Pseudo-exponential-type solutions of wave equations depending on several variables

B. Fritzsche, B. Kirstein, I.Ya. Roitberg, A.L. Sakhnovich

Abstract

Explicit pseudo-exponential-type solutions of linear differential equations depending on two variables and nonlinear wave equations depending on three variables are constructed. S-nodes and S-multinodes are used for that purpose.

Keywords: explicit solution, Bäcklund-Darboux transformation, pseudo-exponential-type potential, nonstationary Dirac equation, Loewner system, Davey-Stewartson equation, generalized nonlinear optics equation, S-node, S-multinode.

1 Introduction

Explicit solutions of linear and nonlinear equations of mathematical physics play an important role in theory and applications. The theory is well-developed for the case of linear equations depending on one variable and nonlinear integrable equations depending on two variables and includes, in particular, various versions of the commutation methods, algebro-geometric methods and Bäcklund-Darboux transformations (see [11, 16, 18, 20, 28, 36, 38, 46] and references therein). In spite of various interesting results on the cases of more variables (see, e.g., [1, 5, 7, 9, 27, 32, 33, 35, 42, 44, 48]), these cases are more complicated and contain also more open problems. The pseudo-exponential-type potentials and solutions, that is, potentials and solutions, which, roughly speaking, rationally depend on exponents (see, e.g., [15, 19].
for the definition of the term *pseudo-exponential potential* are of a special interest. When we deal with rational functions of matrix exponents, rational potentials may appear as a special subcase.

In this paper we apply the $S$-nodes approach from [35] and $S$-multinodes approach from [37] in order to construct explicitly pseudo-exponential-type potentials and solutions of several important equations of mathematical physics depending on several variables. $S$-multinodes were introduced in [37] as a certain generalization of the $S$-nodes by L.A. Sakhnovich [39–41] on one hand and commutative colligations by M.S. Livšic [24, 25] on the other hand and were used in [37] in order to construct explicit solutions of the time-dependent Schrödinger equation. We start (see Subsection 2.1) with the construction of the explicit solutions of the nonstationary Dirac equation

$$H\Psi = 0, \quad H := \frac{\partial}{\partial t} + \sigma_2 \frac{\partial}{\partial y} - iV(t, y), \quad V = V^*.$$  \hspace{1cm} (1.1)

Then, we consider in the Subsection 2.2 the well-known Loewner’s system

$$\Psi_x = \mathcal{L}(x, y)\Psi_y,$$  \hspace{1cm} (1.2)

where $\mathcal{L}$ is an $m \times m$ matrix function (and the case $m = 2$ with applications to the hodograph equation was dealt with in the seminal paper [26] by C. Loewner). System (1.1) was studied in the interesting papers [29, 43], see also references therein. For the Loewner’s system, its transformations, generalizations and applications in mechanics, physics and soliton surfaces see, for instance, [10, 22, 26, 30, 31] and references therein.

Section 3 is dedicated to the nonlinear integrable equations.

In our paper, $\Psi_{tx} = \frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \Psi \right) = \frac{\partial^2}{\partial x \partial t} \Psi$, $\sigma(D)$ stands for the spectrum of $D$, $[G, F]$ stands for the commutator $GF - FG$ and $\otimes$ stands for Kronecker product. By $\text{diag}\{b_1, b_2, \ldots, b_m\}$ we denote the diagonal matrix with the entries $b_1, b_2, \ldots$ on the main diagonal.
2 Linear equations: explicit solutions

2.1 Nonstationary Dirac equation

First we note that in the GBDT version \[36, 38\] of the Bäcklund-Darboux transformation (BDT) the solution of the transformed equation is represented in the form \( \Pi^* S^{-1} \), where \( \Pi^* \) is the solution of the initial equation. Here we construct solutions of (1.1) in the same form. Namely, we set

\[
\Pi = CE_A(t, y) \hat{C}, \quad E_A = \exp \{ tA_1 + yA_2 \}, \quad A_1A_2 = A_2A_1, \quad \hat{C} = \begin{bmatrix} g_1^* & g_2^* \end{bmatrix},
\]

