FAMILIES OF THUE EQUATIONS ASSOCIATED WITH A RANK ONE SUBGROUP OF THE UNIT GROUP OF A NUMBER FIELD

CLAUDE LEVESQUE AND MICHEL WALDSCHMIDT

In memory of Klaus Roth

Abstract. Let $K$ be an algebraic number field of degree $d \geq 3$, $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of $K$ into $\mathbb{C}$, $\alpha$ a non-zero element in $K$, $a_0 \in \mathbb{Z}$, $a_0 > 0$ and

$$F_0(X, Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha)Y).$$

Let $\nu$ be a unit in $K$. For $a \in \mathbb{Z}$, we twist the binary form $F_0(X, Y) \in \mathbb{Z}[X, Y]$ by the powers $\nu^a$ ($a \in \mathbb{Z}$) of $\nu$ by setting

$$F_a(X, Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha\nu^a)Y).$$

Given $m > 0$, our main result is an effective upper bound for the size of solutions $(x, y, a) \in \mathbb{Z}^3$ of the Diophantine inequalities

$$0 < |F_a(x, y)| \leq m$$

for which $xy \neq 0$ and $\mathbb{Q}(\alpha\nu^a) = K$. Our estimate is explicit in terms of its dependence on $m$, the regulator of $K$ and the heights of $F_0$ and of $\nu$; it also involves an effectively computable constant depending only on $d$.

§1. Introduction and the main results. Let $d \geq 3$ be a given integer. We denote by $\kappa_1, \kappa_2, \ldots, \kappa_{38}$ positive effectively computable constants which depend only on $d$.

Let $K$ be a number field of degree $d$. Denote by $\sigma_1, \sigma_2, \ldots, \sigma_d$ the embeddings of $K$ into $\mathbb{C}$ and by $R$ the regulator of $K$. Let $\alpha \in K$, $\alpha \neq 0$, and let $a_0 \in \mathbb{Z}$, $a_0 > 0$, be such that the coefficients of the polynomial

$$f_0(X) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha))$$

are in $\mathbb{Z}$. Let $\nu$ be a unit in $K$, not a root of unity. For $a \in \mathbb{Z}$, define the polynomial $f_a(X)$ in $\mathbb{Z}[X]$ and the binary form $F_a(X, Y)$ in $\mathbb{Z}[X, Y]$ by

$$f_a(X) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha\nu^a))$$
and

\[ F_a(X, Y) = Y^d f_a(X/Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha \nu^a)Y). \]

Define

\[ \lambda_0 = a_0 \prod_{i=1}^{d} \max\{1, |\sigma_i(\alpha)|\} \quad \text{and} \quad \lambda = \prod_{i=1}^{d} \max\{1, |\sigma_i(\nu)|\}. \]

Let \( m \in \mathbb{Z}, m > 0 \). We consider the family of Diophantine inequalities

\[ 0 < |F_a(x, y)| \leq m, \quad (1) \]

where the unknowns \((x, y, a)\) take their values in the set of elements in \( \mathbb{Z}^3 \) such that \( xy \neq 0 \) and \( \mathbb{Q}(\alpha \nu^a) = K \). It follows from the results in [4] that the set of solutions is finite. However, the proof in [4] relies on Schmidt’s subspace theorem, which is not effective. Here by using lower bounds for linear forms in logarithms, we give an upper bound for \( \max\{\log |x|, \log |y|, |a|\} \), which is explicit in terms of \( m, R, \lambda_0 \) and \( \lambda \) and which involves an effectively computable constant depending only on \( d \).

For \( x \in \mathbb{R}, x > 0 \), we use the notation \( \log^* x \) to denote \( \max\{1, \log x\} \). Here is our main result.

**Theorem 1.** There exists an effectively computable constant \( \kappa_1 > 0 \), depending only on \( d \), such that any solution \((x, y, a)\in \mathbb{Z}^3\) of (1), which verifies \( xy \neq 0 \) and \( \mathbb{Q}(\alpha \nu^a) = K \), satisfies

\[ |a| \leq \kappa_1 \lambda d^{2(d+2)/2} (R + \log m + \log \lambda_0) R \log^* R. \]

Under the assumptions of Theorem 1, with the help of the upper bound

\[ H(F_a) \leq 2^d \lambda_0 \lambda |a| \]

for the (usual) height of the form \( F_a \) (namely the maximum of the absolute values of the coefficients of the form), it follows from the bound (3.2) in [1, Theorem 3] (see also [2, Theorem 9.6.2]) that

\[ \log \max\{|x|, |y|\} \leq \kappa_2 (R + \log^* m + |a| \log \lambda + \log \lambda_0) R (\log^* R), \quad (2) \]

where \( \kappa_2 \) is an explicit constant depending only on \( d \). Combining this upper bound with our Theorem 1 provides an effective upper bound for \( \max\{|x|, \log |y|, |a|\} \).

**Corollary 2.** Under the assumptions of Theorem 1, there exists an effectively computable constant \( \kappa_3 \) depending only on \( d \) such that

\[ \max\{|x|, \log |y|, |a|\} \leq \kappa_3 \lambda d^{2(d+2)/2} (\log \lambda) (R + \log m + \log \lambda_0) R^2 (\log^* R)^2. \]
Our proof of Theorem 1 actually gives a much stronger estimate for $|a|$; see Theorem 3 below. It involves an extra parameter $\mu > 1$ that we now define. For $i = 1, \ldots, d$, set $\nu_i = \sigma_i(\nu)$ and assume

$$|\nu_1| \leq |\nu_2| \leq \cdots \leq |\nu_d|.$$ 

Define

$$\mu = \begin{cases} 
\lambda & \text{if } |\nu_1| = |\nu_{d-1}| \text{ or } |\nu_2| = |\nu_2|, \\
\min \left\{ \frac{|\nu_{d-1}|}{|\nu_1|}, \frac{|\nu_2|}{|\nu_2|} \right\} & \text{if } |\nu_1| < |\nu_2| = |\nu_{d-1}| < |\nu_d|, \\
\frac{|\nu_{d-1}|}{|\nu_2|} & \text{if } |\nu_2| < |\nu_{d-1}|.
\end{cases}$$

Notice that the condition $|\nu_1| = |\nu_{d-1}|$ means $|\nu_1| = |\nu_2| = \cdots = |\nu_{d-1}|$ and that the condition $|\nu_2| = |\nu_d|$ means $|\nu_2| = |\nu_3| = \cdots = |\nu_d|$; using Lemma 12, we deduce that each of these two conditions implies that $d$ is odd, hence that the field $K$ is almost totally imaginary (namely, with a single real embedding)—compare with [9].

**Theorem 3.** There exists a positive effectively computable constant $\kappa_4$, depending only on $d$, with the following property. Let $(x, y, a) \in \mathbb{Z}^3$ satisfy

$$xy \neq 0, \quad [\mathbb{Q}(\alpha \nu^d) : \mathbb{Q}] = d \quad \text{and} \quad 0 < |F_a(x, y)| \leq m.$$ 

Then

$$|a| \leq \kappa_4 \frac{\log \lambda}{\log \mu} (R + \log m + \log \lambda_0 + \log \lambda) R \log \left( \frac{R (\log \lambda)^2}{\log \mu} \right). \quad (3)$$

On the one hand, using Lemma 13 (§3.6), we will prove in §5 that

$$\log \mu \geq \kappa_5 \lambda^{-d^2(d+2)/2} (\log \lambda)^2,$$

which will enable us to deduce Theorem 1 from Theorem 3. On the other hand, thanks to (7), we have $\mu \leq \lambda^2$. Hence the largest possible value of $\mu$ is $\lambda^{\kappa_6}$ with a positive constant $\kappa_6$ depending only on $d$. For the units $\nu$ satisfying such an estimate, the conclusion of Theorem 3 becomes

$$|a| \leq \kappa_7 (R + \log m + \log \lambda_0 + \log \lambda) R (\log R + \log^* \log^* \lambda) \quad (4)$$

with a positive effectively computable constant $\kappa_7$ depending only on $d$. In §2, we give a few examples where this last bound is valid.

