We consider the problem of approximating a function in general nonlinear subsets of \(L^2\) when only a weighted Monte Carlo estimate of the \(L^2\)-norm can be computed. Of particular interest in this setting is the concept of sample complexity, the number of sample points that are necessary to achieve a prescribed error with high probability. Reasonable worst-case bounds for this quantity exist only for particular subsets of \(L^2\), like linear spaces or sets of sparse vectors. For more general subsets, like tensor networks, the currently existing bounds are very pessimistic.

By restricting the model class to a neighbourhood of the best approximation, we can derive improved worst-case bounds for the sample complexity. When the considered neighbourhood is a manifold with positive local reach, the sample complexity can be estimated by the sample complexity of the tangent space and the product of the sample complexity of the normal space and the manifold’s curvature.

**Key words.** sample efficiency · least squares approximation · tensor networks · manifolds · positive reach

**AMS subject classifications.** 15A69 · 41A30 · 62J02 · 65Y20 · 68Q25

1 Introduction

Consider the task of estimating an unknown function from noiseless observations. For this problem to be well-posed we assume that the sought function can be well approximated in some given model class.\(^1\) Given such a model class, it is of particular interest how well a sample-based estimator can approximate the sought function. In investigating this question, many papers rely on a restricted isometry property (RIP) or a RIP-like condition. The RIP asserts, that the sample-based estimate of the approximation error is equivalent to the approximation error for all elements of the model class. Without this equivalence it is easy to conceive circumstances under which a minimizer of the empirical approximation error is arbitrarily far away from the real best approximation. The number of random samples that are required for the RIP to hold with a prescribed probability has been studied extensively for linear spaces [CM17] and sparse-grid spaces [Boh18], for sparse vectors [CRT06; RW16], low-rank matrices and tensors [RSS17; RFP10], as well as for neural networks [GK20] and, only recently, for generic, non-linear model classes [EST22].\(^2\) This work continues the line of thought started in [EST22] and studies how the probability of the RIP depends on the model class.

Although applicable to a wide range of model classes, our deliberations are motivated by model classes of tensor networks. They can be regarded as generalisations of low-rank matrices as well as a subclass of neural networks with identity activation and product pooling.\(^3\) As such tensor networks present one of the simplest non-trivial, non-linear model classes.

Next to this simplicity argument, there exists a practical reason to consider model classes tensor networks: They are commonly used tools in the numerics of high-dimensional parametric PDEs [ENSW18; ESTW19], in uncertainty

\(^1\)Note that, assuming Lagrange duality, this model class may be defined implicitly via regularisation.

\(^2\)As machine learning and statistics are huge and highly-active research areas, this list raises no claim to completeness.

\(^3\)For a more comprehensive discussion the reader is referred to the survey article [GKT13] and the monograph [Hac12].
quantification [Hab20; Wol19], in dynamical systems recovery [GKES19; Goe+20] and recently even in computational finance [BEST21]. But even though this model class exhibits exceptional performance in many practical applications, a theoretical foundation for this observation is still lacking. This stands in sharp contrast to other recent results that utilise weighted sparsity [BBRS15] or precisely tailored linear spaces [CM21]. This paper sets out to rectify this.

Setting. Let $\rho$ be a probability measure on some set $Y$ and consider the space $\mathcal{V} = L^\infty(Y, \rho)$ on which the norms $\| \cdot \| := \| \cdot \|_{L^2(Y, \rho)}$ and $\| \cdot \|_\infty = \| \cdot \|_{L^\infty(Y, \rho)}$ are defined. Given point-evaluations $\{(y^i, u(y^i))\}_{i=1}^m$ of an unknown function $u \in \mathcal{V}$ we want to find a (not necessarily unique) best approximation

$$u_M \in \arg \min_{v \in M} \| u - v \|$$

of $u$ in the model class $M \subseteq \mathcal{V}$. In general however, $u_M$ is not computable and a popular remedy is to estimate

$$\| v \| \approx \| v \|_y := \left( \frac{1}{n} \sum_{i=1}^n w(y^i) |v(y^i)|^2 \right)^{1/2} \quad \text{and} \quad u_M \approx u_{M,y} \in \arg \min_{v \in M} \| u - v \|_y,$$  \hspace{1cm} (2)

where $w$ is a fixed weight function, satisfying $w > 0$ almost everywhere and $\int_Y w^{-1} \, d\rho = 1$, and where $y^i \sim w^{-1} \rho$ for all $i = 1, \ldots, n$. To guarantee that this is well-defined, we assume that point-evaluations $v(y)$ exist for every function $v \in \mathcal{M}$ and all $y \in Y$.

By considering the model class $LM$, even the minimisation of residuals of the form $\| u - Lv \|$ can be incorporated in this setting. The problems that can be phrased this way are hence so ubiquitous that we refrain from listing them.

We prove that the empirical best approximation error $\| u - u_{M,n} \|$ is, in expectation, equivalent to the best approximation error $\| u - u_M \|$.

To do this, we require the restricted isometry property (RIP)

$$\text{RIP}_A(\delta) \implies (1 - \delta)\| u \|^2 \leq \| u \|^2 \leq (1 + \delta)\| u \|^2 \quad \forall u \in A$$

(3)

to be satisfied for the set $A = \{u_M\} - M$ and some $\delta \in (0, 1)$.

Structure. The remainder of the paper is organised as follows.

Section 2 explores how the sample complexity is related to the reciprocal of a generalised Christoffel function, termed the variation function. Calculus rules for the computation of this function are derived and applied to model classes of tensor networks. In this way it is shown that the derived worst-case estimate for the sample complexity of tensor networks behaves asymptotically the same as the sample complexity estimate for the full tensor space in which they are contained. This agrees with observations from matrix and tensor completion where a low-rank matrix or tensor has to satisfy an additional incoherence condition to guarantee a reduced sample complexity [CT10; YZ15].

Since the model class may be very large and contain “highly coherent” elements which can not be recovered with a low sample complexity, we consider the restriction of the model class to a neighbourhood of the best approximation $u_M$.

Section 3 derives upper and lower bounds for the variation function of neighbourhoods of $u_M$ that form manifolds with positive local reach. These bounds relate the variation function of the neighbourhood to those of the tangent and normal spaces at $u_M$ as well as the neighbourhood’s curvature.

Section 4 concludes this work by discussing the results and possible ways forward.

Notation. We denote this set of integers from 1 through $d$ by $[d]$ and define $\text{supp}(v) := \{ j \in [d] : v_j \neq 0 \}$.

For any set $X$, $\mathcal{B}(X)$ denotes the set of all subsets of $X$. If $(X, \rho)$ is a metric space and $Y \subseteq X$, then $\text{cl}(Y)$ denotes the closure of $Y$ in $X$ and $\mathcal{G}(X)$ denotes the set of all non-empty, compact subsets of $X$. $S(x, r) \subseteq X$ denotes the sphere and $B(x, r) \subseteq X$ denotes the open ball of radius $r > 0$ and with center $x \in X$. Since the metric space $X$ should always be clear from context, we do not include it in the notation for $S(x, r)$ and $B(x, r)$. If $X$ is subset of a linear space, the notation $(X)$ denotes the linear span of $X$. Finally, if $X$ is a $C^1$ submanifold of Euclidean space and $x \in X$ then $T_x X$ shall denote the tangent space of $X$ at $x$ and $T^*_x X$ shall denote its orthogonal complemen in $(X)$.

In Theorem 8 we require the concept of a continuous function that operates on sets. The relevant topologies are induced by the following two metrics.

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A Preprint
With this definition we can state the following bound on the probability of
As defined in equation (3), the RIP can be considered for any deterministic choice of
variation function
First observe that
Using the triangle inequality, the assertion now follows, since both
and provides a point-wise bound for the variation of the functions in
bounded by a standard concentration of measure argument. To do this, we define the normed space
empirical best approximation
𝒚
that
is a vector of i.i.d. random variables, the RIP is a random event and Theorem 3 bounds the expected error of the
errors
Considering the Hausdorff distance between these intersections. This metric is called the truncated Hausdorff distance.

In the following we define Cone(\(X\)) := \(\{\lambda x : \lambda > 0, x \in X\}\) and denote by Cone(\(\Psi(M)\)) the set of all cones in \(M\). Since conic set are uniquely defined by their intersection with the unit sphere we can define a more suitable (pseudo-)metric by considering the Hausdorff distance between these intersections. This metric is called the truncated Hausdorff distance.

\[ d_{\text{TH}}(X,Y) := d_{\text{H}}(S(0,1) \cap X, S(0,1) \cap Y), \]

is a pseudometric on Cone(\(\Psi(M)\)).

