Buffon’s problem with a pivot needle

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Abstract

In this paper, we solve Buffon’s needle problem for a needle consisting of two line segments connected in a pivot point.

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1 Introduction

The classical Buffon needle problem asks for the probability that a needle of length $\ell$ thrown at random onto a plane lattice $\mathcal{R}_d$ of parallel lines at a distance $d \geq \ell$ apart will hit one of these lines. This problem was stated and solved by Buffon in his *Essai d’Arithmétique Morale*, 1777 (see e. g. [5, pp. 71-72], [6, pp. 501-502]). If an arbitrary convex body $C$ with maximum width $\leq d$ is used in this experiment, then the hitting probability is given by $u/(\pi d)$, where $u$ denotes the perimeter of $C$. This is the result of Barbier in 1860 [1, pp. 274-275], [6, p. 507]. If $C$ is a needle (line segment), then $u = 2\ell$. If $C$ is an ellipse, then there are elliptic integrals in the formulas of the hitting probabilities, see Duma and Stoka [3].

We consider a needle $N_{a,b}$ consisting of two line segments $C' A'$, $C' B'$ of lengths $a := |C' A'|$ and $b := |C' B'|$, connected in a pivot point $C'$ (see Fig. 1), and assume $a + b \leq d$. The random throw of $N_{a,b}$ onto $\mathcal{R}_d$ is defined as follows: The $y$-coordinate of the point $C'$ is a random variable uniformly distributed in $[0,d]$. The angles $\alpha$ and $\beta$ between the lines of $\mathcal{R}_d$, and segments $C' A'$ and $C' B'$, respectively, are random variables uniformly distributed in $[0,2\pi]$. All three random variables are stochastically independent.

The probability of the event that $N_{a,b}$ hits two lines of $\mathcal{R}_d$ at the same time is equal to zero, even in the case $a + b = d$. The expectation $E(n)$ of the random variable $n =$ number of intersection points between $N_{a,b}$ and $\mathcal{R}_d$ is given by $E(n) = 2(a + b)/(\pi d)$, cp. [4].

Here we are asking for the probabilities $p(i)$, $i \in \{0, 1, 2\}$, of the events that $N_{a,b}$ hits $\mathcal{R}_d$ in exactly $i$ points. We denote by $A$ and $B$ the events that segments $C' A'$ and $C' B'$, respectively, hit one line of $\mathcal{R}_d$. 
2 Hitting probabilities

Theorem. If $a + b \leq d$, then the probabilities $p(i)$ that $N_{a,b}$ hits $R_d$ in exactly $i$ points are given by

\[ p(0) = 1 - \frac{(a + b)(\pi + 2E(k))}{\pi^2 d}, \quad p(1) = \frac{4(a + b)E(k)}{\pi^2 d}, \]

\[ p(2) = \frac{(a + b)(\pi - 2E(k))}{\pi^2 d}, \]

where

\[ E(k) = E(\pi/2, k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \]

is the complete elliptic integral of the second kind with $k^2 = 4ab/(a + b)^2$.

Proof. We observe that the angle $\phi := \angle(C'A', C'B')$ is a random variable uniformly distributed in $[0, 2\pi]$. Due to the result of Barbier, the conditional probability $P(A \cup B | \phi)$ of $A \cup B$ for fixed value of $\phi \in [0, 2\pi]$ is given by $u(\phi)/(\pi d)$, where $u(\phi)$ is the perimeter of the convex hull of $N_{a,b}$. ($N_{a,b}$ hits $R_d$ if and only if its convex hull hits $R_d$.) Using the law of total probability, the probability that $N_{a,b}$ hits $R_d$ is given by

\[ P(A \cup B) = \int_0^{2\pi} P(A \cup B | \phi) \frac{d\phi}{2\pi} = \frac{1}{2\pi^2 d} \int_0^{2\pi} u(\phi) \, d\phi \]

