Combination Networks with End-user-caches: Novel Achievable and Converse Bounds under Uncoded Cache Placement

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Abstract—Caching is an efficient way to reduce network traffic congestion during peak hours by storing some content at the users' local caches. For the shared-link network with end-user-caches, Maddah-Ali and Niesen proposed a two-phase coded caching strategy. In practice, users may communicate with the server through intermediate relays. This paper studies the tradeoff between the memory size M and the network load R for the networks where a server with N files is connected to H relays (without caches), which in turn are connected to K users equipped with caches of M files. When each user is connected to a different subset of r relays, i.e., K = \( \binom{H}{r} \), the system is referred to as a combination network with end-user-caches.

In this work, converse bounds are derived for the practically motivated case of uncoded cache contents, that is, bits of the various files are directly pushed into the user caches without any coding. In this case, once the cache contents and the users’ demands are known, the problem reduces to a general index coding problem. This paper shows that relying on a well-known “acyclic index coding converse bound” results in converse bounds that are not tight for combination networks with end-user-caches. A novel converse bound that leverages the network topology is proposed, which is the tightest converse bound known to date. As a result of independent interest, an inequality that generalizes “acyclic index coding converse bound” results in converse bounds that are not tight for combination networks with end-user-caches.

Several novel caching schemes are proposed, based on the Maddah-Ali and Niesen cache placement. These schemes leverage the structure of the combination network or/and perform interference elimination at the end-users. The proposed schemes are proved: (i) to be (order) optimal for some \((N, M, H, r)\) parameters regimes under the constraint of uncoded cache placement, and (ii) to outperform the state-of-the-art schemes in numerical evaluations.

Index Terms—Coded caching; combination networks; uncoded cache placement; interference elimination.

I. INTRODUCTION

Due to the high variability in network traffic, the network serving period can be divided into peak traffic hours (where traffic is high and the network performance suffers) and off-peak hours (where traffic is low). Caching effectively reduces peak-hour network traffic by storing some content at the users’ local caches during peak-off hours. A caching scheme includes two phases: (i) Placement phase: the server pushes content into the users’ local caches without knowledge of the users’ future demands. The placement is said to be uncoded if bits are directly copied into the cache; (ii) Delivery phase: after each user has requested one file and according to the cache contents, the server transmits packets in order to satisfy the users’ demands. The goal is to minimize the number of transmitted bits (referred to as load) such that any set of demands (referred to as worst-case demands) can be satisfied. The fundamental limits of shared-link networks with end-user-caches were originally studied by Maddah-Ali and Niesen (MAN) in [4], where a server with N files is connected to K users through a shared error-free broadcast link and each user can cache up to M files.

Recently, a more practical case where users may communicate with the server through intermediate relays has gained attention. As the analysis of relay networks with arbitrary topologies is challenging, we focus here on a symmetric version of this general problem known as combination networks with end-user-caches. This model was first proposed in [5], in the context of network coding and is shown in Fig. 1, where a server with N files is connected to H relays (without caches), which in turn are connected to K = \( \binom{H}{r} \) users, where each user is equipped with a cache of M files and is connected to a different subset of r relays. All links are assumed to be error-free and interference-free. The objective is to determine the optimal max-link load \( R^* \), defined as the smallest max-rate (i.e., the maximum number of bits sent on a link normalized by the file size, which is proportional to the overall download time) for the worst-case demands. The main contribution of this paper is to characterize new (order) optimal results on...
the tradeoff between the memory size $M$ and the max-link load $R^*$ under the constraint of uncoded cache placement. This is accomplished by deriving novel converse bounds and achievable schemes that leverage the network topology.

In the rest of this section, we revise relevant past works on cache-aided shared-link networks in Section I-A and on cached-aided combination networks in Section I-B. Section I-C provides the summary of our major contributions.

A. Cache-aided Shared-link Networks

For the shared-link model [4] Maddah-Ali and Niesen (MAN) proposed a coded caching scheme that utilizes an uncoded combinatorial cache construction in the placement phase and a binary linear network code to generate multicast messages in the delivery phase. The achieved worst-case load of the MAN scheme satisfies

$$R_{\text{MAN}}[t] = \frac{K-t}{1+t} = \binom{K}{t+1} \text{for } M = N \binom{K}{t} = N \binom{K-1}{t}, t \in [0 : K],$$

and for $M^{K}_{t} \neq t \in [0 : K]$ one takes the lower convex envelope of the set of points $(M,R) = \left( i \binom{N}{R}, R_{\text{MAN}}[t] \right)$. When a file is requested by multiple users, Yu, Maddah-ali and Avestimehr (YMA) [7] found that among the $(t+1)$ MAN multicast messages, $(K-\min(K,N))$ of them can be obtained as linear combinations of the others. This observation led to the achievable load

$$R_{\text{YMA}}[t] = \binom{K}{t+1} - \binom{K-\min(K,N)}{t+1} \text{for } M = N \binom{K}{t}, t \in [0 : K].$$

In [4] Theorem 2], Maddah-Ali and Niesen also derived a cut-set converse bound, which proved that the load of the MAN caching scheme in [1] is optimal within a factor of 12.

An enhanced converse bound was proposed in [8, Theorem 2] to prove that the load of the YMA caching scheme in [3] is order optimal within a factor of 2. In other words, coded cache placement can at most half the network load compared to the best caching scheme(s) with uncoded cache placement.

As it is difficult to characterize the exact optimality, the optimality under the constraint of uncoded cache placement was originally considered in [9, 10]. By relating the caching problem under the constraint of uncoded placement to the index coding problem [11], the “acyclic index coding converse bound” in [12, Corollary 1] was exploited in [10] to derive a converse bound on the load for the shared-link networks under the constraint of uncoded placement. The same converse bound was also obtained by invoking a genie-based idea in [7]. By comparing this "acyclic index coding converse bound"-based converse bound and the loads in (1) and (2), it was proved that the MAN caching scheme and the YMA caching scheme are exactly optimal under the constraint of uncoded placement when $N \geq K$ and $N < K$, respectively.

B. Cache-aided Combination Network

Past works on coded caching schemes for combination networks with end-user-caches can be divided into two classes, with uncoded or coded cache placement.

In [5] two achievable schemes with the MAN uncoded cache placement were proposed. In the delivery phase, the first scheme uses routing while the second scheme uses Minimum Distance Separable (MDS) codes to deliver the MAN multicast messages to the users. The achieved max-link load in [5] is

$$R_{\text{base}}[t] = \min \left\{ \frac{(1-M/N)}{H}, \frac{(1-M/KM/N)}{r(1 + KM/N)} \right\},$$

for $M = N \frac{t}{K}, t \in [0 : K]$. (3)

An achievable scheme was proposed in [13] for the case where $r$ divides $H$; the idea was to split the combination network into $H$ shared-link networks, in each of which the scheme in [4] is used, so as to achieve the max-link load

$$R_{\text{ZY}}[t'] = \frac{(1-M/N)}{H(1 + \frac{Kt'}{N})}, \text{ for } M = N \frac{t'}{K}, t' \in [0 : Kr/H].$$

(4)

The Placement Delivery Array (PDA) scheme, originally proposed in [14] to reduce the sub-packetization of MAN caching scheme in the shared-link model, was extended to combination networks in [15] for the case where $r$ divides $H$; it was shown to achieve the same load as in (3) and (4) but with lower sub-packetization. For some other specific cases, improved constructions of PDA for combination networks were proposed in [16].

A caching scheme with MDS coded placement was proposed in [17], which achieves the load in [4] but without the constraint that $r$ divides $H$. The same authors of this paper further improved on the coded placement scheme in [17] by proposing an asymmetric coded cache placement in [18], and leveraging the symmetries in the topology to generate and deliver the multicast messages as in [19], which is not discussed here as in this paper we focus on achievable schemes and converse bounds with uncoded cache placement.

The cut-set converse bound for the shared-link model in [4] Theorem 2] was extended to combination networks in [5]. It was proved in [6] that the cut-set converse bound in [5] and the achieved max-link load in (3) are to within a factor of 12.
in the “large memory” regime $M/N \geq 1/(2r)$. However, the gap may be large when the “small memory” regime and this paper aims to address this open problem.

C. Main Contributions

Our main contributions are as follows.

1) Achievability: Based on the MAN placement, we propose four achievable schemes for combination networks with end-user-caches to deliver the MAN multicast messages to the corresponding users:
   a) We first propose a novel delivery scheme, named Direct Independent delivery Scheme (DIS), that exploits the fact that not all the MAN multicast messages are useful to every user.
   b) For $M = N/K$, we propose an Interference Elimination delivery Scheme (IES) that uses interference elimination (a form of interference alignment) to encode the MAN multicast messages such that each user can cancel the interference caused by the MAN multicast messages that are not of interest.
   c) The scheme, named Concatenated Inner Code delivery Scheme (CICS), proposes a coded delivery scheme composed of two steps: (i) in the first step we directly transmit each MAN multicast message to some relay(s), which forward them to their connected users; such messages are simultaneously useful for $t = KM/N$ users and will be used as ‘side information’ in the next step; (ii) in the second step, we deliver the MAN multicast messages through a carefully designed network code that lets the remaining ‘unsatisfied’ users recover their demanded files. In this step, a multiplicative coding gain can be achieved.
   d) By leveraging the multicasting opportunities which are ignored in the CICS, we finally propose a scheme, named Improved Concatenated Inner Code delivery Scheme (ICICS).

2) Converse: For combination networks with end-user-caches, we propose several converse bounds under the constraint of uncoded cache placement when $N \geq K$.

Based on the cut-set strategy in [8], we firstly extend the shared-link converse bound under the constraint of uncoded placement to combination networks with end-user-caches, and propose a converse bound based on the acyclic index coding converse bound [12].

Furthermore, by deriving bounds on the joint entropy of the various random variables that define the problem, we provide a novel converse bound which tightens the acyclic index coding converse bound, and produces the best known converse bound to date, to the best of our knowledge.

As a result of independent interest, an inequality that generalizes the well-known sub-modularity of entropy is derived, which may find applications in other network information theory problems.

3) Optimality: By comparing the proposed achievable schemes and the proposed converse bounds under the constraint of uncoded cache placement, we obtain the (order) optimality results summarized in Table II at the top of the next page. For general $(N,M,H,r)$ with $N \geq K = \binom{H}{r}$, the achieved max-link load in (3) was proved to be order optimal within a factor of 12 when $M \geq \frac{1}{2}$ (i.e., large memory regime) [6]. In this paper, under the constraint of uncoded cache placement, we obtain the order optimality results for $M$ is small, in particular for $\frac{M}{N} \leq \frac{1}{2}$, where the CICS is order optimal within a factor of 2. Hence, the order optimality under the constraint of uncoded cache placement for the regime $\frac{M}{N} \geq \frac{1}{2}$ remains open.

4) Numerical Comparisons: Numerical results show that the proposed bounds outperform the state-of-the-art schemes.

D. Paper Organization

The rest of the paper is organized as follows. Section II presents the system model and some relevant past results. Section III introduces our main results. Section IV provides the proofs of the proposed novel delivery schemes. Section V provides the proofs of the converse bounds. Section VI concludes the paper. Some technical proofs are relegated in Appendix.

E. Notation Convention

Calligraphic symbols denote sets, bold symbols denote vectors, and sans-serif symbols denote system parameters. The main network parameters and notations are given in Table II at the top of the next page. We use $|\cdot|$ to represent the cardinality of a set or the length of a vector; $X_J := \{X_i : i \in J\}$; $[a : b] := \{a, a+1, \ldots, b\}$ and $[n] := \{1 : n\}$; $A \setminus B := \{x \in A : x \notin B\}$; $\arg\max_{x \in X} f(x) := \{x \in X : f(x) = \max_{x \in X} f(x)\}; p(J) := \{p_1(J), \ldots, p_{|J|}(J)\}$ represents a permutation of elements of the set $J$; $\text{RLC}(m, S)$ represents $m$ random linear combinations of the equal-length packets indexed by $S$. We note that $m$ random linear combinations of $|S|$ packets are linearly independent with high probability if operations are done on a large enough finite field; the same can be obtained by using the parity-check matrix of an $(|S|, |S| - m)$ MDS (Maximum Distance Separable) code.

II. System Model and Some Known Results

A. General System Model

Consider the $(H,r,M,N)$ combination network with end-user-caches illustrated in Fig. 1. The server has access to $N$ files denoted by $\{F_1, \ldots, F_N\}$, each composed of $B$ independent and uniformly distributed bits. The server is connected to $H$ relays through $H$ error-free and interference-free links. The
TABLE I: Order optimality results under the constraint of uncoded placement and $N \geq K$.

| Delivery Scheme | Constraint of system parameters | Optimality under the constraint of uncoded cache placement |
|-----------------|---------------------------------|---------------------------------------------------------|
| DIS,CICS, ICICS | $M \leq \frac{H}{2}$ | optimal |
| DIS,CICS, ICICS | $r = H - 1$ | optimal |
| Combining CICS and | $r = H - 2$ | order optimal within a factor of 12 |
| CICS | $M \leq N/K$ | optimal |
| CICS | $M = N/K$, $t/r \rightarrow 0$ | optimal |
| IES | $H \geq 2r$, $M \leq N/K$ | order optimal within a factor of $2t/(2r - 1) \leq 4/3$ |
| IES | $r < \frac{M}{H} < \frac{1}{2}$ | open regime |

TABLE II: Notations and Acronyms

| Notations | Semantics |
|-----------|-----------|
| $H$ | number of relays |
| $r$ | number of relays connected to each user |
| $K$ | number of users |
| $N$ | number of files |
| $M$ | cache size in multiple of the file size |
| $R$ | load |
| $B$ | number of bits per file |
| $h_k$ | set of relays connected to user $k$ |
| $H_{WY}$ | set of relays connected to some user in $W$ |
| $U_h$ | set of users connected to relay $h$ |
| $R_w$ | set of users whose connected relays are all in $J$ |
| $R^*$ | set of relays connected to all the users in $W$ |
| MAN | coded caching scheme proposed by Maddah-Ali and Niesen |
| YMA | coded caching scheme proposed by Yu, Maddah-ali and Avestimehr |
| DIS | Direct Independent delivery Scheme |
| IES | Interference Elimination delivery Scheme |
| CICS | Concatenated Inner Code delivery Scheme |
| ICICS | Improved Concatenated Inner Code delivery Scheme |

Relays are connected to $K = \binom{H}{r}$ users through $rK$ error-free and interference-free links. Each user is connected to a distinct subset of $r$ relays. The set of users connected to the $h$-th relay is denoted by $U_h$, $h \in [H]$. The set of relays connected to $k$-th user is denoted by $H_k$, $k \in [K]$. For each set of users $W \subseteq [K]$, let $H_W = \cup_{h \in W} H_h$. For each set of relays $J \subseteq [H]$ such that $|J| \geq r$, let $K_J := \{k \in [K] : H_k \subseteq J\}$ be the set of users whose connected relays are all in $J$. For each set of users $W \subseteq [K]$, define $R_W = \{h \in [H] : W \subseteq U_h\}$ as the set of relays connected to all the users in $W$. For the network in Fig. 1, we have, for example, $U_2 = \{1, 2, 3\}$, $H_1 = \{1, 2\}$, $K_{\{1, 2, 3\}} = \{1, 2, 4\}$ and $R_{\{1, 2\}} = \{1\}$.

