TWO PREDUALITIES AND THREE OPERATORS OVER ANALYTIC CAMPANATO SPACES

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ABSTRACT. This article is devoted to not only characterizing the first and second preduals of the analytic Campanato spaces ($\mathcal{CA}_p$) on the unit disk, but also investigating boundedness of three operators: superposition ($S^\phi$); backward shift ($S_b$); Schwarzian derivative ($S$), acting on $\mathcal{CA}_p$.

1. Introduction

From now on, $\mathbb{D}$ and $\mathbb{T}$ respectively represent the unit disk and the unit circle in the finite complex plane $\mathbb{C}$. For $p \in (-\infty, \infty)$, $\mathcal{CA}_p$ denotes the Campanato space of all analytic functions $f : \mathbb{D} \to \mathbb{C}$ with radial boundary values $f$ on $\mathbb{T}$ obeying

$$\|f\|_{\mathcal{CA}_p,*} = \sup_{|I| \subseteq \mathbb{T}} \sqrt{|I|^{-p} \int_I |f(\xi) - f_I|^2 |d\xi|} < \infty,$$

where the supremum is taken over all sub-arcs $I \subseteq \mathbb{T}$ with $|I|$ being their arc-lengths, and

$$\begin{align*}
|d\xi| &= |de^{i\theta}| = d\theta \\
|I| &= (2\pi)^{-1} \int_I |d\xi| \\
f_I &= (2\pi|I|)^{-1} \int_I f(\xi) |d\xi|
\end{align*}$$

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Obviously, $\| \cdot \|_{CA_{p,\ast}}$ cannot distinguish between any two $CA_p$ functions differing by a constant, but
$$\|f\|_{CA_p} = |f(0)| + \|\cdot\|_{CA_{p,\ast}}$$
defines a norm so that $CA_p$ is a Banach space. The following table tells us that how $CA_p$ looks like (see, e.g. [29, 30, 31] and their references):

| Index $p$ | Analytic Campanato Space $CA_p$ |
|-----------|--------------------------------|
| $p \in (-\infty, 0]$ | Analytic Hardy space $H^2$ |
| $p \in (0, 1)$ | Holomorphic Morrey space $H^{2,p}$ |
| $p = 1$ | Analytic John-Nirenberg space $BMOA$ |
| $p \in (1, 3)$ | Analytic Lipschitz space $A_{\frac{1}{p-1}}$ |
| $p \in (3, \infty)$ | Complex constant space $\mathbb{C}$ |

Similarly, the little (or vanishing) analytic Campanato space $CA_{0,p}$ consists of all functions in $CA_p$ satisfying
$$\lim_{|I| \to 0} |I|^{-p} \int_I |f(\xi) - f_I|^2 d\xi = 0.$$ 
In particular, one has

| Index $p$ | Little Analytic Campanato Space $CA_{0,p}$ |
|-----------|--------------------------------|
| $p \in (-\infty, 0]$ | Analytic Hardy space $H^2$ |
| $p \in (0, 1)$ | Little Holomorphic Morrey space $H^{2,p}_0$ |
| $p = 1$ | Analytic Sarason space $VMOA$ |
| $p \in (1, 3)$ | Little Analytic Lipschitz space $A_{0,\frac{1}{p-1}}$ |
| $p \in (3, \infty)$ | Complex constant space $\mathbb{C}$ |

According to [29, 30, 34], if
\[
\begin{align*}
\sigma_a(z) &= \frac{a - z}{1 - a\bar{z}}; \\
E(f, a) &= (1 - |a|^2)^{1-p} \int_D |f'(z)|^2 (1 - |\sigma_a(z)|) dm(z).
\end{align*}
\]
then not only the following are equivalent:
- $f \in CA_p$;
- $|f'(z)|^2 (1 - |z|^2) dm(z)$ is a bounded $p$-Carleson measure;
- $|f'(z)|^2 (1 - |\sigma_a(z)|) dm(z)$ is a bounded $p$-Carleson measure;
- $\|f\|_{CA_{p,\ast}} = \sup_{a \in D} [E(f, a)]^{\frac{1}{p}} < \infty$,

but also the following are equivalent:
- $f \in CA_{0,p}$;
- $|f'(z)|^2 (1 - |z|^2) dm(z)$ is a compact $p$-Carleson measure;
- $|f'(z)|^2 (1 - |\sigma_a(z)|) dm(z)$ is a compact $p$-Carleson measure;
- $\|f\|_{CA_{0,p,\ast}} < \infty$ & $\lim_{|a| \to 1} E(f, a) = 0$.

In the above and below, we say that a nonnegative measure $\mu$ on $\mathbb{D}$ is a bounded or compact $(0, \infty) \ni p$-Carleson measure provided
$$\|\mu\|_p = \sup_{I \subseteq \mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty \quad \text{or} \quad \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^p} = 0,$$
where for any arc $I \subseteq \mathbb{T}$ one sets $S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |l| \leq r < 1, \ e^{i\theta} \in I \}$.

Continuing essentially from \cite[Chapter 3]{29}, \cite{30, 31} and \cite{9, 18, 27}, in this paper we study two predualities and three operators associated to the analytic Campanato spaces. More precisely, in \S2 we use \S2.1 - the Choquet integrals and quadratic tent spaces to discover \S2.2 - the predual space of $CA_p$ and \S2.3 - the dual space of $CA_{0,p}$.

And, in \S3 we discuss: \S3.1 - when the superposition $S^\theta$ is bounded on $CA_p$ and $CA_{0,p}$; \S3.2 - the boundedness of the backward shift $S_b$ on both $CA_p$ and $CA_{0,p}$; \S3.3 - the behavior of the Schwarzian derivative $S(f)$ of a univalent function $f$ on $\mathbb{D}$ whenever $\log f'$ is in $CA_p$ or $CA_{0,p}$.

Notation: In this note, we will use $X \lesssim Y$ or $X \gtrsim Y$ to express $X \leq \kappa Y$ or $X \geq \kappa Y$ for some constant $\kappa > 0$. Moreover, $X \asymp Y$ means $X \lesssim Y$ and $X \gtrsim Y$. In addition, $dm(z) = dx dy$ stands for two dimensional Lebesgue area measure.

