Affine group representation formalism for four-dimensional, Lorentzian, quantum gravity

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Abstract
Within the context of the Ashtekar variables, the Hamiltonian constraint of four-dimensional pure general relativity with cosmological constant, $\Lambda$, is re-expressed as an affine algebra with the commutator of the imaginary part of the Chern–Simons functional, $Q$, and the positive-definite volume element. This demonstrates that the affine algebra quantization program of Klauder can indeed be applicable to the full Lorentzian signature theory of quantum gravity with non-vanishing cosmological constant, and it facilitates the construction of solutions to all of the constraints. Unitary, irreducible representations of the affine group exhibit a natural Hilbert space structure, and coherent states and other physical states can be generated from a fiducial state. It is also intriguing that formulation of the Hamiltonian constraint or the Wheeler–DeWitt equation as an affine algebra requires a non-vanishing cosmological constant, and a fundamental uncertainty relation of the form $\Delta V/\langle V \rangle \Delta Q \geq 2\pi \Lambda L^2_{\text{Planck}}$ (wherein $V$ is the total volume) may apply to all physical states of quantum gravity.

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Introduction
Consistent quantization of gravity remains one of the most challenging problems in theoretical physics. To meet the standard criteria for quantization via the canonical approach, a Hilbert space for physical states, consistent with all of the constraints of the theory, is needed. There is one known exact and explicit solution to all of the constraints of pure general relativity (GR) with cosmological constant: the Chern–Simons state $\psi_{\text{CS}}$, which as well exhibits a good semiclassical limit [1–3]. While lying in the simultaneous kernel of all of the constraints for a particular operator ordering, this solution may not meet the rigorous definition of a physical state owing to issues of normalizability and unitarity raised by Witten and others [4]. In this paper, we shall reformulate the Hamiltonian constraint of Lorentzian gravity, and in this context demonstrate that physical states of the Hilbert space must belong to a representation of an affine algebra. Some of the motivation for this work comes from the results of [5], wherein
one can rewrite the Hamiltonian constraint completely in terms of fundamental geometric entities which are the Chern–Simons functional and volume element.

The main quantities of interest for this paper will be the imaginary part of the same Chern–Simons functional, \( Q \), of the Ashtekar connection, and the local and global volume operators \( V(x) \) and \( V = \int V(x) \). It will be demonstrated, within the context of the Ashtekar variables, that the Hamiltonian constraint for pure GR with cosmological constant can be re-expressed as an affine algebra with the commutator of \( Q \) and the volume element. Another topic in quantum gravity which will be important for this paper is the affine quantization program started by Klauder [6]. In this approach, which was originally developed in the metric representation, the spatial 3-metric \( q_{ij} \) must satisfy certain positivity requirements. As emphasized by Klauder, these requirements are best implemented in the quantum theory via the affine group representation wherein the solutions exhibit a natural Hilbert space structure, and wherein coherent states and other physical states can be generated from fiducial states. This paper will demonstrate that the affine algebra quantization program can indeed be applicable to the full theory of quantum gravity and that all physical states must thus come from representations of the affine algebra of the imaginary part of the Chern–Simons functional and the positive-definite volume operator. It is also intriguing that formulation of the Hamiltonian constraint or Wheeler–DeWitt equation as an affine algebra requires a non-vanishing cosmological constant.

The organization of this paper is as follows. In section 1 we provide some background on the Ashtekar variables and their quantization to set the stage. Section 2 recapitulates some results of [5], expressing the Hamiltonian constraint using Poisson brackets involving the Chern–Simons functional \( I_{CS}(A) \) and the local volume element \( V(x) \). Section 3 introduces the relevant basics of the affine group and its quantization, and sets up the classical preliminaries. Section 4 performs the actual quantization using the affine group in conjunction with the relevant steps of the algebraic quantization program due to Ashtekar [7], and solutions of the Hamiltonian constraint as an affine algebra are constructed from the affine group representations. Section 5 is a brief summary of our main results. Further details and identities related to affine group quantization are gathered in appendix A. The noteworthy identity that it is the imaginary part, rather than the full Chern–Simons functional, that is really needed here comes from representations of the affine algebra of the imaginary part of the Chern–Simons functional and the positive-definite volume operator. This paper will demonstrate that the affine algebra quantization program can indeed be applicable to the full theory of quantum gravity and that all physical states must thus come from representations of the affine algebra of the imaginary part of the Chern–Simons functional and the positive-definite volume operator. It is also intriguing that formulation of the Hamiltonian constraint or Wheeler–DeWitt equation as an affine algebra requires a non-vanishing cosmological constant.

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1. The Ashtekar variables

Let \( M \) be a four-dimensional spacetime manifold of topology \( M = \Sigma \times \mathbb{R} \), where \( \Sigma \) is a spatial 3-manifold of a given topology embedded in \( M \). Then define on each \( \Sigma \) the canonical pair \((A^a_i, E^i_a)\), where \( A^a_i \) is an \( SO(3) \) gauge potential, and \( E^i_a \) is a densitized triad of density weight \( -1 \) constructed from undensitized spatial triads \( e^a_i \). These are given by

\[
A^a_i = \Gamma^a_i + \gamma K^a_i, \quad E^i_a = \frac{1}{L^3} \epsilon^{ijk} \epsilon_{abc} e^b_j e^c_k,
\]

where \( \gamma \) is the Barbero–Immirzi parameter, \( \Gamma^a_i \) is the spin connection compatible with \( e^a_i \) and \( K^a_i \) is the triadic form of the extrinsic curvature of \( \Sigma \). Then the action for four-dimensional gravity in the Ashtekar variables can be written in the 3+1 form as [8–10]

\[
I = \int dt \int_\Sigma d^3x \left( \frac{1}{\sqrt{\gamma}} \left[ \dot{E}^i_a A^a_i + A^a_i A^a_i + N^a \mathcal{H}_a \right] + N \mathcal{H} \right).
\]

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3 Index conventions are that symbols from the beginning of the Latin alphabet \( a, b, c, \ldots \) label internal \( SO(3) \) indices, while spatial indices are denoted by \( i, j, k, \ldots \). For internal indices, raised and lowered positions are equivalent, since the group metric is the Euclidean 3-metric \( \delta_{ij} \). For spatial indices, raised and lowered index positions are not equivalent since the spatial 3-metric \( q_{ij} = e^a_i e^a_j \) is not the unit matrix.
The equations of motion for the auxiliary fields $A_i^a, N^i, N$ imply the vanishing of Gauss' law, vector and Hamiltonian constraints, respectively, $G_a, \mathcal{H}_i$ and $\mathcal{H}$ given by

$$G_a = D_i \tilde{E}_a^i = \partial_t \tilde{E}_a^i + \epsilon_{abc} A_b^i \tilde{E}_c^i;$$

$$\mathcal{H}_i = \tilde{E}_a^i F_{ij}^a = \frac{1 + \gamma^2}{\gamma} K_i^a G_a;$$

$$\mathcal{H} = \frac{1}{2G\sqrt{|detq|}} \tilde{E}_a^i \tilde{E}_b^j \left( \epsilon^{ab} F_{ij}^a + \frac{\Lambda}{3} \epsilon^{abc} \epsilon_{ijk} \tilde{E}_k^a - 2(1 + \gamma^2) K_i^a K_i^b \right) + \frac{(1 + \gamma^2)}{G\gamma^2} \left( \frac{\tilde{E}_a^i}{\sqrt{|detq|}} \right) \partial_t G^a.$$ (3)

Equation (2) provides the fundamental nontrivial Poisson bracket $\{A_i^a(x), \tilde{E}_j^a(y)\} = \gamma G \delta^a_b \delta^{(3)}(x, y).$ (4)

which in conjunction with the reality conditions constitutes a basis for computation of the Hamiltonian dynamics of four-dimensional GR and its quantization.

