BERNSTEIN-TYPE TECHNIQUES FOR 2D FREE BOUNDARY GRAPHS

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Abstract. We prove an a-priori bound for the Lipschitz constant of a smooth one-phase free boundary graph $F(u)$ in two dimensions. The function $u$ satisfies an elliptic equation in its positive side, and $|\nabla u| = 1$ on $F(u)$.

1. Introduction

We consider the following one-phase free boundary problem,

\[
\begin{aligned}
\Delta u &= 0, &\text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\
|\nabla u| &= 1, &\text{on } F(u) := \partial \Omega^+(u) \cap \Omega,
\end{aligned}
\]

where $\Omega$ is a domain in $\mathbb{R}^n$. Here, for any non-negative function $v : \Omega \to \mathbb{R}$, we set

$\Omega^+(v) := \{x \in \Omega : v(x) > 0\}$, $\Omega^-(v) := \{x \in \Omega : v(x) = 0\}^\circ$, $F(v) := \partial \Omega^+(v) \cap \Omega$.

Problem (1.1) arises for example in the minimization of the variational integral

\[
J(u) = \int_{\Omega} (|\nabla u|^2 + \chi_{\{u>0\}}) dx,
\]

that appears in many applications (see [AC], [F].)

In [C1],[C2],[C3], the author introduced the notion of “viscosity” solution to (1.1), and developed the theory of existence and regularity of viscosity free boundaries. In particular, the regularity theory is inspired by the regularity theory for minimal surfaces, precisely by the “oscillation decay” method of De Giorgi, according to which if $S$ is a minimal surface in the unit ball $B_1$, and $S$ is the graph of a Lipschitz function, then $S$ is $C^{1,\alpha}$ (hence smooth) in $B_{1/2}$. Analogously, if $F(u)$ is a Lipschitz free boundary in $B_1$, then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$. Higher regularity results of [KN] then yield the local analyticity of $F(u)$ in the interior.

Thus, a natural question arises, that is how to obtain the Lipschitz continuity of a viscosity free boundary. In the theory of minimal surfaces, in the special case of minimal graphs, this is achieved via an a-priori gradient bound for solutions to the minimal surface equation, originally proved in [BMG]. Analogously, an a-priori bound for the Lipschitz constant of smooth free boundary graphs is needed, in order to obtain that viscosity free boundary graphs are smooth in the interior.

In this note we provide this tool in the 2D and 3D case. Our proof is based on the so-called Bernstein technique, which is been widely used in literature (see for example [GT].) A similar approach for minimal surfaces is used in [WX].
Moreover, in the 2D case, our technique is flexible enough to allow us to obtain an a-priori bound for the Lipschitz constant of free boundary graphs for a wider class of problems, to which the regularity theory of [C1] has been extended (see for example [WP]).

In order to state our main result, we introduce some notation. Let

\[ C_{(R,M)} := B_R(0) \times [-M, M] \subset \mathbb{R}^n. \]

Here \( B_r \) denotes a \((n-1)\)-dimensional ball, while a \(n\)-dimensional ball is denoted by \( B_r \). When \( R = 1 \), we simply write \( C_M \) for \( C_{(1,M)} \).

Assume that \( u \) is a classical solution to the following one-phase free boundary problem:

\[
\begin{align*}
\mathcal{F}(D^2 u) &= 0, \quad \text{in} \quad C^+_{M}(u), \\
|\nabla u| &= 1, \quad \text{on} \quad \partial C_{M}(u), \\
u_n &> 0, \quad \text{on} \quad \partial C^+_{M}(u),
\end{align*}
\]

that is, \( u \in C^2(C^+_{M}(u)) \), and \( F(u) \) is a \( C^2 \)-surface. Here \( \mathcal{F} \) is a nonlinear uniformly elliptic operator with ellipticity constants \( 0 < \lambda < \Lambda \), and \( \mathcal{F}(0) = 0 \).

Furthermore, assume that \( 0 \in F(u) \), and that

\[ B_{2\alpha}(0, M/2) \subset C^+_{M}(u), \quad B_{2\alpha}(0, -M/2) \subset C^{-}_{M}(u), \]

for some constant \( \alpha < 1/2 \).

By the implicit function theorem, \( F(u) \) is a smooth graph in the \( x_n \) (vertical) direction. Let us denote by \( \text{Lip}(s) \) the Lipschitz constant of \( F(u) \) over \( B_s(0) \).

In this note we focus on the 2 dimensional case, \( n = 2 \). Our main result is the following a priori bound.

**Theorem 1.1.** Assume \( n = 2 \), and let \( u \) be a solution to (1.2). Then, there exists constants \( C, s \), depending on \( M, \lambda, \Lambda, \alpha \) such that

\[
\frac{|u_1|}{u_2} \leq C \quad \text{on} \quad B_s(0).
\]

In particular \( \text{Lip}(s) \leq C \).

