Global existence and blow-up phenomena for two-component Degasperis-Procesi system and two-component b-family system

Jingjing Liu1∗ Zhaoyang Yin2

1Department of Mathematics and Information Science, Zhengzhou University of Light Industry, 450002 Zhengzhou, China
2Department of Mathematics, Sun Yat-sen University, 510275 Guangzhou, China

Abstract

This paper is concerned with global existence and blow-up phenomena for two-component Degasperis-Procesi system and two-component b-family system. The strategy relies on our observation on new conservative quantities of these systems. Several new global existence results and a new blowup result of strong solutions to the two-component Degasperis-Procesi system and the two-component b-family system are presented by using these new conservative quantities.

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1 Introduction

In this paper, we first consider the Cauchy problem of the following two-component Degasperis-Procesi system (DP2):

\[
\begin{align*}
    m_t + 3m u_x + m_x u + c \rho_x = 0, & \quad t > 0, \ x \in \mathbb{R}, \\
    \rho_t + u \rho_x + 2u_x \rho = 0, & \quad t > 0, \ x \in \mathbb{R}, \\
    u(0, x) = u_0(x), & \quad x \in \mathbb{R}, \\
    \rho(0, x) = \rho_0(x), & \quad x \in \mathbb{R},
\end{align*}
\]

(1.1)

where \( m = u - u_{xx} \) and \( c \) is an arbitrary real constant. The system (1.1) was firstly introduced by Popowicz in [25] as a generalization of the Degasperis-Procesi (DP) equation by use of the Dirac reduction of the generalized, but degenerated Hamiltonian operator of the Boussinesq equation. Since any Lax representation for the system (1.1) has not been discovered so far, the integrability of it is still an open problem [25].

For \( \rho \equiv 0 \), the system (1.1) becomes the DP equation [13], which can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm (CH) shallow water equation [11, 10, 11, 20]. The DP equation was proved

*E-mail address: jingjing830306@163.com (J. Liu); mcsyz@sysu.edu.cn (Z. Yin)
formally integrable by constructing a Lax pair \[9\] and showed has a bi-Hamiltonian structure and an infinite sequence of conserved quantities \[9\]. Moreover, similar to the CH peakons \[1, 7, 8\], it also admits exact peakon solutions. However, solutions of this type are not mere abstractizations, it replicate a feature that is characteristic for the waves of great height—waves of largest amplitude that are exact solutions of the governing equations for irrotational water waves \[3, 4\].

The Cauchy problem and the initial boundary value problem for the DP equation have been studied extensively \[17, 27, 28\]. Local well-posedness of this equation is established in \[27, 28\] for initial data \(u_0 \in H^s(\mathbb{R})\), \(s > \frac{3}{2}\). Interestingly, it has global strong solutions \[23, 29\] and finite time blow-up solutions \[15, 16, 23, 28\]. On the other hand, it has global weak solutions in \(H^1(\mathbb{R})\) \[15, 28, 29\] and global entropy weak solutions belonging to the class \(L^1(\mathbb{R}) \cap BV(\mathbb{R})\) and to the class \(L^2(\mathbb{R}) \cap L^4(\mathbb{R})\), cf. \[2\].

The interest in the DP equation and in the related CH equation lies in that model equations presenting breaking waves as well as peaked traveling waves are of great importance in hydrodynamics \[21\], and the traveling wave solutions of large amplitude to the governing equations for water waves are peaked waves \[3\]. We note that in the KdV regime such model equations do not exist so that one has to consider waves of larger amplitude, which leads to the regime proper for CH and DP equations \[6\]. Although the DP equation is similar to the CH equation in several aspects, these two equations are truly different. One of the novel features of the DP equation different from the CH equation is that it has not only peakon solutions \[9\] and periodic peakon solutions \[29\], but also shock peakons \[24\] and the periodic shock waves \[16\].

Recently, the system (1.1) has been studied in \[14, 26, 30, 31\]. The authors established local well-posedness, derived precise blow-up scenario, proved the existence of strong solutions which blow up in finite time \[26\], investigated traveling wave solutions and self-similar blow-up solutions in \[30\] and \[31\] respectively. In \[14\], the authors studied the geometric properties of DP2 by use of geometric methods, they showed that DP2 can be regarded as geodesic equations on the semidirect product of the diffeomorphism group of the circle \(\text{Diff}(S^1)\) with some space of sufficiently smooth functions on the circle.

Although the existing results for the system (1.1) are rather abundant, there is no global existence result for DP2 so far. The main difficulty is the structure of (1.1) makes the method in \[5\] invalid, which is almost the only method used to prove global existence for two-component shallow water system for some time past. In this paper, we will deal with this problem by finding a new conserved quantity for the system (1.1). Moreover, using this new conserved quantity, we will give a new blow-up result. By comparing the global existence result and the blow-up result, we find our obtained results for the system (1.1) are sharp.

