The relation between the Mandelstam and the Cayley-Hamilton identities

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Abstract

Starting from the characteristic polynomial for ordinary matrices we give a combinatorial deduction of the Mandelstam identities and vice versa, thus showing that the two sets of relations are equivalent. We are able to extend this construction to supermatrices in such a way that we obtain the Mandelstam identities in this case, once the corresponding characteristic equation is known.

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1 Introduction

When any gauge theory is described in terms of Wilson loops, which are traces of group elements associated to parallel transport around closed space-time curves (holonomies), one is faced with the problem of having a non-local and overcomplete set of variables. In other words, Wilson loops are constrained and an important aspect of the kinematics of the problem is just to identify the reduced phase space in the loop space of the problem.

A most dramatic example of this situation occurs in 2+1 Chern-Simons theories which are known to be described by a finite number of true degrees of freedom, which must arise in this process of reduction from the initially infinite dimensional phase space.

An important part of the reduction to the true degrees of freedom is usually performed by using the Mandelstam identities, whose explicit form depends on the dimension $n$ of the group matrices and which provide non-linear constraints among the different Wilson loops. These constraints must be solved in order to exhibit the independent degrees of freedom and this is by no means a simple problem. The Mandelstam identities can be systematically derived from the following identity of $n$-dimensional $\delta$- functions [1],

$$\sum_{P \in S_{n+1}} (-1)^{\pi(P)} \delta_{i_1 P(j_1)} \ldots \delta_{i_{n+1} P(j_{n+1})} = 0, \quad i_k, j_k = 1, \ldots n,$$

where the sum runs over all permutations $P$ of the symmetric group of order $n + 1$ and $\pi(P)$ denotes the parity of the permutation. The expression (1) can be understood as arising from the expansion, in terms of a determinant of $\delta$-functions, of the following product of two completely antisymmetric tensors with $n + 1$ indices

$$\epsilon_{i_1 i_2 \ldots i_{n+1}} \epsilon_{j_1 j_2 \ldots j_{n+1}}.$$
which nevertheless are allowed to take only $n$ values, thus giving zero as the result [2].

Contracting $n + 1$ holonomy matrices with the relation (1) one obtains a trace identity among $n + 1$ Wilson loops. The resulting Mandelstam identity is

$$\sum_{\text{Perm}(1,\ldots,n+1)} (-1)^{\pi(P)} W(M_1,\ldots,M_{n+1}) = 0. \quad (2)$$

If the cycle decomposition of the permutation $P$ is given by $(a(1),\ldots,a(i)) \times (a(i+1),\ldots)\ldots$, then

$$W(M_1,\ldots,M_{n+1}) = Tr(M_{a(1)}\ldots M_{a(i)}) Tr(M_{a(i+1)}\ldots)\ldots$$

In relation with the problem of constructing the reduced phase space in Chern-Simons theories we can find recently the suggestion that the reduction process can be carried over by using non-linear relations of lower degree among the traces, derived from the Cayley-Hamilton identity satisfied by the characteristic polynomial of the group elements involved [3, 4]. In particular, the reduced phase space for one genus of a genus $g$ two-dimensional surface in the case of super de Sitter gravity, which is the Chern-Simons theory of the supergroup $OSp(1|2; C)$, was obtained using a non-linear identity of order four among the supertraces of two basic supermatrices. In this case the supergroup elements are represented by $(2 + 1) \times (2 + 1)$ supermatrices and the characteristic polynomial is of degree 3 [4].

In this work we consider the problem of the equivalence among the two procedures and we prove that the Cayley-Hamilton identities imply those of Mandelstam and viceversa in the case of ordinary matrices. We use the same idea of this proof to give a method for obtaining the Mandelstam identities for supermatrices, starting from the corresponding characteristic polynomial.
These identities will be important in the reduction of the loop variables phase space when dealing with gauge theories defined over a supergroup.

The paper is organized as follows: in Section 2 we start from the characteristic polynomial of an $n \times n$ matrix $M$ and write explicit expressions for the coefficients of it in terms of the traces of powers of $M$. These results are subsequently used in Section 3 to prove the equivalence between the Mandelstam and the Cayley-Hamilton identities. Finally, Section 4 contains our construction of the Mandelstam identities in the case of supermatrices and we present them explicitly for the simple case of $(1+1) \times (1+1)$ supermatrices.

