Grand Unification, Gravitational Waves, and the Cosmic Microwave Background Anisotropy

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Abstract
We re-examine the gravitational wave background resulting from inflation and its effect on the cosmic microwave background radiation. The new COBE measurement of a cosmic background quadrupole anisotropy places an upper limit on the vacuum energy during inflation of \( \approx 5 \times 10^{16} \) GeV. A stochastic background of gravitational waves from inflation could produce the entire observed signal (consistent with the observed dipole anisotropy and a flat spectrum) if the vacuum energy during inflation was as small as \( 1.5 \times 10^{16} \) GeV at the 95% confidence level. This coincides nicely with the mass scale for Grand Unification inferred from precision measurements of the electroweak and strong coupling constants, for the SUSY Grand Unified Theories. Thus COBE could be providing the first direct evidence, via gravitational waves, for GUTs, and supersymmetry. Further tests of this possibility are examined, based on analyzing the energy density associated with gravitational waves from inflation.
The observation by the DMR instrument aboard the COBE satellite of large scale anisotropies in the Cosmic Microwave background (CMB) is probably the most important discovery in cosmology since the discovery of the CMB itself. Such anisotropies cannot have been induced by causal processes which were initiated after the era of recombination and thus represent true primordial fluctuations resulting from physics associated with the initial conditions of the FRW cosmology itself. These initial conditions are likely to have resulted from processes associated with either an inflationary phase or new planck scale physics. Only in the former case can explicit predictions be made and the COBE data on the temperature correlation function is remarkably consistent with a flat Harrison-Zel’dovich spectrum as predicted from inflation.

Inflation predicts at least two sources of Harrison-Zel’dovich type CMB anisotropies. Scalar energy density fluctuations on the surface of last scattering induced by primordial (dark) matter density perturbations will result both in subsequent structure formation and in appropriate dipole, quadrupole, and higher moment anisotropies in the CMB. Based on the observed dipole asymmetry one can determine an upper limit on the expected quadrupole anisotropy in the case of such a flat spectrum. In addition, if the scale of inflation is sufficiently high, long wavelength gravitational waves will be generated during inflation whose re-entry into the horizon can result in a large scale observed quadrupole and higher multipole anisotropies in the CMB today. While inflation is not the only method of generating a stochastic background of horizon-sized waves, it is certainly the most well motivated.

In this work we re-examine gravitational wave generation during inflation and determine the predicted signal in the CMB and compare this with the COBE data. In the process of deriving detailed estimates we update and reconcile various earlier analyses. We present a likelihood function for the probability that inflation at a given scale would result in a quadrupole anisotropy at least as big as that which is observed. We also briefly compare this to the nature and magnitude of the expected quadrupole anisotropy resulting from scalar density perturbations expected from
inflation. Our analysis allows us to place both upper and lower limits on the range of scales for which gravitational waves from inflation could result in all or most of the observed quadrupole anisotropy. These scales are consistent with the scale at which the $SU(3) \times SU(2) \times U(1)$ gauge couplings can be unified, based on a renormalization group extrapolation of low energy data, for SUSY GUT models. We find this coincidence both suggestive and exciting, and consider other possible observational probes of a gravitational wave background at this level. For this purpose we calculate the energy density stored in such a stochastic background today.

Since the work of Starobinsky [6], it has been recognized that a period of exponential expansion in the early universe would lead to the production of gravitational waves. Rubakov and collaborators [7] were the first to use this to limit the scale of inflation and with it the scale of Grand Unification. Since that time analyses designed to help more accurately compute the gravitational wave background and compare predictions to the data have been developed [8, 9, 10, 11]. More recently the the limits on the quadrupole anisotropy of the microwave background had improved. It thus seemed, even before COBE, a propitious time to re-analyze the gravitational wave limits. Many of the analytic techniques and results we derive have appeared in one form or another scattered in the literature, but we have made some effort to check, unify and reconcile the previous methods and in the process correct any errors. Further details can be found in [12].

