The Stäckel systems and algebraic curves.

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Abstract

We show how the Abel-Jacobi map provides all the principal properties of an ample family of integrable mechanical systems associated to hyperelliptic curves. We prove that derivative of the Abel-Jacobi map is just the Stäckel matrix, which determines \( n \)-orthogonal curvilinear coordinate systems in a flat space. The Lax pairs, \( r \)-matrix algebras and explicit form of the flat coordinates are constructed. An application of the Weierstrass reduction theory allows to construct several flat coordinate systems on a common hyperelliptic curve and to connect among themselves different integrable systems on a single phase space.

1 Introduction

In the classical mechanics the arrow from the initial physical variables to the action-angle variables is provided by the separation of variables and then by the Arnold construction of the action-angle representation [1]. The motion in the opposite direction ought to allow us to construct various mechanical integrable systems. However, in the action-angle representation all the mechanical systems with fixed number of degrees of freedom are indistinguishable. To describe some particular integrable system one should present an explicit construction of the initial physical variables as functions on the action-angle variables. This mapping contains all the information about a given integrable system. By using variety of these mappings the different integrable models may be connected together via the common action-angle variables. For instance, mechanical systems may be related to nonlinear equations and to gauge field theory.

As an example, investigation of the finite-gap solutions of the nonlinear problems leads to the introduction of analytic symplectic form \( \Omega_g \) on the Jacobian fibrations and to the definition of the action-angle variables on the complex space of Liouville variables. In [5] it is shown that possible obstructions to the existence of global systems of action-angle variables on symplectic vector bundles are a nontrivial first Chern class and the presence of monodromy at singularities. Introduction of the action-angle representation enables ones to consider mechanical integrable systems as systems associated with these variables on a torus bundle with base \( \mathcal{M} \), moduli space of complex polynomials \( F(\lambda) = \prod_{j=1}^{2g+1} (\lambda - \lambda_j) \), (1.1)

and with a fiber \( J(C) \), the \( g \)-dimensional complex Jacobian of auxiliary curve \( C \) defined by the Abel-Jacobi map \( U \). The fact that action-angle variables could be used for quantization of classical systems leads to introduction of semiclassical geometric phases. This approach results, for instance, in a quantum conditions on the moduli of \( n \)-dimensional Jacobi varieties [6].

By using this Abel-Jacobi map \( \mathcal{U} \) and the Jacobi problem of inversion, the so-called root variables \( \{p_j, q_j\}_{j=1}^{n} \) on an associated Riemann surface \( C \) may be constructed instead of the action-angle variables. In these root variables on the level of integrals of motion the action is represented as a
sum of items depending on one coordinate only, i.e. these variables are separated variables. The corresponding Riemann surface $C$ depends on parameters (moduli), parameterizing the moduli space $\mathcal{M}$ of $C$. In terms of mechanical integrable systems the curve $C$ is interpreted as a time-independent spectral curve, integrals of motion are some specific coordinates on the moduli space $\mathcal{M}$ and Jacobian $J(C)$ is a common level of the involutive integrals of the system.

In what follows, we have to describe appropriate mechanical systems together with their phase space in initial physical coordinates $\{p_j, x_j\}_{j=1}^n$. In particular, separated coordinate systems ought to be orthogonal curvilinear coordinate systems on the flat Riemannian manifold. In this case, these separated coordinate systems are associated to some solutions to the Lamé equation. Recently, the solutions to this equation have been obtained in an explicit form with the help of the "dressing procedure", the Baker-Akhiezer function and the Lie algebraic construction within framework of the inverse problem method.

The main objective this paper is to illustrate how fixed mapping from the action-angle variables to separated variables completely defines all the principal properties of mechanical systems. We shall consider the uniform Stäckel models associated to the Abel-Jacobi map $U$ on the hyperelliptic curve $C$ and the well-known elliptic, parabolic and spherical curvilinear coordinate systems on $\mathbb{R}^n$. Also we discuss relations of these mechanical systems with other integrable models associated to the same algebraic curve.

2 The Stäckel systems

One of the oldest problem of the hamiltonian mechanics is to find the quadratures for the integrable hamiltonian systems. The simplest models integrable in quadratures are the Liouville systems and the Stäckel systems (the Liouville systems are a particular case of the Stäckel systems).

Before proceeding father it is useful to recall the classical work of Stäckel. The system associated with the name of Stäckel is a holonomic system on the phase space $\mathbb{R}^{2n}$, their hamiltonian is

$$H = \sum_{j=1}^n g_j(q_1, \ldots, q_n) \left( p_j^2 + U_j \right). \quad (2.1)$$

Here $\{p_j, q_j\}_{j=1}^n$ are canonical variables in $\mathbb{R}^{2n}$ with the standard symplectic structure and with the following Poisson brackets

$$\Omega_n = \sum_{j=1}^n dp_j \wedge dq_j, \quad \{p_j, q_k\} = \delta_{jk}. \quad (2.2)$$

There is an even stronger version of the Stäckel theorem.

**Theorem 1** For a hamiltonian system with hamiltonian $H$ of the form (2.1) the following assertions are equivalent:

1) The associated Hamilton-Jacobi equation is separable.
2) There exists a nondegenerate $n \times n$ Stäckel matrix $S$, whose elements $s_{kj}$ depend only on $q_j$

$$\det S \neq 0, \quad \frac{\partial s_{kj}}{\partial q_m} = 0, \quad \text{for } j \neq m$$

and such that

$$\sum_{j=1}^n s_{kj}(q_j)g_j(q_1, \ldots, q_n) = \delta_{k1}. \quad (2.3)$$

3) There exist $n$ functionally independent integrals of motion which are quadratic in momenta.

Let $C = [c_{ik}]$ denotes inverse matrix to $S$ such that $c_{jk} = g_j$. Then the Stäckel theorem asserts that there are $n$ first integrals of motion, namely

$$I_k = \sum_{j=1}^n c_{jk}(p_j^2 + U_j), \quad I_1 = H. \quad (2.4)$$
The common level surface of these integrals

$$M_{\alpha} = \left\{ z \in \mathbb{R}^{2n} : I_{k}(z) = \alpha_{k}, \ k = 1, \ldots, n \right\}$$

is diffeomorphic to the $n$-dimensional real torus and one immediately gets

$$p_{j}^{2} = \left( \frac{\partial S}{\partial q_{j}} \right)^{2} = \sum_{k=1}^{n} \alpha_{k} s_{kj}(q_{j}) - U_{j}(q_{j}), \quad (2.5)$$

where $S(q_{1}, \ldots, q_{n})$ is an action function $[1]$. If this real torus is a part of complex algebraic torus, then the corresponding mechanical system is called an algebraic completely integrable system $[18]$.

The Stäckel theorem allows to reduce the solution of the equations of motion to a problem in algebraic geometry. We can regard each expression (2.5) as being defined on the Riemann surface

$$C_{j} : \quad y_{j}^{2} = F_{j}(\lambda), \quad F_{j}(\lambda) = \sum_{k=1}^{n} \alpha_{k} s_{kj}(\lambda) - U_{j}(\lambda), \quad (2.6)$$

which depends on the values $\alpha_{k}$ of integrals of motion. All the pairs of variables $(p_{j}, q_{j})$ lie on these Riemann surfaces (2.6). Considered together, they determine an $n$-dimensional Lagrangian submanifold in $\mathbb{R}^{2n}$

$$C^{(n)} : \quad C_{1}(p_{1}, q_{1}) \times C_{2}(p_{2}, q_{2}) \times \cdots \times C_{n}(p_{n}, q_{n}). \quad (2.7)$$

The associated Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + H(t, \frac{\partial S}{\partial q_{1}}, \ldots, \frac{\partial S}{\partial q_{n}}, q_{1}, \ldots, q_{n}) = 0, \quad \Rightarrow \quad g^{ij} \frac{\partial S}{\partial q_{i}} \frac{\partial S}{\partial q_{j}} = E, \quad (2.8)$$

on the local manifold $\mathcal{V}_{n}$ with diagonal metric $g^{ij} = g_{j}(q_{1}, \ldots, q_{n})$ analytic in the local coordinates $\{q_{j}\}$ has the following additive solution

$$S(q_{1}, \ldots, q_{n}) = \sum_{j=1}^{n} s_{j}(q_{j}), \quad s_{j}(q_{j}) = \int \sqrt{F_{j}(q_{j})} \ dq_{j}, \quad (2.9)$$

with the functions $F_{j}(\lambda)$ defined in (2.6). Coordinates $q_{j}(t, \alpha_{1}, \ldots, \alpha_{n})$ are determined from the equations

$$\sum_{j=1}^{n} \int_{\gamma_{0}(p_{0}, q_{0})}^{\gamma_{j}(p_{j}, q_{j})} \frac{S_{ij}(\lambda)d\lambda}{\sqrt{\sum_{k=1}^{n} \alpha_{k} s_{kj}(\lambda) - U_{j}(\lambda)}} = \beta_{1} = t, \quad (2.10)$$

$$\sum_{j=1}^{n} \int_{\gamma_{0}(p_{0}, q_{0})}^{\gamma_{j}(p_{j}, q_{j})} \frac{S_{kj}(\lambda)d\lambda}{\sqrt{\sum_{k=1}^{n} \alpha_{k} s_{kj}(\lambda) - U_{j}(\lambda)}} = \beta_{k}, \quad k = 2, \ldots, n,$$

where points $\gamma_{j}(p_{j}, q_{j})$ and $\gamma_{0}(p_{0}, q_{0})$ be on the curve $C_{j}$ (2.6). Notice, that bounded motion in this case will not be periodic in general but only conditionally periodic $[17, 11]$. If $\lambda_{0}$ and $\lambda_{j}$ are the turning points determined by the conditions that functions $F_{j}(\lambda)$ (2.6(2.9) vanish, the periods of the motion $w_{jk}$ are equal to

$$w_{jk} = \int_{\lambda_{0}}^{\lambda_{j}} \frac{S_{kj}(\lambda)d\lambda}{\sqrt{F_{j}(\lambda)}}, \quad (2.11)$$

Thus, Stäckel $[17]$ showed that the orthogonal coordinates $\{q_{j}\}_{j=1}^{n}$ permit separation in the Hamilton-Jacobi equation (2.3) if the metric

$$ds^{2} = \sum_{j=1}^{n} g_{jj}(q_{1}, \ldots, q_{n})(dq^{j})^{2}, \quad g_{jj}(q_{1}, \ldots, q_{n}) \equiv g_{j}(q_{1}, \ldots, q_{n}) \quad (2.12)$$
is in the Stäckel form
\[ g_{jj}(q_1, \ldots, q_n) = H_j^2(q_1, \ldots, q_n) = \frac{\det S}{S_j^{11}}, \]  
(2.13)

where \( S_j^{11} \) means the cofactor of \( s_{j1} \) in matrix \( S \) \([2,3]\). Here \( g_{jj} \) is a diagonal metric and \( H_j \) are called the Lamé coefficients. The modern approach to construction of the Lamé coefficients see in \([3,13,14]\).

