A study on immersion and submersion utilizing transversality theorem

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Abstract
In mathematics, submersion is considered as the differential function among differential manifolds, the differential in that function is surjective. Submersion is considered as the basic conceptual function of differential topology. In the case of immersion, it is considered as the differentiable function among differentiable manifolds, the differential in that function is injective. The submersion notion is dual to that of the notion of immersion. In this paper, aspects of generalized Immersion & Submersion via Transversality are explored. The standard material on the notions of Transversality and some definitions and examples that are needed are presented first. In order to get smooth structures to the manifolds and for studying the property of submersion and immersion transversality theorem will be very helpful. In the concept of differential topology, the transversality theorem is also named as Thom transversality theorem. It is used for understanding the properties of the transverse intersection of a smooth family of a smooth map. The Thom transversality mostly relies on Sard's theorem.

Keywords
Smooth Sub manifolds, Transverse, Immersion, Submersion, Regular Value of Smooth function \( f \), Regular Value Theorem.

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1. Introduction

(Gomes & Garcia, 2020) Transversality is the concept of analyzing the intersection of 2 objects. Transversality was introduced by Rene Thom a French mathematician in the 1950s. This was based on his research thesis which contains proof and statement which is a later stage became Thom’s Transversality. In this research modern types of ideas were provided in order to replace the older ones. It was sketched in a simplistic way for better understanding in which certain transformation was done to convert the geometric situation into a reduced form. This may help in the reduction of standard types, and this study is necessary to solve all problems.

(Smania, 2019) For the present scenario, it is important to consider the existing discrete invariants. Determining the list of discrete invariants is the first step but it is considered a difficult process in order to find it the values must be assigned with a continuous set of invariants value for each set. But in the case of linear algebra finding invariants is a simple task because of their structure containing matrices classification, solution for linear forms, and linear independence.

(Conway, 2018) In many problems, the general position appears which is the most common and the easier way to analyze when compared to others. For example, 4 straight lines of a plane where they form a quadrilateral structure when the straight lines meet in pairs with 6 points of intersection. The concept of general position is to accept the 2 separate notions which are openness and density.

Let us consider \( E \) as a set of quadruples in a straight line of a plane where \( A \) is considered as the subset for \( E \) set of quadruples which are in the form of a quadrilateral. The mentioned idea was cleared in 2 steps as follows:

- All the quadruple of \( E \) can be distorted into elements of \( A \) which small modification as \( A \) is dense in \( E \), all the
The notion of transversality, or “general position”, is a fundamental topic in differential topology. The openness and dense subset are capable enough for carrying the program of the general position. But sometimes our assumption may be wrong so it can be extended till the end of validity. The form of general position is strong in the openness set function. The sub-section of the open dense function is known as the residual set. The rational numbers will form a countable number of subsets of \( \mathbb{R} \) in which numbers are taken randomly and the taken numbers will be irrational.

(Beebe, 2020; Bogachev, 2018) Random numbers were selected with an intention so that another way for mathematicizing the concept of the general position. The mentioned concept will hold all except the set of measuring zero sets. The sets which are arising will have 2 properties at a similar time.

(Lerario & Stecconi, 2019) The main objective of this paper is to study the relative positions of 2 sub-manifolds and a sub-manifold map. The notion of general position was found to be transversality which will be discussed in detail in the following sections. The obtained result will be deduced using a basic theorem known as Baire’s Theorem and Sard’s theorem, and utilizing Thom’s transversality theorem.

2. Theorems based on Immersion and Submersion utilizing Transversality

The notion of transversality, or “general position”, is a fundamental topic in differential topology.

Definition 2.1. Suppose \( M \) is a smooth manifold of dimension \( m \) and \( S \) and \( T \) are smooth submanifolds of \( M \) of dimensions \( s \) and \( t \), respectively. We say \( S \) and \( T \) are transverse (or in general position), denoted \( S \cap T \), if for every \( p \in S \cap T \), the vector space spanned by \( T_p S \) and \( T_p T \) is \( T_p M \).

Example 2.2. Here are some examples of one-dimensional sub manifolds in \( \mathbb{R}^2 \): is illustrated in fig 1 as follows

![Figure 1. One-dimensional sub manifolds](image)

Example 2.3. In \( \mathbb{R}^3 \), two 2-submanifolds can intersect in general position in various ways, but two 1 submanifolds are in general position if and only if they are disjoint. This is easy to see from the definition: if \( S \) and \( T \) are 1 - submanifolds then \( T \cap S \) and \( T \cap T \) are 1 -dimensional vector space for any \( x \in S \cap T \). So they cannot generate the 3 -dimensional vector space \( T_x \mathbb{R}^3 = \mathbb{R}^3 \).