The system works in two phases. In the placement phase, user $k \in [K]$ stores information about the $N$ files in its cache of $MB$ bits, where $M \in [0, N]$. We denote the content in the cache of user $k \in [K]$ by $Z_k$ and let $Z := (Z_1, \ldots, Z_K)$. During the delivery phase, user $k \in [K]$ demands file $F_d$, where $d \in [N]$; the demand vector $\mathbf{d} := (d_1, \ldots, d_K)$ is revealed to all nodes. Given $(\mathbf{d}, Z)$, the server sends a message $X_h$ of $B R_h(d, Z)$ bits to relay $h \in [H]$. Then, relay $h \in [H]$ transmits a message $X_h \rightarrow k$ of $B R_h \rightarrow k(d, Z)$ bits to user $k \in U_h$. User $k \in [K]$ must recover its desired file $F_{d_h}$ from $Z_k$ and $(X_h \rightarrow k : h \in H_k$) with high probability for some file size $B$.

The max-link load is defined as

$$R(M) = \max_{h \in [H]} \max_{d \in [N]^K} \{R_h(d, Z, M), R_h \rightarrow k(d, Z, M)\}. \quad (5)$$

Our objective is to determine the minimum max-link load $R^*(M)$ over all possible caching schemes. Obviously, for each relay $h$, the load on the link from the server to $h$ should not be less than the load on each link from relay $h$ to user $k \in U_h$. So in our achievable schemes, we assume that each relay forwards all of its received packets from the server to its connected users.

B. Systems with Uncoded Cache Placement

The minimum max-link load under the constraint of uncoded cache placement is denoted by $R_u^*(M)$. In general, $R_u^*(M) \geq R^*(M)$. In the rest of the paper, we simplify $R_u^*(M)$ and $R^*(M)$ by $R_u^*$ and $R^*$, respectively.

After the uncoded placement phase is concluded, each file can be effectively divided into non-overlapping subfiles depending on which user stores which bit. Let $T_Z := \{F_i : i \in [N], W \subseteq [K]\}$, where $F_i$ is the set of bits of the file $F_i$ uniquely stored by the users in $W$. After the demand vector is revealed, the set of requested subfiles according to the demand vector $\mathbf{d} \in [N]^K$ is denoted by

$$T_{A,D} := \{F_{d_h} : k \in [K], W \subseteq [K], k \notin W\} \subseteq T_Z. \quad (7)$$

In the delivery phase, the delivery of the subfiles in $T_{A,D}$ is equivalent to an index coding problem and can thus be represented by a directed graph $G_{T_{A,D}}$ (i.e., known as side information graph) as follows. Each node in the graph represents one subfile demanded by one user only (if the same file is demanded by multiple users, only one such user is considered at a time). A directed edge from node $i$ to node $i'$ exists if the
In the delivery phase, given the demand vector $d$ in the cut. By using this strategy, the cut-set converse bound with $N$ than the converse bound of the load in a shared-link system.

The max-link load is the second term of (3).

D. Results from [5] for Combination Networks

The authors in [5] proposed two delivery schemes for combination networks.

In the first scheme, for each user $k$, we divide the bits of $F_{dk}$ which are not cached by user $k$ into $r$ non-overlapping and equal-length pieces, and transmit one different piece to each relay in $H_k$, such that the max-link load is the first term of (3).

In the second scheme, the MAN multicast messages in (9) are generated. For each multicast message $W_J$, we divide it into $r$ non-overlapping and equal-length pieces, and then encoded by an $(H, r)$ MDS code. We then transmit one different MDS symbol of $W_J$ to each relay $h \in H$. The max-link load is the second term of (3).

Cut-set Converse Bound: Each time out cut including $[\alpha r] \in [r : H]$ relays and $(\lfloor \alpha r \rfloor)$ users is considered. The total load from the server to the $[\alpha r]$ relays should not be less than the converse bound of the load in a shared-link system with $N$ files and $(\lfloor \alpha r \rfloor)$ users. Then the max-link load is lower bounded by this total load divided by the number of relays in the cut. By using this strategy, the cut-set converse bound in [4] extended to combination networks in [5] to show that the optimal max-link load $R^*(M)$ must satisfy

$$R^*(M) \geq \max_{\alpha \in [1, \frac{H}{r}]} \max_{l \in \min\{N, \lfloor \alpha r \rfloor\}} \frac{1}{l} \left( l - \frac{\lfloor \alpha r \rfloor}{r} \right)^{M}. \tag{10}$$

III. MAIN RESULTS ON COMBINATION NETWORKS WITH END-CACHE-USERS

In this section, we introduce our main results on combination networks with end-cache-users, including novel achievable schemes in Section III-A, novel converse bounds in Section III-B, and finally optimality results in Section III-C.

A. Novel Achievable Schemes

In this paper we propose four novel delivery schemes for combination networks with end-user-caches by using the MAN placement in [8] and multicast message generation in [9]. The novelty of our schemes is on the delivery of the MAN multicast messages. Recall that $R_J$ is the set of relays connected to all the users in $J$. For each $t \in [0 : K]$, we define

$$\mathcal{V}_1 := \{J \subseteq [K] : |J| = t + 1, R_J \neq \emptyset\}, \tag{11a}$$
$$\mathcal{V}_2 := \{J \subseteq [K] : |J| = t + 1, R_J = \emptyset\}, \tag{11b}$$

where $\mathcal{V}_1$ and $\mathcal{V}_2$ represent the families of the subsets with cardinality $t + 1$ of the set of users who have and do not have a common connected relay, respectively.

In this paper, we mainly propose four novel achievable schemes: (i) Direct Independent delivery Scheme (DIS) in Section IV-A, which improves on the second scheme in [5] by observing that the multicast message $W_J$ is only useful to the users in $J$ and thus we need not let all users recover it; (ii) Interference Elimination delivery Scheme (IES) in Section IV-B based on a novel topological interference alignment idea; (iii) Concatenated Inner Code delivery Scheme (CICS) in Section IV-C which includes two steps to deliver each $W_J$ to the users in $J$; (iv) Improved Concatenated Inner Code delivery Scheme (ICICS) in Section IV-D which leverages the ignored multicasting opportunities of the CICS. The achievable loads of the proposed scheme is given in the following theorem.

**Theorem 1 (Achievable schemes).** For the $(H, r, M, N)$ combination network with end-user-caches, the optimal tradeoff under the constraint of uncoded cache placement $(M, R^*_M)$ is upper bounded by the lower convex envelope of the following points:

- $(\frac{N}{K}, R_{\text{DIS}}[t])$ for each $t \in [0 : K]$, where
  $$R_{\text{DIS}}[t] := \frac{\sum_{J \in \mathcal{V}_1} 1 + \sum_{J \in \mathcal{V}_2} |\{h \in [H] : U_h \cap J \neq \emptyset\}|}{H \begin{pmatrix} K \\ t \end{pmatrix}}; \tag{12}$$

- $$\left( \frac{N}{K}, R_{\text{IES}} \right) = \left( \frac{N}{K}, \frac{1}{2^r} \left( K - 1 - \frac{H - r}{r}\right) + \frac{2^r - 1}{2(2^r - 1)} \frac{H - 1}{2(r - 1)} \right), \tag{13}$$

where $2r - 1 = p^r$ or $2r - 1 = pq$ where $p, q$ are different primes and $v$ is a positive integer.
Corollary 1. For the \((H, r, M, N)\) combination network with end-user-caches, it holds that
\[
R_{\text{DIS}} \leq R_{\text{base}}.
\]

Remark 1. As mentioned in Theorem 7 the IES works for the combination networks with \(2r - 1 = p^a q^b\) where \(p, q\) are different primes and \(v\) is a positive integer. The smallest value of \(r\) not satisfying the above condition is 23. If \(r = 23 \text{ and } H = 2r = 46\), in the network there are more than \(8.23 \times 10^{12}\) users, which is not practical. Note that by the proof of Theorem 3 in Appendix A (upon which the above condition is built), the circulant matrix in Theorem 3 may not be invertible when \(r \geq 23\) (i.e., when \(r = 45\)); but this does not imply that Problem 1 in Appendix A does not have a solution for \(r \geq 23\). We conjecture that the IES works in general.

B. Novel Converse bounds

First we use the cut-set strategy of the converse bound in [10] to extend the converse bound in [8] Theorem 2) for the shared-link model to combination networks with end-user-caches.

Theorem 2 (Enhanced cut-set, any placement). For the \((H, r, M, N)\) combination network with end-user-caches, it must satisfy that
\[
R^* \geq \max_{x \in [r:H]} \max_{s \in \left[\min \{N, \binom{r}{t}\}\right]} \max_{\alpha \in [0,1]} \frac{s - 1 + \alpha - \frac{s(x - 1) - l(t - 1) + 2\alpha s}{2(N - l + 1)} M}{x},
\]
where \(l \in [s]\) is the minimum value such that \(\frac{s(x - 1) - l(t - 1) + 2\alpha s}{2(N - l + 1)} \leq 1\).

Proof: Each time we consider a cut with \(x\) relays. The total load transmitted from the server to this \(x\) relays is lower bounded by the load in a shared-link model with \(N\) files, \(\binom{r}{t}\) users and memory of MB bits. So we can use this strategy to extend any converse bound for the shared-link model to combination networks. Theorem 2 uses the enhanced cut-set converse bound in [8] Theorem 2).

Similarly to Theorem 2 we next extend the shared-link converse bound under the constraint of uncoded placement to combination networks.

Theorem 3 (Cut-set, uncoded placement). For the \((H, r, M, N)\) combination network with end-user-caches, the optimal tradeoff under the constraint of uncoded cache placement \((M, R^*_u)\) is lower bounded by the lower convex envelop of
\[
\left(\frac{N}{t} \frac{1}{x} \frac{\binom{r}{t}}{\binom{l+1}{t+1}} - \frac{\max \{\binom{r}{t} - N, 0\}}{\binom{l+1}{t+1}}, \forall t \in \left[0 : \frac{r}{H}\right], x \in [r : H]\right).
\]

We then propose the following converse bound which is tighter than the cut-set bound in Theorem 3. As this result follows straightforwardly from the work for shared-link network done by some of the authors of this paper in [10], here we consider the resulting bound as our ‘baseline’ bound. The proof is in Section V-B.

Theorem 4 (Acyclic index coding, uncoded placement). Consider the \((H, r, M, N)\) combination network with end-user-caches where \(N \geq K = \binom{r}{t}\). For each subset \(Q \subseteq [H]\) such that \(|Q| \in [r : H]\), and each permutation \(p(K_Q)\), we have
\[
|Q| R^*_u \geq \sum_{i \in [|K_Q|]} \sum_{W \subseteq \{p_i(K_Q), \ldots, p_{|W|}(K_Q)\}} x_W, \quad (18a)
\]
\[
x_W := \frac{1}{\text{NB}} \sum_{i \in [H]} |F_{i,W}|, \forall W \subseteq [K], \quad (18b)
\]
\[
\sum_{W \subseteq [K]} x_W = 1, \quad (18c)
\]
\[
\sum_{W \subseteq [K]: i \in W} x_W \leq \frac{M}{N}, \forall i \in [K]. \quad (18d)
\]

Recall that for any uncoded cache placement, \(F_{i,W}\) represents the set of bits uniquely cached by the users in \(W\). Thus \(x_W \text{NB}\) represents the total number of bits uniquely cached by the users in \(W\).

Remark 2. The converse bound in Theorem 4 can be numerically computed by means of a linear programming with variables \((R^*_u, x_W : W \subseteq [K])\) and constraints in (18a)-(18d).

The “baseline” bound in Theorem 3 can be improved as follows (see also examples in Sections V-C and V-D, whose detailed proof can be found in Section V-E). This is a tightened converse bound obtained by leveraging the network topology and a generalized version of the submodularity of entropy.

Theorem 5 (Improved converse, uncoded placement). Consider the \((H, r, M, N)\) combination network with end-user-caches where \(N \geq K = \binom{r}{t}\). For each integer \(b \in [r : H]\), each set of relays \(Q_{\alpha} \subseteq [H]\) with \(|Q_{\alpha}| = b\), each integer \(a \in [b/r]\), each disjoint partition \(Q = Q_{\alpha_1} \cup \ldots \cup Q_{\alpha_a}\) where \(|Q_{\alpha_i}| \geq r + i \in [a]\), and each combination of permutations \(p(K_{Q_{\alpha_1}}), \ldots, p(K_{Q_{\alpha_a}})\), \(p(K_Q \setminus (Q_{\alpha_1} \cup \ldots \cup Q_{\alpha_a}))\), the following must hold
\[
|Q| R^*_u \geq \sum_{i \in [a]} \sum_{j \in |K_{Q_{\alpha_i}}|} \sum_{W \subseteq [K \setminus \cup_{k \in [j]} p_k(K_{Q_{\alpha_k}})}} x_W
\]

+ \sum_{j \in [K \setminus \mathcal{V}]} x_{Wj} + y_{Q},

(19a)
satisfying (18c) and (18d), where \( V := \cup_{i \in [a]} \mathcal{K}_Q \), the variables \( x_{Wj} \) are as in (18b), and for each permutation \( p([K]) \) we have,

\[
\sum_{Q: |Q| = b} y_{Q} \geq \sum_{i \in [K]} \sum_{W \subseteq \mathcal{K} \setminus \cup_{j \in [j]} \{ p_i([K]) \}} c(\{ p_i([K]) \} \cup W, b) x_{W},
\]

(19b)

\[
c(\mathcal{W}_1, l) := \max \left\{ \left( H - \frac{1}{l} \right) - \left[ \sum_{Q \subseteq [H]} : |Q| = l, \mathcal{K}_Q \not\subseteq [K] \setminus \mathcal{W}_1 \right], 0 \right\}.
\]

(19c)

Intuitively, compared to the acyclic index coding converse bound in Theorem 3, the converse bound in Theorem 5 is tighter because of the following two improvements:

1) As discussed in Remark 10, the sum

\[
\sum_{i \in [a]} \sum_{j \in [\mathcal{K}_Q \setminus \{ K \}]} \sum_{W \subseteq \mathcal{K} \setminus \cup_{j \in [j]} \{ p_i([K]) \}} x_{Wj}
\]

(19a)
in (19a) may contain the cycles in the directed graph that represents the equivalent index coding problem, i.e., it may contain more terms than the RHS of (18a).

