Beurling’s theorem for the Hardy operator on $L^2[0, 1]$

Jim Agler and John E. McCarthy

Abstract. We prove that the invariant subspaces of the Hardy operator on $L^2[0, 1]$ are the spaces that are limits of sequences of finite dimensional spaces spanned by monomial functions.

Keywords. Beurling’s theorem, Monomial operators, Invariant subspaces, Hardy operator.

1. Introduction

The space $L^2[0, 1]$ is a cornerstone of analysis. One way to analyze it is to use the exponential functions $e^{itx}$, which have the advantage of being eigenfunctions for differentiation. Another way is to use the monomial functions $x^s$. The Müntz–Szász theorem gives necessary and sufficient conditions for a collection of monomial functions to span $L^2[0, 1]$. Monomials are eigenfunctions for the Hardy operator $H$, defined by

$$Hf(x) = \frac{1}{x} \int_0^x f(t) dt.$$ 

Conversely, if $T$ is a bounded linear operator on $L^2[0, 1]$ that has $x^s$ as an eigenvector whenever $x^s$ is in $L^2[0, 1]$, then $T$ is a function of $H$; specifically, it is of the form $\phi(H)$ for some function $\phi$ that is bounded and analytic on the disk $D(1, 1) = \{z \in \mathbb{C} : |z - 1| < 1\}$ [4].

We shall use $L^2$ to denote $L^2[0, 1]$ throughout. Hardy proved in [9] that $H$ is bounded on $L^2$ (and indeed on $L^p$ for all $p > 1$). For a treatment of $H$ consult the book [13]. What are its invariant subspaces?

Let $\mathbb{S}$ denote the half plane $\{s \in \mathbb{C} : \text{Re}(s) > -\frac{1}{2}\}$. Then if $s \in \mathbb{S}$, the monomial function $x^s$ is in $L^2$, and $Hx^s = \frac{1}{s+1}x^s$; moreover the monomials constitute all the eigenvectors of $H$. Any space that is the linear span of finitely many monomial functions is invariant for $H$. We shall call such a space a finite monomial space. It is the object of this note to prove that every invariant subspace of $H$ is a limit of finite monomial spaces.

Partially supported by National Science Foundation Grant DMS 2054199.
The Hardy operator is unitarily equivalent to $1 - S^*$, where $S$ is the unilateral shift [7]. Its invariant subspaces are therefore described by the celebrated theorem of Beurling [5] which described the invariant subspaces of the shift using the beautiful theory of Hardy spaces of holomorphic functions. Using this theory, Theorem 1.4 below is well-known. It is proved as the Theorem on Finite Dimensional Approximation [16, p.37]. However, the point of this note is to describe the invariant subspaces of $H$ without using any Hardy space theory, just using $L^2$ techniques and functional analysis. Our hope is that this approach will not only illuminate $L^2$ with a new light, but may also generalize to related spaces, such as $L^p$ or weighted $L^p$ spaces.

**Definition 1.1.** For $S$ a finite subset of $S$ we let $\mathcal{M}(S)$ denote the span in $L^2$ of the monomials whose exponents lie in $S$, i.e.,

$$\mathcal{M}(S) = \left\{ \sum_{s \in S} a(s)x^s \mid a : S \to \mathbb{C} \right\}.$$  

We refer to sets in $L^2$ that have the form $\mathcal{M}(S)$ for some finite subset $S$ of $S$ as finite monomial spaces.

**Definition 1.2.** If $\mathcal{M}$ is a subspace of a Hilbert space $H$ and $\{\mathcal{M}_n\}$ is a sequence of closed subspaces, we say that $\{\mathcal{M}_n\}$ tends to $\mathcal{M}$ and write $\mathcal{M}_n \to \mathcal{M}$ as $n \to \infty$ if

$$\mathcal{M} = \{ f \in H \mid \lim_{n \to \infty} \text{dist}(f, \mathcal{M}_n) = 0 \}.$$  

**Definition 1.3.** We say that a subspace $\mathcal{M}$ of $L^2$ is a monomial space if there exists a sequence $\{\mathcal{M}_n\}$ of finite monomial spaces such that $\mathcal{M}_n \to \mathcal{M}$.

Equipped with these definitions, we can now state our main theorem.

**Theorem 1.4.** Let $\mathcal{M}$ be a closed non-zero subspace of $L^2$. Then $\mathcal{M}$ is invariant for $H$ if and only if $\mathcal{M}$ is a monomial space.

One way to construct a monomial space is to take the closed linear span of an infinite set of monomial functions,

$$\mathcal{M} = \vee\{x^{s_k} : k \in \mathbb{N}\}. \quad (1.5)$$

The Müntz–Szász theorem (proved in [15,18] for integer exponents, and in [19] for general real exponents) characterizes when such a space is a proper subspace of $L^2$. See [6] for a thorough treatment.

**Theorem 1.6.** (Müntz–Szász)

$$\vee\{x^{s_k} : k \in \mathbb{N}\} = L^2 \quad \text{if and only if} \quad \sum_k \frac{2\text{Re } s_k + 1}{|s_k + 1|^2} = \infty.$$
Not every monomial space looks like (1.5). It is easy to see that for any $0 < s < 1$, the space $\{f \in L^2 : f = 0 \text{ a.e. on } [0, s]\}$ is invariant for $H$, and hence is a monomial space. (For an explicit construction of finite monomial spaces that converge to this subspace, see [3].)

Our goal is to give a real analysis proof of Theorem 1.4. To do this, we first need some preliminary results. In Sect. 2 we state two theorems about Hilbert spaces that we will use. The first, due to von Neumann in 1929, describes isometries on a Hilbert space. The second, due to Quiggin in 1993, gives a sufficient condition to extend partially defined multipliers of a reproducing kernel Hilbert space without increasing the norm. We apply Quiggin’s theorem to the commutant of the Hardy operator in Sect. 4.

In Sect. 3 we describe the Laguerre basis for $L^2$, the basis obtained by evaluating the Laguerre polynomials on $\log \frac{1}{x}$, which are also the functions obtained by applying $(1 - H^*)^n$ to the constant function 1. In Sect. 5 we deal with multiplicity; this corresponds to generalizing the notion of finite monomial space to allow not just monomials $x^s$, but also functions of the form $(\log x)^m x^s$. In Sect. 6 we prove that certain rational functions are cyclic for $H^*$. Finally in Sect. 7 we prove Theorem 1.4. Our strategy to prove that an invariant subspace $M$ of $H$ is a monomial space is to look at the projection $\eta$ of the constant function 1 onto $M^\perp$, and show that the function $\eta$ uniquely characterizes $M$. We then approximate $\eta$ by functions that arise in a similar way from finite monomial spaces, and show that this proves that the finite monomial spaces converge to $M$ in the sense of Definition 1.2.

