The q-Rung Orthopair Hesitant Fuzzy Uncertain Linguistic Aggregation Operators and Their Application in Multi-Attribute Decision Making

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ABSTRACT This paper combines the q-rung orthopair hesitant fuzzy sets (q-ROHFSs) with the uncertain linguistic variables, and proposes the q-rung orthopair hesitant fuzzy uncertain linguistic sets (q-ROHFULSs). In addition, the Schweizer-Sklar T-norm is introduced, and a multi-attribute decision-making method based on the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar aggregation operators is established. Firstly, based on the Schweizer-Sklar T-norm, the operational properties of q-rung orthopair hesitant fuzzy uncertain linguistic elements are defined, and the score function, accuracy function and ranking method of the q-rung orthopair hesitant fuzzy uncertain linguistic elements are proposed. Secondly, the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Bonferroni mean (BM) operator and geometric Bonferroni mean (GBM) operator, Maclaurin symmetric mean (MSM) operator and dual Maclaurin symmetric mean (DMSM) operator, Maclaurin mean (MM) operator and dual Maclaurin mean (DMM) operator are defined. The calculation formulas of the operators are given, the related properties are studied, and the special forms of the operators are discussed. Finally, a multi-attribute decision-making model based on the q-rung orthopair hesitant fuzzy uncertain linguistic aggregation operators is established, and the feasibility and effectiveness of the decision-making method are demonstrated through calculation examples and comparative analyses.

INDEX TERMS q-Rung orthopair hesitant fuzzy uncertain linguistic sets (q-ROHFULSs), Schweizer-Sklar T-norm, Bonferroni mean (BM) operator, Maclaurin symmetric mean (MSM) operator, Maclaurin mean (MM) operator, multi-attribute decision making (MADM).

I. INTRODUCTION
Multi-attribute decision-making refers to the decision-making problem of sorting alternatives and choosing the best while considering multiple attributes. It is an important part of modern decision-making science and is widely used in performance evaluation, factory site selection, and supplier selection and other fields. In recent years, due to the increasingly prominent uncertainty and ambiguity of the decision-making environment, the corresponding fuzzy multi-attribute decision-making theory has emerged as the times require, and has gradually become a key issue in the field of decision science. Since Zadeh [1] proposed the concept of fuzzy set in 1965, fuzzy set theory has been widely used in marginal detection [2], pattern recognition [3], image reconstruction [4], decision-making [5] and other fields. In addition, many scholars have carried out research and development on fuzzy set theory. On the one hand, on the basis of fuzzy sets, Atanassov considers the membership degree and non-membership degree of elements belonging to the set at the same time, and proposed the concept of intuitionistic fuzzy sets (IFSs) [6]. Yager proposed the Pythagorean fuzzy sets (PFSs) [7], [8], which can describe the situation where the sum of the membership degree and non-membership degree of elements exceeds 1, and the sum of the squares does not exceed 1. However, there are
some situations that Pythagorean fuzzy sets cannot describe. Yager [9] further generalized the Pythagorean fuzzy sets and proposed the concept of q-rung orthopair fuzzy sets (q-ROFSs). The q-rung orthopair fuzzy sets is the generalization of intuitionistic fuzzy sets and Pythagorean fuzzy sets, which can describe fuzzy phenomena more widely. On the other hand, Torra considered the hesitation of decision makers on the basis of fuzzy sets, and proposed the concept of hesitant fuzzy sets (HFSs) [10]. Khan et al. [11] combined the hesitant fuzzy set with the Pythagorean fuzzy set, proposed the Pythagorean hesitant fuzzy sets (PHFSs), gave the distance formula of the Pythagorean hesitant fuzzy element, and a multi-attribute decision-making method based on Pythagorean hesitant fuzzy weighted average operator is constructed. In addition, Liu et al. [12] combined hesitant fuzzy sets and the q-rung fuzzy sets, and proposed the concept of the q-rung orthopair hesitant fuzzy sets (q-ROHFSs), which can better describe the uncertainty and fuzzy phenomena in reality.

However, both Pythagorean hesitant fuzzy sets and q-rung orthopair hesitant fuzzy sets can only quantitatively describe the membership degree and non-membership degree of a fuzzy concept [13]. In practical problems, experts prefer to give a qualitative evaluation of the attributes of the scheme [14]. For example, in the selection of investment options, decision makers often use linguistic variables such as “general” and “higher” to express evaluation opinions regarding the economic benefits of the options.

When linguistic variables express the expert’s decision-making opinions, they can only express their qualitative decision-making opinions and ignore the quantitative decision-making opinions. For this reason, Wang et al. [15] combined linguistic variables with intuitionistic fuzzy sets and proposed the intuitionistic linguistic sets (IFLSs). Since the intuitionistic linguistic term set was put forward, it has attracted many scholars to study and discuss it [16]–[18]. Similarly, Peng and Yang [19] combined linguistic variables with the Pythagorean fuzzy set, and proposed the Pythagorean fuzzy linguistic sets (PFLSs). In addition, Wang et al. [20] combined linguistic variables with q-rung orthopair fuzzy sets and proposed q-rung orthopair fuzzy linguistic sets (q-ROFLSs), and studied the q-rung orthopair fuzzy linguistic aggregation operators and their applications in decision-making. The q-rung orthopair fuzzy linguistic set is a generalization of the intuitionistic linguistic set, which can use linguistic variables to express qualitative opinions, and can also describe the membership degree and non-membership degree of decision attributes to linguistic variables.

In practical problems, uncertain linguistic variables can more fully represent fuzzy data than linguistic variables. Therefore, Liu and Jin [21] combined uncertain linguistic variables and intuitionistic fuzzy sets, and proposed the concept of intuitionistic uncertain linguistic sets (IFULSs). Since the introduction of the intuitionistic uncertain linguistic set, many scholars have carried out research and development on the theory of intuitionistic uncertain linguistic set [22]–[24]. Liu and Liu [25] studied the intuitionistic uncertain linguistic Bonferroni mean aggregation operator and its application. Liu and Wang [26] studied the intuitionistic uncertain linguistic power geometric aggregation operator and its application in project evaluation. Liu [27] combined uncertain linguistic variables and interval intuitionistic fuzzy sets, and proposed the concept of interval intuitionistic uncertain linguistic sets (IVFULSs). Meng and Chen [28] studied the interval intuitionistic uncertain linguistic hybrid weighted average operator and its application in multi-attribute decision-making. In addition, Lu and Wei [29] combined uncertain linguistic variables with Pythagorean fuzzy sets and proposed the Pythagorean uncertain linguistic sets (PFULSs). Geng et al. [30] proposed a multi-attribute group decision-making model of Pythagoras uncertain linguistic set. Wang et al. [31] combined uncertain linguistic variables with q-rung orthopair fuzzy sets and proposed the q-rung orthopair fuzzy uncertain linguistic sets (q-ROFULSs). Liu et al. [32] studied the aggregation operators of the q-rung orthopair fuzzy uncertain linguistic sets and their applications in multi-attribute decision-making.

However, the intuitionistic uncertain linguistic set cannot describe the degree of hesitant of the decision maker. For this reason, drawing on the ideas of Pythagorean uncertain linguistic sets, this paper combines the q-rung orthopair hesitant fuzzy sets with uncertain linguistic sets, and proposes the q-rung orthopair hesitant fuzzy uncertain linguistic sets (q-ROHFULSs). In addition, the Schweizer-Sklar T-norm is introduced into the q-rung orthopair hesitant fuzzy uncertain linguistic sets, and the Schweizer-Sklar Bonferroni mean (BM) operator, Maclaurin symmetric mean (MSM) operator and Maclaurin mean (MM) operator of are defined, and the multiple attributes decision-making methods of the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar aggregation operator are established. In summary, the innovations of this article are: 1) The q-rung orthopair hesitant fuzzy uncertain linguistic set is an extension and generalization of the intuitionistic uncertain linguistic set, which makes up for the lack of q-rung orthopair hesitant fuzzy set and uncertain linguistic set in describing decision information, and it can fully express the true opinions of decision makers. And the q-rung orthopair hesitant fuzzy uncertain linguistic set further considers the hesitation of the decision maker, its application range is wider, and it can describe the decision information of the decision maker more widely. 2) The q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar aggregation operator can flexibly select different parameter values according to the decision-making situation to meet the requirements of different decision-making problems in practice. In addition, the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar MM operators can describe the correlation between decision attributes, fully reflect the decision information, reduce the information loss in the decision, and make the decision result more reasonable and reliable.
The structure of this paper is as follows: In Chapter 2, the definitions and related knowledge of q-rung hesitant fuzzy sets and uncertain linguistic variables are briefly introduced. In Chapter 3, the definition of the q-rung orthopair hesitant fuzzy uncertain linguistic set is given, the score function and the accuracy function of the q-rung orthopair hesitant fuzzy uncertain linguistic set are proposed, and based on the Schweizer-Sklar T-norm, the operation properties of q-rung orthopair hesitant fuzzy uncertain linguistic elements are defined. In Chapter 4, Chapter 5, and Chapter 6, the Schweizer-Sklar BM operator, MSM operator, and MM operator of the q-rung orthopair hesitant fuzzy uncertain linguistic sets are defined. In Chapter 7, a multi-attribute decision-making method based on the q-rung orthopair hesitant fuzzy uncertain linguistic set is proposed and the special forms of the operators are discussed. In Chapter 8, an example of decision-making model based on the q-rung orthopair hesitant fuzzy uncertain linguistic set is proposed and the related properties are further introduced to carry out comparative analysis, which shows the validity and rationality of the decision-making method in this paper. Chapter 9 summarizes the research content.

II. PRELIMINARIES

A. THE Q-RUNG ORTHOPAIR HESITANT FUZZY SET

Based on the HFSs and q-ROFs, Khan et al. [11] proposed the q-rung orthopair hesitant fuzzy sets (q-ROHFSs). Some fundamental theorems regarding the q-ROHFSs are presented as follows.

**Definition 1 [11]:** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a fixed set, then a q-ROHFS on \( X \) can be represented as:

\[
A = \{<x_i, \Gamma_A(x_i), \Psi_A(x_i)>| x_i \in X \} \quad (q \geq 1),
\]

where \( \Gamma_A(x_i) \) and \( \Psi_A(x_i) \) are two sets of values in \([0, 1]\), denoting all the possible membership degrees and non-membership degrees of the element \( x_i \in X \), and they satisfy the following conditions:

\[
\forall x_i \in X, \forall \mu_A(x_i) \in \Gamma_A(x_i), \forall \nu_A(x_i) \in \Psi_A(x_i), \text{ then } 0 \leq \mu_A^q(x_i) + \nu_A^q(x_i) \leq 1 (q \geq 1).
\]

\(<\Gamma_A>x_i, \Psi_A(x_i)>_q \) is called the q-rung orthopair hesitant fuzzy element (q-ROHFE), and for convenience, the q-rung orthopair hesitant fuzzy set is recorded as \( h = <\Gamma_h, \Psi_h>_q \). The overall q-rung orthopair hesitant fuzzy elements are denoted as \( q-ROHFE(X) \).

In particular, when \( |\Gamma_h| = |\Psi_h| = 1 \), the q-rung orthopair hesitant fuzzy set reduces to the q-rung orthopair fuzzy set; when \( q = 1 \), the q-rung orthopair hesitant fuzzy set reduces to the dual hesitant fuzzy set.

**Definition 2 [11]:** Let \( h = <\Gamma_h, \Psi_h>_q \). \( h_1 = <\Gamma_{h_1}, \Psi_{h_1}>_q \) and \( h_2 = <\Gamma_{h_2}, \Psi_{h_2}>_q \) be three q-ROHFEs, for \( \lambda > 0, q \geq 1 \), the operational laws between them are described as follows:

1. \( h_1 \oplus h_2 = \bigcup_{\mu_1 \in \Gamma_1, \mu_2 \in \Gamma_2} \{\mu_1^q + \mu_2^q - \mu_1^q \mu_2^q \}^1/q \),
2. \( h_1 \odot h_2 = \bigcup_{\nu_1 \in \Psi_1, \nu_2 \in \Psi_2} \{(\nu_1^q + \nu_2^q - \nu_1^q \nu_2^q \}^1/q \),
3. \( \lambda h = \bigcup_{\mu \in \Gamma_h} \{(1 - (1 - \mu^q)^1/q \),
4. \( h^\lambda = \bigcup_{\nu \in \Psi_h} \{(1 - (1 - \nu^q)^1/q \),
5. \( h^c = <\Psi_h, \Gamma_h>_q \).

**Definition 3 [11]:** Let \( h = <\Gamma_h, \Psi_h>_q \in q-ROHFE(X) \), the numbers of set \( \Gamma_h \) and \( \Psi_h \) are denoted as \( |\Gamma_h| \) and \( |\Psi_h| \), respectively, then a score function \( S(h) \) can be defined as:

\[
S(h) = \frac{1}{|\Gamma_h|} \sum_{\mu \in \Gamma_h} \mu^q - \frac{1}{|\Psi_h|} \sum_{\nu \in \Psi_h} \nu^q,
\]

\[
H(h) = \frac{1}{|\Gamma_h|} \sum_{\mu \in \Gamma_h} \mu^q + \frac{1}{|\Psi_h|} \sum_{\nu \in \Psi_h} \nu^q.
\]

**Definition 4 [11]:** Let \( h = <\Gamma_h, \Psi_h>_q \in q-ROHFE(X), i = 1, 2 \), then
1. If \( S(h_1) > S(h_2) \), then \( h_1 > h_2 \);
2. If \( S(h_1) = S(h_2) \), then
   a. if \( H(h_1) = H(h_2) \), then \( h_1 = h_2 \);
   b. if \( H(h_1) < H(h_2) \), then \( h_1 < h_2 \).

