ON CLASSICAL INEQUALITIES OF TRIGONOMETRIC AND
HYPERBOLIC FUNCTIONS

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract

This article is the collection of the six research papers, recently written by the authors. In these papers authors refine the inequalities of trigonometric and hyperbolic functions such as Adamović-Mitrinović inequality, Cusa-Huygens inequality, Lazarević inequality, Huygens inequality, Jordan’s inequality, Carlson’s inequality, Wilker’s inequality, Redheffer’s inequality, Wilker-Anglesio inequality, Becker-Stark inequality, Kober’s inequality, Shafer’s inequality and Shafer-Fink’s inequality. The relation between trigonometric and hyperbolic functions has been built too. In the last paper, the sharp upper and lower bounds for the classical beta function has been established by studying the Jordan’s inequality.

For the introduction of the above mentioned inequalities we refer to reader to see the book “Analytic Inequalities, Springer-Verlag, Berlin, 1970” by D.S. Mitrinović. Also for the historical background of the inequalities reader is referred to see A. Fink: An Essay on the History of Inequalities, J. Math. Anal. Appl. 249 (2000) 118–134. and the references therein.

2010 Mathematics Subject Classification: 26D05, 26D07, 26D99, 33B15. Keywords and phrases: Trigonometric functions, hyperbolic functions, inequalities, monotonicity theorem, Jordan’s inequality, Cusa-Huygens inequality, Redheffer’s inequality, Wilker’s inequality and Becker-Stark inequality, Carlsons inequality, Shafer’s inequality, inverse trigonometric and hyperbolic functions, Kober’s inequality, new proofs, Euler gamma function, beta function.
List of papers

I. On certain old and new trigonometric and hyperbolic inequalities.
II. On Jordan’s, Redheffer’s and Wilker’s inequality.
III. On Carlson’s and Shafer’s inequalities.
IV. On Jordan’s, Cusa-Huygens and Kober’s inequality.
V. On an inequality of Redheffer.
VI. On the inequalities for Beta function.
ON CERTAIN OLD AND NEW TRIGONOMETRIC AND
HYPERBOLIC INEQUALITIES

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. In this paper we study the two sided inequalities of the trigonometric
and hyperbolic functions.

1. Introduction

Since last one decate many authors got interest to study the following inequalities

\[(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3},\]

for \(0 < |x| < \pi/2\). The first inequality is due to D.D. Adamović and D.S.
Mitrinović [10, p.238], and the second one was obtained by N. Cusa and C. Huygens
[22]. The hyperbolic version of (1.1) is given as

\[(\cosh x)^{1/3} < \frac{\sinh x}{x} < \frac{\cosh x + 2}{3}.\]

for \(x \neq 0\). The first inequality in (1.2) was obtained by Lazarević [10, p. 270], and
the second inequality is called sometime hyperbolic Cusa-Huygens inequality [15].

There are many results on the refinement of the inequalities (1.1) and (1.2) in the
literature, e.g., see [3, 11, 12, 13, 15, 16, 19, 17, 20, 21, 22] and the references therein.
In this paper, we give the simple proofs of some known results, and establish the new
inequalities as well as. The main result of this paper reads as follows.

1.3. Theorem. For \(x \in (0, \pi/2)\), the following inequalities

\[\frac{x^{3/2}}{2(\sin x)^{1/2}} < \tan \frac{x}{2} < \frac{x^{2/\alpha}}{2(\sin x)^{2/\alpha - 1}}\]

hold true, where \(\alpha = 2\log(\pi/2)/\log(2) \approx 1.30299\).

1.5. Theorem. For \(x \in (0, \pi/2)\), the following inequalities

\[\frac{x}{\tan(x/2)} - 1 < \exp \left(\frac{1}{2} \left(\frac{x}{\tan x} - 1\right)\right) < \frac{\sin x}{x} < \exp \left(\left(\log \frac{\pi}{2}\right) \left(\frac{x}{\tan x} - 1\right)\right)\]

hold true.

1.6. Theorem. For \(x \in (0, \infty)\), the following inequalities

\[\frac{x}{\tanh(x/2)} - 1 < \exp \left(\frac{1}{2} \left(\frac{x}{\tanh x} - 1\right)\right) < \frac{\sinh x}{x} < \exp \left(\left(\frac{x}{\tanh x} - 1\right)\right)\]
hold true.

The second inequality in Theorem 1.5 and 1.6 is known [19]. For these, however, here a new method of proof is offered.

2. Preliminaries

In this section we give few lemmas, they will be used in the proof of theorems. For $|x| < \pi$, the following power series expansions can be found in [8, 1.3.1.4 (2)-(3)],

\[
\cot x = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},
\]

and

\[
\coth x = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1},
\]

where $B_{2n}$ are the even-indexed Bernoulli numbers, see [7, p. 231]. The following expansions can be obtained directly from (2.2) and (2.3),

\[
\frac{1}{(\sin x)^2} = -(\cot x)' = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| (2n-1)x^{2n-2},
\]

\[
\frac{1}{(\sinh x)^2} = -(\coth x)' = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} (2n-1)|B_{2n}| x^{2n-2}.
\]

The following result is due to Biernacki and Krzyż [4], which will be very useful in studying the monotonicity of certain power series. This result was also simply proved in [6] by Heikkala et al.

2.6. Lemma. For $0 < R \leq \infty$. Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $\{a_n/b_n\}$ is increasing (decreasing) and $b_n > 0$ for all $n$, then the function $A(x)/B(x)$ is also increasing (decreasing) on $(0, R)$.

In [2], Anderson et al. proved the following result, which is sometime called Monotone l'Hôpital rule.
2.7. Lemma. For $-\infty < a < b < \infty$, let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and be differentiable on $(a, b)$. Let $g(x) \neq 0$ on $(a, b)$. If $f'(x)/g'(x)$ is increasing (decreasing) on $(a, b)$, then so are
\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]
If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

2.8. Lemma. (1) The function
\[
f(x) = x + \sin x - \frac{x^2}{\tan(x/2)}
\]
is strictly increasing from $(0, \pi/2)$ onto $(0, c)$, $c = (4 + 2\pi - \pi^2)/4 \approx 0.10336$. In particular,
\[
1 + \frac{\sin x}{x} - \frac{c}{x} < \frac{x}{\tan(x/2)} < 1 + \frac{\sin x}{x},
\]
for $x \in (0, \pi/2)$.

(2) The function
\[
g(x) = \frac{1}{x} \left( \frac{\cos x}{\sin x} - 1 \right) \left( \frac{\cos x}{\sin x} + \frac{1}{\sin x} \right)
\]
is strictly increasing from $(0, \pi/2)$ onto $(-4/3, -4/\pi)$.

Proof. For the proof of (1), we get
\[
f'(x) = \frac{1}{2} \left( \frac{1}{\sin(x/2)} \right)^2 (x - \sin x)^2 > 0,
\]
the limiting values are clear. For (2), derivation gives
\[
g'(x) = \frac{1}{x^2(1 - \cos x)} \left( x + \sin x - \frac{x^2}{\tan(x/2)} \right),
\]
which is positive by part (1), and limiting values follow from l'Hôpital rule. This completes the proof. \(\square\)

2.9. Lemma. (1) The function
\[
f(x) = \cos x - \left( 1 - \frac{x^2}{3} \right)^{3/2}
\]
is strictly decreasing from $(0, \pi/2)$ onto $(0, c_1)$, where $c_1 = -(12 - \pi^2)^{3/2}/(24\sqrt{3}) \approx -0.07480$. In particular, for $x \in (0, \pi/2)$
\[
(2.10) \quad \left( 1 - \frac{x^2}{3} \right)^{3/2} - c_1 < \cos x < \left( 1 - \frac{x^2}{3} \right)^{3/2},
\]
(2) For $x \in (0, \pi/2)$, we have

\[ \frac{x}{4} \left( \frac{5}{2} - \frac{x}{2 \tan x} \right) < \tan \frac{x}{2}. \]  

(3) For $x \in (0, \infty)$, we have

\[ \frac{x}{4} \left( \frac{5}{2} - \frac{x}{2 \tanh x} \right) < \tanh \frac{x}{2}. \]

Proof. By utilizing the inequality \((1 - x^2/6) < (\sin x)/x\) from [9, Theorem 3.1], we get

\[ f'(x) = x(1 - x^2/3)^{1/2} - \sin x \]
\[ < x \left( (1 - x^2/3)^{1/2} - (1 - x^2/6) \right) = x g(x), \]

which is negative, because

\[ g'(x) = \frac{x}{3} \left( 1 - \frac{1}{(1 - x^2/3)^{1/2}} \right) < 0, \]

and \(g(x) = 0\). This implies that \(f\) is decreasing, and limiting values are clear.

For the proof of part (2), we use the identity \(\tan(x/2) = (1 - \cos x)/\sin x\) and write the inequality \((2.11)\) as

\[ \sin x \frac{x}{x} < \frac{8(1 - \cos x) + x^2 \cos x}{5x^2}, \quad x \in (0, \pi/2). \]

Let

\[ h(x) = 5x \sin x - 8(1 - \cos x) + x^2 \cos x, \quad x \in (0, \pi/2). \]

Applying the inequality \((\cos x)^{1/3} < \sin x/x\), we get

\[ h'(x) = 3x \cos x - (3 - x^2) \sin x < x(3 \cos x - (3 - x^2)(\cos x)^{1/3}) \]
\[ = 3x(\cos x)^{1/3} \left( (\cos x)^{2/3} - \left( 1 - \frac{x^2}{3} \right) \right), \]

which is negative by part (1), and \(h(0) = 0\). The proof of part (3) is similar to the proof of part (2), if we use the identity \(\tanh x/2 = (\cosh x - 1)/\sinh x\). \(\square\)

3. Proofs of the theorems

3.1. Proof of Theorem 1.3. Let \(f(x) = f_1(x)/f_2(x)\), where

\[ f_1(x) = \log(x/\sin x), \quad f_2(x) = \log(1/\sqrt{(1 + \cos x)/2}), \]

and clearly \(f_1(x) = f_2(x) = 0\). We get

\[ \frac{f_1'(x)}{f_2'(x)} = -\frac{1}{x} \left( \frac{\cos x}{\sin x} - 1 \right) \left( \frac{\cos x}{\sin x} + \frac{1}{\sin x} \right), \]
which is strictly decreasing by Lemma 2.7 and by l’Hôpital rule \( \lim_{x \to 0} f(x) = 4/3 \) and \( \lim_{x \to \pi/2} f(x) = 2 \log(\pi/2)/\log(2) = 1.30299 \ldots = \alpha \). This implies the following inequalities

\[
(3.2) \quad \left( \frac{1 + \cos x}{2} \right)^{4/3} < \frac{\sin x}{x} < \left( \frac{1 + \cos x}{2} \right)^{\alpha}.
\]

By utilizing the identity \( \tan x/2 = \frac{\sin x}{1 + \cos x} \) we get (1.4). □

In [18], it is proved that

\[
(3.3) \quad \frac{\sin x}{x} < \sqrt{\frac{1 + \cos x}{2} \frac{(1 + \cos x)^{1/2} + 2}{3} = \frac{t^{1/2}(t^{1/2} + 2)}{3} = \frac{t + 2t^{1/2}}{3},}
\]

if one denotes \( t = (\cos x + 1)/2 \). The right side of (3.2) cannot be compared with inequality (3.3), and this inequality improves the Cusa-Huygens inequality \( \sin x/x < (\cos x + 2)/3 = (2t + 1)/3 \).

The following theorem is known [20]. Here another proof appears, but both proofs use Lemma 2.7.

3.4. Theorem. The function

\[
f(x) = \frac{\log((\sinh x)/x)}{\log((\cosh(x)/2))}
\]

is strictly decreasing from \((0, \infty)\) onto \((4/3, 2)\). In particular,

\[
\left( \frac{1 + \cosh x}{2} \right)^{2/3} < \frac{\sinh x}{x} < \left( \frac{1 + \cosh x}{2} \right).
\]

Proof. Let \( f_1(x) = \log((\sinh x)/x), \quad f_2(x) = \log((\cosh x)/x), \quad \text{and} \quad f_1(0) = f_2(0) = 0. \)

We get

\[
\frac{f_1'(x)}{f_2'(x)} = 2g_1(x),
\]

where

\[
g_1(x) = x \left( 1 + \frac{\cosh x}{\sinh x} \right) \frac{1}{(\sinh x)^2} \left( \frac{\cosh x}{x} + \frac{\sinh x}{x} \right).\]

Differentiation gives,

\[
g_1'(x) = \frac{1}{2x^2} g_2(x),
\]

where

\[
g_2(x) = 2 \frac{\cosh(x/2)}{\sinh(x/2)} + \frac{x}{(\sinh(x/2))^2} \left( 1 - x \frac{\cosh(x/2)}{\sinh(x/2)} \right),
\]

which is positive because

\[
g_2'(x) = \frac{x}{2(\sinh(x/2))^4} (x(2 + \cosh x) - 3 \sinh x) > 0
\]
by inequality \((\sinh x)/x < (2 + \cosh x)/3\), and \(g_2\) tends to 0 as \(x\) tends to 0. This implies that \(f'_1/f'_2\) is increasing. Hence by Lemma 2.7, \(f\) is increasing, and limiting values follow from l'Hôpital rule. This completes the proof. □

3.5. **Corollary.** For \(x \in (0, \infty)\), we have

\[
(cosh x)^{1/3} < \left(\frac{1 + \cosh x}{2}\right)^{2/3} < \frac{\sinh x}{x} < \frac{1 + \cosh x}{2} < \left(\frac{2 + \cosh(x/2)}{3}\right)^4.
\]

**Proof.** The proof of the first inequality is trivial, because it can be simplified as \(0 < (1 - \cosh x)^2\). The second and third inequality follow from Theorem 3.4. For the proof of the fourth inequality, it is enough to prove that the function

\[
h(y) = 3 \left(\frac{y + 2}{3}\right)^{1/4} - \left(\frac{y + 1}{2}\right)^{1/2} - 2
\]

is negative for \(y \in (0, \infty)\). We get

\[
h'(x) = \frac{3^{3/4}}{4(1 + y)^{1/2}} \left(h_1(y) - \frac{\sqrt{2}}{3^{3/4}}\right),
\]

where

\[
h_1(x) = \frac{(1 + y)^{1/2}}{(2 + y)^{3/4}}.
\]

Now we prove that \(h_1\) is negative, we get

\[
(\log(h_2(y)))' = -\frac{y - 1}{4(2 + 3y + y^2)} < 0,
\]
and \(h_2(1) = \sqrt{2}/3^{3/4}\). This implies that \(h\) is decreasing, and \(h\) tends to 0 when \(y\) tends to 1, hence the proof of fourth inequality is completed. □

3.6. **Proof of Theorem 1.5.** It is well known that \(L(a, b) < (a + b)/2\), where \(L(a, b) = (b - a)/(\log b - \log a)\) denotes the Logarithmic mean of two distinct real numbers \(a\) and \(b\). Particularly, \(L(t, 1) < (t + 1)/2\), for \(t > 1\). This inequality can be written as \(\log t > 2(t - 1)/(t + 1)\). Letting \(t = \tan(x/2)/(x - \tan(x/2)) > 1\) and using (2.11), we get

\[
\log \left(\frac{\tan(x/2)}{x - \tan(x/2)}\right) > \frac{2(\tan(x/2) - x)}{x} > \frac{\tan x - x}{2 \tan x},
\]

this implies the proof of first inequality.

For the proof of second and third inequality, we define \(f(x) = f_1(x)/f_2(x)\), \(x \in (0, \pi/2)\), where \(f_1(x) = \log(x/\sin x)\), \(f_2(x) = 1 - x/\tan x\), and clearly \(f_1(0) = f_2(0) = 0\). Differentiation with respect \(x\) gives

\[
\frac{f'_1(x)}{f'_2(x)} = \frac{A(x)}{B(x)},
\]
where
\[ A(x) = 1 - x \frac{\cos x}{\sin x} \quad \text{and} \quad B(x) = \frac{x^2}{\sin x^2} - x \frac{\cos x}{\sin x}. \]

By (2.1) and (2.4), we get
\[ A(x) = \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} x^{2n} = \sum_{n=1}^{\infty} a_n x^{2n}, \]
and
\[ B(x) = \sum_{n=1}^{\infty} \frac{2^{2n}2n}{(2n)!} B_{2n} x^{2n} = \sum_{n=1}^{\infty} b_n x^{2n}. \]

The function \( c_n = a_n/b_n = 1/(2n) \) is decreasing in \( n = 1, 2, \ldots \). This implies from Lemma 2.6 that \( f_1'/f_2' \) is decreasing, and the function \( f \) is decreasing in \( x \in (0, \pi/2) \) by Lemma 2.7. By l'Hôpital rule, we get \( \lim_{x \to 0} f(x) = 1/2 \) and \( \lim_{x \to \pi/2} f(x) = \log(\pi/2) \), this completes the proof of the second and the third inequality. \( \square \)

3.7. **Proof of Theorem 1.6.** The proof of the first inequality is similar to the proof of the first inequality of Theorem 1.5 if we use (2.12). For the proof of second and third inequality, we define the function \( g(x) = g_1(x)/g_2(x) \), \( x \in (0, \infty) \), where \( g_1(x) = \log(\sinh(x)/x) \) and \( g_2(x) = x/\tanh(x) - 1 \), and \( g_1(0) = g_2(0) = 0 \). Differentiating \( g \) with respect \( x \) we get
\[ g'(x) = \frac{A(x)}{B(x)}, \]
where \( A(x) = 1 - x \coth(x) \) and \( B(x) = (x/\sinh x)^2 - x \coth x \). Now the monotonicity of \( g \) follows from (2.2), (2.5), Lemma 2.6 and 2.7. Applying the l'Hôpital rule, we get \( \lim_{x \to 0} g(x) = 1/2 \) and \( \lim_{x \to \pi/2} g(x) = 1 \), this completes the proof. \( \square \)

The second inequality of Theorem 1.5 can be written as
\[ \log \left( \frac{x}{\sin x} \right) < \frac{\sin x - x \cos x}{2 \sin x}, \]
and this appears in \[19\]. The similar is true for the second inequality of Theorem 1.6 which appears in \[19\] as
\[ \log \left( \frac{\sinh x}{x} \right) < \frac{x \cosh x - \sinh x}{2 \sinh x}. \]

3.8. **Corollary.** The following relations hold true,

1. \( x \tan \left( \frac{x}{2} \right) < \log \left( \frac{1}{\cos x} \right) < \frac{x}{2} \tan x, \quad x \in (0, \pi/2), \)
2. \( \frac{x}{2} \tanh x < \log(\cosh x) < \frac{x^2}{2}, \quad x \in (0, \infty), \)
3. \( \frac{x}{2} \tanh x < \log(\cosh x) < \frac{\sinh x \tanh x}{2}, \quad x \in (0, \infty). \)
Proof. Applying the Hadamard inequalities (see [17]) to the convex function $t \tan t$, we get:

$$x \tan \left( \frac{x + 0}{2} \right) = x \tan(x/2) < \int_0^x \tan t \, dt < \frac{x(tan + tan)}{2} = x \tan(x/2),$$

so (1) follows. For part (2), it is sufficient to prove that the functions $m(x) = \log(cosh x) - (x/2) \tanh x$ and $n(x) = x^2/2 \log(cosh x)$ are strictly increasing for $x > 0$. This follows from $m'(x) = (\sin x - x/\cos x)/(2 \cos x) > 0$ and $n'(x) = 1 - 1/(\cosh x)^2 > 0$. It is interesting to note that, the right side of (2) cannot be compared to right side of (3). □

3.9. Lemma. The following inequalities hold:

1. $\log \left( \frac{1}{\cos x} \right) < \frac{1}{2} \sin x \tan x$, $x \in \left( 0, \frac{\pi}{2} \right)$,
2. $\log \left( \frac{x}{\sin x} \right) > \frac{x - x \cos x}{2x}$, $x \in \left( 0, \frac{\pi}{2} \right)$,
3. $\log \left( \frac{\sinh x}{x} \right) < \frac{x \cosh x - \sinh x}{2x}$, $x > 0$,
4. $\log \left( \frac{\sinh x}{x} \right) > \frac{x \cosh x - \sinh x}{2 \sinh x}$, $x > 0$.