2. Some results from operator theory

An operator $V$ defined on a Hilbert space $\mathcal{H}$ is called an isometry if it preserves norms; a co-isometry is the adjoint of an isometry. An isometry $V$ is called pure if $\cap_{n=0}^{\infty} \text{ran}(V^*)^n = 0$. The von Neumann-Wold decomposition describes the structure of isometries [20, 21]. We state it not in its most general form, but in a way that will be useful below.

**Theorem 2.1.** (von Neumann–Wold)

(i) Every isometry is the direct sum of a unitary operator and a pure isometry.

(ii) If $V$ is a pure isometry on the space $\mathcal{H}$, and $M = \ker V^*$, then $\mathcal{H} = \vee\{V^j m : m \in M\}$. The dimension of $M$ is called the multiplicity of $V$.

(iii) The spaces $V^j M$ form an orthogonal decomposition of $\mathcal{H}$.

(iv) If $V$ is a pure isometry of multiplicity 1 and $f$ is any non-zero vector in $\mathcal{H}$ then

$$\mathcal{H} = \vee\{V^i f, (V^*)^j f : i, j \geq 0\}.$$

We shall also need a result on extending the adjoints of multiplication operators, due to Quiggin [17]. We say that a sesquilinear form $\ell(x, y)$ has one positive square if for any finite set of points $\{x_1, \ldots, x_N\}$, the self-adjoint $N$-by-$N$ matrix $\ell(x_i, x_j)$ has one positive eigenvalue.
Theorem 2.2. (Quiggin): Let \((\mathcal{H}, k)\) be a reproducing kernel Hilbert space on a set \(X\). A sufficient condition that every bounded operator \(T\) defined on \(\mathcal{H}\) for some subset \(X_0 \subseteq X\) that has the form

\[Tk_x = \alpha(x)k_x, \quad x \in X_0\]

extend to a bounded operator \(\tilde{T}: \mathcal{H} \to \mathcal{H}\) that has the form

\[\tilde{T}k_x = \tilde{\alpha}(x)k_x, \quad x \in X_0\]

and satisfies \(\|\tilde{T}\| = \|T\|\) is that the reciprocal \(\frac{1}{k(x,y)}\) has exactly one positive square.

In the form stated, the converse to Quiggin’s theorem is not true. However, if one requires norm-preserving extensions in the vector-valued case too, then the condition that \(\frac{1}{k(x,y)}\) has one positive square is both necessary and sufficient. This was proved by McCullough [14] in a different context, and put in a unified context in [1]. See also the paper by Knese [11] for an elegant proof of necessity, and [2] for a discussion in a book.

3. The Laguerre basis for \(L^2\)

The following identity is a special case of one in [10]. In our case, it is easily proved by checking on polynomials; see e.g. [3].

Lemma 3.1. Let \(f \in L^2\). Then

\[\|f\|^2 = \|(1 - H)f\|^2 + |\int_0^1 f(x)dx|^2\]

Consequently, \(1 - H\) is a co-isometry with a one dimensional kernel, which consists of the constant functions. As

\[(1 - H)^k x^n = \left(\frac{n}{n + 1}\right)^k x^n,\]

we see that \(1 - H^*\) is a pure isometry of multiplicity 1. Let us state this for future use.

Proposition 3.2. (Brown, Halmos, Shields) The operator \((1 - H^*)\) is a pure isometry of multiplicity one.

Proposition 3.2 was first proved in [7]. If we apply powers of \((1 - H^*)\) to the constant function 1, we get a useful orthonormal basis. This was first found explicitly in [7], and developed further in [12].

Lemma 3.3.

\[(H^*)^j 1 = (-1)^j \frac{(\log x)^j}{j!}\]  \hspace{1cm} (3.4)
Proof. We proceed by induction. Clearly, (3.4) holds when \( j = 0 \). Assume \( j \geq 0 \) and (3.4) holds. Then

\[
(H^*)^{j+1} 1 = H^*((H^*)^j 1)
\]

\[
= (-1)^j \frac{1}{j!} H^*(\log x)^j
\]

\[
= (-1)^j \frac{1}{j!} \int_x^1 \frac{(\log t)^j}{t} dt
\]

\[
= (-1)^j \frac{1}{j!} \int_{\log x}^0 u^j du
\]

\[
= (-1)^{j+1} \frac{(\log x)^{j+1}}{(j + 1)!}.
\]

\[\square\]

Lemma 3.5.

\[
(1 - H^*)^n 1 = \sum_{j=0}^n \binom{n}{j} \frac{(\log x)^j}{j!}
\]

Proof. By Lemma 3.3,

\[
(1 - H^*)^n 1 = \sum_{j=0}^n (-1)^j \binom{n}{j} (H^*)^j 1
\]

\[
= \sum_{j=0}^n (-1)^j \binom{n}{j} ((-1)^j \frac{(\log x)^j}{j!})
\]

\[
= \sum_{j=0}^n \binom{n}{j} \frac{(\log x)^j}{j!}.
\]

\[\square\]

From Lemma 3.5 and part (iii) of the von Neumann–Wold theorem, it follows that the functions

\[
e_n(x) = \sum_{j=0}^n \binom{n}{j} \frac{(\log x)^j}{j!}
\]

are orthonormal. To see that they are complete, note that their closed linear span \( \mathcal{M} \) is invariant under \( H^* \) and contains the function 1. Since the constant functions are the kernel of the pure co-isometry \( 1 - H \), this means \( \mathcal{M} = L^2 \).
by the von Neumann–Wold Theorem 2.1. So we have proved the following result, which was first proved in [7] and [12].

**Theorem 3.7.** (Brown, Halmos, Shields) The functions $e_n$ defined by (3.6) form an orthonormal basis for $L^2$.

The Laguerre polynomials are the polynomials

$$p_n(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x)^j j!.$$  

These are orthogonal polynomials for $L_2[0, \infty)$ with the weight function $e^{-x}$. As $e_n(x) = p_n(\log \frac{1}{x})$, the change of variables $t = \log \frac{1}{x}$ is an alternative way to prove that $e_n$ are orthonormal.

The functions $e_n$ are generalized eigenvectors of $H$ at 1. Later we shall need the following.

**Proposition 3.8.** Let $s \in S$. The $(n + 1)^{st}$ generalized eigenvector of $H$ with eigenvalue $\frac{1}{s + 1}$ is in the linear span of \{ $x^s, (\log x)x^s, \ldots, (\log x)^nx^s$ \}.

