Precise sinusoidal signal extraction from noisy waveform in vibration calibration

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Received 28 September 2021, revised 2 May 2022
Accepted for publication 4 May 2022
Published 26 May 2022

Abstract

Precise extraction of sinusoidal vibration parameters is essential for the dynamic calibration of vibration sensors, such as accelerometers. However, several standard methods have not yet been optimized for large background noise. In this work, signal processing methods to extract small vibration signals from noisy data in the case of accelerometer calibration are discussed. The results show that spectral leakage degrades calibration accuracy. Three methods based on the use of a filter, window function, and numerical differentiation are investigated to reduce the contribution of the calibration system noise. We demonstrate the effectiveness of these methods with theoretical calculations, simulations, and experiments. The uncertainty of microvibration calibration in the National Metrology Institute of Japan is reduced by two orders of magnitude using the proposed methods. We demonstrate the effectiveness of these methods with theoretical calculations, simulations, and experiments. The uncertainty of microvibration calibration in the National Metrology Institute of Japan is reduced by two orders of magnitude using the proposed methods. We recommend to use a combination of numerical differentiation and either a time-domain window function or a bandpass filter for most accelerometer microvibration sensitivity calibrations. For suppressing the effects of line noise, adjusting the data set length is also effective.

Keywords: signal processing, vibration calibration, sine approximation method, laser interferometer, accelerometer, micro vibration

(Some figures may appear in colour only in the online journal)

1. Introduction

Microvibration measurement is required in various fields, such as infrastructure health monitoring [1] or satellite performance analysis [2]. Several types of sensors have been developed, such as high-sensitivity accelerometers, broadband seismometers, or low-noise microelectromechanical system accelerometers (e.g., [3–5]). The calibration of the sensor frequency response is essential for the reliability of microvibration measurements. As the applications of microvibration measurement increase, measuring responses to small input vibrations is becoming increasingly important.

Accurate extraction of amplitude and phase from a sinusoidal waveform is required for the calibration of accelerometer sensitivity according to standardized methods [6]. The target accelerometer is sinusoidally vibrated by a vibration exciter, and the amplitudes and phases of the sensor voltage signal and reference displacement signal are estimated and compared to calibrate the sensitivity and phase shift of the accelerometer. The estimation accuracy is essential for the calibration uncertainty. The requirement for primary calibration is about 0.1% and 0.1° for the amplitude and phase, respectively [7, 8]. In typical calibrations [7, 8], a large vibration amplitude (e.g., 10 m s\(^{-2}\)) is applied. To obtain the response to a microvibration with an amplitude down to about 10\(^{-3}\) m s\(^{-2}\), the applied vibration amplitude should also be small (on the same order of magnitude) because the linearity of the response is not ensured in general. The extraction of such a small vibration signal usually suffers from the background noise of the calibration system, which originates from the background vibration or electrical noise.

For accelerometer calibration, amplitude and phase extraction are performed by the sine approximation method (SAM) in ISO16063-11 [6]. However, the SAM is not optimized for
real cases with large background noise, as we have demonstrated in this paper. In addition to the background noise reduction of the calibration system, the effect of the background noise in signal processing should be minimized. Some of the calibration institutes may empirically apply a digital filter to deal with the problem, although an unified approach has not yet been established. Another signal processing method using correlation has also been proposed [9]. In this paper, we discuss the limitations of the SAM and its optimization through theoretical investigations, simulations, and experiments. These investigations are important to calibrate accelerometers under a large noise or with a small vibration amplitude. The results can be applied not only to accelerometer calibration but also to other signal processing settings that require accurate sinusoidal parameter estimation.

The remainder of this paper is organized as follows: section 2 summarizes the mathematical background of the SAM and its verification with simulations. Section 3 proposes optimization methods, including filtering, changing the window function, and numerical differentiation. Section 4 applies the proposed method to vibration calibration in the National Metrology Institute of Japan (NMIJ).

2. Contribution of system noise to sinusoidal signal extraction

2.1. Mathematical framework of the conventional SAM

An overview of the accelerometer calibration system is depicted in figure 1. The accelerometer is sinusoidally vibrated by the vibration exciter, and its output signal \( V_\text{s} \) (sensor signal) is recorded along with the reference displacement signal \( r \) (reference signal) measured by the laser interferometer. The sensor and reference signals have the units of voltage [V] and displacement [meters (m)], respectively. Throughout this paper, signal processing is assumed to be digital signal processing using discretely sampled data. The SAM specified in ISO16063-11 [6] determines the sensitivity modulus and phase shift from two waveforms. The amplitude and phase of each waveform at vibration frequency \( f_\text{v} \) are extracted in the following process. Here, a continuous waveform \( x(t) \) is sampled within \( t_0 < t < t_0 + T \) at \( t_n = t_0 + nT/N \) \((n = 0, 1, \ldots, N - 1)\), where \( t_0 \) is the start time of the measurement, \( T \) is the time length, and \( N \) is the number of data. The sampling frequency is given by \( f_s = N/T \). The recorded data \( x_n \equiv x(t_n) \) is modeled as

\[
x_n = b_0 + b_1 \cos(2\pi f_s t_n) + b_2 \sin(2\pi f_s t_n) + \epsilon_n, \tag{1}
\]

where \( b_0, b_1, \) and \( b_2 \) are the fitting parameters, and \( \epsilon_n \) is the residual from the model. In this study, we assume that the sampling is sufficiently faster than the vibration frequency, \( f_s \gg f_\text{v} \), and the data length is an integer multiple of the vibration period, \( T = N_c / f_\text{v} \) \((N_c: \text{integer})\). The former condition is achievable using a commercially available digitizer that has a sampling rate of \( >10 \text{ MHz}\). A typical target frequency for the vibration calibration is below \( 20 \text{ kHz}\); hence, it is easy to set \( f_s / f_\text{v} > 500 \). With such a sampling rate, the difference between \( f_\text{v}T \) and an integer value can be kept \(< 1/500 \) by setting an appropriate number of record data because the ratio of the temporal resolution to the vibration period is \( f_s / f_\text{v} \). As explained in appendix A, a difference of this magnitude has only a negligible effect on the estimated parameters. Therefore, \( f_sT \) can be treated as an integer without introducing any significant error. Under these conditions, the optimal parameters that minimize the residual \( \sum \epsilon_n^2 \) are given by

\[
\begin{align*}
b_1 &= \frac{2}{N} \sum_{n=0}^{N-1} x_n \cos(2\pi f_s t_n), \\
b_2 &= \frac{2}{N} \sum_{n=0}^{N-1} x_n \sin(2\pi f_s t_n),
\end{align*}
\tag{2}
\]

which are identical to the Fourier coefficients of the data. The derivation of equation (2) based on [10] is given in appendix A. Here, the estimated complex amplitude for the variable \( x \) is defined as \( \hat{x}_{\text{est}} \equiv b_1 + ib_2 \) for convenience. The amplitudes are computed for both the sensor signal, \( V_\text{s} \), and the reference signal, \( x_r \), to calculate the sensitivity modulus and phase delay, which are given by

\[
\begin{align*}
S_{\text{cal}} &= \frac{|V_{\text{s, est}}|}{(2\pi f_s)^2 |x_{r, \text{est}}|} \quad \text{and} \\
\Delta \phi_{\text{cal}} &= \arg(V_{\text{s, est}}) - \arg(x_{r, \text{est}}) - \pi,
\end{align*}
\tag{3}
\]

respectively.