2. Two Predualities

2.1. Choquet integrals and tent spaces. The $(0, 1] \ni p$-dimensional capacity of $E \subseteq \partial \mathbb{D}$ is defined by

$$\Lambda_p^{(\infty)}(E) = \inf \left\{ \sum_{j=1}^{\infty} |I_j|^p : E \subseteq \bigcup_{j=1}^{\infty} I_j \right\},$$

where the infimum is taken over all coverings of $E$ by countable families of open arcs $I_j \subseteq \mathbb{T}$. According to \cite{1}, $\Lambda_p^{(\infty)}$ is a monotone, countably subadditive set function on the class of all subsets of $\mathbb{T}$ which vanishes on the empty set, and the Choquet integral of a nonnegative function $f$ on $\mathbb{T}$ against $\Lambda_p^{(\infty)}$ is defined by

$$\int_{\mathbb{T}} f \ d\Lambda_p^{(\infty)} = \int_0^{\infty} \Lambda_p^{(\infty)}(\{\xi \in \mathbb{T} : f(\xi) > t\}) \ dt.$$

Following \cite[Chapter 4]{29}, let

$$N(\omega)(\xi) = \sup \{\omega(z) : |z - \xi| < 1 - |z|^2 \} \quad \forall \quad \xi \in \mathbb{T},$$

be the nontangential maximal function of $\omega$, and define

$$\|\omega\|_{LN^1(\Lambda_p^{(\infty)})} = \int_{\mathbb{T}} N(\omega) \ d\Lambda_p^{(\infty)}.$$

**Definition 2.1.** Let $p \in (0, 1]$.

- The space $T_0^p$ consists of all Lebesgue measurable functions $f$ on $\mathbb{D}$ with

$$\|f\|_{T_0^p} = \sup_{I \subseteq \mathbb{T}} \left( |I|^{-p} \int_{T(I)} |f(z)|^2 (1 - |z|^2) dm(z) \right)^{\frac{1}{2}} < \infty,$$

where the supremum runs over all open subarcs $I$ of $\mathbb{T}$ and

$$T(E) = \{re^{i\theta} \in \mathbb{D} : \text{dist}(e^{i\theta}, \mathbb{T} \setminus E) > 1 - r \}$$

is the tent over the set $E \subseteq \mathbb{T}$.

- The space $T_1^p$ consists of all Lebesgue measurable functions $f$ on $\mathbb{D}$ with

$$\|f\|_{T_1^p} = \inf_{\omega} \left( |I|^{-p} \int_{\mathbb{D}} |f(z)|^2 (\omega(z))^{-1} (1 - |z|^2)^{-1} dm(z) \right)^{\frac{1}{2}} < \infty,$$
where the above infimum is taken over all nonnegative functions $\omega$ on $D$ with $\|\omega\|_{L^{N/(N_p)}} \leq 1$.

- A function $a$ on $D$ is called a $T_p^1$-atom if there exists a subarc $I$ of $T$ such that $a$ is supported in the tent $T(I)$ and satisfies

$$\int_{T(I)} |a(z)|^2 (1 - |z|^2)^{-1} dm(z) \leq |I|^{-p}.$$ 

**Lemma 2.2.** Let $p \in (0, 1]$.

- If $\sum_{j=1}^{\infty} \|f_j\|_{T_p^1} < \infty$, then $f = \sum_{j=1}^{\infty} f_j \in T_p^1$ with $\leq 2 \sum_{j=1}^{\infty} \|f_j\|_{T_p^1}$.
- $f \in T_p^1$ if and only if there is a sequence of $T_p^1$-atom $\{a_j\}$ and an $l^1$-sequence $\{\lambda_j\}$ such that $f = \sum_{j=1}^{\infty} \lambda_j a_j$. Moreover,

$$\|f\|_{T_p^1} \approx \|f\|_{T_p^1} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\},$$

where the infimum is taken over all possible atomic decompositions of $f \in T_p^1$.

Consequently, $T_p^1$ is a Banach space under the norm $\|f\|_{T_p^1}$.

- $T_p^{\infty} = \{T_p^1\}^*$ under the pairing $<f, g> = \frac{1}{\pi} \int_D f\overline{g} dm$.

**Proof.** This follows easily from a slight modification of [29] Lemma 4.3.1 & Theorem 4.3.2. □

### 2.2. First predual.

When $1 < p \leq 3$, Duren, Romberg and shierds [13] gave the predual space of the analytic Lipschitz $A^{\infty}_{p-1}$ is the Hardy space $H^{\infty}_{1/p}$. For $p = 1$, Fefferman [14] established the well-known result $(H^1)^* = BMOA$. For $p = 0$, $(H^2)^* = H^2$. For $p \in (0, 1)$, note that $BMOA \subset H^{2-p} \subset H^2$, hence the predual of the analytic Morrey space should be an analytic function space between the analytic Hardy spaces $H^2$ and $H^1$. To work out this predual space, we need the following lemma.

**Lemma 2.3.** [27] For $p, \eta \in (0, 2)$, $a > \frac{2-\eta}{2}$, $b > \frac{1+\eta}{2}$, and a Lebesgue measurable function $f$ on $D$, let

$$T_{a,b}f(z) = \int_D \frac{(1 - |w|^2)^{b-1}}{|1 - \overline{w}z|^{a+b}} f(w) dm(w), \quad z \in D.$$ 

If $|f(z)|^2 (1 - |z|^2)^\eta dm(z)$ is a $p$-Carleson measure, then $|T_{a,b}f(z)|^2 (1 - |z|^2)^{2a+\eta-2} dm(z)$ is also a $p$-Carleson measure.

Below is a description of the first predual space of the analytic Morrey space.

**Theorem 2.4.** For $p \in (0, 1]$ let $BA_p$ be the class of all analytic functions $f$ on $D$ satisfying $f(0) = 0$ and

$$\|f\|_{BA_p} = \inf_{\omega} \left( \int_D |f''(z)|^2 (\omega(z))^{-1} (1 - |z|^2) dm(z) \right)^{1/2} < \infty,$$
where the infimum is taken over all nonnegative functions $\omega$ on $\mathbb{D}$ with $\|\omega\|_{\mathcal{LN}^1(\mathbb{C}_p)} \leq 1$. Then $\mathcal{CA}_p$ is isomorphic to the dual of $\mathcal{BA}_p$ under the pairing

$$\langle f, g \rangle = \int_{\mathbb{T}} f(\xi)\overline{g}(\xi)\frac{|d\xi|}{2\pi} \quad \forall \ (f, g) \in \mathcal{BA}_p \times \mathcal{CA}_p.$$ 

That is, $\mathcal{BA}_p^* = \mathcal{CA}_p$. Moreover,

$$\|f\|_{\mathcal{BA}_p} \approx \sup \{\langle f, g \rangle : g \in \mathcal{CA}_p \land \|g\|_{\mathcal{CA}_p} \leq 1\} \quad \forall \ f \in \mathcal{BA}_p.$$

**Proof.** On the one hand, assume that $f \in \mathcal{BA}_p$ and $g \in \mathcal{CA}_p$. According to the function-theoretic characterization of $\mathcal{CA}_p$ stated in §1 we have

$$d\mu(z) = |g'(z)|^2(1 - |z|^2)dm(z)$$

is a $p$-Carleson measure on $\mathbb{D}$; that is

$$\|\mu\|_p = \sup_{I \subset \mathbb{T}} \mu(S(I))|I|^{-p} \leq \|g\|_{\mathcal{CA}_p}^2.$$

