ON THE DIMENSION OF VOISIN SETS IN THE MODULI SPACE OF ABELIAN VARIETIES

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Abstract. We study the subsets $V_k(A)$ of a complex abelian variety $A$ consisting in the collection of points $x \in A$ such that the zero-cycle $\{x\} - \{0_A\}$ is $k$-nilpotent with respect to the Pontryagin product in the Chow group. These sets were introduced recently by Voisin and she showed that $\dim V_k(A) \leq k - 1$ and it is 0 for a very general abelian variety of dimension at least $2k - 1$.

We study in particular the locus $V_{g,2}$ in the moduli space of abelian varieties of dimension $g$ with a fixed polarization where $V_2(A)$ is positive dimensional and we prove that an irreducible subvariety $Y \subset V_{g,2}$, $g \geq 3$, such that for a very general $A \in Y$ there is a curve in $V_2(A)$ generating $A$ satisfies $\dim Y \leq 2g - 1$. The hyperelliptic locus shows that this bound is sharp.

1. Introduction

Claire Voisin in [V3] defines the subset $V_k(A)$ of a complex abelian variety $A$ consisting in the collection of points $x \in A$ such that the zero-cycle $\{x\} - \{0_A\}$ is $k$-nilpotent with respect to the Pontryagin product in the Chow group:

$$V_k(A) := \{x \in A \mid (\{x\} - \{0_A\})^k = 0 \text{ in } CH_0(A)_\mathbb{Q}\}.$$

Here we have denoted by $\{x\}$ the zero-cycle of degree 1 corresponding to the point $x \in A$. These are naturally defined sets in the sense that they exist in all the abelian varieties, are functorial and move in families. Moreover they are related with the gonality of the abelian variety itself (the minimal gonality of a curve contained in $A$) in a natural way.

We consider the following subsets of the moduli space of abelian varieties of dimension $g$ with a polarization of type $\delta$:

$$V_{g,k,l} = \{A \in \mathcal{A}^\delta_g \mid \dim V_k(A) \geq l\}.$$

Since the sets $V_k$ are naturally defined, then $V_{g,k,l}$ is a union of countably many closed subvarieties of $\mathcal{A}^\delta_g$. Hence it makes sense to ask about its dimension. Put $V_{g,k} := V_{g,k,1}$. For a abelian subvariety $B \subset A$ the inclusion

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$V_k(B) \subset V_k(A)$ holds and a well-known theorem of Bloch implies that $B = V_k(B)$ if $\dim B + 1 \leq k$, hence in this situation $B \subset V_k(A)$. These are in some sense “degenerated examples”. In this paper we concentrate in the case $k = 2$ and we take care of the non-degenerate case, that is we will assume that $V_2(A)$ contains some curve generating the abelian variety $A$.

Our main result is:

**Theorem 1.1.** Let $g \geq 3$ and consider an irreducible subvariety $Y \subset V_{g,2}$ such that for a very general $A \in Y$ there is a curve in $V_2(A)$ generating $A$. Then $\dim Y \leq 2g - 1$.

This result is sharp due the existence of the hyperelliptic locus in the moduli space of principally polarized abelian varieties. In fact, the motivation for this study is to understand the geometrical meaning of the positive dimensional components in $V_2$. Our result gives some evidence that there is a link between the existence of hyperelliptic curves in abelian varieties and the fact that $V_2$ is positive dimensional. We remark that the statement of the theorem (1.1) was suggested by the the analogous result in [NP] concerning hyperelliptic curves.

Section 2 is devoted to give some preliminaries and some useful properties of the loci $V_k(A)$ focusing specially in the case $k = 2$. A remarkable property is that $V_2(A)$ is the preimage of the orbit of the image of the origin with respect to rational equivalence in the Kummer variety $Kum(A)$. We also prove the following interesting facts (see Corollaries (2.11) and (2.12)):

**Proposition 1.2.** For any abelian variety $A$ the inclusion

$$V_k(A) + V_l(A) \subset V_{k+l-1}(A)$$

holds for all $1 \leq k, l \leq g$. Moreover if $C$ is a hyperelliptic curve of genus $g$, and $J(C)$ be its Jacobian variety, then for all $1 \leq k \leq g + 1$, we have that $\dim V_k(JC) = k - 1$ (the maximal possible value).

The rest of the paper is devoted to the proof of the main theorem. The beginning follows closely the same strategy as in [NP] since we reduce to prove the vanishing of certain adjoint form. The novelty here is that we prove this vanishing by using the action of a family of rationally trivial zero-cycles as in the classical papers of Mumford and Rojiman.

More precisely we proceed by contradiction: we can assume that there is relative map $f : C \to A$ of curves in abelian varieties over a base $U$ of dimension at least $2g$ and such that $f(C_y) \subset V_2(A_y)$ generates the abelian variety $A_y$ for all $y \in U$. Then, by using deformation of differential forms as in [NP] we compute the so-called adjunction form in a generic point of the family. This technique can be traced-back to [CP], where this procedure is introduced for the first time. In section 4 we prove that the vanishing of the adjoint form implies the desired contradiction with the hypothesis on the dimension (see Proposition 4.3). We prove this vanishing in section 5.
in the following way: we construct a family of zero-cycles \( Z \) on the family of abelian varieties (after a convenient base change). Since all the curves belong to the \( V_2 \) sets, all the zero-cycles are rationally trivial and then the cycle \( Z \) is rationally trivial in a Zariski open set. Then we look to the action of the cycle on the 2-differential forms extending a little bit a classical construction by Mumford and Roitman to the case of a family of projective varieties over an analytical base. This extension is done separately in section 3. In our opinion section 3 is interesting on its own and could have other applications. The nice fact is that doing convenient choice the adjoint form is recovered very directly from this action and this allows to finish the proof. We point out that this last part might be surely done by using the infinitesimal invariant of the family of \( Z \) (as, for instance, in [VI]), we have chosen instead this strategy since the adjoint form emerges cleanly from the action of the cycle on the holomorphic 2-forms which makes the proof very natural. On the other hand it should be interesting to have a more algebraic proof in the spirit of Bloch-Shrinivas (see [BS]).

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2. Preliminaires on the subsets \( V_k \) of an abelian variety \( A \)

2.1. On the dimension of \( V_k \). The most part of this subsection comes from [V3], where the sets \( V_k(A) \) appear for the first time.

