CLASSIFICATION OF SIMPLE MODULES OVER DEGENERATE DOUBLE AFFINE HECKE ALGEBRAS OF TYPE A

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Abstract. We study a class of representations over the degenerate double affine Hecke algebra of $\mathfrak{gl}_n$ by an algebraic method. As fundamental objects in this class, we introduce certain induced modules and study some of their properties. In particular, it is shown that these induced modules have unique simple quotient modules under certain conditions. Moreover, we show that any simple module in this class is obtained as such a simple quotient, and give a classification of all the simple modules.

Introduction

Double affine Hecke algebras and their degenerate (or graded) version are introduced by Cherednik [Ch1], and successfully applied to the theory of symmetric polynomials [Ch2, Ch3].

The purpose of this paper is to give an algebraic approach to the study of the representation theory of the degenerate double affine Hecke algebra of type $A$. In particular, we give a classification of simple modules of a certain class, which is studied in [BEG] for double affine Hecke algebras from the geometric viewpoint.

Let $\check{H}_n$ denote the degenerate affine Hecke algebra of $\mathfrak{gl}_n$. The algebra $\check{H}_n$ has a commutative subalgebra $S(\hat{h})$. Here $S(\hat{h})$ denotes the symmetric algebra of the vector space $\hat{h} = \bigoplus_{i=1}^n \mathbb{C}e_i' \oplus \mathbb{C}c$ with $c$ central in $\check{H}_n$. Note that a locally $S(\hat{h})$-finite $\check{H}_n$-module admits generalized weight space decomposition with respect to the action of $\hat{h}$. We study the category $\mathcal{O}(\check{H}_n)$ consisting of finitely generated, locally $S(\hat{h})$-finite $\check{H}_n$-modules whose (generalized) weights are integral.

Since $c$ is a center, it follows that the category is decomposed into a direct sum of subcategories $\mathcal{O}_\kappa(\check{H}_n)$ ($\kappa \in \mathbb{Z}$), where $\mathcal{O}_\kappa(\check{H}_n)$ denotes the full subcategory consisting of modules on which $c$ acts as a scalar multiple by $\kappa$.

We will give a classification of all simple modules in $\mathcal{O}_\kappa(\check{H}_n)$ with $\kappa \neq 0$. (We do not treat the case $\kappa = 0$, which is rather special.) Let us sketch our approach.
We introduce a certain set of parameters $S$. For each parameter $(\lambda, \mu) \in S$, we introduce an $\tilde{H}_n$-module $\tilde{M}(\lambda, \mu)$, which is induced from a certain one-dimensional module of a parabolic subalgebra of $\tilde{H}_n$, and investigate their properties (§6).

The first main result in this paper is Theorem 7.2, which gives a sufficient condition on $(\lambda, \mu)$ ensuring that $\tilde{M}(\lambda, \mu)$ has a unique simple quotient module, denoted by $\tilde{L}(\lambda, \mu)$. Define $S^+$ as the subset of $S$ consisting of all parameters satisfying the conditions in Theorem 7.2. Then, we can get a correspondence from $S^+$ to the set of isomorphism classes of simple objects in $\mathcal{O}_\kappa(\tilde{H}_n)$.

We prove that any simple module in $\mathcal{O}_\kappa(\tilde{H}_n)$ can be obtained as a simple quotient $\tilde{L}(\lambda, \mu)$ for some $(\lambda, \mu) \in S^+$ (Theorem 8.1), that is, the correspondence above is surjective. Furthermore, we write down when two parameters in $S^+$ give isomorphic simple modules (Theorem 8.2). This completes the classification of simple objects in $\mathcal{O}_\kappa(\tilde{H}_n)$. It turns out that the set of isomorphism classes of simple objects in $\mathcal{O}_\kappa(\tilde{H}_n)$ is indexed by isomorphism classes of $n$-dimensional nilpotent representations of the (cyclic) quiver of type $A^{(1)}_\kappa$.

We give detailed proofs for all the statements above by an algebraic and rather direct method with the help of some fundamental results on the representation theory of the (degenerate) affine Hecke algebra.

We treat the degenerate double affine Hecke algebra in this paper, but it is easy to modify the arguments to obtain the same results for the double affine Hecke algebra of $\mathfrak{gl}_n$ provided that a certain parameter (often denoted by $q$) is not a root of one. Note that the classification of simple modules over the affine Hecke algebra can be deduced to the same problem for the degenerate affine Hecke algebra and vice versa [Lu1], but the corresponding rigorous statement has not been established (as far as the author knows).

A similar class of representations over (non-degenerate) double affine Hecke algebras of general type is studied by a geometric method in the preprint [Va], where the classification of simple modules and certain Jordan-Hölder multiplicity formulas are obtained by means of the theory of perverse sheaves and equivariant K-theory. In particular, Vasserot’s result gives a geometric proof of our classification for the double affine Hecke algebra. It should be also mentioned that Cherednik announces the classification of simple modules over the double affine Hecke algebra of type $A$ by an alternative algebraic approach in the preprint [Ch4].
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1. Affine root system

Through this paper, we use the notation

\[ [i, j] = \{i, i + 1, \ldots, j\} \]

for \( i, j \in \mathbb{Z} \) with \( i < j \).

Fix \( n \in \mathbb{Z}_{>0} \). Let \( \tilde{h} \) be an \((n+2)\)-dimensional vector space over \( \mathbb{C} \) with the basis \( \{\epsilon_i^\vee, \epsilon_2^\vee, \ldots, \epsilon_n^\vee, c, d\} \): \( \tilde{h} = \bigoplus_{i=1}^{n} \mathbb{C}\epsilon_i^\vee \oplus \mathbb{C}c \oplus \mathbb{C}d \).

Introduce the non-degenerate symmetric bilinear form \((\ | \) \) on \( \tilde{h} \) by

\[
\begin{align*}
(\epsilon_i^\vee | \epsilon_j^\vee) &= \delta_{ij}, \quad (\epsilon_i^\vee | c) = (\epsilon_i^\vee | d) = 0, \\
(c | d) &= 1, \quad (c | c) = (d | d) = 0.
\end{align*}
\]

Put \( h = \bigoplus_{i=1}^{n} \mathbb{C}\epsilon_i^\vee \) and \( \dot{h} = h \oplus \mathbb{C}c \).

Let \( \tilde{h}^* = \bigoplus_{i=1}^{n} \epsilon_i \oplus \mathbb{C}c^* \oplus \mathbb{C}\delta \) be the dual space of \( \tilde{h} \), where \( \epsilon_i, c^* \) and \( \delta \) are the dual vectors of \( \epsilon_i^\vee, c \) and \( d \) respectively.

We identify the dual space \( \dot{h}^* \) (resp. \( h^* \)) of \( \dot{h} \) (resp. \( h \)) as a subspace of \( \tilde{h}^* \) via the identification \( \dot{h}^* = \tilde{h}^*/\mathbb{C}\delta \cong h^* \oplus \mathbb{C}c^* \) (resp. \( h^* = \tilde{h}^*/(\mathbb{C}c^* \oplus \mathbb{C}\delta) \cong \bigoplus_{i=1}^{n} \mathbb{C}\epsilon_i \)).

The natural pairing is denoted by \( (\ | ) : \tilde{h}^* \times \tilde{h} \to \mathbb{C} \). There exists an isomorphism between \( \tilde{h}^* \) and \( h \) such that \( \epsilon_i \mapsto \epsilon_i^\vee, \delta \mapsto c \) and \( c^* \mapsto d \). We denote by \( \zeta^\vee \in \dot{h} \) the image of \( \zeta \in \tilde{h}^* \) under this isomorphism.

Put \( \alpha_{ij} = \epsilon_i - \epsilon_j \) \((1 \leq i \neq j \leq n)\) and \( \alpha_i = \alpha_{ii+1} \) \((1 \leq i \leq n - 1)\). Then

\[
\begin{align*}
R &= \{\alpha_{ij} \mid 1 \leq i \neq j \leq n\}, \quad R^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq n\}, \\
\Pi &= \{\alpha_1, \ldots, \alpha_{n-1}\}
\end{align*}
\]

give the system of roots, positive roots and simple roots of type \( A_{n-1} \) respectively.

Put \( \alpha_0 = -\alpha_{1n} + \delta \), and define the system \( \dot{R} \) of roots, \( \dot{R}^+ \) of positive roots and \( \dot{\Pi} \) of simple roots of type \( A_{n-1}^{(1)} \) by

\[
\begin{align*}
\dot{R} &= \{\alpha + k\delta \mid \alpha \in R, \ k \in \mathbb{Z}\}, \\
\dot{R}^+ &= \{\alpha + k\delta \mid \alpha \in R^+, \ k \in \mathbb{Z}_{\geq 0}\} \cup \{-\alpha + k\delta \mid \alpha \in R^+, \ k \in \mathbb{Z}_{>0}\}, \\
\dot{\Pi} &= \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}.
\end{align*}
\]
2. Affine Weyl group

Let $Q$ denote the root lattice $\bigoplus_{i=1}^{n-1} \mathbb{Z} \alpha_i$ and let $P$ denote the weight lattice $\bigoplus_{i=1}^{n} \mathbb{Z} \epsilon_i$ of $\mathfrak{gl}_n$. Let $W_n$ denote the Weyl group of $\mathfrak{gl}_n$, which is isomorphic to the symmetric group $\mathfrak{S}_n$.

The extended affine Weyl group $\hat{W}_n$ (resp. the affine Weyl group $\check{W}_n$) of $\mathfrak{gl}_n$ is defined as the semidirect product of $W_n$ and $P$ (resp. $Q$) with the relation $w \cdot t_\eta \cdot w^{-1} = t_{w(\eta)}$, where $w$ and $t_\eta$ are the elements in $\hat{W}$ corresponding to $w \in W$ and $\eta \in P$ (resp. $Q$). In the following, we simply denote $\hat{W} = \hat{W}_n$, $\check{W} = \check{W}_n$ and $W = W_n$.

Let $s_\alpha \in W$ denote the reflection corresponding to $\alpha \in R$. For an affine root $\beta = \alpha + k\delta \in \check{R}$ ($\alpha \in R$, $k \in \mathbb{Z}$), define the corresponding affine reflection by $s_\beta = t_{-k\alpha} \cdot s_\alpha$.

Put $s_i = s_{\alpha_i}$ for $i \in \{0, n-1\}$ and put $\pi = t_{\epsilon_1} \cdot s_1 \cdots s_{n-1}$. The following fact is well-known.

**Proposition 2.1.** (i) The group $\hat{W}$ is isomorphic to the group defined by the following generators and relations:

- **generators**: $s_i$ ($i \in \{0, n-1\} \cong \mathbb{Z}/n\mathbb{Z}$), $\pi^{\pm 1}$.
- **relations**: $s_i^2 = 1$,
  $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ ($i \in \mathbb{Z}/n\mathbb{Z}$),
  $s_is_j = s_js_i$ ($i - j \not\equiv \pm 1 \mod n$),
  $\pi s_i = s_{i+1}\pi$ ($i \in \mathbb{Z}/n\mathbb{Z}$),
  $\pi^{\pi^{-1}} = \pi^{-1}\pi = 1$.

(ii) The subgroup $\check{W}^\circ$ is generated by the simple reflections $s_0, s_1, \ldots, s_{n-1}$.

The action of $\hat{W}$ on $\check{h}$ is given by the following formulas:

- $s_\alpha (h) = h - \langle \alpha|h \rangle \alpha^\vee$ ($\alpha \in \check{R}$, $h \in \check{h}$),
- $\pi (\epsilon_i^\vee) = \epsilon_{i+1}^\vee$ ($i \in \{1, n-1\}$), $\pi (\epsilon_n^\vee) = \epsilon_1^\vee - c$, (2.1)
- $\pi (c) = c$, $\pi (d) = d$.