If $\omega$ is a positive function on $\mathbb{D}$ satisfying

$$\int_{\mathbb{T}} N(\omega)(\xi)d\Lambda_\omega^\omega(\xi) \leq 1,$$

then, by the Hardy-Littlewood identity and the Cauchy-Schwarz inequality we obtain

$$\left| \frac{1}{2\pi} \int_{\mathbb{T}} f(e^{i\theta})\overline{g}(e^{i\theta})d\theta \right|$$

$$= \left| f(0)\overline{g}(0) + \frac{2}{\pi} \int_{\mathbb{D}} f'(z)\overline{g}'(z)\log \frac{1}{|z|}dm(z) \right|$$

$$\leq \int_{\mathbb{D}} |f'(z)||g'(z)||(1 - |z|^2)dm(z)$$

$$\leq \left( \int_{\mathbb{D}} |g'(z)|^2\omega(z)(1 - |z|^2)dm(z) \right)^{\frac{1}{2}} \left( \int_{\mathbb{D}} |f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) \right)^{\frac{1}{2}}$$

$$\leq \left( \int_{\mathbb{D}} (\omega(z))d\mu(z) \right)^{\frac{1}{2}} \left( \int_{\mathbb{D}} |f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) \right)^{\frac{1}{2}}$$

$$\leq \|\mu\|_p^{\frac{1}{2}} \left( \int_{\mathbb{D}} |f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) \right)^{\frac{1}{2}}$$

$$\leq \|g\|_{\mathcal{CA}_p} \left( \int_{\mathbb{D}} |f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) \right)^{\frac{1}{2}}$$

Hence,

$$|\langle f, g \rangle| \leq \|g\|_{\mathcal{CA}_p}\|f\|_{\mathcal{BA}_p},$$

namely, $\mathcal{CA}_p \subset \mathcal{BA}_p^*$.

On the other hand, suppose $L \in \mathcal{BA}_p^*$. If

$$D(f)(z) = f'(z)(1 - |z|^2) \quad \forall \ z \in \mathbb{D},$$

then $D$ is an isometric map from $\mathcal{BA}_p$ into $\mathcal{T}_p^1$. Since $\mathcal{T}_p^1$ is a Banach space under $\|\cdot\|_{\mathcal{T}_p^1}$, it follows from the Hahn-Banach Theorem and Lemma 2.2 that one can select a function

\[ \text{...} \]
thereby finding

\[ L(f) = \int_{\mathbb{D}} f'(z)(1 - |z|^2)\overline{h(z)}\,dm(z) \quad \forall \quad f \in \mathcal{B}\mathcal{A}_p. \]

Since

\[ \int_{\mathbb{T}} N(\omega)(\xi)\,d\Lambda_{\omega}(\xi) \leq 1 \implies (1 - |z|^2)^p \omega(z) \leq 1 \quad \forall \quad z \in \mathbb{D}, \]

one has

\[ \int_{\mathbb{D}} |f'(z)(1 - |z|^2)^2\,dm(z) \leq \int_{\mathbb{D}} |f'(z)(1 - |z|^2)^{1+p}\,dm(z) \leq \int_{\mathbb{D}} |f'(z)(\omega(z))^{-1}(1 - |z|^2)\,dm(z) \]

holds for any \( \omega \) used in \( \mathcal{B}\mathcal{A}_p \). Thus we utilize the reproducing formula in [24, p.120] or [34, p.81] to achieve

\[ f'(z) = \frac{2}{\pi} \int_{\mathbb{D}} \frac{f'(w)(1 - |w|^2)}{(1 - \overline{w}z)^3}\,dm(w) \quad \forall \quad f \in \mathcal{B}\mathcal{A}_p, \]

thereby finding

\[
\begin{align*}
L(f) &= \frac{1}{\pi} \int_{\mathbb{D}} f'(z)(1 - |z|^2)\overline{h(z)}\,dm(z) \\
&= \frac{1}{\pi} \int_{\mathbb{D}} f'(w) \left( \frac{2}{\pi} \int_{\mathbb{D}} \overline{h(z)} \frac{(1 - |z|^2)}{(1 - \overline{w}z)^3}\,dm(z) \right) (1 - |w|^2)\,dm(w) \\
&= \frac{1}{\pi} \int_{\mathbb{D}} f'(w) \overline{g}'(w)(1 - |w|^2)\,dm(w),
\end{align*}
\]

where

\[ g(w) = \frac{2}{\pi} \int_{0}^{\infty} \left( \int_{\mathbb{D}} \frac{h(z)(1 - |z|^2)}{(1 - utz)^3}\,dm(z) \right) du. \]

Note that

\[ g(0) = 0 \quad \text{&} \quad \left| \int_{\mathbb{D}} f'(w) \overline{g}'(w)(1 - |w|^2)\,dm(w) \right| \approx \left| \int_{\mathbb{D}} f'(w) \overline{g}'(w) \log \left( \frac{1}{|w|} \right)\,dm(w) \right|. \]

In terms of the Hardy-Littlewood identity, we get \( |L(f)| \approx |\langle f, g \rangle| \). Note also that

\[ d\mu(z) = |h(z)|^2(1 - |z|^2)\,dm(z) \]

is a \( p \)-Carleson measure on \( \mathbb{D} \). So, Lemma [2.3] is employed to derive that

\[ |g'(w)|^2(1 - |w|^2)\,dm(z) \]

is also a \( p \)-Carleson measure on \( \mathbb{D} \), and then \( g \in \mathcal{C}\mathcal{A}_p \). Therefore, \( \mathcal{B}\mathcal{A}_p^* = \mathcal{C}\mathcal{A}_p \). Furthermore, applying a consequence of the Hahn-Banach Extension Theorem (see [11 p.48]) to \( \mathcal{B}\mathcal{A}_p \), we see that if \( f \in \mathcal{B}\mathcal{A}_p \) is a nonzero then there exists \( L \in \mathcal{B}\mathcal{A}_p^* = \mathcal{C}\mathcal{A}_p \) such that

\[ ||L|| = 1 \quad \text{&} \quad ||f||_{\mathcal{B}\mathcal{A}_p} = L(f) = \langle f, g \rangle. \]

With the help of the foregoing argument, we can find a function \( g \in \mathcal{C}\mathcal{A}_p \) such that

\[ ||g||_{\mathcal{C}\mathcal{A}_p} \leq 1 \quad \text{&} \quad L(f) = \langle f, g \rangle \quad \forall \quad f \in \mathcal{B}\mathcal{A}_p. \]
This clearly implies that
\[ \|f\|_{\mathcal{B}A_p} \approx \sup \{|\langle f, g \rangle| : g \in \mathcal{C}A_p \& \|g\|_{\mathcal{C}A_p} \leq 1\} \quad \forall \ f \in \mathcal{B}A_p. \]

2.3. Second predual. It is well-known that
\[
\begin{align*}
  \{\mathcal{H}^2\}^{**} & = \mathcal{H}^2; \\
  \{\mathcal{VMOA}\}^{**} & = \mathcal{BMOA}; \\
  \{\mathcal{A}_{0,p}\}^{**} & = \mathcal{A}_{p/2} \quad \forall \ \ p \in (1, 3).
\end{align*}
\]

So, it remains to see whether the above identification can be extended to the analytic Morrey space. To do so, we need two lemmas.