**Proof.** We want to prove

$$\text{Ker} \left( H - \frac{1}{s + 1} \right)^{n+1} = \vee \{ x^s, (\log x)x^s, \ldots, (\log x)^nx^s \}. \tag{3.9}$$

This is true when $n = 0$, since

$$Hx^s = \frac{1}{s + 1}x^s. \tag{3.10}$$

Differentiate both sides of (3.10) with respect to $s$. We get

$$H(\log x)x^s = \frac{1}{s + 1}(\log x)x^s - \frac{1}{(s + 1)^2}x^s. \tag{3.11}$$

Now we proceed by induction. The inductive hypothesis is that

$$H(\log x)^nx^s = \frac{1}{s + 1}(\log x)^nx^s + \sum_{j=0}^{n-1} c_j(s)(\log x)^j x^s \tag{3.12}$$

for some functions $c_j$. We have proved (3.12) for $n = 0$ and 1. (The $n = 1$ case we proved just for expositional clarity). Assume the hypothesis holds up to $n$. Differentiate (3.12) with respect to $s$ and we get

$$H(\log x)^{n+1}x^s = \frac{1}{s + 1}(\log x)^{n+1}x^s - \frac{1}{(s + 1)^2}(\log x)^nx^s$$

$$+ \sum_{j=0}^{n-1} c_j'(s)(\log x)^j x^s + c_j(s)(\log x)^{j+1}x^s.$$ 

Thus by induction, (3.12) holds for all $n$, and hence so does (3.9).
4. Commutant lifting for the Hardy operator

Suppose $T : L^2 \rightarrow L^2$ commutes with $H$. Then it must have the same eigenvectors, and so be a monomial operator of the form

$$T : x^s \mapsto \alpha(s)x^s. \quad (4.1)$$

When is such an operator bounded?

**Theorem 4.2.** The operator $T$ commutes with $H$ and has norm at most $M$ if and only if $T$ is of the form (4.1) and, for any finite set $\{s_i\}_{i=1}^N \subset \mathbb{S}$, the matrix

$$\left( \frac{M^2 - \alpha(s_i)\alpha(s_j)}{1 + \bar{s}_i + s_j} \right)_{i,j=1}^N \quad (4.3)$$

is positive semidefinite.

$T$ may be defined by (4.1) just on some subspace of $L^2$. The positivity of (4.3) on this set is necessary and sufficient to lift $T$ from the span of $\{x^{s_i}\}$ to an operator on all of $L^2$ that commutes with $T$ and has the same norm. Without loss of generality we can take $M = 1$.

**Theorem 4.4.** Suppose that for some subset $S_0 \subseteq \mathbb{S}$ there is an operator

$$T : \vee\{x^s : s \in S_0\} \rightarrow \vee\{x^s : s \in S_0\}$$

$$T : x^s \mapsto \alpha(s)x^s.$$  

A necessary and sufficient condition for $T$ to extend to an operator from $L^2$ to $L^2$ that commutes with $H$ and has norm at most one is that for every finite set $\{s_i\} \subseteq S_0$, we have

$$\left( \frac{1 - \alpha(s_i)\alpha(s_j)}{1 + \bar{s}_i + s_j} \right) \geq 0.$$  

Notice that Theorem 4.2 is a special case of Theorem 4.4, so we shall just prove the latter theorem.

**Proof of Theorem 4.4.** Necessity: We have that $1 - T^*T \geq 0$. Therefore

$$\langle (1 - T^*T)x^{s_i}, x^{s_i} \rangle = \left( \frac{1 - \alpha(s_i)\alpha(s_j)}{1 + \bar{s}_i + s_j} \right)$$

is a positive semi-definite matrix for any subset of $S_0$.

Sufficiency: Suppose that (4.5) is positive semi-definite for every finite subset of $S_0$. Then $T$ commutes with $H|_{\vee\{x^s : s \in S_0\}}$. Let us define a kernel on $\mathbb{S}$ by

$$k(s, t) = \int_0^1 x^t x^s \, dx$$

$$= \frac{1}{1 + t + \bar{s}}.$$
The reciprocal of $k$ is the sesquilinear form

$$\ell(s,t) = \left( \frac{1}{2} + t \right) + \left( \frac{1}{2} + s \right)$$

$$= \frac{1}{2} \left( \frac{3}{2} + \bar{s} \right) \left( \frac{3}{2} + t \right) - \frac{1}{2} \left( \frac{1}{2} - \bar{s} \right) \left( \frac{1}{2} - t \right).$$

So for any $N \geq 2$ the matrix $[\ell(s_i, s_j)]_{i,j=1}^N$ is a rank 2 symmetric matrix, with one positive and one negative eigenvalue. By Theorem 2.2, $T$ extends to an operator of norm 1 on all of $L^2$ that has each $x^s$ as an eigenvector, and hence commutes with $H$.

### 5. Monomial spaces with multiplicity

If one takes the two dimensional monomial spaces $\mathcal{M}(s, s+h)$ and lets $h \to 0$, the spaces converge to the two-dimensional space spanned by $x^s$ and $\frac{\partial}{\partial s} x^s = (\log x)x^s$. So if we have a multi-set $S = \{s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots, s_n\}$, where each $s_j$ appears $m_j$ times, we will define

$$\mathcal{M}(S) = \forall\{x^{s_1}, (\log x)x^{s_1}, \ldots, (\log x)^{m_1-1}x^{s_1}, \ldots, x^{s_n}, (\log x)x^{s_n}, \ldots, (\log x)^{m_n-1}x^{s_n}\}. \quad (5.1)$$

We shall call a set of the form (5.1) a generalized finite monomial space.

**Proposition 5.2.** Every generalized finite monomial space is a limit of finite monomial spaces.

**Proof.** Fix $m \geq 2$. Let

$$\mathcal{M}_1 = \forall\{x^s, (\log x)x^s, \ldots, (\log x)^{m-1}x^s\}.$$

Let $\omega$ be a primitive $m$th root of unity, and let $h$ be a small positive number. Let

$$\mathcal{M}_2 = \forall\{x^{s+\omega^j h} : 0 \leq j \leq m-1\}.$$

We shall prove that there is a constant $C$, which depends on $s$ and $m$ but not $h$, so that

$$f \in \mathcal{M}_1 \Rightarrow \text{dist}(f, \mathcal{M}_2) \leq C\|f\|h$$

$$f \in \mathcal{M}_2 \Rightarrow \text{dist}(f, \mathcal{M}_1) \leq C\|f\|h. \quad (5.4)$$

As every generalized monomial space of the form (5.1) is the sum of finitely many spaces of the form $\mathcal{M}_1$, this will prove the proposition.

In the proof we shall use $C$ for a constant that depends on $m$ but not $h$, and which may change from one line to the next.

Proof of (5.3).

