Divergence of Lubkin’s series for a quantum subsystem’s mean entropy

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Abstract. In 1978, Lubkin proposed a method of approximating the mean von Neumann entropy for a subsystem of a finite-dimensional quantum system in an overall pure state by expanding the entropy as a series in terms of the mean trace of powers of the system’s reduced density operator, but the convergence of this series was never established. We find an exact closed form expression for the mean traces, which enables us to prove that the series converges if and only if the system’s dimension $m \leq 2$. This brings into question the validity of Lubkin’s approximation.
1. Introduction

This paper is a comment on a previous paper by Lubkin [4], in which he considered the von Neumann entropy of an $m$-dimensional subsystem $A$ of an $mn$-dimensional quantum system $S$ when $S$ is in a pure state. For a given pure state of $S$ (represented by the density operator $\hat{\rho}^{mn}$), the entropy is given in terms of the reduced density operator $\hat{\rho}^{mn}_m$ of $A$ as

$$S_{m,n} = -\text{Tr}[\hat{\rho}^{mn}_m \ln \hat{\rho}^{mn}_m].$$

Lubkin was concerned specifically with the mean entropy $\langle S_{m,n} \rangle$ of a random pure state of $S$. Considering the pure state of $S$ instead as a normalised vector $|x\rangle$ in the Hilbert space of $S$ (denoted $\mathcal{H}_S$), such that $\hat{\rho}^{mn} = |x\rangle\langle x|$, he defined the mean $\langle S_{m,n} \rangle$ with respect to the natural invariant measure on the unit sphere in $\mathcal{H}_S$, which he referred to as the Haar measure. He attempted to find an approximation of $\langle S_{m,n} \rangle$ by proposing the use of the Taylor series expansion of the logarithm, giving (in our notation)

$$\langle S_{m,n} \rangle = \ln m + \sum_{r=1}^{\infty} \frac{m^r}{r(r+1)} (-1)^r \langle \text{Tr}[(\hat{\rho}^{mn}_m - \hat{\rho}_0)^{r+1}] \rangle,$$

where $\hat{\rho}_0 = 1/\sqrt{m}$. He then showed that

$$\langle \text{Tr}[(\hat{\rho}^{mn}_m - \hat{\rho}_0)^2] \rangle = \frac{m^2 - 1}{mn + 1}$$

and, based on the assumption that (1) converged, truncated the series after the first two terms to propose the approximation

$$\langle S_{m,n} \rangle \approx \ln m - \frac{1}{2} \frac{m^2 - 1}{mn + 1}$$

for $n \gg m$.

The purpose of this paper, however, is to show that (1) in fact only converges when $m \leq 2$, and diverges absolutely otherwise. We do so by finding closed-form expressions for the series terms in (1), given in (18), and looking at their behaviour as $r \to \infty$. From this it becomes clear that the series diverges rapidly when $m > 2$, indicating that there is no reason to expect truncations of it to be good approximations of the entropy.

This result is somewhat unexpected, however, as further work has been done since which appears to support the validity of (2). In particular, Page proposed the exact formula

$$\langle S_{m,n} \rangle = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}$$

for the entropy [5], which was later proven by a number of methods [2, 6, 8], and from this work he also derived the approximation

$$\langle S_{m,n} \rangle \approx \ln m - \frac{m}{2n}$$

for $1 \ll m \leq n$, which he points out does agree with Lubkin’s guess in the overlap ($1 \ll m \ll n$) of its range of validity with the range of validity ($n \gg m$, as stated earlier) given by Lubkin for his guessed approximation. However, as we concluded above, there appears to be no evidence to support the validity of Lubkin’s guess outside the overlap ($1 \ll m \ll n$).
2. Preliminaries

Determining where (1) does and does not converge requires studying the large-$r$ limit of $\langle \text{Tr}[(\hat{\rho}_m^{mn} - \hat{\rho}_0)^r] \rangle$. In order to evaluate this, it is easiest to first evaluate $\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle$, which we do in this section using a method based on work by Lloyd and Pagels [3], Page [5] and Sen [8].

**Theorem 1.**

$$\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{\Gamma(mn)}{r\Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + r - k)\Gamma(n + r - k)}{k!\Gamma(r - k)\Gamma(m - k)\Gamma(n - k)}$$  \hspace{1cm} (3)

for any real $r$.

**Proof.** To begin with, we assume that $n \geq m$. $\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle$ is necessarily symmetric under exchange of $m$ and $n$, as $\hat{\rho}_m^{mn}$ and $\hat{\rho}_n^{mn}$ (the reduced density operator of the ‘other part’ of $S$, which Lubkin refers to as a “reservoir”) always have the same eigenvalues when $S$ is in a pure state [7, 4], meaning that $\text{Tr}[(\hat{\rho}_m^{mn})^r] = \text{Tr}[(\hat{\rho}_n^{mn})^r]$ for any $r$, so it will still be possible to derive the behaviour when $n < m$ from these results by exchanging $m$ and $n$ (we will do so at the end of this proof).