The main Theorem in the theory of transversality is the following:

**Theorem 2.4.** Suppose \( S^t, T^t \subset M^m \) are all compact, smooth manifolds embedded in Euclidean space. Then there exists a smooth \( \varepsilon \) - isotopy \( \varphi : M \rightarrow M \), ie, an isotopy that moves all points by a distance less than \( \varepsilon \), such that \( \varphi(T) \) is transverse to \( S \).

There are two special classes of smooth maps between manifolds which are particular interest in differential topology.

**Definition 2.5.** A smooth map \( f : M^m \rightarrow N^n \) is an immersion at a point \( p \in M \) if the differential \( D_p f : T_p M \rightarrow T_{f(p)} N \) is injective. We say \( f \) is an immersion at every point \( p \in M \). Clearly it is only possible to define an immersion if \( m \leq n \).

**Proposition 2.6.** If \( f : M \rightarrow N \) is an immersion at a point \( p \in M \) then there exists a neighborhood \( U \subset M \) of \( p \) such that \( f \) is an immersion at every \( x \in U \).

**Proof.** The differential \( D_p f \) is injective if and only if in the \( n \times m \) matrix \( A \) representing \( D_p f \), there is an \( m \times m \) minor with nonzero determinant. Since the determinant is a continuous function (in fact, it is smooth), this same minor has nonzero determinant on a neighborhood of \( p \). Hence \( D_x f \) will be injective on this neighborhood.

**Example 2.7.** 1. Suppose \( M \subset N \) is a submanifold. Then by definition for every \( x \in M \), there is a neighborhood \( U \subset N \) of \( x \) and a diffeomorphism \( \varphi : U \rightarrow \varphi(U) \subset \mathbb{R}^n \) such that \( \varphi(U \cap N) \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^m \times \mathbb{R}^{n-m} \). Consider the embedding \( i : M \rightarrow N \). For any \( x \in M, D_x i : T_x M \rightarrow T_x N \), is an injection, so \( i \) is an immersion.

For example, if \( m \leq n \) then the canonical inclusion \( \mathbb{R}^m \hookrightarrow \mathbb{R}^n \) \((x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_m, 0, \ldots, 0)\) is an immersion.

2. There are immersions that are not embeddings. For example, the degree \( n \) maps \( f_n : S^1 \rightarrow S^1, z \mapsto z^n \) are immersions (the tangent line to the circle is nondegenerate everywhere) but when \( |n| \geq 2, f_n \) is not one-to-one.

3. A parameterized differential curve is the image of a map \( g : [0, 1] \rightarrow \mathbb{R}^n \) such that \( g'(t) \neq 0 \) for all \( t \). The map \( g \) is an immersion and in many cases may not be an embedding.

Next we define the related notion of a submersion.

**Definition 2.8.** A smooth \( f : M^m \rightarrow N^n \) is a submersion at a point \( p \in M \) if the differential \( D_p f : T_p M \rightarrow T_{f(p)} N \) is surjective. We say \( f \) is a submersion if it is a submersion at every point \( p \in M \) clearly it is only possible to define a submersion if \( m \geq n \).
Proposition 2.9. If \( f : M \to N \) is a submersion at a point \( p \in M \) then there exists a neighborhood \( U \subset M \) of \( p \) such that \( f \) is a submersion at every \( x \in U \).

Proof. The differential \( D_p f \) is surjective if and only if in the \( n \times m \) matrix \( A \) representing \( D_p f \), there is an \( m \times m \) minor with nonzero determinant. Since the determinant is a continuous function (infact, it is smooth), this same minor has nonzero determinant on a neighborhood of \( p \). Hence \( D_p f \) will be surjective on this neighborhood.

Example 2.10. 1. The coordinate projections \( \mathbb{R}^{n+m} \to \mathbb{R}^n, (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \mapsto (x_1, \ldots, x_n) \) for all \( n, m \geq 0 \) are canonical examples of submersions. More generally, for any smooth manifolds \( M, N \), the coordinate projections \( M \times N \to M \) and \( M \times N \to N \) are submersions.

