SOME IMPROVEMENTS ON THE $L_p$ INEQUALITIES FOR DIFFUSION PROCESSES

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Abstract. In this paper, we give some improvements on the $L_p(0 < p < \alpha)$ inequalities for diffusion processes. We obtain smaller constants in the $L_p$ inequalities and derive that the growth rates of the constants, as $p \to 0^+$, grows like $O\left(\frac{1}{p^\alpha}\right)$, instead of the exponential of $\frac{1}{p}$. Finally, we apply the improved inequalities to the Ornstein-Uhlenbeck processes, Bessel processes and reflected Brownian motion with drift and get better constants.

1. Introduction

Diffusion processes are a class of stochastic processes with wide applications. Many mathematical models in engineering and finance are related to diffusion processes. In the applications, the $L_p$ inequalities for diffusion processes are basic tools and the constants in the inequalities are also important when the estimations should be exact. $L_p$ inequalities and Davis-type inequalities for diffusion processes have been extensively studied for a long time. Gordon[5], Burkholder[2], Rosenkrantz and Sawyer[13] and DeBlassie[3, 4] studied $L_p$ inequalities for Bessel processes. Graversen and Peskir [6, 7], Peskir and Shiryaev[10] and Botnikov[1] established $L_p$ inequalities and Davis-type inequalities for Ornstein-Uhlenbeck processes and reflected Brownian motion with drift. Yan and Zhu[15] introduced the condition $S(\gamma, K_1, K_2)$ and established $L_p$ inequalities for diffusion processes satisfying the condition $S(\gamma, K_1, K_2)$. The $L_p(0 < p < \alpha)$ inequalities established before are all based on the well-known domination inequalities or domination principles, which established by Lenglart[8] and improved by Yor[12, 16].

Diffusion processes satisfying the condition $S(\gamma, K_1, K_2)$ are more general. They contain Bessel processes, Ornstein-Uhlenbeck processes and reflected Brownian motion with drift and others as special cases. In this paper, we study the constants in the $L_p(0 < p < \alpha)$ inequalities for diffusion processes satisfying the condition $S(\gamma, K_1, K_2)$. We obtain smaller constants in the $L_p$ inequalities and derive that the growth rates of the constants, as $p \to 0^+$, grows like $O\left(\frac{1}{p^\alpha}\right)$, instead of the exponential of $\frac{1}{p}$.

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Let $X = (X_t, \mathcal{F}_t)_{t \geq 0}$ be a diffusion process given by
\[
dX_t = \mu (X_t) \, dt + \sigma (X_t) \, dB_t, \quad X_0 = x_0,
\]
where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion starting at zero and $\mu(x), \sigma(x)$ are continuous functions.

If $\sigma(x) > 0$, for $x \neq 0$ and there exist constants $\gamma, K_2 \geq K_1 > 0$ such that
\[
K_1 |x|^{\gamma} \sigma^2(x) \leq |\mu(x)| \leq K_2 |x|^{\gamma} \sigma^2(x), \quad -\infty < x < \infty, \tag{2}
\]
we say that the diffusion process $X$ satisfies the condition $S(\gamma, K_1, K_2)$ if $\mu(x)$ and $\sigma(x)$ satisfy the condition $S(\gamma, K_1, K_2)$.

Let $f(x)$ be a positive function and $g(x)$ be a positive increasing function. We denote $f(x) = O(g(x))$, as $x \to 0$, if there exists a constant $C > 0$ such that
\[
\limsup_{x \to 0} \frac{f(x)}{g(x)} \leq C.
\]

For a stochastic process $X = (X_t)_{t \geq 0}$ and a stopping time $\tau$, we write
\[
X^*_\tau = \sup_{s \leq \tau} |X_s|, \quad X^*_\infty = \sup_{s < \infty} |X_s|.
\]

Let $F(x)$ be the solution of the equation
\[
\sigma^2(x) \frac{d^2 y}{dx^2} + 2 \mu(x) \frac{dy}{dx} = 2 \varphi(x) \tag{3}
\]
such that $y(0) = 0, y'(0) = 0$, where $\mu(x), \sigma(x), \varphi(x)$ are continuous functions on $R$ and $\varphi(x) \geq 0$.

For a nonnegative continuous function $\varphi$, define
\[
J_t = \int_0^t \varphi (X_s) \, ds, \quad t \geq 0,
\]
then $(J_t)_{t \geq 0}$ is a continuous increasing process.

