Petite Unification of Quarks and Leptons: Twenty Two Years After

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A recent surge of interest in the novel ideas of Large Extra Dimensions and their implications, such as the early unification of quarks and leptons, has prompted us to revive a paper [1] written twenty two years ago. In that paper, we provided a general discussion of quark-lepton unification characterized by the gauge group $G_S \otimes G_W$ with two couplings $g_S$ and $g_W$ and by the unification mass scales $M = 10 \text{ TeV} - 1000 \text{ TeV}$. The constraint from $\sin^2 \theta_W$ restricts the choices for $G_W$ and our favorite model for the Petite Unification (PUT) was chosen to be $SU(4)_{PS} \otimes SU(2)^4$. In the present paper, we review the main results of [1] and propose two new models based on the groups $SU(4)_{PS} \otimes SU(2)^2$ and $SU(4)_{PS} \otimes SU(3)^2$ for which the consistency with the measured value of $\sin^2 \theta_W (M_Z^2)$ determines the unification scale to be roughly $1 \text{ TeV}$ and $3 - 10 \text{ TeV}$, respectively. The implications of this very early unification is the existence of new quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $\mathcal{O}(250 \text{ GeV})$. Interestingly, in these models the rare decay $K_L \rightarrow \mu e$ is automatically absent at tree level and the one-loop contributions are consistent with the experimental upper bound for this decay. On the other hand the original $SU(4)_{PS} \otimes SU(2)^4$ model can only be made consistent with the measured value of $\sin^2 \theta_W (M_Z^2)$ and the unification scale $M = \mathcal{O}(1 \text{ TeV})$, provided there exist at least nine ordinary quark and lepton generations, with four generations in the case of the supersymmetric version. Moreover, the solution to the $K_L \rightarrow \mu e$ problem is not as natural as in the two other scenarios. We comment on the recent papers on early unification in the context of Large Extra Dimensions.

I. INTRODUCTION

Twenty two years ago, we have proposed alternatives to popular Grand Unified models such as $SU(5)$ [2, 3] or $SO(10)$ [4, 5], based on a less ambitious program which aimed at unifying quarks and leptons at some energy scale $M$ which is not too much greater than the electroweak scale [1]. We assumed that the Standard Model (SM), $SU(3)_{c} \otimes SU(2)_L \otimes U(1)_Y$, which has three independent couplings, $g_3$, $g_2$ and $g'$, is embedded into a gauge theory $G_S \otimes G_W$, which is characterized by two independent couplings $g_S$ and $g_W$, at a “petite unification” scale $M$ which can be as small as $M \approx 10^{5 \pm 1} \text{ GeV}$, namely the TeV region. We further assumed that $G_S$ and $G_W$ are either simple or pseudosimple (a direct product of simple groups with identical couplings). Our approach was a “bottom up” one, that is to say we used the available inputs from the “low energy” to constrain the choices of $G_S$ and $G_W$. We used $\sin^2 \theta_W$ and the known fermion representations as inputs. It turned out that the choices of $G_W$ are quite restricted. Furthermore, if $G_S$ is chosen to be $SU(4)$, the Petite Unification (PUT) was discussed at length in our paper.

In the $SU(4)_{PS} \otimes [SU(2)^4]$ model the value of $\sin^2 \theta_W$ at the unification scale $M \gg M_Z$ turns out to be $\sin^2 \theta_W = 1/4$, very close to its experimental value that is now very precisely known: $\sin^2 \theta_W (M_Z^2) = 0.23113(15)$ [7]. For $M = 100 \text{ TeV}$ the inclusion of $\mathcal{O}(\alpha)$ corrections and the renormalization group evolution led in 1981 to $\sin^2 \theta_W (M_Z^2) \approx 0.22$, still consistent with the data of 1981. As we will show below with the present value of the QCD coupling constant, $\alpha_s (M_Z^2)$, the consistency with the measured very precise value of $\sin^2 \theta_W (M_Z^2)$ requires in this model the unification scale $M$ to be as low as 330 GeV. This is clearly unacceptable as the lower bound on the right-handed gauge boson mass is $M_{WH} \geq 800 \text{ GeV}$ [7]. The scale $M$ can be raised to 1 TeV by adding six additional standard fermion generations with masses $\mathcal{O}(250 \text{ GeV})$ or making the model supersymmetric, in which case two new fermion generations suffice. However in the simplest version of this model the rare decay $K_L \rightarrow \mu e$ proceeds at the tree level and its rate with $M = 1 \text{ TeV}$ exceeds the experimental upper bound by many orders of magnitude. A possible solution to these difficulties, as advocated recently in [8], is to introduce one Large Extra Dimension to obtain acceptable values for $\sin^2 \theta_W (M_Z^2)$ and $\text{Br}(K_L \rightarrow \mu e)$ with
$M = \mathcal{O}(1 - 10)$ TeV and the usual three fermion genera-
tions. We will discuss other alternatives in this paper.

In the present paper we would like to propose two possibly
more attractive PUT groups

$$\text{PUT}_1 = SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_H \otimes SU(2)_R$$

and

$$\text{PUT}_2 = SU(4)_{PS} \otimes SU(3)_L \otimes SU(3)_H$$

that were listed in our PUT classification of 1981, but
were not analyzed by us in detail. In these models $\sin^2 \theta_W$ equals 1/3 and 3/8, respectively but a very fast
renormalization group evolution allows to obtain correct
$\sin^2 \theta_W(M_Z^2)$ with $M = 1$ TeV and $M = 3.3$ TeV, re-
spectively when the spontaneous breakdown of the PUT
groups to the Standard Model group proceeds in one
step. Moreover, the fast renormalization group evolution
combined with the very precise experimental value for
$\sin^2 \theta_W(M_Z^2)$ determines these unification scales within
10 – 15%. If the breakdowns of $SU(4)_{PS}$ and of $G_W$ are
allowed to appear at two different scales $M$ and $\tilde{M} < M$,
these two scales have to be close to 1 TeV in the case
of PUT$_1$ but can differ up to an order of magnitude in the
case of PUT$_2$ with roughly $3 \leq M \leq 10$ TeV and
0.8 $\leq \tilde{M} \leq 3$ TeV.

These two scenarios for early unification of quark and
leptons have three interesting properties:

- In addition to the standard three generations of
quarks and leptons, new three generations of un-
conventional quarks and leptons with charges up
to 4/3 (for quarks) and 2 (for leptons) and masses
$\mathcal{O}(250 \text{ GeV})$ are automatically present. The hori-
 zontal groups $SU(2)_H$ and $SU(3)_H$ connect the
standard fermions with the unconventional ones.

- The placement of the ordinary quarks and leptons
in the fundamental representation of $SU(4)_{PS}$ is
such that there are no tree-level transitions be-
tween ordinary quarks and leptons mediated by the
$SU(4)_{PS}$ gauge bosons. This prevents rare decays
such as $K_L \rightarrow \mu e$ from acquiring large rates, even
when the masses of these gauge bosons are in the
few TeV’s range.

- There are new contributions to flavour changing
neutral current processes (FCNC) involving stan-
dard quarks and leptons that are mediated by the
horizontal $SU(2)_H$ and $SU(3)_H$ weak gauge bosons
and the new unconventional quarks and leptons.
However, they appear first at the one–loop level
and can be made consistent with the existing ex-
perimental bounds.

Our paper is organized as follows. In Sec. II, we
review the steps that lead to the three choices for $G_W$
mentioned above and we summarize the most important
formulas. In particular we derive the general expression
for $\sin^2 \theta_W$ and discuss its relation to $\sin^2 \theta_W(M_Z^2)$. In
section III we present in detail the fermion content of
the selected groups. The results of the renormalization
group analysis of $\sin^2 \theta_W$, in the scenarios in question,
is presented in Sec. IV and in Sect. V the rare decay
$K_L \rightarrow \mu e$ is briefly discussed. Here we emphasize that
while in the $SU(4)_{PS} \otimes SU(2)^4$ scenario, it is very dif-
ficult to satisfy the experimental bound on $K_L \rightarrow \mu e$ when
$M = \mathcal{O}(1 \text{ TeV})$, the presence of GIM–like mechanism in
the remaining two scenarios allows to satisfy this bound
without any unnatural conditions on the mass spectrum
of new quarks and leptons and related CKM-like mixing
matrix. Similar comments apply to FCNC processes.

In Sec. VI we compare our work of 1981 and the one
presented here with the recent papers on the early uni-
fication of quarks and leptons in the context of Large Ex-
tra Dimensions [8, 9]. As a matter of fact the $SU(3)_W$
model of Dimopoulos and Kaplan [9] is just one of the
cases considered by us in [1] and the analysis in [8] is
the generalization of our $SU(4)_{PS} \otimes [SU(2)]^4$ model to
extra dimensions. Finally, in Sec. VII we summarize the
main results of our paper and offer some perspectives
for the future work. Detailed analysis of $K_L \rightarrow \mu e$
and other phenomenological implications of the PUT groups
discussed here will be presented elsewhere.

II. PETITE UNIFICATION REVISITED

A. Preliminaries

The objective, then and now, is to unify quarks and
leptons at an intermediate scale in the TeV range. We
assume, then and now, that $SU(3) \otimes SU(2)_L \otimes U(1)_Y$
is embedded in $G = G_S(g_S) \otimes G_W(g_W)$, where $g_S$ and $g_W$
denote the corresponding couplings. Furthermore, $G_S$
and $G_W$ are assumed to be either simple or pseudosim-
ple, i.e., a direct product of simple groups with identical
couplings. The pattern of symmetry breaking is assumed to be

$$G \xrightarrow{M} G_1 \xrightarrow{\tilde{M}} G_2 \xrightarrow{M_Z} SU(3)_c \otimes U(1)_{EM},$$

where

$$G_1 = SU(3)_c(g_S) \otimes \bar{G}_S(g_S) \otimes G_W(g_W),$$

and

$$G_2 = SU(3)_c(g_S) \otimes SU(2)_L(g_2) \otimes U(1)_Y(g').$$

We assume $M_Z < \tilde{M} \leq M$. In principle, $G$ can be broken
down directly to $G_2$, but to be more general, the pattern
(4) was assumed in [1]. Furthermore, in accordance with
our petite-unification idea, we require

- $M$ and $\tilde{M}$ to be at most a few orders of magnitude
larger than $M_Z$. 


• the weak hypercharge $U(1)_Y$ group to merge into both $\hat{G}_S$ and $G_W$ at $\hat{M}$,

• $SU(3)$, and $\hat{G}_S$ to be unbroken subgroups of $G_S$ so that their generators are unbroken generators of $G_S$.

The second requirement allows us to put quarks and leptons into identical representations of the weak group $G_W$ and consequently make the quarks and leptons to be indistinguishable when the strong interactions are turned off. The last requirement implies

$$g_3(M^2) = \hat{g}_S(M^2) = g_S(M^2).$$

(7)

## B. $\sin^2 \theta_W$ and the choices of $G_W$

We will next summarize the salient points of our earlier paper concerning the restrictions imposed on $G_W$ from the value of $\sin^2 \theta_W$. We will focus, in particular, on the case where $G_W = [SU(N)]^k$ and use $\sin^2 \theta_W$ to constrain the pair $(N, k)$. Furthermore, we have argued in [1] that the most economical choice for $G_S$ is $SU(4)$ à la Pati-Salam although we have presented there a more general discussion. In the following we shall then deal principally with the groups

$$G = SU(4)_P S \otimes [SU(N)]^k, \quad \hat{G}_S = U(1)_S.$$  

(8)

To derive $\sin^2 \theta_W$, we write the generators $T_{3L}$ and $T_0$ of $SU(2)_L$ and $U(1)_Y$ respectively, in terms of the generators of $G_S$ and $G_W$. As usual, one has for the electric charge generator $Q$

$$Q = T_{3L} + T_0,$$

(9)

where $T_{3L}$ and $T_0$ are diagonal generators of $SU(2)_L$ and $U(1)_Y$, respectively. They can be written as

$$T_{3L} = \sum_\alpha C_{\alpha W} T_{\alpha W}^0,$$

(10)

and

$$T_0 = \sum_\alpha C_{\alpha W} T_{\alpha W}^0 + C_ST_{15},$$

(11)

where $T_{\alpha W}^0$ and $T_{15}$ are the diagonal generators of $G_W$ and $SU(4)_PS$ respectively, with $T_{\alpha W}^0$ being the generators of the $SU(2)$ disjoint subgroups of $G_W$. Also, $C_{\alpha W}$ and $C_{\alpha W}$ are orthogonal to each other.