(2.1)

where \( \hat{C} \) is an \( N \times 2 \) matrix, \( g_1^* \) and \( g_2^* \) are columns of \( \hat{C} \), \( A_1 \) and \( A_2 \) are \( N \times N \) matrices and \( C \) is an \( n \times N \) matrix (\( n, N \in \mathbb{N} \)). We assume that the equalities

\[
g_1A_1^* - ig_2A_2^* = 0, \quad g_2A_1^* + ig_1A_2^* = 0
\]

(2.2)

hold. From (2.1) and (2.2), we easily see that

\[
H_0 \Pi^* = 0, \quad H_0 := \frac{\partial}{\partial t} + \sigma_2 \frac{\partial}{\partial y}.
\]

(2.3)

Matrices \( A_1, A_2, R, \nu_1, \nu_2 \) and \( \hat{C} \) form a symmetric 2-node if \( A_1 \) and \( A_2 \) commute and the following identities are valid:

\[
A_kR + RA_k^* = \hat{C}\nu_k\hat{C}^* \quad (k = 1, 2); \quad R = R^*, \quad \nu_k = \nu_k^*.
\]

(2.4)

It is immediate that the matrix function

\[
S(t, y) = S_0 + CE_A(t, y)RE_A(t, y)^* C^* \quad (S_0 = S_0^*)
\]

(2.5)

satisfies equations \( \frac{\partial}{\partial t} S = \Pi \nu_1 \Pi^* \) and \( \frac{\partial}{\partial y} S = \Pi \nu_2 \Pi^* \). These equations and equation (2.3) yield the proposition below.

**Proposition 2.1** Let relations (2.1), (2.2), (2.4) and (2.5) hold and assume that \( \nu_1 = \sigma_2 \), \( \nu_2 = -I_2 \). Then, in the points of invertibility of \( S \), we have

\[
H \left( \Pi(t, y)^* S(t, y)^{-1} \right) = 0, \quad V := i \left( \Pi^* S^{-1} \Pi \sigma_2 - \sigma_2 \Pi^* S^{-1} \Pi \right),
\]

(2.6)

where \( H \) has the form (1.1).
The important part of the problem is to find the cases where the conditions of Proposition 2.1 hold.

**Example 2.2** Set \( g_2 = -i g_1 J \), \( A_1 = D = \text{diag}\{D_1, D_2\} \) (where \( D_1 \) and \( D_2 \) are \( n_1 \times n_1 \) and \( n_2 \times n_2 \) diagonal blocks of the diagonal matrix \( D \), \( n_1 + n_2 = n \), \( \sigma(D_k) \cap \sigma(-D_k^*) = \emptyset \) for \( k = 1, 2 \)), \( A_2 = DJ \) and

\[
J := \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}, \quad R = \begin{bmatrix} R_{11} & 0 \\ 0 & R_{22} \end{bmatrix}.
\]

We uniquely define \( R_{11} \) and \( R_{22} \) by the matrix identities

\[
D_1 R_{11} + R_{11} D_1^* = -g_1^*(I_n + J)g_1, \quad D_2 R_{22} + R_{22} D_2^* = g_1^*(I_n - J)g_1.
\]

Then the conditions of Proposition 2.1 hold.

Thus, according to Proposition 2.1 and Example 2.2, each vector \( g_1 \) and diagonal matrix \( D \) (such that \( \sigma(D_k) \cap \sigma(-D_k^*) = \emptyset \)) determine a family of pseudo-exponential-type potentials and explicit solutions of (1.1).

### 2.2 Loewner’s system

Direct calculation proves the following proposition.

**Proposition 2.3** Let \( m \times m \) and \( m \times n \), respectively, matrix functions \( \Lambda_1 \) and \( \Lambda_2 \) satisfy a linear differential equation

\[
\Lambda_x = q_1(x, y)\Lambda_y + q_0(x, y)\Lambda.
\]

Then, in the points of invertibility of \( \Lambda_1 \), the matrix function \( \Psi = \Lambda_1^{-1}\Lambda_2 \) satisfies the Loewner equation (1.2), where

\[
\mathcal{L} = \Lambda_1^{-1}q_1\Lambda_1.
\]