In Theorem 1, the hypothesis that $\nu$ is not a root of unity cannot be omitted. Here is an example with $\alpha = \zeta_n$ and $a_0 = m = 1$. Let $\Phi_n(X)$ be the cyclotomic polynomial of index $n$ and degree $\varphi(n)$ (Euler totient function). Let $\zeta_n$ be
a primitive $n$th root of unity. Set $f_0 = \Phi_n$ and $u = \zeta_n$. For $a \in \mathbb{Z}$ with $\gcd(a, n) = 1$, the irreducible polynomial $f_a$ of $\zeta_n^a$ is nothing else than $f_0$. Hence, if the equation

$$F_0(x, y) = \pm 1$$

has a solution $(x, y) \in \mathbb{Z}^2$ with $xy \neq 0$, then for infinitely many $a \in \mathbb{Z}$ the twisted Thue equation $F_a(x, y) = \pm 1$ has also the solution $(x, y)$, since $F_a = F_0$. For instance, when $n = 12$, we have $\Phi_{12}(X) = X^4 - X^2 + 1$ and the equation

$$x^4 - x^2y^2 + y^4 = 1$$

has the solutions $(1, 1), (-1, 1), (1, -1), (-1, -1)$.

Let us compare the results of the present paper with our previous work.

The main result of [5], which deals only with cubic equations which are not totally real, is a special case of Theorem 3: the “constants” in [5] depend on $\alpha$ and $\nu$, while here they depend only on $d$. The main result of [6] deals with Thue equations twisted by a set of units which is not supposed to be a group of rank 1, but it involves an assumption (namely that at least two of the conjugates of $\nu$ have a modulus as large as a positive power of $|\nu|$) which we do not need here. Our Theorem 3 also improves the main result of [7]: we remove the assumption that the unit is totally real (besides, the result of [7] is not explicit in terms of the heights and regulator). We also notice that part (iii) of [8, Theorem 1.1] follows from our Theorem 3. The main result of [9] does not assume that the twists are done by a group of units of rank 1, but it needs a strong assumption which does not occur here, namely that the field $K$ has at most one real embedding.

§2. Examples. The lower bound $\mu \geq \lambda^{k_6}$ quoted in §1 is true:

- when $d = 3$ and the cubic field $K$ is not totally real;
- for the Salem numbers;
- for the roots of the polynomials in the families giving the simplest fields of degree three (see [8]), and also the simplest fields of degrees four and six; and
- when $|\nu_1| = |\nu_2|$ and $|\nu_{d-1}| = |\nu_d|$ with $d \geq 4$. In particular when $-\nu$ is a Galois conjugate of $\nu$ (which means that the irreducible polynomial of $\nu$ is in $\mathbb{Z}[X^2]$).

Here is an example of this last situation. Let $\epsilon$ be an algebraic unit, not a root of unity, of degree $\ell \geq 2$ and conjugates $\epsilon_1, \epsilon_2, \ldots, \epsilon_\ell$. Let $h \geq 2$ and let $d = \ell h$. For $a \in \mathbb{Z}$, define

$$F_a(X, Y) = \prod_{i=1}^\ell (X^h - \epsilon_i^a Y^h). \quad (5)$$

Let $R$ be the regulator of the field $\mathbb{Q}(\epsilon^{1/h})$. From Theorem 3 we deduce the following corollary.

**Corollary 4.** Let $m \geq 1$. If the form $F_a$ in (5) is irreducible and if there exists $(x, y) \in \mathbb{Z}^2$ with $xy \neq 0$ and $|F_a(x, y)| \leq m$, then

$$|a| \leq k_8 (R + \log m + \log |\epsilon|) \log^* (R \log |\epsilon|).$$
Proof. Without loss of generality, assume $|\epsilon_1| \leq |\epsilon_2| \leq \cdots \leq |\epsilon_\ell|$, so that $|\epsilon_\ell| = |\epsilon|$. Let $\zeta$ be a primitive $h$th root of unity. Let $\nu = \epsilon_1^{-1/h}$. We apply Theorem 3 with $\alpha = \zeta^a$, $a_0 = 1$, $\lambda_0 = 1$, $\lambda_\ell \leq |\epsilon|$, $F_0(X, Y) = (X^h - Y^h)^\ell$ and $\upsilon_{ih+j} = \zeta^{j-1}\epsilon_{i+1}^{1/h}$ ($0 \leq i \leq \ell - 1$, $1 \leq j \leq h$).

From $|\upsilon_1| = |\upsilon_2| = |\epsilon|^{1/h} < 1$ and $|\upsilon_{d-1}| = |\upsilon_d| = |\epsilon_\ell|^{1/h}$ we deduce

$$\mu = \left|\epsilon_\ell \epsilon_1^{-1/h}\right| = \left|\upsilon_d / \upsilon_1\right|$$

and using (7) we conclude

$$\log \mu \geq \frac{2}{d-1} \log \lambda. \quad \Box$$

A variant of this proof is to take $\alpha = 1$, $\lambda_0 = 1$, $F_0(X, Y) = (X - Y)^d$, and to use the fact that $\zeta^a$ is also a primitive $h$th root of unity since $F_a$ is irreducible.

Remark. There are cases where $\mu$ is very small when compared to $\lambda$. Let $D$ be an integer $\geq 2$. Consider the algebraic number field $K = \mathbb{Q}(\omega)$ where $\omega = \sqrt[d]{D^d - 1}$. The number $\nu = D - \omega$ is a Bernstein–Hasse unit of $K$. When $d$ is fixed, $\lambda$ is larger than $\kappa_9 D^{d-1}$, while $\mu$ is bounded above by $\kappa_{10}$. In this example, when $d$ is odd, the field $K$ is almost totally imaginary in the sense of [9] and our proof yields the estimate (4). However, when $d$ is even, we are not able to prove the estimate (4); the estimate (3) has one extra factor $\log \lambda$.

§3. Auxiliary results.

3.1. Mahler measure, house and height. When $f$ is a polynomial in one variable of degree $d$ with coefficients in $\mathbb{Z}$ and leading coefficient $c_0 > 0$, the Mahler measure of $f$ is

$$M(f) = c_0 \prod_{i=1}^d \max\{1, |\gamma_i|\},$$

where $\gamma_1, \gamma_2, \ldots, \gamma_d$ are the roots of $f$ in $\mathbb{C}$.

Let us recall that the logarithmic height $h(\gamma)$ of an algebraic number $\gamma$ of degree $d$ is $(1/d) \log_M(\gamma)$ where $M(\gamma)$ is the Mahler measure of the irreducible polynomial of $\gamma$. We have

$$M(f) \leq \sqrt{d + 1} H(f) \quad \text{and} \quad H(f) \leq 2^d M(f) \quad (6)$$

(see [12, Annex to Ch. 3], Inequalities Between Different Heights of a Polynomial, pp. 113–114; see also [2, §1.9]). The second upper bound in (6) could be replaced by the sharper one

$$H(f) \leq \left(\frac{d}{\lfloor d/2 \rfloor}\right) M(f),$$

but we shall not need it.