Eqs. (10,11) form the basis for the derivation of $\sin^2 \theta_W$. In [1], we discussed two cases which were called (a) the “unlocked standard model” where the generators of $SU(2)_L$ are the unbroken generators of $G_W$, and (b) the “locked standard model” where the generators of $SU(2)_L$ are the unbroken combination of generators belonging to several disjoint $SU(2)$ subgroups of $G_W$. We showed that case (a) (the “unlocked standard model”) is the most economical one and this is one we will choose to concentrate on in the present paper. The reader is encouraged to consult [1] for a more general discussion. Therefore for case (a), one has

$$T_{3L} = T_{3W}^0,$$

(12)

where $T_{3W}^0$ is a diagonal generator of one of $SU(2)$ subgroups of $G_W$. This implies that $C_{1W} = 1$ with all other coefficients in (10) equal to zero. In consequence, in the “unlocked standard model” scenario, one is now in a position to derive $\sin^2 \theta_W$, taking into account the pattern (4). First, we present a formula for the renormalized value of $\sin^2 \theta_W$ at the one-loop level. We will comment on its generalization to two loops in Sec. IV. From

$$\frac{1}{e^2(M_Z^2)} = \frac{1}{[g_2(M_Z^2)]^2} + \frac{1}{[g(M_Z^2)]^2},$$

(13)

$$g_2(M^2) = g_W(M),$$

(14)

$$\frac{1}{[g_2(M^2)]^2} = \frac{\sum_\alpha C_{\alpha W}^2}{[g_W(M^2)]^2} + \frac{C_S^2}{[g_S(M^2)]^2},$$

(15)

and using the $\overline{MS}$ definition for $\sin^2 \theta_W$, namely

$$\sin^2 \theta_W(M_Z^2) = \frac{e^2(M_Z^2)}{g_2^2(M_Z^2)},$$

(16)

one obtains the master formula [1]

$$\sin^2 \theta_W(M_Z^2) = \sin^2 \theta_W^0 [1 - \alpha_S(M_Z^2) - 8\pi \times \omega(M_Z^2)(K \ln \frac{\hat{M}}{M_Z} + K' \ln \frac{\hat{M}}{M})],$$

(17)

where

$$\alpha(M_Z^2) \equiv \frac{e^2(M_Z^2)}{4\pi}, \quad \alpha_S(M_Z^2) \equiv \frac{g^2_S(M_Z^2)}{4\pi},$$

(18)

$$\sin^2 \theta_W^0 = \frac{1}{1 + C_{1W}^2},$$

(19)

with $C_{1W}^2 = \sum_\alpha C_{\alpha W}^2$, and

$$K = b_1 - C_{1W}^2b_2 - C_S^2b_3,$$

(20)

$$K' = C_S^2(b - \hat{b}_3).$$

(21)

Here, $b_1$, $b_2$, $b_3$, and $\hat{b}$ are the one-loop coefficients of the beta functions for $U(1)_Y$, $SU(2)_L$, $SU(3)_c$, and $U(1)_S$, respectively with $\hat{b}_3 \neq b_3$ due to possible contributions of new particles with masses larger than $\hat{M}$. Explicit expressions for these coefficients are given in section IV. We will see there that in the case of the new groups in (2) and (3), the presence of new particles with masses
$\mathcal{O}(250 \text{ GeV})$ will require the introduction of the appropriate threshold corrections in $K$.

Neglecting the contributions of new particles to $K$ and $K'$ for a moment and using the \textsc{ms} values \cite{7}

\begin{equation}
1/\alpha(M_Z^2) = 127.934(27), \quad \alpha_S(M_Z^2) = 0.1172(20) \tag{22}
\end{equation}

we find

\begin{equation}
\sin^2 \theta_W(M_Z^2) = R \sin^2 \theta_W^0, \tag{23}
\end{equation}

where

\begin{equation}
R = 1 - 0.067C_S^2 - 0.014C_S^2 \ln \frac{M}{M_Z} \nonumber \end{equation}
\begin{equation}
- (0.009 + 0.004C_W^2 + 0.009C_S^2) \ln \frac{M}{M_Z}. \tag{24}
\end{equation}

We observe that $\sin^2 \theta_W(M_Z^2)$ is a sensitive function of $C_W^2$, present in particular in $\sin^2 \theta_W^0$, and of $C_S^2$ in the term $C_S^2 \alpha(M_Z^2)/\alpha_S(M_Z^2)$ and in the renormalization group corrections. The renormalization of $\sin^2 \theta_W$ increases with increasing $C_S^2$ but of course depends also strongly on the values of $b_i$s, that in turn depend on the content of the fermion representations and their weak and strong charges.

As $\sin^2 \theta_W(M_Z^2)$ is known with a very high precision,

\begin{equation}
\sin^2 \theta_W(M_Z^2)_{\text{exp}} = 0.23113(15), \tag{25}
\end{equation}

and $C_W$ and $C_S$ in the case of $SU(4) \otimes [SU(N)]^k$ can take only special values, only certain pairs $(C_W,C_S)$ are allowed if we are interested in the unification scales $M \leq 1000 \text{TeV}$ and in particular $M \leq 10 \text{TeV}$. We will now briefly describe the steps that led us in [1] to the acceptable choices of $(C_W,C_S)$.

The crucial quantity to be considered first is $\sin^2 \theta_W^0$, which is determined at the petite unification scale $M$. For $G_W = [SU(N)]^k$, a given pair of $(N,k)$ will determine $C_W$ and hence $\sin^2 \theta_W^0$ through Eq. (19) which can also be written as

\begin{equation}
\sin^2 \theta_W^0 = \frac{1}{1 + C_W^2} = \frac{\text{Tr}T_{3k}^W}{\text{Tr}Q^2} |_{\text{adjoint}}, \tag{26}
\end{equation}

where the last term in (26) reflects the fact that the adjoint representation of $G_W$ is a singlet of $G_S$. It is then sufficient to evaluate (26) by simply examining the adjoint representation.

Since quarks and leptons are assumed to be in separate (but identical) representations of $G_W$, the gauge bosons of $G_W$ have integer charges. Assuming next a permutation symmetry among the $SU(N)$'s in $G_W$, and allowing for arbitrary integer charges for the gauge bosons one finds \cite{1}.

\begin{equation}
\sin^2 \theta_W^0 = \frac{N}{k \text{Tr}(Q_W^k)|_{\text{adj}}}, \quad \text{Tr}(Q_W^k)|_{\text{adj}} = \sum_{i=1}^{\alpha} i^2 n_i, \tag{27}
\end{equation}

where $\text{Tr}(Q_W^k)|_{\text{adj}}$ is for each $SU(N)$, $n_i$ is the number of gauge bosons with $|Q| = i$, and $\alpha$ is the maximal gauge-boson charge involved. Since the adjoint representation can be constructed from the product of the fundamental representation $N$ and its conjugate $\bar{N}$, one can compute $n_i$ by looking at the charge distribution of the fundamental representation, namely

\begin{equation}
[\tilde{Q}_W, \cdots, \tilde{Q}_W, \tilde{Q}_W - 1, \cdots, \tilde{Q}_W - \alpha, \cdots, \tilde{Q}_W - 1], \tag{28}
\end{equation}

where $\tilde{Q}_W$ is an eigenvalue of $Q_W$.

The detailed analysis in [1] has shown that

- Gauge bosons with charges $\pm 3$ or higher corresponding to $N \geq 4$ are excluded since one can derive the inequality $\sin^2 \theta_W \leq 1/(12 - (8/N)) \leq 1/10$ which rules out this case.

- For doubly charged gauge bosons, the maximal allowed number is two ($\pm 2$) leading to $\text{Tr}(Q_W^2) = 4N$ for any $SU(N)$ with $N \geq 3$. For $k = 1$ this gives $\sin^2 \theta_W^0 = 1/4$. However, as shown in [1], in this case $C_S = 8/3$, implying through (24) $\sin^2 \theta_W(M_Z^2) = 0.205$ even without including the renormalization group effects that decrease it even further. As a consequence, scenarios with $G_W = SU(3), SU(4), \ldots$, having two doubly charged gauge bosons are inconsistent with the data.

We thus obtain an important result:

- the only charges of weak gauge bosons that are consistent with the measured value of $\sin^2 \theta_W(M_Z^2)$ within the petite unification framework with the gauge group $SU(4) \otimes [SU(N)]^k$ are 0 and $\pm 1$.

Consequently the formula (27) simplifies to

\begin{equation}
\sin^2 \theta_W^0 = \frac{N}{kn_1}, \tag{29}
\end{equation}

where $n_1$ is the number of weak gauge bosons with $Q = \pm 1$ in $SU(N)$.

In order to find $n_1$ let us consider first the class (i) of fermion representations that transform under $G_W$ as

\begin{equation}
(f, 1, 1, \cdots, 1), \quad (f, 1, 1, \cdots, 1). \tag{30}
\end{equation}

Each entry in (30) corresponds to the group $\tilde{G}$ in the product $G_W = \tilde{G} \otimes \tilde{G} \cdots \otimes \tilde{G}$. That is quarks and leptons transform nontrivially under one of the groups $\tilde{G}$ and are singlets under the rest. The fundamental representation for the group $\tilde{G}$ has then a charge distribution

\begin{equation}
[\tilde{Q}_W, \cdots, \tilde{Q}_W, \tilde{Q}_W - 1, \cdots, \tilde{Q}_W - 1], \tag{31}
\end{equation}

with $r_0 + r_1 = N$. The tracelessness condition for the charge operator $Q_W$ gives the eigenvalues

\begin{equation}
\tilde{Q}_W = 1 - \frac{r_0}{N}, \quad \tilde{Q}_W - 1. \tag{32}
\end{equation}
Moreover we find
\[ n_1 = 2r_0r_1 = 2r_0(N - r_0) \quad (33) \]
and consequently a very useful formula
\[ \sin^2 \theta_W^0 = \frac{N}{2k_0(N - r_0)} = \frac{1}{1 + C_W^2}, \quad (34) \]
that can be used to calculate \( \sin^2 \theta_W^0 \) and \( C_W^2 \) for given \( N, k \) \( r_0 \). This formula is equivalent to the formulae given in [1] but is more transparent. The results for \( \sin^2 \theta_W^0 \) are given in Table I, where also the values of the charges \( Q_{iW} \) in the fundamental representation obtained by means of (32) are given. We observe a correlation between the values of \( \sin^2 \theta_W^0 \) for given \((N, k)\) and the weak charges of quarks and leptons. This correlation implies eventually the correlation between \( \sin^2 \theta_W^0 \) and electric charges of quarks and leptons that follows from
\[ Q = Q_S + Q_W = C_S T_{15} + Q_W \quad (35) \]
where \( T_{15} \) is the diagonal generator of \( SU(4)_{PS} \) that commutes with \( SU(3)_c \). We will return to this correlation below.

