Overcharging a Black Hole and Cosmic Censorship

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Abstract

We show that, contrary to a widespread belief, one can overcharge a near extremal Reissner-Nordström black hole by throwing in a charged particle, as long as the backreaction effects may be considered negligible. Furthermore, we find that we can make the particle’s classical radius, mass, and charge, as well as the relative size of the backreaction terms arbitrarily small, by adjusting the parameters corresponding to the particle appropriately. This suggests that the question of cosmic censorship is still not wholly resolved even in this simple scenario. We contrast this with attempting to overcharge a black hole with a charged imploding shell, where we find that cosmic censorship is upheld. We also briefly comment on a number of possible extensions.
1 Introduction

The question of cosmic censorship remains one of the most important open questions in classical general relativity. Since Penrose proposed the idea in 1969 \cite{Penrose1969}, numerous papers have been produced addressing the question of whether the cosmic censorship conjecture holds or not. We shall not attempt to review those developments here; for some of the recent reviews and lists of references, see e.g. \cite{Wald2004}, \cite{Penrose1999}, \cite{Dias2008}. Rather, we will consider a specific attempt to violate cosmic censorship. To that end, we will first give a brief discussion of what the conjecture implies in this context and therefore what might constitute providing a counter-example to it.

The basic idea of cosmic censorship is roughly as follows: Starting from regular, generic initial conditions, and evolving the system using classical general relativity, we cannot obtain a singularity which can be observed from “far away” (or more precisely, from the future null infinity)\footnote{This is referred to as the “weak” cosmic censorship. The “strong” version drops the “far away” proviso, namely, it asserts that we cannot evolve to a timelike singularity. In the following, we will consider just the “weak” version.}. On the other hand, we know from the singularity theorems that singularities are a rather generic occurrence in general relativity. For instance, we can obtain a singularity from regular and generic initial data (with “reasonable” properties, such as the energy condition) just by concentrating a sufficient amount of matter into a given compact region of space. The statement of cosmic censorship then implies that one cannot destroy a black hole (i.e. its horizon) just by throwing in more “reasonable” matter. We will attempt to test the validity of this statement by considering a particular example.

In his seminal paper, Wald \cite{Wald1974} attempted to destroy a Kerr-Newman black hole by throwing in particles which “supersaturate” extremality (i.e. which, when captured by the black hole, would violate the bound $M^2 \geq Q^2 + a^2$, where $M$ is the mass, $Q$ the charge, and $a$ the angular momentum per unit mass of the resulting configuration). The Kerr-Newman metric with such a bound violated does not possess an event horizon, and therefore describes a naked singularity. Wald presented gedanken experiments of an extremal black hole capturing particles with high charge and angular momentum to mass ratio. However, he found that precisely those particles which would exceed extremality will not be captured by the black hole.

In the spirit of \cite{Wald1974}, we present a mechanism whereby a black hole does “capture” a particle such that the parameters of the final configuration correspond to a naked singularity. The crucial difference between our set-up
and that of § 4 is that we are starting with a black hole arbitrarily close to, but not exactly at, extremality.

While our calculation of the “test” process (i.e. neglecting backreaction effects) unequivocally indicates that the particle falls into the singularity and overcharges the black hole, the implications about cosmic censorship cannot be easily assessed without first estimating the backreaction effects. We attempt to do so in this paper. Although we find these to be small, a definite conclusion with regards to cosmic censorship is not yet at hand.

We contrast this situation with that of attempting to overcharge a black hole with an imploding charged shell. In such a case of exact spherical symmetry, we can determine the equation of motion exactly–i.e. we do not need to consider further backreaction effects or be confined to just “test” processes. However, we find that in this case the self-repulsion of the shell prevents it from collapsing past the horizon of the black hole.

The outline of this paper is as follows: In section 2 we describe the basic idea for overcharging a black hole. In section 3, we discuss the actual mechanism, and provide a numerical example. Section 4 is devoted to the discussion of the simpler case of an imploding charged shell. Various extensions and other examples of potential cosmic censorship violation are presented in section 5, and we discuss and summarize our results in section 6.

2 Basic Idea

The basic general idea for “overcharging” a black hole is the following: We start with a near-extremal Reissner-Nordström black hole with charge $Q$ and mass $M$. The goal is to send in (radially) a charged particle with charge $q$, rest mass $m$, and energy $E$, such that $m < E < q < M$, satisfying the following conditions:

1. The particle falls past the radial coordinate, $r_+$, corresponding to the initial horizon of the black hole.
2. The final configuration exceeds extremality, $Q + q > M + E$.
3. The particle may be treated as a test particle; that is, the “backreaction” effects are negligible.

(Whereas conditions 1. and 2. would in principle suffice to overcharge a black hole, condition 3. renders our calculations meaningful and feasible.)
Then, after the completion of such a process, the solution should approach a spacetime described by Reissner-Nordström metric with $M_{\text{final}} \leq M + E < Q + q \equiv Q_{\text{final}}$, which corresponds to a naked singularity rather than a black hole. In such a case, our process seems to “destroy the horizon of a black hole”, in violation of cosmic censorship.

It has long been known (\cite{5},\cite{6},\cite{7}) that one cannot destroy a black hole using this process in a quasistationary manner, that is, by allowing the black hole mass and charge to change just by infinitesimal amounts. Namely, if one considers strictly test processes, to overcharge a black hole, one must first reach extremality. But once extremality is reached, any further changes would at best maintain it. The crucial point here is that in our process, the change in black hole’s parameters is finite (albeit small), so that we are not constrained to starting with an extremal black hole in order to overcharge it.

3 The Mechanism

In this section we discuss how well we can satisfy conditions 1.–3. Namely, for a suitable choice of parameters $(M, Q)$ corresponding to the black hole, we wish to find the parameters $(m, E, q)$ describing the particle which satisfy all three conditions. The specification of these 5 parameters determines the motion of the particle.

We proceed in two steps. If we can find parameters which satisfy all three conditions, then in particular, the particle’s motion will be described arbitrarily well by the test charge equation of motion. Hence we can show that conditions 1. and 2. hold (for a suitable choice of parameters) using the test particle approximation (i.e. assuming condition 3.), and then attempt to show the self-consistency of the approach by arguing that the backreaction effects can be arbitrarily small.

Such calculation however still does not suffice to argue that all three conditions are satisfied simultaneously, due to the perhaps somewhat subtle reason that a small change in the equation of motion can produce large changes in the trajectory. In fact, we will see that even though the test approximation leads to conditions 1. and 2. being fulfilled and condition 3. is satisfied to a desired degree of accuracy, using a better approximation to the particle equation of motion leaves the fulfilment of condition 1. in

\footnote{We write $M_{\text{final}} \leq M + E$ in order to include the possibility that some of the mass is carried away by gravitational radiation before the system settles down to its final state.}
question.

Nevertheless, studying conditions 1. and 2. separately may be of some interest on its own. Although as pointed out in the Introduction, we cannot make claims about real cosmic censorship violation without taking backreaction into account, the “test” analysis will serve to provide a comparison with earlier works such as [8], and will be used as a springboard for the analysis of the backreaction effects.