2 Worst case bounds

Using the restricted isometry property, defined in equation (3), we can show an equivalence of the errors \(\|u - u_{M,y}\|\) and \(\|u - u_M\|\).

**Theorem 3.** Let \(u \in \mathcal{V}\) and recall the definitions (1), (2) and (3). If RIP_{\{u_M\} - M}(\delta) holds, then

\[ \|u - u_M\| \leq \|u - u_{M,y}\| \leq \left(1 + \frac{\sqrt{2}}{\sqrt{1-\delta}}\right)\|u - u_{M}\|_{w,\infty}. \]

**Proof.** First observe that \(u_{M,y} \in M\) and therefore \(u_{M - u_{M,y}} \in \{u_M\} - M\). In the event RIP_{\{u_M\} - M}(\delta), we can use the triangle inequality and the definition of \(u_{M,y}\) to deduce

\[ \|u_{M} - u_{M,y}\| \leq \frac{1}{\sqrt{1-\delta}}\|u_{M} - u_{M,y}\|_{y} \]
\[ \leq \frac{1}{\sqrt{1-\delta}}\left[\|u_{M} - u\|_{y} + \|u - u_{M,y}\|_{y}\right] \]
\[ \leq \frac{2}{\sqrt{1-\delta}}\|u - u_{M}\|_{y}. \]

Using the triangle inequality, the assertion now follows, since both \(\|\cdot\|\) and \(\|\cdot\|_{y}\) are dominated by \(\|\cdot\|_{w,\infty}\). \(\square\)

As defined in equation (3), the RIP can be considered for any deterministic choice of \(y \in Y^n\). But since we assume that \(y\) is a vector of i.i.d. random variables, the RIP is a random event and Theorem 3 bounds the expected error of the empirical best approximation \(u_{M,y}\) under the condition that RIP_{\{u_M\} - M}(\delta) holds. The probability of this event can be bounded by a standard concentration of measure argument. To do this, we define the normed space

\[ \mathcal{V}_{w,\infty} := \{v \in \mathcal{V} : \|v\|_{w,\infty} < \infty\} \quad \text{where} \quad \|v\|_{w,\infty} := \text{ess sup}_{y \in Y} \sqrt{w(y)}|v(y)|. \]

The variation function of a subset \(A \subseteq \mathcal{V}\) is then given by

\[ \mathcal{R}_A(y) := \sup_{a \in U(A)} |a(y)|^2 \quad \text{where} \quad U(A) := \left\{ \frac{a}{\|a\|} : u \in A \setminus \{0\} \right\} \]

and provides a point-wise bound for the variation of the functions in \(A\) relative to their norm.

With this definition we can state the following bound on the probability of RIP_{\delta}(\delta).
Theorem 4 (Theorem 2.7 and Corollary 2.10 in [EST22]). For any \( A \subseteq V \) and \( \delta > 0 \), there exists a constant \( C > 0 \), independent of \( n \), such that

\[
\mathbb{P}\left[ \text{RIP}_A(\delta) \right] \geq 1 - C \exp\left(\frac{-\delta^2}{4 \| A \|_{W,\infty}^2}\right).
\]

If \( \dim((A)) < \infty \), then \( C \) depends only polynomially on \( \delta \) and \( \| R_A \|_{W,\infty}^{-1} \).

Theorem 4 guarantees that the probability of RIP_A(\( \delta \)) increases, when the value of \( \| R_A \|_{W,\infty} \) decreases. Moreover, combining Theorem 3 and Theorem 4 allows us to bound the probability with which the best approximation \( u_M \) of a function \( u \) may be recovered exactly in a given model class \( \mathcal{M} \).

Corollary 5. Recall the definition of \( U \) from equation (4) and assume that the \( \| \cdot \|_{W,\infty} \)-covering number of \( U(\{ u_M \} - \mathcal{M}) \) can be bounded by \( \nu(U(\{ u_M \} - \mathcal{M}), r) \leq C (cr)^{-M} \) for some \( c, C, M > 0 \). Moreover, let

\[
n \geq 2 \left( M \ln \left( \frac{8 \sqrt{K}}{c \delta} \right) - \ln \left( \frac{p}{2C} \right) \right) \left( \frac{K}{\delta} \right)^2
\]

for \( K := \| R_{\{ u_M \} - \mathcal{M}} \|_{W,\infty} \) and some \( p \in (0, 1) \) and \( \delta \in (0, 1) \). Then

\[
\| u - u_M \| \leq \| u - u_M \| \leq \left( 1 + \frac{2}{\sqrt{1 - \delta}} \right) \| u - u_M \|_{W,\infty}
\]

hold with a probability of at least \( 1 - p \).

Remark 6. Theorem 4 also provides worst-case bounds for deterministic algorithms. If \( \mathbb{P}\left[ \text{RIP}_M(\delta) \right] > 0 \), we can find \( y \in Y^M \) such that \( \text{RIP}_M(\delta) \) is satisfied. Thus, the conditions for Theorem 3 are satisfied for any \( u \in \mathcal{M} \) and hence there exists a deterministic algorithm to exactly recover any \( u \in \mathcal{M} \), using \( n \) function evaluations.

The variation function also allows us to compute the optimal sampling density of a set \( A \) as stated in the subsequent theorem.

Theorem 7 (Theorem 3.1 in [EST22]). \( R_A \) is \( \rho \)-measurable and

\[
\| w R_A \|_{L^\infty(Y, \rho)} \geq \| R_A \|_{L^1(Y, \rho)}
\]

for any weight function \( w \). The lower bound is attained by the weight function \( w = \frac{\| R_A \|_{L^1(Y, \rho)}}{S_A} \).

The subsequent theorem provide us with calculus rules for the computation of \( R \) which we will frequently use in the remainder of this work.

Theorem 8 (Basic properties of \( R \)). Let \( A, B \subseteq V_{w,\infty} \) and \( \mathcal{A} \subseteq \Psi(V_{w,\infty}) \). Then the following statements hold.

1. \( R_{\cup \mathcal{A}} = \sup_{A \in \mathcal{A}} R_A \), where \( \cup \mathcal{A} := \bigcup_{A \in \mathcal{A}} A \).
2. \( R_{\emptyset} = \psi_{A,\emptyset} \).
3. \( R \) : \( \mathcal{C}(V_{w,\infty} \setminus \{0\}) \rightarrow V_{w,\infty} \) is continuous with respect to the Hausdorff metric.
4. \( R \) : \( \Psi(V_{w,\infty} \setminus B(0, r)) \rightarrow V_{w,\infty} \) is continuous with respect to the Hausdorff pseudometric for all \( r > 0 \).
5. \( R \) : \( \text{Cone}(\Psi(V_{w,\infty})) \rightarrow V_{w,\infty} \) is continuous with respect to the truncated Hausdorff pseudometric.
6. If \( A \perp B \), then \( R_{A \oplus B} \leq R_A + R_B \). If \( A \) and \( B \) are linear spaces, then equality holds.
7. If \( A \perp B \), then \( R_{A \oplus B} = R_A \cdot R_B \).
8. If \( A \) and \( B \) are linear spaces, then \( R_{A \otimes B} = R_A \cdot R_B \).

Here the sum (\( + \)), product (\( \cdot \)), the stochastic independence (\( \perp \)), and the \( L^2(Y, \rho) \)-orthogonality (\( \bot \)) of the sets \( A \) and \( B \) have to be understood element-wise.
allows estimating the variation constant numerically. The properties of the variation function also induce analogous properties of the norm $\|R_A\|_{w,\infty}$. Theorem 8.6 for example, implies that for any $d$-dimensional linear space $A$ spanned by the $L^2(Y,\rho)$-orthonormal basis $\{B_k\}_{k \in [d]}$,

$$\|R_A\|_{w,\infty} = \|R_{\langle B_1 \rangle \oplus \cdots \oplus \langle B_d \rangle}\| \leq \sum_{k \in [d]} \|B_k\|_{w,\infty}^2.$$  

(5)

Finally note that, by virtue of Theorem 7, Theorem 8 provides calculus rules for the computation of optimal weight functions.

**Remark 9.** A common misconception is that the probability bound in Theorem 4 relies primarily on the metric entropy (cf. [CG17]) of the model class. This however is not true, since $\|R_A\|_{w,\infty}$ can not be controlled by the metric entropy of $A$. To see this, consider any set $B$ for which $U(B)$ is compact. By continuity, there exists $b^* \in B$ such that $\|R_{\langle b^* \rangle}\|_{w,\infty} \geq \|R_{\langle b \rangle}\|_{w,\infty}$ for all $b \in B$. Thus, $\|R_{\langle b^* \rangle}\|_{w,\infty} \geq \|R_A\|_{w,\infty}$ for any subset $A \subseteq B$, independent of its metric entropy.

