Abstract

We consider the problem of testing pattern-freeness: given a string $I$ and a fixed pattern $J$ of length $k$ over a finite alphabet $\Sigma$, decide whether $I$ is $J$-free (has no occurrence of $J$) or alternatively, one has to modify $I$ in at least an $\epsilon$-fraction of its locations to obtain a string that is $J$-free. The two-dimensional analog where one is given a 2D-image and a fixed pattern of dimension $k$ is studied as well. We show that other than a small number of specific patterns, for both one-dimensional and two-dimensional cases, there are simple one-sided testers for this problem whose query complexity is $O(1/\epsilon)$. The testers work for any finite alphabet, with complexity that does not depend on the template dimension $k$, which might depend on $n$.

For the one-dimensional case, testing pattern-freeness is a specific case of the problem of testing regular languages. Our algorithm improves upon the query complexity of the known tester of Alon et. al. [2], removing a polynomial dependency on $k$ as well as a polylogarithmic factor in $1/\epsilon$. For the two-dimensional case, it is the first testing algorithm we are aware of. The pattern-freeness property belongs to the more general class of matrix properties, studied by Fisher and Newman in [11, 12]. They provide a 2-sided tester, doubly exponential in $1/\epsilon$, for matrix properties that can be characterized by a finite set of forbidden induced submatrices and pose the problem of testing such properties for tight submatrices (with consecutive rows and columns) as an open problem. Our algorithm provides a strong tester for this class of properties, in which the forbidden set is of size 1.

The dependence of our testers on $\epsilon$ is tight up to constant factors since any tester erring with probability at most $1/3$ must make $\Omega(1/\epsilon)$ queries to $I$. The analysis of our testers builds upon novel combinatorial properties of strings and images, which may be of independent interest.
1 Introduction

Pattern matching is the algorithmic problem of finding occurrences of a fixed pattern in a given string. This problem appears in many settings and has applications in diverse domains such as computational biology, computer vision, natural language processing and web search. There has been extensive research concerned with developing algorithms that search for patterns in strings, resulting with a wide range of efficient algorithms [9, 17, 13]. The two-dimensional setting where one searches for a 2D pattern in a 2D image has received much attention as well [5, 6].

Given a string $I$ of length $n$ and a pattern $J$ of length $k \leq n$, any algorithm which determines whether $J$ occurs in $I$ has running time $\Omega(n)$ [10, 19]. One may wonder if there are circumstances in which such a task can be performed in sublinear (namely $o(n)$) time.

The field of property testing [14, 20] deals with decision problems regarding discrete objects (e.g., graphs, functions, images) that either have a certain property $P$ or are far from having $P$. In the context of pattern matching, we are given a string $I$ of length $n$ and a fixed pattern $J$ of length $k \leq n$. We say a string $I$ is $\epsilon$-far from being $J$-free if one needs to change at least an $\epsilon$-fraction of the locations of $I$ in order to ensure that the resulting string after these changes, $I'$, is $J$-free (namely $I'$ may not contain a string of $k$-consecutive characters that equals $J$). Alternatively, $I$ is $\epsilon$-close to being $J$-free, if it can be made $J$-free by changing at most $\epsilon n$ locations in it. Here $\epsilon$ is in $(0, 1)$ and both the string and the pattern are over a fixed alphabet $\Sigma = \{0, 1, \ldots, q\}$ for an integer $q \geq 1$. The two-dimensional version of the problem is defined similarly. Namely, we are given an $n \times n$ image $I$ and a $k \times k$ pattern $J$, both over a fixed alphabet $\Sigma$. An image $I$ contains $J$-free if there do not exist $i, j \in [1, \ldots, n-k+1]$ such that the $k \times k$ image induced by rows $i, i+1, \ldots, i+k-1$ and columns $j, j+1, \ldots, j+k-1$ is identical to $J$. An image is $\epsilon$-far from being $J$-free, if one needs to change at least $\epsilon n^2$ locations in $I$ for the resulting image to be $J$-free.

We consider the pattern-freeness problem where one needs to distinguish between the case that a given string (or image) $I$ is $J$-free for a fixed pattern $J$ or $\epsilon$-far from being $J$-free. A tester $Q$ for the pattern-freeness problem is a randomized algorithm that is given access to a string $I$, as well as the length of the string $n$ and a proximity parameter $\epsilon \in (0, 1)$. Furthermore, $Q$ needs to distinguish with probability at least $2/3$ between the case that $I$ is $J$-free and the case that $I$ is $\epsilon$-far from being $J$-free. The query complexity of $Q$ is the number of queries it makes to $I$. Following [2], we say $Q$ is a strong tester with one-sided error, if its query complexity depends only on the proximity parameter $\epsilon$ but not on $n$ and additionally, if the probability that $Q$ classifies a $J$-free string as containing $J$ is 0. Given the existence of such a tester we will that pattern-freeness is strongly testable.

The main question we address is whether pattern-freeness is strongly testable. One motivation for this question is to identify settings where the pattern matching problem can be solved in time $o(n)$. Devising efficient testers for pattern-freeness along with having a better understanding of strings that are $\epsilon$-far from being $J$-free might prove useful for practical applications. In particular, it might be possible to run much faster testing algorithms that quickly validate that a string is not $J$-free before running more time-consuming pattern-matching algorithms. Finally, our study of strings and images that are $\epsilon$-far from being $J$-free has lead to observations regarding strings and images, which might be of independent interest.

We present a tester $T$ for the pattern-freeness problem, whose performance is summarized in the following Theorem. For $k \geq 2$, let $E_k$ be the set of strings $\{0^{k-1}1, 1^{k-1}0, 01^{k-1}\}$. Here and throughout the paper, for a character $\sigma \in \Sigma$ and an integer $m$, we denote by $\sigma^m$ the character $\sigma$ concatenated $m$ times.
Theorem 1. [pattern-freeness testable in 1D] Suppose that $J$ is a pattern of length $k$ and that $J \not\in E_k$. Then being $J$-free is strongly testable by a tester $T$ whose query complexity is $O(1/\epsilon)$. Furthermore, if $|\Sigma| > 2$ then $T$ is a strong tester for pattern-freeness for any fixed pattern over $\Sigma$.

The query complexity of the tester above is tight up to a constant multiplicative factor. If we treat $k$ as a constant, this follows from a theorem of Alon et al. [2] (Proposition 3.3) which shows that testing the language of all strings avoiding a fixed pattern $J$ (in fact any “nondegenerate regular language”) requires query complexity $\Omega(1/\epsilon)$. Indeed, the language of strings that avoid a fixed pattern $J$ is regular.\footnote{We give a self contained proof for a $\Omega(1/\epsilon)$ lower bound (in Theorem 3 of Appendix A) which is more general in the sense that it covers the case when $k$ may depend on $n$ and it extends to the 2D setting discussed later as well.}

Our tester $T$ depends crucially on a characterization of strings that are $\epsilon$-far from being $J$-free as strings that contain a large number ($\Omega(\epsilon n)$) of nonoverlapping copies of $J$. For this characterization to hold, it is necessary to restrict $J$ (not belonging to $E_k$) in the case of binary strings, as can be seen by considering $J = 01^{k-1}$ and a string $I = 0^{n/2}1^{n/2}$ for an even integer $n$. The string $I$ contains only a single occurrence of $J$, yet it is $\Omega(1)$-far from being $J$-free.\footnote{Testing pattern freeness with respect to the strings in $E_k$ has not been studied in this work.}

Theorem 1 generalizes to the 2-dimensional (2D) setting. We say a $k \times k$ pattern is almost homogeneous if all of its locations are identical other than exactly one of the corners. The set of almost homogeneous $k \times k$ patterns is a generalization of the set $E_k$ from the one-dimensional (1D) case. Our tester $T_2$ for the 2D version, with a 2D pattern $k \times k$ pattern $P$ works similarly to the tester $T$ for the 1D case. We get the following Theorem:

Theorem 2. [pattern-freeness testable in 2D] Let $P$ be a two-dimensional pattern of size $k \times k$ that is not almost homogeneous where $k > 11$. Then being $P$-free is strongly testable by a tester $T_2$ with query complexity of $O(1/\epsilon)$. Furthermore, if $|\Sigma| > 2$ then $T_2$ is a strong tester for pattern-freeness for every fixed pattern over $\Sigma$.\footnote{As in the one-dimensional case, the assumption about almost homogeneity is essential for the success of our tester, as we show in the sequel.}

Our testers for both the 1D and 2D cases work (with query complexity $O(1/\epsilon)$) for arbitrary alphabets. This feature may be useful, as often alphabets encountered in applications are quite large (e.g. in the case of discretized grey-scale images). More important is the fact that our testers’ query complexity does not depend on the template dimension $k$, which often might not be assumed to be a small constant or even independent of $n$. This is the case, e.g., with template matching in images for detection applications, where the template size depends on the image resolution.

2 Related works

Our work is inspired by the study of testing subgraph-freeness (see, for example, [1, 3]). This line of work examines how one can test quickly whether a given graph $G$ is $H$-free or $\epsilon$-far from being $H$-free, where $H$ is a fixed subgraph. In this problem, a graph is $\epsilon$-far from being $H$-free if at least an $\epsilon$-fraction of its edges and non-edges need to be altered in order to ensure that the resulting graph does not contain $H$ as a (not necessarily induced) subgraph. A recurring idea in these works is that if $G$ is $\epsilon$-far from being $H$-free, it necessarily contains a large number of copies of $H$. Perhaps the best example for this phenomena is the triangle removal lemma which asserts that for every
\(\epsilon \in (0, 1)\), there exists \(\delta = \delta(\epsilon) > 0\) such that if an \(n\)-vertex graph \(G\) is \(\epsilon\)-far from being triangle free, then \(G\) contains at least \(\delta n^3\) triangles (see e.g., [4] and the reference within).

Alon et. al. show [2] that regular languages over \(\{0, 1\}\) are strongly testable. Testing pattern-freeness (1-dimensional, binary alphabet, constant pattern length \(k\)) is a special case of the former, since the language of all strings avoiding a fixed pattern is regular. The query complexity of their tester is \(O\left(\frac{c}{\epsilon} \cdot \ln^3\left(\frac{1}{\epsilon}\right)\right)\), where \(c\) is a constant that depends on the minimal size of a DFA \(A_L\), that accepts the regular language \(L\). It is shown in [2] that \(c\) can be taken to be \(O(s^3)\) where \(s\) is the size of \(A_L\). In the case of the regular language considered here a simple pumping-Lemma inspired argument shows that \(s \geq \Omega(k)\). Hence the upper bound on testing pattern freeness implied by their algorithm is \(O\left(\frac{k^3}{\epsilon} \cdot \ln^3\left(\frac{1}{\epsilon}\right)\right)\). Our 1D tester solves a very restricted case of the problem the tester of [2] deals with, but it achieves a better query complexity of \(O(1/\epsilon)\) in this setting. Moreover, our tester is much simpler and can be applied in the more general 2-dimensional and non-binary settings, or when the pattern length \(k\) is allowed to grow as a function of the string length \(n\).

Building on sophisticated applications of the regularity lemma, Fisher and Newman [11, 12] provide a 2-sided tester, doubly exponential in \(1/\epsilon\), for testing whether a 2-dimensional matrix is free from containing a (forbidden) fixed set of induced sub-matrices. This is part of a more general study of testing properties that can be defined by first order formulae over the syntax containing the standard-grid partially ordered set (poset) relations. The property of 2D pattern-freeness that we study is explicitly mentioned in their work (they consider the more general case of testing whether a matrix is far from containing \(r\) patterns, where \(r\) might be greater than 1). While this property of being free of a set of tight (over consecutive rows and columns) sub-matrices can be defined by a \(\forall\exists\)-poset that uses 'same row/col' and 'successive row/col' relations, it is unclear how to use this characterization to devise a tester for matrices avoiding tight patterns, and the question of finding such a tester was left as an open question. The case where the forbidden set consists of a single submatrix is exactly the 2D pattern-freeness property, which we show to be strongly testable.

The 2D part of our work adds to a growing literature concerned with testing properties of images [18, 21, 8]. Ideas and techniques from the property testing literature have recently been used in the fields of computer vision and pattern recognition [15, 16].

### 3 Testing pattern freeness

We focus initially on the 1-dimensional case to illustrate the ideas behind our testers and their correctness proofs. These proofs involve the analysis of combinatorial properties of strings that are \(\epsilon\)-far from being \(J\)-free. Note that our testers can handle general finite alphabets. They are based on lemmas, that are introduced in this section strictly for binary alphabets and which are generalized in Appendix B to larger ones.

A key observation used in our proofs is the following lemma which we term the modification Lemma. Roughly speaking, this lemma shows that other than exceptional cases (the set \(E_k\)), given a string \(I\) and an occurrence of \(J\) in \(I\), it is possible to change the occurrence of \(J\) in a single location without creating a new occurrence of \(J\).

**Lemma 1.** [1D modification lemma] Let \(I := I[1\ldots n]\) be a string and let \(J\) be a pattern of length \(k\) over a binary alphabet \(\Sigma\). Suppose there is an index \(\ell \in [1, n - k + 1]\) such that \(I[\ell], \ldots, I[\ell + k - 1]\) is equal to \(J\). Then assuming \(J \notin E_k\), it is possible to change \(I\) in one of the locations \(I[\ell], \ldots, I[\ell + k - 1]\), without creating a new occurrence of \(J\) in the resulting string \(I'\).
As we are unaware of existing proofs for the 1D or 2D modification lemmas, we provide their self-contained proofs. The main idea behind the proofs is to assume towards a contradiction that for a given occurrence of a pattern \( J \) in a string \( I \), no matter which of its locations we change, we get a new occurrence of \( J \) in \( I \). This generally implies the existence of pairs or larger sets of newly created occurrences. Such co-occurrences, some with significant overlap, imply a range of periodicity constraints on the pattern \( J \) which can be leveraged to show that the assumption is impossible, unless \( J \) belongs to the set \( E_k \). For more details, refer to Section 4.

Using the modification Lemma, we can immediately prove that a string of length \( n \) that is far from being \( J \)-free contains many occurrences of \( J \).

**Lemma 2.** Let \( I \) be a string of length \( n \) over a binary alphabet \( \Sigma \), and let \( J \) be a string of length \( k \) not belonging to \( E_k \). If \( I \) is \( \epsilon \)-far from being \( J \)-free, then \( I \) contains at least \( en \) occurrences of \( J \).

**Proof.** Assume towards a contradiction that \( I \) contains strictly less than \( en \) occurrences of \( J \). Iteratively pick an occurrence of \( J \) in the string \( I \). By the modification Lemma, there is a location \( r \) in the occurrence of \( J \), such that flipping the value at \( J[r] \) does not create a new occurrence of \( J \). Change \( J \) at \( r \) and repeat until no occurrences of \( J \) remain. In this process, \( I \) has been made \( J \)-free by modifying less than \( en \) locations, contradicting the fact that \( I \) is \( \epsilon \)-far from being \( J \)-free. \( \square \)

In spite of Lemma 2, the existence of \( en \) occurrences of \( J \) does not imply that \( I \) is \( \epsilon \)-far from being \( J \)-free. The string \( 1^n \) contains \( n-k+1 \) occurrences of \( J := 1^k \), yet it is roughly \( 1/k \)-far from being \( J \)-free. Nonetheless, we will show later that strings that are \( \epsilon \)-far from being \( J \)-free can be characterized in terms of the number of non-overlapping occurrences of \( J \) that they contain.

