Quantum information processing and quantum logic:
towards mutual illumination

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Abstract

Quantum information and computation may serve as a source of useful axioms and ideas for the quantum structures project of characterizing and classifying types of physical theories, including quantum mechanics and classical mechanics. The axiomatic approach of quantum structures may help isolate what aspects of quantum mechanics are responsible for what aspects of its greater-than-classical information processing power, and whether more general physical theories may escape some common limitations of classical and quantum theories. Also, by helping us understand how existing quantum algorithms work, quantum structures analyses may suggest new quantum protocols exploiting general features of quantum mechanics. I stress the importance, for these matters, of understanding open and closed-system dynamics, and the structure of composite systems in general frameworks for operational theories, such as effect algebras, convex sets, and related structures.

1 Introduction

A main concern of the field of “quantum logic” and related algebraic structures has long been to understand the differences between quantum and classical physical theories by looking for axiomatic, often largely algebraic characterizations of the two types of theories. Usually, appealing axioms having an intuitively understandable meaning have been sought. One might hope to find a nice set $S$ of axioms which both kinds of theories satisfy, and two other sets $Q$ and $C$, such that $Q \cup S$ gives classical theory while $C \cup S$ gives quantum theory. Such a characterization of the two types of theory would help one understand, what they have in common and how they differ. Changes to either the shared axiom set or the distinctive classical or quantum axioms might help one understand what other kinds of physical theories are possible. Such alternative kinds of theories might be intermediate between the two types (perhaps keeping $S$ but substituting some other set for $C$ or $Q$), or diverging from both on fundamental
issues (perhaps differing on $S$). I think it is fair to say that quantum logic and related fields have made much progress towards this goal over the years, but that no characterization is yet widely recognized to have met it, although some approaches have come close. Of course, the success of one approach would not mean others should be abandoned: it is possible, even likely, that there are various useful characterizations of quantum mechanics, classical mechanics, and how they differ.

Over the last fifteen or twenty years, the fields of quantum information theory (QIT) and quantum information processing (QIP) have emerged. QIP involves physicists, computer scientists, mathematicians and others in investigating the fact that in a world governed by quantum theory, one can do some tasks of classical information processing faster or better than in a world governed by classical theory. But in investigating these matters, one quickly comes to appreciate the importance of quantum analogues of classical tasks such as data compression and the transmission of states through a noisy channel, in which the data or messages are themselves conceived of as quantum in nature—classical information. This has given rise to QIT, a quantum generalization of information theory, which is an analogue of the classical theory in which not just the performance of classical tasks (possibly in superior fashion) by quantum means is considered, but also the performance of tasks which are themselves quantum and may or may not have an analogue in the classical world.

The prime examples of classical information processing by quantum means are quantum computation and quantum cryptography. Quantum computation allows some tasks, such as factoring integers [1], [2], to be done in polynomial time whereas no polynomial time classical algorithm is known despite intense effort to find one. It also allows brute-force search (for instance, for solutions to problems in NP) to be done in the square root of the classical time [3], [4], although this cannot of course speed up an exponential search to a polynomial one. And quantum cryptography allows (with the aid of an authenticated classical channel or a small amount of classical shared key to use in authenticating classical communication) the distribution of a large amount of key material in such a way that eavesdropping is virtually certain to be detected [5]—a classically impossible task. (The amount of key distributed can be much larger than the amount required for the classical authentication in the protocol.)

This has raised the question: what aspects of quantum mechanics are responsible for these better-than-classical feats of information processing? Here there is scope for fruitful interaction between the fields of quantum information and quantum structures. On the one hand, the axiomatic approach of QS suggests varying the axiom set and investigating the information-processing power of the resulting theories. On the other hand, information theory and information-processing considerations may suggest new and physically meaningful axioms, for use in the quantum structures characterization program.
In this paper, I will investigate some of the most promising possibilities for this interaction, and in some cases sketch how I think it might play out. I will also, though tangentially, address the possible usefulness and limitations of such an approach as a way of gaining perspective on possible new physics, for instance quantum gravity.

2 An operational approach

An important part of QS is concerned with what I will call “operational structures.” This approach goes back at least to the work of Mackey [6]. I would include in it the work of Ludwig and his school, the convex approach with important contributions from Ludwig, Davies, Edwards, Araki, Gudder, Pulmannová, Beltrametti, and many others, and the “test space” and “effect algebras” approach of Foulis and his coworkers and students, Gudder, Schroock, and (using initially different terminology) the Italian schools of Giuntini et. al., dalla Chiara, and the Slovak school of Dvurecenskij, Pulmannová and others. (I also think the “Geneva school” may have evolved in this direction, and there are category-theoretic approaches which may also be related.) I will use the term “operational structures” for structures describing an abstraction of a physical theory in terms of operations or measurements one may perform on a system, and the probabilities for outcomes of these procedures.

