THE POKROVSKI-TALAPOV PHASE TRANSITION
AND QUANTUM GROUPS

Haye Hinrichsen

Universität Bonn, Physikalisches Institut
Nussallee 12, W-5300 Bonn 1, FRG

Vladimir Rittenberg

Theory Division, CERN
CH-1211 Geneva 23, Switzerland

Abstract

We show that the XY quantum chain in a magnetic field is invariant under a two parameter deformation of the $SU(1/1)$ superalgebra. One is led to an extension of the braid group and the Hecke algebras which reduce to the known ones when the two parameter coincide. The physical significance of the two parameters is discussed. When both are equal to one, one gets a Pokrovski-Talapov phase transition. We also show that the representation theory of the quantum superalgebras indicates how to take the appropriate thermodynamical limits.
1 Introduction

There were several attempts to extend the one-parameter quantum algebras to multiparameter ones [1]. As shown however by Reshetikhin [2] the link polynomials depend only on one parameter. One can state this result in a different way: if one has a one-dimensional quantum chain which is invariant under a multiparameter quantum algebra, one can do a similarity transformation which eliminates all the parameters but one. As will be shown in this paper, the situation is different in the case of quantum superalgebras. We will start with a physical example. Consider the quantum chain

\[ H = \Delta_q \sum_{i=1}^{L} \sigma_i^z + \frac{\Delta_\eta}{2} \sum_{i=1}^{L-1} [(1 + u)\sigma_i^x \sigma_{i+1}^x + (1 - u)\sigma_i^y \sigma_{i+1}^y] + B + S, \]  

(1)

where \( \sigma^x, \sigma^y \) and \( \sigma^z \) are Pauli matrices inserted in the i-th position of the Kronecker product

\[ \sigma_i^k = 1 \otimes 1 \otimes \ldots \otimes \sigma_i^k \otimes \ldots \otimes 1 \otimes 1 \]  

(i = 1, 2, \ldots, L)  

(2)

\[ [\sigma_i^k, \sigma_j^l] = 0. \]  

(i \neq j)

\( \Delta_q, \Delta_\eta \) and \( u \) are parameters, \( B \) and \( S \) are boundary and surface terms respectively. This chain appears in the domain wall theory of two-dimensional commensurate-incommensurate phase transitions [3, 10] and in Glauber’s kinetic Ising model [4]. In order to make contact with quantum algebras we will first make an important change of notations, choose \( B = 0 \) (no periodic boundary conditions!) and fix \( S \) by

\[ \Delta_q = \frac{q + q^{-1}}{2}, \quad \Delta_\eta = \frac{\eta + \eta^{-1}}{2}, \quad u = \frac{\eta - \eta^{-1}}{\eta + \eta^{-1}} \]  

(3)

\[ S = \frac{1}{2} (q^{-1} \sigma_1^z + q \sigma_L^z). \]

With this change of notations we have

\[ H = H(q, \eta) = \sum_{i=1}^{L-1} H_i(q, \eta) \]  

(4)

\[ H_i(q, \eta) = \frac{1}{2} [\eta \sigma_i^x \sigma_{i+1}^x + \eta^{-1} \sigma_i^y \sigma_{i+1}^y - q \sigma_i^z - q^{-1} \sigma_{i+1}^z]. \]

A detailed discussion of the properties of the chain given by eq. (4) will be given elsewhere [5], here we are going to mention only a few. First, there are the symmetry properties

\[ H(q, \eta) \cong H(q^{-1}, \eta) \cong H(q, \eta^{-1}) \cong H(\eta, q). \]  

(5)

The "equality" among the Hamiltonians implies that the spectra are identical. The first two equalities are obvious but not the last one which reminds of duality transformations of quantum chains [1]. In the continuum limit, one has the following phase structure [3, 5]:
\[ \Delta_q \leq 1, \quad \Delta_\eta \leq 1: \text{massless-incommensurate} \]
\[ \Delta_q \leq 1, \quad \Delta_\eta > 1 \text{ or } \Delta_q > 1, \quad \Delta_\eta \leq 1: \text{massive incommensurate} \]
\[ \Delta_q > 1, \quad \Delta_\eta > 1, \quad \Delta_q \neq \Delta_\eta: \text{massive} \]
\[ \Delta_q = \Delta_\eta, \quad (\Delta_q > 1): \text{critical Ising type} \]
\[ \Delta_q = \Delta_\eta = 1: \text{Pokrovsky-Talapov phase transition} \]

We will return to the physical picture later on in the text. It is by now clear that the properties of the chain depend on both parameters \( q \) and \( \eta \).