2. Suppose that \( f_1, \ldots, f_k : M \to \mathbb{R} \) are smooth functions on a smooth manifold \( M^m \) and define their vanishing locus

\[
Z = \{ p \in M | f_i(p) = 0 \text{ for all } 1 \leq i \leq k \}
\]

Define the map \( f = (f_1, \ldots, f_k) : M \to \mathbb{R}^k \). Then \( Z \) is the preimage of the point \( 0 = (0, \ldots, 0) \in \mathbb{R}^k \). Since each \( f_i \) is smooth, \( D_p f_i : T_p M \to \mathbb{R} \) is linear. Moreover, one can see that \( D_p f_i = (D_p f_1, \ldots, D_p f_k) : T_p M \to \mathbb{R}^k \) is surjective if and only if \( D_p f_1, \ldots, D_p f_k \) are linearly independent on \( T_p M \). Therefore \( f : M \to \mathbb{R}^k \) is a submersion if and only if the \( D_p f_i \) are linearly independent at every \( p \in M \), and in this case, the regular value Theorem implies \( Z \) is a smooth sub manifold of \( M \) of dimension \( m - k \). when the \( f_i \) are all polynomials, \( Z \) is called an algebraic variety.

The next theorem is fundamental in the study of immersions and submersions. Intuitively, it says that every immersion is locally a coordinate inclusion (as in Example 1) and every submersion is locally a coordinate projection (as in Example 4).

Theorem 2.11. Let \( f : M \to N \) be smooth map and take \( p \in M \).

1. If \( f \) is an immersion at \( p \) then there exist coordinate \( (U, \varphi) \) around \( p \) and \( (V, \psi) \) around \( f(p) \) such that in these local coordinates, the map \( \psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n \) is of the form \( \bar{x} \mapsto (\bar{x}, \bar{0}) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \)

2. If \( f \) is a submersion at \( p \) then there exist coordinate charts \( (U, \varphi) \) around \( p \) and \( (V, \Psi) \) around \( f(p) \) such that in these local coordinates, \( \varphi \circ f \circ \Psi^{-1} \) is of form \( (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \mapsto (x_1, \ldots, x_n) \)

2.4.1 Proof:

Proof. We prove (1) and remark that the proof of (2) is similar. Consider any charts \( (U, \varphi) \) and \( (V, \psi) \) around \( p \) and \( f(p) \), respectively. Then at \( \varphi(p) \in \varphi(U) \subset \mathbb{R}^m \), the differential of \( h = \psi \circ \varphi^{-1} \) is injective by the chain rule. In particular, \( \text{im } D_{\varphi(p)} h \) is an \( m \) - dimensional subspace of \( \mathbb{R}^n \). We may compose with an isometry \( L \) (a rotation) of \( \mathbb{R}^n \). We may compose with an isometry \( L \) (a rotation) of \( \mathbb{R}^n \) to arrange it so that \( D_{\varphi(p)} h \) lies in \( \text{ht subspace } \mathbb{R}^m \times \{0\} \subset \mathbb{R}^n \). Replacing \( (V, \psi) \) with \( (V, L \psi) \), we may just assume that this situation occurs to begin with.

Now the differential \( D_{\varphi(p)} h \) takes the tangent space at \( \varphi(p) \) into \( \mathbb{R}^m \times \{0\} \). We next change the coordinates so that the entire image of \( U \) lies in \( \mathbb{R}^m \times \{0\} \). First define map \( A : \mathbb{R}^n \to \mathbb{R}^n \) so that for all \( x \in \mathbb{R}^m, A(x, \bar{0}) = h(\bar{x}) \). Extend this to all of \( \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m} \) by \( A(\bar{x}, \bar{y}) = h(\bar{x}) + \bar{y} \). Then \( A \) is diffeomorphism so the charts \( (U, \varphi) \) and \( (A^{-1}(V), A^{-1} \psi) \) satisfy the conclusion of the theorem.

If \( f \) is both an immersion and a submersion at \( p \in M \) then of course \( p \) is a regular point of \( f \). The following theorem generalizes the regular value theorem.

Theorem 2.12. Suppose \( N_{11}^1 \) and \( N_{22}^2 \) are submanifolds of a smooth manifold \( M \) such that \( N_1 \cap N_2 \). Then \( N_1 \cap N_2 \) is a submanifold of \( M \) of dimension \( n_1 + n_2 - m \). Moreover, for every \( p \in N_1 \cap N_2 \) there exist coordinate systems \( N_1 \) and \( N_2 \) such that \( N_1 \) corresponds to \( \mathbb{R}^{n_1} \times \{0\} \) and \( N_2 \) to \( \{0\} \times \mathbb{R}^{n_2} \).

Lemma 2.13. For a smooth function \( f : M \to N \) where \( \text{dim} M \geq \text{dim} N \), if \( y \) is a regular value of \( f \) then \( \text{ker} (D_x f) = T_x (f^{-1}(y)) \) for every point \( x \in f^{-1}(y) \).

Proof. It’s easy to see that \( \text{ker} (D_x f) \supseteq T_x (f^{-1}(y)) \) but both are \((n-m)\) - dimensional vector spaces so they must be equal.

