Comment on “Nucleon spin-averaged forward virtual Compton tensor at large $Q^2$”

Michael C. Birse and Judith A. McGovern

Theoretical Physics Division,
School of Physics and Astronomy,
The University of Manchester,
Manchester, M13 9PL, UK

In recent work, Hill and Paz apply the operator product expansion to forward doubly virtual Compton scattering. The resulting large-$Q^2$ form of the amplitude $W_1(0,Q^2)$ is compatible with the one we obtain by extrapolation of low-$Q^2$ results from a chiral effective field theory, providing support for our approach. That paper also presents a result for the two-photon contribution to the Lamb shift in muonic hydrogen that has a much larger uncertainty than in previous work. We show that this arises from the inclusion of the proton pole term in the subtracted dispersion relation for $W_1$, not from the treatment of the non-pole Born term.

In Ref. [1], Hill and Paz use the operator product expansion (OPE) to determine the large-$Q^2$ behaviour of the amplitude for forward doubly virtual Compton scattering (VVCS) from a nucleon. This work extends and corrects an older treatment by Collins [2]. The result is important because it can help to constrain the uncertainty in the proton radius extracted from measurements of the Lamb shift of muonic hydrogen [3, 4].

In particular, the result provides the large-$Q^2$ form of the subtraction term in one of the dispersion relations used in that analysis [3]. (See Ref. [6] for a summary of the theory of the Lamb shift in muonic hydrogen.) Previous work by the current authors [7] used chiral effective field theory ($\chi$EFT) to determine the low-$Q^2$ form of the subtraction term and then extrapolated this to higher $Q^2$ by matching it on to the asymptotic $1/Q^2$ behaviour of the OPE. The resulting asymptotics of the $1/Q^2$ term are compatible with those from the QCD calculations of Ref. [1], providing support for the approach taken in Ref. [7].

Ref. [1] also presents results for the two-photon contribution to the muonic Lamb shift obtained with a simple interpolation between their OPE result at large $Q^2$ and the low-energy theorem (LET) governing the behaviour as $Q^2 \to 0$. This gives an energy that is compatible with the previous result, but with an uncertainty that is an order of magnitude larger. The authors of Ref. [1] suggest that this additional uncertainty arises from the “sticking-in form factors” ansatz but, as we discuss below, it arises in the most part from their treatment of the pole terms of the Born amplitude.

To recap, most of the two-photon contributions to the muonic Lamb shift can be determined from empirical input using the methods outlined by Pachucki [5]. The Born (elastic) pieces can be written as integrals over proton electromagnetic form factors. Dispersion relations can be used to express the inelastic pieces in terms of integrals over structure functions. However the dispersion relation for $W_1(\nu, Q^2)$ (the amplitude multiplying $-g_{\mu\nu} + q_\mu q_\nu/q^2$ in the usual tensor decomposition) requires a subtraction. The resulting term in the energy has the form of an integral over $W_1(0, Q^2)$. This zero-energy limit of the amplitude for forward VVCS cannot be measured directly and is one of the main sources of uncertainty in extractions of the proton radius. Its small $Q^2$ behaviour is constrained by an LET [8, 10] and can be calculated in $\chi$EFTs [7, 11, 12].

The large-$Q^2$ behaviour of $W_1$ can be obtained from QCD using the OPE [2]. The calculation in Ref. [1] shows that $Q^2W_1(0, Q^2)/(2M_p^2) \sim 0.27-0.37$ for $Q \gtrsim 5 \text{ GeV}^2$. For comparison, the extrapolation of $\chi$EFT in Ref. [7] gives a central value of 1.3, but with a wide uncertainty band: 0.2–23. Although the central value differs from the OPE by a factor of 3 to 4, these results are consistent within that large uncertainty. Moreover, in the corresponding contribution to the Lamb shift, the integral over $W_1$ is heavily weighted to low $Q^2$ so that differences at high-$Q^2$ have only a small effect. This indicates that a smooth interpolation between $\chi$EFT and the OPE should lead to results that lie within the uncertainties of Ref. [7]. For example, even reducing the entire contribution from $Q^2 > 0.3 \text{ GeV}^2$ by a factor of 4 would alter the Lamb shift by only about 0.3 $\mu$eV.

The bottom line of Ref. [1] for the two-photon contribution to the muonic Lamb shift is $\Delta E(2P-2S) = +30 \pm 13 \mu$eV. For comparison, the result of Ref. [7] is $+33 \pm 2 \mu$eV. The central values are similar, but the uncertainty in Ref. [1] is a factor of 6 larger.