We now perform a Jordan-Wigner transformation. First write \( \sigma_z^j = -i \sigma_x^j \sigma_y^j \) and next define

\[
\tau_{i,j}^{x,y} = \exp \left( \frac{i \pi}{2} \sum_{k=1}^{j-1} (\sigma_k^x + 1) \right) \sigma_j^{x,y} \tag{6}
\]

\[ \{ \tau_i^x, \tau_j^y \} = \{ \tau_i^y, \tau_j^x \} = \{ \tau_i^x, \tau_j^y \} = 0. \quad (i \neq j) \tag{7} \]

Using Eq. (4) and (6) we get

\[
H_j = i \left[ \eta \tau_j^y \tau_{j+1}^x - \eta^{-1} \tau_j^x \tau_{j+1}^y + q \tau_j^x \tau_j^y + q^{-1} \tau_{j+1}^x \tau_{j+1}^y \right]. \tag{8}
\]

We now observe the following important identity

\[ [T^X, H(q, \eta)] = [T^Y, H(q, \eta)] = 0 \tag{9} \]

with

\[
T^X = \Delta(\tau^x) = \alpha \frac{L+1}{2} \sum_{j=1}^{L} \alpha^j \tau_j^x \]
\[
T^Y = \Delta(\tau^y) = \beta^{-\frac{L+1}{2}} \sum_{j=1}^{L} \beta^j \tau_j^y \tag{10}
\]

\[
\{ T^X, T^X \} = 2[L]_\alpha, \quad \{ T^Y, T^Y \} = 2[L]_\beta, \quad \{ T^X, T^Y \} = 0, \tag{11}
\]

where \( L \) is the length of the chain and

\[ \alpha = \frac{q}{\eta}, \quad \beta = q\eta, \quad [L]_\lambda = \frac{\lambda^L - \lambda^{-L}}{\lambda - \lambda^{-1}}. \tag{12} \]

The equalities (8) come from the existence of a fermionic zero mode for any \( q \) and \( \eta \). The equations (10) together with the coproduct (9) give a representation of a Hopf algebra. Before we prove this statement let us consider the case \( \alpha = \beta = q \).
2 The $\eta = 1$ case. Mathematics.

We first notice that in this case $S^z = \frac{1}{L} \sum_{i=1}^L \sigma_i^z$ also commutes with $H(q, \eta)$. We now remind the reader the $U_\alpha[SU(1/1)]$ algebra [7]. With $A^\pm = \frac{1}{2}(T^X \pm iT^Y)$ we have

$$\{A^\pm, A^\pm\} = 0, \quad \{A^+, A^-\} = [E]_\alpha, \quad [S^z, A^\pm] = \pm A^\pm$$

(12)

$$[E, S^z] = [E, A^\pm] = 0$$

with the coproduct

$$\Delta(\alpha, A^\pm) = \alpha^{E/2} \otimes A^\pm + A^\pm \otimes \alpha^{-E/2}$$

(13)

$$\Delta(\alpha, S^z) = S^z \otimes 1 + 1 \otimes S^z$$

$$\Delta(\alpha, E) = E \otimes 1 + 1 \otimes E.$$

The fermionic representations correspond to take $E = 1, \ S^z = \frac{1}{L} \sigma_z, \ A^\pm = a^\pm$ and $\{a^+, a^-\} = 1$ in Eq (13). In this representation $E$ in Eq. (12) is equal to $L$ (the number of sites). Comparing now (12), (13) with Eqs. (9,10) we observe [8] that the quantum chain (4) with $\eta = 1$ is invariant under $U_\alpha[SU(1/1)]$ transformations. It was also shown by Saleur [8] that the quantities $U_j = \Delta_q - H_j(q,1)$ are the generators of the Hecke algebra

$$U_j^2 = 2 \Delta_q U_j$$

$$U_j U_{j+1} U_j - U_j = U_{j+1} U_j U_{j+1} - U_{j+1}$$

(14)

$$U_i U_{i+j} = U_{i+j} U_i.$$  \hspace{1cm} (j \neq 1)

