Scalar-Tensor Theory of Gravity on $M_4 \times Z_2$ Geometry

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(Dedicated to the late Prof. Robert H. Dicke)

Abstract

In the Brans-Dicke (BD) theory on $M_4 \times Z_2$ geometry the geometrical meaning of the torsion is clarified. The BD theory on $M_4 \times Z_2$ is rederived by taking into account of a new isometry condition.

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I. Introduction

Recently we have shown\[1\] that the pure Einstein action on $M_4 \times \mathbb{Z}_2$ geometry exactly leads to the Brans-Dicke (BD) theory\[2\] in 4-dimensional space-time $M_4$, where a scalar field is coupled to gravity. Here the $\mathbb{Z}_2$ is a discrete space with two points. We have used the geometrical theory on $M_4 \times \mathbb{Z}_2$, which was previously proposed by two of authors\[3\] without recourse to the noncommutative geometry (NCG) of Connes\[4\]. On this manifold $M_4 \times \mathbb{Z}_2$ there are three kinds of Riemann curvature tensors. We have clarified the geometrical meaning of them.

In order to calculate the Riemann curvature we should express the affine connections $\Gamma_{LMN}$ in terms of the metric $G_{MN}$ in this space. This space formally can be regarded as the 5-dimensional Kaluza-Klain like space where the fifth continuous dimension is replaced by two points. On $M_4$ usually one uses the isometry condition that any inner product of vectors is invariant under the parallel-transportation of vectors. This leads to the covariant constancy of the metric $g_{\mu\nu}$

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\lambda\nu} + \Gamma_{\nu\lambda\mu}. \quad (1.1)$$

As well known, if the affine connection is symmetric, $\Gamma_{\mu\lambda\nu} = \Gamma_{\mu\nu\lambda}$, then Eq.(1.1) is used to express $\Gamma$ in terms of $g_{\mu\nu}$. Even on $M_4 \times \mathbb{Z}_2$ we can do the same thing. In this case we have two kinds of equations

$$\partial_\lambda G_{MN} = \Gamma_{M\lambda N} + \Gamma_{N\lambda M}, \quad (1.2)$$

$$\partial_r G_{MN} = \Gamma_{MrN} + \Gamma_{NrM} + \Gamma_{KrM} \Gamma^K \triangle^r z, \quad (1.3)$$

where $M = (\mu, r)$, $\partial_r$ is a derivative on $\mathbb{Z}_2$, and $\triangle^r z = z(g + r) - z(g)$ defined below. The point is that Eq.(1.3) is valid for any $\triangle^r z$ as explained in the text. Since $\partial_r \triangle^r z = -2$, there are contributions from the last term. In the previous work\[1\] we have not taken into account of such term. This term is so important as to eliminate the Riemann tensor of the third type, and to yield the BD kinetic term.

The purpose of this paper is to reconsider the scalar-tensor theory of gravity on $M_4 \times \mathbb{Z}_2$, by using the new isometry condition (1.3). We also consider the geometrical meaning of torsion on $M_4 \times \mathbb{Z}_2$, which has not so far been discussed enough. To begin we need the equivalence assumption proposed in the previous work. This is stated as follows: the manifold $M_4 \times \mathbb{Z}_2$ may be regarded as a pair of $M_4$, each at the point $e$ or $r$ on $\mathbb{Z}_2$. The physics on these two $M_4$-pieces should be equivalent to each other. We assume that the equivalence is attained by a limiting process with some parameter $\varepsilon$ which tends to zero. As a technical
tool to express the limiting process we introduce a coordinate $z(g)$ on $\mathbb{Z}_2$ ($g = e$ or $r$) such that a difference

$$\Delta^r z(g) = z(g + r) - z(g) \sim \varepsilon \quad (1.4)$$

is proportional to the limiting process parameter $\varepsilon$. Let $f(z(g))$ be any function on $\mathbb{Z}_2$. Since $f(z(g))$ is a linear function of $z(g)$, one may put

$$f(z(g)) = A + B z(g), \quad (1.5)$$

where $A$ and $B$ are some constants. From this we find that its Taylor expansion is cut off only up to the first order $\Delta^r z(g)$

$$f(z(g + r)) = f(z(g)) + B \Delta^r z(g). \quad (1.6)$$

Now, the equivalence indicates that

$$f(x, z(g + r)) \rightarrow f(x, z(g)) \text{ as } \varepsilon \rightarrow 0, \quad (1.7)$$

where the coordinate $x$ on $M_4$ is inserted.

