Regularizing effect and decay results for a parabolic problem with repulsive superlinear first order terms

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ABSTRACT. We want to analyse both regularizing effect and long, short time decay concerning parabolic Cauchy-Dirichlet problems of the type
\[
\begin{align*}
 u_t - \text{div} (A(t,x)|\nabla u|^{p-2}\nabla u) &= \gamma |\nabla u|^q & \text{in } Q_T, \\
 u &= 0 & \text{on } (0,T) \times \partial \Omega, \\
 u(0,x) &= u_0(x) & \text{in } \Omega.
\end{align*}
\]

We assume that \(A(t,x)\) is a coercive, bounded and measurable matrix, the growth rate \(q\) of the gradient term is superlinear but still subnatural, \(\gamma > 0\), the initial datum \(u_0\) is an unbounded function belonging to a well precise Lebesgue space \(L^\sigma(\Omega)\) for \(\sigma = \sigma(q,p,N)\).

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1. Introduction

The main goal of this work is proving regularity and decay results regarding solutions of a class of parabolic equations with superlinear (and subquadratic) growth. The model we consider is the following:
\[
\begin{align*}
 u_t - \text{div} (A(t,x)|\nabla u|^{p-2}\nabla u) &= \gamma |\nabla u|^q & \text{in } Q_T, \\
 u &= 0 & \text{on } (0,T) \times \partial \Omega, \\
 u(0,x) &= u_0(x) & \text{in } \Omega,
\end{align*}
\] (1.1)

where \(\Omega\) is a bounded subset of \(\Omega \subset \mathbb{R}^N, N \geq 2\), \(Q_T = (0,T) \times \Omega\) is the parabolic cylinder, \(1 < p < N\) and \(q < p\).

The problem in (1.1) collects all the basic features which motivate our incoming study. Let us spend some words on the elements appearing in (1.1).

The matrix \(A(t,x)\) is supposed to be bounded, coercive with only measurable coefficients. Then, the lack of regularity in the divergence operator prevents us to apply classical regularity estimates and we need to develop a suitable nonlinear theory. In particular, this means that nonlinear operators in divergence form are admitted as well.

The initial datum \(u_0\) is supposed to be an unbounded function belonging to Lebesgue spaces and the lack of boundedness implies that we cannot invoke maximum principles.

The \(q\) power of the gradient makes such growth to be superlinear (in some sense) but still subnatural \(q < p\). To fix ideas, we assume that \(q\) is strictly greater than a certain critical value \(q_c\) which splits the

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interval $0 < q < p$ between sublinear growths if $0 < q \leq q_c$ and superlinear ones $q_c < q < p$. Finally, the coefficient $\gamma$ is assumed to be strictly positive and then it gives a repulsive nature to the r.h.s.: roughly speaking, the gradient term in the r.h.s. "fights against" the coercitivity of the l.h.s.

Let us give a brief overview on the literature behind problems of \eqref{1.1} type. As far as the case with Laplace operator in \eqref{1.1} is concerned, regularizing effects and long time decays are dealt with in \cite{5, 3, 4, 23} regarding different notions of solutions (classical, mild and weak ones). In particular, when the initial datum is supposed to be continuous or simply bounded, decay estimates are proved when the gradient rate is positive $q > 0$ with both repulsive and attractive nature (i.e. $\gamma > 0$ and $\gamma < 0$, respectively, in \eqref{1.1}). In particular, \cite[Theorem 1.2]{5} and \cite[Lemma 3.2]{23} show that, if $1 < q \leq 2$, then the $L^\infty$-norms of both solutions and gradients decay to zero for large times with exponential rates:

$$\|u(t)\|_{L^\infty(\Omega)} \leq Ke^{-\lambda_1 t},$$

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \leq K(1 + t^{-\frac{1}{2}})e^{-\lambda_1 t},$$

being $\lambda_1$ the first eigenvalue of the Laplace operator with homogeneous Dirichlet boundary conditions. Note that this decay is sharp since it is satisfied also by the heat equation. We underline that the authors of \cite{5, 23} can apply Bernstein’s estimates, as well as linear semigroup theory or heat kernel estimates, which are not allowed in our general setting because of the assumptions on the matrix $A(t, x)$ in \eqref{1.1}.

As already anticipated, our aim is dealing with unbounded data in Lebesgue’s spaces

$$u_0 \in L^v(\Omega) \quad \text{for} \quad v \geq 1, \quad \text{(1.2)}$$

and thus, due to the presence of a superlinear term in \eqref{1.1}, an explanation on the admissible values of $v$ is in order to be given. We underline that the need of taking care of the data regularity is due to the superlinear setting and does not depend on the nature of the superlinearity itself. For instance, we refer to \cite{9} where the superlinearity has the form $|u|^p$, $q > 1$ and to \cite{17} in our case.

As shown in \cite{3, 4} when $p = 2$ and in \cite{17} for $1 < p < N$ in a more general context, we need to fix

$$v \geq \max\left\{1, \frac{N(q - (p - 1))}{p - q}\right\}$$

in (1.2) in order to get an existence result when a superlinear growth in the gradient term occurs. The same compatibility condition was already observed in \cite{3} for the Cauchy problem with $p = 2$. We remark that, when $q$ is superlinear, nonexistence counterexamples are proved if $1 \leq v < \frac{N(q - (p - 1))}{p - q}$ in \cite[Subsection 3.2]{3} for the Cauchy problem with Laplace operator in \eqref{1.1} and in \cite[Section 7]{18} as far as the Cauchy-Dirichlet problem with $p = 2$ in \eqref{1.1} is concerned.

A nonlinear approach, aimed at studying the regularity and the behaviour in time of solutions of \eqref{1.1} with $p = 2$, is contained in \cite{18}. In particular, the main step relies on the proof of an a priori estimate for the level set function $G_k(u) = (|u| - k)_+$, $\text{sign}(u)$ which has the form

$$\sup_{t \in (0, T)} \|G_k(u(t))\|_{L^{\frac{N(q-1)}{N(q-1)}}(\Omega)} + \|\nabla \left[1 + |G_k(u)|^{\frac{N(q-1)}{N(q-1)}}\right]\|_{L^2(\Omega)}^2 \leq M$$

for $2 - \frac{N}{N+1} < q < 2$,

where $k$ is taken large enough to have $\|G_k(u_0)\|_{L^{\frac{N(q-1)}{N(q-1)}}(\Omega)}$ suitable small and with $M = M(\|u_0\chi_{|u_0| > k}\|_{L^v(\Omega)})$.

Observing the inequality above, we deduce two important facts: first, we have that (morally) the function $G_k(u)$ acts like a subsolution of the coercive problem

$$\begin{cases}
    u_t - \text{div} a(t, x, u, \nabla u) = 0 & \text{in } Q_T, \\
    u = 0 & \text{on } (0, T) \times \partial \Omega, \\
    u(0, x) = u_0(x) & \text{in } \Omega,
\end{cases} \quad \text{(P_c)}$$

and so we expect that $G_k(u)$ inherits the own features of (P_c); moreover, looking at the energy term, we foresee that a well precise power $|u|^{\beta}$, $\beta = \beta(p, q, N)$, plays a certain role in the study of \eqref{1.1}.

We are going to comment this last observation. Dealing with a general superlinear setting, then one has to require some regularity on the solutions in order to have the problem well posed. In this sense, we
refer to [2, 25] in the elliptic framework and [18, 16] in the parabolic one. More precisely, a comparison result is proved in [18, Section 6] when the solution \( u \) belongs to the regularity class

\[
\left\{ u \text{ solving (1.1)} : |u|^\frac{N(q-1)}{2(q-2)} \in L^2(0,T;H^1_0(\Omega)) \right\}
\]

while nonuniqueness occurs (see [16, Appendix A]) if

\[
\left\{ u \text{ solving (1.1)} : |u|^\rho \in L^2(0,T;H^1_0(\Omega)) \quad \text{with} \quad \rho < \frac{N(q-1)}{2(2-q)} \right\}.
\]

See also [15, Example 1.1] for an analogous observation in the elliptic framework. In the same spirit, we quote [1] where (1.1) is studied with \( q = p = 2 \) and, due to the natural growth, the right class in which one has to study the problem is given by

\[
\left\{ u \text{ solving (1.1)} \text{ with } q = p = 2 : (e^q - 1) \in L^2(0,T;H^1_0(\Omega)) \right\}.
\]

We now recall some well known facts concerning coercive problems. Let us focus on \((P_r)\) for a while. We assume that \( a(t,x,u,\xi) : (0,T) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) verifies classical Leray-Lions structure conditions (see also \((A)\)) and \( u_0 \in L^p(\Omega) \), \( v \geq 1 \).

We stress on the relation between the parameter \( p \) and the Lebesgue summability \( v \) of the initial datum. If we consider values of \( p \) that are smaller than the threshold \( \frac{2N}{N+v} \), \( v > 1 \), then we cannot expect any regularizing effect (see [25, Theorem 1.2]).

On the contrary, as \( p > \frac{2N}{N+v} \) and \( v \geq 1 \), then a regularizing effect occurs. Indeed, we have that (see [25, Theorems 1.3])

\[
\|u(t)\|_{L^1(\Omega)} \leq \frac{\|u_0\|_{L^1(\Omega)}}{h_0} \quad \text{a.e.} \quad t \in (0,T),
\]

for \( c = c(\alpha, r, p, v, N) \),

\[
h_0 = \frac{v[2N - p(N + r)]}{r[2N - p(N + v)]} \quad \text{and} \quad h_1 = \frac{N(v - r)}{r[2N - p(N + v)]}.
\]

Furthermore, the case \( r = \infty \) ([25, Theorem 1.4], [24] and also [11] when \( p = 2 \) and \( v \geq 2 \)) is admitted and the decay estimate is given by

\[
\|u(t)\|_{L^\infty(\Omega)} \leq \frac{\|u_0\|_{L^\infty(\Omega)}}{h_1} \quad \text{a.e.} \quad t \in (0,T),
\]

with \( c = c(\alpha, p, v, N) \) and where the exponents follow from the limits

\[
\lim_{r \to \infty} h_0 = \frac{pv}{p(N+v) - 2N} \quad \text{and} \quad \lim_{r \to \infty} h_1 = \frac{N}{p(N+v) - 2N}.
\]

Note that the above estimates, beyond the regularizing effect, can be read as decay estimates too. However, it is well known that (1.4) is not sharp in the sense that it can be refined with respect to great and small values of \( t \) in bounded domains (see [13] and also the last part of [24, Corollary 2.1] for \( p > 2 \) and [14] as \( p = 2 \)).

