Three-Loop Electroweak Correction to the Rho Parameter in the Large Higgs Mass Limit

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Abstract

We present an analytical calculation of the leading three-loop radiative correction to the $\rho$-parameter in the Standard Model in the large Higgs mass limit. This correction, of order $g^6 m_H^4 / M_W^4$, is opposite in sign to the leading two-loop correction of order $g^4 m_H^2 / M_W^2$. The two corrections cancel each other for a Higgs mass of approximately 480 GeV. The result shows that it is extremely unlikely that a strongly interacting Higgs sector could fit the data of electroweak precision measurements.

Keywords: Rho parameter, Higgs boson, Multi-loop calculations

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1 Introduction

The Standard Model of electroweak physics is in good agreement with the data. The only particle of the Standard Model, that has not been detected so far is the Higgs boson. Direct searches give a lower bound of $m_H = 114.4$ GeV [1]. The precision of present day experiments even makes it possible to put limits on the Higgs mass through its influence in radiative corrections. At the one-loop level, such radiative corrections grow logarithmically with the Higgs boson mass [2]. Fits to the data imply a light Higgs boson. However the situation is not completely satisfactory, as there is difference of about $2.9 \sigma$ between the two most precise determinations of the effective electroweak mixing angle, $\sin^2 \theta_{\text{eff}}^\text{lept}$ i.e. the one based on the measurement of the $b$-quark forward-backward asymmetry $A_{FB}^{0,b}$ at LEP on one hand, and the one based on measurements of the leptonic asymmetry parameter $A_L$ at SLD on the other [3, 4]. The value of the Higgs mass preferred by the b-quark data is around 0.5 TeV, while the leptonic asymmetry data and the $W$-boson mass point to a value which is slightly below the lower bound from the direct searches [5, 6]. With the recent measurements of the W-mass and top mass from the Tevatron this is well within statistics. Still there is the logical possibility that the Higgs boson is very heavy ($\approx 1$ TeV) and strongly interacting. Since the Higgs self-coupling grows like $m_H^2$ higher loop effects can play a role and cancel against the one-loop leading log($m_H$) effects. At the two-loop level such radiative corrections have been calculated, also allowing for anomalous Higgs-boson self-couplings [7, 8]. Inclusion of these two-loop corrections does not make it possible to fit the data with a heavy Higgs boson. However in a recent paper it was shown that the two-loop correction is accidentally anomalously small and therefore important effects might first appear only at the three-loop level [9]. The situation can only be clarified by an explicit three-loop calculation. This three-loop calculation for one of the precision variables, the so-called $\rho$-parameter, is the subject of this paper.

The electroweak $\rho$-parameter is a measure of the relative strengths of neutral and charged-current interactions in four-fermion processes at zero momentum transfer [10]. After defining the Fermi constant $G_F$ by means of the effective interaction that describes muon decay,

$$\mathcal{L}^{CC} = -\frac{G_F}{\sqrt{2}} [\bar{\nu}_\mu \gamma^\mu (1 + \gamma_5) \mu] [\bar{e} \gamma_\mu (1 + \gamma_5) \nu_e]$$

one can define $\rho$ and the sine of the weak mixing angle $s_W = \sin \theta_W$ as the parameters that appear in the effective interaction that describes the scattering of neutrinos and anti-neutrinos by electrons,

$$\mathcal{L}^{NC} = -\frac{\rho G_F}{2\sqrt{2}} [\bar{\nu}_\mu \gamma^\mu (1 + \gamma_5) \nu_\mu] [\bar{e} \gamma_\mu (1 - 4s_W^2 + \gamma_5) e] .$$

In the Standard Model, at tree level, it is related to the $W$ and $Z$ boson masses by

$$\rho = \frac{M_W^2}{c_W M_Z^2} = 1 ,$$

1
where \( c_W = \cos \theta_W \). This relation gets modified by radiative corrections

\[
\rho = \frac{1}{1 - \Delta \rho}
\]

which are sensitive to the existence of heavy particles in the Standard Model, in particular the top quark and the Higgs boson. The dominant contributions in the large top mass limit come from corrections to the \( W \) and \( Z \) boson propagators involving \( t \) and \( b \)-quark loops. They are proportional to \( m_t^{2L} \), where \( L \) is the number of loops, and have recently been calculated up to three loops [11, 12].