For real $\gamma$ the connection $A_i^a$ is real and there is no need to implement reality conditions in the Ashtekar–Barbero formalism with generic real $\gamma$. On the other hand, the Hamiltonian constraint in this case is difficult to implement on account of the presence of extrinsic curvature squared and other terms involving $\gamma$. For $\gamma = \pm i$, the aforementioned extra terms vanish but the connection $A_i^a$ becomes complex. While this yields a simple, polynomial Hamiltonian constraint, one must impose reality conditions in order to obtain real GR of Lorentzian signature $^5$. From now on, we will restrict consideration to $\gamma = i$ for concreteness, and the (anti-)self-dual case will constitute the fundamental basis for the results of this paper.

1.1. Dirac quantization procedure

There is no unique prescription for the extrapolation from classical to quantum theory. A standard approach to canonical quantization proceeds according to the Heisenberg prescription, wherein one promotes the dynamical variables to quantum operators $(A_i^a, \tilde{E}_j^a) \to (\hat{A}_i^a, \hat{\tilde{E}}_j^a)$ and defines a set of unit vectors $|\psi\rangle \in H_{\text{Kin}}$ on which the operators act. These state vectors form a kinematic Hilbert space $H_{\text{Kin}}$, which is the Hilbert space at the level prior to the implementation of any constraints. All Poisson brackets would become promoted to $\frac{1}{i\hbar}$ times quantum commutators, and so equation (4) would become (for $\gamma = i$) promoted to equal time commutation relations

$$[\hat{A}_i^a(x, t), \hat{\tilde{E}}_j^b(y, t)] = -i\hbar G \delta_i^b \delta^{(3)}(x, y).$$ (5)

The initial value constraints would become promoted to operator constraints $\hat{G}_a, \hat{\mathcal{H}}_i$, and $\hat{\mathcal{H}}$ for a prescribed operator ordering. According to the Dirac procedure for constrained systems $^{[11]}$, the physical states $|\psi\rangle \in H_{\text{Phys}}$ are those elements of $H_{\text{Kin}}$ which are annihilated by all of the quantum constraints

$$\hat{G}_a(x)|\psi\rangle = \hat{\mathcal{H}}_i(x)|\psi\rangle = \hat{\mathcal{H}}(x)|\psi\rangle = 0.$$ (6)

This amounts to finding a set of gauge- and diffeomorphism-invariant functionals lying in the kernel of the quantum Hamiltonian constraint, which admits a Hilbert space structure.

In this paper, we will proceed via reformulation of the Hamiltonian constraint and its consequent restriction on physical states. This reveals the primacy of the Chern–Simons functional (or rather its imaginary part) and the volume operators, and that they form an

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4 We have made the identification $G = \pi G_{\text{Newton}}/c^3$, to avoid carrying numerical factors around.

5 We shall nevertheless show that the generators of the affine algebra will still be Hermitian.
affine algebra which must be satisfied by all solutions of the local Hamiltonian constraint \( H(x) = 0 \) or the Wheeler–DeWitt equation. We will then quantize this algebra and construct an associated Hilbert space respecting the constraints.

2. Reformulation of the Hamiltonian constraint

From the spatial curvature \( F_{ij}^a \), let us define the Ashtekar magnetic field of the connection \( A_i^a \), an object of density weight 1 given by

\[
\tilde{B}_{ij}^a = \frac{1}{\hbar} \tilde{\epsilon}^{ijk} F_{jk}^a.
\]

When the densitized lapse function \( N = N/\sqrt{|\det q|} \) is treated as a fundamental auxiliary field, then the Hamiltonian constraint is a polynomial constraint of density weight 2, given by (we omit the double-tilde on \( N \) for simplicity; it should be clear from the context)

\[
\delta I \over \delta N(x) = 0 \Leftrightarrow H(x) = \epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ij}^a(x) \tilde{E}_{kj}^b(x) + \Lambda \frac{1}{3} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ij}^a(x) \tilde{E}_{kj}^b(x) \tilde{E}_{kl}^c(x) = 0.
\]

One of the steps in [5] in reformulation of the Hamiltonian constraint is that the Chern–Simons functional \( I_{CS}[A] \) of the connection \( A_i^a \) has the Poisson bracket

\[
\{I_{CS}[A], \tilde{E}_k^a(x)\} = \int_\Sigma d^3y \tilde{B}_k^a(y) [A^b_j(y), \tilde{E}_k^a(x)] = iG \tilde{B}_k^a(x).
\]

The curvature term of (8) can thus be written using this Poisson bracket by contracting (9) with two factors of \( \tilde{E}_a^i \) in antisymmetric combination and using the definition of determinants, yielding

\[
\{I_{CS}[A], \det \tilde{E}(x)\} = \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ij}^a(x) \tilde{E}_{jk}^b(x) [I_{CS}[A], \tilde{E}_i^a(x)] = \frac{iG}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ij}^a(x) \tilde{E}_{jk}^b(x) \tilde{E}_k^c(x).
\]

Let us define a local volume squared operator \( V^2(x) \) given by

\[
V^2(x) = \frac{1}{2} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ij}^a(x) \tilde{E}_{jk}^b(x) \tilde{E}_k^c(x) = \det \tilde{E}(x).
\]

Then using (10), equation (8) can be written in the following form:

\[
\epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ij}^a(x) \tilde{E}_{jk}^b(x) \tilde{E}_k^c(x) + 2 \Lambda \det \tilde{E}(x) = \frac{2i}{G} \{I_{CS}[A], V^2(x)\} + 2 \Lambda V^2(x) = 0.
\]

Equation (11) is a polynomial constraint, a highly desirable feature from the standpoint of the quantum theory. Indeed, the quantum Wheeler–DeWitt equation defined from the fundamental commutation relations (5) will have symmetric factor-ordering (omitting the hats for simplicity) [5]:

\[
-\frac{2}{\hbar G} \{I_{CS}[A], V^2(x)\} + 2 \Lambda V^2(x) = \frac{1}{3} \epsilon_{ijk} \epsilon^{abc} \left( \tilde{E}_{ij}^a \tilde{E}_{kb} \tilde{B}_{ac} + \tilde{E}_{ia} \tilde{B}_{jb} \tilde{E}_{kc} + \tilde{B}_{ia} \tilde{E}_{jb} \tilde{E}_{kc} \right) + \Lambda \frac{1}{3} \epsilon_{ijk} \epsilon^{abc} \tilde{E}_{ia} \tilde{E}_{kj} \tilde{E}_{lc} =: H = 0.
\]