For some special kinds of similarity transformations of \( \mathcal{L} \) see also [20, formulas (5.10a) and (5.27)]. Pseudo-exponential-type \( \Psi \) and \( \mathcal{L} \) are constructed in the next proposition.
Proposition 2.4  Introduce $m \times m$ and $m \times n$, respectively, matrix functions $\Lambda_1$ and $\Lambda_2$ by the equalities

\[
\Lambda_i = C_i E_A(x, y, i) \tilde{C}_i \quad (i = 1, 2); \\
E_A(x, y, i) := \exp\{x\tilde{A}_i + y\tilde{\tilde{A}}_i\}, \quad \tilde{A}_i := D \otimes A_i, \quad \tilde{\tilde{A}}_i := I_m \otimes A_i, \\
D = \text{diag}\{d_1, \ldots, d_m\}, \quad \hat{C}_i := \sum_{k=1}^m (e_k e_k^*) \otimes (e_k^* c_i),
\]

where $A_i$ are $l_i \times l_i$ matrices, $c_i$ are $m \times l_i$ matrices, $\tilde{C}_1$ is an $N_1 \times m$ matrix, $\tilde{C}_2$ is an $N_2 \times n$ matrix, $N_i = ml_i$ and $l_i \in \mathbb{N}$. Here $\otimes$ is Kronecker product, $e_k$ is a column vector given by $e_k = \{\delta_{jk}\}_{j=1}^m$ and $\delta_{jk}$ is Kronecker’s delta.

Then, in the points of invertibility of $\Lambda_1$, the matrix functions

\[
\Psi = \Lambda_1^{-1} \Lambda_2 \quad \text{and} \quad \mathcal{L} = \Lambda_1^{-1} D \Lambda_1
\]

satisfy (1.2).

Proof. It is easy to see that $\Lambda_1$ and $\Lambda_2$ given by (2.9) satisfy equation $\Lambda_x = D \Lambda_y$. Now, Proposition 2.4 follows from Proposition 2.3. ■

In a similar (to the construction of $\Lambda_i$ in the proposition above) way, matrix functions $\Pi$ satisfying (3.20) are constructed in (3.23)–(3.25).

3 Nonlinear integrable equations

Among 2+1-dimensional integrable equations, Kadomtsev-Petviashvili, Davey-Stewartson (DS) and generalized nonlinear optics (also called $N$-wave) equations are, perhaps, the most actively studied systems. $S$-nodes were applied to the construction and study of the pseudo-exponential, rational and lump solutions of the Kadomtsev-Petviashvili equations in [35]. Here we investigate the remaining two equations from the three above.

3.1 Davey-Stewartson equations

The Davey-Stewartson equations are well-known in wave theory (see, e.g., [6,12,14,21] and references therein). Since Davey-Stewartson equations (DS
I and DS II) are natural multidimensional generalizations of the nonlinear Schrödinger equations (NLS), their matrix versions should also be of interest (similar to matrix versions of NLS, see, e.g., [4]). The matrix DS I has the form

\[ iu_t - (u_{xx} + u_{yy})/2 = uq_1 - q_2 u, \]  
(3.1)

\[ (q_1)_x - (q_1)_y = \frac{1}{2}((u^*u)_y + (u^*u)_x), \quad (q_2)_x + (q_2)_y = \frac{1}{2}((uu^*)_y - (uu^*)_x), \]  
(3.2)

where \( u, q_1 \) and \( q_2 \) are \( m_2 \times m_1, m_1 \times m_1 \) and \( m_2 \times m_2 \) matrix functions, respectively (\( m_1 \geq 1, m_2 \geq 1 \)). We note that another matrix version of the Davey-Stewartson equation, where \( m_1 = m_2 \), was dealt with in [23]. It is easy to see that in the scalar case \( m_1 = m_2 = 1 \) equations (3.1) and (3.2) are equivalent, for instance, to [20, p. 70, system (2.23)] (after setting in (2.23) \( \varepsilon = \alpha = 1 \)).

