Vector multiplets in $N = 2$ supersymmetry and its associated moduli spaces

Antoine Van Proeyen

Instituut voor theoretische fysica
Universiteit Leuven, B-3001 Leuven, Belgium

ABSTRACT

An introduction to $N = 2$ rigid and local supersymmetry is given. The construction of the actions of vector multiplets is reviewed, defining special Kähler manifolds. Symplectic transformations lead to either isometries or symplectic reparametrizations. Writing down a symplectic formulation of special geometry clarifies the relation to the moduli spaces of a Riemann surface or a Calabi-Yau 3-fold. The scheme for obtaining perturbative and non-perturbative corrections to a supersymmetry model is explained. The Seiberg-Witten model is reviewed as an example of the identification of duality symmetries with monodromies and symmetries of the associated moduli space of a Riemann surface.

1 Lectures given in the 1995 Trieste summer school in high energy physics and cosmology.
2 Onderzoeksleider NFWO, Belgium; E-mail: Antoine.VanProeyen@fys.kuleuven.ac.be

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Instituut voor theoretische fysica
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1. Introduction

When discussing supersymmetry, it is natural to consider extended supersymmetries. In the early days of supergravity, one realized immediately that the
maximal extended theory in 4 dimensions has $N = 8$ extended supersymmetry. The
idea was therefore that this should be the ultimate theory. But it turned out that
8 supersymmetries restrict the theory too much, such that physically interesting
theories could not be constructed. Any phenomenological theory was based on simple $N = 1$. Of course then the first extension, $N = 2$, was considered. It was found
that the Kähler structure known from $N = 1$ appears again, but in a restricted form
[1, 2].

The appearance of geometries, mentioned above, is related to the spinless
fields appearing in these supergravity theories. They define a map from the $d$-
dimensional Minkowskian space-time to some ‘target space’ whose metric is given
by the kinetic terms of these scalars. Supersymmetry severely restricts the possible
target-space geometries. The type of target space which one can obtain depends
on $d$ and on $N$, the latter indicating the number of independent supersymmetry
transformations. The number of supersymmetry generators (‘supercharges’) is thus
equal to $N$ times the dimension of the (smallest) spinor representation. For realistic
supergravity this number of supercharge components cannot exceed 32. As 32 is
the number of components of a Lorentz spinor in $d = 11$ space-time dimensions, it
follows that realistic supergravity theories can only exist for dimensions $d \leq 11$. For
the physical $d = 4$ dimensional space-time, one can have supergravity theories with
$1 \leq N \leq 8$. 

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Table 1: Restrictions on target-space manifolds according to the type of supergravity theory. The rows are arranged such that the number $\kappa$ of supercharge components is constant. $\mathcal{M}$ refers to a general Riemannian manifold, $SK$ to ‘special Kähler’, $VSR$ to ‘very special real’ and $Q$ to quaternionic manifolds.

| $\kappa$ | $d = 2$ | $d = 3$ | $d = 4$ | $d = 5$ | $d = 6$ |
|---------|---------|---------|---------|---------|---------|
| $N = 1$ | $\mathcal{M}$ | $Kahler$ | $Kahler$ | $Kahler$ | $Kahler$ |
| $N = 2$ | $N = 2$ | $N = 2$ | $N = 2$ | $N = 2$ | $N = 1$ |
| $N = 4$ | $Q$ | $Q$ | $SK \oplus Q$ | $VSR \oplus Q$ | $Q$ |
| $N = 8$ | $SO(6, n)$ | $SU(1, 1)$ | $U(1)$ | $\rightarrow$ |
| $N = 16$ | $SO(3, n)$ | $SU(2)$ | $\rightarrow$ |
| $N = 32$ | $E_7$ | $SU(8)$ | $\rightarrow$ |

As clearly exhibited in table 1, the more supercharge components one has, the more restrictions one finds. When the number of supercharge components exceeds 8, the target spaces are restricted to symmetric spaces. For $\kappa = 16$ components, they are specified by $n$, the number of vector multiplets. This row continues to $N = 1$, $d = 10$. Beyond 16 supercharge components there is no freedom left. The row with 32 supercharge components continues to $N = 1$, $d = 11$. Here I treat the case of 8 supercharge components. This is the highest value of $N$ where the target space is not restricted to be a symmetric space, although supersymmetry has already fixed a lot of its structure. This makes this row rather interesting. For physical theories we are of course mostly interested in $N = 2$ in $d = 4$. Below I will give some more motivations for studying $N = 2$ in $d = 4$. First I review the physical contents of these theories.

Table 2: Multiplets of $N = 2$, $d = 4$

| spin | pure SG | $n$ vector m. | $s$ hypermult. |
|------|---------|---------------|---------------|
| 2    | 1       | 1             |               |
| 3/2  | 2       | 2             |               |
| 1    | 1       | $n$           |               |
| 1/2  | 2$n$    | 2$s$          |               |
| 0    | 2$n$    | 4$s$          |               |

The physical fields which occur in $N = 2$, $d = 4$ are shown in table 2. In rigid supersymmetry there can be vector multiplets and hypermultiplets. In the former, the vector potentials, which describe the spin-1 particles, are accompanied by complex scalar fields and doublets of spinor fields. The vectors can gauge a group, and all these fields then take values in the associated Lie algebra. As I will show in section 2, the manifold of the scalar fields is Kählerian. The hypermultiplets
contain a multiple of 4 scalars, and one can construct a quaternionic structure on this space. In rigid supersymmetry this space is hyperKählerian. When coupling to supergravity an extra spin-1 field, called the graviphoton, appears. We will see in section 6 what the consequences are of mixing the vectors in the vector multiplets with the graviphoton. In supergravity the space of the scalars of hypermultiplets becomes a quaternionic manifold. The total scalar manifold is factorized into the quaternionic and the Kähler manifold. The latter is of a particular type $[4]$, called special $[4]$. Recently the special Kähler structure received a lot of attention, because it plays an important role in string compactifications. Also quaternionic manifolds appear in this context, and also here it is a restricted class of special quaternionic manifolds that is relevant. In lowest order of the string coupling constant these manifolds are even ‘very special’ Kähler and quaternionic, a notion that will be defined in section 6.

A further motivation for $N = 2$ theories comes from the compactification of superstring theories. It turns out that $N = 2$ often appears in this context. The amount of supersymmetry one gets in compactifications of superstrings depend on the type of superstring one starts with, and on the compactification manifold, see table 3. The compactification manifolds are classified here by the number of left

|   | Heterotic | Type II |
|---|-----------|---------|
| (2,0) | $N = 1$ | $N = 1$ |
| (2,1) | $N = 1$ | $N = 2$ |
| (2,2) | $N = 2$ | $N = 2$ |
| (4,0) | $N = 2$ | $N = 3$ |
| (4,4) | $N = 2$ | $N = 4$ |

and right handed world-sheet supersymmetries. (Note that possibly only a part of the compactification manifold has the world-sheet supersymmetries mentioned in the first column). Calabi–Yau manifolds have (2,2) supersymmetry, while $K3 \times T^2$ is at least a (4,0) compactification. It is clear from this table that $N = 2$ often appears.

Another motivation for $N = 2$ comes from topological theories. These can be constructed by ‘twisting $N = 2$’ theories $[3, 5, 6]$. In the simplest version the original $SO(4)_{\text{spin}} = SU(2)_L \otimes SU(2)_R$ Lorentz group is deformed to $SO(4)'_{\text{spin}} = SU(2)_L \otimes SU(2)'_R$ where $SU(2)'_R$ is the diagonal subgroup of $SU(2)_L \otimes SU(2)_R$, and $SU(2)_L$ is the group rotating the 2 supersymmetry charges. Then one generator of $N = 2$ modifies the BRST transformations, and an $R$–symmetry which is present in $N = 2$ modifies the ghost number such that the cohomology at zero ghost number of the BRST gives the interesting topological configurations. In $[8]$ the ghost number assignments for this procedure has been improved by splitting Vafa’s BRST operator $[6]$ in a BRST

$^*$The terminology ‘special Kähler’ is used for this structure which one finds in the context of supergravity. The manifolds obtained in rigid supersymmetry are usually called ‘rigid special Kähler manifolds’.
and anti-BRST. There are also more complicated twisting procedures \[7\] which deform also the $SU(2)_L$. In any case the correlation functions of $N = 2$ theories can be connected to topological quantities. The application of this procedure in compactified string theories has been discussed in \[9\].

The final motivation I present, has to do with the recent ideas about dualities. This is the main issue in the lectures below. I already mentioned that the restrictions from the presence of 2 supersymmetries still allow enough freedom for the manifold to be not restricted to a symmetric space. The definition of these manifolds then depends on some arbitrary functions, as the Kähler potential in $N = 1$. We will see in the coming lectures that the special Kähler manifolds are determined by a holomorphic function, which is called a prepotential. This restriction was crucial to obtain exact quantum results for $N = 2$ theories in the famous Seiberg-Witten papers last year \[10\].

The general idea for obtaining such exact results by connecting the $N = 2$ theory to moduli of surfaces is as follows. The $N = 2$ theories (Seiberg and Witten considered rigid $N = 2$ supersymmetry) have a potential that is zero for arbitrary values of some (massless) scalars. The value of these scalars therefore parametrize the vacua of these theories. The aim is then to find an effective quantum theory for these scalars when the massive states are integrated out of the path integral. This effective action gets perturbative and non-perturbative contributions. It will be invariant under a set of duality symmetries, which are a subset of symplectic transformations as I will discuss in section \[3\]. After getting clues from the perturbative results, Seiberg and Witten have connected the scalar fields to moduli of a Riemann surface, and conjecture that the full quantum theory can be obtained from a metric in the space of moduli. The duality transformations of the effective quantum theory are now represented either as symmetries of the defining equation of the Riemann surface or as transformations obtained by encircling singular points in moduli space, called monodromies. These singular points correspond in the effective field theory to vacua where other states become massless, and therefore at these points the starting setup was not valid, leading to the singularity.

In supergravity theories the same ideas can be applied. The main difference is now that the connection should be made to moduli of Calabi-Yau 3-folds \[11\]. Moreover, in this case the surface is not just a geometrical tool, but can be seen as target space of the dual theory in the context of string compactifications.

For that purpose consider compactifications of the heterotic string on $K3 \times T^2$ manifolds on the one hand, and type II strings (type IIA for definiteness) compactified on a Calabi-Yau manifold on the other hand. Both lead to $N = 2$ theories with vector and hypermultiplets. For the Calabi-Yau manifold in type IIA one obtains $h^{1,1}$ neutral vectormultiplets and $h^{1,2} + 1$ neutral hypermultiplets. (The Hodge numbers are interchanged for type IIB). Now there are important facts of $N = 2$ which come to help to get information on the quantum theories. First, the scalar ‘dilaton’ field of the original superstring action becomes part of a vector multiplet when compactifying a heterotic string, while it becomes part of a hypermultiplet when compactifying a type II string. This can already be seen from counting the number
of fields in the compactified theory. Secondly, this dilaton field $S$ arranges the perturbation theory. Its expectation value is $\langle S \rangle = \frac{\theta}{\pi} + \frac{i}{g}$, where $\theta$ is the theta-angle and $g$ is the coupling constant, which should appear in all quantum corrections. The third information is that in $N = 2$ there are no couplings between the scalars of vector multiplets and those of neutral hypermultiplets. Combining these 3 facts, one concludes that in the compactified heterotic string the hypermultiplet manifold is not quantum corrected as it can not depend on the coupling constant. For the compactified type II string this holds for the vector multiplets. The further assumption is the validity of the duality hypothesis, called ‘second quantised mirror symmetry’ [12]. This states that the quantum theories of the mentioned heterotic and type II compactified theories are dual to each other. This hypothesis is used to get information about the vector multiplet couplings which can be obtained through dimensional reduction of the heterotic string. These are thus related to the quantum theory of the type II theory, which is by the previous arguments the same as the classical theory. That classical theory is the one of the moduli of Calabi-Yau manifolds. Therefore we have here a string-based relation between the quantum theory of vector multiplet couplings and the moduli space of Calabi-Yau manifolds.

Section 2 will treat rigid $N = 2$ supersymmetry. I will pay most attention to the vector multiplets, explaining their description in superspace. Their action is determined by a holomorphic prepotential. Duality symmetries (symplectic transformations) are first shown for general couplings of scalars and vectors in section 3. Then I specify to the case of $N = 2$. There are two kind of applications, either as isometries of the manifolds (symmetries of the theory), or as equivalence relations of prepotentials (pseudo-symmetries). As an example, I will look at the duality symmetries of the Seiberg-Witten model when the perturbative quantum corrections are taken into account. In section 4 I will give another definition of the geometry of the scalar manifold in a coordinate independent way, and covariant for symplectic transformations, which paves the way for the comparison with the theory of the space of moduli of a Riemann surface (section 5). This moduli space is conjectured to describe the full quantum theory for the massless fields. Section 5.3 will contain this theory for the simplest example of Seiberg and Witten [10]. In section 6 I will exhibit how the structure gets more rich in supergravity, where the space is embedded in a projective space. This structure was found by starting from the superconformal tensor calculus. The symplectic formulation and connection to Calabi-Yau moduli will be explained shortly, mainly by making the analogy with the rigid case. Finally also the basic facts about special quaternionic and classification of homogeneous special manifolds are recalled.

In a first appendix the conventions are explained and some useful formulae are given. The second appendix contains a translation table for conventions used in the $N = 2$ literature. The final two appendices are related to technical issues for section 3: the volume form in pseudo homogeneous spaces and some main facts of elliptic integrals.
2. Rigid $N = 2$

2.1. The $N = 2$ algebra

Supersymmetry by definition means that the supersymmetry operator $Q_\alpha$ squares to the momentum operator $P_\mu$. For several supersymmetries, labelled by $i$, the algebra is

$$\{Q_\alpha, Q_\beta\} = -i (\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \delta_{ij}. \quad (2.1)$$

The factor $i$ is introduced for consistency with hermitian conjugation with an hermitian $P_\mu$. See the appendix for hermitian conjugation of the spinors, where it is also explained that the position of the index $i$ indicates the chirality. The hermitian conjugates of the generators

$$Q^{\dagger\alpha i} \equiv (Q_\alpha)^\dagger = -iQ^{i T}C\gamma_0 \quad (2.2)$$

then satisfy

$$\{Q_{\alpha i}, Q^{\dagger j\beta}\} = P_\mu \delta_{ij} (\gamma^\mu \gamma_0)_{\alpha\beta}. \quad (2.3)$$

With $P_0$ the energy, or the mass, this exhibits the positive energy statements in supersymmetry. To see this, select two values of $\alpha$ for which $Q_\alpha$ are considered as annihilation operators, and $Q^{i\alpha}$ are then creation operators (they are related by (2.2) to the other two $Q_\alpha$).

The Haag-Lopuszanski-Sohnius theorem [13] restricts the symmetries which a field theory with non-trivial scattering amplitudes can have. It restricts then also the algebra of the supersymmetries, but still allows central charges in extended supersymmetry. This means that we may still have (for $N = 2$)

$$\{Q_{\alpha i}, Q_{\beta j}\} = C^{-1}_{\alpha\beta} \epsilon_{ij} Z. \quad (2.4)$$

For higher extended supersymmetry one could have $Z_{ij}$ antisymmetric, which for $N = 2$ reduces to the mentioned form. For the hermitian conjugates this implies

$$\{Q^{\dagger\alpha i}, Q^{\dagger j\beta}\} = -\epsilon_{ij} C^{\alpha\beta} Z^\dagger. \quad (2.5)$$

Remark that this theorem concerns algebras of symmetries in the sense that the structure constants are constant, and not field-dependent 'structure functions' as in the 'soft algebras' that are used in field representations of supersymmetry. The theorem should then apply for the vacuum expectation values of the structure functions.

