ABSTRACT. A birational transformation of the three-dimensional quadric is constructed. The inequalities of Fano are not fulfilled for this transformation.

MSC : 14E07, 14J45.
Fano’s inequality is also false for three-dimensional quadric

Marat Gizatullin*

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Introduction

It was shown in [3] that Fano’s inequalities are false for three-dimensional projective space. The goal of this article is to show the same thing for the case of three-dimensional quadric. Our considerations are parallel to [3], and in a sense, an additional text on the subject is not necessary, but if there exists an uncountable set of papers based on the wrong inequality, then it is worthy to publish at least two different texts explaining the truth.

Let $X$ be a non-singular threefold such that Pic($X$) = $\mathbb{Z}$, and the anti-canonical class ($-K_X$) is ample. If ($-K_X$) = $rH$ for a generator $H$ of the Picard group, then $X$ is said to be Fano threefold of index $r$ (and of the first kind). According to [1], Fano’s inequality is the statement:

For any birational transformation,

$$f: X \dasharrow X$$

defined by a linear system of degree $d > 1$ (the degree is the number defined by $f(mH) = dmH$) either there exists a point $P \in X$ such that

$$\text{mult}_P(f(|H|)) > 2d/r,$$

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*Department of Mathematics, Technical University Federico Santa María, Avenida España, No. 1680, Casilla 110-V, Valparaíso, Chile. e-mail mgizatul@mat.utfsm.cl
or there exists an irreducible curve $C \subset X$ such that
\[ \text{mult}_C(f(|H|)) > 2d/r. \]

Let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface in the four-dimensional projective space. $Q$ is a Fano threefold of index 3. Therefore, for this threefold, one can rewrite Fano’s inequality as follows.

For any birational transformation, \[ f: Q \dashrightarrow Q \]
defined by five homogeneous polynomials of the same degree $d$ and without a common two-dimensional set of zeros on $Q$,
\[ x'_i = f_i(x_0, x_1, x_2, x_3, x_4), \quad i = 0, 1, 2, 3, 4, \]
either there exists a point $P \in Q$ such that
\[ \text{mult}_P(f_i) > 2d/3 \]
for every $i = 0, 1, 2, 3, 4$, or there exists an irreducible curve $C \subset Q$ such that
\[ \text{mult}_C(f_i) > d/3 \]
for every $i$.

The goal of my article is to show that these inequalities do not take place for a certain birational transformation of degree $d=13$. That is, I write down the formulas for a birational transformation of degree 13 such that for the forms $f_0, \ldots, f_4$ defining the transformation, for any point $P \in Q$ and for any irreducible curve $C \subset Q$ one can see that
\[ \min_i(\text{mult}_P(f_i)) \leq 8 \quad \text{and} \quad \min_i(\text{mult}_C(f_i)) \leq 4. \]

**Construction of the example**

Let us consider the homogeneous coordinates $x_0, x_1, x_2, x_3, x_4$ for $\mathbb{P}^4$ as the normalized coefficients of a binary quartic form $F(T_0, T_1)$,
\[ F(T_0, T_1) = x_0 T_0^4 + 4x_1 T_0^3 T_1 + 6x_2 T_0^2 T_1^2 + 4x_3 T_0 T_1^3 + x_4 T_1^4, \]
that is we consider \( \mathbb{P}^4 \) as the projectivization of the vector space of binary quartics. Let \( I = I(x_0,x_1,x_2,x_3,x_4) \) be the quadratic (or polar ) invariant of the binary quartic,

\[
I = x_0 x_4 - 4 x_1 x_3 + 3 x_2^2, 
\]

\( J = J(x_0,x_1,x_2,x_3,x_4) \) be the cubic invariant (or the Hankel determinant ) of the quartic,

\[
J = \begin{vmatrix}
 x_0 & x_1 & x_2 \\
 x_1 & x_2 & x_3 \\
 x_2 & x_3 & x_4
\end{vmatrix}.
\]

\( I \) and \( J \) are ground (or basic) invariants of binary quartic, therefore, if both of them vanish simultaneously on a quartic, then the quartic is a null-form (or unstable form), that is it has a triple linear factor.

We define our quadric \( Q \) with the help of the quadratic invariant, \( Q: I(x_0,x_1,x_2,x_3) = 0 \).

