Weyl’s theorem for paranormal closed operators

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Abstract
In this article, we discuss a few spectral properties of paranormal closed operator (not necessarily bounded) defined in a Hilbert space. This class contains closed symmetric operators. First, we show that the spectrum of such an operator is non-empty and give a characterization of closed range operators in terms of the spectrum. Using these results, we prove the Weyl’s theorem: if \( T \) is a densely defined closed paranormal operator, then \( \sigma(T) \setminus \omega(T) = \pi_{00}(T) \), where \( \sigma(T) \), \( \omega(T) \) and \( \pi_{00}(T) \) denote the spectrum, the Weyl spectrum and the set of all isolated eigenvalues with finite multiplicities, respectively. Finally, we prove that the Riesz projection \( E_\lambda \) with respect to any non-zero isolated spectral value \( \lambda \) of \( T \) is self-adjoint and satisfies \( R(E_\lambda) = N(T - \lambda I) = N(T - \lambda I)^* \).

Keywords Closed operator · Fredholm operator · Minimum modulus · Paranormal operator · Riesz projection and Weyl’s theorem

Mathematics Subject Classification 47A10 · 47A53 · 47B20

1 Introduction

One of the most important and well-studied class in operator theory is the class of normal operators. The spectral theorem for normal operators assures the existence of non-trivial invariant subspace and also reveals the complete structure of the operator. Thus this class led to several generalizations, one among such is the class of paranormal operators.
The class of bounded paranormal operators was first studied by Istrătescu [14], who named it as the class $N$. Further, Furuta [7] introduced the term paranormal operator.

Many authors studied bounded paranormal operators, for example [1,7,14,15,26]. In particular, Ando [1] gave a characterization of bounded paranormal operators. Istrătescu [14] proved that the class of normaloid operators is a generalization of paranormal operators. We have the following inclusion relation between some subclasses and a generalized class of bounded paranormal operators.

$$\text{Normal} \subseteq \text{Hyponormal} \subseteq \text{Paranormal} \subseteq \text{Normaloid}.$$  

The above inclusion relations are proper. For more details, we refer to [7,12]. The definition of bounded paranormal operators is extended to unbounded operators by Daniluk [6], where he discussed about closability of unbounded paranormal operators.

In this article, we are going to deal with densely defined closed paranormal operators in a Hilbert space $H$ and prove the following results.

Let $T$ be a densely defined closed paranormal operator in $H$. Then

1. spectrum of $T$ is non-empty.
2. Every isolated spectral value of $T$ is an eigenvalue.
3. In addition, if $N(T) = N(T^*)$, then
   a. range of $T$ is closed if and only if 0 is an isolated spectral value of $T$.
   b. The minimum modulus, $m(T)$ is equal to the distance of 0 from spectrum of $T$.
4. $T$ satisfies the Weyl’s Theorem i.e. $\sigma(T) \setminus \omega(T) = \pi_00(T)$. Here $\omega(T)$ is the Weyl’s spectrum and $\pi_00(T)$ consists of all isolated eigenvalues of $T$ with finite multiplicity.
5. If $\lambda$ is a non-zero isolated spectral value of $T$, then the Riesz projection $E_\lambda$ with respect to $\lambda$ is self-adjoint and satisfies $R(E_\lambda) = N(T - \lambda I) = N(T - \lambda I)^*$.

Results (2) and (3) are well known in the literature for self-adjoint operators. For unbounded self-adjoint operators, simple proofs of these results are given by Kulkarni et. al. [18], without using the spectral theorem. For the bounded case, three elementary proofs are given in [17]. Also (1) is well known for self-adjoint and normal operators, refer [13, Lemma 8.6, Page 102] for more details.

Weyl’s theorem and self-adjointness of Riesz projection with respect to an isolated spectral value of an operator are studied for many different class of operators. For some non-normal operators (hyponormal and Toeplitz operators), this was established by Coburn [5]. Further Uchiyama [26], extended it to bounded paranormal operators using Ando’s characterization [1] for paranormal operators. But Ando’s characterization is not available for unbounded paranormal operators, the techniques of bounded operators does not work in our case. Hence we try to prove (4) and (5), using a different approach.

Anuradha and Karuna [11] proved that closed hyponormal operators satisfy Weyl’s theorem. In [10], the same authors gave a few necessary conditions under which the orthogonal direct sum of densely defined closed operators satisfy the Weyl’s Theorem.
In this article, we prove that densely defined closed paranormal operators satisfy Weyl’s theorem and discuss a few more spectral properties of this class along with the Riesz idempotent.

This article is divided into four sections. In the second section, we set up some notations and known results which we will be using throughout the article. In the third section, we discuss some spectral properties of densely defined closed paranormal operators. In the fourth section, we prove Weyl’s theorem for densely defined closed paranormal operators.

2 Notations and preliminaries

In this article we consider complex Hilbert spaces, which are denoted by $H$, $H_1$, $H_2$ etc. The inner product and the induced norm are denoted by $\langle \ldots \rangle$ and $\| \cdot \|$, respectively.

We denote the space of all linear operators on $H$ by $\mathcal{L}(H)$ and the space of all bounded linear operators by $\mathcal{B}(H)$. For $T \in \mathcal{L}(H)$, the domain, null space, and range space of $T$ are denoted by $D(T)$, $N(T)$ and $R(T)$, respectively. If $\overline{D(T)} = H$, then $T$ is called a densely defined operator.

If $T \in \mathcal{L}(H)$ and $M$ is a closed subspace of $H$, then $M$ is said to be invariant under $T$, if for every $x \in D(T) \cap M$, $Tx$ is in $M$. We denote the identity operator on $M$ by $I_M$, the orthogonal projection on $M$ by $P_M$. The unit sphere of $M$ is $S_M := \{ x \in M : \|x\| = 1 \}$. The restriction of $T$ to $M$ is an operator $T|_M : M \cap D(T) \to H$ defined by $T|_M x = Tx$, $\forall x \in D(T) \cap M$. If $M$ is invariant under $T$, then $T|_M$ is an operator from $D(T) \cap M$ into $M$.

