A polar decomposition for quantum channels
(with applications to bounding error propagation in quantum circuits)

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Inevitably, assessing the overall performance of a quantum computer must rely on characterizing some of its elementary constituents and, from this information, formulate a broader statement concerning more complex constructions thereof. However, given the vastitude of possible quantum errors as well as their coherent nature, accurately inferring the quality of composite operations is generally difficult. To navigate through this jumble, we introduce a non-physical simplification of quantum maps that we refer to as the leading Kraus (LK) approximation. The uncluttered parameterization of LK approximated maps naturally suggests the introduction of a unitary-decoherent polar factorization for quantum channels in any dimension. We then leverage this structural dichotomy to bound the evolution – as circuits grow in depth – of two of the most experimentally relevant figures of merit, namely the average process fidelity and the unitarity. We demonstrate that the leeway in the behavior of the process fidelity is essentially taken into account by physical unitary operations.

I. INTRODUCTION

Just like evaluating a piano doesn’t involve playing all possible pieces of music, characterizing a computer (classical or quantum) doesn’t involve running all infinitely many circuits. The natural procedure to characterize both these devices is to gather information on a restricted number of components, and based on that information make conclusions on the quality of more involved constructions (melodies, chords, circuits, magic state injections, etc). When considering the tuning of a piano, the extrapolation is not much of a problem; imperfections are typically tied to specific keys, and they don’t tend to propagate over the keyboard as the music goes on, and unless there is some resonant effect, the errors don’t coherently interfere. Hence, the quality of individual keys generally guarantees playability. In this sense, the characterization of a piano is similar to that of a classical computer: the well-behaved stochasticity of the noise eases the passage between an assertion of components quality to a broader assertion on the performance of more complex operations. This statement can be phrased the other way around: a limited range of behaviors simplifies the search for imperfections.

In contrast, when characterizing a quantum computer, the jump from a characterization of elementary operations to a quantified assertion on the overall device performance is more knotty; errors can coherently interfere and propagate through the entire device via multi-qubit operations. This thorny situation can be quantified, for instance, by bounding the behavior of the average process fidelity (hereafter the fidelity and its counterpart, the infidelity), an experimentally important figure of merit which captures the overlap between an implemented operation and its target. More precisely, one may ask: “What are the best and worst fidelities of a circuit given a knowledge of the fidelity of its components?” When dealing with a classical scenario, we would expect the difference between the best and worst cases to remain insignificant (remember the piano analogy). In a quantum scenario, however, it is known that the largest discrepancy, which is achieved by unitary errors, grows quickly (quadratically) in the circuit depth (see, for instance, Carignan-Dugas et al [1]). Not so surprisingly, the best case corresponds to a unitary cancellation, and the worst case corresponds to a coherent buildup. This lead to another question: “What if we are guaranteed that the individual errors are not unitary?” In particular, what if we measure the degree to which the error operations are unitary, known as the unitarity[2], an experimental figure of merit which captures the coherence in the noise? Previous work has given partial answers to this question: Carignan-Dugas et al [1] derive bounds that fall back to the “piano analogy” when the unitarity is minimal; additionally, they provide examples of quantum channels that saturate their bound in the intermediate regime where errors are neither purely unitary nor purely stochastic, but still unital and acting on a single qubit coupled with a system of arbitrary (but finite) size[3]. In this paper, we generalize that bound to all dimensions and show its near saturation (i.e. to second order in the infidelity or better) and also account for non-unital processes. That is, we provide a closely saturated bound for all finite-dimensional quantum channels. While this is already an interesting result, the tools that we develop to generalize the bound help us answering a far

1 They attribute all the error dynamics on the qubit; the intuitive geometric picture offered by parameterization of processes acting on the Bloch sphere allows showing the saturation of the bound for unital channels. The bound in the non-unital case included a dimensional factor which prevented its saturation.
more fundamental question. In previous work, the saturation was shown through a handful of examples. Now, we provide a complete descriptive answer to:

**What is the set of mechanisms responsible for the discrepancy between the best and the worst fidelity of a circuit?**

This would not be much of a fundamental question if the answer didn’t also unravel an important dichotomy in classifying quantum errors. Indeed, given the intricate geometry of quantum states [3], the answer could have included some obscure blend of non-intuitive mechanisms, leaving us with yet another resignation in the attempt to intuitively reason about quantum dynamics. Although, for once, this is not the case: the discrepancy between the best and worst fidelity is, to high precision, entirely taken into account by unitary dynamics. Even more surprisingly, the unitary dynamics itself is precisely the product of the “unitary factors” of individual circuit components. Yes, as we later demonstrate, every non-catastrophic channel can be decomposed as a physical unitary followed (or preceded) by a decoherent channel. For realistic errors, the unitary is unique and is referred to as the coherent factor. This factorization is analog to the so-called matrix polar decomposition and, as we will show, directly stems from it. The uniqueness of the coherent factor might puzzle the skeptical reader. Indeed, for example, how should we unambiguously define such factor in the case of an error which consists of a mixture of near-identity unitaries (i.e. $A(\rho) = \sum_i p_i U_i \rho U_i^\dagger$, where $U_i \approx I_d$)? Should it be the unitary operation with the highest weight? Should it relate with some kind of ensemble average over the associated Hamiltonians? To systematically answer this type of question, we introduce the leading Kraus (LK) approximation, a sub-parameterization of quantum channels which, among other things, exposes a natural definition for the coherent and decoherent factors of a channel.

What allows us to really profit from the channel polar decomposition is the surprising property that the LK approximation, despite its seemingly bare structure, closely captures the evolution of the fidelity and unitarity in circuits. That is, we can mathematically replace all the channels in a circuit by their respective LK approximation and still expect to accurately bound its fidelity and unitarity. Working with the uncluttered structure offered by the LK approximation helped us identify and rule out pathological error scenarios, which we refer to as “extremal”. For all realistic noisy channel, we derive the following observations (they hold to high precision):

i. The infidelity (the counterpart to the fidelity) of a channel can be split into two terms:
   (a) a coherent infidelity, which corresponds to the infidelity of the coherent factor to the target channel;
   (b) a decoherent infidelity, which corresponds to the infidelity of the decoherent factor to the identity.

ii. The decoherent infidelity of a channel is in one-to-one correspondence with its unitarity. Moreover, the decoherent infidelity corresponds to the minimum infidelity of the channel after the application of a unitary (the coherent infidelity is correctable through a composition with a unitary).

iii. The unitarity of a composite channel is a decay function expressed in terms of individual channels’ unitarity.

iv. The fidelity of the composition of decoherent channels is a decay function expressed in terms of individual channels’ fidelity.

v. The fidelity of a general composition is upper bounded by a decay dictated by the decoherent factors (hence by the unitarity of individual components).

vi. The discrepancy between the upper and the lower bound of the fidelity is captured by the fidelity of the composition of the coherent factors (to the target circuit).

These realizations are directly applicable to the analysis and development of process characterization methods. The fidelity of various error processes can be robustly and efficiently estimated through a scalable experimental protocol known as randomized benchmarking (RB) [4–7] and a family of generalizations thereof [2, 8–24]. To remain efficient as quantum devices grow larger, RB experiments only extract partial information about specific sets of components. A known challenge is to leverage this limited view to formulate a more rounded understanding of the device. By looking at the fidelity of well-designed compositions, it should be possible to extract other figures of merit attached to quantum processes. The idea is that since maps dictate the evolution of the fidelity, conversely, the evolution of the fidelity can tell us information about the maps. The relationship between the maps and the fidelity is generally obscure, but the LK approximation allows seeing through it more clearly.

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2 Given wise sense equable errors, see definition 4.
3 See definition 1.
4 Realistic errors are properly defined in section V A and are more formally referred to as “equable”. The equable assumption corresponds to ruling out two types of errors. 1) Extreme dephasing effects between a small set of states and the rest of the systems. 2) Extreme Hamiltonian alterations.
We structure the paper as follows. In section II, we introduce important characterization figures of merit – the average process fidelity and the unitarity – and relate them with the Kraus operator formalism. In section III, we define the LK approximation and present its aptitude in capturing important characteristics of evolving quantum circuits. In section IV, based on the emergent mathematical structure of LK approximated channels, we show the existence of a channel polar unitary-decoherent decomposition. In section V, we make use of the approximation to demonstrate key behavioral aspects of quantum circuits based on partial knowledge of their components.

For the sake of conciseness, most demonstrations are pushed to the appendix. Moreover, in the main text, certain results have been abridged by gathering higher order terms under the acronym “H.O.T.”. The complete expressions – which are not any more insightful than their abbreviated analog – are provided in the appendix.

II. CHANNEL PROPERTIES CAPTURED BY THE LEADING KRAUS OPERATOR

A quantum channel is a completely-positive (CP), trace-preserving (TP) map acting on \( M_d(\mathbb{C}) \). Given a quantum channel \( \mathcal{A} : M_d(\mathbb{C}) \to M_d(\mathbb{C}) \), the Choi matrix of \( \mathcal{A} \) is defined as \[ \text{Choi}(\mathcal{A}) := \sum_{ij} E_{ij} \otimes \mathcal{A}(E_{ij}) , \] (Choi matrix)

where \( E_{ij} := e_i e_j^\dagger \),

and \( e_i \) are canonical orthonormal vectors. The Choi matrix is positive semi-definite iff \( \mathcal{A} \) is CP, and has trace \( d \) if \( \mathcal{A} \) is TP or unital. Since \( \text{Choi}(\mathcal{A}) \geq 0 \), it has a spectral decomposition of the form

\[ \text{Choi}(\mathcal{A}) := d^2 \sum_{i=1}^{d^2} \text{col}(A_i) \text{col}^\dagger(A_i) , \]

\[ = \sum_{i=1}^{d^2} \|A_i\|_2^2 \text{col}(\bar{A}_i) \text{col}^\dagger(\bar{A}_i) , \]

where \( \text{col}(A) \in \mathbb{C}^{d^2} \) denotes the column vectorization of a matrix \( A \in M_d(\mathbb{C}) \) and \( \| \cdot \|_p \) denotes the Schatten \( p \)-norm. The eigenvectors \( \text{col}(\bar{A}_i) \) are orthonormal, an without loss of generality the eigenvalues are ordered with respect to the Frobenius norm (Schatten 2-norm):

\[ \|A_1\|_2^2 \geq \|A_2\|_2^2 \geq \cdots \geq \|A_{d^2}\|_2^2 \geq 0 . \]

Given a spectral decomposition like eq. (2), we can express the channel’s action on states \( \rho \in M_d(\mathbb{C}) \) as \[ \mathcal{A}(\rho) = \sum_{i=1}^{d^2} A_i \rho A_i^\dagger , \] (Kraus decomposition)

with

\[ \langle A_i, A_j \rangle = \|A_i\|_2^2 \delta_{ij} , \]

where the usual Hilbert-Schmidt inner product is used. Notice that the TP condition implies that \( \sum_i (\|A_i\|_2^2/d) = 1 \). The matrices \( A_i \in M_d(\mathbb{C}) \) are referred to as (ordered) canonical Kraus operators. In this work, \( A_1 \) (which is associated with the highest Choi matrix eigenvalue \( \|A_1\|_2^2 \)) will deserve special attention, and is attributed the title of “leading Kraus (LK) operator”. In general, \( A_1 \) might be non-unique when the spectrum of the Choi matrix is degenerate. However, in this work we focus on non-catastrophic channels (definition 1), for which \( A_1 \) is unique.

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5 A channel \( \mathcal{A} \) is unital iff \( \mathcal{A}(I_d) = I_d \).

6 \( \text{col}(A) := \sum_{ij} A_{ij} e_j \otimes e_i \)
Given an operation $A$ and a target unitary channel $U(\rho) = U\rho U^\dagger$ we can compare the overlap of their outputs given specific inputs $M \in M_d(\mathbb{C})$ through the $M$-fidelity:

$$f_M(A, U) := \frac{\langle A(M), U(M) \rangle}{\|M\|_2^2}.$$  \hfill (6)

The so-called average gate fidelity is obtained by averaging the $M$-fidelities uniformly\(^7\) over all physical pure states $|\psi\rangle\langle\psi|$

$$F(A, U) := \mathbb{E}_{\text{Haar}} f_{|\psi\rangle\langle\psi|}(A, U).$$  \hfill (7)

The average infidelity $r$ is simply a shorthand for $1 - F$. Instead of averaging over quantum states, we could also average uniformly over all operators $M \in M_d(\mathbb{C})$. More precisely, given any orthogonal operator basis $\{B_i\}$ for $M_d(\mathbb{C})$, we can uniformly average over the $M$-fidelities $f_{B_i}$, which yields the average process fidelity\(^7\)

$$\Phi(A, U) := \mathbb{E}_{\{B_i\}} f_{B_i}(A, U).$$  \hfill (8)

Compared to $\Phi$, $F$ puts a slightly higher weight over the identity component $I_d$. The TP condition enforces this special component to take a fixed value, $f_{I_d} = 1$. Hence the two quantities are closely related via \(^{27}\):

$$F(A, U) = \frac{d\Phi(A, U) + 1}{d + 1}. \hfill (9)$$

$F(A, U)$ is the overlap between the output state $A(\rho)$ of an implemented channel $A$ and its ideal output $U(\rho)$, averaged over all physical pure input states $|\psi\rangle\langle\psi|$. While $F(A, U)$ conveys a more graspable interpretation, it will remain easier here to work with $\Phi(A, U)$ since it ties with the Kraus operators through

$$\Phi(A, U) = \sum_{i=1}^{d^2} \left| \frac{A_i}{\sqrt{d}} , \frac{U}{\sqrt{d}} \right|^2 = \sum_{i=1}^{d^2} \left( \|A_i\|_2^2 / d \right) \left| \frac{\bar{A}_i}{d} , \frac{U}{\sqrt{d}} \right|^2. \hfill (10)$$

Since $\{\bar{A}_i\}$ forms an orthonormal basis and $\|U/d\|_2 = 1$, it follows that

$$\sum_{i=1}^{d^2} \left| \frac{\bar{A}_i}{d} , \frac{U}{\sqrt{d}} \right|^2 = 1. \hfill (11)$$

If $\|A_i\|_2^2/d$ can be thought as the “weights” of the Kraus operators, $\left| \frac{\bar{A}_i}{d} , \frac{U}{\sqrt{d}} \right|^2$ can be thought as normalized overlaps with the target $U$.