In §2 we consider the geometrical meaning of torsion. In §3 the isometry condition is required in order to derive Eqs.(1.2) and (1.3). In §4 we calculate three kind of Riemann curvature tensors and derive the BD theory. The final section is devoted to concluding remarks.
II. Torsion on $M_4 \times Z_2$

We consider the tangent space $T(p)$ at a point $P$ on $M_4 \times Z_2$ with local coordinates $(x^\mu, g_\nu), x^\mu \in M_4$ and $g = \{e($unit element$), r\} \in Z_2$. Let the origin of $T(x, g)$ be $O(x, g)$. At a point $(x^\mu + \Delta x^\mu, g)$ very close to $P$ we have also another tangent space $T(x + \Delta x, g)$, whose origin is $O(x + \Delta x, g)$. We consider a mapping of the origin $O(x + \Delta x, g)$ from $T(x + \Delta x, g)$ onto $T(x, g)$. The mapped point is denoted by a notation $U(x, x + \Delta x, g)O(x + \Delta x, g)$. A covariant difference between $U(x, x + \Delta x, g)O(x + \Delta x, g)$ and $O(x, g)$ defines a vector $e_\mu(x, g)$ on $T(x, g)$

$$\Delta_x O(x, g) = U(x, x + \Delta x, g)O(x + \Delta x, g) - O(x, g) = e_\mu(x, g)\Delta x^\mu.$$  \hspace{1cm} (2.1)

In the same way the mapping of the origin $O(x, g + r)$ from $T(x, g + r)$ onto $T(x, g)$ is given by the notation $U(x, g, g + r)O(x, g + r)$. The covariant difference between the mapped point and $O(x, g)$ defines another vector $e_r(x, g)$ on $T(x, g)$

$$\Delta_r O(x, g) = U(x, g, g + r)O(x, g + r) - O(x, g) = e_r(x, g)\Delta^r z(g),$$ \hspace{1cm} (2.2)

where $z(g)$ is the coordinate corresponding to $g$ and

$$\Delta^r z(g) = z(g + r) - z(g)$$ \hspace{1cm} (2.3)

is proportional to the limiting process parameter $\varepsilon$. A set of vectors

$$e_N(x, g) = \{e_\mu(x, g), e_r(x, g), g = (e, r) \in Z_2\}$$ \hspace{1cm} (2.4)

supplies a basis on $T(x, g)$. In this paper we do not consider the direct mapping of the origin $O(x + \Delta x, g + r)$ from $T(x + \Delta x, g + r)$ to $T(x, g)$.

Let us then consider the mapping of the basis $e_N(x + \Delta x, g)$ from $T(x + \Delta x, g)$ onto $T(x, g)$. The mapped basis which is denoted by $e_N^H(x + \Delta x, g)$ is given by a rotation $H_N^M(x, x + \Delta x, g)$ of $e_N(x, g)$

$$e_N^H(x + \Delta x, g) = e_M(x, g)H_N^M(x, x + \Delta x, g).$$ \hspace{1cm} (2.5)

We define the affine connection $\widehat{\Gamma}_{N\mu}^M(x, g)$ by

$$H_N^M(x, x + \Delta x, g) = \delta_N^M + \widehat{\Gamma}_{N\mu}^M(x, g)\Delta x^\mu + O(\Delta x)^2.$$ \hspace{1cm} (2.6)
Substituting this into the above equation we have
\[ e^H_N(x + \Delta x, g) = e_N(x, g) + \hat{\Gamma}_{N\mu}(x, g)\Delta x^\mu + O(\Delta x)^2, \tag{2.7} \]
where
\[ \hat{\Gamma}_{N\mu}(x, g) \equiv e_M(x, g)\hat{\Gamma}^M_{N\mu}(x, g). \tag{2.8} \]
This equation defines the covariant difference of \( e_N(x, g) \) along \( M_4 \)
\[ \Delta x e_N(x, g) \equiv e^H_N(x + \Delta x, g) - e_N(x, g) = \hat{\Gamma}_{N\mu}(x, g)\Delta x^\mu + O(\Delta x)^2. \tag{2.9} \]
In the same way we have the covariant difference of \( e_N \) along \( Z_2 \)
\[ \Delta_r e_N(x, g) \equiv e^H_N(x, g + r) - e_N(x, g) = \hat{\Gamma}_{N\nu}(x, g)\Delta \nu z(g), \tag{2.10} \]
where
\[ \hat{\Gamma}_{N\nu}(x, g) \equiv e_M(x, g)\hat{\Gamma}^M_{N\nu}(x, g). \tag{2.11} \]
In the previous paper\[1\] we have shown that the rotation matrix \( H^M_N(x, g, g + r) \) has the
form, \( H^M_N(x, g, g + r) = \delta^M_N + \hat{\Gamma}^M_{N\nu}(x, g)\Delta \nu z(g) \), having no term of \( O(\Delta r z)^2 \), i.e., its Taylor
expansion is cut off only up to the first order of \( \Delta r z \). This sharply differs from \( H^M_N(x, x + \Delta x, g) \), which is expanded into an infinite power series of \( \Delta x \). (See Appendix.)

