A Simple Model for Lensing by Black Holes in Galactic Nuclei

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1 INTRODUCTION
Albert Einstein, in his 1911 paper On the Influence of Gravitation on the Propagation of Light, applied the principle of equivalence to the deflection of light rays by the sun. This marks the beginning of our modern understanding of gravitational lensing. After the full prediction from General Relativity had been confirmed by Eddington’s observations of the 1919 solar eclipse, it was only in 1936 that Einstein considered, rather pessimistically, the deflection due to stars. Zwicky’s (1937) insight one year later that galaxies were much better candidates was finally corroborated by the discovery of the first doubly-lensed quasar in 1979 by Walsh, Carswell and Weymann.

Nowadays, over a 100 multiply imaged gravitational lens systems are known. In many of these, a background quasar or radio galaxy is lensed into doublets or quadruplets by an early-type galaxy. Provided the density distribution at the centre of the lensing galaxy is weakly singular (that is, no worse than \( r^{-1} \)), then gravitational lensing yields an odd number of images, generally three or five. If the singularity is stronger than this, then there is an even number of images, as the central image is absent. Most early-type galaxies are believed to harbour supermassive black holes in their centres. The existence of a black hole – provided it is not too massive – can create an additional faint central image.

The present paper is a mainly analytical study of the Plummer lens with a central black hole and in the presence of external shear. In \( x^2 \), we briefly review the properties of the spherical Plummer lens and show that its images obey a magnification invariant. This model is extended in \( x^3 \) to include a central point mass to which external shear is added in \( x^4 \).

2 THE PLUMMER MODEL
Plummer’s (1911) model is the simplest self-consistent solution of the collisionless Boltzmann equation with finite total mass. Although the density in early-type galaxies decreases less steeply than \( r^{-5} \) at large radii, the Plummer model still gives a good description of the inner parts, which control the lensing properties.

For an isotropic stellar system, the distribution function \( f \) depends on the (negative) total particle energy \( E \) only. The Plummer model is then a polytrope defined by the distribution function

\[
f(E) / (E^{7/2})
\]

For a static system, energy conservation \( E = -\frac{1}{2} v^2 \) holds and, as a consequence, Poisson’s equation for a self-consistent, self-gravitating, spherical and isotropic system

\[
\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) Z = 16 \pi G f(E) \nu^2 dv
\]

implies the mass density (e.g. Binney & Tremaine 1994, p. 225)

\[
= \frac{1}{2} \frac{r^2}{\epsilon_0} v^2 \Rightarrow \frac{r^2}{\epsilon_0} = \frac{2}{5} \Rightarrow M = 4 \epsilon_0^3
\]

and total mass \( M = 4 \epsilon_0^3 \).

Key words: gravitational lensing, black holes

ABSTRACT
The lensing properties of the Plummer model with a central point mass and external shear are derived, including the image multiplicities, critical curves and caustics. This provides a simple model for a flattened galaxy with a central supermassive black hole. For the Plummer model with black hole, the maximum number of images is 4, provided the black hole mass is less than an upper bound which is calculated analytically. This introduces a method to constrain black hole masses by counting images, thus applicable at cosmological distance. With shear, the maximum number of images is 6 and we illustrate the occurrence of an astroid caustic and two metamorphoses.

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2.1 Lens Equation

The angular diameter distances from the observer to the lens plane $L$, the source plane $S$ and from $L$ to $S$ be denoted by $D_{L0}, D_{S0}$ and $D_{SL}$, respectively. We use the conventional dimensionless coordinates $x$ in $L$ and $y$ in $S$ (see e.g., Schneider et al. 1992, p. 245). The relative surface density $\rho$ is now obtained by projecting onto $L$,

$$\rho = 1 + \frac{x^2}{x_0^2}, \quad x_0 = \frac{M}{r_0}$$

where $x = x_r x^j y$ such that the origin coincides with the lens centre, $x_0 = x_0 = D_{L0}$. The deflection potential $\phi$ is

$$\phi = \frac{x_0^2}{2} \ln \left(1 + \frac{x^2}{x_0^2}\right)$$

The lens equation is then

$$y = x \times \rho = x \frac{x_0}{1 + \left(\frac{x^2}{x_0^2}\right)^2}$$

The image and caustic properties of the Plummer lens are briefly summarized below, for comparison with our results in Section 4.

