Phase Transition with the Berezinskii–Kosterlitz–Thouless Singularity in the Ising Model on a Growing Network

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We consider the ferromagnetic Ising model on a highly inhomogeneous network created by a growth process. We find that the phase transition in this system is characterised by the Berezinskii–Kosterlitz–Thouless singularity, although critical fluctuations are absent and the mean-field description is exact. Below this infinite order transition, the magnetization behaves as \( \exp(-\text{const}/\sqrt{T_c-T}) \). We show that the critical point separates the phase with the power-law distribution of the linear response to a local field and the phase where this distribution rapidly decreases. We suggest that this phase transition occurs in a wide range of cooperative models with a strong infinite-range inhomogeneity.

Note added.—After this paper had been published, we have learnt that the infinite order phase transition in the effective model we arrived at was discovered by O. Costin, R.D. Costin and C.P. Grünfeld in 1990. This phase transition was considered in the following papers:

[1] O. Costin, R.D. Costin and C.P. Grünfeld, Infinite-order phase transition in a classical spin system, J. Stat. Phys. 59, 1531 (1990);
[2] O. Costin and R.D. Costin, Limit probability distributions for an infinite-order phase transition model, J. Stat. Phys. 64, 193 (1991);
[3] M. Bundaru and C.P. Grünfeld, On a phase transition in a one-dimensional non-homogeneous model, J. Phys. A 32, 875 (1999);
[4] S. Romano, Computer simulation study of one-dimensional lattice spin models with long-range inhomogeneous interactions, Mod. Phys. Lett. B 9, 1447 (1995).

We would like to note that Costin, Costin and Grünfeld treated this model as a one-dimensional inhomogeneous system. We have arrived at the same model as a one-replica ansatz for a random growing network where expected to find a phase transition of this sort based on earlier results for random networks (see the text). We have also obtained the distribution of the linear response to a local field, which characterises correlations in this system. We thank O. Costin and S. Romano for indicating these publications of 90s.

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The ferromagnetic Ising model on lattices has an ordinary second-order phase transition. Above the upper critical dimension of the model, the critical fluctuations are absent, and the mean-field description of this transition is exact. In particular, this takes place if couplings between all spins are equal—finite-range interactions. In this Letter we report the finding of a phase transition with the Berezinskii–Kosterlitz–Thouless (BKT) singularity in the Ising model on a growing network, which is infinitely-dimensional as most of networks. This transition is quite unusual for an infinitely-dimensional system as well for a cooperative model with the order parameter of discrete symmetry.

Recall that in “ordinary” continuous phase transitions, pair correlations of an order parameter show a slow, power-law space decay only at the critical point, \( T_c \), and decay exponentially both in the low- and high-temperature phases. This behavior was observed in the Ising model on equilibrium complex networks (for percolation and for disease spreading on equilibrium networks, see Refs. [11] and [12], respectively). In contrast, the BKT phase transition separates the phase with rapidly decreasing correlations and the critical phase with correlations decaying by a power law. The contact of these two phases is characterised by specific dependences. For example, the order parameter behaves as \( M(T) \sim \exp(-\text{const}/\sqrt{T_c-T}) \), and the phase transition is of infinite order.

Normally, the BKT transition is realized in systems with two-component order parameters of continuous symmetry at a lower critical dimension. Also, this anomalous phase transition is present in a few low-dimensional systems (e.g., the Luttinger liquid) which actually can be reduced to above indicated ones. There is one more interesting situation, where the BKT singularity emerges. It was observed that in some growing networks, near the birth point of the giant connected component, its relative size behaves similarly to the magnetization near the BKT transition.
The model.—We find an exact solution of the following cooperative model. A network grows up to a large size, and interacting spins are considered on the resulting net. The interaction between the nearest neighbor spins and and interacting spins are considered on the resulting cooperative model. A network grows up to a large size, where spins magnetic field. Actually, we reduce our problem to a system—coincide. We will show afterwards that the mean-field solution is equivalent to our growing network. The numbers on the edges show the values of the Ising couplings between the spins at the corresponding vertices.

