Tempered fractional Brownian and stable motions
of second kind

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Abstract

Meerschaert and Sabzikar [12], [13] introduced tempered fractional Brownian/stable motion (TFBM/TFSM) by including an exponential tempering factor in the moving average representation of FBM/FSM. The present paper discusses another tempered version of FBM/FSM, termed tempered fractional Brownian/stable motion of second kind (TFBM II/TFSM II). We prove that TFBM/TFSM and TFBM II/TFSM II are different processes. Particularly, large time properties of TFBM II/TFSM II are similar to those of FBM/FSM and are in deep contrast to large time properties of TFBM/TFSM.

Keywords: tempered fractional Brownian/stable motion; tempered fractional Brownian/stable noise; tempered fractional integration; local and global self-similarity

1 Introduction

Meerschaert and Sabzikar [13] introduced tempered fractional stable motion (TFSM) $Z_{H,\alpha,\lambda} = \{Z_{H,\alpha,\lambda}(t), t \in \mathbb{R}\}$ for $0 < \alpha \leq 2$, $H > 0$, $\lambda > 0$ as stochastic integral

$$Z_{H,\alpha,\lambda}(t) := \int_\mathbb{R} \left( (t - y)^{H - \frac{\alpha}{2}} e^{-\lambda(t-y)_+} - (-y)^{H - \frac{\alpha}{2}} e^{-\lambda(-y)_+} \right) M_\alpha(dy), \quad (1.1)$$

with respect to $\alpha$-stable Lévy process $M_\alpha$. A particular case of TFSM termed the \textit{tempered fractional Brownian motion} (TFBM) corresponding to $\alpha = 2$ and $M_2 = B$ (a standard Brownian motion) was studied in Meerschaert and Sabzikar [12]. Note that for $\lambda = 0$ (and $H \in (0,1)$) TFSM/TFBM agree with fractional stable/Brownian motion (FSM/FBM), see [16]. The role of the tempering by exponential factor in (1.1) manifests in the dependence properties of the increment process $Y_{H,\alpha,\lambda} = \{Y_{H,\alpha,\lambda}(t) := Z_{H,\alpha,\lambda}(t + 1) - Z_{H,\alpha,\lambda}(t), t \in \mathbb{Z}\}$ called \textit{tempered fractional stable noise} (TFSN) and \textit{tempered fractional Gaussian noise} (TFGN) in the Gaussian case $\alpha = 2$. In particular, for small $\lambda > 0$ the autocovariance function of TFGN closely resembles that of fractional Gaussian noise (FGN) on an intermediate scale, but then it eventually falls off exponentially. On the other hand, the spectral density of TFGN vanishes at the origin for all $H > 0$ exhibiting a strong anti-persistent behavior, see [12].

In this paper we study a closely related but different tempered process called \textit{tempered fractional Brownian/stable motion of second kind} (TFBM II/TFSM II) which is defined by replacing the integrand in (1.1) by

$$h_{H,\alpha,\lambda}(t; y) := (t - y)^{H - \frac{\alpha}{2}} e^{-\lambda(t-y)_+} - (-y)^{H - \frac{\alpha}{2}} e^{-\lambda(-y)_+} + \lambda \int_0^t (s - y)^{H - \frac{\alpha}{2}} e^{-\lambda(s-y)_+} ds, \quad y \in \mathbb{R}.$$
The corresponding $\alpha$-stable process, denoted by $Z_{H,\alpha,\lambda}^H = \{Z_{H,\alpha,\lambda}^H(t), t \in \mathbb{R}\}$ is defined for all $H > 0, 1 < \alpha \leq 2, \lambda > 0$ and has stationary increments similarly as $Z_{H,\alpha,\lambda}$. The change of the integrand results in a drastic change of large-time behavior of the increment process

$$Y_{H,\alpha,\lambda}^H = \{Y_{H,\alpha,\lambda}^H(t) := Z_{H,\alpha,\lambda}^H(t + 1) - Z_{H,\alpha,\lambda}^H(t), t \in \mathbb{Z}\}$$

called tempered fractional Gaussian/stable noise of second kind (TFGN II/TFSN II). The spectral density of TFGN II $Y_{H,2,\lambda}^H$ decays as a power function for frequencies $|\omega| > \lambda$ but remains bounded and separated from zero near zero frequency, making TFGN II a realistic model in turbulence and other applied areas. See Figures 1, 2 and Remark 3.2.

One of the main motivation for our introducing and studying is the fact that these processes appear as the limits of the partial sums process of tempered stationary linear processes with discrete time and small tempering parameter $\lambda_N \sim \lambda/N$ tending to zero together with the sample size $N$. This problem is discussed in a parallel paper Sabzikar and Surgailis [17] where we prove that the limit behavior of such partial process essentially depends on how fast the tempering parameter tends to zero, resulting in different limits in the strongly tempered ($\lim_{N \to \infty} \lambda_N/N = 0$), weakly tempered ($\lim_{N \to \infty} \lambda_N/N = \infty$), and moderately tempered ($\lim_{N \to \infty} \lambda_N/N \in (0, \infty)$) situations.

Let us describe the main results of this paper. Section 2 provides the basic definitions and properties of TFBM II/TFSM II. The latter include the spectral representation and the covariance function of TFBM II, relation to tempered fractional calculus (see [11]), and the relation between TFSM and TFSM II. Theorem 2.10 establishes local and global asymptotic self-similarity of TFSM and TFSM II. It shows that TFSM and TFSM II are very different processes; indeed, the former process is stochastically bounded and the latter is stochastically unbounded as $t \to \infty$. Section 3 discusses the dependence properties of stationary processes TFSN II and TFGN II. We obtain the asymptotic behavior of the bivariate characteristic function of TFSN II which can be compared to the corresponding results for TFSN in Meerschaert and Sabzikar in [13] and for FSN in Astrauskas et al. [2].

In what follows, $C$ denotes generic constants which may be different at different locations. We write $\overset{\text{i.d.}}{\Rightarrow}$ and $\overset{\text{i.d.}}{=} = \text{f.d.d.}$ for weak convergence and equality of finite-dimensional distributions, respectively. $\mathbb{R}_+ := (0, \infty), (x)_+ := \max(\pm x, 0), x \in \mathbb{R}$, $\int := \int_{\mathbb{R}}$.