Equation (12) is also the Weyl ordering of \( \tilde{E} \) and \( \tilde{B} \). Moreover, the constraint algebra for a symmetric ordering of the constraints can be shown to close formally, which is necessary for consistency of Dirac quantization. Thus this form of the Hamiltonian constraint has some nice properties. It has been pointed out in [12] that background independent field theories are ultraviolet self-regulating if the constraint weight is equal to 1 but not for other density weights.
This suggests that a more appropriate form for the Hamiltonian constraint for quantization is a density weight 1 constraint. A density weight 1 Hamiltonian constraint can be accomplished by treating the lapse $N$ rather than the densitized lapse $\tilde{N}$ as the basic auxiliary field in the action (2). So vis-à-vis (8), variation of the lapse $N(x)$ yields

$$[\det \widetilde{E}(x)]^{-\frac{1}{2}} \left( \varepsilon_{ijk} \epsilon^{abc} \tilde{E}^i_{a} \tilde{E}^j_{b} \tilde{R}^k_{c} + \frac{\Lambda}{3} \varepsilon_{ijk} \epsilon^{abc} \tilde{E}^i_{a} \tilde{E}^j_{b} \tilde{E}^k_{c} \right)$$

$$= -\frac{2i}{G} [\det \widetilde{E}(x)]^{-\frac{1}{2}} ([I_{CS}[A], \det \widetilde{E}(x)] + iGA \det \widetilde{E}(x)) = 0. \quad (13)$$

For reasons related to diffeomorphism invariance which will come up later, we will need to write the Hamiltonian constraint in terms of the square root of $\det \widetilde{E}$. Subject to non-vanishing $\det \widetilde{E}$, (13) yields the weight 1 Hamiltonian constraint as

$$[I_{CS}[A], \sqrt{\det \widetilde{E}(x)}] = -\frac{iGA}{2} \sqrt{\det \widetilde{E}(x)}. \quad (14)$$

Equation (14) is however non-polynomial in terms of the fundamental variable $\tilde{E}_a$ on account of the presence of the square root, which is an undesirable feature from the standpoint of quantization, but it is polynomial in the local volume operator

$$V(x) = \sqrt{\frac{1}{8} \varepsilon_{ijk} \epsilon^{abc} \tilde{E}^i_{a}(x) \tilde{E}^j_{b}(x) \tilde{E}^k_{c}(x)}, \quad (15)$$

with the absolute value sign put in so that $V(x)$ is real both for positive and negative triad orientations (while $V(x)$ is of density weight 1, we will omit the tilde symbol over it for notational simplicity. This should not lead to any confusion, and the proper density weight should be understood from the context). In terms of $V(x)$, the preferred local density weight 1, Hamiltonian constraint can now be written (in $V$ and $A$) as

$$[I_{CS}[A], V(x)] = -\frac{iGA}{2} V(x) \forall x. \quad (16)$$

3. Affine group and quantum gravity with Ashtekar variables

The importance of the affine group for quantum gravity was first pointed out by Klauder in [6, 13, 14], and the general concepts of continuous representation theory for the affine group have been developed by Klauder and Aslaksen in [15]. It is well known, from the Stone–von Neumann theorem, that there is only one irreducible representation up to unitary equivalence of canonical, self-adjoint operators $p$ and $q$ satisfying the Weyl form of the canonical commutation relations $[q, p] = -i$. This representation, equivalent to the Schrödinger representation, implies that the spectrum of both $p$ and $q$ cover the whole real line. The affine commutation relation for a single degree of freedom takes the form [15]

$$[D, q] = -iq, \quad (17)$$

where $D = (pq + qp)/2$ denotes the dilation operator and $q$ is the operator being dilated. It has been established that there exist two and only two unitarily inequivalent, irreducible representations $\pi$ of the affine group $\text{Aff}(\mathbb{R})$: one representation $\pi^+$ for which the spectrum of $q$ is positive and another $\pi^-$ for which it is negative. This provided the motivation for a multidimensional generalization of the affine group provided by Klauder, where the dilated objects become replaced with operators corresponding to the spatial 3-metric $q_{ij}$. As noted in [6], the metric must satisfy certain positivity requirements which must be reflected in the quantum theory. Appendix A is a supplement on the measure, inner product and coherent states associated with the affine group.
A comparison of (16) with (17) reveals that the local Hamiltonian constraint (16) is nothing other than the Lie algebra of affine transformations of the straight line. More precisely, it is an infinite number of affine Lie algebras, one Lie algebra \( \text{aff}(x) \) per spatial point \( x \). The Chern–Simons functional \( I_{CS}[A] \) plays the role of the dilator, and \( V(x) \) plays the role of the object being dilated. By applying the affine quantum gravity concept to the Ashtekar variables, we will ultimately construct physical Hilbert spaces based upon the representation \( \pi^+ \), thereby endowing the local volume operator for \( V(x) \) with a positive spectrum. There is an important caveat though: \( I_{CS} \) is not Hermitian due to complex (anti-)self-dual Ashtekar variables, but we shall show that in (16) only the imaginary part of the Chern–Simons functional, \( Q \), is relevant (the proof is relegated to appendix B), and thus the affine representation correspondence can be made exact.

We have shown the local density weight 1 Hamiltonian constraint \( H(x) = 0 \), as shown in (16), can classically be written as a Poisson bracket

\[
\{ -i I_{CS}[A], V(x) \} = - \left( \frac{G \Lambda}{2} \right) V(x) \forall x.
\]

With the following definitions for the real and the imaginary parts of the Chern–Simons functional,

\[
Y = \text{Re}[I_{CS}[A]]; \quad Q = \text{Im}[I_{CS}[A]].
\]

it follows that at the classical level, the Hamiltonian constraint (18) is

\[
- i [Y, V(x)] + [Q, V(x)] = - \left( \frac{G \Lambda}{2} \right) V(x).
\]

We will now make use of a remarkable identity

\[
\{ -i I_{CS}[A], V(x) \} = \{ \text{Im}[I_{CS}[A]], V(x) \},
\]

namely that the Poisson bracket of \( V(x) \) with the Chern–Simons functional \( I_{CS}[A] \) is the same as the Poisson bracket of \( V(x) \) with its imaginary part. This is proven in appendix B. This means that the first term on the left-hand side of (20) is zero, which implies the following fundamental Poisson brackets:

\[
\{ Q, V(x) \} = - \left( \frac{G \Lambda}{2} \right) V(x); \quad [Y, V(x)] = 0.
\]

Thus the imaginary part \( Q \) of the Chern–Simons functional \( I_{CS} \) admits an affine Lie algebraic structure with \( V(x) \) for each \( x \), while the real part \( Y \) commutes with \( V(x) \).

Noting that \( Q = \frac{1}{2} (I_{CS} - I_{CS}^\dagger) = Q^\dagger \), and \( V^\dagger = V \), it follows from equation (12) that (in the quantum operators we omit the hats for simplicity, they will be restored when necessary)

\[
[Q, V^2(x)] = -2i \lambda V^2(x); \quad \lambda = \frac{\hbar G \Lambda}{2}.
\]

In order to formulate the Hamiltonian constraint as a weight 1 scalar density and to make contact with the affine representation, we postulate

\[
\{ Q, V(x) \} = -i \lambda V(x)
\]

as the equation of the Hamiltonian constraint defining physical states. It holds at the classical Poisson bracket level, and it is also a sufficient condition for (12) and (23).