Define now

$$A_{\alpha i} = Q_{\alpha i} + e^{i\theta} \epsilon_{ij} C^{-1}_{\alpha\beta} Q^{j\beta}, \quad (2.6)$$

where $e^{i\theta}$ is an arbitrary phase factor. The hermitian conjugates are (C is taken to be unitary)

$$A^{\dagger\alpha i} = Q^{\dagger\alpha i} + e^{-i\theta} \epsilon^{ij} Q_{j\beta} C^{\beta\alpha} = e^{-i\theta} \epsilon^{ij} A_{j\beta} C^{\beta\alpha}. \quad (2.7)$$
Consider now their anticommutators on a state for which \( P_\mu = \delta^0_\mu M \):

\[
\{A_{\alpha i}, A_{\beta j}\} = \epsilon_{ij} \gamma^\mu (Z + 2e^{i\theta}M + Z^\dagger e^{2i\theta}) \tag{2.8}
\]

or

\[
\{A_{\alpha i}, A^{\dagger \beta j}\} = \delta_j^i \delta^\beta_\alpha (2M + Ze^{-i\theta} + Z^\dagger e^{i\theta}) . \tag{2.9}
\]

As the left hand side is a positive definite operator, the right hand side should be positive for all \( \theta \), which shows that \( M \geq |Z| \). This is an important result relating the masses to the central charges. One can now also show that if the equality is satisfied for a state, then that state is invariant under some supersymmetry operation.

I will not require the presence of central charges, but we will find that they are needed for certain representations.

2.2. Multiplets

The superfield which is most useful for our purposes is the chiral superfield. The \( N = 2 \) superspace is built with anticommuting coordinates \( \theta^i_\alpha \), and \( \theta_{\alpha i} \), where again the position of the index \( i \) indicates the chirality. A chiral superfield is defined by a constraint \( D^\alpha \Phi = 0 \), where \( D^\alpha \) indicates a (chiral) covariant derivative in superspace. I will not give a detailed definition of covariant derivatives in superspace as this will not be necessary for the following. The superfield \( \Phi \) is complex, and can be expanded as

\[
\Phi = A + \theta^i_\alpha \Psi_i^\alpha + C^{\alpha \beta} \theta^i_\alpha \eta^i_{\beta j} B_{ij} + \epsilon_{ij} \theta^i_\alpha \theta^j_\beta F^{\alpha \beta} + \ldots .
\]

\( B_{ij} \) is symmetric and so is \( F^{\alpha \beta} \), which can then, due to the chirality and the symmetry, be written as \( F^{\alpha \beta} = \sigma^{\alpha \beta} F^{-ab} \), where \( F^{-ab} \) is an arbitrary antisymmetric antiselfdual tensor (the selfdual part occurs in \( \Phi \)).

One can define a chiral multiplet also without superspace. Then one starts from a complex scalar \( A \) and demands that it transforms under supersymmetry only with a chiral supersymmetry parameter (no terms with \( \epsilon \)):

\[
\delta(\epsilon) A = \bar{\epsilon} \Psi_i . \tag{2.10}
\]

The spinor \( \Psi_i \) is a new field, and one then takes for \( \Psi \) the most general transformation law compatible with the supersymmetry algebra. First, I translate the anticommutator of the generators \( Q_{\alpha i} \), (2.7), to a commutator of \( \delta(\epsilon) \equiv \bar{\epsilon} Q_i + \epsilon Q^i \). The operator \( P_\mu \) is represented on fields as \( P_\mu = -i \partial_\mu \). This leads to

\[
[\delta(\epsilon_1), \delta(\epsilon_2)] = \bar{\epsilon}_{2i} \gamma^\mu \epsilon_{1i} \partial_\mu + \bar{\epsilon}_{2i} \gamma^\mu \epsilon_{1i} \partial_\mu , \tag{2.11}
\]

where the second term is the hermitian conjugate of the first, or could also be denoted as \(-1 \leftrightarrow 2 \).

Compatibility with this algebra restricts \( \delta \Psi \) to

\[
\delta(\epsilon) \Psi_i = \bar{\epsilon} A \epsilon_i + \frac{1}{2} B_{ij} \epsilon^j + \frac{1}{2} \sigma_{ab} F^{-ab} \epsilon_i \epsilon^j , \tag{2.12}
\]

where \( B_{ij} \) is an arbitrary symmetric field as in the superspace, and \( F^{ab} \) is antisymmetric (and the (anti)selfduality is automatic, see (A.12)). This most general
transformation law defines the next components of the multiplets. If we continue with the most general transformation laws for $B$ and $F$ we find again new arbitrary fields, and finally the procedure ends defining the chiral multiplet with as free components

$$(A, \Psi_i, B_{ij}, F_{ab}, \Lambda_i, C).$$

(2.13)

$\Lambda_i$ is a spinor and $C$ a complex scalar. Note that we did not allow central charges in the algebra. Allowing these would already change (2.12), see [14], and lead to more components.

In $N = 2$ the minimal multiplets have 8+8 real components. The chiral multiplet has 16+16 components and is a reducible multiplet. The vector multiplet is an irreducible 8+8 part of this chiral multiplet. The others form a ‘linear multiplet’, see below. The reduction is accomplished by an additional constraint, which in superspace reads

$$D_{\alpha(i}D_{\beta)j}\Phi C^{\alpha\beta} = \epsilon_{ik}\epsilon_{jl}D_{(k}^{(i}\Phi D_{l)}^{j)}\Phi C^{\alpha\beta},$$

(2.14)

where $\Phi$ is the complex conjugate superfield, containing the complex conjugate fields. In components, this is equivalent to the condition

$$L_{ij} \equiv B_{ij} - \epsilon_{ik}\epsilon_{jl}\tilde{B}^{kl} = 0,$$

(2.15)

and the equations which follow from this by supersymmetry. These are

$$\phi_i \equiv \bar{\phi}\Psi_i - \epsilon^{ij}\Lambda_j = 0,$$

$$E^a \equiv \partial_b^*F^{ab} = 0,$$

$$H \equiv C - 2\partial_a\bar{\phi}^a\bar{\Lambda} = 0.$$

(2.16)

The equation (2.15) is a reality condition. It leaves in $B_{ij}$ only 3 free real components. The other equations define $\Lambda$ and $C$ in terms of $\Psi$ and $A$, and the remaining one is a Bianchi identity for $F_{ab}$, which implies that this is the field strength of a vector potential. The constrained multiplet is therefore called a ‘vector multiplet’, which is thus a multiplet consisting of independent fields (we give new names for the independent fields of a vector multiplet) $(X; \Omega_i; Y_{ij}; W_{\mu})$ with transformation laws

$$\delta(\epsilon)X = \bar{\epsilon}^i\Omega_i,$$

$$\delta(\epsilon)\Omega_i = \bar{\phi}X\epsilon_i + \frac{1}{2}Y_{ij}\epsilon^j + \frac{1}{2}\sigma_{ab}F^{-ab}\epsilon_{ij}\epsilon^j,$$

$$\delta(\epsilon)Y_{ij} = 2\bar{\epsilon}(\phi\Omega_{ij}) + 2\epsilon_{ik}\epsilon_{jl}\epsilon^{(k}\phi\Omega^{l)},$$

$$\delta(\epsilon)W_{\mu} = \bar{\epsilon}^i\gamma_{\mu}\Omega_{ij}\epsilon^j + \epsilon^i\gamma_{\mu}\Omega_{ij}\epsilon_{ij}.$$  

(2.17)

Here, the vector is abelian, and $F_{\mu\nu} = 2\partial_{[\mu}W_{\nu]}$. Let me remark that these $N = 2$ multiplets exist in dimensions up to $d = 6$. In 6 dimensions, the vector multiplet has no scalars, and consists of $(\Omega_i; Y_{ij}; W_{\mu})$, where now $\mu$ runs over 6 values. We may understand the complex scalar $X$ in four dimensions as the fifth and sixth coordinate of the vector in 6 dimensions.

†Therefore you find no trace of these in table 1.

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Now, I extend the vector multiplet to the non-abelian case. This means that the vectors gauge a group $G$ with generators $T_A$ and coupling constant $g$. The index $A$ runs over $n = \dim G$ values. I then attach an index $A$ to all fields of the vector multiplets, and write the fields in the adjoint as e.g. $X = X^A T_A$, and (with parameters $y^A$)

$$
\delta_G(y) X = g[y,X] = g y^A X^B [T_A, T_B] ; \quad \delta_G(y) W_\mu = \partial_\mu y + g[y, W_\mu] .
$$

(2.18)

First we could try to replace all derivatives with derivatives covariant under the gauge group, e.g. $D_\mu = \partial_\mu - g \delta_G(W_\mu)$. However it can not be that simple. Indeed, replacing all derivatives by covariant derivatives gives also a covariant derivative in (2.11). But then the Jacobi identities are not satisfied any more. Indeed, applying a third supersymmetry on that commutator, this one can also act on the $W_\mu$ contained in the covariant derivative $D_\mu$. That gives a term with $\Omega$:

$$
[\delta(\epsilon_3), [\delta(\epsilon_1), \delta(\epsilon_2)]] = -(\bar{\epsilon}_2^\mu \gamma^\nu \epsilon_{1i} + h.c.) \, \delta_G(\epsilon_{3k} \gamma_\mu \Omega_{j}^k \epsilon_{j} + h.c.) ,
$$

(2.19)

which violates the Jacobi identity in $d = 4$. In 6 dimensions the Jacobi identity is satisfied, due to a well-known Fierz identity. Therefore in 6 dimensions, the substitution of ordinary derivatives by covariant derivatives is sufficient. Reducing that to 4 dimensions creates terms with the $X$ field. Therefore the algebra in 4 dimensions is not just (2.11), with a covariant derivative, but contains an $X$–dependent gauge transformation:

$$
[\delta(\epsilon_1), \delta(\epsilon_2)] = \bar{\epsilon}_2^\mu \gamma_\mu \epsilon_{1i} D_\mu - 2 \delta_G (X \bar{\epsilon}_{1i} \epsilon_{2j} \epsilon^{ij}) + h.c. .
$$

(2.20)

The algebra has become ‘soft’, i.e. the structure functions depend on fields. If the fields have zero expectation value, then the algebra reduces in the vacuum to (2.1). However, if some $X^A$ get a non–vanishing vacuum expectation value, the right hand side of (2.4) is non–zero, and there is a central charge $Z = -2 \langle X^A \rangle T_A$.

To realise the new algebra, the transformation rules of the fields have to be modified by $g$–dependent terms. E.g. for the spinor of the vector multiplet we now find

$$
\delta(\epsilon) \Omega_i = \partial X_{\epsilon_i} + \frac{1}{2} Y_{ij} \epsilon^j + \frac{1}{2} \sigma^{ab} F_{ab}^{-} \epsilon_{ij} \epsilon^j - g(X, \bar{X}) \epsilon_{ij} \epsilon^j .
$$

(2.21)

For a more complete treatment of $N = 2$ we should also consider other multiplets. The physical hypermultiplet can be described by scalar multiplets consisting of $2s$ complex scalar fields, their central charge transformed fields, which are auxiliary, and $s$ fermions. An alternative is the linear multiplet. The constraints (2.13) and (2.16) transform as a linear multiplet. $E_a$ is then the field strength of an antisymmetric tensor. There is also a non–linear multiplet, a ‘relaxed hypermultiplet’, or descriptions in harmonic superspace. Similarly there is an alternative for the vector multiplet where one of the scalars is described by an antisymmetric tensor, or another description of massive vectormultiplets. In these lectures I will restrict myself to the vector multiplet as described above.
2.3. Action for the vector multiplet

In superspace, actions can be obtained as integrals over the full or chiral superspace. So we can get an action from integrating over a chiral superfield. Remembering that vector multiplets are chiral superfields, we can form new chiral superfields by taking an arbitrary holomorphic function of the vector multiplets. If the superfield defining the vector multiplet is $\Phi$ (which thus satisfies also (2.14)), then we take the integral (and add its complex conjugate)

$$\int d^4x \int d^4\theta \ F(\Phi) + \text{h.c.} ,$$

for an arbitrary holomorphic function $F$.

The superfield $F(\Phi)$ has by definition as lowest component $A = F(X)$. The further components are then defined by the transformation law, which gives, comparing with (2.10)

$$\Psi_i = F_A \Omega_i^A ,$$

Calculating the transformation of $\Psi_i$, one finds $B_{ij}$ and $F_{ab}$. . . . In components, the integral (2.22) is the highest component of the superfield, $C$ in the notation of (2.13). This leads to (The standard convention these days is to start with $A = -iF(X)$, which I now also use):

$$\mathcal{L}_F = -iF_A D_a \dot{D}^a X^A + \frac{i}{4} F_{AB} \bar{F}_{-B} + iF_{AB} \bar{\Omega}_i^A \varphi \Omega_i^B$$

$$-\frac{i}{8} F_{AB} Y_{ij} X^{ij} B + \frac{i}{4} F_{ABC} Y^{ij} A \bar{\Omega}_i^A \Omega_j^B \Omega_k^C$$

$$-\frac{i}{4} F_{ABC} \epsilon^{ij} \bar{\Omega}_i^A \sigma \cdot F^{ij} \Omega_j^C + \frac{i}{16} F_{ABCD} \epsilon^{ij} \epsilon^{kl} \bar{\Omega}_i^A \Omega_j^B \Omega_k^C \Omega_l^D$$

$$-ig \epsilon^{ij} F_{AB} \bar{\Omega}_i^A \Omega_j^B + ig F_A \bar{\Omega}_i^A \Omega_j^B - ig^2 F_A \left( [\bar{\Omega}_i^A, \Omega_j^B] + i \varepsilon^{ij} + ig \right) + \text{h.c.}$$

The first terms of the action give kinetic terms for the scalars $X$, the vectors, and the fermions $\Omega$. The following term says that $Y_{ij}$ is an auxiliary field that can be eliminated by its field equation.

Most of the discussion about the supersymmetry actions is only about their bosonic part. The presence of the fermions and supersymmetry has given a restriction of the bosonic action, but can then be forgotten for many considerations. So let me describe what we have obtained here for the bosonic part.
First, I write a general formulation for the bosonic sector of a theory with scalar fields $z^\alpha$, and vector fields labelled by an index $\Lambda$. If there are no Chern-Simons terms (these do occur for non-abelian theories if $\delta F \neq 0$), one can write a general expression

$$L_0 = -g_{\alpha\bar{\beta}}D_\mu z^\alpha D^\mu \bar{z}^{\bar{\beta}} - V(z)$$  \hspace{1cm} (2.26)

$$L_1 = \frac{1}{4}\text{Im} \left( N_{\Lambda\Sigma}(z) F^{\Lambda^\mu\nu} F_{\mu\nu} \Sigma \right) = \frac{1}{4}\left( \text{Im} N_{\Lambda\Sigma} \right) F^{\Lambda^\mu\nu} F_{\mu\nu} \Sigma - \frac{i}{8} \left( \text{Re} N_{\Lambda\Sigma} \right) \epsilon_{\mu\nu\rho\sigma} F^{\Lambda^\mu\nu} F_{\rho\sigma} \Sigma .$$

$g_{\alpha\bar{\beta}}$ is the (positive definite) metric of the target space, while $\text{Im} N$ is a (negative definite) matrix of the scalar fields, whose vacuum expectation value gives the gauge coupling constants, while that of $\text{Re} N$ gives the so-called theta angles. $V$ is the potential.

