Limit Theorems For Sequences of Tempered Stable and Related Distributions

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Abstract

In this paper we define the closure under weak convergence of the class of $p$-tempered $\alpha$-stable distributions. We give necessary and sufficient conditions for convergence of sequences in this class. Moreover, we show that any element in this class can be approximated by the distribution of a linear combination of elementary $p$-tempered $\alpha$-stable random variables.

Key words: Tempered Stable Distributions; Limit Theorems; Infinite Divisibility; Thorin Class; Goldie-Steutel-Bondesson Class; Weak Convergence

1 Introduction

Tempered stable distributions were introduced in Rosiński 2007 [13] as a class of models that are similar to stable distributions in some central region, but they have lighter tails. Such models have been used successfully in a variety of areas including physics, biostatistics, and mathematical finance (see the references in [7]). A discussion of why such models come up in applications is given in [8].

In [7] the wider class of $p$-tempered $\alpha$-stable distributions ($TS^p_\alpha$), where $p > 0$ and $\alpha < 2$, was introduced. Rosiński’s class corresponds to the case when $p = 1$ and $\alpha \in (0, 2)$. Tempered infinitely divisible distributions defined in [5] are another subclass corresponding to the case when $p = 2$ and $\alpha \in [0, 2)$. If we allow the distributions to have a Gaussian part, then we would have the class $J^p_\alpha$ defined in [10]. This, in turn, contains important subclasses including the Thorin class (when $p = 1$ and $\alpha = 0$), the Goldie-Steutel-Bondesson class (when $p = 1$ and $\alpha = -1$), the class of type M distributions (when $p = 2$ and $\alpha = 0$), and the class of type G distributions (when $p = 2$ and $\alpha = -1$). For more information about these classes see the references in [3] and [4].

In this paper we discuss the possible weak limits of $TS^p_\alpha$ distributions. It turns out that this class is not closed under weak convergence. We introduce the class of extended $p$-tempered $\alpha$-stable distributions ($ETS^p_\alpha$), which is the smallest class that contains $TS^p_\alpha$.
and is closed under weak convergence. For \( \alpha \leq 0 \) it corresponds to the class \( J_{\alpha,p} \), but for \( \alpha \in (0, 2) \) we have \( J_{\alpha,p} \subsetneq ET S^p_{\alpha} \).

We also show that every \( d \)-dimensional \( ET S^p_{\alpha} \) distribution can be approximated by the distribution of a linear combination of 1-dimensional elementary \( TS^p_{\alpha} \) random variables. Elementary \( TS^p_{\alpha} \) distributions, more commonly called “smoothly truncated Lévy flights,” form a well studied subclass of \( TS^p_{\alpha} \). For \( p = 1 \) and \( \alpha \in (0, 2) \) the problem of simulation from such distributions is considered in, e.g. [2] and [9]. Combining this with our results allows for approximate simulation of \( ET S^p_{\alpha} \) random vectors in \( d \)-dimensions. We will consider this in a future work.

This paper is organized as follows. In Section 2 we define the class \( ET S^p_{\alpha} \). We take a detour in Section 3 to discuss a compactification of \( \mathbb{R}^d \). In Section 4 we define a measure on this compactification, which determines many properties of \( ET S^p_{\alpha} \) distributions. Then, in Section 5, we give our main limit theorem. Finally, in Section 6, we show that the class \( ET S^p_{\alpha} \) is, in fact, the closure of \( TS^p_{\alpha} \) under weak convergence.

Before proceeding, recall that the characteristic function of an infinitely divisible distribution \( \mu \) on \( \mathbb{R}^d \) can be written as

\[
\hat{\mu}(z) = \exp \left\{ C\mu(z) \right\}
\]

where

\[
C\mu(z) = -\frac{1}{2} \langle z, Az \rangle + i \langle b, z \rangle + \int_{\mathbb{R}^d} \left( e^{i(z,x)} - 1 - \frac{i}{1 + |x|^2} \right) M(dx),
\]

(1.1)

\( A \) is a Gaussian covariance matrix, \( M \) is a Lévy measure, and \( b \in \mathbb{R}^d \). The measure \( \mu \) is uniquely identified by the Lévy triplet \( (A, M, b) \), and we write \( \mu = \text{ID}(A, M, b) \). For details about infinitely divisible distributions, the reader is referred to [14]. We use the notation \( S^{d-1} = \{ x \in \mathbb{R}^d : |x| = 1 \} \), and we write \( \mathcal{B}(\mathbb{R}^d) \) to denote the Borel sets on \( \mathbb{R}^d \).

2 Extended Tempered Stable Distributions

We begin by recalling the definition of \( p \)-tempered \( \alpha \)-stable distributions given in [7].

**Definition 2.1.** Fix \( \alpha < 2 \) and \( p > 0 \). An infinitely divisible probability measure \( \mu \) is called a **\( p \)-tempered \( \alpha \)-stable distribution** if it has no Gaussian part and its Lévy measure is given by

\[
M(B) = \int_{S^{d-1}} \int_0^\infty 1_B(ru) q(r^p, u) r^{-\alpha-1} dr \sigma(du), \quad B \in \mathcal{B}(\mathbb{R}^d),
\]

(2.1)

where \( \sigma \) is a finite Borel measure on \( S^{d-1} \) and \( q : (0, \infty) \times S^{d-1} \rightarrow (0, \infty) \) is a Borel function such that for all \( u \in S^{d-1} \) \( q(\cdot, u) \) is completely monotone and

\[
\lim_{r \to \infty} q(r, u) = 0.
\]

(2.2)

We denote the class of \( p \)-tempered \( \alpha \)-stable distributions by \( TS^p_{\alpha} \). If, in addition,

\[
\lim_{r \downarrow 0} q(r, u) = 1
\]

(2.3)

for every \( u \in S^{d-1} \) then \( \mu \) is called a **proper \( p \)-tempered \( \alpha \)-stable distribution**.
In [7] it was shown that $M$ is the Lévy measure of a $p$-tempered $\alpha$-stable distribution if and only if there is a Borel measure $R$ on $\mathbb{R}^d$ such that

$$M(B) = \int_{\mathbb{R}^d} \int_0^\infty 1_B(tx) t^{-1-\alpha} e^{-t^p} dt R(dx), \quad B \in \mathcal{B}(\mathbb{R}^d).$$

(2.4)

We call $R$ the Rosiński measure of the $p$-tempered $\alpha$-stable distribution. Moreover, $R$ is the Rosiński measure of some $p$-tempered $\alpha$-stable distribution if and only if $R(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (|x|^2 \wedge |x|^\alpha) R(dx) < \infty \quad \text{if } \alpha \in (0, 2),$$

$$\int_{\mathbb{R}^d} (|x|^2 \wedge [1 + \log^+ |x|]) R(dx) < \infty \quad \text{if } \alpha = 0,$$

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) R(dx) < \infty \quad \text{if } \alpha < 0,$$

(2.5)

where $\log^+ |x| = 1_{|x| \geq 1} \log |x|$. We now extend $T S_p^\alpha$ to the smallest class that contains it and is closed under weak convergence. To do this we must allow for a Gaussian part and remove the assumption that (2.2) holds.

