SPATIAL CORRELATION OF CONDUCTION ELECTRONS IN METAL WITH COMPLICATED GEOMETRY OF THE FERMI SURFACE.

D.I. Golosov, M.I. Kaganov.

P.L. Kapitza Institute for Physical Problems, Russian Academy of Sciences, Vorobyovskoye sh. 2, Moscow 117334 Russia

ABSTRACT

The “density-density” correlation function of conduction electrons in metal is investigated. It is shown, that the asymptotic behaviour of the CF depends on the shape and the local geometry of the Fermi surface. In particular, the exponent of power law which describes the damping of Friedel oscillations at \( r \to \infty \) (\(-4\) for an isotropic Fermi gas) is determined by local geometry of the FS. The applications of the obtained results to calculations of the CF in a metal near the electron topological transition and of the RKKY exchange integral are considered as well.

Classification numbers: 71.45G, 71.25P.

1. Introduction.

In this paper we investigate the ”density-density” correlation function (CF) of conduction electrons in a metal at \( T = 0 \).

It is well-known, that correlation function of electron gas can be written as

\[
\nu(\vec{r}) \equiv \frac{1}{\langle n \rangle} \langle \Delta n(\vec{r}_1) \Delta n(\vec{r}_2) \rangle - \delta(\vec{r}_1 - \vec{r}_2) = -\frac{2}{\langle n \rangle} \int n_p \exp \left( ipr/\hbar \right) \frac{d^3p}{(2\pi\hbar)^3} |^2 \cdot \vec{r} = \vec{r}_2 - \vec{r}_1. \tag{1}
\]

Here the corner brackets stand for average, \( \Delta n(\vec{r}) = n(\vec{r}) - \langle n \rangle \) is the departure of electron density \( n(\vec{r}) \) from its average value \( \langle n \rangle \),

\[
n_p = \begin{cases} 1, \epsilon(\vec{p}) < \epsilon_F \\ 0, \epsilon(\vec{p}) > \epsilon_F \end{cases}
\]

- Fermi distribution function, \( \vec{p} \)- momentum, \( \epsilon_F \) - Fermi energy. Note, that equation (1) is valid for any (not necessary isotropic) dispersion law \( \epsilon(\vec{p}) \).

Let us remind, that for \( \epsilon(\vec{p}) = p^2/2m \), equation (1) leads to the following expression for the \( CF[1] \):

\[
\nu(\vec{r}) \approx \frac{3\hbar}{2\pi^2p_F^2r^4} \cos^2 \left( \frac{p_F r}{\hbar} \right), \quad r \gg \hbar/p_F \tag{3}
\]
(\(p_F\) - Fermi momentum), which contains Friedel oscillations (FO) \(^2\) with the wave number \(2p_F/\hbar\).

One can easily determine the wave numbers of FO (if the direction of radius vector \(\vec{r}\) is given) also in the case of an arbitrary dispersion law. Indeed, the long-wavelength behaviour of the \(CF\) (equation (1)) is determined by the singularities of the integral

\[
S(p_{||}) = \int n_{p_{||}} d^2p_\perp, \quad p_{||} = \vec{p} \cdot \vec{r}/r, \quad p_\perp = \vec{p} - p_{||} \vec{r}/r. \tag{4}
\]
as a function of \(p_{||}\). Obviously, \(S(p_{||})\) is nothing but the square of the section of the Fermi surface (FS) formed by a plane perpendicular to vector \(\vec{p}\) and located at the distance \(p_{||}\) from the origin in \(\vec{p}\)-space. Therefore the singularities we are interested in correspond to the tangencies of the FS to planes perpendicular to \(\vec{r}\). The nature of these singularities depends on the local geometry of the FS near tangency points.

Therefore the wave numbers \(FO \Delta k_{ij}\) are the distances in \(\vec{p}\)-space between planes tangent to the FS and perpendicular to \(\vec{r}\) (see Fig. 1). It is well-known, that the wave numbers \(\Delta k_{ij}\) correspond also to Migdal - Kohn singularities \(^3\) in the spectrum of phonons propagating in the direction \(\vec{r}/r\), as well as to singularities in the longitudinal plasmon spectrum (cf. \(^4\) etc.). Thus, the vectors \(\Delta k_{ij} \vec{r}/r\) define the distinguished points in the reciprocal space; since the present paper is devoted to the spatial correlation of electrons, we shall study the effects that occur in the real space.