To quantify the coherence of a quantum channel, one could wonder how much the Bloch vectors (the traceless component of quantum states \(^{28}\)) are contracted. For instance, consider the unitarity, which is the squared length ratio of the Bloch vectors before and after the action of the channel $A$, averaged over all physical Bloch vector inputs corresponding to pure states $|\psi\rangle\langle\psi| - I_d/d$ \(^2\):

$$u(A) := \mathbb{E}_{\text{Haar}} \frac{\|A(\psi)\langle\psi| - I_d/d\|_2^2}{\|\psi\rangle\langle\psi| - I_d/d\|_2^2}. \hfill (12)$$

Let’s extend the domain of $\Phi$ to include a new function of $A$:

$$\Upsilon(A) := \Phi(A^\dagger A, I) = \Phi(A, A) = \sum_{i,j=1}^{d^2} \left| \frac{A_i^\dagger A_j}{\sqrt{d}} , \frac{I}{\sqrt{d}} \right|^2 = \sum_{i,j=1}^{d^2} \left| \frac{A_i}{\sqrt{d}} , \frac{A_j}{\sqrt{d}} \right|^2 = \sum_{i=1}^{d^2} \left( \frac{\|A_i\|_2^2}{d} \right)^2. \hfill (13)$$

Straightforward calculations closely relate the unitarity to $\Upsilon$ via

$$u(A) = \frac{d^2 \Upsilon(A) - 1}{d^2 - 1}. \hfill (14)$$

\(^7\) For unitaries, we used the calligraphic font to denote the channel and the non-calligraphic one to denote its associated $d \times d$ unitary matrix.

\(^8\) “Uniformly” is to be read here as “with respect to the Haar measure”.

\(^9\) For the readers familiar with the $\chi$-matrix, $\Phi(A, U)$ is a way to express the so-called $X_{00}$ element. Of course, the $\chi$-matrix has to be defined with respect to an orthonormal operator basis $\{B_i\}$ with $B_0 = U$. Some might also be more familiar with the notion of entanglement fidelity, which is again $\Phi$. 
(Notice that the notation alludes to the connection between greek and latin alphabets; it relates “phi” to “F” and “upsilon” to “u”.)

We are ready to express a first result:

**Lemma 1**

Consider a CPTP map $\mathcal{A}$ with ordered canonical Kraus decomposition

$$\mathcal{A}(\rho) = \sum_{i=1}^{d^2} A_i \rho A_i^\dagger.$$  

Then,

$$0 \leq \Upsilon(\mathcal{A}) - \left( \frac{\|A_1\|_2^2}{d} \right)^2 \leq (1 - \Upsilon(\mathcal{A}))^2. \hspace{1cm} (15)$$

**Proof.** $\Upsilon(\mathcal{A})$ can be expanded as a sum over $d^2$ terms:

$$\Upsilon(\mathcal{A}) = \sum_i \left( \frac{\|A_i\|_2^2}{d} \right)^2. \hspace{1cm} (16)$$

Using Hölder’s inequality on the RHS yields

$$\Upsilon(\mathcal{A}) \leq \max_i \frac{\|A_i\|_2^2}{d} \sum_j \frac{\|A_j\|_2^2}{d} = \frac{\|A_1\|_2^2}{d}. \hspace{1cm} (17)$$

Using this lower bound on $\|A_1\|_2$, we get

$$\Upsilon(\mathcal{A}) = \left( \frac{\|A_1\|_2^2}{d} \right)^2 + \sum_{i \neq 1} \left( \frac{\|A_i\|_2^2}{d} \right)^2 \leq \left( \frac{\|A_1\|_2^2}{d} \right)^2 + \sum_{i \neq 1} \left( \frac{\|A_i\|_2^2}{d} \right)^2 \leq \left( \frac{\|A_1\|_2^2}{d} \right)^2 + (1 - \Upsilon(\mathcal{A}))^2 \leq \left( \frac{\|A_1\|_2^2}{d} \right)^2 + (1 - \Upsilon(\mathcal{A}))^2 \hspace{1cm} (eq. (17))$$

From eq. (16), it follows that $\left( \frac{\|A_1\|_2^2}{d} \right)^2 \leq \Upsilon(\mathcal{A})$, which completes the proof. \hfill $\Box$

It follows from eq. (15) that $\Upsilon(\mathcal{A}) > 1/2$ is a sufficient condition to guarantee the uniqueness of $A_1^{10}$. This partially motivates the following definition:

**Definition 1: non-catastrophic channels**

A channel $\mathcal{A}$ is said to be non-catastrophic if it overlaps enough with its targeted unitary channel $\mathcal{U}$:

$$\Phi(\mathcal{A}, \mathcal{U}) > 1/2, \hspace{1cm} (19)$$

and if it doesn’t greatly contract the Bloch vectors:

$$\Upsilon(\mathcal{A}) > 1/2. \hspace{1cm} (20)$$

\[10\] Indeed, it implies that $\|A_1\|_2^2/d > 1/\sqrt{2} > 1/2$. 

The condition described by eq. (19) allows us to express our second result:

**Lemma 2**

Consider a non-catastrophic channel \( \mathcal{A} \) with unitary target \( \mathcal{U} \) and ordered canonical Kraus decomposition

\[
\mathcal{A}(\rho) = \sum_{i=1}^{d^2} A_i \rho A_i^\dagger.
\]

Then,

\[
0 \leq \Phi(\mathcal{A}, \mathcal{U}) - \left| \frac{A_1}{\sqrt{d}} \cdot \frac{U}{\sqrt{d}} \right|^2 \leq (1 - \Upsilon(\mathcal{A}))(1 - \Phi(\mathcal{A}, \mathcal{U})) .
\]  

(21)

**Proof.** Using Hölder’s inequality on the RHS of eq. (10), we have

\[
\Phi(\mathcal{A}, \mathcal{U}) \leq \max_i \left| \left< \bar{A}_i, U/\sqrt{d} \right> \right|^2 \sum_{j=1}^{d^2} (\|A_j\|_2^2/d) = \max_i \left| \left< \bar{A}_i, U/\sqrt{d} \right> \right|^2 .
\]  

(22)

For non-catastrophic channels, it must be that

\[
\max_i \left| \left< \bar{A}_i, U/\sqrt{d} \right> \right|^2 = \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2,
\]  

(23)

since any other arrangement would lead to a \( \Phi \) of at most \( 2\Phi(1 - \Phi) \), which is impossible for \( \Phi > 1/2 \). Hence, eq. (22) can be reexpressed into

\[
1 - \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2 \leq 1 - \Phi(\mathcal{A}, \mathcal{U}),
\]

which yields the following:

\[
\Phi(\mathcal{A}, \mathcal{U}) = \left( \|A_1\|_2^2/d \right) \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2 + \sum_{i \neq 1} \left( \|A_i\|_2^2/d \right) \left| \left< \bar{A}_i, U/\sqrt{d} \right> \right|^2
\]

\[
\leq \left( \|A_1\|_2^2/d \right) \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2 + \sum_{i \neq 1} \left( \|A_i\|_2^2/d \right) \left[ \left| \left< \bar{A}_i, U/\sqrt{d} \right> \right|^2 \right]
\]

\[
= \left( \|A_1\|_2^2/d \right) \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2 + \left( 1 - \left( \|A_1\|_2^2/d \right) \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2 \right) \left( 1 - \left( \Upsilon(\mathcal{A}) \right) \left( 1 - \Phi(\mathcal{A}, \mathcal{U}) \right) \right).
\]  

(eq. (17))

From eq. (10) we also have

\[
\left( \|A_1\|_2^2/d \right) \left| \left< \bar{A}_1, U/\sqrt{d} \right> \right|^2 \leq \Phi(\mathcal{A}, \mathcal{U}),
\]

which completes the proof.

The LK operator alone provides a very accurate approximation of \( 1 - \Phi \) and \( 1 - \Upsilon \). This only begins a list of realizations regarding the role of LK operators in quantum dynamics. As we will see, they also contain most of the information necessary to describe the **evolution** of \( \Phi \) and \( \Upsilon \).

### III. THE LK APPROXIMATION AND TWO EVOLUTION THEOREMS

The last section naturally suggests the following channel approximation as a mean to partially characterize non-catastrophic quantum dynamics:
Consider a channel $\mathcal{A} : M_d(\mathbb{C}) \to M_d(\mathbb{C})$ with ordered canonical Kraus decomposition

$$\mathcal{A}(\rho) = \sum_{i=1}^{d^2} A_i \rho A_i^\dagger.$$  \hfill (24)

We define its leading Kraus (LK) approximation as:

$$\mathcal{A}^*(\rho) = A_1 \rho A_1^\dagger.$$  \hfill (25)

Notice that $\mathcal{A}^*$ is always CP ($\text{Choi}(\mathcal{A}^*) \geq 0$), but is TP iff $\mathcal{A}$ is unitary. Hence, $\mathcal{A}^*$ fails to be generally physical. However, as we will see, it closely describes the dynamics of certain physical quantities, so one may qualify this map as “quasi-dynamical”. The general specification of a map acting on a $d$-dimensional quantum system requires roughly $d^4$ parameters, and due to the intricate geometry of quantum states, the parameterization of its range of action is quite convoluted. In contrast, the LK approximation is remarkably transparent: it is fully parameterized by $d$ parameters, and due to the intricate geometry of quantum states, the parameterization of its range of action is quite convoluted. In contrast, the LK approximation is remarkably transparent: it is fully parameterized by $d$ matrices with spectral radius smaller than 1 (contractions) and Frobenius norm greater than $d/\sqrt{2}$ \footnote{This last constraint only prevents catastrophic noise scenarios.}. If the noise is non-catastrophic, every quantum map has a corresponding LK approximation, and every $d \times d$ linear contraction corresponds to at least one quantum operator.

Given $m$ channels $\mathcal{A}_i$, we denote the composition $\mathcal{A}_m \circ \mathcal{A}_{m-1} \circ \cdots \circ \mathcal{A}_2 \circ \mathcal{A}_1$ as $\mathcal{A}_{m:1}$. Replacing every element of the composition by its LK approximation, $\mathcal{A}_m^* \circ \mathcal{A}_{m-1}^* \circ \cdots \circ \mathcal{A}_2^* \circ \mathcal{A}_1^*$, is noted as $\mathcal{A}_{m:1}^*$. In general, the composition operation doesn’t commute with the LK approximation, that is $\mathcal{A}_{m:1}^* \neq (\mathcal{A}_{m:1})^*$. To put it in other words, the LK operator of a circuit is generally not the multiplication of the LK operators of its elements. However, while $\mathcal{A}_{m:1}$ provides an incomplete description of $\mathcal{A}_{m:1}$, they still might share some comparable characteristics. That is, there might exist some function $f : \text{CP maps} \to \mathbb{R}$ for which $f(\mathcal{A}_{m:1}^*) \approx f(\mathcal{A}_{m:1})$. As we show, not only there exist such functions, but some correspond to important experimental figures of merit. From the previous section, we know that $\Phi(\mathcal{A}, \mathcal{U}) \approx \Phi(\mathcal{A}^*, \mathcal{U})$ and $\Upsilon(\mathcal{A}) \approx \Upsilon(\mathcal{A}^*)$. What may be more surprising are the following two theorems:

**Theorem 1: The unitarity of a circuit barely changes after LK approximating its elements**

Consider $m$ non-catastrophic channels $\mathcal{A}_i$ with respective unitary targets $\mathcal{U}_i$ and suppose that the composition $\mathcal{A}_{m:1}$ is also non-catastrophic. Then,

$$0 \leq \Upsilon(\mathcal{A}_{m:1}) - \Upsilon(\mathcal{A}_{m:1}^*) \leq (1 - \Upsilon(\mathcal{A}_{m:1}))^2 \leq (1 - \Upsilon(\mathcal{A}_{m:1}^*))^2.$$  \hfill (26)

**Theorem 2: The process fidelity of a circuit barely changes after LK approximating its elements**

Consider $m$ non-catastrophic channels $\mathcal{A}_i$ with respective unitary targets $\mathcal{U}_i$ and suppose that the composition $\mathcal{A}_{m:1}$ is also non-catastrophic. Then,

$$0 \leq \Phi(\mathcal{A}_{m:1}, \mathcal{U}_{m:1}) - \Phi(\mathcal{A}_{m:1}^*, \mathcal{U}_{m:1}) < \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(\mathcal{A}_k^*)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \Upsilon(\mathcal{A}_j^*)) \right) (1 - \Phi(\mathcal{A}_{m:1}^*, \mathcal{U}_{m:1})).$$  \hfill (27)

and

$$0 \leq \Phi(\mathcal{A}_{m:1}, \mathcal{U}_{m:1}) - \Phi(\mathcal{A}_{m:1}^*, \mathcal{U}_{m:1}) < \frac{5}{4} \left( \sum_{k=1}^{m} (1 - \Upsilon(\mathcal{A}_k^*)) \right)^2 + \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(\mathcal{A}_k^*)) \right)^3$$

$$+ \left( \sum_{j=1}^{m} (1 - \Upsilon(\mathcal{A}_j^*)) \right) (1 - \Phi(\mathcal{A}_{m:1}^*, \mathcal{U}_{m:1})).$$  \hfill (28)
\( \mathcal{A}^\ast \) differs from the veritable channel \( \mathcal{A} \) in many ways as shown by comparing various \( M \)-fidelities \( f_M(\mathcal{A}_{m:1}, \mathcal{U}_{m:1}) \) with \( f_M(\mathcal{A}_{m:1}^\ast, \mathcal{U}_{m:1}) \) (see fig. 1). Of course, some kind of discrepancy is expected since the LK approximation contains only \( d^2 \) parameters instead of \( \sim d^4 \). Essentially, the LK operators closely dictate the evolution of the average of \( M \)-fidelities \( \Phi = \mathbb{E} f_M \) (see eq. (8)), while the other Kraus operators add or subtract to specific \( M \)-fidelities \( f_M \) in such a way that the sum of those variations almost exactly cancels.