In our case the $z^\alpha$ can be chosen to be $X^A$ (special coordinates), and the index $\Lambda$ is also $A$. We obtain

$$G_{AB}(X, \bar{X}) = 2 \text{Im} F_{AB} = \partial_A \partial_B K(X, \bar{X}) \quad \text{with} \quad K(X, \bar{X}) = i(\bar{F}_A(\bar{X})X^A - F_A(X)X^A)$$

$$N_{AB} = \bar{F}_{AB} ; \quad V = ig^2 F_A \left\{ X^A, \bar{X}^A \right\} + h.c. \quad (2.27)$$

The metric in target space is thus Kählerian. For $N = 1$ the Kähler potential could have been arbitrary. We find here that the presence of two independent supersymmetries implies that this Kähler metric, and even the complete action, depends on a holomorphic prepotential $F(X)$. Two different functions $F(X)$ may correspond to equivalent equations of motion and to the same geometry. It is easy to see that

$$F \approx F + a + q_A X^A + c_{AB} X^A X^B ,$$ \hspace{1cm} (2.28)

where $a$ and $q_A$ are complex numbers, and $c_{AB}$ real. But more relations can be derived from the symplectic transformations that we discuss in section 3.

As a simple example, used also in [10], consider the $N = 2$ susy-YM theory with gauge group $SU(2)$. For invariant holomorphic function, we can use

$$F = \alpha X^A X^A ,$$ \hspace{1cm} (2.29)

where now $A = 1, 2, 3$, and $\alpha$ is a complex number. As, according to (2.28) the real part of $\alpha$ does not contribute to the action, we can take $\alpha = i$ (positive imaginary part for positive kinetic energies). The potential of this theory is

$$V = 4g^2 \left| \epsilon_{ABC} \bar{X}^B X^C \right|^2 .$$ \hspace{1cm} (2.30)

This shows that it remains zero in valleys where e.g. $X^A = a \delta^A_3$. Note that different values of $a$ give different masses for the vectors, so the value of this 'modulus' is physically relevant. For $a = 0$ all the vectors are massless.

The description of the action as it follows from superspace is not manifestly invariant under reparametrizations of the target space. Indeed, the superfield constraint (2.14) allows only real linear transformations of the superfields, and thus of the $X^A$. A description of the scalar manifold covariant under target space reparametrizations will be given in section 4.1. Its formulation is inspired by the symplectic transformations which we will find in section 3.
3. Symplectic transformations

The symplectic transformations are a generalization of the electro-magnetic
duality transformations. I first recall the general formalism for arbitrary actions
with coupled spin-0 and spin-1 fields [26], and then come to the specific case of
$N = 2$.

3.1. Duality symmetries for the vectors

Consider a general action of the form $L_1$ in (2.26) for abelian spin-1 fields.
The field equations for the vectors are

$$
0 = \frac{\partial L}{\partial W_\mu} = 2 \partial_\nu \left( \frac{\partial L}{\partial F^{\mu\nu}_+} + \frac{\partial L}{\partial F^{-\mu\nu}_+} \right)
$$

(3.1)

I define

$$
G^{\mu\nu}_+ \equiv 2i \frac{\partial L}{\partial F^{\mu\nu}_+} = N\Sigma F^{\mu\nu} + \Sigma^{\mu\nu};
G^{-\mu\nu}_+ \equiv -2i \frac{\partial L}{\partial F^{-\mu\nu}_+} = \bar{N}\Sigma F^{-\mu\nu}.
$$

(3.2)

Observe that these relations are only consistent for symmetric $N$. So far, this is an
obvious remark, as in (2.26) we can choose $N$ to be symmetric. The equations for
the field strengths can then be written as

$$
\partial^\mu \text{Im } F^{\mu\nu}_+ = 0 \quad \text{Bianchi identities}
\partial_\mu \text{Im } G^{\mu\nu}_+ = 0 \quad \text{Equations of motion (3.3)}
$$

This set of equations is invariant under $GL(2m, \mathbb{R})$ transformations:

$$
\left( \begin{array}{c}
\tilde{F}^+
\\tilde{G}^+
\end{array} \right) = S \left( \begin{array}{cc}
F^+
G^+
\end{array} \right) = \left( \begin{array}{cc}
A & B \\
C & D
\end{array} \right) \left( \begin{array}{c}
F^+
G^+
\end{array} \right).
$$

(3.4)

However, the $G_{\mu\nu}$ are related to the $F_{\mu\nu}$ as in (3.2). The previous transformation
implies

$$
\tilde{G}^+ = (C + D\tilde{N})F^+ = (C + D\tilde{N})(A + B\tilde{N})^{-1} \tilde{F}^+
$$

(3.5)

$$
\rightarrow \quad \tilde{N} = (C + D\tilde{N})(A + B\tilde{N})^{-1}
$$

(3.6)

As remarked above, this tensor should be symmetric:

$$
\rightarrow \quad (A + B\tilde{N})^T (C + D\tilde{N}) = (C + D\tilde{N})^T (A + B\tilde{N})
$$

(3.7)

which for a general $\tilde{N}$ implies

$$
A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = 1.
$$

(3.8)

These equations express that $S \in Sp(2m, \mathbb{R})$:

$$
S^T \Omega S = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

(3.9)
Some remarks are in order. First, these transformations act on the field strengths. They generically rotate electric into magnetic fields and vice versa. Such rotations, which are called duality transformations, because in four space-time dimensions electric and magnetic fields are dual to each other in the sense of Poincaré duality, cannot be implemented on the vector potentials, at least not in a local way. Therefore, the use of these symplectic transformations is only legitimate for zero gauge coupling constant. From now on, we deal exclusively with Abelian gauge groups.

Secondly, the Lagrangian is not an invariant if \( C \neq 0 \) and \( B = 0 \):

\[
\text{Im} \, \tilde{F}^+ \Lambda \tilde{G}^+ \Lambda = \text{Im} \, (F^+ G_+) + \text{Im} \, (2F^+(CTB)G_+ + F^+(CTA)F^+ + G_+(D^TB)G_+) .
\]  

(3.10)

If \( C \neq 0, B = 0 \) it is invariant up to a four–divergence, as \( \text{Im} \, F^+ F^+ = -\frac{i}{4} \epsilon^{\mu
u\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \) and the matrices are real. Thirdly, the transformations can also act on dyonic solutions of the field equations and the vector \( \left( \frac{q_{\alpha}}{q_{\nu}} \right) \) of magnetic and electric charges transforms also as a symplectic vector. The Schwinger-Zwanziger quantization condition restricts these charges to a lattice (see also lectures of J. Harvey [27]). Invariance of this lattice restricts the symplectic transformations to a discrete subgroup:

\[
S \in Sp(2m, \mathbb{Z}) .
\]

(3.11)

Finally, the transformations with \( B \neq 0 \) will be non–perturbative. This can be seen from the fact that they do not leave the purely electric charges invariant, or from the fact that (3.6) shows that these transformations invert \( N \), which plays the role of the gauge coupling constant.

3.2. Pseudo–symmetries and proper symmetries

The transformations described above, change the matrix \( N \), which are gauge coupling constants of the spin-1 fields. This can be compared to diffeomorphisms of the scalar manifold \( z \to \hat{z}(z) \) which change the metric (which is the coupling constant matrix for the kinetic energies of the scalars) and \( N \):

\[
\hat{g}_{\alpha\beta}(\hat{z}(z)) \frac{\partial \hat{z}^\alpha}{\partial z^\gamma} \frac{\partial \hat{z}^\beta}{\partial z^\delta} = g_{\gamma\delta}(z) ; \quad \hat{N}(\hat{z}(z)) = \hat{N}(z) .
\]

Diffeomorphisms and symplectic reparametrizations are ‘Pseudo–symmetries’:

\[
D_{\text{pseudo}} = Diff(\mathcal{M}) \times Sp(2m, \mathbb{R}) .
\]

(3.12)

They leave the action form invariant, but change the coupling constants and thus are not invariances of the action.

If \( \hat{g}_{\alpha\beta}(z) = g_{\alpha\beta}(z) \) then the diffeomorphisms become isometries of the manifold, and proper symmetries of the scalar action. If these isometries are combined with symplectic transformations such that

\[
\tilde{N}(z) = N(z) ,
\]

(3.13)

then this is a proper symmetry. These are invariances of the equations of motion (but not necessarily of the action as not all transformations can be implemented locally
on the gauge fields). To extend the full group of isometries of the scalar manifold to proper symmetries, one thus has to embed this isometry group in $Sp(2m; \mathbb{R})$. In general, but not always \cite{23}, this seems to be realized in supersymmetric theories.

The simplest case is with one abelian vector. Then $N = S$ is a complex field, and the action is

$$\mathcal{L} = \frac{1}{4} (\text{Im } S) F_{\mu \nu} F^{\mu \nu} - \frac{i}{8} (\text{Re } S) \epsilon^{\mu \nu \rho \sigma} F_{\mu \nu} F_{\rho \sigma}.$$  

The set of Bianchi identities and field equations is invariant under symplectic transformations, transforming the field $S$ as

$$\tilde{S} = \frac{C + DS}{A + BS} \quad \text{where} \quad AD - BC = 1.$$  

If the rest of the action, in particular the kinetic term for $S$, is also invariant under this transformation, then this is a symmetry. These transformations form an $Sp(2; \mathbb{R}) = SL(2, \mathbb{R})$ symmetry. The $SL(2, \mathbb{R})$ subgroup is generated by

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$  

$$\tilde{S} = S + 1 \quad \tilde{S} = -\frac{1}{S}. \quad (3.14)$$

Note that Im $S$ is invariant in the first transformation, while the second one replaces Im $S$ by its inverse. Im $S$ is the coupling constant. Therefore, the second transformation, can not be a perturbative symmetry. It relates the strong and weak coupling description of the theory.

For another example, namely $S$ and $T$ dualities in this framework, see \cite{30}.

3.3. Symplectic transformations in $N = 2$

I now come back to rigid $N = 2$ supersymmetry. There are $n$ vectors, and the indices $\Lambda$ of the general theory are now the $A$-indices. In the formulas of section 2 the scalars, $z^\alpha$ in the general theory, are the $X^A$. The matrix $\tilde{N}_{AB}$ is now a well defined function. It was given in (2.27) as

$$\tilde{N}_{AB} = F_{AB} = \frac{\partial F_A}{\partial X^B}. \quad (3.15)$$

The last expression shows how we can obtain the transformation (3.6) for $N$. Indeed, identify \cite{2}

$$V \equiv \begin{pmatrix} X^A \\ F_A \end{pmatrix}, \quad (3.16)$$

as a symplectic vector, i.e. transforming under $Sp(2n, \mathbb{R})$ as $(F^A_{\mu \nu}, G_{A \mu \nu})$ in (3.4):

$$\tilde{X}^A = A^A_{\mu B} X^B + B^{AB} F_B, \quad (3.17)$$

This leads to

$$\tilde{N}_{AB} = \frac{\partial \tilde{F}_B}{\partial X^A} = (C + D\tilde{N})_{BC} \frac{\partial X^C}{\partial X^A} = \left[ (C + D\tilde{N}) (A + B\tilde{N})^{-1} \right]_{BA}, \quad (3.18)$$
which is the transformation we want for $N$. So by identifying $V$ as a symplectic vector, the structural relation (3.15) is preserved by (3.6). One may wonder whether (3.17) is consistent with the definition that $F_A$ is the derivative of a scalar function $F$. This requires that $F_A$ is the derivative of a new function $\tilde{F}(\tilde{X})$ w.r.t. the $\tilde{X}^A$. The integrability condition for the existence of $\tilde{F}$ can be seen from (3.18) to be the condition that $\tilde{N}$ is symmetric. We saw already in section 3.1 that this is just the condition that the transformation is symplectic.

Note that the argument for the existence of $\tilde{F}$ only applies if the mapping $X^A \to \tilde{X}^A$ is invertible, such that the $\tilde{X}^A$ are again independent coordinates. Hence, we need that

$$\frac{\partial \tilde{X}^A}{\partial X^B} = A^A_B + B^{AC} F_{CB}(X)$$

(3.19)

is invertible (the full symplectic matrix is always invertible). This we should anyway demand as $F_{CB} = \bar{N}_{CB}$, and the inverse of this matrix appears thus already from the very beginning in (3.6). I put some emphasis on this point, because this can be violated in supergravity.

As argued in the general theory, the transformations induced by (3.6) should extend to the other parts of the action. From (2.27) it is clear that the Kähler potential is a symplectic invariant. The fermionic sectors were checked in [2, 31, 32].

Hence we obtain a new formulation of the theory, and thus of the target-space manifold, in terms of the function $\tilde{F}(\tilde{X})$.

We have to distinguish two situations:

1. The function $\tilde{F}(\tilde{X})$ is different from $F(\tilde{X})$, even taking into account (2.28). In that case the two functions describe equivalent classical field theories. We have a pseudo symmetry. These transformations are called symplectic reparametrizations [33]. Hence we may find a variety of descriptions of the same theory in terms of different functions $F$.

2. If a symplectic transformation leads to the same function $F$ (again up to (2.28)), then we are dealing with a proper symmetry. As explained above, this invariance reflects itself in an isometry of the target-space manifold. Henceforth these symmetries are called ‘duality symmetries’, as they are generically accompanied by duality transformations on the field equations and the Bianchi identities.

E.g. the symplectic transformations with

$$S = \begin{pmatrix} 1 & 0 \\ C & \mathbb{1} \end{pmatrix}$$

(3.20)

do not change the $X^A$ and give $\tilde{F} = F + \frac{1}{4} C_{AB} X^A X^B$. So these give proper symmetries for any symmetric matrix $C_{AB}$. The symmetry of $C$ is required for $S$ to be symplectic. In the quantum theory $C$ will be restricted.

3.4. Example: Perturbative duality symmetries of the Seiberg-Witten model

The 1-loop theory of the Seiberg-Witten model [10] gives a non-trivial example. As classical theory Seiberg and Witten took the $SU(2)$ theory (2.29). I mentioned already that the potential is flat in one direction. The one-loop contributions give a ‘quantum theory’ for the massless field $X = X^3$. This has been
calculated in [34]. It leads to an effective theory with

\[ F(X) = \frac{i}{2\pi} X^2 \log \frac{X^2}{\Lambda^2}, \]

(3.21)

where \( \Lambda \) is the renormalization mass scale. Consider now the transformation defined by

\[ S = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}. \]

(3.22)

This leads to (I write \( F_A \) for the derivative w.r.t. the one variable \( X \))

\[ \begin{align*}
\tilde{X} &= -X = X e^{i\pi} \\
\tilde{F}_A &= -2X - F_A(X) \\
&= -2X - \frac{i}{\pi} X \left( \log \frac{X^2}{\Lambda^2} + 1 \right) = F_A(\tilde{X}(X)).
\end{align*} \]

(3.23)

Therefore this transformation leaves the function \( F \) invariant and is thus another (apart from (3.20)) duality symmetry. This transformation corresponds to going around the singular point \( X = 0 \) for the square of \( X \), which corresponds to the Casimir of the original theory. In this way we cross the branch cut of the function \( F \), and this transformation is thus a ‘monodromy’, as will be explained in section 5.4.

We thus find that the perturbative duality group is generated by

\[ \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}, \]

and

\[ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

(3.24)

A non-renormalization theorem says that in \( N = 2 \) there are no perturbative corrections beyond 1 loop.