If the eigenvalues of $\hat{\rho}_m^{mn}$ are labelled $\{p_1, \ldots, p_m\}$, then

$$\text{Tr}[(\hat{\rho}_m^{mn})^r] = \sum_{i=1}^{m} p_i^r,$$

which is valid for any real $r$. This is important, as it means that taking the mean of this expression only requires performing an integral over the space of possible combinations of eigenvalues (i.e. the space $(\mathbb{R}^+)^m$). Lloyd and Pagels [3] proved that (when $n \geq m$) the joint probability distribution over the eigenvalues which is equivalent to Lubkin’s Haar measure is

$$P(p_1, \ldots, p_m) dp = \Delta^2(p_1, \ldots, p_m) \delta \left(1 - \sum_{i=1}^{m} p_i\right) \prod_{k=1}^{m} p_k^{n-m} dp_k,$$

where

$$\Delta(p_1, \ldots, p_m) = \prod_{1 \leq i < j \leq m} (p_j - p_i) = \begin{vmatrix} 1 & p_1 & \cdots & p_1^{n-1} \\ 1 & p_2 & \cdots & p_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & p_m & \cdots & p_m^{n-1} \end{vmatrix}$$  \hspace{1cm} (4)

is the Vandermonde determinant of the eigenvalues. Page then showed how this could be used to construct an eigenvalue integral expression for $\langle S_{m,n} \rangle$ by integrating $\text{Tr}[\hat{\rho}_m^{mn} \ln \hat{\rho}_m^{mn}]$ using this distribution [5]. We will now apply the same method to $\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle$. First we write

$$\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{1}{\Lambda} \int \delta \left(1 - \sum_{i=1}^{m} p_i\right) \Delta^2(p_1, \ldots, p_m)$$

$$\times \prod_{k=1}^{m} p_k^{n-m} dp_k \sum_{i=1}^{m} p_i^r,$$
where $\Lambda$ is a normalisation factor which is a function of $m$ and $n$ defined such that
\[
\langle \text{Tr}(\hat{\rho}_m^{mn})^0 \rangle = m.
\]
Next we multiply this by the factor
\[
\frac{1}{\Gamma(mn + r)} \int_0^{\infty} \lambda^{mn+r-1} e^{-\lambda} d\lambda,
\]
which equals unity by the definition of the gamma function and perform the coordinate substitution $q_i = \lambda p_i$. Some rearrangement gives
\[
\langle \text{Tr}(\hat{\rho}_m^{mn})^r \rangle = \frac{1}{\Delta(mn + r)} \int_0^{\infty} e^{-\lambda} d\lambda \int \delta\left(\lambda - \sum_{i=1}^{m} q_i\right) \Delta^2(q_1, \ldots, q_k) \times \prod_{k=1}^{m} q_k^{n-m} dq_k \sum_{i=1}^{m} q_i^r.
\]
Next, as the integral is symmetric under exchange of any two $q_i$, we can remove the summation and simply write that
\[
\langle \text{Tr}(\hat{\rho}_m^{mn})^r \rangle = \frac{m}{\Delta(mn + r)} \int \Delta^2(q_1, \ldots, q_m) \prod_{k=1}^{m} q_k^{n-m} e^{-q_k} dq_k \cdot \sum_{i=1}^{m} q_i^r.
\]

The remainder of the proof follows the same procedure used by Sen to prove Page’s conjectured entropy formula [3]. He observed that, as the determinant of a matrix is unchanged by addition of multiples of its columns onto each other, the definition of the Vandermonde determinant given in [4] can be rewritten as
\[
\Delta(q_1, \ldots, q_m) = \begin{vmatrix} L_0^\alpha(q_1) & L_1^\alpha(q_1) & \cdots & L_{m-1}^\alpha(q_1) \\ L_0^\alpha(q_2) & L_1^\alpha(q_2) & \cdots & L_{m-1}^\alpha(q_2) \\ \vdots & \vdots & \ddots & \vdots \\ L_0^\alpha(q_m) & L_1^\alpha(q_m) & \cdots & L_{m-1}^\alpha(q_m) \end{vmatrix},
\]

where $L_i^\alpha$ are generalised Laguerre polynomials
\[
L_i^\alpha(q) = \frac{e^q}{q^\alpha} (-1)^i \frac{d^i}{dq^i} (e^{-q} q^{\frac{i}{\alpha}})
\]
for some real factor $\alpha$, defined such that $L_i^\alpha(q)$ is an order $i$ polynomial in $q$ where the coefficient of $q^i$ is unity (this and any other properties of $L_i^\alpha$ used in this paper are taken from [3]).

Using the expansion of the determinant in terms of the Levi-Civita symbol, we can write
\[
\Delta^2(q_1, \ldots, q_m) = \varepsilon_{i_1i_2\ldots i_m} \varepsilon_{j_1j_2\ldots j_m} \prod_{\sigma=1}^{m} L_{i_\sigma}^\alpha(q_\sigma) L_{j_\sigma}^\alpha(q_\sigma).
\]

† Technically $\langle \text{Tr}(\hat{\rho}_m^{mn})^0 \rangle = a_0$ where $a_0 = \min(m, n)$, in keeping with the required symmetry of $\langle \text{Tr}(\hat{\rho}_m^{mn})^0 \rangle$. However, as we are only looking at the cases where $m \leq n$ to begin with, it is sufficient to say for the moment that $\langle \text{Tr}(\hat{\rho}_m^{mn})^0 \rangle = m$. 