An operator $T \in \mathcal{L}(H)$ is said to be closed if for any sequence $(x_n) \subseteq D(T)$ with $x_n \to x$ and $Tx_n \to y$ then $x \in D(T)$ and $Tx = y$.

Lemma 2.1 [19, Lemma 3.3] Let $T \in \mathcal{L}(H)$ be a densely defined closed operator. Then $D(T) \cap N(T)^\perp = N(T)^\perp$.

If $S$ and $T$ are two closed operators, then $S$ is called an extension of $T$ (or $T$ is a restriction of $S$), if $D(T) \subseteq D(S)$ and $Sx = Tx$ for all $x \in D(T)$. This is often denoted as $T \subseteq S$. Consequently, $S = T$ if and only if $D(S) = D(T)$ and $Sx = Tx$ for all $x \in D(S) = D(T)$.

A densely defined closed operator $T \in \mathcal{L}(H)$ is said to be self-adjoint if $T^* = T$ and normal if $TT^* = T^*T$.

If $T \in \mathcal{B}(H)$, then $T$ is said to be hyponormal if $TT^* \leq T^*T$. Equivalently, $T$ is hyponormal if $\|T^*x\| \leq \|Tx\|$ for all $x \in H$.

Definition 2.2 [8,20] Let $T \in \mathcal{L}(H)$ be a closed operator. Then

1. the minimum modulus of $T$ is defined by $m(T) := \inf \{ \|Tx\| : x \in S_{D(T)} \}$.
2. the reduced minimum modulus of $T$ is defined by $\gamma(T) := \inf \{ \|Tx\| : x \in S_{D(T) \cap N(T)^\perp} \}$.

By the definition, it is clear that $m(T) \leq \gamma(T)$.

The following characterization of closed range operators is frequently used in the article.
Theorem 2.3 [2, Page 334] For a densely defined closed operator \( T \in \mathcal{L}(H) \), the following are equivalent.

1. \( R(T) \) is closed.
2. \( R(T^*) \) is closed.
3. \( \gamma(T) > 0 \).
4. \( T_0 = T|_{D(T) \cap N(T)^\perp} \) has a bounded inverse.

If \( T \in \mathcal{L}(H) \) is a densely defined closed operator and \( N(T) = \{0\} \), then the inverse operator, \( T^{-1} \) is the linear operator from \( H \) to \( H \) with \( D(T^{-1}) = R(T) \) and \( T^{-1}(Tx) = x \) for all \( x \in D(T) \). In particular if \( T \) is a bijection, then by the closed graph theorem it follows that \( T^{-1} \in \mathcal{B}(H) \). In addition, if \( T \) is normal then \( T \) has a bounded inverse if and only if \( m(T) > 0 \).

Definition 2.4 [23, Page 365] If \( T \in \mathcal{L}(H) \) is a closed operator, then the resolvent set of \( T \) is defined by

\[
\rho(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is invertible and } (T - \lambda I)^{-1} \in \mathcal{B}(H) \}
\]

and \( \sigma(T) = \mathbb{C} \setminus \rho(T) \) is called the spectrum of \( T \).

Note that \( \sigma(T) \) is a closed subset of \( \mathbb{C} \). Moreover \( \sigma(T) \) can be empty set or the whole complex plane \( \mathbb{C} \) (For more details, we refer to [22,23]).

The spectrum of \( T \) decomposes as the disjoint union of the point spectrum \( \sigma_p(T) \), the continuous spectrum \( \sigma_c(T) \) and the residual spectrum \( \sigma_r(T) \), where

\[
\sigma_p(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not injective} \},
\sigma_r(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is injective but } R(T - \lambda I) \text{ is not dense in } H \},
\sigma_c(T) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T)).
\]

The spectral radius of \( T \in \mathcal{B}(H) \) is defined by

\[
r(T) := \sup\{ |\lambda| : \lambda \in \sigma(T) \}.
\]

An operator \( T \in \mathcal{B}(H) \) is said to be normaloid, if \( r(T) = \|T\| \).

Definition 2.5 [24, Page 156] A densely defined closed operator \( T \in \mathcal{L}(H) \) is called Fredholm if \( R(T) \) is closed, \( \text{dim}(N(T)) \) and \( \text{dim}(R(T)^\perp) \) are finite.

In this case, \( \text{ind}(T) = \text{dim}(N(T)) - \text{dim}(R(T)^\perp) \) is called the index of \( T \).

Remark 2.6 If \( T \in \mathcal{L}(H) \) is a densely defined closed Fredholm operator and \( K \) is a compact operator, then \( T + K \) is also Fredholm and \( \text{ind}(T + K) = \text{ind}(T) \).

Recall that a linear operator \( T \in \mathcal{L}(H) \) is compact, if \( T \) maps every bounded set in \( H \) to a pre-compact set in \( H \). For more details about compact and Fredholm operators, we refer to [24].
Definition 2.7 [24, Page 172] If \( T \in \mathcal{L}(H) \) is a densely defined closed operator, then the \textit{Weyl’s spectrum} of \( T \) is defined by

\[
\omega(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Fredholm operator of index 0} \}
\]

and \( \pi_{00}(T) = \{ \lambda \in \sigma_p(T) : \lambda \text{ is isolated with } \dim(N(T - \lambda I)) < \infty \} \).

Suppose \( T \in \mathcal{L}(H) \) is a densely defined closed operator with \( \sigma(T) = \sigma \cup \tau \), where \( \sigma \) is contained in some bounded domain \( \Delta \) such that \( \partial \Delta \cap \tau = \emptyset \). Let \( \Gamma \) be the boundary of \( \Delta \), then

\[
E_{\sigma} = \frac{1}{2\pi i} \int_{\Gamma} (zI - T)^{-1}dz
\]

is called the \textit{Riesz projection} with respect to \( \sigma \).

