Admissible heuristics for obstacle clearance optimization objectives

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Abstract
Obstacle clearance in state space is an important optimization objective in path planning because it can result in safe paths. This technical report presents admissible solution- and path-cost heuristics for this objective, which can be used to improve the performance of informed path planning algorithms.

1 Preliminaries
Let $X$ be a state space, $X_{\text{invalid}} \subseteq X$ be the subset of invalid states, and $X_{\text{valid}} := X \setminus X_{\text{invalid}}$ be the subset of valid states. Let $\sigma: [0, l] \rightarrow X_{\text{valid}}$ be a path, i.e., a continuous function parameterized by its arc length, $l < \infty$, and let $\Sigma$ denote the set of all valid paths. Let $\delta: X_{\text{valid}} \rightarrow (0, \infty)$ be the clearance of a valid state and let $c: \Sigma \rightarrow [0, \infty)$ be the reciprocal clearance cost of a path,

$$
\delta(x) := \min_{x' \in X_{\text{invalid}}} \|x - x'\| \quad c(\sigma) := \int_{0}^{l} \frac{1}{\delta(\sigma(t))} \, dt.
$$

(1)

It is assumed that no state on a path of interest has clearance of exactly zero.

Lemma 1 (An upper bound on state clearance). Let $\sigma \in \Sigma$ be a path with arc length $l$. Let $\sigma(t_1) \in X$ be the state $0 \leq t_1 \leq l$ along this path, and let $\delta_1$ be the known clearance of this state, $\delta_1 := \delta(\sigma(t_1))$. The clearance of any state on the same path, $\delta(\sigma(t))$, is then upper bounded by

$$
\delta(\sigma(t)) \leq \delta_1 + |t_1 - t|.
$$

(2)

Proof. (Figure 1) Let $x_1' \in X_{\text{invalid}}$ be one of the closest invalid states of the state $x_1 := \sigma(t_1)$,

$$
x_1' := \arg \min_{x' \in X_{\text{invalid}}} \|x_1 - x'\| \quad \Rightarrow \quad \delta_1 = \|x_1 - x_1'\|.
$$

Because $\sigma(t)$ is parametrized by arc length, any state on the path is at most $|t_1 - t|$ away from the state $x_1$ and therefore at most $\|x_1 - x_1'\| + |t_1 - t|$ away from the state $x_1'$. Since $x_1'$ is an invalid state, the distance $\delta_1 + |t_1 - t|$ provides an upper bound on the true clearance of any state on the path.

2 Solution-cost heuristics
This section presents two lower bounds on the cost of an optimal path between two states. These bounds can be used as admissible cost-to-go heuristics in informed planners, such as Batch Informed Trees [BIT*; 1], Advanced BIT* [ABIT*; 2], and Adaptively Informed Trees [AIT*; 3], if a lower bound on the arc length of the optimal path between two states is known, e.g., the Euclidean distance.

Lemma 2 (An admissible solution-cost heuristic when the clearance of one end state is known). Let $\sigma \in \Sigma$ be a path whose arc length, $l$, is lower bounded by $l_{\text{lb}} \leq l$. Let the clearance of the start or goal state of this path be known and denoted by $\delta_1 := \delta(\sigma(0 \text{ or } l))$. The reciprocal clearance cost $c(\sigma)$ of the path $\sigma$ is then lower bounded by

$$
c(\sigma) \geq \ln \left( \frac{\delta_1 + l_{\text{lb}}}{\delta_1} \right).
$$

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Figure 1: An illustration of Lemma 1. Any state $\sigma(t)$ on the path $\sigma$ can not be farther from the state $\sigma(t_1)$ than $|t_1 - t|$ and must be within the light gray shaded area. The clearance $\delta(\sigma(t))$ of any state on the path can therefore not be larger than the clearance $\delta_1 := \delta(\sigma(t_1))$ of the state $\sigma(t_1)$ plus the distance from that state, i.e., $\delta(\sigma(t)) \leq \delta_1 + |t_1 - t|$.

**Proof.** The bounds are computed by setting $t_1 = 0$ or $t_1 = l$ in Lemma 1 and replacing the clearance function in the integrand of the reciprocal clearance cost (1) with the upper bound on the clearance (2).