It is convenient to write the metric in the $k = 0$ Robertson-Walker form

$$ds^2 = R^2(\tau) \left( -d\tau^2 + d\vec{x}^2 \right)$$

where $d\tau = dt/R(t)$ is the conformal time. In a universe which undergoes a period of exponential inflation, followed by a radiation dominated epoch and then a matter dominated phase, $R(\tau)$ and $\dot{R}(\tau)$ can be matched at the transition points, assuming that the transitions between phases are sudden (this approximation is sufficient for our purposes) [13]. Because of the matching conditions $\tau$ is discontinuous across the transitions. We define $\tau_1$ to be the (conformal) time of radiation-matter
equality, $\tau_2$ to be the end of inflation. The Hubble constant during inflation, $H$, and vacuum energy density $V_0$ driving the inflation are related by

$$H^2 = \frac{8\pi}{3} \frac{V_0}{m_{Pl}^2} = \frac{8\pi}{3} m_{Pl}^2 v$$

(2)

where we introduce the notation $v \equiv V_0/m_{Pl}^4$.

A classical gravitational wave in the linearized theory is a ripple on the background space-time

$$g_{\mu \nu} = R^2(\tau) (\eta_{\mu \nu} + h_{\mu \nu})$$

where

$$\eta_{\mu \nu} = \text{diag}(-1, 1, 1, 1), \quad h_{\mu \nu} \ll 1$$

(3)

In transverse traceless (TT) gauge the two independent polarization states of the wave are denoted as $+, \times$. In the linear theory the TT metric fluctuations are gauge invariant (they can be related to components of the curvature tensor). Write

$$h_{\mu \nu}(\tau, \vec{x}) = h_{\lambda}(\tau; \vec{k}) e^{i\vec{k} \cdot \vec{x}} \epsilon_{\mu \nu}(\vec{k}; \lambda)$$

(4)

where $\epsilon_{\mu \nu}(\vec{k}; \lambda)$ is the polarization tensor and $\lambda = +, \times$. The equation for the amplitude $h_{\lambda}(\tau; \vec{k})$ is obtained by requiring the perturbed metric (3) satisfy Einstein’s equations to $O(h)$. As was first noted by Grishchuk [14] the equation of motion for this amplitude is then identical to the massless Klein-Gordon equation for a plane wave in the background space-time. In this way, one finds each polarization state of the wave behaves as a massless, minimally coupled, real scalar field, with a normalization factor of $\sqrt{16\pi G}$ relating the two.

The spectrum of gravitational waves generated by quantum fluctuations during the inflationary period can be derived by a sequence of Bogoluibov transformations relating creation and annihilation operators defined in the various phases: inflationary, radiation and matter dominated [13, 11]. The key idea is that for long wavelength (c.f. the horizon size) modes the transitions between the phases are sudden and the universe will remain in the quantum state it occupied before the transition (treating each of the transitions as instantaneous is a good approximation for all but the highest frequency graviton modes). However the creation and annihilation operators that describe the
particles in the state are related by a Bogoluibov transformation, so the quantum expectation value of any string of fields is changed. A calculation of the quantum \( n \)-point functions suffices to find the spectrum of classical gravitational waves today since the statistical average of the ensemble of classical waves can be related to the corresponding quantum average.

A stochastic spectrum of classical gravitational waves (in terms of comoving wavenumber \( \vec{k} \)) in the expanding universe has the form

\[
h_\lambda(\tau; \vec{k}) = A(k)a_\lambda(\vec{k}) \left[ \frac{3j_1(k\tau)}{k\tau} \right] \quad \text{with} \quad \lambda = +, \times
\]

where the term \([\cdots]\) is a real solution of the Klein-Gordon equation in a matter dominated universe. \( a_\lambda(\vec{k}) \) is a random variable with statistical expectation value (normalized to simplify the relation between \( A(k) \) and the energy density per logarithmic frequency interval as we later describe)

\[
\langle a_\lambda(\vec{k})a_{\lambda'}(\vec{q}) \rangle = k^{-3}\delta(3)(\vec{k} - \vec{q})\delta_{\lambda\lambda'} \quad (6)
\]

