Comments on HKT supersymmetric sigma models and their Hamiltonian reduction

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Received 12 December 2014, revised 24 March 2015
Accepted for publication 8 April 2015
Published 1 May 2015

Abstract

Using complex notation, we present new simple expressions for two pairs of complex supercharges in HKT (‘hyper-Kähler with torsion’) supersymmetric sigma models. The second pair of supercharges depends on the holomorphic antisymmetric ‘hypercomplex structure’ tensor $I_{\alpha\beta}$ which plays the same role for the HKT models as the complex structure tensor for the Kähler models. When the Hamiltonian and supercharges commute with the momenta conjugate to the imaginary parts of the complex coordinates, one can perform a Hamiltonian reduction. The models thus obtained represent a special class of quasicomplex sigma models introduced recently by Ivanov and Smilga (2013 \textit{SIGMA} \textbf{9} 069)

Keywords: sigma models, supersymmetric mechanics, Kähler geometry

1. Introduction

Supersymmetric quantum mechanical (SQM) models describing the motion of a supersymmetric particle on a curved manifold have been studied since \cite{2}. Most of these problems represent a reformulation of classical problems of differential geometry. In particular, the model analyzed in \cite{2} boils down to the well-known de Rham complex.

The powerful supersymmetry formalism allows one to reproduce known mathematical results in a simple way. In this regard, one can mention the famous Atiyah–Singer theorem \cite{3}. A pure mathematical proof of this theorem is rather complicated. On the other hand, its
supersymmetric proof using the functional integral formalism [4] (see also [5, 6]) is transparent and beautiful.

But supersymmetry also makes it possible to derive new results. In particular, it allows one to construct new differential geometry structures not studied before by mathematicians. For example, the SQM model studied in [2] involving an extra potential is now called, ‘Witten deformation of the de Rham complex’. There are other deformations of the classical de Rham and Dolbeault complexes involving torsions [7, 8]. The hyper-Kähler with torsion (HKT) models (the subject of the present paper) were first introduced by physicists in the supersymmetric sigma model framework [9] (see also the earlier papers [10–13] where some elements of the HKT structure were displayed) and only then were described in purely mathematical terms [14, 15]. The lesser known CKT and OKT (‘Clifford Kähler with torsion’ and ‘octonionic Kähler with torsion’) models [16–18] are still awaiting full appreciation by mathematicians. The same is true of the recently discovered quasicomplex sigma models.

To find one’s way in this multitude of models, one needs road maps. We noticed in [19] that all of these models can be obtained from the trivial flat Dolbeault model with

\[ Q = \psi \sigma, \quad \bar{Q} = \bar{\psi} \bar{\sigma}, \quad H = \bar{\sigma} \pi \]

by means of two operations: (i) similarity transformation of complex supercharges and (ii) Hamiltonian reduction. In particular, a similarity transformation

\[ Q \to e^{i R} Q e^{-i R}, \quad \bar{Q} \to e^{-i R} \bar{Q} e^{i R}, \]

with \( R = \omega_{\mu} \bar{\psi} \psi \), applied to (1.1) gives a model describing a nontrivial Dolbeault complex. If the metric

\[ h_{\mu \bar{\nu}} = \left( e^{i \omega_{\mu} e^{-i \omega}} \right)_{\mu \bar{\nu}} \]

thus obtained does not depend on the imaginary parts of the complex coordinates \( z^m \), the momenta \( \pi_m = \bar{\sigma} m \) commute with the Hamiltonian and one can perform a Hamiltonian reduction giving a model with half as many bosonic degrees of freedom \( \{ \text{Re}(z^m) \} \). If the Hermitian metric (1.3) involves an imaginary part,

\[ h_{\mu \bar{\nu}} = \frac{1}{2} \left( g_{(\mu \bar{\nu})} + i h_{(\mu \bar{\nu})} \right), \]

we obtain a quasicomplex model [1] (the origin of the factor 1/2 in (1.4) will be clarified later). If \( h_{(\mu \bar{\nu})} = 0 \), we obtain a usual de Rham model of [2].

Both Dolbeault and de Rham models can have extended supersymmetries. The de Rham model with an extra pair of supercharges can be formulated for Kähler even-dimensional manifolds [20–22]. Mathematicians know this model as the Kähler–de Rham complex. There are also \( \mathcal{N} = 8 \) supersymmetric (i.e. including eight different real supercharges) de Rham models with three extra pairs of supercharges and defined on hyper-Kähler manifolds. A Dolbeault model with an extra pair of supercharges is called an HKT model\(^7\). If its metric does not depend on \( \text{Im}(z^m) \), one can perform a Hamiltonian reduction.

Our main observation is that the model thus obtained belongs to the class of quasi-complex models representing their special type. It enjoys \( \mathcal{N} = 4 \) supersymmetry.

\(^6\) The Hamiltonian can, of course, commute with any number of momenta. The corresponding Hamiltonian reductions give different models, some of which were discussed in [19]. In this paper, we will discuss only the Hamiltonian reduction with respect to all imaginary parts of \( z^m \).

\(^7\) HKT stands for hyper-Kähler with torsion. This name is probably a little bit misleading because these manifolds are not hyper-Kähler and not even Kähler, but a better one has not been invented.
The explicit component expressions for the HKT supercharges were derived in [24]. However, they were written in terms of real coordinates. To perform the Hamiltonian reduction described above, we need first to represent them in complex form. If they are expressed in proper terms, the corresponding expressions turn out to be very simple (see equation (3.18) below). This representation makes manifest the kinship between the mathematical structure of the HKT models and the structure of Kähler–de Rham models. The latter are characterized by the presence of a closed Kähler form. The components of this form define the complex structure tensor $I_{MN}$. Similarly, an HKT manifold is characterized by the presence of a closed holomorphic $(2, 0)$-form. Its components define a holomorphic tensor $\tilde{I}_{mn}$ which may be called a hypercomplex structure tensor.

The plan of the paper is the following. Section 2 presents a mathematical introduction where we translate many facts known to mathematicians into a language understandable to physicists. In section 3, after recalling how simple expressions for the supercharges can be derived in $\mathcal{N} = 2$ models (the main idea is to treat the fermions with world indices rather than the fermions with tangent space indices as basic dynamical variables), we present new nice generic expressions for the complex HKT supercharges as well as the supercharges obtained after their Hamiltonian reduction.

In section 4 (the central section of the paper), we discuss the Hamiltonian reduction procedure invoking the superfield formalism. A generic Dolbeault $\mathcal{N} = 2$ model is expressed via $(2, 2, 0)$ chiral superfields. When the metric depends only on the real parts of the coordinates, one can perform the Hamiltonian reduction with respect to the imaginary parts. The reduced model is described in terms of $(1, 2, 1)$ multiplets—the imaginary parts of the coordinates are traded for auxiliary fields. Likewise, $\mathcal{N} = 4$ HKT models are described using $(4, 4, 0)$ multiplets that involve four real or two complex coordinates. After reduction, the imaginary parts of the latter are traded for auxiliary fields and we are led to $(2, 4, 2)$. Generically, one obtains a deformed Kähler–de Rham complex which involves extra ‘quasicomplex’ terms. At the superfield level, such models involve, besides the familiar Kähler potential term, a holomorphic $F$-term of some special form (see equation (4.38)).

We emphasize that this type of Hamiltonian reduction differs from the Hamiltonian reduction for hyper-Kähler manifolds [26–28] and HKT manifolds [29] studied earlier. In [26–29], the reduction related models of the same type: hyper-Kählerian models to hyper-Kählerian ones and HKT ones to HKT ones. In our case, the reduction changes the geometry: a Dolbeault model gives after reduction a quasicomplex de Rham model and an HKT model gives a quasicomplex Kähler model.

Brief conclusions are drawn in the last section.

In appendix A, we discuss in detail how Hamiltonian reduction is described in Lagrangian component formalism. In appendix B, we present complete component Lagrangians of the original HKT theory with several interacting $(4, 4, 0)$ multiplets and of the quasicomplex Kähler–de Rham theory with several interacting $(2, 4, 2)$ multiplets. In appendix C, we give some technical details concerning establishing the correspondence between a generic HKT model admitting reduction and its reduced Kähler–de Rham quasi-complex daughter.

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8 Throughout the paper, the real tensor indices are denoted by upper-case italic letters $M, N, \ldots$, while lower-case italic letters $m, \bar{m}, \ldots$ are reserved for the holomorphic and antiholomorphic complex indices.

9 Unfortunately, the papers written by mathematicians and by theorists doing mathematical physics are written in rather different languages, even when they are devoted to basically the same subject. More often than not, they are not understandable to those from other disciplines, and translation is necessary.

10 We follow the notation of [25], such that the numerals count the numbers of physical bosonic, physical fermionic and auxiliary bosonic fields.
2. Two definitions of HKT manifolds, and their equivalence

We assume that the reader is familiar with the geometry of Kähler and hyper-Kähler manifolds. For a physicist reader, we can recommend the excellent review [30]. The basic facts are the following.

- A Kähler manifold is characterized by an antisymmetric complex structure tensor $I_{MN}$. The property $I_{MN} I^{MN} = -\delta^N_M$ holds. $I_{MN}$ is covariantly constant; $\nabla_P I_{MN} = 0$. It follows that the Kähler form $\Omega = I_{MN} dx^M \wedge dx^N$ is closed; $d\Omega = 0$.