## 2 Preliminaries

It is well known that the coefficients of the characteristic polynomial of an $n \times n$ matrix $M$ can be written in terms of the traces of the powers of $M$, up to order $Tr(M^n)$. If the characteristic equation of the matrix $M$ is given by

$$P(x) = x^n + a_1 x^{n-1} + \ldots + a_n$$

and if $s_k = r^n_1 + \ldots + r^n_n$ is the sum of the $k$-th powers of the roots of $P(x)$, then the Newton equations give a recursive method to calculate the coefficients $a_i$

$$a_1 + s_1 = 0,$$
$$2a_2 + a_1 s_1 + s_2 = 0,$$
$$\vdots$$
$$na_n + a_{n-1} s_1 + \ldots + s_n = 0.$$  \hspace{1cm} (4)

For any matrix $M$, $s_i$ will be the trace of the $i$-th power of $M$, i.e. $s_i = Tr(M^i)$. 

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An explicit solution of the recursion equations (4) is given by the following expression

\[ a_i = \sum_{\alpha_1 + \ldots + \alpha_s = i} \frac{(-1)^s s_{\alpha_1}\ldots s_{\alpha_s}}{(\alpha_1 + \alpha_2 + \ldots + \alpha_s)(\alpha_2 + \ldots + \alpha_s)\ldots(\alpha_s)}, \]  

(5)

where the sum is made over all the unordered distinct partitions of \( i \) and \( s \) denotes the total number of terms in the partition. The above expression can be demonstrated by using induction, with the recurrence

\[ a_{i+1} = \sum_{k=0}^{i} -\frac{a_k s_{i+1-k}}{i+1}, \]  

(6)

obtained from Eqs. (4). Substituting the proposed solution (5) for all of the \( a_k, k < i \), we obtain

\[ a_{i+1} = \sum_{\alpha_0 + \alpha_1 + \ldots + \alpha_s = i+1} \frac{(-1)^{s+1} s_{\alpha_0+1-k} s_{\alpha_1}\ldots s_{\alpha_s}}{(i+1)k(k-\alpha_1)\ldots(\alpha_s)}. \]  

(7)

Calling \( \alpha_0 = \alpha_{i+1-k} \) we realize that the double sum in Eq. (7) is just a way of considering all the partitions of \( i+1 \) into \( s+1 \) elements which start with a given \( \alpha_0 \) where \( \alpha_0 + \alpha_1 + \ldots + \alpha_s = i+1 \). There are just \( i+1 \) of them. Besides \( k = i+1 - \alpha_0 = \alpha_1 + \ldots + \alpha_s \) and so on. This leads to the final result

\[ a_{i+1} = \sum_{\alpha_0+\alpha_1+\ldots+\alpha_s=i+1} \frac{(-1)^{s+1} s_{\alpha_0} s_{\alpha_1}\ldots s_{\alpha_s}}{(\alpha_0 + \alpha_1 + \ldots + \alpha_s)(\alpha_1 + \ldots + \alpha_s)\ldots(\alpha_s)}. \]  

(8)

Now, it can be shown that expression (5) can be rewritten in the more convenient form

\[ a_i = \sum_{k_1 \alpha_1 + \ldots + k_m \alpha_m = i} \frac{(-1)^{k_1+\ldots+k_m} s_{k_1\alpha_1}\ldots s_{k_m\alpha_m}}{k_1! s_{k_1}! s_{k_2}!\ldots s_{k_m}! \alpha_1! \alpha_2! \ldots \alpha_m! k_1}, \]  

(9)

where the sum is now made over all ordered partitions of \( i = \alpha_1 + \ldots + \alpha_1 + \ldots + \alpha_m \) and \( k_1 + k_2 + \ldots + k_m \) is the total number of terms in the partition.
In reference [2] we can find alternative expressions for the coefficients of the characteristic polynomial of an arbitrary matrix $M$ given in terms of a recursively defined $n$-entry symbol $\{M_1, M_2 \ldots M_n\}$, where one sets $M_1 = M_2 = \ldots = M_n = M$. The Mandelstam identity for $n \times n$ matrices is given by $\{M_1, M_2 \ldots M_{n+1}\}=0$ in this formulation.

As an example of the above expression (9) consider the partition $i = a + b$ in Eq.(5). Then, permuting $a$ and $b$ so that they become ordered we can rewrite the original sum as

$$\frac{s_as_b}{i \cdot a} + \frac{s_as_b}{i \cdot b} = \frac{s_as_b}{a \cdot b},$$

which corresponds to expression (4) with $k_1 = k_2 = 1, \alpha_1 = a, \alpha_2 = b$. Nevertheless, if $a = b$ the two original unordered partitions are the same one, and we can not count them twice. The final result in this case is

$$\frac{s_a^2}{2a^2},$$

where $k_1 = 2$ and $\alpha_1 = a$.

