REALISING A FINITE GROUP AS A SUBGROUP OF A
PRODUCT OF TWO GROUPS OF PERMUTATION MATRICES

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Abstract. In this paper we prove that any finite group of order $n$ can be viewed as the group of the solutions of a certain matrix equation $XB = BY$, where the unknowns $X, Y$ are two permutation matrices of order $n$ and $(1 + k)n + 2$ respectively and where $k \in \mathbb{N}$ is given by Cayley’s theorem. Moreover, we show that $G$ is isomorphic to a certain subgroup formed by permutation matrices of order $(1 + k)n$ obtained by permuting all the rows of the identity matrix $I_{(1+k)n}$.

1. Introduction

Let $\mathcal{P}(n)$ denote the group of permutation matrices of degree $n$. For a given matrix $B$, let us consider the group $\Omega_B$ of the pairs $(X, Y) \in \mathcal{P}(n) \times \mathcal{P}(n)$ which are solutions of the matrix equation $XB = BY$. Obviously, $\Omega_B$ is finite group as $\mathcal{P}(n)$ and $\mathcal{P}(m)$ are finite and it is worth noting that if $\lambda \in \mathbb{Q}$ and $(X, Y) \in \Omega_B$, then the pair $(\lambda X, \lambda Y)$ needs not be in $\Omega_B$ although that we have $(\lambda X)B = B(\lambda Y)$ because $\lambda X, \lambda Y$ are not permutation matrices for $\lambda \neq 1$.

A subgroup $H$ of $\mathcal{P}(n)$ is called realisable if each element $M \in H$ is obtained by permuting the rows of the identity matrix $I_n$ using a permutation $\tau \in S_n$ satisfying $\tau(i) \neq i$ for all $1 \leq i \leq n$. Here $S_n$ denotes the symmetric group of order $n$.

Recall that, by Cayley’s theorem, any finite group $G$ of order $n$ is isomorphic to a realisable subgroup, denoted by $C_G$, of $\mathcal{P}(n)$ via the map

$$G = \{g_1, \ldots, g_n\} \to S_n \cong \mathcal{P}(n), \quad g_j \mapsto \sigma_j = \begin{pmatrix} g_1 & g_2 & \cdots & g_n \\ g_j & \sigma_j(g_2) & \cdots & \sigma_j(g_n) \end{pmatrix} \leftrightarrow M_j$$

where $M_j$ is the matrix obtained by permuting all the rows of the identity matrix $I_n$ using $\sigma_j$.

Following the idea developed in [8] and inspired by the works done in [5, 7] regarding the so-called Kahn’s realisability problem of groups (see [9, 12] for more details), this paper is devoted to answer the question whether a given finite group $G$ can occur as a group on the form $\Omega_B$ and whether $G$ can be embedded in $\mathcal{P}(m)$, where $m > n$, as a realisable subgroup. For this purpose we shall assign to $G$ a matrix $B_G$ and a realisable subgroup $A_G$ of $\mathcal{P}(1 + kn + 2)$, where the number $k$ is given by the decomposition of the permutation $\sigma_2$ into product of disjoint cycles, i.e., $\sigma_2 = \tau_1 \tau_2 \cdots \tau_k$ and we shall define $\Omega_{B_G}$ as a certain subgroup of $A_G \times C_G$.

The group $A_G$ and the matrix $B_G$ are defined using the framework of rational homotopy theory [11] and the ideas developed in [6, 8]. More precisely, $A_G$ is defined in
terms of the cohomology of a certain a free commutative cochain \( \mathbb{Q} \)-algebra associated with the group \( G \) and \( B_G \) is related to its differential.

Thus, in this paper we establish the following result

**Theorem 1.** For any finite group \( G \) of order \( n \), there exists a matrix \( B_G \) such that \( G \) is isomorphic to the group \( \Omega_{B_G} \) of the solutions of the matrix equation \( XB_G = B_G Y \), where the unknowns \( X, Y \) are two permutation matrices belonging to the groups \( A_G \) and \( C_G \) respectively.