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If this is substituted into (5) with $\alpha = n - m$, then the orthogonality relation

$$\int_0^\infty q^{n-m} e^{-q} L_i^{n-m}(q) L_j^{n-m}(q) dq = i! \Gamma(n - m + j + 1) \delta_{ij}$$

causes any terms to vanish which don’t satisfy $i_\sigma = j_\sigma$ for all $2 \leq \sigma \leq m$ (it also follows then that $i_1 = j_1$ by a process of elimination). Collecting together only the non-zero terms, we are left with

$$\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{m!}{\Lambda \Gamma(mn + r)} \sum_{i=0}^{m-1} \int_0^\infty q_i^{n-m+r} e^{-q_i} [L_i^{n-m}(q_i)]^2 dq_i$$

$$\times \prod_{j \neq i} \int_0^\infty q_j^{n-m} e^{-q_j} [L_j^{n-m}(q_j)]^2 dq_j$$

$$= \frac{m!}{\Lambda \Gamma(mn + r)} \prod_{j=0}^{m-1} j! \Gamma(n - m + j + 1)$$

$$\times \sum_{i=0}^{m-1} \int_0^\infty q_i^{n-m+r} e^{-q_i} [L_i^{n-m}(q_i)]^2 dq_i$$

$$\times \frac{\Gamma(mn)}{\Gamma(mn + r)} \frac{\prod_{j=0}^{m-1} j! \Gamma(n - m + j + 1)}{i! \Gamma(n - m + i + 1)}.$$

We fix $\Lambda$ now by looking at the special case $r = 0$, where

$$\langle \text{Tr}[(\hat{\rho}_m^{mn})^0] \rangle = m \cdot \frac{m!}{\Lambda \Gamma(mn)} \prod_{j=0}^{m-1} j! \Gamma(n - m + j + 1) = m,$$

meaning that we can write

$$\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{\Gamma(mn)}{\Gamma(mn + r)} \frac{\prod_{j=0}^{m-1} j! \Gamma(n - m + j + 1)}{i! \Gamma(n - m + i + 1)} \int_0^\infty q_i^{n-m+r} e^{-q_i} [L_i^{n-m}(q_i)]^2 dq_i.$$ (7)

The remaining integrals do not match the orthogonality relation due to the additional $q^r$ factor, but we can still evaluate them as finite sums using two additional identities given in [8]:

$$L_i^\alpha(q) = \sum_{k=0}^i \left( \begin{array}{c} i \\ k \end{array} \right) (-1)^k \frac{\Gamma(i + \alpha + 1)}{\Gamma(i + \alpha - k + 1)} q^{i-k}$$

$$= \sum_{k=0}^i \left( \begin{array}{c} i \\ k \end{array} \right) (-1)^{i-k} \frac{\Gamma(i + \alpha + 1)}{\Gamma(k + \alpha + 1)} q^k$$ (8)

and

$$\int_0^\infty q^{\beta-1} e^{-q} L_i^\alpha(q) dq = (-1)^i (1 - \beta + \alpha)_i \Gamma(\beta)$$

$$= \frac{\Gamma(\beta - \alpha)}{\Gamma(\beta - \alpha - i)} \Gamma(\beta),$$ (9)

where $(1 - a)_i = (1 - a)(2 - a) \ldots (i - a) = (-1)^i \Gamma(a)/\Gamma(a - i)$ is the Pochhammer symbol representing the rising factorial. We simplify (7) by substituting (8) in place.
of one of the $L_i^{n-m}$ terms and then evaluating the integral using (9). This gives

$$\langle \text{Tr}[i^{\hat{\rho}_m^{mn}}] \rangle = \frac{\Gamma(mn)}{r \Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{\Gamma(r - k) \Gamma(m + r - k) \Gamma(n + r - k)}{k! \Gamma(r - k) \Gamma(m - k) \Gamma(n - k)}.$$

(10)

The steps in this rearrangement are:

(i) Swap the order of the two summations using $\sum_{i=0}^{m-1} \sum_{k=0}^{n} (-1)^i \frac{\Gamma(n - m + r + k + 1) \Gamma(r + k + 1)}{k! \Gamma(i - k + 1) \Gamma(n - m + k + 1) \Gamma(r - i + 1)}$

(ii) Evaluate the sum over $i$ using the identity

$$\sum_{i=0}^{a} \frac{(-1)^i}{i+1} \frac{(-1)^n}{r(a+1)\Gamma(r-a)}$$

(see Lemma 3 in Appendix B).

(iii) Replace $k$ with $m - k - 1$.