We will derive the general result (9) by permuting over the numbers in a given partition, and then dividing by the order of the permutation group that leaves that partition unchanged.

To begin with we prove that, for a given unordered partition with $s$ terms, the sum over the permutations of the numbers $\alpha_1, \ldots, \alpha_s$ in the denominator of Eq.(5) gives

$$P_s = \sum_{\text{Perm}\{\alpha_1, \ldots, \alpha_s\}} \frac{1}{(\alpha_{j(1)} + \alpha_{j(2)} + \ldots + \alpha_{j(s)}) (\alpha_{j(2)} + \ldots + \alpha_{j(s)}) \ldots (\alpha_{j(s)})}$$

$$= \frac{1}{\alpha_1 \ldots \alpha_s}.$$  \hfill (10)

Here $j(i)$ is the number in which $i$ is permuted in the permutation $j$. This can be shown again by induction calculating the sum over permutations of $s + 1.$
elements in the following way. We split the sum into \( s + 1 \) terms where the \( k-\text{th} \) term has \( \alpha_k \) fixed at the first position while the permutation over the remaining \( s \) values of \( \alpha_s \neq \alpha_k \) is calculated according to the expression (10).

The result of this calculation is

\[
P_{s+1} = \frac{1}{\alpha_1 + \alpha_2 \ldots + \alpha_{s+1}} + \frac{1}{\alpha_2 \alpha_3 \ldots \alpha_{s+1}} + \frac{1}{\alpha_2 + \alpha_1 \ldots + \alpha_{s+1}} \ldots + \frac{1}{\alpha_1 \alpha_3 \ldots \alpha_{s+1}}.
\]

Now let us observe that the denominator containing the sum of the \( \alpha_k \)'s is common for all the terms. Besides, \( \alpha_k \) is the only factor missing in the denominator containing products in the \( k \)-th term so that each of these terms can be rewritten as \( \alpha_k/\alpha_1 \ldots \alpha_{k-1} \alpha_k \alpha_{k+1} \ldots \alpha_{s+1} \) where the new denominator is again common for all the terms. Thus the summation reduces to those of the \( \alpha_k \)'s and the result follows.

Next we notice that if \( \alpha_p \) is repeated \( k_p \) times in the unordered partition of \( i \) into \( s \) elements, the product \( \alpha_1 \alpha_2 \ldots \alpha_s \) reduces to \( \alpha_1^{k_1} \alpha_2^{k_2} \ldots \alpha_m^{k_m} \) where \( k_1 \alpha_1 + \ldots + k_m \alpha_m = i \). Finally the \( k_p! \) permutations of these repeated terms produce the same partition of \( i \) so that we must divide by this factor not to overcount and the sign is given by the number of terms of the partition, in this case \( s = k_1 + \ldots + k_s \). It is worthwhile observing that each \( a_i \) is an homogeneous function of \( M \) of order \( i \).

3 The relation between the Mandelstam and the Cayley-Hamilton identities

Let us consider the Cayley-Hamilton identity

\[
\sum_{i=0}^{n} a_i M^{n-i} = P_M(M) \equiv T_1(M) = 0,
\]

(12)
where we have introduced the notation $P_M$ for the characteristic polynomial associated with the matrix $M$, and $a_0 = 1$. The coefficients $a_i$ have the explicit form calculated in Eq.(9). Now, we can write the following identity

$$P_{M_1+M_2}(M_1 + M_2) = 0,$$  \hspace{1cm} (13)

where the subscript $M_1 + M_2$ in $P$ is to emphasize that this substitution is also performed in the coefficients $a_i$ of the characteristic polynomial through Eq.(9). Now we can use Eq.(12) for $M_1$ and $M_2$ together with Eq.(13) to obtain a reduced identity

$$T_2(M_1, M_2) = P_{M_1+M_2}(M_1 + M_2) - P_{M_1}(M_1) - P_{M_2}(M_2) \hspace{1cm} (14)$$