(i) First take $s = 0$. By Taylor’s theorem, for any unimodular number $\tau$ and any $x > 0$ we have

$$|x^{\tau h} - \sum_{n=0}^{m-1} \frac{(\tau h)^n}{n!} (\log x)^{n-1}| \leq \frac{h^m}{m!} (\log x)^m x^{-h}. \quad (5.5)$$
Consider the function \( f(x) = (\log x)^n \), for some \( n \leq m - 1 \). We shall approximate this by the function \( g \in \mathcal{M}_2 \) given by

\[
g(x) = \frac{1}{m} \frac{n!}{h^n} \sum_{j=0}^{m-1} \bar{\omega}^j x^{\omega^j h}.
\]

The choice of arguments for the coefficients means that if one adds together the Taylor series for each \( x^{\omega^j h} \), all the terms cancel except for the ones that are \( n \mod m \) one, so

\[
|g(x) - (\log x)^n| \leq C h^m (\log x)^{m+n} x^{-h} \tag{5.6}
\]

where \( C \) is independent of \( x \). Integrating the square of (5.6) we get that \( \text{dist}((\log x)^n, \mathcal{M}_2) \leq C h^m \). As the functions \((\log x)^n\) form a basis for \( \mathcal{M}_1 \), we deduce that (5.3) holds.

(ii) For general \( s \), the above argument shows that for each function \( x^s (\log x)^n \) there is a function \( g \) in \( \mathcal{M}_2 \) that satisfies the pointwise estimate

\[
|g(x) - x^s (\log x)^n| \leq C h^m (\log x)^{m+n} x^{\text{Re } s - h}.
\]

As long as \( h \) is small enough that \( \text{Re } s - h > -\frac{1}{2} \), we again can deduce (5.3).

Proof of (5.4).

(i) First take \( s = 0 \). From (5.5), we get that \( \text{dist}(x^{\omega^j h}, \mathcal{M}_1) \leq C h^m \). So the result will follow if we prove that whenever \( \sum c_i x^{\omega^i h} \) is in the unit ball of \( \mathcal{M}_2 \), then \( c_i = O(\frac{1}{h^{m-1}}) \). This in turn will follow if we can show that

\[
\text{dist}(x^{\omega^i h}, \mathcal{M}_2) \geq C h^{m-1} \tag{5.7}
\]

for some non-zero \( C \), as this proves that the functions \( x^{\omega^i h} \) are not too colinear. For definiteness, we will prove (5.7) for \( \ell = 0 \). Let \( G(i,j) \) denote the Gram matrix with \((i,j)\) entry \( \langle x^{\omega^i h}, x^{\omega^j h} \rangle = \frac{1}{1 + \omega^i h + \omega^j h} \). Then

\[
\text{dist}(x^h, \mathcal{M}_1) \geq \text{det} G(i,j)_{i,j=0}^{m-1} / \text{det} G(i,j)_{i,j=1}^{m-1} \tag{5.8}
\]

By Cauchy’s formula for determinants

\[
\text{det} \left( \frac{1}{1 + \omega^i h + \omega^j h} \right) = \frac{\prod_{j<i} |\omega^i h - \omega^j h|^2}{\prod_{i,j} (1 + \omega^i h + \omega^j h)}.
\]

Putting this into (5.8), we get

\[
\text{dist}(x^h, \mathcal{M}_1) \geq \frac{h^{2m-2} |\omega| - 1|^2}{(1 + 2h) \prod_{i=1}^{m-1} |1 + (1 + \omega^i) h|^2}.
\]

This equation yields (5.7) for \( \ell = 0 \), and by symmetry for all \( \ell \).

(ii) For general \( s \in \mathbb{S} \), a similar argument gives \( \text{dist}(x^{s+\omega^i h}, \mathcal{M}_1) \leq C h^m \), and

\[
\text{dist}(x^{s+h}, \mathcal{M}_1) \geq \frac{h^{2m-2} |\omega| - 1|^2}{(1 + 2\text{Re } s + 2h) \prod_{i=1}^{m-1} |1 + 2\text{Re } s + (1 + \omega^i) h|^2}.
\]

\( \square \)
With more work, one can improve (5.4) to $O(h^m)$, but we do not need a sharper estimate.

**Corollary 5.9.** Any space that is a limit of generalized finite monomial spaces is a monomial space.

### 6. Some cyclic vectors for $H^*$

We know from Proposition 3.2 that the spectrum of $H$ is $\mathbb{D}(1,1)$, and for $\lambda \in \mathbb{D}(1,1)$ that $H - \lambda$ is Fredholm with index 1. It follows that $1 + sH$ and $1 + sH^*$ are invertible if and only if $s \in \mathbb{S}$.

**Lemma 6.1.** If $s \in \mathbb{S}$, then
\[
x^s = (1 + sH^*)^{-1}1.
\]

**Proof.** We have
\[
\langle (1 + sH^*)x^s, x^t \rangle = \langle x^s, (1 + s\frac{1}{t+1}x^t) \rangle = \frac{1}{t+1} = \langle 1, x^t \rangle.
\]

**Lemma 6.2.** Suppose $f(x) = \sum_{j=0}^N c_j x^{s_j}$, where each $s_j \in \mathbb{S}$. If $f$ is not orthogonal to any monomial $x^t$ for $t \in \mathbb{S}$, then $f$ is cyclic for $H^*$.

**Proof.** By Lemma 6.1, we have
\[
f(x) = \sum_{j=0}^N c_j (1 + s_j H^*)^{-1}1.
\]

Define a rational function $r(z)$ by
\[
r(z) = \sum_{j=0}^N c_j \frac{1}{1 + s_j z},
\]
and let $p, q$ be polynomials with no common factors and $r = p/q$. The zeroes of $q$ are at the points $\{-\frac{1}{s_j} : 1 \leq j \leq N\}$. We have
\[
f = p(H^*)q(H^*)^{-1}1.
\] (6.3)

Claim: $p$ has no roots in $\mathbb{D}(1,1)$.

Indeed, suppose $p(z_0) = 0$ for some $z_0 \in \mathbb{D}(1,1)$. Let $t_0 = \frac{1 - z_0}{z_0} \in \mathbb{S}$. Factor $p$ as $p(z) = (z - z_0)\tilde{p}(z)$. Then
\[
\langle f, x^{t_0} \rangle = \langle (H^* - z_0)\tilde{p}(H^*)q(H^*)^{-1}1, x^{t_0} \rangle
\]
\[
= \langle \tilde{p}(H^*)q(H^*)^{-1}1, (\frac{1}{t_0 + 1} - z_0)x^{t_0} \rangle
\]
\[
= 0.
\]

This would contradict the assumption that $\langle f, x^t \rangle \neq 0$ for all $t \in \mathbb{S}$. 
Since \( q(H^*) \) is invertible, \( f \) is cyclic if and only if \( p(H^*)1 \) is cyclic. We now factor \( p(z) = \sum_{j=0}^{N} c_j x^{s_j} \), where each \( s_j \in S \), and \( f \neq 0 \). Let

\[
T = \{ t \in S : \langle f, x^t \rangle = 0 \}.
\]

Then \( T \) is finite, and we write it as

\[
T = \{ t_1, \ldots, t_m \},
\]

counted with multiplicity. Let \( z_i = \frac{1}{1+t_i} \) for \( 1 \leq i \leq m \). Then

\[
f = \prod_{i=1}^{m} (H^* - z_i)g,
\]

(6.5)

where \( g \) is cyclic for \( H^* \).

