Anomalous Spontaneous Symmetry Breaking in non-Hermitian Systems with Biorthogonal $Z_2$-symmetry

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Landau’s spontaneous symmetry breaking theory is a fundamental theory that describes the collective behaviors in many-body systems. It was well known that for usual spontaneous symmetry breaking in Hermitian systems, the order-disorder phase transition with gap closing and spontaneous symmetry breaking occur at the same critical point. In this paper, we generalized the Landau’s spontaneous symmetry breaking theory to the cases in non-Hermitian (NH) many-body systems with biorthogonal $Z_2$ symmetry and tried to discover certain universal features. We were surprised to find that the effect of the NH terms splits the spontaneous biorthogonal $Z_2$ symmetry breaking from a (biorthogonal) order-disorder phase transition with gap closing. The sudden change of similarity for two degenerate ground states indicates a new type of quantum phase transition without gap closing accompanied by spontaneous biorthogonal $Z_2$ symmetry breaking. We will take the NH transverse Ising model as an example to investigate the anomalous spontaneous symmetry breaking. The numerical results were consistent with the theoretical predictions.

Landau’s spontaneous symmetry breaking theory is a fundamental theory and plays an important role in modern particle physics and condensed matter physics. For a fundamental theory and plays an important role in modern particle physics and condensed matter physics. For a process with spontaneous symmetry breaking, the model of the system obeys symmetries, but the ground state (or vacuum) does not exhibit the same symmetry. It is perturbations that select one from the degenerate ground states. A lot of universal features become physics consequences of the spontaneous symmetry breaking, including the order-disorder phase transition, the universal critical phenomenon, the symmetry-protected degeneracy of ground states, ...

On the other hand, because non-Hermitian (NH) systems show quite different properties with their Hermitian counterparts, it attracts massive researches from different fields in recent years. For a special type of NH systems, there may also exist real spectra. For parity-time ($PT$) symmetric models that is invariant under the combined action of the $P$ and $T$ operations, the energy spectra are real in $PT$-symmetric phase. In $PT$-symmetric phase, the system always obeys similarities, i.e., it could be deformed to a Hermitian system under a NH similarity transformation (ST). For example, for certain NH transverse Ising models with imaginary external field, one can apply a certain ST and map the original NH transverse Ising model to a Hermitian model with the same energy spectra.

In this paper, we ask the following questions, "How spontaneous symmetry occur in many-body NH systems?" and "Do there exist new universal features for the spontaneous symmetry breakings in NH systems?" Motivated by above questions, we investigate a special class of NH systems with the non-unitary $Z_2$ symmetry (or the so-called biorthogonal $Z_2$-symmetry) and try to develop the theory for non-Hermitian spontaneous symmetry breaking (NHSSB). The universal features of NHSSB for this class of NH systems are explored.

Biorthogonal $Z_2$-symmetry in NH systems: Firstly, we discuss the global $Z_2$-symmetry in Hermitian systems. For a Hermitian system with $Z_2$-symmetry, the Hamiltonian $\hat{H}_{Z_2}$ is invariant under the group operation $g \in Z_2$, i.e., $U(g)\hat{H}_{Z_2}U^{-1}(g) = \hat{H}_{Z_2}$, where $U(g)$ is the unitary operator (with $\det(U(g)) = 1$) representing the operation $g$ on the Hilbert space. In general, the local degrees of freedom that has a nonvanishing ground-state expectation value for a system with $Z_2$-symmetry can be phenomenologically described by a $1/2$ (pseudo)-spin operator $\hat{\tau}^z_i$, i.e., $(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ or $(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. The group operation is $U(g)\hat{\tau}^z_iU^{-1}(g) = -\hat{\tau}^z_i$.

Next, we introduce the NH generalization for global $Z_2$-symmetry in a NH system – biorthogonal $Z_2$-symmetry.