Finally, if either \( \frac{2N}{N+v} \leq p < 2 \) and \( v > 1 \) or \( \frac{2N}{N+1} < p < 2 \) and \( v = 1 \), then extinction in finite time occurs (see [25, Theorems 1.5 & 1.6]), i.e. there exists a time \( T \) such that

\[
u(t,x) = 0 \quad \forall t \geq T.
\]

2. Assumptions

Let us present the problem we are going to study in its generality.

We consider the following parabolic Cauchy-Dirichlet problem

\[
\begin{cases}
  u_t - \text{div } a(t,x,u,\nabla u) = H(t,x,\nabla u) & \text{in } Q_T, \\
  u = 0 & \text{on } (0,T) \times \partial\Omega, \\
  u(0,x) = u_0(x) & \text{in } \Omega,
\end{cases}
\]

(P)
assuming that the vectorial valued function \(a(t, x, u, \xi) : (0, T) \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n\) satisfies classical Leray-Lions structure assumptions, namely

\[
\exists \alpha > 0 : a|\xi|^{p} \leq a(t, x, u, \xi) \cdot \xi, \quad (A1)
\]
\[
\exists \lambda > 0 : |a(t, x, u, \xi)| \leq \lambda (|u|^{p-1} + |\xi|^{p-1} + h(t, x)) \quad \text{where} \ h \in L^{p'}(Q_T), \quad (A2)
\]
\[
(a(t, x, u, \xi) - a(t, x, u, \eta)) \cdot (\xi - \eta) > 0, \quad (A3)
\]

for almost every \((t, x) \in Q_T\), for every \(u \in \mathbb{R}\) and for every \(\xi, \eta \in \mathbb{R}^N\) with \(\xi \neq \eta\).

As far as the r.h.s. is concerned, we assume that it grows at most as a power of the gradient for almost every \(u \in \Omega\)

\[
\text{we can consider } L^p(\Omega) \text{ with } \sigma = \frac{N(q - (p - 1))}{p - q}. \quad (2.2)
\]

Then, if we have

\[
\max \left\{ \frac{p}{2}, \frac{p(N + 1) - N}{N + 2} \right\} < q < p \quad \text{with} \quad 1 < p < N \quad (Q_c)
\]

in (H), we need to ask at least the following summability on the initial datum:

\[
u = \max \{1, \sigma\}, \quad \sigma = \frac{N(q - (p - 1))}{p - q}.
\]

As the \(q\) rate gets slower but keeps superlinear, i.e.

\[
\max \left\{ \frac{p}{2}, \frac{p(N + 1) - N}{N + 2} \right\} < q < p - \frac{N}{N + 1} \quad \text{with} \quad \frac{2N}{N + 1} < p < N, \quad (Q_t)
\]

we can consider \(L^1(\Omega)\) data (see (2.2)):

\[
u = \sigma = 1 \quad \text{if} \quad q = p - \frac{N}{N + 1} \quad \text{and} \quad v = \sigma = 2 \quad \text{if} \quad q = p - \frac{N}{N + 2}. \quad (ID_1)
\]

We require \(\frac{2N}{N + 1} < p\) in order to give sense to \((Q_t)\).

The growth rates in \((Q_t)\) would allow us to deal even with measures data, since \(\frac{N(q - (p - 1))}{p - q} < 1\). For further comments in this sense, we refer to [3, Theorem 2.2]. However, we choose \(L^1(\Omega)\) data in order to keep ourselves in the Lebesgue framework.

The particular case \(q = p - \frac{N}{N + 1}\) with \(p > \frac{2N}{N + 2}\) will be commented later with its own assumptions and, at this moment, we just observe that such a \(q\) value is critical in the sense that it implies that the value of \(\sigma\) in (2.2) is exactly 1. Note that such a \(q\) growth represents the changing point between \(L^{p'}(\Omega)\) and \(L^1(\Omega)\) data.

Some words on the relation between the ranges of both \(p\) and \(q\), aimed at clarifying the data setting, are in order to be given. Let us set

| \[u_0 \in L^{p'}(\Omega),\] |
| \[u_0 \in L^1(\Omega).\] |

Colours legend

We sketch out our \(q\) intervals on the real lines below with respect to the value of \(p\), highlighting the cases \(q = p - \frac{N}{N + 1}\) and \(q = p - \frac{N}{N + 2}\) since they represent, respectively, the \(L^1(\Omega)\) and the \(L^{p'}(\Omega)\) thresholds of the data (i.e. \(v = \sigma = 1\) if \(q = p - \frac{N}{N + 1}\) and \(v = \sigma = 2\) if \(q = p - \frac{N}{N + 2}\)).
The case $2 \leq p < N$

As far as the cases $p - \frac{2N}{N+1} < \frac{p}{2}$ and $p - \frac{N}{N+2} < \frac{p}{2}$ are concerned, we have

Looking at the real lines above we deduce that

$$q > \frac{p}{2} \iff p > \max \left\{ \frac{2N}{N+\sigma'}, \frac{2N}{N+1} \right\} = \frac{2N}{N+\nu'} \quad \nu' \text{ in (2.2)},$$

which, roughly speaking, means that we have an existence result in the superlinear setting if and only if we have $p$ great enough. Note that the $p$ threshold $\frac{2N}{N+1}$ is the same as the coercive case ($P_c$). This means that we cannot fall in the range $1 < p \leq \frac{2N}{N+1}$ if we want to keep the superlinear character of ($P$).

We synthesise the above comments saying that if we are in the superlinear framework and a solution of ($P$) exists, then such a solution regularizes.

We collect in the figure below our incoming decay results.

\[\begin{align*}
S &= \text{regularizing effect } L^r - L^r \text{ for } u \\
U &= \text{long time decay } L^r - L^\infty \text{ for } u \\
E &= \text{extinction for } u \text{ (see (1.4))}
\end{align*}\]

Regularizing effect estimates and long time decays w.r.t. $p$ and $\sigma$

We point out that obtaining decays results in superlinear settings is not obvious: for instance, solutions of the superlinear power problem

$$u_t - \Delta u = |u|^q \quad \text{with } q > 1$$

may blow up in finite time (see [10, 21]).
The existence of solutions of (1.1) has been proved in [17, Theorems 4.5 & 5.4]. We underline that dealing with solutions which enjoy (RC) is crucial since it determines the well posedness class of (P). We note also that, if \( \sigma \geq 2 \) (i.e. (3.1) hold), then \( \beta \geq 1 \) and so (RC) provides us with a stronger information than only
knowing \( u \in L^p(0,T;W^{1,p}_0(\Omega)) \).

In order to deal with our current framework, we here define the function \( \theta_n(\cdot) \) as below:

\[
\theta_n(v) = \begin{cases} 
1 & |v| \leq n, \\
\frac{2n - |v|}{n} & n < |v| \leq 2n, \\
0 & |v| > 2n.
\end{cases}
\] (3.3)

The function \( \theta_n(v) \)

Note that \( \theta_n(v) \) is compactly supported and converging to 1.

### 3.1. \( L^p(\Omega) - L^p(\Omega) \) regularity.

Our first result contains the key point of our next ones and we will refer to this particular step as the \( \delta \) argument. Roughly speaking, we prove that a contraction in the \( L^p \)-norm, \( \sigma > 1 \) as in (ID\(_2\)), holds for the level set function \( G_k(u(t)) \) provided that this is initially \( (t = 0) \) not too big (i.e., \( k \) is large). We underline that, when dealing with the \( G_k(\cdot) \) function, no smallness conditions on the initial datum are assumed, but eventually it is enough to take a large \( k \). An analogous \( \delta \) argument has already been used in [18] where (P) is studied under the assumptions in Section 2 when \( p = 2 \).

**Lemma 3.2.** Assume (ID\(_n\)), (A1)–(A2) with \( p > \frac{2N}{N+p} \) and (H) with (Q\(_\sigma\)). Moreover, let \( u \) be a solution of (P) in the sense of Definition 3.1. Then, there exists a positive value \( \delta_0 \) such that, for every \( k > 0 \) and for every \( \delta < \delta_0 \) satisfying

\[
\|G_k(u_0)\|_{L^p(\Omega)} < \delta,
\]

we have

\[
\|G_k(u(t))\|_{L^p(\Omega)} < \delta \quad \forall t \in [0,T].
\]

**Proof.** We claim that the function \( S'(\cdot) = S'_{n,\varepsilon}(\cdot) \) defined as

\[
S'_{n,\varepsilon}(G_k(u)) \varphi = \int_0^{T_{n}(G_k(u))} (\varepsilon + |v|)^{\sigma-3}|v| \, dv
\]

with \( \varphi = 1 \)

can be taken in (3.2). Indeed, even if it is not compactly supported, the regularity assumption (RC) allows us to proceed by standard arguments for renormalized solutions (i.e., beginning with \( \theta_k(G_k(u)) \) \( S_{n,\varepsilon}(G_k(u)) \varphi \) where \( \theta_k(\cdot) \) is defined in (3.3), recalling (A1) and (H) and then letting \( h \to \infty \)). Then, thanks also to the growth assumption in (H), we get

\[
\int_{\Omega} S_{n,\varepsilon}(G_k(u(t))) \, dx + \alpha \int_0^t \|\nabla \Phi_k(T_n(G_k(u(s))))\|_{L^p(\Omega)}^p \, ds
\]

\[
\leq \int_{\Omega} S_{n,\varepsilon}(G_k(u_0)) \, dx + \gamma \int_{Q_T} |\nabla G_k(u)|^q \left( \int_0^{G_k(u)} (\varepsilon + |z|)^{\sigma-3}|z| \, dz \right) \, dx \, ds,
\]

where \( \Phi_k(\varphi) = \int_0^\varphi (\varepsilon + |z|)^{\sigma-2} \frac{z^\gamma}{\gamma} \, dz \). The definition of \( \Phi_k(\cdot) \) allows us to estimate the second term in the above r.h.s. as

\[
\gamma \int_{Q_T} |\nabla G_k(u)|^q \left( \int_0^{G_k(u)} (\varepsilon + |z|)^{\sigma-3}|z| \, dz \right) \, dx \, ds
\]

\[
\leq \gamma \int_{Q_T} |\nabla \Phi_k(G_k(u))|^q |\Phi_k(G_k(u))|^{p-q} |G_k(u)|^{q-p+1} \, dx \, ds
\]
where the last step is due to Hölder’s inequality with indices \( \left( \frac{1}{p}, \frac{q}{q-(p-1)} \right) \) (we recall that \( q > \max \left\{ \frac{p}{q}, \frac{N}{N-1} \right\} > p - 1 \)). An application of the Hölder inequality with \( \left( \frac{p}{q}, \frac{p}{p-(q-(p-1))} \right) \), Sobolev’s embedding and the definition of \( \sigma \) (we just recall here that \( \sigma = N(q-(p-1))/(p-q) \)) give us

\[
\int_{\Omega} S_{n,\varepsilon}(G_k(u(t))) \, dx + \alpha \int_0^t \| \nabla \Phi_k(T_n(G_k(u(s)))) \|_{L^p(\Omega)}^p \, ds
\]

\[
\leq \int_{\Omega} S_{n,\varepsilon}(G_k(u_0)) \, dx + \gamma C_S \left( \sup_{x \in (0,t)} \| G_k(u(s)) \|_{L^{p,q}(\Omega)}^p \right) \int_0^t \| \nabla \Phi_k(G_k(u(s))) \|_{L^p(\Omega)}^p \, ds.
\]

