Parametric identification of differential-difference models of heat transfers in the systems of bodies

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Abstract. The combined inverse heat conduction problem (IHCP) was solved to restore the boundary conditions of heat exchange, in particular, unsteady heat flux, while simultaneously refining the thermophysical properties of the object material by parametric identification of the differential - difference models (DDM) of heat transfer in the object of study. The differential-difference model is a system of first order ordinary differential equations with respect to the state vector. The type of feedback, control, and measurement matrices included in the DDM is established. A priori parametrization and parametric identification of the heat transfer model was performed. The parametrization involves approximation of the desired heat flux by B-splines of the first order for each piecewise-linear segment. With parametric identification, the discrepancy between the model and experimental values of the parameters is minimized using a recurrent linear Kalman filter. Taking into account the parametrization and parametric identification, the time-varying heat flux at the body boundary was restored and its thermal and physical properties, in particular, thermal conductivity, were simultaneously refined. The uncertainty of restoration of parameters based on the Gram matrix is estimated. The boundaries of confidence areas for determining the desired parameters based on the sensitivity functions, which are obtained by solving the heat transfer equations, are established. The results of model experiments, in which initial estimates of the desired parameters were set twice less than the true ones, are given.

1 Introduction.
Thermal regime of various bodies under the influence of the environment or internal sources depends significantly on the thermophysical characteristics (TPC) of materials. This is especially evident at high temperatures and the non-stationary nature of the impact, in particular, when determining the parameters of thermal protection of aircraft, heat exchange in fluidized beds when burning low-grade fuel, dispersed materials, etc. In heat metering, reference data are used as information on thermal characteristics. However, for new materials, multicomponent complex alloys, ceramics, composites, and many others, the thermal characteristics are either unknown or differs significantly depending on the technology of their production, which leads to uncertainty in the measurement results.

When restoring the heat exchange boundary conditions, taking into account the uncertainties caused by possible differences in the thermal characteristics of materials, a rather complex scientific problem arises.
It is promising to use methods for solving combined IHCP, in which the tasks of simultaneous solving of the IHCP boundary and coefficient are set.

The paper discusses the method of restoring the unsteady heat flow with simultaneous refinement of thermal conductivity of the material of the object of study. For this, the combined inverse heat conduction problem is solved. Since IHCP is an incorrectly posed problem of mathematical physics, to obtain a sustainable solution, a method that is related to the compilation of a differential-difference model of heat transfer, parametrization of the problem, parametric identification using the recurrent digital Kalman filter (FC) has been selected. The literature [1-7] shows the successful use of FC in solving various IHCP. However, using real-time FC, there are the issues that require additional research. In particular, these are the issues related to determination of the trusted domain of the desired parameters. The following are the tasks associated with the uncertainty of the results of both computational and experimental methods. For this purpose, the Gram matrix is used, on the basis of which the limits of application of the method in question and the possibility of planning an experiment to achieve a given uncertainty are established.

2 Method of restoring the boundary conditions of heat transfer with simultaneous refinement of thermal and physical properties.

The method is based on IHCP parameterization with subsequent parametric identification of the differential-difference model (DDM) of heat transfer in the object of study, which is a system of first order ordinary differential equations for the temperature state vector \( T(\tau) = [t_i(\tau)]_{i=1}^n \).

In general, the DDM has the form [2]:

\[
\frac{d}{d\tau} T(\tau) = F(\tau) T(\tau) + G(\tau) U(\tau),
\]

where \( F \) and \( G \) are the feedback and control matrices; \( U(\tau) \) is the control vector. The feedback and control matrices are temperature dependent and equation (1) is non-linear.

In the object, either temperatures at individual points, or their differences, or mean volume temperatures are measured, and this is reflected in the measurement matrix \( H \) of the universal measurement model

\[
Y_k = H T_k + \varepsilon_k,
\]

where \( Y_k \) is the measurement vector, \( \varepsilon_k \) is the vector of random errors.