This result is identical in form to (3), but we have only assumed it to be valid for $n \geq m$ so far. It is easy to see that it is also valid for $n < m$ though; as stated earlier, $\langle \text{Tr}[i^{\hat{\rho}_m^{mn}}] \rangle$ is necessarily symmetric under exchange of $m$ and $n$. Applying this to (10) gives

$$\langle \text{Tr}[i^{\hat{\rho}_m^{mn}}] \rangle = \frac{\Gamma(mn)}{r \Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{\Gamma(r - k) \Gamma(m + r - k) \Gamma(n + r - k)}{k! \Gamma(r - k) \Gamma(m - k) \Gamma(n - k)}$$

when $n < m$. But because $1/\Gamma(n - k)$ is an entire function with respect to $k$ with zeroes at all integers $k \geq n$, the upper bound of the $k$-summation can be raised without changing the result. Thus, the limit $n - 1$ can be increased back to $m - 1$ to give the completely general equation

$$\langle \text{Tr}[i^{\hat{\rho}_m^{mn}}] \rangle = \frac{\Gamma(mn)}{r \Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{\Gamma(r - k) \Gamma(m + r - k) \Gamma(n + r - k)}{k! \Gamma(r - k) \Gamma(m - k) \Gamma(n - k)}.$$

§ This is guaranteed to be true for non-integer $r$. Some ambiguity can arise for integer $r$ due to divergent terms in the numerator, but as the integer cases can be treated as the limits of sequences of non-integer $r$, and the limit of $\langle \text{Tr}[i^{\hat{\rho}_m^{mn}}] \rangle$ is well defined in any such case, it is true for integer $r$ as well.
This derivation followed essentially the same procedure as that used by Sen to prove Page’s exact entropy result \[8\]. Sen based his proof on evaluating \(\langle \text{Tr}[\hat{\rho}_m^{mn} \ln \hat{\rho}_m^{mn}] \rangle\) directly using the same basic method, and the method we used here can also be used to get the same result using the fact that

\[
\hat{\rho}_m^{mn} \ln \hat{\rho}_m^{mn} = \lim_{r \to 1} \frac{\partial}{\partial r} (\hat{\rho}_m^{mn})^r,
\]

which is applicable as \(3\) is valid for any real value \(r\). See Appendix A for a demonstration of this.

Next, a further rearrangement of \(3\) is required to put it into a form suitable for use in Lubkin’s series.

**Corollary 1.**

\[
\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{\Gamma(mn) r!}{r! \Gamma(mn + r)} \sum_{k=0}^{m-1} \binom{r-1}{k} \binom{m}{k+1} \binom{n+r-k-1}{n-1}
\]

for integer \(r \geq 1\).

**Proof.** First, as only integer \(r\) is needed from this point onwards, \(3\) can be restated as

\[
\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{\Gamma(mn) r!}{r! \Gamma(mn + r)} \sum_{k=0}^{m-1} \binom{r-1}{k} \binom{m}{k+1} \frac{(-1)^k \Gamma(m+r-k) \Gamma(n+r-k)}{\Gamma(m-k) \Gamma(n-k)} \Gamma(m-k) \Gamma(n-k).
\]

This derivation then relies on the fact that

\[
\frac{\Gamma(m+r-k)}{\Gamma(m-k)} = \left. \frac{\partial^r (-1)^r}{\partial u^r u^{m-k}} \right|_{u=1}^{l=0},
\]

which follows from simple repeated differentiation. We substitute this twice into (11) to give

\[
\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{\Gamma(mn)}{r! \Gamma(mn + r)} \frac{\partial^r}{\partial u^r} \frac{\partial^r}{\partial v^r} \sum_{k=0}^{m-1} \binom{r-1}{k} \binom{m}{k+1} \frac{(-1)^k}{u^{m-k} v^{n-k}} \bigg|_{u,v=1}^{l=0},
\]

and expand out the derivatives to give

\[
\langle \text{Tr}[(\hat{\rho}_m^{mn})^r] \rangle = \frac{\Gamma(mn)}{r! \Gamma(mn + r)} \sum_{k=0}^{r-1} \binom{r}{k} \binom{r-1}{r-k-1} \binom{n+r-k-1}{n-1} \left. \frac{\partial^r}{\partial u^r} \frac{\partial^r}{\partial v^r} \right|_{u,v=1}^{l=0},
\]

\[
\times \binom{n+k-1}{n} \binom{l}{l} \binom{r-1}{l} \frac{1}{(r-l-1)!} \frac{1}{(m-r-k-l-1)!} \frac{(1-uv)^{r-k-l-1}}{u^{m+r-k-l} v^{n+r-k-l}} \bigg|_{u,v=1},
\]

\[
= \frac{\Gamma(mn)}{r! \Gamma(mn + r)} \sum_{k=0}^{r-1} \binom{r}{k} \binom{r}{r-k-1} \binom{n+r-k-1}{n-1} \left. \frac{\partial^r}{\partial u^r} \frac{\partial^r}{\partial v^r} \right|_{u,v=1}^{l=0},
\]

