Modularity of minor-free graphs

Michał Lason¹ | Małgorzata Sulkowska²,³

¹Institute of Mathematics of the Polish Academy of Sciences, Warszawa, Poland
²Université Côte d’Azur, CNRS, Inria, I3S, France
³Department of Fundamentals of Computer Science, Wrocław University of Science and Technology, Wrocław, Poland

Correspondence
Michał Lason, Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warszawa, Poland. Email: michalason@gmail.com

Abstract
We prove that a class of graphs with excluded minor and with the maximum degree of smaller order than the number of edges is maximally modular, that is, for every ε > 0, the modularity of any graph in the class with sufficiently many edges is at least 1 − ε.

KEYWORDS
Cheeger’s inequality, edge separator, minor-free graph, modularity, sparse graph

1 INTRODUCTION

1.1 Modularity

Modularity is a well-established parameter measuring the presence of a community structure in the graph. It was introduced by Newman and Girvan in 2004 [22]. Nowadays, it is widely used as a quality function for community detection algorithms. The most popular heuristic clustering algorithms (see Louvain [6] or Leiden [24]) find a partition using a modularity-based approach. The definition of modularity is based on a comparison between the density of edges inside communities one observes in the graph and the density of edges one would expect if the edges of the graph were wired randomly preserving the degree sequence. We make this precise just below.

Consider a simple undirected graph $G$ with $|V(G)| = n$ and $|E(G)| = m$. Whenever the context is clear, we write $V$ and $E$ for $V(G)$ and $E(G)$, respectively. For a vertex $v \in V$, denote by $\text{deg}(v)$ the degree of $v$ in $G$. For a subset of vertices $S \subseteq V$ define $E(S)$ to be the set of edges in $G$ with both end-vertices in $S$ and let $\text{deg}(S) = \sum_{v \in S} \text{deg}(v)$. The modularity of $G$ is defined as follows.

Definition 1 (Modularity [22]). Let $G$ be a graph with at least one edge. For a partition $\mathcal{A}$ of $G$ into induced subgraphs define its modularity score on $G$ to be...
\[ \text{mod}_A(G) = \sum_{A \in \mathcal{A}} \left( \frac{|E(A)|}{|E(G)|} - \left( \frac{\deg(V(A))}{\deg(V(G))} \right)^2 \right). \]

Modularity of \( G \) is given by
\[ \text{mod}(G) = \max_{\mathcal{A}} \text{mod}_A(G). \]

Conventionally, a graph with no edges has modularity equal to 0. For a given partition \( \mathcal{A} \), the value \( \sum_{A \in \mathcal{A}} \frac{|E(A)|}{|E(G)|} \) is called the edge contribution (or the coverage) while \( \sum_{A \in \mathcal{A}} \left( \frac{\deg(V(A))}{\deg(V(G))} \right)^2 \) is the degree tax. A single summand of the modularity score is the difference between the fraction of edges within \( A \) and the expected fraction of edges within \( A \) if we considered a random multigraph on \( V \) with the degree sequence given by \( G \). It is easy to check that \( 0 \leq \text{mod}(G) < 1 \), and also that adding or deleting isolated vertices from the graph does not impact its modularity.

In practice, the problem of community detection very often concerns complex networks, that is, graphs modeling real-life systems. Since most complex networks are sparse, it is natural to investigate which classes of sparse graphs exhibit high modularity. Our paper addresses exactly this question-modularity of commonly considered subclasses of nowhere dense graphs, which is a class introduced by Nešetřil and Ossona de Mendez in [21]. The notion of nowhere dense graphs appears to be a very robust concept of sparsity. In particular, it generalized many familiar classes of sparse graphs that were not comparable before, for example, graphs with excluded minor and graphs with bounded degree or with bounded local treewidth.

### 1.2 Related work

For a concise, up-to-date summary of the modularity of various classes of graphs check the appendix of [18] by McDiarmid and Skerman from 2020. For very recent results on random graphs consult [8, 13].