4. Symplectic formulation of rigid special geometry

I mentioned already at the end of section 2 that one can formulate the rigid special geometry in a reparametrization invariant way [35, 36, 37, 38]. In the previous section we saw that the symplectic group plays an important role. The formulations which I will present below are reparametrization invariant and manifestly symplectic covariant. I give several equivalent formulations, which are appropriate for making the connection to the moduli space of Riemann surfaces in section 5.

4.1. Coordinate independent description of rigid special geometry

The Kähler space is now parametrized by some holomorphic coordinates \( z^\alpha \) with \( \alpha = 1, \ldots, n \). The variables \( X^A \) from above are holomorphic functions of the \( z \). But, as announced, I want to formulate it in a symplectic invariant way, so I just take the symplectic vector (3.16) to be a function of the \( z^\alpha \), or, in other words, I define \( n \) symplectic sections \( V(z) \). I mentioned already that the Kähler potential is symplectic invariant. I rewrite now (2.27) as

\[ K(z, \bar{z}) = i \langle V(z), \bar{V}(\bar{z}) \rangle \equiv i V^T(z) \Omega \bar{V}(\bar{z}), \]

(4.1)
where I defined a symplectic inner product. The metric is then

$$g_{\alpha\beta}(z, \bar{z}) = \partial_{\alpha}\partial_{\bar{\beta}}K(z, \bar{z}) = i \langle U_\alpha(z), \bar{U}_{\bar{\beta}}(\bar{z}) \rangle,$$

(4.2)

where I defined

$$U_\alpha \equiv \left( \partial_{\alpha}X^A \right) \equiv \left( e^A_\alpha \right).$$

(4.3)

One restriction which I mentioned already is the holomorphicity, which can be expressed as

$$\partial_{\bar{\beta}}U_\alpha = 0; \quad \partial_{\beta}\bar{U}_\alpha = 0.$$

(4.4)

In section 2 the coordinates were the $X^A$. These are now called 'special coordinates'. So special coordinates are the ones where $z^\alpha = X^A$, or $e^A_\alpha = \delta^A_\alpha$.

This does not yet contain the full definition of rigid special geometry. Indeed, nothing of the above corresponds to the fact that the $F_A$ are in special coordinates the derivatives of a function $F$. I also did not yet give the matrix $N_{AB}$ in general coordinates.

I have 2n coordinates in my symplectic vector, which depend on n coordinates $z^\alpha$. In the special coordinates the $F_A$ are functions of the $X^A$, and I argued in section 3.3 that symplectic transformations leave these as independent coordinates. Therefore we can write

$$\partial_{\alpha}F_A = \frac{\partial F_A}{\partial X^B} \partial_{\alpha}X^B.$$

(4.5)

The matrix relating the lower to the upper components in (4.3) is thus the one which is $N_{AB}$ in special coordinates. I take this as definition of $N$ in general coordinates:

$$h_{A\alpha} \equiv N_{AB} e^B_\alpha.$$

(4.6)

I stressed already in the general theory that this matrix $N$ should be symmetric. It is clear here that this is equivalent to the integrability condition for the existence of the scalar function $F$. So therefore, the main equation defining special geometry is the symmetry requirement of $N$. Multiplying the above equation by $e^A_{\beta}$, this condition is equivalent (as in these arguments we assumed $e^B_\beta$ to be invertible) to

$$e^A_{\beta} h_{A\alpha} - h_{A\beta} e^A_\alpha = \langle U_\alpha, U_{\bar{\beta}} \rangle = 0.$$

(4.7)

This is a symplectic invariant condition. This condition together with (4.2) and (4.4) define the rigid special geometry.

4.2. Alternative definition in matrix form and curvature

I still want to rewrite this definition in a matrix form which will be useful for comparing with the moduli of Riemann surfaces. First I rewrite (4.2) and (4.7) as the matrix equation

$$\mathcal{V} \Omega \mathcal{V}^T = -i \Omega,$$

(4.8)

where $\mathcal{V}$ is the $2n \times 2n$ matrix

$$\mathcal{V} \equiv \begin{pmatrix} U^T_\alpha & \bar{U}^T_{\bar{\alpha}} \end{pmatrix} \equiv \begin{pmatrix} e^A_\alpha \ h_{A\alpha} \\ g^{\alpha\beta} e^A_\beta \bar{g}^{\alpha\bar{\beta}} h_{A\bar{\beta}} \end{pmatrix},$$

(4.9)
and $U^\alpha \equiv g^{\alpha \beta} U_\beta$. The result (4.8) states that $\mathcal{V}$ is isomorphic to a symplectic matrix. Therefore $\mathcal{V}$ is invertible and we can define $Sp(2n)$ connections $A_\alpha$ and $\bar{A}_\alpha$ such that

$$D_\alpha \mathcal{V} = A_\alpha \mathcal{V}, \quad D_{\bar{\alpha}} \mathcal{V} = \bar{A}_{\bar{\alpha}} \mathcal{V}.$$  \hspace{1cm} (4.10)

Here we introduced covariant derivatives where the Levi-Civita connections appear because there are $\alpha$ indices hidden in $\mathcal{V}$:

The values of these two connections can be computed from multiplying the above equation with $\Omega$:

$$- A_\alpha i = D_\alpha \mathcal{V} \Omega \mathcal{V}^T \Omega = \left( \langle D_\alpha U_\beta, U_\gamma \rangle \langle D_\alpha U_\beta, \bar{U}_\gamma \rangle \right) \left( \langle D_\alpha \bar{U}_\beta, U_\gamma \rangle \langle D_\alpha \bar{U}_\beta, \bar{U}_\gamma \rangle \right).$$ \hspace{1cm} (4.11)

By (4.4) and the covariant derivative of (4.2) we know already that only the upper left component of the last matrix can be non-zero:

$$C_{\alpha \beta \gamma} \equiv - i \langle D_\alpha U_\beta, U_\gamma \rangle.$$ \hspace{1cm} (4.12)

Later we want to use the matrix $\mathcal{V}$ as a starting point. So I give now the definition of special geometry from this point of view. We then start with a square matrix $\mathcal{V}$ whose components are denoted as

$$\mathcal{V} \equiv \left( \begin{array}{cc} e^A_{\alpha} & h_{A\alpha} \\ \bar{e}^A_{\bar{\alpha}} & \bar{h}_{A\bar{\alpha}} \end{array} \right).$$ \hspace{1cm} (4.14)

Here, the requirements for special geometry are (4.8) and (4.10) with (4.13) in which $C$ is a symmetric tensor. Although the metric is not yet given, the connection is determined by the equations themselves. Indeed, the upper left component of (4.10) multiplied by $e^A_{\alpha}$, the inverse of $e^A_{\alpha}$, gives

$$\Gamma^\gamma_{\alpha \beta} = \bar{\Gamma}^\gamma_{\alpha \beta} - \bar{e}^A \partial_{\beta} e^A_{\alpha} \epsilon^\delta_{\alpha \beta} \epsilon^\gamma_\delta$$ \hspace{1cm} \text{with} \hspace{1cm} \bar{\Gamma}^\gamma_{\alpha \beta} = e^A_{\bar{\alpha}} \partial_{\beta} e^A_{\bar{\alpha}}.$$ \hspace{1cm} (4.15)

The differential equations give a lot of information. The conditions (4.10) imply that the combined connection consisting of $A$ and the Levi-Civita connections must be flat. The integrability conditions are

$$[D_\gamma - A_\gamma, D_\delta - A_\delta] \mathcal{V} = 0; \quad [D_\gamma - A_\gamma, D_\delta - A_\delta] \mathcal{V} = 0.$$ \hspace{1cm} (4.16)

The upper component of the first one is

$$U^\beta R^\alpha_{\beta \alpha \gamma \delta} + U^\beta D_\delta C_{\gamma \alpha \beta} + C_{\gamma \alpha \epsilon} e^\epsilon_{\beta} U_\beta = 0.$$ \hspace{1cm} (4.17)
where I used the definition of the curvature tensor

\[ [\mathcal{D}_\gamma, \mathcal{D}_\delta] U_\beta = U_\alpha R^\alpha_{\beta\gamma\delta} . \] (4.18)

Using the symplectic orthogonalities (4.2) and (4.7) this equation can be split. The first relation we obtain is that the Riemann curvature is given by

\[ R^\alpha_{\beta\gamma\delta} = -C^\beta_{\gamma\epsilon} \bar{C}^\epsilon_{\alpha\delta} . \] (4.19)

Secondly, the tensor \( C^\alpha_{\beta\gamma} \) satisfies the following two conditions (the second one follows from the second condition in (4.16))

\[ \mathcal{D}_{\bar{\alpha}} C^\beta_{\gamma\delta} = 0 . \] (4.20)

The last one implies that \( C^\alpha_{\beta\gamma} \) can be written as the third covariant derivative of some scalar function. To make the connection with section 2, (4.12) can be worked out using (4.6) and (2.27) to obtain

\[ C^\alpha_{\beta\gamma} = i e^A_{\alpha} e^B_{\beta} e^C_{\gamma} F_{ABC} . \] (4.21)

### 4.3. Holomorphic equations

Apart from the coordinate invariance, the above formulation is also invariant under gauge transformations \([37]\)

\[ \mathcal{V}' = S^{-1} \mathcal{V} ; \quad \mathcal{A}'_\alpha = S^{-1} (\mathcal{A}_\alpha - \mathcal{D}_\alpha) S ; \quad \mathcal{A}'_{\bar{\alpha}} = S^{-1} (\mathcal{A}_{\bar{\alpha}} - \mathcal{D}_{\bar{\alpha}}) S \] (4.22)

for \( S \) a symplectic element: \( \Omega S^T = \Omega \). Such a transformation can be used to reduce the equations to holomorphic ones.

Indeed, perform the symplectic transformation

\[ S = \begin{pmatrix} \delta^\alpha_{\beta} & 0 \\ \bar{e}^A_{\alpha} e^\beta_A & \delta^\beta_{\alpha} \end{pmatrix} , \] (4.23)

which gives

\[ \mathcal{V}' = \begin{pmatrix} e^A_A & h_{\alpha A} \\ f^{\alpha\beta} e^\beta_A & \bar{h}^\alpha_A \end{pmatrix} . \] (4.24)

The value of \( \bar{h}' \) does not matter, but it is important to note that \( f^{\alpha\beta} \) is antisymmetric (its value is \( f^{\alpha\beta} \equiv \bar{e}^A_{[\alpha} e^\beta_A \)). This is sufficient to obtain from the constraints (4.8) that

\[ f^{\alpha\beta} = 0 ; \quad \bar{h}^\alpha_{\alpha} = -i e^\alpha_A \quad \Rightarrow \quad \mathcal{V}' = \begin{pmatrix} e^A_A & h_{\alpha A} \\ 0 & -i e^\alpha_A \end{pmatrix} . \] (4.25)

The remaining part of (4.8) is (1.7).

For the differential equations, consider first \( \mathcal{A}'_{\bar{\alpha}} \). From (4.22) it follows that only the lower left component is non–zero. However, then (4.10) with (4.25) imply that this component should also be zero, leaving us with

\[ \mathcal{D}_{\bar{\alpha}} \mathcal{V}' = \partial_{\bar{\alpha}} \mathcal{V}' = 0 . \] (4.26)
So here the holomorphicity becomes apparent. Remains the calculation of $\mathcal{A}'_{\alpha}$. Again (4.10) and (4.25) imply that the lower left component is zero. Further it is easy to check that the upper right component does not change. It is then sufficient to calculate the upper left component, and the lower right follows from consistency:

$$\mathcal{A}'_{\alpha} = \begin{pmatrix} \gamma_{\alpha\beta} & C_{\alpha\beta\gamma} \\ 0 & X_{\alpha\gamma} \end{pmatrix} \quad \text{with} \quad X_{\alpha\beta} = C_{\alpha\beta\delta} \bar{e}^{\delta A} e^A$$

(4.27)

(The symmetrization symbol in the expression of $X$ is superfluous in view of $f^{\alpha\beta} = 0$).

If one writes the differential equation with an ordinary rather than a covariant derivative, then the diagonal elements simplify:

$$\left( \partial_{\alpha} - \hat{\mathcal{A}}_{\alpha} \right) \nu' = 0 \quad \text{with} \quad \hat{\mathcal{A}}_{\alpha} = \begin{pmatrix} \hat{\Gamma}_{\alpha\beta} & C_{\alpha\beta\gamma} \\ 0 & \hat{\Gamma}_{\alpha\gamma} \end{pmatrix},$$

(4.28)

and $\hat{\Gamma}$, which in fact follows from this equation, given in (4.13). So in this formulation special geometry is determined by (4.28) on the holomorphic matrix (4.25), which moreover should satisfy (4.7).

The differential equations can be combined to a second order equation. E.g. for $n = 1$, if we write the upper components of $\nu'$ generically as $f(z)$ and the lower component as $g(z)$, the differential equations are

$$(\partial - \hat{\Gamma})f + Cg = 0; \quad (\partial + \hat{\Gamma})g = 0.$$  

(4.29)

Combining these we have for $f$:

$$(\partial + \hat{\Gamma})C^{-1}(\partial - \hat{\Gamma})f = 0.$$  

(4.30)

5. Special geometry in the moduli space of Riemann surfaces

After identifying objects in $N = 2$ supersymmetry with objects in the moduli space of Riemann surfaces, I will show that there are differential equations (Picard-Fuchs equations) which are related to the equations defining the special geometry. These will be solved to lead to a symplectic section compatible with the constraints of special geometry. As explained in the introduction, this symplectic section is conjectured to yield the full perturbative and non-perturbative quantum theory of some models. In particular, I will explain the Seiberg-Witten model from this point of view and discuss its duality symmetries.

5.1. The period matrix

The matrix $\nu$ of special geometry will be identified with the period matrix of Riemann surfaces (up to a gauge transformation (1.22) on the left and a symplectic constant transformation on the right). This will depend on the moduli which define deformations of the surface. These moduli will be identified with the scalars of special geometry. In this first subsection, the dependence on the moduli will not yet be essential. That will be the subject of section 5.2.
The period matrix is

\[ \tilde{\mathbf{V}} = \begin{pmatrix} \tilde{U}_a^	op \\ \tilde{\mathcal{G}}_a \end{pmatrix} = \begin{pmatrix} \int_{A^a} \gamma_a \\ \int_{A^a} \tilde{\gamma}_a \\ \int_{B^a} \gamma_a \\ \int_{B^a} \tilde{\gamma}_a \end{pmatrix}, \tag{5.1} \]

where \( \gamma_a \) is a basis of \( H^{1,0} \), i.e. the closed and non-exact one-forms, while \( \tilde{\gamma}_a \) are a basis of \( (0,1) \) forms. On a genus \( n \) surface there are \( n + n \) such forms. These are integrated in (5.1) along a canonical homology basis of 1-cycles with intersection numbers:

\[ A_A \cap A_B = 0 ; \quad B_A \cap B_B = 0 ; \quad A_A \cap B_B = -B_B \cap A_A = \delta_B^A. \tag{5.2} \]

The relation (5.8) follows now from the identity for two 1-forms \( \lambda \) and \( \chi \)

\[ \sum_A \left[ \int_{A^A} \lambda \cdot \int_{B_A} \chi - \int_{B^A} \lambda \cdot \int_{A^A} \chi \right] = \int \int \lambda \wedge \chi, \tag{5.3} \]

where the integral at the right hand side is over the full Riemann surface. The product of two \( (1,0) \) forms is then obviously zero, and the basis of \( (0,1) \)-forms can be chosen then such that (5.8) is satisfied.