One can also come to the same conclusions if one considers the Fourier component of the \(CF\)

\[
\nu(\vec{k}) = -\frac{2}{32\pi^2} \frac{r^3}{r^4} \sum_{ij} A_i^* A_j \exp(i\Delta k_{ij} r). \tag{6}
\]

\(^2\)Also, the quantities \((\Delta k_{ij})^{-1}\) are proportional to the periods of oscillations of the sound absorption coefficient in a magnetic field \(^5\).
Here \( r_e = \langle n \rangle^{-1/3} \) is the average distance between electrons, and indexes \( i, j \) enumerate the tangency points. Factors \( A_i \) (which, in general, functions of \( r \) as well as of \( \vec{r}/r \)) depend on the local geometry of the FS at the corresponding tangency points and have a dimensionality of inverse length. Due to the central symmetry of the FS, expression (6) is real (obviously, \( \Delta k_{ij} = -\Delta k_{ji} \)).

At \( i = j, \Delta k_{ij} = 0 \) and corresponding terms in the CF decrease monotonously as \( r \) increases. These terms are due to the singularity (namely, discontinuity of the first derivative) of \( \nu(\vec{k}) \) at \( \vec{k} = 0 \).

Usually, one can write down the validity condition for equation (6) as

\[
\Delta k_{ij} r \gg 1 \text{ for any } i \neq j.
\]

Note that the wave numbers \( \Delta k_{ij} \) (as well as their total number) depend on the direction of radius vector \( \vec{r} \).

So far we discussed the electron gas with an arbitrary dispersion law. The co-ordinate dependency of the CF of conduction electrons in a real metal is not exhausted by equation (1). In order to describe this case, one should take into account the influence of periodic potential of crystal lattice not only on the spectrum but also on the electron wave functions: wave functions that should be used now are the Bloch waves (instead of plane waves). This topic is considered in detail in our previous article [7]. The qualitative results of this straightforward though somewhat cumbersome consideration are listed below.

1. Due to the broken translation invariance, the co-ordinate dependency of the CF of conduction electrons in a real metal is not reduced to the dependency only on the difference of its arguments \( \vec{r} = \vec{r}_2 - \vec{r}_1 \). If the value of \( \vec{r} \) is fixed, \( \nu(\vec{r}_1, \vec{r}_2) \) is a periodic function of \( \vec{r}_1 \):

\[
\nu(\vec{r}_1, \vec{r}_2) = \nu(\vec{r}_1, \vec{r} + \vec{r}_1) = \nu(\vec{r}_1 + \vec{a}, \vec{r} + \vec{r}_1 + \vec{a}), \quad (7)
\]

where \( \vec{a} \) is any lattice period.

2. Let us keep \( \vec{r} = \text{const} \) and average \( \nu(\vec{r}_1, \vec{r}_2) \) over \( \vec{r}_1 \). Denote the obtained quantity by \( \nu(\vec{r}) \). Its asymptotic behaviour will be similar to the one described by equation (6). Wave numbers of FO are to be determined in the same way - the only difference is that now one should imagine the FS as a periodic surface in the reciprocal space (not restricted to the first Brillouin zone alone!). Therefore, the expression for \( \nu(\vec{r}) \) should contain summation over reciprocal lattice vector \( \vec{b} \); terms with \( \vec{b} \neq 0 \) (Umklapp terms) oscillate with a wave number \( \Delta k_{ij} + \vec{b}/r \).

3. CF contains a term, whose oscillation wavelength is the inverse diameter of the FS. Note, however, that if the FS is open, there can exist such a region of radius vector \( \vec{r} \) directions, where \( \nu(\vec{r}) \) does not oscillate at all.

Let us now consider the dependency of the CF on the local geometry of the FS. For the sake of simplicity, we shall restrict ourselves to the analysis of equations (1) and (6); this approach may be easily generalized for the case of conduction electrons in a real metal [7].
2. The Effect of Local Geometry of Fermi Surface on Correlation Function.

From equation (6), it follows that each term in the CF contains the factors \( A_i \), which depend on the local geometry of the FS at each of the two contributing tangency points. The expressions for \( A_i(\vec{r}) \) for various cases are listed below. We shall not describe here the calculations - one might find such a description in [7].