We use the following growing network:
(i) The growth starts with a single vertex \( t = 0 \).
(ii) At each time step, we add a new vertex and attach it to one of “older” vertices.
(iii) For simplicity, we suggest a specific annealing. For an edge born at time \( t \), the end of the edge at vertex \( t \) is fixed, and the second end can be found at each of the vertices in the range \( 0 \leq \tau < t \) with equal probability. Characteristic times for the jumps of this end between the vertices \( 0 \leq \tau < t \) are assumed to be not greater than those of the spin relaxation.

One can show that the resulting model is equivalent to the ferromagnetic Ising model on the deterministic graph shown in Fig. 1. In this system, the spin on a vertex, which was born at time \( t \), has equal coupling \( 1/t \) to each of spins on the older vertices. The Hamiltonian of the model is:

\[
\mathcal{H} = -\sum_{0 < i < j < t} s_i s_j \frac{H_i}{j} - \sum_{i=0}^{t} H_i s_i ,
\]

where spins \( s_i = \pm 1 \), and \( H_i \geq 0 \) is an inhomogeneous magnetic field. Actually, we reduce our problem to a system with a strong deterministic infinite-range disorder.

Mean-field treatment.—Let us first use a mean-field ansatz. We will show afterwards that the mean-field solution is exact. For the sake of brevity, we use the following simple mean-field treatment. We assume small fluctuations of spins from their mean-field values \( m_i \): \( s_i s_j \rightarrow m_i m_j + m_s (s_j - m_j) + m_j (s_i - m_i) \). Substituting this relation into Eq. (1) gives a linear effective mean-field Hamiltonian. With this Hamiltonian, it is easy to obtain the partition function \( Z = \sum_{\{s_i : s_i = \pm 1\}} e^{-\beta \mathcal{H}([s_i])} \) \((\beta \equiv 1/T)\) and the free energy \( F = -\beta^{-1} \ln Z \):

\[
F = t \int_{0}^{1} dx \int_{x}^{1} \frac{dy}{y} m(x)m(y) - \frac{t}{\beta} \ln 2 - t \beta^{1/3} \int_{0}^{1} dx \ln \cosh \left\{ \beta \left[ \frac{1}{x} \int_{0}^{x} dy m(y) + \int_{x}^{1} \frac{dy}{y} m(y) + H(x) \right] \right\} .
\]

(2)

FIG. 1: Deterministic system of interacting spins which is equivalent to our growing network. The numbers on the edges show the values of the Ising couplings between the spins at the corresponding vertices.

Here we assumed that \( t \) is large and passed to the continuum limit: \( m_i = m(x = i/t) \). Expression (2) together with the relation \( m(x) = -(1/t) \delta F/\delta H(x) \) allows us to obtain the equation for the mean local magnetization:

\[
m(x) = \tanh \left\{ \beta \left[ \frac{1}{x} \int_{0}^{x} dy m(y) + \int_{x}^{1} \frac{dy}{y} m(y) + H(x) \right] \right\} .
\]

(3)

The exact derivation of the free energy.—The free energy can be found exactly. We compare the free energies of the network at times \( t-1 \) and \( t \). For brevity, here we consider only the homogeneous magnetic field. The form of the Hamiltonian (1) results in the relation

\[
e^{-\beta F_i(H)} = \sum_{s_i = \pm 1} e^{-\beta F_{i-1}(H+s_i/t)} e^{\beta H s_i} ,
\]

that is,

\[
e^{-\beta[F_i(H)-F_{i-1}(H)]} \rightarrow \frac{e^{-\beta F_i(H)/t}}{t} = \sum_{s_i = \pm 1} \exp \left[ -\frac{\beta \partial F_i(H)}{\partial H} s_i \right] + \beta H s_i.
\]

(4)

