On bifurcations and stability of central configurations in the planar circular restricted four-body problem

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Abstract. The restricted four-body problem is considered. That is, we consider motion of an infinitely small body (particle) under the Newtonian gravitational attraction of three bodies (primaries). It is assumed that the primaries move in circular orbits, forming a stable equilateral Lagrange triangle. It is supposed that four bodies move in a plane. There exist relative equilibriums of the particle in the rotating with the primaries coordinate system. In such an equilibrium the particle forms a central configuration with the primaries. In the case of small mass of a primary the bifurcations of the central configurations are investigated, as well as conditions of linear stability for the central configurations is obtained. The study is performed analytically by using small parameter method.

1. Introduction

In the restricted four-body problem the motion of a small body (particle) in the Newtonian gravitational field of three bodies (primaries) is considered. The particle has negligibly-small mass, so that it does not affect the motion of the primaries, whose move on trajectories obtained as a solution of the three body-problem. The restricted four-body problem was studied in numerous papers and it continues to attract the attention of researchers in the field of celestial mechanics and satellite dynamics. In particular, of especial interest is a remarkable type of motion of the bodies, when the resultant attracting force acting on a body is a central force directed to the system mass center. Such a type of motion is called central configuration. In classical three body problem the central configuration are known as Lagrange and Euler points of librations. The central configuration in the restricted four-body problem is possible when primaries are located in libration points and all four bodies move in Keplerian orbits.

The problem of existence and bifurcations of the central configurations has been considered in [1, 2]. The linear stability study for the central configurations was performed in [3-6]. In the case equal masses of two primaries rigorous conclusions on the stability in the sense of Lyapunov have been obtained for the central configurations by using normal forms method and KAM theory [7, 8].

In this paper we suppose that three primaries rotate as a rigid equilateral triangle and the particle moves in the plane of primaries motion. In this case the particle has relative equilibrium positions in the rotating with the primaries coordinate system. In the relative equilibrium positions mentioned the
particle forms a central configuration with the primaries. We study possible scenarios of bifurcations of these central configurations as well as perform a linear analysis of their stability.

2. Formulation of the problem
We denote by $m_i$ ($i = 1, 2, 3$) the masses of primaries $M_i$ respectively. To describe the motion of a particle $P$, we use the coordinate system $Oxyz$ with the origin in the middle of the segment connecting the bodies $M_2$ and $M_3$. The axes $Ox$ and axes $Oy$ pass through $M_1$ and $M_3$, respectively, as shown in figure 1. The axis $Oz$ complements the coordinate system to the right, orthogonal triple.

![Coordinate system](image)

Figure 1. Coordinate system.

The equations of motion of the particle $P$ can be written in the form of Hamilton's equations

$$
\frac{d\xi}{d\tau} = \frac{\partial H}{\partial p_\xi}, \quad \frac{d\eta}{d\tau} = \frac{\partial H}{\partial p_\eta}, \quad \frac{dp_\xi}{d\tau} = -\frac{\partial H}{\partial \xi}, \quad \frac{dp_\eta}{d\tau} = -\frac{\partial H}{\partial \eta}.
$$

(1)

The dimensionless coordinates $\xi, \eta$ are introduced by means of formulae $x = r\xi, y = r\eta$, where $x$ and $y$ are coordinates of the particle in system $Oxyz$ and $r = \text{const}$ is the distance between the primaries. The dimensionless time $\tau$ is introduced by means of formula $\tau = \omega t$, where $\omega^2 = f(m_1 + m_2 + m_3)r^{-3}$ and $f$ is the gravitational constant. Let us note that $\omega$ is the angular velocity of Lagrange triangular configuration.

