CONSTRUCTION OF MULTI-BUBBLE BLOW-UP SOLUTIONS TO THE 
\(L^2\)-CRITICAL HALF-WAVE EQUATION

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ABSTRACT. In this paper, we consider the \(L^2\)-critical half-wave equation in one space dimension. Given arbitrary \(K\) distinct points, we construct multi-bubble blow-up solutions with the mass concentrating at these points separately. This result exhibits an interesting multi-bubble structure in the nonlocal setting, where some new ingredients are required to overcome the difficulty arising from the strong interaction between the bubbles and the interplay between the nonlocal dispersion and a localization procedure.

1. Introduction

1.1. Setting of the problem. In this paper, we are concerned with the \(L^2\)-critical half-wave equation in one space dimension:

\[
\begin{aligned}
    i\partial_t u &= D u - |u|^2 u, \\
    u(t_0, x) &= u_0(x) \in H^2(\mathbb{R}), \quad u: I \times \mathbb{R} \to \mathbb{C}.
\end{aligned}
\]  

(1.1)

Here, \(I \subseteq \mathbb{R}\) is an interval containing the initial time \(t_0 < 0\), and the square root of Laplacian operator \(D = (-\Delta)^{\frac{1}{2}}\) can be defined via Fourier transform as

\[
\widehat{Df}(\xi) = |\xi|\hat{f}(\xi).
\]  

(1.2)

The evolution problem (1.1) has been first introduced by Laskin [31] in the context of quantum mechanics, generalizing the Feynman path integral from the Brownian-like to Lévy-like quantum mechanical paths. It also arises in various other physical settings, including turbulence phenomena, wave propagation, continuum limits of lattice systems and models for gravitational collapse in astrophysics, see e. g. [6, 13, 17, 28, 32] and references therein.

Apart from its applications in physics, equation (1.1) is of considerable interest from the PDE point of view. It can be seen as a canonical model for an \(L^2\)-critical PDE with nonlocal dispersion given by a fractional power of the Laplacian. On one hand, equation (1.1) shares similar structures with the classic \(L^2\)-critical focusing nonlinear Schrödinger equations

\[
\begin{aligned}
    i\partial_t u + \Delta u + |u|^4 u &= 0, \\
    (t, x) &\in \mathbb{R} \times \mathbb{R}^d,
\end{aligned}
\]  

(1.3)

so that it can exhibit rich nondispersive dynamics, such as solitary waves, turbulence phenomena, and singularity formation. On the other hand, several properties of (1.3) which are key elements in the quantitative descriptions of long time dynamics no longer hold for (1.1) due to the nonlocal nature of the operator, such as the Galilean invariance, pseudo-conformal invariance and the exponential decay property of the ground state. The aim of this paper is to the explore the existence of multi-bubble structure in the nonlocal setting, where the effect of the strong interaction between bubbles along with the nonlocal dispersion makes the analysis rather subtle.

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We first recall some known results for the half-wave equation. The Cauchy problem for (1.1) is locally well-posed in the energy space $H^\frac{1}{2}(\mathbb{R})$, see e.g. [29]. For every initial datum $u_0 \in H^\frac{1}{2}(\mathbb{R})$, there exists a unique solution $u \in C([t_0, T); H^\frac{1}{2}(\mathbb{R}))$ to (1.1) with $t_0 < T \leq +\infty$ denoting the maximal forward time of existence. In addition, the following blow-up alternative property holds:

$$ T < +\infty \quad \text{implies} \quad \lim_{t \to T^-} \|u(t)\|_{H^\frac{1}{2}} = \infty. $$

The property for the backward direction is similar. Moreover for $s > \frac{1}{2}$, additional $H^s$ regularity on the initial data is propagated by the flow. We should also mention that the well-posedness theory of the general fractional Schrödinger equation (1.10) has attracted much attention in recent years, see e.g. [3, 7, 10, 18, 21, 22, 24, 25] and references therein.

Equation (1.1) also admits a number of symmetries and conservation laws. It is invariant under the translation, scaling, phase rotation, i.e., if $u$ solves (1.1), then so does

$$ v(t, x) = \Lambda_0^\frac{1}{2} u(\lambda_0 t + t_0, \lambda_0 x + x_0) e^{i\gamma_0}, $$

with $v(0, x) = \Lambda_0^\frac{1}{2} u_0(\lambda_0 x + x_0) e^{i\gamma_0}$, where $(\lambda_0, t_0, x_0, \gamma_0) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. In particular, the $L^2$-norm of solutions is preserved under the symmetries above, and thus (1.1) is called the $L^2$-critical equation.

The conservation laws related to (1.1) contain

- **Mass**: $M(u(t)) := \int_{\mathbb{R}} |u(t)|^2 dx = M(u_0)$.
- **Energy**: $E(u(t)) := \frac{1}{2} \int_{\mathbb{R}} |u(t)|^2 dx - \frac{1}{4} \int_{\mathbb{R}} |\nabla u(t)|^2 dx = E(u_0)$.
- **Momentum**: $P(u) := \text{Im} \int_{\mathbb{R}} \nabla u \bar{u} dx = P(u_0)$.

The long time dynamics of solutions to (1.1) is closely related to the ground state $Q$, which is the unique positive even solution to

$$ DQ + Q - Q^3 = 0, \quad Q \in H^\frac{1}{2}(\mathbb{R}). \quad (1.4) $$

The existence of $Q$ follows from standard variational arguments, however the uniqueness of $Q$ is an intricate problem since shooting arguments and ODE techniques are not applicable to the nonlocal operator, see [15, 16]. As a particularly intriguing feature of the fractional Laplacian, the ground state only exhibits a slow algebraic decay ([15, 27]), i.e.,

$$ Q(x) \sim \frac{1}{1 + |x|^2}. \quad (1.5) $$

Moreover, by following the approach of Weinstein [42], the ground state is related with the best constant of a sharp Gagliardo-Nirenberg type inequality

$$ \|f\|_{L^4}^4 \leq \frac{\|f\|_{L^2}^2 \|D^\frac{s}{2} f\|_{L^2}^2}{2 \|Q\|_{L^2}^2}, \quad \forall f \in H^s(\mathbb{R}). $$

Thus a standard argument shows that, if the initial datum has a subcritical mass, i.e., $\|u_0\|_{L^2}^2 < \|Q\|_{L^2}^2$, then the corresponding solution exists globally in time. However, unlike (1.3), where Dodson [12] proved that the subcritical mass condition actually implies scattering at both time direction, in the fractional case, there exist non-scattering traveling waves with arbitrarily small mass to (1.1), see e.g. [4, 29].
When the initial datum has a critical mass, i.e., \( \|u_0\|_{L^2}^2 = \|Q\|_{L^2}^2 \), the corresponding solution may form singularities in finite time. However, different from the \( L^2 \)-critical NLS (1.3), the existence of finite time blow-up solution with this minimal mass is a nontrivial result since the absence of the pseudo-conformal invariance for (1.1). The construction of minimal mass blow-up solution to (1.1), was initiated by Krieger, Lenzmann and Raphaël [29], in which the authors provide a robust dynamical approach for the construction of minimal blow-up elements in a setting of nonlocal dispersion. This robust modulation argument has been developed by Raphaël and Szeftel [39] for inhomogeneous nonlinear Schrödinger equations. Recently, the construction of minimal blow-up solutions to \( L^2 \)-critical half-wave equations has been generalized by Georgiev and Li [19] in two dimensions, and [20] in three dimensions with an additional radial assumption. Moreover, these minimal mass blow-up solutions admit the blow-up speed that \( \|D^2 u(t)\|_{L^2} \sim \frac{1}{t} \) as \( t \to T^- \).

When the mass of initial data is slightly above the critical mass, a different blow-up scenario has been observed by Lan [30]. In [30] the author studies the general \( L^2 \)-critical fractional Schrödinger equation (1.10), and proves that for \( s \) close to 1, there exist blow-up solutions with the blow-up rate \( \|D^s u(t)\|_{L^2} \leq C \sqrt{\log(t)} \) as \( t \to T^- \). In addition, different from the minimal mass blow-up solutions mentioned above, this kind of solutions is proved to be stable with respect to the perturbation of the initial data. In some sense, they are similar to the log-log blow-up solutions for \( L^2 \)-critical NLS (1.3) which have been extensively studied by Merle and Raphaël [35, 36, 37]. However for the half-wave equation (1.1), it is still unclear that if this kind of blow-up solutions exist.

For even larger mass of initial data, the formation of singularity is expected to be more complicated. For \( L^2 \)-critical fractional Schrödinger equation (1.10), general blow-up criteria for radial and non-radial initial data have been established in [5] and [11] respectively. However, to the best of our knowledge, it seems that no explicit examples of blow-up solutions to (1.1) have been constructed in this regime.

1.2. Main result. In the paper, we construct multi-bubble blow-up solutions concentrating at arbitrary \( K \) different points. This result provide a new example of blow-up solutions to (1.1) which possess even larger mass \( K\|Q\|_{L^2}^2 \). The proof combines the robust modulation method in [29] for the construction of single bubble blow-up solutions in the nonlocal setting and the localization procedure in [40] for the construction of multi-bubble blow-up solutions in the local setting. The idea, roughly speaking, is to use the single bubbles as building blocks and glue them together. However, to carry out the analysis, one will face two main difficulties. One is the strong interaction between the different bubbles, which arises from the algebraic decay property of the ground state (1.5). The other one, which is a more canonical problem in the nonlocal setting, is the interplay of the nonlocal operator and the localization functions. So that, compared with the constructions in [29] and [40], some new ingredients are needed.

The main theorem of this paper is the following.

**Theorem 1.1.** Let \( \{x_k\}_{k=1}^K \) be arbitrary \( K \) distinct points in \( \mathbb{R} \), \( \{\theta_k\}_{k=1}^K \subseteq \mathbb{R}^K \) and \( \omega > 0 \). There exists \( t_0 < 0 \) and a solution \( u \in C([t_0, 0); H^2(\mathbb{R})) \) of (1.1) which blows at time \( T = 0 \) with the asymptotic expansion

\[
  u(t, x) - \sum_{k=1}^K \frac{1}{\lambda_k^2(t)} Q \left( \frac{x - \alpha_k(t)}{\lambda_k(t)} \right) e^{i \gamma_k(t)} \to 0 \quad \text{in} \quad L^2(\mathbb{R}), \quad \text{as} \quad t \to 0^-,
\]

(1.6)
where the parameters satisfy
\[
\lambda_k(t) = \omega t^2 + O(t^{4-2s}), \quad \alpha_k(t) = x_k + O(t^{3-s}), \quad \gamma_k(t) = \frac{1}{\omega |t|} + \theta_k + O(t^{1-2s}), \quad 1 \leq k \leq K, \quad (1.7)
\]
for \( \kappa \in \mathbb{R} \) being any small positive number. Moreover, it follows that
\[
\|u\|_{L^2}^2 = K\|Q\|_{L^2}^2 \quad (1.8)
\]
and the mass concentrates exactly at these \( K \) points
\[
u^2(t) \rightharpoonup K \sum_{k=1}^{K} \|Q\|_{L^2}^2 \delta_{x=x_k}, \quad \text{as} \quad t \to 0^- . \quad (1.9)
\]

**Remark 1.2.**

(1) Theorem 1.1 extends the construction of multi-bubble solutions to the \( L^2 \)-critical NLS (1.3), which was initiated by Merle [34]. In the local setting, bubbling phenomena have been extensively studied in the past decades. We refer the reader to [14] for the construction of multi-bubbles with log-log blow-up rate, [33] for the construction of multi-bubbles concentrating at the same point and e.g. [8, 26, 38, 40] for the construction of multi-bubbles in more general situations. However, to our knowledge, only few papers have dealt with the construction of multi-bubbles in the nonlocal setting.

(2) Equation (1.1) is a special case of the following \( L^2 \)-critical fractional nonlinear Schrödinger equations
\[
i\partial_t u = (\Delta)^s u - |u|^\frac{4s}{d} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad s \in \left[\frac{1}{2}, 1\right). \quad (1.10)
\]
Here in this paper, we focus on the case that \( s = \frac{1}{2} \) and \( d = 1 \), this is because the restriction is critical with respect to several aspects of the problem. On one hand, the absence of any smoothing properties for the propagator \( e^{-itD} \) is a delicate issue. On the other hand, when \( s = \frac{1}{2} \) and \( d = 1 \), (1.10) admits the strongest interaction between the bubbles, which brings challenge to the construction of multi-bubble solutions. In fact, the interactions are determined by the decay rate of the ground state to the fractional elliptic problem
\[
(\Delta)^s Q + Q - |Q|^\frac{4s}{d} Q = 0, \quad (1.11)
\]
such that
\[
Q(x) \sim \frac{1}{1 + |x|^{d+2s}}. \quad (1.12)
\]
Thus the construction can be extended to the generalized problem (1.10) (with weaker interactions), except for the technical problem of the nonsmoothness of nonlinearity.

(3) From (1.7), one can see that the asymptotic frequencies of the scaling parameters \( \lambda_k \) are assumed to be the same, which is a technical assumption for simplicity. In fact, we can construct solutions such that the asymptotic frequencies of the scaling parameters are different, i.e. \( \lambda_k(t) \sim \omega_k t^2 \), provided that the variations are small. We refer the reader to [40] for similar assumptions on the frequencies.

**1.3. Comments on the proof.** As mentioned above, the proof adapts the strategy developed in [29] and [40], which involves a soft compactness argument using the reversibility of the flow, and a monotone functional, particularly constructed in the context of the multi-bubble case, to integrate the flow backwards from the singularity.

In the following, we briefly explain how difficulties arise in the adaption of the modulation argument.
The first issue is the strong interaction between the bubbles. According to the slow algebraic decay of the ground state (1.5), an additional error roughly of the size $O(t^\delta)$ appears in the analysis, compared to the single bubble case [29]. With the error term of this leading order, we construct approximate blow-up profiles $Q_k$ solving equations, $1 \leq k \leq K$,

$$-rac{i}{2} b_k^2 \partial_{x_k} Q_k - i b_k v_k \partial_{v_k} Q_k + i b_k \Lambda Q_k - iv_k Q_k' - D Q_k - Q_k + |Q_k|^2 Q_k = O(b_k^4 + v_k^2).$$

Let’s mention that this approximation is weaker than that in the single bubble case [29], see Remark 3.2. This error also limits the best decay estimate one can obtain for the remainder $R$, see Remark 5.2.

The other main problem is the interplay between the nonlocal dispersion with the localization procedure introduced in [40]. When studying the localized quantities such as the localized mass and the localized momentum, we will frequently use the pointwise characterization of the nonlocal operator along with the integration by parts formula in order to decouple the nonlocal operator with localization functions. The most technical part of the proof lies in the monotonicity of the generalized energy. In order to decouple the generalized energy into $K$ localized parts with sufficiently high order error, we have to treat the nonlocal operator and localization functions very carefully and take advantage of the decay coming from the derivatives of the cut-off function.

The rest of this paper is organized as follows. In Section 2 we collect some preliminary lemmas that will used in the analysis. In Section 3 we first construct approximate blow-up profiles, then we establish the geometrical decomposition for the solutions and obtain modulation estimates for the parameters. Section 4 is devoted to establish bootstrap estimates for the remainder and parameters. To this end, we study several localized quantities and an important monotone functional. In Section 5, we prove Theorem 1.1 by using the bootstrap estimates and a compactness argument. Finally, the proof of Lemma 2.8 is contained in Appendix A.

1.4. Notation and conventions. $\hat{f}$ denotes the Fourier transform of function $f$. The general fractional Laplacian operator $D^s = (-\Delta)^{\frac{s}{2}}$ for $s \geq 0$ can be defined via Fourier transform as

$$\widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

We use the standard (non)homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R})$ and $H^s(\mathbb{R})$ respectively, for $s \in \mathbb{R}$. $L^p(\mathbb{R})$ denotes the space of $p$-integrable complex-valued functions, $\langle \nu, w \rangle := \int_{\mathbb{R}} \nu(x) \overline{w}(x) dx$ denotes the inner product of the Hilbert space $L^2(\mathbb{R})$, and $\mathcal{S}(\mathbb{R})$ denotes the Schwartz space. We also use the notation $\langle x \rangle := (1 + |x|^2)^{1/2}$.

In the following, we sometimes use the multi-variable calculus notation such as $\nabla f = f'$ and $\Delta f = f''$ for functions $f : \mathbb{R} \to \mathbb{R}$ to improve the readability of certain formulae derived below.

We use $f = O(g)$ to denote $|f| \leq C|g|$, where the positive constant $C$ is allowed to depend on universally fixed quantities only. Through out the paper, the positive constant $C$ may change from line to line.

2. Preliminaries

In this section, we introduce some notations and collect some preliminaries used in this paper.

2.1. Localization functions. We first introduce the localization functions which will be frequently used in the construction. Let $\{x_k\}_{k=1}^K$ be the $K$ distinct points defined as in Theorem 1.1 and denote

$$\sigma := \frac{1}{12} \min_{1 \leq k \leq K-1} \{ (x_{k+1} - x_k) \} > 0.$$ (2.1)
Let $\Phi : \mathbb{R} \to [0, 1]$ be a smooth function such that $|\Phi'(x)| \leq C\sigma^{-1}$ for some $C > 0$, $\Phi(x) = 1$ for $x \leq 4\sigma$ and $\Phi(x) = 0$ for $x \geq 8\sigma$. The localization functions $\Phi_k$, $1 \leq k \leq K$, are defined by

\[
\Phi_1(x) := \Phi(x - x_1), \quad \Phi_K(x) := 1 - \Phi(x - x_{K-1}), \\
\Phi_k(x) := \Phi(x - x_k) - \Phi(x - x_{k-1}), \quad 2 \leq k \leq K - 1.
\] (2.2)

Note that, the partition of unity holds: $1 \equiv \sum_{k=1}^{K} \Phi_k$.

2.2. **The fractional Laplacian operator.** Next we recall some fundamental properties of the fractional Laplacian operator. The following lemma provides a pointwise characterization of the fractional Laplacian operator, will be fundamental.

**Lemma 2.1** ([9]). Let $s \in (0, 1)$ and $f \in \mathcal{S}(\mathbb{R})$,

\[
(-\Delta)^s f(x) = C(s)P.V. \int_{\mathbb{R}} \frac{f(x + y) - f(x)}{|y|^{1+2s}} dy = -\frac{1}{2} C(s) \int_{\mathbb{R}} \frac{f(x + y) + f(x - y) - 2f(x)}{|y|^{1+2s}} dy,
\] (2.3)

where the normalization constant is given by

\[
C(s) = \left( \int_{\mathbb{R}} \frac{1 - \cos x}{|x|^{1+2s}} dx \right)^{-1}.
\] (2.4)

The equivalence of the above definition of the fractional Laplacian with (1.13) can also be found in [9]. We then recall the following lemma, which provides a formula for the fractional Laplacian of the product of two functions, and will be frequently used later.

**Lemma 2.2** ([2]). Let $s \in (0, 1)$ and $f, g \in \mathcal{S}(\mathbb{R})$, then we have

\[
(-\Delta)^s (fg) - (-\Delta)^s f g = C(s) \int \frac{(x+y)(g(x+y) - g(x)) - f(x+y)(g(x+y) - g(x-y))}{|y|^{1+2s}} dy,
\] (2.5)

and

\[
(-\Delta)^s (fg) - (-\Delta)^s f g - f(-\Delta)^s g = -C(s) \int \frac{(f(x+y) - f(x))(g(x+y) - g(x))}{|y|^{1+2s}} dy,
\] (2.6)

where $C(s)$ is defined as in (2.4).

In the following lemma, we collect some Sobolev inequalities that will be used to control high order terms in the expansion of nonlinearity.

**Lemma 2.3.** (i) For $0 < s < \frac{1}{2}$,

\[
H^{s}(\mathbb{R}) \hookrightarrow L^{2^{-s}}(\mathbb{R}).
\]

(ii) For $s = \frac{1}{2}$,

\[
H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^{p}(\mathbb{R}), \quad \forall p \in [2, +\infty).
\]

(iii) For $s > \frac{1}{2}$,

\[
H^{s}(\mathbb{R}) \hookrightarrow L^{p}(\mathbb{R}), \quad \forall p \in [2, +\infty].
\]

(iv) For $s_1 \leq s \leq s_2$, the following interpolation estimate holds

\[
\|u\|_{H^{s}(\mathbb{R})} \leq \|u\|_{H^{s_1}(\mathbb{R})}^{1-\theta} \|u\|_{H^{s_2}(\mathbb{R})}^\theta \quad \text{with} \quad s = (1-\theta)s_1 + \theta s_2.
\]
(v) For $s > \frac{1}{2}$ and $f \in H^s(\mathbb{R})$, there exists $C > 0$ depending on $s$ such that

$$
\|f\|_{L^\infty} \leq C \|f\|_{H^s_{\text{w}}} \left( \ln(2 + \frac{\|f\|_{H^s_{\text{w}}}}{\|f\|_{H^s_{\text{w}}}}) \right)^{\frac{1}{2}}.
$$

(2.7)

The proofs of the standard Sobolev embeddings and interpolation estimate (i) – (iv) can be found in textbooks, see [1] for example. And we refer the reader to [29, Appendix D] for the proof of (v).

We will often use the following fractional chain rule without explicit mentioning.

**Lemma 2.4.** ([1]) Let $0 < s \leq 1$ and $1 \leq p, p_1, p_2 < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Then for any $f \in C^1$, the exists $C > 0$ such that

$$
\|D^s f(u)\|_{L^{p_2}} \leq C \|f'(u)\|_{L^{p_1}} \|D^s u\|_{L^{p_2}}.
$$

We recall the following commutator estimate that will be often used.

**Lemma 2.5.** ([29, Appendix E]) For $0 \leq s \leq 1$, and $f, g \in \mathcal{S}(\mathbb{R})$,

$$
\|D^s(fg) - f D^s g\|_{L^2} \leq C \min\{\|D^s f\|_{L^2}, \|\hat{g}\|_{L^1}, \|\hat{f}\|_{L^1}, \|\hat{g}\|_{L^2}\}.
$$

(2.8)

We also recall the classic interpolation theory in the following lemma.

**Lemma 2.6.** ([41, Lemma 23.1]) Let $f, g \in \mathcal{S}(\mathbb{R})$, $T : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be a linear operator with the following property holds:

$$
|\langle f, T g \rangle| \leq C\|f\|_{L^2}\|g\|_{H^s} \quad \text{and} \quad |\langle f, T g \rangle| \leq C\|f\|_{H^s}\|g\|_{L^2},
$$

for some $C > 0$. Then we have

$$
|\langle f, T g \rangle| \leq C\|f\|_{H^s_{\text{w}}}\|g\|_{H^s_{\text{w}}}.
$$

2.3. **Coercivity of linearized operators.** We first recall the fact that the ground state is a smooth function with the decay estimate [15, 27]

$$
|Q(x)| + |\Lambda Q(x)| + |\Lambda^2 Q(x)| \leq \frac{C}{1 + |x|^2},
$$

(2.9)

for some $C > 0$, with the $L^2$-critical scaling operator defined by

$$
\Lambda f(x) := \frac{1}{2} f + x f''.
$$

Let $L = (L_+, L_-)$ be the linearized operator around the ground state, defined by

$$
L_+ := D + I - 3Q^2, \quad L_- := D + I - \hat{Q}^2.
$$

(2.10)

The generalized null space of $L$ is spanned by $\{Q, G_1, S_1, \nabla Q, \Lambda Q, \rho\}$, where $S_1$ is the unique even solution to

$$
L_- S_1 = \Lambda Q,
$$

(2.11)

$G_1$ is the unique odd solution to

$$
L_- G_1 = -\nabla Q,
$$

(2.12)

and $\rho$ is the unique even solution to

$$
L_+ \rho = S_1.
$$

(2.13)

Thus, we have the following algebraic property:

$$
L_+ \nabla Q = 0, \quad L_+ \Lambda Q = -2Q, \quad L_+ \rho = S_1,
$$

$$
L_- Q = 0, \quad L_- G_1 = -\nabla Q, \quad L_- S_1 = \Lambda Q.
$$

(2.14)
For any complex-valued $H^\frac{1}{2}$ function $f = f_1 + if_2$ in terms of the real and imaginary parts, we set
\[(Lf, f) := \int f_1 \overline{f}_1 dx + \int f_2 \overline{f}_2 dx,\]
and define the scalar products along all the unstable directions in the null space
\[\text{S cal}(f) := \langle f_1, Q \rangle^2 + \langle f_1, G_1 \rangle^2 + \langle f_1, S_1 \rangle^2 + \langle f_2, \nabla Q \rangle^2 + \langle f_2, \Lambda Q \rangle^2 + \langle f_2, \rho_1 \rangle^2.\] (2.15)

Let’s first recall the following coercivity estimate, which is an extension of the standard coercivity estimate ([42, Theorem 2.5]) in the context of NLS.