Here we took into account the fact that at large \( t \), the ratio \( F_i(H)/t \) approaches a \( t \)-independent limit. Using the relation for the (relative) full magnetization \( M(H) = \int_{0}^{1} dx m(x) = -(1/t) \partial F(H)/\partial H \), we get the exact form of the free energy:

\[
F = -t \beta^{-1} \ln \{ 2 \cosh[\beta (H + M(H))] \}
\]

(5)

at \( t \rightarrow \infty \). One can check that free energy expressions (5)—the mean-field one—and (6)—the exact expression—coincide. Indeed, substituting Eq. (4) into the relation (2) and making partial integration, we arrive at the free energy exactly in form (5). In this sense, the mean-field treatment of this problem is exact.

Analysis of the equation for the magnetization.—Let us consider Eq. (5). From this equation, one can see that the assumption \( m(x) \neq 0 \) at some \( x \) immediately leads to the following behavior of \( m(x) \) near \( x = 0 \):

\[
m(x \sim 0) \approx 1 - A x^{2\beta},
\]

(6)

where \( A \) depends on \( \beta \) and \( H \). If \( H = 0 \), this behavior is realized only in the low-temperature phase, and \( m(x) = 0 \) above \( T_c \). If \( H > 0 \), \( m(x = 0) = 1 \) at any temperature. (We will see that the critical point, \( T_c \), exists.) On the boundary, Eq. (5) readily gives

\[
m(1) = \tanh\{\beta [M + H(1)]\}.
\]

For finding this profile, it is convenient to pass to a differential equation. For brevity, here we assume that \( H = 0 \). We introduce a new variable, \( n(z) \):

\[
n(-\ln x) = \beta \left[ \frac{1}{x} \int_{0}^{x} dy m(y) + \int_{x}^{1} \frac{dy}{y} m(y) \right] ,
\]

(8)

so

\[
m(x) = \tanh n(-\ln x).
\]

(9)
Differentiating Eq. (8) and using Eq. (9) gives the second order differential equation
\[
\frac{dn(z)}{dz} - \frac{d^2 n(z)}{dz^2} = \beta \tanh n(z)
\]  
(10)
with the boundary conditions: (i) \((dn/dz)(z=0) = n(z=0)\) [note that \(n(z=0) = \beta M = \beta \int_0^1 dy m(y)\)] and (ii) \(n(z \to \infty) \approx \beta z + \text{const.} \ z\) is related to \(x = i/t\): \(z = -\ln x, \) so \(0 \leq z < \infty, \) where \(z = 0\) corresponds to \(x = 1.\) Boundary conditions (i) and (ii) follow from definition [5] and relation (11), respectively. At each value of \(\beta,\) there is a single solution of Eq. (10) with these boundary conditions, which allows one to get \(M.\)

Equation (10) can be transformed into a first order differential equation. For this, we pass from variables \((t, n(t))\) to \((n, w(n)),\) where \(w = \beta^{-1}(dn/dt). \) \([n\) varies from 0 to \(\infty,\) while \(w(n)\) takes values between 0 and 1.\]

This gives the equation
\[
w \frac{dw}{dn} = \beta^{-1}(w - \tanh n)
\]
(11)
for \(w(n)\) with the following boundary conditions: (i) \(w[n(z=0)] = \beta^{-1} n(z=0)\) [recall that \(\beta^{-1}n(z=0) = M\)] and (ii) \(w(n \to \infty) = 1.\) Here, boundary condition (i) on the line \(w = \beta^{-1}n\) corresponds to that at \(x = i/t = 1.\) Asymptotic boundary condition (ii) corresponds to the limit \(i/t \to 0.\) Knowing \(w(n)\) one can easily get \(m(x)\).