Another limit one can consider is the large Higgs mass limit. Here, one might expect to find corrections of order \( m_H^2 \), since the \( W \) and \( Z \) self-energies contain terms of this order, but it turns out that such terms are screened in low energy observables such as the \( \rho \)-parameter [2]. The leading one-loop terms in \( \Delta \rho \) depend logarithmically on \( m_H \) [13], and the leading two-loop correction is proportional to \( m_H^2 \) [7, 14]. In this paper, we present the leading three-loop correction \( \Delta \rho^{(3)} \), which is of order \( m_H^4 \).

Radiative corrections to four-fermion processes include self-energy corrections to the gauge boson propagators and vertex and box corrections. In principle, all of these can affect \( \rho \) and should be taken into account. However, if the fermion masses are neglected, the Higgs boson does not couple directly to the fermions. In this case, the one-loop vertices and boxes do not depend on \( m_H \). Two and three-loop vertices and boxes can only contribute to the leading order term of \( \Delta \rho^{(3)} \) if they contain one-loop subgraphs of order \( m_H^2 \) or two-loop subgraphs of order \( m_H^4 \). However, it is possible to choose a renormalization scheme in which all terms of order \( m_H^2 \) \((m_H^4)\) in the relevant one-loop (two-loop) subgraphs are absorbed into the renormalization factors [7, 15]. In such a scheme, vertices and boxes do not contribute to \( \Delta \rho \) at the leading order in \( m_H \), and one is left with just the corrections coming from the transversal \( W \) and \( Z \) self-energies,

\[
\Delta \rho = \frac{\Sigma^{ZZ}(0)}{M_Z^2} - \frac{\Sigma^{WW}(0)}{M_W^2}.
\]

In section 2 we describe the calculation of the bare three-loop gauge boson self-energies, which we calculate in the full electroweak Standard Model. The calculation is simplified by the fact that they are only required at external momentum \( p = 0 \), which reduces the diagrams involved to vacuum diagrams. Diagrams containing fermion loops are omitted, since after renormalization, they do not give any contributions of order \( m_H^4 \). We simplify the calculation further by systematically expanding all diagrams in powers of \( m_H \) for \( m_H \gg M_W, M_Z \), using the method of asymptotic expansions [16]. In this way, all terms are factorized into integrals depending only on the large scale \( m_H \) on the one hand, times integrals that depend on the small mass scales \( M_W \) and \( M_Z \) on the other. The factorized expressions are then reduced to a set of independent master integrals using integration by parts identities. This reduction is performed exactly in \( d = 4 - \varepsilon \) dimensions. Higgs tadpole contributions and would-be Goldstone boson self-energies, used to determine the renormalization constants, are calculated by the same method. An attractive
feature of this approach is that it enables one to check Ward identities and observe cancellations explicitly at the level of master integrals.

The renormalization is discussed in section 3, and the final result is presented in section 4

2 Large mass expansion and reduction to master integrals

All our diagrams are generated by the program QGRAF [17]. The rest of the calculation is done mainly in FORM [18]. For the three-loop one particle irreducible $W$ and $Z$ self-energies, for example, there are 104340 and 82985 diagrams, respectively. They can be divided into 80 different topologies. After setting the external momentum $p = 0$, the diagrams can be expressed in terms of scalar vacuum integrals. We use the following notations

\[
I_6(m^2_1, m^2_2, m^2_3, m^2_4, m^2_5, m^2_6; n_1, n_2, n_3, n_4, n_5, n_6) = \int d^d k_1 d^d k_2 d^d k_3 P(k_1; m_1)^{n_1} P(k_2; m_2)^{n_2} P(k_3; m_3)^{n_3} P(k_1 + k_2; m_4)^{n_4} P(k_2 + k_3; m_5)^{n_5} P(k_1 + k_2 + k_3; m_6)^{n_6}
\]