2.2 Image and Caustic Properties

The determinant of the Jacobian of the Plummer lens map is

$$\det J = 1 - \frac{2}{1 + \left(\frac{x^2}{x_0^2}\right)^2} - 1 + \frac{2}{1 + \left(\frac{x^2}{x_0^2}\right)^2} \left(\frac{x_0^2}{1 + \left(\frac{x^2}{x_0^2}\right)^2}\right)^2$$

The origins of $L$ and $S$ are critical and caustic points, respectively. For a point source at the origin $y = 0$, the lens equation implies a circular image, the Einstein ring, with radius $x = x_0$ and infinite magnification. Otherwise, without loss of generality, let the source position be $y = (y_1, y_2)$, whence $x_2 = 0$ and

$$x_1^3 y_1 x_1^2 + (1 \times 0) x_1^3 x_1 \times (0)^2 y_1 = 0$$

Thus, there are one or three images, depending on the source position $y_1$. The images lie on a straight line through the lens centre.

For $y_1 \neq 1; J \neq 1$ (the identity matrix), so there is always one image at large $y_1$, as expected.

The critical curves can be obtained from (8) with $\det J = 0$ resulting in

$$x_{crit,1} = x_0 \frac{p_1}{S_0} 1$$

$$x_{crit,2} = x_0 \frac{0}{2} 1 + \frac{8}{2} 1 + \frac{0}{2}$$

There are two critical circles in $L$ if $S_0 > 1$. Applying the lens equation then shows that the caustic corresponding to $x_{crit,1}$ is the origin point in $S$, whereas the other caustic is circular, as expected by symmetry. Hence three images can occur by crossing this caustic inwards. However, if $S_0 \leq 1$ then only a critical and caustic point exist and the number of images is one for all $y \neq 0$.

2.3 Magnification Invariant

Witt & Mao’s (1995) paper introduced the notion of a lensing invariant. They considered a binary lens of two point masses. Their main result is that the sum of the signed magnifications of the maximum number of five images is unity, namely

$$\sum_{i=1}^{5} \mu_i = 1$$

Here, $\mu_i$ is the absolute magnification of the image, while $p_1$ is the parity. This result holds good irrespective of the position of the source, provided it remains within the central caustic giving rise to the maximum number of images. Subsequently, Rhie (1997) and Witt & Mao (2000) found further examples of lens models with invariants. A general theory of such invariants using single variable complex analysis (Hunter & Evans 2001; Evans & Hunter 2002; An & Evans 2006) was developed. This approach is used here to investigate the general circularly symmetric lens and, in particular, the Plummer model.

2.3.1 Circular Symmetric Lenses

The standard complexity is introduced (e.g. Petters, Levine & Wambsganss 2001, chap. 15.1). Let $L_c, S_c$ be the complex lens and source planes, respectively, and define the complex coordinates $z = x_1 + iy_1, \quad 2 L_c, \quad y_1 + iy_2, \quad 2 S_c$.

Figure 1. Plummer lens with central point mass, $S_0 = 2; a_0 = 0 \neq a_2$. (Left: critical circles in $L$, right: caustic circles in $S$. Number labels indicate number of images.)
Let the complex conjugate be denoted by a bar. The complex lens equation is rendered

\[ z = z \frac{df}{ds} (z) ; \]

(12)

where the deflection factor \( f \) is

\[ f (s) = \frac{2 \frac{dI}{ds}}{I} ; \]

(13)

Note that, by definition, \( f \) is a real-valued function so that \( f = f^\star \).

The conjugate of the lens equation is therefore

\[ z = z \frac{df}{ds} (z) \]

(14)

and so

\[ z = \bar{z} ; \]

(15)

The geometrical meaning of (15) is plain: due to circular symmetry, we can without loss of generality orientate the real axes in such a way that the source position \( s \) is real and hence, by (15), the image positions as well: lens centre, point source and all images are collinear. Also, (15) allows us to eliminate \( z \) from the lens equation (12) so that standard, one-dimensional complex analysis may be applied. It can be recast thus to give the imaging equation

\[ I (z_i) = 0 ; \]

(16)

defined so that image positions \( z_i \) correspond to the roots of

\[ I (z_i) = z \frac{df}{ds} \bar{z}_i^2 ; \]

(17)

As the lens model \( f \) may have isolated poles in the complex plane, the imaging equation is meromorphic.