**Lemma 2.5.** For \((p, r) \in (0, 1) \times (0, 1)\) and \(f \in \mathcal{B}A_p\) let \(f_r(z) = f(rz)\). Then
\[
\lim_{r \rightarrow 1} \|f_r - f\|_{\mathcal{B}A_p} = 0 \quad \& \quad \|f_r\|_{\mathcal{B}A_p} \leq \|f\|_{\mathcal{B}A_p}.
\]

Thus the polynomials are dense in \(\mathcal{B}A_p\).

**Proof.** Choosing \(\omega(z) = 1\) in the definition of \(\mathcal{B}A_p\), one gets that any bounded analytic function \(f\) with \(f(0) = 0\) must be in \(\mathcal{B}A_p\) and hence \(f_r \in \mathcal{B}A_p\) for any \(0 < r < 1\). Suppose now \(f \in \mathcal{B}A_p\). An application of Poisson’s formula to \(f\) gives
\[
f_r(z) = \frac{1}{2\pi} \int_{\mathbb{T}} f(z\zeta) \frac{1 - r^2}{|1 - rz|^2} |d\zeta|.
\]

Derivating both sides of the above equality with respect to \(z\) and using Minkowski’s inequality, we have
\[
|f'_r(z)| \leq \left( \frac{1}{2\pi} \int_{\mathbb{T}} |f'(z\zeta)||\frac{1 - r^2}{|1 - rz|^2}|d\zeta| \right)^{1/2}.
\]

For any \(\epsilon > 0\), by the definition of \(\mathcal{B}A_p\), there is a nonnegative \(\omega_e\) on \(\mathbb{D}\) such that
\[
\left( \int_{\mathbb{D}} |f'(z)|^2 (\omega_e(z))^{-1} (1 - |z|^2) dm(z) \right)^{1/2} < \|f\|_{\mathcal{B}A_p} + \epsilon
\]

According to the rotation invariance of \(\omega_e\), one has that for any \(\zeta \in \mathbb{T}\)
\[
\left( \int_{\mathbb{D}} |f'(z\zeta)|^2 (\omega_e(z\zeta))^{-1} (1 - |z|^2) dm(z) \right)^{1/2} < \|f\|_{\mathcal{B}A_p} + \epsilon
\]

Using the inequalities (2.1)–(2.2) and Fubini’s theorem, we obtain
\[
\begin{align*}
\|f_r\|_{\mathcal{B}A_p}^2 & \leq \int_{\mathbb{D}} |f'_r(z)|^2 (\omega_e(z))^{-1} (1 - |z|^2) dm(z) \\
& \leq \frac{1}{2\pi} \left( \int_{\mathbb{T}} \left( \int_{\mathbb{D}} |f'(z\zeta)|^2 (\omega_e(z\zeta))^{-1} (1 - |z|^2) dm(z) \right) \frac{1 - r^2}{|1 - rz|^2} |d\zeta| \right)^{1/2} \\
& \leq \frac{1}{2\pi} \int_{\mathbb{T}} \|f\|_{\mathcal{B}A_p}^2 + \epsilon \frac{1 - r^2}{|1 - rz|^2} |d\zeta| \\
& = (\|f\|_{\mathcal{B}A_p} + \epsilon)^2.
\end{align*}
\]
Letting $\epsilon \to 0$ in the above estimates, one obtains $\|f_r\|_{B\mathcal{A}_p} \leq \|f\|_{B\mathcal{A}_p}$.

In the sequel we prove $\lim_{r \to 1} \|f_r - f\|_{B\mathcal{A}_p} = 0$. For any $0 < \delta < 1$, we have

$$\int_{D} |f_r'(z) - f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) = \int_{|z| \leq \delta} |f_r'(z) - f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) + \int_{1 - \delta < |z| \leq 1} |f_r'(z) - f'(z)|^2(\omega(z))^{-1}(1 - |z|^2)dm(z) = Int_1 + Int_2.$$

Note that $f, f_r \in B\mathcal{A}_p$. So, for any $\epsilon > 0$ there exists $\eta > 0$ such that

$$1 - \eta < \delta < 1 \implies Int_2 < \epsilon^2.$$

Now, fixing some $\delta$ and noticing that $f_r$ is uniformly convergent to $f$ on the compact set $\{z \in \mathbb{C} : |z| \leq \delta\}$, one gains a number $\lambda > 0$ such that

$$1 - \lambda < r < 1 \implies Int_1 < \epsilon^2.$$

Thus, putting all together gives $\lim_{r \to 1} \|f_r - f\|_{B\mathcal{A}_p} = 0$. □

A modification of the techniques used in [26] produces the following density result for $C\mathcal{A}_{0,p}$.

**Lemma 2.6.** For $(p, r) \in (0, 1) \times (0, 1)$ let $f \in C\mathcal{A}_p$ with $f_r(z) = f(rz)$. Then the following are equivalent:

- $f \in C\mathcal{A}_{0,p}$, i.e., $\lim_{|T| \to 0} \sqrt{|T|^{-p} \int |f(\xi) - f||d\xi|} = 0$;
- $\lim_{r \to 1} \|f_r - f\|_{C\mathcal{A}_p} = 0$;
- $f$ belongs to the closure of all polynomials in the norm $\| \cdot \|_{C\mathcal{A}_p}$;
- For any $\epsilon > 0$ there is a $g \in C\mathcal{A}_{0,p}$ such that $\|g - f\|_{C\mathcal{A}_p} < \epsilon$.