One of the main $L_p$ inequalities established by Yan and Zhu[15] is the following. Let $X$ be a diffusion process given by (1), starting at zero such that the condition $S(\gamma, K_1, K_2)$ be satisfied and $\mu(x) \leq 0$, for $x \geq 0$. Assume that $\varphi$ satisfies the following condition
\[
N_1 |x|^{\gamma - 1} \sigma^2(x) \leq |\varphi(x)| \leq N_2 |x|^{\gamma - 1} \sigma^2(x) e^{N_3 |x|^{\gamma + 1}/\gamma + 1},
\]
with constants $N_i > 0, (i = 1, 2, 3)$. If either $X_t \geq 0$ or the function $F(x)$ is an even function, then the following inequality
\[
c_{p, \gamma} \left\| \ln^{1+\gamma} (1 + J_t) \right\|_p \leq \left\| X^*_t \right\|_p \leq C_{p, \gamma} \left\| \ln^{1+\gamma} (1 + J_t) \right\|_p \tag{4}
\]
holds, for $0 < p < \gamma + 1$ and any stopping time $\tau$. 
As in the theory of martingale inequalities, one interesting problem for $L_p$ inequalities of diffusion processes is to find the best constants or the growth rates of the constants in the inequalities as $p \to \infty$ and $p \to 0^+$. Since the $L_p$ inequalities established by the domination inequality are only the type of inequalities for $0 < p < \alpha$, we study the growth rates of the constants as $p \to 0^+$. Simple calculations show that all the growth rates of the constants obtained by the approach of domination inequality are

$$\frac{1}{c_{p,\gamma}} = O\left(2^{\frac{1}{p}}\right), \quad C_{p,\gamma} = O\left(2^{\frac{1}{p}}\right), \quad p \to 0^+. $$

This approach yields only the exponential of $\frac{1}{p}$ estimate, for $\frac{1}{c_{p,\gamma}}$ and $C_{p,\gamma}$.

Recently, Ren and Shen[11] established an improved domination inequality. By this inequality, we can show that the growth rates of the constants depend on the function $\varphi$ and

$$\frac{1}{c_{p,\gamma}} = O\left(\frac{1}{p^{1/\gamma}}\right), \quad C_{p,\gamma} = O\left(\frac{1}{p^{1/\gamma}}\right), \quad p \to 0^+. $$

This gives us more information for the growth rates of the constants in the $L_p$ inequalities, as $p \to 0^+$.

Throughout this paper, we shall use the standard notions of general theory of stochastic processes and thus consider stochastic processes with cadlag paths. We suppose that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space with filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions, $B = (B_t)_{t \geq 0}$ is a standard Brownian motion with $B_0 = 0$. A stochastic process $A = (A_t)_{t \geq 0}$ is called an increasing process if it is adapted to the family $(\mathcal{F}_t)$, whose paths are positive, increasing, finite and right continuous on $[0, +\infty)$. An adapted positive cadlag process $X$ is called dominated by an adapted increasing process $A$ with $A_0 \geq 0$, if

$$E(X_\tau) \leq E(A_\tau),$$

for any bounded stopping time $\tau$.

For a continuous increasing function $H(x)$ from $R_+$ to $R_+$ with $H(0) = 0$, set

$$\tilde{H}(x) = H(x) + x \int_x^\infty \frac{dH(u)}{u}. $$

Ren and Shen established the following improved domination inequality.

**Lemma 1.** Let $X$ be an adapted positive cadlag process and be dominated by a predictable increasing process $A$ with $X_0 = A_0 = 0$, $H(x)$ be a continuous increasing function from $R_+$ to $R_+$ with $H(0) = 0$. Then for any stopping time $\tau$ and any $0 < \lambda \leq 1$, the following inequality holds

$$E[H(X^\tau_\tau)] \leq E\left[H + \lambda \tilde{H}\left(\frac{A_\tau}{\lambda}\right)\right].$$

Using this improved domination inequality, we give some improvements on the $L_p$ inequalities for diffusion processes and, as applications, obtain some new $L_p$ inequalities for the Ornstein-Uhlenbeck processes, Bessel processes and the reflected Brownian motion with drift.
2. Main results and proofs

Let $F(x)$ be the solution of the equation (3)
\[ \sigma^2(x) \frac{d^2y}{dx^2} + 2\mu(x) \frac{dy}{dx} = 2\varphi(x), \quad y(0) = 0, y'(0) = 0. \]

Then
\[ F(x) = \int_0^x e^{-\int_0^t \frac{2\mu(u)}{\sigma^2(u)} du} dt \int_0^t \frac{2\varphi(s)}{\sigma^2(s)} e^{\int_0^s \frac{2\mu(u)}{\sigma^2(u)} du} ds. \quad (7) \]

Since $\mu(x), \sigma(x), \varphi(x)$ are continuous functions on $R$ and $\varphi(x) \geq 0$, $F(x)$ is a continuous increasing function on $R_+$ with $F(0) = 0$.

If $\mu(x)$ is an odd function, $\sigma^2(x)$ and $\varphi(x)$ are even functions, $F(x)$ is an even function.

Denote by $H(x) = F^{-1}(x)$ the inverse of $F(x)$ on $R_+$. Let $H_p(x) = [H(x)]^p = [F^{-1}(x)]^p$, for $p > 0$. Then $H_p(x)$ is a continuous increasing function from $R_+$ to $R_+$ with $H_p(0) = 0$ and for $x \geq 0$, $p > 0$

\[ H_p[F(x)] = x^p. \]

For $x \geq 0$, define the function $\tilde{H}_p(x)$ by
\[ \tilde{H}_p(x) = H_p(x) + x \int_x^\infty \frac{dH_p(u)}{u}, \quad p > 0. \]

We have the following inequality.