The second equation, (4.10), will be the subject of the next subsection. First, I introduce a description of the Riemann surface (with its moduli) as an elliptic curve, and give an expression for the elements in \( \tilde{\mathbf{V}} \). The formulations given here are chosen such that they can easily be generalized to the Calabi-Yau case. I use a 3-dimensional weighted projective complex space \((\mathbb{Z}, X, Y)\). The surface is defined as the points of this space where a pseudo-homogenous holomorphic polynomial vanishes. Due to the homogeneity we can view this in the weighted projective space, as a 1 complex dimensional surface, which is the Riemann surface. For genus 1, which will become the main example, I use the polynomial

\[ 0 = W(X, Y, Z; u) = -Z^2 + \frac{1}{4} (X^4 + Y^4) + \frac{u}{2} X^2 Y^2, \tag{5.4} \]

in the projective space where \( Z \) has weight 2, and \( X \) and \( Y \) have weight 1. There is one complex parameter (modulus) \( u \). The holomorphic function \( W \) can also be seen as a Landau-Ginzburg superpotential, but I will not use that connection here.

To obtain the forms, we make use of the Griffiths mapping \([40]\), which relates the holomorphic forms to elements of the chiral ring

\[ \mathcal{R}(\mathcal{W}) \overset{\text{def}}{=} \mathfrak{V}[X, Y, Z]/\partial \mathcal{W}, \tag{5.5} \]

i.e. polynomials of a certain degree in the variables \( X, Y, Z \), where the derivatives of \( \mathcal{W} \) are divided out. The 1-forms on the Riemann surface \( \mathcal{M} \) are represented as

\[ \gamma|_{\mathcal{M}} = \int_{\Gamma} \frac{\gamma \wedge d\mathcal{W}}{\mathcal{W}} = \int_{\Gamma} P_{k+1}^{\mathcal{W}}(X, Y, Z) \mathcal{W}^{k+1} \omega, \tag{5.6} \]

(everything can depend on the moduli \( u \)) where \( \Gamma \) is a 1-cycle around the surface \( \mathcal{M} \) in the 4-dimensional space (6-dimensions from the complex \( X, Y, Z - 2 \) from
homogeneity). The first equality is based on a generalisation of the residue theorem. In the second equality exact forms were discarded because of the integration over a cycle (see e.g. [11] sect. 5.9 for more details). $P^\gamma_{k|\nu}(X,Y,Z)$ is a pseudo homogeneous polynomial of a degree such that (5.6) has weight zero. That degree is denoted as $k|\nu$, where $\nu$ is the degree of $W$. In the example (5.4) we have $\nu = 4$, and we will see that $\omega$ has degree 4, so that

$$k|\nu = k|4 = 4k.$$  \tag{5.7}

The 'volume form' $\omega$ is [42]

$$\omega = 2(X\, dY \wedge dZ + Y\, dZ \wedge dX + 2\, Z\, dX \wedge dY).$$  \tag{5.8}

The important property of $\omega$ is that for any meromorphic function $P^\Lambda(X,Y,Z)$

$$\omega \, \partial_\lambda P^\Lambda(X,Y,Z) = d\Theta$$  \tag{5.9}

for some form $\Theta$ if the degrees of $\partial_\lambda P^\Lambda$ and $\omega$ add up to zero (I introduced the notation $X^\Lambda$ for $X,Y,Z$). The proof of this statement is given in appendix C for any pseudo-homogeneous space. In fact, all the above can be generalised for more variables (see below for Calabi-Yau) and for manifolds defined by the intersection of the vanishing locus of several polynomials.

If $P^\gamma_{k|\nu}(X,Y,Z)$ has terms proportional to $\partial_\lambda W$, then in (5.3) they occur as proportional to $\partial_\lambda W^{-k}$. So, using (5.9), they can be removed by adding a total differential to the integrand (which will decrease $k$ and the degree of the remaining $P$). The total differential is integrated over a cycle and thus does not contribute. This shows that for the polynomials $P^\gamma$, we suffice by a basis of the elements of the ring (5.5) with the correct weights. In the chiral ring there is a maximal weight, which therefore restricts also $k$. In general the upper limit of $k$ is the dimension (over $\mathbb{C}$) of the manifold. In our example in the chiral ring $X^3, Y^3$ and any $Z$ terms can be removed in this way, and the maximal element is thus $X^3Y^2$, i.e. of weight 4. Comparing with (5.7), the maximal $k$ is 1, as it should be on a Riemann surface. In fact, in general for manifolds of complex dimension $N$, the polynomials $P^\gamma_{k|\nu}$ are associated to the elements in

$$H^{N,0} \oplus H^{N-1,1} \oplus \cdots \oplus H^{N-k,k}.$$  \tag{5.10}

In our case, we thus have that for $k = 0$ the polynomial has degree 0, and can thus just be a constant. It is associated with the unique $(1,0)$ form on the genus 1 surface. For $k = 1$ there is only $X^2Y^2$, and together they are associated to all the 1-forms. Exercise: check that for genus $n$ surface, defined by $W = -Z^2 + X^{2n+2} + Y^{2n+2}$ one obtains indeed $n$ $(1,0)$ and $n$ $(0,1)$ forms.

We now integrate these forms over the $A$ or $B$ cycles to obtain the elements of the period matrix. The product of that cycle with $\Gamma$ in (5.4) is a 2-cycle which will be generically denoted by $C$. So we obtain (I have not imposed here a normalization condition as in (4.8))

$$\tilde{\varphi} = \begin{pmatrix} \int_C \frac{1}{W(X,Y,Z;u)} \omega \\ \int_C \frac{1}{W^2(X,Y,Z;u)} \omega \end{pmatrix}.$$  \tag{5.11}
5.2. Picard–Fuchs equations

The forms and periods depend on the moduli $u$. As we will see in the example, the derivatives of this period matrix can be re-expressed linearly in terms of the periods such that we get the differential equations

$$\partial_u \hat{\nabla} = \hat{A}_u \hat{\nabla}.$$  \hspace{1cm} (5.12)

These differential equations are called Picard-Fuchs equations. They are similar to the equations (4.10) of rigid special Kähler manifolds, and therefore complete the identification of special geometry with the geometry of the moduli space of the Riemann surface. In section 5.3 we will see that these differential equations allow us to calculate the period matrix. The main tool to derive the Picard-Fuchs equations is (5.9).

Higher genus surfaces have more than $n$ complex moduli. To make the connection with special geometry, one then has to define a surface depending on exactly $n$ moduli $u^\alpha$. The choice of these $n$ moduli is non-trivial [43].

I will concentrate on the case of a genus 1 surface which has one complex modulus [10]. If I take the derivative of (5.11) with respect to $u$, then this derivative acts on the denominator, and

$$\partial_u W = \frac{1}{2} X^2 Y^2.$$  \hspace{1cm} (5.13)

This gives

$$(1 - u^2) X^4 Y^4 = X Y^4 \partial_X W - u X^2 Y^3 \partial_Y W.$$  \hspace{1cm} (5.14)

We thus obtain (5.12) where the $2 \times 2$ matrix connection $\hat{A}_u$ is given by:

$$\hat{A}_u = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{u^2} & \frac{-u}{2u^2} \end{pmatrix}. \hspace{1cm} (5.15)$$

The differential equations corresponding to the 'A' and 'B'-cycles are of course identical. For both I can combine the upper and lower equations to a second order differential equation for the upper component, which I denote generically by $f(u)$:

$$\left( \frac{d^2}{du^2} - \frac{2u}{1 - u^2} \frac{d}{du} - \frac{1}{41 - u^2} \right) f(u) = 0.$$  \hspace{1cm} (5.16)

Comparing this equation with (4.30), we can already conclude that $C \propto (1 - u^2)^{-1}$, which determines the curvature (up to a constant).
5.3. Solutions

Solving the differential equation \((5.16)\) will lead us now to the symplectic vector, determining again an \(N = 2\) rigid supersymmetric theory. As mentioned in the introduction, that theory is conjectured by Seiberg and Witten to be the full quantum theory for the modulus \(u\).

If we change variable, setting \(w = (1 + u)/2\), \((5.16)\) becomes a hypergeometric equation of parameters \(a = 1/2, b = 1/2, c = 1\). The corresponding hypergeometric functions are elliptic integrals. Some general facts about these are given in appendix \([D]\). We choose as two linear independent solutions (writing, as in section \(3.4\), \(F_A\) for the one component of that vector).

\[
\begin{align*}
U_u &= \left( \frac{\partial_u X}{\partial_u F_A} \right) = f^{(1)}(u) = \frac{2}{\pi} \left[ K \left( \frac{1-u}{2} \right) + i K \left( \frac{1+u}{2} \right) \right] = \frac{2}{\pi} \sqrt{\frac{2}{1+u}} K \left( \frac{2}{1+u} \right), \\
U_u &= f^{(2)}(u) = \frac{2}{\pi} \sqrt{\frac{2}{1+u}} K \left( \frac{2}{1-u} \right)
\end{align*}
\]

where we understand here and further \(i e\) on the positive real axis for the definition of \(K\) as given in appendix \([D]\). As it is obvious, \(f^{(1)}(u)\) and \(f^{(2)}(u)\) just provide a basis of two independent solutions. The reason why precisely \(f^{(1)}(u)\) and \(f^{(2)}(u)\) are respectively identified with \(\partial_u X\) and \(\partial_u F_A\) is given by the boundary conditions imposed at infinity. When \(u \to \infty\), the special coordinate \(X(u)\) must approach the value it has in the original microscopic \(SU(2)\) gauge theory. The parameter \(u\) corresponds to a gauge–invariant quantity in the microscopic theory, so it is associated to the gauge invariant polynomial \(X^1 X^1 + X^2 X^2 + X^3 X^3\). This is now restricted to the Cartan subalgebra, i.e. \(u\) is proportional to the square of \(X \equiv X^3\). Correspondingly the boundary condition at infinity for \(X(u)\) is (choosing a convenient normalisation)

\[
X(u) \approx 2 \sqrt{2u} + \ldots \quad \text{for } u \to \infty.
\]

At the same time when \(u \to \infty\) the non perturbative rigid special geometry must approach its perturbative limit defined by the prepotential \((3.21)\). Combining eq.\((5.18)\) and \((3.21)\) we obtain

\[
F_A(u) \approx \frac{1}{\pi} 2 \sqrt{2u} \log u + \ldots \quad \text{for } u \to \infty.
\]

I now show that these boundary conditions are realized by the choice of \((5.17)\). Integrating the latter, using \((D.4)\) and \((D.3)\), gives

\[
V = \left( \begin{array}{l}
X(u) = \frac{2}{\pi} \int_{u_0}^u \sqrt{\frac{2}{1+t}} K \left( \frac{2}{1+t} \right) dt = \frac{8}{\pi} \sqrt{\frac{1+u}{2}} E \left( \frac{2}{1+u} \right) + \text{const.} \\
F_A(u) = \frac{2}{\pi} i \int_{u_0}^u K \left( \frac{1+u}{2} \right) dt = -4 \sqrt{\frac{1-u}{2}} (1-u) B \left( \frac{1+u}{2} \right) + \text{const.}
\end{array} \right)
\]

Choosing zero for the integration constants, the result \((5.20)\) coincides with the integral representations originally given by Seiberg and Witten \([10]\). Indeed, performing the substitutions \(y^2 = (1 - x)/2\) in the integral representation of the first one, and \(y^2 (u - 1) = x - 1\) in the second, leads to

\[
V = \left( \begin{array}{l}
X(u) = 2a(u) = \frac{2}{\pi} \sqrt{\frac{u-x}{u}} \int_{-1}^1 \sqrt{\frac{u-x}{u}} dx \\
F_A(u) = 2 a_D(u) = 2i \sqrt{\frac{u-x}{u}} \int_1^u \sqrt{\frac{u-x}{u}} dx.
\end{array} \right)
\]
It is clear that the first one leads to (5.18). For the second one, a substitution $x = uz$ exhibits the asymptotic behaviour as in (5.19) \[\square\].

5.4. Duality symmetries

The duality symmetry group $\Gamma_D$ consists of the symmetries of the potential $\mathcal{W}$ and the group generated by the monodromies around singular points. All these are given by integer valued symplectic matrices $\gamma \in \Gamma_D \subset Sp(2n, \mathbb{Z})$ that act on the symplectic vector $U_u$. Given the geometrical interpretation (5.1) of these vectors they correspond to changes of the canonical homology basis respecting the intersection matrix (5.2).

Let us start by the symmetries of the defining equation. The symmetry group $\Gamma_W$ can be defined by considering linear transformations $X \to M_x(u)X$ of the quasi–homogeneous coordinate vector $X = (X, Y, Z)$. For preservation of the weights $M_x(u)$ should be block diagonal. $\Gamma_W$ consists of the transformations such that

$$\mathcal{W}(M_x(u)X; u) = f_x(u)\mathcal{W}(X; u_x'(u)) \quad \text{and} \quad \omega(M_x(u)X) = g_x(u)\omega(X),$$

where $u_x'(u)$ is a (generally non–linear) transformation of the moduli and $f_x(u)$ and $g_x(u)$ are overall rescalings of the superpotential and the volume form that depend both on the moduli $u$ and on the chosen transformation $x$. In the supergravity case all elements of $\Gamma_W$ are duality symmetries. However, here the symplectic vector $U_u$, the upper component of (5.11), satisfies

$$U_u(u_x'(u)) = \int_{C} \frac{f_x(u)\omega(M_x(u)X)}{g_x(u)} S_x u_x(u) = \frac{f_x(u)}{g_x(u)} S_x U_u(u).$$

In the last step, a change of integration variables was performed, which transforms also the $A$ and $B$ cycles, leading to the symplectic matrix $S_x$. If $\frac{f_x(u)}{g_x(u)}$ is a constant, then the right hand side gives also a solution to the differential equations, and we have a duality symmetry. Therefore, only $\Gamma^0_W \subset \Gamma_W$ given by the transformations that have a constant $\frac{f_x(u)}{g_x(u)}$ acts as an isometry group for the moduli space \[\square\].

For (5.4), the symmetry group is $\Gamma_W = D_3$ \[\square\], defined by the following generators and relations

$$B^2 = 1 \quad , \quad C^3 = 1 \quad , \quad (CB)^2 = 1$$

(5.24)

with the following action on the homogeneous coordinates and the modulus $u$ \[\square\]

$$M_B = \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \quad u_B'(u) = -u \quad ; \quad f_B(u) = 1 \quad ; \quad g_B = i$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 \\ -i & 0 & \sqrt{1+u} \\ 0 & 1 & 0 \end{pmatrix} \quad ; \quad u_C'(u) = \frac{u^2-1}{u^2+1} \quad ; \quad f_C(u) = \frac{1+u}{1-u} \quad ; \quad g_C = i\sqrt{\frac{1+u}{1-u}}.$$

(5.25)

From the value of $f_x$ and $g_x$ we thus see that only the $\mathbb{Z}_2$ cyclic group generated by $B$ is actually realized as an isometry group of the rigid special Kählerian metric.

\[\square\] The full group connects 6 points as in (D.8): $\pm u, \pm \frac{i}{\sqrt{1+u}}, \pm \frac{1}{\sqrt{1+u}}$. 