- A generic complex manifold also involves an antisymmetric complex structure tensor $I$, but the standard covariant derivative $\nabla_P I_{MN}$ (with symmetric Christoffel symbols) does not necessarily vanish. $I$ should, however, satisfy certain integrability conditions:

$$\nabla_P \nabla^P = -2I_{PQ} I^{PQ}.$$  (2.1)

Equation (2.1) amounts to the vanishing of the so-called Nijenhuis tensor [12]. It is necessary to be able to define (anti)holomorphic coordinates $z^m, \bar{z}^\alpha$ with a Hermitian metric, $dx^2 = 2h_{mn} dz^m dz^n$ on the whole manifold. In addition, if (2.1) does not hold, nilpotent supercharges cannot be constructed. When the complex coordinates are chosen, the tensor $I_{MN}$ has the following nonzero components:

$$I_{m^m} = -I_{m^m} = -i \delta_m^m, \quad I_{m^m} = -I_{m^m} = i \delta_m^m.$$  (2.4)

It follows that $I_{m^m} = -I_{m^m} = -i h_{mn}$.

- As was mentioned, a standard covariant derivative of $I_{MN}$ does not generically vanish. However, for any $I$ satisfying the conditions above, one can define an affine connection

$$\hat{\Gamma}^M_{NK} = g^{MP} C_{LK}$$  (2.5)

with the torsion tensor $C_{LK}$ antisymmetric under $N \leftrightarrow K$ such that $\hat{\nabla}_P I_{MN} = 0$. If one requires the tensor $C_{LK}$ to be totally antisymmetric, such a connection is unique and is called the Bismut connection [31]. Explicitly,

$$C_{MNK}^{(Bismut)}(I) = I_{MN}^P I_{PQ}^R (\nabla_P I_{QR} + \nabla_Q I_{RP} + \nabla_R I_{QP}).$$  (2.6)

In complex coordinates, this tensor involves only the components of the type (2, 1) and (1, 2). The explicit expressions are [5]

$$C_{mnp} = C_{\bar{m}\bar{n}p} = C_{\bar{m}mn} = \partial_m h_{\bar{n}p} - \partial_\bar{n} h_{mp},$$

$$C_{mnp} = C_{\bar{m}\bar{n}p} = C_{\bar{m}m\bar{n}} = \partial_\bar{m} h_{np} - \partial_n h_{\bar{m}p}.$$  (2.7)

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11 Our index policy is the following. (i) Capital italic letters denote the indices in $\mathbb{R}^N$. (ii) Lowercase italic letters are reserved for the indices of holomorphic variables. (iii) In most cases, but not always, the indices of antiholomorphic variables are marked with a bar ($\bar{z}^\alpha$ etc). (iv) For reasons which will become clear later, the holomorphic indices in section 4.2.3 are Greek characters.

12 The Nijenhuis tensor is defined as

$$N_{JK} = I_{MN} \partial_M t^I_{JK} - I_{MN} \partial_M t^I_{JN}.$$  (2.2)

Its vanishing may be expressed as a condition:

$$\partial_M I_{MN}^P = I_{MN}^P \partial_P I_{MN}^P.$$  (2.3)

One can observe that one can also replace the usual derivatives in (2.3) with covariant ones. Lowering the index $P$ then gives (2.1).
A hyper-Kähler manifold has three different antisymmetric covariantly constant complex structures $I, J, K$ satisfying the quaternion algebra

$$I^2 = J^2 = K^2 = -1, \quad IJ = K, \quad JK = I, \quad KI = J.$$  \hfill (2.8)

Finally, we define a hypercomplex manifold as a manifold with three integrable quaternionic complex structures whose standard covariant derivatives do not necessarily vanish. The real dimension of a hypercomplex manifold is an integer multiple of 4—the same as for the hyper-Kähler manifolds.

We go over now to the HKT manifolds. There are two equivalent definitions:

**Definition 1.** An HKT manifold is a hypercomplex manifold where the complex structures satisfy an additional constraint: they are covariantly constant with one and the same torsionful Bismut affine connection:

$$C_{MNK}(I) = C_{MNK}(J) = C_{MNK}(K).$$  \hfill (2.9)

**Definition 2.** An HKT manifold is a hypercomplex manifold where the $(2, 0)$-form

$$\omega = \Omega_J + i\Omega_K = (J + iK)_{MN} \, dx^M \wedge dx^N$$  \hfill (2.10)

(we will shortly see that it is holomorphic with respect to $I$) is closed;

$$\partial_I \omega = 0.$$  

We will give a proof here for half of the equivalence theorem (see e.g. [15] for the other half). Taking (2.9) as a basic definition (suggested originally in [9]), we construct the closed holomorphic $(2, 0)$-form. The existence of such a form was first proven in [32]. We follow here the much more physicist-user-friendly [14].

As a first step, we introduce two operators associated with the complex structure $I$ and acting on $n$-forms. The operator $\iota$ is defined according to

$$\iota_1 \omega = \omega_{M_1 \cdots M_n} \, dx^{M_1} \wedge \cdots \wedge dx^{M_n},$$ 

then

$$\iota_2 \omega = n\omega_{N_1 \cdots N_n} \, (I_{M_1}^N) \, dx^{M_1} \wedge \cdots \wedge dx^{M_n}. $$  \hfill (2.11)

For a form $\omega_{p,q}$ with $p$ holomorphic and $q$ antiholomorphic indices,

$$\iota_{p,q} \omega = i(p - q) \omega_{p,q}. $$ \hfill (2.12)

Another operator $\omega \rightarrow I\omega$ is defined as

$$I\omega = I_{M_1}^{N_1} \cdots I_{M_n}^{N_n} \omega_{N_1 \cdots N_n} \, dx^{M_1} \wedge \cdots \wedge dx^{M_n}. $$ \hfill (2.13)

When acting on a form of the type $(p, q)$, it multiplies $\omega$ by the factor $i^{p - q}$.

Finally, besides using the usual exterior derivative $d$, we introduce the operator

$$d_I = [d, i].$$

Representing $d$ as the sum of the holomorphic and antiholomorphic (with respect to $I$) exterior derivatives, $d = \partial_I + \partial_J$, and using (2.12), we readily derive $d_I = i(\partial_I - \partial_J)$ (and hence $\partial_I = (d + i\partial_J)/2$). For an integrable $I$, complex coordinates can be chosen such that the
complex structure matrix (2.4) is constant. In this case, we can write a simple explicit expression for $d_I$:  
\[ d_I\omega = I_M^\mathcal{S} \partial_\mathcal{S}\omega_{N_1 \ldots N_n} \, dx^M \wedge dx^{N_1} \wedge \cdots \wedge dx^{N_n}. \]  
(2.14)

We now prove some simple lemmas.

**Proposition 1.**  
\[ d_I\omega = (-1)^n I \, d(\mathcal{I} \omega), \]  
(2.15)

where $n$ is the order of the form.

**Proof.** Choose the complex coordinates. Consider the rhs of (2.15) and use the complex expression (2.4) for $I$. The components $I_M^\mathcal{S}$ are thus constant and the partial derivatives do not act upon them. The form $d(\mathcal{I} \omega)$ has the order $n + 1$ and, according to (2.13), the expression $I \, d(\mathcal{I} \omega)$ has altogether $(n + 1) + n = 2n + 1$ factors of $I$. This involves $n$ pairs giving $I^2 = -1$ (this compensates for the factor $(-1)^n$) and we are left with just one unpaired factor. We obtain the expression (2.14). In contrast to the case for (2.14), the rhs of (2.15) has a tensorial form and is valid with any choice of coordinates.

**Proposition 2.** The form (2.10) has the type $(2, 0)$ with respect to $I$.

**Proof.** Using the definition (2.11) and the properties (2.8), it is easy to derive $\omega = 2i\omega$.

**Proposition 3.** For any complex manifold,
\[ d_I\omega = \frac{1}{3} C_{MPQ} \, dx^P \wedge dx^Q \wedge dx^\mathcal{I}. \]  
(2.16)

**Proof.** Choosing complex coordinates and bearing in mind (2.4), (2.14) and (2.7), we derive
\[ d_I\omega = C_{mnp} \, dz^m \wedge dz^n \wedge dz^q + C_{mpq} \, dz^m \wedge dz^p \wedge dz^q, \]
which coincides with (2.16).

**Corollary 1.** For the HKT manifolds where the Bismut torsions for $I, J, K$ coincide,
\[ d_I\Omega_J = d_J\Omega_I = d_K\Omega_K. \]  
(2.17)

**Proposition 4.** Let $I, J, K$ be quaternion complex structures. Then
\[ I\Omega_I = \Omega_I, \quad J\Omega_J = \Omega_J, \quad K\Omega_K = \Omega_K, \quad J\Omega_I = K\Omega_J = -\Omega_I, \quad I\Omega_J = K\Omega_I = -\Omega_J, \quad I\Omega_K = J\Omega_K = -\Omega_K. \]  
(2.18)

**Proof.** Let us prove the relation $J\Omega_I = -\Omega_I$. By definition,
\[ J\Omega_I = I_M^\mathcal{S} J_S^Q I_P^Q \, dx^M \wedge dx^\mathcal{I}. \]
On the other hand,
\[ J_M^P J_S^Q I_P^Q = -K_{MQ} J_S^Q = -iJ_{MS}. \]
Other relations are proved similarly.

**Remark 1.** The condition (2.17) can be rewritten bearing in mind (2.15) and the first line in (2.18) as
\[ I \, d\Omega_J = J \, d\Omega_J = K \, d\Omega_K. \] (2.19)

We are now ready to prove the main theorem:

**Theorem 1.**
\[ \partial_J (\Omega_J + i\Omega_K) = 0. \] (2.20)

**Proof.** The real and imaginary parts of (2.20) give Cauchy–Riemann conditions of a kind:
\[ d\Omega_J - d_J\Omega_K = 0, \quad d\Omega_K + d_J\Omega_J = 0. \] (2.21)
Consider the first relation. We obtain
\[ d_J\Omega_K \triangleq I \, d(I\Omega_K) \overset{\text{remark}}{=} -I \, d\Omega_K = -JK \, d\Omega_K \overset{\text{remark}}{=} -J^2 \, d\Omega_J = d\Omega_J. \]
The number ‘1’ above the equals sign means by virtue of proposition 1, and so on.
The relation \( d\Omega_K + d_J\Omega_J = 0 \) is proved similarly.

