Vibration of Workpieces during Aggressive Turning Operations

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Abstract. Turning is a widely used metal cutting operation. Because of the dynamic interaction between the cutter and the workpiece, vibration is excited during turning operations. If the vibration is severe, there will be degradation of surface quality of the workpiece, excessive wear and breakage of the cutter. Chatter can also take place. This paper presents a dynamic model for aggressive turning operations and considers the influence on vibration of the time-dependent reduction of workpiece diameter. The workpiece is modelled as a beam rotating about its longitudinal axis and the cutter provides a moving load. Simulated numerical examples are presented.

1. Introduction
The dynamic model of a rotating beam subjected to different loads has long been studied. Nevertheless, vibration remains a major problem in turning. Lee et al [1] and Katz et al [2] studied vibration of a rotating shaft as a beam based on Euler, Rayleigh, and Timoshenko beam theories under a transverse load moving at constant speed. Huang and his co-workers [3-4] investigated the response of the rotating cylindrical shell and spinning orthotropic beams subjected to harmonic moving loads. Argento and Scott [5] examined a rotating Timoshenko beam excited by a distributed surface force traveling with acceleration. Vibration of a rotating Timoshenko with general boundary conditions subjected to a moving load was investigated by Zu and Han [6]. Sinha [7] studied parametric stability of a rotor-disk system under a nonmoving dynamic axial load. Lee [8] introduced the axial force that was constant and nonmoving and found it had a significant effect on the magnitude of the dynamic response. Zibdeh and Juma [9] considered the moving load as a random force. El-Saeidy [10] analyzed boundary conditions of nonlinear stiffness due to clearance in rolling. He included bending moments in moving load and used the finite element method to study the rotating members subjected to moving load. Sheu and Yang [11] researched the dynamic response of a spinning Rayleigh beam with mass eccentricity but without moving load. Dassanayake and Suh [12] investigated a 3D model as a system of three rotors to simulate workpiece-tool deflections in response to a nonlinear regenerative force with a method of rotor dynamics. He studied different stages of stability for the workpiece and the tool. Ouyang and Wang [13] recently developed a dynamic model for the vibration of a rotating Rayleigh beam subjected to a three-directional load moving in the axial direction. They found that the
bending moment induced by the axial force component could have a significant influence on the dynamic response of the beam.

Chatter prediction has a long history that began with works by Tobias and Fishwick [14] and Tlusty and Polacek [15]. Analytical solutions for turning chatter are mainly in regenerative chatter: as the cutter cuts a workpiece surface, the undulations generated in the previous revolution preserve the tool–workpiece vibrations which are closely coupled to the cutting forces. Refs [16] and [17] explained regenerative chatter with a simplified scenario in turning operation. There are several dynamics models of regenerative chatter [18, 19].

Sims [20] developed a new analytical tuning methodology for chatter suppression with vibration absorbers from a regenerative chatter perspective. Chen and Tsao [21] built two models of the workpiece considered as a flexible beam and the tool simplified as a one degree-of-freedom mass-spring-damper system for an orthogonal cutting. They discussed a stability analysis of regenerative chatter for a cantilever beam in turning process.

One disadvantage of the aforementioned moving-load models and regenerative chatter models is that time-dependent reduction of diameter, variable mass and stiffness of the workpiece are neglected, even though chips are removed during a turning operation. Different depths of cut (DOCs) were shown to affect the fundamental natural frequency of a workpiece [22]. Thus, the time-dependent reduction of the shaft diameter would influence the dynamic response and should be studied.

In this paper, a dynamic model is presented for vibration of a rotating shaft as a Rayleigh beam with a time-dependent reducing diameter subjected to a three-directional force that moves in the axial direction. The dynamic response is analysed at different cutting conditions and ratios of radius to length of the shaft. The model represents a simplified tuning operation in metal cutting.

2. Dynamic model

Turning operation has two moving components, a cutter and a workpiece that is fixed to the spindle and pin-mounted at the tailstock. The shaft has a rotary motion while the cutter moves along a straight line. As a result of this operation, the shaft has three obvious sections: un-machined, being-machined and machined, as shown in Figure 1, where \( l \) is the full length of the shaft, \( l_1 \) is the length of unmachined section of shaft, \( l_2 \) is the total length of unmachined and being-machined section of the shaft, and \( s(t) \) is the variable length from the spindle end to the location of the cutter.