It follows that the action of $t_\eta$ ($\eta \in P$) is given by

$$t_\eta (h) = h + \langle \delta|h \rangle \eta^\vee - \left( \langle \eta|h \rangle + \frac{1}{2} \langle \eta|\eta \rangle \delta|h \rangle \right) c.$$

The dual action on $\check{h}^*$ is given by

- $s_\alpha (\zeta) = \zeta - \langle \alpha|\zeta \rangle \alpha$ ($\alpha \in \check{R}$, $\zeta \in \check{h}^*$),
- $t_\eta (\zeta) = \zeta + \langle \delta|\zeta \rangle \eta - \left( \langle \eta|\zeta \rangle + \frac{1}{2} \langle \eta|\eta \rangle \delta|\zeta \rangle \right) \delta$ ($\eta \in P$, $\zeta \in \check{h}^*$),
- $\pi (\epsilon_i) = \epsilon_{i+1}$ ($i \in \{1, n-1\}$), $\pi (\epsilon_n) = \epsilon_1^\vee - \delta$,
- $\pi (c^*) = c^*$, $\pi (\delta) = \delta$. 


With respect to these actions, the inner products on \( \hat{\mathfrak{h}} \) and \( \hat{\mathfrak{h}}^* \) are \( \hat{W} \)-invariant.

Note that the subspace \( \hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C} c \) is preserved by \( \hat{W} \), and that the dual action of \( \hat{W} \) on \( \mathfrak{h}^* \) (called the affine action) is given by

\[
\begin{align*}
    s_\alpha (\zeta) &= \zeta - (\alpha | \zeta) \alpha, \\
    t_\eta (\zeta) &= \zeta + (\delta | \zeta) \eta
\end{align*}
\] (2.2)

for \( \zeta \in \mathfrak{h}^* \), \( \alpha \in \hat{R} \) and \( \eta \in P \).

For \( w \in \hat{W} \), set

\[
R(w) = \hat{R}^+ \cap w^{-1} \hat{R}^-,
\]

where \( \hat{R}^- = \hat{R} \setminus \hat{R}^+ \). The length \( l(w) \) of \( w \in \hat{W} \) is defined as the number \# \( R(w) \) of the elements in \( R(w) \). For \( w \in \hat{W} \), an expression \( w = \pi^k \cdot s_{j_1} \cdots s_{j_m} \) is called a reduced expression if \( m = l(w) \). It can be seen that

\[
R(w) = \{ s_{j_m} \cdots s_{j_2} (\alpha_{j_1}), s_{j_m} \cdots s_{j_2} (\alpha_{j_2}), \ldots, \alpha_{j_m} \} \tag{2.3}
\]

if \( w = \pi^k \cdot s_{j_1} \cdots s_{j_m} \) is a reduced expression.

The partial ordering \( \preceq \) is defined in the Coxeter group \( \hat{W}^\circ \) as follows: \( w \preceq w' \iff w \) can be obtained as a subexpression of a reduced expression of \( w' \). Extend this ordering \( \preceq \) to the partial ordering in \( \hat{W} \) by \( \pi^k w \preceq \pi^k w' \iff k = k' \) and \( w \preceq w' (k, k' \in \mathbb{Z}, w, w' \in \hat{W}^\circ) \).

Let \( I \) be a subset of \([0, n-1]\). Put

\[
\begin{align*}
    \hat{\Pi}_I &= \{ \alpha_i \mid i \in I \} \subseteq \hat{\Pi}, \\
    \hat{W}_I &= \langle s_i \mid i \in I \rangle \subseteq \hat{W}, \\
    \hat{R}_I &= \{ \alpha \in \hat{R} \mid s_\alpha \in \hat{W}_I \}.
\end{align*}
\]

Note that \( \hat{W}_I \) is the parabolic subgroup corresponding to \( \hat{\Pi}_I \). Define

\[
\hat{W}^I = \{ w \in \hat{W} \mid R(w) \subset \hat{R}^+ \setminus (\hat{R}^+ \cap \hat{R}_I) \}.
\]

The following fact is well-known.

**Proposition 2.2.** For any \( w \in \hat{W} \), there exist a unique \( w_1 \in \hat{W}^I \) and a unique \( u \in \hat{W}_I \), such that \( w = w_1 \cdot u \). Their length satisfy \( l(w) = l(w_1) + l(u) \). In particular, the set \( \hat{W}^I \) gives a complete set of representatives in the coset \( \hat{W}/\hat{W}_I \).

In the case \( I \subseteq [1, n-1] \), we can define \( \hat{W}_I (= \hat{W}_I) \) and \( W^I \subseteq W \) analogously, and similar statements as Proposition 2.2 hold for them.

Put \( P^- = \{ \xi \in P \mid (\xi | \alpha) \leq 0 \text{ for any } \alpha \in R^+ \} \).

**Lemma 2.3.** Let \( \eta \in P \). Let \( y \) be a shortest element of \( W \) such that \( y(\eta) \in P^- \). Then \( R(y) = \{ \alpha \in R^+ \mid (\eta | \alpha) > 0 \} \).
Proof. Let $\alpha \in R(y) = R^+ \cap y^{-1} R^-$. Then $(\eta | \alpha) = (y(\eta) | y(\alpha)) \geq 0$. If $(\eta | \alpha) = 0$ then $s_\alpha(\eta) = \eta$. Hence we have $ys_\alpha(\eta) \in P^-$ and $l(ys_\alpha) < l(y)$. This contradicts to the choice of $y$. Therefore we have $(\eta | \alpha) > 0$ and hence $R(y) \subseteq \{ \alpha \in R^+ | (\eta | \alpha) > 0 \}$. It is easier to show the opposite inclusion. \hfill $\Box$

We denote by $y_\eta$ the (unique) shortest element of $W$ such that $y_\eta(\eta) \in P^-$. The following proposition follows from Proposition 2.2 and Lemma 2.3. We omit the detailed proof.

**Lemma 2.4.** ([AST]) (i) We have $\dot{W}^{[1, n-1]} = \{ t_\eta \cdot y_\eta^{-1} | \eta \in P \}$.

(ii) For a subset $I \subset [1, n-1]$, we have $\dot{W}^I = \dot{W}^{[1, n-1]} \cdot W^I$.

Moreover, $l(w) = l(t_\eta \cdot y_\eta^{-1}) + l(u)$ for $w = t_\eta \cdot y_\eta^{-1} \cdot u$ ($\eta \in P$, $u \in W^I$).

3. **Degenerate double affine Hecke algebra**

Let $\mathbb{C}[\dot{W}]$ denote the group algebra of $\dot{W}$ and let $S(\dot{h})$ denote the symmetric algebra of $\dot{h} = h \oplus \mathbb{C}c$.

The degenerate double affine Hecke algebra was introduced by Cherednik [Ch1].

**Definition 3.1.** The degenerate double affine Hecke algebra $\check{H}_n$ of $\mathfrak{gl}_n$ is the unital associative $\mathbb{C}$-algebra defined by the following properties:

(i) As a $\mathbb{C}$-vector space,

$$\check{H}_n = \mathbb{C}[\dot{W}] \otimes S(\dot{h}).$$

(ii) The natural inclusions $\mathbb{C}[\dot{W}] \hookrightarrow \check{H}_n$ and $S(\dot{h}) \hookrightarrow \check{H}_n$ are algebra homomorphisms (the images of $w \in \dot{W}$ and $h \in \dot{h}$ will be simply denoted by $w$ and $h$).

(iii) The following relations hold in $\check{H}_n$:

$$s_\alpha h - s_\alpha(h)s_\alpha = -\langle \alpha | h \rangle \quad (\alpha \in \hat{R}, h \in \dot{h}), \quad (3.1)$$

$$\pi h = \pi(h)\pi \quad (h \in \dot{h}). \quad (3.2)$$

By definition, the element $c \in \check{H}_n$ belongs to the center $Z(\check{H}_n)$ of $\check{H}_n$. For $\kappa \in \mathbb{C}^*$, we set $\check{H}_n(\kappa) = \check{H}_n / \langle c - \kappa \rangle$.

**Definition 3.2.** Define the degenerate affine Hecke algebra $\check{H}_n$ as the subalgebra of $\check{H}_n$ generated by the elements in $W$ and the elements in $\dot{h}$:

$$\check{H}_n = \mathbb{C}[W] \otimes S(\dot{h}) \subset \check{H}_n.$$
Proposition 3.3. For $w \in \hat{W}$ and $h \in \hat{\mathfrak{h}}$, we have

$$hw = w \left( w^{-1}(h) + \sum_{\alpha \in R(w)} \langle w(\alpha) | h \rangle s_\alpha \right).$$

In particular, $hw = w \cdot w^{-1}(h) + \sum_{w' \prec w} c_{w'w}w'$ for some $c_{w'} \in \mathbb{C}$.

Proof. The statement is shown by the induction on $l(w)$ using the fact that $R(s_iw) = R(w) \cup \{ w^{-1}(\alpha) \}$ if $l(s_iw) = l(w) + 1$. □

It is easy to verify the following proposition directly (see e.g. [AST, Lul1]).

Proposition 3.4. (i) The center of $\hat{H}_n$ is given by $Z(\hat{H}_n) = \mathbb{C}[c]$.

(ii) The center of $\hat{H}_n$ is given by

$$Z(\hat{H}_n) = \{ \xi \in S(\mathfrak{h}) \mid w(\xi) = \xi \text{ for all } w \in W \}.$$ 

For $i \in \mathbb{Z}$, we introduce the following notations:

$$\epsilon_i = \epsilon_i^+ - k\delta \in \hat{\mathfrak{h}}^*, \quad \epsilon_i^\vee = \epsilon_i^- - kc \in \hat{\mathfrak{h}},$$

where $i = i + kn$ with $i \in \{1, n\}$ and $k \in \mathbb{Z}$.

Put $\alpha_{ij} = \epsilon_i - \epsilon_j$ (and $\alpha_{ij}^\vee = \epsilon_i^\vee - \epsilon_j^\vee$) for any $i, j \in \mathbb{Z}$. Note that

$$\alpha_{ij} \in \hat{R} \quad \iff \quad i \not\equiv j \pmod{n},$$

$$\alpha_{ij} \in \hat{R}^+ \quad \iff \quad i \not\equiv j \pmod{n} \text{ and } i < j.$$

Let $J = \{ j_1, j_2, \ldots, j_m \}$ be a subset of $\mathbb{Z}$ such that $\alpha_{j_aj_b} = \epsilon_{j_a} - \epsilon_{j_b} \in \hat{R}$ for $a \neq b$. Then, in particular, we have $m \leq n$.

Define $\hat{H}_J$ to be the subalgebra of $\hat{H}_n$ generated by

$$\epsilon_{j_1}^\vee, \epsilon_{j_2}^\vee, \ldots, \epsilon_{j_m}^\vee, \ s_{j_{i+1}j_i}, \ s_{j_{i+2}j_i}, \ldots, \ s_{j_{m-1}j_m}. $$

The following lemma will be used later.

Lemma 3.5. Let $J = \{ j_1, j_2, \ldots, j_m \}$ be a subset of $\mathbb{Z}$ such that $j_a \not\equiv j_b \pmod{n}$ for $a \neq b$. Suppose that $j_1 < j_2 < \cdots < j_m$. Then, the algebra $\hat{H}_J$ is isomorphic to the degenerate affine Hecke algebra $\hat{H}_m$ of $\mathfrak{gl}_m$.

Proof. Note that $\alpha_{j_aj_{a+1}} \in \hat{R}^+$ for all $a \in \{1, m - 1\}$. Using Proposition 3.3, it is verified that there exists an algebra homomorphism $\hat{H}_m \to \hat{H}_J$ such that $\epsilon_{j_i}^\vee \mapsto \epsilon_{j_i}^\vee$ (for $i \in \{1, m\}$) and $s_i \mapsto s_{\alpha_{j_{i+1}j_i}}$ (for $i \in \{1, m - 1\}$).

This gives an isomorphism.