The Hamiltonian function reads

$$
H = \frac{1}{2}(p_x^2 + p_y^2) + p_x\eta - p_y\xi = -\frac{\sqrt{3}(1 - \mu_2 - \mu_3)}{2}p_\xi - \frac{\mu_2 - \mu_3}{\rho_1}p_\eta - \frac{\mu_2}{\rho_2} + \frac{\mu_3}{\rho_3},
$$

(2)

where

$$
\rho_1 = \left(\xi^2 + \left(\eta - \frac{\sqrt{3}}{2}\right)^2\right)^{1/2}, \quad \rho_2 = \left(\left(\xi + \frac{1}{2}\right)^2 + \eta^2\right)^{1/2}, \quad \rho_3 = \left(\left(\xi - \frac{1}{2}\right)^2 + \eta^2\right)^{1/2}.
$$

(3)

$$
\mu_2 = \frac{m_2}{m_1 + m_2 + m_3}, \quad \mu_3 = \frac{m_3}{m_1 + m_2 + m_3}.
$$

(4)

It is worth noting that the coordinates of the system mass center in dimensionless variables $\xi, \eta$ read

$$
\xi_c = -\frac{\mu_2 - \mu_3}{2}, \quad \eta_c = \frac{\sqrt{3}(1 - \mu_2 - \mu_3)}{2}.
$$

(5)

In what follows we suppose that the Lagrange triangle configuration of the primaries $M_i$ is stable. That is, the Routh's necessary stability condition [9] given by the inequality

$$
1 + 27(\mu_2^2 + \mu_2\mu_3 + \mu_3^2 - \mu_2 - \mu_3) > 0
$$

(6)

holds. Without loss of any generality, we assume that the mass of the body $M_1$ is greater than the masses of the bodies $M_2$ and $M_3$. Then the parameters $\mu_2$ and $\mu_3$ lie in the interval
The relative equilibrium positions of practical \( P \) in the rotating coordinate system \( Oxyz \) are described by the following solution of system (1).

\[
\xi = \xi_*, \quad \eta = \eta_*, \quad p_\xi = -\eta_* + \frac{\sqrt{3}}{2} (1 - \mu_2 - \mu_3), \quad p_\eta = \xi_* + \frac{1}{2} (\mu_2 - \mu_3),
\]

where \( \xi_* \) and \( \eta_* \) satisfy the following equations

\[
\frac{\xi + \frac{\mu_2 - \mu_3}{2}}{\rho_1^2} - \frac{(1 - \mu_2 - \mu_3)}{\rho_1^2} \frac{\xi}{\rho_1^2} \xi - \frac{\mu_2}{\rho_2^2} \left( \frac{\xi + \frac{1}{2}}{\rho_2^2} \right) - \frac{\mu_3}{\rho_2^2} \left( \xi - \frac{1}{2} \right) = 0,
\]

\[
\eta - \frac{\sqrt{3} (1 - \mu_2 - \mu_3)}{2} - \frac{(1 - \mu_2 - \mu_3)}{\rho_3^2} \left( \eta - \frac{\sqrt{3}}{2} \right) - \frac{\mu_2}{\rho_3^2} \eta - \frac{\mu_3}{\rho_3^2} \eta = 0.
\]

In this paper, we investigate the bifurcation nature of relative equilibriums mentioned and perform a linear stability analysis of the corresponding central configurations.

3. Bifurcation of relative equilibriums

In the limiting cases, when one of the masses of the primaries \( m_2 \) or \( m_3 \) vanishes, the restricted four-body problem considered here degenerates into a restricted three-body problem, and the central configurations pass into collinear (Eulerian) \( L_1, L_2, L_3 \) or triangular (Lagrangian) \( L_4, L_5 \) libration points. The central configurations in these limiting cases are shown in figure 2 and figure 3. The primaries are indicated by black points and the possible relative equilibrium positions of the particle are indicated by blue points.

We denote by \( P_{ij} \) the relative equilibrium position, which degenerates to the libration point \( L_i^{(2)} \) at \( \mu_2 \to 0 \) and it degenerates to the libration point \( L_j^{(3)} \) at \( \mu_3 \to 0 \).