## 2 Definition and properties of TFSM II and TFBM II

For $0 < \alpha \leq 2$, let $\{M_\alpha(t)\}_{t \in \mathbb{R}}$ be an $\alpha$-stable Lévy process with stationary independent increments and characteristic function

$$Ee^{i\theta M_\alpha(t)} = e^{-\sigma|\theta|^\alpha t \left(1 - i\beta \tan(\pi \alpha/2) \text{sign}(\theta)\right)}, \quad \theta \in \mathbb{R}$$

where $\sigma > 0$ and $\beta \in [-1, 1]$ are the scale and skewness parameters, respectively. For $\alpha = 2$, $M_2(t) = \sqrt{2\sigma}B(t)$, where $B$ is a standard Brownian motion with variance $E B^2(t) = t$. Stochastic integral $I_\alpha(f) \equiv \int f(x) M_\alpha(dx)$ is defined for any $f \in L^\alpha(\mathbb{R})$ as $\alpha$-stable random variable with characteristic function

$$Ee^{i\theta I_\alpha(f)} = \exp\{-|\theta|^\alpha \int |f(x)|^\alpha \left(1 - i\beta \tan(\pi \alpha/2) \text{sign}(\theta f(x))\right)dx\}, \quad \theta \in \mathbb{R}$$

see e.g. [16, Chapter 3].

Note the function $y \mapsto h_{H,\alpha,\lambda}(t; y) : \mathbb{R} \to \mathbb{R}$ in (1.2) satisfies $h_{H,\alpha,\lambda}(t; \cdot) \in L^\alpha(\mathbb{R})$ for any $t \in \mathbb{R}$ and any $\lambda > 0, 1 < \alpha \leq 2, H > 0$ and also for $\lambda = 0, 0 < \alpha \leq 2, H \in (0, 1)$. We will use the following integral representation of (1.2):

For $H > \frac{1}{\alpha}$:

$$h_{H,\alpha,\lambda}(t; y) = (H - \frac{1}{\alpha}) \int_0^t (s - y)^{H - \frac{1}{\alpha} - 1} e^{-\lambda(s-y)} ds.$$ (2.3)
For $0 < H < \frac{1}{\alpha}$:
\[
 h_{H,\alpha,\lambda}(t; y) = (H - \frac{1}{\alpha}) \begin{cases} 
 f_0^t (s-y)^{H-\frac{1}{\alpha} - 1} e^{-\lambda(s-y)+} ds, & y < 0, \\
 - \int_{t}^{\infty} (s-y)^{H-\frac{1}{\alpha} - 1} e^{-\lambda(s-y)+} ds + \lambda \frac{1}{\alpha} - H \Gamma(H - \frac{1}{\alpha}), & y \geq 0.
\end{cases}
\]
(2.4)

**Definition 2.1** Let $M_\alpha$ be $\alpha$-stable Lévy process in (2.1), $1 < \alpha \leq 2$ and $H > 0$, $\lambda > 0$. The stochastic process
\[
 Z^H_{H,\alpha,\lambda}(t) := \int h_{H,\alpha,\lambda}(t; y) M_\alpha(dy), \quad t \in \mathbb{R}
\]
will be called tempered fractional stable motion of second kind (TFSM II). A particular case of (2.5) corresponding to $\alpha = 2$
\[
 B^H_{H,\lambda}(t) := \frac{1}{\Gamma(H + \frac{1}{2})} \int h_{H,2,\lambda}(t; y) B(dy)
\]
will be called tempered fractional Brownian motion of second kind (TFBM II).

**Remark 2.2** TFSM II can be also defined for $0 < \alpha \leq 1$, however this definition uses a different kernel from (1.2) for $0 < H < \frac{1}{\alpha} - 1$. The reason is that the tempered fractional derivative $D_{\alpha}^{\kappa,\lambda}$ in (2.1) takes a different form for $\kappa > 1$, see ([11], Remark 2.12).

**Remark 2.3** For $\lambda > 0$ and the same Lévy process $M_\alpha$, TFSM in (1.1) and TFSM II in (2.6) are related as
\[
 Z^H_{H,\alpha,\lambda}(t) = Z_{H,\alpha,\lambda}(t) + \lambda C_{H,\alpha,\lambda}(t),
\]
(2.7)
where
\[
 C_{H,\alpha,\lambda}(t) := \int_0^t Z_{H,\alpha,\lambda}(s) ds + tC^0_{H,\alpha,\lambda}
\]
(2.8)
and
\[
 C^0_{H,\alpha,\lambda} := \int (-y)^{H-\frac{1}{\alpha} - 1} e^{-\lambda(-y)+} M_\alpha(dy)
\]
(2.9)
is a well-defined $\alpha$-stable r.v. Relation (2.7) shows that $Z^H_{H,\alpha,\lambda}$ and $Z_{H,\alpha,\lambda}$ have similar path properties, particularly, path properties of $Z^H_{H,\alpha,\lambda}$ can be derived from the path properties of $Z_{H,\alpha,\lambda}$ studied in [13].

Recall that for any function in $L^p(\mathbb{R})$, where $1 \leq p < \infty$, the (positive and negative) tempered fractional integrals are defined by
\[
 I^\kappa_{\pm} f(y) := \frac{1}{\Gamma(\kappa)} \int f(s)(y-s)^{\kappa-1} e^{-\lambda(y-s)\pm} ds, \quad \kappa > 0
\]
(2.10)
and tempered fractional derivatives are
\[
 D^\kappa_{\pm} f(y) := \lambda^\kappa f(y) + \frac{\kappa}{\Gamma(1 - \kappa)} \int \frac{f(y) - f(s)}{(y-s)^{\kappa+1}} e^{-\lambda(y-s)\pm} ds, \quad 0 < \kappa < 1.
\]
(2.11)

The following proposition shows that TFSM II can be written as a stochastic integral of tempered fractional integral (derivative) of the indicator function of the interval $[0, t]$. In contrast, the corresponding expression for TFSM in [11] is more complicated and involves a linear combination of tempered integrals and derivatives of the indicator function.