The analysis of Eq. (11) is similar to that of an equation of this type in Ref. [16]. At small \(n,\) one can substitute \(\tanh n\) by \(n\) on the right-hand side of Eq. (11), so we have \(w dw/dn = \beta^{-1}(w - n)\). The solutions of this equation can be presented in an analytical form. A physically reasonable non-zero solution must cross the ordinate axis (and the \(w = \beta^{-1}n\) line) at non-negative \(w.\) This solution exists if \(\beta \geq 1/4.\) There is a critical point, \(\beta_c = 1/4,\) where the solution is
\[
w_c(n, \beta = 1/4) = 2n[1 - f(n)]
\]
(12)
with \(f(n)\) satisfying \(f(n \to 0) \to 0\) and the relation: \(\ln[n f(n)] + 1/f(n) = \ln c.\) Here the constant \(c = 1.554 \ldots\) ensures that \(w_c(n)\) fits the corresponding solution of Eq. (11) which approaches 1 as \(n \to \infty.\) The form of the critical solution at small \(n\) indicates the presence of the BKT singularity.

Near \(T_c,\) the solution of Eq. (11) is close to the critical one. In this range, the asymptotics of the solution at small \(n\) satisfies the relation:
\[
-\frac{1}{\sqrt{4\beta - 1}} \arctan \left[ \frac{2 \beta w(n)/n - 1}{\sqrt{4\beta - 1}} \right] - \ln \sqrt{n^2 - w(n)n + \beta w^2(n)} = \text{const}
\]
(13)
(\(\beta > 1/4.\)) This asymptotics and the solution at large \(n\) can be sewed together (see details in the full version of the present work). For obtaining the dependence of the full (relative) magnetization on \(\beta\) near the critical \(\beta_c = 1/4,\)

we use the following procedure. (i) We substitute the boundary condition \(w(n = \beta M) = M\) into relation (13). After expansion of the arctangent, we obtain the left-hand side of the relation below:

\[
-\frac{\pi/2}{\sqrt{4\beta - 1}} + 1 - \ln \left[ \frac{M(\beta)}{4} \right] = \frac{\pi/2}{\sqrt{4\beta - 1}} \left[ 1 - \frac{w(n)}{2n} \right] - \ln \left[ n - \frac{w(n)}{2} \right] \to \frac{\pi/2}{\sqrt{4\beta - 1}} - \ln c.
\]

(ii) On the other hand, near \(\beta = 1/4,\) in the region \(\beta M \ll n \ll 1,\) the main contribution of Eq. (13) gives the right-hand side of the equality above. We also use the fact that the solution must approach the critical one as \(\beta \to 1/4.\) So, we obtain the full magnetization near \(\beta_c = 1/4:\)

\[
M(\beta) \cong 4e \exp \left( - \frac{\pi}{2 \sqrt{\beta - 1/4}} \right),
\]
(14)
where \(4e = 16.90 \ldots\) is Euler’s number. Note that this BKT behavior is a direct result of the specific singular form of Eq. (11) at small \(n\) and \(w.\) The behaviors of the magnetization and other main thermodynamic quantities near the phase transition are shown in Fig. 2.

By using Eq. (11), one can also find the coefficient of
the term $x^{2\beta}$ in relation (7). Near $T_c$, at small $x$,

$$m(x) \equiv 1 - 2 e^{\beta[(2\pi/\sqrt{\beta-1/4})-1.06]} x^{2\beta}. \quad (15)$$

Here $H = 0$. That is, as the temperature approaches $T_c$, $m(x)$ decreases with $x$ more and more rapidly.

**Specific heat and susceptibility.**—Substituting result (13) into formula (9) for the free energy readily gives the specific heat, $t C(T) = -T \partial^2 F/\partial T^2$. $C(T > T_c) = 0$, as is usual for mean-field theories. If $T < T_c$,

$$C(T) = \frac{(\pi ce)^2}{8(\beta-1/4)^3} \exp \left(-\frac{\pi}{\sqrt{\beta-1/4}}\right), \quad (16)$$

where $(\pi ce)^2/8 = 22.01 \ldots$

Similarly, one can consider the case of a non-zero homogeneous magnetic field. Here we present the resulting expressions for the magnetic susceptibility:

$$\chi(\beta > 1/4) = \beta^{-1} - 1,$n
$$\chi(\beta < 1/4) = -(1 - \sqrt{1 - 4\beta})/(1 + \sqrt{1 - 4\beta}). \quad (17)$$

There is a finite jump of the susceptibility at the phase transition point: $\chi[\beta = (1/4)^{-}] = 1$ and $\chi[\beta = (1/4)^{+}] = 3$ [see Fig. 2(c)].

