ON THE SPECTRAL RIGIDITY OF EINSTEIN-TYPE KÄHLER MANIFOLDS

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Abstract. We are concerned in this article with a classical question in spectral geometry dating back to McKean-Singer, Patodi and Tanno: whether or not the constancy of holomorphic sectional curvature of a complex n-dimensional compact Kähler manifold can be completely determined by the eigenvalues of its p-Laplacian for a single integer p? We treat this question in this article under two Einstein-type conditions: cohomologically Einstein and Fano Einstein. Building on our previous work, we show that for cohomologically Einstein Kähler manifolds this is true for all but finitely many pairs \((p, n)\). As a consequence, the standard complex projective spaces can be characterized among cohomologically Einstein Kähler manifolds in terms of a single spectral set in all these cases. Moreover, in the case of \(p = 0\), we show that the complex projective spaces can be characterized among Fano Kähler-Einstein manifolds only in terms of the first nonzero eigenvalue with multiplicity, which has a similar flavor to a recent celebrated result due to Kento Fujita.

1. Introduction and main results

Let \((M, g)\) be an \(m\)-dimensional connected, closed and oriented Riemannian manifold, \(\Omega^p(M) (0 \leq p \leq m)\) the space of smooth exterior \(p\)-forms on \(M\), \(d : \Omega^p(M) \to \Omega^{p+1}(M)\) the operator of exterior differentiation, and \(d^* : \Omega^p(M) \to \Omega^{p-1}(M)\) the formal adjoint of \(d\) relative to the Riemannian metric \(g\). Here \(\Omega^0(M) := 0\) provided that \(p = -1\) or \(m + 1\).

We have, For each \(0 \leq p \leq m\), the following second-order self-adjoint elliptic operator, the Laplacian acting on \(p\)-forms:

\[
\Delta_p := dd^* + d^* d : \Omega^p(M) \to \Omega^p(M).
\]

It is well-known that \(\Delta_p\) has an infinite discrete sequence of eigenvalues and each of them is repeated as many times as its multiplicity indicates. These \(\lambda_{k,p}\) are called the spectra of \(\Delta_p\). Put

\[
\text{Spec}^p(M, g) := \{\lambda_{1,p}, \lambda_{2,p}, \ldots, \lambda_{k,p}, \ldots\},
\]

which is called the spectral set of \(\Delta_p\). Duality and Hodge theory tell us that \(\text{Spec}^p(M, g) = \text{Spec}^{m-p}(M, g)\) and \(0 \in \text{Spec}^p(M, g)\) if and only if the \(p\)-th Betti number \(b_p(M) \neq 0\) and its multiplicity is precisely \(b_p(M)\).

An important problem in spectral geometry is to investigate how the geometry of \((M, g)\) can be reflected by its spectra \(\{\lambda_{k,p}\}\). In general the spectra \(\{\lambda_{k,p}\}\) are not able to determine a
manifold up to an isometry, as Milnor has constructed in [Mi64] two non-isometric Riemannian structures on a 16-dimensional manifold such that for each \( p \) the spectral sets \( \text{Spec}^p(\cdot) \) with respect to these Riemannian metrics are the same. Nevertheless, we may still ask to what extent the spectra \( \{\lambda_{k,p}\} \) encode the geometry of \((M, g)\).

Recall that, for any positive integer \( N \), the famous Minakshisundaram-Pleijel asymptotic expansion formula, which is the integration on the asymptotic expansion of the heat kernel for Laplacian, tells us
\begin{equation}
\text{Trace}(e^{-t\Delta_p}) = \sum_{k=0}^{\infty} \exp(-\lambda_{k,p}t) = \frac{1}{(4\pi t)^{\frac{m}{2}}} \left( \sum_{i=0}^{N} a_{i,p}t^i + O(t^{N-\frac{m}{2}+1}), \quad t \downarrow 0 \right),
\end{equation}
Here \( \text{Vol}(M, g) \) is the volume of \((M, g)\) and \( a_{i,p} \) \( (i \geq 1) \) are certain functions of the curvature, which are completely determined by the spectral set \( \text{Spec}^p(M, g) \). The coefficients \( a_{1,0} \) and \( a_{2,0} \) were calculated by Berger and McKea-Singer ([Be68], [MS67]) and then in [Pa70] Patodi explicitly determined \( a_{1,p} \) and \( a_{2,p} \) for all \( p \).

When \((M, g)\) is flat, i.e., has constant sectional curvature \( c = 0 \), then \( a_{i,p} = 0 \) for all \( p \) and \( i \geq 1 \) as these \( a_{i,p} \) are functions of the curvature. McKean and Singer raised in [MS67] a converse question: if \( a_{i,0} = 0 \) for all \( i \geq 1 \), then whether or not \((M, g)\) is flat? They proved in [MS67] that this is true if the dimension \( m \leq 3 \). Patodi further showed in [Pa70] that this is true if \( m \leq 5 \) and is false when \( m > 5 \) by constructing counterexamples ([Pa70, p. 283] or [Pa96, p. 65]). This means that in general the vanishing of \( a_{i,p} \) \( (i \geq 1) \) for only one single value \( p = 0 \) is not enough to derive the flatness. Nevertheless, applying the explicit expressions of \( a_{1,p} \) and \( a_{2,p} \) determined by himself in [Pa70], Patodi showed that whether or not \((M, g)\) is of constant sectional curvature \( c \) is completely determined by the quantities \( \{a_{i,p} \mid i = 0, 1, 2, \quad p = 0, 1\} \), i.e., by the spectral sets \( \text{Spec}^0(M, g) \) and \( \text{Spec}^1(M, g) \) ([Pa70, p. 281] or [Pa96, p. 63]).

