The optical Möbius strip cavity: Tailoring geometric phases and far fields

Jakob Kreismann\textsuperscript{1} and Martina Hentschel\textsuperscript{1}

\textsuperscript{1}Institute for Physics, Theoretical Physics II/Computational Physics Group, Technische Universität Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany

(Dated: September 5, 2018)

The Möbius strip, a long sheet of paper whose ends are glued together after a 180° twist, has remarkable geometric and topological properties. Here, we consider dielectric Möbius strips of finite width and investigate the interplay between geometric properties and resonant light propagation. We show how the polarization dynamics of the electromagnetic wave depends on the topological properties, and demonstrate how the geometric phase can be manipulated between 0 and π through the system geometry. The loss of the Möbius character in thick cavities and for small twist segment lengths allows one to manipulate the polarization dynamics and the far-field emission, and opens the venue for applications.

PACS numbers: 42.55.Sa, 42.60.Da, 42.25.Ja

Optical microcavities have attracted a lot of interest in the past two decades \textsuperscript{1}2. Especially, three-dimensional (3D) optical microcavities are an interesting object of research, see e.g. \textsuperscript{3}. Unlike the two-dimensional case, the polarization is not fixed to transverse magnetic (TM) or transverse electric (TE). Rather, it may change, and the resulting polarization evolution is a characteristic system property. The polarization dynamics of light in terms of spin-orbit interaction has been investigated in recent years, see e.g. \textsuperscript{4}7 where the polarization evolution was determined by geometric phases arising from optical spin-orbit interaction. These spin-orbit interactions are based on light propagating in smoothly inhomogeneous or anisotropic materials. Earlier works demonstrate the appearance of geometric phases in coiled fibers \textsuperscript{8}10, and the Pancharatnam phase of beams whose polarization state undergoes certain changes \textsuperscript{11}, verified in Mach-Zender interferometry \textsuperscript{12}13.

In this paper, we show the presence and the consequences of the Pancharatnam phase in a 3D microcavity, namely the dielectric Möbius strip. The Möbius-strip is a fascinating and special surface, not only from the mathematical point of view. It can be created by joining the ends of a half-twisted strip to form a loop. The Möbius-strip is a surface with only one side and non-orientable, hence it is not possible to make a consistent choice of a surface normal vector at every point.

Möbius strips have been realized in a wire-loaded copper cavity \textsuperscript{14}, in a electric circuit resonator \textsuperscript{15}, by coupled resonators \textsuperscript{16}, in twisted ribbon crystal materials \textsuperscript{17}, as plasmonic resonances in nano metallic strips \textsuperscript{18}, or in metamaterials \textsuperscript{19}. Quantum mechanics on a Möbius ring has been studied in \textsuperscript{20}. Furthermore, optical (cavity-free) polarization Möbius-strips were investigated using focused light \textsuperscript{21}24 or light scattered by nanoparticles \textsuperscript{25}. A theoretical investigation of plasmonic resonances in metallic Möbius nanorings and dielectric Möbius nanorings has been reported in \textsuperscript{29} and \textsuperscript{27}.

Here, we will investigate a pure dielectric Möbius-strip cavity, discuss how the geometry has to be modified in order to tune the topological phase, and discuss the far-field properties of these modified structures. We consider a cavity of radius \( R = 2.29 \mu m \), height \( h = 1.75 \mu m \), and wall thickness \( w = 0.4 \mu m \) with a homogeneous and isotropic refractive index \( n = 3.3 \) (c.f. FIG. \textsuperscript{1}). We choose the parameters such that a thin strip with \( w < h < R \) results. Furthermore, we are interested in resonances where the medium wavelength is smaller than the wall thickness, \( w < \lambda/n \). We perform full 3D finite difference time domain (FDTD) wave calculations in order to compute the electromagnetic field, its polarization direction, the light intensity and the far fields using the free software package MEEP \textsuperscript{28}.

Polarization dynamics in the standard Möbius strip. A three-dimensional pure dielectric ring cavity can support TE-like and TM-like polarization modes \textsuperscript{29}. In the following analysis, we focus on the TE-like modes where the electric field \( E \) is almost transverse to the propagation direction \( k \cdot E = 0 \) and remains parallel to the wall of the Möbius ring cavity during its evolution along the strip. An example of the resulting, generalized whispering-gallery-(WG-)type resonance is shown in Fig. \textsuperscript{1}a). In a local coordinate system \((x', y', z')\) as depicted in Fig. \textsuperscript{1}a), where \((x', y')\) represent the transverse coordinates (in radial, \( R \), and vertical direction) and \( z' \) is the longitudinal coordinate (along the propagation direction), the polarization orientation of the electric field \( E \) can be described by the complex unit vector

\[
P = (p_{x'}, p_{y'}) ,
\]

where \( p_{x'} \cdot p = 1 \). The longitudinal component \( p_{x'} \) is omitted due to the almost transverse electric field vector. In case of linearly polarized light, the components \( p_{x'} \) and \( p_{y'} \) are purely real.