3. **Supercharges and reduced supercharges**

3.1. Dolbeault, de Rham, Kähler–de Rham, and quasicomplex systems

The classical supercharges of the best known de Rham SQM sigma model are usually presented in the form
\[ Q = \psi^M (P_M - i\Omega_{M,AB}\psi_A\psi_B), \]
\[ \bar{Q} = \bar{\psi}^M (P_M - i\Omega_{M,AB}\bar{\psi}_A\psi_B), \] (3.1)
where \( A, B \) are the tangent space indices, \( \psi_A = e_{AM}\psi^M, \quad g_{MN} = \epsilon_{AM}\epsilon_{AN} \), and
\[ \Omega_{M,AB} = \epsilon_{AN} \left( \partial_{ME} e_B^N + F_{MTR}^N e_B^R \right) \] (3.2)
are spin connections. The ‘flat’ fermion variables \( \psi_A, \bar{\psi}_A \) constitute, together with \( x^M, P_M \), the orthogonal canonically conjugated pairs.

For our purposes, it is more convenient to express the supercharges in terms of fermionic variables carrying world indices. The commutation relations are in this case more
complicated:
\[
\begin{align*}
\{x^M, \Pi_N\}_{\text{P.B.}} &= \delta^M_N, \\
\{\Pi_M, \psi^N\}_{\text{P.B.}} &= -\frac{1}{2} \partial_M g^NQ \psi_Q, \\
\{\pi_M, \psi^N\}_{\text{P.B.}} &= -\frac{1}{2} \partial_M g^NQ \psi_Q.
\end{align*}
\]  
(3.3)

([ } \text{ P.B. indicates a Poisson bracket.}) On the other hand, the expressions for the supercharges become much simpler: [1],
\[
Q = \psi^M \left( \Pi_M - \frac{i}{2} \partial_M g_{NP} \psi^N \psi^P \right),
\]
\[
Q = \psi^M \left( \Pi_M + \frac{i}{2} \partial_M g_{NP} \psi^P \psi^N \right). 
\]  
(3.4)

Note that the momenta \(P_M\) and \(\Pi_M\) are not the same. \(P_M\) is the variation of the Lagrangian over \(x^M\) while keeping \(\psi^A\) and \(\psi^A\) fixed. And \(\Pi_M\) is the variation of the Lagrangian over \(x^M\) while keeping \(\psi^M\) and \(\psi^M\) fixed. These two canonical momenta are related as follows [8]:
\[
P_M = \Pi_M + \frac{i}{2} \left( \partial_M e_{AP} e_{AQ} - \left( \partial_M e_{AQ} \right) e_{AP} \right) \psi^P \psi^Q.
\]  
(3.5)

The covariant quantum supercharges that act on the wavefunctions normalized with the measure \(\mu = \sqrt{\det(g)} \ d^3x\) have the same functional form as the operators \(\Pi_M = -i \partial/\partial x^M\) and \(\psi^M = g^{MN} \partial/\partial \psi^N\).

For Kähler manifolds, the de Rham complex can be extended to involve an extra pair of supercharges. Expressed in the same terms as in (3.4), they acquire a very simple form: [19],
\[
R = \psi^M \left( \Pi_M - \frac{i}{2} \partial_M h_{NP} \psi^N \psi^P \right),
\]
\[
R = \psi^M \left( \Pi_M + \frac{i}{2} \partial_M h_{NP} \psi^P \psi^N \right). 
\]  
(3.6)

Similar simple expressions can be derived for the supercharges of the Dolbeault complex:
\[
Q = \sqrt{2} \psi^m \left( \Pi_m - \frac{i}{2} \partial_m h_{\bar{m}} \psi^{\bar{m}} \psi^\bar{m} \right),
\]
\[
Q = \sqrt{2} \psi^m \left( \Pi_m + \frac{i}{2} \partial_m h_{\bar{m}} \psi^{\bar{m}} \psi^\bar{m} \right). 
\]  
(3.7)

When \(h_{mn}\) does not depend on \(\text{Im}(z^\alpha)\), one can perform a Hamiltonian reduction with the identification \(\Pi_m \equiv \Pi_{\bar{m}} \rightarrow \Pi_M/2\) [13]. If the \(h_{mn}\) are real, we obtain the de Rham supercharges (3.4) [14]. For a generic Hermitian metric (1.4), we obtain the supercharges of a quasicomplex model:

13 This implies the convention
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]
for each complex coordinate.

14 When carrying out this derivation, we have to take into account (1.4) and bear in mind that the canonical de Rham fermions \(\psi^M\) carrying the world indices and satisfying (3.3) have an additional factor \(1/\sqrt{2}\) compared to \(\psi^m\).
\[ Q = \psi^M \left[ \Pi_M - \frac{i}{2} \partial_M \left( g_{NP} + i b_{NP} \right) \psi^N \bar{\psi}^p \right], \]
\[ \bar{Q} = \bar{\psi}^M \left[ \Pi_M + \frac{i}{2} \partial_M \left( g_{NP} - i b_{NP} \right) \psi^N \bar{\psi}^p \right]. \tag{3.8} \]

### 3.2. HKT supercharges

The expressions for the four real supercharges in an HKT model were derived in \cite{24}. They are
\[ Q = \psi^M \left( P_M - \frac{1}{2} \Omega_{M,AB} \psi^A \psi^B + \frac{i}{12} C_{MNP} \psi^N \psi^p \right), \tag{3.9} \]
\[ Q^a = \psi^m \left( P_m - \frac{1}{2} \Omega_{m,AB} \psi^A \psi^B - \frac{i}{4} C_{MNP} \psi^N \psi^p \right), \tag{3.10} \]
where the \( \psi^M \) are here real fermions with \( \Omega^M_N \) and \( I^a = \{ I, J, K \} \).

We choose now complex coordinates \( x^M = \{ z^m, \bar{z}^\alpha \} \) and construct the complex combinations
\[ S = \frac{Q + i Q^l}{2}, \quad \bar{S} = \frac{Q - i Q^l}{2}, \quad R = \frac{Q^2 + i Q^3}{2}, \quad \bar{R} = \frac{Q^2 - i Q^3}{2}. \tag{3.11} \]

A short calculation gives
\[ S^{\text{HKT}} = \sqrt{2} \psi^m \left[ P_m - i \Omega_{m,kl} \psi^k \bar{\psi}^l \right], \]
\[ \bar{S}^{\text{HKT}} = \sqrt{2} \psi^m \left[ \bar{P}_m - i \bar{\Omega}_{m,kl} \psi^k \bar{\psi}^l \right], \tag{3.12} \]
\[ R^{\text{HKT}} = \sqrt{2} \psi^m \left[ \bar{P}_m - i \left( \Omega_{m,kl} + \frac{1}{2} C_{m,kl} \right) \psi^k \bar{\psi}^l \right], \]
\[ \bar{R}^{\text{HKT}} = \sqrt{2} \psi^m \left[ P_m - i \left( \bar{\Omega}_{m,kl} + \frac{1}{2} C_{m,kl} \right) \psi^k \bar{\psi}^l \right]. \tag{3.13} \]

where \( \Omega_{m,kl} = \Omega_{m,ab} e^a_k e^b_l \) and \( I = J + i K \).

It is noteworthy that in the expressions for \( S \) and \( \bar{S} \), the torsions \( C_{m,kl} \) and \( C_{m,kl} \) cancelled, so \( S, \bar{S} \) represent usual Dolbeault supercharges (cf (3.15) of \cite{5}). The torsions enter, however, in \( R \) and \( \bar{R} \). For hyper-Kähler manifolds, there are no torsions and the expressions (3.13) simplify.

Substituting the explicit expressions for \( \Omega \) and \( C \) via vielbeins:
\[ \Omega_{m,kl} = e^b_k \partial_m e^{a_l}, \quad \bar{\Omega}_{m,kl} = -e^b_k \partial_m e^{a_l}, \quad C_{m,kl} = -e^a_k \partial_m e^{b_l} + e_{(m} e^{a_l e^{b_k)}, \tag{3.14} \]
\[ \frac{1}{2} C_{m,kl} = -e^b_k \partial_m e^{a_l} + e_{(m} e^{a_l e^{b_k}, \quad \frac{1}{2} C_{m,kl} = -e^a_k \partial_m e^{b_l} + e_{(m} e^{a_l e^{b_k}, \tag{3.15} \]
we derive for the supercharges
\[ S^{\text{HKT}} = \sqrt{2} \psi^m \left[ P_m - i \left( e^b_k \partial_m e^{a_l} \right) \psi^k \bar{\psi}^l \right], \]
\[ \bar{S}^{\text{HKT}} = \sqrt{2} \psi^m \left[ \bar{P}_m + i \left( e^a_k \partial_m e^{b_l} \right) \psi^k \bar{\psi}^l \right]. \tag{3.16} \]
At the last step, we go over from the momenta $P_m, \bar{P}_m$ to the momenta $\Pi_m, \bar{\Pi}_m$ (which are relevant when $\psi^m$ and $\bar{\psi}^m$ rather than $\psi^a$ and $\bar{\psi}^a$ are treated as fundamental dynamic variables), according to (3.5). The supercharges take the nice simple form

$$S^{\text{HKT}} = \sqrt{2} \psi^m \left[ \Pi_m - \frac{i}{2} (\partial_m h_{\ell \ell}) \psi^k \bar{\psi}^l \right],$$

$$\bar{S}^{\text{HKT}} = \sqrt{2} \bar{\psi}^m \left[ \bar{\Pi}_m + \frac{i}{2} (\bar{\partial}_m \bar{h}_{\ell \ell}) \bar{\psi}^k \bar{\psi}^l \right].$$