\[
\times \binom{n+k-1}{n} \binom{l}{l} \binom{r-1}{l} \frac{1}{(r-l-1)!} \frac{1}{(m-r-k-l-1)!} \frac{(1-uv)^{r-k-l-1}}{u^{m+r-k-l} v^{n+r-k-l}} \bigg|_{u,v=1}.
\]
Taking the limit of $u, v \to 1$ removes all terms except those where $r - k - l - 1 = 0$, and by collecting together the various factorial terms we get
\[
\langle \text{Tr}\left[\left(\hat{\rho}_{mn}^{r}\right)^{r}\right]\rangle = \frac{\Gamma(mn)!}{\Gamma(mn + r)} \sum_{k=0}^{r-1} \binom{m}{k+1} \binom{r - 1}{k} \binom{n + r - k - 1}{n - 1}
\]
\[
= \frac{\Gamma(mn)!}{\Gamma(mn + r)} \sum_{k=0}^{m-1} \binom{m}{k+1} \binom{r - 1}{k} \binom{n + r - k - 1}{n - 1}.
\]
In the final step here we use the same method for changing limits that was used at the end of Theorem 1, which relies on the fact that $\binom{a}{b} = 0$ when $b > a$, meaning that
\[
\binom{m}{k+1} \binom{r - 1}{k} = 0
\]
when either $k > m$ or $k > r$.

(11) has the notable property that all terms in the summation are positive, whereas (3) was an alternating summation. However, the main property which motivated us to use this form is the prefactor, specifically the $r!$ term, the importance of which will become evident in Theorem 2.

3. Example case: $m = n = 2$

Before looking at the series for general dimensions in the next section, it will be beneficial to first look at the case $m = n = 2$. In this case the terms in (11) can be evaluated explicitly in a simple closed form. To begin with, if we substitute $m = n = 2$ into (11) we get
\[
\langle \text{Tr}\left[\left(\hat{\rho}_{2}^{2}\right)^{r}\right]\rangle = \frac{6r!}{(r + 3)!} (r^2 + r + 2).
\]
The general binomial expansion of $\langle \text{Tr}\left[\left(\hat{\rho}_{mn}^{r} - \hat{\rho}_{0}^{r}\right)^{r}\right]\rangle$ is
\[
\langle \text{Tr}\left[\left(\hat{\rho}_{mn}^{r} - \hat{\rho}_{0}^{r}\right)^{r}\right]\rangle = \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^{r-k}}{m^{r-k}} \langle \text{Tr}\left[\left(\hat{\rho}_{mn}^{k}\right)^{k}\right]\rangle,
\]
which in this case means that
\[
\langle \text{Tr}\left[\left(\hat{\rho}_{2}^{2} - \hat{\rho}_{0}^{r}\right)^{r}\right]\rangle = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \frac{1}{2^{r-k}} \frac{6k!}{(k + 3)!} (k^2 + k + 2).
\]
The next few steps in particular demonstrate the procedure which will be used on the general case in the next section. First, we rearrange the above using the identity
\[
\binom{r}{k} \frac{k!}{(k + N)!} = \frac{r!}{(r + N)!} \binom{r + N}{k + N},
\]
giving
\[
\langle \text{Tr}\left[\left(\hat{\rho}_{2}^{2} - \hat{\rho}_{0}^{r}\right)^{r}\right]\rangle = \frac{6r!}{(r + 3)!} \sum_{k=0}^{r} \binom{r + 3}{k} (-1)^{r-k} \frac{1}{2^{r-k}} (k^2 + k + 2)
\]
\[
= - \frac{6r!}{(r + 3)!} \sum_{k=3}^{r+3} \binom{r + 3}{k} (-1)^{r-k} \frac{k(k-1) - 4k + 8}{2^{r-k+3}}.
\]
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Then, we replace \( k(k - 1) - 4k + 8 \) with

\[
\left( \frac{\partial^2}{\partial u^2} - 4 \frac{\partial}{\partial u} + 8 \right) u^k \bigg|_{u=1},
\]

and finally the summation is replaced with the difference of two sums, one from zero to \( r - 3 \) (which is a complete binomial expansion), and the other from zero to two. This gives

\[
\langle \text{Tr}[(\hat{\rho}_2^2 - \hat{\rho}_0)^r] \rangle = \frac{-6r!}{(r + 3)!} \left( \frac{\partial^2}{\partial u^2} - 4 \frac{\partial}{\partial u} + 8 \right) \left( \frac{(-1)^r}{2^{r+3}} (1 - 2u)^{r+3} \right)
\]

\[
- \sum_{k=0}^{2} \binom{r + 3}{k} (-1)^{r-k} \frac{u^k}{2^{r+k+3}} \bigg|_{u=1}
\]

\[
= \frac{3[1 + (-1)^r]}{2^r(r + 3)}.
\]

Substituting this into (1) then gives

\[
\langle S_{2,2} \rangle = \ln 2 - \sum_{r=1}^{\infty} \frac{3[1 - (-1)^r]}{2r(r + 1)(r + 4)}.
\]

At this point it is clear that the series will converge for \( m = n = 2 \), by comparison with the series expansion of \( \zeta(3) \) (where \( \zeta \) is the Riemann zeta function). This series converges to

\[
\langle S_{2,2} \rangle = \frac{1}{3},
\]

which agrees with Page’s explicit formula [5].