Here we focus on the modularity of commonly considered subclasses of nowhere dense graphs. First, recall the definition of maximally modular class of graphs introduced by de Montgolfier, Soto, and Viennot in 2011 (see [20]).

**Definition 2** (Maximally modular class of graphs [20]). A class of graphs \( \mathcal{C} \) is maximally modular if for every \( \varepsilon > 0 \) there exists \( M_\varepsilon \) such that, whenever \( G \) is a graph from \( \mathcal{C} \) with \( m \geq M_\varepsilon \) edges, then \( \text{mod}(G) > 1 - \varepsilon \).

Note that a maximally modular class of graphs must have the maximum degree of smaller order than the number of edges. Indeed, if there exists a vertex of degree \( \alpha m \) in \( G \), then the degree tax is at least \( \alpha^2/4 \). Hence, the maximum degree of smaller order than the number of edges is a necessary condition for a class of graphs to be maximally modular.

The early results on the modularity of sparse graphs are due to Bagrow who observed that some families of trees are maximally modular (consult [4]). In 2018 McDiarmid and Skerman [17] formulated the following sufficient condition for a class of graphs to be maximally modular. By \( \Delta(G) \) and \( \text{tw}(G) \) we denote, respectively, the maximum degree in \( G \) and the treewidth of \( G \).
Corollary 3 [17, Corollary 12]. For $m = 1, 2, \ldots$ let $G_m$ be a graph with $m$ edges. If $\text{tw}(G_m) \cdot \Delta(G_m) = o(m)$ then $\text{mod}(G_m) \rightarrow 1$ as $m \rightarrow \infty$.

The above result is tight in a sense that $o(m)$ cannot be replaced by $O(m)$. To justify the optimality of Corollary 3 McDiarmid and Skerman present two examples. First, let $G$ be a star $K_{1,m}$ (with treewidth 1 and maximum degree $m$). Then, $\text{tw}(G) \cdot \Delta(G) = m$ and, by [20], $\text{mod}(G) = 0$. Second, let $G$ be a random cubic graph on $n$ vertices (thus with $m = 3n/2$ edges). Then, $\text{tw}(G) \cdot \Delta(G) = O(m)$ and with high probability $\text{mod}(G) \leq 0.79$ (see [13, 17]).

Thus, Corollary 3 implies that any class of graphs $\mathcal{F}$ such that, for all $G \in \mathcal{F}$, $\text{tw}(G) = O(1)$ and $\Delta(G) = o(|E(G)|)$, is maximally modular. On the other hand, the example of a random cubic graph gives that classes of bounded degree graphs are not necessarily maximally modular. These conclusions already lead to the classification presented in Figure 1.

By Corollary 3 one can obtain also some results for random planar graphs and partial results for graphs of bounded genus.

Indeed, the random planar graph $G_n$ on $n$ vertices has $\text{tw}(G_n) = O(\sqrt{n})$. This follows from the separator theorem for graphs of bounded genus [11] (see Section 3) and a recent result by Dvorák and Norin [10] establishing the linear dependence between the treewidth and the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{Maximally and nonmaximally modular subclasses of nowhere dense graphs. All classes of graphs are considered to have the maximum degree of smaller order than the number of edges. (Background picture by Felix Reidl, Source: https://tcs.rwth-aachen.de/~reidl/).}
\end{figure}
separation number for graphs of bounded genus. Next, with high probability $|E(G_n)| = \Theta(n)$ (here the upper bound is deterministic, while the lower bound holds with high probability) and, by [16], with high probability $\Delta(G_n) = O(\log n)$. Thus, by Corollary 3, random planar graphs are maximally modular with high probability.

Similarly, again by [10, 11], for graphs of bounded genus $\text{tw}(G_m) = O(\sqrt{m})$. Thus, a class of bounded genus graphs satisfying the additional assumption $\Delta(G_g) = o(\sqrt{m})$ is maximally modular.