Generally, the estimated parameters depend on the start time, \( t_0 \), because the background noise is different for each measurement. Using equation (2), the complex amplitude \( \hat{x}_{\text{est}} \) can be modified as

\[
\hat{x}_{\text{est}}(t_0) = \hat{x}_{\text{est}}(t_0) - \left( \frac{2}{N} \sum_{n=0}^{N-1} x_n e^{2\pi i n t_0} \right)
= \frac{2}{N} \int_{-\infty}^{\infty} w(t - t_0) x(t) e^{2\pi i t_0} \sum_{n=-\infty}^{\infty} \delta(t - t_n) dt
= \frac{2}{N} \int_{-\infty}^{\infty} \mathcal{F}[w(t - t_0)x(t)](f) \cdot \mathcal{F}[\delta(t)](f_s - f) df.
\tag{4}
\]

Here,

\[
w(t - t_0) = w_s(t - t_0) = \begin{cases} \frac{1}{2} (t_0 < t < t_0 + T) \\
0 \quad \text{(otherwise)}
\end{cases}
\tag{5}
\]
are a rectangular window function and a comb function, respectively. $F[y(t)](f)$ denotes the Fourier transform of the function $y(t)$. In the second line of equation (4), the discrete sampling and finite measurement time range are expressed using the comb and window functions, respectively. The third line can be derived using the convolution theorem. Since the comb function has a period of 1, it can be expanded as a Fourier series:

$$\delta_c(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT)$$

Equation (4) then becomes

$$\hat{x}_{est}(t_0) = \frac{2}{T} \sum_{m=-\infty}^{\infty} F[w(t - t_0)x(t)](f \mp mfs)e^{2\pi imf_0}.$$  (9)

Here, we assume that the sampling frequency is sufficiently higher than that of the frequency components in $x(t)$. This can be achieved by applying a proper anti-aliasing filter before sampling the signal and using a digitizer with a sufficiently high sampling rate. Under these conditions, the Fourier transform of the windowed signal, $F[w(t - t_0)x(t)](f)$, is small in $|f| > fs/2$; hence, the $m \neq 0$ terms in equation (9) are negligible, and the estimated amplitude of $x(t)$ can be approximated by the $m = 0$ term as follows:

$$\hat{x}_{est}(t_0) \approx \frac{2}{T} F[w(t - t_0)x(t)](f_0)$$

where sinc function is defined as $\text{sinc}(z) = \sin(\pi z)/\pi z$. Although the amplitude of the purely sinusoidal vibration $x_{0}(t) = x_0 \sin(2\pi f_0 t)$ is calculated to be $|\hat{x}_{est}| = 2x_0$, the background noise at $f_0$, or the leakage from $f \neq f_0$, can be a problem in real-world cases.

2.2. Comparison with discrete Fourier transformation

The amplitude of a sinusoidal signal can be estimated using discrete Fourier transformation (DFT) as discussed in [11]. As shown above, the SAM is equivalent to the DFT under the following conditions:

- The sampling frequency is sufficiently higher than the vibration frequency: $fs \gg f_0$.
- The measurement time is an integer multiple of the vibration period: $T = Nf$.  

In the first condition is satisfied, the SAM corresponds to the first line of equation (4), which is equivalent to the discrete-time Fourier transformation at $f_0$ for the data windowed in $[t_0, t_0 + T]$. If the second condition is also satisfied, $f_0$ is included in the frequency series of the DFT, and equation (4)
is equivalent to the DFT. As explained previously, it is possible to satisfy these conditions using commercially available equipment.

As a consequence of the equivalence of the SAM and DFT, the discussion later in this paper based on equation (11) holds for both methods. Therefore, regardless of whether the SAM or DFT is used, the effects of spectral leakage from the random noise cannot be avoided. In some previous studies, it has been stated that the spectral leakage can be eliminated by using the DFT. This is true for the periodic components, such as the harmonics of the vibration, whose frequencies are included in the DFT. For the periodic components, such as the harmonics of the vibration, whose frequencies are included in the DFT: \( f_n = n/T \) (\( n = 0, 1, \ldots, N - 1 \)). However, since random noise is aperiodic in the time domain (in other words, it has a continuous spectrum in the frequency domain), in general, its contribution cannot be treated properly within the framework of the DFT and leakage from \( f \neq f_n \) occurs.

2.3. SAM with noise

Figure 2 shows the signal flow model for accelerometer calibration shown in figure 1. A similar discussion is applicable to other experiments conducted to measure the amplitude ratio between two vibration timeseries. The error sources considered in figure 2 are divided into four types:

- Independent random background noise of each device \( (n_x, n_y) \)
- Common random noise to both sensor and reference signals \( (n_x) \)
- Independent line noise or harmonics of each device \( (l_x, l_y) \)
- Common line noise or harmonics to both sensor and reference signals \( (l_x) \).

In the calibration process, sinusoidal vibration \( x_0 = \hat{x}_0 \sin(2\pi f_n t) \) is applied. The random vibration noise \( n_x \) and line noise \( l_x \) are added due to the background vibration, electrical noise of the system, or the distortion of the waveform. Here, \( l_x \) includes the line noise, such as the power supply noise, which appears at constant frequencies, and the harmonics of the input vibration, which appear at \( 2f_n, 3f_n, \) and so on. Then, the reference signal measures the displacement of the waveform, while the sensor to be calibrated responds to the second derivative of it. Each output signal contains the independent random and line noise; \( n_x, l_x, n_y, \) and \( l_y \). The sensor outputs the signal with the sensitivity modulus of \( S \). The recorded signals are the sum of these contributions, as shown in figure 2. The calculated amplitudes of the recorded signals \( V_{x, \text{est}} \) and \( \hat{x}_{x, \text{est}} \) are affected by the noise components as

\[
\hat{V}_{x, \text{est}} = S(\hat{x}_{0, \text{est}} + \hat{n}_{x, \text{est}} + \hat{\bar{l}}_{x, \text{est}} + \hat{n}_{x, \text{est}} + \hat{l}_{x, \text{est}}) \quad (13)
\]

\[
\hat{x}_{x, \text{est}} = \hat{x}_{0, \text{est}} + \hat{n}_{x, \text{est}} + \hat{\bar{l}}_{x, \text{est}} + \hat{n}_{x, \text{est}} + \hat{l}_{x, \text{est}}. \quad (14)
\]

which results in \( S_{\text{cal}} \neq S \) due to the second or later terms. Note that the harmonics generated through a non-linear process, such as the non-linearity of the sensor, are not independent of the input signal; for example, \( l_x \) can be correlated to \( l_y \) in general. However, we assumed that the error sources in figure 2 were independent of each other for simplicity. The random noise is discussed in section 2.4, and the line noise and harmonics are discussed in section 2.5.

2.4. Effect of random noise

We consider the general case for random noise \( n(t) \), which is characterized with (one-sided) power spectral density (PSD) \( G(f) \). The standard deviation of the estimated amplitude for the noise, \( \sqrt{\langle |n_{\text{est}}|^2 \rangle} \), is the standard uncertainty of the amplitude of \( x(t) = x_0(t) + n(t) \). From equation (11), the real and imaginary parts of \( n_{\text{est}}(t) \) can be written as

\[
\text{Re}[n_{\text{est}}(t)] = \frac{1}{T} \int_{-\infty}^{\infty} \tilde{W}(f) \left( \tilde{N}(f - f) + \tilde{N}^*(f + f) \right) e^{-2\pi ift} df. \quad (15)
\]

\[
\text{Im}[n_{\text{est}}(t)] = \frac{1}{T} \int_{-\infty}^{\infty} \tilde{W}(f) \left( \tilde{N}(f - f) - \tilde{N}^*(f + f) \right) e^{-2\pi ift} df. \quad (16)
\]

\( \tilde{N}(f) \) is the Fourier spectrum of \( n(t) \). The PSD corresponding to the Fourier spectrum \( \tilde{N}(f - f) \pm \tilde{N}^*(f + f) \) is given by \( G(|f - f|) + G(|f + f|) \). Therefore, the variances of equations (15) and (16) are both given by the integral of \( \tilde{W}(f)^2 \left[ G(|f - f|) + G(|f + f|) \right] / T^2 \). This is because, according to the Wiener–Khinchin theorem or to Parseval’s theorem, the variance of a random signal is equal to the integral of its PSD. Since \( \text{Re}[n_{\text{est}}(t)] \) and \( \text{Im}[n_{\text{est}}(t)] \) are independent from each other because of the randomness of \( n(t) \), the amplitude estimation uncertainty under the random noise is derived as

\[
\sqrt{\langle |n_{\text{est}}|^2 \rangle} = \sqrt{\langle \text{Re}[n_{\text{est}}]^2 \rangle} = \sqrt{\langle \text{Im}[n_{\text{est}}]^2 \rangle} = \sqrt{\int_0^\infty \frac{\tilde{W}(f)^2}{T^2} \left[ G(|f - f|) + G(|f + f|) \right] df}
\]

\[
= \sqrt{\int_0^\infty \frac{\tilde{W}(f)^2}{T^2} G(|f - f|) df + \int_{-\infty}^0 \frac{\tilde{W}(-f)^2}{T^2} G(|f + f|) df}
\]

\[
= \sqrt{\int_{-\infty}^\infty \frac{\tilde{W}(f)^2}{T^2} G(|f - f|) df}
\]

\[
= \sqrt{\int_{-\infty}^\infty \frac{\tilde{W}(f + f)^2}{T^2} G(|f|) df}. \quad (17)
\]
Here, we have used $|\tilde{W}(-f)| = |\tilde{W}(f)|$. Equation (17) represents the uncertainty in the estimate of the vibration amplitude under random noise that has a PSD given by $G(f)$. The uncertainty follows a Gaussian distribution.