Waves which are still well outside the horizon at the time of matter-radiation equality \((k\tau_1 \ll 2\pi)\) will give the largest contribution to the CMB anisotropy today. Calculating the Bogoluibov coefficients by matching the field and its first derivative at \( \tau_2, \tau_1 \) in the limit \( k\tau \ll 2\pi \) one can derive the prediction for the \((k\text{-independent})\) spectrum of long-wavelength gravitational waves generated by inflation \( [4, 12] \)

\[
A^2(k) = \frac{H^2}{\pi^2 m_P^2} = \frac{8}{3\pi^2} v \quad (7)
\]

To make contact with observations one must consider the effect such a spectrum will have on the CMBR. If one expands the CMBR temperature anisotropy in spherical harmonics

\[
\frac{\delta T}{T}(\theta, \phi) = \sum_{lm} a_{lm} Y_{lm}(\theta, \phi) \quad (8)
\]

one can present the prediction of a given spectrum of gravitational waves in terms of the \( a_{lm} \). The temperature fluctuation due to a gravitational wave \( h_{\mu\nu} \) can be found using the Sachs-Wolfe formula

\[
\frac{\delta T}{T} = -\frac{1}{2} \int_0^\infty d\Lambda \frac{\partial h_{\mu\nu}(\tau, \vec{x})}{\partial \tau}\hat{x}^\mu\hat{x}^\nu \quad (9)
\]
where \( \Lambda \) is a parameter along the unperturbed path and the lower (upper) limit of integration represents the point of emission (reception) of the photon.

It is standard to project out a multipole and calculate the rotationally symmetric quantity

\[
\langle a_l^2 \rangle \equiv \left\langle \sum_m |a_{lm}|^2 \right\rangle
\]  

(10)

After some algebra, utilizing identities for spherical polynomials and Bessel functions (i.e. see [12]), one finds for waves entering the horizon during the matter dominated era (the results are very insensitive to this restriction since the \( k \)-integral is dominated by waves entering the horizon today: \( k \approx 2\pi/\tau_0 \))

\[
\langle a_l^2 \rangle = 36\pi^2(2l+1)(l+2)\int_{l-2}^{2\pi/\tau_1} kdk \ A^2(k)|F_l(k)|^2
\]  

(11)

where the function \( F_l(k) \) is defined as \( \tau = \tau_0 - r \)

\[
F_l(k) \equiv \int_0^{\tau_0-r} dr \left( \frac{d}{d(k\tau)} \frac{j_l(k\tau)}{k\tau} \right) \left[ \frac{j_{l-2}(kr)}{(2l-1)(2l+1)} + \frac{2j_l(kr)}{(2l-1)(2l+3)} + \frac{j_{l+2}(kr)}{(2l+1)(2l+3)} \right]
\]  

(12)

Accounting for the factor of two difference between definitions of \( A^2(k) \) this agrees with the result of [9], and differs by \( \approx 2 \) with the earlier result of [8].

The calculation of the expectation value \( \langle a_l^2 \rangle \) is not the end of the story however. One must also consider the statistical properties of \( a_l^2 \) [8 15 16 12]. Given that each of the \( a_{lm} \) are independent Gaussian random variables the probability distribution for each \( a_l^2 \), with mean \( \langle a_l^2 \rangle \), is of the \( \chi^2 \) form. One can calculate the confidence levels for \( a_l^2 \) in terms of the incomplete gamma function. We find for the for the quadrupole, \( \langle a_2^2 \rangle = 7.74\nu \) and \( a_2^2/\langle a_2^2 \rangle = .63,.32,.23 \) at the 68, 90 and 95% (lower) confidence levels respectively.