**THEOREM 1.** Let $X$ be a diffusion process given by (1), starting at zero, $F(x)$ be the solution of the equation (3), $\mu(x), \sigma(x), \varphi(x)$ be continuous functions and $\varphi(x) \geq 0$, $J_t = \int_0^t \varphi(X_s) ds$. If either $X_t \geq 0$ or the function $F(x)$ is an even function and
\[ \tilde{H}_p(x) \leq C_p H_p(x), \quad x \geq 0, \quad (8) \]

for some $p > 0$, then for $0 < \lambda \leq 1$
\[ \frac{1}{\lambda C_p + 1} E \left[ H_p(\lambda J_\tau) \right] \leq E \left[ (X^+_\tau)^p \right] \leq (\lambda C_p + 1) E \left[ H_p \left( \frac{J_\tau}{\lambda} \right) \right] \quad (9) \]
hold for any stopping time $\tau$, where $C_p$ is a constant.

**Proof.** Since $F(x)$ is the solution of the equation (3), $X_t \geq 0$ or the function $F(x)$ is an even function, by Itô formula we have
\[ F(|X_t|) = F(X_t) = F(X_0) + \int_0^t \varphi(X_s) ds + \int_0^t \sigma(X_s) F'(X_s) dB_s. \]

By the optional sampling theorem of martingale theory, we get
\[ E \left[ F(|X_\tau|) \right] = E \left[ \int_0^\tau \varphi(X_s) ds \right] = E \left( J_\tau \right), \quad (10) \]
for any bounded stopping time $\tau$. This shows that $F(|X_t|)$ is dominated by $J_t$ and $J_t$ is dominated by $F(X_t^\ast)$. By Lemma 1, we have

$$E[(X_t^\ast)^p] = E[H_p(F(X_t^\ast))^p] \leq E\left[(H_p + \lambda \tilde{H}_p)\left(\frac{J_t}{\lambda}\right)\right] \leq (\lambda C_p + 1) E\left[H_p\left(\frac{J_t}{\lambda}\right)\right],$$

for $0 < \lambda \leq 1$. By the same approach for $X_t = \lambda J_t$ and $A_t = \tilde{\lambda} F(X_t^\ast)$ in Lemma 1, we have

$$E[H_p(\lambda J_t)] \leq E\left[H_p + \lambda \tilde{H}_p\right] (F(X_t^\ast)) \leq (\lambda C_p + 1) E[(X_t^\ast)^p].$$

This completes the proof of Theorem 1. \(\square\)

By Theorem 1, we can establish some $L_p$ inequalities for diffusion processes satisfying the inequality (8) and study the constants in these $L_p$ inequalities as $p \to 0^+$. 

2.1. \(\varphi(x) = N x^\nu \sigma^2(x)\)

**Theorem 2.** Let $X$ be a diffusion process given by (1), starting at zero, $\mu(x)$ and $\sigma(x)$ be continuous functions with $x|\mu(x)| = K \sigma^2(x)$, $\varphi(x) = N x^\nu \sigma^2(x)$ ($\nu > -1$), for some constants $K > 0$, $N > 0$ and $\mu(x) \geq 0$ for $x \geq 0$ or $\mu(x) \leq 0$ and $\nu + 1 > 2K$. $J_t = \int_0^t \varphi(X_s) \, ds$. If $X_t \geq 0$ or $F(x)$ is even, then for $0 < p < 2 + \nu$ and any stopping time $\tau$, we have

$$\frac{a}{c_{p,v}} \left\| J_t^{\frac{1}{2+v}} \right\|_p \leq \|X_t\|_p \leq ac_{p,v} \left\| J_t^{\frac{1}{2+v}} \right\|_p,$$

where $a$ is an absolute constant and

$$c_{p,v} = \left(\frac{2 + \nu}{2 + \nu - p}\right)^{\frac{1}{p}} \left(\frac{2 + \nu}{p}\right)^{\frac{1}{2+v}} = O\left(\frac{1}{p^{\frac{1}{2+v}}}\right), \quad p \to 0^+.$$

**Proof.** Let $F(x)$ be the solution of the equation (3)

$$\sigma^2(x) \frac{d^2 y}{dx^2} + 2 \mu(x) \frac{dy}{dx} = 2 \varphi(x), y(0) = y'(0) = 0.$$