First let $t_1 = 0$ (Figure 2a). Then by Lemma 1

$$\delta(\sigma(t)) \leq \delta_1 + t,$$

and the lower bound on the cost is

$$c(\sigma) = \int_0^t \frac{1}{\delta(\sigma(t))} \, dt \geq \int_0^t \frac{1}{\delta_1 + t} \, dt$$

$$= [-\ln(\delta_1 + t)]_0^t$$

$$= \ln(\delta_1 + l) - \ln(\delta_1)$$

$$= \ln \left( \frac{\delta_1 + l}{\delta_1} \right)$$

$$\geq \ln \left( \frac{\delta_1 + l_{\text{lb}}}{\delta_1} \right).$$

Similarly, let $t_1 = l$ (Figure 2b). Then by Lemma 1

$$\delta(\sigma(t)) \leq \delta_1 + l - t,$$

and the lower bound on the cost is again

$$c(\sigma) = \int_0^l \frac{1}{\delta(\sigma(t))} \, dt \geq \int_0^l \frac{1}{\delta_1 + l - t} \, dt$$

$$= [-\ln(\delta_1 + l - t)]_0^l$$

$$= -\ln(\delta_1) - (-\ln(\delta_1 + l))$$

$$= \ln \left( \frac{\delta_1 + l}{\delta_1} \right)$$

$$\geq \ln \left( \frac{\delta_1 + l_{\text{lb}}}{\delta_1} \right).$$

**Theorem 1** (An admissible solution-cost heuristic when the clearance of both end states is known). Let $\sigma$ be a path whose arc length, $l$, is lower bounded by $l_{\text{lb}} \leq l$. Let $\sigma(t_1 = 0) \in X$ and $\sigma(t_2 = l) \in X$ be its start and goal states with clearances $\delta_1$ and $\delta_2$, respectively. The reciprocal clearance cost $c(\sigma)$ of the path $\sigma$ is then lower bounded by

$$c(\sigma) \geq \ln \left( \frac{(\delta_1 + \delta_2 + l_{\text{lb}})^2}{4\delta_1\delta_2} \right).$$
Proof. (Figure 2c) The clearance of any state \( \sigma (t), 0 \leq t \leq l \), on the path is upper bounded by both clearances \( \delta_1 \) and \( \delta_2 \) according to Lemma 1,

\[
\delta (\sigma (t)) \leq \min \{ \delta_1 + |t_1 - t|, \delta_2 + |t_2 - t| \} = \min \{ \delta_1 + t, \delta_2 + l - t \}
\]

A lower bound on the reciprocal clearance cost \( c(\sigma) \) of the path \( \sigma \) can therefore be computed by

\[
c(\sigma) = \int_0^l \frac{1}{\delta (\sigma (t))} \, dt \geq \int_0^l \frac{1}{\min \{ \delta_1 + t, \delta_2 + l - t \}} \, dt.
\]  \( (3) \)

Since \( \delta_1 + t \) is strictly monotonically increasing and \( \delta_2 + l - t \) is strictly monotonically decreasing, the two bounds must intersect at some point, \( t_e \).

\[
\delta_1 + t_e = \delta_2 + l - t_e \quad \Rightarrow \quad t_e = \frac{\delta_2 - \delta_1 + l}{2}.
\]

This intersection must lie within the integration limits because by Lemma 1 we have

\[
\delta (\sigma (t)) \leq \delta_2 + l - t \quad \Rightarrow \quad \delta (\sigma (0)) \leq \delta_2 + l \quad \Rightarrow \quad \delta_1 \leq \delta_2 + l \quad \Rightarrow \quad t_e \geq 0
\]

\[
\delta (\sigma (t)) \leq \delta_1 + t \quad \Rightarrow \quad \delta (\sigma (l)) \leq \delta_1 + l \quad \Rightarrow \quad \delta_2 \leq \delta_1 + l \quad \Rightarrow \quad \delta_2 - \delta_1 \leq l \quad \Rightarrow \quad t_e \leq l.
\]