(6)

\[
I_5(m^2_1, m^2_2, m^2_3, m^2_4, m^2_5; n_1, n_2, n_3, n_4, n_5) = \int d^d k_1 d^d k_2 d^d k_3 P(k_1; m_1)^{n_1} P(k_2; m_2)^{n_2} P(k_3; m_3)^{n_3} P(k_1 + k_2; m_4)^{n_4} P(k_2 + k_3; m_5)^{n_5}
\]

(7)

\[
I_4(m^2_1, m^2_2, m^2_3, m^2_4; n_1, n_2, n_3, n_4) = \int d^d k_1 d^d k_2 d^d k_3 P(k_1; m_1)^{n_1} P(k_2; m_2)^{n_2} P(k_3; m_3)^{n_3} P(k_1 + k_2 + k_3; m_4)^{n_4}
\]

(8)

\[
I_2(m^2_1, m^2_2, m^2_3; n_1, n_2, n_3) = \int d^d k_1 d^d k_2 P(k_1; m_1)^{n_1} P(k_2; m_2)^{n_2} P(k_1 + k_2; m_3)^{n_3}
\]

(9)

\[
I_1(m^2_1; n_1) = \int d^d k_1 P(k_1; m_1)^{n_1}
\]

(10)

with

\[
P(k; m) = \frac{1}{k^2 + m^2}.
\]
These integrals correspond to the diagrams presented in Figure 1. In addition to the integrals (6)–(10), integrals with scalar products $k_i \cdot k_j$ in the numerator appear.

In general, the propagators in the diagrams can depend on three different non-zero masses: $m_H$, $M_W$ and $M_Z$. The Higgs mass also appears in the scalar 3 and 4-point vertices via $\lambda = g^2 m_H^2 / (4 M_W^2)$. We extract the leading $m_H$ dependent terms by performing an asymptotic large mass expansion, considering $m_H$ to be large and $M_W$ and $M_Z$ to be small. Here, we describe the procedure using the language of expansion by regions [16].

The expansion is constructed by considering different regions in loop momentum space, distinguished by the set of propagator momenta which are large or small in those regions. In each region, a Taylor expansion of all propagators in the small masses and in the small momenta of that region is performed. Typically, the expansion generates extra scalar products of loop momenta in the numerator and higher powers of denominators, as compared to the original diagrams. However, in each term, the dependence of the integrand on the large and small parameters is factorized. The resulting expression is then integrated over the whole loop momentum space.

For the three-loop vacuum topology $I_6$, there are 15 regions in loop momentum space to consider.

1. The region where the momenta in all propagators are large. In this case, the Taylor expansion yields three-loop vacuum integrals in which all masses are either zero, or equal to $m_H$.

2. Six regions where one momentum, e.g. $k_1$, is small, while the others are large. Here, the Taylor expansion leads to products of two-loop vacuum integrals depending on $m_H$, times one-loop vacuum integrals depending on $M_W$ or $M_Z$.

3. Three regions where the momenta in two non-adjacent propagators, e.g. $k_1$ and $k_3$, are small, and the momenta in the other propagators are large.

4. Four regions where the momenta in three propagators that are connected to a common vertex, e.g. $k_1$, $k_2$ and $k_1 + k_2$, are small, and the momenta in the other propagators are large. In these regions and in the regions of type 3, the Taylor expansion leads to products of one-loop vacuum integrals depending on $m_H$, times two-loop vacuum integrals depending on $M_W$ and $M_Z$.

5. The region where all momenta are small. In this region, the Taylor expansion gives three-loop vacuum integrals depending only on $M_W$ and $M_Z$. These two-scale integrals are the most complicated ones that appear in the expansion. So far, only a few of the corresponding master integrals have been calculated in the literature [19].

Not all regions give non-vanishing contributions for all diagrams. In many cases, after the Taylor expansion we are left with scale-less integrals, which are zero in
dimensional regularization. Which regions actually do contribute depends on the distribution of large masses in the diagrams concerned. For example, in case 2 above, the region where \( k_1 \) is small only gives a non-vanishing contribution when \( m_1 \) is a small mass.

