Abstract

We discuss a new class of coordinate systems for a plane, which provide an analytical representation of arbitrary straightline, and then define the form of potential on the plane, under which the equations of motion of a mass point are solvable. In both cases we use the Hamiltonian approach, reducing the problem to the Hamilton-Jacobi equation. The coordinate systems in question are defined in a somewhat unusual way, and, correspondingly, the potentials we obtain, also possess some unusual properties. Since the equations of motion are solvable within this potential, the second integral of motion is defined too, and, like the coordinate system and potential, it possesses some nonstandard properties.

1 Introduction

All known integrals of motion of conservative mechanical systems can be obtained as constants appearing as the result of variables separation in the Hamilton-Jacobi equation for given system. They always are either linear or quadric functions of generalized velocities of the system; the earlier are provided by its symmetry while the latter appear when system has no symmetry group. No other forms of integrals of motion are known. Therefore it is interesting to explore possibility of existence of systems possessing integrals of motion of other form.

In this paper we make an attempt to find out systems with other integrals of motion in particular case of two degrees of freedom. The systems under consideration describe motion of particle on plane under action of given potential. The work is organized as follows. As new class of desired systems is found we pass to the next stage and search for potentials which can be included into the equation without losing separability of the equation. Finally we discuss properties of the potentials satisfying these conditions.

To show what do we mean by new class of coordinate systems, let us start with the commonplace definition of coordinate transformation. This definition reads that coordinate transformation \( \{x^i\} \rightarrow \{y^a\} \) is just introducing some functions \( y^a \) of coordinates

\[
y^a = y^a(x^i)
\]
which satisfy well-known conditions, and it suffices to know that the inverse transformation

\[ x^i = x^i(y^a) \]  

exists. From practical point of view this is not sufficient, because, besides definition of new coordinates (1) one needs explicit form of metric in the new system. Transformation of the metric requires that the equations (1) are solved analytically such that explicit form of old coordinates \( x^i \) as functions of new ones is known. In other words, the difference between commonplace definition and practical needs is that the earlier assumes only existence of the functions \( x^i(y^a) \) and the latter require them to be given explicitly. This means that, in fact, very few kinds of coordinate systems can be used practically, because analytical solvability of the equations (1) means that they actually can be reduced to algebraic equations of not more than second order, otherwise they are not solvable. However, all coordinate systems of this sort are known and solvability of Hamilton-Jacobi equation has been studied in all of them. Therefore, in order to find out new usable coordinate systems we try to broaden the scope and consider systems which cannot be defined analytically as in the equation (1). By new class of coordinate systems we mean systems defined such an unusual way.

There is strong interdependence between solvability of the equations (1) and that of free motion of a particle. If, for example, \( \{x^1\} \) are Cartesian coordinates, solutions of the equations (1) specify straights which, at the same time, are trajectories of free motion of a particle. Therefore, if equations (1) cannot be solved analytically, solvability of equations of dynamics is doubtful because, otherwise, Cartesian coordinates can be restored from straights obtained as trajectories. On the other hand, solvability means possibility to express solution in terms of known objects which does not depend on approach to the problem. Therefore in this work solvability of both equations (1) and equations of dynamics are considered to be the same.

2 Strange coordinate systems and their metrics

All coordinate transformations used till now specify new coordinates \( y^a \) analytically as certain functions of the old ones. Another possibility is to define new coordinates geometrically. To do it specify two foliations of plane or its domain with two families of curves and introduce two numerical parameters each of which labels curves of one family. These two parameters are new coordinates which, however, are not defined as functions of old ones, moreover, when defining them we did not need any ‘old’ coordinates to be introduced previously. In fact, we use here the natural way of introducing coordinates on plane for the first time, as if no coordinates on plane have been introduced before. As it is done, our task is to obtain metric of this system from geometric properties of the foliations and definition of the parameters introduced.