Being \( \frac{p-q}{q} < 1 \) and thanks to (ID\(_\varepsilon\)) and (Q\(_\varepsilon\)), we deduce that \( \int_{\Omega} S_{n,\varepsilon}(G_k(u(t))) \, dx < \infty \) uniformly in \( n \) and for fixed \( \varepsilon \). In particular, we gain the boundedness of \( \| G_k(u) \|_{L^{\infty}(0,T;L^p(\Omega))} \). Such a result, combined with (RC) and (ID\(_\varepsilon\)), allows us to consider the limit for \( n \to \infty \) in the previous inequality getting

\[
\int_{\Omega} S_{\varepsilon}(G_k(u(t))) \, dx \leq \int_{\Omega} S_{\varepsilon}(G_k(u_0)) \, dx + \gamma C_S \left( \sup_{x \in (0,t)} \| G_k(u(s)) \|_{L^{p,q}(\Omega)}^p \right) \int_0^t \| \nabla \Phi_k(G_k(u(s))) \|_{L^p(\Omega)}^p \, ds,
\]

where \( S_{\varepsilon}(x) = \int_0^\varepsilon \left( \int_0^r (r + |z|)^{-\frac{p}{q}} |z| \, dz \right) \, dy \). In particular, thanks again to (RC), we deduce the convergence to zero of \( \int_{\Omega} S_{\varepsilon}(G_k(u(t))) \, dx \) for \( k \to \infty \) which provides us the one of \( \int_{\Omega} |G_k(u(t))|^p \, dx \to 0 \) for \( k \to \infty \).

Then, the continuity regularity \( u \in C([0,T];L^p(\Omega)) \) follows combining this last convergence with [20, Theorem 1.1] (which implies that \( u \in C([0,T];L^1(\Omega)) \)), the decomposition in (2.3) and by an application of the Vitali Theorem.

The \( \delta \) argument.

Let us focus on (3.4). We choose a value \( \delta_0 \) such that \( 0 < \gamma C_S \delta_0^\frac{p-q}{q} < \alpha \) and a value \( k_0 \) large enough so that

\[
\| G_k(u_0) \|_{L^{p,q}(\Omega)}^p < \delta \quad \forall k \geq k_0
\]

for fixed \( \delta < \delta_0 \).

Moreover, always considering \( k \geq k_0 \), we set

\[
T^* := \sup \{ s \in [0,T] : \| G_k(u(t)) \|_{L^{p,q}(\Omega)}^p \leq \delta \quad \forall t \leq s \}
\]

and we have that \( T^* > 0 \) due to the continuity regularity just proved and to (3.5). Choosing \( t \leq T^* \) in (3.4) and recalling the definition of \( \delta \), we manage to absorb the r.h.s. obtaining

\[
\int_{\Omega} S_{\varepsilon}(G_k(u(t))) \, dx + \left( \alpha - \gamma C_S \delta_0^{\frac{p-q}{q}} \right) \int_{\Omega} |\nabla \Phi_k(G_k(u))|^p \, dx \, ds \leq \int_{\Omega} S_{\varepsilon}(G_k(u_0)) \, dx.
\]

Moreover, since the convergence \( S_{\varepsilon}(G_k(u(s))) \xrightarrow{\varepsilon \to 0} \| G_k(u(s)) \|_{L^{p,q}(\Omega)}^p \) holds, (3.6) provides us with the contraction

\[
\int_{\Omega} |G_k(u(t))|^p \, dx \leq \int_{\Omega} |G_k(u_0)|^p \, dx \quad \forall k \geq k_0.
\]

The inequality (3.7) can be extended to the whole interval \([0,T]\) reasoning by contradiction. Let us suppose that \( T^* < T \). Then, the definition of \( T^* \) and (3.5) lead to

\[
\delta = \int_{\Omega} |G_k(u(T^*))|^p \, dx \leq \int_{\Omega} |G_k(u_0)|^p \, dx < \delta \quad \forall k \geq k_0
\]

which is in contrast with the definition of \( T^* \) because of continuity \( u \in C([0,T];L^p(\Omega)) \).

We here state an important consequence which derives from the \( \delta \) argument above.
**Corollary 3.3.** Assume $u_0 \in L^\infty(\Omega)$, (A1)–(A2) and (H) with $(Q_c)$. Moreover, let $u$ be a solution of (P) in the sense of Definition 3.1. Then, we have that $u \in L^\infty(Q_T)$. Moreover, the following contraction estimate holds:

$$\|u(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \quad \forall t \in [0, T].$$

**Proof.** The assertion can be easily deduced taking $k_0 = \|u_0\|_{L^\infty(\Omega)}$ in (3.7).

Roughly speaking, this contraction result implies that if one manages to prove that $u$ is bounded at a certain time $\tau$, then it keeps bounded and the $L^\infty$-norm decreases in the time variable.

**Lemma 3.4.** Assume (ID$_c$), (A1)–(A2) with $p > \frac{2N}{N+\rho}$ and (H) with $(Q_c)$. Moreover, let $u$ be a solution of (P) in the sense of Definition 3.1 and consider $\Phi : \mathbb{R} \to \mathbb{R}$ be a $C^2$ convex function such that

$$\Phi'(0) = 0 \quad \text{and} \quad \Phi''(\xi) \leq c(1 + |\xi|)^{p-2},$$

for some constant $c > 0$. Then the function $t \to \int_\Omega \Phi(u(t)) \, dx$ belongs to $W^{1,1}(0, T)$ and satisfies

$$\frac{d}{dt} \int_\Omega \Phi(u(t)) \, dx + \int_\Omega a(t, x, u, \nabla u) \cdot \nabla \Phi'(u) \, dx = \int_\Omega H(t, x, \nabla u) \Phi'(u) \, dx$$

a.e. in $t \in (0, T)$.

**Proof.** We omit the proof since it is very similar to the one proposed in [18, Lemma 3.1]. We just observe that the growth assumption (3.8) plays the role of (RC). In particular, (3.8) is needed to justify the choice of $S_\delta'(|\cdot|) \phi = \Phi'_\delta(|\cdot|)$, $\phi = 1$, in (3.2) (i.e., we begin with $S_\delta'(-\sigma) = \theta_{\delta}(\cdot)\Phi'_\delta(\cdot)$, where $\theta_{\delta}(\cdot)$ has been defined in (3.3); then, thanks to (A1), (H), we let $\delta \to 0$).

Here we propose the generalization of [18, Proposition 3.2] in which the $L^p(\Omega) - L^\infty(\Omega)$ long time decay of (P) is proved with $p = 2$ in (A), (ID$_{\nu}$) and $(Q_c)$.

**Proposition 3.5.** Assume (ID$_{\nu}$), (A1)–(A2) with $p > \frac{2N}{N+\rho}$ and (H) with $(Q_c)$. Moreover, let $u$ be a solution of (P) in the sense of Definition 3.1. Then, for $k$ sufficiently large (say $k \geq k_0$ with $k_0$ as in Lemma 3.2), we have that

$$\frac{d}{dt} \int_\Omega |G_k(u(t))|^p \, dx + \frac{\sigma}{\beta^p} \left( a - \gamma \|u\|_{L^\infty(\Omega)} \right) \int_\Omega |\nabla [G_k(u)]|^p \, dx \leq 0$$

a.e. $t \in (0, T)$, for all $k \geq k_0$ (see (3.6)).

Furthermore, for $\lambda = \frac{\sigma}{\beta^p} \left( a - \gamma \|u\|_{L^\infty(\Omega)} \right) < 0$ and $k \geq k_0$, we have that

- if $2 < p < N$, then $\|G_k(u(t))\|_{L^p(\Omega)}$ decreases in the time variable and the following polynomial decay holds:

$$\|G_k(u(t))\|_{L^p(\Omega)} \leq \left( \|G_k(u_0)\|_{L^\infty(\Omega)} + \lambda \|u_0\|_{L^\infty(\Omega)} \right)^{\frac{1}{2}} \quad \forall t \geq 0;$$

- if $\frac{2N}{N+\rho} < p < 2$, there exists a positive time $T$ such that

$$G_k(u) = 0 \quad \forall t \geq T.$$

In particular, such a value $T$ is given by

$$T = \frac{\sigma}{(2 - p)\lambda} \|G_k(u_0)\|_{L^\infty(\Omega)}^{2-p}.$$  

**Proof.** The inequality in (3.10) follows combining Lemma 3.4 with Lemma 3.2. Indeed, invoking Lemma 3.4 with $\Phi'_\delta(G_k(u)) = (\sigma - 1) \int_0^{|G_k(u)|} (w + w^p)^{-2} \, dw$ and reasoning as in Lemma 3.2 (see (3.6)), we obtain

$$\frac{d}{dt} \int_\Omega \Phi'_\delta(G_k(u(t))) \, dx + \left( a - \gamma \|u\|_{L^\infty(\Omega)} \right) \int_\Omega |\nabla \Phi'_\delta(G_k(u))|^p \, dx \leq 0$$

and (3.10) is recovered once we let $\epsilon$ vanish.

We go further observing that, by definitions of $\sigma$ and $\beta$, we have that

$$\sigma < \beta \Leftrightarrow p > \frac{2N}{N+\sigma}.$$
and thus, thanks to Sobolev's embedding and to Lebesgue's spaces inclusion, we can estimate from below as follows:

\[
0 \geq \frac{d}{dt} \|G_k(u(t))\|_{L^p(\Omega)} + \frac{\sigma}{\beta^p} \left( \alpha - \gamma c_\delta \delta \frac{\lambda^p}{\mu} \right) \|\nabla [G_k(u(t))]^\beta\|_{L^p(\Omega)}^p \\
\geq \frac{d}{dt} \|G_k(u(t))\|_{L^p(\Omega)} + \lambda \|G_k(u(t))\|_{L^p(\Omega)} \tag{3.11}
\]
for every \( k \geq k_0 \) and where \( \lambda = \frac{\beta^p}{\mu} \left( \alpha - \gamma c_\delta \delta \frac{\lambda^p}{\mu} \right) |\Omega| \). We set

\[
y(s) = \|G_k(u(s))\|_{L^p(\Omega)}^p
\]
and rewrite (3.11) as

\[
y'(s) + \lambda y(s) \frac{\beta^p}{\mu} \leq 0 \quad \forall k \geq k_0. \tag{3.12}
\]

We now split the rest of the proof with respect to the cases \( p > 2 \) and \( p < 2 \).