Let us keep the direction of radius vector \( \vec{r} \) fixed and consider the factor \( A_i \), corresponding to the \( i^{th} \) tangency point.

(i) If it is an elliptic point, then

\[
A_{\text{ell}} = 2G^{-1/2},
\]

where \( G \) is the Gaussian curvature (product of the principal curvatures [8]) of the FS.

(ii) For a hyperbolic tangency point, one obtains

\[
A_{\text{hyp}} = \pm 2i |G|^{-1/2},
\]

where the sign depends on the direction of radius vector (whether \( \vec{r} \) or \(-\vec{r}\)).

In these cases, \( A \) does not depend on \( r \), and the amplitude of FO corresponding to a pair of elliptic and/or hyperbolic points of tangency, decreases at \( r \to \infty \) as \( r^{-4} \).

(iii) The domains of elliptic and hyperbolic points on the FS are separated by the lines of parabolic points. Up to very few exceptions (Na, K, Cs, Rb and Bi), the FS of real metals are rather complex and do contain the lines of parabolic points [6,9].

In the generic case, one can choose the local coordinates \( \xi_1, \xi_2 \) on the FS in such a way, that the departure \( \xi_3 \) of the FS from the plane, tangent at parabolic point, reads:

\[
\xi_3(\xi_1, \xi_2) \approx C\xi_1^3 - B\xi_2^2.
\]

(10)

(Here we restrict ourselves to the leading order in \( x, y \)). We shall assume, for convenience, that \( B > 0 \) and \( C > 0 \).

If the tangency point is parabolic, then

\[
A_{\text{par}} \approx \frac{2K(\sin \pi/12)\Gamma(5/6)}{3^{1/4}C^{1/3}B^{1/2}\pi}r^{1/6} \propto r^{1/6}.
\]

(11)

Here \( K \) - elliptic integral, \( \Gamma \) - Euler's gamma-function;

\[
K(\sin \pi/12)\Gamma(5/6) \approx 1.5.
\]

Thus we immediately notice, that if parabolic tangency points appear at a given direction of \( \vec{r} \) then the amplitude of the corresponding term in the CF decreases
as $r^{-11/3}$, i.e. slower than in the usual case. Due to the fact that parabolic points on the FS are not isolated, but form continuous lines, there exist cones of radius vector directions corresponding to the presence of parabolic tangency points⁹.

Let us now denote by $\Delta \theta$ the angle between the normal at the parabolic point and the projection of radius vector onto the plane perpendicular to the line of parabolic points (Fig.2). At $\Delta \theta < 0$, there are no tangency points near the parabolic points. At $\Delta \theta > 0$, however, there exist both elliptic and hyperbolic tangency points and both of them approach parabolic points as $\Delta \theta \to 0$. Therefore in the CF there appears an additional long-wavelength term which has the wave number

$$2 \cdot 3^{-3/2} C^{-1/2} (\Delta \theta)^{3/2}.$$ 

Due to the vicinity of a parabolic point, equations (8) and (9) should be modified (one may not take into account terms quadratic in $x$, $y$ only). Thus, for an elliptic point (Fig.2, $x < 0$), we have

$$A_{\text{ell}} \approx B^{-1/2}(3C\Delta \theta)^{-1/4}(1 - \frac{5iC^{1/2}}{16 \cdot 3^{1/2} \cdot \Delta \theta^{3/2} \cdot r})^{1}, \quad (12)$$

and for a hyperbolic point, we have

$$A_{\text{hyp}} \approx iB^{-1/2}(3C\Delta \theta)^{-1/4}(1 + \frac{C^{1/2}}{2\pi 3^{1/2} \Delta \theta^{3/2}^{3/2}} (1 \frac{5i}{8}) \frac{1}{r}). \quad (13)$$

This distinction between elliptic and hyperbolic points is not so surprising. One has to remember that the singularities of $\nu(\vec{k})$ corresponding to elliptic and hyperbolic tangency points are somewhat different.