3.1 Motion of a test charge

We consider the motion of a test particle of charge \( q \), mass \( m \), and four-velocity \( u^a \), in a fixed background Reissner-Nordström spacetime with parameters \( M \) and \( Q \), and an electrostatic potential \( A_a \). In the ingoing Eddington coordinates, \((v, r, \theta, \varphi)\), the Reissner–Nordström metric is

\[
\text{ds}^2 = -g(r) \, dv^2 + 2 \, dv \, dr + r^2 \, d\Omega^2
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2 \) and \( g(r) \equiv 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \). For \( Q \leq M \), the metric describes a black hole with charge \( Q \) and mass \( M \). In this case we can write \( g(r) = \frac{(r-r_-)(r-r_+)}{r^2} \) with the event and Cauchy horizon given by \( r_{\pm} = M \pm \sqrt{M^2 - Q^2} \). Hence \( r_- \leq Q \leq M \leq r_+ \). Extremality corresponds to \( Q = M \), whereas for \( Q > M \), we have a naked singularity. The electrostatic potential \( A_a \) has the only nonvanishing component \( A_v(r) = -\frac{Q}{r} \).

The equation of motion for a test charge is given by

\[
u^a \nabla_a u^b = \frac{q}{m} F^b_{\ c} u^c \]

with \( F_{ab} \equiv 2\nabla_{[a} A_{b]} \). For radial trajectories, we can write \( u^a = \dot{v} \left( \frac{\dot{v}}{E} \right)^a + \dot{r} \left( \frac{\dot{r}}{E} \right)^a \) where \( \dot{r} \equiv \frac{d}{d\tau} \) and \( \tau \) is the affine parameter along the worldline. Since \( \left( \frac{\dot{v}}{E} \right)^a \) is a Killing field, the particle’s “energy”, \( E \equiv -\left( \frac{\dot{v}}{E} \right)^a (mu_a + qA_a) = m [g(r) \dot{v} - \dot{r}] - qA_v \) is a constant of motion along the particle’s worldline. The particle follows a timelike trajectory, so that \(-1 = u^a u_a = -g(r) \dot{v}^2 + 2\dot{v} \dot{r} = -\frac{1}{m} (E + qA_v) \dot{v} + \dot{r} \). Eliminating \( \dot{v} \) from these expressions and solving for \( \dot{r} \), we obtain the radial equation of motion in a particularly useful form:

\[
\dot{r}^2 = \frac{1}{m^2} \{ E + qA_v(r) \}^2 - g(r).
\]

Since \( \dot{r} = 0 \) corresponds to a turning point, the particle “falls in” if \( \dot{r}^2 \) never vanishes outside the horizon,

\[
\dot{r}^2 > 0 \quad \forall \ r \geq r_+.
\]
Thus, if we find suitable parameters \( m, E, \) and \( q, \) such that eq. (4) is satisfied, then condition 1. will hold. Since \( g(r) \) is positive outside the horizon, eqs. (3) and (4) imply that \( [E + qA_v(r)]^2 > m^2 g(r) > 0 \quad \forall \ r \geq r_+. \) This imposes the necessary condition on the energy: \( E > -q A_v(r) = \frac{qQ}{r}. \) (Note that this relation also follows from the above definition of the energy, since \( g(r) \dot{v} - \dot{r} > 0 \) initially.) We can simplify this condition by noting that \(-q A_v(r)\) attains its maximal value at \( r = r_+. \) Replacing \( r \) with \( r_+ \) above, we have

\[
E > \frac{qQ}{r_+}.
\]  
(5)

Condition 1. thus provides a lower bound on \( E. \) On the other hand, condition 2. provides an upper bound:

\[
E < Q + q - M.
\]  
(6)

Thus, in order to find \( E. \) satisfying both conditions, \( q. \) must satisfy the inequality \( \frac{qQ}{r_+} < Q + q - M, \) or

\[
q > r_+ \left( \frac{M - Q}{r_+ - Q} \right) = \frac{r_+ - Q}{2}.
\]  
(7)

Note that for a non-extremal black hole, we can always find sufficiently large \( q. \) such that there exists a finite range of allowed energies \( E. \) Furthermore, the allowed range of admissible \( E. \)'s grows linearly with \( q. \) On the other hand, if we start with an extremal black hole, \( Q = M = r_+, \) then we obtain \( E > q \) from eq. (5), and \( E < q \) from eq. (6), which means that conditions 1. and 2. cannot be satisfied simultaneously—i.e. we cannot overcharge an extremal black hole.

The above choice of \( q \) and \( E. \) automatically ensures that condition 2. is satisfied. To address condition 1., we must return to eq. (3). Since \( g(r) \) is bounded and \( [E + qA_v(r)]^2 > 0 \quad \forall \ r \geq r_+, \) we can always find \( m. \) such that \( \dot{r}^2 > 0. \) In particular, we must pick \( m < (E - qQ/r)/\sqrt{g(r)} \quad \forall \ r \geq r_+. \) Analytically, the RHS attains its minimum at

\[
\dot{r}_m = Q \left( \frac{Mq - QE}{Qq - ME} \right) > r_+,
\]  
(8)

where the inequality follows from eq. (5). Upon substitution, we find that \( m. \) must satisfy the bound

\[
m < Q \sqrt{\frac{-E^2 + 2MqEq - q^2}{M^2 - Q^2}}.
\]  
(9)
Note that the positivity of the radical follows automatically from the above conditions:

\[- E^2 + 2 \frac{M}{Q} Eq - q^2 = 2 Eq \left( \frac{M}{Q} - 1 \right) - (q - E)^2 \]

\[> (M - Q)^2 \left[ 2r_+ \frac{M - Q}{(r_+ - Q)^2} - 1 \right] = 0. \tag{10} \]

This choice of \( m \) ensures that eq.(10) holds.

To summarize, given the background spacetime with parameters \((Q, M)\), we can satisfy conditions 1. and 2. by choosing the parameters \( m, E, \) and \( q \) as follows:

1. Pick \( q \) which satisfies eq.(7) with the given values of \( Q \) and \( M \).
2. Select \( E \) satisfying eqs. (5) and (6) with the chosen \( q, Q, \) and \( M \).
3. Finally, choose \( m \) according to eq.(9) using the other parameters.

This prescription guarantees that if the particle follows the test charge equation of motion (as dictated by condition 3.), then it falls past the horizon and “overcharges” the black hole. Also note that the parameters necessary for this process are not too restricted: \( q \) is bounded only from below, the range of allowed values of \( E \) grows linearly with \( q \), and \( m \) is bounded only from above. Although condition 3. will restrict these parameters further, their “generic” nature will be preserved.

It may be of interest to consider what happens to the particle after it crosses the horizon. In particular, does it fall into the singularity, or does it “bounce” and reemerge from the “horizon” (possibly into another asymptotically flat region of the spacetime)? In order for the particle to fall to \( r = 0 \), it must satisfy \( \dot{r}(r) < 0 \ \forall r \), and in particular, \( \dot{r}^2(r_m) > 0 \) with \( r_m \) denoting the minimum of \( \dot{r}^2(r) \). In the above construction we only required \( \dot{r}^2(r) > 0 \ \forall r \geq r_+ \). However, eq.(10) implies that \( \left( E - \frac{M}{r} \right) \) could only vanish at radius \( r_0 > Q > r_- \). For \( r_0 < r_+ \) we then have \( g(r_0) < 0 \), i.e. \( \dot{r}^2(r_0) > 0 \), whereas \( r_0 > r_+ \) corresponds to the case discussed above. In either situation, eq.(10) then guarantees that \( \dot{r}^2 > 0 \ \forall r \), which means that

\[3 \text{ Of course, as will be discussed below, we expect the backreaction effects to take over sufficiently close to the singularity, so the actual behavior may be very different from that predicted by the test equation of motion.} \]
once the particle passes the horizon, it also falls into the singularity. To see this more clearly, it will be convenient to rewrite eq.\((3)\) as

\[
\dot{r}^2 = \left( \frac{E^2}{m^2} - 1 \right) - \frac{2M}{r} \left( \frac{Q E q}{M m^2} - 1 \right) + \frac{Q^2}{r^2} \left( \frac{q^2}{m^2} - 1 \right).
\]

(11)

The minimum is attained at

\[
r_m = Q \left( \frac{q^2}{m^2} - 1 \right) \left( \frac{E q}{m^2} - \frac{M}{Q} \right)^{-1},
\]

(12)

with the corresponding value

\[
\dot{r}^2(r_m) = \left( \frac{q^2}{m^2} - 1 \right)^{-1} \left\{ - \left( \frac{q - E}{m} \right)^2 + 2 \left( \frac{M}{Q} - 1 \right) \left[ \frac{E q}{m^2} - \frac{1}{2} \left( \frac{M}{Q} + 1 \right) \right] \right\}.
\]

(13)

Eq.\((9)\) then dictates that \(\dot{r}^2(r_m) > 0\).