The signed flux magnification of the image at \( z_i \) is given by (e.g., Petters et al. 2001, p. 85)

\[ \lambda_i = \frac{1}{\det J (z_i)} ; \]

(18)

where the Jacobian \( J \) and its inverse \( J^{-1} \) in complex form can be written as (e.g., Petters et al. 2001, p. 506)

\[
J = \begin{bmatrix}
0 & \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\
\frac{\partial x}{\partial z} & 0 & \frac{\partial y}{\partial z} \\
\frac{\partial y}{\partial x} & \frac{\partial x}{\partial y} & 1
\end{bmatrix} ; \quad J^{-1} = \begin{bmatrix}
\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & 0 \\
\frac{\partial x}{\partial z} & \frac{\partial y}{\partial z} & 0 \\
\frac{\partial y}{\partial x} & \frac{\partial x}{\partial y} & 1
\end{bmatrix}.
\]

Requiring that \( JJ^{-1} = 1 \), it follows that

\[ \det J = \frac{\partial z}{\partial x} \frac{\partial z}{\partial x} ; \]

(19)

By standard calculus, we find that the sum of the signed magnifications of the \( n \) images is

\[ \sum_{i=1}^{n} \lambda_i = \frac{1}{\det J (z_i)} \sum_{i=1}^{n} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} ; \]

(20)

By using (12) in the denominator and (15) in the numerator of (20), the following simplification is obtained

\[ \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \frac{z_i}{\bar{z}_i} ; \]

(21)

We now seek a complex-valued function with residues of the form (19). Consider the integral of

\[ F (z_i) = \frac{z}{(z + 1I)i} ; \]

(22)

along a contour \( C \) enclosing all \( N_{\text{max}} \) roots \( z_i \). Assuming that the roots of the imaging equations are simple, the integrand \( F \) has a simple pole at each root of \( z \) and hence residues

\[ \text{res} (F \, dz_i) = \frac{z_i}{\bar{z}_i} (z_i) ; \]

(23)

by Taylor expansion of \( I \) about \( z_i \), as required. Now from the residue theorem, we find

\[ \sum_{i=1}^{n} \lambda_i = \frac{1}{2 \pi i} \int_{C} \frac{z}{(z + 1I)i} dz = \frac{1}{2 \pi i} \int_{C} \frac{1}{(z + 1I)i} dz = 1 ; \]

(24)

Furthermore, two assumptions are made which both restrict the functions \( f \). First, no branch cuts to ensure singlevaluedness of \( I \) are to occur, such that the complex plane is analytic everywhere outside \( C \). Secondly, it is assumed that the deflection angle becomes negligible at large distance from the lens centre. This amounts to \( x \rightarrow 0 \) as \( z \rightarrow 1 \), hence \( zf \rightarrow 0 \) and therefore \( I \rightarrow z \) as \( \gamma j \rightarrow 1 \). Then by Cauchy’s theorem, \( C \) can be distorted to a circle at infinity \( C_\infty \)

\[ \sum_{i=1}^{n} \lambda_i = \frac{1}{2 \pi i} \int_{C_\infty} \frac{z}{(z + 1I)i} dz = \frac{1}{2 \pi i} \int_{C} \frac{dz}{z} = 1 ; \]

(25)

This result is a magnification invariant for the maximum number of roots \( N_{\text{max}} \) of the imaging equation.

Although each image corresponds to a complex root of the imaging equation, there may exist roots which do not correspond to physical images and which are therefore called spurious roots. To understand their impact on (25), consider two of their properties. First, by choosing without loss of generality \( (21) \), it is seen that both \( z \) and \( z \) are solutions. So, all spurious roots occur as complex conjugate pairs. Secondly, from (16) and (21), we see that the magnifications of a pair of conjugate spurious roots are also complex conjugate. This implies that the total contribution from spurious roots to the signed magnification sum is always real.