Main Proof:

Fix \( p \in N_1 \cap N_2 \). Since \( N_1 \) is a submanifold, there exists a neighborhood \( U_1 \subset M \) of \( p \) and a diffeomorphism \( \varphi_1 : U_1 \to \varphi(U_1) \subset \mathbb{R}^m \) such that \( \varphi_1 (U_1 \cap N_1) \subset \mathbb{R}^{n_1} \times \{0\} \). Define the orthogonal projection \( \pi_1 : \mathbb{R}^m \to \mathbb{R}^{m-n_1} \) and set \( \psi_1 \circ \varphi_1 \circ U_1 \to \mathbb{R}^{m-n_1} \). Then 0 is a regular value of \( \psi_1 \) so by the regular value theorem, \( N_1 \cap U_1 = \psi_1^{-1}(0) \) is a submanifold of \( M \) of dimension \( n_1 \). Performing the same construction with \( N_2 \) gives us a map \( \psi_2 : \pi_2 \circ \varphi_2 \circ U_2 \to \mathbb{R}^{m-n_2} \) such that \( U_2 \cap N_2 = \psi_2^{-1}(0) \) is a submanifold of \( M \) of dimension \( n_2 \). Set \( U = U_1 \cap U_2 \) so that \( \psi_1 \) and \( \psi_2 \) are both defined on \( U \). Define a map

\[
\Psi : U \to \mathbb{R}^{m-n_1} \times \mathbb{R}^{m-n_2} = \mathbb{R}^{2m-n_1-n_2}
\]

\[x \mapsto (\psi_1(x), \psi_2(x))\]

Now \( N_1 \cap N_2 \cap U = \psi_1^{-1}(0,0) \) so it suffices to show that \( (0,0) \) is a regular value of \( \Psi \), for then \( N_1 \cap N_2 \) will be a submanifold of \( M \) of dimension \( m - (2m - n_1 - n_2) = n_1 + n_2 - m \).

Consider the differential maps \( D_p \psi_1 : T_p M \to \mathbb{R}^{m-n_1} \) for any \( p \in \psi_1^{-1}(0,0) \). Then \( D_p \psi_1 \) is surjective onto \( \mathbb{R}^{m-n_1} \) and \( D_p \psi_2 \) is surjective onto \( \mathbb{R}^{m-n_2} \).

By Lemma, \( \text{ker} (D_p \psi_1) \cap \text{ker} (D_p \psi_2) = T_p N_1 \cap T_p N_2 \). But by transversality, \( T_p N_1 \) and \( T_p N_2 \) span \( T_p M \). So

\[\text{dim } \text{ker} (D_p \Psi) \geq n_1 + n_2 - m.\]
However, it is always true that \( \dim \ker (D_p \psi) \geq n_1 + n_2 - m \), so we have \( \dim \ker (D_p \psi) = n_1 + n_2 - m \). Hence by rank-nullity,

\[
\dim \ker (D_p \psi) = m - (n_1 + n_2 - m) = 2m - n_1 - n_2
\]

So \( D_p \psi \) is surjective. This shows that \((0, 0)\) is regular value of \( \psi \). So we conclude that \( N_1 \cap N_2 \) is a submanifold of the desired dimension.

For the second statement, use the fact that \( N_1 \cap N_2 \) is a submanifold to choose a neighborhood \( V \subset M \) containing \( p \) and a diffeomorphism \( \phi : V \rightarrow \phi(V) \subset \mathbb{R}^m \) such that \( \phi(V \cap N_1 \cap N_2) \subset \mathbb{R}^{n_1 + n_2 - m} \times \{0\} \).

Extend the target to \( \mathbb{R}^{n_1 + n_2 - m} \) by projection and abuse notation by letting \( f \) denote this function \( V \rightarrow \mathbb{R}^{n_1 + n_2 - m} \). Consider the map

\[
f : U \rightarrow \mathbb{R}^{m-n_2} \times \mathbb{R}^{n_1 + n_2 - m} \times \mathbb{R}^{m-n_1} = \mathbb{R}^m
\]

\[
x \mapsto (\psi_2(x), \phi(x), \psi_1(x))
\]

We claim that \( f \) is a diffeomorphism satisfying the conditions of the theorem. First, consider the differential \( D_p f : T_p M \rightarrow T_p N_1 \cap T_p N_2 = T_p (N_1 \cap N_2) \).