(3.18)

$$R^{\text{HKT}} = \sqrt{2} \psi^a I_n^m \left[ \bar{\Pi}_m - \frac{i}{2} (\bar{\partial}_m \bar{h}_{\ell \ell}) \psi^k \bar{\psi}^l \right],$$

$$\bar{R}^{\text{HKT}} = \sqrt{2} \bar{\psi}^a \bar{I}_n^m \left[ \Pi_m + \frac{i}{2} (\partial_m h_{\ell \ell}) \bar{\psi}^k \bar{\psi}^l \right].$$

(3.19)

We observe a remarkable similarity with (3.4), (3.6). For an HKT manifold, the matrix $I_n^m$ plays the same role as the usual complex structure for the Kähler–de Rham complex. It can thus be called the matrix of hypercomplex structure. The form $I_{mn} \, dz^m \wedge \bar{d}z^n$ is closed, as dictated by (2.20).

When $h_{mn}$ does not depend on the imaginary coordinate parts, one can perform the Hamiltonian reduction. As an HKT manifold is a complex manifold of a special kind, we obtain after reduction a quasicomplex model of a special kind. The reduced supercharges are

$$S^{\text{quasi}} = \psi^M \left[ \Pi_M - \frac{i}{2} \partial_M (g_{KL} + i b_{KL}) \psi^K \bar{\psi}^L \right],$$

$$\bar{S}^{\text{quasi}} = \bar{\psi}^M \left[ \bar{\Pi}_M + \frac{i}{2} \partial_M (g_{KL} + i b_{KL}) \bar{\psi}^K \bar{\psi}^L \right].$$

(3.20)

$$R^{\text{quasi}} = \psi^N I_N^M \left[ \Pi_M - \frac{i}{2} \partial_M (g_{KL} + i b_{KL}) \psi^K \bar{\psi}^L \right],$$

$$\bar{R}^{\text{quasi}} = \bar{\psi}^N \bar{I}_N^M \left[ \bar{\Pi}_M + \frac{i}{2} \partial_M (g_{KL} + i b_{KL}) \bar{\psi}^K \bar{\psi}^L \right].$$

(3.21)

When the imaginary part of the metric $b_{KL}$ vanishes, the supercharges (3.20), (3.21) boil down to the Kähler supercharges (3.4), (3.6). When it does not, we are dealing with the Kähler quasicomplex model, to be discussed in more detail in the next section.

4. Hamiltonian reduction and superfields

Hamiltonians of supersymmetric systems are expressed in components, and Hamiltonian reduction is usually described in components too—see the component expressions for the reduced supercharges (3.8), (3.20), (3.21) in the previous section. But it is interesting and instructive to see what it corresponds to in the Lagrangian superfield formulation.
4.1. Dolbeault → quasicomplex de Rham

The Dolbeault complex is described using a set of chiral complex $(2, 2, 0)$ superfields $Z^m$ [5]. They are expressed in components as

$$Z^m = z^m + \sqrt{2} \theta \psi^m - i \theta \bar{\psi}^m. \quad (4.1)$$

The corresponding supersymmetry transformations are

$$\delta z^m = -\sqrt{2} \psi^m, \quad \delta \psi^m = i \sqrt{2} \bar{\psi}^m, \quad (4.2)$$

$$\delta \bar{z}^m = \sqrt{2} \bar{\psi}^m, \quad \delta \bar{\psi}^m = -i \sqrt{2} \psi^m. \quad (4.3)$$

We now set $\psi^m = \sqrt{2} \chi^m$, $z^m = x^m + iy^m$ to obtain

$$\delta x^m = -i \chi^m + \bar{\epsilon} \bar{\chi}^m, \quad \delta \epsilon^m = i (\epsilon x^m + \bar{\epsilon} \bar{\chi}^m), \quad (4.4)$$

$$\delta \bar{\chi}^m = i (\bar{\epsilon} \bar{x}^m - \bar{\epsilon} \bar{y}^m), \quad \delta \bar{\epsilon}^m = -i (\bar{\epsilon} \bar{x}^m + \bar{\epsilon} \bar{y}^m). \quad (4.5)$$

and observe that (4.2) coincides with the transformation law for a $(1, 2, 1)$ real superfield:

$$X^m \equiv X^M = x^M + \partial y^M + \bar{\chi}^M \bar{\theta} + B^M \partial \bar{\theta} \quad (4.6)$$

if we make the identification $y^m = \text{Im}(z^m) \equiv B^m$.

The Lagrangian of the pure Dolbeault sigma model (without a gauge field) is expressed via chiral superfields as

$$L = -\frac{1}{4} \int d\theta d\bar{\theta} \ h_{\text{mat}}(Z, \bar{Z}) DZ^m \bar{D} \bar{Z}^\theta, \quad (4.7)$$

with the Hermitian metric $h_{\text{mat}}$ and

$$D_{\theta} = \partial_{\theta} - i \partial_{\bar{\theta}}, \quad \bar{D}_{\bar{\theta}} = -\partial_{\bar{\theta}} + i \partial_{\theta}. \quad (4.8)$$

If the metric does not depend on $\text{Im}(z^m)$, one can perform the Hamiltonian reduction. The reduced Lagrangian should be expressed via the superfields $X^M$ as

$$L_{\text{reduced}} = -\frac{1}{2} \int d\theta d\bar{\theta} \ \left[ g_{\text{MN}}(X) + ib_{\text{MN}}(X) \right] DX^M \bar{D}X^N, \quad (4.9)$$

where $g/2$ and $b/2$ are the real and imaginary parts of the Hermitian metric $h$, according to (1.4). Heuristically, (4.9) is obtained from (4.7) by making the substitution $Z^m, \bar{Z}^m \rightarrow 2X^M$, while taking into account (1.4). When $h_{\text{mat}}$ is real, this is the usual de Rham model. When it involves an antisymmetric imaginary part, we arrive at the quasicomplex de Rham model of [1].

The fact that the reduction of (4.7) gives (4.9) looks very natural. It can be accurately derived in the following way. (i) Express the Lagrangian (4.7) in components. (ii) It is invariant under the shifts $y^m \rightarrow y^m + c^m$ (a corollary of the fact that the Hamiltonian commutes with the corresponding canonical momentum). In other words, it does not explicitly depend on $y^m$, but only on $\bar{y}^m$. (iii) Substitute $\sqrt{2} \chi^M$ for $\psi^m$ and $B^M$ for $y^m$. The result coincides with the component expansion of (4.9).

A more detailed justification of this procedure at the component level is given in appendix A.

15 The observation that the supertransformation laws for the multiplets with the same net number of fermionic and bosonic components, but with a different distribution of the latter among the dynamic and auxiliary fields, coincide under such identification was made a long time ago in [33, 34]. This was discussed in the Hamiltonian reduction context in [35] and in a gauging approach (when the Hamiltonian commutes with $\text{Im}(\Pi_m)$, one can impose the first-class constraint $\text{Im}(\Pi_m) = 0$ and treat the system as a gauge one) in [36].
4.2. HKT → quasicomplex Kähler

Consider now the $\mathcal{N} = 4$ supersymmetric HKT model. The Lagrangian is expressed via linear $\mathcal{N} = 4$ supermultiplets of the type $(4, 4, 0)$ [16, 38–40]. A $(4, 4, 0)$ multiplet lives in the $\mathcal{N} = 4$ superspace with the coordinates $(t, \theta^{ik})$, $(\overline{\theta}^{ik}) = -\epsilon_{ij} \epsilon_{k\ell} \theta^{j\ell} \equiv -\theta_{k\ell}$, where $i = 1, 2$ and $k' = 1, 2$ are doublet indices of the $SU(2)$ and $SU_R(2)$ groups respectively, which form the full automorphism group $SO(4) = SU_L(2) \times SU_R(2)$ of the $\mathcal{N} = 4$ superalgebra. Each multiplet carries a 4-vector or two spinor indices. Its component decomposition is

$$X^{il} = x^{il} - \theta^{ik} \alpha_k^{\ell} + 4 \theta^{ik} \theta_{k\ell} \overline{x}^{\ell},$$

and so it encompasses four real bosonic component fields $(\overline{x}^{ik}) = -\epsilon_{ij} \epsilon_{k\ell} x^{j\ell}$ and four real fermionic component fields $(\overline{\chi}^{ik}) = -\overline{\epsilon_{ij} \epsilon_{k\ell} x^{j\ell}}$.

The set $x^{ik}$ satisfying the pseudoreality condition can be represented as four real coordinates:

$$x^M = \frac{1}{2} \left( \sigma^M \right)_{k\ell} x^{ik}, \quad \sigma^M = (\bar{\sigma}, i),$$

or else as two complex coordinates $v^m, m = 1, 2$:

$$x^{ik} = \left( \begin{array}{c} v^2 \\ \bar{v}^1 \\ v^1 \\ -v^2 \end{array} \right).$$

The same holds for the superfields $\chi^{ik}$.