B. UNCERTAIN LINGUISTIC VARIABLES

The linguistic term set \( S = \{s_0, s_1, s_2, \ldots, s_r\} \) is composed of an odd number of elements, that is, \( r \) should be an even number. For any linguistic term set \( S \), the following conditions should be met [33]:

1. If \( i > j \), then \( s_i > s_j \), it means that \( s_i \) is superior to \( s_j \);
2. There is a negative operator \( \text{neg}(s_i) = s_j \), making \( j = r - i \);
3. If \( s_i \geq s_j \), it means that \( s_i \) is not inferior to \( s_j \), then \( \max(s_i, s_j) = s_i \);
4. If \( s_i \leq s_j \), it means that \( s_i \) is not better than \( s_j \), then \( \min(s_i, s_j) = s_i \).

For any one linguistic term set \( S = \{s_0, s_1, s_2, \ldots, s_r\} \), there is a strictly monotonically increasing relationship between element \( s_i \) and its subscript \( i \). Therefore, the function \( f: s_i = f(i) \) can be defined, and the function \( f(i) \) is a strictly monotonically increasing function with the subscript \( i \).
Extending the discrete linguistic term set \( S = \{s_0, s_1, s_2, \ldots, s_p\} \) into a continuous linguistic term set \( \tilde{S} = \{s_l | l \in [0,p], p \geq \gamma \} \), the extended linguistic term set still satisfies a strict monotonic increasing relationship.

**Definition 4** [34]: Let \( S = \{s_0, s_1, s_2, \ldots, s_p\} \) be a linguistic term set, and \( \tilde{S} = \{s_l | l \in [0,p], p \geq \gamma \} \) be an extended linguistic term set. Assume \( \tilde{s} = [s_0, s_1, s_2, \ldots, s_\gamma] \) and \( \theta \leq \tau \), then \( s_0 \) and \( s_\tau \) are the lower and upper limits of \( \tilde{s} \), respectively. Then \( \tilde{s} \) is called the uncertain linguistic variable, and \( \tilde{S} \) is the set of all uncertain linguistic variables.

**Definition 5** [34]: If \( \tilde{s}_1, \tilde{s}_2 \in \tilde{S} \) and \( m > 0 \), the operational laws between them are defined as follows:

1. \( \tilde{s}_1 \oplus \tilde{s}_2 = [s_0, s_1, s_2, \ldots, s_\gamma] = [s_0+\theta, s_1+\tau, \ldots, s_\gamma+\gamma] \); 
2. \( \tilde{s}_1 \otimes \tilde{s}_2 = \{s_l | l \in [0,p], p \geq \gamma \} \); 
3. \( \tilde{s}_1 \cong \tilde{s}_2 = [s_0, s_1, s_2, \ldots, s_\gamma] = [s_0, s_1, s_2, \ldots, s_\gamma] \); 
4. \( \tilde{s}_1^m = \{s_l | l \in [0,p], p \geq \gamma \} \); 
5. \( \tilde{s}_1/\tilde{s}_2 = [s_0, s_1, s_2, \ldots, s_\gamma] = [s_0/\theta, s_1/\tau, \ldots, s_\gamma/\gamma] \), where \( \theta \neq 0, \tau \neq 0 \).

**Theorem 1** [35]: If \( \tilde{s}_1, \tilde{s}_2 \in \tilde{S} \) and \( m > 0 \), then the operational properties between them are defined as follows:

1. \( m(\tilde{s}_1 \otimes \tilde{s}_2) = m\tilde{s}_1 \otimes m\tilde{s}_2 \); 
2. \( m(\tilde{s}_1 \oplus \tilde{s}_2) = m\tilde{s}_1 \oplus m\tilde{s}_2 \).

### III. THE Q-RUNG ORTHOPAIR HSITANT FUZZY UNCERTAIN LINGUISTIC SETS

By combining the uncertain linguistic sets with q-ROHFSs, we propose the concept of q-ROHFULs.

**Definition 6**: Let \( [s_{\theta}(x), s_{\tau}(x)] \in \tilde{s} \) and \( x \) is a given domain, then a q-ROHFUL is defined as follows:

\[ A = \{x, [s_{\theta}(x), s_{\tau}(x)] \in \tilde{s} \in \tilde{x} \}) \]

where \( \Gamma(x) \) and \( \Psi_A(x) \) are two sets of values in \([0, 1]\), denoting all the possible membership degrees and non-membership degrees of the element to the uncertain linguistic variable \([s_{\theta}(x), s_{\tau}(x)] \), respectively. And they satisfy the following conditions:

\[ \forall x \in X, \Psi_{A}(x) \in \Gamma(x), \Psi_{A}(x) \in \Psi_A(x) \text{, then } 0 \leq \mu_A(x) + \nu_A(x) \leq 1 \text{ (q} \geq 1) \]

\[ [s_{\theta}(x), s_{\tau}(x)] \in \tilde{s} \text{, } \Gamma(x) \text{, } \Psi_A(x) \text{ is called q-rung orthopair hesitant uncertain linguistic element (q-ROHULE), where } [s_{\theta}(x), s_{\tau}(x)] \text{ is the linguistic part of the q-rung orthopair hesitant uncertain linguistic element, and } \Gamma(x) \text{ and } \Psi_A(x) \text{ is the q-rung orthopair hesitant fuzzy part of uncertain linguistic element.} \]

For convenience, the q-rung orthopair hesitant uncertain linguistic element is recorded as \( h = [s_{\theta_1}, s_{\tau_1}] \in \tilde{h} \), \( \Psi_h > q \). The overall q-rung orthopair hesitant uncertain linguistic elements is recorded as \( q - ROHULE(X) \).

**Definition 7**: Let \( h = [s_{\theta}, s_{\tau}] \in \tilde{h} \), \( \Psi逅 > q \), \( h = [s_{\theta_1}, s_{\tau_1}] \in \tilde{h} \), \( i \in [1, 2] \) be any three q-ROHFULs, then

\[ h_1 \leq h_2 \Leftrightarrow s_{\theta_1} \leq s_{\theta_2}, s_{\tau_1} \leq s_{\tau_2}, \mu_i \leq \mu_2, \nu_i \geq \nu_2; \]

where \( \mu_i \in \Gamma_i, \nu_i \in \Psi_i, i = 1, 2 \).

\[ h^q = [s_{y_\gamma}, s_{y_\tau}] \in \Psi_{\gamma} \geq > \]

To compare two q-ROHFULs, we provide the concepts of score function and accuracy function of a q-ROHULE.

**Definition 8**: Let \( h = [s_{\theta_1}, s_{\tau_1}] \in \tilde{h} \), \( \Psi_h > \), be any a q-ROHFUL, the numbers of set \( \Gamma_h \) and \( \Psi_h \) are denoted as \( |\Gamma_h| \) and \( |\Psi_h| \), respectively, then the score function \( S(h) \) can be defined as:

\[ S(h) = \frac{\theta_h + \nu_h}{4} (1 + \frac{1}{|\Gamma_h|} \sum_{\mu \in \Gamma_h} \sum_{\nu \in \Psi_h} v^q) \]

The accuracy function \( H(h) \) can be defined as:

\[ H(h) = \frac{\theta_h + \nu_h}{2} (1 + \frac{1}{|\Gamma_h|} \sum_{\mu \in \Gamma_h} \sum_{\nu \in \Psi_h} v^q) \]

Based on the above concepts, a comparison law for q-ROHFULs can be provided.

**Definition 9** [36]: Let \( x, y \in [0, 1] \), then the Schweizer-Sklar T-norm and Schweizer-Sklar T-conorm are special cases of T-norm and T-conorm.

**Definition 10**: Let \( h = [s_{\theta}, s_{\tau}], \Gamma_h, \Psi_h \in \tilde{h} \), \( \Psi_{h_1} > q \), be any three q-ROHFULs, then the operational rules of q-ROHFULs based on the Schweizer-Sklar T-norm and T-conorm can be defined as follows:

\[ (1) \ h_1 \oplus h_2 \]

\[ = \bigcup_{\mu_{\theta_1} \in \Gamma_1, \nu_{\theta_1} \in \Psi_1, \mu_{\theta_2} \in \Gamma_2, \nu_{\theta_2} \in \Psi_2} [s_{\theta_1+\theta_2}, s_{\tau_1+\tau_2}] \]
\[
\{1-(1-\mu_{h_1}^q)^q + (1-\mu_{h_2}^q)^q - 1\}^{1/r/q} > q
\]
\[
\{(u_{h_1}^q)^q + (u_{h_2}^q)^q - 1\}^{1/r/q} > q
\]  
(15)

(2) \[ h_1 \oplus h_2 = \bigcup_{\mu_{h_1} \in \Gamma_{h_1}, \mu_{h_2} \in \Psi_{h_2}} \left< s_{\theta_1, \theta_2}, s_{\tau_1, \tau_2} \right> \]
\[ \{(\mu_{h_1}^q)^q + (\mu_{h_2}^q)^q - 1\}^{1/r/q} > q \]
\[ \{(1-(1-\mu_{h_1}^q)^q + (1-\mu_{h_2}^q)^q - 1\}^{1/r/q} > q \]

(16)

(3) \[ \lambda h = \bigcup_{\mu_{h} \in \Gamma_{h}} \left< s_{\lambda \theta}, s_{\lambda \tau} \right>, \{(\lambda \mu_{h}^q)^q - (\lambda - 1)\}^{1/r/q} > q \]

(17)

(4) \[ h^\lambda = \bigcup_{\mu_{h} \in \Gamma_{h}} \left< s_{\theta^\lambda}, s_{\tau^\lambda} \right>, \{(\lambda \mu_{h}^q)^q - (\lambda - 1)\}^{1/r/q} > q \]

(18)

Theorem 2: Let \[ h = \left< s_{\theta}, s_{\tau} \right>, \Gamma_{h}, s_{\theta}, s_{\tau} > q \], \[ h_j = \left< s_{\theta}, s_{\tau} \right>, \Gamma_{h}, s_{\theta}, s_{\tau} > q \] \( (j = 1, 2) \) be any three q-ROHFULEs, \[ q \geq 1, \lambda > 0, \lambda_1 > 0, \lambda_2 > 0 \], then

(1) \[ h_1 \oplus h_2 = h_2 \oplus h_1 \];
(2) \[ h_1 \otimes h_2 = h_2 \otimes h_1 \];
(3) \[ \lambda (h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2 \];
(4) \[ (h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda \];
(5) \[ (\lambda_1 + \lambda_2) h = \lambda_1 h \oplus \lambda_2 h \];
(6) \[ h^{(\lambda_1 + \lambda_2)} = h^{\lambda_1} \otimes h^{\lambda_2} \];
(7) \[ h \oplus (h_1 \oplus h_2) = h \oplus (h_1 \oplus h_2) \];
(8) \[ (h \otimes h_1) \otimes h_2 = h \otimes (h_1 \otimes h_2) \].

Proof: The proof process of (3) and (7) is given below.

(3) \[ h_1 \oplus h_2 = \bigcup_{\mu_{h_1} \in \Gamma_{h_1}, \mu_{h_2} \in \Psi_{h_2}} \left< s_{\theta_1+\theta_2}, s_{\tau_1+\tau_2} \right> \]
\[ \{(1-(1-\mu_{h_1}^q)^q + (1-\mu_{h_2}^q)^q - 1\}^{1/r/q} > q \]
\[ \{(u_{h_1}^q)^q + (u_{h_2}^q)^q - 1\}^{1/r/q} > q \]

(19)

\[ \lambda (h_1 \oplus h_2) = \bigcup_{\mu_{h} \in \Gamma_{h}} \left< s_{\theta}, s_{\tau} \right>, \{(\lambda \mu_{h}^q)^q - (\lambda - 1)\}^{1/r/q} > q \]

(20)

In the same way, we can get

\[ \lambda h_1 \oplus \lambda h_2 = \bigcup_{\mu_{h_1} \in \Gamma_{h_1}, \mu_{h_2} \in \Psi_{h_2}} \left< s_{\theta_1+\theta_2}, s_{\tau_1+\tau_2} \right> \]
\[ \{(1-(\lambda_1-\mu_{h_1}^q)^q + (\lambda_1-\mu_{h_2}^q)^q - (2\lambda_1-1)\}^{1/r/q} > q \]
\[ \{(\lambda u_{h_1}^q)^q + (\lambda u_{h_2}^q)^q - (2\lambda_1-1)\}^{1/r/q} > q \]

Therefore, it is proved that \( \lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2 \) is established.