Both formulae (12) and (13) are valid at

$$\Delta \theta \gg (r^2/C)^{1/3} \quad (14)$$

Suppose that at $\Delta \theta < 0$ there exist $2N$ tangency points (here we are taking into account the symmetry of the FS) and, therefore,

$$2N - 1 + \frac{N(N - 1)}{2} = \frac{N(N + 3)}{2} - 1$$

different wave numbers of FO. At $\Delta \theta = 0$ there appear two new (parabolic) tangency points, and the amount of FO wave numbers increases by $N + 2$. Then at $\Delta \theta > 0$, $\Delta \theta < (r^2/C)^{1/3}$ the crossover occurs (the contribution of each parabolic tangency point turns into the sum of contributions of elliptic and

⁹If the FS contains parabolic points then the singularities of kinetic characteristics of the metal are enhanced for the corresponding directions of the quasimomentum vector [10,11].
hyperbolic tangency points) and finally at $\Delta \theta \gg (r^2/C)^{1/3}$ there are $2(N+2)$ tangency points and $(N+2)(N+5)/2 - 1$ FO wave numbers.

Let us consider the Gauss map of the $FS$(Fig. 3a). The Gauss map is a mapping of the surface onto the unit sphere, induced by the normal at each point of the surface (the direction of the normal gives a point of the sphere, thus giving the image of an original point of the surface) [8]. The number of tangency points changes by 4 as the point, representing the direction of the radius vector, crosses the line corresponding to the directions of the normals at parabolic points. This is the only way for this number to be changed. Suppose that for the direction of a radius vector lying within the area 1 (see Fig. 3a) the total number of the tangency points (on the whole $FS$) is $2N$. Then for $\vec{r}/r$ situated on the lines BE or ED there are $2(N+1)$ tangency points (two parabolic points have been added). The number of tangency points for $\vec{r}/r$ lying in areas 2 and 4, on lines AE or EC, and in area 3 are given by $2(N+2), 2(N+3),$ and $2(N+4)$ respectively. The angle $\theta$ appearing on Fig. 2 is just the distance between point $\vec{r}/r$ and the image of the line of parabolic points on the Gauss map.

(iv) Points at which the $FS$ becomes flat [12] manifest another type of local geometry. The $FS$ with such a point is a boundary case between convex surfaces and surfaces with a'crater'. Near this flattening point, the departure of the $FS$ from the tangent plane is a fourth-order form of the local coordinates within the surface. If the tangency point is a point of such a kind, then

$$A_{fl} \propto r^{1/2}. \quad (15)$$

Let us assume for definiteness, that the departure of the $FS$ from the plane tangent to the $FS$ at the flattening point, is given in the leading order by

$$\xi_3(\xi_1, \xi_2) \approx D(\xi_1^2+\xi_2^2) = D\xi_\perp^2. \quad (16)$$

Then as the (elliptic) tangency point approaches the flattening point, the Gaussian curvature decreases and

$$A_{el} \approx 25/6 r^{-1/2} D^{-1/3} \Delta \theta^{-2/3}, \quad (17)$$

Here $\Delta \theta$ is now the angle between the radius vector and the normal to the $FS$ at the flattening point. Formula (17) is valid at

$$\Delta \theta \gg D^{1/4} r^{-3/4}, \quad (18)$$

where the intermediate region is the region of crossover between the dependencies (15) and (17) (see Fig. 3b).

(v) Suppose that the lines of parabolic points on the $FS$ cross at some point. In real metals such a crossing occurs rather often; for example, the $FS$ of the form shown in Fig. 4 is encountered in $Mo, W$, paramagnetic $Cr$[6, 9, 11].

The departure of the $FS$ from the plane tangent to the $FS$ in this point of crossing, is given by a third-order form and we have
In fact, this is a flattening point of another kind. The important feature of flattening tangency points of any type is that they correspond to isolated points on the Gauss map (see Fig.3).

(vi) Let us assume that the FS contains finite flat elements. If the radius vector is parallel to the normal to such an element of the FS, it leads to the contribution of this "tangency area" in the CF:

\[ A_{\text{plane}} \propto S_{\text{plane}} r, \tag{20} \]

where \( S_{\text{plane}} \) is the square of this flat element.

In fact, the FS of such a kind can hardly be stable: in this case, Peierls transition [13] with the transformation of the lattice structure should become energetically favourable (at least at sufficiently low temperatures). It is easy to notice that there should be some correlation between extremely slow damping of \( FO \) as \( r \to \infty \) (here \( \nu \propto r^{-2 \cos(\Delta k_{\text{plane}} r)} \)) and the possibility of Peierls transition.

