A SELBERG INTEGRAL TYPE FORMULA FOR AN $\mathfrak{sl}_2$
ONE-DIMENSIONAL SPACE OF CONFORMAL BLOCKS

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Abstract. For distinct complex numbers $z_1, \ldots, z_{2N}$, we give a polynomial $P(y_1, \ldots, y_{2N})$ in the variables $y_1, \ldots, y_{2N}$, which is homogeneous of degree $N$, linear with respect to each variable, $\mathfrak{sl}_2$-invariant with respect to a natural $\mathfrak{sl}_2$-action, and is of order $N - 1$ at $(y_1, \ldots, y_{2N}) = (z_1, \ldots, z_{2N})$.

We give also a Selberg integral type formula for the associated one-dimensional space of conformal blocks.

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1. Introduction

According to a general principle in [MV], if a KZ-type equation has a one-dimensional space of solutions, then the hypergeometric integrals representing the solutions can be calculated explicitly, see demonstrations of that principle in [TV], [FSV], [W]. In this note we give another example of that type.

We consider the bundle of the $\mathfrak{sl}_2$ conformal blocks at level one on the Riemann sphere. That bundle is of rank one. Our first result is a formula for a generator of a fiber of that bundle, see Theorem 3.2. Namely, for distinct complex numbers $z = (z_1, \ldots, z_{2N})$, we give a polynomial $P(y_1, \ldots, y_{2N})$ in the variables $y = (y_1, \ldots, y_{2N})$, which is homogeneous of degree $N$, linear with respect to each variable, $\mathfrak{sl}_2$-invariant with respect to a natural $\mathfrak{sl}_2$-action, and is of order $N - 1$ at $y = z$.

The conformal block bundle has a KZ connection. The flat sections of the KZ connection have representations in terms of multidimensional hypergeometric integrals, see [SV], [FSV1], [FSV2]. The formula of Theorem 3.2 for a generator of a conformal block space allows us to calculate those integrals explicitly, see Theorem 5.1. The result is a Selberg integral type formula (5.5).

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2. Spaces of Conformal Blocks

2.1. Conformal blocks. Consider the complex Lie algebra $\mathfrak{sl}_2$ with generators $e, f, h$ and relations $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. For a nonnegative integer $m$, denote by $V_m$ the irreducible $\mathfrak{sl}_2$-module with highest weight $m$.

Let $\ell, m_1, \ldots, m_n, N$ be given nonnegative integers such that
\begin{equation}
\ell > 0, \quad 0 \leq m_1, \ldots, m_n, m_1 + \cdots + m_n - 2N \leq \ell.
\end{equation}

Denote $p = \ell + 1 + 2N - m_1 - \cdots - m_n$, $V = V_{m_1} \otimes \cdots \otimes V_{m_n}$.

For each $a = 1, \ldots, n$, denote by $e_a : V \to V$ the linear operator acting as $e$ on the $a$-th factor and as the identity on all the other factors.

Let $z = (z_1, \ldots, z_n)$ be a collection of distinct complex numbers. Denote
\begin{equation}
W(z) = \{ v \in V \mid hv = (\sum_{a=1}^n m_a - 2N)v, ev = 0, (\sum_{a=1}^n z_a e_a)^p v = 0 \}.
\end{equation}

The vector space $W(z)$ is called the space of conformal blocks at level $\ell$, see [FSV1], [FSV2].

**Remark.** This definition is nonstandard. Usually the space of conformal blocks is defined if one has $n$ distinct points on a Riemann surface and $n$ irreducible representations of an affine Lie algebra, see [KL]. If the Riemann surface is the Riemann sphere, then one can describe the space of conformal blocks in terms of finite dimensional representations of the corresponding finite dimensional Lie algebra. That description is one of two main results of [FSV1] and [FSV2]. We take that description as our definition.

2.2. KZ connection. Denote
\[ X_n = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_a \neq z_b \text{ for all } a \neq b \}. \]

The trivial vector bundle $\eta : V \times X_n \to X_n$ has a KZ connection,
\[ \frac{\partial}{\partial z_a} - \frac{1}{\ell + 2} \sum_{b \neq a} \Omega^{(a,b)} z_a - z_b, \quad a = 1, \ldots, n, \]
where $\Omega = \frac{1}{2} h \otimes h + e \otimes f + f \otimes e$ and $\Omega^{(a,b)} : V \to V$ is the linear operator acting as $\Omega$ on the $a$-th and $b$-th factors and as the identity on all the other factors.