The following analysis is motivated by Pancharatnam’s...
Let us consider the horizontally polarized initial state \(|\Psi\rangle = (\Psi^x, \Psi^y, \Psi^z)\). Each such \(|\Psi\rangle\) can be illustrated as a point on the so-called Poincaré sphere with spherical coordinates \((\theta, \phi)\) that describe the phase difference between \(\Psi^x\) and \(\Psi^y\), \(\theta = \text{arg}(\Psi^x/\Psi^y) + \pi/2\), and the orientation, \(\phi\), of the principal axis of the polarization ellipse with respect to \(x’\)-axis, respectively. The poles of this sphere, \(\theta = 0(\pi)\) correspond to vanishing \(|\Psi^+\rangle (|\Psi^-\rangle\) components, and represent left (right) circularly polarized light. Points on the equator, \(\phi = \pi/2\), indicate linear polarization into various directions \(\phi\) where always \(|\Psi^+\rangle = |\Psi^-\rangle\).

As the light propagates in the Möbius-strip, the polarization orientation changes from vertical at \(|A\rangle\) in FIG. 2a to horizontal \(|B\rangle\) and back again. This evolution is represented by a closed curve along the equator of the Poincaré sphere, as depicted in Fig. 2a. Most importantly, we notice the number of intensity maxima \(N = 19\) to be an odd number which corresponds to a fractional azimuthal mode number \(m = N/2 = 9.5\). To understand this unusual behavior, we recall the condition of resonance formation in a ring-like resonator. It is well known that the formation of a resonance requires a phase difference of \(2\pi\) of two interfering waves, e.g., the clockwise (CW) and counter-clockwise (CCW) propagating waves. Let us consider the horizontally polarized initial state \(|A\rangle\), cf. FIG. 2b). Starting from this state, waves propagate in CW and CCW direction. After half a round trip along the cavity (indicated by the green and red curved arrow, cf. Fig. 2b), both waves interfere and the resulting intensity \(I\) is given by:

\[
I = |e^{-i\phi_d}|^2 |B_{CCW}\rangle + e^{i\phi_d} |B_{CW}\rangle|^2 \nonumber
\]

\[
= 2 + 2 |\langle B_{CCW}|B_{CW}\rangle| \cos(\text{arg}(\langle B_{CCW}|B_{CW}\rangle) - 2\phi_d), \tag{3}
\]

where \(\phi_d = nk_d R \pi\) is the dynamical phase of half a round trip. It is related to the azimuthal number \(m\) by \(m = nk_d R\) that counts the number of medium wavelengths \(\lambda/n\) to match the length of resonator \(2\pi R\). In order to realize the resonance condition (maximum intensity), the argument of the cosine has to be an integer multiple \(q\) of \(2\pi\):

\[
\text{arg}(\langle B_{CCW}|B_{CW}\rangle) - 2\phi_d = q2\pi. \tag{4}
\]

**Geometric phase.** As pointed out by Pancharatnam \[11\] and reformulated by Berry \[30\], \(\text{arg}(\langle B_{CCW}|B_{CW}\rangle) = \Gamma/2\) is the geometric phase equal to the half of the solid angle spanned by the curve on the Poincaré sphere. Inserting the dynamical phase \(\phi_d\) and
FIG. 3. a) Upper row shows three Möbius strips with different lengths of the twisted segments as indicated, and the intensity plots of the generalized WG-type modes with the corresponding polarization orientation. For all three cases, the twisting region is big enough for the solid angle to span the full hemisphere (lower insets). b) When the lengths of twisted segments are shorter, the polarization change will happen in a contracted region and therefore faster. The spanned solid angle is much less than $2\pi$. The lower row insets display the corresponding maximum intensity profiles. The two maxima around $0^\circ$ begin to merge as the solid angle decreases. c) Resonance wave number $kR$ and corresponding azimuthal mode number $m = q + \Gamma/(4\pi)$ for Möbius strips with varying geometry (blue) and a corresponding ring strip resonator (red).

rearranging yields the azimuthal mode number $m$:

$$m = q + \frac{\Gamma}{4\pi}.$$  

(5)

The azimuthal mode number is a function of the solid angle $\Gamma$ on the Poincaré sphere and can take integer, half-integer or arbitrary-fractional values if $\Gamma$ equals 0 (usual ring resonator), $2\pi$ (usual Möbius strip, cf. Fig. 2) or any other angle (modified Möbius strip, cf. Fig. 3), respectively.