**Proposition 2.4** Let $H > 0$, $1 < \alpha \leq 2$, and $\lambda > 0$. Then
\[
 I^H_{\pm} \chi_{[0, t]}(y) = \Gamma(H + 1 - \frac{1}{\alpha})^{-1} h_{H,\alpha,\lambda}(t; y), \quad H > \frac{1}{\alpha},
\]
(2.12)
\[
 D^H_{\pm} \chi_{[0, t]}(y) = \Gamma(H + 1 - \frac{1}{\alpha})^{-1} h_{H,\alpha,\lambda}(t; y), \quad 0 < H < \frac{1}{\alpha}.
\]
(2.13)
Thus, TFSM II for $1 < \alpha \leq 2, H > 0$ can be defined as

$$Z_{H,\alpha,\lambda}^I(t) = \Gamma(H + 1 - \frac{1}{\alpha}) \left\{ \begin{array}{ll}
\int_{-\infty}^{H-\frac{1}{2}-\lambda} 1_{[0,t]}(y) M_\alpha(dy), & H > \frac{1}{\alpha}, \\
\int_{H-\frac{1}{2}-\lambda}^{\infty} 1_{[0,t]}(y) M_\alpha(dy), & 0 < H < \frac{1}{\alpha}.
\end{array} \right. \quad (2.14)$$

**Proof.** Apply the tempered fractional operator $F_\kappa \hat{f}$ in (2.10) for $\kappa = H - \frac{1}{\alpha}, \lambda > 0$ and $f := 1_{[0,t]}$ to see that

$$\hat{1}_{-\frac{1}{\alpha} - \lambda} 1_{[0,t]}(y) = \frac{1}{\Gamma(H - \frac{1}{\alpha} + 1)} \int_0^{t} \left[ (t-y)_{+}^{H-\frac{1}{2}} e^{-\lambda(t-y)+} - (y)_{+}^{H-\frac{1}{2}} e^{-\lambda(y)+} + \lambda \int_0^{t} (s-y)_{+}^{H-\frac{1}{2}} e^{-\lambda(s-y)+} ds \right]$$

$$= \frac{1}{\Gamma(H - \frac{1}{\alpha} + 1)} h_{H,\alpha,\lambda}(t;y),$$

see (1.2). This proves (2.12) and (2.13) follows similarly. \(\square\)

**Remark 2.5** Note for $H = 1/\alpha$, $Z_{1/\alpha,\alpha,\lambda}^I = M_\alpha$ is a stable Lévy process for any $\lambda \geq 0$ which follows from (2.14) and also from (2.5) by exchanging the order of integration:

$$Z_{1/\alpha,\alpha,\lambda}^I(t) = \int_{-\infty}^{t} e^{-\lambda(t-y)} M_\alpha(dy) - \int_{-\infty}^{0} e^{-\lambda(y)+} M_\alpha(dy) \quad (2.15)$$

$$+ \lambda \int_0^{t} ds \int_{-\infty}^{t} e^{-\lambda(s-y)} M_\alpha(dy) = \int_0^{t} M_\alpha(dy) = M_\alpha(t).$$

On the other hand,

$$Z_{1/\alpha,\alpha,\lambda}^I(t) = Y_{\alpha,\lambda}(t) - Y_{\alpha,\lambda}(0) \quad (2.16)$$

where $Y_{\alpha,\lambda}(t) := \int_{-\infty}^{t} e^{-\lambda(t-y)} M_\alpha(dy), t \in \mathbb{R}$ is stationary Ornstein-Uhlenbeck process.

The next proposition gives the spectral domain representation of TFBM II.

**Proposition 2.6** Let $H > 0, \lambda > 0$. The TFBM II $B_{H,\lambda}^I$ in (2.6) has the spectral domain representation

$$B_{H,\lambda}^I(t) \overset{\text{idd}}{=} \frac{1}{\sqrt{2\pi}} \int \frac{e^{i\omega t} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H} \hat{B}(d\omega), \quad (2.17)$$

where $\hat{B}$ is an even complex-valued Gaussian white noise, $\overline{\hat{B}}(dx) = \hat{B}(-dx)$, with zero mean and variance $E|\hat{B}(dx)|^2 = dx$.

**Proof.** From (2.12), $h_{H,2,\lambda}(t;y) = \Gamma(H + \frac{1}{2}) 1_{-\frac{1}{2} - \lambda} 1_{[0,t]}(y)$ for $H > \frac{1}{2}$ and then

$$\widehat{h_{H,2,\lambda}}(t;\omega) = \Gamma(H + \frac{1}{2}) \mathcal{F}[1_{-\frac{1}{2} - \lambda} 1_{[0,t]}](\omega) = \frac{\Gamma(H + \frac{1}{2})}{\sqrt{2\pi}} \frac{e^{i\omega t} - 1}{i\omega} (\lambda + i\omega)^{\frac{1}{2}-H},$$

where we used the Fourier transform of tempered fractional integrals (see Lemma 2.6 in [11]). In the case $0 < H < \frac{1}{2}$ the same expression for $\widehat{h_{H,2,\lambda}}(t;\omega)$ follows by a similar argument. Then (2.17) follows from Parseval’s formula for stochastic integrals, viz., $\int h_{H,2,\lambda}(t;y) B(dy) \overset{d}{=} \int \widehat{h_{H,2,\lambda}}(t;\omega) \hat{B}(d\omega)$, see Proposition 7.2.7 in [16]. \(\square\)

**Remark 2.7** Meerschaert and Sabzikar [12] obtained spectral representation of TFBM:

$$B_{H,\lambda}(t) = \Gamma(H + \frac{1}{2}) \frac{1}{\sqrt{2\pi}} \int \frac{e^{i\omega t} - 1}{(\lambda + i\omega)^{H+\frac{1}{2}}} \hat{B}(d\omega) \quad (2.18)$$

where $H > 0, \lambda > 0$. By comparing spectral densities, it can be shown that for $\lambda > 0$ and $\sigma > 0$, $B_{H,\lambda}$ and $\sigma B_{H,\lambda}^I$ are different processes (indeed, the coincidence of these spectral densities would imply that $|(\lambda + i\omega)/\omega|^2$ is a constant function of $\omega$ which is possible if and only if $\lambda = 0$.)
The next proposition summarizes basic properties of TFSM II $Z_{H,\alpha,\lambda}^H$.

**Proposition 2.8** (i) $Z_{H,\alpha,\lambda}^H$ in (2.5) is well-defined for any $t \geq 0$ and $1 < \alpha \leq 2, H > 0, \lambda > 0$, as a stochastic integral in (2.2).

(ii) $Z_{H,\alpha,\lambda}^H$ in (2.5) has stationary increments and $\alpha$-stable finite-dimensional distributions. Moreover, it satisfies the following scaling property:

$$\{Z_{H,\alpha,\lambda}^H(bt)\}_{t \in \mathbb{R}} \overset{f.d.d.}{=} \{b^H Z_{H,\alpha,\lambda}^H(t)\}_{t \in \mathbb{R}}, \quad \forall \ b > 0. \tag{2.19}$$

(iii) $Z_{H,\alpha,\lambda}^H$ in (2.5) has a.s. continuous paths if either $\alpha = 2, H > 0$, or $1 < \alpha < 2, H > 1/\alpha$ hold.

