The Kato square root problem on irregular open sets

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What is the Kato square root problem?
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- $H_0^{1,2}(O) \subseteq \mathcal{V} \subseteq H^{1,2}(O)$ closed subspace
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- $H_0^{1,2}(O) \subseteq V \subseteq H^{1,2}(O)$ closed subspace
- $A \in L^\infty(O; \mathbb{C}^{d \times d})$
- define sesquilinear form

\[ a(u, v) := \int_O A \nabla u \cdot \overline{\nabla v} \, dx \quad (u, v \in V) \]

- $A$ coercive in Gårding’s sense

\[ \Re a(u, u) \gtrsim \| \nabla u \|_{L^2(O)}^2 \quad (u \in V) \]
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- \( L \) realization of \( a \) in \( L^2(O) \).
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- \( L \) realization of \( a \) in \( L^{2}(O) \).

Problem

*For which spaces \( \mathcal{V} \) do we have \( \text{D}(L^{1/2}) = \mathcal{V} \) with equivalent norms?*
Theorem (Egert, Haller-Dintelmann, Tolksdorf ’14 & ’16)

Suppose:
- $O$ bounded domain
- $O$ is $d$-regular
- $\partial O$ is $(d - 1)$-regular.
- $D \subseteq \partial O$ is $d - 1$-regular
- $\overline{\partial O \setminus D}$ admits bi-Lipschitz charts

Then the Kato property holds for $\mathcal{V} = H^{1,2}_D(O)$. 
What is known for mixed boundary conditions?

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Suppose:

- $O$ is bounded domain
- $O$ is $d$-regular
- $\partial O$ is $(d - 1)$-regular.
- $D \subseteq \partial O$ is uniformly $(d - 1)$-regular
- $\overline{\partial O \setminus D}$ uniformly admits bi-Lipschitz charts

Then the Kato property holds for $\mathcal{V} = H^{1,2}_D(O)$.

Aim: only demand for boundary regularity!

- inspection of proof: no connectedness
- better interpolation theory (joint work with M. Egert): no boundedness
What is known for mixed boundary conditions?

**Theorem (B., Egert, Haller-Dintelmann, Tolksdorf ’14, ’16 & ’19)**

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**Aim:** only demand for boundary regularity!

- inspection of proof: no connectedness
- better interpolation theory (joint work with M. Egert): no boundedness
- localization and thickening of $O$: no $d$-regularity
Thickening of $O$

For simplicity: pure Dirichlet boundary conditions

- For example: $O$ is an unbounded cusp domain $\rightsquigarrow$ not $d$-regular
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- For example: $O$ is an unbounded cusp domain $\rightsquigarrow$ not $d$-regular
- $O := \mathbb{R}^d \setminus \partial O$ is $d$-regular and contains $O$
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**Question**

How do the Kato problems on $O$ and $O$ relate?

**Idea:** relate functional calculi of $L$ and $L$
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L^1$

1. $QL \subseteq LQ$ for good projection $Q$

Calculate with good projection $Q$ and $u \in D(QL) = D(L)$:

$$a(Qu, v)$$
"Localization" of the functional calculus

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Calculate with good projection $Q$ and $u \in D(QL) = D(L)$:

$$a(Qu, v) = \int\nabla Qu \cdot \nabla v$$
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L$

1. $QL \subseteq LQ$ for *good* projection $Q$

Calculate with *good* projection $Q$ and $u \in D(QL) = D(L)$:

$$a(Qu, v) = \int O A \nabla Qu \cdot \nabla v = \int O A \nabla u \cdot \nabla Qv$$
“Localization” of the functional calculus

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Calculate with good projection $Q$ and $u \in D(QL) = D(L)$:

$$a(Qu, v) = \int_{\Omega} A \nabla Q u \cdot \nabla v = \int_{\Omega} A \nabla u \cdot \nabla Q v$$

$$= (Lu | Qv)_{L^2}$$
“Localization” of the functional calculus

Idea: relate functional calculi of \( L \) and \( \mathcal{L} \)