The second representation (via two complex coordinates) is convenient when performing the Hamiltonian reduction. We may adopt the representation $v^m = x^m + i y^m$ and express the laws of supersymmetry transformations via $x^m$ and $y^m$. Like for the case for the $\mathcal{N} = 2$ superfields, one can be convinced that these laws coincide with the supersymmetry transformations for the $(2, 4, 2)$ multiplet if we identify $y^m$ with the auxiliary fields $B^M$ (see [41] for the discussion of the reduction $(4, 4, 0) \rightarrow (2, 4, 2)$ in superfield language using the gauging procedure). Thus, to perform the Hamiltonian reduction using the Lagrangian language, one should merely make the substitution $y^m \rightarrow B^M$ in the component expression for the Lagrangian.

A wide class of HKT models are described via the superfield Lagrangian involving $n$ $(4, 4, 0)$ linear multiplets:

$$L = \int d^4 \theta \mathcal{L} (X_\alpha),$$

where $\alpha = 1, \ldots, n$ is the flavor index.

4.2.1. The four-dimensional model. Considered as the simplest example the model with only one multiplet, $n = 1$.

The simplest HKT metric is a conformally flat metric in four dimensions:

$$\text{d}x^2 = G(x) \text{d}x^M \text{d}x^M.$$
The complex structures can be chosen as
\[
I_M^N = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix},
J_M^N = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix},
K_M^N = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
\end{pmatrix},
\]
(4.15)

These are self-dual matrices expressible via ‘t Hooft symbols. The characteristic for an HKT manifold closed holomorphic form is
\[
\omega = \Omega_J + i\Omega_K = 2G(x)\,dv^1 \wedge dv^2,
\]
i.e. \( I_{mn} = G(x)\,\epsilon_{mnp} \). The complex supercharges (3.18) with this hypercomplex structure matrix were written down in [19, 24]. When \( G \) depends only on \( \text{Re}(\psi_m) \), one can perform a Hamiltonian reduction. After that, we are left with only one complex coordinate, \( z = \text{Re}(\psi) + i \text{Re}(\tilde{\psi}) \); the complex metric involves only one component and is real. We obtain the usual \( \mathcal{N} = 4 \) Kähler model on a manifold of real dimension 2.

Note that in [1] a certain nontrivial quasicomplex two-dimensional model was constructed and studied. It was observed that the spectrum of this model involves degenerate quartets and enjoys \( \mathcal{N} = 4 \) supersymmetry. The corresponding metric cannot be obtained, however, by a Hamiltonian reduction of an HKT metric, and the observed extended supersymmetry has a different origin.

4.2.2. The 4n-dimensional model. When \( n > 1 \), the situation becomes more complicated and more interesting. The eight-dimensional model described via two linear \((4, 4, 0)\) multiplets was studied in detail in [18]. We consider here a model with an arbitrary number \( n \) of such multiplets. Anticipating a subsequent reduction, it is convenient to use the complex notation and describe each of them as an \( \mathcal{N} = 4 \) superfield \( V_m^a(t; \theta, \eta, \tilde{\theta}, \tilde{\eta}) \), where the \( \theta, \eta, \tilde{\theta}, \tilde{\eta} \) are odd coordinates of the \( \mathcal{N} = 4 \) superspace and \( m = 1, 2 \). The fields \( V_m^a \) are related to \( \psi^a \) as in (4.12).

To make things quite transparent, we can give a representation in terms of \( \mathcal{N} = 2 \) superfields. Each superfield \( V_m^a \) is expressed via two \((2, 2, 0)\) complex superfields \( V_m^a \) and their conjugates. Its expansion in \( \eta, \tilde{\eta} \) reads [18]
\[
V_m^a = V_m^a + \eta \epsilon_m^{ab} \bar{V}_a^{b} - i\eta \bar{\epsilon}_m^{ab} \bar{V}_a^{b}, \quad \bar{V}_a^{b} = \bar{V}_a^{b} - \bar{\eta} \epsilon_a^{bc} \bar{V}_b^{c} + i\bar{\eta} \bar{\epsilon}_a^{bc} \bar{V}_b^{c},
\]
(4.17)
with \( D \equiv D_{\theta} \) and \( \bar{D} \equiv \bar{D}_{\theta} \) defined in (4.8). The \( V_m^a(\theta, \tilde{\theta}) \) are chiral superfields; \( D_{\theta} V_m^a = 0, \bar{D}_{\theta} \bar{V}_a^m = 0 \). They have a standard component expansion:
\[
\nu_m^a = \nu_m^a + \sqrt{2} \theta \psi_m^a - i\theta \bar{\psi}_m^a, \quad \bar{\nu}_a^m = \bar{\nu}_a^m - \sqrt{2} \bar{\theta} \psi_a^m + i\bar{\theta} \bar{\psi}_a^m.
\]
(4.18)

The superfield action is
\[
S = \frac{1}{4} \int dt \, d\theta \, d\tilde{\theta} \, d\eta \, d\tilde{\eta} \, L \left( V_m^a, \bar{V}_a^m \right) = \frac{1}{4} \int dt \, d\theta \, d\tilde{\theta} \left( \mathcal{L}_{\psi} \right) \, D_v^a \, \bar{D}_{\bar{\psi}}^b.
\]
(4.19)
where

\[ \Delta_{mn}^{\alpha\beta} \mathcal{L} \equiv \frac{\partial^2 \mathcal{L}(V, \bar{V})}{\partial V^m_{\alpha} \partial \bar{V}^\beta} + \epsilon_{mk} \epsilon_{nl} \frac{\partial^2 \mathcal{L}(V, \bar{V})}{\partial V^k_{\alpha} \partial \bar{V}^l_{\beta}}. \]  

(4.20)

Note that, for \( \alpha = \beta \),

\[ \Delta_{mn}^{\alpha\alpha} = \delta_{mn} \frac{\partial^2 \mathcal{L}}{\partial V^k_{\alpha} \partial \bar{V}^k_{\alpha}} \]  

(here with no summation with respect to \( \alpha \)).

(4.21)

The rhs of equation (4.19) expresses the action in terms of the fields. It is invariant under ‘hidden’ supersymmetry transformations [37]:

\[ \delta_\eta V^m_\alpha = -\epsilon_\eta \epsilon^{mn} \bar{D} \bar{V}^n_\alpha, \quad \delta_\eta \bar{V}^m_\alpha = \bar{\epsilon}_\eta \epsilon^{mn} D V^n_\alpha. \]  

(4.22)

Integrating it further over \( d^2\theta \), we obtain the component Lagrangian. Its bosonic part reads

\[ L_b = \left( \Delta_{mn}^{\alpha\beta} \mathcal{L} \right) \psi^m_\alpha \bar{\psi}^\beta. \]  

(4.23)

The Lagrangian (4.23) implies the target space metric

\[ ds^2 = \left( \Delta_{mn}^{\alpha\beta} \mathcal{L} \right) d\psi^m_\alpha d\bar{\psi}^\beta. \]  

(4.24)

The closed holomorphic form is

\[ \Omega = \frac{1}{2} \epsilon_{mk} \frac{\partial^2 \mathcal{L}}{\partial V^k_{\alpha} \partial \bar{V}^k_{\beta}} d\psi^m_\alpha \wedge d\bar{\psi}^\beta. \]  

(4.25)

4.2.3. The reduced model and its superfield description.

When \( L \) is real, the metric (4.24) is Hermitian, but not necessarily symmetric and real. To make it possible to perform the Hamiltonian reduction, we have to impose an extra constraint: for the metric to be independent of \( \text{Im}(\psi^m_\alpha) \). This implies also certain constraints on \( \mathcal{L} \). A generic admissible form of \( \mathcal{L} \) will be written out and discussed in appendix C. Here we write out a restricted ansatz for \( \mathcal{L} \), generating the metric with a constant part antisymmetric under \( \alpha \leftrightarrow \beta \):

\[ \mathcal{L} = 4\mathcal{K} - \frac{1}{2} C_{\alpha\beta} \left( \psi^1_\alpha \bar{\psi}^1_\beta - \psi^2_\alpha \bar{\psi}^2_\beta \right), \]  

(4.26)

with a real function \( \mathcal{K} \) generating the real symmetric part of the target space metric (4.24) that does not depend on \( \text{Im}(\psi^m_\alpha) \), and a real constant antisymmetric \( C_{\alpha\beta} = -C_{\beta\alpha} \) (the coefficients are chosen for further convenience).

Consider the second term in (4.26). Its contribution to the bosonic kinetic Lagrangian reads

\[ L_{\text{bos}}^{\text{red}} = -2C_{\alpha\beta} \left( \dot{\psi}^1_\alpha \bar{\psi}^1_\beta - \dot{\psi}^2_\alpha \bar{\psi}^2_\beta \right). \]  

(4.27)

18 See appendix B for the complete expression.
in the reduced Lagrangian. The presence of the structure (4.28) is characteristic of quasicomplex de Rham models—see equation (B.8). In our case, we are dealing with an \( \mathcal{N} = 4 \) model, a deformation of the Kähler–de Rham complex not studied before. To reveal the Kählerian nature of the reduced model, we introduce new complex coordinates (made up from the real parts of \( \alpha_{vm} \) and of \( B_M^a \)):

\[
z^n = x^n_a + i x^n_a, \quad A^a = B^a_\alpha + i B^a_\beta.
\]

(4.29)

In these variables, the full bosonic Lagrangian of the reduced model has the form

\[
L_b = h_{ab}(\zeta, \bar{\zeta}) \left( \dot{z}^a \dot{\zeta}^\beta + A^a \dot{\bar{\zeta}}^\beta \right) - C_{[ab]} \left( \dot{z}^a A^\beta + \dot{\zeta}^\beta \bar{A}^a \right),
\]

(4.30)

where

\[
h_{ab} = \frac{\partial^2 \mathcal{K}}{\partial z^a \partial \bar{z}^\beta}.
\]

(4.31)

Besides involving the familiar first term with the Kähler metric, the Lagrangian (4.30) also involves an extra term involving \( C_{[ab]} \). In modifies the full complex metric obtained after excluding the auxiliary fields \( A^a, \bar{A}^\beta \):

\[
h_{ab} = h_{ab} + C_{[ab]} h^{\beta \alpha} C_{[\bar{\rho} \bar{\beta}]},
\]

(4.32)

where \( h^{\beta \alpha}h_{\beta \beta} = \delta^\alpha_\beta, \ h_{\alpha \alpha}h^{\beta \beta} = \delta^\beta_\alpha \).