(iv) The variance and covariance of TFSM II $B_{H,\alpha}^H (H > 0, \lambda > 0)$ has the form

$$C_t^2 = \mathbb{E}[ (B_{H,\alpha}^H(t))^2 ] = \frac{1}{2\pi} \int \left| \frac{\omega^2 - 1}{\iota \omega} \right|^{\frac{1}{2} - H} d\omega$$

$$= -\frac{2\Gamma(H)\lambda^{-2H}}{\sqrt{\pi} \Gamma(H - 1/2)} \left[ 1 - 2F_3 \left( \{1, -1/2\}, \{1 - H, 1/2, 1\}, \lambda^2 t^2 / 4 \right) \right]$$

$$+ \frac{\Gamma(1 - H)}{\sqrt{\pi} H 2^{2H} \Gamma(H + 1/2)} 2F_3 \left( \{1, H - 1/2\}, \{1, H + 1, H + 1/2\}, \lambda^2 t^2 / 4 \right), \tag{2.20}$$

and

$$\text{Cov} \left[ B_{H,\alpha}^H(t), B_{H,\alpha}^H(s) \right] = \frac{1}{2} \left[ C_t^2 + C_s^2 - C_{t-s}^2 \right], \quad s, t \in \mathbb{R}, \tag{2.21}$$

where $C_t^2$ is given in (2.20) and $2F_3$ is the generalized hypergeometric function. In particular,

$$\mathbb{E}(B_{H,0}^H(t))^2 = \frac{t^{2H}}{\Gamma(2H + 1/2)} \int_{-\infty}^{t} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right)^2 ds$$

$$= \frac{t^{2H}}{\sqrt{\pi} H 2^{2H} \Gamma(H + 1/2)}, \quad 0 < H < 1. \tag{2.22}$$

**Proof.** (i) Follows from $h_{H,\alpha,\lambda}(t, \cdot) \in L^0(\mathbb{R})$, see above, also [13].

(ii) Stationarity of increments follows from the invariance properties $h_{H,\alpha,\lambda}(t + T; y) = h_{H,\alpha,\lambda}(T; y) = h_{H,\alpha,\lambda}(t; y - T)$ and $\{M_\alpha(y + T) - M_\alpha(T)\} \overset{\text{f.d.d.}}{=} \{M_\alpha(y)\}, \forall T \geq 0$. Similarly, property (2.19) follows from the scaling properties $h_{H,\alpha,\lambda}/h(t; y) = b^{H-1} h_{H,\alpha,\lambda}(t; y)$ and $\{M_\alpha(bt)\} \overset{\text{f.d.d.}}{=} \{b^{1/\lambda} M(t)\}, \forall b > 0$.

(iii) We use the Kolmogorov criterion, see ([3], Theorem 12.4). Since $\lambda > 0$ is fixed, we can assume $\lambda = 1$ w.l.g. First, let $\alpha = 2, H > 0$. Then $Z_{H,\alpha,1}^H = C B_{H,1}^H$ is a Gaussian process with stationary increments. Accordingly, it suffices to prove $\mathbb{E}[|B_{H,1}^H(t)|^p] \leq C t^\gamma$ for some $p > 0, \gamma > 1$ and all $0 < t < 1$. By Gaussianity, $\mathbb{E}[|B_{H,1}^H(t)|^p] \leq C(\mathbb{E}[|B_{H,1}^H(t)|^2])^{p/2}$. We have

$$\mathbb{E}[|B_{H,1}^H(t)|^2] = C \int_{-\infty}^{t} h_{H,2,1}^1(t; y) dy = C(I_1 + I_2),$$

where

$$I_1 = \int_{-\infty}^{t} h_{H,2,1}^2(t; y) dy \leq C \int_{0}^{2t} (t-y)^{2(H-\frac{1}{2})} dy + C t^2 \int_{0}^{2t} s^{H-\frac{1}{2}} ds \leq C t^{2H}$$

and

$$I_2 = \int_{-\infty}^{t} h_{H,2,1}^2(t; y) dy \leq C \int_{0}^{\infty} ((t+y) H^{-\frac{1}{2}} e^{-t-y} - y H^{-\frac{1}{2}} e^{-y})^2 dy$$

$$+ C \int_{0}^{t} \left( \int_{0}^{t} (s+y)^{H-\frac{1}{2}} e^{-s-y} ds \right) dy = C(I_2 + I_2').$$

Using $|(t+y) H^{-\frac{1}{2}} e^{-t-y} - y H^{-\frac{1}{2}} e^{-y}| \leq |e^{-t-1} - e^{-y}| (t+y) H^{-\frac{1}{2}} - y H^{-\frac{1}{2}}$, we obtain $I_2' \leq C t^2$ and, similarly, $I_2' \leq C t^2$, implying $I_1 + I_2 \leq C(t^{2H} + t^2)$ and $\mathbb{E}[|B_{H,1}^H(t)|^p] \leq C(t^{2H} + t^2)^{p/2}$. Hence,
the above inequality is satisfied with \( p > (1/H) \lor 1 \).

Next, let \( 1 < \alpha < 2, H > \frac{1}{\alpha} \). Similarly as above, it suffices to prove \( \mathbb{E}|Z_{H,\alpha,1}^H(t)|^p \leq Ct^\gamma \) for some \( 1 < p < \alpha, \gamma > 1 \) and all \( 0 < t < 1 \). According to well-known moment inequality, \( \mathbb{E}|Z_{H,\alpha,1}^H(t)|^p = C \int_{-\infty}^t |h_{H,\alpha,1}(t; y)|^p dy = C(I_1 + I_2) \), where \( I_1 = \int_{-\infty}^t |h_{H,\alpha,1}(t; y)|^p dy \leq C \int_0^{2t}(t-y)^p(H - \frac{1}{2})dy + Ct^p(\int_0^{2t} s^{H - \frac{1}{2}} ds)^p \leq Ct^{1+p(H - \frac{1}{2})} \) where \( \gamma = 1 + p(H - \frac{1}{2}) > 1 \) due to \( H > \frac{1}{\alpha} \). The bound \( I_2 = \int_{-\infty}^t |h_{H,\alpha,1}(t; y)|^p dy \leq Ct^p \) can proved similarly as in the case \( \alpha = 2 \) above. This proves part (iii).

(iv) (2.20) follows from the spectral representation in (2.17) and the formula for integral transform in (Prudnikov et al. [15], p.379), see also ([1], Lemma 2.1). In turn, (2.21) follows from (2.20) and stationarity of increments of \( B_{H,\lambda}^H \). Proposition 2.8 is proved.