Definition 1 – Biorthogonal $Z_2$-symmetry: For a NH Hamiltonian $\hat{H}_{Z_2}$ (with $\hat{H}_{Z_2}^\dagger \neq \hat{H}_{Z_2}$), there exists a global non-unitary $Z_2$ symmetry, i.e., $\tilde{g} \in Z_2 : U(\tilde{g})\hat{H}_{Z_2}U^{-1}(\tilde{g}) = \hat{H}_{Z_2}$ (or $U(\tilde{g}^{-1})\hat{H}_{Z_2}^\dagger U^{-1}(\tilde{g}^{-1})$). Here, $U(\tilde{g})$ (or $U(\tilde{g}^{-1})$) is the non-unitary operator (with $\det(U(\tilde{g})) \neq 1$ or $\det(U(\tilde{g}^{-1})) \neq 1$) representing $\tilde{g}$ (or $\tilde{g}^{-1}$) on the Hilbert space that obeys $U(\tilde{g}) \cdot U(\tilde{g}^{-1}) = 1$.

For a (pseudo)-spin operator with nonvanishing ground-state expectation value $\tau^z_i$, we have $U(\tilde{g})\tau^z_iU^{-1}(\tilde{g}) = -\tau^z_i$ (or $U(\tilde{g}^{-1})\tau^z_iU^{-1}(\tilde{g}^{-1}) = -\tau^z_i$).

In general, except for a possible unitary transformation, the group operation for biorthogonal $Z_2$-symmetry can be transformed into a unitary one $U(g)$ for usual

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global $Z_2$-symmetry by a similarity transformation, i.e., $\tilde{U}(\tilde{g}) = S^{-1}U(\tilde{g})S$. That means the NH system with biorthogonal $Z_2$-symmetry $(\tilde{U}(\tilde{g})H_{Z_2}\tilde{U}^{-1}(\tilde{g}) = H_{Z_2})$ obeys global similarities (STs), $S^{-1}\tilde{H}_{Z_2}S = H_{Z_2}$, where $S$ is the operator for the (NH) ST and $H_{Z_2}$ is a (Hermitian or NH) Hamiltonian obeying global $Z_2$-symmetry, i.e., $\tilde{U}(\tilde{g})H_{Z_2}\tilde{U}^{-1}(\tilde{g}) = H_{Z_2}$. If the local degrees of freedom that has a nonvanishing ground-state expectation value is denoted by a pseudo-spin operator $\tau^z_i$, we can phenomenologically derive $S$ to be $S(\beta) = \prod_i S_i(\beta)$ with $S_i(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta} \end{pmatrix}$. Here, $\beta$ denotes the non-Hermiticity.

In the following parts, in order to be more obvious, we denote the non-unitary group operator $\tilde{g}$ by $g^\beta$ and $S$ by $S(\beta)$. And when $\beta = 0$, the NH model turns into a Hermitian one.

**Universal features for non-Hermitian spontaneous symmetry breaking:** Firstly, as shown in Fig.1(a), we summarize the universal features for usual spontaneous $Z_2$-symmetry breaking in Hermitian systems: 1) The energy gap for bulk states is closed ($\Delta m = 0$) at $\lambda = \lambda_c$, (with $\lambda$ is a tunable parameter). For example, $A$ is the strength of a transverse field in transverse Ising model; 2) A quantum phase transition (QPT) occurs at $\lambda = \lambda_c$ from ordered phase with $\langle \text{vac}|A|\text{vac} \rangle = A_0 \neq 0$ to disordered phase $\langle \text{vac}|A|\text{vac} \rangle = A_0 = 0$. Here, $|\text{vac}\rangle$ denotes ground state and $A$ is an operator with a nonvanishing ground-state expectation value which changes sign under $Z_2$ group, $U(\beta)A U^{-1}(\beta) = -A$. In general, $A$ is denoted by a pseudo-spin operator $\tau^z_i$; 3) The ground state degeneracy $D$ changes suddenly from 2 in ordered phase to 1 in disordered phase.

