Eigenvalues of the Breit equation

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\[
\left[ (\vec{\alpha}_1 \vec{p} + \beta_1 m)_{\alpha\alpha'} \delta_{\beta\beta'} + \delta_{\alpha\alpha'} (\vec{\alpha}_2 \vec{p} + \beta_2 M)_{\beta\beta'} - \frac{e^2}{r} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \right] \Psi_{\alpha'\beta'} = E \Psi_{\alpha\beta},
\]

in which only the static Coulomb potential is considered, have been found. Here a detailed discussion of the simple cases, \(1^1S_0, m = M,\) and \(m \neq M,\) is given, deriving the exact energy eigenvalues. An \(\alpha^2\) expansion is used to find radial wave functions. The leading term is given by the classical Coulomb wave function. The technique used here can be applied to other cases.

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1. Introduction and the Breit equation

1.1. Introduction

The Breit equation [1–3] has traditionally [4] been considered to be singular, and no attempt has ever been made to solve it. Instead, the Pauli approximation or a generalized Foldy–Wouthuysen transformation has been applied to derive the effective Hamiltonian \(H_{\text{eff}},\) consisting of the Schrödinger Hamiltonian

\[
H_0 = -\frac{1}{2} \left( \frac{1}{m} + \frac{1}{M} \right) \vec{p}^2 - \frac{e^2}{r}
\]

and many other relativistic correction terms [4–6]. The perturbation method was then used to evaluate the eigenvalues (binding energies) in a power series with \(\alpha\) as high as possible (or as high as necessary for experimental verification of quantum electrodynamics).

Here we shall try to solve the Breit equation directly. The radial wave function for the simplest case, \(1^1S\) leptonium states with equal masses, is given by the confluent Heun function. The author is not familiar with handling this function. Instead, an approximate method using an \(\alpha^2\) expansion should be used to find the wave functions. Its leading term is given by the Coulomb wave function.

However, the exact eigenvalues are found in singlet states with equal, \(m = M,\) and unequal, \(m \neq M,\) mass cases. We shall study the simplest case, \(1^1S,\) equal lepton masses, \(m = M,\) in greater detail here. The technique used here shall be applied to other cases, \(1^1(l), \; 3^1(l),\) and \(3^P_0\) including \(m \neq M,\) to find the exact eigenvalues.

\(^1\)This author contributed mainly to this work.

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More complicated cases, i.e., $m \neq M$, described in Sect. 2.3 below, and triplet cases, shall be described in due course.

1.2. Basic equation

The Breit equation for two leptons (Dirac particles) with charge and mass $(-e, m)$ and $(+e, M)$ interacting through the (static) Coulomb potential $-e^2/r = -\alpha/r$ ($\alpha = 1/137$) is given by

$$\left[ (\tilde{a}_1 \tilde{p} + \beta_1 m)_{\mu \nu} \delta_{\rho \sigma} + \delta_{\mu \nu} ( -\tilde{a}_2 \tilde{p} + \beta_2 M)_{\rho \sigma} - \frac{e^2}{r} \delta_{\mu \nu} \delta_{\rho \sigma} \right] \Psi_{\nu \sigma} = E \Psi_{\mu \rho}. \quad (1)$$

This equation holds in the center of mass (CM) system. The total energy of the system $E$ can be expressed as

$$E = \sqrt{m^2 - q^2} + \sqrt{M^2 - q^2}. \quad (2)$$

The momentum operator in (1) can be written as

$$\tilde{p} = \frac{1}{i} \tilde{\nabla} = \frac{1}{i} \frac{\partial}{\partial \tilde{r}}, \quad (3)$$

introducing the operator $\tilde{r}$ canonically conjugate to $\tilde{p}$. Then the (static) Coulomb potential (the contribution from one longitudinal photon exchange between two oppositely charged point Dirac particles) can be written as $-e^2/r = -\alpha/r$. $\tilde{a}_j, \beta_j (j = 1, 2)$ are the usual $4 \times 4$ Dirac matrices for particles 1 $(-e, m)$ and 2 $(+e, M)$, respectively. $\Psi_{\alpha \beta}$ is a $4 \times 4$ Dirac spinor wave function.

We shall try to solve

$$\left[ (\tilde{a}_1 \frac{1}{i} \tilde{\nabla} + \beta_1 m) + ( -\tilde{a}_2 \frac{1}{i} \tilde{\nabla} + \beta_2 M) - \frac{\alpha}{r} \right] \Psi = E \Psi, \quad (4)$$

and find the eigenvalue(s)

$$E = \sqrt{m^2 - q^2} + \sqrt{M^2 - q^2}, \quad (2)$$

which are specified by the discrete real value(s) $q_n$.