Henceforth, we shall restrict our attention to the uniform Stäckel systems, where all the potentials \( U_j(q_j) = U(q_j) \) and curves \( C_j \) \([2,10]\) are equal. Variables \( \{s_k, w_k\} \) \([2,3,2,11]\) on a single curve \( C \) are the action-angle variables for the uniform Stäckel systems. To construct the metric \( g_j(q_1, \ldots, q_n) \) and the potentials \( U(q_j) \) in an explicit form we shall identify periods \( w_k \) \([2,11]\) with periods of the Abel differentials on a common hyperelliptic curve \( C \) \([2,3]\) along the elements of a homology basis \([1, 4]\). In this case definition of the separated variables \( \{q_j\} \) \([2,10]\) leads to the Jacobi inversion problem. In the next Section we prove that the Stäckel matrix \( S \) \([2,3]\) is completely defined by the derivative of the Abel-Jacobi map \( U \) on \( C \) at generic point (so-called Brill-Noether matrix).

3 Uniform Stäckel systems and algebraic curves

To begin with let us briefly recall some necessary facts about the action-angle variables on the Jacobian \( J(C) \) \([1, 3, 4]\). The main ingredient of this construction is a universal configuration space, which is the moduli space \([10]\) of all algebraic curves with fixed jets of local coordinates at a fixed number of punctures. This concept is closely related to the notion of the Baker-Akhiezer function on admissible curves \([14]\) and to the theory of algebraic completely integrable systems \([18]\).

Let us consider a genus \( g \) Riemann surface \( C \) with \( N \) ordered punctures \( P_j \) and with two special Abelian integrals \( y \) and \( \lambda \) with poles of order at most \( l = (l_j)_{j=1}^N \) and \( m = (m_j)_{j=1}^N \) at the punctures. The universal configuration space \( \mathcal{M}_g(l,m) \) can then be defined as a moduli space of \( C \) under certain constraints on the set of algebraic geometrical data \([14, 4]\). In this case the space \( \mathcal{M}_g(l,m) \) is a complex manifold with only orbifold singularities. To introduce the local coordinates on \( \mathcal{M}_g(l,m) \) we cut apart the Riemann surface \( C \) along a homology basis \( A_i, B_j \) \( j = 1, \ldots, g \) with canonical intersection matrix

\[ A_i \circ A_j = B_i \circ B_j = 0, \quad A_i \circ B_j = \delta_{ij}. \]  
(3.1)

By selecting cuts from \( P_i \) to other \( P_j \) for each \( 2 \leq j \leq N \) one gets a well-defined branch of the Abelian integrals \( y \) and \( \lambda \). Among the complete set of local coordinates on \( \mathcal{M}_g(l,m) \) the following moduli are distinguished

\[ s_j = \oint_{A_j} y d\lambda, \quad j = 1, \ldots, g. \]  
(3.2)

The universal configuration space \( \mathcal{M}_g(l,m) \) is a base space for a hierarchy of fibrations \( C^{(k)}(l,m) \) of particular interest to us. These are the fibrations whose fiber above each point of \( \mathcal{M}_g(l,m) \) is the \( k \)-th symmetric power \( S^k(C) \) of \( C \). This fiber \( C^{(k)}(l,m) \) is the set of all effective divisors \( D = \gamma_1 + \cdots + \gamma_k \) (the \( \gamma_j \)’s may not be mutually distinct) of \( \deg k \) of \( C \), i.e. \( C^{(k)}(l,m) \) can be identified with the set of all unordered \( k \)-tuples \( \{\gamma_1, \ldots, \gamma_k\} \), where \( \gamma_j \)’s are arbitrary elements of \( C \).

Let \( D \) be the open set in \( \mathcal{M}_g(l,m) \), where the zero divisors of \( dy \) and \( d\lambda \) do not intersect. Fixing all the local coordinates on \( \mathcal{M}_g(l,m) \) except \( s_j \) \([3,2]\) one can determine a smooth \( g \)-dimensional foliation of \( D \), independent of the choice we made to define the coordinates themselves \([4]\). Hereafter, by abuse of notation, one leaf of this foliation, is denoted just by \( M \) and \( C^{(k)} \) means the above fibrations restricted to \( M \).

Let \( dS = y d\lambda \) be a meromorphic 1-form on \( C \) with the special Abelian integrals \( y \) and \( \lambda \), which have fixed expansions near the punctures \( P_j \) \([4]\). It means that we have imposed the certain constraints on the algebraic geometrical data (according to \([3,2]\) we used admissible data). These constraints ensure the existence of global system of action-angle variables and the presence of the corresponding symplectic form \([4]\). The fact that we impose some constraints provides us with additional properties of \( dS \). Namely, generating 1-form \( dS \) possesses the property

\[ \frac{\partial dS}{\partial s_j} = \frac{\partial y d\lambda}{\partial s_j} = dw_j, \quad j = 1, \ldots, g, \]  
(3.3)
the derivative of the Abel-Jacobi map $U$ is the so-called Brill-Noether matrix. Hence, for any generic divisor $D = \gamma_1 + \cdots + \gamma_g$ on $C$ the standard 2-form on $C^{(g)}$

\[ \Omega_g = \delta \left( \sum_{j=1}^g y(\gamma_j) d\lambda(\gamma_j) \right) = \sum_{j=1}^g \delta y(\gamma_j) \wedge d\lambda(\gamma_j) = \sum_{j=1}^g ds_j \wedge dw_j, \tag{3.4} \]

is a desired holomorphic symplectic form $\Omega_g$ on $C^{(g)}$. The set of variables $\{s_j, w_j\}_{j=1}^g$ are the complete set of action-angle variables on $J(C)$. These action-angle variables $\{s_j, w_j\}$ have been obtained by generalizing the definition of actions introduced for integrable systems on tori in the form of periods of holomorphic differentials $dw_j$ along the elements of a homology basis in $D$.

Now we turn to the uniform St"ackel systems. The corresponding algebraic curve (2.4) is a hyperelliptic curve given by an equation of the form

\[ C : \quad y^2 = \prod_{i=1}^{2g+1} (\lambda - \lambda_i), \tag{3.5} \]

and puncture $P$ is the point at infinity $\lambda = \infty$. Recall that the moduli $\lambda_j$ of $C$ are integrals of motion (2.4). Solution to the inverse Jacobi problem and associated Abel-Jacobi map on $C$ relate a set of the action-angle variables and the separated variables.

Variables of separation $q_j(t)$ give solution to the inverse Jacobi problem (2.10). The associated Abel-Jacobi map $U : \text{Div}(C) \to J(C)$ is restricted to Lagrangian submanifold $C^{(k)}$.

Note that whenever we discuss the Abel-Jacobi map, we shall tacitly assume that a base point $\gamma_0$ on $C$ has already been fixed in an appropriate position.

Suppose that point $D = \gamma_1 + \cdots + \gamma_k$, $k \leq g$ belongs to $C^{(k)}$. The differential of the Abel-Jacobi map (3.6) at the point $D$ is a linear mapping from the tangent space $T_D(C^{(g)})$ of $C^{(g)}$ at the point $D$ into the tangent space $T_{U(D)}(J(C))$ of $J(C)$ at the point $U(D)$.

\[ U_D : \quad T_D(C^{(k)}) \to T_{U(D)}(J(C)). \]

Now suppose that $D$ is a generic divisor, and $z_j$ is a local coordinate on $C$ near the point $\gamma_j$. Then $(z_1, \ldots, z_k)$ yields a local coordinate system near the point $D$ in $C^{(k)}$. Let $dw_k$ ($k = 1, \ldots, g$) is a basis for a space $H_1(C)$ of holomorphic differentials on $C$, and near $\gamma_j$

\[ dw_k = \phi_{kj}(z_j) dz_j, \]

where $\phi_{kj}(z_j)$ is holomorphic. It follows that the Abel-Jacobi map $U$ can be expressed near $D$ as

\[ U(z_1, \ldots, z_k) = \left( \sum_{j=1}^k \int_{\gamma_0}^{z_j} \phi_{1j}(z_j) dz_j, \ldots, \sum_{j=1}^k \int_{\gamma_0}^{z_j} \phi_{gj}(z_j) dz_j \right). \]

Hence

\[ U_D^* = \begin{pmatrix} \phi_{11}(\gamma_1) & \cdots & \phi_{g1}(\gamma_1) \\ \vdots & \ddots & \vdots \\ \phi_{1k}(\gamma_k) & \cdots & \phi_{gk}(\gamma_k) \end{pmatrix}, \tag{3.7} \]

is the so-called Brill-Noether matrix.

**Theorem 2** Transpose Brill-Noether matrix $U_D^*$ on the genus $g \geq n$ hyperelliptic curve $C$, which is the derivative of the Abel-Jacobi map $U$ at generic divisor $D$, $\deg D = n$, is equal to the St"ackel matrix $S$ for the uniform St"ackel system on $\mathbb{C}^{2n}$ with metric

\[ g_{jj}(q_1, \ldots, q_n) = \frac{\det S}{S^{ij}}. \]
At generic point \( D \), \( \deg D = g \) matrix \( S = U_D^T \) is regular matrix satisfying the Stäckel theorem.