The paper is organized as follows. In Section 2, we prove a technical Lemma, which is dimension independent. In Section 3, for expository purposes, we present the proof of Theorem 1.1 in the case when \( \mathcal{F}(D^2 u) = \Delta u \). Then, in Section 4, we proceed to show the proof of Theorem 1.1 in the general case. In Section 5, we obtain the analogue of Theorem 1.1 in the case when \( u \) is a \( p \)-harmonic function in its positive phase. Finally, we conclude this note with some remarks about the applicability of the method of the proof to problems in higher dimensions.

2. A PRELIMINARY LEMMA.

Here and henceforth, \( C, C' \) will denote constants depending possibly on \( M, \lambda, \Lambda \) and \( \alpha \).
We start with the following technical Lemma, which holds in any dimension.

**Lemma 2.1.** Let $u$ be a solution to (1.2). Then, there exist constants $C, C'$, such that

1. $|\nabla u| \leq C'$ on $C(\alpha, M/2)$,
2. $u \geq C$ on $B_{\alpha}(0, M/2)$.

**Proof.** (1) We start by showing that if $x_0 \in C(\alpha, M/2)(u)$, and $d = \text{dist}(x_0, F(u))$, then

$$(2.1) \quad u(x_0) \leq Cd.$$ 

Let $v(x) = \frac{1}{d}u(x_0 + dx)$ be the rescale of $u$ in $B_d(x_0)$. Then $v \geq 0$ solves a uniformly elliptic equation

$$\mathcal{G}(D^2v) = 0 \quad \text{in} \quad B_1(0),$$

with $\mathcal{G}$ having the same ellipticity constants as $F$, and $\mathcal{G}(0) = 0$. Hence, by Harnack’s inequality (see [CC])

$$v \geq cv(0) \quad \text{in} \quad B_{1/2}(0).$$

Let us choose $\beta < 0$ such that, the radially symmetric function

$$g(x) = \frac{cv(0)}{2^{-\beta} - 1}(|x|^\beta - 1)$$

satisfies $\mathcal{G}(D^2g) \geq 0$ in the annulus $B_1 \setminus B_{1/2}, g = 0$ on $\partial B_1$ and $g = cv(0)$ on $\partial B_{1/2}$. Then, by the maximum principle

$$v \geq g \quad \text{in} \quad B_1 \setminus B_{1/2}.$$ 

Now, let $x_1 \in \partial B_1(0)$ be such that $v(x_1) = 0$. Then, since $\nabla v(x) = \nabla u(x_0 + dx)$, and $u$ solves (1.2), we have $|\nabla v(x_1)| = 1$. Let $\nu$ be the inward normal to $\partial B_1$ at $x_1$. Then, at $x_1$,

$$1 = |\nabla v(x_1)| \geq v_\nu \geq g_\nu \geq Cv(0),$$

which yields (2.1). Using Harnack’s inequality and elliptic regularity (see [CC]) we obtain the desired claim.

(2) Let $g$ be a radially symmetric function such that $F(D^2g) \leq 0$, in the annulus $B_\alpha(0, -M/2) \setminus B_{\alpha/2}(0, -M/2)$ and: 
\[
\begin{align*}
g &= C & \text{on } \partial B_\alpha(0,-M/2), \\
g &= 0 & \text{on } \partial B_{\alpha/2}(0,-M/2), \\
|\nabla g| &< 1 & \text{on } \partial B_\alpha(0,-M/2).
\end{align*}
\]

Since \( F \) is uniformly elliptic and \( F(0) = 0 \), such kind of supersolution can be obtained by a similar formula as in part (1).

Since \( u = 0 \) on \( B_{2\alpha}(0,-M/2) \), then \( g > u \) on the annulus between \( B_\alpha(0,-M/2) \) and \( B_{\alpha/2}(0,-M/2) \). Let \( g_t \) be a family of translates of \( g \) in the positive vertical direction. Then, the first touching point \( x_0 \) of \( g_t \) and \( u \) must occur where \( g_t = C \), and \( |x_0'| \leq \alpha \), within distance \( \alpha/2 \) from \( \partial B_\alpha(0,M/2) \). Moreover, \( x_0 \) occurs before \( F(g_t) \) coincides with \( \partial B_\alpha(0,M/2) \). Therefore, since \( u \) is monotone increasing in the vertical direction, \( u \geq C \) at some point \( x \in \partial B_\alpha(0,M/2) \). Harnack’s inequality implies the desired statement, \( u \geq C \) on \( B_\alpha(0,M/2) \). □

3. Proof of Theorem 1.1. The Laplace operator.

Let us define a smooth positive function \( g(x_1, u) \) over the trapezoid \( T(h,a) = \{(x_1, u) \in \mathbb{R}^2 : 0 < u < h, -u - a < x_1 < u + a \} \), with \( h, a > 0 \), which satisfies the following properties on \( \partial T \):

(i) \( g(x_1, u) > 0 \), on \( B := \{(x_1, u) : u = 0, -a < x_1 < a\} \);
(ii) \( g(x_1, u) = 0 \), on \( \partial T \setminus B \);
(iii) \( \beta g_{x_1} + g_u \geq 0 \), on \( B \), for all \( |\beta| \leq 1 \).