In the limiting cases a bifurcation of the relative equilibrium positions place. In particular, the four relative equilibrium positions \( P_{51}, P_{52}, P_{54} \) and \( P_{55} \) approach the primary \( M_2 \) as \( \mu_2 \) tends to zero and coincide with libration point \( L_4^{(2)} \), when \( \mu_2 = 0 \). In this limiting case the particle \( P \) occupies the position of primary \( M_2 \) and forms a triangular Lagrangian configuration with the primary \( M_1 \) and \( M_3 \).
Similarly, the four relative equilibrium positions $P_{15}$, $P_{25}$, $P_{45}$ and $P_{55}$ approach the primary $M_3$ as $\mu_3$ tends to zero and coincide with libration point $L_5^{(3)}$, when $\mu_3 = 0$. In this limiting case the particle $P$ occupies the position of primary $M_3$ and forms a triangular Lagrangian configuration with the primary $M_1$ and $M_2$ (see figure 3).

To study the question on bifurcation of central configurations in more detail we apply the small parameter method. Let us first consider the limiting case $\varepsilon = \mu_2^{1/3}$. For $\varepsilon = 0$ the system of algebraic equations (9) have the following particular solution corresponding to Lagrangian triangular libration point $L_5^{(2)}$

$$\xi = -\frac{1}{2}, \quad \eta = 0.$$  \hfill (10)

First, let us show, that at $\varepsilon \neq 0$ the solution of system (9) can be obtained in the form of convergent series in powers of small parameter $\varepsilon$. To this end we perform the following change of variables

$$\xi = -\frac{1}{2} + \tilde{\xi} \varepsilon, \quad \eta = \tilde{\eta} \varepsilon. \hfill (11)$$

In the new variables the equations (9) read

$$3\mu_3 \tilde{\xi} + \frac{3(1 - \mu_3)}{4} (\tilde{\xi} + \sqrt{3} \tilde{\eta}) - \frac{\tilde{\xi}}{(\tilde{\xi}^2 + \tilde{\eta}^2)^{3/2}} + O(\varepsilon) = 0,$$

$$\frac{3\sqrt{3}(1 - \mu_3)}{4} (\tilde{\xi} + \sqrt{3} \tilde{\eta}) - \frac{\tilde{\eta}}{(\tilde{\xi}^2 + \tilde{\eta}^2)^{3/2}} + O(\varepsilon) = 0.$$  \hfill (12)

At $\varepsilon = 0$ the algebraic system (12) has four solutions of the following form

$$\tilde{\xi}_* = \delta_1 \left( \frac{2}{(z^2 + 1)^{3/2}} \left( \frac{3 + \sqrt{3} \sqrt{9\mu_3^2 - 9 \mu_3 + 3 \delta_2}}{3 \mu_3 - 1} \right) \right)^{1/3}, \quad \tilde{\eta}_* = z \tilde{\xi}_*.$$  \hfill (13)

where $\delta_1 = \pm 1$ and $\delta_2 = \pm 1$.

The Jacobian of the system (12) calculated at $\varepsilon = 0$ on solutions (13) reads

$$J = -\frac{27(3\mu_3^2 - 3\mu_3 + 1)}{(z^2 + 1)} (\sqrt{3}z + 1).$$  \hfill (15)

By taking into account the formula (14) it is easy to show that the expression (15) can vanish only for $\mu_3 = 0$ or $\mu_3 = 1$. Hence, with the only exception of special case $\mu_3 = 0$ the Jacobian of the system (12) is not equal to zero. Thus, in the accordance with the implicit function theorem the system (12) has at $\varepsilon \neq 0$ four solutions analytically depending on small parameter $\varepsilon$. That is, the original system (9) has four solutions. By using of the method of analytical continuation the above solutions can be constructed in the form of the following series

$$\xi = -\frac{1}{2} + \xi_1 \varepsilon + \xi_2 \varepsilon^2 + \xi_3 \varepsilon^3 + O(\varepsilon^4), \quad \eta = \eta_1 \varepsilon + \eta_2 \varepsilon^2 + \eta_3 \varepsilon^3 + O(\varepsilon^4),$$  \hfill (16)

where $\xi_1 = \xi_*, \eta_1 = \eta_*$.  

