THE HODGE GROUP AND ENDMORPHISM ALGEBRA OF AN ABELIAN VARIETY

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Abstract. This is an English translation of the author’s 1981 note in Russian, published in a Yaroslavl collection. We prove that if an Abelian variety over \( \mathbb{C} \) has no nontrivial endomorphisms, then its Hodge group is \( \mathbb{Q} \)-simple.

In this note we prove that if an Abelian variety \( A \) over \( \mathbb{C} \) has no nontrivial endomorphisms, then its Hodge group \( \text{Hg} \, A \) is a \( \mathbb{Q} \)-simple algebraic group. Actually a slightly more general result is obtained. The note was inspired by Tankeev’s paper [1]. The author is very grateful to Yu.G. Zarhin for useful discussions.

Let \( A \) be an Abelian variety over \( \mathbb{C} \). Set \( V = H_1(A, \mathbb{Q}) \). Denote by \( T^1 \) the compact one-dimensional torus over \( \mathbb{R} \): \( T^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). Denote by \( \varphi : T^1 \to \text{GL}(V_\mathbb{R}) \) the homomorphism defining the complex structure in \( V_\mathbb{R} = H_1(A, \mathbb{R}) \). By definition, the Hodge group \( \text{Hg} \, A \) is the smallest algebraic subgroup \( H \subset \text{GL}(V) \) defined over \( \mathbb{Q} \) such that \( H_\mathbb{R} \supset \text{im} \, \varphi \). Denote by \( \text{End} \, A \) the ring of endomorphisms of \( A \), and set \( \text{End}^0 \, A = \text{End} \, A \otimes \mathbb{Z} \, \mathbb{Q} \).

Theorem. Let \( A \) be a polarized Abelian variety. Let \( F \) denote the center of \( \text{End}^0 \, A \), and let \( F_0 \) denote the subalgebra of fixed points in \( F \) of the Rosati involution induced by the polarization. Set \( G = \text{Hg} \, A \) and denote by \( r \) the number of factors in the decomposition of the commutator subgroup \( G' = (G, G) \) of \( G \) into an almost direct product of \( \mathbb{Q} \)-simple groups. Then \( r \leq \dim_{\mathbb{Q}} F_0 \).

Corollary 1. If \( F = \mathbb{Q} \) (in particular, if \( \text{End}^0 \, A = \mathbb{Q} \)), then \( \text{Hg} \, A \) is a \( \mathbb{Q} \)-simple group.

Before proving the Theorem and deducing Corollary 1, we describe the necessary properties of the Hodge group.

Proposition (see [2]). Let \( A \) be an Abelian variety over \( \mathbb{C} \). Then
(a) The Hodge group \( G = \text{Hg} \, A \) is a connected reductive group.
(b) The centralizer \( K \) of \( \text{im} \, \varphi \) in \( G_\mathbb{R} \) is a maximal compact subgroup of \( G_\mathbb{R} \).
(c) \( G'_R \) is a group of Hermitian type (i.e., its symmetric space admits a structure of a Hermitian symmetric space).

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(d) The algebra $\text{End}^0 A$ is the centralizer of $Hg A$ in $\text{End} V$.

(e) For any polarization $P$ of $A$, the Hodge group $Hg A$ respects the corresponding nondegenerate skew-symmetric form $\psi_P$ on the space $V$.

ceeded to the Proposition, it follows that $C \subset F^*$. Hence, under the hypotheses of the corollary we have $C \subset Q^*$. From assertion (b) of the Proposition it follows that $C_R$ is a compact group, hence, $C$ is a finite group and $G = G'$. By virtue of the Theorem, $r = 1$ and $G$ is a $Q$-simple group.

**Lemma 1.** Let $H$ be a normal subgroup of $G$ defined over $Q$ such that the $R$-group $H_R$ is compact. Then $H \subset C$ (where $C$ denotes the center of $G$).

**Proof.** By assertion (b) of the Proposition we have $H_R \subset K$, where $K$ is the centralizer of $\im \varphi$ in $G_R$. Therefore, $\im \varphi$ is contained in the centralizer (defined over $Q$) $Z(H)$ of $H$ in $G$. Then it follows from the definition of the Hodge group that $G \subset Z(H)$. Hence $H \subset C$. □

**Lemma 2.** Consider the natural representation $\rho$ of the group $G'_R$ in the vector space $V_R$. Denote by $r_\rho$ the number of pairwise nonequivalent summands in the decomposition of $\rho$ into a direct sum of $R$-irreducible representations. Then $r_\rho = \dim_Q F_0$.

**Proof.** We set $A = \text{End}^0 A \otimes_Q R$ and write the decomposition

$$A_1 + \cdots + A_r,$$

of the semisimple $R$-algebra $A$ into a sum of simple $R$-algebras. By the assertion (d) of the Proposition we have $r_\rho = r_A$. Furthermore, it is known (see [3, Section 21]) that the Rosati involution acts on the center $F_i$ of the algebra $A_i$ trivially if $F_i = R$, and as the complex conjugation if $F_i = C$. It follows that

$$F_0 \otimes_Q R = \mathbb{R} + \cdots + \mathbb{R}$$

($r_A$ summands) whence $\dim_Q F_0 = r_A$. Thus $\dim_Q F_0 = r_A = r_\rho$. □

**Proof of the theorem.** Denote by $r_{nc}$ the number of noncompact groups in the decomposition

$$G'_R = G_1 \cdot G_2 \cdot \cdots \cdot G_N$$

of the group $G'_R$ in an almost direct product of simple $R$-groups. It is known from results of Satake [4, Theorem 2] that for each $R$-irreducible representation $\rho'$ in the decomposition of the representation $\rho$ into a direct sum of $R$-irreducibles, there exist not more that one noncompact group $G_i$ ($1 \leq i \leq N$) such that the restriction $\rho'|_{G_i}$ is nontrivial. Therefore, $r_{nc} \leq r_\rho$.

Taking in account Lemma 2, we obtain that $r_{nc} \leq \dim_Q F_0$.

Further, since $G'_R$ is of Hermitian type, we see that all the groups $G_1, \ldots, G_N$ are of Hermitian type as well, and hence they are absolutely simple. Consider the action of the Galois group $\text{Gal} (\overline{Q}/Q)$ on the set of the simple factors $G_1, \ldots, G_N$. The orbits of the Galois group bijectively correspond to $Q$-simple normal subgroups of $G'$. By Lemma 1 each orbit contains at
least one noncompact group $G_i$. Thus the number of orbits, i.e., the number $r$ of $\mathbb{Q}$-simple normal subgroups of $G'$, does not exceed the number $r_{nc}$ of noncompact groups among $G_1, \ldots, G_N$. We obtain that $r \leq r_{nc} \leq \dim_{\mathbb{Q}} \mathcal{F}_0$, which completes the proof of the theorem.

Corollary 2. Assume that $\text{End}^0 A = \mathbb{Q}$. Write the decomposition

$$\rho_C = \rho_1 \otimes \cdots \otimes \rho_N$$

of the irreducible representation $\rho_C$ of the semisimple group $G_C$ in the vector space $V_C$ into a tensor product of irreducible representations $\rho_i$ of the universal coverings $\tilde{G}_{iC}$ ($i = 1, \ldots, N$) of the simple factors $G_{iC}$ of $G_C$. Then each of the representations $\rho_i$ respects a nondegenerate skew-symmetric bilinear form, and the number $N$ is odd.

Proof. Indeed, by Corollary 1 the Galois group permutes transitively the groups $G_{iC}$ and the representations $\rho_i$. Now Corollary 2 follows from assertion (e) of the Proposition.

References

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