Example 3.6. We have $\hat{H}_{[1,n]} = \hat{H}_n$ by definition. More generally, we have $\hat{H}_{[j,j']} \cong \hat{H}_{j'-j+1}$ for any $j, j' \in \mathbb{Z}$ such that $j' - j \in [1, n - 1]$. 

4. SET OF PARAMETERS AND AFFINE WEYL GROUP

We introduce some sets of parameters, which will index representations of $\hat{H}_n$.

Fix $p \in \mathbb{Z}_{>0}$. Put $C_p(n) = \{(r_1, \ldots, r_p) \in (\mathbb{Z}_{\geq 0})^p \mid \sum_{i=1}^p r_i = n\}$, and set

$$I_p = \{(\lambda, \mu) \in I_p \mid \lambda_i - \mu_i > 0 \text{ for all } i \in [1, p]\}. \quad (4.2)$$

We denote the extended affine Weyl group of $\mathfrak{gl}_p$ by $\hat{S}_p$ instead of $\hat{W}_p$ in order to avoid confusion. The elements of $\hat{S}_p$ corresponding to $s_i, \pi$ and $t_{\epsilon_i}$ are denoted by $\sigma_i, \varpi_p$ and $t_{\epsilon_i}$ respectively. ($\{\epsilon_i\}_{i=1}^p$ is the generators of the weight lattice $P_p$ of $\mathfrak{gl}_p$: $P_p = \oplus_i^p \mathbb{Z} \epsilon_i$.)

We put $\hat{S}_1 = \langle \varpi_1 \rangle$ for convenience. The subgroups corresponding to $\hat{W}_p$ and $W_p$ are denoted by $\hat{S}_p$ and $S_p$ respectively:

$$\hat{S}_p = \langle \sigma_0, \sigma_1, \ldots, \sigma_{p-1} \rangle, \quad S_p = \langle \sigma_1, \ldots, \sigma_{p-1} \rangle.$$  

For $\kappa \in \mathbb{C}$, there exists an action of $\hat{S}_p$ on the set $\mathbb{C}^p$ which is given by

$$\sigma_i \circ \lambda = (\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_p) \quad (i \in [1, p-1]),$$

$$\sigma_0 \circ \lambda = (\lambda_p + \kappa - p + 1, \lambda_2, \ldots, \lambda_{p-1}, \lambda_1 - \kappa + p - 1),$$

$$\varpi_p \circ \lambda = (\lambda_p + \kappa - p + 1, \lambda_1 + 1, \ldots, \lambda_{p-1} + 1)$$

for $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p) \in \mathbb{C}^p$. It follows that the action of $t_{\epsilon_i}$ is given by

$$t_{\epsilon_i} \circ \lambda = (\lambda_1, \ldots, \lambda_i + \kappa, \ldots, \lambda_p) \quad (i \in [1, p]).$$

If $\kappa \in \mathbb{Z}$, then this action preserves $\mathbb{Z}^p$ and induces an action of $\hat{S}_p$ on $I_p$ and $I^*_p$ via

$$w \circ (\lambda, \mu) = (w \circ \lambda, w \circ \mu). \quad (4.3)$$

In the following, we always assume $\kappa \in \mathbb{Z}$.

For $\lambda \in \mathbb{C}^p$, put

$$[\lambda]_0 = \kappa - p + 1 - \lambda_1 + \lambda_p,$$

$$[\lambda]_i = \lambda_i - \lambda_{i+1} + 1 \quad (i \in [1, p-1]).$$

Remark 4.1. It is natural to describe the action $\circ$ and the numbers $[\lambda]_i$ in terms of the root system of type $A_{p-1}^{(1)}$: Put $\mathfrak{h}_p^* = \bigoplus_{i=1}^p \mathbb{C} e_i \oplus \mathbb{C} c^*$, where notations are analogous to the $A_{n-1}^{(1)}$ case, and regard $\lambda \in \mathbb{C}^p$ as
an element of \( \hat{h}^* \) by \( \lambda = \sum \lambda_i e_i + (\kappa - p)e^* \). Then we have
\[
w \circ \lambda = w(\lambda + \rho) - \rho \quad (w \in \hat{S}_p),
\]
\[
[\lambda]_i = \langle \lambda + \rho \mid \alpha_i^\vee \rangle \quad (i \in [0, p-1]),
\]
where \( \rho = \sum_{i=1}^p (-i + 1)e_i + pc^* \).

Set
\[
\mathcal{D}_p = \{ \lambda \in \mathbb{Z}^p \mid [\lambda]_i \geq 0 \text{ for all } i \in [1, p-1] \},
\]
\[
\hat{\mathcal{D}}_{p,\kappa} = \{ \lambda \in \mathbb{Z}^p \mid [\lambda]_i \geq 0 \text{ for all } i \in [0, p-1] \}.
\]

The following fact is well-known:

**Lemma 4.2.** Let \( \kappa \in \mathbb{Z}_{>0} \).

(i) \( \mathcal{D}_p \) is a fundamental domain for the action of \( S_p \) on \( \mathbb{Z}^p \).

(ii) \( \hat{\mathcal{D}}_{p,\kappa} \) is a fundamental domain for the action of \( \hat{S}_p \) on \( \mathbb{Z}^p \).

The proof of the following lemma is similar to the proof of Lemma 2.3.

**Lemma 4.3.** Let \( \kappa \in \mathbb{Z}_{>0} \) and \( \lambda \in \mathbb{Z}^p \). Let \( w \) be the shortest element in \( \hat{S}_p^\circ \) such that \( w \circ \lambda \in \hat{\mathcal{D}}_{p,\kappa} \). Then, we have \( [\sigma_{i_{k+1}} \sigma_{i_{k+2}} \cdots \sigma_{i_l} \circ \lambda]_l < 0 \) for each \( k \in [1, l] \), where \( w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_l} \) is a reduced expression of \( w \).

For \( \mu \in \mathbb{Z}^p \), set
\[
\mathcal{D}^\mu_p = \{ \lambda \in \mathbb{Z}^p \mid [\lambda]_i \geq 0 \text{ for any } i \in [1, p-1] \text{ such that } [\mu]_i = 0 \},
\]
\[
\hat{\mathcal{D}}^\mu_{p,\kappa} = \{ \lambda \in \mathbb{Z}^p \mid [\lambda]_i \geq 0 \text{ for any } i \in [0, p-1] \text{ such that } [\mu]_i = 0 \}.
\]

Put
\[
\mathcal{I}_p^+ = \{ (\lambda, \mu) \in \mathcal{I}_p \mid \mu \in \mathcal{D}_p, \lambda \in \mathcal{D}^\mu_p \},
\]
\[
\hat{\mathcal{I}}_{p,\kappa}^+ = \{ (\lambda, \mu) \in \hat{\mathcal{I}}_p \mid \mu \in \hat{\mathcal{D}}_{p,\kappa}, \lambda \in \hat{\mathcal{D}}^\mu_{p,\kappa} \},
\]
\[
\mathcal{I}_p^+ = \mathcal{I}_p^* \cap \mathcal{I}_p^+, \quad \hat{\mathcal{I}}_{p,\kappa}^+ = \mathcal{I}_p^* \cap \hat{\mathcal{I}}_{p,\kappa}^+.
\]

It is easy to show the following lemma:

**Proposition 4.4.** Let \( \kappa \in \mathbb{Z}_{>0} \).

(i) \( \mathcal{I}_p^+ \) (resp. \( \mathcal{I}_p^{++} \)) is a fundamental domain for the action (4.3) of \( \mathcal{S}_p \) on \( \mathcal{I}_p \) (resp. \( \mathcal{I}_p^* \)).

(ii) \( \hat{\mathcal{I}}_{p,\kappa}^+ \) (resp. \( \hat{\mathcal{I}}_{p,\kappa}^{++} \)) is a fundamental domain for the action (4.3) of \( \hat{\mathcal{S}}^\circ_p \) on \( \hat{\mathcal{I}}_p \) (resp. \( \hat{\mathcal{I}}_p^* \)).

(iii) \( \varpi_p \) preserves the sets \( \mathcal{I}_{p,\kappa}^+ \) and \( \mathcal{I}_{p,\kappa}^{++} \) respectively.
5. REPRESENTATIONS OF DEGENERATE AFFINE HECKE ALGEBRAS

We review some facts on the representation theory of the degenerate affine Hecke algebra \( \hat{H}_n \) for later use.

For an \( \hat{H}_n \)-module \( N \) and \( \zeta \in \mathfrak{h}^* \), define the weight space \( N_\zeta \) and the generalized weight space \( N^\text{gen}_\zeta \) of weight \( \zeta \) by

\[
N_\zeta = \{ v \in N \mid hv = \langle \zeta|h \rangle v \text{ for any } h \in \mathfrak{h} \},
\]

\[
N^\text{gen}_\zeta = \{ v \in N \mid (h - \langle \zeta|h \rangle)^k v = 0 \text{ for any } h \in \mathfrak{h}, \text{ for some } k \in \mathbb{Z}_{>0} \}.
\]

Denote by \( P(N) \) the set of all weights of \( N \):

\[
P(N) = \{ \zeta \in \mathfrak{h}^* \mid N_\zeta \neq \{0\} \} = \{ \zeta \in \mathfrak{h}^* \mid N^\text{gen}_\zeta \neq \{0\} \}.
\]

Define \( \mathcal{O}(\hat{H}_n) \) to be the category consisting of all finite-dimensional \( \hat{H}_n \)-modules \( N \) such that \( N = \bigoplus_{\zeta \in P} N^\text{gen}_\zeta \), i.e. \( P(N) \subseteq P \).

Let \((\lambda, \mu) \in \mathcal{I}_p\). Put

\[
n_0 = 0, \quad n_i = \sum_{j=1}^i (\lambda_j - \mu_j) \quad (i = [1, p]). \quad (5.1)
\]

Set \( I_{\lambda, \mu} = [1, n-1] \setminus \{n_1, n_2, \ldots, n_{p-1}\} \). Then \( W_{I_{\lambda, \mu}} = W_{\lambda_1 - \mu_1} \times W_{\lambda_2 - \mu_2} \times \cdots \times W_{\lambda_p - \mu_p} \). We denote \( W_{\lambda - \mu} = W_{I_{\lambda, \mu}} \) and \( W^\lambda = W_{I^\lambda} \).

Define \( \hat{H}_{\lambda - \mu} \) as the subalgebra of \( \hat{H}_n \) generated by the elements in \( W_{\lambda - \mu} \) and the elements in \( S(\mathfrak{h}) \):

\[
\hat{H}_{\lambda - \mu} = \mathbb{C}[W_{\lambda - \mu}] \otimes S(\mathfrak{h}) = \hat{H}_{[n_0 + 1, n_1]} \hat{H}_{[n_1 + 1, n_2]} \cdots \hat{H}_{[n_{p-1} + 1, n_p]} \subseteq \hat{H}_n,
\]

where \( \hat{H}_{[i, j]} \) is as in Lemma 3.5.

Define \( \zeta_{\lambda, \mu} \) to be the element of \( \mathfrak{h}^* \) such that

\[
\langle \zeta_{\lambda, \mu} | e_j^\vee \rangle = \mu_i - i + j - n_{i-1} \quad \text{for} \ j \in [n_{i-1} + 1, n_i]. \quad (5.2)
\]

Note, in particular, that we have

\[
\langle \zeta_{\lambda, \mu} | e_{n_{i-1}+1}^\vee \rangle = \mu_i - i + 1,
\]

\[
\langle \zeta_{\lambda, \mu} | e_{n_i}^\vee \rangle = \lambda_i - i \quad (5.3)
\]

if \( n_{i-1} < n_i \). There exists a one-dimensional representation \( \mathbb{C}1_{\lambda, \mu} \) of \( \hat{H}_{\lambda - \mu} \) such that

\[
w1_{\lambda, \mu} = 1_{\lambda, \mu} \quad \text{for all} \ w \in W_{\lambda - \mu},
\]

\[
h1_{\lambda, \mu} = \langle \zeta_{\lambda, \mu} | h \rangle 1_{\lambda, \mu} \quad \text{for all} \ h \in \mathfrak{h}. \quad (5.4)
\]

Define the induced representation \( \hat{M}(\lambda, \mu) \) of \( \hat{H}_n \) associated with \((\lambda, \mu)\) by

\[
\hat{M}(\lambda, \mu) = \hat{H}_n \otimes \hat{H}_{\lambda - \mu} \mathbb{C}1_{\lambda, \mu}.
\]
Clearly, $\hat{M}(\lambda, \mu) \cong \mathbb{C}[W/W_{\lambda-\mu}]$ as a $W$-module.