4. Algebraic quantization

We shall now proceed with the quantization of the theory, bringing in the relevant steps from the algebraic quantization program [7, 16].
(i) **Step 1.** Our first step will be to identify an appropriate set \( S \) of \( SO(3) \) gauge-invariant classical objects, which is closed under Poisson brackets and under complex conjugation
\[
S = \{ I, Q, V(x) \}_{x \in \Sigma}.
\] (25)

Encoded in the requirement that the set \( S \) be closed under Poisson brackets is the solution to the local Hamiltonian constraint (24). Combined with the fact that \( S \) is invariant under the identity-connected component of complex \( SO(3) \) transformations, this will address the Gauss’ law and Hamiltonian constraints. We will address the diffeomorphism constraint shortly.

(ii) **Step 2.** Each function in \( S \) will be regarded as an elementary classical variable which is to have an unambiguous quantum analogue. Since \( Q \) and \( V(x) \) are composite objects constructed respectively from purely commuting coordinate \( A_i^a(x) \) and momentum \( \hat{E}^i_a(x) \) variables from the original Ashtekar phase space, their quantum analogues will be free of ordering ambiguities.

With each element \( I, Q, V(x) \in S \) we associate an abstract operator \( \hat{I}, \hat{Q}, \hat{V}(x) \), which defines the free associative algebra \( B_{\text{aux}} \) generated by these elementary quantum operators. Upon this we impose the commutation relation, consistent with the Heisenberg–Dirac promotion of classical Poisson brackets to quantum commutators
\[
[\hat{Q}, V(x)] = -\left(\frac{G A}{2}\right) V(x) \quad \longrightarrow \quad \frac{1}{i\hbar} [\hat{Q}, \hat{V}(x)] = -\left(\frac{G A}{2}\right) \hat{V}(x).
\] (26)

Then using the definitions (19), the quantum Hamiltonian constraint can be written as an infinite number of affine commutation relations, one affine commutation relation per spatial point \( x \):
\[
[\hat{Q}, \hat{V}(x)] = -\frac{\hbar}{i} \hat{V}(x); [\hat{Q}, \hat{Q}] = [\hat{V}(x), \hat{V}(y)] = 0.
\] (27)

The following exponentiated point-wise relations for any real numbers \( a \) and \( b \) can be immediately written down from the commutation relations (27):
\[
e^{-i a \hat{Q} \hat{V}(x)} e^{i b \hat{Q} \hat{V}(x)} = e^{-i a \hat{V}(x)} e^{i b \hat{V}(x)} = \hat{Q} e^{i b \hat{V}(x)} = \hat{Q} + \lambda b \hat{V}(x).
\] (28)

Since \( V(x) \) is a locally defined object of density weight 1, it is not diffeomorphism invariant. It transforms under spatial diffeomorphisms parametrized by any smooth vector field \( \xi^i \in C^\infty(\Sigma) \) as
\[
\delta\xi V(x) = \xi^i (\partial_i V(x)) \neq 0,
\] (29)

which means that the local Hamiltonian constraint \( H(x) = 0 \) is also not diffeomorphism invariant and is of density weight 1. But we want our states to be solutions of the diffeomorphism constraint \( H(x) = 0 \) in addition to the Gauss’ law constraint \( G_{\text{aux}}(x) = 0 \), while at the same time lying in the kernel of \( H(x) \). That is, we want physical states, or elements of the physical Hilbert space \( \mathcal{H}_{\text{phys}} \). For diffeomorphism-invariant statements, it will be appropriate to construct a global total volume functional \( V \) from the local function \( V(x) \) given by
\[
V = \int_{\Sigma} d^3x V(x).
\] (30)

Note that \( V \), the volume of 3-space \( \Sigma \), is a diffeomorphism-invariant quantity, and so is \( \hat{Q} \).

On integrating (27) over \( \Sigma \), and then \( [\hat{Q}, \hat{V}] = -\frac{\hbar}{i} \hat{V} \), the global form of an affine algebra involving \( \hat{Q} \) and the total volume \( V \) must also hold. So the following exponentiated
relations, a weaker form of (28) in relation to the Hamiltonian constraint $H(x)$, are also true:

$$e^{-i\lambda\hat{\mathcal{V}}^\dagger} e^{i\lambda\hat{\mathcal{V}}} = e^{-i\lambda\hat{\mathcal{V}}^\dagger} e^{i\lambda\hat{\mathcal{V}}} = \hat{\mathcal{Q}} + \lambda\hat{b}\hat{\mathcal{V}}.$$  \hfill (31)

(iii) Step 3. Next, we introduce an involution operation $\ast$ on $B_{\text{aff}}$, which defines an algebra $B_{\text{aff}}^\ast$. So one must have $(A^\ast)^\ast = A$, $(B^\ast)^\ast = B$, and $(AB)^\ast = A^\ast B^\ast$ and similarly for the products in opposite order. The quantum analogues of $A$ and $B$ must be self-adjoint operators such that

$$A^\ast \rightarrow \hat{A}^\dagger; B^\ast \rightarrow \hat{B}^\dagger; (\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$$  \hfill (32)

for all $A, B$. Applying this to (27) one sees that

$$[\hat{\mathcal{Q}}^\dagger, \hat{\mathcal{V}}^\dagger(x)] = -i\hbar\hat{\mathcal{V}}^\dagger(x); [\hat{\mathcal{Q}}^\dagger, \hat{\mathcal{V}}^\dagger] = [\hat{\mathcal{V}}^\dagger(x), \hat{\mathcal{V}}^\dagger(y)] = 0.$$  \hfill (33)

The result is that the set of elementary observables $\mathcal{S}$ is not only closed under Poisson brackets and complex conjugation, but also is consistent with the involution operation $\ast$. The existence of self-adjoint operators $\hat{\mathcal{Q}}$ and $\hat{\mathcal{V}}$ requires that a physical Hilbert space $\mathcal{H}_{\text{phys}}$ be defined, such that the $\dagger$ operation acts by Hermitian conjugation. We use the term ‘physical’ since all of the constraints will have inherently been implemented.

(iv) Step 4. Ultimately we will construct a linear $\ast$-representation $\pi$ of the abstract algebra $B_{\text{aff}}^\ast$ via linear operators on the physical Hilbert space $\mathcal{H}_{\text{phys}}$. For the remaining steps of the quantization procedure we will proceed along a different path to the one presented in [16], as we will now be addressing the quantization of the Hamiltonian constraint and the physical states from a different interpretation.

It could be argued that $\mathcal{S}$ is not ‘large enough’ since it essentially consists merely of three elements. After all, there are an infinite number of sufficiently regular functions on the original Ashtekar phase space $(A^a, E^a)$ which cannot be obtained as a sum of products of elements of $\mathcal{S}$. On the other hand, $\mathcal{S}$ is ‘large enough’ to admit nontrivial, unitary representations of its associated Lie algebra on a natural Hilbert space in which all physical states satisfy, and arise from, the affine algebra.