Definition 2.1. Let \( A \) be an abelian variety and denote by \( CH_0(A) \) the Chow group of zero-cycles in \( A \) with rational coefficients. We also denote by \( * \) the Pontryagin product in the Chow group. Given a point \( x \in A \), we put \( \{x\} \) for the class of \( x \) in \( CH_0(A) \). Then we define the Voisin sets (cf. [V3]):

\[
V_k = V_k(A) := \{x \in A \mid (\{x\} - \{0_A\})^* k = 0\}.
\]

It is known that the set \( V_k \) is a countable union of closed subvarieties of \( A \). A typical way to obtain points in \( V_k \) is given by the following Proposition:

Proposition 2.2. ([V3] Prop.(1.9)) Assume that \( \{x_1\} + \cdots + \{x_k\} = k \{0_A\} \) in \( CH_0(A) \) for some points \( x_i \in A \), then for all \( i \) we have \( x_i \in V_k \).

Since the orbits (w.r.t. rational equivalence) \( |k \{0_A\}| \) are hard to compute there are only a few examples of positive dimensional components in \( V_k(A) \) that we can construct from this Proposition. The simplest instance of this comes from a \( k \)-gonal curve \( C \) contained in \( A \) such that there is a totally ramified point \( p \) for the degree \( k \) map \( f : C \to \mathbb{P}^1 \). Translating we can assume that \( p \) is the origin \( 0_A \) and then the fibers of \( f \) provide a 1-dimensional component in the symmetric product (\( k \) times) of \( A \). Therefore, by the Proposition above, we obtain that \( C \) is contained in \( V_k(A) \). Observe that the sets \( V_k \) are invariant under isogenies, hence these positive dimensional components also appear in many other abelian varieties. In particular for any integer \( n \) we have that \( n_* V_2(A) \subset V_2(A) \).

Remark 2.3. We have the following properties:

a) All the abelian varieties containing hyperelliptic curves have positive dimensional components in $V_2$.

b) A very well known theorem of Bloch (see [B]) implies that

$$V_{g+1}(A) = A,$$

hence the natural filtration

$$V_1(A) = \{0\} \subset V_2(A) \subset \ldots \subset V_g(A) \subset V_{g+1}(A) = A$$

has at most $g$ steps. It is natural to ask how is the behaviour of the dimension of $V_k(A)$, with $k \leq g$, for very general abelian varieties and which geometric properties of $A$ codify these sets.

c) Assume that $B \subset A$ is an abelian subvariety, then $V_k(B) \subset V_k(A)$.

In particular, if $k \geq \dim B + 1$, then $B \subset V_k(A)$. For instance: all the elliptic curves in $A$ passing through the origin are contained in $V_2(A)$.

d) Let $C$ be a smooth quartic plane curve and let $p$ a flex point with tangent $t$, then $t \cdot C = 3p + q$ and the projection from $q$ provides a collection of zero-cycles of degree 3 in $JC$ rationally equivalent to $3\{0\} \subset V_3(JC)$. Using isogenies we get that there are in fact a countably number of curves in $V_3(A)$ for a very general abelian variety of dimension 3.

The following is proved in Theorem (0.8) of [V3] by using some ideas from [Mü] and improving the techniques of [P]:

**Theorem 2.4.** Let $A$ be an abelian variety of dimension $g$. Then:

a) $\dim V_k(A) \leq k - 1$.

b) If $A$ is very general and $g \geq 2k - 1$ we have that $\dim V_k(A) = 0$.

In the specific case of $V_2(A)$ we have the following properties.

**Proposition 2.5.** Let $A$ be an abelian variety and let $Kum(A)$ be its Kum-mer variety.

a) We have the equality: $V_2(A) = \{x \in A \mid \{x\} + \{-x\} = 2\{0\}\}$.

b) Let $\alpha : A \rightarrow Kum(A)$ be the quotient map. Then

$$V_2(A) = \alpha^{-1}(\{y \in Kum(A) \mid \{y\} \sim_{rat} \{\alpha(0A)\}\}).$$

**Proof.** Part a) follows from the observation that

$$(\{x\} - \{0\})^2 = 2\{x\} - 2\{x\} + \{0\} = 0$$

is equivalent, translating with $-x$, to $\{x\} + \{-x\} = 2\{0\}$.

To prove b) we first see that $x \in V_2(A)$ if and only if $\{\alpha(x)\} \sim_{rat} \{\alpha(0A)\}$. Indeed, assume that $\{x\} + \{-x\} = 2\{0\}$, applying $\alpha$ we get that $2\{\alpha(x)\} \sim_{rat} 2\{\alpha(0A)\}$. Since $\text{Alb}(Kum(A)) = 0$ the Chow group has no torsion (see [R2]) therefore $\{\alpha(x)\} \sim_{rat} \{\alpha(0A)\}$. In the opposite direction, if $\{\alpha(x)\} \sim_{rat} \{\alpha(0A)\}$ we apply $\alpha^*$ at the level of Chow groups and
we obtain that $x \in V_2(A)$. Hence $V_2(A)$ is the pre-image by $\alpha$ of the points rationally equivalent to $\alpha(0_A)$. 

\[ \Box \]

### 2.2. Relation with the Chow ring.

In this part we collect some computations on 0-cycles on abelian varieties which are more or less implicit in [Be1] and [V3].

Let us recall first some facts on the Chow group (with rational coefficients) of an abelian variety $A$ of dimension $g$ which are proved in [Be2]. Let us define the subgroups:

\[ \text{CH}^g(A) := \{ z \in \text{CH}^g(A) \mid k_*(z) = k^g z, \quad \forall k \in \mathbb{Z} \} \]

Then:

\[ \text{CH}^g(A)_{\mathbb{Q}} = \text{CH}^g_0(A) \oplus \text{CH}^g_1(A) \oplus \ldots \oplus \text{CH}^g_g(A) \]

Moreover $\text{CH}^g_0(A) = \mathbb{Q}\{0_A\}$ and $I = \bigoplus_{s \geq 1} \text{CH}^g_s(A)$ is the ideal, with respect to the Pontryagin product, of the zero-cycles of degree 0. It is known that $I^*r = \bigoplus_{s \geq r} \text{CH}^g_s(A)$ and that $I^*2$ is the kernel of the albanese map:

\[ \text{CH}^g(A)_{\mathbb{Q}} \longrightarrow A, \]

sending a zero cycle $\sum n_i(a_i)$ to the sum $\sum n_i a_i$ in $A$. Another useful property is that $\text{CH}^g_s(A) * \text{CH}^g_t(A) = \text{CH}^g_{s+t}(A)$.