The induced module $\hat{M}(\lambda, \mu)$ is not irreducible in general. We will use the following criterion for the irreducibility in the case $p = 2$. See e.g. [Ze1] or [Su1, Su2] for the proof.

**Lemma 5.1.** Let $(\lambda, \mu) \in I_2$ with $\lambda = (\lambda_1, \lambda_2)$, $\mu = (\mu_1, \mu_2)$.

(i) $\hat{M}(\lambda, \mu)$ is reducible if and only if one of the following conditions hold:

(a) $\mu_2 \leq \mu_1 \leq \lambda_2 + 1$ and $\lambda_2 \leq \lambda_1$.

(b) $\mu_1 \leq \mu_2 \leq \lambda_1 + 1$ and $\lambda_1 \leq \lambda_2$.

In each case, there exist the following exact sequences:

(a) $0 \to \hat{L}(\sigma \circ \lambda, \mu) \to \hat{M}(\lambda, \mu) \to \hat{L}(\lambda, \mu) \to 0$.

(b) $0 \to \hat{L}(\sigma \circ \lambda, \sigma \circ \mu) \to \hat{M}(\lambda, \mu) \to \hat{L}(\lambda, \sigma \circ \mu) \to 0$.

Here $\sigma = \sigma_1 \in S_2$.

(ii) If $\hat{M}(\lambda, \mu)$ is irreducible, then $\hat{M}(\lambda, \mu) \cong \hat{M}(\sigma \circ \lambda, \sigma \circ \mu)$.

The following lemma follows from Proposition 3.3.

**Lemma 5.2.** We have

$$P(\hat{M}(\lambda, \mu)) = W^{\lambda-\mu}\zeta_{\lambda, \mu} := \{ w(\zeta_{\lambda, \mu}) \mid w \in W^{\lambda-\mu} \},$$

$$\dim \hat{M}(\lambda, \mu)^{\text{gen}} := \sharp \{ w \in W^{\lambda-\mu} \mid w(\zeta_{\lambda, \mu}) = \xi \} \text{ for } \xi \in \mathfrak{h}^*. $$

In particular, we have $\dim \hat{M}(\lambda, \mu)^{\text{gen}} = \sharp \left( W^{\lambda-\mu} \cap W[\zeta_{\lambda, \mu}] \right)$, where $W[\xi] = \{ w \in W \mid w(\xi) = \xi \}$ for $\xi \in \mathfrak{h}^*$.

Let $(\lambda, \mu) \in I_p$. Take integers $a_1 < a_2 < \cdots < a_k$ such that

$$\{a_1, a_2, \ldots, a_k\} = \{ a \in [1, p] \mid |\lambda|_a \neq 0 \text{ or } |\mu|_a \neq 0 \}.$$ 

Put $X_{\lambda, \mu} = [1, n] \setminus \{ n_{a_1}, n_{a_2}, \ldots, n_{a_k} \}$, where $n_i$ is as in (5.1).

**Lemma 5.3.** Let $(\lambda, \mu) \in I^+_p$. Then $W^{\lambda-\mu} \cap W[\zeta_{\lambda, \mu}] \subseteq W_{X_{\lambda, \mu}}$.

**Proof.** Take integers $b_1 < b_2 < \cdots < b_t$ such that

$$\{b_1, b_2, \ldots, b_t\} = \{ a \in [1, p] \mid |\mu|_a \neq 0 \},$$

and put $Y_{\mu} = [1, n] \setminus \{ n_{b_1}, n_{b_2}, \ldots, n_{b_t} \}$. Then $W_{\lambda-\mu} \subseteq W_{X_{\lambda, \mu}} \subseteq W_{Y_{\mu}}$.

Let $w \in W^{\lambda-\mu} \cap W[\zeta_{\lambda, \mu}]$. First, we prove $w \in W_{Y_{\mu}}$.

It is enough to show $w([1, n_{b_i}]) = [1, n_{b_i}]$ for all $i \in [1, l]$. Suppose $w([1, n_{b_i}]) \neq [1, n_{b_i}]$. Let $m$ be the smallest number such that $m \in [1, n_{b_i}]$ and $w^{-1}(m) \notin [1, n_{b_i}]$. Then it follows from $w \in W^{\lambda-\mu}$ that $w^{-1}(m) = n_{b_{i-1}} + 1$ for some $b > b_i$. Since $w \in W[\zeta_{\lambda, \mu}]$, we have

$$\mu_b - b + 1 = \langle \zeta_{\lambda, \mu} \mid w^{-1}(\epsilon_{m}^\vee) \rangle = \langle \zeta_{\lambda, \mu} \mid \epsilon_{m}^\vee \rangle \geq \mu_{b_i} - b_i + 1.$$
This implies \( \mu|_{b_i} = 0 \) as \( (\lambda, \mu) \in \mathcal{I}_p^+ \), and this is a contradiction. Therefore we have \( w \in W_{Y'_{\mu}} \).

Next, let us see \( w \in W_{X_{\lambda,\mu}} \). Suppose that \( w([1, n_a]) \neq [1, n_a] \) for some \( i \in [1, k] \). Let \( m' \) be the largest number such that \( m' \in [1, n_a] \) and \( w(m') \notin [1, n_a] \). Then we have \( m' = n_a \) for some \( a > a_i \). By similar arguments as above, we have \( [\lambda]_{a_i} = 0 \), and this is a contradiction. Hence we have \( w \in W_{X_{\lambda,\mu}} \). □

In [Su2], Lemma 5.3 is used to reduce the proof of the following proposition to the special case

\[
\lambda = (m, m + 1, \ldots, m + p - 1), \quad \mu = (0, 1, \ldots, p - 1)
\]

with \( mp = n \):

**Proposition 5.4.** (Lemma 5.2 in [Su2]) Let \( (\lambda, \mu) \in \mathcal{I}_p^+ \). Then we have \( \hat{M}(\lambda, \mu)_{\lambda,\mu} = \mathbb{C}1_{\lambda,\mu} \).

As a direct consequence of Proposition 5.4, we have the following.

**Theorem 5.5.** ([Ro, Su2]) Let \( (\lambda, \mu) \in \mathcal{I}_p^+ \). Then \( \hat{M}(\lambda, \mu) \) has a unique simple quotient, which we denote by \( \hat{L}(\lambda, \mu) \).

Let \( \operatorname{Irr}(O(\hat{H}_n)) \) denote the set of isomorphism classes of irreducible modules in \( O(\hat{H}_n) \). Then, by Theorem 5.5, we have correspondences \( \mathcal{I}_p^{*+} \rightarrow \operatorname{Irr}(O(\hat{H}_n)) \) \((p \in [1, n])\) given by \( (\lambda, \mu) \mapsto \hat{L}(\lambda, \mu) \).

The classification of simple modules described below is originally obtained by Zelevinsky [Ze1] (see also [Ro]) for the affine Hecke algebra. An alternative algebraic proof of the classification using Theorem 5.5 is given in [Su2].

**Theorem 5.6.** (See [Su2]-§6.) The correspondence

\[
\bigcup_{p=1}^{n} \mathcal{I}_p^{*+} \rightarrow \operatorname{Irr}(O(\hat{H}_n))
\]

which maps \( (\lambda, \mu) \in \mathcal{I}_p^{*+} \) \((p \in [1, n])\) to \( \hat{L}(\lambda, \mu) \) is a bijection.

Recall that \( \mathfrak{S}_p \) acts on \( \mathcal{I}_p^* \) and that \( \mathcal{I}_p^{*+} \) is a fundamental domain for this action (Proposition 4.4): \( \mathcal{I}_p^{*+} \cong \mathcal{I}_p^*/\mathfrak{S}_p \). Hence Theorem 5.6 asserts that there exists a one to one correspondence

\[
\bigcup_{p=1}^{n} \mathcal{I}_p^*/\mathfrak{S}_p \leftrightarrow \operatorname{Irr}(O(\hat{H}_n)).
\]
6. Induced representations of $\tilde{H}_n$

Let $\kappa \in \mathbb{Z}$. We consider representations of $\tilde{H}_n(\kappa)$, namely, representations of $\tilde{H}_n$ on which $c \in \mathbb{Z}(\tilde{H}_n)$ acts as a constant integer $\kappa$.

For an $\tilde{H}_n$-module $N$ and a weight $\zeta \in \hat{h}^*$, we use the same notations as those for $\dot{H}_n$-modules to denote the weight space, the generalized weight space and the set of all weights of $N$:

$$N_\zeta = \{ v \in N \mid hv = \langle \zeta | h \rangle v \text{ for any } h \in \hat{h} \},$$
$$N_\zeta^{\text{gen}} = \{ v \in N \mid (h - \langle \zeta | h \rangle)^k v = 0 \text{ for any } h \in \hat{h}, \text{ for some } k \in \mathbb{Z}_{>0} \},$$
$$P(N) = \{ \zeta \in \hat{h}^* \mid N_\zeta \neq \{0\} \}.$$

If $N$ is an $\tilde{H}_n(\kappa)$-module, then any weight of $N$ is of the form $\zeta + \kappa c^*$ for some $\zeta \in \hat{h}^*$. Put $P_\kappa = P + \kappa c^* \subseteq \hat{h}^*$. Note that $P_\kappa$ is preserved under the action (2.2) of $\tilde{W}$.

**Definition 6.1.** Define $O_\kappa(\tilde{H}_n)$ to be the full subcategory of the category of finitely generated $\tilde{H}_n(\kappa)$-modules consisting of those $\tilde{H}_n(\kappa)$-modules $N$ such that

(i) $N$ is locally $S(\hat{h})$-finite,

(ii) $P(N) \subseteq P_\kappa$.

(Note the the condition (i) ensures the generalized weight space decomposition $N = \bigoplus_{\zeta \in \hat{h}^*} N_\zeta^{\text{gen}}$.)

**Remark 6.2.** A similar category for the (non-degenerate) double affine Hecke algebra of general type is studied from a geometric viewpoint in [BEG], where Deligne-Langlands-Lusztig type conjecture concerning the classification of simple modules is proposed.

**Remark 6.3.** There exists an algebra automorphism on $\tilde{H}_n$ such that

$$s_i \mapsto -s_i, \ (i \in [0, n-1]), \ \pi \mapsto \pi, \ \epsilon_i^\vee \mapsto -\epsilon_i^\vee \ (i \in [1, n]), \ c \mapsto -c.$$

This gives a categorical equivalence $O_\kappa(\tilde{H}_n) \cong O_{-\kappa}(\tilde{H}_n)$.

In the rest of this paper, we mostly consider the case $\kappa \in \mathbb{Z}_{>0}$.