(a) Amplitude damping evolution:

(b) Random \( \mathcal{A}_i \) with high coherence levels:

FIG. 1: (If the videos are not accessible, try a better PDF viewer.) Comparison between \( \mathcal{A}_{m:1} \) and \( \mathcal{A}_{m:1}^\ast \) for different dynamical evolutions. On the left side, we represented the Bloch sphere with a scaled radius to accentuate decoherent effects. If \( 1 - \| \vec{r} \|_2 \geq 10^{-3} \), then

\[
\vec{r} \rightarrow \frac{-\log_{10}(1 - \| \vec{r} \|_2)}{3} \frac{\vec{r}}{\| \vec{r} \|_2}.
\]

The bar plot on the right illustrates the evolution of different \( M \)-infidelities (the four Pauli unitaries), together with the evolution of \( 1 - \Phi \). In a), the noise model consists in amplitude damping towards the \( |0 \rangle \) state. In b), each \( \mathcal{A}_i \) is sampled from a distribution of noisy operations with high coherence level (the notion of coherence level is defined in section V).
The evolution theorems presented in this section will greatly help classify different types of error\footnote{An error channel simply refers to a channel with identity target $I$.}. Indeed, they allow tying behavioral signatures in the evolution of $Y$ and $\Phi$ to more digestible error profiles. In particular, the two theorems further motivate, as shown in section \ref{sec:IV}, the definition of a natural dichotomy in quantum channels (itself introduced in section \ref{sec:IV}).

IV. A POLAR DECOMPOSITION FOR QUANTUM CHANNELS

Due to the intricate geometry of $d$-dimensional quantum states \cite{3}, quantum processes can be delicate to dissect. One of the main reasons the single qubit Bloch sphere is frequently invoked stems from the simple picture it offers:

i. There is a clear bijection between quantum states and the Bloch ball \cite{28}.

ii. The action on the Bloch vectors can be decomposed into a positive semi-definite contraction $|M| \leq I_3$, followed by orthogonal matrix $R \in O(3)$, which corresponds to a physical unitary $U \in SU(2)$, added to a translational vector $\vec{t}$ (the non-unital vector) \cite{29,31}:

$$\vec{v} \rightarrow R|M| \vec{v} + i\vec{t},$$

where $|M|$ denotes $(M^\dagger M)^{1/2}$. $M = R|M|$ is referred to the unital matrix.

Not every contraction $|M|$ is physical; for instance, transforming the Bloch sphere into a disk violates CP-ness (the “no pancake” theorem). A thorough analysis of CPTP maps acting on $M_2(\mathbb{C})$ is provided in \cite{30}. For higher dimensions, the Bloch sphere imagery falls apart in many ways:

i. The generalized Bloch space is not a $(d^2 - 1)$-ball (with respect to the 2-norm on $\mathbb{R}^{d^2 - 1}$) \cite{3}/

ii. If we express the action on the Bloch vector as in eq. (30) where $R \in O(d^2 - 1)$ and $|M| \geq 0$, we realize that

(a) $R$ generally doesn’t correspond to a physical unitary operation in $SU(d)$ (the unitary map defined by $\vec{v} \rightarrow R\vec{v}$ is not necessarily CP).

(b) $|M|$ is not necessarily a contraction. Its spectrum is optimally upper-bounded by $\sqrt{d/2}$ for even dimensions and $(\frac{1}{i+1} + \frac{1}{i+1})^{-1/2}$ for odd dimensions \cite{32}.

The polar decomposition of the unital matrix $M$ generally splits it into two nonphysical constituents. Essentially, the unitary factor of $M$ ($R \in O(d^2 - 1)$) s.t. $R^{-1}M \geq 0$ can’t generally be interpreted as a physically meaningful unitary operation. To see this, consider the following canonical Kraus decomposition:

$$A_1 = \begin{pmatrix} \cos(\alpha) & 0 & 0 \\ 0 & \cos(\alpha/2)e^{i\alpha^2/2} & 0 \\ 0 & 0 & \cos(\alpha/2)e^{-i\alpha^2/2} \end{pmatrix}; \quad A_2 = \begin{pmatrix} \sin(\alpha) & 0 & 0 \\ 0 & -\sin(\alpha/2)e^{i(\alpha+\alpha^2/2)} & 0 \\ 0 & 0 & -\sin(\alpha/2)e^{-i(\alpha+\alpha^2/2)} \end{pmatrix}.$$ \hspace{1cm} (31)

The spectrum of the associated unital part $M$ is a subset of the spectrum of $A_1^\dagger \otimes A_1 + A_2^\dagger \otimes A_2 \in M_{d^2}(\mathbb{C})$. By expanding up to order $\alpha^4$, it is straightforward to show that the phase factors of $M$ are all $\approx 1$ except for a single conjugate pair $\phi_{\pm} \approx \exp(\pm 3\alpha^3/2)$. This single pair can’t be factored into any unitary process since any non-trivial $U^* \otimes U$ contains at least two conjugate pairs. Hence, trying to cancel the rotating component of the spiraling action (see fig. 2) induced on $\vec{v}_{\pm}$ by $\phi_{\pm}$ would merely relocate the spiraling motion on an other pair of eigenvectors $\vec{v}'_{\pm}$ (or on multiple other pairs). To put it simply, spiraling is inherent to some decoherent processes. To explicitly show this, we constructed an example in which the rotation factors in the spirals couldn’t be accounted for by any physical unitary (without creating more spirals).

Separating a quantum channel $\mathcal{A}$ into a composition of a physical unitary $U$ and a decoherent operation $\mathcal{D}$ (i.e. $\mathcal{A} = U \circ \mathcal{D}$ or $\mathcal{A} = \mathcal{D} \circ U$) demands a more careful surgery. Allocate too much rotation factors to the process $U$
and it fails to remain physical; allocate too little and the allegedly decoherent constituent $\mathcal{D}$ may still contain some physically reversible motion. In fact, depending on the definition of decoherence, it is not even clear if such surgery is even possible. Here, we propose a definition of decoherence that allows to easily decompose any non-catastrophic quantum channel into a composition of a unitary channel with a decoherent one.

Consider a channel $\mathcal{A}$. Its LK operator $A_1 \in M_d(\mathbb{C})$ can be factored into a $d \times d$ unitary component $U$ multiplied with a positive semi-definite contraction $0 < |A_1| \leq I_d$, i.e. $A_1 = U|A_1|$. This polar decomposition provides a geometric understanding of the range of action of LK approximated channels on the space of quantum states. The absence of phase factors in the spectrum of $|A_1|$ motivates the following definition:

**Definition 3: decoherent channel**

A non-catastrophic channel $\mathcal{A}$ is said to be decoherent if its LK operator is positive semi-definite:

$$A_1 \geq 0 .$$

(32)

From this definition immediately follows a unitary-decoherent decomposition for quantum channels:

**Theorem 3: Unitary-decoherent polar decomposition for quantum channels**

Any non-catastrophic quantum channel $\mathcal{A}$ can be express as a composition of a unitary channel $U$ with an decoherent channel $\mathcal{D} = U^\dagger \circ \mathcal{A}$ (or $\mathcal{D}' = \mathcal{A} \circ U$):

$$\mathcal{A} = U \circ \mathcal{D} ,$$

(33a)

$$\mathcal{A} = \mathcal{D}' \circ U .$$

(33b)

In terms of LK approximation, we have:

$$\mathcal{A}^\star (\rho) = A_1 \rho A_1^\dagger = U|A_1|\rho|A_1|^\dagger U^\dagger ,$$

(34a)

$$\mathcal{D}^\star (\rho) = |A_1|\rho|A_1|^\dagger ,$$

(34b)

$$\mathcal{D}'^\star (\rho) = U|A_1|U^\dagger \rho U|A_1|^\dagger U^\dagger .$$

(34c)

**Proof.** Under the composition $U^\dagger \circ \mathcal{A}$, the canonical Kraus operators $\{A_i\}$ of $\mathcal{A}$ are mapped to $\{U^\dagger A_i\}$, since it preserves their orthonormality. Given the polar decomposition $A_1 = U|A_1|$, it follows that the LK operator of $U^\dagger \circ \mathcal{A}$ is positive semi-definite.

---

**FIG. 2:** Representation of the spiraling action of a normal matrix acting on a $2 \times 2$ subspace. The polar decomposition, in this case, separates the azimuthal and radial components of the action. Quantum dynamics on $d > 2$ can generate spiraling actions on the Bloch space for which the rotation factor can’t be interpreted as a physical unitary operation. In this sense, spiraling, despite generating some rotating action, is inherent to some decoherent dynamics.
While the proof nearly trivially follows from definition \[3\], it remains to show that decoherent channels as we defined them deserve such an appellation. An interesting angle to initially justify our definition of decoherence is to observe its contribution in the Gorini-Kossakowski-Sudarshan-Lindblad equation \[33, 34\]. Consider a time evolution dictated by instantaneous CPTP channels\[14\] with (possibly time-dependent) canonical Kraus operators \(\{A_k(t,dt)\}:
\[
\rho(t + dt) = \sum_k A_k(t,dt)\rho(t)A_k^\dagger(t,dt) .
\]
Since \(dt\) is infinitesimal, the instantaneous LK operator \(A_1(t,dt)\) must be close to \(I\), and can be expressed as
\[
A_1(t,dt) = \exp\left(-iH(t)dt - P(t)dt\right) = I - iH(t)dt - P(t)dt + O(dt^2) ,
\]
where \(H(t)\) is Hermitian and \(P(t)\) is positive semi-definite. The TP condition can be expressed as
\[
\sum_k A_k^\dagger(t,dt)A_k(t,dt) = I ,
\]
which combined with eq. \[36\] yields
\[
P(t)dt = \frac{1}{2} \sum_{k \neq 1} A_k^\dagger(t,dt)A_k(t,dt) + O(dt^2) .
\]
This enforces the remaining instantaneous Kraus operators \(A_{k \neq 1}(t,dt)\) to scale as \(\sqrt{dt}\), and leaves us with
\[
\frac{d}{dt}\rho(t) = -i[H(t),\rho(t)] + \sum_{k \neq 1} L_k(t)\rho(t)L_k^\dagger(t) - \frac{1}{2} \sum_{k \neq 1} L_k^\dagger(t)L_k(t),\rho(t)\right) ,
\]
where
\[
L_k(t) := \lim_{dt \to 0} \frac{A_k(t,dt)}{\sqrt{dt}} ,
\]
and \([A,B] := AB - BA, \{A,B\} := AB + BA\) are respectively the so-called commutator and anticommutator. The fact that \(\{A_k(t,dt)\}\) are canonical (hence orthogonal) at every moment in time implies that \(\langle A_1(t,dt), A_{k \neq 1}(t,dt)\rangle = 0\), which by using eq. \[36\] results in
\[
\text{Tr} A_{k \neq 1}(t,dt) = -idt \text{Tr} H(t)A_{k \neq 1}(t,dt) + dt\text{Tr} P(t)A_{k \neq 1}(t,dt) + O(dt^2\sqrt{dt}) .
\]
This together with eq. \[40\] implies that
\[
\text{Tr} L_k(t) = 0 .
\]
Notice that the Lindblad operators featuring in a master equation generally do not have a zero trace, but since the master eq. \[39\] is derived from instantaneous canonical Kraus operators, they do. That is, for every Lindblad master equation, there exists an alternate one, giving rise to the same dynamics, for which the Lindblad operators have a zero trace. This is an important feature for what follows. Let’s re-express eq. \[39\] as a differential equation acting on the column-vectorized states, \(\text{col}(\rho)\).

Using the property \(\text{col}(ABC) = C^T \otimes A \text{col}(B)\), we have
\[
\frac{d}{dt}\text{col}(\rho(t)) = \begin{bmatrix}
-i (1 \otimes H(t) - H^T(t) \otimes 1) - \frac{1}{2} \sum_{k \neq 1} \left(1 \otimes L_k^\dagger(t)L_k(t) + (L_k^\dagger(t)L_k(t))^T \otimes 1\right) + \sum_{k \neq 1} L_k^\dagger(t) \otimes L_k(t) \end{bmatrix} \text{col}(\rho(t)) .
\]
\[14\] This corresponds to the so-called Markovian regime.
A quick calculation suffices to show that the three indicated terms are mutually orthogonal. This means that their respective actions have no overlap. The first term should be familiar as it corresponds to the generator of unitary evolution. The remaining two terms are often referred to as the relaxation or decoherent part of the Lindbladian \cite{35, 36}. This integrates well with our notion of decoherence since the instantaneous channels are decoherent if and only if the Hamiltonian is null at every moment in time:

\[
\exp(-iH(t)dt - P(t)dt) \geq 0 \iff H(t) = 0. 
\] (44)

To formulate it otherwise, the Lindbladian consists solely of a decoherent part orthogonal to any commutator if and only if the instantaneous channels are decoherent. An additional interesting remark is that the LK approximation applied to the instantaneous channels essentially eliminates the term \(iii\), leaving only the commutator (term \(i\)) and the anticommutator (term \(ii\)). In particular, the master equation with LK approximated instantaneous decoherent channels consists of an anticommutator only:

\[
\frac{d}{dt}\rho(t) = -\{P(t), \rho(t)\}. 
\] (45)

When considered as infinitesimal perturbations from the identity, the channels that we refer to as “decoherent” correspond to the generators of the familiar class of decoherent master equations. While our notion of decoherence connects with previous physics literature in the infinitesimal case, it remains to show that our definition is also appropriate without taking such limit. As we will see in the next section, not only the definition is appropriate, but the polar decomposition that results from it (i.e. theorem 3) unravels a series of interesting realizations about \(\Phi(\mathcal{A}_{m:1}, \mathcal{U}_{m:1})\) and \(\Upsilon(\mathcal{A}_{m:1})\).

V. BEHAVIORAL SIGNATURES OF COHERENCE AND DECOHERENCE

The introduction in the previous section of the dichotomy between coherence and decoherence, together with the demonstration of a polar decomposition for quantum channels wasn’t void of ulterior motives. In this section, we leverage the intrinsic differences between coherent and decoherent channels to explore the behavior of the average process fidelity and the unitarity as circuits grow in depth. Before we begin such investigation, however, let’s first make a side step to define various classes of operations which will harmonize with our notion of decoherence.

A. Extremal dephasers, extremal unitaries, and equable error channels

The non-catastrophic condition still leaves room for pathological noise scenarios. We highlight two extreme (unrealistic) types of channel; the first is of decoherent nature, and the second is purely unitary.