In this new identity every term is a homogeneous function of $M_1$ and $M_2$ of order $n$ and additionally, $M_1$ and $M_2$ appear at least once in every term of $T_2(M_1, M_2)$. This means that $T_2(M_1, M_2 = 0)$ and $T_2(M_1 = 0, M_2)$ are identically zero, as can be verified from (14). Moreover, we consider that $T_2(M_1, M_2)$ is fully expanded using the distributivity of the trace and of the matrix product with respect to matrix addition. In a similar fashion, we can construct

$$T_3(M_1, M_2, M_3) = P_{M_1+M_2+M_3}(M_1 + M_2 + M_3)|_{red}$$

$$= P_{M_1+M_2+M_3}(M_1 + M_2 + M_3) - T_2(M_1, M_2) - T_2(M_1, M_3)$$

$$- T_2(M_2, M_3) - T_1(M_1) - T_1(M_2) - T_1(M_3) = 0, \hspace{1cm} (15)$$

which is a sum of null terms. Again, $T_3$ is identically zero when any of the $M_i$’s is set equal to zero. We have introduced the subscript $|_{red}$ to indicate an identity which has been reduced in such a way that every matrix involved is present at least once in each term after the identity is fully expanded. In
other words \( P_{M_1+M_2+M_3} (M_1 + M_2 + M_3) \) defined in Eq. (15) can be directly constructed by expanding the corresponding characteristic polynomial and discarding all terms in which any one of the three matrices is missing. Now, let us define the order \( o(M_i) \) of the matrix \( M_i \) in a monomial of the form

\[
aTr(M_1^{\alpha_1} M_2^{\alpha_2} \ldots) Tr(M_1^{\beta_1} M_2^{\beta_2} \ldots) \ldots M_1^{\gamma_1} M_2^{\gamma_2} \ldots M_1^{\delta_1} M_2^{\delta_2} \ldots, \tag{16}
\]

where some of the exponents may be equal to zero, as the sum of all of the exponents \( \alpha_i, \ldots \) that appear associated with \( M_i \). That is to say, \( o(M_i) = \alpha_i + \beta_i + \ldots + \gamma_i + \delta_i + \ldots \). The construction of the identity (15) guarantees that \( o(M_i) \geq 1 \) for every matrix. It is easy to see that the expanded expression consists of terms like the one we propose in (16).

We can continue in a similar fashion to (15) and construct reduced identities of always increasing order, reminding ourselves that we must subtract all the lower order identities at our disposal. This produces a new identity for every order \( i = 1, \ldots, n \).

\[
T_k(M_1, \ldots, M_k) = P_{M_1+\ldots+M_k} (M_1 + \ldots + M_k) - \sum_{i<k} T_i(M_{s_1}, \ldots, M_{s_i}), \tag{17}
\]

where the sum is carried out over all subsets \( \{s_1, \ldots, s_i\} \) of \( \{1, \ldots, k\} \).

Now, \( T_{n+1}(M_1, \ldots, M_{n+1}) \) together with all higher order identities are identically zero, since \( o(M(i)) \geq 1 \) for \( i = 1, \ldots, n + 1 \) would imply that a general term of order \( n + 1 \) is available in the characteristic polynomial, but we know that the Cayley-Hamilton identity is only of order \( n \). The identity (17) of order \( n \) is very interesting, because every matrix must be of order one which means that this particular expression is linear in each of its components. We will show that \( T_n(M_1, \ldots, M_n) \) is proportional to the Mandelstam identity for \( n \times n \) matrices.
The identity formally reads

\[ T_n(M_1, \ldots, M_n) = P_{M_1 + \ldots + M_n}(M_1 + \ldots + M_n)|_{\text{red}} = 0, \tag{18} \]

which can be written as

\[ \sum_{i=0}^{n} a_i(M_1 + \ldots + M_n)(M_1 + \ldots + M_n)^{n-i}|_{\text{red}}, \tag{19} \]

with \(a_0 = 1\). Now, let us consider in this expression the contribution

\[ \frac{(-1)^{k_1 + \ldots + k_s} s_{\alpha_1}^{k_1} s_{\alpha_2}^{k_2} \ldots s_{\alpha_s}^{k_s}}{\alpha_1! \alpha_2! \alpha_3! \ldots \alpha_s! \alpha_s!} M_{j(1)} M_{j(2)} \ldots M_{j(n-i)} \tag{20} \]

where the \(n-i\) numbers \(j(1), \ldots, j(n)\) are part of a permutation of the set \(\{1, \ldots, n\}\), and naturally \(k_1 \alpha_1 + \ldots + k_s \alpha_s + n - i = n\). Now, each of the remaining matrices \(M_{j(n-i+1)}, \ldots, M_{j(n)}\) which are contained in the corresponding traces, must appear only once in each of the expanded terms of (20). So, after this reduction the general term will be of the form

\[ \underbrace{\text{\(\alpha_1\) terms}}_{\text{k_1-terms}} \frac{\text{Tr}(M_{j(n-i+1)} M_{j(n-i+2)} \ldots M_{j(n-i+\alpha_1)}) \text{Tr}(M_{j(n-i+\alpha_1+1)} \ldots M_{j(n-i+2\alpha_1)}) \ldots \text{Tr}(M_{j(n-\alpha_s+1)} \ldots M_{j(n)}) M_{j(1)} \ldots M_{j(n-i)}}{\text{k_1-terms}} \]

with the coefficient given in (21).