Möbius ring cavities with half-integer $m$. For the moment, we discuss the case of $\Gamma = 2\pi$ where the upper hemisphere of the Poincaré sphere is spanned by the polarization evolution, cf. Fig. 2. Hence, the azimuthal mode number $m$ is half-integer $m = q + \frac{1}{2}$ and as a consequence, a standing wave must fit an half-integer number of medium wavelengths along the circumference of the Möbius strip: $2\pi R = (q + 1/2)\lambda/n$. The number of intensity maxima is given by $N = 2m = 2q + 1$ which yields an odd number and readily explains our above findings, c.f. FIG. 1. In other words, the change of the polarization orientation during the propagation represents an adiabatic parallel transport of linear polarization which causes a geometric phase. This geometric phase is quantified by (one-half of) the corresponding solid angle in the parameter space, in our case the wave polarization parameter space which is governed by the same algebra as spin-1/2 particles [30].

More importantly, the additional geometric phase of $\pi$ does not cause a phase jump as it might seem at the first sight in FIG. 1 a) and c). Rather, the two adjacent blue areas and the blue-red area result from the non-orientability of the strip. The TE-like (TM-like) mode in a) (c)) has a minimum (maximum) at the position where the strip is glued together and therefore we see two areas with the same color (one blue-red area). Note that, the electric field vectors (black arrows) show indeed a continuous evolution without any phase jump.

We would like to point out once more that all these optical modes confined in a thin and unperturbed Möbius strip have half-integer azimuthal mode number $m$. In contrast, quantum mechanical modes on a Möbius strip (as a solution of the Schrödinger equation, see [20]) exhibit both integer and half-integer azimuthal quantum numbers. In our case, the optical states undergo parallel transport of polarization which is different from the quantum mechanical states where the azimuthal quantum number depends on an even or odd transverse quantum number [20]. Furthermore, studies of propagation of polarized light in twisted waveguides show that the geometric phase can be linked to the torsion of the trajectory of the light, see [31] and [32]. However, in our work, we considered a narrow Möbius strip where the light propagates on an undistorted and nearly planar (central) ring trajectory. Therefore, we conclude that the geometric phase arises purely from the twisting of the strip.

At this point, a remark on geometric phases in electronic mesoscopic transport is in order. The Aharonov-Bohm effect [33, 34] is a well-understood example. A closer relation to the present subject provide geometric phases that occur in magnetotransport through inhomogeneous magnetic fields, see e.g. [35, 36] and references therein. If the magnetic field is sufficiently strong (adiabatic case), the geometric phase corresponds to one half (corresponding to electronic spin) the solid angle spanned by the evolution in parameter (magnetic field) space [37]. However, for the non-adiabatic case, i.e., relatively weaker magnetic fields, the geometric phase is smaller than the original solid angle [38]. In all cases, the geometric phase...
but now, the distance between two points on the Poincaré sphere increases as the polarization change occurs faster on a smaller portion of the trajectory. This is illustrated in Fig. 3. As a consequence of Pancharatnam’s connection, the states (points) on the Poincaré sphere have to be connected by geodesic arcs. Thus, the spanned solid angle remains $2\pi$, unless the distance of two states along the equator exceeds $\pi$. This case shows up for the first time at a segment extent of $\Phi_{TW} = 32^\circ$, c.f. FIG. 3b). Now, the geodesic polygon spans an angle much less than $2\pi$ and generates a geometric phase less than $\pi$, and consequently the azimuthal mode number $m$ will be some value between 9 and 9.5, c.f. equation (1).

It is interesting to study the transition to non-adiabatic transport in terms of intensity. This process involves the merging of two neighboring intensity maxima and disappearance of the zero/minimum in between them, cf. Fig. 3b). As the geometric phase is decreased, we see a smooth transition from $N = 19$ and $m = 9.5$ to $N = 18$ and $m = 9$, cf. Fig. 3c). This is comparable to the smooth transition between the adiabatic and non-adiabatic situation in electronic magnetotransport. If the twist segment length is very small, it shall act like a small perturbation of a ring resonator. Indeed, we find that the resonance wave numbers do approach the one of the conventional WG mode with $m = 9$ of the corresponding ring (cylinder) resonator.

b) Changing the wall thickness $w$. In a next step, we analyzed the polarization state of the resonances as a function of the wall thickness $w$ of the Möbius-strip. Interestingly, at a critical thickness $w/h$ that depends on the resonance chosen, the odd number $N$ switches to an even number, as depicted in Fig. 4. Furthermore, the polarization then remains pointing in almost the same direction during the whole propagation cycle along the Möbius strip, and hence the waves seem to completely ignore the Möbius topology.