2) In (19a), there is an additional non-negative term \( y_{Q} \), which further tightens the acyclic index coding converse bound in (18a). Intuitively, Theorem 4 uses the cut-set strategy, that is, each time we select a set of relays \( Q \) and only consider the users in \( \mathcal{K}_Q \) (i.e., whose connected relays are all in \( Q \)). The total load from the server to these relays is lower bounded by the load for the shared-link caching model containing \( [\mathcal{K}_Q] \) users. However, there are some coded messages transmitted to the relays in \( Q \) which are only useful to the users in \( [K] \setminus Q \). The non-negative term \( y_{Q} \) characterizes the entropy of these messages.

Remark 3. The converse bound in Theorem 5 can be numerically computed by means of a linear program with variables \( (R^*_u, x_{Wj}) : W \subseteq [K], y_{Q} : Q \subseteq [H] \) and constraints in (19a), (19b), (18c) and (18d). Notice that the computation complexity orders of Theorems 4 and 5 are no more than that of the linear programming including \( \mathcal{O}(2^K) \) variables and \( \mathcal{O}(HK) \) constraints.

C. Optimality Results

We first compare the cut-set converse bound in Theorem 3 and the converse bound in Theorem 5 to the aforementioned proposed schemes, in order to derive the following optimality results under the constraint of uncoded cache placement. The detailed proof can be found in Appendix D.

Theorem 6 (Exact Optimality Under Uncoded Placement). For the \( (H, r, M, N) \) combination network with end-user-caches where \( N \geq K = \binom{M}{r} \),

1) Case \( M \leq \frac{N}{r} \). The optimal tradeoff under the constraint of uncoded cache placement \( (M, R^*_u) \) is the lower convex envelop of the following corner points

\[
\left( \frac{Nt}{K} - \frac{t}{H(t+1)} \right) \quad \text{for } t \in \left[ 0 : \left[ \frac{r}{H-r} - 1 \right] \right],
\]

(20)

which is achieved by the DIS, the CICS and the ICICS.

2) Case \( r = H - 1 \). The optimal tradeoff under the constraint of uncoded cache placement \( (M, R^*_u) \) is the lower convex envelop of the following corner points

\[
\left( \frac{Nt}{K} - \frac{t}{H(t+1)} \right) \quad \text{for } t \in \left[ 0 : K - 2 \right],
\]

(21a)

\[
\left( \frac{Nt}{K} - \frac{t}{H(K-1)} \right) \quad \text{for } t \in \left[ 1 : K - 1 \right],
\]

(21b)

which is achieved by the DIS, the CICS and the ICICS.

3) Case \( M \leq \frac{N}{r} \). The optimal max-link load under the constraint of uncoded cache placement is

\[
R^*_u = \frac{K}{H - 2H/N} \quad \text{when } H < 2r,
\]

(22a)

\[
R^*_u = \frac{K(H - 1) - (KH + H - K - 1)KM}{2H(H-1)} \quad \text{when } H = 2r,
\]

(22b)

which is achieved by the IES.

We then compare the cut-set converse bound in Theorem 3 with the achieved max-link loads of the CICS in (24a) and of the IES in (13) to derive the following order optimality results, whose detailed proof is in Appendix E.

Theorem 7 (Order Optimality Under Uncoded Placement). For the \( (H, r, M, N) \) combination network with end-user-caches where \( N \geq K = \binom{M}{r} \),

1) Case \( M = \frac{N}{r} \) where \( t \in [0, K] \). The multiplicative gap between the memory sharing of the DIS and of the scheme in (15) and the converse bound under the constraint of uncoded cache placement in Theorem 2 is within a factor of \( \min\{1, r/t + 1\} \).

2) Case \( M = \frac{N}{r} \) where \( t \in [0, K] \). The multiplicative gap between the CICS and the converse bound under the constraint of uncoded cache placement in Theorem 3 is within a factor of \( 1 + [ t ] / r \).

3) Case \( M \leq \frac{N}{r} \). The CICS is order optimal under the constraint of uncoded cache placement within a factor of 2.

4) Case \( H > 2r \) and \( M \leq \frac{N}{r} \). The IES is order optimal under the constraint of uncoded cache placement within a factor of \( \frac{2r}{H} \leq 4 \).

Remark 4. By using the technique in (9), in the regimes identified by Theorem 2 the mentioned schemes are optimal within the reported factor multiplied by 2 when no constraint is imposed on the placement phase.

Remark 5. We remark that for the small cache size regime given by \( M \leq \frac{N}{r} \), the proposed CICS is optimal within a factor no more than two under the constraint of uncoded placement. By combining the order optimality result in (25) (i.e., the order optimality within a factor of 12 when \( M \geq \frac{1}{2} \)) with the one
in Theorem 7 for general (H, r, M, N) with N ≥ K, the order optimality under the constraint of uncoded cache placement for the regime \( \frac{r}{K} < \frac{M}{N} < \frac{1}{2r} \) remains open.

**Remark 6.** Our proposed delivery schemes can be extended to “decentralized systems” where users are not allowed to coordinate in the placement phase. With the placement phase in [21], the delivery is divided into K + 1 rounds and in the i-round where \( i \in \{0 : K\} \) we can use the proposed achievable schemes in this paper with parameter \( t = i \).

**D. Numerical Evaluations**

Finally, in Table III at the top of the next page and Fig. 2 we provide numerical results for \( H = 4, r = 2 \) and \( N = K = (H) = 6 \), to illustrate the proposed achievable bounds and converse bounds compared to the state-of-the-art. It can be seen that our results improve on the literature. In addition, when \( M \in \{0, 1\} \), the proposed IES coincides with the proposed converse bound in Theorem 5. For the case \( H \in \{4 : 8\}, K = N, r = 2 \) and \( M = 1 \), Fig. 3 shows that IES outperforms all of the other schemes.

**IV. Novel Achievable Delivery Schemes**

In this section, we introduce our proposed delivery schemes corresponding to the achievable bounds in Section III-A. The placement phase is the same as the MAN placement described in Section II-C. Hence, we focus on each memory \( M = \frac{N}{K} \) where \( t \in \{0 : K\} \).

**A. Direct Independent delivery Scheme (DIS)**

Recall that \( \mathcal{R}_J \) is the set of relays connected to all the users in \( J \), and that the definitions of \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are given in (11a) and (11b), respectively. The DIS is based on the observation that each MAN multicast message \( W_J \) is only useful to the users in \( J \); therefore, if \( \mathcal{R}_J \neq \emptyset \), i.e., there exists at least one relay that is connected to all the users in \( J \), it is enough to transmit \( W_J \) only to the relays in \( \mathcal{R}_J \); this observation motivates the next steps.

**Step 1:** For each \( J \in \mathcal{V}_1 \), we divide \( W_J \) into \( |\mathcal{R}_J| \) non-overlapping pieces with equal length and directly transmit each different piece to the relays in \( \mathcal{R}_J \).

**Step 2:** For each \( J \in \mathcal{V}_2 \), we divide \( W_J \) into \( r \) non-overlapping and equal-length pieces, which are then encoded by an \( \{|\{h \in [H] : U_h \cap J \neq \emptyset\}|, r\} \) MDS code. We then transmit one different MDS symbol to each relay in \( \{h \in [H] : U_h \cap J \neq \emptyset\} \). Each user in \( J \) is connected to \( r \) relays, and thus can receive \( r \) MDS symbols of \( W_J \) to recover \( W_J \).

By Step 1 and Step 2, each user \( k \in [K] \) can recover all the multicast messages \( W_J \) where \( k \in J \) and then recover the subfile \( F_{d_k, J}(k) \). The resulting load is in (12).

It is not difficult to prove that the DIS is better than the second scheme in [5], which transmits \( |\mathcal{V}_2|/r \) bits to each relay \( h \in [H] \). The first scheme in [5] is equivalent to using \( |\mathcal{J}|/|\mathcal{W}_J| \) bits to transmit \( W_J \). However, in the DIS, we use at most \( |\mathcal{J}|/|\mathcal{W}_J| \) bits to transmit \( W_J \), only when for each pair of users \( k_1, k_2 \in J \) we have \( H_{k_1} \cap H_{k_2} = \emptyset \), we use \( |\mathcal{J}|/|\mathcal{W}_J| \) bits. Hence, the max-link load of the DIS is no larger than the one in [5], as we state in Corollary 1.

**B. Interference Elimination delivery Scheme (IES) for the Case \( t = 1 \)**

The IES is for \( M = N/K \) (i.e., \( t = 1 \)) only. Notice that when \( H < 2r \) and \( M = N/K \), \( \mathcal{V}_2 \) contains the sets \( J \) where no relay is connected to all the users in \( J \) with \( |J| = t + 1 = 2 \). So we have \( \mathcal{V}_2 = \emptyset \).

**Delivery of multicast messages in \( \mathcal{V}_1 \):** For each \( J \in \mathcal{V}_1 \) with \( V_1 \in [11a] \), we divide \( W_J \) into \( |\mathcal{R}_J| \) non-overlapping and equal-length pieces, i.e., \( W_J = \{W_{J,h} : h \in \mathcal{R}_J\} \); we transmit \( W_{J,h} \) to each relay \( h \in \mathcal{R}_J \). Note that each relay \( h \in \mathcal{R}_J \) is connected to all of the users in \( J \). Hence, if relay \( h \) forwards \( W_{J,h} \) to the users in \( J \), each user in \( J \) can recover...
$W_J$. The link load to deliver the coded multicast messages in $V_1$ to each relay is \( \frac{1}{M} (K - 1 - (H_u - 2) \left\lfloor \frac{t}{M} \right\rfloor) \).

The key idea of the IES is using an interference elimination scheme to transmit the messages $W_J$ where $J \in V_2$. We examine three examples to highlight the key idea.

**Example 1** ($H = 2r, r = 2$). Consider the combination network with end-user-caches in Fig. 1 where $H = 4, r = 2, M = 1$ and $K = N = 6$. Assume that $d = (1:6)$, so that

$V_1 = \{[1, 2], [1, 3], [1, 4], [1, 5], [2, 3], [2, 4], [2, 6], [3, 5], [3, 6], [4, 5], [4, 6], [5, 6]\}$,

$V_2 = \{[1, 6], [2, 5], [3, 4]\}$.

For the messages in $V_1$, the server transmits

$W_{(1, 2)}, W_{(1, 3)}, W_{(2, 3)}$ to relay 1,

$W_{(1, 4)}, W_{(1, 5)}, W_{(4, 5)}$ to relay 2,

$W_{(2, 4)}, W_{(2, 6)}, W_{(4, 6)}$ to relay 3,

$W_{(3, 5)}, W_{(3, 6)}, W_{(5, 6)}$ to relay 4.

Then we use an interference elimination scheme to transmit the messages $W_J$ where $J \in V_2$ and $V_2 = \{[1, 6], [2, 5], [3, 4]\}$. Consider user 1, who is connected to relays 1 and 2, and only needs to recover $W_{(1, 6)}$ (i.e., $W_{(2, 5)}$ and $W_{(3, 4)}$ are interference). In order to perform the IES we transmit (with operations on some finite field $\mathbb{F}_2$)

$X_1 := -W_{(1, 6)} + W_{(2, 5)} + W_{(3, 4)}$ to relay 1,

$X_2 := +W_{(1, 6)} - W_{(2, 5)} - W_{(3, 4)}$ to relay 2,

$X_3 := -W_{(1, 6)} + W_{(2, 5)} - W_{(3, 4)}$ to relay 3,

$X_4 := -W_{(1, 6)} - W_{(2, 5)} + W_{(3, 4)}$ to relay 4,

and the relays then forward what they received to the users. Then, user 1 computes $X_1 + X_2 = 2W_{(1, 6)}$ and recovers $W_{(1, 6)}$. Similarly, user 2 computes $X_1 + X_3 = 2W_{(2, 5)}$, and so on. With this, all users recover the missing subfiles of the demanded file. As the length of $W_J$ equals to $B/6$, goes to infinity as $B \to \infty$, we can divide each $W_J$ into $P$ sub-packets with length $B/(6P)$ such that we can do operations among multicast messages on a finite field of size $3$. Notice that each file is composed of $6P$ sub-packets.

The link load to transmit the multicast messages in $V_2$ to each relay is $P/(6P) = 1/6$, while the link load to transmit the multicast messages in $V_1$ to each relay is $1/2$. Hence, the max-link load of this scheme is $2/3 = 1/2 + 1/6$, coinciding with the cut-set converse bound under the constraint of uncoded placement in (17), while that of the IICS (ICICS) and the schemes in [5], [7] are 0.6875, 5/4, and 1, respectively.

In the next example, we generalize the IES to transmit $W_J$ where $J \in V_2$ to the case $H > 2r$ and $r = 2$.

**Example 2** ($H > 2r, r = 2$). Consider the combination networks with end-user-caches where $H = 5, r = 2, M = 1$, and $K = N = 10$. Assume that $d = (1:10)$, so that

$U_1 = [1:4], \quad U_2 = [1, 5, 6, 7], \quad U_3 = [2, 5, 8, 9], \quad U_4 = [3, 6, 8, 10], \quad U_5 = [4, 7, 9, 10], \quad V_2 = \{[1, 8], [1, 9], [1, 10], [2, 6], [2, 7], [2, 10], [3, 5], [3, 7], [3, 9], [4, 5], [4, 6], [4, 8], [5, 10], [6, 9], [7, 8]\}$.

The link load to transmit the multicast messages in $V_1$ to each relay is $3/5$. For $W_J$ where $J \in V_2$, we expand on the IES introduced in Example 1.

For each subset of relays $B \subseteq [H]$ with cardinality $|B| = 2r = 4$, the set of users connected to $r = 2$ of the chosen relays is denoted as $P_B$ and we also let

$\mathcal{T}_B := \{J \in V_2 : J \subseteq P_B\}$, (23)
be the set of multicast messages in $\mathcal{V}_2$ which are useful to some users in $\mathcal{P}_B$ and not useful to the users not in $\mathcal{P}_B$. We then use the scheme in Example 1 to transmit the codewords for the multicast messages in $\mathcal{T}_B$ to the relays in $\mathcal{B}$.

For example, for $\mathcal{B} = \{1, 2, 3, 4\}$ we have $\mathcal{P}_{\{1,2,3,4\}} = \{1, 2, 3, 5, 6, 8\}$ and $\mathcal{T}_{\{1,2,3,4\}} = \{\{1, 8\}, \{2, 6\}, \{3, 5\}\}$: we transmit

\[
+ W_{\{1,8\}} + W_{\{2,6\}} + W_{\{3,5\}} \text{ to relay 1,}
+ W_{\{1,8\}} - W_{\{2,6\}} - W_{\{3,5\}} \text{ to relay 2,}
- W_{\{1,8\}} + W_{\{2,6\}} - W_{\{3,5\}} \text{ to relay 3,}
- W_{\{1,8\}} - W_{\{2,6\}} + W_{\{3,5\}} \text{ to relay 4.}
\]

Hence, each multicast message $W_J$ where $J \in \mathcal{T}_{\{1,2,3,4\}}$ can be recovered by the users in $J$. We proceed similarly to transmit the codewords for the multicast messages in $\mathcal{T}_B$ to the relays $\mathcal{B}$ such that each user in $J$ can recover $W_J$ where $J \in \mathcal{T}_B$.