Let \( 2 < p < N \): in this way, we have that \( \frac{\beta p}{\mu} = \frac{\sigma + \rho - 2}{\sigma} > 1 \) and then Gronwall's type Lemma (see, e.g., [24, Lemma 3.1]) provides us with

\[
y(t) \leq \left( y(0) - \frac{2 \rho - 2}{\sigma} t \right) \frac{\sigma}{\rho} \quad \forall t \in (0, T), \forall k \geq k_0.
\]

Having \( \frac{2N}{N + \sigma} < p < 2 \) guarantees that \( \frac{\beta p}{\mu} < 1 \) and (3.12) gives us

\[
y(t) \leq y(0) - \frac{\rho - 2}{\sigma} t \quad \forall t \in (0, T), \forall k \geq k_0
\]
from which we deduce that \( y(t) = 0 \) if \( t \geq T = \frac{\sigma}{\rho (2 - \rho)} y(0) \frac{2 \rho - 2}{\rho} \).

The assertions follow recalling the definitions of \( y(\cdot) \) and \( \lambda \).

\[\square\]

3.2. The regularizing effect \( L^p(\Omega) - L^r(\Omega) \)

**Proposition 3.6.** Assume (ID), (A1)–(A2) with \( p > \frac{2N}{N + \sigma} \) and (H) with (Q, r) and let \( u \) be a solution of (P) in the sense of Definition 3.1. Then

\[
u \in C((0, T); L^r(\Omega)) \quad \text{for} \quad r > \sigma. \tag{3.13}
\]

Moreover, there exists a value \( k_0 \), independent of \( r \), such that the regularizing effect can be expressed through the decay estimate

\[
\|G_k(u(t))\|_{L^r(\Omega)} \leq c \frac{\|G_k(u_0)\|_{L^r(\Omega)}^{\frac{2N(p-2) + \rho}{p}}}{t^{\frac{2N(p-2) + \rho}{p}}} \quad \forall t \in (0, T), \forall k \geq k_0, \tag{3.14}
\]

where \( c = c(\gamma, r, q, p, a, N) \). Furthermore we have the short time decay

\[
\|u(t)\|_{L^r(\Omega)} \leq \frac{C}{t^{\frac{N(p-2) + \rho}{p}}} \quad \forall t \in (0, t_0) \tag{3.15}
\]

where \( C = C(\gamma, r, q, p, a, N, t_0, u_0, |\Omega|) \).

**Proof.** We set \( \Phi(\cdot) = S(\cdot) \) in (3.9), with \( S \in W^{2,\infty}(\mathbb{R}) \) satisfying

\[
0 \leq S''(v) \leq c(\epsilon + |v|)^{\beta - 1 - \frac{1}{2}} |v| = c(\epsilon + |v|)^{\rho - 3} |v| \tag{3.16}
\]
and

\[
\frac{S'(v)}{(S''(v))^\frac{\rho-\alpha}{\alpha}} \leq L \int_0^v (S''(y))^\frac{\rho-\alpha}{\alpha} dy, \tag{3.17}
\]
for some positive constants \( c, L \). Again, we justify such a choice of \( S(\cdot) \) reasoning as in Lemma 3.2 and taking advantage of (3.16), since this last condition plays the same role of (RC).

Then, letting \( S(\cdot) = S(G_k(u(t))) \) and recalling (H), we have that the following differential inequality

\[
\frac{d}{dt} \int_\Omega S(G_k(u(t))) \, dx + \alpha \int_\Omega |\nabla G_k(u)|^p S''(G_k(u)) \, dx \leq \gamma \int_\Omega |\nabla G_k(u)|^\rho S'(G_k(u)) \, dx \tag{3.18}
\]
holds a.e. \( t \in (0, T) \).
We now define $\Psi(G_k(u)) = \int_0^{G_k(u)} (S''(y))^{\frac{1}{\rho}} dy$ and use (3.17) in (3.18), obtaining
\[
\frac{d}{dt} \int_{\Omega} S(G_k(u(t))) dx + \alpha \int_{\Omega} |\nabla \Psi(G_k(u))|^p dx \\
\leq \gamma L \int_{\Omega} |\nabla G_k(u)|^q (S''(G_k(u)))^{\frac{q}{p}} \left( \int_0^{G_k(u)} (S''(z))^{\frac{p}{q}} dz \right) dx
\]
from which, being
\[
\int_0^{G_k(u)} (S''(v))^{\frac{p}{q}} dv \leq \left( \int_0^{G_k(u)} (S''(v))^{\frac{1}{\rho}} dv \right) ^{\frac{q}{p}} |G_k(u)|^{q-(p-1)}
\]
by Hölder’s inequality with \( \left( \frac{1}{p-q}, \frac{1}{q-(p-1)} \right) \), we get
\[
\frac{d}{dt} \int_{\Omega} S(G_k(u(t))) dx + \alpha \int_{\Omega} |\nabla \Psi(G_k(u))|^p dx \\
\leq \gamma L \int_{\Omega} |\nabla \Psi(G_k(u))|^q (\Psi(G_k(u)))^{p-q} |G_k(u)|^{q-(p-1)} dx.
\]
Another application of Hölder’s inequality with indices \( \left( \frac{q}{p}, \frac{p}{p-q}, \frac{N}{p-q} \right) \) and Sobolev’s embedding give us
\[
\frac{d}{dt} \int_{\Omega} S(G_k(u(t))) dx + \alpha \int_{\Omega} |\nabla \Psi(G_k(u))|^p dx \leq L_\gamma c_S \sup_{t \in (0,T)} \|G_k(u(t))\|^{q-(p-1)} \int_{\Omega} |\nabla \Psi(G_k(u))|^p dx
\]
and then, invoking Lemma 3.2 with \( k_0 \) sufficiently large in order to have \( \alpha > L_\gamma c_S \delta^{\frac{p-q}{N}} \), we finally get
\[
\frac{d}{dt} \int_{\Omega} S(G_k(u(t))) dx + c_1 \int_{\Omega} |\nabla \Psi(G_k(u))|^p dx \leq 0 \quad \forall k \geq k_0 \tag{3.19}
\]
where \( c_1 = \alpha - L_\gamma c_S \delta^{\frac{p-q}{N}} \).

We now fix a value \( r > \sigma \) and define
\[
S'(v) = S'_{n,\sigma}(v) = \int_0^v (\epsilon + |y|)^{\sigma-3} |y| T_n(y)^{1-\sigma} dy \quad \text{if } 1 < \sigma < 2, \tag{3.20}
\]
\[
S'(v) = S'_n(v) = \int_0^v |T_n(y)|^{r-2} dy \quad \text{if } \sigma \geq 2. \tag{3.21}
\]
Note that, for fixed \( n \), (3.20)–(3.21) are admissible choices of \( S'(.) \) since they verify both (3.16) and (3.17).

Our current goal is characterising the relation between
\[
S_{n,\sigma}(G_k(u)) \quad \text{and} \quad \Psi_n(G_k(u)) = \Psi_{n,\sigma}(G_k(u)) = \int_0^{G_k(u)} (S'_{n,\sigma}(y))^{\frac{1}{p}} dy \quad \text{when } 1 < \sigma < 2,
\]
\[
S_n(G_k(u)) \quad \text{and} \quad \Psi_n(G_k(u)) = \Psi_n(G_k(u)) = \int_0^{G_k(u)} (S'_n(y))^{\frac{1}{p}} dy \quad \text{when } \sigma \geq 2,
\]
in order to rewrite (3.19) only in terms of \( S_{n,\sigma}(G_k(u)) \) and \( S_n(G_k(u)) \). To this aim, we split the rest of the proof with respect to the value of \( \sigma \).

Let us consider the case \( 1 < \sigma < 2 \) first. We start with an estimate of the test function (3.20) itself. Let \( \omega \in (0,1) \) to be fixed later. Then, by Hölder’s inequality with \( \left( \frac{1}{p-\omega}, 1 - \frac{1}{p-\omega} \right) \), we get
\[
\int_0^y (\epsilon + |z|)^{\sigma-3} |z| T_n(z)^{1-\sigma} dz \\
\leq \left( \int_0^y (\epsilon + |z|)^{\sigma-3} |z| T_n(z)^{1-\sigma} dz \right)^{\omega p^*} \left( \int_0^y (\epsilon + |z|)^{\sigma-3} |z| T_n(z)^{1-\sigma} \frac{N-p N_0}{N-p-N_0} dz \right)^{1-\omega p^*}.
\]
Since it holds that \((\epsilon + |z|)^{\sigma-3}|z|T_n(z)^{r-\sigma} \leq |z|^{-2}\) being \(\sigma < 2\), we improve the inequality above as

\[
\int_0^y (\epsilon + |z|)^{\sigma-3}|z|T_n(z)^{r-\sigma} \, dz 
\leq \left( \int_0^y (\epsilon + |z|)^{\sigma-3}|z|T_n(z)^{r-\sigma} \, dz \right)^{\frac{1}{\rho}} |y|^{r-1 - \frac{N\sigma}{N(r-\sigma+p-2)+p\rho}} |\Psi_n(\epsilon)| \left| y^{r-1 - \frac{N\sigma}{N(r-\sigma+p-2)+p\rho}} \right| .
\]

Finally, we fix \(\omega = \frac{(r-\sigma)(N-p)}{N(r-\sigma+p-2)+p\rho}\) in order to have \(r - 1 - \frac{N\sigma}{N-\sigma} (r - 2 + p) = \sigma (1 - \omega) - 1\) and conclude saying that the previous steps and the definition of \(S_n(\cdot)\) in (3.20) lead us to

\[
S_{n,k}(G_k(u(s))) \leq c(r) |\Psi_n(\epsilon)(G_k(u(s)))| p^\omega \int_0^{G_k(u(s))} |y|^\epsilon(1-\omega)^{-1} \, dy 
\leq c(r) |\Psi_n(\epsilon)(G_k(u(s)))| p^\omega |G_k(u(s))|^\epsilon(1-\omega) ,
\]

so we get

\[
\int_\Omega S_{n,k}(G_k(u(s))) \leq c(r) \left( \int_\Omega |\Psi_n(\epsilon)(G_k(u(s)))| p^\omega \right) \left\| G_k(u_0) \right\|^{\epsilon(1-\omega)}_{L^p(\Omega)} \tag{3.22}
\]

as desired. The inequality in (3.22) implies that (3.19), read in terms of \(S_n(\cdot)\) and \(\Psi_n(\cdot)\), can be estimated from below as

\[
\frac{d}{ds} \int_\Omega S_{n,k}(G_k(u(s))) \, dx + c_2 \left( \frac{\int_\Omega S_{n,k}(G_k(u(s))) \, dx}{\| G_k(u_0) \|^{\epsilon(1-\omega)}_{L^p(\Omega)}} \right)^\frac{p}{p} \leq 0 \tag{3.23}
\]

a.e. \(s \in (0, T]\), for all \(k \geq k_0\) and with \(c_2\) depending on \(\alpha, \gamma, N, q, p\) and \(r\).