The new COBE observations can be summarized for our purposes as as a value for the rms quadrupole moment. If one fits to a Harrison-Zel’dovich spectrum the quoted value is [1]

\[
Q_{\text{rms-PS}} \equiv \left( \frac{a_2^2}{4\pi} \right)^{1/2} = \frac{(16.7 \pm 4)\mu K}{2.73K} \Rightarrow a_2^2 = (4.7 \pm 2) \times 10^{-10}.
\]  

(13)
Note that the quoted error on $Q_{\text{rms}} - P_S$ is Gaussian, while the distribution of $a_2^2$ is $\chi^2$. This implies that in proceeding from the inferred value of $a_2^2$ to a value of $v$ we must be careful to properly take into account the resultant statistics which will be far from Gaussian. In particular the mode of the distribution will be lower than the mean (as is noted in [1]). From (13) the quadrupole moment is consistent with gravitational waves resulting from a mean value of $v = 6.1 \times 10^{-11}$. To determine the uncertainty on $v$ we have performed a simple Monte-Carlo analysis to find the distribution for $v$ (see figure 1). Based on this we can determine both upper and lower limits on the value of $v$ consistent with the observations and also find the most probable value of $v$. We find

$$
3.7 \times 10^{-10} > v > 2.5 \times 10^{-12} \quad 95\% \ CL \\
1.5 \times 10^{-10} > v > 2.3 \times 10^{-11} \quad 68\% \ CL
$$

with a maximum likelihood value of $v \approx 4 \times 10^{-11}$.

These limits as quoted require some interpretation. First the 95% upper limit $v < 3.7 \times 10^{-10}$ provides a strict upper limit on the scale of inflation $\approx (v^{1/4} M_{Pl}) = 5.2 \times 10^{16}$GeV assuming that the contribution to the quadrupole moment from scalar density perturbations is insignificant. This could be increased slightly in the unlikely case that a comparable quadrupole moment from density perturbations existed and happened to cancel out that due to gravitational waves to some degree.

In this regard it is worthwhile considering what magnitude of quadrupole moment is expected from scalar density perturbations from inflation. By requiring that the induced dipole due to long wavelength modes not greatly exceed the observed dipole anisotropy one can put an upper limit on the overall magnitude of a flat spectrum of perturbations at horizon crossing and from this an upper limit on all higher multipoles. At the 90% confidence level an upper limit of $a_2^2 \approx 2 \times 10^{-10}$ has been derived [14]. Equivalently, fitting observed clustering to a primordial fluctuations spectrum [16] one can predict a value of $a_2^2$. Such an analysis yields best fit values in the range $a_2^2 \approx (1.9 - 9.9) \times 10^{-11}$. While these estimates are probably consistent with the COBE observation, they also suggest that a major fraction of the observed anisotropy may be due to gravitational waves. Note that [13, 10] both the gravitational wave induced anisotropy and the density fluctuation induced anisotropy...
yield similar distributions of moments at least up to \( l \approx 10 \). Thus measurements of the correlation function cannot easily serve to distinguish between these possibilities at this stage.

What scale of inflation, \( M \), in GeV, do the above limits correspond to? From (14) we find, at the 95\% confidence level, \( 1.5 \times 10^{16} \text{GeV} < M < 5.2 \times 10^{16} \text{GeV} \), with the best fit value \( 2.9 \times 10^{16} \text{GeV} \). On the other hand using data from precision electroweak measurements at LEP on the strong and weak coupling constants one finds, for minimal \( SU(5) \) SUSY models with SUSY breaking between \( M_Z \) and \( 1 \text{TeV} \), that coupling constant unification can occur at a single GUT scale, \( M_X \) in the range \( M_X \approx (1 - 3.6) \times 10^{16} \text{GeV} \) [17]. Unfortunately there are no explicit compelling SUSY or GUT inflationary scenarios with which one can compare, but generically, unless there is fine tuning, or hierarchies, in a GUT scenario \( V_0 \approx \kappa M^4 \) where \( \kappa \approx .01 - 1 \) (for example in a Coleman-Weinberg \( SU(5) \) model \( \kappa = 9/32 \pi^2 \)). Thus the energy scale of inflation consistent with the observed quadrupole anisotropy coming from gravitational waves can coincide with the estimated scale of SUSY Grand Unification as inferred from extrapolation of low energy couplings. We find this possibility both plausible and exciting. At the very least it is quite promising that COBE through the quadrupole anisotropy is sensitive to gravitational waves from inflation at interesting scales.