**Lemma 2.7. (Coercivity estimate)** There exists positive constant $C > 0$, such that
\[(Lf, f) \geq C||f||^2_{H^\frac{1}{2}} - \frac{1}{C}\text{S cal}(f), \quad \forall f \in H^\frac{1}{2}(\mathbb{R}).\] (2.16)

The proof is based on the non-degeneracy of the linearized operator ([15]) and a variational argument, see [29, Lemma B.4] for more details. For the construction of multi-bubbles, we also establish a localized version of coercivity estimate in the following lemma.

**Lemma 2.8. (Localized coercivity estimate)** Let $0 < a < 1$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a positive smooth radial function, such that $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x) = |x|^{-d}$ for $|x| \geq 2$, $0 < \phi \leq 1$. Set $\phi_A(x) := \phi(\frac{x}{A})$, $A > 0$. Then, there exists $C > 0$ such that for $A$ large enough we have
\[\int (|f|^2 + |D^\frac{1}{2}f|^2)\phi_A - 3Q^2 f_1^2 - Q^2 f_2^2 dx \geq C \int (|D^\frac{1}{2}f|^2 + |f|^2)\phi_A dx - \frac{1}{C}\text{S cal}(f), \quad \forall f \in H^\frac{1}{2}(\mathbb{R}).\] (2.17)

The proof of Lemma 2.8 is postponed to Appendix A.

### 2.4. Decoupling estimates.

In many situations, the interactions between different bubbles, and the interplay between a bubble and localization functions with the support away from the center of the bubble, can be reduced to the following estimates.

**Lemma 2.9.** Assume that $f(x), g(x) \in C^\infty(\mathbb{R})$ have the decay estimate $|f(x)| + |g(x)| \leq \frac{C_1}{1 + |x|^\alpha}$, for some constant $C_1 > 0$, and $h(x) \in C^\infty(\mathbb{R})$ with $0 \leq h(x) \leq 1$ and $h(0) = 0$. And let $\epsilon > 0$ be a small positive number. Then there exists $C > 0$ such that
\[
| \int f(x)g(x + \frac{1}{\epsilon})dx | \leq C\epsilon^2,
\]
\[
\left( \int f\left(\frac{x}{\epsilon}\right)^2 h(x)dx \right)^\frac{1}{2} \leq C\epsilon^2.
\]

### 3. Modulation estimates

In this section, we first introduce the construction of the approximate blow-up profiles. Then we establish the standard geometrical decomposition for solutions to (1.1) and obtain preliminary estimates for modulation equations of parameters. This is the starting point to carry out the modulation analysis.
3.1. **Approximate blow-up profiles.** Since the absence of pseudo-conformal invariance to (1.1), one cannot obtain an explicit blow-up solution directly from the ground state (1.4). Thus in [29], a high order approximate blow-up profile $Q^p_k$ has been constructed to replace $Q$. In this subsection, we introduce the approximate blow-up profiles in the context of multi-bubble case, which will be the building block to construct approximate solutions later. For convenience, we identify a complex-valued function $f : \mathbb{R} \to \mathbb{C}$ with the vector valued function $f : \mathbb{R} \to \mathbb{R}^2$ as

$$f = \begin{bmatrix} \text{Re} f \\ \text{Im} f \end{bmatrix}.$$

Assume $b_k$ and $v_k$ are small positive real number, $1 \leq k \leq K$. Then we have the existence of approximate blow-up profiles $Q_k$ in the following lemma.

**Lemma 3.1** (Approximate blow-up profiles). For $1 \leq k \leq K$, there exist smooth functions $Q_k(x)$ of the form

$$Q_k := Q + b_k R_{1,0} + v_k R_{0,1} + b_k v_k R_{1,1} + b_k^2 R_{2,0} + b_k^2 R_{3,0},$$

satisfy equations

$$-\frac{i}{2} b_k^2 \partial_t Q_k - i b_k v_k \partial_t Q_k + i b_k \Lambda Q_k - i v_k Q_k' - DQ_k - Q_k + |Q_k|^2 Q_k = -\Psi_k. \quad (3.1)$$

Here, the functions $\{R_{i,j}\}_{1 \leq i \leq 3, 0 \leq j \leq 1}$ have the following symmetry structure

$$R_{1,0} = \begin{bmatrix} 0 \\ \text{even} \end{bmatrix}, \quad R_{0,1} = \begin{bmatrix} 0 \\ \text{odd} \end{bmatrix}, \quad R_{1,1} = \begin{bmatrix} \text{odd} \\ 0 \end{bmatrix}, \quad R_{2,0} = \begin{bmatrix} \text{even} \\ 0 \end{bmatrix}, \quad R_{3,0} = \begin{bmatrix} 0 \\ \text{even} \end{bmatrix},$$

and regularity and decay bounds

$$||R_{i,j}||_{H^m} + ||\Lambda R_{i,j}||_{H^m} + ||\Lambda^2 R_{i,j}||_{H^m} \leq C(m), \quad m \in \mathbb{N},$$

$$|R_{i,j}| + |\Lambda R_{i,j}| + |\Lambda^2 R_{i,j}| \leq C(x)^{-2}, \quad x \in \mathbb{R},$$

Moreover, the remainder terms $\Psi_k$ satisfy, for $m \in \mathbb{N}$ and $\nu = 0, 1, 2$,

$$||\Psi_k||_{H^m} \leq C(m)(b_k^4 + v_k^2), \quad |\Psi_k^{(\nu)}(x)| \leq C(b_k^4 + v_k^2)(x)^{-2}, \quad x \in \mathbb{R}. \quad (3.2)$$

**Remark 3.2.** The parameters $b_k$ and $v_k$ are assumed to be small. In fact as we shall see in the bootstrap procedure in Section 4, the parameters $b_k, v_k$ depend on $t$, satisfying a prior bounds that $b_k(t) \sim |t|$ and $v_k(t) \sim b_k^2 \sim t^2$, for $t$ close to 0.

Let’s mention the difference between the approximate profiles $Q_k$ with the one constructed in [29, Proposition 4.1]. In our case, we expand $Q_k$ to the order roughly of $O(b_k^5)$, so that it solves equation (3.1) with the remainder $\Psi_k$ of the order $O(b_k^4)$. However in [29], the approximate profile $Q^p_k$ is expanded to higher order to solve equation (3.1) with the remainder of the order $O(b_k^5)$. The reason is that, in our case the strong interaction between different bubbles contribute dominant errors of order $O(t^4)$, which determines that it is not necessary for a better approximation of $\Psi_k$, see the estimate (4.60) for example.

The proof of Lemma 3.1 is similar (and simpler) with the one of Proposition 4.1 in [29], thus is omitted here for simplicity.

**Remark 3.3.** As we can see from [29], the choice of the functions $\{R_{i,j}\}$ can be

$$R_{1,0} = \begin{bmatrix} 0 \\ S_1 \end{bmatrix}, \quad R_{0,1} = \begin{bmatrix} 0 \\ G_1 \end{bmatrix}, \quad R_{1,1} = \begin{bmatrix} G_2 \\ 0 \end{bmatrix}, \quad R_{2,0} = \begin{bmatrix} S_2 \\ 0 \end{bmatrix}, \quad R_{3,0} = \begin{bmatrix} 0 \\ S_3 \end{bmatrix}.$$
with \( S_1 \) and \( G_1 \) being defined as in (2.11) and (2.12), \( G_2 \) being the unique odd solution to
\[
L_s G_2 = G_1 - \Lambda G_1 + \nabla S_1 + 2 S_1 G_1 Q,
\]
\( S_2 \) being the unique even solution to
\[
L_s S_2 = \frac{1}{2} S_1 - \Lambda S_1 + S_1^2 Q,
\]
and \( S_3 \) being the unique odd solution to
\[
L_s S_3 = -S_2 + \Lambda S_2 + 2 S_1 S_2 Q + S_1^3.
\]
Thus for \( 1 \leq k \leq K \), we have the expansion of \( Q_k \)
\[
Q_k(x) = Q(x) + ib_k S_1(x) + iv_k G_1(x) + b_k v_k G_2(x) + b_k^2 S_2(x) + b_k^3 S_3(x),
\]
and a simple calculation yields that
\[
\|Q_k\|^2_{L^2} = \|Q\|^2_{L^2} + O(b_k^4 + v_k^2).
\]

### 3.2. Geometrical decomposition and modulation equations.

For \( 1 \leq k \leq K \), define the modulation parameters by \( P_k := (\lambda_k, b_k, v_k, \alpha_k, \gamma_k) \in \mathbb{R}^5 \). We also set \( P := (P_1, \ldots, P_K) \in \mathbb{R}^{5K} \). Similarly, let \( \lambda := (\lambda_1, \ldots, \lambda_K) \in \mathbb{R}^K \) and \( \lambda := \sum_{k=1}^K \lambda_k \). Similar notations are also used for the remaining parameters. Let \( \omega > 0 \), \( T < 0 \) and define
\[
U(T, x) := \sum_{k=1}^K U_k(T, x), \quad \text{with} \quad U_k(T, x) := \lambda_k^{-\frac{1}{2}}(T)Q_k \left( T, \frac{x - \alpha_k(T)}{\lambda_k(T)} \right) e^{i\gamma_k(T)},
\]
where \( Q_k \) is defined as in Lemma 3.1 and
\[
P_k(T) = \left( \frac{\omega^2}{4} T^2, -\frac{\omega^2}{2} T, \frac{\omega^2}{4} T^2, x_k, -\frac{4}{\omega^2 T} + \theta_k \right),
\]
with \( x_k \) and \( \theta_k \) be defined as in Theorem 1.1.

**Proposition 3.4.** (Geometrical decomposition) Let \( u(t) \) be a solution to (1.1) with \( u(T) = U(T) \). For \( T \) sufficiently close to 0, there exist \( T_* < T \) and unique modulation parameters \( P \in C^1([T_*, T]; \mathbb{R}^{5K}) \), such that \( u \) can be geometrically decomposed into a main blow-up profile and a remainder
\[
u(t, x) = U(t, x) + R(t, x), \quad t \in [T_*, T],
\]
where the main blow-up profile
\[
U(t, x) = \sum_{k=1}^K U_k(t, x), \quad \text{with} \quad U_k(t, x) = \lambda_k^{-\frac{1}{2}}(t)Q_k \left( t, \frac{x - \alpha_k(t)}{\lambda_k(t)} \right) e^{i\gamma_k(t)},
\]
Moreover, for \( 1 \leq k \leq K \), the following orthogonality conditions hold on \( [T_*, T] \):
\[
\text{Re}(\overline{S}_k(t), R(t)) = 0, \quad \text{Re}(\overline{G}_k(t), R(t)) = 0,
\]
\[
\text{Im}(\nabla U_k(t), R(t)) = 0, \quad \text{Im}(\Lambda_k U_k(t), R(t)) = 0, \quad \text{Im}(\overline{\rho}_k(t), R(t)) = 0.
\]
Here, \( \Lambda_k := \frac{1}{2} + (x - \alpha_k) \cdot \nabla \) and
\[
\overline{S}_k(t, x) = \lambda_k^{-\frac{1}{2}} S_1(t, \frac{x - \alpha_k}{\lambda_k}) e^{i\gamma_k}, \quad \overline{G}_k(t, x) = \lambda_k^{-\frac{1}{2}} G_1(t, \frac{x - \alpha_k}{\lambda_k}) e^{i\gamma_k}, \quad \overline{\rho}_k(t, x) = \lambda_k^{-\frac{1}{2}} \rho_k(t, \frac{x - \alpha_k}{\lambda_k}) e^{i\gamma_k}
\]
where \( \rho_k = \rho + i\varrho_k \) with \( \rho \) being defined as in (2.13) and \( \varrho_k \) solving
\[
L_s \varrho_k = 2b_k S_1 \rho Q + b_k \Lambda \rho - 2b_k S_2 + 2v_k G_1 \rho Q + v_k \cdot \nabla \rho + v_k G_2.
\]
Remark 3.5. The geometrical decomposition in Proposition 3.4 is actually a local version, since the backwards time \( T_* \) may depend on \( T \). However, as we shall see later, by virtue of the bootstrap estimates in Theorem 4.1 below, we can establish the geometrical decomposition on a time interval \([t_0, T]\) with \( t_0 \) independent of \( T \), for \( T \) being sufficiently close to 0.

The proof of Proposition 3.4 is based on a standard fixed point argument. For the proof of Proposition 3.4 in the single bubble case, we refer the reader to [29]. The argument can be easily extended to the multi-bubble case (see e.g., [40] for the proof of a similar result in the context of NLS), thus the details of the proof is omitted for simplicity.

For \( 1 \leq k \leq K \), denote the vector of modulation equations by

\[
\text{Mod}_k := |\dot{\lambda}_k + b_k| + |\lambda_k b_k + \frac{1}{2} b_k^2| + |\dot{\lambda}_k - v_k| + |\lambda_k v_k + b_k v_k| + |\lambda_k \dot{v}_k - 1|,
\]

with \( \dot{g} := \frac{d}{dt} g \) for any \( C^1 \) function \( g \). Define \( \text{Mod} := \sum_{k=1}^{K} \text{Mod}_k \).

In the following proposition, we establish estimates for these modulation equations.

Proposition 3.6. Let \( u \) be a solution to (1.1), and the geometrical decomposition holds on \([T_*, T]\) with \( T \) close to 0. Assume the uniform smallness bound for the parameters and the remainder

\[
\lambda(t) + b(t) + v(t) + \sum_{k=1}^{K} |\alpha_k(t) - x_k| + \|R(t)\|_{H^1}^2 \ll 1.
\]

Then, there exists \( C > 0 \), such that for any \( t \in [T_*, T] \),

\[
\text{Mod}(t) \leq C \left( \sum_{k=1}^{K} |\text{Re}(U_k, R)| + \lambda \|R\|_{L^2} + \|R\|_{L^2}^2 + \|\lambda\|_{L^2}^2 \right) + \lambda^2 + b^2 + v^2. \tag{3.11}
\]

Remark 3.7. The terms \( \text{Re}(U_k, R) \) arising in the right hand side of (3.11) correspond to the remaining unstable direction, which has not been controlled through the geometrical decomposition. By exploring the localized mass, these terms will be treated in Lemma 4.3. It’s worth pointing out that unlike the single bubble case as in [29], these terms will dominate the bound for the modulation equations, which exhibits a new feature in the multi-bubble case.

Proof of Proposition 3.6. The proof is similar to the one of [40, Proposition 4.3] in the context of NLS. For the reader’s convenience, we provide the estimate for the modulation equations \( \dot{\lambda}_k + b_k \), as an example to illustrate the main arguments. Without loss of generality, we take \( k = 1 \).

By inserting the decomposition (3.8) into (1.1), we obtain the equation for the remainder \( R \)

\[
iR_t - DR + \sum_{k=1}^{K} (2|U_k|^2R + U_k^2\overline{R}) = \sum_{j=1}^{4} H_j, \tag{3.12}
\]

where

\[
H_1 = -\sum_{k=1}^{K} (i\partial_t U_k - DU_k + |U_k|^2U_k), \quad H_2 = -(|U|^2U - \sum_{k=1}^{K} |U_k|^2U_k),
\]

\[
H_3 = -(2|U|^2R + U^2\overline{R} - 2 \sum_{k=1}^{K} |U_k|^2R - \sum_{k=1}^{K} U_k^2\overline{R}), \quad H_4 = -(\overline{UR}^2 + 2U|R|^2 + |R|^2R).
\]
Moreover, by (1.4) and (3.9), for $1 \leq k \leq K$, we have the following identity
\begin{align*}
  i\partial_t U_k - DU_k + |U_k|^2 U_k = & \ e^{i\nu_k} \lambda_k^{-\frac{L}{2}} \left( i(\lambda_k \dot{v}_k + b_k v_k) \partial_v Q_k + i(\lambda_k \dot{b}_k + \frac{1}{2} b_k^2) \partial_b Q_k - i(\dot{\lambda}_k - v_k) Q'_k \right) \\
  & \ - i(\lambda_k + b_k) \Lambda Q_k - (\lambda_k \dot{v}_k - 1) Q - \Psi_k(t, \frac{x - \alpha_k}{\lambda_k}). \tag{3.13}
\end{align*}

Taking the inner product of (3.12) with $\Lambda_1 U_1$ and then taking the real part, we get
\begin{equation}
  - \text{Im}(R, \Lambda_1 U_1) + \text{Re}(-D^2 + 2|U_1|^2 R + U_1^2 \overline{R}, \Lambda_1 U_1) = \sum_{j=1}^{5} \text{Re}(\langle H_j, \Lambda_1 U_1 \rangle), \tag{3.14}
\end{equation}

with
\begin{equation}
  H_5 = \sum_{k=2}^{K} (2|U_k|^2 R + U_k^2 \overline{R}).
\end{equation}

We first consider the left hand side of (3.14). Using the orthogonality conditions (3.10), the identity (3.13) for $k = 1$ and the renormalization
\begin{equation}
  R(t, x) = \lambda_k^{-\frac{1}{2}} \varepsilon_k(t, \frac{x - \alpha_k}{\lambda_k}) e^{i\nu_k}, \quad \text{with} \quad \varepsilon_k = \varepsilon_{k,1} + i\varepsilon_{k,2},
\end{equation}

we get
\begin{align*}
  & - \text{Im}(R, \Lambda_1 U_1) = \text{Im}(R, \partial_t \Lambda_1 U_1) = - \text{Im}(\Lambda R, \partial_t U_1) \\
  & = - \lambda_1^{-1} \text{Re}(\langle \lambda_1, -Q_1 - \frac{i}{2} b_1^2 \partial_b Q_1 - ib_1 v_1 \partial_v Q_1 + ib_1 \lambda Q_1 - iv_1 Q'_1 \rangle + O(\lambda^{-1} \text{Mod} ||R||_{L^2})) \\
  & = \lambda_1^{-1} \left( - \text{Re}(\langle \varepsilon_1, \Lambda Q_1 \rangle) + b_1 \text{Im}(\langle \varepsilon_1, \Lambda^2 Q_1 \rangle) + O((\text{Mod} + b_1^2 + v_1)||R||_{L^2}) \right). \tag{3.16}
\end{align*}

Using again the renormalization (3.15) and the expansion of $Q_1$ (3.5), it follows that
\begin{align*}
  & \text{Re}(-D^2 + 2|U_1|^2 R + U_1^2 \overline{R}, \Lambda_1 U_1) = \lambda_1^{-1} \text{Re}(-D\varepsilon_1 + 2|Q_1|^2 \varepsilon_1 + Q_1^2 \overline{\varepsilon_1}, \Lambda Q_1) \\
  & = \lambda_1^{-1} \text{Re}(-D\varepsilon_1 + 2|Q_1|^2 \varepsilon_1 + 2ib_1 S_1 \overline{Q_1} + Q_1^2 \overline{\varepsilon_1}, \Lambda Q + ib_1 \Lambda S_1) + O(\lambda_1^{-1} (b_1^2 + v_1)||R||_{L^2}). \tag{3.17}
\end{align*}

Combining (3.16) and (3.17), we then get
\begin{align*}
  & - \text{Im}(R, \Lambda_1 U_1) + \text{Re}(-D^2 + 2|U_1|^2 R + U_1^2 \overline{R}, \Lambda_1 U_1) \\
  & = \lambda_1^{-1} \left( - \langle L_{-1} \varepsilon_{1,1}, \Lambda Q \rangle - b_1 \langle L_{-1} \varepsilon_{1,2}, \Lambda S_1 \rangle + 2b_1 \langle \varepsilon_{1,2}, Q_1 \Lambda Q_1 \rangle + b_1 \langle \varepsilon_{1,2}, \Lambda^2 Q_1 \rangle \right) \\
  & + O(\lambda_1^{-1} (\text{Mod} + b_1^2 + v_1)||R||_{L^2}). \tag{3.18}
\end{align*}

By using the commutator formula $[D, \Lambda] = D$ and $L_{-1} S_1 = \Lambda Q$, the following identity holds
\begin{equation}
  L_{-1} \Lambda S_1 = - S_1 + 2Q \Lambda Q S_1 + \Lambda Q + \Lambda^2 Q.
\end{equation}

This fact along with (3.18) and using $L_{-1} \Lambda Q = -Q$ yields that
\begin{align*}
  & - \text{Im}(R, \Lambda_1 U_1) + \text{Re}(-D^2 + 2|U_1|^2 R + U_1^2 \overline{R}, \Lambda_1 U_1) \\
  & = \lambda_1^{-1} \left( \langle \varepsilon_{1,1}, Q \rangle + b_1 \langle \varepsilon_{1,2}, S_1 \rangle - b_1 \langle \varepsilon_{1,2}, \Lambda Q \rangle \right) + O(\lambda_1^{-1} (\text{Mod} + b_1^2 + v_1)||R||_{L^2}) \\
  & = \lambda_1^{-1} \left( \text{Re}(\langle \varepsilon_1, Q_1 \rangle) + b_1 \langle \varepsilon_{1,2}, S_1 \rangle + O(\lambda_1^{-1} (\text{Mod} + b_1^2 + v_1)||R||_{L^2}) ight) \\
  & = \lambda_1^{-1} \left( \text{Re}(\langle R, U_1 \rangle) + O(\lambda_1^{-1} (\text{Mod} + b_1^2 + v_1)||R||_{L^2}) \right). \tag{3.19}
\end{align*}

Here, we use the fact that $b_1 \langle \varepsilon_{1,2}, \Lambda Q \rangle = \text{Im}(\langle \varepsilon_1, \Lambda Q_1 \rangle) + b_1^2 \langle \varepsilon_{1,1}, \Lambda S_1 \rangle = O(b_1^2 ||R||_{L^2})$, which follows from the orthogonality conditions (3.10) that $\text{Im}(\langle \varepsilon_1, \Lambda Q_1 \rangle) = \text{Im}(\langle R, \Lambda_1 U_1 \rangle) = 0.$
We then consider the right hand side of (3.14). The first term can be expanded as
\[
\text{Re}(H_1, \Lambda_1 U_1) = -\text{Re}(i\partial_t U_1 - DU_1 + |U_1|^2 U_1, \Lambda_1 U_1) - \sum_{k=2}^{K} \text{Re}(i\partial_t U_k - DU_k + |U_k|^2 U_k, \Lambda_1 U_1).
\]
Recall that \( L_- > 0 \) on \( Q^+ \), so we can define \( e_1 := \langle S_1, \Lambda Q \rangle = \langle S_1, L_- S_1 \rangle > 0 \). This fact along with the identity (3.13) for \( k = 1 \) and (3.2), yields that
\[
-\text{Re}(i\partial_t U_1 - DU_1 + |U_1|^2 U_1, \Lambda_1 U_1) = \lambda_1^{-1} \left( e_1(\lambda_1 \dot{b}_1 + \frac{1}{2} b_1^2) + O(b_1 \text{Mod} + b_1^4 + \nu_1^2) \right).
\]
And the remainder can be bounded by applying Lemma 2.9
\[
|\sum_{k=2}^{K} \text{Re}(i\partial_t U_k - DU_k + |U_k|^2 U_k, \Lambda_1 U_1)| \leq C \lambda \text{Mod}.
\]
Thus we have
\[
\text{Re}(H_1, \Lambda_1 U_1) = \lambda_1^{-1} \left( e_1(\lambda_1 \dot{b}_1 + \frac{1}{2} b_1^2) + O(b_1 \text{Mod} + b_1^4 + \nu_1^2) \right). \tag{3.20}
\]
Then the second term can be bounded by applying Lemma 2.9
\[
\text{Re}(H_2, \Lambda_1 U_1) = O(\lambda). \tag{3.21}
\]
Moreover, the third and the fifth terms which are linear in \( R \) can also be bounded by applying Lemma 2.9
\[
\text{Re}(H_3, \Lambda_1 U_1) + \text{Re}(H_5, \Lambda_1 U_1) = O(\|R\|_{L^2}). \tag{3.22}
\]
Finally, the forth term in which is nonlinear in \( R \) can be bounded by using Sobolev embeddings
\[
\text{Re}(H_4, \Lambda_1 U_1) = O(\lambda^{-1}\|\epsilon_1\|_{L^2}^3 + \lambda^{-1}\|\epsilon_1\|_{L^2}^3) = O(\lambda^{-1}\|R\|_{L^2}^2 + \|R\|_{H^2}^3). \tag{3.23}
\]
Inserting (3.19) and (3.20) to (3.23), we get
\[
|\lambda_1 \dot{b}_1 + \frac{1}{2} b_1^2| \leq C(\|\text{Re}(U_1, R)\| + \lambda\|R\|_{L^2} + \|R\|_{L^2}^2 + \lambda\|R\|_{H^2}^3 + b_1 \text{Mod} + \lambda^2 + b_1^4 + \nu_1^2),
\]
thus it yields that
\[
\sum_{k=1}^{K} |\lambda_k \dot{b}_k + \frac{1}{2} b_k^2| \leq C(\sum_{k=1}^{K} \|\text{Re}(U_k, R)\| + \lambda\|R\|_{L^2} + \|R\|_{L^2}^2 + \lambda\|R\|_{H^2}^3 + b_\text{Mod} + \lambda^2 + b_1^4 + \nu_1^2).
\]
Taking the inner products of equation (3.12) with \( i\tilde{S}_k, i\tilde{G}_k, \nabla U_k, \rho_k \), respectively, and then taking the real parts, the remaining four modulation equations can be estimated similarly. Summing these estimates together yields (3.11), thus we complete the proof of Proposition 3.6. \( \square \)

## 4. Bootstrap estimates

The target of this section is to establish bootstrap estimates (Theorem 4.1) for the remainder and parameters in the geometrical decomposition, which will be the key element to construct multi-bubble blow-up solutions in the next section. Although the strategy is in some sense similar with the previous ones as in [40], the reader will find that the nonlocal operator together with the multi-bubble structure brings challenge to the analysis, which makes this section the most technical part of the paper.