We point out that the filtration $V_1(A) \subset V_2(A) \subset \ldots \subset A$ is, in some sense, induced by the filtration $I \subset I^*2 \subset \ldots \subset \text{CH}^g(A)_{\mathbb{Q}}$. Indeed, given a point $x \in A$ we use the notation:

\[ (1) \quad \{x\} = \{0_A\} + x_1 + \ldots + x_g, \quad x_i \in \text{CH}^i_1(A). \]

Then we have:

**Proposition 2.6.** For all $x \in A$, $x$ belongs to $V_k(A)$ if and only if

\[ x_k = \ldots = x_g = 0. \]

In particular $x \in V_2(A)$ if and only if $\{x\} - \{0_A\} \in \text{CH}^g_1(A)$.

**Proof.** We apply to (1) the multiplication by $l$ in $A$:

\[ \{lx\} = \{0_A\} + lx_1 + \ldots + l^g x_g. \]

Using this we have:

\[ (\{x\} - \{0_A\})^*k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \{(k-i)x\} = \]

\[ = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \{0_A\} + (k-i)x_1 + \ldots + (k-i)^gx_g = \]

\[ = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \{0_A\} + \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)x_1 + \ldots \]

\[ + \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^gx_g. \]
Now we use the following formulas (see the proof of Lemma 3.3 in [V3] or prove them by induction):
\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^l = 0 \quad \text{if } l < k
\]
\[
\sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^k = k!
\]

Therefore we obtain that:
\[
(\{x\} - \{0_A\})^k = k! x_k + \ldots
\]
and similarly \((\{x\} - \{0_A\})^l = l! x_l + \ldots\) for any \(l \geq k\). Hence \(x \in V_k(A)\) if and only if \(x \in V_l(A)\) for all \(l \geq k\) if and only if \(x_k = \ldots = x_g = 0\).

We have several consequences of this Proposition and of its proof:

**Corollary 2.7.** With the same notations we have that \(x_k = \frac{1}{k!} x_1^k\). Hence \(\{x\} = \exp(x_1)\). Moreover \(x \in V_k(A)\) if and only if \(x_k = x_1^k = 0\).

**Proof.** We have seen along the proof of the Proposition that \((\{x\} - \{0_A\})^k = k! x_k + \ldots\) higher degree terms.

Computing directly we get that \((\{x\} - \{0_A\})^k = (x_1 + \ldots + x_k)^k = x_1^k + \ldots\) higher degree terms.

By comparing both formulas we obtain the equality. \(\square\)

**Remark 2.8.** Notice that our computations are somewhat related with that of [Be1]. Indeed define as in section 4 of loc. cit. the map
\[
\gamma : A \longrightarrow I, \quad a \mapsto \{0\} - \{a\} + \frac{1}{2} (\{0\} - \{a\})^2 + \frac{1}{3} (\{0\} - \{a\})^3 + \ldots
\]
this is a morphism of groups. Then, with our notations, \(\gamma(x) = -x_1\). In particular the image of \(\gamma\) belongs to \(CH_1^g(A)\).

**Corollary 2.9.** Let \(a, b \in A\) be two points such that
\[
n\{a\} + m\{b\} = (n + m)\{0_A\}
\]
for some integers \(1 \leq n, m\). Then \(a, b \in V_2(A)\).

**Proof.** Decomposing as before:
\[
\{a\} = \{0\} + a_1 + \frac{1}{2} a_1^2 + \ldots \quad \{b\} = \{0\} + b_1 + \frac{1}{2} b_1^2 + \ldots
\]
the equality of the statement implies that \(na_1 + mb_1 = 0 = na_1^2 + mb_1^2\). Then \(b_1 = -\frac{a}{m} a_1\) and thus \(na_1^2 + \frac{a^2}{m^2} a_1^2 = 0\), so \(a_1^2 = b_1^2 = 0\) and \(a, b \in V_2(A)\). \(\square\)

**Corollary 2.10.** Let \(\varphi : A \longrightarrow B\) be an isogeny, then \(\varphi^{-1}(V_k(B)) \subset V_k(A)\). In particular \(\varphi(V_k(A)) = V_k(B)\) and \(\varphi^{-1}(V_k(B)) = V_k(A)\).
Proof. Since we work with Chow groups with rational coefficients it is clear that for an integer \( n \neq 0 \) the map \( n_* : CH^0_k(A) \to CH^0_k(A) \) is bijective. Let \( \psi : B \to A \) be an isogeny such that \( \psi \circ \varphi = n \), we deduce that \( \varphi_* : CH^0_k(A) \to CH^0_k(B) \) is injective. Let \( x \in \varphi^{-1}(V_k(B)) \) and set \( \{ x \} = \{ 0_A \} + x_1 + \ldots + x_g \). By hypothesis

\[
\varphi(\{ x \}) = \{ 0_B \} + \varphi_*(x_1) + \ldots + \varphi_*(x_g) \in V_k(B).
\]

Hence \( \varphi_*(x_k) = 0 \) and \( x_k = 0 \). Therefore \( x \in V_k(A) \) and we have done. \( \square \)

Another interesting consequence of this characterization is the following property:

Corollary 2.11. For any \( 0 \leq k, l \leq q \) we have that

\[
V_k(A) + V_l(A) \subset V_{k+l-1}(A).
\]

Proof. Let \( x \in V_k(A) \), \( y \in V_l(A) \). Then \( \{ x \} = \{ 0_A \} + x_1 + \ldots + x_{k-1} \) and \( \{ y \} = \{ 0_A \} + y_1 + \ldots + y_{l-1} \). Since \( x_j * y_j \in CH^0_{i+j}(A) \) we obtain

\[
\{ x + y \} = \{ x \} * \{ y \} = \{ 0_A \} + (x_1 + y_1) + (x_2 + y_2) + \ldots + (x_{k-1} + y_{l-1}).
\]

Thus \( x + y \in V_{k+l-1}(A) \). \( \square \)

As an application we have:

Corollary 2.12. Let \( C \) be a hyperelliptic curve of genus \( g \). Then

\[
\dim V_k(JC) = k - 1
\]

for \( 1 \leq k \leq g \), that is the maximal possible dimension is attained.