The second system we will consider in this paper is the following two-component b-family system \[19\]

\[
\begin{align*}
    m_t &= um_x + k_1u_xm + k_2\rho \rho_x, \quad t > 0, \ x \in \mathbb{R}, \\
    \rho_t &= k_3(u \rho)_x, \quad t > 0, \ x \in \mathbb{R}, \\
    m(0, x) &= m_0(x), \quad x \in \mathbb{R}, \\
    \rho(0, x) &= \rho_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

where \(m = u - u_{xx}\) and there are two cases about this system: (i) \(k_1 = b, k_2 = 2b\) and \(k_3 = 1\); (ii) \(k_1 = b + 1, k_2 = 2\) and \(k_3 = b\) with \(b \in \mathbb{R}\). The system (1.2) includes the well-known two-component CH system and two-component DP system.
For $\rho \equiv 0$, the system (1.2) becomes the b-family equation
\[ u_t - u_{txx} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx} \] (1.3)
where $b$ is arbitrary real constant, which can be derived as the family of asymptotically equivalent shallow water wave equations that emerge at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation \[11\] [12]. For the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated \[11\] [12]. The Cauchy problem of the equation (1.3) has been studied on the line and on the circle for $b \in \mathbb{R}$ in \[18\] [34] and \[32\] respectively. They established the local well-posedness, described the precise blow-up scenario, proved the equation has strong solutions which exist globally in time and blow up in finite time \[18\] [34] [32]. Moreover, infinite propagation speed for (1.3) was also investigated in \[44\].

For $\rho \neq 0$, $b \in \mathbb{R}$, the system (1.2) has been studied in \[22\] [33]. It has been shown that the system (1.2) is locally well-posed and has blow-up solutions in finite time \[22\]. In \[33\], by separation method, the authors obtained a class of self-similar solutions and determined the local or global behavior of them by the corresponding Emden equation. However, we find that the global existence of strong solutions to the system (1.2) has not been resolved up to now for lack of necessary conservation law.

In this paper, we will study the blow-up phenomena and the global existence of strong solutions for DP2 (1.1) and the two-component b-family system (1.2). Our paper is organized as follows. In Section 2, we recall some useful lemmas for DP2 (1.1) and present some new global existence results for it. In Section 3, we discuss the blow-up phenomena of strong solutions to (1.1). In Section 4, we study the global existence of strong solutions to the system (1.2).

**Notation**
Given a Banach space $Z$, we denote its norm by $\| \cdot \|_Z$. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. We denote by $\ast$ the convolution.

## 2 Global existence for DP2

In this section, we will recall some useful lemmas for DP2 (1.1) firstly. Then, we will show that there exist strong solutions to the system which exists globally in time.

Using the Green’s function $p(x) \triangleq \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$ and the identity $(1 - \partial_x^2)\^{-1}f = p \ast f$ for all $f \in L^2(\mathbb{R})$, we can rewrite the system (1.1) as follows:

\[
\begin{aligned}
\begin{cases}
  u_t + uu_x = -\partial_x(1 - \partial_x^2)^{-1}(\frac{3}{2}u^2 + \frac{3}{2}\rho^2) = -\partial_x p \ast (\frac{3}{2}u^2 + \frac{3}{2}\rho^2), & t > 0, x \in \mathbb{R}, \\
  \rho_t + u\rho_x + 2u_x\rho = 0, & t > 0, x \in \mathbb{R} \\
  u(0, x) = u_0(x), & x \in \mathbb{R}, \\
  \rho(0, x) = \rho_0(x), & x \in \mathbb{R},
\end{cases}
\end{aligned}
\] (2.1)

or the equivalent form:

\[
\begin{aligned}
\begin{cases}
  u_t - u_{txx} + 4uu_x = 3u_xu_{xx} + uu_{xxx} - c\rho x_x, & t > 0, x \in \mathbb{R}, \\
  \rho_t + u\rho_x + 2u_x\rho = 0, & t > 0, x \in \mathbb{R} \\
  u(0, x) = u_0(x), & x \in \mathbb{R}, \\
  \rho(0, x) = \rho_0(x), & x \in \mathbb{R}.
\end{cases}
\end{aligned}
\] (2.2)
Lemma 2.1 [26] Suppose that \( z_0 \triangleq \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, \; s > \frac{3}{2} \). There exists a maximal existence time \( T = T(\| z_0 \|_{H^s \times H^{s-1}}) > 0 \), and a unique solution \( z \triangleq \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (2.1) such that
\[
z = z(\cdot, z_0) \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2}).
\]
Moreover, the solution depends continuously on the initial data, i.e. the mapping \( z_0 \mapsto z(\cdot, z_0) : H^s \times H^{s-1} \rightarrow C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2}) \)
is continuous. Furthermore, the maximal existence time \( T \) may be chosen independent of \( s \).

Lemma 2.2 [26] Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1} \) with \( s > \frac{3}{2} \) and \( T > 0 \) be the maximal existence time of the corresponding solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (2.1). Then the corresponding solution \( z \) blows up in finite time if and only if
\[
\liminf_{t \to T} \inf_{x \in \mathbb{R}} \{ u_x(t, x) \} = -\infty.
\]

Consider now the following initial value problem
\[
\begin{aligned}
q_t &= u(t, q), \quad t \in [0,T), \\
q(0, x) &= x, \quad x \in \mathbb{R}.
\end{aligned}
\tag{2.3}
\]

Lemma 2.3 [17, 26] Let \( z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, \; s > \frac{3}{2}, \) and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = (u, \rho) \) to (2.1). Then problem (2.3) has a unique solution \( q \in C^1([0,T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with
\[
q_x(t, x) = \exp \left( \int_0^t u_x(s, q(s, x)) \, ds \right) > 0, \quad \forall (t, x) \in [0,T) \times \mathbb{R}.
\]
Besides,
\[
\rho(t, q(t, x)) q_x^2(t, x) = \rho_0(x), \quad \forall (t, x) \in [0,T) \times \mathbb{R}.
\tag{2.4}
\]

Lemma 2.4 [34] Suppose that \( \Psi(t) \) is twice continuously differential satisfying
\[
\begin{aligned}
\Psi''(t) &\geq C_0 \Psi'(t) \Psi(t), \quad t > 0, \; C_0 > 0, \\
\Psi(t) &> 0, \quad \Psi'(t) > 0.
\end{aligned}
\]
Then \( \Psi(t) \) blows up in finite time. Moreover the blow-up time can be estimated in terms of the initial datum as
\[
T \leq \max \left\{ \frac{2}{C_0 \Psi(0)}, \frac{\Psi(0)}{\Psi'(0)} \right\}.
\]
Next, we will give two important lemmas to the proof of our results in this section and the next section.