In this paper I will sometimes adopt a version of the operational point of view which guides much of my own thinking in the area. I will call it the “procedural” point of view since “operational” is already used in various senses in quantum structures. On this view, which I associate particularly with Ludwig and his followers, but which has roots in the work of Mackey or even earlier, the “measurements” which may ultimately be given some formal structure such as that of effect algebra, test space, etc., are to be viewed as suitable equivalence classes of operational procedures. Any operation which we can perform on a system, including any sequence of two operations in a row, any interaction with an environment followed by an operation or measurement on environment, system, or both, etc., should be included in the set of procedures from which equivalence classes are generated to give us an effect algebra (or whatever structure we get). I am investigating a framework in which one begins with experimental procedures having different possible outcomes, and divides out of the set of outcomes a relation of “probabilistic equivalence” (two outcomes are equivalent if they have the same probability in all physically preparable states). One dubs the equivalence classes thus obtained “effects”, because—as I will show elsewhere—one can embed them into an effect algebra with natural relations (induced by the quotient map) to the Boolean algebras generated by the experimental outcomes themselves. This effect algebra may in general contain additional effects which are not equivalence classes of the original outcomes. If these are not to be
countenanced, one obtains a structure I call a “weak effect algebra”: all axioms of an effect algebra hold, except that the associative law is weakened so as not to require the existence and equality of both sides of 
\((a \oplus b) \oplus c = a \oplus (b \oplus c)\)
whenever one side exists; rather, equality is to hold whenever both sides are defined. This notion of probabilistic equivalence has been criticized, (e.g. in \([7]\), where it is attributed to Bohr, and \([8]\)) but I believe it is nevertheless useful. One complaint has been that outcomes which are equivalent in this respect may lead to different probabilities for subsequent measurements \([8]\). This is correct, but in the framework I am proposing, just means that many different conditional dynamics can correspond to a given effect. A complete theory, including dynamics and conditional dynamics, will need structure beyond that of effects, but effect algebras are nevertheless an important conceptual structure, probably adequate for statics and providing a good starting point for the introduction of dynamical structure. For example, see \([1]\) for a vigorous start on the project of introducing at least reversible dynamics on effect algebras (via their ambient unigroups), the last three sections of \([10]\) for similar considerations on test spaces, and \([11]\) for explicit discussion of conditional dynamics in effect algebras with some convex structure. The construction of effects I have mentioned may well have been carried out previously, by Ludwig, as part of a convex spaces approach to deriving quantum mechanics. I arrived at it as an abstraction of the lovely presentation by Ludwig’s student, Kraus, of the quantum case\([12]\).

It is natural to suppose that apparatus involving “dice” may be used to prepare (as a mathematical idealization) any convex combination of states, and also to perform any convex combination of measurement procedures. Even though for some physical systems we may not have much freedom to actually perform different preparations, dice or no dice, the supposition may apply to many laboratory systems, and we might argue from some sort of homogeneity considerations that the restrictions, if any, on theory derived from considering convex combination in laboratory systems reveal general structure which we should suppose distant systems, more difficult to manipulate, also to exhibit. In an effect algebra, requiring that measurements can be convexly combined induces a notion of multiplication of an effect by nonnegative real number (and thus also a notion of convex combination of effects); since the orthosum \(\oplus\) in most nice effect algebras is a restriction of the sum on a partially ordered Abelian group into which the effect algebra is embedded, one thereby gets a real vector space structure for the effects. \(\ominus\) This goes a long way towards enforcing a representation of probabilities as linear functionals on a real vector space of effects, possibly a step on the way to a Born-like probability rule representing probabilities as \(\text{tr } XT\) for linear operators \(X\) representing effects and \(T\) representing states. Obviously, careful treatments must keep track of the ordering aspect of the relevant structures as well, and an appropriate representation theorem is proved in \([14]\).

\(^1\)It is not completely clear to me yet how important convexity is for the introduction of vector-space structure—Foulis and Bennett \([13]\) show the existence, for any field \(K\) and effect algebra \(L\), of a “universal vector space” for \(L\) over \(K\).
The general operational approach may turn out to be a limiting, or limited, way of viewing physical theories. It seems rather suited, however, to quantum mechanics, particularly in a Copenhagen style, and also adequate to treat classical mechanics in its information-processing aspect. The limitations arise because this sort of theory takes “operations,” such as making a measurement on a system, as basic terms in formulating a physical theory. It views a physical theory as a description of the behavior of a “system,” a part of the world viewed as susceptible to the performance of operations on it by, presumably, another “part of the world,” the observer or experimenter ([15], p.). It seems, inherently, to renounce the attempt to view the entire world as one structure, and physical theory as describing that structure. Inasmuch as many have suggested quantum theory shows that project to be hubris, this is perhaps desirable. Moreover, one might in some theories be able to view the entire world, if desired, as a system of the kind the theory describes, with the “procedures,” or at least those viewed as measurements, imagined performed by some external observer. Classical theory may be such a case. Many believe that quantum mechanics is not. However, Everett’s “relative-state” interpretation (the inspiration for what is sometimes called the “many-worlds interpretation”) attempts such an “objective” interpretation of quantum mechanics. Some, notably Rovelli and Smolin [17], have proposed making the impossibility of such an external view an axiom for an acceptable physical theory.