Proof. Choosing a Weierstrass point to define the Abel-Jacobi map we can assume that the curve \( C \) is contained in \( V_2(JC) \), using inductively the previous Corollary, we have that

\[
C + \{ k-1 \} + C = W^0_{k-1}(C) \subset V_k(JC).
\]

\( \square \)

For instance, for a Jacobian of a curve \( C \) of genus 3 we have in the hyperelliptic case that \( \dim V_2(JC) = 1 \) and \( \dim V_3(JC) = 2 \). If instead \( C \) is a generic quartic plane curve we have that \( \dim V_2(JC) = 0 \) by Theorem \( \{ 2.3 \} \) and \( \dim V_3(JC) \geq 1 \) by Remark \( \{ 2.3 \} d \). Denoting as in that remark \( p \) a flex point and \( q \) its residual point we have also the following: for any \( x \in C \) such that the tangent to \( C \) in \( x \) goes through \( q \), \( x \in V_2(C) \) (we identify \( C \) with the Abel-Jacobi image in \( JC \) using \( p \)). Indeed: there exists a \( y \in C \) with \( 2x + y + q \sim 3p + q \), hence in \( JC \) there is a relation of the form \( 2\{ x \} + \{ y \} = 3\{ 0 \} \) and then Corollary \( \{ 2.9 \} \) implies that \( \{ x \} , \{ y \} \in V_2(JC) \).

Assume now that \( C \) is a quartic plane with a hyperflex, that is a point \( p \) such that \( \mathcal{O}_C(1) \cong \mathcal{O}_C(4p) \). This condition defines a divisor in \( \mathcal{M}_3 \). Consider as above \( C \) identified with the image of the Abel-Jacobi map via \( p \). Then for any bitangent \( 2x + 2y \) we have \( \{ x \} , \{ y \} \in V_2(JC) \). Also, with the same argument, for a standard flex \( q \) we have \( \{ q \} \in V_2(JC) \). Everything suggests
that the points in “$C \cap V_2(JC)$” could have some geometrical meaning. Notice that it also remain open to determine if the dimension of $V_3(JC)$ is 1 or 2 for a generic quartic plane curve $C$.

3. Mumford–Roitman construction in the analytical case

In the paper [Mu] Mumford proves that the existence of non-trivial holomorphic 2-forms on a smooth projective complex surface $F$ implies that the Chow ring of zero-cycles $CH^2(F)$ is infinite dimensional. The main tool he uses is the following: given a $(2,0)$ form $\Omega$ on $F$ he considers the form

$$\Omega_n := \sum_{i=1}^{n} pr_i^* (\Omega) \in H^0(F^n, \Omega^2_{F^n}),$$

where $pr_i$ is $i$-th projection, which is invariant under the action of the symmetric group $S_n$. Then for any morphism $f : S \to F^{(n)}$, where $S$ is a smooth variety and $F^{(n)}$ is the symmetric product, there is a natural functorial $(2,0)$-form $f^*(\Omega_n)$ on $S$. Morally this is the pull-back by $f$ of the quotient of $\Omega_n$ by $S_n$. More precisely let $\tilde{S} = (S \times F^{(n)})_{\text{red}}$ be the reduced fiber product, then there is a diagram

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{\tilde{f}} & F^n \\
\tilde{p} & \downarrow & \downarrow p \\
S & \xrightarrow{f} & F^{(n)}. \\
\end{array}$$

The form $f^*(\Omega_n)$ is the unique regular $(2,0)$-form in $S$ such that

$$\tilde{p}^*(f^*(\Omega_n)) = \tilde{f}^*(\Omega_n)$$

is torsion in $\Omega^2_S$ (see [Mu, §1]). The main Theorem that Mumford proves is that if $f(s) \sim_{\text{rat}} f(t)$ for all $s, t \in S$, then $f^*(\Omega_n) = 0$.

Roitman proves in [R1] that the same construction works for $(p,0)$ forms in any dimension and applies also to the study of zero-cycles. We remind here the setup: let $X$ be a smooth complex projective variety of dimension $\geq 2$ and let $\Omega \in H^0(X, \Omega^2_X)$. With an analogous construction he defines the holomorphic form $f^*(\Omega_n)$ for any $f : S \to X^{(n)}$, where $S$ is a smooth variety (see [R1] Theorem 3). Then he proves that if $f_1, f_2 : S \to X^{(n)}$ are morphisms such that $f_1(s) \sim_{\text{rat}} f_2(s)$ for all $s \in S$ then $f_1^*(\Omega_n) = f_2^*(\Omega_n)$. The following result is a variation of these ideas.

**Proposition 3.1.** Let $p : X \to Y$ be a projective fibration, where $X$ and $Y$ are smooth and connected. Let $F, G : Y \to X^{(n)}_{\text{rat}} \subset X^{(n)}$ be two holomorphic sections of $\mathcal{X}^{(n)}_{\text{rat}} \to Y$, where $X^{(n)}_{\text{rat}}$ is the relative symmetric product and $X^{(n)}$ is the usual symmetric product. Assume that $F(s) \sim_{\text{rat}} G(s)$ seen as a zero-cycle in $X_s := p^{-1}(s)$, for all $s \in Y$. Then for a generic point $y \in Y$, for any smooth analytic surface $\Gamma \subset Y$ passing through $y$, and for any holomorphic 2-form $\Omega$ in $X_{\Gamma} = p^{-1}(\Gamma)$ we have that $F^*(\Omega_n) = G^*(\Omega_n)$.
Proof. We recall the spreading out of rational equivalence in our case, in Mumford spirit, as in [V2] pag. 292. The spreading out argument guarantees that there is a generically finite (dominant) map \( q : Y' \to Y \) (which is a Zariski open set of an irreducible subvariety of a relative Hilbert scheme and a diagram

\[
\begin{array}{ccc}
Y' & \xrightarrow{F',G'} & X_Y^{(n)} \\
\downarrow{q} & & \downarrow{\tilde{q}_n} \\
Y & \xrightarrow{F,G} & X^{(n)}
\end{array}
\]

such that \( F'(Y') - G'(Y') = \tilde{q}_n^{-1}(F(Y)) - \tilde{q}_n^{-1}(G(Y)) \) is rationally equivalent to zero as a cycle in \( X' := X \times_Y Y' \). That means that there is a finite number of universal subschemes \( \mathcal{C}_l \) of relative dimension 1, Weil divisors \( \mathcal{D}_l \) of \( \mathcal{C}_l \) and rational functions \( \phi_l \in \mathbb{C}(\mathcal{C}_l) \) such that \( \mathcal{D}_l = n_l \cdot \text{div}(\phi_l) \), for \( n_l \) the normalization map, and \( F'(Y') - G'(Y') = \sum_l \mathcal{D}_l \).