The two-particle system can be classified by the total angular momentum and its z-component and parity. In so doing, it is sufficient to specify the spin-angular parts of the large–large components $\Psi_{\alpha \beta}$ ($\alpha, \beta = 1, 2$) as

$$^1_l l, \quad (l = 0, 1, 2, \ldots)$$

$$^3(l - 1)l + ^3(l + 1)l, \quad (l = 1, 2, \ldots)$$

$$^3_l l, \quad (l = 1, 2, \ldots)$$

where $l$ is the quantum number for the orbital angular momentum. The spin-angular parts of the other (large–small, small–large, small–small) components of $\Psi_{\alpha \beta}$ are completely fixed by those of the large–large components.

We shall describe the singlet case in Sect. 2 in detail. Other cases can be handled similarly.
2. The singlet cases

2.1. $^1(l)_l$, general

We shall first discuss the simplest case of $^1(l)_l$: $\Psi_{\alpha\beta}$ can be taken as follows:

| $\alpha$ | $\beta$ | $\Psi_{\alpha\beta}$ |
|---------|---------|----------------------|
| 1       | 1       | $F(r)|^1(l)_l >$     |
| 2       | 2       | $i(G(r)|^3(l + 1)_l > + \tilde{G}(r)|^3(l - 1)_l >)$ |
| 1       | 3       | $i(H(r)|^3(l + 1)_l > + \tilde{H}(r)|^3(l - 1)_l >)$ |
| 3       | 1       | $K(r)|^1(l)_l >$     |
| 4       | 2       | $K(r)|^3(l + 1)_l >$ |
| 3       | 3       | $K(r)|^3(l - 1)_l >$ |
| 4       | 4       | $K(r)|^3(l + 1)_l >$ |

where $|^1(l)_l >$ and $|^3(l \pm 1)_l >$ are normalized spin-angular wave functions (the $z$-component of the total angular momentum has been omitted). The Breit equation (4) in Sect. 1 gives the following set of differential equations for the radial wave functions $F(r)$, $K(r)$, $G(r)$, $\tilde{G}(r)$, $H(r)$, and $\tilde{H}(r)$:

\[
\left( E + \frac{\alpha}{r} - m - M \right) F = \sqrt{\frac{l+1}{2l+1}} \left( \frac{d}{dr} + \frac{l+2}{r} \right) (G + H) - \sqrt{\frac{l}{2l+1}} \left( \frac{d}{dr} - \frac{l-1}{r} \right) (\tilde{G} - \tilde{H}),
\]

\[
\sqrt{l} \left( \frac{d}{dr} + \frac{l+2}{r} \right) (G - H) + \sqrt{l+1} \left( \frac{d}{dr} - \frac{l-1}{r} \right) (\tilde{G} - \tilde{H}) = 0,
\]

\[
\left[ E + \frac{\alpha}{r} - (m + M) \right] F = \left[ E + \frac{\alpha}{r} + (m + M) \right] K,
\]

\[
\left( E + \frac{\alpha}{r} + m - M \right) H = -\sqrt{\frac{l+1}{2l+1}} \left( \frac{d}{dr} - \frac{l}{r} \right) (F + K) = \left( E + \frac{\alpha}{r} - m + M \right) G,
\]

\[
\left( E + \frac{\alpha}{r} + m - M \right) \tilde{H} = \sqrt{\frac{l}{2l+1}} \left( \frac{d}{dr} + \frac{l+1}{r} \right) (F + K) = \left( E + \frac{\alpha}{r} - m + M \right) \tilde{G}.
\]

Therefore, we have the relations:

\[
\left( E + \frac{\alpha}{r} + m - M \right) H = \left( E + \frac{\alpha}{r} - m + M \right) G,
\]

\[
\left( E + \frac{\alpha}{r} + m - M \right) \tilde{H} = \left( E + \frac{\alpha}{r} - m + M \right) \tilde{G}.
\]

For later convenience, we introduce the dimensionless quantities:

\[
\rho = 2qr,
\]

\[
y = \frac{E}{2\alpha q} = \frac{\sqrt{M^2 - q^2} + \sqrt{m^2 - q^2}}{2\alpha q} (> 0),
\]

\[
\lambda = \frac{M + m + E}{2\alpha q} (> 0),
\]

\[
\nu = \frac{M + m - E}{2\alpha q} (> 0).
\]

If the eigenvalue $E$ is a proper relativistic generalization of the Schrödinger case (as we shall see is the case), $y$ and $\lambda$ are large quantities of the order of $1/\alpha^2$, while $\nu$ is of the order of unity.
For $\rho \to \infty (r \to \infty)$, all radial wave functions should behave like
\[
F(\rho) \to \rho^n(1 + O(1/\rho))e^{-\rho/2}.
\] (8)

This is because $e^{-\rho/2} = e^{-qr}$ and $E = \sqrt{m^2 - q^2} + \sqrt{M^2 - q^2}$. Notice that, if we consider $E$, the total CM energy $\sqrt{m^2 + p^2} + \sqrt{M^2 + p^2}$, to be larger than $M + m$, we would expect that the radial wave functions should contain a factor $e^{ipr}$. If we analytically continue $p$ to imaginary values, we get the factor $e^{-qr}$ and the energy expression (2) for bound states.