There is a simple geometric explanation for this apparent counterintuitive behavior: increasing or decreasing $w$ does not change the topology of the Möbius strip, but it changes its volume. Note that there is a torus volume inside the Möbius strip that is not affected by the twisting of the strip and changing the geometry. If the Möbius strip’s thickness is of the order of the wavelength inside the medium, the main distribution of the electric field is located inside the invariant torus volume. As a result, a thick Möbius strip is equivalent to a (slightly perturbed) torus or cylinder.

Far fields of the Möbius strip.

In the following, we investigate the far fields of the standard and the modified Möbius strips. FIG. 5a) - d) and f) show the far-field intensity plotted as a 3D parametric plot as a function of the far-field angles $\chi$ and $\theta$ as indicated in the figure. The standard Möbius strip (c.f. FIG. 5a) ) displays a very complex far-field emission that originates from the continues twisting of the strip wall. As the length of the twisted segment $\Phi_{TW}$ is reduced, bidirectional far-field emission emerges characterized by two main lobes along the y-axis, c.f. FIG. 5c) and d).
The origin of this bidirectional emission is the refractive output at the strongly twisted segment that represents a region of very high curvature. Most of the electromagnetic energy carried by clockwise and counter-clockwise propagating waves leaves the Möbius ring cavity at this region as schematically illustrated by two red arrows in FIG. 5(e). A remarkable feature of all far fields presented in FIG. 5(a) - d) is that the emission is limited to the x-y-plane (θ = 0) because the waves propagate in an almost planar ring. For comparison, we show the far field of an usual ring resonator in FIG. 5(f) that reveals planar and isotropic emission with 18 lobes.

Conclusions and application potential. We have conducted full 3D FDTD wave calculations of thin dielectric Möbius strips in order to compute their optical modes, their polarization properties and far fields. The topological properties of the (conventional) Möbius strip result in a half-integer azimuthal mode number m because of an extra, geometric, phase of π provided by the Pancharatnam phase that arises from the change of the polarization state (from vertical to horizontal and back again) on the Poincaré sphere upon propagation of light through the dielectric Möbius strip. This behavior represents an example of adiabatic parallel transport of linear polarization which causes a geometric phase. Pure transverse light is governed by the same algebra as spin-1/2 particles, and thus the geometric phase is given by half of the solid angle spanned on the Poincaré sphere. A half-integer m resonance is formed if the full upper hemisphere is spanned.

We have presented ways to tune the value of m to an arbitrary value corresponding to a smaller solid angle enclosed on the Poincaré sphere. We have shown that this can be realized by a) decreasing the twist segment length of the Möbius strip so that the change of the polarization state occurs very fast, or by b) increasing the wall thickness w. The limiting case in both situations is a conventional WG-mode resonance of the corresponding ring resonator, either because the Möbius twist is shrunk to a small perturbation as in a) or because an invariant torus formed around the central line of the Möbius strip possesses sufficient volume to guide a WG-mode as in b). In other words, we find and illustrate that, and how, topological properties can be easily manipulated via the system geometry in optical Möbius strip cavities.

This opens a broad venue for applications based, for example, on opto-mechanical interactions that will be useful, e.g., in sensor design. For example, when placing a dielectric Möbius strip on a substrate, there will be an evanescent, tapered-fiber-like coupling in the region where the Möbius strip lies flat on the substrate. This coupling will affect the cavity Q-factor. We find that this change is larger than an order of magnitude in the conventional Möbius geometry, and much smaller in Möbius geometries with very small twist segment lengths, such that a change in the cavity Q-factor could be directly related to the system geometry and mechanical properties. This will be the subject of further studies. Our findings might also be useful in separating the geometric and spin-orbit-coupling contributions in the polarization evolution in more complex systems such as in [7].

ACKNOWLEDGMENTS

This work was partly supported by Emmy-Noether programme of the German Research Foundation (DFG).