The link load to transmit the multicast messages in $\mathcal{V}_2$ to each relay is $2/5$. Hence, the max-link load of this scheme is $3/5 + 2/5 = 1$, while that of the CICS (ICICS) and the schemes in [5, 17] are 1.05, 2.25, and 1.5, respectively. $\square$

In the final example, we generalize the IES to transmit $W_J$ where $J \in \mathcal{V}_2$ to any $r \geq 2$.

**Example 3** ($H \geq 2r, r > 2$). Consider the combination networks with end-user-caches where $H = 6$, $r = 3$, $K = N = 20$, and $M = 1$. Assume that $d = (1 : 20)$ so that

$\mathcal{U}_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$,
$\mathcal{U}_2 = \{1, 2, 3, 4, 11, 12, 13, 14, 15, 16\}$,
$\mathcal{U}_3 = \{1, 5, 6, 7, 11, 12, 13, 17, 18, 19\}$,
$\mathcal{U}_4 = \{2, 5, 8, 9, 11, 14, 15, 17, 18, 20\}$,
$\mathcal{U}_5 = \{3, 6, 8, 10, 12, 14, 16, 17, 19, 20\}$,
$\mathcal{U}_6 = \{4, 7, 9, 10, 13, 15, 16, 18, 19, 20\}$,
$\mathcal{V}_2 = \{1, 20\}$.

We focus on the transmission of the multicast messages in $\mathcal{V}_2$. If $H = 2r$ (the case $H > 2r$ can be dealt as in Example 2), there are $2^r - 1$ elements in $\mathcal{V}_2$. We partition $\mathcal{V}_2$ into $(2^r - 1)/2$ groups where each group contains $2r - 1 = 5$ elements and some constraint should be satisfied (to be clarified soon). The existence of such a partition is discussed in Appendix A. In this example the groups could be chosen as

$\mathcal{G}_1 = \{1, 20\}$,
$\mathcal{G}_2 = \{2, 19\}$,
$\mathcal{G}_3 = \{3, 18\}$,
$\mathcal{G}_4 = \{4, 17\}$,
$\mathcal{G}_5 = \{5, 16\}$,
$\mathcal{G}_6 = \{6, 15\}$,
$\mathcal{G}_7 = \{7, 14\}$,
$\mathcal{G}_8 = \{8, 13\}$,
$\mathcal{G}_9 = \{9, 12\}$,
$\mathcal{G}_{10} = \{10, 11\}$.

Recall that each multicast message $W_J$ where $J \in \mathcal{G}_1$ is useful to the users in $J$ and is interference to the users in $\cup_{J_i \in \mathcal{G}_1 \setminus J \neq J_i}$. Our objective is to eliminate the interference caused by $W_J$ to those users who do not need it. To achieve our objective, we construct the linear code as follows.

We focus on $W_{\{1,20\}}$ (assumed to be the first element in the vector $\mathcal{M}_1$). Firstly, on some finite field, we let

$$\sum_{i \in [6]} a_{1,i,1} = 0.$$  \hfill (25a)

The reason for the choice in (25a) will become clear later. Then, for each $J_i \in \mathcal{G}_1 \setminus \{\{1, 20\}\}$ we eliminate the interference caused by $W_{\{1,20\}}$ to users in $J_i = \{k_1, k_2\}$ (where $k_1$ is the user connected to relay 6) by letting \footnote{For each pair of users $J_i = \{k_1, k_2\}$, we can choose any one of them (denoted by $k'$) and write the equation $\sum_{i \in \mathcal{G}_1} a_{1,i,1} = 0$ which can also lead to $\sum_{i \in \mathcal{G}_1} a_{1,i,1} = 0$. Thus, the interference caused by $W_{\{1,20\}}$ to these two users is eliminated. Here we choose the user connected to relay 6, i.e., $k' = k_1$.}

$$\sum_{i \in \mathcal{H}_{k_1}} a_{1,i,1} = 0.$$  \hfill (25b)

It can be seen that each set $J \in \mathcal{V}$ contains two users $k_1$ and $k_2$ for which $\mathcal{H}_{k_1} \cap \mathcal{H}_{k_2} = \emptyset$ and $\mathcal{H}_{k_1} \cup \mathcal{H}_{k_2} = \{H\}$, therefore, from (25a) and (25b), we have

$$\sum_{i \in \mathcal{H}_{k_2}} a_{1,i,1} = 0.$$  \hfill (25c)

Hence, if users $k_1$ and $k_2$ sum their received codewords from their connected relays, they can eliminate the interference caused by $W_{\{1,20\}}$. This construction is repeated for all $J_i \in \mathcal{G}_1 \setminus \{\{1, 20\}\}$. Last, as user 20 requires $W_{\{1,20\}}$, we let

$$\sum_{i \in \mathcal{H}_{20}} a_{1,i,1} = s : s \neq 0.$$  \hfill (25d)

As $\mathcal{H}_{1} \cap \mathcal{H}_{20} = \emptyset$ and $\mathcal{H}_{1} \cup \mathcal{H}_{20} = \{H\}$, from (25a) and (25d), it can be seen that

$$\sum_{i \in \mathcal{H}_{1}} a_{1,i,1} = -s \neq 0.$$  \hfill (25e)

Hence, if user 1 and 20 sum their received codewords from their connected relays, they can recover $W_{\{1,20\}}$.

To summarize, by collecting all the constraint in (25) we obtain the following system of equations to solve

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
a_{1,1,1} \\
a_{1,2,1} \\
a_{1,3,1} \\
a_{1,4,1} \\
a_{1,5,1} \\
a_{1,6,1} \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
s \\
\end{bmatrix}.$$  \hfill (26)

So the group division must satisfy the constraint that the first matrix in (26) is full-rank. Thus if we let for example $s = -3$, we have $a_{1,1,1}; a_{1,2,1}; a_{1,3,1}; a_{1,4,1}; a_{1,5,1}; a_{1,6,1} = [1; 2; 3; 1; 2; 1; 4; 1; 5; 1; 6; 1]$. Similarly, we can get all the elements in $\mathcal{A}_1$.\footnote{For each pair of users $J_i = \{k_1, k_2\}$, we can choose any one of them (denoted by $k'$) and write the equation $\sum_{i \in \mathcal{G}_1} a_{1,i,1} = 0$ which can also lead to $\sum_{i \in \mathcal{G}_1} a_{1,i,1} = 0$. Thus, the interference caused by $W_{\{1,20\}}$ to these two users is eliminated. Here we choose the user connected to relay 6, i.e., $k' = k_1$.}
Hence, for all the multicast messages $W_{\mathcal{J}}$ where $\mathcal{J} \in \mathcal{G}_1$, we transmit (note that the following operations can be done on a finite field with size larger than 7)

$$
\begin{bmatrix}
X_{1,1} & X_{2,1} & X_{3,1} & X_{4,1} & X_{5,1} & X_{6,1} \\
1 & +1 & -2 & +1 & -2 & -2 \\
1 & +1 & +1 & +1 & +1 & -2 \\
-2 & +1 & -1 & -2 & +1 & +1 \\
+1 & +1 & -2 & +1 & +1 & +1 \\
+1 & +1 & +1 & -2 & -2 & +1 \\
1 & +1 & +1 & -2 & +1 & +1 \\
+1 & +1 & +1 & -2 & +1 & -1 \\
\end{bmatrix}
\begin{bmatrix}
W_{(1,20)} \\
W_{(2,19)} \\
W_{(3,18)} \\
W_{(5,16)} \\
W_{(7,14)} \\
\end{bmatrix}.
$$

Similarly, for all the multicast messages $W_{\mathcal{J}}$ where $\mathcal{J} \in \mathcal{G}_2$, we transmit

$$
\begin{bmatrix}
X_{1,2} & X_{2,2} & X_{3,2} & X_{4,2} & X_{5,2} & X_{6,2} \\
-2 & +1 & +1 & +1 & -2 & +1 \\
+1 & -2 & +1 & -1 & -2 & +1 \\
+1 & +1 & -2 & +1 & +1 & +1 \\
+1 & +1 & +1 & -2 & -2 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & +1 & +1 & +1 \\
+1 & +1 & +1 & -2 & +1 & +1 \\
\end{bmatrix}
\begin{bmatrix}
W_{(4,17)} \\
W_{(6,15)} \\
W_{(8,13)} \\
W_{(9,12)} \\
W_{(10,11)} \\
\end{bmatrix}.
$$

Each user $k \in \mathcal{J}$ can recover $W_{\mathcal{J}}$ where $\mathcal{J} \in \mathcal{G}_u$, $u \in \{1, 1\}$, by summing the received codewords (corresponding to $\mathcal{G}_u$) from the relays in $\mathcal{H}_k$.

We can now compute the max-link load. The link load to transmit the multicast messages in $\mathcal{V}_1$ to each relay is $3/2$. The link load to transmit the multicast messages in $\mathcal{V}_2$ to each relay is $1/10$. Thus, the max-link load of this scheme is $3/2 + 1/10 = 8/5$ coinciding with the improved converse bound under the constraint of uncoded placement in Theorem 5 while that of the CICS (ICICS) and the schemes in [3, 17] are 1.6111, 19/6 ≈ 3.1667, and 29/12 ≈ 2.41667, respectively.

We now generalize the scheme to transmit the messages $W_{\mathcal{J}}$ where $\mathcal{J} \in \mathcal{V}_2$, described in the above examples to the general case of $t = 1$ and $H \geq 2r$. We use the MAN cache placement and let $W_{\mathcal{J}} = \bigoplus_{j \in \mathcal{J}} F_{d_j, \mathcal{J} \setminus \{j\}}$ for each $\mathcal{J} \subseteq [K]$ where $|\mathcal{J}| = 2$. Notice that since $t = 1$, $\mathcal{V}_2$ contains all the sets of two users $k$ and $k'$ where $\mathcal{H}_k \cap \mathcal{H}_{k'} = \emptyset$.

a) Transmission of $\mathcal{V}_2$ for $H = 2r$: If $H = 2r$, we have $|\mathcal{V}_2| = \binom{2r}{r}/2 = \binom{2r-1}{r-1}$. We partition $\mathcal{V}_2$ into $\binom{2r-1}{r-1}/(2r-1)$ groups, each containing $2r-1$ elements (the group division is explained in Appendix A).

For each group $\mathcal{G}_g$, $g \in \left\{\binom{2r-1}{r-1}/(2r-1)\right\}$, the vector $M_g$ of dimension $(2r-1) \times 1$ lists the multicast messages $W_{\mathcal{J}}$ where $\mathcal{J} \in \mathcal{G}_g$. Let $m_{g,j}$ denote the $j$th element of $M_g$. We design a linear code, on some finite field of sufficiently large size, to transmit $M_g$, whose coding matrix is denoted by $A_g$. Let the element on $i$th row and $j$th column of $A_g$ be denoted by $a_{g,i,j}$, the $i$th row by $A_{g,i}$; the message $a_{g,i,j} \cdot M_g$ is transmitted to relay $i \in [H]$.

We construct the $j$th column of $A_g$, whose elements are the coefficients for the multicast messages $m_{g,j}$, as follows. First, we let

$$\sum_{i \in [H]} a_{g,i,j} = 0. \tag{27a}$$

Then, for each $\mathcal{J}_1 \in \mathcal{G}_g \setminus \{\mathcal{J}\}$, we let

$$\sum_{i \in \mathcal{H}_k} a_{g,i,j} = 0, \quad \text{where } k_1 \text{ is the user in } \mathcal{J}_1 \text{ connected to relay } H. \tag{27b}$$

where $k_1$ is the user in $\mathcal{J}_1$ connected to relay $H$. Assume that $k_2$ is the other user in $\mathcal{J}_1$, where $\mathcal{H}_{k_1} \cap \mathcal{H}_{k_2} = \emptyset$ and $\mathcal{H}_{k_1} \cup \mathcal{H}_{k_2} = [H]$, from (27a) and (27b) it can be seen that

$$\sum_{i \in \mathcal{H}_{k_2}} a_{g,i,j} = 0. \tag{27c}$$

Users $k_1$ and $k_2$ can eliminate the interference caused by $W_{\mathcal{J}}$ by summing their received codewords for $\mathcal{G}_g$ from their connected relays. Last, we let

$$\sum_{i \in \mathcal{H}_{k_3}} a_{g,i,j} = s, \quad \text{where } s \neq 0, \tag{27d}$$

where $k_3$ is the user in $\mathcal{J}$ who is connected to relay $H$. Assume $k_4$ is the other user in $\mathcal{J}$, where $\mathcal{H}_{k_3} \cap \mathcal{H}_{k_4} = \emptyset$ and $\mathcal{H}_{k_3} \cup \mathcal{H}_{k_4} = [H]$, from (27a) and (27d) it can be seen that

$$\sum_{i \in \mathcal{H}_{k_4}} a_{g,i,j} = -s. \tag{27e}$$

Hence, if the users in $\mathcal{J}$ sum their received codewords for $\mathcal{G}_g$ from their connected relays, they can recover $W_{\mathcal{J}}$.

By collecting all the equations in (27), we can write the following system of equations

$$C_g \times \begin{bmatrix} a_{g,1,j} \\ \vdots \\ a_{g,H-1,j} \\ a_{g,H,j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ s \end{bmatrix}, \quad j \in [2r-1]. \tag{28}$$

In Appendix A we show how the proposed group division results in a matrix $C_g$ that is full-rank. With this, we design the $j$th column of $A_g$, and then the whole matrix of $A_g$.

Finally, each user $k \in \mathcal{J}$ can recover $W_{\mathcal{J}}$ for $\mathcal{J} \in \mathcal{G}_g$ by summing the received codewords (corresponding to $\mathcal{G}_g$) from the relays in $\mathcal{H}_k$. Having recovered the multicast messages $W_{\mathcal{J}}$ where $k \in \mathcal{J}$, user $k$ then decodes $F_{d_k,\mathcal{J} \setminus \{k\}}$.