Obviously, the factorizable integrals coming from regions 2, 3 and 4 are easy to deal with, since they are simply products of one and two-loop vacuum integrals. The \( I_6 \) integrals coming from region 1 can be classified into ten different kinds, according to the distribution of massless and massive denominators they contain. For two of these categories, we follow the integration-by-parts \([20]\) reduction algorithm obtained by Broadhurst \([21]\), to reduce them to master integrals. Reductions for the eight other categories have been indicated by Avdeev \([22]\), but here, we prefer to use our own reduction routines. As a cross-check, we have also performed the reduction of these single-scale integrals using the automatic integral reduction package AIR \([23]\).

The master integrals themselves are known, some in terms of \( \Gamma \) functions, the others as expansions in \( \epsilon \) \([24, 25]\).

The three-loop integrals from region 5 appear to present a more difficult problem, since some of them depend on two mass scales: \( M_W \) and \( M_Z \). It turns out that, in the three-loop gauge boson self-energies up to the order \( m_H^4 \), they are all of the \( I_4 \) topology (sometimes with scalar products in the numerator). However, by using symmetry relations between such integrals, we find that they cancel out of the gauge boson self-energies at this order in \( m_H \), provided we sum over all diagrams.

The gauge boson self-energies we need are the “full” self-energies, consisting of a one-particle irreducible part plus terms containing Higgs tadpole insertions, which, in turn, include tadpole insertions themselves, as shown in Figures 2 and 3.

We have checked several Ward identities that are satisfied by the unrenormalized
self-energies. In particular, we have checked that the photon self-energy at zero momentum $\Sigma_{A A, 3 \text{-loop,full}}^T(0)$ vanishes at order $m^4_H$. For the would-be Goldstone bosons we have checked that $\Sigma_{\phi, 3 \text{-loop,full}}(0)$ and $\Sigma_{\phi^0, 3 \text{-loop,full}}(0)$ both vanish at order $m^6_H$. These identities are only valid for the self-energies including tadpole contributions.

3 Renormalization

We write the Standard Model Lagrangian without fermions as $\mathcal{L} = \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{fix}} + \mathcal{L}_{FP}$. The invariant part is

$$\mathcal{L}_{\text{inv}} = -\frac{1}{4} W_{\mu \nu} W^{\mu \nu} - \frac{1}{4} B_{\mu \nu} B^{\mu \nu} - (D_\mu \Phi)^\dagger (D^\mu \Phi) - \frac{1}{2} \lambda (\Phi^\dagger \Phi)^2 - \mu \Phi^\dagger \Phi$$  \hspace{1cm} (11)
where $W_{a\mu}$ and $B^{\mu\nu}$ denote the curvatures of the $SU(2)_L$ and $U(1)_Y$ gauge fields $W^a_\mu$ and $B^\mu$,

$$\Phi = \frac{1}{\sqrt{2}} \left( H + \sqrt{2} v + i\phi^0 \right)$$

is the Higgs doublet, and

$$D_\mu\Phi = \left( \partial_\mu - \frac{ig}{2} W_{a\mu}^{a} - \frac{ig'}{2} B_\mu \right) \Phi$$

its covariant derivative. The fields $\phi^\pm$, $W^{\pm}_{\mu}$, $Z_\mu$ and $A_\mu$ are defined by

$$\phi^\pm = \frac{1}{\sqrt{2}} \left( \phi^1 \pm i\phi^2 \right),$$

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}} \left( W_{\mu}^1 \pm iW_{\mu}^2 \right),$$

$$Z_\mu = c_W W_{\mu}^3 - s_W B_\mu,$$

$$A_\mu = s_W W_{\mu}^3 + c_W B_\mu,$$

and the coupling constants $g'$ and $g$ are related to each other by

$$g' = -g \frac{s_W}{c_W}.$$

The gauge fixing term is given by

$$\mathcal{L}_{fix} = -C^+ C^- - \frac{1}{2} (C^Z)^2 - \frac{1}{2} (C^A)^2$$
with

\begin{align}
C^\pm &= -\partial_\mu W^\pm,\mu + \xi M_W \phi^\pm \\
C^Z &= -\partial_\mu Z^\mu + \xi M_Z \phi^0 \\
C^A &= -\partial_\mu A^\mu
\end{align}

(20)

In equations (20)–(21), we have introduced a gauge parameter \(\xi\). The standard 't Hooft Feynman gauge fixing term corresponds to \(\xi = 1\). Finally, the Faddeev Popov Lagrangian \(\mathcal{L}_{FP}\) is derived from the variation of \(\mathcal{L}_{fix}\) under gauge transformations.