**Lemma 2.5** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, \ s > \frac{3}{2}, \) and let \( T \) be the maximal existence time of the solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (2.1) with the initial data \( z_0 \). Then we have

\[
e^{-t \int_0^t \frac{(cpp_x)(s,q(s,x))}{m(s,q(s,x))} ds} m(t,q(t,x))q_x^3(t,x) = m_0(x), \quad \forall (t,x) \in [0,T) \times \mathbb{R}. \tag{2.5}
\]

**Proof.** Applying Lemma 2.1 and a simple density argument, it suffices to consider the case \( s = 3 \). Differentiating the left-hand side of (2.5) with respect to \( t \), in view of the relation (2.3) and the first equation of the system (1.1), we obtain

\[
\partial \left( e^{-t \int_0^t \frac{(cpp_x)(s,q(s,x))}{m(s,q(s,x))} ds} m(t,q(t,x))q_x^3(t,x) \right)
\]

\[= e^{-t \int_0^t \frac{(cpp_x)(s,q(s,x))}{m(s,q(s,x))} ds} \frac{\partial t}{m(t,q(t,x))} \cdot m(t,q(t,x))q_x^3(t,x)\]

\[+ e^{-t \int_0^t \frac{(cpp_x)(s,q(s,x))}{m(s,q(s,x))} ds} \cdot (m_t + m_x u)q_x^3(t,x) + e^{-t \int_0^t \frac{(cpp_x)(s,q(s,x))}{m(s,q(s,x))} ds} \cdot m(t,q(t,x)) \cdot 3u_x q_x^2(t,x)\]

\[= e^{-t \int_0^t \frac{(cpp_x)(s,q(s,x))}{m(s,q(s,x))} ds} q_x^3(t,x)(cpp_x + m_t + m_x u + 3mu_x) = 0\]

This completes the proof of the lemma. \( \square \)

**Lemma 2.6** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, \ s > \frac{3}{2} \) and let \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) be the corresponding solution to (1.1) with the initial data \( z_0 \). If \( m_0 = (u_0 - u_{0,xx}) \in L^1 \), then

\[
\int_\mathbb{R} u(t,x)dx = \int_\mathbb{R} u_0(x)dx = \int_\mathbb{R} m_0(x)dx = \int_\mathbb{R} m(t,x)dx.
\]

**Proof.** As in the proof of Lemma 2.5 it suffices to prove the above lemma for \( s = 3 \). Since \( m_0 = (u_0 - u_{0,xx}) \in L^1 \) and \( u_0 = (1 - \partial_x^2)^{-1}m_0 = p * m_0 \), by Young's inequality we have

\[
\|u_0\|_{L^1} = \|p * m_0\|_{L^1} \leq \|p\|_{L^1} \|m_0\|_{L^1} \leq \|m_0\|_{L^1}.
\]

By the first equation in (2.1), we have

\[
u_t + uu_x = -\partial_x p * \left( \frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right).
\]

It follows that

\[
\frac{d}{dt} \int_\mathbb{R} u(t,x)dx = \int_\mathbb{R} u_t(t,x)dx = \int_\mathbb{R} \left( -uu_x - \partial_x p * \left( \frac{3}{2} u^2 + \frac{c}{2} \rho^2 \right) \right) dx = 0
\]

Thus \( \int_\mathbb{R} u(t,x)dx = \int_\mathbb{R} u_0(x)dx \). Moreover,

\[
\int_\mathbb{R} m(t,x)dx = \int_\mathbb{R} (u - u_{xx})dx = \int_\mathbb{R} u(t,x)dx = \int_\mathbb{R} u_0(x)dx \]

\[= \int_\mathbb{R} (u_0 - u_{0,xx})dx = \int_\mathbb{R} m_0(x)dx.
\]
This completes the proof of the lemma. □

Then, we will give two global existence results for DP2.

**Theorem 2.1** Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, \ s > \frac{3}{2} \) and let \( T \) be the maximal existence time of the solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (2.1) with the initial data \( z_0 \). If \( m_0 = (u_0 - u_{0,xx}) \in L^1 \) and \( m_0 = u_0 - u_{0,xx} \) does not change sign on \( \mathbb{R} \), then \( T = +\infty \). Moreover, \( \|u_x(t,\cdot)\|_{L^\infty} \leq \|m_0\|_{L^1} \).

**Proof.** Again we assume \( s = 3 \) to prove this theorem. Firstly, we assume that \( m_0 \geq 0 \) on \( \mathbb{R} \). Then, Lemma 2.5 implies \( m(t,x) \geq 0 \) for all \((t,x) \in [0,T) \times \mathbb{R} \). By \( u = (1 - \partial_t^2)^{-1} m = p \ast m \), we have \( u(t,x) \geq 0 \). By Lemma 2.6, we get

\[
-u_x(t,x) + \int_{-\infty}^{x} u(t,\xi) d\xi = \int_{-\infty}^{x} (u - u_{\xi})(t,\xi) d\xi = \int_{-\infty}^{x} m(t,\xi) d\xi \\
\leq \int_{-\infty}^{+\infty} m(t,x) dx = \int_{\mathbb{R}} m_0(x) dx = \|m_0\|_{L^1}.
\]