4.1. Construction of the physical Hilbert space $\mathcal{H}_{\text{phys}}$

Having put in place the necessary elements, we will now proceed with the construction of the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Let us define for our fiducial vector $|\eta\rangle = |0, 0\rangle$, a gauge- and a diffeomorphism-invariant states lying in the kernel of the quantum constraints $|0, 0\rangle \in \text{Ker} \hat{\mathcal{C}}$, where $\hat{\mathcal{C}} = \{ \hat{G}_c(x), \hat{H}(x), \hat{H}(x) \}$ are the Gauss’ law, diffeomorphism and Hamiltonian constraints. By $\text{Ker} |\hat{H}(x)\rangle$, we mean that the Hamiltonian constraint acting on $|\eta\rangle$ is given by the affine commutation relation

$$\lambda\hat{\mathcal{V}}(x)|0, 0\rangle = [\hat{\mathcal{Q}}, \hat{\mathcal{V}}(x)]|0, 0\rangle,$$  \hfill (34)

which is a restatement of $\hat{H}(x) = 0$. In this interpretation, $|0, 0\rangle$ plays the role of a ‘seed’ from which other physical states can be obtained. A generic basis (denoted by $|\alpha\rangle$) of a representation space of the affine group has resolution of unity, $\int \mu d\alpha |\alpha\rangle\langle \alpha| = I$, with respect to an appropriate measure $\mu$ (see, for instance the discussion in appendix A). The fiducial state can be expanded as $|0, 0\rangle = \int d\alpha C(\alpha)|\alpha\rangle$ for some coefficients $C(\alpha)$. Since $\hat{Q}$ is a Hermitian generator, $e^{-i\omega\hat{Q}}$, where $\omega$ is an arbitrary real-valued dimensionless numerical constant, is a unitary affine group element whose representation is $D^\omega_{\alpha\alpha'} = |\alpha'\rangle e^{-i\omega\hat{Q}}|\alpha\rangle$. It follows that
lies in the kernel of all of the constraints.

\[ [\hat{Q}, \hat{V}(x)]e^{-i\hat{Q}}|0, 0\rangle = \int \mu \, \text{d}x' [\hat{Q}, \hat{V}(x')]|a\rangle \int \mu \, \text{d}x' \langle a' | e^{-i\hat{Q}}|a\rangle |a, 0\rangle = \int \mu \, \text{d}x' [\hat{Q}, \hat{V}(x')]|a\rangle \int \mu \, \text{d}x' D_{a' a}^{\hat{Q}}|a\rangle |0, 0\rangle = \int \mu \, \text{d}x' \lambda \hat{V}(x')|a\rangle \int \mu \, \text{d}x' \langle a' | e^{-i\hat{Q}}|a\rangle |0, 0\rangle = \lambda \hat{V}(x) e^{-i\hat{Q}}|0, 0\rangle, \tag{35} \]

where we have used the fact that an affine representation \( |\alpha\rangle \) satisfies, by definition, \([\hat{Q}, \hat{V}(x')]|a\rangle = \lambda \hat{V}(x')|a\rangle \quad \forall \alpha'. \) But (35) demonstrates that \(|a, 0\rangle = e^{-i\hat{Q}}|0, 0\rangle\) which is diffeomorphism-invariant since \([\hat{H}, \hat{Q}] = 0\) as well.

\[ \lambda \hat{V}(x)|a, 0\rangle = [\hat{V}(x), -i\hat{Q}]|a, 0\rangle, \tag{36} \]

which is none other than the vanishing of the local Hamiltonian constraint \(\hat{H}(x)\) acting on the dilated state \(|a, 0\rangle\). So given that \(|0, 0\rangle \in \text{Ker} \, \hat{C}\), it follows that \(|a, 0\rangle \in \text{Ker} \, \hat{C}\) as well, i.e. \(|a, 0\rangle \in \mathcal{H}_{\text{phys}}\).

Using the unitary group element \(e^{-ib\hat{V}(x)}\) at each spatial point, and noting that \([\hat{V}(x), \hat{V}(y)]\) commutes \(\forall x, y \in \Sigma\), the global volume \(V = \lim_{\Delta x \to 0} \sum_{x \in \Sigma} V(x) \Delta x\), corresponds to the unitary group element \(e^{-ib\hat{V}} = \lim_{\Delta x \to 0} \prod_{x \in \Sigma} e^{-ib\hat{V}(x)} \Delta x\), where \(b\) is an arbitrary real-valued numerical constant of mass dimension \([b] = 3\). This can be used to construct the diffeomorphism-invariant state, translated in the carrier space of the affine group, \(|0, b\rangle = e^{-ib\hat{V}}|0, 0\rangle\). Similar considerations will then yield

\[ [\hat{Q}, \hat{V}(x)]e^{-i\hat{Q}}|0, 0\rangle = \int \mu \, \text{d}x' [\hat{Q}, \hat{V}(x')]|a\rangle \int \mu \, \text{d}x' \langle a' | e^{-i\hat{Q}}|a\rangle |0, 0\rangle = \int \mu \, \text{d}x' [\hat{Q}, \hat{V}(x')]|a\rangle \int \mu \, \text{d}x' D_{a' a}^{\hat{Q}}|a\rangle |0, 0\rangle = \int \mu \, \text{d}x' \lambda \hat{V}(x')|a\rangle \int \mu \, \text{d}x' \langle a' | e^{-i\hat{Q}}|a\rangle |0, 0\rangle = \lambda \hat{V}(x) e^{-i\hat{Q}}|0, 0\rangle, \tag{37} \]

which leaves us with

\[ \lambda \hat{V}(x)|0, b\rangle = [\hat{V}(x), -i\hat{Q}]|0, b\rangle, \tag{38} \]

which again expresses the vanishing of the local Hamiltonian constraint \(\hat{H}(x)\) acting on the translated state \(|0, b\rangle\). As \([\hat{H}, \hat{V}] = 0\), given that \(|0, 0\rangle \in \text{Ker} \, \hat{C}\), then \(|0, b\rangle \in \text{Ker} \, \hat{C}\) as well.

4.2. General quantum affine group element

Having illustrated the idea for physical states corresponding to transformations by \(a\) and by \(b\) individually, we will now consider the two transformations applied together. Define the general affine coherent state

\[ |a, b\rangle = U(a, b)|0, 0\rangle = e^{-i\hat{Q}} e^{-i\hat{V}}|0, 0\rangle. \tag{39} \]

By using the resolution of unity, and invoking the linearity of the action of the group elements \(D_{a' a}^{\hat{Q}}\) and \(D_{a' a}^{\hat{V}}\), it follows that

\[ \lambda \hat{V}(x)|a, b\rangle = [\hat{Q}, \hat{V}(x)]|a, b\rangle \quad \text{and} \quad e^{-i\hat{V}}|a, b\rangle = [\hat{Q}, e^{-i\hat{V}}]|a, b\rangle \]

also holds for the coherent state \(|a, b\rangle\). The result is that given a fiducial state \(|0, 0\rangle \in \text{Ker} \, \hat{C}\), it follows that an arbitrary diffeomorphism-invariant coherent state \(|a, b\rangle = U(a, b)|\eta\rangle \in \mathcal{H}_{\text{phys}}\) lies in the kernel of all of the constraints.
The coherent nature with respect to the uncertainty in $Q$ and the total volume $V$, and other properties, of these states are discussed in appendix A. In particular, the uncertainty relation which depends on the cosmological constant is $(\Delta \hat{V})^2 (\Delta \hat{Q})^2 \geq \frac{\Delta^2}{\Delta^2 x} (\hat{V})^2$.