**Proof.** Write \( f = p(H^*)q(H^*)^{-1}1 \) as in (6.3). Let \( p^\cup(z) := \frac{p(z)}{z} \). Then

\[
\langle f, x^t \rangle = \langle p(H^*)q(H^*)^{-1}1, x^t \rangle = \langle 1, p^\cup(H)q^\cup(H)^{-1}1 \rangle = \langle 1, \frac{p^\cup(1)}{q^\cup(1)} x^t \rangle = \frac{p(1)}{q(1)} \langle 1, x^t \rangle.
\]

So \( t \in T \) if and only if \( \frac{1}{1+t} \) is a root of \( p \) in \( D(1,1) \), proving that \( T \) is finite, and that the roots of \( p \) that lie in \( D(1,1) \) are exactly the points \( \{ z_i : 1 \leq i \leq m \} \). (Multiplicity is handled by Proposition 3.8). Factor \( p \) as \( p(z) = \prod_{i=1}^{m} (z - z_i) \tilde{p}(z) \), where \( \tilde{p} \) has no roots in \( D(1,1) \). Let \( g = \tilde{p}(H^*)q(H^*)^{-1}1 \). Then \( g \) is cyclic, and (6.5) holds.

Later we will need the next lemma.

**Lemma 6.6.** Let \( z \in D(1,1) \). Then

\[
(H^* - z)[(\bar{z} - 1)H^* - \bar{z}]^{-1}
\]

is an isometry.

**Proof.** This follows by calculation, using the fact that \( 1 - H^* \) is isometric.
7. Proof of Theorem 1.4

Sufficiency is obvious. For necessity, let $\mathcal{M}$ be a proper closed subspace of $L^2$ that is invariant for $H$. We must show that it is a monomial space.

**Lemma 7.1.** Let $\mathcal{M}$ be a finite dimensional subspace of $L^2$, of dimension $n+1$, that is invariant for $H$. Then $\mathcal{M}$ is a generalized finite monomial space, i.e. there exist $n + 1$ points $s_0, \ldots, s_n$, with multiplicity allowed, so that $\mathcal{M} = \vee \{ x^{s_i} : 0 \leq i \leq n \}$.

**Proof.** Consider $H |_{\mathcal{M}}$, which leaves $\mathcal{M}$ invariant. The space $\mathcal{M}$ is spanned by the eigenvectors and generalized eigenvectors of $H$ that lie in $\mathcal{M}$. Suppose the corresponding eigenvalues are $s_j$, with multiplicity $m_j$. By Proposition 3.8, the generalized eigenvectors are of the form $x^{s_j}, (\log x)x^{s_j}, \ldots, (\log x)^{m_j-1}x^{s_j}$. Therefore $\mathcal{M}$ is the generalized finite monomial space corresponding to the exponents $s_j$ with multiplicity $m_j$. □

To prove the full theorem, we use the idea of wandering subspace, due to Halmos [8]. Let $k_0 = \min \{ k : e_k \notin \mathcal{M} \}$. Write $N$ for $\mathcal{M}^\perp$. Write $e_{k_0} = \xi + \eta$, where $\xi \in \mathcal{M}$ and $\eta \in N$. The assumption that $e_{k_0} \notin \mathcal{M}$ means $\eta \neq 0$. Let $u = \frac{\eta}{\|\eta\|}$.

**Lemma 7.2.** We have $u \perp (1 - H^*)N$.

**Proof.** Let $f \in N$. Then

$$\langle u, (1 - H^*)f \rangle = \langle \|\eta\| e_{k_0}, (1 - H^*)f \rangle = \|\eta\| \langle (1 - H)e_{k_0}, f \rangle.$$  

If $k_0 = 0$, then $(1 - H)e_{k_0} = 0$. If $k_0 > 0$, then $(1 - H)e_{k_0} = e_{k_0-1} \in \mathcal{M}$. Either way, the inner product with $f$ is 0. □

Define an operator $R : L^2 \to L^2$ in terms of the orthonormal basis $e_n$ from (3.6) by

$$R : e_n \mapsto (1 - H^*)^n u.$$  

(7.3)

**Lemma 7.4.** The operator $R$ defined by (7.3) is an isometry from $L^2$ onto $N$.

**Proof.** The functions $\{(1 - H^*)^n u : n \geq 0\}$ form an orthonormal set. Indeed, by Proposition 3.2 and Lemma 7.2, if $m \geq n$ then

$$\langle (1 - H^*)^m u, (1 - H^*)^n u \rangle = \langle (1 - H^*)^{m-n} u, u \rangle = \delta_{m,n}.$$  

As $R$ maps an orthonormal basis to an orthonormal set, it must be an isometry onto its range.

---

1A wandering subspace for an operator $T$ on $\mathcal{H}$ is a space $\mathcal{M} \subset \mathcal{H}$ such that $\{ T^j \mathcal{M} : j \geq 0 \}$ are orthogonal and span $\mathcal{H}$. In particular, by the von Neumann Wold theorem, the kernel of the adjoint forms a wandering subspace for any isometry.
We know that the range of $R$ is contained in $\mathcal{N}$. To see that it is all of $\mathcal{N}$, observe that by Lemma 7.2, we have that
$$\forall\{(1-H)^m u : m \geq 1\}$$
is contained in $\mathcal{N}^\perp = \mathcal{M}$. As $1-H$ is a pure isometry of multiplicity 1, by Theorem 2.1 for any non-zero vector $f$ the vectors
$$\{(1-H)^m f, (1-H^*)^n f : m, n \geq 0\}$$span $L^2$. Therefore in particular, $\forall\{(H^*)^n u : n \geq 0\}$ and $\forall\{H^m u : m \geq 1\}$span $L^2$, so
$$\mathcal{N} = \forall\{(H^*)^n u : n \geq 0\}$$
$$\mathcal{M} = \forall\{H^m u : m \geq 1\}.$$
□

Let us calculate $T = R^*$, the adjoint of $R$.