The notion of “holomorphic sectional curvature” ("HSC" for short) in Kähler geometry is the counterpart of that of “sectional curvature” in Riemannian geometry and so it is natural to consider a similar question on Kähler manifolds. Note that if two Riemannian manifolds have the same spectral set \( \text{Spec}^p(\cdot) \) for some \( p \), then due to the asymptotic formula (1.3) they necessarily have the same dimension. Note also that for an \( m \)-dimensional Riemannian manifold we only need to consider the spectral sets \( \text{Spec}^p(\cdot) \) for \( p \leq \frac{m}{2} \) as \( \text{Spec}^p(\cdot) = \text{Spec}^{m-p}(\cdot) \).
In view of these two basic facts, we can now pose the following question in the Kähler version, which was initiated by Tanno in [Ta73].

**Question 1.1.** Suppose that \((M_1, g_1, J_1)\) and \((M_2, g_2, J_2)\) are two complex \( n \)-dimensional compact Kähler manifolds such that \( \text{Spec}^p(M_1, g_1) = \text{Spec}^p(M_2, g_2) \) for a fixed \( p \) with \( p \leq n \). Is it true that \((M_1, g_1, J_1)\) is of constant HSC \( c \) if and only if \((M_2, g_2, J_2)\) is so?

Recall that, up to a holomorphic isometry, \((\mathbb{C}P^n(c), g_0, J_0)\), the standard complex \( n \)-dimensional projective space equipped with the Fubini-Study metric with positive constant
HSC c, is the unique complex n-dimensional compact Kähler manifold with positive constant HSC c by the classical uniformization theorem. So we also have the following spectral characterization problem for \((\mathbb{C}P^n(c), g_0, J_0)\), which was first explicitly proposed by B.Y. Chen and Vanhecke in [CV80].

**Question 1.2.** Suppose that \((M, g, J)\) is a compact Kähler manifolds such that \(\text{Spec}^p(M, g) = \text{Spec}^p(\mathbb{C}P^n(c), g_0)\) for a fixed \(p\) with \(p \leq n\). Is it true that \((M, g, J)\) is holomorphically isometric to \((\mathbb{C}P^n(c), g_0, J_0)\)?

Clearly a positive answer to Question 1.1 implies that to Question 1.2. Tanno showed in [Ta73, p. 402] that Question 1.1 is true for \((p = 0, n \leq 5)\) and \((p = 0, n \leq 6)\) provided that the constant HSC \(c \neq 0\). In [Ta74, p. 129] he further showed that Question 1.1 is true for \((p = 1, 8 \leq n \leq 51)\). Consequently, Question 1.2 is also true in these cases ([Ta73, Theorem D], [Ta74, p. 129]). Chen and Vanhecke showed in [CV80] that Question 1.2 is true for \((p = 2, n\) except \(n = 8)\). Besides these results, Question 1.2 was also treated in some other literature and various results were claimed (see Remark 1.8) but unfortunately their proofs contain various mistakes and/or gaps, which have been clarified recently in [Li18, §2.3]. The purpose of the work in [Li18] is two-folds: to clarify some gaps in previously existing literature related to Question 1.2, and to settle Question 1.2 down affirmatively for each **positive** and even \(p\) in all dimensions \(n\) with at most two exceptions ([Li18, Theorem 1.3]).

As mentioned above, without any extra condition, a single spectral set is in general **not** enough to derive the constancy of sectional curvature in the Riemannian case. Nevertheless, Sakai showed that, with the condition of \((M, g)\) being Einstein, \(\text{Spec}^p(M, g)\) is indeed enough to derive the desired conclusion ([Sa71, Theorem 5.1]). The purpose of the present work is to treat this similar question for Kähler manifolds. Recall that a compact Kähler manifold is called **cohomologically Einstein** if its first Chern class and Kähler class are proportional. With this notion understood, building on the work in [Li18], our **first main result** in this article is the following

**Theorem 1.3.** Suppose that \((M_1, g_1, J_1)\) and \((M_2, g_2, J_2)\) are two complex n-dimensional compact Kähler manifolds such that

\[ a_{i,p}(M_1, g_1) = a_{i,p}(M_2, g_2), \quad i = 0, 1, 2, \]

for a fixed \(p\) with \(p \leq n\), \((M_1, g_1, J_1)\) is cohomologically Einstein, and \((M_2, g_2, J_2)\) is of constant HSC \(c\). Then \((M_1, g_1, J_1)\) is of constant HSC \(c\) if the pair \((p, n)\) satisfies one of the following conditions:

1. \(p = 0\) and \(n \geq 1\);
2. \(p = 1\) and \(n \geq 6\);
3. \(p = 2\) and \(n \neq 8\);
4. \(p \geq 3\) and \(p^2 - 2np + \frac{n(2n-1)}{3} \neq 0\).

**Remark 1.4.** Here the reason that the exceptional cases \((p = 1, n < 6)\) be not able to be dealt with is due to the negativity of some quantity related to \(p\) and \(n\) in these cases, which is required to be positive in our proof. The requirement that

\[ p^2 - 2np + \frac{n(2n-1)}{3} \neq 0 \]

arises from a constant in front of the expression \(a_{1,p}\). (see (2.5))
It turns out in [Li18, §5.2] that the positive integer solutions \((p, n)\) to the equation 
\[ p^2 - 2np + \frac{n(2n-1)}{3} = 0 \]
with \(p \leq n\) are precisely parametrized by positive integers \(k\), denoted by \((p_k, n_k)\), and satisfy the following recursive formula

\[
\begin{align*}
(p_1, n_1) &= (1, 3), \\
(p_{k+1}, n_{k+1}) &= (8n_k - 5p_k + 1, 19n_k - 12p_k + 3).
\end{align*}
\]

Direct calculations show that

\((p_2, n_2) = (20, 48), (p_3, n_3) = (285, 675), (p_4, n_4) = (3976, 9408), (p_5, n_5) = (55385, 131043), \ldots\),

whose distributions become more and more sparse as \(k \to \infty\).