Thus, \( u_x(t,x) \geq -\|m_0\|_{L^1} \). It follows from Lemma 2.2 that \( T = +\infty \). On the other hand,

\[
u_x(t,x) - \int_{-\infty}^{x} u(t,\xi) d\xi = -\int_{-\infty}^{x} m(t,\xi) d\xi \leq 0.
\]

It follows that

\[
u_x(t,x) \leq \int_{-\infty}^{x} u(t,\xi) d\xi \leq \int_{\mathbb{R}} u(t,x) dx = \int_{\mathbb{R}} m_0(x) dx = \|m_0\|_{L^1}.
\]

Therefore, \( \|u_x(t,\cdot)\|_{L^\infty} \leq \|m_0\|_{L^1} \). In the case when \( m_0 \leq 0 \) on \( \mathbb{R} \), one can repeat the above proof to get the desired result. □

**Theorem 2.2** Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, \ s > \frac{3}{2} \) and let \( T \) be the maximal existence time of the corresponding solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to (2.1) with the initial data \( z_0 \). Assume \( c \leq 0 \) and there exists \( x_0 \in \mathbb{R} \) such that

\[
\begin{align*}
m_0(x) &= u_0(x) - u_{0,xx}(x) \leq 0 & \text{if } x \leq x_0, \\
m_0(x) &= u_0(x) - u_{0,xx}(x) \geq 0 & \text{if } x \geq x_0,
\end{align*}
\]

and \( m_0(x) \in L^1 \). Then the solution \( z \) exists globally in time.

**Proof.** We only assume \( s = 3 \) to prove the above theorem. Since \( q(t,\cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with \( q_x(t,x) > 0 \) for all \((t,x) \in [0,T) \times \mathbb{R} \), it follows from the assumptions of the theorem that

\[
\begin{align*}
m(t,x) &\leq 0 & \text{if } x \leq q(t,x_0), \\
m(t,x) &\geq 0 & \text{if } x \geq q(t,x_0),
\end{align*}
\]

(2.6)
and $m(t, q(t, x_0)) = 0$. Firstly, we claim that
\[
\int_{-\infty}^{Q(t, x_0)} u(t, x)dx \geq -\|m_0\|_{L^1} \quad \text{and} \quad \int_{q(t, x_0)}^{+\infty} u(t, x)dx \leq \|m_0\|_{L^1}.
\]
By the first equation in the system (2.2), we have $m_t = -4uu_x + 3u_xu_{xx} + uu_{xxx} - c\rho \rho_x$. Thus
\[
\frac{d}{dt} \int_{-\infty}^{Q(t, x_0)} m(t, x)dx = \int_{-\infty}^{Q(t, x_0)} m(t, x)dx + m(t, q(t, x_0))q_t(t, x_0)
\]
\[
= \int_{-\infty}^{Q(t, x_0)} m(t, x)dx = \int_{-\infty}^{Q(t, x_0)} (-4uu_x + 3u_xu_{xx} + uu_{xxx} - c\rho \rho_x)dx
\]
\[
= (-2u^2 + u_x^2 + uu_{xx} - \frac{c}{2}\rho^2)\big|_{q(t, x_0)}^{Q(t, x_0)}
\]
\[
= -2u^2(t, q(t, x_0)) + u_x^2(t, q(t, x_0)) + (uu_{xx})(t, q(t, x_0)) - \frac{c}{2}\rho^2(t, q(t, x_0))
\]
\[
\geq -u^2(t, q(t, x_0)) + u_x^2(t, q(t, x_0)), \quad (2.7)
\]
here we used $0 = m(t, q(t, x_0)) = u(t, q(t, x_0)) - uu_{xx}(t, q(t, x_0))$. Next, we will claim that $u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0)) \leq 0$. Since $u = p \ast m$,
\[
u(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^\xi m(t, \xi) d\xi + \frac{e^x}{2} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi
\]
and
\[
u_x(t, x) = \frac{e^{-x}}{2} \int_{-\infty}^{x} e^\xi m(t, \xi) d\xi + \frac{e^x}{2} \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi.
\]
From these two equations, we obtain
\[
u(t, x) + u_x(t, x) = e^x \int_{x}^{+\infty} e^{-\xi} m(t, \xi) d\xi, \quad (2.8)
\]
and
\[
u(t, x) - u_x(t, x) = e^{-x} \int_{-\infty}^{x} e^\xi m(t, \xi) d\xi. \quad (2.9)
\]
From (2.8)-(2.9), we deduce that
\[
u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0)) = \int_{q(t, x_0)}^{+\infty} \nu(t, x)dx \int_{-\infty}^{q(t, x_0)} \nu^2(t, x)dx \leq 0.
\]
Thus
\[
\frac{d}{dt} \int_{-\infty}^{Q(t, x_0)} m(t, x)dx \geq 0.
\]
It then follows that
\[
\int_{-\infty}^{Q(t, x_0)} m(t, x)dx \geq \int_{-\infty}^{x_0} m_0(x)dx = -\int_{-\infty}^{x_0} |m_0(x)|dx \geq -\int_{-\infty}^{+\infty} |m_0(x)|dx = -\|m_0\|_{L^1}.
\]
Note that
\[ \int_{-\infty}^{q(t,x_0)} u(t,x)dx = \int_{-\infty}^{q(t,x_0)} m(t,x)dx + u_x(t,q(t,x_0)). \]
Since
\[ u_x(t,q(t,x_0)) = -\frac{e^{-q(t,x_0)}}{2} \int_{-\infty}^{q(t,x_0)} e^\xi m(t,\xi)d\xi + \frac{e^{q(t,x_0)}}{2} \int_{q(t,x_0)}^{+\infty} e^{-\xi} m(t,\xi)d\xi \geq 0, \]
it follows that
\[ \int_{-\infty}^{q(t,x_0)} u(t,x)dx \geq \int_{-\infty}^{q(t,x_0)} m(t,x)dx \geq -\|m_0\|_{L^1}. \]
A similar argument produces
\[ \int_{-\infty}^{+\infty} u(t,x)dx \leq \|m_0\|_{L^1}. \]
If \( x \leq q(t,x_0) \), then \( m(t,x) \leq 0 \) and \( u(t,x) = (p \ast m)(t,x) \leq 0 \). In view of
\[ u_x(t,x) - \int_{-\infty}^{x} u(t,\xi)d\xi = -\int_{-\infty}^{x} m(t,\xi)d\xi \geq 0, \]
we have
\[ u_x(t,x) \geq \int_{-\infty}^{x} u(t,\xi)d\xi \geq \int_{-\infty}^{q(t,x_0)} u(t,x)dx \geq -\|m_0\|_{L^1}. \]
If \( x \geq q(t,x_0) \), then \( m(t,x) \geq 0 \) and \( u(t,x) = (p \ast m)(t,x) \geq 0 \). In view of
\[ u_x(t,x) + \int_{x}^{+\infty} u(t,\xi)d\xi = \int_{x}^{+\infty} m(t,\xi)d\xi \geq 0, \]
we obtain
\[ u_x(t,x) \geq -\int_{x}^{+\infty} u(t,\xi)d\xi \geq -\int_{q(t,x_0)}^{+\infty} u(t,x)dx \geq -\|m_0\|_{L^1}. \]
Therefore, \( u_x(t,x) \geq -\|m_0\|_{L^1} \) for all \( (t,x) \in [0,T) \times \mathbb{R} \). Lemma 2.2 implies \( T = +\infty \). □