**Definition 2.2.** Fix $\alpha < 2$ and $p > 0$. An infinitely divisible probability measure $\mu$ is called an extended $p$-tempered $\alpha$-stable distribution if its Lévy measure is given by (2.1) where $\sigma$ is a finite Borel measure on $S_d$ and $q : (0, \infty) \times S_d \rightarrow (0, \infty)$ is a Borel function such that for all $u \in S_d$ $q(r, u)$ is completely monotone. We denote the class of extended $p$-tempered $\alpha$-stable distributions by $ET S_p^\alpha$.

**Remark 2.3.** When $\alpha \leq 0$ (2.2) is necessary to ensure that $M$ is a Lévy measure. Thus, whenever $\alpha \leq 0$ an $ET S_p^\alpha$ distribution is just a $T S_p^\alpha$ distribution with a Gaussian part.

**Remark 2.4.** Since the sum of completely monotone functions is completely monotone, it follows that both the class $T S_p^\alpha$ and the class $ET S_p^\alpha$ are closed under taking convolutions.

By Bernstein’s Theorem (see e.g. [6]) the complete monotonicity of $q(\cdot, u)$ implies that there is a measurable family $\{Q_u\}_{u \in S_d}$ of Borel measures on $[0, \infty)$ such that

$$q(r, u) = \int_{[0, \infty)} e^{-rs} Q_u(ds).$$

(2.6)

Letting

$$q_1(r, u) = \int_{(0, \infty)} e^{-rs} Q_u(ds)$$

(2.7)

gives

$$q(r^p, u) = q_1(r^p, u) + Q_u(\{0\}).$$

(2.8)
Thus, the Lévy measure of a distribution in \( ETS^p_\alpha \) is given by

\[
M(B) = \int_{\mathbb{R}^{d-1}} \int_0^\infty 1_B(ru) q_1(r^p, u) r^{-\alpha - 1} dr \sigma(du) \\
+ \int_{\mathbb{R}^{d-1}} \int_0^\infty 1_B(ru) r^{-\alpha - 1} dr Q_\alpha(\{0\}) \sigma(du), \quad B \in \mathcal{B}(\mathbb{R}^d). \tag{2.9}
\]

Note that, by Remark 2.3, when \( \alpha \leq 0 \) we have \( Q_\alpha(\{0\}) = 0 \). Since \( q_1(\cdot, u) \) is completely monotone for all \( u \in \mathbb{S}^{d-1} \) and it satisfies (2.2), \( M \) is the sum of the Lévy measure of a p-tempered \( \alpha \)-stable distribution and (when \( \alpha \in (0, 2) \)) an \( \alpha \)-stable distribution with spectral measure \( Q_\alpha(\{0\}) \sigma(du) \).

Remark 2.5. This implies that a distribution is in \( ETS^p_\alpha \) if and only if it can be written as the convolution of a Gaussian distribution, an element of \( TS^p_\alpha \), and (when \( \alpha \in (0, 2) \)) an \( \alpha \)-stable distribution.

Note that \( M \) is defined in terms of two measures \( R(du) \) and \( Q_\alpha(\{0\}) \sigma(du) \). To make it easier to work with we combine these into one measure, which we will define on a particular compactification of \( \mathbb{R}^d \).

3 A Compactification of \( \mathbb{R}^d \)

In this section we develop a compactification of \( \mathbb{R}^d \) with a sphere at infinity. Vague convergence of Radon measures on this space will be fundamental to our main results, therefore we give a detailed discussion.

Let \( \mathbb{R}^d_0 = \mathbb{R}^d \setminus \{0\} \) and note that for \( x \in \mathbb{R}^d_0 \) we have \( x = |x| \frac{x}{|x|} \). Thus we can uniquely identify every element of \( \mathbb{R}^d_0 \) with an element of \( (0, \infty) \times \mathbb{S}^{d-1} \). Let \( \mathbb{R}^d_0 = (0, \infty) \times \mathbb{S}^{d-1} \) and \( \overline{\mathbb{R}^d_0} = \mathbb{R}^d_0 \cup \{0\} \). For simplicity of notation define \( \mathbb{I}^{d-1} = \{\infty\} \times \mathbb{S}^{d-1} \) and \( \infty u = (\infty, u) \).

We introduce the functions \( \xi : \overline{\mathbb{R}^d} \mapsto \mathbb{S}^{d-1} \cup \{0\} \) and \( \vartheta : \overline{\mathbb{R}^d} \mapsto [0, \infty] \) as follows. Let \( \xi(0) = \vartheta(0) = 0 \). If \( x \in \mathbb{R}^d_0 \) then \( x = (r, u) \) and we define \( \xi(x) = u \) and \( \vartheta(x) = r \). For simplicity, we sometimes write \( |x| := \vartheta(x) \), and when \( x \in \mathbb{I}^{d-1} \) we take \( |x|^{-1} = 1/|x| = 0 \).

Let \( \overrightarrow{\mathbb{R}^d} \) and \( \overleftarrow{\mathbb{R}^d} \) denote, respectively, the usual convergence on \([0, \infty] \) and on \( \mathbb{R}^d \). If \( x, x_1, x_2, \ldots \in \mathbb{R}^d_0 \), we will write \( x_n \to x \) when \( \vartheta(x_n) \xrightarrow{\mathbb{R}^d} \vartheta(x) \) and \( \xi(x_n) \xrightarrow{\mathbb{S}^{d-1}} \xi(x) \). Let \( \tau_0 \) be the class of subsets of \( \mathbb{R}^d_0 \) such that \( A \in \tau_0 \) if and only if for any \( x \in A \) and any \( x_1, x_2, \ldots \in \mathbb{R}^d_0 \) with \( x_n \to x \) there is an \( N \) such that for all \( n \geq N \), \( x_n \in A \). It is straightforward to show that \( \tau_0 \) is a topology. In this topology compact sets are closed sets that are bounded away from 0. The Borel \( \sigma \)-algebra on \( \mathbb{R}^d_0 \) is the \( \sigma \)-algebra generated by \( \tau_0 \); we denote it by \( \mathcal{B}(\mathbb{R}^d_0) \).

To define convergence of a sequence in \( \overleftarrow{\mathbb{R}^d} \), we first define convergence to a point \( x \neq 0 \) as before. For \( x_1, x_2, \ldots \in \overleftarrow{\mathbb{R}^d} \) we write \( x_n \to 0 \) when \( \vartheta(x_n) \xrightarrow{\mathbb{R}^d} 0 \). Note that if \( x, x_1, x_2, \ldots \in \overleftarrow{\mathbb{R}^d} \), then...
\(\mathbb{R}^d \setminus \mathbb{T}^{d-1}\) then \(x_n \to x\) if and only if \(x_n \xrightarrow{d} x\). We define a topology \(\tau\) on \(\mathbb{R}^d\) in a manner analogous to the previous case. Here the compact sets are the closed sets. The Borel \(\sigma\)-algebra on \(\mathbb{R}^d\) is the \(\sigma\)-algebra generated by \(\tau\); we denote it by \(\mathcal{B}(\mathbb{R}^d)\).

For notational convenience, throughout this paper we identify Borel measures on \(\mathbb{R}^d\) with Borel measures on \(\mathbb{R}\) that place no mass at zero. Likewise, we identify Borel measures on \(\mathbb{R}^d\) with Borel measures on \(\mathbb{R}\) that place no mass on \(\mathbb{R}^{d-1}\).

