4D Einstein-Gauss-Bonnet Gravity Generated By Invisible Extra Dimensions

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Abstract

We show that within the recent formulation of gravity in presence of extra dimensions of vanishing proper length [1], the Gauss-Bonnet and higher Lovelock terms become dynamically nontrivial in four dimensions. The effective theory so obtained exhibits at the most a quadratic curvature nonlinearity. In absence of torsion, this formulation reflects a pure metric theory with no higher than second derivatives. The formalism is independent of any compactification, requires no (classical) regularization of divergences and is generally covariant. This is in contrast with a recent proposal by Glavan and Lin, and also with prescriptions based on singular rescaling of couplings applied to a Kaluza-Klein reduced or a conformally modified action. The vacuum field equations lead to an FLRW cosmology containing nonsingular bounce and self-accelerating Universe models and to spherically symmetric black holes superceding Schwarzschild.

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I. INTRODUCTION

The Gauss-Bonnet and higher Lovelock densities, as they are, do not affect the gravitational dynamics in four dimensions [2, 3]. This implies that there is no natural way to define a purely four dimensional metric theory beyond Einstein’s where such nonlinear Lovelock terms could induce a dynamical signature without introducing higher than second derivatives in the field equations.

Apparently, the best one could do is to define a Kaluza-Klein reduction [4, 5] of a higher \((D > 4)\) dimensional Einstein Gauss-Bonnet action [6–8]. In general, this leads to scalar-vector-tensor theories of gravity where the non-metric propagating modes encode the dynamical effect of extra dimensions. Recently though, it had been claimed that by introducing a singular rescaling of the Gauss-Bonnet coupling and then imposing a \(D \to 4\) limit, one could reinterpret the original \(D \geq 5\) dimensional solutions of Einstein-Gauss-Bonnet action [9] as four dimensional ‘solutions’ of a new metric theory beyond Einstein’s [10]. However, such a construction, apparently formulated on an action which admits no four dimensional limit, is not generally covariant, not unique (depends on the dynamics of extra dimensions even though these are treated as fiducial in the limit), and has a number of conceptual drawbacks essentially stemming from the ill-defined limit [11–13]. In retrospect, a complete consistency of such a proposal inspite of the manifest violation of Lovelock’s theorem would imply that there should be no essential distinction between dynamical and topological densities in a given number of dimensions. Such a feature would be fairly disturbing. The configurations discussed by Glavan and Lin have subsequently been observed to be solutions to the Horndeski class [14] in four dimensions, which exhibits an additional propagating scalar contrary to the original claim [15]. These theories could be derived using a Kaluza-Klein reduction on a maximally symmetric space whose breathing mode precisely corresponds to this scalar. As an alternative method to obtain such four dimensional effective corrections induced by the Gauss-Bonnet term, a conformal scaling prescription has been invoked [16, 17]. However, the physical interpretation of these methods, based on a singular rescaling of the coupling and an infinite subtraction in a classical action (‘regularization’), remains unclear. The Kaluza-Klein procedure happens to lead to non-unique four dimensional limits [15]. The conformal trick, while being independent of compactification, requires two different metrics along with the corresponding dynamical terms in a (bimetric) action to begin with and ends...
up with a single one after the regularized limit is imposed. Moreover, the issue whether the discontinuous limit $D \to 4$ applied to $(D - 4)$-dimensional tensors could make a real sense without leading to ambiguous final results, as the extra dimensions are eventually forced to be treated as fictitious, is a rather subtle one.

In this context of a singular limit invoked to define an eventual extra dimensional space of zero volume as above, one should also note that in general, the solutions for a $D$-dimensional gravity theory with an $n < D$ dimensional subspace of vanishing size (could be represented by a noninvertible $D$-metric with $n$ zero eigenvalues) is not equivalent to one where these same $n$ directions have a nonvanishing (even if infinitesimal) proper size. These two sectors are related by singular diffeomorphisms.

Thus, a new framework to find an effective four dimensional theory is necessary, if the formalism has to reflect genuine dynamical effects induced by higher Lovelock terms without depending on the dynamics associated with extra dimensions, or requiring any singular rescaling of couplings along with a regularization of the resulting divergence at the classical level. If such a formulation could admit a purely gravitational sector associated with field equations having no higher than second derivatives, that should signify some progress.

Here we elucidate such a framework, developed around an extra dimensional formulation introduced recently within a first order gravity theory [1]. A dimensional reduction of a five dimensional action with Einstein and Gauss-Bonnet terms is set up, where the additional dimension represents a vanishing eigenvalue of the (non-invertible) five-vielbein. The extra dimension have no dynamics of its own, and cannot be detected in principle. The resulting emergent theory obtained by solving the fundamental field equations is shown to inherit nonlinearities induced by the Gauss-Bonnet term. A generalization to the case with more than one invisible dimension is also presented. This reveals the remarkable fact that Lovelock terms higher than Gauss-Bonnet do not affect the four dimensional emergent theory already obtained. The question of any regularization in a discontinuous limit $D \to 4$ does not arise. To emphasize, the resulting metric theory is inequivalent to Kaluza-Klein reduced theories obtained from a higher dimensional Einstein-Gauss-Bonnet (Lovelock) action as well as to the ones obtained through the singular $D \to 4$ prescription, which exhibit additional degrees of freedom that do not decouple from the metric.

The relevance of our formalism could go beyond the motive of constructing an effective Einstein-Gauss-Bonnet metric theory. We note that the only nonlinear correction induced
by the full Lovelock series in the effective four dimensional action is quadratic, and is unique in form. The inherent economy in the emergent structure is expected to reduce the ambiguity pervading higher curvature gravity theories in general. Among the non-Einsteinian signatures seeded by the nonlinearities in this vacuum theory, the emergence of nonsingular bouncing cosmology in absence of a spatial curvature and a bare cosmological constant is intriguing. In the case of (static) spherical symmetry, the Schwarzschild solution of vacuum Einstein gravity is superceded by a four parameter solution consisting of an emergent mass and charge, exhibiting curvature singularity.

In the next section, we begin with a discussion of the basic five dimensional action along with the associated equations of motion. This is followed by analyses of the emergent four dimensional theory resulting from their solutions, characterized by spacetimes with constant curvature and vanishing torsion (in higher dimension), respectively. The implications for FLRW cosmology and static spherically symmetric spacetimes are discussed. The generalization to higher order Lovelock densities in presence of additional dimensions of zero proper length is also worked out. The final section contains a summary and criticism of our work, along with some perspective into possible future investigations.

II. THE FUNDAMENTAL THEORY

The fundamental five-dimensional Lagrangian density is given by:

\[ \mathcal{L}(\hat{e}, \hat{w}) = \epsilon^{\mu\nu\alpha\beta\gamma} \hat{e}_{IJKLM} \left[ \frac{\alpha}{2} \hat{R}_{\mu\nu}^{IJ}(\hat{w})\hat{R}_{\alpha\beta}^{KL}(\hat{w})\hat{e}_{\gamma}^{M} + \frac{\zeta}{3} \hat{R}_{\mu\nu}^{IJ}(\hat{w})\hat{e}_{\alpha}^{K}\hat{e}_{\beta}^{L}\hat{e}_{\gamma}^{M} + \frac{\beta}{5} \hat{e}_{\mu}^{I}\hat{e}_{\nu}^{J}\hat{e}_{\alpha}^{K}\hat{e}_{\beta}^{L}\hat{e}_{\gamma}^{M} \right], \]

(1)

where the vielbein \( \hat{e}_{\mu}^{I}(x) \) and the super-connection \( \hat{w}_{\mu}^{IJ}(x) \) (\( \mu \equiv [t, x, y, z, v] \), \( I \equiv [0, 1, 2, 3, 4] \)) are the independent fields. \( \zeta \) and \( \alpha \) define the gravitational and Gauss-Bonnet coupling, respectively and \( \beta \) is the bare cosmological constant. These have dimensions: \( \zeta \sim M^{-1}L^{2} \), \( \alpha \sim M \), \( \beta \sim ML^{-4} \). The \( SO(4, 1) \) field-strength is defined as \( \hat{R}_{\mu\rho}^{LM}(\hat{w}) = \partial_{[\rho}\hat{w}_{\mu]}^{LM} + \hat{w}_{[\rho}^{LK}\hat{w}_{\mu]}^{LM} \), the internal metric being defined as \( \eta_{IJ} = [-1, 1, 1, 1, 1] \).