Consider the subbundle of conformal blocks with fiber $W(z) \subset V$. It is well known that this subbundle is invariant with respect to the KZ connection [KZ].

3. Conformal blocks at level one

In the rest of the paper we assume that \begin{equation} \ell = 1, \quad n = 2N, \quad m_1 = \cdots = m_{2N} = 1. \end{equation}

These numbers satisfy conditions (2.1). Then
\[ W(z_1, \ldots, z_{2N}) = \{ v \in (V_1)^{\otimes 2N} \mid hv = 0, ev = 0, (\sum_{a=1}^n z_a e_a)^2 v = 0 \}. \]
Theorem 3.1. Under assumptions (3.1), the space of conformal blocks is one-dimensional.

Proof. The fusion ring of $\mathfrak{sl}_2$ at level $\ell = 1$ is a free $\mathbb{Z}$-module with generators $[V_0], [V_1]$ and commutative associative multiplication:

$$[V_0] \cdot [V_0] = [V_0], \quad [V_0] \cdot [V_1] = [V_1], \quad [V_1] \cdot [V_1] = [V_0].$$

The dimension of $W(z_1, \ldots, z_{2N})$ is the coefficients of $[V_0]$ in the decomposition of $[V_1]^{2N}$ in terms of the generators. Clearly $[V_1]^{2N} = [V_0].$

Let us realize the tensor product $(V_1)^{\otimes 2N}$ as the vector space of polynomials

$$p(y) = p(y_1, \ldots, y_{2N})$$

of degree not greater than one with respect to each variable $y_1, \ldots, y_{2N}$. The Lie algebra $\mathfrak{sl}_2$ acts on this space in the standard way, in particular, $\varepsilon$ acts as $\sum_{a=1}^{2N} \partial/\partial y_a$.

By [R], Theorem 4.3 (cf. [LV], Lemma 1.3), under assumptions (3.1), the subspace $W(z) \subset (V_1)^{\otimes 2N}$ consists of polynomials $p(y)$, which are $\mathfrak{sl}_2$-invariant, homogeneous of degree $N$ and of order at least $N - 1$ at $y = z$. (The formulation of Theorem 4.3 in [R] has a misprint: $\phi$ should vanish to order $J - k$ rather than $J - k - 1$).

Introduce a polynomial $P(y; z)$ in the variables $y$ depending on the parameters $z$,

$$(3.2) \quad P(y; z) = \det_{1 \leq a, b \leq N} \frac{y_a - y_{N+b}}{z_a - z_{N+b}} = \sum_{\sigma \in S_N} (-1)^\sigma \prod_{a=1}^{N} \frac{y_{\sigma(a)} - y_{N+a}}{z_{\sigma(a)} - z_{N+a}}. $$

Theorem 3.2. For a fixed $z$, the polynomial $P(y; z)$ in the variables $y$ is $\mathfrak{sl}_2$-invariant, is homogeneous of degree $N$, has degree not greater than one with respect to each variable $y_1, \ldots, y_{2N}$ and has order $N - 1$ at $y = z$; therefore, this polynomial is a generator of the space $W(z)$ of conformal blocks.

Proof. Each difference $y_i - y_j$ is $\mathfrak{sl}_2$-invariant. Hence $P(y, z)$ is $\mathfrak{sl}_2$-invariant. We have $P(z; z) = 0$ since it is the determinant of a matrix with all entries equal to one. The fact that $P(y; z)$ has zero of order $N - 1$ at $y = z$ is proved similarly. \hfill $\Box$

Denote

$$(3.3) \quad A(z) = \prod_{1 \leq a \leq N < b \leq 2N} (z_b - z_a)^{1/2} \prod_{1 \leq a < b \leq N} (z_b - z_a)^{-1/2} \prod_{N < a < b \leq 2N} (z_b - z_a)^{-1/2}. $$

Theorem 3.3. Consider the trivial bundle $\eta : (V_1)^{\otimes 2N} \times X_{2N} \to X_{2N}$ and its section

$$(3.4) \quad s : z \mapsto A(z)P(y; z) \in W(z).$$

Then this is a flat section of the KZ connection for $\ell = 1$.