1. Extremal dephasers

For a channel \(\mathcal{A}\) to be non-catastrophic, the singular values of its LK operator \(\sigma_i(A_1)\) must nearly average to 1, but nothing else constrains their distribution. Consider a 10-qubit error \(\mathcal{A}\) that essentially acts as identity on all operators in \(M_2(\mathbb{C})\), but cancels any phase between \(|0\rangle\) and \(|i\rangle\) for \(i \neq 0\) (that is, \(|0\rangle\langle i|, |i\rangle\langle 0| \to 0\) for \(i \neq 0\)). It is easily shown that the LK operator is \(A_1 = \sum_{i \neq 0} |i\rangle\langle i|\); this is an instance of what we call an “extremal dephaser”. An extremal dephaser is defined as a channel for which there exists a singular value \(\sigma_j \in \{\sigma_i(A_1)\}\) (in our example, it is \(\sigma_0 = 0\)) that deviates from the average by much more than the average perturbation:

\[
|\mathbb{E}_i[\sigma_i] - \sigma_j| \gg 1 - \mathbb{E}_i[\sigma_i]. 
\] (46)

To obey eq. (46), channels must involve excessively strong depthing mechanisms between a small number of states and the rest of the system.\(^{15}\) Let’s come back to our example: a quick calculation shows that \(\mathcal{A}\) has an infidelity of around \(O(2^{-10}) = O(10^{-3})\): extremal dephasers can have a high average fidelity; they are not ruled out by the non-catastrophic assumption. However, based on realistic grounds, one might discard such scenarios by assuming that

\(^{15}\) Relative to other decoherent mechanisms.

\(^{16}\) This is entirely different than: “excessively strong dephasing mechanisms between a small subsystem and the rest of the system”, which we already discarded through the non-catastrophic assumption.
the perturbations of the singular values $|\mathbb{E}_i[\sigma_i] - \sigma_j|$ remain comparable to the average perturbation $\mathbb{E}[1 - \sigma_j(A_1)]$. Indeed, most physically motivated noise mechanisms – such as unitary, amplitude damping and stochastic channels\footnote{A stochastic channel has (up to constant factors) unitary operations as canonical Kraus operators and has a LK operator proportional to the identity. Examples of orthogonal unitary bases include the Heisenberg-Weyl operators, and the n-fold tensor product of Paulis. Standard dephasing channels are a special case of stochastic channels where the unitaries are simultaneously diagonalizable (i.e. they all commute).} – perturb the singular values of $A_1$ in a rather homogeneous way (see table I).

2. Extremal unitaries

The same argument that was made about the singular values of $A_1 = U|A_1\rangle\langle 1|U$, which are the eigenvalues of its positive semidefinite factor $|A_1\rangle\langle 1|$, can be made for the eigenvalues of the unitary factor $U$. To mimic our previous example, consider a 10-qubit unitary error $U$ that essentially acts as identity on operators in $M_d(\mathbb{C})$, but maps $|0\rangle\langle i| \rightarrow -|0\rangle\langle i|$ and project the eigenvalues on the real axis (this is easy to picture on an Argand diagram):

$$| \mathbb{E}_i[\text{Re}\{\lambda_i\}] - \text{Re}\{\lambda_j\} | \gg 1 - \mathbb{E}_i[\text{Re}\{\lambda_i\}] = 1 - \text{Tr}U/d. \quad (47)$$

To obey eq. (47), the unitary error must result from a strong alteration made to the targeted Hamiltonian. Indeed, as a simple Taylor expansion can confirm, small perturbations from the intended Hamiltonian cannot yield an extremal unitary error. Just as for extremal dephasers, extremal unitaries can have a high average fidelity, yet can be reasonably discarded. The perturbations $1 - \text{Re}\{\lambda_i\}$ are expected to be comparable to the average perturbation $1 - \mathbb{E}_i[\text{Re}\{\lambda_i\}]$ (here, $\text{Tr}U \in \mathbb{R}_+$).

3. Equable error channels

In this paper, we qualify as “equable” the non-catastrophic error channels $A = U \circ D$ for which the factors $D$ and $U$ are not extremal. Notice that the equability assumption ensures a unique polar decomposition since the LK operator is guaranteed to be full rank.

While ruling out extremal error channels seems reasonable, we also define a weaker condition based on the variance of the perturbations:

**Definition 4: Wide sense equable (WSE) error channels**

Consider a non-catastrophic error channel $A = U \circ D$ with LK operator $A_1 = U|A_1\rangle\langle 1|U$. Let $\{\sigma_i\}$ be the singular values of $A_1$ and $\{\lambda_i\}$ be the eigenvalues of $U$ for which the phase is fixed such that $\text{Tr}U \in \mathbb{R}_+$. We define the WSE decoherence constant $\gamma_D$ as:

$$\text{Stand}[\sigma_i] = \gamma_D \mathbb{E}[1 - \sigma_i]. \quad (48)$$

We define the WSE coherence constant $\gamma_U$ as:

$$\text{Stand}[\text{Re}\{\lambda_i\}] = \gamma_U \mathbb{E}[1 - \text{Re}\{\lambda_i\}]. \quad (49)$$

A non-catastrophic error channel is said to be equable in the wide sense if

$$\gamma_D \ll 1/\sqrt{\mathbb{E}[1 - \sigma_i]}, \quad (50)$$

$$\gamma_U \ll 1/\sqrt{\mathbb{E}[1 - \text{Re}\{\lambda_i\}]. \quad (51)}$$

\begin{align}
|\mathbb{E}[\sigma_i] - \sigma_j| &\leq \gamma_D (1 - \mathbb{E}[\sigma_i]) \quad (52a) \\
|\mathbb{E}_i[\text{Re}\{\lambda_i\}] - \text{Re}\{\lambda_j\}| &\leq \gamma_U \mathbb{E}_i[1 - \text{Re}\{\lambda_i\}] \quad (52b)
\end{align}
for $\gamma_D, \gamma_U$ obeying eqs. (50) and (51), trivially implies equability in the wide sense. The WSE condition is weaker: for instance, there could be some singular values $\sigma_i(A_1)$ that widely differ from the average, but not enough of them to increase the variance in a substantial manner. Let’s modify our previous extremal dephaser example by adding some depolarizing component to $A$. Let $A_1 = \sum_i \epsilon_i |i\rangle \langle i|$. A quick calculation confirms that despite being an extremal dephaser, this channel is still equable in the wide sense.

The reason to invoke the WSE condition rather than simply ruling out extremal errors is twofold:

i. Wide sense equability is sufficient for deriving our results.

ii. Since the definitions of the WSE constants $\gamma_D, \gamma_U$ rely on simple statistics (i.e. variances and means), it seems conceivable to efficiently bound them experimentally (it’s always desirable to not discard verifiability if possible).

| Error channel      | Type of error          | LK operator | Coherence level, $r_{coh}/r$ |
|--------------------|------------------------|-------------|-------------------------------|
| Depolarizing       | Decoherent, WSE        | $A_1 \propto I$ | $O(r)$                      |
| Standard dephasing | Decoherent, WSE        | $A_1 \propto I$ | $O(r)$                      |
| Stochastic         | Decoherent, WSE        | $A_1 \propto I$ | $O(r)$                      |
| Amplitude damping  | Decoherent, realistically WSE | $A_1 \geq 0$ | $O(r)$                      |
| Unitary            | Coherent, realistically WSE | $A_1 = U$ | 1                           |
| General WSE        | Contains a coherent and decoherent factor | $A_1 = U|A_1|$ | $\frac{2^2-1}{|\text{Tr} U|^2} + O(r)$ |

TABLE I: Categorization of different well-known error channels. Many canonical error mechanisms fall under the “decoherent” appellation, except for unitary errors, of course. The coherence level is negligible for decoherent channels, and 1 for coherent errors. In the intermediate regime, the coherence level can vary between 0 and 1. It only makes sense to discuss about the coherence level when errors are WSE.

B. Reasoning about the unitarity

Now that we have defined (wide sense) equable errors, we are ready to express the following result:

**Theorem 4: The unitarity of composite channels**

Consider $m$ non-catastrophic channels $A_i$. Then $\Upsilon(A_{m:1})$ has the following properties:

$$\Upsilon(A_{m:1}) \leq \min_k \Upsilon(A_k) + (1 - \Upsilon(A_{m:1}))^2$$

(Quasi-monotonicity)

$$1 - \Upsilon(A_{m:1}) \leq \sum_k (1 - \Upsilon(A_k)) + 2 \sum_k (1 - \Upsilon(A_k))^2 + 2 \sum_k (1 - \Upsilon(A_k))^3 + \sum_k (1 - \Upsilon(A_k))^4.$$  

(Quasi-subadditive property)

Those inequalities are almost saturated by extremal channels. If we introduce the WSE decoherence constants $\gamma_D(A_i) \leq \gamma_D$, we obtain:

$$|\Upsilon(A_{m:1}) - \prod_i \Upsilon(A_i)| \leq \sum_{k=1}^m (1 - \Upsilon(A_k))^2 + (1 - \Upsilon(A_{m:1}))^2 + \frac{4\gamma_D}{\prod_i \sqrt{\Upsilon(A_k)}} \left(1 - \prod_i \sqrt{\Upsilon(A_k)}\right)^2 + \text{H.O.T.},$$

(53)

which guarantees quasi-multiplicativity of $\Upsilon$ when the errors are WSE.

Of course, those results can be immediately translated in terms of unitarity by using eq. (14). Without using the LK approximation, showing the monotonicity of the unitarity can be difficult, since quantum channels aren’t contractive maps; going to the LK picture fixes this issue since Kraus operators are contractions. Quasi-multiplicativity is another way of stating that the unitarity of a composition essentially behaves as a multiplicative decay involving the unitarity of individual components:

$$u(A_{m:1}) \approx \frac{d^2 \prod_{i=1}^m \Upsilon(A_i) - 1}{d^2 - 1}.$$  

(54)
Equation (54) should be seen as a staple of wide sense equability; deviations from this behavior indicates the presence of extremal dephasers. Another signature of wide sense equability can be seen through the following result:

**Theorem 5: Maximal average process fidelity through unitary correction**

Consider a non-catastrophic channel \( \mathcal{A} = \mathcal{V} \circ \mathcal{D} \) with unitary target \( \mathcal{U} \). Then, the maximal unitary correction of \( \mathcal{A} \) (in terms of \( \Phi \)) is approximately bounded by the interval \([\Upsilon(\mathcal{A}), \sqrt{\Upsilon(\mathcal{A})}]\):

\[
\max_{W \in SU(d)} \Phi(W \circ \mathcal{A}, \mathcal{U}) \leq \sqrt{\Upsilon(\mathcal{A})} + (1 - \Upsilon(\mathcal{A}))^2 + (1 - \Upsilon(\mathcal{A}))^3, \tag{55a}
\]

\[
\max_{W \in SU(d)} \Phi(W \circ \mathcal{A}, \mathcal{U}) \geq \Upsilon(\mathcal{A}) - (1 - \Upsilon(\mathcal{A}))^2. \tag{55b}
\]

Moreover, if we introduce the WSE decoherence constant \( \gamma_D \), we obtain:

\[
\max_{W \in SU(d)} \Phi(W \circ \mathcal{A}, \mathcal{U}) \geq \sqrt{\Upsilon(\mathcal{A})} - (1 - \Upsilon(\mathcal{A}))^2 - \gamma_D \left(1 - \sqrt{\Upsilon(\mathcal{A})}\right)^2. \tag{56}
\]

If the error attached to \( \mathcal{A} \) is WSE, then the interval virtually collapses and \( \max_{W \in SU(d)} \Phi(W \circ \mathcal{A}, \mathcal{U}) \approx \sqrt{\Upsilon(\mathcal{A})} \).

A quasi-maximal choice of unitary correction consists in \( W = \mathcal{U} \circ V^\dagger \).

Essentially, wide sense equability ensures a quasi-one-to-one correspondence between the maximal average gate fidelity (through a unitary correction) and the unitarity:

\[
\max_{V \in SU(d)} F(\mathcal{V} \circ \mathcal{A}, \mathcal{U}) \approx \frac{\sqrt{(d^2 - 1)u(\mathcal{A}) + 1} + 1}{d + 1}. \tag{57}
\]

**C. Justifying our notion of decoherence**

Let’s now return to our allegedly decoherent channels. Typically, quantum error channels are said to enact decoherently if they exhibit a non-reversible deterioration. In turn, coherent error channels correspond to a mishandling of information - which can in principle be reverted - rather than a loss of information. An additional expected property of decoherent operations is that they shouldn’t allow for coherent buildsups such as in the case accumulating over-rotations. Given \( m \) non-catastrophic unitary channels \( \mathcal{U}_i \approx I \) with

\[
U_i = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix},
\]

the infidelity grows faster than linearly (let the composition \( \mathcal{U}_{m:1} \) be non-catastrophic so that \( m\theta \leq \pi/4 \)) \( \Box \):

\[
1 - \sqrt{\Phi(\mathcal{U}_{m:1}, I)} = 1 - \cos (m\theta_i) \geq m (1 - \cos(\theta)) = \sum_i \left(1 - \sqrt{\Phi(\mathcal{U}_i, I)}\right). \tag{59}
\]

As an intuitive pair of properties of our so-called decoherent channels, we show that

i. The average process fidelity of decoherent error channels cannot be substantially recovered by any unitary (quasi-monotonicity).

ii. The evolution of the infidelity of a circuit composed of decoherent operations is (approximately) at most additive in the individual infidelities. There is no substantial coherent buildup.
Theorem 6: The average process fidelity of decoherent compositions

Consider \( m \) non-catastrophic decoherent channels \( D_i \) and any non-catastrophic unitary channel \( U \). Then,

\[
\Phi(U \circ D_m:1, I) \leq \min_k \Phi(D_k, I) + \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(D_k^*)) \right)^2 + \frac{1}{2} \left( \sum_{j=1}^{m} (1 - \Upsilon(D_j^*)) \right)^2 (1 - \Phi(U D_m^*:1, U m:1))
\]

(Quasi-monotonicity)

\[
1 - \Phi(D_m:1, I) \leq \sum_k (1 - \Phi(D_k, I)) + \sum_k (1 - \Phi(D_k, I))^2 + \sum_k (1 - \Phi(D_k, I))(1 - \Upsilon(D_k)) + \text{H.O.T.}
\]

(Quasi-subadditive property)

Moreover, by introducing the WSE condition we get:

Theorem 7: The average process fidelity of a composition of WSE decoherent errors

Consider \( m \) non-catastrophic, decoherent channels \( D_i \) (with target \( I \)) with WSE decoherence constants \( \gamma_D(D_i) \leq \gamma_D \). Then, \( \Phi(D_m:1, I) \) is bounded as follows:

\[
\left| \Phi(D_m:1, I) - \prod_{i=1}^{m} \Phi(D_i, I) \right| \leq \sum_k (1 - \Upsilon(D_k))(1 - \Phi(D_k, I)) + \sum_k (1 - \Upsilon(D_k^*)) (1 - \Phi(D_m^*:1, I))
\]

\[
+ \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(D_k^*)) \right)^2 \frac{\gamma_D}{\prod_{i=1}^{m} \Phi(D_i^*, I)} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(D_i^*, I)} \right)^2 + \text{H.O.T.}
\]

(61)

If the channels are WSE, \( \Phi(D_m:1, I) \) is essentially multiplicative.