If fermions transform as (class(ii)) \((f, \tilde{f})\) under any pair \( \tilde{G} \otimes G \) in \( G_W \) and are singlets under the rest, that is in the symbolical notation of (30) one has
\[ (f, \tilde{f}, 1, \ldots, 1), \quad (36) \]
the charge distribution is a \( N \times N \) matrix with \( r_0 + r_1 = N \) columns and \( r_0 + r_1 = N \) rows (see Eq. (4.10) of [1]). This matrix looks like:
\[ \begin{pmatrix} Q_W & \ldots & Q_W & Q_W - 1 & \ldots & Q_W - 1 \\ \vdots & \ldots & \vdots & \ldots & \ldots & \vdots \\ Q_W & \ldots & Q_W & Q_W - 1 & \ldots & Q_W - 1 \\ Q_W + 1 & \ldots & Q_W + 1 & Q_W & \ldots & Q_W \\ \vdots & \ldots & \vdots & \vdots & \ldots & \vdots \\ Q_W + 1 & \ldots & Q_W + 1 & Q_W & \ldots & Q_W \end{pmatrix}, \quad (37) \]
where the rows refer to \( f \) and the columns to \( \tilde{f} \). The eigenvalues of \( Q_W \) are now [1]
\[ Q_{W} = \frac{r_0' - r_0}{N}, \quad Q_{W} \pm 1. \quad (38) \]

It turns out that from the point of view of \( \sin^2 \theta_W \) only the cases \( r_0' = r_0 \) and consequently \( r_1' = r_1 = N - r_0 \) are of interest to us implying \( Q_{W} = 0, \pm 1 \) as shown in Table I. Moreover the formula (34) also applies here.

Whether the groups listed in Table I give the acceptable \( \sin^2 \theta_W (M_Z^2) \) depends also on \( C_S^2 \) as discussed before. In fact it has been shown in [1] that if \( G_S \) was chosen to be the Pati-Salam \( SU(4) \) with each standard quark \( SU(3)_c \) triplet put with a lepton into the same fundamental representation of \( SU(4) \) and the electric charges of quarks and leptons are restricted to
\[ Q_q = \frac{d}{3} + n, \quad Q_l = n', \quad n, n' \text{ integer}, \quad d = 1, 2, \quad (39) \]
then many of the possibilities given in Table I can be eliminated. The choice in (39) allows to include at least quarks and leptons with ordinary charges. Indeed under the latter assumption one can show that \( \tilde{Q}_{W}^i \) should be multiples of 1/4, in fact
\[ \tilde{Q}_{W}^i = \frac{1}{4}(3Q_q^i + Q_l^i). \quad (40) \]

Consequently a number of possibilities listed in Table I can be eliminated only by this requirement. For the remaining cases that satisfy (40) we find using
\[ Q_q^i = \frac{C_S}{2\sqrt{6}} + \tilde{Q}_q^i, \quad Q_l^i = -\frac{3C_S}{2\sqrt{6}} + \tilde{Q}_l^i, \quad (41) \]
the expression for \( C_S^2 \) in terms of quark and lepton electric charges
\[ C_S^2 = \frac{1}{6}(3Q_q^i - 3Q_l^i)^2. \quad (42) \]

One word of caution is in order here. The previous statements related to (39) refer only to scenarios in which the only representations present are of a single class, i.e. (i) or (ii). In the case where both classes are needed, as will be the case of PUT1, we should broaden the restriction (39) in the following sense. First, the value of \( C_S^2 \) should be chosen judiciously depending on \( \sin^2 \theta_W \). Once it is chosen, the charges of the fermions are determined depending on their representations under \( G_W \) and are given by Eq. (35), namely \( Q = C_S T_{15} + Q_W \). As we have discussed earlier and shown in Table 1, representations \((f, \tilde{f}, 1, \ldots)\) have \( Q_W = \pm 1/2 \) and representations \((f, \tilde{f}, 1, \ldots)\) have \( Q_W = 0, \pm 1 \). Obviously, when a scenario contains both classes of representations, it will be unavoidable to have quarks and leptons with “funny” charges in addition to the familiar ones. As we will discuss below in the context of PUT1, as long as some of these “funny” fermions belong to a vector-like representation of one of the \( G_W \) gauge groups, they can be very massive, in the sense that their masses are not proportional to the SM electroweak scale. The obvious caution that one has to take is that, in a mixed case, at least one of the representations has to contain SM fermions.

With the condition on \( Q_q^i, Q_l^i \) in (39) the lowest values for \( C_S^2 \) are found to be
\[ C_S^2 = \frac{1}{6}, \quad \frac{2}{3}, \quad \frac{8}{3}. \quad (43) \]
The next value \( C_S^2 = 25/6 \) and higher values would require very small \( \sin^2 \theta_W (M_Z^2) \) and rather high quark and lepton charges. In Table II we list \( \tilde{Q}_{W}^i \) of Table I which satisfies Eq. (40) along with the corresponding quark and lepton charges.
TABLE I: The values of $\sin^2 \theta_W$ for the weak groups $G_W = SU(N)^s$ and different fermion representations.

| $G_W$  | $r_0$ | $\sin^2 \theta_W$ | $\tilde{Q}_W$ | $(f,1)+(1,f)$ | $(f,f)$ |
|--------|-------|---------------------|----------------|--------------|---------|
| $[SU(2)]^3$ | 1     | 0.333               | $\pm \frac{3}{7}$ | 0,±1        |         |
| $[SU(2)]^4$ | 1     | 0.250               | $\pm \frac{7}{2}$ | 0,±1        |         |
| $[SU(3)]^2$ | 1     | 0.375               | $\frac{2}{7}, -\frac{1}{7}$ | 0,±1 |         |
| $[SU(3)]^3$ | 1     | 0.250               | $\frac{2}{7}, -\frac{1}{7}$ | 0,±1 |         |
| $[SU(4)]^2$ | 2     | 0.250               | $\pm \frac{1}{2}$ | 0,±1        |         |
| $[SU(5)]^2$ | 1     | 0.313               | $\frac{4}{5}, -\frac{1}{5}$ | 0,±1 |         |
| $[SU(6)]^2$ | 1     | 0.300               | $\frac{5}{7}, -\frac{1}{7}$ | 0,±1 |         |
| $SU(7)$  | 3     | 0.292               | $\frac{4}{7}, -\frac{1}{7}$ | 0,±1 |         |
| $[SU(7)]^2$ | 1     | 0.292               | $\frac{5}{7}, -\frac{1}{7}$ | 0,±1 |         |
| $SU(8)$  | 3     | 0.267               | $\frac{3}{4}, -\frac{1}{4}$ | 0,±1 |         |
| $SU(8)$  | 4     | 0.250               | $\pm \frac{3}{7}$ |         |         |

lepton charges, as well as the values of $C_S^2$. Although, for completeness, we also list the case $C_S^2 = 1/3$ in table II, it has been shown in [1] that it corresponds to a weak group $SU(4)_1 \otimes SU(4)_2$ which has $\sin^2 \theta_W = 0.286$. Because of the low value of $C_S^2$, one needs $M > 10^8$ GeV in order to obtain the correct value of $\sin^2 \theta_W (M_Z^2)$ and consequently this scenario does not fit into our framework.

We now classify the $G_W$ groups listed in table I in terms of their possible agreements with $\sin^2 \theta_W (M_Z^2)$. As seen from tables I and II only the values $C_S^2 = 2/3, 8/3$ have to be considered. We can make then the following observations.

(a) Groups which can have $C_S^2 = 2/3$ are those for which $\tilde{Q}_W = \pm 1/2$ which corresponds to representations which contain only conventionally charged quarks and leptons, as can be seen from Table II. From Table I, these weak groups are $[SU(2)]^3, [SU(2)]^4, [SU(4)]^2$ and $SU(8)$, with $\sin^2 \theta_W = 0.333, 0.25, 0.25, 0.25$, respectively. For $[SU(2)]^3$, one would need a petite unification scale substantially larger than 1000 TeV because $C_S^2 = 2/3$ is too small to bring $\sin^2 \theta_W = 0.333$ down to $\sin^2 \theta_W (M_Z^2) \sim 0.23$. (We shall however come back to this group in the discussion below.) The promising groups in this class of models are, in order of complexity, $[SU(2)]^4, [SU(4)]^2$ and $SU(8)$, all of which have $\sin^2 \theta_W = 0.25$. In particular, the group $[SU(2)]^4$ was our favorite choice in [1]. The renormalization group (RG) analysis of these models will be discussed in Sec. IV.

(b) Groups that have $C_S^2 = 8/3$ are those with $\tilde{Q}_W = 0, \pm 1$ which corresponds to representations having quark charges as high as ±4/3 and lepton charges as high as ±2 in addition to the standard charges. Because of the high value for $C_S^2$, we need those groups for which $\sin^2 \theta_W > 0.3$. From Table II, one can see that only three groups satisfy this criterion: $[SU(2)]^3, [SU(3)]^2$, and $[SU(5)]^2$, with $\sin^2 \theta_W = 0.333, 0.375, 0.313$, respectively. The implications of the first two of these models through a RG analysis will be discussed in Sec. IV.

In summary, we have arrived at two classes of weak gauge groups $G_W$ which with $G_S = SU(4)_P$, might satisfy the experimental constraint on $\sin^2 \theta_W (M_Z^2)$:

- $[SU(2)]^4, [SU(4)]^2, SU(8)$, (44)

which have only conventionally charged quarks and leptons in the fundamental representations in (30), $C_S^2 = 2/3$ and $\sin^2 \theta_W = 0.25$.

- $[SU(2)]^3, [SU(3)]^2, [SU(5)]^2$, (45)

which contain extra quarks and leptons with higher charges (±4/3 and ±2) placed together with the standard quark and leptons in the representations (36). See also Table II. These groups have respectively higher initial $\sin^2 \theta_W = 0.333, 0.375, 0.313$ and $C_S^2 = 8/3$.

### III. FERMION CONTENT OF SELECTED GROUPS

#### A. Preliminaries

In this section we will present in detail the fermion content of three groups, $PUT_0$, $PUT_1$ and $PUT_2$ as defined in (1), (2) and (3), respectively. As we shall see in the next section, these three groups seem to be the best candidates for a successful Petite Unification consistent with the measured value of $\sin^2 \theta_W$. The values for $\sin^2 \theta_W$ in these three scenarios are 1/4, 1/3 and 3/8, respectively.
with the latter being very reminiscent of the quintessential $SU(5)$ value. Our analysis of the previous section implies then that the only chance to satisfy the $\sin^2 \theta_W$ constraint is to choose for these three groups $C_S^2$ equal to 2/3, 8/3 and 8/3, respectively. In other words, as one can deduce from Table 2, we should have class (i) representation i.e. $(4,2,1,1,1)$, $(4,1,2,1,1)$, $(4,1,1,1,1)$, $(4,1,1,1,2)$ for $SU(4)_S \otimes [SU(2)]^4$, and class (ii) representation i.e. $(4,3,3)$ for $SU(4)_S \otimes [SU(3)]^2$. On the other hand we will show that, for $SU(4)_S \otimes [SU(2)]^3$, both classes are involved.

While the value of $C_S^2$ is an important ingredient in the relation between $\sin^2 \theta_W$ and $\sin^2 \theta_W(M_Z^2)$, the values of the renormalization group coefficients $b_i$ that enter $K$ and $K'$ in (20) and (21) are equally important. In order to find these values in the scenarios considered, it is necessary to identify the fermion representations and the relevant charges with respect to the SM group and $U(1)_S$. This is what we intend to do next.