Proof. Letting

$$t(x) = \frac{\sin x \tan x}{2} - \log \left( \frac{1}{\cos x} \right),$$

we get $t'(x) = \sin x(\cos x - 1)^2/(\cos x)^2 > 0$. Hence $t$ is strictly increasing, this implies (1). As the logarithmic mean of $t$ and 1 is $L(t, 1) = (t - 1)/\log t$, from $\log t > 1 - 1/t$ for $t > 1$, we get $L(t, 1) < t$. Putting $t = x/\sin x$, we obtain $\log(x/\sin x) > 1 - (\sin x)/x$. Thus, it is sufficient to prove that

$$1 - \frac{\sin x}{x} > \frac{\sin x}{2x} - \frac{\cos x}{2},$$

which is equivalent to write $\sin x)/x < (\cos x + 2)/3$. This is a known inequality, namely the so-called Cusa-Huygens inequality [11], thus (2) follows. For the proof of (3) we use the similar method as in the circular case: By inequality $\log t < t - 1$ for $t > 1$ applied to $t = (\sinh x)/x$, to deduce (3) it is sufficient to prove that:

$$\frac{\sinh x}{x} - 1 < \frac{\cosh x}{2} - \frac{\sinh x}{2x}.$$ 

But this is equivalent to $(\sinh x)/x < (\cosh x + 2)/3$, which is known as the hyperbolic Cusa-Huygens inequality [12]. For the proof of (4), see [20]. □

The last inequality of Theorem 1.6 can be written as

$$\log \left( \frac{\sinh x}{x} \right) < \frac{x \cosh x - \sinh x}{\sinh x}.$$ 

(3.10)

We observe that the inequality (3) in Lemma 3.9 can not be compared with (3.10).
3.11. **Theorem.** The following functions

\[ F_1(x) = \frac{\log(x/\sin x)}{\log(\cosh x)}, \quad F_2(x) = \frac{\log(\cosh x)}{\log(\cos x)}, \quad F_3(x) = \frac{\log(x/\sin x)}{\log(\cos x)} \]

are strictly increasing in \(x \in (0, \pi/2)\).

The proof of the functions \(F_1\) can be found in [16]. The proof of \(F_2\) is a consequence of Corollary 3.8, by considering the derivative

\[ F_2'(x) = \tanh x \log(\cos x) + \cos x \log(\cosh x) > 0. \]

The proof for the \(F_3\) can be found in [12], here we give a simple proof. Let \(F_3(x) = u(x)/v(x)\), the proof follows if \((u(x)/v(x))'\) is negative or one has to show that \(u'(x)v(x) < u(x)v'(x)\). This is equivalent to

\[ \log \left( \frac{1}{\cos x} \right) \left( \frac{\sin x - x \cos x}{x \sin x} \right) < \log \left( \frac{x}{\sin x} \right) \tan x. \]

The inequality (3.12) follows immediately from Lemma (3.9) (1) and (2). This completes the proof.

3.13. **Corollary.** For \(x \in (0, \pi/2)\), one has

\[ \frac{3 \sin x}{2 + \cos x} < \frac{8 \sin(x/2) - \sin x}{3} < x < \frac{\sin x}{((\cos x + 1)/2)^{2/3}}. \]

**Proof.** By applying the Mitrinovic-Adamovic inequality \((\sin t)/t > (\cos t)^{1/3}\) for \(t = x/2\), we get

\[ \left( \frac{2 \sin(x/2) \cos(x/2)}{x} \right)^3 > (\cos x/2)^4 \]

by multiplying both sides with \(\cos(x/2)\). Since \(\cos(x/2) = (1 + \cos x)^{1/2}\), we get the inequality:

\[ (\sin x)/x > ((\cos x + 1)/2)^{2/3}. \]

By putting \(x/2\) in the Cusa-Huygens inequality \((\sin x)/x < (\cos x + 2)/3\), we get

\[ \frac{\sin(x/2)}{x/2} < \frac{\cos(x/2) + 2}{3}. \]

Clearly, \((\cos x - 1)^2 > 0, t = \cos(x/2)\), this implies

\[ \frac{\cos t + 2}{3} < \frac{3}{4 - \cos t}. \]

By (3.14) and (3.15), one has

\[ \sin(x/2) < \frac{3x}{2(4 - \cos(x/2))}, \]

this is stronger than the Cusa-Huygens inequality \((\sin x)/x < (2 + \cos x)/3\). Indeed, we will prove that:

\[ \frac{3 \sin x}{2 + \cos x} < \frac{1}{3} (8 \sin(x/2) - \sin x). \]
By letting $t = \cos(x/2)$ in (3.16), after some simple transformations, we have to prove that $2t^3 - 8t^2 + 10t - 4 < 0$, this is equivalent to $(t - 1)^2(t - 2) < 0$, which is obvious. □

3.17. Remark. The first two inequalities of Corollary [3.13] offer an improvement of the Cusa-Huygens inequality, while the last inequality is in fact the first relation of (3.2), with an elementary proof. The authors have found in a book on history of number π, that the second inequality of Corollary [3.13] was discovered by geometric and heuristic arguments (see [5]) by C. Huygens.

References

[1] M. Abramowitz, I. Stegun, eds.: Handbook of mathematical functions with formulas, graphs and mathematical tables, National Bureau of Standards, Dover, New York, 1965.
[2] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: Monotonicity Rules in Calculus, Amer. Math. Month. Vol. 113, No. 9 (2006), pp. 805–816.
[3] G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen: Monotonicity of some functions in calculus, preprint, https://www.math.auckland.ac.nz/Research/Reports/Series/538.pdf.
[4] M. Biernacki, J. Krzyż: On the monotonicity of certain functionals in the theory of analytic functions, Ann. Univ. Mariae. Curie-Skłodowska 2 (1955) 134–145.
[5] F.T. Campan: The history of number pi (Romanian), second ed., 1977, Albatros ed., Romania.
[6] V. Heikkala, M. K. Vamanamurthy, and M. Vuorinen: Generalized elliptic integrals, http://arxiv.org/pdf/math/0701436v2.pdf.
[7] K. Ireland, M. Rosen: A Classical Introduction to Modern Number Theory, 2nd ed., Springer-Verlag, New York, Berlin, Heidelberg, 1990.
[8] A. Jeffrey: Handbook of Mathematical Formulas and Integrals, 3rd ed., Elsevier Academic Press, 2004.
[9] R. Klen, M. Visuri and M. Vuorinen: On Jordan type inequalities for hyperbolic functions, J. Ineq. Appl., vol. 2010, pp. 14.
[10] D.S. Mitrinovic: Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[11] E. Neuman: Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions, Advances in Inequalities and Applications, 1 (2012), No. 1, 1–11.
[12] E. Neuman: On some means derived from the Schwab-Borchardt mean II, JMI accepted.
[13] E. Neuman: Inequalities Involving Hyperbolic Functions and Trigonometric Functions, Bull. Int. Math. Vir. Int. Vol. 3, 2(2012), 87–92.
[14] E. Neuman: Inequalities involving jacobian elliptic trigonometric and hyperbolic functions, J. Ineq. Spec. Funct. Vol. 3, 2(2012), 16–21.
[15] E. Neuman and J. Sándor: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Inequal. Appl. Vol. 13, Number 4 (2010), 715–723.
[16] E. Neuman and J. Sándor: Optimal inequalities for hyperbolic and trigonometric functions, Bull. Math. Analysis Appl. 3(2011), no. 3, 177–181.
[17] J. Sándor: On new refinements of Kobers and Jordans trigonometric inequalities, Notes Number Theory Discrete Math. Vol. 19, 2013, No. 1, 73–83.
[18] J. Sándor: On certain inequalities for means III, Arch.Math.(Basel), 76(2001), 34–40.
[19] J. Sándor: *Two sharp inequalities for trigonometric and hyperbolic functions*, MIA 15(2012), no. 2, 409–413.

[20] J. Sándor: *Sharp Cusa-Huygens and related inequalities*, Notes Number Theory Discrete Math. Vol. 19, 2013, No. 1, 50–54.

[21] J. Sándor and R. Oláh-Gál: *On Cusa-Huygens type trigonometric and hyperbolic inequalities*, Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 145–153.

[22] J. Sándor and M. Bencze: *On Huygens trigonometric inequality*, RGMIA Res. Rep. Collection, 8 (2005), No. 3, Art. 14.
ON JORDAN’S REDHEFFER’S AND WILKER’S INEQUALITY

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. In this paper, the authors offer new Jordan, Redheffer and Wilker type inequalities, along with refinements and converses. Connections with Euler’s gamma function are pointed out, too.

1. Introduction

In the recent years, the refinements of the inequalities of trigonometric functions such as Adamović-Mitrinović, Cusa-Huygens, Jordan inequality, Redheffer inequality, Becker-Stark inequality, Wilker inequality, Huygens inequality, and Kober inequality have been studied extremely by numerous authors, e.g., see [4, 5, 10, 18, 19, 23, 24, 25, 26, 32, 33, 34, 35, 36] and the references therein. Motivated by these rapid studies, in this paper we make a contribution to the subject by refining the Cusa-Huygens, Jordan and Redheffer inequality, and establish a Wilker type inequality. In all cases, we give the upper and lower bound of \( \sin(x)/x \) in terms of simple functions. Meanwhile, we give some Redheffer type inequalities for trigonometric functions, which refine the existing results in the literature.

For the representation of trigonometric functions in terms of gamma function, we define for \( x, y > 0 \) the classical gamma function \( \Gamma \), the digamma function \( \psi \) and the beta function \( B(\cdot, \cdot) \) by

\[
\Gamma(x) = \int_0^\infty e^{-t}t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},
\]

respectively. We denote the trigamma function \( \psi' \) by \( K \), and defined as,

\[
(1.1) \quad K(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}, \quad K'(x) = -2 \sum_{n=0}^{\infty} \frac{1}{(x+n)^3}.
\]

We refer to reader to see [30, 31, 32, 33] for more properties and relations of \( K \). The functions \( \Gamma \) and \( \psi \) satisfy the recurrence relation

\[
(1.2) \quad \Gamma(1+z) = z\Gamma(z), \quad \psi(1+z) = z\psi(z).
\]

The following relation

\[
(1.3) \quad \psi(1+z) - \psi(z) = \frac{1}{z}
\]

follows from (1.2). Differentiating both sides of (1.3), one has

\[
(1.4) \quad K(1+z) - K(z) = -\frac{1}{z^2}.
\]
The following Euler’s reflection formula \[1, 6.1.17\]
\[
\Gamma(t)\Gamma(1 - t) = \frac{\pi}{\sin(\pi t)}, \quad 0 < t < 1,
\]
can be written as
\[
\frac{x}{\sin(x)} = \Gamma \left(1 + \frac{x}{\pi}\right) \Gamma \left(1 - \frac{x}{\pi}\right) = B \left(1 + \frac{x}{\pi}, 1 - \frac{x}{\pi}\right), \quad 0 < x < \pi.
\]
The logarithmic differentiation to both sides of (1.5) gives the following reflection formula,
\[
\psi(1 - t) - \psi(t) = \frac{\pi}{\tan(\pi t)}.
\]
Replacing \(t\) by \(t + 1/2\) in (1.5), we get
\[
\frac{x}{\cos(x)} = \frac{x}{\pi} \Gamma \left(\frac{1}{2} + \frac{x}{\pi}\right) \Gamma \left(\frac{1}{2} - \frac{x}{\pi}\right), \quad 0 < x < \frac{\pi}{2}.
\]
Next we recall the Adamović-Mitrinović \[21, p.238\] and Cusa-Huygens \[36\] inequalities
\[
(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < |x| < \frac{\pi}{2},
\]
For the refinement of (1.9), see \[18, 23, 25, 35, 36, 52\].
Our first main result refines the inequalities in (1.9) as follows:

**1.10. Theorem.** For \(x \in (0, \pi)\), the following inequalities hold,
\[
2 - \frac{(x/\pi)^3 + \cos(x)}{3} < \frac{\sin(x)}{x} < 2 - \frac{(x/\pi)^4 + \cos(x)}{3}.
\]

In \[47\], Wilker asked a question to find the largest constant \(c\) such that the inequality
\[
a(x) > c b(x), \quad 0 < |x| < \frac{\pi}{2},
\]
holds true, where
\[
a(x) = \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} - 2, \quad \text{and} \quad b(x) = x^3 \tan(x).
\]
The inequality (1.12) is known as the Wilker inequality in the literature. Anglesio \[45\] proved that the ratio \(a(x)/b(x)\) is decreasing in \(x \in (0, \pi/2)\), and he answered the question by showing the following two sided inequality,
\[
2 + \frac{8}{45} x^3 \tan(x) < \left(\frac{\sin(x)}{x}\right)^2 + \frac{\tan(x)}{x} < 2 + \frac{16}{\pi^4} x^3 \tan(x),
\]
with the best possible constants $8/45$ and $16/\pi^4$. For the new proofs and refinement of (1.13), see [13, 28, 51, 53, 56]. Thereafter the following Wilker type inequalities,

$$3 + \frac{3}{20} x^3 \tan(x) < 2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} < 3 + \frac{16}{\pi^4} x^3 \tan(x), \ 0 < x < \frac{\pi}{2},$$

$$2 + \frac{2}{45} x^3 \sin(x) < \left( \frac{x}{\sin(x)} \right)^2 + \frac{x}{\tan(x)} < 2 + \left( \frac{2}{\pi} - \frac{16}{\pi^3} \right) x^3 \sin(x), \ 0 < x < \frac{\pi}{2},$$

$$\left( \frac{\sinh(x)}{x} \right)^2 + \frac{\tanh(x)}{x} > 2 + \frac{8}{45} x^3 \tanh(x), \ x > 0,$$

$$\left( \frac{x}{\sinh(x)} \right)^2 + \frac{x}{\tanh(x)} < 2 + \frac{2}{45} x^3 \sinh(x), \ x > 0,$$

were established by Chen-Sándor [11], Wang [51], Zhu [57], and Sun-Zhu [58], respectively.

We establish an other Wilker type inequality by giving the following result.

1.14. Theorem. For $x \in (0, \pi)$, we have

$$3 + \left( \frac{x}{\pi} \right)^4 \frac{x}{\sin(x)} < 2 \frac{x}{\sin(x)} + \frac{x}{\tan(x)} < 3 + \left( \frac{x}{\pi} \right)^3 \frac{x}{\sin(x)}.$$

The following inequality

$$(1.15) \quad 2 \frac{x}{\sin(x)} + \frac{x}{\tan(x)} > 2, \quad 0 < |x| < \frac{\pi}{2},$$

has been recently established by Wu and Srivastava [49], which is sometime known as the second Wilker inequality [25]. Inequality (1.15) can be written as,

$$\frac{\sin(x)}{x} < \frac{1}{2} \left( \frac{x}{\sin(x)} + \cos(x) \right), \quad 0 < |x| < \frac{\pi}{2}.$$

It was shown in [25] that this inequality is weaker than the Cusa-Huygens inequality (1.9), as follows:

$$\frac{\sin(x)}{x} < 2 + \frac{\cos(x)}{3} < \frac{1}{2} \left( \frac{x}{\sin(x)} + \cos(x) \right), \quad 0 < |x| < \frac{\pi}{2},$$

here second inequality is equivalent to

$$(1.16) \quad 3 \frac{x}{\sin(x)} + \cos(x) > 4, \quad 0 < |x| < \frac{\pi}{2}.$$

Recently, Mortici [22] refined this inequality (1.16) as follows:

$$(1.17) \quad 3 \frac{x}{\sin(x)} + \cos(x) > 4 + \frac{1}{10} x^4 + \frac{1}{210} x^6, \quad 0 < |x| < \frac{\pi}{2}.$$

We establish the following sharp result as a counterpart of (1.17).
1.18. **Theorem.** For \( x \in (0, \pi) \), we have

\[
2 - \frac{x^4}{\pi^4} < 3 \frac{\sin(x)}{x} - \cos(x) < 2 - \frac{x^3}{\pi^3}.
\]

The well-know Jordan’s inequality [21] states,

\[
\frac{\pi}{2} \leq \frac{\sin(x)}{x}, \quad 0 < x \leq \frac{\pi}{2},
\]
equality with \( x = \pi/2 \).

In [12], Debnath and Zhao refined the Jordan’s inequality as below,

\[
dZ_l(x) = \frac{2}{\pi} + \frac{1}{12\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin(x)}{x},
\]

\[
DZ_l(x) = \frac{2}{\pi} + \frac{1}{3\pi^3}(\pi^2 - 4x^2) \leq \frac{\sin(x)}{x},
\]
for \( x \in (0, \pi/2) \), equality in both inequalities with \( x = \pi/2 \).

The following inequality

\[
O_l(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3} \left( x - \frac{\pi}{2} \right)^2 \leq \frac{\sin(x)}{x},
\]
for \( x \in (0, \pi/2) \) is due to Özban [27], equality with \( x = \pi/2 \).

In [53], Zhu proved that

\[
\frac{\sin(x)}{x} \leq \frac{2}{\pi} + \frac{(\pi - 2)(\pi^2 - 4x^2)}{\pi^3} = Z_u(x),
\]
for \( x \in (0, \pi/2) \).

For the following inequalities

\[
J_l(x) = \frac{2}{\pi} + \frac{\pi^4 - 16x^4}{2\pi^5} \leq \frac{\sin(x)}{x} < \frac{2}{\pi} + \frac{(\pi - 2)(\pi^4 - 16x^4)}{\pi^5} = J_u(x),
\]
for \( x \in (0, \pi/2) \), see [14].

In [15] Theorem 1.3, Klén et al. proved that

\[
1 - \frac{x^2}{6} < \frac{\sin(x)}{x} < 1 - \frac{2x^2}{3\pi^2},
\]
for \( x \in (-\pi/2, \pi/2) \). The following inequality refines the second inequality in (1.24),

\[
\frac{\sin(x)}{x} \leq \frac{1 - x^2/\pi^2}{\sqrt{1 + 3(x/\pi)^4}}
\]
for \( x \in (0, \pi) \), see [19].

Our first main result reads as follows:

1.26. **Theorem.** For \( x \in (0, \pi/2) \), we have

\[
C_l(x) < \frac{\sin(x)}{x} < C_u(x),
\]
where
\[ C_l(x) = \left( 1 - \frac{x^2}{\pi^2} \right)^{\pi^2/6} \quad \text{and} \quad C_u(x) = \left( 1 - \frac{x^2}{\pi^2} \right)^{3/2}. \]

The right side of the following theorem can not be compared with the corresponding side of Theorem 1.26, and the second inequality is weaker than the above result.

1.27. Theorem. For \( x \in (0, \pi) \), we have
\[ D_l(x) < \sin(x) < D_u(x), \]
where
\[ D_l(x) = 1 - \left( \frac{x}{\pi} \right)^2 \left( 2 - \frac{x}{\pi} \right) \quad \text{and} \quad D_u(x) = 1 - \frac{1}{2} \left( \frac{x}{\pi} \right)^2 \left( 3 - \left( \frac{x}{\pi} \right)^2 \right). \]

It is not difficult to see that Theorem 1.27 refines the inequalities (1.24) and (1.25).