Therefore, as shown in Fig.1(a), for usual spontaneous $Z_2$-symmetry breaking in Hermitian systems $(\tilde{U}(\beta)H_{Z_2}\tilde{U}^{-1}(\beta) = H_{Z_2})$, there are two phases: ordered phase and disordered phase. In the ordered phase $(A_0 \neq 0)$, the ground states have two-fold degeneracy. Under perturbations that select one from the degenerate ground states (DGSs), spontaneous $Z_2$-symmetry breaking occurs, i.e., $\tilde{U}(\beta)|\text{vac} \rangle \neq |\text{vac} \rangle$; in disordered phase $(A_0 = 0)$, $Z_2$-symmetry is unbroken, i.e., $\tilde{U}(\beta)|\text{vac} \rangle = |\text{vac} \rangle$.

To illustrate the universal features for the NHSSB in NH systems with biorthogonal $Z_2$-symmetry, we introduce a concept – spontaneous biorthogonal $Z_2$-symmetry breaking, together with two theorems (the detailed proof is given in supplementary materials):

**Definition 2 – Spontaneous biorthogonal $Z_2$-symmetry breaking:** For non-Hermitian systems with biorthogonal $Z_2$-symmetry $(\tilde{U}(\beta)H_{Z_2}\tilde{U}^{-1}(\beta) = H_{Z_2})$, the ground states (or vacuum) shows the same symmetry $\tilde{U}(\beta)\langle \text{vac}|\text{vac} \rangle = \pm \langle \text{vac}|\text{vac} \rangle$. However, with an additional perturbation $\tilde{H}_{Z_2} \rightarrow \tilde{H}_{Z_2} + \delta \tilde{H}$, the ground states (or vacuum) does not exhibit the original symmetry, i.e., $\tilde{U}(\beta)\langle \text{vac}^R(\beta)|\text{vac}^L(\beta) \rangle \neq \pm \langle \text{vac}^R(\beta)|\text{vac}^L(\beta) \rangle$.

**Theorem 1:** Near the biorthogonal order-disorder QPT for the NH system described by $H_{Z_2}$ with biorthogonal $Z_2$-symmetry, under the conditions $\tilde{H}_{Z_2} = S^{-1}\tilde{H}_{Z_2}S = H_{Z_2}$ and $H_{Z_2} = \tilde{H}_{Z_2}$, the universal critical phenomenon for biorthogonal order parameter is the same to that of the Hermitian model $H_{Z_2}$. Here $S(\beta)$ is a global ST. A biorthogonal order parameter is defined by calculating the expectation value for the DGSs in the biorthogonal set $\langle \text{vac}^R(\beta)\tilde{A}\tilde{A}^R(\beta) \rangle$ and $\langle \text{vac}^L(\beta)\tilde{A}\tilde{A}^L(\beta) \rangle$ [13], i.e., $\langle \text{vac}^R(\beta)|\tilde{A}\tilde{A}^R(\beta)\rangle = 1$, $\langle \text{vac}^L(\beta)|\tilde{A}\tilde{A}^L(\beta)\rangle = 1$ for the two DGSs in biorthogonal order.