A Borel measure on \(\mathbb{R}^d\) is called a **Radon measure** if it is finite on any subset that is bounded away from 0. Note that all Lévy measures and all Rosiński measures are Radon measures on \(\mathbb{R}^d\). We now define vague convergence on the spaces \(\mathbb{R}^d, \mathbb{R}\), and \(\mathbb{R}\). All three follow from the general definition of vague convergence on a topological space (see e.g. Chapter 3 in [12]).

**Definition 3.1.** Let \(\mu_0, \mu_1, \mu_2, \ldots\) be Radon measures on \(\mathbb{R}^d_0\). We write \(\mu_n \xrightarrow{v} \mu_0\) on \(\mathbb{R}^d_0\) if for all continuous, real-valued functions \(f\) on \(\mathbb{R}^d\) vanishing on a neighborhood of zero

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) \mu_n(\mathrm{d}x) = \int_{\mathbb{R}^d} f(x) \mu_0(\mathrm{d}x). 
\]

(3.1)

**Definition 3.2.** Let \(\mu_0, \mu_1, \mu_2, \ldots\) be finite Borel measures on \(\mathbb{R}^d\). We write \(\mu_n \xrightarrow{v} \mu_0\) on \(\mathbb{R}^d\) if for all continuous, real-valued functions \(f\) on \(\mathbb{R}^d\) holds.

**Definition 3.3.** Let \(\mu_0, \mu_1, \mu_2, \ldots\) be Borel measures on \(\mathbb{R}^d_0\) such that for all \(0 < a < b < \infty\) we have \(\mu_n([a, b]) < \infty\). If (3.1) holds for any continuous, real-valued function \(f\) vanishing on a neighborhood of zero and on a neighborhood of infinity we write \(\mu_n \xrightarrow{v} \mu_0\) on \(\mathbb{R}^d_0\).

It is not difficult to show that \(\mathbb{R}^d_0\) with the topology \(\tau_0\) and \(\mathbb{R}^d\) with the topology \(\tau\) are locally compact Hausdorff spaces with a countable basis. This implies that a number of standard results about vague convergence on \(\mathbb{R}^d\) extend to these spaces. In particular, Theorems 3.12 and 3.16 in [12], imply that a version of the Portmanteau Theorem and Helly’s Selection Theorem hold. The latter can be formulated as follows.

**Proposition 3.4.** Let \(\{\mu_n\}\) be a sequence of Borel measures on \(\mathbb{R}^d\) with \(\sup \mu_0(\mathbb{R}^d) < \infty\). There exists a subsequence \(\{\mu_{n_k}\}\) and a finite Borel measure \(\mu_0\) on \(\mathbb{R}^d\) such that \(\mu_{n_k} \xrightarrow{v} \mu_0\) on \(\mathbb{R}^d\).

We now give a useful characterization of vague convergence on \(\mathbb{R}^d\) for the special case when none of the measures place mass on \(\mathbb{T}^{d-1}\). Let \(C^b\) be the class of Borel functions mapping \(\mathbb{R}^d\) into \(\mathbb{R}\), which are continuous and bounded on \(\mathbb{R}^d \setminus \mathbb{T}^{d-1}\). We make no assumption about their behavior on \(\mathbb{T}^{d-1}\).

**Lemma 3.5.** Let \(\mu_0, \mu_1, \mu_2, \ldots\) be finite Borel measures on \(\mathbb{R}^d\) such that \(\mu_n(\mathbb{T}^{d-1}) = 0\) for all \(n\). Then \(\mu_n \xrightarrow{v} \mu_0\) on \(\mathbb{R}^d\) if and only if \(\int_{\mathbb{R}^d} f(x) \mu_n(\mathrm{d}x) \to \int_{\mathbb{R}^d} f(x) \mu_0(\mathrm{d}x)\) for all \(f \in C^b\).

**Proof.** Assume that \(\mu_n \xrightarrow{v} \mu_0\) on \(\mathbb{R}^d\), let \(H = \{T \in (0, \infty) : \mu_0(|x| = T) = 0\}\), and fix \(f \in C^b\). This means that there is a \(K\) such that \(|f(x)| \leq K\) for all \(x \in \mathbb{R}^d\). Without loss of
generality assume that \( f(x) \geq 0 \). From the Portmanteau Theorem (Theorem 3.12 in [12]) it follows that for all \( T \in H \)

\[
\lim_{n \to \infty} \int_{|x| \leq T} f(x) \mu_n(dx) = \int_{|x| \leq T} f(x) \mu_0(dx).
\]

Thus

\[
\liminf_{n \to \infty} \int_{\mathbb{R}^d} f(x) \mu_n(dx) \geq \lim_{H \ni T \uparrow \infty} \liminf_{n \to \infty} \int_{|x| \leq T} f(x) \mu_n(dx) = \lim_{H \ni T \uparrow \infty} \int_{|x| \leq T} f(x) \mu_0(dx) = \int_{\mathbb{R}^d} f(x) \mu_0(dx),
\]

where the last equality follows by dominated convergence. Since \( \mu_0(\mathbb{R}^{d-1}) = 0 \), for any \( \delta > 0 \) there is a \( T_\delta \in H \) with \( \mu_0(|x| \geq T_\delta) \leq \delta/K \). Thus

\[
\int_{\mathbb{R}^d} f(x) \mu_n(dx) \leq \int_{|x| \leq T_\delta} f(x) \mu_n(dx) + K \mu_n(|x| > T_\delta)
\]

\[
= \int_{|x| \leq T_\delta} f(x) \mu_0(dx) + K \mu_0(|x| > T_\delta)
\]

\[
\leq \int_{\mathbb{R}^d} f(x) \mu_0(dx) + \delta.
\]

Since this holds for all \( \delta > 0 \), \( \limsup_{n \to \infty} \int_{\mathbb{R}^d} f(x) \mu_n(dx) \leq \int_{\mathbb{R}^d} f(x) \mu_0(dx) \), and hence

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} f(x) \mu_n(dx) = \int_{\mathbb{R}^d} f(x) \mu_0(dx).
\]

The other direction follows from Definition 3.2.

We conclude this section by recalling a standard result about convergence of infinitely divisible distributions in terms of vague convergence of their Lévy measures. The following is a variant of Theorem 3.1.16 and Corollary 2.1.17 in [11].

**Proposition 3.6.** Let \( \mu_n = ID(A_n, M_n, b_n) \). If \( \mu_n \xrightarrow{\mathcal{W}} \mu \) then \( \mu = ID(A, M, b) \). Moreover, \( \mu_n \xrightarrow{\mathcal{W}} \mu \) if and only if \( M_n \xrightarrow{\mathcal{V}} M \) on \( \mathbb{R}^d \), \( b_n \to b \), and

\[
\lim_{\epsilon, \delta \downarrow 0} \lim_{n \to \infty} \left( A_n + \int_{|x| \leq \epsilon} xx^T M_n(dx) \right) = A. \tag{3.2}
\]

The result remains true if (3.2) is replaced by

\[
\lim_{\epsilon, \delta \downarrow 0} \liminf_{n \to \infty} \left( A_n + \int_{|x| \leq \epsilon} xx^T M_n(dx) \right) = \lim \sup_{\epsilon, \delta \downarrow 0} \lim_{n \to \infty} \left( A_n + \int_{|x| \leq \epsilon} xx^T M_n(dx) \right) = A. \tag{3.3}
\]

In the above and throughout, convergence of matrices should be interpreted as pointwise convergence of the components.
4 Extended Rosiński Measure

We now return to our discussion of the Lévy measures of extended $p$-tempered $\alpha$-stable distributions. Recall that the Lévy measure of such a distribution can be given by (2.10).