1.3 | Our results

We prove the following.

**Theorem 4.** A class of graphs with excluded minor and with the maximum degree of smaller order than the number of edges is maximally modular.

Since classes of graphs with bounded genus (in particular, the class of planar graphs) are subclasses of graphs with excluded minor, the above theorem resolves, in the positive, all three questions marked in Figure 1 that remained unsolved. This way we achieve a complete classification of maximally modular classes among all commonly considered subclasses of nowhere dense graphs with maximum degree of smaller order than the number of edges.

Our proof uses tools of spectral graph theory, in particular, the so-called Cheeger’s Inequality and a recent important result by Biswal, Lee, and Rao [5] for graphs with excluded minor.

2 | PROOF

We begin by presenting tools of spectral graph theory and results for minor-free graphs that will be used in this section.

Let $G$ be a simple undirected graph. Denote by $A$ its adjacency matrix and by $D$ its diagonal degree matrix. Recall that $L = D - A$ is the Laplacian matrix of $G$. When $G$ has no isolated vertices, then $L = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}$ is the normalized Laplacian matrix of $G$. Let $\lambda_i(L)$ and $\lambda_i(L)$ denote $i$th smallest eigenvalue of $L$ and $L$, respectively. We concentrate on the second smallest eigenvalues, $\lambda_2(L)$ and $\lambda_2(L)$, as they are linked to the size of possibly small and balanced edge cuts in $G$.

In our proof, we justify the existence of such sufficiently small and reasonably balanced edge cuts in the considered graphs. This will be done by Cheeger’s inequality, which was first established for manifolds in 1970 by Cheeger [7], while the graph version is due to Alon [1] and Alon and Milman [2].

**Theorem 5** (Cheeger’s inequality [2]). Let $G$ be a graph without isolated vertices. Then,

$$\frac{\lambda_2(L)}{2} \leq \min_{\emptyset \neq S \subseteq V} \frac{|E(S, V \setminus S)|}{\min\{\deg(S), \deg(V \setminus S)\}} \leq \sqrt{2\lambda_2(L)},$$

where $E(S, V \setminus S)$ denotes the set of edges in $G$ between $S$ and $V \setminus S$. 


One of the first results for minor-free graphs (due to Mader [15]) shows that they have at most a linear number of edges in terms of the number of vertices.

**Theorem 6** [15]. A class of graphs with excluded minor has at most a linear number of edges. That is, for every graph $H$, there exists a constant $c_H$ such that every graph on $n$ vertices without an $H$-minor has at most $c_H n$ edges.

A recent important result by Biswal, Lee, and Rao for minor-free graphs gives an upper bound for the second smallest eigenvalue of the Laplacian matrix.

**Theorem 7** [5, Theorem 5.3]. If $G$ is a $K_h$-minor-free graph and $|V| \geq c_1 h^2 \log h$, then $$\lambda_2(L) \leq c_2 \frac{\Delta(G) h^6 \log h}{|V|}$$ for some positive absolute constants $c_1$, $c_2$.

**Remark 8.** For graphs without isolated vertices $\lambda_2(L) \leq \lambda_2(L')$ holds.

**Proof.** We have $L = D^{-\frac{1}{2}} LD^{-\frac{1}{2}} = (D^{-\frac{1}{2}} LD^{-\frac{1}{2}}) D^{-1} = L'D^{-1}$. The matrices $L$ and $L'$ are similar, hence have the same spectrum. Recall that $\lambda_1(L) = \lambda_1(L') = 0$, so also $\lambda_2(L') = 0$.