**Theorem 2:** For the NH system in biorthogonal ordered phase, with increasing the non-Hermiticity $\beta$ the biorthogonal $Z_2$-symmetry could be spontaneously broken that is accompanied by the sudden change of state-similarity $\langle \text{vac}^R(\beta)|\text{vac}^L(\beta)\rangle$ for the two DGSs $\langle \text{vac}^R(\beta)\tilde{A}\tilde{A}^R(\beta) \rangle$. Here, $\langle \text{vac}^R(\beta)|\text{vac}^L(\beta)\rangle$ is satisfied self-normalization condition, i.e., $\langle \text{vac}^R(\beta)|\text{vac}^L(\beta)\rangle = 1$. As shown in Fig.1(b), in general, for NH systems with biorthogonal $Z_2$-symmetry, there are three phases: 1)
a biorthogonal order (BO) with $A^{bi} \neq 0$, in which the biorthogonal $Z_2$-symmetry are spontaneously broken simultaneously, i.e., $\hat{U}(g^2) [\text{vac}(\beta)] = \pm [\text{vac}(\beta)]$. In the inset in Fig.1(b), we can effectively use a figure of double-well potential with a hidden well to represent this ordered phase (there is no quantum states in the hidden well); 2) a BO with $A^{bi} \neq 0$, in which the biorthogonal $Z_2$-symmetry are unbroken, i.e., $\hat{U}(g^2) [\text{vac}(\beta)] = \pm [\text{vac}(\beta)]$. In the inset in Fig.1(b), we can effectively use a figure of double-well potential with two hidden wells to represent this ordered phase (there is only one ground state in the double-well system); 3) a disordered phase with $A^{bi} = 0$, in which the biorthogonal $Z_2$-symmetry are unbroken, i.e., $\hat{U}(g^2) [\text{vac}(\beta)] = [\text{vac}(\beta)]$. In the inset in Fig.1(b), we can effectively use a figure of single-well potential to represent the disordered phase.

In Fig.1(b), the biorthogonal order-disorder QPT with gap closing occurs at $\lambda = \lambda_c$. However, another QPT without gap closing occur at $\lambda = \lambda_2$, at which the biorthogonal $Z_2$-symmetry are spontaneously breaking. We can say it is the effect of NH terms that splits the spontaneous biorthogonal $Z_2$-symmetry breaking (at $\lambda = \lambda_2$) from the biorthogonal order-disorder QPT (at $\lambda = \lambda_c$) with $\lambda_2 \neq \lambda_c$.

**Example – One dimensional transverse Ising model with biorthogonal $Z_2$-symmetry:** We use a transverse Ising (TI) model with biorthogonal $Z_2$ symmetry as an example to show the universal features of NHSSB for NH systems.

The Hamiltonian of one dimensional (1D) TI model with biorthogonal $Z_2$ symmetry is given by

$$\hat{H}_{NTI}^{\beta} = \sum_i (-J \sigma_i^x \sigma_{i+1}^x + h \sigma_i^y + ih^z \sigma_i^z)$$

where $J > 0$ is ferromagnetic Ising coupling constant between two nearest neighbor spins and $h^y$ is the strength of a real transverse field along $y$-axis, $h^z$ is the strength of an imaginary transverse field along $z$-axis. In this paper, the coupling parameter $J$ is set to be unit, $J = 1$.

The group element of the biorthogonal $Z_2$-symmetry is defined by a non-unitary operator $\hat{U}(g^2) = S(\beta) [\prod_i (\sigma_i^y)] S(\beta)^{-1}$, where $\det(\hat{U}(g^2)) \neq 1$ (or $\det(\hat{U}^{-1}(g^2)) \neq 1$) and $\hat{U}(g^2) \hat{U}^{-1}(g^2) = 1$. Here $S(\beta) = \prod_i S_i(\beta)$ is the operator of a global NH ST on spin system.

The similar transformation $S_\beta$ is defined as a non-Hermiticity $\beta = \ln(h^{a \rightarrow b}) \cdot \hat{H}_{NTI}^{\beta} \cdot \hat{H}_{NTI}^{\beta}$ and the non-Hermiticity

$$\hat{H}_{NTI}^{\beta} = \sum_i (-J \sigma_i^x \sigma_{i+1}^x + h \sigma_i^y + ih^z \sigma_i^z)$$

under a global inverse ST, $\hat{H}_{NTI}^{\beta}$ is deformed into a Hermitian one, i.e.,

$$\hat{H}_{NTI}^{\beta} = S^{-1}(\beta) \hat{H}_{NTI}^{\beta} S(\beta) = \sum_i (-J \sigma_i^x \sigma_{i+1}^x + h \sigma_i^y + ih^z \sigma_i^z).$$

A $PT$ spontaneous symmetry breaking occurs at $|h^y| = |h^z|$. For the case of $|h^y| > |h^z|$, the energy spectra for excitations are all real; For the case of $|h^y| < |h^z|$, the energy spectra for the excitations become complex.