Aside from quantum gravity aspects, the main interest the Rovelli-Smolin approach holds for me is, ironically, that it suggests a framework in which quantum mechanics is good for describing things from the point of view of subsystems, but not appropriate for the entire universe, but in which nevertheless there exists a mathematical structure—something like a topological quantum field theory (TQFT)—in which these local subsystem points of view are coordinated into an overall mathematical structure which, while its terms may be radically different from those we are used to, may still be viewed as in some sense “objective.” However, it is far from clear yet that this can be done while avoiding the more grotesque aspects (proliferating macroscopic superpositions viewed as objectively existing) and remaining conceptual issues (how to identify a preferred tensor factorization, and/or preferred bases, in which to identify “relative states”) of the Everett interpretation.

In an interesting article (brought to my attention by Chris Fuchs) which relates it to the interpretation of quantum mechanics Bilodeau [10] has termed a similar distinction one between “dynamical” and “geometric” theories.
3 Statics versus dynamics, composite systems, and the “local” nature of operational theories

A large part of quantum structures has concentrated on the statics of states and measurements or propositions. It is certainly reasonable to examine statics on its own, all the more so because in some cases the concern is with theories which may become part of a generalization of quantum theory, a model, say for quantum gravity or a field-theoretic approach, rather than a simple dynamical theory in which time is an external parameter. In such a situation, it may be desirable to develop a static algebraic structure first, and then to integrate it into a dynamical, spacetime, or some more general physical framework. The $\mathbb{C}^*$-algebraic approach to quantum field theory [18], [19], [20] is one example of this, in which the local structures are taken to be $\mathbb{C}^*$-algebras, and they are to be coordinated into “nets” consistent with relativistic spacetime structure. The external-time Hilbert space dynamics of nonrelativistic quantum mechanics is another, more straightforward one. Rovelli-Smolin is an attempt to coordinate “local” algebraic structure (Hilbert space) into larger structures exhibiting some properties of gravitation and quantum field theory. However, it could also turn out (in my view this is very likely) that the nature of the appropriate local structure requires input from the global structure or the dynamics. For one thing, we should arguably be able to take an “external” view of the operations performed by one part of the global structure on another, as dynamical interactions between two systems considered as a single composite system, say. It could even turn out that the composite of A and B is not described by quite the same type of local theory as are A and B themselves. In fact, although in TQFT’s the local theory type is the same (complex inner product spaces), only in special cases does composition of systems induce the tensor product of the local systems.

For information-processing or computation, both dynamical considerations and composite systems are of the utmost importance. Since the environment which induces noise in a system or the apparatus used by an information-processing agent must be considered together with the system, a notion of composite system is needed. And notions of composition are basic to computational complexity, where the question may be how many bits or qubits are needed, as a function of the size of an instance of a problem (number of bits needed to write down an integer to be factored, say) to solve that instance. Indeed, the very notion of Turing computability is based on a factorization of the computer’s state space (as a Cartesian product of bits, or of some higher-arity systems), in terms of which a “locality” constraint can be imposed. The constraint is, roughly, that only a few of these subsystems can interact in one “time-step.” The analogous quantum constraint allows only a few qubits to interact at a time. In general operational models, some notion of composition of systems, such as a tensor product, together with a theory describing what dynamics can be implemented on a
subsystem, could allow for circuit or Turing-machine models involving “bits” or other local systems of a nature more general than quantum or classical systems. There may be much to learn from a study of computational complexity in such general systems. For example, I think we are likely to find intuitively meaningful, very general properties of operational physical theories, shared by quantum and classical mechanics but also by a wider class of theories, which forbid, for conceptually clear reasons, exponential speed-ups of brute-force search. These properties may be linked to other properties of theories: for example, the second law of thermodynamics (impossibility of a \textit{perpetuum mobile}), or the impossibility of instantaneous signalling between subsystems of a composite system. Richard Jozsa has suggested that the impossibility of such speedups could serve as a constraint on proposed new physics. In this regard, Lloyd and Abrams\cite{21} demonstration that theories of “nonlinear quantum mechanics” do permit such exponential speedups is relevant. It is particularly interesting in light of the fact that nonlinear quantum mechanics also appears inconsistent with the second law of thermodynamics \cite{2}. Moreover, Polchinski \cite{23} has argued that a class of nonlinear modifications of quantum mechanics (which includes Weinberg’s \cite{24} proposal) either allow superluminal signalling (because they allow instantaneous signalling between subsystems, which may be spacelike-separated) or, if they do not allow superluminal signalling, allow something he calls an “Everett phone.” This latter phenomenon involves a sequence of spin measurements, in which the outcome of a later spin-measurement, conditional on the first spin-measurement having resulted in spin up, depends on what the observer \textit{would have} proceeded to measure if the first measurement had resulted in spin down. Polchinski discusses the Everett phone in the framework of the “relative states” interpretation of quantum mechanics. It would appear to be inconsistent with a more standard view of quantum mechanics, for example as an “operational theory” of the sort we have been considering. On such a view, the probability of a measurement result is taken to be independent of the context in which it is measured—i.e., of the other outcomes tested. (Such context-independence is automatic on the procedural point of view described above: it is enforced by the definition of effects as probabilistic equivalence classes of measurement outcomes.)