Suppose these matrices act linearly on $\mathbb{R}^n$, instead of a typical action, by $x \to x^T L$. Then, the kernels of $L$ and $L'$ coincide. Now, we restrict linear maps $L$ and $L'$ to the subspace, denoted by $\ker$, orthogonal to these kernels. Since $D^{-1}$ is a shrinking linear map (i.e., the module of every eigenvalue is at most 1) we get that the module of the smallest eigenvalue of $L_{\ker} L$ is less or equal to the module of the smallest eigenvalue of $L'_{\ker} L'$. Since these eigenvalues are real and nonnegative, $\lambda_2(L) \leq \lambda_2(L') = \lambda_2(L)$.

Below we prove that any graph from a class of graphs with excluded minor and the maximum degree of smaller order than the number of edges can be divided into components with a low degree sum by removing only a sublinear number of edges.

To do so, we will iteratively delete a small set of edges dictated by Cheeger’s Inequality (Theorem 5) until the degree sum of each component drops below the desired level. Combining Cheeger's Inequality with the upper bound on the second smallest eigenvalue of the Laplacian matrix for $K_h$-minor-free graphs (Theorem 7) ensures that these edge cuts are sufficiently small.

**Proposition 9.** For every graph $H$ and $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that the following is true. Let $G$ be a graph without minor $H$, with $m$ edges, and maximum degree at most $\delta m$. A weight $w(v) = \frac{\deg(v)}{\deg(V)}$ is assigned to every vertex $v$ of $G$. Then, there is a set of no more than $\varepsilon m$ edges in $G$ whose deletion creates a graph in which the total weight of every connected component is smaller than $\varepsilon$.

**Proof.** Fix a graph $H$ and $\varepsilon > 0$, and let $h = |H|$. Now, choose $\delta > 0$ such that

$$\frac{\varepsilon^2 \frac{1}{2} \frac{1}{\delta} c_H}{2} \geq c_1 h^2 \log h \text{ and } \left( \left| \log \frac{1}{\varepsilon} \right| + 2 \right) \frac{1}{2} c_1 c_H \frac{1}{\varepsilon} h^6 \log h < \varepsilon.$$  

Suppose $G(V, E)$ is a graph without minor $H$, with $m$ edges, and maximum degree at most $\delta m$. In particular, $m \geq \frac{1}{\delta}$. We will show that the statement of the proposition holds
for $G$, $\varepsilon$, and $\delta$. Notice that without loss of generality we may assume that $G$ does not have isolated vertices, as these do not impact weights.

Consider the following procedure which takes as input an induced subgraph $G'(V', E')$ of $G$ with $2|E'| = \deg_{G}(V') \geq \frac{\varepsilon}{2} \deg(V) = \varepsilon m$. Notice that by Theorem 6 we have $|V'| = |E'| |V'| \geq \frac{\varepsilon}{2} m \cdot \frac{1}{c_{H}} + \frac{\varepsilon}{2} \frac{1}{c_{H}} \geq c_{1} h^{2} \log h$. Clearly, $G'$ is $H$-minor-free, hence also $K_{6}$-minor-free, thus by Theorem 7 since $|V'| \geq c_{1} h^{2} \log h$, we have that $\lambda_{2}(L') \leq c_{2}^{m-h^{6}} \log h = c_{2}^{m-h^{6}} \frac{\delta h^{6}}{\varepsilon} \log h \leq c_{2}^{m-h^{6}} \frac{2}{\varepsilon} \delta h^{6} \log h$. Thus, by Theorem 5 and Remark 8, we get that for some proper subset $S$ of $V'$:

$$|E'(S, V' \setminus S)| \leq \min\{\deg_{G}(S), \deg_{G}(V' \setminus S)\} \sqrt{2c_{2} c_{H} \frac{2}{\varepsilon} \delta h^{6} \log h}.$$  

(1)

The procedure deletes edges $E'(S, V' \setminus S)$ from $G'$, and returns as output all connected components of the resulting graph. These are also induced subgraphs of $G$.