25
The transformation \( u \to -u \) acts on \( U_u \) as
\[
U_u(-u) = -i \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} U_u(u) \quad \Rightarrow \quad B = -i S_B = -i \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} .
\] (5.26)

This transformation has a physical interpretation as R-symmetry. It is precisely the requested R-symmetry for the topological twist \([9]\).

The monodromy group generators correspond to analytic continuation of the solutions around the singular points and have a matrix action on the solution two-vector \( U_u \to T_Q U_u \), where \( Q \) denotes any of the singular values of \( u \), and \( T_Q \) is a \( 2 \times 2 \) matrix. One can see that the polynomial (5.4) degenerates for \( u = 1, -1 \) and \( \infty \). Of course, one can also find these singular points by considering the solution vector (5.17). Under the action of the group \( \Gamma_0 \) generated by \( B: u \to -u \), the singular points fall in two orbits: \( \{1, -1\} \) and \( \{\infty\} \).

Let us derive the structure of the monodromy group by direct evaluation. The monodromies will be defined going around the singularity in counter clockwise direction. The calculation of the monodromy around \( u = \infty \) is similar to (3.23). The matrix \( S (3.22) \) gives now
\[
T(\infty) = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix} .
\] (5.27)

The monodromy around \( u = 1 \) is found from (D.11) and \( K(1-y') = K(1-y) \):
\[
U_u(1 + re^{2\pi i}) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} U_u(u) \quad \text{for} \quad 0 < r < 2 \quad \Rightarrow \quad T(1) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} .
\] (5.28)

Finally around \( u = -1 \) we have for the counter clockwise direction
\[
U_u(-1 - r + i\epsilon) = \begin{pmatrix} K(-\frac{r}{2}) + iK(1+\frac{r}{2} - i\epsilon) \\ iK(1+\frac{r}{2} - i\epsilon) \end{pmatrix} = U_u(-1 - r - i\epsilon) + 2 \begin{pmatrix} K(-\frac{r}{2}) \\ iK(-\frac{r}{2}) \end{pmatrix}
\]
\[
\Rightarrow \quad T(-1) = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} .
\] (5.29)

We could have restricted ourselves to the computation of the monodromy matrix in \( u = 1 \), i.e. \( T(1) \), and \( B \). Indeed, since \( B \) permutes the singular points \( 1, -1 \) among themselves the monodromy around any of them can be obtained from the monodromy at the other point by conjugation with \( B \). Explicitly, from considering the paths in moduli space one has
\[
T(-1) = BT(1)B = S_B T(1)S_B^{-1} .
\] (5.30)

Furthermore, the product of the monodromy matrices in all the singular points must give the identity, as a contour encircling all the singularities is homotopic to zero. Hence the monodromy around the point \( u = \infty \) is obtained from:
\[
T(\infty) T(1) T(-1) = 1 .
\] (5.31)

\( T_{\infty} \) and \( T(1) \) generate the \( \Gamma(2) \subset \Gamma = PSL(2,\mathbb{Z}) \) group. This is the normal principal congruence subgroup of order 2 of the modular group of the torus \( \Gamma \):
\[
\gamma \in \Gamma(2) : \quad \gamma = \begin{pmatrix} 1 + 2q & 2p \\ 2r & 1 + 2s \end{pmatrix} ; \quad 2(qs - pr) + q + s = 0 .
\] (5.32)
It is known that $D_3 \sim PSL(2, \mathbb{Z}_2) \sim \Gamma / \Gamma(2)$. So as the full $D_3$ is not realized as duality symmetries, the final duality group is not $\Gamma$, but the subgroup which can be generated by $S_B$ and $T_{(1)}$. The perturbative part is the subgroup where the upper right component is zero. It is generated by (3.24), i.e. by $T(\infty)$ and $-S_B T^{-1}_{(1)}$.

6. Supergravity

Also for supergravity, I am first giving the original construction of the actions, before re-expressing the result in a symplectic invariant way. We will see that here the symplectic formulation is even more necessary than in the rigid case. Then I explain the connection to moduli of Calabi-Yau manifolds.

I treat here only the vector multiplets. For more complete reviews of tensor calculus, the multiplets, and construction of the actions, see [45]. About Calabi-Yau manifolds, there is a fast increasing literature. For a recent introduction I refer to [41].

6.1. Vector multiplets coupled to supergravity

To construct the action of vector multiplets coupled to supergravity, I use the superconformal tensor calculus. This method keeps the fruits of the superspace approach while avoiding a lot of technical difficulties with many constraints on large superfields. No superfields are introduced to covariantize the interactions of the matter multiplets. This is replaced by defining the multiplets within a larger algebra. The formulation can be compared with the description of non-abelian vector multiplets which I gave at the end of section 2.2. There, a full superspace approach was replaced by defining the vector multiplet in the algebra (2.20). Still I keep working with multiplets, which shows the structure of actions (and potentials) better due to the presence of auxiliary fields. Of course, there is also a drawback. One is never sure that the constructions one thinks of are the most general one. In some cases the most general result is not found yet. As an example I mention here that it remains a challenge to construct the most general couplings of hypermultiplets in superspace or tensor calculus.

The group which I am going to use for defining the multiplets is the superconformal group. This group is bigger than the super-Poincaré group, which we want to have as final invariance of the actions. The method consists of first constructing actions invariant under the full superconformal group, and then afterwards choose explicit gauge fixings. Fields which are just introduced to allow later gauge choices are called compensating fields. The method is called ’gauge equivalence’ (see the reviews [10, 15]). It has the advantage of showing more structure in the theory. In $d = 4$, $N = 2$ the superconformal group is

\[ SU(2,2|N = 2) \supset SU(2,2) \otimes U(1) \otimes SU(2) \, . \]  

(6.1)

The bosonic subgroup, which I exhibited, contains, apart from the conformal group $SU(2,2) = SO(4,2)$, also $U(1)$ and $SU(2)$ factors. The Kählerian nature of vector multiplet couplings and the quaternionic nature of hypermultiplet couplings is directly related to the presence of these two groups. The dilatations, special conformal
transformations, $U(1) \otimes SU(2)$, and an extra $S$–supersymmetry in the fermionic sector will be broken by gauge fixings. In that way we just keep the super-Poincaré invariance.

To describe theories as exhibited in table [2], the following multiplets are introduced: (other possibilities, leading to equivalent physical theories, also exist, see [24, 46, 45]). The Weyl multiplet contains the vierbein, the two gravitinos, and auxiliary fields. I introduce $n + 1$ vector multiplets:

$$ (X^I, \chi^I, A^I_\mu) \quad \text{with} \quad I = 0, 1, \ldots, n. \quad (6.2) $$

The extra vector multiplet labelled by $I = 0$ contains the scalar fields which are to be gauge–fixed in order to break dilations and the $U(1)$, the fermion to break the $S$–supersymmetry, and the vector which corresponds to the physical vector of the supergravity multiplet in table [2]. Finally, there are $s + 1$ hypermultiplets, one of these contains only auxiliary fields and fields used for the gauge fixing of $SU(2)$. For most of this paper I will not discuss hypermultiplets ($s = 0$).

In the first step we have to build a superconformal invariant action. To do so similarly to the construction in rigid superspace (2.22), the highest component of the chiral superfield $F(\Phi)$ should have weight 4 under dilatations. This implies that the lowest component $F(X)$ should have Weyl weight 2. The weight of fields in a vector multiplet is fixed due to the presence of gauge vectors which should have weight zero. This fixes the Weyl weight of the scalars $X^I$ to 1. Combining the above facts leads to the important conclusion that for the coupling of vector multiplets to supergravity, one again starts from a holomorphic prepotential $F(X)$, this time of $n + 1$ complex fields, but now it must be a homogeneous function of degree two [2].

In the resulting action appears $-\frac{1}{2}i(\bar{X}^IF_I - X^I\bar{F}_I)eR$, where $R$ is the space–time curvature. To have the canonical kinetic terms for the graviton, it is therefore convenient to impose as gauge fixing for dilatations the condition

$$ i(X^IF_I - \bar{F}_IX^I) = 1. \quad (6.3) $$

Therefore, the physical scalar fields parametrize an $n$-dimensional complex hypersurface, defined by the condition (6.3), while the overall phase of the $X^I$ is irrelevant in view of the $U(1)$ in (6.1) which we have not fixed.

The embedding of this hypersurface can be described in terms of $n$ complex coordinates $z^\alpha$ by letting $X^I$ be proportional to some holomorphic sections $Z^I(z)$ of the projective space $\mathbb{P}^n$. The $n$-dimensional space parametrized by the $z^\alpha$ ($\alpha = 1, \ldots, n$) is a Kähler space; the Kähler metric $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z, \bar{z})$ follows from the Kähler potential

$$ K(z, \bar{z}) = -\log \left[ i\bar{Z}^I(\bar{z}) F_I(Z(z)) - iZ^I(z) \bar{F}_I(\bar{Z}(\bar{z})) \right], \quad \text{where} \quad (6.4) $$

$$ X^I = e^{K/2}Z^I(z), \quad \bar{X}^I = e^{K/2}\bar{Z}^I(\bar{z}). $$

The resulting geometry is known as special Kähler geometry [1, 2, 4].
A convenient choice of inhomogeneous coordinates \( z^\alpha \) are the *special* coordinates, defined by
\[
\frac{X^\alpha}{X^0}, \quad \text{or} \quad Z^0(z) = 1, \quad Z^\Lambda(z) = z^\alpha. \tag{6.5}
\]

In the general form of the spin-1 action (2.26), the indices \( \Lambda \) are now replaced by \( I \) running over \( m = n + 1 \) values, as the graviphoton is included. The matrix \( N \) is given by
\[
N_{IJ}(z, \bar{z}) = \bar{F}_{IJ} + 2i \frac{(\text{Im} F_{IK})(\text{Im} F_{JK})X^KX^L}{(\text{Im} F_{LM})X^KX^M}. \tag{6.6}
\]
The kinetic energy terms of the scalars are positive definite in a so-called 'positivity domain'. This is given by the conditions \( g_{\alpha \beta} > 0 \) and \( e^{-K} > 0 \). Then it follows that \( \text{Im} N_{IJ} < 0 \) \[47\]. Note that the fact that the positivity domain is non-empty, restricts the functions \( F \) which can be used. In [30] some examples are discussed.

### 6.2. Symplectic transformations

To extend the duality transformations on the vectors to the scalar sector, one introduces also here a \( 2m = 2(n + 1) \) component vector
\[
V = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \tag{6.7}
\]
which transforms as a symplectic vector. Note that now the upper part of this vector can not be used as independent coordinates, as there are only \( n \) scalars. However, in the formulation described in the previous subsection the lower part still depends on the upper components in the sense that \( F_I(z, \bar{z}) = F_I(X(z, \bar{z})) \), as \( F_I = \frac{\partial F(X)}{\partial X^I} \). When symplectic reparametrizations are performed, this can change. Indeed, now the matrix \( \frac{\partial X^I}{\partial X^J} \), see (3.19), may be non-invertible [32] while the inverse of \( (A + BN) \) still exists because \( N_{IJ} \neq F_{IJ} \). This implies that the proof of existence of a new function \( \tilde{F}(\tilde{X}) \) is invalidated in these cases. However, the transformed symplectic vector still describes the same theory. It is clear that as the new \( X^I \) are not independent, one can not choose special coordinates (6.5) in terms of the new symplectic vectors. So as in this new formulation the function \( F(X) \) does not exist, one can not immediately obtain this formulation from superspace or tensor calculus. This shows that in supergravity, even more than in rigid symmetry, we need a formulation which does not start from the function \( F \), but is essentially symplectic covariant. This will be treated in section 6.3.

In [30] we gave a few simple examples of symplectic reparametrisations and duality symmetries, including such a transformation to a formulation without a function \( F(X) \). Another important example of this phenomenon is the description of the manifold
\[
\frac{SU(1, 1)}{\text{U}(1)} \otimes \frac{SO(r, 2)}{SO(r) \otimes SO(2)}. \tag{6.8}
\]
This is the only special Kähler manifold which is a product of two factors [18]. Therefore it is also determined that in the classical limit of the compactified heterotic string, where the dilaton does not mix with the scalars of the other vector.
multiplets, the target space should have that form. The first formulation of these spaces used a function $F$ of the form [17]

$$F = \frac{d_{ABC}X^AX^BX^C}{X^0}. \quad (6.9)$$

In fact, such a form of $F$ is what one expects for all couplings which can be obtained from $d = 5$ supergravity [19]. Such manifolds are called ‘very special Kähler manifolds’. In such a formulation for (6.8) the $SO(r,2)$ part of the duality group sits not completely in the perturbative part of the duality group, i.e. one needs $B \neq 0$ in the duality group to get the full $SO(2,r)$. However, from the superstring compactification one expects $SO(2,r;\mathbb{Z})$ as a perturbative ($T$-duality) group.

By making a symplectic transformation this can indeed be obtained [32]. After that symplectic transformation one has a symplectic vector $(X^I,F_I)$ satisfying

$$X^I \eta_{IJ} X^J = 0; \quad F_I = S \eta_{IJ} X^J, \quad (6.10)$$

where $\eta_{IJ}$ is a metric for $SO(2,r)$. The first constraint comes on top of the constraint (6.3), and thus implies that the variables $z$ can not be chosen between the $X^I$ only. Indeed, $S$ occurs only in $F_I$.

6.3. Symplectic definition of special geometry

The symplectic formulation of special geometry was first given in the context of a treatment of the moduli space of Calabi-Yau three-folds [4, 35, 50]. The formulation which we present is based on [37], and the equivalence with our previous formulation was explained in detail in [38].

The definition can again be given on the basis of a matrix $\mathcal{V}$. As I already said, the symplectic vectors now have $2(n+1)$ components. Similarly $\mathcal{V}$ will be a $2(n+1)$ by $2(n+1)$ matrix. It consists of the following rows of symplectic vectors:

$$\mathcal{V} = \begin{pmatrix} \bar{V}^T \\ U^T \\ \bar{V}^T \end{pmatrix}, \quad (6.11)$$

which again should satisfy (4.8). Furthermore there are the differential equations (4.10). In [38] the covariant derivatives in that equations included connection for Kähler transformations. Here, I include this connection in the matrix $\mathcal{A}$. This leads to

$$\mathcal{A}_\alpha = \begin{pmatrix} \frac{1}{2} \partial_\alpha K & 0 & 0 & 0 \\ 0 & -\frac{1}{2} \delta^\beta_\alpha \partial_\alpha K & 0 & C_{\alpha\beta\gamma} \\ 0 & \delta^\alpha_\beta & -\frac{1}{2} \partial_\alpha K & 0 \\ \delta^\alpha_\gamma & 0 & 0 & \frac{1}{2} \delta_\gamma^\beta \partial_\alpha K \end{pmatrix},$$

$$\mathcal{A}_{\bar{\alpha}} = \begin{pmatrix} -\frac{1}{2} \partial_\bar{\alpha} K & 0 & 0 & 0 \\ 0 & \frac{1}{2} \delta^\beta_{\bar{\alpha}} \partial_\bar{\alpha} K & g_{\bar{\alpha}\beta} & 0 \\ 0 & 0 & \frac{1}{2} \partial_{\bar{\alpha}} K & 0 \\ 0 & C_{\bar{\alpha}\gamma} & 0 & -\frac{1}{2} \delta_\gamma^\beta \partial_\bar{\alpha} K \end{pmatrix}. \quad (6.12)$$
Here $K$ is the Kähler potential. Again the system of equations is invariant under constant symplectic transformations acting from the right on $V$ and gauge transformations of the form (4.22) on the left. The latter can be used again to obtain holomorphic equations \[37\]. E.g. with $S = \exp \left( \frac{1}{2} \text{diag}(-K, K, K, -K) \right)$ one removes the Kähler connections from $A_{\alpha}$ and these equations thus imply holomorphicity for the vector $V$ in that basis, whose first components are then in fact the $Z^I$ from (6.4).