Note that this construction can be restricted to \( \Gamma' := q^{-1}(\Gamma) \). The functions \( \phi_l \) define in a natural way holomorphic maps:

\[ f_l : \Gamma' \times \mathbb{P}^1 \to X^{(n_l)}, \quad f_l(v, t) = \phi_l^{-1}(t) \cap X_v^{(n_l)}. \]

The argument continues as in Mumford and Roitman (see [Mu] and [R1]). Let \( \tilde{\Omega} \) be a 2 form in \( X'_{\mathbb{P}^1} \), and consider the 2-form \( f_l^*(\Omega_{n_l}) \) in \( W := \Gamma' \times \mathbb{P}^1 \). From the decomposition of the cotangent sheaf of \( W \):

\[ \Omega_W^1 = \pi_1^*\Omega_{\mathbb{P}^1} \oplus \pi_2^*\Omega_{\mathbb{P}^1} \]

we obtain that

\[ \Omega_W^2 = \pi_1^*\Omega_{\mathbb{P}^1} \oplus (\pi_1^*\Omega_{\mathbb{P}^1} \oplus \pi_2^*\Omega_{\mathbb{P}^1}). \]

Using this last equality, the form \( f_l^*(\Omega_{n_l}) \) has a decomposition as \( f_l^*(\Omega_{n_l}) = \alpha + \beta \), we want to prove that \( \beta = 0 \). Indeed, let us consider a generic point \( s \in \Gamma' \) and a tangent vector \( v \in T_{\Gamma'}(s) \). The contraction \( \beta(v) \) in \((s,t)\) is a \((1,0)\)-form on \( \{v\} \times \mathbb{P}^1 \) hence it is zero. Therefore, if \((v_1,w),(v_2,w)\) is a basis of the tangent vector space of \( W \) in \((s,t)\), we have that \( \beta(v_1 \wedge w) = \beta(v_2 \wedge w) = 0 \). Hence \( \beta = 0 \). Thus \( f_l^*(\Omega_{n_l}) = \alpha = \pi_2^*(\gamma) \) for some \((2,0)\)-form on \( \Gamma' \). By construction we obtain that \( f_l^*(\Omega_{n_l})|_{\Gamma' \times \{t\}} = \tilde{\gamma} \) for all \( t \in \mathbb{P}^1 \). Therefore we always obtain \( \tilde{\gamma} \) following the chain of \( \mathbb{P}^1 \), summing up using the natural morphisms \( X^{(n)} \times X^{(m)} \to X^{(n+m)} \) (see [R1] Section3).

Hence at the end \( (F'_{\Gamma'})^*(\tilde{\Omega}_n) = (G'_{\Gamma'})^*(\tilde{\Omega}_n) \). If \( \tilde{\Omega} = q^*(\Omega) \), then, by diagram \( 2 \) and since \( q \) is dominant \( F_{\Gamma}^*(\Omega_n) = G_{\Gamma}^*(\Omega_n) \).\]
Proposition 3.2. With these notations and for all \((2, 0)\)-form \(\Omega\) on \(X\) we have that:

\[
F^*(\Omega_n) = f_1^*(\Omega) + \ldots + f_n^*(\Omega) \quad \text{and} \quad G^*(\Omega_n) = g_1^*(\Omega) + \ldots + g_n^*(\Omega).
\]

4. The locus of abelian varieties with positive dimensional \(V_2\) and the adjoint form

The results given above show that for a very general abelian variety \(A\) of dimension \(g \geq 3\) the subset \(V_2(A)\) is 0-dimensional (hence countable). Observe that \(V_{2,2} = \mathcal{A}_2^4\), since there are hyperelliptic curves in any abelian surface. On the other hand the abelian varieties isogenous to a product of an elliptic curve by an abelian variety of dimension \(g - 1\) are trivially contained in \(V_{g,2}\). This gives somehow “degenerated” components in \(V_{g,2}\) of codimension \(g - 1\). Other components of high dimension can be constructed similarly: assume that \(A\) contains an abelian surface \(S\). Then \(S\) is isogenous either to a 2-dimensional Jacobian or to a product of elliptic curves, and therefore there is a curve in \(V_2(S) \subset V_2(A)\). This trick can be extended to higher dimensions: consider \(A\) isogenous to an abelian variety that contains a hyperelliptic Jacobian, then \(V_2(A)\) has trivially positive dimension. We avoid these degenerated components by imposing in the statement of the theorem that \(V_2(A)\) generates \(A\) as a group.

The main result of this note, as mentioned in the Introduction, is the following bound for non-degenerated components of \(V_{g,2}\).

Theorem 4.1. Let \(g \geq 3\) and let \(\mathcal{Y} \subset V_{g,2}\) be an irreducible subvariety such that for a very general \(A \in \mathcal{Y}\) there is a curve contained in \(V_2(A)\) generating \(A\). Then \(\dim \mathcal{Y} \leq 2g - 1\).

Observe that the hyperelliptic locus \(\mathcal{H}_g\) of curves of genus \(g\) is contained in \(V_{g,2}\), hence the bound is sharp. In fact, by pulling-back the hyperelliptic locus to the Siegel half-space, then applying an element of \(Sp(2g, \mathbb{Q})\) and finally going down to \(\mathcal{A}_g\) we can construct countably many components in \(V_{g,2}\) of dimension \(2g - 1\).

The goal of this section is to reduce the proof of Theorem 4.1 to the vanishing of a certain adjoint form that we will define below.