Now, we run a process on a set $T$ of induced subgraphs of $G$. We begin with the set $T$ consisting of the graph $G$ and repeatedly apply the above procedure to elements of $T$ (i.e., induced subgraphs $G'$ of $G$) satisfying $\deg_{G_{i}}(V') \geq \frac{\varepsilon}{2} \deg(V)$. Notice that since the degree of each element of the output is smaller than the degree of the input, the process has to end. Keep in mind that at any step of the process the vertex sets of elements of $T$ form a partition of the set $V$. Moreover, after the first step of the process elements of $T$ are connected induced subgraphs of $G$. Suppose that at the end of the process the set $T$ consists of connected induced subgraphs $G_{1}, ..., G_{t}$. Denote by $D$ the set of edges deleted during the process.

To count how many edges were deleted during the process in total, assign in a single procedure that started with $G'$ edges $E'(S, V' \setminus S)$ to vertices in $S$ proportionally to their $G'$-degree when $\deg_{G_{i}}(S) \leq \deg_{G_{i}}(V' \setminus S)$ and to $V' \setminus S$ otherwise. Notice that the assigned value may involve fraction of edges, thus does not necessarily need to be an integer. Now, by the inequality (1) in this single run of the procedure every vertex $v$ in $S$ got assigned at most $\deg_{G_{i}}(v) \sqrt{2c_{2} c_{H} \frac{2}{\varepsilon} \delta h^{6} \log h}$ deleted edges when $\deg_{G_{i}}(S) \leq \deg_{G_{i}}(V' \setminus S)$ and 0 otherwise. Notice that every single vertex $v$ gets assigned nonzero deleted edges at most $\left\lfloor \log_{2} \frac{1}{\varepsilon} \right\rfloor + 1 = \left\lfloor \log_{2} \frac{1}{\varepsilon} \right\rfloor + 2$ times. Indeed, it happens when the degree of the induced subgraph to which $v$ belongs gets at least halved. This degree starts from $\deg(V)$ and ends just after dropping below $\frac{\varepsilon}{2} \deg(V)$. Therefore, summing over all vertices, the total number of deleted edges $|D|$ is at most

$$\left\lfloor \log_{2} \frac{1}{\varepsilon} \right\rfloor + 2 \right\rfloor 2m \sqrt{2c_{2} c_{H} \frac{2}{\varepsilon} \delta h^{6} \log h} < \varepsilon m.$$

Now, connected components of the graph $G \setminus D$ are the $G_{i}$'s. Every $G_{i}$ has weight equal to $\frac{1}{\deg(V)} \deg_{G}(V(G_{i})) \leq \frac{1}{\deg(V)} (\deg_{G}(V(G_{i})) + |D|) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. 

\textbf{Theorem 4.} Let $C$ be a class of graphs excluding a fixed minor $H$ and with the maximum degree of smaller order than the number of edges—that is, for every $\delta > 0$ there exists $m_{s}$
such that if $m \geq m_\delta$ and $G$ is a graph from $C$ with $m$ edges, then $\Delta(G) \leq \delta m$. Then, for every $\varepsilon > 0$ there exists $M_\varepsilon$ such that if $m \geq M_\varepsilon$ and $G$ is a graph from $C$ with $m$ edges, then $\mod(G) \geq 1 - \varepsilon$.

Proof. Fix a class of graphs $C$ as in the assumption and fix $\varepsilon > 0$. Now, choose $\delta := \delta(\varepsilon)$ such that the assertion of Proposition 9 holds. We will show that if $m \geq M_\varepsilon := m_\delta$ and $G$ is a graph from $C$ with $m$ edges, then $\mod(G) > 1 - \varepsilon$.

Indeed, then by Proposition 9 there is a set of no more than $\frac{\varepsilon}{2}m$ edges in $G$ whose deletion creates a graph in which the total weight of every connected component is less than $\frac{\varepsilon}{2}$. Now, let $A$ be the set of those connected components and let $D$ be the set of deleted edges.