At this point we finish our consideration of the local geometry of the FS. Note, that we have arrived to the remarkable conclusion: the exponent, which determines the damping of \( FO \) as \( r \to \infty \), depends on the local geometry at the points of tangency. Thus, one might build up a hierarchy of types of the FS local geometry with respect to the long-distance behaviour of the CF:

1. Spherical FS: Amplitude of FO decreases as \( r^{-4} \); this exponent is given, generally, by elliptic or hyperbolic tangency points and thus corresponds to a generic direction in a 3D metal.

2. Cylindrical FS: Amplitude of FO decreases as \( r^{-3} \). This is the case of a 2D metal, or the case of a toroidal FS cavity (see sect.5 of Ref. [7] [4]).

3. Flat FS. Amplitude of FO decreases as \( r^{-2} \) (one could imagine it as a FS of an 1D metal).

From our consideration, it follows that parabolic points (along with crossing points of the lines of parabolic points) should be placed "between" spherical and cylindrical FS.

3. Some Applications.

Let us consider a metal near the electron topological transition (ETT) point [14]. The ETT is the restructuring of the FS which occurs when \( \epsilon_F = \epsilon_{cr}(\epsilon_{cr} \) - the value of electron energy that corresponds to Van Hove singularity). Depending on whether this singularity is due to an extremum or to a saddle point of the electron dispersion law, a new cavity of the FS appears (disappears), or a "neck" connecting two FS cavities is formed (disrupted). The difference

\[ A_{\text{or}} \propto r^{1/3}. \tag{19} \]

Point at which the FS becomes flat also gives the \( r^{-3} \) law (see Eqn (15)).
\[ z = \epsilon_f - \epsilon_{cr} \] depends on the applied pressure (or, for example, impurity concentration), so that the ETT can be realized experimentally (see Ref. [15] and references there). The ETT causes distinctive singularities of thermodynamical and kinetic quantities [14-16].

Near the ETT point, long-wavelength terms appear in the CF due to the existence of small diameters of the FS. If the ETT results in the appearance of a new cavity of the FS at \( z > \theta \), then
\[
\nu_{\text{ETT}}(\bar{r}) \approx -\frac{1}{n^2 + \pi^2 \bar{r}^2} \left[ \sum_{i=1}^{3} \frac{\beta m_1 m_2 m_3}{m_i \cos^2 \theta_i} \right] \cos^2 \left( (2\pi r \sum_{i=1}^{3} m_i \cos^2 \theta_i) \right) / h. \tag{21}
\]
Here \( m_1, m_2, m_3 \) are effective masses and the axes \( \hat{x}_1, \hat{x}_2, \hat{x}_3 \) (where \( x_i = r \cos \theta_i \)) are the main axes of the tensor of inverse effective masses, so that the electron dispersion law near the point of extremum can be written as
\[
z \approx \frac{(\Delta p_1)^2}{2m_1} + \frac{(\Delta p_2)^2}{2m_2} + \frac{(\Delta p_3)^2}{2m_3}. \tag{22}
\]
If (instead of an extremum) one has a saddle point of the dispersion law, one should write instead of (22):
\[
z \approx \frac{(\Delta p_1)^2 + (\Delta p_2)^2}{2m_\perp} + \frac{(\Delta p_3)^2}{2m_\parallel} + \frac{\beta}{4m_\parallel^2}(\Delta p_3)^4 \tag{23}
\]
(we assume that the FS near the extremal point \( \Delta \bar{p} = \theta \) possesses the rotational symmetry with respect to the \( p_3 \) axis and is also symmetric with respect to the \( \Delta p_3 = \theta \) plane). In this case, the "neck" of the FS, which exists at \( z > 0 \), is ruptured at \( z < 0 \) (Fig.5). Therefore, at \( |z| < \epsilon_f \) there exist long-wavelength \( (\Delta k \approx (2m_\parallel |z|)^{1/2}/h \ll k_F) \) terms in the CF. The issue of significance here is that the angular dependence of these terms "rotates" by \( \pi/2 \) as the sign of \( z \) changes.