From now on, we assume (for binary alphabets) that \( J \not\in E_k \) in the 1D case and that \( J \) is not almost homogeneous in the 2D case. Lemma 2 suggests a simple tester \( T' \) for pattern-freeness. \( T' \) first queries \( 2/\epsilon \) locations randomly in the first \( n-k+1 \) locations of a given string \( I \). The tester then examines whether there is an occurrence of \( J \) starting at any of these locations. If an occurrence is found - \( T' \) rejects (the string is not \( J \)-free). Otherwise, it accepts and labels the string as being \( J \)-free. Clearly \( T' \) is a one-sided tester, since \( T' \) can only err if it fails to find an occurrence of \( J \). Furthermore, in the \( \epsilon \)-far case, \( T' \) will fail to find an occurrence of \( J \) with probability at most \((1-(1-\epsilon)/k)^2 < 1/3 \). The query complexity of \( T' \) is \( 2k/\epsilon \).

Looking more closely into the tester \( T' \) suggests that it is wasteful. In order to check whether \( J \) occurs at a single starting location in \( I \), \( T' \) queries \( k \) locations. Instead, one could randomly choose a location \( i \) and check all \( k \)-substrings starting in locations \( i, i+1, i+2, \ldots, i+k-1 \) for equality to \( J \). This enables a more efficient query scheme - by examining \( 2k-1 \) locations we can locate the existence of \( J \) in \( k \) starting locations. This suggests the tester \( T \), specified in Algorithm 1.

**Algorithm 1** The Tester \( T \): Tests if a String \( I \) is \( J \)-free

**Input:** Strings \( I \) and \( J \) of lengths \( n \) and \( k \); Precision parameter \( \epsilon \in (0,1) \);

let \( \ell = \lceil Ck \rceil \) for some large enough constant \( C \);

repeat \( \ell \) times:

1: Choose an integer \( i \in [1, n-2k+1] \) uniformly at random

2: Query all substrings of length \( k \) starting at locations \( i, i+1, \ldots, i+k-1 \)

3: if \( J \) was found among the queried substrings

\[ \text{return } I \text{ contains } J \]

return \( I \) is \( J \)-free
The increased efficiency of $T$ (when compared to $T'$) is apparent: whereas an occurrence of $J$ is discovered by a random query of $T'$ with probability \( \frac{1}{n-k-1} \), the probability that it is discovered for a single choice of a random position $i$ by the tester $T$ is \( \frac{k}{n-2k+1} \). However, there are two problems in leveraging the enhanced efficiency of $T$. First, an implicit assumption which needs to be justified is that the amplification parameter $\ell$ can be chosen properly. Namely, we need to show the existence of a constant $C$ (not depending on $k, \epsilon, n$) for which $l$ is a large enough integer. Second, in contrast to $T'$, there are dependencies in the locations queried by $T$ due to overlaps, making the analysis of the behavior of $T$ in the $\epsilon$-far case less obvious.

The first difficulty is addressed by the following lemma:

**Lemma 3.** [Any string is $O(1/k)$-close to being $J$-free] If $I$ is $\epsilon$-far from being $J$-free then $\epsilon \leq 2/k$.

An immediate implication of Lemma 3 is that if $I$ is $\epsilon$-far from being $J$-free (for $|J| = k$), then $\epsilon = O(1/k)$. We stress that here and elsewhere, we did not optimize the constants hiding in the $O$-notation.

The second difficulty is handled by showing that if $I$ is $\epsilon$-far from being $J$-free then not only does it contain $\epsilon n$ occurrences of $J$ (Lemma 2), but it must contain $\Omega(\epsilon n)$ non-overlapping occurrences of $J$. That is,

**Lemma 4.** [\(\Omega(\epsilon n)\) non-overlapping pattern occurrences] If $I$ is $\epsilon$-far from being $J$-free then $I$ must contain $\epsilon n/2$ non-overlapping copies of $J$.

Lemma 4 gives a necessary and sufficient characterization of length strings that are $\epsilon$-far from being $J$-free: such strings contain $\Omega(\epsilon n)$ nonoverlapping occurrences of $J$.

The proofs of Lemmas 3 and 4 rely on a constructive procedure that alters a string $I$ that is $\epsilon$-far from being $J$-free. It iteratively picks an occurrence of $J$ inside it as well as a bit to flip that removes the occurrence from $I$ without adding a new one, based on the modification lemma. The properties above follow from the analysis of the number of and the spacing between occurrences chosen. The proofs are deferred to Section 5.

If $I$ is $J$-free, $T$ returns the correct answer with probability 1. Having the above properties, we can now upper bound the error of the tester $T$ in the $\epsilon$-far case.

**Lemma 5.** [Correctness of Tester $T$] Suppose that a string $I$ is $\epsilon$-far from being $J$-free. Then the probability that $T$ classifies $I$ as being $J$-free is at most $1/3$.

**Proof.** Since $T$ examines $k$ consecutive starting locations per choice of random location, for a given occurrence of $J$ there are exactly $k$ possible locations $i$ (the $k$ locations that precede the occurrence of $J$) whose choice would result with $T$ finding the occurrence. Furthermore, as there are at least $\epsilon n/2$ non-overlapping occurrences, by Lemma 4, there are at least $\frac{k \epsilon n}{2(n-k-1)}$ locations that result with an occurrence of $J$ being found. Therefore (assuming $k = o(n)$), the probability $i$ falls in one of these locations is $\frac{k \epsilon n}{2(n-k-1)} > \frac{k \epsilon}{2}$. With the choice of $\ell = C \frac{k \epsilon}{kt}$ for large enough $C$, the probability that $T$ fails to find an occurrence of $J$ is at most $(1 - k \epsilon/2)^\ell < 1/3$ (observe that $k \epsilon/2 < 1$ by Lemma 3), as desired.

The observations above imply that $T$ is strong tester for pattern freeness. Clearly, the query complexity of $T$ is $O(1/\epsilon)$. Its query complexity is tight up to a multiplicative constant, as is shown by the matching lower bound (Theorem 3 of Appendix A).
3.1 The 2-dimensional setting

For the 2-dimensional case, we provide a modification lemma (Lemma 8 in Section 4.1) similar to Lemma 1 showing that it is possible to modify an occurrence of a pattern (that is not almost-homogeneous) at a single location without creating any new occurrences. We apply it to show that an image that is \( \epsilon \)-far from being free of a given pattern must contain \( \epsilon n^2 \) occurrences of the pattern. Appendix D is devoted to the proof of the 2D modification lemma. It is significantly more complicated compared to the 1D setting, requiring an intricate case-analysis.

As in the 1-dimensional case, to achieve query complexity of \( O(1/\epsilon) \) we need the following two-properties:

**Lemma 6.** [Any image is \( O(1/k^2) \)-close to being \( P \)-free] If \( I \) is \( \epsilon \)-far from being \( P \)-free then \( \epsilon \leq 20/k^2 \).

**Lemma 7.** [\( \Omega(\epsilon n^2) \) non-overlapping pattern occurrences] If \( I \) is \( \epsilon \)-far from being \( P \)-free then \( I \) must contain \( \epsilon n^2/35 \) non-overlapping copies of \( P \).

The lemmas are proved through a constructive procedure that is similar to the 1D case, although there are some complicating factors that arise from 2-dimension that require different reasoning. We defer the proofs to Appendix C. Using Lemmas 6 and 7 it is straightforward to generalize the tester \( T \) (and the proof of correctness) to the 2-dimensional case.

4 Modification lemmas

Here we prove the 1-dimensional modification lemma (Lemma 1) for a binary alphabet. When the size of the alphabet is larger than 2, a pigeonhole argument allows for a much simpler proof of the 1-dimensional modification lemma. See appendix B for details.

**Proof.** We shall assume that \( k > 2 \) otherwise the claim is trivial, since for \( k = 2 \) the only options for \( J \) (not belonging to \( E_k \)) are the patterns 00 and 11, for which no new occurrence can be created by flipping any bit. Assume first w.l.o.g. that \( J[1] = 0 \) and note that there are at most \( 2 \cdot (k - 1) \) length-\( k \) substrings of \( I \) (\( k - 1 \) from each side) that overlap a given occurrence of \( J \), excluding \( J \) itself (there may be less than \( k - 1 \) overlaps, of the occurrence of \( J \) if \( J \) is occupies any of the first or last \( k - 1 \) locations of \( I \)). Now, assume towards a contradiction, that flipping any single bit of the occurrence \( J \) creates a new occurrence of \( J \) in \( I \). Take any two bit locations in \( J \) and we have that flipping each of them creates a new occurrence of \( J \). Since different bits are flipped, it is trivial to see that these two new occurrences of \( J \) must be distinct (at different locations in \( I \)). This implies that the flipping of the \( k \) bits of \( J \) (one at a time) creates \( k \) distinct (differently located) new occurrences of \( J \) in \( I \). Clearly, for one of the sides of \( I \), w.l.o.g. the left side, at least \( \lceil k/2 \rceil \) occurrences are created.

These \( \lceil k/2 \rceil \) locations are positioned among the \( k - 1 \) possible consecutive locations (from left to right), which begin \( r \) bits before the original \( J \) for \( r = 1, 2, \ldots, k - 1 \). We do a case analysis considering two possible complementary situations. The first is that there are two new occurrences that are consecutive (i.e. which begin \( r - 1 \) and \( r \) bits before \( J \), for some \( r \in 2 \ldots k - 1 \)). Otherwise, the only way of having \( \lceil k/2 \rceil \) occurrences over the \( k - 1 \) positions is when \( k - 1 \) is odd (\( k \) is even) using all of the ‘odd’ values for \( r \): 1, 3, 5, ..., \( k - 1 \).
Figure 1: Illustrations for the case 'two consecutive new occurrences'. Blue and red arrows denote equality, while green arrows denote inequality. Note that the constraints are symmetric and the direction of the arrow is just used to show the direction in which we apply each specific constraint.

Case 'two consecutive new occurrences' Please follow the illustrations in Figure 1. Denote the new occurrences by \( J_1 \) and \( J_2 \) and assume that they start \( r - 1 \) and \( r \) bits before the string \( J \). The first clear constraints are:

\[
J = J_1 \quad \text{and} \quad J_1 = J_2. \tag{1}
\]

Now assume that \( J_1 \) was created by flipping the \( i_1 \)th bit of \( J \) and that \( J_2 \) was created by flipping the \( i_2 \)th bit of \( J \). This introduces the new constraints:

\[
J[i] = J_1[i + r], \quad i = 1, \ldots, k - r \quad (i \neq i_1) \quad \text{and} \quad J[i_1] \neq J_1[i_1 + r] \tag{2}
\]

\[
J[i] = J_2[i + r - 1], \quad i = 1, \ldots, k - r - 1 \quad (i \neq i_2) \quad \text{and} \quad J[i_2] \neq J_2[i_2 + r - 1] \tag{3}
\]

Since \( J_1 \) and \( J_2 \) are consecutive, another set of constraints is given by:

\[
J_1[i] = J_2[i - 1], \quad i = 2, \ldots, k \quad (i \neq i_1 + r \quad \text{and} \quad i \neq i_2 + r) \tag{4}
\]

\[
J_1[i_1 + r] \neq J_2[i_1 + r - 1] \quad \text{and} \quad J_1[i_2 + r] \neq J_2[i_2 + r - 1] \tag{5}
\]

Some of the above constraints are depicted as arrows in Figure 1: diagonal blue (1); long vertical red and green (2); long vertical green (3); short vertical red (4); short vertical green(5);

A first observation is that necessarily \( i_1 < i_2 \). This means that the 'flipping bit' of the first (from left) created occurrence (which is \( J_1 \)) must appear first (from the left). This will be shown by contradiction, assuming that \( i_2 < i_1 \) and following Figure 1(a). First, looking at the short red and blue arrows and recall we assumed that \( J[1] = 0 \). This 0 value propagates along the prefixes of \( J_1 \) and \( J_2 \), till it reaches the first short green arrow at the earlier of the indices \( i_1 \) and \( i_2 \). However, due to the diagonal blue arrows, the value of \( J_2 \) at that location must be 0 and therefore that of \( J_1 \)
must be 1. However, the value of $J$ at that location has to be 0 (like that of $J_2$), due to the long blue arrows from constraint (1). Therefore, this index must be $i_1$ and not $i_2$.

We now claim that the flipping indices $i_1$ and $i_2$ must be consecutive, i.e. $i_1 = i_2 - 1$. We assume that they are not, and reach the contradiction depicted in Figure 1(b). As claimed in the previous argument, we have that $J_1[i] = 0$ for all $i = 1, \ldots, i_1 - r + 1$ and $J_1[i_1 + r] = 1$. Similarly, that $J_2[i] = 0$ for all $i = 1, \ldots, i_1 - r + 1$. Both $J_1$ and $J_2$ are specified till (inclusive) the flipping index $i_1$ in $J$. Looking at the next index ($i_1 + 1$ in $J$) within $J_1$ and $J_2$ (these are the indices $i_1 + 1 + r$ in $J_1$ and $i_1 + r$ in $J_2$), assuming that $i_2 > i_1 + 1$, we obtain that $J_2[i_1 + r] = 1$ (using the right-most diagonal blue arrow in the illustration) and therefore that $J[i_1 + 1] = 1$. However, $J[i_1 + 1] = J_1[i_1 + 1 - r]$ (diagonal long blue arrow) and $J_1[i_1 + 1 - r]$ is known to be 0, leading to contradiction.

The consecutive flipping indices $i_1$ and $i_2$ are in the range $1 \leq i_1, i_2 \leq k - r$. If $i_1 = k - r$ and $i_2 = k - r + 1$ (i.e. these are the last locations in $J_1$ and $J_2$ respectively) then we get the possible configuration $J = S_1 = 0^{k-1}.1$ (seen in Figure 1(d)) and we are done. Otherwise, assuming $i_1 < k - r$, we will show a contradiction, seen in Figure 1(c). Following the rest of the short arrows, we get that $J_1$ and $J_2$ (and therefore $J$ too) are of the form $0^{i_1+r-1} \cdot 1 \cdot 0^{k-(i_1+r)}$ with a non empty suffix of zeros (since $k - (i_1 + r) > 0$). In this case, we denote by $i_\ast = i_1 + r + 1$ the index in $J$ that follows the (single) occurrence of a 1. We claim that flipping this bit at location $i_\ast$ could not possibly create any new occurrence of $J$, in contrast to the claim assumption. The reason is that flipping this bit creates a new 1 bit following the original 1 bit. A new occurrence of $J$ (which has a non empty 0s prefix followed by a single 1 bit followed by a non empty 0s suffix) that intersects $J$ at location $i_\ast$ could therefore not exist.