**Firstly,** we study the QPT at gap closing. In this part, we focus on the $PT$ symmetric phase, $|h^y| > |h^z|$. Under NH (inverse) ST, the energy levels $E_n(\beta)$ of $\hat{H}_{NTI}^{\beta}$ and $\hat{H}_{NTI}^{\beta}$ are the same those of the Hamiltonian model $E_n(\beta) = 0$ of $\hat{H}_{NTI}^{\beta}$, i.e., $E_n(\beta) = E_n(-\beta) = E_n(\beta) = 0$.

As a result, the QPT with the gap closing for $\hat{H}_{NTI}^{\beta}$ is same to that for $\hat{H}_{NTI}^{\beta}$ that is obtained as $J = |h|^{\beta}$. See Fig.2(a) from exact diagonal numerical calculation for 1D NH TI model with $N = 16$, in which the dotted red lines come from theoretical prediction.

**Next,** we study the biorthogonal order-disorder phase transition. The biorthogonal order parameter is defined by the expectation value in the ground states $[\text{vac}(\beta)]$ and $[\text{vac}(\beta)]$, i.e.,

$$\frac{1}{N} \sum_i \langle \text{vac}(\beta) | \sigma^z_i | \text{vac}(\beta) \rangle = A^{bi}.$$  

In the region of $J > |h|$, $A^{bi} \neq 0$, there exists BO; In the region of $J < |h|$, $A^{bi} \neq 0$, the ground state is a disordered state. The biorthogonal order-disorder phase transition occurs at $J = |h|$ that coincides the QPT from the gap closing. Fig.2(b) show the biorthogonal order parameter from exact diagonal numerical calculation for 1D NH TI model with $N = 16$, in which the dotted red lines come from theoretical prediction.

**Thirdly,** we study the spontaneous $Z_2$-symmetry breaking in this NH TI model.

We concentrate the DGSs in biorthogonal ordered phase with $A^{bi} \neq 0$. Because the order parameter changes sign under the biorthogonal $Z_2$ transformation, i.e., $[\text{vac}(\beta)] \hat{U}^{-1}(g^2) \hat{H}_{NTI}^{\beta} \hat{U}(g^2) [\text{vac}(\beta)] = -A^{bi} \neq A^{bi}$, there must exist two DGSs, $[\text{vac}(\beta)]$ and $[\text{vac}(\beta)]$. Under the global biorthogonal $Z_2$-symmetric transformation, we have $\hat{U}(g^2) |\text{vac}(\beta)] = \pm |\text{vac}(\beta)]$.

To show the NHSSB, we add a tiny longitudinal field on site $i_0$, $\hat{H}_{NTI}^{\beta} \rightarrow (\hat{H}_{NTI}^{\beta} + \delta \hat{H})$, where $\delta \hat{H} = h^z \sigma_i^z$ with $h^z \ll J$.

To quantitatively demonstrate the NHSSB, we introduce the effective Hamiltonian for the DGSs, $\hat{H}_{GS} = \hat{H}_{NTI}^{\beta}$, $J_{11} = +$, $I_{11} = +$, $J_{12} = +$, $I_{12} = +$, $J_{21} = -$, $I_{21} = -$, $J_{22} = -$, $I_{22} = -$. The basis of the DGSs for $\hat{H}_{GS} = \hat{H}_{NTI}^{\beta}$ under biorthogonal set.

In general, we have $|\text{vac}(\beta) = 0)\rangle = c_{1,1}^L |\text{vac}(\beta) = 0)\rangle + c_{1,2}^L |\text{vac}(\beta) = 0)\rangle$ which $c_{1,1}^L$ and $c_{1,2}^L$ are the complex parameters, respectively.