We use the remainder of this section to compute the variation function that is associated with a generic model class of tensor networks $M$. We do this by proving the sequence of inequalities

$$R_{\langle u(M) \rangle} = R_{\langle u(M) \rangle} \geq R_{\langle u(M) \rangle} - M \geq R_M = R_{\langle u(M) \rangle} - M.$$  

(6)

The first and the last equality hold, since $\{u(M)\} = \langle M \rangle = \{u(M)\} - \langle M \rangle$. The remaining relations follow from Theorem 8.1, Proposition 10 and Proposition 12.

Equation 6 indicates that (asymptotically) the same number of sample points are necessary to guarantee recovery in a model class of tensor networks $M$ as for the ambient space $\langle M \rangle$. Since, by equation (5), $\|R_{\langle M \rangle}\|_{w,\infty} \geq \dim(\langle M \rangle)$ grows exponentially with the order of the tensors, this model class may hence be infeasible for the recovery of certain tensors. Numerical evidence for this claim is provided in Figure 1. Although the number of parameters of $M$ is typically exponentially smaller than the number of parameters of the ambient space $\langle M \rangle$, the observed behaviour is not surprising. In the setting of low-rank matrix and tensor recovery it is well known, that the sought tensor has to satisfy an additional incoherence condition to be recoverable with few sample points (cf. [CT10; YZ15]).

![Figure 1: Two phase diagrams for the recovery of multivariate polynomials in the tensor product basis of Legendre polynomials. For every order $M$ and number of samples $n$, the mean error is computed as the relative L2-error of the approximation, averaged over 20 independent realisations. A hard-thresholding algorithm (cf. [ENSW18]) was used for recovery. Note that the optimal coefficient tensor $C \in (\mathbb{R}^5)^{\otimes M}$ is always of rank 1.](image-url)

(a) The sought function is defined by $C_{k_1,\ldots,k_M} = 1$. (b) The sought function is $\exp(y_1 + \cdots + y_M)$.

We use the remainder of this section to prove the Propositions 10 and 12, that are used in the derivation of equation (6).

**Proposition 10.** Let $M$ be conic and symmetric and let $v \in V$. Then $R_{\langle v \rangle} - M = R_{\langle v \rangle + \text{cl}(M)}$.

To prove Proposition 10 we need the following lemma.

**Lemma 11.** $\text{cl}(A + B) \supseteq \text{cl}(A) + \text{cl}(B)$ for all sets $A$ and $B$.

**Proof.** Let $a \in \text{cl}(A)$ and $b \in \text{cl}(B)$. Then there exist sequences $\{a_k\} \in A$ and $\{b_k\} \in B$ such that $a_k \to a$ and $b_k \to b$. Since $a_k + b_k \in A + B$ we have $a + b = \lim_{k \to \infty} (a_k + b_k) \in \text{cl}(A + B)$. $\square$
Proof of Proposition 10. By Theorem 8.1, Lemma 11 and Theorem 8.2, it holds that
\[ R_{\{v\}} - \mathcal{M} \leq R_{\mathcal{M}}(\{v\} - \mathcal{M}) \leq R_{\mathcal{M}} \cap (\mathcal{M} \cap (\{v\} - \mathcal{M})) = R_{\mathcal{M}}(\{v\} - \mathcal{M}). \]
Since $\mathcal{M}$ is conic and symmetric, it holds that
\[ \text{Cone}(\{v\}) - \mathcal{M} = \text{Cone}(\{v\} - \mathcal{M}) \]
and consequently $R_{\text{Cone}(\{v\})} - \mathcal{M} = \max\{R_{\text{Cone}(\{v\})}, R_{\mathcal{M}}(\mathcal{M})\}$. Since Theorem 8.7 implies that $R_{\mathcal{M}} \cap \mathcal{M} = R_{\mathcal{M}}(\mathcal{M})$ for any set $\mathcal{M}$ and all subsets $\mathcal{C} \subseteq \mathcal{M}$, it follows that $R_{\text{Cone}(\{v\})} - \mathcal{M} = R_{\mathcal{M}}(\mathcal{M})$. □

Proposition 12. For any model class of tensor networks of fixed order $\mathcal{M}$, it holds that $R_{\mathcal{M}} = R_{\mathcal{M}}(\mathcal{M})$.

Proof. Let $\mathcal{M} \subseteq L^2(\mathcal{V}_1, \rho_1) \otimes \cdots \otimes L^2(\mathcal{V}_M, \rho_M)$ be a set of tensor networks of order $M$ with arbitrary but fixed architecture and rank constraints and define the vector spaces $\mathcal{V}_m \subseteq L^2(\mathcal{V}_m, \rho_m)$ such that
\[ \langle \mathcal{M} \rangle = \bigotimes_{m=1}^M \mathcal{V}_m. \]
Recall that the set of rank–1 tensors (cf. [Hit27]) is defined as $\mathcal{T}_1 := \{v_1 \otimes \cdots \otimes v_M : v_m \in \mathcal{V}_m \text{ for all } m\} = \mathcal{V}_1 \cdots \mathcal{V}_M$, i.e. the set of element-wise products of functions in $\mathcal{V}_1, \ldots, \mathcal{V}_M$. Since every element in $\mathcal{T}_1$ can be approximated arbitrarily well in $\mathcal{M}$, Theorem 8.2 and Theorem 8.1 imply
\[ R_{\mathcal{M}} = R_{\mathcal{C} \mathcal{M}(\mathcal{M})} \geq R_{\mathcal{T}_1} \]
and by Theorem 8.7 and 8.8, it holds that
\[ R_{\mathcal{T}_1} = R_{\mathcal{V}_1 \cdots \mathcal{V}_M} = R_{\mathcal{V}_1} \cdots \mathcal{V}_M = R_{\mathcal{V}_M} = \mathcal{M}. \]
Combining equation (7) and (8) and employing Theorem 8.1 a final time yields the chain of inequalities $R_{\mathcal{M}}(\mathcal{M}) \geq R_{\mathcal{M}} \geq R_{\mathcal{T}_1} = R_{\mathcal{M}}(\mathcal{M})$, which concludes the proof. □

3 Restriction to neighbourhoods

The chain of inequalities (6) illustrates a problem that is not specific to the model class of tensor networks. Since Theorem 4 provides a worst-case bound for the entire model class, it must take into account every single element of $A = \{u_M\} - \mathcal{M}$ — even those that are very far away from $u_M$. In light of Theorem 8.1, it stands to reason that this problem could be eliminated by restricting the model class $\mathcal{M}$ to a neighbourhood $\mathcal{N} \subseteq \mathcal{M}$ of $u_M$.

For model classes that can be locally linearised in $u_M$, we can compute upper and lower bounds for $R_{\{u_M\}} - \mathcal{N}$. To formalise this we utilise the concept of reach. First introduced by Federer [Fed59], the local reach $\text{rch}(A, v)$ of a set $A \subseteq \mathcal{V}$ at a point $v \in A$ is the largest number such that any point at distance less than $\text{rch}(A, v)$ from $v$ has a unique nearest point in $A$.

Definition 13. For any subset $A \subseteq \mathcal{V}$, let $P_A : \mathcal{V} \to A$ be the best approximation operator $P_A v := \arg \min_{a \in A} \|v - a\|$ and denote by $\text{dom}(P_A)$ the domain on which it is well defined. Then $\text{rch}(A, v) := \sup\{r \geq 0 : B(v, r) \subseteq \text{dom}(P_A)\}$.

Intuitively, if the local reach $\text{rch}(A, v)$ of a set $A$ is larger than $R > 0$ at every point $v \in A$, then a ball of radius $R$ can roll freely around $A$ without ever getting stuck [CFP12]. Moreover, if $A$ is a differentiable manifold, then the rolling ball interpretation of the reach also provides a measure of the curvature for $A$ that generalises the concept of an osculating circle for planar curves. The distance of the manifold $A$ to its tangent space $v + T_v A$ is hence inversely proportional to the reach.

Proposition 14. Let $\mathcal{N}$ be a manifold, $u \in \mathcal{N}$ and assume that $R := \text{rch}(\mathcal{N}, u) > 0$. Then there exists $0 < r^* \leq R$ such that the following two statements hold.

1. For all $r \in (0, \infty)$ and every $v \in \mathcal{N} \cap B(u, r)$ there exists $w \in (u + T_v \mathcal{N}) \cap B(u, r)$ with $\|v - w\| \leq \frac{\|v - u\|^2}{2r}$.

