The formula for the resistance force during slow motion of a sphere in an ideal gas

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Abstract. In this article the formula for the resistance force during slow motion of a sphere in an ideal equilibrium gas is discussed, taking into account the velocity distribution of molecules. Using the density function of the velocity distribution of molecules hitting the surface of the body during a certain period of time, an expression for the resistance force is obtained. The resulting expression is compared with the well-known expression in the Stokes law.

1. Introduction
For the force of resistance to the motion of a spherical body in a viscous fluid or gas in modeling the motion of particles, the expression in the form of the Stokes law is widely used, which, as is commonly believed, is valid for small values of the Reynolds number \( Re \ll 1 \) [1, 2]:

\[
F_s = 6\pi \mu RU,
\]

where \( \mu \) is the dynamic viscosity of the fluid (gas), \( R \) is the radius of the sphere, \( U \) is the speed of the sphere, \( Re = \frac{\rho UR}{\mu} \) is the Reynolds number, \( \rho \) is the liquid (gas) density.

In this paper, the force of resistance to movement is considered as the resultant of pressure force caused by the impact of molecules on the surface of the body.

In order to simplify, the impacts of molecules are considered in the approximation of the model of elastic spheres. When modeling the collisions, a mechanical model of an ideal gas in an equilibrium state is used. The simulation method is based on the application of molecular dynamics methods, which are widely used at present in physical modeling [3-6].

2. Modeling methods
As it is known, in the equilibrium state of a gas, the relative fraction of molecules in the coordinates of velocities (in a rectangular coordinate system \( X, Y, Z \)) is described by the Maxwell distribution, which has, for example, the following form for the \( Z \) axis:

\[
\frac{N(v_{1z} < v_z < v_{2z})}{N_0} = \int_{v_{1z}}^{v_{2z}} f(v_z) dv_z,
\]

with the distribution density

\[
f(v_z) = \frac{m}{2\pi kT} \exp \left( -\frac{mv_z^2}{2kT} \right),
\]

where \( m \) is the mass of the molecule, \( T \) is the temperature, \( k \) is the Boltzmann constant. For the coordinate axes \( X, Y \) expressions are similar to (1) and (2).
Let the wall is parallel to the coordinate plane \((XY)\), the \(Z\) axis is perpendicular to the wall and the total number of molecules which hit the wall in the time interval \(\Delta t\) is equal to \(N_1\). Then the fraction of molecules, which hit the wall and whose velocity coordinate \(v_Z\) satisfies the relation \(v_{1Z} < v_Z < v_{2Z}\), is determined from equality

\[
\frac{N_1}{N_1} \{v_{1Z} < v_Z < v_{2Z}\} = \int_{v_{1Z}}^{v_{2Z}} f_1(v_Z)dv_Z,
\]

(3)

where \(f_1(v_Z)\) is the corresponding distribution density function. According to [7]

\[
f_1(v_Z) = \left(\frac{m}{kT}\right) v_Z \exp\left\{-\frac{m v_Z^2}{2kT}\right\}.
\]

(4)

To find the pressure force on the surface of the body, it is necessary to find the change in the normal component of the momentum \(P_n\) of the molecules which hit the small area \(dS\) during the time interval \(\Delta t\). If the component of the molecule velocity, normal to the surface, is equal to \(v_n\), then in the case of the elastic collision of one molecule the change in the normal component of impulse is \(\Delta P_n = 2mv_n\), therefore

\[
\Delta P_n = \int_0^{+\pi} n\Delta t 2mv_n f_1(v_n)dv_n dS.
\]

(5)

If the sphere moves in an equilibrium gas at a low speed, then in the first approximation one can neglect the change in the velocity distribution function of molecules.

Let the velocity of the sphere is equal to \(U\) and is directed along the \(Z\) axis. Then in the frame of reference where the sphere is at rest, for the density function of the velocity distribution of molecules along the \(Z\) axis, the following expression is true:

\[
f(v_Z) = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left\{-\frac{m(v_Z + U)^2}{2kT}\right\}.
\]

(6)

The form of the density functions for the other components of the velocity vector does not change. Therefore, for the total change of the normal component of the momentum of molecules along the \(Z\) axis obtained from (5) and taking into account (6), the expression for the entire surface of the sphere can be obtained:

\[
\Delta P_z = 2n\Delta t \int_0^{+\pi/2} mv_n^2 \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left\{-\frac{m(v_n + U\cos\theta)^2}{2kT}\right\}dv_n cos\theta 2\pi R\sin\theta d\theta,
\]

(7)

where \(R\) is the radius of the target, equal to the sum of the sphere radius and the molecule radius, \(\theta\) is the angle between the \(Z\) axis and the direction of the normal \(n\) to the surface (figure 1). When finding the integral (7), one can use the decomposition

\[
\exp(-x) \approx 1 - x + \frac{1}{2} x^2,
\]

that is valid for small \(x\) values.
As a result, the following ratio is obtained:

$$\Delta P_c = 4\pi R^2 \Delta t m n \left( \frac{m}{2\pi kT} \right)^{3/2} 4 \frac{kT}{U} \frac{kT}{m}. \quad (8)$$

3. Results and discussion

Introducing the average velocity of molecules $\langle V \rangle = \left( \frac{8kT}{\pi m} \right)^{1/2}$, the gas density $\rho = mn$ and determining the magnitude of the force as $F = \left| \frac{\Delta P_c}{\Delta t} \right|$, the following expression can be obtained from (8):

$$F = \frac{4}{3} \pi R^2 \rho \langle V \rangle U. \quad (9)$$

For an ideal gas, the viscosity is determined by the expression

$$\mu = \frac{1}{3} \rho \langle V \rangle \lambda, \quad (10)$$

where $\lambda$ is the mean free path of molecules. The expression (9) in view of (10) can be written as

$$F = F_s \frac{2}{3} \frac{R}{\lambda}. \quad (11)$$

Thus, expressions (1) and (9) are comparable in the case when the sphere radius is comparable with the mean free path of molecules.

4. Conclusion

In this paper, the formula was obtained for the resistance force during slow motion of a sphere in an ideal gas using the methods of the kinetic theory of gas. For the case when the size of the sphere is comparable to the mean free path of the molecules, this formula produces results comparable to the Stokes law formula.

References

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