(7) \[ h \oplus h_1 = \bigcup_{\mu_{h} \in \Gamma_{h}, \mu_{h_1} \in \Psi_{h_1}} \left< s_{\theta + \theta_1, s_{\tau + \tau_1}} \right>, \]
\[ \{(1-(1-\mu_{h_1}^q)^q + (1-\mu_{h_2}^q)^q - 1\}^{1/r/q} > q \]
\[ \{(u_{h_1}^q)^q + (u_{h_2}^q)^q - 1\}^{1/r/q} > q \]

(21)

\[ (h \oplus h_1) \oplus h_2 = \bigcup_{\mu_{h} \in \Gamma_{h}, \mu_{h_1} \in \Psi_{h_1}, \mu_{h_2} \in \Psi_{h_2}} \left< s_{\theta + \theta_1 + \theta_2, s_{\tau + \tau_1 + \tau_2}} \right>, \]
\[ \{(1-(1-\mu_{h_1}^q)^q + (1-\mu_{h_2}^q)^q - (2\lambda_1-1)\}^{1/r/q} > q \]
\[ \{(u_{h_1}^q)^q + (u_{h_2}^q)^q - (2\lambda_1-1)\}^{1/r/q} > q \]

In the same way, we can get

\[ h \oplus (h_1 \oplus h_2) = \bigcup_{\mu_{h} \in \Gamma_{h}, \mu_{h_1} \in \Psi_{h_1}, \mu_{h_2} \in \Psi_{h_2}} \left< s_{\theta + \theta_1 + \theta_2, s_{\tau + \tau_1 + \tau_2}} \right>, \]
\[ \{(1-(1-\mu_{h_1}^q)^q + (1-\mu_{h_2}^q)^q - (2\lambda_1-1)\}^{1/r/q} > q \]
\[ \{(u_{h_1}^q)^q + (u_{h_2}^q)^q - (2\lambda_1-1)\}^{1/r/q} > q \]

Therefore, it is proved that \( (h \oplus h_1) \oplus h_2 = h \oplus (h_1 \oplus h_2) \) is established.

IV. THE Q-RUNG ORTHOPAIR HESITANT FUZZY UNCERTAIN LINGUISTIC SCHWEIZER-SKLAR BONFERRONI MEAN OPERATOR

A. q-ROHFULSSBM OPERATOR

Bonferroni mean (BM) operator [37] can describe the relationship between any two decision attributes. Therefore, BM operator can meet the practical needs of multi-attribute decision making, and is widely used in information aggregation and multi-attribute decision making [38]–[40].

In this article, the Bonferroni mean operator is introduced into the q-rung orthopair hesitant fuzzy uncertain linguistic set, and the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Bonferroni mean operator and geometric Bonferroni mean operator are defined based on the Schweizer-Sklar T-norm.

Definition 11: Let \( h_j = \left< s_{\theta_j}, s_{\tau_j} \right>, \Gamma_{h}, \Psi_{h} > q \in q - \text{ROHFULE}(X) \) \( (j = 1, 2, \ldots, n) \), then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Bonferroni mean (q-ROHFULSSBM) operator as follow:

\[ q - \text{ROHFULSSBM}(h_1, h_2, \ldots, h_n) = \left( \frac{1}{n(n-1)} \sum_{i=j=1, i\neq j}^{n} \left< h_i^t \otimes h_j^t \right> \right)^{1/t} \]

where \( s, t \geq 0 \) and \( s + t > 0 \).
Theorem 3: Let \( h_j = \langle \hat{s}_j, \hat{t}_j \rangle, \Gamma_{h_j}, \Psi_{h_j} \rangle \in q-ROHFULE(X)(j = 1, 2, \ldots, n) \), then the aggregate results of Definition 11 is still a q-ROHFULE, and (19), as shown at the bottom of the page.

Proof: 1) when \( n = 2 \), by the operational law in Definition 10, we have

\[
(h_1)^s \otimes (h_2)^t = \bigcup_{\mu_1 \in \Gamma_1, \psi_1 \in \Psi_1} \bigcup_{\mu_2 \in \Gamma_2, \psi_2 \in \Psi_2} \left[ \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \bigg\},
\]

Then

\[
\bigoplus_{i,j=1\atop i \neq j} (h_i)^s \otimes (h_j)^t = ((h_1)^s \otimes (h_2)^t) \oplus ((h_2)^s \otimes (h_1)^t) = \bigcup_{\mu_1 \in \Gamma_1, \psi_1 \in \Psi_1} \bigcup_{\mu_2 \in \Gamma_2, \psi_2 \in \Psi_2} \left[ \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \bigg\},
\]

Thus, result is true for \( n = 2 \).

2) If Equation (19) holds for \( n = k \), that is

\[
\bigoplus_{i,j=1\atop i \neq j} (h_i)^s \otimes (h_j)^t = \bigcup_{\mu_1 \in \Gamma_1, \psi_1 \in \Psi_1} \bigcup_{\mu_j \in \Gamma_j, \psi_j \in \Psi_j} \left[ \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \bigg\},
\]

\[
q - ROHFULSSBM(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_1 \in \Gamma_1, \psi_1 \in \Psi_1} \bigcup_{\mu_j \in \Gamma_j, \psi_j \in \Psi_j} \left[ \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \left\{ \left( 1 - \left( t \mu_j^{qr} + s \mu_i^{qr} - s - t + 1 \right) \right)^{1/r} \right\}^{1/q} - 1 \right] \bigg\},
\]

(19)
3) when \( n = k + 1 \), we have

\[
\begin{align*}
\sum_{i=1}^{k+1} (h_i^1 \otimes h_i^2) \\
= \left( \sum_{i=1}^{k} (h_i^1 \otimes h_i^2) \right) + (h_{k+1}^1 \otimes h_{k+1}^2) \\
= \left( \sum_{i=1}^{k} (h_i^1 \otimes h_i^2) + (h_{k+1}^1 \otimes h_{k+1}^2) \right) \\
= \left( \sum_{i=1}^{k} (h_i^1 \otimes h_i^2) + (h_{k+1}^1 \otimes h_{k+1}^2) \right) \\
= \left( \sum_{i=1}^{k} (h_i^1 \otimes h_i^2) + (h_{k+1}^1 \otimes h_{k+1}^2) \right) \\
\end{align*}
\]

Thus, Equation (19) holds for \( n = k + 1 \).

Therefore, Equation (19) holds for all \( n \), which completes the proof.

It can be proved by mathematical induction that

\[
\sum_{i=1}^{n} (h_i^1 \otimes h_i^2) = \sum_{i=1}^{n} (h_i^1 \otimes h_i^2) + (h_{n+1}^1 \otimes h_{n+1}^2)
\]

Then, it can be proved that Theorem 3 is true.

Theorem 4 (Idempotence): Let \( h_j = [s_{j}, \tau_{j}], \Gamma_{h_j}, \Psi_{h_j} \in q - ROHFULX(X), j = 1, 2, \ldots, n \), if \( h_j = h = [s_{h}, \tau_{h}], \Gamma, \Psi \) is satisfied for any \( h_j \), then

\[
q - ROHFULSSBM(h_1, h_2, \ldots, h_n) = h.
\]

Proof:

\[
\begin{align*}
\left( \frac{1}{n(n-1)} \right)^{1/(s+t)} \\
= \left( \sum_{i=1}^{n} \left(1 - \left( t\mu_{ij}^{qr} + s\mu_{ij}^{qr} - s - t + 1 \right)^{1/r} \right) \right)^{1/q} \\
= \left( \sum_{i=1}^{n} \left(1 - \left( t\mu_{ij}^{qr} + s\mu_{ij}^{qr} - s - t + 1 \right)^{1/r} \right) \right)^{1/q} \\
\leq q - ROHFULSSBM(h_1, h_2, \ldots, h_n)
\end{align*}
\]

It is proved that \( q - ROHFULSSBM(h_1, h_2, \ldots, h_n) = h \).

Theorem 5 (Monotonicity): Let \( h_j = [s_{j}, \tau_{j}], \Gamma_{h_j}, \Psi_{h_j} \in q - ROHFULX(X), h_j' = [s_{j}', \tau_{j}'], \Gamma_{h_j}', \Psi_{h_j}' \in q - ROHFULX(X), j = 1, 2, \ldots, n \), if \( \forall j = 1, 2, \ldots, n \), \( h_j \leq h_j' \), then

\[
q - ROHFULSSBM(h_1, h_2, \ldots, h_n) \leq q - ROHFULSSBM(h_1', h_2', \ldots, h_n')
\]

Proof: Since \( \forall j = 1, 2, \ldots, n \), \( h_j \leq h_j' \) there are \( \mu_j' \geq \mu_j, v_j' \leq v_j \), then \( 1 - (t\mu_j^{qr} + s\mu_j^{qr} - s - t + 1)^{1/r} \geq 0 \).
1 - \left(1 - \left(1 - \left(1 - \left(1 - \left(1 - \left(1 - \left(\frac{1}{s + t} \right)^r \right)^{1/r} \right)^{1/r} \right)^{1/r} \right)^{1/r} \right)^{1/r} \right)^{1/r} \right) \leq q - ROHFULSSBM(h_1, h_2, \ldots, h_n) \\

\text{Theorem 6 (Boundedness): Let } h^- = \min \{s \in \mathbb{R}^+ \text{ such that } s \leq \frac{1}{s + t} \}, \ \Psi^{-} > h^+ = \min \{s \in \mathbb{R}^+ \text{ such that } s \geq \frac{1}{s + t} \}, \text{ where } s = \min(\bar{s}), \ \bar{s}^+ = \max(\bar{s}), \ \bar{s}^- = \min(\bar{s}), \text{ and let } h^+ = \max(\bar{s}), \ \bar{s}^- = \min(\bar{s}). \text{ Then } h^- \leq q - ROHFULSSBM(h_1, h_2, \ldots, h_n) \leq h^+.

\text{Proof: Based on Theorem 4 and 5, we have}

q - ROHFULSSBM(h_1, h_2, \ldots, h_n) \geq q - ROHFULSSBM(h^-, h^-, \ldots, h^-) = h^-

and

q - ROHFULSSBM(h_1, h_2, \ldots, h_n) \leq q - ROHFULSSBM(h^+, h^+, \ldots, h^+) = h^+

Then, we have

h^- \leq q - ROHFULSSBM(h_1, h_2, \ldots, h_n) \leq h^+.

\text{Theorem 7 (Commutativity): Let } h_j = \langle s_\theta, s_\eta \rangle, \ \Gamma, \ \Psi h_j > q \in q - ROHFULX(X), \ \bar{h}_j = \langle s_\theta, s_\eta \rangle, \ \bar{\Gamma}, \ \bar{\Psi} h_j > q \in q - ROHFULX(X), j = 1, 2, \ldots, n. \text{ Suppose } (\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n) \text{ is any permutation of } (h_1, h_2, \ldots, h_n), \text{ then}

q - ROHFULSSBM(h_1, h_2, \ldots, h_n) = q - ROHFULSSBM(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n)

\text{Proof: Since } (\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n) \text{ is any permutation of } (h_1, h_2, \ldots, h_n), \text{ then}

\left(\frac{1}{n(n-1)} \sum_{j=1}^{n} \left(\frac{1}{s + t} \right)^r \right)^{1/r} = \left(\frac{1}{n(n-1)} \sum_{j=1}^{n} \left(\frac{1}{s + t} \right)^r \right)^{1/r}

\text{thus,}

q - ROHFULSSBM(h_1, h_2, \ldots, h_n) = q - ROHFULSSBM(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n)

By giving different values of the parameters } r, s \text{ and } t, \text{ we get the following special cases.}

\text{Case 1: When } r = 0, \text{ then the q-ROHFULSSBM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Bonferroni mean (q-ROHFULBM) operator which can be presented as:}

q - ROHFULSSBM(h_1, h_2, \ldots, h_n)

\text{Case 2: When } s \to 0, \text{ then the q-ROHFULSSBM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar generalized arithmetic average operator (q-ROHFULSSGAA) operator which can be presented (20), as shown at the bottom of the next page.}

\text{Case 3: When } t = 1 \text{ and } s \to 0, \text{ then the q-ROHFULSSBM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar arithmetic average operator (q-ROHFULSSAA) operator which can be presented (21), as shown at the bottom of the next page.}

q - ROHFULSSAA(h_1, h_2, \ldots, h_n)
Definition 12: Let $h_j = <[s_{ij}, r_{ij}], \Gamma_{h_j}, \Psi_{h_j}> \in q - ROFHULE(X)(j = 1, 2, \ldots, n)$, then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar weighted Bonferroni mean (q-ROHFULSSWB) operator as follow:

$$q - ROHFULSSWB(h_1, h_2, \ldots, h_n)$$

$$= \left( \frac{1 + \sum_{i,j=1}^{n} (w_i h_i)^t \otimes (w_j h_j)^t}{n(n-1)} \right)^{1/q},$$

where $s, t \geq 0$ and $s + t > 0$. $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $h_j(j = 1, 2, \ldots, n)$, which satisfies $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

Theorem 8: Let $h_j = <[s_{ij}, s_{ij}], \Gamma_{h_j}, \Psi_{h_j}> \in q - ROFHULE(X)(j = 1, 2, \ldots, n)$, and their weight vector $w = (w_1, w_2, \ldots, w_n)^T$ satisfies $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$, then the aggregate results of Definition 12 is still a q-ROHFUL, and (23), as shown at the bottom of the next page.

Proof: The proof process of Theorem 8 is similar to that of Theorem 3. Replace $h_i$ and $h_j$ of Theorem 3 with $w_i h_i$ and $w_j h_j$, where

$$nw_i h_i = \bigcup_{\mu_i \in \Gamma_{h_i}, \nu_i \in \Psi_{h_i}} \left\{ \left( \frac{s}{1 - \sum_{j=1}^{n} (1 - (1 - \nu_i)^{q}(1 - \nu_j)^{q})} \right)^{1/q} \right\},$$

$$nw_j h_j = \bigcup_{\mu_j \in \Gamma_{h_j}, \nu_j \in \Psi_{h_j}} \left\{ \left( \frac{s}{1 - \sum_{j=1}^{n} (1 - (1 - \nu_j)^{q}(1 - \nu_j)^{q})} \right)^{1/q} \right\}.$$ 

Thus, it can be proved that Theorem 8 is valid.

In addition, similar to the q-ROHFULSSWB operator, it can be proved that the q-ROHFULSSWB operator satisfies monotonicity and boundedness.