The main result of this section is the following.
Theorem 4.1 (Bootstrap estimates). Let \( \kappa, \delta \in (0, \frac{1}{2}) \) and \( \kappa + 2\delta < 1 \). There exists a uniform backwards time \( t_0 < 0 \) such that the following holds. Let \( u \) be a solution to (1.1) and admits the geometrical decomposition on \([T_*, T]\) as in Proposition 3.4, with \( t_0 < T_\ast \). Assume that for all \( t \in [T_*, T] \), \( 1 \leq k \leq K \), the following bounds for the remainder and parameters hold:

\[
\|D^2 R(t)\|_{L^2} \leq |t|^{2-k}, \quad \|R(t)\|_{L^2} \leq |t|^{3-k}, \quad \|D^{\frac{1}{2}+\delta} R\|_{L^2} \leq |t|^{1-k-2\delta}, \tag{4.1}
\]

\[
|\lambda_k(t) - \omega^2 4 t^2| \leq |t|^{4-2k}, \quad |b_k(t) + \omega^2 2 t| \leq |t|^{3-2k}, \tag{4.2}
\]

\[
|\alpha_k(t) - x_k| \leq |t|^{3-k}, \quad |v_k(t) - \omega^2 4 t^2| \leq |t|^{4-2k}, \tag{4.3}
\]

\[
|\gamma_k(t) + \frac{4}{\omega^2 t} - \theta_k| \leq |t|^{1-k}. \tag{4.4}
\]

Then, the bounds can be improved such that, for all \( t \in [T_*, T] \) and \( 1 \leq k \leq K \),

\[
\|D^2 R(t)\|_{L^2} \leq \frac{1}{2}|t|^{2-k}, \quad \|R(t)\|_{L^2} \leq \frac{1}{2}|t|^{3-k}, \quad \|D^{\frac{1}{2}+\delta} R\|_{L^2} \leq \frac{1}{2}|t|^{1-k-2\delta}. \tag{4.5}
\]

\[
|\lambda_k(t) - \omega^2 4 t^2| \leq \frac{1}{2}|t|^{4-2k}, \quad |b_k(t) + \omega^2 2 t| \leq \frac{1}{2}|t|^{3-2k}, \tag{4.6}
\]

\[
|\alpha_k(t) - x_k| \leq \frac{1}{2}|t|^{3-k}, \quad |v_k(t) - \omega^2 4 t^2| \leq \frac{1}{2}|t|^{4-2k}, \tag{4.7}
\]

\[
|\gamma_k(t) + \frac{4}{\omega^2 t} - \theta_k| \leq \frac{1}{2}|t|^{1-k}. \tag{4.8}
\]

According to (1.1), we have the equation for the remainder \( R \)

\[
i\partial_t R - DR + |u|^2 u - |U|^2 U = -\eta, \tag{4.9}
\]

where the error \( \eta \) satisfies

\[
\eta = i\partial_t U - DU + |U|^2 U. \tag{4.10}
\]

Define a quantity of the remainder

\[
X(t) := \|D^\frac{1}{2} R(t)\|_{L^2}^2 + t^2 \|R(t)\|_{L^2}^2.
\]

From bounds (4.1)-(4.4), we have the following a priori bounds for the modulation parameters, the modulation equations and the error \( \eta \), which will be frequently used in the proof of Theorem 4.1.

Lemma 4.2. By the assumptions in Theorem 4.1, there exists a positive constant \( C \) independent of \( R \) such that for all \( t \in [T_*, T] \), the following holds:

(i) For the modulation parameters, \( 1 \leq k \leq K \),

\[
C t^2 \leq \lambda_k(t) \leq \frac{t^2}{C}, \quad C |t| \leq b_k(t) \leq \frac{|t|}{C}, \quad C t^2 \leq v_k(t) \leq \frac{t^2}{C}. \tag{4.11}
\]

(ii) For the modulation equations

\[
\Mod(t) \leq C |t|^{4-k}. \tag{4.12}
\]

(iii) For the error \( \eta \)

\[
\|\eta^{(\nu)}(t)\|_{L^2} \leq C |t|^{-2-\nu}(\Mod + t^4), \quad \nu = 0, 1, 2. \tag{4.13}
\]
Then by using the identity (3.13), we have
\[
u = \sum_{k=1}^{K} (i\partial_t U_k - DU_k + |U_k|^2 U_k) + (|U|^2 U - \sum_{k=1}^{K} |U_k|^2 U_k) = I + II.
\]
Then by using the identity (3.13), we have
\[
|I^{(v)}| \leq C \sum_{k=1}^{K} \lambda_k^{-\frac{3}{2}+\nu} (\text{Mod}_k \langle \frac{x-a_k}{\lambda_k} \rangle^{-2} + |\Psi_k|),
\]
which along with (3.2) and (4.11) yields that
\[
\|I^{(v)}\|_{L^2} \leq C|t|^{-2-2\nu}(\text{Mod} + t^4).
\]
And from Lemma 2.9 and (4.11), we have
\[
\|II^{(v)}\|_{L^2} \leq C|t|^{-2-2\nu} t^4.
\]
Combing the above estimates, the bound (4.13) is thus obtained. \qed

4.1. **Localized mass and momentum.** We first study the quantities \(\int |u|^2 \Phi_k dx\), which we call the *localized mass*, to obtain the estimates for the unstable directions \(\text{Re} \langle U_k, R \rangle\), \(1 \leq k \leq K\). Note that these quantities are no longer conserved.

**Lemma 4.3 (Estimate of \(\text{Re} \langle U_k, R \rangle\)).** For all \(t \in [T_*, T]\) and \(1 \leq k \leq K\), we have
\[
2\text{Re} \int \bar{U}_k R dx + \int |R(t)|^2 \Phi_1 dx = O(|t|^{4-\nu}). \tag{4.14}
\]

**Remark 4.4.** The instable directions \(\text{Re} \langle U_k, R \rangle\) appear in serval important estimates, such as the coercivity of the linearized operator (2.16) and the estimate for the modulation equations (3.11). However, unlike the single bubble case for (1.1) nor the multi-bubble case for the local problem (1.3), the analysis to obtain (4.14) is more delicate. This reflects an interesting feature of the multi-bubble structure of the nonlocal problem.

**Proof of Lemma 4.3.** Without loss of generality, we prove for \(k = 1\). Using (3.8), Lemma 2.9 and Lemma 4.2, we expand the localized mass
\[
\int |u|^2 \Phi_1 dx = \int |U|^2 \Phi_1 dx + \int |R|^2 \Phi_1 dx + 2\text{Re} \int \bar{U}_1 R dx + O(|t|^3 \|R\|_{L^2}),
\]
which yields that
\[
2\text{Re} \int (\bar{U} R)(t) dx + \int |R(t)|^2 \Phi_1 dx = \int |u(T)|^2 \Phi_1 dx - \int |U(t)|^2 \Phi_1 dx + | \int |u(t)|^2 \Phi_1 dx - \int |u(T)|^2 \Phi_1 dx | - C|t|^3 \|R\|_{L^2}.
\]
where in the last step, we use the boundary condition \(u(T) = U(T)\).

By using Lemma 2.9, (3.6) and Lemma 4.2, for any \(t \in [T^*, T]\), we have
\[
\int |U(t)|^2 \Phi_1 dx = \int |U_1(t)|^2 \Phi_1 dx = \int |U_1(t)|^2 dx + O(t^4) = \|Q\|_{L^2}^2 + O(t^4),
\]
which yields the estimate for the first term in (4.15) that, for some $C > 0$,

$$\left| \int |U(T)|^2 \Phi_1 dx - \int |U(t)|^2 \Phi_1 dx \right| \leq C t^4. \quad (4.16)$$

The second term in (4.15), which is the evolution of the localized mass, can be estimated by using equation (4.9). In fact, we have

$$\frac{d}{dt} \int |u|^2 \Phi_1 dx = 2 \text{Im} \int \overline{u} Du \Phi_1 dx$$

$$= 2 \text{Im} \int \overline{U} D U_1 \Phi_1 dx + 2 \text{Im} \int (\overline{U} D R + \overline{R} D U) \Phi_1 dx + 2 \text{Im} \int \overline{R} D R \Phi_1 dx =: I + II + III. \quad (4.17)$$

The first term on the right hand side of (4.17) can be decomposed as

$$I = 2 \text{Im} \int \overline{U} D U_1 \Phi_1 dx + 2 \sum_{k=2}^{K} \text{Im} \int (\overline{U} D U_k + \overline{U}_k D U_1) \Phi_1 dx + 2 \sum_{k,l=2}^{K} \text{Im} \int \overline{U}_k D U_l \Phi_1 dx$$

$$=: I_1 + I_2 + I_3. \quad (4.18)$$

Since $D$ is self-adjoint, then integration by parts and using the identity (2.3), we obtain

$$|I_1| = |\text{Im} \int U_1(D(\overline{U} \Phi_1) - D \overline{U} \Phi_1) dx|$$

$$= |\text{Im} \int U_1(x) dx \int \frac{\overline{U_1}(x+y)(\Phi_1(x+y) - \Phi_1(x)) - \overline{U_1}(x-y)(\Phi_1(x) - \Phi_1(x-y))}{|y|^2} dy|$$

$$\leq |\text{Im} \int_{|x-x_1| \leq 3 \sigma} dxdy| + |\text{Im} \int_{|x-x_1| \geq 3 \sigma} dxdy| + |\text{Im} \int_{|x-x_1| \leq \frac{3}{2} \sigma} dxdy| + |\text{Im} \int_{|x-x_1| \geq \frac{3}{2} \sigma} dxdy|$$

$$=: I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}.$$

Using the definition of $\Phi_1$, we have $I_{1,1} = 0$. By using Taylor’s expansion, there exists some $|\theta| \leq 1$ such that,

$$|f(x+y)(\Phi_1(x+y) - \Phi_1(x)) - f(x-y)(\Phi_1(x) - \Phi_1(x-y))|$$

$$\leq |y|^2 (\|\nabla f(x + \theta y)\|\|\nabla \Phi_1\|_{L^\infty} + |f(x)|\|\nabla^2 \Phi_1\|_{L^\infty})$$

$$\leq C |y|^2 (\|\nabla f(x + \theta y)\| + |f(x)|). \quad (4.19)$$

Thus, we get

$$I_{1,2} \leq C \int_{|x-x_1| \geq 3 \sigma} \|U_1(x)(|\nabla U_1(x + \theta y)| + |U_1(x)|) dxdy$$

$$\leq C \lambda_1^{-\frac{1}{2}} \int_{|x-x_1| \geq 3 \sigma} |Q_1(\frac{x - \alpha_1}{\lambda_1})(\lambda_1^{\frac{1}{2}} |\nabla Q_1(\frac{x + \theta y - \alpha_1}{\lambda_1})| + \lambda_1^{-\frac{1}{2}} |Q_1(\frac{x - \alpha_1}{\lambda_1})|) dxdy.$$  

Moreover, we may take $T_\ast$ close to 0 such that $|x_1 - \alpha_1(t)| \leq \sigma$. Thus for $|x - x_1| \geq 3 \sigma$, $|y| \leq \sigma$,

$$|x + \theta y - \alpha_1(t)| \geq \frac{1}{3} |x - x_1| \quad \text{and} \quad |x - \alpha_1(t)| \geq \frac{2}{3} |x - x_1|.$$  

So by applying the decay property of $Q$ (2.9), we get

$$I_{1,2} \leq C \lambda_1^2 \int_{|x-x_1| \geq 3 \sigma} |x - x_1|^{-4} dx \leq C t^4.$$
We may take $T_*$ even close to 0 such that $|\alpha_1(t) - x_1| < \frac{C}{T}$. So for $|x - x_1| \leq \frac{C}{T}$, $|y| \geq \sigma$, we have $|x + y - \alpha_1(t)| > \frac{C}{T}$, and by applying (2.9),

$$I_{1,3} \leq C \int_{|x-x_1| \leq \frac{C}{T}} |U_1(x)| dx \int_{|y| \geq \sigma} \frac{|U_1(x + y)|}{|y|^2} dy$$

$$\leq C \lambda_1^{-1} \int_{|x-x_1| \leq \frac{C}{T}} |Q_1(\frac{x - \alpha_1}{\lambda_1})| dx \int_{|y| \geq \sigma} |Q_1(\frac{x + y - \alpha_1}{\lambda_1})||y|^{-2} dy$$

$$\leq C \sigma^{-1} \lambda_1^{-1} \int_{|x-x_1| \leq \frac{C}{T}} |Q_1(\frac{x - \alpha_1}{\lambda_1})| \sigma^{-2} dx \leq C \gamma^4.$$

For $|x - x_1| \geq \frac{C}{T}$, we have $|x - \alpha_1(t)| \geq \frac{C}{T}$. So it follows that,

$$I_{1,4} \leq C \int_{|x-x_1| \geq \frac{C}{T}} |U_1(x)| dx \int_{|y| \geq \sigma} |y|^{-2} dy$$

$$\leq C \sigma^{-1} \lambda_1^{-\frac{k}{2}} \int_{|x-x_1| \geq \frac{C}{T}} |Q_1(\frac{x - \alpha_1}{\lambda_1})| dx \leq C \gamma^3.$$

Thus we have

$$|I_1| \leq C \gamma^3. \quad (4.20)$$

The estimate of $I_2$ shall take advantage of the approximate equation of $Q_k$ (3.1). In fact combing equation (3.1), Lemma 3.1 and Lemma 4.2, we can find that

$$DQ_k(y) = -Q_k(y) + |Q_k(y)|^2 Q_k(y) + O(|t|^2), \quad 1 \leq k \leq K. \quad (4.21)$$

Using renormalization, for $1 \leq k \leq K$, we get

$$\text{Im} \int (\overline{U_1}DU_k + \overline{U_k}DU_1) \Phi_1 dx$$

$$= (\lambda_1 \lambda_k)^{-\frac{k}{2}} \text{Im} \int \overline{Q_1(\frac{x - \alpha_1}{\lambda_1})} DQ_k(\frac{x - \alpha_k}{\lambda_k}) \Phi_1 dx + \lambda_1^{-1} \int \overline{Q_k(\frac{x - \alpha_k}{\lambda_k})} DQ_1(\frac{x - \alpha_1}{\lambda_1}) \Phi_1 dx$$

$$= (\lambda_1 \lambda_k)^{-\frac{k}{2}} \frac{4}{\omega \ell^2} \text{Im} \int \overline{Q_1(\frac{x - \alpha_1}{\lambda_1})} DQ_k(\frac{x - \alpha_k}{\lambda_k}) \Phi_1 + \overline{Q_k(\frac{x - \alpha_k}{\lambda_k})} DQ_1(\frac{x - \alpha_1}{\lambda_1}) \Phi_1 dx$$

$$+ (\lambda_1 \lambda_k)^{-\frac{k}{2}} (\lambda_1^{-1} - \frac{4}{\omega \ell^2}) \text{Im} \int \overline{Q_1(\frac{x - \alpha_1}{\lambda_1})} DQ_k(\frac{x - \alpha_k}{\lambda_k}) \Phi_1 dx$$

$$+ (\lambda_1 \lambda_k)^{-\frac{k}{2}} (\lambda_1^{-1} - \frac{4}{\omega \ell^2}) \text{Im} \int \overline{Q_k(\frac{x - \alpha_k}{\lambda_k})} DQ_1(\frac{x - \alpha_1}{\lambda_1}) \Phi_1 dx$$

$$=: I_{2,1} + I_{2,2} + I_{2,3} \quad (4.22)$$

The identity (4.21) implies that $DQ_k(y) = -Q_k(y) + O(|t|^2 + \langle y \rangle^{-2})$. Inserting this into $I_{2,1}$, we observe the algebraic cancellation of the leading order term

$$\text{Im} \int \overline{Q_1(\frac{x - \alpha_1}{\lambda_1})} Q_k(\frac{x - \alpha_k}{\lambda_k}) \Phi_1 + \overline{Q_k(\frac{x - \alpha_k}{\lambda_k})} Q_1(\frac{x - \alpha_1}{\lambda_1}) \Phi_1 dx = 0.$$

Using this fact and following a similar calculation as above, we get

$$|I_{2,1}| \leq C \gamma^3.$$

To estimate $I_{2,2}$ and $I_{2,3}$, we shall additionally apply the a prior bound (4.2). So it yields that

$$|I_{2,2}| + |I_{2,3}| \leq C \gamma^{4-2x}.$$
Back to (4.22) and sum \( k \) from 2 to \( K \), we thus obtain

\[
|I_2| \leq C|t|^3. \tag{4.23}
\]

By taking \( T_\epsilon \), close to 0, it follows from (4.21) and Lemma 4.2 that

\[
|I_3| \leq C \sum_{k=2}^{K} \lambda_k^{-\frac{1}{2}} \lambda_l^{-\frac{1}{2}} \int_{|x-x_k| \leq 3\sigma, |x-x_l| \leq 4\sigma} |Q_k(x-x_k)||DQ_l(x-x_l)| \, dx \leq C t^4. \tag{4.24}
\]

Inserting (4.20), (4.23) and (4.24) into (4.18), we arrive at

\[
|I| \leq C|t|^3. \tag{4.25}
\]

Regarding the second term on the right hand side of (4.17), we first apply the integration by parts formula to get

\[
\text{Im} \int (\overline{U}D + \overline{R}DU) \Phi_1 \, dx = \sum_{k=1}^{K} \text{Im} \int (\overline{U_k}D + \overline{R}DU_k) \Phi_1 \, dx = \sum_{k=1}^{K} \int R(D(\overline{U_k} \Phi_1) - D\overline{U_k} \Phi_1) \, dx.
\]

Then by Lemma 2.2, for \( 1 \leq k \leq K \),

\[
|II| = |\text{CIm} \int R(x) \, dx \int \frac{\overline{U_k}(x+y)(\Phi_1(x+y) - \Phi_1(x)) - \overline{U_k}(x-y)(\Phi_1(x) - \Phi_1(x-y))}{|y|^2} \, dy| \\
\leq |\text{Im} \int \int_{|x-x_1| \leq 3\sigma, |x-x_2| \leq 3\sigma, |x-y| \leq \sigma} dx \, dy| + |\text{Im} \int \int_{|x-x_1| \geq 3\sigma, |x-x_2| \geq 3\sigma, |y| \leq \sigma} dx \, dy| + |\text{Im} \int \int_{|y| \geq \sigma} dx \, dy| \\
=: II_1 + II_2 + II_3.
\]

It’s easy to see that \( |x+y-x_1| \leq 4\sigma \) for \( |x-x_1| \leq 3\sigma \cup |x-x_2| \leq 3\sigma, |y| \leq \sigma \). Thus we get \( II_1 = 0 \) according to the definition of \( \Phi_1 \). Moreover, by (4.19), we get

\[
II_2 \leq \int_{|x-x_1| \leq 3\sigma} |R(x)| \int_{|y| \leq \sigma} |\nabla U_k(x+y)| + |U_k(x)| \, dy \\
\leq \int_{|x-x_1| \leq 3\sigma} |R(x)| \int_{|y| \leq \sigma} \lambda_k^{-\frac{1}{2}} |\nabla Q_k(x+y - \alpha_k)| + \lambda_k^{-\frac{1}{2}} |Q_k(x+y - \alpha_k)| \, dy.
\]

We may take \( T_\epsilon \) close to 0 such that \( |x_k - \alpha_k(t)| \leq \sigma \). Thus for \( |x-x_k| \geq 3\sigma \) and \( |y| \leq \sigma \),

\[
|x+y-x_1| \geq \frac{1}{3}|x-x_k| \quad \text{and} \quad |x-y-x_1| \geq \frac{2}{3}|x-x_k|.
\]

So by applying (2.9), we get

\[
II_2 \leq C\sigma \lambda_k^{-\frac{1}{2}} \int_{|x-x_1| \geq 3\sigma} |R(x)||x-x_k|^{-2} \, dx \leq C t^4 |R|_{L^2}.
\]

The rest term can be estimated as

\[
II_3 \leq C \int_{|y| \geq \sigma} |y|^{-2} \, dy \int |R(x)||U_k(x+y)| + |U_k(x-y)|| \, dx \leq C \sigma^{-1} |R|_{L^2}.
\]

Thus we get

\[
|II| \leq C |R|_{L^2}. \tag{4.26}
\]

At last, we consider the third term on the right hand side of (4.17). Integration by parts,

\[
III = 2\text{Im} \int \overline{R}D\Phi_1 \, dx = \text{Im} \int R(D(\overline{\Phi}_1) - D\overline{\Phi}_1) \, dx,
\]
and by using Lemma 2.5, we obtain
\[ |III| \leq C||R||_{L^2}||D(\tilde{R}\Phi_1) - D\tilde{R}\Phi_1||_{L^2} \leq C||R||_{L^2}||D\tilde{\Phi}_1||_{L^2}. \]

Then we show that
\[ \|D\Phi\|_{L^1} < \infty. \] (4.27)
In fact, by (2.3), we infer from the identity
\[ D\Phi(x) = -\frac{1}{2} C(s) \int_{\mathbb{R}} \frac{\Phi(x+y) + \Phi(x-y) - 2\Phi(x)}{|y|^2} dy \]
that \( |D\Phi(x)| \leq \langle x \rangle^{-1} \), which implies \( D\Phi \in L^2(\mathbb{R}) \), \( D\tilde{\Phi} \in L^2(\mathbb{R}) \) and particularly \( D\tilde{\Phi} \in L^1_{loc}(\mathbb{R}) \). Moreover, since \( \Phi'' \in C^\infty_c(\mathbb{R}) \), we have \( (D\Phi)'' = D(\Phi'') \in \mathcal{S} \). Thus for \( |\xi| > 0 \), there holds the decay estimate
\[ |D\tilde{\Phi}(\xi)| \leq C \frac{|(D\Phi)''(\xi)|}{|\xi|^2} \leq C \frac{1}{|\xi|^2}, \]
which yields (4.27), as claimed. Recall the definition of \( \Phi_1 \), we thus obtain the estimate for III
\[ |III| \leq C||R||_{L^2}^2. \] (4.28)

Combining (4.25), (4.26) and (4.28) together, we get
\[ \left| \frac{d}{dt} \int |u|^2 \Phi_1 dx \right| \leq C(|r|^3 + ||R||_{L^2} + ||R||_{L^2}^2). \]

Then by integrating from \( t \) to \( T \), and using the a priori bound (4.1), the second part of the right hand side of (4.15) can be estimated as
\[ |\int |u(t)|^2 \Phi_j dx - \int |u(T)|^2 \Phi_j dx| \leq C|t|^{4-k}. \] (4.29)

Finally, we can conclude the proof of (4.14), by inserting (4.16) and (4.29) in to (4.15). \( \Box \)

We then study the quantities \( \text{Im} \int \nabla u(t)\overline{\Phi}_1 dx \), which we call the localized momentum, to obtain the estimates for the parameters \( v_k \), \( 1 \leq k \leq K \).