Using the simple relation between \( F \) and \( \Phi \) (eq. (9)) we come to this observation: the average gate fidelity of a composition of non-catastrophic decoherent WSE channels behaves almost exactly as a multiplicative decay in the average process fidelity of individual components, that is

\[
F(D_m:1, I) \approx \frac{d \prod_{i=1}^{m} \Phi(D_i) + 1}{d + 1}.
\]

(62)

The decay becomes exact with the depolarizing channel \( \mathcal{P}_p(\rho) = p\rho + (1-p)(\text{Tr } \rho)\mathbb{I}_d/d \), which is a celebrated example of decoherent operation.

D. The coherence level

Let’s extend theorem 7 by appending a coherent operation to the decoherent composition:
The transition from eq. (64) to eq. (65) simply involves using the approximation (1
errors).
The channel average infidelity of a channel can be split into a sum of a coherent and decoherent terms (given WSE fidelity of a circuit. Consider
and polar decomposition
equality (in the WSE scenario) for all dimensions using the polar decomposition of LK operators.
qubit case in [37] using the polar decomposition of the action on Bloch sphere. Here, we have shown the (approximate)
equality – which is much more valuable since it provides an upper bound on
level introduced (under different appellations) in [37, 38]. In those previous works, the RHS of eq. (65) is generally
Similarly, the decoherence level is defined as
(65) motivates the definition of coherence level as the fraction of the infidelity that is associated to coherence. It can be obtained by combining
the infidelity and the unitarity through:

(67)

Similarly, the decoherence level is defined as \( r_{\text{decoh}} / r \). Equation (65) strengthen the insight behind notion of coherence level introduced (under different appellations) in [37, 38]. In those previous works, the RHS of eq. (63) is generally depicted as a lower bound on the infidelity, which can be reduced to \( r_{\text{decoh}} \) through a unitary correction. The (approximate) equality – which is much more valuable since it provides an upper bound on \( r \) – is shown for single qubit case in [37] using the polar decomposition of the action on Bloch sphere. Here, we have shown the (approximate) equality (in the WSE scenario) for all dimensions using the polar decomposition of LK operators.

E. Bounding the worst and best case fidelity of a circuit

Now, let’s revisit theorem 8 for general circuit depth \( m \). This will allow us to identify the worst and best case fidelity of a circuit. Consider \( m \) channels \( A_i \) with target \( U_i \) and polar decomposition \( D_i \circ V_i \). The circuit \( A_{m:1} \) can be

---

\( ^{18} \) The transition from eq. (64) to eq. (65) simply involves using the approximation \( (1 - \delta_1)(1 - \delta_2) \approx 1 - \delta_1 - \delta_2 \) for small \( \delta_i \).
re-expressed as
\[
A_{m:1} = V_{m:1} \circ (V_{m:1})^\dagger \circ D_{m} \circ V_{m:1} \circ (V_{m-1:1})^\dagger \circ D_{m-1} \cdots D_{2} \circ (V_{2:1}) \circ V_{1}^\dagger \circ D_{1} \circ V_{1} = V_{m:1} \circ D'_{m:1},
\]
where \(D'_k := (V_{k:1})^\dagger \circ D_k \circ V_{k:1}\) are decoherent channels with the same fidelity as \(D_k\). This means that:
\[
\Phi(A_{m:1}, U_{m:1}) = \Phi(V_{m:1} \circ D'_{m:1}, U_{m:1}) \mathop{\approx}^{\text{thm.8}} \Phi(V_{m:1}, U_{m:1}) \prod_{i=1}^{m} \Phi(D_i, I) \mathop{\approx}^{\text{thm.5}} \Phi(V_{m:1}, U_{m:1}) \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)}.
\]
In this last expression, we clearly see that the evolution of \(\Phi\) is factored into a decoherent decay multiplied by a function \(\Phi(V_{m:1}, U_{m:1})\) which captures the fidelity of a purely coherent process. This is already an interesting realization: since the decoherent decay is fixed, all the freedom in the evolution of the fidelity is contained in the coherent factors. An assessment concerning the circuit’s average process fidelity must rely on a characterization of coherent effects. Since we know that such effects are correctable through composition, we first get:

**Theorem 9: Maximal average process fidelity of channel compositions**

Consider \(m\) non-catastrophic channels \(A_i\) with respective unitary targets \(U_i\) and polar decompositions \(A_i = V_i \circ D_i\). Let the WSE decoherence constants be \(\gamma_D(D_i) \leq \gamma_D\). Then, the maximal unitary correction of the composition \(A_{m:1}\) is bounded as follows:

\[
\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) - \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)} \leq \frac{5}{4} \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \Upsilon(A_j)) \right) \left( 1 - \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)} \right)
\]

\[
+ \sum_{k=1}^{m} (1 - \Upsilon(A_k))^2 + \frac{2\gamma_D}{\prod_{i=1}^{m} \sqrt{\Upsilon(A_i)}} \left( 1 - \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)} \right)^2 + \text{H.O.T.}
\]

\[(70a)\]

\[
\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) - \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)} \geq - \gamma_D \sum_{k=1}^{m} (1 - \Upsilon(A_k))^2 - \sum_{k=1}^{m} (1 - \Upsilon(A_k))^2
\]

\[
- \frac{\gamma_D}{\prod_{i=1}^{m} \sqrt{\Upsilon(A_i)}} \left( 1 - \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)} \right)^2 + \text{H.O.T.}
\]

\[(70b)\]

For WSE errors, the maximal unitary correction of the composition \(A_{m:1}\) is essentially \(\prod_{i=1}^{m} \sqrt{\Upsilon(A_i)}\). A quasi-optimal choice of unitary correction is \(W = U_{m:1} \circ (V_{m:1})^\dagger\).

In short, the average gate fidelity of a composite circuit is upper bounded by a decaying envelope which is closely prescribed by the decoherent factors of its individual components:

\[
\max_{W \in SU(d)} F(W \circ A_{m:1}, U_{m:1}) \approx \frac{d \prod_{i=1}^{m} \Phi(D_i, I) + 1}{d + 1} \approx \frac{d \prod_{i=1}^{m} \sqrt{\Upsilon(A_i)} + 1}{d + 1}.
\]

\[(71)\]

This unforgiving behavior harmonizes well with the more typical comprehension of decoherence as a limiting process. To find the worst possible \(\Phi(A_{m:1}, U_{m:1})\), it suffices to use a lower bound for the coherent factor \(\Phi(V_{m:1}, U_{m:1})\). This is partially done in [1], where the inequality

\[
\Phi(V_{m:1}, U_{m:1}) \geq \cos^2 \left( \sum_{i=1}^{m} \arccos \left( \sqrt{\Phi(V_i, U_i)} \right) \right)
\]

\[(72)\]

is shown to be saturated in even dimensions. For odd dimensions, we find the following saturated bound:

\[
\Phi(V_{m:1}, U_{m:1}) \geq \left( \frac{(d - 1) \cos \left( \sum_{i=1}^{m} \arccos \left( \frac{d \sqrt{\Phi(V_i, U_i)} - 1}{d - 1} \right) \right) + 1}{d} \right)^2.
\]

\[(73)\]
Proof. The generalization to odd dimensions almost immediately follows by looking at the saturation case in even dimensions, which consists of commuting unitary errors of the form
\[
\begin{pmatrix}
\cos(\theta_i) & -\sin(\theta_i) \\
\sin(\theta_i) & \cos(\theta_i)
\end{pmatrix} \otimes I_{d/2}.
\] (74)

In the odd dimension case, it suffices to always pick the global phase to fix the first eigenvalue of \(V_{m:1}(U_{m:1})^{-1}\) to 1. The minimization over \(|\text{Tr} V_{m:1}(U_{m:1})|\) then falls back to the even dimensional case, since the saturation case has a real trace.

By using \(\Phi(V_i, U_i) \approx \Phi(A_i, U_i)/\sqrt{\gamma(A_i)}\) we can formulate a quasi-saturated assessment about the average process fidelity of the circuit \(A_{m:1}\) given a partial information about its components \(A_i\) (in the WSE scenario):
\[
\cos^2 \left( \sum_{i=1}^{m} \arccos \left( \frac{\Phi(A_i, U_i)}{\sqrt{\gamma(A_i)}} \right) \right) \prod_{i=1}^{m} \sqrt{\gamma(A_i)} \leq \Phi(A_{m:1}, U_{m:1}) \approx \prod_{i=1}^{m} \sqrt{\gamma(A_i)},
\]
\[
\begin{cases}
(Even \ dimensions) \\
(Odd \ dimensions)
\end{cases}
\]

The terms in the cosine function are very close to what was defined as “coherence angles” in \(\Pi\). Their sum can be interpreted as a coherent buildup. In some sense, the coherence angle is just another way to go about the notion of coherence level: it ties \(r_{\text{coh}}\) to an optimal rotation angle.

VI. CONCLUSION

In this work, we investigated a quasi-dynamical sub-parameterization of quantum channels that we referred to as the LK approximation. A remarkable realization is that this reduced picture still allows to closely follow the evolution of two important figures of merit, namely the average process fidelity and the unitarity.

Working with a simplified portrait sets aside superfluous subtleties and typically grants new mathematical properties to the object of consideration\(^{19}\). In our case, LK approximated mappings can be parameterized as contractions in \(M_d(\mathbb{C})\); this set of matrices offers a much more intelligible categorization of error scenarios than the more abstruse full process matrix parameterization. Any matrix \(A \in M_d(\mathbb{C})\) has a polar decomposition \(U|A|\) where \(|A| \geq 0\) and \(U\) is unitary. \(U\) corresponds to a purely coherent physical operation \(U(\rho) = U\rho U^\dagger\), whereas the positive contraction \(|A|\) is the LK operator belonging to what we classify as a decoherent channel. In a nutshell, the polar decomposition in \(M_d(\mathbb{C})\) translates into a coherent-decoherent factorization for quantum channels. We leveraged this dichotomy between types of noise to derive fundamental principles of behavior concerning our two considered figures of merit. Among other properties, we demonstrated, up to high precision, the general monotonicity of the unitarity as well as the monotonicity of the average process fidelity of circuits with decoherent components.

To pursue our analysis further, we introduced the wide sense equable parameters \(\gamma_D, \gamma_U\), which are defined through the LK parameterization. Wide sense equable (WSE) error channels, for which \(\gamma_D, \gamma_U\) are not too high, includes all realistic noise models (and more). Under the WSE condition, we make multiple interesting connections between individual channels and compositions thereof:

i. The infidelity of any channel can be decomposed into a sum of two terms: a decoherent infidelity and a coherent one (respectively tied to the decoherent/coherent components of the channel).

\(^{19}\) It is well known to physicists that approximating a cow as a spherical entity has its own advantages.
ii. The unitarity, as well as for the fidelity of circuits with decoherent elements, obey decay laws. Both these decays are closely dictated by the unitarity of individual components alone.

iii. The decoherent decay (that is, the decay prescribed by the decoherent factors of the circuit components) forms an upper bound to the total average process fidelity. Any substantial deviation from this upper bound is due to coherent effects alone (which gives us a lower bound).

This work was primarily cast as a stepping-stone to formulate assessments about the performance of circuits based on partial knowledge of their constituents. While we do provide some assertion formulas, we want to emphasize that the more fundamental introduction of the LK approximation should also benefit the development of further characterization schemes. Indeed, the simple parameterization offered by the LK approximation facilitates the identification of specific noise signatures.

| Concept | Definition | Notes |
|---------|------------|-------|
| Non-catastrophic channel | $\Phi(A, \mathcal{U}), \Upsilon(A) > 1/2$ | - Guarantees a unique LK operator. - Achieved given an acceptable level of control. |
| LK operator, $A_1$ | Highest weight canonical Kraus operator, $A_1$ | - Contains remarkable information about $\Phi, \Upsilon$. |
| LK approximation, $\mathcal{A}^*$ | $\mathcal{A}^*(\rho) = A_1 \rho A_1^\dagger$ | - Replacing channels by their LK approximation in a circuit barely affects its fidelity and unitarity. |
| Decoherent channel | $A_1 \geq 0$ | - Every non-catastrophic channel has a coherent-decoherent decomposition $A = \mathcal{U}_A \circ D_A = D_A' \circ \mathcal{U}_A$. - This definition of decoherence generalizes the notion of decoherence in the Lindblad picture. |
| Extremal dephaser (channel) | $\exists \sigma_j \in \{\sigma_i(A_1)\}$ s.t. $|E[|\sigma_j|] - |\sigma_j|| \gg 1 - E[|\sigma_j|]$ | - Strongly dephases a small set of states from the rest of the system. Since the set of states is small, extremal dephasers can still have high fidelity. |
| Extremal unitary (channel) | Let $\text{Tr} U \in \mathbb{R}_+$. $\exists \lambda_j \in \{\lambda_i(U)\}$ s.t. $|E[\text{Re}\{\lambda_j]\}] - |\text{Re}\{\lambda_j]\}| \gg 1 - E[|\text{Re}\{\lambda_j]\}]$ | - Strongly dephases a small set of states from the rest of the system. Since the set of states is small, extremal dephasers can still have high fidelity. |
| WSE decoherence constant, $\gamma_D$ | $\text{Stand}[\sigma_i(A_1)] = \gamma_D E[1 - |\sigma_i(A_1)|]$ | - For WSE channels, $\gamma_D \ll 1/\sqrt{E[1 - |\sigma_i|]}$. |
| WSE coherence constant $\gamma_U$ of unitary error $\mathcal{U}$ | Let $\text{Tr} U \in \mathbb{R}_+$. $|\text{Stand}[\text{Re}\{\lambda_i(U)\}]| = \gamma_U E[1 - |\text{Re}\{\lambda_i(U)\}|]$ | - For WSE channels, $\gamma_U \ll 1/\sqrt{E[1 - |\text{Re}\{\lambda_i\}|]}$. |
| Equable channel | Non-catastrophic, no extremal errors (dephasers and unitaries). | - Excludes pathological behaviors induced by extremal errors. - Should apply to all realistic scenarios. - Equable implies WSE. |
| Wide sense equable (WSE) channel | $\gamma_D \ll 1/\sqrt{E[1 - |\sigma_i|]}$, $\gamma_U \ll 1/\sqrt{E[1 - |\text{Re}\{\lambda_i}\}|}$. | - Ensures the quasi-correspondence: $\Phi(D_A, I) \approx \sqrt{\Upsilon(A)} \approx \max_{W \in SU(d)} \Phi(W \circ A, \mathcal{U})$. - Ensures the simple decay of the unitarity: $\Upsilon(A_{m_1}) \approx \prod \Upsilon(A_i)$. |
| Average gate fidelity, $F(A, \mathcal{U})$ | $E_{\text{Haast}} f_1 \langle \psi | \langle \psi | (A, \mathcal{U})$ | - Is the overlap between noisy and ideal outputs averaged over all physical inputs. |
| Unitarity, $u(A)$ | $E_{\text{Haast}} |\langle A|\psi\rangle\langle\psi| A|\rangle|^2 / |\psi\rangle\langle\psi| |A| |^2$ | - Is the average contraction factor of the squared norm of the physical Bloch vectors. |
| $\Phi(A, \mathcal{U})$ | $(d+1) F(A, \mathcal{U}) - 1$ | - For non-catastrophic channels, $\Phi(A_{m_1}, \mathcal{U}_{m_1}) \approx \Phi(A_{m_1}, \mathcal{U}_{m_1})$. - For channels $A_i = \mathcal{V}_i \circ D_i$ with WSE errors, $\Phi(A_{m_1}, \mathcal{U}_{m_1}) \approx \Phi(V_{m_1, i} \circ U_{m_1}) \prod \Phi(D_i, I)$. |
| $\Upsilon(A)$ | $(d^2 - 1) u(A) + 1$ | - For non-catastrophic channels, $\Upsilon(A_{m_1}) \approx \Upsilon(A_{m_1})$. - In the WSE scenario, $\Upsilon(A_{m_1}) \approx \prod \Upsilon(A_i)$. |
| Infidelity, $r(A, \mathcal{U})$ | $1 - F(A, \mathcal{U})$ | - For a channel $A = \mathcal{V} \circ D$, (with WSE error) $r = r_{\text{coh}} + r_{d\text{eoh}} + O(r^2)$, where $r_{\text{coh}} = r(V, \mathcal{U})$ and $r_{d\text{eoh}} = r(D, I)$. |
| Coherence level | $r_{\text{coh}}/r$ | - Quantifies the proportion to which the error is coherent. |