Because the tree level \(\rho\)-parameter is independent of the parameters in the Lagrangian, \(\Delta \rho^{(3)}\) is not affected by any three-loop counterterms. Therefore, we only need to renormalize the model up to two loops. Our renormalization scheme is similar to the one used in ref. [7]. However, in line with our calculation of the unrenormalized diagrams, the renormalization factors are systematically expanded in powers of \(m_H\), and only terms that can give a contribution of order \(m_H^4\) to \(\Delta \rho^{(3)}\) are retained, i.e. terms of order \(m_H^2\) in one-loop counterterms, and terms of order \(m_H^4\) in two-loop counterterms. This implies that we do not renormalize \(g\) or \(c_W\) at all.

In the Lagrangian, we express \(\lambda, \mu, v\) and \(M_Z\) in terms of the four independent parameters \(g, c_W, m_H\) and \(M_W\) by \(\mu = -\frac{1}{2}m_H^2, \lambda = g^2 m_H^2/(4M_W^2), v = \sqrt{\frac{2M_W}{g}}, \) and \(M_Z = M_W/c_W\). Then, the masses \(m_H\) and \(M_W\), and the Higgs and would-be Goldstone fields are replaced with bare masses and fields

\begin{align}
  m_H^{(0)} &= Z_m m_H \\
  M_W^{(0)} &= Z_M M_W \\
  H^{(0)} &= Z_H H \\
  \phi^{\pm,(0)} &= Z_H \phi^{\pm} \\
  \phi^{0,(0)} &= Z_H \phi^0
\end{align}

(23)

(24)

(25)

(26)

(27)

The gauge parameter \(\xi\) is renormalized in such a way as to compensate the effect of the above renormalizations in the gauge fixing term:

\[\xi^{(0)} = Z_H^{-1} Z_M^{-1} \xi\]

(28)

We choose the renormalized \(\xi\) to be equal to one. Each \(Z\)-factor is written as

\[Z = 1 - \delta^{(1)} - \delta^{(2)}\]

(29)

The renormalization constants \(Z_M\) and \(Z_H\) are fixed by imposing conditions on the renormalized \(W\) and \(\phi\) self-energies

\[\Sigma_T^{WW,ren}\big|_{p^2=0} \sim 0\]

(30)
and
\[ \frac{\partial}{\partial p^2} \Sigma^{\phi, \text{ren}}|_{p^2=0} \sim 0, \]  
where the notation \( X \sim 0 \) means that \( X \) does not contain any terms of order \( m_H^2 \) at one loop, or of order \( m_H^4 \) at two loops. These renormalizations remove all the \( m_H^2 \) and \( m_H^4 \) terms from the one and two-loop gauge boson self-energies, the \( \phi \) self-energies, and the mixings between \( \phi \)'s and gauge bosons. The Higgs mass renormalization constant \( Z_m \) is fixed by demanding that
\[ \text{Re} \Sigma^{HH, \text{ren}}|_{p^2+m_H^2=0} \sim 0. \]  

Solving equations (30)–(31), we find the following expressions in terms of vacuum integrals.

\[
\delta^{(1)}_H = \frac{1}{i(2\pi)^d M_W^2} g^2 I_1(m_H^2; 1) \frac{\varepsilon}{8(\varepsilon - 4)}
\]  

(33)

\[
\delta^{(1)}_M = \frac{1}{i(2\pi)^d M_W^2} g^2 I_1(m_H^2; 1) \frac{6 - \varepsilon}{4(\varepsilon - 4)}
\]  