Note that the uncertainty in the phase estimation (in radians) under random noise is identical to the relative uncertainty in the amplitude:

$$u(\text{arg}[\hat{x}_{\text{est}}]) = u \left( \arctan \left( \frac{\text{Im}[x_0 + \hat{n}_{\text{est}}]}{\text{Re}[x_0 + \hat{n}_{\text{est}}]} \right) \right) = \sqrt{\frac{\langle [\hat{n}_{\text{est}}]^2 \rangle}{x_0}} = \frac{u(\hat{x}_{\text{est}})}{x_0}. \quad (18)$$

This is because the distribution of $\hat{n}_{\text{est}}(t_0)$ is isotropic in the complex plane. The component with the same argument as $x_0$ is the amplitude error, and the orthogonal component is the phase error. Therefore, we investigated only the amplitude uncertainty using equation (17) for the random noise in the following subsections and section 3 for simplicity as the same results are applicable to the phase. Their equivalence is confirmed in section 4.

### 2.4.1. Independent random noise for the sensor and reference.

The sensor and reference signals contain independent random background noise $n_s$ and $n_r$. They contribute to the measurement result via $\hat{n}_{s,\text{est}}$ and $\hat{n}_{r,\text{est}}$, which have random values for different measurements. Their standard deviations are the standard uncertainty of the amplitude $V_s$ and $x_r$. Their expressions $\langle [\hat{n}_{s,\text{est}}]^2 \rangle$ and $\langle [\hat{n}_{r,\text{est}}]^2 \rangle$ are given by equation (17) using the PSDs $G_s(f)$ and $G_r(f)$. Consequently, the relative standard uncertainty of the calibration sensitivity in the absence of the other noise sources is

$$\frac{u(S_{\text{cal}})}{S} = \sqrt{\left( \frac{u(\hat{V}_{s,\text{est}})}{S[x_0]^{0.5}} \right)^2 + \left( \frac{u(\hat{x}_{r,\text{est}})}{x_0^{0.5}} \right)^2}. \quad (19)$$

$$= \frac{1}{(2\pi f_v)^2 x_0} \times \int_{-\infty}^{\infty} \frac{|\tilde{W}(f - f_v)|^2}{T^2} \{ G_s(f) + (2\pi f_v)^2 G_r(|f|) \} \, df. \quad (20)$$

In the limit of long measurement time $T$, the window function is asymptotically identical to the Dirac delta function as $\lim_{T \to \infty} |\tilde{W}(f)|^2 = T\delta(f)$; hence, the amplitude estimation uncertainty is

$$\sqrt{\langle [\hat{n}_{s,\text{est}}]^2 \rangle} \rightarrow \frac{G_s(f_v)}{T}. \quad (21)$$

This relation indicates that the uncertainty is determined by the noise spectrum at the vibration frequency and is inversely proportional to the square-root of the measurement time. Equation (21) gives the theoretical limit of measurements, which cannot be avoided unless the background noise of the system is reduced.

In reality, spectral leakage happens as shown in equation (17). This means that even if the background noise is small at $f_v$, the overall signal-to-noise ratio (S/N) can be degraded by the noise in the other frequency band. The amount of spectral leakage is determined by the shape of the window function. Therefore, the leakage can be reduced by changing the window function $w(t)$ or filtering the signal $V_s$ and $x_r$ to suppress the noise at $f \neq f_v$. The details of these modifications are discussed in section 3.

To validate the calculations above, a simulation of amplitude estimation was performed (figure 3). The excited waveform was $x_0 = x_0 \sin(2\pi f_v t)$ with $(2\pi f_v)^2 x_0 = 1$ m s$^{-2}$ and $f_v = 1$ Hz. The data length was set to $T = 100$ s. The noise $n_r$ was randomly generated 300 times, and the vibration amplitude was estimated from each $x_0 + n_r$ using equation (2). The standard deviation of the estimated amplitudes $\hat{x}_{\text{est}}$ relative to the true amplitude $x_0$ was calculated and compared to the theoretical expectation from equation (17). For the PSD of the noise, $G_s(f)$, two cases were considered as examples: the flat spectrum $(2\pi f_v)^2 G_s(f) \approx 10^{-2}$ (m s$^{-2}$) Hz$^{-1/2}$ and the frequency-dependent spectrum. The latter had the same noise levels as the former at the vibration frequency, $(2\pi f_v)^2 G_s(f) \approx 10^{-2}$ (m s$^{-2}$) Hz$^{-1/2}$, while having a larger noise in $f \neq f_v$. Figure 3 shows the simulation results, which agree with the calculation obtained using equation (17). The result demonstrates the importance of broadband noise suppression.

### 2.4.2. Common random noise for the sensor and reference.

The vibration noise of the exciter and reference sensors is random, the overall signal-to-noise ratio (S/N) can be degraded by the noise in the other frequency band. The amount of spectral leakage is determined by the shape of the window function. Therefore, the leakage can be reduced by changing the window function or filtering the signal $V_s$ and $x_r$ to suppress the noise at $f \neq f_v$. The details of these modifications are discussed in section 3.

To validate the calculations above, a simulation of amplitude estimation was performed (figure 3). The excited waveform was $x_0 = x_0 \sin(2\pi f_v t)$ with $(2\pi f_v)^2 x_0 = 1$ m s$^{-2}$ and $f_v = 1$ Hz. The data length was set to $T = 100$ s. The noise $n_r$ was randomly generated 300 times, and the vibration amplitude was estimated from each $x_0 + n_r$ using equation (2). The standard deviation of the estimated amplitudes $\hat{x}_{\text{est}}$ relative to the true amplitude $x_0$ was calculated and compared to the theoretical expectation from equation (17). For the PSD of the noise, $G_s(f)$, two cases were considered as examples: the flat spectrum $(2\pi f_v)^2 G_s(f) \approx 10^{-2}$ (m s$^{-2}$) Hz$^{-1/2}$ and the frequency-dependent spectrum. The latter had the same noise levels as the former at the vibration frequency, $(2\pi f_v)^2 G_s(f) \approx 10^{-2}$ (m s$^{-2}$) Hz$^{-1/2}$, while having a larger noise in $f \neq f_v$. Figure 3 shows the simulation results, which agree with the calculation obtained using equation (17). The result demonstrates the importance of broadband noise suppression.

In the limit of long measurement time $T$, the window function is asymptotically identical to the Dirac delta function as $\lim_{T \to \infty} |\tilde{W}(f)|^2 = T\delta(f)$; hence, the amplitude estimation uncertainty is

$$\sqrt{\langle [\hat{n}_{s,\text{est}}]^2 \rangle} \rightarrow \frac{G_s(f_v)}{T}. \quad (21)$$

This relation indicates that the uncertainty is determined by the noise spectrum at the vibration frequency and is inversely proportional to the square-root of the measurement time. Equation (21) gives the theoretical limit of measurements, which cannot be avoided unless the background noise of the system is reduced.

In reality, spectral leakage happens as shown in equation (17). This means that even if the background noise is small at $f_v$, the overall signal-to-noise ratio (S/N) can be degraded by the noise in the other frequency band. The amount of spectral leakage is determined by the shape of the window function. Therefore, the leakage can be reduced by changing the window function or filtering the signal $V_s$ and $x_r$ to suppress the noise at $f \neq f_v$. The details of these modifications are discussed in section 3.