Theorem 1.3 and the recursive formula (1.4) imply the following result

**Corollary 1.5.** Suppose that \((M_1, g_1, J_1)\) and \((M_2, g_2, J_2)\) are two complex \(n\)-dimensional compact Kähler manifolds such that \(\text{Spec}^p(M_1, g_1) = \text{Spec}^p(M_2, g_2)\) for a fixed \(p\) with \(p \leq n\), \((M_1, g_1, J_1)\) is cohomologically Einstein, and \((M_2, g_2, J_2)\) is of constant HSC \(c\). Then \((M_1, g_1, J_1)\) is of constant HSC \(c\) if the pair \((p, n)\) satisfies one of the following cases:

1. \((p = 0, \text{all dimensions } n)\);
2. \((p = 1, \text{all dimensions } n \geq 6)\);
3. \((p = 2, \text{all dimensions } n \text{ with at most one exception } n = 8)\);
4. \((p \geq 3 \text{ and } p \not\in \{p_k \mid k \geq 2\}, \text{all dimensions } n)\);
5. \((p = p_k, \text{all dimensions } n \text{ with at most one exception } n = n_k) \ (k \geq 2)\).

Here \((p_k, n_k) \ (k \geq 2)\) are determined by (1.4).

Consequently, Corollary 1.5 can be carried over to yield the same result when the HSC \(c > 0\), which amounts to \((M_2, g_2, J_2) = (\mathbb{C}P^n(c), g_0, J_0)\). In this situation we can, however, do one more case. Note that the exceptional case \((p = 2, n = 8)\) is not able to be dealt with in Corollary 1.5 due to the vanishing of a coefficient in the proof, which would be clear later (Lemma 3.4). Nevertheless, thanks to a recent breakthrough due to Fujita ([Fu18]) solving a long-standing conjecture in complex geometry, the difficulty in this exceptional case for \((\mathbb{C}P^n(c), g_0, J_0)\) can be successfully overcome, which has been explained in [Li18] and shall be briefly reviewed again at the end of Section 3, after the proof of Theorem 1.3 as well as Corollary 1.5. In summary, we have the following partial affirmative answer towards Question 1.2.

**Corollary 1.6.** Suppose that \((M, g, J)\) is a complex \(n\)-dimensional compact cohomologically Einstein Kähler manifolds such that \(\text{Spec}^p(M, g) = \text{Spec}^p(\mathbb{C}P^n(c), g_0)\) for a fixed \(p\) with \(p \leq n\). Then \((M, g, J)\) is holomorphically isometric to \((\mathbb{C}P^n(c), g_0, J_0)\) if the pair \((p, n)\) satisfies one of the following cases:

1. \((p = 0, \text{all dimensions } n)\);
2. \((p = 1, \text{all dimensions } n \geq 6)\);
3. \((p = 2, \text{all dimensions } n)\);
4. \((p \geq 3 \text{ and } p \not\in \{p_k \mid k \geq 2\}, \text{all dimensions } n)\);
(5) \((p = p_k, \text{ all dimensions } n \text{ with at most one exception } n = n_k) \) \((k \geq 2)\).

Here \((p_k, n_k) \) \((k \geq 2)\) are determined by (1.4).

When \(p\) is even and positive and \(\text{Spec}^p(M, g) = \text{Spec}^p(\mathbb{C}P^n(c), g_0)\), it turns out in [Li18, Lemma 4.3] that the condition of \((M, g, J)\) being cohomologically Einstein is automatically satisfied due to the Hard Lefschetz theorem. Also note that the positive integers \(p_k\) determined by the recursive formula (1.4) are even if and only if \(k\) are even. Hence we have the following affirmative answer to Question 1.2 for the following \((p, n)\) without any extra condition, which is precisely the main result in [Li18].

**Corollary 1.7.** Assume that \(p\) is even, positive and \(p \leq n\). Then

1. for \(p = 2\), Question 1.2 holds in all dimensions \(n\);
2. for \(p \geq 4\) and \(p \not\in \{p_{2k} \mid k \geq 1\}\), Question 1.2 holds in all dimensions \(n\);
3. for \(p = p_{2k}\), Question 1.2 holds in all dimensions \(n\) with at most one exception \(n = n_{2k} \) \((k \geq 1)\).

Here \((p_{2k}, n_{2k}) \) \((k \geq 1)\) are determined by (1.4).

**Remark 1.8.** As previously mentioned, Chen and Vanhecke settled Question 1.2 in [CV80] for the cases \((p = 2, \text{ all } n \text{ except } n = 8)\) in Corollary 1.7. The exceptional case \((p = 2, n = 8)\) left in [CV80] was treated by Goldberg in [Go84]. The main result in Corollary 1.6 was also claimed by Gauchman and Goldberg in [GG86, Theorem 1]. Unfortunately the proofs in [Go84] and [GG86] contain several gaps, which have been clarified in [Li18] (cf. [Li18, §2.3, Remark 4.2]). Nevertheless, the proofs in [Go84] and [GG86] still contain invaluable ideas, which, together with the recent result of Fujita in [Fu18], inspired our work [Li18].