**Remark 2.9** (i) Property (2.19), as well as other properties in Proposition 2.8 (ii), are also shared by the TFSM in (1.1), see ([13], Prop.2.3). It is related to the scaling property for two-parameter processes introduced in [7].

(ii) For \( H > 1/2 \), the covariance function of TFBM II \( B_{H,\lambda}^H \) admits the integral representation

\[
\mathbb{E}B_{H,\lambda}^H(t)B_{H,\lambda}^H(s) = C(H, \lambda) \int_0^t \int_0^s |u - v|^{H-1} K_{H-1}(\lambda|u-v|) dv du,
\]

where \( C(H, \lambda) = \frac{1}{\sqrt{\pi} \Gamma(H - \frac{1}{2})(2\lambda)^{H-1}} \) and \( K_{H-1} \) is the modified Bessel function of second kind. Formula (2.23) follows from [6, p.344] using the representation of \( h_{H,2,\lambda} \) in (2.3). For \( H > 1 \) the integrand in (2.23), viz.,

\[
\frac{1}{\sqrt{\pi} \Gamma(H - \frac{1}{2})(2\lambda)^{H-1}} |u - v|^{H-1} K_{H-1}(\lambda|u-v|)
\]

is the Matérn covariance function (in one dimension) with shape parameter \( \nu = H - 1 > 0 \), scale parameter \( \lambda > 0 \), and variance parameter 1, see e.g. ([4], (1.1)). Note that the integral in (2.23) diverges when \( 0 < H < 1/2 \). A related integral albeit more complex representation of the covariance function of \( B_{H,\lambda}^H \) can be obtained for \( 0 < H < 1/2 \), too, but we do not include it in the present paper.

The following theorem discusses local and global scaling properties of TFSM and TFSM II.

**Theorem 2.10** Let \( 1 < \alpha \leq 2, 0 < H < 1 \) and \( \lambda > 0 \).

(i) As \( b \to \infty \)

\[
b^{-1/\alpha} Z_{H,\alpha,\lambda}^H(bt) \xrightarrow{fdd} c_{H,\alpha,\lambda} M_{\alpha}(t) \quad \text{and} \quad Z_{H,\alpha,\lambda}^H(bt) \xrightarrow{fdd} C_{H,\alpha,\lambda}^+ - C_{H,\alpha,\lambda}^-,
\]

where \( c_{H,\alpha,\lambda} = \lambda^{\frac{1}{\alpha} - H} \Gamma(1 + H - \frac{1}{\alpha}) \) and \( C_{H,\alpha,\lambda}^+, C_{H,\alpha,\lambda}^- \) are independent copies of \( \alpha \)-stable random variable \( C_{H,\alpha,\lambda}^0 \) in (2.9).

(ii) As \( b \to 0 \)

\[
b^{-H} Z_{H,\alpha,\lambda}^H(bt) \xrightarrow{fdd} c_{H,\alpha} Z_{H,\alpha,0}^H(t) \quad \text{and} \quad b^{-H} Z_{H,\alpha,\lambda}^H(bt) \xrightarrow{fdd} c_{H,\alpha} Z_{H,\alpha,0}(t),
\]

where \( c_{H,\alpha} \) is defined in (2.33) below and \( Z_{H,\alpha,0} = Z_{H,\alpha,0}^H \) is fractional stable motion.

**Proof.** For simplicity of notation we shall assume that \( \beta = 0 \) (\( M_{\alpha} \) is symmetric) and \( \sigma = 1 \) in (2.1). The proof in the general case is analogous.

(i) Consider the first relation in (2.25). It suffices to prove the convergence of characteristic functions

\[
\mathbb{E} \exp \{ ib^{-\frac{1}{\alpha}} \sum_{j=1}^m \theta_j (Z_{H,\alpha,\lambda}^H(bt_j) - Z_{H,\alpha,\lambda}^H(bt_{j-1})) \} \to \mathbb{E} \exp \{ i \sum_{j=1}^m \theta_j (M_{\alpha}(t_j) - M_{\alpha}(t_{j-1})) \}, \quad b \to \infty
\]
for any $0 = t_0 < t_1 < \cdots < t_m, \theta_j \in \mathbb{R}, m = 1, 2, \ldots$, or

$$
    b^{-1} \int_{\mathbb{R}} \sum_{j=1}^{m} \theta_j (h_{H,\alpha,\lambda}(bt_j;y) - h_{H,\alpha,\lambda}(bt_{j-1};y)) |^\alpha dy \to \sum_{j=1}^{m} |\theta_j|^\alpha (t_j - t_{j-1})
$$

(2.27)
as $b \to \infty$. Consider (2.27) for $m = 1, t_1 = 1$ (the proof in the general case seems similar). This follows from

$$
    b^{-1} \int_{0}^{b} |h_{H,\alpha,\lambda}(b;y)|^\alpha dy \to c^{\alpha}_{H,\alpha,\lambda}
$$

and

$$
    b^{-1} \int_{-\infty}^{0} |h_{H,\alpha,\lambda}(b;y)|^\alpha dy \to 0.
$$

(2.28)

Note for each $y > 0$

$$
    c_{H,\alpha,\lambda} = \lim_{b \to \infty} h_{H,\alpha,\lambda}(b;y) = \lim_{b \to \infty} \lambda \int_{0}^{b} (s-y)^{H-\frac{\lambda}{2}} e^{-\lambda(s-y)} ds.
$$

(2.29)

Since $b^{-1} \int_{0}^{b} |h_{H,\alpha,\lambda}(c;y)|^\alpha dy \leq 2(1 + \lambda) \int_{0}^{\infty} z^{H-1} e^{-\lambda z} dz < \infty$ for all $b \geq 1$, the first convergence in (2.28) follows from (2.29) and the dominated convergence theorem. The second convergence in (2.28) follows by noting that

$$
    \limsup_{b \to \infty} \int_{-\infty}^{0} |h_{H,\alpha,\lambda}(b;y)|^\alpha dy < \infty
$$

(2.30)

This proves the first convergence in (2.25).