We are now in a position to consider the torsion on \( M_4 \times Z_2 \). There are three kinds of
torsion in this space. They will be shown to be
\[ T^{(1)} = [\Delta_{1x}, \Delta_{2x}]O(x, g) = [\hat{\Gamma}_{\nu\mu}(x, g) - \hat{\Gamma}_{\mu\nu}(x, g)]\Delta x^\nu \Delta_{2x}^\mu, \tag{2.12} \]
\[ T^{(2)} = [\Delta_x, \Delta_r]O(x, g) = [\hat{\Gamma}_{\nu\mu}(x, g) - \hat{\Gamma}_{\mu\nu}(x, g)]\Delta x^\mu \Delta r z(g), \tag{2.13} \]
\[ T^{(3)} = [\Delta_r \Delta_r + 2\Delta_r]O(x, g) = -\hat{\Gamma}_{rr}(x, g)(\Delta r z(g))^2. \tag{2.14} \]

The first torsion \( T^{(1)} \) is known to be of the
conventional type. For the latter convenience let us derive this formula. First we
consider two sequential mappings of the origin \( O(P_3) \) from \( T(P_3) \) onto \( T(P) \) along two
paths \( C_1 \) and \( C_2 \) depicted in Fig.1, where co-
ordinates of \( P, P_1, P_2 \) and \( P_3 \) are

![Fig.1.](image-url)
\[ P = (x, g), \]
\[ P_1 = (x + \Delta_1 x, g), \]
\[ P_2 = (x + \Delta_2 x, g), \]
\[ P_3 = (x + \Delta_1 x + \Delta_2 x, g). \]

They are given by
\[ C_1 = U(P, P_1)U(P_1, P_3)O(P_3), \]  \hspace{1cm} (2.15)
\[ C_2 = U(P, P_2)U(P_2, P_3)O(P_3). \]  \hspace{1cm} (2.16)

The difference between \( C_1 \) and \( C_2 \) gives just the torsion
\[ T^{(1)} = C_1 - C_2. \]  \hspace{1cm} (2.17)

Noting
\[ \Delta_2 x O(P) = U(P, P_2)O(P_2) - O(P) = e_\nu(x, g)\Delta_2 x^\nu, \]  \hspace{1cm} (2.18)
and from Eq. (2.9) one gets
\[ \Delta_1 x \Delta_2 x O(P) = U(P, P_1)U(P_1, P_3)O(P_3) - U(P, P_1)O(P_1) \]
\[ -U(P, P_2)O(P_2) + O(P) \]
\[ = \Delta_1 x e_\nu(x, g)\Delta_2 x^\nu \]
\[ = \tilde{\Gamma}_{\nu\mu}(x, g)\Delta_1 x^\mu \Delta_2 x^\nu. \]  \hspace{1cm} (2.19)

Thus we find
\[ (\Delta_1 x \Delta_2 x - \Delta_2 x \Delta_1 x)O(P) = [U(P, P_1)U(P_1, P_3) - U(P, P_2)U(P_2, P_3)]O(P_3) = T^{(1)}. \]  \hspace{1cm} (2.20)

Namely, \( [\Delta_1 x, \Delta_2 x]O(P) \) gives just the first torsion \( T^{(1)} \), (2.12).