We localize on the box \( C(\alpha, M/2) \). Denote by \( \Omega \) be the intersection of \( C(\alpha, M/2) \) with the set \( S := \{(x_1, x_2) : (x_1, u(x_1, x_2)) \in T\} \). Notice that, in view of (2) in Lemma 2.1, by choosing the width \( a \) and height \( h \) of the trapezoid \( T \) sufficiently small, we can guarantee that \( \Omega \subset C(\alpha, M/2) \).

Define

\[
H(x) = G(x)e^{x_2 \log \left( \frac{|u_1|}{u_2} \right)},
\]

with

\[
G(x) = g(x_1, u(x)).
\]

Let

\[
H(\overline{x}) = \max_{\Omega} H(x),
\]

and assume by contradiction that \( H(\overline{x}) \geq N \geq 1 \), for some large constant \( N \) to be chosen later. Hence,

\[
|u_1| \geq \log \left( \frac{|u_1|}{u_2} \right) = \frac{H}{Ge^{x_2}} \geq CN, \text{ at } \overline{x},
\]

and also, using (1) in Lemma 2.1

\[
\frac{u_2}{G} \leq C \frac{|u_1|}{H} \leq C' \frac{u_2}{N}, \text{ at } \overline{x}.
\]
Furthermore, either of the following two possibilities holds:

1. \( \mathbf{x} \in F(u) \),
2. \( \mathbf{x} \in \Omega \).

Let us start by showing that (1) cannot occur. Indeed, in this case, we would have

\[
(\partial_n \log |H|)(\mathbf{x}) \leq 0,
\]

where \( \nu = (u_1(\mathbf{x}), u_2(\mathbf{x})) \) denotes the inner normal direction to \( F(u) \) at \( \mathbf{x} \). Hence, at \( \mathbf{x} \), we would have

\[
\begin{align*}
  & u_1 \left\{ \frac{G_1}{G} + \frac{1}{\log \left( \frac{|u_1|}{u_2} \right)} \left[ \frac{u_{11} - u_{21}}{u_1 - u_2} \right] \right\} + u_2 \left\{ \frac{G_2}{G} + 1 + \frac{1}{\log \left( \frac{|u_1|}{u_2} \right)} \left[ \frac{u_{12} - u_{22}}{u_1 - u_2} \right] \right\} = \\
  & \left( u_1 \frac{G_1}{G} + u_2 \frac{G_2}{G} + u_2 + \frac{1}{\log \left( \frac{|u_1|}{u_2} \right)} \left[ u_{11} - u_{22} + u_{12} \left( \frac{u_2}{u_1} - \frac{u_1}{u_2} \right) \right] \right) \leq 0.
\end{align*}
\]

Let us show that the quantity in the square bracket is zero along \( F(u) \). The free boundary condition says that

\[
(3.4) \quad u_1^2 + u_2^2 = 1 \quad \text{on} \quad F(u).
\]

Thus, differentiating this condition along the tangential direction \((u_2, -u_1)\) we obtain,

\[
(3.5) \quad u_2 (u_1 u_{11} + u_2 u_{21}) - u_1 (u_1 u_{12} + u_2 u_{22}) = 0 \quad \text{on} \quad F(u).
\]

Hence, we deduce that

\[
(3.6) \quad u_1 u_2 (u_{11} - u_{22}) + u_{12} (u_2^2 - u_1^2) = 0 \quad \text{on} \quad F(u).
\]

Therefore,

\[
\left[ u_{11} - u_{22} + u_{12} \left( \frac{u_2}{u_1} - \frac{u_1}{u_2} \right) \right](\mathbf{x}) = 0,
\]

and (3.3) reads,

\[
(3.7) \quad (u_1 \frac{G_1}{G} + u_2 \frac{G_2}{G} + u_2)(\mathbf{x}) \leq 0.
\]

On the other hand,

\[
(3.8) \quad (u_1 \frac{G_1}{G} + u_2 \frac{G_2}{G})(\mathbf{x}) = \frac{(u_1 g_{x_1} + g_u u_1) + g_u u_2^2}{G}(\mathbf{x}) = \frac{(u_1 g_{x_1} + g_u)(\mathbf{x})}{G(\mathbf{x})} \geq 0,
\]
according to property (iii) in the definition of $g$, and the free boundary condition (3.4). The inequality (3.8) together with the fact that $u_2 > 0$, contradicts (3.7).

Remark. We remark that this argument is independent of the particular equation which is satisfied by $u$ in its positive phase. Moreover, it is easily generalized to higher dimensions.

Now, we proceed to showing that by choosing $N$ sufficiently large, we obtain a contradiction also in case (2). In this case we would have,

\[(3.9) \quad (\partial_i \log |H|)(\overline{x}) = 0, \quad i = 1, 2,\]

and

\[(3.10) \quad \Delta (\log |H|)(\overline{x}) \leq 0.\]

For brevity, we denote by

\[
L = \left( \frac{1}{\log \left( \frac{|u_1|}{u_2} \right)} \right)(\overline{x}),
\]

hence, according to (3.1)

\[(3.11) \quad L \leq C/N.\]

Then, (3.9) reads,

\[(3.12) \quad \left( \frac{G_i}{G} + \delta_{i2} + L \left[ \frac{u_{1i}}{u_1} - \frac{u_{2i}}{u_2} \right] \right)(\overline{x}) = 0, \quad i = 1, 2.\]

In order to use (3.10), let us compute,

\[(3.13) \quad \partial_{ii}(\log |H|) = \frac{G_{ii}}{G} - \frac{G_i^2}{G^2} + L \left[ \frac{u_{1ii}}{u_1} - \frac{u_{1i}^2}{u_1^2} - \frac{u_{2ii}}{u_2} + \frac{u_{2i}^2}{u_2^2} \right] - L^2 \left[ \frac{u_{1i}}{u_1} - \frac{u_{2i}}{u_2} \right]^2, \quad i = 1, 2.