In this section, we will put it into a form that is easier to work with. First, let $\nu$ be a Borel measure on $\mathbb{R}^d$ such that if $B \in \mathcal{B}(\mathbb{R}^d)$ then if $\alpha \in (0, 2)$

$$\nu(B) = \int_{\mathbb{R}^d} 1_B(x) \left(|x|^2 \wedge |x|^\alpha\right) R(dx) + \int_{\mathbb{R}^{d-1}} 1_B(\infty x) Q_x(\{0\}) \sigma(dx) \quad (4.1)$$

and

$$\nu(B) = \left\{ \begin{array}{ll}
\int_{\mathbb{R}^d} 1_B(x) \left(|x|^2 \wedge [1 + \log^+ |x|]\right) R(dx) & \text{if } \alpha = 0 \\
\int_{\mathbb{R}^d} 1_B(x) \left(|x|^2 \wedge 1\right) R(dx) & \text{if } \alpha < 0
\end{array} \right. \quad (4.2)$$

Note that $\nu(\{0\}) = 0$ and by (2.5) $\nu$ is a finite measure. In particular, $\nu$ is a Radon measure on $\mathbb{R}^d$. We will call it the extended Rosiński measure. From $\nu$ we get $R$ back by

$$R(dx) = \left\{ \begin{array}{ll}
|\alpha|^{-1} \nu_\mathbb{R}^d(dx) & \text{if } \alpha \in (0, 2) \\
|\alpha|^{-1} \nu_{\mathbb{R}^d}(dx) & \text{if } \alpha = 0 \\
|\alpha|^{-1} \nu_{\mathbb{R}^d}(dx) & \text{if } \alpha < 0
\end{array} \right. \quad (4.3)$$

where $\nu_\mathbb{R}^d$ is the restriction of $\nu$ to $\mathbb{R}^d$.

**Remark 4.1.** Let $\nu$ be any finite Borel measure on $\mathbb{R}^d$ with $\nu(\{0\}) = 0$. For any $p > 0$ and $\alpha \in (0, 2)$, $\nu$ is the extended Rosiński measure of some distribution in $ETS^p_\alpha$. If, in addition, $\nu(T^{d-1}) = 0$ then for any $p > 0$ and $\alpha < 2$, $\nu$ is the extended Rosiński measure of some distribution in $ETS^p_\alpha$.

**Proposition 4.2.** For a fixed $\alpha < 2$ and $p > 0$, the extended Rosiński measure $\nu$ is uniquely determined by the Lévy measure of the extended $p$-tempered $\alpha$-stable distribution.

**Proof.** This follows from the fact that $\nu$ is uniquely determined by $R$ and $Q_a(\{0\}) \sigma(du)$. In [7] it was shown that $R$ is uniquely determined by the Lévy measure of the $p$-tempered $\alpha$-stable part, and Remark 14.4 in [14] says that $Q_a(\{0\}) \sigma(du)$ is uniquely determined by the Lévy measure of the $\alpha$-stable part. \qed

**Definition 4.3.** A distribution in $ETS^p_\alpha$ with Gaussian part $A$, extended Rosiński measure $\nu$, and shift $b$ is denoted by $ETS^p_\alpha(A, \nu, b)$.

We conclude this section by giving a representation of the Lévy measure of a distribution in $ETS^p_\alpha$ in terms of its extended Rosiński measure. Fix $\mu \in ETS^p_\alpha$ with Lévy measure $M$ given by (2.10), and let $f$ be any Borel function, which is integrable with respect to $M$. If $\alpha < 0$ then

$$\int_{\mathbb{R}^d} f(x) M(dx) = \int_{\mathbb{R}^d} \int_0^\infty f(tx) t^{-1-\alpha} e^{-tv} dt \frac{1}{1 \wedge |x|^2} \nu(dx), \quad (4.4)$$

\[7\]
if $\alpha = 0$ then
\[
\int_{\mathbb{R}^d} f(x)M(dx) = \int_{\mathbb{R}^d} \int_{0}^{\infty} f(tx)t^{-1}\nu(dx) dt \frac{1}{|x|^2 \land [1 + \log^+ |x|]} \nu(dx), \quad (4.5)
\]
and if $\alpha \in (0, 2)$ then
\[
\int_{\mathbb{R}^d} f(x)M(dx) = \int_{\mathbb{R}^{d-1}} \int_{0}^{\infty} f(tx)t^{-1-\alpha} dtQ_x(\{0\})\sigma(dx) \\
+ \int_{\mathbb{R}^d} \int_{0}^{\infty} f(tx)t^{-1-\alpha} e^{-t^p} dtR(dx) \\
= \int_{\mathbb{R}^{d-1}} \int_{0}^{\infty} f(t\xi(x))t^{-1-\alpha} e^{-(t/|x|)^p} dt\nu(dx) \\
+ \int_{\mathbb{R}^d} \int_{0}^{\infty} f(t\xi(x))t^{-1-\alpha} e^{-(t/|x|)^p} dt|x|^\alpha R(dx) \\
= \int_{\mathbb{R}^d} \int_{0}^{\infty} f(t\xi(x))t^{-1-\alpha} e^{-(t/|x|)^p} 1 \land |x|^{2-\alpha} dt\nu(dx). \quad (4.6)
\]

5 Sequences of Extended Tempered Stable Distributions

We can now state our main result.

**Theorem 5.1.** Fix $\alpha < 2$, $p > 0$, and let $\mu_n = ETS^p_n(A_n, \nu_n, b_n)$. If $\mu_n \xrightarrow{w} \mu$ then $\mu = ETS^p_n(A, \nu, b)$. Moreover, $\mu_n \xrightarrow{w} \mu$ if and only if $\nu_n \xrightarrow{w} \nu$ on $\mathbb{R}^d$, $b_n \xrightarrow{w} b$, and
\[
\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} (A_n + H^\epsilon_n) = A, \quad (5.1)
\]
where
\[
H^\epsilon_n = \int_{|x| < \sqrt{\epsilon}} \frac{x x^T}{|x|^2} \int_{0}^{e|x|-1} t^{-1-\alpha} e^{-t^p} dt\nu_n(dx). \quad (5.2)
\]
The result remains true if $(5.1)$ is replaced by
\[
\lim_{\epsilon \downarrow 0} \liminf_{n \to \infty} (A_n + H^\epsilon_n) = \limsup_{\epsilon \downarrow 0} \lim_{n \to \infty} (A_n + H^\epsilon_n) = A. \quad (5.3)
\]