We proceed by contradiction, hence we assume that there exists an irreducible component of \(V_{g,2}\) of dimension \(\geq 2g\). By 2.4 we have \(\dim V_2 \leq 1\), hence \(V_2(A)\) contains curves for all \(A \in \mathcal{Y}\) and we assume by hypothesis that there is at least one of these curves generating the abelian variety. By a standard argument (involving the properness and countability of relative Chow varieties and the existence of universal families of abelian varieties up
Lemma 4.2. For any \( \cdot \) of the target space \( 3 + 2 \omega \). Hence, if \( E \) restriction to \( \mathcal{CP} \). Observe that \( \xi H \in \) the connection map \( \Phi : U \to Y \) isomorphism \( C \) where the parameter space \( U \) contained in \( V_2(A_y) \) and generating \( A_y \) followed by the inclusion. We can also assume that \( \mathcal{C} \to U \) has a section and then that \( f \) induces a map of families of abelian varieties \( F : \mathcal{C} \to A \) over \( U \).

We fix a generic point \( y \) in \( U \) and we denote by \( T \) the tangent space of \( U \) at \( y \). Observe that \( T \hookrightarrow Sym^2 H^{1,0}(A_y)^* \). Moreover the surjective map \( F_\xi \) induces an inclusion of \( W_y := H^{1,0}(A_y) \to H^0(C_y, \omega_{C_y}) \). Let \( D \) be the base locus of the linear system generated by \( W_y \), therefore \( W_y \subset H^0(C_y, \omega_{C_y}(-D_y)) \). Lemma 3.1 in [NP] states that for a generic two dimensional subspace \( E \) of \( W_y \) the base locus of the pencil attached to \( E \) is still \( D_y \). As in the proof of Theorem 1.4 [NP] there exists a map sending \( \xi \in T \), seen as a symmetric map \( \xi : W_y = H^{1,0}(A_y) \to H^{1,0}(A_y)^* \), to its restriction to \( E \). We compare the dimension of \( T (= \dim \mathcal{Y} \geq 2g) \) and that of the target space \( 3 + 2 \cdot (g - 2) = 2g - 1 \) and we conclude the following:

**Lemma 4.2.** For any \( E \subset W_y \) there exists \( \xi \in T \) killing all the forms in \( E \). Hence, if \( \omega_1, \omega_2 \) is a basis of \( E \), then \( \xi \cdot \omega_1 = \xi \cdot \omega_2 = 0 \).

We want to compute the adjunction class for these forms as defined in [CP]. Observe that \( \xi \) can be seen via \( f \) as an infinitesimal deformation of \( C_y \). We denote by \( F_\xi \) the rank 2 vector bundle on \( C_y \) attached to \( \xi \) via the isomorphism \( H^1(C_y, T_{C_y}) \cong Ext^1(\omega_{C_y}, \mathcal{O}_{C_y}) \). By definition there is a short exact sequence of sheaves:

\[
0 \to \mathcal{O}_{C_y} \to F_\xi \to \omega_{C_y} \to 0.
\]

The connection map \( H^0(C_y, \omega_{C_y}) \to H^1(C_y, \mathcal{O}_{C_y}) \) is the cup-product with \( \xi \in H^1(C_y, T_{C_y}) \). Then the forms \( \omega_1, \omega_2 \) generating \( E \) lift to sections \( s_1, s_2 \in H^0(C_y, F_\xi) \). The adjoint form of \( E \) is defined as the image of

\[
s_1 \wedge s_2 \in \Lambda^2 H^0(C_y, F_\xi)
\]

in \( H^0(C_y, \Lambda^2 F_\xi) \cong H^0(C_y, \omega_{C_y}) \). We have done some choices (the base in \( F_\xi \) and the liftings \( s_1, s_2 \)), but the **adjoint class** \( \text{adj}_\xi(E) \) in \( H^0(C_y, \omega_{C_y})/E \) is well-defined.

**Proposition 4.3.** If the adjoint class \( \text{adj}_\xi(E) \) vanishes then \( \xi \) belongs to the kernel of \( d\Phi : T \to T_{A_y}(A_y) = Sym^2 H^{1,0}(A_y)^* \).
Proof. According to Theorem 1.1.8 in [CP], the adjoint form vanishes if and only if the image of $\xi$ in

$$H^1(C_y, T_{C_y}(D)) \cong Ext^1(\omega_{C_y}(-D), \mathcal{O}_{C_y})$$

is zero. This says that the corresponding extension is trivial, so the short exact sequence in the first row of the next diagram splits (i.e. $i^* F_\xi = \mathcal{O}_{C_y} \oplus \omega_{C_y}(-D)$):

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_{C_y} & \longrightarrow & i^* F_\xi & \longrightarrow & \omega_{C_y}(-D) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{C_y} & \longrightarrow & F_\xi & \longrightarrow & \omega_{C_y} & \longrightarrow & 0
\end{array}
$$

which implies that the connecting homomorphism in the associated long exact sequence of cohomology $H^0(C_y, \omega_{C_y}(-D)) \longrightarrow H^1(C_y, \mathcal{O}_{C_y})$ is trivial. Therefore $\xi \cdot H^0(C_y, \omega_{C_y}(-D)) = 0$ and in particular $\xi \cdot W_y = 0$. This says that $\xi$ is in the kernel of $d\Phi_y$. $\square$

Hence, since $d\Phi$ is injective in a generic point, we are reduced to prove the vanishing of the adjoint form to reach a contradiction.

5. Proof of the main Theorem

The next step is to construct a family of rationally trivial zero-cycles in the family of abelian varieties and to study the action of this family viewed as a correspondence on the differential 2-forms. The key point is that on one hand this action is trivial due to Proposition 3.1 and on the other hand it relates naturally with the adjoint form. This will give the vanishing of the adjoint form that we need to end the proof. We present this strategy step by step in the rest of this section.