[1] K. J. Vahala, Nature 424, 839 (2003)
[2] A. J. C. R. K. Chang, ed., Optical Processes in Microcavities (World Scientific, Singapore, 1996).
[3] J. Kreismann, S. Sinzinger, and M. Hentschel, Phys. Rev. A 95, 011801 (2017)
[4] K. Y. Bliokh, F. J. Rodriguez-Fortuno, F. Nori, and A. V. Zayats, Nat Photon 9, 796 (2015)
[5] F. Cardano and L. Marrucci, Nat Photon 9, 776 (2015)
[6] S. Slussarenko, A. Alberucci, C. P. Jisha, B. Piccirillo, E. Santamato, G. Assanto, and L. Marrucci, Nat Photon 10, 571 (2016)
[7] L. B. Ma, S. L. Li, V. M. Fomin, M. Hentschel, J. B. Gtte, Y. Yin, M. R. Jorgensen, and O. G. Schmidt, Nat. Commun. 7, 10983 (2016)
[8] A. Tomita and R. Y. Chiao, Phys. Rev. Lett. 57, 937 (1986)
[9] R. Y. Chiao and Y.-S. Wu, Phys. Rev. Lett. 57, 933 (1986)
[10] F. D. M. Haldane, Opt. Lett. 11, 730 (1986)
[11] S. Pancharatnam, Proceedings of the Indian Academy of Sciences - Section A 44, 247 (1956)
[12] H. Jiao, S. R. Wilkinson, R. Y. Chiao, and H. Nathel,
[13] R. Y. Chiao, A. Antaramian, K. M. Ganga, H. Jiao, S. R. Wilkinson, and H. Nathel, Phys. Rev. Lett. 60, 1214 (1988).

[14] S. J. Cooke and J. M. Pond, Microwave and Optical Technology Letters 31, 6 (2001).

[15] D. J. Ballon and H. U. Voss, Phys. Rev. Lett. 101, 247701 (2008).

[16] L.-T. Wu, R.-P. Guo, T.-J. Cui, and J. Chen, Journal of Optics 19, 025101 (2017).

[17] S. Tanda, T. Tsuneta, Y. Okajima, K. Inagaki, K. Yamaya, and N. Hatakenaka, Nature 417, 397 (2002).

[18] Y. Zeng, Z.-Y. Wang, Y. Wu, L.-S. Lu, Y.-X. Wang, S.-J. Shi, and Q. Qiu, Chinese Physics B 26, 037303 (2017).

[19] C.-W. Chang, M. Liu, S. Nam, S. Zhang, Y. Liu, G. Bartal, and X. Zhang, Phys. Rev. Lett. 105, 235501 (2010).

[20] Z. Li and L. R. Ram-Mohan, Phys. Rev. B 85, 195438 (2012).

[21] I. Freund, Opt. Lett. 35, 148 (2010).

[22] I. Freund, Opt. Lett. 36, 4506 (2011).

[23] I. Freund, Opt. Lett. 39, 727 (2014).

[24] T. Bauer, P. Banzer, E. Karimi, S. Orlov, A. Rubano, L. Marrucci, E. Santamato, R. W. Boyd, and G. Leuchs, Science 347, 964 (2015).

[25] A. Garcia-Etxarri, ACS Photonics 4, 1159 (2017).

[26] Y. Yin, S. Li, V. Engemaer, E. Saei Ghareh Naz, S. Giudicatti, L. Ma, and O. G. Schmidt, Laser & Photonics Reviews 11, n/a (2017).

[27] S. L. Li, L. B. Ma, V. M. Fomin, S. Böttner, M. R. Jorgensen, and O. G. Schmidt, arXiv:1311.7158 (2013), arXiv:1311.7158 [physics.optics].

[28] A. F. Oskooi, D. Roundy, M. Ibanescu, P. Bermel, J. D. Joannopoulos, and S. G. Johnson, Computer Physics Communications 181, 687 (2010).

[29] J. D. Jackson, ed., Classical Electrodynamics (Wiley, New York, 1998).

[30] M. V. Berry, Journal of Modern Optics 34, 1401 (1987).

[31] I. I. Satija and R. Balakrishnan, Physics Letters A 375, 3582 (2009).

[32] J. Tromp and F. A. Dahlen, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 437, 329 (1992).

[33] Y. Aharonov and D. Bohm, Phys. Rev. 115, 485 (1959).

[34] S. Olariu and I. I. Popescu, Rev. Mod. Phys. 57, 339 (1985).

[35] Y.-S. Yi, T.-Z. Qian, and Z.-B. Su, Phys. Rev. B 55, 10631 (1997).

[36] D. Frustaglia, M. Hentschel, and K. Richter, Phys. Rev. Lett. 87, 256602 (2001).

[37] M. V. Berry, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 392, 45 (1984).

[38] M. Hentschel, H. Schomerus, D. Frustaglia, and K. Richter, Phys. Rev. B 69, 155326 (2004).