Thus, according to (3.10), using that $u_1, u_2$ are harmonic functions, we get

\[(3.14) \quad \frac{\Delta G}{G} - \frac{\left| \nabla G \right|^2}{G^2} + L \left[ -\frac{\left| \nabla u_1 \right|^2}{u_1^2} + \frac{\left| \nabla u_2 \right|^2}{u_2^2} \right] - L^2 \sum_{i=1}^{2} \left[ \frac{u_{1i}}{u_1} - \frac{u_{2i}}{u_2} \right]^2 \leq 0, \quad \text{at } \overline{x}.\]
We wish to prove that if \( N \) is large enough, the quantity \( L \left| \nabla u_2 \right|_{u_2}^2 \) is very large and hence it dominates all the summands in (3.14). Toward this aim, let us start by proving that if \( N \) is sufficiently large, then

\[
\frac{|G_2|}{G} (\bar{x}) \leq \frac{1}{2},
\]

which combined with (3.12) when \( i = 2 \), implies:

\[
\frac{1}{2} \leq L \left| \frac{u_{12}}{u_1} - \frac{u_{22}}{u_2} \right| \leq \frac{3}{2}.
\]

Indeed, from the definition of \( G \) and (3.2) we obtain immediately,

\[
\frac{|G_2|}{G} (\bar{x}) = \frac{|g_2|}{G} (\bar{x}) \leq \frac{C}{N}.
\]

Therefore, for \( N \) large enough, (3.15), hence (3.16) hold. Since \( |u_1(\bar{x})| \geq u_2(\bar{x}) \), we deduce immediately from (3.16),

\[
|\nabla u_2|_{u_2}^2 \geq u_2^2 + u_2^2 \geq \frac{1}{2} \left[ \frac{u_{12}}{u_1} - \frac{u_{22}}{u_2} \right]^2 \geq \frac{C}{L^2},
\]

which together with (3.11) gives that \( L \left| \nabla u_2 \right|_{u_2}^2 \) is very large, for \( N \) large. In particular, according to (3.17), for \( N \) large we have,

\[
\left| \frac{G_2}{G} (\bar{x}) \right|^2 \leq L^2 \left| \nabla u_2 \right|_{u_2}^2.
\]

Moreover, (3.12) for \( i = 1 \) implies

\[
\left| \frac{G_1}{G} \right|^2 = L^2 \left[ \frac{u_{11}}{u_1} - \frac{u_{21}}{u_2} \right]^2 \leq 2L^2 \left( \frac{u_{11}}{u_1} + \frac{u_{21}}{u_2} \right) \leq 2L^2 \frac{\left| \nabla u_2 \right|_{u_2}^2}{u_2^2},
\]

where in order to obtain (3.20) we have used that \( |u_1(\bar{x})| \geq u_2(\bar{x}) \), together the fact that since \( u \) is a solution to \( \Delta u = 0 \), then \( u_{11}^2 = u_{22}^2 \), and in particular

\[
\left| \nabla u_1 \right|^2 = \left| \nabla u_2 \right|^2.
\]

Thus, combining (3.19) and (3.20) we get,

\[
\frac{|\nabla G|}{G^2} \leq 3L^2 \frac{\left| \nabla u_2 \right|_{u_2}^2}{u_2^2},
\]

and combining (3.18) and (3.20) we also get,
Now, combining (3.14) with (3.1), (3.21), (3.22), and (3.23), we obtain

\[
\frac{\Delta G}{G} - 6L^2\left|\nabla u_2\right|^2 - L^2\left[\frac{1}{N^2} + 1\right] \leq 0, \quad \text{at } \bar{x}.
\]

Therefore, for \( L \) sufficiently small, that is \( N \) sufficiently large, we get

\[
\frac{\Delta G}{G} + \frac{L}{2}\left|\nabla u_2\right|^2 \leq 0, \quad \text{at } \bar{x}.
\]

On the other hand, since \( \Delta u = 0 \), we have

\[
\frac{\Delta G}{G} = \frac{\left[g_{x_1x_1} + 2g_{x_1u} + g_{uu} |\nabla u|^2\right]}{G} \geq -\frac{C}{G}.
\]

Moreover, since \( H(x) \geq N \), we have

\[
\frac{\Delta G}{G}(\bar{x}) \geq -\frac{C}{G(\bar{x})} \geq -\frac{C}{LN}.
\]

Therefore, combining (3.18), (3.25) and (3.27) we obtain

\[
-\frac{C}{LN} + \frac{C'}{L} \leq 0,
\]

and we reach a contradiction for \( N \) large. \( \square \)