Actually they correspond to a quotient of this algebra since the generators satisfy also the relations [9]

$$U_j U_{j+2} U_{j+1} (2\Delta_q - U_j)(2\Delta_q - U_{j+2}) = 0.$$  \hspace{1cm} (15)

The generators $\tilde{R}_j = \frac{2-q^{-1}}{2} + H_j(q,1)$ satisfy the braiding relations

$$\tilde{R}_j \tilde{R}_{j\pm 1} \tilde{R}_j = \tilde{R}_{j\pm 1} \tilde{R}_j \tilde{R}_{j\pm 1}$$

(16)

with

$$\tilde{R}_j^2 = (q - q^{-1}) \tilde{R}_j + 1.$$  \hspace{1cm} (17)

Considering the matrices $R_j = P \tilde{R}_j$ ($P$ is the graded permutation operator) we have (see Eq. (13))

$$R \Delta(\alpha) R^{-1} = \Delta(\alpha^{-1}).$$  \hspace{1cm} (18)
3 The $\eta = 1$ case. Physics.

Before pursuing our mathematical developments let us pause and discuss some physical implications. First we notice the very unusual role of the operator $E$ for the quantum chain. It does not behave like an usual symmetry operator in quantum mechanics (like the angular momentum) which commutes with the Hamiltonian and helps in its diagonalisation. Since $E$ simply counts the number of sites it plays a different role that we clarify now. From Eq. (12) we see that for $\alpha$ generic ($\alpha \neq e^{i\pi s}$), $U_\alpha[SU(1/1)]$ has two-dimensional irreducible representations and one one-dimensional irreducible representation where $A^\pm = E = S^z = 0$. If $\alpha$ is not generic ($\alpha = e^{i\pi s}$), notice that

$$\{A^+, A^-\} = \frac{\sin(\frac{\pi rL}{s})}{\sin(\frac{\pi r}{s})}$$

and that for $L = ns$ one has only one-dimensional irreducible representations. This implies that for a given value of $q$, changing $L$ one can reach pathological situations. As shown in Ref. [5], if $L = ns$ one has not only one zero mode but two which makes the degeneracies larger and not smaller as one would expect from the fact that we have only one-dimensional irreducible representations. In order to avoid this type of problems and to keep the normalisations of the zero-mode operator (i.e. $A$ and $A^+$), if one wants to take the thermodynamical limit, one has to take sequences like

$$L = ns + t \quad (t = 0, 1, \ldots, s - 1; \ n \in \mathbb{Z}_+)$$

and the results will depend on the sequence. The necessity of taking sequences for the quantum chain (11) with periodic boundary conditions is already known [11] but now we understand its origin. The same observation applies when we have two parameters (see Eq. (9)) and one or both of them are not generic [5].

A more detailed discussion of the physical meaning of the parameter $q$ as well as the connection of the model with the experimental data [13] can be found in Ref. [5].

4 The $\eta \neq 1$ case.

As suggested by Eqs. (11,12), we define the two parameter deformation of the $SU(1/1)$ algebra as follows:

$$\{T^X, T^X\} = 2 [E]_\alpha, \quad \{T^Y, T^Y\} = 2 [E]_\beta$$
$$\{T^X, T^Y\} = 0 \quad \{E, T^X\} = \{E, T^Y\} = 0$$

