The attractors in sequence processing neural networks

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The average length and average relaxation time of attractors in sequence processing neural networks are investigated. The simulation results show that a critical point of \( \alpha \), the loading ratio, is found. Below the turning point, the average length is equal to the number of stored patterns; conversely, the ratio of length and numbers of stored patterns, grow with an exponential dependence \( \exp (A \alpha) \). Moreover, we find that the logarithm of average relaxation time is only linearly associated with \( \alpha \) and the turning point of coupling degree is located for examining robustness of networks.

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I. INTRODUCTION

The dynamic behavior of Hopfield model\(^1\), a global symmetric coupling neural network, is relatively simple: the system relaxes to a fixed point corresponding to a stable patterns\(^2\). The latest result of its maximal stored capacity \( \alpha_s \) given by Volk\(^3\) is 0.143 ± 0.002. Because the synaptic connections in real biological neural networks have a highly degree of asymmetry and a real human idea consists of a set of patterns, many modification of Hopfield model have been designed\(^4\text{-}^6\). In general, an asymmetric connection strength may arise to recall a series of patterns or even chaos behavior, and cannot be studied by conventional statistical methods because it disobey detailed balance.

In this paper, we study the behavior of attractors in a sequence processing model\(^5\), a fully connected Ising spin model, through numerical simulation with deterministic parallel dynamics. Moreover, the robustness is examined by locating critical coupling degrees since the loss of synaptic connections in the human brain may occur because of brain damage\(^7\).

II. MODEL DEFINITION

We consider a sequence processing neural network model described by the following equations\(^4\text{-}^6\)

\[
s_i(t+1) = \text{sgn} \left( \sum_j J_{ij} s_j(t) \right) \tag{1}
\]

where Ising spin \( s_i(t) \in \{-1, 1\} \) represents the firing state or resting state and sgn \( (x) \) is updating function corresponding with signum function which if \( x > 0 \), we get 1; if \( x < 0 \), we get \(-1\). The neurons evolve their states simultaneously with deterministic parallel dynamics and the time step is 1 in all our work.

The synaptic connection matrix \( J \) can be given by\(^4,8\)

\[
J_{ij} = \frac{1}{N} \sum_{\mu=1}^{q} c_{ij}^\mu \xi_{ij}^\mu \tag{2}
\]

for a diluted model consisting of \( N \) neurons. The \( q \) vectors \( \xi^\mu = \{\xi_1^\mu, \xi_2^\mu, \ldots, \xi_N^\mu\} \in \{-1, 1\}^N \) are randomly and independently stored patterns in neural networks. It means the stored patterns are organized in one \( q \)-period cycle. It can be introduced by definition \( \xi_{ij}^{\mu+1} = \xi_{ij}^\mu \) in (2). The dilution factors \( c_{ij} = 0 \) or 1 are independent identically distributed random variables. They are selected by

\[
\text{if } z \leq d, \text{ then } c_{ij} = 1; \text{ else } c_{ij} = 0 \tag{3}
\]

where \( z \in [0,1] \) is a random number and \( d \in [0,1] \) is the coupling degree of networks. For \( c_{ij} = 1 \) or \( d = 1 \), the model restore to a standard (Hopfield-Hebb) sequence processing model.
Comparing with standard Hopfield model, the synapses of equation (2) are insufficient for the generation of stable fixed point in phase space, since they combine pattern $\mu$ with pattern $\mu + 1$. Nevertheless, the transition from one pattern to another is so fast that it possible get into a stationary limit cycle for the modulo-$q$ checking of $\mu$ in $\xi^\mu_i$.

III. THE PERFORMANCE PARAMETERS

For measuring the retrieving ability of this model, in a similar way to the definition in Hopfield models, the overlaps of state $s(t)$ and the stored patterns with the initial arbitrary pattern $s(0) \in \{-1, 1\}^N$ is defined by

$$m^\mu(t) = \frac{1}{N} \sum_{j=1}^{N} \xi^\mu_j s_j(t), \quad \mu = 1, \ldots, q \quad (4)$$

For example, in 100 neurons with 10 stored patterns in a cycle, we obtained a pattern sequence as $8 \rightarrow 7 \rightarrow 6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 10 \rightarrow 9 \rightarrow 8 \rightarrow 7 \cdots$ (see Fig. 1a). It is easy to find that the series is equal to a cycle of stored patterns with 10 patterns for different retrieving times.

There are another two important parameters for describing long time behaviors of networks: the cycle size $p$ and the relaxation time $r$. They are defined by

$$p = \min_n [s(t + n) = s(t)] \quad (5)$$

$$r = \min_n [s(p + n) = s(n)] \quad (6)$$

Here, $p$ is the periodicity of attractors and $r$ is the time the system converging into the $p$-period cycles.

