Einstein Energy-Momentum Complex for a Phantom Black Hole Metric

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We calculate the energy distribution associated with a static spherically symmetric non-singular phantom black hole metric in Einstein's prescription in general relativity. As required for the Einstein energy-momentum complex, we perform the calculations in quasi-Cartesian coordinates. We also calculate the momentum components and obtain a zero value, as expected from the geometry of the metric.

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Since the beginning of Einstein’s theory of general relativity (GR), there have been several known important issues, e.g., the missing matter cosmology problem and the energy-momentum localization in curved spacetimes, which are still in doubt and possess non-specific solutions. Nevertheless, there have been a wide range of attempts by several researchers to beat these problems using various hypotheses and tools. In GR (that is, in curved spacetimes), the partial derivative of the usual local conservation equation $T_{i,k} = 0$ valid in Minkowski spacetime is replaced by a covariant derivative. Here $T^k_i$, which is the energy-momentum tensor of matter and all non-gravitational fields, no longer satisfies $T_{i,k} = 0$ in the presence of a gravitational field. The contribution from the gravitational field is now required to construct an energy-momentum expression which satisfies a local conservation law.

Einstein solved this problem and suggested an expression for energy-momentum distribution (for further detail, see Ref. [1]). Although his work was criticized by a few physicists (e.g., notably by Pauli), he justified that his energy-momentum complex provides convincing results for the total energy and momentum of isolated systems. Later, many physicists have suggested alternative expressions for the energy-momentum distribution, including Tolman,[2] Landau et al.,[3] Weinberg,[4] Papapetrou,[5] and Bergmann et al.[6] The main problem with all of these definitions is that they are coordinate-dependent. One can, however, obtain meaningful results for the total energy and momentum of isolated systems only when calculations are performed in quasi-Cartesian coordinates. These complexes are also pseudo-tensors due to the fact that they are not tensorial objects. Although Komur[7] and Penrose[8] constructed coordinate-independent definitions of energy and angular momentum, their approaches were restricted to only a limited class of metrics. Therefore, the coordinate-independent approaches become in fact worse.

In 1990, Virbhadra’s seminal studies shook the notion that energy-momentum complexes could give sensible results only for the total energy of isolated systems. Virbhadra[9] showed that several energy-momentum complexes give the same energy-distribution for the Kerr–Newman metric. In this context, Virbhadra et al.[10] studied many spacetimes and obtained energy distributions. Nathan (an eminent collaborator with Albert Einstein) et al.[11] studied the energy and momentum distributions in Einstein–Rosen cylindrically gravitational waves. To their great surprise several complexes produced the same results, even though this metric is not asymptotically flat. Aguirregabiria et al.[12] showed that several coordinate-dependent definitions give the same energy and momentum distribution for any metric of Kerr–Schild class. In 1999, Virbhadra[13] proved that various energy-momentum complexes coincide, not only for the Kerr–Schild class metrics but also for a class of solutions that are much more general, which includes asymptotically flat as well as non-flat spacetimes.

The problem of energy-momentum localization in GR gained a new point of view with the results of the study by Virbhadra et al. Using Einstein energy-momentum complex, Rosen[12] found that the total energy is zero for a closed homogeneous isotropic universe described by a Friedmann–Robertson–Walker (FRW) metric. Johri et al.,[15] using the Landau and Lifshitz energy-momentum complex, showed that the total energy of a FRW spatially closed universe is zero at all times, irrespective of the equations of state of the cosmic fluid. They also showed that the total energy enclosed within any finite volume of the spatially flat FRW universe is zero at all times.

Many researchers have tried to solve the energy-momentum localization problem using different spacetimes and different energy-momentum prescriptions and they have obtained a plethora of important results (see Refs. [16–18] and the references therein). Xulu studied several spacetimes (asymptotically flat as well as non-flat). He obtained the energy distri-
buton of a charged dilaton black hole and Melvin’s magnetic universe. Xulu investigated the Papapetrou and Weinberg complexes for the anisotropic Bianchi type-I space time.

Radinschi used Landau–Lifshitz and Papapetrou energy-momentum solutions for Bianchi type $V_{Ib}$ spacetime. Later, she obtained the results for the same metric using Tolman, Bergmann–Thompson and Møller energy-momentum complexes. Loi and Vargas studied energy localization for Bianchi I and II universes in teleparallel gravity. Aydogdu investigated Einstein and Landau, and Lifshitz energy-momentum complexes for Bianchi type-II universe in GR. Aydogdu and Salti used Einstein Bergmann–Thompson prescriptions for Bianchi type-V metric in general relativity and teleparallel gravity. Andrade et al.\textsuperscript{[19]} obtained a conserved energy-momentum gauge current in the context of a gauge theory for the translation group.