4. PROOF OF THEOREM 1.1 NON-LINEAR OPERATORS.

The proof follows the lines of the case when \( F(D^2u) = \Delta u \). Precisely, with the same notation as in Section 3, we assume by contradiction that \( H(\bar{x}) \geq N \geq 1 \), for some large constant \( N \) to be chosen later. Hence, the following three bounds hold

\[
\frac{|u_1|}{u_2}(\bar{x}) \geq CN,
\]

\[
\frac{u_2}{G}(\bar{x}) \leq \frac{C}{N},
\]

\[
\frac{1}{G}(\bar{x}) \leq \frac{C}{LN}.
\]

Furthermore, according to the argument in Section 2 (and the remark following it), the maximum must be achieved in the interior, that is \( \bar{x} \in \Omega \).
Now, we proceed to showing that by choosing \( N \) sufficiently large, we obtain a contradiction. Since \( H \) achieves a maximum at \( \mathbf{x} \) we have,

\[
(\partial_i \log |H|)(\mathbf{x}) = 0, \quad i = 1, 2,
\]

and

\[
\mathcal{L}(\log |H|)(\mathbf{x}) \leq 0,
\]

where

\[
\mathcal{L}(v) = \sum_{i,j=1}^{2} a_{ij}v_{ij},
\]

is the linearized operator associated to \( \mathcal{F}(D^2v) \). Again, we denote by

\[
L = \left( \frac{1}{\log \left( \frac{|u_1|}{u_2} \right)} \right)(\mathbf{x}).
\]

In order to use (4.5), let us compute,

\[
\partial_{ij}(\log |H|) = \frac{G_{ij}}{G} \frac{G_i G_j}{G^2} + L \left[ \frac{u_{11j}}{u_1} - \frac{u_{1j}u_{1i}}{u_1^2} - \frac{u_{2ij}}{u_2} + \frac{u_{2j}u_{2i}}{u_2^2} \right]
\]

\[
- L^2 \partial_j \left( \log \left( \frac{|u_1|}{u_2} \right) \right) \partial_i \left( \log \left( \frac{|u_1|}{u_2} \right) \right), \quad i,j = 1,2.
\]

Thus, (4.5) reads,

\[
\frac{\mathcal{L}G}{G} - \frac{1}{G^2} \sum_{i,j=1}^{2} a_{ij}G_i G_j + L \left[ \sum_{i,j=1}^{2} a_{ij} \left( \frac{u_{2j}u_{2i}}{u_2^2} - \frac{u_{2j}u_{2i}}{u_2^2} \right) \right]
\]

\[
- L^2 \sum_{i,j=1}^{2} a_{ij} \partial_j \left( \log \left( \frac{|u_1|}{u_2} \right) \right) \partial_i \left( \log \left( \frac{|u_1|}{u_2} \right) \right) \leq 0, \quad \text{at } \mathbf{x},
\]

where we have used that \( u_1, u_2 \) are solutions to the linearized equation \( \mathcal{L}v = 0 \). Then, by the uniform ellipticity of \( \mathcal{L} \) we derive the following inequality

\[
\frac{\mathcal{L}G}{G} - \Lambda \frac{|\nabla G|^2}{G^2} + L \left[ \lambda \frac{|\nabla u_2|^2}{u_2^2} - \Lambda \frac{|\nabla u_1|^2}{u_1^2} \right]
\]

\[
- L^2 \left[ \left( \frac{u_{11}}{u_1} - \frac{u_{21}}{u_2} \right)^2 + \left( \frac{u_{12}}{u_1} - \frac{u_{22}}{u_2} \right)^2 \right] \leq 0, \quad \text{at } \mathbf{x}.
\]

Again, we wish to prove that the quantity \( L \frac{|\nabla u_2|^2}{u_2^2} \) is very large, and it dominates all the negative summands in (4.8). The same argument as in Section 3 gives that
Moreover, although \(3.21\) is no longer valid, we know that \(u\) is a solution to \(F(D^2 u) = 0\), and \(F(0) = 0\). Therefore \(u\) solves a linear equation with uniformly bounded coefficients, and we get

\[
(4.10) \quad |\nabla u_1|^2 \leq C|\nabla u_2|^2,
\]

for some constant \(C\) depending on the ellipticity constants \(\lambda, \Lambda\). Thus, we conclude as in the previous section, that the following two bounds hold:

\[
(4.11) \quad \frac{|\nabla G|^2}{G^2} \leq CL^2 \frac{|\nabla u_2|^2}{u_2^2},
\]

\[
(4.12) \quad L^2 \left( \left[ \frac{u_{11}}{u_1} \right] - \frac{u_{21}}{u_2} \right)^2 + \left[ \frac{u_{12}}{u_1} - \frac{u_{22}}{u_2} \right]^2 \leq CL^2 \frac{|\nabla u_2|^2}{u_2^2}.
\]