We will see now how this system is expressed in terms of the \((2, 4, 2)\) superfields. In contrast to the \( n = 1 \) case, where the extra terms in (4.30) were absent, and the reduced model is a well-known Kähler–de Rham model, with the Lagrangian representing a superspace integral of the Kähler potential, when \( n \geq 2 \), the superfield Lagrangian is somewhat complicated.

We consider a set of \( n \) chiral \((2, 4, 2)\) multiplets \( Z^a(\tau; \theta, \bar{\theta}, \eta, \bar{\eta}) \) satisfying the constraints

\[
\bar{D}_\eta Z^a = 0, \quad \bar{D}_\bar{\eta} Z^a = 0,
\]

(4.33)

where \( D_\eta, \bar{D}_{\bar{\eta}} \) are defined by (4.8) and

\[
D_\eta = \partial_\eta - i \eta \partial_\tau, \quad \bar{D}_{\bar{\eta}} = -\partial_{\bar{\eta}} + i \bar{\eta} \partial_{\bar{\tau}}.
\]

(4.34)

It is convenient to represent \( Z^a \) via \( \mathcal{N} = 2 \) superfields [5]: the usual chiral \((2, 2, 0)\) superfield \( Z^a(\theta, \bar{\theta}) \) and the superfield \( \Phi^a(\theta, \bar{\theta}) \) of the type \((0, 2, 2)\),

\[
Z^a = Z^a + \sqrt{2} \eta \Phi^a - i \eta \bar{\eta} Z^a,
\]

(4.35)

where

\[
Z^a = z^a + \sqrt{2} \partial_\theta \phi^a - i \partial_\bar{\theta} \phi^a, \quad \bar{D}_\eta Z^a = 0,
\]

\[
\Phi^a = \bar{\phi}^a + \sqrt{2} \partial_\bar{\theta} \bar{A}^a - i \partial_\theta \bar{\phi}^a, \quad \bar{D}_\bar{\eta} \Phi^a = 0.
\]

(4.36)

In (4.36), (4.37), the dynamical fields \( z^a \) and complex auxiliary fields \( A^a \) are bosonic whereas \( \phi^a, \bar{\phi}^a \) are fermionic.

Now, the standard Kähler model is described via the action \( -\int dt d\theta d\bar{\theta} d\eta d_{\bar{\eta}} \mathcal{K}(Z, \bar{Z}) \). We note that one can add to this expression \( F \)-terms of a certain particular
The action is expressed in terms of the \( \mathcal{N} = 2 \) superfields (4.36), (4.37) as follows:

\[
S = -\frac{1}{4} \int dt \, dq \, \bar{h}_{\dot{\alpha} \dot{\beta}}(Z, \bar{Z}) \left( \bar{D}Z^\alpha \bar{D}Z^\beta - 2 \Phi^{\alpha \dot{\beta}} \bar{\Phi}^{\dot{\alpha}} \right) + \frac{1}{\sqrt{2}} \left[ \int dt \, dq \, C_{[\alpha \beta]}(Z) \Phi^{\alpha} \bar{Z}^\beta + \text{c.c.} \right].
\]  

(4.39)

where

\[
C_{\alpha \beta} = F_{\alpha \beta} + Z^\gamma \partial_\gamma F_{\alpha \beta}.
\]  

(4.40)

The full component expression for this Lagrangian is written out in appendix B. Its bosonic part (B.4) depends on the holomorphic antisymmetric tensor \( C_{[\alpha \beta]} \). The expression (4.30) corresponds to the particular choice of constant real \( F_{\alpha \beta} = C_{\alpha \beta} \).

Alternatively, one can express the Lagrangian of this deformed Kähler model via an even number of real \((1, 2, 1)\) multiplets (obviously, \((2, 4, 2) = (2, 2, 0) + (0, 2, 2) = (1, 2, 1) + (1, 2, 1)\)). The model represents then a particular case of the quasicomplex de Rham model (4.9), with the metric \( g_{MN} \) and the real antisymmetric tensor \( b_{[MN]} \) having a particular form depending on the Hermitian \( h_{\alpha \beta} \) and the holomorphic \( C_{[\alpha \beta]} \) in equation (4.30).

5. Conclusions

We list again here the most significant original observations made in this paper.

(1) We derived the new simple representation (3.18) for the HKT supercharges. In contrast to the case in [24], the supercharges are expressed via complex coordinates and the fermion variables with world (rather than the tangent space) indices. The second pair of supercharges involves the holomorphic matrix \( I_{mn} \) of hypercomplex structure.

(2) We presented the new quasicomplex Kähler–de Rham model (4.38) where the Lagrangian involves, in addition to the standard Kähler structure, extra \( F \)-terms of a certain particular form.

(3) We have shown that models of this kind are obtained after a Hamiltonian reduction of HKT models. We discussed and justified the known recipe, according to which the canonical velocities corresponding to the variables subject to reduction in the original Lagrangian should be replaced by the auxiliary fields, \( \gamma^m \rightarrow B^M \). At the beginning of section 4, we showed how the supertransformation laws of the original multiplet and the reduced multiplet match. In appendix A, we explore how the Hamiltonian reduction works at the Lagrangian component level for a wide class of systems (not necessarily supersymmetric).

\[\text{19 This is specific to } d = 1. \text{ In } d \geq 2 \text{ field theories, it does not apply. Probably this is why such a structure was not considered before.} \]
It would be interesting to see how this procedure works for CKT and OKT models. What kinds of models would be obtained as a result of their Hamiltonian reduction? This question is being studied now.

Acknowledgments

We are indebted to Evgeny Ivanov for illuminating discussions.

SF acknowledges support from the Russian Foundation for Basic Research grants 12-02-00517, 13-02-90430 and a grant from the IN2P3-JINR Program. He would like to thank SUBATECH, Université de Nantes, for the warm hospitality enjoyed in the course of this study.

Appendix A. Hamiltonian reduction in component Lagrangian language

As was discussed in the main text, the reduced component Lagrangian is obtained from the original Lagrangian by trading time derivatives of the coordinates subject to reduction (in our case, time derivatives of the imaginary coordinate parts) by auxiliary fields. We will illustrate here how this works by means of an explicit calculation. Namely, we compare the reduced Hamiltonians obtained by (i) Hamiltonian reduction from the original one and (ii) Legendre transformation from the reduced Lagrangian, and show that they coincide.

Our starting point is the complex sigma model with the coordinates

\[ z^m = x^m + iy^m, \quad \bar{z}^m = x^m - iy^m. \] (A.1)

The metric tensor \( h_{mn} \) is Hermitian, but not necessarily real:

\[ h_{mn} = \frac{1}{2} \left( g_{(mn)} + i b_{(mn)} \right). \] (A.2)

For the case where the real tensors \( g_{(mn)} \) and \( b_{(mn)} \), and other structures in the Hamiltonian, do not depend on the imaginary parts \( y^m \), we can perform the Hamiltonian reduction. Disregard for simplicity the fermion variables (the recipe \( \dot{y} \rightarrow \dot{y} \) works actually not only for the SQM case where it is justified by comparing the supertransformation laws before and after reduction, but also for purely bosonic systems) and consider the Lagrangian

\[ L = h_{mn} \dot{z}^m \dot{\bar{z}}^n + G_m \dot{z}^m + \bar{G}_m \dot{\bar{z}}^m - V, \] (A.3)

where \( G_m, \bar{G}_m \) and \( V \) do not depend on the imaginary parts \( y^m \).

The corresponding Hamiltonian is

\[ H = (\bar{\pi} \bar{h} - \bar{\bar{G}})(h^{-1})^{mn} (\pi_m - G_m) + V. \] (A.4)

We now adopt the representation

\[ \pi_m = \frac{1}{2} \left( p_m^{(x)} - \dot{p}_m^{(y)} \right), \quad \bar{\pi}_m = \frac{1}{2} \left( p_m^{(x)} + \dot{p}_m^{(y)} \right), \] (A.5)

and perform the reduction. The reduced Hamiltonian is

\[ H^{\text{red}} = \frac{1}{4} (h^{-1})^{NM} \left( p_N - 2\bar{G} \right) \left( p_M - 2G \right) + V, \] (A.6)

We need not be concerned with their nature, although one can also note that, in the Dolbeault model that we are mostly interested in here (HKT models represent particular cases), \( G_m \) is associated with the gauge potential. In the full Lagrangian that also includes fermions, \( G_m \) contains in addition a term bilinear in fermions.
where \((h^{-1})^{NM} \equiv (h^{-1})^{mN}\). We are now using capital italic indices and are no longer displaying the superscript \(^{\alpha}\) for \(p\).