It can be seen that the link load to transmit the multicast messages in $\mathcal{V}_2$ to each relay is $\binom{2r-1}{r-1}/(2r-1)K$.

b) Transmission of $\mathcal{V}_2$ for $H > 2r$. For each set of relays $B$ where $|B| = 2r$, we find the users who are connected to $r$ of the chosen relays, and denote them by $P_B$. We can see that $|P_B| = \binom{2r}{r}$. It can be also seen that $|T_B| = \binom{2r}{r}/2 = \binom{2r-1}{r-1}$ where $T_B$ is defined in (23). So we can use the same scheme for the case $H = 2r$ to transmit the multicast messages in $T_B$ to each relay in $B$ with load $\binom{2r-1}{r-1}/(2r-1)K$. Each user $k \in P_B$ can recover $W_{\mathcal{J}}$ where $\mathcal{J} \in T_B$ and $k \in \mathcal{J}$. Notice that as $\mathcal{V}_2$ contains all the sets of two users $k$ and $k'$ where $\mathcal{H}_k \cap \mathcal{H}_{k'} = \emptyset$, we have $\bigcup_{B \subseteq [H]; |B|=2r} T_B = \mathcal{V}_2$. Hence, after considering all the sets of relay $B$ where $|B| = 2r$, each user $k$ can recover all the multicast messages $W_{\mathcal{J}}$ where $k \in \mathcal{J}$ and $\mathcal{J} \in \mathcal{V}_2$, and then decodes $F_{d_k,\mathcal{J} \setminus \{k\}}$. 

$$
\begin{bmatrix}
W_{(1,20)} \\
W_{(2,19)} \\
W_{(3,18)} \\
W_{(5,16)} \\
W_{(7,14)} \\
\end{bmatrix}.$$
**Performance:** The max-link load to transmit the multicast messages in $V_2$ is $\frac{2r-1}{(2r-1)K} \frac{H(K-1)}{2r-1}$. In Appendix A we show that when $2r-1 = p^q$, where $p$ and $q$ are as in the statement of Theorem 1, we can partition $V_2$ in such a way that the IES is doable. By summing the loads for the two classes of multicast messages, the achieved max-link load of the IES is given in [13].

**C. Concatenated Inner Code delivery Scheme (CICS)**

At a high level, the CICS contains two delivery steps. We directly transmit each MAN multicast message to some relays in the first step such that each $W_J$ can be recovered by a subset of users in $J$; these messages can be seen as side information for the second step. In the second step, we design linear combinations of the MAN multicast messages such that the remaining users in $J$ recover $W_J$. We illustrate this idea by means of one example first.

**Example 4.** Consider the combination networks with end-user-caches in Fig. 1 where $H = 4$, $r = 2$, $K = N = 6$, and $M = t = 3$. Let $d = (1 : 6)$. For each $J \subseteq [6]$ where $|J| = t + 1 = 4$, the MAN multicast message $W_J$ in [6] contain $B$ bits, because each file was split into $P(2) = 20$ equal-length parts. We now look in detail into the two-step delivery of the CICS.

**First step:** For each $J \subseteq [6]$ of size $|J| = 4$, we compute the set of relays each of which is connected to the largest number of users in $J$, $S_J := \arg \max_{h \in [H]} |U_h \cap J|$. Then partition $W_J$ into $|S_J|$ non-overlapping and equal-length pieces, $W_J = (W_{J|h})$, and transmit $W_{J|h}$ to each relay $h \in S_J$.

For example, for $J = \{1, 2, 3, 4\}$, relay 1 is connected to three users in $J$ (users 1, 2, and 3) and relay 2 is connected to two users in $J$ (users 1, 4), relay 3 is connected to two users in $J$ (users 2, 4), and relay 4 is connected to one user in $J$ (user 3). So we have $S_{\{1,2,3,4\}} = \arg \max_{h \in [H]} |U_h \cap J| = 1$. Therefore, we have $W_{\{1,2,3,4\}} = \{W_{\{1,2,3,4\},1}\}$, which contains $B/20$ bits.

As another example, for $J = \{1, 2, 5, 6\}$, relay 1 is connected to two users in $J$ (users 1 and 2), relay 2 is connected to two users in $J$ (users 1 and 5), relay 3 is connected to two users in $J$ (users 2 and 6), and relay 4 is connected to two users in $J$ (users 5 and 6). So we have $S_J = \arg \max_{h \in [H]} |U_h \cap J| = 4$. Therefore, we have $W_{\{1,2,5,6\}} = \{W_{\{1,2,5,6\},1}, W_{\{1,2,5,6\},2}, W_{\{1,2,5,6\},3}, W_{\{1,2,5,6\},4}\}$, where $W_{\{1,2,5,6\},i}$, $i \in [4]$, contains $B/80$ bits.

After considering all the sets $J \subseteq [K]$ where $|J| = 4$, we see that the server has sent to relay 1 (and similarly for all other relays) the following messages

$$
\begin{align*}
W_{\{1,2,3,4\},1}, & W_{\{1,2,3,5\},1}, W_{\{1,2,3,6\},1}, W_{\{1,2,5,6\},1}, \\
W_{\{3,4,5\},1}, & W_{\{3,4,6\},1}, W_{\{2,3,4,5\},1},
\end{align*}
$$

for a total of $3 \frac{B}{20} + 3 \frac{B}{80} = \frac{38B}{160}$ bits; these messages are then transmitted by relay 1 to the users in $U_1$.

**Second step:** Recall that $W_{\{1,2,3,4\}} = \{W_{\{1,2,3,4\},1}\}$. In the first step, $W_{\{1,2,3,4\},1}$ is transmitted to relay 1 and can be recovered by users 1, 2, 3. In the second step, we aim to transmit $W_{\{1,2,3,4\},1}$ to user 4, while considering it as the side information for users 1, 2, 3. For user 4, we partition $W_{\{1,2,3,4\},1}$ into $r = 2$ equal-length parts and denote $W_{\{1,2,3,4\},1} = \{W_{\{1,2,3,4\},1}, h \in H_4\}$. We then let user 4 recover the first part $W_{\{1,2,3,4\},1,2}$ from relay 2, and the second part $W_{\{1,2,3,4\},1,3}$ from relay 3. As user 1 connected to relay 2 knows $W_{\{1,2,3,4\},1,2}$, we assign $W_{\{1,2,3,4\},1,2} \rightarrow \mathcal{P}_{2,1}(1)$, representing the set of bits needed to be recovered by user 4 (the first entry in the subscript) from relay 2 (the superscript) and already known by user 1 (the second entry in the subscript) who is also connected to relay 2. As user 2 connected to relay 3 knows $W_{\{1,2,3,4\},1,3}$, we assign $W_{\{1,2,3,4\},1,3} \rightarrow \mathcal{P}_{3,2}(1)$, representing the set of bits needed to be recovered by user 4 (the first entry in the subscript) from relay 3 (the superscript) and already known by user 2 (the second entry in the subscript) who is also connected to relay 3.

Recall that $W_{\{1,2,5,6\}} = \{W_{\{1,2,5,6\},1}, W_{\{1,2,5,6\},2}, W_{\{1,2,5,6\},3}, W_{\{1,2,5,6\},4}\}$. Let us focus on $W_{\{1,2,5,6\},1}$ which is directly transmitted to relay 1 in the first step and can be recovered by users 1, 2, 3. In the second step, we aim to transmit $W_{\{1,2,5,6\},1}$ to users 5, 6, while considering it as the side information for users 1, 2, 3. For user 5, we partition $W_{\{1,2,5,6\},1}$ into $r = 2$ equal-length parts and denote $W_{\{1,2,5,6\},1} = \{W_{\{1,2,5,6\},1,k} : k \in H_5\}$ where $H_5 = \{2, 4\}$. We then let user 5 recover the first part $W_{\{1,2,5,6\},1,2}$ from relay 2, and the second part $W_{\{1,2,5,6\},1,3}$ from relay 3. As user 1 connected to relay 2 knows $W_{\{1,2,5,6\},1,2}$, we assign $W_{\{1,2,5,6\},1,2} \rightarrow \mathcal{P}_{2,1}(5)$. As user 3 connected to relay 4 knows $W_{\{1,2,5,6\},1,3}$, we assign $W_{\{1,2,5,6\},1,3} \rightarrow \mathcal{P}_{3,2}(5)$. For user 6, we partition $W_{\{1,2,5,6\},1}$ into $r = 2$ equal-length parts (i.e., $W_{\{1,2,5,6\},1} = \{W_{\{1,2,5,6\},1,k} : k \in H_6\}$), and assign $W_{\{1,2,5,6\},1,3} \rightarrow \mathcal{P}_{3,2}(6), W_{\{1,2,5,6\},1,4} \rightarrow \mathcal{P}_{3,3}(6)$. We perform the same procedure for the other pieces in $W_{\{1,2,5,6\}}$.

After considering all the pieces of the multicast messages $W_J$, for relay 1 we have (and similarly for all other relays)

$$
\begin{align*}
\mathcal{P}_{1,2}(1) = & \{W_{\{1,2,4,6\},1,1}', W_{\{1,2,5,6\},3,1}', W_{\{1,3,4,6\},3,1}'\}, \\
\mathcal{P}_{1,3}(3) = & \{W_{\{1,3,5,6\},1,1}', W_{\{1,2,5,6\},4,1}', W_{\{1,3,4,6\},4,1}'\}, \\
\mathcal{P}_{2,1}(1) = & \{W_{\{1,2,4,5\},1,1}', W_{\{1,2,5,6\},2,1}', W_{\{2,3,4,5\},2,1}'\}, \\
\mathcal{P}_{2,3}(3) = & \{W_{\{1,2,3,5\},1,1}', W_{\{1,2,5,6\},4,1}', W_{\{2,3,4,5\},4,1}'\}, \\
\mathcal{P}_{3,1}(1) = & \{W_{\{1,3,4,5\},1,1}', W_{\{1,3,4,6\},2,1}', W_{\{2,3,4,5\},2,1}'\}, \\
\mathcal{P}_{3,2}(3) = & \{W_{\{1,3,4,5\},1,1}', W_{\{1,3,4,6\},3,1}', W_{\{2,3,4,5\},3,1}'\},
\end{align*}
$$

all such $\mathcal{P}$’s contains $3B/80$ bits. Finally the server transmits to relay 1

$$
\begin{align*}
\mathcal{P}_{1,2}(1) \oplus \mathcal{P}_{2,1}(1), & \mathcal{P}_{1,3}(3) \oplus \mathcal{P}_{3,1}(1), \mathcal{P}_{2,3}(3) \oplus \mathcal{P}_{3,2}(3),
\end{align*}
$$
be XORed with the other \( r - 1 \) pieces with the same length (each of which is demanded by one user in \( Q^k_{h,h'} \)) and then be transmitted to relay \( h' \).

**Performance:** For each MAN multicast message \( W_j \), the server directly transmits \( W^{S_j}_{h,h'} \) to relay \( h \in S_j \) in the first step for a total of \( |W_j| = B/|J| \) bits. In the second step, for relay \( h \in S_j \), \( |J \cap U_h| \) users should recover \( W^{S_j}_{h,h'} \). For this purpose, the server transmits \( W^{S_j}_{h,h'} \) to each user \( k \in J \setminus U_h \) for \( h' \in H_k \) in one linear combination with other \( r - 1 \) pieces of the same length (equal to \( B/|J \cap U_h| \)). Hence, the total link load to transmit \( W_j \) is \( 1 + |J \cap U_h|/r \). By the symmetry of network, the number of transmitted bits to each relay is the same as in (14).

### D. Improved Concatenated Inner Code delivery Scheme (ICICS)

The main idea in the ICICS is to leverage the multicasting opportunities which are ignored in the CICS. We also consider the network in Example 4 to highlight the improvement provided by the ICICS compared to the CICS.

**Example 5.** Consider the same network as in Example 4 where \( H = 4, r = 2, K = N = 6, \) and \( M = 3 \). We also let \( d = (1 : 6) \). The MAN cache placement and the MAN multicast messages generation are used, such that each MAN multicast message in (9) contains \( B/20 \) bits. As the CICS, the delivery of the ICICS also contains two steps.

**Improved first step of the ICICS:** This step is the same as the CICS with the exception that, each MAN multicast message \( W_j \) is divided into \( |W_j|/P \) packets where each packet contains \( P \) bits, for some large enough length \( P \), which is possible since \( B \) can be taken arbitrary large and where \( P \) is such that the resulting packets can be seen as elements on a finite field of suitable size.

**Improved second step of the ICICS:** In the CICS, recall that both \( W^{1.2.5.6}_{1.2.5.6.4.1} \) and \( W^{4.1}_{1.2.5.6.4.1} \) are from \( W^{1.2.5.6.4} \) and should be transmitted to relay 4. It can be seen that the CICS treats \( W^{1.2.5.6.4} \) and \( W^{4.1}_{1.2.5.6.4.1} \) (demanded by user 1 and 2, respectively) as two independent pieces. More precisely, the CICS transmits \( W^{4.1}_{1.2.5.6.4.1} \) in one XORRed message to relay 1 to let user 1 recover; transmits \( W^{4.2}_{1.2.5.6.4.1} \) in another XORRed message to relay 1 to let user 2 recover.

Instead, we can leverage the following multicasting opportunity. We put \( RLC(W^{1.2.5.6.4}_{1.2.5.6.4})(/2P) \) in \( X^{1.2.5.6.4}_{1.2.3} \), where \( X^{1.2.3} \) is the set of packets needed to be recovered by users in \( \{1,2\} \) (first part of the subscript) from relay 1 (superscript) and already known by the users in \( \{3\} \) (second part of the subscript) who are also connected to relay 1 (superscript). The number of packets in \( X^{1.2.3} \) is \( |X^{1.2.3}| = 1 \). We then encode the messages at relay 1 as

\[
X^{1.2.3} = X^{1.2.3} \oplus X^{1,2}(1) \oplus X^{1.2}(3) \oplus X^{1.3}(1) \oplus X^{1.3}(2) \oplus X^{1.3}(3) \oplus X^{1.3}(2) \oplus X^{1,3}(2) \oplus X^{1,3}(3)
\]

where we used the same convention as before when it comes to ‘summing’ sets. We also send

\[
RLC(2[X^{1.2,3}(1)] / P, X^{1.2,3}(3) \cup X^{1.3,2}(2) \cup X^{1.2,3}(1))
\]
to relay 1. Note that the users in \{1, 2, 3\} know \(|X^1_{\{2,3\},\{1\}}|/P\) packets in \(X^1_{\{1,2\},\{3\}} \cup X^1_{\{1,3\},\{2\}} \cup X^1_{\{2,3\},\{1\}}\). So if the server transmits \(|X^2_{\{2,3\},\{1\}}|/P\) random linear combinations of those packets to relay 1, each user in \{1, 2, 3\} can recover all the packets in \(X^2_{\{1,2\},\{3\}} \cup X^2_{\{1,3\},\{2\}} \cup X^2_{\{2,3\},\{1\}}\) with high probability provided that \(B \rightarrow \infty\).

The max-link load of the ICICS is \(\frac{15}{4} + \frac{17}{8} = 2.9375\), which is less than the max-link load of the CICS (equal to 0.3); for the same set of parameters, the max-link loads of the schemes in [3], [17] are 0.375 and 1/3, respectively.

In general, the pseudo-code for this improved delivery can be found in Algorithm 1 next.