**Proof.** It is enough to verify that

$$f \in C\mathcal{A}_{0,p} \implies \lim_{r \to 1} \|f_r - f\|_{C\mathcal{A}_p} = 0.$$

Suppose now $f \in C\mathcal{A}_{0,p}$. Then

$$\lim_{|\epsilon| \to 1} E(f_r - f, a) = 0$$

holds for any fixed $r \in (0, 1)$. Meanwhile,

$$\lim sup_{r \to 1} E(f_r - f, a) = 0.$$

Also, it is not hard to establish
\[ \|f\|_{\mathcal{C}\mathcal{A}_{p,\ast}} = \sup_{a \in \mathbb{D}} \{E(f, a)\}^{\frac{1}{2}} \]
\[ = \sup_{a \in \mathbb{D}} \left(1 - |a|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |(f_r)'(z)|^2 (1 - |\sigma_a(z)|) dm(z)\right)^{\frac{1}{2}} \]
\[ \leq \frac{1}{2\pi} \int_{\mathbb{T}} \sup_{a \in \mathbb{D}} \left(1 - |a|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |f'(z\xi)|^2 (1 - |\sigma_a(z)|) dm(z)\right)^{\frac{1}{2}} \frac{1 - r^2}{|1 - r\xi|^2} |d\xi| \]
\[ \leq \|E(f, a)\|^{\frac{1}{2}} \]
\[ = \|f\|_{\mathcal{C}\mathcal{A}_{p,\ast}}. \]

Summing up, one finds that
\[ \lim_{r \to 1} \|f_r - f\|_{\mathcal{C}\mathcal{A}_{p,\ast}} = 0 \quad \text{and} \quad \lim_{r \to 1} \|f_r - f\|_{\mathcal{C}\mathcal{A}_p} = 0, \]
as desired. \qed

Below is the second preduality for the analytic Morrey space.

**Theorem 2.7.** Let \( p \in (0, 1] \). Then \( \mathcal{B}\mathcal{A}_p \) is isomorphic to the dual of \( \mathcal{C}\mathcal{A}_{0,p} \) under the pairing
\[ \langle f, g \rangle = \int_{\mathbb{T}} f(\xi)g(\xi) \frac{|d\xi|}{2\pi} \quad \forall \quad (f, g) \in \mathcal{C}\mathcal{A}_{0,p} \times \mathcal{B}\mathcal{A}_p. \]
That is, \( [\mathcal{C}\mathcal{A}_{0,p}]^* = \mathcal{B}\mathcal{A}_p \). Thus \( [\mathcal{C}\mathcal{A}_{0,p}]^{**} = \mathcal{C}\mathcal{A}_p \).

**Proof.** On the one hand, for \( g \in \mathcal{B}\mathcal{A}_p \) let
\[ S_g(f) = T_f(g) = \int_{\mathbb{T}} f(\xi)g(\xi) \frac{|d\xi|}{2\pi} \quad \forall \quad f \in \mathcal{C}\mathcal{A}_{0,p}. \]
Then \( S_g \) is linear. Also
\[ |S_g(f)| \leq \|f\|_{\mathcal{C}\mathcal{A}_p} \|g\|_{\mathcal{B}\mathcal{A}_p}. \]
Thus
\[ S_g \in [\mathcal{C}\mathcal{A}_{0,p}]^* \quad \text{and} \quad \|S_g\| \leq \|g\|_{\mathcal{B}\mathcal{A}_p}. \]
Conversely, suppose \( S \in [\mathcal{C}\mathcal{A}_{0,p}]^* \). For each \( n \in \{0, 1, 2, \ldots\} \) let
\[ \phi_n(z) = z^n \quad \forall \quad z \in \mathbb{D}. \]
Then \( \phi_n \in \mathcal{C}\mathcal{A}_{0,p} \) for all \( n \). According to [30] Lemma 1, an elementary calculation shows
\[ 1 \leq \|\phi_n\|_{\mathcal{C}\mathcal{A}_{p,\ast}} = \sup_{a \in \mathbb{D}} \left(1 - |a|^2\right)^{\frac{1}{2}} \|\phi_n \circ \sigma_a - \phi_n(a)\|_{H^p} \leq 2, \]
where \( \sigma_a(z) = \frac{a - z}{1 - az} \). Set
\[ b_n = S(\phi_n) \quad \forall \quad n = 0, 1, 2, \ldots \]
Then we see that
\[ |b_n| = |S(\phi_n)| \leq \|S\| \|\phi_n\|_{\mathcal{C}\mathcal{A}_p} \leq \|S\|. \]
It follows that the power series \( \sum_{n=0}^{\infty} b_n z^n \) has its radius of convergence greater than or equal to 1. Define
\[
g(z) = \sum_{n=0}^{\infty} b_n z^n \quad \forall \quad z \in D.
\]
Then \( g \) is analytic in \( D \). For \( 0 < r < 1 \), set
\[
g_r(z) = g(rz) \quad \forall \quad z \in D.
\]
We are going to show that
\[
(2.3) \quad g \in \mathcal{B}_p \quad \text{with} \quad \|g\|_{\mathcal{B}_p} \lesssim ||S||
\]
and that
\[
(2.4) \quad S = S_g.
\]
In terms of Lemma 2.5, (2.3) is equivalent to
\[
\|g_r\|_{\mathcal{B}_p} \lesssim ||S|| \quad \forall \quad r \in (0, 1).
\]
Theorem 2.4 shows that
\[
\|T_f\| \approx \|f\|_{C\mathcal{A}_p}.
\]
Taking \( f \in \mathcal{B}\mathcal{A}_p \) and setting \( f(z) = \sum_{n=0}^{\infty} a_n z^n \), one obtains
\[
f_r(z) = f(rz) = \sum_{n=0}^{\infty} a_n r^n z^n \quad \forall \quad z \in D.
\]
Clearly,
\[
\sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n} a_k \phi_k \to f_r \quad \text{as} \quad n \to \infty
\]
uniformly in \( \overline{D} \) and, hence, in \( \mathcal{B}\mathcal{A}_p \), which, since \( S \in [C\mathcal{A}_{0,p}]^* \), implies
\[
S(\sum_{k=0}^{n} a_k r^k \phi_k) = \sum_{k=0}^{n} a_k \bar{b}_k r^k \to S(f_r) \quad \text{as} \quad n \to \infty.
\]
That is,
\[
(2.5) \quad \sum_{k=0}^{\infty} a_k \bar{b}_k r^k = S(f_r).
\]
Now (2.5) can be written as
\[
S(f_r) = T_{g_r}(f).
\]
Using Lemmas 2.5, 2.6, we have \( \|f_r\|_{\mathcal{B}\mathcal{A}_p} \lesssim \|f\|_{\mathcal{C}\mathcal{A}_p} \). Thus,
\[
|T_f(g_r)| = |T_{g_r}(f)| = |S(f_r)| \leq ||S|| \|f_r\|_{C\mathcal{A}_p} \lesssim ||S|| \|f\|_{C\mathcal{A}_p}.
\]
From Theorem 2.4 it follows that
\[
\|g_r\|_{\mathcal{B}\mathcal{A}_p} = \sup_{f \in \mathcal{C}\mathcal{A}_p \setminus \{0\}} \frac{|T_f(g_r)|}{\|T_f\|} \lesssim \sup_{f \in \mathcal{C}\mathcal{A}_p \setminus \{0\}} \frac{|T_f(g_r)|}{\|f\|_{C\mathcal{A}_p}}.
\]
Therefore,
\[
\|g_r\|_{\mathcal{B}\mathcal{A}_p} \lesssim ||S||.
\]
However, Lemma 2.5 tells us that
\[ \|g\|_{\mathcal{B}_p} \lesssim \|S\|, \]
which completes the proof of (2.3).

