A Higher Probability of Detecting Lensed Supermassive Black Hole Binaries by LISA

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ABSTRACT
Gravitational lensing of gravitational waves (GWs) is a powerful probe of the matter distribution in the universe. Here we revisit the wave-optics effects induced by dark matter (DM) halos on the GW signals of merging massive black hole binaries (MBHBs), and we study the possibility of discerning these effects using the Laser Interferometer Space Antenna (LISA). In particular, we include the halos in the low-mass range of $10^5 - 10^8 M_{\odot}$ since they are the most numerous according to the cold DM model. We simulate the lensed signals corresponding to a wide range of impact parameters, and we find distinguishable deviation from the standard best-fit GW templates even when the impact parameter is as large $y \approx 50$. Consequently, we estimate that over $(0.1 - 1.6)\%$ of the MBHBs in the mass range of $10^5.0 - 10^6.5 M_{\odot}$ and the redshift range of $4 - 10$ should show detectable wave-optics effects. This probability is one order of magnitude higher than that derived in previous works. The uncertainty comes mainly from the mass function of the DM halos. Not detecting any signal during the LISA mission would imply that DM halos with $10^7 - 10^8 M_{\odot}$ are less numerous than what the cold DM model predicts.

Key words: gravitational waves — gravitational lensing: strong — dark matter

1 INTRODUCTION
Gravitational lensing events are unique probes of the distribution of matter in the universe (Schneider et al. 1992). Like light, gravitational waves (GWs) could also be lensed by intervening matter (Lawrence 1971; Cyrranski 1974; Sonnabend 1979; Marković 1993). The prospect of detecting the lensing of GWs is promising given the increasing number of GW events discovered in the recent years by the Laser Interferometer Gravitational-wave Observatory (LIGO) and the Virgo detectors (Abbott et al. 2016, 2019; Abbott et al. 2020).

One major difference between GW and light is that the former usually has a much longer wavelength. For example, LIGO/Virgo are sensitive to the GWs with a wavelength of $O(10^6)$ km. It is comparable to or longer than the characteristic sizes of many astrophysical objects, such as stars or intermediate-massive black holes (IMBHs). If lensed by these objects, the GWs in the LIGO/Virgo band would behave like light in the wave-optics limit (Ohanian 1974; Bontz & Haugan 1981; Nakamura 1998; Nakamura & Deguchi 1999). In this case, wave diffraction would modify the amplitude and phase of the GWs, producing a characteristic “beating pattern” in the frequency domain of the waveform (Takahashi & Nakamura 2003). In addition, the wave-optics effect could also smear the plane of GW polarization (Cusin & Lagos 2020) and produce beat patterns in the time-domain waveform (Yamamoto 2005; Hou et al. 2020). These effects, in principle, could allow LIGO/Virgo to detect massive stars, IMBHs, and the dense cores of globular clusters and dark-matter (DM) halos (Moylan et al. 2008; Cao et al. 2014; Takahashi 2017; Christian et al. 2018; Dai et al. 2018; Diego et al. 2019; Jung & Shin 2019; Liao et al. 2019; Meena & Bagla 2020; Oguri & Takahashi 2020; Mishra et al. 2021; Wang et al. 2021). However, so far no strong evidence of lensing effects has been officially reported by LIGO/Virgo (Hannuksela et al. 2019), suggesting that the lensing probability is relatively low.

The Laser Interferometer Space Antenna (LISA) is a future space-based mission aiming at detecting the GWs in the milli-Hertz (mHz) band (Amaro-Seoane et al. 2017). One of its major targets is the merger of two massive black holes (MBHs), preferentially in the mass range of $10^4 \sim 10^5 M_{\odot}$. Because of the superb sensitivity, LISA could detect MBH mergers up to a redshift of 20 with a signal-to-noise ratio (SNR) as high as $10^2 \sim 10^3$ (Amaro-Seoane et al. 2017). Such a high redshift suggests that gravitational lensing by the large-scale structure is no longer negligible for LISA (Takahashi 2006; Yoo et al. 2007). The long wavelength and high SNR also indicate that the diffraction effects in the wave-optics limit, which is relatively weak for LIGO/Virgo sources, may become significant for LISA.