As a corollary we derive

**Corollary 2.** Any finite group \( G \) of order \( n \) is isomorphic to a realisable subgroup of \( P((1 + k)n) \).

### 2. Main results

#### 2.1. Definition of the group \( A_G \)

Let us start by recalling the main construction in [S] on which this work is based. Indeed, let \( G = \{g_1, g_2, \ldots, g_n\} \) be a finite group of order \( n \) and let \( S_n \) the symmetric group. By Cayley’s theorem there is a monomorphism

\[
\Psi : G \to S_n, \quad g_j \mapsto \sigma_j : g_k \mapsto g_j g_k, \quad 1 \leq k \leq n
\]

For \( 2 \leq j \leq n \), write \( \sigma_j = \begin{pmatrix} 1 & 2 & \cdots & n \\ j & \sigma_j(2) & \cdots & \sigma_j(n) \end{pmatrix} \) and let:

\[
\sigma_2 = \begin{pmatrix} 1 & 2 \sigma_2(2) & \cdots & \sigma_2^{n-2}(2) \\ \sigma_2(2) & \cdots & \sigma_2^n(2) \end{pmatrix} \begin{pmatrix} i_1 \sigma_2(i_1) & \cdots & \sigma_2^{k_1}(i_1) \\ i_2 \sigma_2(i_2) & \cdots & \sigma_2^{k_2}(i_2) \end{pmatrix} \cdots \begin{pmatrix} i_k \sigma_2(i_k) & \cdots & \sigma_2^{k_k}(i_k) \end{pmatrix}
\]

the decomposition of \( \sigma_2 \) into a product of cycles.

Recall that in [S] we constructed a free commutative cochain \( \mathbb{Q} \)-algebra

\[
(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{j \in G}, \partial)
\]

where the degrees of the elements in this graded algebra are

\[
|x_1| = 8, \quad |x_2| = 10, \quad |w_j| = 40
\]

and where the differential is given by:

\[
\partial(x_1) = \partial(x_2) = \partial(w_j) = 0, \quad \partial(y_1) = x_1^3 x_2, \quad \partial(y_2) = x_1^2 x_2^2, \quad \partial(y_3) = x_1 x_2^3
\]

\[
\partial(z_j) = w_j^3 + w_j w_{\sigma_j+1(1)} x_2^4 + \sum_{\tau=1}^{k} w_j w_{\sigma_{j+1}(\tau)} x_2^4 + u + x_1^{15}, \quad 1 \leq j \leq n - 1
\]

\[
\partial(z_n) = w_n^3 + w_n w_1 x_1^4 + \sum_{\tau=1}^{k} w_n w_{\tau} x_2^4 + u + x_1^{15}
\]

(1)

where \( u = y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_2^3 x_2 + y_2 y_3 x_1^4, \) and we proved that

\[
\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{j \in G}) \cong G
\]

where \( \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{1 \leq j \leq n}) \) denotes the group of self homotopy cochain equivalences of \( \Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{j \in G}) \) (see [S] for more details).

Now let \( V^{119} = \mathbb{Q}\{z_1, \ldots, z_n\} \) be the vector space spanned by the set \( \{z_1, \ldots, z_n\} \). Recall that \( |z_i| = 119 \) for every \( 1 \leq i \leq n \). In ([S], Proposition 3.9), it is shown that

\[
\mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{z_j, w_j\}_{j \in G}) \cong D_{40}^{119}
\]
where $\mathcal{D}_{40}^{119}$ is the subgroup of $\text{aut}(V^{119}) \times \mathcal{E}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\})_{g_j \in G})$ consisting of the couples $(\xi, [\alpha])$ making the following diagram commutes:

\[
\begin{array}{c}
V^{119} \\
\downarrow b \\
\Gamma_G^{120} \\
\downarrow H^{120}(\alpha) \\
\Gamma_G^{120} \end{array}
\xrightarrow{\xi} 
\begin{array}{c}
V^{119} \\
\downarrow b \\
\Gamma_G^{120} \\
\downarrow H^{120}(\alpha) \\
\Gamma_G^{120} \end{array}
\]

(2)

where $\Gamma_G^{120} = H^{120}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\})_{g_j \in G})$ and where $b$ is defined by

\[
b(z_i) = \partial(\hat{z}_i), \quad 1 \leq j \leq n
\]

(3)

Here $\partial(\hat{z}_i)$ is the cohomology class of $\partial(z_i)$ in $H^{120}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\})_{g_j \in G})$.