We also note in passing that the conclusion reached above about extremal black holes becomes even more transparent in this formulation: for \(Q = M\), \(r_m > r_+\), and \(\dot{r}^2(r_m) < 0\) since the second term on RHS of eq.\((11)\) vanishes.

Hence, the test charge equation of motion dictates that with a suitable choice of parameters, the particle not only passes the initial horizon of the black hole, but also that it crashes into the singularity at \(r = 0\). If this were indeed an accurate description of the actual process, then it would be difficult to see how the cosmic censorship could survive.

### 3.2 Backreaction effects

The proposed mechanism for overcharging a black hole can only be valid if we can trust our conclusions, in particular, if the use of the test charge equation of motion for our particle is justified. It has been shown (e.g., by [8], [9]) that in some instances, the “backreation” effects do become important, so that the test particle calculation can no longer be trusted. Thus, to check whether such effects arise in our case, we have to examine the backreaction.

In the following, we first present rough plausibility arguments for the negligibility of the backreaction effects, followed by a more systematic analysis. Finding that condition 3. may be fulfilled to an arbitrary degree of accuracy, we then reexamine condition 1. using a better approximation to the equation of motion. Since a full analysis of the corrected solution in the general situation is beyond the scope of this work, we present a numerical example in section 3.3.
There are several different meanings of “backreaction”, depending on the context. First, the presence of the particle modifies the spacetime metric, so that the approximation of the particle moving on a fixed background spacetime need no longer be valid. As a rough indicator of the magnitude of this effect, we may compare the relative sizes of the background electromagnetic stress-energy tensor with the electromagnetic stress tensor of the particle; if the latter is much smaller than the former, the approximation of the particle moving on a fixed background spacetime should be a good one.

Assuming the particle may be thought of as a uniformly charged ball of dust (in particular, that the internal stresses are not significant\(^4\)), the particle’s stress tensor is maximal at its “surface” at radius \(r_q\), where \(|T_q^{ab}| \sim F_{q}^2 \sim \left(\frac{q}{r_q^2}\right)^2\). In contrast, the electromagnetic stress tensor of the black hole at radius \(r\) is \(|T_{BH}^{ab}| \sim F_{BH}^2 \sim \left(\frac{Q}{r^2}\right)^2 \sim \frac{1}{M^2}\). (The last relation holds only in the vicinity of the black hole, but that is the regime we are mainly interested in.) Thus, to satisfy \(|T_q^{ab}| \ll |T_{BH}^{ab}|\), we need to allow \(r_q^2 \gg qM\). At the same time, we have the obvious requirement that \(r_q \ll M\), i.e. the particle must be smaller than the black hole in order for us to talk meaningfully of it falling into the black hole. Also, to follow the same equation of motion, its constituent parts must be subjected to the same background curvature. As we will show explicitly below, we can satisfy both conditions for sufficiently small \(q\).

Since there exists a consistent choice of \(r_q\) satisfying \(\sqrt{qM} \ll r_q \ll M\), and since the equation of motion is independent of this choice of \(r_q\), we will henceforth treat the particle as moving on a fixed background spacetime. Strictly speaking, this argument may not be sufficient to justify neglecting this form of backreaction (the reason will become more clear at the end of this section); however, further analysis is beyond the scope of this paper.

In the following, we will concentrate on a somewhat weaker form of “backreaction”, involving only a modification of the particle’s equation of motion on a fixed background spacetime. (Hence the treatment of this latter form of backreaction is contingent upon the negligibility of the former effects.)

To this end, we may use the results obtained recently by Quinn & Wald [10] (and previously by e.g. [11], later corrected by [12]), who cal-

\(^4\) We believe that this is a physically reasonable assumption, since we can first consider a particle satisfying these conditions, then take the black hole mass to be much bigger that the particle mass or charge, and then set the black hole charge \(Q\) to be near enough to extremality such that our conditions are satisfied.
ulate the corrections to the test particle equation of motion due to its “self field” effects, on a fixed spacetime. Using $\frac{q^2}{m}$ as a small expansion parameter, derive the corrected equation of motion for a small spherical charged body to the leading order:

$$a^a \equiv u^b \nabla_b u^a = \frac{q}{m} F^{ab} u_b + \frac{2}{3} \frac{q^2}{m} (\frac{q}{m} u^c \nabla_c F^{ab} u_b + \frac{q^2}{m^2} F^{ab} F_{bc} u^c - \frac{q^2}{m} u^a F^{bc} u_c F_{bd} u^d)$$

$$+ \frac{1}{3} \frac{q^2}{m} \left( R^a_{\ b} u^b + u^a R_{bc} u^b u^c \right) + \frac{q^2}{m} u_b \int_{-\infty}^{\tau} \nabla^{[b} (G^-)^{a]} u_c (\tau') d\tau'$$  \hspace{1cm} (14)

Here $F^{ab}$ is the background Maxwell tensor (corresponding to the electromagnetic field of the black hole, but not the particle’s self-field), $R^{ab}$ is the Ricci curvature tensor for the background metric, and $(G^-)^{ab}$ is the retarded Green’s function for the vector potential in the Lorentz gauge.

The first term on the RHS corresponds to the test particle limit (cf. eq.(2)). The next three terms describe the Abraham-Lorentz radiation damping, the following two local curvature terms are present to preserve the conformal invariance of the acceleration, and the last term is the “tail term”, which arises due to the failure of the Huygen’s principle in curved spacetime.\footnote{There are also $O(m)$ corrections such as the gravitational radiation corrections (which \cite{[10]} treat separately), but since $q > m$, i.e. $\frac{q^2}{m} > m$, we expect these terms to be subleading.} Note that all the correction terms contain a factor of $\frac{q^2}{m}$.

Hence we clearly desire to make $\frac{q^2}{m} \ll M$. We may interpret this physically as making the “classical radius” of the particle small compared to the size of the black hole. (The “classical radius” is defined by $r_{cl} \equiv \frac{q^2}{m}$, corresponding to the mass $m$ of a (spherical) particle given entirely by its electrostatic self-energy, $\frac{q^2}{r_{cl}}$.) The condition $\frac{q^2}{m} \ll M$ is then imposed by the previous argument: By the positive energy condition, we require the electrostatic self-energy of the particle to be smaller than or equal to its rest mass, so that $\frac{q^2}{r_q} < m$, or $r_{cl} < r_q$. Hence to satisfy $r_q \ll M$, we need $\frac{q^2}{m} = r_{cl} \ll M$.

