A Generalized result of Output Stabilizability

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Abstract

Output stabilizability of a class of infinite dimensional linear systems is studied in this paper. A criterion for the system to be output stabilizable by a linear bounded feedback $u = Fx$, $F \in L(Z, \mathbb{R}^p)$ will be given.

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1 Introduction

In this note, inspired by the result in [2] for output stabilizability of the diffusion equation, we proposed a new output stabilizability criterion for a class of infinite dimensional linear systems with multi-actuators and multi-sensors. The system we consider is described by the abstract differential equation

$$\begin{cases}
\dot{x} = Ax + Bu \\
x(0) = x_0
\end{cases} \tag{1}$$

where $A$ generates a strongly continuous semigroup $(S(t))_{t \geq 0}$ on $Z$ (state space); $U$ is the control space and the control function $u(.) \in L^2(0, T; U)$; $B \in L(U, Z)$; $U$ and $Z$ are supposed to be a separable Hilbert spaces. The system (1) is augmented by the output equation

$$y = Cx \tag{2}$$

where $C \in L(Z, Y)$, $Y$ is the observation (output) space, a separable Hilbert space, $y(.) \in L^2(0, T; Y)$. 

The system we shall characterize its output stabilizability is assumed to be controlled via $p$ actuators $(\Omega_i, g_i)_{1 \leq i \leq p}$ and takes the form

$$\frac{\partial z}{\partial t}(\xi, t) = \Delta z + kz + \sum_{i=1}^{p} g_i(\xi)u_i(t) \quad \text{in} \: \Omega \times (0, T),$$

with boundary conditions

$$z(\xi, t) = 0 \quad \text{in} \: \partial \Omega \times (0, T),$$

and the initial condition

$$z(\xi, 0) = z_0(\xi) \quad \text{in} \: \Omega,$$

with the output function given by

$$y(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix},$$

where

$$y_i(t) = \int_{D_i} f_i(\xi)z(\xi, t)d\xi,$$

and $\Delta$ is the Laplacian operator, $\Omega$ is bounded and open in $\mathbb{R}^n$ with smooth boundary $\partial \Omega$, $g_i \in L^2(\Omega_i)$, $\Omega_i \subset \Omega$, $\Omega_i \cap \Omega_j = \emptyset$, $k > 0$ and $(D_i, f_i)_{1 \leq i \leq q}$ is a suite of sensors with $D_i \subset \Omega$ and $f_i \in L^2(D_i)$. The above system (3)-(7) is a special form of (1)-(2) where $Z = L^2(\Omega)$, $A = \Delta + kI$, $D(A) = H^1_0(\Omega) \cap H^2(\Omega)$ and

$$B \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} = \sum_{i=1}^{p} g_i u_i.$$ 

Clearly the output function (6) may be written in the form

$$y(t) = Cx(t).$$

If the associated eigenfunctions are $\varphi_{nj}$ then

$$S(t)x = \sum_{n=1}^{\infty} \exp(\mu_n t) \sum_{j=1}^{r_n} \langle x, \varphi_{nj} \rangle \varphi_{nj}$$

where $r_n$ is the multiplicity of the eigenvalue $\mu_n$. 
In this work the case when the eigenvalue are \( \mu_n \) with multiplicity \( r_n \) is treated. Our results extend and complete those established in [2].

This paper is organized as follows: We recall in section 2, the notions of approximate controllability, state and output stabilizability for infinite dimensional systems defined in Hilbert spaces.

In section 3, we give a generalization of the results presented in [2].

2 Preliminaries

We consider the system \((S)\) augmented by the output equation \((E)\) defined respectively by (1) and (2).

**Definition 2.1** We say that the system \((S)\) (or the pair \((A,B)\)) is approximately controllable if \( \mathcal{N} = \{0\} \).

Where \( \mathcal{N} = \bigcap_{t \geq 0} \ker B^*S^*(t) \).

\( \mathcal{L} = \mathcal{N}^\perp \) and \( \mathcal{N} \) are called, the controllable and uncontrollable subspaces of the system \((S)\), respectively.