1 Our $h$ is the same as in [2], it corresponds to the logarithm of the $h$ in [1].
Let \( \upsilon \) be a unit of degree \( d \) and conjugates \( \upsilon_1, \ldots, \upsilon_d \) with

\[
|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_d|,
\]

so that \( |\upsilon| = |\upsilon_d| \). Let \( \lambda = M(\upsilon) \) and let \( s \) be an index in \( \{1, \ldots, d-1\} \) such that

\[
|\upsilon_1| \leq |\upsilon_2| \leq \cdots \leq |\upsilon_s| \leq 1 \leq |\upsilon_{s+1}| \leq \cdots \leq |\upsilon_d|.
\]

We have

\[
\lambda = M(\upsilon) = |\upsilon_{s+1} \cdots \upsilon_d| \leq |\upsilon_d|^{d-s} \leq |\upsilon_d|^{d-1}
\]

and

\[
M(\upsilon^{-1}) = |\upsilon_1 \cdots \upsilon_s|^{-1} = M(\upsilon) = \lambda
\]

with

\[
\lambda \leq |\upsilon_1|^{-s} \leq |\upsilon_1|^{-(d-1)}.
\]

Therefore we have

\[
\lambda^{1/(d-1)} \leq |\upsilon_d| \leq \lambda \quad \text{and} \quad \lambda^{-1} \leq |\upsilon_1| \leq \lambda^{-1/(d-1)}.
\] (7)

3.2. An elementary result. For the convenience of the reader, we include the following elementary result—similar arguments are often used without explicit mention in the literature.

**Lemma 5.** Let \( U \) and \( V \) be positive numbers satisfying \( U \leq V \log^* U \). Then \( U < 2V \log^* V \).

**Proof.** If \( \log U \leq 1 \), the assumption is \( U \leq V \) and the conclusion follows. Assume \( \log U > 1 \). Then \( \log U \leq \sqrt{U} \), hence the hypothesis of the lemma implies \( U \leq V \sqrt{U} \) and therefore we have \( U \leq V^2 \). We deduce

\[
\log U \leq 2 \log V,
\]

hence

\[
U \leq V \log U \leq 2V \log V.
\]

3.3. Diophantine tool. In this section only, the positive integer \( d \) is not restricted to \( d \geq 3 \).

The main tool is the following Diophantine estimate ([6, Proposition 2], [12, Theorem 9.1] or [2, Theorem 3.2.4]), the proof of which uses transcendental number theory.

**Proposition 6.** Let \( s \) and \( D \) be two positive integers. There exists an effectively computable positive constant \( \kappa(s, D) \), depending only upon \( s \) and \( D \), with the following property. Let \( \eta_1, \ldots, \eta_s \) be non-zero algebraic numbers generating a number field of degree \( \leq D \). Let \( c_1, \ldots, c_s \) be rational integers and let \( H_1, \ldots, H_s \) be real numbers \( \geq 1 \) satisfying

\[
H_i \geq h(\eta_i) \quad (1 \leq i \leq s).
\]

Let \( C \) be a real number with \( C \geq 2 \). Suppose that one of the following two statements is true:
(i) \[ C \geq \max_{1 \leq j \leq s} |c_j| \]

or

(ii) \[ H_j \leq H_s \text{ for } 1 \leq j \leq s \text{ and } \]

\[ C \geq \max_{1 \leq j \leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}. \]

Suppose also \( \eta_1^{c_1} \cdots \eta_s^{c_s} \neq 1 \). Then

\[ |\eta_1^{c_1} \cdots \eta_s^{c_s} - 1| > \exp\{ -\kappa(s, D) H_1 \cdots H_s \log C \}. \]

The statement (ii) of Proposition 6 implies the statement (i) by permuting the indices so that \( H_j \leq H_s \) for \( 1 \leq j \leq s \); however, we find it more convenient to use part (i) so that we can use the estimate without permuting the indices.

We will use Proposition 6 several times. Here is a first consequence.

**Corollary 7.** Let \( d \geq 1 \). There exists an effectively computable constant \( \kappa_{11} \), which depends only on \( d \), with the following property. Let \( K \) be a number field of degree \( d \). Let \( \alpha_1, \alpha_2, \upsilon_1, \upsilon_2 \) be non-zero elements in \( K \) and let \( a \) be a non-zero integer. Set \( \gamma_1 = \alpha_1 \upsilon_1^a \) and \( \gamma_2 = \alpha_2 \upsilon_2^a \). Let \( \lambda_0 \) and \( \lambda \) satisfy

\[ \max\{h(\alpha_1), h(\alpha_2)\} \leq \log \lambda_0, \quad \max\{h(\upsilon_1), h(\upsilon_2)\} \leq \log \lambda \]

and assume \( \gamma_1 \neq \gamma_2 \). Define

\[ \chi = (\log^* \lambda)(\log^* \lambda) \log^* \left( |a| \min \left\{ 1, \frac{\log^* \lambda}{\log^* \lambda_0} \right\} \right). \]

Then

\[ |\gamma_1 - \gamma_2| \geq \max\{|\gamma_1|, |\gamma_2|\} e^{-\kappa_{11} \chi}. \]

**Proof.** By symmetry, without loss of generality, we may assume \( |\gamma_2| \geq |\gamma_1| \).

Set

\[ s = 2, \quad \eta_1 = \frac{\upsilon_1}{\upsilon_2}, \quad \eta_2 = \frac{\alpha_1}{\alpha_2}, \quad c_1 = a, \quad c_2 = 1, \]

\[ H_1 = 2 \log^* \lambda, \quad H_2 = 2 \log^* \lambda_0, \quad C = \max \left\{ 2, |a| \min \left\{ 1, \frac{H_1}{H_2} \right\} \right\}. \]

The conclusion of Corollary 7 follows from Proposition 6 (via part (i) if \( H_1 \geq H_2 \), via part (ii) otherwise), thanks to the relation

\[ |\eta_1^{c_1} \eta_2^{c_2} - 1| = |\gamma_2|^{-1} |\gamma_1 - \gamma_2|. \]
3.4. **Lower bound for the height and the regulator.** For the record, we quote Kronecker’s theorem and its effective improvement.

**Lemma 8.**

(a) If a non-zero algebraic integer $\alpha$ has all its conjugates in the closed unit disc $\{z \in \mathbb{C} \mid |z| \leq 1\}$, then $\alpha$ is a root of unity.

(b) More precisely, given $d \geq 1$, there exists an effectively computable positive constant $\kappa_{12}$, depending only on $d$, such that, if $\alpha$ is a non-zero algebraic integer of degree $d$ satisfying $h(\alpha) < \kappa_{12}$, then $\alpha$ is a root of unity.

**Proof.** Voutier (1996) refined an earlier estimate due to Dobrowolski (1979) by proving that the conclusion of part (b) in Lemma 8 holds with

$$
\kappa_{12} = \begin{cases} 
\log 2 & \text{if } d = 1, \\
\frac{2}{d \log(3d)^3} & \text{if } d \geq 2. 
\end{cases}
$$

See for instance [2, Proposition 3.2.9] and [12, §3.6].

**Lemma 9.** There exists an explicit absolute constant $\kappa_{13} > 0$ such that the regulator $R$ of any number field of degree $\geq 2$ satisfies $R > \kappa_{13}$.