B. q-ROHFULSSGBM OPERATOR

Next, we introduce the geometric Bonferroni mean operator [41] into the q-rung orthopair hesitant fuzzy uncertain linguistic set, and define the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric Bonferroni mean operator.

Definition 13: Let $h_j = <[s_{ij}, s_{ij}], \Gamma_{h_j}, \Psi_{h_j}> \in q - ROFHULE(X)(j = 1, 2, \ldots, n)$, then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric Bonferroni mean

$$q - ROHFULSSGBA(h_1, h_2, \ldots, h_n)$$

$$= \left( \frac{1 + \sum_{i,j=1}^{n} (w_i h_i)^t \otimes (w_j h_j)^t}{n(n-1)} \right)^{1/q},$$

$$= \left( \frac{1}{n} \sum_{j=1}^{n} h_j \right)^{1/q} \bigcup_{\mu_i \in \Gamma_{h_i}, \nu_i \in \Psi_{h_i}} \left\{ \left( \frac{s}{1 - \sum_{j=1}^{n} (1 - (1 - \nu_i)^{q}(1 - \nu_j)^{q})} \right)^{1/q} \right\}.$$ 

(20)

(21)
(q-ROHFULSSGBM) operator as follow:

\[
q - \text{ROHFULSSGBM}(h_1, h_2, \ldots, h_n) = \left( \frac{\sum_{i,j=1}^{n} (sh_i + th_j)}{s + t} \right)^{\frac{1}{n^{\frac{1}{m-1}}}}
\]

where \( s, t \geq 0 \) and \( s + t > 0 \).

**Theorem 9:** Let \( h_j = <s_{0j}, s_{1j}>, \Gamma_{h_j}, \Psi_{h_j} > \in q - ROHFULE(X)(j = 1, 2, \ldots, n) \), then the aggregate results of Definition 13 is still a q-ROHFULE, and (24), as shown at the bottom of the next page.

**Proof:** The proof process of Theorem 9 is similar to that of Theorem 3.

Moreover, it can be proved that the q-ROHFULSSGBM operator satisfies commutativity, boundedness, monotonicity and idempotence.

By giving different values of the parameters \( r, s \) and \( t \), we get the following special cases.

**Case 1:** When \( r = 0 \), then the q-ROHFULSSGBM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic geometric Bonferroni mean (q-ROHFULGBM) operator which can be presented (25), as shown at the bottom of the next page.

**Case 2:** When \( s \to 0 \), then the q-ROHFULSSGBM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar generalized geometric average operator (q-ROHFULSSGGA) operator which can be presented (26), as shown at the bottom of the next page.

**Case 3:** When \( t = 1 \) and \( s \to 0 \), then the q-ROHFULSSGBM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric average operator (q-ROHFULSSGA) operator which can be presented as:

\[
q - \text{ROHFULSSGA}(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_j \in \Gamma_j} \left[ s \left( \frac{\sum_{i,j=1}^{n} (nw_i \theta_j)^r (nw_j \theta_i)^r}{\sum_{i,j=1}^{n} (nw_i \theta_j)^r (nw_j \theta_i)^r} \right)^{\frac{1}{r}} \right], \]

where

\[
\begin{cases}
\{ s^{\frac{1}{r}}, s, \} & \text{if } \frac{1}{s+t} \leq 1 \\
\left( s^{\frac{1}{r}}, s, \frac{1}{s+t} \right)^{\frac{1}{r}} & \text{if } 1 < \frac{1}{s+t} \leq s-t+1 \\
\left( s^{\frac{1}{r}}, 1, \frac{1}{s+t} \right)^{\frac{1}{r}} & \text{if } s-t+1 < \frac{1}{s+t}
\end{cases}
\]

Next, the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar weighted geometric Bonferroni mean operator is proposed.

**Definition 14:** Let \( h_j = <s_{0j}, s_{1j}>, \Gamma_{h_j}, \Psi_{h_j} > \in q - ROHFULE(X) (j = 1, 2, \ldots, n) \), then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic
Schweizer-Sklar weighted geometric Bonferroni mean (q-ROHFULSSWGBM) operator as follow:

\[
q - \text{ROHFULSSWGBM}(h_1, h_2, \ldots, h_n)
= \left( \frac{n}{\sum_{i,j=1 \atop i \neq j}^n (s_{h_i}^{p_{ij}} + s_{h_j}^{p_{ij}})} \right)^{1/(s+t)},
\]

where \(s, t \geq 0\) and \(s + t > 0\).

The weight vector of \(h_j(\mathbf{r}) = \{w_1, w_2, \ldots, w_n\}^\top\) is \(w_j \in [0, 1]\) and \(\sum_{j=1}^n w_j = 1\).

**Theorem 10:** Let \(h_j = \{s_{r}, s_{t}\}, \Gamma_{h_j}, \Psi_{h_j} \in q - \text{ROHFULE}(\mathbf{r})\), \(1, 2, \ldots, n\), and their weight vector \(w = (w_1, w_2, \ldots, w_n)^\top\) satisfies \(w_j \in [0, 1]\) and \(\sum_{j=1}^n w_j = 1\), then...
the aggregate results of Definition 14 is still a q-ROHFULE, and (28), as shown at the bottom of the page.

Proof: The proof process of Theorem 10 is similar to that of Theorem 9. Replace \( h_i \) and \( h_j \) of Theorem 9 with \( h_i^{nw_i} \) and \( h_j^{nw_j} \), where

\[
\begin{align*}
    h_i^{nw_i} &= \bigcup_{\mu_k \in \Gamma h_i} \left\{ x_{ij}^{nw_i}, s_{ij}^{nw_i} \right\} \left( \frac{(nw_i \mu_i^{q} - (nw_i - 1))^{1/q}}{1} \right) q \\
    h_j^{nw_j} &= \bigcup_{\mu_k \in \Gamma h_j} \left\{ x_{ij}^{nw_j}, s_{ij}^{nw_j} \right\} \left( \frac{(nw_j \mu_j^{q} - (nw_j - 1))^{1/q}}{1} \right) q
\end{align*}
\]

Thus, it can be proved that Theorem 10 is valid.

In addition, similar to the q-ROHFULSSGBM operator, it can be proved that the q-ROHFULSSWGBM operator satisfies monotonicity and boundedness.

V. THE Q-RUNG ORTHOPAIR HESITANT FUZZY UNCERTAIN LINGUISTIC SCHWEIZER-SKLAR MACLAURIN SYMMETRIC MEAN OPERATOR

A. q-ROHFULSSMSM OPERATOR

The Maclaurin symmetric mean (MSM) operator [42] can describe the correlation between multiple decision attributes. The Maclaurin symmetric mean operator is introduced into the q-rung orthopair hesitant fuzzy uncertain linguistic set. Based on Schweizer-Sklar T-norm, the q-rung orthopar

\[
q - \text{ROHFULSSWGBM}(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_k \in \Gamma h_i} \left\{ x_{ij}^{nw_i}, s_{ij}^{nw_i} \right\} \left( \frac{(nw_i \mu_i^{q} - (nw_i - 1))^{1/q}}{1} \right) q
\]

\[
\begin{align*}
    &= \left\{ 1 - \left( 1 - \frac{1}{s + t} \right) \left( 1 - \left( 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} (1 - (s(1 - (nw_i \mu_i^{q} - nw_i + 1))^{1/r}))^{1/r} \right) \right) \right\}^{-1} \left( 1 - \left( 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} (1 - (s(1 - (nw_i \mu_i^{q} - nw_i + 1))^{1/r}))^{1/r} \right) \right) \right\}^{-1/q} \\
    &+ t(1 - (nw_j \mu_j^{q} - nw_j + 1)^{1/r})^{r} - s - t + 1 \left( 1 - \left( 1 - \frac{1}{s + t} \right) \left( 1 - \left( 1 - \frac{1}{n(n-1)} \sum_{i,j=1}^{n} (1 - (s(1 - (nw_i \mu_i^{q} - nw_i + 1))^{1/r}))^{1/r} \right) \right) \right) \right\}^{-1/q} \\
    &= \left\{ 1 - \left( 1 - \frac{1}{s + t} \right) \sum_{i,j=1}^{n} (1 - (s(1 - (nw_i \mu_i^{q} - nw_i + 1))^{1/r}))^{1/r} \right) \right\}^{-1/q} \left( 1 - \left( 1 - \frac{1}{s + t} \right) \sum_{i,j=1}^{n} (1 - (s(1 - (nw_i \mu_i^{q} - nw_i + 1))^{1/r}))^{1/r} \right) \right\}^{-1/q}
\end{align*}
\]

(28)
Proof: According to the operational laws of q-ROHFULEs, we get

\[ \bigoplus_{j=1}^{k} h_{ij} = \bigcup_{\mu_j \in \Gamma_j, \nu_j \in \Psi_j} \left\{ \left[ s, s, \prod_{j=1}^{k} h_{ij}, \prod_{j=1}^{k} h_{ij} \right], \left\{ \left( \sum_{j=1}^{k} (\mu_j^q)^r - (k-1) \right)^{1/r} \right\} \right\} , \]

Moreover, similar to Theorem 3, it is proved by mathematical induction that

\[ \bigoplus_{1 \leq i_1 \leq \ldots \leq i_k \leq n} (\bigotimes_{j=1}^{k} h_{ij}) \]

\[ = \bigcup_{\mu_j \in \Gamma_j, \nu_j \in \Psi_j} \left\{ \left[ s, \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} h_{ij}, \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} h_{ij} \right], \left\{ \left( \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left( 1 - \left( \sum_{j=1}^{k} (\mu_j^q)^r - (k-1) \right)^{1/r} \right)^{1/r} \right)^{1/r} \right\} \right\} . \]

Therefore,

\[ \left( \frac{1}{C_n^k} \bigoplus_{1 \leq i_1 \leq \ldots \leq i_k \leq n} (\bigotimes_{j=1}^{k} h_{ij}) \right)^{1/k} \]

\[ = \bigcup_{\mu_j \in \Gamma_j, \nu_j \in \Psi_j} \left\{ \left[ s, \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} h_{ij}, \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} h_{ij} \right], \left\{ \left( \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left( 1 - \left( \sum_{j=1}^{k} (\mu_j^q)^r - (k-1) \right)^{1/r} \right)^{1/r} \right)^{1/r} \right\} \right\} . \]

**Theorem 2 (Idempotence):** Let \( h_j = \langle [s_{ij}, s_{ij}], \Gamma_j, \Psi_j \rangle \), if \( h_j = h = \langle [s_{ij}, s_{ij}], \Gamma, \Psi \rangle \) is satisfied for any \( h_j \), then

\[ HFULSSM(h_1, h_2, \ldots, h_n) = h. \]

**Theorem 3 (Monotonicity):** Let \( h_j = \langle [s_{ij}, s_{ij}], \Gamma_j, \Psi_j \rangle \), if \( h_j = h = \langle [s_{ij}, s_{ij}], \Gamma_j, \Psi_j \rangle \) and \( h_j' = \langle [s_{ij}', s_{ij}], \Gamma_j', \Psi_j' \rangle \) are satisfied for any \( h_j \), then

\[ q - ROHFULSSM(h_1, h_2, \ldots, h_n) \leq q - ROHFULSSM(h_1', h_2', \ldots, h_n') \]

\[ q - ROHFULSSM(h_1, h_2, \ldots, h_n) \]

\[ = \bigcup_{\mu_j \in \Gamma_j, \nu_j \in \Psi_j} \left\{ \left[ s, \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} h_{ij}, \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} h_{ij} \right], \left\{ \left( \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} \left( 1 - \left( \sum_{j=1}^{k} (\mu_j^q)^r - (k-1) \right)^{1/r} \right)^{1/r} \right)^{1/r} \right\} \right\} . \]
Theorem 14 (Boundedness): Let \( h^- = \langle [s_0^-, s^-_0], \Gamma^-, \Psi^- \rangle \), \( h^+ = \langle [s^+, s^+_0], \Gamma^+, \Psi^+ \rangle \), where \( s_0^- = \min(\bar{S}), s_0^+ = \max(\bar{S}) \), \( \Gamma^- = \min(\mu) \), \( \Gamma^+ = \min(\mu) \) when \( j = 1, 2, \ldots, n \), \( \Psi^- = \min(\nu) \), \( \Psi^+ = \max(\nu) \) when \( j = 1, 2, \ldots, n \). Then

\[
h^- \leq q - \text{ROHFULSSMSM}(h_1, h_2, \ldots, h_n) \leq h^+.
\]

Theorem 15 (Commutativity): Let \( h_j = \langle \bar{s}_0, s_j \rangle, \Gamma_{h_j}, \Psi_{h_j} \rangle \), \( q \in q - \text{ROHFUL}(X) \), \( \bar{h}_j = \langle [s_0 \bar{s}_j, s_0], \Gamma_{\bar{h}_j}, \Psi_{\bar{h}_j} \rangle \), \( q \in q - \text{ROHFUL}(X) \), \( j = 1, 2, \ldots, n \). Suppose \( \{h_1, h_2, \ldots, h_n\} \) is any permutation of \( \{h_1, h_2, \ldots, h_n\} \), then

\[
q - \text{ROHFULSSMSM}(h_1, h_2, \ldots, h_n) = q - \text{ROHFULSSMSM}(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n).
\]

By giving different values of the parameters \( r \) and \( k \), we get the following special cases.

Case 1: When \( r = 0 \), then the q-ROHFULSSMS operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Maclaurin symmetric mean (q-ROHFULSSM) operator which can be presented (30), as shown at the bottom of the page.