Consider the second convergence in (2.25) for $t = 1$. It suffices to show that

$$
    \int_{-\infty}^{b} \left| g_{H,\alpha,\lambda}(b;y) \right|^\alpha dy \to 2 \int_{0}^{\infty} y^{H-1} e^{-\lambda y} dy = \frac{2 \Gamma(H\alpha)}{(\lambda\alpha)^{H\alpha}}
$$

(2.31)
as $b \to \infty$. Here,

$$
    g_{H,\alpha,\lambda}(t;y) := (t-y)^{H-\frac{\lambda}{2}} e^{-\lambda(t-y)^{+}} - (t+y)^{H-\frac{\lambda}{2}} e^{-\lambda(y)^{+}}, \quad y \in \mathbb{R}
$$

(2.32)
is the kernel of TFSM (1.1). Relation (2.31) follows from $\int_{0}^{b} \left| g_{H,\alpha,\lambda}(b;y) \right|^\alpha dy \to \Gamma(H\alpha)/(\lambda\alpha)^{H\alpha}$ and $\int_{-\infty}^{0} \left| g_{H,\alpha,\lambda}(b;y) \right|^\alpha dy \to \Gamma(H\alpha)/(\lambda\alpha)^{H\alpha}$ using the dominated convergence theorem. The general finite-dimensional convergence follows analogously. This proves part (i).

(ii) As in part (i), we restrict the proof of (2.26) to one-dimensional convergence at $t = 1$ since the general case follows analogously. Consider the first convergence in (2.26). It suffices to show

$$
    b^{-\alpha H} \int_{\mathbb{R}} \left| h_{H,\alpha,\lambda}(b;y) \right|^\alpha dy \to c_{H,\alpha}^{\alpha} = \int_{\mathbb{R}} \left| h_{H,\alpha,\lambda}(1;y) \right|^\alpha dy
$$

(2.33)
as $b \to 0$. By change of variables, we have $b^{-\alpha H} \int_{\mathbb{R}} \left| h_{H,\alpha,\lambda}(b;y) \right|^\alpha dy = \int_{\mathbb{R}} \left| h_{H,\alpha,\lambda}(b;1;y) \right|^\alpha dy$ where $h_{H,\alpha,\lambda}(b;1;y) \to h_{H,\alpha,\lambda}(1;y)$ ($b \to 0$) for each $y \in \mathbb{R}, y \neq 1$ and the convergence of integrals in (2.33) can be justified by the dominated convergence theorem. The second convergence in (2.26) for $t = 1$ follows similarly from $b^{-\alpha H} \int_{\mathbb{R}} \left| g_{H,\alpha,\lambda}(b;y) \right|^\alpha dy \to c_{H,\alpha}^{\alpha}$ and we omit the details. Theorem 2.10 is proved.

3 Dependence properties of TFSN II

Recall the definition of tempered fractional stable noise (TFSN II) $Y_{H,\alpha,\lambda}^{H}$ in (1.2), which is a stationary process with discrete time $t \in \mathbb{Z}$. The following proposition obtains the spectral representation and spectral density of this process, or TFGN II

$$
    Y_{H,\alpha,\lambda}^{H} = \{Y_{H,\alpha,\lambda}^{H}(t) := B_{H,\lambda}^{H}(t+1) - B_{H,\lambda}^{H}(t), \quad t \in \mathbb{Z}\}
$$

(3.1)
in the Gaussian case $\alpha = 2$. 

7
Proposition 3.1 TFGN II in (3.2) has spectral representation

\[ Y_{H,\lambda}(t) = \frac{1}{\sqrt{2\pi}} \int e^{i\omega t} \frac{e^{i\omega} - 1}{i\omega} \left( \frac{i}{\lambda^2} \right)^{\frac{1}{2}} \tilde{B}(\omega), \quad t \in \mathbb{Z} \]  

(3.2)

and spectral density

\[ h(\omega) = \frac{1}{2\pi} \left\{ \left( \frac{e^{i\omega} - 1}{i\omega} \right)^2 \lambda^2 + \omega^2 \right\}^{\frac{1}{2} - H} \frac{1}{\lambda^2 + \omega^2 \left( \lambda^2 + (\omega + 2\pi\ell)^2 \right)^{H - \frac{1}{2}}} \right\}, \quad \omega \in [-\pi, \pi]. \]  

(3.3)

Proof. (3.2) is immediate from (2.17). Whence it follows that the covariance

\[ r(j) = \frac{1}{2\pi} \int e^{i\omega j} \left| \frac{e^{i\omega} - 1}{i\omega} \right|^2 \lambda^2 + \omega^2 \left( \lambda^2 + \omega^2 \right)^{\frac{1}{2} - H} d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega j} \left| e^{i\omega} - 1 \right|^2 \sum_{\ell \in \mathbb{Z}} \left( \omega + 2\pi\ell \right)^{-2} \left( \lambda^2 + (\omega + 2\pi\ell)^2 \right)^{\frac{1}{2} - H} d\omega, \quad j \in \mathbb{Z} \]  

(3.4)

implying (3.3) and the proposition. \(\square\)

Note that the spectral density in (3.3) is bounded, continuous and separated from zero on the whole interval \([-\pi, \pi]\).

Particularly, for any \( H, \lambda > 0 \)

\[ h(0) = \lim_{\omega \to 0} h(\omega) = \lambda^{1-2H}/2\pi \in (0, \infty). \]

Using the popular terminology (see e.g. [5], Definition 3.1.4) we may say that TFGN II \( Y_{H,\lambda} \) has short memory in the frequency domain for any \( H, \lambda > 0 \) while TFGN has negative memory in the frequency domain since its spectral density vanishes at \( \omega = 0 \). The distinctions in low frequency spectrum between the two processes are illustrated in Figures 1 and 2.

Remark 3.2 In the celebrated Kolmogorov’s model of turbulence ([8], [9]), the spectral density of turbulent velocity data in the inertial range is proportional to \( \omega^{-5/3} \) for moderate frequencies \( \omega \) and is bounded at low frequencies. Such behavior can be exhibited by both TFGN and TFGN II with \( H = 4/3 \), see (3.3). Meerschaert et al. [14] validate ARTFIMA(0, \( d, \lambda, 0 \)) model on turbulent water velocities in the Great Lakes region. They find that this model effectively captures both the correlation properties and the underlying probability distribution of this data, with \( d = 5/6 \) corresponding to Kolmogorov’s law, and small tempering parameter \( \lambda = 0.006 \). Since ARTFIMA(0, \( d, \lambda, 0 \)) is closely related to TFGN II, see [17], we expect that the latter model can be also successfully applied for modeling of turbulent data.