**B. $SU(4)_{PS} \otimes [SU(2)]^4$**

This scenario has been already worked out in detail in [1] and we will only recall the most important points. The weak group

$$[G_W]_0 = SU(2)_L \otimes SU(2)_R \otimes SU(2)_L \otimes SU(2)_R \quad (46)$$

consists of the standard weak gauge group of the Pati-Salam model and its “mirror group” $SU(2)_L \otimes SU(2)_R$ necessary to obtain the correct $\sin^2 \theta_W$. In the original Pati-Salam model [6] one has $\sin^2 \theta_W = 1/2$, that is much too high for an early unification with $C_S^2 = 2/3$. We will return to it in section IV.

Let us denote by $l_L$, the usual left-handed lepton $SU(2)_L$ doublet, and by $q_L$ the left-handed quark doublet. The $SU(2)_R$ doublets are denoted by $l_R$ and $q_R$. Similarly, the $SU(2)_L$ doublets will be denoted by $\tilde{l}_L$ and $\tilde{q}_L$. Consequently, each generation of $SU(4)_{PS} \otimes [SU(2)]^4$ can be written as

$$\Psi_L = (q_L, l_L) = (4, 2, 1, 1, 1) \ , \quad (47)$$

$$\Psi_R = (q_R, l_R) = (4, 1, 2, 1, 1) \ , \quad (48)$$

$$\tilde{\Psi}_L = (\tilde{q}_L, \tilde{l}_L) = (4, 1, 1, 2, 1) \ , \quad (49)$$

$$\tilde{\Psi}_R = (\tilde{q}_R, \tilde{l}_R) = (4, 1, 1, 1, 2) \ . \quad (50)$$

$\Psi_L$ and $\tilde{\Psi}_R$ are what we call “mirror fermions”.

Note that in this scenario the weak charges in each $SU(2)$ representation are

$$Q_W = (1/2, -1/2) \quad (51)$$

and with $C_S^2 = 2/3$,

$$Q^q_l = \frac{1}{6} + Q_W^l, \quad Q^q_l = \frac{1}{2} + Q_W^l \ . \quad (52)$$

Consequently only conventional electric charges are present and they are the same for the ordinary and mirror fermions. However, the latter are $SU(2)_L$ (as well as $SU(2)_R$) singlets.

Now, in order to have a “Petite Unification” with only two independent couplings, $g_S$ and $g_W$, the four gauge couplings of $[SU(2)]^4$ have to be equal to each other above the scale $M$. Consequently the mirror fermions have to be lighter than $M$. Below $M$, the masses of mirror fermions and possible extra generations are however unconstrained, although the detailed spectrum depends on the Higgs system used to generate the fermion masses. As discussed in [1], the appropriate Higgs scalars which could give masses to the normal and mirror fermions can transform as $(1,2,2,1,1)$ and $(1,1,1,1,2)$, respectively. We refer for details to [1], where a possible breakdown mechanism for the gauge group $SU(4)_{PS} \otimes [SU(2)]^4$ is discussed. Needless to say, it is a quite complicated task to generate fermion masses in general and we leave it for the future.

Experimentally, it is safe to assume that any long-lived new quarks, if they exist, should have a mass larger than $200 \text{ GeV}$[10, 11]. For new leptons, the experimental lower bounds are weaker ($45,90 \text{ GeV}$ for stable and unstable neutral heavy leptons, respectively and $100 \text{ GeV}$ for the charged leptons [7]).

Now, the possible extra generations of ordinary fermions couple to the SM Higgs field. This normally means that they cannot be much heavier than, say, $200 \text{ GeV}$ and the $SU(2)$ doublet partners have to be approximately degenerate in mass to be consistent with the electroweak precision studies. We will assume that they have masses $\mathcal{O}(250 \text{ GeV})$. On the other hand, as the mirror fermions and the relevant Higgs system are singlets under $SU(2)_L$, the latter restriction is absent. In fact as already found in [1], it is more favourable from the point of view of the RG analysis that the mirror fermion masses are close to $M$ so that their contributions to $K$ in (20) can be neglected.

Finally, let us recall that in this model the ordinary quark and leptons are coupled to each other by the heavy PS gauge bosons with masses $\mathcal{O}(M)$ and electric charges $\pm 2/3$. The detailed presentation of the $SU(4)_{PS}$ gauge boson sector can be found in [1], where also the implications of these quark-lepton couplings for very rare or forbidden decays have been analyzed. We will update this analysis in Sec. V.

**C. $SU(4)_{PS} \otimes [SU(2)]^3$**

From Table 1, we see that $\sin^2 \theta_W^0 = 1/3$ in this case and one should have $C_S^2 = 8/3$. What are the appropriate fermion representations? As usual, the requirements are
simply that these representations are anomaly-free under $SU(4)_S \otimes [SU(2)]^3$, and that they appear in a sufficient number so as to ensure the equality of the three “weak” couplings above $\hat{M}$. The most economical way to satisfy these requirements is to have the following fermion content for each generation which also gives a rather interesting physical interpretation of $[SU(2)]^3$:

(a) $(4,2,2,1)_L$,  
(b) $(4,1,2,2)_R$,  
(c) $(4,2,1,1)_L$, $(4,2,1,1)_R$,  
(d) $(4,1,1,2)_L$, $(4,1,1,2)_R$.

This is clearly a situation in which one has mixed representations of classes (i) and (ii). Before addressing the issues of charges, let us first verify whether (a)-(d) are anomaly-free. If (a) and (b) represent the same particles but with opposite chiralities, then they are anomaly-free when combined. Also, (c) and (d) are separately anomaly-free. In addition, the number of degrees of freedom for (a)-(d) combined is exactly what one needs to guarantee the equality of the $G_W$ couplings above $\hat{M}$.

The physical interpretation of $[SU(2)]^3$ is now clear, namely

$$[G_W]_1 = SU(2)_L \otimes SU(2)_H \otimes SU(2)_R.$$  

As we will show below $SU(2)_H$ is the “horizontal” gauge group which links conventionally charged SM fermions to the unconventionally charged ones. To clearly see these features, let us write down explicitly the charge structure of the fermions in (a)-(d). First we look at (a) and (b).

In accordance with (37), $Q_W$ for (a) and (b) is simply given by

$$Q_W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with the columns and the rows representing $SU(2)_{L,R}$ and $SU(2)_H$ doublets, respectively.

With $C^3_S = 8/3$, the electric charges of the quarks and leptons are then given by

$$Q^i_q = 1/3 + \tilde{Q}^i_w, \quad Q^i_l = -1 + \tilde{Q}^i_w$$

and consequently with (54), these charges are

$$Q_q = \begin{pmatrix} 1/3 & 4/3 \\ -2/3 & 1/3 \end{pmatrix},$$

for the quarks and

$$Q_l = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix},$$

for the leptons. Notice that one now has quarks and leptons with unconventional charges, 4/3 and 2.

For (c) and (d), one has $Q_W = \pm 1/2$ as in (51). But since the charges of fermions are still given by (55), one now has the following charge assignments for the vector-like quarks and leptons: 5/6, −1/6 for the quarks, and −1/2, −3/2 for the leptons. These are the “funny” charges mentioned in the previous section. Let us remember that these are vector-like fermions and, therefore, can possess large masses which are not connected to the electroweak scale, nor to the scale of $SU(2)_R$ breaking. We shall come back to this point in the RG analysis.

To facilitate the discussion, we now present the following notations for the above quarks and leptons, for each generation. We have (with the electric charges shown in parentheses):

$$\psi^q_{L,R} = \begin{pmatrix} u(2/3) \\ d(-1/3) \end{pmatrix}_{L,R};$$  

$$\tilde{Q}_{L,R} = \begin{pmatrix} \tilde{U}(4/3) \\ \tilde{D}(1/3) \end{pmatrix}_{L,R};$$  

$$\psi^l_{L,R} = \begin{pmatrix} \nu(0) \\ l(-1) \end{pmatrix}_{L,R};$$  

$$\tilde{L}_{L,R} = \begin{pmatrix} \tilde{l}_u(-1) \\ \tilde{l}_d(-2) \end{pmatrix}_{L,R};$$  

$$\tilde{Q}'_{L,R} = \begin{pmatrix} \tilde{U}'(5/6) \\ \tilde{D}'(1/6) \end{pmatrix}_{L,R};$$  

$$\tilde{L}'_{L,R} = \begin{pmatrix} \tilde{l}'_u(-1/2) \\ \tilde{l}'_d(-3/2) \end{pmatrix}_{L,R}. $$

In order to put these $SU(2)$ doublets into representations (a)–(d), we note that the following field transforms like a 2 which is equivalent to a 2 of $SU(2)_L$:

$$i\tau_2 \psi_{L,R}^{q,a} = \begin{pmatrix} d^*(1/3) \\ -a^*(-2/3) \end{pmatrix}_{L,R},$$

with $\tau_2$ being an $SU(2)_{L,R}$ generator.

Using the above definitions, one can write

$$(4,2,2,1)_L = [(i\tau_2 \psi_{L}^{q,a}, \tilde{Q}_L), (\tilde{L}_L, \psi^l_L)],$$  

$$(4,1,2,2)_R = [(i\tau_2 \psi_{R}^{q,a}, \tilde{Q}_R), (\tilde{L}_R, \psi^l_R)],$$

and

$$(4,2,1,1)_L,R = [\tilde{Q}_{L,R}, \tilde{L}'_{L,R}].$$

Three remarks are in order here.
• First, the fermions in (62, 63) are vector-like and, in consequence, can have gauge-invariant bare masses which can be much larger than the electroweak scale.

• Second, the placement of the quarks and leptons in (60, 61) is such that there are no tree-level transitions between ordinary quarks and leptons mediated by the SU(4)_{PS} gauge bosons. Indeed, in contrast to the previous scenario the electric charges of the PS gauge bosons are now ±4/3 and as seen for instance in (60), (56) and (57) these gauge bosons couple a left-handed ordinary anti-down-quark with charge 1/3 to a new heavy −1 charge lepton and a left-handed ordinary charged lepton with charge −1 to a new heavy 1/3 charge quark. Analogous comments apply to anti-up-quarks and neutrinos.

• Third, as seen explicitly in (56) and (57), the horizontal SU(2)_{H} weak gauge bosons couple the ordinary quarks and leptons to new heavy quarks and leptons, respectively and consequently there are no dangerous tree level flavour changing neutral current (FCNC) transitions between the ordinary quarks and between the ordinary leptons mediated by the SU(2)_{H} bosons.

As we shall see, the second property will prevent rare decays such as \( K_L \to \mu e \) from acquiring large rates, even for the masses of the PS gauge bosons as low as 1 TeV. Similar comments apply to horizontal SU(2)_{H} gauge bosons with respect to FCNC transitions.

D. \( SU(4)_{ps} \otimes [SU(3)]^2 \)

In this scenario the weak gauge group is

\[ [G_W]^2 = SU(3)_L \otimes SU(3)_H \]  

(64)

with the SM SU(2)_{L} group being the subgroup of SU(3)_{L}. As we will show below the "horizontal" gauge group SU(3)_{H} similarly to SU(2)_{H} in the previous scenario links conventionally charged SM fermions to the unconventionally charged ones.

As we have discussed above, \( \sin^2 \theta_W = 3/8 \) in this model and \( C_3^S = 8/3 \) is required. The appropriate fermion representations that are together anomaly free, are then (4, 3, 3) and (4, 3, 3). The “weak charge” matrices are now written as

\[ Q_W = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \]  

(65)

for (4, 3, 3), and

\[ Q_W = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \]  

(66)

for (4, 3, 3), both with eigenvalues 0, ±1. The charges for the fermions are given by (55) as in the previous scenario, but as only representation of class ii) are present the fermions with “funny” charges are absent. We will soon see that the rows in (65) and (66) correspond to SU(3)_{L} triplets with the SU(2)_{L} doublets occupying the first two entries in these triplets. The columns in (65) and (66) correspond to SU(3)_{H} triplets.