**Remark 5.2.** The extended Rosiński measure does not contribute to the Gaussian part if and only if
\[
\lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \text{tr} H^\epsilon_n = \lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} \int_{|x| < \sqrt{\epsilon}} \int_{0}^{e|x|-1} t^{-1-\alpha} e^{-t^p} dt\nu_n(dx) = 0. \quad (5.4)
\]
Since for any $\epsilon \in (0, 1)$
\[
\int_{|x| < \epsilon} \nu_n(dx) \int_{0}^{1} t^{-1-\alpha} e^{-t^p} dt \leq \text{tr} H^\epsilon_n \leq \int_{|x| < \sqrt{\epsilon}} \nu_n(dx) \int_{0}^{\infty} t^{-1-\alpha} e^{-t^p} dt,
\]
\[8\]
\[ (\ref{5.4}) \text{ holds if and only if } \]
\[
\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \int_{|x| < \epsilon} \nu_n(dx) = 0. \tag{5.5}
\]

**Remark 5.3.** Let \( R_n \) be the Rosiński measure corresponding to \( \nu_n \) and let \( R \) be the Rosiński measure corresponding to \( \nu \). When \( \alpha \leq 0 \) the condition \( \nu_n \to \nu \) on \( \mathbb{R}_0^d \) is equivalent to the condition \( R_n \to R \) on \( \mathbb{R}_0^d \) and when \( \alpha = 0 \)
\[
\lim_{N \to \infty} \int_{|x| > N} \log |x| R_n(dx) = 0 \tag{5.6}
\]
or when \( \alpha < 0 \)
\[
\lim_{N \to \infty} R_n(|x| > N) = 0. \tag{5.7}
\]

When \( \alpha \in (0, 2) \) the limit does not have an \( \alpha \)-stable part if and only if
\[
\lim_{N \to \infty} \int_{|x| > N} |x|^\alpha R_n(dx) = 0. \tag{5.8}
\]

To facilitate the proof of Theorem \( \ref{5.1} \) we begin with several lemmas.

**Lemma 5.4.** Fix \( \alpha < 2 \) and \( p > 0 \). If \( s \in \mathbb{R} \) with \( |s| \leq 1 \) then
\[
\int_0^\infty (\cos (ts) - 1) t^{-1-\alpha} e^{-t^p} dt \leq -\frac{11}{24} s^2 \int_0^1 t^{1-\alpha} e^{-t^p} dt. \tag{5.9}
\]

**Proof.** We have
\[
\int_0^\infty (\cos (ts) - 1) t^{-1-\alpha} e^{-t^p} dt \leq \int_0^1 (\cos (ts) - 1) t^{-1-\alpha} e^{-t^p} dt
\]
\[
\leq \int_0^1 \left( \frac{s^4 t^4}{24} - \frac{s^2 t^2}{2} \right) t^{-1-\alpha} e^{-t^p} dt \leq -\frac{11}{24} s^2 \int_0^1 t^{1-\alpha} e^{-t^p} dt,
\]
where the second line follows by the Taylor expansion of cosine and the remainder theorem for alternating series.

**Lemma 5.5.** Let the sequence \( \{\mu_n\} \) be as in Theorem \( \ref{5.1} \).

1. If \( \mu_n \overset{w}{\to} \mu \) for some probability measure \( \mu \) then \( \sup \nu_n(\mathbb{R}^d) < \infty \).
2. If \( \nu_n \overset{w}{\to} \nu \) on \( \mathbb{R}_0^d \) for some finite measure \( \nu \) then for any \( \delta > 0 \), \( \sup \nu_n(|x| \geq \delta) < \infty \).
3. If \( \ref{5.3} \) holds with some matrix \( A \) then for any \( \delta > 0 \), \( \sup \nu_n(|x| < \delta) < \infty \).

**Proof.** The second part follows immediately from Definition \( \ref{3.1} \). To show the first part, assume that \( \mu_n \overset{w}{\to} \mu \) and get \( R_n \) from \( \nu_n \) by \( \ref{4.3} \). Lemma \( \ref{5.4} \) implies that for \( |z| \leq 1 \)
\[
|\hat{\mu}_n(z)| \leq \left| \exp \left\{ \int_{\mathbb{R}^d} \int_0^\infty \left( e^{it\langle x, z \rangle} - 1 - \frac{t\langle x, z \rangle}{1 + |z|^2} \right) t^{-1-\alpha} e^{-t^p} dt R_n(dx) \right\} \right|
\]
\[
\leq \exp \left\{ \int_{|x| \leq 1} \int_0^\infty (\cos(t\langle x, z \rangle) - 1) t^{-1-\alpha} e^{-t^p} dt R_n(dx) \right\}
\]
\[
\leq \exp \left\{ -\frac{11}{24} \int_0^1 t^{-1-\alpha} e^{-t^p} dt \int_{|x| \leq 1} \langle x, z \rangle^2 R_n(dx) \right\},
\]

where the first inequality follows by the fact that we can write \( \mu_n \) as the convolution of a Gaussian, an element of \( TS^p_{\alpha} \), and (when \( \alpha \in (0, 2) \)) an \( \alpha \)-stable distribution. By Proposition 2.5 in [14] \( |\hat{\mu}_n(z)| \rightarrow |\hat{\mu}(z)| \) uniformly on compact sets, and for some \( b > 0 \), \( |\hat{\mu}(z)| > b \) on a neighborhood of zero. Thus, on this neighborhood, for large enough \( n \),

\[
b < \exp \left\{ -\frac{11}{24} \int_0^1 t^{1-\alpha} e^{-\nu_p} dt \int_{|x| \leq 1} \langle x, z \rangle^2 R_n(dx) \right\},
\]

which implies that \( \sup_{|x| \leq 1} \langle x, z \rangle^2 R_n(dx) < \infty \) for every \( z \in \mathbb{R}^d \), and hence

\[
\sup \nu_n(|x| \leq 1) < \infty.
\]

By Proposition 3.6 \( \mu \) is infinitely divisible. Let \( M_n \) be the Lévy measure of \( \mu_n \) and let \( M \) be the Lévy measure of \( \mu \). Let \( f_1 \) be a non-negative, continuous, bounded, real-valued function vanishing on a neighborhood of zero such that \( f_1(y) = 1 \) for \( |y| \geq 1 \). When \( \alpha \in (0, 2) \) by (4.6)

\[
\int_{\mathbb{R}^d} f_1(x)M_n(dx) = \int_{\mathbb{R}^d} \int_0^\infty f_1(\xi(x)t)t^{-1-\alpha} \frac{e^{-r(t/|x|)^p}}{1 \wedge |x|^{2-\alpha}} dt \nu_n(dx)
\geq \int_{\mathbb{R}^d} \int_1^{d-1} t^{-1-\alpha} dt \nu_n(dx)
+ \int_{|x| \geq 1} \int_1^2 t^{-1-\alpha} e^{-r(t/|x|)^p} dt \nu_n(dx)
\geq \alpha^{-1} \nu_n(t^{d-1}) + e^{-2p} \frac{2^\alpha - 1}{\alpha^2 \nu_n(\infty > |x| \geq 1)}
\geq e^{-2p} \frac{2^\alpha - 1}{\alpha^2 \nu_n(|x| \geq 1)}.
\]