Next, we discuss dependence properties of TFSN II with \( 1 < \alpha < 2 \). Since this process has infinite variance, other numerical characteristics extending the notion of covariance must be used to characterize the decay rate of dependence.
It follows from (2.5) that
\[
\text{Proof.}
\]
For concreteness, assume
\[
\text{Let } Z_{\omega, h, \alpha, \lambda}(t_1) \text{ and } Y_{\omega, h, \alpha, \lambda}(t_2) \text{ as } t_2 - t_1 \to \infty. \text{ Astrauskas et al. [2], Samorodnitsky and Taqqu [16], Koul and Surgailis [10] discussed the asymptotics as } t \to \infty \text{ of the bivariate characteristic function}
\]
\[
\rho_Y(t_1, t_2) = E[e^{i(t_1 Y(t_1) + t_2 Y(t_2))}] - E[e^{i t_1 Y(t_1)}]E[e^{i t_2 Y(t_2)}], \quad t_1, t_2 \in \mathbb{R}
\]
for some stationary processes \( Y = \{Y(t), t \in \mathbb{Z}\} \) with infinite variance, including fractional stable noise (FGN) and long memory moving average with infinite variance innovations. Meerschaert and Sabzikar [13] studied the asymptotic behavior of (3.5) for TFSN \( Y_{\omega, h, \alpha, \lambda}(t) = Z_{\omega, h, \alpha, \lambda}(t + 1) - Z_{\omega, h, \alpha, \lambda}(t), t \in \mathbb{Z}, \) where \( Z_{\omega, h, \alpha, \lambda} \) is the TFSM in (1.1). Given two real-valued functions \( f(t), g(t) \) on \( \mathbb{R}, \) we will write \( f(t) \asymp g(t) \) if \( C_1 \leq |f(t)/g(t)| \leq C_2 \) for all \( t > 0 \) sufficiently large, for some \( 0 < C_1 < C_2 < \infty. \) ([13], Theorem 2.6) proved that for \( 1 < \alpha < 2, \frac{1}{\alpha} < H < 1 \)
\[
r_Y(t; \theta_1, \theta_2) \asymp e^{-\lambda t^{H-\frac{1}{\alpha}}}, \quad t \to \infty.
\]
Below, we prove the following result about the behavior of (3.5) for TFSN II which is another indication that TFSM and TFSM II are different processes.

**Theorem 3.3** Let \( Z_{H, \alpha, \lambda}^{H II} \) be a TFSM II in (2.5) with SSO noise \( M_\alpha, \) \( 1 < \alpha \leq 2, \frac{1}{\alpha} < H < 1, \lambda > 0, \) and \( Y_{H, \alpha, \lambda}^{H II}(t) \) be TFSM II in (1.2). Then for any \( \theta_1, \theta_2 \neq 0 \)
\[
r_{Y_{H, \alpha, \lambda}^{H II}}(t; \theta_1, \theta_2) \asymp e^{-\lambda t^{H-\frac{1}{\alpha}}}, \quad t \to \infty.
\]

**Proof.** For concreteness, assume \( \theta_1 \theta_2 > 0, \) the case \( \theta_1 \theta_2 < 0 \) being analogous. Write \( r(t) = r_{Y_{H, \alpha, \lambda}^{H II}}(t; \theta_1, \theta_2), Y(t) = Y_{H, \alpha, \lambda}^{H II}(t), h(t; x) = h_{H, \alpha, \lambda}(t; x) \) for brevity. Then \( r(t) = K(e^{-t} - 1), \) where \( K = K(\theta_1, \theta_2) = E[e^{i \theta_1 Y(0)}]E[e^{i \theta_2 Y(0)}] > 0 \)
\[
I(t) = I(t; \theta_1, \theta_2) := ||\theta_1 Y(t) + \theta_2 Y(0)||_\alpha^\alpha - ||\theta_1 Y(t)||_\alpha^\alpha - ||\theta_2 Y(0)||_\alpha^\alpha.
\]
Clearly, it suffices to prove (3.7) for \( I(t), \) viz.,
\[
I(t) \asymp e^{-\lambda t^{H-\frac{1}{\alpha}}}, \quad t \to \infty.
\]
It follows from (2.5) that \( Y(t) = \int_t^\infty g(t; x) M_\alpha(dx) \) where \( g(t; x) = h(t + 1; x) - h(t; x). \) From the representation (2.3) we obtain
\[
g(t; x) = (H - \frac{1}{\alpha}) \int_x^{t+1} (s - x)_+^{H-\frac{1}{\alpha}-1} e^{-\lambda (s-x)_+} ds, \quad -\infty < x < t.
\]
We have

\[ I(t) = \int_{-\infty}^{1} \{ |\theta_1 g(t; x) + \theta_2 g(0; x)|^{\alpha} - |\theta_1 g(t; x)|^{\alpha} - |\theta_2 g(0; x)|^{\alpha} \} \, dx \]

Then \( g(t; x) > 0 \), \( x < t + 1 \) and for any \( -\infty < x < 1 < t \) we obtain

\[
\frac{g(t; x)}{g(0; x)} = \frac{\int_{t}^{t+1} (s-x)^{H-\frac{1}{\alpha}-1} e^{-\lambda(s-x)} \, ds}{\int_{0}^{1} (s-x)^{H-\frac{1}{\alpha}-1} e^{-\lambda(s-x)} \, ds} = \frac{\int_{t}^{t+1} (s-x)^{H-\frac{1}{\alpha}-1} e^{-\lambda s} \, ds}{\int_{0}^{1} (s-x)^{H-\frac{1}{\alpha}-1} e^{-\lambda s} \, ds} \leq e^{-\lambda(t-t_1)} \frac{g^0(t; x)}{g^0(0; x)}
\]

where

\[
g^0(t; x) := (H - \frac{1}{\alpha}) \int_{t}^{t+1} (s-x)^{H-\frac{1}{\alpha}-1} \, ds = (t+1-x)^{H-\frac{1}{\alpha}} - (t-x)^{H-\frac{1}{\alpha}}.
\]

Similarly,

\[
\frac{g(t; x)}{g(0; x)} = \frac{\int_{t}^{t+1} (s-x)^{H-\frac{1}{\alpha}-1} e^{-\lambda s} \, ds}{\int_{0}^{1} (s-x)^{H-\frac{1}{\alpha}-1} e^{-\lambda s} \, ds} \geq e^{-\lambda(t+1)} \frac{g^0(t; x)}{g^0(0; x)}
\]

implying

\[
C_1 e^{-\lambda t} \frac{g^0(t; x)}{g^0(0; x)} \leq \frac{g(t; x)}{g(0; x)} \leq C_2 e^{-\lambda t} \frac{g^0(t; x)}{g^0(0; x)} \tag{3.11}
\]

for some constants \( 0 < C_1 < C_2 < \infty \) independent of \( -\infty < x < 1 < t \). Then \( I(t) = I_1(t) - I_2(t) \), where

\[
I_1(t) := \int_{-\infty}^{1} |\theta_2 g(0; x)|^{\alpha} \left\{ 1 + \frac{\theta_1}{\theta_2} \frac{g(t; x)}{g(0; x)} \right\} \, dx, \quad I_2(t) := \int_{-\infty}^{1} |\theta_1 g(t; x)|^{\alpha} \, dx.
\]