For $\mu(x) \geq 0$, from (7) we have

$$F(x) = \frac{2N}{(2K + \nu + 1)(2 + \nu)} x^{2 + \nu},$$

$$H_p = \left(\frac{(2K + \nu + 1)(2 + \nu)}{2N}\right)^{\frac{p}{2+v}} x^{\frac{p}{2+v}}, \quad p > 0,$$

$$\tilde{H}_p(x) = \frac{2 + \nu}{2 + \nu - p} H_p(x), \quad 0 < p < 2 + \nu.$$
For $\mu(x) \leq 0$ and $v + 1 > 2K$,
\[
F(x) = \frac{2N}{(v - 2K + 1)(2 + v)} x^{2 + v},
\]
\[
H_p = \left( \frac{(v - 2K + 1)(2 + v)}{2N} \right)^{\frac{p}{2 + v}} x^{\frac{p}{2 + v}}, \quad p > 0,
\]
\[
\tilde{H}_p(x) = \frac{2 + v}{2 + v - p} H_p(x), \quad 0 < p < 2 + v.
\]
Let
\[
a = \max \left\{ \left( \frac{(2K + v + 1)(2 + v)}{2N} \right)^{\frac{1}{2 + v}}, \left( \frac{(v - 2K + 1)(2 + v)}{2N} \right)^{\frac{1}{2 + v}} \right\}.
\]
By Theorem 1, we obtain
\[
E[(X^*_t)^p] \leq E \left[ \left( \frac{\lambda}{2 + v - p} + 1 \right) \cdot a^p \cdot \left( J_{\tau} \right)^{\frac{p}{2 + v}} \right]
= a^p \cdot \left( \frac{\lambda}{2 + v - p} + 1 \right) \cdot \lambda^{\frac{p}{2 + v}} E \left( J_{\tau}^{\frac{p}{2 + v}} \right).
\]
Let
\[
\phi_{p,v}(\lambda) = \lambda^{\frac{p}{2 + v}} \left( \frac{2 + v}{2 + v - p} + 1 \right),
\]
then $\phi_{p,v}$ is strictly decreasing in $(0, \frac{p}{2 + v})$, and strictly increasing in $(\frac{p}{2 + v}, 1]$. $\phi_{p,v}$ takes its minimum at $\lambda = \frac{p}{2 + v}$ and yields the desired inequality
\[
E[(X^*_t)^p] \leq a^p \cdot \frac{2 + v}{2 + v - p} \cdot \left( \frac{2 + v}{p} \right)^{\frac{p}{2 + v}} E \left( J_{\tau}^{\frac{p}{2 + v}} \right).
\]
For the left hand, by Theorem 1
\[
E \left( a^p \cdot J_{\tau}^{\frac{p}{2 + v}} \right) \leq \lambda^{\frac{p}{2 + v}} \left( \frac{2 + v}{2 + v - p} + 1 \right) E[(X^*_t)^p].
\]
Take $\lambda = \frac{p}{2 + v}$, we get
\[
a^p E \left( J_{\tau}^{\frac{p}{2 + v}} \right) \leq \frac{2 + v}{2 + v - p} \cdot \left( \frac{2 + v}{p} \right)^{\frac{p}{2 + v}} E[(X^*_t)^p].
\]
This completes the proof of Theorem 2. \qed

**Remark 1.** Since
\[
\frac{2 + v}{2 + v - p} \cdot \left( \frac{2 + v}{p} \right)^{\frac{p}{2 + v}} < \phi_{p,v}(1) = \frac{4 + 2v - p}{2 + v - p}
\]
and when $v = 0$, we have
\[
\frac{2}{2-p} \cdot \left( \frac{2}{p} \right)^{\frac{q}{p}} < \frac{4-p}{2-p}.
\]
We obtain some constants smaller than the constants widely used.

2.2. $N_1|x|^\gamma \sigma^2(x) \leq \varphi(x) \leq N_2|x|^\gamma \sigma^2(x)e^{N_3|x|^{\gamma+1}/\gamma+1}$

**Theorem 3.** Let $X$ be a diffusion process given by (1), starting at zero, $\mu(x)$ and $\sigma(x)$ be continuous functions satisfying the condition $S(\gamma,K_1,K_2), N_1|x|^\gamma \sigma^2(x) \leq \varphi(x) \leq N_2|x|^\gamma \sigma^2(x)e^{N_3|x|^{\gamma+1}/\gamma+1}$, for some constants $\gamma > -1, K_i > 0 (i = 1,2), N_i > 0 (i = 1,2,3)$ and $\mu(x) \leq 0$, for $x \geq 0$. $J_t = \int_0^t \varphi(X_s) \, ds$. If $X_t \geq 0$ or $F(x)$ is even, then for $0 < p < 2 + \gamma$ and any stopping time $\tau$, we have
\[
\frac{a_1}{c_{p,\gamma}} \left\| \ln^{\frac{1}{1+\gamma}} \left( 1 + J_{\tau}^{\frac{1}{1+\gamma}} \right) \right\|_p \leq \|X^*_\tau\|_p \leq a_2c_{p,\gamma} \left\| \ln^{\frac{1}{1+\gamma}} \left( 1 + J_{\tau}^{\frac{1}{1+\gamma}} \right) \right\|_p,
\]
where $a_1$ and $a_2$ are absolute constants and
\[
c_{p,\gamma} = \left( \frac{2 + \gamma}{2 + \gamma - p} \right)^{\frac{p}{\gamma}} \cdot \left( \frac{2 + \gamma}{\gamma} \right)^{\frac{1}{2+\gamma}} = O \left( \frac{1}{p^{\frac{1}{\gamma}}} \right), \quad p \to 0^+.
\]