### 2.3.2 The Particular Case of the Plummer Model

The Plummer lens equation (7) can be rewritten as

\[ \frac{z}{1 + z} ; \]

(26)

From the definition of the deflection factor \( f \), it follows that

\[ f (z) = \frac{z}{1 + z} ; \]

(27)

Hence \( zf \rightarrow 0 \) as \( \gamma j \rightarrow 1 \) and \( f \) involves no roots of \( z \) and hence no branch cut. The imaging equation is of third degree in \( z \) and, since the maximum number of images is three, there are no spurious roots. Thus, the Plummer model has a magnification invariant

\[ \sum_{i=1}^{n} \lambda_i = 1 ; \]

(28)

In other words, the sum of the signed magnifications \( \sum_{i=1}^{n} \lambda_i \) of the images is an invariant of the model. This result appears not to have been realized before.
3 THE PLUMMER MODEL WITH BLACK HOLE

3.1 Lens Equation

The standard Plummer lens is now modified by introducing a point mass \( m \) as a model for a black hole at the centre. Using dimensionless coordinates in \( L \) as before, its relative surface density is

\[
\Sigma_H(w) = \frac{m}{D_L w^2} = G^1(w),
\]

whence the total deflection potential is

\[
\phi(x) = \pi \left( \frac{1}{\Sigma_H(w) \ln |y|} \right) x y^2 w = \frac{\pi a}{x^2} + \ln x;
\]

where \( a = m = D_L x_{crit}^2 \). The lens equation follows immediately,

\[
y = x + \frac{\phi x}{1 + (\phi x / a)} \frac{a x}{x^2}
\]

(32)

using (6).

3.2 Image and Caustic Properties

This lens equation shows that the point mass introduces an additional image at \( x! \), \( a y = y^2 \) as \( y! : 1 \) : the minimum number of images is two. Moreover, eq (32) is of fourth degree in \( x \), so a maximum of four images is expected. Some further insight can be gained analytically. With the substitution

\[
\frac{x_1}{x_0} = \cos \theta \quad \frac{x_2}{x_0} = \sin \theta \quad a = \frac{a}{x_{0}^2} = \frac{m}{x_{0}^2 \ c_m^2}
\]

where \( x > 0 \), the Jacobian can be constructed from the partial derivatives,

\[
\frac{\partial y_1}{\partial x_1} = 1 + 2 \frac{\cos^2 \theta}{(1 + \theta)^2} + \frac{a_2}{(2 \cos \theta)^2}
\]

\[
\frac{\partial y_1}{\partial x_2} = \frac{2 \sin \theta \cos \theta}{(1 + \theta)^2} + \frac{a_2 \cos \sin}{2}
\]

\[
\frac{\partial y_2}{\partial x_1} = \frac{2 \sin \theta \cos \theta}{(1 + \theta)^2} + \frac{a_2 \cos \sin}{2}
\]

\[
\frac{\partial y_2}{\partial x_2} = 1 + 2 \frac{\sin^2 \theta}{(1 + \theta)^2} + \frac{a_2}{(2 \sin \theta)^2}
\]

The radii of the two critical circles are given by \( \det J = 0 \), which yields:

\[
1 = \frac{0}{1 + \frac{a_0}{x_{crit,1}}} \quad \frac{a_0}{x_{crit,1}} = 0
\]

(34)

so that

\[
\frac{\sqrt{1 + \frac{a_0}{x_{crit,1}}}}{1 + \frac{a_0}{x_{crit,2}} \frac{a_0}{x_{crit,2}}} = \frac{\sqrt{1 + \frac{a_0}{x_{crit,1}}}}{1 + \frac{a_0}{x_{crit,2}} \frac{a_0}{x_{crit,2}}} = 0
\]

(35)

and

\[
\frac{0}{1 + \frac{a_0}{x_{crit,2}} \frac{a_0}{x_{crit,2}}} + \frac{2 \frac{a_0}{x_{crit,2}}}{(1 + \frac{a_0}{x_{crit,2}})^2} + \frac{a_0}{x_{crit,2}} = 0
\]

(36)

The critical circle (35) has precisely one solution for all positive \( a \). Applying the lens equation shows that it maps to a caustic point at the origin of \( S \). If \( a > 1 \), there is hence one critical circle in \( L \) at \( x_{crit,1} \) and one caustic point in \( S \); the number of images is always two for \( y \neq 0 \). If \( a > 1, x_{crit,2} \) introduces two further critical circles in \( L \) and two caustics in \( S \), in contrast to one for the Plummer model. Although (35) is a third degree polynomial in \( \frac{a_0}{x_{crit,2}} \) and could in principle be solved analytically, it is, however, more instructive to display the critical circles and caustics numerically. Figure illustrates this case with three critical circles and two caustic circles, giving rise to an annular domain with four images inside and two outside of it.