For $(\lambda, \mu) \in I_p$, put $\zeta_{\lambda,\mu}^\kappa = \zeta_{\lambda,\mu} + \kappa c^* \in P_\kappa$, where $\zeta_{\lambda,\mu}$ is given in (5.2). We regard the one-dimensional $\tilde{H}_{\lambda-\mu}$-module $\mathbb{C}1_{\lambda,\mu}$ (defined by (5.4)) as an $(\tilde{H}_{\lambda-\mu} \otimes \mathbb{C}[c])$-module by letting $c$ act as a constant integer $\kappa$; we have

$$w1_{\lambda,\mu} = 1_{\lambda,\mu} \quad \text{for all } w \in W_{\lambda-\mu},$$
$$h1_{\lambda,\mu} = \langle \zeta_{\lambda,\mu}^\kappa | h \rangle 1_{\lambda,\mu} \quad \text{for all } h \in \hat{h}.$$
Define an $\tilde{H}_n$-module $\tilde{M}(\lambda, \mu)$ by
\[\tilde{M}(\lambda, \mu) = \tilde{H}_n \otimes_{\tilde{H}_n \otimes \mathbb{C}[c]} \mathbb{C}1_{\lambda, \mu}.\]

Clearly, $\tilde{M}(\lambda, \mu) \cong \tilde{H}_n(\kappa) \otimes_{\tilde{H}_n} M(\lambda, \mu)$ as an $\tilde{H}_n(\kappa)$-module,
\[\cong \mathbb{C}[P] \otimes M(\lambda, \mu)\] as an $\tilde{H}_n$-module,
\[\cong \mathbb{C}[\tilde{W}/W_{\lambda-\mu}]\] as a $\tilde{W}$-module.

By Proposition 3.3, we have the following:

**Proposition 6.4.** We have
(i) $P(\tilde{M}(\lambda, \mu)) = \tilde{W}^{\lambda-\mu}\zeta_{\lambda, \mu}^\kappa := \left\{ w(\zeta_{\lambda, \mu}^\kappa) \mid w \in \tilde{W}^{\lambda-\mu} \right\}$.
(ii) $\dim \tilde{M}(\lambda, \mu)^{\text{gen}} = \sharp \left\{ w \in \tilde{W}^{\lambda-\mu} \mid w(\zeta_{\lambda, \mu}^\kappa) = \xi \right\}$ for all $\xi \in \mathfrak{h}^*$, and it is finite if $\kappa \neq 0$. In particular $\dim \tilde{M}(\lambda, \mu)^{\text{gen}} = \sharp \left( \tilde{W}^{\lambda-\mu} \cap \tilde{W}[\zeta_{\lambda, \mu}^\kappa] \right)$, where $\tilde{W}[\xi] = \{ w \in \tilde{W} \mid w(\xi) = \xi \}$ for $\xi \in \mathfrak{h}^*$.

From Proposition 6.4, it follows that $\tilde{M}(\lambda, \mu)$ is an object of $\mathcal{O}_\kappa(\tilde{H}_n)$ if $\kappa \in \mathbb{Z}$.

**Proposition 6.5.** Let $(\lambda, \mu) \in \mathcal{I}_p$. Then $\tilde{M}(\lambda, \mu) \cong \tilde{M}(\varpi_p \circ (\lambda, \mu))$.

**Proof.** Put $\lambda' = \varpi_p \circ \lambda = (\lambda_p + \kappa - p + 1, \lambda_1 + 1, \ldots, \lambda_{p-1} + 1)$ and $\mu' = \varpi_p \circ \mu = (\mu_p + \kappa - p + 1, \mu_1 + 1, \ldots, \mu_{p-1} + 1)$. Put $m = \lambda_p - \mu_p$.

If $m = 0$, then $\tilde{M}(\lambda', \mu') \cong \tilde{M}(\lambda, \mu)$ because $\zeta_{\lambda', \mu'}^\kappa = \zeta_{\lambda, \mu}^\kappa$ and $W_{\lambda'-\mu'} = W_{\lambda-\mu}$.

Suppose $m \neq 0$. Set $v = \pi^m 1_{\lambda, \mu} \in \tilde{M}(\lambda, \mu)$ (note that $\pi \neq \varpi_p$).

It can be checked that the weight of $v$ is $\pi^m (\zeta_{\lambda, \mu}^\kappa) = \zeta_{\lambda', \mu'}^\kappa$, and that $wv = v$ for $w \in W_{\lambda'-\mu'}$. Hence, there exists a unique $\tilde{H}_n$-homomorphism $\psi : \tilde{M}(\lambda', \mu') \rightarrow \tilde{M}(\lambda, \mu)$ such that $\psi(1_{\lambda', \mu'}) = v = \pi^m 1_{\lambda, \mu}$. Similarly, there exists a unique homomorphism $\psi' : \tilde{M}(\lambda, \mu) \rightarrow \tilde{M}(\lambda', \mu')$ such that $\psi'(1_{\lambda, \mu}) = \pi^{-m} 1_{\lambda', \mu'}$. Now, it is easy to see that $\psi$ is an isomorphism with the inverse $\psi'$.

**Proposition 6.6.** Let $(\lambda, \mu) \in \mathcal{I}_p$ and $w \in \tilde{S}_p^\circ$. Suppose $w \circ \mu = \mu$. Then $\tilde{M}(\lambda, \mu) \cong \tilde{M}(w \circ \lambda, \mu)$.

**Proof.** It is enough to prove the statement when $w$ is a simple reflection.

Let us prove the statement in the case $w = \sigma_1$ first. Define $(\overline{\lambda}, \overline{\mu}) \in \mathcal{I}_2$ and $(\underline{\lambda}, \underline{\mu}) \in \mathcal{I}_{p-2}$ by
\[\overline{\lambda} = (\lambda_1, \lambda_2), \quad \overline{\mu} = (\mu_1, \mu_2), \quad \underline{\lambda} = (\lambda_3, \ldots, \lambda_p), \quad \underline{\mu} = (\mu_3, \ldots, \mu_p).\]
Putting \( n' = \lambda_1 - \mu_1 + \lambda_2 - \mu_2 \), we have
\[
\tilde{M}(\lambda, \mu) \cong \tilde{H}_n(\kappa) \otimes H_{\nu'} \otimes \tilde{H}_{n-\nu'} \left( \tilde{M}(\lambda, \mu) \otimes \tilde{M}(\lambda, \mu) \right).
\]

Since \( \mu_1 - \mu_2 + 1 = 0 \), it follows from Lemma 5.1 that \( \tilde{M}(\lambda, \mu) \) is simple and \( \tilde{M}(\lambda, \mu) \cong \tilde{M}(\sigma_1 \circ \lambda, \sigma_1 \circ \mu) \). Hence \( \tilde{M}(\lambda, \mu) \cong \tilde{M}(\sigma_1 \circ \lambda, \mu) \).

Next, suppose \( \sigma_j \circ \mu = \mu \) \( (j \in [0, p-1]) \). Then, we have \( \sigma_1 \omega_p^{-1-j} \circ \mu = \omega_p^{-1-j} \circ \mu = \omega_p^{-1-j} \circ \mu \). Using Proposition 6.5, we have
\[
\tilde{M}(\lambda, \mu) \cong \tilde{M}(\omega_p^{-1-j} \circ \lambda, \omega_p^{-1-j} \circ \mu)
\cong \tilde{M}(\sigma_1 \omega_p^{-1-j} \circ \lambda, \omega_p^{-1-j} \circ \mu)
\cong \tilde{M}(\omega_p^{-1-j} \sigma_1 \omega_p^{-1-j} \circ \lambda, \mu) = \tilde{M}(\sigma_j \circ \lambda, \mu). \quad \Box
\]

7. Uniqueness of Simple Quotient

We give a sufficient condition for an induced module \( \tilde{M}(\lambda, \mu) \) to have a unique simple quotient module.

Fix \( \kappa \in \mathbb{Z}_{>0} \). Let \( \mathcal{I}_p \) and \( \tilde{\mathcal{I}}_{p,\kappa}^+ \) be as in (4.1) and (4.5) respectively.

**Proposition 7.1.** (cf. Proposition 2.5.3. in [AST]) Let \( \kappa \in \mathbb{Z}_{>0} \) and \( (\lambda, \mu) \in \tilde{\mathcal{I}}_{p,\kappa}^+ \). Then \( \tilde{M}(\lambda, \mu)_{\kappa_{\lambda, \mu}} = \mathbb{C}1_{\lambda, \mu} \).

**Proof.** We denote \( \zeta_{\kappa_{\lambda, \mu}} \) simply by \( \zeta \) till the end of the proof.

It is enough to prove the statement in the case \( (\lambda, \mu) \in \mathcal{I}_{p,\kappa}^+ \).

First, suppose \( [\mu]_0 = \kappa - p + 1 - (\mu_1 - \mu_p) > 0 \). Take \( u \in \tilde{W}^{\lambda-\mu} \cap \tilde{W}[\zeta] \), where \( \tilde{W}[\zeta] = \{ w \in \tilde{W} \mid w(\zeta) = \zeta \} \).

By Lemma 2.4, we can write \( u = t_{y_{\eta}(0)} \cdot w = y_{\eta}^{-1} \cdot t_{y_{\eta}(0)} \cdot w \) \( (\eta \in P, w \in W^{\lambda-\mu}) \).

Suppose \( \eta \neq 0 \). Setting \( e^\nu = \sum_{i=1}^{n} e_i^\nu \), we have
\[
\langle \zeta | e^\nu \rangle = \langle u(\zeta) | e^\nu \rangle = \langle t_{y_{\eta}(0)} w(\zeta) | e^\nu \rangle = \langle \zeta | e^\nu \rangle + \kappa \langle y_{\eta}(\eta) | e^\nu \rangle,
\]
and thus \( \langle y_{\eta}(\eta) | e^\nu \rangle = 0 \). This implies that \( r := -\langle y_{\eta}(\eta) | e^\nu \rangle \) is a positive integer as \( y_{\eta}(\eta) \in P^- \). Since \( t_{y_{\eta}(0)} w(\zeta) = y_{\eta}(\zeta) \), we have
\[
0 = \langle t_{y_{\eta}(0)} w(\zeta) - y_{\eta}(\zeta) | e^\nu \rangle = \langle w(\zeta) | e^\nu \rangle + \langle y_{\eta}(\eta) | e^\nu \rangle \kappa - \langle y_{\eta}(\zeta) | e^\nu \rangle.
\]

Put \( n_0 = 0 \) and \( n_i = \sum_{j=i}^{n} (\lambda_j - \mu_j) \) \( (i \in [1, p]) \) as before. Then \( w^{-1}(1) = n_{a-1} + 1 \) for some \( a = [1, p] \) since \( w \in W^{\lambda-\mu} \). Let \( b \) be the number such that \( n_{b-1} < y_{\eta}^{-1}(1) \leq n_b \). From the definition (5.2) of \( \zeta \), it follows that
\[
\langle \zeta | e^\nu_{w^{-1}(1)} \rangle = \mu_a - a + 1 \quad \text{and} \quad \langle \zeta | e^\nu_{y_{\eta}^{-1}(1)} \rangle = \mu_b - b + y_{\eta}^{-1}(1) - n_{b-1}.
\]
Now, (7.1) leads
\[ 0 = r\kappa - (\mu_a + 1 - a) + \mu_b - b + y^{-1}(1) - n_{b-1} \]
\[ \geq \kappa - (\mu_a - a) + (\mu_b - b) \]
\[ \geq \kappa - (\mu_1 - 1) + (\mu_p - p) > 0. \]
This is a contradiction. Hence \( \eta = 0 \) and thus \( u \in W \). Therefore
\[ \dot{W}^I \cap \dot{W}^{[\zeta_{\lambda,\mu}]} = \dot{W}^I \cap W^{[\zeta_{\lambda,\mu}]} \]  
(7.2)
This implies \( \ddot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}}^\text{gen} = \dot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}}^\text{gen} \) and thus
\[ \ddot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} = \dot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} = \mathbb{C}_{1_{\lambda,\mu}} \]
by Proposition 5.4.