The solutions (16) describe the relative equilibrium positions $P_{54}$, $P_{52}$, $P_{45}$ and $P_{55}$, which are emanate at $\varepsilon \neq 0$ from the triangular libration point $L_5^{(2)}$. We also note that in formulas (13), (14) $\delta_1 = 1$, $\delta_2 = 1$ correspond to relative equilibrium $P_{54}$; $\delta_1 = 1$, $\delta_2 = -1$ correspond to relative equilibrium $P_{55}$; $\delta_1 = -1$, $\delta_2 = 1$ correspond to relative equilibrium $P_{52}$; $\delta_1 = -1$, $\delta_2 = -1$ correspond to relative equilibrium $P_{54}$.
Similarly, for $\mu_2 \ll 1$, the relative equilibrium positions $P_{15}, P_{25}, P_{45}$ and $P_{55}$ can be constructed in the form of the following series
\begin{equation}
\xi = \frac{1}{2} + \xi_1 \varepsilon + \xi_2 \varepsilon^2 + \xi_3 \varepsilon^3 + O(\varepsilon^4), \quad \eta = \eta_1 \varepsilon + \eta_2 \varepsilon^2 + \eta_3 \varepsilon^3 + O(\varepsilon^4),
\end{equation}
where $\varepsilon = \mu_3^{1/3}$, $\xi_1$ and $\eta_1$ read
\begin{equation}
\xi_1 = \delta_1 \left(\frac{2}{(z^2 + 1)^{3/2} \left(3 + \sqrt{3} \sqrt{9\mu_2^2 - 9\mu_2 + 3 \delta_2}\right)}\right)^{1/3}, \quad \eta_1 = 2 \xi_1,
\end{equation}
where $\delta_1 = 1$, $\delta_2 = 1$ correspond to relative equilibrium $P_{25}$; $\delta_1 = 1$, $\delta_2 = -1$ correspond to relative equilibrium $P_{45}$; $\delta_1 = -1$, $\delta_2 = 1$ correspond to relative equilibrium $P_{15}$; $\delta_1 = -1$, $\delta_2 = -1$ correspond to relative equilibrium $P_{55}$.

For small values of the parameter $\mu_2$, the relative equilibriums $P_{5j}$ are located in narrow domains connected to point $L_5^{(2)}$. These domains are shown in figure 4. At $\mu_2 \ll 1$ the boundaries of the above regions can be obtained analytically in a form of convergent series with respect to small parameter $\mu_2^{1/3}$. These boundaries are given in table 1.
Table 1. Boundaries of possible locations for relative equilibriums $P_{5j}$.

| $P_{51}$ | $\xi + \frac{1}{2} - \frac{\sqrt{3}}{3} \eta + O\left(\mu_2^{1/3}\right) = 0$, $\xi + \frac{1}{2} + k_1 \eta + O\left(\mu_2^{1/3}\right) = 0$ |
| $P_{52}$ | $\xi + \frac{1}{2} + \frac{\sqrt{3}}{3} \eta + O\left(\mu_2^{1/3}\right) = 0$, $\xi + \frac{1}{2} + k_1 \eta + O\left(\mu_2^{1/3}\right) = 0$ |
| $P_{54}$ | $\xi + \frac{1}{2} + \sqrt{3} \eta + O\left(\mu_2^{1/3}\right) = 0$, $\xi + \frac{1}{2} + k_2 \eta + O\left(\mu_2^{1/3}\right) = 0$ |
| $P_{55}$ | $\xi + \frac{1}{2} + \sqrt{3} \eta + O\left(\mu_2^{1/3}\right) = 0$, $\xi + \frac{1}{2} + k_2 \eta + O\left(\mu_2^{1/3}\right) = 0$ |

The constant coefficients in formulas of table 1 read

$$k_1 = \sqrt{23}(2\sqrt{2} + 3) - 2\sqrt{3}(4 + 3\sqrt{2}) \approx -0.60119039,$$
$$k_2 = \sqrt{23}(3 - 2\sqrt{2}) + 2\sqrt{3}(3\sqrt{2} - 4) \approx 1.66336660.$$  

Let us note that the values $k_1$ and $k_2$ are close to $-\sqrt{3}/3$ and $\sqrt{3}$, respectively. That is why the regions of existence of relative equilibriums are very narrow.