The “modern” quantum formalism of “Positive Operator Valued Measures” — resolutions of unity, into positive operators $F_i \geq 0$, $\sum_i F_i = I$ — and completely positive maps on the space of operators on a Hilbert space as representing the dynamics of a system conditional on getting measurement result $i$, is the stock in trade of the quantum information processing theorist. The CP-maps represent the most general physical process that an encoder, decoder, or programmer initially independent of the system can hope to get a system to undergo. CP-maps may also represent the effect of an interaction with an environment on a quantum system. Frequently, one looks for the optimal CP-map with respect to some information-processing criterion, or shows that no CP-map with certain information-processing properties exists.

There are several equivalent characterizations of CP-maps. Write $B(H)$ for the
set of positive operators on a (finite-dimensional) Hilbert space $H$. The definition of complete positivity says that the map not only takes positive operators to positive operators, but that any extension of the map by the identity map on an additional system preserves the positivity of operators on the extended system. That is, a CP-map $A$ satisfies $(\rho^{AB} \geq 0) \rightarrow I^B \otimes A^A(\rho^{AB}) \geq 0$. If such a map is trace preserving, it represents a general open-system dynamics not conditioned on any measurement outcome or additional information about the $B$ system. If it is trace-decreasing, it may represent dynamics conditional on a measurement outcome (the trace of the output operator when a trace-one operator is put in, is the measurement probability). An equivalent characterization of TPCP maps is that the map’s action on $B(A)$ is obtained by considering a reversible dynamics, i.e. a unitary operation, on $A$ and some environment $E$, but considering its effect only on $A$. An equivalent characterization for trace-decreasing maps is as the effect, on $B(A)$, of unitary interaction between $A$ and $E$ (so that elements of $B(A)$ are conjugated by $U^{AE}$) followed by the action of an operator $G^E \otimes I^A$ (also by conjugation), representing a measurement made on the environment. (This corresponds to an effect $G^I G$ on the environment, and the trace-decreasing condition imposes $G^I G \leq I$. It is easily seen that $G^I G$ is all that matters to the operation realized on the system.) For arbitrary CP-maps (with no trace constraints), $G$ may be an arbitrary operator on $E$.) Still another characterization of CP-maps, this one referring only to the system $A$, but having little prima facie intuitive motivation, is that their action may be written $A(X) = \sum_i A_i X A_i^\dagger$. Trace-preservation is the condition $\sum_i A_i^\dagger A_i = I$.

A notable example of the introduction of additional structure is Gudder and Greechie’s introduction of the sequential product on effect algebras, which abstracts some properties of the product $A \ast B = A^{1/2} B A^{1/2}$ on effect algebras. We have $\text{tr} A^{1/2} B A^{1/2} \rho = \text{tr} B^{1/2} A^{1/2} \rho A^{1/2} B^{1/2}$, which is the probability of measuring the effect $B$ after obtaining the effect $A$ from a prior measurement, when the conditional dynamics of the first measurement are the “square-root dynamics” $\rho \mapsto A^{1/2} \rho A^{1/2}$. The square-root dynamics is a natural generalization of the projection postulate and “Lüder’s rule” to effects that are not projections, and it shares important features with them. For instance, when a given resolution of
unity is measured, the square-root dynamics are the conditional dynamics that cause the least expected disturbance to an initial pure state picked uniformly from Hilbert space \[25, 26\].

4 Composite systems, and tensor products

As mentioned above, I believe that the “procedural” approach to operational theories leads naturally to theories whose statics is described by an effect algebra. This notion is simultaneously an abstraction of the unit interval of operators \(0 \leq e \leq I\) on a (say, finite-dimensional) Hilbert space, and the space of positive valued functions \(f\) bounded above by unity on a (for convenience, let us say finite) set. (The latter represents possibly “fuzzy” classical measurement outcomes.) An effect algebra is a set on which is defined a partial binary operation \(\oplus\), sometimes called the orthosum. (In effect algebras of Hilbert-space operators, \(\oplus\) is just addition of operators, except that it is not defined when \(X + Y > I\), since then \(X + Y\), while positive is not in the unit interval. The operation is commutative and associative in the strong sense that when the effect referred to by one side of the commutative or associative law is defined, so are the effects mentioned on the other side, and they are equal. Additionally the algebra contains special elements 0 and 1. These satisfy \(x \oplus 0 = x\), and \(x \oplus y = x \rightarrow y = 0\). Importantly, for every effect \(x\) there is exactly one other effect, its orthosupplement \(x^\perp\), which can be added to it to get \(I\).

The category of effect algebras, \(E\mathcal{A}\), presents aspects which are quite fascinating and appealing, but some of those same aspects also suggest that we may need to focus on some subcategory of \(E\mathcal{A}\), especially in order to focus on differences between quantum and classical information processing. A consideration of composite systems allows us to define relatively natural notions of tensor product of effect algebra. For effect algebras, the is due to Dvurecenskij [28]; for a nice alternative construction of this tensor product see Wilce [29][30]. These constructions are similar to Foulis and Randall’s [31], [3], [32] pioneering construction of the tensor product of (most) test spaces and Foulis and Bennett’s construction [33] for orthoalgebras. This notion may be derived (in one of several ways) by essentially requiring that it include all elements of the cartesian product of the two factors, that the partial operation “\(\oplus\)” of the algebra be such that all locally implementable measurements (consisting in doing a measurement on one system and, conditional on the result of that measurement, doing a measurement on the other system) are measurements in the product algebra. All states on the new effect algebra will automatically add up to one on such measurements. Remarkably, the tensor product may also be derived

\[3\] This may be done using the standard category-theoretic definition of tensor product in terms of universal properties of bimorphisms; for QIT theorists unfamiliar with this construction, [27] Chapters 1 and 7, are an excellent introduction.