4.3. Normalizability and inner product

If a fiducial state $|0, 0\rangle$ satisfies the admissibility condition (A.6), then all states $|a, b\rangle$ are unitarily related to $|0, 0\rangle$, and form an overcomplete basis of physical coherent states.\footnote{These results and the results which follow are independent of the specific carrier representation space of the affine group.} Let the fiducial state be normalized such that

$$\langle 0, 0 | 0, 0 \rangle = 1.$$ \hspace{1cm} (41)

We would like to find the inner product between two states $|a, b\rangle$ and $|a', b'\rangle$. This is given by

$$\langle a', b' | a, b \rangle = \langle 0, 0 | e^{i\hat{a}^\dagger \hat{Q}} e^{\hat{b}^\dagger \hat{Q}} e^{-i\hat{a} \hat{Q}} e^{-i\hat{b} \hat{Q}} | 0, 0 \rangle$$

$$= \langle 0, 0 | e^{i\hat{a}^\dagger \hat{Q}} e^{-i\hat{a}^\dagger \hat{Q}} e^{i\hat{b} \hat{Q}} e^{-i\hat{b} \hat{Q}} | 0, 0 \rangle.$$ \hspace{1cm} (42)

Proceeding from (42) and using the trick $I = e^{-\hat{A}^\dagger \hat{A}}$, we have

$$\langle a', b' | a, b \rangle = \langle 0, 0 | e^{i\hat{a} \hat{Q}} e^{-i\hat{a} \hat{Q}} e^{i\hat{b}^\dagger \hat{Q}} e^{-i\hat{b}^\dagger \hat{Q}} \exp[i\hat{b} e^{-\lambda (\hat{a}^\dagger - \hat{a})} \hat{V}] | 0, 0 \rangle$$

$$= \langle 0, 0 | e^{i\hat{a} \hat{Q}} \exp[i\hat{b} e^{-\lambda (\hat{a}^\dagger - \hat{a})} \hat{V}] | 0, 0 \rangle = \langle 0, 0 | a - a', b - b' e^{\lambda (\hat{a}^\dagger - \hat{a})} \rangle.$$ \hspace{1cm} (43)

As (43) shows, the overlap between two states is equivalent to performing two affine group transformations in opposite directions, and the coherent states will thus have unit norm.

5. Summary and discussion

The affine quantization program in conjunction with the relevant steps of the algebraic quantization procedure yield a natural Hilbert space and facilitate the construction of gravitational coherent physical states $|a, b\rangle$ and other physical states which satisfy all the constraints of four-dimensional Lorentzian GR.

While we have performed a quantization of a Poisson-closed algebra consisting essentially of three elements $Q, V(x)$ and $I$ (the identity operator), it is important to note that these objects do not separate the points of the full classical phase space of GR. Nevertheless, they are of utmost importance to the Hamiltonian constraint, and through them an affine representation admitting natural Hilbert space structures consistent with the full Lorentzian theory can be defined. The total volume $V$, $I$ and $Q$ operators as diffeomorphism- and gauge-invariant entities form a very limited subset of invariant phase space elements. However, the group theoretical aspect of the affine quantization program enables the infusion of new results and techniques from wavelet transform theory into quantum gravity, wherein the powerful coherent state machinery becomes available. In this approach, one can dispense with direct reference to the quantum operators $\hat{V}(x), \hat{V}$ and $\hat{Q}$, and think of the carrier space in terms of coherent states encoding the group action. The results of this paper depend only on the existence of fiducial vectors satisfying the admissibility condition, and are independent of the specific representation or polarization of the carrier space. Within the context of this work, all physical states of quantum gravity with cosmological constant must come from representations of the affine algebra, since it is the full local Hamiltonian constraint, and
not just a mini-superspace version, that has been reformulated as an affine algebra. We have demonstrated that the affine quantization program of Klauder can indeed be applicable to four-dimensional full Lorentzian signature quantum gravity. It is also intriguing that the formulation of the Hamiltonian constraint as an affine algebra is predicated upon a non-vanishing cosmological constant (current observational bounds place $\Lambda_{\text{Planck}} \sim 10^{-120}$), and a fundamental uncertainty relation of the form $\langle \Delta \hat{V} \rangle \langle \Delta \hat{Q} \rangle \geq \frac{1}{2} = 2\pi \Lambda_{\text{Planck}}$ may govern all physical states of quantum gravity.

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Appendix A. Measure, inner product and coherent states associated with the affine group

The action of the affine group $G_{\text{aff}}$ on the real numbers has the following matrix representation:

$$U(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \tag{A.1}$$

which has a natural action on itself by left-matrix multiplication via the group multiplication law

$$U(a, b)U(a', b') = U(aa', b + ab'). \tag{A.2}$$

The affine group has invariant left Haar and right Haar measures $d\mu_l$ and $d\mu_r$, respectively, given by

$$d\mu_l(a, b) = \frac{da \wedge db}{a^2}; \quad d\mu_r(a, b) = \frac{da \wedge db}{a}. \tag{A.3}$$

While these two measures are equivalent in the sense of measure theory, they are not the same measure. Hence the affine group is not a unimodular group. With respect to the unitary representation of the group, one sees, from exponentiation of (17), that the parametrization of the general group element as the linear operator

$$U(a, b) = e^{-iD\ln a} e^{-ibq} \tag{A.4}$$

correctly reproduces the group multiplication law (A.2).\footnote{Note that group composition for the unitary form (A.4) must be performed in the opposite order as the matrix version (A.2).}

Using any normalized fiducial vector $|\eta\rangle$ with $\langle \eta | \eta\rangle = 1$, one can define an overcomplete basis of unit vectors as

$$|a, b\rangle = U(a, b)|\eta\rangle. \tag{A.5}$$

So for each fiducial vector $|\eta\rangle$, there exists a set of states labeled by $a$ and $b$. Since there is a large class of possible fiducial vectors $|\eta\rangle$, one has certain freedom in the choice of Hilbert spaces (one could go further, utilizing the admissibility condition for fiducial vectors in order to construct reproducing kernel Hilbert spaces (see e.g. [14])).