At \( g > n \) we have to consider restriction of the Abel-Jacobi map \((3.6)\) onto \( C^{(n)} \). In this case symplectic form \( \Omega_n \) on \( C^{(n)} \) is an appropriate projection of \( \Omega_g \) \((3.4)\) and \( C^{(n)} \) be a Lagrangian submanifold in the phase space \( C^{2n} \). The separated variables \( \{p_j, q_j\}_{j=1}^n \) are constructed from the first \( 2n \) action-angle variables \((3.4)\) only and the action differential \( dS = \sum_{j=1}^n p_j dq_j \) give rise to an \( n \)-dimensional chart of the whole space \( \mathcal{H}_1(C) \). The corresponding \( n \times n \) Stäckel matrix is the left upper \( n \times n \) block of the general matrix \( S = U_D^T \) and, therefore, unless otherwise indicated, we assume \( n = g \).

As an example, let us consider some basis for \( \mathcal{H}_1(C) \), for instance

\[
dw_j = \frac{\lambda^{j-1}}{g(\lambda)} d\lambda, \quad j = 1, \ldots, g.
\]

By choosing this basis we fix a basis of action-angle variables \((3.2-3.4)\). To solve the Jacobi inversion problem \((2.10)\) one gets variables of separation

\[
p_j = y(\gamma_j), \quad q_j = \lambda(\gamma_j), \quad j = 1, \ldots, g
\]

for a generic point \( D = \gamma_1 + \cdots + \gamma_g \) on \( C \), which coincides with divisor of simple poles of the corresponding Baker-Akhiezer function \( C^{(g)} \). In the real case (when \( p_j \) and \( q_j \) are real), the separated variables \( q_j \) \((3.9)\) (so-called root variables) vary along cycles \( A_j \) \((3.1)\) over basic cuts on \( C \) and, therefore, our problem is defined on \( g \)-dimensional real torus. The holomorphic symplectic form \( \Omega_g \) on \( C^{(g)} \) coincides with standard ones \((2.2)\) and a fiber \( C^{(g)} \) be a complex Lagrangian submanifold of the phase space \( C^{2g} \) \((2.7)\)

\[
C^{(n)} \equiv S^n(C) : (C(\lambda) \times C(\mu) \times \cdots \times C(\nu))/\sigma_n, \quad n \leq g,
\]

where \( \sigma_n \) is the permutation group on \( n \) letters.

Recall, that derivative \( U_D^T \) bears a great resemblance to the usual Gauss mapping. The map \( U_D^T \) induces a canonical mapping from \( C \) into the \((g - 1)\)-dimensional projective space \( C \rightarrow \mathbb{P}^{g-1} \). On the other hand, the canonical mapping is defined the derivative of the Abel-Jacobi map. For a hyperelliptic curve \( C \) of genus \( g \geq 2 \), the canonical map \( C \rightarrow \mathbb{P}^{g-1} \) is the composition of the double covering map \( C \rightarrow \mathbb{P}^1 \), sending \((y, \lambda)\) to \( \lambda \), with the Veronese map \( \mathbb{P}^1 \rightarrow \mathbb{P}^{g-1} \) given by a basis for the polynomial ring of degree \( g - 1 \). With respect to the basis of \( \mathcal{H}_1(C) \) \((3.8)\), the canonical map of \( C \) has an extremely simple expression

\[
(y, \lambda) \rightarrow \lambda \rightarrow [\lambda^{g-1}, \lambda^{g-2}, \ldots, \lambda, 1].
\]

By using this map we introduce the \( g \times g \) matrix

\[
S(\lambda, \mu, \ldots, \nu) = \begin{pmatrix}
\lambda^{g-1} & \mu^{g-1} & \cdots & \nu^{g-1} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda & \mu & \cdots & \nu \\
1 & 1 & \cdots & 1
\end{pmatrix}.
\]

\((3.11)\)

determined on a Lagrangian submanifold \((3.10)\). For a generic point \( D = \gamma_1 + \cdots + \gamma_g \) in \((3.9)\) the Stäckel matrix is equal to

\[
S(q_1, q_2, \ldots, q_g) = S(\lambda, \mu, \ldots, \nu)|_{\lambda=q_1, \mu=q_2, \ldots, \nu=q_g}; \quad S_{kj}(q_j) = \lambda^{g-k}|_{\lambda=q_j}.
\]

\((3.12)\)

Recall, that the diagonal metric \( g_{jj} \) is completely determined by the corresponding Stäckel matrix \((2.13)\). Nevertheless, we introduce another equivalent definition of the metric. Substituting the Stäckel matrix \((3.12)\) in the algebraic equation \((2.3)\) one gets

\[
\sum_{j=1}^g S_{kj}(q_j) g_{jj}(q_1, q_2, \ldots, q_g) = \delta_{k1} =
\]

\((3.13)\)

\[\sum_{j=1}^g \text{Res}|_{\lambda=q_j} \frac{\lambda^{k-1}}{e(\lambda)} = \frac{1}{2\pi i} \int_C \frac{\lambda^{k-1}}{e(\lambda)}.\]
where by definition
\[ g_{jj}(q_1, q_2, \ldots, q_n) = \text{Res}_{\lambda=q_j} \frac{1}{e(\lambda)}, \]

Here we introduced function \( e(\lambda) \), which has zeroes at the points \( q_j \) giving solution of the inverse Jacobi problem.

In general, function \( e(\lambda) \) with \( g \) zeroes, which are solution of inverse Jacobi problem, is expressed in the Riemann theta-function
\[ e(\lambda) = \theta(\mathcal{U}(\gamma_1, \ldots, \gamma_g) - \beta - K), \quad \mathcal{U}(\gamma_1, \ldots, \gamma_g) = \mathcal{U}(\gamma_1) + \cdots + \mathcal{U}(\gamma_g). \tag{3.14} \]

Here \( K \) is a vector of the Riemann constants and \( \beta = (\beta_1, \ldots, \beta_g) \in \mathbb{C}^g \) is a fixed vector. The principal properties of the function \( e(\lambda) \) (3.14) are considered in \[3\].

**Proposition 1** Function \( e(\lambda) \) on \( \mathbb{C} \) with \( g \) zeroes \( (q_j, q_j) \) giving solution to the Jacobi inversion problem is completely defined the metric \( g_{jj}(q_1, q_2, \ldots, q_n) \) (3.13) for a uniform Stäckel system.

We prove this proposition in the polynomial ring only. In this case
\[ e(\lambda) = \prod_{k=1}^{g} (\lambda - q_k), \tag{3.15} \]

and
\[ g_{jj}(q_1, q_2, \ldots, q_g) = \text{Res}_{\lambda=q_j} e^{-1}(\lambda) = \frac{1}{\prod_{j\neq k} (q_j - q_k)}. \tag{3.16} \]

To prove (3.13) for this metric, it suffices to consider the following integral
\[ \frac{1}{2\pi i} \oint_C \frac{\lambda^k}{e(\lambda)} = \sum_{j=1}^{g} \text{Res}_{\lambda=q_j} \frac{\lambda^k}{e(\lambda)} = - \text{Res}_{\lambda=\infty} \frac{\lambda^k}{e(\lambda)} = \delta_{k,g-1} \tag{3.17} \]

where \( C \) encloses all \( q_j \).

Function \( e(\lambda) \) is defined on the universal configuration space, i.e. it is independent on the moduli \( \lambda_j \) of \( \mathcal{C} \) (integrals of motion) and on a choice of the basis of holomorphic differentials in \( \mathcal{H}_1(\mathcal{C}) \). For instance, in the polynomial ring let us consider a set of the equivalent Stäckel matrices with the following entries (3.17)
\[ \mathbf{S}_{kj}(\lambda)|_{q_j} = \phi_{kj}(\lambda)|_{q_j}, \quad \phi_{kj}(\lambda) = \lambda^{g-k} + a_1^{(j)} \lambda^{g-k-1} + \cdots + a_{g-k}^{(j)}, \tag{3.18} \]

where polynomials \( \phi_{kj} \) form various basises for the polynomial ring of degree \( g-1 \). Substituting (3.18) in (3.13) and (3.17) one obtains at once universal solution \( e(\lambda) \) (3.18). Below we shall see that the hamiltonian \( H \) (2.1) with the diagonal metric \( g_{jj} \) (3.13) is closely related to the distinguished puncture \( P \) at infinity \( \lambda = \infty \) on the hyperelliptic curve \( \mathcal{C}(1) \). The different Stäckel matrices (3.18) correspond to the distinct sets of the integrals of motion in the involution for a single hamiltonian \( H \). The completeness and functional independence of these integrals directly follows from the completeness and independence of the basis elements (3.13) for a polynomial ring.

Finally, we look at other fibrations \( \mathcal{C}(n) \) at \( n \neq g \). At \( g > n \), to construct the metric \( g_{jj}(q_1, \ldots, q_n) \) on \( \mathcal{C}(n) \), we expand the initial curve \( \mathcal{C} \) (3.17) by
\[ y^2 = \prod_{i=1}^{2g+1} (\lambda - \lambda_i) = U_{2g+1}(\lambda) + \prod_{i=1}^{2n+1} (\lambda - \tilde{\lambda}_i), \quad n \leq g. \tag{3.19} \]

Here \( U_{2g+1}(\lambda) \) is an at most \( 2g + 1 \) order polynomial, which is regarded as a potential in (2.6). The \( n \times n \) Stäckel matrix and the corresponding function \( e(\lambda) \) may be associated to the auxiliary genus \( n \) curve
\[ \tilde{\mathcal{C}} : \quad \tilde{y}^2 = \prod_{i=1}^{2n+1} (\lambda - \tilde{\lambda}_i). \tag{3.20} \]
Function $e(\lambda)$ are independent on the moduli of $C$ \[3.19\] and, therefore, uniform potential $U_{2g+1}$ in \[3.19\] has an arbitrary form and decomposition \[3.19\] determines the highest power of the polynomial $U(\lambda)$ only.