Now, if \( f \in \mathcal{CA}_0 \), according to Lemma 2.6 and the continuity of \( S \), one has that
\[ S(f) = \lim_{r \to 1} S(f_r) = \lim_{r \to 1} \sum_{k=0}^{\infty} \overline{b_k} a_k r^k. \]
Hence, the proof of (2.4) is completed.

As an immediate consequence of the second preduality established above, the following covers the corresponding \( \mathcal{BMOA} \)-result in [7, 8, 25] and \( \text{Lip}_{\alpha} \)-result in [17].

**Corollary 2.8.** Let \( p \in [0, 3) \) and \( f \in \mathcal{CA}_p \). Then
\[ \text{dist}(f, \mathcal{CA}_0, \mathcal{CA}_p) = \lim_{|I| \to 0} \left( |I|^{-p} \int_I |f - f_I|^2 |d\xi| \right)^{\frac{1}{p}}. \]

**Proof.** Let
\[
\begin{aligned}
X &= \mathcal{H}^2 \setminus \mathbb{C}; \\
Y &= \mathcal{H}^1; \\
L &= \{ L_I : L_I = \chi_I |I|^{-p} (f - f_I) \quad \& \quad \emptyset \neq I \subseteq \mathbb{T} \text{ is an arc} \},
\end{aligned}
\]
where \( \chi_I \) is the characteristic function of \( I \) and \( f_I = |I|^{-1} \int_I f(\xi) |d\xi| \). Now, if
\[
\begin{aligned}
f &\in \mathcal{CA}_p; \\
f_n &= f \ast P_{1-\frac{1}{n}} \quad \forall \ n = 1, 2, 3, \ldots; \\
P_r(\theta) &= \frac{1-r^2}{|e^{i\theta} - r|^2},
\end{aligned}
\]
then an application of the second preduality \([\mathcal{CA}_0, \mathcal{CA}_p]^{**} = \mathcal{CA}_p \) (cf. Theorem 2.7) and Lemma 2.6 gives that \( \mathcal{CA}_p \) and \( \mathcal{CA}_0, \mathcal{CA}_p \) satisfy Assumption A of [20, Theorem 2.3], and hence one has
\[ \text{dist}(f, \mathcal{CA}_p) = \lim_{|I| \to 0} \left( |I|^{-p} \int_I |f - f_I|^2 |d\xi| \right)^{\frac{1}{p}}. \]

\[ \square \]

### 3. Three operators

#### 3.1. Superposition

Denote by \( \mathcal{H}(\mathbb{D}) \) the space of analytic functions on \( \mathbb{D} \). If \( \mathcal{X} \) and \( \mathcal{Y} \) are two subspaces of \( \mathcal{H}(\mathbb{D}) \) and \( \phi \) is a complex-valued function \( \mathbb{C} \) such that \( \phi \circ f \in \mathcal{Y} \) whenever \( f \in \mathcal{X} \), we say that \( \phi \) acts by superposition from \( \mathcal{X} \) into \( \mathcal{Y} \). If \( \mathcal{X} \) and \( \mathcal{Y} \) contain the linear functions, then \( \phi \) must be an entire function. The superposition \( \mathcal{S}^\phi : \mathcal{X} \to \mathcal{Y} \) with symbol \( \phi \) is then defined by \( \mathcal{S}^\phi(f) = \phi \circ f \). A basic question is when \( \mathcal{S}^\phi \) map \( \mathcal{X} \) into \( \mathcal{Y} \) continuously? This question has been studied for many distinct pairs \((\mathcal{X}, \mathcal{Y})\) - see e.g. [4, 5, 15, 29]. In this section, we are interested in the analytic Morrey space and its little one, and have the following result which extends the case of \( p \in (1, 3] \) in [32].

**Theorem 3.1.** Let \( p \in [0, 1) \). Then \( \mathcal{S}^\phi \) is bounded on \( \mathcal{CA}_p \) or \( \mathcal{CA}_0, \mathcal{CA}_p \) if and only if \( \phi(z) = az + b \) for some \( a, b \in \mathbb{C} \).
Proof. Note that (cf. [30, 31]) if \( p \in [0, 2) \) then
\[
\begin{cases}
  f_0(z) = (1 - |b|^2)^{1/2} / (1 - \bar{b}z) \implies \sup_{b \in \mathbb{D}} \| f_0 \|_{C_A p} < \infty; \\
  f \in C_A p \implies |f'(z)| \leq \frac{\| f \|_{C_A p}}{(1 - |z|^2)^{1/2}} \quad \forall \quad z \in \mathbb{D}.
\end{cases}
\]

Firstly, if \( S^\phi \) is bounded on \( C_A p \), then an application of Lemma 2.6 yields that \( S^\phi \) is bounded on \( C_A 0, p \).

Secondly, if \( S^\phi \) is bounded on \( C_A p \), then for \( f \in C_A p \) one has
\[
| (S^\phi(f))'(z) | = | \phi'(f(z)) | | f'(z) | \leq \frac{\| f \|_{C_A p}}{(1 - |z|^2)^{1/2}} \quad \forall \quad z \in \mathbb{D}.
\]
Choosing the following \( C_A p \)-function
\[
f_{\theta, b}(z) = e^{\theta z} (1 - |b|^2)^{1/2} / (1 - \bar{b}z)
\]
in the last inequality, one gets
\[
| \phi'(f_{\theta, b}(z)) | | f_{\theta, b}'(z) | = | \phi'(f_{\theta, b}(z)) | \left( \frac{|b|(1 - |b|^2)^{1/2}}{|1 - \bar{b}z|^2} \right) \leq \frac{\| f_{\theta, b} \|_{C_A p}}{(1 - |z|^2)^{1/2}}.
\]
Note that
\[
\sup_{\theta, b} \| f_{\theta, b} \|_{C_A p} < \infty.
\]
So there is a positive \( M \) independent of \( (\theta, b) \) such that
\[
\sup_{\theta, b} \| f_{\theta, b} \|_{C_A p} \leq M.
\]
In particular, setting \( b = z \) yields
\[
| \phi'(f_{\theta, b}(z)) | | f_{\theta, b}'(z) | = \sup_{b \in \mathbb{D}} | \phi'(f_{\theta, b}(z)) | |b| \leq M \implies | \phi'(e^{\theta z} (1 - |b|^2)^{1/2}) | \leq M \quad \forall \quad b \in \mathbb{D}.
\]
Now letting \( |b| \to 1 \) in the last estimate and noticing \( p \in [0, 1) \), we obtain that the entire function \( \phi \) is bounded on \( \mathbb{C} \). An application of the maximum principle yields that \( \phi \) must be a linear function.