Theorem 2.8 [9, Theorem 2.1, Page 326] Suppose \( T \in \mathcal{L}(H) \) is a densely defined closed operator with \( \sigma(T) = \sigma \cup \tau \), where \( \sigma \) is contained in some bounded domain \( \Delta \) and \( E_{\sigma} \) is the operator defined in Eq. (2.1). Then

1. \( E_{\sigma} \) is a projection.
2. The subspaces \( R(E_{\sigma}) \) and \( N(E_{\sigma}) \) are invariant under \( T \).
3. The subspace \( R(E_{\sigma}) \) is contained in \( D(T) \) and \( T|_{R(E_{\sigma})} \) is bounded.
4. \( \sigma(T|_{R(E_{\sigma})}) = \sigma \) and \( \sigma(T|_{N(E_{\sigma})}) = \tau \).

In particular, if \( \lambda \) is an isolated point of \( \sigma(T) \), then there exist a positive real number \( r \) such that \( \{ z \in \mathbb{C} : |z - \lambda| \leq r \} \cap \sigma(T) = \{ \lambda \} \). If we take \( \Gamma = \{ z \in \mathbb{C} : |z - \lambda| = r \} \), then the Riesz projection with respect to \( \lambda \) is defined by

\[
E_{\lambda} = \frac{1}{2\pi i} \int_{\Gamma} (zI - T)^{-1}dz.
\]

For more details about Riesz projection, we refer to [9,21].

Definition 2.9 [6, Definition 1.1] An operator \( T \in \mathcal{L}(H) \) is called paranormal if

\[
\|Tx\|^2 \leq \|T^2x\|\|x\|, \ \forall x \in D(T^2).
\]

Equivalently, \( T \) is paranormal if \( \|Tx\|^2 \leq \|T^2x\|, \ \forall x \in S_{D(T^2)}. \)

If \( T \in \mathcal{B}(H) \), then Eq. (2.3) holds for every \( x \in H \). More details about bounded paranormal operators can be found in [1,7,15,16,26]. Here we list out a few properties of paranormal operators that we need later.

Lemma 2.10 [16, Page 98] Let \( T \in \mathcal{B}(H) \) be a paranormal operator and \( M \) be a closed subspace of \( H \) which is invariant under \( T \). Then \( T|_M \) is also paranormal.

Theorem 2.11 [15, Theorem 1] If \( T \in \mathcal{B}(H) \) is paranormal, then

1. \( T \) is normaloid.
2. \( T^{-1} \) is paranormal, if \( T \) is invertible.
3. If \( \sigma(T) \) lies on the unit circle, then \( T \) is unitary.
3 Spectral properties

In this section, we study some spectral properties of densely defined closed paranormal operators.

Recall that a densely defined closed operator $T \in \mathcal{L}(H)$ is called symmetric, if $T \subseteq T^*$. Now we show that every symmetric operator is paranormal.

**Proposition 3.1** Let $T \in \mathcal{L}(H)$ be a densely defined symmetric closed operator. Then $T$ is paranormal.

**Proof** It is easy to see from the following inequality.

\[ \|Tx\|^2 = \langle T^*Tx, x \rangle = \langle T^2x, x \rangle \leq \|T^2x\|\|x\|, \forall x \in D(T^2). \]

Here we discuss some basic results related to unbounded paranormal operators, which are often used in the article.

**Proposition 3.2** Let $T \in \mathcal{L}(H)$ be a densely defined closed paranormal operator. Then the following holds.

1. If $M$ is a closed invariant subspace of $T$, then $T|_M$ is paranormal.
2. If $0 \notin \sigma(T)$, then $T^{-1}$ is paranormal.
3. $\sigma(T)$ is non-empty.

**Proof** Proof of (1): As $M$ is invariant under $T$, we have

\[
D(T^2|_M) = D(T^2) \cap M \\
= \{x \in D(T) : Tx \in D(T)\} \cap M \\
= \{x \in D(T) \cap M : Tx \in D(T) \cap M\}, \text{ since } T(D(T) \cap M) \subseteq M \\
= \{x \in D(T|_M) : Tx \in D(T|_M)\} \\
= D((T|_M)^2).
\]

Thus $T^2|_M = (T|_M)^2$. Now the result follows from the below inequality;

\[
\|T|_Mx\|^2 = \|Tx\|^2 \leq \|T^2x\| = \|T^2|_Mx\| = \|(T|_M)^2x\|, \forall x \in S_{D((T|_M)^2)}.
\]

Proof of (2): Existence of $T^{-1}$ implies $R(T) = H$ and consequently $R(T^2) = H$. As $T$ is paranormal, we get $N(T^2) = N(T)$, so $T^2$ is bijective and $(T^2)^{-1}$ exists. Also $D((T^{-1})^2) = H = R(T^2)$. If $y \in H$, then there exist $x \in D(T^2)$, such that $y = T^2x$. Now

\[
\|T^{-1}y\|^2 = \|Tx\|^2 \leq \|T^2x\|\|x\| \\
= \|y\|\|T^{-2}y\|.
\]

Hence $T^{-1}$ is paranormal.
Proof of (3): On the contrary, assume that $\sigma(T) = \emptyset$. Then $T$ is invertible and $T^{-1} \in \mathcal{B}(H)$.

First, we show that $\sigma(T^{-1}) = \{0\}$. For any complex number $\lambda \neq 0$, consider the operator $S = \lambda^{-1}T(T - \lambda^{-1}I)^{-1}$. Here $S$ can also be written as the sum of two bounded operators, $S = \lambda^{-1}(I + \lambda^{-1}(T - \lambda^{-1}I)^{-1})$, so $S$ is bounded. By a simple computation we can show that $S$ is the bounded inverse of $\lambda I - T^{-1}$. Thus $\sigma(T^{-1}) \subseteq \{0\}$. As $T^{-1} \in \mathcal{B}(H)$, this implies $\sigma(T^{-1})$ is non-empty, so we conclude that $\sigma(T^{-1}) = \{0\}$.

By part (2), $T^{-1}$ is bounded paranormal operator and consequently normaloid by Theorem 2.11. Hence $\|T^{-1}\| = 0$, which implies $T^{-1} = 0$, a contradiction. Hence $\sigma(T)$ is non-empty. \qed

Note that in Proposition 3.2 part (1), we only used the fact that $T^{-1}$ is normaloid. Thus we can make the following statement.