Recent analysis of type Ia supernovae, cosmic microwave background anisotropy and mass power spectrum observations favor the negative values of the equation of state parameter $\omega$ for dark energy.\textsuperscript{[20]} Considering the equation of state parameter of accustomed quintessence models with positive kinetic energy, it is not possible to derive the aforesaid order of $\omega$. Therefore, many researchers\textsuperscript{[21]} have studied phantom field models with negative kinetic energy to achieve this regime of $\omega$. If this candidate of dark energy is part of the real field content of the large scale structure of the Universe, then it is natural to look for its manifestation in the study of black holes. The exact solution of black holes in a phantom field is called a phantom black hole. Gao et al.\textsuperscript{[22]} discussed the cosmological aspects of the phantom black hole and phantom field. Babichev et al.\textsuperscript{[23]} studied the accretion of phantom fluid onto a black hole. Bronnikov et al. investigated the physics of neutral phantom black holes and presented some interesting results.\textsuperscript{[24]} Ding et al.\textsuperscript{[20]} studied the influence of phantom fields on strong gravitational lensing.

The purpose of this work is to calculate the energy and momentum distributions in a phantom black hole spacetime in Einstein’s prescription. Here we use the convention that Latin indices take values from 0 to 3 and Greek indices run from 1 to 3. As usual in general relativity studies, we also use $G_5 = 1$ and $c = 1$ units.

The Bronnikov–Fabris phantom black hole metric,\textsuperscript{[25]} expressed by Ding et al.\textsuperscript{[20]} in a neat form, is

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - (r^2 + p^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$

with

$$f(r) = 1 - \frac{3M}{p} \left[ (\frac{\pi}{2} - \arctan \frac{r}{p}) \left(1 + \frac{r^2}{p^2}\right) - \frac{r}{p}\right],$$

where $M$ is a mass parameter defined in the usual way and $p$ is a positive constant termed as the phantom constant\textsuperscript{[26]} (symbol $p$ is meant for phantom). Ding et al.\textsuperscript{[20]} explained that for $M = 0$, the metric represents an Ellis wormhole.

To calculate energy and momentum components, we transform the line element Eq.(1) to quasi-Cartesian coordinates $t, x, y, z$ using the following transformation

$$x = r \sin \theta \cos \phi,$$
$$y = r \sin \theta \sin \phi,$$
$$z = r \cos \theta.$$  

The line element Eq.(1) becomes

$$ds^2 = f(r)dt^2 - \frac{r^2 + p^2}{r^2} \left( dx^2 + dy^2 + dz^2 \right) - \frac{r^2}{f(r)} \left( \frac{xdy + ydz}{r} \right)^2,$$

where

$$r = \sqrt{x^2 + y^2 + z^2}.$$  

The determinant of the metric tensor $g_{ik}$ is

$$g = - \left(1 + \frac{p^2}{r^2}\right),$$

and the 10 independent contravariant components of the symmetric energy-momentum tensor ($g^{ik} = k^i$ for all values of $i, k$) are

$$g^{00} = \frac{1}{f},$$
$$g^{11} = -\left(p^2 + r^2\right)f x^2 - (r^2 - x^2)r^2, \quad \frac{f}{(p^2 + r^2)^2},$$
$$g^{22} = -\left(p^2 + r^2\right)f y^2 - (r^2 - y^2)r^2, \quad \frac{f}{(p^2 + r^2)^2},$$
$$g^{33} = -\left(p^2 + r^2\right)f z^2 - (r^2 - z^2)r^2, \quad \frac{f}{(p^2 + r^2)^2},$$
$$g^{01} = 0,$$
$$g^{02} = 0,$$
$$g^{03} = 0,$$
$$g^{12} = \frac{(r^2 - (p^2 + r^2)f)xy}{p^2 + r^2},$$
$$g^{13} = \frac{(r^2 - (p^2 + r^2)f)yz}{p^2 + r^2},$$
$$g^{23} = \frac{(r^2 - (p^2 + r^2)f)xz}{p^2 + r^2}. $$

A thorough study by Virbhadra\textsuperscript{[14]} revealed that the Einstein energy-momentum complex gives the most reliable energy distribution and, therefore, we will use the same definition in this work. The energy-momentum complex of Einstein is\textsuperscript{[31]}

$$\Theta_{k}^{i} = \frac{1}{16\pi} H_{k}^{ij},$$

where

$$H_{i}^{kl} = -H_{i}^{lk} = \frac{g_{lm}}{\sqrt{g}} \left[ -g \left(g^{kn}g^{lm} - g^{ln}g^{km}\right)\right].$$

Here $\Theta_{0}^{0}$ and $\Theta_{0}^{3}$ denote the energy and momentum density components, respectively (Virbhadra\textsuperscript{[14]} mentioned that, although the energy-momentum complex

020402-2
found by Tolman differs in form from the Einstein energy-momentum complex, they are both equivalent in import).