Next, suppose that \( [\mu_0] = 0 \). Then there exists \( j \) such that \( [\mu_j] = \mu_j - \mu_{j+1} + 1 > 0 \). Put
\[ \lambda' = \varpi_p^{-j} \circ \lambda, \quad \mu' = \varpi_p^{-j} \circ \mu. \]
It is easy to check that \( (\lambda', \mu') \in \check{I}_{p,\kappa}^+ \) and \( [\mu'_0] = [\mu_j] \geq 0 \). Moreover we have \( \zeta_{\lambda',\mu'} = \pi^{n-j}(\zeta_{\lambda,\mu}). \) The linear automorphism \( v \mapsto \pi^{n-j}v \) on \( \dot{M}(\lambda, \mu) \) gives an isomorphism \( \ddot{M}(\lambda, \mu)_{\zeta_{\lambda',\mu'}} \cong \dot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} \). On the other hand, we have an \( \dot{H}_n \)-isomorphism \( \ddot{M}(\lambda, \mu) \cong \dot{M}(\lambda', \mu') \) by Proposition 6.5, and thus we have \( \ddot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} \cong \dot{M}(\lambda', \mu')_{\zeta_{\lambda,\mu}} \). Therefore,
\[ \dim \ddot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} = \dim \dot{M}(\lambda', \mu')_{\zeta_{\lambda,\mu}} = 1. \]

**Theorem 7.2.** Let \( \kappa \in \mathbb{Z}_{>0} \) and \( (\lambda, \mu) \in \check{I}_{p,\kappa}^+ \). Then \( \ddot{M}(\lambda, \mu) \) has a unique simple quotient module, which we denote by \( \dddot{L}(\lambda, \mu) \).

**Proof.** Let \( N \) be a proper submodule of \( \ddot{M}(\lambda, \mu) \). By Proposition 7.1, we have \( \ddot{M}(\lambda, \mu)_{\zeta_{\lambda,\mu}} = \mathbb{C}_{1_{\lambda,\mu}} \). This implies \( N_{\zeta_{\lambda,\mu}} = \{0\} \) since \( 1_{\lambda,\mu} \) is a cyclic vector of \( \dot{M}(\lambda, \mu) \). Hence \( \zeta_{\lambda,\mu} \notin P(N) \). Therefore the sum of the all proper submodules of \( \ddot{M}(\lambda, \mu) \) is the maximal proper submodule of \( \ddot{M}(\lambda, \mu) \).

The condition \( (\lambda, \mu) \in \check{I}_{p,\kappa}^+ \) in Theorem 7.2 can be relaxed by means of Proposition 6.6.

**Corollary 7.3.** Let \( \kappa \in \mathbb{Z}_{>0} \). Let \( (\lambda, \mu) \in \check{I}_p \) and \( \mu \in \check{D}_{p,\kappa} \). Then \( \ddot{M}(\lambda, \mu) \) has a unique simple quotient module.
8. Classification of simple modules

Let $\kappa \in \mathbb{Z}_{>0}$. Let $\text{Irr}(\mathcal{O}_\kappa(\hat{H}_n))$ be the set of isomorphism classes of all simple modules in $\mathcal{O}_\kappa(\hat{H}_n)$. Through Theorem 7.2, we can construct a correspondence

$$\tilde{\Phi} : \bigsqcup_{p=1}^n \hat{\mathcal{T}}_{p,\kappa}^+ \to \text{Irr}(\mathcal{O}_\kappa(\hat{H}_n))$$

(8.1)

by $(\lambda, \mu) \mapsto \tilde{\mathcal{L}}(\lambda, \mu)$. The proofs of the following two theorems are given later.

**Theorem 8.1.** Let $\kappa \in \mathbb{Z}_{>0}$. Let $K$ be a simple module in $\mathcal{O}_\kappa(\hat{H}_n)$. Then there exists $p \in [1, n]$ and $(\lambda, \mu) \in \hat{\mathcal{I}}_{\kappa+p}^*$ such that $K \cong \tilde{\mathcal{L}}(\lambda, \mu)$. In other words, the correspondence $\tilde{\Phi}$ is surjective.

**Theorem 8.2.** Let $\kappa \in \mathbb{Z}_{>0}$. Let $(\lambda, \mu) \in \hat{\mathcal{I}}_{\kappa+p}^*$ and $(\beta, \gamma) \in \mathcal{I}_{\kappa+q}^*$. Then, the following are equivalent:

(a) $\tilde{\mathcal{M}}(\lambda, \mu) \cong \tilde{\mathcal{M}}(\beta, \gamma)$.
(b) $\tilde{\mathcal{L}}(\lambda, \mu) \cong \tilde{\mathcal{L}}(\beta, \gamma)$.
(c) $p = q$ and $\tilde{\mathcal{L}}(\beta, \gamma) = \varpi_p^r \circ (\lambda, \mu)$ for some $r \in \mathbb{Z}$.

By Theorem 8.2 (or Proposition 6.5), the correspondence $\tilde{\Phi}$ factors $\bigsqcup_{p=1}^n \hat{\mathcal{T}}_{p,\kappa}^+ / \langle \varpi_p \rangle$, and we get the following.

**Corollary 8.3.** The correspondence $\tilde{\Phi} : \bigsqcup_{p=1}^n \hat{\mathcal{T}}_{p,\kappa}^+ \to \text{Irr}(\mathcal{O}_\kappa(\hat{H}_n))$ above induces a bijection

$$\Phi : \bigsqcup_{p=1}^n \hat{\mathcal{T}}_{p,\kappa}^+ / \langle \varpi_p \rangle \to \text{Irr}(\mathcal{O}_\kappa(\hat{H}_n)).$$

Recall that $\hat{\mathcal{T}}_{p,\kappa}^+ \cong \mathcal{T}_p^*/\hat{\mathcal{S}}_p^\circ$ (Proposition 4.4) and $\hat{\mathcal{T}}_{p,\kappa}^+ / \langle \varpi_p \rangle \cong \mathcal{T}_p^*/\hat{\mathcal{S}}_p$. Hence, we have a natural one to one correspondence

$$\bigsqcup_{p=1}^n \mathcal{T}_p^*/\hat{\mathcal{S}}_p \leftrightarrow \text{Irr}(\mathcal{O}_\kappa(\hat{H}_n)).$$

(8.2)

**Remark 8.4.** We treat the degenerate double affine Hecke algebra in this paper, but it is easy to modify the arguments to obtain the same results for the double affine Hecke algebra provided that a certain parameter (often denoted by $q$) of the algebra is not a root of one. In particular, the classification of simple modules over the double affine Hecke algebra of $\mathfrak{gl}_n$ follows.
Remark 8.5. (i) For (non-degenerate) double affine Hecke algebras of general type, a geometric proof of the classification of simple modules has been given by Vasserot in the preprint [Va].

Our parameterization by \((\lambda, \mu)\) is related to Vasserot’s parameterization by \(\sigma = (\sigma_{a,b})\) in [Va]–§8 through

\[
\sigma_{a,b} = \sharp \left\{ i \in [1, p] \mid a = \mu_i - i + 1, \ b = \lambda_i - i \right\} \quad (a, b \in \mathbb{Z}, \ a \leq b).
\]

(cf. Remark 8.7.)

(ii) In the preprint [Ch4], Cherednik announces a similar classification for the double affine Hecke algebra of type \(A\) by an alternative algebraic approach.

Remark 8.6. In [AST, Ch4], another class of representations have been studied, that is, \(S(\mathfrak{h})\)-semisimple modules. They form a subcategory of \(O_\kappa(\hat{H}_n)\), and the classification of simple modules in this category is given in [Ch4].

The method and results developed in our present paper are also effective for the study of \(S(\mathfrak{h})\)-semisimple modules. We have obtained an alternative proof of the classification of simple modules and some concrete results on the structure of simple modules of this class. These results will be presented in the forthcoming paper.

Remark 8.7. It is known that the set \(\bigcup_{p=1}^n \mathcal{T}_p^* / \mathfrak{S}_p \cong \text{Irr}\mathcal{O}(\hat{H}_n)\) (Theorem 5.6) is naturally indexed by isomorphism classes of nilpotent representations of the quiver of type \(A\) [Ze2].

It can be seen that the set \(\bigcup_{p=1}^n \mathcal{T}_p^* / \mathfrak{S}_p\) above is indexed by isomorphism classes of nilpotent representations of the cyclic quiver:

Let \(Q_\kappa\) be the quiver of type \(A^{(1)}_\kappa\) with the cyclic orientation, i.e. the set of vertices is \(\mathbb{Z}/\kappa\mathbb{Z}\) and the set of morphisms consists of the arrows \(i \to i + 1 \quad (i \in \mathbb{Z}/\kappa\mathbb{Z})\). Let \(S_n\) be the set of isomorphism classes of \(n\)-dimensional nilpotent representations of \(Q_\kappa\).

Let \(Z\) be the set of all pairs \((a, b)\) of integers such that \(a \leq b\) and \((a, b)\) is defined up to simultaneous translation by a multiple of \(\kappa\): \((a, b) \sim (a + m\kappa, b + m\kappa), \ m \in \mathbb{Z}\). It is known that isomorphism classes of indecomposable finite-dimensional representations of \(Q_\kappa\) are indexed by elements of \(Z\), and any finite-dimensional nilpotent representation of \(Q_\kappa\) is decomposed into a sum of indecomposable representations (see [Lu2] for details). Let \(V(a, b)\) denote the indecomposable representation corresponding to \((a, b) \in Z\).

Then, the correspondences \(\mathcal{T}_p^* \to S_n \quad (p \in [1, n])\) defined by

\[
(\lambda, \mu) \mapsto \sum_{i \in [1, p]} V(\mu_i - i + 1, \lambda_i - i)
\]
give rise to a bijection $\bigsqcup_{p=1}^n \mathcal{I}_p^*/\hat{\mathbb{S}}_p \to S_n$.

9. Proof of Theorem 8.1

In this section, we will give a proof of Theorem 8.1, which asserts the surjectivity of the correspondence $\Phi$ in Corollary 8.3. For this purpose, we need to introduce some notations.

Fix $\kappa \in \mathbb{Z}_{>0}$. For an $\hat{H}_n$-module $N$ and $(\lambda, \mu) \in \mathcal{I}_p$, set

$$N_{[\lambda,\mu]} = \{v \in N_{\zeta_{\lambda,\mu}} \mid wv = v \text{ for } w \in W_{\lambda-\mu}\}.$$  

Set

$$\mathcal{B}_p(N) = \{\mu \in \mathbb{Z}^p \mid \exists \lambda \in \mathbb{Z}^p \text{ such that } (\lambda, \mu) \in \mathcal{I}_p^* \text{ and } N_{[\lambda,\mu]} \neq 0\}.$$  

Example 9.1. If $p = n$ then $\mathcal{I}_n^* = \{(\lambda, \mu) \in \mathbb{Z}^n \times \mathbb{Z}^n \mid \lambda = (\mu_1 + 1, \mu_2 + 1, \ldots, \mu_n + 1)\}$. Hence, for $(\lambda, \mu) \in \mathcal{I}_n^*$, we have $\zeta_{\lambda,\mu} = \sum_{i=1}^n (\mu_i - i + 1)\varepsilon_i + \kappa \sigma^*$ and $N_{[\lambda,\mu]}$ is nothing but the weight space $N_{\zeta_{\lambda,\mu}}$. In particular $\|\mathcal{B}_n(N)\| = \|P(N)\| > 0$.

For $\mu \in \mathbb{Z}^p$, we put

$$[\mu]_0 = \kappa - p + 1 - (\mu_1 - \mu_p), \quad [\mu]_i = \mu_i - \mu_{i+1} + 1 \quad (i \in [1, p-1])$$

as before.

Lemma 9.2. Let $K$ be a simple module in $\mathcal{O}_n(\hat{H}_n)$ and let $p$ be the minimum integer such that $\mathcal{B}_p(K) \neq \emptyset$. Suppose that $[\mu]_i < 0$ for $\mu \in \mathcal{B}_p(K)$ and $i \in [0, p-1]$. Then $\sigma_i \circ \mu \in \mathcal{B}_p(K)$.

Proof. Let $\mu \in \mathcal{B}_p(K)$. Let $\lambda$ be such that $(\lambda, \mu) \in \mathcal{I}_p^*$ and $K_{[\lambda,\mu]} \neq 0$. Put $n_0 = 0, n_j = \sum_{k=1}^j (\lambda_k - \mu_k)$ ($j \in [1, p]$).