The second torsion \( T^{(2)} \) will be derived in the same way as above if we consider two mappings of the origin \( O(x + \Delta x, g + r) \) from \( T(x + \Delta x, g + r) \) onto \( T(x, g) \) along two paths \( C_3 \) and \( C_4 \) depicted in Fig.2. They are given by
\[ C_3 = U(x, x + \Delta x, g)U(x + \Delta x, g, g + r)O(x + \Delta x, g + r), \]  \hspace{1cm} (2.21)
\[ C_4 = U(x, g, g + r)U(x, x + \Delta x, g + r)O(x + \Delta x, g + r). \]  \hspace{1cm} (2.22)
The difference between $C_3$ and $C_4$ gives the second torsion $T^{(2)}$. After the similar calculation as $T^{(1)}$ we have

$$T^{(2)} = C_3 - C_4 = (\Delta_x \Delta_r - \Delta_r \Delta_x)O(x, g),$$

which is just Eq.(2.13).

Finally the third torsion $T^{(3)}$ will be obtained if we consider the mappings of the origin $O(x, g)$ from $T(x, g)$ onto $T(x, g + r)$ and again onto the same $T(x, g)$. The paths of the mappings are depicted in Fig.3. The difference between the mapped point and $O(x, g)$ gives the torsion $T^{(3)} = U(x, g, g + r)U(x, g + r, g)O(x, g) - O(x, g)$. (2.24)

In order to obtain the explicit formula for $T^{(3)}$, let us take the covariant difference of both sides of (2.2)

$$\Delta_r \Delta_r O(x, g) = \Delta_r [U(x, g, g + r)O(x, g + r) - O(x, g)]$$

$$= U(x, g, g + r)[U(x, g + r, g)O(x, g) - O(x, g + r)]$$

$$- U(x, g, g + r)O(x, g + r) + O(x, g)$$

$$= U(x, g, g + r)U(x, g + r, g)O(x, g)$$

$$- 2U(x, g, g + r)O(x, g + r) + O(x, g).$$

Hence we have

$$(\Delta_r \Delta_r + 2\Delta_r)O(x, g) = U(x, g, g + r)U(x, g + r, g)O(x, g) - O(x, g) = T^{(3)}.$$ (2.26)

Namely, the left-hand side quantity is equal to the third torsion. From the last equation of (2.2) and (2.10) we find

$$\Delta_r \Delta_r O(x, g) = \Delta_r [e_r(x, g)\Delta^r z(g)]$$

$$= e_r^H(x, g + r)\Delta^r z(g + r) - e_r(x, g)\Delta^r z(g)$$

$$= [e_r(x, g) + \Gamma_{rr}(x, g)\Delta^r z(g)]\Delta^r z(g + r) - e_r(x, g)\Delta^r z(g)$$

$$= -\Gamma_{rr}(x, g)(\Delta^r z(g))^2 - 2e_r(x, g)\Delta^r z(g).$$ (2.27)
Here we have used $\Delta^r z(g + r) = z(g) - z(g + r) = -\Delta^r z(g)$. Thus, finally we get

$$T^{(3)}(x, g) = -\hat{\Gamma}_{rr}(x, g)(\Delta^r z(g))^2. \quad (2.28)$$

The first and second torsions vanish when the affine connection $\hat{\Gamma}_{MN}(x, g)$ is symmetric with respect to $M$ and $N$, $\hat{\Gamma}_{MN} = \hat{\Gamma}_{NM}$. However, the third torsion $T^{(3)}$ remains generally finite.
III. The isometry condition

The manifold $M_4 \times Z_2$ can be regarded as the Kaluza-Klein like space where the fifth continuous dimension is replaced by two discrete points $z(e)$ and $z(r)$. The line element $\Delta s$ of this space is assumed to be

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu + \lambda^2(x) \Delta z^2 = G_{MN}(x) \Delta x^M \Delta x^N,$$

where

$$\Delta x^N = (\Delta x^\mu, \Delta x^r \equiv \Delta z = z(r) - z(e)),$$

and $G_{MN}(x)$ is regarded as the five-dimensional metric of $M_4 \times Z_2$.

Here we have considered a simple case that the four-dimensional metric $g_{\mu\nu}(x)$ and the scalar field $\lambda(x)$ are independent of $z(g)$ on $Z_2$ and are functions only of $x \in M_4$. The first and second kinds of torsions $T^{(1)}$ and $T^{(2)}$ are assumed to be zero, i.e.,

$$\hat{\Gamma}^N_{\mu\nu}(x, g) = \hat{\Gamma}^N_{\nu\mu}(x, g), \quad \hat{\Gamma}^N_{\mu r}(x, g) = \hat{\Gamma}^N_{r\mu}(x, g),$$

whereas the third kind of torsion $T^{(3)}$ is not necessarily vanished

$$\hat{\Gamma}^N_{r r}(x, g) \neq 0.$$

The metric is defined by the inner product

$$G_{MN}(x, g) = G_{MN}(x) = e_M(x, g) \cdot e_N(x, g).$$

Let the manifold $M_4 \times Z_2$ be isometric, that is, any inner product of vectors is invariant under the parallel-transportation or the mapping of vectors. Then we have

$$G_{MN}(x + \Delta x, g) = e_M(x + \Delta x, g) \cdot e_N(x + \Delta x, g) = e_M^H(x + \Delta x, g) \cdot e_N^H(x + \Delta x, g).$$