**Proposition 3.3** If $T \in \mathcal{L}(H)$ is a densely defined closed operator such that $T^{-1}$ is normaloid, then $\sigma(T) \neq \emptyset$.

The following result is well known for bounded operators and can be easily extended to unbounded operators.

**Lemma 3.4** Let $T \in \mathcal{L}(H)$ be a densely defined closed operator and $\lambda$ be an isolated point of $\sigma(T)$. Then $N(T - \lambda I) \subseteq R(E_\lambda)$, where $E_\lambda$ is the Riesz projection with respect to $\lambda$ defined in Eq. (2.2).

**Proof** Let us consider

$$S := \frac{1}{2\pi i} \int_{\Gamma} (z - \lambda)^{-1}(zI - T)^{-1}dz,$$

where $\Gamma$ is the boundary of the disc $D = \{z \in \mathbb{C} : |z - \lambda| \leq r\}$ for which $D \cap \sigma(T) = \{\lambda\}$. For any $z \in \rho(T)$,

$$(z - \lambda)^{-1}(zI - T)^{-1}(T - \lambda I) = (z - \lambda)^{-1}(zI - T)^{-1}[T - zI + zI - \lambda I] = -(z - \lambda)^{-1}I_{D(T)} + (zI - T)^{-1}I_{D(T)}.$$

(3.1)

We know that

$$- \int_{\Gamma} (z - \lambda)^{-1}I_{D(T)}dz + \int_{\Gamma} (zI - T)^{-1}I_{D(T)}dz$$

is well defined. Integrating Eq. (3.1) on $\Gamma$, we get $S(T - \lambda I) = -I_{D(T)} + E_\lambda|_{D(T)}$. If we take any $x \in N(T - \lambda I)$, then $(-I_{D(T)} + E_\lambda|_{D(T)})x = 0$. Consequently $x = E_\lambda x \in R(E_\lambda)$. Hence $N(T - \lambda I) \subseteq R(E_\lambda)$. \qed

Now we discuss about isolated spectral values of paranormal operators.

**Proposition 3.5** Let $T$ be a densely defined closed paranormal operator. If $\lambda$ is an isolated point of $\sigma(T)$, then $N(T - \lambda I) = R(E_\lambda)$. 

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Proposition 3.6 Let $T \in \mathcal{L}(H)$ be a densely defined closed paranormal operator and $\lambda$ be an isolated point of $\sigma(T)$. Then $N(E_\lambda) = R(T - \lambda I)$.

Proof By Theorem 2.8, $\lambda \notin \sigma(T|_{N(E_\lambda)})$. This implies that $R((T - \lambda I)|_{N(E_\lambda)}) = N(E_\lambda)$ and consequently $N(E_\lambda) \subseteq R(T - \lambda I)$.

Let $y \in R(T - \lambda I)$. There exist $x \in D(T)$ such that $y = (T - \lambda I)x$. Since $H = R(E_\lambda) + N(E_\lambda)$ and $R(E_\lambda) \cap N(E_\lambda) = \{0\}$, we have

$$x = u + v,$$

where $u \in R(E_\lambda)$, $v \in N(E_\lambda)$.

It follows from Proposition 3.5, that $u \in N(T - \lambda I) \subseteq D(T)$ and consequently $v = x - u \in D(T)$. As we know from Theorem 2.8 that $N(E_\lambda)$ is invariant under $T$, we have

$$y = (T - \lambda I)x = (T - \lambda I)v \in (T - \lambda I)(N(E_\lambda)) \subseteq N(E_\lambda).$$

Hence $R(T - \lambda I) \subseteq N(E_\lambda)$. This proves the result. \[\square\]

The following results are consequences of Proposition 3.6 which gives a characterization for closed range paranormal operators.

Corollary 3.7 Suppose $T \in \mathcal{L}(H)$ is a densely defined closed paranormal operator. If $0$ is an isolated point of $\sigma(T)$, then $R(T)$ is closed.

In general the converse of Corollary 3.7 is not true. We have the following example to illustrate this.

Example 3.8 Let $T : \ell_2(\mathbb{N}) \rightarrow \ell_2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots), \text{ for all } (x_n) \in \ell_2(\mathbb{N}).$$

Then $\sigma(T) = \{z \in \mathbb{C} : |z| \leq 1\}$, $R(T) = \ell_2(\mathbb{N}) \setminus \text{span}\{e_1\}$. Here $R(T)$ is closed but $0$ is not an isolated point of $\sigma(T)$. Clearly, $T$ is a paranormal operator.

Next result gives a sufficient condition under which the converse of Corollary 3.7 is also true.
Theorem 3.9 Let $T \in \mathcal{L}(H)$ be a densely defined closed paranormal operator with $N(T) = N(T^*)$ and $0 \in \sigma(T)$. Then 0 is an isolated point of $\sigma(T)$ if and only if $R(T)$ is closed.

Proof The forward implication is clear by Corollary 3.7.

For the reverse implication, assume that $R(T)$ is closed. Consider $T_0 = T|_{N(T)^\perp} : N(T)^\perp \cap D(T) \to N(T)^\perp$. Clearly $T_0$ is injective and $R(T_0) = R(T)$ is closed. Also $R(T_0) = N(T^*)^\perp = N(T)^\perp$, consequently $T_0$ is bijective and $T_0^{-1} \in \mathcal{B}(N(T)^\perp)$.

Thus $0 \notin \sigma(T_0)$. Applying [25, Theorem 5.4, Page 289], $\sigma(T) \subseteq \{0\} \cup \sigma(T_0)$. Since $0 \in \sigma(T)$, we have $\sigma(T) = \{0\} \cup \sigma(T_0)$ and hence 0 is an isolated point of $\sigma(T)$.

Note that Theorem 3.9 does not hold if we drop the condition $N(T) = N(T^*)$.

Consider the operator $T$ defined in Example 3.8. Clearly $N(T) = \{0\} \neq \text{span}\{e_1\} = N(T^*)$, and $R(T)$ is closed but 0 is not an isolated point of $\sigma(T)$.