First, let us prove the statement when $i \in [1, p-1]$. Suppose $[\mu]_i = \mu_i - \mu_{i+1} + 1 < 0$. Put $\lambda(i) = (\lambda_1, \lambda_{i+1}) \in \mathbb{Z}^2$ and $\mu(i) = (\mu_i, \mu_{i+1}) \in \mathbb{Z}^2$.

Consider the subalgebra $A := \hat{H}_{[n_1+1, n_4+1]}$ of $\hat{H}_n$, which we identify with $\hat{H}_{n_1+1-n_4-1}$ through Lemma 3.5 and Example 3.6.

Take $v \in K_{[\lambda,\mu]} \setminus \{0\}$ and consider the $A$-module $N := Av$. Then $N$ is a surjective image of $\check{M}(\lambda(i), \mu(i))$.

If $\check{M}(\lambda(i), \mu(i))$ is irreducible, then it follows from Lemma 5.1 that $N \cong \check{M}(\lambda(i), \mu(i)) \cong \check{M}(\lambda(i)', \mu(i'))$, where $\lambda(i)' = (\lambda_{i+1} - 1, \lambda_i + 1)$ and $\mu(i)' = (\mu_{i+1} - 1, \mu_i + 1)$. Hence there exists $s \in A \subset \hat{H}_n$ such that $sv \in N_{[\lambda(i)',\mu(i)']} \setminus \{0\}$. Clearly, $sv \in K_{[\sigma_0 \circ \lambda, \sigma_0 \circ \mu]} \setminus \{0\}$. Hence $\sigma_0 \circ \mu \in \mathcal{B}_p(K)$.

Suppose that $\check{M}(\lambda(i), \mu(i))$ is reducible. Then, by Lemma 5.1, there exists an exact sequence

$$0 \to \check{L}(\lambda(i)', \mu(i)') \to \check{M}(\lambda(i), \mu(i)) \to \check{L}(\lambda(i), \mu(i)) \to 0.$$
Since $N$ is a surjective image of $\dot{M}(\lambda(i), \mu(i))$, it is isomorphic to either $\dot{M}(\lambda(i), \mu(i))$ or $\dot{L}(\lambda(i), \mu'(i))$. If $N \cong \dot{L}(\lambda(i), \mu'(i))$, then we have $K[\lambda, \sigma \circ \mu] \neq 0$. (Note that $(\lambda, \sigma \circ \mu) \in \mathcal{T}_p^+$ by the assumption of $p$.)

If $N \cong \dot{M}(\lambda(i), \mu(i))$, then $N$ contains a submodule $\dot{L}(\lambda(i), \mu'(i))$ and thus $K[\sigma \circ \lambda, \sigma \circ \mu] \neq 0$. In both cases, we have $\sigma_i \circ \mu \in \mathcal{B}_p(K)$.

Next, let us prove the statement for $i = 0$. Suppose $[\mu]_0 = \kappa + p - 1 - \mu_1 + \mu_p < 0$.

Consider the subalgebra $A' := \hat{H}_{n+1} \cdots \hat{H}_1$ of $\hat{H}_n$, which is identified with $\hat{H}_{n+1-n-1}$ through Lemma 3.5.

Put $\lambda(0) = (\lambda_p + \kappa - p + 1, \lambda_1 + 1)$ and $\mu(0) = (\mu_p + \kappa - p + 1, \mu_1 + 1)$. Take $v \in K[\lambda, \mu] \setminus \{0\}$ and consider the $A'$-module $N' := A'v$. Then $N'$ is a surjective image of the $\hat{H}_{n+1-n-1}$-module $\dot{M}(\lambda(0), \mu(0))$. By similar arguments as in the case $i \in [1, p]$, we have either $K[\sigma \circ \lambda, \sigma \circ \mu] \neq 0$ or $K[\lambda, \sigma \circ \mu] \neq 0$. Therefore $\sigma_0 \circ \mu \in \mathcal{B}_p(K)$.

\begin{lemma}
Let $\kappa \in \mathbb{Z}_{>0}$. Let $K$ be a simple module in $\mathcal{O}_\kappa(\hat{H}_n)$. If $\mathcal{B}_p(K) \cap \mathcal{D}_{\kappa, \lambda} \neq 0$, then there exists $(\lambda, \mu) \in \mathcal{T}_{p, \kappa}^+$ such that $K \cong \dot{L}(\lambda, \mu)$.
\end{lemma}

\begin{proof}
By the assumption, there exists $(\lambda', \mu) \in \mathcal{T}_{p, \kappa}^+$ such that $\mu \in \mathcal{D}_{p, \kappa}$ and $K[\lambda', \mu] \neq 0$. Take $v \in K[\lambda', \mu] \setminus \{0\}$. Since $K = \hat{H}_n v$, it is a surjective image of $\dot{M}(\lambda', \mu)$ and thus $K \cong \dot{L}(\lambda', \mu)$ by Theorem 7.2. Noting that $\mu \in \mathcal{D}_{p, \kappa}$, we can find $w \in \mathcal{S}_p$ such that $w \circ \mu = \mu$ and $(w \circ \lambda', \mu) \in \mathcal{T}_{p, \kappa}^+$. Put $\lambda = w \circ \lambda'$. Now, Proposition 6.6 implies $\dot{L}(\lambda, \mu) \cong \dot{L}(\lambda', \mu) \cong K$.
\end{proof}

\begin{proof}[Proof of Theorem 8.1]
Let $K$ be a simple module in $\mathcal{O}_\kappa(\hat{H}_n)$. Take the smallest integer $p$ such that $\mathcal{B}_p(K) \neq 0$.

By Lemma 9.3, it is enough to prove that $\mathcal{B}_p(K) \cap \mathcal{D}_{p, \kappa} \neq 0$. Take $\mu \in \mathcal{B}_p(K)$, and let $\mu^+$ denote the unique element in $\{w \circ \mu\}_{w \in \mathcal{S}_p} \cap \mathcal{D}_{p, \kappa}$. Take the shortest $w \in \mathcal{S}_p^+$ such that $w \circ \mu = \mu^+$. Let $w = \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_l}$ be a reduced expression. Then by Lemma 4.3, we have $[\sigma_{i_{k+1}} \sigma_{i_{k+2}} \cdots \sigma_{i_l} \circ \mu]_{i_k} < 0$ for $k \in [1, l]$. Now, Lemma 9.2 implies $\mu^+ \in \mathcal{B}_p(K) \cap \mathcal{D}_{p, \kappa}$.
\end{proof}

10. PROOF OF THEOREM 8.2

We will give a proof of Theorem 8.2, which asserts the injectivity of the correspondence $\Phi$ in Corollary 8.3.

\begin{lemma}
Let $(\lambda, \mu) \in \mathcal{T}_{p, \kappa}^+$ and $s_i \in W_{\lambda-\mu}$. Then $s_i(\zeta_{\lambda, \mu}) \notin P(\dot{M}(\lambda, \mu))$.
\end{lemma}
Proof. We will give a proof only for the case $|\mu|_0 > 0$. Other cases can be shown similarly, using the same argument using Proposition 6.5 as in the proof of Theorem 7.2.

Assume that there exists $s_i \in W_{\lambda-\mu}$ such that $s_i(\zeta_{\lambda,\mu}^\kappa) \in P(\hat{M}(\lambda, \mu)) = \hat{W}^{{\lambda-\mu}}\zeta_{\lambda,\mu}^\kappa$ (Proposition 6.4). Then $s_i(\zeta_{\lambda,\mu}^\kappa) = x(\zeta_{\lambda,\mu}^\kappa)$ for some $x \in \hat{W}^{{\lambda-\mu}}$. Putting $w = s_ix$, we have $w \in \hat{W}^{{\lambda-\mu}} \cap \hat{W}[\zeta_{\lambda,\mu}^\kappa]$ because $R(s_i x) = R(x) \cup \{x^{-1}(\alpha_i)\}$ or $R(s_i x) = R(x) \setminus \{-x^{-1}(\alpha_i)\}$.

Recall that we have proved $\hat{W}^{{\lambda-\mu}} \cap \hat{W}[\zeta_{\lambda,\mu}^\kappa] = \hat{W}^{{\lambda-\mu}} \cap \hat{W}[\zeta_{\lambda,\mu}^\kappa]$ when $|\mu|_0 > 0$ in the proof of Theorem 7.2 (see (7.2)). Hence $w \in \hat{W}^{{\lambda-\mu}} \cap \hat{W}[\zeta_{\lambda,\mu}^\kappa]$.

First, we consider the case where

$$\lambda = (m, m+1, \ldots, m+p-1), \quad \mu = (0, 1, \ldots, p-1) \text{ with } mp = n.$$ 

Set $I_j = \{k \in [1, n] | \langle \zeta_{\lambda,\mu}^\kappa, \varepsilon_k^\gamma \rangle = \langle \zeta_{\lambda,\mu}^\kappa, \varepsilon_j^\gamma \rangle \}$ for $j \in [1, n]$. Then, in particular, we have $I_{i+1} = \{k + 1 | k \in I_i\}$. By induction on $k$, we have $w(k + 1) = w(k) + 1$ for all $k \in I_i$. Taking $k = w^{-1}(i) \in I_i$, we have $s_i w(k) = i + 1$ and $s_i w(k + 1) = i$. This implies $x = s_i w \notin \hat{W}^{{\lambda-\mu}}$. This is a contradiction. The same contradiction is deduced for general $(\lambda, \mu)$ through Lemma 5.3.

Let $\xi \in P_\kappa$. Then, there exists a unique element $(\xi^L, \xi^R) \in \bigsqcup_{p=1}^n T_p^*$ for which the following two conditions hold:

$$s_i \in W_{\xi^R - \xi^L} \iff \langle \xi | \alpha_i^\vee \rangle = -1 \quad (i \in [1, n-1]).$$

$$\zeta_{\xi^L, \xi^R}^\kappa = \xi. \quad (10.1)$$

(10.2)

We denote $h(\xi) = p$ if $(\xi^L, \xi^R) \in T_p^*$.

In §6, we defined $\zeta_{\lambda,\mu}^\kappa \in P_\kappa$ for each $(\lambda, \mu) \in T_p$. The correspondence $\xi \mapsto (\xi^L, \xi^R)$ is a left inverse of $(\lambda, \mu) \mapsto \zeta_{\lambda,\mu}^\kappa$ in the following sense:

Lemma 10.2. If $(\lambda, \mu) \in \hat{T}_{p,\kappa}^* +$ then $((\zeta_{\lambda,\mu}^\kappa)^L, (\zeta_{\lambda,\mu}^\kappa)^R) = (\lambda, \mu)$ and $h(\zeta_{\lambda,\mu}^\kappa) = p$.

Proof. The statement follows easily from the definition of $(\xi^L, \xi^R)$.  

Definition 10.3. For a subset $S$ of $P_\kappa$. Define $C(S)$ to be the subset of $S$ consisting of all elements $\xi \in S$ satisfying the following conditions:

(C1) If $\langle \xi | \alpha_i^\vee \rangle < 0$ for $i \in [0, n-1]$ then $s_i(\xi) \notin S$.

(C2) $(\xi^L, \xi^R) \in \bigsqcup_{p=1}^n \hat{T}_{p,\kappa}^* +$.

For $(\lambda, \mu) \in \hat{T}_{p,\kappa}^* +$, put $C_{\lambda,\mu} = C(P(\hat{L}(\lambda, \mu)))$.

Lemma 10.4. Let $(\lambda, \mu) \in \hat{T}_{p,\kappa}^* +$. Then $\zeta_{\lambda,\mu}^\kappa \in C_{\lambda,\mu}$. 

\textbf{Proof.} It is obvious that \(\zeta^\kappa_{\lambda,\mu}\) satisfies (C2) by Lemma 10.2. By \((\lambda, \mu) \in \tilde{\mathcal{I}}_{p,\kappa}^*\), we have
\[
\langle \zeta^\kappa_{\lambda,\mu} \mid \alpha^\vee_i \rangle < 0 \iff \langle \zeta^\kappa_{\lambda,\mu} \mid \alpha^\vee_i \rangle = -1 \iff s_i \in W_{\lambda-\mu}.
\]
Now, it follows from Lemma 10.1 that \(\zeta^\kappa_{\lambda,\mu}\) satisfies (C1).