GBDT version of the Bäcklund-Darboux transformation for the matrix DS I was constructed in [34]. When the initial DS I equation (in GBDT for DS I from [34, Theorem 5]) is trivial, that is, \( u_0 \equiv 0 \) and \( Q_0 \equiv 0 \) in [34], Theorem 5 takes the form:

**Proposition 3.1** Let an \( n \times m \) (\( n \in \mathbb{N}, m = m_1 + m_2 \)) matrix function \( \Pi \) and an \( n \times n \) matrix function \( S \) satisfy equations

\[ \Pi_x = \Pi_y j, \quad \Pi_t = -i \Pi_{yy} j, \quad j := \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \]  
(3.3)

\[ S_y = -\Pi \Pi^*, \quad S_x = -\Pi_j \Pi^*, \quad S_t = i(\Pi_y j \Pi^* - \Pi_j \Pi_y^*). \]  
(3.4)

Partition \( \Pi \) into \( n \times m_1 \) and \( n \times m_2 \), respectively, blocks \( \Phi_1 \) and \( \Phi_2 \) (i.e., set \( \Pi =: [\Phi_1 \quad \Phi_2] \)).

Then, the matrix functions

\[ u = 2\Phi_2^* S^{-1} \Phi_1, \quad q_1 = \frac{1}{2}u^*u - 2(\Phi_1^* S^{-1} \Phi_1)_y, \quad q_2 = -\frac{1}{2}uu^* + 2(\Phi_2^* S^{-1} \Phi_2)_y \]  
(3.5)

satisfy (in the points of invertibility of \( S \)) DS I system (3.1), (3.2).
Introduce $\Phi_1$, $\Phi_2$ and $S$ via relations

$$
\Phi_1(x, t, y) = C_1 E_1(x, t, y) \hat{C}_1, \quad E_1(x, t, y) := \exp\{(x + y) A_1 - i t A_1^2\}; \quad (3.6)
$$

$$
\Phi_2(x, t, y) = C_2 E_2(x, t, y) \hat{C}_2, \quad E_2(x, t, y) := \exp\{(x - y) A_2 + i t A_2^2\}; \quad (3.7)
$$

$$
S(x, t, y) = S_0 + C_1 E_1(x, t, y) R_1 E_1(x, t, y)^* C_1^* 
- C_2 E_2(x, t, y) R_2 E_2(x, t, y)^* C_2^* \quad (S_0 = S_0^*), \quad (3.8)
$$

where $C_1$ and $C_2$ are $n \times N$ matrices; $A_1$, $A_2$, $R_1 = R_1^*$ and $R_2 = R_2^*$ are $N \times N$ matrices; $\hat{C}_1$ and $\hat{C}_2$ are $N \times m_1$ and $N \times m_2$, respectively, matrices; $S_0$ is an $n \times n$ matrix and the following identities hold:

$$
A_1 R_1 + R_1 A_1^* = -\hat{C}_1 \hat{C}_1^*, \quad A_2 R_2 + R_2 A_2^* = -\hat{C}_2 \hat{C}_2^*. \quad (3.9)
$$

It is immediate from (3.6)–(3.9) that $\Pi = [\Phi_1 \quad \Phi_2]$ and $S$ satisfy relations (3.3) and the first two relations in (3.4). In order to prove the third equality in (3.4), we note that

$$
(C_1 E_1 R_1 E_1^* C_1^*)_t = -i C_1 E_1 (A_1^2 R_1 - R_1 (A_1^2)^*) E_1^* C_1^*
- i C_1 E_1 (A_1 (A_1 R_1 + R_1 A_1^*) - (A_1 R_1 + R_1 A_1^*) A_1^*) E_1^* C_1^*
- i ((\Phi_1)_y \Phi_1^* - \Phi_1 (\Phi_1^*)_y). \quad (3.10)
$$

Here we used (3.6) and the first identity in (3.9).

In a similar way we show that

$$
(C_2 E_2 R_2 E_2^* C_2^*)_t = i ((\Phi_2)_y \Phi_2^* - \Phi_2 (\Phi_2^*)_y). \quad (3.11)
$$

Equalities (3.8), (3.10) and (3.11) yield the last equality in (3.4). Hence, the conditions of Proposition 3.1 are valid, and so we proved the following proposition.

**Proposition 3.2** Let $\Phi_1$, $\Phi_2$ and $S$ be given by the formulas (3.6)–(3.8) and assume that (3.9) holds. Then, the matrix functions $u$, $q_1$ and $q_2$ given by (3.5) satisfy (in the points of invertibility of $S$) DS I system (3.1), (3.2).