As in the rigid case, one can obtain equations similar to (4.20) (corrected by the Kähler connection) and the curvature:

\[
R^\alpha_{\beta\gamma\delta} = 2\delta^\alpha_{(\beta}\delta^{\gamma)} - C_{\beta\gamma\epsilon} \tilde{C}^{\alpha\epsilon} ; \quad \tilde{C}^{\alpha\beta\epsilon} \partial_{\delta} (e^{-K}C_{\alpha\beta\gamma}) = 0 ; \quad \mathcal{D}[\alpha e^{K}C_{\beta\gamma\delta}] = 0 . \quad (6.13)
\]

6.4. Calabi-Yau manifolds

Calabi-Yau manifolds are complex manifolds with Ricci tensor which is exact, i.e. $R_{\alpha\beta} = \partial_{\alpha}A_{\beta} - \partial_{\beta}A_{\alpha}$. For our purposes we need 3-folds (i.e. real dimensional 6, related to the reduction of 6 dimensions of the superstring). The Hodge diamond gives the number of closed and non-exact forms of all holomorphic types. E.g. for the Riemann surface of genus $n$ it looks like

\[
\begin{array}{cccc}
   & h_{00} & h_{01} & 1 \\
1 & h_{10} & h_{11} & n \\
 & h_{11} & 1 & n \\
\end{array}
\]

(6.14)

For the Calabi-Yau manifolds this Hodge diamond is already fixed for a large part:

\[
\begin{array}{cccc}
   & 0 & 1 & 0 \\
0 & h_{11} & h_{12} & 0 \\
1 & h_{12} & h_{21} & 1 \\
0 & h_{21} & h_{22} & 0 \\
 & h_{22} & 0 & 1 \\
\end{array}
\]

(6.15)

A large class of them can be obtained as the vanishing locus of a quasi-homogeneous polynomial in projective space \[51\], similar to the description of Riemann surfaces in section 6.4. For the connection with special geometry one now considers all 3-forms. Identifying $h_{12} = h_{21} = n$, there are $2(n + 1)$ of these. The integrals over a canonical basis of 3-cycles defines then the period matrix. One can obtain again differential equations (Picard-Fuchs equations) by differentiating with respect to the moduli of the surface, which are to be compared with the defining equations of special geometry as given above. As in the case of Riemann surfaces, the duality symmetries are defined by the monodromies around singular points and symmetries of the defining equation. For the latter we do not have to restrict now to those having constant rescaling factors \[39\].

6.5. Special quaternionic manifolds and homogeneous special manifolds

The $c$ map \[33\] gives a mapping from a special Kähler to a quaternionic manifold. It is induced by reducing an $N = 2$ supergravity action in $d = 4$ space-time dimensions to an action in $d = 3$ space-time dimensions, by suppressing the
dependence on one of the (spatial) coordinates. The resulting \( d = 3 \) supergravity theory can be written in terms of \( d = 3 \) fields and this rearranges the original fields such that the number of scalar fields increases from \( 2n \) to \( 4(n + 1) \). This map is also obtained in string theory context by changing from a type IIA to a type IIB string or vice-versa.

This leads to the notion of ‘special quaternionic manifolds’, which are those manifolds appearing in the image of the \( c \) map. Similarly, very special real manifolds are the manifolds defined by coupling (real) scalars to vector multiplets in 5 dimensions \([13]\) (characterised by a symmetric tensor \( d_{ABC} \)). Very special Kähler manifolds \([29]\) are induced as the image under the \( r \) map (dimensional reduction from 5 to 4 dimensions) and very special quaternionic manifolds as the image of the \( c\circ r \) map.

It turns out that these very special manifolds contain all known homogeneous non–symmetric Kähler and quaternionic spaces. The classification of the homogeneous spaces is related to the enumeration of all realizations of real Clifford algebras, see \([31]\), or the summary in \([38]\). In these papers also all the continuous isometries of homogenous special and of very special manifolds are given.

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Appendix A. Conventions

I use the metric signature \((- + ++)\). The curved indices are denoted by \( \mu, \nu, \ldots = 0, \ldots, 3 \) and the flat ones by \( a, b, \ldots \). (Anti)symmetrization is done with weight one: \( A_{[ab]} = \frac{1}{2} (A_{ab} - A_{ba}) \) and \( A_{(ab)} = \frac{1}{2} (A_{ab} + A_{ba}) \).

I define

\[
\epsilon^{\mu\nu\rho\sigma} = \sqrt{-g} e_0^\mu e_0^\nu e^\rho e^\sigma ; \quad \epsilon^{0123} = i , \tag{A.1}
\]

where the former implies that the latter is true for flat as well as for curved indices. I introduce the self–dual and anti–self dual tensors

\[
F_{ab}^\pm = \frac{1}{2} (F_{ab} \pm \dual F_{ab}) \quad \text{with} \quad \dual F^{ab} = \frac{i}{2} \epsilon^{abcd} F_{cd} . \tag{A.2}
\]

The gamma and sigma matrices are defined by

\[
\gamma_a \gamma_b = \eta_{ab} + 2\sigma_{ab} ; \quad \gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 , \tag{A.3}
\]

which implies that \( \frac{i}{2} \epsilon^{abcd} \sigma_{cd} = -\gamma^a \sigma^{ab} \). The following realization brings you to the 2-component formalism: (here \( \alpha = 1, 2, 3 \), but this is not used anywhere else)

\[
\gamma_0 = \begin{pmatrix} 0 & 1_2 \\ i_2 & 0 \end{pmatrix} ; \quad \gamma_\alpha = \begin{pmatrix} 0 & -i\sigma_\alpha \\ i\sigma_\alpha & 0 \end{pmatrix} ; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.4}
\]
The matrices $\gamma_\alpha$ and $\gamma_5$ are hermitian, while $\gamma_0$ is antihermitian. There is a charge conjugation matrix $C$ such that

$$C^T = -C ; \quad C\gamma_\alpha C^{-1} = -\gamma_\alpha^T .$$

(A.5)

This fixes $C$ up to a (complex) constant. One can fix the proportionality constant (up to a phase) by demanding $C$ to be unitary, so that $C^* = -C^{-1}$. In the representation (A.4) we can choose

$$C = \begin{pmatrix} \epsilon_{AB} & 0 \\ 0 & \epsilon^{AB} \end{pmatrix} ,$$

(A.6)

where $\epsilon$ is the antisymmetric symbol with $\epsilon_{AB} = -\epsilon^{AB} = 1$ for $A = 1, B = 2$.

Majorana spinors are spinors $\chi$ which satisfy the ‘reality condition’ which says that their ‘Majorana conjugate’ is equal to the ‘Dirac conjugate’

$$\bar{\chi} \equiv \chi^T C = -i\chi^\dagger \gamma_0 \equiv \bar{\chi} C \quad \text{or} \quad \chi^C \equiv -i\gamma_0 C^{-1} \chi^* = \chi .$$

(A.7)

The factor $-i$ is just a conventional choice as the phase of $C$ is arbitrary. Majorana spinors can thus be thought as spinors $\chi_1 + i\chi_2$, where $\chi_1$ and $\chi_2$ have real components, but these are related by the above condition. There exists a Majorana representation where the matrices $\gamma_\mu$ are real, and with a convenient choice of the phase factor of $C$ the Majorana spinors are just real.

In the 2-component formulation, indices are raised or lowered with $\epsilon$ in a NW-SE convention

$$\chi^A = \epsilon^{AB} \chi_B ; \quad \chi_A = \chi^B \epsilon_{BA} ,$$

(A.8)

which implies $\epsilon^{AB} \epsilon_{BC} = -\delta^A_C$, and the Majorana condition for a spinor $(\zeta^A, \zeta_\dot{A})$ is then $\zeta_A = (\zeta_\dot{A})^*$.

I often use Weyl spinors, where the left and right chiral spinors are defined as

$$\chi_L = \frac{1}{2}(1 + \gamma_5)\chi ; \quad \chi_R = \frac{1}{2}(1 - \gamma_5)\chi .$$

(A.9)

In extended supergravity the chirality is indicated by the position of the $i,j$ index (index running over $1, \ldots N$ for $N$-extended supergravity). To choice of chirality for the spinor with an upper (lower) index can change for each spinor. It is chosen conveniently on the first occurrence of the spinor. E.g. in this paper:

$$\epsilon^i = \gamma_5 \epsilon^i \quad Q^i = -\gamma_5 Q^i \quad \theta^i = \gamma_5 \theta^i \quad \Lambda^i = -\gamma_5 \Lambda^i \quad \Omega^i = -\gamma_5 \Omega^i \quad \phi^i = \gamma_5 \phi^i .$$

(A.10)

Note that these Weyl spinors are not Majorana. A spinor which is Majorana and Weyl with the above definitions would only be possible in $d = 2$ mod 8. In the two-component representation, the left Weyl spinors are just $(\zeta^A, 0)$.

Hermitian conjugation on a bispinor reverses by definition the order of the spinors. To perform h.c. in practice it is easier to replace it by charge conjugation. For any bispinor one has

$$(\chi M \lambda)^\dagger = (\chi^C M^C \lambda^C) ,$$

where $C$ was defined for the spinors in (A.7), and for the matrix $M$ one has $M^C = -\gamma_0 C^{-1} M^* C \gamma_0$. The gamma matrices are inert under this transformation, but $\gamma_5$
changes sign due to the $i$ in its definition (A.3). The Majorana spinors are inert under $C$, but the Weyl spinors change chirality: $\epsilon^c = -i \epsilon^c$. Therefore h.c. effectively replaces $i$ by $-i$, interchanges upper and lower indices, and $\epsilon^{\mu\nu\rho\sigma}$ changes sign, or self–dual becomes anti–self dual.

In exceptional cases I use spinor indices. Then by definition $\bar{\chi}_\lambda = \chi^\alpha \lambda_\alpha$, where $\chi^\alpha = \chi_\beta C^{\beta\alpha}$ is now the counterpart of (A.7). Its inverse is $C^{-1}_{\alpha\beta}$. The $\gamma$–matrices are written as $(\gamma^\mu)_{\alpha\beta}$.

Exercises or useful formulae.

Here all spinors are fermionic and the left or right projections of Majorana spinors.

$$
\sigma^{ab} F_{ab} \epsilon^i = \sigma^{ab} F_{ab}^- \epsilon^i \\
\bar{\epsilon}^i = \bar{\epsilon}^i \gamma_5 \\
\xi_L \alpha_L = \lambda_L \xi_L \\
\lambda_L \bar{\chi}_R = -\frac{1}{2} \gamma^m (\bar{\chi}_R \gamma_m \alpha_L) \\
\lambda_L \bar{\chi}_R = -\frac{1}{2} \gamma^m (\bar{\chi}_R \gamma_m \alpha_L) + \frac{1}{2} \sigma^{ab} (\bar{\chi}_R \sigma_{ab} \alpha_L)
$$

(A.12)

Show that in $N = 2$ in a commutator $[\delta(\epsilon_1), \delta(\epsilon_2)]$ only the following index structures and their hermitian conjugates can appear:

$$
\bar{\epsilon}_1 \epsilon^i_2; \quad \bar{\epsilon}_1 \gamma_5 \epsilon_2 j; \quad \bar{\epsilon}_1 \epsilon_2^{ab} \epsilon^j_2.
$$

(A.13)

Appendix B. Normalisations

Unfortunately the normalization of $F$ and various other functions vary in the $N = 2$ literature. In table 4, I compare the notations of various articles (in supergravity). The first column is the notation used here, in [30, 38], and most of it also in [22]. The column ‘old’ refers to the articles [1, 2, 14, 23, 24, 29, 31, 33, 45, 46, 47]. Note that the first row shows that also the convention of the space-time metric has changed. The freedom of the real parameter $\alpha$, indicated in the second column, can be repeated in all columns, but looks most useful in this case.

The symplectic matrices compare as follows between the notations here (left hand side) and in the ‘old’ notation (right hand side):

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} =
\begin{pmatrix}
U & 2\alpha^2 Z \\
\frac{1}{2\alpha} W & V
\end{pmatrix}
$$

(B.1)

Appendix C. The volume form

I consider here a pseudo homogeneous space in $N + 2$ dimensions with coordinates $X^\Lambda$, where the weight of $X^\Lambda$ is $\beta_\Lambda$. I will prove that the volume form

$$
\omega = X^{\alpha_1} dX^{\Sigma_0} \wedge \cdots \wedge dX^{\Sigma_N} \epsilon_{\Sigma_0 \cdots \Sigma_N},
$$

(C.1)

where I introduced $X^{\alpha_1} = \beta_0 X^{\Omega}$, satisfies (5.9) when the overall degree of its left hand side is zero. The degree of $\omega$ is $\gamma = \sum \beta_\Lambda$, and therefore the degree of $\mathcal{P}^{\Lambda}$ is

34
α_A = β_A - γ. This implies, using the notation with primes,

\[ X'^\Omega \partial_\Omega P^A(X) = \alpha_A P^A \] (C.2)

(no sum over \Lambda in r.h.s.). Let me take

\[ \Theta = -(N + 1) P^A X'^\Omega dX'^{\Sigma_1} \wedge \cdots \wedge dX'^{\Sigma_N} \epsilon_{\Lambda_1 \cdots \Lambda_N} \] (C.3)

The differential gives

\[ d\Theta = -(N + 1) (\partial_\Sigma P^A) X'^\Omega dX'^{\Sigma_0} \wedge \cdots \wedge dX'^{\Sigma_N} \epsilon_{\Lambda_0 \cdots \Lambda_N} \\
- (N + 1) P^A dX'^{\Sigma_0} \wedge dX'^{\Sigma_1} \wedge \cdots \wedge dX'^{\Sigma_N} \epsilon_{\Lambda_0 \cdots \Lambda_N} \] (C.4)

The last term involves

\[ dX'^{\Sigma_0} \wedge dX'^{\Sigma_1} \wedge \cdots \wedge dX'^{\Sigma_N} = \beta_{\Sigma_0} dX^{\Sigma_0} \wedge \cdots \wedge dX^{\Sigma_N} \] (C.5)

By symmetrization, the \beta_{\Sigma_0} becomes replaced by the average \beta of those in the differentials, which is all but \Lambda. Thus, taking into account the Levi-Civita tensor, it can be replaced by \( \frac{\gamma - \beta}{N + 1} = -\frac{\alpha N}{N + 1} \). On the first term of d\Theta, I use the Schouten identity in the \( N + 3 \) lower indices. This gives

\[ d\Theta = (\partial_\Lambda P^A) \omega \]

\[ + (X'^\Omega \partial_\Omega P^A + \alpha_A P^A) dX^{\Sigma_0} \wedge \cdots \wedge dX^{\Sigma_N} \epsilon_{\Lambda_0 \cdots \Lambda_N} \] (C.6)

which, combined with \( \text{C.2} \), gives the desired result.
Appendix D. Some formulas about elliptic integrals.