The singlet case becomes very simple when the two masses are equal, $M = m$. First,
\[
H = G, \quad \tilde{H} = \tilde{G},
\]

\[
K = \frac{1 - \nu \rho}{1 + \lambda \rho} F
\] (9)

Then, the combination
\[
F + K = \tilde{h}(\rho)e^{-\rho/2}
\] (10)

obeys the differential equation
\[
\tilde{h}'' + \tilde{h}' \left\{ -1 + \frac{2}{\rho} + \frac{1}{\rho(1 + y\rho)} \right\} + \tilde{h} \left[ \left( \frac{\alpha^2 y}{2} - 1 \right) \frac{1}{\rho} + \frac{\alpha^2}{4 \rho^2} - \frac{l(l + 1)}{\rho^2} - \frac{1}{2\rho(1 + y\rho)} \right] = 0.
\] (11)

2.2. $^1S_0, M = m$

For the $^1S_0$, i.e., $l = 0$, case we have the following equation for $\tilde{h}$:
\[
\tilde{h}'' + \tilde{h}' \left\{ -1 + \frac{2}{\rho} + \frac{1}{\rho(1 + y\rho)} \right\} + \tilde{h} \left[ \left( \frac{\alpha^2 y}{2} - 1 \right) \frac{1}{\rho} + \frac{\alpha^2}{4 \rho^2} - \frac{1}{2\rho(1 + y\rho)} \right] = 0.
\] (11)

If we ignore $\alpha^2$ and higher-order terms in (11) and notice that $\alpha^2 y$ is of the order of unity, we find
\[
\tilde{h}'' + \tilde{h}' \left\{ -1 + \frac{2}{\rho} \right\} + \tilde{h} \left( \frac{\alpha^2 y}{2} - 1 \right) \frac{1}{\rho} = 0,
\] (12)

which is precisely the Schrödinger equation for our issue and $\tilde{h}$ in this approximation is given by $F(-\frac{\alpha^2 y}{2} + 1, 2, \rho) = F(1 - n, 2, \rho)$, apart from the normalization constant. $\frac{\alpha^2 y}{2} - 1$ must be equal to an integer $n - 1 = 0, 1, 2, \ldots$

Equation (11) shows that $\rho = 0$ is the regular singular point while $\rho = \infty$ is an irregular singular point.

Near $\rho = 0$, $\tilde{h}(\rho)$ can be expressed as
\[
\tilde{h}(\rho) = \rho^s h(\rho),
\]

where
\[
s = -1 + \sqrt{1 - \frac{\alpha^2}{4}}
\] (13)

Another solution $s = -1 - \sqrt{1 - \frac{\alpha^2}{4}}$ is unacceptable from the square-integrability of the wave function $\Psi_{\alpha \beta}$. $h(\rho)$ obeys the equation:
\[
\tilde{h}'' + \tilde{h}' \left\{ -1 + \frac{2 + 2s}{\rho} + \frac{1}{\rho(1 + y\rho)} \right\} + h \left\{ \left( \frac{\alpha^2 y}{2} - 1 - s \right) \frac{1}{\rho} - \left( \frac{1}{2} + sy \right) \frac{1}{\rho(1 + y\rho)} \right\} = 0.
\] (14)
$h(\rho)$ can be expanded in a Taylor series near $\rho = 0$:

$$h(\rho) = \sum_{n=0}^{\infty} h_n \rho^n,$$

(15)

$$h_1 = \frac{1}{3 + 2s} \left\{ \frac{-\alpha^2 y}{2} + 1 + s + \frac{1}{2} + sy \right\} h_0, \quad (n + 1)(n + 3 + 2s) \frac{1}{y} h_{n+1}
+ \left\{ n(n + 1 + 2s) - \frac{1}{y} (n - \frac{\alpha^2 y}{2} + 1 + s + \frac{1}{2} + sy) \right\} h_n
- \left\{ n - 1 - \frac{\alpha^2 y}{2} + 1 + s \right\} h_{n-1} = 0.$$

(16)

It can be shown that $h(\rho)$ cannot be a polynomial:

$$h(\rho) = \sum_{n=0}^{N} h_n \rho^n \quad (N < \infty)$$

from the recurrence formulae (16). So the sum in Eq. (15) must extend to $n = \infty$, a sharp difference from the Schrödinger case. Note that the expansion (15) holds only in the tiny region $\rho < 1/y \sim O(\alpha^2)$.