with the coproduct

$$\Delta(\alpha, \beta; T^X) = \alpha^{E/2} \otimes T^X + T^X \otimes \alpha^{-E/2}$$
$$\Delta(\alpha, \beta; T^Y) = \beta^{E/2} \otimes T^Y + T^Y \otimes \beta^{-E/2}$$
$$\Delta(\alpha, \beta; E) = E \otimes 1 + 1 \otimes E.$$
Notice that $S_z$ does not appear in the algebra anymore. We denote this quantum algebra by $U_{\alpha,\beta}[SU(1/1)]$. It is a Hopf algebra for the same reasons as the $U_{\alpha}[SU(1/1)]$. If we take the fermionic representations $E = 1$, $\tau^x = (a^+ + a^-)$, $\tau^y = -i(a^+ - a^-)$, from Eq. (22) we derive Eqs. (14-18). The quantum chain $H(q, \eta)$ is thus invariant under the quantum algebra $U_{\alpha,\beta}[SU(1/1)]$. We would like to see what replaces the relations (14-18) when we have two parameters. We first notice a remarkable identity satisfied by the $H_j(q, \eta)$

$$[H_j H_{j\pm 1} - H_{j\pm 1} H_j + (\nu - 1)(H_j - H_{j\pm 1})] (H_j - H_{j\pm 1}) = \mu$$

$$H_j^2 = \nu,$$

where

$$\nu = \left(\frac{\alpha + \alpha^{-1}}{2}\right) \left(\frac{\beta + \beta^{-1}}{2}\right) = \left(\frac{q + q^{-1}}{2}\right)^2 + \left(\frac{\eta + \eta^{-1}}{2}\right)^2 - 1$$

$$\mu = \left(\frac{\alpha + \alpha^{-1}}{2} - \frac{\beta + \beta^{-1}}{2}\right)^2 = 4 \left(\frac{q - q^{-1}}{2}\right)^2 \left(\frac{\eta - \eta^{-1}}{2}\right)^2.$$}

We can now define a generalised Hecke algebra taking

$$U_i = \sqrt{\nu} - H_i(p, q)$$

$$(U_i U_{i\pm 1} U_i - U_{i\pm 1} U_i U_{i\pm 1} - U_i + U_{i\pm 1}) (U_i - U_{i\pm 1}) = \mu$$

$$U_i^2 = 2 \sqrt{\nu} U_i.$$}

Notice that when $\eta = 1$, $\mu = 0$ and we have representations which coincide with those of the original Hecke algebra (For a detailed discussion of the representation theory of (25) see Ref. [16]). We did not have the patience to find the equivalent of Eq. (15) which gives the quotient of the generalised Hecke algebra (25) corresponding to the chain given by Eq. (4). Another quotient is however suggested by the structure of Eq. (25):

$$(U_i U_{i\pm 1} U_i - U_i) (U_i - U_{i\pm 1}) = \frac{\mu}{2}.$$}

For $\mu = 0$ one gets in this case representations of the Temperley-Lieb algebra $U_i U_{i\pm 1} U_i = U_i$. We now turn our attention to the generalised braid group algebra. We take $\tilde{R}_i = H_i(q, \eta) + \sqrt{\nu - 1}$ and get

$$(\tilde{R}_i \tilde{R}_{i\pm 1} \tilde{R}_i - \tilde{R}_{i\pm 1} \tilde{R}_i \tilde{R}_{i\pm 1}) (\tilde{R}_i - \tilde{R}_{i\pm 1}) = \mu$$

$$\tilde{R}_i^2 = 1 + \sqrt{\nu - 1} \tilde{R}_i.$$}
In the basis where the $\sigma^z_i$ are diagonal (see Eq. (4)) we have

$$\hat{R}_i = \begin{pmatrix}
\sqrt{\nu - 1} + \frac{q+q^{-1}}{2} & 0 & 0 & \frac{\eta - \eta^{-1}}{2} \\
0 & \sqrt{\nu - 1} - \frac{q-q^{-1}}{2} & 0 & \frac{\eta + \eta^{-1}}{2} \\
0 & \frac{\eta - \eta^{-1}}{2} & \sqrt{\nu - 1} + \frac{q-q^{-1}}{2} & 0 \\
\frac{\eta - \eta^{-1}}{2} & 0 & 0 & \sqrt{\nu - 1} - \frac{q+q^{-1}}{2}
\end{pmatrix}. \tag{29}$$