**Case ’new occurrences take all ’odd’ overlap positions’** We will show that no configuration is possible in this case, by focusing on the two last overlap occurrences, overlapping by $k - 1$ and $k - 3$ bits. Recall that we assume that $k > 2$ and therefore $k \geq 4$ (since in this case $k$ is known to be even). We denote by $J_1$ and $J_2$ the corresponding new occurrences of $J$. The constraints in this case are very similar to the ones from Case 1, and are shown by colored arrows in Figure 2, using the same conventions. Again, note by $i_1$ the index of $J$ whose flipping creates the occurrence $J_1$. First, assume that $i_1 = 1$ and follow the illustration in Figure 2(a), reaching a clear contradiction. Similarly, in the complementary case of $i_1 > 1$ - follow the illustration in Figure 2(b), reaching a clear contradiction.

![Figure 2: Illustration for the case ’new occurrences take all ’odd’ overlap positions’.](image)

In this case there is no possible configuration. Here the new occurrences $J_1$ and $J_2$ overlap $J$ by $k - 1$ and $k - 3$ bits.
4.1 The 2-dimensional setting

We present here the 2D modification lemma over a binary alphabet. As mentioned above, its proof leans on some similar ideas from the 1-dimensional proof as well as on exploiting different complex 2D periodicity constraints that arise from the assumption that the flipping of any pattern pixel creates a new occurrence of the pattern. As the analysis required for the proof is rather long, it was deferred to Section D of the appendix. We note that certain parts of our current proof require that the dimension of the pattern $k$ is greater than 11. We suspect that the lemma holds for smaller values of $k$, though we know it does not hold for $k = 2$, for which there exist examples of an occurrence of a (not almost-homogeneous) pattern $P$ in an image, such that every change to $P$ will create a new occurrence of $P$.

Lemma 8. [2D modification lemma] Let $I$ be an $n \times n$ binary image (over $\{0, 1\}$). Let $P$ be a binary $k \times k$ pattern, for some $k > 11$, which appears in $I$. Assume further that $P$ is almost-homogeneous (homogeneous except one value in one of the corners). Then it is possible to flip a single bit in the occurrence of $P$, without creating any new occurrence of $P$ in $I$.

For completeness, we provide a ‘negative’ example showing that the exclusion of almost-homogeneous patterns is indeed an exception to the 2D modification lemma. Take a pattern $P$ that is all zero except at its bottom-right pixel and place it in the center of an image $I$, that has a boundary of $k - 1$ rows and columns around $P$. Take this boundary to be all zeros, except the bottom-right pixel of $I$ which is set to be one. Clearly, flipping any bit of $P$ creates a new copy of itself.

5 Properties of $\epsilon$-far strings

We prove here Lemmas 3 and 4 from Section 3, that describe properties of strings that are $\epsilon$-far from being $J$-free, for a given binary pattern $J$ of length $k$. These lemmas are an important part of the proof of the correctness of our proposed tester $T$. Note that no attempt has been made to further optimize the obtained constants.

We begin with some preliminary definitions.

Definition 1. [$\alpha$-independence] For a given $\alpha \in [0, 1]$, two length $k$ substrings $J_1$ and $J_2$ of a string $I$ are $\alpha$-independent if they overlap by at most $(1 - \alpha)k$ locations. Furthermore, a set of substrings $\{J_i\}$ is $\alpha$-independent if any pair of substrings in the set is $\alpha$-independent. A set that is $1$-independent contains no pairwise overlaps and will be said to be non-overlapping.

One of our goals is to find a large non-overlapping set of occurrences of $J$ in $I$ (Lemma 4). Recall that Lemma 2 guaranteed the existence of $en$ occurrences and notice that Lemma 4 guarantees a similar size non-overlapping set of occurrences, which is needed for our final tester $T$. In the procedure below, we will iteratively construct a 1/2-independent set of occurrences of $J$ in $I$, from which we can pick every second element to get a non-overlapping set.

Definition 2. [cycle length] For a pattern $J$ of length $k$, we say it has a cycle of length $t$, if $t$ is an integer in $[k]$ such that for every pair of locations $i_1, i_2 \in [k]$ where $i_1 \equiv i_2 \pmod{t}$ it holds that $J[i_1] = J[i_2]$.

Claim 1. [Shifted occurrences imply a cyclic pattern] If $I$ contains two overlapping occurrences of $J$, at a relative offset of $t \in [k]$ locations, then $J$ has a cycle of length $t$. 

10
If we denote the two occurrences by \( J_1 \) and \( J_2 \) (where \( J_1 \) precedes \( J_2 \) by \( t \) locations), the claim follows by noticing that \( J_2[i] = J_1[i + t] \) for any \( i \in [k - t] \), while clearly \( J = J_1 = J_2 \).

**The procedure** The following iterative procedure which we call \textsc{Remove-Occurs} constructs a set of occurrences \( IS \) as well as a set of bits to flip \( F \). It halts when the flipping of bits in \( F \) makes the string \( I \) become \( J \)-free.

- Let \( IS \leftarrow \phi, F \leftarrow \phi \)
- Repeat as long as \( I \) still has occurrences of the pattern \( J \)
  - Let \( J^* \) be the leftmost occurrence of \( J \) in \( I \); Add \( J^* \) to \( IS \);
  - Set \( t \) to be the offset between \( J^* \) and the next leftmost occurrence of \( J \) in \( I \), while setting \( t = k \) if such a next occurrence doesn’t exist. (It implies that \( J \) has a cycle of length \( t \)).
  - **case 1:** \( t \geq k/2 \) // far or no overlapping occurrence
    * Let \( i \) be a location in \( I \), whose flipping changes \( J^* \), but does not create any new occurrence of \( J \) in \( I \) (the existence of \( i \) is guaranteed by the 1D Modification Lemma)
  - **case 2:** \( t < k/2 \) // close overlapping occurrence
    * Let \( i \) be the location in \( I \) that corresponds to the \((k - t)\)th bit of \( J^* \).
  - Flip the bit at location \( i \) in \( I \); Add \( i \) to \( F \);

**Claim 2.** The bit flips performed by \textsc{Remove-Occurs} do not introduce new pattern occurrences.

*Proof.* The bit chosen to be flipped in 'case 1' of the procedure was specifically chosen in a way that its flipping does not introduce new occurrences (possible due to the 1D modification Lemma).

In 'case 2', the two leftmost occurrences are at an offset of \( t \) (overlap of \( k - t \)), where \( t < k/2 \). This implies that the pattern \( J \) is cyclic with cycle length \( t \). Since \( t < k/2 \), we can look at 3 special locations in \( J^* \): the \((k - 2t)\)th, \((k - t)\)th and \( k \)th locations. Notice that they are spaced at intervals of \( t \) and therefore have the same value, while the middle location is the one that is flipped by the procedure, changing its value to be the (binary) opposite of the identical values of the two others. If we denote these three special locations by \( l, m \) and \( r \) (left, middle and right), any new occurrence created by the flipping of the bit at location \( m \) must contain at least one of the locations \( l \) or \( r \). The result would be that it contains \( m \) and one of \( l \) or \( r \), which are at a distance of \( t \) and have different values - in contradiction to the fact that the pattern has a cycle of length \( t \).

**Claim 3.** The set of occurrences \( IS \) is 1/2-independent.

*Proof.* We know from Claim 2 that the flips done by the procedure do not create any new occurrences of the pattern \( J \) in \( I \). This implies, since at each round we take into \( IS \) the leftmost remaining occurrence of \( J \) in \( I \), that the occurrence added to \( IS \) at each round is to the right of those added in previous rounds. Therefore, to prove the claim it will suffice to show that the occurrences added to \( IS \) at two consecutive rounds overlap by at most \( k/2 \) bits. Denote by \( J_i^* \) and \( J_{i+1}^* \) the occurrences that are added to \( IS \) at rounds \( i \) and \( i + 1 \). If at round \( i \) we are in 'case 1' then the 'next' occurrence (chosen in round \( i + 1 \)) is \( t \geq k/2 \) bits away, as required. If at round \( i \) we are in 'case 2' we choose to flip the \((k - t)\)th bit of \( J_i^* \). If the next pattern occurrence started to
the left of this bit flip location, it would not exist any more as a result of the bit flip. The result, after the flipping, is that the next occurrence location must be to the right of the flip location, i.e. at an offset of at least $k-t$ bits from $J_i^*$, which is at least $k/2$. Notice that in both cases we needed the fact that no new occurrences are created by the bit flips at round $i$, in order to only consider occurrences that existed prior to that bit flip.

Claim 4. If $I$ is $\epsilon$-far from being $J$-free, then the number of iterations $m$ of REMOVE-OCCURS until it halts (that is, $|IS| = |F| = m$) is at least $\epsilon n$ and at most $2n/k$.

Proof. The procedure iterates $m$ times, flipping a single bit of $I$ at each iteration and stops only when $I$ becomes $J$-free. If $I$ is $\epsilon$-far from being $J$-free, by the definition it implies that the size of the set $F$ (which equals the number of iterations) is at least $\epsilon n$. On the other hand, from Claim 3 we know that the set $IS$ is $1/2$-independent, namely that its elements (occurrences of length $k$) overlap by less than $k/2$. It follows that $|IS| \leq 2n/k$.

Lemma 3 is an immediate corollary of Claim 4, since the procedure shows that $I$ can be made $J$-free by flipping not more than $2n/k$ bits. Lemma 4 is an immediate corollary of Claims 3 and 4. Simply take every second occurrence from $IS$.

6 Conclusions

We have provided algorithms for testing whether strings and 2D images do not contain a fixed pattern. Our testing algorithms are simple and their query complexity is optimal up to constant multiplicative factors.

There are several questions that arise from this work. It would be interesting to devise property testing algorithms for the approximate pattern matching case, where one needs to decide if a string (image) does not contain a pattern of small hamming distance from a fixed pattern $J$. Another question is how to test whether a string is free from a fixed set of patterns $J_1, \ldots, J_r$. It may be also of interest to study testing algorithms for the approximate pattern matching case, with distance measures between patterns differing from the Hamming distance (e.g., $\ell_1$ distance for grey-scale patterns). Our current proof ideas do not seem to directly apply in these settings and it appears that new ideas are required.

Acknowledgements We are grateful to Swastik Kopparty for numerous useful comments. We are thankful to Sofya Raskhodnikova for providing useful feedback on an early version of this work.

References

[1] N. Alon. Testing subgraphs in large graphs (2002). Random structures and algorithms, 21(34):359-370.

[2] N. Alon, M. Krivelevich, I. Newman and M. Szegedy (2001). Regular languages are testable with a constant number of queries, SIAM Journal on Computing, 30, 1842–1862.

[3] N. Alon, E. Fischer, M. Krivelevich and M. Szegedy (2000). Efficient testing of large graphs. Combinatorica, 20, 451–476.

[4] N. Alon and J. Spencer (2008). The Probabilistic Method. Wiley.
[5] A. Amir, G. Benson (1998). Two-Dimensional Periodicity in Rectangular Arrays. *SIAM Journal on Computing* 27, 90-106.

[6] A. Amir, G. Benson, M. Farach (1994). An Alphabet Independent Approach to Two-Dimensional Pattern Matching. *SIAM Journal on Computing* 23, 313-323.

[7] Y. Bar-Hillel, M. Perles, and E. Shamir (1964). On formal properties of simple phrase structure grammars. In Y. Bar-Hillel, editor, *Language and Information: Selected Essays on Their Theory and Application*, 116–150. Addison-Wesley, Reading, Massachusetts.

[8] P. Berman, M. Murzabulatov, S. Raskhodnikova (2015). Constant-Time Testing and Learning of Image Properties. *arXiv preprint* 1503.01363.

[9] R.S. Boyer and J.S. Moore (1977). A fast string searching algorithm. *Comm. ACM*, 20(10), 762-772.

[10] R. Cole (1991). Tight Bounds on the Complexity of the Boyer-Moore String Matching Algorithm. *SODA*, 224–233.

[11] E. Fischer, and I. Newman (2001). Testing of matrix properties. *STOC*, 286–295

[12] E. Fischer and I. Newman (2007). Testing of matrix-poset properties. *Combinatorica*, 27(3), 293–327.

[13] Z. Galil and J. I. Seiferas (1983). Time-Space-Optimal String Matching. *J. Comput. Syst. Sci*, 26(3), 280–294.

[14] O. Goldreich, S. Goldwasser and D. Ron (1998). Property testing and its connection to learning and approximation. *JACM*, 45, 653–750.

[15] I. Kleiner, D. Keren, I. Newman, O. Ben-Zwi (2011). Applying Property Testing to an Image Partitioning Problem. *IEEE Trans. Pattern Anal. Mach. Intell*, 33(2), 256–265.

[16] S. Korman, D. Reichman, G. Tsur and S. Avidan. Fast-Match: Fast Affine Template Matching. International Journal of Computer Vision, to appear.

[17] D. E. Knuth, J. H. Morris Jr. and V. R. Pratt (1977). Fast Pattern Matching in Strings. *SIAM J. Comput.* 6(2): 323–350.

[18] S. Raskhodnikova (2003). Approximate testing of visual properties. *RANDOM*, 370-381.

[19] R. L. Rivest (1977). On the Worst-Case Behavior of String-Searching Algorithms. *SIAM J. Comput.* 6(4): 669–674.

[20] R. Rubinfeld and M. Sudan (1996). Robust characterization of polynomials with applications to program testing. *SIAM J. Comput.* 25, 252–271.

[21] G. Tsur and D. Ron (2014). Testing properties of sparse images. *ACM Transactions on Algorithms* 4.

[22] A. C. Yao (1977). Probabilistic computation, towards a unified measure of complexity. *FOCS*, 222–227.
A Lower bound

Here we show that any tester that makes \( o_r(\frac{1}{\epsilon^2}) \) queries in an attempt to test if a fixed pattern appears in a string, must err with probability greater than \( 1/3 \). While the lower bound of \[2\] already entails a lower bound of \( \Omega(1/\epsilon) \) for testing pattern-freeness (as for any fixed pattern \( J \) the language consisting of all strings not containing \( J \) is regular), we give here a self-contained proof for two reasons. First, our proof extends to the case where \( k \), the length of \( J \), is allowed to depend on \( n \). Second, it is not hard to adapt the proof to the 2D case and demonstrate a lower bound of \( \Omega(1/\epsilon) \) on the query complexity of testing pattern freeness of images.

**Theorem 3. [Lower bound of \( \Omega(1/\epsilon) \) queries]** Suppose \( k \) is even and consider the pattern \( J := 0^{k/2-1}10^{k/2} \). Any tester that distinguishes with probability at least \( 2/3 \) between the case that a string \( I \) is \( J \)-free and the case that \( I \) is \( \epsilon \)-far from being \( J \)-free, makes at least \( \frac{1}{13\epsilon} \) queries to \( I \).

Note: we show that the lower bound above applies for wide ranges of possible values for \( \epsilon \): ranging from \( \epsilon = O_n(1/n) \) to \( \epsilon = \Omega_k(1/k) \). We did not attempt to optimize the constant \( \frac{1}{13\epsilon} \) in the Theorem above. Furthermore, to ease readability we avoid using floor/ceiling signs.

**Proof.** By Yao’s Principle \[22\], it suffices to construct two distributions \( B \) and \( C \) over length \( n \) strings where all strings in \( B \) are \( J \)-free and all strings in \( C \) are \( \epsilon \)-far from being \( J \)-free, with the property that any deterministic algorithm \( D \) making less than \( \frac{1}{13\epsilon} \) queries to \( I \) sampled from \( A = \frac{1}{2} B + \frac{1}{2} C \) (namely we sample from \( B \) with probability \( 1/2 \) and from \( C \) otherwise), must err with probability greater than \( 1/3 \). We now explain how to construct these distributions.