In fact, the values of $p$ and $r$ are dependent upon the initial patterns $s(0)$ and the coupling matrix $J$ or the stored patterns $\xi^\mu$. So, for a given value $p$, we can choose many connection matrices through randomly generating a large number of stored patterns. Moreover, the average size of cycles and the average relaxation time can be defined by

$$\langle p \rangle = \frac{1}{M} \sum_{n=1}^{M} p(n) \quad (7)$$

$$\langle r \rangle = \frac{1}{M} \sum_{n=1}^{M} r(n) \quad (8)$$

in which $p(n)$ and $r(n)$ are defined by (5) and (6) for the $n$-th sample.

Now, viewing the stored patterns series as a whole, our interest is the overlaps between $s(t)$ and the stored cycle, not with a simple pattern in the stored series. As a special case, only for $p = q$, we introduced the average cycle overlaps

$$\langle m \rangle (T) = \frac{1}{q} \sum_{i=1}^{q} \max m^\mu(Tq - i + 1), \quad \mu = 1, \ldots, q, \quad T = 1, 2, \ldots \quad (9)$$

in which $T$ is the refreshed time scale. Figure 1b shows that the system plotted in Figure 1a is successful in forming $\langle m \rangle = 1$ for $q = 10$, but there is a very small error with $\langle m \rangle = 0.963$ for $q = 20$.

IV. SIMULATION RESULTS

For searching the features of period and relaxation time of attractors, we set the $c_{ij} = 1$ corresponding to the standard sequence processing networks in fig. 2 to 4. We focus attention on the relations $\langle p \rangle$ vs. $\alpha$ and $\langle r \rangle$ vs. $\alpha$ in which the loading ratio $\alpha$ is defined by $q = \alpha N$. From comparing the value of $M$, the numbers of samples, given by 1000 or 200, we find simulation results of $\langle p \rangle$ vs. $\alpha$ have only a very small error. Considering our computer device, the value of $N$ ranges from 50 to 150 and the value of $M$ is 200 in the following calculation.
Obviously there is a turning point $\alpha_c$ dividing the curve of $\langle p \rangle$ vs. $\alpha$ into two parts (see Fig. 2). In the first part with $\alpha < \alpha_c$, we find stable behaviors of cycles with $\langle p \rangle \approx \alpha N$ equal to $q$, the number of stored patterns. Beyond this point, the curve gradually changes into an exponential dependence of $\langle p \rangle / (\alpha N) \propto \exp (A \alpha)$. Here, $A$ is a constant in the range from 8.26 to 46.74 for $N = 50$ to 150. The value of $\langle p \rangle$ increases so drastically that no results of $\alpha > 0.35$ are plotted. Additionally, the $\alpha_c$ are in the range 0.13 to 0.17, and respectively the formation ratio $k/M = 0.80$ to 0.85, where $k$ is the number of $q$-period attractors with $\langle m \rangle \geq 0.90$ and $M = 200$ is the number of samples (see Fig. 3).

Figure 4 shows the average relaxation times in fully coupled networks. Similarly, an exponential dependence $\langle r \rangle \propto \exp (B \alpha)$ was observed with $B = 12.01$, 18.98 and 29.07 for $N = 50, 80$ and 120, respectively.

In order to investigate the robustness of networks within the limits of $\alpha_c$, simulation with diluted connection have been carried out through changing the values of coupling degree $d$ or setting $c_{ij}$ belong to $\{0,1\}$ randomly. In Fig. 5, in the same way as figure 2, it is easy to see that there exists a critical point $d_c = 0.15, 0.35$ and 0.60 for $\alpha = 0.05, 0.10$ and 0.15. For stronger dilution, the average periodicity grows towards a plateau higher than $\alpha N$.

In conclusion, for small systems with spin neurons, it can be seen that the ratio of the average size of attractors $\langle p \rangle$ and the numbers of stored patterns is 1 as long as the loading ratio $\alpha > \alpha_c$, and then it increases strongly following an exponential law $\exp (A \alpha)$. The more the system size increases, the more the ratio grows. From analyzing the average cycle overlaps $\langle m \rangle$ (Fig. 3), there is a narrow quickly decreasing region near the turning point of periodicity. The maximal stored capacity$^9$ at $\alpha_c = 0.23$ for $N = 100$, corresponding to $k/M = 0.66$, is much larger than at $\alpha_c = 0.32$ (when $k/M = 0.005$), under the condition that $\langle m \rangle \geq 0.90$. Moreover, the average relaxation time depends exponentially on the loading ratio and it raises faster with larger systems.

Additionally, through the simulation of the coupling degree $d$ in networks, it is clear that there is a narrow transition region between small and large $\langle p \rangle / (\alpha N)$ centered on an inflection point $d_c$. Moreover, the larger loading ratio corresponding to the larger value of $d_c$ means its robustness of stored information is more unstable. Beyond our expectation, one result is a stable large attractor for highly diluted systems, another is that possibly this difference from the definition of asymmetric connection strength by Gutfr...
Fig. 1a
Fig. 1b
Fig. 2

\[ \log(\langle p \rangle / (\alpha N)) \]
Fig. 3

\[ \alpha_{s} = 0.23 \]

\[ \alpha = 0.32 \]
Fig. 4
Fig. 5