**Proof.** According to a result of Friedman (1989—see [2, (1.5.3)]) the conclusion of Lemma 9 holds with $\kappa_{13} = 0.2052$.

3.5. **A basis of units of an algebraic number field.** Let us quote Lemma 1 of [1]. See also [2, Proposition 4.3.9]. The result is essentially due to Siegel [11].

**Proposition 10.** Let $d$ be a positive integer with $d \geq 3$. There exist effectively computable constants $\kappa_{14}, \kappa_{15}, \kappa_{16}$ depending only on $d$, with the following property. Let $K$ be a number field of degree $d$, with unit group of rank $r$. Let $R$ be the regulator of this field. Denote by $\varphi_1, \varphi_2, \ldots, \varphi_r$ a set of $r$ embeddings of $K$ into $\mathbb{C}$ containing the real embeddings and no pair of conjugate embeddings. Then there exists a fundamental system of units $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_r\}$ of $K$ which satisfies the following:

(i) $\prod_{1 \leq i \leq r} h(\epsilon_i) \leq \kappa_{14} R$;

(ii) $\max_{1 \leq i \leq r} h(\epsilon_i) \leq \kappa_{15} R$;

(iii) the absolute values of the entries of the inverse matrix of

$$(\log |\varphi_j(\epsilon_i)|)_{1 \leq i, j \leq r}$$

do not exceed $\kappa_{16}$.

The next result is [10, Lemma A.15].
Lemma 11. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be an independent system of units for $K$ satisfying the condition (ii) of Proposition 10. Let $\beta \in \mathbb{Z}_K$ with $N_{K/\mathbb{Q}}(\beta) = m \neq 0$. Then there exist $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ and $\tilde{\beta} \in \mathbb{Z}_K$ with conjugates $\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_d$, satisfying

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{17} R} \leq |\tilde{\beta}_j| \leq |m|^{1/d} e^{\kappa_{17} R} \quad \text{for } j = 1, \ldots, d.$$  

The conclusion of Lemma 11 can be written

$$|\log(|m|^{-1/d} |\tilde{\beta}_j|)| \leq \kappa_{17} R \quad \text{for } j = 1, \ldots, d.$$  

3.6. Estimates for the conjugates.

Lemma 12. Let $\gamma$ be an algebraic number of degree $d \geq 3$. Let $\gamma_1, \gamma_2, \ldots, \gamma_d$ be the conjugates of $\gamma$ with $|\gamma_1| \leq |\gamma_2| \leq \cdots \leq |\gamma_d|$.

(a) If $|\gamma_1| < |\gamma_2|$ and $\gamma_2 \in \mathbb{R}$, then $|\gamma_2| < |\gamma_3|$.

(b) If $|\gamma_{d-1}| < |\gamma_d|$ and $\gamma_{d-1} \in \mathbb{R}$, then $|\gamma_{d-2}| < |\gamma_{d-1}|$.

Proof. (a) The conditions $|\gamma_1| < |\gamma_2| \leq |\gamma_i|$ for $3 \leq i \leq d$ imply that $\gamma_i$ is real and that $-\gamma_1$ is not a conjugate of $\gamma_1$. Hence the minimal polynomial of $\gamma$ is not a polynomial in $X^2$. Assume $|\gamma_2| = |\gamma_3|$. Since $-\gamma_2$ is not a conjugate of $\gamma_2$, we deduce $\gamma_3 \notin \mathbb{R}$, hence $d \geq 4$. We may assume $\gamma_4 = \gamma_3$. Let $\sigma$ be an automorphism of $\overline{\mathbb{Q}}$ which maps $\gamma_2$ to $\gamma_1$; via $\sigma$, let $\gamma_j$ be the image of $\gamma_3$ and $\gamma_k$ the image of $\gamma_4$. From

$$\gamma_2^2 = \gamma_3 \gamma_4$$

we deduce $\gamma_2^2 = \gamma_j \gamma_k$ and $|\gamma_2|^2 = |\gamma_j \gamma_k|$. This is not possible since $|\gamma_j| > |\gamma_1|$ and $|\gamma_k| > |\gamma_1|$.

(b) We deduce (b) from (a), by using $\gamma \mapsto 1/\gamma$ (or by repeating the proof, mutatis mutandis). \hfill \Box

Remark. Here is an example showing that the assumptions of Lemma 12 are sharp. The polynomial $X^4 - 4X^2 + 1$ is irreducible, its roots are

$$u_1 = \sqrt{2 - \sqrt{3}}, \quad u_2 = -u_1, \quad u_3 = 1/u_1 = \sqrt{2 + \sqrt{3}}, \quad u_4 = -u_3$$

with

$$u_1 = |u_2| < u_3 = |u_4|.$$  

More generally, if $h \geq 2$ is a positive integer and $\epsilon$ is a quadratic unit with Galois conjugate $\epsilon'$ and if $\epsilon^{1/h}$ has degree $2h$, then it has $h$ conjugates of absolute value $|\epsilon|^{1/h}$ and $h$ conjugates of absolute value $|\epsilon'|^{1/h}$. See also §2.

Lemma 13. Let $\nu$ be an algebraic unit of degree $d \geq 3$. Set $\lambda = M(\nu)$. Let $\nu'$ and $\nu''$ be two conjugates of $\nu$ with $|\nu'| < |\nu''|$. Then

$$\log \left| \frac{|\nu''|}{|\nu'|} \right| \geq \kappa_{18} \lambda^{-(d^3 + 2d^2 - d + 2)/2}.$$
We will deduce Lemma 13 from [3, Theorem 1] which states the following.

**Lemma 14 (Gourdon and Salvy [3]).** Let $P$ be a polynomial of degree $d \geq 2$ with integer coefficients and with Mahler measure $M(P)$. If $\alpha'$ and $\alpha''$ are two roots of $P$ with $|\alpha'| < |\alpha''|$, then

$$|\alpha''| - |\alpha'| \geq \kappa_{19} M(P)^{-d(d^2+2d-1)/2}$$

with

$$\kappa_{19} = \frac{\sqrt{3}}{2} \left( \frac{d(d+1)/2}{d+1} \right)^{d(d+1)/4-1}.$$

**Proof of Lemma 13.** We apply Lemma 14 to the minimal polynomial of $\nu$. To conclude the proof of Lemma 13, we use the bounds $|\nu'| \leq \lambda$ and

$$\log(1 + x) \geq \frac{x}{2} \quad \text{for} \ 0 \leq x \leq 1 \ \text{with} \ x = \frac{|\nu'|}{|\nu|} - 1. \quad \square$$

§4. **Proof of Theorem 3.** In order to prove Theorem 3 with the assumption $|F_a(x, y)| \leq m$, it suffices to consider the equation $F_a(x, y) = m$ with $m \neq 0$.

Let $(a, x, y, m) \in \mathbb{Z}^4$ satisfy $m \neq 0$, $xy \neq 0$, $[\mathbb{Q}(\alpha v^a) : \mathbb{Q}] = d$ and $F_a(x, y) = m$.

Without loss of generality, we may restrict $(a, y)$ to $a \geq 0$ (otherwise, replace $\nu$ by $\nu^{-1}$) and to $y > 0$ (otherwise replace $F_a(X, Y)$ by $F_a(X, -Y)$).