Although we will show explicitly how to make $\frac{q^2}{m}$ arbitrarily small, we can see that a consistent way to choose $q$ and $m$ should exist, based on the following heuristic argument: For the test particle approximation, it is clear that we need to take $q \ll Q$. This in turn means that we must start near extremality, so that the range of allowed $E$’s is small. As is apparent from eqs. (6) and (7), as $Q \to M$, any allowed $E \to q$. Setting $E \sim q$ in eq.(9),
we see that conditions 1. and 2. then only constrain the ratio $\frac{q}{m}$, i.e.

$$\frac{q}{m} \lesssim \sqrt{g(r)} \quad \forall \quad r \geq r_+, \quad \text{or} \quad \frac{q}{m} \gtrsim \sqrt{\frac{1}{2} \left(\frac{M}{Q} + 1\right)},$$

but do not impose any further constraints on $\frac{q^2}{m}$. We thus expect that if we start with the black hole near enough to extremality, so that we can take $q$ to be very small, then we will satisfy $q \frac{q}{m} \ll M$. A more careful analysis below shows that this is indeed the case (and a numerical example is given in section 3.3).

However, although the expansion parameter is small, we still need to check that the actual correction terms themselves are small. To that end, we must consider eq.(14) in greater detail.

We find that for radial motion in a static potential, the third and fourth terms on the RHS of eq.(14) cancel:

$$F_{ab} F_{bc} u^c = u^a F_{bc} u^c F_{bd} u^d.$$  

Correspondingly, the curvature terms (i.e. fifth and sixth terms) also cancel,

$$R^a_b u^b + u^a R_{bc} u^b u^c = 2 \left(F^{ab} F_{bc} u^c - u^a F^{bc} u_c F_{bd} u^d\right) = 0.$$  

The tail term requires a bit more analysis, since it involves the Green’s function, and therefore is a-priori nonlocal. However, since it would not arise in a flat background spacetime, and we can set the background curvature as small as we need by letting $M$ be large, we expect there to be a regime in which the leading order effect is given only in terms of the local physics. Indeed, as [10] point out, for sufficiently small spacetime curvature and slow motion of the body, “one would expect the ‘tail term’ to become effectively local, since the contributions to the ‘tail term’ arising from portions of the orbit distant from the present position of the particle should become negligible.” We will thus assume that the full tail term is roughly bounded by its local part; specifically we assume that the full integral is not much larger than the largest value of the integrand (multiplied by a unit $\Delta \tau$ to make the units comparable).

The local expression for the Green’s function in curved spacetime may be obtained from e.g. [11]. In the current case of vanishing Ricci scalar, we may write this local contribution as $\nabla^{[b}(G^{-})^{a]c} = \nabla^{[b}R^{a]c}$. In terms of the Maxwell tensor and the components of the 4-velocity, we can then write this, with the actual factors appearing in the equation of motion as

$$-\frac{q^2}{m} u_b u_c \nabla^{[a} R^{b]c} = \left(\frac{q}{m} F^{ab} u_b\right) \left[-q f'(g\dot{v} - \dot{r})\right],$$

11
where we have employed the notation $f(r) \equiv F^{rr} = \frac{Q}{r}$, so $f'(r) = \frac{df}{dr} = -\frac{2Q}{r^2}$. Since $(g \dot{v} - \dot{r}) = (E - \frac{Q^2}{r})/m$, which is small near the horizon and $O(1)$ further away, and $f' \sim O(1/M)$ at $r \sim r_+$ and falls off as $r^{-3}$, the whole term should have at most a contribution $\sim \frac{Q}{r^4}$ near the horizon, which is indeed tiny.

Finally, let us consider the second term, which we will refer to as the “radiation damping” term. Based on the above discussion, we expect this to provide the dominant correction to the test particle equation of motion. We can rewrite the radiation damping in the form

$$2 \frac{q^3}{3 m^2} u^c \nabla_c (F^{ab}) u_b = \left( \frac{q}{m} F^{ab} u_b \right) \left[ 2 \frac{q^2}{3 m} \left( \frac{\dot{r} f'}{f} + g' \dot{v} \right) \right]$$

(18)

where $g'(r) = \frac{dg}{dr} = \frac{2M}{r} - \frac{2Q^2}{r^2}$, and we may conveniently express $\dot{v} = \frac{1}{g} (\sqrt{r^2 + g} - \sqrt{r^2})$. Again, we expect that the factor $\left( \frac{\dot{r} f'}{f} + g' \dot{v} \right) \sim O(1/M)$ near the horizon and falls off with $r$. This will be explicitly confirmed momentarily. The relative contribution of the radiation damping term to the equation of motion will then scale as $\sim \frac{q^2}{m M}$, which may be arbitrarily small, as we argued above.

To summarize, we may write the corrected equation of motion as

$$u^b \nabla_b u^a = \frac{q}{m} F^{ab} u_b \left( 1 + h(r) + \text{“tail”} \right)$$

(19)

where $h(r) \equiv \frac{2}{3} \frac{q^2}{m} \left( \frac{\dot{r} f'}{f} + g' \dot{v} \right)$, and the local (dominant) contribution of the “tail” is given by eq.(17). We have given heuristic arguments for why we expect these two backreaction terms to be subleading, and in fact, arbitrarily small. This suggests that we may indeed select the parameters of the particle such that they satisfy condition 3.

We now proceed to a more systematic way of showing explicitly how, in the regime where the test approximation fulfills conditions 1. and 2., condition 3. may also be satisfied to any desired degree of accuracy. Namely, we use a linearized expansion around the test particle limit. We may normalize $M \equiv 1$ and set $Q \equiv 1 - 2\varepsilon^2$, treating $\varepsilon$ as a small parameter. Following the previous approach we can then reexpress all the other quantities in terms of $\varepsilon$. The lower bound on $q$ imposed by eq.(7) becomes

$$q > \varepsilon + \varepsilon^2 - O(\varepsilon^3).$$

6 In fact, $\varepsilon \ll 1$ is required by conditions 2. and 3. and the condition that we are starting with a near-extremal black hole, $Q < M$.

7 Although we are quoting the relevant quantities only up to the subleading order, it is necessary to keep higher orders for the intermediate calculations.
Thus we let \( q \equiv a \varepsilon \) with \( a > 1 + \varepsilon \). This allows us to find the corresponding bounds on \( E \), using eq. (5), which yields \( E > a \varepsilon - 2a \varepsilon^2 + O(\varepsilon^3) \), and eq. (6), which yields \( E < a \varepsilon - 2 \varepsilon^2 \). Correspondingly, we let \( E \equiv a \varepsilon - 2b \varepsilon^2 \) with \( 1 < b < a(1 + \varepsilon) \). Finally, from eq. (9) we obtain the upper bound on \( m \):

\[
m < \sqrt{a^2 - b^2 \varepsilon - ab \sqrt{a^2 - b^2 \varepsilon^2} + O(\varepsilon^3)},
\]

which allows us to set \( m \equiv c \varepsilon \) with \( c < \sqrt{a^2 - b^2 (1 + ab a^2 - b^2 \varepsilon)} \).

In summary, to satisfy conditions 1. and 2., we may select the parameters of the particle (to the leading order, i.e. for \( 0 < \varepsilon \ll 1 \)) as follows:

\[
M \equiv 1 \quad (20) \\
Q \equiv 1 - 2 \varepsilon^2 \quad (21) \\
q \equiv a \varepsilon \quad \text{with} \quad a > 1 \quad (22) \\
E \equiv a \varepsilon - 2b \varepsilon^2 \quad \text{with} \quad 1 < b < a \quad (23) \\
m \equiv c \varepsilon \quad \text{with} \quad c < \sqrt{a^2 - b^2} \quad (24)
\]

We are now in the position to discuss condition 3. more explicitly. First, we may check that the effect of the particle on the spacetime metric is arbitrarily small near the horizon. In particular, the condition \( \sqrt{qM} \ll r_q \ll M \) may be achieved by picking, for instance, \( r_q \sim \varepsilon^{1/4} \). Then the particle is small, and its stress energy is also small:

\[
T_q/T_{BH} \sim \varepsilon. 
\]

More importantly, the classical radius of the particle is small:

\[
\frac{q^2}{m} = \frac{a^2}{c} \varepsilon. \quad (25)
\]

This means that the expansion used in eq. (14) is a valid one.