(34)

\[
\delta^{(2)}_H = \left[ \frac{1}{i(2\pi)^d M_W^2} g^2 \right]^2 \left\{ I_1(m_H^2; 1)^2 \left( \frac{13}{128} + \frac{3\varepsilon}{64} + \frac{3}{16(\varepsilon - 4)} + \frac{1}{8(\varepsilon - 4)^2} \right) 
+ m_H^2 I_2(m_H^2, m_H^2, m_H^2; 1, 1, 1) \left( \frac{3\varepsilon}{64} + \frac{9}{64} + \frac{3}{8(\varepsilon - 4)} \right) 
+ m_H^2 I_2(m_H^2, 0, 0; 1, 1, 1) \left( \frac{\varepsilon}{64} + \frac{11}{64} + \frac{3}{4(\varepsilon - 4)} \right) \right\}
+ \frac{1}{i(2\pi)^d M_W^2} I_1(m_H^2; 1) \left( \frac{1}{4} + \frac{\varepsilon}{8} + \frac{1}{(\varepsilon - 4)} \right) \delta^{(1)}_m
\]  

(35)

\[
\delta^{(2)}_M = \left[ \frac{1}{i(2\pi)^d M_W^2} g^2 \right]^2 \left\{ I_1(m_H^2; 1)^2 \left( \frac{13}{32} - \frac{3\varepsilon}{32} + \frac{1}{8(\varepsilon - 4)^2} \right) 
+ m_H^2 I_2(m_H^2, m_H^2, m_H^2; 1, 1, 1) \left( -\frac{3\varepsilon}{32} + \frac{3}{8} + \frac{3}{8(\varepsilon - 4)} \right) 
+ m_H^2 I_2(m_H^2, 0, 0; 1, 1, 1) \left( -\frac{\varepsilon}{32} + \frac{5}{16} + \frac{3}{4(\varepsilon - 4)} \right) \right\}
+ \frac{1}{i(2\pi)^d M_W^2} I_1(m_H^2; 1) \left( 1 - \frac{\varepsilon}{4} + \frac{1}{(\varepsilon - 4)} \right) \delta^{(1)}_m
\]  

(36)

In order to determine the renormalization constant \( \delta^{(2)}_m \), we use analytical results for the two-loop on-shell Higgs self-energy from ref. [26]. Some care is needed here,
because the bare parameters \( M_{W0} \) and \( m_{H0} \) used in ref. [26] do not correspond exactly to our bare parameters \( m_{H0}^{(0)} \), \( M_{W0}^{(0)} \). This is due to the fact that we do not introduce a counterterm \( \delta v^2 \) for the Higgs tadpole, but instead, explicitly include tadpole contributions. Their \( m_{H0}/m_{H} \) corresponds to our \( Z_{m}Z_{H}/Z_{\Delta M} \). Taking this difference into account, we find

\[
\delta_{m}^{(1)} = \left( \frac{g^2}{16\pi^2} \right) \left( \frac{m_{H}^2}{4\pi} \right)^{-\frac{\varepsilon}{2}} \Gamma \left( 1 + \frac{\varepsilon}{2} \right) \left( 1 + \frac{\varepsilon}{2} \right) \frac{m_{H}^2}{M_{W}^2} \left\{ -\frac{3}{4\varepsilon} - \frac{9}{8} + \frac{3}{16}\pi\sqrt{3} \right. \\
+\varepsilon \left( -\frac{3}{32}\pi\sqrt{3}\log 3 + \frac{3}{16}\pi\sqrt{3} + \frac{1}{16}\pi^2 + \frac{3}{8}\sqrt{3}C - \frac{21}{16} \right) \right\},
\]

(37)

\[
\delta_{m}^{(2)} = \left( \frac{g^2}{16\pi^2} \right)^2 \left( \frac{m_{H}^2}{4\pi} \right)^{-\varepsilon} \Gamma^2 \left( 1 + \frac{\varepsilon}{2} \right) \frac{m_{H}^4}{M_{W}^4} \left\{ -\frac{45}{32}\pi^2 + \frac{1}{\varepsilon} \left( \frac{27}{64}\pi\sqrt{3} - \frac{189}{64} \right) \right. \\
-\frac{807}{256} - \frac{27}{128}\pi\sqrt{3}\log 3 + \frac{57}{32}\pi\sqrt{3} - \frac{39}{32}\pi C - \frac{261}{512}\pi^2 - \frac{9}{32}\sqrt{3}C + \frac{63}{32}\zeta(3) \right\}. 
\]

(38)