![Figure 1. Shaft configuration](image)

Referring to Figure 2, the circular shaft is rotating around its longitudinal axis, and is subjected to a three-directional cutting force along the \( x \) axis. A rectangular initial frame is used as the coordinate system. \( F_x \), \( F_y \) and \( F_z \) are the three perpendicular components of the cutting force acting on the surface of the shaft in the \( x, y \) and \( z \) directions, respectively, which together play the important role of the shaft material removal. The deflections of the rotating shaft in the \( y \) and \( z \) directions caused by the cutting force components are represented by \( v \) and \( w \).

As a result of the translating axial force \( F_x \), a bending moment \( M_z \) is produced as [13]

\[
M_z = -F_x r_2
\]
where \( r_1 \) is radius of the un-machined section, \( r \) is radius of the machined section, and \( r_2 = (r_1 + r)/2 \) is average radius of the being-machined section.

On the basis of Rayleigh beam theory, the strain energy of the sectioned shaft is [23]:

\[
V = \frac{1}{2} \int_0^{l_1} E I_1 \left( \frac{\partial^2 v}{\partial x^2} \right)^2 + \frac{1}{2} \int_0^{l_2} E I_2 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx + \frac{1}{2} \int_0^{l_3} E I_3 \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \, dx + \frac{1}{2} \int_1^{l_1} \left( \frac{\partial^2 v}{\partial x^2} \right)^2 \, dx \tag{2}
\]

where \( E \) is Young’s modulus, \( I_1 = \pi r_1^4/4 \), \( I_2 = \pi r^4/4 \) and \( I_3 = \pi r^4/4 \) are the second moments of area of the three sections of the shaft, respectively. The unmachined length of the shaft is variable as \( l_1 = l - ut \), where \( u \) is feed speed of the cutter or the cutting force moving along the longitudinal axis in the opposite direction of the \( x \) axis. In Figure 1, variable \( l_2 = ut + t_0 \) is total length of the unmachined section length plus the being-machined one, where \( t_0 = a_p \tan(\phi_c) \) is the length of the being-machined section, \( a_p = r_1 - r \) is DOC, and \( \phi_c \) is the side cutting edge angle of cutter. It should be pointed out that the strain energy of the chips is ignored in this paper.

Excluding the kinetic energy of the chips removed, the kinetic energy of the rotating shaft is [14]

\[
T = \frac{1}{2} \int_0^{l_1} \rho A_1 \left( \frac{\partial v}{\partial t} \right)^2 + \frac{1}{2} \int_0^{l_2} \rho A_2 \left( \frac{\partial w}{\partial t} \right)^2 + \frac{1}{2} \int_0^{l_3} \rho A_3 \left( \frac{\partial w}{\partial t} \right)^2 + \frac{2\Omega^2}{l_1} \int_0^{l_1} \rho A_1 \left( \frac{\partial v}{\partial x} \right)^2 \, dx + \frac{2\Omega^2}{l_1} \int_0^{l_2} \rho A_2 \left( \frac{\partial w}{\partial x} \right)^2 \, dx + \frac{2\Omega^2}{l_3} \int_0^{l_3} \rho A_3 \left( \frac{\partial w}{\partial x} \right)^2 \, dx \tag{3}
\]

where \( \rho \) is the mass density of the shaft, \( A_1 = \pi r_1^2 \), \( A_2 = \pi (r_1 + r)^2/4 \), \( A_3 = \pi r^2 \) are the cross sectional areas of the un-machined, machining, and machined sections, respectively. \( \Omega \) is rotary speed of the shaft.