Obviously, one can see that
\[ dz_l(x) < DZ_l(x) < O_l(x), \quad x \in (0, 1.19540), \]
\[ O_l(x) < C_l(x), \quad x \in (0, 0.92409), \]
\[ C_u(x) < Z_u(x), \quad x \in (0, 1.09447), \]
\[ D_u(x) < Z_u(x), \quad x \in (0, 0.95784). \]

We see that our result refines (1.21) and (1.22) in the given interval of \( x \).

The following result is the consequence of Theorem 1.27.

1.28. Theorem. For \( y \in (0, 1) \), we have
\[ (1) \quad B(x,y) < \frac{1}{x} \frac{1}{y} < \frac{x+y}{1+xy}, \quad 0 < x < 1, \]
\[ (2) \quad \frac{1}{xy} \frac{x+y}{1+xy} < B(x,y), \quad 1 < x < \infty. \]

It is worth to mention that the part (1) of Theorem 1.28 recently appeared in [16] as a one of the main results. The proof of the claim is based on [16, Lemma 2.5], and the proof of the lemma is invalid.

The following Redheffer inequality [30]
\[ \frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x}, \quad 0 < x \leq \pi, \]
was proved by Williams [48]. Chen et al. [10] obtained the three Redheffer-type inequalities for \( \cos x \), \( \cosh x \) and \( \frac{\sinh x}{x} \). Sun and Zhu [19, 46] proved the Redheffer-type two-sided inequalities for trigonometric and hyperbolic functions. The inequalities appeared in [19] were refined in [55], and read as follows:
\begin{equation}
\left( \frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}} \right)^{2/3} < \frac{\sin(x)}{x} < \left( \frac{\pi^2 - x^2}{\sqrt{\pi^4 + 3x^4}} \right), \quad 0 < x < \pi,
\end{equation}

\begin{equation}
\left( \frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}} \right)^{2/3} < \cos(x) < \left( \frac{\pi^2 - 4x^2}{\sqrt{\pi^4 + 48x^4}} \right)^{3/4}, \quad 0 < x < \frac{\pi}{2},
\end{equation}

\begin{equation}
\left( \frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2} \right)^{1/2} < \frac{\tan(x)}{x} < \left( \frac{\sqrt{\pi^4 + 48x^4}}{\pi^2 - 4x^2} \right)^{\pi^2/6}, \quad 0 < x < \frac{\pi}{2}.
\end{equation}

One can see easily that the first inequality in Theorem 1.27 refines the (1.29). Our following theorem refines (1.30), as well as Cusa-Huygens inequality, and also gives a new upper bound for the right side of (1.31).

\textbf{1.33. Theorem.} For \(x \in (0, \pi/2)\) one has

\begin{equation}
\frac{\sin(x)}{x} < \frac{\pi^2 - x^2}{\pi^2 + \alpha x^2} < \frac{2 + \cos(x)}{3} < \frac{\pi^2 - x^2}{\pi^2 + \beta x^2},
\end{equation}

with best possible constants \(\alpha = \pi^2/6 - 1 \approx 0.644934\) and \(\beta = 1/2\).

\textbf{1.35. Theorem.} The function

\[ g(t) = \frac{\log((1 + t^2)/(1 - t^2))}{\log(1/\cos(\pi t/2))} \]

is strictly decreasing from \((0, 1)\) on \((\alpha, \beta)\). In particular, for \(x \in (0, \pi)\)

\[ \cos(x/2)^\alpha < \frac{\pi^2 - x^2}{\pi^2 + x^2} < \cos(x/2)^\beta, \]

with the best possible constants \(\alpha = 16/\pi^2 \approx 1.62114\) and \(\beta = 1\).

In the following theorem we refine the inequalities given in (1.31).

\textbf{1.36. Theorem.} The following function

\[ h(t) = \frac{1 - 4t^2}{t^2 \cos(\pi t)} - \frac{1}{t^2} \]

is strictly increasing from \((0, 1/2)\) onto \((1/2, (\pi^2 - 8)/2, 16/\pi - 4)\). In particular, for \(x \in (0, \pi/2)\)

\[ \frac{\pi^2 - 4x^2}{\pi^2 + \alpha x^2} < \cos(x) < \frac{\pi^2 - 4x^2}{\pi^2 + \beta x^2}, \]

with the best possible constants \(\alpha = 16/\pi - 4 \approx 1.09296\) and \(\beta = (\pi^2 - 8)/2 \approx 0.934802\).

The paper is organized into three sections as follows. Section 1, contains the introduction and the statements of our main results. In Section 2, we give some lemmas, which will be used in our proofs. Section 3 is consists of the proofs of the main results and some corollaries.
The following result is sometime called the Monotone l’Hôpital rule, which is due to Anderson et al. [4].

2.1. Lemma. For \(-\infty < a < b < \infty\), let \(f, g : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\), and be differentiable on \((a, b)\). Let \(g'(x) \neq 0\) on \((a, b)\). If \(f'(x)/g'(x)\) is increasing (decreasing) on \((a, b)\), then so are

\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

If \(f'(x)/g'(x)\) is strictly monotone, then the monotonicity in the conclusion is also strict.

2.2. Lemma. For \(a \in (0, 1)\) and \(x > 0\), the following inequality holds,

\[
1 - \frac{a}{x + a} < \psi(1 + x) - \psi(x + a).
\]

Proof. It is well-known that the function \(f(x) = x\psi(x), x > 0\) is strictly convex [4, Theorem 6]. This implies that,

\[
f(ar + (1 - a)s) < af(r) + (1 - a)f(s), \quad r, s > 0, \ a \in (0, 1).
\]

Setting \(r = 1 + x\) and \(s = x\) in the above inequality, we get

\[
(a + x)\psi(a + x) < (a + ax)\psi(1 + x) + (1 - a)x\psi(x),
\]

now the proof follows easily if we replace \(\psi(x) = \psi(1 + x) - 1/x\). \(\blacksquare\)

2.3. Lemma. For \(x \in (1, \infty)\) and \(y \in (0, 1)\), we have

\[
\frac{\Gamma(1 + x)\Gamma(1 + y)}{\Gamma(1 + x + y)} > \frac{1}{1 + xy}.
\]

Proof. Let

\[
f_y(x) = \log(\Gamma(1 + x)) + \log(\Gamma(1 + y)) - \log(\Gamma(1 + x + y)) + \log((1 + xy))
\]

for \(x \in (1, \infty)\) and \(y \in (0, 1)\), clearly \(f_y(1) = 0\). Differentiating \(f\) with respect to \(x\) and using the formula (1.3), we get

\[
f'_y(x) = \frac{y}{1 + xy} + \psi(1 + x) - \psi(1 + x + y)
\]

\[
= \frac{y}{1 + xy} - \frac{1}{x + y} + \psi(1 + x) - \psi(x + y)
\]

\[
> \frac{y}{1 + xy} - \frac{1}{x + y} + \frac{1 - y}{x + y} = \frac{y(1 - y)(x - 1)}{(x + y)(1 + xy)} > 0,
\]

by Lemma 2.2. Thus, \(f\) is strictly increasing, and \(f_y(x) > f_y(1) = 0\), this implies the proof of part (1). For the proof of part (2), see [10, (3.2)]. \(\blacksquare\)
2.4. Remark. It is easy to see that the convexity of the function \( x \mapsto \log(x\Gamma(x)) \), \( x > 0 \), implies the following inequality,

\[
\frac{1}{xy} \left( \frac{x + y}{2} \right)^2 < \frac{\Gamma(x)\Gamma(y)}{\Gamma((x + y)/2)^2}.
\]

Replacing \( x \) by \( 1 - x/\pi \) and \( y \) by \( 1 - x/\pi \) in (2.5), and applying (1.5), we get

\[
\frac{\sin(x)}{x} < \frac{\pi^2 - x^2}{\pi^2}, \quad 0 < x < \pi.
\]

This improves the following inequality

\[
\frac{\sin(x)}{x} \leq \left( \frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^{1/2}, \quad 0 < x \leq \pi,
\]

which was proved in [13].

3. PROOF OF THE MAIN RESULTS

In this section we will give the proofs of the main results highlighted in the first section, as well as some corollaries are being established.

Proof of Theorem 1.10. The proof of the first inequality is trivial. For the proof of the second and the third inequality, we define

\[
f(x) = \frac{2 + \cos(x) - 3 \sin(x)/x}{x^4},
\]

for \( x \in (0, \pi) \). We will prove that \( f \) is strictly decreasing from \((0, \pi)\) onto \((1/\pi^4, 1/60)\). One has,

\[
x^6 f(x) = -x^2 \sin(x) - 7x \cos(x) + 15 \sin(x) - 8x = f_1(x),
\]

\[
f_1'(x) = 5x \sin(x) + 8 \cos(x) - x^2 \cos(x) - 8,
\]

\[
f_2'(x) = -3 \sin(x) + 3x \cos(x) + x^2 \sin(x), \quad f_1''(x) = x(x \cos(x) - \sin(x)) < 0.
\]

Thus \( f_1'(0) = 0, f_1'(x) < f_1'(0) = 0, \) and \( f_1(x) < f_1(0) = 0 \). The limiting values can be achieved by l'Hôpital rule.

Similarly, for the proof of the first inequality we will prove that the function

\[
g(x) = \frac{2 + \cos(x) - 3 \sin(x)/x}{x^3}
\]

is strictly increasing from \((0, \pi)\) onto \((0, 1/\pi^3)\). One has

\[
x^5 g'(x) = -(x^2) \sin(x) - 6x \cos(x) + 12 \sin(x) - 6x = h(x),
\]

\[
h'(x) = 4x \sin(x) - x^2 \cos(x) + 6 \cos(x) - 6,
\]

\[
h''(x) = 2x \cos(x) + x^2 \sin(x) - 2 \sin(x), \quad h'''(x) = x^2 \cos(x).
\]

Thus \( h''(x) \) is positive in \( x \in (0, \pi/2) \) and negative in \( x \in (\pi/2, \pi) \). Since \( h''(\pi/2) = \pi^2/4 - 2 > 0 \) and \( h''(\pi) = -2\pi < 0 \), and \( h''(x) \) is strictly increasing (resp. decreasing) in \( x \in (0, \pi/2) \) (resp. \( x \in (\pi/2, \pi) \)). There exists a unique \( x_0 \) in \((\pi/2, \pi)\) such that \( h''(x_0) = 0 \). Thus \( h''(x) > 0, x \in (0, x_0) \) and \( h''(x) < 0, x \in (x_0, \pi) \). This implies
that $h'(x)$ is strictly increasing in $x \in (0, x_0)$ and strictly decreasing in $x \in (x_0, \pi)$. As $h'(0) = 0$, one has $h'(x_0) > 0$. By $h'(\pi) = \pi^2 - 12 < 0$, we get that there exists a unique $x_1$ in $(x_0, \pi)$ such that $h'(x_1) = 0$. We get that $h(x)$ is strictly increasing in $(0, x_1)$, and decreasing in $(x_1, \pi)$, with $h(0) = 0$ and $h(\pi) = 0$. Thus $h(x) > h(0) = 0$, $x \in (0, x_1)$ and $h(x) > h(\pi) = 0$, $x \in (x_1, \pi)$. Hence, in all cases, one has $h(x) > 0$. This completes the proof of the second inequality.

It follows from the proof of the second inequality of Theorem 1.10 that $f < 1/60$ and $f_1 < 0$. This implies the following result.

**3.1. Corollary.** For $x \in (0, \pi)$, we have

\[
(1) \quad M_1 = \frac{2 - x^4/60 + \cos(x)}{3} < \frac{\sin(x)}{x} < \frac{8 + 7 \cos(x)}{15 - x^2} = M_2,
\]

\[
(2) \quad 2 \frac{x}{\sin(x)} + \frac{x}{\tan(x)} < 3 + \frac{x^4}{60 \sin(x)},
\]

\[
(3) \quad \frac{8/7}{\sin(x)} + \frac{x}{\tan(x)} > \frac{15 - x^2}{7}.
\]

Recently, the following inequalities appeared in 52.

\[
(3.2) \quad Y_1 = \cos \left( \frac{x}{\sqrt{x}} \right)^a < \frac{\sin(x)}{x} < \cos \left( \frac{x}{\sqrt{x}} \right)^{5/3} = Y_a
\]

for $x \in (0, \pi/2)$ with $a = \log(2/\pi)/\log(\cos(\sqrt{5} \pi/10)) \approx 1.67141$. By using Mathematica Software® [31], one can see that

$Y_1 < M_1$, $x \in (0, 1.06580)$ and $Y_a - M_2 \in (0, -0.0009)$, $x \in (0, \pi/2)$.

**Proof of Theorem 1.14 & 1.18.** The proof of both theorems follow immediately from Theorem 1.10

**Proof of Theorem 1.26.** Let us consider the application

\[
f(x) = \log \left( \frac{x}{\sin(x)} \right) - c \log(1/(1 - x^2/\pi^2)),
\]

for $x \in (0, \pi/2)$. A simple computation gives

\[
x \sin(x)(\pi^2 - x^2)f'(x) = (\sin x - x \cos x)(\pi^2 - x^2) - 2cx^2 \sin(x) = g(x).
\]

One has

\[
g'(x)/x = (\pi^2 - 2 - 4c) \sin(x) + (2 - 2c)x \cos(x) - x^2 \sin(x) = h(x).
\]

Finally,

\[
h'(x) = (\pi^2 - 6c) \cos(x) - (4 - 2c)x \sin(x) - x^2 \cos(x).
\]

Now, if we select $c = \pi^2/6$, then, as $\pi^2 - 6c = 0$ and $4 - 2c = 4 - \pi^2/3 > 0$, we get $h'(x) < 0$, so this finally leads to $f(x) < 0$. Hence, the first inequality follows.
For the proof of the second inequality, let \( c = 3/2 \), then one has

\[
g'(x)/x = (\pi^2 - 8) \sin(x) - x \cos(x)(x^2) \sin(x) = h(x).
\]

Since \( h(\pi/2) < 0 \), \( h(\pi/4) > 0 \), there exists an \( x' \) in \((\pi/4, \pi/2)\) such that \( h(x') = 0 \).

We’ll show that \( x' \) is unique. One has \( h(x)/\sin(x) = \pi^2 - 8 - s(x) \), where \( s(x) = x^2 + x/\tan(x) \). Now,

\[
s'(x) \sin(x)^2 = 2x \sin(x)^2 + \cos(x) \sin(x) - x = p(x).
\]

Here \( p'(x) = 4x \sin(x) \cos(x) > 0 \), which shows that \( p(x) > p(0) = 0 \), so \( s'(x) > 0 \), finally: \( s(x) \) is a strictly increasing function. Thus the equation \( s(x) = \pi^2 - 8 \) has a single root (which is \( x' \)), so \( h(x) > 0 \) for \( x \in (0, x') \) and \( h(x) < 0 \) for \( x \in (x', \pi/2) \).

Thus \( g \) is increasing, resp. decreasing in the above intervals, and \( g(0) = g(\pi/2) = 0 \), so \( g(x) > 0 \) for \( x \in (0, \pi/2) \). This completes the proof of the second inequality. \( \square \)

3.3. Corollary. For \( x \in (0, \pi/2) \) one has

\[
\frac{\pi^2 - x^2 - \pi^2 x^2/3}{\pi^2 - x^2} < \frac{x}{\tan(x)} < \frac{\pi^2 - 4x^2}{\pi^2 - x^2}.
\]

Proof. After simplification the derivative of the function

\[
f_c(x) = \log \left( \frac{x}{\sin(x)} \right) - c \log(1/(1 - x^2/\pi^2)),
\]

can be written as

\[
f_c'(x) = \frac{1}{x} - 2c \pi x^2 - x^2 - \frac{1}{\tan(x)}.
\]

By the proof of Theorem 1.26, \( f_{\pi^2/6}'(x) < 0 \) and \( f_{3/2}'(x) > 0 \). Clearly, \( f_{\pi^2/6} = 0 = f_{3/2} \).

Now the proof of the inequalities is obvious.

The second inequality in Corollary 3.3 improves the first inequality in (1.32).

Proof of Theorem 1.27. For \( t \in (0, 1) \), let

\[
f(t) = \frac{\pi t (t^3 + 1) - \sin(\pi t)}{\pi t^3}, \quad g(t) = \frac{\pi t (2 + t^4) - 2 \sin(\pi t)}{\pi t^3}.
\]

By Theorem 1.10 we get

\[
f'(x) = \frac{3\pi t^3 + \pi (t^3 + 1) - \pi \cos(\pi t)}{\pi t^3} - \frac{3(\pi t (t^3 + 1) - \sin(\pi t))}{\pi t^4}
\]

\[
= \frac{3}{t^3} \left( \frac{\sin(\pi t)}{\pi t} - \frac{2 - t^3 + \cos(\pi t)}{3} \right) > 0,
\]

and

\[
g'(x) = 2t - \frac{2}{t^2} \left( \frac{\cos(\pi t)}{t} - \frac{\sin(\pi t)}{\pi t^2} \right) + \frac{4}{t^3} \left( \frac{\sin(\pi t)}{\pi t} - 1 \right)
\]

\[
= \frac{2}{t^3} \left( \frac{\sin(\pi t)}{\pi t} - \frac{2 - t^4 + \cos(\pi t)}{3} \right) < 0.
\]
Thus, the functions $f$ and $g$ are strictly increasing and decreasing in $t \in (0, 1)$, respectively. Hence, the proof follows easily if we use the inequalities,

$$f(t) < \lim_{t \to 1} = 2, \quad g(t) > \lim_{t \to 1} g(t) = 3,$$

and replace $t$ by $x/\pi$. □

**Proof of Theorem 1.28** Utilizing (1.3), the first inequality in Theorem 1.27 is equivalent to

$$\frac{1 + x/\pi - x/\pi}{\Gamma(1 + x/\pi)\Gamma(1 - x/\pi)} > \frac{(1 - x/\pi)(1 + x/\pi - (x/\pi)^2)}{\Gamma(1 + x/\pi - x/\pi)}.$$

Replacing $x$ by $x/\pi$ and $y$ by $1 - x/\pi$ we get (1). Similarly, the second inequality in Theorem 1.27 can be written as,

$$\frac{1 + x/\pi + 1 - x/\pi}{(1 - (x/\pi)^2)(2 + (x/\pi)^2)} < \frac{\Gamma(1 + x/\pi)\Gamma(1 - x/\pi)}{\Gamma(1 + x/\pi - x/\pi)}.$$

If we replace $x$ by $1 + x/\pi$ and $y$ by $1 - x/\pi$ then we get the proof of part (2) for $1 < x < 2$. The rest of proof follows from Lemma 2.3. □

**Proof of Theorem 1.33** Let $f(t) = f_1(t)/f_2(t)$, $t \in (0, 1/2)$, where

$$f_1(t) = 1 - 3t^2 - \cos(\pi t) \quad \text{and} \quad f_2(t) = t^2(2 + \cos(\pi t)).$$

A simple calculation gives

$$f_1'(t) = -6t - \pi \sin(\pi t) < 0, \quad \text{and} \quad f_2'(t) = t(4 + 2 \cos(\pi t) - \pi t \sin(\pi t)) > 0.$$

Thus, $f$ is the product of two positive strictly decreasing functions, this implies that $f$ is strictly decreasing in $t \in (0, 1/2)$. Applying l’Hôpital rule, we get $\lim_{t \to 1/2} f(t) = 1/2 < f(t) < \alpha = \lim_{t \to 0} f(t) = \pi^2/6 - 1$. Here the first inequality implies

$$\frac{3(1 - t^2)}{2 + \cos(\pi t)} - 1 < \alpha t^2,$$

which is equivalent to

$$\frac{2 + \cos(\pi t)}{3} > \frac{1 - t^2}{1 + \alpha t^2}.$$

Letting $\pi t = x \in (0, \pi/2)$, we get the second inequality of (1.34), and the third inequality of (1.34) follows similarly. For the proof of the first inequality, see [2, (2.5)]. □

**3.4. Corollary.** For $x \in (0, 1)$, we have

$$\frac{4}{\pi} \frac{t}{1 - t^2} < \tan \left(\frac{\pi t}{2}\right) < \frac{\pi}{2} \frac{t}{1 - t^2}.\tag{3.5}$$
Proof. Let \( f(t) = t/(\tan(\pi t/2)(1 - t^2)), \ t \in (0, 1/2). \) We get
\[
f'(t) = \frac{(1 + t^2) \sin(\pi t) - \pi t(1 - t^2)}{2(1 - t^2)^2 \sin(\pi t/2)^2},
\]
which is positive by (1.29). By l'Hôpital rule, we get \( \lim_{t \to 0} f(t) = 2/\pi < f(t) < \pi/4 = \lim_{t \to 1/2} f(t). \) This completes the proof. \( \square \)

For \( 0 < t < 1, \) letting \( x = (\pi t)/2 \) in (3.5), we get
\[
8 \pi^2 - 4x^2 < \frac{\tan(x)}{x} < \frac{\pi^2}{\pi^2 - 4x^2},
\]
which is so-called Becker-Stark inequality \([8]\). Thus, Corollary 3.4 gives a simple proof of Becker-Stark inequality. It is easy to see that the second inequality in Corollary 3.3 improves the first inequality in (3.6).