9
from the requirement that it contain as a subset the cartesian product \( E \times F \), exhibit no instantaneous influence in either direction, and that it be a universal object satisfying these requirements

Beautiful as this construction is, when we apply it to Hilbert space effect algebras, we see that it does not give the effect algebra of the tensor product Hilbert space. For the algebra of sharp Hilbert-space effects (projectors), this was noted by [32], [34], but it clearly extends to the algebra of all effects. There are states on this tensor product reproducing the statistics of all the quantum states, including the entangled ones, as far as tensor product measurements are concerned. But there exist other states, given by tracing with Hermitian operators that are positive on pure tensors but negative on some entangled outcomes, with statistics different from the quantum ones even for product measurements. “Entangled measurements” (perhaps we could abstract this notion!) are not available in this structure, so negativity on entangled outcomes does not prevent these from being legitimate states on the tensor product of the effect algebras. An independent construction of what is essentially the Dvurečenskij tensor product of two Hilbert space effect algebras appears in [35], [36].

It would be fascinating to investigate the computational power of a model starting with, say, Hilbert effect algebras of a given dimensionality and building up composite systems via the DW tensor product. The competition between the availability of more states and the availability of fewer measurements makes this particularly interesting. But it also suggests the need for a new kind of operational structure—a new category, perhaps—that will give us the correct notion of tensor product in the quantum-mechanical case as well as the classical case. A variety of additional requirements have been imposed on effect algebras—convex structure, “S-domination,” existence of a sequential product—and these seem to get us closer to the Hilbert space structure (while still leaving open the possibility of classical effect algebras). But probably none is quite satisfactory yet, either for for characterizing the Hilbert space effect algebras, or for the somewhat different project I have suggested here, of creating a category (call it little category \( \mathcal{Z} \)) with the property (call it VOOM!) that the subcategory of Hilbert-space effect algebras is closed under formation of the \( \mathcal{Z} \) tensor product. With collaborators, I am attempting to find such a \( \mathcal{Z} \) which is a subcategory of the category of effect algebras.

In either \( \mathcal{EA} \) or \( \mathcal{Z} \), we could proceed to investigate general open-system conditional dynamics. On the other hand, dynamics could provide a source of requirements helping to define the category \( \mathcal{Z} \). For instance, we could require that the first two definitions of complete positivity, when appropriately generalized to the category \( \mathcal{Z} \), coincide. We could investigate the generalized measurement problem in \( \mathcal{Z} \). For starters, we could require that any measurement \( M \) that can be done on \( A \in \mathcal{Z} \) can also be done by interacting \( A \) (probably reversibly) with \( B \) and measuring \( B \) (in the sense that both the measurement probabilities and the conditional dynamics associated with \( M \) can be achieved thus), and vice
versa, that any procedure of the type: reversibly interact and then measure the apparatus $B$ corresponds to a measurement in $Z$.

Sequential measurements raise the question of the relation of the operations considered to the parameter of time. Up to now, I have considered operations independently of the length of time they take; and that is perhaps reasonable, since the time they take may depend on how they are implemented. But clearly this is an important question for complexity theory. A related question is whether the set of operations implementable at one time is the same as that implementable later. Perhaps this is the operational version of the question of time translation invariance. In both classical and quantum theory the set of operations is time-translation invariant. In a general setting, this property is likely to make for a more tractable and probably more interesting theory. Possibly we will want to get by, in a truly operational theory, without an absolute notion of time: rather, we might define time by using some subsemigroup of dynamical operations as a clock (cf. e.g. [37]).

5 Distinguishability

An important tool in quantum information theory, and QIP theory, has been measures of distinguishability of two, possibly mixed, quantum states. A copious supply of such measures may be obtained in a general operational setting using a strategy which has proved useful in quantum information theory. We start by considering classical measures of distinguishability of probability distributions. Say our operational theory consists of a subcategory $\mathcal{F}$ of the category of effect algebras, together with a convex set $\mathcal{S}$ of states on algebras in this subcategory, and a set of possible dynamical evolutions on this subcategory (probably a convex subset of the morphisms of the subcategory). We simply define the distinguishability of two states $\rho, \omega \in \mathcal{S}$ as the maximum over effect-tests $\Sigma$ (sets of effects $e_i$ such that $\bigoplus_i e_i = 1$) of the distance between the classical probability distributions $p_{\Sigma,\rho}$ and $p_{\Sigma,\omega}$ induced by $\rho$ and $\omega$ on the outcomes in $\Sigma$:

$$D_{\mathcal{F}}(\omega, \rho) := \max_{\Sigma} D_{cl}(p_{\Sigma,\rho}, p_{\Sigma,\omega}) .$$

The question then arises: when $D_{cl}$ is nonincreasing under classical dynamics, is the induced $D_{\mathcal{F}}$ also nonincreasing, under the notion of dynamical evolution incorporated into $\mathcal{F}$? I suspect a counterexample can be found even in the quantum case. There has been extensive study of the distance measures which are contractive under the morphisms appropriate to the operational category of standard nonrelativistic quantum systems (i.e., unital completely positive linear maps on operators interpreted as observables, or, dually, trace-preserving completely positive linear maps on the state space of such systems). Petz [38]
and Lesniewski and Ruskai [39] are particularly noteworthy. It may well be that the contractiveness of an appropriate set of distance measures is a principle that (combined with a tensor product structure of system composition which may well, as it does in the cases of orthoalgebras and effect algebras, automatically prohibit instantaneous inter-system influence) renders exponential speedup of brute-force search impossible.