From (31, 65), the three fundamental representations of SU(3)_{L} have the weak charge distributions: (0, 1, 1), (−1, 0, 0), and (−1, 0, 0). This corresponds to the electric charge distributions: (1/3, 4/3, 4/3), (−2/3, 1/3, 1/3), and (−2/3, 1/3, 1/3) for the quarks and (−1, 0, 0), (−2, −1, −1), and (−2, −1, −1) for the leptons.

In short, each (4, 3, 3) representation will have the following fermion content:

\[ \Psi_1 = \begin{pmatrix} [(1/3, 4/3, 4/3), (−1, 0, 0)], \\ [(−2/3, 1/3, 1/3), (−2, −1, −1)], \\ [(−2/3, 1/3, 1/3), (−2, −1, −1)] \end{pmatrix}. \]  

(67)

Similarly, each (4, 3, 3) representation has the following fermion content:

\[ \Psi_2 = \begin{pmatrix} [(1/3, −2/3, −2/3), (−1, −2, −2)], \\ [(4/3, 1/3, 1/3), (0, −1, −1)], \\ [(4/3, 1/3, 1/3), (0, −1, −1)] \end{pmatrix}. \]  

(68)

To appreciate the physical meaning of \( \Psi_1 \) and \( \Psi_2 \), it is best to express them explicitly in terms of various particles. In particular, we would like to clearly distinguish these particles from their SM counterparts, which possess usual electric charges. The notation used below should not be confused with the ones used in Section III. One has

\[ \psi^C_{L,R} \equiv C\psi^R_{L,R}C^{-1} = C\overline{\psi}^T_{R,L}, \]  

(69)

where \( C = i\gamma^2\gamma^0 \).

First, we start with the (4, 3, 3) representation. We shall first list normal quarks and leptons, followed by those which possess unusual electric charges. We shall first list normal quarks and leptons, followed by those which possess unusual electric charges. The notations used below should not be confused with the ones used in Section III. One has

\[ \psi^C_{L} = \begin{pmatrix} \frac{u(2/3)}{d(-1/3)} \end{pmatrix} \]  

(70a)

and

\[ \nu_{L} = \begin{pmatrix} \nu(0) \end{pmatrix} \]  

(70b)
\[
Q_L = \left( \frac{U(-1/3)}{D(-4/3)} \right)_L; \quad D^*_L(4/3) = C \bar{D}^T_R, \tag{70c}
\]

\[
L_{1L} = \left( \frac{l_{u1}(2)}{l_{d1}(1)} \right)_L; \quad l_{u1,L}^c(-1) = C \bar{I}^T_{d1,R}, \tag{70d}
\]

\[
L_{2L} = \left( \frac{l_{u2}(2)}{l_{d2}(1)} \right)_L, \tag{70e}
\]

\[
\tilde{\psi}_L^q = \left( \frac{\tilde{u}(2/3)}{\tilde{d}(-1/3)} \right)_L; \quad d'_{L}(-1/3), \tag{70f}
\]

\[
l'_L(+1). \tag{70g}
\]

In the above, we have put particles in SU(2)_L doublets and singlets. To put these fields into the representation (4, 3, 3), we shall need the following SU(2)_L doublets obtained from above:

\[
i\tau_2 L^*_{1L} = \left( \frac{l_{d1}(-1)}{-l_{u1}(-2)} \right)_L, \quad i\tau_2 Q^*_L = \left( \begin{array}{cc}
D^*(4/3) \\
-U^*(1/3)
\end{array} \right)_L, \tag{71}
\]

\[
i\tau_2 \tilde{\psi}^q_L = \left( \begin{array}{c}
d^*(1/3) \\
-u^*(-2/3)
\end{array} \right)_L, \quad i\tau_2 \tilde{\psi}^q_L = \left( \begin{array}{c}
\tilde{d}^*(1/3) \\
-\tilde{u}^*(-2/3)
\end{array} \right)_L, \tag{71}
\]

where \(\tau_2\) is a generator of SU(2)_L. One can now write (4, 3, 3) in terms of specific fields, namely

\[
\Psi_1 = \{(i\tau_2 Q^*_L, D^*_L), (\psi^q_L, \nu^q_L), [i\tau_2 \psi^q_L, d^*_L], [i\tau_2 \tilde{\psi}^q_L, d^*_L] \} \tag{72}
\]

From (72), one can identify the SM fields, namely \(\psi^q_L, i\tau_2 \psi^q_L, \nu^q_L, d^*_L\). However, this representation is incomplete in that the right-handed charged lepton and up-quark fields are missing. This is where the (4, 3, 3) representation comes in. The meaning of the non-SM fields appearing in (72) will be elucidated below.

For the (4, 3, 3) representation, one can look at (68) to find the appropriate fields. To this end, let us introduce

\[
Q'_{L,R} = \left( \begin{array}{c}
U(-1/3) \\
D(-4/3)
\end{array} \right)_{L,R}; \quad U^c_L(1/3) = C U^T_R; \tag{73e}
\]

\[
d'_{R}(-1/3). \tag{73f}
\]

From the above equations, one can immediately identify the following vector-like fields: \(L_{2L,R}, Q'_{L,R}, \tilde{\psi}'_{L,R}, l'_{u1,L}, l'_{d1,L}\) and \(d'_{L,R}\).

Next, in order to match the charge assignments of (68), we define the following SU(2)_L doublets, using the ones defined in (71):

\[
\tilde{\psi'}^c_L = C \tilde{\psi}^L_T = \left( \begin{array}{c}
\tilde{\nu}_L^c(0) \\
\tilde{l}_L^c(+1)
\end{array} \right), \tag{74a}
\]

\[
i\tau_2 L^*_{2L} = i\tau_2 C L^T_{2R} = \left( \begin{array}{cc}
l_{d2,L}^c(-2) \\
-l_{u2,L}^c(-1)
\end{array} \right), \tag{74b}
\]

\[
i\tau_2 Q^*_{L} = i\tau_2 C Q^T_R = \left( \begin{array}{c}
D^*_L(4/3) \\
-U^*_L(1/3)
\end{array} \right), \tag{74c}
\]

\[
i\tau_2 \tilde{\psi}^q_L = \tilde{\psi}^L_T = \left( \begin{array}{c}
\tilde{d}_L^c(1/3) \\
-\tilde{u}_L^c(-2/3)
\end{array} \right). \tag{74d}
\]

\[
i\tau_2 \tilde{\psi'}^c_L = i\tau_2 C \tilde{\psi}^T_R = \left( \begin{array}{c}
\tilde{d}^c_L(1/3) \\
-\tilde{u}'_L^c(-2/3)
\end{array} \right). \tag{74e}
\]

The representation (4, 3, 3) can now be written explicitly as

\[
\Psi_2 = \{(i\tau_2 \tilde{\psi'}^c_L, u^c_L), (i\tau_2 L^*_{2L}, l^c_{u1,L}), [i\tau_2 Q^*_{L}, U^c_L), (\tilde{\psi}^c_L, \tilde{l}^c_L)], [i\tau_2 \tilde{\psi}^q_L, d^*_L](\tilde{\psi}^c_L, \tilde{l}^c_L)) \}. \tag{75}
\]

Several remarks are in order here. First, the (4, 3, 3) and (4, 3, 3) representations, as described by \(\Psi_1\) and \(\Psi_2\), together form an anomaly-free representation of the group SU(4)_S \(\otimes |SU(3)|^2\). Second, the particle content described in (70) and (73) has the following features:

- There are two types of families with SM transformations under SU(2)_L, i.e. left-handed doublets and right-handed singlets: one contains the SM quarks and leptons and the other one contains unconventional quarks and leptons with charges up to 4/3 (for the quarks) and 2 (for the leptons). The unconventional fields are \(Q_L, D^*_L, U^c_L, L_{1L}, l'_{u1,L}, l'_{d1,L}\) and \(d'_{L,R}\). The (normal and unconventional) quarks and leptons couple to the SM Higgs field. This normally means that their masses cannot be much heavier than, say, 200 GeV.
• There are, in addition, two families of quarks and leptons, \((\psi^q, \bar{\psi}^q)_{L,R}\) and \((Q', L_2)_{L,R}\), with normal and unconventional charges which are vector-like under \(SU(2)_L\). This means that their masses come from sources other than the SM Higgs field and they can be much heavier than the first two types of families mentioned above.

• Next, there are two vector-like \(SU(2)_L\)-singlets with charge +1 for the lepton-color-singlet \((l_{L,R})\) and charge \(-1/3\) for the quark-like color-triplet \((d_{L,R})\). They also can acquire large masses.

Finally as in the previous scenario we have two phenomenologically very relevant properties that can be clearly seen in (72, 75):

• The placement of the quarks and leptons in (72, 75) is such that there are no tree-level transitions between ordinary quarks and leptons mediated by the \(SU(4)_R\) gauge bosons. Also here the electric charges of the PS gauge bosons are \(\pm 4/3\).

• The horizontal \(SU(3)_H\) weak gauge bosons couple the ordinary quarks and leptons to new heavy quarks and leptons, respectively and consequently there are no dangerous tree level flavour changing neutral current (FCNC) transitions between the ordinary quarks and between the ordinary leptons mediated by the \(SU(3)_H\) bosons.

IV. RG ANALYSIS OF SIN^2\theta_W

A. Preliminaries

In 1981 the values of \(\sin^2\theta_W(M_Z^2)\) and \(\alpha_s(M_Z^2)\) were rather poorly known. As of 2003 we know them with a very high precision as given in (22) and (25) with \(\alpha_s(M_Z^2)\) substantially smaller than in 1981 so that the \(O(\alpha/\alpha_s)\) correction in (17) plays now a bigger role. In this section we will update our 1981 renormalization group analysis of PUT_0 and generalize it to the additional scenarios considered in the previous section.

The master formula for \(\sin^2\theta_W(M_Z^2)\) in (17) has been obtained in the one-loop approximation, whereas the values of \(\sin^2\theta_W(M_Z^2)_{\text{exp}}, \alpha_s(M_Z^2)\) and \(\alpha(M_Z^2)\) have been extracted from various data including higher order QCD and electroweak corrections. Strictly speaking we should then generalize (17) to include two-loop contributions. This would be indispensable in the case of GUTS where \(\mu\) varies from \(M_Z\) to \(10^{16}\) GeV and the change of the gauge couplings in this range is substantial. On the other hand in the case of early unification, the changes of the couplings between \(M_Z\) and \(\tilde{M}, M\) that are in the TeV's range are rather small and the two-loop contributions to (17) are insignificant. In what follows we will therefore use the one-loop formula (17), relegating the RG analysis at two-loop level to a future paper.

While \(M\) and \(\tilde{M}\) differ in principle from each other, with \(M \geq \tilde{M}\), we will first set \(\tilde{M} = M\). Consequently the last term in (17) is absent and only the coefficient \(K\) has to be calculated. On the other hand in the scenarios considered, there are new particles with masses below \(M\) and their contributions to (17) have to be taken into account. Now, as discussed in the previous section, all new particles with non-trivial properties under \(SU(2)_L\) which are not vector-like cannot have masses much larger than 200 GeV. In the RG analysis we will set all these masses to be equal to a single scale \(M_F\) with

\[ M_F = (250 \pm 50) \text{ GeV} \] (76)

and we will assume that all the remaining new particles have masses very close to \(M\) so that their contributions to (17) can be neglected.

