Frequency Analysis of Chaotic Flow in Transition to Turbulence in Taylor-Couette System with Small Aspect Ratio

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Abstract. Taylor-Couette flow with small aspect ratio has characteristics such as the different vortex structure, because of a boundary layer of the upper and lower wall and the acceleration of the inner cylinder. In this study, the mechanism of Taylor-Couette system with the small aspect ratio is measured and analyzed by using an ultrasound measurement and a numerical simulation. The process of transition to turbulent flow is observed by using a spectra analysis in a radial and an axial direction. The experimental and numerical results confirmed the characteristics of the broadband component in Taylor-Couette system.

Key Words: Taylor-Couette flow, Small aspect ratio, Chaotic wavy vortex flow, Numerical simulation

1. Introduction.
Many researchers have been studying about the experiments and theoretic of Taylor-Couette flow (TCF), since it can be reproduced under the most fundamental conditions to understand the turbulent transition process of a fluid. In the previous studies, effect of the aspect ratio (Γ) which defined with the clearance between coaxial cylinder and the working fluid height was considered. Benjamin et. al. [1] built up the fundamental research with small aspect ratio of TCF. It is also expected as a stirring flow with the less local shear. This flow system can be used in blood filtration and bioreactor microorganism culturing devices. Therefore it is suggested to be compact due to the circumstances such as the place where it will be applied, so it is important to elucidate the flow problem with small aspect ratio in TCF. In the study of infinite cylinder TCF, Fenstermacher et al. [2] and David Andereck et al. [3] report the frequency analysis of the TCF velocity, and reveal the TCF transition with the changing of Reynolds number (Reynoldss number : Re). In the process transition from Taylor vortex flow (TVF), Wave vortex flow (WVF), Modulate wavy vortex flow (MWV) to turbulence are ordered to the Reynolds number. Especially in the turbulent state, existence of broadband component: B has been reported. This is also studied by Takeda et. al. [4][5], they report that there is a stage called chaotic wavy vortex flow (CWV) before the transition to turbulence, which has the B component and the fundamental frequencies.

In this research, we are studying a normal 2-cell mode with the aspect ratio is 3 and stable behaviour caused of the upper and lower boundary conditions. Nakamura et.al. [6] report that TCF with small aspect ratio has instability such as branches of the number of cells comparing to TCF with large aspect ratio. And it has been reported that the CWV state continues relatively even at large Reynolds number under this equipment condition. According to Kageyama et. al. [7], by numerical simulation and experiment of TCF with small aspect ratio, it is stated that the critical Reynolds number which divides laminar flow and turbulent flow is \( Re_{crit} \approx 5 \times 10^5 \). Our consideration is the fixed
upper and lower boundaries that prevent turbulence of TCF. The purpose of calculating the velocity field of TCF by numerical simulation is to estimate the cause of modulation of TCF by deriving the radial velocity component that is difficult to be measured by experiment.

2. Methodology

2.1. Experiment Method to Measure Velocity Field Using Ultrasound Doppler Velocity Profiler (UVP).

The experimental apparatus was filled with glycerin aqueous solution (68% C₃H₈O₃, Wako Pure Chemical Ltd.) and 80 μm nylon particles were added as the tracer. The inner cylinder was rotated to generate TCF in the normal 2-cell mode. An ultrasonic transducer (TDX) was installed 7 mm from the surface of the inner cylinder to measure the velocity in the axial direction. We selected to install it at 7 mm from the surface of the inner cylinder because at that condition the characteristic of the cell type vortex of TCF appeared well.