From eqs. (18) and (17) we can also estimate the relative sizes of the radiation damping term and the bound on the tail term. The radiation damping term becomes:

\[
h(r) = \frac{2}{3} \frac{q^2}{m} \left( \frac{\dot{r}(r)}{f(r)} + g'(r) \dot{v}(r) \right)
= \frac{4a^2}{3c^2r} \left\{ \left[ \sqrt{a^2 - c^2} + \frac{a - 2\sqrt{a^2 - c^2}}{r} \varepsilon + \frac{2b}{\sqrt{a^2 - c^2}} \left[-a - \frac{a - \sqrt{a^2 - c^2}}{r - 1} \right] \varepsilon^2 \right] \right\}.
\]

Although one may worry about the divergence of the term \( \propto \frac{1}{r-1} \) in the subleading order as \( r \to 1 \), one can check that in fact, at the horizon,
\( r - 1 \approx 2\varepsilon \), so \( h \) is still small: \( h(r_+) = \mathcal{O}(\varepsilon) \). Furthermore, the local contribution from the “tail” term is

\[
-qf'(g^\dot{v} - \dot{r}) = \frac{2a^2}{c} \left( \frac{r - 1}{r^4} \right) \varepsilon,
\]

(27)

which again is small. In fact, even though this appears to be of the same order in \( \varepsilon \) as the radiation damping term, far from the black hole, it will be subleading since it contains a relative factor of \( \sim \frac{1}{r} \), whereas near the black hole, it actually becomes \( \frac{4a^2}{c} \varepsilon^2 \). This confirms our expectation that the radiation damping term provides the dominant correction to the equation of motion.

The \( \varepsilon \)-analysis verifies that not only the expansion parameter, but also the “self-field” correction terms themselves in eq. (14) may be taken arbitrarily small. Hence, by letting \( \varepsilon \to 0 \), condition 3. is satisfied in a self-consistent way, to desired accuracy (parameterized by \( \varepsilon \)). However, this still is not a sufficient justification for neglecting the backreaction effects in our context, because the difference between the equations of motion leading to vastly different outcomes (i.e. particle falling in or bouncing) is also tiny. To see that explicitly, we now return to the first two conditions. Although conditions 1. and 2. are of course automatically guaranteed to hold in the test approximation by the construction process, it will be useful to determine the leading order in \( \varepsilon \) at which this becomes evident. This will illuminate the reason for the condition 1. possibly ceasing to hold in the first-order correction to the equation of motion.

First, using eq. (13), we have

\[
\dot{r}^2(r_m) \sim 4 \left( \frac{a^2 - b^2 - c^2}{a^2 - c^2} \right) \varepsilon^2 - 8 \left( \frac{ab}{a^2 - c^2} \right) \varepsilon^3.
\]

(28)

By the conditions on the coefficients \( a, b, c \) imposed in eqs. (22), (23), (24), the coefficient of the leading order in \( \varepsilon \) is positive, so that for small \( \varepsilon \), \( \dot{r}^2(r_m) > 0 \). This means that according to the test particle equation of motion, the particle falls past the horizon (and in fact, reaches the singularity), so that condition 1. is indeed satisfied in this approximation. We also note that the particle overcharges the black hole:

\[
(M + E) - (Q + q) = -2(b - 1) \varepsilon^2 < 0
\]

(29)

by eq. (23). This reconfirms condition 2.
We may now see why the test particle equation of motion is not necessarily indicative of the actual motion, despite the fact of the backreaction terms being small. Whereas the particle “slows down” to $|\dot{r}(r_m)| \sim O(\varepsilon)$, the characteristic size of backreaction is also $O(\varepsilon)$, so that the backreaction effects may well “destroy” the process. Furthermore, we note that the black hole is overcharged by only $O(\varepsilon^2)$, though it does not seem meaningful to compare this estimate directly with the other quantities we discussed.

We can also see how the backreaction could prevent the particle from being captured by the black hole from a more physical standpoint: As it falls towards the black hole, the particle loses energy to electromagnetic and gravitational radiation. We may see this explicitly by evaluating the change in energy along the particle’s worldline: for $u^b \nabla_b u^a = \frac{q}{m} F^{ab} u_b (1 + \hbar(r)),$

$$u^b \nabla_b E = u^b \nabla_b \left[ - \left( \frac{\partial}{\partial v} \right)^a (m u_a + q A_a) \right] = -q \hbar(r) f(r) \dot{v}(r) \quad (30)$$

which is negative, since both the radiation damping and the (full) tail term give a positive contribution to $\hbar(r)$, and $q$, $f$, and $\dot{v}$ are all positive. This energy loss becomes stronger as the particle approaches the horizon, so that near the horizon, the particle may no longer have enough energy to overcome the electrostatic repulsion of the black hole.

To see then whether or not the process survives, we would like to solve the corrected equation of motion, namely

$$u^b \nabla_b u^a = \frac{q}{m} F^{ab} u_b + \frac{2}{3} \frac{q^3}{m^2} u^c \nabla_c (F^{ab}) u_b + \text{“tail”} \quad (31)$$

where again the local contribution to the “tail” is $-\frac{q^2}{m} u_b u_c \nabla^a R^{bc}$. This is a rather complicated (integro-)differential equation, so solving it exactly is intractable in the present situation. However, working in the regime of interest where the correction terms are small, we can achieve considerable simplification by using a linearized approximation.

The derivation of the test equation of motion, eq.(3), depended only on the definition of the “energy” and the expression for the 4-velocity $u^a$ of the particle, so the calculation is identical in the present case, when $E$ is no longer a constant of motion. Hence we can write:

$$\dot{r}^2(r) = \frac{1}{m^2} \left[ E(r) - \frac{qQ}{r} \right]^2 - g(r). \quad (32)$$

We can find $E(r)$ by solving $\frac{dE}{dr} = \frac{1}{r^2} \frac{dE}{d\tau} = \frac{1}{\tau} u^b \nabla_b E \approx -q h(r) f(r) \dot{v}(r) / \dot{r}(r)$

(We have neglected the “tail term” contribution, since the above analysis
indicated this to be subleading.) Integrating, we obtain the energy as a function of $r$:

$$E(r) = E - q \int_r^\infty \frac{h(r') f(r') \dot{v}(r')}{|\dot{r}(r')|} dr'.$$  (33)

We now make the linearized approximation to $E(r)$ by letting $h$, $\dot{v}$, and $\dot{r}$ in the integrand be the expressions used in the test approximation (i.e. dependent on the constant $E$ only).

Although the resulting expression is still too complicated to determine its behavior analytically, we may now find the corrected $\dot{r}^2(r)$ numerically. However, this constrains us to consider only specific examples, so we cannot analyse the general behavior. Such a calculation is discussed in section 3.3 below. We find that in that case, $\dot{r}^2(r)$ does become negative at some $r_0 > r_+$, which means that the particle actually “bounces” before reaching the horizon, so that the corrected process now fails to destroy the horizon of the black hole.

### 3.3 Numerical example

Here we discuss a specific example, obtained by following the method outlined in section 3.1. It will be convenient to choose our mass unit to correspond to the (initial) black hole mass, i.e. we will set $M \equiv 1$. (All other quantities are then treated as dimensionless.)