We fix \((\lambda, \mu) \in \tilde{\mathcal{I}}_{p,\kappa}^*\) for a while. Let \(\xi \in \mathcal{C}_{\lambda,\mu}\). Put \(q = h(\xi)\) and put
\[
n_0 = 0, \quad n_i = \sum_{j=1}^i (\lambda_j - \mu_j) \quad (i \in [1, p]),
\]
\[
m_0 = 0, \quad m_i = \sum_{j=1}^i (\xi^L_j - \xi^R_j) \quad (i \in [1, q]).
\]

Noting that \(\mathcal{C}_{\lambda,\mu} \subseteq P(\tilde{L}(\lambda, \mu)) \subseteq P(\tilde{M}(\lambda, \mu)) = \tilde{W}^{\lambda-\mu}\mathcal{C}_{\lambda,\mu}\), take \(w \in \tilde{W}^{\lambda-\mu}\) such that \(\xi = w(\zeta^\kappa_{\lambda,\mu})\).

\textbf{Lemma 10.5.} If \(\alpha_i \in R_{\lambda-\mu} \ (i \in [1, n-1])\) then \(w(\alpha_i) = \alpha_l\) for some \(l \in [0, n-1]\).

\textit{Proof.} Let \(\alpha_i \in R_{\lambda-\mu}\). We have \(w(\epsilon_i) = \epsilon_{j'} + k'\delta\) and \(w(\epsilon_{i+1}) = \epsilon_j + k\delta\) for some \(j, j' \in [1, n]\) and \(k, k' \in \mathbb{Z}\). We have \(w(\alpha_i) = \epsilon_j - \epsilon_{j'} + (k' - k)\delta \in \hat{R}\) and
\[
-1 = \langle \zeta^\kappa_{\lambda,\mu} \mid \alpha^\vee_i \rangle = \langle \xi \mid \epsilon_{j'} - \epsilon_j \rangle + (k' - k)\kappa.
\]

Since \(\alpha_i \in R_{\lambda-\mu}\) and \(w \in \tilde{W}^{\lambda-\mu}\), we have \(w(\alpha_i) \in \tilde{R}^+\). Note that \(w(\alpha_i) \in \tilde{H}\) if and only if \(j' - j + (k' - k)n = -1\). We assume that \(j' - j + (k' - k)n \neq -1\) and will deduce a contradiction.

First, suppose \(\langle \xi \mid \alpha^\vee_{j-1} \rangle = -1\). Let \(A\) be the subalgebra of \(\tilde{H}_n\) generated by \(\{\epsilon_{j'} - k'\epsilon, \epsilon_{j+1} - k\epsilon, \epsilon_{j'} + k, \epsilon_j + k, s_{\alpha'}, s_{\alpha_{j-1}}\}\), where \(\alpha' = \epsilon_{j'} - \epsilon_{j-1} + (k' - k)\delta \in \hat{R}^+\). Then, it follows from Lemma 3.5 that \(A\) is isomorphic to the degenerate affine Hecke algebra \(\tilde{H}_3\) of \(\mathfrak{gl}_3\).

Let \(v \in \tilde{L}(\lambda, \mu)\). Then we have
\[
\alpha^\vee v = 0, \quad \alpha^\vee_{j-1} v = -v, \quad s_{\alpha_{j-1}} v = v.
\]

The subspace \(A v \subseteq \tilde{L}(\lambda, \mu)\) is regarded as an \(A\)-module, and it is a surjective image of the induced module \(\tilde{M}(\lambda, \mu)\) over \(\tilde{H}_3\) with \(\lambda = (z+1, z+3), \mu = (z, z+1)\), where \(z = \langle \xi \mid \epsilon_{j-1}^\vee \rangle\). By Lemma 5.1, \(\tilde{M}(\lambda, \mu)\) is simple and thus \(A v \cong \tilde{M}(\lambda, \mu)\). It follows from Lemma 5.2 that \(s_{\alpha_{j-1}}(\xi)\) is a weight of \(A v\), and hence \(s_{\alpha_{j-1}}(\xi) = s_{j-1}(\xi) \in P(\tilde{L}(\lambda, \mu))\). Combined with the assumption \(\langle \xi \mid \alpha^\vee_{j-1} \rangle = -1\), this contradicts to the condition (C1).
Therefore we must have $\langle \xi \mid \alpha_{j-1}^\vee \rangle \neq -1$. In this case, we have $s_{j-1} \notin W_{\mathcal{L}_{-\xi^R}}$. This implies $\epsilon_j = \epsilon_{m_{a-1}+1}$ for some $a \in [1, q]$. Let $b \in [1, q)$ be the number such that $j' \in [m_{b-1}+1, m_b]$. Then, using (10.5), we have
\[
\xi^R - b + 1 = \langle \xi \mid \epsilon_{m_{a-1}+1}^\vee \rangle \leq \langle \xi \mid \epsilon_{j'}^\vee \rangle = \langle \xi \mid \epsilon_{m_{a-1}+1}^\vee \rangle - (k' - k)\kappa - 1
\]
\[
= \xi^R_a - a + 1 - (k' - k)\kappa - 1.
\]
Note that $k' - k \geq 0$ as $w(\alpha_i) \in \hat{R}^+$. If $b \leq a$, then the inequality $\xi^R - b + 1 < \xi^R - a + 1$ contradicts to the condition $(\xi^L, \xi^R) \in \mathcal{I}^{\ast +}_{q, \kappa}$.

If $b > a$, then we have $k' - k > 0$ and $(\xi^R_a - a + 1) - (\xi^R - b + 1) \geq 1 + (k' - k)\kappa > \kappa$. This is a contradiction too. Therefore we have $j' - j + (k' - k)n = -1$, and hence $w(\alpha_i) = \alpha_{j-1} \in \mathcal{W}$.

\[\square\]

**Lemma 10.6.** (i) For each $i \in [1, q]$, we have
\[
w^{-1}(\epsilon_{m_{i-1}+1}^\vee) = \epsilon_{n_{a_i}+1}^\vee + k_i c,
\]
\[
w^{-1}(\epsilon_{m_i}^\vee) = \epsilon_{n_1}^\vee + l_i c
\]
for some $a_i, b_i \in [1, p]$ and $k_i, l_i \in \mathbb{Z}$. In particular, we have
\[
\xi^R_i - i = \mu_{a_i} - a_i + k_i \kappa, \quad \xi^L_i = -i = \lambda_{b_i} - b_i + l_i \kappa \quad (i \in [1, q]).
\]

(ii) If $q = p$ then $a_i = b_i$ and $k_i = l_i$ in (10.6)(10.7) for all $i \in [1, p]$.

(iii) The correspondences $[1, q] \rightarrow [1, p]$ given by $i \mapsto a_i$ and $i \mapsto b_i$ are injective. In particular $q \leq p$.

**Proof.** (i) There exist $r \in [1, n]$ and $k \in \mathbb{Z}$ such that $w^{-1}(\epsilon_{m_{i-1}+1}^\vee) = \epsilon_{r}^\vee + kc$. Suppose $n_{a-1} + 1 < r \leq n_a (a \in [1, p])$. Then $a_{r-1} \in R_{\Lambda - \mu}$ and $\langle \xi \mid \alpha_{m_{i-1}}^\vee \rangle = \langle \xi^\kappa_{\mu} \mid \alpha_{r-1} \rangle = -1$. This contradicts to the condition (10.1) and we have (10.6). Similarly, (10.7) follows.

(ii) The statement follows easily from Lemma 10.5.

(iii) Suppose that there exist $i, j \in [1, q]$ such that
\[
w^{-1}(\epsilon_{r}^\vee + kc) = \epsilon_{n_{a-1}+1}^\vee + k_i c, \quad w^{-1}(\epsilon_{m_j-1+1}^\vee) = \epsilon_{n_{a-1}+1}^\vee + k_j c
\]
for some $a \in [1, p]$. Then, we have $w^{-1}(\alpha) = 0$ for a root $\alpha = \epsilon_{m_{i-1}+1} - \epsilon_{m_j-1+1} - (k_i - k_j)j$. This is a contradiction and thus the correspondence $i \mapsto a_i$ is injective.

\[\square\]

Now, we show the key lemma to the proof of Theorem 8.2.

**Lemma 10.7.** Let $(\lambda, \mu) \in \mathcal{I}_{p, \kappa}^{\ast +}$. Let $\xi \in \mathcal{C}_{\lambda, \mu}$ and suppose that $h(\xi) = \max \{h(\zeta) \mid \zeta \in \mathcal{C}_{\lambda, \mu}\}$. Then $h(\xi) = p$ and $(\xi^L, \xi^R) = \varpi^p \circ (\lambda, \mu)$ for some $r \in \mathbb{Z}$.
Proof. We have $h(\xi) = \max\{h(\zeta) \mid \zeta \in \mathcal{C}_{\lambda,\mu}\} = p$ by Lemma 10.4 and Lemma 10.6-(iii).

Take $w \in W^{\lambda-\mu}$ such that $\xi = w(\zeta^\kappa)$. By Lemma 10.6-(i)(ii), there exist $a_i \in [1, p]$ and $k_i \in \mathbb{Z}$ such that

$$w^{-1}(\epsilon^\vee_{m_i-1+1}) = \epsilon^\vee_{n_{a_i-1+1}} + k_i c, \quad w^{-1}(\epsilon^\vee_{m_i}) = \epsilon^\vee_{n_{a_i}} + k_i c$$

for each $i \in [1, p]$.

Let $y$ be the element of $\mathcal{S}_p$ such that $y(i) = a_i$ ($i \in [1, p]$), and put $\nu = \sum_{i=1}^p k_i e_i$ ($\nu$ is an element of the weight lattice of $\mathfrak{gl}_p$). Put $x = t_\nu y \in \hat{\mathcal{S}}_p$. Then, we have $(\xi^L, \xi^R) = x \circ (\lambda, \mu)$. Since $\hat{\mathcal{T}}_{q,\kappa}$ is a fundamental domain for the action of $\hat{\mathcal{S}}^+_p$ on $\mathcal{I}^*_p$, the condition $(\xi^L, \xi^R) \in \hat{\mathcal{T}}_{q,\kappa}$ implies $(\xi^L, \xi^R) = \varpi^r \circ (\lambda, \mu)$ for some $r \in \mathbb{Z}$. \hfill \Box

Proof of Theorem 8.2.
The implication (a)$\Rightarrow$(b) is clear, and (c)$\Rightarrow$(a) follows from Proposition 6.5. Let us prove (b)$\Rightarrow$(c), which completes the proof of Theorem 8.2.

Suppose $\tilde{L}(\lambda, \mu) \cong \tilde{L}(\beta, \gamma)$. Then we have $P(\tilde{L}(\lambda, \mu)) = P(\tilde{L}(\beta, \gamma))$ and $\mathcal{C}_{\lambda,\mu} = \mathcal{C}_{\beta,\gamma}$. By Lemma 10.4, we have $\zeta^\kappa_{\beta,\gamma} \in \mathcal{C}_{\beta,\gamma} = \mathcal{C}_{\lambda,\mu}$. By Lemma 10.2, we have $((\zeta^\kappa_{\beta,\gamma})^L, (\zeta^\kappa_{\beta,\gamma})^R) = (\beta, \gamma)$ as $(\beta, \gamma) \in \mathcal{I}^*_q$. On the other hand, Lemma 10.7 implies $q = p$ and $((\zeta^\kappa_{\beta,\gamma})^L, (\zeta^\kappa_{\beta,\gamma})^R)) = \varpi^r \circ (\lambda, \mu)$ for some $r \in \mathbb{Z}$. Therefore we have $(\beta, \gamma) = \varpi^r \circ (\lambda, \mu)$. \hfill \Box

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