At $n > g$ the above holomorphic symplectic form $\Omega_n$ on the leaves $\mathcal{M}$ is degenerate. However, a non-degenerate form on $C^{(n)}$ may be obtained by restricting $C^{(n)}$ to the larger leaves $\tilde{\mathcal{M}}$ of the foliation \[3.4\]. The leaves $\mathcal{M}$ correspond to the level sets of all the local coordinates except to holomorphic $s_j$ \[3.3\] and to some additional $(n - g)$ coordinates associated to meromorphic differentials $d\tilde{w}_j$ in \[3.3\] \[3.4\]. In fact, to construct the action-angle variables we have to add several meromorphic differentials to holomorphic angle variables. Thus, at $n > g$ the symplectic 2-form $\Omega_n$ on $C^{(n)}$ is meromorphic \[3.4\].

As an example, at $n = g + 1$, we can add one local coordinate in the neighborhood of puncture $P$ at infinity \[3.3\]. This additional coordinate occurs in the Stäckel matrix and in the metric in the following way

\[
S^{(g+1)}(\lambda, \mu, \ldots, \nu) = q_0 S^{(g)}(\lambda, \mu, \ldots, \nu),
\]

\[
e(\lambda) = q_0 \prod_{j=1}^n (\lambda - q_j) \quad g_{00} = \operatorname{Res}_{\lambda=\infty} \frac{\lambda^{g-1}}{g(\lambda)}.
\]

At $n > g$ these systems with meromorphic form $\Omega_n$ possess several reductions of the additional meromorphic coordinates, for instance $q_0 = \text{const}$ in \[3.24\] \[20\].

Above formulas are well adjusted for generalization. If the curve $C$ \[3.3\] is substituted by

\[
C : \quad y^2 = F(\lambda) = \frac{P_1(\lambda)}{Q_m(\lambda)} = \frac{\prod_{j=1}^{2g+1}(\lambda - \lambda_j)}{\prod_{k=1}^{c}(\lambda - \delta_k)}, \quad m \leq 2g + 1,
\]

where $\{\delta_k\}$ is a set of $m$ arbitrary constant, one gets

\[
S_{kj}(\lambda)|_{\lambda=q_j} = \frac{\phi_{kj}(\lambda)}{Q_m(\lambda)}|_{q_j}, \quad e(\lambda) = \frac{\prod_{j=1}^g(\lambda - q_j)}{Q_m(\lambda)}.
\]

Note, that the algebraic equation \[3.13\] is covariant with respect to the transformations

\[
S \to R^{-1}(\lambda)S, \quad e(\lambda) \to R^{-1}(\lambda)e(\lambda),
\]

that leads to the general form of the metric

\[
g_{jj}(q_1, q_2, \ldots, q_n) = \operatorname{Res}_{\lambda=q_j} \left( \frac{Q_m(\lambda)R(\lambda)}{\prod_{j=1}^g(\lambda - q_j)} \right).
\]

associated to the curve $C$. We shall use this freedom to consider the standard curvilinear coordinate systems \[11\] \[12\] \[19\].

So, the hyperelliptic genus $g$ curve $C$ may be associated to a family of the uniform Stäckel systems on the phase space $C^{2g+1}$ by using the Abel-Jacobi map $U$, its differential $U_D$ and their restrictions on $C^{(n)}$. Diagonal metric $g_{jj}(q_1, q_2, \ldots, q_n)$ \[2.13\] in the hamiltonian \[2.1\] is completely defined by number of degrees of freedom $n$ and potential $U(\lambda)$ is at most $2g + 1$ order arbitrary polynomial.

On the other hand, one fixed metric $g_{jj}(q_1, q_2, \ldots, q_n)$ may be associated to an infinite set of the hyperelliptic curves $C$. The corresponding hamiltonian systems differ from each other by the power and by the form of polynomial potentials $U(\lambda)$ \[3.19\]. Among these systems we must to distinguish systems on $C^{(n)}$ at $n > g$ \[3.19\] for which the number of degrees of freedom $n$ is more than genus $g$ of the associated curve $C$. In this case the corresponding symplectic 2-form on $C$ is meromorphic \[3.4\]. In the next section, we shall identify these systems with the degenerate or superintegrable systems \[21\]. Recall, that for degenerate system the number of independent integrals of motion is more than number degrees of freedom.
4 The Lax representations

Let us recall that the key idea, which has started the modern age in the study of classical integrable systems, is to bring them into the Lax form. All the properties of the uniform St"ackel systems may be recovered from the properties of the Abel map. Nevertheless, now we want to obtain the Lax representations for all the uniform St"ackel systems associated to the hyperelliptic curve $C(y, \lambda)$ \([3,5]\).

We consider construction of the Lax representation as a necessary intermediate step to study quantum counterparts of the St"ackel systems.

In the simplest case the Lax matrices $L(\lambda)$ or $L(y)$ are defined as the matrix valued functions on bare spectral curves $F_\lambda$, $\lambda \in F_\lambda$ \([1,3]\) or $F_y$, $y \in F_y$, while the full spectral curve $C(y, \lambda)$ is given by the Lax eigenvalue equations

$$C: \quad \det (L(\lambda) - y) = 0, \quad \det (L(y) - \lambda) = 0.$$  \hspace{1cm} (4.1)

As a result, $C$ arises as a ramified covering over the bare spectral curve $F_\lambda$ or $F_y$ \([22]\).

Till now a delicate questions is how to construct the Lax matrices $L(\lambda)$ or $L(y)$ for a given integrable system. The one integrable system may be associated to the different curves and one curve $C$ may be associated to the different mechanical integrable system on a common phase space. As an example, the $n$-particles Toda lattice can be equivalently formulated in terms of two different Lax representations \([23]\) associated to the single hyperelliptic curve $C$.

Here we consider equation for a general algebro-geometric symplectic structure associated to the spectral curve $C$ of the given Lax representation $L$

$$\Omega_n = - \sum_\alpha \text{Res}_{\alpha} \frac{\langle \delta \psi^+ \wedge \delta L \psi \rangle}{\langle \psi^+ \psi \rangle}$$  \hspace{1cm} (4.2)

proposed in \([4]\). Here $\Omega_n$ is the restriction of the algebro-geometrical symplectic form \([3,4]\) on $C$ generated by two differentials $dy$ and $d\lambda$ having poles at punctures $P_\alpha$. Functions $\psi$ and $\psi^+$ are the Baker-Akhiezer function on $C$ and it’s dual function. If we fix some 2-form $\Omega_n$ and the Baker-Akhiezer functions $\psi$, $\psi^+$ on a given curve $C$, then one can attempt to recover the associated Lax matrix $L$.

For the particular St"ackel systems the $2 \times 2$ Lax matrices \([20,24]\) and the corresponding vector Baker-Akhiezer function $\vec{\psi}$ associated to natural vector fields on the Jacobian of any hyperelliptic curve are known. On the other hand, we know the general scalar Baker-Akhiezer function $\psi$ on $C$ defined by its analitical properties on $C$, which corresponds to geodesic systems with diagonal metric \([4]\).

Note, here we have the vector Baker-Akhiezer function $\vec{\psi}$, which is the eigenfunction of the matrix $L$ associated to the curve $C$, and scalar Baker-Akhiezer $\psi$, which is completely defined by analitical properties on the same curve $C$.

For the uniform St"ackel systems let us identify the preassigned symplectic structure $\Omega$ \([2,2]\) with the symplectic structure \([3,4]\) defined on a hyperelliptic algebraic curve $C$. Next we try to recover Lax matrix for a geodesic motion under the following additional assumptions:

1) $L(\lambda)$ is a generic $2 \times 2$ matrix associated to a spectral hyperelliptic curve $C$ of genus $g = [(n-1)/2]$.

2) The associated vector Baker-Akhiezer function $\vec{\psi}$ has a constant normalization $\vec{\alpha}$ \([23]\)

$$\langle \vec{\alpha}, \vec{\psi} \rangle = \alpha_1 \psi_1 + \alpha_2 \psi_2 = 1, \quad \vec{\alpha} = (0,1).$$

3) The first component of $\vec{\psi}$ in \([1,2]\) is proportional to the unique Baker-Akhiezer function $\psi$ on $C$ with fixed analytical properties \([4]\).

At the first assumption $n$ is a number of integrals of motion, which are moduli of $C$ ($n = 2g + 1$) and, therefore, form $\Omega_n$ in \([1,2]\) is a restriction of meromorphic symplectic form $\Omega_n$ \([3,4]\) to the minimal $n$-dimensional leaf $\mathcal{M}$ \([3]\) for integrable systems on $\mathbb{C}^2$. The second assumption allows us to reduce vector Baker-Akhiezer function to scalar one. In this case, solution of \([1,2]\) is completely defined by the function $\psi$ on $C$ only. At first we present this particular solution in term of the function $e(\lambda)$ associated to the Abel map $\mathcal{M}$. Introduce function $e(\lambda)$ and its time derivative

$$e(\lambda) = \prod_{j=1}^n (\lambda - q_j), \quad m \leq n, \quad e_x(\lambda) = \{H, e(\lambda)\}.$$  \hspace{1cm} (4.3)
where \( \{\delta_k\} \) is a set of \( m \) arbitrary constant and \( H \) be a hamiltonian of the geodesic motion

\[
H = \sum_{j=1}^{n} g_{jj}(q_1, \ldots, q_n)p_j^2, \quad g_{jj}(q_1, \ldots, q_n) = \text{Res}_{\lambda=q_j} \frac{1}{e(\lambda)}.
\]

(4.4)