3.3 Maximum Black Hole Mass

If the point mass \( a_0 \) increases while the Plummer mass \( \sigma \) is kept fixed, the former will eventually dominate the latter, for some \( a_0 > a_{m, ax} \). In consequence, two images will occur for all \( y \neq 0 \), as is plain from the lens equation for negligible \( \sigma \). The annulus in \( S \) resulting in four images will hence disappear, as shown in Figure 2

Let us take without loss of generality \( y = (y_1, 0) \), \( x_0 = 0 \), then (22) requires for the presence of four images \( x > 1 \) and four intersections of

\[
1 = \frac{0}{1 + \frac{a_0}{x_{crit,1}}} \quad \frac{a_0}{x_{crit,1}} = 0
\]

(37)

where \( x = x_1 = x_0 \). At the threshold \( x = 0; a_0 = a_{m, ax} \), the graphs of left-hand side and right-hand side match in first and second derivatives,

\[
1 = \frac{0}{1 + \frac{a_0}{x_{crit,1}}} \quad \frac{a_0}{x_{crit,1}} = 0
\]

(38)

\[
0 = \frac{2 \frac{a_0}{x_{crit,1}}}{(1 + \frac{a_0}{x_{crit,1}})^2} + \frac{2 \frac{a_0}{x_{crit,1}}}{(1 + \frac{a_0}{x_{crit,1}})^2}
\]

(39)

Eliminating \( a_{m, ax} \), they yield a third degree polynomial in \( \sqrt{x} \) which becomes, after the standard substitution \( q = \sqrt{g} + 1, q^3 + 3 q^2 + 4 q = 0 \):

\[
q^3 + 3 q^2 + 4 q = 0
\]

(40)

Since its cubic determinant is positive, there is one real solution which can be obtained from Cardano's formula,

\[
q = \left( \frac{2}{(0 + 4)^{1/3}} \right) \left( \frac{p}{(0 + 4)^{1/3}} \right)
\]

(41)

to give, using (19) and (20),

\[
a_{m, ax} = \frac{(q^3 + 4 q) (q^3 + 4 q)}{4 (3q)}
\]

(42)

In physical units, the maximum mass of the point mass for four images relative to the Plummer model is

\[
\frac{m_{m, ax}}{M} = \frac{a_{m, ax}}{a_{m, ax}}
\]

(43)

Now consider this as a model of a galaxy with a supermassive black hole. Since many parameters are free, let us take, as a typical example, the total galaxy mass to be \( M = 10^{12} M \) and scale length \( r_0 = 10 \) kpc. Let the lens and source be at redshifts \( z = 0.5 \) and \( z = 3 \), respectively, in an Einstein-de Sitter universe with \( H = 70 \) km s \(^{-1} \) Mpc \(^{-1} \). This is a sufficient approximation to the Concordance Cosmology for the purpose of estimating parameters here, especially because the source distance at higher redshift almost cancels out. A well-known expression for angular diameter distances (e.g., Peacock 2003, p. 103), then yields \( D_{so} \), \( D_{sl} \), \( D_{lo} \). Hence from the definitions (35), we have:

\[
\frac{x_0}{x_0} = 4 \frac{M}{D_{so} D_{SL} D_{LO}} = 1.28 > 1
\]

(44)

so that four images can occur, provided the mass of the central black hole is less than

\[
\frac{m_{m, ax}}{M} = 4 \times 10^{-3} \quad m_{m, ax} = 4 \times 10^{-3} M
\]

(45)
from [13]. In fact, this result is of the same order of magnitude as the masses inferred from observations: the presently surmised median mass of supermassive black holes is \((1.3 \pm 0.9) \times 10^9\) of the bulge mass, and galaxy M87 is thought to have a supermassive black hole of mass \((3 \pm 1) \times 10^9\) M\(_\odot\) [Haring & Rix 2004]. This indicates that by ascertaining a fourth lensed image, a relevant stellar kinematics.