The virtual work done by components of cutting forces \( F_r \) and \( F_z \) and moment \( M_z \) is

\[
\delta W = F_y \delta v(s, t) + F_z \delta w(s, t) + M_z \frac{\partial \delta v}{\partial x} \bigg|_{x=s} \tag{4}
\]

3. Mathematical formulation and calculation

It is assumed that the deflection of the shaft can be expressed as

\[
v(x, t) = \sum_{i=1}^{n} \varphi_i(x) \alpha_i(t) = \varphi^T \alpha, \quad w(x, t) = \sum_{i=1}^{n} \varphi_i(x) \beta(t) = \varphi^T \beta \tag{5}
\]
where \( \varphi^T = \{ \varphi_1, \varphi_2, ..., \varphi_n \} \), \( \alpha^T = \{ \alpha_1, \alpha_2, ..., \alpha_n \} \), \( \beta^T = \{ \beta_1, \beta_2, ..., \beta_n \} \) and \( \varphi_i(x) \) is the spatial function that meets the boundary condition of the shaft and is taken to be the \( i \)-th mode shape of the stationary shaft in the analysis, \( \alpha_i (t) \) and \( \beta_i (t) \) are corresponding modal coordinates.

After Eq. (5) is substituted into Eq. (2) and Eq. (3), one gets
\[
V = \frac{1}{2} \int_{l_1} l_1 E_1 \left[ \alpha^T \varphi'' \alpha + \beta^T \varphi'' \beta \right] dx + \frac{3}{2} \int_{l_2} l_2 E_2 \left[ \alpha^T \varphi'' \alpha + \beta^T \varphi'' \beta \right] dx + \frac{1}{2} \int_{l_3} E_3 \left[ \alpha^T \varphi'' \alpha + \beta^T \varphi'' \beta \right] dx
\]
(6)
and
\[
T = \frac{1}{2} \int_{l_0} l_0 \left\{ \rho A_1 (\alpha^T \varphi \varphi^T \varphi + \beta^T \varphi \varphi^T \beta) + \rho \left[ \{ \alpha^T \varphi \varphi^T \varphi + \beta^T \varphi \varphi^T \beta \} + 2 \Omega \{ \alpha^T \varphi \varphi^T \varphi - \beta^T \varphi \varphi^T \beta \} \right] \right\} dx + \frac{1}{2} \int_{l_1} l_2 \left\{ \rho A_2 (\alpha^T \varphi \varphi^T \varphi + \beta^T \varphi \varphi^T \beta) + \rho \left[ \{ \alpha^T \varphi \varphi^T \varphi + \beta^T \varphi \varphi^T \beta \} + 2 \Omega \{ \alpha^T \varphi \varphi^T \varphi - \beta^T \varphi \varphi^T \beta \} \right] \right\} dx + \frac{1}{2} \int_{l_2} l_3 \left\{ \rho A_3 (\alpha^T \varphi \varphi^T \varphi + \beta^T \varphi \varphi^T \beta) + \rho \left[ \{ \alpha^T \varphi \varphi^T \varphi + \beta^T \varphi \varphi^T \beta \} + 2 \Omega \{ \alpha^T \varphi \varphi^T \varphi - \beta^T \varphi \varphi^T \beta \} \right] \right\} dx
\]
(7)

The Lagrange’s equations of motion can be derived as
\[
\rho K_1 \ddot{\alpha} + \rho K_1 \ddot{\beta} + [E K_3 - F_e B_3] \ddot{\alpha} + \rho \Omega (K_2 \ddot{\beta} + 2 K_2 \ddot{\beta}) = F_e \varphi (s(t)) + M_e \varphi'(s(t))
\]
\[
\rho K_1 \ddot{\beta} + \rho K_1 \ddot{\beta} + [E K_3 - F_e B_3] \ddot{\beta} - \rho \Omega (K_2 \ddot{\alpha} + 2 K_2 \ddot{\alpha}) = F_e \varphi (s(t))
\]
(8)
where
\[
K_1 = A_1 A_1 + A_2 A_2 + A_3 A_3 + I_1 B_1 + I_2 B_2 + I_3 B_3
\]
\[
K_2 = I_1 B_1 + I_2 B_2 + I_3 B_3
\]
\[
K_3 = I_1 C_1 + I_2 C_2 + I_3 C_3
\]
and
\[
A_1 = \int_0^{l_1} \varphi(x) \varphi^T(x) dx, B_1 = \int_0^{l_1} \varphi'(x) \varphi^T(x) dx, C_1 = \int_0^{l_1} \varphi''(x) \varphi^T(x) dx
\]
\[
A_2 = \int_{l_1}^{l_2} \varphi(x) \varphi^T(x) dx, B_2 = \int_{l_1}^{l_2} \varphi'(x) \varphi^T(x) dx, C_2 = \int_{l_1}^{l_2} \varphi''(x) \varphi^T(x) dx
\]
\[
A_3 = \int_{l_2}^{l_3} \varphi(x) \varphi^T(x) dx, B_3 = \int_{l_2}^{l_3} \varphi'(x) \varphi^T(x) dx, C_3 = \int_{l_2}^{l_3} \varphi''(x) \varphi^T(x) dx
\]
\[
B_4 = \int_{s(t)}^{l_0} \varphi'(x) \varphi^T(x) dx \approx B_3
\]