We fix a parameter \( t \) and consider five following forms of degree 13.

\[
(f_t)_0 = x_0 J^4, \\
(f_t)_1 = x_1 J^4 + t x_0^4 J^3, \\
(f_t)_2 = x_2 J^4 + 2 t x_1 x_0^3 J^3 + t^2 x_0^7 J^2, \\
(f_t)_3 = x_3 J^4 + 3 t x_2 x_0^3 J^3 + 3 t^2 x_1 x_0^6 J^2 + t^3 x_0^{10}, \\
(f_t)_4 = x_4 J^4 + 4 t x_3 x_0^3 J^3 + 6 t^2 x_2 x_0^6 J^2 + 4 t^3 x_1 x_0^9 J + t^4 x_0^{13}.
\]

These five forms define a one-parameter family of rational maps
\[
g_t: \mathbb{P}^4 \rightarrow \mathbb{P}^4.
\]

If \( t = 0 \), all five forms have a common factor \( J^4 \). After cancelling this, we see that \( g_0 \) is the identity transformation. For our example, we need nonzero values of \( t \). If \( t \) is not zero, then it is clear that the five forms have no nonconstant common factor. Moreover,

\[
J\left( (f_t)_0, (f_t)_1, (f_t)_2, (f_t)_3, (f_t)_4 \right) = J(x_0,x_1,x_2,x_3,x_4)^{13}.
\]
In fact, this identity expresses the invariance of the Hankel determinant under triangular transformation of variables $T_0, T_1$. Using the latter identity, it is not hard to see that

$$(f_{-t})_i ((f_{t})_0, (f_{t})_1, (f_{t})_2, (f_{t})_3, (f_{t})_4) = x_i J^{42},$$

that is,

$$g_{(-t)} \circ g_t = \text{the identity transformation}.$$ 

Thus $g_t$ is rationally invertible and is a Cremona transformation. More generally,

$$g_s \circ g_t = g_{s+t},$$

and we get a one-parameter group of Cremona transformations. These transformations induce biregular automorphisms of an affine open subset of the projective space, the complement of the determinantal cubic hypersurface $J = 0$. Indeed, the above formula of the determinant transformation proves this (moreover, one can see below the exact calculation of the fundamental points of such a transformation).

Moreover

$$I(x'_0, x'_1, x'_2, x'_3, x'_4) = I(x_0, x_1, x_2, x_3, x_4) \cdot J^{12},$$

this identity expresses the invariance of the quadratic invariant under triangular transformation of variables $T_0, T_1$. Thus, the quadric $Q$ is invariant by the transformations.

**Remark.**

The formulas for $g_t$ can be generalized and some formulas for an infinite dimensional family of automorphisms of the complement to the determinantal hypersurface (or induced automorphisms of the complement of the determinantal hypersurface in the quadric) can be written. The generalization is similar to formulas written down on page 8 of the Max-Planck Institute preprint [2] and looks as follows. If $\phi_m(x, y)$ is any binary form of degree $m, \phi = \phi_m(x^3_0, J)$, then one can define the following transformations.

- $x'_0 = x_0 J^{4m}$,
- $x'_1 = x_1 J^{4m} + x_0 \phi J^{3m}$,
- $x'_2 = x_2 J^{4m} + 2x_1 \phi J^{3m} + x_0 \phi^2 J^{2m}$,
- $x'_3 = x_3 J^{4m} + 3x_2 \phi J^{3m} + 3x_1 \phi^2 J^{2m} + x_0 \phi^3 J^m$,
- $x'_4 = x_4 J^{4m} + 4x_3 \phi J^{3m} + 6x_2 \phi^2 J^{2m} + 4x_1 \phi^3 J^m + x_0 \phi^4$. 

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It is clear that
\[ J(x'_0, x'_1, x'_2, x'_3, x'_4) = J_{12}^{m+1}, \]
\[ I(x'_0, x'_1, x'_2, x'_3, x'_4) = I(x_0, x_1, x_2, x_3, x_4) \cdot J^{12m}. \]

Let us return to the family of transformations \( g_t \). We fix a nonzero value of the parameter \( t \), for example, \( t = 1 \), and consider the corresponding Cremona transformation
\[
\begin{align*}
x'_0 &= x_0 J^4, \\
x'_1 &= x_1 J^4 + x_0^4 J^3, \\
x'_2 &= x_2 J^4 + 2x_1 x_0^3 J^3 + x_0^7 J^2, \\
x'_3 &= x_3 J^4 + 3x_2 x_0^3 J^3 + 3x_1 x_0^6 J^2 + x_0^{10} J, \\
x'_4 &= x_4 J^4 + 4x_3 x_0^3 J^3 + 6x_2 x_0^6 J^2 + 4x_1 x_0^9 J + x_0^{13}.
\end{align*}
\]

First of all, we find the points \( P \in \mathbb{P}^4 \) where each form on the right hand side has positive multiplicity (that is, the set of all common zeros of these right hand sides, or, in other words, the fundamental points of the transformation).