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Figure 3. Simulated results for the standard deviation of the amplitude estimates for two different types of noise spectrum: flat noise (left-hand column) and frequency-dependent noise (right-hand column). The simulated waveforms (top row), PSDs of these waveforms (center row), and histograms of the estimated amplitudes (bottom row) are shown. Standard deviations expected from equation (17) are shown in the PSD graphs, and the simulated values of the standard deviations are shown next to the histograms.

To validate the calculation and show the order of the uncertainty contribution, we performed a simulation similar to that shown in section 2.4.1. The excited acceleration amplitude was fixed to \((2\pi f_v)^2 \hat{x}_0 = 1 \text{ m s}^{-2}\) for frequency varying from 0.1 Hz to 300 Hz. The data length was set to \(T = 100/f_v\). The background vibration spectrum in the acceleration unit, \((2\pi f_v)^2 \sqrt{G_x(f_v)}\), of the calibration system in NMIJ is shown in figure 4. The smoothed spectrum model was used for the simulation. The noise below 0.1 Hz and above 500 Hz was assumed to be proportional and inversely proportional to the frequency, respectively, along the line of figure 4; however, at those frequencies, the contribution to the simulated result was small. The common vibration noise \(n_x\) was randomly generated 20 times at each frequency, and the amplitude of the reference displacement, \(\hat{x}_r\), was estimated from each \(\hat{x}_0 + n_x\) using equation (2). Then, the same time series were numerically differentiated twice to prepare the sensor signal \(S(\ddot{x}_0 + \ddot{n}_r)\), which was used to estimate the amplitude \(\hat{V}_{s, \text{est}}\). The standard deviation of the estimated amplitude ratio \(\hat{V}_{s, \text{est}}/(2\pi f_v)^2 \hat{x}_r, \text{est}\) relative to the true sensitivity modulus \(S\) was calculated at each frequency and compared with the theoretical result obtained from equation (23). The results are shown in figure 4. Equation (23) explains the simulated standard deviation of the sensitivity modulus. The contribution ranged from 0.1% to 1% around 100 Hz, which is not ignorable in accelerometer calibration.

2.5. Effect of line noise and harmonics

The error components include both the line noise and the harmonics of the input vibration having constant amplitudes and phases. The difference between them is that the frequency of the former is fixed and independent of \(f_v\), while the latter always appears at integer multiples of \(f_v\). Their contributions can be treated in the same way using equation (11). Here, we consider the general case of the line noise \(\hat{l}(t) = l(t) \sin(2\pi f_l t + \phi_l)\) added to a sinusoidal wave \(\hat{x}_0(t) = \hat{x}_0 \sin(2\pi f_v t)\). Although the actual line noise is not monofrequent, i.e., it has a finite bandwidth, we consider monofrequent noise here for simplicity. The complex amplitude estimated from \(x(t) = x_0(t) + l(t)\) is calculated from...
Figure 4. Amplitude spectral density of the vibration noise in the NMIJ’s low-frequency vibration exciter showing the measured spectrum (green) and model (black) used for the calculations (left). Standard deviation of the estimated sensitivity under random vibration noise showing the results based on a simulation (red circles) and obtained using equation (23) (dashed blue line) (right). A rectangular window is used.

Figure 5. Uncertainty in the estimates of the amplitude (left) and phase (right) due to the line noise for \( \hat{l}/\hat{x}_0 = 10^{-3} \), as obtained using a rectangular window. The open circles show the simulated results, the solid magenta and cyan lines show the theoretical uncertainty calculated using equations (27) and (28), respectively, and the black lines represent approximate values derived using equation (29).

Figure 6. Uncertainty in estimates of the sensitivity (left) and phase delay (right) due to the common line noise, \( l_x \), for \( \hat{l}/\hat{x}_0 = 10^{-3} \), as obtained using a rectangular window. The values obtained using the approximation given in equation (33) are shown as the black lines, the values obtained using the precise formulas are shown as the red and blue lines, and the simulated values are plotted as the open circles.

The amplitude and phase estimation errors are then

\[
\begin{align*}
\frac{|\hat{x}_{\text{est}}|}{\hat{x}_0} - 1 &= \frac{\hat{l}}{\hat{x}_0} \left( \frac{\hat{W}(f_v - f_l)}{T} e^{-i\phi_l} - \frac{\hat{W}(f_v + f_l)}{T} e^{i\phi_l} \right), \\
\arg[\hat{x}_{\text{est}}] - \frac{\pi}{2} &= \frac{\hat{l}}{\hat{x}_0} \left( \frac{\hat{W}(f_v - f_l) + \hat{W}(f_v + f_l)}{T} \right) \sin(\phi_l).
\end{align*}
\] (24)

In actual measurements, the phase \( \phi_l \) of the line noise relative to \( x_0 \) is usually random for each measurement and is uniformly distributed from 0 to 2\( \pi \). In such a case, their standard uncertainties are given by

\[
\begin{align*}
\frac{u(|\hat{x}_{\text{est}}|)}{\hat{x}_0} &= \frac{\hat{l}}{2\hat{x}_0} \sqrt{\hat{W}(f_v - f_l) - \hat{W}(f_v + f_l)}, \\
\frac{u(\arg[\hat{x}_{\text{est}}])}{\hat{x}_0} &= \frac{\hat{l}}{\sqrt{2\hat{x}_0}} \sqrt{\hat{W}(f_v - f_l) + \hat{W}(f_v + f_l)},
\end{align*}
\] (25) (26) (27) (28)

In equation (11) as

\[
\hat{x}_{\text{est}} = \hat{x}_0 + \hat{l} \left( \frac{\hat{W}(f_v - f_l)}{T} e^{-i\phi_l} - \frac{\hat{W}(f_v + f_l)}{T} e^{i\phi_l} \right).
\] (24)
Figure 7. Summary of the signal-processing methods used for sensitivity calibration including (0) the conventional SAM and the proposed methods, (a) bandpass filtering, (b) multiplication of the window function, and (c) differentiation of the reference displacement signal.

The uncertainty follows a U-shaped distribution. Since $\tilde{W}(f)$ has large value around $f \simeq 0$, the uncertainty is large when $f_v \simeq f_i$. In this case, equations (27) and (28) are approximated as

$$
\frac{u(\hat{x}_{est})}{\hat{x}_0} \simeq u(\text{arg}[\hat{x}_{est}]) \simeq \frac{i}{\sqrt{2\pi}x_0} \frac{|\tilde{W}(f_v - f_i)|}{T}.
$$

(29)

Notably, the harmonics of $f_v$ do not affect the amplitude estimation through the spectral leakage when the rectangular window is used because $\tilde{W}(f_v \pm Nf_v) = 0$ ($N$: integer) if the length $T$ is integer multiples of the vibration period. It is the same for the Hanning window or some other types of windows. Note that it does not mean that the harmonics do not become an error source in any case. If the sensor or reference interferometer has non-linearity, the harmonics $l_x$ is nonlinearly converted with the fundamental wave $x_0$ to the line noise $l_x$ or $l_r$ at $f_v$ and can affect the estimated amplitude even if $T$ is properly selected. To evaluate such a contribution, detailed information about the input harmonics $l_x$ and the non-linearity of the sensor are necessary, which is out of the scope of this work. As far as the harmonics $l_x$, $l_r$, and $l_t$ are independent from each other (i.e., the sensor is strictly linear), the contribution of the harmonics can be easily suppressed by the proper choice of $T$. Therefore, we mainly discuss the line noise in the remainder of this article.
2.5.1 Independent line noise for the sensor and reference. The standard uncertainties of sensitivity under the independent line noise of the sensor and reference, \( l_s = \hat{l}_s \sin(2\pi f_{1,sl} + \phi_{l,s}) \) and \( l_r = \hat{l}_r \sin(2\pi f_{1,rl} + \phi_{l,r}) \), are given by the square-root of the sum of their uncertainty contributions. Using equation (29), it is approximated as

\[
\frac{u(S_{cal})}{S} = \frac{1}{\sqrt{2}} \left( \frac{\hat{l}_s}{(2\pi f_{1,sl})} \right)^2 |W(f_r - f_{0,s})| + \left( \frac{l_r}{(2\pi f_{1,rl})} \right)^2 |W(f_r - f_{0,r})|. \tag{30}
\]

Here, we assumed that \( \phi_{l,s} \) and \( \phi_{l,r} \) are not correlated to each other. If the line noise originates from the power supply noise at 50 Hz or 60 Hz, or from their harmonics, it is likely that \( \phi_{l,s} \) and \( \phi_{l,r} \) are correlated and \( f_{1,s} = f_{1,r} \), whereas the line amplitudes will be different. In such situations, the sum of squares in equation (30) is replaced with a simple sum:

\[
\frac{u(S_{cal})}{S} = \frac{1}{\sqrt{2}} \left( \frac{\hat{l}_s + (2\pi f_{1,sl})^2 \hat{l}_r}{(2\pi f_{1,rl})} \right) |W(f_r - f_{0,s})| / T. \tag{31}
\]

The corresponding accurate expressions are given by replacing \( W(f_r - f_{0,s}) \) with \( W(f_r - f_{0,s}) - W(f_r + f_{0,s}) \) for sensitivity and \( W(f_r - f_{0,s}) + W(f_r + f_{0,s}) \) for phase delay.