The correction to the \( \rho \)-parameter can now be written as

\[
\Delta \rho^{(3)} = \frac{\Sigma_{T,Z,Z,\text{3-loop,ren}}^{ZZZ}(0)}{M_{Z}^2} - \frac{\Sigma_{T,W,W,\text{3-loop,ren}}^{WZ}(0)}{M_{W}^2}. 
\]

(39)

The counterterms \( \delta_{m}^{(1)} \), \( \delta_{m}^{(2)} \) are substituted into equation (39) in two steps. In the first step, \( \delta_{H}^{(1)} \), \( \delta_{H}^{(2)} \), \( \delta_{M}^{(1)} \) and \( \delta_{M}^{(2)} \) are replaced with the expressions (33)–(36), (keeping \( \delta_{m}^{(1)} \) and \( \delta_{m}^{(2)} \) as symbols). This step is done before performing any expansions in \( \varepsilon \). At this point, all master integrals depending on \( M_{W} \) or \( M_{Z} \), which originate from small momenta regions, cancel out exactly, so that only master integrals depending on \( m_{H} \) are left. In the second step, the counterterms \( \delta_{m}^{(1)} \) and \( \delta_{m}^{(2)} \) are replaced with (37)–(38) and the master integrals are expanded in \( \varepsilon \), using results from refs. [24, 25]. In the expansions, singular terms of order \( 1/\varepsilon^j \), \( j = 1, 2, 3, 4 \) appear and cancel each other. The order \( \varepsilon \) term of \( \delta_{m}^{(2)} \) is not needed, because the coefficient of \( \delta_{m}^{(2)} \) in \( \Delta \rho^{(3)} \) is finite in the limit \( \varepsilon \to 0 \). Similarly, \( \delta_{m}^{(1)} \) is not needed beyond the term of order \( \varepsilon \) given in eq. (37).

As a check on the renormalization procedure, we have verified that the renormalized three-loop photon self-energy, the photon-Z mixing self-energy, and the \( \phi \) self-energies vanish at zero external momentum.

4 Results and conclusion

Finally, combined with the one-loop [13] and two-loop [7] terms, the three-loop correction to the \( \rho \)-parameter reads

\[
\rho = 1 + \Delta \rho^{(1)} + \Delta \rho^{(2)} + \Delta \rho^{(3)},
\]

(40)
with

\[ \Delta \rho^{(1)} = -\frac{3}{4} \frac{g^2}{16\pi^2} \frac{s_W^2}{c_W^2} \log \left( \frac{m_H^2}{M_W^2} \right), \tag{41} \]

\[ \Delta \rho^{(2)} = \left( \frac{g^2}{16\pi^2} \right)^2 \frac{s_W^2}{c_W^2} \frac{m_H^2}{M_W^2} \left( -\frac{21}{64} + \frac{9}{32} \sqrt{3} + \frac{3}{32} \pi^2 - \frac{9}{8} \pi \sqrt{3} \right) \]
\[ = \left( \frac{g^2}{16\pi^2} \right)^2 \frac{s_W^2}{c_W^2} \frac{m_H^2}{M_W^2} \left( 0.1499 \right), \tag{42} \]

\[ \Delta \rho^{(3)} = \left( \frac{g^2}{16\pi^2} \right)^3 \frac{s_W^2}{c_W^2} \frac{m_H^4}{M_W^4} \left( -\frac{21}{512} + \frac{729}{512} \pi \sqrt{3} - \frac{3391}{4608} \pi^2 - \frac{9}{16} \pi C \right) \]
\[ - \frac{1577}{2304} \pi^3 \sqrt{3} - \frac{9109}{69120} \pi^4 + \frac{99}{16} \sqrt{3} \log 3 \ C \]
\[ - \frac{297}{32} \sqrt{3} \text{Li}_3(2\pi/3) - \frac{399}{16} \sqrt{3} \ C + \frac{3043}{128} \zeta(3) \]
\[ + \frac{567}{32} C^2 + \frac{109}{8} \left( U_{3,1} - 36 V_{3,1} \right) \]
\[ = \left( \frac{g^2}{16\pi^2} \right)^3 \frac{s_W^2}{c_W^2} \frac{m_H^4}{M_W^4} \left( -1.7282 \right). \tag{43} \]

