Bi-Lipschitz bijections of $\mathbb{Z}$

Itai Benjamini and Alexander Shamov

February 4, 2014

Abstract

It is shown that every bi-Lipschitz bijection from $\mathbb{Z}$ to itself is at a bounded $L_\infty$ distance from either the identity or the reflection. We then comment on the group-theoretic properties of the action of bi-Lipschitz bijections.

1 Introduction

Definition 1. A bi-Lipschitz bijection between two metric spaces $(X, \rho_X)$ and $(Y, \rho_Y)$ is a bijective map $f : X \to Y$, such that there are $0 < C_1 \leq C_2 < +\infty$, such that for all $x_1, x_2 \in X$

$$C_1 \rho_X (x_1, x_2) \leq \rho_Y (f(x_1), f(x_2)) \leq C_2 \rho_X (x_1, x_2).$$

Recall the definition of the Lipschitz constant of a map:

$$\|f\|_{\text{Lip}} := \sup_{x_1 \neq x_2} \frac{\rho_Y (f(x_1), f(x_2))}{\rho_X (x_1, x_2)}.$$

A map $f$ is Lipschitz if and only if $\|f\|_{\text{Lip}}$ is finite, and bi-Lipschitz if and only if it is bijective and both $\|f\|_{\text{Lip}}$ and $\|f^{-1}\|_{\text{Lip}}$ are finite.

While the real line $\mathbb{R}$ admits a large family of bi-Lipschitz bijections, e.g. including any increasing function with derivative bounded away from 0 and $\infty$, bi-Lipschitz bijections of $\mathbb{Z}$ turn out to be much more rigid. Namely, we have

Theorem 1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a bi-Lipschitz bijection ($\mathbb{Z}$ is equipped with its usual metric, namely $\rho(x, y) := |x - y|$). Then either

$$\sup_{x \in \mathbb{Z}} |f(x) - x| < +\infty$$

or

$$\sup_{x \in \mathbb{Z}} |f(x) + x| < +\infty.$$

More precisely,

$$f(x) = \pm x + \text{const} + r(x),$$

$$|r(x)| \leq \|f\|_{\text{Lip}} \|f^{-1}\|_{\text{Lip}}.$$
This result extends to spaces that are bi-Lipschitz isomorphic to $\mathbb{Z}$, like, for instance, products $\mathbb{Z} \times G$ with a finite graph $G$, equipped with the graph metric.

The reason for different behavior of $\mathbb{Z}$ vs. $\mathbb{R}$ is that unlike $\mathbb{R}$, $\mathbb{Z}$ cannot be “squeezed and stretched”. In the proof below one of the arguments is a cardinality estimate. It is quite obvious that this argument fails in the continuum, and indeed for $\mathbb{R}$ the statement is just wrong. However, the analogy is restored if we equip our space with a measure and require the bijection to be measure preserving. This motivates the following

**Question 1.** Let $f : \mathbb{Z}^2 \to \mathbb{Z}^2$ be a bi-Lipschitz bijection. Can it be extended to a bi-Lipschitz Lebesgue measure preserving bijection $g : \mathbb{R}^2 \to \mathbb{R}^2$?

Note that the two dimensional grid $\mathbb{Z}^2$ admits many bi-Lipschitz bijections. For example, let $g : \mathbb{Z} \to \mathbb{Z}$ be a Lipschitz function. Then $F(x, y) := (x, y + g(x))$ is a bi-Lipschitz bijection of $\mathbb{Z}^2$. This shows that a naive generalization of Theorem 1 fails for $\mathbb{Z}^2$: not every bi-Lipschitz bijection is at a bounded distance from an isometry.

For background on metric geometry see e.g. [1]. The group of bijections from $\mathbb{Z}$ to $\mathbb{Z}$ within a bounded $L_\infty$ distance to the identity recently appeared in [2].

## 2 Proof of Theorem 1

The key to the result is to understand how the image sets $f ((-\infty, x])$ may look like.

The “picture” above illustrates what we are going to prove. ○’s are used to denote $y \in \mathbb{Z}$ such that $y \notin f ((-\infty, x])$, and •’s for $y \in f ((-\infty, x])$. In the sequel we denote the constant $\|f\|_{\text{Lip}} \|f^{-1}\|_{\text{Lip}}$ by $C$.

**Lemma 1.** One of the following two cases occurs: either

$$(-\infty, [f(x) - C/2]) \subset f ((-\infty, x]) \subset (-\infty, [f(x) + C/2])$$

or

$$[[f(x) + C/2], +\infty) \subset f ((-\infty, x]) \subset [[f(x) - C/2], +\infty).$$

for all $x \in \mathbb{Z}$.

**Proof.** Let $y \neq f(x)$ be such that $y \notin f ((-\infty, x])$ and $y + 1 \notin f ((-\infty, x])$ (i.e. $y$ is the position of a “•” on the “picture”). Then since $y \in f ((-\infty, x])$, it follows that $f^{-1}(y) < x$. In the same way since $y + 1 \notin f ((-\infty, x])$, we have $f^{-1}(y + 1) > x$. From the Lipschitz property of $f$ it follows that

$$x - f^{-1}(y) \geq \frac{|f(x) - y|}{\|f\|_{\text{Lip}}}.$$
\[
f^{-1}(y + 1) - x \geq \frac{|f(x) - y - 1|}{\|f\|_{\text{Lip}}}.
\]

Therefore,
\[
f^{-1}(y + 1) - f^{-1}(y) \geq 2 \frac{|f(x) - y - \frac{1}{2}|}{\|f\|_{\text{Lip}}}.
\]

Now from the Lipschitz property of \(f^{-1}\) it follows that
\[
1 = (y + 1) - y \geq \frac{f^{-1}(y + 1) - f^{-1}(y)}{\|f^{-1}\|_{\text{Lip}}} \geq 2 \frac{|f(x) - y - \frac{1}{2}|}{\|f\|_{\text{Lip}} \|f^{-1}\|_{\text{Lip}}}.
\]

In other words, the distance between \(f(x)\) and any \(\cdot \circ \) is bounded by \(\frac{1}{2}C\).