Let us now pass to practical implementation of this program. Consider a simple arch of curve on plane. By simplisity we mean that it is smooth and has no intersections with straights tangent to it. Hereafter we call it the basic curve of the coordinate system. Consider now all straights tangent to the curve and starting from each its point in (locally) one direction. These straights constitute foliation of a domain on plane, whose boundary consists of the basic curve and two rays starting from its endpoints. Now we build a coordinate system for this
domain using the rays as coordinate lines. For this end we label each ray with the value of angle $\psi$ between the ray and some fixed direction on the plane. Thus, $\psi$ is one of new coordinates. To introduce the second coordinate, note that all rays $\psi = \text{const}$ are orthogonal to all evolvents of the curve all rays are tangent to, hence, in order to obtain an orthogonal coordinate system we must use foliation of the domain with the evolvents. Now it remains to parametrize the family of evolvents with a numerical parameter and find out the metric of coordinate system defined this way.

Evolvents of a curve on plane $[1, 2]$ are known to be equidistant lines (parallels) $[3]$. Therefore it is natural to label them with the length parameter $R$. If this parameter is defined properly it specifies a function on the domain, which satisfies the Hamilton-Jacobi equation for straightlines. Gradients of $R$ and $\psi$ as functions on plane are orthogonal and radius of curvature of evolvent $R = \text{const}$ equals $R - l(\psi)$ where $l(\psi)$ depends only on shape of the basic curve. Square gradient of the function $\psi$ is inverse square of radius of curvature:

$$<dR, dR> = 1, <dR, \psi> = 0, <d\psi, d\psi> = [R - l(\psi)]^{-1}.$$  \hfill (3)

These equalities specify the metric of the coordinate system $\{R, \psi\}$.

### 3 Arbitrary straightlines and Cartesian coordinates

Each solution of Hamilton-Jacobi equation specifies one congruence of straightlines. We use this fact to obtain analytical representation of arbitrary straightline in the coordinates $\{R, \psi\}$. In this case the equation has the form

$$<dS, dS> = 1,$$  \hfill (4)

and, due to the equations (3) can be written as

$$\left(\frac{\partial S}{\partial R}\right)^2 + \frac{1}{[R - l(\psi)]^2} \left(\frac{\partial S}{\partial \psi}\right)^2 = 1.$$  \hfill (5)

In order to separate variables in Hamilton-Jacobi equation one assumes that the function to be found has the form of sum of single variable functions each of which depends only on one coordinate. In this case this means that the desired function must be taken in the form $S(R, \psi) = f(R) + g(\psi)$. Note that this is the only known method of separating variables in this equation. As the equation just obtained cannot be separated this way, to solve it we have to think out a new ansatz about the desired function, because the standard one does not work in this case. Nevertheless, it will be shown below that this equation reduces to some ordinary differential equation, and obtain the form of the desired function under which variables separate in metrics like (6). This form will be used when separating variables in more general case of particle motion in some potential.

We reduce the equation (4) to an ordinary differential equation in two steps. First, we introduce a 1-form $\pi$ of unit norm, and second, we require that this form is closed:

$$<\pi, \pi> = 1, \, d\pi = 0.$$  \hfill (5)
Then, as such a 1-form is obtained, we put $dS = \pi$ that gives solution of the equation (4). To obtain a norm one 1-form we introduce an orthonormal frame of 1-forms $\{\nu^a\}$. Its form follows from the equations (3): 
\[\nu^1 = dR, \ \nu^2 = [R - l(\psi)]d\psi,\] (6)
and exterior derivatives of $\nu^a$'s are
\[d\nu^1 = 0, \ d\nu^2 = dR \wedge d\psi \equiv [R - l(\psi)]^{-1}\nu^1 \wedge \nu^2.\] (7)