First, notice that

$$\sum_{A \in A} \frac{|E(A)|}{|E(G)|} = \frac{|E(G)| - |D|}{|E(G)|} \geq 1 - \frac{\varepsilon}{2}.$$ 

Second, we have

$$\sum_{A \in A} \left( \frac{\deg(V(A))}{\deg(V(G))} \right)^2 = \sum_{A \in A} w(A)^2 < \sum_{A \in A} \frac{\varepsilon}{2} w(A) = \frac{\varepsilon}{2}.$$ 

Concluding,

$$\mod(G) \geq \sum_{A \in A} \left( \frac{|E(A)|}{|E(G)|} - \left( \frac{\deg(V(A))}{\deg(V(G))} \right)^2 \right) > 1 - \varepsilon.$$ 

\[\square\]

3 SEPARATORS

Consider a graph $G$ and a weight function $w : V \to \mathbb{R}_+$. For $S \subseteq V$ define a weight of $S$ by $w(S) = \sum_{v \in S} w(v)$. The graph $G$ has a vertex (edge) separator of size $s$ if there exists a vertex (edge) cutset of size $s$ dividing $G$ into two parts with weights not exceeding $w(V)$. A class of graphs is said to have $f(n)$ vertex (edge) separator if every graph from this class with $n$ vertices has a vertex (edge) separator of size $f(n)$. The existence of reasonably small vertex (edge) separators is crucial for the divide-and-conquer approach.

In 1979 Lipton and Tarjan proved that planar graphs have $O(\sqrt{n})$ vertex separator [14]. The result was generalized to $O(\sqrt[3]{\delta n})$ vertex separator for graphs of genus $g$ by Gilbert, Hutchinson, and Tarjan in 1984 [11]. Later, Alon, Seymour, and Thomas showed that a graph excluding an $h$-vertex minor $H$ has a vertex separator of order $O(h^{3/2}/\sqrt{n})$ and also conjectured that the order could be reduced to $O(h\sqrt{n})$ [3]. The conjecture was proved by Kawarabayashi and Reed in 2010 [12]. Since the genus of a complete graph on $h$ vertices is of order $\Theta(h^2)$, this is a natural extension of the result from [11].
Not as much attention so far was paid to edge separators. In [9] Diks, Djidjev, Sýkora, and Vrto used the result of Miller [19] to prove the existence of $O(\sqrt{\Delta(G)n})$ edge separator for planar graphs. The existence of $O(g\Delta(G)n)$ edge separator for graphs of genus $g$ is due to Sýkora and Vrto [23]. However, to the best of our knowledge, the question whether graphs excluding an $h$-vertex minor have an edge separator of size $O(h\sqrt{\Delta(G)n})$ remains open. Having this question resolved in the positive would facilitate our proof. Indeed, then we could just use a series of such edge separators to recursively divide the graph and arrive at the partition from Proposition 9.

ACKNOWLEDGMENT
We would like to thank Fiona Skerman for suggesting the problem.

ORCID
Michał Lason &copy; http://orcid.org/0000-0003-4830-2270
Małgorzata Sulkowska &copy; http://orcid.org/0000-0001-7745-2951