3 Blow-up phenomena of DP2

In this section, we will prove that there exists smooth initial data such that the corresponding solution to the system (1.1) with \( c \geq 0 \) does not exit globally in time.

**Theorem 3.1** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, \ s > \frac{3}{2}, \) and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (1.1). Assume \( c \geq 0 \) and there exists \( x_0 \in \mathbb{R} \) such that \( m_0(x_0) = \rho_0(x_0) = 0 \),
\[
\begin{cases}
  m_0(x) = u_0(x) - u_{0,xx}(x) > 0 & \text{if } x < x_0, \\
  m_0(x) = u_0(x) - u_{0,xx}(x) < 0 & \text{if } x > x_0
\end{cases}
\]
Then, the solution $z$ blows up in finite time.

**Proof.** The technique used here is inspired from [23]. As in the proof of Lemma 2.5 it suffices to prove the above theorem for $s = 3$.

Since $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with $q_x(t, x) > 0$ for all $(t, x) \in [0, T) \times \mathbb{R}$, it follows from the assumptions of the theorem that

$$\begin{cases} 
m(t, x) > 0 & \text{if } x < q(t, x_0), \\
m(t, x) < 0 & \text{if } x > q(t, x_0) 
\end{cases}$$

and $m(t, q(t, x_0)) = 0$. Moreover, by (2.4) and the condition $\rho_0(x_0) = 0$, we get $\rho(t, q(t, x_0)) = 0$. Define

$$f(t, x) = e^{-x} \int_{-\infty}^{t} e^{\xi} m(t, \xi) d\xi, \quad (t, x) \in [0, T) \times \mathbb{R}$$

and

$$g(t, x) = e^{x} \int_{t}^{+\infty} e^{-\xi} m(t, \xi) d\xi, \quad (t, x) \in [0, T) \times \mathbb{R}.$$  

Take

$$M(t) = f(t, q(t, x_0)) = e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{x} m(t, x) dx, \quad t \in [0, T)$$

and

$$N(t) = g(t, q(t, x_0)) = e^{q(t,x_0)} \int_{q(t,x_0)}^{+\infty} e^{-x} m(t, x) dx, \quad t \in [0, T).$$

By (3.1), we obtain $M(t) > 0$, $N(t) < 0$ and $M(t)N(t) < 0$. Moreover, by (2.8)-(2.9), we have

$$M(t) = u(t, q(t, x_0)) - u_x(t, q(t, x_0)), \quad N(t) = u(t, q(t, x_0)) + u_x(t, q(t, x_0)).$$

Thus

$$u^2(t, q(t, x_0)) - u_x^2(t, q(t, x_0)) = M(t)N(t) < 0.$$  

Next, we will prove that

$$\frac{dM(t)}{dt} \geq -\frac{1}{2} M(t)N(t) > 0 \quad \text{and} \quad \frac{dN(t)}{dt} \leq \frac{1}{2} M(t)N(t) < 0.$$  

(3.4)

By (2.3) and the first equation in (1.1), we have

$$\frac{dM(t)}{dt} = -q(t, x_0)e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{x} m(t, x) dx + e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{x} m_t(t, x) dx$$

$$= -u(t, q(t, x_0))e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{x} m(t, x) dx$$

$$+ e^{-q(t,x_0)} \int_{q(t,x_0)}^{+\infty} e^{x} (-2mu_x - (mu)_x - c^2 \rho_x) dx. \quad (3.5)$$