**Proof.** Let $F(x)$ be the solution of the equation (3). Since $\mu(x)$ and $\sigma(x)$ satisfy condition $S(\gamma,K_1,K_2)$ and $\mu(x) \leq 0$, for $x \geq 0$,

\[-K_2x^\gamma \sigma^2(x) \leq \mu(x) \leq -K_1x^\gamma \sigma^2(x).
\]

Yan and Zhu [15] proved that there exist constants $c_i > 0, d_i > 0 (i = 1,2)$ depending only on $\gamma, K_i, N_i$ such that
\[
c_1 \int_0^x t^{1+\gamma}e^{dt^{1+\gamma}} \, dt \leq F(x) \leq c_2 \int_0^x t^{1+\gamma}e^{dt^{1+\gamma}} \, dt \quad (x \geq 0).
\]
From this inequality, we can show that there exist constants $\eta_2 \geq \eta_1 > 0$ depending only on $\gamma$ such that
\[
\left( e^{\eta_1t^{1+\gamma}} - 1 \right)^{\frac{1+\gamma}{2+\gamma}} \leq \int_0^x t^{1+\gamma}e^{dt^{1+\gamma}} \, dt \leq \left( e^{\eta_2t^{1+\gamma}} - 1 \right)^{\frac{1+\gamma}{2+\gamma}}
\]
holds, for all $x \geq 0$. Thus there exist constants $b_2 \geq b_1 > 0$ depending only on $\gamma, K_i, N_i$ such that
\[
\left( e^{b_1t^{1+\gamma}} - 1 \right)^{\frac{1+\gamma}{2+\gamma}} \leq F(x) \leq \left( e^{b_2t^{1+\gamma}} - 1 \right)^{\frac{1+\gamma}{2+\gamma}},
\]
for $x \geq 0$. Hence, for $x \geq 0$ and $0 < p < \infty$, we have
\[
\left( \frac{1}{b_2} \right)^{\frac{p}{p^*}} \ln^{\frac{p}{p^*}} \left( 1 + x^{\frac{1}{2+\gamma}} \right) \leq H_p(x) \leq \left( \frac{1}{b_1} \right)^{\frac{p}{p^*}} \ln^{\frac{p}{p^*}} \left( 1 + x^{\frac{1}{2+\gamma}} \right).
\]
(13)
For the functions of type
\[ H_p(x) = A^{\frac{p}{\gamma}} \ln^{\frac{p}{\gamma}} \left( 1 + Bx^{1+\gamma} \right), \]
for some constants \( A > 0, B > 0 \) and \( 0 < p < \infty \). Let
\[ G_p(x) = \frac{x}{H_p(x)} \int_x^\infty \frac{dH_p(u)}{u}. \]
Elementary calculations show that
\[ 0 \leq G_p(x) \leq \frac{p}{2 + \gamma - p}, \]
for all \( x \geq 0 \) and \( 0 < p < 2 + \gamma \). Hence, we get
\[ \tilde{H}_p(x) = H_p(x) + x \int_x^\infty \frac{dH_p(u)}{u} \leq \frac{2 + \gamma}{2 + \gamma - p} H_p(x), \]
for all \( x \geq 0 \) and \( 0 < p < 2 + \gamma \). By Theorem 1, for \( 0 < \lambda \leq 1 \), we have
\[ E [(X^*_\tau)^p] \leq (\lambda C_p + 1) E \left[ H_p \left( \frac{J_\tau}{\lambda} \right) \right] \]
\[ \leq \left( \lambda \frac{2 + \gamma}{2 + \gamma - p} + 1 \right) \cdot \left( \frac{1}{b_1} \right)^{\frac{p}{\gamma+\gamma}} E \left[ \ln^{\frac{p}{\gamma+\gamma}} \left( 1 + \left( \frac{J_\tau}{\lambda} \right)^{\frac{1+\gamma}{\gamma}} \right) \right] \]
\[ = \lambda^{\frac{p}{\gamma+\gamma}} \left( \frac{2 + \gamma}{2 + \gamma - p} + 1 \right) \cdot \left( \frac{1}{b_1} \right)^{\frac{p}{\gamma+\gamma}} E \left[ \ln^{\frac{p}{\gamma+\gamma}} \left( 1 + J_\tau^{\frac{1+\gamma}{\gamma}} \right) \right]. \]
Let
\[ a_2 = \left( \frac{1}{b_1} \right)^{\frac{1}{\gamma+\gamma}}, \quad \phi_{p,\gamma}(\lambda) = \lambda^{\frac{p}{\gamma+\gamma}} \left( \lambda \frac{2 + \gamma}{2 + \gamma - p} + 1 \right), \]
then \( \phi_{p,\gamma} \) takes its minimum at \( \lambda = \frac{p}{2 + \gamma} \) and yields the desired inequality
\[ E [(X^*_\tau)^p] \leq a_2^p \cdot \frac{2 + \gamma}{2 + \gamma - p} \cdot \left( \frac{2 + \gamma}{p} \right)^{\frac{p}{\gamma+\gamma}} E \left[ \ln^{\frac{p}{\gamma+\gamma}} \left( 1 + J_\tau^{\frac{1+\gamma}{\gamma}} \right) \right]. \]
For the left hand, by Theorem 1
\[ E \left[ H_p \left( \lambda J_\tau \right) \right] \leq (\lambda C_p + 1) E [(X^*_\tau)^p]. \]
From (13), we have
\[ \left( \frac{1}{b_2} \right)^{\frac{p}{\gamma+\gamma}} E \left[ \ln^{\frac{p}{\gamma+\gamma}} \left( 1 + J_\tau^{\frac{1+\gamma}{\gamma}} \right) \right] \leq \lambda^{\frac{p}{\gamma+\gamma}} \left( \frac{2 + \gamma}{2 + \gamma - p} + 1 \right) E [(X^*_\tau)^p]. \]
Let \( a_1 = \left( \frac{1}{b_2} \right)^{\frac{1}{\gamma+\gamma}} \) and take \( \lambda = \frac{p}{2 + \gamma} \), we get
\[ a_1^p E \left[ \ln^{\frac{p}{\gamma+\gamma}} \left( 1 + J_\tau^{\frac{1+\gamma}{\gamma}} \right) \right] \leq \frac{2 + \gamma}{2 + \gamma - p} \cdot \left( \frac{2 + \gamma}{p} \right)^{\frac{p}{\gamma+\gamma}} E [(X^*_\tau)^p]. \]
This completes the proof of Theorem 3. \( \square \)
2.3. \( N_1 |x|^{\gamma - 1} \sigma^2(x) \leq \varphi(x) \leq N_2 |x|^{\gamma - 1} \sigma^2(x) e^{N_3 |x|^{\gamma + 1}/\gamma + 1} \)