**Lemma 4.5** (Estimate of \( v_k \)). For all \( t \in [T_*, T] \) and \( 1 \leq k \leq K \), we have
\[ \left| \frac{v_k(t)}{\lambda_k(t)} - 1 \right| = O(|t|^{2-k}). \] (4.30)

**Proof.** WLOG, we prove for \( k = 1 \). By (3.8), we first expand the localized momentum
\[ \text{Im} \int \nabla u(t)\overline{\Phi}_1 dx = \text{Im} \int \nabla U(t)\overline{U}(t)\Phi_1 dx + \text{Im} \int \nabla R(t)\overline{R}(t)\Phi_1 dx + \text{Im} \int (\nabla U(t)\overline{R}(t) + \nabla R(t)\overline{U}(t))\Phi_1 dx. \] (4.31)

By Lemma 2.9 and Lemma 4.2, we have for \( t \in [T_*, T] \)
\[ |\text{Im} \int \nabla U(t)\overline{U}(t)\Phi_1 dx - \text{Im} \int \nabla U_1(t)\overline{U}_1(t) dx| \leq C|t|^3, \] (4.32)

and
\[ |\text{Im} \int (\nabla U(t)\overline{R}(t) + \nabla R(t)\overline{U}(t))\Phi_1 dx - 2\text{Im} \int \nabla U_1(t)\overline{R}(t) dx| \leq C|t||R||_{L^2}, \]
Thus inserting (4.32)-(4.34) into (4.31), we obtain

\[ |\text{Im} \int (\nabla U(t) \overline{R}(t) + \nabla R(t) \overline{U}(t)) \Phi_1 dx| \leq C |t| \|R\|_{H^2}. \]  

(4.33)

Moreover, by using Lemma 2.6 and a standard density argument, one can easily get

\[ |\text{Im} \int \nabla R(t) \overline{R}(t) \Phi_1 dx| \leq C \|R\|^2_{H^\frac{1}{2}}. \]  

(4.34)

Thus inserting (4.32)-(4.34) into (4.31), we obtain

\[ \text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx - \text{Im} \int \nabla U_1(t) \overline{U_1}(t) dx = O(|t|\|R\|_{L^2} + \|R\|^2_{H^\frac{1}{2}} + |t|^3). \]  

(4.35)

On one hand, we obtain from (4.35) that

\[ |\text{Im} \int \nabla U_1(t) \overline{U_1}(t) dx - \text{Im} \int \nabla U_1(T) \overline{U_1}(T) dx| \]

\[ \leq |\text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx - \text{Im} \int \nabla u(T) \overline{u}(T) \Phi_1 dx| + |\text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx - \text{Im} \int \nabla U_1(t) \overline{U_1}(t) dx| \]

\[ + |\text{Im} \int \nabla U_1(T) \overline{U_1}(T) dx - \text{Im} \int \nabla U_1(T) \overline{U_1}(T) dx| \]

\[ \leq |\text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx - \text{Im} \int \nabla u(T) \overline{u}(T) \Phi_1 dx| + O(|t|\|R\|_{L^2} + \|R\|^2_{H^\frac{1}{2}} + |t|^3). \]  

(4.36)

On the other hand, by (3.5), Lemma 4.2 and using the fact \( \frac{1}{2} \langle S_1, S_1 \rangle + \langle Q, T_2 \rangle = 0 \), we get

\[ \text{Im} \int \nabla U_1(t) \overline{U_1}(t) dx = \lambda_1^{-1}(i) \text{Im} \int \nabla Q_1(t) \overline{Q_1}(t) dx = p_1 \frac{\nu_1(T)}{\lambda_1(T)} + O(T^2) \]  

(4.37)

where \( p_1 := 2 \langle L . G_1, G_1 \rangle > 0 \). Particularly, by (3.7), it follows that

\[ \text{Im} \int \nabla U_1(T) \overline{U_1}(T) dx = p_1 \frac{\nu_1(T)}{\lambda_1(T)} + O(T^2) = p_1 + O(T^2). \]  

(4.38)

Combing (4.36), (4.37) and (4.38) together, we find that

\[ p_1 \frac{\nu_1(T)}{\lambda_1(T)} - 1 | \leq |\text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx - \text{Im} \int \nabla u(T) \overline{u}(T) \Phi_1 dx| + O(|t|\|R\|_{L^2} + \|R\|^2_{H^\frac{1}{2}} + |t|^2). \]  

(4.39)

The evolution of the localized momentum on the right side of (4.39) can be estimated by using equation (4.9). Integrating by parts, we get

\[ \frac{d}{dt} \text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx = -\text{Re} \int \nabla \overline{u}(D(u \Phi_1) - Du \Phi_1) dx - \frac{1}{2} \int |u|^4 \nabla \Phi_1 dx \]

\[ = - \sum_{k=1}^{K} \text{Re} \int \nabla \overline{U_k}(D(U_k \Phi_1) - DU_k \Phi_1) dx - \sum_{k=1}^{K} \text{Re} \int \nabla \overline{R}(D(U_k \Phi_1) - DU_k \Phi_1) dx \]

\[ = - \sum_{k=1}^{K} \text{Re} \int \nabla \overline{U_k}(D(R \Phi_1) - DR \Phi_1) dx - \text{Re} \int \nabla \overline{R}(D(R \Phi_1) - DR \Phi_1) dx - \frac{1}{2} \int |u|^4 \nabla \Phi_1 dx \]

\[ =: I + II + III + IV + V. \]  

(4.40)
We first estimate $I$. For $1 \leq k \leq K$, we have

$$\text{Re} \int \nabla U(D(U_k \Phi_1) - DU_k \Phi_1) \, dx$$

$$= \text{Re} \int \nabla U_k(D(U_k \Phi_1) - DU_k \Phi_1) \, dx + \sum_{i \neq k} \text{Re} \int \nabla U_i(D(U_i \Phi_1) - DU_i \Phi_1) \, dx$$

$$=: I_1 + I_2. \quad (4.41)$$

$I_1$ follows the estimate of (4.20) in the proof of Lemma 4.3. In fact, by the formula (2.5), we get

$$|\text{Re} \int \nabla U_k(D(U_k \Phi_1) - DU_k \Phi_1) \, dx|$$

$$= |C| \text{Re} \int \nabla U_k(x) \, dx \int \frac{U_k(x + y)(\Phi_1(x + y) - \Phi_1(x)) - U_k(x - y)(\Phi_1(x) - \Phi_1(x - y))}{|y|^2} \, dy$$

$$\leq |\text{Re} \int_{|x - x_k| \leq 3\sigma, |y| \leq \sigma} dxdy| + |\text{Re} \int_{|x - x_k| \leq 3\sigma, |y| \leq \sigma} dxdy| + |\text{Re} \int_{|x - x_k| \leq \frac{3\sigma}{2}, |y| \geq \sigma} dxdy|. \quad (4.42)$$

Then each term can be estimated by following the estimate of (4.20) from line to line. Let’s mention that, compare with (4.20), the derivative contributes an additional $|t|^{-2}$ fact, which implies that

$$|I_1| \leq C|t|.$$ 

$I_2$ follows the estimate of (4.23). An integration by parts yields that, for $l \neq k$,

$$\text{Re} \int \nabla U_l(D(U_k \Phi_1) - DU_k \Phi_1) \, dx = -\text{Re} \int D\nabla U_l \nabla U_k \Phi_1 + \nabla U_l DU_k \Phi_1 \, dx - \text{Re} \int D\nabla U_k \nabla \Phi_1 \, dx.$$ 

By inserting the identity (4.21) into the above formula, we observe the algebraic cancellation of the leading order term

$$\text{Re} \int \frac{\nabla \Phi(x - \alpha_l)}{\lambda_l} \nabla Q_k(x - \alpha_k) \Phi_1 + \nabla \Phi(x - \alpha_l) \nabla Q_k(x - \alpha_k) \Phi_1 \, dx.$$ 

Then we can obtain similarly as (4.23) that

$$|I_2| \leq C|t|,$$

where an additional $|t|^{-2}$ fact also appears. Back to (4.41) and sum $k$ from 1 to $K$, it yields the bound for $I$

$$|I| \leq C|t|. \quad (4.42)$$

Integrating by parts and using the formula (2.5) again, it follows that, $1 \leq k \leq K$,

$$|\text{Re} \int \nabla R(D(U_k \Phi_1) - DU_k \Phi_1) \, dx + \text{Re} \int \nabla U_k(D(R \Phi_1) - DR \Phi_1) \, dx|$$

$$= |2\text{Re} \int R(D\nabla U_k \Phi_1) - D\nabla U_k \Phi_1 \, dx + \text{Re} \int R(D(U_k \nabla \Phi_1) - DU_k \nabla \Phi_1) \, dx|$$

$$= |2C| \text{Re} \int R(x) \, dx \int \frac{\nabla U_k(x + y)(\Phi_1(x + y) - \Phi_1(x)) - \nabla U_k(x - y)(\Phi_1(x) - \Phi_1(x - y))}{|y|^2} \, dy$$

$$+ |C| \text{Re} \int R(x) \, dx \int \frac{U_k(x + y)(\nabla \Phi_1(x + y) - \nabla \Phi_1(x)) - U_k(x - y)(\nabla \Phi_1(x) - \nabla \Phi_1(x - y))}{|y|^2} \, dy$$

$$\leq |\text{Im} \int_{|x - x_k| \leq 3\sigma} dxdy| + |\text{Im} \int_{|x - x_k| \leq 3\sigma, |y| \leq \sigma} dxdy| + |\text{Im} \int_{|y| \geq \sigma} dxdy|.$$


Then each term can be estimated by following the estimate of (4.26). Similarly, compare with (4.26), an additional $|t|^{-2}$ fact appear in the estimate. Thus we can get

$$|II + III| \leq \sum_{k=1}^{K} |\text{Re} \int \nabla R(D(U_k \Phi_1) - DU_k \Phi_1)dx| + |\text{Re} \int \nabla U_k(D(R \Phi_1) - DR \Phi_1)dx| \leq C \frac{|R|_{L^2}}{r^2}. \quad (4.43)$$

$IV$ can be estimate by applying Lemma 2.6. In fact by denoting $T_g := \nabla((D(g \Phi_1) - Dg \Phi_1))$, an integration by parts implies

$$\langle f, Tg \rangle = - \int \nabla \overline{f}(D(g \Phi_1) - Dg \Phi_1)dx.$$ 

On one hand, from Lemma 2.5 and (4.27), it follows that

$$|\langle f, Tg \rangle| \leq \|\nabla f\|_{L^2} \|g\|_{L^2} \|D \Phi_1\|_{L^1} \leq C \|f\|_{H^1} \|g\|_{L^2}.$$ 

On the other hand, integration by parts and using the commutator formula $[D, \nabla] = 0$,

$$\langle f, Tg \rangle = \int \overline{f}(D(\nabla g \Phi_1 + g \nabla \Phi_1) - \nabla g \Phi_1 - Dg \nabla \Phi_1)dx.$$ 

From Lemma 2.5, (4.27) and using the fact that $\nabla \Phi \in C_c^\infty(\mathbb{R})$,

$$|\langle f, Tg \rangle| \leq |\int \overline{f}(D(\nabla g \Phi_1) - \nabla g \Phi_1)dx| + |\int \overline{f}(D(g \nabla \Phi_1) - Dg \nabla \Phi_1)dx|$$

$$\leq \|f\|_{L^2} \|\nabla g\|_{L^2} \|D \Phi_1\|_{L^1} + \|f\|_{L^2} \|g\|_{L^2} \|D \nabla \Phi_1\|_{L^1}$$

$$\leq C \|f\|_{H^2} \|g\|_{H^2}. \quad (4.44)$$

So it yields from Lemma 2.6 that

$$|\langle f, Tg \rangle| \leq C \|f\|_{H^2} \|g\|_{H^2}. \quad (4.44)$$

By taking $f = g = R$, it follows that $IV = \text{Re} \langle f, Tg \rangle$, thus we obtain

$$|IV| \leq C \|R\|_{H^2}^2. \quad (4.45)$$

Moreover, by using (2.9) and Sobloev embeddings, it yields that

$$|V| \leq C \sum_{k=1}^{K} \int |U_k|^4 \nabla \Phi_1 dx + \|R\|_{L^2}^4 \leq C(|t|^2 + \|R\|_{H^2}^4). \quad (4.45)$$

Inserting (4.42), (4.43), (4.44) and (4.45) into (4.40), we get

$$\left|\frac{d}{dt} \text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx\right| \leq C \left(\|R(t)\|_{L^2}^2 + \|R(t)\|_{H^2}^2 + |t|\right),$$

which along with the a prior estimate (4.1) implies that

$$\left|\text{Im} \int \nabla u(t) \overline{u}(t) \Phi_1 dx - \text{Im} \int \nabla u(T) \overline{u}(T) \Phi_1 dx\right| \leq \int_T^t |\frac{d}{ds} \text{Im} \int \nabla u(s) \overline{u}(s) \Phi_1 dx| ds$$

$$\leq C \int \left(\|R(s)\|_{L^2}^2 + \|R(s)\|_{H^2}^2 + |s|\right) ds \leq C |t|^{2-\kappa}.$$ 

Thus we conclude that

$$p_i \frac{|\nu_i(t)|}{A_i(t)} - 1 \leq C |t|^{2-\kappa},$$

and finish the proof. \qed
4.2. **Monotone functional.** This subsection is devoted to prove the monotonicity of the generalized energy (4.46), which is a combination of an energy part and a localized virial part. The monotonicity of the functional allow one to integrate the flow backwards from the singularity, so that it is the key ingredient in the proof of the bootstrap estimates of the remainder (4.5).

It should be mentioned that, inspired by [40] in the context of NLS, the generalized energy in this paper includes also the localization functions \( \Phi \) in an appropriate way, which is different from the single bubble case in [29].

Let \( \chi(x) : \mathbb{R} \to \mathbb{R} \) be a smooth even function, satisfying \( \chi'(x) = x \) if \( 0 \leq x \leq 1 \), \( \chi'(x) = 3 - e^{-x} \) if \( x \geq 2 \), with the convexity condition

\[
\chi''(x) \geq 0 \quad \text{for} \quad x \geq 0.
\]

Let \( \chi_A(x) := A^2 \chi(\tfrac{x}{A}) \) for \( A > 0 \), and let \( f(u) := |u|^2 u \), \( F(u) := \frac{1}{4}|u|^4 \). We define the generalized energy by

\[
\mathcal{Z}(R) := \frac{1}{2} \int |D^{\frac{1}{2}}R|^2 + \sum_{k=1}^K \frac{1}{\lambda_k} |R|^2 \Phi_k dx - \text{Re} \int F(u) - F(U) - f(U)\overline{R} dx + \sum_{k=1}^K \frac{b_k}{2} \text{Im} \int (\nabla \chi_A)(\frac{x - \alpha_k}{\lambda_k}) \cdot \nabla R \overline{R} \Phi_k dx.
\]  

(4.46)

The key monotonicity property of the generalized energy is formulated in Theorem 4.6 below.

**Proposition 4.6.** For all \( t \in [T_*, T] \), there exists positive constants \( C \) and \( C(A) \) such that, for \( A \) large enough,

\[
\frac{d\mathcal{Z}}{dt} \geq C \frac{||R(t)||_{L^2}^2}{|t|^3} + O(C(A) \left( \ln(2 + ||R||_{H^2}^{-1}) \right)^{\frac{1}{2}} X(t) + |t|^{3-\kappa}).
\]

In order to prove Theorem 4.6, we separate \( \mathcal{Z} = \mathcal{E} + \mathcal{U} \), where \( \mathcal{E} \) denotes the energy part

\[
\mathcal{E}(R) := \frac{1}{2} \int |D^{\frac{1}{2}}R|^2 + \sum_{k=1}^K \frac{1}{\lambda_k} |R|^2 \Phi_k dx - \text{Re} \int F(u) - F(U) - f(U)\overline{R} dx,
\]

and \( \mathcal{U} \) denotes the localized virial part

\[
\mathcal{U}(R) := \sum_{k=1}^K \frac{b_k}{2} \text{Im} \int (\nabla \chi_A)(\frac{x - \alpha_k}{\lambda_k}) \cdot \nabla R \overline{R} \Phi_k dx.
\]

Below we treat \( \mathcal{E} \) and \( \mathcal{U} \) separately in Propositions 4.7 and 4.8.

**Proposition 4.7.** For all \( t \in [T_*, T] \), there exists some positive constant \( C \) such that

\[
\frac{d\mathcal{E}}{dt} \geq \sum_{k=1}^K \left( \frac{b_k}{2\lambda_k^2} ||R_k||_{L^2}^2 - \frac{b_k}{2\lambda_k} \text{Re} \int 2|U_k|^2|R_k|^2 + \overline{U_k} R_k^2 dx \right.
\]

\[
- b_k \text{Re} \int \frac{x - \alpha_k}{\lambda_k} \cdot \nabla \overline{U_k} (U_k R_k^2 + 2U_k |R_k|^2) dx
\]

\[
- C(|t|^{3-\kappa} + \frac{||R||_{L^2}^2}{|t|^{3-\kappa}} + X(t)).
\]
Proof. In view of (4.9), we obtain
\[
\frac{d\mathcal{E}}{dt} = -\sum_{k=1}^{K} \frac{\lambda_k}{2\lambda_k^2} \int |R|^2 \Phi_k dx - \sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \langle R_k, DR \rangle - \sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \langle 2|U|^2 R + U^2 \overline{R}, R_k \rangle \\
- \Re \langle 2|U|^2 + \overline{U} R^2 + |R|^2 R, \partial_t U \rangle - \sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \langle 2|U|^2 + \overline{U} R^2 + |R|^2 R, R_k \rangle \\
+ \Im \langle DR + \sum_{k=1}^{K} \frac{1}{\lambda_k} R_k - (|u|^2 u - |U|^2 U), \eta \rangle =: \sum_{j=1}^{6} \mathcal{E}_j. \tag{4.47}
\]

(i) Estimate of $\mathcal{E}_1$. From Lemma 4.2, we get $|\frac{dr + \alpha u}{t^2}| \leq C_{\text{mod}} \leq C|t|^{-k}$, thus it follows that
\[
\mathcal{E}_1 = \sum_{k=1}^{K} \frac{b_k}{2\lambda_k^2} \int |R|^2 \Phi_k dx - \frac{\lambda_k + \beta_k}{2\lambda_k^2} \int |R|^2 \Phi_k dx \geq \sum_{j=1}^{K} \frac{b_k}{2\lambda_k^2} \|R_k\|_{L^2}^2 - C|t|^{-k}X(t). \tag{4.48}
\]

(ii) Estimate of $\mathcal{E}_2$. Using the fact that $\sum_{k=1}^{K} R_k = R$ and $\Im \langle R, DR \rangle = 0$, we compute
\[
|\mathcal{E}_2| = \left| \sum_{k=1}^{K} \left( \frac{1}{\lambda_k} - \frac{1}{\lambda_k \omega t^2} \right) \Im \langle R_k, DR \rangle \right| \leq \sum_{k=1}^{K} \left| \frac{\lambda_k - \omega t^2}{\lambda_k \omega t^2} \right| |\Im \langle R_k, DR \rangle|.
\]
From Lemma 4.2 and use the a prior bound (4.2), we get
\[
|\mathcal{E}_2| \leq C|t|^{-2k} \|R\|_{L^2}^2 \leq CX(t). \tag{4.49}
\]

(iii) Estimate of $\mathcal{E}_3 + \mathcal{E}_4$. We first treat $\mathcal{E}_3$ along with the terms in $\mathcal{E}_4$ which are quadratic in $R$. From (2.9) and Lemma 2.9, we have
\[
\sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \langle 2|U|^2 + U^2 \overline{R}, R_k \rangle = \sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \langle 2|U_k|^2 R_k + U_k^2 \overline{R}, R_k \rangle + O(\|R\|_{L^2}^2) \\
= \sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \int U_k^2 \overline{R_k} dx + O(\|R\|_{L^2}^2),
\]
\[
\Re \langle 2|U|^2 + \overline{U} R^2, \partial_t U \rangle = \sum_{k=1}^{K} \Re \langle 2U_k |R_k|^2 + \overline{U}_k R_k^2, \partial_t U_k \rangle + O(\|R\|_{L^2}^2). \tag{4.50}
\]
From (3.13), Lemma 3.1 and Lemma 4.2, a direct calculation shows that, $1 \leq k \leq K$,
\[
\partial_t U_k(x) = \frac{i}{\lambda_k} U_k + \frac{b_k}{2\lambda_k} U_k + b_k \frac{x - \alpha_k}{\lambda_k} \cdot \nabla U_k - v_k \cdot \nabla U_k + O(\lambda_k^\frac{1}{2} f(x - \frac{\alpha_k}{\lambda_k})),
\]
for some $|f(y)| \leq C(y)^{-2}$. Thus insert this into (4.50) and use Lemma 4.2, we get
\[
\Re \langle 2|U|^2 + \overline{U} R^2, \partial_t U \rangle \\
= \sum_{k=1}^{K} \frac{1}{\lambda_k} \Im \int U_k^2 \overline{R_k} dx + \sum_{k=1}^{K} \frac{b_k}{2\lambda_k} \Re \int 2|U_k|^2 |R_k|^2 + \overline{U}_k R_k^2 dx \\
+ \sum_{k=1}^{K} b_k \Re \int \frac{x - \alpha_k}{\lambda_k} \cdot \nabla U_k (\overline{U}_k R_k^2 + 2U_k |R_k|^2) dx + O(r^{-2} \|R\|_{L^2}^2). \tag{4.51}
\]
Then we consider the remaining cubic term in $\mathcal{C}_4$. Using equation (4.10),

$$|\langle |R|^2 R, \partial_t U \rangle| \leq \int |R|^3 (|DU| + |U|^3 + |\eta|) dx \leq C \sum_{k=1}^{K} \int |R|^3 (|DU_k| + |U_k|^3) dx + \int |R|^3 |\eta| dx$$