Similarly when \( \alpha = 0 \) by (4.5)

\[
\int_{\mathbb{R}^d} f_1(x)M_n(dx) = \int_{\mathbb{R}^d} \int_0^\infty f_1(xt)t^{-1} e^{-\nu_p} dt \frac{\nu_n(dx)}{|x| \wedge (1 + \log t/|x|)}
\geq e^{-\nu_p} \int_{|x| \geq 1} \int_{|x|^{-1}}^e t^{-1} dt \frac{\nu_n(dx)}{1 + \log |x|} = e^{-\nu_p} \nu_n(|x| \geq 1),
\]

and when \( \alpha < 0 \) by (4.4)

\[
\int_{\mathbb{R}^d} f_1(x)M_n(dx) = \int_{\mathbb{R}^d} \int_0^\infty f_1(xt)t^{-\alpha} e^{-\nu_p} dt \frac{1}{1 \wedge |x|^2} \nu_n(dx)
\geq \nu_n(|x| \geq 1) \int_1^\infty t^{-1-\alpha} e^{-\nu_p} dt.
\]

Proposition 3.6 implies that the left side converges to \( \int_{\mathbb{R}^d} f_1(x)M(dx) \) in all three cases. Thus, since \( \int_{\mathbb{R}^d} f_1(x)M(dx) < \infty \), we have \( \sup \nu_n(|x| \geq 1) < \infty \).
The third part follows from the fact that \((\ref{eq:2})\) implies that for any \(\varepsilon > 0\)

\[
\limsup_{n \to \infty} \operatorname{tr} H_n^\varepsilon = \limsup_{n \to \infty} \int_{|x| < \sqrt{\varepsilon}} \int_0^{e|x|^{-1}} t^{1-\alpha} e^{-tp} dt \nu_n(dx)
\]

\[
\geq \limsup_{n \to \infty} \int_0^{\sqrt{\varepsilon}} t^{1-\alpha} e^{-tp} dt \int_{|x| < \sqrt{\varepsilon}} \nu_n(dx),
\]

and hence \(\sup_n \nu_n(|x| < \sqrt{\varepsilon}) < \infty\). 

**Lemma 5.6.** Let the sequence \(\{\mu_n\}\) be as in Theorem \([5.7]\) and let \(M_n\) be the Lévy measure of \(\mu_n\). If \(\sup \nu_n(\mathbb{R}^d) < \infty\) then

\[
\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \left( A_n + \int_{|x| \leq \varepsilon} xx^T M_n(dx) \right) = \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \left( A_n + \int_{|x| < \sqrt{\varepsilon}} \frac{xx^T}{|x|^2} \int_0^{e|x|^{-1}} t^{1-\alpha} e^{-tp} dt \nu_n(dx) \right),
\]

\((5.10)\)

whenever at least one of the limits exists. The result remains true if we replace \(\lim_{n \to \infty}\) by \(\liminf_{n \to \infty}\) or \(\limsup_{n \to \infty}\).

**Proof.** We give the proof for the case when \(\alpha \in (0, 2)\) only. The other cases are similar. We can write

\[
\int_{|x| \leq \varepsilon} xx^T M_n(dx) = \int_{\mathbb{R}^{d-1}} \int_0^\varepsilon \xi(x)[\xi(x)]^T t^{1-\alpha} dt \nu_n(dx)
\]

\[+ \int_{|x| \geq 1} \int_0^{e|x|^{-1}} xx^T t^{1-\alpha} e^{-tp} dt |x|^{-\alpha} \nu_n(dx)
\]

\[+ \int_{|x| > \sqrt{\varepsilon}} \int_0^{e|x|^{-1}} xx^T t^{1-\alpha} e^{-tp} dt |x|^{-2} \nu_n(dx)
\]

\[+ \int_{|x| < \sqrt{\varepsilon}} \int_0^{e|x|^{-1}} xx^T t^{1-\alpha} e^{-tp} dt |x|^{-2} \nu_n(dx)
\]

\[=: I_1^{n,\varepsilon} + I_2^{n,\varepsilon} + I_3^{n,\varepsilon} + I_4^{n,\varepsilon}.
\]

With \(C := \sup_n \nu_n(\mathbb{R}^d) < \infty\) we have

\[
\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \operatorname{tr} I_1^{n,\varepsilon} = \lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \nu_n(\mathbb{R}^{d-1}) \frac{e^{2-\alpha}}{2 - \alpha} = 0,
\]

\[
\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \operatorname{tr} I_2^{n,\varepsilon} \leq \lim_{\varepsilon \downarrow 0} C \frac{e^{2-\alpha}}{2 - \alpha} = 0,
\]

and

\[
\lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \operatorname{tr} I_3^{n,\varepsilon} \leq \lim_{\varepsilon \downarrow 0} \limsup_{n \to \infty} \int_{|x| > \sqrt{\varepsilon}} \int_0^{\sqrt{\varepsilon}} t^{1-\alpha} dt \nu_n(dx)
\]

\[\leq \lim_{\varepsilon \downarrow 0} C \frac{e^{1-\alpha/2}}{2 - \alpha} = 0.
\]
This completes the proof.

\textbf{Proof of Theorem 5.1.} Let $M_n$ be the Lévy measure of $\mu_n$.

Assume that $\mu_n \to \mu$. By Proposition 3.6 $\mu$ is infinitely divisible with some Lévy triplet $(A,M,b)$ such that $b_n \to b$, $M_n \to M$ on $\mathbb{R}_+^d$, and (3.2) holds. Combining this with Lemmas 5.5 and 5.6 gives (5.1) which implies (5.3). It remains to show that there is an extended Rosiński measure $\nu$ such that $\mu = ETS^p(A,\nu,b)$ and $\nu |_{\mathbb{R}_+^d} \to \nu$ on $\mathbb{R}_+^d$.

By Lemma 5.3 $\sup \nu_n(\mathbb{R}^d) < \infty$. Thus, Proposition 3.1 implies that there is a finite Borel measure $\tilde{\nu}$ on $\mathbb{R}^d$ and a subsequence $\{\nu_{n_j}\}$ such that $\nu_{n_j} \to \tilde{\nu}$ on $\mathbb{R}^d$. Let $\nu$ be a finite Borel measure on $\mathbb{R}^d$ such that $\nu|_{\mathbb{R}_+^d} = \tilde{\nu}|_{\mathbb{R}_+^d}$ and $\nu(\{0\}) = 0$. Clearly $\nu_{n_j} \to \nu$ on $\mathbb{R}_+^d$. Let $f$ be any continuous non-negative function on $\mathbb{R}^d$ such that there are $\epsilon, K > 0$ with $f(x) = 0$ whenever $|x| \leq \epsilon$ and $f(x) \leq K$ for all $x \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ define

$$g_\alpha(x) = \left\{ \begin{array}{ll}
\int_0^\infty f(\xi(x)t)t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{|\xi(x)|^p} \, dt & \alpha \in (0,2) \\
\int_{|x|}^\infty f(\xi(x)t)t^{-1} \frac{e^{-(t/|x|)^p}}{|\xi(x)|^p} \, dt & \alpha = 0 \\
\int_{|x|}^\infty f(\xi(x)t)t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{|\xi(x)|^p} \, dt & \alpha < 0.
\end{array} \right. \quad (5.11)$$

We will show that

$$\lim_{j \to \infty} \int_{\mathbb{R}^d} g_\alpha(x)\nu_{n_j}(dx) = \int_{\mathbb{R}^d} g_\alpha(x)\tilde{\nu}(dx). \quad (5.12)$$

Assuming that this holds and observing that $g_\alpha(0) = 0$ gives

$$\int_{\mathbb{R}^d} f(x)M(dx) = \lim_{j \to \infty} \int_{\mathbb{R}^d} f(x)M_{n_j}(dx) = \lim_{j \to \infty} \int_{\mathbb{R}^d} g_\alpha(x)\nu_{n_j}(dx)$$

$$= \int_{\mathbb{R}^d} g_\alpha(x)\tilde{\nu}(dx) = \int_{\mathbb{R}^d} g_\alpha(x)\nu(dx).$$

This implies that $M$ is the Lévy measure of a $p$-tempered $\alpha$-stable distribution with extended Rosiński measure $\nu$. Since Lévy measures are unique, this proves that the class $ETS^p(A,\nu,b)$ is closed under weak convergence. Moreover, since, by Proposition 4.2, $\nu$ is uniquely determined by $M$, $\nu_n \to \nu$ on $\mathbb{R}^d$.