We integrate the inequality in (3.23) between \(0 < s \leq t\), getting

\[
\int_\Omega G_k(u(t)) \left( \int_0^t (\epsilon + |z|)^{\sigma-3}|z|T_n(z)^{r-\sigma} \, dz \right)^{\frac{1}{\rho}} \, dv \, dx \leq \frac{c_2}{t^\frac{N(r-\sigma)}{N(p-2)+p\rho}} \| G_k(u_0) \|^{\epsilon(1-\omega)}_{L^p(\Omega)}
\]

Note that \(\frac{N(r-\sigma)}{N(p-2)+p\rho} > 0\) since \(\frac{2N}{N+p} < p\) and \(r > \sigma\).

We finally apply the Fatou Lemma on \(n\) and on \(\epsilon\) in the previous inequality so that, recalling the definition of \(\omega\), we obtain

\[
\| G_k(u(t)) \|_{L^p(\Omega)} \leq c_2 \| G_k(u_0) \|^{\frac{N(p-2)+p\rho}{N(p-2)+p\rho}}_{L^p(\Omega)} \quad \text{a.e. } t \in (0, T), \forall k \geq k_0. \tag{3.24}
\]

We now deal with the case \(\sigma \geq 2\). We rewrite \(r = p^\sigma \frac{r-2+\mu}{\rho} \omega + \sigma (1 - \omega)\) where, as in the previous case, \(\omega = \frac{(r-\sigma)(N-p)}{N(r-\sigma+p-2)+p\rho} \in (0, 1)\). An application of Holder’s inequality with \(\left( \frac{1}{\omega}, \frac{1}{1-\omega} \right)\), combined with the inequality

\[
\int_0^y |T_n(z)|^{p^\sigma \frac{r-2+\mu}{\rho} - 1} \, dz \leq c(r) \left( \int_0^y |T_n(z)|^{p^\sigma \frac{r-2+\mu}{\rho} - 1} \, dz \right)^\omega \left( \int_0^y |T_n(z)|^{(\sigma-1)\frac{1}{1-\sigma} - 1} \, dz \right)^{1-\omega}
\]

gives us

\[
\int_0^y |T_n(z)|^{r-2} \, dz \leq \left( \int_0^y |T_n(z)|^{p^\sigma \frac{r-2+\mu}{\rho} - 1} \, dz \right)^\omega \left( \int_0^y |T_n(z)|^{(\sigma-1)\frac{1}{1-\sigma} - 1} \, dz \right)^{1-\omega} 
\leq c(r) \left( \int_0^y |T_n(z)|^{\frac{r-2+\mu}{\rho}} \, dz \right)^\omega |y|^{\sigma(1-\omega)} - 1
\leq c(r) |\Psi_n(y)| p^\omega |y|^\epsilon(1-\omega)^{-1} - 1,
\]
from which, recalling (3.21), we deduce
\[
S_n(G_k(u(s))) = \int_0^y |T_n(z)|^{-2} \, dz \, dy
\]
\[
\leq c(r) |\Psi_n(G_k(u(s)))|^\rho \omega^\gamma |\Psi_n(G_k(u(s)))|^{-\gamma(1-\omega)} \, dy
\]
\[
\leq c(r) |\Psi_n(G_k(u(s)))|^\rho \omega^\gamma |G_k(u(s))|^{-\gamma(1-\omega)}.
\]
This step, together with Holder’s inequality with \((\frac{1}{\omega}, \frac{1}{\rho})\) and the monotonicity of \(\|G_k(u(s))\|_{L^r(\Omega)}\) for large values of \(k\) (see Proposition 3.5), implies that
\[
\int_\Omega S_n(G_k(u(s))) \, dx \leq c(r) \int_\Omega \left( |\Psi_n(G_k(u(s)))|^\rho \omega^\gamma |\Psi_n(G_k(u(s)))|^{-\gamma(1-\omega)} \right) \, dx
\]
\[
\leq c(r) \left( \int_\Omega |\Psi_n(G_k(u(s)))|^\rho \, dx \right)^\omega \left( \int_\Omega |\Psi_n(G_k(u(s)))|^{-\gamma(1-\omega)} \, dx \right)^{1-\omega} \left( \int_\Omega |G_k(u(s))|^{\gamma(1-\omega)} \, dx \right)^{\gamma(1-\omega)} \left( \int_\Omega |G_k(u_0)|^{\gamma(1-\omega)} \, dx \right)^{1-\gamma(1-\omega)}
\]
and we get again an estimate from below for \(\int_\Omega |\Psi_n(G_k u)|^\rho \, dx\) in terms of \(\int_\Omega S_n(G_k(u(s))) \, dx\). Then (3.19), read in terms of \(S_n(\cdot)\) and \(\Psi_n(\cdot)\), can be estimated from below as
\[
\frac{d}{ds} \int_\Omega S_n(G_k(u(s))) \, dx + c_3 \left( \int_\Omega S_n(G_k(u(s))) \, dx \right)^{\frac{p}{\rho}} \leq 0
\]
a.e. \(s \in (0, T]\), for all \(k \geq k_0\) and with \(c_3\) depending on \(a, \gamma, N, q, p\) and \(r\), thanks also to Sobolev’s embedding.
The inequality in (3.24), with a possibly different constant depending on \(a, L, \gamma, N, q, p, k_0\) and \(r\), follows reasoning as before.

The decomposition (2.3) implies that we also have
\[
\|u(t)\|_{L^r(\Omega)} \leq \|G_{k_0}(u(t))\|_{L^r(\Omega)} + k_0' |\Omega|
\]
\[
\leq c \|G_{k_0}(u_0)\|_{L^r(\Omega)} + k_0' |\Omega|
\]
\[
\leq c \left( \int_\Omega |\Psi_n(G_k(u(s)))|^\rho \, dx \right)^\omega \left( \int_\Omega |\Psi_n(G_k(u(s)))|^{-\gamma(1-\omega)} \, dx \right)^{1-\omega} \left( \int_\Omega |G_k(u(s))|^{\gamma(1-\omega)} \, dx \right)^{\gamma(1-\omega)} \left( \int_\Omega |G_k(u_0)|^{\gamma(1-\omega)} \, dx \right)^{1-\gamma(1-\omega)} + k_0' |\Omega|
\]
for a.e. \(t \in (0, T]\), \(c\) depending on \(a, \gamma, N, q, p\) and \(r\) and where \(\delta\) is a constant depending on the equi-integrability of \(u_0\) in \(L^s(\Omega)\) (see Lemma 3.2). Then, we deduce that the decay
\[
\|u(t)\|_{L^r(\Omega)} \leq C \delta \left( \int_\Omega |\Psi_n(G_k(u(s)))|^\rho \, dx \right)^{\omega} \left( \int_\Omega |\Psi_n(G_k(u(s)))|^{-\gamma(1-\omega)} \, dx \right)^{1-\omega} \left( \int_\Omega |G_k(u(s))|^{\gamma(1-\omega)} \, dx \right)^{\gamma(1-\omega)} \left( \int_\Omega |G_k(u_0)|^{\gamma(1-\omega)} \, dx \right)^{1-\gamma(1-\omega)}
\]
holds for small times and with positive constant \(C = C(\gamma, r, q, p, a, N, t_0, u_0, |\Omega|)\).

The continuity regularity in (3.13) follows invoking the Vitali’s Theorem and so do (3.14)–(3.15). \(\square\)

**Remark 3.7.** We claim that the previous Proposition 3.6 implies that
\[
u \in L^{\infty}((\tau, T]; L^r(\Omega)) \cap L^{r-2+p}(\tau, T; L^p) \cap L^{r-2+p}(\tau, T; L^{r-2+p}(\Omega)) \quad \text{for } \tau > 0 \tag{3.25}
\]
since
- the regularity \(u \in L^{\infty}((\tau, T]; L^r(\Omega))\) directly follows from (3.13);
- the regularity \(u \in L^{r-2+p}(\tau, T; L^p) \cap L^{r-2+p}(\Omega))\) is due to the definitions of \(S'(G_k(u))\) in (3.20)–(3.21), since both implies that
\[
(1 + |G_k(u)|)^{\frac{r-2+p}{p}} \in L^p(0, T; W^{1,p}(\Omega)).
\]
In particular, considering also the limit in \( n \to \infty \) and in \( \varepsilon \to 0 \) in (3.19), we have that
\[
\frac{d}{dt} \int |G_k(u(t))|^r \, dx + C_1 \int_{\Omega} |\nabla (1 + |G_k(u(t))|)|^{\frac{\gamma + p - 2}{p}} \, |G_k(u(t))|^\gamma 
\leq 0 \quad \forall k \geq k_0,
\] (3.26)
where \( C_1 = C_1(\alpha, \gamma, N, p, q, r) \).

This fact implies that the function \( t \to \int_{\Omega} |u(t)|^r \, dx \) belongs to \( W^{1,1}(0, T) \) since, once we know (3.25), then we can reason as in LEMMA 3.4.

3.3. Long time decay results. So far, the generalization of [18] to the case \( p \neq 2 \) strictly follows the methods adopted in this work. However, once we get interested in the \( L^\infty \)-regularity, we change approach. More precisely, in [18] it is shown that the analogies between (P) (when \( p = 2 \)) and superlinear power problems (see, for instance, [21]) can be exploited to reason through a Moser type iteration argument, gaining the boundedness of solutions for positive times. The general case \( p \neq 2 \) could be reasonably dealt with a similar argument. However, we choose to apply the results contained in [24].