On the other hand, the reduced Lagrangian is obtained from (A.3) by making the substitution \(\dot{y}^m \rightarrow B^M (B^M \text{ being the real auxiliary fields}), and reads

\[
L^\text{red} = h_{MN} \left( \dot{x}^M + iB^M \right) \left( \dot{\bar{x}}^N - iB^N \right) + G_M \left( \dot{x}^M + iB^M \right) + G_M \left( \dot{\bar{x}}^M - iB^M \right) - V. \tag{A.7}
\]

Adopting the representation

\[
G_M = \frac{1}{2} \left( R_M + iM_M \right), \quad G_M = \frac{1}{2} \left( R_M - iM_M \right). \tag{A.8}
\]

and excluding \(B^M\), we obtain

\[
L^\text{red} = \frac{1}{2} G_{MN} \dot{x}^M \dot{\bar{x}}^N + \left[ R_M + b_{MK} \left( g^{-1} \right)^{KN} M_N \right] \dot{x}^M - \frac{1}{2} \left( g^{-1} \right)^{MN} M_M M_N - V, \tag{A.9}
\]

where

\[
G_{MN} = g_{MN} + b_{MK} \left( g^{-1} \right)^{KL} b_{LN}. \tag{A.10}
\]

Bearing in mind that the tensor \((h^{-1})^{NM}\) entering (A.6) is expressed as

\[
(h^{-1})^{NM} = 2 \left[ \left( g^{-1} \right)^{NM} - i \left( g^{-1} \right)^{NK} b_{KL} \left( g^{-1} \right)^{LM} \right]. \tag{A.11}
\]

it is a straightforward exercise to verify that (A.6) and (A.9) are related to each other by the standard Legendre transformation.

Appendix B. Component Lagrangians

B.1. The multiplets \((4, 4, 0)\)

The superfield action (4.19) of the interacting linear \((4, 4, 0)\) multiplets yields the component Lagrangian \(L = L_b + L_{2i} + L_{4i}\),

\[
L_b = \left( \Delta^{ab}_{m} \mathcal{L} \right) \bar{\psi}^m_a \psi^b, \tag{B.1}
\]

\[
L_{2i} = \frac{1}{2} \left( \Delta^{ab}_{m} \mathcal{L} \right) \left( \psi^m_a \bar{\psi}^b - \bar{\psi}^m_a \psi^b \right) + i \left( \partial \Delta_{m}^{ab} \mathcal{L} \right) \bar{\psi}^m_a \psi^b \bar{k} - i \left( \partial \Delta_{m}^{ab} \mathcal{L} \right) \psi^m_a \bar{\psi}^b \bar{k} + \frac{1}{2} \left( \psi^m_a \partial \bar{\psi}^b - \bar{\psi}^m_a \partial \psi^b \right) \left( \Delta^{ab}_{m} \mathcal{L} \right) \psi^m_a \bar{\psi}^b, \tag{B.2}
\]

\[
L_{4i} = \frac{1}{4} \epsilon_{a m} \epsilon_{b l} \left( \Delta^{ab}_{m} \Delta^{a b} \mathcal{L} \right) \psi^m_a \psi^b \bar{\psi}^l \bar{\psi}^l, \tag{B.3}
\]

where \(a_m = \partial \psi^m_a / \partial \alpha^a_m\) and \(a_l = \partial \psi^b_l / \partial \alpha^b_l\), and \(\Delta^{ab}_{m} \mathcal{L}\) is defined by (4.20).

B.2. The multiplets \((2, 4, 2)\)

The component Lagrangian \(\tilde{L} = \tilde{L}_b + \tilde{L}_{2i} + \tilde{L}_{4i}\) of the superfield action (4.38) of the interacting linear chiral \((2, 4, 2)\) multiplets has the following form:
\[ L_b = \left( \partial_\alpha \partial_\beta \mathcal{K} \right) \left( \partial^a \phi^a + A^a \tilde{A}^a \right) - C_{\alpha \alpha \beta} \tilde{\phi}^a - \tilde{C}_{\alpha \alpha \beta} \tilde{\phi}^a, \]  

(B.4)

\[ \tilde{L}_{2f} = \frac{1}{2} \left( \partial_\alpha \partial_\beta \mathcal{K} \right) \left( \phi^a \phi^a - \phi^\alpha \phi^a + \tilde{\phi}^a \phi^a - \tilde{\phi}^\alpha \phi^a \right) \]

\[- \frac{1}{2} C_{\alpha \beta} \tilde{\phi}^a \tilde{\phi}^a + \frac{1}{2} \tilde{C}_{\alpha \beta} \tilde{\phi}^a \tilde{\phi}^a \]

\[- \frac{1}{2} \left( \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \mathcal{K} \right) \tilde{\phi}^a \tilde{\phi}^a \]

\[ - \frac{1}{2} \phi^\alpha \phi^a \]

\[ - \frac{1}{2} \tilde{\phi}^\alpha \tilde{\phi}^a \]

\[ \left( \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \mathcal{K} \right) \tilde{\phi}^a \tilde{\phi}^a \]

\[ - \frac{1}{2} \phi^\alpha \phi^a \tilde{\phi}^a \tilde{\phi}^a \]

\[ - \frac{1}{2} \tilde{\phi}^\alpha \tilde{\phi}^a \tilde{\phi}^a \tilde{\phi}^a \]

\[ + \frac{1}{2} \phi^\alpha \phi^a \tilde{\phi}^a \tilde{\phi}^a \]

\[ + \frac{1}{2} \tilde{\phi}^\alpha \tilde{\phi}^a \tilde{\phi}^a \tilde{\phi}^a, \]  

(B.5)

\[ L_{4f} = - \left( \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \mathcal{K} \right) \phi^a \phi^a \tilde{\phi}^a \tilde{\phi}^a, \]  

(B.6)

where \( \partial_\alpha \equiv \partial / \partial \tilde{z}^\alpha \), \( \partial_\beta \equiv \partial / \partial \tilde{z}^\beta \) and \( C_{\alpha \beta} = \partial_\alpha (z^\gamma \partial_\gamma \tilde{z}^\beta) \), \( \tilde{C}_{\alpha \beta} = \partial_\alpha (\tilde{z}^\gamma \partial_\gamma \tilde{z}^\beta) \). Note the identities

\[ \partial_\alpha C_{\alpha \beta} + \partial_\beta C_{\alpha \gamma} + \partial_\gamma C_{\alpha \beta} = 0, \quad \partial_\alpha \tilde{C}_{\alpha \beta} + \partial_\beta \tilde{C}_{\alpha \gamma} + \partial_\gamma \tilde{C}_{\alpha \beta} = 0. \]  

(B.7)

It is instructive to compare the Lagrangian (B.4)–(B.6) with the Lagrangian of a generic quasicomplex \( \mathcal{N} = 2 \) model derived in [1]. The expression for the Lagrangian (4.9) in the components of (4.6) reads

\[ L = \frac{1}{2} \mathcal{g}_{MN} \left( \mathcal{X}^M \mathcal{X}^N + B^M B^N \right) + b_{[MN]} \mathcal{X}^M B^N + \frac{i}{2} \mathcal{g}_{MN} \left( \mathcal{X}^N \mathcal{V} \mathcal{X}^M - \mathcal{V} \mathcal{X}^N \mathcal{X}^M \right) \]

\[- \frac{1}{2} b_{[MN]} \left( \mathcal{X}^N \mathcal{X}^M - \mathcal{X}^M \mathcal{X}^N \right) - \frac{1}{2} \partial_r \partial_\tilde{Q} \left( \mathcal{g}_{MN} + i b_{[MN]} \right) \mathcal{X}^M \mathcal{X}^N + \mathcal{X}^Q \mathcal{X}^M \mathcal{X}^N \]

\[ + G_{MPQ} \mathcal{X}^M \mathcal{X}^P \mathcal{X}^Q - \frac{1}{2} \left( \partial_M b_{[MPQ]} + \partial_P b_{[QMP]} \right) \mathcal{X}^M \mathcal{X}^N, \]  

(B.8)

\[ G_{MPQ} = \Gamma_{MPQ} - \frac{i}{2} \left( \partial_M b_{[PQ]} + \partial_P b_{[QM]} + \partial_Q b_{[MP]} \right), \]  

(B.9)

with \( \Gamma_{MPQ} \) being the standard Christoffel forms for \( \mathcal{g}_{MN} \);

\[ \Gamma_{MPQ} = \frac{1}{2} \left[ \partial_P \mathcal{g}_{[MQ]} + \partial_Q \mathcal{g}_{[MP]} - \partial_M \mathcal{g}_{[PQ]} \right]. \]  

(B.10)

and

\[ \mathcal{V} \mathcal{X}^M = \mathcal{X}^M + \Gamma_{NQ}^M \mathcal{X}^N \mathcal{X}^Q. \]  

(B.11)

One can observe that the Lagrangian (B.8) involves among other terms the \( b \)-dependent four-fermion term \( \sim b^a \tilde{b}^a \tilde{b}^a \) and the terms \((\partial b) F \tilde{F} \), which do not have a counterpart in (B.5), (B.6). One can explicitly show that, in the \( \mathcal{N} = 4 \) case for particular \( b_{[MN]} \) depending on holomorphic \( C_{\alpha \beta} \), these contributions do indeed vanish.
Appendix C. Reduction of general HKT models

We will construct here a generic form of the HKT prepotential in (4.13) allowing reduction, and show that the bosonic action of the reduced model coincides with (4.30) with generic $\alpha\beta$.