Case 2: When \( k = 1 \), then the q-ROHFULSSMS operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar arithmetic average (q-ROHFULSSAA) operator which can be presented as:

\[
q - \text{ROHFULSSAA}(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_j \in \Gamma_j} \left[ \left\{ \sum_{j=1}^{n} \frac{\mu_j}{n} \right\}^{1/2}, \sum_{j=1}^{n} \frac{\mu_j}{n} \right\}^{1/2} \right\}
\]

Case 3: When \( k = 2 \), then the q-ROHFULSSMS operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Bonferroni mean (q-ROHFULSSBM) operator.

Case 4: When \( k = n \), then the q-ROHFULSSMS operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric average (q-ROHFULSSGA) operator which can be presented (32) and (33), as shown at the bottom of the next page.

In actual decision-making, the weight of each attribute may be different. For this reason, the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar weighted Maclaurin symmetric mean (q-ROHFULSSWMSM) operator is proposed.

Definition 16: Let \( h_j = \langle \bar{s}_0, s_j \rangle, \Gamma_{h_j}, \Psi_{h_j} \rangle \), \( q \in q - \text{ROHFUL}(X) \), \( j = 1, 2, \ldots, n \), then we can define the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar weighted Maclaurin symmetric mean (q-ROHFULSSWMSM) operator as follow:

\[
q - \text{ROHFULSSWMSM}(h_1, h_2, \ldots, h_n) = \left( \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{C^k_\mu}{C^n_\mu} (w_{ij}) \right)^{1/k},
\]

where \( k \) is a parameter and \( k = 1, 2, \ldots, n \), \( i_1, i_2, \ldots, i_n \) are \( k \) integer values taken from the set \( \{1, 2, \ldots, n\} \) of \( n \) integer values, \( C^k_\mu \) represents the binomial coefficient and \( C^n_\mu \). And \( w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( h_j(j = 1, 2, \ldots, n) \), which satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

Theorem 16: Let \( h_j = \langle \bar{s}_0, s_j \rangle, \Gamma_{h_j}, \Psi_{h_j} \rangle \in q - \text{ROHFUL}(X) \), \( j = 1, 2, \ldots, n \), and their weight vector \( w = (w_1, w_2, \ldots, w_n)^T \) satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), then the aggregate results of Definition 16 is still a q-ROHFUL, and (34), as shown at the bottom of the next page.

Proof: The proof process of Theorem 16 is similar to that of Theorem 11. Replace \( h_j \) of Theorem 11 with \( w_{ij} h_j \),

\[
q - \text{ROHFUL}(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_j \in \Gamma_j} \left[ \left\{ \sum_{j=1}^{n} \frac{\mu_j}{n} \right\}^{1/2}, \sum_{j=1}^{n} \frac{\mu_j}{n} \right\}^{1/2} \right\}
\]

(30)
where

\[
\begin{align*}
nw_i h_j &= \bigcup_{\mu_{h_j} \in \Gamma_{h_j}} \left\{ \left[ s_{nw_i \theta_j}, s_{nw_i \tau_j} \right] \right\}, \\
&= \left\{ (1 - (nw_j(1 - \mu^q_{h_j} - (nw_j - 1))_1/r)^{1/q}) \right\}, \\
&= \left\{ (nw_j v_{h_j}^{qr} - (nw_j - 1))^{1/qr} \right\}
\end{align*}
\]

Thus, it can be proved that Theorem 16 is valid.

Moreover, it can be proved that the q-ROHFULSSWMSM operator satisfies monotonicity and boundedness.

**B. q-ROHFULSSDMSM OPERATOR**

Next, the dual Maclaurin symmetric mean operator is introduced into the q-rung orthopair hesitant fuzzy uncertain linguistic set, and the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar dual Maclaurin symmetric mean (q-ROHFULSSDMSM) operator is defined.

**Definition 17:** Let \( h_j = \langle [s_q, s_r], \Gamma_{h_j}, \Psi_{h_j} \rangle \in q - ROHFULE(X) (j = 1, 2, \ldots, n) \), then we can

\[
\begin{align*}
q &- ROHFULSSGA(h_1, h_2, \ldots, h_n) \\
&= \bigcup_{\mu_i \in \Gamma_i, v_j \in \Psi_j} \left\{ \left[ s_{\mu_i \theta_j}, s_{\mu_i \tau_j} \right] \right\}, \\
&= \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu^q_{\theta_j})^{1/q} \right) \right\}, \\
&= \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu^q_{\tau_j})^{1/q} \right) \right\}
\end{align*}
\]

\[
\begin{align*}
q &- ROHFULSSBM(h_1, h_2, \ldots, h_n) \\
&= \bigcup_{\mu_i \in \Gamma_i, v_j \in \Psi_j} \left\{ \left[ s_{\mu_i \theta_j}, s_{\mu_i \tau_j} \right] \right\}, \\
&= \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu^q_{\theta_j})^{1/q} \right) \right\}, \\
&= \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu^q_{\tau_j})^{1/q} \right) \right\}
\end{align*}
\]

\[
\begin{align*}
q &- ROHFULSSWMSM(h_1, h_2, \ldots, h_n) \\
&= \bigcup_{\mu_i \in \Gamma_i, v_j \in \Psi_j} \left\{ \left[ s_{\mu_i \theta_j}, s_{\mu_i \tau_j} \right] \right\}, \\
&= \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu^q_{\theta_j})^{1/q} \right) \right\}, \\
&= \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu^q_{\tau_j})^{1/q} \right) \right\}
\end{align*}
\]
defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar dual Maclaurin symmetric mean (q-ROHFULDDMSM) operator as follow:

\[
q - ROHFULDDMSM(h_1, h_2, \ldots, h_n) = \frac{\prod_{1 \leq i_1 \leq \ldots \leq i_k \leq n} (\sum_{j=1}^{k} h_{ij}^q)^{1/k}}{k},
\]

where \( k \) is a parameter and \( k = 1, 2, \ldots, n, i_1, i_2, \ldots, i_n \) are \( k \) integer values taken from the set \( \{1, 2, \ldots, n\} \) of \( n \) integer values, \( C_k^n \) represents the binomial coefficient and \( C_n^k = \frac{n!}{k!(n-k)!} \).

**Theorem 17:** Let \( h_j = <[s_{ij}, \tau_{ij}], \Gamma_{h_j}, \Psi_{h_j}> \in q - ROHFULDDM(j = 1, 2, \ldots, n) \), then the aggregate results of Definition 17 is still a q-ROHFULD, and (35), as shown at the bottom of the page.

**Proof:** The proof process of Theorem 17 is similar to that of Theorem 11.

Moreover, it can be proved that the q-ROHFULDDMSM operator satisfies commutativity, boundedness, monotonicity and idempotence.

By giving different values of the parameters \( r, k \), we get the following special cases.

**Case 1:** When \( r = 0 \), then the q-ROHFULDDMSM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic dual Maclaurin symmetric mean (q-ROHFULDDMSM) operator which can be presented (36), as shown at the bottom of the page.

**Case 2:** When \( k = 1 \), then the q-ROHFULDDMSM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric average (q-ROHFULDDMSM) operator which can be presented as:

\[
q - ROHFULDDMSM(h_1, h_2, \ldots, h_n)
\]

**Case 3:** When \( k = 2 \), then the q-ROHFULDDMSM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric Bonferroni mean (q-ROHFULDDMSM) operator.

**Case 4:** When \( k = n \), then the q-ROHFULDDMSM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar arithmetic average...
(q-ROHFULSSAA) operator which can be presented as (39),
as shown at the bottom of the page.

\[
q - \text{ROHFULSSAA}(h_1, h_2, \ldots, h_n) = \underset{\mu_j \in \Gamma_j, v_j \in \Psi_j}{\bigcup} \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu_j^q)^{1/r} \right)^{1/q} \right\}^{1/\alpha},
\]

Definition 18: Let \( h_j = [s_j, \tau_j], \gamma_j, \Psi_j > \in q - \text{ROHFULE}(X) \) \( j = 1, 2, \ldots, n \), then we can
define the q-rung orthopair hesitant fuzzy uncertain linguistic
Switchez-Sklar weighted geometric Bonferroni mean
(q-ROHFULSSWGBM) operator as follow:

\[
q - \text{ROHFULSSWGBM}(h_1, h_2, \ldots, h_n)
\]

\[
= \underset{\mu_j \in \Gamma_j, v_j \in \Psi_j}{\bigcup} \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu_j^q)^{1/r} \right)^{1/q} \right\}^{1/\alpha},
\]

The proof process of Theorem 18 is similar to
that of Theorem 17. Replace \( h_j \) of Theorem17 with

\[
q - \text{ROHFULSSGBM}(h_1, h_2, \ldots, h_n)
\]

\[
= \underset{\mu_j \in \Gamma_j, v_j \in \Psi_j}{\bigcup} \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu_j^q)^{1/r} \right)^{1/q} \right\}^{1/\alpha},
\]

\[
q - \text{ROHFULSSWDMSM}(h_1, h_2, \ldots, h_n)
\]

\[
= \underset{\mu_j \in \Gamma_j, v_j \in \Psi_j}{\bigcup} \left\{ \left( \frac{1}{n} \sum_{j=1}^{n} (\mu_j^q)^{1/r} \right)^{1/q} \right\}^{1/\alpha},
\]

where \( s, t \geq 0 \) and \( s + t > 0 \), \( w = (w_1, w_2, \ldots, w_n)^T \) is the
weight vector of \( h_j \) \( j = 1, 2, \ldots, n \), which satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

Theorem 18: Let \( h_j = [s_j, \tau_j], \gamma_j, \Psi_j > \in q - \text{ROHFULE}(X) \) \( j = 1, 2, \ldots, n \), and their weight vec-
tor \( w = (w_1, w_2, \ldots, w_n)^T \) satisfies \( w_j \in [0, 1] \) and
\( \sum_{j=1}^{n} w_j = 1 \), then the aggregate results of Definition18 is still
a q-ROHFULE, and (40), as shown at the bottom of the
page.
where
\[ m_{w_{j}}h_{j} = \bigcup_{\mu_{h_{j}}\in\Gamma_{h_{j}}} \left\{ \frac{(mw_{j}y_{j}-mw_{j}y_{j})}{(1-(mw_{j}y_{j}-mw_{j}y_{j}))^{1/q}} \right\} \]

Thus, it can be proved that Theorem 18 is valid.

Moreover, it can be proved that the q-ROHFULSSWDSMM operator satisfies monotonicity and boundedness.

VI. THE Q-RUNG ORTHOPAIR HESITANT FUZZY UNCERTAIN LINGUISTIC SCHWEIZER-SKLAR MACLAURIN MEAN OPERATOR

A. q-ROHFULSSMM OPERATOR

Maclaurin mean (MM) operator [43] can describe the relationship between all decision attributes. The Maclaurin mean operator is introduced into the q-rung orthopair hesitant fuzzy uncertain linguistic set. Based on the Schweizer-Sklar T-norm, the q-rung orthopair hesitant fuzzy uncertain linguistic Maclaurin mean (q-ROHFULSSMM) operator and the q-rung orthopair hesitant fuzzy uncertain linguistic dual Maclaurin mean (q-ROHFULSSDMM) operator are defined.

Definition 19: Let \( h_{j} = <[s_{0}, s_{1}], \Gamma_{h_{j}}, \Psi_{h_{j}} > \in q – ROHFULE(X) \) \((j = 1, 2, \ldots, n)\), then we can define the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Maclaurin mean (q-ROHFULSSMM) operator as follow:

\[ q – ROHFULSSMM(h_{1}, h_{2}, \ldots, h_{n}) = \left( \bigoplus_{\vartheta(j)\in S_{n}} \left( \bigotimes_{j=1}^{n} S_{j}^{\vartheta(j)} \right)^{\frac{1}{q}} \right) ^{-1/p}, \]

where \( \vartheta(j) \) \((j = 1, 2, \ldots, n)\) is any a permutation of \((1, 2, \ldots, n)\), and \( S_{n} \) represents the set of all permutation of \((1, 2, \ldots, n)\). \( P = (p_{1}, p_{2}, \ldots, p_{n}) \in R^{n} \) is a parameter vector.

Theorem 19: Let \( h_{j} = <[s_{0}, s_{1}], \Gamma_{h_{j}}, \Psi_{h_{j}} > \in q – ROHFULE(X) \) \((j = 1, 2, \ldots, n)\), then the aggregate results of Definition 19 is still a q-ROHFULE, and (41), as shown at the bottom of the page.

Proof: According to the operational laws of q-ROHFULEs, we get

\[ \bigotimes_{j=1}^{n} H_{\vartheta(j)}^{p_{j}} = \bigcup_{\mu_{p_{j}}\in\Gamma_{p_{j}}} \left\{ [s_{0}, \prod_{j=1}^{n} p_{j}]^{\frac{1}{q}} \right\}, \]

where

\[ \left\{ \begin{array}{l}
\left( \sum_{j=1}^{n} (p_{j} \mu_{\vartheta(j)}^{q} (p_{j}-1)) \right) - (n-1) ^{1/r} \left( 1 - \sum_{j=1}^{n} (1 - v_{\vartheta(j)}^{q} (p_{j}-1)) (n-1) ^{1/r} \right) ^{1/q} \\
\end{array} \right\} \]

Moreover, similar to Theorem 3, it can be proved by mathematical induction that

\[ \bigoplus_{\vartheta(j)\in S_{n}} \left( \bigotimes_{j=1}^{n} S_{j}^{\vartheta(j)} \right) ^{\frac{1}{q}} \]

(41)
Thus, this proves that Theorem 19 is valid.