Combining \(4.8\) with \(4.11, 4.11, 4.12\), we obtain

\[
(4.13) \quad \frac{\mathcal{L}G}{G} - CL^2 \frac{|\nabla u_2|^2}{u_2^2} + L \left( \lambda \frac{|\nabla u_2|^2}{u_2^2} - CL \frac{|\nabla u_2|^2}{N^2 u_2^2} \right) \leq 0, \quad \text{at } \varpi.
\]

Hence, for \(N\) sufficiently large, that is \(L\) small enough,

\[
(4.14) \quad \frac{\mathcal{L}G}{G} + CL \frac{|\nabla u_2|^2}{u_2^2} \leq 0, \quad \text{at } \varpi.
\]

We now proceed to estimate \(\mathcal{L}G/G\). Using \(4.2, 4.3\) and \(4.10\), we get

\[
(4.15) \quad \frac{|\mathcal{L}G|}{G} = \frac{1}{G} \sum_{i,j=1}^{2} a_{ij} \left[ g_{xx,ij} \delta_{i1} \delta_{j1} + 2g_{xu,ij} u_j + g_{uu,ij} u_j + g_{iu,ij} \right] \leq \frac{1}{G} (C + C|\nabla u_2|) \leq \frac{C}{LN} + \epsilon L \frac{|\nabla u_2|^2}{u_2^2} + \frac{1}{4\epsilon L} \frac{u_2^2}{G^2} \leq \frac{C}{LN} + \epsilon L \frac{|\nabla u_2|^2}{u_2^2} + \frac{1}{4\epsilon L} \frac{C}{N^2}.
\]

Hence, for \(\epsilon\) small, combining \(4.14\) with \(4.9\), and \(4.15\) we get,

\[
(4.16) \quad -\frac{C}{LN} - \frac{C}{LN^2} + \frac{C}{L} \leq 0,
\]

that is a contradiction for \(N\) large enough.
5. The p-Laplace operator.

In this section, we generalize the a-priori bound in Theorem 1.1 to the case when \( u \) is a classical solution to the following one-phase free boundary problem:

\[
\begin{align*}
\text{div}(|\nabla u|^{p-2}\nabla u) &= 0, & \text{in } C_M^+(u), \\
|\nabla u| &= 1, & \text{on } F(u), \\
u_2 &> 0, & \text{on } C_M^-(u),
\end{align*}
\]

with \( 1 < p < \infty \). The setting will be the same as in Section 1, that is we assume that \( 0 \in F(u) \), and

\[
B_{2\alpha}(0, M/2) \subseteq C_M^+(u), \quad \text{and} \quad B_{2\alpha}(0, -M/2) \subseteq C_M^-(u),
\]

for some constant \( \alpha < 1/2 \).

Here and henceforth, \( C, C' \) will denote constants depending possibly on \( M, p \) and \( \alpha \). The following technical Lemma still holds.

**Lemma 5.1.** Let \( u \) be a solution to (5.1). Then, there exist constants \( C, C' \), such that

1. \( |\nabla u| \leq C' \) on \( C_{M/2}(\alpha, M/2) \),
2. \( u \geq C \) on \( B_\alpha(0, M/2) \).

We wish to prove the following result.

**Theorem 5.2.** There exists constants \( C, s \), depending on \( M, \lambda, \Lambda, \alpha, p \) such that

\[
\frac{|u_1|}{u_2} \leq C \quad \text{on } B_s(0).
\]

In particular \( \text{Lip}(s) \leq C \).

**Proof.** The proof follows the lines of the proof of Theorem 1.1. Again, we introduce the function \( H \) and we assume that it achieves a large maximum at \( \overline{x} \), which according to the argument in Section 2 must be an interior point, i.e. \( \overline{x} \in \Omega \). Then,

\[
\partial_i \log |H|(\overline{x}) = 0, \quad i = 1, 2,
\]

and

\[
\mathcal{L}(\log |H|)(\overline{x}) \leq 0,
\]

where

\[
\mathcal{L}(v) = \sum_{i,j=1}^2 a_{ij}v_{ij}, \quad a_{ij} = \delta_{ij} + (p - 2)\frac{v_i v_j}{|\nabla v|^2}.
\]

In particular, \( \mathcal{L} \) is uniformly elliptic with constants \( \lambda = \min\{1, p-1\}, \Lambda = 1 + |p-2| \). Notice that since \( |\nabla u| > 0 \), \( u \) is a solution to \( \mathcal{L}v = 0 \). Again, we denote by
\[ L = \left( \frac{1}{\log \left( \frac{|u_1|}{u_2} \right)} \right)(x). \]

According to (5.3), from formula (4.6) and from the uniform ellipticity of \( L \), we derive the following inequality

\[
\begin{align*}
\frac{L^2 G}{G^2} - \Lambda G \frac{\nabla G}{G^2} + L \left[ \lambda \frac{\nabla u_2}{u_2}^2 - \Lambda \frac{\nabla u_1}{u_1}^2 \right] + L \left[ \sum_{i,j=1}^{2} a_{ij} \left( \frac{u_{1ij}}{u_1} - \frac{u_{2ij}}{u_2} \right)^2 \right] \\
- \lambda^2 \left[ \left( \frac{u_{11}}{u_1} - \frac{u_{21}}{u_2} \right)^2 + \left( \frac{u_{12}}{u_1} - \frac{u_{22}}{u_2} \right)^2 \right] &\leq 0, \text{ at } x.
\end{align*}
\]