2. For all $r \in (0, r^*)$ and every $w \in (u + T_u \mathcal{N}) \cap B(u, r)$ there exists $v \in \mathcal{N} \cap B(u, r)$ with $\|w - v\| \leq \frac{\|u - w\|^2}{2r}$.

A very elegant proof of this statement that relies only on fundamental geometric arguments can be found in Appendix B. With all of this at our disposal we can now provide a first upper bound for the variation function $R_{\{u_M\} - \mathcal{N}}$. 

6
Theorem 15. Let \( r > 0 \) and \( N \subseteq M \cap B(u_M, r) \) be a manifold with \( R := \text{rch}(N, u_M) \geq r \). Then
\[
\mathcal{R}_{\{u_M\}-N} \leq \left( \frac{\sqrt{\mathcal{R}_{\varepsilon_{u_M} M}^2 + \frac{r^2}{2R} \sqrt{\mathcal{R}_{\varepsilon_{u_M} M}^2}}}{2} \right)^2.
\]

Proof. Recall that \( \mathcal{R}_{\{u_M\}-N} = \sup_{v \in N} \mathcal{R}_{\{u_M\}-v} \). We start by bounding \( \mathcal{R}_{\{u_M\}-v} \) for a fixed \( v \in N \). For this define
\[
\alpha_T = \|P(u_M - v)\| \quad \text{and} \quad b_T = \alpha_T^{-1} P(u_M - v),
\]
\[
\alpha_N = \|Q(u_M - v)\| \quad \text{and} \quad b_N = \alpha_N^{-1} Q(u_M - v),
\]
where \( P \) denotes the orthogonal projection onto \( T_{u_M} N \) and \( Q := 1 - P \). This means that \( (u_M - v)^2 = (\alpha_T b_T + \alpha_N b_N)^2 \leq (|\alpha_T| |b_T| + |\alpha_N| |b_N|)^2 \) and \( |u_M - v| = \sqrt{\alpha_T^2 + \alpha_N^2} \leq r \). Since \( |u_M - v| \leq r \leq R \), Proposition 14.1 implies
\[
|\alpha_T| \leq |\frac{\alpha_T^2}{2R} + \alpha_N^2| \quad \text{and} \quad |\alpha_N| \leq \frac{\alpha_T^2}{2R} \quad \text{and} \quad |\alpha_N| \leq \frac{\alpha_T^2}{2R}.
\]
Consequently, \( \mathcal{R}_{\{u_M\}-v} = \frac{(u_M-v)^2}{|u_M-v|^2} \leq (|b_T| + \frac{r}{2R} |b_N|)^2 = (\sqrt{R^2(b_T)} + \frac{r}{2R} \sqrt{R(b_N)})^2 = (\sqrt{R^2(b_T)} + \frac{r}{2R} \sqrt{R^2(b_N)})^2. \)
Taking the supremum over \( v \in N \), we obtain
\[
\mathcal{R}_{\{u_M\}-N} \leq \left( \frac{\sup_{v \in N} \mathcal{R}_P(b_T-v) + \frac{r}{2R} \sqrt{\mathcal{R}_Q(b_N-v)}}{2} \right)^2 = \left( \frac{\sqrt{\mathcal{R}_P(b_T)} + \frac{r}{2R} \sqrt{\mathcal{R}_Q(b_N)}}{2} \right)^2.
\]
To conclude the proof, observe that \( P(\{u_M\} - N) = T_{u_M} N \) and \( Q(\{u_M\} - N) \subseteq T_{u_M} M \). \( \square \)

This theorem provides an upper bound to the variation function in a neighbourhood of the best approximation \( u_M \) and supports the intuition that the probability of finding the best approximation increases when the search is restricted to a small neighbourhood of the best approximation. The variation function of the tangent space captures the regularity of the model class in the proximity of the best approximation and provides a lower bound for the variation function of any neighbourhood of \( u_M \). This establishes \( \mathcal{R}_{\varepsilon_{u_M} M} \) as a generalised measure of incoherence.

Remark 16. This is, presumably, the reason for the practical success of many algorithms for low-rank approximation. Suppose that the initial guess \( v_0 \) satisfies \( v_0 \in N \) for some neighbourhood \( N \) of \( u_M \). Then, with high probability, every iterate \( v_k \) will satisfy \( v_k \in N \), if the variation function \( \mathcal{R}_{\{u_M\}-N} \) is sufficiently small. The preceding theorem gives sufficient conditions for this to happen, namely that

1. the initial guess is close to the best approximation (the neighbourhood \( N \) is small),
2. the best approximation is sufficiently incoherent (the norm \( \|\mathcal{R}_{\varepsilon_{u_M} M}\|_{w, \infty} \) is small) and
3. the local reach of the manifold is positive.

All three assumptions are necessary but challenging to guarantee in practice. Specifically, the first assumption depends significantly on the problem and the additional information available. We discuss the validity of the remaining two assumptions on the three Examples 20–22 at the end of this section.

We now prove that bound from Theorem 15 is sharp when the the radius \( r \) of the neighbourhood goes to zero. This again follows from the fact that the distance of the manifold to its tangent space can be bounded with the help of the reach. We start by formulating the following simple corollary from Proposition 14.

Corollary 17. Let \( N \) be a manifold and \( u \in N \). Moreover, assume that \( R := \text{rch}(N, u) > 0 \). Then, for any \( r \leq R \),
\[
d_H(N \cap B(u, r), (u + T_u N) \cap B(u, r)) \leq \frac{r^2}{2R}.
\]
Since both $N \cap B(v, r)$ and $(v + T_v N) \cap B(v, r)$ are contained in the ball $B(v, r)$, they have to converge with a rate of at least $r$. The key message of the Corollary 17 is hence the convergence at a faster rate of $r^2$. This means that this convergence is preserved even after a rescaling of the sets. This is made explicit with the help of the normalisation operator $U$ from equation (4) in the subsequent proposition.

**Proposition 18.** Let $N$ be a manifold and $u \in N$. Moreover, assume that $R = \text{rch}(N, u) > 0$. Then, for any $r \leq R$,

$$d_H(U(N \cap B(u, r) - u), U(N \cap B(u, r) - u)) \leq \frac{r}{R}.$$ 

The proof of both statements can be found in Appendix C. Using this convergence and the continuity of the variation function, we can deduce the following corollary.

**Theorem 19.** Let $N \subseteq M$ be a manifold and neighbourhood of $u_M$ and assume that $R := \text{rch}(N, u_M) > 0$. Then

$$\mathcal{R}_{u_M} \subseteq \mathcal{R}_{u_M} - N.$$  

**Proof.** We start by proving that

$$\lim_{r \to 0} \mathcal{R}_{u_M} - N \cap B(u_M, r) = \mathcal{R}_{u_M} - N. \quad (9)$$

Recall from the proof of Theorem 8.3–8.5 that $\mathcal{R}_* = F \circ U$, where $F = \text{sq} \circ \sup \circ \text{abs}$ and $\text{sq}$, sup, and abs are defined in Equations (10)–(12). By Lemmas 24–26 $F : \mathcal{V}((\mathcal{V}_w, \omega)) \to \mathcal{V}_{w^2, \infty}$ is continuous with respect to the Hausdorff pseudometric. The continuity of $F$ and Proposition 18 then imply

$$\lim_{r \to 0} F(U(N \cap B(u_M, r) - u_M)) = F(\lim_{r \to 0} U(N \cap B(u_M, r) - u_M)) = F(U(N \cap B(u_M, N)).$$

Theorem 8.1 implies $\mathcal{R}_{u_M} - N \cap B(u_M, r) \subseteq \mathcal{R}_{u_M} - N$. After taking the limit $r \to 0$, the assertion follows from equation (9).

We conclude this section with three examples that illustrate the conditions from Remark 16 on several well-known model classes. For linear spaces, sparse vectors and low-rank matrices, we compute the local reach as well as the variation function of the tangent space.

**Example 20** (Linear spaces). Let $Y$ be some set and $\rho$ be a probability measure on $Y$. Moreover, let $M$ be the linear space spanned by the $L^2(Y, \rho)$-orthonormal basis $\{b_k\}_{k \in [d]}$ with $d \in \mathbb{N}$. Since the best approximation in $M$ is unique for any element in $\mathcal{V}$, it holds that $\text{rch}(M, v) = \infty$ for every $v \in M$. Furthermore, for any $v \in M$, it holds that $T_v M = M$. The variation function of any neighbourhood $N \subseteq M$ of $v$ is hence given by

$$\mathcal{R}_{\{v\} - N} = \mathcal{R}_{\{v\} - M} = \mathcal{R}_{T_v M} = \mathcal{R}_{(b_1) \oplus \cdots \oplus (b_d)} = b_1^2 + \cdots + b_d^2.$$

In the special case $Y = [-1, 1]$ and $\rho = \frac{1}{2} \text{dx}$ and for the basis of Legendre polynomials, this results in the lower bound $\|\mathcal{R}_{T_v M}\|_{w, \infty} = d^2$ for the weight function $w \equiv 1$. Using the optimal weight function, given by Theorem 7, yields the bound $\|\mathcal{R}_{T_v M}\|_{w, \infty} = d$.