Substituting (2.9) into (3.6) this is reduced to

$$\partial_N G_{MN} = \hat{\Gamma}^M_{MN} + \hat{\Gamma}^M_{NX}, \quad \hat{\Gamma}^M_{MN} \equiv G_{MK} \hat{\Gamma}^K_{MN}.$$
Since any function $f(z(g))$ on $Z_2$ is a linear function of $z(g)$ (see (1.5)), the derivative on $Z_2$, $\partial_r f \equiv \partial f/\partial z(g)$, can be defined without taking the limit $\Delta^r z(g) \to 0$ and is always independent of $g$. Namely, Eq.(3.8) is valid for any $\Delta^r z$. Of course, the left-hand side of (3.8) is zero from the assumption that $G_{MN}$ is independent of $g$.

Noting $\partial_r \Delta^r z(g) = -2$ and differentiating the right-hand side of (3.8) with respect to $z(g)$, we have

$$0 = \partial_r \hat{\Gamma}_{MrN} + \partial_r \hat{\Gamma}_{Nrm} - 2\hat{\Gamma}_{Krm}(\hat{\Gamma}^K_{rN} + \partial_r(\hat{\Gamma}_{Krm}\hat{\Gamma}^K_{rN}))\Delta^r z.$$

(3.9)

Taking the limit $\Delta^r z \to 0$ we get

$$(\partial_r \hat{\Gamma}_{MrN} + \partial_r \hat{\Gamma}_{Nrm})_0 = 2\hat{\Gamma}_{Krm}\hat{\Gamma}^K_{rN}, \quad (\Delta^r z = 0)$$

(3.10)

where the suffix 0 and $\Gamma$ without the hat show that these quantities are independent of $g$. In the same limit $\Delta^r z \to 0$ Eqs.(3.7) and (3.8) tend to

$$\partial_\lambda G_{MN} = \Gamma_{M\lambda N} + \Gamma_{N\lambda M}, \quad (\Delta^r z = 0)$$

(3.11)

$$\partial_\lambda G_{MN} = \Gamma_{MrN} + \Gamma_{NrM}, \quad (\Delta^r z = 0)$$

(3.12)

Note that from (3.8) or (3.9) one cannot derive $\hat{\Gamma}_{Krm}\hat{\Gamma}^K_{rN} = 0$ or $\partial_r(\hat{\Gamma}_{Krm}\hat{\Gamma}^K_{rN}) = 0$ because the other terms contain $\Delta^r z$. From (3.11) and (3.12) and from (3.3) one obtains the expression of $\Gamma_{LMN}$ in terms of $G_{MN}$

$$\Gamma_{LMN} = \frac{1}{2}(\partial_M G_{LN} + \partial_N G_{LM} - \partial_L G_{MN}).$$

(3.13)

Since the metric $G_{MN}$ is given by (3.1) we have

$$\Gamma_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}),$$

(3.14)

$$\Gamma_{rr\mu} = -\Gamma_{\mu rr} = \lambda \partial_\mu \lambda,$$

(3.15)

$$\Gamma_{r\mu\nu} = \Gamma_{\mu r\nu} = \Gamma_{\mu \nu r} = 0,$$

(3.16)

$$\Gamma_{rrr} = 0.$$  

(3.17)

1 Since any function $f(g)$ on $Z_2$ has a form, $f(g) = A + Bz(g)$, we have

$$f(e) = \frac{1}{2}[f(e) + f(r)] + \frac{1}{2}[f(e) - f(r)] = \overline{T} - \frac{1}{2}B\Delta z(e), \Delta z(e) = z(r) - z(e),$$

$$f(r) = \overline{T} - \frac{1}{2}B\Delta z(r), \Delta z(r) = z(e) - z(r),$$

so that

$$f(g) = \overline{T} - \frac{1}{2}B\Delta^r z(g), \Delta^r z(g) = z(g + r) - z(g).$$

Namely, the function $f(g)$ contains $\Delta^r z(g)$. 