**Lemma 7.5.** The adjoint of $R$ is given by the operator
$$T : x^s \mapsto (1+s)\langle x^s, u \rangle x^s.$$  \hfill (7.6)

**Proof.** We have
$$\langle R^* x^s, e_n \rangle = \langle x^s, (1-H^*)^n u \rangle = \langle (1-H)^n x^s, u \rangle = \left(\frac{s}{s+1}\right)^n \langle x^s, u \rangle.$$We also have by Lemma 3.5
$$\langle T x^s, e_n \rangle = (1+s)\langle x^s, u \rangle \langle x^s, (1-H^*)^1 1 \rangle = (1+s)\langle x^s, u \rangle \langle (1-H)^n x^s, 1 \rangle = \left(\frac{s}{s+1}\right)^n \langle x^s, u \rangle.$$Therefore $T = R^*$. \hfill □

We want to approximate $\mathcal{M}$ by monomial spaces. We shall do this by approximating $u$ by linear combinations of monomials. Since $R$ is an isometry, $T$ is a co-isometry, and since each eigenvector of $H$ is an eigenvector of $T$, it follows that $T$ commutes with $H$. This means by Theorem 4.4 that for each $N$, the matrix
$$\left(1-(i+1)(j+1)\langle u, x^i \rangle \langle x^j, u \rangle \right)_{i,j=0}^N \geq 0.$$We shall assume for the remainder of this section that $N$ is large enough that $\langle u, x^i \rangle \neq 0$ for some $i \leq N$. Let $C_N \geq 1$ be the largest number $C$ so that
$$\left(1-C^2(i+1)(j+1)\langle u, x^i \rangle \langle x^j, u \rangle \right)_{i,j=0}^N \geq 0.$$
The hypothesis on $N$ means $C_N$ is finite, and $\lim_{N \to \infty} C_N = 1$. Define $\tilde{T}_N$ by

$$\tilde{T}_N : x^i \mapsto C_N(i + 1) (x^i, u) x^i, \quad 0 \leq i \leq N.$$ 

By Theorem 4.4, this extends to an operator $T_N$ that maps $L^2$ to $L^2$, commutes with $H$, and has norm equal to 1. So $T_N$ is of the form

$$T_N : x^s \mapsto \alpha_N(s) x^s.$$ 

(7.7)

**Lemma 7.8.** The function $\alpha_N(s)$ is a rational function of degree at most $N$, and maps $\mathbb{S}$ to $\mathbb{D}$.

**Proof.** We know that

$$\alpha_N(i) = C_N(i + 1) (u, x^i), \quad 0 \leq i \leq N,$$ 

(7.9)

Let $\gamma$ be a non-zero vector in the kernel of

$$\left( \frac{1 - C_N^2(i + 1)(j + 1) (u, x^i) (x^j, u)}{1 + i + j} \right)^{N}_{i,j=0} \geq 0.$$ 

By Theorem 4.2, the matrix (4.3) has to be positive semidefinite when we augment the set $\{0, \ldots, N\}$ by any other point $s$. This means by Lemma 7.11 that the first $N + 1$ entries in the last column of the extended $(N + 2)$-by-$(N + 2)$ matrix must be orthogonal to $\gamma$, so

$$\sum_{i=0}^{N} \frac{1 - \alpha_N(i) \alpha_N(s)}{1 + i + s} \gamma_i = 0.$$ 

This equation yields

$$\left( \sum_{i=0}^{N} \frac{\alpha_N(i) \gamma_i}{1 + i + s} \right) \alpha_N(s) = \sum_{i=0}^{N} \frac{\gamma_i}{1 + i + s}.$$ 

(7.10)

Let $R(s)$ denote the right-hand side of (7.10), and $L(s)$ denote the coefficient of $\alpha_N(s)$ on the left. Both $R$ and $L$ are rational functions, vanishing at infinity, with simple poles exactly in the set

$$\{-1 - i : \gamma_i \neq 0\}.$$ 

Their ratio $\alpha_N = R/L$, therefore, is a rational function with poles at the zero set of $L$, and zeroes on the zero set of $R$. The degree will be at most $N$, since they both have zeroes at infinity.

As $\|Tx^s\| = |\alpha_N(s)||x^s| \leq \|x^s\|$, we have $\alpha_N : \mathbb{S} \to \mathbb{D}$. □

We used the following lemma, whose proof is elementary linear algebra.

**Lemma 7.11.** Suppose $A$ is a positive semi-definite matrix, and $\gamma$ is a non-zero vector in the kernel of $A$. If there is a vector $\beta$ and a constant $c$ so that

$$\begin{pmatrix} A & \beta \\ \beta^* & c \end{pmatrix} \geq 0,$$

then $\langle \beta, \gamma \rangle = 0$. 

\begin{flushright}
\textcopyright Birkhäuser
\end{flushright}
Lemma 7.12. Let $\alpha$ be a rational function of degree $N$ with all its poles in the set $\{s : \text{Re } (s) < -\frac{1}{2}\}$, and with no pole at $\infty$.

(i) If $\alpha(-1) \neq 0$, then there exists a sequence $\{s_0, \ldots, s_N\}$, with multiplicity allowed, and a function $u_N$ in the generalized finite monomial space $\mathcal{M}(\{s_0, s_1, \ldots, s_N\})$, so that

$$\alpha(s) = (1 + s) \langle x^s, u_N \rangle \quad \forall s \in \mathbb{S}. \quad (7.13)$$

Moreover we can take $s_0 = 0$.

(ii) If $\alpha(-1) = 0$, then there exists a sequence $\{s_1, \ldots, s_N\}$, with multiplicity allowed, and a function $u_N$ in the generalized finite monomial space $\mathcal{M}(\{s_1, \ldots, s_N\})$, so that

$$\alpha(s) = (1 + s) \langle x^s, u_N \rangle \quad \forall s \in \mathbb{S}. \quad (7.14)$$

Proof. Expand $\alpha(s)/(s + 1)$ by partial fractions to get

$$\frac{\alpha(s)}{s + 1} = \sum_{j=1}^{p} \sum_{r=1}^{m_j} \frac{(r-1)!c_j^r}{(s - \lambda_j)^r}. \quad (7.13')$$

There is no constant term, since the left-hand side vanishes at $\infty$. We can assume that $c_j^{m_j} \neq 0$ for each $j$. In case (i), there is a pole, which we denote $\lambda_1$, at $-1$, and $\sum_{j=1}^{p} m_j = N + 1$. In case (ii) there is no pole at $-1$, and $\sum_{j=1}^{p} m_j = N$.