Proof of Theorem 1.35. Write \( g(t) = g_1(t)/g_2(t), \ 0 < t < 1, \) where
\[
g_1(t) = \log \left( \frac{1 + t^2}{1 - t^2} \right), \ g_2(t) = \log \left( \frac{1}{\cos(\pi t/2)} \right).
\]
We get,
\[
g'_1(t) = g'_2(t) = \frac{8}{\pi (1 - t^4) \tan(\pi t/2)} = \frac{8}{\pi} g_3(t).
\]
One has,
\[
g'_3(t) = \frac{\sin(\pi t)(1 + 3t^4) - \pi t(1 - t^4)}{2 \sin(\pi t/2)^2(1 - t^4)^2},
\]
which is negative by (1.30). Clearly \( g_1(0) = 0 = g_2(0), \) thus \( g \) is strictly decreasing by Lemma 2.1. Using l'Hôpital rule, we get \( \lim_{t \to 0} g(t) = 16/\pi^2 > g(t) > 1 = \lim_{t \to 1} g(t). \) Replacing \( \pi t \) by \( x, \) we get the desired inequalities. \( \square \)

3.7. Corollary. For \( x \in (0, \pi), \) one has
\[
\left( \frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^{4/(3\alpha)} < \cos \left( \frac{x}{2} \right)^{4/3} < \frac{\sin(x)}{x} < \left( \frac{\pi^2 - x^2}{\pi^2 + x^2} \right)^{4/(3\alpha)},
\]
where \( \alpha \) and \( \beta \) are as in Theorem 1.35.
Proof. The proof of the first inequality follows from Theorem 1.35 and the second inequality is also well known \([25]\). The third inequality is just (1.30). \( \square \)

3.8. Theorem. For \( x \in (0, \pi/2), \) we have
\[
\frac{2}{\pi} + \frac{\pi - 2}{\pi} \cos(x) < \frac{\sin(x)}{x},
\]
\[
\left( \frac{1 + \cos(x)}{2} \right)^{2/3} < \frac{\sin(x)}{x} < \frac{2 \cdot 2^{2/3}}{\pi} \left( \frac{1 + \cos}{2} \right)^{2/3} < \frac{4}{\pi} \frac{1 + \cos(x)}{2}.
\]
Proof. The inequality (3.9) may be rewritten as
\[ f(x) = \pi \sin(x) - (\pi - 2)x \cos(x) - 2x > 0, \]
for \( x \in (0, \pi/2) \). Clearly, \( f(0) = f(\pi/2) = 0 \). On the other hand,
\[ f(x) = 2(\cos(x) - 1) + (\pi - 2)x \sin(x) = -4 \sin(x/2)^2 + 2(\pi - 2)x \sin(x/2) \cos(x/2). \]
This implies that
\[ f'(x)/(4 \sin(x/2) \cos(x/2)) = (\pi - 2)/2 - \tan(x/2) = g(x) \]
for \( x > 0 \). As \( \tan(x/2) \) is strictly increasing, there is a unique \( x' \) in \( (0, \pi/2) \) such that \( g(x') = 0 \). Also, \( g(x) > 0 \) for \( x \in (0, x') \) and \( g(x) < 0 \) for \( x \in (x', \pi/2) \); i.e. \( x' \) is a maximum point of \( f(x) \). This gives \( f(x) > f(0) = 0 \) for \( x \in (0, x') \) and \( f(x) > f(\pi/2) = 0 \) for \( x \in (x', \pi/2) \). This completes the proof of (3.9).

For the proof of (3.10), let
\[ j(x) = \log \left( \frac{x}{\sin(x)} - \frac{2}{3} \frac{1 + \cos(x)}{\sin(x)} \right), \]
x \( \in (0, \pi/2) \). One has
\[
j'(x) = \frac{1 - x \cot(x)}{x} - \frac{2 \sin(x)}{3 + \cos(x)} \\
= \frac{1}{x} - \cot(x) - \frac{2}{3} \frac{1 - \cos(x)}{\sin(x)} = j_1(x),
\]
which is negative, because the inequality \( j_1(x) < 0 \) can be written as \( \sin(x)/x < (2 + \cos(x))/3 \), which is so-called Cusa-Huygens inequality [21]. Thus, \( j(x) \) is strictly decreasing, and
\[
\lim_{x \to 0} j(x) = 0 > j(x) > \log(\pi) - (5/3) \log(2) = \lim_{x \to \pi/2} j(\pi/2) \approx -0.010515.
\]
By these inequalities we get
\[
\cos(x/2)^{4/3} < \frac{\sin(x)}{x} < \exp((5/3) \log(2) - \log(\pi)) \cos(x/2)^{4/3}.
\]
The proof of the first and second inequality is completed, and the proof of the third inequality is trivial. \( \square \)

The inequality (3.9) improves the following one
\[
(3.11) \quad 1 - 2 \frac{\pi - 2}{\pi^2} x < \frac{\sin(x)}{x}, \quad 0 < x < \pi,
\]
which was proved in [38] as an application of the concavity of \( \sin(x)/x \). Indeed, the inequality
\[
1 - 2 \frac{\pi - 2}{\pi^2} x < \frac{2}{\pi} + \frac{\pi - 2}{\pi} \cos(x),
\]
is equivalent to
\[
\cos(x) > 1 - \frac{2}{\pi}x, \quad 0 < x < \frac{\pi}{2},
\]
which is Kober’s inequality, see [21, 37].

**Proof of Theorem 1.36.** For \( x \in (0, \pi/2) \), let
\[
f(x) = \frac{x}{\sin(x)} \frac{x(\pi - x)}{(2x - \pi)^2} - \frac{4x^2}{(2x - \pi)^2}.
\]
We get,
\[
f'(x) = \frac{16\pi^2 \cos(x)}{(2x - \pi)^3 \sin(x)^2} g(x),
\]
where
\[
g(x) = \tan(x) - \frac{\sin(x)^2}{\cos(x)} - x.
\]
One has,
\[
g'(x) = \frac{\sin(x)^2}{\cos(x)} (\sin(x)^2 - \sin(x) - 2) < 0,
\]
as \( 0 < \sin(x) < 1 \), and \( g(x) = 0 \). Thus, \( g < 0 \), and in result \( f'(x) < 0 \), this implies that \( f \) is strictly decreasing. By l’Hôpital rule we get
\[
\lim_{x \to \pi/2} f(x) = \frac{\pi^2}{2} - \frac{8}{\pi} < f(x) < \lim_{x \to 0} f(x) = \frac{16}{\pi} - 4,
\]
this implies the proof. \( \square \)

Replacing \( x \) by \( \pi/2 - x \) in the inequalities of Theorem 1.36 we get the following corollary as a result.

**3.13. Corollary.** For \( x \in (0, \pi/2) \), we have
\[
\frac{16(\pi - x)}{4\pi^2 + \alpha (2x - \pi)^2} < \frac{\sin(x)}{x} < \frac{16(\pi - x)}{4\pi^2 + \beta (2x - \pi)^2},
\]
where \( \alpha \) and \( \beta \) are as in Theorem 1.36.

**References**

[1] M. Abramowitz, and I. Stegun, eds.: *Handbook of mathematical functions with formulas, graphs and mathematical tables*, National Bureau of Standards, Dover, New York, 1965.

[2] D. Aharonov, and U. Elias: *Improved inequalities for trigonometric functions via Dirichlet and Zeta functions*, Math. Ineq. Appl. vol. 16, no. 3 (2013), 851–859.

[3] H. Alzer: *On some inequality of gamma and psi function*, Math. Comp. Vol. 66, 217 (1997), 373–389.

[4] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: *Monotonicity Rules in Calculus*, Amer. Math. Month. Vol. 113, No. 9 (2006), pp. 805–816.
[5] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: Monotonicity of some functions in calculus, preprint, https://www.math.auckland.ac.nz/Research/Reports/Series/538.pdf.
[6] G.D. Anderson, M. Vuorinen, and X. Zhang: Topics in special functions III., http://arxiv.org/abs/1209.1696.
[7] Á. Baricz: Redheffer Type Inequality for Bessel Functions, JIPAM. Vol. 8, 1, Art 11, 2007.
[8] M. Becker, and E.L. Strak: On a hierarchy of quasipolynomial inequalities for tan(x), Univ. Beograd. Publ. Elektrotehn Fak Ser Mat Fiz. No. 602-633 (1978), p. 133–138.
[9] M. Biernacki, J. Krzyż: On the monotonicity of certain functionals in the theory of analytic functions, Ann. Univ. Mariae. Curie-Skłodowska 2 (1955) 134–145.
[10] C.-P. Chen, J.-W. Zhao, and F. Qi: Three inequalities involving hyperbolically trigonometric functions, RGMIA Res. Rep. Coll. 6 (3) (2003) 437–443. Art. 4.
[11] C.-P. Chen, J. Sándor: Inequality chains for Wilker, Huygens and Lazarević type inequalities, http://ajmaa.org/RGMIA/papers/v15/v15a11.pdf.
[12] L. Debnath, and C.-J. Zhao: New strengthened Jordan’s inequality and its applications, Appl. Math. Lett., vol. 16, no. 4, 2003, pp. 557–560.
[13] B.-N. Guo, B.-M. Qiao, F. Qi, and W. Li: On new proofs of Wilker’s inequalities involving trigonometric functions, Math. Ineq. Appl. 6 (2003) 19–22.
[14] W.-D. Jiang, and H. Yun: Sharpening of Jordans Inequality and its Applications, J. Ineq. Pure and Appl. Math. 7(3) Art. 102, 2006.
[15] V. Heikkala, M.K. Vamanamurthy, and M. Vuorinen: Generalized elliptic integrals, Comput. Methods and Function Theory April 2009, Vol. 9, Issue 1, pp 75–109.
[16] P. Iványi: On a beta function inequality, J. Math. Inequal., 6, 3 (2012) 333–341.
[17] A. Jeffrey: Handbook of Mathematical Formulas and Integrals, 3rd ed., Elsevier Academic Press, 2004.
[18] R. Klén, M. Visuri, and M. Vuorinen: On Jordan type inequalities for hyperbolic functions, J. Ineq. Appl., vol. 2010, pp. 14.
[19] J.-L. Li, and Y.-L. Li: On the Strengthened Jordan’s Inequality, J. Ineq. Appl. 2007, Art. ID 74328, pp. 8.
[20] L. Li, and J. Zhang: A new proof on Redheffer-Williams’ inequality, Far East J. Math. Sci., 56(2011), no.2, 213–217.
[21] D.S. Mitrinovic: Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[22] C. Mortici: The natural approach of Wilker-Cusa-Huygens inequalities, Math. Inequal. Appl., 14, 3 (2011) 535–541.
[23] E. Neuman: Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions, Advances in Inequalities and Applications, 1 (2012), No. 1, 1–11.
[24] E. Neuman: Inequalities Involving Hyperbolic Functions and Trigonometric Functions, Bull. Int. Math. Vir. Int. Vol. 2(2012), 87–92.
[25] E. Neuman, and J. Sándor: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Inegal. Appl. Vol. 13, Number 4 (2010), 715–723.
[26] E. Neuman, and J. Sándor: Optimal inequalities for hyperbolic and trigonometric functions, Bull. Math. Anal. Appl. 3(2011), no. 3, 177–181.
[27] A. Y. Özbak: A new refined form of Jordan’s inequality and its applications, Appl. Math. Lett., vol. 19 (2006), 155–160, 2006.
[28] I. Pinelis: L’Hôpital Rules for Monotonicity and the Wilker-Anglesio Inequality, Amer. Math. Monthly 111 (10) (2004) 905–909.
[29] F. Qi: Extensions and sharpenings of Jordan’s and Kober’s inequality, Journal of Mathematics for Technology (in Chinese), 4 (1996), 98–101.
[30] R. REDHEFFER: Problem 5642, Amer. Math. Monthly 76 (1969) 422.
[31] H. RUSKEEPÄÄ: Mathematica® Navigator. 3rd ed. Academic Press, 2009.
[32] J. SÁNDOR: On new refinements of Kober’s and Jordan’s trigonometric inequalities, Notes Number Theory Discrete Math. Vol. 19, 2013, No. 1, 73–83.
[33] J. SÁNDOR: Two sharp inequalities for trigonometric and hyperbolic functions, MIA 15(2012), no.2, 409–413.
[34] J. SÁNDOR: Sharp Cusa-Huygens and related inequalities, Notes Number Theory Discrete Math. Vol 19, 2013, No. 1, 50–54.
[35] J. SÁNDOR, AND R. OLÁH-GÁL: On Cusa-Huygens type trigonometric and hyperbolic inequalities, Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 145–153.
[36] J. SÁNDOR, AND M. BENČZE: On Huygens’ trigonometric inequality, RGMIA Res. Rep. Collection, 8 (2005), No. 3, Art. 14.
[37] J. SÁNDOR: On new refinements of Kober’s and Jordan’s trigonometric inequalities, Notes Number Th. Discr. Math. 19(2013), no.1, 73–83.
[38] J. SÁNDOR: On the concavity of \( \sin(x)/x \), Octogon Math. Mag., 13(2005), no.1, 406–407.
[39] J. SÁNDOR: Trigonometric and hyperbolic inequalities, http://arxiv.org/abs/1105.0859
[40] J. SÁNDOR: Some integral inequalities, Elem. Math. 43(1988), 177–180.
[41] J. SÁNDOR: Sur la fonction Gamma, P.C.R. Math. Pures Neuchâtel, Serie I, 21(1989), 4–7.
[42] J. SÁNDOR: On the Gamma function II, P.C.R. Math. Pures Neuchâtel, Serie I, 28(1997), 10–12.
[43] J. SÁNDOR: On the Open problem OQ. 532, Octogon Math. Mag. 9(2001), No.1B, 569–570.
[44] J. SÁNDOR: On the gamma function III, P.C.R. Math. Pures Neuchâtel, Serie 2, 19(2001), 33–40.
[45] J.S. SUMNER, A.A. JAGERS, M. VOWE, AND J. ANGELESIO: Inequalities involving trigonometric functions, Amer. Math. Monthly 98 (1991), 264–267.
[46] J. Sun and L. Zhu: Six new Redheffer-type inequalities for circular and hyperbolic functions, Comput. Math. Appl. 56, 2 (2008) 522–529.
[47] J.B. Wilker: Problem E3306, Amer. Math. Monthly 96 (1989) 55.
[48] J.P. Williams: Solution of problem 5642, Amer. Math. Monthly 76 (1969) 1153–1154.
[49] S.-H. Wu and H.M. Srivastava: A weighted and exponential generalization of Wilker’s inequality and its applications, Integral Transforms and Spec. Funct. 18 (2007), No. 8, 525–535.
[50] Z.-H. Yang: Refinements of a two-sided inequality for trigonometric functions, J. Math. Inequal., 7, 4 (2013) 601–615
[51] L. Zhang and L. Zhu: A new elementary proof of Wilker’s inequalities, Math. Ineq. Appl. vol. 11 2008, No. 1, 149–151.
[52] L. Zhu: Sharpening Redheffer-type inequalities for circular functions, App. Math. Lett. 22 (2009), 743–748.
[53] L. Zhu: Sharpening Jordan’s inequality and the Yang Le inequality, Appl. Math. Lett., 19 (2006), 240243.
[57] L. Zhu: *On Wilker-type inequalities*, Math. Ineq. Appl. Vol. 10 (2007) No. 4, 727–731.
[58] Z. Sun and L. Zhu: *On new Wilker-type inequalities*, ISRN Math. Anal. Vol. 2011, Article ID 681702, 7 pp.
ON CARLSON’S AND SHAFER’S INEQUALITIES

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. In this paper authors refine the Carlson’s inequalities for inverse cosine function, and the Shafer’s inequalities for inverse tangent function.

1. Introduction

During the past fifteen years, numerous authors have studied various inequalities for trigonometric functions [4, 15, 17, 18, 20, 21, 32, 33, 34, 35, 38, 44]. Thus, some classical and also more recent inequalities, such as inequalities of Jordan, Cusa-Huygens, Shafer-Fink, and Wilker have been refined and generalized. One of the key methods in these studies has been so called monotone l’Hospital Rule from [4] and an extensive survey of the related literature is given in [5]. This Rule is formulated as Lemma 2.1 and it will be also applied here. Motivated by these studies, in this paper we make a contribution to the topic by sharpening the Carlson’s and Shafer’s inequalities, and our inequalities refine the existing results in the literature.

We start our discussion with the following well-known inequalities,

(1.1) \[ \cos(t)^{1/3} < \frac{\sin(t)}{t} < \frac{\cos(t) + 2}{3}, \]

for \( 0 < |t| < \pi/2 \). The first inequality is due to by D.D. Adamović and D.S. Mitrinović [19, p.238], while the second inequality was obtained by N. Cusa and C. Huygens [14]. These inequalities can be written as,

\[ \frac{3\sin(t)}{2 + \cos(t)} < t < \frac{\sin(t)}{\cos(t)^{1/3}}. \]

For the further studies and refinements of inequalities in (1.1), e.g., see [5, 15, 20, 21, 32, 33, 39] and the references therein. For the easy references we recall the following inequalities

(1.2) \[ \cos \left( \frac{t}{2} \right)^{4/3} < \frac{\sin(t)}{t} < \cos \left( \frac{t}{3} \right)^{3}, \]

the first inequality holds for \( t \in (0, \pi/2) \) [21], while the second one is valid for \( t \in (-\sqrt{27/5}, \sqrt{27/5}) \), and was proved by Klén et al. [15]. The first inequality in (1.2) refines the following one

\[ t < \frac{2\sin(t)}{1 + \cos(t)}, \quad 0 < x < \frac{\pi}{2}, \]

which was constructed in [26] by using the Chebyshev’s integral inequality.
The Oppenheim’s problem [22, 23, 19] states that: to determine the greatest $a_2$ and least $a_3$ as a function of $a_1 > 0$, such that the following inequalities

\[(1.3) \quad \frac{a_2 \sin(x)}{1 + a_1 \cos(x)} < x < \frac{a_3 \sin(x)}{1 + a_1 \cos(x)},\]

hold for all $x \in (0, \pi/2)$. A partial solution of this problem was given by Oppenheim and Carver [23], they showed that (1.3) holds for all $a_1 \in (0, 1/2)$ and $x \in (0, \pi/2)$ when $a_2 = 1 + a_1$ and $a_3 = \pi/2$. In 2007, Zhu [45, Theorem 7] solved the Oppenheim’s problem completely by proving that the inequalities in (1.3) hold if $a_1$, $a_2$ and $a_3$ are as follows:

1. if $a_1 \in (0, 1/2)$, then $a_2 = 1 + a_1$ and $a_3 = \pi/2$,
2. if $a_1 \in (1/2, \pi/2 - 1)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = \pi/2$,
3. if $a_1 \in (\pi/2 - 1, 2/\pi)$, then $a_2 = 4a_1(1 - a_1^2)$ and $a_3 = 1 + a_1$,
4. if $a_1 > 2/\pi$, then $a_2 = \pi/2$ and $a_3 = 1 + a_1$,

where $a_2$ and $a_3$ are the best possible constants in (1) and (4), while $a_3$ is the best possible constant in (2) and (3). Thereafter, the Carver’s solution was extended to the Bessel functions for the further results by Baricz [6, 7]. On the basis of computer experiments we came up that the following lower and upper bounds for $x$,

\[(1.4) \quad \frac{\pi/2 \sin(x)}{1 + (2/\pi) \cos(x)} < x < \frac{\pi \sin(x)}{2 + (\pi - 2) \cos(x)},\]

are the best possible bounds, and can be obtained from case (4) and (3), respectively.