Thirdly, if \( \phi(z) = az + b \) for some \( a, b \in \mathbb{C} \), then \( \phi'(z) = a \), and hence \( S^\phi(f) = af + b \). Hence \( S^\phi \) is bounded on \( C_A p \).

3.2. Backward shift. For a function \( z \mapsto f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( C_A p \), the backward shift operator \( S_b f \) is defined as
\[
S_b(f)(z) = \frac{f(z) - f(0)}{z} = a_1 + a_2 z + a_3 z^2 + ....
\]
As well-known, the backward shift operator plays an important role in the general study of bounded linear operators on Hilbert spaces; see [10]. Moreover, the Hardy space \( H^p \) is invariant under \( S_b \); see [12] for \( p > 1 \) and [3] for \( p \in (0, 1] \). The backward shift operator for \( BMOA \) and Lipschitz spaces have been considered in [33]. Hence, it remains to handle the action of \( S_b \) on the analytic Morrey space \( C_A 0, p < 1 \) and its little one. The following result indicates the behavior of \( S_b \) on \( C_A p \) is particularly good.

**Theorem 3.2.** Let \( p \in (0, 1) \). Then \( S_b \) is a bounded operator on both \( C_A p \) and \( C_A 0, p \).
Proof. Since the polynomials are dense in $CA_{0,p}$ (cf. Lemma 2.6), it suffices to prove that boundedness of $S_b$ on $CA_p$. To do so, suppose $f \in CA_p$.

The decay of $f'$ ensures
\[
\int_\mathbb{D} |f'(z)|(1-|z|^2) dm(z) \leq \|f\|_{CA_p} \int_\mathbb{D} (1-|z|^2)^{-1} dm(z) < \infty.
\]

Hence $f'$ is in the weighted Bergman space $L^1_u(\mathbb{D}, (1-|z|^2) dm(z))$ on $\mathbb{D}$. Using the reproducing kernel formula on [$24$, p.120]:
\[
f'(z) = 2\pi^{-1} \int_\mathbb{D} f'(w)(1-|w|^2) \frac{dw}{(1-\overline{w}z)^3} dm(w) \quad \forall \ z \in \mathbb{D},
\]
one obtains
\[
S_b(f)(z) = \int_0^1 f'(tz) dt = 2\pi^{-1} \int_\mathbb{D} f'(w)(1-|w|^2) \left( \int_0^1 \frac{dt}{(1-\overline{w}z)^3} \right) dm(w).
\]

Differentiating the above equality with respect to $z$, one gets
\[
(S_b(f))'(z) = 6\pi^{-1} \int_\mathbb{D} \left( \int_0^1 \frac{t\overline{w}}{(1-t\overline{w}z)^4} dt \right) f'(w)(1-|w|^2) dm(w).
\]

Also, it is not hard to estimate
\[
\left| \int_0^1 \frac{t\overline{w}}{(1-t\overline{w}z)^4} dt \right| \leq \frac{1}{|1-\overline{w}z|^3}.
\]

A combination of (3.1) and (3.2) yields
\[
|S_b(f)'(z)| \leq \int_\mathbb{D} \frac{|f'(w)| (1-|w|^2)}{|1-\overline{w}z|^3} dm(w).
\]

Note that $f \in H^{2,p}$. So $|f'(z)|^2 (1-|z|^2) dm(z)$ is a $p$-Carleson measure. From the Lemma 2.3 it follows that $|S_b(f)'(z)|^2 (1-|z|^2) dm(z)$ is also a $p$-Carleson measure. It is obvious that $S_b(f)$ is analytic on $\mathbb{D}$. Thus,
\[
S_b(f) \in CA_p \quad \text{with} \quad \|S_b(f)\|_{CA_p} \leq \|f\|_{CA_p}.
\]

\[\Box\]

3.3. Schwarzian derivative. Let $f$ be a conformal mapping from $\mathbb{D}$ into a simply connected domain in $\mathbb{C}$. We say that $f(\mathbb{D})$ is a $\mathcal{X}$ domain whenever $\log f'$ belongs to an analytic function space $\mathcal{X}$ on $\mathbb{D}$.

Recall that $\mathcal{B}$ is the Bloch space of all analytic functions $f$ in $\mathbb{D}$ with
\[
\sup_{z \in \mathbb{D}} (1-|z|^2)|f'(z)| < \infty
\]
and $\mathcal{B}_0$ is the little Bloch space consisting of all functions $f \in \mathcal{B}$ with
\[
\lim_{|z| \to 1} (1-|z|^2) |f'(z)| = 0.
\]

Pommerenke [22] proved that $f \in \mathcal{B}$ if and only if there is a constant $a \in \mathbb{C}$ and a univalent function $f$ such that $g = a \log f'$. So, it is interesting to characterize such a
univalent $f$ that log $f' \in X$ and then to establish whether this condition implies some nice geometric properties of the image domain $f(D)$. To be more precise, recall that the Schwarzian derivative of a univalent (analytic) function $f$ on $D$ is defined by

$$S(f) = \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2.$$  

This fully nonlinear operator plays an important role in geometric function theory, conformal field theory, differential equations and others. Interestingly, if $f$ is univalent on $D$, then

$$|f''(z)/f'(z)| (1 - |z|^2) \leq 6 \quad \& \quad (1 - |z|^2)^2 |S(f)(z)| \leq 6 \quad \forall \quad z \in D.$$  

Conversely, if

$$|f''(z)/f'(z)| (1 - |z|^2) \leq 1 \quad \text{or} \quad (1 - |z|^2)^2 |S(f)(z)| \leq 2 \quad \forall \quad z \in D,$$

then $f$ is univalent on $D$, for more details see [23]. Moreover, $S(f)$ vanishes identically if and only if $f$ is a Möbius mapping. Recently, there have been some results linking the Schwarzian derivative of a univalent analytic function to the characterization of some analytic function spaces. According to Astala-Zinsmeister [6] and Pérez-González-Rättyä [21], one has

$$\begin{cases} \log f' \in \mathcal{BMOA} \iff |S(f)(z)|^2 (1 - |z|^2)^3 dm(z) \text{ is a bounded Carleson measure on } D; \\ \log f' \in \mathcal{VMOA} \iff |S(f)(z)|^2 (1 - |z|^2)^3 dm(z) \text{ is a vanishing Carleson measure on } D. \end{cases}$$

Further, $Q_p$-domains and even more general $F(p, q, s)$-domains were studied in [16, 19, 29, 35]. The above review actually leads to a consideration of the case of analytic Campanato spaces.