Equation (A.5) bears an analogy to continuous wavelet transform theory [17], where $|a, b\rangle$, a set of coherent states, plays the role of the wavelet transform of a mother wavelet (signal).
\(|\eta\rangle\). A necessary condition is that \(|\eta\rangle\) satisfy a certain admissibility condition predicated on its existence as an element of the set \(\psi \in L^2(\mathcal{R}, dz)\). The admissibility condition is
\[
c_\psi = 2\pi \int_{-\infty}^{\infty} \frac{dx}{|\xi|} |\phi(x)|^2 < \infty, \tag{A.6}\]
where \(\phi\) is the Fourier transform of \(\psi\). Note that the unitary action of (A.4) on \(|\eta\rangle\) translates into the language of functions into
\[
\psi'(z) = U(a, b)\psi = |a|^{-1/2}\psi\left(\frac{z-b}{a}\right). \tag{A.7}\]
where \(b \in \mathcal{R}\) and \(a \neq 0\). That \(U(a, b)\) is unitary is evident in the fact that it preserves the Hilbert space norm
\[
\|\psi\|^2 = \int_{-\infty}^{\infty} dz|\psi'(z)|^2. \tag{A.8}\]
In the context of quantum gravity, the square integrability of the representation \(U(a, b)\) implies the existence of fiducial vectors \(\psi\) for which the matrix element \(\langle U(a, b)\psi|\psi\rangle\) is square integrable as a function of the labels \(a\) and \(b\) with respect to the left Haar measure
\[
\int_{G_{af}} d\mu(a, b)|\langle U(a, b)\psi|\psi\rangle|^2 < \infty. \tag{A.9}\]
Additionally, since the Fourier transform is a linear isometry, it follows that the Fourier transformed version of (A.4) also provides a unitary representation of the affine group in its action on \(\phi\). Both representations are irreducible.

In this paper, we will use the ‘affine conjugate’ pair \((\hat{V}, \hat{Q})\), where \(V\) is the integral of \(V(x)\) over all space (the volume operator) to describe quantum gravity. Our application is somewhat different from that described by the affine conjugate pair \((\hat{q}_{ij}, \hat{r}_{ij})\) due to Klauder, namely the spatial metric and the field \(\hat{r}_{ij}\) related to the conjugate momentum \(\hat{r}^{kl}\) of the spatial metric. Closure of the constraint algebra in canonical quantum gravity described by \((\hat{q}_{ij}, \hat{r}^{kl})\) produces second-class constraints [18]. This is unlike the case as described in the Ashtekar variables \((A_a^i, \hat{E}^i_a)\), where the constraint algebra produces first class constraints (at least at the classical level). Hence for this paper, we will use the affine commutator of \(\hat{Q}\) and \(\hat{V}\).

In the following, we want to construct the coherent state framework according to the article [14] by using the ‘affine conjugate’ pair \((\hat{Q}, \hat{V})\), as another application for the affine representation. We also can mimic the treatment of the spatial matrix field \(g\) and the momentum matrix field \(\pi\) to describe this topic. Articles [13, 18] provide further descriptions. Here, the commutation relation for \((\hat{V}, \hat{Q})\) is suitable for the one-dimensional affine algebra, \([\hat{V}, -i\hat{Q}] = \lambda\hat{V}\). This algebra follows from integration over all spatial points of the local fundamental commutation relation \([\hat{V}(x), -i\hat{Q}] = \lambda\hat{V}(x)\), which is the local Hamiltonian constraint \(H(x) = 0\).

One can write the following realization of \(Q\) (for ease of discussion here we absorb \(1/\lambda\) in the definition of \(Q\):
\[
\hat{Q} = \frac{1}{2}[\hat{\partial}\hat{V} + \hat{V}\hat{\partial}], \quad \hat{\partial} = -i\frac{\partial}{\partial\hat{V}}, \tag{A.10}\]
in terms of \(V\) which is also equal to
\[
\hat{Q} = \frac{1}{2}\left[-i\frac{\partial}{\partial\hat{V}} + \hat{V}\left(-i\frac{\partial}{\partial\hat{V}}\right)\right] = -\frac{i}{2} - i\hat{V}\frac{\partial}{\partial\hat{V}}. \tag{A.11}\]

With respect to a polarization on \(V\)-space one has the inner product
\[
\langle \phi|\psi\rangle = \int_0^\infty\phi(\hat{V})^*\psi(\hat{V}) d\hat{V},
\]
consistent with the positivity of \( V \) as required by the affine algebra. The operator \( \hat{Q} \) generates the unitary dilations in the representation space:

\[
e^{-i\hat{Q}\hat{\psi}}(\hat{V}) = e^{-\frac{i}{\hbar}\hat{Q}}(\hat{V}e^{\frac{i}{\hbar}\hat{Q}})
\]

\[
\|e^{-i\hat{Q}\hat{\psi}}\| = \|\hat{\psi}\|.
\]

From the definition for unitary operators \( U(a, b) = e^{-i\hat{Q}} e^{-i\hat{Q}} \) subject to the composition rule

\[
U(a', b')U(a, b) = U(a' + a, b + b'),
\]

one can construct a set of coherent states:

\[
|a, b\rangle = U(a, b)|\eta\rangle,
\]

where \(|\eta\rangle\) is an unspecified normalized fiducial vector in the representation space. The family of coherent states provides a resolution of unity in the form

\[
N^{-1} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} da \, e^{i\langle a, b\rangle} |a, b\rangle \langle a, b| = I,
\]

where \( N = 2\pi \int_{0}^{\infty} d\hat{V} \hat{V}^{-1} |\eta(\hat{V})|^{2} < \infty \) is finite.

For minimum uncertainty states, one takes the Heisenberg uncertainty principle into consideration. In general quantum mechanics, any two observable operators \( \hat{A}, \hat{B} \) are consistent with the uncertainty relation

\[
\langle (\Delta \hat{A})^{2} \rangle \langle (\Delta \hat{B})^{2} \rangle \geq \frac{1}{4} \langle [\hat{A}, \hat{B}] \rangle^{2}.
\]

But on the affine representation space, any two self-adjoint operators can be consistent with the general affine uncertainty relation (here, the two operators are \( \hat{V} \) and \( \hat{Q} \))

\[
\langle (\Delta \hat{V})^{2} \rangle \langle (\Delta \hat{Q})^{2} \rangle \geq \frac{\lambda^{2} \langle (\hat{V})^{2} \rangle}{4}.
\]

We can use the general analysis just like that in [14] to prove the affine uncertainty relation by taking a normalized fiducial vector of form \( \eta_{1}(\hat{V}) = C_{1}(\alpha', \beta')\hat{V}^{\alpha} e^{-\beta\hat{V}} \), where \( \alpha', \beta' \) are positive real coefficients and \( C_{1}(\alpha', \beta') \) is the normalization constant.

**Appendix B. Proof that the real part of the Chern–Simons functional Poisson-commutes with the volume element**

Consider the Chern–Simons functional: \( I_{CS}[A] = \frac{1}{2\pi} \int_{M} [A^{\mu} \wedge dA_{\mu} + \frac{1}{4} \varepsilon^{abc} A_{a} \wedge A_{b} \wedge A_{c}] \). Using the definition (1), and defining the one form \( k^{a} = k_{a}^{\mu} dx_{\mu} \), expansion of the Chern–Simons functional in \( k \) and \( \Gamma \) leads straightforwardly to the expression

\[
I_{CS}[A] = I_{CS}[\Gamma] + \gamma \int_{M} R_{a}^{\mu} \wedge k^{a} + \frac{\gamma^{2}}{2!} \int_{M} k^{a} \wedge (D^{k})_{a} + \frac{\gamma^{3}}{3!} \int_{M} \varepsilon^{abc} k_{a} \wedge k_{b} \wedge k_{c},
\]

where \( R_{a}^{\mu} = d\Gamma_{a} + \frac{1}{2} \varepsilon_{a}^{bc} \Gamma_{b} \wedge \Gamma_{c} \) is the curvature 2-form of the connection 1-form \( \Gamma^{a} \).