For LISA, the lenses which produce the diffraction effects are mainly low-mass dark-matter (DM) halos, as well as the subhalos in massive main halos (Takahashi & Nakamura 2003; Takahashi 2004). Takahashi & Nakamura (2003) considered the DM halos in the mass range of $10^9 \sim 10^{12} M_{\odot}$ and estimated that the lensing probability for each MBH merger in the LISA band is about $10^{-4} \sim 10^{-3}$. However, the cold DM (CDM) model predicts that the most abundant halos are those in the mass range of $10^6 \sim 10^8 M_{\odot}$ (e.g. Cooray & Sheth...
2 METHOD

2.1 Lensing Model

For simplicity, we assume a singular isothermal sphere (SIS) profile for our DM halos and subhalos. In this case, we can follow Takahashi & Nakamura (2003) to model the diffraction of GWs. Using a more realistic Navarro-Frenk-White (NFW) profile usually leads to a slightly smaller magnification factor (Takahashi & Nakamura 2003; Gil Choi et al. 2021).

The basic picture of lensing of GWs is illustrated in Figure 1, where $D_L$, $D_S$, and $D_{LS}$ denote the angular distances to the lens, to the source, and their difference. All these quantities are measured in the frame of the observer.

The magnification factor is defined as

$$ F(\omega, \eta) := \frac{\phi_{obs}^L(\omega, \eta)}{\phi_{obs}(\omega, \eta)}, $$

(1)

where $\phi_{obs}^L(\omega, \eta)$ and $\phi_{obs}(\omega, \eta)$ are the lensed and unlensed gravitational wave amplitudes in the Fourier space, $\omega$ is the observed angular frequency of the GW, and $\eta$ is the position vector on the source plane (“impact parameter” hereafter). Taking the cosmological redshift into account, the magnification factor can be calculated with

$$ F(\omega, \eta) = \frac{\omega(1+z_L)}{2\pi i D_L D_{LS}} \int d^2 \xi \exp[i\omega(1+z_L) t_d(\xi, \eta)], $$

(2)

where $z_L$ is the redshift of the lens and $t_d$ is the time delay caused by lensing. The time delay can be computed with

$$ t_d(\xi, \eta) = \frac{D_L D_{LS}}{2 D LS} \left( \frac{\xi}{D_L} - \frac{\eta}{D_S} \right)^2 - \hat{\phi}(\xi) + \hat{\phi}_m(\eta), $$

(3)

where $\hat{\phi}_m(\eta)$ denotes the arrival time of the unlensed GW, which is

$$ \eta = \frac{x}{\xi_0}; \quad y = \frac{D_L}{\xi_0 D_S} \eta, $$

(5)

with $\eta_0$ the Einstein radius. In the SIS model, it can be calculated with $\eta_0 = 4\pi\sigma_\star^2 D_L D_{LS}/(c^2 D_S)$, where $\sigma_\star$ is the velocity dispersion of the lens. The corresponding dimensionless frequency is

$$ w = 4GML/(1+z_L) \omega/c^3, $$

(6)

where $M_L$ is the so-called “lens mass”, which is defined as the mass enclosed by a circle of the Einstein radius in the lens plane. In the SIS model, we have $M_L = 4\pi \sigma_\star^2 D_L D_{LS}/(c^2 D_S)$ (Appendix A).

Using the above nondimensional quantities, the time delay can be rewritten as

$$ F(x, y) = \frac{w}{2\pi i} \int d^2 x \exp[iwT(x, y)], $$

(8)

where $T(x, y) = \int d^2 \xi \exp[i\omega(1+z_L) t_d(\xi, \eta)]$. It follows that the nondimensional amplification factor is

$$ F(w, y) = -i w e^{i y^2/2} \int_0^\infty dx x J_0(wx) \times \exp[i \omega(1+z_L) t_d(\xi, \eta)]. $$

(9)

In the SIS model the last equation can be calculated with

$$ F(w, y) = -i w e^{i y^2/2} \int_0^\infty dx x J_0(wx) \times \exp[i \omega(1+z_L) t_d(\xi, \eta)]. $$

(9)

The corresponding amplification factor $|F|$ and phase change factor $\theta_F$ are

$$ |F| = \sqrt{F F^*}, \quad \theta_F = -i \ln|F/F|, $$

(10)

where $F^*$ is the complex conjugate of $F$.

Figure 2 shows the dependence of $|F|$ and $\theta_F$ on the dimensionless

![Physical picture of GW lensing. The vectors $\eta$ and $\xi$ denote the two positions on the source plane and on the lens plane. For illustrative purposes the two vectors are aligned but in principle they are not. The angular diameter distances $D_L$, $D_S$, and $D_{LS}$ are measured in the rest frame of the observer.](image-url)
frequency \( w \) and impact factor \( y \). We notice three results which are important for the later estimation of the lensing probability.