The minimum in (3) can therefore be written as

\[
\min \{ \delta_1 + t, \delta_2 + l - t \} = \begin{cases} \delta_1 + t & \text{if } t \leq t_e, \\ \delta_2 + l - t & \text{otherwise}, \end{cases}
\]

and the definite integral (3) can be evaluated to

\[
c(\sigma) = \int_0^l \frac{1}{\delta (\sigma (t))} \, dt \geq \int_0^{t_e} \frac{1}{\min \{ \delta_1 + t, \delta_2 + l - t \}} \, dt
\]

\[
= \int_0^{t_e} \frac{1}{\delta_1 + t} \, dt + \int_{t_e}^l \frac{1}{\delta_2 + l - t} \, dt
\]

\[
= [\ln (\delta_1 + t)]_0^{t_e} + [-\ln (\delta_2 + l - t)]_{t_e}^l
\]

\[
= \ln \left( \frac{\delta_1 + \delta_2 - \delta_1 + l}{2} \right) - \ln (\delta_1) + \left( -\ln (\delta_2) - \left( -\ln \left( \frac{\delta_2 + l - \delta_2 - \delta_1 + l}{2} \right) \right) \right)
\]

\[
= \ln \left( \frac{\delta_1 + \delta_2 + l}{2\delta_1} \right) + \ln \left( \frac{\delta_1 + \delta_2 + l}{2\delta_2} \right)
\]

\[
\geq \ln \left( \frac{(\delta_1 + \delta_2 + l)^2}{4\delta_1 \delta_2} \right).
\]

\[ \Box \]
3 Path-cost heuristics

Informed sampling-based planning algorithms, such as BIT*, ABIT*, and AIT*, also use path-cost heuristics, i.e., estimates of the unknown cost of known paths, for example when ordering their search queues. The solution-cost heuristics of Section 2 can be made more accurate for known paths by sampling additional states along the path and computing their clearance. This improves performance if the evaluation of the true edge cost is computationally expensive.

Lemma 3 (An admissible path-cost heuristic when the clearance of any state on the path is known). Let $\sigma \in \Sigma$ be a path with arc length $l$. Let $\sigma(t_1) \in X$ be the state $0 \leq t_1 \leq l$ along this path, and let $\delta_1 := \delta(\sigma(t_1))$ be the known clearance of this state. The reciprocal clearance cost $c(\sigma)$ of the path $\sigma$ is then lower bounded by

$$c(\sigma) \geq \ln \left( \frac{\delta_1 + t_1}{\delta_1} \right) + \ln \left( \frac{\delta_1 + l - t_1}{\delta_1} \right).$$

Proof. (Figure 3a) The lower bound is computed by replacing the clearance function in the integrand of the reciprocal clearance cost (1) with the upper bound on the clearance (2),

$$c(\sigma) = \int_{0}^{l} \frac{1}{\delta(\sigma(t))} \, dt \geq \int_{0}^{l} \frac{1}{\delta_1 + |t_1 - t|} \, dt$$

$$= \int_{0}^{t_1} \frac{1}{\delta_1 + t_1 - t} \, dt + \int_{t_1}^{l} \frac{1}{\delta_1 + t - t_1} \, dt.$$  

$$= -\ln (\delta_1 + t_1 - t_1)_{t_1} + \ln (\delta_1 + t_1)_{t_1}$$

$$= -\ln (\delta_1) - (-\ln (\delta_1 + t_1)) + \ln (\delta_1 + l - t_1) - \ln (\delta_1)$$

$$= \ln \left( \frac{\delta_1 + t_1}{\delta_1} \right) + \ln \left( \frac{\delta_1 + l - t_1}{\delta_1} \right).$$

Note that this reduces to the result of Lemma 1 when $t_1 = 0$ or $t_1 = l$.

Theorem 2 (An admissible path-cost heuristic when the clearance of multiple states on the path is known). Let $\sigma \in \Sigma$ be a path with arc length $l$. Let $0 \leq t_1 < t_2 < \cdots < t_n \leq l$ be a sequence of $n$ numbers between 0 and $l$ whose associated states on the path have known clearance, $\delta_i := \delta(\sigma(t_i))$ for $i = 1, 2, 3, \ldots, n$. The reciprocal clearance cost $c(\sigma)$ of the path $\sigma$ is then lower bounded by

$$c(\sigma) \geq \ln \left( \frac{\delta_1 + t_1}{\delta_1} \right) + \sum_{i=1}^{n-1} \ln \left( \frac{\delta_i + \delta_{i+1} + t_{i+1} - t_i}{4\delta_i\delta_{i+1}} \right) + \ln \left( \frac{\delta_n + l - t_n}{\delta_n} \right).$$