Consider a black hole with $Q = 0.99999$, i.e. $M - Q = 10^{-5}$. Suppose the particle has charge $q = 3 \times 10^{-3}$, mass $m = 1.8 \times 10^{-3}$, and is thrown radially inward with energy $E = 2.9897 \times 10^{-3}$. In terms of the $\varepsilon$-expansion notation used above, we have: $\varepsilon \approx 0.0022$, $a \approx 1.34$, $b \approx 1.03$, and $c \approx 0.805$. (As one may check explicitly, the conditions imposed in eqs. (22)–(24) are all satisfied.) We then have the following results:

First, the test particle equation of motion yields $\dot{r}^2 > 0 \ \forall r$; in fact, $\dot{r}^2(r_m) \approx 2.36 \times 10^{-6}$, i.e. the particle is captured by the black hole. (A full plot of $\dot{r}^2(r)$ near its minimum is given in Fig.1.)

Second, the final total mass is $M_{\text{final}} \leq M + E = 1.0029897$, whereas the final total charge is $Q_{\text{final}} = Q + q = 1.0029900$. Thus, the charge of the final configuration is greater than the mass, $M_{\text{final}} - Q_{\text{final}} \leq -3 \times 10^{-7}$. Although the process does not overcharge the black hole significantly, this was to be expected from the fact that we consider only “nearly test” processes.

Next, consider the relative size of the backreaction: Certainly all the parameters describing the particle are much smaller than those corresponding
to the black hole: $q, E, m \ll M$. Also, if we let the actual radius of the particle be, say, $r_q \approx 0.1$, then the corresponding stress energy tensor is small: $T_q \approx 0.09T_{BH}$. More importantly, the classical radius, (i.e. the expansion parameter for the self-field corrections), is tiny: $r_{cl} = \frac{q^2}{m} \approx 0.005 \ll 1$. We may also check that the relative sizes of the self-field terms are $\ll 1$; the maximum attained by the radiation damping is $\sim h(r_+^2) \approx 0.0086$, while the local part of the tail term reaches the maximal value of $\approx 0.001$.

Finally, we may compute the actual first order correction to $r^2(r)$ as given by eqs. (32) and (33). The corrected $r^2(r)$ is plotted in Fig.2. We see that now $r^2$ does become negative, indicating that the particle does not fall past the horizon.

Since the first-order estimate of the backreaction effects suggests that unlike the test case, the particle actually bounces, one is led to ask what happens if we modify the parameters somewhat, in particular what if we increase the particle’s initial energy. Since the upper bound on the energy was determined by the condition that the particle overcharges the black hole, it suffices to require that only the energy the particle has at the horizon, plus any additional energy that falls in from the radiation, needs to satisfy the given bound. Assuming that some radiation escapes to infinity, the starting energy may then be higher.

To determine whether this relaxation is sufficient to allow the particle to fall in, we consider the “best case scenario,” in which all the radiation escapes, so that the energy which contributes to the final mass of the black hole is just given by the energy the particle has at the horizon. Then we may use eq.(33) to determine what energy the particle would have had at infinity. Again, to avoid solving an integral equation, we use the approximation

$$E_\infty = E(r_+) + q \int_{r_+}^{\infty} \frac{h(r')(f(r')\hat{v}(r'))}{|\hat{r}(r')|} dr'$$

(34)

with the $E$ appearing in $\dot{r}$ and $\hat{v}$ being the (constant) energy the particle has at the horizon, which we define $E(r_+) \equiv Q + q - M$. If the particle bounces even in this set-up, then backreaction cannot be overcome just by throwing in the particle harder.

Using eq.(34), we find that $E_\infty \approx 0.0122$, which is considerably larger than the final required energy. Our computations suggest that if this were indeed the starting energy, then the particle would fall past the horizon, after all. The corresponding plot of $\dot{r}^2(r)$ is shown in Fig.3. (However, given the numerical accuracy required for this calculation, one might question the
sufficiency of the linearized approximation of the backreaction. In particular, the minimum of \( \dot{r}^2 \approx 6 \times 10^{-9} < \varepsilon^3 \). Furthermore, if none of the escaped radiation contributed to the mass of the black hole, the black hole would be (barely) overcharged. However, since most of the lost energy is radiated near the horizon, such a constraint does not seem too realistic. Also, the first-order approximation which we used is not necessarily reliable near the horizon, since \( \frac{\dot{h}f}{|f|} \sim \mathcal{O}(\varepsilon^2) \) which becomes \( \sim \mathcal{O}(1) \) at the horizon.

To summarize, we found that according to the test approximation, the particle will be captured by the black hole. This would mean that the process overcharges the black hole, leaving cosmic censorship in question. The first order correction to the equation of motion indicates that this will no longer hold, because the particle will bounce. However, since this conclusion depends crucially on the energy loss near the horizon, where the linearized approximation may not be valid, and since our further calculation suggests that increasing the initial energy of the particle may allow the particle to fall past the horizon, we conclude that a more thorough analysis would be required to reach a definitive answer.

### 4 Imploding Charged Shell

In the previous section, we saw that even though the backreaction effects may be kept arbitrarily small as compared to the “test” case, the size of the process we are interested in may be smaller, so that the backreaction could easily prevent the process from happening, or at least the final outcome is not readily analysable.

To avoid such complications arising from the backreaction, we are led to consider a simpler case in which these effects disappear, namely the case of exact spherical symmetry \([13]\). Spherically symmetric spacetimes have of course been studied extensively in the past (for review of some of the studies related to cosmic censorship, see e.g. \([2],[4]\)), the main appeal being that in such cases, we may use the generalized Birkhoff’s theorem and the appropriate matching conditions to determine the dynamics exactly.

The spherically symmetric analog of an infalling charged particle is an imploding charged shell. Note that here we do not need to confine our considerations to “test” shells. Hence in this section, we consider the conditions for overcharging a black hole with an imploding shell.

Pioneering efforts in this direction include \([14],[15],[16]\). In the special case of an imploding charged shell in an otherwise empty spacetime, \( Q = \)}
\( M = 0 \), it was shown \[16\] that the shell can not collapse to form a naked singularity. However, to the author’s best knowledge, the more general case of a shell imploding onto an already existing black hole has not been fully analysed in the context of cosmic censorship.

Proceeding in parallel to section 3.1, we denote the shell’s total charge, rest mass, and energy by \( q \), \( m \), and \( E \), respectively. The spacetime inside the shell will be given by the Reissner-Nordström metric, eq.(1), for the black hole with \( g(r) = g_{\text{in}}(r) \equiv 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \), whereas outside the shell, the geometry will be described by eq.(1) with \( g(r) = g_{\text{out}}(r) \equiv 1 - \frac{2(M+E)}{r} + \frac{(Q+q)^2}{r^2} \), which, for \( Q + q > M + E \) corresponds to a “naked singularity” spacetime. The radial coordinate is continuous across the shell; the position of the shell will be denoted by \( R \).