The theorem can be proved by a direct calculation. A different proof see in Section 5.1.

Remark. The flat section $s$ is multivalued with the monodromy equal to -1 around each hyperplane $z_a = z_b$. Hence, the KZ connection on the $\mathfrak{sl}_2$ conformal block bundle at level one is unitarizable. The unitarity of the KZ connection on the $\mathfrak{sl}_2$ conformal block bundle at any level is proved in [R], cf [LV].
4. An integral representation for conformal blocks

4.1. The master and weight functions. Introduce a scalar function
\[
\Phi(t; z) = \Phi(t_1, \ldots, t_{2N}; z_1, \ldots, z_{2N})
\]
\[
= \prod_{1 \leq a < b \leq 2N} (z_a - z_b)^{1/6} \prod_{1 \leq i < j \leq N} (t_j - t_i)^{2/3} \prod_{i=1}^{2N} \prod_{a=1}^{N} (t_i - z_a)^{-1/3}
\]
and a \((V_1)^{\otimes 2N}[0]\)-valued rational function,
\[
\omega(t; z) = \sum_{1 \leq a_1 < \cdots < a_N \leq 2N} \sum_{\sigma \in S_N} \prod_{i=1}^{N} \frac{y_{a_i}}{t_{\sigma_i} - z_{a_i}}.
\]
The functions \(\Phi\) and \(\omega\) are called the master and weight functions, respectively, see [SV].

4.2. The local system. Denote
\[
Y_{2N} = \{(t; z) = (t_1, \ldots, t_{2N}; z_1, \ldots, z_{2N}) \in \mathbb{C}^{3N} \mid z_a \neq z_b \text{ for all } a < b; t_i \neq t_j \text{ for all } i < j; z_a \neq t_i \text{ for all } a, i\}.
\]
The master function \(\Phi\) defines on \(Y_{2N}\) a one-dimensional local system \(\mathcal{L}\). The horizontal sections of \(\mathcal{L}\) are generated by the univalued branches of the multivalued holomorphic function \(\Phi\).

The projection \(\tau : Y_{2N} \to X_{2N}, (t; z) \mapsto z\) is topologically trivial. Let \(\tau_N\) be the associated homological vector bundle with fiber \(H_N(\tau^{-1}(z), \mathcal{L}|_{\tau^{-1}(z)})\). The vector bundle \(\tau_N\) has a canonical Gauss-Manin connection.

4.3. The integral representation.

Theorem 4.1 ([SV], [FSV1], [FSV2]). Let \(\gamma\) be a horizontal section of the homological bundle \(\tau_N\). Then
\[
(4.1) \quad I_{\gamma}(z) = \int_{\gamma(z)} \Phi(t; z) \omega(t; z) dt_1 \wedge \cdots \wedge dt_N
\]
is a horizontal section of the KZ connection. Moreover, this section takes values in the conformal block spaces.

Corollary 4.2. For any horizontal section \(\gamma\), there exists \(c_\gamma \in \mathbb{C}\) such that \(I_{\gamma}(z) = c_\gamma s(z)\), where \(s\) is the horizontal section defined in (3.4).

5. An example of a horizontal family \(\gamma\)

5.1. Euler’s beta function. Let \(z_a < z_b\) be real numbers and \(\alpha, \beta\) positive numbers. For \(t \in (z_a, z_b)\), we fix \(\arg(t - z_a) = 0, \arg(t - z_b) = \pi, \arg(z_b - z_a) = 0\). Then
\[
(5.1) \quad \int_{z_a}^{z_b} (t - z_a)^{\alpha-1}(t - z_b)^{\beta-1} dt = -e^{\pi i \beta} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} (z_b - z_a)^{\alpha + \beta - 1}.
\]
The right hand side of (5.1) is a holomorphic function of \(\alpha, \beta \in \mathbb{C} - \{0, -1, \ldots\}\). We define the integral in the left hand side of (5.1) for \(\alpha, \beta \in \mathbb{C} - \{0, -1, \ldots\}\) by analytic continuation.
Fix $\alpha, \beta \in \mathbb{C}$. The function $(t-z_a)^{\alpha-1}(t-z_b)^{\beta-1}$ defines on $\mathbb{C} - \{z_a, z_b\}$ a one-dimensional local system $\ell$, whose sections are generated by the univalued branches of that function. It is easy to see that if $\alpha, \beta, \alpha + \beta \in \mathbb{C} - \mathbb{Z}$, then there is a unique cycle $\gamma(z_a, z_b; \alpha, \beta) \in H_1(\mathbb{C} - \{z_a, z_b\}, \ell)$ such that