As in the proof of Theorem 3, we can give the following theorem.

**THEOREM 4.** Let \( X \) be a diffusion process given by (1), starting at zero, \( \mu(x) \) and \( \sigma(x) \) be continuous functions satisfying the condition \( S(\gamma, K_1, K_2), N_1 |x|^{\gamma - 1} \sigma^2(x) \leq \varphi(x) \leq N_2 |x|^{\gamma - 1} \sigma^2(x) e^{N_3 |x|^{\gamma + 1}/\gamma + 1} \), for some constants \( \gamma > 0, K_i > 0 (i = 1, 2), N_i > 0 (i = 1, 2, 3) \) and \( \mu(x) \leq 0, \) for \( x \geq 0, J_t = \int_0^t \varphi(X_s) \, ds. \) If \( X_t \geq 0 \) or \( F(x) \) is even, then for \( 0 < p < 1 + \gamma \) and any stopping time \( \tau, \) we have

\[
\frac{a_1}{c_{p, \gamma}} \left\| \ln \frac{1}{1 + J_\tau} \right\|_p \leq \| X^*_\tau \|_p \leq a_2 c_{p, \gamma} \left\| \ln \frac{1}{1 + J_\tau} \right\|_p,
\]

where \( a_1 \) and \( a_2 \) are absolute constants and

\[
c_{p, \gamma} = \left( \frac{1 + \gamma}{1 + \gamma - p} \right)^{1/p} \left( \frac{1 + \gamma}{p} \right)^{1/\gamma} = O \left( \frac{1}{p^{\gamma/\gamma}} \right), \quad p \to 0^+.
\]

If \( F(x) \) is not even, by the method of Peskir[9], define the functions \( F_1(x) \) and \( F_2(x) \) on \( R_+ \) as follows

\[
F_1(x) = \max \{ F(-x), F(x) \}, \quad F_2(x) = \min \{ F(-x), F(x) \}
\]

and let \( H_i(x) = F_i^{-1}(x) \) be the inverse of \( F_i(x), \) for \( x \geq 0, \) \( H_i p(x) = [F_i^{-1}(x)]^p, \) for \( p > 0 (i = 1, 2). \)

\[
\tilde{H}_i p(x) = H_i p(x) + x \int_x^\infty \frac{dH_i p(u)}{u}, \quad p > 0.
\]

As in the proof of Theorem 1, we can give the following inequality.