Similarly, for small values of the parameter $\mu_3$, the relative equilibrium positions $P_{j5}$ are located in narrow domains connected to point $L_5^{(3)}$. These domains are shown in figure 5. At $\mu_3 \ll 1$ the boundaries of the above regions can be obtained analytically in a form of convergent series with respect to small parameter $\mu_3^{1/3}$. These boundaries are given in table 2.

Table 2. Boundaries of possible locations for relative equilibriums $P_{j5}$.

| $P_{15}$ | $\xi - \frac{1}{2} + \frac{\sqrt{3}}{3} \eta + O\left(\mu_3^{1/3}\right) = 0$, $\xi - \frac{1}{2} + k_1 \eta + O\left(\mu_3^{1/3}\right) = 0$ |
| $P_{25}$ | $\xi - \frac{1}{2} - \frac{\sqrt{3}}{3} \eta + O\left(\mu_3^{1/3}\right) = 0$, $\xi - \frac{1}{2} + k_1 \eta + O\left(\mu_3^{1/3}\right) = 0$ |
| $P_{45}$ | $\xi - \frac{1}{2} + \sqrt{3} \eta + O\left(\mu_3^{1/3}\right) = 0$, $\xi - \frac{1}{2} + k_2 \eta + O\left(\mu_3^{1/3}\right) = 0$ |
| $P_{55}$ | $\xi - \frac{1}{2} + \sqrt{3} \eta + O\left(\mu_3^{1/3}\right) = 0$, $\xi - \frac{1}{2} + k_2 \eta + O\left(\mu_3^{1/3}\right) = 0$ |

The constant coefficients in formulas of table 2 read

$$k_1 = 2\sqrt{3}(4 + 3\sqrt{2}) - \sqrt{23}(2\sqrt{2} + 3) \approx 0.60119039,$$
$$k_2 = 2\sqrt{3}(4 - 3\sqrt{2}) + \sqrt{23}(2\sqrt{2} - 3) \approx -1.66336660.$$  

As above, we note that the values $k_1$ and $k_2$ are close to $\sqrt{3}/3$ and $-\sqrt{3}$, respectively. That is why the regions of existence of relative equilibriums are very narrow.

4. Stability of central configuration

In this section we perform a linear analysis of stability of central configurations corresponding to relative equilibriums $P_{5j}$ and $P_{j5}$ ($j = 1, 2, 4, 5$).
Let us study the stability of the relative equilibriums \( \gamma_{1842} \gamma_{3037} \gamma_{2873} \) and \( \gamma_{3037} \gamma_{2873} \gamma_{1842} \). To this end we introduce local canonical variables \( \gamma_{1869} \gamma_{2869}, \gamma_{1869} \gamma_{2870}, \gamma_{1868} \gamma_{2869} \) and \( \gamma_{1868} \gamma_{2870} \) in a neighborhood of the above relative equilibriums 
\[
\gamma_{2022} = \gamma_{2022}^* + \gamma_{1869} \gamma_{2869}, \quad \gamma_{2015} = \gamma_{2015}^* + \gamma_{1869} \gamma_{2870}, \quad \gamma_{1868} \gamma_{3093} = -\gamma_{2015}^* + \sqrt{3} \frac{1}{2} (1 - \gamma_{2020} \gamma_{2870} - \gamma_{2020} \gamma_{2871}) + \gamma_{1868} \gamma_{2869}, \quad \gamma_{1868} \gamma_{3086} = \gamma_{2022}^* + \frac{1}{2} (\gamma_{2020} \gamma_{2870} - \gamma_{2020} \gamma_{2871}) + \gamma_{1868} \gamma_{2870},
\]
where \( \gamma_{2022}^* \) and \( \gamma_{2015}^* \) are given by formulas (16) for the relative equilibriums \( \gamma_{1842} \gamma_{2873} \gamma_{3037} \) and by formulas (17) for the relative equilibriums \( \gamma_{1842} \gamma_{3037} \gamma_{2873} \).