Next, split \( I_1(t) = \int_{-\infty}^{0} \cdots + \int_{0}^{1} \cdots =: I'_1(t) + I''_1(t) \). Note \( g^0(t; x) \leq g^0(0; x) \) for \( x < 0 \), \( t > 0 \) and therefore \( \sup_{x < 0} \frac{g(t; x)}{g^0(0; x)} \leq C_2 e^{-\lambda t} \to 0 \) as \( t \to \infty \). Therefore and from (3.11) we obtain for \( x < 0 \)

\[
C_1 e^{-\lambda t} \frac{g^0(t; x)}{g^0(0; x)} \leq \frac{g(t; x)}{g(0; x)} \leq C_2 e^{-\lambda t} \frac{g^0(t; x)}{g^0(0; x)} \tag{3.12}
\]

This implies

\[
I'_1(t) \asymp e^{-\lambda t} J(t), \quad \text{where} \quad J(t) = \int_{-\infty}^{0} \frac{g(0; x)}{g^0(0; x)} \frac{g^0(t; x)}{g^0(0; x)} \, dx. \tag{3.13}
\]

Let us prove that

\[
J(t) \asymp t^{H-\frac{1}{\alpha}-1}. \tag{3.14}
\]

Split \( J(t) = \int_{-\infty}^{0} + \int_{0}^{1} \cdots =: J_1(t) + J_2(t) \). Then since \( g(0; x) \leq g^0(0; x) \), \( x < 0 \) we have that \( J_2(t) \leq C \int_{0}^{1} ((t+1+x)H-\frac{1}{\alpha} - (t+x)H-\frac{1}{\alpha}) \, dx \leq Ct^{1-H-\frac{1}{\alpha}}. \) Next, using \( C_1 z^{H-\frac{1}{\alpha}-1} \leq (z+1)^{H-\frac{1}{\alpha}} - z^{H-\frac{1}{\alpha}} \leq C_2 z^{H-\frac{1}{\alpha}-1}, z \geq 1 \) we obtain

\[
J_1(t) \asymp \int_{2}^{\infty} e^{-\lambda x} \left((1+x)H-\frac{1}{\alpha} - xH-\frac{1}{\alpha}\right)^{\alpha-1} ((t+1+x)H-\frac{1}{\alpha} - (t+x)H-\frac{1}{\alpha}) \, dx
\]

\[
\asymp \int_{2}^{\infty} e^{-\lambda x} x^{H-\frac{1}{\alpha}-1} (t+1+x)^{H-\frac{1}{\alpha}} \, dx
\]

proving (3.14). Next, let us show that

\[
I''_1(t) \leq C e^{-\lambda t} t^{H-\frac{1}{\alpha}} \tag{3.15}
\]

Indeed, from (3.11) and the definition of \( g(0; x) \) and \( g^0(t; x) \) it follows that

\[
I''_1(t) \leq C \int_{0}^{1} (1-x)^{(H-\frac{1}{\alpha})\alpha - 1} \left( 1 + C \frac{e^{-\lambda t} t^{H-\frac{1}{\alpha}}}{(1-x)^{H-\frac{1}{\alpha}}} \right)^{\alpha - 1} \, dx \leq C e^{-\lambda t} t^{H-\frac{1}{\alpha} - 1},
\]

proving (3.15). Relation

\[
I_1(t) \asymp e^{-\lambda t} t^{H-\frac{1}{\alpha}-1} \tag{3.16}
\]
follows from (3.13), (3.14), (3.15) and the fact that $I_1'(t) \geq 0, I_2'(t) \geq 0$, due to $\theta_1 \theta_2 > 0$.

It remains to show

$$I_2(t) = o(e^{-\lambda t^{H-\frac{1}{\alpha}-1}}).$$  \hspace{1cm} (3.17)

Split $I_2(t) = \int_{-\infty}^{0} \cdots + \int_{0}^{1} \cdots = I_2^0(t) + I_2^1(t)$. According to (3.11), $I_2^0(t) \leq C e^{-\alpha \lambda t} \tilde{J}(t)$ where

$$\tilde{J}(t) = \int_{-\infty}^{0} \left( \frac{g^0(t;x)}{g^0(0;x)} \right)^{\alpha} g^0(0;x) dx \leq J(t)$$

since $g^0(t;x) \leq g^0(0;x), x < 0$, with $J(t)$ as in (3.13), (3.14). Then, $I_2^0(t)$ satisfies (3.17) since $\alpha > 1$. Finally, since $g^0(t;x) \leq C e^{-\lambda t^{\frac{1}{\alpha}-1}}, 0 < x < 1$ so $I_2^0(t) \leq C e^{-\alpha \lambda t(H-\frac{1}{\alpha}-1)} = o(e^{-\lambda t^{H-\frac{1}{\alpha}-1}})$, ending the proof of (3.17) and (3.9). Theorem 3.3 is proved. \ \Box

Remark 3.4 Note that in the case $\lambda = 0$, the exponent $H-\frac{1}{\alpha}-1$ in (3.7) does not agree with the corresponding exponent for (untempered) fractional stable noise in Astrauskas et al. [2], including the Gaussian case $\alpha = 2$. Particularly, for fractional Gaussian noise the covariance function as well as the function $r_Y(t)$ in (3.5) decay at rate $t^{2H-2}$, see [16]. The reason is that (3.7) holds for fixed $\lambda > 0$, or the tempered fractional noise alone.

Remark 3.5 In the Gaussian case $\alpha = 2$ we have that (3.8) is proportional to the covariance of TFGN II: $I(1,1;t) = \text{Cov}(Y_{H,\lambda}^H(0),Y_{H,\lambda}^H(t))$. Therefore (3.9) implies

$$\text{Cov}(Y_{H,\lambda}^H(0),Y_{H,\lambda}^H(t)) = e^{-\lambda t^{H-\frac{3}{2}}}, \text{ } t \to \infty,$$

for $\lambda > 0, 1/2 < H < 1$.

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