$$
\int_{\gamma(z_a, z_b; \alpha, \beta)} (t-z_a)^{\alpha-1}(t-z_b)^{\beta-1} dt = \int_{z_a}^{z_b} (t-z_a)^{\alpha-1}(t-z_b)^{\beta-1} dt.
$$

This cycle will be called a Pochhammer cycle.

5.2. An example of a horizontal section $\gamma$ of the homological bundle $\tau_N$. Assume that $z \in \mathbb{R}^{2N}$ and

$$
(5.2) \quad z_1 < z_{N+1} < z_2 < z_{N+2} < \cdots < z_N < z_{2N}.
$$

First, we define $\gamma(z)$ as an oriented product of intervals,

$$
(5.3) \quad \gamma(z) = \{(t; z) \mid t_a \in (z_a, z_{N+a}), \, a = 1, \ldots, N\}
$$

with the standard orientation of each of the intervals $(z_a, z_{N+a})$.

To define the integral in (4.1), we fix on $\gamma(z)$ the arguments of all factors of the master function $\Phi$ as follows. We set $\text{arg}(t_j - t_i) = 0$ for $j > i$. We set $\text{arg}(z_a - z_b) = 0$ if $z_a > z_b$ and $\text{arg}(z_a - z_b) = \pi$ if $z_a < z_b$. We set $\text{arg}(t_i - z_a) = 0$ if $t_i > z_a$ and $\text{arg}(t_i - z_a) = \pi$ if $t_i < z_a$. This assignment of arguments determines the integral $I_{\gamma}(z)$ in (4.1) and determines a horizontal section $\gamma$ of the homological bundle $\tau_N$ for real $z$ satisfying the above conditions. We extend this horizontal section to other values of $z$ by continuity.

Remark. Strictly speaking we have defined $\gamma(z)$ as a cell with a coefficient in $\mathcal{L}$. The boundary of that cell lies in the union of hyperplanes of the singularities of the master function $\Phi$. Nevertheless, using the remark in Section 5.1, we can represent the same function $I_{\gamma}(z)$ as an integral over the product of the corresponding Pochhammer cycles

$$
\gamma(z_1, z_{N+1}; -1/3, -1/3) \times \cdots \times \gamma(z_N, z_{2N}; -1/3, -1/3),
$$

that is, we can represent $I_{\gamma}(z)$ as an integral over an element of $H_N(\tau^{-1}(z), \mathcal{L}|_{\tau^{-1}(z)})$ and that element depends on $z$ horizontally.

5.3. A Selberg integral type formula. Assume that $z \in \mathbb{R}^N$ satisfies (5.2). Let $s : z \mapsto A(z)P(y; z)$ be the section defined in (3.1). The function $A(z)$ is multivalued. We fix its univalued branch over the set of $z$’s satisfying (5.2) by setting $\text{arg}(z_a - z_b) = 0$ if $z_a > z_b$ and $\text{arg}(z_a - z_b) = \pi$ if $z_a < z_b$.