We can now apply this method to the terms in the general series.

4. The general case

The convergence of the series is trivial to establish when \( m = 1 \), so we will state that first:

**Lemma 1.** The series (1) is trivial, and so converges absolutely, when \( m = 1 \).

*Proof.* When \( m = 1 \), the reduced density operator \( \hat{\rho}_1^{1,n} \) is necessarily just the one-dimensional identity (it acts on a one-dimensional Hilbert space and its trace is unity, and the identity is the only operator that satisfies these conditions). This also means that \( \hat{\rho}_1^{1,n} = \hat{\rho}_0 \), so

\[
\langle \text{Tr}[(\hat{\rho}_s - \hat{\rho}_0)^r] \rangle = 0
\]

for any \( r \geq 1 \). The terms in (1) are therefore trivially zero, giving

\[
\langle S_{1,n} \rangle = \ln 1 = 0.
\]

For cases where \( m \geq 2 \), the following result relating to the convergence of series will also be required:
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Lemma 2. If a sequence \( A_r \) is defined for all integers \( r \geq 1 \) with the form

\[ A_r = \sum_{i=1}^{N} A_{r,i} \]

for a finite constant \( N \), and there exists a second sequence \( B_r \) such that

\[ \lim_{r \to \infty} \left| \frac{A_{r,i}}{B_r} \right| \leq c \delta_{ik} \]  \hspace{1cm} (13)

for finite \( c \) and \( 1 \leq k \leq N \), then

\[ \sum_{r=1}^{\infty} A_r \]  \hspace{1cm} (14)

converges absolutely to a finite value if and only if

\[ \sum_{r=1}^{\infty} B_r \]  \hspace{1cm} (15)

also converges.

Proof. Given (13), it is easy to see that

\[ \lim_{r \to \infty} \left| \frac{A_r}{B_r} \right| = \sum_{i=1}^{N} c \delta_{ik} = c, \]

which is finite. It then follows from the limit comparison test that (14) converges absolutely if and only if (15) converges absolutely.

We now have all the necessary tools to establish the conditions under which the general series converges and diverges.

Theorem 2. The series (1) converges if and only if \( m \leq 2 \).

Proof. Only cases where \( m \geq 2 \) need be considered here due to Lemma (1), so we will assume during this proof that \( m \geq 2 \).

(11) states that

\[ \langle \text{Tr}[\hat{\rho}_m^m]^r \rangle = \frac{\Gamma(mn)r!}{\Gamma(mn+r)} \sum_{k=0}^{m-1} \binom{m}{k+1} \binom{r}{k} \left( \binom{n+r-k-1}{n-1} \right) \]

for any integer \( r \geq 1 \). In addition it is known that

\[ \langle \text{Tr}[\hat{\rho}_m^0]^r \rangle = a_0, \]

where \( a_0 = \min(m,n) \) (see the footnote on page 4). Therefore,

\[ \langle \text{Tr}[\hat{\rho}_m^m - \hat{\rho}_0^m]^r \rangle \]

\[ = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \frac{1}{m^{r-k}} \langle \text{Tr}[\hat{\rho}_m^m] \rangle \]

\[ = \frac{(-1)^r a_0}{m^r} + \sum_{k=1}^{r} \binom{r}{k} (-1)^{r-k} \frac{1}{m^{r-k}} \frac{\Gamma(mn)k!}{\Gamma(mn+k)} \]

\[ \times \sum_{s=0}^{m-1} \binom{m}{s+1} \binom{k-1}{s} \left( \frac{n+k-s-1}{n-1} \right) \]
\[
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\]

\[
= \frac{(-1)^{r-1} a_0}{m^r} + \frac{\Gamma(mn)r!}{\Gamma(mn + r)} \sum_{k=1}^{r} \frac{r + mn - 1}{k + mn - 1} \frac{(-1)^{r-k}}{m^{r-k}}
\]

\[
\times \sum_{s=0}^{m-1} \binom{m}{s} \binom{k-1}{s} \frac{(n + k - s - 1)}{n - 1}
\]

\[
= \frac{(-1)^r a_0}{m^r} + \frac{\Gamma(mn)r!}{\Gamma(mn + r)} \sum_{k=mn}^{r+mn-1} \frac{(r + mn - 1)}{k} \frac{(-1)^{r-k+mn-1}}{m^{r-k+mn-1}}
\]

\[
\times \sum_{s=0}^{m-1} \binom{m}{s} \binom{k-mn}{n} \left( \frac{n+k-mn-s}{n-1} \right),
\]

(16)

The various binomial coefficients at the end can be simplified using the identity

\[
\frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial u^{n-1}} \left[ \frac{u^n}{s!} \frac{\partial^s}{\partial u^s} u^{k-mn} \right] \bigg|_{u=1} = \binom{k-mn}{s} \binom{n+k-mn-s}{n-1}.
\]