**Algorithm 1** Improved Concatenated Inner Code delivery Scheme (ICICS)

1. **input:** \(F_i, W\) where \(i \in [N]\), \(W \subseteq [K]\) and \(|W| = t\); \(c_{\text{init}}\) initialization: \(X^h_{W_1,W_2} = \emptyset\) for each \(h \in [H]\), \(W_1 \subseteq \mathcal{U}_h\) and \(W_2 \subseteq \mathcal{U}_h\ \setminus W_1\);
2. for each \(J \subseteq [K]\) where \(|J| = t + 1\),
   a) let \(W_J = \bigcap_{j \in J} F_{d_j,J} (j);\) divide \(W_J\) into \(|W_J|/P\) packets where each packet contains \(P\) bits, for some large enough length \(P\);
   b) \(S_J = \text{argmax}_{h \in [H]} |\mathcal{B}_h \cap J|;\) divide all packets in \(W_J\) into \([S_J]\) non-overlapping parts with equal length, \(W_J = \{W^j_{S_J,h} : h \in [H]\}\);
   c) for each \(h \in S_J,\)
      i) transmit \(W^j_{S_J,h}\) to relay \(h\);
      ii) for each \(h' \in [H]\) where \(U_{h'} \cap (J \setminus U_h) \neq \emptyset\), assign \(|W^j_{S_J,h'}|/(rP)\) random linear combinations of packets in \(W^j_{S_J,h}\) to \(A_{h} \cup (J \setminus U_h)\) and \(B = \{j \in \mathcal{U}_h \cap \mathcal{U}_{h'} : h \subseteq H_A \cup \{h\}\}\);
3. for each \(h \in [H]\),
   a) for each \(J' \subseteq \mathcal{U}_h\), let \(L^h_{J'} = \text{RLC}(c, C)\), where \(c = \max_{k \in J'} |X^h_{W_1,W_2}|\) and \(c' = \max_{k \in \mathcal{U}_h \cap J' \subseteq \mathcal{U}_h; L^h_{J'}\text{ is unknown to } k} \sum_{J' \subseteq \mathcal{U}_h} |L^h_{J'}|\);

**Remark 8.** We now analyze the actual file split level for our proposed schemes. In the IES, we need not to divide \(W_J\) in the DIS, we divide each \(W_J\) into \(t\) non-overlapping and equal-length pieces. In the CICS, we start by dividing \(W_J\) into \([S_J]\) non-overlapping and equal-length pieces and then divide each obtained piece into \(t\) non-overlapping and equal-length parts. Hence, \(W_J\) is divided into \([S_J]|t|\) pieces. In the ICICS, we first divide \(W_J\) into \([S_J]\) non-overlapping and equal-length pieces and then divide each obtained piece into \(t\) non-overlapping and equal-length parts. Hence, \(W_J\) is divided into at most \(|H| - 1\) non-overlapping and equal-length parts.

V. PROOFS OF THE NOVEL CONVERSE BOUNDS UNDER THE CONSTRAINT OF UNCODED CACHE PLACEMENT

In the rest of the paper, for a set of subfiles \(S \subseteq T_{A,Z}\) where \(T_{A,Z}\) is given in (7), we denote by \(H(S)\) the joint entropy of the subfiles in \(S\), and by \(H(Y|S^c)\) the entropy of a random variable \(Y\) conditioned on the subfiles in \(S^c := T_{A,Z} \setminus S\)
A. Preliminaries

We start this section by extending the “acyclic index coding converse bound” for shared-link network from [10] to combination networks.

**Proposition 1.** Consider a combination network with end-user-caches, where the cache placement $Z$ is uncoded and the demands in $d$ are distinct. For a set of relays $\mathcal{J} \subseteq [H]$, and for an acyclic set of subfiles $S \subseteq \mathcal{T}_{d,Z}$ in the directed graph $G_{\mathcal{T}_{d,Z}}$ that are demanded by the users in $K_{\mathcal{J}}$, the following must hold

$$H(S) \leq H(X_{\mathcal{J}}|S^c) + B_B \varepsilon_B + \varepsilon_B. \tag{30a}$$

$$\leq |\mathcal{J}| R^*_B \varepsilon_B + B_B. \tag{30b}$$

**Proof:** The entropy of the subfiles in $S$ is bounded as

$$H(S) = H(S|S^c) = H(X_{\mathcal{J}}|S) \leq H(X_{\mathcal{J}}|S^c) + H(S|X_{\mathcal{J}}, S^c) \leq H(X_{\mathcal{J}}|S^c) + B_B \varepsilon_B \tag{31b}$$

where in (31b) we use the independence of the subfiles and the fact that $X_{\mathcal{J}}$ is function of $T_{d,Z}$, in (31d) we use the fact that $S$ is acyclic and Fano’s inequality (where $\lim_{B \rightarrow \infty} \varepsilon_B = 0$), and in (31f) we use the definition of $R^*_B$.

Proposition 1 may not be tight when $|\mathcal{J}| R^*_B$ in (31f) is strictly larger than $H(X_{\mathcal{J}}|S^c)$ in (31d). In the following, we tighten the bound in Proposition 1.

**Proposition 2.** Consider a combination network with end-user-caches, where the cache placement $Z$ is uncoded and the demands in $d$ are distinct. For a set of relays $\mathcal{J} \subseteq [H]$, and for two sets of subfiles $S_0, S_2 \subset \mathcal{T}_{d,Z}$ that are acyclic in the graph $G_{\mathcal{T}_{d,Z}}$, where $S_1$ includes some subfiles demanded by the users in $K_{\mathcal{J}}$ and $S_2$ includes some subfiles demanded by the users in $[K] \setminus K_{\mathcal{J}}$ but not cached by the users in $K_{\mathcal{J}}$, we have

$$H(S_1) + H(X_{\mathcal{J}}|S_2^c) \leq |\mathcal{J}| R^*_B \varepsilon_B + 2B_B. \tag{32}$$

**Proof:** The entropy of the subfiles can be bounded as

$$H(S_1, S_2) \leq H(X_{[H]}|S_1 \cup S_2^c) + B_B \varepsilon_B \tag{33a}$$

$$= H(X_{\mathcal{J}}|S_1 \cup S_2^c) + H(X_{[H]} \setminus X_{\mathcal{J}}|S_1 \cup S_2^c) + B_B \varepsilon_B \tag{33b}$$

$$\leq |\mathcal{J}| R^*_B + H(X_{[H]} \setminus X_{\mathcal{J}}|S_1 \cup S_2^c) + B_B \varepsilon_B \tag{33c}$$

where (33a) is from (30b); (33d) is from Fano’s inequality (where $\lim_{B \rightarrow \infty} \varepsilon_B = 0$), and from the fact that $S_1$ is acyclic and $S_2$ does not include the side information of the user requesting $S_1$; (33e) is because

$$H(S_2) = H(S_2|S_2^c)$$

This concludes the proof.

Finally, we generalize the well-known sub-modularity of entropy.

**Proposition 3.** Let $\mathcal{Y}$ be a set of random variables, and $M$ be a set of mutually independent random variables (but not necessary independent of $\mathcal{Y}$). If $Y_1, Y_2 \subseteq \mathcal{Y}$ and $M_1, M_2 \subseteq M$, then the following must hold

$$H(Y_1|M_1) + H(Y_2|M_2) \geq H(Y_1 \cup Y_2|M_1 \cup M_2) + H(Y_1 \cap Y_2|M_1 \cap M_2). \tag{34}$$

**Proof:** Without loss of generality, assume $M_1 = \{M_0, M_1\}$ and $M_2 = \{M_0, M_2\}$, where $M_0, M_1, M_2$ are independent random variables. We have

$$H(Y_1|M_0, M_1) + H(Y_2|M_0, M_2) \geq H(Y_1 \cup Y_2|M_0, M_1, M_2) + H(Y_1 \cap Y_2|M_0, M_1, M_2) \tag{35a}$$

where (35c) follows from

$$I(Y_1; Y_2|M_0, M_1, M_2) + I(Y_1; Y_2|M_0, M_1) + I(Y_2; M_1|M_0, M_2) \leq I(Y_1; Y_2|M_0, M_1, M_2) + I(Y_2; M_1|M_0, M_2) \tag{35b}$$

by [10] Lemma 1] the set $f(d, S', v)$ forms an acyclic set in the directed graph $G_{\mathcal{T}_{d,Z}}$. Fix one $Q \subseteq [H]$ with $|Q| \in \{r : H\}$, and one permutation $\mathcal{P}(Q)$. For each demand vector $d$
with distinct demands, Proposition 1 with $J = Q$ and $S = f(d, |K|, p(K_Q))$ provides a converse bound on $R^*_1$. By summing all the so obtained bounds for the fixed $Q$ and $p(K_Q)$, we arrive at (18a).

C. First example on how to improve Theorem 4

Our first improvement to Theorem 4 is explained by way of an example – see also Remark 10. Consider the combination network with end-user-caches in Fig. 1 where $H = 4$, $r = 2$, $K = N = 6$, and $M = 2$. Consider the demand vector $d = (1, \ldots, 6)$. Choose a set of relays $Q$ and divide $Q$ into several disjoint subsets, each of which has a length not less than $r = 2$. In this example, we let $Q = [H] = [4]$ and divide $Q$ into $Q_1 = \{1,2\}$ and $Q_2 = \{3,4\}$; so $K_{Q_1} = \{1\}$ and $K_{Q_2} = \{6\}$. We then consider the three permutations $p(K_{Q_1}) = (1)$, $p(K_{Q_2}) = (6)$ and $p(K_{Q_1} \cup K_{Q_2}) = (2,3,4,5)$. Recall the definition of $f(\cdot, \cdot, \cdot)$ given in (37) and let

$$B_1 = f(d, |K|, p(K_{Q_1})) = \{F_{1,W} : W \subseteq [2 : 6]\}, \quad (38a)$$

$$B_2 = f(d, |K|, p(K_{Q_2})) = \{F_{6,W} : W \subseteq [1 : 5]\}, \quad (38b)$$

$$B_3 = f(d, |K| \setminus (K_{Q_1} \cup K_{Q_2}), p(K_{Q_1} \cup K_{Q_2})) = \{F_{i,W} : i \in [2 : 5], W \subseteq [i + 1 : 5]\}. \quad (38c)$$

By using Proposition 2 with $(J, S_1, S_2) = (Q_1, B_1, B_3)$ we get

$$H(B_1) \leq |Q_1|R^*_1B - H(X_{Q_1}|B_3^c) + 2B_{EB}. \quad (39)$$

and with $(J, S_1, S_2) = (Q_2, B_2, B_3)$ we get

$$H(B_2) \leq |Q_2|R^*_1B - H(X_{Q_2}|B_3^c) + 2B_{EB}. \quad (40)$$

We sum (39) and (40) to obtain

$$H(B_1, B_2) \leq |Q|R^*_1B - \left[H(X_{Q_1}|B_3^c) + H(X_{Q_2}|B_3^c)\right] + 4B_{EB} \quad (41a)$$

$$\leq |Q|R^*_1B - H(X_Q|B_3^c) + 4B_{EB} \quad (41b)$$

$$\leq |Q|R^*_1B - H(B_3) + 5B_{EB}, \quad (41c)$$

where (41c) follows from (38a). With the above mentioned choice of permutations and $B \gg 1$, the bound in (41c) becomes

$$4R^*_u \geq \sum_{W \subseteq [2 : 6]} |F_{1,W}|/B + \sum_{W \subseteq [1 : 5]} |F_{6,W}|/B$$

$$+ \sum_{i \in [2 : 5]} \sum_{W \subseteq [i + 1 : 5]} |F_{i,W}|/B. \quad (42)$$

If we list all the inequalities in the form of (42) for all the possible demands where users request distinct files, and sum them all together, then we obtain (the definition of $x_W$ is in (18b))

$$4R^*_u \geq \sum_{W \subseteq [2 : 6]} x_W + \sum_{W \subseteq [1 : 5]} x_W + \sum_{i \in [2 : 5]} \sum_{W \subseteq [i + 1 : 5]} x_W. \quad (43)$$

We then consider all the possible disjoint partitions of $Q$, and for each partition we consider all the possible combinations of permutations to write bounds as in the form of (43).

For $Q$ with $|Q| \leq 3$, as $Q$ can not be divided into two sets each of which has length not less than $r = 2$, we directly use the bound in (18a). With the file length and memory size constraints in (18c) and (18d), we can compute the converse bound by a linear programming with the above mentioned constraints and with variables $(R^*_u, x_W : W \subseteq [K] = [6])$.

By solving the linear programming numerically, the converse bound on $R^*_u$ given by the above method is $7/17 \approx 0.411$, while Theorem 4 gives $9/23 \approx 0.391$.

Remark 10. Notice that in (11c), $|Q|R^*_uB \geq H(B_1, B_2, B_3)$ where $B_1 \cup B_2$ forms a directed circle. The technique in this example provides a tighter converse bound compared to Theorem 4 because it allows to deal with cycles in the directed graph that represents the equivalent index coding problem.

D. Second example on how to improve Theorem 4

Our second improvement to Theorem 4 is explained by way of an example – see also Remark 11. For a set $S$ and a vector $p$, where each element of $S$ is also an element in $p$, we define $g(S, p)$ as the vector obtained by removing the elements not in $S$ from $p$, e.g., $g(\{1,2,3\}, (2,4,1,3)) = (2,1,3)$.