Since both density perturbations and gravitational wave anisotropies resulting from inflation result in a flat spectrum for the CMB anisotropy, with a great similarity in all multipoles up to at least \( l = 9 \), it will be difficult from CMB measurements alone to verify whether or not the observed signal is due gravitational waves. How might one hope then to distinguish between these possibilities? The simplest way would be to probe for evidence of a flat spectrum of gravitational waves in regions of smaller wavenumber. At present, the most sensitive gravitational wave detector at shorter wavelengths (periods of \( O(\text{years}) \)) is also astrophysical in origin, and is based on timing measurements of millisecond pulsars [18, 19, 20]. On smaller wavelengths still terrestrial probes, such as the proposed LIGO gravity wave detector [21], are currently envisaged.

The sensitivity of all such detectors is based on the mean energy density per logarithmic fre-
quency interval in gravitational waves. For waves which come inside the horizon during the matter dominated era we can utilize (3) and (6). Averaging over many wavelengths/periods, summing over helicities, and also taking the stochastic average, we find

$$k \frac{d \rho_g}{dk} = \frac{k_{phys}^2}{2G} A^2(k) \left[ \frac{3j_1(k\tau)}{k\tau} \right]^2$$

(15)

where $k_{phys} = k/R(\tau)$. For wavelengths much smaller than the horizon ($k\tau \gg 2\pi$) this goes as $R^{-4}(\tau)$ as expected. The time evolution factor $(3j_1(k\tau)/k\tau)^2$ is crucial, and in fact implies that the energy density in gravitational waves also redshifts considerably as it comes inside the horizon.

Thus the energy density at horizon crossing is considerably smaller than the asymptotic value when the wave is well outside the horizon, a fact which has not been stressed before to our knowledge. In any case dividing by the critical density today we find for waves just coming inside the horizon,

$$\left( \Omega_g \right)_{hc} \approx \begin{cases} \frac{16v}{9} & \frac{2}{3\pi}(H_{infl}/M_{Pl})^2 \quad RD \\ \frac{v}{\pi^2} & \frac{3}{8\pi^2}(H_{infl}/M_{Pl})^2 \quad MD \end{cases}$$

(16)

The result at horizon crossing in a radiation dominated epoch was calculated using the appropriate Bogolubov coefficients. This results in the factor of $3j_1(k\tau)/k\tau$ above being changed to $j_0(k\tau)$.

Waves which come inside the horizon during the radiation dominated era will redshift with one extra power of $R$ compared to matter during the matter dominated era. Thus their contribution to $\Omega$ today will be suppressed compared to their contribution at horizon crossing by the factor $\rho_{rad}/\rho_c = 4 \times 10^{-5}h^{-2}$, where the Hubble constant today is $100h\text{km/sec/Mpc}$. As a result, we find that such waves today, taking $v < 3.7 \times 10^{-10}$, form a stochastic background with $\Omega_g < 2.6 \times 10^{-14}h^{-2} < 10^{-13}; \ (h > 0.5)$.

The waves for which the millisecond pulsar timing and future interferometer measurements are sensitive entered the horizon during the radiation dominated era. The present limit, at the 68% confidence level, from pulsar timing data is $\Omega_g < 9 \times 10^{-8}$ [20]. This limit can improve in principle with the measuring time to the fourth power [18, 19] but, even in the most optimistic case, dedicated observations with many pulsars, and better clocks, over a period of perhaps a century would be
required to uncover such a signal. The expected energy density is also about 2 orders of magnitude below the optimum projected capabilities of future terrestrial detectors.

Thus, prospects look grim, without some significant technological advances, for detecting such a gravitational wave background directly anywhere but in the microwave background signal. Barring a very refined measurement of high multipoles in the CMB anisotropy we may have to await confirmation at accelerators, or perhaps proton decay detectors, before we can determine whether COBE has discovered the first evidence for GUTs, supersymmetry, and at the very least, gravitational waves.

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Figure Captions

Figure 1: The distribution for the scale of inflation \( v = (V_0/M_P l^4) \) as determined by Monte Carlo, using the COBE measurements and assuming the observed quadrupole anisotropy is due to gravitational waves.