The distribution \( B \) is just the single string \( 0^n \) sampled with probability \( 1 \). For the distribution \( C \) we take the string \( I = 0^n \) and divide it to \( n/k \) disjoint intervals of length \( k \): \( I_1, \ldots, I_{n/k} \). Then, we sample randomly a subset of \( 2\epsilon n \) of these intervals. For each length \( k \) interval in the subset, we choose one of the last (right side) \( k/2 \) locations and flip it to be a 1.

Clearly any string from \( B \) is \( J \)-free with probability 1, while every string in \( C \) contains exactly \( 2\epsilon n \) occurrences of \( J \). Furthermore, as changing a single location in a string \( I \) sampled from \( C \) can remove at most \( 2 \) occurrences of \( J \), such a string \( I \) is \( \epsilon \)-far from being \( J \)-free.

Consider a deterministic tester \( D \) that performs at most \( \frac{1}{13\epsilon} \) queries to the string \( I \). Since \( D \) is deterministic, in the case that it encounters only zeros during its queries, it has to either declare the string \( I \) as being \( J \)-free or not. The first option is that \( D \) rejects (declares ‘\( \epsilon \)-far from \( J \)-free’) when it sees only zeros. In this case \( D \) errs with probability \( 1 \) if \( I \) was chosen from \( B \), hence \( D \) errs with overall probability of at least \( 1/2 \). We next handle the second case, where \( D \) accepts (declares ‘free’) when it sees only zeros. We show that in this case \( D \) errs with probability greater than \( 1/3 \), by focusing on the distribution \( C \) (i.e. the \( \epsilon \)-far case).

Being deterministic, when \( D \) sees only zeros it queries a fixed set \( X \) of locations, where we chose \( |X| = \frac{1}{13\epsilon} \). All we need to show is that under the random process of the distribution \( C \), the fixed set of locations \( X \) does not contain a 1 w.p. \( > 2/3 \). This will suffice, since all strings from \( C \) (which is chosen w.p. \( 1/2 \)) are \( \epsilon \)-far from being \( J \)-free, and therefore \( D \) will fail w.p. \( > 1/2 \cdot 2/3 = 1/3 \).

Let \( X_i = X \cap I_i \) be the set of locations from \( X \) that are in interval \( I_i \). An interval \( I_i \) has \( k/2 \) locations where a 1 could be placed (the right half of the interval) and we can assume that the set \( X_i \) is contained in the right half of \( I_i \) (since the left half is zero w.p. \( 1 \) hence would be pointless to query by any algorithm). Clearly, \( Pr[X_i \text{ contains a } 1] = Pr[I_i \text{ contains a } 1] \cdot Pr[\text{the 1 is within } X_i] = \frac{2\epsilon n}{k} \cdot \frac{|X_i|}{k/2} = 4\epsilon |X_i| \), leading to \( Pr[X \text{ contains a } 1] \leq \sum_{i=1}^{n/k} 4\epsilon |X_i| = 4\epsilon |X| < 1/3 \), by union bound. \( \square \)
As noted, Theorem 3 can be generalized to the two-dimensional settings (details omitted).

B Larger alphabets

Here we deal with proving that pattern-freeness is strongly-testable in the case where $|\Sigma| \geq 3$, for an arbitrary pattern $J$, for both 1D strings and 2D images. It is not hard to verify that in order to prove the testability result it suffices to prove our respective modification lemmas in this setting\footnote{Lemmas 3 and 4 and their proofs made no specific use of the alphabet being binary, other than the assumption that the modification lemma can be applied (and same the same holds for the 2D equivalents)}. These can be done by building on our results for the binary case.

We first deal with the one-dimensional case.

**Lemma 9. [1D modification lemma]** Let $I$ be a string of length $n$ and let $J$ be a pattern of length $k$ over an alphabet $\Sigma$ of cardinality at least 3. For any given occurrence of $J$ in $I$, it is possible to change $J$ at a single location, without creating a new occurrence of $J$ in the resulting string $I'$.

**Proof.** The string $I$ contains at most $2k - 2$ length-$k$ sub-strings that overlap with $J$. There are at least 2 ways to change each of the $k$ locations in $J$ and it is impossible for any 2 distinct changes to create the same new occurrence of $J$ in $I$. By the pigeonhole principle we get that it is impossible that every change in $J$ will create a new occurrence of $J$, as desired.

We now move to the two-dimensional case.

**Lemma 10. [2D modification lemma, $|\Sigma| \geq 3$]** Let $I$ be an $n \times n$ image over an alphabet $\Sigma$ of cardinality at least 3. Let $P$ be a $k \times k$ pattern over $\Sigma$, for some $k > 11$, which appears in $I$. Then it is possible to change a single location in the occurrence of $P$, without creating any new occurrence of $P$ in $I$.

**Proof.** The essence of the argument is to reduce the non-binary case to a binary case which is not almost homogeneous. The ability to flip a single binary bit without creating new instances implies the ability to do so in the original alphabet. More specifically, $\Sigma$ is partitioned into two non empty alphabets $\Sigma_1$ and $\Sigma_2$ that will be soon defined. Then, given a pattern $I$ we simply map every character in $\Sigma_1$ to 0 and every character in $\Sigma_2$ to 1.

If not all of the characters of $\Sigma$ appear at the corners of $I$ then we simply map the set of all the characters appearing in any of the corners to 0 and rest of the alphabet to 1.

Otherwise either 3 or 4 different characters occupy the four corners of $I$. If 4 appear then simply map 2 of them to 0 and the rest of the alphabet to 1. Clearly the resulting image is not almost homogeneous. If 3 characters appear on the corners then one of them, say $\sigma$, appears in two of the corners. In this case simply map $\sigma$ to 0 and $\Sigma \setminus \{\sigma\}$ to 1. The resulting pattern cannot be almost homogeneous as well.

C Properties of $\epsilon$-far images

Recall the two Lemmas from Section 3, regarding properties of images that are $\epsilon$-far from being $P$-free, for a given $k \times k$ binary pattern $P$. These lemmas were used for the proof of the correctness
of our proposed tester \( T \). Here again, more evidently than in the 1-dimensional case, we preferred the simplicity of the proofs over attempting to further optimize the constants.

**Lemma 11.** [Any image is \( O(1/k^2) \)-close to being \( P \)-free] If \( I \) is \( \varepsilon \)-far from being \( P \)-free then \( \varepsilon \leq 20/k^2 \).

**Lemma 12.** [\( \Omega(n^2) \) non-overlapping pattern occurrences] If \( I \) is \( \varepsilon \)-far from being \( P \)-free then \( I \) must contain \( en^2/35 \) non-overlapping copies of \( P \).

The following definitions and claim generalize those of the 1-dimensional case.

**Definition 3.** [\( \alpha \)-independence] For a given \( \alpha \in [0,1] \), two \( k \times k \) sub-images \( P_1 \) and \( P_2 \) of an image \( I \) are \( \alpha \)-independent if they overlap by at most \((1-\alpha)k^2\) locations. Furthermore, a set of sub-images \( \{P_i\} \) is \( \alpha \)-independent if any pair of sub-images in the set is \( \alpha \)-independent. A set that is \( 1 \)-independent contains no pairwise overlaps and will be said to be non-overlapping.

**Definition 4.** [cycle size] For the \( k \times k \) pattern \( P \), we say it has a cycle of size \((t_j,t_i)\), if \( t_j \) and \( t_i \) are integers in \([k]\) such that for every pair of locations \((j_1,i_1),(j_2,i_2)\in[k]\times[k]\) where \( j_1 \equiv j_2 \pmod{t_j} \) and \( i_1 \equiv i_2 \pmod{t_i} \) it holds that \( P[j_1,i_1]=P[j_2,i_2] \).

**Claim 5.** [Shifted occurrences imply a cyclic pattern] If \( I \) contains two overlapping occurrences of \( P \), at a relative offset of \((t_j,t_i)\in[k]\times[k]\) locations, then \( P \) has a cycle of size \((t_j,t_i)\).

This claim is straightforward, following the same logic as the correctness of Claim 1 (its 1-dimensional equivalent).

The following basic property is needed for the 2-dimensional case. It limits the number of new occurrences that can be created as a result of flipping a single bit in the middle of a pattern occurrence. It is easy to show that not more than 4 can be created. Possibly, a tighter result could be obtained (i.e. showing that not even 4 can be created), but it would require a more complex proof and would affect our result only up to a constant, independent of \( \varepsilon \) and \( k \).

**Claim 6.** [Central bit flip creates few new occurrences in images] Let \( P_0 \) be an occurrence of a \( k \times k \) pattern \( P \) in a binary image \( I \). Flipping the central pixel of \( P_0 \) (the pixel of \( I \) that corresponds to \( P[(k+1)/2,(k+1)/2] \), assuming \( k \) is odd\(^5\)) does not create more than 4 new occurrences of \( P \) in the image \( I \).

**Proof.** Assume that more than 4 new occurrences are created. Since these occurrences overlap (at the bit flip location), a pigeonhole argument implies that there must be two of them that are shifted (one from the other) by less than \( k/2 \), in both vertical and horizontal directions. Denote these by \( P_1 \) and \( P_2 \), assuming without loss of generality that \( P_2 \) is at an offset of \( t=(t_j,t_i) \) from \( P_1 \), for some integers \( 0 \leq t_j,t_i < k/2 \), not both zero.

By Claim 5 we can conclude that the pattern \( P \) (and hence each of the occurrences) has a cycle of size \( t=(t_j,t_i) \). Let \( x=(x_j,x_i) \) be the index in \( I \) of the (central) flipped bit in \( P_0 \). Now, consider the index \( x'=(x_j+t_j,x_i+t_i) \), which is also in \( P_0 \), since \(|t_j|,|t_i| < k/2 \). The occurrence \( P_2 \) overlaps both locations \( x \) and \( x' \) (since both new occurrences \( P_1 \) and \( P_2 \) overlap the bit flip location \( x \) and \( P_2 \) is shifted by \( t=(t_j,t_i) \) from \( P_1 \), which overlaps \( x \)).

On one hand we know that \( I[x]=I[x'] \) (before the bit flip), since both locations belong to \( P_0 \), which has a cycle of size \( t=(t_j,t_i) \). On the other hand, \( I[x] \neq I[x'] \), since these locations both belong to \( P_2 \) and must be equal after flipping \( I[x] \) as \( P_2 \) has a cycle length of \( t=(t_j,t_i) \). This leads to a contradiction. \( \square \)

\(^5\)if \( k \) is even the same can be shown by taking any of the 4 off-center pixels

16
The procedure for images  The following iterative procedure REMOVE-OCCURRENCES-2D constructs a set of occurrences $IS$ as well as a set of bits to flip $F$. Different from the 1D procedure, new pattern occurrences might be created as a result of the sequential flipping of image bits. We therefore differentiate between the original occurrences and those that are created during the procedure and did not exist in the original image $I$. In Step A, the procedure iteratively collects a set $IS$ of original occurrences, which we will later show to be $1/2$-independent. After an occurrence is added to $IS$ one of its bits is flipped. We keep iterating until no original occurrences of $P$ remain. Note that as a result of the flipping, new occurrences might be created. In step B, we will continue flipping bits, one per remaining (newly created) occurrence without creating new ones, until $I$ becomes $P$-free.

- Let $IS \leftarrow \phi$, $F \leftarrow \phi$

- **Step A:** Repeat as long as there are original occurrences of $P$ in $I$
  - Let $P^*$ be an original occurrence of $P$ in $I$
  - Add $P^*$ to $IS$
  - Let $(j, i)$ be the location in $I$ that corresponds to the central pixel of $P$ (i.e. the pixel $P[(k + 1)/2, (k + 1)/2]^6$).
  - Flip the bit at location $(j, i)$ in $I$.
  - Add $(j, i)$ to $F$

- **Step B:** Repeat as long as there are occurrences of $P$ in $I$ (that were created in Step A)
  - Let $P^*$ be an occurrence of $P$ in $I$
  - Let $(j, i)$ be a location in $I$, whose flipping changes $P^*$, but does not create any new occurrence of $P$ in $I$ (existence by 2D Modification Lemma)
  - Flip the bit at location $(j, i)$ in $I$.
  - Add $(j, i)$ to $F$

Claim 7. The set of occurrences $IS$ is $1/2$-independent.

**Proof.** This is a straightforward consequence of the choice to flip the central pixel of each occurrence added to $IS$. At any round, an existing original occurrence $P^*$ cannot contain a location of a bit that was flipped in previous rounds, hence it may overlap any previously added occurrence by less than $k^2/2$ pixels. \qed

Claim 8. If $I$ is $\epsilon$-far from being $P$-free, the procedure returns a set $IS$ of size at least $\epsilon n^2/5$ as well as a set $F$ of size at most $20n^2/k^2$.

**Proof.** Denote by $N_A$ and $N_B$ the number of iterations REMOVE-OCCURRENCES-2D performs in steps A and B respectively. In particular, the set of returned original occurrences $IS$ is of size $N_A$ and the set of flipped bits $F$ is of size $N_A + N_B$.

---

6assuming that $k$ is odd, an assumption that can easily be dropped
Since \( I \) was \( \epsilon \)-far from being \( P \)-free to start with and it became \( P \)-free after flipping \(|F|\) bits, we can conclude that \( N_A + N_B = |F| \geq \epsilon n^2 \).

Now, by Claim 6, each bit flip in Step A creates at most 4 new occurrences of \( P \) in \( I \). Since the bit flips in Step B do not create any new occurrences (we chose them in that particular way using the modification lemma), it holds that \( N_B \leq 4N_A \).

Regarding the set \( IS \), on one hand we have that \(|IS| = N_A \geq (N_A + N_B)/5 \geq \epsilon n^2/5 \), as required.

On the other hand, since \( IS \) is \( 1/2 \)-independent (by Claim 7), a simple packing argument (of \( 1/2 \)-independent \( k \times k \) pattern occurrences into an \( n \times n \) image) implies that \( N_A = |IS| \leq 4n^2/k^2 \). This limits the size of \( F \) to be: \( |F| = N_A + N_B \leq 5N_A \leq 20n^2/k^2 \).

Lemma 11 is an immediate corollary of Claim 8, since the procedure shows that \( I \) can be made \( P \)-free by flipping not more than \( 20n^2/k^2 \) bits.

We show that \( IS \) contains a non-overlapping subset of size at least \(|IS|/7 \). Lemma 12 is a corollary of Claims 7 and 8 as \(|IS| \geq \epsilon n^2/5 \).

To show that \( IS \) contains a non-overlapping subset of size \(|IS|/7 \), we associate each occurrence in \( IS \) with the coordinates \((j, i)\) of its top left pixel. We then order the occurrences by increasing order of \( i \) (occurrences at the top of the image first), breaking ties according to an increasing order of \( j \) (from left to right). Now we build a non-overlapping subset from the ordered list in the following iterative manner. Pick the first occurrence in the list. Remove this occurrence and all occurrences that overlap it from the list. Repeat till the list is empty.