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On the other hand, from (3.10) with \( M = \mu \) and \( N = \nu \) and from (3.16) we see
\[
(\partial_r \hat{\Gamma}_{\mu \nu r} + \partial_r \hat{\Gamma}_{\nu r \mu})_0 = 2 \Gamma_{K \rho} \Gamma^K_{r \nu} = 2 \Gamma_{r \rho} \Gamma^r_{r \nu},
\]
and from (3.7)
\[
\partial_\nu G_{\mu r} = \hat{\Gamma}_{\mu \nu r} + \hat{\Gamma}_{r \nu \mu} = 0.
\]
The equation (3.19) yields
\[
\hat{\Gamma}_{\mu \nu r} = - \hat{\Gamma}_{r \nu \mu}
\]
and
\[
\hat{\Gamma}_{r \mu \nu} = - \hat{\Gamma}_{r \nu \mu} = - \hat{\Gamma}_{r \nu \mu},
\]
so that
\[
\hat{\Gamma}_{\mu \nu r} = \hat{\Gamma}_{\nu r \mu}.
\]
From (3.20) one can see that the first and second terms in (3.18) are equal to each other, hence we have
\[
(\partial_r \hat{\Gamma}_{\mu \nu r})_0 = \Gamma_{r \mu r} \Gamma^r_{r \nu} = - (\partial_r \hat{\Gamma}_{r \mu \nu})_0.
\]
Other useful relations come from (3.10)
\[
(\partial_r \hat{\Gamma}_{r r r})_0 = \Gamma_{K r} \Gamma^K_{r r} = \Gamma_{r r} \Gamma^r_{r r},
\]
\[
(\partial_r \hat{\Gamma}_{r r r} + \partial_r \hat{\Gamma}_{r r r})_0 = 2 \Gamma_{K r} \Gamma^K_{r r} = 0.
\]
In the next section we use Eqs.(3.14)-(3.17) and (3.21) -(3.23) to calculate the curvature.
IV. Curvature

Corresponding to three kinds of torsions there are also three kinds of curvature tensors. They are given by

\[
\begin{align*}
\Delta_{x_1}x_1, \Delta_{x_2}x_2 & = e_K R^K_{\mu} \Delta_{x_1}x_1 \Delta_{x_2}x_2, \\
R^K_{\mu} & = \partial_{x_1} \hat{F}^K_{x_1} \partial_{x_2} \hat{F}^K_{x_2} + \hat{F}^K_{x_1} \hat{F}^J_{x_2} - \hat{F}^K_{x_2} \hat{F}^J_{x_1}, \\
\Delta_{x_1}x_1, \Delta_{x_2}x_2 & = e_K R^K_{\mu} \Delta_{x_1}x_1 \Delta_{x_2}x_2, \\
R^K_{\mu} & = \partial_{x_1} \hat{F}^K_{x_1} \partial_{x_2} \hat{F}^K_{x_2} + \hat{F}^K_{x_1} \hat{F}^J_{x_2} - \hat{F}^K_{x_2} \hat{F}^J_{x_1}, \\
(\Delta_{x_1}x_1 + 2\Delta_{x_2}x_2) & = e_K R^K_{\mu} (\Delta_{x_1}x_1 + 2\Delta_{x_2}x_2), \\
R^K_{\mu} & = -\partial_{x_1} \hat{F}^K_{x_1} \partial_{x_2} \hat{F}^K_{x_2} + \hat{F}^K_{x_1} \hat{F}^J_{x_2} - \hat{F}^K_{x_2} \hat{F}^J_{x_1}.
\end{align*}
\]

Their geometrical meanings have been previously clarified [1]. In the limit \(\Delta_{x_1}x_1 \rightarrow 0\), by using Eqs. (3.11) and (3.12) three curvature tensors above can be rewritten in forms

\[
\begin{align*}
R_{MN\mu} & = \partial_{x_1} \Gamma_{MN\mu} - \partial_{x_2} \Gamma_{MN\mu} - \Gamma_{J\mu M} \Gamma_{J_{x_1}x_1} + \Gamma_{J\mu M} \Gamma_{J_{x_2}x_2}, \\
R_{MN\mu} & = \partial_{x_1} \Gamma_{MN\mu} - (\partial_{x_2} \Gamma_{MN\mu})_0 - \Gamma_{J\mu M} \Gamma_{J_{x_1}x_1} + \Gamma_{J\mu M} \Gamma_{J_{x_2}x_2}, \\
R_{MN\rho} & = -(\partial_{x_1} \Gamma_{\rho x_1})_0 + \Gamma_{J_{x_1}x_1} \Gamma_{J_{x_2}x_2}.
\end{align*}
\]