Now we expand the Hamiltonian (2) in a power series with respect to the new canonical variables \( \gamma_{1834} = \gamma_{1834}^* \gamma_{2870} + \gamma_{1834}^* \gamma_{2871} + \gamma_{1834}^* \gamma_{2872} + \cdots \).

The quadratic part \( \gamma_{1834}^* \gamma_{2870} \) of the Hamiltonian has the following form 
\[
\gamma_{1834}^* \gamma_{2870} = \frac{9}{4} \mu_m \left( \frac{z^2 - 2}{(z^2 + 1)^{5/2}} \xi_1^3 + O(\varepsilon) \right) + \frac{1}{4} \left( a q_1^2 + b q_2^2 \right) + c q_1 q_2 + p_1 q_2 - p_2 q_1,
\]
where the coefficients \( a, b, c \) reads 
\[
a = -\frac{9}{4} \mu_m + \frac{1}{4} \frac{z^2 - 2}{(z^2 + 1)^{5/2}} \xi_1^3 + O(\varepsilon), \quad b = \frac{9}{4} \mu_m - \frac{5}{4} \frac{2z^2 - 1}{(z^2 + 1)^{5/2}} \xi_1^3 + O(\varepsilon), \quad c = (-1)^{m+1} \frac{3}{4} (\mu_m - 1) \sqrt{3} \frac{3z}{(z^2 + 1)^{5/2}} \xi_1^3 + O(\varepsilon).
\]

In formulas (25) \( m=2 \) in the case of relative equilibriums \( P_{5j} \), and \( m=3 \) in the case of relative equilibriums \( P_{5j} \). The values \( \xi_1 \) and \( z \) are given by formulas (13), (14) for \( P_{5j} \) and given by formulas (18), (19) for \( P_{5j} \).

The characteristic equation of linear system reads 
\[
\lambda^4 + (a + b + 2) \lambda^2 + ab - c^2 - a - b + 1 = 0.
\]

The relative equilibrium is stable in the linear approximation if the following conditions hold \([10]\)
\[
a + b + 2 > 0, \quad ab - c^2 - a - b + 1 > 0, \quad a^2 + b^2 + 4c^2 - 2ab + 8(a + b) > 0.
\]

In order to check the conditions (27) we have numerically calculated the leading terms of the coefficients \( a, b, c \) by using formulas (25). It appears that at \( \mu_2 \ll 1 \) the relative equilibriums \( P_{51}, P_{52} \) and \( P_{54} \) are unstable, whereas the relative equilibrium \( P_{55} \) is stable for \( 0 < \mu_3 < 0.011943 \) and unstable for \( \mu_3 > 0.011943 \). Similar, at \( \mu_3 \ll 1 \) the relative equilibriums \( P_{15}, P_{25} \) and \( P_{45} \) are unstable. The relative equilibrium \( P_{55} \) is stable for \( 0 < \mu_2 < 0.011943 \) and unstable for \( \mu_2 > 0.011943 \).

Now we recall that by our assumption Routh’s necessary condition is fulfilled. That is the Lagrange triangle formed by primaries is stable in linear approximation. Hence, in linear approximation both the problem on stability of central configurations and the problem on stability of the corresponding relative equilibriums are equivalent. It means that the above-obtained conclusions on stability of relative equilibriums are also valid for the corresponding central configurations.

Conclusions
The results of our study can be summarized as is follows. The bifurcation of central configurations in the restricted four-body problem was investigated. Regions of existence of possible relative equilibriums of the particle are constructed in an analytical form.

It has been shown that at \( \mu_2 \ll 1 \) the relative equilibriums \( P_{51}, P_{52}, P_{54} \) located in vicinity of libration point \( l_5^{(2)} \) are unstable, and the relative equilibrium \( P_{55} \) can be both stable and unstable.
depending on value of parameter $\mu_3$. A similar result was obtained at $\mu_3 \ll 1$ in vicinity of libration point $L_5^{(3)}$ the relative equilibriums $P_{15}, P_{25}, P_{45}$ are unstable, and the relative equilibrium $P_{55}$ can be both stable and unstable depending on value of parameter $\mu_2$. The conclusions on stability of relative equilibriums are also valid for the corresponding central configurations.

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