PROPOSITION 3.8. Assume (ID\( \alpha \)), (A1)–(A2) with \( p > \frac{2N}{N + q} \) and (H) with \( (Q_r) \) and let \( u \) be a solution of (P) in the sense of DEFINITION 3.1. Then the function \( G_k(u) \) satisfies the decays of the coercive problem \( (P_k) \) for \( k \) suitable large, i.e.
\[
\|G_k(u(t))\|_{L^\infty(\Omega)} \leq c \frac{\|G_k(u_0)\|_{L^\infty(\Omega)}}{h_1^p} \quad \forall k \geq k_0, \forall t > 0,
\] (3.27)
with \( h_1 = \frac{N}{N(p - 2) + p\sigma}, \quad h_0 = \frac{p\sigma}{N(p - 2) + p\sigma} \)
and where \( c \) is a constant depending on \( N, q, p, \alpha, \gamma \) and on some fixed value \( r > \sigma \). Furthermore, if \( p > 2 \), we have the following universal bound:
\[
\|G_k(u(t))\|_{L^\infty(\Omega)} \leq \frac{C}{r - \sigma} \quad \forall k \geq k_0, \forall t > 0,
\] (3.28)
where \( C \) is a positive constant depending on \( \alpha, \gamma, N, p, q, |\Omega|, u_0 \) and on \( r \).

PROOF. Consider the differential inequality (3.26) in REMARK 3.7 and integrate between \( \tau < s < t \). Then, thanks to Sobolev’s inequality, we have
\[
\int_{\Omega} |G_k(u(t))|^r \, dx - \int_{\Omega} |G_k(u(\tau))|^r \, dx + C_1 c_3 \int_{\tau}^t \left( \int_{\Omega} |G_k(u)|^{\frac{r - \sigma - 2}{p'}} \, dx \right) ds \leq 0,
\]
where \( k \geq k_0, k_0 \) has been fixed as in PROPOSITION 3.6 and \( C_1 \) is the same constant appearing in (3.26) (we recall that \( C_1 \) depends on \( \alpha, \gamma, N, p, q \) and \( r \)).
The inequality above still holds if we consider \( G_h(G_k(u)) \) instead of \( G_k(u) \). We point out that \( h \) is an arbitrary positive fixed value but we always need to take \( k \geq k_0 \) as in PROPOSITION 3.6.

So far, we already know that
\[
u \in C([0, T]; L^\sigma(\Omega)) \cap C((\tau, T]; L^\sigma(\Omega)) \cap L^{\sigma + p - 2}(\tau, T; L^{\frac{\sigma + p - 2}{p}}(\Omega)),
\]
\[
\sigma < r < p^{\frac{r - \sigma}{r + p - 2}},\quad \frac{\sigma r}{r + p - 2} < r + p - 2 < \frac{r - \sigma}{r^{\sigma + p - 2}} \quad \text{being} \quad \frac{2N}{N + h} < p,
\]
\[
\|G_h(G_k(u(t)))\|_{L^\sigma(\Omega)} \leq \|G_k(u_0)\|_{L^\sigma(\Omega)} \quad \forall h > 0, \forall k \geq k_0,
\]
\[
\int_{\Omega} |G_h(G_k(u(t)))|^r \, dx - \int_{\Omega} |G_h(G_k(u(\tau)))|^r \, dx + C_1 c_3 \int_{\tau}^t \|G_h(G_k(u(s)))\|^{\frac{r + p - 2}{L^{\sigma + p - 2}(\Omega)}} \, ds \leq 0 \quad \forall h > 0, \forall k \geq k_0.
\] (3.29)
Since the constant \( C_1 \) in (3.29) does not depend on \( h \), we apply [24, Theorem 2.1] to \( G_k(u) \) and deduce (3.27).
If \( p > 2 \), then \( r < r + p - 2 \) and thus we can invoke again \([24, \text{Theorem 2.2}]\) gaining the universal bound in \((3.28)\) where \( \sigma \) is a positive constant depending on \( a, \gamma, N, p, q, r \), and \( |\Omega| \).

We point out that \( \frac{1}{p-2} \) does not depend on the summability of the initial datum. Moreover, being \( \frac{1}{p-2} > h_1 \), then \((3.28)\) gives a stronger decay than \((3.27)\) for great values of \( t \). Summarizing, we can say that if \( p > 2 \), then

\[
\|G_k(u(t))\|_{L^\infty(\Omega)} \leq c \min \left\{ \frac{\|G_k(u(\tau))\|_{L^q(\Omega)}^{h_0}}{\eta_1}, \frac{1}{t^{p-2}} \right\} \quad \forall t \in (0, T), \forall k \geq k_0.
\]

As a consequence of the decay above and \((2.3)\), we gain the boundedness for positive times of the solution \( u \).

So far, we have that \( G_k(u) \) behaves as solutions of the coercive problem \((P)\) if \( k \) is large enough. This is not surprising since \( G_k(u) \) satisfies a differential inequality of the type \((3.10)\), of course for great value of \( k \). The next Proposition provides us with the long time decay of the \( L^\infty \)-norm of the whole solution.

**Proposition 3.9.** Assume \((ID)\), \((A1)-(A2)\) with \( p > \frac{2N}{N+\sigma} \) and \((H)\) with \((Q_c)\). Moreover, let \( u \) be a solution of \((P)\) in the sense of Definition 3.1. Then, we have

\[
\lim_{t \to \infty} \|u(t)\|_{L^\infty(\Omega)} = 0.
\]

**Proof.** We skip the proof of this result and say that, once we have Lemma 3.2 and the decay in \((3.27)\), then it can be proved as in \([18, \text{Proposition 3.10}]\). □

**Remark 3.10 (A new smallness condition).** We claim that the results proved so far for \( G_k(u) \) hold for the whole solution \( u \) as well, up to consider large values of \( t \). Indeed, by Proposition 3.9, it is now sufficient to replace the smallness of \( \|G_k(u(t))\|_{L^\infty(\Omega)} \) for great \( k \) with the one of \( \|u(t)\|_{L^\infty(\Omega)} \) for large \( t \) and then taking \( k_0 = 0 \) in Lemma 3.2.

**Proposition 3.11.** Assume \((ID)\), \((A1)-(A2)\) with \( p > \frac{2N}{N+\sigma} \) and \((H)\) with \((Q_c)\). Moreover, let \( u \) be a solution of \((P)\) in the sense of Definition 3.1. Then, if \( \tau \) is sufficiently large such that \( \alpha - \gamma \sigma \|u(\tau)\|_{L^\infty(\Omega)}^{\frac{p-2}{p}} > 0 \) and for \( \lambda = \frac{c_0 \sigma}{p} \left( \alpha - \gamma \sigma \|u(\tau)\|_{L^\infty(\Omega)}^{\frac{p-2}{p}} \right)^{-\frac{p-2}{p}}, \Omega^{-\frac{N(p-2)+pr}{N}} \), we have that

- if \( 2 < p < N \), then \( \|u(t)\|_{L^\infty(\Omega)} \) is decreasing in the time variable for \( t > \tau \) and the following polynomial decay holds:

  \[
  \|u(t)\|_{L^\infty(\Omega)} \leq \left( \|u(\tau)\|_{L^\infty(\Omega)}^{-(p-2)} + \lambda \frac{p-2}{\sigma}(t-\tau) \right)^{-\frac{1}{p-2}} \quad \forall t \geq \tau;
  \]

- if \( \frac{2N}{N+\sigma} < p < 2 \), then there exists a positive time \( \overline{T} \) such that

  \[
  u = 0 \quad \forall t \geq \tau + \overline{T}.
  \]

In particular, we can consider

\[
\overline{T} = \frac{\sigma}{\lambda(2-p)} \|u(\tau)\|_{L^\infty(\Omega)}^{\frac{2-p}{p}}.
\]

**Proof.** We omit the proof since it is very similar to the one of Proposition 3.5, up to replacing the smallness condition in Lemma 3.2 with the one proposed in Remark 3.10. □

**Proposition 3.12.** Assume \((ID)\), \((A1)-(A2)\) with \( p > \frac{2N}{N+\sigma} \) and \((H)\) with \((Q_c)\). Moreover, let \( u \) be a solution of \((P)\) in the sense of Definition 3.1. Then

\[
u \in C((0, T); L'(\Omega)) \quad \text{for} \quad r > \sigma.
\]
Furthermore, there exists a value $\tau$ such that the regularizing effect can be expressed through the decay estimate
\[
\|u(t)\|_{L^r(\Omega)} \leq C \frac{\|u(\tau)\|_{L^r(\Omega)}^{\frac{r}{N(p-1)+r}}}{{(t-\tau)^{-\frac{N(p-1)+r}{r}}}} \quad \forall t > \tau,
\]
where $c = c(\gamma, r, q, p, a, N, |\Omega|)$.

**Proof.** We omit the proof since, thanks to Remark 3.10, it is very similar to the one of Proposition 3.6, up to replacing the smallness condition in Lemma 3.2 with the one proposed in Remark 3.10. \hfill $\square$

**Theorem 3.13.** Assume (A1)–(A2), (ID$_c$) and (H) with (Q$_c$). Moreover, let $u$ be a solution of (P) in the sense of Definition 3.1. Then, the following polynomial decays hold for $2 < p < N$
\[
\|u(t)\|_{L^\infty(\Omega)} \leq \begin{cases} 
C \|u(\tau)\|_{L^r(\Omega)}^{\frac{p^*}{p^*+p-2}} (t-\tau)^{-\frac{N(p-1)+r}{r}} & \forall t > \tau, \\
C_T (t-\tau)^{-\frac{N(p-1)+r}{r}} & \forall t > \tau,
\end{cases}
\] (3.30)
where $h_0, h_1$ are defined in (3.27), $C$ is a positive constant depending on $q, p, N, r, a, |\Omega|$ and $u_0$ whether $C_T$ depends also on $\tau$.

Even if this result immediately follows by Proposition 3.8 and Remark 3.10, we present a short guideline which puts in evidence the use of Proposition 3.9.

**Proof.** We verify that the assumptions of [24, Theorem 2.1 & 2.2] hold. We already know that
\[
u \in C([0, T]; L^\sigma(\Omega)) \cap C((\tau, T]; L^r(\Omega)) \cap L^{r+p-2}(\tau, T; L^{p^*+p-2}(\Omega))
\]
with
\[
\sigma < r < p^* \frac{r + p - 2}{p}, \quad \frac{r - \sigma}{1 - \frac{N(p-1)+r}{r}} < r + p - 2 < p^* \frac{r + p - 2}{p}
\]
and that
\[
\|G_k(u(t))\|_{L^r(\Omega)} \leq \|G_k(u(\tau))\|_{L^r(\Omega)} \quad \forall k \geq 0
\]
are satisfied thanks to Lemma 3.2.