Theorem 3.10 Let $T \in \mathcal{L}(H)$ be a densely defined closed paranormal operator. If $N(T) = N(T^*)$, then $m(T) = d(0, \sigma(T))$, the distance between 0 and $\sigma(T)$.

Proof We will prove this result by considering the following two cases, which exhaust all the possibilities.

Case (1): $T$ is not injective. Clearly $m(T) = 0$ and $0 \in \sigma_p(T)$. Hence $m(T) = 0 = d(0, \sigma(T))$.

Case (2): $T$ is injective. It suffices to show that $\gamma(T) = d(0, \sigma(T))$ because $m(T) = \gamma(T)$.

First assume that $\gamma(T) = 0$. It follows from Theorem 2.3 that $R(T)$ is not closed and consequently $0 \in \sigma_c(T)$. Thus $d(0, \sigma(T)) = 0 = \gamma(T)$.

Now assume that $\gamma(T) > 0$. As a consequence of Theorem 2.3, $R(T)$ is closed. Note that $0 \notin \sigma(T)$, otherwise Theorem 3.9 and Proposition 3.5 implies that $0 \in \sigma_p(T)$. But this is not true, as $T$ is injective. Thus $0 \notin \sigma(T)$ and $T^{-1}$ is bounded paranormal operator, by Proposition 3.2. Hence $T^{-1}$ is normaloid and [18, Proposition 2.12] implies that

$$\gamma(T) = \frac{1}{\|T^{-1}\|} = \frac{1}{r(T^{-1})} = \sup\{||\lambda|| : \lambda \in \sigma(T^{-1})\} = \inf\{||\lambda|| : \lambda \in \sigma(T)\} = d(0, \sigma(T)).$$

This completes the proof.

As a consequence of Theorem 3.10 we have the following result.

Corollary 3.11 If $T \in \mathcal{L}(H)$ is a densely defined closed paranormal operator and $N(T) = N(T^*)$, then $\gamma(T) = d(T) := \inf\{||\lambda|| : \lambda \in \sigma(T) \setminus \{0\}\}$. 

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Proof Consider the operator $T_0 = T|_{N(T)^\perp} : D(T) \cap N(T)^\perp \to N(T)^\perp$. By Proposition 3.2 and Theorem 3.10, $T_0$ is paranormal and

$$\gamma(T) = m(T_0) = d(0, \sigma(T_0)) = d(T).$$

This proves the result. \qed

Remark 3.12 Next example illustrates the following facts.

1. Theorem 3.10 does not hold if $N(T) \neq N(T^*)$.
2. It is well known that the residual spectrum of a closed densely defined normal operator is empty. But this is not true in the case of paranormal operators.

Example 3.13 Let $T : \ell_2(\mathbb{N}) \to \ell_2(\mathbb{N})$ be defined by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, 2x_2, 3x_3, \ldots),$$

where $D(T) = \{(x_1, x_2, x_3, \ldots) \in \ell_2(\mathbb{N}) : \sum_{i=1}^{\infty} \|i x_i\|^2 < \infty\}$.

As $C_{00}$, the space of all complex sequences consisting of at most finitely many non zero terms is a subset of $D(T)$ and is dense in $\ell_2(\mathbb{N})$, we can conclude that $T$ is densely defined. Hence $T^*$ is well defined. Note that $T$ is a closed operator. We can show that

$$T^*(x_1, x_2, x_3, \ldots) = (x_2, 2x_3, 3x_4, \ldots)$$

with $D(T^*) = \{(x_n) \in \ell_2(\mathbb{N}) : \sum_{i=2}^{\infty} \|(i-1)x_i\|^2 < \infty\}$.

For any $x = (x_n) \in D(T^2)$, we have

$$\|Tx\|^2 = \sum_{i=1}^{\infty} \|i x_i\|^2 \leq \sum_{i=1}^{\infty} (i+1)i \|x_i\|^2 \leq \left(\sum_{i=1}^{\infty} ((i+1)i)\|x_i\|^2\right) \left(\sum_{i=1}^{\infty} \|x_i\|^2\right) = \|T^2x\| \|x\|.$$

Hence $T$ is paranormal.

Since $\|Tx\| \geq \|x\|$ for all $x \in D(T)$ and $\|Te_1\| = \|e_1\|$, we get $m(T) = 1$. Also it can be easily verified that $T$ is injective, $R(T) = \ell_2(\mathbb{N}) \setminus \text{span}\{e_1\}$ is closed but $R(T) \neq H$, so $0 \in \sigma(T)$. Hence $d(0, \sigma(T)) = 0 \neq 1 = m(T)$. 

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Now we will show that $\sigma(T) = \mathbb{C}$. To prove this, we show that $T - \lambda I$ is injective and $N(T - \lambda I)^* \neq \{0\}$, for all $\lambda \in \mathbb{C}$.

Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $(T - \lambda I)x = 0$ for some $x = (x_n) \in D(T)$. Then

$$(-\lambda x_1, x_1 - \lambda x_2, 2x_2 - \lambda x_3, \ldots) = 0.$$  

Equating component-wise we get $x = 0$. This implies that $T - \lambda I$ injective.

Let $\lambda \in \mathbb{C} \setminus \{0\}$ and $(T - \lambda I)x = 0$ for some $x = (x_n) \in D(T)$. Then

$$(y_2 - \bar{\lambda}y_1, 2y_3 - \bar{\lambda}y_2, 3y_4 - \bar{\lambda}y_3, \ldots) = 0.$$  

From this we get

$$y = \left(1, \bar{\lambda}, \frac{\bar{\lambda}^2}{2!}, \frac{\bar{\lambda}^3}{3!}, \ldots\right) y_1. \tag{3.2}$$

If $\lambda = 0$, then $N(T^*) = \text{span}\{e_1\}$. If $\lambda \neq 0$, then we will show that $y$ obtained in Eq. (3.2) belongs to $N(T - \lambda I)^*$. Consider $z_n = \frac{\lambda^{2n}}{(n!)^2}$. Then

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{|\lambda|^2}{(n + 1)^2} \to 0 \text{ as } n \to \infty.$$  

By the ratio test we conclude that $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, that is $\sum_{n=1}^{\infty} \left(\frac{|\lambda|^n}{n!}\right)^2 < \infty$. Thus $y \in \ell_2(\mathbb{N})$. On the similar lines we can show that $\sum_{i=1}^{\infty} \left(\frac{|\lambda|^n}{(n-1)!}\right)^2 < \infty$. Hence $N(T - \lambda I)^* \neq \{0\}$.