Moreover, it is shown that if $(\xi, [\alpha]) \in \mathcal{D}_{40}^{119}$, then there exists a unique permutation

\[
\sigma_s = \begin{pmatrix}
1 & s & 2 & \cdots & n \\
\end{pmatrix}
\]

such that

\[
\xi(z_j) = z_{\sigma_s(j)}, \quad \alpha(w_j) = w_{\sigma_s(j)}, \quad \alpha = \text{id}, \quad \text{on} \quad x_1, x_2, y_1, y_2, y_3.
\]

(4)

Thus, there is an isomorphism $\Psi : \mathcal{D}_{40}^{119} \to G$ defined by $\Psi((\xi, [\alpha])) = g_s$, where the element $g_s$ corresponds to the permutation $\sigma_s$, given in (2), via Cayley's theorem.

Set $u = y_1 y_2 x_1^2 x_2^2 - y_1 y_3 x_2^3 x_2 + y_2 y_3 x_1^6$. As the following set of generators

\[
\Sigma = \left\{ w_1^3 : \cdots ; w_n^3 ; \ w_{w_{\sigma_{j+1}(1)}} x_2^4 ; \ w_{w_{\sigma_{j+1}(i)}} x_2^4 ; \ w_{w_{\sigma_{j+1}(i)}} x_2^4 : \ u ; \ x_1^{15} \right\}
\]

(5)

where $1 \leq j \leq n$ and $1 \leq \tau \leq k$, is linearly independent in the vector space

\[
\Gamma_G^{120} = H^{120}(\Lambda(x_1, x_2, y_1, y_2, y_3, \{w_j\})_{g_j \in G})
\]

it follows that $\Sigma$ can be chosen, according the formulas (1) and (3), as a basis for the vector space $b(V^{119}) \subseteq \Gamma_G^{120}$. Notice that

\[
\dim b(V^{119}) = \text{cardinal}(\Sigma) = (1 + k)n + 2
\]

(6)

Thus, if $B_G$ denotes the the matrix of order $((1 + k)n + 2) \times n$ which is associated to the linear map $b$ defined in (3) with respects to the basis is $\Sigma$, then we can write

\[
B_G = \begin{bmatrix}
I_n \\
M \\
D
\end{bmatrix}
\]

where $D = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1
\end{bmatrix}$,

where the matrix $M = [m_{ij}]$ is defined by

\[
m_{ij} = \begin{cases}
1, & \text{if } i \in \{\sigma_{j+1}(1), \sigma_{j+1}(i_1), \ldots, \sigma_{j+1}(i_k)\}, \\
0, & \text{otherwise}.
\end{cases}
\]

Consequently, taking into construction (1), the matrices associated to the linear maps $\xi$ and the restriction of the linear map $H^{120}(\alpha)$ to $b(V^{119})$, given in the diagram (2)
and corresponding to the element \((\xi, [\alpha]) \in D_{40}^{119}\), can be written, respectively, as

\[
C_{g_s} = \sigma_s I_n \quad \text{and} \quad A_{g_s} = \begin{bmatrix} \sigma_s I_n & 0 & 0 \\ 0 & \overline{A}_{g_s} & 0 \\ 0 & 0 & I_2 \end{bmatrix}.
\]  

where

\[
\sigma_s I_n = \begin{bmatrix} c_{i,j} \end{bmatrix}_{1 \leq i, j \leq (k+1)n}, \quad c_{i,j} = \begin{cases} 1, & \text{if } i = \sigma_s(j) \\ 0, & \text{otherwise} \end{cases},
\]

and where

\[
\overline{A}_{g_s} = \begin{bmatrix} a_{n+i,n+j} \end{bmatrix}_{1 \leq i, j \leq (k+1)n}, \quad a_{n+i,n+j} = \begin{cases} 1, & \text{if } i = \sigma_s(j) \\ 0, & \text{otherwise} \end{cases}.
\]

Here \(\sigma_s\) is the permutation corresponds to \(g_s\) via Cayley’s theorem.