The difference between this inequality and (4.8), consists in the presence on the term \( L \sum_{i,j=1}^{2} a_{ij} \left[ \frac{u_{1ij}}{u_1} - \frac{u_{2ij}}{u_2} \right] \), which appears since \( u_1 \) and \( u_2 \) are not solutions to \( L v = 0 \). Thus, if we show that

\[
(5.5) \quad \left| \sum_{i,j=1}^{2} a_{ij} \left( \frac{u_{1ij}}{u_1} - \frac{u_{2ij}}{u_2} \right) \right| \leq \lambda \left( \frac{\nabla u_2}{u_2} \right)^2\]

we obtain that

\[
(5.6) \quad 0 = \partial_k(Lu) = \sum_{i,j=1}^{2} a_{ij} u_{ijk} + \sum_{i,j=1}^{2} \partial_k(a_{ij}) u_{ij},
\]

and we reach a contradiction as in the previous section.

In order to prove (5.4), we start by differentiating the equation \( Lu = 0 \). We get,

\[
(5.7) \quad \sum_{i,j=1}^{2} a_{ij} u_{ijk} + 2(p-2) \sum_{i,j=1}^{2} \left( \frac{u_{ijk}}{|\nabla u|^2} - u_{ij} \frac{\sum_{l=1}^{2} u_{il} u_{lk}}{|\nabla u|^4} \right) u_{ij} = 0.
\]

Now, using (5.7), set \( \epsilon = \frac{\lambda}{16(p-1)} \) we obtain that
BERNSTEIN-TYPE TECHNIQUES FOR 2D FREE BOUNDARY GRAPHS

\[ \left| \frac{1}{u_1} \sum_{i,j=1}^{2} a_{ij} u_{1ij} - \frac{1}{u_2} \sum_{i,j=1}^{2} a_{ij} u_{2ij} \right| = \]

\[ \frac{2p-1}{|\nabla u|^2} \left| u_{11}^2 - u_{22}^2 + u_{12} \left( \frac{u_2}{u_1} - \frac{u_1}{u_2} \right) (u_{11} + u_{22}) \right| \leq \]

\[ 2 \frac{p-1}{|\nabla u|^2} \left[ u_{11}^2 + u_{22}^2 + 2\epsilon u_{12} \left( \frac{u_2^2}{u_1^2} + \frac{u_1^2}{u_2^2} \right) + \frac{1}{2\epsilon} (u_{11}^2 + u_{22}^2) \right] \leq \]

\[ C \frac{1}{\epsilon} (p-1) \frac{\left| \nabla u_2 \right|^2}{N^2 u_2^2} + 4\epsilon \frac{p-1}{u_2^2} |\nabla v_2|^2 \leq \]

\[ \frac{\lambda |\nabla u_2|^2}{2 u_2^2} \]

as long as \( N \) is large enough. \( \square \)

6. A priori bound for 3D free boundary graphs.

In this section we extend our result in 3D, for the case when \( u \) is harmonic in its positive phase. We intend to highlight the difficulties which arise when trying to adapt our technique to higher dimensions.

Assume that \( u \) is a classical solution to the following one-phase free boundary problem:

\[
\begin{aligned}
\Delta u &= 0, \quad \text{in } C^+_M(u), \\
|\nabla u| &= 1, \quad \text{on } F(u), \\
u_3 &> 0, \quad \text{on } C^-_M(u).
\end{aligned}
\]

Furthermore, assume that \( 0 \in F(u) \), and that

\[ B_{2\alpha}(0, M/2) \subseteq C^+_M(u), \quad \text{and} \quad B_{2\alpha}(0, -M/2) \subseteq C^-_M(u), \]

for some constant \( \alpha < 1/2 \).

**Theorem 6.1.** There exists constants \( C, s \), depending on \( M, \alpha \) such that

\[ \text{Lip}(s) \leq C. \]

**Proof.** Let \( g = g(r, u) \) be the function introduced in Section 3. One can easily construct such function, so that it satisfies the following condition:

(iv) \( |\nabla g| \leq Cg \) near \( \partial T \setminus B \).

We localize on the box \( C_{(\alpha, M/2)} \). Denote by \( \Omega \) be the intersection of \( C^+_M(u) \) with the set \( S := \{ x : (|x'|, u(x)) \in T \} \). Again, in view of (2) in Lemma 2.1 \( \Omega \subseteq C_{(\alpha, M/2)} \).