Note that $(e^x(u - u_x))' = e^x(u - u_{xx}) = e^x m$. We have

$$e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^{x} m(t, x) dx = e^{-q(t,x_0)} \left( e^x(u - u_x) \right)_{-\infty}^{q(t,x_0)}$$

$$= u(t, q(t, x_0)) - u_x(t, q(t, x_0)). \quad (3.6)$$

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Note that \((e^{x}(uu_{x} - u_{x}^{2}))') = e^{x}(uu_{x} + uu_{xx} - 2u_{x}u_{xx})\). We obtain
\[
\int_{-\infty}^{q(t,x_{0})} e^{x}(-2mu_{x} - (mu)_{x} - c\rho_{x})\, dx \\
= -2 \int_{-\infty}^{q(t,x_{0})} e^{x}(u - u_{xx})u_{x}\, dx + \int_{-\infty}^{q(t,x_{0})} e^{x}mu\, dx - c \int_{-\infty}^{q(t,x_{0})} e^{x}\frac{\rho^{2}}{2}\, dx \\
= -2 \int_{-\infty}^{q(t,x_{0})} e^{x}uu_{x}\, dx + \int_{-\infty}^{q(t,x_{0})} e^{x}(u^{2} - uu_{xx})\, dx + \frac{c}{2} \int_{-\infty}^{q(t,x_{0})} e^{x}\rho^{2}\, dx \\
= - \int_{-\infty}^{q(t,x_{0})} e^{x}uu_{x}\, dx - \int_{-\infty}^{q(t,x_{0})} e^{x}(uu_{x} + uu_{xx} - 2u_{x}u_{xx})\, dx \\
+ \int_{-\infty}^{q(t,x_{0})} e^{x}u^{2}\, dx + \frac{c}{2} \int_{-\infty}^{q(t,x_{0})} e^{x}\rho^{2}\, dx \\
= \frac{3}{2} \int_{-\infty}^{q(t,x_{0})} e^{x}u^{2}\, dx - \frac{1}{2} e^{q(t,x_{0})}u^{2}(t, q(t, x_{0})) - e^{q(t,x_{0})}u(t, q(t, x_{0}))u_{x}(t, q(t, x_{0})) \\
+ e^{q(t,x_{0})}u_{x}^{2}(t, q(t, x_{0})) + \frac{c}{2} \int_{-\infty}^{q(t,x_{0})} e^{x}\rho^{2}\, dx \tag{3.7}
\]
Substituting (3.6)-(3.7) into (3.5), we have
\[
\frac{dM(t)}{dt} \\
= -u(t, q(t, x_{0}))(u(t, q(t, x_{0})) - u_{x}(t, q(t, x_{0}))) + \frac{3}{2} e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}u^{2}\, dx \\
- \frac{1}{2} e^{-q(t,x_{0})}u^{2}(t, q(t, x_{0})) - u(t, q(t, x_{0}))u_{x}(t, q(t, x_{0})) + u_{x}^{2}(t, q(t, x_{0})) + \frac{c}{2} e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}\rho^{2}\, dx \\
= u_{x}^{2}(t, q(t, x_{0})) - \frac{3}{2} e^{-q(t,x_{0})}u^{2}(t, q(t, x_{0})) + \frac{1}{2} e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}(u^{2} - u_{x}^{2})\, dx \\
+ e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}(u^{2} + \frac{1}{2}u_{x}^{2})\, dx + \frac{c}{2} e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}\rho^{2}\, dx \\
\geq u_{x}^{2}(t, q(t, x_{0})) - u^{2}(t, q(t, x_{0})) + \frac{1}{2} e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}(u^{2} - u_{x}^{2})\, dx \\
= -M(t)N(t) + \frac{1}{2} e^{-q(t,x_{0})} \int_{-\infty}^{q(t,x_{0})} e^{x}(u^{2} - u_{x}^{2})\, dx \\
\geq -M(t)N(t) + \frac{1}{2} M(t)N(t) = -\frac{1}{2} M(t)N(t) > 0,
\]
here we used
\[
e^{-x} \int_{-\infty}^{x} e^{y}(u^{2} + \frac{1}{2}u_{y}^{2})\, dy \geq \frac{1}{2} u^{2}
\]
and
\[ e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^x (u^2(t,x) - u_x^2(t,x)) \, dx \geq M(t)N(t) \quad \text{(cf. (4.17) in [23])}. \]  
(3.8)

By the assumptions of the theorem and the following fact
\[(e^{-x}(u + u_x))' = e^{-x}(-u + u_{xx}) = -e^{-x}m, \]
\[(e^{-x}(uu_x + u_x^2))' = e^{-x}(-uu_x + uu_{xx} + 2u_xu_{xx}), \]
\[e^x \int_{x}^{+\infty} e^{-y}(u^2 + \frac{1}{2}u_y^2) \, dy \geq \frac{1}{2}u^2\]

and
\[e^{q(t,x_0)} \int_{q(t,x_0)}^{+\infty} e^{-x}(u^2(t,x) - u_x^2(t,x)) \, dx \geq M(t)N(t) \quad ((4.21) \text{in [23]}). \]
(3.9)

A similar proof implies that
\[\frac{dN(t)}{dt} \leq \frac{1}{2}M(t)N(t) < 0.\]