Consider the combination network with end-user-caches in Fig. 1 where $H = 4$, $r = 2$, $K = N = 6$, and $M = 2$. Consider the demand vector $d = (1, \ldots, 6)$. For an integer $b \in [r : H] = [2 : 4]$, e.g., say $b = 3$, consider each set of relays with cardinality $b$. Consider a permutation $p_{K_Q}$ and a permutation $p([K])$. We apply Proposition 2 with $J = Q$ so as to obtain

$$|Q|BR^*_u \geq H(X_Q) \geq H(S_1) + H(X_Q|S_2^c) + 2B_{EB}. \quad (44a)$$

$$S_1 = f(d, |K|, p(K_Q)), \quad (44b)$$

$$S_2 = S_Q := f(d, |K| \setminus K_Q, g([K] \setminus K_Q, p([K]))). \quad (44c)$$

If $Q = \{1,2,3\}$ and thus $K_Q = \{1,2,4\}$, we have

$$g([K] \setminus K_Q, p([K])) = g(\{3,5,6\}, \{1,2,4\}) = (3,5,6),$$

$$H(X_Q|S_2^c) = H\left(X_{\{1,2,3\}}|f(d, \{3,5,6\}, \{3,5,6\})\right).$$

We sum all the inequalities in the form of (44a) for all the possible demands where the users request distinct files. With $B \gg 1$, we have

$$|Q|BR^*_u \geq \sum_{j \in [|K_Q|]} \sum_{W \subseteq [j \cup K \setminus \{p_K(Q_Q)\}]} x_W + y_{Q_p([K])}, \quad (45a)$$

$$y_{Q_p([K])} := \frac{1}{BK!} \sum_{d,d_i \neq d_j, i \neq j} H(X_Q|S_2^c), \text{ with } S_Q \text{ in (44c).} \quad (45b)$$

For $Q = \{1,2,3\}$ and $Q = \{1,2,4\}$, we have

$$H(X_{\{1,2,3\}}|A^c) + H(X_{\{1,2,4\}}|B^c) \geq H(X_{\{1,2,3,4\}}|\{F_{6,8}\}^c) + H(X_{\{1,2\}}|A^c \cap B^c),$$

$$\geq H(F_{6,8}) + H\left(X_{\{1,2\}}|A^c \cap B^c, A := f(d, \{3,5,6\}, (3,5,6))\right), \quad (46a)$$

$$\geq H(F_{6,8}) + H\left(X_{\{1,2\}}|A^c \cap B^c, A := f(d, \{3,5,6\}, (3,5,6))\right). \quad (46b)$$
where to get (46a) we used Proposition 3 and the fact that $A^e \cup B^e = \{F_6,\theta\}^e$. Notice that using Proposition 3 we can not bound the sum of the two terms in the LHS (left hand side) of (46a). By using Proposition 3 we have the term $\sum_{\{F_6,\theta\}} H(X_{\theta} | \{(B_{a+1} \cup S_Q)^c\})$ and all the relays connected to user 6 demanding $F_{\theta,6}$ are in $\{1,2,3,4\}$, such that we can use Proposition 3 to bound this term by $H(F_{\theta,6})$. Similarly,

$$ \sum_{Q \subseteq [H]:|Q| = b} y_{Q,p([K])} \geq \frac{1}{K!B} \sum_{d_i : d_i \neq d_j, i \neq j} H(F_{1,\ldots,\theta,6}) = 6x_{\theta}. \quad (47) $$

We then consider each permutation $p(K_Q)$ for $Q \subseteq [K]$ with $|Q| = b = 3$ to write inequalities in the form of (45a). With the constraints in (18c), (18d) and (47) we can compute a converse bound on $R_B^*$ by solving a linear programming which gives 13/12, while Theorem 4 gives 17/16. Notice that in general we should consider each permutation $p([K])$ to write constraints in the form of (47), but in this example it is enough to consider one permutation. In order to reduce the number of variables, the constraint in (45a) is equivalent to the following

$$ |Q| R_{a}^* \geq \sum_{j \in [K|Q|]} \sum_{W \subseteq [K]\{j\} \cup \{a\}} x_W + y_{Q}, \quad (48a) $$

$$ y_Q := \max_{p([K])} y_{Q,p([K])}, \quad (48b) $$

satisfying for each permutation $p([K])$,

$$ \sum_{Q \subseteq [H]:|Q| = b} y_Q \geq \sum_{Q \subseteq [H]:|Q| = b} y_{Q,p([K])}, \quad (49) $$

where $y_{Q,p([K])}$ is defined in (45b).

**Remark 11.** In Theorem 7 for each set $Q$ we have the constraint in (48a) but without $y_Q$. The above example shows that the sum of all the $y_Q$'s, where $|Q| = b$, is positive, thus the converse bound in (48a) is tighter than that in Theorem 4.

**E. Proof of Theorem 5**

We now generalize and combine the two approaches illustrated in Sections 4.C and 4.D to improve the converse bound based on the acyclic index coding converse in Theorem 4.

We first focus on a permutation of users $p([K]),$ a subset of relays $Q \subseteq [H]$ (assume that $|Q| = b$). Consider a demand vector where users demand different files, and a partition $Q = Q_1 \cup \cdots \cup Q_r$. Let $p(K_Q_1), \ldots, p(K_{Q_r}), p(K_Q \setminus K_{Q_1} \cup \cdots \cup K_{Q_r})$ be a combination of permutations. We write $B_i := f(d, [K], p(K_{Q_i}))$ for $i \in [a]$; $B_{a+1} := f(d, [K], p(K_{Q_1} \cup \cdots \cup K_{Q_r}))$; $S_Q := f(d, [K], q([K] \setminus K_Q, p([K])))$. For each $i \in [a]$, we write an inequality in the form of (32) by $J = Q_i,

$$ S_1 = B_i, \quad S_2 = B_{a+1} \cup S_Q. \quad (50a) $$

The sum of all of the $a$ inequalities to obtain

$$ \sum_{i \in [a]} \sum_{y=1}^{2} H(B_i) \leq |Q| R_B^* - \sum_{i \in [a]} H(X_{Q_i}|(B_{a+1} \cup S_Q)^c) + 2aB \varepsilon_B \quad (50b) $$

$$ = |Q| R_B^* - H(X_{Q_i}|S_Q^c) + 2aB \varepsilon_B \quad (50c) $$

where from (50a) to (50b) the submodularity of entropy is used, and from (50b) to (50c) we use Fano's inequality and the fact that $B_{a+1}$ is acyclic and $S_Q$ does not include the side information of the user requiring $B_{a+1}$.

We list all the inequalities in the form of (50e) for all the possible demands where users demand different files, and sum them to obtain

$$ |Q| R_B^* \geq \sum_{i \in [a]} \sum_{y=1}^{2} \sum_{W \subseteq [K]\{a\}} x_W + y_Q, \quad (51) $$

From (51) and the definition $y_Q := \max_{p([K])} y_{Q,p([K])},$ we have

$$ |Q| R_B^* \geq \sum_{i \in [a]} \sum_{y=1}^{2} \sum_{W \subseteq [K]\{a\}} x_W + y_Q, \quad (52) $$

We then focus on the term $y_Q$. Choose an integer $b \in \{r\}$ and a permutation $p([K]).$ In order to prove (19b), we aim to bound

$$ \sum_{Q \subseteq [H]:|Q| = b} y_Q \geq \sum_{Q \subseteq [H]:|Q| = b} y_{Q,p([K])}, \quad (53) $$

Consider two sets of relays with cardinality $b$, $J_1$ and $J_2$. We can use Proposition 3 to bound

$$ H(X_{J_1} | S_{J_1}^c) + H(X_{J_2} | S_{J_2}^c) \geq \left( \sum_{J_1 \cup J_2} H(S_{J_1}^c, S_{J_2}^c) + H(X_{J_1 \cup J_2} (S_{J_1} \cup S_{J_2})^c) \right). \quad (54) $$

Each time we use Proposition 3 we call the first term in the RHS result as $U$-term and the second term as $\cap$-term.
We then add \( H(X_{J \cup} | S_{\mathcal{F}}) \) to the RHS of (54), where \( J_3 \) is the third term in the sum \( \sum_{Q \subseteq |H|} |Q| = b \cdot H(X_Q | S_{\mathcal{Q}}) \). We use Proposition 3 to bound the sum of the \( \cup \)-term of (54) and \( H(X_{J \cup} | S_{\mathcal{F}}) \). We put the \( \cup \)-term in the new RHS result at the first position, and use Proposition 3 again to bound the sum of the \( \cap \)-term in the RHS result of (54) and the \( \cap \)-term in the new RHS result. The \( \cup \)-term of the last one is put at the second position while the \( \cap \)-term is at the third position. So after considering three sets, we now have three terms. Similarly, each time we consider the \( j^\text{th} \) set with cardinality \( b \), we use Proposition 3 to bound the sum of the term in the first position of the last iteration and \( H(X_{J \cup} | S_{\mathcal{F}}) \). The \( \cup \)-term of the result is put at the first position in this iteration. The \( \cap \)-term of the result should be added to the term at the second position at the last iteration. We perform this procedure until the term in the last position at the last iteration. We describe the above iterative procedure in Algorithm 2.

Notice that when we apply Proposition 3 to bound a sum of two terms, the \( \cap \)-term of the result may be 0. When we use Proposition 3 to bound the sum of \( 0 \) and one term, the result is also the sum of this term (seen as the \( \cup \)-term) and 0 (seen as the \( \cap \)-term). We should also notice that after each iteration, by assuming that the term at the \( i^\text{th} \) position is \( H(X_{\mathcal{G}_{i+1}} | I_{i+1}) \) and the term at the \( i^\text{th} \) position is \( H(X_{\mathcal{G}_{i+1}} | I_{i+1}) \) where \( i_1 < i_2 \), we can see that \( \mathcal{G}_{i_2} \subseteq \mathcal{G}_{i_1} \) and \( T_{i_2} \subseteq T_{i_1} \) of the result.

Algorithm 2 Iterative Procedure by using Proposition 3

1) input: \( H(X_{J \cup} | S_{\mathcal{F}}) \) where \( i \in \left( \binom{H}{b} \right) \) (each \( J_i \) is a distinct set of relays with cardinality \( b \)); initialization: \( t = 2 \);
2) use Proposition 3 to bound \( H(X_{J_1} | S_{\mathcal{F}}) + H(X_{J_2} | S_{\mathcal{F}}) \); let \( L_{t,1} \) be the \( \cup \)-term and \( L_{t,2} \) be the \( \cap \)-term;
3) use Proposition 3 to bound \( L_{t,1} + H(X_{J_{t+1}} | S_{\mathcal{F}_{t+1}}) \); let \( L_{t+1,1} \) be the \( \cup \)-term and \( T_{\cap} \) be the \( \cap \)-term;
4) for \( i = 2, \ldots, t \), use Proposition 3 to bound \( L_{i,1} + T_{i,1} \) and let \( L_{t+1,1} \) be the \( \cup \)-term and \( T_{\cap} \) be the \( \cap \)-term;
5) let \( L_{t+1,1} = T_{\cap} \);
6) if \( i < \binom{H}{b} - 1 \), then \( t = t + 1 \) and go to 3);
7) output: \( \sum_{i \in \left( \binom{H}{b} \right)} L_{\binom{H}{b}} \).

After considering all the sets of relays with cardinality \( b \), we have a summation including \( \binom{H}{b} \) terms. In the end, for an acyclic set of subfiles \( \mathcal{S} \), by using Proposition 1 we have
\[
H(X_{[H]} | S_{\mathcal{F}}) \geq H(S).
\] (55)

Hence, we can bound this summation by a sum of the lengths of subfiles, then we obtain
\[
\sum_{Q \subseteq |H|} |Q| = b 
\geq \sum_{i \in |K|} \sum_{W \subseteq |K| \cup \cup \cup |p_i([K])|} c(|p_i([K]) | \cup |W, b | |F_{dp_i([K]), W}|,
\] (56)

where \( c(W_1, l) := \binom{H-1}{l-1} - |\{Q \subseteq |H| : |Q| = l, K_{\mathcal{Q}} \subseteq |K| \setminus W_1\}| \), which will be proved in the following. We focus on \( |F_{dp_i([K]), W}| \), where \( W \subseteq |K| \setminus \{p_1([K]), \ldots, p_i([K])\} \). For a set of relays \( Q \) with cardinality \( h \), in the term \( H(X_Q | S_{\mathcal{Q}}) \), we can see that \( F_{dp_i([K]), W} \subseteq \mathcal{Q} \) if and only if \( K_{\mathcal{Q}} \cap (\{p_i([K]) | \cup |W) = \emptyset \) (in other words, \( K_{\mathcal{Q}} \subseteq \{p_i([K]) | \cup |W) \)). Focus on one relay \( h \in |H| \). When we apply Proposition 3 to bound the sum of the \( \cup \)-term by the sum of two new terms, if among the two terms in the LHS of Proposition 3 one term includes \( X_h \) not knowing \( F_{dp_i([K]), W} \) and the other does not include \( X_h \) knowing \( F_{dp_i([K]), W} \), we can see that the number of terms in the RHS of Proposition 3 including \( X_h \) not knowing \( F_{dp_i([K]), W} \) decreases by 1 compared to the LHS (the number of terms in the RHS not including \( X_h \) but knowing \( F_{dp_i([K]), W} \) also decreases by 1 compared to the LHS); otherwise, the number of terms in the RHS including \( X_h \) not knowing \( F_{dp_i([K]), W} \) does not change compared to the LHS. In addition, it can be checked that in any case when we use Proposition 3 the number of terms in the RHS not including \( X_h \) but knowing \( F_{dp_i([K]), W} \) is not less than
\[
\max \left\{ |\{Q \subseteq |H| : |Q| = b, h \in |H|, K_Q \subseteq |K| \setminus (\{p_i([K]) | \cup |W)\}| \right\} 
\geq \max \left\{ \left[ \frac{H - 1}{l - 1} \right] - \left| \{Q \subseteq |H| : |Q| = l, K_Q \subseteq |K| \setminus W_1\} \right| \right\}. \tag{57}
\]

In addition, after the final iteration, the term at the \( i^\text{th} \) position is \( H(X_{\mathcal{G}_{i+1}} | I_{i+1}) \) and the term at the \( i^\text{th} \) position is \( H(X_{\mathcal{G}_{i+2}} | I_{i+2}) \) where \( i_1 < i_2 \), we can see that \( \mathcal{G}_{i+2} \subseteq \mathcal{G}_{i+1} \) and \( T_{i+2} \subseteq T_{i+1} \). So by (56) we have, \( c(W_1, l) = \binom{H-1}{l-1} - \left| \{Q \subseteq |H| : |Q| = l, K_Q \subseteq |K| \setminus W_1\} \right| \) as defined in (19).

Finally, from (53), (56), and the value of \( c(W_1, l) \), we obtain (19b). After proving (52) and (19b), the proof of Theorem 5 is completed.

Remark 12. If \( N < K \), we can not find a demand vector where each user has a distinct demand. In this case, one should consider many subsystems with only \( \min(N,K) = N \) users with distinct demands (as we did in [20] for shared-link model), which is not conceptually more difficult but requires a somewhat heavier notation.

Remark 13. It can be seen that the proposed strategies to tighten the acyclic index coding converse bound in combination networks do not rely on the symmetry of the network. Thus we can also use these strategies in general relay networks.

VI. CONCLUSIONS

In this paper we investigated the combination networks with end-user-caches. For the achievability part, we proposed four delivery schemes with the MAN cache placement to deliver the MAN multicast messages through the network. For the converse part, we extended the acyclic index coding converse bound for the shared-link model to combination networks and improved it by leveraging network topology. The proposed achievable and converse bounds were shown to be better than
the state-of-the-art bounds. In addition, for the combination networks with end-user-caches where \( N \geq K \), compared to the existing order optimality results for the high memory size regime, we additionally obtained the order optimality under the constraint of uncoded cache placement for the small memory size regime, such that the remaining open case is when \( \frac{M}{N} < \frac{1}{2\pi} \).

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APPENDIX A

DISCUSSION OF THE GROUP DIVISION OF THE INTERFERENCE ELIMINATION SCHEME

To get the coefficients \( \{a_{i,j,k}; \ldots ; a_{i,j,N}\} \) in equation (28), \( C_g \) should be full-rank for each group \( \mathcal{G}_g \). We should solve the following problem to ensure the feasibility where we introduce an integer \( k = r - 1 \) such that \( 2k + 2 = 2r \).