First, the amplification factor \( |F| \) in general decreases with increasing impact parameter \( y \). However, even when \( y \) is relatively large, e.g., \( y = 90 \), the amplification factor converges to the value in the geometric limit and is not 1, and the phase-change factor does not vanish. The implication is that even though the wave-optics effect is weak, it may still be detectable if the SNR of the event is sufficiently large. We will study the criterion for detecting such a weak signal in the later sections. The previous works, however, normally adopt an upper limit between \( y = 1 \) and 10 to estimate the lensing probability (e.g., Takahashi & Nakamura 2003). Such a small value could cause an underestimation of the number of lensing events.

Second, when \( y \) is fixed, both \( |F| \) and \( \theta_F \) could vary significantly due to the change of \( w \). In particular, the critical value of \( y \), above which the diffraction effect becomes undetectable, depends on \( w \), which, according to Equation (6), depends on the lens mass, lens redshift, and the GW frequency. We will take such a dependence into account in the following sections. These results indicate that it is oversimplified to use a single value of \( y \) to estimate the lensing probability in the diffraction limit, as is often the case in the previous works.

Third, the peaks of the amplification factor and the phase-change angles shift to smaller \( w \) as \( y \) increases. As a result, for large impact parameters, i.e., \( y = 10 - 150 \), the wave-optics effect appears the most significant at a small value of frequency, e.g., \( w \sim (10^{-5} - 10^{-3}) \). Such a small dimensionless frequency corresponds to a low-mass lens according to Equation (6), which is about

\[
M_L(1 + z_L) = 800 M_\odot \left( \frac{w}{10^{-4}} \right) \left( \frac{f}{10^{-3}\text{Hz}} \right)^{-1}.
\]

The corresponding halo mass is also small, about \( 10^5 - 10^7 M_\odot \) according to Appendix A. The above relationships suggest that the majority of the diffraction events detected by LISA should be induced by small halos, because (i) the lensing probability increases with \( y^2 \) and (ii) when \( y \) is large only small halos produce strong diffraction effect.

### 2.2 DM Halos and Subhalos

The lenses of our interest are those DM halos as small as \( 10^5 - 10^7 M_\odot \). The last section has shown that they induce an observable diffraction effect to the mHz GWs in the LISA band. Two types of DM halos fall in this mass range.

The first type reside in the low-density regions of the universe. They predominate the low-mass end of the mass function of ordinary DM halos (e.g. Wang et al. 2020). To compute the number density of these halos, we adopt the Sheth Tormen halo mass function \( d\nu/dM_h \) (see Cooray & Sheth 2002, for a review), where \( n \) denotes the number density of halos in unit of \( \text{Mpc}^{-3} \) and \( M_h \) is the halo mass. Note that by convention \( d\nu/dM_h \) has a unit of \( \text{Mpc}^{-3} \). The second type of DM halos fall in our interested mass range are the substructures of those massive DM halos. These substructures are often referred to as “subhalos”. Numerical simulations show that given the mass \( M_h \) of a main halo, the masses of the subhalos follow a power-law distribution with a universal power-law index (e.g. Gao et al. 2004a,b; Diemand et al. 2004; Libeskind et al. 2005; Giocoli et al. 2008)). Following Han et al. (2016), we write the mass function of the subhalos as

\[
\frac{dN(< R)}{dm} = A(R) \frac{M(< R)}{m_0} \left( \frac{m}{m_0} \right)^{-\alpha},
\]

where \( N(< R) \) is the number of subhalos within a radius of \( R \) of the main halo, \( m \) is the mass of the subhalo, \( A(R) \) is a normalization factor, \( M(< R) \) is the total mass enclosed by the radius \( R \), and \( m_0 = 10^{10} M_\odot \) and \( \alpha = 0.96 \) are constants nearly independent of the halo mass or redshift. In the later calculation, we are mainly interested in the number of subhalos within the virial radius \( R_{\text{vir}} \) of the main halo, regardless of their spatial distribution within the main halo. Therefore, we should replace \( R \) with \( R_{\text{vir}} \) when using Equation (12). By construction, the total mass within the virial radius is \( M(< R_{\text{vir}}) = M_h \). This leaves \( A(R_{\text{vir}}) \) the only quantity that is undetermined. We notice that Figure 15 of Han et al. (2016) gives the value of \( A(< R_{\text{vir}}) \) which shows that at \( R = R_{\text{vir}} \) the value converges to 0.01 for a wide range of halo mass, from \( M_h = 10^{12} h^{-1} M_\odot \) to \( 10^{15} h^{-1} M_\odot \). For this reason, we adopt \( A(< R_{\text{vir}}) = 0.01 \) for our later calculations.