Thus, in the Lax equation for a geodesic motion

\[
L_x(\lambda) = \{H, L\} = [L, A],
\]

matrices \( L \) and \( A \) are given by

\[
L(\lambda) = \begin{pmatrix} -e_x/2 & e \\ -e_xx/2 & e_x/2 \end{pmatrix} \lambda = \begin{pmatrix} f & e \\ h & -f \end{pmatrix} \lambda,
\]

(4.5)

\[
A(\lambda) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

The hamiltonian \( H \) (4.4) is equal to a highest reside at the distinguished Weierstrass point on \( C \) at infinity \( \lambda = \infty \)

\[
H = -\text{Res}_{\lambda=\infty} \lambda^{-m} \det L(\lambda).
\]

(4.6)

where the full spectral curve \( C \) is equal to

\[
C : \quad y^2 = F(\lambda) \equiv \det L(\lambda),
\]

(4.7)

\[
F(\lambda) = -h^2(\lambda) - e(\lambda)f(\lambda) = \frac{e \cdot e_{xx}}{2} - \frac{e_x^2}{4}.
\]

By definition zeroes of \( e(\lambda) \) be separation variables and conjugated variables \( p_j \) are given by

\[
p_j = h(\lambda)|_{\lambda=q_j}, \quad h(\lambda) = -e_x/2 = e(\lambda) \sum_{j=1}^{n} g_{jj}(q_1, \ldots, q_n)\frac{p_j}{\lambda - q_j}.
\]

(4.8)

In accordance with [25] pairs of separation variables \( q_j, p_j \) lie on the spectral curve \( C \)

\[
y^2(\gamma_j) = p_j^2 = h^2(\lambda)|_{\lambda=q_j} = -F(\lambda = q_j) = -F(\lambda)|_{\gamma_j}.
\]

As usual, rational function \( F(\lambda) \) admits some different representations

\[
F(\lambda) = \frac{\sum_{j=1}^{n} I_j \lambda^{n-j}}{\prod_{k=1}^{m} (\lambda - \delta_k)} = \sum_{k=1}^{m} \frac{J_k}{(\lambda - \delta_k)} + \sum_{k=m+1}^{n} J_k \lambda^{n-k-1}.
\]

(4.9)

Here \( \{I_j\}_{j=1}^{n} \) and \( \{J_k\}_{k=1}^{n} \) are two sets of independent integrals of motion in the involution. The first set of integrals \( \{I_j\}_{j=1}^{n} \) in [13] corresponds to the Stäckel matrix \( \{3,11\} \). The set of equivalent Stäckel matrices \( \{3,18\} \) relate to another decompositions of a numerator of \( F(\lambda) \) \( \{13\} \). The second set of integrals \( \{J_k\}_{k=1}^{n} \) in \( \{4,9\} \) is associated to an expansion near punctures \( \{\delta_k, \infty\} \) on \( C \).

The spectral curve \( C \) \( \{4,1\} \) is a time-independent curve and, therefore,

\[
\{H, F(\lambda)\} = 0, \quad \Rightarrow \quad \partial_\lambda^2 e(\lambda) = e_{xxx} = 0.
\]

(4.10)

Thus, in fact [26], we consider the polynomial solutions \( e(\lambda) = \prod (\lambda - q_j(t)) \) to the equation \( \{4,10\} \) and describe the hamiltonian dynamics of their zeroes \( q_j(t) \) (recall, that \( \partial_\lambda \) means derivative by time).

Substituting function \( e(\lambda) \) \( \{13\} \) and hamiltonian \( H \) \( \{4,4\} \) into \( \{4,10\} \) one gets the equations in the metric \( g_{jj} = H_j^2 \) \( \{2,12\} \). If we introduce so-called rotation coefficients

\[
\beta_{ij} = \frac{\partial H_j}{\partial H_i}, \quad i \neq j,
\]

(4.11)
these equations may be reduced to the following equations \[13\]

\[
\partial_k \beta_{ij} = \beta_{ik} \beta_{kj}, \quad i \neq j \neq k,
\]

\[
\partial_i \beta_{ij} + \partial_j \beta_{ji} + \sum_{m \neq i, j} \beta_{mi} \beta_{mj} = 0, \quad i \neq j,
\]

(4.12)

where the notation \(i \neq j \neq k\) means that indices \(i, j, k\) are distinct.

Of course, these equations may be obtained without any Lax representation by using definition \([2.13]\) of the metric, properties of the Abel-Jacobi map and preassigned asymptotic behavior of \(e(\lambda)\) at the distinguished point \(\lambda = \infty\).

The equations (4.12) are equivalent to the vanishing conditions of all a’priory non-trivial components of the curvature tensor \([13, 14, 15]\). Therefore, using (4.12) we conclude that local Riemannian submanifold \((V_n, g|_{V_n})\) of the Riemannian manifold \((\mathbb{C}^n, g)\) is a flat manifold whose metric is diagonal with respect to the coordinates \(\{q_j\}\). Imposing some additional restrictions on the space of solutions to (4.2)\([20]\), one could get the Bourlet type equations \([15]\) related to another Riemannian manifolds of constant curvature.

To construct more general solutions to (4.2) associated to hyperelliptic curve \(C\) of higher genus we begin with calculation of the Poisson bracket relations for the initial Lax matrix \(L(\lambda)\). It allows us to identify the space of solutions to equation (4.2) with the loop algebra \(L(sl(2))\) in fundamental representation \([23]\) and then to use the representation theory of the underlying algebra \(sl(2)\) \([27]\).

**Theorem 3** The Poisson bracket relations for the matrix \(L(\lambda)\) (4.3) are closed into the following \(r\)-matrix algebra at \(m \leq n\) only

\[
\left\{ L(\lambda), L(\mu) \right\} = \left[ r_{12}(\lambda, \mu), L(\lambda) \right] - \left[ r_{21}(\lambda, \mu), L(\mu) \right].
\]

(4.13)

Here the standard notations are introduced:

\[
\hat{L}(\lambda) = L(\lambda) \otimes I, \quad \hat{L}(\mu) = I \otimes L(\mu),
\]

(4.14)

\[
r_{12}(\lambda, \mu) = \frac{\Pi}{\lambda - \mu} \quad r_{21}(\lambda, \mu) = \Pi r_{12}(\mu, \lambda) \Pi,
\]

and \(\Pi\) is the permutation operator of auxiliary spaces \([23]\).

The Poisson bracket relations for the Lax matrix \(L(\lambda)\) (4.13) are preassigned by the initial symplectic structure (3.4). It is necessary to calculate two brackets only

\[
\{ e(\lambda), e(\mu) \} = 0,
\]

(4.15)

and

\[
\{ h(\lambda), e(\mu) \} = \left\{ e(\lambda) \sum_{j=1}^{n} g_{jj} (q_1, \ldots, q_n) p_j \frac{\Pi_{j=1}^{n} (\lambda - q_j)}{\Pi_{j=1}^{n} (\lambda - \delta_k)} \right\}
\]

\[
= -e(\lambda) e(\mu) \sum_{j=1}^{n} \frac{g_{jj}}{(\lambda - q_j)(\mu - q_j)}
\]

\[
= e(\lambda) e(\mu) \sum_{j=1}^{n} \left( \frac{g_{jj}}{\lambda - q_j} - \frac{g_{jj}}{\mu - q_j} \right) = \frac{1}{\lambda - \mu} \left[ e(\mu) - e(\lambda) \right],
\]

(4.16)

where we used a standard decomposition of rational function

\[
e^{-1}(\lambda) = \sum_{j=1}^{n} \frac{g_{jj}}{\lambda - q_j}, \quad g_{jj} = \text{Res}_{\lambda = q_j} e^{-1}(\lambda).
\]
Another Poisson brackets may be directly derived from these brackets and by definition of the entries of the Lax matrix $L(\lambda)$ \eqref{eq:4.15} via derivative of the single function $e(\lambda)$

\[
\begin{align*}
\{h(\lambda), h(\mu)\} &= 0, \\
\{f(\lambda), e(\mu)\} &= \partial_{\lambda} \{h(\lambda), e(\mu)\} = \frac{2}{\lambda - \mu} [h(\lambda) - h(\mu)] , \\
\{f(\lambda), h(\mu)\} &= -\frac{1}{2} \partial_{\lambda}^2 \{h(\lambda), e(\mu)\} = \frac{1}{\lambda - \mu} [f(\lambda) - f(\mu)] , \\
\{f(\lambda), f(\mu)\} &= -\frac{1}{2} \partial_{\lambda}^2 \{h(\lambda), e(\mu)\} = 0 ,
\end{align*}
\tag{4.17}
\]

To derive the first bracket we have to combine second and first derivatives of the brackets \eqref{eq:4.11} and \eqref{eq:4.10}, respectively. At the last bracket one substitutes the equation of motion \eqref{eq:4.10}.

If, contrary to our geometric conventions, the order of polynomial $Q_m(\lambda)$ is more then order of polynomial $P_l$ in \eqref{eq:3.22}, i.e. if $m > n$ in the metric \eqref{eq:4.13}, then rational function $e(\lambda)$ admits another representation

\[
e^{-1}(\lambda) = \sum_{j=1}^{n} \frac{g_{jj}}{\lambda - q_j} + \xi(\lambda, q_1, \ldots, q_n),
\]

where remainder $\xi(\lambda)$ is a certain polynomial. Substituting this function $e(\lambda)$ into \eqref{eq:4.17} one gets

\[
\frac{\partial \xi(\lambda, q_1, \ldots, q_n)}{\partial \lambda} = 0.
\]

This constraint to remainder $\xi(\lambda, q_1, \ldots, q_n)$ directly follows from the symmetry of the last Poisson bracket in \eqref{eq:4.17}.

The $r$-matrix algebra \eqref{eq:4.13} is so-called linear case of the $r$-matrix algebras corresponds to integrable systems, which are modelled on coadjoint orbits of Lie algebra $sl(2)$. The $r$-matrix in \eqref{eq:4.17} is a standard rational $r$-matrix on $\mathcal{L}(sl(2))$. The general form of the function $e(\lambda)$ \eqref{eq:3.14} leads to the elliptic and trigonometric $r$-matrices \cite{28, 29}.