REFERENCES
1. N. Alon, Eigenvalues and expanders, J. Combin. Theory Ser. B. 6 (1986), 83–96.
2. N. Alon and V. Milman, $\lambda_1$, Isoperimetric inequalities for graphs, and superconcentrators, J. Combin. Theory Ser. B. 38 (1985), no. 1, 73–88.
3. N. Alon, P. Seymour, and R. Thomas, A separator theorem for nonplanar graphs, J. Amer. Math. Soc. 3 (1990), no. 4, 801–808.
4. J. P. Bagrow, Communities and bottlenecks: Trees and treelike networks have high modularity, Phys. Rev. E. 85 (2012), 066118.
5. P. Biswal, J. R. Lee, and S. Rao, Eigenvalue bounds, spectral partitioning, and metrical deformations via flows, J. ACM. 57 (2010), no. 3, 13:1–13:23.
6. V. D. Blondel, J. L. Guillaume, R. Lambiotte, and E. Lefebvre, Fast unfolding of communities in large networks, J. Stat. Mech.—Theory E. 2008 (2008), no. 10, P10008.
7. J. Cheeger, A lower bound for the smallest eigenvalue of the Laplacian. In Problems in analysis, pp. 195–199. Princeton University Press, Princeton, N.J., 1970.
8. J. Chellig, N. Fountoulakis, and F. Skerman, The modularity of random graphs on the hyperbolic plane, J. Complex Networks. 10 (2021), no. 1, cnab051.
9. K. Diks, H. N. Djidjev, O. Sýkora, and I. Vrto, Edge separators of planar and outerplanar graphs with applications, J. Algorithm. 14 (1993), no. 2, 258–279.
10. Z. Dvorák and S. Norin, Treewidth of graphs with balanced separations, J. Combin. Theory Ser. B. 137 (2019), 137–144.
11. J. R. Gilbert, J. P. Hutchinson, and R. E. Tarjan, A separator theorem for graphs of bounded genus, J. Algorithm. 5 (1984), no. 3, 391–407.
12. K. Kawarabayashi and B. Reed, A separator theorem in minor-closed classes, 2010 IEEE 51st Annual Symposium on Foundations of Computer Science, IEEE, 2010, pp. 153–162.
13. L. Lichev and D. Mitsche, On the modularity of 3-regular random graphs and random graphs with given degree sequences, Random Structures Algorithms. (2022). https://doi.org/10.1002/rsa.21080
14. R. J. Lipton and R. E. Tarjan, A separator theorem for planar graphs, SIAM J. Discrete Math. 36 (1979), no. 2, 177–189.
15. W. Mader, Homomorphieeigenschaften und mittlere kantendichte von graphen, Math. Ann. 174 (1967), no. 4, 265–268.
16. C. McDiarmid and B. Reed, On the maximum degree of a random planar graph, Combin. Probab. Comput. 17 (2008), no. 4, 591–601.
17. C. McDiarmid and F. Skerman, *Modularity of regular and treelike graphs*, J. Complex Netw. 6 (2018), no. 4, 596–619.
18. C. McDiarmid and F. Skerman, *Modularity of Erdős–Rényi random graphs*, Random Struct Algor. 57 (2020), no. 1, 211–243.
19. G. L. Miller, *Finding small simple cycle separators for 2-connected planar graphs*, J. Comput. Syst. Sci. 32 (1986), no. 3, 265–279.
20. F. de Montgolfier, M. Soto, and L. Viennot, *Asymptotic modularity of some graph classes*, In T. Asano, S.-I. Nakano, Y. Okamoto, and O. Watanabe, eds., Algorithms and Computation, ISAAC 2011, Yokohama, Japan, December 5–8, 2011, Vol., 7074, Lect. Notes Comput. Sci., Springer, Berlin, Heidelberg, 2011, pp. 435–444.
21. J. Nešetřil and P. Ossona de Mendez, *On nowhere dense graphs*, Eur. J. Combin. 32 (2011), no. 4, 600–617.
22. M. E. J. Newman and M. Girvan, *Finding and evaluating community structure in networks*, Phys. Rev. E. 69 (2004), no. 2, 026113.
23. O. Sýkora and I. Vrto, *Edge separators for graphs of bounded genus with applications*, Theor. Comput. Sci. 112 (1993), no. 2, 419–429.
24. V. A. Traag, L. Waltman, and N. J. van Eck, *From Louvain to Leiden: Guaranteeing well-connected communities*, Sci. Rep. UK. 9 (2019), 5233.

**How to cite this article:** M. Lasoń and M. Sulkowska, *Modularity of minor-free graphs*, J. Graph Theory. 2023;102:728–736. [https://doi.org/10.1002/jgt.22896](https://doi.org/10.1002/jgt.22896)