PECULIARITIES IN THE STRUCTURE OF TWO-PARTICLE STATES WITHIN THE BETHE-SALPETER APPROACH

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1 Introduction

The two-fermion bound system is an attractive subject of atomic (positronium, hydrogen atom) and sub-atomic physics (deuteron, mesons). Despite these systems are rather simple the study of two-particle bound states is challenging and still remains a source of progress in quantum theory. Last decade significant efforts were undertaken to describe such objects, and progress has been achieved in both, solving the corresponding equations [1]-[5] and calculating the experimentally measured observables [6]-[10].

The homogeneous Bethe-Salpeter (BS) equation nowadays remains a powerful tool to investigate the relativistic bound state problem. Recently the problem of rigorous treating the BS equations received renewed interest, and several successful attempts were made to reconsider solving algorithms for the BS equation [11] and to make its reduction to Light Front form more transparent [4]. A good example of a consistent study of this subject can be found in Ref. [12].

In fact, our understanding of the mathematical properties of bound states within any relativistic approach is far from perfection. The BS equation itself is a quite complicated mathematical object, and the technical problem of solving it is still of principal value. In the present paper we propose an efficient and promising method to solve the BS equation for fermions involving interaction kernels in the form of one boson exchange supplemented with the corresponding form factors. It is based on employing the basis of hyperspherical harmonics for expanding the partial amplitudes and kernels. We show that this new technique allows one to utilize many advantages in understanding the BS approach. Basically, the current study is inspired by the results reported in [3]. We explore the structure of $0^+$ and $1^+$ bound states for different couplings studying in details the convergence of solutions and corresponding eigenvalues. In particular, on a basis of the introduced rigorous method to solve the BS equation it becomes possible to analyze in details the problem of stability of bound states within the BS approach. It is worth mentioning that this phenomenon was also considered in [3] in the framework of Light-Front Dynamics, and it is completely similar to the bound state collapse in non relativistic quantum mechanics for potentials behaving like $\sim 1/r^2$.

2 Homogeneous Bethe-Salpeter equation for fermions

In the present study, we generalize the method described in [11,13,14] to homogeneous BS equations for spinor particles. To provide a clear illustration, we present here the results of
solving the equation for the vertex function

\[ \mathcal{G}(p) = i \int \frac{d^4k}{(2\pi)^4} V(p, k) \Gamma(1) S(k_1) \mathcal{G}(k) \tilde{S}(k_2) \tilde{\Gamma}(2) \]  

(1)

in the \(^1S_0\) channel with \(V(p, k) = g^2/[(p - k)^2 - \mu^2 + i\varepsilon]\) as one meson exchange interaction kernel for scalar, pseudoscalar and vector meson exchanges with corresponding vertices \(\Gamma(i), i = 1, 2\). The meaning of the introduced quantities \(p, k, k_1, k_2\) is the following: \(k_{1,2} = P/2 \pm k\), \(k = (k_0, k, p = (p_0, p)\) are the relative 4-momenta, and \(P = (M, 0)\) is the total 4-momentum of the bound state in its c.m.s. Spinor propagators of constituent particles with equal masses \(m\) are

\[ S(k) = \frac{\hat{k} + m}{k^2 - m^2 + i\varepsilon}, \quad \tilde{S}(k) = CS(k)^T C = \frac{\hat{k} - m}{k^2 - m^2 + i\varepsilon}, \]

with \(C\) standing for charge conjugation matrix, \(C = i\gamma_0\gamma_2\), \(\hat{k} = \gamma_\mu k^\mu\). In general, the vertices include the meson-nucleon form factor, which will be further considered as

\[ F(q^2) = \frac{\Lambda^2 - \mu^2}{\Lambda^2 - q^2}. \]  

(2)

In (1) the BS vertex function \(\mathcal{G}(p)\) being the matrix 4 \(\times\) 4 should be expanded over the proper set of matrices for the given channel. The standard choice for them is the set of the \(\rho\)-spin angular momentum vector harmonics \(\Gamma_\alpha(p)\), where index \(\alpha\) includes not only \(LSJ\) momenta but also \(\rho\)-spin quantum numbers, which are denoted by ++, −−, e and o [15]. For convenience in our further considerations we introduce another equivalent set of spin-angular matrices instead of \(\rho\)-spin basis. For \(^1S_0\) channel the following functions may be chosen:

\[ T_1(p) = \frac{1}{2}\gamma_5, \quad T_2(p) = \frac{1}{2}\gamma_0\gamma_5, \quad T_3(p) = -\frac{(p, \gamma)}{2|p|}\gamma_0\gamma_5, \quad T_4(p) = -\frac{(p, \gamma)}{2|p|}\gamma_5. \]  

(3)

This basis is orthonormal, i.e.

\[ \int d\Omega_p Tr[T_m(p)T_n^+(p)] = \delta_{mn}, \]

and the partial expansion can be written as

\[ \mathcal{G}(p_0, p) = \sum_n g_n(p_0, |p|) T_n(p), \quad g_n(p_0, |p|) = \int d\Omega_p Tr[\mathcal{G}(p_0, p)T_n^+(p)]. \]  

