HYPERPLANE SECTIONS OF PROJECTIVE BUNDLE ASSOCIATED TO THE TANGENT BUNDLE OF $\mathbb{P}^2$.

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Abstract. In this note we give a complete description of all the hyperplane section of the projective bundle associated to the tangent bundle of $\mathbb{P}^2$ under its natural imbedding in $\mathbb{P}^7$.

Keywords: Projective bundle; natural imbedding; Hyperplane section.

1. Introduction

Let $\mathbb{P} (T_{\mathbb{P}^2})$ be the projective bundle associated to the tangent bundle of $T_{\mathbb{P}^2}$. This three fold is naturally imbedded in $\mathbb{P}^7$. The aim of this short note is to discribe explicitly all possible type of hyperplane sections of this imbedding. Existence of such a complete discription is very rare.

We have the following:

Theorem 1.1. A hyperplane section of $\mathbb{P} (T_{\mathbb{P}^2})$ in its natural imbedding in $\mathbb{P}^7$ is either

i) an irreducible surface of degree six in $\mathbb{P}^6$.

or

ii) union of two degree three surfaces.

More over when the hyperplane section is irreducible then it either a non-singular Del Pezzo surface or a singular surface which is singular along a rational curve of multiplicity two or three.

2. Proof of the Theorem

Throughout this note we use standered notations see for example [GH] or [Ha]. The proof of the Theorem depends on the following:

Theorem 2.1. Let $s$ be a non-zero section of the tangent bundle $T_{\mathbb{P}^2}$ of $\mathbb{P}^2$. Then the scheme $V$ of zeros of $s$ is one of the following type:

a) consists of three distinct points,

b) consists of two points with multiplicity two at one of the points,

c) consists of a single point with multiplicity three,

d) union of a line and a point not on the line,
Proof: Let $X, Y$ and $Z$ be the linearly independent section of the of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$. We consider $X, Y$ and $Z$ as the variables giving the homogeneous coordinates on $\mathbb{P}^2$. On $\mathbb{P}^2$ we have the Euler sequence

\begin{equation}
0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow T_{\mathbb{P}^2} \rightarrow 0,
\end{equation}

where the map $\mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^3$ is given by $1 \mapsto (X, Y, Z)$. Any section of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ gives a section of $T_{\mathbb{P}^2}$ by projection. Since the group $H^1(\mathcal{O}_{\mathbb{P}^2}) = 0$ every section of $T_{\mathbb{P}^2}$ is a image of a section of $\mathcal{O}_{\mathbb{P}^2}(1)^3$. Any section of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ is a ordered triple $(f_1, f_2, f_3)$ where $f_i$ $(1 \leq i \leq 3)$ are linear forms in the variables $X, Y, Z$. Two sections $(f_1, f_2, f_3)$ and $(g_1, g_2, g_3)$ of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ maps to same section of $T_{\mathbb{P}^2}$ if the difference of these two section is a scalar multiple of $(X, Y, Z)$. Let $s$ be a non-zero section of $T_{\mathbb{P}^2}$ and $(f_1, f_2, f_3)$ be a section of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ which maps to $s$ under the surjection given by the Euler sequence (1). Clearly the scheme $V$ of zeros of $s$ is equal to subscheme of $\mathbb{P}^2$ on which the the two sections $(X, Y, Z)$ and $(f_1, f_2, f_3)$ of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ are dependent, namely the scheme defined by the vanishing of the two by two minors of the the matrix

\[
\begin{pmatrix}
X & Y & Z \\
 f_1 & f_2 & f_3 
\end{pmatrix}
\]

i.e., the scheme $V$ defined by the common zeros of the polynomials

\begin{equation}
X f_2 - Y f_1, X f_3 - Z f_1, Y f_3 - Z f_2.
\end{equation}

Since the second Chern class $c_2(\mathcal{O}_{\mathbb{P}^2}(1)^3) = 3$, we see that if $V$ is finite then it must be one of type a),b) or c). More over as $s$ is non zero we see that $V$ is defined by at least two linearly independent quadrics and hence it cannot contain a conic as subscheme. On the other hand if the restriction of the sections $(f_1, f_2, f_3)$ and $(X, Y, Z)$ to a line $\ell$ are linearly dependent then line $\ell \subset V$. In this case we see that $V$ is of the type d) or e) as at least two of the quadrics in (2) are linearly independent. \hfill \Box