A variation of (1) with respect to the connection and the vielbein leads to the following

1 Even though physically reasonable requirements such as ghost freedom etc. have been considered earlier [18], these still allow for an infinity of terms in the equations of motion, in contrast with our formalism.
field equations in vacuum, respectively:

\[ \epsilon^{\mu\nu\alpha\beta\gamma\epsilon_{\alpha\beta}} \epsilon_{\epsilon J K L M} \left[ \zeta \hat{e}_\mu^I \hat{e}_\nu^J + \alpha \hat{R}_{\mu\nu}^{IJ}(\hat{w}) \right] \hat{D}_\alpha(\hat{w}) \hat{e}_\beta^K = 0, \quad (2a) \]

\[ \epsilon^{\mu\nu\alpha\beta\gamma\epsilon_{\alpha\beta}} \epsilon_{\epsilon J K L M} \left[ \zeta \hat{e}_\mu^I \hat{e}_\nu^J \hat{R}_{\alpha\beta}^{KL}(\hat{w}) + \frac{\alpha}{2} \hat{F}_{\mu\nu}^{IJ}(\hat{w}) \hat{F}_{\alpha\beta}^{KL}(\hat{w}) + \beta \hat{e}_\rho^I \hat{e}_\sigma^J \hat{e}_\alpha^K \hat{e}_\beta^L \right] = 0 \quad (2b) \]

Here we have defined \( \hat{D}_\mu(\hat{w}) \) as the gauge-covariant derivative with respect to the super-connection \( \hat{w}_\mu^{IJ} \) and have used the Bianchi identity \( \hat{D}_\mu(\hat{w}) \hat{R}_{\nu\alpha}^{IJ}(\hat{w}) = 0 \) in obtaining the connection equations \( (2a) \).

In general, both the Lagrangian density \( (1) \) as well as these equations above are well-defined for invertible as well as non-invertible vielbein. The invertible spacetime solutions, upon a compactification of the fifth dimension, is equivalent to the well-known Kaluza-Klein reduction of the five dimensional Einstein Gauss-Bonnet action. Here, however, we shall be concerned with solutions with a degenerate vielbein (one zero eigenvalue), leading to a solution space completely inequivalent to the invertible five-metric case.\(^2\) As is obvious from the analysis below, one does not require any additional stabilization mechanism for the extra dimension, which itself cannot be detected directly as it is not associated with any genuine dynamics.

The connection equations \( (2a) \) are solved by:

a) \( \zeta \hat{e}_\mu^I \hat{e}_\nu^J + \alpha \hat{R}_{\mu\nu}^{IJ}(\hat{w}) = 0; \)

b) \( \hat{D}_\alpha(\hat{w}) \hat{e}_\beta^K = 0. \)

In what follows next, we shall explore these cases and present the complete solutions leading to the four dimensional gravity theory, before elucidating specific applications.

We adopt a notation similar to ref. \[1\], where the notion of zero length extra dimensions had been introduced. The zero eigenvalue of the vielbein \( \hat{e}_\mu^I \) could be chosen to lie along the fifth dimension \( \nu \):

\[ \hat{e}_\nu^I = 0, \]

leading to:

\[ \hat{e}_\mu^I = \begin{bmatrix} \hat{e}_a^i & 0 \\ 0 & 0 \end{bmatrix} \quad (3) \]

\(^2\) The mutual inequivalence of the invertible and non-invertible spacetime solutions in first order gravity in four dimensions have been noted and explored in references \[21, 22\].
Here the world indices are $\mu \equiv (a, v) \equiv (t, x, y, z, v)$ and the internal indices are $I \equiv (i, 4) = (0, 1, 2, 3, 4)$.

The emergent tetrad fields $e_a^i$ (invertible) define the effective four-dimensional spacetime. Their inverse are denoted as $e^a_i$ (not the same as $\hat{e}_i^a$ which do not exist in our case):

$$e_a^i e^b_i = \delta^b_a, \quad e_i^a e_j^a = \delta^i_j.$$

In terms of these, the corresponding 4-metric is defined as $g_{ab} = e_i^a e_i^b$ which are the only nontrivial components of the fundamental five-metric ($\hat{g}_{va} = \hat{g}_{av} = 0 = \hat{g}_{vv}$). The four dimensional epsilon symbols are derived from the five dimensional ant isymmetric tensor densities as:

$$\epsilon^{vaabcd} = \epsilon^{abcd}, \quad \epsilon_{ijkl} = \epsilon_{ijkl}.$$

III. CONSTANT CURVATURE SPACETIME SOLUTIONS

Here we consider the class (a) of spacetime solutions:

$$\zeta \hat{e}_i^a \hat{e}_j^b + \alpha \hat{R}^{ij}_{\mu \nu} (\hat{w}) = 0$$

The requirement that such spacetimes solve the vielbein equations of motion (2b) implies the following constraint among couplings:

$$\zeta^2 - 2\alpha \beta = 0.$$ (5)

Using the non-invertible vielbein (3), decomposition of the five dimensional equations (4) leads to:

$$\hat{R}_{ai}^{\quad i4} = \partial_{[a} \hat{w}_{vk]}^{i4} + \hat{w}_{[a}^{ik} \hat{w}_{v]k}^{i4} = 0,$$

$$\hat{R}_{ai}^{\quad ij} = \partial_{[a} \hat{w}_{vk]}^{ij} + \hat{w}_{[a}^{ik} \hat{w}_{v]k}^{ij} + \hat{w}_{[a}^{i4} \hat{w}_{v]4}^{ij} = 0,$$

$$\hat{R}_{ai}^{\quad i4} = \partial_{[a} \hat{w}_{bj]}^{i4} + \hat{w}_{[a}^{ik} \hat{w}_{bj]k}^{i4} = 0,$$

$$\hat{R}_{ai}^{\quad ij} + \frac{\zeta}{2\alpha} e^i_{[a} c^j_{b]} = 0 = \hat{R}_{ab}^{ij} (\hat{w}) + \hat{D}_{[a} (\hat{w}) K^i_{b]}^{ij} + K_{[a}^{ik} K_{b]k}^{ij} + \hat{w}_{[a}^{i4} \hat{w}_{b]4}^{ij} + \frac{\zeta}{2\alpha} e^i_{[a} c^j_{b]}$$ (6)

In the last equation above we have used the fact that the connection components $\hat{w}_{ai}^{ij}$ could in general be decomposed as:

$$\hat{w}_{ai}^{ij} = \hat{w}_{ai}^{ij} (e) + K_{ai}^{ij},$$

where $\hat{w}_{ai}^{ij} (e) = \frac{1}{2} [e^b_i \partial_{[a} c^j_{b]} - e^b_j \partial_{[a} c^i_{b]} - e^c_i e^b_j \partial_{[a} c^b_{c]}]$, $\hat{D}_{[a} (\hat{w}) c^i_{b]} = 0$ and $K_{ai}^{ij} = -K_{ai}^{ji}$ is the contortion tensor in the emergent spacetime. To emphasize, the covariant derivative $\hat{D}_{a} (\hat{w})$ is defined with respect to the connection $\hat{w}_{ai}^{ij}$.
Solution for connection fields:

In order to unravel the full implications of the above equations of motion, we first analyze the Bianchi identities: \( \epsilon^{\mu \nu \alpha \beta} \epsilon_{IJKLM} \dot{e}_\mu^I \dot{e}_\nu^J \dot{R}_{\alpha \beta}^{JK} = 0 \). For the spacetime solutions (4), these reduce to:

\[
\epsilon^{\mu \nu \alpha \beta} \epsilon_{IJKLM} \dot{e}_\mu^I \dot{e}_\nu^J \dot{\hat{D}}_{\alpha}^K \dot{\hat{e}}_{\beta}^L = 0
\]  

(7)

These are linear in the connection fields, whose solutions are found below.