In the Hermitian limit $\beta \rightarrow 0$, the effective Hamiltonian for the DGSs $\hat{H}_{GS}$ is obtained as $\hat{H}_{GS} = \Delta c^2 + \epsilon t^2$.
where $\Delta$ is the energy splitting from quantum tunneling effect that defined by $\Delta = \langle \text{vac}^R | H_{\text{NTI}}^\beta | \text{vac}^L \rangle = \frac{(h^2-j^2)}{j}(-\frac{h}{j})^N$ and $\varepsilon = h^z$ is the energy difference between the two DGSs. For the NH TI model, after considering a global ST on the two DGSs $S_{\text{GS}}(\beta N) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta N} \end{pmatrix}$, the effective Hamiltonian for the DGSs is obtained as $\tilde{H}_{\text{GS}}^{\beta N} = S_{\text{GS}}(\beta N)\tilde{H}_{\text{GS}}S_{\text{GS}}^{-1}(\beta N) = \Delta^+ + \Delta^{-} + e^{\gamma} \varepsilon^2$ where $\Delta^+ = \Delta e^{\beta N} = \frac{(h^2-j^2)}{j}(-\frac{h}{j})^N$ and $\Delta^- = \Delta e^{-\beta N} = \frac{(h^2-j^2)}{j}(-\frac{h}{j})^N$. In thermodynamic limit $N \to \infty$, although $\Delta \to 0$, there exists the competition between the exponential decay of $\Delta$ with the size of the system from quantum tunneling effect and the exponential increase of $e^{\pm\beta N}$ with the size of the system from NH similarity effect. Therefore, in thermodynamic limit there exist two phases: one phase is $|\Delta e^{\pm\beta N}| \to 0$, the other is $|\Delta e^{\pm\beta N}| \to \infty$. At $|\Delta e^{\pm\beta N}| = 1$ (or $\pm |h|^2 \pm |h^z| = 1$), the QPT occurs. The QPT induced by perturbations at $\pm |h|^2 \pm |h^z| = 1$ is accompanied by the sudden change of state-similarity for the DGSs $|\text{vac}_-^{\beta N} \rangle \sim |\text{vac}_+^{\beta N} \rangle$. Fig.2(c) are the numerical results for the state-similarity for the two DGSs. The dotted red lines in Fig.2(c) is shown from the theoretical prediction, i.e., $\pm |h|^2 \pm |h^z| = 1$ [62].

On the one hand, in the region of $|\Delta e^{\pm N\beta}| \to 0$, the effective Hamiltonian for the DGSs is reduced into $\tilde{H}_{\text{GS}}^{\beta N} \to \epsilon \cdot \varepsilon^2$. The two DGSs are $|\text{vac}_+^{L/R}(\beta) \rangle = |\text{vac}_+^{L/R}(\beta = 0) \rangle$ and $|\text{vac}_-^{L/R}(\beta) \rangle = e^{-\beta N} |\text{vac}_-^{L/R}(\beta = 0) \rangle$. Now, the state-similarity of the two DGSs is zero, i.e., $|\text{vac}_-^{\beta N} \rangle \langle \text{vac}_+^{\beta N} | \to 0$. In the thermodynamic limit $N \to \infty$, due to the normalization factor for $|\text{vac}_+^{L/R}(\beta) \rangle$ vanishes, i.e., $|\text{vac}_+^{L/R}(\beta) \rangle | \text{vac}_-^{L/R}(\beta) \rangle = e^{-2\beta N}$, the quantum state $|\text{vac}_-^{L/R}(\beta) \rangle$ disappears and the ground state degeneracy $D$ becomes 1. In particular, the biorthogonal $Z_2$-symmetry are spontaneously broken simultaneously, i.e., $U(g^{\beta}) |\text{vac}_+^{\beta N} \rangle \neq |\text{vac}_-^{\beta N} \rangle$. On the other hand, in the region of $|\Delta e^{\pm N\beta}| \to \infty$, the effective Hamiltonian for the DGSs is reduced into $\tilde{H}_{\text{GS}}^{\beta N} \to \Delta e^{\pm\beta N} \varepsilon^2$ or $\Delta e^{\pm\beta N} \varepsilon$. The two DGSs are $|\text{vac}_+^{L/R}(\beta) \rangle = e^{-\beta N} |\text{vac}_+^{L/R}(\beta = 0) \rangle$ or $|\text{vac}_-^{\beta N} \rangle = |\text{vac}_-^{L/R}(\beta = 0) \rangle$.