The first right hand side vanishes if either \( x_0 = 0 \), or \( J = 0 \), or simultaneously \( x_0 = 0, J = 0 \). If \( x_0 = 0 \), but \( J \neq 0 \), then using other four formulas, one sees that for other four coordinates of a fundamental point, the equalities \( x_1 = x_2 = x_3 = x_4 = 0 \) take place. This case is not a point of \( \mathbb{P}^4 \).

The case \( J = 0 \), but \( x_0 \neq 0 \) is also impossible for a fundamental point.

Thus, the fundamental points consist of the solutions of the following system of equations
\[ x_0 = 0, \quad J = 0, \]
in other words, the set of fundamental points is the hyperplane section (by the hyperplane \( H_0 \) defined by \( x_0 = 0 \)) of the Hankel determinantal threefold \( D \).

The determinantal hypersurface \( D \) has double points lying on the twisted quartic \( T \) parameterized by
\[ x_0 = t_0^4, \quad x_1 = t_0^3 t_1, \quad x_2 = t_0^2 t_1^2, \quad x_3 = t_0 t_1^3, \quad x_4 = t_1^4. \]

More precisely, the singular locus of the determinantal hypersurface is \( T \), and \( \text{mult}_P J = 2 \) for every \( P \in T \). Therefore any point not belonging to \( T \) has multiplicity one on the determinantal.
The next step. Let us consider the fundamental points of the induced transformation $f|Q$, that is intersection

$$Q \cap D \cap H_0.$$ 

Intersection $Q \cap D$ consists of the quartic null-forms. Binary quartic null-form $q$ has triple linear factor: $q = b(T_0, T_1) \cdot a^3(T_0, T_1)$. If such unstable binary quartic belongs to the hyperplane $H_0$, then the first coefficient $x_0$ of the quartic is zero, $T_1$ is a divisor of the unstable quartic. One can distinguish three cases:

- either the triple factor $a^3$ is not proportional to $T_1^3$, and $b$ is proportional to $T_1$,
- or $a = T_1$, and $b$ is not proportional to $T_1$,
- or $q = T^4$.

Case 1: $q = T_1(uT_0 + vT_1)^3$, $u \neq 0$.

The set $B$ of these points is isomorphic to the affine line, and it is clear that for this curve

$$\text{mult}_B(x_1J^4 + x_0^4J^3) = 4,$$

because $B \neq T$, and the points of $B$ are out of the singularities of $D$.

Case 2: $q = (uT_0 + vT_1)T_1^3$, $u \neq 0$.

The set $C$ of these points is isomorphic to the affine line, and it is clear that for this curve

$$\text{mult}_C(x_3J^4 + 3x_2x_0^3J^3 + 3x_1x_0^6J^2 + x_0^{10}J) = 4.$$ 

Case 3: $q = T_1^4$ $q$ is a double point of hypersurface $D$, and

$$\text{mult}_q(x_4J^4 + 4x_3x_0^3J^3 + 6x_2x_0^6J^2 + 4x_1x_0^9J + x_0^{13}) = 8.$$ 

References

[1] V.A. Iskovskikh, *Birational automorphisms of three-dimensional algebraic varieties*. Current problems in mathematics, VINITI, Moscow, vol. 12, (1979), 159-239.
[2] M.Gizatullin. *Examples of m-algebras*, Max-Planck-Institut für Mathematik, Preprint Series, 2000 (50).

[3] M.Gizatullin. *Fano’s inequality is a mistake*, E-preprint math.arXiv.org, math AG/0202069.

The Department of Mathematics,
Technical University Federico Santa María,
Avenida España, No. 1680, Casilla 110-V,
Valparaíso, Chile

e-mail mgizatul@mat.utfsm.cl