**Example 21** (Sparse vectors). Let $Y$ be some set and $\rho$ be a probability measure on $Y$. For $d \in \mathbb{N} \cup \{\infty\}$, let $\{b_k\}_{k \in [d]}$ be an $L^2(Y, \rho)$-orthonormal basis and let $\{\omega_k\}_{k \in [d]}$ satisfy $\omega_k \geq \|b_k\|_{w, \infty}$ for all $k \in [d]$. For ease of notation we identify every $v \in \langle b_1, \ldots, b_d \rangle$ with its coefficient vector $v \in \mathbb{R}^d$. The model class of $\omega$-weighted $s$-sparse functions can then be defined as

$$M := \{v \in \langle b_1, \ldots, b_d \rangle : \|v\|_{\omega, 0} \leq s\} \quad \text{with} \quad \|v\|_{\omega, 0} := \sqrt{s \sum_{k \in \text{supp}(v)} \omega_k^2}.$$ 

Now consider the neighbourhood $N := M \cap B(v, r)$ for some $v \in M$ with $\|v\|_{\omega, 0}^2 + \omega_k^2 > s^2$ for all $k \in [d] \setminus \text{supp}(v)$ and $r \leq \min\{|v_k| : k \in \text{supp}(v)\}$. Since

$$N = \{w \in \langle b_1, \ldots, b_d \rangle : \text{supp}(w) = \text{supp}(v) \text{ and } \|v - w\|_2 \leq r\}$$

is a ball of radius $r$ in a $|\text{supp}(v)|$-dimensional linear space, it holds that $\text{rch}(N, v) = \infty$. We have, furthermore, that $T_v N = \langle b_k : k \in \text{supp}(v) \rangle$ and hence

$$\mathcal{R}_{T_v N} = \sum_{k \in \text{supp}(v)} \|b_k\|_{L^2}^2 \leq \sum_{k \in \text{supp}(v)} \omega_k^2 \leq s^2.$$
In contrast to the previous example, this already yields a mild improvement over the bound $\mathcal{R}_{\mu(\mathcal{M})} \leq \mathcal{R}_{\mathcal{M}} \leq 2\tilde{s}^2$. Moreover, we see that the condition that $N$ is a manifold with $\text{rch}(N, v) > 0$ becomes non-trivial. If the $\omega$-weighted sparsity $s$ is not completely exhausted by $v$, then $N$ may not be a manifold or the best approximation in $N$ may not be uniquely defined and the local reach vanishes. This shows, that the sparsity of the sought function $u$ should not be overestimated, when defining the model class $\mathcal{M}$.

**Example 22** (Low-rank matrices). Let $Y = Y_L \times Y_R$ be some set and $\rho = \rho_L \otimes \rho_R$ be a product probability measure on $Y$. For $m \in \{L, R\}$, let $\{b_{m,k}\}_{k \in [d]}$ be an $L^2(Y_m, \rho_{m})$-orthonormal basis and let $B_L := b_{L,k} \otimes b_{R,l}$ be the corresponding tensor product basis. As in the previous example, we identify every $v \in \langle B_{11}, \ldots, B_{dd} \rangle$ with its coefficient matrix $v \in \mathbb{R}^{d \times d}$. Now we can define the model class

$$
\mathcal{M} := \{v \in \langle B_{11}, \ldots, B_{dd} \rangle : \text{rank}(v) \leq R\}
$$

of functions with coefficient matrices of rank at most $R$. Let $v \in \mathcal{M}$ be of rank $R$ and consider the neighbourhood $\mathcal{N} := \mathcal{M} \cap B(v, r)$ for some $r \leq \sigma_R(v)$. Here, $\sigma_k(v)$ denotes the $k^{\text{th}}$ largest singular value of $v$ and $B(v, r)$ denotes the $\|\cdot\|$-ball of radius $r$ around $v$. Then $\mathcal{N}$ corresponds to a $\|\cdot\|_{\rho_0}$-ball of radius $r$ in the $C^{\infty}$-manifold of rank-$R$ matrices and $\text{rch}(N, v) \geq \frac{r}{2}$. To see this, let $w$ be any matrix with $\|v - w\|_{\rho_0} \leq \frac{r}{2}$ and observe that

$$
\sigma_R(w) \geq \sigma_R(v) - \frac{r}{2} > 0 \quad \text{and} \quad \sigma_R(w) - \sigma_{R+1}(w) \geq \sigma_R(v) - \frac{r}{\sqrt{2}} > 0.
$$

This means, that rank$(w) \geq R$ and that its best rank-$R$ approximation, given by the truncated singular value decomposition, is uniquely defined. This approximation, denoted by $x$, lies in $\mathcal{N}$, since $\|v - x\| \leq ||v - w\| + ||w - x\| \leq \frac{r}{2} + \frac{r}{2}$. Let $v = w_L \text{diag}(\sigma)w_R^T$ be the singular value decomposition of $v$. The tangent space at $v$ can then be written as $T_v \mathcal{N} = \mathcal{W}_L \oplus \mathcal{W}_R$ with $\mathcal{W}_L := \langle w_L \rangle \otimes \langle b_{R,1}, \ldots, b_{R,d} \rangle$ and $\mathcal{W}_R := \langle w_L \rangle^\perp \otimes \langle w_R \rangle$, where $\langle w_L \rangle$ denotes the linear span of the columns of $w_L$ and $\langle w_L \rangle^\perp$ denotes its orthogonal complement in $\langle b_{L,1}, \ldots, b_{L,d} \rangle$. Thus

$$
\mathcal{R}_{T_v \mathcal{N}} = \mathcal{R}_{\mathcal{W}_L} + \mathcal{R}_{\mathcal{W}_R}, \\
\mathcal{R}_{\mathcal{W}_L} := \mathcal{R}_{\{w_L\}} \mathcal{R}_{\langle b_{R,1}, \ldots, b_{R,d} \rangle} \text{ and } \\
\mathcal{R}_{\mathcal{W}_R} := \mathcal{R}_{\{w_L\}^\perp} \mathcal{R}_{\langle w_R \rangle} \leq \mathcal{R}_{\langle b_{L,1}, \ldots, b_{L,d} \rangle} \mathcal{R}_{\langle w_R \rangle}.
$$

For the sake of simplicity, assume that $\|b_{m,k}\|_{\rho_\infty} = 1$ for all $m \in \{L, R\}$ and $k \in [d]$. Then we obtain the bound $\mathcal{R}_{T_v \mathcal{N}} \leq d(\mathcal{R}_{\{w_L\}} + \mathcal{R}_{\{w_R\}})$. This shows that, when $R$ is fixed, the variation function $\mathcal{R}_{T_v \mathcal{N}}$ grows only linearly with $d$ while $\mathcal{R}_{\mathcal{M}}$ grows quadratically. This reduction is even more pronounced for model classes of higher order tensors.

This is the first example which truly benefits from restricting the model class to a neighbourhood of the best approximation. But as in the previous example, we see that the condition that $N$ is a manifold with positive local reach is not trivial. If the rank of $v$ is strictly smaller than $R$, then $N$ may not be a manifold or the best approximation in $N$ may not be uniquely defined and the local reach vanishes. This shows that the rank of $u$ must not be overestimated, when defining the model class $\mathcal{M}$. It is easy to see, that otherwise $\mathcal{R}_{\{u_m\} \neq N} = \mathcal{R}_{\mathcal{M}}$ for all neighbourhoods $N$. Many successful algorithms for low-rank approximation already account for this by starting with an initial guess of rank 1 and successively increasing the rank while testing for divergence on a validation set.

### 4 Discussion

This work is an extension of the theory developed in [EST22]. Theorem 3 recalls a bound for the empirical best approximation error in the event that a certain norm equivalence is satisfied and the subsequent Theorem 4 bounds the probability of this event. The fact that this bound depends mainly on a weighted $L^\infty$-norm of the variation function motivates the remainder of this paper, in which several properties of the variation function are derived.