The form $\bar{F}_a(X, Y) = a_0^{d-1} F_a(X, Y)$ has coefficients in $\mathbb{Z}$, and if we set $\bar{x} = a_0 x$, $\bar{y} = y$, $\bar{m} = a_0^{d-1} m$ we have $\bar{F}_a(\bar{x}, \bar{y}) = \bar{m}$ with $(\bar{x}, \bar{y}) \in \mathbb{Z}^2$. Therefore, there is no loss of generality to assume $a_0 = 1$.

Theorem 3 includes the assumption that $\nu$ is not a root of unity, hence $\lambda > 1$. More precisely, it follows from part (b) of Lemma 8 that

$$\log \lambda \geq \kappa_{12}.$$

In particular, we have

$$\log^* \lambda \leq \max \left\{ 1, \frac{1}{\kappa_{12}} \right\} \log \lambda,$$

an inequality which can be written

$$\log^* \lambda \leq \kappa_{20} \log \lambda \quad (8)$$

with an effectively computable constant $\kappa_{20} > 0$.

From Lemma 9, we deduce that $R > \kappa_{13}$. Therefore, there is no loss of generality to assume that, for a sufficiently large constant $\kappa_{21}$, we have

$$a \geq \kappa_{21} (\log |m| + (\log^* \lambda_0) \log^* \log \lambda). \quad (9)$$

This hypothesis will frequently be used, sometimes without explicit mention.

---

2 This reference was kindly suggested to us by Bugeaud.
By assumption, $\mathbb{Q}(\alpha \nu^a) = K$. If some conjugate $\sigma_j(\alpha \nu^a)$ of $\alpha \nu^a$ is real, then it follows that $\sigma_j(K) \subset \mathbb{R}$, hence the embedding $\sigma_j$ is real, and $\alpha$ and $\nu_j$ are both real. We also notice that if $\sigma_j(\nu) = -\sigma_i(\nu)$ with $i \neq j$, then it follows that $\nu$ and $-\nu$ are conjugate, hence the irreducible polynomial of $\nu$ belongs to $\mathbb{Z}[X^2]$.

Recall that $\nu_i = \sigma_i(\nu)$ ($i = 1, \ldots, d$) and that 

$$|\nu_1| \leq |\nu_2| \leq \cdots \leq |\nu_d|.$$ 

Let us write $\alpha_i$ for $\sigma_i(\alpha)$ ($i = 1, \ldots, d$). Let 

$$\gamma = \alpha \nu^a$$ 

and $$\beta = x - \gamma y.$$ 

Since $a_0 = 1$, it follows that $\alpha$, $\beta$ and $\gamma$ are algebraic integers in $K$. For $j = 1, 2, \ldots, d$, define $\gamma_j$ and $\beta_j$ by 

$$\gamma_j = \sigma_j(\gamma) = \alpha_j \nu_j^a, \quad \beta_j = \sigma_j(\beta) = x - \alpha_j \nu_j^a y = x - \gamma_j y.$$ 

The assumption $F_a(x, y) = m$ yields $\beta_1 \beta_2 \ldots \beta_d = m$. Let $i_0 \in \{1, 2, \ldots, d\}$ be an index such that 

$$|\beta_{i_0}| = \min_{1 \leq i \leq d} |\beta_i|.$$ 

We define $\Psi_1, \Psi_2, \ldots, \Psi_d$ by the following conditions:

$$\beta_i = \begin{cases} 
\gamma_i y \Psi_i & \text{for } 1 \leq i < i_0, \\
\gamma_i y \Psi_i & \text{for } i_0 < i \leq d,
\end{cases}$$ 

and 

$$\beta_{i_0} = \frac{m}{\gamma d^{-1}} \frac{\gamma_1 \gamma_2 \cdots \gamma_{i_0-1}}{\gamma_{i_0}^{i_0-2}} \Psi_{i_0}.$$ 

We split the proof into several steps.

**Step 1.** We start by proving that 

$$|x| \leq 2\lambda_0 \lambda^a y$$ (10) 

and that there exists an effectively computable positive constant $\kappa_{22}$ depending only on $d$ such that 

$$e^{-\kappa_{22}x} \leq |\Psi_i| \leq e^{\kappa_{22}x} \quad (i = 1, 2, \ldots, d)$$ (11) 

with 

$$\chi = (\log^* \lambda_0)(\log \lambda) \log \left( a \min \left\{ 1, \frac{\log \lambda}{\log^* \lambda_0} \right\} \right).$$ 

From the estimate (11) we will deduce 

$$|\beta_{i_0}| < |\beta_i|$$ 

for $i \neq i_0$, which implies $\alpha_{i_0} \in \mathbb{R}$ and $\nu_{i_0} \in \mathbb{R}$. 

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Remark. The estimate (11) can be written as follows:

$$|\log(|\beta_i| y^{-1} \max(|\gamma_i^{-1}|, |\gamma_{i_0}^{-1}|))| \leq \kappa_{22} \chi$$

for $i \neq i_0$ and

$$|\log\left(|\beta_{i_0}| \frac{y^{d-1}}{|m|} \cdot |\gamma_1^{-1} \cdots \gamma_{i_0-1}^{-1} \gamma_{i_0-2}^{-1}| \right) | \leq \kappa_{22} \chi.$$

Proof of (10) and (11). We have

$$|x| = |\beta_{i_0} + \gamma_{i_0} y| \leq |\beta_{i_0}| + |\gamma_{i_0}| y. \quad (12)$$

From $|\beta_{i_0}| \leq |\beta_i|$ for $i = 1, 2, \ldots, d$ and $\beta_1 \cdots \beta_d = m$, we deduce $|\beta_{i_0}| \leq |m|^{1/d}$, hence

$$|x| \leq |m|^{1/d} + |\gamma_{i_0}| y \leq |m|^{1/d} + \lambda_0 \lambda^a y.$$

Using the assumption (9), we check $|m|^{1/d} \leq \lambda_0 \lambda^a y$, whereupon the inequality (10) is secured.

We also have

$$|\beta_{i_0}|^{d-1} \max_{1 \leq i \leq d} |\beta_i| \leq |m|. \quad (13)$$

For $i = 1, 2, \ldots, d$, we write

$$\beta_i = \beta_{i_0} + y(\gamma_{i_0} - \gamma_i). \quad (14)$$

We have

$$|\alpha_1 \alpha_2 \cdots \alpha_d| \geq 1$$

(recall $a_0 = 1$), hence

$$\frac{1}{\lambda_0} \leq |\alpha_i| \leq \lambda_0 \quad \text{for } i = 1, 2, \ldots, d.$$

We choose an index $j_0 \neq i_0$ as follows.