Indeed, for example, at \( z < \theta \) and \( \bar{r} \parallel \bar{p}_3 \), due to the small distance between the two sheets of the FS, there exist long-wavelength component of the FO with corresponding \( \Delta k = (2m_\parallel |z|)^{1/2}/h \) and amplitude
\[
m_\perp^2 |z| r^{-4}/(2\hbar^2 \pi^4 < n > m_\parallel).
\]
There are no long-wavelength oscillations at \( \bar{r} \perp \bar{p}_3 \). Vice versa, for \( z > \theta \), the long-wavelength oscillations are absent at \( \bar{r} \parallel \bar{p}_3 \). For \( z > \theta \) and \( \bar{r} \perp \bar{p}_3 \), due to small diameter of the FS "neck", the following long-wavelength term in the CF appears:
\[
\nu_{\text{ETT}}(r) \approx -\frac{m_\parallel^2 z}{\pi^4 < n > r^4 \hbar^2} \sin^2 \left( r (2m_\perp |z|)^{1/2} \right) \times \bar{r} \perp \bar{p}_3. \]
At \( \beta z > \theta \), the fourth-order term in (29) gives rise to the appearance of the lines of parabolic points on the FS near \( \bar{p} = \bar{p}_{cr} \) (see Fig.5). Thus for

\[^5\text{Note that parabolic points on the FS themselves are not peculiar from the point of view}\]
some directions of \( \vec{r} \) forming angle \( \theta_{\text{par}} \) with \( p_3 \) axis, the amplitude of the long-wavelength term in the \( CF \) behaves as \( r^{-11/3} = r^{-4} \nu^{2/3} \) instead of \( r^{-4} \). In the neighbourhood of these peculiar directions, there exists an additional small wave number of \( FO \), which is due to neighbouring tangency points pairs (elliptic and hyperbolic). At \( \theta = \theta_{\text{par}} \), each pair "sticks together", giving the parabolic tangency point and then disappearing.

On the whole, the ETT, being a local "event" in reciprocal space, in coordinate space results in dramatic restructuring of the long-wave length terms in the \( CF \).

Obviously, the knowledge of the \( CF \) is useful when one considers various collective phenomena in metals: screening, structure of spin glasses etc. Let us mention for example the RKKY interaction [17-19]. This indirect exchange interaction between the site (or impurity) spins is caused by exchange interaction between the localized spins and conduction electrons:

\[
\mathcal{H} = (J/ < n >) \sum_i \vec{\sigma}_i \vec{S}_i \delta(\vec{r} - \vec{r}_i). \tag{24}
\]

Here \( J \) - exchange constant, \( \vec{\sigma} \) - Pauli matrices, \( \vec{S}_i \) - spin of the ion located in the site \( \vec{r}_i \). The RKKY exchange integral reads

\[
J^{\text{RKKY}}(\vec{r}) = -2(\frac{J}{< n >})^2 \int \frac{\nu(q)}{\epsilon(q) - \epsilon(q')} \exp \left[ i(\vec{q} - \vec{q}')\vec{r}/\hbar \right] \frac{d^3q}{(2\pi\hbar)^3}, \tag{25}
\]

and its Fourier component,

\[
J^{\text{RKKY}}(\vec{k}) = -2(\frac{J}{< n >})^2 \int \frac{\nu(q)}{\epsilon(q) - \epsilon(q - \vec{k})} \frac{d^3q}{(2\pi\hbar)^3} \tag{26}
\]

This kind of integral appears also in studying of the electron-phonon interaction. It has singularities at the same points as does the \( CF \) (except \( k = 0 \)). These singularities are called Migdal - Kohn singularities [3] if the electron velocity vectors in the two contributing tangency points are antiparallel. On the other hand, if they are parallel, then the singularity is called Taylor singularity [20] and these two cases should be treated separately.

It turns out, that at least for the great majority of cases there exists a quite simple relationship between the asymptotics of \( \nu(\vec{r}) \) and \( J^{\text{RKKY}}(\vec{r}) \). Indeed, for a fixed direction of the radius vector

\[
J^{(\Delta k)}_{\text{RKKY}} \propto r^{\nu(\Delta k)}(r), \quad \Delta k \neq 0 \tag{27}
\]

where \( J^{(\Delta k)}_{\text{RKKY}} \) and \( r^{\nu(\Delta k)}(r) \) stand for the terms in \( J^{\text{RKKY}} \) and the \( CF \) respectively, oscillating with the wave number \( \Delta k \).