We take the graded permutation matrix $P$

$$P = \begin{pmatrix}
1 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 0 & 1
\end{pmatrix}. \tag{30}$$

and define the matrix $R_i = P\hat{R}_i$. We now write the coproduct (22) in the original language of Pauli matrices

$$\Delta(\alpha, \beta; T^X) = \alpha^{-1/2}(\sigma^x \otimes 1) - \alpha^{1/2}(\sigma^y \otimes \sigma^y)$$

$$\Delta(\alpha, \beta; T^Y) = \beta^{-1/2}(\sigma^x \otimes 1) - \beta^{1/2}(\sigma^x \otimes \sigma^x)$$

$$\Delta(\alpha, \beta; E) = 1 \otimes 1 + 1 \otimes 1. \tag{31}$$

It is trivial to check that similar to Eq. (18) we get

$$R \Delta(\alpha, \beta) R^{-1} = \Delta(\alpha^{-1}, \beta^{-1}). \tag{32}$$

5 Are more-parameter deformations possible?

In this section we would like to show that for $U_{\alpha,\beta}[SU(1/1)]$ one can introduce more than two parameters (as in the Lie algebra case when we had more than one). The most general chain which has a zero mode for all its values of the parameters is

$$H_i = \frac{1}{2} \left\{ \frac{\Theta + \Theta^{-1}}{2}(\eta \zeta \sigma^x_i \sigma^x_{i+1} + \eta^{-1} \zeta^{-1} \sigma^y_i \sigma^y_{i+1}) + \frac{\Theta - \Theta^{-1}}{2}(\eta \zeta^{-1} \sigma^x_i \sigma^y_{i+1} + \eta^{-1} \zeta \sigma^y_i \sigma^x_{i+1}) + q \sigma^z_i + q^{-1} \sigma^z_{i+1} \right\}. \tag{33}$$

$H_i$ depends on four parameters. One can check however that Eq. (23) holds with

$$\nu = \frac{q^2 + q^{-2}}{4} + \frac{(\eta^2 - \eta^{-2})(\zeta^2 - \zeta^{-2})}{8} + \frac{(\eta^2 + \eta^{-2})(\zeta^2 + \zeta^{-2})(\Theta^2 + \Theta^{-2})}{16},$$
\[ \mu = \left( \frac{(\eta^2 - \eta^{-2})(\zeta^2 - \zeta^{-2})}{4} + \frac{(\eta^2 + \eta^{-2})(\zeta^2 + \zeta^{-2})(\Theta^2 + \Theta^{-2})}{8} - 1 \right) \times \left( \frac{(q - q^{-1})^2}{2} + \frac{(\Theta - \Theta^{-1})^2(\frac{q}{\zeta} - \frac{q}{\eta})^2}{8} \right) \]  

which means that we are back to two parameters. This means that there is a similarity transformation which connects the Hamiltonian with four parameters and the one with two (see Eq. (34)). In order to illustrate this point we consider the "two-parameter deformation" of Ref. [14]. It corresponds to the choice

\[ \zeta = e^{i\pi/4}, \quad \eta = e^{-i\pi/4}, \quad q = \sqrt{QP}, \quad \theta = \sqrt{\frac{Q}{P}} \]  

in Eq. (33) where Q and P are the two parameters given in [14]. From Eq. (34) we get \( \mu = 0 \) which implies that we are back to the \( U_\alpha[SU(1/1)] \) case. From Eq. (33) we derive

\[ \tilde{R}_i = \frac{1}{2}(\sqrt{QP} - \frac{1}{\sqrt{QP}}) + \sqrt{\frac{Q}{P}} \sigma_i^+ \sigma_i^- + \sqrt{\frac{P}{Q}} \sigma_i^- \sigma_i^+ + \frac{1}{2} \sqrt{QP} \sigma_i^z + \frac{1}{2} \sqrt{QP} \sigma_i^z. \]  

We now do the similarity transformation [15]

\[ \sigma_i^+ \to (\sqrt{QP})^{i-1} \sigma_i^+, \quad \sigma_i^- \to (\sqrt{PQ})^{i-1} \sigma_i^-, \quad \sigma_i^z \to \sigma_i^z \]  

and recover Eq. (4) with \( \eta = 1 \) and \( q = \sqrt{QP} \), which means that the two-parameter deformation is a one-parameter deformation.

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