Recently, Qi et al. [28] have given a new proof of Oppenheim’s problem, and deduced the following inequalities,

\[(1.5) \quad \frac{(\pi/2) \sin(x)}{1 + (2/\pi) \cos(x)} < x < \frac{(\pi + 2) \sin(x)}{\pi + 2 \cos(x)},\]

for $x \in (0, \pi/2)$. It is obvious that

\[((\pi - 2) - 4)(1 - \cos(x)) < 0,\]

which is equivalent to

\[\frac{\pi \sin(x)}{2 + (\pi - 2) \cos(x)} < \frac{(\pi + 2) \sin(x)}{\pi + 2 \cos(x)}.\]

This implies that the second inequality of (1.4) is better than the corresponding inequality of (1.5).

Our first main result reads as follows, which refines the inequalities in (1.4).

1.6. **Theorem.** For $x \in (0, \pi/2)$, we have

\[(1.7) \quad C_\alpha < x < C_\beta\]
where
\[ C_\alpha = \frac{8\sin(x/2) - \sin(x)}{\alpha} \quad \text{and} \quad C_\beta = \frac{8\sin(x/2) - \sin(x)}{\beta}, \]
with the best possible constants \( \alpha = 3 \) and \( \beta = (8\sqrt{2} - 2)/\pi \approx 2.96465 \).

By using Mathematica Software\textsuperscript{® \[30\]}, one can see that Theorem 1.6 refines the inequalities in (1.4) as follows:
\[ Z_l(x) < C_\alpha(x), \quad \text{for} \quad x \in (0, 1.28966), \]
\[ Z_u(x) < C_\beta(x), \quad \text{for} \quad x \in (0, 0.980316). \]

It is worth to mention that the first inequality in Theorem 1.6 was discovered heuristically by Huygens \[9\], here we have given a proof.

In 1970, Carlson \[10\] established the following inequalities,
\[
(1.8) \quad \frac{6(1-x)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}} < \arccos(x) < \frac{4^{1/3}(1-x)^{1/2}}{(1+x)^{1/6}},
\]
0 < \( x < 1 \). These inequalities are known as Carlson’s inequalities in the literature. Thereafter, several authors studied these inequalities, and gave some generalization and partial refinement, e.g., see \[11, 12, 40, 43\]. It is interesting to observe that the Adamović-Mitrinović and Cusa-Huygens inequality (1.1) implies the second and the first inequality of (1.8), respectively, with the transformation \( x = \arccos(t) \), 0 < \( t < \pi/4 \).

For 0 < \( x < 1 \), Guo and Qi \[12, 40\] gave the following inequalities,
\[
(1.9) \quad \frac{\pi (1-x)^{1/2}}{2(1+x)^{1/6}} < \arccos(x) < \frac{(1/2 + \sqrt{2})(1-x)^{1/2}}{2\sqrt{2} + (1+x)^{1/2}},
\]
\[
(1.10) \quad \frac{4^{1/\pi}(1-x)^{1/2}}{(1+x)\left(4-\pi)/(2\pi)\right)} < \arccos(x) < \frac{\pi(1-x)^{1/2}}{2(1+x)\left(4-\pi)/(2\pi)\right)}.
\]
They concluded that these inequalities doesn’t refine (1.8) in the whole interval (0, 1) of \( x \).

Chen et al. \[11\] established the lower bound for \( \arccos(x) \) as follows,
\[
(1.11) \quad \frac{\pi (1-x)^{(\pi+2)/\pi^2}}{2(1+x)^{(\pi-2)/\pi^2}} < \arccos(x), \quad 0 < x < 1.
\]
The inequality (1.11) refines the first inequality of (1.8) for \( x \in (0, 0.345693) \).

In \[43\], Zhu proved that for \( p \geq 1 \) and \( x \in (0, 1) \)
\[
(1.12) \quad \frac{2 \cdot 3^{1/p} \sqrt{1-x}}{((2\sqrt{2})^p + (\sqrt{1+x})^p)^{1/p}} < \arccos(x) < \frac{2\pi \sqrt{1-x}}{((2\sqrt{2})^p + (\pi^p - 2^2\pi)(\sqrt{1+x})^p)^{1/p}},
\]
inequalities reverse for \( p \in [0, 4/5] \).

We give the following theorem, which refines the Carlson’s inequality, see Figure 1.
1.13. **Theorem.** For $x \in (0, 1)$,

\[(1.14) \quad \frac{1}{3} \left( 8 \sqrt{2 - \sqrt{2} \sqrt{1 + x}} \right) < \arccos(x) < \frac{2^{11/6} \sqrt{1 - x}}{(2 + \sqrt{2} \sqrt{1 + x})^{2/3}}. \]

We see that Theorem 1.13 refines the inequalities in (1.12) by using the Mathematica Software® [30].

In 1967, Shafer [36] proposed the following elementary inequality

\[(1.15) \quad \frac{3x}{1 + 2 \sqrt{1 + x^2}} < \arctan(x), \quad x > 0. \]

This inequality was proved by Grinstein, Marsh and Konhauser by different ways in [37].

In 2009, Qi et al. [25] refined the inequality (1.15) as follows,

\[(1.16) \quad \frac{(1 + a)x}{a + \sqrt{1 + x^2}} < \arctan(x) < \frac{(\pi/2)x}{4 + \sqrt{1 + x^2}}, \quad x > 0, \quad -1 < a < 1/2, \]

\[\frac{4a(1 + a^2)x}{a + \sqrt{1 + x^2}} < \arctan(x) < \frac{\max\{\pi/2, 1 + a\}x}{a + \sqrt{1 + x^2}}, \quad x > 0, \quad 1/2 < a < 2/\pi. \]

Recently, Alirezaei [2] has sharpened the Shafer’s inequality (1.15) by giving the following bounds for $\arctan$,

\[(1.17) \quad \frac{x}{4/\pi^2 + \sqrt{(1 - 4/\pi^2)^2 + 4x^2/\pi^2}} < \arctan(x) < \frac{x}{1 - 6/\pi^2 + \sqrt{(6/\pi^2)^2 + 4x^2/\pi^2}} \]

for $x \in \mathbb{R}$. Graphically, it is shown that the maximum relative errors of the obtained bounds are approximately smaller than 0.27% and 0.23% for the lower and upper bound, respectively.

Our next result refines the bounds given in (1.17), which is illustrated in Figure 2.

1.18. **Theorem.** For $x \in (0, 1)$, we have

\[(1.19) \quad \frac{1}{3} \left( 4 \sqrt{2} \sqrt{1 - \frac{1}{\sqrt{1 + x^2}}} - \frac{x}{\sqrt{1 + x^2}} \right) < \arctan(x) < \frac{2^{2/3}x}{\sqrt{1 + x^2} \left( 1 + 1/\sqrt{1 + x^2} \right)^{2/3}}. \]
2. Preliminaries

For easy reference, we recall the following Monotone l’Hôpital rule due to Anderson et al. [1, Theorem 2], which has been extremely used in the literature.

2.1. Lemma. For \(-\infty < a < b < \infty\), let \(f, g : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\), and be differentiable on \((a, b)\). Let \(g'(x) \neq 0\) on \((a, b)\). If \(f'(x)/g'(x)\) is increasing (decreasing) on \((a, b)\), then so are

\[
\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

If \(f'(x)/g'(x)\) is strictly monotone, then the monotonicity in the conclusion is also strict.

2.2. Lemma. The function

\[
f(x) = 4x \sin(x) + (4 - x^2) \cos(x) - x^2
\]

is strictly decreasing from \((0, \pi/2)\) onto \((a, 4)\), \(a = \pi(8 - \pi)/4 \approx 3.81578\). In particular,

\[
\frac{\pi(8 - \pi)/4 + x^2 - (4 - x^2) \cos(x)}{4x^2} \frac{\sin(x)}{x} < \frac{4 + x^2 - (4 - x^2) \cos(x)}{4x^2}
\]

for \(x \in (0, \pi/2)\).

Proof. By differentiating and using the identities \(\sin(x) = 2 \sin(x/2) \cos(x/2)\) and \(1 - \cos(x) = 2 \sin(x/2)^2\) we get

\[
f'(x) = x(2 \cos(x) + x \sin(x) - 2) = 2 \sin(x/2)(x \cos(x/2) - 2 \sin(x/2)) < 0.
\]

Hence \(f\) is strictly decreasing in \(x \in (0, \pi/2)\), and the limiting values can be obtained easily. \(\square\)

2.3. Lemma. The following function

\[
f(x) = \frac{\sin(x) - x \cos(x)}{2 \sin(x/2) - x \cos(x/2)}
\]

is strictly decreasing from \((0, \pi/2)\) onto \((b, 4)\), \(b = 2\sqrt{2}/(4 - \pi) \approx 3.81578\). In particular,

\[
\frac{2\sqrt{2}}{4 - \pi} \left(2 \sin \left(\frac{x}{2}\right) - x \cos \left(\frac{x}{2}\right)\right) < \sin(x) - x \cos(x) < 4 \left(2 \sin \left(\frac{x}{2}\right) - x \cos \left(\frac{x}{2}\right)\right),
\]

for \(x \in (0, \pi/2)\).

Proof. We get

\[
f'(x) = \frac{x \sin(x)}{2 \sin(x/2) - x \cos(x/2)} - \frac{x \sin(x/2)(\sin(x) - x \cos(x))}{2(2 \sin(x/2) - x \cos(x/2))^2}
\]

\[
= \frac{x \sin(x/2)(x(2 + \cos(x)) - 3 \sin(x))}{4 - (4x \sin(x) + (4 - x^2) \cos(x) - x^2)},
\]
which is negative by the second inequality of (1.1) and Lemma 2.2. This implies that \( f \) is strictly decreasing \( x \in (0, \pi/2) \), and by applying l’Hôpital rule we get the limiting values.

2.4. Lemma. The following function

\[
g(x) = \frac{8 \sin(x/2) - \sin(x)}{x}
\]

is strictly decreasing from \( (0, \pi/2) \) onto \( (\alpha, 3) \), \( \beta = (8\sqrt{2} - 2)/\pi \approx 2.96465 \). Also, the function

\[
f(z) = \frac{8 \sin(z)}{6z + \sin(2z)}
\]

is strictly decreasing from \( (0, \pi/4) \) onto \( (1, \gamma) \), \( \gamma = 8\sqrt{2} / (2 + 3\pi) \approx 0.99028 \).

Proof. We get

\[
g'(x) = \frac{4 \cos(x/2) - \cos(x)}{x} - \frac{8 \sin(x/2) - \sin(x)}{x^2}
\]

which is negative by Lemma 2.3. Thus, \( g \) is strictly decreasing in \( x \in (0, \pi/2) \), and the limiting values follow from the l’Hôpital rule.

Next, let \( f = f_1(z)/f_2(z) \), \( z \in (0, \pi/4) \), where \( f_1(z) = 8 \sin(z) \) and \( f_2(z) = 6z + \sin(2z) \). We get

\[
\frac{f_1'(z)}{f_2'(z)} = \frac{4 \cos(z)}{1 + \cos(z)^2} = f_3(z).
\]

One has,

\[
f_3'(z) = -\frac{\sin(z)^3}{(3 + \cos(2z))^2} < 0.
\]

Clearly, \( f_1(0) = f_2(0) = 0 \), hence by Lemma 2.1 \( f \) is strictly decreasing, and we get

\[
\lim_{z \to \pi/4} f(z) = 8\sqrt{2} / (2 + 3\pi) \approx 0.99028 < f(z) < \lim_{z \to 0} f(z) = 1.
\]

This implies the proof if we let \( z = x/2 \).

3. Proof of Theorems

Proof of Theorem 1.6. The proof follows easily from Lemma 2.4.

3.1. Corollary. For \( x \in (0, \pi/2) \), we have

\[
\frac{8 \sin(x/2) - \sin(x)}{\beta} < \frac{8 \sin(x/2) - \beta \sin(x)}{\gamma},
\]

where \( \beta \) and \( \gamma \) are as in Lemma 2.4.
ON CARLSON’S AND SHAFER’S INEQUALITIES

Proof. For $x \in (0, \pi/2)$, let $f(x) = \sin(x/2)/\sin(x)$. One has

$$f'(x) = \frac{\sin(x/2)^3}{\sin(x)^2} > 0.$$ 

Hence, $f$ is strictly increasing, and

$$\frac{1}{2} = \lim_{x \to 0} f(x) < f(x) < \lim_{x \to \pi/2} f(x) = \frac{1}{\sqrt{2}}.$$ 

We observe that

$$\frac{\sin(x/2)}{\sin(x)} < \frac{1}{\sqrt{2}} = \frac{2 - 8\sqrt{2} + 3\pi}{16 - 2\sqrt{2} - 3\sqrt{2}\pi},$$

which is equivalent to

$$\frac{(16 - 2\sqrt{2} - 3\sqrt{2}\pi)\sin(x/2) + (2 - 8\sqrt{2} + 3\pi)\sin(x)}{24\sqrt{2} - 6} > 0.$$ 

This is equivalent to the desired inequality. □

Proof of Theorem 1.13. Let $x = \cos(2t)$ for $0 < t < \pi/4$. Then $\arccos(x)/2 = t$, and clearly $0 < x < 1$. From (1.2) and (1.7) we have

$$8\sin(t/2) - \sin(t) < \frac{2^{2/3} \sin(t)}{(1 + \cos(t))^{2/3}},$$

for $t \in (0, \pi/2)$. Replacing $\cos(t)$, $\sin(t)$ and $t$ by $\sqrt{(1 + x)/2}$, $\sqrt{(1 - x)/2}$ and $\arccos(t)/2$, respectively, in (3.2), we get

$$\frac{8((1 - \sqrt{(1 + x)/2})/2)^{1/2} - \sqrt{(1 - x)/2}}{3} < \frac{\arccos(x)}{2} < \frac{2^{2/3} \sqrt{(1 - x)/2}}{(1 + \sqrt{(1 + x)/2})^{2/3}}.$$ 

After implication we get the desired inequality. □

Proof of Theorem 1.18. Next, let $x = \tan(t)$, $t \in (0, \pi/2)$ and $x \in (0, 1)$. Then $t = \arctan(x)$, and by using the identity $1 + \tan(t)^2 = \sec(t)^2$ we get

$$\sin(t) = \frac{x}{\sqrt{1 + x^2}} = m, \quad \sin\left(\frac{t}{2}\right) = \left(\frac{\sqrt{1 + x^2} - 1}{2\sqrt{1 + x^2}}\right) = n.$$ 

We get the desired inequalities if we replace, $t$, $\sin(t)$, $\sin(t/2)$ by $\arctan(x)$, $m$, $n$, respectively, in (3.2). □

For the comparison of the bounds of $\arccos(x)$ and $\arctan(x)$ given in (1.8) and (1.14) with the corresponding bounds appear in Theorem 1.13 and 1.18, we use the graphical method, see Figure 1 and 2.

The proof of the following corollary is the analogue of the proofs of the previous theorems, and can be obtained by utilizing (14).

3.3. Corollary. For $x \in (0, 1)$, we have
We denote the left-hand sides of (1.8) and (1.14) by $C_{\text{low}}$ and $N_{\text{low}}$, respectively, while the right-hand sides by $C_{\text{up}}$ and $N_{\text{up}}$, respectively. It is clear that (1.14) refines the Carlson’s inequality (1.8).

We denote the lower and upper bound of (1.19) by $B_{\text{low}}$ and $B_{\text{up}}$, respectively, while the corresponding bounds of (1.17) are denoted by $A_{\text{low}}$ and $A_{\text{up}}$. The differences $B_{\text{low}} - A_{\text{low}}$, $A_{\text{up}} - B_{\text{up}}$ are positive, this implies that the inequalities in (1.19) are better than the corresponding inequalities of (1.17).

\begin{align*}
(1) \quad \frac{\pi}{2} \sqrt{\frac{1-x}{2+(2/\pi)\sqrt{1+x}}} < \arccos(x) < \frac{\pi}{2} \sqrt{\frac{1-x}{(\pi/2-1)\sqrt{1+x}}} \\
(2) \quad \frac{\pi/2 x}{2/\pi + \sqrt{1+x^2}} < \arctan(x) < \frac{\pi x}{(\pi-2) + \sqrt{1+x^2}}
\end{align*}

References

[1] M. Abramowitz, and I. Stegun, eds.: Handbook of mathematical functions with formulas, graphs and mathematical tables. National Bureau of Standards, Dover, New York, 1965.

[2] G. Alirezaei: A sharp double inequality for the inverse tangent Function, http://arxiv.org/pdf/1307.4983.pdf.

[3] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: Inequalities of quasi-conformal mappings in the space, Pacific J. Math. Vol. 160 1993 No. 1, 1–20.