Under these assumptions, the following replacement should be made in (17): 

\[ K \ln \frac{M}{M_Z} \rightarrow K_{n_G=3} \ln \frac{M_F}{M_Z} + K_{\text{total}} \ln \frac{M}{M_F} \] (77)

where

\[ K_{n_G=3} = [b_1 - C^2_W b_2 - C^2_S b_3]_{n_G=3} , \] (78)

with \(b_1\)'s receiving only contributions from the ordinary three generations \((n_G)\) of quarks and leptons and the SM Higgs doublet. On the other hand

\[ K_{\text{total}} = [b_1 - C^2_W b_2 - C^2_S b_3]_{\text{total}} , \] (79)

includes all particles with masses below \(M\).

With \(\tilde{M} = M, M_F\) given in (76), \(\alpha_s(M_Z^2)\) and \(\alpha(M_Z^2)\) known experimentally and \(C^2_S, C^2_W\) and \(b_1\) fixed (see below) in each scenario we can determine the value of \(M\) that is consistent with the experimental value \(\sin^2\theta_W(M_Z^2)_{\text{exp}}\) in (25). This is what we will do first. Subsequently we will analyze the general case with \(M \leq \tilde{M}\). In the next section we will investigate whether the values of \(M\) determined here are consistent with bounds on rare decays.

B. \(SU(4)_R \otimes [SU(2)]^4\)

In this scenario

\[ \sin^2\theta_W^0 = \frac{1}{4}, \quad C^2_W = 3, \quad C^2_S = \frac{2}{3} \] (80)

and

\[ b_1 = \frac{1}{48\pi^2} \left[ \frac{20}{3} n_G + \frac{1}{2} \right] , \] (81)

\[ b_2 = \frac{1}{48\pi^2} \left[ 4n_G + \frac{1}{2} - 22 \right] , \] (82)
where $\sigma = \frac{1}{8\pi^2} [4n_G - 33]$ (83)

with $n_G = 3$ in $K_{n_G = 3}$ and $n_G \geq 3$ in $K_{\text{total}}$. The “1/2” is the contribution of the Higgs doublet.

We find then

$$M \lesssim 330 \text{ GeV}, \quad n_G = 3$$

(84)

that is clearly excluded. Including new generations of ordinary fermions with masses $O(M_F)$ allows to increase $M$ as seen in the following formula

$$\sin^2 \theta_W(M_Z^2) = 0.2389 - 0.0065 \ln \frac{M_F}{M_Z} - 0.0001 P \ln \frac{M}{M_F}$$

where

$$P = 87 - 8n_G$$

(86)

As the coefficient in front of the last logarithm in (85) must be very small in order to obtain the correct $\sin^2 \theta_W(M_Z^2)$, the result for $M$ in this scenario is rather sensitive to the input parameters, in particular $n_G$ and $M_F$. However, requiring $M_F \geq 200 \text{ GeV}$ and $M \geq 800 \text{ GeV}$ we find the lowest acceptable value for $n_G$ to be $n_G = 9$.

On the other hand making the model supersymmetric and setting as an example the masses of all SUSY particles equal to $M_F$, one finds

$$[b_1]_{\text{total}} = \frac{1}{48\pi^2} [10n_G + 3]$$

(87)

$$[b_2]_{\text{total}} = \frac{1}{48\pi^2} [6n_G + 3 - 18]$$

(88)

$$[b_3]_{\text{total}} = \frac{1}{48\pi^2} [6n_G - 27]$$

(89)

This gives the formula (85) with

$$P = 66 - 12n_G$$

(90)

and the lowest acceptable value for $n_G$ to be $n_G = 4$. For $n_G = 3$ we find $M \lesssim 550 \text{ GeV}$ that is excluded.

Whether this model is supersymmetric or not, the compatibility of this scenario with the experimental value of $\sin^2 \theta_W(M_Z^2)_{\text{exp}}$ requires, for $M \geq 800 \text{ GeV}$, many new particles around the $M_F$ scale.

The RG analysis of $SU(4)^2$ and $SU(8)$ proceeds in a similar manner but as these groups are very large we will not consider them further.

C. $SU(4)_R \otimes [SU(2)]^3$

In this scenario

$$\sin^2 \theta_W^0 = \frac{1}{3}, \quad C_W^2 = 2, \quad C_S^2 = \frac{8}{3}$$

(91)

and $[b_1]_{n_G = 3}$ are simply given by (81)–(83). Above $M_F$ new generations of quarks and leptons with unconventional electric charges contribute and we find

$$[b_1]_{\text{total}} = \frac{1}{48\pi^2} \left[ \frac{20}{3} n_G + \frac{1}{2} + \frac{116}{3} n_G^{\text{new}} \right]$$

(92)

$$[b_2]_{\text{total}} = \frac{1}{48\pi^2} \left[ 4(n_G + n_G^{\text{new}}) + \frac{1}{2} - 22 \right]$$

(93)

$$[b_3]_{\text{total}} = \frac{1}{48\pi^2} [4(n_G + n_G^{\text{new}}) - 33]$$

(94)

with

$$n_G^{\text{new}} = n_G$$

(95)

We note in particular the large contribution of the new fermions to $b_1$ that is related to high charges of these fermions. This gives for $n_G = 3$

$$\sin^2 \theta_W(M_Z^2) = 0.2740 - 0.0132 \ln \frac{M_F}{M_Z} - 0.0215 \ln \frac{M}{M_F}$$

(96)

We observe that the coefficients of the logarithms are much larger than in the previous scenario and the correct value of $\sin^2 \theta_W(M_Z^2)$ can be found with low unification scale and $n_G = 3$ in spite of the much higher value of $\sin^2 \theta_W^0$. Scanning $\alpha_s(M_Z^2)$ and $M_F$ in the ranges (22) and (76), respectively, and requiring (at the two $\sigma$ level)

$$0.23083 \leq \sin^2 \theta_W(M_Z^2) \leq 0.23143$$

(97)

we find

$$M = (1.00 \pm 0.14) \text{ TeV}, \quad n_G = 3$$

(98)

with lower values for $n_G > 3$. Thus in this scenario additional generations of ordinary quarks and leptons are disfavoured although $n_G = 5$ would still give $M \gtrsim 800 \text{ GeV}$.

D. $SU(4)_R \otimes [SU(3)]^2$

In this scenario

$$\sin^2 \theta_W^0 = \frac{3}{8}, \quad C_W^2 = \frac{5}{3}, \quad C_S^2 = \frac{8}{3}$$

(99)

and $b_i$ coefficients are the same as in the last scenario. In this case (96) is replaced by

$$\sin^2 \theta_W(M_Z^2) = 0.3083 - 0.0144 \ln \frac{M_F}{M_Z} - 0.0243 \ln \frac{M}{M_F}$$

(100)

and we find

$$M = (3.30 \pm 0.47) \text{ TeV}, \quad n_G = 3$$

(101)

with lower values for $n_G > 3$. For instance for $n_G = 4$ and $n_G = 5$, $M$ is found for the central values of input parameters in the ballpark of 3.0 TeV and 2.6 TeV, respectively.
E. $SU(4)_{PS} \otimes [SU(2)]^2$

Finally, let us consider the original Pati-Salam model [6]. Here

$$\sin^2 \theta_W = \frac{1}{2}, \quad C_W^2 = 1, \quad C_S^2 = \frac{2}{3}$$  (102)

and $b_i$ coefficients are the same as in the $SU(4)_{PS} \otimes [SU(2)]^4$ scenario. This gives

$$M \approx (5 \cdot 10^{10}) \text{ TeV}, \quad n_G = 3$$  (103)

with higher values for $n_G > 3$. Clearly this model is not an early unification model.

F. The case of $\tilde{M} \neq M$

Let us finally consider the general case $\tilde{M} \leq M$ with $\tilde{M} \geq 800$ GeV as required by the lower limit of right-handed gauge boson masses in the case of $[SU(2)]^4$ and $[SU(2)]^3$ scenarios. The latter restriction is absent in the case of $SU(3)^2$ but as we will see below in this case $\tilde{M}$ has to be above 1 TeV if we want $M \leq 10$ TeV.

For $\tilde{M} \leq M$ the last logarithm in (77) is replaced as follows

$$K_{total} \ln \frac{M}{M_F} \rightarrow K_{total} \ln \frac{\tilde{M}}{M_F} + K' \ln \frac{M}{M}$$  (104)

with $K'$ defined in (21).

Now, the values of $\tilde{b}$ and of $b_3$ relevant for the evolution of the couplings $\tilde{g}_S$ and $g_3$ for scales above $\tilde{M}$ include contributions from all fermions present in the model, that is also the vector-like ones. However, as $SU(3)_c$ and $U(1)_S$ are subgroups of $SU(4)_{PS}$, the contributions of all fermions to $\tilde{b}$ and of $b_3$ are equal to each other at the one-loop level and consequently we find

$$K' = C_S^2 \frac{33}{48\pi^2}$$  (105)

for all non-supersymmetric scenarios considered here with 33 replaced by 27 in the case of Supersymmetry.

In the case of $PUT_0$ the factor $C_S^2 33 = 22$ in $K'$ should be compared with 15 present in $K_{total}$ for $n_G = 9$. Consequently the evolution between $\tilde{M}$ and $M$ is essentially the same as between $M_F$ and $M$ and making $\tilde{M} \neq M$ will not help to increase the value of $M$. It will even lower it.

In the case of $PUT_1$ the factor $C_S^2 33 = 88$ in $K'$ should be compared with $311/2$ present in $K_{total}$. Therefore lowering $\tilde{M}$ to 800 GeV allows for central values of all parameters to increase $M$ from 3.3 TeV in (101) to as high as 9.9 TeV.

In fig. 1 we show the allowed regions in the space $(M, \tilde{M})$ that have been obtained by varying $\alpha_S(\tilde{M}_Z^2)$, $M_F$ and $\sin^2 \theta_W(\tilde{M}_Z^2)$ in the ranges (22), (76) and (97), respectively. For a given $\tilde{M}$, the maximal value of $M$ is found for the minimal $\sin^2 \theta_W(\tilde{M}_Z^2)$ and maximal values of $M_F$ and $\alpha_S(\tilde{M}_Z^2)$. The minimal value of $M$ is found for the maximal $\sin^2 \theta_W(\tilde{M}_Z^2)$ and minimal values of $M_F$ and $\alpha_S(\tilde{M}_Z^2)$. The vertical boundary lines at $M = 800$ GeV have been set as discussed above and the boundary lines on the right represent the case $\tilde{M} = M$ considered previously. See the ranges in (98) and (101).

We observe that even when $\tilde{M} \neq M$, the two scales have to be rather close to 1 TeV in the $SU(2)^3$ scenario. On the other hand a much larger allowed region is obtained in the case of the $SU(3)^2$ scenario where $M$ and $\tilde{M}$ can differ even by an order of magnitude. However, we find that if $M$ is required to be less than 10 TeV, the scale $\tilde{M}$ has to be larger than $\sim 1.1$ TeV.

![FIG. 1: The allowed ranges for the $SU(2)^3$ and $SU(3)^2$ scenarios as discussed in the text.](image)

G. Summary

We observe that whereas the $SU(2)^4$ scenario requires new generations of ordinary quarks and leptons in order to be consistent with the experimental value of $\sin^2 \theta_W(M_Z^2)$ and $M > 800$ GeV, in the case of the scenarios $SU(2)^3$ and $SU(3)^2$, the correct value of $\sin^2 \theta_W(M_Z^2)$ in the case of $\tilde{M} = M$ can be obtained with $n_G = 3$ for $M \approx 1$ TeV and $M \approx 3.3$ TeV, respectively. In fig. 2 we show $\sin^2 \theta_W(M_Z^2)$ as a function of $M$ for
the $SU(2)^4$ scenario with $n_G = 9$ and for the scenarios $SU(2)^3$ and $SU(3)^2$ with $n_G = 3$. To this end we have set $\alpha_s(M_Z^2)$ and $M_F$ to their central values. The curve for the supersymmetric scenario $SU(2)^4$ with $n_G = 4$ is rather similar to the non-supersymmetric case with $n_G = 9$ shown in the figure. The large sensitivity to $M_F$ in the case of the $SU(2)^4$ scenario is shown by the curve with $M_F = 200$ GeV.