**Remark 3.3** It is easy to see that if $\sigma(A_1) = \sigma(A_2) = 0$, then $\Phi_1$, $\Phi_2$ and $S$ are rational matrix functions. Thus, if $\sigma(A_1) = \sigma(A_2) = 0$, the solutions $u$, $q_1$ and $q_2$ of the DS I system, which are constructed in Proposition 3.2, are also rational matrix functions.
The compatibility condition $w_{tx} = w_{xt}$ of the auxiliary systems

$$w_x = \pm ij w_y + jV w, \quad w_t = 2ij w_{yy} \pm 2jV w_y \pm jQ w, \quad (3.12)$$

$$V = \begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} q_1 & u_y \mp iu_x \\ u_y^* \pm iu_x^* & -q_2 \end{bmatrix}, \quad (3.13)$$

$$q_k(x, t) = -q_k(x, t)^* \quad (k = 1, 2) \quad (3.14)$$

is equivalent (for the case that the solution $w$ is a non-degenerate matrix function) to the matrix DS II equation

$$u_t + i(u_{xx} - u_{yy}) = \mp (q_1 u - u q_2), \quad (3.15)$$

$$(q_1)_x \mp i(q_1)_y = (uu^*)_y \mp i(uu^*)_x, \quad (q_2)_x \pm i(q_2)_y = (u^* u)_y \pm i(u^* u)_x. \quad (3.16)$$

As we see from $(3.12)$–$(3.16)$, there are two versions of auxiliary systems and corresponding DS II equations. After setting $m_1 = m_2 = 1$ (and setting also $\varepsilon = 1$, $\alpha = \mp i$ in [20, p. 70, system (2.23)]), like for the scalar DS I case, equations $(3.15)$ and $(3.16)$ are equivalent to [20, p. 70, (2.23)].

**Open problem.** Use the approach from Proposition 3.1 in order to construct explicit pseudo-exponential solutions of the matrix DS II.

We note that various results on DS II, including BDT results, are not quite analogous to the results on DS I (see, e.g., [20]).

### 3.2 Generalized nonlinear optics equation

The integrability of the generalized nonlinear optics equation (GNOE)

$$[D, \xi_t] - [\tilde{D}, \xi_x] = [[D, \xi], [[\tilde{D}, \xi] + D\xi y \tilde{D} - \tilde{D}\xi y D], \quad (3.17)$$

$$\xi(x, t, y)* = B\xi(x, t, y)B, \quad B = \text{diag}\{b_1, b_2, \ldots, b_m\}, \quad b_k = \pm 1; \quad (3.18)$$

$$D = \text{diag}\{d_1, d_2, \ldots, d_m\} > 0, \quad \tilde{D} = \text{diag}\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_m\} > 0 \quad (3.19)$$

was dealt with in [34]. This system is a generalization of the well-known $N$-wave (nonlinear optics) equation $[D, \xi_t] - [\tilde{D}, \xi_x] = [[D, \xi], [[\tilde{D}, \xi] first studied in [45] (see also [2]). GBDT version of the Bäcklund-Darboux transformation for GNOE was (as well as the GBDT version for the equations) constructed in [34]. When the initial system in GBDT for GNOE from [34, Theorem 4] is trivial (i.e., $\xi_0 \equiv 0$), Theorem 4 takes the form:
Proposition 3.4  Let an \( n \times m \) matrix function \( \Pi \) and an \( n \times n \) matrix function \( S \) satisfy equations

\[
\begin{align*}
\Pi_x &= \Pi_y D, \quad \Pi_t = \Pi_y \tilde{D}, \\
S_y &= -\Pi B \Pi^*, \quad S_x = -\Pi BD \Pi^*, \quad S_t = -\Pi B \tilde{D} \Pi^*.
\end{align*}
\]  

(3.20)

Then the matrix function

\[
\xi = \Pi^* S^{-1} \Pi B
\]

(3.22)
satisfies (in the points of invertibility of \( S \)) GNOE (3.17) and reduction condition (3.18).