The equation of motion for a charged shell is given by

\[
\sqrt{g_{\text{in}}(r)} + \dot{R}^2 - \sqrt{g_{\text{out}}(r)} + \dot{R}^2 = -\frac{m}{r}
\]

where \( \dot{R} \equiv \frac{dR}{d\tau} \) and for an infalling shell we can again write \( \dot{R}^2 \) as a function of \( r \). Solving for \( \dot{R}^2(r) \), we obtain (cf. eq.(11))

\[
\dot{R}^2 = \left( \frac{E^2}{m^2} - 1 \right) - \frac{2M}{r} \left[ \left( \frac{Q}{M} - \frac{E}{m^2} - 1 \right) + \frac{E}{2M} \left( \frac{q^2}{m^2} - 1 \right) \right] + \frac{Q^2}{r} \left[ \frac{q^2}{m^2} - 1 \right] \left[ 1 + \frac{q}{Q} + \frac{q^2 - m^2}{4Q^2} \right].
\]

(36)

We can also write the equation of motion in a form of eq.(13), but with an effective potential \( A_{\text{shell}}(r) \equiv -\frac{Q}{r} - \frac{q^2}{2r} + \frac{m^2}{2r} \) which includes not only the electromagnetic interaction with the black hole, but also the electromagnetic self-repulsion and the gravitational self-attraction of the shell. (The factor of 2 in the denominator of the latter two terms comes from the standard “Coulomb” potential of a shell evaluated on the shell itself.) Hence

\[
\dot{R}^2 = \frac{1}{m^2} \left[ E - \frac{qQ}{r} - \frac{q^2}{2r} + \frac{m^2}{2r} \right]^2 - g_{\text{in}}(r).
\]

(37)

We now show that the conditions imposed on the parameters are not mutually consistent. As before, we require that the shell overcharges the black hole, \( M > Q \) and \( M + E < Q + q \), and also that the parameters of the shell satisfy \( 0 < m < E < q \). A shell which falls past the horizon \( r_+ \) of the
black hole has $\dot{R}(r) > 0 \ \forall r \geq r_+$, which is the necessary (and sufficient) condition for creating a naked singularity. There are two physically distinct possibilities of how this could be satisfied.

First, the shell may implode all the way down to the singularity, which means $\dot{R}(r) > 0 \ \forall r$ and in particular,

$$\dot{R}(r_m) > 0$$

(38)

where $r_m$ is the coordinate where $\dot{R}$ achieves its minimum. Solving $\frac{d\dot{R}}{dr}(r_m) = 0$ for $r_m$ and writing $\dot{R}(r_m)$ in a convenient form, we have:

$$\dot{R}(r_m) = \frac{-m^4 + 2m^2[\omega - 2EM - 2(M^2 - Q^2)] - \omega(\omega - 4EM) - 4E^2Q^2}{(q^2 - m^2)[(q + 2Q)^2 - m^2]}$$

(39)

where, for compactness of notation, we have defined $\omega \equiv q(q + 2Q)$. Since the denominator is always positive, in order to satisfy eq. (38), we must find a suitable combination of the parameters $(m, E, q, Q, M)$ such that the numerator is positive. Imposing this as a condition on $m^2$, we must have

$$\omega - 2EM - 2(M^2 - Q^2) - \omega(\omega - 4EM) - 4E^2Q^2 = 4(M^2 - Q^2)(E^2 + 2EM + (M^2 - Q^2) - \omega) > 0.$$  

(40)

However, this is exactly inconsistent with the “overcharging” conditions ($M > Q$ and $Q + q > M + E \Rightarrow \omega > E^2 + 2EM + (M^2 - Q^2)$), which means that eq. (38) can not be satisfied with any suitable parameters corresponding to overcharging the black hole. Thus, a naked singularity cannot result from a charged shell imploding to the central singularity.

The second, less stringent, way to create a naked singularity is for the shell to implode past the horizon and “bounce”, i.e. $\dot{R}(r_{\text{bounce}}) = 0$ with $r_{\text{bounce}} < r_+$. In this case, even though the shell does not itself crash into the singularity, it does destroy the horizon of the black hole, thereby exposing the singularity that already exists. Since $\dot{R}(r)$ has only one minimum, it is a monotonically increasing function $\forall r > r_{\text{bounce}}$, and in particular,

$$\frac{d\dot{R}}{dr}(r_+) > 0.$$  

(41)

It is not clear how this process might occur in a more realistic black hole collapse scenario, such as the black hole forming, say, from a charged ball of dust, as opposed starting with an eternal black hole: There does not seem to be a consistent Penrose-Carter diagram which would correspond to this process.
Eq. (41) is the analog of eq. (38), and upon simplifications, we obtain the corresponding analog of eq. (39),

\[
\frac{d\dot{R}^2}{dr}(r_+) = \frac{-m^4 + 2m^2[\omega - Er_+ - 2r_+\sqrt{M^2 - Q^2}] - \omega(\omega - 2Er_+)}{2m^2r_+^3}.
\]

(42)

which we require to be positive.

Now, the situation is a bit more complicated than in the previous case, because we can find some \(m^2\) and \((E, q, Q, M)\) corresponding to overcharging the black hole such that the RHS of eq. (42) is positive. However, the physical constraints on \(m^2\), namely \(0 < m^2 < E^2\), unfortunately turn out to be inconsistent with these solutions. Although the analytic proof is not as apparent as above, numerical calculations indicate that this is indeed the case.

This means that the shell must bounce before reaching the horizon, so that it cannot create a naked singularity. We note that in the special case of \(Q = M = 0\), (i.e. a charged shell in an otherwise empty spacetime) it is now easy to see that the shell cannot implode to form a naked singularity. In such case eq. (36) simplifies to

\[
\dot{R}^2 = \left(\frac{E^2}{m^2} - 1\right) - \frac{E}{r} \left(\frac{q^2}{m^2} - 1\right) + \frac{m^2}{4r^2} \left(\frac{q^2}{m^2} - 1\right)^2,
\]

(43)

which attains minimum at \(r_m = \frac{q^2 - m^2}{2E}\), at which point \(\dot{R}^2(r_m) = -1 < 0\). Therefore, the shell bounces at \(r_{\text{bounce}} > r_m > 0\), and there is no singularity anywhere.

What conclusions may one draw from this negative result? There is an important difference between the shell and the point particle: the shell experiences its own self-repulsion, whereas the point particle does not. On the other hand, the particle still does experience the “self-field” effects from the backreaction, which may prevent it from falling past \(r_+\). We found that attempting to circumvent these backreaction effects by considering the case of exact spherical symmetry unfortunately failed to produce the desired effect.

5 Extensions

The particular mechanism for possible violation of cosmic censorship discussed in section 4 was motivated primarily by its algebraic simplicity. How-
ever, we can imagine a number of possible generalizations or extensions which could also lead to similar results.

Following up on the lesson learned in the previous section, one would be tempted to try to find a “compromise” between the shell and the particle, which would have the virtues of both. Namely, one would like to find a case in which the backreaction effects would be small enough to preserve the outcome of the process, and at the same time, the self-repulsion would not be too great to prevent the process from happening in the first place. One might hope that such a case could be achieved with a symmetric configuration of \( N \) particles distributed uniformly around the black hole and infalling according to the same equation of motion. The previous two cases we discussed in sections 3 and 4 correspond to the special cases of \( N = 1 \) and \( N \to \infty \), respectively.

Unfortunately, a definite conclusion about the backreaction effects in such set-up has not yet been reached. Nevertheless, it is interesting to note some exploratory results of such study. We have considered explicitly the cases of \( N = 2, 4, \) and \( 6 \) particles. To find the electromagnetic force that each particle feels, one may use the expression derived by [17] for the potential of a point charge in Reissner-Nordström spacetime, along with the superposition principle. One finds that according to the test process, one may still find a finite (though more constrained) range of parameters whereby all the particles fall past the horizon and overcharge the black hole. In this scenario, one is actually helped by causality: Each particle “feels” the other ones as if they were much further away from the black hole. Indeed, one may find cases in which each “process” by itself (i.e. each particle falling into the black hole in the field of the others) is perfectly allowed in all regimes (test as well as macroscopic) since by itself this doesn’t overcharge the black hole; only the combined effect of all processes taken together leads to overcharging. However, we expect that such configurations still do not reduce the backreaction effects sufficiently, though more detailed calculations would be needed to verify it.