Next we discuss $\rho = \infty$, which is an irregular singular point, assuming the form

$$h(\rho) = e^{\lambda \rho} \rho^k \left[ c_0 + \frac{c_1}{\rho} + \frac{c_2}{\rho^2} + \cdots \right] \quad \text{at } \rho \to \infty,$$

(17)

where $\lambda, k, c_n$ are constant. Introducing (17) into the differential equation (14), we find that $\lambda$ is either 0 or 1 while $k$ is undetermined. From the square-integrability of the wave function, we must choose $\lambda$ to be 0 and $k < \infty$.

$k$ turns out to be

$$k = \frac{\alpha^2 y}{2} - 1 - s,$$

(18)

which appeared in (14); its value is not fixed from the situation at $\rho \to \infty$. Further discussion will be given in Sect. 2.2.4.

2.2.1. Detailed discussion of $^1S_0$, $m = M$. Note again

$$F + K = e^{-\rho/2} \rho^k h(\rho).$$

$h(\rho)$ obeys the differential equation (14),

$$\frac{d^2 h}{d\rho^2} + \frac{dh}{d\rho} \left\{ -1 + \frac{\gamma}{\rho} \right\} + h \left\{ \frac{N}{\rho} \right\} = \left\{ -\frac{1}{\rho(1 + y\rho)} \left( \frac{d}{d\rho} - \delta \right) \right\} h,$$

where

$$\gamma = 2 + 2s, \quad s = -1 + \sqrt{1 - \frac{\alpha^2}{4}},$$

$$\delta = \frac{1}{2} + sy,$$

$$N = \frac{\alpha^2 y}{2} - 1 - s,$$

$$\rho = 2q r,$$

$$y = \frac{2\sqrt{m^2 - q^2}}{2 \alpha q} = \frac{E}{2 \alpha q}$$

where $\alpha^2 y$ and $sy$ are of the order of unity $O(1)$.
The r.h.s. of Eq. (19) is of the order of $1/y$ or $O(\alpha^2) \simeq 10^{-4}$. As a first step we shall ignore the r.h.s. of Eq. (19). Then $h$ can be solved immediately:

$$h \simeq F(-N, \gamma, \rho). \quad (20)$$

Normalizability of the radial wave function requires $N$ equal to a positive integer:

$$N = \frac{\alpha^2 y}{2} - 1 - s = n - 1, \quad n = 1, 2, 3, \ldots. \quad (21)$$

$n$ is the principal quantum number. Equation (20) is the relativistic extension form of the non-relativistic Coulomb wave function:

$$F(-N, \gamma, \rho) \begin{cases} N = \frac{\alpha^2 y}{2} - 1 - s \to n - 1, \\ \gamma = 2 + 2s \to 2. \end{cases} \quad (22)$$

The energy eigenvalues derived from condition (21) are

$$E_n = 2 \sqrt{m^2 - q^2} = 2m \sqrt{1 - \frac{1}{1 + (\alpha y)^2}}$$

$$= 2m \sqrt{1 - \frac{\alpha^2}{(\alpha^2 y)^2 + \alpha^2}}$$

$$= 2m \sqrt{1 - \frac{\alpha^2}{(2(n + s))^2 + \alpha^2}}. \quad (23)$$

When $n = 1$,

$$E_1 = 2m \sqrt{1 - \frac{\alpha^2}{4}}. \quad (24)$$

The approximate solution $F(-N, \gamma, \rho)$ is correct, ignoring terms of the order $O(1/y) \simeq \alpha^2 \simeq 10^{-4}$. We write

$$h(\rho) = F(-N, \gamma, \rho) + f(\rho). \quad (25)$$

When $n = 1$, $F(0, \gamma, \rho) = 1$,

$$f(0) = 0. \quad (26)$$

We shall introduce

$$h = F + f,$$

into Eq. (19), then we find

$$\frac{d^2 f}{d\rho^2} + \frac{df}{d\rho} \left\{ -1 + \frac{\gamma}{\rho} \right\} + f \frac{N}{\rho} = \left\{ -\frac{1}{\rho(1 + y\rho)} \left( \frac{d}{d\rho} - \delta \right) \right\} (F + f), \quad (27)$$

whose solution gives $h = F + f$.