Note that \( A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3, K_1, K_2 \) and \( K_3 \) are all time varying matrices. Eq. (8) will be solved by a fourth-order Runge-Kutta method with a very small time step.

The cutting force is variable with the cutting condition, which can be established from turning experiments [24–25]. To determine the cutting forces under cutting condition, dry turning operations were carried out on a lathe CA6140. The lathe equipped with dynamometer YDX—III9702 (made by a research group at Dalian University of Technology) was used for measuring the cutting force components. The cutter was type PSSNR2525M12 and tool insert was coated cemented carbide. Cutting angles of the tool are: side cutting edge angle \( \gamma_c = 45^\circ \), inclination angle \( \lambda = 0^\circ \), and normal rake angle \( \gamma_n = 10^\circ \). The workpiece was 48 mm in diameter and 650 mm in length. The material of the shaft was AISI 1045: its hardness is HB190. The cutting condition and the experimental results are shown in Table 1. The rotational speed, feed rate and depth of cut are carefully chosen in different values as in the subsequent numerical analysis under different cutting condition. In order to simply the analytical model, the influence of vibration on the area of the cutting layer \( A_c \) is neglected in this paper.
4. Numerical results and analysis

The material and geometric properties of the circular shaft are $l = 1.0 \text{ m}$, $E = 2.07 \times 10^{11} \text{ Pa}$ and $\rho = 7700 \text{ kg/m}^3$. Considering the chucked-pinned boundary condition of the shaft in the turning operation, the normalized modes are [26]

$$
\varphi_i(x) = \frac{\sin(\beta_i l) - \sinh(\beta_i l)}{\cos(\beta_i l) - \cosh(\beta_i l)} \left(\cos(\beta_i x) - \cosh(\beta_i x)\right) - (\sin(\beta_i x) - \sinh(\beta_i x))
$$

where $\beta_i l = [3.9266, 7.0686, 10.2102, 13.3518, 16.4934, \ldots]$ ($i = 1, 2, 3, \ldots$). The natural frequency of the shaft (when the shaft is not rotating) is $\omega_i = \frac{(\beta_i l)^2}{E I_3 / (\rho A_3 l)}$. Nondimensional parameter $\gamma = a_p / r$ is used, where $a_p$ is depth of cut. Five modes of the shaft are found to produce satisfactory results and hence are used. The numerical results of the dynamic responses $v_p$ and $w_p$ of the shaft at the moving cutter location (in the $y$ and $z$ direction respectively) are depicted in Figures 3-8. Two radii of the circular shaft ($r = 0.095 \text{ m}$ and $r = 0.019 \text{ m}$) are studied. Please note that the values of $r$ refer to the radius of the shaft after turning and the radius of the original unmachined shaft is $r + \text{DOC}$ when different DOCs are studied. So a bigger DOC for a shaft of the same radius actually is thicker before cutting.