We define the superfields

$$
\mathcal{V}_a^m = \lambda_a^m + i\phi_a^m, \quad \tilde{\mathcal{V}}^a = \lambda_a^1 + i\phi_a^3,
$$

with the bosonic component fields

$$
v_a^m = x_a^m + i\phi_a^m, \quad z^a = x_a^1 + i\phi_a^3, \quad \xi^a = y_a^1 + i\phi_a^2. \quad (C.1)
$$

We consider now the operator $\Delta_{12}^{\alpha\beta}$ entering (4.20) and express it in terms of $z, z, \xi, \bar{\xi}$:

$$
2\Delta_{12}^{\alpha\beta} \equiv \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} + \frac{\partial^2}{\partial z^\beta \partial \bar{z}^\alpha} + i \left( \frac{\partial^2}{\partial \xi^\alpha \partial \bar{\xi}^\beta} - \frac{\partial^2}{\partial \xi^\beta \partial \bar{\xi}^\alpha} \right),
$$

$$
2\Delta_{12}^{\alpha\beta} = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} + \frac{\partial^2}{\partial z^\beta \partial \bar{z}^\alpha} + i \left( \frac{\partial^2}{\partial \xi^\alpha \partial \bar{\xi}^\beta} - \frac{\partial^2}{\partial \xi^\beta \partial \bar{\xi}^\alpha} \right),
$$

$$
2\Delta_{12}^{\alpha\beta} = \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta} + \frac{\partial^2}{\partial z^\beta \partial \bar{z}^\alpha} + i \left( \frac{\partial^2}{\partial \xi^\alpha \partial \bar{\xi}^\beta} - \frac{\partial^2}{\partial \xi^\beta \partial \bar{\xi}^\alpha} \right).
$$

The second term in (4.26) is expressed via $\mathcal{Z}, \tilde{\mathcal{Z}}, \Xi, \tilde{\Xi}$ as

$$
-\frac{1}{2} C_{\alpha\beta} \left( \mathcal{Z}^\alpha \Xi^\beta + \tilde{\mathcal{Z}}^\alpha \tilde{\Xi}^\beta \right). \quad (C.4)
$$

It is linear in $\xi, \tilde{\xi}$, but the result of the action of (C.3) on (C.4) gives a constant not depending on the imaginary parts of $v_a^m$ entering $\xi, \tilde{\xi}$.

We now generalize (C.4) by introducing the following term in the prepotential:

$$
\Delta \mathcal{L} = -\frac{1}{2} \left[ F_{\alpha\beta}(\mathcal{Z}) \mathcal{Z}^\alpha \Xi^\beta + \tilde{F}_{\alpha\beta}(\tilde{\mathcal{Z}}) \tilde{\mathcal{Z}}^\alpha \tilde{\Xi}^\beta \right]. \quad (C.5)
$$

It is not difficult to observe that only the mixed terms in (C.3) involving both $z$ and $\xi$ derivatives give a nonzero result when acting on (C.5). The result does not depend on $\xi, \tilde{\xi}$ and is expressed in the form (4.30). QED.
References

[1] Ivanov E A and Smilga A V 2013 Quasicomplex N = 2, d = 1 supersymmetric sigma models SIGMA 9 069
[2] Witten E 1981 Dynamical breaking of supersymmetry Nucl. Phys. B 188 513
Witten E 1982 Supersymmetry and Morse theory J. Diff. Geom. 17 661
[3] Atiyah M F and Singer I M 1968 The index of elliptic operators Ann. Math. 87 484
Atiyah M F and Singer I M 1968 The index of elliptic operators Ann. Math. 87 546
Atiyah M F and Singer I M 1971 The index of elliptic operators Ann. Math. 93 119
Atiyah M F and Singer I M 1971 The index of elliptic operators Ann. Math. 93 139
[4] Alvarez-Gaume L 1983 Supersymmetry and the Atiyah-Singer index theorem Commun. Math. Phys. 90 161
Friedan D and Windey P 1984 Supersymmetric derivation of the Atiyah–Singer index and the chiral anomaly Nucl. Phys. B 235 395
[5] Ivanov E A and Smilga A V 2012 Dirac operator on complex manifolds and supersymmetric quantum mechanics Int. J. Mod. Phys. A 27 493
[6] Smilga A V 2012 Supersymmetric proof of the Hirzebruch–Riemann–Roch theorem for non-Kähler manifolds SIGMA 8 003
[7] Braden H W 1985 Sigma models with torsion Phys. Lett. B 163 171
Rohm R and Witten E 1986 The antisymmetric tensor field in superstring theory Ann. Phys. 170 454 NY
Kimura T 2007 Index theorems of torsional geometries J. High Energy Phys. JHEP08(2007)048
[8] Fedoruk S A, Ivanov E A and Smilga A V 2012 Real and complex supersymmetric d = 1 sigma models with torsions Int. J. Mod. Phys. A 27 1250146
[9] Howe P S and Papadopoulos G 1996 Twistor spaces for HKT manifolds Phys. Lett. B 379 80
[10] Ivanov E A, Krivonos S O and Leviant V M 1988 A new class of superconformal $\sigma$ models with the Wess-Zamino action Nucl. Phys. B 304 601
[11] Spindel P, Sevrin A, Troost W and van Proeyen A 1988 Extended supersymmetric sigma models on group manifolds: I. The complex structures Nucl. Phys. B 308 662
[12] Delia C F and Valenti G 1993 New geometry from heterotic supersymmetry Class. Quantum Grav. 10 1201
[13] Papadopoulos G 1995 Elliptic monopoles and (4, 0) supersymmetric sigma models with torsion Phys. Lett. B356 249
[14] Grantcharov G and Poon Y S 2000 Geometry of hyper-Kähler connections with torsion Commun. Math. Phys. 213 19
[15] Verbitsky M 2002 Hyperkähler manifolds with torsion, supersymmetry and Hodge theory Asian J. Math. 6 679
[16] Gibbons G W, Papadopoulos G and Steffe K S 1997 HKT and OKT geometries on soliton black hole moduli spaces Nucl. Phys. B 508 623
[17] Hull C M 1999 The geometry of supersymmetric quantum mechanics arXiv:hep-th/9910028
[18] Fedoruk S A, Ivanov E A and Smilga A V 2014 N = 4 mechanics with diverse (4, 4, 0) multiplets: explicit examples of HKT, CKT, and OKT geometries J. Math. Phys. 55 052302
[19] Smilga A V 2013 Taming the zoo of supersymmetric quantum mechanical models J. High Energy Phys. JHEP05(2013)119
[20] Zamino B 1979 Supersymmetry and Kähler manifolds Phys. Lett. 87B 203
[21] Davis A C, Macfarlane A J, Popat P and van Holten J W 1984 The quantum mechanics of the supersymmetric nonlinear $\sigma$ model J. Phys. A: Math. Gen. 17 2945
[22] Macfarlane A J and Popat P 1984 The quantum mechanics of the $\mathcal{N} = 2$ extended supersymmetric nonlinear $\sigma$ model J. Phys. A: Math. Gen. 17 2955
[23] Alvarez-Gaume L and Freedman D Z 1981 Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model Commun. Math. Phys. 80 443
[24] Smilga A V 2012 Supercharges in the HKT supersymmetric sigma models J. Math. Phys. 53 122105
[25] Pashnev A and Toppan F 2001 On the classification of N-extended supersymmetric quantum mechanical systems J. Math. Phys. 42 5257
[26] Hitchin N J, Karlhede A, Lindstrom U and Roček M 1987 Hyperkähler metrics and supersymmetry Commun. Math. Phys. 108 535
[27] Gibbons G W, Rychenkova P and Goto R 1997 Hyper-Kahler quotient construction of BPS monopole moduli spaces Commun. Math. Phys. 186 585
[28] Selivanov K G and Smilga A V 2003 Effective Lagrangian for 3d N = 4 SYM theories for any gauge group and monopole moduli spaces J. High Energy Phys. JHEP09(2003)073
[29] Grantcharov G, Papadopoulos G and Poon Y S 2002 Reduction of HKT structures J. Math. Phys. 43 3766
[30] Candelas P 1987 Lectures on Complex Manifolds University of Texas
[31] Mavromatos N E 1988 A note on the Atiyah–Singer index theorem for manifolds with totally antisymmetric H torsion J. Phys. A: Math. Gen. 21 2279
Bismut J-M 1989 A local index theorem for non-Kähler manifolds Ann. Math. 284 681
[32] Gauduchon P 1997 Hermitian connections and Dirac operators Boll. Unione Math. Ital. 11B 257
[33] Berezovoj V P and Pashnev A I 1991 Three-dimensional N = 4 extended supersymmetrical quantum mechanics Class. Quantum Grav. 8 2141
[34] Gates S J Jr and Rana L 1995 Ultramultiplets: a new representation of rigid 2-d, N = 8 supersymmetry Phys. Lett. B342 132
[35] Bellucci S, Krivonos S, Marrani A and Orazi E 2006 ‘Root’ action for N = 4 supersymmetric mechanics theories Phys. Rev. D 73 025011
[36] Delduc F and Ivanov E 2006 Gauging N = 4 supersymmetric mechanics Nucl. Phys. B753 211
[37] Delduc F and Ivanov E 2012 N = 4 mechanics of general (4, 4, 0) multiplets Nucl. Phys. B 855 815
[38] Coles R A and Papadopoulos G 1990 The geometry of the one-dimensional supersymmetric nonlinear sigma models Class Quantum Grav. 7 427
[39] Ivanov E and Lechtenfeld O 2003 N = 4 supersymmetric mechanics in harmonic superspace J. High Energy Phys. JHEP12(2003)027
[40] Ivanov E, Krivonos S and Lechtenfeld O 2004 N = 4, d = 1 supermultiplets from nonlinear realizations of D (2, 1, α) Class. Quantum Grav. 21 1031
[41] Delduc F and Ivanov E 2007 The common origin of linear and nonlinear chiral multiplets in N = 4 mechanics Nucl. Phys. B787 176