For every $\lambda \in \mathbb{C}$, $N(T - \lambda I) = \{0\}$ and $R(T - \bar{\lambda}I) = (N(T - \lambda I)^*)^\perp \neq \ell_2(\mathbb{N})$. Hence we conclude that $\lambda \in \sigma_r(T)$, and $\sigma(T) = \mathbb{C}$.

We also have $\gamma(T) = 1 \neq 0 = d(T)$. From this we conclude that Corollary 3.11 is also not true if the condition, $N(T) = N(T^*)$ is dropped.

### 4 Weyl’s theorem for paranormal operators

In this section we show that a densely defined closed paranormal operator $T$ satisfy the Weyl’s theorem. We also prove that the Riesz projection $E_\lambda$ with respect to any non zero isolated spectral value $\lambda$ of $T$ is self-adjoint.

If $H = H_1 \oplus H_2$ is a Hilbert space and $T \in \mathcal{L}(H)$ is a closed operator, then $T$ has the block matrix representation

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}, \tag{4.1}$$

where $T_{ij} : D(T) \cap H_j \to H_i$ is defined by $T_{ij} = P_{H_i}T|_{D(T)\cap H_j}$ for $i, j = 1, 2$. Here $P_{H_i}$ is an orthogonal projection onto $H_i$. 

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For \((x_1, x_2) \in (H_1 \cap D(T)) \oplus (H_2 \cap D(T)),\)

\[ T(x_1, x_2) = (T_{11}x_1 + T_{12}x_2, T_{21}x_1 + T_{22}x_2). \]

**Remark 4.1** Let \(T\) be as defined in Eq. (4.1). If \(H_1 = N(T) \neq \{0\}\) and \(H_2 = N(T)^\perp\), then

\[ T = \begin{bmatrix} 0 & T_{12} \\ 0 & T_{22} \end{bmatrix}. \quad (4.2) \]

1. If \(T\) is densely defined closed operator then by Lemma 2.1, \(T_{22} \in \mathcal{L}(N(T)^\perp)\) is also densely defined closed operator.
2. It can be easily checked that \(R(T_{22}) = R(T) \cap N(T)^\perp\). If \(R(T)\) is closed, then \(R(T_{22})\) is closed in \(N(T)^\perp\).

We say a closed operator \(T \in \mathcal{L}(H)\) satisfy the Weyl’s theorem if the Weyl’s spectrum, \(\omega(T)\) consists of all spectral values of \(T\) except the isolated eigenvalues of finite multiplicity. That is, \(\sigma(T) \setminus \omega(T) = \pi_{00}(T)\).

In [5] Coburn proved that any bounded hyponormal and Toeplitz operator satisfies the Weyl’s theorem. This was extended by Uchiyama [26] to bounded paranormal operators. Here we are going to prove this for unbounded paranormal operators.

**Theorem 4.2** Let \(T \in \mathcal{L}(H)\) be a densely defined closed paranormal operator. Then \(\sigma(T) \setminus \omega(T) = \pi_{00}(T)\).

**Proof** Let \(\lambda \in \sigma(T) \setminus \omega(T)\). So, we have \(\dim(N(T - \lambda I)) = \dim(N(T - \lambda I)^*) < \infty\) and \(R(T - \lambda I)\) is closed.

On \(H = N(T - \lambda I) \oplus N(T - \lambda I)^\perp\), \(T - \lambda I\) can be decomposed as

\[ T - \lambda I = \begin{bmatrix} 0 & T_{12} \\ 0 & T_{22} - \lambda I_{N(T - \lambda I)^\perp} \end{bmatrix}, \]

where \(T_{22} = P_{N(T - \lambda I)^\perp} T|_{N(T - \lambda I)^\perp}\). By Remark 4.1, \(T_{22} - \lambda I_{N(T - \lambda I)^\perp}\) is a densely defined closed operator with domain \(D(T - \lambda I) \cap N(T - \lambda I)^\perp\) and \(R(T_{22} - \lambda I_{N(T - \lambda I)^\perp})\) is closed.

As \(N(T - \lambda I)\) is finite dimensional, this implies \(T_{12}\) is finite rank operator and by Remark 2.6, \(\text{ind}(T - \lambda I) = \text{ind}(T_{22} - \lambda I_{N(T - \lambda I)^\perp}) = 0\).

Since \(N(T_{22} - \lambda I_{N(T - \lambda I)^\perp}) = \{0\}\) and \(\text{ind}(T_{22} - \lambda I_{N(T - \lambda I)^\perp}) = 0\), we get \(N(T_{22} - \lambda I_{N(T - \lambda I)^\perp}^*) = \{0\}\) and consequently \(R(T_{22} - \lambda I_{N(T - \lambda I)^\perp}) = N(T - \lambda I)^\perp\). Thus \(T_{22} - \lambda I_{N(T - \lambda I)^\perp}\) has bounded inverse and hence \(\lambda \notin \sigma(T_{22})\). As \(\sigma(T) \subseteq \{\lambda\} \cup \sigma(T_{22})\), this implies that \(\lambda\) is an isolated point of \(\sigma(T)\). Hence \(\lambda \in \pi_{00}(T)\).