From the component \( \gamma = v, L = l, M = m \) of eq. (7), we obtain:

\[
\epsilon^{abcd} \epsilon_{ijkl} e_a^i e_b^j \dot{D}_c e_d^k = 0 = \epsilon^c_k \dot{D}_b e^k_c \Rightarrow \epsilon^a_k K_{ai} = 0 = K_{ai},
\]  

(8)

which forces four of the twenty four contortion components to vanish. Next, the component \( \gamma = d, L = l, M = m \) reads:

\[
\epsilon^{abcd} \epsilon_{ijkl} e_a^i e_b^j \dot{D}_c e_d^k = 0 = \epsilon^c_k \dot{D}_b e^k_c \Rightarrow \epsilon^a_k K_{ai} = 0 = K_{ai}.
\]  

(9)

Finally, for \( \gamma = d, L = l, M = m \) we find:

\[
\epsilon^{abcd} \epsilon_{ijkl} e_a^i e_b^j \dot{D}_c e_d^k = 0 = \epsilon^c_k \dot{D}_b e^k_c \Rightarrow \epsilon^a_k \dot{D}_b e^k_c = 0 = \dot{D}_b e^a_c.
\]  

(10)

From this, it follows that the determinant of the tetrad is \( v \)-independent: \( \partial_v e = 0 \) and that \( \dot{\hat{w}}_{ij}^v \) is a pure gauge:

\[
\dot{\hat{w}}_{ij}^v = -\epsilon^a_j \partial_v e^i_a
\]  

As a consequence, the emergent 4D metric \( g_{ab} \) exhibits no dependence on the fifth coordinate \( v \): \( \partial_v g_{ab} = \partial_v (e_a^i e_b^i) = e_{ai} \dot{D}_v e_b^i + (a \leftrightarrow b) = 0 \). All these facts together imply that any possible dependence of the tetrad on \( v \) must be a gauge artifact and might just be gauged away by choosing the trivial gauge:

\[
\dot{\hat{w}}_{ij}^v = 0.
\]  

(12)
Four dimensional effective theory:

Inserting these solutions for the connection fields found above into the first three equations in the set (6), we obtain the following constraints:

\[ \hat{R}^{i4}_{av} = 0 = \partial_v M_{ik}^{i}, \quad \hat{R}^{ij}_{av} = 0 = \partial_v K_a^{ij}, \]
\[ \hat{R}^{i4}_{ab} = 0 = \bar{D}_{[a} (\bar{w}) M_{b]}^{i} + K_a^{ij} M_{b]}^{ij} \]  \hspace{1cm} (13)

where we have defined: \( M_i^a = M_{ij} e_{aj} \). As expected from the arguments presented in the previous paragraph, the fields \( M_{ij}, K_a^{ij} \), which are the only emergent fields other than the tetrad, also exhibit no dependence on the fifth coordinate associated with the null eigenvalue. The last equation, however, is intriguing in its own right and could be interpreted further. Since the tensor \( M_{ij} \) is symmetric, this equation could be solved for the contortion in terms of \( M_{ij} \) itself along with its inverse and first derivative. Note that the field \( M_{ab} = M_{ij} e^i_a e^j_b \) could then be seen as a second emergent metric. From the last equation in (6), it is evident that both metrics would be dynamical in general. Hence, this formulation admits an interpretation as an emergent bimetric theory of gravity. However, here we shall not explore this interesting direction any further, as it is beyond the scope and purpose of this article. Rather, we assume that there is only a single emergent metric and hence the tensors \( g_{ab} \) and \( M_{ab} \) must be proportional:

\[ M_i^a = \lambda e_i^a, \quad \lambda \equiv const. \]  \hspace{1cm} (14)

This, when inserted back into eq.(13), implies:

\[ K_a^{ij} e_{b]j} = 0 \Rightarrow K_a^{ij} = 0. \]  \hspace{1cm} (15)

In other words, the four dimensional torsion must vanish.

Using the results above, the only remaining equation of motion, given by the last one in the set (6), finally simplifies to:

\[ \bar{R}^{ij}_{ab} (\bar{w}(e)) + \left[ \frac{\zeta}{2\alpha} - \lambda^2 \right] e^i_a e^j_b = 0. \]  \hspace{1cm} (16)

These solutions represent maximally symmetric four dimensional spacetimes. The curvature is positive, negative or zero provided: \( \frac{\zeta}{2\alpha} < \lambda^2, \frac{\zeta}{2\alpha} > \lambda^2 \) or \( \frac{\zeta}{2\alpha} = \lambda^2 \), respectively. The only vestige of the Gauss-Bonnet contribution is contained in this spacetime curvature through the coupling \( \alpha \).
This completely defines the four dimensional gravity theory resulting from the solution (4), described by a single emergent metric and no other propagating fields.

IV. VANISHING TORSION SPACETIME SOLUTIONS

As discussed earlier, the connection equations exhibit yet another important class of solutions, which exhibits lesser symmetry but underlies a richer structure than the previous case:

\[ \hat{D}_{[\alpha}(\hat{w})e_{\beta]}^I = 0. \]  
(17)

Let us proceed to find the general solutions of the above.

Rewriting eq.(17) in terms of the four dimensional emergent fields, we obtain the following solutions for the five dimensional spin-connection components:

\[ \hat{D}_{[\alpha}(\hat{w})e_{\beta]}^i = 0 \Rightarrow K_{ij} = \hat{w}_{ij}^a - \bar{w}_{ij}^a(e) = 0, \]
\[ \hat{D}_{[\alpha}(\hat{w})e_{\beta]}^4 = 0 \Rightarrow \hat{w}_{v}^4i = e^{ak}Q^j_{ik} \equiv Q^i_a [Q^j_{ik} = Q^{ki}], \]
\[ \hat{D}_{[\alpha}(\hat{w})e_{\beta]}^i = 0 \Rightarrow \hat{w}_{v}^{ij} = 0, \]
\[ \hat{D}_{[\alpha}(\hat{w})e_{\beta]}^i = 0 \Rightarrow \hat{w}_{v}^{ij} = -\epsilon^a_j \partial_v e_a^i. \]  
(18)

Exactly as in the earlier section, the last equation implies that \( \hat{w}_{v}^{ij} \) is a pure gauge. Once again, we choose the physical gauge \( \hat{w}_{v}^{ij} = 0 \) which eliminates the spurious \( v \)-dependence of the tetrad. The only nontrivial connection component is encoded by the symmetric emergent field \( Q_k^l \) (or equivalently, \( Q^i_a \)).