We now state our second main result closely related to Question 1.2 in this article as well as the main result in [Fu18]. The case \(p = 0\) is particularly interesting as the spectral set \(\text{Spec}^0(\cdot)\) consists of the eigenvalues of the Laplacian on functions, and so it is more important to see if Questions 1.1 and 1.2 are true when \(p = 0\). As previously noted, we only know from [Ta73] that they hold in low dimensions. As is well-known among the set \(\text{Spec}^0(\cdot)\) the first nonzero eigenvalue, which is \(\lambda_{2,0}\) in our notation of (1.2), plays fundamental roles in various aspects in differential geometry. With this fact in mind, applying an integral formula of Bochner type essentially due to Lichnerowicz ([Lic69]) as well as a result of Tanno in [Ta69], we shall show the following second main result in this article.

**Theorem 1.9.** Assume that \((M, g, J)\) is a complex \(n\)-dimensional Fano Kähler-Einstein manifold such that its scalar curvature is normalized to be that of \((\mathbb{C}P^n(c), g_0, J_0)\), i.e., \(s_g = s_{g_0} = n(n + 1)c\). If the first nonzero eigenvalue \(\lambda_{2,0}(M, g)\) and its multiplicity of \((M, g, J)\) are the same as those of \((\mathbb{C}P^n(c), g_0, J_0)\), then \((M, g, J)\) is holomorphically isometric to \((\mathbb{C}P^n(c), g_0, J_0)\).

**Remark 1.10.** As previously mentioned, our treatment of the exceptional case \((p, n) = (2,8)\) in Corollaries 1.6 and 1.7 relies on Fujita’s recent result [Fu18, Theorem 1.1], which characterizes the standard complex projective spaces among Fano Kähler-Einstein manifolds in terms of their volumes when their metrics are normalized with the same constant scalar curvatures (cf. [Li18, Theorem 2.2]). Therefore Theorem 1.9 can also be compared to this result as another characterization in terms of the first nonzero eigenvalues with multiplicity.
The rest of this article is structured as follows. We recall in Section 2 some necessary notation and integral formulas set up in [Li18] and prove the main result, Theorem 1.3, in Section 3. Section 4 is then devoted to the proof of Theorem 1.9.

2. Preliminaries

In this section we shall recall some necessary notation and integral formulas involving in the curvature on compact Kähler manifolds, which rely on the tools developed in [Li18], and we refer the reader to [Li18] for more related details.

Assume now that $(M, g, J)$ is a compact Kähler manifold with complex dimension $n \geq 2$, i.e., $J$ is an integrable complex structure and $g$ a $J$-invariant Riemannian metric. Define

$$
\begin{align*}
\omega &:= \frac{1}{2\pi} g(J\cdot, \cdot), \text{ the Kähler form of } g, \\
\text{Ric}(g) &:= \text{the Ricci tensor of } g, \\
\text{Ric}(\omega) &:= \frac{1}{2\pi} \text{Ric}(g)(J\cdot, \cdot), \text{ the Ricci form of } g, \\
s_g &:= \text{Trace}_g \text{Ric}(g), \text{ the scalar curvature of } g, \\
\tilde{\text{Ric}}(\omega) &:= \text{Ric}(\omega) - \frac{s_g}{2\pi} \omega, \text{ the traceless part of } \text{Ric}(\omega).
\end{align*}
$$

It is well-known that the Ricci form $\text{Ric}(\omega)$ is a closed form representing the first Chern class $c_1(M)$, $g$ is Einstein if and only if $\tilde{\text{Ric}}(\omega) \equiv 0$, and, in our notation of $\omega$,

$$
\begin{align*}
\int_M c_1(M) \wedge [\omega]^{n-1} &= \int_M \text{Ric}(\omega) \wedge \omega^{n-1} = \frac{1}{2n} \int_M s_g \cdot \omega^n, \\
\int_M c_1^2(M) \wedge [\omega]^{n-2} &= \int_M \text{Ric}(\omega)^2 \wedge \omega^{n-2} = \int_M \left( \frac{n-1}{4n} s_g^2 - |\tilde{\text{Ric}}(\omega)|^2 \right) \cdot \frac{\omega^n}{n(n-1)}, \\
\int_M s_g d\text{vol} &= \frac{(2n-1)!}{p!(2n-p)!} [p^2 - 2np + \frac{n(2n-1)}{3}] \int_M s_g d\text{vol},
\end{align*}
$$

With the notation understood, we have the following integral formulas (cf. [Li18, Lemma 3.5, Lemma 4.1]).

**Lemma 2.1.** Suppose that $(M, g, J)$ is a compact Kähler manifold with complex dimension $n \geq 2$. Then

$$
\begin{align*}
\int_M c_1(M) \wedge [\omega]^{n-1} &= \int_M \text{Ric}(\omega) \wedge \omega^{n-1} = \frac{1}{2n} \int_M s_g \cdot \omega^n, \\
\int_M c_1^2(M) \wedge [\omega]^{n-2} &= \int_M \text{Ric}(\omega)^2 \wedge \omega^{n-2} = \int_M \left( \frac{n-1}{4n} s_g^2 - |\tilde{\text{Ric}}(\omega)|^2 \right) \cdot \frac{\omega^n}{n(n-1)}, \\
\int_M s_g d\text{vol} &= \frac{(2n-1)!}{p!(2n-p)!} [p^2 - 2np + \frac{n(2n-1)}{3}] \int_M s_g d\text{vol},
\end{align*}
$$
and

\[ a_{2,p}(M, g) = \int_M \left\{ \left( \frac{2}{n(n+1)} \lambda_1 + \frac{1}{2n} \lambda_2 + \frac{1}{n+2} \lambda_3 \right) s_g^2 + \left( \frac{16}{n+2} \lambda_1 + 2 \lambda_2 \right) |\tilde{\text{Ric}}(\omega)|^2 + 4 \lambda_1 |B|^2 \right\} \text{dvol}, \]

where $|\tilde{\text{Ric}}(\omega)|^2$ and $|B|^2$ are their pointwise squared norms and

\[
\begin{cases}
\lambda_1 = \frac{1}{180} (2n)_p - \frac{1}{12} (2n-2)_{p-1} + \frac{1}{2} (2n-4)_{p-2}, \\
\lambda_2 = -\frac{1}{180} (2n)_p + \frac{1}{2} (2n-2)_{p-1} - 2 (2n-4)_{p-2}, \\
\lambda_3 = \frac{1}{72} (2n)_p - \frac{1}{6} (2n-2)_{p-1} + \frac{1}{2} (2n-4)_{p-2}.
\end{cases}
\]