**THEOREM 5.** Let \( X \) be a diffusion process given by (1), starting at zero, \( F(x) \) be the solution of equation (3), \( \mu(x), \sigma(x), \varphi(x) \) be continuous functions and \( \varphi(x) \geq 0, J_t = \int_0^t \varphi(X_s) \, ds. \) If

\[
\tilde{H}_i p(x) \leq C_{ip} H_i p(x), \quad i = 1, 2,
\]

for some \( p > 0 \) and all \( x \geq 0. \) Then for \( 0 < \lambda \leq 1, \) we have

\[
\frac{1}{\lambda C_{1p} + 1} E [H_{1p}(\lambda J_\tau)] \leq E [(X^*_\tau)^p] \leq (\lambda C_{2p} + 1) E \left[ H_{2p} \left( \frac{J_\tau}{\lambda} \right) \right],
\]

for any stopping time \( \tau, \) where \( C_{1p} \) and \( C_{2p} \) are constants.

As in the Theorem 2, Theorem 3 and Theorem 4, similar inequalities can also be established.
3. Applications

3.1. $L_p$ inequalities for the Ornstein-Uhlenbeck process

Let $V = (V_t)_{t \geq 0}$ be an Ornstein-Uhlenbeck velocity process solving the Langevin equation

$$dV_t = -\beta V_t dt + dB_t,$$

(18)

with $V_0 = 0$, where $\beta > 0$ and $B = (B_t)_{t \geq 0}$ is a standard Brownian motion. $\mu(x) = -\beta x$ is an odd function and $\sigma^2(x) = 1$ is an even function. The Ornstein-Uhlenbeck process satisfies the condition $S(1, \beta, \beta)$. Graversen and Peskir [7] introduced the functional

$$I_t = \int_0^t e^{\beta V_t^2} dr$$

and established the following Davis-type inequality

$$\frac{A_1}{\sqrt{\beta}} E \left[ \sqrt{\ln (1 + \beta I_t)} \right] \leq E \left( \sup_{0 \leq r \leq t} |V_r| \right) \leq \frac{A_2}{\sqrt{\beta}} E \left[ \sqrt{\ln (1 + \beta I_t)} \right],$$

(19)

for any stopping time $\tau$ with $A_1 = \frac{1}{3}$ and $A_2 = 3$.

Take $\phi(x) = e^{\beta x^2}$, then $F(x) = \frac{1}{\beta} \left( e^{\beta x^2} - 1 \right)$ is an even function. By Theorem 3, with more accurate calculation, for $0 < p < 2$ and any stopping time $\tau$, we have

$$\frac{1}{c_p \sqrt{\beta}} \left\| \ln^{\frac{1}{p}} (1 + \beta I_\tau) \right\|_p \leq \left\| V_\tau^p \right\|_p \leq \frac{c_p}{\sqrt{\beta}} \left\| \ln^{\frac{1}{p}} (1 + \beta I_\tau) \right\|_p,$$

(20)

with $c_p = \left( \frac{2}{2-p} \right)^{\frac{1}{p}} \sqrt{\frac{2}{p}} = O \left( \frac{1}{\sqrt{p}} \right)$, as $p \to 0^+$.

If $p = 1$, we get the inequality of Graversen and Peskir (19) with smaller constants $A_1 = \frac{1}{2\sqrt{2}}$ and $A_2 = 2\sqrt{2}$.

Take $\phi(x) = 3\beta^2 |x| e^{\beta x^2}$, $J_t = \int_0^t \phi(X_s) ds$. Then $F(x)$ is an even function and for $x \geq 0$

$$F(x) = 3\beta^2 \int_0^x t^2 e^{\beta t^2} dt,$$

$$\left( e^{2\beta x^2} - 1 \right)^{\frac{3}{2}} \leq F(x) \leq \left( e^{\beta x^2} - 1 \right)^{\frac{3}{2}}.$$

By Theorem 3, for $0 < p < 2$ and any stopping time $\tau$, we have

$$\frac{1}{c_p \sqrt{\beta}} \left\| \ln^{\frac{1}{p}} \left( 1 + J_\tau^{\frac{3}{2}} \right) \right\|_p \leq \left\| V_\tau^p \right\|_p \leq \sqrt{\frac{3}{2}} \frac{c_p}{\sqrt{\beta}} \left\| \ln^{\frac{1}{p}} \left( 1 + J_\tau^{\frac{3}{2}} \right) \right\|_p,$$

(21)

with $c_p = \left( \frac{3}{3-p} \right)^{\frac{1}{p}} \left( \frac{3}{p} \right)^{\frac{1}{3}} = O \left( \frac{1}{\sqrt{p}} \right)$, as $p \to 0^+$. 
3.2. $L_p$ inequalities for Bessel processes

Let $\delta \geq 0$ and $x \geq 0$. The unique strong solution of the stochastic differential equation

$$dY_t = \delta dt + 2\sqrt{|Y_t|}dB_t, \quad Y_0 = x$$

(22)

is called a squared Bessel process of dimension $\delta$, started at $x$ and the process $Z = \sqrt{|Y|}$ is called a Bessel process of dimension $\delta$.