Knowing the mass function of subhalos in one main halo, we can calculate the mass function density at a given redshift for all the subhalos of the same mass with

\[
\frac{d\nu_{\text{sub}}}{dm}(m, z_L) = \int \frac{dM_h}{dM_h} \frac{dN}{dM_h} \frac{1}{ln m m}.
\]

Such a quantity is useful for our later calculation of the lensing probability. Correspondingly, the total mass function density contributed by the halos and subhalos of a mass of \( M_h \) is

\[
\xi_{\text{lens}}(M_h, z) = \frac{d\nu_{\text{sub}}}{dM_h} (M_h, z) + \frac{d\nu}{dM_h}(M_h, z).
\]
Figure 3. The mass function of the small halos which could produce diffraction effects as a function the halo mass (upper panel) or redshift (lower panel).

Note that to find the lens mass $M_L$ corresponding to a halo mass $M_h$, the relationship derived in Appendix A is applied.

Figure 3 shows the mass function density predicted by Equation (14). We can see that the mass function density has little evolution from redshift $z = 2$ to 6 (upper panel), and it is more sensitive to the halo mass (lower panel). We note that in general halos are more numerous than subhalos. Nevertheless, we include subhalos in the calculation for completeness.

2.3 Calculation of the Lensing Probability

To calculate the lensing probability, we have to specify (i) the number of lenses of different masses at each redshift and (ii) the solid angle these lenses cover in which we can detect the diffraction of GW.

For (i), we start with the halo mass function density derived in the previous section, $\xi(M_h, z)$. Since the SIS model predicts a unique relationship between the lens mass and halo mass, $M_h(M_L, z_L, z_S)$ (see Appendix A), we can rewrite $\xi_l$ as a function of the lens mass, i.e., $\xi_l(M_h(M_L, z_L, z_S), z_L)$. Using this new mass function, we can calculate the number of lens in the mass range $(M_L, M_L + dM_L)$ and redshift bin $(z_L, z_L + dz_L)$ per unit solid angle using the equation

$$d^3N_{\text{lens}}(M_L, z_L, z_S) = \xi_l(M_h(M_L, z_L, z_S), z_L) \times \chi(z_L) \frac{dz}{dz} \frac{dM_h}{dM_L},$$

where $\chi(z)$ is the comoving distance for redshift $z$.

Figure 4 shows the result of Equation (15) integrated over a redshift range of $[0, z_{L\max}]$ and above a certain lens mass. The source is assumed to be at $z_S = 4$. It is clear that the number of lenses in a solid angle increases with redshift, and small lenses (e.g., $M_L \sim 10^5 M_\odot$) are the most numerous. Therefore, we expect that small halos contribute most of the lensing events. We have considered the lenses as small as $10^5 M_\odot$ because they correspond to a halo mass of about $10^5 M_\odot$ (see Appendix A).

As for (ii), suppose $\gamma_{\text{crit}}$ is the critical impact parameter in the source plane within which the effect due to the diffraction of GW is detectable. In the lens plane, the critical impact parameter corresponds to an angular size of

$$\theta(M_S, M_L, z_L, z_S) = \gamma_{\text{crit}} / D_L.$$

We have written $\theta$ as a function of the source mass $M_S$ and redshift $z_S$ to highlight the dependence of $\gamma_{\text{crit}}$ on the “loudness” of the source. Therefore, the lensing effect is detectable within a solid angle of

$$\pi \theta^2 = \pi \gamma^2 \times \frac{4GM_LD_L}{c^2D_SD_S}$$

(17)

towards the lens, where we have used the relation

$$\xi_0 = 2 \frac{GM_LD_L}{c^2D_S}$$

(18)

from the SIS model. Summing up all possible lenses between the source and the observer, we derive the lensing probability—the probability of detecting the diffraction effect in a given GW source—as

$$P = \int_0^{z_S} dz_L \int \pi \theta^2 \frac{d^3N_{\text{lens}}(M_L, z_L, z_S)}{dM_L dz_L d\Omega} dM_L.$$

(19)

In principle, the integration should be performed over all possible lens masses. In practice, we restrict the integration within a mass range $[M_{L\min}, M_{L\max}]$. The upper and lower limits are functions of lens redshift $z_L$, which should be determined by evaluating the prominence of the diffraction effect. Only those lenses producing a detectable diffraction effect should be counted. The following subsection explains how we quantify the detectability of the diffraction effect.