When the line noise frequency is equal to the vibration frequency, \( W(f_r - f_{0,s}) = T \). Therefore, the relative amplitude estimation uncertainty of the reference is determined by only the amplitude ratio:

\[
\frac{u(x_{r,est})}{x_0} = \frac{\hat{l}_r}{\sqrt{2}\hat{x}_0}, \tag{32}
\]

and the similar relation for the sensor signal. This is the fundamental limit of uncertainty from the line noise. Although the leakage from \( f \neq f_s \) can be reduced by proper signal processing, the contribution at \( f = f_s \) cannot be avoided unless the line noise amplitude is reduced.

The line noise contribution was confirmed by simulation (figure 5). Fixing \( \hat{l}_r / \hat{x}_0 = 10^{-3} \), we estimated the amplitude and phase of \( x(t) = x_0(t) + \hat{l}_r(t) \) for various \( f_r / f_s \) and \( \phi_l \). The data length was fixed to \( T = 100 / f_s \) at each frequency. The standard deviations for different \( \phi_l \) were calculated from the simulated values and compared with equations (27) and (28). Figure 5 shows the simulation results. Theoretical calculations explained the uncertainty well. The approximated formula, equation (29), also accurately estimated the uncertainty around the line noise frequency \( f_s \), where the line noise contribution becomes important. At frequencies away from \( f_s \), equation (29) led to over-/under-estimation by a few times.

2.5.2 Common line noise for the sensor and reference. The standard uncertainty of the sensitivity under the common line noise \( l_c = \hat{l}_c \sin(2\pi f_{1,sl} + \phi_{l,s}) \) was calculated similarly as equation (22). Using equation (29), it was approximated as

\[
\frac{u(S_{cal})}{S} \simeq u(\Delta\phi_{cal}) \simeq \frac{\hat{l}_c}{2\hat{x}_0} \frac{|f_r^2 - f_{1,sl}^2|}{f_r^2} \left| W(f_r - f_{1,sl}) \right| / T. \tag{33}
\]

Again, accurate expressions are given by replacing \( W(f_r - f_{1,sl}) \) with \( W(f_r - f_{1,sl}) - W(f_r + f_{1,sl}) \). The effect of the line noise was simulated again. The common line noise amplitude was fixed as \( \hat{l}_c / \hat{x}_0 = 10^{-3} \), and the data length was fixed to \( T = 100 / f_s \) at each frequency. The sensor and reference signals \( V_s = S(\hat{x}_0 + \hat{l}_c) \) and \( x_r = \hat{x}_0 + \hat{l}_c \) were prepared for different line phase \( \phi_{l,s} \). The standard deviations of the estimated sensitivity and phase for different \( \phi_{l,s} \) were calculated. The simulation results are shown in figure 6. The results agreed with the theoretical calculations using equation (33). Unlike the independent line noise, the uncertainty contribution of the common line noise is not concentrated around \( f_s \approx f_{1,sl} \), because the peak of \( W(f_r - f_{1,sl}) \) is canceled by the factor \( f_r^2 - f_{1,sl}^2 \) as shown in equation (33), when the rectangular window is used.

3. Reduction of noise contribution in calibration

The calibration uncertainty considered in this paper is explained by equations (20), (23), (30), and (33). These equations show how the spectral leakage contributes to the calibration in the conventional acceleration calibration with the SAM. As already mentioned, the background random/line noise of the sensing parts (sensor and interferometer) at \( f_s \) is a fundamental limit of signal processing. In this section, we aimed to minimize the leakage from \( f \neq f_s \). In the following subsections, we discuss three signal processing modification methods including filtering, changing the window function, and numerical differentiation to align the unit of the measurement. These processes are summarized in figure 7. For the line noise, the selection of the data length \( T \) is also discussed. The investigations about the three modifications are mainly focused on the random noise, although the same methods are applicable to the line noise. These methods are simulated for both random and line noises in appendix B.

3.1 Filtering the signal

One of the simplest ways to reduce the leakage is filtering the signal before amplitude estimation. A bandpass filter (BPF) centered at \( f_s \) reduces the noise PSD \( G(f \neq f_s) \) in equations (20) and (23) and the line noise at \( f \neq f_s \). Depending on the background noise spectrum, the low-pass or high-pass filters can also be used. Such filtering is already adopted in the calibration process in NMIJ, although the shape of the filter has been empirically determined. The filtering changes
Figure 9. Simulated settling times for different orders of BPF (second, fourth, sixth, and eighth) and for different $Q$-factor values (1, 2, 5, 10, and 20) (left). The filter gain of the sixth-order BPFs used for the calculation (right).

Figure 10. The amount of spectral leakage in the vibration amplitude estimation for $T = 30/f_v$ (top) and $100/f_v$ (bottom) for the three different cases of a conventional SAM (red), sixth-order bandpass filtering (orange), and a Hanning window (green). The inset shows an enlargement of the region around $f/f_v = 1$.

In this study, a digital Butterworth BPF was applied to the discretely sampled data set. The first-order filter was applied $n$ times to produce the $n$th order gain. The similar noise reduction is generally possible using the other implementations of the digital filter, although there are some quantitative differences between them. The use of an analog BPF was not considered in this study as this would have required additional calibration of the filter.

The disadvantage of filtering is that it takes time until the waveform becomes stable after starting the excitation. An example is shown in figure 8. A sixth-order BPF was applied to the noisy waveform to extract the frequency component around $f_v$ with a $Q$-factor ($Q$) of $\approx 2$. Here, the $Q$-factor is the ratio of the center frequency $f_v$ over the bandwidth of the filter. In the example, the amplitude of the filtered data became stable after about eight vibration cycles. Consequently, the first several vibration cycles of the data could not be used for amplitude estimation, which increased the measurement time, especially at low frequencies.

Figure 9 shows the simulated settling time for different orders and $Q$-factors of the BPF. The settling time is defined as the time length until the filtered waveform amplitude settles within $\pm 0.1\%$ of the true amplitude. The settling time is roughly proportional to the $Q$-factor, and increases with the order of the filter. Thus, a trade-off exists between the noise reduction ability of the filter and discarded measurement time. Although it may be possible to compensate for the amplitude change associated with this settling, it may be difficult to perform a simulation appropriate to the shape of the filter and the input signal. In the remainder of this paper, we discuss the use of a sixth-order BPF with $Q = 1$ to keep the settling time within 10 vibration cycles.

3.2. Changing the window function

Another way to improve the vibration amplitude estimation accuracy is changing the window function $w(t)$. The rectangular window, which is implicitly used in the conventional SAM, is well-known for large spectral leakage. The other window functions such as a Hanning window can reduce $\tilde{W}(f)$ at $f \neq f_v$. Window functions are widely used in fast Fourier transform applications, and their use in the vibration calibration is mentioned in [6, 12]. Here, the effect of the window function is reviewed from the viewpoint of noise reduction and compared with other methods.