**TABLE II**: Summary of the main concepts addressed in this paper.

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[1] A. Carignan-Dugas, J. J. Wallman, and J. Emerson, arXiv e-prints, arXiv:1610.05296 (2016), arXiv:1610.05296 [quant-ph]
[2] J. J. Wallman, C. Granade, R. Harper, and S. T. Flammia, New J. Phys. 17, 113020 (2015), arXiv:arXiv:1503.0786
[3] I. Bengtsson and K. Zyczkowski, Geometry of Quantum States: An Introduction to Quantum Entanglement (Cambridge University Press, 2006).
[4] J. Emerson, R. Alicki, and K. Życzkowski, Journal of Optics B: Quantum and Semiclassical Optics 7, S347 (2005), arXiv:0503243 [quant-ph]
[5] C. Dankert, R. Cleve, J. Emerson, and E. Livine, (2006), arXiv:0606161v1
[6] E. Magesan, J. M. Gambetta, and J. Emerson, Physical Review Letters 106, 180504 (2011)
[7] E. Magesan, J. M. Gambetta, and J. Emerson, Physical Review A 85, 042311 (2012)
[8] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. B. Blakestad, J. D. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland, Physical Review A 77, 012307 (2008)
[9] E. Magesan, J. M. Gambetta, B. R. Johnson, C. A. Ryan, J. M. Chow, S. T. Merkel, M. P. da Silva, G. A. Keefe, M. B. Rothwell, T. A. Ohki, M. B. Ketchen, and M. Steffen, Physical Review Letters 109, 080505 (2012), arXiv:1204.4550
[10] J. M. Gambetta, A. D. Córcoles, S. T. Merkel, B. R. Johnson, J. A. Smolin, J. M. Chow, C. A. Ryan, C. Rigetti, S. Poletto, T. A. Ohki, M. B. Ketchen, and M. Steffen, Phys. Rev. Lett. 109, 240504 (2012), arXiv:1204.6308 [quant-ph]
[11] J. F. Gaebler, A. M. Meier, T. R. Tan, R. Bowler, Y. Lin, D. Hanneke, J. D. Jost, J. P. Home, E. Knill, D. Leibfried, and D. J. Wineland, Phys. Rev. Lett. 108, 260503 (2012)
[12] C. Granade, C. Ferrie, and D. G. Cory, New Journal of Physics 17, 1 (2014), arXiv:1404.5275v1
[13] R. Barends, J. Kelly, A. Veltia, A. Meignant, A. G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, I.-C. Ho, E. Jeffrey, C. Neill, P. J. J. O’Malley, J. Mutus, C. Quintana, P. Roushan, D. Sank, J. Wenner, T. C. White, A. N. Korotkov, A. N. Cleland, and J. M. Martinis, Phys. Rev. A 90, 030303 (2014)
[14] J. J. Wallman, M. Barnhill, and J. Emerson, arXiv e-prints, arXiv:1412.4126 (2014), arXiv:1412.4126 [quant-ph]
[15] A. Carignan-Dugas, J. J. Wallman, and J. Emerson, Physical Review A 92, 060302 (2015), arXiv:1508.06312 [quant-ph]
[16] J. J. Wallman, M. Barnhill, and J. Emerson, Phys. Rev. Lett. 115, 060501 (2015), arXiv:1510.01272 [quant-ph]
[17] S. Sheldon, L. S. Bishop, E. Magesan, S. Filipp, J. M. Chow, and J. M. Gambetta, Phys. Rev. A 93, 012301 (2016)
[18] A. W. Cross, E. Magesan, L. S. Bishop, J. A. Smolin, and J. M. Gambetta, npj Quantum Information 2, 16012 (2016), arXiv:1510.02720 [quant-ph]
[19] J. Combes, C. Granade, C. Ferrie, and S. T. Flammia, arXiv e-prints, arXiv:1702.03688 (2017), arXiv:1702.03688 [quant-ph]
[20] A. K. Hashagen, S. T. Flammia, D. Gross, J. J. Wallman, arXiv e-prints, arXiv:1801.06121 (2018), arXiv:1801.06121 [quant-ph]
[21] W. G. Brown and B. Eastin, Physical Review A 97, 062323 (2018), arXiv:1801.04042 [quant-ph]
[22] D. Sticcc França and A.-L. Hashagen, arXiv e-prints, arXiv:1803.03621 (2018), arXiv:1803.03621 [quant-ph]
[23] J. Helsen, X. Xue, L. M. K. Vandersypen, and S. Wehner, arXiv e-prints, arXiv:1806.02048 (2018), arXiv:1806.02048 [quant-ph]
[24] T. J. Proctor, A. Carignan-Dugas, K. Rudinger, E. Nielsen, R. Blume-Kohout, and K. Young, arXiv e-prints, arXiv:1807.07975 (2018), arXiv:1807.07975 [quant-ph]
[25] M.-D. Choi, Linear Algebra and its Applications 10, 285 (1975)
[26] K. Kraus, A. Böhm, J. Dollard, and W. Wootters, States, effects, and operations: fundamental notions of quantum theory : lectures in mathematical physics at the University of Texas at Austin Lecture notes in physics (Springer-Verlag, 1983).
[27] M. A. Nielsen, Physics Letters A 303, 249 (2002)
[28] F. Bloch, Phys. Rev. 70, 460 (1946)
[29] A. Fujiwara and P. Algoet, Phys. Rev. A 59, 3290 (1999)
[30] M. B. Ruskai, S. Szarek, and E. Werner, Linear Algebra and its Applications 347, 159 (2002)
[31] P. S. Bourdon and H. T. Williams, Phys. Rev. A 69, 022314 (2004)
[32] D. Pérez-Garcia, M. M. Wolf, D. Petz, and M. B. Ruskai, Journal of Mathematical Physics 47, 083506 (2006) [math-ph/0601063]
[33] G. Lindblad, Communications in Mathematical Physics 48, 119 (1976)
[34] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Journal of Mathematical Physics 17, 821 (1976) https://aip.scitation.org/doi/pdf/10.1063/1.522979
[35] R. Ernst, G. Bodenhausen, and A. Wokaun, Principles of Nuclear Magnetic Resonance in One and Two Dimensions International series of monographs on chemistry (Clarendon Press, 1987).
[36] T. F. Havel, Journal of Mathematical Physics 44, 534 (2003) [arXiv:quant-ph/0201127 [quant-ph]]
[37] G. Feng, J. J. Wallman, B. Buonacorsi, F. H. Cho, D. K. Park, T. Xin, D. Lu, J. Baugh, and R. Laflamme, Physical Review Letters 117, 260501 (2016), arXiv:1603.03761 [quant-ph]
[38] C. H. Yang, K. W. Chan, R. Harper, W. Huang, T. Evans, J. C. C. Hwang, B. Hensen, A. Laucht, T. Tanttu, F. E. Hudson, S. T. Flammia, K. M. Itoh, A. Morello, S. D. Bartlett, and A. S. Dzurak, arXiv e-prints, arXiv:1807.09500 (2018), arXiv:1807.09500 [cond-mat.mes-hall]
[39] B.-Y. Wang and M.-P. Gong. Linear Algebra and its Applications 184, 249 (1993)
Appendix A: A noteworthy trace inequality

This section is dedicated to demonstrating a useful trace inequality.

Lemma 3: Noteworthy trace inequality

Let $A, B \in M_d(\mathbb{C})$ be Hermitian matrices with eigenvalues of at most $\rho_A, \rho_B$ respectively. Then,

$$\frac{\text{Tr} AB}{d} \geq \rho_B \frac{\text{Tr} A}{d} + \rho_A \frac{\text{Tr} B}{d} - \rho_A \rho_B . \quad (A1)$$

Proof. We first show this inequality for positive semi-definite matrices with eigenvalues of at most 1, under the condition that

$$d < \lfloor \text{Tr} A \rfloor + \lfloor \text{Tr} B \rfloor + 2 . \quad (A2)$$

In such case, the inner product is minimized by the sum of eigenvalues paired in opposite order [39] (it’s a matrix equivalent to the Hardy-Littlewood rearrangement inequality):

$$\frac{\text{Tr} AB}{d} \geq \frac{1}{d} \sum_i \lambda^\uparrow_i(A) \lambda^\downarrow_i(B) . \quad (A3)$$

This is in turn minimized when both $\{\lambda_i(A)\}$ and $\{\lambda_i(B)\}$ are maximized in terms of strong majorization. Since the eigenvalues are between zero and 1, both majorizations have a simple form:

$$\lambda_i(A) = \begin{cases} 1 & \text{for } i \leq \lfloor \text{Tr} A \rfloor \\ \text{Tr} A - \lfloor \text{Tr} A \rfloor & \text{for } i = \lfloor \text{Tr} A \rfloor + 1 \\ 0 & \text{otherwise} \end{cases} \quad (A4)$$

$$\lambda_i(B) = \begin{cases} 1 & \text{for } i \leq \lfloor \text{Tr} B \rfloor \\ \text{Tr} B - \lfloor \text{Tr} B \rfloor & \text{for } i = \lfloor \text{Tr} B \rfloor + 1 \\ 0 & \text{otherwise} \end{cases} \quad (A5)$$

With such spectrum and the condition $d < \lfloor \text{Tr} A \rfloor + \lfloor \text{Tr} B \rfloor + 2$, we are ensured that

$$\frac{1}{d} \sum_i \lambda^\uparrow_i(A) \lambda^\downarrow_i(B) = \frac{\text{Tr} A}{d} + \frac{\text{Tr} B}{d} - 1 , \quad (A6)$$

which, together with eq. (A3), yields eq. (A1) in this simpler case.

Now, consider the general case of Hermitian matrices $A, B$ with eigenvalues of at most $\rho_A, \rho_B$ respectively. Let $A = (A + n_A \mathbb{I}) - n_A \mathbb{I}, B = (B + n_B \mathbb{I}) - n_B \mathbb{I}$, for $n_A, n_B \in \mathbb{R}_+$, and consider the following expansion:

$$\frac{\text{Tr} AB}{d} = \frac{\text{Tr}(A + n_A \mathbb{I})(B + n_B \mathbb{I})}{d} - n_A \frac{\text{Tr}(B + n_B \mathbb{I})}{d} - n_B \frac{\text{Tr}(A + n_A \mathbb{I})}{d} + n_A n_B$$

$$= (\rho_A + n_A)(\rho_B + n_B) \frac{\text{Tr} \left( \frac{A + n_A \mathbb{I}}{\rho_A + n_A} \right) \left( \frac{B + n_B \mathbb{I}}{\rho_B + n_B} \right)}{d} - n_A \frac{\text{Tr} B}{d} - n_B \frac{\text{Tr} A}{d} - n_A n_B \quad (A7)$$

Now, let’s pick $n_A, n_B$ large enough so that

i. $A + n_A \mathbb{I}, B + n_B \mathbb{I} \geq 0$ ,

ii. $d < \lfloor \text{Tr} \left( \frac{A + n_A \mathbb{I}}{\rho_A + n_A} \right) \rfloor + \lfloor \text{Tr} \left( \frac{B + n_B \mathbb{I}}{\rho_B + n_B} \right) \rfloor + 2$ .

For i we can simply pick $n_A \geq \min \lambda(A), n_B \geq \min \lambda(B)$. To see why ii is also possible, realize that

$$\lim_{n_A \to \infty} \left[ \text{Tr} \left( \frac{A + n_A \mathbb{I}}{\rho_A + n_A} \right) \right] = d , \quad (A8)$$
meaning that there exists a finite $n_A$ such that ii is fulfilled. Moreover, realize that the maximum eigenvalue of both $\frac{A+n_A I}{\rho_A+n_A}$ and $\frac{B+n_B I}{\rho_B+n_B}$ is upper-bounded by 1 by construction. Combining all this, we get

$$\frac{\text{Tr}(A+n_A I)}{\rho_A+n_A d} \frac{B+n_B I}{\rho_B+n_B} \geq \frac{\text{Tr}(A+n_A I)}{\rho_A+n_A d} + \frac{\text{Tr}(B+n_B I)}{\rho_B+n_B d} - 1,$$

since this corresponds to our initial simpler case. Substituting eq. (A9) into eq. (A7) and simplifying, we get eq. (A1) which completes the proof.