(4)

After the partial expansion and upon performing the Wick rotation we can expand the partial vertex functions \(g_n\) and the partial interaction kernels in hyperspherical functions using the formula [11]

\[ V_E(p, k) = -\frac{1}{(p - k)^2} \frac{\mu^2}{E} = -2\pi^2 \sum_{nlm} \frac{1}{n+1} V_n(\bar{\rho}, \bar{k}) Z_{nlm}(\omega_p) Z_{nlm}^*(\omega_k), \]

\[ V_n(a, b) = \frac{4}{(A_+ - A_-)^2} \left( \frac{A_+ - A_-}{A_+ + A_-} \right)^n, \quad A_\pm = \sqrt{(a \pm b)^2 + \mu^2}, \]  

(5)

where \((k, p)_E \equiv k_4 p_4 + (k, p)\), \(\bar{k} = \sqrt{k_4^2 + \bar{k}^2}\) is the 4-dimensional absolute value, and \(\omega_k = (\chi, \theta, \phi)\) are the angles of vector \(k = (k_4, k)\) in 4-dimensional Euclidean space. The
The dimension of the transition to the new variables, the weights of the Gaussian mesh etc.

Table 1: Pseudo-probabilities of partial components in the state with given $M$, i.e. their contributions to normalization condition.

| $\alpha$ | $M$ | $P_{++}(LFD)$ | $P_{--}$ | $P_{o}(LFD)$ | $P_c$ |
|----------|------|--------------|----------|--------------|-------|
| 1.194    | 1.937| 1.012        | $-1.18 \cdot 10^{-3}$ | $-6.63 \cdot 10^{-3}$ | $-4.37 \cdot 10^{-3}$ |
| 1.592    | 1.892| 1.020        | $-2.99 \cdot 10^{-3}$ | $-1.07 \cdot 10^{-2}$ | $-6.92 \cdot 10^{-3}$ |
| 1.989    | 1.842| 1.030        | $-6.22 \cdot 10^{-3}$ | $-1.46 \cdot 10^{-2}$ | $-9.41 \cdot 10^{-3}$ |
| 2.149    | 1.820| 1.034        | $-8.11 \cdot 10^{-3}$ | $-1.61 \cdot 10^{-2}$ | $-1.03 \cdot 10^{-2}$ |
| 2.308    | 1.798| 1.039        | $-1.05 \cdot 10^{-2}$ | $-1.75 \cdot 10^{-2}$ | $-1.12 \cdot 10^{-2}$ |
| 2.348    | 1.788| 1.041        | $-1.25 \cdot 10^{-2}$ | $-1.80 \cdot 10^{-2}$ | $-1.16 \cdot 10^{-2}$ |
| 2.352    | 1.5  | 1.210        | $-0.19$   | $-1.24 \cdot 10^{-2}$ | $-7.77 \cdot 10^{-3}$ |

By using these decompositions one can obtain the final system of 1-dimensional integral equations for the coefficient functions $g_3(\tilde{p})$. This system will be explicitly shown and discussed in details separately [16]. What is important now, is that its numerical analysis does not require large computer resources. This set can be easily transformed to the system of linear equations. For this aim, firstly, the infinite summation over hyperspherical components should be limited to some finite value $N_{max}$. Secondly, to calculate the integrals a reliable integration scheme is required. Applying Gaussian quadrature formula, one can get the system of linear equations with the sought functions defined in the mesh points [13],

$$X = \lambda AX,$$

where $\lambda = g^2$, and the column

$$X^T = \left(\left[\{g^{j}_{1}(\tilde{p}_{i})\}_{i=1}^{N_G}\right]_{j=1}^{N_{max}}, \ldots, \left[\{g^{j}_{N_c}(\tilde{p}_{i})\}_{i=1}^{N_G}\right]_{j=1}^{N_{max}}\right)$$

represents the sought solution in the form of a group of sets of partial wave components $g^{j}_{\alpha}, \alpha = 1, \ldots, N_c; j = 1, \ldots, N_{max}$ specified on the integration mesh of order $N_G$. For $0^+$ ($1^+$) state we have $N_c = 4$ ($N_c = 8$). The matrix $A$ is obtained as a product of partial kernels, the Jakobian of the transition to the new variables, the weights of the Gaussian mesh etc. The dimension of $A$ is $N \times N$, where $N = N_c N_G N_{max}$.

### 3 Results

In this short communication we are able to present only the most indicative results of the numerical treatment of the BS equations within our method. First of all, the set of
Figure 1: Dependence of the coupling constant $\alpha = g^2/4\pi$ from the mass of the bound state $M$ in different approaches.