A crucial point in this paper is the observation that... any model class of tensor networks exhibits the same variation function as the linear space in which it is embedded. This indicates that a feasible empirical best approximation is impossible when considering the entire model class of tensor networks and motivates our investigation of the variation function in neighbourhoods of the best approximation in Theorem 15 and 19. If the neighbourhood is a manifold with positive local reach, these theorems estimate the variation function of the neighbourhood by that of the tangent space. This confirms the supposition that the probability of a successful empirical approximation increases if the approximation algorithm remains in a sufficiently small and regular neighbourhood of the best approximation. As a consequence, we can formulate a heuristic argument for the empirically observed success of many approximation algorithms on tensor networks.
However, as illustrated in Examples 20, 21 and 22, the conditions for Theorem 15 and 19 are not trivial to check and seem unrealistic to guarantee in most practical settings. Going forward, we hence advocate for exploring algorithms that automatically remain in a subclass of sufficiently small variation function. A tried and tested method to achieve this is to explore a nested sequence of model classes.

Acknowledgements

Our code made use of the following Python packages: numpy [Har+20], scipy [Vir+20], and matplotlib [Hun07].

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A Appendix: Proof of Theorem 8

1. Follows directly from the definition.

2. To see that $\mathcal{R}_A = \mathcal{R}_{cl(A)}$ let $a \in cl(A) \setminus \{0\}$. Then there exists a sequence $\{a_k\} \in A \setminus \{0\}$ such that $a_k \to a$.

Due to the continuity of $a \mapsto a(y)^2/\|a\|^2$ on $A \setminus \{0\}$ it follows that

$$\mathcal{R}_{(a)}(y) = \frac{|a(y)|^2}{\|a\|^2} = \lim_{k \to \infty} \frac{|a_k(y)|^2}{\|a_k\|^2} = \lim_{k \to \infty} \mathcal{R}_{(a_k)}(y)$$

And since $\mathcal{R}_{(a_k)} \leq \mathcal{R}_A$ for all $k = 1, \ldots, \infty$ and we can conclude $\mathcal{R}_{(a)} \leq \mathcal{R}_A$. The assertion follows with 8.1 since $\mathcal{R}_{\{a\}} \leq \mathcal{R}_{cl(A)} = \sup_{a \in cl(A)} \mathcal{R}_{(a)} \leq \mathcal{R}_A$. 

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3.-5. In all three cases we can write \( R \ast = sq \circ sup \circ abs \circ U \) with

\[
\begin{align*}
\text{sq}: & \mathcal{V}_{w,\infty} \to \mathcal{V}_{w^2,\infty}, & \text{sq}(v)(y) := v(y)^2, \\
\text{abs}: & \mathcal{V}_{w,\infty} \to \mathcal{V}_{w,\infty}, & \text{abs}(v)(y) := |v(y)|, \\
\text{sup}: & \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w,\infty}, & \sup(v) := \sup_{v \in V} v(y), \text{ and} \\
\text{inf}: & \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w,\infty}, & \inf(v) := \inf_{v \in V} v(y).
\end{align*}
\]

This allows us to prove the continuity of \( sq \circ sup \circ abs : \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w^2,\infty} \) and \( U \) individually. The main difference between Theorems 8.3, 8.4 and 8.5 then comes from the domain of \( U \).

We proceed by showing that \( sq \circ sup \circ abs : \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w^2,\infty} \) is continuous with respect to the Hausdorff pseudometric, which also implies the continuity of \( sq \circ sup \circ abs : \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w^2,\infty} \) with respect to the Hausdorff metric.

To do this we require the following four lemmata.

**Lemma 23.** Let \((M_1, \delta_1)\) and \((M_2, \delta_2)\) be metric spaces, let \( f : M_1 \to M_2 \) and define \( f(X) := \{ f(x) : x \in X \} \) for any \( X \in M_1 \). If \( f \) is uniformly continuous, then \( f : \mathcal{V}(M_1) \to \mathcal{V}(M_2) \) is uniformly continuous with respect to the Hausdorff pseudometric.

**Proof.** Recall, that \( d_{\mathcal{H}}(X, Y) \leq \varepsilon \) means that

\[
\forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon \quad \text{and} \quad \forall y \in Y \exists x \in X : d(y, x) \leq \varepsilon.
\]

Let \( \varepsilon > 0 \). Since \( f \) is uniformly continuous there exists \( \delta > 0 \) such that \( d_1(x, y) < \delta \) implies \( d_2(f(x), f(y)) < \varepsilon \). We now show that \( d_{\mathcal{H}}(U, V) < \delta \) implies \( d_{\mathcal{H}}(f(U), f(V)) < \varepsilon \).

For this let \( f_u \in f(U) \) and choose \( u \in U \) such that \( f(u) = f_u \). Since \( d_{\mathcal{H}}(U, V) < \delta \) there exists \( v \in V \) such that \( d_1(u, v) < \delta \) and consequently \( d_2(f(u), f(v)) < \varepsilon \), by uniform continuity. This means that for every \( f_u \in f(U) \) there exists \( f_v \in f(V) \) such that \( d_2(f_u, f_v) < \varepsilon \). Since this argument remains valid if the roles of \( U \) and \( V \) are reversed we can conclude that \( d_{\mathcal{H}}(f(U), f(V)) < \varepsilon \).

**Lemma 24.** \( \text{abs} : \mathcal{V}_{w,\infty} \to \mathcal{V}_{w,\infty} \) is Lipschitz continuous with constant 1. \( \text{abs} : \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}(\mathcal{V}_{w,\infty}) \) is uniformly continuous with respect to the Hausdorff pseudometric.

**Proof.** The first assertion follows by the reverse triangle inequality, \( ||v(y)| - |w(y)|| \leq |v(y) - w(y)| \). The second assertion follows by Lemma 23, since Lipschitz continuity implies uniform continuity.

**Lemma 25.** \( \text{sup} : \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w,\infty} \) is Lipschitz continuous with constant 1.

**Proof.** Let \( U, V \in \mathcal{V}(\mathcal{V}_{w,\infty}) \). To show that \( \|\sup(U) - \sup(V)\|_{w,\infty} \leq d_{\mathcal{H}}(U, V) \), fix \( y \in Y \) and assume w.l.o.g. that \( \sup(U)(y) \geq \sup(V)(y) \). Then

\[
\sqrt{w(y)}|\sup(U)(y) - \sup(V)(y)| = \sqrt{w(y)} \sup_{u \in U} \inf_{v \in V} (u(y) - v(y)) \\
\leq \sup_{u \in U} \inf_{v \in V} \|u - v\|_{w,\infty} \\
\leq d_{\mathcal{H}}(U, V).
\]

Taking the supremum over \( y \in Y \) proves the assertion.

**Lemma 26.** \( \text{sq} : \mathcal{V}_{w,\infty} \to \mathcal{V}_{w^2,\infty} \) is continuous.

**Proof.** Fix \( v \in \mathcal{V}_{w,\infty} \) and let \( w \in \mathcal{V}_{w,\infty} \) be arbitrary. Then

\[
\|v^2 - w^2\|_{w^2,\infty} = \|v v - v w + v w - w w\|_{w^2,\infty} \\
\leq \|v(v - w)\|_{w^2,\infty} + \|w(v - w)\|_{w^2,\infty} \\
\leq (\|v\|_{w,\infty} + \|w\|_{w,\infty}) \|v - w\|_{w,\infty}.
\]
Observe that, due the reverse triangle inequality, \( \|v - w\|_{w,\infty} \leq \delta \) implies \( \|w\|_{w,\infty} \leq \|v\|_{w,\infty} + \delta \). This proves continuity, since for any \( \varepsilon \) we can choose \( \delta \) such that \( \|v - w\|_{w,\infty} \leq \delta \) implies
\[
\|v^2 - w^2\|_{w^2,\infty} \leq (2\|v\|_{w,\infty} + \delta)\delta \leq \varepsilon.
\]
\( \square \)

As a composition of continuous functions, the continuity of \( \text{sq} \circ \text{sup} \circ \text{abs} : \mathcal{V}(\mathcal{V}_{w,\infty}) \to \mathcal{V}_{w^2,\infty} \) is guaranteed by Lemmas 24 to 26. We now proceed by proving the continuity of \( U \) in the settings of Theorem 8.3, 8.4 and 8.5 individually.

3. To prove this we need the subsequent lemma.

**Lemma 27.** Let \((M_1, d_1)\) and \((M_2, d_2)\) be metric spaces, let \( f : M_1 \to M_2 \) and define \( f(X) := \{ f(x) : x \in X \} \) for any \( X \in M_1 \). If \( f \) is continuous, then \( f : \mathcal{C}(M_1) \to \mathcal{C}(M_2) \) is continuous with respect to the Hausdorff metric.