- If $|\nu_{i_0}| \leq \lambda^{1/(2(d-1))}$, we take $j_0 = d$ so that, with the help of (7), we have $|\nu_{j_0}| \geq \lambda^{1/(d-1)}$, whereupon with the help of (9) we obtain

$$\frac{|\gamma_{i_0}|}{|\gamma_{j_0}|} < \frac{1}{2}.$$

- If $|\nu_{i_0}| > \lambda^{1/(2(d-1))}$, we take $j_0 = 1$ so that, again with the help of (7), we have $|\nu_{j_0}| \leq \lambda^{-1/(d-1)}$, whereupon with the help of (9) we obtain

$$\frac{|\gamma_{j_0}|}{|\gamma_{i_0}|} < \frac{1}{2}.$$
In both cases, we deduce
\[ \|\gamma_j - \gamma_i\| \geq \frac{1}{2} \max\{\|\gamma_j\|, \|\gamma_i\|\} \geq \frac{\lambda a/(2(d-1))}{2\lambda_0} \]
and therefore, using (9) again together with (13) and (14), we obtain
\[ |\beta_j| \geq |\gamma_j - \gamma_i| y - |\beta_i| \geq \frac{\lambda a/(2(d-1)) y}{2\lambda_0} - |m|^{1/d} \geq \lambda a/(2d) y. \]

Since \( \max_{1 \leq i \leq d} |\beta_i| \geq \lambda a/(2d) y \), from (13) we deduce
\[ |\beta_i| \leq \left( \frac{|m|}{y \lambda a/(2d)} \right)^{1/(d-1)}. \] (15)

In particular, thanks to (9), we have
\[ |\beta_i| \leq \frac{1}{2}. \] (16)

Using the assumption \( |x| \geq 1 \) together with (12), we deduce
\[ \frac{|x|}{2} \leq |\gamma_i| y \leq |x| + |\beta_i| \leq \frac{3|x|}{2}. \] (17)

Let \( i \neq i_0 \). The upper bound
\[ |\gamma_i - \gamma_i_0| \leq 2 \max\{ |\gamma_i_0|, |\gamma_i| \} \]
is trivial, while the lower bound
\[ |\gamma_i - \gamma_i_0| \geq \max\{ |\gamma_i_0|, |\gamma_i| \} e^{-\kappa_{23} \chi} \] (18)
follows from (8) and from Corollary 7. We first use the lower bound
\[ |\gamma_i - \gamma_i_0| \geq |\gamma_i_0| e^{-\kappa_{23} \chi}. \]

Using (17), we obtain
\[ |\gamma_i - \gamma_i_0| \geq \frac{1}{2} e^{-\kappa_{23} \chi} \geq \frac{2}{y} e^{-\kappa_{24} \chi} \] (19)
with \( \kappa_{24} > 0 \). Using the contrapositive of Lemma 5 with
\[ U = a \frac{\log \lambda}{\log \lambda_0}, \quad V = \frac{1}{\kappa_{25}} \log \lambda, \]
we deduce from (9) that
\[ \chi \leq \kappa_{25} a \log \lambda. \]
Recall that $\kappa_{21}$ is sufficiently large, hence $\kappa_{25}$ is sufficiently small. Now from (15), the inequality $|m| \leq e^{a|1/\kappa_{21}}$ and (19) we deduce

$$|\beta_{i0}| \leq |m|^{1/(d-1)} \lambda^{-a/(2d(d-1))} \leq \lambda^{-\kappa_{26}a} \leq e^{-\kappa_{24}x} \leq \frac{1}{2} y|\gamma_i - \gamma_{i0}|.$$ 

Therefore, for $i \neq i_0$, using (14), we deduce

$$\frac{1}{2} y|\gamma_i - \gamma_{i0}| \leq |\beta_i| \leq \frac{3}{2} y|\gamma_i - \gamma_{i0}|.$$

Using once more (18), we obtain (11) for $i \neq i_0$. We also deduce

$$|\beta_{i0}| > \lambda^{-a/(2d)}$$

for $i \neq i_0$. (20)

Recall

$$N(\gamma) = \gamma_1 \gamma_2 \cdots \gamma_d = N(\alpha)N(\nu)^a = \pm N(\alpha)$$

and

$$N(\beta) = \beta_1 \beta_2 \cdots \beta_d = m.$$ 

The estimate (11) for $i = i_0$ follows from the relations

$$\frac{m}{\beta_{i0}} = \prod_{i \neq i_0} \beta_i = y^{d-1} \gamma_{i0}^{-1} \gamma_{i0+1} \cdots \gamma_d \prod_{i \neq i_0} \Psi_i$$

and

$$\frac{N(\gamma)}{\gamma_{i0}^{-1} \gamma_{i0+1} \cdots \gamma_d} = \frac{\gamma_1 \cdots \gamma_{i0-1}}{\gamma_{i0}^{i-2}}.$$ 

From (9) and (11), we deduce

$$|\beta_{i0}| < \frac{|m|}{y^{d-1}} |\gamma_1| e^{\kappa_{22}x} \leq \lambda^{-a/(2d)},$$

hence from (20) we infer $|\beta_{i0}| < |\beta_i|$ for $i \neq i_0$. It follows that $\beta_{i0}$ is real, and therefore $\gamma_{i0}, \alpha_{i0}$ and $\nu_{i0}$ also.

Step 2. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be a basis of the group of units of $K$ given by Proposition 10. From Lemma 11, it follows that there exists $\tilde{\beta} \in \mathbb{Z}_K$ and $b_1, b_2, \ldots, b_r$ in $\mathbb{Z}$ with

$$\beta = \tilde{\beta} \epsilon_1^{b_1} \epsilon_2^{b_2} \cdots \epsilon_r^{b_r}$$

and

$$|m|^{1/d} e^{-\kappa_{17}R} \leq |\tilde{\beta}| \leq |m|^{1/d} e^{\kappa_{17}R}$$

for $i = 1, 2, \ldots, d$.

We set

$$B = \kappa_{27}(R + a \log \lambda + \log y)$$

(21) with a sufficiently large constant $\kappa_{27}$. We want to prove that

$$\max_{1 \leq i \leq r} |b_i| \leq B.$$
Proof. We consider the system of $d$ linear forms in $r$ variables with real coefficients

$$L_i(X_1, X_2, \ldots, X_r) = \sum_{j=1}^{r} X_j \log |\sigma_i(\epsilon_j)|, \quad (i = 1, 2, \ldots, d).$$

The rank is $r$. By Proposition 10(ii),

$$\log |\sigma_i(\epsilon_j)| \leq \kappa_{28} R.$$ 

For $i = 1, 2, \ldots, d$, define $e_i = L_i(b_1, b_2, \ldots, b_r)$. We have

$$e_i = \log |\sigma_i(\beta/\tilde{\beta})| = \log |\beta_i/\tilde{\beta}_i|,$$

hence, using the inequality $|m| \leq e^{a/\kappa_{29}}$ and (11), we deduce

$$|e_i| \leq \kappa_{29}(R + a \log \lambda + \log y).$$

Computing $b_1, b_2, \ldots, b_r$ by means of the system of linear equations

$$L_i(b_1, b_2, \ldots, b_r) = e_i \quad (i = 1, 2, \ldots, d)$$

and using Proposition 10(iii), we deduce

$$\max_{1 \leq j \leq r} |b_j| \leq \kappa_{30} \max_{1 \leq i \leq d} |e_i| \leq B.$$ 

Step 3. From the inequality (3.2) in [1, Theorem 3] (see also [2, Theorem 9.6.2]), thanks to (9), we deduce the following upper bound for $|x|$ and $|y|$ in terms of $a$, $\lambda$, $\lambda_0$, $m$ and $R$: there exists a positive effectively computable constant $\kappa_{31}$ depending only on $d$ such that

$$\log \max\{|x|, y\} \leq \kappa_{31} R(\log^* R)(R + a \log \lambda). \quad (22)$$

Step 4. Assume $c\gamma_i\beta_j \neq \gamma_k\beta_\ell$ for some indices $i, j, k, \ell$ in $\{1, \ldots, d\}$ and some $c \in \{1, -1\}$. Then there exists $\kappa_{32} > 0$ such that

$$\left| \frac{c\gamma_i\beta_j}{\gamma_k\beta_\ell} - 1 \right| \geq \exp \left\{ -\kappa_{32}(\log \lambda)(R + \log |m| + \log \lambda_0 + \log \lambda)R \right. 
\times \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right) \right\}.$$ 