Firstly, the effect of decreasing shaft radius on the dynamic response of the shaft is studied. Three cases are considered: (1) the radius of the shaft throughout this particular analysis is taken to be the radius of the shaft after machining ($\gamma_0$) and is a constant, i.e., $r_1 = r_2 = r = 0.019 \text{ m}$; (2) the radius of the shaft throughout this particular analysis is taken to be the radius of the original unmachined shaft ($\gamma_1$) and is constant, i.e. $r = r_2 = r_1 = 0.022 \text{ m}$; (3) the radius of the shaft is time-varying with DOC=3 mm ($\gamma = 0.158$), i.e. $r(t) = r_1 - a_p$. Figure 3 shows the numerical results obtained at $\Omega = 1250 \text{ rpm}$, feed speed $f = 0.3 \text{ mm/rev}$ with $M_z$ ignored. The values of the cutting force used in the analysis are from Table 1. Clearly consideration of time-dependent reduction of shaft radius makes a noticeable difference in the dynamic response, as shown in Figure 3.

![Figure 3](image-url)
Secondly, different magnitudes of DOC are simulated (DOC=3 mm for γ = 0.032, DOC=2 mm for γ = 0.021 and DOC=1 mm for γ = 0.01) with \( f = 0.3 \) mm/rev and \( \Omega = 1250 \) rpm, but without \( M_c \). For \( r = 0.095 \) m, \( v_p \) and \( w_p \) are shown in Figures 4(a) and (b). For \( r = 0.019 \) m, they are shown in Figure 5. From Figure 3, reducing shaft diameter has a clear and similar influence on the dynamic response. The bigger \( \gamma \) (\( \gamma = d_p/r \)) used, which means larger DOC (the shaft radius after cutting \( r \) is constant) and thus bigger cutting force, the greater the deflection, as shown in Figures 4 and 5.

**Figure 4. Dynamic response of the shaft (\( r = 0.095 \) m) subjected to three moving forces at different DOCs (no \( M_c, \Omega = 1250 \) rpm, \( f = 0.3 \) mm/rev)**

Next, \( M_c \) is considered and the same data are used. At \( r = 0.095 \) m, numerical results of \( v_p \) and \( w_p \) are shown in Figure 6(a) and (b), respectively. Again different DOCs have a noticeable influence on dynamic response of the shaft in the \( y \) and \( z \) axes. Meanwhile, \( M_c \) induces a significant fluctuation of the deflection in the \( y \) axis of Figure 6(a) than the deflection in \( z \) axis of Figure 6(b), this is because \( M_c \) acts in the \( xoy \) plane (refer to Figure 1) and directly bends the shaft in that plane (causing more deflection in the \( y \) direction). On the other hand, using the same cutting condition and \( r = 0.019 \) m, numerical results also show that DOCs have a significant effect on dynamic response of the shaft shown in Figure 7. The effect of \( M_c \) is smaller as \( M_c \) itself is smaller for a smaller radius of shaft.

**Figure 5. Dynamic response due to three moving forces at different DOCs: \( r = 0.019 \) m (no \( M_c, \Omega = 1250 \) rpm, \( f = 0.3 \) mm/rev)**

**Figure 6. Dynamic response due to three moving forces at different DOCs: \( r = 0.095 \) m (with \( M_c, \Omega = 1250 \) rpm, \( f = 0.3 \) mm/rev)**
When $\Omega$ is decreased to 630 rpm, and the other data remain the same and $r=0.095$ m is used, numerical results are depicted in Figure 8, from which different DOCs also have a considerable effect on dynamic response $v_p$ and $w_p$ of the large radial shaft in comparison with Figure 6.

$M_e$ is seen to add lower-amplitude higher-frequency oscillation, which is similar to the finding of [13]. The method can be extended to deal with workpieces modelled as Timoshenko beam [27].

5. Conclusions
This paper studies vibration of turning operations. The workpiece is modelled as a clamped-pinned circular Rayleigh beam rotating about its longitudinal axis and the cutter provides a moving load of three perpendicular force components. The dynamic model includes the influence on vibration of the time-dependent reduction of workpiece diameter as a result of material being removed during turning. The numerical results show that the reduction of the workpiece has a significant effect on the dynamic response and the vibration magnitude is greater than that of the unmachined workpiece but smaller than that of the fully machined workpiece, and hence should be considered. It is also found that a bigger depth of cut produces greater vibration magnitude and the moment induced by the axial component of the cutting force results in a big moving-load effect by adding lower-amplitude higher-frequency oscillation to the dynamic response, especially in the range of realistic low feed speeds.

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