If \( v \in \ker (D_p \psi_2) \cap \ker (D_p \psi_1) = \ker (N_1 \cap N_2) \) then \( \phi(v) \in \mathbb{R}^{n_1 + n_2 - m} \), but \( \psi \) is a diffeomorphism so in this case \( v = 0 \). This implies that \( D_p f \) is nonsingular, and thus an isomorphism. Finally, since \( N_1 \cap U = \psi_1^{-1}(0) \) and \( N_2 \cap U = \psi_2^{-1}(0) \), it follows that \( f(N_1 \cap U) \subset \mathbb{R}^{n_1} \times \{0\} \) and \( f(N_2 \cap U) \subset \{0\} \times \mathbb{R}^{n_2} \).

**Example 2.14.** The torus \( T = S^1 \times S^1 \) does not admit an immersion into \( \mathbb{R}^2 \), but the punctured torus \( T^\ast \) does have this property. Consider the torus as the quotient space of a square, and expand the removed disk so that it almost captures all of the interior of the square is shown in fig 2. The resulting quotient space \( T^\ast = T \setminus D \) looks like interlocking bands joined together at one small square. Alternatively, these bands can be viewed as thickened strips around a meridian and longitude of the torus. To obtain an immersion \( f : T^\ast \rightarrow \mathbb{R}^2 \), simply embed \( T^\ast \) into \( \mathbb{R}^3 \) in a way such that there are no vertical tangent planes to \( T^\ast \) - note that this is not possible with the unpunctured torus, but removing a disk makes it possible.

Then define \( f : T^\ast \rightarrow \mathbb{R}^2 \) to be the restriction of the projection \( \mathbb{R}^3 \rightarrow \mathbb{R}^2 \), \((x, y, z) \mapsto (x, y)\) to \( B \) by construction, for every point \( p \in T^\ast \) the tangent plane \( T_p T^\ast \) is spanned by two vectors \( v_1 = (x_1, y_1, z) \) and \( v_2 = (x_2, y_2, z) \) with at least one of \( x_1, y_1 \) nonzero and at least one of \( x_2, y_2 \) nonzero. The differential \( d_p f \) takes these vectors to \( d_p f (v_1) = (x_1, y_1) \) and \( d_p f (v_2) = (x_2, y_2) \) in \( T_f (x) \subset \mathbb{R}^2 \), since \( v_1 \) and \( v_2 \) were assumed to span \( T_f T^\ast \), \((x_1, y_1) \) and \((x_2, y_2) \) cannot be parallel. Therefore \( d_p f (v_1) \) and \( d_p f (v_2) \) span \( \mathbb{R}^2 \), so \( d_p f \) is an isomorphism and in particular injective. Hence \( f \) is an immersion.

### 3. The Analytic Case

The analytic condition is described using it is considered as the stratified set. Since \( A_0 \) is defined by analytic conditions (at least in a suitable chart), it is a stratified set. It suffices to bound the codimension of the stratum \( S \cdots A_0 \) of maximal dimension. Let us consider the restricted projection

\[
(\pi^p p - 1 | s) : SJ^{p-1}(X, Y)
\]

and the associated rank map \( S \ni a \mapsto \text{rank} d_{x_0} (\pi^p p - 1 | s) \in \{0, 1, \ldots, \min \{\dim S, \dim J^{p-1}(X, Y)\}\} \).

Let us also consider an open subset \( U \) of \( S \) such that the rank map is constant on \( U \). Such a subset exists: for instance we can take as \( U \) the preimage of the maximum value attained by the map (this preimage is open because the rank map is lower semicontinuous).

It follows from the constant-rank theorem that, up to further restricting \( U \) if necessary, \( \pi^p p - 1 \) is a submanifold of \( J^{p-1}(X, Y) \). Let us call \( V \) this manifold, We claim that \( U \subset \tilde{V} \).

Where \( \tilde{V} \subset J^p(X, Y) \) is defined as

\[
\tilde{V} = \{f \in J^p(X, Y) : J^{p-1} f \text{ is not transverse to } V \text{ at } x\}. \]

Since the conclusion is assumed to be true for \( p = 1 \), we have \( \text{codim} V^{n+1} + 1 \), hence the claim implies \( \text{codim} U^{n+1} \). Since \( U \) is open in \( S \) and \( S \) is the stratum of maximal dimension, we get

\[
\text{codim} A_0 = \text{codim} S = \text{codim} U \geq n + 1
\]

which proves the proposition. Let us now prove the claim \( U \subset \tilde{V} \). Given any \( a = j^p x f \in U \), we have

\[
T \pi^p p - 1 a V = (\pi^p p - 1 | s) (T_p U) \subset \pi^p p - 1 (T_p A)
\]

Which implies that \( a \in \tilde{V} \) as desired.

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