But we have to add all the terms of the permutations of \(\{M_{j(n-i+1)}, \ldots, M_{j(n)}\}\) inside the traces that leave the general term invariant. It is easy to see that this group of permutations is of order \(\alpha_1^{k_1} k_1! \alpha_2^{k_2} k_2! \ldots \alpha_s^{k_s} k_s!\), since it counts the permutations of the \(k_1\) different \(s_{\alpha_1}\) where a term can be found (\(k_1!\)), and it also counts the cyclic groups that leave the trace invariant, for example

\[ \text{Tr}(M_1 \ldots M_d) = \text{Tr}(M_2 \ldots M_d M_1) \tag{22} \]
and all their cyclic permutations \((\alpha_1^{k_1})\). After this reduction, we obtain an identity where we are only considering terms corresponding to inequivalent permutations under the trace and where all factors in (20) have cancelled except for the sign. Next we multiply this expression by \(M_{n+1}\) and trace it. The result is

\[
\sum_{i} k_1\alpha_1 + \ldots + k_s\alpha_s = i \quad \alpha_1 > \alpha_2 > \ldots > \alpha_s
\]

\[
(-1)^{k_1+\ldots+k_s} \sum \left[ \underbrace{Tr(\ldots M\ldots)}_{\alpha_1 \text{ terms}} \right] \cdot \underbrace{Tr(\ldots M\ldots)\ldots}_{k_1 \text{ terms}} \cdot \underbrace{Tr(\ldots M\ldots)}_{\alpha_s \text{ terms}} \cdot \underbrace{Tr(\ldots M\ldots)}_{(n+1-i) \text{ terms}} \cdot \underbrace{M_{n+1}}_{1 \text{ term}} = 0, \tag{23}
\]

where we emphasize again that the sumation is made only over the inequivalent cycles of each partition. Now we rewrite (23) in terms of the partitions of \(n+1\) elements into \(k_1 + k_2 + \ldots + k_s + k_{s+1}\) cycles with \(k_{s+1} = 1\), \(\alpha_{s+1} = n+1-i\) and where \(k_1\alpha_1 + \ldots + k_s\alpha_s + \alpha_{s+1} = n+1\). For fixed \(i\), the parity exponent in (23) can be expressed as

\[
(-1)^{k_1+\ldots+k_s} = (-1)^{n-\pi((k_1\alpha_1)+\ldots+(k_s\alpha_s)+(n+1-i))}, \tag{24}
\]

where \(\pi((k_1\alpha_1)+\ldots+(k_s\alpha_s)+(n+1-i))\) is the parity of the permutation of \((n+1)\) elements, that corresponds to the above mentioned cycle decomposition. This comes about because the parity of each cycle \(m\) is \(\alpha_m + 1\) so that the whole parity of the particular decomposition we are considering is

\[
\pi = \sum_{\text{cycles}} (\alpha_m + 1) = n + k_1 + \ldots + k_s,
\]

from where the above result follows. Consider again the cycle decomposition of a permutation of \(n+1\) elements in the form \((k_1\alpha_1)+\ldots+(k_s\alpha_s)+(n-i+1)\), and let us further order the
cycles in such a way that the \( n + 1 \)-th term belongs to the last member of the decomposition. Then we obtain a term exactly of the form we have previously considered. Since in every permutation of \( n + 1 \) elements, the \((n+1)\)-th term belongs to some cycle of order \( n + 1 - i \) for some \( i \), in practice we are summing over all the different permutations of \( n + 1 \) elements. In this way we recover the expression (2) for the Mandelstam identities, except for the sign, that in our case is \((-1)^n\) times the one chosen in Ref. [1].