Substituting this into (16), we can rearrange the summation over \( k \) by the same procedure used in Section 3, giving

\[
\langle \text{Tr}[(\hat{\rho}^{mn}_m - \hat{\rho}_0)^r] \rangle = \frac{(-1)^r a_0}{m^r} + \frac{\Gamma(mn)r!}{\Gamma(mn + r)} \sum_{s=0}^{m-1} \binom{m}{s} \frac{1}{(n-1)!}
\]

\[
\times \frac{\partial^{n-1}}{\partial u^{n-1}} \left[ \frac{u^n}{s!} \frac{\partial^s}{\partial u^s} \left( \frac{1}{u - 1/m} \right)^{r+mn-1} - \sum_{k=0}^{m-1} \binom{r + mn - 1}{k} (-1)^{r-k+mn-1} \frac{u^{k-mn}}{m^{r-k+mn-1}} \right] \bigg|_{u=1},
\]

and then expanding the two derivatives gives

\[
\langle \text{Tr}[(\hat{\rho}^{mn}_m - \hat{\rho}_0)^r] \rangle = \frac{(-1)^r a_0}{m^r} + \sum_{s=0}^{m-1} \binom{m}{s} \frac{(-1)^s}{(n-1)!s!} \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{n!(-1)^q}{(q+1)!}
\]

\[
\times \sum_{k=0}^{q+s} \binom{q+s}{k} (-1)^k \frac{(q+s+mn-k-1)!}{(r+mn-k-1)!} \frac{1}{m} \left( 1 - \frac{1}{m} \right)^{r+mn-k-1} - \sum_{k=0}^{mn-1} \binom{mn-1}{k} (-1)^{q+s}(q+s+mn-k-1)! \frac{(q+s+mn-k-1)!}{m^{r-k+mn-1}}
\]

(17)

Finally, this gives the exact form for the terms in (11) (labelled \( T_r \) for simplicity) as

\[
T_r = \frac{(-1)^r m^r}{r(r+1)} \langle \text{Tr}[(\hat{\rho}^{mn}_m - \hat{\rho}_0)^{r+1}] \rangle
\]

\[
= \frac{a_0}{mr(r+1)} + \sum_{s=0}^{m-1} \binom{m-1}{s} \frac{m(-1)^s}{(s+1)!} \sum_{q=0}^{n-1} \binom{n-1}{q} \frac{n!(-1)^{r+q}}{(q+1)!}
\]

\[
\times \sum_{k=0}^{q+s} \binom{q+s}{k} (-1)^k \frac{(r-1)!((q+s+mn-k-1))}{(r+mn-k-1)!} \frac{(m-1)^{r+mn-k}}{m^{mn-k}}
\]
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\[-\sum_{k=0}^{mn-1} \binom{mn-1}{k} \frac{(r-1)!(q+s+mn-k-1)!}{(r+mn-k)!} \frac{(-1)^{mn-k}}{m^{mn-k}}.\]

This expression is now compatible with Lemma 2 as it gives each term in Lubkin’s series as a sum over a fixed number of terms. Therefore, to determine if (1) converges we only need to determine what the limiting behaviour of the dominant term in (18) is.

When \(m = 2\), the limiting behaviour is given by the sequence \(B_r^{(2)} = r^{-2}\), which is comparable only to \((r-1)!/(r+2n-k)!\) with \(k = 2n-1\), as well as to the \(a_0\) term. When \(m > 2\), the limiting behaviour is given by

\[B_r^{(m)} = \frac{(m-1)^r}{r^{(m-1)(n-1)+2}},\]

which is comparable to \((r-1)!(m-1)^r/(r+mn-k)!\) when \(k = m+n-2\). It therefore follows from the fact that

\[\sum_{r=1}^{\infty} B_r^{(2)} = \frac{1}{\zeta(2)}\]

converges absolutely but

\[\sum_{r=1}^{\infty} B_r^{(m)}\]

does not (in addition to Lemma 2, that (1) converges if and only if \(m \leq 2\).

5. Discussion and Conclusions

Lubkin’s original derivation of his approximation was based on the assumption that (1) converged quickly to a finite value, allowing truncations of the series to be used as approximations. However, we have now proved that the only cases of significance where the series converge are those where \(m = 2\), and even then it does so slowly. Lubkin’s method may therefore still be of use in this case, but its usefulness is limited to single-qubit systems.

In all other cases, the rapid divergence of the series means that there is no evidence to support the general validity of Lubkin’s approximation in the entirety of the stated limit \(m \ll n\). The only evidence which exists to support its validity is comparison with other approximations, such as the formula

\[\langle S_{m,n} \rangle \simeq \ln m - \frac{m}{2n},\]

valid when \(1 \ll m \leq n\), which was derived by Page independently of Lubkin’s work, but this only supports Lubkin’s approximation in the range where Page’s approximation is itself valid, and says nothing about its general validity.