**Remark 2.1.** From (7), it is clear to see that \(A_{g_s}\) is a permutation matrix. Recall that the commutativity of the diagram (2) implies that

\[A_{g_s}B_G = B_GC_{g_s}, \quad \forall g_s \in G\]  

**Definition 2.2.** Let \(G = \{g_1, \ldots, g_n\}\) be a group, we define the following two sets

\[A_G = \{A_{g_s}, \ g_s \in G\} \quad \text{and} \quad \Omega_G = \{(A_{g_s}, C_{g_s}) \in A_G \times C_G, \ g_s \in G\}\]

**Theorem 2.3.** The sets \(A_G\) and \(\Omega_G\) are groups isomorphic to \(G\).

**Proof.** First let us prove that \(A_G\) is a group. Indeed, let \(A_{g_s}, A_{g_r} \in A_G\). By (8) there exist two matrices \(C_{g_s}, C_{g_r}\) such that

\[A_{g_s}B_G = B_GC_{g_s} \quad \text{and} \quad A_{g_r}B_G = B_GC_{g_r}\]

therefore \(A_{g_s}A_{g_r}B_G = A_{g_s}B_GC_{g_r} = B_GC_{g_s}C_{g_r}\), it follows that \(A_{g_s}A_{g_r} \in A_G\). Here we use the that fact that

\[A_{g_s}A_{g_r} = A_{g_sg_r} \quad \text{and} \quad C_{g_s}C_{g_r} = C_{g_sg_r}\]  

Next let \(A_{g_s} \in A_G\). As \(A_{g_s}B_G = B_GC_{g_s}\) and \(A_{g_s}, C_{g_s}\) are invertible, we deduce that \(B_GC_{g_s}^{-1} = (A_{g_s})^{-1}B_G\) implying that \((A_{g_s})^{-1} \in A_G\). Notice that \(A_{g_s}^{-1} = A_{g_s^{-1}}\).

Next, using the same arguments, it is easy to check that the set \(\Omega_G\) is a group. Finally, it is clear that the two maps \(\chi : G \rightarrow A_G\) and \(\varphi : G \rightarrow \Omega_G\), defined by \(\chi(g_s) = A_{g_s}\) and \(\varphi(g_s) = (A_{g_s}, C_{g_s})\) respectively, are isomorphisms of groups.

**2.2. Realisable subgroups.**

**Definition 2.4.** A subgroup \(H\) of \(\mathcal{P}(n)\) is called realisable if each element \(M \in H\) is obtained by permuting the rows of the identity matrix \(I_n\) using a permutation \(\tau \in S_n\) satisfying \(\tau(i) \neq i\) for all \(1 \leq i \leq n\).

Let \(G = \{g_1, \ldots, g_n\}\) be a group. Based on the formula (7), let us define the following matrix

\[M_{g_s} = \begin{bmatrix} \sigma_s I_n & 0 \\ 0 & \overline{A}_{g_s} \end{bmatrix}, \quad g_s \in G\]  

**Theorem 2.5.** If \(H_G = \{M_{g_s}, \ g_s \in G\}\), then \(H_G\) is a realisable subgroup of \(\mathcal{P}((1 + k)n)\) isomorphic to \(G\).
2.3. Examples. In the following examples we illustrate our study by determining all the groups introduced in this paper for the cyclic group $\mathbb{Z}_4$ and the Klein group $\mathbb{V}$.