A simple pigeonhole argument shows that any choice of an occurrence can remove at most 6 other occurrences. See Figure 3, starting with the case of odd \( k \) on the left. The black square is the occurrence chosen, with its central pixel (\( k \) is odd) shown in the middle. Looking at all remaining overlapping occurrence that overlap the chosen occurrence, which will be be removed from the list - their central pixel must be within the red dashed region. The red region is restricted in the specific way due to the ordering of the occurrences from top to bottom and left to right breaking ties as well as the requirement to be at a vertical and horizontal offset of at least \( k/2 \) from the chosen occurrence. This region has been divided into 6 rectangular sub-regions, with dimensions at most \((k + 1)/2 \). The existence of more than a single occurrence with center in a single rectangle, would contradict the \( 1/2 \)-independence of \( IS \), since the vertical and horizontal offset between occurrences needs to be at least \( k/2 \). The same analysis works for the even \( k \) case, where we look at possible locations for the bottom-right off-center pixel.

D  Proof of 2-dimensional modification lemma

We provide here the proof of the 2D modification lemma (Lemma 8) over a binary alphabets, which we recite here for completeness. We start with an outline of the proof which explains the division into the three main cases, whose proof is given in the following sections of the appendix.

Lemma 13. [2D modification lemma] Let \( I \) be an \( n \times n \) binary image (over \( \{0, 1\} \)). Let \( P \) be a binary \( k \times k \) pattern, for some \( k > 11 \), which appears in \( I \). Assume further that \( P \) is almost-homogeneous (homogeneous except one value in one of the corners). Then it is possible to flip a single bit in the occurrence of \( P \), without creating any new occurrence of \( P \) in \( I \).
We put forth several preliminaries. We will use the following coordinate system for two dimensional patterns: with the top-left (TL) corner labeled by (1, 1), the bottom-left (BL) by (1, k), the top-right (TR) by (k, 1) and the bottom-right (BR) by (k, k). Namely, coordinates increase from left to right and from top to bottom. The value of the coordinate (i, j) in a pattern P is denoted by P[i, j]. For a fixed j ∈ [k], the horizontal vector P(j) represents the jth row of P. A pattern is homogeneous if all its pixels have the same value and homogeneous except at one pixel if all its pixels except one have the same value. As a reminder, a pattern is almost homogeneous if it is homogeneous except for one of its corner pixels. Similarly, we will refer to the homogeneity of a row of a pattern by restricting the definition to the pixels values in the specific row.

Given two patterns P_1 and P_2 located in an n × n image, a vector (r, m) is the offset of P_2 with respect to P_1 if shifting P_1 by (r, m) would result with P_1 completely overlapping P_2. We refer to r as the horizontal component of the offset and to m as the vertical component of the offset. Furthermore, we will refer extensively to offsets that are axis aligned or diagonal. Formally, an offset will be called pure horizontal of size t if (r, m) = (±t, 0), pure vertical of size t if (r, m) = (0, ±t), pure pos-diagonal of size t if (r, m) = (±t, ±t) or pure neg-diagonal of size t if (r, m) = (±t, ±t).

Lastly, a 1-dimensional bit string S = [S[1], S[2], ..., S[p]] has a cycle length of t if S[i] = S[i + t] for every i ∈ [p − t]. We will be interested in the cycle length of rows, columns and diagonals of the pattern P. In particular, a row/column/diagonal is homogeneous, if it has cycle length of 1.

Proof. In this 2-dimensional setting, there are k^2 possible bits in the pattern P that could be flipped. We will be assuming that flipping any of the k^2 pattern bits creates a new pattern occurrence. As in the 1D case, these k^2 new occurrences are distinct and the entire proof deals with analyzing the constraints that such a large number of new occurrences (all residing in a limited area) impose on the structure of the pattern. The goal is to show that the pattern must be almost homogeneous.
Please refer to the illustration on the right. The \( k \times k \) pattern \( P \) is shown in blue, within the \( n \times n \) image \( I \). Several \( k \times k \) sub-images that overlap \( P \) are shown in red. We define a region \( M \) to be the collection (or union) of bottom-left pixels of any \( k \times k \) sub-image of \( I \) that overlaps \( P \) non-trivially. The large region \( M \), shown in gray, is a square of dimension \( 2k-1 \) with a missing single pixel at its center, which is at the bottom left pixel of the pattern \( P \). The central pixel does not belong to \( M \), since it corresponds to the sub-image \( P \), which overlaps itself trivially.

**Lemma 14.** One of the 2 following options hold:

1. \( \exists \) a pair of new occurrences that are horizontally, vertically or diagonally offsetted by 1 pixel.

2. \( \exists \) a pair of new occurrences that are horizontally offsetted by 2 pixels and \( \exists \) a pair of new occurrences that are vertically offsetted by 2 pixels.

**Proof.** The proof uses a very simple counting argument, over the partition of the region \( M \), shown on the right. Let us begin by counting the number of parts. The number of \( 2 \times 2 \) sub-images (squares) is \( (k-1)^2 \), the number of \( 2 \times 1 \) or \( 1 \times 2 \) rectangles is \( 2(k-1) \) and there is a single \( 1 \times 1 \) part (pixel) at the bottom right corner. Overall there are \( k^2 \) parts.

Since we have \( k^2 \) new occurrences, if there are two in the same part we obtain a pure (horizontal or vertical or diagonal) offset of one pixel. Otherwise, there must be a single new occurrence in each of the parts. Looking at the three parts shown highlighted at the bottom right corner, it is clear that if there are not any neighboring occurrences (pure horizontal or vertical offsets of 1 pixel) then there must exist both pure horizontal and pure vertical offsets of 2 pixels. \( \square \)

As a consequence of Lemma 14, the proof of the 2D modification lemma will be divided into the 3 cases illustrated in Figure 4 which are explained next. The first case, ‘Horizontal and vertical 2-pixel offset’, corresponds to option 2 of Lemma 14 and is fully proved in Appendix D.2. Next, option 1 of Lemma 14 can be subdivided according to whether the offset is horizontal, vertical or diagonal. The horizontal and vertical cases are symmetric (by 90° rotation) and therefore, we will
prove the case ‘Horizontal 1-pixel offset’ in Appendix D.3. The remaining case, ‘Diagonal 1-pixel offset’ is proved in Appendix D.4.

D.1 Some general lemmas

We give here two lemmas that are useful in the different cases of proof of the 2D modification lemma.

First, we prove a simple lemma, that translates a pure offset between a pair of new occurrences into a strong constraint on the structure of the pattern $P$. This lemma will be used in all 3 cases of the proof.

For a 1-dimensional vector $v$ of length $N$, we say that it is cyclic with cycle length $t$, if $i_1 \equiv i_2 \pmod{t} \Rightarrow v[i_1] = v[i_2]$. We will treat rows, columns, pos-diagonals (diagonals with indices of the form $(i + l, \pm l)_{l=0}^m$) and neg-diagonals (diagonals with indices of the form $(i + l, \pm l)_{l=0}^m$) as 1-dimensional vectors. Furthermore, we say that $v$ is cyclic with cycle length $t$ except at a set of indices $X \subset [t]$, if $i_1 \equiv i_2 \pmod{t}$ and $i_1 \notin X \pmod{t} \Rightarrow v[i_1] = v[i_2]$.

Lemma 15. Assume there exist a pair of newly created occurrences $P_1$ and $P_2$, which are a result of flipping 2 pixels of $P$ at locations $l_1 = (l_{1x}, l_{1y})$ and $l_2 = (l_{2x}, l_{2y})$ respectively. Furthermore, assume that $P_1$ and $P_2$ have a pure horizontal offset of size $t$ between one another. Then:

(a) All rows of $P$, except for the one or two that contain $l_1$ and $l_2$ are cyclic with cycle length $t$.

(b) If $l_1$ and $l_2$ are on separate rows then for $i = 1, 2$, the row of $P$ that contains $l_i$ is cyclic with cycle length $t$, except at $l_i^x \pmod{t}$.

(c) If $l_1$ and $l_2$ are on the same row then the row that contains them is cyclic with cycle length $t$, except at $\{l_1^x, l_2^x\} \pmod{t}$.

Note: The lemma could be phrased equivalently by replacing horizontal with any of vertical/pos-diagonal/neg-diagonal and rows with columns/pos-diagonals/neg-diagonals respectively.

Proof. We will prove this simple lemma for the pure horizontal case and then claim that similar ideas apply to the horizontal or diagonal case. Assume that $P_2$ is shifted from $P_1$ to the right by $t$ pixels. Notice that this shift creates an overlap constraint between them, at all the overlapping pixels except those at the bit flip locations $l_1$ and $l_2$. Formally, for any row and column indices $i$
and \( j \) s.t. \([i \notin \{l_1^y, l_2^y\} \text{ and } j \in [t]\) or \([i = l_i^y \text{ and } j \notin l_i^x \text{ (mod } t) \) for some \( i = 1, 2 \), it holds that:

\[
P_1[j,i] = P_2[j,i] = P_1[j+t,i] = P_2[j+t,i] = \cdots = P_1[j + \lfloor \frac{k-j}{t} \rfloor \cdot t, i] = P_2[j + \lfloor \frac{k-j}{t} \rfloor \cdot t, i] \quad (6)
\]

The equalities at the odd locations follow from the constraint \( P_1 = P_1 = P_2 \), while the ones at the even locations follow from the fact that \( P_2 \) is at a pure offset of \( t \) pixels to the right from \( P_1 \).

Equation (6) proves items (a), (b) and (c) for the case of a horizontal shift. It can be verified that an equivalent version of it could be written for the vertical and diagonal cases as well.

Second, another useful constraint, complementary to that of Lemma 14, which is also proved by a counting argument.

**Lemma 16.** Among the \( k^2 \) new occurrences of the pattern \( P \), if there does not exist a pair of occurrences that are at a pure horizontal or vertical offset of 1 or 2 pixels, then there exists a pair of new occurrences that are at a pure horizontal offset of 3 pixels as well as a pair of new occurrences that are at a pure vertical offset of 3 pixels.

**Proof.** This proof is similar to that of Lemma 14, using a different partition of the square region \( M \), whose dimension is \( 2k - 1 \). Assume first that \( 2k - 1 \) is a multiple of both 3 and 4, and partition \( M \) into rectangles of height 4 and width 3. The number of such rectangles in \( M \) is \((2k - 1)^2/12 \). This number is strictly larger than \( k^2/3 \) and therefore there must be one of the rectangles with at least 4 new occurrences. Considering that no two occurrences are at a pure horizontal or vertical offset of size 1 or 2, no more than one occurrence can lie on each row of the rectangle, meaning that there must be 2 on the same column, with a pure vertical offset of 3. It is readily verified that this constraint happens for any value of \( k \) larger than 11, a constraint that we assumed in the modification lemma. Furthermore, taking a height 3 width 4 rectangle gives the pure horizontal offset of 3 pixels.

D.2 The ‘Horizontal and vertical 2-pixel offset’ case

We are in the case depicted on the left of Figure 4, namely that there are two pairs of overlapping new occurrences, one pair which we denote by \( P_1^H \) and \( P_2^H \) that has a pure horizontal offset of 2 pixels and another pair \( P_1^V \) and \( P_2^V \) with a pure vertical offset of 2 pixels.

Applying Lemma 15 to this setting gives very strong constraints over the structure of the pattern \( P \). Informally, the pattern \( P \) is a repetitive \( 2 \times 2 \) tiling with a single tile

\[
S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}
\]

(last row or column might use respective tile parts, if \( k \) is odd) except at possibly 4 pixels that are the intersection of (up to) two rows and (up to) two columns.

These constraints are shown in Figure 5 and are formalized in the following Claim.

**Claim 9.** There exist row indices \( r_1 \) and \( r_2 \) (possibly \( r_1 = r_2 \)) and column indices \( c_1 \) and \( c_2 \) (possibly \( c_1 = c_2 \)) such that for any pair of coordinate locations \((x_1, y_1)\) and \((x_2, y_2)\) that do not belong to \( \{(r_1, c_1), (r_1, c_2), (r_2, c_1), (r_2, c_2)\} \), if \( (x_1, y_1) \equiv (x_2, y_2) \text{ (mod } 2) \) then \( P[x_1, y_1] = P[x_2, y_2] \).

22
Proof. The claim follows almost directly, by applying item (c) of Claim 15 on both the rows and the columns of $P$ using the known offset, in both directions, of size $t = 2$. The row indices $r_1$ and $r_2$, for example, are the $y$-coordinates $l^y_1$ and $l^y_2$ of the bit flip locations $l_1$ and $l_2$ that correspond to the pair of horizontally offsetted new occurrences $P^H_1$ and $P^H_2$.

Take a pair of coordinate locations $(x_1,y_1)$ and $(x_2,y_2)$ as defined in the claim. They are characterized by not being in the intersection of the special rows $r_1$ and $r_2$ and columns $c_1$ and $c_2$. Since we assume the pattern dimension $k \geq 6$ ($k > 5$) there must be a coordinate $(x_3,y_3)$ in $P$ such that $(x_1,y_1) \equiv (x_2,y_2) \equiv (x_3,y_3) \pmod{2}$ as well as $x_3 \notin \{c_1,c_2\}$ and $y_3 \notin \{r_1,r_2\}$. We will show that $P[x_1,y_1] = P[x_3,y_3] \pmod{2}$ (and similarly $P[x_2,y_2] = P[x_3,y_3] \pmod{2}$ holds), proving the Claim.

Clearly, it holds that $y_1 \notin \{r_1,r_2\}$ or $x_1 \notin \{c_1,c_2\}$. Assume, w.l.o.g. that $y_1 \notin \{r_1,r_2\}$. By the horizontal application of Lemma 15, the $y_1$th row of $P$ is cyclic with cycle length $t = 2$, thus $P[x_1,y_1] \equiv P[x_3,y_1] \pmod{2}$. And once again, since $x_3 \notin \{c_1,c_2\}$, by the vertical application of Lemma 15, the $x_3$th column of $P$ is cyclic with cycle length $t = 2$, thus $P[x_3,y_1] = P[x_3,y_3] \pmod{2}$, which leads to $P[x_1,y_1] = P[x_3,y_3] \pmod{2}$.

We have shown that there are 4 locations that can potentially deviate from the repetitive $2 \times 2$ structure. We will examine the several options regarding which of the 4 locations actually deviate from the repetitive square. We will call these pixels deviating pixels. The number of deviating pixels is anywhere between zero and four.

For proving Lemma 8, we will assume towards contradiction that the flipping of any pixel $p \in P$ creates a new occurrence $Q$ of $P$ (which must overlap $P$ at $p$). Throughout the proof, we will be making specific choices of the pixel $p$ in order to rule out many different options and eventually to conclude that $P$ must be almost homogeneous.

For the moment, we will assume that the tile $S$ is not homogeneous (all zeros or all ones). We postpone the handling of the case that $S$ is homogeneous to the end of the proof.