For the squared Bessel process $Y$ and Bessel process $Z$ started at 0, Yan and Zhu[15] established the following inequalities

$$\frac{\delta}{a_p} \|\tau\|_p \leq \|Y^*_\tau\|_p \leq a_p\delta \|\tau\|_p, \quad 0 < p < 1,$$

with $a_p = \left(\frac{2-p}{1-p}\right)^{\frac{1}{p}} = O\left(\frac{1}{p}\right)$ as $p \to 0^+$.

$$\frac{\sqrt{\delta}}{b_p} \|\sqrt{\tau}\|_p \leq \|Z^*_\tau\|_p \leq b_p\sqrt{\delta} \|\sqrt{\tau}\|_p, \quad 0 < p < 2,$$

with $b_p = \left(\frac{4-p}{2-p}\right)^{\frac{1}{p}} = O\left(\frac{1}{p}\right)$ as $p \to 0^+$.

Since $Y$ and $Z$ are positive processes, we get the following inequalities from Theorem 2 with $\phi(x) = 1$

$$\frac{\delta}{c_p} \delta \|\tau\|_p \leq \|Y^*_\tau\|_p \leq c_p\delta \|\tau\|_p, \quad 0 < p < 1,$$

(23)

with $c_p = \left(\frac{1}{1-p}\right)^{\frac{1}{p}} = O\left(\frac{1}{p}\right)$ as $p \to 0^+$.

$$\frac{\sqrt{\delta}}{d_p} \|\sqrt{\tau}\|_p \leq \|Z^*_\tau\|_p \leq d_p\sqrt{\delta} \|\sqrt{\tau}\|_p, \quad 0 < p < 2,$$

(24)

$$d_p = \left(\frac{2}{2-p}\right)^{\frac{1}{p}} \sqrt{\frac{2}{p}} = O\left(\frac{1}{\sqrt{p}}\right) \text{ as } p \to 0^+.$$

**REMARK 2.** The constants obtained by Lenglart domination inequalities are:

$$a_p = \left(\frac{2-p}{1-p}\right)^{\frac{1}{p}} = \left(2 + \frac{p}{1-p}\right)^{\frac{1}{p}} = O\left(2^\frac{1}{p}\right), \quad p \to 0^+,$$

$$b_p = \left(\frac{4-p}{2-p}\right)^{\frac{1}{p}} = \left(2 + \frac{p}{2-p}\right)^{\frac{1}{p}} = O\left(2^\frac{1}{p}\right), \quad p \to 0^+.$$

The growth rate of $a_p$ and $b_p$, as $p \to 0^+$, are the exponential of $\frac{1}{p}$.

**REMARK 3.** The constants we obtained are $O\left(\frac{1}{p}\right)$ or $O\left(\frac{1}{\sqrt{p}}\right)$. The growth rates of constants as $p \to 0^+$ are substantially improved.
3.3. \( L_p \) inequalities for reflected Brownian motion with drift

Let \( X = (X_t)_{t \geq 0} \) be the strong solution of the SDE

\[
dX_t = -\mu \text{sgn}(X_t) \, dt + dB_t, \quad X_0 = 0, \tag{25}
\]

where \( \mu > 0 \) and \( B = (B_t) \) is a standard Brownian motion. \( |X| = (|X_t|)_{t \geq 0} \) is a realization of the reflected Brownian motion with drift \( -\mu \). \( \mu(x) = -\mu \text{sgn}(x) \) and \( \sigma(x) = 1 \) satisfies the condition \( S(0, \mu, \mu) \). Take \( \varphi(x) = 1 \), then \( F(x) \) is an even function and for \( x \geq 0 \)

\[
F(x) = \frac{1}{2\mu^2} (e^{2\mu x} - 2\mu x - 1).
\]

Since

\[
\frac{1}{\mu^2} \left( e^{\frac{\mu x}{2}} - 1 \right)^2 \leq F(x) \leq \frac{1}{\mu^2} (e^{\mu x} - 1)^2,
\]

\[
\frac{1}{\mu} \ln (1 + \mu \sqrt{x}) \leq H(x) \leq \frac{2}{\mu} \ln (1 + \mu \sqrt{x}).
\]

As in the proof of Theorem 3, we can obtain the following inequality

\[
\frac{1}{\mu c_p} \left\| \ln (1 + \mu \sqrt{\tau}) \right\|_p \leq \left\| X^*_\tau \right\|_p \leq \frac{2c_p}{\mu} \left\| \ln (1 + \mu \sqrt{\tau}) \right\|_p, \quad 0 < p < 2,
\tag{26}
\]

for any stopping time \( \tau \) of \( X \). And \( c_p = \left( \frac{2}{\sqrt{p}} \right)^{\frac{1}{2}} \sqrt{\frac{2}{p}} = O \left( \frac{1}{\sqrt{p}} \right) \) as \( p \to 0^+ \).

For \( p = 1 \), we get the following inequality

\[
\frac{1}{2\sqrt{2\mu}} E \left[ \ln (1 + \mu \sqrt{\tau}) \right] \leq E \left( X^*_\tau \right) \leq \frac{4\sqrt{2}}{\mu} E \left[ \ln (1 + \mu \sqrt{\tau}) \right].
\tag{27}
\]

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