2.4 Signal and Matched Filtering

The magnification factor derived in Section 2.1 is a function of GW frequency. To use it, we need to first derive the unlensed GW signal
in the frequency domain. This is done by a Fourier transformation,

\[ \tilde{h}(f) = \int e^{2\pi ift} h(t) dt, \tag{20} \]

of the GW strain \( h(t) \) in the time domain.

For illustrative purposes, we show in Figure 5 the characteristic strain of three MBH mergers (solid curves). We assume equal-mass mergers with zero eccentricity, zero spins and zero inclination for simplicity and the total masses are \( M_S = 10^4 \), \( 10^5 \), and \( 10^6 \) \( M_\odot \) respectively. The source redshift is fixed at \( z_S = 4 \) in these examples. The waveforms are generated using the “IMRPhenomC” model in the PyCBC package (Santamaría et al. 2010; Nitz et al. 2020), excluding the effect of BH spin. As is mentioned in Dai et al. (2018), spin and eccentricity could also introduce diffraction-like waveform modulation. However, they also found that for misaligned spin, the induced pattern is more densely packed at low frequencies while the diffraction effect is spread across the frequency domain. The amplitude modulation due to spin is also much higher than its phase modulation, which is distinctive from the diffraction effect. As for eccentricity, it induces high harmonics, which is a feature absence from diffraction effect. Based on these differences, we assume that the modulation of the waveform by spin and eccentricity can be modeled in the future. Here we focus only on the diffraction effect. For the following calculation of unlensed template, we only vary the \( M_S \) and \( z_S \) parameters. Because we assume circular orbits for the MBH binaries, the merger time is about \( 1.7 f_{\text{mHz}}^{-8/3} M_4^{-5/3}(1+z_S) \) years (Peters & Mathews 1963), where \( f_{\text{mHz}} \) is the GW frequency in unit of mHz and \( M_4 \) is the total BH mass in unit of \( 10^4 M_\odot \). It is shorter than the canonical lifetime of LISA (5 years) except in the case of the smallest BHs.

We now integrate the characteristic strain in Figure 5 to derive the SNR of each merger. The calculation takes advantage of an inner product (Finn 1992; Cutler & Flanagan 1994) which is defined as

\[ \langle h_1 | h_2 \rangle = 2 \int_0^\infty \frac{\tilde{h}_1(f) \tilde{h}_2(f) + \tilde{h}_1(f) \tilde{h}_2^*(f)}{S_h(f)} df, \tag{21} \]

where \( h_1 \) and \( h_2 \) are two waveforms, \( S_h(f) \) is the one-sided power spectral density for LISA (from Robson et al. 2019), and the star symbols denote the complex conjugates. The SNR of a signal \( h \) is defined as \( \text{SNR} := \sqrt{\langle h | h \rangle} \).

Figure 6 shows the SNR of different MBH mergers at different redshift. We see that when the total mass is higher than about \( 10^5 M_\odot \) and the source redshift is lower than 10, the source in general has a SNR much higher than 10. These events are detectable by LISA (Amaro-Seoane et al. 2017). In the following we study the lensing signals of these events.

The strain of the lensed signals are shown in Figure 5 as the dotted lines. In the calculation, we assumed that the lens has a mass of \( M_L = 10^4 M_\odot \) and is at a redshift of \( z_L = 2 \). The impact parameter is set to \( y = 1 \) to maximize the effect in these examples. In this case, we can discern by eye that the lensed signals differ from the unlensed ones.

In more general cases, the impact parameters are much larger than 1 so that the diffraction effects are much more difficult to discern by eye (e.g., see Figure 2). Therefore, we employed the matched-filtering technique to quantify the deviation of a lensed signal from a waveform in the template bank. Suppose \( h_1 \) is the lensed signal and \( h_2 \) is an unlensed template, the difference \( \delta h := h_1 - h_2 \) is discernible when the SNR of the difference is larger than 1, i.e., \( \langle \delta h | \delta h \rangle > 1 \) (see Lindblom et al. 2008, for a proof).

In our work, \( h_2 \), the unlensed waveform, is generated from the aforementioned PyCBC package. We explore the parameter space of the template bank until we find the minimum value of \( \langle \delta h | \delta h \rangle \). If this minimum (\( \langle \delta h | \delta h \rangle \)) is still greater than 1, we deem the lensing signal detected. We call the corresponding waveform the "best fit". Note that the best-fit MBH binaries may differ from the real ones because of lensing.