By (2.3) and the first equation in (2.1), we have
\[\frac{du_x(t,q(t,x_0))}{dt} = u_{tx}(t,q(t,x_0)) + uu_{xx}(t,q(t,x_0)) \]
\[= -u_x^2(t,q(t,x_0)) + \frac{3}{2}u^2(t,q(t,x_0)) - p * (\frac{3}{2}u^2 + \frac{c}{2}p^2)(t,q(t,x_0)) \]
\[= -u_x^2(t,q(t,x_0)) + \frac{3}{2}u^2(t,q(t,x_0)) - p * (u^2 + \frac{1}{2}u_x^2)(t,q(t,x_0)) \]
\[+ \frac{1}{2}p * (u_x^2 - u^2)(t,q(t,x_0)) - \frac{c}{2}p * p^2(t,q(t,x_0)) \]
\[\leq -u_x^2(t,q(t,x_0)) + u^2(t,q(t,x_0)) + \frac{1}{2}p * (u_x^2 - u^2)(t,q(t,x_0)),\]

here we used \(p * (u^2 + \frac{1}{2}u_x^2) \geq \frac{1}{2}u^2\). By \(p(x) \leq \frac{1}{2}e^{-|x|}\) and (3.8)-(3.9), we obtain
\[p * (u_x^2 - u^2)(t,q(t,x_0)) \]
\[= \int_{-\infty}^{+\infty} \frac{1}{2}e^{-q(t,x_0) - x}(u_x^2(t,x) - u^2(t,x)) \, dx \]
\[= \frac{1}{2}e^{-q(t,x_0)} \int_{-\infty}^{q(t,x_0)} e^x (u_x^2(t,x) - u^2(t,x)) \, dx \]
\[+ \frac{1}{2}e^{q(t,x_0)} \int_{q(t,x_0)}^{+\infty} e^{-x}(u_x^2(t,x) - u^2(t,x)) \, dx \]
\[\leq -M(t)N(t) = u_x^2(t,q(t,x_0)) - u^2(t,q(t,x_0)).\]

Thus
\[\frac{du_x(t,q(t,x_0))}{dt} \leq -\frac{1}{2}(u_x^2(t,q(t,x_0)) - u^2(t,q(t,x_0))),\]

it follows that
\[\quad - u_x(t,q(t,x_0)) \geq \frac{1}{2} \int_{\tilde{t}}^{\tilde{t}} (u_x^2(s,q(s,x_0)) - u^2(s,q(s,x_0))) \, ds - u_0(x). \quad (3.10)\]
Combining (3.4) with (3.10), note that \(u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0)) = -M(t)N(t) > 0\), we get

\[
\frac{d}{dt}(u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0))) = \frac{d}{dt}(-M(t)N(t)) = -M'(t)N(t) - M(t)N'(t) \geq \frac{1}{2}M(t)N^2(t) - \frac{1}{2}M^2(t)N(t) = \frac{1}{2}M(t)N(t)(N(t) - M(t)) = -u_x(t, q(t, x_0))(u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0))) \geq \frac{1}{2}(u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0))) \left(\int_0^t (u_x^2(s, q(s, x_0)) - u^2(s, q(s, x_0)))ds - 2u_0'(x)\right).
\]

Let \(\Psi(t) = \int_0^t (u_x^2(s, q(s, x_0)) - u^2(s, q(s, x_0)))ds - 2u_0'(x_0)\), then we obtain \(\Psi''(t) \geq \frac{1}{2}\Psi'(t)\Psi(t)\). Moreover, \(\Psi'(t) = u_x^2(t, q(t, x_0)) - u^2(t, q(t, x_0)) = -M(t)N(t) > 0\), which implies \(\Psi(t)\) is strictly increasing, thus \(\Psi(t) > \Psi(0) = -2u_0'(x_0) = M(0) - N(0) > 0\). By Lemma 2.4, we get \(\Psi(t)\) blows up in finite time, which implies \(\limsup_{t \to T} ||u_x(t, \cdot)||_{L^\infty} = +\infty\). This completes the proof of the theorem by use of Sobolev’s imbedding theorem.

**Remark 3.1** By comparing the assumptions of Theorem 2.2 and Theorem 3.1, we obtain that the shape of \(m_0(x) = u_0(x) - u_{0,xx}(x)\) determines the character of the corresponding solution to some extent. Moreover, our obtained results for the system are new and sharp.

### 4 Global existence for the two-component b-family system

In this section, we will give some new global existence results for the two-component b-family system with \(b \in \mathbb{R}\). Since the research method of this problem is the same as that in Section 2, we only give the results and the key points in the proof.

With \(p(x) := \frac{1}{2}e^{-|x|}, x \in \mathbb{R}\), we can rewrite the system (1.2) as follows:

\[
\begin{cases}
  u_t - uu_x = \partial_x p \ast \left(\frac{b_1}{2}u^2 + \frac{3}{2}k_1 u_x^2 + \frac{b_2}{2} \rho^2\right), & t > 0, x \in \mathbb{R}, \\
  \rho_t - k_3 u \rho_x = k_3 u_x \rho, & t > 0, x \in \mathbb{R}, \\
  u(0, x) = u_0(x), & x \in \mathbb{R}, \\
  \rho(0, x) = \rho_0(x), & x \in \mathbb{R},
\end{cases}
\tag{4.1}
\]

Next, we give some important lemmas.

**Lemma 4.1** [22] Given \(z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2\), there exist a maximal \(T = T(\|z_0\|_{H^s \times H^{s-1}}) > 0\), and a unique solution \(z = \begin{pmatrix} u \\ \rho \end{pmatrix}\) to (4.1) such that

\[
z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}).
\]
Moreover, the solution depends continuously on the initial data, i.e. the mapping
\[ z_0 \rightarrow z(\cdot, z_0) : H^s \times H^{s-1} \rightarrow C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}) \]
is continuous.