Alternatively, we may use an approximate method: $F$ is the order of unity, while $f(\rho)$ is the order of $O(1/y) \simeq O(\alpha^2)$, so that $f$ in the r.h.s. of Eq. (27) can be ignored. This equation immediately gives the solution

$$f(\rho) = \int_0^\rho \frac{e^{\rho d\rho}}{(F)^2 \rho^2} \int_0^\rho e^{-\sigma} F(-N, \gamma, \sigma) \sigma d\sigma \left[ -\frac{1}{\sigma(1 + y\sigma)} \left( \frac{d}{d\sigma} - \delta \right) \right] F.$$

It is difficult to perform integration here, but a computer shall easily do the job.
Equation (21) give the exact eigenvalues for our problem. The reason for this is as follows. When \( \rho \to \infty \), the function \( h(\rho) = F(\rho) + f(\rho) \) behaves like (see Sect. 2.2.4)

\[
h(\rho) = \rho^\beta \left( c_0 + \frac{c_1}{\rho} + \cdots \right), \quad \beta = N.
\]

As will be described in Sect. 2.2.4, \( F(\rho) \) contributes to \( \rho^\beta \), while \( f(\rho) \) does not, i.e.,

\[
\frac{f(\rho)}{\rho^\beta} \simeq O\left( \frac{1}{y\rho} \right), \quad \text{when} \ \rho \to \infty.
\]

Therefore, Eq. (21) gives the exact eigenvalues.

To find \( f(\rho) \), we may use \( 1/y \) expansion, or \( \alpha^2 \) expansion, since \( y \) is of the order of \( 1/\alpha^2 \).

Introducing

\[
h(\rho) = \sum_{v=0}^{\infty} f^{(v)}(\rho),
\]

\[
f^{(0)}(\rho) = F(\rho) = F(-N, \gamma; \rho), \quad N = n - 1,
\]

\[
f(\rho) = \sum_{v=1}^{\infty} f^{(v)}(\rho),
\]

where \( f^{(v)}(\rho) \) is of the order of \( (1/y)^v \sim O(\alpha^{2v}) \), and the differential operators

\[
D^{(0)} = \frac{d^2}{d\rho^2} + \left( -1 + \frac{\gamma}{\rho} \right) \frac{d}{d\rho} + \frac{N}{\rho},
\]

\[
D^{(1)} = -\frac{1}{\rho(1+y\rho)} \left( \frac{d}{d\rho} - \delta \right),
\]

where \( D^{(v)} \) is of the order of \( (1/y)^v \) (here \( v = 0 \) and 1), the differential equation (19) will give the following series of equations:

\[
D^{(0)} f^{(0)} = 0,
\]

\[
D^{(0)} f^{(v)}(\rho) = D^{(1)} f^{(v-1)}(\rho), \quad (v = 1, 2, \ldots).
\]

It is easy to see that

\[
f^{(0)}(\rho) + f^{(1)}(\rho) = F(\rho) + f^{(1)}(\rho)
\]

shall be good if the terms of the order of \( \alpha^4 \sim 10^{-8} \) are ignored.

These techniques can be applied to other singlet and triplet cases with \( m \neq M \).

2.2.2. \( f(\rho) \) for the ground state of \( ^1S_0, m = M, \) near \( \rho = 0 \) \( (\rho < 1/y) \). The function \( h(\rho) \) for \( n = 1 \) is given by

\[
h(\rho) = F(0, \gamma, \rho) + f(\rho) = 1 + f(\rho), \quad f(0) = 0.
\]

(28)

The Taylor expansion of \( f(\rho) \) near \( \rho = 0 \) is given by

\[
f(\rho) = \sum_{n=1}^{\infty} f_n \rho^n;
\]
the $f_n$ are equal to $h_n (n \geq 1)$ and are given by (16). Noting that
\[ s \div -\frac{\alpha^2}{8}, \quad \alpha^2 y \div 2, \]
\[ sy \div -\frac{1}{4}, \quad \delta \div \frac{1}{4}, \]
\[ N \div 0, \quad \gamma \div 2, \]
on one finds
\[ f_0 = 0, \quad f_1 = \frac{1}{12}, \quad f_2 = -\frac{1}{48} y f_1 = -\frac{1}{48} y, \]
and in general
\[ f_{n+1} = \frac{(-y)^{n/2}}{(n+1)(n+2)(n+3)}, \]
\[ f(\rho) = \sum_{n=1}^{\infty} f_n \rho^n = \sum_{n=1}^{\infty} \frac{(-y)^{n-1}}{n(n+1)(n+2)} \frac{1}{2} \rho^n \]
\[ = -\frac{1}{2y} \left[ (1 + y \rho)^2 \log \left( \frac{1}{1 + y \rho} \right) + \frac{3}{4} + \frac{1}{2y} \rho \right], \]
which holds at $-\frac{1}{y} < \rho < \frac{1}{y}$: $f(\rho)$ is of the order of $O(1/y) = O(\alpha^2)$ and this $y f(\rho)$ is good within the errors of $O(1/y) = O(\alpha^2)$. General cases $n > 1$ can be treated analogously.

2.2.3. $h(\rho)$ for the ground state of $^1S_0$, $m = M$, near $\rho = -1/y$, mathematical curiosity.

Equation (14) has a regular singular point at $\rho = -1/y$. The regular solution near $\rho = -1/y$ is given by
\[ h(x) = \sum_{n=2}^{\infty} g_n x^n, \quad (29) \]
where $x = \rho + \frac{1}{y}$.