We are being optimistic in adopting this criterion because we
assume that the deviation of the waveform from the theoretical waveform comes completely from the lensing effect. However, when the SNR is high, the criterion can be satisfied due to other factors, such as the presence of other signals, inaccurate waveform template, and the non-Gaussianity/non-stationarity of the noise. Nevertheless, it provides a practical criterion by which we can select from our simulations the lensing events which contain possibly discernible diffraction features.

3 DIFFERENCE BETWEEN THE LENSED AND THE BEST-FIT WAVEFORMS

The significance of the diffraction effect on the lensing signal depends on five parameters. Two of them are related to the source, i.e., the total mass of the MBH binary \( M_S \) and the source redshift \( z_S \). Two are related to the lens, i.e., the lens mass \( M_L \) and redshift \( z_L \). The final one is the impact parameter \( y \). In this section, we choose a grid of typical values for these five parameters and we investigate how the variation of their values affects \( \langle \delta h | \delta h \rangle \). More specifically, we choose \( M_S = (10^{5.0}, 10^{5.5}, 10^{6.0}, 10^{6.5}) \ M_\odot; \)
\( M_L = (16, 80, 160, 800, 1600, 8000, 16000) \ M_\odot; \)
\( z_S = (4, 6, 8, 10); \)
\( z_L \)  depends on \( z_S \) and we choose \( z_L = (0, 1/5, 2/5, 3/5, 4/5) z_S \) unless mentioned otherwise.

Figure 7 shows the dependence of \( \langle \delta h | \delta h \rangle \) on the source mass \( M_S \). Comparing it with the lower panel of Figure 6, we find that the inner product behaves similarly as the SNR of the unlensed signal. The reason is that higher SNR normally makes the deviation between the lensed signal and the best-fit template more discernible.

The dependence of \( \langle \delta h | \delta h \rangle \) on the lens mass \( M_L \) is shown in Figure 8. In this example, with a high impact parameter \( y = 40 \), we find that \( \langle \delta h | \delta h \rangle \) first increase and then decrease with the lens mass. The peak corresponds to a lens whose Einstein radius is comparable to the wavelength of the GW. We can also understand the result through the dimensionless frequency \( w \) and the corresponding amplification factor. On one hand, when \( M_L \) is small, \( w \) is small. According to Figure 2, the lensing effect is small. On the other hand, when \( M_L \) is particularly large so that \( w \) approaches unity, the system enters the geometric-optics regime where at any frequency the GW is amplified by the same factor \( \sqrt{1 + 1/y} \). The wave-optics effect, which is a frequency-dependent amplification of GWs, diminishes in this case.

Figure 9 shows the inner product as a function of the lens redshift \( z_L \). We find that \( \langle \delta h | \delta h \rangle \) increases as the lens redshift approaches the source redshift. This behavior is caused by the fact that the lensing effect is in general stronger when the lens and the source are closer. We also find that the sources at higher redshift in general produce a smaller \( \langle \delta h | \delta h \rangle \). This result stems from the decrease of the SNR as the source redshift increases. Note that the effect of lens redshift should be similar, but relatively weak, compared to the effect induced by the lens mass. This is because they affect \( w \) through the product \( (1 + z_L)M_L \). While \( z_L \) can only vary by a factor of a few, \( M_L \) can change by orders of magnitude.

Finally, we show the dependence of \( \langle \delta h | \delta h \rangle \) on the impact pa-
4 LEARNING PROBABILITY

Having investigated the dependence of the inner product \(\langle \delta h | \delta h \rangle\) on the parameters \(M_S, z_S, M_L, z_L, y\), we can now include the halo mass function and calculate the probability that a MBH binary in the LISA band has \(\langle \delta h | \delta h \rangle > 1\) due to the diffraction effect. We denote this probability as \(P(\langle \delta h | \delta h \rangle > 1)\), and the expression can be derived from Equation (19).

Given \((M_S, z_S, M_L, z_L)\), we first calculate the critical impact parameter \(y_{\text{crit}}\) which produces exactly \(\langle \delta h | \delta h \rangle = 1\). Figure 11 shows the critical impact parameter as a function of the lens redshift and lens mass. In these examples, we have chosen \(M_S = 10^{5.5} M_{\odot}\) and varied \(z_S\). We can see that in a large redshift range, \(y_{\text{crit}}\) has a value around 40, much higher than the value of \(y_{\text{crit}} = 3\) as has been chosen by the previous studies (e.g. Takahashi & Nakamura 2003). Moreover, sources at lower redshifts have higher \(y_{\text{crit}}\) since the higher SNR makes it easier to discern the wave-optics effect. In this work we have calculated the \(y_{\text{crit}}\) for a total number of \(4 \times 7 \times 4 \times 5 = 560\) grid points, covering the four-dimensional parameter space of \((M_S, z_S, M_L, z_L)\). Then by interpolation, we construct the function \(y_{\text{crit}}(M_S, z_S, M_L, z_L)\) which we will use in the following calculation of the lensing probability.