[4] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: Monotonicity Rules in Calculus. Amer. Math. Month. Vol. 113, No. 9 (2006), pp. 805–816.
[5] G.D. Anderson, M. Vuorinen, and X. Zhang: Topics in special functions III., http://arxiv.org/abs/1209.1696.
[6] Á. Baricz: Functional inequalities involving Bessel and modified Bessel functions of the first kind, Exposition. Math., Vol. 26 2008 No. 3, 279–293.
[7] Á. Baricz: Some inequalities involving generalized Bessel functions, Math. Ineq. Appl., Vol. 10, 2007 No. 4, pp. 827–842.
[8] Á. Baricz, and L. Zhu: Extension of Oppenheims problem to Bessel functions, J. Inequal. Appl. (2007), Article ID 82038, pp. 7.
[9] F.T. Campan: The history of number pi (Romanian), second ed., 1977, Albatros ed., Romania.
[10] B.C. Carlson: Inequality for a symmetric elliptic integral, Proc. Amer. Math. Soc. 25 (3), 1970, 698–703.
[11] C.-P. Chen, W.-S. Cheung, and W. Wang On Shafer and Carlson Inequalities, J. Ineq. Appl. Vol. 2011, Article ID 840206, 10 pp.
[12] B.-N. Guo, and F. Qi: Sharpening and generalizations of Carlson’s inequality for the arc cosine function, Hacettepe Journal of Mathematics and Statistics Vol. 39 (3) (2010), 403–409.
[13] B.-N. Guo, Q.-M. Luob, and F. Qi: Sharpening and generalizations of Shafer–Fink double inequality for the arc sine function, Filomat 27:2 (2013), 261–265 DOI 10.2298/FIL1302261G.
[14] C. Huygens: Oeuvres Completes 18881940, Société Hollondaise des Science, Haga.
[15] R. Klen, M. Visuri, and M. Vuorinen: On Jordan type inequalities for hyperbolic functions, J. Ineq. Appl. Vol. 2010, Article ID 362548, pp. 14.
[16] L. Larsson: A new Carlson type inequality, Math. Ineq. Appl. Vol. 6, No. 1 (2003), 55–79.
[17] B.J. Malesevic: One method for proving inequalities by computer, J. Ineq. Appl. (2007) 8. Article ID 78691.
[18] B.J. Malesevic: An application of λ–method on inequalities of Shafer–Fink type, Math. Ineq. Appl. 10 (3) (2007) 529–534.
[19] D.S. Mitrinovic: Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[20] E. Neuman and J. Sándor: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Ineq. Appl. Vol. 13, Number 4 (2010), 715–723.
[21] E. Neuman, and J. Sándor: Optimal inequalities for hyperbolic and trigonometric functions, Bull. Math. Analysis Appl. 3(2011), No. 3, 177–181.
[22] C. S. Ogilvy, A. Oppenheim, V. F. Ivanoff, L. F. Ford Jr., D. R. Fulkerson, and V. K. Narayan Jr.: Elementary problems and solutions: problems for solution: E1275-E1280, Amer. Math. Monthly, Vol. 64, No. 7, pp. 504–505, 1957.
[23] A. Oppenheim: E1277, Amer. Math. Monthly 64 (1957), No. 6, 504.
[24] A. Oppenheim and W. B. Carver: Elementary problems and solutions: E1277, Amer. Math. Monthly, Vol. 65, 1958, No. 3, pp. 206–209.
[25] F. Qi, Sh.-Q. Zhang, and B.-N. Guo: Sharpening and generalizations of Shafer’s inequality for the arc tangent function, J. Ineq. Appl. 2009, Article ID 930294, pp., 10.
[26] F. Qi, L.-H. Cui, and S.-L. Xu: Some inequalities constructed by Tchebysheff’s integral inequality, Math. Inequal. Appl. 2 (1999), No. 4, 517–528
[27] F. Qi, S.-Q. Zhang, and B.-N. Guo: Sharpening and generalizations of Shafer’s inequality for the arc tangent function, J. Ineq. Appl. Vol. 2009.
[28] F. Qi, Q.-M. Luo, and B.-N. Guo: A simple proof of Oppenheim’s double inequality relating to the cosine and sine functions, J. Math. Inequal., 6, 4 (2012) 645–654.
[29] R. Redheffer: Problem 5642, Amer. Math. Monthly 76 (1969) 422.
[30] H. Ruskeepää: Mathematica® Navigator. 3rd ed. Academic Press, 2009.
[31] J. Sándor: On new refinements of Kober’s and Jordan’s trigonometric inequalities, Notes Number Theory Discrete Math. Vol. 19, 2013, No. 1, 73–83.
[32] J. Sándor: Two sharp inequalities for trigonometric and hyperbolic functions, Math. Inequal. Appl., 15, 2 (2012) 409–413.
[33] J. Sándor: Sharp Cusa-Huygens and related inequalities, Notes Number Theory Discrete Math. Vol. 19, 2013, No. 1, 50–54.
[34] J. Sándor, and R. Oláh-Gál: On Cusa-Huygens type trigonometric and hyperbolic inequalities, Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 145–153.
[35] J. Sándor: Trigonometric and hyperbolic inequalities, http://arxiv.org/abs/1105.0859
[36] R. E. Shafer: Problem E1867, Amer. Math. Monthly, 74 (6) (1967), 726–727.
[37] R. E. Shafer, L. S. Grinstein, D. C. B. Marsh, and J. D. E. Konhauser: Problems and solutions: solutions of elementary problems: E1867, Amer. Math. Monthly, Vol. 74, No. 6, pp. 726–727, 1967.
[38] J. Sun, and L. Zhu: Six new Redheffer-type inequalities for circular and hyperbolic functions, Comput. Math. Appl. 56, 2 (2008) 522–529.
[39] Y. Lv, G. Wang and Y. Chua: A note on Jordan type inequalities for hyperbolic functions, Appl. Math. Lett. 25 (2012) 505–508.
[40] J.-L. Zhao, C.-F. Wei, B.-N. Guo, and F. Qi: Sharpening and generalizations of Carlson’s double inequality for the arc cosine function, Hacettepe Journal of Mathematics and Statistics Vol. 41 (2012) 2, 201–209.
[41] L. Zhu: On Shafer–Fink inequalities, Math. Ineq. Appl., 8 (4) (2005) 571–574.
[42] L. Zhu: On Shafer–Fink-type inequality, J. Ineq. Appl. (2007) 4. Article ID 67430.
[43] L. Zhu: A source of inequalities for circular functions, Comput. Math. Appl. 58 (2009) 1998–2004.
[44] L. Zhu: Sharpening Redheffer-type inequalities for circular functions, App. Math. Lett. 22 (2009) 743–748.
[45] L. Zhu: A solution of a problem of Oppenheim, Math. Inequal. Appl. 10 (2007), No. 1, 57–61.
ON JORDAN’S, CUSA-HUYGENS AND KOBER’S INEQUALITY

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. In this paper, authors refine the classical inequalities of the trigonometric functions, such as Jordan’s inequality, Cusa-Huygens inequality and Kober’s inequality.

1. Introduction

The study of the classical inequalities of the trigonometric functions such as Adamović-Mitrinović inequality, Cusa-Huygens inequality, Jordan inequality, Redheffer inequality, Becker-Stark inequality, Wilker inequality, Huygens inequality, and Kober inequality has got big attention of the numerous authors. Since last ten years, the huge number of papers on the refinement and the generalization of these inequalities have appeared, e.g. see [2, 3, 8, 9, 11, 12, 13, 16, 17, 18, 20] and the references therein. Motivated by these studies, in this paper we refine Jordan’s, Kober’s and Cusa-Huygens inequality, and our results refine the existing results in the literature.

The well-know Jordan’s inequality [10] states,

\[ \frac{\pi}{2} \leq \frac{\sin(x)}{x}, \quad 0 < x \leq \frac{\pi}{2}, \]
equality with \( x = \pi/2 \).

In 2003, Debnath and Zhao [6] refined the inequality (1.1) as below,

\[ d_1(x) = \frac{2}{\pi} + \frac{1}{12\pi}(\pi^2 - 4x^2) \leq \frac{\sin(x)}{x}, \]
for \( x \in (0, \pi/2) \), equality in both inequalities with \( x = \pi/2 \). Thereafter, another proof of the inequality (1.3) was given by Zhu in [22].

In 2006, Özban [14] proved the following inequality,

\[ o(x) = \frac{2}{\pi} + \frac{1}{\pi^3}(\pi^2 - 4x^2) + \frac{4(\pi - 3)}{\pi^3} \left( x - \frac{\pi}{2} \right)^2 \leq \frac{\sin(x)}{x}, \]
for \( x \in (0, \pi/2) \), equality with \( x = \pi/2 \).

In the same year, the following refinement of (1.1) was proved by Jiang and Yun [7],

\[ j(x) = \frac{2}{\pi} + \frac{\pi^4 - 16x^4}{2\pi^5} < \frac{\sin(x)}{x}, \quad 0 < x < \frac{\pi}{2}. \]
for $x \in (0, \pi/2)$, equality with $x = \pi/2$.

In [21], Zhang et al. gave the following inequality,

$$zw(x) = \frac{3}{\pi} - \frac{4}{\pi^3}x^2 < \frac{\sin(x)}{x}, \quad 0 < x < \frac{\pi}{2}.$$  

It is easy to see that $d_1(x) < d_2(x)$, $d_2(x) = zw(x)$ and $j(x) < d_2(x) < o(x)$ for $x \in (0, \pi/2)$.

Our first main result refines the inequality (1.4) as follows:

1.7. Theorem. For $x \in (0, \pi/2)$, we have

$$o(z) \leq 1 + \frac{16(\pi - 3)}{\pi^4}x^3 - \frac{4(3\pi - 8)}{\pi^3}x^2 \leq \frac{\sin(x)}{x},$$

equality in both inequalities holds with $x = \pi/2$.

In literature, the following inequalities

$$\left(\cos x\right)^{1/3} < \frac{\sin x}{x} < \frac{\cos x + 2}{3}, \quad 0 < \left| x \right| < \frac{\pi}{2},$$

are known as Adamović-Mitrinović inequality [10, p.238] and Cusa-Huygens [18] inequality, respectively. For the refinement of (1.8), e.g. see [8, 11, 13, 17, 18, 20] and the bibliography of these papers. Most of the refinements of (1.8) are involving very complicated upper and lower bound of $\sin(x)/x$. In the following theorem we refine (1.8) by giving the upper and lower bound of $\sin(x)/x$ in terms of simple functions, and these functions are also independent of the exponent.

1.9. Theorem. For $x \in (0, \pi/2)$, we have

$$\frac{1 + \cos(x)}{2 - \alpha x^2} < \frac{\sin(x)}{x} < \frac{1 + \cos(x)}{2 - \beta x^2},$$

with the best possible constants $\alpha = 1/6 \approx 0.166667$ and $\beta = 2(4 - \pi)/\pi^2 \approx 0.17396$.

In 1944, Kober [10, 3.4.9] established the following inequalities,

$$1 - 2\frac{x}{\pi} < \cos(x) < 1 - 2\frac{x}{\pi},$$

the first inequality is valid for $x \in (0, \pi/2)$, while the second one holds for $x \in (\pi/2, \pi)$. In literature, these inequalities are known as Kober’s inequalities.

By studying the function $x \mapsto (1 - \cos(x))/x$, $x \in (0, \pi/2)$, Sándor [19] refined the Kober’s inequality as follows:

$$\cos(x) < 1 - \frac{2}{\pi}x - \frac{2(\pi - 2)}{\pi^2}\left(x - \frac{\pi}{2}\right), \quad 0 < x < \frac{\pi}{2},$$

(1.11)

$$1 - \frac{x^2}{2} < \cos(x) < 1 - \frac{4x^2}{\pi^2}, \quad 0 < x < \frac{\pi}{2}.$$  

(1.12)

In [21], the following refinement appeared,

$$1 - \frac{4 - \pi}{\pi}x - \frac{2(\pi - 2)}{\pi^2}x^2 < \cos(x) < 1 - \frac{4}{\pi^2}x^2, \quad 0 < x < \frac{\pi}{2}.$$  

(1.13)
Applying Taylor series expansion one can have

\[(1.14) \quad 1 - \frac{x^2}{2} < \cos(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}, \quad 0 < x < \frac{\pi}{2}.\]

On the basis of Mathematica Software® \[15\], we conclude that our next result
refines the above Kober’s inequalities as follows:

1.15. Theorem. For \(x \in (0, \pi/2)\), we have

\[
1 - \frac{x^2/2}{1 + x^2/12} < \cos(x) < 1 - \frac{24x^2/(5\pi^2)}{1 + 4x^2/(5\pi^2)}.
\]

2. Preliminaries and proof of main result

The following lemma will be used in the proof of theorems.

2.1. Lemma. (1) The function

\[f_1(x) = \cos(x)^{2/3} + \frac{x^2}{3}\]

is strictly decreasing from \((0, \pi/2)\) onto \((\pi^2/12, 1)\). In particular, for \(x \in (0, \pi/2)\)

\[
\left(\frac{\pi^2 - 4x^2}{12}\right)^{3/2} < \cos(x) < \left(1 - \frac{x^2}{3}\right)^{3/2},
\]

equivalently,

\[
\frac{x^2 \cos(x)^{1/3} + 3 \cos(x)}{3} \ < \cos(x)^{1/3} \ < \frac{4}{\pi^2} \left(\frac{x^2 \cos(x)^{1/3} + 3 \cos(x)}{3}\right).
\]

(2) The function

\[f_2(x) = 1 + \cos(x)^{4/3} - 2 \frac{\sin(x)^2}{x^2}\]

is strictly increasing from \((0, \pi/2)\) onto \((0, 1 - 8/\pi^2)\). In particular, for \(x \in (0, \pi/2)\)

\[
\frac{\cos(x)^{4/3} + 8/\pi^2}{2} < \frac{\sin(x)^2}{x^2} < \frac{1 + \cos(x)^{4/3}}{2},
\]

and

\[
\frac{4(1 - \cos(x))}{x^2} - 1 < \cos(x)^{4/3} < \frac{4(1 - \cos(x))}{x^2} + \frac{1}{2^{2/3}} - \frac{16}{\pi^2}.
\]

Proof. By Adamović-Mitrinović inequality, we get

\[f_1'(x) = \frac{2x}{3} - \frac{2 \sin(x)}{3 \cos(x)^{1/3}} = \frac{2x}{3} \left(1 - \frac{\sin(x)/x}{\cos(x)^{1/3}}\right) < 0.\]

Since, \(f_1\) is strictly decreasing in \(x \in (0, \pi/2)\), and the limiting values are clear.

Next for the proof of part (2), we get

\[f_2'(x) = -\frac{4 \sin(x)}{x^3} \left(\frac{x^3 \cos(x)^{1/3}}{3} + x \cos(x) - \sin(x)\right) > -\frac{4 \sin(x)}{x^3} (\cos(x)^{1/3} - \sin(x)/x) > 0,
\]
by part (1). Thus, \( f_2 \) is strictly increasing in \( x \in (0, \pi/2) \), and the limiting values follow easily. Last inequalities follow if we consider \( f_2(x/2), x \in (x, \pi/2) \). □

Proof of Theorem 1.9. For \( x \in (0, \pi/2) \), let

\[
g_1(x) = \frac{(1 + \cos(x))}{x \sin(x)} - \frac{2}{x^2}.
\]

By Lemma 2.1(2) we get

\[
g_1'(x) = \frac{4}{x^3} - \frac{1}{x} - \frac{1 + \cos(x)}{x^2 \sin(x)} - \frac{(1 + \cos(x)) \cos(x)}{x \sin(x)}
\]

\[
= \frac{4(1 - \cos(x))}{x^2} - \frac{1 - \sin(x)/x}{x(1 + \cos(x))} < \frac{\cos(x)^{4/3} - \sin(x)/x}{x(1 + \cos(x))} < 0.
\]

Since \( g_1 \) is strictly decreasing in \( x \in (0, \pi/2) \), and by applying l'Hôpital rule, we get the limiting values. This completes the proof. □

2.2. Lemma. The function

\[
f_3(x) = \frac{x - \sin(x)}{x^3}
\]

is strictly decreasing and concave from \( (0, \pi/2) \) onto \( (1/\pi^2, 1/6) \). In particular, for \( x \in (0, \pi/2) \)

\[
1 - \frac{x^2}{6} < \frac{\sin(x)}{x} < 1 - \frac{x^2}{\pi^2}.
\]

Proof. One has,

\[
f_3'(x) = \frac{1 - \cos(x)}{x^3} - \frac{3x - \sin(x)}{x^4} = \frac{3 \sin(x)/x - 2 + \sin(x)}{x^4},
\]

which is negative by Cusa-Huygens inequality, hence \( f_3 \) is strictly decreasing in \( x \in (0, \pi/2) \). Again

\[
f_3''(x) = \frac{2 \cos(x) + x \sin(x) - 2}{x^5} - \frac{4}{x^5}(3 \sin(x) - x(2 + \cos(x)))
\]

\[
= \frac{6x(1 + \cos(x)) - (12 - x^2) \sin(x)}{x^5},
\]

which is negative by Theorem 1.9 this implies the concavity of the function \( f_3 \). □

Proof of Theorem 1.7. Using the concavity of the function \( f_3(x), x \in (0, \pi/2) \), the tangent line on point \((\pi/2, f_3(\pi/2))\) is above the graph of \( f_3(x) \) on \((0, \pi/2)\). The equation of the tangent line is

\[
y = \frac{4(\pi - 2)}{\pi^3} + \frac{16(3 - \pi)}{\pi^4}(x - \pi/2).
\]

After some computations, we get the desired inequality. The first inequality is equivalent to

\[\frac{-4(\pi - 3)(\pi - 2)x^2}{\pi^4} < 0,\]
which is obvious. This completes the proof. □

Proof of Theorem 1.15. For \( x \in (0, \pi/2) \), let

\[
f(x) = \frac{x^2 (5 + \cos(x))}{1 - \cos(x)}.
\]

Differentiation gives

\[
f'(x) = \frac{2x(5 - g(x))}{(\cos(x) - 1)^2},
\]

where

\[
g(x) = \cos(x)(4 + \cos(x)) + 3x \sin(x).
\]

We get

\[
g'(x) = 3x \cos(x) - (1 + 2 \cos(x)) \sin(x)
= x \cos(x) \left( 3 - \left( \frac{2 \sin(x)}{x} + \frac{\tan(x)}{x} \right) \right),
\]

which is negative by so-called Huygens inequality \([13]\]

\[
2 \frac{\sin(x)}{x} + \frac{\tan(x)}{x} > 3, \quad 0 < x < \frac{\pi}{2}.
\]

Thus, \( g \) is decreasing and \( \lim_{x \to 0} g(x) = 5 \), and as a result \( f' > 0 \). This implies that \( f \) is strictly increasing. Applying l'Hôpital rule we get

\[
12 = \lim_{x \to 0} f(x) < f(x) < \lim_{x \to 0} f(x) = \frac{5\pi^2}{4} \approx 12.33701,
\]

which is equivalent to

\[
\frac{6}{1 + x^2/12} - 5 < \cos(x) < \frac{6}{1 + 4x^2/(5\pi^2)} - 5.
\]

This implies the desired inequalities. □

References

[1] M. Abramowitz, and I. Stegun, eds.: Handbook of mathematical functions with formulas, graphs and mathematical tables, National Bureau of Standards, Dover, New York, 1965.
[2] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: Monotonicity Rules in Calculus, Amer. Math. Month. Vol. 113, No. 9 (2006), pp. 805–816.
[3] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: Monotonicity of some functions in calculus, preprint, https://www.math.auckland.ac.nz/Research/Reports/Series/538.pdf.
[4] G.D. Anderson, M. Vuorinen, and X. Zhang: Topics in special functions III., http://arxiv.org/abs/1209.1696.
[5] C.-P. Chen, J. Sándor: Inequality chains for Wilker, Huygens and Lazarević type inequalities, J. Math. Inequal., 8, 1 (2014) 55–67.
[6] L. Debnath, and C.-J. Zhao: New strengthened Jordan’s inequality and its applications, Appl. Math. Lett., vol. 16, no. 4, 2003, pp. 557–560.
[7] W.-D. Jiang, and H. Yun: Sharpening of Jordans Inequality and its Applications, J. Ineq. Pure and Appl. Math. 7(3) Art. 102, 2006.

[8] R. Klén, M. Visuri, and M. Vuorinen: On Jordan type inequalities for hyperbolic functions, J. Ineq. Appl., vol. 2010, pp. 14.

[9] J.-L. Li, and Y.-L. Li: On the Strengthened Jordan’s Inequality, J. Ineq. Appl. 2007, Art. ID 74328, pp. 8.

[10] D.S. Mitrinovic: Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[11] E. Neuman: Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions, Advances in Inequalities and Applications, 1 (2012), No. 1, 1–11.

[12] E. Neuman: Inequalities Involving Hyperbolic Functions and Trigonometric Functions, Bull. Int. Math. Vir. Int. Vol. 2(2012), 87–92.

[13] E. Neuman, and J. Sándor: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Inequal. Appl. Vol. 13, Number 4 (2010), 715–723.

[14] A. Y. Özban: A new refined form of Jordan’s inequality and its applications, Appl. Math. Lett., vol. 19 (2006), 155–160, 2006.

[15] H. Ruskeepää: Mathematica® Navigator. 3rd ed. Academic Press, 2009.

[16] J. Sándor: Sharp Cusa-Huygens and related inequalities, Notes Number Theory Discr. Math. Vol. 19, 2013, No. 1, 50–54.