Other concepts applicable to classical probability, such as entropy, can similarly be extended to the noncommutative case. Entropy may prove a bit tricky because, in the quantum-mechanical case, it needs to be minimized over fine-grained, sharp measurements (rank-one projectors), not just all measurements. Thus one might need to define sharp and finegrained measurements in $\mathcal{E}_A$; [40] and [41] considered these notions; also, the notions of principal and characteristic elements (cf. [9]) may be relevant.

### 6 Information-disturbance tradeoffs and quantum cryptography

One of the most remarked-on differences between quantum and classical mechanics is that in quantum mechanics our gaining information about the identity of an unknown state in general disturbs it. This difference is between the quantum analogue of a classical task, and the classical task. (Should one encode classical information into quantum states so that it can be reliably read out, as different states of an orthonormal basis, for instance, the information about which of these states obtains can be extracted without disturbing them.) Closely related is the impossibility of cloning an unknown quantum state. (If you could make an independent copy of an unknown quantum state, you could measure it and extract information without disturbing the original.) Quantitative expressions of this information-disturbance relationship as a monotonic tradeoff have been developed, for example, in [42], [25], [26] and [43]. Such tradeoffs could be studied in more general settings. The information-disturbance tradeoff lies at the heart of the possibility of eavesdropping-proof quantum distribution of classical secret key. Such protocols use nonorthogonal states (or similar devices involving entanglement) to encode prospective key-material. An eavesdropper who gains information about the key will reveal herself by the disturbance she causes to these states (some of which Alice and Bob sacrifice, by measuring them and discussing the results in order to detect disturbance). After they are satisfied there has been no eavesdropping, Alice and Bob are able to use the states, via some measurements and some discussion, to generate shared, secret, classical key.

At one time, some thought that quantum mechanics could be used for another purpose that is known to be classically impossible without relying on assumed
computational limitations of the parties: bit commitment. In a bit commitment protocol, Alice can perform some action which assures Bob that Alice has committed to a choice of value (0 or 1) for a bit. At some later time, Alice can perform an action which reveals that bit to Bob. In a successful protocol, nothing Bob can do allows him to gain more than a negligible amount of information about the bit before Alice reveals it, and nothing Alice can do allows her more than a negligible probability of being able to change the bit after she has committed to it. This is impossible in quantum mechanics, as in classical mechanics. The impossibility of bit-commitment is another candidate for the axiom-set $S$, of information-processing limitations shared by quantum and classical mechanics. This was suggested by Gilles Brassard; Chris Fuchs and Brassard also speculated that the combination of no-bit-commitment and eavesdropping-proof key distribution might single out quantum mechanics. This intriguing suggestion is spun into a thought-provoking fantasy in.

7 Axioms revisited

In the axiomatic approach to characterizing classical and quantum theories mechanics, one meets several types of axioms. Some, usually in the shared set $S$, set up a general framework for operational theories: axioms for effect algebras, convexity, and the like. Others might be dubbed “conceptual,”. One task conceptual axioms frequently do is distinguish classical from quantum mechanics. These are axioms like the existence, in some abstract sense, of superpositions of states or the requirement, within the convex approach, that the state space be a Choquet simplex (that the decomposition of states into extremal states be unique, which holds classically but not quantum-mechanically). Many axioms abstracted from quantum information considerations will fall into this class—axioms such as the existence of information-disturbance tradeoffs for ensembles of states with certain properties, or axioms about the behavior of measures of distinguishability of states. The “conceptual” content of these axioms varies from extremely operational and task-related (possibility/impossibility of specific types of protocols) through an intermediate level (conceptual properties viewed as probably responsible for certain tasks, e.g. information-disturbance tradeoffs) to relatively abstract and mathematical notions (such as the Choquet simplex axiom, or perhaps no-cloning) which nevertheless naturally arise when one looks at states abstractly as information.