Another solution near $x = 0 = \rho - \frac{1}{y}$ contains the log of solution (29).

The recurrence formulae for $g_n$ are
\[ 3 g_3 = (2(1 + y)y + 2 - \gamma) g_2, \quad (30) \]
\[ -(n + 1)(n - 1) g_{n+1} + \{n(n + \gamma y)\delta\} g_n - y(n - 1 - N) g_{n-1}. \quad (31) \]

Ignoring the $O(1/y)$ terms, one finds
\[ g_3 = 2y g_2, \quad g_4 = \frac{3}{2} y g_3 = 3y^2 g_2. \]

Taking $g_2 = 1$,
\[ g_{n+1} = \frac{ny}{(n - 1)} g_n = ny^{n-1}, \quad (n > 1), \]
so $h(\rho)$ turns out to be
\[ h(\rho) = \sum_{n=2}^{\infty} g_n x^n y^{n-1} = \sum_{n=1}^{\infty} n x^{n+1} y^n \]
\[ = \frac{x^2 y}{1 - yx} = \frac{(1 + y \rho)^2}{y^2 \rho}. \]

This expression is valid over the unphysical range $-2/y < \rho < 0$ and good within $\alpha^2$. 

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2.2.4. Behavior of $h(\rho)$ at $\rho \to \infty$. $\rho = \infty$ is the irregular singular point, so that $h(\rho)$ can be expressed in the form (17), as described in Sect. 2.2.

We shall give the recurrence formulae here for the case of $^{1}S$, $m = M$:

$$h(\rho) = \rho^\beta \left( c_0 + \frac{c_1}{\rho} + \frac{c_2}{\rho^2} + \cdots \right),$$

$$c_1 = \left\{-\beta(\beta + \gamma - 1) + \frac{\delta}{\gamma}\right\} c_0,$$

$$(n + 1)\gamma c_{n+1} + (n - \beta)(n + 1 - \gamma - \beta)\gamma - n + \delta)c_n + (n - 1 - \beta)(n - \gamma - 1 - \beta)c_{n-1} = 0,$$

where

$$\gamma = 2 + 2s, \quad \delta = \frac{1}{2}, \quad \beta = N = \frac{\alpha^2 y}{2} - 1 - s = n - 1.$$  

$N$ has already been fixed in Eq. (21).

We repeat that $\rho^\beta$ is the contribution from $F(\rho)$, and $f(\rho) = h(\rho) - F(\rho)$ does not contribute to the $\rho^\beta$ term. Furthermore, it is easy to show that

$$\frac{f(\rho)}{\rho^\beta} \sim O\left(\frac{1}{y\rho}\right) \text{ when } \rho \to \infty.$$  

This shows that Eq. (21) does in fact give the correct (or exact) eigenvalues.

2.2.5. Comparison of the Breit case with the Dirac Coulomb solution. For simplicity we shall only discuss the ground states of $^{2}S_{1/2}$ (Dirac) and $^{1}S_{0}$ (Breit).

The Dirac equation with Coulomb potential [7,8] reads

$$\left\{ \frac{1}{i} \vec{\alpha} \vec{\nabla} + \beta m - \frac{\alpha}{r} \right\} \Psi = E \Psi,$$

$$E = \sqrt{m^2 - q^2}. \quad (32)$$

$\Psi$ for $^{2}S_{1/2}$, spin up, can be written as

$$\Psi = \begin{cases} F(r)Y_{00}\frac{\sigma_{3}+1}{2} & s_{1/2} \text{ spin up} \\ K(r)(\frac{\alpha}{\vec{r}})\frac{\sigma_{3}+1}{2} & p_{1/2} \text{ spin up} \end{cases},$$

$$\sigma_{3} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y_{00}(\theta, \phi) = \frac{1}{\sqrt{2\pi}}.$$  

$\vec{\sigma} \cdot \vec{r} = \frac{1}{r}(\vec{\sigma} \cdot \vec{r})$.  