**Proof.** \( f : \mathcal{C}(M_1) \to \mathcal{C}(M_2) \) is well-defined since the image of a compact set under a continuous function is compact. Now recall, that \( d_H(X, Y) \leq \varepsilon \) means that
\[
\forall x \in X \exists y \in Y : d(x, y) \leq \varepsilon \quad \text{and} \quad \forall y \in Y \exists x \in X : d(y, x) \leq \varepsilon.
\]
Let \( \varepsilon > 0 \) and \( U \in \mathcal{C}(M_1) \). Since \( f \) is continuous in every \( u \in U \) there exists a \( \delta_u > 0 \) that guarantees
\[
d_1(u, \tilde{u}) < \delta_u \Rightarrow d_2(f(u), f(\tilde{u})) < \frac{\varepsilon}{2}.
\]
Now define the sets \( N_u := \{ \tilde{u} \in M_1 : d_1(u, \tilde{u}) < \frac{\varepsilon}{2} \} \). Since \( u \in N_u \), the family \( \{N_u\}_{u \in U} \) defines a covering of \( U \) and since \( U \) is compact there exists a finite subcovering \( \{N_{u_i}\}_{i=1,\ldots,n} \). Choose \( \delta := \min_{i=1,\ldots,n} \frac{\delta_{u_i}}{2} \) and note, that \( \delta \) has to be positive, since it is the minimum of finitely many positive numbers. Now let \( V \in \mathcal{C}(M_1) \) such that \( d_H(U, V) \leq \delta \).

First, we show that
\[
\forall f_v \in f(V) \exists f_u \in f(U) : d_2(f_v, f_u) \leq \varepsilon.
\]
For this let \( v \in V \) be any element that satisfies \( f(v) = f_v \). Since \( d_H(U, V) \leq \delta \) there exists \( u \in U \) with \( d_1(u, v) \leq \delta \). Moreover, by definition of the covering \( \{N_{u_i}\}_{i=1,\ldots,n} \), there exists \( u_i \) such that \( d_1(u, u_i) \leq \frac{\delta_{u_i}}{2} \). Using the triangle inequality, we thus obtain
\[
d_1(u, v) \leq d_1(u_i, u) + d_1(u_i, v) \leq \frac{\delta_{u_i}}{2} + \delta \leq \delta_{u_i}
\]
and the definition of \( \delta_{u_i} \) finally yields \( d_2(f_v, f(u_i)) \leq \frac{\varepsilon}{2} \leq \varepsilon \).

Now we show that
\[
\forall f_v \in f(V) \exists f_u \in f(U) : d_2(f_u, f_v) \leq \varepsilon.
\]
Analogously to the argument from above let \( u \in U \) be any element that satisfies \( f(u) = f_u \). Since \( d_H(U, V) \leq \delta \) there exists \( v \in V \) with \( d_1(u, v) \leq \delta \) and by the definition of the covering there exists also a \( u_i \) with \( d_1(u, u_i) \leq \frac{\delta_{u_i}}{2} \). We can now estimate
\[
d_2(f(u), f(v)) \leq d_2(f(u), f(u_i)) + d_2(f(u_i), f(v)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
which holds by the definition of \( \delta_{u_i} \) and because
\[
d_1(u, v) \leq d_1(u_i, u) + d_1(u_i, v) \leq \frac{\delta_{u_i}}{2} + \delta \leq \delta_{u_i}.
\]
\( \square \)

Since the function \( u \mapsto u/\|u\| \) is continuous on \( \mathcal{V}_{w,\infty} \setminus \{0\} \) the function \( U : \mathcal{C}(\mathcal{V}_{w,\infty} \setminus \{0\}) \to \mathcal{C}(S(0, 1) \cap \mathcal{V}_{w,\infty}) \) is continuous by Lemma 27.

4. Let \( r > 0 \). Since the function \( u \mapsto u/\|u\| \) is uniformly continuous on \( \mathcal{V}_{w,\infty} \setminus \text{B}(0, r) \) the function \( U : \mathcal{C}(\mathcal{V}_{w,\infty} \setminus \text{B}(0, r)) \to \mathcal{C}(S(0, 1) \cap \mathcal{V}_{w,\infty}) \) is uniformly continuous by Lemma 23.

5. By definition of the truncated Hausdorff distance, \( U : \text{Cone}(\mathcal{V}(\mathcal{V}_{w,\infty})) \to \mathcal{V}(S(0, 1) \cap \mathcal{V}_{w,\infty}) \) is Lipschitz continuous with constant 1.
6. Every \( v \in A + B \) can be written as \( v = v^\top \alpha \) for some \( \alpha \in \mathbb{R}^2 \) and \( v \in (A \times B) \setminus \{0\} \). Moreover, \( A \perp B \) implies that \( \|v\|^2 = \alpha^\top D(v)\alpha \) with \( D(v) := \text{diag}(\|v_1\|, \|v_2\|) \). Now define \( C_{v,y} = D(v)^{-1}v(y) \) and observe that

\[
\mathcal{R}_{A+B}(y) \leq \sup_{v \in (A \times B) \setminus \{0\}} \sup_{\alpha \in \mathbb{R}^2 \setminus \{0\}} \frac{|v^\top D(v)^2 \alpha|}{\alpha^\top D(v)\alpha} = \sup_{v \in (A \times B) \setminus \{0\}} \sup_{\beta \in \mathbb{R}^2 \setminus \{0\}} \frac{|\beta^\top C_{v,y}C_{v,y}^\top \beta|}{\beta^\top \beta}.
\]

Note that the first inequality is indeed an equality, if \( A \) and \( B \) are linear spaces.

7. Let \( a \in A \) and \( b \in B \). Since \( a \perp b \) also \( a^2 \perp b^2 \) and consequently \( \|a \cdot b\|^2 = \mathbb{E}[a^2b^2] = \mathbb{E}[a^2]\mathbb{E}[b^2] = \|a\|^2\|b\|^2 \). Now recall that \( \mathcal{R}_A(y) = \sup_{a \in A \setminus \{0\}} a(y)^2 \). Thus

\[
\mathcal{R}_{A \cap B}(y) = \sup_{a \in A \cap B} \frac{(a \cdot b)(y)^2}{\|a \cdot b\|^2} = \sup_{a \in A \cap B} \frac{a(y)^2 \cdot b(y)^2}{\|a\|^2\|b\|^2} = \mathcal{R}_A(y) \cdot \mathcal{R}_B(y).
\]

8. A direct consequence of Theorem 8.6 is the following lemma.

**Lemma 28.** Let \( \{P_j\}_{j \in J} \) be an orthonormal basis for \( A \). Then \( \mathcal{R}_A(y) = \sum_{j \in J} P_j(y)^2 \). □

Now let \( \{P_{A,j}\}_{j \in J} \) be an orthonormal basis of \( A \) and \( \{P_{B,k}\}_{k \in K} \) be an orthonormal basis of \( B \). Then \( \{P_{A,j} \otimes P_{B,k}\}_{j \in J, k \in K} \) is an orthonormal basis for \( A \otimes B \) and by Lemma 28

\[
\mathcal{R}_{A \otimes B}(y) = \sum_{j \in J} \sum_{k \in K} P_{A,j}(y)^2 \cdot P_{B,k}(y)^2 = \left(\sum_{j \in J} P_{A,j}(y)^2\right) \left(\sum_{k \in K} P_{B,k}(y)^2\right) = \mathcal{R}_A(y) \cdot \mathcal{R}_B(y).
\]

### B Appendix: Proof of Proposition 14

The first statement indeed characterizes the reach of a set and an easily accessible proof can be found in [BCY18, Theorem 7.8 (2)]. We reiterate this proof in the following, since the proof of the second statement relies on similar arguments.

1. Let \( v \in N \cap B(u, r) \). Then there exists a unique best approximation of \( v \) in \( u + T_u N \) which we denote by \( w \). By the Pythagorean theorem, it holds that \( \|v - w\|^2 = \|v - w\|^2 - \|v - w\|^2 \leq r^2 \) and thus \( w \in (u + T_u N) \cap B(u, r) \). To show that \( \|v - w\| \leq C\|u - v\|^2 \), we consider the intersection of the sets \( N \) and \( u + T_u N \) with the plane spanned by \( u, v \) and \( w \). Since all three points lie in this plane their relative distances are preserved and it suffices to consider this two-dimensional problem from here on. Let \( D \) be the disk of radius \( R \) that is tangent to \( T_u N \) at \( u \) and whose center \( c \) is on the same side of \( u + T_u N \) as \( v \). This is illustrated in Figure 2.