In summary, in Fig.2(d), we plot the global phase diagram for the 1D NH TI model: I is $PT$ symmetric phase with BO and spontaneous biorthogonal $Z_2$ symmetry breaking, II is $PT$ symmetry breaking phase with BO and spontaneous biorthogonal $Z_2$ symmetry breaking, III is $PT$ symmetry breaking phase without BO and with biorthogonal $Z_2$ symmetry, IV is $PT$ symmetry breaking phase with BO and with biorthogonal $Z_2$ symmetry, V is $PT$ symmetric phase with BO and with biorthogonal $Z_2$ symmetry, VI is $PT$ symmetric phase without BO and with biorthogonal $Z_2$ symmetry.

For 1D TI model with biorthogonal $Z_2$ symmetry, we can use Jordan-Wigner transformation to map the original spin model $\tilde{H}_{\text{NTI}}^{\beta}$ to a NH superconducting model and obtain the correspondence exact results. That means the TI model with biorthogonal $Z_2$ symmetry is an exactly solvable spin model.

We generalize the Jordan-Wigner transformation to the NH case by considering global ST, i.e., $\tilde{c}_j = S(\beta) \tilde{c}_j S^{-1}(\beta) = S(\beta) \prod_{k=1}^{n-1} (\sigma_k^z - i\sigma_k^x) | S^{-1}(\beta) = S^{-1}(\beta) \prod_{k=1}^{n-1} \sigma_k^x | S(\beta) = e^{i\beta \sigma_x}$, and $\tilde{c}_j = S(\beta) | S^{-1}(\beta) = S^{-1}(\beta) \prod_{k=1}^{n-1} \sigma_k^x | S(\beta) = e^{i\beta \sigma_x} \tilde{c}_j$. From the Jordan-Wigner transformation, one can see that the fermions get an additional "imaginary" vector $k_0$, i.e., $k_0 = i\beta = i\ln(h^{-\beta N})$. The resulting fermionic Hamiltonian corresponding to $\tilde{H}_{\text{NTI}}^{\beta}$ becomes $\tilde{H}_F^{\beta} = \sum_{k>0} \psi_k^\dagger (\hat{H}_F^{\beta}) \psi_k = \langle \psi_{\hat{c}_k} | (\hat{c}_k, \hat{c}_k^\dagger) \rangle$ where

$$H_F^{\beta} = (-J \cos k + h^y) \sigma^z + (-J \sin k + i(h^z)) \sigma^y.$$ (3)

The fermion Hamiltonian $H_F^{\beta}$ can be "renormalized" by an inverse ST $S^{-1}(\beta)$ and becomes $\tilde{H}_F^{\beta = 0}$. 

FIG. 2: (Color online) (a) The energy difference between the lowest two energy levels from exact diagonal numerical calculations for 1D NH TI model with $N = 16$; (b) The biorthogonal order parameter from exact diagonal numerical calculation for 1D NH TI model with $N = 16$; (c) The state-similarity for the two degenerate ground states $|\text{vac}_+^{\beta N} \rangle \sim |\text{vac}_+^{\beta N} \rangle$ from exact diagonal numerical calculation for 1D NH TI model with $N = 16$; (d) The global phase diagram for theoretical predictions.
\[ \sum_{k>0} \psi_k^\dagger (\tilde{H}_F^\beta=0) \psi_k \left( \psi_k^\dagger = (\tilde{c}_k^\dagger, \tilde{c}_k) \right) \] where 
\[ \tilde{H}_F^\beta=0 = (-J \cos k + h)\sigma^z + (-J \sin k)\sigma^y \] (with \( h = \sqrt{|h^y|^2 - |h^z|^2} \)). The detailed discussion about this issue is given in supplementary materials.