It is likely, however, that in addition to such axioms a characterization of quantum and classical mechanics will involve what I will call “regularity axioms.” Probably these form a continuum with “conceptual” axioms, rather than a clear dichotomy, but the polarity is still relevant. Regularity axioms may impose a high degree of symmetry on structures, possibly through the introduction of a group action on them. Another example of regularity appears in
Hardy’s derivation of quantum mechanics. Here, the convex structures approach (redeveloped independently in a simple finite-dimensional version) gets one, as it has often been used to do, representations of probabilities as linear functionals. Then two key notions of “dimensionality” $N$ of a system (defined in an operational way not immediately dependent on the vector space notion, though of course in the end related to it) and of “number of degrees of freedom” $K$ are introduced. $K$ represents the number of measurement outcomes whose probabilities one needs to know in order to know the entire state (the probabilities of all measurement outcomes). In quantum mechanics, $K$ is the square of the dimension: one needs the probabilities of a set of effects which are linearly independent in the real vector space of Hermitian operators. Hardy postulates that a theory (or theory-type) must have a “universal” relation between $K$ and $N$, specifying one as a function of the other. I consider this an example of a “regularity” axiom, and it goes a very long way towards pinning us down to a choice between classical and quantum mechanics. Hardy’s composite system axiom, that $K$ and $N$ for composite systems are the products of those for the subsystems, has the flavor of regularity, but it may also arise conceptually, through as part of the notion of composite system—perhaps through an intimate relation of this notion with the category-theoretic one of tensor product. This is the sort of thing that deserves more investigation. In fact, one may sometimes discover alternative formulations of a regularity axiom, or of its conjunction with other axioms, that give it a more conceptual interpretation. The prospect of such discoveries is yet another reason for pursuing the axiomatic approach to operational theories.

I suspect that even after much conceptual content has been wrung from axioms initially motivated by regularity or symmetry, regularity and symmetry axioms will still remain an essential part of axiomatic formulations of quantum and classical mechanics as well as generalizations and alternatives. Investigation of the “irregular” structures that arise when they are relaxed will be an important part of understanding their meaning, and the meaning of the conceptual axioms that impose less regularity. For example, while finite-dimensional quantum and finite classical systems both obey Hardy’s universal $K - N$ relationship, the category of quantum systems with superselection sectors does not. (Of course, in other senses this is still a fairly “regular” category of theories, and I am not aware of a completely conceptual characterization of it.)

8 Conclusion

I hope to have convinced both the quantum structures community and the more abstractly inclined sectors of the quantum computation/information community that there are important contributions to both fields to be made by collaboration and exchange of tools, methods, and ideas. A crisper delineation
of what makes certain quantum protocols work, using notions from quantum logic, may not only lead to clearer understanding of these protocols but may suggest new ones. And considerations from quantum information, whether they involve the abstract approach of considering various kinds of operational theory as embodying different kinds of “information,” or the more concrete approach of considering the power of different kinds of operational theories to do particular tasks of classical information-processing, provide ideas for use in the quantum structures project of characterizing, classifying, and inventing types of operational physical theories. Critical to this project will be understanding the interplay between composition of physical systems, and the dynamics of those systems, in order to generalize the “modern” quantum mechanical formalism of effect-valued measures and completely positive maps, so effective in quantum information theory, to more general operational settings.

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References

[1] P. W. Shor, “Algorithms for quantum computation: discrete logarithms and factoring,” Proc. 37th ann. symp. on the foundations of computer science, pp. 56–65, 1994.

[2] P. W. Shor, “Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer,” SIAM J. Comp., pp. 1484–1509, 1997.

[3] Lov Grover, “A fast quantum mechanical algorithm for database search,” Proceedings of the 28th Annual ACM Symposium on the Theory of Computing (STOC), pp. 212–219, May 1998.

[4] Lov Grover, “Quantum mechanics helps in searching for a needle in a haystack,” Physical Review Letters, pp. 325–328, July 1997.

[5] C. H. Bennett and G. Brassard, “Quantum cryptography: Public-key distribution and coin-tossing,” in Proceedings of IEEE International Con-
ference on Computers, Systems, and Signal Processing, Bangalore, India. 1985, pp. 175–179, IEEE Press.

[6] G. W. Mackey, The mathematical foundations of quantum mechanics, W. A. Benjamin, New York, 1963.

[7] R. M. Cooke and J. Hilgevoord, “A new approach to equivalence in quantum logic,” in Current issues in quantum logic, E. Beltrametti and B. van Fraassen, Eds., New York and London, 1981, Plenum.

[8] R. Wright, “Spin manuals: empirical logic talks quantum mechanics,” in Mathematical foundations of quantum theory, A. Marlow, Ed., New York, 1978, pp. 177–254, Academic Press.

[9] D. J. Foulis, “Representations on uni-groups,” in Current research in operational quantum logic, B. Coecke, D. Moore, and A. Wilce, Eds., Dordrecht, 2000, Kluwer.

[10] A. Wilce, “Test spaces and orthoalgebras,” in Current research in operational quantum logic, B. Coecke, D. Moore, and A. Wilce, Eds., Dordrecht, 2000, Kluwer.

[11] S. Pulmannová, “Quantum logics and instruments,” Int. J. Theor. Phys., vol. 37, pp. 163–174, 1998.

[12] K. Kraus, States, Effects, and Operations, Springer-Verlag, Berlin, 1983.

[13] D. J. Foulis and M. K. Bennett, “Effect algebras and unsharp quantum logics,” Found. Phys., vol. 24, pp. 1325–1346, 1994.

[14] S. Gudder and S. Pulmannová, “Representation theorem for convex effect algebras,” Commentationes Mathematicae Universitatis Carolinae, vol. 39, pp. 645–659, 1998.

[15] D. J. Foulis, “Mathematical metascience,” J. Natural Geometry, vol. 13, pp. 1–50, 1998.

[16] D. Bilodeau, “Physics, machines, and the hard problem,” J. Consciousness Studies, vol. 3, pp. 386–401, 1996.