Removing the equality $\bar{M} = M$ and lowering $\bar{M}$ to 800 GeV, has essentially no impact on the value of $M$ in the case of the $SU(2)^4$ scenario. An increase of $M$ by at most 300 GeV is found in the case of the $SU(2)^3$ scenario, implying that in this model $M$ and $\bar{M}$ are forced to be of the same order of magnitude and in the ballpark of 1 TeV. On the other hand in the $SU(3)^2$ scenario $M$ can be by an order of magnitude larger than $\bar{M}$ and be as high as 12 TeV. The allowed regions are shown in fig. 1.

\[ \sin^2 \theta_W(M_Z^2) \text{ as a function of } M \text{ in various scenarios. The horizontal band represents the experimental value. The dashed curve (} n_G = 9) \text{ is obtained by using } M_F = 200 \text{ GeV, while the other three curves are obtained by using } M_F = 250 \text{ GeV.} \]

\[ \text{FIG. 2: } \sin^2 \theta_W(M_Z^2) \text{ as a function of } M \text{ in various scenarios. The horizontal band represents the experimental value. The dashed curve (} n_G = 9) \text{ is obtained by using } M_F = 200 \text{ GeV, while the other three curves are obtained by using } M_F = 250 \text{ GeV.} \]

\[ d \leftrightarrow e \text{ and } s \leftrightarrow \mu. \text{ That is no generation mixing. With this hypothesis, we obtained an effective Lagrangian for the subprocess } d + \mu \rightarrow e + s \text{ of the form} \]

\[ \mathcal{L}_{eff}^{d \mu \rightarrow e s} = \sqrt{2} G_S \sum_i (d_i \gamma_\mu \bar{\mu} \gamma^\mu s_i + \text{h.c.)}, \]

where the sum is over color and where

\[ \frac{g_5^2}{2m_G^2} = \sqrt{2} G_S. \]

In (107), the quantity $m_G$ represents a typical mass of the PS gauge bosons and is comparable to the scale $M$.

In [1] we have made the estimate of the branching ratio for $K_L \rightarrow \mu^+ e^-$ by comparing this decay with $K_L \rightarrow \mu \bar{\mu}$. However, it will be more convenient to calculate $Br(K_L \rightarrow \mu^+ e^-)$ directly. Making the Fierz transformation in (106) and neglecting the axial-vector-current contribution as in [1], we find the amplitude

\[ A(K_L \rightarrow \mu^+ e^-) = i F_K G_S \frac{m_K^2}{m_s + m_d} \left[ \frac{m_{\bar{e}} e^{-7}}{m_{\mu} e^{12}} \right] \]

\[ \text{where } m_K \text{ is the kaon mass, } F_K \text{ the kaon decay constant and } m_{s,d} \text{ are the current quark masses. Neglecting the electron mass we find} \]

\[ Br(K_L \rightarrow \mu^+ e^-) = \frac{\pi}{2} \frac{\alpha^2}{m_G^4} m_K F_K^2 \tau(K_L) \sqrt{1 - \frac{m_K^2}{m_K^2}} \]

\[ \times \left[ \frac{m_K^2}{m_s + m_d} \right]^2 \]

\[ \text{Using } F_K = 160 \text{ MeV, } m_s + m_d = 140 \text{ MeV and the values for } m_K, \tau(K_L) \text{ and } m_{\mu} \text{ from [7] we find} \]

\[ Br(K_L \rightarrow \mu^+ e^-) = 4.7 \cdot 10^{-12} \left( \frac{\alpha_S(m_G)}{0.1} \right)^2 \]

\[ \times \left[ \frac{1.8 \cdot 10^3 \text{ TeV}}{m_G} \right]^4 \]

\[ \text{to be compared with the experimental bound [7]}\]

\[ Br(K_L \rightarrow \mu e) < 4.7 \times 10^{-12}. \]

Now, $\alpha_S(m_G) = \alpha_3(m_G)$ and as the presence of new particles at scales lower than $m_G$ slows down the running of the QCD coupling constant, $\alpha_3(m_G)$ with $m_G = O(1 \text{ TeV})$ is not significantly different from 0.1. We conclude then that in a scenario with no generation mixing and tree level contributions, the branching ratio $Br(K_L \rightarrow \mu^+ e^-)$ with $m_G = O(1 \text{ TeV})$ violates the experimental bound by at least thirteen orders of magnitude!

Let us then consider the presence of possible mixing among generations. To be correct, we first denote the $T_{3L} = -1/2$ quarks by $D_0 = (d_0, s_0, b_0)$ and similarly by $L_0 = (e_0, \mu_0, \tau_0)$ for the leptons, with the subscripts 0
referring to the eigenstates before mass mixing. A typical 
\( SU(4)/(SU(3) \otimes U(1)_{B-L}) \) current would be of the 
form \( J'_{LQ} = D_0^\nu L_0 \). Notice that this discussion only 
applies to the case \( SU(4)_{PS} \otimes [SU(2)]^4 \) where tree-level 
SM lepto-quark transitions can occur. If we now diagonal-
ize the mass matrices for the down quark and for the 
charged lepton sectors, we can express \( D_0 \) and \( L_0 \) in 
terms of the mass eigenstates as follows: \( D = U_D D_0 \) 
and \( L = U_L L_0 \). The above current can be rewritten as 
\( J'_{LQ} = D_0^\nu U_D U_L^\dagger L \). One now has the quark-lepton 
mixing matrix \( V_{LQ} = U_D U_L^\dagger \) involved in all quark lep-
ton transitions. In consequence, what should appear on 
the right-hand sides of (106) and (109) are extra factors 
\( V_{ed}^* V_{\mu s} \) and \( |V_{ed}^* V_{\mu s}|^2 \), respectively. Here \( V_{ed} \) and \( V_{\mu s} \) are 
matrix elements of \( V_{LQ} \).

In the absence of a convincing model of fermion masses, 
there is no reason to rule out the possibility that the mix-
ing coefficient \( |V_{ed} V_{\mu s}|^2 \) could be of order \( 10^{-13} \), but such 
a very strong suppression appears rather strange and un-
natural. Moreover, as \( V_{LQ} \) is a unitary matrix not all of 
it elements can be set to zero and consequently even 
if the \( K_L \to \mu e \) bound can be satisfied in this manner, 
other elements of \( V_{LQ} \) that are relevant for lepton flavour 
violation in B decays could be too large. Clearly the pres-
ence of more than three generations and consequently of 
many free parameters in \( V_{LQ} \) could help but such a fine 
tunning in essentially all processes is rather ad hoc.

We conclude therefore that an early unification of 
quark and leptons requires either the absence of tree level 
contributions to \( K_L \to \mu e \) and to analogous very rare 
decays or the presence of new suppression mechanism in 
addition to \( |V_{ed} V_{\mu s}|^2 \) considered above.

We shall now discuss the implication of these findings 
on the three candidates presented in the previous section,

namely \( SU(4)_{PS} \otimes [SU(2)]^4 \), \( SU(4)_{PS} \otimes [SU(2)]^3 \), 
and \( SU(4)_{PS} \otimes [SU(3)]^2 \).

\section{B. \( SU(4)_{PS} \otimes [SU(2)]^4 \)}

In this scenario, the decay \( K_L \to \mu e \) takes place at 
tree level and the RG analysis above has shown that the 
FPUT scale is typically around 1 TeV or less in order to 
agree with the experimental value for \( \sin^2 \theta_W (M_Z^2) \). 
Consequently, as just discussed, this scenario is ruled out 
unless additional suppression mechanisms in addition to 
\( |V_{ed} V_{\mu s}|^2 \) can be invoked.

This could come from aspects of physics of Large 
Extra Dimensions for example. One could add, for instance, 
an extra spatial dimension (for the purpose at hand) and 
denote it, for simplicity, by \( y \). It has been shown that the 
compactification of this extra dimension on an orbifold 
\( S_1/Z_2 \) gives rise to chiral zero modes in four dimensions 
[12]. In [8], it was proposed that \( SU(4)_{PS} \) is broken by 
boundary conditions. As a consequence, a quartet which 
contains a quark and a lepton can only have one chiral 
zero mode which could be either a quark or a lepton, with 
the other one being a heavy partner. Since SM particles

\begin{equation}
C_{ql} = \int \xi_q(y) \xi_l(y) dy, \tag{112}
\end{equation}

in the effective coupling, where \( \xi_q(y) \) and \( \xi_l(y) \) represent 
the wave functions along \( y \) of the quark and lepton chiral 
zero modes respectively. When the quarks and leptons 
are localized far away from each other along \( y \), the factor 
\( C_{ql} \) can be exponentially small [13]. If this scenario is 
correct then the bound (111) can easily be satisfied for 
this model if \( |V_{cd} V_{\mu s} C_{de} C_{sp}|^2 \) is of order \( 10^{-13} \). Even 
if \( |V_{cd} V_{\mu s}|^2 \) were of the order of unity, it is not hard 
for the other branch to satisfy \( |C_{de} C_{sp}|^2 \sim 10^{-6} \).

We observe then that the constraint from \( K_L \to \mu e \) has 
severe implications on the \( SU(4)_{PS} \otimes [SU(2)]^4 \) model because of the low FPUT scale as required by the fit to 
the value of \( \sin^2 \theta_W (M_Z^2) \). It implies either or both of 
the following scenarios: 1) The mass matrices are such 
that \( |V_{cd} V_{\mu s}|^2 \) is very small; and/or 2) The existence of a 
suppression mechanism coming from the physics of Large 
Extra Dimensions.

\section{C. \( SU(4)_{PS} \otimes [SU(2)]^3 \)}

As we have seen in Section III, the particle content 
of this group is rather interesting. The SM fermions 
belong to \( (4, 2, 2, 1)_L = [(i \tau_2 \psi^q_L; \bar{Q}'_L), (L_L, \psi^1_L)] \) 
and \( (4, 1, 2, 2)_R = [(i \tau_2 \psi^q_R; \bar{Q}'_R), (L_R, \psi^1_R)] \). From 
this fermion content, one can see that the \( SU(4)/(SU(3) \otimes 
U(1)_{B-L}) \) gauge bosons with electric charges \( \pm 4/3 \) link 
the normal quarks \( i \tau_2 \psi^q_{L,R} \) with the higher charged 
leptons \( \bar{L}_{L,R} \), and the normal leptons \( \psi^1_{L,R} \) with the higher 
charged quarks \( \bar{Q}'_{L,R} \). What this implies is that, at tree 
level, there is NO transition between normal quarks and 
normal leptons. However, it can occur at the one-loop 
level through a box diagram with two PS boson ex-
changes \( (M_{PS} = O(M)) \) and new heavy quarks \( (\bar{Q}) \) and 
new heavy leptons \( (\bar{L}) \) that have masses \( O(M_F) \) with 
\( M_F \) given in (76). \( \bar{Q} \) and \( \bar{L} \) appear in three generations 
and the mixing between these generations is given by 
3 \times 3 matrices to be denoted by \( U \) and \( V \), repectively. 
In the case of degenerate masses of \( \bar{Q} \) and \( \bar{L} \), the GIM 
mechanism is at work and the decay \( K_L \to \mu e \) is absent.
However, GIM mechanism remains to be powerful also when the masses are non-degenerate but all in the range $200 - 300$ GeV. In this case it provides a suppression factor of $O(10^{-4})$ at the level of the branching ratio. With the typical loop factor $(16\pi^2)^{-2} \approx 4 \cdot 10^{-5}$, the upper bound on the relevant mixing factors $|V_{td}V_{ts}^*|^2 |U_{jd}U_{js}^*|^2$ coming from $K_L \to \mu e$ amounts then roughly to $O(10^{-4})$ and can be easily satisfied.