Thus, for a geodesic motion \eqref{eq:4.4} the Lax representation \eqref{eq:4.3} with arbitrary poles $\{\delta_k\}_{k=1}^m$ \eqref{eq:4.3} may be regarded as a generic point at the loop algebra $\mathcal{L}(sl(2))$ in fundamental representation after an appropriate completion \cite{28}. Since, to construct the Lax representation for a potential motion with the fixed metric $g_{jj}(q_1, \ldots, q_n)$ \eqref{eq:4.13} we can use the outer automorphism of the space of infinite-dimensional representations of $sl(2)$ proposed in \cite{27}.

Applying this automorphism of the underlying algebra $sl(2)$ directly to the Lax representation $L(\lambda)$ \eqref{eq:4.3} on $\mathcal{L}(sl(2))$ we obtain a family of the new Lax pairs

\[
L'(\lambda) = L(\lambda) - \sigma_{-} \cdot [\phi(\lambda)e^{-1}(\lambda)]_{N} , \quad \sigma_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,
\tag{4.18}
\]

\[
A'(\lambda) = A - \sigma_{-} \cdot [\phi(\lambda)e^{-2}(\lambda)]_{N} = \begin{pmatrix} 0 & u_N(\lambda) \\ u_N(\lambda) & 1 \end{pmatrix} .
\]

Here $\phi(\lambda)$ is a function on spectral parameter and $[z]_N$ means restriction of $z$ onto the ad$^+_N$-invariant Poisson subspace of the initial $r$-bracket \cite{27, 29, 30}. For the rational $r$-matrix \eqref{eq:4.14} we can use the linear combinations of the following Taylor projections

\[
[z]_N = \left[ \sum_{k=-\infty}^{+\infty} z_k \lambda^k \right]_N \equiv \sum_{k=0}^{N} z_k \lambda^k ,
\tag{4.19}
\]

or the Laurent projections \cite{27, 29}.

The mappings \eqref{eq:4.18} from the representation of the loop algebra $\mathcal{L}(sl(2))$ to representations of the universal enveloping algebra $U(\mathcal{L})$ play the role of a dressing procedure allowing to construct the Lax matrices $L'_N(\lambda)$ for an infinite set of new integrable systems starting from the single known Lax
matrix $L(\lambda)$ associated to one integrable model. This mapping preserves the metric $g_j(q_1, \ldots, q_n)$ in the hamiltonian \cite{24}, but changes the potential $U(q_j)$ and associated curve $C$.

New Lax matrix $L'(\lambda)$ \cite{14}\ref{14} obeys the linear $r$-bracket \cite{11}, where constant $r_{ij}$-matrices substituted by $r'_{ij}$-matrices depending on dynamical variables \cite{27, 29}.

$$r_{12}(\lambda, \mu) \rightarrow r'_{12} = r_{12} - \frac{([\phi(\lambda)e^{-2}(\lambda)]_N - [\phi(\mu)e^{-2}(\mu)]_N)}{(\lambda - \mu)} \cdot \sigma_- \otimes \sigma_+.$$ \hspace{0.5cm} (4.20)

We have to distinguish systems on $\mathcal{C}^{(n)}$ at $n > g$ \cite{31} for which the number of degrees of freedom $n$ is more than genus $g$ of the associated curve $C$. According to \cite{21} the corresponding symplectic form is meromorphic. In this case the action differential $dS = yd\lambda$ give rise to a whole space $H_1(C)$ and, in addition, several meromorphic differentials on $C$. We can identify these systems with the degenerate or superintegrable systems \cite{21}.

**Theorem 4** The complete set of noncommutative integrals of motion for the degenerate uniform St"ackel systems with meromorphic symplectic form $\Omega_g$ is determined by the generalized spectral surface

$$\mathcal{C}(y, \lambda, \mu) : \quad \det (yI + \Pi L'(\lambda) \otimes L'(\mu)) = 0.$$

Here we used the outer product of the $2 \times 2$ Lax matrices $L'(\lambda)$ with $L'(\mu)$ and $\Pi$ means $4 \times 4$ permutation matrix in $\mathbb{C}^2 \times \mathbb{C}^2$. Equation of motion for the matrix $L(\lambda, \mu) = \Pi L'(\lambda) \otimes L'(\mu)$ is equal to

$$\frac{d}{dt} L(\lambda, \mu) = L(\lambda, \mu) A(\lambda, \mu) - \Pi A(\lambda, \mu) \Pi^{-1} L(\lambda, \mu),$$

$$A(\lambda, \mu) = A(\lambda) \otimes I + I \otimes A(\mu),$$

where matrix $A(\lambda)$ is a second Lax matrix and $I$ is a unit matrix.

It is easy to derive from \cite{14}, that $n > g$ iff $n \geq N$, where $N$ is a highest power in the Taylor projection \cite{4}. In this case the corresponding $r$-matrix \cite{4}\ref{4} preserves the simple pole at the puncture $P$ at $\lambda = \infty$ and the associated second Lax matrix $A'$ remains a constant in spectral sense $\frac{\partial A(\lambda)}{\partial \lambda} = 0$ under the mapping \cite{14}.

Thus, for the degenerate systems $A(\lambda, \mu) = \Pi A(\lambda, \mu) \Pi^{-1}$ and equation \cite{4}\ref{4} takes the standard Lax form and it proves the theorem.

As usual, spectral curve $\mathcal{C}$ \cite{3} of $L'(\lambda)$ is a generating function of the involutive family of integrals of motion. Substituting functions $\phi(\lambda) = \lambda^m Q_m^{-1}(\lambda)^{U_N(\lambda)}$ into $L'(\lambda)$ \cite{14}\ref{14} one gets their spectral curve in the form

$$\mathcal{C} : \quad g^2 = F'(\lambda) = \det L'(\lambda) = U_N(\lambda) + \sum_{j=1}^{n} I_j' \lambda^{n-j} \prod_{k=1}^{m} (\lambda - \delta_k),$$

where $\{I_j\}$ are integrals of motion. It is a time-independent curve and, therefore,

$$\frac{dF'(\lambda)}{dt} = 0, \quad \Rightarrow \quad \left[ \frac{1}{4} \partial^3_x + u_N(\lambda) \partial_x + \frac{1}{2} u_{N,\lambda}(\lambda) \right] \cdot \epsilon(\lambda) = 0.$$ \hspace{0.5cm} (4.22)

Of course, this equation may be obtained directly in framework of symplectic geometry \cite{31}. Let us briefly explain an origin of this equation in the theory of nonlinear equation, that allows us to relate scalar Baker-Akhiezer function $\psi$ and function $\epsilon(\lambda)$.

The same algebro-geometrical symplectic form $\Omega_g$ \cite{3}\ref{3} on hyperelliptic curve $\mathcal{C}$ \cite{3}\ref{3} leads directly to a hamiltonian structure for soliton equations \cite{3, 4}. As an example, we consider the KdV equation associated to hyperelliptic curve \cite{3}\ref{3} with one puncture $P (N = 1)$ at infinity $\lambda = \infty$ and at $l = 1, m = 2$ \cite{3}. Let us select one leaf of foliation corresponded to $d\lambda$ with all zero periods

$$\int_{\mathcal{C}} d\lambda = 0$$
for an arbitrary cycle $C$. In this case, the Abelian integral $\lambda(P)$ is a single-valued function, with only a pole of second order at $P$ ($m = 2$). For finite-gap solutions of the KdV equations, moduli $s_j$ are canonically conjugated with respect to the Gardner-Faddeev-Zakharov symplectic structure to angle variables $w_j$ (see [8] and references within). Thus, the uniform Stäckel systems have a common set of the action-angle variables with solutions of the KdV equations.

Starting with this set of variables we consider general algebro-geometric equation (4.22) for nonlinear systems. Solution of the equation (4.25) in a ring of second order differential operators with the standard Baker-Akhiezer function $\psi$ on $C(3.5)$ is well known [2, 3, 4, 9]. The associated Shrödinger operator has the form

$$\mathcal{L}(\lambda) = -\frac{\partial^2}{\partial x^2} + u(x, t, \lambda),$$

where $\lambda$ is a parameter. In some simple cases, such as the KdV equation, this parameter $\lambda$ appears as an eigenvalue and one ultimately equates the potential $u$ with a solution of the nonlinear equation itself. Let us look for a solution $\mathcal{A}(\lambda)$ of the Lax system in the ring of differential operators

$$\mathcal{L}(\lambda)\psi = 0,$$

(4.24)

of the form

$$\mathcal{A}(\lambda) = e(\lambda) \frac{\partial}{\partial x} - \frac{1}{2} \frac{\partial e(\lambda)}{\partial x},$$

(4.25)

Substituting the given form of $\mathcal{A}$ into the Lax system, one gets

$$\frac{\partial u}{\partial t} = -2 \left[ \frac{1}{4} \frac{\partial^2}{\partial x^2} + u(\lambda) \frac{\partial}{\partial x} + \frac{1}{2} u_x(\lambda) \right] \cdot e(\lambda).$$

(4.26)

Equation (4.26) is called the generating equation. For a different choices of the form of $e(\lambda)$ and $u(\lambda)$, this procedure leads to different hierarchies of integrable equations, as an example to the KdV, nonlinear Shrödinger and sine-Gordon hierarchies [8, 9] or to the Dym hierarchy [33]. If we consider the solutions of the equation (4.26) in the form of polynomial (3.15), then the roots $q_j$ of $e(\lambda)$ define the root variables and as a result finite-gap solutions of the problem of geodesic (see [8, 9, 33] and references within).

Substitution of the special form of second operator $\mathcal{A}(\lambda)$ (4.25) into the Lax system (4.24) allows us to eliminate the Baker-Akhiezer function $\psi$ and to construct $2 \times 2$ Lax matrix in $e(\lambda)$. In fact, we replace the Baker-Akhiezer function $\psi$ on $C$ to the mutually disjoint function $e(\lambda)$ on $C$, which has a transparent mechanical interpretation (3.15). Recall, that function $e(\lambda)$ is defined as function with zeroes, which give solution to the Jacobi inversion problem [8] on the hyperelliptic curve $C$.