In particular, if $g$ is of constant HSC $c$, then $s_g = n(n+1)c$ and thus (2.6) becomes

\[ a_{2,p}(M, g) = \int_M \left\{ \left( \frac{2}{n(n+1)} \lambda_1 + \frac{1}{2n} \lambda_2 + \lambda_3 \right) [n(n+1)c]^2 \text{dvol}, \quad (g: \text{constant HSC } c). \right. \]

Remark 2.2.

1. (2.3) and (2.4) are essentially due to Apte in [Ap55]. For more details and remarks on (2.3) and (2.4), we refer the reader to [Li18, Remark 3.6].

2. The explicit formulas for $a_{2,p}$ as well as $a_{1,p}$ was calculated by Patodi ([Pa70, p. 277] or [Pa96, p. 59]) in terms of various norms in the Riemannian setting. The relations between various norms arising from the curvature in Riemannian and Kähler manifolds were carefully investigated in [Li18, §3.1] and the current formula (2.6) in the Kähler version was obtained in [Li18, Lemma 4.1].

3. The factorial $!$ and binomial symbol \( \binom{t}{n} \) in (2.5) and (2.7) are understood to be 1, 1 and 0 if respectively $t = 0$, $v = 0$ and $v < 0$.

3. Proof of Theorem 1.3

With the preliminaries in Section 2 in hand, we can now proceed to prove Theorem 1.3.

We always assume in the sequel that the two complex $n$-dimensional compact Kähler manifolds $(n \geq 2)$ $(M_1, g_1, J_1)$ and $(M_2, g_2, J_2)$ satisfy the conditions assumed in Theorem 1.3. Namely, $(M_1, g_1, J_1)$ is cohomologically Einstein, $(M_2, g_2, J_2)$ is of constant HSC $c$ and $a_{i,p}(M_1, g_1) = a_{i,p}(M_2, g_2)$ for $i = 0, 1, 2$. Denote by the symbols $s_{g_i}$, $\omega_i$, $B_i$, etc. the corresponding quantities on $(M_i, g_i, J_i)$ ($i = 1, 2$).

The first observations are the following facts deriving from $a_{i,p}(M_1, g_1) = a_{i,p}(M_2, g_2)$ for $i = 0, 1$.

Lemma 3.1. Assume that $p^2 - 2np + \frac{n(2n-1)}{3} \neq 0$. Then

1. \[ \int_{M_1} \left\{ s_{g_1}^2 - [n(n+1)c]^2 \right\} \text{dvol} \geq 0, \quad \text{with equality if and only if the scalar curvature } s_{g_1} = n(n+1)c \text{ is a constant.} \]

2. \[ \int_{M_1} |\tilde{\text{Ric}}(\omega_1)|^2 = \frac{n-1}{4n} \int_{M_1} \left\{ s_{g_1}^2 - [n(n+1)c]^2 \right\} \text{dvol}. \]
Proof. Under the assumptions and via (2.5), we have

\begin{equation}
\text{Vol}(M_1, g_1) = \text{Vol}(M_2, g_2), \quad \int_{M_1} s_{g_1} \text{dvol} = \int_{M_2} n(n+1) \text{dvol}.
\end{equation}

Therefore,

\begin{align*}
\int_{M_1} s_{g_1}^2 \text{dvol} & \geq \frac{(\int_{M_1} s_{g_1} \text{dvol})^2}{\text{Vol}(M_1, g_1)} = \frac{(\int_{M_2} n(n+1) \text{dvol})^2}{\text{Vol}(M_2, g_2)} \quad \text{(3.3)} \\
& = \int_{M_2} [n(n+1)c]^2 \text{dvol} \\
& = \int_{M_1} [n(n+1)c]^2 \text{dvol}, \quad \text{(3.3)}
\end{align*}

where the equality holds if and only if \( s_{g_1} \) is a constant and hence \( s_{g_1} = n(n+1)c \). This completes the first part in this lemma. For the second part, note in (2.2) that \( \omega^n_i \) \((i = 1, 2)\) are volume forms up to a universal constant and \((M_1, g_1, J_1)\) being cohomologically Einstein means that \( c_1(M_1) \in \mathbb{R}[\omega_1] \). Therefore

\begin{align*}
\int_{M_1} \left( \frac{n-1}{4n} s_{g_1} - |\tilde{\text{Ric}}(\omega_1)|^2 \right) \cdot \frac{\omega^n}{n(n-1)} = & \int_{M_1} c_1^2(M_1) \wedge [\omega_1]^{n-2} \quad \text{(2.4)} \\
= & \frac{(\int_{M_1} c_1(M_1) \wedge [\omega_1]^{n-1})^2}{\int_{M_1} \omega^n_1} \quad \text{(c1}(M_1) \in \mathbb{R}[\omega_1]) \\
= & \frac{(\int_{M_1} s_{g_1} \omega^n_1)^2}{4n^2 \int_{M_1} \omega^n_1} \quad \text{(2.3)} \\
= & \frac{\left(\int_{M_2} n(n+1)c \omega^n_2\right)^2}{4n^2 \int_{M_2} \omega^n_2} \quad \text{(3.3)} \\
= & \frac{\left[n(n+1)c\right]^2}{4n^2} \int_{M_2} \omega^n_2 \\
= & \frac{\left[n(n+1)c\right]^2}{4n^2} \int_{M_1} \omega^n_1. \quad \text{(3.3)}
\end{align*}

Now rewriting (3.4) by singling out the term \(|\tilde{\text{Ric}}(\omega_1)|^2\) yields the desired equality (3.2). \(\square\)

Together with (3.2) in Lemma 3.1, the assumed condition \(a_{2,p}(M_1, g_1) = a_{2,p}(M_2, g_2)\) yields the following key equality.