**Theorem 20 (Idempotence):** Let $h_j = <s_{q_j}, t_{q_j}, \Gamma_{h_j},$ \(\Psi_{h_j} \rangle \in q - ROHFULX(X)$, $j = 1, 2, \ldots, n$, if $h_j = h = <s_q, t_q, \Gamma, \Psi>$ is satisfied for any $h_j$, then

$$q - ROHFULSSMM(h_1, h_2, \ldots, h_n) = h.$$  

**Theorem 21 (monotonicity):** Let $h_j = <s_{q_j}, t_{q_j}, \Gamma_{h_j},$ \(\Psi_{h_j} \rangle \in q - ROHFULX(X)$, $h'_j = <s'_{q_j}, t'_{q_j}, \Gamma'_{h_j},$ \(\Psi'_{h_j} \rangle \in q - ROHFULX(X)$, $j = 1, 2, \ldots, n$, if $\forall j = 1, 2, \ldots, n, h'_j \leq h_j$, then

$$q - ROHFULSSMM(h_1, h_2, \ldots, h_n) \leq q - ROHFULSSMM(h'_1, h'_2, \ldots, h'_n).$$

**Theorem 22 (Bounding):** Let $h^- = <s^-_{q_j}, t^-_{q_j}, \Gamma^-,$ \(\Psi^- \rangle, h^+ = <s^+_{q_j}, t^+_{q_j}, \Gamma^+,$ \(\Psi^- \rangle, \text{where } s^-_{q_j} = \min(\bar{s})_{q_j}, s^+_{q_j} = \max(\bar{s})_{q_j}, \Gamma^- = \min(\bar{\mu})_{j = 1, 2, \ldots, n}, \Gamma^+ - \max(\bar{\mu})_{j = 1, 2, \ldots, n}, \Psi^- = \min(\bar{\psi})_{j = 1, 2, \ldots, n}, \Psi^+ = \max(\bar{\psi})_{j = 1, 2, \ldots, n}. \text{Then} \ h^- \leq q - ROHFULX(X, h_1, h_2, \ldots, h_n) \leq h^+.$$

**Theorem 23 (Commutativity):** Let $h_j = <s_{\theta_j}, t_{\theta_j}, \Gamma_{h_j},$ \(\Psi_{h_j} \rangle \in q - ROHFULX(X)$, $h'_j = <s'_{\theta_j}, t'_{\theta_j}, \Gamma_{h'_j},$ \(\Psi_{h'_j} \rangle \in q - ROHFULX(X)$, $j = 1, 2, \ldots, n$. Suppose $(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n)$ is any permutation of $(h_1, h_2, \ldots, h_n)$, then

$$q - ROHFULSSMM(h_1, h_2, \ldots, h_n) = q - ROHFULSSMM(\bar{h}_1, \bar{h}_2, \ldots, \bar{h}_n).$$

By giving different values of the parameters $r, k$, we get the following special cases.

**Case 1:** When $r = 0$, then the $q$-ROHFULSSMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Maclaurin mean (q-ROHFULMM) operator which can be presented (42), as shown at the bottom of the page.

**Case 2:** When $P = (1, 0, \ldots, 0)$, then the $q$-ROHFULSSMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar arithmetic average (q-ROHFULSSAA) operator which can be presented (43), as shown at the bottom of the page.

**Case 3:** When $P = (1, 1, 0, \ldots, 0)$, then the $q$-ROHFULSSMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Bonferroni mean (q-ROHFULSSBM) operator which can be presented (44), as shown at the bottom of the next page.

**Case 4:** When $P = (1, 1, \ldots, 1, 0, \ldots, 0)$, then the $q$-ROHFULSSMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Maclaurin symmetric mean (q-ROHFULSSSM) operator which can be presented (45), as shown at the bottom of the next page.

**Case 5:** When $P = (1, 1, \ldots, 1)$ or $P = (1/n, 1/n, \ldots, 1/n)$, then the $q$-ROHFULSSMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric average
(q-ROHFULSSGA) operator which can be presented as:

\[ q - \text{ROHFULSSGA}(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_i \in \Gamma_i, \nu_j \in \Psi_j} \left[ s \begin{pmatrix} \Pi_{i \neq j} (q_j) & \Pi_{i \neq j} (r_j) \end{pmatrix} \right]^{1/2} \left[ s \begin{pmatrix} \Pi_{i \neq j} (q_j) & \Pi_{i \neq j} (r_j) \end{pmatrix} \right]^{1/2}, \]

\[ \left\{ \left( 1 - \left( \frac{1}{n} \sum_{i=1 \atop i \neq j}^{n} (1 - v_j^q)^r \right)^{1/r} \right)^{1/q} \right\} \] (46)

Definition 20: Let \( h_j = \langle s_{\theta j}, s_{\tau j}, \Gamma_{h_j}, \Psi_{h_j} \rangle \in q - \text{ROHFULE}(X) \), then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar weighted Maclaurin mean (q-ROHFULSSWM) operator as follow:

\[ q - \text{ROHFULSSWM}(h_1, h_2, \ldots, h_n) \]

\[ = \bigcup_{\mu_i \in \Gamma_i, \nu_j \in \Psi_j} \left[ s \begin{pmatrix} \Pi_{i \neq j} (q_j) & \Pi_{i \neq j} (r_j) \end{pmatrix} \right]^{1/2} \left[ s \begin{pmatrix} \Pi_{i \neq j} (q_j) & \Pi_{i \neq j} (r_j) \end{pmatrix} \right]^{1/2}, \]

\[ \left\{ \left( \frac{1}{2} \left( 1 - \left( \frac{1}{n(n-1)} \sum_{i,j=1 \atop i \neq j}^{n} (1 - \mu_j^q)^r + (\mu_j^q - 1)^r \right)^{1/r} \right)^r \right)^{1/q} \right\} \] (44)

\[ q - \text{ROHFULSSM}(h_1, h_2, \ldots, h_n) \]

\[ = \bigcup_{\mu_i \in \Gamma_i, \nu_j \in \Psi_j} \left[ s \begin{pmatrix} \Pi_{i \neq j} (q_j) & \Pi_{i \neq j} (r_j) \end{pmatrix} \right]^{1/2} \left[ s \begin{pmatrix} \Pi_{i \neq j} (q_j) & \Pi_{i \neq j} (r_j) \end{pmatrix} \right]^{1/2}, \]

\[ \left\{ \left( \frac{1}{k} \left( 1 - \frac{1}{C_k^n} \sum_{1 \leq h \leq \cdots \leq k \leq n} (1 - \sum_{j=1}^{k} (\mu_j^q r - (k-1))^r)^{1/r} \right)^{1/q} \right)^{1/r} \right\} \] (45)

where \( \theta(j) = 1, 2, \ldots, n \) is any a permutation of \((1, 2, \ldots, n)\), and \( S_n \) represents the set of all permutation of \( (1, 2, \ldots, n) \). \( P = (p_1, p_2, \ldots, p_n) \in R^n \) is a parameter vector. And \( w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( h(j) = 1, 2, \ldots, n \), which satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

**Theorem 24:** Let \( h_j = \langle s_{\theta j}, s_{\tau j}, \Gamma_{h_j}, \Psi_{h_j} \rangle \in q - \text{ROHFULE}(X) \), and their weight vector \( \varphi = (w_1, w_2, \ldots, w_n)^T \) satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), then the aggregate results of Definition 20 is still a q-ROHFULE, and (47), as shown at the bottom of the next page.

**Proof:** The proof process of Theorem 24 is similar to that of Theorem 19. Replace \( h_{\theta j} \) of Theorem 19 with \( n w_{\theta j} h_{\theta j} \). Thus, it can be proved that Theorem 24 is valid.
Moreover, it can be proved that the q-ROHFULSSWMM operator satisfies monotonicity and boundedness.

**B. q-ROHFULSSDMM OPERATOR**

In this section, the dual Maclaurin mean operator is introduced into the q-rung orthopair hesitant fuzzy uncertain linguistic set, and the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar dual Maclaurin mean (q-ROHFULSSDMM) operator is defined.

**Definition 21:** Let \( h_j = <[s_{θ_j}, s_{r_j}], Γ_{h_j}, Ψ_{h_j}> \in q - ROHULE (X) (j = 1, 2, ..., n) \), then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar Maclaurin mean(q-ROHFULSSDMM) operator as follow:

\[
q - ROHFULSSDMM\ (h_1, h_2, ..., h_n) = \left( \bigotimes_{\theta \in S_n} \bigoplus_{j=1}^{n} (p_j h_{\theta (j)}) \right)^{\frac{1}{n}},
\]

\[
q - ROHFULSSWMM\ (h_1, h_2, ..., h_n)
\]

\[
= \bigcup_{\mu_{\theta (j)} \in \Gamma_{\theta (j)}, \upsilon_{\theta (j)} \in \Psi_{\theta (j)}} \left\{ \sum_{\pi \in \Pi_{S_n}} \left[ \prod_{j=1}^{n} p_j \right]^{\frac{1}{n}} \sum_{\theta \in S_n} \left[ \prod_{j=1}^{n} (p_j \tau_{\theta (j)}) \right]^{\frac{1}{n}}, \right\}.
\]

\[
\left\{ \left( 1 - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right)^{1/r} - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right\}^{1/q}
\]

\[
= \left\{ \left( 1 - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right)^{1/r} - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right\}^{1/q}
\]

\[
= \left\{ \left( 1 - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right)^{1/r} - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right\}^{1/q}
\]

\[
= \left\{ \left( 1 - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right)^{1/r} - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right\}^{1/q}
\]

\[
= \left\{ \left( 1 - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right)^{1/r} - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right\}^{1/q}
\]

\[
= \left\{ \left( 1 - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right)^{1/r} - \left( \sum_{j=1}^{n} (p_j (1 - \mu_{\theta (j)}) - p_j + 1) \right) \right\}^{1/q}
\]
where $\vartheta(j) (j = 1, 2, \ldots, n)$ is any a permutation of $(1, 2, \ldots, n)$, and $S_n$ represents the set of all permutation of $(1, 2, \ldots, n)$. $P = (p_1, p_2, \ldots, p_n) \in R^n$ is a parameter vector.

Theorem 25: Let $h_j = \langle S_{\theta_j}, s_{\tau_j}, \Gamma_{h_j}, \Psi_{h_j} \rangle \in q - ROHFULE(X) (j = 1, 2, \ldots, n)$, then the aggregate results of Definition 21 is still a q-ROHFULE, and (48), as shown at the bottom of the previous page.

Proof: The proof process of Theorem 25 is similar to that of Theorem 3.

Moreover, it can be proved that the q-ROHFULSSDMM operator satisfies commutativity, boundedness, monotonicity and idempotence.

By giving different values of the parameters $r, k$, we get the following special cases.

Case 1: When $r = 0$, then the q-ROHFULSSDMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic dual Maclaurin mean (q-ROHFULDMM) operator which can be presented (49), as shown at the bottom of the page.

Case 2: When $P = (1, 0, \ldots, 0)$, then the q-ROHFULSSDMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric average (q-ROHFULSSGA) operator which can be presented (50), as shown at the bottom of the page.
Case 3: When \( P = (1, 1, 0, \ldots, 0) \), then the q-ROHFULSSMM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar geometric Bonferroni mean (q-ROHFULSSGBM) operator which can be presented (51), as shown at the bottom of the previous page.

Case 4: When \( P = (1, 1, \ldots, 1, 0, \ldots, 0) \), then the q-ROHFULSSDM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar dual Maclaurin symmetric mean (q-ROHFULSSDMSM) operator which can be presented (52), as shown at the bottom of the page.

Case 5: When \( P = (1, 1, \ldots, 1, 1/n) \) or \( P = (1/n, 1/n, \ldots, 1/n) \), then the q-ROHFULSSDM operator reduces to the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar arithmetic average (q-ROHFULSSAA) operator which can be presented as:

\[
q - \text{ROHFULSSAA}(h_1, h_2, \ldots, h_n) = \bigcup_{\varphi_j \in \Psi_j} \left\{ \left[ s \left( \frac{1}{n} \sum_{j=1}^{n} \phi_j \right) \right], \left\{ \begin{array}{l}
1 - \left( \frac{1}{n} \sum_{j=1}^{n} \left( 1 - \mu_j^q \right) \right)^{1/r} \\
1 - \left( \frac{1}{n} \sum_{j=1}^{n} \left( 1 - \nu_j^q \right) \right)^{1/r} \\
1^{1/q}
\end{array} \right\} \right\} (53)
\]

Definition 22: Let \( h_j = <[s_{j1}, s_{j2}], \Gamma_{h_j}, \Psi_{h_j}> \in q - \text{ROHFUL}(X) \) (j = 1, 2, \ldots, n), then we can defined the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar weighted dual Maclaurin mean (q-ROHFULSSWDMM) operator as follow:

\[
q - \text{ROHFULSSWDMM}(h_1, h_2, \ldots, h_n)
\]

where \( \vartheta(j) = 1, 2, \ldots, n \) is any a permutation of \( (1, 2, \ldots, n) \), and \( S_n \) represents the set of all permutation of \( (1, 2, \ldots, n) \). \( P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \) is a parameter vector. And \( w = (w_1, w_2, \ldots, w_n)^T \) is the weight vector of \( h_j(j = 1, 2, \ldots, n) \), which satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \).