From Sobolev embeddings, the a priori bound (4.1) and Lemma 4.2, it follows

$$|\langle |R|^2 R, \partial_t U \rangle| \leq C \sum_{k=1}^{K} (\|U_k\|_{H^\frac{1}{2}} \|U_k\|_{H^\frac{3}{2}} + \|R\|_{L^2} \|U_k\|_{L^6}^3) + C \|\eta\|_{L^2} \|R\|_{L^6}^3$$

$$\leq C \sum_{k=1}^{K} (\|R\|_{L^2}^\frac{3}{2} \|R\|_{H^\frac{1}{2}}^\frac{3}{2} \|U_k\|_{H^\frac{3}{2}}^\frac{3}{2} + \|R\|_{L^2} \|R\|_{H^\frac{1}{2}}^\frac{3}{2} \|U_k\|_{L^6}^3) + C \|\eta\|_{L^2} \|R\|_{L^2} \|R\|_{H^\frac{1}{2}}^2$$

$$\leq C \sum_{k=1}^{K} (\lambda_k^2 \|R\|_{L^2} \|R\|_{H^\frac{1}{2}} + \lambda_k^{-1} \|R\|_{L^2} \|R\|_{H^\frac{1}{2}}^2 + CX(t)) \leq CX(t).$$

Collecting (4.50), (4.51) and (4.52) together, we get

$$\mathcal{C}_3 + \mathcal{C}_4 = - \sum_{k=1}^{K} \left( \frac{b_k}{2\lambda_k} \text{Re} \int 2|U_k|^2 |R_k|^2 + \overline{U_k} R_k^2 dx + b_k \text{Re} \int \frac{X - \alpha_k}{\lambda_k} \cdot \nabla U_k (\overline{U_k} R_k^2 + 2U_k |R_k|^2) dx \right) + O(X(t)).$$

(ii) Estimate of $\mathcal{C}_5$. The terms in $\mathcal{C}_5$ are cubic and higher order in $R$, they can also be bounded by using Sobolev embeddings, (4.1) and Lemma 4.2:

$$|\mathcal{C}_5| = \left| \sum_{k=1}^{K} \lambda_k^{-1} \text{Im} (2U |R|^2 + \overline{U} R^2 + |R|^2 R, R_k) \right|$$

$$\leq C \sum_{k=1}^{K} \lambda_k^{-1} \int |UR^2| dx + \int |R|^4 dx \leq C \sum_{k=1}^{K} \lambda_k^{-1} (\|U\|_{L^2} \|R\|_{L^6}^3 + \|R\|_{L^4}^4)$$

$$\leq C \sum_{k=1}^{K} \lambda_k^{-1} (\|R\|_{L^2} \|R\|_{H^\frac{1}{2}}^2 + \|R\|_{L^2}^2 \|R\|_{H^\frac{1}{2}}^2) \leq CX(t).$$

(ii) Estimate of $\mathcal{C}_6$. We expand the nonlinearity and use integration by parts to get

$$\text{Im}(\partial R + \sum_{k=1}^{K} \frac{1}{\lambda_k} R_k - (|u|^2 u - |U|^2 U), \eta)$$

$$= \text{Im}(R, \partial \eta + \sum_{k=1}^{K} \frac{1}{\lambda_k} \eta \Phi_k - 2|U|^2 \eta + U^2 \eta) - \text{Im}(2U |R|^2 + \overline{U} R^2 + |R|^2 R, \eta).$$

First, we estimate the second part on the right hand side above, in which the terms are quadratic and higher order in $R$. From Sobolev embeddings, (4.1) and Lemma 4.2, we have

$$|\text{Im}(2U |R|^2 + \overline{U} R^2 + |R|^2 R, \eta)| \leq \|\eta\|_{L^2} (\|U\|_{L^6} \|R\|_{L^2}^2 + \|R\|_{L^6}^3)$$

$$\leq C \|\eta\|_{L^2} (\|U\|_{L^2}^2 \|R\|_{H^\frac{1}{2}}^3 + \|R\|_{L^2} \|R\|_{H^\frac{1}{2}}^2)$$

$$\leq C \|\eta\|_{L^2} X(t) \leq CX(t).$$
The first part on the right hand side of (4.55), which contains linear terms in $R$, should be treated more carefully. In view of (3.13) and (4.10), we write $\eta = \sum_{k=1}^{K} \eta_k + \bar{\eta}$ with

$$\eta_k = e^{\nu \alpha_k} \lambda_k^{-\frac{3}{2}} \left( - (\lambda_k \nu_k + b_k \nu_k) G_1 - (\lambda_k b_k + \frac{1}{2} b_k^2) S_1 - i (\alpha_k - \nu_k) \cdot \nabla Q ight) - i (\lambda_k + b_k) \Lambda Q - (\lambda_k \dot{\nu}_k - 1) \langle \lambda_k \rangle (t, \frac{x - \alpha_k}{\lambda_k}).$$

so according to (2.9), Lemma 3.1 and Lemma 4.2, we have for $\nu = 0, 1$

$$|\eta_k(x)| \leq C \lambda_k^{-\frac{3}{2}} \left< \frac{x - \alpha_k}{\lambda_k} \right>^{-2} \text{Mod}_k,$$

and

$$||\bar{\eta}^{(\nu)}||_{L^2} \leq C |t|^{-2-2\nu} (|t| \text{Mod} + |t|^4).$$

Moreover, from (4.57), Lemma 2.9 and Lemma 4.2, we obtain

$$\langle R, 2|U|^2 \eta_k + U^2 \bar{\eta}_k \rangle - \langle R, 2|U|^2 \bar{\eta}_k + U^2 \eta_k \rangle \leq C \text{Mod}_k ||R||_{L^2} \leq C |t|^{-2k},$$

$$||R, 1 \lambda_k \eta \Phi_k ||_{L^2} \leq \lambda_k^{-1} ||R||_{L^2} (||\eta_k (1 - \Phi_k)||_{L^2} + \sum_{k \neq k} ||\eta_k \Phi_k||_{L^2}) \leq C \lambda_k^{-\frac{3}{2}} \text{Mod}_k ||R||_{L^2} \leq C |t|^{-2k}.$$

Thus the first part can be expand as

$$\text{Im}(R, D \eta + \sum_{k=1}^{K} \frac{1}{\lambda_k} \eta_k \Phi_k - 2|U|^2 \eta + U^2 \bar{\eta})$$

$$= \sum_{k=1}^{K} \text{Im}(R, D \eta_k + \frac{1}{\lambda_k} \eta_k - 2|U|^2 \eta_k + U^2 \bar{\eta}_k) + \text{Im}(R, D \bar{\eta} + \sum_{k=1}^{K} \frac{1}{\lambda_k} \bar{\eta} \Phi_k - 2|U|^2 \bar{\eta} + U^2 \eta) + O(|t|^{-2k}).$$

The contribution of $\bar{\eta}$ can be estimated by using (4.58),

$$|\text{Im}(R, D \bar{\eta} + \sum_{k=1}^{K} \frac{1}{\lambda_k} \bar{\eta} \Phi_k - 2|U|^2 \bar{\eta} + U^2 \eta)| \leq C (||\bar{\eta}||_{H^1} + |t|^{-2} ||\eta||_{L^2}) ||R||_{L^2}$$

$$\leq C \left( \frac{||R||^2_{L^2}}{t^2} + \frac{||R||^2_{L^2}}{|t|^3} + |t|^{-3-k} \right).$$

To estimate the terms involving with $\eta_k$, we use the renormalization (3.15) along with the fact that $Q_k(y) = Q(y) + O(t|y|^{-2})$, to get

$$\text{Im}(R, D \eta_k + \frac{1}{\lambda_k} \eta_k - 2|U|^2 \eta_k + U^2 \bar{\eta}_k)$$

$$= -\lambda_k^{-2} \left( (\lambda_k \nu_k + b \nu_k)(\epsilon_{k,2}, L_{-} G_1) + (\lambda_k b_k + \frac{1}{2} b_k^2)(\epsilon_{k,2}, L_{-} S_1) - (\alpha_k - \nu_k) \cdot (\epsilon_{k,1}, L_{-} \nabla Q) - (\lambda_k + b_k)(\epsilon_{k,1}, L_{+} \Lambda Q) + (\lambda_k \dot{\nu}_k - 1)(\epsilon_{k,2}, L_{-} Q) + O(|t| \text{Mod}_k ||R||_{L^2}) \right).$$
Then from the algebraic property of the linearized operator (2.14), and by taking the advantage of the orthogonality conditions (3.10) and Lemma 4.3, an additional $|t|$ factor is gained:

$$|\text{Im}(R, D\eta_k + \frac{1}{\lambda_k} \eta_k - 2|U_k|^2 \eta_k + U_k^2 \eta_k)|$$

$$\leq C\lambda_k^{-2}\text{Mod}_k((|\epsilon_{k,2}, \nabla Q| + |\langle\epsilon_{k,2}, \Lambda Q\rangle| + |\langle\epsilon_{k,1}, Q\rangle| + |t||R||L_2^2\rangle)$$

$$\leq C\left(\frac{|R|}{t^2} + |t|^{3-2\kappa}\right).$$

Inserting (4.60) and (4.61) into (4.59), we get

$$|\text{Im}(R, D\eta + \sum_{k=1}^{K} \frac{1}{\lambda_k} \eta_k - 2|U|^2 \eta + U^2 \eta)| \leq C(|t|^{3-k} + \frac{|R|}{t^2})$$

and combining this with (4.56), the estimate of $E_6$ is thus obtain

$$|E_6| = |\text{Im}(DR + \sum_{k=1}^{K} \frac{1}{\lambda_k} R_k - (|u|^2 u - |U|^2 U), \eta)| \leq C(|t|^{3-k} + \frac{|R|}{t^2} + X(t)).$$

In conclusion, plugging estimates (4.48), (4.49), (4.53), (4.54) and (4.62) into (4.47) we finish the proof of Proposition 4.7.

**Proposition 4.8.** For all $t \in [T, T]$, there exists a positive constant $C(A)$ such that

$$\frac{d\mathcal{U}}{dt} = \sum_{k=1}^{K} \left( \frac{b_k}{2\lambda_k} \text{Re}\langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) R_k, DR_k\rangle + b_k \text{Re}\langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \nabla R_k, DR_k\rangle \right)$$

$$\quad + b_k \text{Re}\langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \nabla U_k, 2U_k|R_k|^2 + U_k^2 R_k^2 \rangle$$

$$\quad + O(C(A)\left(\ln(2 + \frac{|R|}{|t|^2})\right)^{\frac{1}{2}} X(t) + \frac{|R|}{|t|^{3-k}} + |t|^{3-k}).$$

**Proof.** Integration by parts, we get

$$\frac{d\mathcal{U}}{dt} = \sum_{k=1}^{K} \frac{b_k}{2} \text{Im}\langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \cdot \nabla R, R_k\rangle + \sum_{k=1}^{K} \frac{b_k}{2} \text{Im}\langle\partial_t(\nabla \chi A(\frac{x-\alpha_k}{\lambda_k})) \cdot \nabla R, R_k\rangle$$

$$\quad + \sum_{k=1}^{K} \frac{b_k}{2} \text{Im}\langle\Delta \chi A(\frac{x-\alpha_k}{\lambda_k}) R_k, \partial_t R\rangle + \sum_{k=1}^{K} \frac{b_k}{2} \text{Im}\langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \cdot (\nabla R_k + \nabla R \Phi_k), \partial_t R\rangle$$

$$\quad : = \mathcal{U}_1 + \mathcal{U}_2 + \mathcal{U}_3 + \mathcal{U}_4.$$

(i) **Estimate of $\mathcal{U}_1$ and $\mathcal{U}_2$.** Direct calculation shows that for $1 \leq k \leq K$,

$$\frac{b_k}{2} \langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \cdot \nabla R, R_k\rangle = \frac{\lambda_k b_k + \frac{1}{2} b_k^2}{2\lambda_k} \langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \cdot \nabla R, R_k\rangle - \frac{b_k^2}{4\lambda_k} \langle\nabla \chi A(\frac{x-\alpha_k}{\lambda_k}) \cdot \nabla R, R_k\rangle,$$

and

$$\partial_t(\nabla \chi A(\frac{x-\alpha_k}{\lambda_k})) = -\nabla^2 \chi A(\frac{x-\alpha_k}{\lambda_k}) \cdot (\frac{x-\alpha_k}{\lambda_k} \cdot \frac{x-\alpha_k}{\lambda_k} - \frac{x-\alpha_k}{\lambda_k} + \frac{\alpha_k - v_k}{\lambda_k} + \frac{v_k}{\lambda_k}).$$
From Lemma 2.6 and Lemma 4.2, there exists some constant $C(A) > 0$ such that

\[
|\mathcal{U}_1| \leq C \left(1 + \frac{Mod}{t^2}\right) \sum_{k=1}^{K} \left| \int \nabla \chi A(x - \alpha_k^2) \Phi_k \nabla R \Phi \, dx \right|
\]

\[
\leq C \sum_{k=1}^{K} \left( \| \nabla \chi A \|_{L^\infty} \| D^\frac{3}{2} R \|_{L^2}^2 + \lambda_k^{-1} \| \nabla^2 \chi A \|_{L^\infty} \| R \|_{L^2}^2 \right) \leq C(A)X(t),
\]

(4.64)

\[
|\mathcal{U}_2| \leq C(|t| + Mod) \sum_{k=1}^{K} \left| \frac{b_k}{\lambda_k} \int \left(1 + \frac{x - \alpha_k}{\lambda_k}\right) \nabla^2 \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \Phi_k \nabla R \Phi \, dx \right|
\]

\[
\leq C \sum_{k=1}^{K} \left( \| (1 + y) \nabla^2 \chi A \|_{L^\infty} \| D^\frac{3}{2} R \|_{L^2}^2 + \lambda_k^{-1} \| (1 + y) \nabla^3 \chi A \|_{L^\infty} \| R \|_{L^2}^2 \right) \leq C(A)X(t).
\]

(4.65)

(ii) Estimate of $\mathcal{U}_3$ and $\mathcal{U}_4$. Using equation (4.9), we first get, for $1 \leq k \leq K$,

\[
\frac{b_k}{2 \lambda_k} \text{Im} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, \partial R \rangle + \frac{b_k}{2} \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k + \nabla R \Phi_k, \partial R \rangle
\]

\[
= \frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR \rangle + \frac{b_k}{2} \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k + \nabla R \Phi_k, DR \rangle
\]

\[
- \frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, f(u) - f(U) \rangle - \frac{b_k}{2} \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k + \nabla R \Phi_k, f(u) - f(U) \rangle
\]

\[
- \frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, \eta \rangle - \frac{b_k}{2} \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k + \nabla R \Phi_k, \eta \rangle.
\]

(4.66)

In the next step, we focus on estimating the terms in (4.66) that are quadratic in $R$. The target is to decouple the interaction between the remainders. In fact, we claim that there exists $C(A) > 0$ such that, for $1 \leq k \leq K$,

\[
\frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR \rangle - \frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR_k \rangle = O(C(A)X(t)),
\]

(4.67)

\[
\frac{b_k}{2} \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k + \nabla R \Phi_k, DR \rangle - b_k \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k, DR_k \rangle = O(C(A)X(t)).
\]

(4.68)

Once the above claims are proved, it yields that for $1 \leq k \leq K$,

\[
\frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR \rangle + \frac{b_k}{2} \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k + \nabla R \Phi_k, DR \rangle
\]

\[
= \frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR_k \rangle + b_k \text{Re} \langle \nabla \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) \nabla R_k, DR_k \rangle + O(C(A)X(t)).
\]

Let’s point out that Lemma 2.9 is not applicable to decouple the interaction between the remainders $R$ and $R_k$ in (4.67) and (4.68). It turns out that we shall take full advantage of integration by parts formula and the decay of the derivatives of the cut-off function $\nabla \chi$.

First we prove (4.67). By using $R = \sum_{k=1}^{K} R_k$, we have

\[
\frac{b_k}{2 \lambda_k} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR \rangle - \frac{b_k}{2 \lambda_k} \text{Re} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR_k \rangle = \sum_{l \neq k} \frac{b_l}{2 \lambda_l} \langle \Delta \chi A \left(\frac{x - \alpha_k}{\lambda_k}\right) R_k, DR_l \rangle.
\]
To estimate the remainder above, we find that for \( l \neq k \),
\[
\frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) R_k, DR_l \rangle
\]
\[
= \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) R_k, DR_l \rangle + \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) R_k, DR_l - DR_l \rangle,
\]
with
\[
\left| \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) R_k, DR_l \rangle \right| \leq C(A) \frac{b_k}{\lambda_k} \|DR_l\|_{L^1}\|R\|_{L^2}^2 \leq C(A) |t|^{-1} \|R\|_{L^2}^2, \tag{4.69}
\]
where we have applied Lemma 2.5, Lemma 4.2 and (4.27). Denote \( \Delta \chi_{A,k}(x) := \Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) \Phi_k(x) \).

Splitting \( D = D^+D^- \) with \( D^+ \) being self-adjoint, and performing integrating by parts, we then get
\[
\frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) R_k, DR_l \rangle
\]
\[
= \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_{A,k}(x), |D^+R|^2 \rangle + \frac{b_k}{2\lambda_k} \text{Re}\langle D^+(\Delta \chi_{A,k}) R_l - D^+ R \Delta \chi_{A,k} \Phi_k, D^+ R \rangle
\]

Here a key observation is that \( \Delta \chi(y) \) decays exponentially fast for \( |y| \) large, so it yields that
\[
\left| \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_{A,k}(x), |D^+R|^2 \rangle \right| \leq C |t|^{-1} \int_{|x-x_0| \geq 4\sigma} |\Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right)||D^+R|^2 dx
\]
\[
\leq C(A) |t|^{-1} e^{-\frac{\xi}{\lambda_k}} \|D^+R\|_{L^2}^2 \leq C(A)X(t). \tag{4.70}
\]

Moreover, by Lemma 2.5, we have
\[
\left| \frac{b_k}{2\lambda_k} \text{Re}\langle D^+(\Delta \chi_{A,k}) R_l - D^+ R \Delta \chi_{A,k} \Phi_k, D^+ R \rangle \right|
\]
\[
\leq C |t|^{-1} \|D^+R\|_{L^2} \|D^+(\Delta \chi_{A,k}) R_l - D^+ R \Delta \chi_{A,k} \Phi_k\|_{L^2}
\]
\[
\leq C |t|^{-1} \|D^+R\|_{L^2} \|R\|_{L^2} \|D^+(\Delta \chi_{A,k}) \Phi_k\|_{L^1}.
\]

We then show that \( \|D^+(\Delta \chi_{A,k}) \Phi_k\|_{L^1} \leq C(A) \). Using the exponential decay for \( \nabla^n \chi(y) \) for \( n \geq 2 \),
\[
|D^+(\Delta \chi_{A,k}) \Phi_k(\xi)| \leq C \lambda_k^{-1} \int_{|x-x_0| \geq 4\sigma} |\Delta \chi \left( \frac{x - \alpha_k}{A\lambda_k} \right)| dx \leq C.
\]

And for \( |\xi| \geq 0 \), it follows that
\[
|\Delta \chi_{A,k} \Phi_k(\xi)| \leq C \left| \frac{\xi}{|\xi|^2} \right| |\nabla^2(\Delta \chi_{A,k}) \Phi_k(\xi)| \leq C \left| \frac{\xi}{|\xi|^2} \right| \int \nabla^2(\Delta \chi_{A,k}) \Phi_k(\xi)| dx
\]
\[
\leq C \left| \frac{\xi}{|\xi|^2} \right| \int_{|x-x_0| \geq 4\sigma} \lambda_k^{-2} \nabla^4 \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right) + |\Delta \chi_A \left( \frac{x - \alpha_k}{A\lambda_k} \right)| dx
\]
\[
\leq C(A) e^{-\frac{C}{|\xi|^2}} \lambda_k^2 \leq C(A) |\xi|^{-2}.
\]

for some \( C(A) > 0 \). So we get
\[
\|D^+(\Delta \chi_{A,k}) \Phi_k\|_{L^1} = \int |\xi| \frac{d|d\xi|}{|\xi|^2} \leq C(A),
\]
Combing the estimates (4.69), (4.70) and (4.71) together, we obtain

\[ |b_k|^{\frac{1}{2}} \text{Re}(D^\frac{1}{2}(\Delta \chi_{A,k} \Phi_k R) - D^\frac{1}{2} R \Delta \chi_{A,k} \Phi_k, D^\frac{1}{2} R)| \leq C(A)t^{-\frac{1}{2}}||D^\frac{1}{2} R||_{L^2}||R||_{L^2} \leq C(A)X(t). \] (4.71)

and thus

\[ \frac{b_k}{2\lambda_k}(\Delta \chi_{A,k}(x - \alpha_k)R_k, DR) = \frac{b_k}{2\lambda_k} \text{Re}(\Delta \chi_{A,k}(x - \alpha_k)R_k, DR) + O(C(A)X(t)), \] (4.72)

Combining the estimates (4.69), (4.70) and (4.71) together, we obtain

\[ \frac{b_k}{2\lambda_k}(\Delta \chi_{A,k}(x - \alpha_k)R_k, DR) = \frac{b_k}{2\lambda_k} \text{Re}(\Delta \chi_{A,k}(x - \alpha_k)R_k, DR) + O(C(A)X(t)), \] (4.73)

and prove (4.67), as claimed.