Inner cylinder radius is \( R_{in} = 50 \) mm, Outer cylinder radius is \( R_{out} = 75 \) mm, inner and outer cylinder clearance is \( d = 25 \) mm, inner and outer cylindrical height is \( H = 75 \) mm. It is defined that radius ratio is \( \eta = \frac{R_{in}}{R_{out}} \), aspect ratio is \( \Gamma = \frac{H}{d} \), and the Reynolds number is \( Re = \frac{dW_{in}}{\nu} \), with assumption that the inner cylinder speed is \( W_{in} \), \( \nu \) is the viscosity of the glycerin aqueous solution. This experimental condition is shown in table 1.

| Table 1 Experimental conditions |
|--------------------------------|
| Boundary condition | Fixed |
| Cell mode | Normal-2cell |
| Aspect ratio | \( \Gamma \) | 3 |
| Radius ratio | \( \eta \) | 0.667 |
| Reynolds number | \( Re \) | 500-4000 |

2.2. Numerical Simulation Model and Condition.

The numerical simulation model is shown in Fig. 1. The radial direction is \( r \) (the direction of the outer cylinder is positive from the inner cylinder), the axial direction is \( z \) (the direction from the lower boundary end to the upper boundary end is positive) and the circumferential direction is set to \( \theta \) (CCW rotation is positive). The analysis conditions are shown in table 2.
Table 2 Numerical simulation conditions

|                  |       |
|------------------|-------|
| Boundary condition | Non-Slip |
| Cell mode        | Normal-2cell |
| Aspect ratio     | $\Gamma$ 3 |
| Radius ratio     | $\eta$ 0.667 |
| Reynolds Number  | $Re$ 500-4000 |
| Grid num.        | $r \times z \times \theta$ 50x150x160 |
| Time step        | $\Delta t$ 0.01 |

2.3. Numerical Simulation Method.

Numerical simulation method used to solve the Navier-Stokes equation by setting the fluid inside the model as an incompressible viscous fluid. In addition, the Navier-Stokes equations represent the length as the inner cylinder radius: $R_{in}$, and speed as the inner cylinder circumferential velocity: $W_{in}$, and normalize all the parameters. As an example, the radial component of the normalized cylindrical coordinate system in Navier-Stokes equation is shown in equation (1) and the continuous equation is shown in equation (2).

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{w}{r} \frac{\partial u}{\partial \theta} - \frac{w^2}{r} + v \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial r} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial w}{\partial \theta} + \frac{\partial^2 u}{\partial z^2} \right) \quad \ldots(1)
\]

\[
\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial v}{\partial z} + \frac{1}{r} \frac{\partial w}{\partial \theta} = 0 \quad \ldots(2)
\]
For the analysis algorithm, SMAC method (Simplified Marker and Cell Method) is used. This scheme is explained by the radial expression. Regardless of the known velocity field satisfies the continuous equation or not, predictive value of $\tilde{u}$ is obtained by the Euler explicit method as shown in Equation (3). $n$ in the equation refer to the time.

$$
\tilde{u} \approx u^n + \Delta t \left( -u^n \frac{\partial u^n}{\partial r^n} - \frac{w^n \partial u^n}{r^n \partial \theta^n} + \frac{w^n \partial u^n}{z^n} \right)
- \frac{\partial p^n}{\partial r^n} + \frac{1}{Re} \left( \frac{\partial^2 u^n}{\partial r^n^2} + \frac{\partial u^n}{\partial r^n} - \frac{u^n}{r^n^2} \right) + \frac{1}{r^n^2} \frac{\partial^2 w^n}{\partial \theta^n} - 2 \frac{r^n^2}{\partial z^n} \right)...
\tag{3}
$$

However, this predicted value does not satisfy the continuous equation. Therefore introduce a vortex-less velocity field $u'$ and obtain a corrected speed $u^{n+1}$ so as to satisfy the continuous equation from the equation 4.

$$
u^{n+1} = \tilde{u} + u' \quad \tag{4}
$$

In addition, scalar potential: $\phi$ exists since $u'$ is not vortexed. Therefore, the composite speed $V^{n+1}$ of $u^{n+1}$, $v^{n+1}$, and $w^{n+1}$ that omit the intermediate process are expressed as equation (5), when divergence take place and converges to 0.