5.1. The family of zero-cycles and the attached correspondence.

To start with we pull-back the families of curves and abelian varieties to $C$ itself:

$$
\begin{array}{ccc}
\mathcal{A}_C & \longrightarrow & \mathcal{A} \\
\pi \downarrow & & \pi \downarrow \\
\mathcal{C} & \longrightarrow & \mathcal{U}.
\end{array}
$$

Now we define a family of zero-cycles in $\mathcal{A}_C$ parametrized by $\mathcal{C}$. Let $s_+ : \mathcal{C} \longrightarrow \mathcal{A}_C$ be the section given by the map $f$: 
Put $Z^+ := s_+(C)$. Analogously, by considering $-1_A \circ f$, where $-1_A$ is the relative $-1$ map on the family of abelian varieties we define a section $s_- : C \rightarrow A$ of $\pi$ and a cycle $Z^- := s_-(C)$. Finally the zero section $0_A$ gives a section $s_0$ and a cycle $Z_0$. Set $Z = Z^+ + Z^- - 2Z_0$, a cycle on $A_C$. In fact $Z^+ + Z^- - 2Z_0 \subseteq \text{Sym}^2 A_C$. Observe that $Z_t, t \in \mathcal{C}$ is the 0-cycle $Z_t = \{f(t)\} + \{-f(t)\} - 2\{0_A\}$. As above a generic point $y \in \mathcal{U}$ is fixed. Our aim is to construct a curve $y \in B \subset \mathcal{U}$ and a rank 2 vector bundle $E$ on $B$ such that $E$ is a subbundle of the restriction of $\pi^*_s \Omega^1_{\mathcal{A}^1_U}$ to $B$. Moreover we want that the tangent vector to $B$ at any point $b \in B$ (near $y$) kills $E_b$. To do it we consider the incidence family:

$I \subset \text{Grass}(2, \pi^*_s \Omega^1_{\mathcal{A}^1_U}) \times_U \mathbb{P}(T_U) \rightarrow \mathcal{U}$

given by the pairs $(V_y, [\xi_y])$ such that $\xi_y \cup V_y = 0$. Since the dimension of $\mathcal{U}$ is at least $2g$, Lemma 4.2 implies that the projection of $I$ over $\text{Grass}(2, \pi^*_s \Omega^1_{\mathcal{A}^1_U})$ is surjective. Denote by $\tilde{I}$ the image of the projection of $I$ in $\mathbb{P}(T_U)$, observe that $\tilde{I}$ is still surjective on $\mathcal{U}$. Let $(y, [\xi_y])$ be a generic element in the fiber over the fixed point $y$ and choose a section (maybe shrinking the open set to guarantee the existence of this section) going through $y$. Taking the pull-back of the section to the tangent space we construct a vector field on $\mathcal{U}$ such that the value at $y$ is the vector $\xi_y$. We find, by integrating the vector field, the curve $B \subset \mathcal{U}$ through $y$. 

5.2. Integration procedure. As above a generic point $y \in \mathcal{U}$ is fixed. Our aim is to construct a curve $y \in B \subset \mathcal{U}$ and a rank 2 vector bundle $E$ on $B$ such that $E$ is a subbundle of the restriction of $\pi^*_s \Omega^1_{\mathcal{A}^1_U}$ to $B$. Moreover we want that the tangent vector to $B$ at any point $b \in B$ (near $y$) kills $E_b$. To do it we consider the incidence family:
Now the restriction to $B$ of the universal bundle $S$ on the Grassmannian gives a rank 2 vector bundle on $B$, $E_B$. That is:

\[
\begin{array}{ccc}
E_B & \rightarrow & S \\
\downarrow & & \downarrow \\
B & \hookrightarrow & \text{Grass}(2, \pi_\ast \Omega^1_A) \\
\downarrow & & \downarrow \\
U. & & \\
\end{array}
\]

Moreover we know that at each point $p$ of $B$ and for any tangent vector $\xi \in T_B(p)$, the cup product of $\xi$ with the sections of $E_B$ is zero.

**Remark 5.1.** The reader can observe that, by the genericity of the choices, the 2-dimensional vector space corresponding to the fiber of $E_B$ at a generic point $b$, seen as a subspace of:

\[W_b := H^{1,0}(A_b) \subset H^0(C_b, \omega_{C_b}),\]

satisfies that the base locus of $|E_{B,b}|$ is the same as the base locus of $|W_b|$.

From now on we restrict our family over $U$ to a family over $B$. Moreover we denote by $\Gamma$ the family of curves $C_b$ over $B$, hence $\dim \Gamma = 2$ and

\[h : \Gamma \rightarrow B\]

is a fibration. By construction all the fibers of $h$ are smooth. Pulling back to $B$ the family of abelian varieties and the family of curves we get the diagram:

\[
\begin{array}{ccc}
A_B & \rightarrow & A \\
\downarrow & & \downarrow \\
\Gamma & \rightarrow & C \\
\downarrow & & \downarrow \\
B & \rightarrow & U.
\end{array}
\]

**5.3. The adjoint form in families.** The construction of the adjoint form (or the Massey product) in families has been developed in several places. We refer to [G] and [PT, Section 3] for the details. Attached to the fibration $h : \Gamma \rightarrow B$ defined in the diagram (3) there is an exact sequence

\[
0 \rightarrow h^\ast(\omega_B) \rightarrow \Omega^1_\Gamma \rightarrow \Omega^1_{\Gamma/B} \rightarrow 0.
\]

Since $h$ is a smooth map the sheaf $\Omega^1_{\Gamma/B}$ is locally free. Thus taking determinants $\Omega^1_{\Gamma/B} \cong \omega_{\Gamma/B}$. By applying the functor $h_\ast$ to (4) we obtain:

\[
0 \rightarrow \omega_B \rightarrow h_\ast \Omega^1_\Gamma \rightarrow h_\ast \Omega^1_{\Gamma/B} \xrightarrow{\partial} R^1 h_\ast O_\Gamma \otimes \omega_B \rightarrow \ldots
\]
The connection map $\partial$ corresponds to the cup product with the class of the infinitesimal deformation at each point. Let us consider the kernel $K_\partial$. Under our hypothesis this is a locally free sheaf on $B$ and we have:

$$K_\partial \subset h_*\Omega^1_B \cong h_*\omega^1_B.$$  

Since $B$ is a disk the exact sequence

$$0 \longrightarrow \omega_B \longrightarrow h_*\Omega^1_B \longrightarrow K_\partial \longrightarrow 0$$

splits and the splitting pointwise corresponds to the choice of a lifting of the differential forms killed by the infinitesimal deformation. Then the adjoint map is the following

$$\text{adj} : \Lambda^2 K_\partial \otimes T_B \hookrightarrow \Lambda^2 h_*\Omega^1_B \otimes T_B \rightarrow h_* \left( \Lambda^2 \Omega^1_B \right) \otimes T_B \cong h_*\omega^1_B,$$

where the first inclusion is a consequence of the splitting in (6) and depends on it. The last isomorphism comes from contraction with tangent vectors.