**Theorem 3.3.** Let $f$ be a univalent function in $D$.

- Case $p \in (0, 1)$: if $\log f' \in \mathcal{CA}_p$ ($\log f' \in \mathcal{CA}_0, p$) then $|S(f)(z)|^2 (1 - |z|^2)^3 dm(z)$ is a bounded (vanishing) $p$-Carleson measure and $\log f' \in \mathcal{B}$. Conversely, if $|S(f)(z)|^2 (1 - |z|^2)^3 dm(z)$ is a bounded (vanishing) $p$-Carleson measure and $\log f' \in \mathcal{B}_0$, then $\log f' \in \mathcal{CA}_p$ ($\log f' \in \mathcal{CA}_0, p$).

- Case $p \in (1, 3)$: $\log f' \in \mathcal{CA}_p$ ($\log f' \in \mathcal{CA}_0, p$) if and only if $|S(f)(z)|^2 (1 - |z|^2)^3 dm(z)$ is a bounded (vanishing) $p$-Carleson measure and $\log f' \in \mathcal{B}_0$.

**Proof.** It is enough to verify boundedness.

Under $p \in (0, 1)$, we make the following consideration. For simplicity, let

$$P(f)(z) = (\log f')(z)' = \frac{f''(z)}{f'(z)}.$$  

Then

$$S(f)(z) = (P(f))(z)' - \frac{1}{2} (P(f)(z))^2.$$  

Note that log $f' \in \mathcal{CA}_p$ and $|P(f)(z)|^2 (1 - |z|^2) dm(z)$ is a $p$-Carleson measure if and only if $|(P(f))(z)|^2 (1 - |z|^2)^3 dm(z)$ is a $p$-Carleson measure. So, applying the Cauchy-Schwarz inequality, one obtains

$$|S(f)(z)|^2 \leq 2 |(P(f))(z)|^2 + |P(f)(z)|^4 \leq |P(f)(z)|^2 + |P(f)(z)|^4.$$
Using Pommerenke’s result in [22], one always has \( \log f' \in \mathcal{B} \), that is,
\[
\sup_{z \in \mathbb{D}} |P(f)(z)(1 - |z|^2)| = \sup_{z \in \mathbb{D}} |(\log f')(z)'|(1 - |z|^2) < \infty.
\]

This in turns implies that
\[
|S(f)(z)|^2(1 - |z|^2)^3 \leq |(P(f))'(z)|^2(1 - |z|^2)^3 + |P(f)(z)|^2(1 - |z|^2),
\]
thereby giving that \( |S(f)(z)|^2(1 - |z|^2)^3 dm(z) \) is a bounded \( p \)-Carleson measure.

For the converse part, set
\[
I_a = \int_{\mathbb{D}} |(P(f))'(z)|^2(1 - |z|^2)^3 |\sigma'_a(z)|^p dm(z).
\]

Note that
\[
S(f)(z) = (P(f))'(z) - \frac{1}{2}(P(f)(z))^2.
\]

So
\[
I_a \leq \int_{\mathbb{D}} \left( 2|S(f)(z)|^2(1 - |z|^2)^3 + \frac{1}{2}|P(f)(z)|^4(1 - |z|^2)^3 \right) |\sigma'_a(z)|^p dm(z).
\]

If \( \log f' \in \mathcal{B}_0 \), then for any \( \epsilon > 0 \), there exists \( 0 < r_\epsilon < 1 \) such that
\[
|P(f)(z)|(1 - |z|^2) < \epsilon \quad \text{as} \quad |z| > r_\epsilon.
\]

Hence, there exists some \( \kappa > 0 \) depending only on \( p \) such that
\[
\int_{|z| > r_\epsilon} |P(f)(z)|^4(1 - |z|^2)^3 |\sigma'_a(z)|^p dm(z)
\leq \epsilon^2 \int_{\mathbb{D}} |P(f)(z)|^2(1 - |z|^2)^3 |\sigma'_a(z)|^p dm(z)
\leq \epsilon^2 \kappa \int_{\mathbb{D}} |(P(f))'(z)|^2(1 - |z|^2)^3 |\sigma'_a(z)|^p dm(z)
= \epsilon^2 \kappa I_a.
\]

At the same time, note that
\[
|P(f)(z)|(1 - |z|^2) \leq 6 \quad \& \quad (1 - |z|^2)|\sigma'_a(z)| = 1 - |\sigma_a(z)|^2.
\]

So one obtains
\[
\int_{|z| \leq r_\epsilon} |P(f)(z)|^4(1 - |z|^2)^3 |\sigma'_a(z)|^p dm(z)
\leq 6^4 \int_{|z| \leq r_\epsilon} (1 - |z|^2)^{-1}|\sigma'_a(z)|^p dm(z)
\leq 6^4 \int_{|z| \leq r_\epsilon} (1 - |z|^2)^{-1-p}(1 - |\sigma_a(z)|^2)^p dm(z)
\leq \frac{6^4}{(1 - r_\epsilon^2)^{1+p}}.
\]

Hence
\[
\int_{\mathbb{D}} |P(f)(z)|^4(1 - |z|^2)^3 |\sigma'_a(z)|^p dm(z) \leq \epsilon^2 \kappa I_a + \frac{6^4}{(1 - r_\epsilon^2)^{1+p}}.
\]
Upon choosing $0 < \epsilon$ to be so small that $\epsilon < \sqrt{2/\kappa}$, one gets from (3.3) and (3.4) that
\[
(2 - \epsilon^2 \kappa) I_a \leq 4 \int_D |S(f)(z)|^2 (1 - |z|^2)^3 |d\sigma'_a(z)|^p \, dm(z) + \frac{6^4}{(1 - r^2)^{1+\nu}}.
\]
Since $|S(f)(z)|^2 (1 - |z|^2)^3 \, dm(z)$ is a bounded $p$-Carleson measure and $2 - \epsilon^2 \kappa > 0$, it follows that
\[
\sup_{a \in D} \int_D |(P(f))'(z)|^2 (1 - |z|^2)^3 |d\sigma'_a(z)|^p \, dm(z) < \infty.
\]
Hence $|(P(f))'(z)|^2 (1 - |z|^2)^3 \, dm(z)$ is a bounded $p$-Carleson measure. Consequently, $\log f' \in CA_\wp$.

Under $p \in (1, 3)$, one has that $CA_\wp$ is equal to $A_{Q_{\frac{1}{a^2}}}$ comprising all analytic functions on $D$ with
\[
\sup_{z \in D} (1 - |z|^2)^{\frac{1}{a^{2\nu}}} |f'(z)| < \infty.
\]
Hence $CA_\wp \subset B_0$. Therefore, the argument for the case $p \in (0, 1)$ can be utilized to complete the proof of the case $p \in (1, 3)$. \hfill \Box

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