However, this approach has two main difficulties: first, it requires knowledge of \(z_0\) and hence the mass and size of the lens, and secondly, the fourth image may be missed or be misinterpreted. Bowman et al. (2004) have recently considered the lensing properties of a cored and a broken power-law isothermal sphere with central point mass. Because of their diverging masses, these models are only valid centrally and hence a black hole mass bound from lensing, as described here, is not discussed. Based on their numerical simulation of lensing cross-sections, the authors conclude that observations of the fourth image, as characteristic lensing signature, in a galaxy cluster. For simplicity, let us consider a constant external mass surface density \(\Sigma_{\text{ext}}\) and shear \(\gamma_{\text{ext}}\). Let the Cartesian coordinate axes be in the principal directions so that the perturbation of the lens equation is in diagonal form (e.g., Schneider et al. 1992, p. 250)

\[
y = \frac{\Sigma_{\text{ext}} + \gamma_{\text{ext}}}{\Sigma_{\text{ext}}} x
\]

(49)

where \(x, y\) are the components of the total external shear. The lens equation (32) is now modified thus

\[
y_1 = x_1 + \frac{\alpha x_1}{1 + (\frac{x_1}{S})^2} \frac{\alpha x_1}{x^2}; x_1
\]

(50)

\[
y_2 = x_2 + \frac{\alpha x_2}{1 + (\frac{x_2}{S})^2} \frac{\alpha x_2}{x^2}; x_2
\]

(51)

For \(\gamma_{\text{ext}}\), the imaging equation is a fifth order polynomial in \(x, y\). However, as we will see in the next section, the maximum number of images of a Plummer model with black hole and shear is 6. Hence in general, there are spurious roots to the imaging polynomial and a magnification invariant does not exist.

4 THE PLUMMER LENS WITH BLACK HOLE AND SHEAR

4.1 Lens Equation

Circular symmetry is however normally broken in gravitational lensing. This may be caused by flattening of the lensing galaxy itself or by the presence of an external gravitational field, for instance, in a galaxy cluster. For simplicity, let us consider a constant external mass surface density \(\Sigma_{\text{ext}}\) and shear \(\gamma_{\text{ext}}\). Let the Cartesian coordinate axes be in the principal directions so that the perturbation of the lens equation is in diagonal form (e.g., Schneider et al. 1992, p. 250)

\[
y = \frac{\Sigma_{\text{ext}} + \gamma_{\text{ext}}}{\Sigma_{\text{ext}}} x
\]

(49)

so again \(z f \neq 0\) as \(z = 1\) and \(f\) has no branch cut. The maximum number of images is four. Likewise, the imaging equation is of fourth degree in \(z\) so there are no spurious roots. Therefore, the magnification invariant holds for the Plummer model with central point mass as well.

\[
X^4 \prod_{j=1}^{4} = 1
\]

(48)

Although the central black hole has yielded a further image, it is still true that the sum of the signed magnifications is unity.

4.2 Image and Caustic Properties

4.2.1 Astroid Caustic

When \(\gamma_{\text{ext}} > 0\), eqns (50) and (51) remain circularly symmetric with three critical circles, two caustic circles and a critical and caustic point as in \(x_3\).

Consider now the case \(\gamma_{\text{ext}} > 0\), that is, the external mass surface density \(\Sigma_{\text{ext}} > \Sigma_{\text{ext}}\). Since, by construction,
There is now one pair of cusp points on each of the $y_1, y_2$-axes giving rise to an astroid caustic. By crossing the astroid inwards, the number of images increases from two to four. If the point source is at $y = 0$, an additional fifth image at the centre of $L$ is produced, since by \(50\) and \(51\) $x = 0$ is a solution. Note that, in this case, circular symmetry is broken and an Einstein ring image cannot form. The astroid increases in size as $1; 2$ become more different, eventually intersecting the already existing caustic curves. Caustic domains are thereby created in which the image multiplicity increases from a maximum of four to six. Such a configuration of critical and caustic curves is illustrated in the upper and middle panel of figure 9.