1. \( \mathcal{Q}L \subseteq \mathcal{L}Q \) for \textit{good} projection \( Q \)

Calculate with \textit{good} projection \( Q \) and \( u \in D(\mathcal{Q}L) = D(L) \):

\[
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Calculate with *good* projection $Q$ and $u \in D(QL) = D(L)$:

$$a(Qu, v) = \int_{\Omega} A \nabla Qu \cdot \nabla v = \int_{\Omega} A \nabla u \cdot \nabla Qv$$

$$= (Lu | Qv)_{L^2} = (QLu | v)_{L^2}$$

hence: $Qu \in D(L)$ and $LQu = QLu$
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L$

1. $QL \subseteq LQ$ for good projection $Q$
2. decomposition of functional calculus and operator domains

- $Q_1$ good projection
- $L_1$ and $L_2$ the restrictions of $L$ to $Q_1L^2(O)$ and $(1 - Q_1)L^2(O)$

Then

$$u \in D(f(L)) \iff Q_1u \in D(f(L_1)) \text{ and } Q_2u \in D(f(L_2))$$

with

$$f(L)u = f(L_1)Q_1u + f(L_2)Q_2u.$$
“Localization” of the functional calculus
Idea: relate functional calculi of $L$ and $L$

1. $QL \subseteq LQ$ for *good* projection $Q$ ✓
2. decomposition of functional calculus and operator domains ✓
3. $Q_1 = 1_O$ is a *good* projection

- multiplication operators commute with each other
“Localization” of the functional calculus

Idea: relate functional calculi of \( L \) and \( L_1 \)

1. \( QL \subseteq LQ \) for good projection \( Q \)
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- Multiplication operators commute with each other
- \( \nabla Q \varphi = Q \nabla \varphi \) for \( \varphi \in C_0^\infty (O) \)
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L^1$

1. $QL \subseteq LQ$ for good projection $Q$ ✓
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3. $Q_1 = 1_O$ is a good projection

- Multiplication operators commute with each other
- $\nabla Q \varphi = Q \nabla \varphi$ for $\varphi \in C^\infty_0(O)$
- $\nabla Q = Q \nabla$ on $H^{1,2}_0(O)$ by density
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L$

1. $QL \subseteq LQ$ for good projection $Q$ ✓
2. decomposition of functional calculus and operator domains ✓
3. $Q_1 = 1_O$ is a good projection ✓
4. putting it all together

Kato for $L$ implies

$$Q_1 H_0^{1.2}(O) = Q_1 D(L^{1/2}) = D(L^{1/2}_1)$$
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L^1$

1. $QL \subseteq LQ$ for good projection $Q$
2. decomposition of functional calculus and operator domains
3. $Q_1 = 1_O$ is a good projection
4. putting it all together

Kato for $L$ implies

$$Q_1 H^1,2_0(\mathcal{O}) = Q_1 D(L^{1,2}) = D(L^{1,2})$$

and for $u \in D(L^{1,2})$ we get

$$\|L^{1,2}_1 u\|_{L^2} = \|L^{1,2} u\|_{L^2} \approx \|u\|_{H^1,2_0}$$
“Localization” of the functional calculus

Idea: relate functional calculi of $L$ and $L_1$

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Kato for $L$ implies

$$Q_1H_0^{1,2}(O) = Q_1D(L^{\frac{1}{2}}) = D(L^{\frac{1}{2}})$$

and for $u \in D(L^{\frac{1}{2}})$ we get

$$\|L_1^{\frac{1}{2}}u\|_{L^2} = \|L^{\frac{1}{2}}u\|_{L^2} \approx \|u\|_{H_0^{1,2}}$$

identify: $L^2(O) \sim Q_1L^2(O)$ and $H_0^{1,2}(O) \sim Q_1H_0^{1,2}(O)$

$\leadsto L = L_1$
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4. putting it all together
Now, it’s time for conference dinner!

S. Bechtel, R. Haller-Dintelmann. *The Kato square root problem on irregular open sets*. Available on arXiv.