Consider the Poisson bracket of the volume functional and the Chern–Simons functional (for brevity we suppress the label of spatial points \( x \)). The results which follow equally apply to the global volume functional \( V \), just as they apply to the local \( V(x) \). Note also that \( \{F, \sqrt{\det E} \} = \frac{1}{2\sqrt{\det E}} \{F, \det E \} \) for all \( F \). To wit,

\[
\{V, I_{CS}[A]\} = \langle V, I_{CS}[\Gamma] \rangle + \gamma \int_{M} \{V, R_{a}^{\mu} \wedge k^{a}\} + \frac{\gamma^{2}}{2!} \int_{M} \{V, k^{a} \wedge (D^{k})_{a}\}
\]

\[
+ \frac{\gamma^{3}}{3!} \int_{M} \{V, \varepsilon^{abc} k_{a} \wedge k_{b} \wedge k_{c}\}
\]

\[
= \langle V, I_{CS}[\Gamma] \rangle + \frac{\gamma^{2}}{2!} \int_{M} k^{a} \wedge (D^{k})_{a} + \{V, \gamma \int_{M} R_{a}^{\mu} \wedge k^{a} + \frac{\gamma^{3}}{3!} \int_{M} \varepsilon^{abc} k_{a} \wedge k_{b} \wedge k_{c}\}
\]

\[
= \langle V, \text{Re}(I_{CS}[A]) \rangle + \{V, \text{Im}(I_{CS}[A])\} \quad (\gamma = \pm i)
\]

\[
= \{V, \text{Im}(I_{CS}[A])\}.
\]
The result can be explained by the following observations:

1. The term \(\{V, I_{CS}[\Gamma]\}\) is zero because \(\Gamma\) is a function only of \(\vec{E}\), thus Poisson-commuting with \(V\).

2. The term \(\frac{\gamma^2}{2!} \int_M \{V, k^a \wedge (D^F k)_a\}\) also vanishes. We note that

\[
\frac{\gamma^2}{2!} \int_M \{V, k^a \wedge (D^F k)_a\} \propto \left\{ \epsilon_{lmn} \epsilon_{abc} \vec{E}^{id} \vec{E}^{mb} \vec{E}^{nc} \int \epsilon^{ijk} k_a D^f_{ij} k^j \right\}
\]

\[
= \int \epsilon_{lmn} \epsilon_{abc} \epsilon^{ijk} \left( \vec{E}^{id} \vec{E}^{mb} \vec{E}^{nc} \right) k_a D^f_{ij} k^j
\]

\[
= \int \epsilon_{lmn} \epsilon_{abc} \epsilon^{ijk} \left( \vec{E}^{id} k_a D^f_{ij} k^j + \vec{E}^{mb} k_a D^f_{ij} k^j + \vec{E}^{nc} k_a D^f_{ij} k^j \right)
\]

\[
\text{We now focus on the Poisson bracket} \{\vec{E}^{id}, k_a D^f_{ij} k^j\}, \text{namely}
\]

\[
\{\vec{E}^{id}, k_a D^f_{ij} k^j\} = [\vec{E}^{id}, k_a D^f_{ij} k^j] + k_a [\vec{E}^{id}, D^f_{ij} k^j] = 0.
\]

Note that

\[
[\vec{E}^{id}, D^f_{ij} k^j] = D^f_{ij} [\vec{E}^{id}, k^j] \propto \delta^i_j \delta^f_{ij} \delta^a \delta^3.
\]

We have used the shorthand notation \(\delta^a \delta^3 \equiv \delta^a(x, y)\). From this Poisson bracket

\[
\{\vec{E}^{id}, D^f_{ij} k^j\} = \vec{E}^{id} (D^f_{ij} k^j) - (D^f_{ij} k^j) \vec{E}^{id} \propto \delta^i_j \delta^a \delta^3 \vec{E}^{id} = \vec{E}^{id} (D^f_{ij} k^j) - \delta^i_j \delta^a \delta^3 \vec{E}^{id} = \vec{E}^{id} (D^f_{ij} k^j) - \delta^i_j \delta^a \delta^3 \vec{E}^{id},
\]

so we have

\[
[\vec{E}^{id}, k_a D^f_{ij} k^j] = \{\vec{E}^{id}, k_a D^f_{ij} k^j\} + k_a [\vec{E}^{id}, D^f_{ij} k^j] = 0.
\]

The three terms in equation (B.1) will each yield the same result, which can be seen by the relabeling of indices. So it suffices to illustrate the calculation for one term. In this process, we will also use \(\{\vec{E}^{id}, k_{jb}\} \propto \delta^i_j \delta^a \delta^3\).

Consider the term \(\int \epsilon_{lmn} \epsilon_{abc} \epsilon^{ijk} \vec{E}^{id} k_a D^f_{ij} k^j \vec{E}^{mb} \vec{E}^{nc} \) in equation (B.1):

\[
\int \epsilon_{lmn} \epsilon_{abc} \epsilon^{ijk} \vec{E}^{id} k_a D^f_{ij} k^j \vec{E}^{mb} \vec{E}^{nc}
\]

\[
\propto \int \epsilon_{lmn} \epsilon_{abc} \epsilon^{ijk} \delta^a \delta^3 (D^f_{ij} k^j) \vec{E}^{mb} \vec{E}^{nc} + \int \epsilon_{lmn} \epsilon_{abc} \epsilon^{ijk} k_a \delta^3 (D^f_{ij} k^j) \vec{E}^{mb} \vec{E}^{nc}
\]

\[
\propto \int \epsilon^{ijk} \delta^3 (D^f_{ij} k_{ja}) \vec{E}^a + \int (D^f_{ij} \delta^3) (\epsilon^{ijk} k_a \vec{E}^j)
\]

\[
= - \int \delta^3 D^f_{ij} (\epsilon^{ijk} k_{ja}) \vec{E}^a + \int (D^f_{ij} \delta^3) \epsilon^{ijk} k_a \vec{E}^j
\]

\[
\propto - \int \delta^3 D^f \left[ \epsilon_{da} \vec{E}^{da} \vec{E}^{jda} k^j \right] + \int (D^f \delta^3) [\epsilon_{da} \vec{E}^{da} \vec{E}^{jda} k^j]
\]

\[
= \int \delta^3 D^f \left[ \epsilon_{da} \vec{E}^{da} \vec{E}^{jda} k^j \right] - \int (D^f \delta^3) [\epsilon_{da} \vec{E}^{da} \vec{E}^{jda} k^j] = 0.
\]

where \(\Gamma_a = \epsilon_{ad} \vec{E}^{jda} k^j\) is just the Gauss law constraint in the ADM triad formulation of gravity. So equation (B.1) provides no contribution and

\[
\frac{\gamma^2}{2!} \int_M \{\vec{V}, k^a \wedge (D^F k)_a\} = 0
\]

as desired.

3. Therefore, the result is that \(\{\vec{V}, \text{Im}(I_{CS}[A])\} = \{\vec{V}, I_{CS}[A]\}\) since \(\{\vec{V}, \text{Re}(I_{CS}[A])\} = 0\).
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