The effect of the window function is compared with filtering in figure 10. The factor $|W(f_v - f)\tilde{F}(f)|/T$, which determines the amount of the leakage in both the random noise and line noise, is plotted for $T = 30/f_v$ and $100/f_v$ in three cases: using the rectangular window (conventional SAM), by applying the sixth-order BPF ($Q = 1$) with the rectangular window, and using the Hanning window without filtering. As the figures show, the Hanning window can reduce the leakage more effectively than the BPF around the vibration frequency. The reduction ratio of the Hanning window in $0.7 < f/f_v < 1.5$ is approximately equal to the second-order
Figure 11. Standard deviation of the calibrated sensitivity (top) and phase delay (bottom) calculated from the measurement repeated five times with \((2\pi f_s) x_0 \approx 10^{-2} \text{ m/s}^2\) using the calibration system at the NMIJ. The results obtained using different signal-processing methods are plotted: the conventional SAM (solid red circles), a sixth-order BPF (solid orange squares), a Hamming window (solid green triangles), numerical differentiation of \(x_t\) (open blue circles), a BPF and a Hamming window (light green solid diamonds), differentiation and a BPF (open brown squares), and differentiation and a Hanning window (open purple triangles). The theoretical limit due to the background noise of the system is shown by the solid gray line.

BPF with \(Q \approx 15\) and 50 for \(T = 30/f_v\) and 100/f_v, respectively. For such filters, a large amount of measurement time is wasted due to the settling time, as shown in figure 9, while changing the window function does not require discarding the data. On the other hand, the BPF has a better reduction ratio at frequencies away from \(f_s\), although the details depend on the design of the filter. The window functions also increase the leakage from a very close frequency to \(f_s\) compared to the rectangular window, as shown in the inset in figure 10. In other words, the frequency resolution is compromised by changing the window function. Therefore, when there is a large low-frequency noise or the calibration frequency is close to a large line noise (e.g., calibration at 49 Hz when there is a large power supply line noise at 50 Hz), bandpass filtering with the rectangular window can be a better choice. In the scope of vibration frequency determination, the frequency resolution of signal processing is usually not important in vibration calibration because the frequency is accurately controlled based on reference frequency standards.

3.3. Numerical differentiation

For the common vibration noise \(n_v\), the factor \((2\pi f_s)^2 - (2\pi f_v)^2\) is included as in equation (23). Similarly, the uncertainty from the common line noise, equation (33), contains the factor \(|f_s^2 - f_v^2|\). These factors originate from the difference of the measured physical quantity between the sensor and the reference interferometer. The accelerometer measures the acceleration, while the interferometer measures the displacement. In the conventional SAM for the reference signal, the displacement amplitude is estimated from \(x_t\); then, the acceleration amplitude is calculated by multiplying \((2\pi f_s)^2\). Although this gives an accurate estimation for a purely sinusoidal wave, the background noise degrades calibration accuracy.

A natural way to avoid such an effect is converting the reference displacement signal into acceleration by numerical differentiation prior to the amplitude estimation. This process ideally eliminates the factor \((2\pi f_s)^2 - (2\pi f_v)^2\) in equation (23) and \(|f_s^2 - f_v^2|\) in equation (33). Therefore, the common noise sources \(n_v\) and \(l_v\) no longer interfere with the calibration. In practice, the reference displacement data, \([x_{est}] (n = 0, 1, \ldots, N - 1)\), are converted to the reference acceleration, \(a_{r, est}\), as follows:

\[
a_{r, est} = (x_{est} + 2 x_{est} + x_{est} - 1) f_s^2,
\]

where \(f_s\) is the sampling frequency. The amplitudes are then estimated using \([V_{est}]\) and \([a_{est}]\), and the sensitivity is calculated as \(S_{cal} = |V_{x, est}/a_{r, est}|\), instead of using equation (3). A similar process is applicable to the velocity sensor (e.g., seismometer) by performing a single differentiation. The essential point is to align the units of the compared signals, e.g., to acceleration.

The systematic error in numerical differentiation also requires attention. The transfer function of the numerical differentiation (equation (34)) from the displacement to acceleration is given by

\[
\frac{\hat{a}_{r, est}}{x_{est}} = (e^{2\pi i f_s/\lambda} - 2 + e^{-2\pi i f_s/\lambda}) f_s^2,
\]

in the absence of noise. Here, \(e^{2\pi i f_s/\lambda}\) represents the temporal shift by \(1/f_s\) for the frequency component at \(f_s\). On the other hand, the transfer function of the second-order derivative of the continuous signal is given by \(-(2\pi f_s)^2\). Therefore, the numerical differentiation underestimates the amplitude by the factor of \((\sin(f_s/\lambda))^2\), which needs to be corrected. Note that this factor differs from 1 by \(<0.02\%\) for \(f_s < 0.01 f_v\). If the sampling frequency is sufficiently faster than the vibration frequency, typically by 100 times, the error is ignorable.

The differentiation process of the displacement signal relatively enlarges the high-frequency noise in the time domain. This is because the sinusoidal signal amplitude \(x_0\) and the noise \(G(f)\) are multiplied by \(\sim -(2\pi f_s)^2\) and \(\sim -(2\pi f_v)^2\), respectively; hence, the contribution of the noise at frequency \(f\) to the signal-to-noise ratio in the amplitude estimate is changed by a factor of \((f_s/f_v)^2\). Additional processing such as low-pass filtering or changing the window function is recommended.
**Figure 12.** Combinations of numerical differentiation and a filter or window. First, the reference signal is differentiated, and the same window or filter is then applied to both signals. After these pre-processing, the amplitude of each signal is estimated and compared to calculate the sensitivity.

**Table 1.** Summary of the effectiveness (+++; contribution is strongly suppressed, ++: contribution is reduced, −−: ineffective) and the advantages/disadvantages of the proposed signal-processing methods.

| Method        | Effect on $n, l, n, l, l, l$ | Advantage                                      | Disadvantage                                      |
|---------------|-------------------------------|-----------------------------------------------|--------------------------------------------------|
| Filter        | +    +   +   +   +            | Large reduction ratio at $f$ away from $f_v$  | Data is wasted during the settling time          |
|               |                               | Design is flexible                             |                                                  |
| Window        | +    +   +   +   +            | Large reduction ratio at $f$ close to $f_v$    | Low-frequency noise remains                      |
| Differentiation | ++  ++   −   −            | Common noise is eliminated                     | High-frequency reference noise is amplified       |
|               |                               | Low-frequency reference noise is reduced       |                                                  |
| Adjusting $T$ | −    + +   −   + +          | Line noise is strongly suppressed              | Not always possible within reasonable $T$       |

To avoid increasing noise contributions from $f > f_v$. Alternatively, the low-frequency noise at $f < f_v$, which is often included as a drift component in the reference interferometer signal, is reduced by the differentiation process.

### 3.4 Adjusting data length

The line noise contribution can be strongly suppressed by setting proper data length $T$ so that both $f_s - f_i$ and $f_s + f_i$ are at the zero points of $\hat{W}(f)$. Such an adjustment was mentioned in [12] for the purpose of power line noise rejection. Here, more general conditions for the elimination of the line noise are discussed. The Fourier transform of the rectangular window, $\hat{W}_r(f)$, is equal to zero at $fT = N$ ($N$: integer). Recalling that $f_sT$ is set to an integer to suppress the harmonics, as mentioned in section 2.5, $f_iT$ also needs to be an integer. Consequently, if $T$ is an integer multiple of the inverse of the greatest common divisor of $f_v$ and $f_i$, both the harmonics and line noise can be suppressed. For example, in the case of $f_i = 50$ Hz and $f_v = 49.2$ Hz, the greatest common divisor frequency is 0.4 Hz; hence $T$ should be the multiple of 2.5 s. The condition is similar for the Hanning window, although it has two fewer zero points around $f_i$ than the rectangular window, as shown in figure 10. In usual vibration calibration, $f_v$ is selected from the one-third octave band specified in ISO 266:1997 [14], and the frequencies are rounded to 0.5 Hz increment around the main line noise frequency (50 Hz). In such a case, $T$ should be multiple of 2 s. If there are large line noises at several frequencies, it will be good to set $T$ so that the closest line noise to $f_v$ is suppressed.

One of the possible drawbacks of adjusting the data length is that the measurement time can be somewhat longer in some cases due to the limited choice of $T$. If the elimination is impossible within a reasonable measurement time, the line noise contribution needs to be reduced using other methods.
proposed in sections 3.1–3.3. Note that the adjustment is not necessary at every frequency because the line noise contribution becomes a problem only around the line frequency. Therefore, the total measurement time of calibration does not substantially increase over a wide frequency range.