It is well-known, that magnetic order induced by the RKKY interaction (namely, the helicity vector) depends on the geometry of the \( FS \)[18]. Thus, the helicity vector is expected to be determined by the small diameter of the \( FS \)[21], of electron dispersion law. Generally, the lines of parabolic points may have nothing to do with ETT or Van Hove singularities (see sect.2 above).
Therefore the knowledge of the FS and the CF seems to allow one to make some predictions about the magnetic arrangement. We expect, in particular, that the ETT may manifest itself in changes of magnetic ordering.

4. Conclusion.

In this paper, we neglected temperature effects, electron scattering and electron-electron interaction. The first two effects lead to the appearance of an exponentially decreasing factor in the CF as $r \to \infty$. The third (i.e. Fermi-liquid interaction), probably leads to the renormalization of the coefficients (see, for example, [22]). Anyway, the dependence of the asymptotic behaviour of the correlation function on the FS geometry (including the local geometry) survives after taking these effects into account. This dependence should affect collective phenomena in metals. Indeed, one has to know the CF to construct theories of alloying, screening, exchange magnetism etc. We have investigated some features of the CF that seem to be interesting from this point of view. Among them are:

1. The existence of the cones of peculiar directions, corresponding to the lines of parabolic points on the FS. For these directions of the radius-vector, the CF shows some specific features:
   - (i) the attenuation of FO is determined by a factor of $r^{-11/3}$ instead of usual $r^{-4}$; we doubt if this small difference has any observable effect.
   - (ii) the number of FO periods changes as the direction of the radius vector crosses these cones.
   - (iii) On one side of such a cone, one of FO periods appears to become large (it goes to infinity as the direction of $\vec{r}$ approaches the cone).

2. The existence of the isolated peculiar directions. These correspond to the crossings of the lines of parabolic points on the FS (flattening points).

3. The restructuring of the CF due to electron topological transition.

We have also provided one example showing how an asymptote of the RKKY exchange integral depends on the asymptotic behaviour of the CF.

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Figure Captions

**Fig.1** Rather simple $FS$ ("dumb-bell") for the radius vector $\vec{r} \parallel \vec{r}_1$ generates 8 different wave numbers of Friedel oscillations (due to the symmetry):

\[ \Delta k_{12} = \Delta k_{56}, \Delta k_{13} = \Delta k_{46}, \Delta k_{14} = \Delta k_{25} = \Delta k_{36}, \]
\[ \Delta k_{15} = \Delta k_{26}, \Delta k_{23} = \Delta k_{45}, \text{and} \Delta k_{24} = \Delta k_{35}. \]

Circled numbers enumerate the tangency points for this case. The total number of wave numbers depends on the direction of $\vec{r}$: for $\vec{r} \parallel \vec{r}_2$ there is the only wave number $\Delta k$.

**Fig.2.** Parabolic point on the $FS$ (placed at the origin of local co-ordinates $\xi_1, \xi_2, \xi_3$, where $\xi_3$ is parallel to the normal). For a small value of angle $\theta > 0$, there are two tangency points in the vicinity of the parabolic point: elliptic (with $\xi_1 < 0$) and hyperbolic (with $\xi_1 > 0$), and, correspondingly, the small
wave number $\Delta k \propto \theta^{3/2}$. As $\theta \to +0$, these tangency points approach each other; at $\theta = 0$ they coincide at the parabolic point; at $\theta < 0$, there are no tangency points in the vicinity of this parabolic point (see also Fig. 3a).

Fig. 3. Gauss maps of the elements of the FS.

3a. The image of the part of the FS with two lines of parabolic points $AC$ and $BD$ intersecting in the point $E$. Each point in areas 2 and 4 is the image of the two points from the original part of the FS, each point in the area 3 - the image of four points and each point in area 1 does not have any original there. Shaded areas correspond to crossover (the inequality (14) is broken there). This obviously is only one of the two kinds of intersections of the lines of parabolic points.

3b. Gauss map of the part of the FS containing the point at which the FS becomes flat. The number of tangency points is constant for $\vec{r}/r$ laying in this region of directions. The shaded area corresponds to crossover (inequality (18) is broken there).

Fig. 4 Crossing of the lines of parabolic points on the FS[11].

4a. Fragment of the FS("octahedron") in a metal belonging to the molybdenum group.

4b. Crossing of lines of parabolic points on this cavity of the FS(view from above).

Fig. 5 Disappearance (a) and appearance (b) of the lines of parabolic points (marked by the thick lines) during the "neck" formation [16].