Theorem 26: Let \( h_j = <[s_{j1}, s_{j2}], \Gamma_{h_j}, \Psi_{h_j}> \in q - \text{ROHFUL}(X) \) (j = 1, 2, \ldots, n), and their weight vector \( w = (w_1, w_2, \ldots, w_n)^T \) satisfies \( w_j \in [0, 1] \) and \( \sum_{j=1}^{n} w_j = 1 \), then the aggregate results of Definition 22 is still a q-ROHFUL, and (54), as shown at the bottom of the next page.

Proof: The proof process of Theorem 26 is similar to that of Theorem 25. Replace \( h_{\vartheta(j)} \) of Theorem 25 with \( h_{\vartheta(j)}^{nm} \). Thus, it can be proved that Theorem 26 is valid.

Moreover, it can be proved that the q-ROHFULSSWDMM operator satisfies monotonicity and boundedness.

VII. MODELS FOR MADM WITH q-ROHFUL5s
For a multi-attribute decision-making problem, let the scheme set as \( A = \{A_1, A_2, \ldots, A_l\} \), and the attribute set as \( M = \{M_1, M_2, \ldots, M_n\} \). Experts are invited to evaluate each plan. The expert’s evaluation information is expressed in the form of q-rung orthopair hesitant fuzzy uncertain linguistic elements. And \( w = (w_1, w_2, \ldots, w_n)^T \) is the weight set of each attribute, satisfying \( w_j \geq 0 \) and \( \sum_{j=1}^{n} w_j = 1 \). \( H = (h_{ij})_{l \times n} \) is the q-rung orthopair hesitant fuzzy uncertain linguistic decision matrix, and \( h_{ij} \) is the q-rung orthopair hesitant fuzzy uncertain linguistic evaluation value of the scheme \( A_k \) about the attribute \( M_j \). The specific decision steps are as follows:

Step 1: Carry out the standardized processing of data, convert all cost-type data into profit-type data, so that the obtained data is uniformly transformed into a standard q-rung
hesitant fuzzy uncertain linguistic decision matrix.

\[
H = (h_{kj})_{n \times m} = \begin{cases} 
  h_{kj}, & M_j \text{ is profit-type data} \\
  (h_{kj})^c, & M_j \text{ is cost-type data}
\end{cases}
\]

where \( h^c \) is the complement of q-rung orthopair orthopair hesitant fuzzy uncertain linguistic element \( h \).

Step 2: Calculate the q-rung orthopair hesitant fuzzy uncertain linguistic aggregation value of each scheme.

Use the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Skla WMM operator or the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Skla WDMM operator to aggregate the data values of each scheme under different attributes, and obtain the q-rung orthopair hesitant fuzzy uncertain linguistic value of each scheme.

q-ROHFULSSWMM(h_{k1}, h_{k2}, \ldots, h_{kn}) \quad \text{or} \quad q-ROHFULSSWDMM(h_{k1}, h_{k2}, \ldots, h_{kn}).

Step 3: The score function \( S(h_k) \) and accuracy function \( H(h_k) \) of the q-rung orthopair hesitant fuzzy uncertain linguistic value of each scheme are calculated.

Step 4: According to the score function \( S(h_k) \) and the accuracy function \( H(h_k) \) of each solution, the candidate solutions are sorted.

Step 5: Make the final decision based on the ranking results.

\[
q - ROHFULSSWDM(h_1, h_2, \ldots, h_n) = \bigcup_{\mu_{\varnothing(0)} \in \bar{\Gamma}_{\varnothing(0)}} \left\{ \frac{1}{\sum_j p_j} \left( \prod_{j=1}^n \left( \sum_{\varnothing(0) \in S_j} (p_j \mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1 \right)^{1/r} \right)^{1/q} \right\},
\]

\[
\left\{ 1 - \left( \frac{1}{\sum_j p_j} \left( \prod_{j=1}^n \left( \sum_{\varnothing(0) \in S_j} (p_j (1 - (n \mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1)^{1/r}) \right)^{1/q} \right) \right)^{1/r} \right}^{1/q} - \left( \frac{1}{\sum_j p_j} - 1 \right)^{1/r},
\]

\[
\left\{ \frac{1}{\sum_j p_j} \left( 1 - \left( \frac{1}{n!} \sum_{\varnothing(0) \in S_j} \left( 1 - \left( \sum_{j=1}^n (p_j (1 - (n\mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1)^{1/r}) \right)^{1/q} \right) \right)^{1/r} \right}^{1/q} \right\},
\]

\[
\left\{ 1 - \left( \frac{1}{\sum_j p_j} \left( \prod_{j=1}^n \left( \sum_{\varnothing(0) \in S_j} (p_j (1 - (n\mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1)^{1/r}) \right)^{1/q} \right) \right)^{1/r} \right}^{1/q} - \left( \frac{1}{\sum_j p_j} - 1 \right)^{1/r},
\]

\[
\left\{ \frac{1}{\sum_j p_j} \left( 1 - \left( \frac{1}{n!} \sum_{\varnothing(0) \in S_j} \left( 1 - \left( \sum_{j=1}^n (p_j (1 - (n\mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1)^{1/r}) \right)^{1/q} \right) \right)^{1/r} \right}^{1/q} \right\},
\]

\[
S(h_k) = \frac{1}{\sum_j p_j} \left( \prod_{j=1}^n \left( \sum_{\varnothing(0) \in S_j} (p_j (1 - (n\mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1)^{1/r}) \right)^{1/q} \right),
\]

\[
H(h_k) = \frac{1}{\sum_j p_j} \left( \prod_{j=1}^n \left( \sum_{\varnothing(0) \in S_j} (p_j (1 - (n\mu_{\varnothing(0)}^{\text{qr}} - w_{\varnothing(0)}) + 1)^{1/r}) \right)^{1/q} \right).
\]

VIII. EXAMPLE APPLICATION

A. NUMERICAL EXAMPLE

There is a doctoral enrollment quota for a certain major in a university, and there are currently 5 candidate students A1, A2, A3, A4, A5 entering the retest stage. The five students were evaluated from the four attributes of written test M1, interview performance M2, foreign language level M3, and scientific research level M4.

This major invites experts with rich professional background, experience and knowledge in related fields to evaluate the candidates. There are many evaluation factors. For comprehensive and accurate evaluation, q-rung orthopair hesitant fuzzy uncertain linguistic elements are used to reflect the evaluation information of experts.

In order to better express the expert’s evaluation of the selected objects, an evaluation linguistic term set \( S = \{x_0, s_1, s_2, s_3, s_4, s_5, s_6\} \) is given, which respectively represent the evaluation as “very bad”, “poor”, “poor”, “fair”, “good”, “good”, “well”. The evaluation information of the experts on the five candidates is shown in Table 1. The weight of each attribute of the scheme is \( w = (0.32, 0.22, 0.20, 0.26)^T \).

Since written test situation \( M_1 \), interview performance \( M_2 \), foreign language level \( M_3 \), and scientific research level \( M_4 \) are all profit data, there is no need to standardize the data.

According to the expert’s decision matrix, the \( q - ROHFULSSWMM \) operator is used to aggregate the elements.
of different schemes under each attribute, and the q-rung hesitant fuzzy uncertain linguistic aggregation value $h_k$ of each scheme is obtained.

It is easy to verify that each q-rung orthopair hesitant fuzzy uncertain linguistic element in Table 1 satisfies the $q \geq 3$. Without loss of generality, taking $P = (1, 1, 1)$ and $q = 3$, $r = -1$, we can get

$$h_1 = \left[\begin{array}{c}
\{s_1, s_2\}, \{0.8, 0.9\}, \{0.2, 0.3\}, \\
\{s_2, s_3\}, \{0.7, 0.8\}, \{0.3, 0.4\}, \\
\{s_3, s_4\}, \{0.6, 0.7\}, \{0.3, 0.4\}
\end{array}\right].
$$

Then, the score function $S(h_k)$ of the q-rung orthopair hesitant fuzzy uncertain linguistic value of each scheme is calculated, which is $S(h_1) = 2.9958$, $S(h_2) = 2.1403$, $S(h_3) = 1.9316$, $S(h_4) = 2.2048$, and $S(h_5) = 1.8345$ respectively.

Finally, according to the score function $S(h_k)$ of each scheme, the ranking is $A_1 \succ A_4 \succ A_2 \succ A_3 \succ A_5$. It can be seen that the most suitable candidate for admission is $A_1$.

When the $q$-ROHFULSSWMM operator is used, take $P = (1, 1, 1, 1)$, $q = 3$, and $r = -1$, so that the $q$-ROHFULSSWMM operator aggregates the elements of each scheme in the decision matrix under different attributes, and obtains the q-rung orthopair hesitant fuzzy uncertain linguistic value of each scheme.

$$h_1 = \left[\begin{array}{c}
0.7167, 0.7575, 0.7534, \\
0.7845, 0.7505, 0.7823, 0.7791, 0.8041, 0.7800, \\
0.8048, 0.8023, 0.8224, 0.8205, 0.8120, 0.8189, \\
0.8358, 0.2294, 0.2343, 0.2623, 0.2710, 0.2353, \\
0.2408, 0.2727, 0.2829, 0.2478, 0.2547, 0.2969, \\
0.3116, 0.2560, 0.2638, 0.3146, 0.3353
\end{array}\right].
$$

Then, the score function $S(h_k)$ of the q-rung orthopair hesitant fuzzy uncertain linguistic value of each scheme is calculated, which are $S(h_1) = 3.5098$, $S(h_2) = 2.5126$, $S(h_3) = 2.1511$, $S(h_4) = 2.7939$, and $S(h_5) = 2.3450$ respectively.

Finally, according to the score function $S(h_k)$ of each scheme, the ranking is $A_1 \succ A_4 \succ A_2 \succ A_5 \succ A_3$. It can be seen that the most suitable candidate for admission is $A_1$.

### B. Sensitivity Analysis of Parameters

The following discusses the influence of the parameter vector $P$ in the $q$-ROHFULSSWMM operator and the $q$-ROHFULSSWMM operator on the decision result. The parameter vector $P$ in the $q$-ROHFULSSWMM operator and the $q$-ROHFULSSWMM operator takes different values, and the score functions and ranking results of each scheme obtained are shown in Table 4 and Table 5.

From the results in Table 2 and Table 3, it can be concluded that although the scoring function and ranking result of each scheme will change when the parameter

| 1 | {s_4, s_5}, {0.8, 0.9}, {0.2, 0.3} | {s_4, s_4}, {0.7, 0.8}, {0.3, 0.4} | {s_5, s_5}, {0.7, 0.8}, {0.2, 0.3} | {s_3, s_5}, {0.6, 0.8}, {0.3, 0.4} |
| 2 | {s_4, s_4}, {0.6, 0.7}, {0.3, 0.4} | {s_4, s_5}, {0.5, 0.7}, {0.4, 0.5} | {s_3, s_4}, {0.6, 0.7}, {0.5, 0.6} | {s_3, s_4}, {0.5, 0.6}, {0.4, 0.5} |
| 3 | {s_4, s_4}, {0.4, 0.5}, {0.4, 0.6} | {s_2, s_4}, {0.5, 0.6}, {0.4, 0.5} | {s_4, s_5}, {0.4, 0.6}, {0.5, 0.6} | {s_4, s_5}, {0.4, 0.6}, {0.5, 0.6} |
| 4 | {s_4, s_4}, {0.7, 0.8}, {0.3, 0.4} | {s_2, s_3}, {0.6, 0.8}, {0.3, 0.4} | {s_3, s_4}, {0.6, 0.7}, {0.5, 0.6} | {s_5, s_5}, {0.6, 0.7}, {0.5, 0.6} |
| 5 | {s_4, s_4}, {0.5, 0.8}, {0.5, 0.7} | {s_3, s_4}, {0.7, 0.8}, {0.3, 0.4} | {s_5, s_5}, {0.4, 0.5}, {0.5, 0.6} | {s_2, s_4}, {0.4, 0.5}, {0.5, 0.6} |
vector $P$ takes different values, the best candidate for admission is always $A_1$, which shows the decision result reliability. Moreover, the $q$–ROHFULSSWMM operator and the $q$–ROHFULSSWDMM operator consider the correlation between all decision attributes, which can fully reflect the decision information and reduce the loss of information in the decision.

In addition, when $P = (1, 0, 0, 0)$, the decision attributes in the operator are independent of each other; when $P = (1, 1, 0, 0)$, the operator can describe the correlation between any two decision attributes; when $P = (1, 1, 1, 0)$, the operator can consider the correlation between multiple attributes. Therefore, the decision maker can flexibly select the parameter vector $P$ to deal with a variety of different types of decision-making problems, making it more widely applicable.

Below, the influence of the parameter $q$ and parameter $r$ in the $q$–ROHFULSSWMM operator on the decision result is further discussed.

It can be seen from Table 5 that with the continuous decrease of the parameters $r$ in the $q$–ROHFULSSWMM operator, the score function of each scheme is also continuously reduced, that is, the result of the decision is increasingly pessimistic. Therefore, the value of the parameter $r$ can be regarded as the degree of pessimism and optimism when the expert makes a decision. If the expert is pessimistic about the decision result, assign a smaller value to the parameter $r$ in the $q$–ROHFULSSWMM operator. Conversely, if the expert is optimistic about the decision result, assign a lower value to the parameter $r$ in the $q$–ROHFULSSWMM operator.

Combining Table 2, Table 3, Table 4 and Table 5, it can be concluded that although the $q$–ROHFULSSWMM operator and the $q$–ROHFULSSWDMM operator are used to aggregate the attribute values of each scheme, and with the continuous changes of the parameters $q$ and $r$ in the operator, the scoring function of the proposed programs is different, and the ranking results of each program are slightly changed, the best candidates for doctoral admission are $A_1$, which shows that the decision-making method in this article is effective and reasonable.