[17] J. Sándor, and R. Oláh-Gál: On Cusa-Huygens type trigonometric and hyperbolic inequalities, Acta Univ. Sapientiae, Mathematica, 4, 2 (2012) 145–153.

[18] J. Sándor, and M. Bencze: On Huygens’ trigonometric inequality, RGMIA Res. Rep. Collection, 8 (2005), No. 3, Art. 14.

[19] J. Sándor: On new refinements of Kober’s and Jordan’s trigonometric inequalities, Notes Number Th. Discr. Math. 19(2013), no.1, 73–83.

[20] Z.-H. Yang: Refinements of a two-sided inequality for trigonometric functions, J. Math. Inequal., 7, 4 (2013) 601–615

[21] X. Zhang, G. Wang, and Y. Chu: Extensions and sharpenings of Jordan’s and Kober’s inequalities, J. Ineq. Appl. Pure. Math. Vol. 7, Issue 2, Art. 63, 2006.

[22] L. Zhu: Sharpening Jordan’s inequality and the Yang Le inequality, Appl. Math. Lett., 19 (2006), 240243.
ON AN INEQUALITY OF REDHEFFER

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. We offer two new proofs of famous Redheffer’s inequality, as well establish two converse inequalities for it. Also a hyperbolic analogue is pointed out.

2010 Mathematics Subject Classification: 26D05, 26D07, 26D99.
Keywords: Trigonometric and hyperbolic functions, Redheffer’s inequality, Cusa-Huygens inequality.

1. Introduction

In 1969, Redheffer [9] proposed the following inequality

\[ \frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin x}{x}, \quad x \in \mathbb{R}, \]

which was proved by Williams [13] in the same year. In literature, this inequality is known as Redheffer’s inequality. By using the infinite product and induction method, Williams verified this inequality also in 1969, [14]. Motivated by his work, many developments such as generalizations, refinements and applications took place, e.g., see [5].

Thereafter some Redheffer-type inequalities for other trigonometric, hyperbolic and Bessels function were established, e.g., see [3, 4, 15, 12] and the references therein.

Recently a new proof of (1.1) has appeared in [6], where authors are using the Lagrange mean value theorem, combined with induction, which is very complicated for the reader.

The inequality (1.1) is valid for all \( x \in \mathbb{R} \). It is immediate that we may assume \( x > 0 \) and \( x \in (0, \pi) \) as for \( x > \pi \), we may let \( x = \pi + t \) for \( t > 0 \), then inequality (1.1) becomes

\[ \frac{\sin t}{t} < \frac{2\pi^2 + 3\pi t + t^2}{2\pi^2 + 2\pi t + t^2}. \]

This is obvious, as right side is greater than one, and left side less than one. Thus, we may consider \( x \in (0, \pi) \).

So far, all the authors have given the proof of (1.1) by using the induction method. From our proof, it is obvious that the induction method is not needed. In this paper we give new interesting proofs of (1.1), which are based on the elementary calculus. The authors think that this proof could be one from the “Book” (See [2]).
2. New Proof of Inequality and Its Converse

2.1. Lemma. For all \( x \in (0, \pi) \) one has

\[
x + \sin(x) > x^2 \frac{\cos(x/2)}{\sin(x/2)}.
\]

Proof. Let \( h_1(x) = x + \sin(x) - x^2 \cos(x/2)/\sin(x/2) \). Then

\[
h_1'(x) = 1 + \cos(x) - \frac{2x \sin(x) - x^2}{2 \sin(x/2)^2} = \frac{\sin(x)^2 + x^2 - 2x \sin(x)}{2 \sin(x/2)^2} > 0,
\]

where we have used \( 1 + \cos(x) = 2 \cos(x/2)^2 \) and \( 2 \sin(x/2) \cos(x/2) = \sin(x) \). As \( h_1(x) > h_1(0) = 0 \), the result follows. \( \square \)

2.2. Theorem. The following inequalities

\[
\frac{\pi^2 - x^2}{\pi^2 + x^2} \leq \frac{\sin(x)}{x} < \frac{12 - x^2}{12 + x^2}
\]

hold for \( x \in (0, \pi] \).

Proof. Let

\[
f_1(x) = \frac{x^2(x + \sin(x))}{x - \sin(x)},
\]

for \( x \in (0, \pi] \). After some elementary computations, one has

\[
\frac{(x - \sin(x))^2}{2x} f'(x) = x^2(1 + \cos(x)) - \sin(x) (x + \sin(x)) = g_1(x).
\]

As \( 1 + \cos(x) = 2 \cos(x/2)^2 \), and \( \sin(x) = 2 \sin(x/2) \cos(x/2) \), we get

\[
g_1'(x) = 2 \cos\left(\frac{x}{2}\right) \left[ x^2 \cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right) (x + \sin(x)) \right]
= 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) \left[ x^2 \cos(x/2) \sin(x/2) - (x + \sin(x)) \right] < 0
\]

by Lemma 2.1. Thus \( f_1'(x) < 0 \), and \( f_1 \) is strictly decreasing in \( x \in (0, \pi] \). We get

\[
f_1(x) > \lim_{x \to \pi} f_1(x) = \pi^2,
\]

which is equivalent to

\[
x + \sin(x) > \frac{\pi^2}{x^2},
\]

thus the first inequality in (2.3) follows. The second inequality in (2.3) follows similarly from \( f_1(x) < \lim_{x \to 0} f_1(x) = 12 \). \( \square \)

The right side of (2.3) improves the known inequality:

\[
\frac{\sin(x)}{x} < 1 - \frac{x^2}{\pi^2}, \quad 0 < x < \frac{\pi}{2}
\]

Indeed, this follows by \( x^2 + 12 < 2\pi^2 \), which is true, as \( \pi^2/4 + 12 < 2\pi^2 \) becomes \( 48 < 7\pi^2 \).
2.4. Proposition. The second inequality in (2.3) refines the relation
\[ \frac{\sin(x)}{x} < \frac{\cos(x) + 2}{3}, \quad 0 < x < \frac{\pi}{2}, \]
so-called Cusa-Huygens inequality [8, 11].

Proof. It is equivalent to prove that,
\[ (12 + x^2) \cos(x) + 5x^2 - 12 = s(x) > 0. \]
One has successively:
\[ s'(x) = 2x(5 + \cos(x)) - (12 + x^2) \sin(x) - 12, \]
\[ s''(x) = -4x \sin(x) - (10 + x^2) \cos(x) + 10, \]
\[ s'''(x) = x^2 \sin(x) + 6(\sin(x) - x \cos(x)) > 0, \]
as \( \sin(x) > x \cos(x) \) (i.e. \( \tan(x) > x \)). Thus we get
\[ s''(x) > s''(0) = 0, \quad s'(x) > s'(0) = 0, \]
and finally \( s(x) > s(0) = 0. \) \( \square \)

3. An other proof of (1.1)

3.1. Lemma. Let \( g(x) = f(x)/\sin(x) \) for \( x \in (0, \pi) \). Then \( \sin(x)^2 g'(x) = h(x) \), and the sign of \( h'(x) \) depends on the sign of \( F(x) = f(x) + f''(x) \).

Proof. One has
\[ \sin(x)^2 g'(x) = f'(x) \sin(x) - f(x) \cos(x) = h(x), \]
and
\[ h'(x) = (f(x) + f''(x)) \sin(x) = F(x) \sin(x). \]
As \( \sin(x) > 0 \) for all \( x \in (0, \pi) \), the result follows. \( \square \)

3.2. Theorem. For \( x \in (0, \pi) \), we have
\[ \frac{x^2}{\pi^2 + x^2} < \frac{\sin(x)}{x} < c_1 \frac{x^2}{\pi^2 + x^2}, \]
where \( c_1 = 1.07514. \)

Proof. Let \( g(x) = f(x)/\sin(x) \) for \( x \in (0, \pi) \), where
\[ f(x) = \frac{x(\pi^2 - x^2)}{\pi^2 + x^2}. \]
One has
\[ f'(x) = \frac{\pi^4 - 4\pi^2 x^2 - x^4}{(\pi^2 + x^2)^2}, \]
so we get
\[ F(x) = f(x) + f''(x) = \frac{x p(x)}{(\pi^2 + x^2)^3}, \]
i.e., by Lemma 3.1 the sign of \( h'(x) \) depends on the sign of \( p(x) \). Here
\[
h(x) = f'(x) \sin(x) - f(x) \cos(x) = \frac{x(\pi^4 - x^4) \cos(x) + (-x^4 - 4\pi^2x^2 + \pi^4) \sin(x)}{(x^2 + \pi^2)^2} = \frac{k(x)}{(x^2 + \pi^2)^2}
\]
and
\[
p(x) = -x^6 - \pi^2 x^4 + \pi^2 x^2 (\pi^2 + 4) - 12\pi^4 + \pi^6.
\]
Here \( h(0) = h(\pi) = 0 \) and \( h(x_0) = 0 \).

Elementary computation gives \( k(2\pi/3) < 0 \), while with the use of a computer we get \( k(17\pi/24) > 0 \). So \( k(x) \) has a root \( x_0 \) between \( 2\pi/3 \approx 2.0944 \) and \( 17\pi/24 \approx 2.22529 \).

Now, by letting \( y = x^2 \), we get
\[
p(x) = q(y) = -y^3 - \pi^2 y^2 + \pi^2 y(\pi^2 + 4) + \pi^6 - 12\pi^4.
\]
Since
\[
q'(y) = -3y^2 - 2\pi^2 y + \pi^2 (\pi^2 + 4),
\]
and \( y > 0 \), the only root of \( q'(y) = 0 \) is \( y^* = (2\pi\sqrt{3+\pi^2+3-\pi^2})/3 \), which lies between \( \pi \) and \( \pi^2 \). One can verify that \( q(0) < 0 \) and \( q(\pi) > 0 \). As \( q'(y) > 0 \) for \( y \in (0,y^*) \) and \( < 0 \) in \( y \in (y^*,\pi^2) \), and as \( q(\pi^2) < 0 \), we get the following: \( q \) is increasing from \( q(0) \) to \( q(y^*) \) and decreasing from \( q(y^*) \) to \( q(\pi^2) \).

Thus there exist only two roots in \((0,\pi^2)\) to \( q(y) \), let them \( z_1 \) and \( z_2 \). Clearly, \( z_1 \) is in \((0,\pi)\) and \( y_2 \) in \((y^*,\pi^2)\). More precisely, as \( q(6) < 0 \) and \( q(7) > 0 \), one finds that \( y_2 > 6 \). These imply that \( q(y) < 0 \) in \((0,z_1)\), \( > 0 \) in \((z_1,z_2)\) and \( < 0 \) in \((z_2,\pi^2)\). In terminology of \( P(x) \), we get that \( p(x) < 0 \) in \((0,\sqrt{z_1})\), \( > 0 \) in \((\sqrt{z_1},\sqrt{z_2})\), and \( < 0 \) in \((\sqrt{z_2},\pi)\).

In \((0,\sqrt{z_1})\) clearly \( h'(x) < 0 \), so \( h(x) < h(0) = 0 \); similarly in \((z_2,\pi)\) one has \( h'(x) < 0 \), so \( h(x) > h(\pi) = 0 \). Remains the interval \((\sqrt{z_1},\sqrt{z_2})\). As \( z_1 < \pi \) and \( z_2 > 6 \), we get that
\[
\sqrt{z_1} < \sqrt{\pi} < 2 < x_0 < \sqrt{6} \approx 2.44949 < \sqrt{z_2},
\]
so we find that \( x_0 \) lies between \( \sqrt{z_1} \) and \( \sqrt{z_2} \). Then clearly \( h(x) < h(x_0) = 0 \) in \((\sqrt{z_1},x_0)\), and \( h(x) > h(x_0) = 0 \) in \((x_0,\sqrt{z_2})\), so all is done.

Thus the minimum point of \( g \) is at \( x_0 \). As \( g(x) \) tends to 1 when \( x \) tends to 0 or \( \pi \), thus the Redheffer’s inequality follows. On the other hand, we get also \( g(x) \geq g(x_0) \), i.e. the best possible converse to Redheffer’s inequality. Now, with the aid of a computer one can find the more precise approximation \( x_0 \approx 2.12266 \), giving \( g(x_0) \approx 0.93012 = 1/\epsilon_1 \) so the converse to the Redheffer’s inequality holds true. □

3.3. Lemma. For \( a = 2.175 \), the function
\[
Q(x) = -3ax^{2a+2} - 2a\pi^a x^{a+2} + 2a(a-1)(2a-1)\pi^a x^a + a\pi^{2a} x^a - 2a(a+1)(a-2)\pi^{2a}
\]
has exactly two roots \( y_1 \) and \( y_2 \) in \((0,\pi)\).
Proof. We have $Q(1/2) < 0$, $Q(\pi/2) > 0$, and $Q(\pi) < 0$, so $Q$ has two roots $y_1$ in $(1/2, \pi/2)$, resp. $y_2$ in $(\pi/2, \pi)$. To show that $Q$ has no other zeros, we have to consider $Q'(x) = xR(x)$, where

$$R(x) = -3a(2a + 2)x^{2a} - 2a(a + 2)\pi^a x^a - 2a^2(a - a)(2a - 1)\pi^a x^{a-2} + 2a\pi^{2a}.$$ 

One has further

$$R'(x) = 2a^2x^{a-3}T(x),$$

here

$$T(x) = -3(2a + 2)x^{a+2} - (a + 2)\pi^a x^{2a} - 2a^2(a - 1)(a - 2)(2a - 1)\pi^a.$$ 

Since $a > 2$, we get $T(x) < 0$, so $R'(x) < 0$. One has $Q'(x) = xR(x)$, where $R'(x) < 0$. Since $R(0) > 0$ and $R(\pi) < 0$, and $R(x)$ is strictly decreasing, $R(x) = 0$ can have exactly one root $r$ in $(0, \pi)$. Therefore, $Q(x)$ has exactly one extremal point. Since $Q(0) < 0$ and $Q(\pi) < 0$ and $Q$ takes also positive values, clearly $Q(r)$ will be a maximum of $Q(x)$. This shows that $Q$ has exactly two roots in $(0, \pi)$: one in $(0, r)$ and the other one in $(r, \pi).$ 

\[\Box\]

3.4. Theorem. For $x \in (0, \pi)$, the following inequality holds

$$\frac{\sin(x)}{x} < \frac{\pi^a - x^a}{\pi^a + x^a},$$

where $a = 87/40 = 2.175$.

Proof. Inequality can be written as $g(x) = f(x)/\sin(x) > 1$, where

$$f(x) = x\left(\frac{\pi^a - x^a}{\pi^a + x^a}\right).$$

First of all, similarly to the proof of Theorem 3.2, one has

$$f(x) = \frac{\pi^{2a} - 2a\pi^a x^a - x^{2a}}{(\pi^a + x^a)^2},$$

so we get

$$h(x) = \frac{K(x)}{(\pi^a + x^a)^2},$$

where

$$K(x) = (\pi^2 - 2a\pi^a x^a - x^{2a})\sin(x) - x(\pi^{2a} - x^{2a})\cos(x).$$

One finds

$$F(x) = \frac{xP(x)}{(\pi^a + x^a)^2},$$

Where

$$P(x) = -x^{3a} - \pi^a x^{2a} + 2a(a - 1)\pi^a x^{a-2} + \pi^{2a} x^{a-2} - 2a(a + 1)\pi^{2a} x^{a-2} + \pi^{3a}.$$
Now the proof of Theorem runs as follows: Since \( K(1/2) > 0 \) and \( K(\pi/4) < 0 \), \( K(\pi/2) > 0 \) and \( K(2\pi/3) < 0 \), we get \( x_1 \) in \((1/2, \pi/4)\) such that \( k(x_1) = 0 \) and \( x_2 \) in \((\pi/2, 2\pi/3)\) such that \( K(x_2) = 0 \). As \( P(0) > 0 \) and \( P(\pi/4) < 0 \). One has

\[
P'(x) = xQ(x),
\]

where \( Q(x) \) is as in Lemma 3.3 It follows from Lemma 3.3 that that, \( Q(x) < 0 \) for \( x \) in \((0, y_1)\) and \((y_2, \pi)\), and \( Q(x) > 0 \) for \( x \) in \((y_1, y_2)\). This shows that \( P(x) \) is strictly decreasing in \((0, y_1)\) and \((y_2, \pi)\) and strictly increasing in \((p_1, p_2)\). This implies that \( P(x) \) has a unique root \( p_1 \) in \((0, y_1)\), as well as a unique \( p_2 \) in \((y_1, y_2)\) and \( p_3 \) in \((y_2, \pi)\). By approximate computation we can see that \( p_1 < 2/3, p_2 > 1 \) and \( p_3 > 2, \) and \( p_1 < p_2 < p_3 \). This shows that for the roots \( x_1 \) and \( x_2 \) of function \( f(x) \) one has that \( x_1 \) is in \((p_1, p_2)\) and \( x_2 \) in \((p_2, p_3)\). As \( h(0) = h(x_1) = h(x_2) = h(\pi) = 0 \), we get the following;

1. for \( x \in (0, p_1) \) one has \( h(x) > h(0) = 0 \),
2. for \( x \in (p_1, x_1) \) one has \( h(x) > h(x_1) = 0 \),
3. for \( x \in (x_1, p_2) \) one has \( h(x) < h(x_1) = 0 \),
4. for \( x \in (p_2, x_2) \) one has \( h(x) < h(x_2) = 0 \),
5. for \( x \in (x_2, p_3) \) one has \( h(x) > h(x_2) = 0 \),
6. for \( x \in (p_3, \pi) \) one has \( h(x) > h(\pi) = 0 \).

From the above it follows that \( x_1 \) is a local minimum point, while \( x_2 \) a local maximum point of \( g(x) \). Clearly, \( g(x_1) > \lim g(x) \), when \( x \) tends to zero, = 1 and \( \lim g(x) \) at \( x = \pi \) is \( a/2 > 1 \). This completes the proof.

\[\square\]

4. A HYPERBOLIC ANALOGUE

4.1. Lemma. For \( x \in (0, \infty) \),

\[x + \sinh(x) > x^2 \coth(x/2).
\]

Let \( h_2(x) = x + \sinh(x) - x^2 \coth(x/2) \).

**Proof.** One has

\[
h_2'(x) = \frac{\sinh(x) - x^2}{2 \sinh(x/2)^2} > 0,
\]

and \( h_2(x) > \lim_{x \to 0} h_2(x) = 0 \). Thus, inequality holds. \[\square\]

4.2. Theorem. For \( x \in (0, \infty) \), we have

\[\frac{\sinh(x) + x}{\sinh(x) - x} > \frac{12}{x^2}.
\]

**Proof.** Let

\[
f_2(x) = \frac{x^2(\sinh(x) + x)}{\sinh(x) - x},
\]

\[\square\]
for $x \in (0, \infty)$. We get
\[
\frac{(\sinh(x) - x)^2}{2x} f'_2(x) = (\sinh(x/2))((x + \sinh(x)) - x^2 \cosh(x/2)^2),
\]
which is positive by Lemma 4.1. Thus, $f_2$ is strictly increasing in $x \in (0, \infty)$, and the inequality (4.3) follows from $f_2(x) > \lim_{x \to 0} f_2(x) = 12$. Therefore, we have an analogue of the second inequality in the circular case, and this is (4.3). When $x^2 < 12$, then (4.3) becomes
\[
\frac{\sinh(x)}{x} < \frac{12 + x^2}{12 - x^2}.
\]
□

It is interesting to observe that
\[
\frac{12 + x^2}{12 - x^2} < \frac{\pi^2 + x^2}{\pi^2 - x^2},
\]
if $x < \pi$. Indeed this becomes: $\pi^2 < 12$.

References

[1] M. Abramowitz, I. Stegun, eds.: Handbook of mathematical functions with formulas, graphs and mathematical tables, National Bureau of Standards, Dover, New York, 1965.