**Lemma 4.2** [22] Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1 \), and let \( T \) be the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (4.1). Then the solution \( z \) blows up in finite time if and only if
\[ \limsup_{t \to T} \| u_x(t, \cdot) \|_{L^\infty} = +\infty. \]

Consider now the following initial value problem
\[ \left\{ \begin{array}{l}
q_t = u(t, -k_3 q), \quad t \in [0, T), \\
q(0, x) = x, \quad x \in \mathbb{R},
\end{array} \right. \] (4.2)

**Lemma 4.3** [22] Let \( u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}), s \geq 2 \). Then Eq.(4.2) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with
\[ q_x(t, x) = \exp \left( \int_0^t -k_3 u_x(s, -k_3 q(s, x)) ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \]

Besides,
\[ \rho(t, -k_3 q(t, x))q_x(t, x) = \rho_0(-k_3 x), \quad \forall \ (t, x) \in [0, T) \times \mathbb{R}. \] (4.3)

Similar to the proof of Lemmas 2.5-2.6, we have the following two important lemmas.

**Lemma 4.4** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2 \), and let \( T \) be the maximal existence time of the solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (4.1) with the initial data \( z_0 \). Then we have
\[ e^{\int_0^t (-k_3 \rho_0 x(q(s, x)) \rho_0(x) ds) m(t, q(t, x))} m(t, x) = m_0(x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}. \] (4.4)

**Lemma 4.5** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2 \) and \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) is the corresponding solution to (4.1) with the initial data \( z_0 \). If \( m_0 = (u_0 - u_{0, xx}) \in L^1 \), then
\[ \int \mathbb{R} u(t, x) dx = \int \mathbb{R} u_0(x) dx = \int \mathbb{R} m_0(x) dx = \int \mathbb{R} m(t, x) dx. \]

Similar to the proof of Theorem 2.1, we obtain the first global existence result for any \( b \in \mathbb{R} \).
Theorem 4.1 Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1 \), \( b \in \mathbb{R} \) and let \( T \) be the maximal existence time of the solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (4.1) with the initial data \( z_0 \). If \( m_0 = (u_0 - u_{0,xx}) \in L^1 \) and \( m_0 = u_0 - u_{0,xx} \) does not change sign in \( \mathbb{R} \), then \( T = +\infty \). Moreover, \( \| u_x(t, \cdot) \|_{L^\infty} \leq \| m_0 \|_{L^1} \).

Theorem 4.2 Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1 \), \( k_1 \leq 1 \), \( k_2 \geq 0 \) and let \( T \) be the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to (4.1) with the initial data \( z_0 \). If there exists \( x_0 \in \mathbb{R} \) such that

\[
\begin{cases}
  m_0(x) = u_0(x) - u_{0,xx}(x) \leq 0 & \text{if } x \leq x_0, \\
  m_0(x) = u_0(x) - u_{0,xx}(x) \geq 0 & \text{if } x \geq x_0
\end{cases}
\]

and \( m_0(x) \in L^1 \). Then the solution \( z \) exists globally in time.

Proof. Similar to the proof of Theorem 2.2, by \( k_1 \leq 1 \) and \( k_2 \geq 0 \) we have

\[
\int_{-\infty}^{q(t,x_0)} u(t,x)dx \geq \int_{-\infty}^{q(t,x_0)} m(t,x)dx \geq -\| m_0 \|_{L^1}
\]

and

\[
\int_{q(t,x_0)}^{+\infty} u(t,x)dx \leq \int_{q(t,x_0)}^{+\infty} m(t,x)dx \leq \| m_0 \|_{L^1}.
\]

If \( x \leq q(t,x_0) \), then \( m(t,x) \leq 0 \) and \( u(t,x) = (p \ast m)(t,x) \leq 0 \). In view of

\[
u_x(t,x) - \int_{-\infty}^{x} u(t,\xi)d\xi = -\int_{-\infty}^{x} m(t,\xi)d\xi \geq 0,
\]

we have

\[
u_x(t,x) \geq \int_{-\infty}^{x} u(t,\xi)d\xi \geq \int_{-\infty}^{q(t,x_0)} u(t,x)dx \geq -\| m_0 \|_{L^1}.
\]

On the other hand

\[
-u_x(t,x) \geq -u_x(t,x) + \int_{-\infty}^{x} u(t,\xi)d\xi = \int_{-\infty}^{x} m(t,\xi)d\xi \\
\geq \int_{-\infty}^{q(t,x_0)} m(t,x)dx \geq -\| m_0 \|_{L^1}.
\]

Thus \( \| u_x(t, \cdot) \|_{L^\infty} \leq \| m_0 \|_{L^1} \).

If \( x \geq q(t,x_0) \), then \( m(t,x) \geq 0 \) and \( u(t,x) = (p \ast m)(t,x) \geq 0 \). In view of

\[
u_x(t,x) + \int_{x}^{+\infty} u(t,\xi)d\xi = \int_{x}^{+\infty} m(t,\xi)d\xi \geq 0,
\]

we obtain

\[
u_x(t,x) \geq -\int_{x}^{+\infty} u(t,\xi)d\xi \geq -\int_{q(t,x_0)}^{+\infty} u(t,x)dx \geq -\| m_0 \|_{L^1}.
\]
On the other hand

\[-u_x(t, x) \geq -u_x(t, x) - \int_x^{+\infty} u(t, \xi) d\xi = -\int_x^{+\infty} m(t, \xi) d\xi \geq -\int_{q(t, x_0)}^{+\infty} m(t, x) dx \geq -\|m_0\|_{L^1}.
\]

Thus \(\|u_x(t, \cdot)\|_{L^\infty} \leq \|m_0\|_{L^1}\). It follows that \(\|u_x(t, \cdot)\|_{L^\infty} \leq \|m_0\|_{L^1}\) for all \((t, x) \in [0, T) \times \mathbb{R}\). Lemma 4.2 implies \(T = +\infty\).

\[\square\]

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