There are two main reasons for this large uncertainty. The first is that the subtraction is applied to a dispersion relation that includes the proton poles [1, 14], in contrast

1 There are minor differences in the input data. Ref. [1] takes the magnetic polarisability $\beta_M$ from the PDG [16] and a dipole ansatz for the elastic form factors. Ref. [7] takes $\beta_M$ from a $\chi$EFT analysis [17] and the elastic contributions from the work of Carlson and Vanderhaeghen [18], who considered several empirical parametrisations of the form factors. Both works take the inelastic contribution from Ref. [12].
to other treatments that use a dispersion relation for the amplitude with the Born terms removed \cite{5,7,13}. If the proton form factors were known with sufficient accuracy, the results from both approaches would be the same, but this is not the case in practice.

At low $\nu$ and $Q^2$, the forward VVCS amplitude can be written in the form \cite{6}

$$W_1(\nu, Q^2) = \frac{2Q^4}{Q^4 - 4M^2\nu^2} G^2_M(Q^2) - 2F^2_D(Q^2)$$

$$+ \frac{2M}{\alpha} [Q^2 \beta_M + \nu^2(\alpha_E + \beta_M)] + \cdots \tag{1}$$

up to terms of fourth order in $\nu$ and $Q$. Here $\alpha_E$ and $\beta_M$ are the electric and magnetic polarisabilities of the proton \cite{17}, and

$$F_D(Q^2) = \frac{G_E(Q^2) + \frac{Q^2}{M^2} G_M(Q^2)}{1 + \frac{Q^2}{4M^2}} \tag{2}$$

is its Dirac form factor. The non-pole term involving $F^2_D$ (the second term in Eq. (1)) has been the subject of some controversy and we return to it below: for now we focus on the pole term.

Setting $\nu = 0$ and expanding $W_1$ to order $Q^2$ gives the LET in the form in Refs. \cite{1,14}:

$$W_1(0, Q^2) = 2\kappa(1 + \kappa) - \frac{2}{3} (1 + \kappa)^2 r^2_M Q^2$$

$$+ \frac{2}{3} r^2_E Q^2 - \frac{\kappa}{M^2} Q^2 + \frac{2M}{\alpha} \beta_M Q^2 + \cdots \tag{3}$$

where $r_E$ and $r_M$ are the charge and magnetic radii of the proton and $\kappa$ is its anomalous magnetic moment. The slope of this with respect to $Q^2$ controls the low-$Q^2$ contribution to the subtraction term (cf. Eq. (37) of Ref. \cite{1}). Using the same input as in Ref. \cite{1}, the various pieces of the LET contribute to the slope as follows: Born pole (second term of Eq. (3)) $-80.4 \pm 7.9$ GeV$^{-2}$, Born non-pole (third and fourth terms) $+11.1 \pm 0.2$ GeV$^{-2}$, magnetic polarisability, $+8.4 \pm 1.3$ GeV$^{-2}$.

As noted above, the dispersion relation used in Ref. \cite{1} includes the Born pole. All three pieces listed above contribute to its subtraction. In this case, both the slope of $W_1$ and its error are dominated by the subtraction of the pole term. The rather poorly determined magnetic radius of the proton appears multiplied by a large factor containing the square of the magnetic moment. It thus contributes significantly to the large uncertainty found in Ref. \cite{1}.

In addition, the Born pole term in Eq. (3) leads to a large factor multiplying the poorly-known form factor of the subtraction term. Even with the constraints from $\chi$EFT, this form factor is one of the main contributions to the uncertainty in Ref. \cite{7}. In their interpolation between the low-$Q^2$ regime and their OPE, the authors of Ref. \cite{1} use no theoretical input on terms of higher-order in $Q^2$, assuming that the order-$Q^4$ term has a typical hadronic scale but leaving even its sign unfixed. This leads to a somewhat larger relative error on their result for the subtraction term than that in Ref. \cite{5}. Multiplying this by the slope of $W_1$ including the pole contribution enhances the absolute error by nearly an order of magnitude. Combined with the contribution from the magnetic radius just discussed, this leads to the large overall uncertainty in the two-photon energy found in Ref. \cite{1}.

These uncertainties associated with the pole contribution to $W_1(0, Q^2)$ are absent in approaches that use a dispersion relation for $W_1$ with the pole removed \cite{5,7,13}. In addition, the low-$Q^2$ form factor for the subtraction has been calculated using $\chi$EFT in Ref. \cite{7}, including the order-$Q^4$ term that is left undetermined in Ref. \cite{1}. This further constrains the subtraction term in the region that makes an important contribution to Lamb shift.