Define

\[ H(x) = G(x)e^{\alpha \delta} \log \left( \frac{|\nabla u_2|}{u_3} \right), \]
with
\[ G(x) = g(|x'|, u(x)). \]
Let
\[ H(x) = \max_{\Omega} H(x), \]
and assume by contradiction that \( H(x) \geq N \geq 1 \), for some large constant \( N \) to be chosen later. Without loss of generality we can assume that \( |\nabla_x u(x)| = u_1(x) \), hence in particular
\[ (6.2) \quad \partial_i(|\nabla_x u|)(x) = u_{1i}(x). \]
Also,
\[ (6.3) \quad \frac{u_1}{u_3} \geq \frac{H}{Ge^{x_3}} \geq CN, \text{ at } x. \]
By the same argument as in 2D, one can deduce that \( x \) is an interior point.
We proceed to showing that by choosing \( N \) sufficiently large, we obtain a contradiction. We have,
\[ (6.4) \quad (\partial_i \log |H|)(x) = 0, \quad i = 1, 2, 3 \]
and
\[ (6.5) \quad \Delta (\log |H|)(x) \leq 0. \]
For brevity, we denote by
\[ L = \left( \frac{1}{\log \left( \frac{u_1}{u_3} \right)} \right)(x), \]
hence, according to (6.3)
\[ (6.6) \quad L \leq C/N. \]
Then, (6.4) reads,
\[ (6.7) \quad \left( \frac{G_i}{G} + \delta_{i3} + L \left[ \frac{u_{1i}}{u_1} - \frac{u_{3i}}{u_3} \right] \right)(x) = 0, \quad i = 1, 2, 3. \]
In particular, at \( x \),
\[ (6.8) \quad L^2 \sum_{i=1}^{3} \left[ \frac{u_{1i}}{u_1} - \frac{u_{3i}}{u_3} \right]^2 \leq \frac{|\nabla G|^2}{G^2} + 1. \]
In order to use (6.3), let us compute at \( x \),
\[ \partial_i(\log |H|) = \frac{G_{ii}}{G} - \frac{G^2}{G^2} + L \left[ \frac{\partial_i |\nabla x'u|}{|\nabla x'u|} - \frac{u_{1i}^2}{u_1^2} u_1^3 + \frac{u_{3i}^2}{u_3^2} u_3^3 \right] - L^2 \left[ \frac{u_{ii}^2}{u_1^2} u_1^3 - \frac{u_{3i}^2}{u_3^2} u_3^3 \right]^2, \quad i = 1, 2, 3, \]

and,

\[ \Delta \frac{\nabla |x'u|}{|\nabla x'u|} = \sum_{i=1}^{3} \frac{u_{ii}^2}{u_1^2}. \]

Thus, according to (6.5), using (6.8) together with the fact that \( u_3 \) is harmonic, we get

\[ \frac{\Delta G}{G} - 2 \frac{\nabla G^2}{G^2} - 1 + L \left[ \sum_{i=1}^{3} \frac{(u_{ii}^2 - u_{ii}^3)}{u_1^2} + \frac{|\nabla u_3|}{u_3^2} \right] \leq 0, \quad \text{at } \mathfrak{F}. \]

With similar computations as in 2D, one has that \( L \frac{|\nabla u_3|^2}{u_3^2} \) is very large, and it dominates all the other summands. Precisely

\[ \frac{|\nabla u_3|^2}{u_3^2} \geq \frac{1}{2} \left[ \frac{u_{13}^2}{u_1} - \frac{u_{33}^2}{u_3} \right]^2 \geq \frac{C}{L^2}. \]

Now, from (6.9), using that \( u \) is harmonic, we obtain at \( \mathfrak{F} \),

\[ \frac{\Delta G}{G} - 2 \frac{\nabla G^2}{G^2} - 1 + L \left[ \frac{u_{13}^2}{u_1} + \frac{u_{33}^2}{u_3} + 2 \frac{u_{11} u_{33}}{u_1^2} + \frac{u_{31}^2}{u_3^2} \right] \leq 0. \]

On the other hand, (6.7) for \( i = 1 \) gives,

\[ L \left| \frac{u_{11}}{u_1} \right| \leq L \left| \frac{u_{13}}{u_3} \right| + \left| \frac{G_1}{G} \right| \quad \text{at } \mathfrak{F}. \]

Hence, using that \( u_1/u_3 \geq 1/L \) at \( \mathfrak{F} \), we have

\[ 2L \left| \frac{u_{11} u_{33}}{u_1^2} \right| \leq L \left[ \frac{u_{31}^2}{u_3^2} + \frac{u_{33}^2}{u_3^2} \right] + \left[ \frac{1}{\epsilon} \frac{G_1^2}{G^2} + \frac{L}{u_3^2} \right] \quad \text{at } \mathfrak{F}. \]

Combining this estimate with (6.11) we obtain, \( (\epsilon = 1/2) \)

\[ \frac{\Delta G}{G} - 4 \frac{\nabla G^2}{G^2} - 1 + L \left[ \frac{u_{33}^2}{u_3^2} - \frac{u_{13}^2}{u_1^2} + \frac{u_{13}^2}{u_3^2} + \frac{u_{31}^2}{u_3^2} \right] \leq 0, \quad \text{at } \mathfrak{F}. \]

Hence, for \( N \) large enough, using (6.10) we get
\[ \frac{\Delta G}{G} - 4\frac{|\nabla G|^2}{G^2} - 1 + \frac{C}{L} \leq 0, \text{ at } \tau. \]

We can now reach a contradiction as in the 2 dimensional case, using that, according to property (iv) of \( g \), we have \( |\nabla G|^2/G^2 \leq C/G \text{ at } \tau. \)

\[ \square \]

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