[2] M. Aigner and G. M. Ziegler: Proofs from THE BOOK, 4th ed. 2010, VIII, 274 p. 250 illus.

[3] Á. Baricz: Redheffer Type Inequality for Bessel Functions, JIPAM. Vol. 8, 1, Art 11, 2007.

[4] C.-P. Chen, J.-W. Zhao, F. Qi: Three inequalities involving hyperbolically trigonometric functions, RGMIA Res. Rep. Coll. 6 (3) (2003) 437–443. Art. 4.

[5] J. Kuang: Applied inequalities, Shandong Science and Technology Press, Ji’nan City, China, third ed., 2004

[6] L. Li and J. Zhang: A new proof on Redheffer-Williams’ inequality, Far East J. Math. Sci. 56(2011), no. 2, 213–217.

[7] D.S. Mitrinović: Analytic Inequalities, Springer-Verlag, Berlin, 1970.

[8] E. Neuman and J. Sándor: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Inequal. Appl. Vol. 13, Number 4 (2010), 715–723.

[9] R. Redheffer: Problem 5642, Amer. Math. Monthly 76 (1969) 422.

[10] R. Redheffer: R. Redheffer, Correction, Amer. Math. Monthly 76 (4) 1969, 422.

[11] J. Sándor and M. Bencze: On Huygens trigonometric inequality, RGMIA Res. Rep. Collection, 8 (2005), No. 3, Art. 14.

[12] J. Sun and L. Zhu: Six new Redheffer-type inequalities for circular and hyperbolic functions, Comput. Math. Appl. 56, 2 (2008) 522–529.

[13] J.P. Williams: Solution of problem 5642, Amer. Math. Monthly 76 (1969) 1153–1154.

[14] J.P. Williams: J.P. Williams, A delightful inequality, Amer. Math. Monthly 76(10) (1969), 1153–1154

[15] L. Zhu: Sharpening Redheffer-type inequalities for circular functions, App. Math. Lett. 22 (2009) 743–748.
ON THE INEQUALITIES FOR BETA FUNCTION

BARKAT ALI BHAYO AND JÓZSEF SÁNDOR

Abstract. Here authors establish the sharp inequalities for classical beta function by studying the inequalities of trigonometric sine function.

1. Introduction

For $x, y > 0$, the classical gamma function $\Gamma$, the digamma function $\psi$ and the beta function $B(. , .)$ are defined by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

respectively. The study of these functions has become crucial because of its dynamic applications in the field of various branches of engineering and mathematics \cite{6}. Since last half century numerous authors has given the several functional inequalities of these function by using different approaches, e.g., see \cite{3, 4, 7, 12}. In this paper, we establish the inequalities for beta function by studying the well-known Jordan’s inequality \cite{9, 10, 11}.

The functions $\Gamma$ and $\psi$ satisfy the following recurrence relation

$$\Gamma(1 + x) = x\Gamma(x), \quad \psi(1 + x) = x\psi(x). \tag{1.1}$$

Weierstrass gave the following infinite production definitions of gamma function and the sine function

$$\frac{1}{\Gamma(x)} = xe^{\gamma x} \prod_{n=1}^\infty \left(1 + \frac{x}{n}\right) e^{-x/e}, \quad \sin(\pi x) = \pi x \prod_{n \neq 0} \left(1 - \frac{x}{n}\right) e^{x/n},$$

where $\gamma$ is the Euler-Mascheroni constant \cite{1} defined by

$$\gamma = \lim_{x \to \infty} \left(\sum_{k=1}^{\infty} \frac{1}{k} - \log(n)\right) \approx 0.57721.$$

These definitions gives the following relation

$$\Gamma(t)\Gamma(1 - t) = \frac{\pi}{\sin(\pi t)}, \quad t \notin \mathbb{Z}, \tag{1.2}$$

which is known as the Euler’s reflection formula \cite{11 6.1.17}. We refer to reader to see \cite{7} for the historical background and the properties of gamma and beta function.
Dragomir et al. [2] established the following inequality

\begin{equation}
B(x, y) \leq \frac{1}{xy}, \quad x, y \in (0, 1),
\end{equation}

which was refined by Alzer [2] as follows

\begin{equation}
\frac{1}{xy} \left( 1 - a \frac{1 - x}{1 + x} - y \right) < B(x, y) < \frac{1}{xy} \left( 1 - b \frac{1 - x}{1 + x} - y \right), \quad x, y \in (0, 1),
\end{equation}

with the best possible constants \( a = 2\pi^2 / 3 - 4 \approx 2.57973 \) and \( b = 1 \). Recently, the second inequality in (1.3) was refined by Ivády [8]

\begin{equation}
\frac{1}{xy} (x + y - xy) \leq B(x, y) \leq \frac{1}{xy} \frac{x + y}{1 + xy}, \quad x, y \in (0, 1).
\end{equation}

1.6. **Lemma.** Let \( g(x) = f(x) / \sin(x) \) for \( x \in (0, \pi) \). Then \( \sin(x)^2 g'(x) = h(x) \), and the sign of \( h'(x) \) depends on the sign of \( F(x) = f(x) + f''(x) \).

**Proof.** One has

\[ \sin(x)^2 g'(x) = f'(x) \sin(x) - f(x) \cos(x) = h(x), \]

and

\[ h'(x) = (f(x) + f''(x)) \sin(x) = F(x) \sin(x). \]

As \( \sin(x) > 0 \) for all \( x \in (0, \pi) \), the result follows. \( \square \)

1.7. **Theorem.** For \( x \in (0, 1) \), we have

\begin{equation}
\frac{3(1 - t)}{\pi t^2 - \pi t + \pi} < \frac{\sin(\pi t)}{\pi t} < \frac{\pi(1 - t)}{\pi t^2 - \pi t + \pi},
\end{equation}

\begin{equation}
1 - (2 - t)^2 < \frac{\sin(\pi t)}{\pi t} < \frac{16}{5\pi} (1 - (2 - t)^2).
\end{equation}

**Proof.** Let \( g(x) = f(x)/\sin(x) \) for \( x \in (0, \pi) \), where

\[ f(x) = (\pi x - x^2)/(\pi^2 - \pi x + x^2). \]

We get

\[ \frac{(\pi^2 - \pi x + x^2)^3}{x(\pi - x)} F(x) = (\pi^2 - \pi x + x^2)^2 - 6\pi^2 = A(x) B(x), \]

where \( B(x) > 0 \) always, and \( A(x) = x^2 - \pi x + \pi^2 - \pi \sqrt{6} \). The roots of equation \( A(x) = 0 \) are \( x_1 = (\pi - \sqrt{4 \cdot 6^{1/2} \pi^2 - \pi^2})/2 \) which is in \((0, \pi/2)\), and \( x_2 = (\pi + \sqrt{4 \cdot 6^{1/2} \pi^2 - \pi^2})/2 \) which is in \((\pi/2, \pi)\). Let \( x \in (0, x_1) \), then \( A(x) > 0 \), so \( F(x) > 0 \), giving \( h'(x) > 0 \). This implies \( h(x) > h(0) = 0 \), so \( g'(x) > 0 \) by Lemma 1.6. Again, let \( x \in [x_1, \pi/2] \), then \( A(x) \geq 0 \), giving \( F(x) \leq 0 \), i.e. \( h'(x) \leq 0 \). This implies \( h(x) > h(\pi/2) = 0 \). So \( g'(x) \geq 0 \) here too. We have proved that \( g'(x) > 0 \) for all \( x \in (0, \pi/2) \). Let now \( x \) in \((\pi/2, x_2)\). Then \( A(x) < 0 \), so \( h'(x) < 0 \), implying \( h(x) < h(\pi/2) = 0 \). For \( x \) in \([x_2, \pi]\) one has \( h'(x) \geq 0 \), so \( h(x) \leq h(\pi) = 0 \). Therefore, for all \( x \) in \((\pi/2, \pi)\) one has \( h(x) < 0 \), i.e. \( g'(x) < 0 \) here. In both cases we
had \( g'(x) = 0 \) only for \( x = \pi/2 \). Consequently, the function \( g \) is strictly increasing in \((0, \pi/2)\) and strictly decreasing decreasing \((\pi/2, \pi)\), and attains maximum 1/3 at \( x = \pi/2 \) as well as \( g \) tends to 1/\(\pi\) when \( x \) tends to 0 or \( \pi \). This implies the proof of (1.8) if we let \( x = \pi t \).

For the proof of (1.9), let
\[
  f(x) = \frac{\sin(x)}{x(x^3 - 2\pi x^2 + \pi^3)}.
\]

A simple calculation gives
\[
  (x k(x))^2 f'(x) = g(x),
\]
where \( k(x) = x^3 - 2\pi x^2 + \pi^3 \) and
\[
  g(x) = \cos(x). (x^4 - 2\pi x^3 + \pi^3 x) - \sin(x). (4x^3 - 6\pi x^2 + \pi^3).
\]
It is immediate that \( g(0) = g(\pi/2) = 0 \). One has \( g'(x) = -x \sin(x) h(x) \), with
\[
  h(x) = x^3 - 2\pi x^2 + 12x + \pi^3 - 12\pi.
\]

Here \( h(0) = \pi(\pi^2 - 12) < 0, h(\pi/2) = 5\pi^3/8 - 6\pi > 0 \) as \( 5\pi^2 > 48 \) and \( h(\pi) = 0 \). Further \( h'(x) = 3x^2 - 4\pi x + 12, \) and \( h''(x) = 2(3x - 2\pi) \). Here \( \pi/2 < 2\pi/3 < \pi \). The roots of \( h'(x) = 0 \) are \( x_1 = (2\pi - 2\sqrt{\pi^2 - 9})/3 \approx 1.47271, \) which is in \((0, \pi/2)\), and \( x_2 = (2\pi + 2\sqrt{\pi^2 - 9})/3 \approx 2.71608, \) which is in \((2\pi/3, \pi)\). Therefore, \( h(x) \) is strictly increasing in \((0, x_1)\), and \((x_2, \pi)\), while strictly decreasing in \((x_1, x_2)\).

Let \( x \in (0, \pi/2) \), then as \( h(0) < 0, h(\pi/2) > 0, \) \( h \) has a single root \( x_0 \), and a maximum point in \( x_1 \). Thus \( h(x) < 0 \) in \((0, x_0)\), and \( h(x) > 0 \) in \((x_0, \pi/2)\). Therefore, \( g'(x) > 0 \) for \( x \in (0, x_0) \) and \( g'(x) < 0 \) in \((x_0, \pi/2)\). Thus \( g(x) > g(0) = 0 \) in \((0, x_0)\) and \( g(x) > g(\pi/2) = 0 \) in \((x_0, \pi/2)\). In all cases, \( g(x) > 0 \) for \( x \in (0, \pi/2) \). This means that, \( f(x) \) is strictly increasing in \((0, \pi/2)\).

When \( x \) is in \((\pi/2, \pi)\), the proof runs as above, by remarking that by \( h(2\pi/3) < 0, \) there exists a unique \( x_0^{*} \in (\pi/2, \pi) \) such that \( h(x_0^{*}) = 0 \). Since \( h(x) > 0 \) in \((x_0^{*}, \pi)\) and \( h(x) < 0 \) in \((x_0^{*}, \pi)\) we get that \( g(x) < g(\pi/2) = 0 \) in \((\pi/2, x_0^{*})\), while \( g(x) < g(\pi) = 0 \) in \((x_0^{*}, \pi)\), so in all case \( g(x) < 0 \), when \( x \) is in \((\pi/2, \pi)\). Thus \( f(x) \) is strictly decreasing in \((\pi/2, \pi)\). This completes the proof.

The inequalities in (1.8) and (1.9) are not comparable. From the proof of (1.9) we get the following corollary.

1.10. **Corollary.** For \( x \in (0, \pi/2) \), we have
\[
  \frac{x^3 - 2\pi x^2 + \pi^3}{4x^3 - 6\pi x^2 + \pi^3} > \frac{\tan(x)}{x},
\]
inequality reverses for \( x \in (\pi/2, \pi) \).

1.11. **Theorem.** For we have
\[
  (1) \quad \frac{\alpha \ x + y}{xy \ 1 + xy} < B(x, y) < \frac{\beta \ x + y}{xy \ 1 + xy}, \quad x \in (0, 1) \text{ with } y = 1 - x,
\]
with the best possible constants \( \alpha = 5\pi/16 \approx 0.98175 \) and \( \beta = 1 \),
\[ B(x, y) < \frac{x + y}{xy 1 + xy}, \quad x, y \in (0, 1), \]

inequality reverses for \( x > 1 \).

\textbf{Proof.} Utilizing (1.2), the first inequality in (1.9) can be written as

\[ t(1 - t)(1 + t(1 - t)) < \frac{1}{\Gamma(t) \Gamma(1 - t)}, \]

which is equivalent to

\[ \frac{t(1 - t)(1 + t(1 - t))}{t + 1 - t} < \frac{\Gamma(t + 1 - t)}{\Gamma(t) \Gamma(1 - t)}. \]

Letting \( x = t \) and \( y = 1 - t \), we get the first inequality. The second inequality follows similarly from the second inequality of (1.9). This completes the proof. \( \square \)

1.12. \textbf{Remark.} The inequality

\[ 1 - \frac{z}{\pi} < \frac{\sin(z)}{z}, \quad z \in (0, \pi), \]

can be written as

\[ \Gamma \left(1 + \frac{z}{\pi}\right) \Gamma \left(1 - \frac{z}{\pi}\right) < \frac{1}{1 - z/\pi}, \]

by (1.2). This implies (1.3) if we let \( x = z/\pi \) and \( y = 1 - z/\pi \).

1.13. \textbf{Lemma.} We have

\[ (1) \quad \psi(1 + x) - \psi(x + y) < \frac{1 - y}{x + y - xy}, \quad x > 1, \ y \in (0, 1), \]

\[ (2) \quad \psi(2 - x) - \psi(1 + x) < \frac{1 - 2x}{1 - (1 - x)x}, \quad x \in (0, 1/2), \]

inequality reverses for \( x \in (1/2, 1) \).

\textbf{Proof.} For \( x > 1 \) and \( y \in (0, 1) \), we define

\[ g_x(y) = \psi(1 + x) - \psi(x + y) - \frac{1 - y}{x + y - xy}. \]

Differentiating with respect to \( y \) we get

\[ g_x''(y) = -\frac{2(1 - x)^2(1 - y)}{(x(1 - y) + y)^3} - \frac{2(1 - x)}{(x(-y) + x + y)^2} - \psi''(x + y) \]

\[ = \frac{2x - 2}{(x(1 - y) + y)^3} - \psi''(x + y) > 0, \]

since \( \psi''(x + y) < 0 \). Thus, \( g_x \) is convex in \( y \), clearly \( g_x(0) = f_x(1) = 0 \). This implies that the function \( g_x \) lies under the line segment joining origin and the point \((1, 0)\). Hence the proof is obvious now.

For (2), write

\[ f(x) = \psi(2 - x) - \psi(1 + x) - \frac{1 - 2x}{1 - (1 - x)x}, \quad x \in (0, 1). \]
One has
\[ f''(x) = \left( \frac{2(2x-1)^2}{(1-(1-x)x)^3} - \frac{2}{(1-(1-x)x)^2} \right) (2x-1) \]
\[ - \frac{4(2x-1)}{(1-(1-x)x)^2} + \psi''(2-x) - \psi''(x+1) \]
\[ = \frac{2(x-2)(x+1)(2x-1)}{((x-1)x+1)^3} + \psi''(2-x) - \psi''(x+1). \]

Clearly, the function \( \psi'' \) is increasing and negative. So, it is not difficult to see that \( f'' \) is positive for \( x \in (0, 1/2) \), and negative for \( x \in (1/2, 1) \). This implies the convexity and concavity of \( f \) in \( x \in (0, 1/2) \) and \( x \in (1/2, 1) \), respectively. Clearly, \( f(0) = f(1/2) = f(1) = 0 \). This completes the proof. \( \square \)

1.14. **Theorem.** We have

1. \( \frac{1}{xy}(x + y - xy) > B(x,y), \quad x > 1, \ y \in (0,1), \)
   inequality reverses for \( x \in (0,1) \),

2. \( B(x,y) < \frac{\pi}{3xy}(x + y - xy), \quad x \in (0,1), \) with \( y = 1 - x \).

**Proof.** The inequality in (1) can be written as

\[ h_y(x) = \log(\Gamma(1+x)) + \log(\Gamma(1+y)) - \log(\Gamma(x+y)) + \log(x+y-xy) > 0. \]

Clearly, \( h_y(1) = 0 \). Differentiation with respect to \( x \) yields

\[ h'_y(x) = \psi(1+x) - \psi(x+y) - \frac{1-y}{x+y-xy} = g_x(y), \]

which is negative by Lemma 1.13(1). thus the function \( h_y(x) \) is decreasing in \( x > 1 \), this implies (1). For part (2), let

\[ h(x) = \log \left( \frac{1}{3} \pi (1 - (1-x)x) \right) - \log(\Gamma(2-x)) - \log(\Gamma(x+1)), \]

clearly \( h(1/2) = 0 \). One has,

\[ h'(x) = \psi(2-x) - \psi(1+x) - \frac{1-2x}{1-(1-x)x}, \]

which is positive in \( x \in (0, 1/2) \) and negative in \( x \in (1/2, 1) \) by Lemma 1.13 (2). This implies that \( h \) is decreasing in \( x \in (0, 1/2) \) and increasing in \( x \in (1/2, 1) \). Thus the proof follows. \( \square \)
References

[1] M. Abramowitz, I. Stegun, eds.: Handbook of mathematical functions with formulas, graphs and mathematical tables. National Bureau of Standards, Dover, New York, 1965.
[2] H. Alzer Some beta function inequalities, Proc. of the Royal Soc.of Edinburgh 133A (2003), 731–745.
[3] H. Alzer Some gamma function inequalities, Math. Comp. 60(1993), 337–346.
[4] H. Alzer On some inequalities for the gamma and psi functions, Math. Comp. 66(1997), 373-389.
[5] G.D. Anderson, M.K. Vamanamurthy, and M. Vuorinen: Inequalities of quasi-conformal mappings in the space, Pacific J. Math. Vol. 160, No. 1, 1993.
[6] G. Andrews, R. Askey, and R. Roy: Special Functions, Encyclopedia of Mathematics and its Applications, Vol. 71, Cambridge Univ. Press, 1999.
[7] S.S. Dragomir, R.P. Agarwal, and N.S. Barnett Inequalities for beta and gamma functions via some classical and new integral inequalities, J. Inequal. Appl. 5 (2000), 103–165.
[8] P. Ivády: On a beta function inequality, J. Math. Inequal., 6, 3 (2012) 333-341
[9] R. Klen, M. Visuri, M. Vuorinen: On Jordan type inequalities for hyperbolic functions, J. Ineq. Appl. Vol. 2010, Art. ID 362548, pp. 14.
[10] D.S. Mitrinović: Analytic Inequalities, Springer-Verlag, Berlin, 1970.
[11] E. Neuman and J. Sándor: On some inequalities involving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker, and Huygens inequalities, Math. Inequal. Appl. Vol. 13, Number 4 (2010), 715–723.
[12] S.-L. Qiu and M. Vuorinen: Some properties of the gamma and psi functions with applications, Math. Comp., 74 (250) (2004), pp. 723–742.
[13] J. Spanier and K.B. Oldham, An atlas of functions, Hemisphere Publishing, Washington, 1987.

Department of Mathematical Information Technology, University of Jyväskylä, 40014 Jyväskylä, Finland
E-mail address: bhayo.barkat@gmail.com

Babes-Bolyai University Department of Mathematics Str. Kogalniceanu nr. 1 400084 Cluj-Napoca, Romania
E-mail address: jsandor@math.ubbcluj.ro