According to [4], we can decompose the state space \( Z \) as \( \mathcal{L} \oplus \mathcal{N} \) and then the system (1)-(2) can be written as:

\[
\begin{align*}
&\dot{x}_1 = A_{11}x_1 + B_1u \\
&\dot{x}_2 = A_{22}x_2 \\
y &= y_1 + y_2
\end{align*}
\]

where \( y_i = C_ix_i \), for \( i = 1,2 \).

**Definition 2.2** The system \((S)\) is said to be exponentially stabilizable if there is an \( F \in L(Z,U) \) such that the semigroup \( S_{A+BF}(t) \) is exponentially asymptotically stable.

Where \( S_{A+BF}(t) \) is the semigroup generated by \( A + BF \).

**Definition 2.3** The system \((S)\) augmented by the output equation \((E)\) is output stabilizable by a bounded feedback if there is an \( F \in L(Z,U) \) such that the output \( y(t) \) of the closed system

\[
\dot{x}(t) = (A+BF)x(t), \ x(0) = x_0
\]

is exponentially stable, i.e., \( y(t) \) converges to zero when \( t \to \infty \), for every \( x_0 \in Z \). See e.g.,[1],[3], [4].
3 Main Results

We need the following lemmas in the proof of our proposition.

Lemma 3.1 The uncontrollable subspace $\mathcal{N}$ of the system (3)-(7) is of the following form

$$\mathcal{N} = \text{span}\left\{\sum_{j=1}^{r_n} \alpha_j \varphi_{nj} / B_n^* v = 0, \quad v = (\alpha_1, \ldots, \alpha_{rn})^T\right\}$$

(13)

where $B_n = (\langle g_i, \varphi_{nj} \rangle_{L^2(\Omega_i)}), \ 1 \leq i \leq p, \ 1 \leq j \leq r_n$ and $\text{span}\{e_m, m \in I\}$ denotes the closed subspace generated by the vectors $e_m, m \in I$. $T$ means transpose.

Proof: As in the proof of Lemma 3.2 in [2], we have $B^*S^* (t) x = 0$ if and only if

$$\langle E(\mu_n) x, g_i \rangle = 0, \text{ for all } n \geq 1, \ i = 1, \ldots, p$$

(14)

where

$$E(\mu_n) = \sum_{j=1}^{r_n} \langle \cdot, \varphi_{nj} \rangle \varphi_{nj}$$

(15)

Noting that it is easy to see that

$$J = \{n / \text{rank} B_n < r_n\} = \{n / \ker B_n^* \neq \{0\}\}.$$  \hfill (16)

Let $x \in E(\mu_{n_0}) \mathcal{N}, x \neq 0$, for a certain $n_0 \in J$. Then

$$B_{n_0}^* v_{n_0} = 0,$$

(17)

with $v_{n_0} = \left(\langle x, \varphi_{n_01}\rangle, \ldots, \langle x, \varphi_{n_0r_n}\rangle\right)^T \neq 0$.

This shows that

$$\mathcal{N} \subset \text{span} \left\{\sum_{j=1}^{r_n} \alpha_j \varphi_{nj} / B_n^* v = 0, \quad v = (\alpha_1, \ldots, \alpha_{rn})^T\right\}$$

The remaining part of the proof is easy to establish and will be omitted here.

From the previous Lemma we deduce the following consequence

Lemma 3.2 The controllable subspace $\mathcal{L}$ of the system (3)-(7) is given by

$$\mathcal{L} = \text{span} \left\{\sum_{j=1}^{r_n} \alpha_j \varphi_{nj} / (\alpha_1, \ldots, \alpha_{rn})^T \in \text{Im} B_n\right\}$$

(18)

where $B_n = (\langle g_i, \varphi_{nj} \rangle_{L^2(\Omega_i)}), \ 1 \leq i \leq p, \ 1 \leq j \leq r_n$. 

We are now in position to prove the main result of this section.