$$
\text{div} V^{n+1} = \text{div} (\tilde{V}^n + \text{grad} \phi) = 0 \quad \tag{5}
$$

From the equation (5), Poisson equation for the scalar potential $\phi$ is described as the following equation

$$
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = -\frac{1}{r} \frac{\partial}{\partial r} (r \tilde{u}) + \frac{1}{r} \frac{\partial \tilde{w}}{\partial \theta} + \frac{\partial \tilde{v}}{\partial z} \quad \tag{6}
$$

By calculating this Poisson equation iteratively, the vortex-less velocity field is obtained from the scalar potential, and the velocity at the next time is obtained from the predicted value and the modified value. According to the program in this study, the iteration number is set to 10 for the convenience of shortening calculation time, and the 10th calculated value is taken as a solution regardless of the continuous equation is satisfied or not. The sufficiently small value of the data is used to the continuous formula. For discretization, Adams-Bashforth method is used for time progression with the second order accuracy, the central difference with the fourth order accuracy is used for convection term in the space, and central difference with the second order accuracy is used for viscosity term and pressure term as explained respectively in equation (7), (8), and (9).

$$
u^{n+1} = u^n - \Delta t \frac{p_{i-1} - p_i}{\Delta r} + \Delta t \frac{3}{2} (A_r + B_r)^{n-1} - (A_r + B_r)^{n-1} \quad \tag{7}
$$

$$
f_i' = \frac{-f_{i-1/2} + f_{i+1/2}}{\Delta} \quad \tag{8}
$$
\[ f'_j = \frac{f_{j-3/2} - 27f_{j-1/2} + 27f_{j+1/2} - f_{j+3/2}}{24\Delta} \]  

...(9)

2.4. Frequency Analysis.

For the frequency analysis, in order to discriminate the transition process of TCF from the frequency with respect to the flow velocity, Fast Fourier Transform (FFT) is applied to the velocity data obtained by UVP and numerical simulation. Then the oscillation frequency and power spectral density (PSD) are obtained at each Reynolds number. PSD is the amplitude obtained by FFT squared and divided by frequency resolution. The number of data is 2048, and the frequency is normalized by the rotation speed of the inner cylinder. In order to confirm the accuracy of numerical simulation, FFT is applied to the axial flow velocity value on the same line as on the measurement line in UVP. Also, there are two evaluation methods. The first method is to evaluate the change in the vibration state accompanying the increase in the Reynolds number by taking the average of the vibration frequencies on the measurement or calculation line. The second method is by taking the vibration frequency of each channel in the axial direction, the vibration state at each position in the axial direction is evaluated in detail. The results are summarized in a color plot that showing the intensity of the PSD.

3. Results and discussion

Fig. 2 shows the PSD distributions map of the frequency components on the axial velocity in space. Fig. 2-(a) is the experimental data with the UVP method and Fig. 2-(b) is the numerical. The vertical axis indicates the normalized frequency \( f^* \) and the horizontal axis is the Reynolds number (\( Re \)). Red color shows the higher PSD value and the blue one is the lower. In Fig.2-(a), a sharp frequency around \( f^*=2.1 \) can be observed continuously from \( Re=700 \). This is the flow transition from TVF to WVF. However, this continuous line is disappeared at \( Re=1000 \), then at the same time, new three types of the sharp peaks lines are appeared at \( f^*=1.8, 1.7 \) and around 0 respectively. This means another flow transition has been occurred from the WVF to the MWV around \( Re=1000 \). The first peak which was observed at the beginning of the MWV keeps its existence clearly until \( Re=4000 \). This phenomenon is observed in both the experimental and the numerical results, which can be assumed as the fundamental frequency in the flow.

Fig. 2 Variation of PSD result of the velocity in z direction on Re - f* plane. (a) The velocity measured by experiment and (b) the velocity calculated by numerical analysis.