We are left with the proof of
\[
\int_\Omega |G_k(u(t))|^r \, dx - \int_\Omega |G_k(u(\tau))|^r \, dx + \varepsilon \int_\tau^t \|G_k(u(s))\|^{r+p-2} \, ds \leq 0
\] (3.31)
for all $k \geq 0$ and where constant $\varepsilon$ does not depend neither on $k$ nor on the solution. To this aim, we choose $|G_k(u)|^{r-2}G_k(u)$, $r > \sigma$, as test function. Then, we have
\[
\frac{1}{r} \frac{d}{ds} \int_\Omega |G_k(u(s))|^r \, dx + \alpha (r-1) \int_\Omega |\nabla G_k(u(s))|^p |G_k(u(s))|^{r-2} \, dx \\
\leq \gamma \int_\Omega |\nabla G_k(u(s))|^q |G_k(u(s))|^{r-1} \, dx
\]
We apply Hölder’s inequality with indices $\left(\frac{p}{q}, \frac{p}{p-q}\right)$ in the r.h.s. obtaining
\[
\int_\Omega |\nabla G_k(u(s))|^q |G_k(u(s))|^{r-1} \, dx = \int_\Omega |\nabla G_k(u(s))|^q |G_k(u(s))|^\frac{2}{p}(r-2) |G_k(u(s))|^{r-1-\frac{2}{p}(r-2)} \, dx \\
\leq \left( \int_\Omega |\nabla G_k(u(s))|^p |G_k(u(s))|^{r-2} \, dx \right)^{\frac{2}{p}} \left( \int_\Omega |G_k(u(s))|^{p^* \frac{p}{p-2} (r-2)} \, dx \right)^{-\frac{p-2}{p}}.
\]
Then, since the equality $\frac{p}{p-\eta}(r - 1 - \frac{\eta}{p}(r - 2)) = p^{r+p-2} + \frac{p-\eta}{\eta}$ holds by definition of $\sigma$, the $L^\infty((\tau, T) \times \Omega)$ regularity and then the Poincaré inequality give us

$$
\left( \int_\Omega |\nabla G_k(u(s))|^p |G_k(u(s))|^{-2} dx \right)^{\frac{1}{p}} \leq \|u(s)|^q_{L^\infty(\Omega)} \left( \int_\Omega |\nabla G_k(u(s))|^p |G_k(u(s))|^{-2} dx \right)^{\frac{1}{p}} \leq c_p \|u(s)|^q_{L^\infty(\Omega)} \int_\Omega |\nabla G_k(u(s))|^p |G_k(u(s))|^{-2} dx.
$$

Thus, for $\tau$ large enough such that $\bar{\varepsilon} = \alpha(r-1) - \gamma c_p \|u(\tau)|^{q-1}_{L^\infty(\Omega)} > 0$, we have

$$
\frac{d}{ds} \int_\Omega |G_k(u(s))|^p dx + \frac{\bar{\varepsilon} r^p}{(r+p-2)\tau} \int_\Omega |\nabla |G_k(u(s))|^p_{L^\infty(\Omega)} |p| dx \leq 0 \quad \forall k \geq 0.
$$

Having $r > \sigma$, an integration in the time variable for $\tau < s \leq T$ provides us with (3.31) with $\bar{\varepsilon} = \frac{\bar{\varepsilon} r^p}{(r+p-2)\tau}c_p$ in (3.31). Finally, being $p > 2$, then $r < p^{r+p-2}/p$ and so we invoke [24, Theorems 2.1 & 2.2] getting (3.30).

\[ \Box \]

**Remark 3.14 (The critical case $q = p - \frac{N}{N+1}$).** Some remarks on the critical growth case $q = p - \frac{N}{N+1}$ are in order to be given. Beyond assuming the Leray-Lions structure conditions in (A) and the growth assumption (H), we deal with this case taking into account initial data satisfying

$$
u_0 \in L^{1+\omega}(\Omega), \quad \omega > 0.
$$

As already observed, such value of $q$ implies that the value $\sigma$ in (ID$_v$) is equal to 1. However, due to the criticality of this case, we have to ask for more than just $L^1(\Omega)$ data. In this way, we are allowed to consider solutions as in Definition 3.1, with (RC) replaced by

$$
(1 + |u|)^{-\frac{1}{\omega}} u \in L^p(0, T; W^{1,p}_0(\Omega)).
$$

Then, we proceed as before and we prove that (3.30) holds with $\sigma = 1 + \omega$.

## 4. The growth range with $L^1(\Omega)$ data

We now deal with the case of superlinear growth (Q$_t$) that we recall being

$$
\frac{2N}{N+1} < p < N \quad \text{and} \quad \max \left\{ \frac{p}{2} \left( \frac{p(N+1)-N}{N+2} \right) \right\} < q < p - \frac{N}{N+1}.
$$

We are going to deal with this case assuming $L^1(\Omega)$ data. In particular, since we can no longer require (RC), we will ask for the asymptotic energy condition

$$
\lim_{n \to \infty} \frac{1}{n} \int_{[-n,n]^2} a(t,x,u,\nabla u) \cdot \nabla u = 0. \quad \text{(ET)}
$$

We consider the following notion of solution.

**Definition 4.1.** We say that a function $u \in T_{0}^{1,p}(Q_T)$ is a solution of (P) if satisfies (ET) and

$$
H(t,x,u,\nabla u) \in L^1(Q_T),
$$

$$
- \int_{Q_T} S(u_0) \varphi(0,x) dx + \int_{Q_T} -S(u) \varphi _1 + a(t,x,u,\nabla u) \cdot \nabla (S'(u) \varphi) dx ds = \int_{Q_T} H(t,x,\nabla u) S'(u) \varphi dx ds
$$

for every $S \in W^{2,\infty}(\mathbb{R})$ such that $S'(-\cdot)$ has compact support and for every test function $\varphi \in C_c^\infty([0,T] \times \Omega)$ such that $S'(\cdot) \varphi \in L^p(0, T; W^{1,p}_0(\Omega))$ (i.e. $S'(\cdot) \varphi$ is equal to zero on $(0, T) \times \partial \Omega$).

The existence of solutions of (1.1) has been proved in [17, Theorem 6.5].
4.1. $L^1 - L^1$ and Marcinkiewicz regularities. As seen in Section 3, the crucial step relies on a $\delta$ argument which allows us to move the attention from (P) to its “coercive version”, i.e. (P) read in terms of $G_k(u)$. However, due to the low regularity of the initial data, we lose the purely contractive relation between $\|G_k(u(t))\|_{L^1(\Omega)}$ and $\|G_k(u_0)\|_{L^1(\Omega)}$.

**Lemma 4.2.** Assume (ID), (A1)–(A2) with $p > \frac{2N}{N+1}$ and (H) with (Q). Moreover, let $u$ be a solution of (P) in the sense of Definition 4.1. Then, for every $k > 0$ so that

$$\|G_k(u_0)\|_{L^1(\Omega)} < \delta,$$

where $\delta > 0$ is arbitrary fixed, we have

$$\|G_k(u(t))\|_{L^1(\Omega)} < c \delta^{\frac{1}{2}} \quad \forall t \in [0, T],$$

for some positive constant $c$ depending on $|\Omega|$, $N$, $p$ and $q$.

Before proving Lemma 4.2, we recall some standard regularity results in renormalized settings with $L^1$-data.

**Proposition 4.3.** Assume (ID), (A1)–(A2) with $p > \frac{2N}{N+1}$ and (H) with (Q). Moreover, let $u$ be a solution of (P) in the sense of Definition 4.1. Then we have that

$$u \in C([0, T]; L^1(\Omega)) \cap M^{\frac{p(N+1)-N}{N}}(Q_T)$$

and

$$|\nabla u| \in M^{\frac{p(N+1)-N}{N}}(Q_T). \quad (4.2)$$

**Proof.** The Marcinkieicz regularities follow from [7, 8].

As far as the continuity of $u(t)$ in $L^1(\Omega)$ is concerned, let $S'(u) \varphi = \frac{T_{\varphi}(G_k(u))}{\varphi}$, $\varphi = 1$ and $\omega > 0$, in (4.1). Again, we note that such a test function can be made rigorous up to be multiplied by $\theta_n(G_k(u))$ and recalling the asymptotic condition (ET). Then the limit for $\omega \to 0$ provides us with the inequality

$$\int_{\Omega} |G_k(u(t))| \, dx \leq \int_{\Omega} |G_k(u_0)| \, dx + \gamma \int_{Q_T} |\nabla G_k(u)|^q \, dx \, dt. \quad (4.3)$$

The gradient regularity in (4.2) and (4.3) imply that $\|G_k(u(t))\|_{L^1(\Omega)} \to 0$ when $k \to \infty$. Since we already know from [20, Theorem 1.1] that $T_k(u) \in C([0, T]; L^1(\Omega))$ for every $k > 0$, then the Vitali-Lebesgue Theorem implies the continuity regularity $u \in C([0, T]; L^1(\Omega))$. \hfill \Box

**Proof of Lemma 4.2.** We set

$$S'_n(u) \varphi = \left[ 1 - \frac{1}{1 + |G_k(u)|^p} \right] \text{sign}(u) \quad \text{with} \quad b = \frac{(p-q)(N+1)}{N} - 1,$$

$\varphi = 1$ in (4.1). Note that $0 < b < \frac{p}{q}$ by (Q). We justify the above choice reasoning as in Proposition 4.3. Then, recalling (ET), we get

$$\int_{\Omega} S_n(u(t)) \, dx + ab \int_{Q_T} \frac{|\nabla G_k(u)|^p}{(1 + |G_k(u)|)^{p+1}} \, dx \, ds \leq \gamma \int_{Q_T} |\nabla G_k(u)|^q S'_n(u) \, dx \, ds. \quad (4.4)$$

We are going to deal with the integral in the r.h.s.. An application of Young’s inequality with indices $\left( \frac{p}{q}, \frac{p}{p-q} \right)$ gives us

$$\gamma \int_{Q_T} |\nabla G_k(u)|^q S'_n(u) \, dx \, ds \leq \frac{ab}{2} \int_{Q_T} \frac{|\nabla G_k(u)|^p}{(1 + |G_k(u)|)^{p+1}} \, dx \, ds + c \int_{Q_T} (1 + |G_k(u)|)^{\frac{q(p+1)}{p}} \frac{1}{(S'_n)^\frac{q(p+1)}{pq}} \, dx \, ds \leq \frac{ab}{2} \int_{Q_T} \frac{|\nabla G_k(u)|^p}{(1 + |G_k(u)|)^{p+1}} \, dx \, ds + c \int_{Q_T} (1 + |G_k(u)|)^{\frac{q(p+1)}{p}} \frac{1}{\frac{q(p+1)}{pq}} |G_k(u)| \, dx \, ds,$$
being \( S'_n(u) < \frac{|G_k(u)|}{(1 + |G_k(u)|)} \) because \( b < 1 \) and for \( c = c(\alpha, \gamma, q, p, N) \). Then, we improve (4.4) with

\[
\int_{\Omega} S_n(u(t)) \, dx + \frac{\alpha b}{2} \int_{Q_1} \frac{|\nabla G_k(u)|^p}{(1 + |G_k(u)|)^{b+1}} \, dx \, ds \\
\leq c \int_{Q_1} \left( 1 + |G_k(u)| \right)^{-\frac{q(b+1)}{p-q}} |G_k(u)| \, dx \, ds + \int_{\Omega} |G_k(u_0)| \, dx.
\]