Conversely, let \(\lambda \in \pi_{00}(T)\). Now consider the Riesz projection \(E_\lambda\) with respect to \(\lambda\). By Theorem 2.8 and Proposition 3.6, \(\lambda \notin \sigma(T|_{N(E_\lambda)})\) and

\[ R(T - \lambda I) = R((T - \lambda I)|_{N(E_\lambda)}) = N(E_\lambda). \]
Since $\lambda \notin \sigma(T|_{N(E_\lambda)})$, we have that $R((T - \lambda I)|_{N(E_\lambda)}) = N(E_\lambda)$. Hence $R(T - \lambda I)$ is closed. Also $((T - \lambda I)|_{N(E_\lambda)})^{-1} \in \mathcal{B}(N(E_\lambda))$. Thus we get

$$\dim N(T - \lambda I)^* = \dim(R(T - \lambda I)^\perp)$$
$$= \dim(N(E_\lambda)^\perp)$$
$$= \dim(R(E_\lambda))$$
$$= \dim(N(T - \lambda I)).$$

Note that $\dim(N(E_\lambda)^\perp) = \dim(R(E_\lambda))$ but the spaces, $N(E_\lambda)^\perp$ and $R(E_\lambda)$ need not be the same. Hence $T - \lambda I$ is Fredholm operator of index zero. This proves our result. $\square$

As a consequence of Theorem 4.2 and Proposition 3.1, we have the following result.

**Corollary 4.3** If $T \in \mathcal{L}(H)$ is a densely defined closed symmetric operator, then $T$ satisfy the Weyl’s theorem.

**Theorem 4.4** Let $T \in \mathcal{L}(H)$ be a densely defined closed paranormal operator and $\lambda$ be a non-zero isolated point of $\sigma(T)$. Then the Riesz projection $E_\lambda$ with respect to $\lambda$ satisfy

$$R(E_\lambda) = N(T - \lambda I) = N(T - \lambda I)^*.$$

Moreover $E_\lambda$ is self-adjoint.

**Proof** Let $\lambda$ be a non-zero isolated point of $\sigma(T)$. By Theorem 2.8 and Proposition 3.6, $\lambda \notin \sigma(T|_{N(E_\lambda)})$ and $R(T - \lambda I) = N(E_\lambda)$. That means $(T - \lambda I)|_{N(E_\lambda)} : N(E_\lambda) \cap D(T) \rightarrow N(E_\lambda) = R(T - \lambda I)$ is a bijection. Also $(T - \lambda I)|_{N(T - \lambda I)^\perp \cap D(T)} : N(T - \lambda I)^\perp \cap D(T) \rightarrow R(T - \lambda I)$ is a bijection, we have $N(E_\lambda) \cap D(T) \subseteq N(T - \lambda I)^\perp \cap D(T)$.

Now we claim that $N(E_\lambda) \cap D(T) = N(T - \lambda I)^\perp \cap D(T)$. Let $x \in N(T - \lambda I)^\perp \cap D(T)$ and

$$E_\lambda x = u + v, \text{ where } u \in N(T - \lambda I), \ v \in N(T - \lambda I)^\perp.$$

Operating $E_\lambda$ on both sides, we get

$$u + v = E_\lambda x = u + E_\lambda v.$$

This implies $E_\lambda v = v \in R(E_\lambda) \cap N(T - \lambda I)^\perp = \{0\}$, by Proposition 3.5. From this we conclude that $E_\lambda x = u = E_\lambda u$, that is $x - u \in N(E_\lambda) \cap D(T) \subseteq N(T - \lambda I)^\perp \cap D(T)$. As $x \in N(T - \lambda I)^\perp$, we get $u \in N(T - \lambda I) \cap N(T - \lambda I)^\perp = \{0\}$. Consequently $E_\lambda x = 0$. So $N(T - \lambda I)^\perp \cap D(T) \subseteq N(E_\lambda) \cap D(T)$. Hence $N(T - \lambda I)^\perp \cap D(T) = N(E_\lambda) \cap D(T)$. 

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By Lemma 2.1 and Proposition 3.6, we get

\[
N(T - \lambda I)^\perp = \overline{N(T - \lambda I)^\perp \cap D(T)} \\
= \overline{N(E_{\lambda}) \cap D(T)} \\
= \overline{R(T - \lambda I) \cap D(T)} \\
= (N(T - \lambda I)^*)^\perp \cap D(T) \\
\subseteq (N(T - \lambda I)^*)^\perp.
\]

Hence \(N(T - \lambda I)^* \subseteq N(T - \lambda I)\). By Proposition 3.6, \(N(E_{\lambda})^\perp = R(T - \lambda I)^\perp = N((T - \lambda I)^*) \subseteq N(T - \lambda I) = R(E_{\lambda})\). Hence \(N(E_{\lambda})^\perp \subseteq R(E_{\lambda})\).

If \(x \in R(E_{\lambda})\), then \(x = a + b\) where \(a \in N(E_{\lambda})\) and \(b \in N(E_{\lambda})^\perp\). As \(N(E_{\lambda})^\perp \subseteq R(E_{\lambda})\), we get \(a = x - b \in N(E_{\lambda}) \cap R(E_{\lambda}) = \{0\}\). Thus we get \(N(E_{\lambda})^\perp = R(E_{\lambda})\), which is equivalent to say that \(N(T - \lambda I) = N(T - \lambda I)^*\).

As \(N(E_{\lambda})^\perp = R(E_{\lambda})\), we have that \(E_{\lambda}\) is an orthogonal projection. Hence \(E_{\lambda}\) is self-adjoint. \(\Box\)

Using Birkhoff–James orthogonality, we show that the eigenspace corresponding to different isolated eigenvalues of a paranormal operator are mutually orthogonal. Recall that a subspace \(M\) of a Banach space \(X\) is said to be orthogonal to a subspace \(N\) of \(X\), in the sense of Birkhoff–James, if \(\|m\| \leq \|m + n\|\) for all \(m \in M\) and \(n \in N\). If \(X\) is Hilbert space, then this coincides with the usual concept of orthogonality.

**Corollary 4.5** Let \(T \in \mathcal{L}(H)\) be a densely defined closed paranormal operator. If \(\lambda_1\) and \(\lambda_2\) are two non zero distinct isolated points of \(\sigma(T)\), then \(N(T - \lambda_1 I)\) is orthogonal to \(N(T - \lambda_2 I)\).