To calculate the probability, we replace \(\theta\) in the integrand of Equation (19) with \(\theta(y_{\text{crit}})\) (given by Equation (17)) and integrate to give the probability \(P(y < y_{\text{crit}})\). This probability \(P(y < y_{\text{crit}})\) is equivalent to \(P(\langle \delta h | \delta h \rangle > 1)\). In principle, given the source, i.e., after fixing \(M_S\) and \(z_S\), the \(y_{\text{crit}}\) in the integrand is a function of both \(z_L\) and \(M_L\). In practice, we only consider the lenses more massive than \(M_L = 16 M_{\odot}\) because less massive lenses in general do not produce a sufficiently large inner produce \(\langle \delta h | \delta h \rangle\), as Figure 8 has shown.

The resulting lensing probabilities are given in Table 1. We find that, in general, the probability of detecting the wave-optics effect is about \((0.1 - 1.6)\%\). Although low, such a probability is one order of magnitude larger than that those found in Takahashi & Nakamura (2003). The enhancement is influenced by a combination of larger impact parameters and more numerous lenses, but also the decrease of lens mass, which we will discuss in detail in the next section. We note that the probabilities \(P(M_S, z_S)\) derived here can be used to estimate the number of MBH mergers which show wave-optics effects prominent enough to be detectable by LISA, once the merger rate of MBHs as a function of mass \((M_S)\) and redshift \((z_S)\) is known.

In this work, we did not consider a lens mass higher than \(1.6 \times 10^{4} M_{\odot}\) because such lenses contribute a small fraction (less than \(10\%\)) to the total probability. The reasons are two fold. (i) The corresponding halos mass is greater than \(10^{8} M_{\odot}\). The number density of such halos is low. (ii) As the lens mass exceeds \(1.6 \times 10^{4} M_{\odot}\), the lensing effect will approach the geometric limit, diminishing the detectability of the diffraction effect.
Table 1. Total Lensing Probability for Different Source Parameters

| $P(M_s, z_S)$ | $M_s = 10^5.0 M_\odot$ | $M_s = 10^5.5 M_\odot$ | $M_s = 10^6.0 M_\odot$ | $M_s = 10^6.5 M_\odot$ |
|--------------|----------------|----------------|----------------|----------------|
| $z_S = 4$    | 0.0038         | 0.012          | 0.016          | 0.0059         |
| $z_S = 6$    | 0.0050         | 0.014          | 0.0081         | 0.0024         |
| $z_S = 8$    | 0.0059         | 0.012          | 0.0056         | 0.00095        |
| $z_S = 10$   | 0.0057         | 0.0094         | 0.0036         | 0.00040        |

5 IMPACT OF DM MODELS

We notice that Takahashi & Nakamura (2003) derived a lensing probability of $10^{-3} \sim 10^{-4}$ for the MBH binaries in the LISA band. It is at least one order of magnitude smaller than our estimation. The discrepancy stems from the different ranges of lens mass adopted in these two works.

Takahashi & Nakamura (2003) considered the lenses in the mass range of $M_L = 10^6 \sim 10^8 M_\odot$, which corresponds to a halo mass of $10^5 \sim 10^{12} M_\odot$. Such lenses are already in the geometric-optics limit. Moreover, they assumed a critical impact parameter of $y_{\text{crit}} = 3$. In our model, we considered the lenses in the mass range of $1.6 \times 10^4 \sim 1.6 \times 10^9 M_\odot$. The corresponding halo mass is $10^5 \sim 10^9 M_\odot$. These lenses produce wave-optics effect in the lensing signal, and we have shown that the effect is detectable even for a large impact parameter of $y \sim 40$ (Figure 11).