Proof. (Figure 3b) The proof follows from Lemma 2 and Theorem 1 by splitting the clearance cost into $n + 1$ segments between the states with known clearance,

$$c(\sigma) = \int_{0}^{l} \frac{1}{\delta(\sigma(t))} \, dt$$

$$= \int_{0}^{t_1} \frac{1}{\delta(\sigma(t))} \, dt + \int_{t_1}^{t_2} \frac{1}{\delta(\sigma(t))} \, dt + \cdots + \int_{t_{n-1}}^{l} \frac{1}{\delta(\sigma(t))} \, dt + \int_{t_n}^{l} \frac{1}{\delta(\sigma(t))} \, dt.$$ 

Let the arc lengths of these segments be denoted by

$$l_0 = t_1, l_1 = t_2 - t_1, \ldots, l_{n-1} = t_n - t_{n-1}, l_n = l - t_n.$$ 

The first segment is a path of arc length $l_0$ with known clearance of its end state and therefore lower bounded by Lemma 2,

$$\int_{0}^{t_1} \frac{1}{\delta(\sigma(t))} \, dt \geq \ln \left( \frac{\delta_1 + l_0}{\delta_1} \right) = \ln \left( \frac{\delta_1 + t_1}{\delta_1} \right).$$

(5)
Similarly, the last segment is a path of arc length $l_n$ with known clearance of its start state and therefore also lower bounded by Lemma 2,

$$
\frac{1}{\delta(\sigma(t))} \int_{t_{n-1}}^{t_n} dt \geq \ln \left( \frac{\delta_n + l_n}{\delta_n} \right) = \ln \left( \frac{\delta_n + l - t_n}{\delta_n} \right).
$$

Each of the segments between $t_1$ and $t_n$ can be viewed as a path with known clearance at the end-states and is therefore lower bounded by the result of Theorem 1. Specifically, the segment from $t_i$ to $t_{i+1}$ with $i \in [1, n-1]$ is lower bounded by

$$
\frac{1}{\delta(\sigma(t))} \int_{t_i}^{t_{i+1}} dt \geq \ln \left( \frac{(\delta_i + \delta_{i+1} + l_i)^2}{4\delta_i \delta_{i+1}} \right) = \ln \left( \frac{(\delta_i + \delta_{i+1} + t_{i+1} - t_i)^2}{4\delta_i \delta_{i+1}} \right).
$$

A lower bound on the path cost can be computed by adding the lower bounds (5), (6) and (7)

$$
c(\sigma) = \int_0^l \frac{1}{\delta(\sigma(t))} dt \geq \int_0^{t_1} \frac{1}{\delta(\sigma(t))} dt + \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{\delta(\sigma(t))} dt + \int_{t_n}^{l} \frac{1}{\delta(\sigma(t))} dt \\
\geq \ln \left( \frac{\delta_1 + t_1}{\delta_1} \right) + \sum_{i=1}^{n-1} \left( \ln \left( \frac{(\delta_i + \delta_{i+1} + t_{i+1} - t_i)^2}{4\delta_i \delta_{i+1}} \right) + \ln \left( \frac{\delta_n + l - t_n}{\delta_n} \right) \right).
$$

The accuracy of this heuristic improves as the number of states of known clearance increases (Figures 3b and c). If the start and end states are among the states of known clearance, then the path is a chain of paths whose end states have known clearance and Theorem 2 simplifies to a sum of Theorem 1 over all segments (Corollary 1).

**Corollary 1** (An admissible path-cost heuristic when the clearance of the end states and other states of the path is known). Let $\sigma \in \Sigma$ be a path with arc length $l$. Let $0 = t_1 < t_2 < \cdots < t_n = l$ be a sequence of $n$ numbers between 0 and $l$ whose associated states on the path have known clearance, $\delta_i := \delta(\sigma(t_i))$ for $i = 1, 2, 3, \ldots, n$. The reciprocal clearance cost $c(\sigma)$ of the path $\sigma$ is then lower bounded by

$$
c(\sigma) \geq \sum_{i=1}^{n-1} \left( \ln \left( \frac{(\delta_i + \delta_{i+1} + t_{i+1} - t_i)^2}{4\delta_i \delta_{i+1}} \right) \right).
$$

**Proof.** (Figure 3c) The proof follows from Theorem 2 by setting $t_1 = 0$ and $t_n = l$. \qed
References

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