We first note from (4.6) and (3.22)

\[
\begin{align*}
R_{rrrr} & = -(\partial_{x_1} \Gamma_{rr})_0 + \Gamma_{J_{rr}x_1} \Gamma_{J_{rr}x_1} \\
& = -\Gamma_{rrrr} + \Gamma_{rr\rho} \Gamma_{rr\rho} \\
& = 0,
\end{align*}
\]

hence

\[
R_{rrrr} = G^{rr} R_{rrrr} = 0.
\]

From (4.5) and (3.21) the relevant component \(R_{\nu \rho \mu \nu}\) is reduced to

\[
\begin{align*}
R_{\nu \rho \mu \nu} & = \partial_{x_1} \Gamma_{\nu \rho \mu} - (\partial_{x_2} \Gamma_{\nu \rho \mu})_0 - \Gamma_{J_{\nu \rho \mu}x_1} \Gamma_{J_{\nu \rho \mu}x_1} + \Gamma_{J_{\nu \rho \mu}x_2} \Gamma_{J_{\nu \rho \mu}x_2} \\
& = \partial_{x_1} \Gamma_{\rho \nu \mu} - \Gamma_{\rho \nu \mu} \Gamma_{\rho \nu \mu} - \Gamma_{\nu \rho \mu} \Gamma_{\nu \rho \mu} + \Gamma_{\nu \rho \mu} \Gamma_{\nu \rho \mu} \\
& = \nabla_{x_1} \Gamma_{\nu \rho \mu} \\
& = -\nabla_{x_1} (\lambda \partial_{x_1} \lambda),
\end{align*}
\]

where we have used (3.15) and \(\nabla_{x_1}\) is the covariant derivative in \(M_4\), hence

\[
R_{\rho \nu \mu \nu} = -\nabla^\rho (\lambda \partial_{x_1} \lambda).
\]
In the same way we get
\[
R_{\mu \nu r} = \partial_r \Gamma_\mu \Gamma_\nu - \partial_\mu \Gamma_{r \nu} + \Gamma_{j \mu r} \Gamma^j_{\nu r} - \Gamma_{j r \mu} \Gamma^j_{\nu r} = -\nabla_\mu (\lambda \partial_\nu \lambda),
\]
so that
\[
R^r_{\mu \nu} = G^{rr} R_{\mu \nu r} = -\frac{1}{\lambda^2} \nabla_\nu (\lambda \partial_\mu \lambda).
\]

From (4.4) we have
\[
R_{\rho \sigma \mu \nu} = \partial_\mu \Gamma_{\rho \sigma \nu} - \partial_\nu \Gamma_{\rho \sigma \mu} - \Gamma_{j \mu \rho} \Gamma^j_{\sigma \nu} + \Gamma_{j \nu \rho} \Gamma^j_{\sigma \mu} = \partial_\mu \Gamma_{\rho \sigma \nu} - \partial_\nu \Gamma_{\rho \sigma \mu} - \Gamma_{\lambda \mu \rho} \Gamma^\lambda_{\sigma \nu} + \Gamma_{\lambda \nu \rho} \Gamma^\lambda_{\sigma \mu}.
\]
This gives the 4-dimensional conventional Riemann scalar curvature
\[
R^{(4)} = g^{\mu \nu} R^\rho_{\mu \rho \nu},
\]
which is referred to the first kind of scalar curvature. In addition to \(R^{(4)}\) we have the second and third kinds of scalar curvatures corresponding to (4.5) and (4.6), respectively. For the second kinds of scalar curvatures we have two types, which are defined by
\[
R^{(1)}_{2nd} = g^{\mu \nu} R^r_{\mu \nu r} = -\frac{1}{\lambda^2} \nabla_\rho (\lambda \partial_\rho \lambda),
\]
\[
R^{(2)}_{2nd} = G^{rr} R^p_{r p r} = -\frac{1}{\lambda^2} \nabla_\rho (\lambda \partial_\rho \lambda).
\]
The third kind of scalar curvature is defined by
\[
R_{3rd} = G^{rr} R^r_{r r r} = 0,
\]
which vanishes owing to (4.7).