In the above discussions, we assumed that each line noise is monofrequent, indicating that it is completely periodic. However, in realistic systems, the amplitude or phase of the line noise fluctuates slightly and the frequency component of the line noise has a finite bandwidth. In such cases, the component of the line noise that deviates from the center frequency, \( f_n \), is multiplied by a small but finite window function and contributes to the noise, even if \( f_n \) is adjusted to the zero point of the window function. The amount of noise is determined by the shape of the window function and the bandwidth of the line noise, which depends on the experimental conditions. The details should be analyzed based on the performance of the actual system, which is beyond the scope of this paper.

4. Application to the accelerometer calibration in NMIJ

In NMIJ, the calibration system with a small excitation amplitude \((2\pi f_n)^2 x_0 \approx 10^{-2} \text{ m s}^{-2}\) is under development [13]. The background noise of the calibration system becomes a significant uncertainty source in microvibration calibration. Here, the proposed processing methods are applied to the actual calibration data to reduce uncertainty. The calibration system is shown in figure 1. A servo accelerometer JA-5V [Japan Aviation Electronics Industry, Ltd., \( S \approx 0.1 \text{ V m}^{-2} \text{ s}^{-1} \) (nominal)] was used as the calibration target. Since the calibration uncertainty discussed in this paper mainly affects repeatability, the calibration was repeated 5 times at frequencies from 0.4 Hz to 500 Hz, and their standard deviations were measured at each frequency. The results were compared with the theoretical limits due to the background random noise of \( V_n \) and \( S_n \) (as equation (21)) in our system. The effect of the line noise was limited and not large compared to the random noise; hence, we mainly discuss the random noise in this section. Regarding the independent background noise, \( n_n \) includes the self-noise of the accelerometer and the noise of the signal acquisition system, and \( n_n \) includes the self-noise of the laser interferometer, seismic vibration noise, and signal acquisition system noise. The common background noise \( n_n \) is shown in figure 4. Relatively large drift (1.2 m s\(^{-1}\)), which also behaves as the common background noise, was applied during the measurements above 30 Hz for averaging the cyclic error of the interferometer [15].

The experimentally obtained repeatability with different signal processing methods are shown in figure 11. Note that the total calibration uncertainty is determined by the sum of the repeatability and the other uncertainty components, such as the uncertainty in the laser wavelength. The sixth-order BPF, a Hanning window, numerical differentiation (equation (34)), and their combinations were adopted. With the conventional SAM without any additional processing methods, the repeatability uncertainty was about 100% over a broad frequency range. The calibration results were almost meaningless in such a large uncertainty. By using the BPF, Hanning window, or numerical differentiation, the repeatability was improved by up to four orders of magnitude. Since numerical differentiation could reduce uncertainty, the overall repeatability was thought to be limited by the common vibration noise \( n_n \) or low-frequency noise in the reference displacement signal \( n_n \). At high frequencies above 50 Hz, the noise reduction is insufficient only in the case of the Hanning window because the low-frequency drift is not suppressed enough. The BPF is a better choice in such a case. Applying the BPF before using the Hanning window to reduce the low-frequency components improves the reduction of the noise above 50 Hz; however, since the BPF and Hanning window have similar effects, there is little benefit in combining them below 50 Hz. The combinations of numerical differentiation with BPF or Hanning window (figure 12) were more effective than the single processing method. The results with the combined processing methods are also shown in figure 11, which achieved the theoretical limit by the calibration system noise. As expected from equation (18), the reduction effects are almost the same for the phase delay.

At 0.4 Hz, the repeatability was not improved by the modification of the signal processing. This is because, the repeatability was limited by the noise at \( f_n \) (as equation (21)) even using the conventional SAM. Under such conditions, the application of additional signal-processing methods does not improve the repeatability since these methods are effective only for reducing the spectral leakage from \( f \neq f_n \). Similarly, at 0.63 Hz, the repeatability was not improved by combining the differentiation because the leakage had already been suppressed by the BPF or the Hanning window. At these frequencies, below 1 Hz, the obtained repeatability was slightly higher than the background noise limit. This is probably because the noise level during the calibration was higher than when the noise was measured to calculate the values shown by the gray line in figure 11. The dominant source of noise around 0.3 Hz was microseisms, whose amplitude varies with that of ocean waves and so on. Other sources of noise, such as those due to the air or temperature fluctuations, were not taken into account in this study, and should also be investigated in future work.

Our experiment proved that the signal processing methods proposed in this paper are useful in actual calibration systems. The results agreed with the theoretically expected reduction ability; therefore, the calculations shown in sections 2 and 3 can be used for optimizing signal processing. For example, in the case of the accelerometer calibration in NMIJ, the common vibration noise had a dominant uncertainty contribution. The independent background noise bottomed out near 10 Hz and increased on the low- and high-frequency side; hence, the calibration around 10 Hz was affected by the leakage from the low- and high-frequency ranges. Since both the common and independent noises needed to be reduced, the combination of numerical differentiation and the BPF or Hanning window was required. If the common vibration noise is small, numerical differentiation is not necessary. The spectral leakage may be insignificant if the independent background noise has a flat spectrum; then, the conventional SAM without the BPF or Hanning window may be sufficient. Thus, the processing methods need to be combined based on the calibration system noise.
5. Conclusions

We proposed three signal processing methods to modify the conventional SAM specified in ISO16063-11. The aim was to reduce the spectral leakage based on the theoretical analysis described by equations (20), (23), (30), and (33). The proposed methods included

- Filtering the signal
- Changing the window function
- Differentiating the reference displacement signal.

The proper choice of the data length \( T \) was also discussed to suppress the line noise contribution. Their effect on the reduction of calibration uncertainty was confirmed by both the simulation and experiment. These results showed that, in accelerometer calibration, using the combination of numerical differentiation and the filter or window before amplitude estimation results in a more robust calibration against the background noise compared to using the conventional SAM.

Table 1 summarizes the effectiveness, advantages, and disadvantages of the proposed methods. The line noise and common noise contributions can be suppressed by adjusting the data length and using numerical differentiation for the reference displacement signal. If the independent random noise contribution is significant, additional filtering or window function needs to be combined with numerical differentiation. Note that to avoid the systematic change in the amplitude and phase, the same filter or window should be applied to both the reference and sensor signals; otherwise, these changes must be compensated by calculating the filter gain and so on. Figure 12 shows the recommended combinations of sensitivity calibration. The combination of numerical differentiation and Hanning window is sufficient in standard cases. Although filtering offers flexibility, the data acquired during the settling time needs to be discarded, and instability is a concern for the infinite impulse response filter. Equations (20), (23), (30), and (33) are useful for the optimization based on the system noise characterization.

This work enables accelerometer calibration with a small excitation amplitude relative to the system background noise. Such calibration is required to confirm the sensitivity linearity of accelerometers used for microvibration measurements. Additionally, for accelerometers with high sensitivity, large vibration may not be applied for calibration because they can get saturated with small input vibration. In the calibration at a low frequency (<1 Hz), which is required for broadband seismometers, the excited acceleration amplitude is limited due to the stroke limit of the exciter. The reduction of the background noise is necessary in these cases, and proper signal processing is essential to take full advantage of noise reduction.

Combinations of the proposed signal-processing methods can always be used without significant disadvantages. If the excitation amplitude is sufficiently large, use of the conventional SAM will be sufficient. As shown in equations (20), (23), (30), and (33), the contribution of the noise to the calibration uncertainty is inversely proportional to the excitation amplitude, \( (2\pi f_v)^2 \delta x_0 \). Based on figure 11, the sensitivity deviation using the conventional SAM with \( (2\pi f_v)^2 \delta x_0 = 10 \text{ m s}^{-2} \) is expected to be less than 0.1% in our system. Thus, the proposed signal-processing method is effective when the signal-to-noise ratio is low, and the amplitude deviation is degraded due to the leakage of noise from \( f \neq f_v \).

Although we mainly discussed accelerometer calibration in this work, some of the knowledge obtained also applies to other fields of dynamic sensor calibration, where sinusoidal signal extraction is required. When the sensor output signal is compared with a reference signal, their signals should be in unit of the same physical quantity; otherwise, the common background noise affects the estimation of the amplitude ratio. The proposed signal processing methods for accelerometer calibration can contribute to improving the reliability of those measurements.