This inequality pairs well with the so-called Von-Neumann’s trace inequality, as when $\text{Tr} AB \geq 0$, lemma 3 provides a much better lower bound. To see this, consider the following inequality which is trivially derived from Von’s Neumann’s trace inequality:

**Lemma 4: Flavored Von Neumann’s trace inequality**

Let $A, B \in M_d(\mathbb{C})$ be matrices with spectral radius of at most $\rho_A, \rho_B$ respectively. Then,

$$|\frac{\text{Tr} AB}{d}| \leq \min \left( \rho_B \frac{\text{Tr}|A|}{d}, \rho_A \frac{\text{Tr}|B|}{d} \right).$$  (A10)

Recalling that $\|A\|_2^2 = \text{Tr} A^\dagger A$ and using those two last inequalities, we get the following norm inequality:

**Lemma 5: Norm inequality**

Consider two matrices $A, B$ with spectral radius of at most 1. Then,

$$\frac{\|A\|_2^2}{d} + \frac{\|B\|_2^2}{d} - 1 \leq \frac{\|AB\|_2^2}{d} \leq \min \left( \frac{\|A\|_2^2}{d}, \frac{\|B\|_2^2}{d} \right).$$  (A11)

### Appendix B: Proofs of the main results

#### 1. Notation and remarks

Before we start proving theorems 1 and 2, let’s introduce some handy notation. The $i^{th}$ canonical Kraus operator of a channel $A_j$ is denoted $A_{i}^j$. Let $a \geq b$; we denote

$$A_{i}^{a:b} = A_{i_{a-b-1}}^a A_{i_{a-b}}^{a-1} \cdots A_{i_1}^{b+1} A_{i_1}^b,$$  (B1)

where $\vec{i} \in \mathbb{N}^{a-b+1}$ simply contains indices $i_k \in \{1, \cdots, d^2\}$. Finally we denote $\vec{i} = (1, \cdots, 1)$ for which the dimension is left implicit.

Remark that the set $\{A_{\vec{i}}^{m:1}\}_{\vec{i}}$ consist in a valid Kraus decomposition for the composite channel $A_{m:1}$, and can be used to calculate $\Phi(A_{m:1}, U_{m:1})$ and $\Upsilon(A_{m:1})$ through eqs. (10) and (13) respectively. However, these Kraus operators are generally not orthogonal one another (this is not the canonical decomposition), which prevents the same proof technique as in lemmas 1 and 2.
2. Proof of the evolution theorem 1

Proof. Using Hölder’s inequality, we get

\[ \Upsilon(A_{m:1}) = \sum_i \left( \frac{\|A_i^{m:1}\|_2^2}{d} \right)^2 \]

(B2)

\[ \leq \max_i \frac{\|A_i^{m:1}\|_2^2}{d} \sum_j \frac{\|A_j^{m:1}\|_2^2}{d} \]  

(Hölder ineq.)

\[ = \max_i \frac{\|A_i^{m:1}\|_2^2}{d} . \]  

(TP condition)

One might have a (justified) hunch that \( \arg\max_{\vec{i}} \frac{\|A_i^{m:1}\|_2^2}{d} = \vec{1} \) in non-catastrophic noise scenarios. To show this, consider \( \vec{i} \) with \( i_k \neq 1 \) for some \( k \in \{1, \cdots, m\} \). Using the properties of contractions, we have

\[ \frac{\|A_i^{m:1}\|_2^2}{d} \leq \frac{\|A_k^{m:1}\|_2^2}{d} \]  

(Contractions)

\[ \leq 1 - \frac{\|A_k^{m:1}\|_2^2}{d} \]  

(TP condition)

\[ < 1/2 . \]  

(Non-catastrophic)

Hence, if we suppose \( \arg\max_{\vec{i}} \frac{\|A_i^{m:1}\|_2^2}{d} \neq \vec{1} \), we have

\[ \Upsilon(A_{m:1}) < 1/2 , \]  

(B3)

which cannot be respected if the channel \( A_{m:1} \) is non-catastrophic. Hence, by contradiction we have

\[ \Upsilon(A_{m:1}) \leq \frac{\|A_i^{m:1}\|_2^2}{d} . \]  

(B4)

From there we get

\[ \Upsilon(A_{m:1}) = \left( \frac{\|A_i^{m:1}\|_2^2}{d} \right)^2 + \sum_{\vec{i} \neq \vec{1}} \left( \frac{\|A_i^{m:1}\|_2^2}{d} \right)^2 \]

\[ \leq \Upsilon(A^{*}_{m:1}) + \left( \sum_{\vec{i} \neq \vec{1}} \frac{\|A_i^{m:1}\|_2^2}{d} \right)^2 \]

\[ = \Upsilon(A^{*}_{m:1}) + (1 - \Upsilon(A_{m:1}))^2 . \]  

(TP condition)

(Equation [B4])

From eq. (B2), it should be obvious that

\[ \Upsilon(A_{m:1}) \geq \left( \frac{\|A_i^{m:1}\|_2^2}{d} \right)^2 = \Upsilon(A^{*}_{m:1}) , \]  

(B5)

which completes the proof.

3. Proof of the evolution theorem 2

Proof. We will show that the inequality eq. (28) holds for \( m = 2^n, \forall n \in \mathbb{N} \). This suffices since if \( N < 2^n \), then we can append \( \mathcal{I}_{N-2^n:1} \) to the composition \( A_{N:1} \) so that \( A_{N:1} \circ \mathcal{I}_{N-2^n:1} \) is a composition of length \( 2^n \).Appending \( \mathcal{I}_{N-2^n:1} \) has no effect on eq. (28).
From the definition of $\Phi$, we have that $\Phi(A_{m1}, U_{m1}) - \Phi(A_{m1}', U_{m1}) \geq 0$, so it only remains to derive an upper bound on $\Phi(A_{m1}, U_{m1}) - \Phi(A_{m1}', U_{m1})$. Our approach will be to split the sum as follows:

$$
\frac{1}{d^2} \sum_{i \neq 1, j \neq 1} \left| \left\langle A_i^{m_{i1}+1}, U^{m_1} \right\rangle \right|^2 = \frac{1}{d^2} \sum_{i \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_i^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{j \neq 1} \left| \left\langle A_j^{m_{j1}+1} A_j^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{i \neq 1, j \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_j^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2. \tag{B6}
$$

The double sum (last term) can be bounded via Cauchy-Schwarz inequality followed by the usage of lemma 5.

$$
\frac{1}{d^2} \sum_{i \neq 1, j \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_j^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2 \leq \sum_{i \neq 1, j \neq 1} \frac{\| A_i^{m_{i1}+1} \|_2^2 \| A_j^{\frac{m}{2}+1} \|_2^2}{d^2} \leq \left( 1 - \frac{\| A_i^{m_{i1}+1} \|_2^2}{d^2} \right) \left( 1 - \frac{\| A_j^{\frac{m}{2}+1} \|_2^2}{d^2} \right) \quad \text{(Cauchy-Schwarz ineq.)}
$$

$$
\leq \left( \sum_{i=1}^{\frac{m}{2}+2} \left( 1 - \frac{\| A_i \|_2^2}{d^2} \right) \right) \left( \sum_{j=1}^{m/2} \left( 1 - \frac{\| A_j \|_2^2}{d^2} \right) \right) \quad \text{(TP condition)}
$$

$$
\leq \left( \sum_{i=1}^{\frac{m}{2}+2} \left( 1 - \Psi(A_i) \right) \right) \left( \sum_{j=1}^{m/2} \left( 1 - \Psi(A_j) \right) \right) \quad \text{(Lemma 5)} \tag{B7}
$$

With regards to the first two terms on the RHS of eq. (B6), let’s split them both into three terms once again:

$$
\frac{1}{d^2} \sum_{i \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_i^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2 = \frac{1}{d^2} \sum_{i \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_i^{\frac{m}{2}+1} A_i^{3m_{i1}}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{i \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_i^{\frac{m}{2}+1} A_i^{m_{i1}}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{i \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_i^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2. \tag{B8a}
$$

$$
\frac{1}{d^2} \sum_{j \neq 1} \left| \left\langle A_j^{m_{j1}+1} A_j^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2 = \frac{1}{d^2} \sum_{j \neq 1} \left| \left\langle A_j^{m_{j1}+1} A_j^{\frac{m}{2}+1} A_j^{3m_{j1}}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{j \neq 1} \left| \left\langle A_j^{m_{j1}+1} A_j^{\frac{m}{2}+1} A_j^{m_{j1}}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{j \neq 1} \left| \left\langle A_j^{m_{j1}+1} A_j^{\frac{m}{2}+1}, U^{m_1} \right\rangle \right|^2. \tag{B8b}
$$

The double sums on the RHS of eqs. (B8a) and (B8b) can be upper bounded using the same technique as earlier, which yields

$$
\frac{1}{d^2} \sum_{i \neq 1, j \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_j^{3m_{j1}}, U^{m_1} \right\rangle \right|^2 + \frac{1}{d^2} \sum_{i \neq 1, j \neq 1} \left| \left\langle A_i^{m_{i1}+1} A_j^{m_{j1}}, U^{m_1} \right\rangle \right|^2 \
\leq \left( \sum_{i=1}^{\frac{3m}{2}+1} \left( 1 - \Psi(A_i) \right) \right) \left( \sum_{j=1}^{\frac{m}{2}+1} \left( 1 - \Psi(A_j) \right) \right) + \left( \sum_{i=1}^{\frac{3m}{2}+1} \left( 1 - \Psi(A_i) \right) \right) \left( \sum_{j=1}^{\frac{m}{2}+1} \left( 1 - \Psi(A_j) \right) \right). \tag{B9}
$$
By iterating the same subdivision technique, we end up with
\[
\frac{1}{d^2} \sum_{i \neq 1} \left| \left\langle A^m_1, U^{m:1} \right\rangle \right|^2 \leq \sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} \left( \frac{2^n}{4^i - 2} (2^i - 2j + 2) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \left( \frac{2^n}{4^i - 2} (2^i - 2j + 1) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \right)
+ \frac{1}{d^2} \sum_{j=1}^{m} \left| \left\langle A^{m:j+1} A^{m:i-1:1}, U^{m:1} \right\rangle \right|^2 \tag{B10}
\]

Bounding the first term on the RHS can be done by alternating between the AM-GM inequality and square completions. First let’s perform the AM-GM inequality on the terms of the summation restricted to \( i = n \):
\[
\sum_{j=1}^{2^n-1} (1 - \Upsilon(A_{2^n-2j+2}))(1 - \Upsilon(A_{2^n-2j+1})) \leq \frac{1}{4} \sum_{j=1}^{2^n-1} \left( \sum_{k=2^n-2j+1}^{2^n-2j+2} (1 - \Upsilon(A_k)) \right)^2 \tag{B11}
\]

Then, let’s add in the terms with index \( i = n - 1 \) and complete the squares (taking \( n = 3 \) as an example is recommended):
\[
\sum_{j=1}^{n} \sum_{j=1}^{2^{i-1}} \left( \frac{2^n}{4^i - 2} (2^i - 2j + 2) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \left( \frac{2^n}{4^i - 2} (2^i - 2j + 1) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \right) \leq \frac{1}{4} \sum_{j=1}^{2^n-1} \left( \sum_{k=2^n-2j+1}^{2^n-2j+2} (1 - \Upsilon(A_k)) \right)^2 + \frac{1}{2} \sum_{j=1}^{2^n-2} \left( \sum_{k=2^{n-1}-2j+1}^{2^{n-1}-2j+2} (1 - \Upsilon(A_k)) \right) \left( \sum_{k=2^{n-1}-2j+1}^{2^{n-1}-2j+2} (1 - \Upsilon(A_k)) \right) \right) \tag{eq. B11}
\]

\[
\leq \frac{1}{4} \sum_{j=1}^{2^n-2} \left( \frac{2^{n-1}}{4^i - 2} (2^{n-1} - 2j + 2) \right) \sum_{k=2^{n-1}-2j+1}^{2^{n-1}-2j+2} (1 - \Upsilon(A_k)) \right)^2 + \frac{1}{2} \sum_{j=1}^{2^n-2} \left( \sum_{k=2^{n-1}-2j+1}^{2^{n-1}-2j+2} (1 - \Upsilon(A_k)) \right) \left( \sum_{k=2^{n-1}-2j+1}^{2^{n-1}-2j+2} (1 - \Upsilon(A_k)) \right) \right) \tag{Square completions}
\]

\[
\leq \frac{3}{2} \sum_{j=1}^{2^n-2} \left( \frac{2^{n-1}}{4^i - 2} (2^{n-1} - 2j + 2) \right) \sum_{k=2^{n-1}-2j+1}^{2^{n-1}-2j+2} (1 - \Upsilon(A_k)) \right)^2 \tag{AM-GM ineq.}
\]

Similarly, we can then add in the terms with index \( i = n - 2 \), complete the squares and use the AM-GM inequality on the leftover summation:
\[
\sum_{j=1}^{n} \sum_{j=1}^{2^{i-1}} \left( \frac{2^n}{4^i - 2} (2^i - 2j + 2) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \left( \frac{2^n}{4^i - 2} (2^i - 2j + 1) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \right) \leq \frac{7}{16} \sum_{j=1}^{2^n-3} \left( \sum_{k=4^{n-3}-2j+1}^{4^{n-2}-2j+2} (1 - \Upsilon(A_k)) \right)^2 \tag{B12}
\]

Repeating this procedure until \( i = 1 \), we get
\[
\sum_{i=1}^{n} \sum_{j=1}^{2^{i-1}} \left( \frac{2^n}{4^i - 2} (2^i - 2j + 2) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \left( \frac{2^n}{4^i - 2} (2^i - 2j + 1) \right) \sum_{k=\frac{2^n}{4^i - 2} (2^i - 2j + 1) + 1}^{\frac{2^n}{4^i - 2} (2^i - 2j + 2)} \left( 1 - \Upsilon(A_k) \right) \right) \leq \left( \frac{1}{2} - \frac{1}{2^{n+1}} \right) \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^2 \tag{B13}
\]
The last term on the RHS of eq. (B10) is upper-bounded using an alternate technique. First, we get

\[
\frac{1}{d^2} \sum_{j=1}^{j=m} \left| \langle A^m_{j+1} A_i^j A_i^{j-1:1}, U^{m:1} \rangle \right|^2 \leq \max_{j \neq 1} \left| \langle A^m_{j+1} A_i^j A_i^{j-1:1}, U^{m:1} \rangle \right|^2 \left( \sum_{k=1}^{k=m} \| A^k \|_2^2 \right)^{1/2}
\]  