\[ = \frac{1}{2\pi^2 d} \int_0^{2\pi} [a + b + c(\phi)] \, d\phi = \frac{a + b + c}{\pi d}, \]

where $c := |A'B'|$, and

\[ \bar{c} := \frac{1}{2\pi} \int_0^{2\pi} c(\phi) \, d\phi = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{a^2 + b^2 - 2ab \cos \phi} \, d\phi. \]
Using \( \cos \phi = 2 \cos^2(\phi/2) - 1 \), we have
\[
\tau = \frac{1}{2\pi} \int_0^{2\pi} \sqrt{(a+b)^2 - 4ab \cos^2(\phi/2)} \, d\phi
= \frac{a+b}{2\pi} \int_0^{2\pi} \sqrt{1 - \frac{4ab}{(a+b)^2} \cos^2(\phi/2)} \, d\phi.
\]
For abbreviation we put \( k^2 = \frac{4ab}{(a+b)^2} \). From the inequality \( \sqrt{ab} \leq \frac{(a+b)}{2} \) between the geometric and the arithmetic mean, one finds \( k^2 \leq 1 \), hence \( 0 \leq k \leq 1 \) with \( k = 1 \) only for \( a = b \). With the substitution \( \chi = \phi/2 \) we get
\[
\tau = \frac{a+b}{\pi} \int_0^{\pi} \sqrt{1 - k^2 \cos^2(\chi)} \, d\chi
= \frac{2(a+b)}{\pi} \int_0^{\pi/2} \sqrt{1 - k^2 \cos^2(\chi)} \, d\chi
= \frac{(a+b)(\pi + 2E(k))}{\pi d}.
\]
It follows that
\[
P(A \cup B) = \frac{a+b+\tau}{\pi d} = \frac{(a+b)(\pi + 2E(k))}{\pi^2 d},
\]
\[
P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{2a}{\pi d} + \frac{2b}{\pi d} - \frac{a+b+\tau}{\pi d}
= \frac{(a+b)(\pi - 2E(k))}{\pi^2 d},
\]
and
\[
p(0) = 1 - P(A \cup B) = 1 - \frac{(a+b)(\pi + 2E(k))}{\pi^2 d},
\]
\[
p(1) = P(A \cup B) - P(A \cap B) = \frac{a+b+\tau}{\pi d} - \frac{a+b-\tau}{\pi d} = \frac{2\tau}{\pi d}
= \frac{4(a+b)E(k)}{\pi^2 d},
\]
\[
p(2) = P(A \cap B) = \frac{(a+b)(\pi - 2E(k))}{\pi^2 d}.
\]
This is the result from [2, pp. 57-58]. There it was obtained as special case of the more general result in Corollary 4.2 [2, p. 56].

**Remark.** If the angle \( \phi \) is constant, then we have
\[
P(A \cup B) = \frac{a+b+c}{\pi d} \quad \text{and} \quad P(A \cap B) = \frac{a+b-c}{\pi d}
\]
with \( c = \sqrt{a^2 + b^2 - 2ab \cos \phi} \). This yields
\[
p(0) = 1 - \frac{a+b+c}{\pi d}, \quad p(1) = \frac{2c}{\pi d}, \quad p(2) = \frac{a+b-c}{\pi d},
\]
see Santaló [5, pp. 77-78].
3 Special cases

If $a = b$, we have $k = 1$, $E(1) = 1$, and therefore

$$p(0) = 1 - \frac{2a(\pi + 2)}{\pi^2 d}, \quad p(1) = \frac{8a}{\pi^2 d}, \quad p(2) = \frac{2a(\pi - 2)}{\pi^2 d}.$$ 

If $a \neq 0$ and $b = 0$, then $k = 0$ and $E(0) = \pi/2$, and therefore $P(A \cup B) = P(A) = 2a/(\pi d)$. This is the result of the classical Buffon needle problem.

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