Acknowledgments

This work was partially based on the results obtained from a project commissioned by the New Energy and Industrial Technology Development Organization (NEDO), Japan.

Appendix A. Parameter estimation using the sine approximation method

In this appendix, we derive equation (2) from the principles of the SAM. The SAM is the least-squares fitting method to estimate the amplitude and phase of a sinusoidal signal. As shown in [10], the optimal parameters that minimize the residual in equation (1) are given by

\[
b = (S^T S)^{-1} S^T X,
\]

where

\[
b = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & \cos(2\pi f_v t_0) & \sin(2\pi f_v t_0) \\ \vdots & \vdots & \vdots \\ 1 & \cos(2\pi f_v t_{N-1}) & \sin(2\pi f_v t_{N-1}) \end{pmatrix}, \quad X = \begin{pmatrix} x_0 \\ \vdots \\ x_{N-1} \end{pmatrix}.
\]

The definitions of the parameters \( b_i, f_v, t_0, \) and \( x_n \) are the same as in section 2.1. When the time length of the data, \( T \), is nearly equal to an integer multiple of the vibration period, \( 1/f_v \), as assumed in section 2.1, \( T \) can be written as \( T = (N_c + \delta)/f_v \), where \( N_c \) is an integer and \( \delta \ll 1 \) denotes the difference from this integer. In this case,

\[
\sum_{n=0}^{N-1} e^{2\pi i f_v t_n} = e^{2\pi i f_v t_0} \frac{1 - e^{2\pi i (N_c + \delta)/f_v}}{1 - e^{2\pi i N_c/f_v}} \approx \frac{N\delta}{N_c} e^{i\alpha},
\]

where \( \alpha \equiv 2\pi f_v t_0 \). In the last approximation, the lowest-order terms of \( \delta \) and \( N_c/N \) remain and the higher-order
Appendix B. Simulation of noise reduction using ods for sensitivity estimation, which are summarized in section 2.4. A simulation of amplitude estimation was performed for the two types of random noise discussed in section 2.4 and the line noise mentioned in section 2.5.

B.1. Reduction of independent random noise

First, the independent noise sources, \( n_0 \) and \( n_r \), were investigated. The additional signal processing methods proposed in sections 3.1 and 3.2 were effective in this case and corresponded to (a) and (b) in figure 7, respectively. Here, the amplitude estimation uncertainty of the single signal \( V_1 \) was simulated for simplicity. Using the results, the sensitivity calibration uncertainty can be calculated based on equation (20). A simulation similar to that in section 2.4.1 was performed; the amplitude estimation of sinusoidal waveform with \( 2\pi f_v^2 \Delta t_0 = 1 \text{ m s}^{-2} \) under the background noise as the right column case of figure 3. In this case, the standard deviation of the estimated amplitude increased to 0.83% based on the conventional SAM due to the noise at \( f \neq f_v \), while the S/N at the vibration frequency was 0.1%. Here, the sixth-order Butterworth BPF with \( Q = 1 \) and the Hanning window were applied to the noisy waveform, and the amplitude was estimated from the processed waveforms. The examples of the processed waveforms are shown in figure B1 (upper figure). The data of the first ten vibration cycles were discarded for the filtered waveform, as explained in section 3.1. The simulation was repeated 300 times for the randomly generated noise, and the histogram of the amplitude estimation error is shown in figure B1 (lower figure). For the conventional SAM, BPF, and Hanning window, the simulated standard deviations were 0.91\%, 0.13\%, and 0.15\%, and the expected deviations from equation (17) were 0.83\%, 0.13\%, and 0.15\%, respectively. As expected, the BPF and Hanning window reduce the background noise contributions at \( f \neq f_v \). Although they do not achieve an S/N of 0.1\% at \( f_v \), due to the leakage around \( f_v \), the difference is sufficiently small. Consequently, the calibration uncertainty is also reduced because equation (20) is the sum of the amplitude estimation uncertainties of \( V_1 \) and \( x_r \).

In any cases shown in figure B1, the amplitude estimation errors are distributed around zero, which indicates that there is no bias due to the filter or window. The correction of the filter gain is necessary, depending on the choice of the filter. To avoid the systematic effects on sensitivity calibration, applying...
the same process on both the sensor and reference signals is recommended to cancel out the effect.

B.2. Reduction of common random noise

Second, the common vibration noise, $n_v$, was investigated. The signal processing methods proposed in sections 3.1–3.3 were effective in this case. They correspond to (a)–(c) in figure 7, respectively. The noise spectrum in figure 4 and the excitation amplitude of $(2\pi f_v)^2 x_0 = 1 \text{ m s}^{-2}$ were used for the simulation. In (a) and (b), the same BPF and Hanning window as the previous simulation (figure B1) were used. The amplitudes were estimated using the processed data, and the sensitivity was calculated from equation (3). In (c), the reference displacement data $\{x_{r,n}\}$ $(n = 0, 1, \ldots, N - 1)$ was converted to the reference acceleration $a_{r,n}$ using equation (34). Then, the amplitudes were estimated using $\{V_{s,n}\}$ and $\{a_{s,n}\}$, and the sensitivity was calculated as $S_{cal} = |V_{s, est}/a_{s, est}|$ instead of equation (3). These simulations are repeated 20 times for the randomly generated noise at vibration frequencies $0.1 \text{ Hz} < f_v < 300 \text{ Hz}$. The simulation results and theoretical expectations from equation (23) are shown in figure B2. Both the BPF and Hanning window reduced the relative standard uncertainty of sensitivity from 0.3% to 0.01% and 0.001%, respectively, around 100 Hz. The reduction effect agreed well with the theory. Numerical differentiation is much more effective, as shown in the figure. Therefore, it is recommended to adopt the differentiation process in accelerometer calibration when the background vibration noise is large.

B.3. Reduction of independent line noise

The same methods applied to the random noise are also effective for the line noise. For the independent line noise, $l_v$, $l_s$, and $l_r$, the filter and window function are useful. They correspond to (a) and (b) in figure 7, respectively. The effects of the BPF and Hanning window on the independent line noise were simulated here. The signal $x(t) = x_0(t) + l(t)$ was filtered by the sixth-order BPF with $Q = 1$ or windowed using the Hanning window. The amplitude estimation uncertainties were simulated using the same process described in section 2.5. The result is shown in figure B3. As equation (29) indicates, the reduction effect is similar to the frequency dependence of $|W(f_v - f)\tilde{F}(f)|/T$, shown in figure 10. The Hanning window is more effective than the BPF over the simulated frequency range, where the line noise has a large uncertainty contribution.

To confirm that the proper choice of $T$ can strongly suppress the line noise contribution, as discussed in section 3.4, the dependence on the data length was also simulated. As an example, the calibration frequency was fixed at $f_c = 49.2 \text{ Hz}$, and the data length was changed from 0.1 s to 6 s. The amplitude estimation uncertainty for $x(t) = x_0(t) + l(t)$ was simulated using the same method described in section 2.5.1 (using...
Figure B4. Simulated amplitude estimation uncertainty at \( f_l = 49.2 \) Hz under the line noise at \( f_l = 50 \) Hz (red) and harmonics at \( f_l = 2 f_l \) for different data length \( T \). The line amplitude is \( \hat{l}/\bar{x}_0 = 10^{-3} \) at each frequency.

Figure B5. Sensitivity estimation uncertainty due to the common line noise \( l_i \) with \( l_i/\bar{x}_0 = 10^{-3} \). The simulated uncertainties for four cases using the conventional SAM (red circle), sixth-order BPF (orange square), Hanning window (green triangle), and numerical differentiation (blue cross), are plotted.

B.4. Reduction of common line noise

For the common line noise, \( l_i \), numerical differentiation is effective. The uncertainties with the conventional SAM, sixth-order BPF, Hanning window, and numerical differentiation were simulated and plotted in figure B5. The simulation was performed under the same conditions, as described in section 2.5.2. The same filter, window, and differentiation process described in appendix B.2 were applied. As expected, those signal process modifications reduce the common line noise contributions. Especially, the numerical differentiation eliminates the common line noise contribution over a wide frequency range.

Note that the proper choice of \( T \) is also effective for reducing the common line noise. Therefore, the modification of signal processing is not necessary for line noise reduction, if it is already suppressed by the choice of \( T \).

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