From Eq.(3) and Eq.(4), we obtain the same global phase diagram for the 1D NH TI model as shown in Fig.2(d): The QPT for \( \mathcal{PT} \) spontaneous symmetry breaking from Eq.(4) also occurs at \( |h^y| = |h^z| \); The QPT corresponding to the biorthogonal order-disorder phase transition at \( J = |h| \) is characterized by the gap closing for \( H_F^\beta=0 \) from Eq.(4) (the purple lines in Fig.2(d)). In particular, biorthogonal order-disorder phase transition at \( |h| = J \) in original spin model corresponds to "topological phase transition" for the single body fermion Hamiltonian \( \tilde{H}_F^\beta=0 \). Based on the single body fermion Hamiltonian \( \tilde{H}_F^\beta=0 \), we define a winding number, 
\[ w = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_k \phi(k) \cdot dk \] where \( \phi(k) = \tan^{-1}(d_y/d_x) \) with \( d_y = -J \sin k \) and \( d_x = h - J \cos k \). So, there are two phases: "topological phase" with \( w = 1 \) in the region of \( |h| < J \) and trivial phase with \( w = 0 \) in the region of \( |h| > J \); The QPT corresponding to spontaneous symmetry breaking at \( \pm |h^y| \pm |h^z| = 1 \) is characterized the gap (the gap for the real part of energy levels) closing for \( H_F^\beta \) from Eq.(4) (the blue lines in Fig.2(d)).

**Conclusion and discussion:** In this paper, we develop the theory for non-Hermitian spontaneous symmetry breaking. Universal features of NH many-body systems with biorthogonal \( Z_2 \) symmetry are explored. We find that the effect of NH terms splits the usual spontaneous symmetry breaking (at \( \lambda = \lambda_s \)) from a biorthogonal order-disorder phase transition (at \( \lambda = \lambda_c \) with \( \lambda_s \neq \lambda_c \)). As an exactly solvable spin model, we take the 1D NH transverse Ising model \( H_{NTI}^\beta \) as example to investigate the anomalous spontaneous symmetry breaking.

In addition, we also studied a two dimensional (2D) TI model with biorthogonal \( Z_2 \)-symmetry on square lattice and obtained a global phase diagram that is quite similar to that of the 1D case (Fig.2(d)). And, for 1D or 2D TI model with biorthogonal \( Z_2 \)-symmetry, the NHSSB shows the same universal features. The detailed discussion are shown in supplementary materials. In the future, we generalize the theory for non-Hermitian spontaneous symmetry breaking to other models with discrete or continuum non-unitary symmetries.

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[55] For NH system with complex energy spectra, we assume the ground state \(|\text{vac}_{L/R}(\beta)\rangle\) is the one with lowest real part of energies rather than the one with largest imaginary part of energies.
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[60] This Hamiltonian shows global PT-symmetry, i.e., \(|\mathcal{P}, \hat{H}_{NTI}^\beta| \neq 0 and \mid\mathcal{T}, \hat{H}_{NTI}^\beta\rangle \neq 0, but \mathcal{P}\mathcal{T}, \hat{H}_{NTI}^\beta = 0. Here the time reversal operator \mathcal{T} is defined as \mathcal{T}\mathcal{i}T = -i and the spin rotation operator \mathcal{P} = \Pi_i (i\sigma^z_i).
[61] For the case of |\hbar^y| < |\hbar^z|, the energy spectra for the excitations become complex. Now, we derive the QPT of gap closing at J = |\hbar| by assuming that QPT occurs when the real part of the energy gap is closing.
[62] For the case of |\hbar^y| < |\hbar^z|, the energy spectra for the excitations become complex. Now, we derive the QPT of spontaneous biorthognal Z_2-symmetry breaking at ± |\hbar^y| ± |\hbar^z| = 1 by assuming that the ground states are the quantum states with lowest real parts of energies.