First a remark: I use the notation $K(x)$ for what is usually denoted as $K(k)$ where $x = k^2$, and similar for other elliptic integrals. The elliptic integrals are in general defined as

$$J(x) = \int_{0}^{\pi/2} \frac{\Phi}{\sqrt{1 - x \sin^2 \theta}} d\theta = \int_{0}^{1} \frac{\Phi}{\sqrt{1 - y^2 \sqrt{1 - xy^2}}} dy$$

(D.1)

where $\Phi$ takes different values, see below, the second line is obtained from $y = \sin \theta$, and the square root is positive for $0 < x < 1$. For larger values of $x$ on the real axis, $y$ goes through the pole at $y = 1/k$, and we will have to choose in which way to encircle this point, i.e. whether $x$ is just below or above the real axis. So, the elliptic integrals have a branch cut going from $x = 1$ to $x = +\infty$ along the real line.

The values of $\Phi$ are

$$\Phi = 1 \quad J(x) \to K(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right)$$
$$\Phi = 1 - x \sin^2 \theta = 1 - xy^2 \quad J(x) \to E(x) = \frac{\pi}{2} F\left(\frac{1}{2}, -\frac{1}{2}; 1; x\right)$$
$$\Phi = \cos^2 \theta = 1 - y^2 \quad J(x) \to B(x) = \frac{\pi}{4} F\left(\frac{1}{2}, \frac{1}{2}, 2; x\right)$$

(D.2)

The relation between the functions $\Phi$ implies

$$x B(x) = E(x) - (1 - x) K(x) .$$

(D.3)

The integral of $K(x)$ is

$$\frac{d}{dx} \int_{0}^{x} K(t) \, dt = \int_{0}^{\pi/2} \frac{d\theta}{\sin^2 \theta} \left( \frac{1}{\sqrt{1 - x \sin^2 \theta}} - 1 \right)$$
$$= \left[ \frac{\cos \theta}{\sin \theta} \left( \sqrt{1 - x \sin^2 \theta} - 1 \right) \right]_{0}^{\pi/2} - \int_{0}^{\pi/2} \frac{\cos \theta}{\sin \theta} d\sqrt{1 - x \sin^2 \theta}$$
$$= \int_{0}^{\pi/2} \frac{x \cos^2 \theta}{\sqrt{1 - x \sin^2 \theta}} d\theta = x B(x) .$$

(D.4)

To obtain $K$ for arguments near the real line higher than 1, consider (I still use $0 < x < 1$, and define $k = \sqrt{x} > 0$)

$$\frac{1}{k} K\left(\frac{1}{x} \pm i \epsilon\right) = \int_{0}^{1} \frac{dz}{\sqrt{1 - z^2 \sqrt{1 - \frac{1}{x^2}\pm z^2}}} = \int_{0}^{k} \frac{dz}{\sqrt{1 - k^2 \sqrt{1 - \frac{1}{x^2}z^2}}} + \int_{k}^{1} \frac{dz}{\sqrt{1 - k^2 (\mp i)\sqrt{\frac{1}{x^2}z^2} - 1}} .$$

(D.5)

In the last line, I have split the integral in the region $z < k$ and $z > k$. For the latter, I have gone around the pole at $z = k \mp i \epsilon$, which produces the phase $e^{\mp i\pi/2}$ for the
square root. For the first term I now use the substitution \( y = z/k \), while for the second term I use \( y^2 = 1 - x \). This leads to

\[
K\left(\frac{1}{x} \pm i\varepsilon\right) = k \left( K(x) \pm iK(1-x) \right).
\] (D.6)

Following the same steps for \( E(\frac{1}{x}) \) gives

\[
kE\left(\frac{1}{x} \pm i\varepsilon\right) = xB(x) \mp i(1-x)B(1-x) = \frac{1}{2} \left[ \int_0^x K(t)dt \pm i \int_1^x K(1-t)dt \right]
\]

\[
= \frac{1}{2} \int_0^x \frac{1}{\sqrt{t}} K\left(\frac{1}{x} \pm i\varepsilon\right)dt \mp iB(1) \quad \text{and} \quad B(1) = 1.
\] (D.7)

(D.6) relates \( K \) in 3 points. Using the same relation with \( x \) substituted by one of the other values, relates \( K(x) \) also to its value in the points

\[
x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad 1-\frac{1}{x}, \quad \frac{x}{1-x}.
\] (D.8)

This relation between 6 points shows the \( D_3 \) symmetry. For \( 0 < x < 1 \), the first two values are between 0 and 1, the next two are higher than 1, and the last two are negative. However, in moving \( x \) to the other values, one has to be careful that none of the 3 points \( x, 1-x \) and \( 1/x \) goes through the branch cuts, which would move it to another sheet. There is the branch cut of the elliptic integral, but also the branch cut of \( k = \sqrt{x} \), which goes from 0 to \(-\infty\) along the real axis in this equation. Considering this, one can see that for using (D.6) with the upper sign the point \( x \) should be with negative imaginary part. Otherwise the 3 points are not on the first Riemann sheet. For the lower signs the opposite domain should be used.

With \( 0 < x < 1 \), this matters only for the points \( \frac{1}{x} \) and \( \frac{1}{1-x} \), and for the analytic continuation of \( k = \sqrt{x} \). E.g. for \( x \rightarrow 1-\frac{1}{x} \) this implies that one has to replace \( \sqrt{k} \) by \( \mp i\sqrt{\frac{1}{x}-1} \). Finally it turns out that the independent relations are (D.6), the same with \( x \) replaced by \( 1-x \), and the relations

\[
K\left(1-\frac{1}{x}\right) = \sqrt{x} K(1-x) ; \quad K\left(-\frac{x}{1-x}\right) = \sqrt{1-x} K(x).
\] (D.9)

Concerning the monodromies, the function \( K \) has only a branch point in \( x = 1 \). From (D.6) (using also (D.9)) we have

\[
K\left(\frac{1}{x} - i\varepsilon\right) = K\left(\frac{1}{x} + i\varepsilon\right) - 2i K\left(1-\frac{1}{x}\right).
\] (D.10)

Writing \( y \equiv \frac{1}{x} + i\varepsilon = 1 + r \), we have \( y' \equiv 1 + re^{2\pi i} = \frac{1}{x} - i\varepsilon \), and

\[
K(y') = K(y) - 2i K(1-y).
\] (D.11)

References

1. B. de Wit, P.G. Lauwers, R. Philippe, Su S.-Q. and A. Van Proeyen, Phys. Lett. \textbf{134B} (1984) 37.
2. B. de Wit and A. Van Proeyen, Nucl. Phys. B245 (1984) 89.
3. G. Sierra and P.K. Townsend, in Supergravity and Supergravity 1983, ed. B. Milewski (World Scientific, Singapore, 1983); S. J. Gates, Nucl. Phys. B238 (1984) 349.
4. A. Strominger, Commun. Math. Phys. 133 (1990) 163.
5. E. Witten, Commun. Math. Phys. 117 (1988) 353; 118 (1988) 411.
6. C. Vafa, Mod. Phys. Lett. A6 (1991) 337.
7. D. Anselmi and P. Fré, Nucl. Phys. B392 (1993) 401; B404 (1993) 288; B416 (1994) 255; Phys. Lett. B347 (1995) 247.
8. F. De Jonghe, P. Termonia, W. Troost and S. Vandoren, Phys. Lett. B358 (1995) 246, hep-th/9505174; preprint KUL-TF-95/34, NIKHEF 95-060, hep-th/9510176, to be published in the proceedings of Gauge theories, applied supersymmetry and quantum gravity, Leuven, 1995.
9. M. Billó, R. D’Auria, S. Ferrara, P. Fré, P. Soriani and A. Van Proeyen, R-symmetry and the topological twist of $N=2$ effective supergravities of heterotic strings, preprint hep-th/9505123, to be published in Int. J. Mod. Phys. A.
10. N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19; B431 (1994) 484.
11. A. Ceresole, R. D’Auria and S. Ferrara, Phys. Lett. B339 (1994) 71, hep-th/9408036.
12. S. Kachru and C. Vafa, Nucl. Phys. B450 (1995) 69, hep-th/9505103.
   S. Ferrara, J. A. Harvey, A. Strominger and C. Vafa, Phys. Lett. B361 (1995) 59, hep-th/9505162.
13. R. Haag, J.T. Lopuszański and M. Sohnius, Nucl. Phys. B88 (1975) 257.
14. B. de Wit, J.W. van Holten and A. Van Proeyen, Nucl. Phys. B184 (1981) 77 [E:B222 (1983) 516].
15. P. Fayet, Nucl. Phys. B113 (1976) 135; B149 (1979) 137;
   R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B133 (1978) 275.
16. M.F. Sohnius, Nucl. Phys. B138 (1978) 109;
   B. de Wit, J.W. van Holten and A. Van Proeyen, Phys. Lett. 95B (1980) 51.
17. J. Wess, Acta Phys. Austriaca 41 (1975) 409;
   B. de Wit and J.W. van Holten, Nucl. Phys. B155 (1979) 530;
   B. de Wit, R. Philippe and A. Van Proeyen, Nucl. Phys. B219 (1983) 143.
18. T.L. Curtright and D.Z. Freedman, Phys. Lett. 90B (1980) 71 [E:91B (1980) 487].
19. P.S. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B214 (1983) 519.
   J.P. Yamron and W. Siegel, Nucl. Phys. B263 (1986) 70.
20. A. Galperin, E. Ivanov and V. Ogievetsky, Nucl. Phys. B282 (1987) 74.
21. P. West, M. Sohnius, K.S. Stelle, Phys. Lett. B92 (1980) 123;
   P. Claus, B. de Wit, M. Faux, B. Kleijn, R. Siebelink and P. Termonia,
   The Vector-Tensor Supermultiplet with Gauged Central Charge, preprint Thu-
22. B. de Wit, V Kaplunovsky, J. Louis and D. Lüst, Nucl. Phys. B451 (1995) 53.

23. A. Van Proeyen, in *Developments in the Theory of Fundamental Interactions*, proc. 17th winterschool Karpacz, eds. L. Turko and A. Pekalski, Harwood Acad. Pub., 1981, p. 57.

24. B. de Wit, P. Lauwers and A. Van Proeyen, Nucl. Phys. B255 (1985) 569.

25. B. de Wit, C.M. Hull and M. Roček, Phys. Lett. B184 (1987) 233.

26. S. Ferrara, J. Scherk and B. Zumino, Nucl. Phys. B121 (1977) 393;
   B. de Wit, Nucl. Phys. B158 (1979) 189;
   E. Cremmer and B. Julia, Nucl. Phys. B159 (1979) 141;
   M.K. Gaillard and B. Zumino, Nucl. Phys. B193 (1981) 221.

27. J. Harvey, lectures in this school.

28. C.M. Hull and A. Van Proeyen, Phys. Lett. B351 (1995) 188, hep-th/9503022.

29. B. de Wit and A. Van Proeyen, Phys. Lett. B293 (1992) 94.

30. B. de Wit and A. Van Proeyen, *Special geometry and symplectic transformations*, to be published in the proceedings of the Spring workshop on String theory, Trieste, April 1994 preprint THU-95/25; KUL-TF-95/32, hep-th/9510180.

31. B. de Wit, F. Vanderseypen and A. Van Proeyen, Nucl. Phys. B400 (1993) 463.

32. A. Ceresole, R. D’Auria, S. Ferrara and A. Van Proeyen, Nucl. Phys. B444 (1995) 92, hep-th/9502072.

33. S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2457.

34. P. Di Vecchia, R. Musto, F. Nicodemi and R. Pettorino, Nucl. Phys. B252 (1985) 635.

35. S. Ferrara and A. Strominger, in *Strings ’89*, eds. R. Arnowitt, R. Bryan, M.J. Duff, D.V. Nanopoulos and C.N. Pope (World Scientific, 1989), p. 245;
   P. Candelas and X. C. de la Ossa, Nucl. Phys. B355 (1991) 455,
   P. Candelas, X. C. de la Ossa, P. Green and L. Parkes, Phys. Lett. 258B (1991) 118; Nucl. Phys. B359 (1991) 21.

36. L. Castellani, R. D’Auria and S. Ferrara, Phys. Lett. B241 (1990) 57; Class. Quantum Grav. 7 (1990) 1767,
   R. D’Auria, S. Ferrara and P. Frè, Nucl. Phys. B359 (1991) 705.

37. S. Ferrara and J. Louis, Phys. Lett. B278 (1992) 240;
   A. Ceresole, R. D’Auria, S. Ferrara, W. Lerche and J. Louis, Int. J. Mod. Phys. A8 (1993) 79, hep-th/9204033;
   J. Louis, in *String theory and quantum gravity ’92*, eds. J. Harvey et al., World Scientific, p. 368.

38. B. de Wit and A. Van Proeyen, *Isometries of special manifolds*, Based on
invited talks given at the Meeting on Quaternionic Structures in Mathematics and Physics, Trieste, September 1994; to be published in the proceedings. preprint THU-95/13, KUL-TF-95/13; hep-th/9505097.

39. M. Billó, A. Ceresole, R. D’Auria, S. Ferrara, P. Frè, T. Regge, P. Soriani and A. Van Proeyen, A Search for Non-Perturbative Dualities of Local $N=2$ Yang–Mills Theories from Calabi–Yau Threefolds, preprint SISSA 64/95/EP, POLFIS-TH 07/95, CERN-TH/95-140, UCLA/95/TEP/19, IFUM 508FT, KUL-TF-95/18, to be published in Class. Quantum Grav., hep-th/9506073.

40. P. Griffiths, Ann. Math. 90 (1969) 460.

41. P. Frè and P. Soriani, The $N=2$ Wonderland, from Calabi–Yau manifolds to topological field–theories, World Scientific, 1995.

42. D.R. Morrison, in Essays on Mirror Manifolds, ed. S.T. Yau, International Press, 1992.

43. A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. B344 (1995) 169; and in Physics from the Planck Scale to Electromagnetic Scale, proceedings of the conference in Warsaw 1994, eds. P. Nath, T. Taylor and S. Pokorski, World Scientific (1995), and of the 28th International Symposium on Particle Theory, Wendisch-Rietz, preprint CERN-TH.7538/94, hep-th/9412155; P. Argyres and A. Faraggi, Phys. Rev. Lett. 74 (1995) 3931.

44. A. Giveon and D. J. Smit, Nucl. Phys. B349 (1991) 168; C. Gomez and E. Lopez, Phys. Lett. B357 (1995) 558, hep-th/9505133.

45. A. Van Proeyen, in Supersymmetry and Supergravity 1983, ed. B. Milewski, (World Scientific Publ. Co.), 93 (1983); and in Superunification and Extra Dimensions, eds. R. D’Auria and P. Frè, (World Scientific Publ. Co.), 97 (1986).

46. B. de Wit, in Supergravity ’81, eds. S. Ferrara and J.G. Taylor (Cambridge Univ. Press, 1982).

47. E. Cremmer, C. Kounnas, A. Van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385.

48. S. Ferrara and A. Van Proeyen, Class. Quantum Grav. 6 (1989) 124.

49. M. Günaydin, G. Sierra and P.K. Townsend, Phys. Lett. B133 (1983) 72; Nucl. Phys. B242 (1984) 244, B253 (1985) 573.

50. A. Ceresole, R. D’Auria and S. Ferrara, The symplectic structure of $N=2$ supergravity and its central extension, preprint POLFIS-TH 10/95, CERN-TH/95-244, hep-th/9509160, to be published in the proc. of the Trieste conference on $S$ Duality and Mirror Symmetry, june 1995.

51. P. Candelas, Nucl. Phys. B298 (1988) 458; P. Candelas, A.M. Dale, C.A. Lutken and R. Schimmrigk, Nucl. Phys. B298 (1988) 493; P. Candelas, M. Lynker and R. Schimmrigk, Nucl. Phys. B341 (1990) 383.