The differential equation for $F$ is given by

$$\frac{d^2F}{dr^2} + \left\{ \frac{1}{E + \frac{\alpha}{r} + m} \cdot \frac{\alpha}{r^2} + \frac{2}{r} \right\} \frac{dF}{dr} + \left\{ \left( E + \frac{\alpha}{r} \right)^2 - m^2 \right\} F = 0. \quad (34)$$
We introduce the dimensionless quantities
\[
\rho = 2qr, \quad y = \frac{E + m}{2\alpha q},
\]
and write \( F \) as
\[
F(\rho) = e^{-\rho/2} \rho^2 h(\rho),
\]
where \( s \) is introduced to remove the \( 1/\rho^2 \) terms as usual. Then the differential equation for \( h(\rho) \) is given by
\[
h'' + \left( -1 + \frac{2 + 2s}{\rho} + \frac{1}{\rho(1 + y\rho)} \right) + h \left( \left( \frac{\alpha E}{q} - 1 - s \right) \frac{1}{\rho} - \left( \frac{1}{2} + sy \right) \frac{1}{\rho(1 + y\rho)} \right) = 0,
\]
where \( E = \sqrt{m^2 - q^2} \), binding energy \( B = m - \sqrt{m^2 - q^2} \), and \( y = (E + m)/2\alpha q \). However, \( s \) is different from the Breit case:
\[
s = -1 + \sqrt{1 - \alpha^2}.
\]
The form (35) in the Dirac case looks exactly the same as the Breit case (14). The only differences are given below.

|   | Dirac       | Breit       |
|---|-------------|-------------|
| \( s \) | \( s = -1 + \sqrt{1 - \alpha^2} \) | \( s = -1 + \sqrt{1 - \alpha^2 / 4} \) |
| \( N \) | \( \alpha E / q - 1 - s \) | \( \alpha^2 y / 2 - 1 - s \) |

However, the ground-state solution of (35) in the Dirac case is
\[
h(\rho) = F(0, 2 + 2s, \rho) = 1, \quad \text{for} \quad n = 1.
\]

Both
\[
\left( \frac{\alpha E}{q} - 1 - s \right) = 0 \quad \text{and} \quad \frac{1}{2} + sy = 0
\]
hold, and they give the same value of \( q \),
\[
q = \alpha m.
\]
Therefore,
\[
E = \sqrt{m^2(1 - \alpha^2)}
\]
\[
B = m - E = \left\{ 1 - \sqrt{1 - \alpha^2} \right\} m.
\]

As is well known, the Dirac equation with Coulomb force can be solved for all cases of \( n \) and \( l \) in terms of two confluent hypergeometric functions (multiplied by \( \rho^s e^{-\rho/2} \)).
However, one cannot do

\[ N = 0 \quad \text{and} \quad \delta = 0 \]

simultaneously in the Breit case:

\[ N = 0 \quad \text{means} \quad \frac{\alpha^2 y}{2} - 1 - s = 0, \]

then

\[ \delta = \frac{1}{2} + sy = \frac{1}{4}, \]

while \( \delta = 0 \) demands

\[ \frac{1}{2} - \frac{\alpha^2}{8} y = 0 \]

but

\[ \frac{\alpha^2 y}{2} = 2 \quad 1\text{st excited state!} \]

and \( N \) cannot be zero.

From these situations, the Breit case is more complicated than the Dirac case, as described in this article.

2.2.6. Confluent Heun function. The solution \( h(\rho) \) of differential equation (14) is the confluent Heun function (Prof. T. Rijken kindly informed me of this fact (via his son)); this function has 5 parameters. Unfortunately, the author is not familiar with this function, so that here the physical ground and \( 1/y \) expansion of the wave function \( h(\rho) \) are used.

2.3. \( ^1S_0, M \neq m \)

The unequal mass case \( M \neq m \) can be treated in exactly the same way as in Sect. 2.2 for the case of \( M = m \).

The equation for the quantity \( \tilde{h}(\rho) \),

\[ F + K = \tilde{h}(\rho)e^{-\rho/2}, \]

is given by

\[
\begin{align*}
\tilde{h}'' &+ \tilde{h}' \left[ -\frac{1}{\rho} + \frac{2}{\rho(1 + y\rho)} + \frac{1}{\rho((1 + y\rho)^2 - (\bar{M} - \bar{m})^2\rho^2)} \right] \\
&+ \tilde{h} \left[ -\frac{1}{\rho} + \frac{1}{2\rho(1 + y\rho)} - \frac{1}{\rho((1 + y\rho)^2 - (\bar{M} - \bar{m})^2\rho^2)} \right] \\
&+ \tilde{h} \left[ \frac{1}{4} + \frac{\alpha^2}{4} \left\{ 1 - \frac{(\bar{M} + \bar{m})^2\rho^2}{(1 + y\rho)^2} \right\} \left\{ \frac{(1 + y\rho)^2 - (\bar{M} - \bar{m})^2}{\rho^2} \right\} \right] = 0, \quad (38)
\end{align*}
\]

where \( \bar{M} = M/2\alpha q \) and \( \bar{m} = m/2\alpha q \).