To determine the desired parameters, the DDM is solved. In this case, the assumption is made that the character of the change \( q(\tau) \) is known, which allows one to perform piecewise linear approximation with the required accuracy over the entire interval of its change \( 0, \tau_N \):

\[
q(\tau) = \sum_{j=1}^{r} q_j \varphi_j(\tau),
\]

where \( \varphi_j(\tau) \) is the system of basic functions, \( q_j \) represents a priori unknown coefficients that are combined into the vector of the required parameters \( Q = [q_1 \ldots q_r]^T \) (\( T \) is the transposition sign). B - splines of the 1st order are used as a basic function. Such an approximation \( q(\tau) \) is called IHCP parameterization. Then the problem of restoring \( q(\tau) \) is reduced to the parametric identification of the DDM of heat transfer in the object — successively obtaining the optimal estimates \( \hat{Q}_{z,l} \) of the vector of the required parameters \( Q_z \).

The optimal estimates \( \hat{Q}_{z,l} \) of the vector of the required parameters \( Q_z \) are obtained by minimizing the quadratic residual function by \( Q_z \) [2]:

\[
\Phi(Q_z) = \sum_{k=1}^{l} (Y_k - \hat{Y}_k(Q_z))^T \cdot R^{-1} \cdot (Y_k - \hat{Y}_k(Q_z)),
\]
where $\bar{Y}_k(Q_z)$ is analogue of the measurement vector $Y_k$, calculated by the DDM of heat transfer for different values of the desired parameters $Q_z$ which is called the model measurement vector; $R$ is the covariance matrix of the vector of random errors $\varepsilon_k$ in temperature measurements.

To obtain optimal estimates of $Q_{k+1}$ of the vector $Q$ in $(k+1)$-th instant of time, the FC is used according to the desired parameters [2]:

$$Q_{k+1} = \bar{Q}_k + K_{k+1}[Y_{k+1} - \bar{Y}_{k+1}(Q_k)],$$

$$K_{k+1} = P_k H_k^T (H_k P_k H_k^T + R)^{-1},$$

$$P_{k+1} = P_k - K_{k+1} H_k P_k,$$

where $P_k, P_{k+1}$ are covariance matrices of parameter estimation errors for time instants $\tau_k = k \cdot \Delta \tau$ and $\tau_k = (k + 1) \Delta \tau; H_k$ is the matrix of sensitivity coefficients of the measured temperature to the change of the desired parameters at time $\tau_{k+1}; K_k$ is weight matrix.

2.1 Assessment of the trust area.

To establish the confidence regions of the desired parameters, we discuss briefly the method of jointly solving the boundary and coefficient of the IHCP. Let us consider a sample in the form of a plate with a one-dimensional temperature field when exposed to unknown heat flux $q(\tau)$. In this case, we will consider it known: the temperature on the surface of the sample is $T_0$; the boundary condition on the back side of the sample, as well as, a priori information about the nature of the change $q(\tau)$. Then, as shown in [2], it is possible to perform a piecewise linear B-spline approximation and select the vector of the required parameters $Q_z(z = 1, 2, 3 ... n)$ at each of its $z$-th sections.

$$Q_z = [q_{a_z} \ q_{b_z} \ \lambda_z]^T,$$

where $q_{a_z}, q_{b_z}$ are the flow values at the beginning and at the end of the section of the spline approximation; $\lambda_z$ is thermal conductivity on the $z$-site.

This makes it possible to use the Kalman digital filter algorithm for estimating uncertainty [2], which is essentially recurrent procedure for the generalized least squares method that minimizes the residual function by the vector of the required parameters $\Phi(Q)$.

$$\Phi(Q) = \sum_{k=1}^{N}[Y_k - \bar{Y}_k(Q)]^T \cdot R^{-1} \cdot [Y_k - \bar{Y}_k(Q)],$$

where $Y_k$ is the temperature measurement vector, which includes the vector $\varepsilon$ of random measurement errors; $\bar{Y}_k(Q)$ are model (calculated) values of the measurement vector; $R$ is the covariance matrix of random measurement errors in $Y_k; k$ is discrete time.