(4.5)

The choice of \( b \) implies that \( \frac{q(b+1)}{p-q} = q \frac{N+1}{N} \) which is, in particular, the Gagliardo-Nirenberg exponent of the spaces

\( L^{\infty}(0, T; L^1(\Omega)) \cap L^q(0, T; W_0^1(\Omega)). \)

Since Proposition 4.3 provides us with the above regularities, we are allowed to consider the limit on \( n \to \infty \) in (4.5) getting

\[
\int_{\Omega} S(u(t)) \, dx + \frac{\alpha b}{2} \int_{Q_1} \frac{|\nabla G_k(u)|^p}{(1 + |G_k(u)|)^{b+1}} \, dx \, ds \\
\leq c \int_{Q_1} \left( 1 + |G_k(u)| \right)^{-\frac{q(b+1)}{p-q}} |G_k(u)| \, dx \, ds + \int_{\Omega} |G_k(u_0)| \, dx,
\]

where \( S(u) = \int_{0}^{T} \frac{1}{2} - \frac{1}{(1+[u])^p} \, dy. \)

The above estimate implies that the l.h.s. of (4.6) is bounded and, in particular, that

\( \left( 1 + |G_k(u)| \right)^{-\frac{q(b+1)}{p-q}} |G_k(u)| \in L^{\infty}(0, T; L^p f^{-1} \frac{\lambda}{2}(\Omega)) \cap L^p(0, T; W_0^1, p(\Omega)). \)

Since

\[
\frac{p}{p-1-b} < N + \frac{p}{p-1-b} < p^* \quad \text{and} \quad p < N + \frac{p}{p-1-b}
\]

thanks to (Q.1) and the definition of \( b \), we invoke again Gagliardo-Nirenberg regularity results, obtaining the regularity

\( \left( 1 + |G_k(u)| \right)^{-\frac{q(b+1)}{p-q}} |G_k(u)| \in L^\lambda(\Omega_T) \) where \( \lambda = \frac{N + \frac{p}{p-1}}{N}. \)

In particular, the related Gagliardo-Nirenberg inequality can be estimated as

\[
\int_{0}^{T} \left( 1 + |G_k(u)| \right)^{-\frac{q(b+1)}{p-q}} |G_k(u)| \, dx \, ds \leq c_{GN} \|G_k(u)\|_{L^{\lambda}(0, T; L^1(\Omega))} \int_{Q_T} \left( 1 + |G_k(u)| \right)^{\frac{p}{p-1}} \, dx \, dt.
\]

(4.7)

Let us come back to (4.5). Since

\[
\frac{q(b+1)}{p-q} = \left( \frac{b+1}{p} + 1 \right) \lambda
\]

by definitions of \( b \) and \( \lambda \), we estimate the r.h.s. of (4.5) taking advantage of (4.7) as follows:

\[
\gamma \int_{Q_1} |\nabla G_k(u)|^q S'_n(u) \, dx \, ds \leq \gamma c_{GN} \|G_k(u)\|_{L^{\lambda}(0, T; L^1(\Omega))} \int_{Q_T} \left( 1 + |G_k(u)| \right)^{\frac{p}{p-1}} \, dx \, dt.
\]

We thus deduce

\[
\int_{\Omega} S(u(t)) \, dx + \alpha b \int_{Q_1} \frac{|\nabla G_k(u)|^p}{(1 + |G_k(u)|)^{b+1}} \, dx \, ds \\
\leq \gamma c_{GN} \|G_k(u)\|_{L^{\lambda}(0, T; L^1(\Omega))} \int_{Q_T} \left( 1 + |G_k(u)| \right)^{\frac{p}{p-1}} \, dx \, dt + \int_{\Omega} |G_k(u_0)| \, dx,
\]

where the limit on \( n \to \infty \) has been taken too.

The \( \delta \) argument.

We observe that \( S(v) \geq c_1 \min \{ v, v^2 \} \), where the constant \( c_1 > 1 \) depends only on \( N, p \) and \( q \).

Then, we proceed as in Lemma 3.2 (see the \( \delta \) argument) fixing a small value \( \delta_0 \) so that the inequality

\[
\alpha b - \gamma c_{GN}(c_0 \delta^2) \frac{p}{p-1} > 0 \quad \text{for} \quad c_0 = 2 \max \left\{ \frac{1}{c_1}, (1/|\Omega|/c_1)^{\frac{1}{2}} \right\} \quad \text{and} \quad \delta < \delta_0. \]

Moreover, we let \( k_0 \) large enough so that

\[
\|G_k(u_0)\|_{L^1(\Omega)} < \delta \quad \forall k \geq k_0
\]

(4.8)

and define

\[
T^* := \sup \{ \tau > 0 : \|G_k(u(s))\|_{L^1(\Omega)} \leq c_0 \delta^2, \forall s \leq \tau \} \quad \forall k \geq k_0.
\]
Notice that $T^* > 0$ thanks to the continuity result proved in Theorem 4.3. The above choice of $\delta$ and (4.8) imply
\[
\int_{\Omega} S(G_k(u(t))) \, dx \leq \int_{\Omega} |G_k(u_0)| \, dx < \delta \quad \forall k \geq k_0, \quad \forall t \leq T^*.
\]
Therefore, by definition of $c_1$ and $c_0$, we obtain
\[
\int_{\Omega} |G_k(u(t))| \, dx \leq \int_{\{|G_k(u(t))| > 1\}} |G_k(u(t))| \, dx + |\Omega|^\frac{1}{2} \left( \int_{\{|G_k(u(t))| \leq 1\}} |G_k(u(t))|^2 \, dx \right)^\frac{1}{2}
\]
\[
\leq \frac{1}{c_1} \int_{\Omega} S(G_k(u(t))) \, dx + \left( \frac{|\Omega|}{c_1} \right)^\frac{1}{2} \left( \int_{\Omega} S(G_k(u(t))) \, dx \right)^\frac{1}{2}
\]
for every $t \leq T^*$. Finally, a contradiction argument extends the inequality in (4.9) to the whole time interval. \hfill \Box

**Remark 4.4.** Again, if $u_0 \in L^\infty(\Omega)$, then the $\delta$ argument provides us with a contraction result in $L^\infty(\Omega)$ (see also Corollary 3.3).

**4.2. The regularizing effect $L^1 - L^p$ and long time decays.**

**Proposition 4.5.** Assume (ID$_2$), (A1)–(A2) with $p > \frac{2N}{N+1}$ and (H) with (Q$_1$). Moreover, let $u$ be a solution of ($P$) in the sense of Definition 4.1. Then the claim of Proposition 3.6 holds true.

**Proof.** In order to reason as in Proposition 3.6, we assume (3.17) and modify (3.16) asking for a function $S(\cdot)$ satisfying
\[
S''(v) \leq L(1 + |v|)^{-(2+b)}|v|, \quad b = (p-q)\frac{N+1}{N} - 1.
\]
In this way, we can repeat the argument at the very beginning of the proof of Proposition 3.6, getting (3.18). In particular, Lemma 4.2 provides us with an inequality as in (3.19).

We conclude exhibiting a function which, for fixed $n$, verifies both (3.17) and (4.10):
\[
S'(v) = \int_0^v (1 + |y|)^{-(b+2)}|y|T_n(y)^{r-1+b} \, dz, \quad r > 1.
\]

We conclude this Section observing that, once we have the contraction result of Remark 4.4 as well as the regularizing effect provided by Proposition 4.5, then we are allowed to reason as in Subsection 3.3 getting the same long time decays results as in Theorem 3.13.

**5. Further comments**

**5.1. On the notion of solution.** We here point out that we could consider different notions of solutions than Definitions 3.1 and 4.1. Indeed, as shown in [18], the Definitions 5.1, 5.2 and 5.4 below are strictly related to the ones previously considered.

**Definition 5.1.** A function $u$ is a solution to ($P$) if $u(0) = u_0$,
\[
u \in C([0,T]; L^p(\Omega)) \cap L^p_{loc}(0,T; W^1_{0,p}(\Omega))
\]
and $u$ satisfies the weak formulation
\[
\int_{Q_T} -u \varphi_t + a(t,x,u,\nabla u) \cdot \nabla \varphi \, dx \, dt = \int_{Q_T} H(t,x,\nabla u) \varphi \, dx \, dt
\]
for every $\varphi \in C^\infty_c((0,T) \times \Omega)$.

**Definition 5.2.** A function $u$ is a solution of ($P$) if the regularity condition (RC) holds, $H(t,x,\nabla u) \in L^1(Q_T)$ and $u$ satisfies the weak formulation
\[
-\int_{Q_T} u \varphi_0(0) \, dx + \int_{Q_T} [-\varphi_t u + a(t,x,u,\nabla u) \cdot \nabla \varphi] \, dx \, dt = \int_{Q_T} H(t,x,\nabla u) \varphi \, dx \, dt
\]
for every $\varphi \in C^\infty_c((0,T) \times \Omega)$.

**Proposition 5.3.** We have that Definition 5.1 is equivalent to
Definition 5.2. If the gradient growth occurs with rates as in (3.1) (i.e., we consider the subinterval of \((Q_c)\) such that \(c \geq 2\));

Definition 5.3. If the gradient growth occurs with rates as in \((Q_{c'})\).

Definition 5.4. A function \(u\) is a solution to \((P)\) if \(u(0) = u_0\),

\[
u \in C([0, T]; L^1(\Omega)) \cap L^p_{\text{loc}}(0, T; W^{1,p}_{0}(\Omega))
\]

and \(u\) satisfies the weak formulation

\[
\int_{Q_T} -u \varphi_t + a(t, x, u, \nabla u) \cdot \nabla \varphi \, dx \, dt = \int_{Q_T} H(t, x, \nabla u) \varphi \, dx \, dt
\]

for every \(\varphi \in C_0^\infty((0, T) \times \Omega)\).

Proposition 5.5. We have that Definition 5.4 is equivalent to Definition 4.1.

We omit the proof of the Propositions above since they are a simple generalisation of [18, Propositions 2.2, 2.4 and 4.2].

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