Let us analyze the vielbein equations of motion (2b) next. Note that the components \((\gamma = a, M = 4), (\gamma = a, M = i)\) are identically satisfied upon using the identities \( \hat{R}_{v4}^{ij} = 0 = \bar{R}_{v4}^{ij} \) which follow from the solutions obtained earlier. The \( \gamma = v, M = i \) component reads:

\[ \epsilon^{abcd}\epsilon_{ijkl} \left[ \hat{R}_{ab}^{kl} + \frac{\zeta}{\alpha} e_{a}^{k}e_{b}^{l} \right] \hat{R}_{cd}^{4j} = 0 = \epsilon^{abcd}\epsilon_{ijkl} \left[ \bar{R}_{ab}^{kl} - 2Q_{a}^{l}Q_{b}^{k} + \frac{\zeta}{\alpha} e_{a}^{k}e_{b}^{l} \right] \left( \bar{D}_{c}(\hat{w})Q_{d}^{j} \right) \]  
(19)

Evidently, this is a cubic equation in \( Q_i^a \). As in the previous section, the emergent symmetric tensor \( Q_{ab} = Q^{ij}e_{ai}e_{bj} \) admits an interpretation as a dynamical metric other than \( g_{ab} \). Such a possibility would not be explored any further here.

Let us consider the case where there is only one emergent metric. The solution to eq.(19) satisfying this property is given by:

\[ Q_i^a = \lambda e_i^a, \ \lambda \equiv const. \]  
(20)
Note that this satisfies the identity \( D_c(\bar{\omega})Q^j_{\bar{b}j} = 0 \).

The only remaining component of the field equations (25) is \( \gamma = \nu, M = 4 \), resulting in the following scalar constraint:

\[
\epsilon^{abcd} \epsilon_{ijkl} \left[ \zeta \bar{R}^{ij} e^{k}_c e^l_d + \frac{\alpha}{2} \bar{\nabla}^{ij} \bar{R}^{kl} + \beta e^{ij}_a e^k_c e^l_d \right] = 0 = \epsilon^{abcd} \epsilon_{ijkl} \left[ \phi \bar{R}^{ij} e^{k}_c e^l_d + \frac{\alpha}{2} \bar{R}^{ij} \bar{R}^{kl} + \chi e^{ij}_a e^k_c e^l_d \right],
\]

(21)

where in the last equality we have redefined the coupling constants as: \( \phi = (\zeta - 2\alpha \lambda^2) \) and \( \chi = (\beta - 2\zeta \lambda^2 + 2\alpha \lambda^4) \). The simple equation above, along with the Bianchi identities, constitute the full system of equations of motion. The analysis of Bianchi identities, which eventually demonstrates the \( \nu \)-independence of the nontrivial emergent connection components, is very similar to the one at the earlier section and would not be repeated here.

Note that the maximally symmetric spacetimes of section-III are particular solutions to eq.(21). Hence, the (more general) class of vanishing torsion solutions is what shall concern us in the rest of this work.

Some of the notable features reflected by the four dimensional theory obtained above are outlined below:

(a) The emergent equation of motion (21) has at the most second order derivatives of the metric, even though it inherits nonlinearities induced by the Gauss-Bonnet term (reflected by the second term in the last equality above).

(b) Apart from the emergent metric, there is no other propagating mode.

(c) The Bianchi identity \( \epsilon^{abcd} \bar{D}_b(\bar{\omega}) R^{ij}_{cd} = 0 \) implies that \( \nabla_a \left[ \bar{R}^{ab}(\bar{\omega}(e)) - \frac{1}{2} g^{ab} \bar{R}(\bar{\omega}(e)) \right] = 0 \). This makes the general covariance of the emergent gravity theory explicit (\( \nabla_a \) being the covariant derivative defined with respect to the Christoffel symbol).

(d) The emergent theory has been obtained without introducing any singular rescaling of the coupling constants or any regularization involving a discontinuous limit such as \( D \to 4 \).

(e) The equation of motion (21) could also be derived from a well-defined four dimensional action principle involving an auxiliary (nondynamical) scalar for spacetimes with vanishing torsion. Thus, the emergent theory is indeed geometrical.

(f) The formalism here is inequivalent to a Kaluza-Klein reduction of a higher dimensional Gauss-Bonnet theory or to its dimensionally regularized version discussed in the literature \[6, 10, 15, 17\]. Both of these latter classes are known to contain at least one additional propagating scalar mode which trivially leads to Einstein gravity when taken away.
The fact that the induced four dimensional physics should not depend on the fifth coordinate, which is associated with a zero proper length \( \hat{g}_{\nu\mu} = 0 \) and hence underlies no real dynamics, naturally follows from the original field equations. This may be contrasted with a Kaluza-Klein compactification where the requirement of the independence of the fifth coordinate needs to be imposed.

V. FLRW COSMOLOGY

As a first application of the formalism presented earlier, we shall elucidate the dynamical consequences of the effective field equations (21) in the case of homogeneous and isotropic cosmology. In particular, it is important to isolate the new features, if any, induced by nonlinearities in curvature which are absent in Einstein gravity.

We idealize our considerations by assuming a homogeneous and isotropic form for the emergent spacetime \( (k = 0, \pm 1 \text{ being the spatial curvature}) \):

\[
ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + d\theta^2 + \sin^2 \theta d\phi^2 \right].
\]

Using the identities involving the torsionfree curvature two-form:

\[
\epsilon^{abcd} \epsilon_{ijkl} \epsilon^{ij} \bar{R}^{kl}_{cd} (e) = \frac{24 [\dot{a} + a^2 + k] \dot{a} r^2 \sin \theta}{\sqrt{1 - kr^2}},
\]

the equation of motion (21) becomes:

\[
\dot{\phi} \left[ \frac{\ddot{a} + \dot{a}^2 + k}{a^2} \right] + 2\alpha \left[ \frac{\dot{a}^2 + k}{a^2} \right] \frac{\ddot{a}}{a} + \chi = 0,
\] (22)

Let us first consider the spatially flat case \( (k = 0) \). Introducing the variable \( u = \frac{\dot{a}}{a} \), we rewrite (22) as a first order equation:

\[
\dot{\phi} \left[ \dot{u} + 2u^2 \right] + 2\alpha \left[ \dot{u} + u^2 \right] u^2 + \chi = 0.
\] (23)

Its general solution is given by:

\[
-t + \text{const.} = \sqrt{\frac{\alpha}{2}} \left[ \tan^{-1} \left( \frac{\sqrt{2\alpha u}}{\sqrt{\phi - \sqrt{\phi^2 - 2\alpha \chi}}} \right) + \tan^{-1} \left( \frac{\sqrt{2\alpha u}}{\sqrt{\phi + \sqrt{\phi^2 - 2\alpha \chi}}} \right) \right]
\] (24)
This is a transcendental equation, having no closed form solution for \( a(t) \). However, the approximate solutions in the limit of small and large (with respect to the smallest and largest frequency scales, respectively, among \( \sqrt{\frac{\phi}{2\alpha}} \pm \sqrt{\frac{\phi^2}{4\alpha^2} - \frac{1}{2\alpha}} \) Hubble rate \( u \) are displayed below:

Small \( u \) limit : \( a(t) \approx Ae^{\frac{\mu}{2\alpha}(Bt-t^2)} \)

Large \( u \) limit : \( a(t) \approx C(t-D) \)

where \((A, B)\) and \((C, D)\) are the integration constants. The small \( u \) limit could be expected to describe the late time dynamics, exhibiting what might be called a generalized de-Sitter (anti de-Sitter) behaviour with a nonlinear exponent. The large \( u \) limit, on the other hand, suggests a spatially flat Milne cosmology at early times. Neither of these behaviour have an analogue in the Einsteinian case in general. Note that the theory is well-defined for \( \frac{\phi^2}{2\alpha} - \chi \geq 0 \).