Theorem 5.1. Let $z \in \mathbb{R}^N$ satisfy (5.2). Let $s$ be the section defined in (3.1). Let $\gamma$ be the section of $\tau_N$ defined in Section 5.2. Then $I_{\gamma}(z) = C \cdot s(z)$, where

$$
(5.4) \quad C = e^{-\pi i N^2/3} (3e^{\pi i/6} \Gamma(2/3)^3 \sin(\pi/3)/\pi)^N.
$$
In particular, comparing the coefficients of the monomial $y_1 \cdots y_N$ in the right and left hand sides of the equation $I_\gamma(z) = C s(z)$, we get the following formula,

\( (5.5) \)

\[
\int_{z_1}^{z_{N+1}} \cdots \int_{z_N}^{z_{2N}} \prod_{1 \leq i < j \leq N} (t_j - t_i)^{2/3} \prod_{i=1}^{N} \prod_{a=1}^{2N} (t_i - z_a)^{-1/3} \times \\
\times \sum_{\sigma \in S_N} \prod_{a=1}^{N} \frac{1}{t_{\sigma a} - z_a} \, dt_1 \wedge \cdots \wedge dt_N = C \prod_{1 \leq a \leq N < b \leq 2N} (z_b - z_a)^{1/3} \times \\
\times \prod_{1 \leq a < b \leq N} (z_b - z_a)^{-2/3} \prod_{N < a < b \leq 2N} (z_b - z_a)^{-2/3} \sum_{\sigma \in S_N} (-1)^\sigma \prod_{a=1}^{N} \frac{1}{z_{\sigma a} - z_{N+a}}.
\]

This is a Selberg integral type formula.

5.4. **A proof of Theorems 3.3 and 5.1** The KZ connection on the bundle $\eta$ has regular singularities. Therefore, any horizontal section $\tilde{s}$ of the conformal block subbundle has the form $\tilde{s} : z \mapsto \tilde{A}(z)P(y; z)$, where

\[
\tilde{A}(z) = \prod_{1 \leq a < b \leq 2N} (z_b - z_a)^{\alpha_{a,b}}
\]

for suitable numbers $\alpha_{a,b}$. To prove Theorem 3.3 we need to show that the numbers $\alpha_{a,b}$ are given by formula (3.3).

Assume that $z_{N+1} - z_1, \ldots, z_{2N} - z_N$ all tend to zero. Then

\[
I_\gamma(z) = e^{-\pi i \frac{N(N-1)}{12}} \times \\
\prod_{a=1}^{N} (z_a - z_{N+a})^{1/6} \int_{z_a}^{z_{N+a}} (t_a - z_a)^{-1/3} (t_a - z_{N+a})^{-1/3} \left( \frac{y_a}{t - z_a} + \frac{y_{N+a}}{t - z_{N+a}} \right) dt_a + \ldots
\]

where the dots denote the higher order terms. Calculating the integrals we get

\( (5.6) \)

\[
I_\gamma(z) = e^{-\pi i \frac{N(N-1)}{12}} e^{\pi i \frac{z_N}{6 N \Gamma(2/3)}} (-1)^N \frac{\Gamma(-1/3) \Gamma(2/3) N}{\Gamma(1/3) N} \prod_{a=1}^{N} (z_{N+a} - z_a)^{-1/2} (y_a - y_{N+a}) + \ldots
\]

\[
= e^{-\pi i \frac{N(N-1)}{12}} e^{\pi i z_N} (3 \Gamma(2/3)^3 \sin(\pi/3) / \pi)^N \prod_{a=1}^{N} (z_{N+a} - z_a)^{-1/2} (y_a - y_{N+a}) + \ldots.
\]

Comparing these asymptotics with the asymptotics of $\tilde{A}(z)P(y; z)$ we conclude that $\alpha_{a,N+a} = 1/2$ for all $a$. This statement is in agreement with formula (3.3). Since the order of vanishing of conformal blocks as $z_a - z_b$ tends to zero is the same for all pairs $(a, b)$, we conclude that $\tilde{A}(z) = A(z)$. Theorem 3.3 is proved.

To prove Theorem 5.1 we need to calculate the asymptotics of $A(z)P(y; z)$ as $z_{N+1} - z_1, \ldots, z_{2N} - z_N$ all tend to zero and compare them with the asymptotics of $I_\gamma(z)$. 
Clearly,

\[ A(z)P(y; z) = e^{\pi i \frac{N(N-1)}{4}} e^{\pi i N} \prod_{a=1}^{N} \left( z_{N+a} - z_a \right)^{-1/2} (y_a - y_{N+a}) + \ldots . \]

Hence, \( C \) is given by formula (5.4). Theorem 5.1 is proved.

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