense of multiplying the integrand by the correction factor \( I_{0,0}(\overline{z}, e^{-x}) \) (see (5.4) for explicit expression for \( I_{n,k}(\overline{z}, e^{-x}) \)). Now we have an expression for the equivariant Euler characteristic on \( L_{\mathbb{P}^\ell} \) in terms of the pairing of cohomology classes on \( \mathbb{P}^\ell \). However this correction factor appears to spoil the multiplicative property of (6.4). The underlying reason for this is the appearance of an infinite number of copies of \( \mathbb{P}^\ell \) as components of fixed point set in \( S^1 \)-localization. Thus, the situation is very much different from, for example, the elliptic genus (see e.g. [Se]) where the fixed point set of \( S^1 \) acting on \( LX \) is simply \( X \) itself. It is conceivable that the failure to interpret \( S^1 \)-localization on \( L_{\mathbb{P}^\ell} \) in terms of an extraordinary topological genus implies actually the existence of a meaningful quantum version of the an extraordinary cohomology theory formalism.

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