In this proof we will be inspecting the $2 \times 2$ sub-images (which we call squares) contained in the pattern $P$ (there are $(k-1)^2$ such squares), through the sum of their pixel values (or equivalently the number of 1-pixels within). Denoting $s = a + b + c + d$, we say that a square is sum-average or sum-plus1 or sum-minus1 if the sum of its pixels is $s$ or $s + 1$ or $s - 1$ respectively. A simple fact is that if the pattern $P$ has no deviating pixels then all of its $(k-1)^2$ squares are zero-sum.
For each pixel, we can look at the sum of values of each of the squares it is contained in. Notice that each pixel of $P$ is contained in 1, 2 or 4 squares (depending on whether it is a corner, boundary, or internal pixel). We say that a pixel is equi-sum if all of its containing squares have an identical sum. We say that it is equi-sum-average if it is equi-sum, with sums of $s$ (or equivalently if all its containing squares are sum-average). Similarly, we say it is equi-sum-plus1 or equi-sum-minus1 if it is equi-sum, with sums of $s + 1$ or sums of $s - 1$.

We now define an isolated pixel to be one that is not a (horizontal, vertical or diagonal) neighbor of a deviating pixel. Notice that an isolated pixel can itself be either deviating or non-deviating. The following Claim gives some trivial facts about isolated pixels:

**Claim 10.** The following hold:

(a) Every isolated non-deviating pixel is equi-sum-average. Furthermore, if it is a 0-pixel, flipping it to 1 will make it equi-sum-plus1 and otherwise - flipping it to 0 will make it equi-sum-minus1.

(b) Every isolated deviating pixel is equi-sum-plus1 if it is a 1-pixel or equi-sum-minus1 otherwise.

We can now prove the following simple and useful claim.

**Claim 11.** Let $p$ be an equi-sum-average pixel that is not a boundary pixel of $P$. Assume w.l.o.g. that $p$ is 0-valued. If $q$ is the pixel in the new occurrence $Q$ (of $P$) that is created by the flipping of $p$ to be 1-valued, then $q$ (which clearly must be one-valued) needs to be contained in some sum-plus1 square. (Equivalently, if $p$ is 1-valued, $q$ is contained in a sum-minus1 square).

**Proof.** Since $p$ is not a boundary pixel of $P$, at least one of the squares that contains $q$ must completely overlap $P$. Such a square must be sum-plus1, since $p$ itself is equi-sum-plus1 after its flip from zero to one.

A simple corollary follows directly from this claim:

**Corollary 1.** Under the assumption that the tile $S$ is not homogeneous, the pattern $P$ must contain at least one sum-plus1 square and at least one sum-minus1 square.

**Proof.** Since the tile $S$ is not homogeneous, since the pattern dimension $k$ is large enough and since there are at most 4 deviating pixels limited to two rows and two columns, there must exist a non-boundary zero-pixel $p$ that is isolated (not a neighbor of a deviating pixel) and therefore equi-sum-average. By Claim 11, there must be a sum-minus1 square in $P$. In the same way, the existence of a sum-plus1 square follows.

For the rest of the proof we run a separate analysis for each possible number of deviating pixels. We start with such an analysis, under the assumption that the tile $S$ in not homogeneous, showing that all cases (0, 1, 2, 3 or 4 deviating pixels) lead to a contradiction and thus are not possible. On the other hand, our analysis that follows, for the case of where the tile $S$ is homogeneous shows that the only option is for the pattern to be almost-homogeneous, with a single deviating pixel at a corner of the pattern.

**Case 'zero deviating pixels’** In this case, all the pixels of the pattern $P$ are equi-sum-average. Pick $p$ to be a non-boundary 0-pixel in $P$ and flip it to 1. Such a pixel exists since we assumed that the tile $S$ is not homogeneous, taking into account that the pattern dimension $k$ is greater than 2.

In such a setting, Claim 11 tells us that $P$ must include a sum-plus1 square, which contradicts the fact that all squares in $P$ are zero-sum.
Case 'one deviating pixel’ Assume w.l.o.g. that the single deviating pixel, denote it by $p_1$, is 1-valued. In this case, all the squares of $P$ are sum-average or sum-plus1. The reason is that if we flip $p_1$ to be zero the pattern has no deviations and therefore all of its squares are sum-average. Flipping $p_1$ back to be 1 increase the sum of the squares that contain it to be sum-plus1.

Here, we can pick $p$ to be a non-boundary, isolated (non-neighbor of $p_1$) 1-pixel in $P$, and flip it to 0. Such a pixel exists, again, since we assumed that the tile $S$ is not homogeneous, assuming that $k$ is greater than 5.

In such a setting, Claim 11 tells us that $P$ must include a sum-minus1 square, which contradicts the fact that all squares in $P$ are sum-average or sum-plus1.

Case 'three or four deviating pixels’ In this case, our plan is to show that the deviating pixels all have the same value, namely all ones or all zeros. In addition, we will show that they are isolated (not neighbors of one another). In such a case, we have the same conditions to what we had in the 'one deviating pixel’, namely, that all squares are sum-zero or sum-plus1. Therefore, In the same manner, we pick a 1-pixel $p$, which is non-boundary and isolated (not a neighbor of any of the 3 deviating pixels) in order to get the same contradiction. Here too, the choice is possible for large enough $k$ (e.g. $k > 11$).

In this case, at least three out of the four potential locations have deviating values. This immediately implies that $r_1 \neq r_2$ and $c_1 \neq c_2$ (otherwise there were at most 2 potential deviating locations). Looking back at the horizontally shifted occurrences $P^H_1$ and $P^H_2$, this means that their bit flip locations $l_1$ and $l_2$ were on separate rows. We can therefore use item (b) of Lemma 15 to get that for each of the rows $r_1$ and $r_2$, the two potentially deviating locations have the same $x$-coordinate (mod 2). And using the same consideration for the vertically shifted $P^V_1$ and $P^V_2$ we get that for each of the columns, the two potentially deviating locations have the same $y$-coordinate (mod 2).

This implies that the potential deviation locations have equal coordinates (mod 2). This obviously implies that they are isolated. Also, it implies that the deviating pixels all have the same values, since that deviate from (are the opposite of) a single pixel value, e.g. some non-deviating pixel that has the same coordinates (mod 2).

Case 'two deviating pixels’ If both deviating pixels have the same value, w.l.o.g. one, we are done, in a similar way to the proofs for the cases of 1, 3 and 4 deviating pixels. This is due to the fact, that again, we are able to pick a 1-pixel $p$ that is isolated and non-boundary and flip it to zero. On the other hand, there is no sum-minus1 square in $P$, since the deviating pixels only increased some sums of sum squares, relative to the pattern without the deviations, which has only zero-sum squares.

We will therefore consider two deviating pixels, $p_0$ with value zero and $p_1$ with value one. We will consider two main cases, depending on whether $p_0$ and $p_1$ are neighbors (horizontal, vertical or diagonal) or whether they are not.

Corollary guarantees us that $P$ must contain at one sum-minus1 square and at least one sum-plus1 square. Every sum-minus1 square in $P$ must contain $p_0$ (being the only deviating 0-pixel) and for the same reason every sum-plus1 square in $P$ must contain $p_1$.

This understanding is used to rule out the case that $p_0$ and $p_1$ are neighbors. In such a case, any sum-minus1 and sum-plus1 squares that contain $p_0$ and $p_1$ are only one pixel apart (since $p_0$ and $p_1$ are neighbors). Since $k$ is large enough and the tile $S$ is not homogeneous, we can pick a
0-pixel \( p \) that is at least 2 pixels away from the boundary or from either of the deviating pixels \( p_0 \) and \( p_1 \). Flipping \( p \) to be a 1-pixel creates 4 sum-plus1 squares, that are surrounded by zero-sum squares. It is easily verified that such a flipping does not create a new occurrence of \( P \). The reason being that the one of the sum-plus1 squares that contained \( p_1 \) must overlap with \( p \), but there are no sum-minus1 squares at a distance of 1 pixel for the sum-minus1 square to overlap with.

We are left with the case that \( p_0 \) and \( p_1 \) are non-neighbors. If we denote \( p_0 = (x_0, y_0) \) and \( p_1 = (x_1, y_1) \), we will assume w.l.o.g. that \( x_0 \geq x_1 \) as well as \( y_0 \geq y_1 \) (i.e. that \( p_0 \) is the closer of the two to the bottom-right corner).

The first subcase here is if \( p_0 \) and \( p_1 \) share a row or a column. Assume w.l.o.g. that they are on the same row (\( p_1 \) to the left of \( p_0 \)), at least one pixel apart (being non-neighbors). In such a case, we know that \( p_1 \) and \( p_0 \) are at different location in the row (mod 2), since they are both deviating and have different values and hence they are at least 2 pixels apart. We choose the pixel \( p \) to be flipped, as the pixel that is two pixels to the right of \( p_1 \). The location \( p = (x_1 + 2, y_1) \) in the row is the same as that as of \( p_1 \pmod 2 \) and therefore it is a 0-pixel. The claim now is that a new created occurrence of \( P \) must be at a pure-horizontal shift of at most 2 pixels from the pattern \( P \) and in such a case, at least one of the left-most sum-minus1 squares that contains \( p_0 \) will be overlapped by a square that is not sum-minus1, hence a contradiction. The shift must be pure-horizontal (of maximum 2 pixels), since this is the only way that the the sum-plus1 squares created by the flipping of \( p \) will be overlapped by the sum-plus1 squares that contain \( p_1 \).

The first second subcase here is if \( p_0 \) and \( p_1 \) are on different rows and columns. This case can be shown a in very similar manner to the previous. Since \( p_0 \) and \( p_1 \) do not share a same row or column and are not diagonal neighbors, \( p_0 \) must be either at least 2 rows below \( p_1 \) or at least 2 columns to the right of \( p_1 \). We place the pixel \( p \) to be flipped as a bottom-right diagonal of \( p_1 \) at \( p = (x_1 + 1, y_1 + 1) \). It can be shown that a new occurrence \( P \) must be shifted either downwards or rightwards (or both) by at most 2 pixels, but not upwards or leftwards. This implies that squares that are not sum-minus1 will be overlapped some sum-minus1 squares that contain \( p_0 \).

**Case 'homogeneous tile \( S \)'** We will assume w.l.o.g. that \( S \) is the all zero tile. In such a case the pattern \( P \) is entirely 0-valued, except at the deviating pixels, which are 1-pixels. Recall that the deviating pixels must be at the intersection of the rows \( r_1, r_2 \) and columns \( c_1, c_2 \). We differentiate between five subcases.

In the first, we assume that there exist two (deviating) 1-pixels, on either the same row or the same column, that have at least one (non-deviating) 0-pixel between them. When flipping any such 0-pixel lying between the two 1-pixels, it is readily verified that such a change can not create a new occurrence of \( P \).

In the second, we assume that there exist two 1-pixels that are (horizontal, vertical, or diagonal) neighbors. In such a case, we consider a \( 3 \times 3 \) sub-image of only zeros (one must exist for \( k > 5 \)). Flipping its central pixel to be one cannot create a new occurrence of \( P \).

The first two cases entirely cover the option of 3 or 4 deviating pixels. For the option of 2 deviating pixels, there remains one more case, in which the 2 deviating pixels do not share a column or a row. If we denote these pixels by \( p_1 = (x_1, y_1) \) and \( p_2 = (x_2, y_2) \), we can assume w.l.o.g. that \( x_1 < x_2 \) and \( y_1 < y_2 \). Denote by \( p_2^L = (x_2 - 1, y_2) \) the 0-pixel to the left of \( p_2 \) and by \( p_1^R = (x_1 + 1, y_1) \) the 0-pixel to the right of \( p_1 \). We will flip the pixel \( p_1^R \) to be a 1-pixel and show that no new occurrence of \( P \) is created. If such a new occurrence \( P \) existed, the pixel overlapping \( p_1^R \) must be either \( p_1 \) or \( p_2 \), since these are the only 1-pixels. If it was \( p_2 \) then the 0-pixel \( p_2^L \) overlaps
the 1-pixel \( p_2 \), which is impossible. Similarly, if it was \( p_1 \) then the 0-pixel \( p_2^L \) overlaps \( p_1 \), which is impossible.

A fourth case is that there are no deviating pixels at all, namely that \( P \) is homogeneous. Flipping any pixel in \( P \) cannot create a new occurrence because \( P \) does not contain any pixel with the value needed to overlap the flipped pixel.

The fifth and final case is that of a single deviating pixel, namely that \( P \) is homogeneous except at a single pixel. We complete the proof by showing that in this case \( P \) must be almost homogeneous (i.e. homogeneous with a single different value at one of its corners). This is a situation that will repeat itself in the proof of the other cases in the different appendices and we therefore state it as a Claim.

**Claim 12.** If the pattern \( P \) of dimension \( k \geq 4 \) is homogeneous except at a single pixel then it must be almost homogeneous (i.e. the single different pixel is in one of its 4 corners).

**Proof.** Assume w.l.o.g. that \( P \) is of all zeros, except for a single 1-pixel \( p \). We need to show that \( p \) must be in one of the corners. Assume that \( p \) is on neither of the leftmost or rightmost columns of \( P \). Since the dimension of \( P \) is at least 4, in such a case \( p \) must have a neighbor to its left or to its right that is also not on the leftmost or rightmost columns. It is readily verified that Flipping this 0-pixel to be a 1-pixel does not create a new occurrence of \( P \). This proves that \( p \) must be on either the leftmost or rightmost column of \( P \). Using the exact same consideration on the rows of \( P \), proves that it must also be on either the top or bottom row of \( P \). \( \square \)

**D.3 The ‘Horizontal 1-pixel offset’ case**

We focus here on the horizontal case, in which we assume the new occurrences are shifted horizontally by a single pixel, denoting the left one by \( P_1 \) and the right one by \( P_2 \). We denote the locations in \( P \) whose bit flipping created the occurrences \( P_1 \) and \( P_2 \) by \( l_1 = (l^x_1, l^y_1) \) and \( l_2 = (l^x_2, l^y_2) \). In addition, denote by \( m \) the horizontal offset between \( P_1 \) (or \( P_2 \)) and \( P \) and by \( r \) the vertical offset between \( P_1 \) and \( P \). This setup is depicted in Figure 6.

As in the 1D case, we have that

\[
P = P_1 = P_2
\]  

(7)

In our proof we will repeatedly use the following simple claim.

**Claim 13.** Let \( Q_1 \) and \( Q_2 \) be two \( k \times k \) patterns with nonempty overlap and suppose that \( Q_2 \) is an offset of \( Q_1 \) by \((r, m)\). Suppose without loss of generality that \( m \in [0, k-1] \) and \( r \in [0, k-1] \). Then for every \( j \in [1, k-r] \) and \( i \in [1, k-m] \) it is the case that \( Q_1[j+r, i+m] = Q_2[j, i] \).