**Lemma 3.2.** Assume that \(p^2 - 2np + \frac{n(2n-1)}{3} \neq 0\). Then

\begin{equation}
\text{\begin{array}{l}
\left[ \frac{4n+2}{(n+1)(n+2)} \lambda_1 + \frac{1}{2} \lambda_2 + \lambda_3 \right] \int_{M_1} \left\{ s_{g_1}^2 - [n(n+1)c]^2 \right\} \text{dvol} + 4\lambda_1 \int_{M_1} |B_1|^2 \text{dvol} = 0.
\end{array}}\end{equation}
Proof. The condition $a_{2,p}(M_1, g_1) = a_{2,p}(M_2, g_2)$ and the expressions (2.6) and (2.8) for $M_1$ and $M_2$ tell us that

\[(3.6)\]
\[
0 = \int_{M_1} \left( \frac{2}{n(n+1)} \lambda_1 + \frac{1}{2n} \lambda_2 + \lambda_3 \right) \left( s_{g_1}^2 - [n(n+1)c]^2 \right) d\text{vol}
+ \left( \frac{16}{n+2} \lambda_1 + 2\lambda_2 \right) \int_{M_1} |\tilde{\text{Ric}}(\omega_1)|^2 d\text{vol} + 4\lambda_1 \int_{M_1} |B_1|^2 d\text{vol}
= \int_{M_1} \left( \frac{2}{n(n+1)} \lambda_1 + \frac{1}{2n} \lambda_2 + \lambda_3 \right) \left( s_{g_1}^2 - [n(n+1)c]^2 \right) d\text{vol}
+ \left( \frac{16}{n+2} \lambda_1 + 2\lambda_2 \right) \left( \frac{n-1}{4n} \right) \int_{M_1} s_{g_1}^2 - [n(n+1)c]^2 d\text{vol} + 4\lambda_1 \int_{M_1} |B_1|^2 d\text{vol}
= \left( \frac{4n+2}{(n+1)(n+2)} \lambda_1 + \frac{1}{2} \lambda_2 + \lambda_3 \right) \int_{M_1} s_{g_1}^2 - [n(n+1)c]^2 d\text{vol} + 4\lambda_1 \int_{M_1} |B_1|^2 d\text{vol}.
\]

This yields the desired equality (3.5). \(\square\)

The inequality (3.1) and equality 3.5 allow us to affirmatively solve Question 1.1 in the following situations.

**Lemma 3.3.** If the pair $(p, n)$ satisfies $p^2 - 2np + \frac{n(2n-1)}{3} \neq 0$ and

\[(3.7)\]
\[
\frac{4n+2}{(n+1)(n+2)} \lambda_1 + \frac{1}{2} \lambda_2 + \lambda_3 > 0, \quad \lambda_1 > 0,
\]

then $(M_1, g_1, J_1)$ is of constant HSC $c$.

**Proof.** Put (3.1), (3.5) and Lemma 3.3 together, we deduce that

\[(3.8)\]
\[
\int_{M_1} \left( s_{g_1}^2 - [n(n+1)c]^2 \right) d\text{vol} = 0, \quad B_1 \equiv 0.
\]

The two equalities in (3.8), together with the equality characterization in (3.1), imply that the scalar curvature $s_{g_1} = n(n+1)c$ is constant and the Bochner curvature tensor $B_1$ vanishes. However, it is well-known that the constancy of $s_{g_1}$ and $c_1(M_1) \in \mathbb{R}[\omega_1]$ imply that $g_1$ is necessarily Einstein (cf. [Ti00, p. 19]). Therefore the Kähler metric $g_1$ is Einstein and has vanishing Bochner curvature tensor and hence of constant HSC, whose value is exactly $c$ as $s_{g_1} = n(n+1)c$. \(\square\)

At last, we arrive at the proof of Theorem 1.3 by showing the following technical result.

**Lemma 3.4.**

\[
\left\{ (p, n) \mid 0 \leq p \leq n, \ n \geq 2, \ and \ satisfy \ (3.7) \right\}
= \left\{ (p = 0, \ n \geq 2), \ (p = 1, \ n \geq 6), \ (p = 2, \ n \geq 2 \ and \ n \neq 8), \ (p \geq 3, \ all \ n \geq p) \right\}.
\]

**Proof.** For $p = 0$ and $p = 1$, we can easily check that exactly those $n$ with $n \geq 2$ and $n \geq 6$ respectively satisfy these restrictions. For $n \geq 2$ and $p \in [2, 2n - 2]$, we showed in detail in [Li18, Prop. 4.5, §5.1] that

\[(3.9)\]
\[
\frac{4n+2}{(n+1)(n+2)} \lambda_1 + \frac{1}{2} \lambda_2 + \lambda_3 > 0, \quad \lambda_1 \geq 0,
\]

with $\lambda_1 = 0$ if and only if $(p, n) = (2, 8)$. \(\square\)
Remark 3.5. Although we assume the evenness of $p$ in [Li18, Prop. 4.5] to be compatible with the statement in [Li18, Theorem 1.2], we can see through the proof in [Li18, §5.1] that it plays no role and (3.9) even holds for any real number $p \in [2, 2n - 2]$.