We will now show that (5.12) holds. By Definition 3.2 it suffices to show that $g_\alpha$ is bounded and continuous. When $\alpha \in (0,2)$ the facts that $\int_0^\infty t^{-1-\alpha} \, dt < \infty$ and that $f(\xi(x)t) \frac{e^{-(t/|x|)^p}}{|\xi(x)|^p}$ is uniformly bounded show that $g_\alpha$ is bounded, and by dominated convergence it is continuous on $\mathbb{R}^d$. When $\alpha < 0$,

$$1_{[t>|x|^{-1}]}f(\xi(x)t)t^{-1-\alpha} \frac{e^{-(t/|x|)^p}}{|\xi(x)|^p} \leq Ke^{-(t-\alpha + 1/t)^p} \quad (5.12),$$

which is integrable on $[0,\infty)$. Thus $g_\alpha$ is bounded, and by dominated convergence it is continuous on $\mathbb{R}^d$. 

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When \( \alpha = 0 \), by Lemma 3.5 it suffices to show that \( g_\alpha \) is bounded and continuous only on \( \mathbb{R}^d \). If \( |x| \leq 1 \) then
\[
1_{[t \geq |x|^{-1}]} f(x) t^{-1} e^{-tp} |x|^{-2} \leq 1_{[t \geq 0]} K t^{-2} e^{-tp},
\]
which is integrable with respect to \( t \). If \( |x| \geq 1 \) fix \( \delta \in (0, |x|) \) and let \( x' \) be such that \( |x' - x| < \delta \). Then
\[
1_{[t > |x'|^{-1}]} f(x') t^{-1} e^{-tp} \left[ 1 + \log |x'| \right]^{-1} \leq 1_{[t > |x'|^{-1}]} K t^{-1} e^{-tp} \leq 1_{[t > (|x|+\delta)^{-1}]} K t^{-1} e^{-tp},
\]
which is integrable with respect to \( t \). Thus, by dominated convergence \( g_0 \) is continuous on \( \mathbb{R}^d \). To show that \( g_0(x) \) is bounded, note that when \( |x| \leq 1 \) then, as before
\[
g_0(x) \leq K e^{-2} \int_0^\infty t^{-1} e^{-tp} dt < \infty,
\]
and when \( |x| > 1 \)
\[
g_0(x) \leq K \left[ 1 + \log |x| \right]^{-1} \int_{|x|}^\infty t^{-1} dt + K \int_{|x|}^\infty t^{-1} e^{-tp} dt
\]
\[
= K + K \int_{|x|}^\infty t^{-1} e^{-tp} dt.
\]
Now for the other direction. Let \( M \) be the Lévy measure of \( \mu \). Assume that \( b_n \to b \), \( (5.3) \) holds, and \( \nu_n \to \nu \) on \( \mathbb{R}^d \). Lemma 5.5 implies that \( \sup \nu_n(\mathbb{R}) < \infty \). Thus combining \( (5.3) \) with Lemma 5.6 gives \( (3.2) \). To show that \( M_n \overset{v}{\to} M \) on \( \mathbb{R}^d \) we will show that every subsequence has a further subsequence that does this. Let \( \{n_k\} \) be any increasing sequence in \( N \). By Proposition 3.4 there is a subsequence \( n_{k_j} \) and a finite Borel measure \( \tilde{\nu} \) on \( \mathbb{R}^d \) such that \( \nu_{n_{k_j}} \overset{v}{\to} \tilde{\nu} \) on \( \mathbb{R}^d \). Clearly, \( \nu_{\mathbb{I}_{d_0}} = \tilde{\nu}_{\mathbb{I}_{d_0}} \). Let \( f \) be a continuous nonnegative function on \( \mathbb{R}^d \) satisfying the same assumptions as in the other direction, and define \( g_\alpha \) by \( (3.11) \). Observing that \( g_\alpha(0) = 0 \) gives
\[
\int_{\mathbb{R}^d} f(x) M_{n_{k_j}}(dx) = \int_{\mathbb{R}^d} g_\alpha(x) \nu_{n_{k_j}}(dx)
\]
\[
\to \int_{\mathbb{R}^d} g_\alpha(x) \tilde{\nu}(dx) = \int_{\mathbb{R}^d} g_\alpha(x) \nu(dx) = \int_{\mathbb{R}^d} f(x) M(dx),
\]
where the convergence follows by arguments similar to the other direction. \( \square \)

6 Closure Properties

In this section we will show that \( ETS^\alpha \) is, in fact, the smallest class that contains \( TS^\alpha \) and is closed under weak convergence. In the following let \( N(b, A) \) denote the Gaussian distribution with mean vector \( b \) and covariance matrix \( A \), and let \( S_\alpha(\sigma, b) \) denote the \( \alpha \)-stable distribution with spectral measure \( \sigma \) and shift \( b \). This means that \( N(b, A) = ID(A, 0, b) \) and \( S_\alpha(\sigma, b) = ID(0_{d \times d}, K, b) \) where \( K(B) = \int_{S_{d-1}} \int_0^\infty 1_B(x) t^{-1-\alpha} dt \sigma(dx) \) for \( B \in \mathcal{B}(\mathbb{R}^d) \).
Proposition 6.1. Fix $\alpha < 2$ and $p > 0$.
1. If $\mu = N(0, A)$ then there is a sequence $\{\mu_n\}$ in $TS_p^\alpha$ such that $\mu_n \stackrel{w}{\rightarrow} \mu$.
2. If $\alpha \in (0, 2)$ and $\mu = S_\alpha(\sigma, 0)$ then there is a sequence $\{\mu_n\}$ in $TS_p^\alpha$ such that $\mu_n \stackrel{w}{\rightarrow} \mu$.
3. The class $ETS_p^\alpha$ is the smallest class that contains $TS_p^\alpha$ and is closed under weak convergence. Moreover, this class is closed under taking convolutions.