Figure 6: **Horizontal case - setting.** Blue square is the pattern \( P \). The new occurrences \( P_1 \) and \( P_2 \) (in red) are horizontally shifted by one pixel and were created by flipping the pixel locations \( l_1 \) and \( l_2 \) of \( P \), respectively. \( P_1 \) is offsetted from \( P \) by \( m \) pixels vertically and by \( r \) pixels horizontally.
The following constraints between \( P_1 \) and \( P \) follow immediately from Claim 13.

\[
\begin{align*}
P[j,i] &= P_1[j + r, i + m], & j = 1, \ldots, k - r ; & i = 1, \ldots, [k - m] \quad (i \neq l^y_1) \\
P[l^x_1 + r, l^y_1 + m] &= P_1[l^x_1 + r, l^y_1 + m] \\
1 - P[l^x_1 + r, l^y_1 + m] &= P_1[l^x_1 + r, l^y_1 + m]
\end{align*}
\]

Note that similar versions of (9) and (10) can be written for linking \( P_2 \) and \( P \):

\[
\begin{align*}
P[j, l^y_2] &= P_2[j + r - 1, l^y_2 + m], & j = 1, \ldots, k - r \quad (j \neq l^y_2) \\
1 - P[l^x_2 + r - 1, l^y_2 + m] &= P_2[l^x_2 + r - 1, l^y_2 + m]
\end{align*}
\]

Now, since \( P_1 \) and \( P_2 \) are horizontally consecutive, another set of constraints is given by:

\[
\begin{align*}
P_2[j - 1, i] &= P_2[j - 1, i], & j = 2, \ldots, k ; & i = 1, \ldots, k \quad (j, i) \notin \{(l^x_2 + r, l^y_1 + m), (l^x_2 + r, l^y_2 + m)\} \\
1 - P[l^x_2 + r - 1, l^y_2 + m] &= P_2[l^x_2 + r - 1, l^y_2 + m] \\
1 - P[l^x_2 + r, l^y_2 + m] &= P_2[l^x_2 + r, l^y_2 + m]
\end{align*}
\]

Using some of these constraints, we can easily draw the following claim:

**Claim 14.** For any row index \( i \in \{1, \ldots, k\} \setminus \{l^y_1 + m, l^y_2 + m\} \): The \( i \)-th rows of \( P_1 \), \( P_2 \) and \( P \) (which are equal to one another) are homogeneous.

See Figure 7 for an illustration for Claim 14. Each such row of \( P_1 \) and \( P_2 \) must be homogeneous, since these rows are equal (\( P_1 = P_2 \) (7)) and are shifted by one bit (constraint (13)). An equivalent row of \( P \) is identical, using \( P = P_1 = P_2 \) (7), and therefore homogeneous too.

Let \( l^y_{\min} = \min\{l^y_1, l^y_2\} \) be the index of the top row of the bit change locations \( l_1 \) and \( l_2 \).

**Claim 15.** The following hold:

1. The locations \( l_1 \) and \( l_2 \) occur on the same row. Formally: \( l^y_{\min} = l^y_1 = l^y_2 \).
2. The \((l^y_{\min} + m)\)-th row of \( P \) (and that of \( P_1 \) and \( P_2 \)) is homogeneous except at one pixel.

To prove Claim 15, we take the following steps. First, we rule out the options of \( l^y_1 < l^y_2 \) and \( l^y_2 < l^y_1 \), thereby proving part 1 of the claim. Then, we focus on the case of \( l^y_1 = l^y_2 \) to analyze the \((l^y_{\min} + m)\)-th row of \( P_1 \) (or \( P_2 \)). We will specifically be using some of the constraints enumerated above. The proof consists of a detailed cases analysis, guided by illustrations with the convention that blue and red arrows (constraints) impose same values, while green arrows impose opposite values. The direction of the arrow is the direction in which the constraint is applied (even though all constraints are symmetric).

At the highest level of a detailed case analysis, we differentiate between the two possible settings for pairs of new occurrence patterns in the set \( S_{TL} \), which are horizontally shifted by a single pixel. These two settings are depicted in Figure 7. We will first handle the case of pure horizontal offset (e.g., \( m = 0 \)) as depicted on the left, and then handle the case of non-pure horizontal offset (\( m > 0 \)), shown on the right.
Figure 7: Illustration for Claims 14 and 15. Pattern $P$ (blue) and new occurrences $P_1$ and $P_2$ (red). Homogeneous rows of $P_1$ and $P_2$ (shown on left) and the equivalent (identical) homogeneous rows of $P$ (shown on right). Note that the order between $l_1^y$ and $l_2^y$ can be arbitrary, including equality (in which case there is only one inhomogeneous row.)

**pure horizontal offset** ($m = 0$)

We first rule out the option that $l_1$ and $l_2$ are on separate rows (i.e. $l_1^y \neq l_2^y$). Assuming they were on separate rows, we show how to reach a contradiction by examining the overlapping rows where the bit flip at $l_2$ occurs, namely, $P(l_2^y), P_1(l_2^y)$ and $P_2(l_2^y)$. Under the assumption, the bit flip location $l_1$ does not occur on these rows, meaning that except for the location $l_1$, other locations follow equality constraints only. Please refer to the illustration on the right to follow the proof of the contradiction (which is shown by the ‘x’ value in the illustration).

First, w.l.o.g. assume that $P[l_2^x, l_2^y] = 0$. Using the fact that $P = P_1$ (7) we have that

$$P_1[l_2^x, l_2^y] = P[l_2^x, l_2^y] = 0$$

(long blue arrow)

Now, by applying the following two equalities (that follow from $P_1 = P_2$ (7) and constraint (13)) $r - 1$ times sequentially for $j = l_2^x, \cdots , l_2^x + r - 2$ :

$$P_2[j, l_2^y] = P_1[j, l_2^y] = 0$$ (blue arrow) \hspace{1cm} (16)

$$P_1[j + 1, l_2^y] = P_2[j + 1, l_2^y] = 0$$ (red arrow) \hspace{1cm} (17)

With one more application of (16) for $j = l_2^x + r - 1$, we obtain that:

$$P_2[l_2^x + r - 1, l_2^y] = 0$$

One the other hand (the contradiction), using constraint (12) we obtain that:

$$P_2[l_2^x + r - 1, l_2^y] = 1 - P[l_2^x, l_2^y] = 1$$ (long green arrow)

For the horizontal offset case ($m = 0$), we are left with the case that the flipping bits are on the same row (i.e. $l_1^y = l_2^y$). This is exactly the case that we handled in the 1D section, where we
had two new occurrences shifted by a single pixel. The conclusion was that the only option was for that specific row of $P_1$ (and $P_2$) to be of the form $0^{k-1}1$ (or its negation), which completes the proof for the horizontal offset case.

**non-pure-horizontal offset** ($m > 0$)

Here, we will be using the fact that the $l^r_1$th row of $P$ is homogeneous. This is true by Claim 14, since all its rows except for two ($i \in \{l^r_1 + m, l^r_2 + m\}$) are homogeneous, and the $l^r_{min}$th row is not one of them (since $m \geq 1$, by definition). With out loss of generality we will assume it is all zeros.

**Case $l^r_1 < l^r_2$:** We show this case is impossible by reaching a contradiction. Follow the illustration on the right. We examine the overlapping rows $P(l^r_1)$, $P_1(l^r_1 + m)$ and $P_2(l^r_1 + m)$ (of $P$, $P_1$ and $P_2$). Since the flip location $l_2$ does not occur on these rows, equality constraints will apply, except at the location of $l_1$. By definition, the flipping location $l_1$ is in the intersection of $P_1$ and $P$ and therefore: $l^r_1 \in 1, \ldots, k - r$. In the illustration, this contradiction reached is shown by the ‘$x$’ value, for 3 choices of $l^r_1$ in the range $1, \ldots, k - r$.  

On one hand, using constraint (11), we have:

$$P_2[l^r_1 + r, l^r_1 + m] = P[l^r_1 + 1, l^r_1] = 0 \quad \text{(long red arrows)}$$

and on the other hand (the contradiction), using constraints (7) and (10), we have:

$$P_2[l^r_1 + r, l^r_1 + m] = P_1[l^r_1 + r, l^r_1 + m] = 1 - P[l^r_1, l^r_1] = 1 \quad \text{(green and blue arrows)}$$

The analysis for this case actually shows a more general fact, that if $l_1$ is on a separate row from $l_2$, then the $l^r_2$th row of $P$ cannot be homogeneous. By Claim 14, this inhomogeneous row must be one of $l^r_1 + m$ and $l^r_2 + m$. But as $m > 0$, it has to be $l^r_2 + m$, formally:

$$l^r_1 = l^r_2 + m \quad (18)$$

**Case $l^r_2 < l^r_1$:** In this case, which is similar to the above but slightly more involved, we analyze the overlapping rows $P_1(l^r_2 + m)$, $P_2(l^r_2 + m)$ and $P(l^r_2)$. In this case the flip location $l_1$ does not occur on these rows, so equality constraints will apply, except at the location of $l_2$. In this case, we consider the flipping at location $l_2$ and clearly have that $l^r_2 \in 1, \ldots, k - r + 1$. Here, we look at two complimentary sub-cases (shown in the figure below): $l^r_2 = 1$ (left illustration) and $1 < l^r_2 \leq k - r + 1$ (right illustration).

In the second case (right) a contradiction is reached as follows. Using constraint (12), we have:

$$P_2[l^r_2 + r - 1, l^r_2 + m] = 1 - P[l^r_2, l^r_2] = 1 \quad \text{(long green arrows)}$$

while using constraints (7) and (9), we get (the contradiction):

$$P_2[l^r_2 + r - 1, l^r_2 + m] = P_1[l^r_2 + r - 1, l^r_2 + m] = P[l^r_2 - 1, l^r_2] = 0 \quad \text{(red and blue arrows)}$$
The contradiction reached is shown by the ‘x’ value, for 3 choices of \( l_2 \) in the range \( 2, \ldots, k-r+1 \).

In the first case (left illustration), where \( l_2 = 1 \), we do not reach a contradiction directly, but rather obtain that the specific \((l_2,\text{th})\) rows of \( P_1 \) and \( P_2 \) equal \( 1^r \cdot 0^{k-r} \). Please follow the left illustration above to understand the following derivation.

Using constraints (7) and (9), for \( j = r+1, \ldots, k \) we have that:

\[
P_2[j, l_2 + m] = P_1[j, l_2 + m] = P[j-r, l_2] = 0 \quad \text{(blue and top red arrows)}
\]

Now, using constraint (14) we have that:

\[
P_2[r, l_2 + m] = 1 - P_1[r+1, l_2 + m] = 1 \quad \text{(single green arrow)}
\]

And, using constraint (7) we have that:

\[
P_1[r, l_2 + m] = P_2[r, l_2 + m] = 1 \quad \text{(blue arrow)}
\]

Finally, we are done by applying the following two equalities (that follow from constraints (7) and (13)) sequentially for \( j = r-1, \ldots, 2, 1 \):

\[
P_2[j, l_2 + m] = P_1[j+1, l_2 + m] = 1 \quad \text{(red arrows, bottom)}
\]
\[
P_1[j, l_2 + m] = P_2[j, l_2 + m] = 1 \quad \text{(blue arrows)}
\]

So, we have shown that in the case of \( l_2 = 1 \) (left illustration above) \( P_1(l_2 + m) = P_2(l_2 + m) = 1^r \cdot 0^{k-r} \). This implies (since \( P = P_1 = P_2 \)) that \( P(l_2 + m) = 1^r \cdot 0^{k-r} \). Furthermore, since \( l_2 = l_2 + m \) (equality (18)) we can write \( P(l_2^*) = 1^r \cdot 0^{k-r} \).

We will now use this fact to get to a contradiction by examining the overlapping rows \( P(l_1^*) \), \( P_1(l_1^* + m) \) and \( P_2(l_1^* + m) \), where the flipping at \( l_1 \) occurs. As a reminder, the flip location \( l_2 \) does not occur on these rows. Please refer to the next figure to follow the proof of the contradiction (shown by the ‘x’ value).

Our starting point is that \( P(l_1^*) = 1^r \cdot 0^{k-r} \) and we differentiate between two options, regarding the relative size of the offset \( r \) with respect to the pattern dimension \( k \). In general, \( r \leq k-1 \) since \( P_1 \) and \( P \) overlap horizontally by at least one pixel. We will be examining different cases, shown in the illustration below.

We first deal with the case that \( 2r \geq k \). Since we assume that \( k \geq 3 \), in this case we have that \( r \geq 2 \). This situation is depicted in the top-left. It implies that the relevant section of the \( P(l_1^*) \) (the prefix that overlaps with \( P_1(l_1^* + m) \) or \( P_2(l_1^* + m) \)) is all 1s. In such a case, picking any \( l_1^* \in 1, \ldots, r-1 \) (such a choice is possible since \( r \geq 2 \) in this case), on one hand using constraint (11) we obtain that:

\[
P_2[l_1^* + r, l_1^* + m] = P[l_1^* + 1, l_1^*] = 1 \quad \text{(long red arrow)}
\]
One the other hand (the contradiction), using constraints (7) and (10) we obtain that:

\[ P_2[l_1^r + r, l_2^u + m] = P_1[l_1^r + r, l_2^u + m] = 1 - P[l_1^r, l_1^u] = 0 \]  
(blue and green arrows)

We now take care of the complementary case that \(2r < k\). In this case, we first claim that \(l_1^r \in \{r, r+1\}\). Simply, since if this is not the case (top-right of the illustration) on one hand using constraint (11) we obtain that:

\[ P_2[2r, l_1^u + m] = P[r + 1, l_1^u] = 0 \]  
(long red arrow)

One the other hand (the contradiction), using constraints (7) and (9) we obtain that:

\[ P_2[2r, l_1^u + m] = P_1[2r, l_1^u + m] = P[r, l_1^u] = 1 \]  
(blue and red arrows)

We are left to check the remaining two cases of \(l_1^r \in \{r, r+1\}\). First (bottom-left of the illustration), if \(l_1^r = r\), using constraint (12), we have:

\[ P_2[2r - 1, l_2^u + m] = 1 - P[r, l_2^u] = 0 \]  
(long green arrow)

while using constraints (7) and (9), we get (the contradiction):

\[ P_2[2r - 1, l_2^u + m] = P_1[2r - 1, l_2^u + m] = P[r - 1, l_2^u] = 1 \]  
(red and blue arrows)

Second (bottom-right of the illustration), if \(l_1^r = r + 1\), on one hand using constraint (11) we obtain that:

\[ P_2[2r + 1, l_1^u + m] = P[r + 2, l_1^u] = 0 \]  
(long red arrow)

One the other hand (the contradiction), using constraints (7) and (10) we obtain that:

\[ P_2[2r + 1, l_1^u + m] = P_1[2r + 1, l_1^u + m] = 1 - P[r + 1, l_1^u] = 1 \]  
(blue and green arrows)
Case \( l_1^y = l_2^y \)  This is the remaining possibility. It stands for the case in which the pixel locations \( l_1 \) and \( l_2 \), whose flipping created the occurrences \( P_1 \) and \( P_2 \), are located on the same row of the pattern \( P \). Once again, we look at the overlapping rows \( P(l_{min}^y) \), \( P_1(l_{min}^y + m) \) and \( P_2(l_{min}^y + m) \) and analyze the possible relative horizontal (column) flipping locations, which are \( l_1^x \) and \( l_2^x \). We come to the conclusion that they must be consecutive, with \( l_1^x \) being first (on the left). Furthermore, we show that in this case the respective row \( P_1(l_{min}^y) \) (which is equals the respective row \( P_2(l_{min}^y) \) of \( P_2 \)) must be of the form \( 0^t \cdot 1 \cdot 0^{k-t-1} \) (or \( 1^t \cdot 0 \cdot 1^{k-t-1} \)) for some \( t \in \{1, \ldots, k-1\} \), hence proving part 2 of Claim 15. The proof is illustrated in the figure below.

```
 Starting at the top left, we show that \( l_1^x < l_2^x \) (i.e. \( l_1^x \) comes before \( l_2^x \)), by showing a contradiction otherwise. Next, at the top right, we show that \( l_2^x - l_1^x = 1 \) (i.e. \( l_1^x \) and \( l_2^x \) must be consecutive), again, by showing a contradiction otherwise. Finally, at the bottom, we show the only valid possibilities, in which the respective rows are shown to be homogeneous except at 1 pixel, as required. On the left, \( l_1^x \) and \( l_2^x \) are anywhere in the middle of the rows \( P_1(l_{min}^y) \) and \( P_2(l_{min}^y) \), while on the right they are at the last positions. This concludes the proof of Claim 15.

The current state, combining Claims 14 and 15, is that the pattern \( P \) must have \( k-1 \) homogeneous rows and one row that is homogeneous except at 1 pixel. This is summarized in the illustrations of Figure 8.
```

Figure 8: The single solid black row for each of \( P_1 \), \( P_2 \) and \( P \) is the (identical) ’special’ non-homogeneous row, with the single different valued location marked by the green square.
From this point on, we will not need to use the existence of the consecutive occurrences $P_1$ and $P_2$, but we will rather focus on the pattern $P$ itself. Our next claim will be the following:

**Claim 16.** The pattern $P$ is homogeneous, except for at a single pixel.

Assume w.l.o.g. that the ‘special’ row of $P$, denote it by $T$, is of all 0s except for a single location. We need to show that the rest of the rows (each known to be homogeneous) must be all 0s rows. Assume towards a contradiction that there is some all 1s row in $P$, call it $R$. We will show that there is a location in $R$ whose flipping (to 0) cannot create a new occurrence of $P$. We will be flipping one of the inner locations in $R$ (not the first or last) and therefore the new created overlapping occurrence, denote it by $Q (= P)$, must have both a 0 and a 1 on its row that overlaps $R$. This row of $Q$ therefore must be the ‘special’ (non-homogeneous) row $T$ of $P$.