The sensitivity function matrix included in the Kalman filter has the following form:

$$H_{k+1} = \frac{\partial Y_{k+1}}{\partial Q_z} \bigg|_{Q_z = Q_z, k} = \begin{bmatrix} U_{1,q_{a,k+1}} & U_{1,q_{b,k+1}} & U_{1,\lambda,k+1} \\ \cdots & \cdots & \cdots \\ U_{m,q_{a,k+1}} & U_{m,q_{b,k+1}} & U_{m,\lambda,k+1} \end{bmatrix},$$

where $U_{j,\lambda,k+1}$ is the sensitivity function of the $j$-th dimension ($j = 1, 2, ..., m$) to $\lambda$ at $(k + 1)$ point in time.
The components of the sensitivity matrix are the functions of sensitivity of the j-th dimension $X_j$ to the desired parameter $q_a, q_b$ and $\lambda$ at $(k + 1)$ point in time ($k = 1, 2, \ldots, l$). Their values are calculated by the well-known k-th estimate $\hat{Q}_k$ of the vector of the desired parameters by solving the heat transfer equation.

To calculate $U_{j,\lambda,k+1}$, based on the value of $\hat{Q}_{x,k}$, obtained at the k-th step, you can use the formula:

$$U_{j,\lambda,k+1} = \frac{y_{j,k+1}(\hat{q}_{ak}\hat{q}_{bk}\lambda_k \pm \Delta \lambda) - y_{j,k+1}(\hat{q}_{ak}\hat{q}_{bk}\lambda_k)}{\Delta \lambda}.$$  

Thus, to build the matrix $H_{k+1}$ for $(k + 1)$ point in time, it is necessary to determine the change in time ($k = 1, 2, \ldots, l$) of the sensitivity function values using formulas (7).

The covariance matrix $R(\hat{Q})$, included in the Kalman filter is a characteristic of the accuracy of the estimates of $\hat{Q}$ and has the form:

$$R(\hat{Q}) = \sigma^2 (\sum_{k=1}^l H_k^T \cdot H_k)^{-1}.$$  

where $\sigma^2$ is standard deviation in temperature measurement (the influence of noise in the measurements).

The expression in parentheses is the Gram matrix $A_l$ for the system of vectors of sensitivity functions (it is also an analogue of the Fisher information matrix). In this case, when measuring the temperature on the sample surface

$$A_l(\hat{Q}) = \frac{1}{\sigma^2} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$  

where

$$a_{11} = \sum_{k=1}^n U_{q_{ak}}^2, \quad a_{22} = \sum_{k=1}^n U_{q_{bk}}^2;$$

$$a_{12} = a_{21} = \sum_{k=1}^n U_{q_{ak}} U_{q_{bk}}.$$

After determining the constituent elements of the Gram matrix, it is possible to calculate the absolute uncertainties of measurement $\pm \Delta q_{a,t} \pm \Delta q_{b,t}$ and $\Delta \lambda_z$ for the first section of the spline - heat flow approximation. Since, as mentioned earlier, the Kalman filter is a recurrent procedure, it is not difficult to establish the values of the specified parameters at each site of the spline - approximation and, thus, to establish the confidence region over all measurement intervals of the desired heat flux $q(\tau)$:

$$\Delta q_a = \pm \sigma \frac{-a_{22} B}{\sqrt{a_{12}^2 - a_{11} a_{22}}}, \quad \Delta q_b = \pm \sigma \frac{-a_{11} B}{\sqrt{a_{12}^2 - a_{11} a_{22}}}.$$  

where $B = \alpha \sigma^2_{1-\alpha}$, $\alpha$ is the given probability, $\sigma^2$ is the square of the distribution of values with probability $1 - \alpha$. If $\alpha = 0.95$, then $B = \alpha \sigma^2_{1-0.95} = 5.911$ [4].
As an illustration, the figure shows the results of restoration of a linearly varying heat flux and thermal conductivity of the material of the object under study when setting the initial estimates of the vector of the desired parameters, equal to half true.

![Figure 1. True (1) and restored (2) heat flux and thermal conductivity values.](image)

3 Conclusion.
The method of restoring the surface density of a nonstationary heat flux $q(\tau)$ with simultaneous refinement of the thermal conductivity of the material of the object of study in real time is considered. The method is based on solving a combined IHCP by parametrizing the problem and parametric identification of the DDM. A method for estimating the confidence regions of the unknown parameters based on the use of Gram matrices is considered. The results of one of the model experiments confirming the possibility of using the considered method are given. The method and devices are used in the restoration of the surface density of unsteady heat flux in aero-hydrodynamic shock tubes, when conducting research in dispersed media [5].

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