### 4.2.2 Metamorphoses

As the difference in the shear components increases yet further, $1 > 0$ and $2 < 0$ and the external mass surface density $\omega_{\text{ext}} < 0$. Now critical curves and caustics rejoin, as can be seen in the lower panel of figure 9 as the inner two critical curves in $L$ merge on the $x_2$-axis, the tangent vector vanishes and $det J = 0$. This corresponds to the rejoining of the inner two caustics on the $y_2$-axis to give two cusps, and is an example of a beak-to-beak metamorphosis.

Our model can also serve to illustrate the metamorphosis of an umbilic. Referring to the deflection potential in equation (31) and the lens equation with external shear \(50\), \(51\), the conventional form (e.g. Schneider et al. 1992, p. 162) of the Jacobian $J$ of the lens mapping is,

$$J = \frac{1}{1 + \frac{a_0}{\omega_{\text{ext}}}} \left( \frac{a_1}{\omega_{\text{ext}}} \right)^2$$  \hfill (52)

where

$$1 = \frac{1}{2} \frac{\partial^2}{\partial x_1^2} \frac{\partial^2}{\partial x_2^2} + \omega_{\text{ext}}$$

$$2 = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2}$$

The umbilic point occurs where both eigenvalues $1; 2$ of $J$ vanish,

$$1 = 1 + \frac{a_0}{\omega_{\text{ext}}} = 0$$

$$2 = 1 + \frac{a_1}{\omega_{\text{ext}}} = 0.$$  

These two conditions can now be applied to derive the location of umbilic metamorphoses. Since, by construction, the coordinate system in the lens plane is aligned with the principal directions of external shear, we can set $1 > 2$ and $x_2 = 0$. Then keeping $a_1 = 0$; \(1; 2\) fixed defines a one-parameter family of lens models in $2$. Letting $1 = x_1 = x_0$, we hence obtain the value of $2$ for the metamorphosis from

$$0 \left( \frac{1}{2} \right)^2 \frac{a_0}{\omega_{\text{ext}}} = 1$$

$$0 \left( \frac{1}{2} \right)^2 \frac{a_1}{\omega_{\text{ext}}} = 1$$

Figure 10 illustrates the caustics for a particular Plummer model with central point mass and external shear undergoing metamorphosis at an umbilic point.
Figure 4. Caustics for the Plummer lens with central point mass and external shear, $a_0 = 0.12, a_2 = 0.06; \lambda = 0.8$. The difference between the components of external shear increases left to right as $\lambda$ decreases. Left: $\lambda = 0.1$, middle: $\lambda = 0.2$, where the umbilic metamorphosis occurs, right: $\lambda = 0.9$. Number labels indicate number of images.

5 CONCLUSION

This paper provides an analytic study of the lensing properties of the Plummer model, together with modifications incorporating the effects of a central point mass and external shear. This is a simple model for lensing by a flattened galaxy with a central black hole. The maximum number of images given by the circular Plummer lens is 3. This number rises to 4 if a central point mass is introduced and to 6 if external shear is added as well.

There are three new results of our work. First, a magnification invariant is found to exist for both the Plummer lens alone, and for the Plummer lens with black hole. Even though the introduction of a black hole changes the maximum number of images from 3 to 4, it is nonetheless true that the sum of the signed magnifications remains unity.

The second result is a treatment of the caustic structure for the Plummer lens with black hole and shear. Without shear, there are two caustic circles and a caustic point. This gives three disjoint regions in the source plane where double, quadruple and double imaging respectively occur. The introduction of shear causes the caustic point to unfold as an astroid caustic. The presence of four cusps on an astroid caustic is established, demonstrating the change of image multiplicities as the caustics intersect and metamorphoses occur. For modest shear, the astroid is contained within the two outer caustics. However, as the components of shear $a_1$ and $a_2$ become more different, the size of the astroid increases and additional caustic domains are created in which the image multiplicity is 6.

Based on this discussion of caustic curves, we describe a possible method to obtain an upper limit on the mass of a black hole from gravitational lensing. As this approach relies on counting images alone, it could be used at cosmological distance and therefore as an independent check on mass estimates obtained from other methods, such as stellar kinematics and reverberation mapping. This emphasises the value of lensing surveys for constraining the mass distribution in the inner parts of galaxies.

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