Next we prove (4.68), which is more involved. First, we expand

\[ \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k}(x - \alpha_k)R_k + \nabla R \Phi_k, DR) \]

\[ = \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k} R_k, DR) + \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k} R_k - D^\frac{1}{2} R \nabla \chi_{A,k} \Phi_k, D^\frac{1}{2} R). \] (4.74)

Integration by parts, it yields that

\[ \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k} R_k, DR) \]

\[ = \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k} R_k, |D^\frac{1}{2} R|^2) + \frac{b_k}{2} \text{Re}(D^\frac{1}{2}(\nabla \chi_{A,k} R_k - D^\frac{1}{2} R \nabla \chi_{A,k} \Phi_k, D^\frac{1}{2} R). \] (4.75)

It’s easy to see that there exists $C(A) > 0$ such that

\[ \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k} R_k, |D^\frac{1}{2} R|^2) \leq C(A)t||D^\frac{1}{2} R||_{L^2}, \]

and by Lemma 2.5,

\[ \frac{b_k}{2} \text{Re}(D^\frac{1}{2}(\nabla \chi_{A,k} R_k - D^\frac{1}{2} R \nabla \chi_{A,k} \Phi_k, D^\frac{1}{2} R)\]

\[ \leq C||D^\frac{1}{2} R||_{L^2}||R||_{L^2}||D^\frac{1}{2}(\nabla \chi_{A,k} \Phi_k)||_{L^2}\]

\[ \leq C(A)t||D^\frac{1}{2} R||_{L^2}||R||_{L^2}, \]

where $||D^\frac{1}{2}(\nabla \chi_{A,k} \Phi_k)||_{L^2} \leq C(A)$ is due to the fact that $\nabla \chi_{A,k} \Phi_k \in \mathcal{S}$. Thus we obtain

\[ \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k}(x - \alpha_k)R_k + \nabla R \Phi_k, DR) = O(C(A)X(t)), \] (4.76)

and by inserting (4.74) into (4.73),

\[ \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k}(x - \alpha_k)(\nabla R_k + \nabla R \Phi_k), DR) = \frac{b_k}{2} \text{Re}(\nabla \chi_{A,k}(x - \alpha_k)R_k + \nabla R \Phi_k, DR) + O(C(A)X(t)), \] (4.77)
Denote \( \nabla \chi_{A,h}(x) := \nabla \chi_A(\frac{x - \alpha_k}{\lambda}) \Phi_k(x) \). By using the commutator formula \([D, \nabla] = 0\), (4.74) and integration by parts twice, we get

\[
b_k \text{Re} \left\langle \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) \nabla R \Phi_k, DR \right\rangle
\]

\[
= - \frac{b_k}{\lambda} \text{Re} \left( \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \Phi_k, DR \right) - b_k \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \nabla \Phi_k, DR \right) - b_k \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \Phi_k, \nabla DR \right)
\]

\[
= - \frac{b_k}{\lambda} \text{Re} \left( \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \Phi_k, DR \right) - b_k \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \Phi_k, D \nabla R \right) + O(C(A)X(t))
\]

\[
= - \frac{b_k}{\lambda} \text{Re} \left( \Delta \chi_A, R, DR \right) - b_k \text{Re} \left( D \nabla \chi_A, R \Phi_k, D \nabla R \right) - b_k \text{Re} \left( D \nabla \chi_A, R \Phi_k, DR \right) + O(C(A)X(t))
\]

which implies that

\[
b_k \text{Re} \left( \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \Phi_k, DR \right) = - \frac{b_k}{\lambda} b_k \text{Re} \left( D \nabla \chi_A, R \Phi_k, D \nabla R \right) + O(C(A)X(t)).
\]  

(4.75)

Combining (4.73) and (4.75), it yields that

\[
b_k \frac{1}{2} \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) (\nabla R + \nabla R \Phi_k), DR \right)
\]

\[
= - \frac{b_k}{2\lambda} \text{Re} \left( \Delta \chi_A, R, DR \right) - \frac{b_k}{2} \text{Re} \left( D \nabla \chi_A, R, \nabla R \right) + O(C(A)X(t)).
\]  

(4.76)

Moreover, a similar calculation as in (4.75) shows that

\[
b_k \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) R \Phi_k, DR \right) = - \frac{b_k}{2\lambda} \text{Re} \left( D \nabla \chi_A, R \Phi_k, D \nabla R \right) + O(C(A)X(t)).
\]  

(4.77)

Comparing (4.76) and (4.77), we find that

\[
\frac{b_k}{2} \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) (\nabla R + \nabla R \Phi_k), DR \right) - b_k \text{Re} \left( \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda} \right) \nabla R_k, DR_k \right)
\]

\[
\leq C \frac{b_k}{\lambda} \text{Re} \left( \Delta \chi_A, R, DR_k \right) + C b_k \text{Re} \left( D \nabla \chi_A, R, \nabla R_k \right) + O(C(A)X(t)).
\]  

(4.78)

Thus the proof of (4.68) is now reduced to estimate the terms in (4.78). It is easy to see from (4.72) that

\[
\frac{b_k}{\lambda} \text{Re} \left( \Delta \chi_A, R, DR_k \right) = O(C(A)X(t)).
\]  

(4.79)

Then we consider the term

\[
b_k \text{Re} \left( D \nabla \chi_A, R, \nabla R - \nabla R_k \right)
\]

which equals to

\[
b_k \sum_{i \neq k} \text{Re} \left( D \nabla \chi_A, R, \nabla R_i \right)
\]

\[
= b_k \sum_{i \neq k} \text{Re} \left( D \nabla \chi_A, R, \nabla R \Phi_i \right) + b_k \sum_{i \neq k} \text{Re} \left( D \nabla \chi_A, R, R \nabla \Phi_i \right).
\]  

(4.80)
We first estimate the second term on the right hand side of (4.80). By Hölder’s inequality,
\[ |b_k \text{Re} \langle D(\nabla_{X,A,R}) - \nabla_{X,A,k} D R \nabla \Phi_l \rangle | \leq C b_k \| (D(\nabla_{X,A,R}) - \nabla_{X,A,k} D R) \nabla \Phi_l \|_{L^2} \| R \|_{L^2} \]
Moreover, applying Lemma 2.5, we have for \( l \neq k \),
\begin{align*}
&\| (D(\nabla_{X,A,R}) - \nabla_{X,A,k} D R) \nabla \Phi_l \|_{L^2} \\
&\leq \| D(\nabla_{X,A,R}) \nabla \Phi_l \|_{L^2} + \| D(\nabla_{X,A,k} \nabla \Phi_l) - D(\nabla_{X,A,k} R) \nabla \Phi_l \|_{L^2} \\
&\leq C \| D(\nabla_{X,A,k} \nabla \Phi_l) \|_{L^1} \| R \|_{L^2} + \| D \nabla \Phi_l \|_{L^1} \| \nabla_{X,A,k} R \|_{L^2}.
\end{align*}
(4.81)
Note that, there exists \( C(A) > 0 \) such that
\[ \| \nabla_{X,A,k} \nabla \Phi_l \|_{L^1} (\xi) \leq C \int_{\mathbb{R}^3} \left| \nabla_{X,A} \left( \frac{x - \alpha_k}{\lambda_k} \right) \Phi_k(x) \nabla \Phi_l(x) \right| dx \leq C(A), \]
and the exponential decay property of \( \nabla^4 \chi(x) \) yields that, for \( |\xi| > 1 \),
\begin{align*}
&\| \nabla_{X,A,k} \nabla \Phi_l \|_{L^1} (\xi) \leq \frac{C}{|\xi|^3} \int_{\mathbb{R}^3} \nabla^3 (\nabla_{X,A,k} \nabla \Phi_l) dx \\
&\leq \frac{C}{|\xi|^3} \int \lambda_k^{-3} \left| \nabla_{X,A} \left( \frac{x - \alpha_k}{\lambda_k} \right) \Phi_k(x) \nabla \Phi_l(x) \right| dx \\
&\leq \frac{C}{|\xi|^3} \left( \int_{|x_k| > 4 \pi \lambda_k^{-1}} \lambda_k^{-3} \left| \nabla_{X,A} \left( \frac{x - \alpha_k}{\lambda_k} \right) \right| dx + 1 \right) \\
&\leq C(A)|\xi|^{-3} \lambda_k^{-2} e^{-\frac{\pi}{4 \lambda_k}} + 1 \leq C(A)|\xi|^{-3}.
\end{align*}
So it follows that
\[ \| D(\nabla_{X,A,k} \nabla \Phi_l) \|_{L^1} = \int \left| \xi \right| \| \nabla_{X,A,k} \nabla \Phi_l \|_{L^1} (\xi) d\xi \leq C(A). \]
(4.82)
We also have \( \| D \nabla \Phi_l \|_{L^1} \leq C \), since \( \nabla \Phi(x) \in \mathcal{S} \). Plugging this fact and (4.82) into (4.81), we get
\[ \| (D(\nabla_{X,A,R}) - \nabla_{X,A,k} D R) \nabla \Phi_l \|_{L^2} \leq C(A) \| R \|_{L^2}, \]
and thus
\[ |b_k \text{Re} \langle D(\nabla_{X,A,R}) - \nabla_{X,A,k} D R, R \nabla \Phi_l \rangle | \leq C(A) b_k \| R \|_{L^2}^2. \]
(4.83)
It remains to estimate the first term on the right hand side of (4.80). We claim that
\[ |b_k \text{Re} \langle D(\nabla_{X,A,R}) - \nabla_{X,A,k} D R, \nabla \Phi_l \rangle | \leq C(A) b_k \| R \|_{H^1}^2. \]
(4.84)
In fact by denoting \( T_g := \nabla ((D(\nabla_{X,A,k} g) - \nabla_{X,A,k} D g) \Phi_l) \), we have
\[ \langle f, T_g \rangle = - \langle \nabla \Phi_l, D(\nabla_{X,A,k} g) - \nabla_{X,A,k} D g \rangle. \]
First it follows from Lemma 2.5 that
\[ \langle f, T_g \rangle \leq \| \nabla f \|_{L^2} \| (D(\nabla_{X,A,k} g) - \nabla_{X,A,k} D g) \Phi_l \|_{L^2} \\
\leq \| \nabla f \|_{L^2} \| D(\nabla_{X,A,k} \Phi_l g) - \nabla_{X,A,k} D \Phi_l g \|_{L^2} + \| D(\nabla_{X,A,k} \Phi_l g) - D(\nabla_{X,A,k} g) \Phi_l \|_{L^2} \\
\leq C \| \nabla f \|_{L^2} \left( \| D(\nabla_{X,A,k} \Phi_l g) \|_{L^1} \| g \|_{L^2} + \| D \Phi_l \|_{L^1} \| \nabla_{X,A,k} g \|_{L^2} \right) \\
\leq C(A) \| f \|_{H^1} \| g \|_{L^2}. \]
Then integrating by parts, it yields that
\[ \langle f, T_g \rangle = \langle f \nabla \Phi_l, D(\nabla_{X,A,k} g) - \nabla_{X,A,k} D g \rangle + \langle f \Phi_l, D \nabla(\nabla_{X,A,k} g) - \nabla(\nabla_{X,A,k} D f) \rangle, \]
thus we infer from Lemma 2.5
\[
|\langle f, Tg \rangle| \leq \|f\|_{L^2} \| (D(\nabla X_{A,k}g) - \nabla X_{A,k}Dg) \nabla \Phi \|_{L^2} + \| (D(\nabla X_{A,k}g) - \nabla X_{A,k}Dg) \Phi \|_{L^2} \\
+ \| (D(\nabla X_{A,k}g) - \nabla(\nabla X_{A,k}Dg) \Phi \|_{L^2})
\leq C(A)\|f\|_{L^2}\|g\|_{H^1}.
\]

By taking \( f = g = R \) and using Lemma 2.6, we obtain
\[
|\langle D(\nabla X_{A,k}R) - \nabla X_{A,k}DR, \nabla R \Phi \rangle | = |\langle f, Tg \rangle| \leq C(A)\|f\|_{H^{1/2}}\|g\|_{H^{1/2}} \leq C(A)\|R\|_{H^{1/2}}^2,
\]
which implies (4.84), as claimed.

Combining (4.80), (4.83) and (4.84), we get
\[
|b_k \Re(D(\nabla X_{A,k}R) - \nabla X_{A,k}DR, \nabla R - \nabla R_k) | \leq C(A)\|R\|_{H^{1/2}}^2.
\]
Inserting (4.79) and (4.85) into (4.78), we conclude
\[
\frac{b_k}{2} \Re(D(\nabla X_{A,k}R) - \nabla X_{A,k}DR, \nabla R_k) = O(C(A)X(t)),
\]
and thus (4.68) is proved, as claimed.

Now we are in the position to decouple the interaction between the remainder \( R \) and the nonlinearity \( f(u) \) in (4.66). We expand the nonlinearity \( f(u) \) to get
\[
\frac{b_k}{2 \lambda_k} \Re(\Delta X_A(\frac{x - \alpha_k}{\lambda_k})R_k, f(u) - f(U)) + \frac{b_k}{2} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla R_k + \nabla R \Phi_k, f(u) - f(U))
\]
\[
= \frac{b_k}{2 \lambda_k} \Re(\Delta X_A(\frac{x - \alpha_k}{\lambda_k})R_k, 2|U|^2 R + U^2 \overline{R}) + \frac{b_k}{2} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla R_k + \nabla R \Phi_k, 2|U|^2 R + U^2 \overline{R})
\]
\[
+ \Re(\frac{b_k}{2 \lambda_k} \Delta X_A(\frac{x - \alpha_k}{\lambda_k})R_k + \frac{b_k}{2} \nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla R_k + \nabla R \Phi_k, 2U|R|^2 + \overline{U} R^2 + |R|^2 R)
\]
(4.86)

First we estimate the terms on the right hand side of (4.86) which are quadratic in \( R \). Integration by parts and using Lemma 4.2, it follows that
\[
\frac{b_k}{2 \lambda_k} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla R_k + \nabla R \Phi_k, 2|U|^2 R + U^2 \overline{R})
\]
\[
= \frac{b_k}{2 \lambda_k} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla R_k, 2|U|^2 R + U^2 \overline{R}) + O(|t|^{-1}\|R\|_{L^2}^2)
\]
\[
= -\frac{b_k}{2 \lambda_k} \Re(\Delta X_A(\frac{x - \alpha_k}{\lambda_k})R_k, 2|U|^2 R + U^2 \overline{R}) - \frac{b_k}{2} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla U \Phi_k, 2U|R|^2 + \overline{U} R^2) + O(|t|^{-1}\|R\|_{L^2}^2).
\]
Using this fact, we have
\[
\frac{b_k}{2 \lambda_k} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})R_k, 2|U|^2 R + U^2 \overline{R}) + \frac{b_k}{2} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla R_k + \nabla R \Phi_k, 2|U|^2 R + U^2 \overline{R})
\]
\[
= -\frac{b_k}{2 \lambda_k} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla U \Phi_k, 2U|R|^2 + \overline{U} R^2) + O(|t|^{-1}\|R\|_{L^2}^2)
\]
\[
= -\frac{b_k}{2} \Re(\nabla X_A(\frac{x - \alpha_k}{\lambda_k})\nabla U_k, 2U_k|R_k|^2 + \overline{U}_k R_k^2) + O(|t|X(t)).
\]
(4.87)
where in the last step we apply Lemma 2.9. Then it remains to estimate the higher order terms on the right hand side of (4.86). By using Sobolev embeddings, (4.1) and Lemma 4.2, we get

\[
\frac{b_k}{2\lambda_k} \Re(\Delta_{\lambda_k} X - \frac{\alpha_k}{\lambda_k}) R_k, 2U|R|^2 + \overline{U} R^2 + |R|^2 \Re) \leq C(A) \frac{b_k}{\lambda_k} (|R|_L^3 |U|_L^4 + |R|_{L^2}^4) \\
\leq C(A) \frac{b_k}{\lambda_k} (|R|_L^2 |D^\frac{1}{2} R|_L^2 |U|_L^2 |D^\frac{1}{2} U|_L^2 + |R|_L^2 |D \overline{U} |_L^2) \\
\leq C(A) (\lambda^{-\frac{1}{2}} |R|_L^2 |D^\frac{1}{2} R|_L^2 + \lambda^{-\frac{1}{2}} |R|_L^2 |D \overline{U} |_L^2) \leq C(A) X(t),
\]

and

\[
\frac{b_k}{2} \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k}) \nabla R \Phi_k, 2U|R|^2 + \overline{U} R^2 + |R|^2 \Re \\
\leq C(A) \frac{b_k}{\lambda_k} (|R|_{H^\frac{1}{2}}^3 |\nabla_{\lambda_k} R|_L^2 |U|_L^2 |\nabla \Phi_k|_L^2 + |\nabla_{\lambda_k} \Phi_k|_L^2 R^2 |R|_{H^\frac{1}{2}}^2) \\
\leq C(A) \frac{b_k}{\lambda_k} (|R|_{H^\frac{1}{2}}^3 |\nabla_{\lambda_k} R|_L^2 |U|_L^2 |\nabla \Phi_k|_L^2 + |\nabla_{\lambda_k} \Phi_k|_L^2 R^2 |R|_{H^\frac{1}{2}}^2 + |U|_L^2 |\nabla_{\lambda_k} \Phi_k|_L^2 |\nabla R|_L^2) \\
\leq C(A)|t|^{3-k} \left( |t|^{-1} \left( \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right)^{\frac{1}{2}} + |t|^{-2} \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right) |R|^2_{H^\frac{1}{2}} \\
\leq C(A) \left( \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right)^{\frac{1}{2}} X(t).
\]

Moreover, the a priori estimate (4.1) implies that $|D^\pm \Re R|_{L^2} \leq C$. Combining this fact with the fractional chain rule and (2.7), it yields that

\[
|b_k \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k}) \nabla R \Phi_k, 2U|R|^2 + \overline{U} R^2 + |R|^2 \Re) \\
\leq C(A) \frac{b_k}{\lambda_k} (|R|_{H^\frac{1}{2}}^3 |\nabla_{\lambda_k} R|_L^2 |U|_L^2 |\nabla \Phi_k|_L^2 + |\nabla_{\lambda_k} \Phi_k|_L^2 R^2 |R|_{H^\frac{1}{2}}^2 + |U|_L^2 |\nabla_{\lambda_k} \Phi_k|_L^2 |\nabla R|_L^2) \\
\leq C(A)|t|^{3-k} \left( |t|^{-1} \left( \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right)^{\frac{1}{2}} + |t|^{-2} \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right) |R|^2_{H^\frac{1}{2}} \\
\leq C(A) \left( \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right)^{\frac{1}{2}} X(t).
\]

So it follows from (4.88), (4.89) and (4.90) that

\[
|\Re(\frac{b_k}{2\lambda_k} \Delta_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} R_k + \frac{b_k}{2} \nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} (\nabla R_k + \nabla R \Phi_k), 2U|R|^2 + \overline{U} R^2 + |R|^2 \Re) \\
\leq C(A) \left( \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right)^{\frac{1}{2}} X(t).
\]

Thus by inserting (4.87) and (4.91) into (4.86), we find that

\[
\frac{b_k}{2\lambda_k} \Re(\Delta_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} R_k, f(u) - f(U)) + \frac{b_k}{2} \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} (\nabla R_k + \nabla R \Phi_k), f(u) - f(U)) \\
= -b_k \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k}) \nabla U_k, 2U_k|R_k|^2 + \overline{U_k} R_k^2) + O(C(A) \left( \ln(2 + |R|_{H^\frac{1}{2}}^{-1}) \right)^{\frac{1}{2}} X(t)).
\]

Last, we estimate the terms in (4.66) which involves the remainder \( R \) with the error \( \eta \). Integration by parts yields that

\[
\frac{b_k}{2\lambda_k} \Re(\Delta_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} R_k, \eta) + \frac{b_k}{2} \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} (\nabla R_k + \nabla R \Phi_k), \eta) \\
= -\frac{b_k}{2\lambda_k} \Re(\Delta_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} R_k, \eta) - b_k \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k} R_k, \nabla \eta) - \frac{b_k}{2} \Re(\nabla_{\lambda_k} X - \frac{\alpha_k}{\lambda_k}) R \nabla \Phi_k, \eta).
\]
So by Cauchy-Schwartz inequality and Lemma 4.2, we obtain the estimate
\[
\left| \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) R_k, \eta \rangle + \frac{b_k}{2} \text{Re}\langle \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) (\nabla R_k + \nabla R \Phi_k), \eta \rangle \right|
\leq C(A)b_k(\lambda_k)^{-1}\|R\|_{L^2}^2 \|\eta\|_{L^2} + \|\nabla R\|_{L^2} \|\eta\|_{L^2} + \|R\|_{L^2} \|\eta\|_{L^2}
\leq C(A)\left( \frac{\|R\|_{L^2}^2}{t^2} + \frac{||R||_{L^2}^2}{|t|^{3-\kappa}} + |t|^{3-\kappa} \right) \leq C(A)\left( \frac{\|R\|_{L^2}^2}{t^2} + \frac{||R||_{L^2}^2}{|t|^{3-\kappa}} + |t|^{3-\kappa} \right).
\] (4.93)
Inserting (4.67), (4.68), (4.92) and (4.93) into (4.66), we finally complete the proof of Proposition 4.8.

Combing Propositions 4.7 and 4.8 together, we are now able to prove Proposition 4.6.