We further assume for the moment that $k \geq 4$. Since $T$ is all 0, except at one location, we have that one of its sides starts with two 0s, w.l.o.g the left side (i.e. $T[0] = T[1] = 0$). In this case we flip the $(k-1)$th pixel of $R$ to 0 so it gets the form $1^{k-2}01$. By definition, the new occurrence $Q$ must overlap $P$ at this location. As noted before, focusing on the specific row, $Q$ has a row that equals $T$ that overlaps $R$. Now, $T$ cannot overlap $R$ from the right since it has at least two 0 pixels on its left end that would not fit into $R$ that has only one 0. Similarly, $T$ cannot overlap $R$ from the left since $R$ has at least two 1 pixels on its left end that would not fit into $T$. See the following illustration:

![Illustration of T and R](image)

Now that we have that $P$ has exactly a single 1 pixel, we will show that it has to be in one of the four corners of $P$. This was already proved by Claim 12 from appendix D.2.

**D.4 The ‘Diagonal 1-pixel offset’ case**

In this section we deal with the case in which we assume the new occurrences are shifted diagonally by a single pixel, denoting the top-left one by $P_1$ and the bottom-right one by $P_2$. This case is depicted in Figure 9, which is the equivalent of Figure 6 of the ‘horizontal case’. We will assume that the horizontal and vertical offsets $m$ and $r$ are both non-negative. The case when one is non-negative and one is non-positive is very similar and its proof is omitted.

![Figure 9: Diagonal case - setting](image)

The general plan will be to follow the lines of the ‘horizontal case’ proof, highlighting the differences, while making sure that the equivalent derivations can be made.
Naturally, in this case we will work with the top-left-to-bottom-right diagonals of $P$, $P_1$ and $P_2$, in place of the rows that we used in the 'horizontal case' proof. Instead of $k$ rows of length $k$, we now have $2k - 1$ diagonals of varying lengths ($1, 2, \ldots, k - 1, k, k - 1, \ldots, 2, 1$).

We will be making a slight abuse of notation in order to make the equivalence to the 'horizontal case' easier. First, the bit-flip pixel locations $l_i$ ($i = 1, 2$) will be indexed as usual by the pair $l_i = (l^x_i, l^y_i)$, where $l^y_i$ is the index of the diagonal it occurs in ($l^y_i \in [2k - 1]$), while $l^x_i$ is the location index within the diagonal from top-left to bottom right ($l^x_i \in [q]$, if $q$ is the length of the $l^y_i$th diagonal). In addition, we will denote by $P_1(i)$, $P_2(i)$ and $P(i)$ the $i$th diagonal (instead of the row) of $P_1$, $P_2$ and $P$ respectively.

Notice that in the 'horizontal case' proof we seldom used the fact that the rows have the specific length of $k$ and furthermore, when looking at intersecting rows of $P$, $P_1$ and $P_2$ - we did not use the fact that they were of equal length. These facts will allow us to easily adapt most stages of the 'horizontal case' proof to the current 'diagonal case' one, while others will have to be handled more carefully.

Our starting points is that

$$P = P_1 = P_2$$

and Claim 13 could be rewritten for the diagonal case with indexing capturing diagonal rather than horizontal neighborhood. One difference is that in the 'rows' case the offset of indices between matching rows of $P$ and $P_1$ was $m$, while here the corresponding index offset is $m - r$.

Claim 14 could be restated (and derived) as follows. For any diagonal index $i \in [2k - 1]$; $i \notin \{l^y_1 + m - r, l^y_2 + m - r\}$: The $i$th diagonals of $P_1$, $P_2$ and $P$ (which are equal to one another) are homogeneous.

We can now move to the 'diagonal version' of Claim 15, which will state that the bit flip locations $l_1$ and $l_2$ occur on a single diagonal and that $P$ has a single non homogeneous diagonal, the $(l_{\text{min}} + m - r)$th one, which is homogeneous at all except a single pixel.

Figure 10: Illustration for Claims 14 and 15 (the 'diagonal version of Figure 7). In this 'diagonals' setting, the $2k - 1$ diagonals of $P$ are numbered from top to bottom. The diagonals with indices $l^y_1$ and $l^y_2$ are those where the bit flips occur. **Left:** The 'pure-diagonal-offset' case ($m = r$). **Right:** The 'non-pure-diagonal-offset' case ($m > r$). In both cases, the black intervals show the equivalent groups of homogeneous diagonals for $P_1$ and $P_2$ (top-left) and for $P$ (bottom-right).

The diagonal version of the 'pure horizontal' case, is a 'pure diagonal' shift between $P$, $P_1$ and $P_2$, defined by equal vertical and horizontal offsets, i.e. $m = r$. This setting is depicted on the left side of Figure 10. The other case involves also a shift perpendicular to the direction of the
diagonals and is shown on the right side of the figure. This is the case where the vertical offset is the larger one, i.e. \( m > r \). Note that the complementary case of \( m < r \) is equivalent, by a symmetry argument.

'pure-diagonal-offset' case \((m = r)\)

We use the exact same proof, noticing that in the original proof we examined overlapping rows from \( P, P_1 \) and \( P_2 \) which were of length \( k \), with those of \( P_1 \) and \( P \) shifted by \( r \) pixels. These numbers \( k \) and \( r \) could have been any other positive number. In our case, we examine overlapping diagonals of some fixed length, with those of \( P_1 \) and \( P \) shifted by \( m = r \) pixels.

'non-pure-diagonal-offset' case \((m > r)\)

Following the original proof, we use the fact that the \( l_{\text{min}} \)th diagonal of \( P \) is homogeneous, by our diagonal version of Claim 14.

Case \( l_1^y < l_2^y \) : In this case, we did not use the fact that the overlapping rows \( P(l_1^y), P_1(l_1^y + m) \) and \( P_2(l_1^y + m) \) were of the specific length \( k \), or that \( P(l_1^y) \) and \( P_1(l_1^y + m) \) were shifted by the specific distance of \( r \). Therefore, the same contradiction is reached in the diagonal case, as is depicted on the right.

Case \( l_1^y > l_2^y \) : As in the original proof we examine the overlapping diagonals where the bit flip \( l_2 \) (and not \( l_1 \)) occurs: \( P_1(l_2^y + m - r), P_2(l_2^y + m - r) \) and \( P(l_2) \). We will differentiate between the cases of \( l_2^y = 1 \) versus \( l_2^y > 1 \). As in the original proof, the second case leads directly to a contradiction. In the first case there are some differences worth highlighting in the diagonal case. In the original version we had that \( P_1(l_2^y + m) = P_2(l_2^y + m) = 1^r \cdot 0^{k-r} \). In our diagonal version, the offset between \( P_1(l_2^y + m - r) \) and \( P(l_2^y) \) is not necessarily \( r \), but can be some constant \( R_2 \). In addition, the length of \( P_1(l_2^y + m - r) \) is not necessarily \( k \), but can be some constant \( K_2 \). Therefore, instead of getting that \( P(l_2^y + m - r) = 1^r \cdot 0^{k-r} \) we have that \( P(l_2^y + m - r) = 1^{R_2} \cdot 0^{K_2 - R_2} \), which implies (since \( P = P_1 = P_2 \)) that \( P(l_2^y + m - r) = 1^{R_2} \cdot 0^{K_2 - R_2} \).

Next, in the 'horizontal case' version we differentiated between two options, regarding the relative size of the offset \( r \) with respect to the pattern dimension \( k \), who were shown to be related.
by $2 \leq r \leq k - 1$. In our case, again, if we denote the offset of $P_1(l_1^y + m - r)$ from $P(l_1^y)$ by $R_1$ and the length of $P_1(l_1^y + m - r)$ by $K_1$, we get the setup, as shown in the illustration above.

So, so instead of dividing the analysis into the cases $2r \geq k$ and $2r \geq k$, we divide it into the equivalent cases $R_1 + R_2 \geq K_1$ and $R_1 + R_2 < K_1$, and get exactly the same analyses. The rest of the analysis follows identically.

**Case $l_1^y = l_2^y$:** Finally, this case requires no adaptations for the diagonal case. As mentioned earlier, unlike in the 'horizontal' case, the diagonal lengths and overlap lengths of overlapping diagonals may vary. However, in the $l_1^y = l_2^y$ case of the 'horizontal' case we did not use the fact that row lengths and overlap lengths were fixed. Here too we can conclude that the respective diagonal $P_1(l_{\min}^y)$ (which is equals the respective diagonal $P_2(l_{\min}^y)$) must be of the form $0^t \cdot 1 \cdot 0^{k-t-1}$ (or $1^t \cdot 0 \cdot 1^{k-t-1}$) for some $t \in \{1, \ldots, k - 1\}$, hence proving part 2 of Claim 15.

At this point, combining Claims 14 and 15 implies is that the pattern $P$ must have $2k - 2$ homogeneous diagonals and one diagonal that is homogeneous except at a single pixel. This is summarized in the illustrations of Figure 11, for the pure and non-pure diagonal offset cases.

![Figure 11: (The 'diagonal' version of Figure 8).
A single black diagonal for each of $P_1$, $P_2$ and $P$ is the 'special' non-homogeneous diagonal, with the single different valued location (for each of $P_1$, $P_2$ and $P$) marked by a green square. The sets of homogeneous diagonals of $P_1$ and $P_2$ (top) and $P$ (bottom) are marked with the black intervals.](image)

In order to show that the pattern $P$ is almost homogeneous, we are left to prove that the single non-homogeneous diagonal is the main diagonal with the single 1-pixel in either the top-left or bottom right corner. In addition we need to prove that all the other $2k - 2$ homogeneous diagonals are 0-valued.

We will be using a strong structural constraint (in additional to the diagonal’s constraint), that is a direct consequence of Lemma 16 (which guaranties the existence of two pairs of occurrences offsetted by 3 pixels, one horizontally and one vertically) with Lemma 15. Both Lemmas are from appendix D.1.

The constraint can be formalized by the following claim, which is the exact equivalent of Claim 9 from appendix D.2, only this time for cycles of length $t = 3$ instead of length $t = 2$. 

---

37
Claim 17. There exist row indices \( r_1 \) and \( r_2 \) (possibly \( r_1 = r_2 \)) and column indices \( c_1 \) and \( c_2 \) (possibly \( c_1 = c_2 \)) such that for any pair of coordinate locations \((x_1, y_1)\) and \((x_2, y_2)\) that do not belong to \(\{(r_1, c_1), (r_1, c_2), (r_2, c_1), (r_2, c_2)\}\), if \((x_1, y_1) \equiv (x_2, y_2) \pmod{3}\) then \(P[x_1, y_1] = P[x_2, y_2]\).

Proof. The claim is a restatement of Claim 9 with \(t = 3\) instead of length \(t = 2\). The proof is therefore omitted.

However, this time we have the further constraints on the diagonals, which imply that the pattern \(P\) is a repetitive \(3 \times 3\) tiling (with partial tiles at the boundaries) with a single tile of the structure

\[
S = \begin{bmatrix}
c & a & b \\
b & c & a \\
a & b & c \\
\end{bmatrix}
\]

except for at most 4 locations that might deviate from the tile values. However, every deviation, except at the top-right (TR) and bottom-left (BL) pixels of \(P\) is also a deviation from the homogeneity of the respective diagonal. This implies that there are at most 3 deviations all together - at the TR and BL corners as well as at the single different pixel in the non-homogeneous diagonal.

Our goal is to show that \(a = b = c\). If this is not the case (one of the three is different from the other 2), we now show that there must be three deviating pixels. Specifically, if we number the diagonals from bottom to top, call the diagonals with index 1 (mod 3) 'a'-diagonals, those with index 2 (mod 3) 'b'-diagonals and those with index 0 (mod 3) 'c'-diagonals, then the must be a deviating pixel on one diagonal of each of the three types. The reason for this is as follows. If we take for example an non-deviating a-pixel that is at least 3 pixels away from the boundary of \(P\) and flip its value, it is easily seen that the only option for a new occurrence to exist is by having a deviating a-pixel overlap the flipped pixel. The same holds for \(b\) and \(c\) pixels and we can therefore conclude that there must exist at least one deviating pixel in each of the 3 kinds of diagonals.

Now, since the locations of the deviating pixels are limited to the intersection of two rows and two columns and since 2 out of the 3 locations must be the TR and BL corners, the intersections will have to be the 4 corners of the pattern. This means that the single non-homogeneous diagonal must be the diagonal that contains the top-left (TL) and bottom-right (BR) corners and its single 1-pixels must be in one of these corners, assume w.l.o.g. at the BR one. This situation is depicted in the figure on the right, where the deviating pixels are shown by green squares. However, if we flip the pixel that is 3 pixels above the BR pixel (shown in red in the illustration), as noted before in a newly created occurrence the BR corner will have to overlap the flipped pixel, but this creates a contradiction, as the BL corner (a deviating a-pixel, will overlap the non-deviating a-pixel above it. The contradiction implies that \(a = b = c = 0\).

This means that we are back in the following situation. We have that the pattern \(P\) is all zero, except at the possible 4 deviating locations, at the intersection of up to 2 columns and up to 2 rows. This is exactly the scenario we had in the sub-case 'homogeneous tile \(S\)' from the 'Horizontal and vertical 2-pixel offset' case of appendix D.2. The analysis showed that the only option that does not lead to a contradiction is for a pattern that is almost homogeneous, as required.