Therefore, Page’s formula is much more suited for use as an approximation for the von Neumann entropy of a finite-dimensional quantum system, as its range and validity is known. The only cases where Lubkin’s formula is known to be valid are

\[\|\text{Lemma 2}\|\text{specifically requires it to be comparable to a single term, but the two terms here can be summed to produce a single term which is still comparable with } B_r^{(2)}.\]

\[\|\text{The series also converges when } m = 1, \text{ but the convergence is trivial and the entropy is zero, so this is not of any practical benefit.}\]
cases where Page’s formula is also valid, and then only by comparison of the two, so there is nothing to be gained from using Lubkin’s formula in these cases.

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Appendix A.

In this appendix we re-derive Page’s explicit von Neumann entropy formula [5]. This method demonstrates the parallel between the proof of Theorem 1 and the method previously used by Sen for this purpose [8].

As stated in (3),

\[ \langle \mathrm{Tr}[\hat{\rho}^r]\rangle = \frac{\Gamma(mn)}{r\Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + r - k)\Gamma(n + r - k)}{k!\Gamma(r - k)\Gamma(m - k)\Gamma(n - k)} \]

for general \( r \). We can use this to evaluate the mean von Neumann entropy using the fact that

\[ \langle S_{m,n} \rangle = -\langle \mathrm{Tr}[\hat{\rho}^m \ln \hat{\rho}^m] \rangle \]

\[ = -\lim_{r \to 1} \frac{\partial}{\partial r} \langle \mathrm{Tr}[\hat{\rho}^r]\rangle. \]

First, differentiating gives

\[ -\frac{\partial}{\partial r} \langle \mathrm{Tr}[\hat{\rho}^r]\rangle = \frac{\Gamma(mn)}{r\Gamma(mn + r)} \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(m + r - k)\Gamma(n + r - k)}{k!\Gamma(r - k)\Gamma(m - k)\Gamma(n - k)} \]

\[ \times \left( \psi(mn + r) - \psi(m + r - k) - \psi(n + r - k) + \psi(r - k) + \frac{1}{r} \right). \]

When we take the limit of \( r \to 1 \), the \( 1/\Gamma(r - k) \) factor causes most terms to vanish in cases when \( k > 0 \). The only ones which remain are those where \( \psi(r - k) \) is in the numerator:

\[ \langle S_{m,n} \rangle = -\lim_{r \to 1} \frac{\partial}{\partial r} \langle \mathrm{Tr}[\hat{\rho}^r]\rangle \]

\[ = \psi(mn + 1) - \psi(m + 1) - \psi(n + 1) + \psi(1) + 1 \]

\[ + \frac{1}{mn} \sum_{k=1}^{m-1} \frac{(m - k)(n - k)}{k} \lim_{r \to 1} \frac{(-1)^k \psi(r - k)}{\Gamma(k + 1)\Gamma(r - k)} \]

\[ = \psi(mn + 1) - \psi(m + 1) - \psi(n + 1) + \psi(1) + 1 \]

\[ + \frac{1}{mn} \sum_{k=1}^{m-1} \frac{(m - k)(n - k)}{k}, \]
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where here we use the fact that

\[ \lim_{r \to 1} \frac{\psi(r-k)}{\Gamma(r-k)} = \lim_{r \to 1} \frac{1}{\Gamma(r-k)} \frac{\partial}{\partial r} \Gamma(r-k) \]

\[ = - \lim_{r \to 1} \text{Res}_{x=1-k} \frac{\Gamma(x)}{\Gamma(x-k)} \]

\[ = (-1)^k \Gamma(k) \]

for positive integers \( k \).

Then we use the identity

\[ \psi(b) - \psi(a) = \sum_{k=a}^{b-1} \frac{1}{k} \]

on the pairs \( (\psi(mn+1) - \psi(n+1)) \) and \( (\psi(m-1) - \psi(1)) \), giving

\[ S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2} \]

which agrees with Page’s formula [5].

Appendix B.

Lemma 3.

\[ \sum_{i=0}^{a} \frac{(-1)^i}{\Gamma(i+1) \Gamma(r-i+1)} = \frac{(-1)^a}{r \Gamma(a+1) \Gamma(r-a)} \quad (B.1) \]

for any integer \( a \geq 0 \).

Proof. \( (B.1) \) holds when \( a = 0 \), as both sides of the equation simply equal \( 1/\Gamma(r+1) \) in that case. Now if we assume that \( (B.1) \) holds for \( a = N \geq 0 \), extending the sum to \( a = N+1 \) gives

\[ \sum_{i=0}^{N+1} \frac{(-1)^i}{\Gamma(i+1) \Gamma(r-i+1)} = \frac{(-1)^N}{r \Gamma(N+1) \Gamma(r-N)} + \frac{(-1)^{N+1}}{\Gamma(N+2) \Gamma(r-N)} \]

\[ = \left(1 - \frac{N+1}{r}\right) \frac{(-1)^{N+1}}{\Gamma(N+2) \Gamma(r-N)} \]

\[ = \frac{(-1)^{N+1}}{r \Gamma(N+2) \Gamma(r-N-1)}, \]

which also agrees with \( (B.1) \). It therefore follows by induction that \( (B.1) \) holds for any \( a \geq 0 \). \( \square \)

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