As just discussed, the main uncertainty in their two-photon exchange energy is driven by the way the pole term in the VVCS amplitude is treated in Ref. \cite{1}. There is one quite separate final issue, which concerns the treatment of the non-pole Born term (the $F^2_D$ term in Eq. (1)). This can be generated from a Dirac equation with proton form factors – the procedure that the authors of Ref. \cite{1} refer to as the “sticking-in form factors” ansatz. However the first two terms in the expansion of this term in powers of $Q^2$ are determined by LETs, giving the Thomson limit of real Compton scattering and the third and fourth terms of Eq. (3). Corresponding seagull terms appear in non-relativistic EFTs \cite{18,19}. In contrast, higher-order terms in this expansion are not constrained by LETs. For example, in an EFT, higher-order counterterms will contribute at the same orders in $Q^2$. It is therefore a matter of choice whether or not to remove this non-pole term from $W_1$ along with the Born pole term.

As argued in Ref. \cite{7}, the non-pole Born term follows from Lorentz invariance and so it is natural to treat it together with the pole term. Indeed the separation of the Born amplitude into pole and non-pole pieces is not unique \cite{20}, as it depends on the choice of tensor basis. In any case, removing this piece from $W_1$ simply modifies higher-order terms in the chiral expansion of the subtraction term. Such terms have already been accounted for in the estimated uncertainty on the $\chi$EFT result \cite{7} and changes to them of natural size should fall within that error estimate.

In Appendix B of ref. \cite{1}, it is suggested that there may be larger uncertainties associated with this treatment. This is illustrated by a comparison of results for

---

\(2\) Note that $W_1$ as defined in Ref. \cite{1} differs from $T_1$ in Ref. \cite{5} by a factor of $2M/e^2$.  

---

\(^3\) For example, Carlson and Vanderhaeghen remove only the Born pole from $W_1$ \cite{12}. Although they go on to use an inconsistent LET for their subtraction, they also provide a numerical value for the non-pole Born contribution needed to construct a consistent energy.

---
the subtraction term from Refs. [7] and [12]. This is misleading since the approach of [12] gives a magnetic polarizability with the wrong sign. Indeed Alarcón et al. note that their subtraction term should not be compared with those from other approaches. As stressed in Ref. [7], it is important to work with an EFT at an order (fourth) that implements the LET of Eq. (3). This is needed to get a subtraction with the correct slope in the small $Q^2$ region that is crucial for the Lamb shift.

ACKNOWLEDGMENTS

We are grateful to R. Hill and G. Paz for helpful discussions clarifying their approach. This work was supported by the UK STFC under grant ST/L005794/1.

[1] R. J. Hill and G. Paz, Phys. Rev. D 95, 094017 (2017).
[2] J. C. Collins, Nucl. Phys. B 149, 90 (1979).
[3] R. Pohl et al., Nature 466, 213 (2010).
[4] A. Antognini et al., Science, 339, (2013) 417.
[5] K. Pachucki, Phys. Rev. A 60, 3593 (1999).
[6] A. Antognini, F. Kottmann, F. Biraben, P. Indelicato, F. Nez and R. Pohl, Ann. Phys. 331, 127 (2013).
[7] M. C. Birse and J. A. McGovern, Eur. Phys. J. A 48, 120 (2012).
[8] S. Scherer, A. Yu. Korchin and J. H. Koch, Phys. Rev. C 54, 904 (1996).
[9] D. Drechsel, G. Knochlein, A. Metz and S. Scherer, Phys. Rev. C 55, 424 (1997).
[10] H. W. Fearing and S. Scherer, Few-Body Syst. 23, 111 (1998).
[11] D. Nevado and A. Pineda, Phys. Rev. C 77, 035202 (2008).
[12] J. M. Alarcón, V. Lensky and V. Pascalutsa, Eur. Phys. J. C 74, 2852 (2014).
[13] C. Peset and A. Pineda, Eur. Phys. J. A 51, 32 (2015).
[14] R. J. Hill and G. Paz, Phys. Rev. Lett. 107, 160402 (2011).
[15] C. E. Carlson and M. Vanderhaeghen, Phys. Rev. A 84, 020102 (2011).
[16] C. Patrignani et al. (Particle Data Group), Chin. Phys. C 40, 100001 (2016).
[17] H. W. Griesshammer, J. A. McGovern, D. R. Phillips and G. Feldman, Prog. Part. Nucl. Phys. 67, 841 (2012).
[18] A. V. Manohar, Phys. Rev. D 56, 290 (1997).
[19] N. Fettes, U.-G. Meissner, M. Mojzis and S. Steininger, Ann. Phys. 283, 273 (2000); erratum ibid. 288, 249 (2001).
[20] A. Walker-Loud, C. E. Carlson and G. A. Miller, Phys. Rev. Lett. 108, 232301 (2012).