Proof. First observe that
\[
\lim_{s \to 0} \frac{e^{i(x,z)rs} - 1 - \frac{i(x,z)sr}{1 + |x|^2r^2}}{s^2} = -\frac{1}{2} \langle x, z \rangle^2 r^2.
\]
Let $R = N(0, cA)$, where $c = \left[ \int_0^\infty r^{1-\alpha} e^{-rp} dr \right]^{-1}$. Let $X = (X_1, \ldots, X_d)^T \sim R$ and define
\[
R_n(B) = n^{2-\alpha} \int_{\mathbb{R}^d} 1_B(xn^{-1})R(dx), \quad B \in \mathcal{B}(\mathbb{R}^d).
\]
By (2.5), this is the Rosiński measure of some distribution in $TS_p^\alpha$. If $\mu_n = TS_p^\alpha(R_n, 0)$ then
\[
C_{\mu_n}(z) = \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i(x,z)r} - 1 - \frac{i(x,z)sr}{1 + |x|^2r^2} \right) r^{-1-\alpha} e^{-rp} dr R_n(dx)
\]
\[
= n^{2} \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i(x,z)r/n} - 1 - \frac{i(x,z)sr/n}{1 + |x/n|^2r^2} \right) r^{-1-\alpha} e^{-rp} dr R(dx)
\]
\[
\rightarrow -\frac{1}{2} \int_{\mathbb{R}^d} \langle x, z \rangle^2 R(dx) \int_0^\infty r^{-1-\alpha} e^{-rp} dr
\]
\[
= -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d z_i z_j E[X_i X_j] c^{-1} = -\frac{1}{2} \langle z, Az \rangle,
\]
where the third line follows by dominated convergence. For the second part let
\[
R(B) = \int_{\mathbb{S}^{d-1}} \int_0^\infty 1_B(ut)e^{-t^{\alpha}} dt \sigma(du), \quad B \in \mathcal{B}(\mathbb{R}^d),
\]
and note that
\[
\sigma(B) = \int_{\mathbb{R}^d} 1_B \left( \frac{x}{|x|} \right) |x|^\alpha R(dx), \quad B \in \mathcal{B}(\mathbb{S}^{d-1}).
\]
Let
\[
R_n(B) = n^{-\alpha} \int_{\mathbb{R}^d} 1_B(xn)R(dx), \quad B \in \mathcal{B}(\mathbb{R}^d).
\]
By (2.5), this is the Rosiński measure of some distribution in $TS^p_\alpha$. If $\mu_n = TS^p_\alpha(R_n, 0)$ then

$$
C_{\mu_n}(z) = \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i(x,z)r} - 1 - \frac{i(x,z)r}{1 + |x|^2 r^2} \right) r^{-2-\alpha} e^{-r^p} dr R_n(dx)
$$

$$
= n^{-\alpha} \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i(x,z)rn} - 1 - \frac{i(x,z)rn}{1 + |zn|^2 r^2} \right) r^{-2-\alpha} e^{-r^p} dr R(dx)
$$

$$
= \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i(x,z)t/|x|} - 1 - \frac{i(x,z)t/|x|}{1 + t^2} \right) t^{-1-\alpha} e^{-(t|x|^{-1})^p} dt |x|^\alpha R(dx)
$$

$$
\rightarrow \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i(u,z)t} - 1 - \frac{i(u,z)t}{1 + t^2} \right) t^{-1-\alpha} dt \sigma(du),
$$

where the third line follows by the substitution $t = rn|x|$ and the fourth by dominated convergence. The third part is an immediate consequence of the first two and Remark 2.4.

**Definition 6.2.** For $\alpha < 2$ and $p > 0$, a random vector is called an elementary $p$-tempered $\alpha$-stable random vector on $\mathbb{R}^d$ if it can be written as $Ux$, where $x \in \mathbb{R}^d$ is a nonrandom vector and $U \sim ID(0, M, b)$ is an infinitely divisible random variable on $\mathbb{R}$ with $b \in \mathbb{R}$ and $M(dt) = e^{c|t|} t^{-1-\alpha} e^{-|t|^p} dt$, for some $c > 0$.

By (6.1), for $\lambda \in \mathbb{R}$, we have

$$
E e^{i\lambda U} = \exp \left\{ c \int_0^\infty \left( e^{i\lambda t} - 1 - \frac{i\lambda t}{1 + t^2} \right) t^{-1-\alpha} e^{-t^p} dt + i\lambda b \right\}. \quad (6.1)
$$

Thus for $z \in \mathbb{R}^d$

$$
E e^{i\langle z, Ux \rangle} = \exp \left\{ c \int_0^\infty \left( e^{i\langle z, x \rangle t} - 1 - \frac{i\langle z, x \rangle t}{1 + t^2} \right) t^{-1-\alpha} e^{-t^p} dt + i\langle z, x \rangle b \right\}
$$

$$
= \exp \left\{ \int_{\mathbb{R}^d} \int_0^\infty \left( e^{i\langle y, z \rangle t} - 1 - \frac{i\langle y, z \rangle t}{1 + t^2} \right) t^{-1-\alpha} e^{-t^p} dt R(dy) + i\langle z, x \rangle b \right\},
$$

where $R(dy) = c\delta_x(dy)$. Thus, a random vector is the finite sum of elementary $p$-tempered $\alpha$-stable random vectors if and only if its distribution is an element of $TS^p_\alpha$ with Rosiński measure $R$ having a finite support.

**Theorem 6.3.** Fix $\alpha < 2$ and $p > 0$. The class $ETS^p_\alpha$ is the smallest class of distributions closed under convolution and weak convergence and containing all elementary $p$-tempered $\alpha$-stable distributions. In fact, $\mu \in ETS^p_\alpha$ if and only if there are probability measures $\mu_1, \mu_2, \ldots$ with $\mu_n \rightarrow \mu$ such that each $\mu_n$ is the distribution of the sum of a finite number of independent elementary $p$-tempered $\alpha$-stable random vectors.

For the case when $p = 1$ and $\alpha \in \{-1, 0\}$ this was shown in Theorem F of [3]. There the result followed from the properties of a certain integral representation. A similar representation for the case $\alpha < 2$ and $p > 0$ is given in [10]. However, in the case when $\alpha \in (0, 2)$
the properties of the representation are different. Thus it appears that a proof analogous to that of [3] can only be constructed when $\alpha \leq 0$. Instead, we base our proof on Theorem 5.1.

**Proof.** In light of Proposition 6.1, it suffices to show that we can approximate any distribution in $TS^p_\alpha$. Let $\mu = TS^p_\alpha(R, b)$ and let $\nu$ be its extended Rosiński measure. Let $(\nu_n)$ be any sequence of finite measures on $\mathbb{R}^d$ with a finite support such that $\nu_n(\{0\}) = 0$, $\nu_n((\mathbb{R}^{d-1}) = 0$, and $\nu_n \to \nu$ on $\mathbb{R}^d$ (such measures exist by e.g. Theorem 7.7.3 in [4]). Let $\mu_n = ETS^p_\alpha(0_{d \times d}, \nu_n, b)$. Note that by the Portmanteau Theorem (Theorem 3.12 in [12])

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \nu_n(\{|x| < \epsilon\}) \leq \lim_{\epsilon \to 0} \nu(\{|x| \leq \epsilon\}) = 0,$$

where the final equality follows from the fact that $\nu$ is a finite measure that places no mass at 0. Thus, $\mu_n \to \mu$ by (5.5) and Theorem 5.1.

Since all elementary $p$-tempered $\alpha$-stable distributions are proper, we immediately get the following.

**Corollary 6.4.** $ETS^p_\alpha$ is the smallest class of distributions closed under convolution and weak convergence and containing all proper $p$-tempered $\alpha$-stable distributions.

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