Now, in order to construct the gravity action, we consider that the three scalar curvatures \(R^{(4)}, R^{(1)}_{2nd},\) and \(R^{(2)}_{2nd}\) are all scalar quantities on \(M_4\) and they are never mixed with each other under general coordinate transformations. The most simple gravity action on \(M_4 \times \mathbb{Z}_2\) linear to \(R\)’s, therefore, should be composed of three terms
\[
I = \int_{M_4} \int_{\mathbb{Z}_2} \sqrt{-g} \frac{1}{\lambda^2} [R^{(4)} + c_1 R^{(1)}_{2nd} + c_2 R^{(2)}_{2nd}] = \int_{M_4} \int_{\mathbb{Z}_2} \sqrt{-g} \left[ \frac{1}{\lambda^2} R^{(4)} - (c_1 + c_2) \frac{1}{\lambda^2} \nabla_\rho (\lambda \partial_\rho \lambda) \right].
\]
where $g \equiv \text{det}(g_{\mu\nu})$ and $c_1, c_2$ are real dimensionless arbitrary constants. From (3.1) the dimension of $\lambda$ is $[\lambda] = L(\text{length})$, if $\Delta z$ is dimensionless. Since the action should be dimensionless, the scalar curvatures should be multiplied by $1/\lambda^2$ like (4.17) because $[R's] = L^{-2}$. After partial-integration for the second term in (4.17) and setting $2/\lambda^2 = \phi$, the action summed over $Z_2$ is reduced to

$$I = \int_{M_4} \sqrt{-g} (\phi R^{(4)} - \omega \frac{\partial^\rho \phi \partial_\rho \phi}{\phi}),$$

(4.18)

where $\omega = c_1 + c_2$. This is nothing but the BD theory with the arbitrary BD coupling constant $\omega$. 


V. Concluding remarks

On the basis of the equivalence assumption stated in the introduction and also of the new isometry condition (3.8) we have derived the Brans-Dicke theory on the manifold $M_4 \times Z_2$. In the previous work\[1\] we have not taken into account of the contribution from the last term in (3.8). This term is crucially so important as to eliminate $R_{rrrr}$, the Riemann tensor of the third type (4.3), and to yield the BD kinetic term, which comes out of the Riemann curvature of the second type (4.2).

The BD coupling constant $\omega$ has become to be arbitrary, contrary to the previous work. This comes from the fact that the three scalar curvatures $R^{(4)}$, $R^{(1)}_{2nd}$, and $R^{(2)}_{2nd}$ are all independent scalar quantities on $M_4$ and they are never mixed with each other under general coordinate transformations. This fact allows us to introduce arbitrary parameters $c_1$, $c_2$ into the action, where $\omega$ is given by $\omega = c_1 + c_2$.

We also have clarified the geometrical meaning of torsion in this space. There are three kinds of torsions, $\tilde{\Gamma}^M_{\mu\nu} - \tilde{\Gamma}^M_{\nu\mu}$, $\tilde{\Gamma}^M_{\mu r} - \tilde{\Gamma}^M_{r\mu}$, and $-\tilde{\Gamma}^M_{rr}$. In the BD theory first two torsions should vanish, but the last one remains finite in the limit $\Delta^r z \to 0$, i.e., $-\Gamma^\mu_{rr} = \lambda \partial^\mu \lambda$.

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Appendix

More elegant proof of this is given as follows: Let the coordinate $z(g)$ take values $z(e) = a$ and $z(r) = b$. Then we find the identity

$$z^2(g) = (a + b)z(g) - ab. \tag{A.1}$$

From this any polynomial of $z(g)$ is reduced to a linear function of $z(g)$. Therefore, any function regular on $z(g)$ can be expressed as (1.5), i.e., $f(g) = A + Bz(g)$.

The mapping function $H(g, g + r) = H(z, z')$, $z = z(g)$, $z' = z(g + r)$, is also linear to $z$ and $z'$, i.e.,

$$H(z, z') = A + Bz + B'z' + Czz'. \tag{A.2}$$

Substituting $z' = z + \Delta^rz$ into (A.2) and using $H(z, z) = 1$, we have

$$H(z, z') = 1 + (B' + Cz)\Delta^rz = 1 + \partial_{z'}H(z, z') \big|_{z' = z} \Delta^rz \equiv 1 + \hat{\Gamma}(z)\Delta^rz(g). \tag{A.3}$$

Namely, the Taylor expansion of $H(z, z')$ is cut off up to the first order of $\Delta^rz(g)$. 

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