Observe that the locally free sheaf $E_B \subset \pi_! \Omega^1_{AB}$ constructed by means of the integration is a subsheaf of $K_\partial$ and that the adjoint map restricted to $\Lambda^2 E_B$ gives the map:

$$\text{adj}_E := \text{adj}|_{\Lambda^2 E_B} : \Lambda^2 E_B \otimes T_B \rightarrow h_*\omega^1_B.$$  

By composing with

$$h_*\omega^1_B \longrightarrow (h_*\omega^1_B)/E_B$$

we obtain an “adjoint class map”, which is independent on the lifting and fiberwise gives the adjoint class.

**Remark 5.2.** It is worthy to notice that the lifting to $\Lambda^2 h_*\Omega^1_B \otimes T_B$ given by the splitting (the first step in the adjoint map) can be done in this case by lifting first to $\Lambda^2 \pi_* \Omega^1_{AB} \otimes T_B$ due to the diagram:

$$0 \longrightarrow \omega_B \longrightarrow \pi_* \Omega^1_{AB} \longrightarrow \pi_* \Omega^1_{AB/B} \longrightarrow R^1 \pi_* \mathcal{O}_{AB} \otimes \omega_B$$

$$0 \longrightarrow \omega_B \longrightarrow h_*\Omega^1_B \longrightarrow h_*\Omega^1_{B/B} \longrightarrow R^1 h_* \mathcal{O}_{B} \otimes \omega_B.$$  

Notice that all the vertical arrows are injective since the morphism of relative abelian varieties over $B$, $JT \longrightarrow \mathcal{A}_{B}$ over $B$, is surjective fiber to fiber. This induces

$$\pi_* \Omega^1_{AB} \longrightarrow \pi_* \Omega^1_{AB/B} \cap K_\partial$$

$$h_*\Omega^1_B \longrightarrow K_\partial.$$  

We put everything in a more explicit form: let us consider two linearly independent (at each point of $B$) sections

$$\omega_1, \omega_2 \in H^0(B, E_B) \subset H^0(B, \pi_* \Omega^1_{AB/B}).$$
and let \( s_i \in H^0(B, \pi_*\Omega^1_{A_B}) \) be two liftings given by the split in (6). We identify \( H^0(B, \pi_*\Omega^1_{A_B}) \) with \( H^0(A_B, \Omega^1_{A_B}) \). Then we define \( \Omega := \gamma^*(s_1 \wedge s_2) \in H^0(\Gamma, \Omega^2_{\Gamma}) \), where \( \gamma : \Gamma \to A_B \) is the map in diagram (3). With this identifications the contraction given in the definition of the adjuntion form \( (7) \) can be seen as the contraction with a normal vector to \( \Gamma \).

Summarizing, putting together these subsections 5.2 and 5.3 we have proved the following:

**Proposition 5.3.** Given a general point \( y \in \mathcal{U} \) and a general decomposable \((2,0)\)-form \( \omega_1 \wedge \omega_2 \) on \( A_y \) there exists a disk \( B \) centered at \( y \) contained in \( \mathcal{U} \) such that the form extends to a family of \((2,0)\)-forms on \( A_B \). Let \( \Gamma = \mathcal{C} \times_B \mathcal{U} \), then the pullback \( \Omega \) of this form to \( \Gamma \) contracted with a normal vector to the fiber at \( y \) computes an adjoint form on \( \Gamma_y = C_y \).

5.4. **Connection between the adjoint form and the correspondence.**

In this subsection we finish the proof of the Theorem (4.1).

We come back to the diagram (3) and work over the analytic surface \( \Gamma \):

\[
\begin{array}{c}
A_B \\
g \\
\downarrow \\
A_\Gamma \xrightarrow{\tilde{h}} A_B \\
\downarrow \\
\Gamma \\
\end{array}
\]

and we pull-back the cycle \( Z \) from \( A_\mathcal{C} \) to \( A_\Gamma \) through the map \( \Gamma \to \mathcal{C} \). We still denote by \( Z \) the resulting cycle.

Let \( \Omega' = \tilde{h}^*(s_1 \wedge s_2) \in H^0(A_\Gamma, \Omega^2_{A_\Gamma}) \) and let \( \Omega = \gamma^*(s_1 \wedge s_2) \) be the form given by Proposition 5.3. Combining Propositions (3.1) and (3.2) we get the vanishing:

\[
(s_+)^*(\Omega') + (s_-)^*(\Omega') - 2s_0^*(\Omega') = 0.
\]

Observe that the composition:

\[
\Gamma \xrightarrow{s_+} A_\Gamma \xrightarrow{\tilde{h}} A_B
\]

is simply the original family of curves \( \gamma : \Gamma \to A_B \) over \( B \). Therefore

\[
\Omega = s_+^*(\Omega') \in H^0(\Gamma, \Omega^2_{\Gamma}).
\]

Then, according with Proposition (6.3), contracting with the pull-back of a tangent vector to \( B \) gives a representative of the adjoint class. An almost identical computation can be done with \( s_- \) since \(-1_{A_\Gamma} \) acts trivially on the \((2,0)\)-forms. Finally

\[
\Gamma \xrightarrow{s_0} A_\Gamma \xrightarrow{\tilde{h}} A_B
\]

is the relative 0-map over \( B \) and thus \( s_0^*(\Omega') = 0 \).

Contracting with the pull-back of a tangent vector to \( B \) in \( y \in B \), this gives the vanishing of the adjoint class on \( \Gamma_y = C_y \). Therefore, since due to Remark (5.1) we can apply Proposition (4.3), the tangent vector to \( B \), seen
as a tangent vector to the original family on $\mathcal{U}$ appears in the Kernel of the differential of $\Phi: T \rightarrow \text{Sym}^2 H^{1,0}(A_b)^\ast$. This is a contradiction since $b$ is generic in $\mathcal{U}$ and $\Phi$ is generically finite. This finishes the proof of the main theorem. □

**Remark 5.4.** Since the Voisin set $V_2$ can be seen as the preimage of the rational orbit of the image of the origin in the corresponding Kummer variety our Theorem gives a bound on the dimension of the locus of Kummer varieties where these orbits are positive dimensional and “non-degenerate” (that is, the preimage in the abelian variety generates the abelian variety itself).

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