5 The flat coordinates

According to [14] at $n = g$ the orthogonal curvilinear coordinates $\{p_j, q_j\}_{j=1}^g$ form a generic divisor of the simple poles of the Baker-Akhiezer function $\psi$, which is defined by their analytical properties on $C$. The evaluation of $\psi$ at a set of punctures on $C$ determines the flat coordinates $\{p_j, x_j\}_{j=1}^g$ for the diagonal metric (2.12). It turns out that up to constant factors the Lamé coefficients $H_j$ are equal to the leading terms of the expansion of the same function $\psi$ at the punctures on $C$ [14].

Next we reach the same conclusions by using the function $e(\lambda)$ and the corresponding Lax representation $L(\lambda)$ on $C$. As usual, we reduce the study of algebraic geometrical data to the analysis of the associated geodesic motion. The crucial observation is that the equations of motion in coordinates $\{p_j, x_j\}_{j=1}^g$ on the Riemannian manifolds of constant curvature have a Newton form and the corresponding hamiltonian has a natural form

$$\dot{x}_j = \xi_j(x_1, \ldots, x_n), \quad H = \sum a_{ij} p_i p_j + V(x_1, \ldots, x_n), \quad a_{ij} \in \mathbb{C},$$

(5.1)
where $\xi_j(x_1, \ldots, x_n)$ and potential $V(x_1, \ldots, x_n)$ are functions on coordinates only. Let us introduce new function $B(\lambda)$

$$B^2(\lambda) = e(\lambda) = H^{-2}(\lambda),$$  \hspace{1cm} (5.2)

which is "inverse" to the Lamé coefficients $H_j$ (2.13). One immediately gets

$$F(\lambda) = B^3 B_{xx}, \quad F'(\lambda) = B^3 B_{xx} + B^4 \left[ \frac{\phi(\lambda)}{B^4} \right]_N,$$  \hspace{1cm} (5.3)

These equations have the form of Newton's equations for the function $B$

$$B_{xx} = F(\lambda)B^{-3},$$

$$B_{xx} = F'(\lambda)B^{-3} - B \left[ \frac{\phi(\lambda)}{B^4} \right]_N,$$  \hspace{1cm} (5.4)

To expand function $B(\lambda)$ at the Lourent set

$$B = \sum_{j=0}^{N} x_{N-j} \lambda^j$$

it is easy to prove that coefficients $x_j$ obey the Newton equation of motion (5.4) (see (4.10) and references within [26]). Here we reinterpret the coefficients of the bare curves $F(\lambda)$ and $F'(\lambda)$ in (5.4) not as functions on the phase space, but rather as integration constants. In variables $x_j$ mapping (4.18) affects only on the potential ($x$-dependent) part of the integrals of motion $I_k$. The kinetic (momentum dependent) part of $I_k$ remains unchanged. So, the dressing mapping (4.18) allows us to get over from a free motion on $\mathbb{C}^{2n}$ to a potential motion on $\mathbb{C}^{2n}$.

As an example, from (5.4) we get some well known orthogonal curvilinear coordinates on $\mathbb{R}^n$ (see [11, 12, 13]):

elliptic coordinates $m = n$ in (3.23)

$$e(\lambda, q_1, \ldots, q_n) = \frac{\prod_{j=1}^{n} (\lambda - q_j)}{\prod_{k=1}^{n} (\lambda - \delta_k)} = 1 + \sum_{k=1}^{n} \frac{x_k^2}{\lambda - \delta_k} = B^2(\lambda, x_1, \ldots, x_n)$$

$$\delta_1 < x_1 < \delta_2 \cdots < \delta_n < x_n$$

parabolic coordinates $m = n - 1$ in (3.23)

$$e(\lambda, q_1, \ldots, q_n) = \frac{\prod_{j=1}^{n} (\lambda - q_j)}{\prod_{k=1}^{n-1} (\lambda - \delta_k)} = \lambda - x_n + \sum_{k=1}^{n-1} \frac{x_k^2}{\lambda - \delta_k} = B^2(\lambda, x_1, \ldots, x_n)$$

$$x_1 < \delta_1 < x_2 \cdots < \delta_{n-1} < x_n$$

spherical coordinates $m = n + 1$ see [3.21]

$$e(\lambda, q_0, \ldots, q_n) = \frac{q_0 \prod_{j=1}^{n} (\lambda - q_j)}{\prod_{k=1}^{n+1} (\lambda - \delta_k)} = \sum_{k=1}^{n+1} \frac{x_k^2}{\lambda - \delta_k} = B^2(\lambda, x_1, \ldots, x_{n+1})$$

Curvilinear coordinates $\{q_j\}$ are zeroes of function $e(\lambda)$ and flat coordinates $\{x_j\}$ are residues of $e(\lambda) = B^2(\lambda)$ at the punctures, in accordance with the Baker-Akhiezer function approach [14, 23].

All the separable orthogonal curvilinear coordinate systems in $\mathbb{R}^n$ may be obtained from these coordinate systems [11, 12, 20]. According to [24], all the possible separable in these coordinates potentials, which are polynomials or rational functions of the cartesian coordinates $x_j$, belong to the set of the uniform Stöckel systems. Thus, we can claim that every such mechanical system is embedded into a proposed scheme.
5.1 Quasi-point canonical transformations.

In conclusion, we discuss another parameterizations of the function \( e(\lambda) \). Of course, function \( e(\lambda) \) admits various representations in different variables and we can use this freedom, as an example to solve equations of motion [24]. The considered above parametrization describe the point canonical transformation only. Here we discuss an application of the Weierstrass reduction theory to construct another cartesian coordinates on \( C \).

It is obvious, that the Lax representation

\[
\dot{L}(\lambda) = [L(\lambda), A(\lambda)]
\]

is covariant with respect to the transformation of the first Lax matrix

\[
L(\lambda) \to \phi(\lambda, \lambda_1, \ldots, \lambda_k)L(\lambda)
\]

with an arbitrary function \( \phi(\lambda, \lambda_1, \ldots, \lambda_k) \) on time-independent moduli \( \{\lambda_j\} \) of \( C \) and on spectral parameter \( \lambda \). However, this transformation drastically changes the Poisson bracket relations (1.14) and parameterization of \( L(\lambda) \) in the flat coordinates \( \{p_j, x_j\} \). Hence, in addition to considered above flat coordinates \( \{p_j, x_j\} \), the same function \( e(\lambda, q_1, \ldots, q_n) \) may be associated to another set of flat coordinates. Now we show that to introduce these new variables \( \{q_j\} \) we can use various covering of the initial curve \( C \), as an example, covering listed in [43].

Let us assume that the initial torus \( J(C) = T^{2g} \) may be decomposed in a direct product of several tori

\[
T^{2g} = T^{2g_1} \times \cdots \times T^{2g_k}, \quad \sum_{j=1}^{k} g_j = g.
\]

The corresponding Riemann matrix has a block form \( B = B_1 \times B_2 \cdots B_k \), where \( B_j \) are the \( g_j \times g_j \) Riemann matrices and the corresponding Baker-Akhiezer function on \( C \) is factorized. In this case we can consider curve \( C \) as a \( K \)-sheeted covering of tori \( T^{2g_j} \). Such covers are known to exist for any \( K > 1 \) and for arbitrary tori [23].

First of all, we can introduce the separated variables \( \{q_j\} \) associated to a whole torus \( T^{2g_j} \). For dynamics on \( J(C) = T^{2g} \) the corresponding Lax representations \( L(\lambda) \) [14] are \( 2 \times 2 \) matrices.

Secondly, we can introduce another set of separated variables \( \{\tilde{q}_j\} \) associated to each torus \( T^{2g_j} \) in (5.5). For dynamics splitting on several tori \( T^{2g_j} \), the Lax representations have a block form

\[
L(\lambda) = \left( \begin{array}{cc} L_1 & \\ \vdots & \\ L_k \end{array} \right)(\lambda),
\]

where \( L_j(\lambda) \) are the \( 2 \times 2 \) matrices defined by functions \( e_j(\lambda) \) on the each torus \( T^{2g_j} \) [23]. Two sets of variables \( \{q_j\} \) and \( \{\tilde{q}_j\} \) are related by canonical transformation induced by the covering, that allows us to get \( 2 \times 2 \) Lax matrix instead of matrix (5.7). It means that we have two isomorphic integrable systems with different Lax representations and the corresponding canonical transformation is a quasi-point transformation [36].

To illustrate this construction we take as an example several systems at \( n = 2 \). Starting with an hyperelliptic curve \( C \) of genus \( g = n = 2 \) we define variables \( (p_1, q_1) \) and \( (p_2, q_2) \) on the Lagrangian submanifold \( C(2) \) [39]. The Jacobi inversion problem is the problem of finding these variables from the equations (2.10) with the Stäckel matrix \( S \) given by (3.13). This problem is solved by using the Kleinian \( \wp \)-functions, which are second logarithmic derivatives of the Kleinian \( \sigma \)-function

\[
\wp_{ij} = \frac{\partial \ln \sigma(\beta_1, \beta_2)}{\partial \beta_i \partial \beta_j}, \quad \wp_{22} = q_1 + q_2, \quad \wp_{12} = -q_1 q_2,
\]

(for detail see [55, 57]). The function \( e(\lambda) \) (3.14) on \( C \) with zeroes at the points \( q_1, q_2 \) is equal to

\[
e(\lambda) = \lambda^2 - \wp_{22} \lambda - \wp_{12} = (\lambda - q_1)(\lambda - q_2) = \lambda^2 + 2\lambda x_1 + (2x_2 + x_1^2),
\]

or

\[
e(\lambda) = \frac{(\lambda - q_1)(\lambda - q_2)}{(\lambda - \delta_1)(\lambda - \delta_2)} = 1 + \frac{x_1^2}{\lambda - \delta_1} + \frac{x_2^2}{\lambda - \delta_2},
\]

(5.8)
Here we used the freedom \([3.23]\) and cartesian coordinates \(\{x_j\}\) or \(\{x'_j\}\) are derived from the "inverse" Lamé function \(B(\lambda)\) \([5.2]\). Applying the outer additive automorphism of \(sl(2)\), we can construct the Lax matrices \(L'(\lambda)\) for an infinite set of integrable mechanical systems with the following hamiltonians

\[
H = p_1p_2 + V_N(x_1, x_2),
\]

\[
H = p_1'^2 + p_2'^2 + V'_N(x'_1, x'_2).
\]

Among them, we distinguish the Henon-Heiles systems at \(N = 3\) and the systems with quartic potential at \(N = 4\). For these systems the genus of associated curve \(C\) is equal to the number of degrees of freedom \(g = n = 2\).