**Response to a local magnetic field.**—In networks, instead of correlations in space, one has to consider other expressions for the magnetic susceptibility:

$$H(x, y) = h [\theta(x - (y - \Delta/2)) - \theta(x - (y + \Delta/2))]. \quad (18)$$

$[\theta(x)$ is the theta-function, $h$ and $\Delta$ are small] and find the change of the full magnetization which is induced by this field: $\mu(y) = \int_{0}^{1} dx \ [m(x, y) - m(x)]$. Knowing $\mu(y)$ readily gives the distribution $P(\mu)$ of the response.

Calculations are especially simple at $T > T_c$. One can find $m(x, y)$ by iterating Eq. (3), which gives

$$\mu(y) = \beta h \Delta \frac{2}{1 + \sqrt{1 - 4\beta}} y^{-(1 - \sqrt{1 - 4\beta})/2}. \quad (19)$$

Note the power-law divergence of the linear response as $y \to 0$. Relation (19) results in the power-law response distribution:

$$P(\mu) \propto \mu^{-[1 + 2/(1 - \sqrt{1 - 4\beta})]}. \quad (20)$$

It is important that in contrast to “normal” continuous phase transitions, $P(\mu)$ is a power-law function in the entire phase and not only at $T_c$. As is natural, in the other phase, this distribution is a rapidly decreasing function. At the phase transition point, $P(\mu, \beta = 1/4) \propto \mu^{-3}$.

**Discussion and conclusions.**—Several points must be emphasized.

(i) The spins on the oldest vertices are oriented in most of situations: $m(x = 0) = 1$ even above $T_c$, if any non-zero (positive) magnetic field is applied at least to one spin of the system.

(ii) We considered networks with a specific annealing. The problem of a quenched disorder is more complex. However, there is a quenched situation, to which our results are applicable. Let each new vertex have a large number $N$ — greater than the final size of the network or of this order — of new connections to randomly chosen vertices. Let each of these edges bear the Ising coupling equal to $1/N$. Then we arrive at the situation similar to that is shown in Fig. 1.

(iii) The phase transition found in this paper, as well as the structural transition considered in Refs. [12, 14, 17, 18, 20, 21, 22, 23, 25, 26, 27], seriously differs from the usual BKT transition. In our case, the analogue of the power-law correlations takes place in the normal phase. In contrast, in the traditional BKT transition, the power-law decay of correlations is in the phase with a non-zero order parameter.

(iv) We stress that the more traditional-looking transitions of Refs. [3, 4, 5, 7, 11, 12] are realized in equilibrium networks where all vertices are statistically equivalent. In contrast, the networks, where we observed the transition with the BKT singularity, are specifically inhomogeneous. For our general conclusions, the specific $1/j$ form of the inhomogeneity of the interaction in the Hamiltonian (11) is important only in the region of relatively small $j$. Deviations from this form at larger $j$ do not change the critical behavior. We studied a growing network but the problem has been reduced to the Ising model on a compact system with strong long-range inhomogeneity. We believe that our results are applicable to other systems with inhomogeneity of this kind. Furthermore, the Ising model is only a simple example of cooperative models were the observed transition should be present.

In conclusion, we have solved the ferromagnetic Ising model on a highly inhomogeneous growing net. In this system we have found an infinite order phase transition with the BKT singularity. This transition separates a phase, where the distribution of the response to a local field is power-law, and the phase, where this distribution is rapidly decreasing. We suggest that this transition also occurs in other cooperative models on compact substrates with strong long-range inhomogeneity.

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