The same argument also applies to \(\circ \cdot\): just replace \(y + 1\) by \(y - 1\) everywhere.

This proves that the characteristic function of the set \(f((\infty, x])\) does not change outside the region
\[
\left[ f(x) - \frac{1}{2}C, f(x) + \frac{1}{2}C \right].
\]

Now since both \(f((\infty, x])\) and its complement \(\mathbb{Z} \setminus f((\infty, x]) = f([x + 1, +\infty))\) must be infinite, only two possibilities are left: either \(f((\infty, x])\) is unbounded from below or it is unbounded from above, which obviously corresponds to the two possible conclusions of the lemma.

\textbf{Remark 1.} Actually, by a little more careful application of the same argument one can show that the width of the region where the characteristic function of \(f((\infty, x])\) is nonconstant is bounded by \(\frac{1}{2}C\).

\textbf{Proof of Theorem \[\]} Let’s assume that the images in Lemma \[\] are unbounded from below (the other case can be treated analogously). Let \(x_1, x_2 \in \mathbb{Z}\) be such that \(x_2 - x_1 > C\). Then
\[
f((x_1, x_2]) = f((\infty, x_2]) \setminus f((\infty, x_1]) \subset
\subset (-\infty, f(x_2) + C/2] \setminus (-\infty, f(x_1) - C/2] =
= (f(x_1) - C/2, f(x_2) + C/2].
\]

In the same way
\[
f((x_1, x_2]) \supset (f(x_1) + C/2, f(x_2) - C/2].
\]

Since \(f\) is a bijection, the cardinality of \(f((x_1, x_2])\) must be \(x_2 - x_1\). Therefore,
\[
f(x_2) - f(x_1) - C \leq x_2 - x_1 \leq f(x_2) - f(x_1) + C.
\]

Now if we fix \(x_1 < 0\) and vary \(x_2\), we see that for \(x\) in the interval \([x_1, +\infty)\)
\[
|f(x) - x - \text{const}_{x_1}| \leq C.
\]

Note that \(x_1\) can be arbitrary and the range of possible values of \(\text{const}_{x_1}\) is bounded independently of \(x_1\) (e.g. \(|\text{const}_{x_1}| \leq |f(0)| + C\)), therefore the bound holds on the whole \(\mathbb{Z}\). \(\square\)
3 Corollaries

As pointed out by the referee, our result implies that there is a remarkable
difference between $\mathbb{Z}$ and higher-dimensional lattices in terms of the group-
theoretic properties of the action of bi-Lipschitz bijections. In particular:

**Corollary 1.** The group of bi-Lipschitz bijections of $\mathbb{Z}$ does not contain an
infinite countable subgroup with property (T).

*Proof.* The fact that the wobbling group of $\mathbb{Z}$ – i.e. the group of bijections
that have finite $\ell^\infty$ distance from the identity – does not contain a countable
property (T) subgroup follows from Theorem 4.1 in [3]. On the other hand, by our result, the wobbling group of $\mathbb{Z}$ is an index 2 subgroup of the group of bi-Lipschitz bijections.

Note that Corollary 1 fails for $\mathbb{Z}^d$, $d \geq 3$, since $\text{SL}(d, \mathbb{Z}), d \geq 3$ has property (T) and acts faithfully on $\mathbb{Z}^d$ by bi-Lipschitz bijections. We do not know what happens in the $d = 2$ case.

**Question 2.** Does Corollary 1 hold for $\mathbb{Z}^2$?

Another corollary concerns an amenability-like property:

**Corollary 2.** There is a bi-Lipschitz invariant mean (i.e. finitely additive prob-
ability measure) on $\mathbb{Z}$.

*Proof.* From Lemma 1 it follows that the sets $A_n := [-n, n]$ form a Følner
sequence for the action of bi-Lipschitz bijections – i.e. for any particular bi-
Lipschitz bijection $f$ we have

$$\frac{|f(A_n) \cap A_n|}{|A_n|} \to 1, n \to \infty$$

Therefore, an invariant mean can be obtained by a standard argument, as a
limiting point of the sequence of uniform measures on $A_n$ with respect to the
weak-$*$ topology of $(\ell^\infty)^*$.

On the other hand:

**Proposition 1.** Corollary 2 fails for $\mathbb{Z}^2$.

*Proof.* Let $\mu$ be a bi-Lipschitz invariant mean on $\mathbb{Z}^2$. Then the standard action
of $\text{SL}(2, \mathbb{Z})$ on $\mathbb{Z}^2 \setminus \{0\}$ preserves the mean $\mu$ restricted to $\mathbb{Z} \setminus \{0\}$. This is
impossible, since $\text{SL}(2, \mathbb{Z})$ is nonamenable and acts on $\mathbb{Z}^2 \setminus \{0\}$ with amenable stabilizers.

4 Acknowledgements

The authors wish to thank the referee for suggesting Corollaries 1 and 2 and
the surrounding discussion.
References

[1] Dmitri Burago, Yuri Burago, and Sergei Ivanov, A Course in Metric Geometry. American Mathematical Society (2001).

[2] Kate Juschenko and Nicolas Monod, Cantor systems, piecewise translations and simple amenable groups. Annals of Mathematics 2 (2013), 775-787.

[3] Kate Juschenko and Mikael de la Salle, Invariant means for the wobbling group. Bull. Belg. Math. Soc. Simon Stevin 22 (2015), no. 2, 281290.