**C. COMPARATIVE ANALYSIS**

A comparative analysis is carried out to illustrate the effectiveness and superiority of the proposed method over the method of q-ROFULSs proposed by Liu et al. [32], the method of ULSs developed by Xu [44], the method of q-ROHFSs developed by Xu et al. [45], the method of DHFULSs proposed by Lu and Wei [46], the method of PHFULSs proposed by Shakeel et al. [47], and other methods using the same illustrative example.

1) COMPARATIVE ANALYSIS WITH Q-ROFULS AGGREGATION OPERATOR

The q-rung orthopair fuzzy uncertain linguistic set is a special form of the q-rung orthopair orthopair hesitant fuzzy uncertain linguistic set. When the membership degree and non-membership degree of q-rung orthopair hesitant fuzzy uncertain linguistic set satisfy $|\Gamma_h| = |\Psi_h| = 1$, the q-rung orthopair hesitant fuzzy uncertain linguistic set is transformed into the q-rung orthopair fuzzy uncertain linguistic set. And the possible membership degree and the possible non-membership degree in the q-rung orthopair orthopair hesitant fuzzy uncertain linguistic decision data are respectively averaged. Thus, the q-rung orthopair fuzzy uncertain linguistic decision data is obtained, as shown in Table 6 below.

The q-rung orthopair fuzzy uncertain linguistic weighted geometric mean ($q$–ROFWGA) operator proposed by...
TABLE 4. Ranking of the schemes corresponding to different parameter \( q \).

| \( q \) | \( S(h_1) \) | \( S(h_2) \) | \( S(h_3) \) | \( S(h_4) \) | \( S(h_5) \) | Ranking Results |
|---|---|---|---|---|---|---|
| 3 | 2.9958 | 2.1403 | 1.9316 | 2.2048 | 1.8345 | \( A_1 > A_4 > A_2 > A_3 > A_5 \) |
| 6 | 2.4593 | 2.0189 | 1.9489 | 1.8130 | 1.8437 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |
| 8 | 2.3792 | 2.0028 | 1.9689 | 1.7730 | 1.8226 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |
| 10 | 2.2126 | 1.9828 | 1.9765 | 1.6234 | 1.8104 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |
| 12 | 2.1206 | 1.9678 | 1.9869 | 1.5692 | 1.8056 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |

TABLE 5. Ranking of the schemes corresponding to different parameter \( r \).

| \( r \) | \( S(h_1) \) | \( S(h_2) \) | \( S(h_3) \) | \( S(h_4) \) | \( S(h_5) \) | Ranking Results |
|---|---|---|---|---|---|---|
| -1 | 2.9958 | 2.1403 | 1.9316 | 2.2048 | 1.8345 | \( A_1 > A_4 > A_2 > A_3 > A_5 \) |
| -3 | 2.8386 | 2.1019 | 1.9106 | 1.9556 | 1.8379 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |
| -5 | 2.8097 | 2.0793 | 1.8996 | 1.9331 | 1.7754 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |
| -7 | 2.7756 | 1.9956 | 1.8132 | 1.9201 | 1.7064 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |
| -9 | 2.7259 | 1.9255 | 1.7742 | 1.8965 | 1.6022 | \( A_1 > A_2 > A_4 > A_3 > A_5 \) |

TABLE 6. The q-rung orthopair fuzzy uncertain linguistic decision matrix.

|           | \( M_1 \)                       | \( M_2 \)                       | \( M_3 \)                       | \( M_4 \)                       |
|-----------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 1         | \([s_4, s_5], \{0.85\}, \{0.25\}\) | \([s_4, s_4], \{0.75\}, \{0.35\}\) | \([s_5, s_6], \{0.75\}, \{0.25\}\) | \([s_5, s_5], \{0.7\}, \{0.35\}\) |
| 2         | \([s_4, s_4], \{0.65\}, \{0.35\}\) | \([s_4, s_5], \{0.6\}, \{0.45\}\)  | \([s_5, s_4], \{0.55\}, \{0.45\}\) | \([s_5, s_4], \{0.55\}, \{0.5\}\)  |
| 3         | \([s_3, s_4], \{0.45\}, \{0.5\}\)  | \([s_2, s_4], \{0.55\}, \{0.45\}\)  | \([s_5, s_5], \{0.45\}, \{0.45\}\) | \([s_5, s_5], \{0.5\}, \{0.55\}\)  |
| 4         | \([s_4, s_5], \{0.75\}, \{0.35\}\) | \([s_2, s_5], \{0.7\}, \{0.35\}\)  | \([s_3, s_4], \{0.6\}, \{0.45\}\) | \([s_5, s_5], \{0.65\}, \{0.55\}\) |
| 5         | \([s_4, s_4], \{0.65\}, \{0.6\}\)  | \([s_3, s_4], \{0.75\}, \{0.35\}\) | \([s_3, s_5], \{0.45\}, \{0.55\}\) | \([s_2, s_4], \{0.45\}, \{0.5\}\)  |

Liu et al. [32] is used to aggregate the attribute data, and the q-rung orthopair fuzzy uncertain linguistic evaluation value of each candidate object is obtained. Then, the score function \( S(h_k) \) of the q-rung orthopair fuzzy uncertain linguistic value of each scheme is calculated, which are \( S(h_1) = 3.1350 \), \( S(h_2) = 2.1610 \), \( S(h_3) = 1.9093 \), \( S(h_4) = 2.3990 \), and \( S(h_5) = 1.8482 \) respectively. According to the score function \( S(h_k) \) of each scheme, the ranking is \( A_1 > A_4 > A_2 > A_3 > A_5 \). It can be seen that the most suitable candidate for admission is \( A_1 \).

The comparative analysis shows that the decision-making method based on the \( q \)-ROFWGA operator is consistent with the most suitable candidates obtained by the method in this paper, and the results are both \( A_1 \). However, the score function value of each scheme and the ranking situation of schemes obtained by the \( q \)-ROFWGA operator are different from the method in this paper. The reason is that the q-rung orthopair fuzzy uncertain linguistic set does not take into account the experts’ hesitation on the membership degree and non-membership degree, and cannot fully reflect the expert’s decision information, which will cause the loss of decision information.

2) COMPARATIVE ANALYSIS WITH ULS AGGREGATION OPERATOR

Uncertain linguistic set is a special form of q-rung orthopair hesitant fuzzy uncertain linguistic set. When the membership degree and non-membership degree of the q-rung orthopair hesitant fuzzy uncertain linguistic set satisfy \( h = 1 \), the q-rung orthopair hesitant fuzzy uncertain linguistic set is transformed into an uncertain linguistic set. Taking the membership degree and non-membership degree in the q-rung orthopair hesitant fuzzy uncertain linguistic decision data as \( h = 1 \), the uncertain linguistic decision data is obtained, as shown in Table 7 below.

The uncertain linguistic weighted average (ULWAA) operator proposed by Xu [44] is used to aggregate the attribute data to obtain the uncertain linguistic evaluation value of each candidate object. Then, the score function \( S(h_k) \) of the uncertain linguistic value of each scheme is calculated, which are \( S(h_1) = 2.2300 \), \( S(h_2) = 1.9400 \), \( S(h_3) = 1.9750 \), \( S(h_4) = 1.9950 \), and \( S(h_5) = 1.8150 \) respectively. According to the score function \( S(h_k) \) of each scheme, the ranking is \( A_1 > A_4 > A_3 > A_2 > A_5 \). It can be seen that the most suitable candidate for admission is \( A_1 \).
The comparison shows that the decision-making method based on the ULWAA operator is consistent with the most suitable candidates obtained by the method in this paper, and the results are both $A_1$. However, the uncertain linguistic set only reflects the qualitative decision-making information of the decision maker, ignoring the quantitative decision-making information of the expert, and cannot reflect the membership degree, non-membership degree and hesitation of the expert’s qualitative opinions.

3) COMPARATIVE ANALYSIS WITH Q-ROHFS AGGREGATION OPERATOR

When the qualitative decision information of the decision maker is ignored and only the quantitative decision information of the decision maker is considered, the q-rung hesitant fuzzy uncertain language set degenerates the q-rung orthopair hesitant fuzzy set. The q-rung orthopair hesitant fuzzy decision data are shown in Table 8 below.

Using the q-rung orthopair hesitant fuzzy weighted Heronian mean $(q \text{ – ROHFWH})$ operator proposed by Xu et al. [45] to aggregate all attribute data, the q-rung orthopair hesitant fuzzy evaluation value of each candidate object is obtained. The score function of the q-rung orthopair hesitant fuzzy evaluation value of each scheme is calculated, which are $S(h_1) = 3.5098$, $S(h_2) = 2.5126$, $S(h_3) = 2.1511$, $S(h_4) = 2.7939$, $S(h_5) = 2.3450$. According to the score function of each program, each program is ranked as $A_1 > A_4 > A_2 > A_5 > A_3$. It can be seen that the most suitable candidate for admission is $A_1$.

The comparison shows that the decision-making method based on the $q \text{ – ROHFWH}$ operator is consistent with the most suitable candidates obtained by the method in this paper, and the results are both $A_1$. However, the q-rung orthopair hesitant fuzzy set only quantitatively reflects the decision-making information of the decision maker. In actual decision-making, experts may prefer to use qualitative linguistic variables to express the decision information.

4) COMPARATIVE ANALYSIS WITH PHFULS AND DHFULS AGGREGATION OPERATOR

The q-rung orthopair hesitant fuzzy uncertain linguistic set is a generalization of the dual hesitant fuzzy uncertain linguistic set and the Pythagorean hesitant fuzzy uncertain linguistic set. When $q = 1$, the q-rung orthopair hesitant fuzzy uncertain linguistic set is transformed into the Pythagorean hesitation fuzzy uncertain linguistic set; when $q = 2$, the q-rung orthopair hesitant fuzzy uncertain linguistic set is transformed into the dual hesitant fuzzy uncertain linguistic set. In order to illustrate the effectiveness and feasibility of the decision-making method in this paper, the Pythagorean hesitant fuzzy uncertain linguistic hybrid weighted average (PHFULWA) operator of the literature [46] and the dual hesitant fuzzy uncertain linguistic weighted average (DHFULWAA) operator of the literature [47] are introduced for comparative analysis.

From the data in Table 1, we can see that the element $h_{15}$ is $<[s_4, s_4], [0.5, 0.8], [0.5, 0.7]>$. However, due to $0.8 + 0.7 > 1$ and $0.8^2 + 0.7^2 > 1$, the membership degree and non-membership degree of element $h_{15}$ do not meet the constraints of DHFULSs and PHFULSs, so that the evaluation data cannot be represented by DHFULE and PHFULE. Therefore, as shown in Table 9, the Pythagorean hesitant fuzzy uncertain linguistic hybrid weighted average (PHFULWA) operator and the dual hesitant fuzzy uncertain linguistic weighted average (DHFULWAA) operator cannot be applied to the decision-making problem in this paper.

However, the proposed decision-making method can still be applied to this decision-making problem, because by adjusting the value of $q$, the q-ROFULE can represent the element $<[s_4, s_4], [0.5, 0.8], [0.5, 0.7]>$. Therefore, the decision-making method in this paper has a wider application range than DHFULWAA and PHFULWAA operators.

In addition, according to the results in Table 9, it can be seen that the ranking results of the various schemes obtained by different methods may be slightly different, but the optimal scheme is all $A_1$, which illustrates the effectiveness and rationality of the decision-making method in this paper.

Based on the above comparison and analysis, it can be concluded that the advantages of the proposed decision-making method are:

1) Compared with DHFULSs and PHFULSs, q-ROFULEs can express uncertainty more accurately. In today’s complex decision-making environment, the q-ROFULEs proposed in this paper is a very powerful tool in decision-making.

2) The decision-making method in this paper is an optimization and generalization of existing methods. When $q = 1, 2$, the DHFULWAA and PHFULWAA operators are special cases of the operators proposed in this paper.

3) For the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar aggregation operator, different parameter values can be selected according to the decision-making situation to meet the requirements of different decision-making problems in practice, so it is flexible and general.

4) The q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar MM operator can describe the correlation between the decision attributes, can fully reflect the decision information, reduce the information loss in the decision, and make the decision result more reasonable and reliable.
IX. CONCLUSION

This paper combines the q-rung orthopair hesitant fuzzy sets with uncertain linguistic variables, and proposes the q-rung orthopair hesitant fuzzy uncertain linguistic sets. And the Schweizer-Sklar T-norm is introduced to define the operational properties of the q-rung orthopair hesitant fuzzy uncertain linguistic elements. Then the q-rung orthopair hesitant fuzzy uncertain linguistic Schweizer-Sklar BM operator, MSM operator and MM operators are respectively defined. In addition, a multi-attribute decision-making method based on the Schweizer-Sklar MM operator of the q-rung orthopair hesitant fuzzy uncertain linguistic set is established. The research results of this paper develop the theory of the q-rung orthopair hesitant fuzzy uncertain linguistic aggregation operators, and extend the q-rung orthopair hesitant fuzzy uncertain linguistic multi-attribute decision-making method. In the future, we will study the information measurement and multi-attribute decision-making method of the q-rung orthopair hesitant fuzzy uncertain linguistic set. In addition, the theory of the q-rung orthopair hesitant fuzzy uncertain linguistic set can be expanded by combining other related theories [48], [49], and the aggregation operators of the extended q-rung orthopair hesitant fuzzy uncertain linguistic set and their application in decision-making can be studied.

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