Because \( \|v - w\| = \|u - v\| \sin(\alpha) \), it suffices to bound \( \sin(\alpha) \). To this end, observe that \( \text{rch}(N, u) = R \) implies that the disc \( D \) intersects \( N \) only in \( u \). This means, that \( v \) does not lie in the interior of \( D \) and the line segment \( \overline{uv} = \{u + (1-\lambda)v : \lambda \in [0, 1]\} \) must intersect the boundary of \( D \) at a point \( x \). Since the triangle \( \Delta(u, c, x) \) is isosceles, we have \( \beta = 2\alpha \) and \( \|u - x\| = 2R \sin(\frac{\beta}{2}) = 2R \sin(\alpha) \). Using \( \|u - x\| \leq \|u - v\| \) finally yields

\[
\|v - w\| = \|u - v\| \sin(\alpha) = \|u - v\| \frac{\|u - x\|}{2R} \leq \|u - v\| \frac{\|u - v\|^2}{2R}.
\]

\[
\text{Figure 2}
\]
2. Let \( P : N \rightarrow u + \mathbb{T}_uN \) be the Euclidean projection from \( N \) onto the tangent space of \( N \) at \( u \) and define
\[
r^* := \min \{ \sup \{ r > 0 : (u + \mathbb{T}_uN) \cap B(u, r) \subseteq PN \} \}, \]
Since \( N \) is a neighbourhood of \( u \), this constant is positive and we can consider some fixed \( r \in (0, r^*) \). By definition of \( r^* \), there exists for every \( w \in (u + \mathbb{T}_uN) \cap B(u, r) \) an element \( \bar{v} \in N \) such that \( P_{\mathbb{T}_uN}\bar{v} = w \). Analogously to the proof of Proposition 14.1, we consider the intersection of the sets \( N \) and \( u + \mathbb{T}_uN \) with the plane spanned by \( u, \bar{v} \) and \( w \). Because the distance between these points is preserved by the intersection, we can again consider the resulting two-dimensional problem. Let \( D \) be the disk of radius \( R \) that is tangent to \( \mathbb{T}_uN \) at \( u \) and whose center \( c \) is on the same side of \( u + \mathbb{T}_uN \) as \( \bar{v} \). Since \( r < R \), the neighbourhood \( N \) is a simply connected manifold and the line segment \( \mathbb{c}w = \{ \lambda c + (1 - \lambda)w : \lambda \in [0, 1] \} \) has to intersect \( \mathbb{N} \) at a point \( v \). Finally, let \( x := R \frac{w - \bar{v}}{\|w - \bar{v}\|} + c \) be the intersection of \( \mathbb{c}w \) with the disc \( D \). This is illustrated in Figure 3.

Since \( \|w - v\| \leq \|w - x\| \), it suffices to bound \( \|w - x\| \) which is given by the Pythagorean theorem as \( \|w - x\| = \sqrt{R^2 + \|v - u\|^2} - R \). Defining \( \ell(r) := \sqrt{R^2 + \|v - u\|^2} - R \) and \( \tilde{\ell}(r) := \frac{r^2}{2R} \) we observe that \( \ell(r) \leq \tilde{\ell}(r) \) since \( \ell(0) = 0 = \tilde{\ell}(0) \) and
\[
\ell'(r) = \frac{r}{\sqrt{R^2 + r^2}} \leq \frac{r}{R} = \tilde{\ell}'(r).
\]
This yields \( \|w - v\| \leq \|w - x\| = \ell(\|v - u\|) \leq \tilde{\ell}(\|v - u\|) = \frac{r^2}{2R} \) and concludes the proof.

\[ u + \mathbb{T}_uN \]
\[ u \]
\[ w \]
\[ N \]
\[ \mathbb{c}w \]
\[ D \]
\[ R \]
\[ x \]
\[ c \]
\[ v \]
\[ \bar{v} \]

Figure 3

C Appendix: Proof of Corollary 17 and Proposition 18

Proof of Corollary 17

Also recall that \( d_H(M \cap B(u, r), (u + \mathbb{T}_uM) \cap B(u, r)) \leq \frac{r^2}{2R} \) is equivalent to the conjunction of the following two statements.

1. For every \( v \in M \cap B(u, r) \) there exists a \( w \in (u + \mathbb{T}_uM) \cap B(u, r) \) such that \( \|v - w\| \leq \frac{r^2}{2R} \).
2. For every \( w \in (u + \mathbb{T}_uM) \cap B(u, r) \) there exists a \( v \in M \cap B(u, r) \) such that \( \|v - w\| \leq \frac{r^2}{2R} \).

The statement now follows from Proposition 14.

Proof of Proposition 18

Recall that \( R = \text{rch}(M \cap B(u, r_0)) \) and \( r \leq \min\{r_0, R\} \) and define \( C := (2R)^{-1} \). To prove \( d_H(U(M \cap B(u, r) - u), U(\mathbb{T}_uM)) \leq 2Cr \), note that \( d_H \) is induced by a norm and is therefore absolutely homogeneous and translation invariant. Therefore,
\[
d_H(U(M \cap B(u, r) - u), U(\mathbb{T}_uM)) = \frac{1}{r}d_H(rU(M \cap B(u, r) - u), rU(\mathbb{T}_uM)).
\]
Now define the operator \( U_r(X) := rU(X) \) that scales every element of a set to norm \( r \). The claim follows if \( d_H(U_r(M \cap B(u, r) - u), U_r(\mathbb{T}_uM)) \leq 2Cr^2 \). To prove this we need to show that the following two statements hold.

1. For every \( \hat{v} \in U_r(M \cap B(u, r) - u) \) there exists a \( \hat{w} \in U_r(\mathbb{T}_uM) \) such that \( \|\hat{v} - \hat{w}\| \leq 2Cr^2 \).
2. For every \( \hat{w} \in U_r(\mathbb{T}_uM) \) there exists a \( \hat{v} \in U_r(M \cap B(u, r) - u) \) such that \( \|\hat{v} - \hat{w}\| \leq 2Cr^2 \).
Proof of 1. Let $\hat{v} \in U_r(M \cap B(u, r) - u)$ and let $v \in M \cap B(u, r) - u$ be any element that satisfies $U_r(\{v\}) = \{\hat{v}\}$. In the proof of Theorem 14.1 we have shown that there exists a $w \in T_uM$ that satisfies $\|v - w\| \leq C\|v\|^2$ (cf. Equation (13)). We use this $w$ to define

$$\tilde{v} := \frac{r}{\|v\|}v, \quad \tilde{w} := \frac{r}{\|v\|}w, \quad \text{and} \quad \hat{w} = \frac{r}{\|w\|}w$$

and observe that $\tilde{v} = \hat{v} \in U_r(T_uM)$ and that $\hat{w} \in U_r(T_uM)$. Moreover, $\|\hat{v} - \hat{w}\| \leq \|\tilde{v} - \tilde{w}\| + \|\tilde{w} - \hat{w}\|$ and $\|\tilde{v} - \tilde{w}\| = \frac{r}{\|v\|}\|v - w\| \leq Cr\|v\| \leq Cr^2$. It thus remains to show that $\|\tilde{w} - \hat{w}\| \leq Cr^2$.

To see this we consider the intersection of $M - u$ and $T_uM$ with the plane $\{0, v, w\}$. This is illustrated in Figure 4. Since all the points that we have defined so far reside in this plane, the distances between them are preserved and we can henceforth consider only this two-dimensional problem.

To show $a := \|\tilde{w} - \hat{w}\| \leq \|\tilde{v} - \hat{v}\| =: b$, we consider the triangle $\Delta(\tilde{v}, \tilde{w}, 0)$ and employ the Pythagorean theorem

$$r^2 = (r - a)^2 + b^2.$$ 

Expanding the product and rearranging the terms results in the equation $b^2 = 2ra - a^2$. Since $r \geq a$ also $2ra \geq 2a^2$. Therefore, $b^2 \geq 2a^2 - a^2 = a^2$ which is what we wanted to prove.

Proof of 2. Let $\hat{w} \in U_r(T_uM)$. Since $r \leq R$, Theorem 14.2 guarantees that there exists a $v \in M \cap B(u, r) - u$ such that $\|\hat{w} - v\| \leq Cr^2$. Let $\hat{v} := \frac{r}{\|v\|}v$ and observe that, by the reverse triangle inequality,

$$\|\hat{v}\| - \|v\| \leq \|\hat{w}\| - \|v\| \leq \|\hat{w} - v\| \leq Cr^2.$$

Rearranging the terms and substituting $\|\hat{w}\| = r$ then yields $\|v\| \geq r - Cr^2$. It is now easy to estimate

$$\|v - \hat{v}\| = \left|1 - \frac{r}{\|v\|}\right|\|v\| = r - \|v\| \leq Cr^2.$$

Finally, using the triangle inequality, we obtain $\|\hat{w} - \hat{v}\| \leq \|\hat{w} - v\| + \|v - \hat{v}\| \leq 2Cr^2$. This concludes the proof.