Problem 1: Let \( k \) be a positive integer. We focus on all the \((2k+1)\) subsets of \([2k+2]\) with cardinality \(k+1\) and \(2k+2\) is in each subset (because in (27b) and (27d) we only focus on the users connected to relay \( H \)). We want to divide these subsets into \((2k+1)/(2k+1)\) groups such that each group has \(2k+1\) subsets. For each group \( \mathcal{P}_i \), we create a \((2k+2) \times (2k+2)\) matrix. The first row is all 1. For each subset in this group, we have one row of 0 and 1, where the \( j^t \) element is 1 if and only if \( j \) is in this subset. The condition that a solution must satisfy is that each such matrix is full-rank. \(^8\)

In Appendix A we prove that \((2k+1)/(2k+1)\) is an integer if \( k \) is a positive integer. We provide the following algorithm to construct such groups for Problem 1, which is shown by numerical evaluations to find such groups when \( k \leq 12 \). When \( k > 12 \), this numerical simulation might be infeasible due to the complexity.

Algorithm 3 Group division method for Problem 1
1) input: \( k, \mathcal{P} = \{J \subseteq [2k+2] : |J| = k+1, (2k+2) \in J\}; initialization: \( t_0 = 0; \times times = 10; \mathcal{P}_i = \emptyset \) for \( i \in [(2k+1)/(2k+1)\]
2) for \( i \in [(2k+1)/(2k+1)\]
   a) Test = 0; randomly choose \( 2k+1 \) subsets in \( \mathcal{P} \); create a \((2k+2) \times (2k+2)\) matrix denoted by \( C \) (the first row of \( C \) is all 1 and for each chosen subset, there is one row of 0 and 1, where \( j^t \) element is 1 if and only if \( j \) is in \([2k+2]\) in this subset);
   b) if \( C \) is full-rank, then \( Test = 1 \) and put the chosen subsets in \( \mathcal{P}_i \);
   c) if \( Test = 0 \) and \( t_1 \leq times \), then \( t_1 = t_1 + 1 \) and go to Step 2-a;
   3) if \( \mathcal{P}_i \neq \emptyset \) for all \( i \in [(2k+1)/(2k+1)] \), then Output \( \mathcal{P}_i \) for all \( i \in [(2k+1)/(2k+1)] ; \) else, then go to Step 1;

We can use Algorithm 3 to construct the groups. However, it is hard to prove the existence of the group division satisfying the full-rank condition for the general case. Instead of proving the existence of solution for Problem 1, we introduce Problem 2, the existence of whose solution is easier to analyse. As the number \( 2k+2 \) appears in each subsets of Problem 1, we do not consider the number \( 2k+2 \) in Problem 2. In Appendix C we prove that the we can add \( 2k+2 \) into each subset in the solution of Problem 2 to get one solution of Problem 1.

Problem 2: Let \( k \) be a positive integer. We focus on all the \((2k+1)\) subsets of \([2k+1]\) with cardinality \( k \). We want to divide these subsets into \((2k+1)\)\((2k+1)\) groups such that each group has \(2k+1\) subsets. In each group, the number of subsets containing each number in \([2k+1]\) is the same (equal to \( k \)). We create a \((2k+1) \times (2k+1)\) matrix, called incident matrix. There is one row of 0 and 1 in the incident matrix corresponding to each subset in this group, where the \( j^t \) element in the row is 1 if and only if \( j \) is in this subset. The condition is that each incident matrix is full-rank.

Compared to Problem 1, Problem 2 has an additional constraint, which is that in each group, the number of subsets containing each number in \([2k+1]\) is the same. In Example 3 we have a group division satisfying Problem 1. In addition, if we take out the number \( 2k+2 \) = 6 in each subset, it is a solution for Problem 2. To analyse the existence, we review the following theorem given in \([23, \text{Theorem 1.1}]\).

Theorem 8 (22). Let \( k' \) and \( n' \) be positive integers, and let \( \lambda \) be the smallest non-trivial divisor of \( n' \). Then all the \((n')\) subsets of \([n']\) with cardinality \( k' \) could be divided into \((n')/n'\) non-overlapping groups, where each group includes \( n' \) subsets and its incident matrix is circulant, if and only if \( n' \) is relatively prime to \( k' \), \( \lambda k' > n' \) and \( n' \) divides \((n')\).

A circulant \( n' \times n' \) matrix is uniquely determined by its first row \([c_0,c_1,...,c_{n'-1}]\) and its \( j^t \)th row is obtained by shifting the first row rightwards by \( i - 1 \) where \( i \in [1 : n'] \). In Problem 2, \( n' = 2k+1 \) and \( k' = k \). It is easy to see that \( 2k+1 \) and \( k \) are relatively prime and that \( \lambda \geq 3 \), which leads \( \lambda k' > 2k+1 \). In addition, in Appendix B we prove that \( 2k+1 \) divides \((2k+1)\). Hence, if we choose \( n' = 2k+1 \) and \( k' = k ', \) the conditions in Theorem 8 are satisfied. As the incident matrix of each group is circulant, we can see that in each group, the number of subsets containing each number in \([2k+1]\) is the same. Hence, it remains to analyse the rank of each incident matrix. It was shown in [23, Theorems 1.7 and 1.8] that the following theorem holds.

Theorem 9. Let \( k \) be a positive integer. A \((2k+1) \times (2k+1)\) circulant matrix, where the number of 1 in the first row is \( k \) and the number of 0 in the first row is \( k+1 \), is always
invertible if $2k + 1 = p^v$ or $pq$, where $p, q$ are different primes and $v$ is a positive integer.

**APPENDIX B**

**PROOF:** $(2k^2 + 1)/(2k + 1)$ is an integer.

If $k$ is a positive integer, we have that

$$\frac{1}{2k + 1} \left( \frac{2k^2 + 1}{k} \right) = \frac{1}{k + 1} \left( \frac{2k^2}{k} \right) = \frac{k + 1 - k}{k + 1} \left( \frac{2k}{k} \right)$$

$$= \left( \frac{2k}{k} \right) - \frac{k}{k + 1} \left( \frac{2k}{k} \right) = \left( \frac{2k}{k} \right) - \left( \frac{2k}{k + 1} \right)$$

is an integer.

**APPENDIX C**

**PROOF:** A SOLUTION OF PROBLEM 2 IS A SOLUTION OF PROBLEM 1

For a solution of Problem 2, we focus on any group $g$ and assume that the $(2k + 1) \times (2k + 1)$ incident matrix is $B$. In the following, we will prove that the matrix $C_g$ is also full-rank, where $C_g = \begin{bmatrix} E^T & 1 \\ B & E \end{bmatrix}$ and $E = [1; \ldots; 1]$ with dimension $(2k + 1) \times 1$. $E^T$ is the transpose of $E$. It can be seen that if $C_g$ with dimension $(2k + 1) \times (2k + 2)$ is full-rank for each group $g$, we can add $2k + 2$ into each subset in the solution of Problem 2 to get one solution of Problem 1.

We prove it by contradiction, i.e., assume that $C_g$ is not full-rank. There must exist only one sequence $[a_1, \ldots, a_{2k+1}]$ such that $\sum_{i \in [2k+1]} a_i B_i = E^T$, where $B_i$ represents the $i$th row of $B$. In other words, we have

$$[a_1, \ldots, a_{2k+1}]B = E^T \implies [a_1, \ldots, a_{2k+1}] = E^T B^{-1}.$$ 

For a solution of Problem 2, in each group, the number of subsets containing each number in $[2k + 1]$ is the same (equal to $k$). Hence, $[a_1, \ldots, a_{2k+1}] = [1/k, \ldots, 1/k]$. However, in this case, $\sum_{i \in [2k+1]} a_i \neq 1$, not satisfying the last column of $C_g$. Hence, we prove that $C_g$ is full-rank.

**APPENDIX D**

**PROOF OF THEOREM 5**

**A. Proof of Theorem 5 item 1**

We focus on each corner point with $M = \frac{Nt}{K}$ where $t \in [0 : K]$. In this case, we have $H/\tau < t + 1$.

Achievability. We focus on one set of users $J$ where $|J| > t$. If each relay is connected to at most $t$ users of $J$, then $J$ contains at most $H/\tau$ users. So if $H/\tau < t + 1$, there must be one relay connected to $t + 1$ users of $J$. If $H/\tau < t + 1$, we have $V_2 = \emptyset$, $Q$, for any set $J$ of $t + 1$ users, there exists at least one relay connected to all of these users. So the maximum-link load of the DIS is $\frac{K(1-M/N)}{H(1+KM/N)}$.

Then we focus on the CICS under this case. For each set $J$ of $t + 1$ users, we have $\max_{h \in [H]} |U_h \cap J| = t + 1$. So the total link load to transmit is $|W_J|/|HB|$; thus the link load to transmit all the demanded subfiles of users is also $x$. Hence, the bound in (17) coincides with the above load by taking with $x = H$.

**B. Proof of Theorem 5 item 2**

We focus on each corner point with $M = \frac{Nt}{K}$ where $t \in [0 : K]$.

Achievability. By setting $r = H - 1$ in (11b), we have $V_2 = \emptyset$ when $t \leq K - 2$, i.e., for any set $J$ of $t + 1$ users, there exists at least one relay connected to all of these users. So similar to the previous case, for each $t \in [0 : K - 2]$, the maximum-link load achieved by the DIS or the CICS is $\frac{K-t}{(t+1)H}$. For $t = K$, the link load is 0. For $t \in [K - 2 : K]$, we use memory-sharing between the points $t = K - 2$ and $t = 0$.

Converse. The converse bound in (17) coincides with the lower convex envelop of the above loads by taking $x = H$ when $M \geq N(K - 2)/K$. We prove it by contradiction, i.e., assume that $g$ is also full-rank, there must exist only one sequence $[a_1, \ldots, a_{2k+1}]$ such that $\sum_{i \in [2k+1]} a_i B_i = E^T$, where $B_i$ represents the $i$th row of $B$. In other words, we have

$$[a_1, \ldots, a_{2k+1}]B = E^T \implies [a_1, \ldots, a_{2k+1}] = E^T B^{-1}.$$ 

**C. Proof of Theorem 5 item 3**

Achievability for $H < 2r$. When $t = KM/N = 1$, we have $V_2 = \emptyset$ and the load from the server to all the relays is $(K - t)/(t + 1) = (K - 1)/2$, due to the symmetry, the load from the server to each relay is the same, and thus the link load is $\frac{K-t}{2H}$. When $M = 0$, the load is $K/H$. By memory-sharing between $M = 0$ and $M = KM/N$, we can achieve the load in (12a). Converse for $H < 2r$. The converse bound in (17) coincides with the achievable bound by taking $x = H$ when $0 \leq M \leq N/K$.

Achievability for $H = 2r$. When $0 \leq M \leq N/K$, from (12a) and by memory-sharing between $M = 0$ and $M = KM/N$, the load achieved by the IES is $\frac{K(1-M)}{H(1-K)} \leq \frac{KH + H - K}{2}$. Converse for $H = 2r$. We use the converse bound in Theorem 5.

In Appendix D-D we prove that

$$H(1-H)R_u^* \geq \frac{KH + H - K}{2} - 1 \frac{KM}{N} + \sum_{W \subseteq [K]} z_W x_W,$$ 

where $z_W \geq 0$ is the coefficient of $x_W$ in (18b). Hence, the bound in (19z) coincides with the achievable bound when $0 \leq M \leq N/K$.

**D. Proof of (19z)**

When $H = 2r$, we compute the converse bound in Theorem 5. By choosing $a = 1$, the converse bound in (19z) becomes

$$|Q| R_u^* \geq \sum_{j \in [K]} \sum_{W \subseteq [K] \setminus \cup_{k \in [K]} \{K(K_j)\}} x_W + y_Q.$$ 

We sum all the inequalities as (59) for all the sets of relays $Q$ where $Q = H - 1$ and for all the permutations $p(K_j)$.

In (18a), there are $|K_j| = K/2$ terms of $x_\phi$ and $(K - 1) + \cdots + (K - |K_j|) = (3K/2 - 1)K/4$ terms of $x_\phi$ where $|W| = 1$. Because of the symmetry, from the sum we obtain

$$H(1-H)R_u^* \geq \frac{K}{2} H x_\phi + \frac{H}{4} \frac{3}{2}(K - 1)x_1 + \sum_{W \subseteq [K] : |W| > 1} z_W x_W + \sum_{Q : |Q| = b} y_Q,$$ 

where $z_W \geq 0$ is the coefficient of $x_W$ in (18b). Hence, the bound in (19z) coincides with the achievable bound when $0 \leq M \leq N/K$. Theorem 5.
where \( x_1 = \sum_{W \subseteq [K]} x_W \) and \( v_W \geq 0 \) represents the coefficient of \( x_1 \). Then, we focus on (19a). Notice that \( c_{W_1,H-1} = (\frac{r}{K-1}) - r = r - 1 \) where \( |W_1| = 1 \). So in (19b), the total coefficient of \( x_0 \) is \( K(r-1) \). Then we focus on a set of user \( W_1 : W_1 \subseteq [K] \) where \( |W_1| = 2 \) under the assumption that the two users in \( W_1 \) are \( k_1 \) and \( k_2 \). In (19b), there is only one term with coefficient \( c_{W_1,H-1} \) for each \( W_1 \subseteq [K] \) where \( |W_1| = 1 \). Now we want to compute \( c_{W_1,H-1} \) for each \( W_1 \subseteq [K] \) where \( |W_1| = 2 \). If \( H_{k_1} \cap H_{k_2} = \emptyset \), we have \( c_{W_1,H-1} = \max(H - 1 - H, 0) = 0 \). In addition, there are \( \binom{K}{2} \) such sets. If \( H_{k_1} \cap H_{k_2} = \emptyset \), we have \( c_{W_1,H-1} = \max(H - 1 - H, 0) = 0 \). There are \( \binom{K}{2} \) such sets. Hence, we have

\[
\sum_{W_1 \subseteq [K]} c_{W_1,H-1} = \sum_{i \in [2 : r-1]} \binom{r}{i} \binom{r}{i-1} - \binom{2r}{r} - r + 2
\]

Then, we want to eliminate \( x_0 \) and \( x_1 \) with the help of (18c) and (18d). From (18c), we have

\[
\left( \frac{K}{2} H + K(r-1) \right) x_0 + x_1 = \left( \frac{K}{2} H + K(r-1) \right) \left( 1 - \sum_{W \subseteq [K]} x_W \right)
\]

From (18d), we have

\[
\left( \frac{K}{2} H - KH \right) x_1 = \left( \frac{K}{2} H - KH \right) \left( 1 - \sum_{W \subseteq [K]} x_W \right)
\]
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