Fenstermacher et.al. [2] and Takeda et.al. [4] found a broadband component called B component that was observed in the lower frequency area of MWV. This B component is gradually powered up
with increasing the Reynolds number. In this study, the similar B component was observed in the small aspect ratio of $\Gamma=3$ in both of the experimental and the numerical results. In this experiment, the PSD value of the B component in the lower frequency is gradually increased and widely distributed after $Re=3000$

However the beginning of the B component is anomalous compared with the previous studies in Fenstermacher’s. In $\Gamma=3$, the two fundamental frequencies are much closer in each other where the MWV mode starts, which results in the cause of the growl in lower frequency and it is clearly observed in both Figs. 2-(a) and (b). In the previous studies, some discussions are made that the B component is caused from the growl generated in the MWV mode. In the case of $\Gamma=3$, we found the pair of the fundamental frequencies in MWV just start from the closer frequency values and thus our experiment is easier in estimating the lower frequency growl. In fact from Figs. 2-(a) and (b), the growl generates between the difference of the upper two fundamental frequencies and exists in the lower part of frequency area. But there can be seen a jumping from the growl line at the starting point of the B component with increasing the Reynolds number, which can be certificated in both the experimental and numerical results. It could be discussed that the growl in the lower part of frequency is not causing the B-component. The both results in the experimental and the numerical frequencies correspond well qualitatively until $Re=4000$ including the starting point of the B component. The flow seems to transfer the CWV around $Re=3000$ where the PSD of the B component is much powered up and widely distributed in the lower frequencies, but still we can see the MWV mode in the upper frequencies.

![Variation of power spectra](image)

**Fig. 3** Variation of power spectra of (a) the velocity that measured by experiment and (b) the velocity that calculated by numerical analysis in z direction at $Re = 1000$
Fig. 4 Variation of power spectra of (a) the velocity that measured by experiment and (b) the velocity that calculated by numerical analysis in $z$ direction at $Re = 3000$.

Figs. 3 and 4 show the colored PSD map with the dimensionless $z$ axial coordinate and the normalized frequency from the axial velocity components. The vertical is the $z$ direction and the horizontal is the frequency. In both figure, (a) is from the experimental data with the UVP method, and (b) is the numerical, however the experimental maps are shown in the half part of the space region as $Z/H = 0\sim 0.50$ where the error from the near field of the ultrasound is less influenced.

At $Re=1000$, three peaks of the frequencies are also clearly observed in Fig. 3 with the distribution of $z$ coordinate, which are well corresponded to Fig. 2. The peak of $f^*=1.7$ is seen in the middle area of $z$ direction and the peak of $f^*=1.8$ is in a whole $z$ direction except for the middle area. From this state, the frequency of $f^*=1.7$ is mainly from the outer flow oscillation and the other frequency of $f^*=1.8$ is the self-oscillation in each vortex cell. $f^*=0.1$ is the beat frequency caused by the fundamental frequency between $f^*=1.7$ and 1.8. In comparison with the both experimental and numerical, the frequency distributions in space are also in good agreement with each other. In Fig. 4 at $Re=3000$, a large peak of the frequency is observed in the middle of $z$ direction and the spectrum value is more increased than $Re=1000$, especially in the lower frequency region around $f^*=0\sim 0.5$.

4. Conclusion

Measurement experiment and numerical simulation were carried out with small aspect ratio TCF with $\Gamma=3$ and the following findings were obtained from the frequency analysis of the axial velocity components of TCF.

- Although the broadband (B) component spreads in a high Reynolds number, the fundamental frequencies maintain until $Re=4000$, and at $Re=4000$ or less, the CWV state continues with undeveloped turbulence. The power spectrum of each channel with respect to the axial direction quantitatively agreed with the results of experiment and numerical simulation at $Re=1000$ and 3000.
The broadband component which occurs from upper and lower boundary develops in between the vortex cells. The transition to turbulence of TCF with small aspect ratio is affected by the boundaries and the effect of boundaries is a factor to maintain a chaotic flow state.

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