The inverse Laplace transform of $\alpha(s)/(s + 1)$ is

$$F(t) = \sum_{j=1}^{p} \sum_{r=1}^{m_j} c_j^r t^{r-1} e^{\lambda_j t}. \quad (7.14')$$

Define

$$u_N(x) = \frac{1}{x} F \left( \log \frac{1}{x} \right) = \sum_{j=1}^{p} \sum_{r=1}^{m_j} c_j^r \left( \log \frac{1}{x} \right)^{r-1} x^{-\lambda_j - 1}. \quad (7.15)$$

Then, making the substitution $e^{-t} = x$, we get

$$\langle x^s, u_N \rangle = \int_{0}^{1} x^s u_N(x) dx = \int_{0}^{\infty} e^{-st} F(t) dt = (\mathcal{L}F)(s) = \frac{\alpha(s)}{s + 1}. \quad (7.15')$$

Notice that each point $-1 - \lambda_j$ is in $\mathbb{S}$. We now define the multiset $\{s_0, s_1, \ldots, s_N\}$ (respectively, $\{s_1, \ldots, s_N\}$) by taking $m_j$ copies of the point $-\lambda_j - 1$ for each $j$. □
We shall prove in Lemma 7.19 that case (ii) cannot occur for $\alpha_N$. In the next two lemmas we use the fact that if $\|A\| \leq 1$, then $h \in \ker(1 - AA^*)$ if and only if $\|A^*h\| = \|h\|.$

**Lemma 7.16.** Let $\mathcal{K} = \ker(1 - T_N T_N^*)$. Then $\mathcal{K}$ is $H^*$ invariant.

**Proof.** As $H$ commutes with $T_N$ and $(1 - H)(1 - H^*) = 1$ by Lemma 3.1, we have

$$T_N = (1 - H) T_N (1 - H^*).$$

So if $g \in \mathcal{K}$ then

$$\|g\|^2 = \|T_N^* g\|^2 = \langle (1 - H) T_N^* (1 - H^*) g, T_N^* g \rangle.$$

As $\|1 - H^*\|$ and $\|T_N^*\|$ are both equal to 1, we have

$$\|T_N^* (1 - H^*) g\| = \|(1 - H^*) g\| = \|g\|.$$

Therefore $(1 - H^*) g$ is also in $\mathcal{K}$, and hence $\mathcal{K}$ is $H^*$ invariant. \qed

**Lemma 7.17.** The operator $T_N$ is a co-isometry.

**Proof.** Let $\gamma$ be as in the proof of Lemma 7.8. Let $f(x) = \sum_{j=0}^N \gamma_j x^j$. Then $(1 - T_N^* T_N) f = 0$, so $T_N$ attains its norm on $f$. Let

$$g = T_N f = \sum_{j=0}^N \gamma_j \alpha(j) x^j.$$

As $f = T_N^* T_N f$, we have $g = T_N T_N^* g$.

To prove $T_N$ is a co-isometry, we must show that

$$\mathcal{K} = \ker(1 - T_N T_N^*)$$

is all of $L^2$. By Lemma 7.16, we know that $\mathcal{K}$ is $H^*$ invariant, and it contains the polynomial $g$. If $g$ were not orthogonal to any $x^t$, then we would be done by Lemma 6.2.

As $\langle x^t, g \rangle$ is a non-zero rational function of $t$, it can only have finitely many zeroes in $S$; label these $\{t_1, \ldots, t_m\}$, counting multiplicity. By Lemma 6.4 we have

$$g = \prod_{i=1}^m (H^* - z_i) h_1,$$

where $h_1$ is cyclic for $H^*$ and $z_i = \frac{1}{1 + t_i}$. Let

$$h_2 = \prod_{i=1}^m [(\bar{z}_i - 1) H^* - \bar{z}_i] h_1.$$

Then $h_2$ is cyclic since it is an invertible operator applied to $h_1$. Let

$$r(z) = \prod_{i=1}^m \frac{z - z_i}{(\bar{z}_i - 1) z - \bar{z}_i}.$$
By Lemma 6.6, \( r(H^*) \) is an isometry, and we have \( r(H^*)h_2 = g \). Therefore
\[
\|h_2\| = \|r(H^*)h_2\| \\
= \|T_N T_N^* r(H^*)h_2\| \\
\leq \|T_N^* r(H^*)h_2\| \\
= \|T_N^* h_2\| \\
\leq \|h_2\|.
\]
Therefore \( h_2 \in \mathcal{K} \), and since \( \mathcal{K} \) is \( H^* \) invariant and \( h_2 \) is cyclic, we get that \( \mathcal{K} \) is all of \( L^2 \) and hence \( T_N^* \) is an isometry. \( \square \)

Let \( R_N = T_N^* \). A similar calculation to the proof of Lemma 7.5 yields:

**Lemma 7.18.** The operator \( R_N \) maps \( e_n \) to \( (1 - H^*)^n u_N \).

We are now ready to define \( \mathcal{M}_N \). Let \( \alpha_N \) be as in (7.7). Apply Lemma 7.12 to \( \alpha_N \) to get, in case (i), a space \( \mathcal{M}(\{s_0, s_1, \ldots, s_N\}) \) that contains \( u_N \) given by (7.15) and satisfies (7.13), and in case (ii) a space \( \mathcal{M}(\{s_1, \ldots, s_N\}) \) that contains \( u_N \) given by (7.15) and satisfies (7.14).

We show that Case (ii) of Lemma 7.12 cannot occur.

**Lemma 7.19.** We have \( \alpha_N(-1) \neq 0 \).

**Proof.** Let us assume that we are in Case (ii) of Lemma 7.12. Let \( t_j = -\bar{\lambda}_j - 1 \). In the sequence \( \{s_1, \ldots, s_N\} \) each \( t_j \) appears with multiplicity \( m_j \), and no \( t_j \) is 0. We have
\[
u_N = \sum_{j=1}^{p} \sum_{r=1}^{m_j} \tilde{c}_j^r (-\log x)^{r-1} x^{t_j}.
\]
Since \( R_N \) is isometric by Lemma 7.17, we have \( u_N \) is orthogonal to \( (1 - H^*)^k u_N \) for every \( k \geq 1 \), and hence \( u_N \) is also orthogonal to \( (1 - H)^k u_N \) for every \( k \geq 1 \). For each \( j \geq 1 \), let \( p_j^k \) be a polynomial that vanishes at 0, vanishes at \( t_i \) to order \( m_i \) if \( i \neq j \), and vanishes at \( t_j \) to order \( k \). Since each such polynomial vanishes at zero, we have
\[
\langle u_N, p_j^k(1 - H) u_N \rangle = 0. \tag{7.20}
\]
Consider
\[
P_j^{m_j}(1 - H) u_N = \tilde{c}_j^{m_j} x^{t_j}.
\]
By (7.20), we conclude that \( u_N \perp x^{t_j} \). Similarly \( p_j^{m_j-1}(1 - H) u_N \) equals \( \tilde{c}_j^{m_j} (\log x)x^{t_j} \) plus some multiple of \( x^{t_j} \). Therefore we conclude that \( u_N \) is also orthogonal to \( (\log x)x^{t_j} \). Continuing in this way, we conclude that \( u_N \) is orthogonal to every function in \( \mathcal{M}(\{s_1, \ldots, s_N\}) \). Since \( u_N \) itself is in this space, we conclude that \( u_N = 0 \), a contradiction. \( \square \)

Let \( \mathcal{M}_N = \mathcal{M}(\{s_1, \ldots, s_N\}) \), in other words the space \( \mathcal{M}(\{s_0, s_1, \ldots, s_N\}) \) with the multiplicity at 0 reduced by 1. Here is the final step.
**Lemma 7.21.** The sequence $\mathcal{M}_N$ tends to $\mathcal{M}$.