Now via Lemmas 3.3 and 3.4 the proof of Theorem 1.3 is completed and consequently so is Corollary 1.5.

Let us end our proof of Corollary 1.6 by briefly indicating that how the exceptional case $(p = 2, n = 8)$ can be dealt with in the case of $c > 0$, i.e., in the case of $(M_2, g_2, J_2) = (\mathbb{C}P^8(c), g_0, J_0)$, due to a recent result of Fujita ([Fu18]), which has been explained in detail in [Li18]. If $(p, n) = (2, 8)$, then $\lambda_1 = 0$, i.e., in (3.5) the coefficient in front of the term $\int_{M_1} |B_1|^2 \text{dvol}$ vanishes and from the proof of Lemma 3.3 we can not conclude that $B_1 \equiv 0$ but only conclude that the Kähler metric $g_1$ is Einstein with $s_{g_1} = n(n + 1)c$. Nevertheless, if the constant HSC $c$ in question is positive, then in this case $(M_1, g_1, J_1)$ is a Fano Kähler-Einstein manifold. Then an equivalent form of the main result in [Fu18] (cf. [Li18, Theorem 2.2] and the remarks before it) tells us that $(M_1, g_1, J_1)$ is holomorphically isometric to $(\mathbb{C}P^8(c), g_0, J_0)$.

4. Proof of Theorem 1.9

4.1 Preliminaries on vector fields and 1-forms. Assume throughout this subsection that $(M, g, J)$ is a complex $n$-dimensional compact Kähler manifold. In order to show Theorem 1.9, we need to recall some classical facts and results related to complex-valued vector fields and 1-forms on compact Kähler manifolds.

Due to the Kählerness, we can choose a (locally defined) orthonormal frame field of the Riemannian manifold $(M, g)$ in such a manner: $\{e_i, e_{i+n} = J e_i \mid 1 \leq i \leq n\}$. Then

$$\{u_i := \frac{1}{\sqrt{2}} (e_i - \sqrt{-1} J e_i) \mid 1 \leq i \leq n\}$$

is a $(1,0)$-type unitary frame field. Denote by $\{\theta^i \mid 1 \leq i \leq n\}$ the $(1,0)$-type unitary coframe field dual to $\{u_i\}$.

There is a one-to-one correspondence between complex-valued vector fields $X$ and 1-forms $\xi$ via the Kähler metric $g$ by $\xi(Y) = g(X, Y)$ for all complex vector fields $Y$. We denote by “$X \leftrightarrow \xi$” this correspondence. If we decompose $X$ and $\xi$ into $(1,0)$ and $(0,1)$-types: $X = X^{(1,0)} + X^{(0,1)}$ and $\xi = \xi^{(1,0)} + \xi^{(0,1)}$, then $X^{(1,0)} \leftrightarrow \xi^{(0,1)}$ and $X^{(0,1)} \leftrightarrow \xi^{(1,0)}$. To be more explicit,

$$X^{(1,0)} = \sum_{i=1}^n \alpha_i u_i \leftrightarrow \xi^{(0,1)} = \sum_{i=1}^n \alpha_i \bar{\theta}^i, \quad \alpha_i \in \mathbb{C}.$$

A vector field $X$ is called real holomorphic if it is real-valued and its $(1,0)$-part $X^{(1,0)}$ is holomorphic in the usual sense.

Let $\nabla$ be the complexified Levi-Civita connection on $(M, g, J)$ and write $\nabla = \nabla' + \nabla''$, where $\nabla' = \sum_{i=1}^n \theta^i \otimes \nabla_{u_i}$ and $\nabla'' = \sum_{i=1}^n \bar{\theta}^i \otimes \nabla_{\bar{u}_i}$.

With these notions understood, we collect some well-known facts in the following

Lemma 4.1. Assume that $X$ and $\xi$ are respectively complex-valued vector field and 1-form on $(M, g, J)$.
(1) A (real) killing vector field is real holomorphic. If $X$ is real holomorphic, then so is $JX$. If $X \leftrightarrow \xi$, then $JX \leftrightarrow J\xi$. Here the action of $J$ on 1-forms $\xi$ is canonically defined by $J(\xi)(Y) := -\xi(JY)$ for any vector field $Y$.
(2) If $X$ is of type $(1, 0)$ and $X \leftrightarrow \xi$ ($\xi$ is necessarily a $(0, 1)$-form), then $X$ is holomorphic if and only if $\nabla''\xi = 0$.
(3) If $X$ is real holomorphic and $X \leftrightarrow \xi$, then $\xi$ can be decomposed in a unique manner as
$$\xi = \xi^H + dh_1 + d^c h_2 = \xi^H + dh_1 + J(dh_2),$$
where $\xi^H$ is the harmonic part of $\xi$, $d^c := \sqrt{-1}(\bar{\partial} - \partial)$, and $h_i \ (i = 1, 2)$ are real-valued functions with vanishing integral. Moreover, $X$ is killing if and only if $h_1 = 0$.
(4) If $\xi = \sum_{i=1}^n \alpha_i \theta^i$ is a $(0, 1)$-form, then
$$\Delta_1(\xi) = 2([\nabla'']^* \nabla'' \xi + \sum_{i=1}^n \text{Ric}(e_i, e_i) \alpha_i \theta^i],$$
where $\Delta_1$ is the Laplacian acting on 1-forms in the notation of (1.1), $(\nabla'')^*$ the formal adjoint of $\nabla''$ relative to the metric $g$, and $\text{Ric}(\cdot, \cdot)$ the Ricci tensor of $g$.