**Proof of Proposition 4.6.** By Propositions 4.7 and 4.8, we first get
\[
\frac{d\bar{\mathcal{A}}}{dt} \geq \sum_{k=1}^{K} \left( \frac{b_k}{2\lambda_k} \|R_k\|_{L^2}^2 - \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) R_k, \partial_t R_k \rangle + \frac{b_k}{2} \text{Re}\langle \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) \nabla R_k, \partial_t R_k \rangle \right)
\]
\[+ \frac{b_k}{2\lambda_k} \text{Re}\langle \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) - \frac{x - \alpha_k}{\lambda_k}, \nabla U_k, 2U_k|R_k|^2 + \bar{U}_k R_k^2 \rangle + O(C(A)\left( \ln(2 + \|R\|_{H^1}^{-1}) \right)^{\frac{1}{2}} X(t) + \frac{||R||_{L^2}^2}{t^{3-\kappa}}), \]
\[
\] (4.94)
Following the functional calculus as in the proof of [29, Lemma 6.1], we obtain that for \(1 \leq k \leq K\),
\[
\frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) R_k, \partial_t R_k \rangle + \frac{b_k}{2\lambda_k} \text{Re}\langle \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) \nabla R_k, \partial_t R_k \rangle
\]
\[= \frac{b_k}{2\lambda_k} \int_{s=0}^{\infty} \sqrt{s} \int \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) |\nabla R_{k,s}|^2 dx ds - \frac{b_k}{8\lambda_k} \int_{s=0}^{\infty} \sqrt{s} \int \Delta^2 \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) |R_{k,s}|^2 dx ds,
\] (4.95)
where
\[
R_{k,s}(t, x) = \frac{1}{\sqrt{2\pi s}} \int e^{-\sqrt{s}|x-y|^2} R(t, y) dy,
\]
solves the equation
\[
- \Delta R_{k,s} + sR_{k,s} = \sqrt{\frac{2}{\pi}} R_k, \quad \text{for} \ s > 0.
\] (4.96)
Then by inserting (4.95) into (4.94), we get
\[
\frac{d\bar{\mathcal{A}}}{dt} \geq \sum_{k=1}^{K} \left( \frac{b_k}{2\lambda_k} \|R_k\|_{L^2}^2 - \frac{b_k}{2\lambda_k} \int_{s=0}^{\infty} \sqrt{s} \int \Delta \chi \left( \frac{x - \alpha_k}{\lambda_k} \right) |\nabla R_{k,s}|^2 dx ds
\]
\[+ \frac{b_k}{8\lambda_k} \int_{s=0}^{\infty} \sqrt{s} \int \Delta^2 \chi \left( \frac{x - \alpha_k}{\lambda_k} \right) |R_{k,s}|^2 dx ds - \frac{b_k}{2\lambda_k} \text{Re}\langle \Delta \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) R_k, \partial_t R_k \rangle
\]
\[+ \frac{b_k}{2\lambda_k} \text{Re}\langle \nabla \chi_A \left( \frac{x - \alpha_k}{\lambda_k} \right) - \frac{x - \alpha_k}{\lambda_k}, \nabla U_k, 2U_k|R_k|^2 + \bar{U}_k R_k^2 \rangle + O(C(A)\left( \ln(2 + \|R\|_{H^1}^{-1}) \right)^{\frac{1}{2}} X(t) + \frac{||R||_{L^2}^2}{t^{3-\kappa}}), \]
\[
\] (4.97)
Here we define
\[ R_k = \lambda_k^{\frac{1}{2}} \varepsilon_k(t, \frac{x - \alpha_k}{\lambda_k}) e^{i\eta_k}, \quad \text{with } \varepsilon_k = \varepsilon_{k,1} + i\varepsilon_{k,2}, \quad 1 \leq k \leq K. \] (4.98)

Recall the fact that \( Q_k(y) = Q(y) + |r(y)|^2 \). So it follows from (4.97) that
\[
\frac{d^3}{dt} \geq \sum_{k=1}^{K} \frac{b_k}{2A_k} \left( \| \varepsilon_k \|^2_{L^2} + \int_{s=0}^{+\infty} \sqrt{s} \int \chi''\left(\frac{y}{A}\right) |\nabla \varepsilon_{k,1}|^2 dy ds - \int 3Q^2 \varepsilon_{k,1}^2 + Q^2 \varepsilon_{k,2}^2 dy \right) \\
+ \sum_{k=1}^{K} \frac{b_k}{A_k} \text{Re} \left( \int (A\chi\left(\frac{y}{A}\right) - y) \cdot \nabla Q_k (Q_k e_k^2 + 2Q_k |e_k|^2) dx - \frac{1}{8A_k^2} \int_{s=0}^{+\infty} \sqrt{s} \int \chi'''\left(\frac{y}{A}\right) |\varepsilon_{k,1}|^2 dy ds \right) \\
+ O(C(A) \left( \ln(2 + \|R(t)\|^{-1}_{H^2}) \right)^2) X(t) + \frac{\|R\|^2_{L^2}}{|t|^{3-k}} + |t|^{3-\varepsilon}).
\] (4.99)

Moreover, we infer from the orthogonality conditions (3.10), Lemmas 2.9, 4.2 and 4.3 that, for \( 1 \leq k \leq K \), the instable directions of \( \varepsilon_k \) can be controlled by
\[
S \text{cal}(\varepsilon_k) = O(t^2\|R\|^2_{L^2} + t^{8-2\varepsilon}).
\]

Then we recall a coercivity estimate ([29, Proposition B.1]), which claims that there exists \( C > 0 \) such that, for \( A \) large enough and for all \( f = f_1 + i f_2 \in H^1(\mathbb{R}) \),
\[
\|f\|^2_{L^2} + \int_{s=0}^{+\infty} \sqrt{s} \int \chi''\left(\frac{y}{A}\right) |\nabla f_1|^2 dy ds - \int 3Q^2 f_1^2 + Q^2 f_2^2 dy \geq C \int |f|^2 dx - \frac{1}{C} \text{S \text{cal}(f)},
\]
where \( f_1 \) is defined as in (4.96). So by using this fact, we obtain that, for \( A \) large enough, \( 1 \leq k \leq K \),
\[
\|\varepsilon_k\|^2_{L^2} + \int_{s=0}^{+\infty} \sqrt{s} \int \chi''\left(\frac{y}{A}\right) |\nabla \varepsilon_{k,1}|^2 dy ds - \int 3Q^2 \varepsilon_{k,1}^2 + Q^2 \varepsilon_{k,2}^2 dy \geq C\|\varepsilon_k\|_{L^2} - \frac{1}{C} |t|^{8-2\varepsilon}. \] (4.100)

Recall the definition of the cut-off function \( \chi \), we have \( A\chi\left(\frac{y}{A}\right) - y = 0 \) for \( |y| \leq A \). So it follows from the decay property of \( Q_k(y) \leq C(y)^{-2} \) that
\[
| \int (A\chi\left(\frac{y}{A}\right) - y) \cdot \nabla Q_k (Q_k e_k^2 + 2Q_k |e_k|^2) dy | \leq C_A^{\frac{1}{2}} |y|^2 \|\varepsilon_k\|_{L^2} \leq \frac{C_A}{A} \|R\|^2_{L^2}.
\] (4.101)

Moreover, according to [29, Lemma B.3], the following holds
\[
\left| \int_{s=0}^{+\infty} \sqrt{s} \int \chi'''\left(\frac{y}{A}\right) |\varepsilon_{k,1}|^2 dy ds \right| \leq \frac{C}{A} \|R\|^2_{L^2}. \] (4.102)

Thus inserting (4.100), (4.101) and (4.102) into (4.99), we obtain that there exists \( C > 0 \) such that, for \( A \) large enough,
\[
\frac{d^3}{dt} \geq C \frac{\|R\|^2_{L^2}}{|t|^{3-k}} + O(C(A) \left( \ln(2 + \|R(t)\|^{-1}_{H^2}) \right)^2) X(t) + |t|^{3-\varepsilon}),
\]
and finish the proof of Proposition 4.6. \( \square \)

4.3. **Proof of bootstrap estimates.** First, we establish in the following lemma the lower and upper bound for the generalized energy functional \( \mathcal{S}(t) \).

**Lemma 4.9.** For all \( t \in [T_*, T] \), there exists \( C_1, C_2 > 0 \) such that for \( A \) sufficiently large,
\[
C_1 X(t) - \frac{1}{C_1} t^{6-2\varepsilon} \leq \mathcal{S}(t) \leq C_2 X(t).
\]
Proof. By Lemma 2.6, Sobolev embeddings and Lemma 2.9, we expand the functional

\[ \mathcal{Z}(t) = \frac{1}{2} \text{Re} \int |D^2 R|^2 + \sum_{k=1}^{K} \frac{1}{\lambda_k} |R|^2 \Phi_k - 2 |U_k|^2 |R|^2 - U_k^2 \overline{R}^2 \, dx + o(X(t)) \]

(4.103)

Using the fact that \( ||U(t)||_{L^\infty} \leq Ct^{-2} \), the upper bound is easy to see from (4.103) that for some \( C_2 > 0 \),

\[ \mathcal{Z}(t) \leq C_2 X(t). \]

It remains to prove the lower bound. By using the fact that \( \sum_{k=1}^{K} \Phi_k = 1 \), we rewrite

\[ \sum_{k=1}^{K} \text{Re} \int (|D^2 R|^2 + \frac{1}{\lambda_k} |R|^2) \Phi_k - 2 |U_k|^2 |R|^2 - U_k^2 \overline{R}^2 \, dx \]

(4.104)

where we denote \( \phi_{A,k}(x) := \phi_A(\frac{x-\alpha_k}{\lambda_k}) \) with the cut-off function \( \phi_A \) being defined as in Lemma 2.8.

Recall the renormalization (3.15). On one hand, using \( Q_k(y) = Q(y) + O(|t||\langle y \rangle|^{-2}) \), it follows that

\[ I = \sum_{k=1}^{K} \frac{1}{\lambda_k} \text{Re} \int (|D^2 \epsilon_k|^2 + |\epsilon_k|^2) \phi_A - 2 |Q_k|^2 |\epsilon_k|^2 - Q_k^2 \overline{\epsilon}_k^2 \, dy \]

\[ = \sum_{k=1}^{K} \frac{1}{\lambda_k} \int (|D^2 \epsilon_k|^2 + |\epsilon_k|^2) \phi_A - 2 Q_k^2 \epsilon_{k,1}^2 - Q_k^2 \epsilon_{k,2}^2 \, dy + O(|t|^{-1}||R||^2_{L^2}). \]

Moreover, the orthogonality conditions (3.10), Lemma 4.3 and Lemma 4.2 implies that, the instable directions of \( \epsilon_k \) can be controlled by

\[ \text{Scal}(\epsilon_k) = O(t^2 ||R||^2_{L^2} + t^{8-2\kappa}). \]

Thus, applying Lemma 2.8 and Lemma 4.2, there exists \( C_1 > 0 \) such that for \( A \) sufficiently large,

\[ I \geq \sum_{k=1}^{K} \frac{C_1}{\lambda_k} \int (|D^2 \epsilon_k|^2 + |\epsilon_k|^2) \phi_A \, dy - \frac{1}{C_1} t^{6-2\kappa} + o(X(t)) \]

(4.105)

\[ \geq C_1 \sum_{k=1}^{K} \int (|D^2 R|^2 + \frac{1}{\lambda_k} |R|^2) \phi_{A,k} \, dx - \frac{1}{C_1} t^{6-2\kappa} + o(X(t)). \]
On the other hand, let \( C := \min\{1, C_1\} \), for \( t \) close to 0 such that \( |\alpha_k(t) - x_k| \leq \sigma \), we have after renormalization,

\[
II = \sum_{k=1}^{K} \frac{1}{\lambda_k} \int \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy
\]

\[
\geq \sum_{k=1}^{K} \frac{\tilde{C}}{\lambda_k} \int_{|y| \leq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy
\]

\[
+ \sum_{k=1}^{K} \frac{1}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy.
\]

Here we use the fact that \( \Phi_k(\lambda_k y + \alpha_k) - \phi_A(y) \geq 0 \) for \( |y| \leq \frac{\sigma}{\lambda_k}, 1 \leq k \leq K \). Moreover,

\[
\sum_{k=1}^{K} \frac{1}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy
\]

\[
\geq \sum_{k=1}^{K} \frac{\tilde{C}}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy
\]

\[
- \sum_{k=1}^{K} \frac{1 - \tilde{C}}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) \phi_A(y) dy.
\]

Recall the definition of \( \phi_A \), we have \( |\phi_A(y)| \leq C \lambda_k^a \) for \( |y| \geq \frac{\sigma}{\lambda_k} \), with \( 0 < a < 1 \). So it follows that

\[
\sum_{k=1}^{K} \frac{1}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy
\]

\[
\geq \sum_{k=1}^{K} \frac{\tilde{C}}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy - C \sum_{k=1}^{K} \frac{\lambda_k^a}{\lambda_k} \int \left( |D^2 e_k|^2 + |e_k|^2 \right) dy
\]

\[
\geq \sum_{k=1}^{K} \frac{\tilde{C}}{\lambda_k} \int_{|y| \geq \frac{\sigma}{\lambda_k}} \left( |D^2 e_k|^2 + |e_k|^2 \right) (\Phi_k(\lambda_k y + \alpha_k) - \phi_A(y)) dy + o(X(t)).
\]

Thus by inserting (4.107) into (4.106), we get after renormalization,

\[
II \geq \sum_{k=1}^{K} \tilde{C} \int \left( |D^2 R|^2 + \frac{1}{\lambda_k} |R|^2 \right) (\Phi_k - \phi_A,k) dx + o(X(t)). \tag{4.108}
\]

Plugging (4.105) and (4.108) into (4.104), we finally obtain the lower bound

\[
\Im(t) \geq C \int |D^2 R|^2 + \sum_{k=1}^{K} \frac{1}{\lambda_k} |R|^2 \Phi_k dx - \frac{1}{C} t^{\delta - 2\epsilon} + o(X(t)) \geq CX(t) - \frac{1}{C} t^{\delta - 2\epsilon},
\]

and finish the proof. \( \square \)

Now we are ready to prove Theorem 4.1, using a Gronwall type argument.

**Proof of Theorem 4.1.** (i) Bootstrap estimates of \( X(t) \).
From Proposition 4.6, Lemma 4.9 and taking account $I(T) = 0$, it yields that for all $t \in [T_*, T]$

$$C_1 X(t) - \frac{1}{C_1} t^{6-2\kappa} \leq \mathbb{S}(T) - \int_t^T \frac{d}{ds} d s \leq \mathbb{S}(T) - \int_t^T \frac{d}{ds} d s \leq \int_t^T C(A) \left( \ln(2 + ||R(s)||_{H^2}^{-1}) \right)^{\frac{3}{2}} X(s) + |s|^{-3-k} d s. \quad (4.109)$$

Moreover, by Lemma 4.2, for $s$ close to 0, we have

$$C(A) \left( \ln(2 + ||R(s)||_{H^2}^{-1}) \right)^{\frac{3}{2}} X(s) = o(|s|^{-1} X(s)) \leq \frac{C_1(4-2\kappa)}{8} |s|^{-3-k}. \quad (4.110)$$

Inserting (4.110) into (4.109), it follows that, there exists $t_0$ such that for all $t \in [T_*, T]$ with $T_* \geq t_0$,

$$X(t) \leq \frac{1}{8}|t|^{4-2\kappa} + \frac{1}{C_1(4-\kappa)} |t|^{4-\kappa} \leq \frac{1}{4}|t|^{4-2\kappa},$$

which implies that, for all $t \in [T_*, T]$,

$$||D^{\frac{1}{2}} R(t)||_{L^2} \leq \frac{1}{2}|t|^{2-k}, \quad ||R(t)||_{L^2} \leq \frac{1}{2}|t|^{3-k},$$

and close the bootstrap of $X(t)$ in (4.5).

(ii) Bootstrap estimates of $||D^{\frac{1}{2}+\delta} R||_{L^2}$. First, we can write the equation of $R$

$$i \partial_t R = DR - |R|^2 R - \eta - F, \quad (4.111)$$

with $\eta$ defined as in (4.10) and

$$F = |U + R|^2 (U + R) - |U|^2 U - |R|^2 R.$$

Denote $Y(t) := ||D^{\frac{1}{2}+\delta} R||_{L^2}$. Using the equation (4.111) and integration by parts, we obtain the evolution of $Y(t)$

$$Y'(t) = -2 \text{Im}(D^{\frac{1}{2}+\delta}(|R|^{2} R + \eta + F), D^{\frac{1}{2}+\delta} R).$$

Using $U = \sum_{k=1}^{K} U_k$, it is easy to adapt the estimates in [29, Appendix E] to our multi-bubble case and obtain (for simplicity we omit the details)

$$|Y'(t)| \leq C|t|^{2-2\kappa-4\delta} + C|t|^{-1} Y(t) ||R||_{H^2} \ln \frac{1 + Y^2(t)}{||R||_{H^2}}.$$

Here, we apply the a prior bound that $X(t) \leq |t|^{6-2\kappa}$ and Lemma 4.2. Integrating this from $t$ to $T$, using the boundary condition $Y(T) = 0$ and the a prior bound (4.1), we get that for $t$ close to 0,

$$Y(t) \leq C|t|^{6-2\kappa} \leq \frac{1}{4}|t|^{4-2\kappa-4\delta}. \quad (4.112)$$

So it yields that

$$||D^{\frac{1}{2}+\delta} R||_{L^2} \leq \frac{1}{2}|t|^{1-k-2\delta},$$

and close the bootstrap of $||D^{\frac{1}{2}+\delta} R||_{L^2}$ in (4.5).

(iii) Bootstrap estimates of $\lambda_k$ and $b_k$. From Lemma 4.2, it follows that, for all $t \in [T_*, T]$,

$$\frac{d}{dt} (\lambda_k^{\frac{1}{2}} b_k) = \lambda_k^{\frac{1}{2}} \left( \lambda_k b_k + \frac{1}{2} b_k^2 - \frac{1}{2} (b_k \lambda_k + b_k^2) b_k \right) \leq C|t|^{-3} \text{Mod}_k,$$

which along with (4.12) and the fact $(\lambda_k^{\frac{1}{2}} b_k)(T) = \omega$ yields that

$$|(\lambda_k^{\frac{1}{2}} b_k)(t) - \omega| \leq \int_t^T \frac{d}{ds} (\lambda_k^{\frac{1}{2}} b_k) d s \leq C \int_t^{t_0} |s|^{-3} \text{Mod}_k(s) d s \leq C|t|^{-\kappa}. \quad (4.112)$$
Moreover, a calculation shows that
\[
\left| \frac{d}{dt}(\lambda_k^2(t) + \omega t) \right| = \frac{1}{2} |\lambda_k^{-1}(\lambda_k b_k) - (\lambda_k^{-1} b_k - \omega)| \leq C|t|^{2-\kappa}.
\]
Taking into account \(\lambda_k^2(T) = -\frac{\omega}{2} T\),
\[
|\lambda_k^2(t) + \omega t| \leq \int_t^T \left| \frac{d}{ds}(\lambda_k^2(s) + \omega s) \right| ds \leq C \int_t^0 |s|^{1-\kappa} ds \leq C|t|^{1-\kappa}.
\]
So as long as \(t\) close to 0, we get that
\[
|\lambda_k(t) - \frac{\omega^2}{4} t^2| \leq |\lambda_k^2(t) + \frac{\omega^2}{2} t| |\lambda_k^2(t) - \frac{\omega^2}{2} t| \leq C|t|^{1-\kappa} \leq \frac{1}{2}|t|^{1-2\kappa}, \tag{4.113}
\]
thereby close the bootstrap of \(\lambda_k\) in (4.6).

Similarly, by (4.12) and (4.112), it follows that
\[
|\frac{d}{dt}(b_k(t) + \omega t) - 1| = |\lambda_k^{-1}(\lambda_k b_k + 1) - \frac{1}{2}(\lambda_k^{-1} b_k - \omega)| \leq C|t|^{2-\kappa},
\]
which along with \(b_k(T) = -\frac{1}{2} \omega^2 T\) yields that for \(t\) close to 0,
\[
|b_k(t) + \frac{\omega^2}{2} t| \leq \int_t^T \left| \frac{d}{ds}(b_k(s) + \frac{\omega^2}{2} s) \right| ds \leq C \int_t^0 |s|^{2-\kappa} ds \leq C|t|^{3-\kappa} \leq \frac{1}{2}|t|^{3-2\kappa}.
\]
Thus we close the bootstrap of \(b_k\) in (4.6).

(iv) Bootstrap estimates of \(v_k\) and \(\alpha_k\). From (4.113), Lemma 4.2 and using the estimate of \(v_k\) in Lemma 4.5, it follows that, for \(t\) close to 0,
\[
|v_k - \frac{\omega^2}{4} t^2| \leq |\lambda_k v_k - \lambda_k - 1| + |\lambda_k - \frac{\omega^2}{4} t^2| \leq C|t|^{1-\kappa} \leq \frac{1}{2}|t|^{1-2\kappa}, \tag{4.114}
\]
which closes the bootstrap of \(v_k\) in (4.7).

Moreover from (4.12) and (4.114) and using the fact that \(\alpha_k(T) = x_k\),
\[
|\alpha_k| = |\alpha_k - v_k + v_k| \leq Mod_k(t) + |v_k| \leq Mod_k(t) + Ct^2.
\]
So we infer that for \(t\) close to 0,
\[
|\alpha_k(t) - x_k| \leq \int_t^T |\alpha_k(s)| ds \leq \frac{1}{2}|t|^{3-\kappa},
\]
which closes the bootstrap of \(\alpha_k\) in (4.7).

(iv) Bootstrap estimate of \(\gamma_k\). By (4.113) and Lemma 4.2, a calculation shows that,
\[
|\frac{d}{dt}(\gamma_k(t) + \frac{4}{\omega^2} t - \theta_k)| = |\lambda_k^{-1}(\lambda_k \gamma_k - 1) + \frac{\omega^2 t^2 - 4\lambda_k}{\omega^2 t^2 \lambda_k}| \leq C|t|^{-\kappa},
\]
which yields that for \(t\) close to 0,
\[
|\gamma_k(t) + \frac{4}{\omega^2} t - \theta_k| \leq \int_t^T |\frac{d}{ds}(\gamma_k(s) + \frac{4}{\omega^2} s - \theta_k)| ds \leq C \int_t^0 |s|^{-\kappa} ds \leq C|t|^{1-\kappa} \leq \frac{1}{2}|t|^{1-2\kappa},
\]
thereby we close the bootstrap of \(\gamma_k\) in (4.8). Thus the proof of Theorem 4.1 is complete. □
5. Existence of multi-bubble solutions

We apply a compactness argument to prove Theorem 1.1, while the first step is the construction of approximate solutions. Let \( \{t_n\} \) be an increasing sequence converging to 0. For \( 1 \leq k \leq K \), let

\[
(\lambda_{n,k}, b_{n,k}, v_{n,k}, \alpha_{n,k}, \gamma_{n,k})(t_n) = \left( \frac{\omega^2}{4} t_n^2, -\frac{\omega^2}{2} t_n, \frac{\omega^2}{4} t_n^2, x_k, -\frac{4}{\omega^2 t_n} + \theta_k \right),
\]

and \( Q_{n,k}(t_n, x) \) be the approximate profiles defined in Lemma 3.1 with \( b_{n,k}(t_n), v_{n,k}(t_n) \) replacing \( b_k, v_k \) respectively. The approximate solutions \( u_n \) solve the following equation

\[
\begin{aligned}
\begin{cases}
     i \delta u_n &= Du_n - |u_n|^2 u_n, \\
     u_n(t_n) &= \sum_{j=1}^{K} \lambda_{n,k}^{-\frac{1}{2}}(t_n) Q_{n,k}\left(t_n, \frac{x - \alpha_{n,k}(t_n)}{\lambda_{n,k}(t_n)} \right) e^{i \gamma_{n,k}(t_n)}.
\end{cases}
\end{aligned}
\]

For these solutions, we can establish the following uniform estimates which is a consequence of the bootstrap estimates in Theorem 4.1.

**Theorem 5.1** (Uniform estimates). Let \( \kappa, \delta \in (0, \frac{1}{2}) \) and \( \kappa + 2\delta < 1 \). There exists a uniform backwards time \( t_0 < 0 \) such that, for \( n \) large enough, \( u_n \) admits the geometrical decomposition \( u_n = U_n + R_n \) on \( [t_0, t_n] \), with the main blow-up profile

\[
U_n(t, x) = \sum_{k=1}^{K} U_{n,k}(t, x), \quad \text{with} \quad U_{n,k}(t, x) = \lambda_{n,k}^{-\frac{1}{2}}(t) Q_{n,k}\left(t, \frac{x - \alpha_{n,k}(t)}{\lambda_{n,k}(t)} \right) e^{i \gamma_{n,k}(t)}.
\]

Moreover, the reminder and the modulation parameters satisfy that, for \( t \in [t_0, t_n] \),

\[
\begin{align}
||D^2 R_n(t)||_{L^2} &\leq |t|^{2-\kappa}, & \|R_n(t)\|_{L^2} &\leq |t|^{3-\kappa}, & \|D^{1+\delta} R_n(t)\|_{L^2} &\leq |t|^{1-\kappa-2\delta} \tag{5.3} \\
|\lambda_{n,k}(t) - \frac{\omega^2}{4} t_n^2| &\leq |t|^{4-2\kappa}, & |b_{n,k}(t) + \frac{\omega^2}{2} t| &\leq |t|^{3-2\kappa}, & |\alpha_{n,k}(t) - x_k| &\leq |t|^{3-\kappa}, \tag{5.4} \\
|v_{n,k}(t) - \frac{\omega^2}{4} t_n^2| &\leq |t|^{4-2\kappa}, & |\gamma_{n,k}(t) + \frac{4}{\omega^2 t} - \theta_k| &\leq |t|^{1-2\kappa}, & 1 \leq k \leq K. \tag{5.5}
\end{align}
\]

**Remark 5.2.** In view of Theorem 4.1 and the boundary condition of the parameters (5.1), the proof of Theorem 5.1 follows from a standard continuity argument, see [40] for more details. For simplicity, the details are omitted here. Let’s point out that, the decay estimates for the remainder (5.3) is decided by the contributions coming from the interaction between bubbles. In fact, due to the presence of the strong interaction, we cannot improve the bound to

\[
\|D^{1+\delta} R_n(t)\|_{L^2} = O(|t|^2) \quad \text{and} \quad \|R_n(t)\|_{L^2} = O(|t|).
\]

This important fact brings challenge to the further study of the uniqueness problem of the multi-bubble solutions.