Figure 2: Functions $g_j, j = 1, 2, 3$ for the cut-off $\Lambda = 500$ GeV/c and $g^2 = 30$ (solid line) and $g^2 = 48$ (dashed line).

equations (8) has the solution only if $\text{det}(\lambda A - 1) = 0$. This condition allows us to connect the coupling constant $g^2$ and the mass of bound state $M$, i.e. for any given value of $g^2$ the mass $M$ can be calculated, and vice versa. The results of such calculation for $\alpha = g^2/4\pi$ are shown in Fig. 1 for the case of scalar meson exchange. The solid curve corresponds to our calculations in the BS approach, dashed curve represents the results obtained within Light Front Dynamics [3], dotted one – nonrelativistic calculations for the Yukawa potential.

The spectrum of bound states obtained in this way demonstrates the customary non-relativistic features. Together with the ground states, the exited levels of the system can be found (for an alternative method of calculation see e.g. Ref. [17]). Like in the nonrelativistic picture, solutions of the ground states (i.e. set of partial vertex functions $g_1, \ldots, g_4$) do not have nodes in $|p|$ whereas the excited levels are described by vertex functions having zeroes.

Besides, for any given mass of exchanged meson $\mu$ the bound state (at the ground level) can appear in the considered system only starting from some finite value of the coupling constant $g_{\text{min}}$. For example, at $\mu = 0.15$ GeV we have $g_{\text{min}} = 4.023$, which corresponds to some minimal depth of the potential, where the bound state still exists.

It is obvious, that for weakly bound states for the fixed binding energies $B$ of order of a few MeV coupling constants are approximately equal. Thus, for the value $B = 1$ MeV $\alpha_{BS} = 0.362$, and $\alpha_{LFD} = 0.331$. But for $B$ of order of hundreds MeV coupling constants essentially differ. For example, for the value $B = 100$ MeV we have the ratio $\alpha_{BS}/\alpha_{LFD} \sim 1.3$. The explanation of such a behaviour we found in the role of $^1S_{0^-}$ component in the total BS solution. It is seen from the Table 1 that its contribution to the normalization condition is negligibly small for weakly bound states but increases very rapidly with the increasing $B$, in contrast to the contributions of other partial states. Thus, the role of the $^1S_{0^-}$ component is repulsive, which leads to an increasing coupling constant for the same bound mass $M$ in comparison with the nonrelativistic or LFD formalisms, where ”$^1S_{0^-}$” components are absent.

Another important question to be touched upon is the problem of stability of the bound states within the BS approach. In the present context stability means the existence of the solutions in (1) without any cut off in the vertices. In its turn, such a solution exists only if
it does not depend on the parameters of calculation like $N_{\text{max}}, N_G$, etc. In general, we found that convergence of our method is quite rapid, and it becomes faster for smaller values of the coupling constant. In particular, in the $^1S_0$ channel of the scalar meson exchange kernel it is sufficient to take into account $N_{\text{max}} \sim 4−5$ terms in (6)-(7) for the values of meson mass $\mu > 0.1$ GeV and coupling constant $g^2 < 40$. In the direct calculations it also appears, that the Gaussian mesh with $N_G = 64$ is almost always enough, since an increase of $N_G$ beyond this value practically does not change the results.

Nevertheless, it should be pointed out that at certain conditions convergence of the solutions becomes poor or even breaks. In particular, it is lost at small meson masses $\mu \sim 0$, and similar behaviour was found in [11] for the bound states of scalar particles. In this case, the introduction of form factors allows one to improve the situation. However, in total, we found that in general the equation (1) has stable solutions only for coupling constants $g^2$ below some critical value $g^2_{cr}$, which depends on the type of interaction and the channel considered. It can be found from the numerical calculations, since at coupling constants above some critical value the solution disappears, i.e. it becomes strongly dependent on $N_{\text{max}}, N_G$ and other numerical parameters.

To find the critical value of the coupling constant the dependence of the ground state mass on the cut-off parameter $\Lambda$ at fixed $g^2$ have been investigated. The obtained results are presented in Figs. 3 and 4 for $^1S_0$ and $^3S_1-^3D_1$ channels respectively. It is evident that in the limit $\Lambda \to \infty$ and coupling constants $g^2 < 40$ the $^1S_0$ bound state mass does not depend on $\Lambda$. The same is valid for solutions (vertex functions). More exactly, the critical constant $g^2_{cr}$ for $^1S_0$ state is found to be 40.3. Theoretical estimation of the critical coupling constant can also be performed, and it gives the value $g^2_{cr} = 4\pi^2 \approx 40$. As it is seen from Fig. 4 the similar situation holds for the $^3S_1-^3D_1$ channel, where the critical value is $g^2_{cr} \approx 65$.

It can also be shown that $g^2$ is directly connected to the power of decrease of vertex functions at large $p$, which is shown at Fig. 2. The solid line there corresponds to the solution below $g_{cr}$ with finite normalization, and the dashed one reproduces the solution beyond critical coupling, which seems to have infinite normalization.
4 Conclusion and acknowledgements

A new method of solving the BS equations for the bound states of spinor particles by using the expansion of the vertex functions over the complete set of four-dimensional hyperspherical harmonics is suggested. Within this method the BS equation is treated in a ladder approximation for the cases of scalar, pseudoscalar and vector meson exchanges with corresponding form factors. This method is shown to be effective and stable.

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