In order to construct pseudo-exponential-type solutions \( \xi \), we will consider matrix functions \( \Pi \) and \( S \) of the form (2.1) and (2.5), respectively, where \( E_A \) will depend on three variables and \( N = ml, \ l \in \mathbb{N} \). Namely, we set

\[
\Pi(x, t, y) = CE_A(x, t, y)\tilde{C}, \quad E_A(x, t, y) = \exp\{xA_1 + tA_2 + yA_3\},
\]

(3.23)

\[
A_1 = D \otimes A, \quad A_2 = \tilde{D} \otimes A, \quad A_3 = I_m \otimes A,
\]

(3.24)

\[
\tilde{C} = \sum_{k=1}^m (e_k e_k^*) \otimes (\tilde{c} e_k), \quad e_k = \{\delta_{ik}\}_{i=1}^m \in \mathbb{C}^m, \tag{3.25}
\]

where \( C \) is an \( n \times N \) matrix, \( A \) is an \( l \times l \) matrix, \( N = ml, \ \otimes \) is Kronecker product, \( \tilde{c} \) is an \( l \times m \) matrix, \( e_k \) is a column vector and \( \delta_{ik} \) is Kronecker’s delta. It is immediate that the matrices \( A_k (k = 1, 2, 3) \) commute. Hence, we see that matrices \( A, C \) and \( \tilde{c} \) determine (via (3.23)-(3.25)) matrix function \( \Pi \) satisfying (3.20).

Proposition 3.5  Let relations (3.23)-(3.25) hold and set

\[
S(x, t, y) = S_0 + CE_A(x, t, y)RE_A(x, t, y)^* C^* \quad (S_0 = S_0^*),
\]

(3.26)

where the \( N \times N \) matrix \( R \) \( (N = ml, \ R = R^*) \) satisfies matrix identities

\[
A_1 R + RA_1^* = -\tilde{C} B D \tilde{C}^*, \quad A_2 R + RA_2^* = -\tilde{C} B \tilde{D} \tilde{C}^*,
\]

(3.27)

(3.28)

Then, the matrix function \( \xi = \Pi^* S^{-1} \Pi B \) satisfies (in the points of invertibility of \( S \)) GNOE (3.17) and reduction condition (3.18).
Proof. We mentioned above that $\Pi$ given by (3.23)–(3.25) satisfies (3.20). Moreover, relations (3.23) and (3.26)–(3.28) yield (3.21). Thus, the conditions of Proposition 3.4 are fulfilled. ■

We note that, according to (3.25), the right-hand sides of the equalities in (3.27) and (3.28) are block diagonal matrices with $l \times l$ blocks. Therefore, we will construct block diagonal matrix $R$, which blocks $R_{kk}$ are also $l \times l$ matrices:

$$R = \text{diag}\{R_{11}, R_{22}, \ldots, R_{mm}\}. \quad (3.29)$$

Taking into account (3.24), we see that for $R$ of the form (3.29) identities

$$AR_{kk} + R_{kk}A^* = -b_k(\hat{c}e_k)(\hat{c}e_k)^* \quad (1 \leq k \leq m) \quad (3.30)$$

imply that identities (3.27) and (3.28) hold.

**Corollary 3.6** Let relations (3.23)–(3.25) and (3.30) hold. Then, the matrix function $\xi = \Pi^*S^{-1}\Pi B$, where $S$ is given by (3.26) and (3.29), satisfies (in the points of invertibility of $S$) GNOE (3.17) and reduction condition (3.18).

**Remark 3.7** If $\sigma(A) \cap \sigma(-A^*) = \emptyset$, there exist unique solutions $R_{kk}$ satisfying (3.30). For that case we have also $R_{kk} = R_{kk}^*$ (i.e., $R = R^*$). Clearly, $R_{kk}$ is immediately recovered if $\sigma(A) \cap \sigma(-A^*) = \emptyset$ and $A$ is a diagonal matrix.

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B. Fritzsche,
Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Augustusplatz 10, D-04009 Leipzig, Germany, e-mail: fritzsche@math.uni-leipzig.de

B. Kirstein,
Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Augustusplatz 10, D-04009 Leipzig, Germany, e-mail: kirstein@math.uni-leipzig.de

I. Roitberg,
Fakultät für Mathematik und Informatik, Mathematisches Institut, Universität Leipzig, Augustusplatz 10, D-04009 Leipzig, Germany, e-mail: i.roitberg@yahoo.com

A.L. Sakhnovich,
Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria, e-mail: al.sakhnov@yahoo.com