Proof. This lower bound follows from Proposition 6(ii) with

$$\frac{c\gamma_i\beta_j}{\gamma_k\beta_\ell} = \eta_1^{c_1} \eta_2^{c_2} \cdots \eta_s^{c_s},$$
where \( s = r + 2 \) and

\[
\eta_t = \frac{\sigma_j(\epsilon_t)}{\sigma_\ell(\epsilon_t)} (1 \leq t \leq r), \quad \eta_{r+1} = \frac{\sigma_i(\nu)}{\sigma_k(\nu)}, \quad \eta_{r+2} = \frac{c \sigma_j(\tilde{\beta}) \sigma_i(\alpha)}{\sigma_\ell(\tilde{\beta}) \sigma_k(\alpha)},
\]

\[
c_t = b_t (1 \leq t \leq r), \quad c_{r+1} = a, \quad c_{r+2} = 1,
\]

\[
H_t = \max\{1, 2h(\epsilon_t)\} (1 \leq t \leq r), \quad H_{r+1} = \max\{1, 2 \log \lambda\}, \quad H_{r+2} = \kappa_{33} (R + \log |m| + \log \lambda_0 + \log \lambda),
\]

\[
C = 2 + \frac{2a \log \lambda + RB}{H_{r+2}}.
\]

Using Proposition 10(i) together with part (b) of Lemma 8, we deduce

\[
H_1 H_2 \cdots H_r \leq \kappa_{34} R.
\]

Finally we deduce from the steps 2 and 3 that

\[
\log C \leq \kappa_{35} \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right),
\]

and this secures the lower bound for \(|c(\gamma_1 \beta_j / \gamma_k \beta_l) - 1|\) announced above. \(\square\)

**Step 5.** We will prove Theorem 3 by assuming

\[
\max_{1 \leq i \leq d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} > \mu^{a/4}.
\]

Using (11), we deduce from our assumption

\[
a \leq 4\kappa_{22} \log \mu < \kappa_{22} \chi,
\]

hence

\[
a \leq \frac{4\kappa_{22}}{\log \mu} (\log^* \lambda_0) (\log^* \lambda) \log^* \left( \frac{a \log^* \lambda}{\log^* \lambda_0} \right).
\]

With

\[
U = \frac{a \log^* \lambda}{\log^* \lambda_0} \quad \text{and} \quad V = \frac{4\kappa_{22} (\log^* \lambda)^2}{\log \mu},
\]

we have \( U \leq V \log^* U \), and we conclude that we can use Lemma 5 to deduce

\[
a \leq \frac{8\kappa_{22} (\log^* \lambda_0) (\log^* \lambda)}{\log \mu} \log \left( \frac{4\kappa_{22} (\log^* \lambda)^2}{\log \mu} \right),
\]

and the conclusion of Theorem 3 follows.

In the rest of the paper, we assume

\[
\max_{1 \leq i \leq d} \max\{|\Psi_i|, |\Psi_i|^{-1}\} \leq \mu^{a/4}. \tag{23}
\]
Step 6. Our next goal is to prove the following results.

(a) Assume \(1 \leq i_0 \leq d - 2\) and

\[
\frac{|\nu_{d-1}|}{|\nu_{i_0}|} \geq \sqrt{\mu}.
\]

Then

\[
0 < \left| \frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1 \right| \leq 4\lambda_0^2\mu^{-a/4}.
\]

(b) Assume \(3 \leq i_0 \leq d\) and

\[
\frac{|\nu_{i_0}|}{|\nu_{2}|} \geq \sqrt{\mu}.
\]

Then

\[
0 < \left| \frac{\beta_1}{\beta_2} - 1 \right| \leq 2\lambda_0^2\mu^{-a/4}.
\]

(c) Assume \(2 \leq i_0 \leq d - 1\) and

\[
\min\left\{ \frac{|\nu_{i_0}|}{|\nu_{1}|}, \frac{|\nu_{d}|}{|\nu_{i_0}|} \right\} \geq \mu.
\]

Then

\[
\left| \frac{\gamma_{i_0}\beta_d}{\gamma_d\beta_1} + 1 \right| \leq 4|m|\lambda_0^4\mu^{-a/2}.
\]

Proof. (a) We approximate \(\beta_d\) by \(-\gamma_d y\), \(\beta_{d-1}\) by \(-\gamma_{d-1} y\) and we eliminate \(y\). Since \(\gamma\) has degree \(d\), we have

\[
\beta_d \gamma_{d-1} - \beta_{d-1} \gamma_d = x(\gamma_{d-1} - \gamma_d) \neq 0.
\]

From (17) we deduce \(|x| \leq 2|\gamma_{i_0} y|\) and

\[
|\beta_d \gamma_{d-1} - \beta_{d-1} \gamma_d| \leq 2|\gamma_{i_0}|(|\gamma_d| + |\gamma_{d-1}|) y.
\]

Using \(\beta_{d-1} = \gamma_{d-1} y \Psi_{d-1}\) together with the assumption

\[
|\nu_d| \geq |\nu_{d-1}| \geq \sqrt{\mu}|\nu_{i_0}|,
\]

we deduce

\[
\left| \frac{\gamma_{d-1}\beta_d}{\gamma_d\beta_{d-1}} - 1 \right| \leq \frac{2|\gamma_{i_0}|(|\gamma_{d-1}| + |\gamma_d|)}{|\gamma_{d-1}\gamma_d|} |\Psi_{d-1}|^{-1} \leq 4\lambda_0^2\mu^{-a/2} |\Psi_{d-1}|^{-1}.
\]

The conclusion of (a) follows from (23).

(b) We approximate \(\beta_1\) and \(\beta_2\) by \(x\) and we eliminate \(x\). Since \(\gamma_1 \neq \gamma_2\), we have

\[
|\beta_1 - \beta_2| = |(\gamma_2 - \gamma_1) y| \neq 0.
\]
From $\beta_2 = \gamma_{i_0}y\Psi_2$ and the assumption

$$|\upsilon_1| \leq |\upsilon_2| \leq \mu^{-1/2}|\upsilon_{i_0}|,$$

we deduce

$$\left| \frac{\beta_1}{\beta_2} - 1 \right| \leq \frac{|\gamma_2| + |\gamma_1|}{|\gamma_{i_0}|} |\Psi_2|^{-1} \leq 2\lambda_0^2 \mu^{-a/2} |\Psi_2|^{-1}.$$ 

Again, the conclusion of (b) follows from (23).