[17] L. Smolin, “The Bekenstein bound, topological quantum field theory and pluralistic quantum cosmology,” 1995, LANL ArXiv.org quant-ph/9508064

[18] R. Haag, Local quantum physics, Springer, Berlin, 1992.

[19] R. Haag and D. Kastler, “An algebraic approach to quantum field theory,” J. Math. Phys., vol. 5, pp. 848, 1964.

[20] M. Keyl, “How to describe the space-time structure with nets of C*-algebras,” Int J Theor Phys, vol. 37, pp. 375–385, 1993.
[21] D. S. Abrams and S. Lloyd, “Nonlinear quantum mechanics implies polynomial-time solution for NP-complete and #P problems,” *Phys. Rev. Lett.*, vol. 81, pp. 3992–3995, 1998.

[22] A. Peres, “Nonlinear variants of schrödinger’s equation violate the second law of thermodynamics,” *Phys. Rev. Lett.*, vol. 63, pp. 1114, 1989.

[23] A. Peres, “Weinberg’s nonlinear quantum mechanics and the einstein-podolsky-rosen paradox,” *Phys. Rev. Lett.*, vol. 66, pp. 397–400, 1991.

[24] S. Weinberg, ,” *Phys. Rev. Lett.*, vol. 62, pp. 485, 1989.

[25] Howard Barnum, “Quantum information theory,” 1998, UNM Doctoral Dissertation. Slightly corrected version available electronically at [http://info.phys.unm.edu/papers/papers.html](http://info.phys.unm.edu/papers/papers.html).

[26] H. Barnum, “Information-disturbance tradeoff in quantum measurement on the uniform ensemble (abstract),” *Proc. 2001 IEEE Intl. Symp. on Information Theory*, p. 277, 2001, Electronic version of full paper available from the author.

[27] R. Geroch, *Mathematical Physics*, University of Chicago Press, Chicago, London, 1985.

[28] A. Dvurečenskij, “Tensor product of difference posets and effect algebras,” *Int. J. Theor. Phys.*, vol. 34, pp. 1337–1348, 1995.

[29] A. Wilce, “A note on partial abelian semigroups,” *University of Pittsburgh preprint*, 1994.

[30] A. Wilce, “Perspectivity and congruence in partial abelian semigroups,” *Mathematica Slovaca*, vol. 48, pp. 117–135, 1998.

[31] D. J. Foulis and C. H. Randall, “Empirical logic and tensor products,” in Interpretations and foundations of quantum mechanics: proceedings of a conference hold in Marburg 28-30 May 1979, H. Neumann, Ed., Zürich, 1981, Bibliographisches Institut.

[32] C. H. Randall and D. J. Foulis, “Operational statistics and tensor products,” in Interpretations and foundations of quantum mechanics: proceedings of a conference hold in Marburg 28-30 May 1979, H. Neumann, Ed., Zürich, 1981, Bibliographisches Institut.

[33] M. K. Bennett and D. J. Foulis, “Tensor products of orthoalgebras,” *Order*, vol. 10, pp. 271–282, 1993.

[34] A. Wilce, “Tensor product of frame manuals,” *Int. J. Theor. Phys.*, vol. 29, pp. 805–814, 1990.
[35] C. Fuchs, “Notes on a Paulian idea: Foundational, historical, anecdotal and forward-looking thoughts on the quantum,” LANL arXiv.org e-print archive quant-ph/0105039, 2001.

[36] C. A. Fuchs, “Quantum mechanics as quantum information (and only a little more),” arXiv.org e-print quant-ph/0205039, 2001.

[37] C. Rovelli, “Quantum mechanics without time,” Physical Review D, vol. 42, pp. 2638–2646, 1990.

[38] D. Petz, “Monotone metrics on matrix spaces,” Linear algebra and its applications, vol. 244, pp. 81–96, 1996.

[39] A. Lesniewski and M. B. Ruskai, “Monotone Riemannian metrics and relative entropy on non-commutative probability spaces,” Journal of Mathematical Physics, vol. 40, pp. 5702–5724, 1999.

[40] S. Gudder, “S-dominating effect algebras,” Int. J. Theor. Phys., vol. 37, pp. 915–923, 1998.

[41] M. L. Dalla Chiara, “Unsharp quantum logics,” Int. J. Theor. Phys., vol. 34, pp. 1331–1336, 1995.

[42] C. Fuchs and A. Peres, “Quantum state disturbance vs. information gain: uncertainty relations for quantum information,” Physical Review A, vol. 53, pp. 2038, 1995.

[43] K. Banaszek, “Fidelity balance in quantum operations,” Phys. Rev. Lett., vol. 86, pp. 1366, 2001.

[44] D. Mayers, “The trouble with quantum bit commitment,” LANL e-print archive quant-ph/9603015, 1996.

[45] H.-K. Lo and H. F. Chau, “Is quantum bit-commitment really possible?,” Phys. Rev. Lett., vol. 78, pp. 3410–3413, 1997.

[46] C. A. Fuchs, “Quantum foundations in the light of quantum information,” arXiv.org e-print quant-ph/0106166, to appear in Proceedings of the NATO Advanced Research Workshop on Decoherence and its Implications in Quantum Computation and Information Transfer, ed. A. Gonis, 2001.