As in Sect. 2.2, \( \rho = 0 \) and \( \rho = \infty \) in Eq. (38) are regular and irregular singular points, respectively. Therefore, we can adopt the same procedure as in Sect. 2.2.
To remove the $1/\rho^2$ terms, we put
\[ \tilde{h}(\rho) = \rho^s h(\rho), \]
again finding $s = -1 + \sqrt{1 - \alpha^2/4}$. The differential equation for $h(\rho)$ reads
\[
h'' + h' \left[-1 + \frac{\gamma}{\rho} + \frac{1}{(1 + y\rho)^2 - (\bar{m} - \bar{M})^2 \rho^2} \left\{ \frac{(\bar{m} - \bar{M})^2 \rho}{1 + y\rho} + \frac{1 + y\rho}{\rho} \right\} \right] + h \left[ \left( \frac{\alpha^2 y}{2} - 1 - s \right) \frac{1}{\rho} \right] = 0. \tag{39} \]
We should notice here that $(A) + (B)$ and $(C)$ reduce as follows:
\[
(A) + (B) \rightarrow \frac{-\delta}{\rho(1 + y\rho)}, \quad \delta = \frac{1}{2} + sy, \\
(C) \rightarrow 0, \quad \text{and} \quad (D) = 0,
\]
for the equal masses case, $M = m$. Equation (39) reduces to (14) for the $^1S_0, m = M$ case.

We shall rewrite the term $(D)$ as follows:
\[
(D) = \frac{\alpha^2 (m^2 - \bar{M}^2)^2 y^2}{4 (1 + y\rho)^2} \left( \frac{y\rho}{1 + y\rho} \right)^2,
\]
\[
(E) \cdots = \frac{\alpha^2}{4} \left( \frac{(m^2 - \bar{M}^2)^2 y^2}{y^2} \right) \left[ \left( \frac{y\rho}{1 + y\rho} \right)^2 + \frac{2y}{\rho} \right],
\]
\[
(F) \cdots = -\frac{\alpha^2 y}{2} \left( \frac{(m^2 - \bar{M}^2)^2}{y^2} \right) \frac{1}{\rho}.
\]
The last term $(F)$ shall be added to the first line of the coefficient $h(\rho)$ in (19):
\[
h'' + h' \left[-1 + \frac{\gamma}{\rho} - 2(B)\right] + h \left[ \frac{\alpha^2 y}{2} \left\{ 1 - \left( \frac{m^2 - \bar{M}^2}{y^2} \right)^2 \right\} - 1 - s \right] \frac{1}{\rho} + h[(A) + (B) + (C) + (E)] = 0. \tag{40} \]

$(A), (B), (C),$ and $(E)$ are of the order of $1/y$ or $\alpha^2$. Now we can use the technique described in Sect. 2.3 for $^1S, m = M$:
\[
h(\rho) = F(-N, \gamma, \rho) + f(\rho), \tag{41} \]
\[
N = \frac{\alpha^2 y}{2} \left\{ 1 - \left( \frac{m^2 - \bar{M}^2}{y^2} \right)^2 \right\} - 1 - s,
\]
\[
N = n - 1, \quad n = 1, 2, \ldots, \tag{42} \]
where $n$ is the principal quantum number. The background part $f$ is $O(\alpha^2)$ and can be found by solving
\[
f'' + f' \left\{ -1 + \frac{\gamma}{\rho} \right\} + f \left( \frac{N}{\rho} \right) = \left\{ +2(B) \frac{d}{d\rho} - (A) - (B) - (C) - (E) \right\} F(-N, \gamma, \rho).
\]
Thus the accuracy of the solution $F + f$ is quite good, and terms of the order of $\alpha^4$ are ignored.
The other radial wave functions, $F$, $K$, $G$, $H$, $\tilde{G}$, and $\tilde{H}$, can now be written down.
The energy eigenvalues from Eq. (42),
\[
\frac{\alpha^2 y}{2} \left\{ 1 - \left( \frac{m^2 - \bar{M}^2}{y^2} \right)^2 \right\} = \frac{\alpha E}{4q} \left\{ 1 - \left( \frac{m^2 - M^2}{E^2} \right)^2 \right\} = n + s = \bar{n},
\]
are easily calculated. Therefore, the binding energy $B = M + m - E$ is
\[
B = M + m - E = \sum_{m=1}^{\infty} e_m \left( \frac{\alpha}{2\bar{n}} \right)^{2m},
\]
\[
e_1 = \frac{2mM}{(M + m)},
\]
\[
e_2 = \frac{2mM}{(M + m)} \left\{ 1 - \frac{3mM}{(M + m)^2} \right\},
\]
\[
e_3 = \frac{4mM}{(M + m)} \left\{ 1 - \frac{5mM}{(M + m)^2} + \frac{5(mM)^2}{(M + m)^4} \right\},
\]
and so on. The first term is exactly the well known form of the reduced mass.
It is easy to see that (43) reduces to (23) when $M = m$.
Knowing that $F + K = e^{-\rho/2} \rho^4 h(\rho)$, all relevant radial wave functions, $F$, $K$, $G$, $H$, $\tilde{G}$, and $\tilde{H}$, can easily be derived.

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