**Proof.** Let $t_j = -\bar{\lambda}_j - 1$, with $t_1 = 0$. We have

$$u_N = \sum_{j=1}^{p} \sum_{r=1}^{m_j} e_j^r (-\log x)^{r-1} x^{t_j}.$$ 

As in the proof of Lemma 7.19, we conclude that $u_N$ is orthogonal to $p(1 - H)\mathcal{M}(\{s_0, s_1, \ldots, s_N\})$ for every polynomial $p$ that vanishes at 0. So $u_N$ is a constant multiple of the projection of $e_{m_1-1}$ onto $\mathcal{M}_N^\perp$.

By Lemma 7.4, $T_N^*$ is an isometry from $L^2$ onto $\mathcal{M}_N^\perp$. Therefore the projection $P_{\mathcal{M}_N}$ onto $\mathcal{M}_N$ is given by $1 - T_N^* T_N$. We have

$$T_N : x^i \mapsto (i+1)\langle x^i, u_N \rangle x^i, \quad 0 \leq i \leq N$$

$$= C_N (i+1) \langle x^i, u \rangle x^i, \quad 0 \leq i \leq N.$$ 

As $N \to \infty$, we have $u_N \to u$ weakly and so $T_N \to T$ in SOT. Therefore $P_{\mathcal{M}_N} \to P_{\mathcal{M}} = 1 - T^* T$ in WOT and hence also SOT (since a sequence of projections converges in the SOT if and only if it converges WOT).

If $f \in \mathcal{M}$, then $f_n = P_{\mathcal{M}_N} f$ is a sequence of functions tending to $f$, so $\mathcal{M}$ is contained in the limit of the subspaces. But as each $\mathcal{M}_N \subset \mathcal{M}$, the limit must be exactly $\mathcal{M}$. \hfill \Box

**8. Open question**

Let $1 < p < \infty$, and $p \neq 2$. The Hardy operator is bounded on $L^p[0,1]$, and has $x^s$ as an eigenvector whenever $s \in \mathbb{S}_p = \{ s \in \mathbb{C} : \text{Re}(s) > -\frac{1}{p} \}$.

Any space that is the limit of finite monomial spaces (with powers in $\mathbb{S}_p$) is therefore invariant for $H$. Is every closed subspace of $L^p[0,1]$ that is invariant for $H$ of this form?

**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

**References**

[1] Agler, J., McCarthy, J.E.: Complete Nevanlinna–Pick kernels. J. Funct. Anal. 175(1), 111–124 (2000)
[2] Agler, J., McCarthy, J.E.: Pick Interpolation and Hilbert Function Spaces. Graduate Studies in Mathematics, vol. 44. American Mathematical Society, Providence (2002)

[3] Agler, J., McCarthy, J.E.: A generalization of Hardy’s operator and an asymptotic Müntz–Szász theorem. Expo. Math. 40(4), 920–930 (2022)

[4] Agler, J., McCarthy, J.E.: Monomial operators. Acta Sci. Math. (Szeged) 88(1–2), 371–381 (2022)

[5] Beurling, A.: On two problems concerning linear transformations in Hilbert space. Acta Math. 81, 239–255 (1948)

[6] Borwein, P., Erdélyi, T.: Polynomials and Polynomial Inequalities. Graduate Texts in Mathematics, vol. 161. Springer-Verlag, New York (1995)

[7] Brown, A., Halmos, P.R., Shields, A.L.: Cesàro Operators. Acta Sci. Math. (Szeged) 26, 125–137 (1965)

[8] Halmos, P.R.: Shifts on Hilbert spaces. J. Reine Angew. Math. 208, 102–112 (1961)

[9] Hardy, G.H.: Note on a theorem of Hilbert. Math. Z. 6(3–4), 314–317 (1920)

[10] Kaiblinger, N., Maligranda, L., Persson, L.-E.: Norms in weighted $L^2$-spaces and Hardy operators. In: Function Spaces (Poznań, 1998), volume 213 of Lecture Notes in Pure and Appl. Math., pp. 205–216. Dekker, New York (2000)

[11] Knese, G.: A simple proof of necessity in the McCullough–Quiggin theorem. Proc. Amer. Math. Soc. 148(8), 3453–3456 (2020)

[12] Kruglyak, N., Maligranda, L., Persson, L.-E.: Structure of the Hardy operator related to Laguerre polynomials and the Euler differential equation. Rev. Mat. Complut. 19(2), 467–476 (2006)

[13] Kufner, A., Maligranda, L., Persson, L.-E.: The Hardy inequality. Vydavatelsky Servis, Plzen, (2007). About its history and some related results

[14] McCullough, S. A.: The local de Branges–Rovnyak construction and complete Nevanlinna–Pick kernels. In: Algebraic Methods in Operator Theory, pp. 15–24. Birkhäuser (1994)

[15] Miintz, C.H.: Über den Approximationssatz von Weierstrass. In: H.A. Schwartz Festchrift, pp. 303–312. Berlin (1914)

[16] Nikol’skii, N.K.: Treatise on the Shift Operator: Spectral Function Theory. Grundlehren der Mathematischen Wissenschaften, vol. 273. Springer-Verlag, Berlin (1985)

[17] Quiggin, P.: For which reproducing kernel Hilbert spaces is Pick’s theorem true? Integral Equ. Oper. Theory 16(2), 244–266 (1993)

[18] Szász, O.: Über die Approximation stetiger Funktionen durch lineare aggregate von Potenzen. Math. Ann. 77(4), 482–496 (1916)

[19] Szász, O.: On closed sets of rational functions. Ann. Mat. Pura Appl. 4(34), 195–218 (1953)

[20] von Neumann, J.: Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren. Math. Ann. 102, 49–131 (1929)

[21] Wold, H.: A Study in the Analysis of Stationary Time Series. Almqvist & Wiksells, Uppsala (1938)
Jim Agler  
Department of Mathematics  
University of California at San Diego  
San Diego  
USA

John E. McCarthy(✉)  
Department of Mathematics  
Washington University in St. Louis  
St. Louis  
USA  
e-mail: mccarthy@wustl.edu

Received: December 30, 2022.  
Accepted: February 28, 2023.