Example 2.2. Here we give examples to show all the type mentioned in the theorem (2.1) do occur. Consider the section $(X, bY, cZ)$, $b, c \in \mathbb{C}$, of $\mathcal{O}_{\mathbb{P}^2}(1)^3$ and denote by $s$ the section of $T_{\mathbb{P}^2}$ induced by the surjection

\[
\mathcal{O}_{\mathbb{P}^2}(1)^3 \rightarrow T_{\mathbb{P}^2} \rightarrow 0
\]

given by the exact sequence (1). Note that $a = b = 1$ if and only if corresponds to zero section of $T_{\mathbb{P}^2}$. The scheme $V$ of zeros of the section
s is given by the vanishing of the quadrics
\[(b - 1)XY, (c - 1)XZ, (c - b)YZ.\]

If \(b - 1, c - 1, c - b\) are all non zero, then \(V = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}\) consists of three distinct points and hence of type a) of Theorem (2.1).

If \(b = 0\) and \(c = 1\) then \(V\) as set is the line \(Y = 0\) and the scheme is of type d) of Theorem (2.1).

If \(s\) is the section defined by \((Y, 0, 0)\) then the zero scheme of \(s\) is given by the vanishing of the quadrics \(Y^2\) and \(YZ\) which is of type e) of Theorem (2.1).

If \(s\) is the section defined by \((Y, Z, 0)\) then the zero scheme of \(s\) is given by the vanishing of the quadrics \(XZ - Y^2, YZ\) and \(XZ\) which is of type b) of Theorem (2.1).

Let \(E\) be a vector bundle of rank two on a smooth Projective surface \(X\). Let \(P(E)\) be the projective bundle associated to the bundle of \(E\) and \(\pi : P(E) \to X\) be the natural projection. Let \(\mathcal{O}_{P(E)}(1)\) be the relative ample line bundle quotient of \(\pi^*(E)\) and let \(\phi\) be the natural isomorphism
\[H^0(X, E) \cong H^0(P(E), \mathcal{O}_{P(E)}(1)).\]

**Lemma 2.3.** For a non zero section \(\sigma\) of \(E\) let \(V_0(\sigma)\) (resp. \(V_1(\sigma)\)) denote the zero dimensional (resp. one dimensional) component of the subscheme of \(X\) defined by zeros of \(\sigma\). Let \(s \neq 0\) be a section \(E\) and \(\phi(s)\) be the corresponding section of \(\mathcal{O}_{P(E)}(1)\). The scheme of zeros of a section \(\phi(s)\) is equal to \(W_0 \cup W_1\), where \(W_0\) (resp. \(W_1\)) is a subscheme of \(P(E)\) is isomorphic to blow up of \(X\) along \(V_0(s)\) (resp. is isomorphic to \(P(E|_{V_1(s)})\).

**Proof:** The proof of the lemma is the consequence of the following:

1) If the section \(s\) is non zero at a point \(x\) the subspace generated by it in defines a unique point in the projective line \(P(E_x)\) at which the section \(\phi(s)\) vanishes.

2) If the section \(s\) is zero at along a subscheme \(Z\) of \(X\) then the the subscheme \(\pi^{-1}(Z)\) is a subscheme of the scheme of zeros of \(\phi(s)\).

**Proof of Theorem (1.1):** Note that a hyperplane section of \(\mathbb{P}(T_{\mathbb{P}^2})\) in its natural imbedding in \(\mathbb{P}^7\) is a two a dimensional scheme of degree six and is zero scheme of a non zero section of \(\mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1)\). Let \(s\) be a non zero section of \(T_{\mathbb{P}^2}\) and \(\phi(s)\) the corresponding section \(\mathcal{O}_{\mathbb{P}(T_{\mathbb{P}^2})}(1)\). If the zeros of the section \(s\) is of the form given by a), b) or c) in Theorem
then the we conclude from Lemma (2.3), the zero scheme of $\phi(s)$ is an irreducible surface of degree 6 in $\mathbb{P}^6$ described in the Theorem. If the zeros of the section $s$ is of the form given by e) or f) in Theorem (2.1), then the we conclude from Lemma (2.3) the zero scheme of $\phi(s)$ union of two surfaces. Since Tangent bundle of $\mathbb{P}^2$ restricted to a line in $\mathbb{P}^2$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$, If the zeros of the section $s$ is of the form given by e) or f) in Theorem (2.1), then one of the surfaces is a degree 3 surface and hence the other surface is also of degree three and hence the zero scheme of $\phi(s)$ is of the form given in ii) of the Theorem. □

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