Example 2.6. If $G = \mathbb{Z}_4$, then the monomorphism $\mathbb{Z}_4 = \{g_1, g_2, g_3, g_4\} \rightarrow S_4$ is

g_1 \mapsto id \ , \ g_2 \mapsto \sigma_2 = (1234) \ , \ g_3 \mapsto \sigma_3 = (13)(24) \ , \ g_4 \mapsto \sigma_4 = (1432)

therefore according to (11) the model associated with $\mathbb{Z}_4$ is

\[
\{\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4), \partial\}
\]

where $|x_1| = 8, |x_2| = 10, |w_j| = 40$, and where the differential is given by

\[
\begin{align*}
\partial(x_1) &= \partial(x_2) = \partial(w_j) = 0, \\
\partial(y_1) &= x_1^2 x_2, \\
\partial(y_2) &= x_2^2 x_3, \\
\partial(y_3) &= x_1 x_2^3
\end{align*}
\]

\[
\begin{align*}
\partial(z_1) &= w_1^3 + w_1 w_2 x_2^2 + y_1 y_2 x_2^7 x_2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_2 + x_1^{15} \\
\partial(z_2) &= w_1^3 + w_2 w_3 x_4^2 + y_1 y_2 x_2^7 x_2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_2 + x_1^{15} \\
\partial(z_3) &= w_1^3 + w_3 w_4 x_2^2 + y_1 y_2 x_2^7 x_2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_2 + x_1^{15} \\
\partial(z_4) &= w_1^4 + w_4 w_1 x_4^5 + y_1 y_2 x_2^7 x_2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_2 + x_1^{15}
\end{align*}
\]

For the above construction it is clear that $V^{119} = \mathbb{Q}\{z_1, z_2, z_3, z_4\}$ and by (5) the base $\Sigma$ of the vector space $b(V^{119})$ is given by

\[
\Sigma = \{w_1^3, w_2^3, w_3^3, w_4^3, w_1 w_2 x_2^2, w_2 w_3 x_4^2, w_3 w_4 x_2^2, w_4 w_1 x_4^4, y_1 y_2 x_2^7 x_2 - y_1 y_3 x_1^5 x_2 + y_2 y_3 x_2, x_1^{15}\}
\]

implying that the matrix $B_{\mathbb{Z}_4}$ associated with the linear map $b$, given in (6), is

\[
B_{\mathbb{Z}_4} = 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{pmatrix}
\]

and we have $C_{\mathbb{Z}_4} = \{I_4, C_{(1234)}, C_{(13)(24)}, C_{(1432)}\}$ where

\[
C_{(1234)} = 
\begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad C_{(13)(24)} = 
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad C_{(1432)} = 
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

For instance, $C_{(1234)}$ is simply the permutation matrix obtained by permuting the rows of $I_4$ using the permutation (1234) and likewise $C_{(13)(24)}$ and $C_{(1432)}$.

Next we have $A_{\mathbb{Z}_4} = \{I_{10}, A_{(1234)}, A_{(13)(24)}, A_{(1432)}\}$ where

\[
A_{(1234)} = 
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad A_{(13)(24)} = 
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad A_{(1432)} = 
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
and $H_{Z_4} = \{ f_8, M_{(1234)}, M_{(13)(24)}, M_{(1432)} \}$, where

\[
M_{(1234)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
M_{(13)(24)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
M_{(1432)} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

Recall that $A_{(1234)}$ is the matrix associate to the restriction of the linear map $H^{120}(\alpha)$ to the vector space $b(V^{119})$, where the cochain map $\alpha$ is given 4, with respects to the basis $\Sigma$ in 11. Thus, $A_{(1234)}$ is obtained by using the permutation $\sigma_2 = (1234)$ as follows

\[
\begin{align*}
w_1^3 &\mapsto w_2^3, \quad w_2^3 &\mapsto w_3^3, \quad w_3^3 &\mapsto w_4^3, \quad w_4^3 &\mapsto w_1^3, \\
w_1 w_2 x_2^4 &\mapsto w_2 w_3 x_2^4, \quad w_2 w_3 x_2^4 &\mapsto w_3 w_4 x_2^4, \quad w_3 w_4 x_2^4 &\mapsto w_4 w_1 x_2^4, \quad w_4 w_1 x_2^4 &\mapsto w_1 w_2 x_2^4
\end{align*}
\]

\[
y_1 y_2 x_2^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_2^6 \mapsto y_1 y_2 x_2^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_2^6, \quad x_1^{15} &\mapsto x_1^{15}
\]

and likewise we obtain the matrices $A_{(13)(24)}$ and $A_{(1432)}$. Notice that matrices