Next, we summarize the main cosmological models emerging from the equation of motion (22).

**Non-oscillatory solutions:**

(a) \( k = \pm 1 \): \( a(t) = Ae^{\mu t} + \frac{k}{4\mu}e^{-\mu t}, \mu^2 = -\frac{\phi}{2\alpha} \pm \sqrt{\left(\frac{\phi}{2\alpha}\right)^2 - \frac{\chi}{2\alpha}} \).

(b) \( k = +1 \): \( a(t) = \frac{1}{\mu} \cosh \mu t, \mu^2 = -\frac{\phi}{2\alpha} \pm \sqrt{\left(\frac{\phi}{2\alpha}\right)^2 - \frac{\chi}{2\alpha}} \). This represents a symmetric nonsingular bounce.

(c) \( k = -1 \): \( a(t) = \frac{1}{\mu} \sinh \mu t, \mu^2 = -\frac{\phi}{2\alpha} \pm \sqrt{\left(\frac{\phi}{2\alpha}\right)^2 - \frac{\chi}{2\alpha}} \).

(d) \( k = -1 \): \( a(t) = t, \chi = 0 \). This is the Milne dynamics.

(e) \( k = 0 \): \( a(t) = Ae^{\mu t}, \mu^2 = -\frac{\phi}{2\alpha} \pm \sqrt{\left(\frac{\phi}{2\alpha}\right)^2 - \frac{\chi}{2\alpha}} \) (\( \phi \neq -2\alpha \mu^2 \)). This reflects a pure de-Sitter (anti de-Sitter) behaviour.

(f) \( k = 0 \): \( a(t) = Ae^{\mu t} + Be^{-\mu t}, \mu^2 = -\frac{\phi}{2\alpha} = -\frac{\chi}{2\alpha} \) (\( A, B \) are arbitrary).

Each of the cases from (a) to (e) above have an analogue in standard FLRW cosmology (without matter but with or without \( \Lambda \)). The only difference is, here the inflationary dynamics or a non-singular bounce could be supported even in the absence of a bare cosmological constant \( \beta \), implying the possibility of a self-accelerating Universe. The last case (f) however, have no Einsteinian counterpart. This case also admits a nonsingular bounce for \( A = B \).
Oscillatory solutions:

These models are obtained only for \( k = -1 \) and \( k = 0 \).

(a) \( k = -1 \): \( a(t) = A \cos \nu t + \sqrt{\frac{1}{\nu^2} - A^2} \sin \nu t, \nu^2 = \frac{\phi}{2\alpha} \pm \sqrt{(\frac{\phi}{2\alpha})^2 - \frac{\chi}{4\alpha}} \). This represents a non-singular oscillatory Universe.

(b) \( k = -1 \): \( a(t) = \frac{1}{\nu} \sin \nu t, \nu^2 = \frac{\phi}{2\alpha} \pm \sqrt{(\frac{\phi}{2\alpha})^2 - \frac{\chi}{4\alpha}} \). In each cycle, this oscillatory Universe exhibits a big bang (at \( t = 0 \)), reaches a maximum size (at \( t = \frac{\pi}{2\nu} \)) and recollapses to a singularity (at \( t = \frac{\pi}{\nu} \)).

(c) \( k = 0 \): \( a(t) = A \cos \nu t + B \sin \nu t, \nu^2 = \frac{\phi}{2\alpha} = \frac{\chi}{\phi} \) (\( A, B \) are arbitrary).

The cases (a) are (b) are similar to the FLRW solutions for \( k = -1, \Lambda < 0 \), except the fact that here the solutions could exist even in the absence of a bare cosmological constant. Again, the last case (c) has no analogue in the vacuum Einsteinian case.

To conclude, all the \( k = 0 \) or \( k = +1 \) solutions above are nonsingular. For any value of spatial curvature, the equation of motion admits a bounce followed by a smooth transition to a de-Sitter expansion (contraction) or vice versa. In the absence of torsion, there is no power law Big-bang singularity.

VI. STATIC SPHERICALLY SYMMETRIC SOLUTIONS

Here we shall be concerned with spherically symmetric static solutions of the emergent theory, with the four-metric:

\[
ds^2 = -e^{\mu(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

In terms of the radial functions \( \mu(r), \lambda(r) \), eq. (21) becomes:

\[
\phi \left[ -\frac{1}{2} r^2 \left( \mu'' + \frac{\mu'}{2}(\mu' - \lambda') \right) - r (\mu' - \lambda') + e^\lambda - 1 \right] - \alpha \left[ \left( \mu'' + \frac{\mu'}{2}(\mu' - \lambda') \right) (1 - e^{-\lambda}) + \mu' \lambda' e^{-\lambda} \right] + 3 \chi r^2 e^\lambda = 0, \tag{25}
\]

where we have used the following identities:

\[
e^{\alpha b c d} \epsilon_{ijkl} e^i_a e^j_b \bar{R}_{cd}^{kl}(\bar{\omega}(e)) = 8 \left[ -\frac{1}{2} r^2 \left( \mu'' + \frac{\mu'}{2}(\mu' - \lambda') \right) - r (\mu' - \lambda') + e^\lambda - 1 \right] e^{\frac{\mu'}{2}} \sin \theta,
\]

\[
e^{\alpha b c d} \epsilon_{ijkl} \bar{R}_{ab}^{ij}(e) \bar{R}_{cd}^{kl}(\bar{\omega}(e)) = -16 \left[ \left( \mu'' + \frac{\mu'}{2}(\mu' - \lambda') \right) (1 - e^{-\lambda}) + \mu' \lambda' e^{-\lambda} \right] e^{\frac{\mu'}{2}} \sin \theta.
\]

3 The only propagating mode being a four-tensor, the assumption of staticity should be unnecessary since the Birkhoff’s theorem is expected to hold.
This is the only equation relating two independent functions, reflecting an underdetermined system. Here, we shall focus on the question whether this admits Schwarzschild type solutions, i.e. satisfying \( \mu(r) = -\lambda(r) \), and among them, if there could be black hole spacetimes.

For this case, introducing the new variable \( f(r) = e^{\mu(r)} - 1 = e^{-\lambda(r)} - 1 \), we may rewrite the field equation (25) as:

\[
\phi \left[ \frac{r^2}{2} f'' + 2rf' + f \right] - \alpha \left[ ff'' + f'^2 \right] - 3\chi r^2 = 0 \tag{26}
\]

This has the following solution:

\[
e^{\mu(r)} \equiv 1 + f(r) = 1 + \frac{\phi}{2\alpha} r^2 \pm \frac{1}{2} \sqrt{\left( \frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha} \right) r^4 + 4C_1 r - 4\frac{C_2}{\alpha}}, \tag{27}
\]

\( C_1, C_2 \) being integration constants. We note the formal similarity of the above with the Wiltshire class of metrics \([23]\) defined for \( D \geq 5 \) (the Boulware-Deser-Wheeler class \([9]\) being a special case), which, however, were obtained in a different context and have no four-dimensional analogue.

At the asymptotic limit, the above reduces to:

\[
e^{\mu(r)} \to 1 - \frac{\Lambda_{eff} r^2}{3} \pm \left[ \frac{2M_{eff}}{r} - \frac{Q_{eff}^2}{r^2} \right] \quad (\text{as } r \to \infty)
\]

where we have defined the effective cosmological constant, Schwarzschild ‘mass’ and Reissner-Nordstrom ‘charge’, respectively as: \( \Lambda_{eff} \equiv -\frac{3}{2} \left[ \frac{\phi}{\alpha} \pm \sqrt{\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}} \right] \), \( 2M_{eff} \equiv \left[ \frac{C_1}{\sqrt{\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}}} \right] \), \( Q_{eff}^2 \equiv \left[ \frac{C_2}{\alpha \sqrt{\frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha}}} \right] \), respectively. This solution is characterized by four parameters \( (\phi/\alpha, \Lambda_{eff}, M_{eff}, Q_{eff}) \).