A detailed presentation of this calculation and the analysis of FCNC processes mediated by the $SU(2)_H$ bosons is beyond the scope of this paper and will be presented elsewhere but this discussion shows that in this scenario, the low unification scale required by the value of $\sin^2 \theta_W (M_Z^2)$ is consistent with the present upper bound on $K_L \to \mu e$ and does note pose any problems with FCNC transitions at present.

D. $SU(4)_{PS} \otimes [SU(3)]^2$

The constraint coming from $K_L \to \mu e$ in this model is very similar to the previous one. A look at the fermion content, as shown in (72,75), reveals that the PS gauge bosons once more link normal quarks and leptons to their higher charged counterparts. As a result, there is no tree level contribution to $K_L \to \mu e$. Again this process will occur at one loop, with an analysis similar to the one mentioned above.

VI. COMPARISON WITH THE LITERATURE

In order to make an assessment of our work and compare it with recent attempts at “low scale” unification, we summarize below the essential results which were presented above. The three “simplest” candidates for Petite Unification - a possible nickname could be “Tevunification” - are $SU(4)_{PS} \otimes [SU(2)]^4$, $SU(4)_{PS} \otimes [SU(2)]^3$, and $SU(4)_{PS} \otimes [SU(3)]^2$. As mentioned at various places in the paper, the philosophy of our Petite Unification is to have a unification scale $M \leq 1000 \text{TeV}$ and preferably $M \leq 10 \text{TeV}$.

- **PUT$_1 = SU(4)_{PS} \otimes SU(2)_L \otimes SU(2)_R \otimes SU(2)_L \otimes SU(2)_R$**
  This is the favorite scenario in our 1981 paper [1]. This model has only quarks and leptons (including possible new ones) having standard electric charges. In our update of various numerical results, the conclusions drawn from our analysis can be summarized as follows. In order to obtain the correct value of $\sin^2 \theta_W (M_Z^2)$ and requiring that $M \sim 1 \text{TeV}$, our RG analysis (assuming $\lambda = M$) reveals that we need at least nine generations ($n_G = 9$), with the new generations having masses of order 250 GeV, or $n_G = 4$ if we include supersymmetry. In our RG analysis, the main important assumption which is made is that the masses of all new particles are taken to be of order 250 GeV. No additional assumptions are made about extra new physics other than Petite Unification above the scale $M$ at this stage.

However, this scenario with a PUT scale of order 1 TeV suffers from the problem with the branching ratio for the process $K_L \to \mu e$ which in this scenario can occur at tree level. Several possible remedies were discussed above, in particular in the context of the physics of Large Extra Dimensions.

- **PUT$_2 = SU(4)_{PS} \otimes SU(3)_L \otimes SU(3)_H$**
  In this model the PUT scale is required to be $M \sim 1 \text{TeV}$. In addition to the standard three generations of quark and leptons, new three generations of unconventional quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $O(250 \text{GeV})$ are automatically present. The horizontal groups $SU(2)_H$ connects the standard fermions with the unconventional ones. In addition, there are also very heavy vector-like particles which, however, are irrelevant to the phenomenology discussed in this paper. Furthermore, in this model, the process $K_L \to \mu e$ is forbidden at tree level and appears only at the one-loop level. In consequence, despite the appearance of a low PUT scale, the constraint from $K_L \to \mu e$ can easily be satisfied, in contrast with the $SU(2)_4$ scenario. No additional new physics such as Large Extra Dimensions is needed at this stage.

In summary, PUT$_1$ and PUT$_2$ are able to predict $\sin^2 \theta_W (M_Z^2)$ and to satisfy the constraint on $K_L \to \mu e$ within the perturbative regime. The offshoot of this is the prediction of the existence of three generations of unconventional quarks and leptons with charges up to 4/3 (for quarks) and 2 (for leptons) and masses $O(250 \text{GeV})$.

Having briefly summarized the results of our three “favorite” scenarios, we are now ready to make a comparison with the literature (surely an incomplete task). In particular, we would like to compare our results with those of [8] and [9], whose main focus was to derive $\sin^2 \theta_W$.

Ref. [8] basically generalized our $SU(4)_{PS} \otimes [SU(2)]^4$ model of 1981 to Large Extra Dimensions. This paper
was motivated by the possibility of a TeV scale unification. The first goal there was to obtain a reasonable estimate for $\sin^2 \theta_W(M_Z^2)$ for a unification scale of $O(1 \text{ TeV})$. The second goal was to prevent the process $K_L \rightarrow \mu e$ from acquiring a large branching ratio due to the low unification scale. To reach the first goal, a number of assumptions were made: the size of the cut-off scale where the regime of strong couplings set in (one might wonder whether or not the leading log approximation is still valid), the size of the tree-level boundary corrections, the contribution from the relative running of the $SU(2)$ gauge couplings above the compactification scale. In particular, this last assumption, which is very model-dependent, is crucial in obtaining an agreement with data. We have checked that when supersymmetric contributions to the running of coupling constants are switched on only above 200 GeV and not at $M_Z$ as done in [8] it is not possible to obtain acceptable solutions for the situation in which the $SU(2)$ gauge couplings run parallel to each other as the correct value of the weak mixing angle would require with $n_G = 3$ a compactification scale significantly lower than 1 TeV. On the other hand in a model in which the breakdown of gauge symmetries is accomplished by using boundary conditions, the authors of [8] find a positive contribution to $\sin^2 \theta_W(M_Z^2)$ from scales higher than the compactification scale and the correct value of the mixing angle can be found for the compactification scale $O(2 \text{ TeV})$. In summary, the actual “prediction” for $\sin^2 \theta_W(M_Z^2)$ in this model depends crucially on the assumptions made about various details of the physics of Large Extra Dimensions. The second goal mentioned above is achieved by the orbifold boundary conditions which split a quartet of $SU(4)_{PS}$ into zero and non-zero modes. Since the SM particles are supposed to be surviving zero modes in four dimensions, ordinary quarks and leptons cannot be in the same quartet, similarly to the case of the $SU(2)^3$ and $SU(3)^2$ models considered here. Consequently there are no tree-level transitions between SM quarks and leptons and the $SU(4)/[SU(3) \otimes U(1)_{B-L}]$ gauge bosons can be relatively “light” ($O(1 \text{ TeV})$) without violating the upper bound on the rate of $K_L \rightarrow \mu e$. This model predicts heavy copies of the SM particles with masses of $O(1 \text{ TeV})$.

Ref.[9] proposed to extend the Standard Model $SU(2)_L \otimes U(1)_Y$ to $SU(3) \otimes SU(2) \otimes U(1)$ at some scale $M$ of $O(1 \text{ TeV})$. In this model, $SU(3) \otimes SU(2) \otimes U(1) \rightarrow SU(2)_L \otimes U(1)_Y$ at $M$ which gives the following relations between the couplings of the SM and its parent group, namely

$$\frac{1}{g_2^2} = \frac{1}{g_3^2} + \frac{1}{g^{'2}}, \quad \frac{1}{g^{'2}} = \frac{3}{g_3^2} + \frac{1}{g^{'2}} \quad (113)$$

where the couplings on the right-hand side of these equations belong to those of the parent group while those on the left-hand side are those of the SM. In the limit $\tilde{g}, \tilde{g}' \rightarrow \infty$ (the exact $SU(3)$ limit), one can easily derive $\sin^2 \theta_W = 1/4$. Using the RG equations for $g_2$ and $g'$ to match the value of $\sin^2 \theta_W$ at $M_Z$, Dimopoulos and Kaplan obtained a value for the unification scale $M_0 = 3.75 \text{ TeV}$ in the limit $\tilde{g}, \tilde{g}' \rightarrow \infty$. As mentioned in [8], this prediction is not precise because of these assumptions. Once more, one is facing the problem with strong couplings. Furthermore, unlike the case with the Pati-Salam group or with the quintessential Grand Unified Theories, there is no charge quantization in this scenario. However it is similar in spirit to our 1981 paper [1] in that $\sin^2 \theta_W$ is determined entirely from the weak group although two of the groups in [9] are not so weak after all. Notice that the exact $SU(3)$ limit of [9] giving $\sin^2 \theta_W = 1/4$ is similar to our case of $G_W = SU(3)$ (with two doubly charged gauge bosons) as discussed in [1] and mentioned in Section IIB. In our case, this is ruled out by $\sin^2 \theta_W(M_Z^2)$.

Finally, in addition to [1], there are another two papers within the past three years which dealt with $SU(3) \otimes SU(3)^2$ [14] and $SU(4) \otimes SU(2)^3$ [15] in a very different context.

### VII. CONCLUSIONS

We have revived our previous paper [1] that provided a general discussion of an early quark-lepton unification characterized by the gauge group $G_S \otimes G_W$. As a byproduct we have presented a simple formula (34) for $\sin^2 \theta_W$ in the case of $G_W = SU(N)^k$ that is equivalent to the formula in [1] but is more transparent.

During the last twenty two years the experimental value for $\sin^2 \theta_W(M_Z^2)$ became very precise and the value of $\alpha_s(M_Z^2)$ became not only more precise but also significantly smaller. As a result of these changes, our favourite 1981 scenario, $SU(4)_{PS} \otimes [SU(2)]^4$, cannot be made consistent simultaneously with the data for $\alpha_s(M_Z^2)$ and the lower bound on the masses of right-handed gauge bosons unless six new generations of ordinary quarks and leptons are present. However, with the very low unification scale $O(1 \text{ TeV})$, the improved experimental upper bound on $K_L \rightarrow \mu e$ is violated in this model by many orders of magnitude unless new, not always natural, strong suppression factors are invoked.

Fortunately, we have found two new little unification models for which the situation is much more favourable. These are the models based on the groups $SU(4)_{PS} \otimes [SU(2)]^3$ and $SU(4)_{PS} \otimes [SU(3)]^2$, of which the first one is more appealing in view of its simpler fermion content. The interesting properties of these models, described already briefly in Sec.I and in detail in Sec. III-IV are as follows:

- The correct value of $\sin^2 \theta_W(M_Z^2)$ with the unification scale in the ballpark of 1 TeV and 3–10 TeV, respectively.
- The absence of tree level lepton flavour violation and of tree level FCNC processes. These transitions are generated at one-loop through the exchanges of the heavy PS gauge bosons, new heavy quarks and
leptons with unconventional electric charges (up to 4/3 for quarks and 2 for leptons) and through the exchanges of “horizontal” weak gauge bosons that couple the ordinary quarks and leptons with these new heavy fermions. Due to the GIM–like mechanism the bound on $K_L \rightarrow \mu e$ can easily be satisfied and the FCNC processes put under control.

The rich phenomenology resulting in these two new scenarios will be presented in detail in a forthcoming paper.

Finally, we would like to stress the fact that the physics of our two scenarios, $PUT_1$ and $PUT_2$, stands on its own regardless of whether or not TeV-scale Large Extra Dimensions exist. Even if they do exist, the predictions of $PUT_1$ and $PUT_2$ would be independent of the details of the physics of Large Extra Dimensions.

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