Now, arbitrary norm one 1-form can be represented as follows:
\[\pi = \nu^1 \sin f + \nu^2 \cos f\] (8)
where $f$ is an arbitrary function on the plane. It remains to find it out from the second equation (5):
\[0 = d\pi = -\cos f \nu^1 \wedge df + \sin f df \wedge \nu^2 + \cos f d\nu^2.\] Substituting the equations (6) and (7) gives:
\[\{-[R - l(\psi)]^{-1}\cos f f_\psi - \sin f f_R + [R - l(\psi)]^{-1}\cos f\} \nu^1 \wedge \nu^2 = 0.\]
One particular solution of this equation is evident: $f = \psi + const$. However, since it contains an arbitrary constant no more general solution is needed, and we put
\[dS = \sin(\psi - \psi_0)dR + \cos(\psi - \psi_0)[R - l(\psi)]d\psi.\]
The desired solution is
\[S = R\sin(\psi - \psi_0) - \int l(\psi) \cos(\psi - \psi_0).\]
Note that differentiating this function on the constant $\psi_0$ is equivalent to change of the constant. Due to the Jacobi theorem [4] this means that lines $S = const$ are straightlines, consequently the function $S$ specifies one of Cartesian coordinates or their linear combination. Alternatively, Cartesian coordinates can be introduced as two solutions of the Hamilton-Jacobi equation $x^1$ and $x^2$ with orthogonal gradients $<dx^1, dx^2> = 0$:
\[x^1 = R\sin \psi - \int l(\psi) \cos \psi d\psi, \ x^2 = R\sin \psi + \int l(\psi) \sin \psi d\psi.\] (9)
It is seen now that the coordinates $R$ and $\psi$ cannot be expressed analytically as functions of Cartesian coordinates (9), hence, equation (2) has purely formal meaning.

4 Inclusion of potential

As seen from the result obtained, we have found how to separate Hamilton-Jacobi equation in metrics of the form (3). For this end the function to be found is to be taken in the form
\[S = Rf(\psi) + g(\psi)\] (10)
As this is clear now, let us look for possible form of force function which can be included into the Hamilton-Jacobi equation without losing separability. Hamilton-Jacobi equation for particle of mass $m$ and energy $E$, moving in potential $V(R, \psi)$ has the form

$$\frac{1}{2m} <dS, dS> + V(R, \psi) = E. \quad (11)$$

Assuming that the function to be found has the form (10), we have

$$dS = f(\psi)\nu^1 + \frac{Rf' + g'}{R - l(\psi)}\nu^2$$

and it is seen that condition of separability reads:

$$g(\psi) = -\int l(\psi)f'(\psi)d\psi \quad (12)$$

such that $dS = f(\psi)\nu^1 + f'(\psi)\nu^2$. Then the potential is function of single variable $\psi$, and if we denote $2m[E - V(\psi) \equiv p^2(\psi)$ the Hamilton-Jacobi equation (11) reduces to the following ordinary differential equation:

$$f'^2 + f^2 = p^2. \quad (13)$$

Finally, solution of the equation (11) appears as

$$S = Rf(\psi, C) - \int l(\psi)f'(\psi, C)d\psi$$

where $C$ is arbitrary constant of general solution of the equation (13). Note that this constant plays the role of the second integral of motion. All known integrals of motion can be obtained from the procedure of variables separation in Hamilton-Jacobi equation and are linear or quadratic functions of the particle velocity. This one appears later, after variables are separated and Hamilton-Jacobi equation is reduced to an ordinary differential equation. It is seen that this integral of motion is not linear or quadratic on velocity, moreover, probably, it cannot be expressed in terms of known functions of coordinates and velocity components.

## 5 Conclusion

Thus, we have constructed a class of strange coordinate systems $\{R, \psi\}$ which cannot be expressed in terms of known functions of any standard coordinates on plane. However, Cartesian coordinates can be expressed analytically as functions of $R$ and $\psi$ and this expression is given by the equations (9). The metrics of all coordinate systems of this class has one and the same form (3) and differ only in the form of the function $l(\psi)$ specified by the basic curve of the system. Hamilton-Jacobi equation for straightlines on the plane separates in these metrics, and there exists a class of potentials in which Hamilton-Jacobi equation for a mass point also separates. All potentials of this class depend on only one coordinate $\psi$ and also possess some strange properties. Like the coordinates, these potentials cannot be expressed in terms of known functions of any standard coordinates on plane, and, since equipotential lines are rays $\psi = \text{const}$,
which have envelope, the potentials are non-univalent on the basic curve. The second integral of motion of particle moving in potential of this sort is not of known form, at least, it is certainly not of first or second order on velocity components, and seems to belong to the same class of unknown functions of coordinates and velocities, as the potentials. **Acknowledgment:** The author thanks the Third World Academy of Sciences for financial support and IUCAA for warm hospitality which made this work possible.

**References**

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