The effective asymptotic mass is positive and the charge is real provided \( C_1 < 0 \) \((C_1 > 0)\) and \( \frac{C_2}{\alpha} < 0 \) \((\frac{C_2}{\alpha} > 0)\) for the ‘+’ \((‘-‘)\) branch. As earlier, we assume \( \frac{\phi^2}{\alpha^2} - \frac{2\chi}{\alpha} \geq 0 \) so that the solutions are well-defined.

The metric components are all finite as \( r \to 0 \). However, the emergent Ricci scalar \( \bar{R}(\bar{w}(e)) \) does diverge as \( \frac{1}{r^2} \). To determine whether this curvature singularity lies at a finite proper time, let us consider the radial geodesic equation (timelike). These are given by:

\[
[1 + f(r)] \left( \frac{dt}{d\tau} \right)^2 - \frac{1}{[1 + f(r)]} \left( \frac{dr}{d\tau} \right)^2 = 1, \quad \frac{dt}{d\tau} = \frac{E}{1 + f(r)},
\]

\( E \) being a constant of motion. Hence, the proper time that elapses in reaching \( r = 0 \) from any finite distance \( r = r_* \) is found to be:

\[
\tau_* = \left| \int_{r_*}^{0} \frac{dr}{\sqrt{E^2 - f(r) - 1}} \right|
\]
In general, this cannot be integrated to a closed form. However, it is easy to show that there is at least a finite number of radial trajectories along which the singularity could be reached in finite proper time. Choosing the otherwise arbitrary integration constants as $C_1 = 0 = C_2$, $\phi^2 - 2\alpha \chi = 0$ for simplicity, we find:

$$\tau_* = \frac{1}{\sqrt{\alpha}} \ln \left[ \sqrt{\frac{\alpha}{E^2 - 1} r_*} + \sqrt{1 + \frac{\alpha r_*^2}{E^2 - 1}} \right]$$

This is finite.

The solution \((27)\) represents a black hole spacetime, although different from the Schwarzschild geometry which is the unique spherically symmetric solution in Einsteinian case. In fact, the Schwarzschild spacetime is not an exact solution of the EOM \((25)\). The location of the horizon \((r_h)\) is given by the following quartic equation:

$$\chi r_h^4 + 2\phi r_h^2 - 2\alpha C_1 r_h + 2(\alpha + C_2) = 0,$$

which admits more than one real roots in general (we avoid displaying their closed form expressions for brevity). For $\frac{\phi}{\chi} > 0$, $\frac{\alpha C_1}{\chi} < 0$, $\frac{\alpha + C_2}{\chi} > 0$, all the terms above are monotonic in $r_h$, and hence admits one and only one horizon for any real root that exists. In order to understand whether a particular solution admits a single or double horizon, let us look at \((e^\mu(r))' = \frac{\phi}{\alpha} r \pm \frac{2}{\sqrt{(\frac{\phi^2}{\alpha} - 2\chi)(r^4 + 4C_1 r - 4C_2)}}\). For the ‘+’ branch, this could take both positive and negative values at $r > 0$ for both $\frac{\phi}{\alpha} > 0$ or $\frac{\phi}{\alpha} < 0$ and hence could admit a double horizon. For the ‘−’ branch, the double horizon could occur only if $\frac{\phi}{\alpha} > 0$.

Let us also consider the case $C_1 = 0 = M_{eff}$ separately, as an example which does not come under the category mentioned in the last paragraph (no real root for $\frac{\phi}{\chi} > 0$, $\frac{\alpha + C_2}{\chi} > 0$).

This corresponds to a simpler expression for horizon radius:

$$r_h = \sqrt{-\frac{\phi}{\chi} \pm \sqrt{\frac{\phi^2}{\chi^2} - \frac{2(\alpha + C_2)}{\chi}}}$$

For $\frac{\phi}{\chi} < 0$, $\frac{\alpha + C_2}{\chi} > 0$, there are two horizons where the larger and smaller radius correspond to the + and − branch, respectively. For $\frac{\alpha}{\chi} + \frac{C_2}{\chi} < 0$, there is exactly one horizon given by the + branch. The $\frac{\phi}{\chi} > 0$, on the other hand, exhibits a horizon only if $\frac{\alpha}{\chi} + \frac{C_2}{\chi} < 0$ and does not admit a double horizon.

We may look for more general static spherically symmetric solutions with $\mu(r) \neq -\lambda(r)$ parametrized by a function $g(r)$ as: $\ e^{-\lambda} = g(r)e^{\mu} = g(r)(1 + f(r))$ where $f(r)$ is given by
By definition, $g(r)$ is a solution of the following first order equation obtained using eq. (25) and (26):

$$
\phi \left[ \frac{r^2 f' g'}{4} + r(1 + f) g' - g(1 - g) \right] + \alpha \left[ \frac{g f'}{2} (1 - 3g(1 + f)) + g(1 - g) (f'' + f'') \right] - 3\chi r^2(1 - g) = 0 \quad (29)
$$

However, we have been unable to find any solution of the above other than $g = 1$.

Note that the three-parameter black hole geometries obtained as a (singular) $D \to 4$ limit after regularization from the original $D \geq 5$ dimensional Einstein-Gauss-Bonnet field equations exhibit the same form as the solutions here, with $C_2 = 0 = Q_{\text{eff}}$. From the perspective of field equations, however, such configurations should strictly be seen as either genuinely higher dimensional or as solutions of four dimensional scalar-tensor gravity where the scalar mode does not decouple. In contrast, in our case the geometries are explicit solutions to the vacuum equations of motion of the effective Einstein-Gauss-Bonnet gravity, which does not exhibit any propagating mode other than the four-metric (for vanishing torsion).

VII. HIGHER ORDER LOVELOCK TERMS WITH MORE THAN ONE INVISIBLE EXTRA DIMENSION

In a generic Kaluza-Klein dimensional reduction of the Lovelock series, it is not only the Gauss-Bonnet, but rather, all higher Lovelock terms could contribute to the four dimensional effective action in principle. Thus, there is no natural way to get a finite number of terms in the emergent equations of motion. Here we explore if it is any different in our formulation based on extra dimensions of zero proper length, and consider its generalization to higher dimensions ($D > 5$) in this section.