(c) We approximate $\beta_1$ by $x$, $\beta_d$ by $-y\gamma_d$ and $x$ by $y\gamma_{i_0}$, then we eliminate $x$ and $y$. More precisely we have

$$\beta_1 \gamma_d + \beta_d \gamma_{i_0} = (\gamma_d + \gamma_{i_0}) \beta_{i_0} + \gamma_{i_0}^2 y - \gamma_1 y \gamma_d.$$ 

Hence

$$\frac{\gamma_{i_0} \beta_d}{\gamma_d \beta_1} + 1 = \frac{(\gamma_d + \gamma_{i_0}) \beta_{i_0}}{\gamma_d \beta_1} + \frac{\gamma_{i_0}^2 y}{\gamma_d \beta_1} - \frac{\gamma_1 y}{\beta_1}.$$ 

We have $\beta_1 = \gamma_{i_0} y \Psi_1$. Therefore we have

$$\frac{|\gamma_{i_0}|^2 y}{|\gamma_d \beta_1|} = \frac{|\gamma_{i_0}|}{|\gamma_d|} |\Psi_1|^{-1} \leq \lambda_0^2 \frac{|\upsilon_{i_0}|}{|\upsilon_d|} |\Psi_1|^{-1}$$

and

$$\frac{|\gamma_1| y}{|\beta_1|} = \frac{|\gamma_1|}{|\gamma_{i_0}|} |\Psi_1|^{-1} \leq \lambda_0^2 \frac{|\upsilon_1|}{|\upsilon_{i_0}|} |\Psi_1|^{-1}.$$ 

Finally, from

$$|\beta_{i_0}| \leq \frac{|m|}{y_{d-1}} |\gamma_1 \Psi_{i_0}|$$

we deduce

$$\frac{(|\gamma_d| + |\gamma_{i_0}|) |\beta_{i_0}|}{|\gamma_d \beta_1|} \leq (1 + \lambda_0^2) \frac{|\beta_{i_0}|}{|\beta_1|} \leq (1 + \lambda_0^2) \frac{|m|}{y_{d-1}} \frac{|\gamma_1 \Psi_{i_0}|}{|\Psi_1|} \leq \lambda_0^2 (1 + \lambda_0^2) \frac{|m|}{y_{d-1}} \frac{|\upsilon_1|}{|\upsilon_{i_0}|}.$$ 

Hence from the assumptions

$$|\upsilon_1| \leq \mu^{-1} |\upsilon_{i_0}| \quad \text{and} \quad |\upsilon_{i_0}| \leq \mu^{-1} |\upsilon_d|,$$

we deduce

$$\frac{\gamma_{i_0} \beta_d}{\gamma_d \beta_1} + 1 \leq 4|m| \lambda_0^4 \mu^{-a} \frac{|\Psi_{i_0}|}{|\Psi_1|}.$$ 

The conclusion of (c) follows from (23).
Step 7. (a) Assume $|v_{i_0}| = |v_1|$. Since $v_{i_0} \in \mathbb{R}$, we deduce from Lemma 12 that

$$|v_1| < |v_{d-1}|.$$  

If $|v_2| < |v_{d-1}|$, then

$$\frac{|v_{d-1}|}{|v_{i_0}|} \geq \frac{|v_{d-1}|}{|v_2|} = \mu$$

and we are in case (a) of step 6.

If $|v_2| = |v_{d-1}|$, then $i_0 = 1$, we have

$$\frac{|v_{d-1}|}{|v_1|} \geq \mu$$

and again we are in case (a) of step 6.

(b) Assume $|v_{i_0}| = |v_d|$. Using Lemma 12, we deduce

$$|v_d| > |v_2|.$$  

If $|v_2| < |v_{d-1}|$, then

$$\frac{|v_{i_0}|}{|v_1|} \geq \frac{|v_{d-1}|}{|v_2|} = \mu$$

and we are in case (b) of step 6.

If $|v_2| = |v_{d-1}|$, then $i_0 = d$, we have

$$\frac{|v_d|}{|v_2|} \geq \mu$$

and again we are in case (b) of step 6.

(c) Assume finally $|v_1| < |v_{i_0}| < |v_d|$. In particular we have $2 \leq i_0 \leq d - 1$. Assume that we are neither in case (a) nor in case (b) of step 6. From

$$\frac{|v_{d-1}|}{|v_{i_0}|} < \sqrt{\mu} \quad \text{and} \quad \frac{|v_{i_0}|}{|v_2|} < \sqrt{\mu}$$

we deduce

$$\frac{|v_{d-1}|}{|v_2|} < \mu.$$  

Given the definition of $\mu$, it follows that we have $|v_2| = |v_{d-1}|$. Since $v_{i_0}$ is real, Lemma 12 implies $d = 3$ and therefore $i_0 = 2$, $|v_1| < |v_2| < |v_3|$ and

$$\mu = \min\left\{\frac{|v_3|}{|v_2|}, \frac{|v_2|}{|v_1|}\right\}.$$  

From

$$|v_1| \leq \lambda_0 |v_1|^a \leq \lambda_0 \lambda^{-a/2} < 1, \quad |\beta_2| = |\beta_{i_0}| < 1$$
and
\[ |\gamma_2\beta_3| = |\gamma_2\gamma_3 |y \geq \frac{|y|\Psi_3|}{|\gamma_1|} \geq \lambda_0^{-1}\lambda^{a/2}|\Psi_3| > 1, \]
we deduce \(|\gamma_1\beta_2| < 1 < |\gamma_2\beta_3|\), hence
\[ \gamma_1\beta_2 + \gamma_2\beta_3 \neq 0. \]

There is an element of the Galois group of the Galois closure of the cubic field \( \mathbb{Q}(\upsilon) \) which maps \( \upsilon_1 \) to \( \upsilon_2 \), \( \upsilon_2 \) to \( \upsilon_3 \), \( \upsilon_3 \) to \( \upsilon_1 \). Therefore,
\[ \gamma_2\beta_3 + \gamma_3\beta_1 \neq 0. \]

From part (c) of step 6 we deduce
\[ 0 < \left| \frac{\gamma_2\beta_3}{\gamma_3\beta_1} + 1 \right| \leq 4m\kappa_0^4\mu^{-a/2}. \]

**Step 8.** Combining steps 6 and 7 with step 4 where we choose
\[
\begin{cases}
  i = \ell = d - 1, j = k = d, c = 1 & \text{in case (a)}, \\
  i = k = i_0, j = 1, \ell = d, c = 1 & \text{in case (b)}, \\
  i = i_0, j = k = d, \ell = 1, c = -1 & \text{in case (c)},
\end{cases}
\]
we deduce
\[ a \log \mu \leq \kappa_{36}R(R + \log |m| + \log \lambda_0 + \log \lambda)(\log \lambda) \times \log \left( \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \right). \]

For
\[ U = \frac{Ra \log \lambda}{R + \log |m| + \log \lambda_0 + \log \lambda} \quad \text{and} \quad V = \kappa_{37} \frac{R^2(\log \lambda)^2}{\log \mu}, \]
we have \( U \leq V \log^* U \). Therefore we use Lemma 5 to obtain the conclusion of Theorem 3.

§5. **Proofs of Theorem 1 and of Corollary 2.**

**Proof of Theorem 1.** Since \( d \geq 3 \), under the assumptions of Lemma 13 we have
\[ \log \frac{|\upsilon''|}{|\upsilon'|} \geq \frac{\kappa_{18}(\log \lambda)^2}{\lambda d^2(d+2)/2}. \]

From Lemma 12, we deduce that under the assumptions of Theorem 1 and with the notations of Theorem 3, we have
\[ \log \mu \geq \frac{\kappa_{38}(\log \lambda)^2}{\lambda d^2(d+2)/2}. \]

Hence Theorem 3 implies Theorem 1.

**Proof of Corollary 2.** The conclusion of Corollary 2 follows from Theorem 1 thanks to the upper bound (2).
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Claude Levesque,  
Département de mathématiques et de statistique,  
Université Laval,  
Québec (Québec),  
Canada G1V 0A6,  
Canada  
E-mail: claude.levesque@mat.ulaval.ca

Michel Waldschmidt,  
Sorbonne Universités,  
UPMC Univ Paris 06,  
UMR 7586 IMJ-PRG,  
F-75005 Paris,  
France  
E-mail: michel.waldschmidt@imj-prg.fr