**Proposition 3.3** Suppose there are \( p \) actuators \((\Omega_i, g_i)_{1 \leq i \leq p}\) and \( q \) sensors \((D_i, f_i)_{1 \leq i \leq q}\), then the system (3)-(7) is output stabilizable if and only if

\[
\mu_n < 0, \text{ for all } n \text{ in } K
\]

where

\[
K = \{ n/ \text{Im}T_n \neq \{0\} \text{ and ker } B_n^* \neq \{0\} \}
\]

and \( B_n = (\langle g_i, \varphi_{nj} \rangle_{L^2(\Omega_i)}), T_n = (\langle f_k, \varphi_{nj} \rangle_{L^2(D_k)}), 1 \leq i \leq p, 1 \leq j \leq r_n, 1 \leq k \leq q \).

**Proof:** Similar to the proof of Proposition 3.4 in [2], it suffices to study the stability of the output \( y_2 \) on the observable subspace \( \mathcal{W} \) of the subsystem

\[
\begin{align*}
x_2^1 &= A_{22}^1 x_2^1 \\
x_2^2 &= A_{22}^2 x_2^2 \\
y_2 &= C_2^2 x_2^2
\end{align*}
\]

where

\[
A_{22} = \begin{pmatrix} A_{22}^1 & 0 \\ 0 & A_{22}^2 \end{pmatrix}, \quad C_2 = \begin{bmatrix} 0 & C_2^2 \end{bmatrix},
\]

\[
x_{02} = \begin{bmatrix} x_{01}^2 \\ x_{02}^2 \end{bmatrix} \in \mathcal{M} \oplus \mathcal{W}, \quad \mathcal{W} = \mathcal{M}^\perp, \quad x_2(0) = x_{02}, \quad x_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix} \in \mathcal{L} \oplus \mathcal{N}
\]

and

\[
\mathcal{W} = \text{span}\left\{ \sum_{j=1}^{r_n} \alpha_j \varphi_{nj} \mid v = (\alpha_1, ..., \alpha_{r_n})^T \in \tilde{\mathcal{V}} \right\}
\]

with \( \tilde{\mathcal{V}} = \text{Im}T_n \cap \ker B_n^* \).

The output \( y_2 \) of the subsystem (21) is given by

\[
y_2(t) = \begin{bmatrix} \sum_{n \in K} \exp(\mu_n t) \sum_{j=1}^{r_n} \langle x_0^2, \varphi_{nj} \rangle \langle f_1, \varphi_{nj} \rangle \\ \vdots \\ \sum_{n \in K} \exp(\mu_n t) \sum_{j=1}^{r_n} \langle x_0^2, \varphi_{nj} \rangle \langle f_q, \varphi_{nj} \rangle \end{bmatrix}
\]

where

\[
K = \{ n/ \text{Im}T_n \neq \{0\} \text{ and ker } B_n^* \neq \{0\} \},
\]
The sufficient condition is straightforward. Now we shall prove the converse. Suppose that the output $y_2(t)$ is exponentially stable but for a certain $n_0 \in K$, $\mu_{n_0} \geq 0$, then there are positive $M$ and $\omega$ such that

$$\| y_2(t) \|_{\mathbb{R}^q} \leq M \exp(-\omega t) \| x_0 \| \quad \text{for every } x_0 \in Z$$

(26)

Set $x_0 = \varphi_{n_0}$, in equation (26) where $j \in \{1, ..., r_{n_0}\}$ ($j$ fixed arbitrary)

Then we obtain

$$|\left\langle f_k, \varphi_{n_0} \right\rangle| \leq M \exp\left\{-(\omega + \mu_{n_0})t\right\} \quad \text{for all } t \geq 0, \ k = 1, ..., q.$$  

(27)

Thus $\text{Im} T_{n_0} = \{0\}$ and this contradicts the assumption that $n_0 \in K$.

**Remark 3.4** It is noteworthy that if $p \geq \sup_n r_n$ and $\text{rank } B_n = r_n$, for all $n$, then the approximate controllability is achieved and by virtue of Theorem 7.2 in [3], the system (3)-(7) is output stabilizable.

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