Proof.

(1) The first part is quite well-known (cf. [Mo07, p. 107] or [Ko72, Thm. 4.3]). For the second part, only note that a real vector field $X$ is real holomorphic if and only if $\nabla_JX(Y) = J\nabla_X(Y)$ for all real vector fields $Y$ (cf. [Ma71, p. 6]). For the third part, $X \leftrightarrow \xi$ is equivalent to $\xi(Y) = g(X, Y)$ for any $Y$. Thus
$$(J\xi)(Y) = -\xi(JY) = -g(X, JY) = g(JX, Y).$$
The last equality is due to the $J$-invariance of the Kähler metric $g$.
(2) See [Ko72, Prop. 4.1].
(3) See [Mo07, p. 131] or [Ko72, Thm. 4.4]. Note that $d^c h_2 = J(dh_2)$ is due to the fact that the $(1, 0)$-forms and $(0, 1)$-forms are eigensubspaces of $J$ relative to the eigenvalues $-\sqrt{-1}$ and $\sqrt{-1}$ respectively.
(4) To the author’s best knowledge, the Bochner-type formula (4.1) should be due to Lichnerowicz in [Lic69, §9] (cf. [Ko72, p. 158]). We refer the reader to [Wu88, p. 310] for a thorough treatment on this kind of formulas.

$\square$

4.2. Proof of Theorem 1.9. With Lemma 4.1 in hand, we can now proceed to show Theorem 1.9. It is well-known that the first nonzero eigenvalue $\lambda_{2,0}(C P^n(c), g_0) = (n + 1)c$ whose multiplicity is exactly $n^2 + 2n$. Therefore we know through the assumptions made in Theorem 1.9 that
$$\text{Ric}(g) = \frac{s_g}{2n} g = \frac{(n + 1)c}{2} g$$
and
$$\lambda_{2,0}(g) = (n + 1)c \text{ with multiplicity } n(n + 2).$$

Let $f$ be an eigenfunction with respect to the first nonzero eigenvalue $\lambda_{2,0}(g) = (n + 1)c$, i.e., $\Delta_0 f = (n + 1)c f$. First we have the following claim.

Claim. The real vector field dual to the 1-form $J(df)$ is nontrivial and killing.
Proof.

\[
\Delta_1(df) = (dd^* + d^*d)(df) = (dd^*)f = d(dd^* + d^*d)f = d\Delta_0 f = (n + 1)c(df),
\]

which implies that

(4.4) \[\Delta_1(\bar{\partial}f) = (n + 1)c(\bar{\partial}f)\]

as the Laplacian preserves the types of forms on compact Kähler manifolds.

Denote by \(|\varphi|^2 := \int_M g(\varphi, \bar{\varphi})\) the global squared norm of a form \(\varphi\) on \((M, g, J)\). Now applying the Bochner-type formula (4.1) to the \((0,1)\)-form \(\bar{\partial}f\) and the facts (4.2) and (4.4) yields

\[
\lambda_{2,0}(g)|\bar{\partial}f|^2 = \int_M g(\Delta_1(\bar{\partial}f), \bar{\partial}f) = 2|\nabla''(\bar{\partial}f)|^2 + \frac{s_2}{n}|\bar{\partial}f|^2 \geq \frac{s_2}{n}|\bar{\partial}f|^2 = (n + 1)c|\bar{\partial}f|^2.
\]

Note that \(f\) is a real-valued non-constant function on \(M\) and so \(\bar{\partial}f\) is not identically zero, i.e., \(|\bar{\partial}f|^2 > 0\). Coupling this with (4.5) imply that \(\lambda_{2,0}(g) \geq (n + 1)c\), which, together with (4.3), tells us that the inequality (4.5) is indeed an equality. Therefore \(\nabla''(\bar{\partial}f) = 0\) and thus Lemma 4.1 says that the \((1,0)\)-type complex vector field dual to \(\bar{\partial}f\) is nontrivial and holomorphic. Hence the real vector field dual to \(df\), say \(W\), is nontrivial and real holomorphic. Therefore \(JW\) is also nontrivial and real holomorphic and \(JW \leftrightarrow J(df)\) due to Lemma 4.1. Note that the integral of \(f\) vanishes as \(\Delta_0(f) = (n + 1)cf\). Thus still by Lemma 4.1 we deduce that the real vector field dual to the \((1,0)\)-form \(J(df)\) is nontrivial and killing, which completes the proof of this claim. \(\square\)

Since the multiplicity of \(\lambda_{2,0}(g) = (n + 1)c\) is \(n^2 + 2n\) and so we have \(n^2 + 2n\) linearly independent eigenfunctions \(f_i\) \((1 \leq i \leq n^2 + 2n)\) and hence \(n(n + 2)\) linearly independent killing vector fields \(JW_i\), where \(W_i \leftrightarrow df\) \((1 \leq i \leq n^2 + 2n)\). In summary, we conclude that under the conditions assumed in Theorem 1.9, the dimension of the isometric group of the compact Kähler manifold \((M, g, J)\) is no less than \(n^2 + 2n\).

Recall an old result of Tanno ([Ta69]) that the dimension of the automorphism group of an almost Hermitian manifold preserving both the Hermitian metric and the almost-complex structure is no larger than \(n^2 + 2n\), with equality if and only if it is a standard complex projective space. Note in Lemma 4.1 that those \(n^2 + 2n\) linearly independent killing vector fields \(JW_i\) on the compact Kähler manifold \((M, g, J)\) are automatically real holomorphic, i.e., preserve the complex structure \(J\). Thus we yield the desired conclusion via Tanno's above-mentioned result.

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