Here, $\theta^k_i$ satisfies the covariant local conservation laws

$$\frac{\partial \theta^k_i}{\partial x^j} = 0.$$  \hspace{1cm} (10)

The energy-momentum components are expressed as

$$P_i = \int \int \int \theta^0_i \, dx^1 \, dx^2 \, dx^3,$$  \hspace{1cm} (11)

where $P_\alpha$ gives momentum components $P_1$, $P_2$, $P_3$; and $P_0$ gives the energy. Using Gauss’s theorem, one can obtain

$$P_i = \frac{1}{16\pi} \int \int H^0_\alpha \eta_{\alpha} dS,$$  \hspace{1cm} (12)

where $\eta_\alpha$ is the outward unit normal vector over the infinitesimal surface element $dS$. To obtain energy, we obtain only three components of $H^0_\alpha$,

$$H^{01}_0 = 2x(f p^2 - f r^2 + r^2),$$

$$H^{02}_0 = 2y(f p^2 - f r^2 + r^2),$$

$$H^{03}_0 = 2z(f p^2 - f r^2 + r^2).$$  \hspace{1cm} (13)

We use Eq. (13) in Eq. (12) and obtain the energy distribution

$$E(r) = \frac{f(p^2 - r^2) + r^2}{2r},$$  \hspace{1cm} (14)

where $f$ is defined in Eq. (2), and $E(r)$ is the total (matter plus gravitational field) energy within radius $r$. Similarly, momentum is the total momentum due to both matter and gravitational field.

Similarly, to obtain momentum components, we calculate $H^{01}_1$, $H^{01}_2$, $H^{01}_3$, $H^{02}_1$, $H^{02}_2$, $H^{02}_3$, $H^{03}_1$, $H^{03}_2$, and $H^{03}_3$.

$$H^{01}_1 = H^{01}_2 = H^{01}_3 = 0,$$

$$H^{02}_1 = H^{02}_2 = H^{02}_3 = 0,$$

$$H^{03}_1 = H^{03}_2 = H^{03}_3 = 0.$$  \hspace{1cm} (15)

We use Eq. (15) in Eq. (12). As expected for a static metric, we obtain momentum components

$$P_x = 0, \quad P_y = 0, \quad P_z = 0.$$  \hspace{1cm} (16)

Now, we plot graphs (Figs. 1 and 2) to analyze the nature of the energy distribution $E(r)$ (see Eq. (14)) as the radial distance and phantom constant increase while we keep the mass parameter fixed. In Fig. 1, we plot the ratio of the energy to the mass $E(r)/M$ against the ratio of the radial distance to the mass $r/M$ for four different values of the ratio of phantom constant and the mass parameter $p/M$. We maintain the total mass parameter $M$ constant. These curves have the same horizontal asymptote $E/M = 1$. This shows that as $r \to \infty$, $E(r) \to M$. If we maintain $p$ constant, the energy content $E(r)$ decreases with the decreasing the radial distance $r$. This shows that the phantom field has negative energy. We further find that, with fixed $r/M$, $E/M$ is larger for a larger value of the phantom constant. It is very exciting to note that the decreasing rate of $E(r)$ with increasing $r$ is higher for smaller values of the phantom constant. In the surface plot (Fig. 2), these results are exhibited more elegantly.

**Fig. 1.** The ratio of the energy to the mass $E/M$ versus the ratio of the radial distance to the mass $r/M$ for several values of the ratio of phantom constant to the mass parameter $p/M = 1$ (black), 2 (red), 3 (orange), and 4 (blue). As $r/M$ approaches infinity, and $E/M$ tends to 1.

**Fig. 2.** The ratio of the energy to the mass parameter $E/M$ against the ratio of the radial distance to the mass $r/M$ and the ratio of phantom constant to the mass parameter $p/M$.

There have been some different considerations concerning spherically symmetric systems in the framework of energy momentum localization. Misner et al. (see Ref. [27]) concluded that energy can be localized only for spherically symmetric systems. However, Cooperstock et al. [28] opposed this idea and stated that localizability of energy cannot depend on the geometry of spacetime. The energy-momentum complexes are non-tensorial under general coordinate transformations and are restricted to their uses in quasi-Cartesian coordinates only. Pioneered by Virbhadra, numerous scientists from all over the world have carried out a lot of work showing that energy-momentum complexes are indeed very useful tools in general relativity.

One could ask why should we study energy distri-
bution in a spacetime? The answer is that by knowing energy-momentum distributions we obtain an excellent idea of spacetime. As has already been discussed by others, this gives a good idea of the effective gravitational mass of the object causing a spacetime curvature. In addition, it gives an intuitive feeling of the gravitational lensing in that spacetime. A negative energy region is likely to serve as a convergent lens and a positive energy region is likely to serve as a divergent lens. The analysis of energy distributions in spacetime has helped Virbhadra discover excellent lensing phenomena. For illustration, our analysis in this study proves that a phantom field causes negative energy and this is why when we increase $r$, the energy content $E(r)$ decreases. Thus, our analyses indicate that a phantom field would cause repulsive effects to causal geodesics. This is really an exciting and important result.

In summary, we have calculated the energy and momentum distributions in the Einstein prescription for a phantom black hole metric in quasi-Cartesian coordinates. Further work towards the investigation of the energy-momentum for the phantom black hole spacetime is required. Other prescriptions must be used and compared. These calculations are very lengthy and time consuming, and this work is currently in progress.

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