Beyond $D = 5$, the next nontrivial case is $D = 7$, when the cubic Lovelock term becomes dynamical. The corresponding Lagrangian density is given by:

$$
\mathcal{L}(\hat{e}, \hat{w}) = \epsilon^{\mu\nu\alpha\beta\gamma\delta} \epsilon_{IJKLMNP} \left[ \sigma R^{IJ}_{\mu\nu} (\hat{w}) \hat{R}^{KL}_{\gamma\delta} (\hat{w}) \hat{R}^{MN}_{\alpha\beta} (\hat{w}) + \frac{\alpha}{6} \hat{R}^{IJ}_{\mu\nu} (\hat{w}) \hat{R}^{KL}_{\alpha\beta} (\hat{w}) \hat{R}^{MN}_{\gamma\delta} \right] + \frac{\epsilon^{IJK} \hat{e}^{L} \hat{e}^{M} \hat{e}^{N}}{5} \left[ \hat{e}^{P} \right], \quad (30)
$$
The connection and vielbein equations read, respectively:

\[ \epsilon^{\mu\alpha\beta\gamma\delta\lambda} \epsilon_{IJKLMNP} \left[ 3\sigma \dot{R}_{\mu\nu}^{IJ}(\hat{w}) \dot{R}_{\alpha\beta}^{KL}(\hat{w}) + \alpha \epsilon^{I}_{\mu} \epsilon^{J}_{\nu} \dot{R}_{\alpha\beta}^{KL}(\hat{w}) + \zeta \epsilon^{I}_{\mu} \epsilon^{J}_{\nu} \dot{e}_{\alpha}^{L} e_{\beta}^{M} \right] \left( \dot{D}_{\gamma}(\hat{w}) e_{\delta}^{N} \right) = 0, \]

\[ \epsilon^{\mu\alpha\beta\gamma\delta\lambda} \epsilon_{IJKLMNP} \left[ \sigma \dot{R}_{\mu\nu}^{IJ}(\hat{w}) \dot{R}_{\alpha\beta}^{KL}(\hat{w}) + \alpha \epsilon^{I}_{\mu} \epsilon^{J}_{\nu} \dot{R}_{\alpha\beta}^{KL}(\hat{w}) + \zeta \epsilon^{I}_{\mu} \epsilon^{J}_{\nu} \dot{e}_{\alpha}^{L} e_{\beta}^{M} \right] \left( \dot{R}_{\gamma\delta}^{MN}(\hat{w}) e_{\gamma}^{N} \right) = 0 \]

(31)

(32)

The connection equations of motion (31) are solved by:

\[ \epsilon^{\mu\alpha\beta\gamma\delta\lambda} \epsilon_{IJKLMNP} \left[ 3\sigma \dot{R}_{\mu\nu}^{IJ}(\hat{w}) \dot{R}_{\alpha\beta}^{KL}(\hat{w}) + \alpha \epsilon^{I}_{\mu} \epsilon^{J}_{\nu} \dot{R}_{\alpha\beta}^{KL}(\hat{w}) + \zeta \epsilon^{I}_{\mu} \epsilon^{J}_{\nu} \dot{e}_{\alpha}^{L} e_{\beta}^{M} \right] = 0 \text{ or,} \]

\[ \dot{D}_{[a}(\hat{w}) e_{b]}^{I} = 0. \]

It is straightforward to verify that the first case above, involving a quadratic curvature contribution, reproduces the effective equation of motion (21) after a redefinition of couplings.

The second case, implying vanishing torsion in the \( D \)-dimensional spacetime, needs a more detailed analysis as presented below.

Since the full spacetime now has three dimensions of vanishing proper length associated with the three zero eigenvalues of the seven dimensional vielbein, let us introduce a more general notation as follows. The spacetime and internal indices respectively are decomposed as: \( \mu \equiv (a, \bar{a}) \), \( I \equiv (i, \bar{i}) \) where \( a, i \) are the four dimensional indices and \( \bar{a} \equiv (v_{1}, v_{2}, v_{3}) \), \( \bar{i} \equiv (4, 5, 6) \) are the extra dimensional ones. \( \hat{e}_{a}^{i} \equiv e_{a}^{i} \) denotes the emergent tetrad as before (with emergent inverse tetrads denoted as \( e_{a}^{i} \)), whereas \( \hat{e}_{a}^{\bar{i}} = 0 = \hat{e}_{a}^{\bar{i}} = \hat{e}_{a}^{\bar{i}} \).

With this, the eqs.(33) are solved for the super-connection field components as:

\[ \dot{D}_{[a}(\hat{w}) e_{b]}^{i} = 0 \Rightarrow K_{a}^{i} \equiv \hat{w}_{a}^{ij} - \tilde{w}_{a}^{ij}(e) = 0, \]

\[ \dot{D}_{[a}(\hat{w}) e_{b]}^{\bar{i}} = 0 \Rightarrow \hat{w}_{a}^{\bar{i}} = Q^{i}_{\kappa} e_{\alpha}^{\kappa} \left[ Q^{\bar{i}} = Q^{\bar{i}} \right], \]

\[ \dot{D}_{[a}(\hat{w}) e_{b]}^{i} = 0 \Rightarrow \hat{w}_{a}^{\bar{i}} = 0, \]

\[ \dot{D}_{[a}(\hat{w}) e_{b]}^{i} = 0 \Rightarrow \hat{w}_{a}^{ij} = -e_{a}^{a} \partial_{a} e_{a}^{i}. \]

(34)

The remaining equations \( \dot{D}_{[a}(\hat{w}) e_{b]}^{I} = 0 \) are identically satisfied. Note that the last equation above imply that \( \hat{w}_{a}^{ij} \) is a pure gauge and the tetrad determinant (and hence the emergent metric) is independent of the extra dimensional coordinates, a result analogous to the five dimensional case analyzed in the previous sections. While these could be gauged away: \( \hat{w}_{a}^{ij} = 0 \), the components \( \hat{w}_{a}^{ij}, \hat{w}_{a}^{ij} \) are left arbitrary.
From the above solutions for connection fields, we find that the following field-strength components are trivial:

\[ \hat{R}_{a\bar{a}}{}^{ij} = 0, \quad \hat{R}_{a\bar{a}}{}^{ij} = 0, \quad \hat{R}_{a\bar{a}}{}^{\bar{i}\bar{j}} = 0. \]  

(35)

Further, let us consider the identity:

\[ \hat{R}_{a\bar{a}}{}^{\bar{i}\bar{j}} = \hat{w}_{a}{}^{i\bar{k}} \hat{w}_{a}{}^{\bar{k}} \]  

(36)

This, along with the antisymmetry of \( \hat{R}_{a\bar{a}}{}^{\bar{i}\bar{j}} = -\hat{R}_{a\bar{a}}{}^{\bar{i}\bar{j}} \) forces it to vanish: \( \hat{R}_{a\bar{a}}{}^{\bar{i}\bar{j}} = 0 \). This condition admits three possible solutions:

a) \( \hat{w}_{a}{}^{i\bar{k}} = 0; \)

b) \( \hat{w}_{a}{}^{i\bar{k}} = 0 = \bar{Q}^{i\bar{k}}; \)

c) At least one component of the connection fields \( \hat{w}_{a}{}^{i\bar{k}} \) is nonvanishing. For instance, we may choose: \( \hat{w}_{a}{}^{i\bar{4}} = 0, \hat{w}_{a}{}^{i\bar{k}} = \bar{Q}_{a}^{i\bar{k}} \) (\( \bar{Q}_{a}^{i\bar{k}} \equiv \bar{Q}^{ij}e_{aj}, \bar{Q}^{ij} = Q^{ij} \)) while leaving \( \hat{w}_{a}{}^{i\bar{5}}, \hat{w}_{a}{}^{i\bar{6}} \) arbitrary.

We consider all three cases next.