The constants appearing in \( \Delta \rho^{(3)} \) are defined by \[24, 25\]

\[ U_{3,1} = \frac{1}{2} \zeta(4) + \frac{1}{2} \zeta(2) \log^2 2 - \frac{1}{12} \log^4 2 - \text{Li}_4 \left( \frac{1}{2} \right) \]
\[ = -0.1178759965 \tag{44} \]

\[ V_{3,1} = \sum_{m>n>0} \frac{(-1)^m \cos(2\pi n/3)}{m^3 n} \]
\[ = -0.03901272636 \tag{45} \]

\[ C = \text{Cl}_2 \left( \pi/3 \right) \tag{46} \]

The log-sine integral is defined by

\[ \text{Ls}_3 (\theta) = -\int_0^\theta \sin^2 \theta \log \left| 2 \sin \frac{\phi}{2} \right| d\phi. \tag{47} \]

Some numerical values are shown in Table 1 and in Figure 4 where we have used

\[ g^2 = \frac{e^2}{s_W^2} = \frac{4\pi\alpha}{s_W^2} \tag{48} \]
Table 1: Corrections to $\rho$ as a function of $m_H/M_W$

| $m_H/M_W$ | $\Delta \rho^{(1)}$ | $\Delta \rho^{(2)}$ | $\Delta \rho^{(3)}$ |
|-----------|----------------------|----------------------|----------------------|
| 2         | -0.00078 1.14 $10^{-5}$ | -1.33 $10^{-7}$ |
| 5         | -0.0018 7.14 $10^{-6}$ | -5.20 $10^{-6}$ |
| 6         | -0.0020 0.000010 | -0.000011 |
| 7         | -0.0022 0.000014 | -0.000020 |
| 8         | -0.0024 0.000018 | -0.000034 |
| 9         | -0.0025 0.000023 | -0.000055 |
| 10        | -0.0026 0.000029 | -0.000083 |
| 15        | -0.0031 0.000064 | -0.00042 |
| 20        | -0.0034 0.00011 | -0.0013 |
| 25        | -0.0036 0.00018 | -0.0032 |
| 26        | -0.0037 0.00019 | -0.0038 |
| 27        | -0.0037 0.00021 | -0.0044 |
| 28        | -0.0038 0.00022 | -0.0051 |
| 29        | -0.0038 0.00024 | -0.0059 |
| 30        | -0.0038 0.00026 | -0.0067 |

for the weak coupling constant, with $\alpha = 1/137$ and $s_W^2 = 0.23$. While $\Delta \rho^{(3)}$ is very small for low values of $m_H$, it soon becomes more important than $\Delta \rho^{(2)}$. The two contributions cancel each other at $m_H \approx 6M_W \approx 480$ GeV. $\Delta \rho^{(3)}$ becomes equal to $\Delta \rho^{(1)}$ for $m_H \approx 2$ TeV.

Comparing our result for the large Higgs mass limit with the ones of the large top quark mass limit [11], we can say that for $m_H \approx 500$ GeV, $\Delta \rho^{(3)}$ is already larger than the three-loop pure electroweak correction of order $M_t^6$, which only grows as $m_H^2 \log(m_H^2/M_t^2)$ in the large $m_H$ limit. However, it is still very small compared with the mixed electroweak/QCD correction term of order $\alpha_s M_t^4$.

The original question that motivated this calculation was, whether inclusion of the three-loop corrections with strong interactions could lead to an effect mimicking the one-loop effects of a light Higgs boson. The result of this calculation shows that this is highly unlikely. The sign of the three-loop correction is the same as the one-loop correction. Therefore with increasing Higgs-mass the three-loop term only makes the effect grow faster, instead of the three-loop term partially cancelling the one-loop correction. Therefore the presence of a strongly interacting heavy Higgs-sector appears to be extremely unlikely, and the data can indeed be used as a strong indication for a light Higgs boson sector.
Figure 4: The combined effect of $\Delta \rho^{(1)}$, $\Delta \rho^{(2)}$ and $\Delta \rho^{(3)}$ as a function of $m_{H}/M_{W}$.

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