According to Theorem 5.1, \( \{u_n(t_0)\} \) are uniformly bounded in \( H^{1+\delta}(\mathbb{R}) \), thus there exists \( u_0 \in H^{1+\delta}(\mathbb{R}) \) such that for any \( s \in [0, \frac{1}{2} + \delta] \),

\[
u_n(t_0) \rightharpoonup u_0, \quad \text{weakly in} \ H^s(\mathbb{R}), \quad \text{as} \ n \to \infty. \tag{5.6}
\]

Then we show that

\[
u_n(t_0) \to u_0, \quad \text{strongly in} \ L^2(\mathbb{R}), \quad \text{as} \ n \to \infty. \tag{5.7}
\]
In fact, let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a smooth nonnegative cut-off function such that \( \varphi(x) = 0 \) for \( |x| \leq 1 \) and \( \varphi(x) \equiv 1 \) for \( |x| \geq 2 \). Denote \( \varphi_A(x) := \varphi\left(\frac{x}{A}\right) \) for \( A > 0 \). On one hand, the boundary condition \( u_0(t_n) = \sum_{k=1}^{K} U_{n,k}(t_n) \) and Lemma 2.9 implies that, for all \( A \geq 2 \max_{1 \leq k \leq K} |x_k| \),
\[
    \int |u_n(t_n)|^2 \varphi_A dx \leq \frac{C}{A^3} t_n^6. \tag{5.8}
\]

On the other hand, it follows from the equation (5.2) and integration by parts that
\[
    \frac{d}{dt} \int |u_n|^2 \varphi_A dx = 2 \text{Im} \int \overline{u}_n Du_n \varphi_A dx = \text{Im} \int u_n (D \overline{u}_n \varphi_A - D \overline{u}_n \varphi_A)_dx.
\]

By Lemma 2.5, the conservation law of mass and using the fact that \( \|D \overline{\varphi}_A\|_{L^1} = \frac{1}{A} \|D \varphi\|_{L^1} \), we obtain
\[
    \left| \frac{d}{dt} \int |u_n|^2 \varphi_A dx \right| \leq C \|u_n\|^2_{L^2} \|D \overline{\varphi}_A\|_{L^1} \leq C \|u_n\|^2_{L^2} \|\overline{\varphi}\|_{L^1} \leq \frac{C}{A}.
\]

So by integrating from \( t_0 \) to \( t_n \) and using (5.8), there exists \( C > 0 \) independent of \( n \) such that
\[
    \int |u_n(t_0)|^2 \varphi_A dx \leq \frac{C}{A},
\]

which yields that for all \( n \geq 1 \),
\[
    \|u_n(t_0)\|_{L^2(|x| \geq A)} \to 0, \quad \text{as } A \to \infty. \tag{5.9}
\]

Thus the weak convergence in \( L^2 \) along with (5.9) implies (5.7), as claimed. Combining (5.6) and (5.7) together, we get that for any \( s \in [0, \frac{1}{2} + \delta) \),
\[
    u_n(t_0) \to u_0, \quad \text{strongly in } H^s(\mathbb{R}), \quad \text{as } n \to \infty.
\]

By virtue of (5.7) and the local well-posedness theory, we obtain a unique \( H^4 \)-solution \( u \) to (5.2) on \([t_0, 0]\) satisfying that \( u(t_0) = u_0 \), and
\[
    \lim_{n \to \infty} \|u_n - u\|_{C([t_0, t]; H^4(\mathbb{R}))} = 0, \quad t \in [t_0, 0]. \tag{5.10}
\]

Moreover, for each \( t_0 \leq t < 0 \) fixed, the modulation estimate (3.11) along with Theorem 5.1 yields that the derivatives of parameters \( \mathcal{P}_n \) are uniformly bounded on \([t_0, t]\), and thus \( \mathcal{P}_n \) are equicontinuous on \([t_0, t], n \geq 1 \). Then, by the Arzelà-Ascoli Theorem, \( \mathcal{P}_n \) converges uniformly on \([t_0, t]\) to some subsequence (which may depend on \( t \)). But, using the diagonal arguments one may extract a universal subsequence (still denoted by \( \{n\} \)) such that for some \( \mathcal{P} := (\mathcal{P}_1, \cdots, \mathcal{P}_K) \), where \( \mathcal{P}_k := (\lambda_k, b_k, \nu_k, \alpha_k, \gamma_k) \in C([t_0, t]; \mathbb{R}^5), 1 \leq k \leq K \), and for every \( t \in [t_0, 0] \), one has
\[
    \mathcal{P}_n \to \mathcal{P} \quad \text{in } C([t_0, t]; \mathbb{R}^{5K}). \tag{5.11}
\]

Then, taking into account the uniform estimates (5.4)-(5.5), we obtain that for all \( t \in [t_0, 0] \),
\[
    |\lambda_k(t) - \frac{\omega^2}{4} t^2| \leq |t|^{4-2\kappa}, \quad |b_k(t) + \frac{\omega^2}{2} t| \leq |t|^{3-2\kappa}, \quad |\alpha_k(t) - x_k| \leq |t|^{3-\kappa},
\]
\[
    |\nu_k(t) - \frac{\omega^2}{4} t^2| \leq |t|^{4-2\kappa}, \quad |\gamma_k(t) + \frac{4}{\omega^2 t} - \theta_k| \leq |t|^{1-2\kappa}, \quad 1 \leq k \leq K.
\]

Denote
\[
    U(t, x) = \sum_{k=1}^{K} U_k(t, x), \quad \text{with } U_k(t, x) = \lambda_k^{-\frac{1}{2}} Q_k(t, \frac{x - \alpha_k}{\lambda_k}) \varphi^{\gamma_k}.
\]
It follows from
\[ \|u(t) - U(t)\|_{H^2} \leq \|u(t) - u_n(t)\|_{H^2} + \|U(t) - U_n(t)\|_{H^2} + \|R_n(t)\|_{H^2}, \]
uniform estimate (5.3) and (5.10) that
\[ \lim_{t \to 0} \|u(t) - U(t)\|_{H^2} = 0. \]
So we have completed the proof of Theorem 1.1.

**Appendix A. Proof of Lemma 2.8**

*Proof.* Denote \( \bar{f} := f \phi_A^\frac{1}{2} \) with \( f = f_1 + i f_2 \). It follows that
\[
D^\frac{1}{2} f \phi_A^\frac{1}{2} = D^\frac{1}{2} \bar{f} + \left(D^\frac{1}{2}(f \phi_A^\frac{1}{2}) - D^\frac{1}{2} f \phi_A^\frac{1}{2}\right) \phi_A^\frac{1}{2} =: D^\frac{1}{2} \bar{f} + g. \tag{A.1}
\]
Thus we obtain
\[
\int (|D^\frac{1}{2} f|^2 + |f|^2) \phi_A - 3Q^2 \bar{f}_1^2 - Q^2 \bar{f}_2^2 dx \]
\[= \int |D^\frac{1}{2} \bar{f}|^2 + |\bar{f}|^2 - 3Q^2 \bar{f}_1^2 - Q^2 \bar{f}_2^2 dx + \int (1 - \phi_A^{-1})(3Q^2 \bar{f}_1^2 + Q^2 \bar{f}_2^2)dx + \|g\|^2_{L^2} + 2Re(D^\frac{1}{2} \bar{f}g) =: K_1 + K_2 + K_3 + K_4. \tag{A.2}
\]

*Estimate of \( K_1 \).* We claim that there exists \( C(A) > 0 \) with \( \lim_{A \to +\infty} C(A) = 0 \) such that,
\[ |\text{scal}(\bar{f}) - \text{scal}(f)| \leq C(A)\|\bar{f}\|^2_{L^2}. \tag{A.3}\]
Thus, by Lemma 2.7 and (A.3), there exists \( C > 0 \) such that for \( A \) sufficiently large,
\[ K_1 \geq C\|\bar{f}\|^2_{H^2} - \frac{1}{C}\text{scal}(f). \tag{A.4}\]
In order to prove (A.3), we can rewrite
\[ \langle \bar{f}, Q \rangle = \langle f_1, Q \rangle + \int f_1 Q(\phi_A^\frac{1}{2} - 1)\phi_A^{-\frac{1}{2}} dx, \]
where
\[ \left| \int f_1 Q(\phi_A^\frac{1}{2} - 1)\phi_A^{-\frac{1}{2}} dx \right| \leq C\int_{|x| \geq A} f_1 Q\phi_A^{-\frac{1}{2}} dx \leq C\|\bar{f}\|_{L^2} \left( \int_{|x| \geq A} Q^2 \phi_A^{-1} dx \right)^\frac{1}{2}. \]
Moreover, by using the decay property that \( Q(y) \sim \langle y \rangle^{-2} \), it follows that
\[ \int_{|x| \geq A} Q^2 \phi_A^{-1} dx = \frac{1}{A} \int_{|y| \geq 1} Q^2(Ay)\phi^{-1}(y)dy \leq C\frac{1}{A^5} \int_{|y| \geq 1} \langle y \rangle^{-4}dy \leq \frac{C}{A^5}. \]
Thus we get
\[ |\langle \bar{f}, Q \rangle - \langle f_1, Q \rangle| \leq CA^{-\frac{5}{2}}\|\bar{f}\|_{L^2}. \]
Similar arguments apply also to the remaining five directions, and thus we obtain (A.3), as claimed.

*Estimate of \( K_2 \).* By using the decay property of \( Q \) again, we see that,
\[ |K_2| \leq C \int_{|x| \geq A} \phi_A^{-1} Q^2 |f|^2 dx \leq C\|\phi_A^{-1} Q^2\|_{L^\infty(|x| \geq A)}\|\bar{f}\|^2_{L^2} \leq A^{-4}\|\bar{f}\|^2_{L^2}. \tag{A.5}\]
Estimate of $K_3$ and $K_4$. We claim that there exists $C(A) > 0$ with \( \lim_{A \to +\infty} C(A) = 0 \) such that
\[
\|g\|_{L^2} \leq C(A)\|\tilde{f}\|_{L^2},
\]
thus we can get
\[
|K_3| + |K_4| \leq C(A)\|\tilde{f}\|_{H^1}^2.
\]
In fact, by Lemma 2.2 and the Minkowski inequality, we have
\[
\|g\|_{L^2} = C\left(\int \phi_A(x) \left| \int \frac{\tilde{f}(x+y)(\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x)) - \tilde{f}(x-y)(\phi_A^{-\frac{1}{2}}(x) - \phi_A^{-\frac{1}{2}}(x-y))}{|y|^\frac{3}{2}} dy \right|^2 dx \right)^{\frac{1}{2}}
\]
\[
\leq C \int |y|^{-\frac{3}{2}} \left(\int \phi_A(x) \left(\tilde{f}(x+y)(\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x))\right)^2 dx \right)^{\frac{1}{2}} dy
\]
\[
\leq \int_{|y| \leq \frac{A}{4}} dy \int_{|y| \leq \frac{A}{4}} dx + \int_{|y| \leq \frac{A}{4}} dy \int_{|y| \leq \frac{A}{4}} dx + \int_{|y| \leq \frac{A}{4}} dy \int_{|y| \leq \frac{A}{4}} dx + \int_{|y| \leq \frac{A}{4}} dy \int_{|y| \leq \frac{A}{4}} dx
\]
\[
=: G_1 + G_2 + G_3 + G_4.
\]
Then the proof of (A.6) is reduced to estimate $G_k$, $1 \leq k \leq 4$.

Note that $\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x) = 0$ for $|x| \leq \frac{A}{2}$ and $|y| \leq \frac{A}{4}$, which implies that $G_1 = 0$. By using the mean value theorem, we get for $0 \leq \theta \leq 1$,
\[
\phi_A(x)|\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x)|^2 = \frac{1}{4} \frac{\phi_A(x)}{\phi_A'(x+\theta y)}|\phi_A'(x+\theta y)|^2 |y|^2.
\]
For $\frac{A}{2} \leq |x| \leq 3A$ and $|y| \leq \frac{A}{4}$, we have $\frac{A}{4} \leq |x+\theta y| \leq 4A$, thus it follows that
\[
\phi_A(x)|\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x)|^2 \leq \frac{1}{4A^2} \frac{\phi_A(x)}{\phi_A'(x+\theta y)}|\phi_A'(x+\theta y)|^2 |y|^2 \leq \frac{C}{A^2} |y|^2.
\]
For $|x| \geq 3A$ and $|y| \leq \frac{A}{4}$, we get $\frac{A}{2} \leq |x+\theta y| \leq \frac{3|y|}{2}$, thus
\[
\phi_A(x)|\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x)|^2 \leq \frac{C}{A^2} \frac{\phi_A(x)}{\phi_A'(x+\frac{3|y|}{2})}|\phi_A'(\frac{3|y|}{2})|^2 |y|^2 \leq \frac{C}{A^2} |y|^2.
\]
So we obtain the estimate for $G_2$
\[
G_2 \leq \frac{C}{A} \int_{0}^{\frac{A}{2}} y^{-\frac{1}{2}} dy \|\tilde{f}\|_{L^2} \leq CA^{-\frac{1}{2}} \|\tilde{f}\|_{L^2}.
\]
Regarding $G_3$, we shall use the fact that
\[
\phi_A(x)|\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x)|^2 \leq C\phi_A(x)(\phi_A^{-\frac{1}{2}}(x+y) + \phi_A^{-\frac{1}{2}}(x)).
\]
For $|y| \geq \frac{A}{4}$ and $|x| \leq \frac{3|y|}{2}$, we have $\frac{A}{2} \leq |x+|y| \leq \frac{3|y|}{2}$, so it yields that
\[
\phi_A(x)\phi_A^{-1}(x+y) \leq C\phi_A^{-1}(\frac{3|y|}{2}) \leq C|y|^a,
\]
which implies
\[
\phi_A(x)|\phi_A^{-\frac{1}{2}}(x+y) - \phi_A^{-\frac{1}{2}}(x)|^2 \leq C(1 + |y|^a).
\]
So we get
\[ G_3 \leq C \int_\frac{\pi}{4}^{+\infty} y^{-\frac{1}{2}}(1 + y^2)dy\|\tilde{f}\|_{L^2} \leq CA^{-\frac{1}{2}}\|\tilde{f}\|_{L^2}. \]  

(A.10)

The estimate of \( G_4 \) also relies on (A.9). In fact, for \( |y| \geq \frac{3}{4} \) and \( |x| \geq \frac{|y|}{2} \), we have \( |x + y| \leq 3|x| \), thus it follows that
\[ \phi_A(x)\phi_A^{-1}(x + y) \leq C\phi_A(x)\phi_A^{-1}(3x) \leq C, \]
and
\[ \phi_A(x)\phi_A^{\frac{1}{2}}(x + y) - \phi_A^{\frac{1}{2}}(x) \leq C. \]

So we get
\[ G_4 \leq C \int_\frac{\pi}{4}^{+\infty} y^{-\frac{1}{2}}dy\|\tilde{f}\|_{L^2} \leq CA^{-\frac{1}{2}}\|\tilde{f}\|_{L^2}. \]  

(A.11)

Combing (A.8), (A.10) and (A.11) together, we obtain (A.6), as claimed.

Now by inserting (A.4), (A.5) and (A.7) into (A.2), it yields that
\[ \int (|D^2 f|^2 + |f|^2)\phi_A - 3Q^2 f_1^2 - Q^2 f_2^2 dx \geq C\|\tilde{f}\|_{H^\frac{1}{2}}^2 - C_2\text{scal}(f). \]

To finish the proof, it remains to show that, there exists \( C > 0 \) such that for \( A \) sufficiently large,
\[ \|\tilde{f}\|_{H^\frac{1}{2}}^2 \geq C \int (|D^2 f|^2 + |f|^2)\phi_A dx. \]

Note that \( \|\tilde{f}\|_{L^2}^2 = \int |f|^2\phi_A dx \) and
\[ \|D^2 \tilde{f}\|_{L^2}^2 = \int |D^2 f\phi_A^\frac{1}{2} + (D^2 \tilde{f} - D^2 (f\phi_A^{-\frac{1}{2}}))\phi_A^\frac{1}{2}|^2 dx = \int |D^2 f\phi_A^\frac{1}{2} - g|^2 dx \]
\[ = \int |D^2 f|^2\phi_A dx + \int |g|^2 dx - 2\text{Re} \int D^2 f\phi_A^\frac{1}{2}g dx, \]
where \( g \) is defined in (A.1). So by applying (A.6), it follows that for \( A \) large enough,
\[ \|D^2 \tilde{f}\|_{L^2}^2 + \|\tilde{f}\|_{L^2}^2 \geq C \int (|D^2 f|^2 + |f|^2)\phi_A dx. \]
Thus we conclude the proof of Lemma 2.8. \( \square \)

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References

[1] H. Bahouri, J.Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der Mathematischen Wissenschaften, vol. 343. Springer, Heidelberg (2011).

[2] U. Biccari, M. Warma, E. Zuazua, Local elliptic regularity for the Dirichlet fractional Laplacian, Adv. Nonlinear Stud. 17 (2017), no. 2, 387–409.

[3] J. Bellazzini, V. Georgiev, N. Visciglia, Long time dynamics for semirelativistic NLS and half wave in arbitrary dimension, Math. Ann. 371 (2018), 707–740.

[4] J. Bellazzini, V. Georgiev, E. Lenzmann, N. Visciglia, On traveling solitary waves and absence of small data scattering for nonlinear half-wave equation, Commun. Math. Phys. 372 (2019), 713–732.

[5] T. Boulenger, D. Himmelsbach, E. Lenzmann, Blowup for fractional NLS, J. Funct. Anal. 271 (2016), 2569–2603.

[6] D. Cai, A.J. Majda, D.W. McLaughlin, E.G. Tabak, Dispersive wave turbulence in one dimension, Physica D 152 (2001), 551–572.
[7] A. Choffrut, O. Pocovnicu, Ill-posedness of the cubic nonlinear half-wave equation and other fractional NLS on the real line, *Int. Math. Res. Not.* (2018), no. 3, 699–738.

[8] V. Cömbet, Y. Martel, Construction of multibubble solutions for the critical GKDV equation, *SIAM J. Math. Anal.* 50 (2018), no. 4, 3715–3790.

[9] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhikers guide to the fractional Sobolev spaces, *Bull. Sci. Math.* 136 (2012), no. 5, 521–573.

[10] V.D. Dinh, On the Cauchy problem for the nonlinear semi-relativistic equation in Sobolev spaces, *Discrete Contin. Dyn. Syst.* 38 (2018), no. 3, 1127–1143.

[11] V.D. Dihn, Blow-up criteria for fractional nonlinear Schrödinger equations, *Nonlinear Anal., Real World Appl.* 48 (2019), 117–140.

[12] B. Dodson, Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state, *Adv. Math.* 285 (2015), 1589–1618.

[13] A. Elgart, B. Schlein, Mean field dynamics of boson stars, *Commun. Pure Appl. Math.* 60 (2007), no. 4, 500–545.

[14] C.J. Fan, Log-log blow up solutions blow up at exactly \( m \) points. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 34 (2017), no. 6, 1429–1482.

[15] R. Frank, E. Lenzmann, Uniqueness of nonlinear ground states for fractional Laplacians in \( \mathbb{R} \), *Acta Math.* 210 (2013), no. 2 261–318.

[16] R.L. Frank, E. Lenzmann, L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian, *Commun. Pure Appl. Math.* 69 (2016), 1671–1726.

[17] J. Fröhlich, E. Lenzmann, Blowup for nonlinear wave equations describing boson stars, *Commun. Pure Appl. Math.* 60 (2007), no. 11, 1691–1705.

[18] K. Fujiwara, V. Georgiev, T. Ozawa, On global well-posedness for nonlinear semirelativistic equations in some scaling subcritical and critical cases, *J. Math. Pures Appl.* 136 (2020), 239–256.

[19] V. Georgiev, Y. Li, Nondispersive solutions to the mass critical half-wave equation in two dimensions, *Comm. Partial Differential Equations* 47 (2022), no. 1, 39–88.

[20] V. Georgiev, Y. Li, Blowup dynamics for mass critical half-wave equation in 3D, *J. Funct. Anal.* 281 (2021), no. 7, 109132.

[21] P. Gérard, S. Grellier, Effective integrable dynamics for a certain nonlinear wave equation, *Anal. PDE* 5 (2012), 1139–1155.

[22] P. Gérard, E. Lenzmann, O. Pocovnicu, P. Raphaël, A two-soliton with transient turbulent regime for the cubic half-wave equation on the real line, *Ann. PDE* 4 (2018), 7.

[23] B. Guo and Z. Huo, Global well-posedness for the fractional nonlinear Schrödinger equation, *Comm. Partial Differential Equations* 36 (2010), 247–255.

[24] K. Hidano, C. Wang, Fractional derivatives of composite functions and the Cauchy problem for the nonlinear half wave equation, *Sel. Math.* 25 (2019), 2.

[25] Y. Hong, Y. Sire, On fractional Schrödinger equations in Sobolev spaces, *Commun. Pure Appl. Anal.* 14 (2015), 2265–2282.

[26] J. Jendrej, Y. Martel, Construction of multi-bubble solutions for the energy-critical wave equation in dimension 5, *J. Math. Pures Appl.* 139 (2020), no.9, 317–355.

[27] C.E. Kenig, Y. Martel, L. Robbiano, Local well-posedness and blow-up in the energy space for a class of \( L^2 \) critical dispersion generalized Benjamin-Ono equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28 (2011), 853–887.

[28] K. Kirkpatrick, E. Lenzmann, G. Staffilani, On the continuum limit for discrete NLS with long-range lattice interactions, *Commun. Math. Phys.* 317 (2013), no. 3, 563–591.

[29] J. Krieger, E. Lenzmann, P. Raphaël, Nondispersive solutions to the \( L^2 \)-critical halfwave equation, *Arch. Ration. Mech. Anal.* 209 (2013), no. 1, 61-129.

[30] Y. Lan, Blow-up dynamics for \( L^2 \)-Critical Fractional Schrödinger Equations, *Int. Math. Res. Not.* (2021), rnab086.

[31] N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E* 66 (2002), 056108.

[32] A.J. Majda, D.W. McLaughlin, E.G. Tabak, A one-dimensional model for dispersive wave turbulence, *J. Nonlinear Sci.* 7 (1997), no. 1, 9–44.

[33] Y. Martel, P. Raphaël, Strongly interacting blow up bubbles for the mass critical nonlinear Schrödinger equation. *Ann. Sci. Éc. Norm. Supér.* 51 (2018), no. 3, 701–737.

[34] F. Merle, Construction of solutions with exactly \( k \) blow-up points for the Schrödinger equation with critical nonlinearity, *Comm. Math. Phys.* 129 (1990), no. 2, 223–240.
[35] F. Merle and P. Raphaël, Sharp upper bound on the blow-up rate for the critical nonlinear Schrödinger equation, *Geom. Funct. Anal.* **13** (2003), no. 3, 591–642.

[36] F. Merle, P. Raphaël, The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation, *Ann. of Math.* **161** (2005), no. 1, 157–220.

[37] F. Merle, P. Raphaël, On a sharp lower bound on the blow-up rate for the $L^2$ critical nonlinear Schrödinger equation, *J. Am. Math. Soc.* **19** (2006), no. 1, 37–90.

[38] M. Ming, F. Rousset, N. Tzvetkov, Multi-solitons and related solutions for the water-waves system. *SIAM J. Math. Anal.* **47** (2015), no. 1, 897–954.

[39] P. Raphaël, J. Szeftel, Existence and uniqueness of minimal blow-up solutions to an inhomogeneous mass critical NLS. *J. Amer. Math. Soc.* **24** (2011), no. 2, 471–546.

[40] Y. Su and D. Zhang, On the multi-bubble blow-up solutions to rough nonlinear Schrödinger equations, arXiv: 2012.14037v1.

[41] L. Tartar, An Introduction to Sobolev Spaces and Interpolation Spaces. *Lecture Notes of the Unione Matematica Italiana*, vol. 3. Springer, Berlin (2007).

[42] M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Comm. Math. Phys.* **87** (1983), 567–576.

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