The difference of the lens masses affects the lensing probability in two ways. (i) In our model, the solid angle within in which the lensing signal is detectable is about 100 times smaller than the choice of Takahashi & Nakamura (2003) (assuming $y_{\text{crit}} = 40$), since it is proportional to $y^2_{\text{crit}} M_L$ according to Equation (17). (ii) Our lenses are about 1000 times more numerous than those considered in Takahashi & Nakamura (2003) according to Figure 3. The difference can be more clearly seen in Figure 12. Combining these two consequences, we find that our lensing probability is about 10 times higher than that derived in Takahashi & Nakamura (2003). The case of higher mass source at $z_S = 10$ is somewhat different, because the SNR reduces significantly. Consequently, $y_{\text{crit}}$ decreases, so that the probability is only several times larger than what Takahashi & Nakamura (2003) has derived.

The above comparison suggests that the lensing probability is sensitive to the abundance of small halos. Since different DM models predict very different number density for small halos, we now investigate the dependence of the lensing probability on the lower boundary of the halo mass function. To simulate the effect of different DM models, we cut off the integration of Equation (19) at different lower boundaries $M_{\text{hmin}}$ and count only those halo with $M_h > M_{\text{hmin}}$. The result is shown in Figure 13. We see a sharp cut off around $M_{\text{hmin}} = 10^5 M_\odot$. Compared to Takahashi & Nakamura (2003), our result indicates that the probability of detecting the diffraction effect of a MBH binary in the LISA band is significantly enhanced, because of the numerous small halos in the CDM paradigm. It also implies that if warm DM predominates (e.g. Lovell et al. 2014), the probability of detecting the wave-optics effect would be low.

6 SUMMARY CONCLUSION

In this work, we studied the lensing signals of the MBH binaries in the LISA band. We focused on the wave-optics effect and found that it is produced mainly by the DM halos and subhalos in the mass range of $10^5 \sim 10^8 M_\odot$. Using the matched-filtering technique, we showed that the effect could be discernible by LISA even when the
source has an impact parameter as large as $y = 40$, or even $y = 100$ in some cases. Such a large impact parameter substantially enhances the probability of detecting the diffraction signatures. Despite the large impact parameter, the probability of an event lensed by multiple halos is still low because the Einstein radius is small, normally $10^3 - 10^4$ times smaller than the virial radius of the host halo.

According to our preliminary estimation, if CDM predominates the matter content of the universe, the chance of detecting diffraction effect by LISA is about more than 1%, regarding of the source MBH binaries within the mass range of $10^{5.0} - 10^{5.5} \, M_\odot$. If, on the other hand, warm DM predominates, the chance of detecting the diffraction effect would be diminished by at least one order of magnitude. Therefore, looking for the wave-optics effects in LISA events could help us constrain the DM models.

As a final remark, we note that our model of the lensing signal and the criterion of discerning the diffraction effect are based on ideal assumptions. For example, we do not consider multiple lenses along the line of sight even though we have found a relatively high lensing probability. Moreover, we have assumed that the deviation of the detected signal from the model template is solely due to gravitational lensing, while for real LISA observation, other factors, such as confusion between multiple events or inaccuracy of the wavefront template, could also contribute to the deviation. We will address these caveats in a future work.

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DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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APPENDIX A: RELATION BETWEEN HALO MASS AND LENS MASS IN THE SIS MODEL

We assume that DM halo follows an SIS density profile,

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2},$$  \hspace{1cm} (A1)

where $\sigma_v$ is the velocity dispersion. According to this profile, the total mass of the halo, $M_h$, is related to the virial radius $r_{\text{vir}}(M_h, z_h)$, which is a function of the mass $M_h$ and redshift $z_h$, as

$$M_h = \frac{2\sigma_v^2}{G} r_{\text{vir}}(M_h, z_h).$$  \hspace{1cm} (A2)

The lens mass is defined as the mass enclosed by a circle on the lensing plane with a radius of $\xi_0 = 4\pi \sigma_v^2 D_L D_{LS}/c^2 D_S$, which is known as the Einstein radius. To calculate the lens mass $M_L$, we use the the surface density of an SIS projected on the lensing plane, $\Sigma(\xi) = \sigma_v^2/(2G\xi)$ (Takahashi & Nakamura 2003), and derive

$$M_L = \frac{4\pi^2 \sigma_v^4 D_L D_{LS}}{GD_S c^2}. $$  \hspace{1cm} (A3)

To relate the halo mass to the lens mass, we use Equations (A2) and (A3) to eliminate $\sigma_v$ and we find that

$$M_h = r_{\text{vir}}(M_h, z_h) \sqrt{\frac{M_L D_S c^2}{\pi^2 GD_{LS} D_L}}.$$  \hspace{1cm} (A4)

Figure A1 shows the relationship between these two masses and the dependence on the redshift of the halo.

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