Using the expressions for the field-strength components, all the components of the vielbein equations of motion eq. (32) are identically satisfied except \( (\lambda, P) = (\bar{c}, p) \) and \( (\lambda, P) = (\bar{c}, \bar{p}) \), leading to, respectively:

\[ \epsilon^{abcd\bar{a}\bar{b}} \bar{\epsilon}_{ijkl\bar{j}\bar{k}} \left[ \sigma \hat{R}_{ab}{}^{ij} + \alpha \epsilon^{i\bar{j}} \epsilon^{\bar{k}} \right] \hat{R}_{cd}{}^{k\bar{l}} \hat{R}_{ab}{}^{\bar{k}l} = 0, \]

\[ \epsilon^{abcd\bar{a}\bar{b}} \bar{\epsilon}_{ijkl\bar{j}\bar{k}} \left[ \sigma \hat{R}_{ab}{}^{ij} \hat{R}_{cd}{}^{k\bar{l}} + \frac{\alpha}{2} \epsilon^{i\bar{j}} \epsilon^{\bar{k}} \hat{R}_{cd}{}^{k\bar{l}} + \beta \epsilon^{i\bar{j}} \epsilon^{\bar{k}} \hat{R}_{cd}{}^{k\bar{l}} \right] \hat{R}_{ab}{}^{\bar{i}\bar{j}} = 0. \]  

(37)

Let consider case (c) first, implying \( \hat{R}_{ab}{}^{i\bar{4}} = 0, \hat{R}_{a\bar{b}}{}^{i\bar{4}} = \bar{D}_{a}(\hat{w}) M_{\bar{b}i} \) and leaving \( \hat{R}_{ab}{}^{i\bar{5}}, \hat{R}_{ab}{}^{i\bar{6}}, \hat{R}_{a\bar{b}}{}^{\bar{5}\bar{6}} \) arbitrary. Using these, we obtain from eq. (37):

\[ \epsilon^{abcd} \epsilon_{ijkl} \left[ \sigma \hat{R}_{ab}{}^{ij} + \alpha \epsilon^{i\bar{j}} \epsilon^{\bar{k}} \right] \bar{D}_{c}(\hat{w}) Q_{d}^{i} = 0, \]

\[ \epsilon^{abcd} \epsilon_{ijkl} \left[ \sigma \hat{R}_{ab}{}^{ij} \hat{R}_{cd}{}^{k\bar{l}} + \frac{\alpha}{2} \epsilon^{i\bar{j}} \epsilon^{\bar{k}} \hat{R}_{cd}{}^{k\bar{l}} + \beta \epsilon^{i\bar{j}} \epsilon^{\bar{k}} \hat{R}_{cd}{}^{k\bar{l}} \right] = 0. \]  

(38)

where we have defined \( \epsilon^{abcdv_{1}v_{2}v_{3}} \equiv \epsilon^{abcd} \) and \( \epsilon_{ijklv_{4}v_{5}v_{6}} \equiv \epsilon_{ijkl}. \) The first among the set above is solved by:

\[ \sigma \hat{R}_{ab}{}^{ij} + \alpha \epsilon^{i\bar{j}} \epsilon^{\bar{k}} = 0 \quad \text{or}, \]

\[ Q_{a}^{i} = \lambda \epsilon^{i}. \]  

(39)

(40)

The first solution (39) represents maximally symmetric spacetimes. Using this in the second equation in (38), we obtain: \( \alpha^{2} + 2\sigma \beta = 0. \) This case is thus equivalent to the constant
curvature solutions already discussed in section-III, as reflected by the exact correspondence with equations (4) and (5). The other solution (40), when inserted back into the second equation in (38), precisely reproduces the equation of motion (21) in section-IV after a redefinition of the couplings. Again, this class precisely corresponds to the vanishing torsion solutions for the five dimensional case analyzed in section-IV.

It is now easy to verify that the case (b), which implies $\hat{R}^{kj}_{\bar{a}\bar{b}} = 0$, represents the limit of the above equation of motion (38) where $Q'_a = 0$. Case (a) on the other hand implies $\hat{R}^{i\bar{j}}_{\bar{a}\bar{b}} = 0$, satisfying both the equations in (37) identically. This does not lead to any dynamical content in the emergent theory and may simply be discarded.

From these results, we conclude that a $D = 7$ theory with three additional ‘invisible’ dimensions and hence with a cubic Lovelock term in the action reduces to the five dimensional case where only the Gauss-Bonnet term (along with Einstein) contributes to the effective theory. A generalization of this result to any $D > 7$ is straightforward, and does not change this conclusion. In other words, the pattern that a $D + 2$ dimensional theory ($D \geq 5$) simply reproduces the $D$ dimensional equations of motion keeps repeating itself. To conclude, one does not require to include more than one ‘invisible’ dimension, and consider any higher order Lovelock term other than Gauss-Bonnet, so long as the number of dimensions having a nonvanishing proper length is exactly four.

VIII. CONCLUSIONS

Inspired by the idea of extra dimension of zero proper length, introduced recently within a first order formulation, we have developed a method to generate an effective metric theory of gravity where the otherwise nondynamical higher Lovelock densities leave a dynamical imprint in four dimensions. This formalism is inequivalent to a Kaluza-Klein reduction, as the extra dimension is not associated with a physical evolution and do not lead to any infinite tower of discrete eigenmodes of higher energy. The effective four dimensional equations of motion exhibit nonlinear curvature effects unlike Einstein gravity.

It is striking that the only nonlinear curvature contribution in the emergent equation of motion comes from the Gauss-Bonnet. In other words, increasing the number of ‘invisible’ extra dimensions (of vanishing proper length) has no effect on the emergent theory, as long as the effective spacetime is given by an invertible four tetrad. This signifies an enormous
simplication, particularly when contrasted with approaches based on ghost free higher curvature theories or a Kaluza-Klein dimensional reduction where the effective equations of motion could exhibit an infinity of nonlinear terms in principle.

The five dimensional equations are solved by spacetimes with maximal symmetry and vanishing torsion. Both cases have been worked out in detail. The emergent equations of motion exhibit terms no higher than second derivatives of the metric, the only propagating field. The general covariance of the emergent theories is manifest. Notably, this formulation does not require any singular rescaling of the couplings or any regularization of divergent terms. This is in contrast with a recent proposal by Glavan and Lin [10], as well as with others based on a Kaluza-Klein reduction or conformal scaling [13, 17].

Apart from the general motive of our construction, the details reveal some notable features in the context of homogeneous and isotropic (FLRW) cosmology. For instance, even in absence of a bare cosmological constant (as well as of exotic matter) and any spatial curvature, the cosmological field equations could exhibit nonsingular bounce or inflationary behaviour. This has no analogue in standard (Einsteinian) FRW cosmology. In general, the dynamics admits smooth transitions between a bounce and either an inflationary universe or one undergoing a smooth contraction. That these features could be obtained in vacuum theories with Gauss-Bonnet induced nonlinearities only might turn out to be a more economical alternative for bouncing cosmology models, where it is imperative to choose an appropriate (typically exponential for a slow contraction [19]) scalar potential out of many other possibilities.

Further, we have shown that this formulation admits static, spherically symmetric black hole solutions with curvature singularity lying at a finite proper distance along a radial geodesic. These spacetimes are inequivalent to Schwarzschild geometry in general.

Our studies here are also relevant from the perspective of possible topology changes of lower (four) dimensional spacetimes through degenerate higher dimensional metrics in classical gravity theory or otherwise. Such hidden dimensions might be thought to have played a role during the ‘beginning’ of the Universe, if there has been one. It seems reasonable to speculate that topology changes of spacetime through degenerate vielbeins in higher dimensions could have formed an important part of the (classical or quantum) gravitational dynamics at those times.

We cannot claim our analysis to be complete. While we have answered the question as
to why it should be sufficient to consider only one invisible extra dimension, we have not
touched upon the more fundamental one as to why should the emergent spacetime exhibit
exactly four dimensions with a noninvertible tetrad. We have left the case of nonvanishing
torsion for future work. Nor have we included any matter coupling. In the cosmological context, torsional degrees of freedom could as well introduce effects akin to matter, leaving open the possibility of a richer cosmological dynamics (preferably nonsingular). A detailed Hamiltonian analysis of this theory should also be taken up elsewhere. To conclude, some of the insights gained from this work should serve as sufficient motivation for future investigations along the lines above.

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