One may also imagine a number of generalizations of the single-particle case which compensate for (or at least alleviate) the backreaction effects. In the preceding treatment, we considered the particle moving in a background spacetime with the only forces acting upon it being determined by the background electrostatic field of the black hole and the self-field of the particle. However, one may also consider additional forces acting on the particle. In particular, by rescaling everything to a sufficient size, one may imagine that the “particle” has a little rocket attached to it, so that it may boost itself at
any point of its trajectory. Then the particle could possibly “compensate” for the backreaction effects, by boosting itself to the appropriate energy just before crossing the horizon, so that it would fall in, while still overcharging the black hole.

Perhaps the most interesting extension of the basic mechanism is to consider angular momentum as well as charge as free parameters. For example, we could send the particle(s) toward the horizon spinning, or in a non-radial direction, such that it (they) would impart angular momentum to the black hole. The corresponding equation of motion would of course be far more complicated, but from a physical standpoint, such cases are more realistic and perhaps more “promising”, since extremality might possibly be supersaturated more easily: If the black hole captures the particle(s), both the charge and the angular momentum of the black hole would increase. (Again, we stress that it is crucial to start with a near extremal black hole.)

An interesting limit to consider is the completely uncharged case, namely throwing a spinning particle into a near extremal Kerr black hole. This case has several appealing features. First, we do not have to worry about the electromagnetic self-field backreaction effects, which in the original mechanism were the most constraining and nontrivial effects. Second, this would present a much more physically realistic case, since we believe that near-extremal Kerr black holes actually exist in our universe. Finally, the causal structure inside the horizon of a Kerr black hole is even more interesting than that inside a Reissner-Nordström black hole.

Although we have not examined these latter extensions, which would probably be better addressed by means of numerical simulations, it seems that they might yield a more definitive answer as to whether cosmic censorship is upheld or not.

6 Discussion and Conclusions

We have reexamined the old question of cosmic censorship in the simple context of a charged particle falling into a near extremal Reissner-Nordström black hole. In particular, we tried to analyse whether a particle which would overcharge the black hole could fall past the horizon. As in earlier works which considered this type of set-up (i.e. particles supersaturating extremal bounds of black holes), such as [3,4,9,6,7], the basic motivation was that if the particle is actually captured by the black hole, then it seems
that the resulting final configuration cannot sustain an event horizon, suggesting that a naked singularity results. This conclusion appears even more inescapable if the particle falls all the way into the singularity.

The simplest way to examine the possibility of supersaturating the extremal bound of a black hole is to start at extremality and consider strictly test processes. Unfortunately, such processes cannot lead to destroying the horizon, since the particles which are allowed to fall in would at best maintain extremality (§2, §3, §4, §5). Faced with these results, one is led to start with a near extremal black hole. One might then hope that such a set-up allows the black hole to “jump over” extremality, by capturing a particle with appropriate parameters.

Perhaps somewhat surprisingly, we found that according to the test particle approximation, this can be indeed achieved. Namely, a broad range of configurations can be found in which, according to the test equation of motion, the particle falls into the singularity and the resulting object exceeds the extremal bound. This result, which seems to have been previously overlooked, is important in the context of these earlier attempts to violate cosmic censorship.

However, we now have to contend with backreaction. The natural question then arises, “how good is the test approximation in this context?” We have seen that this issue is somewhat subtle, and in fact it is not yet fully resolved. In particular, the backreaction effects may be kept arbitrarily small as compared to the leading order effects, but this does not suffice. Indeed, we have seen from our numerical example that the particle may lose enough energy due to the backreaction effects that it bounces back to infinity instead of being captured by the black hole.

Although we have not found good examples in which the particle falls past the horizon even when the first-order effects of the backreaction are taken into account, we have not found a solid indication to the contrary. Furthermore, it is not clear that the situation described by our example is in fact generic in terms of the outcome. Also, the possible extensions mentioned in the previous section, which may lead to the particle falling past the horizon after all, have not been analysed.

On the other hand, even if the particle is “captured” by the black hole, the resulting implications for cosmic censorship are no longer as clear as

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9 Technically, in such a situation, there never existed an event horizon, but only the apparent horizon.

10 Then one doesn’t need to worry about the scenario where the particle reemerges, possibly leaving behind a near extremal black hole again.
in the test case. We have seen that the backreaction effects are important already before the particle crosses the horizon; they would become dominant after. This means that inside the horizon, we definitely can not trust the test particle equation of motion. It way well happen that the self-field effects will cause the particle to “bounce” at some radius $r_0 < r_+$, or even that the language we have been using to describe the situation is no longer appropriate. In the former case, the particle reemerges (possibly into the same asymptotically flat region it started from, since there can not have been any horizon “above” the particle), and the inner singularity may never become visible to the outside observer.

Given all these considerations, we are led to conclude that even in the case of a single particle falling into a black hole, a more detailed analysis will be needed to settle the issue of whether or not the cosmic censorship prevails.

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References

[1] R. Penrose, “Gravitational Collapse: the Role of General Relativity,” Riv. Nuovo Cim. 1, 252 (1969)

[2] R. M. Wald, “Gravitational Collapse and Cosmic Censorship,” gr-qc/9710068

[3] C. J. S. Clarke, “A Review of Cosmic Censorship,” Class. Quant. Grav. 10, 1375, (1993)

[4] T. P. Singh, “Gravitational collapse, Black Holes and Naked Singularities,” gr-qc/9805066
[5] R. Wald, “Gedanken Experiments to Destroy a Black Hole,” Ann. Phys. 83, 548, (1974)

[6] J. M. Cohen & R. Gautreau, “Naked Singularities, Event Horizons, and Charged Particles,” Phys. Rev. D19, 2273, (1979)

[7] I. Semiz, “Dyon Black Holes Do Not Violate Cosmic Censorship,” Class. Quant. Grav. 7, 353, (1990)

[8] J. D. Bekenstein & C. Rosenzweig, “Stability of the Black Hole Horizon and the Landau Ghost,” Phys. Rev. D50, 7239, (1994), gr-qc/9406024

[9] T. Needham, “Cosmic Censorship and Test Particles,” Phys. Rev. D22, 791, (1980)

[10] T. C. Quinn & R. M. Wald, “An Axiomatic Approach to Electromagnetic and Gravitational Radiation Reaction of Particles in Curved Spacetime,” Phys. Rev. D56, 3381, (1997), gr-qc/9610053

[11] B. S. DeWitt & R. W. Brehme, “Radiation Damping in a Gravitational Field,” Ann. Phys. 9, 220, (1960)

[12] J. M. Hobbs, “A Vierbein Formalism of Radiation Damping,” Ann. Phys. 47, 141, (1968)

[13] K. Kuchař, private communication

[14] V. de La Cruz & W. Israel, “Gravitational Bounce,” Il. Nuovo Cim. A 51, 744, (1967)

[15] K. Kuchař, “Charged Shells in General Relativity and Their Gravitational Collapse,” Czech. J. Phys. B 18, 435, (1968)

[16] D. G. Boulware, “Naked Singularities, Thin Shells, and the Reissner-Nordström Metric,” Phys. Rev. D8, 2363, (1973)

[17] B. Leaute & B. Linet, “Electrostatics in a Reissner-Nordström spacetime,” Phys. Lett. A58, 5, (1976)
Fig. 1
Fig. 2
Fig. 3