$M_{(1234)}, M_{(13)(24)}, M_{(1432)}$ are constructed from the matrices $A_{(1234)}, A_{(13)(24)}, A_{(1432)}$ using 9 and finally we have

\[
\Omega_{Z_4} = \{ (I_4, I_{12}), (A_{(1234)}, C_{(1234)}), (A_{(13)(24)}, C_{(13)(24)}), (A_{(1432)}, C_{(1432)}) \}
\]

It is also worth noting to point out that the group $M_{Z_4}$, which is isomorphic to $Z_4$ is a realisable subgroup of the group of permutation matrices $\mathcal{P}(8)$.

**Example 2.7.** In this example we use the same analysis and computation as in the example 2.6, but we omit all the details, to determine the two groups $A_{V}, C_{V}, H(V)$ and $\Omega_{V}$ for the Klein group $V$. Indeed, the monomorphism $V = \{ g_1, g_2, g_3, g_4 \} \leftrightarrow S_4$ is given by $g_2 \leftrightarrow (12)(34), g_3 \leftrightarrow (13)(24), g_4 \rightarrow (14)(23)$, so the model associated to $V$ is $(\Lambda(x_1, x_2, y_1, y_2, y_3, w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4, \partial)$ where $|x_1| = 8, |x_2| = 10, |w_j| = 40$, and where the differential is given by

\[
\begin{align*}
\partial(x_1) &= \partial(x_2) = \partial(w_j) = 0, \quad \partial(y_1) = x_1^7 x_2^7, \quad \partial(y_2) = x_2^7 x_1^7, \quad \partial(y_3) = x_1 x_2^2 \\
\partial(z_1) &= w_1^3 + w_1 w_2 x_2^4 + w_1 w_4 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} \\
\partial(z_2) &= w_2^3 + w_2 w_3 x_2^4 + w_2 w_1 x_1^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} \\
\partial(z_3) &= w_3^3 + w_3 w_4 x_2^4 + w_3 w_2 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_1^6 + x_1^{15} \\
\partial(z_4) &= w_4^3 + w_4 w_1 x_2^4 + w_4 w_3 x_2^4 + y_1 y_2 x_1^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_1^6 + x_1^{15}
\end{align*}
\]

If $u = y_1 y_2 x_2^4 x_2^2 - y_1 y_3 x_2^5 x_2 + y_2 y_3 x_2^6$, then the basis $\Sigma$ is given by

\[
\Sigma = \{ x_1^3, x_2^3, w_1^3, w_2^3, w_3^3, w_4^3, w_1 w_2 x_2^4, w_2 w_3 x_2^4, w_3 w_4 x_2^4, w_4 w_1 x_2^4, w_1 w_4 x_2^4, w_2 w_1 x_2^4, w_3 w_2 x_2^4, w_4 w_3 x_2^4, u, x_1^{15} \}
\]

implying that dim $b(V^{119}) = 14$ and the matrix $B_{V}$ is

\[
B_{V} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]
We have $C_V = \{ I_4, C_{(12)(34)}, C_{(13)(24)}, C_{(14)(23)} \}$, where

$$C_{(12)(34)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad C_{(13)(24)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_{(14)(23)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

and $A_V = \{ I_{14}, A_{(12)(34)}, A_{(13)(24)}, A_{(14)(23)} \}$, where

$$A_{(12)(34)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_{(13)(24)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Next we have $H_V = \{ I_{12}, M_{(12)(34)}, M_{(13)(24)}, M_{(14)(23)} \}$, where

$$M_{(12)(34)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{(13)(24)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Notice that $H(V)$ is a realizable subgroup of $P(12)$. Finally, we have

$$\Omega_V = \{ (I_4, I_{14}), (A_{(12)(34)}, C_{(12)(34)}), (A_{(13)(24)}, C_{(13)(24)}), (A_{(14)(23)}, C_{(14)(23)}) \}$$

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