Function \(e(\lambda)\) \([3.13]\) is independent on the moduli of \(C\) and, therefore, the above construction of the integrable systems \([5.3]\) readily gets over on the reducible curve \(C\). To construct this reducible curve, let us take two tori \(T_{g,2}\)

\[
w^2_\pm = \xi(1 - \xi)(1 - k^2_\pm \xi),
\]

with a Jacobi moduli

\[
k^2_\pm = -\frac{(\sqrt{\alpha} \mp \sqrt{\beta})^2}{(1 - \alpha)(1 - \beta)}.
\]

Making the rational order two \((K = 2)\) change of variables

\[
w_\pm = -\sqrt{(1 - \alpha)(1 - \beta)} \frac{\lambda \mp \sqrt{\alpha\beta}}{(\lambda - \alpha)(\lambda - \beta)} y, \quad \xi = \frac{(1 - \alpha)(1 - \beta)}{(\lambda - \alpha)(\lambda - \beta)} \lambda,
\]

one gets hyperelliptic curve

\[
C:\quad y^2 = \lambda(\lambda - 1)(\lambda - \alpha)(\lambda - \beta)(\lambda - \alpha \beta),
\]

which gives a two-sheeted covering of two tori \(T_{g,2}\) \([6.10]\). It is a well-known example of the reduction of hyperelliptic integrals to elliptic ones by using the rational change of variables proposed by Legendre and generalized by Jacobi \([36]\).

The complex torus \(T^2\) is isomorphic to the curve of genus \(g = 1\) given by equation \(w^2 = f(\xi)\). In the above, we have presented the covering for the two odd curves \([5.10]\) at \(\deg(f) = 2g + 1 = 3\). All computations concerning the even curves at \(\deg(f) = 2g + 2 = 4\) give similar covering \([33]\), so we do not present these formulae. The odd and even curves at \(g = 1\) are associated to the integrable cases of the Henon-Heiles system and system with quartic potential, respectively.

Next we can introduce two pairs of variables \((\tilde{p}_1, \tilde{q}_1)\) and \((\tilde{p}_2, \tilde{q}_2)\) being on the tori \(T^2_{g,2}\). Functions \(e_{1,2}(\lambda)\) on \(T^2_{g,2}\) are equal to

\[
e_1(\lambda) = \lambda - \tilde{q}_1, \quad e_2(\lambda) = \lambda - \tilde{q}_2.
\]

Variables \(\{\tilde{p}_j, \tilde{q}_j\}\) are separated cartesian coordinates for the integrable systems on \(T^2_1 \times T^2_2\) with the hamiltonians

\[
H_{3,4} = \tilde{p}_1^2 + \tilde{p}_2^2 + V_{3,4}(\tilde{q}_1) + V_{3,4}(\tilde{q}_2),
\]

which is a sum of two one-dimensional hamiltonians on \(T^2_{g,2}\). The corresponding \(4 \times 4\) Lax representation has a block form \([7.0]\), whose blocks are determined by the functions \(e_{1,2}(\lambda)\) \([5.13]\).

The covering \([5.11]\) induces canonical transformation of variables \(\{\tilde{p}_j, \tilde{q}_j\}\) to \(\{p_j, q_j\}\) \([37]\). These pairs of variables lie on the different curves \(T_{g,2}\) and \(C\), respectively. The common moduli \(\alpha\) and \(\beta\) of these curves are integrals of motion. On the orbit \(O\) \((\alpha = const, \beta = const)\) this canonical transformation \([5.11]\) becomes a point transformation. It is so-called quasi-point transformation \([38]\).

By using change of variables induced by covering \([5.11]\) one can construct the \(2 \times 2\) Lax matrix for the evolution \([5.14]\) splitting on two tori. In variables \(\{\tilde{q}_j\}\) matrix \(L(\lambda)\) is determined by the function

\[
e(\lambda) = \frac{(\lambda - \alpha)(\lambda - \beta)}{(1 - \alpha)(1 - \beta)} \tilde{e}(\lambda), \quad \tilde{e}(\lambda) = (\lambda - \tilde{q}_1)(\lambda - \tilde{q}_2).
\]

In fact, we add two additional zeroes \(\alpha\) and \(\beta\) into the function \(e(\lambda)\) \([3.13]\) on the reducible curve \(C\) \([5.12]\) and, therefore, change parameterization of the Lax matrices in flat coordinates \(\{p_j, x_j\}\).
In general, to introduce new flat coordinates, we can take any tori $T^2_{g_1 g_2}$ of arbitrary genus $g_1 g_2 > 1$ and consider two-dimensional evolution (5.14) splitting on these curves with an arbitrary one-dimensional potentials $V_{g_1 g_2 + 1}(q_j)$. The standard change of variables

$$
\bar{q}_j = \frac{x_1 + x_2}{2}, \quad \Rightarrow \quad \bar{c}(\lambda) = \lambda^2 - \frac{x_1^2 + x_2^2}{4}.
$$

(5.16)

preserves the natural form of the hamiltonians (5.14) for arbitrary potentials $V_{g_1 g_2 + 1}(q_j)$. The equations of motion remain the Newton equations in these variables $\{\bar{p}_j, \bar{x}_j\}$

In the considered above example (5.10) both independent hyperelliptic integrals are reduced to elliptic ones by using a common substitution $\xi \to \lambda$ (5.11). It relates to existence of the second order automorphism of a hyperelliptic curve (5.12) [36]:

$$
\tau : \quad (\lambda, y) \to \left( \frac{\alpha^2 - \lambda}{\sqrt{\alpha^4 - \beta^4}}, \sqrt{\alpha^4 - \beta^4} \right).
$$

(5.17)

It allows us to introduce another parameterization of the function $\bar{c}(\lambda)$ in cartesian coordinates, which preserves the natural form of the hamiltonian. Namely, in addition to (5.16), we can use the following canonical transformation of variables $\{\bar{p}_j, \bar{x}_j\}$ to the cartesian coordinates $\{\hat{p}_j, \hat{x}_j\}$

$$
\bar{c}(\lambda) = \lambda^2 - \frac{Q_+ + Q_-}{x_1} + \left(\frac{Q_+ - Q_-}{x_1}\right)^2.
$$

(5.18)

Here functions $Q_{\pm} (\hat{p}_j, \hat{x}_j)$ are the classical counterparts of the supercharges in two-dimensional SUSY [36] with the following properties

$$
\{H, Q_{\pm}\} = \pm f(\hat{p}_j, \hat{x}_j)Q_{\pm}, \quad \{H, Q_+, Q_-\} = 0.
$$

At $g = 2$ ($N = 3$ or $N = 4$ in (5.14)) these functions $Q_{\pm}$ and $f$ on variables $\{\bar{p}_j, \bar{x}_j\}$ or $\{\hat{p}_j, \hat{x}_j\}$ are listed in [36]. Moduli $\alpha$ and $\beta$ in (5.17) are integrals of motion, therefore, automorphism $\tau$ induces a second quasi-point transformation associated to torus $T^2_{g_1 g_2}$ (5.10).

Two quasi-point transformations (5.11) and (5.18) for the physical variables $\{x'_j\}$, $\{\bar{x}_j\}$ and $\{\hat{x}_j\}$ bind together all the integrable cases of the Henon-Hailes system at $N = 3$ and three integrable cases of the system with quartic potential at $N = 4$. Of course, these systems have a common set of action-angle variables. Moreover, the same variables are associated to the Kowalewski top [3], which is a supersymmetric quantum model as well.

Thus, several supersymmetric models are related to evolution splitting on the tori, when the number of degrees of freedom is equal to the genus $g = n$ of the associated covering curve $C = T_1 \times T_2$. It would be interesting to get a geometrical interpretation of these supersymmetric objects arising from finite-dimensional SUSY quantum mechanics.

6 Conclusion

It is known, that curves $y + y^{-1} = F(\lambda)$ together with the 1-forms

$$
dS^{(4)} = \lambda \frac{dy}{y}, \quad dS^{(5)} = \log \lambda \frac{dy}{y},
$$

are implied by integrable models of the Toda chain family (standard and relativistic models). The corresponding Lax representations are defined on the Poisson-Lie groups with quadratic $r$-matrix algebra. The corresponding mapping from the action-angle variables to separated variables has been proposed in [32].

On the other hand we can consider the umbilic solutions of the KdV equation [3, 33]. These systems are defined on a generalized Jacobi variety of the symmetric product of $n$ logarithmic Riemannian surfaces in place of the Liouville tori. Nevertheless, it is possible to introduce variables that linearize the corresponding hamiltonian flows. These systems may be interpreted as counterparts of the discrete-time Stäckel systems.
Both these sets of models are associated to the change of parameterization of hyperelliptic curve from "plane" parameterization to "annulus" ones ($\lambda \rightarrow \log \lambda$). The crucially interesting lift to the interpretation of $\lambda$ as a coordinate on elliptic curve.

For all these integrable models it would be interesting to estimate the possibility of application of the usual Stäckel approach. On this way we should consider mapping between action-angle variables and separated variables, and should study the differential of this map. In the presented paper there are the Jacobi inversion problem and differential of the Abel-Jacobi map.

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