**Proof** Without loss of generality, assume that \(|\lambda_1| < |\lambda_2|\). For any \(x \in N(T - \lambda_1 I)\) and \(y \in N(T - \lambda_2 I)\), consider the set \(M = \text{span}\{x, y\}\). As \(M\) is invariant subspace for \(T\), it follows that \(T|_M\) is paranormal operator and \(\|T|_M\| = |\lambda_2|\). We have the following.

\[
\| \frac{\lambda_1^n}{\lambda_2^n}x + y \| = \frac{1}{|\lambda_2|^n} \| \lambda_1^n x + \lambda_2^n y \| \\
\leq \|T|_M\|^n \|x + y\| \\
= \|x + y\|.
\]

Taking the limit \(n \to \infty\), we get \(\|y\| \leq \|x + y\|\), for every \(x \in N(T - \lambda_1 I)\) and \(y \in N(T - \lambda_2 I)\). Hence \(N(T - \lambda_2 I)\) is orthogonal to \(N(T - \lambda_1 I)\).
Next, if $|\lambda_1| = |\lambda_2|$, then for every $n \in \mathbb{N}$

$$
\left\| \left( \frac{\lambda_1 + \lambda_2}{2\lambda_2} \right)^n x + y \right\| = \left\| \left( \frac{\lambda_1 + \lambda_2}{2\lambda_2} \right)^n x + (\lambda_2 + \lambda_2)^n y \right\| \\
\leq \frac{1}{(2|\lambda_2|)^n} \sum_{i=0}^{n} \binom{n}{i} |\lambda_2|^i \|\lambda_1^{n-i} x + \lambda_2^{n-i} y\| \\
= \frac{1}{(2|\lambda_2|)^n} \sum_{i=0}^{n} \binom{n}{i} |\lambda_2|^i \| (T|M)^{n-i} (x + y)\| \\
\leq \frac{1}{(2|\lambda_2|)^n} \sum_{i=0}^{n} \binom{n}{i} |\lambda_2|^n \| x + y \| \\
= \| x + y \|.
$$

As $\lambda_1 \neq \lambda_2$, we have $\left| \frac{\lambda_1 + \lambda_2}{2\lambda_2} \right| < 1$. Now as $n \to \infty$ in the above inequality we get that $\| y \| \leq \| x + y \|$. This proves the result.

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**References**

1. Ando, T.: Operators with a norm condition. Acta Sci. Math. (Szeged) **33**, 169–178 (1972)
2. Ben-Israel, A., Greville, T.N.E.: Generalized Inverses, 2nd edn, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, vol. 15. Springer, New York (2003)
3. Baxley, J.V.: On the Weyl spectrum of a Hilbert space operator. Proc. Am. Math. Soc. **34**, 447–452 (1972)
4. Chourasia, N.N., Ramanujan, P.B.: Paranormal operators on Banach spaces. Bull. Austral. Math. Soc. **21**(2), 161–168 (1980)
5. Coburn, L.A.: Weyl’s theorem for nonnormal operators. Mich. Math. J. **13**, 285–288 (1966)
6. Daniluk, A.: On the closability of paranormal operators. J. Math. Anal. Appl. **376**(1), 342–348 (2011)
7. Furuta, T.: On the class of paranormal operators. Proc. Jpn. Acad. **43**, 594–598 (1967)
8. Goldberg, S.: Unbounded Linear Operators: Theory and Applications. McGraw-Hill Book Co., New York (1966)
9. Gohberg, I., Goldberg, S., Kaashoek, M. A.: Classes of Linear Operators, vol. I, Operator Theory: Advances and Applications, vol. 49. Birkhäuser, Basel (1990)
10. Gupta, A., Mamtani, K.: Variants of Weyl’s theorem for direct sums of closed linear operators. Adv. Oper. Theory **2**(4), 409–418 (2017)
11. Gupta, A., Mamtani, K.: Weyl type theorems for unbounded hyponormal operators. Kyungpook Math. J. **55**(3), 531–540 (2015)
12. Halmos, P.R.: A Hilbert Space Problem Book. D. Van Nostrand Co., Inc, Princeton (1967)
13. Hellfer, B.: Spectral Theory and Its Applications, Cambridge Studies in Advanced Mathematics, vol. 139. Cambridge University Press, Cambridge (2013)
14. Istrătescu, V.: On some hyponormal operators. Pac. J. Math. **22**, 413–417 (1967)
15. Istrătescu, V., Saitô, T., Yoshino, T.: On a class of operators. Tôhoku Math. J. (2) **18**, 410–413 (1966)
16. Kubrusly, C.S.: Hilbert Space Operators. Birkhäuser, Boston (2003)
17. Kulkarni, S.H., Nair, M.T.: A characterization of closed range operators. Indian J. Pure Appl. Math. **31**(4), 353–361 (2000)
18. Kulkarni, S.H., Nair, M.T., Ramesh, G.: Some properties of unbounded operators with closed range. Proc. Indian Acad. Sci. Math. Sci. 118(4), 613–625 (2008)
19. Kulkarni, S.H., Ramesh, G.: The carrier graph topology. Banach J. Math. Anal. 5(1), 56–69 (2011)
20. Kulkarni, S.H., Ramesh, G.: On the denseness of minimum attaining operators. Oper. Matrices 12(3), 699–709 (2018)
21. Lorch, E.R.: Spectral Theory. University Texts in the Mathematical Sciences. Oxford University Press, New York (1962)
22. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. I, 2nd edn. Academic Press, New York (1980)
23. Rudin, W.: Functional Analysis. McGraw-Hill Book Co., New York (1973)
24. Schechter, M.: Principles of Functional Analysis. Academic Press, New York (1971)
25. Taylor, A.E., Lay, D.C.: Introduction to Functional Analysis, reprint of the second edition. Robert E. Krieger Publishing Co., Melbourne (1986)
26. Uchiyama, A.: On the isolated points of the spectrum of paranormal operators. Integr. Equ. Oper. Theory 55(1), 145–151 (2006)