Finally, we describe how to derive the Cayley-Hamilton identity from the Mandelstam identity. Our starting point is again the expression (23) where we set the first \( n \) matrices equal to \( M \) while considering \( M_{n+1} = X \) to be an arbitrary \( n \times n \) matrix. Now we must count the number of terms arising from the inequivalent permutations to deduce the general expression (9) for the coefficients of the characteristic polynomial. Let us focus first on \( a_0 \) which arises from the \( i = 0 \) term of the sum (23). After setting the first \( n \) matrices equal to \( M \) we get a factor of \( n! \) for this term. Now let us consider a particular ordered cycle decomposition of \( i \). The last term in (23) contributes with \( \binom{n}{n-i} (n-i)! \) factors of the type \( \text{Tr}(M^n X) \). Now we are left with \( i \) matrices to be distributed among the \( k_s \) remaining cycles. For the \( p \) cycle the corresponding factor is

\[
\frac{1}{k_p!} \binom{i_p}{\alpha_p} (\alpha_p - 1)! \binom{i_p - \alpha_p}{\alpha_p} (\alpha_p - 1)! \cdots \binom{i_p - (k_p-1)\alpha_p}{\alpha_p} (\alpha_p - 1)!,
\]

where \( i_p = i - \sum_r^{p-1} k_r \alpha_r \) is the number of matrices left out after completing the \((p-1)\) cycle. Multiplying all these factors and dividing the product by \( a_0 = n! \) we obtain the result (9) for \( a_i \). One is lead to the final expression for the characteristic polynomial by recalling that the condition \( \sum_i^n a_i \text{Tr}(M^{n-i} X) = 0 \) for all \( X \) implies that \( \sum_i^n a_i M^{n-i} = 0 \). This can be seen by taking one by one \( X = E_{ij} \), the standard elements of the basis of
the matrix algebra of $n \times n$ matrices. We have then shown that the Mandelstam identities and the Cayley-Hamilton identities contain exactly the same information and thus they are equivalent.

4 Mandelstam identities for Supermatrices

The same construction that we have presented in the case of ordinary matrices allows for a generalization providing the Mandelstam identities for the case of supermatrices. A definition of the characteristic polynomial for supermatrices in terms of $Sdet(xI - M)$ together with the corresponding Cayley-Hamilton identity is given in [6]. The latter identity can be written in terms of a finite number of supertraces [7]. One of the differences in this case is that the characteristic polynomial is not monic in general, so that the coefficient $a_0$ is a combination of supertraces. This fact will effectively raise the order of the characteristic polynomial. In any case, these identities are always homogeneous of some degree, let us say $t$, in the matrices, and this allows us to make the following definition for the corresponding Mandelstam identities for supermatrices

$$Str(P_{M_1+\ldots+M_t}(M_1 + \ldots + M_t)|_{\text{red}M_{t+1}}) = 0.$$  \hspace{1cm} (25)

In this case one also obtains a result that is totally symmetric in the first $t$ entries, and one would have to prove that it is symmetric in the $t+1$ entry too. The example we present here possesses complete symmetry in all the indices and we conjecture that it is generally so. The main drawback in the case of supermatrices is the lack of knowledge of a recurrence that would allow us to obtain a closed expression for the coefficients of the corresponding characteristic polynomial in terms of supertraces, in a manner similar to Eq.(3).
As an example, consider the Cayley-Hamilton identity for \((1+1) \times (1+1)\) supermatrices\(^3\):

\[
Str(M)M^2 - (Str(M^2))M + \frac{1}{3}(Str(M^3) - Str(M)^3) = 0, \quad (26)
\]

where \(t = 3\). Using this expression in (25) we obtain

\[
Str(A)(Str(BCD) + Str(CBD)) + Str(B)(Str(ACD) + Str(CAD))
\]
\[
+ Str(C)(Str(ABD) + Str(BAD)) + Str(D)(Str(ABC) + Str(BCA))
\]
\[
-2Str(AB)Str(CD) - 2Str(BC)Str(AD) - 2Str(AC)Str(BD)
\]
\[
-2Str(A)Str(B)Str(C)Str(D) = 0, \quad (27)
\]

which corresponds to a symmetric Mandelstam identity of order four. We have verified this result using Mathematica.

The next simple case corresponds to \((2+1) \times (2+1)\) supermatrices. Here the characteristic polynomial is of degree 3 and order \(t = 7\), which will lead to a Mandelstam identity of order 8. The final result is not very illuminating and thus it is not written.

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Urrutia L F and Morales N, *The Cayley-Hamilton Theorem for Supermatrices*, preprint ICN-UNAM-1993, extended version, submitted for publication.
[7] Kobayashi Y and Nagamashi S, *J. Math. Phys.* **31** (1990) 2736.