(\text{Hölder’s ineq.)}

\[
= \max_{j \neq 1} \left| \langle A^m_{j+1} A_i^j A_i^{j-1:1}, U^{m:1} \rangle \right|^2 \sum_{k=1}^{m} \left( 1 - \frac{\| A^k \|_2^2}{d} \right) \quad (\text{TP condition})
\]

\[
\leq \max_{j \neq 1} \left| \langle A^m_{j+1} A_i^j A_i^{j-1:1}, U^{m:1} \rangle \right|^2 \sum_{k=1}^{m} \left( 1 - \Upsilon(A_k) \right) .
\]  

(B14)

For fixed \(j\), \(\{ A_i^j \}\) forms an orthonormal basis. Since \(\| (A^m_{j+1} A_i^j A_i^{j-1:1}) \|_2^2 \leq 1\) (contractions), we have that, for any \(j\):

\[
\max_{i \neq 1} \left| \langle A_i^j, (A^m_{j+1} A_i^j A_i^{j-1:1}) \rangle \right|^2 \leq 1 - \left| \langle A^m_{j+1} A_i^j A_i^{j-1:1}, U^{m:1} \rangle \right|^2
\]

\[
= 1 - \frac{\| A^m_{j+1} A_i^j A_i^{j-1:1}, U^{m:1} \|}{d \| A_i^j \|_2^2}
\]

\[
= 1 - \frac{\Phi(A^*_m, U^{m:1})}{\sqrt{\Upsilon(A_j)}}
\]

\[
\leq 1 - \Phi(A^*_m, U^{m:1}) .
\]  

(B15)

Combining eqs. (B10) and (B13) to (B15) yields

\[
\Phi(A_m, U^{m:1}) - \Phi(A^*_m, U^{m:1}) \leq \left( \frac{1}{2} - \frac{1}{2^n+1} \right) \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \Upsilon(A_j)) \right) \left( 1 - \Phi(A^*_m, U^{m:1}) \right) .
\]  

(B16)

To obtain eq. (27), it suffices to invoke \(\Upsilon(A_k) \geq \Upsilon(A^*_k)\) and \(1/2 - 1/2^{n+1} < 1/2\). To obtain eq. (28), we substitute \(\Phi(A^*_m, U^{m:1})\) by its lower bound:

\[
\Phi(A_m, U^{m:1}) - \Phi(A^*_m, U^{m:1}) \leq \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \Upsilon(A_j)) \right) \left( 1 - \Phi(A^*_m, U^{m:1}) \right)
\]

\[
+ \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^3 + \left( \sum_{j=1}^{m} (1 - \Upsilon(A_j)) \right)^2 \left( 1 - \Phi(A^*_m, U^{m:1}) \right) \quad (\text{B17})
\]

\[
< \frac{5}{4} \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \Upsilon(A_j)) \right) \left( 1 - \Phi(A^*_m, U^{m:1}) \right)
\]

\[
+ \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \Upsilon(A_k)) \right)^3 . \quad (\text{Non-catastrophic } \Rightarrow \Phi(A^*_m) > 1/4)
\]
4. Proof of theorem 6

Proof. First, we derive an upper bound for $\Phi(U \circ A_{m,1}, I)$:

$$\Phi(U \circ A_{m,1}, I) \leq \Phi(U \circ A_{m,1}^*, I) + \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \gamma(A_k^*)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \gamma(A_j^*)) \right) \left( 1 - \Phi(UA_{m,1}^*U, U_{m,1}) \right)$$

(= Theorem 3)

$$= \left| \frac{\text{Tr} U |A| \cdots |A|}{d} \right|^2 + \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \gamma(A_k^*)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \gamma(A_j^*)) \right) \left( 1 - \Phi(UA_{m,1}^*U, U_{m,1}) \right)$$

(= Lemma 4)

$$\leq \min_k \left| \frac{\text{Tr} |A_k| \cdots |A_k|}{d} \right|^2 + \frac{1}{2} \left( \sum_{k=1}^{m} (1 - \gamma(A_k^*)) \right)^2 + \left( \sum_{j=1}^{m} (1 - \gamma(A_j^*)) \right) \left( 1 - \Phi(UA_{m,1}^*U, U_{m,1}) \right),$$

(= Lemma 3)

which yield to the quasi-monotonicity statement. Now, we derive a lower bound for $\Phi(A_{m,1}, I)$:

$$1 - \sqrt{\Phi(A_{m,1}, I)} \leq 1 - \sqrt{\Phi(A_{m,1}^*, I)}$$

(= Theorem 3)

$$= 1 - \left| \frac{\text{Tr} |A| \cdots |A|}{d} \right|$$

(= Lemma 3)

$$\leq \sum_k \left( 1 - \sqrt{\Phi(A_k^*, I)} \right).$$

(B18)

Using $1 - x/2 - x^2/2 \leq \sqrt{1 - x} \leq 1 - x/2$ for $x \in [0, 1]$, we get

$$1 - \Phi(A_{m,1}, I) \leq \sum_k (1 - \Phi(A_k, I)) + \sum_k (1 - \Phi(A_k, I))^2 + \sum_k (1 - \Phi(A_k, I)) (1 - \gamma(A_k))$$

$$+ 2 \sum_k (1 - \Phi(A_k, I))^2 (1 - \gamma(A_k)) + \sum_k (1 - \Phi(A_k, I))^2 (1 - \gamma(A_k))^2,$$

(B19)

which corresponds to the quasi-subadditive property.

5. Proof of theorem 7

Proof. Given $m$ decoherent channels $A_i$ with respective LK operators $A_i^*$, we first want to bound the behavior of

$$\sqrt{\Phi(A_{m,1}^*, I)} = \frac{\text{Tr} |A| \cdots |A|}{d}$$

as function of individual $\sqrt{\Phi(A_i^*, I)}$. Let’s express the LK operators as $|A_i| = \sqrt{\Phi(A_i^*, I)} I_d + \Delta_i$, and apply a telescopic expansion:

$$\frac{\text{Tr} |A| \cdots |A|}{d} = \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} + \sum_{k=1}^{m} \frac{\text{Tr} |A| \cdots |A| \Delta_k}{d} \prod_{i=1}^{k-1} \sqrt{\Phi(A_i^*, I)}$$

(Telescopic sum)

$$= \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} + \left( \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right) \sum_{k=1}^{m} \frac{\text{Tr} \Delta_k}{d} / \sqrt{\Phi(A_k^*, I)}$$

$$+ \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \frac{\text{Tr} |A| \cdots |A| |\Delta_k \Delta_k \cdots \Delta_k|}{d} \prod_{i=1}^{\ell-1} \sqrt{\Phi(A_i^*, I)}$$

(Telescopic sum, again)
By construction, $\text{Tr} \Delta_i = 0$, which leaves us with

$$
\left| \sqrt{\Phi(A_{m;1}, \mathcal{I})} - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, \mathcal{I})} \right| \leq \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \frac{\text{Tr} |A_1^m| \cdots |A_{\ell+1}^m| \Delta_i \Delta_k}{d} \prod_{i \neq k}^{\ell-1} \sqrt{\Phi(A_i^*, \mathcal{I})}
$$

$$(\text{Triangle ineq.)}
$$

$$
\leq \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \frac{\text{Tr} |A_1^m| \cdots |A_{\ell+1}^m| \Delta_i \Delta_k}{d} \sum_{i \neq k}^{\ell-1} \sqrt{\Phi(A_i^*, \mathcal{I})}
$$

$$(\Phi(A_i^*, \mathcal{I}) \leq 1)
$$

$$
\leq \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \frac{||A_1^m| \cdots |A_{\ell+1}^m| \Delta_i \Delta_k||_2}{\sqrt{d}} \frac{||A_k||_2}{\sqrt{d}}
$$

$$(\text{Cauchy-Schwarz ineq.)}
$$

$$
\leq \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \frac{||A_i||_2 ||A_k||_2}{\sqrt{d}}
$$

$$(\text{Contractions})
$$

$$
\leq \gamma_D \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \left( 1 - \sqrt{\Phi(A_i^*, \mathcal{I})} \right) \left( 1 - \sqrt{\Phi(A_k^*, \mathcal{I})} \right)
$$

$$(\text{WSE decoh. cst})
$$

$$
\leq \gamma_D \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \left( 1 - \sqrt{\Phi(A_i^*, \mathcal{I})} \right) \left( 1 - \sqrt{\Phi(A_k^*, \mathcal{I})} \right) + \gamma_D \frac{1}{2} \sum_{k=1}^{m} \left( 1 - \sqrt{\Phi(A_k^*, \mathcal{I})} \right)^2
$$

$$(\text{Adding a positive term})
$$

$$
= \gamma_D \frac{1}{2} \left( \sum_{k=1}^{m} \left( 1 - \sqrt{\Phi(A_k^*, \mathcal{I})} \right) \right)^2
$$

$$(B21)
$$

$$
= \gamma_D \frac{1}{2} \prod_{i=1}^{m} \frac{1}{\Phi(A_i^*, \mathcal{I})} \left( \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, \mathcal{I})} \sum_{k=1}^{m} \left( 1 - \sqrt{\Phi(A_k^*, \mathcal{I})} \right) \right)^2
$$

$$(B22)
$$

$$
\leq \gamma_D \frac{1}{2} \prod_{i=1}^{m} \frac{1}{\Phi(A_i^*, \mathcal{I})} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, \mathcal{I})} \right)^2
$$

$$(B23)
$$

On the last line, we used the fact that

$$
\left( \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, \mathcal{I})} \right) \left( 1 - \sqrt{\Phi(A_k^*, \mathcal{I})} \right) \leq \prod_{i=1}^{k-1} \sqrt{\Phi(A_i^*, \mathcal{I})} \prod_{i=1}^{k} \sqrt{\Phi(A_i^*, \mathcal{I})} - \prod_{i=1}^{k} \sqrt{\Phi(A_i^*, \mathcal{I})}.
$$

$$(B24)
$$

A few straightforward algebraic manipulations on eq. \text{(B23)} yield

$$
\left| \Phi(A_{m;1}, \mathcal{I}) - \prod_{i=1}^{m} \Phi(A_i^*, \mathcal{I}) \right| \leq \frac{\gamma_D}{\prod_{i=1}^{m} \Phi(A_i^*, \mathcal{I})} \left( \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, \mathcal{I})} \right)^2
$$

$$
+ \frac{\gamma_D^2}{4 \prod_{i=1}^{m} \Phi(A_i^*, \mathcal{I})^2} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, \mathcal{I})} \right)^4
$$

$$(B25)
$$

Using a simple telescopic expansion and lemma \text{[2]} we have

$$
\prod_{i=1}^{m} \Phi(A_i, \mathcal{I}) - \prod_{i=1}^{m} \Phi(A_i^*, \mathcal{I}) = \sum_{k} \left( \prod_{i=k+1}^{m} \Phi(A_i, \mathcal{I}) \right) \left( \Phi(A_k, \mathcal{I}) - \Phi(A_k^*, \mathcal{I}) \right) \left( \prod_{i=1}^{k-1} \Phi(A_i^*, \mathcal{I}) \right)
$$

$$
\leq \sum_{k} \left( 1 - \gamma(A_k) \right) \left( 1 - \Phi(A_k, \mathcal{I}) \right).
$$

$$(B26)
$$

Applying this together with theorem \text{[2]} on the LHS of eq. \text{(B25)} yields eq. \text{(61)}. 

\square
6. Proof of theorem 4

Proof. First, we derive an upper bound for $\Upsilon(\mathcal{A}_{m;1})$:

$$
\Upsilon(\mathcal{A}_{m;1}) \leq \Upsilon(\mathcal{A}_{m;1}^*) + (1 - \Upsilon(\mathcal{A}_{m;1}))^2 \quad \text{(Theorem 1)}
$$

$$
= \left( \frac{\|A^{m:1}_{1}\|_2^2}{d} \right)^2 + (1 - \Upsilon(\mathcal{A}_{m;1}))^2
$$

$$
\leq \min_k \left( \frac{\|A_k\|_2^2}{d} \right)^2 + (1 - \Upsilon(\mathcal{A}_{m;1}))^2 \quad \text{(Lemma 5)}
$$

$$
\leq \min_k \Upsilon(\mathcal{A}_k) + (1 - \Upsilon(\mathcal{A}_{m;1}))^2, \quad \text{(Lemma 1)}
$$

which corresponds to the quasi-monotonicity statement. We then derive a lower bound on $\Upsilon(\mathcal{A}_{m;1})$:

$$
1 - \sqrt{\Upsilon(\mathcal{A}_{m;1})} \leq 1 - \sqrt{\Upsilon(\mathcal{A}_{m;1}^*)} \quad \text{(Theorem 1)}
$$

$$
= 1 - \frac{\|A^{m:1}_{1}\|_2^2}{d}
$$

$$
\leq \sum_k \left( 1 - \frac{\|A_k\|_2^2}{d} \right) \quad \text{(Lemma 5)}
$$

$$
\leq \sum_k \left( 1 - \sqrt{\Upsilon(\mathcal{A}_k)} - (1 - \Upsilon(\mathcal{A}_k))^2 \right) \quad \text{(Lemma 1)}
$$

Using $1 - x/2 - x^2/2 \leq \sqrt{1-x} \leq 1 - x/2$ for $x \in [0,1]$, we get

$$
1 - \Upsilon(\mathcal{A}_{m;1}) \leq \sum_k (1 - \Upsilon(\mathcal{A}_k)) + 2 \sum_k (1 - \Upsilon(\mathcal{A}_k))^2 + 2 \sum_k (1 - \Upsilon(\mathcal{A}_k))^3 + \sum_k (1 - \Upsilon(\mathcal{A}_k))^4, \quad \text{(B27)}
$$

which corresponds to the quasi-subadditive property. To derive the approximate multiplicativity statement, let’s factor the decoherent channels into their (left) polar decomposition $\mathcal{A}_i = \mathcal{D}_i \circ \mathcal{V}_i$. By relabeling $(\mathcal{V}_{i:1})^{-1} \circ \mathcal{D}_i \circ \mathcal{V}_{i:1} = \mathcal{D}_i'$, we have

$$
\sqrt{\Upsilon(\mathcal{A}_{m;1}^*)} = \sqrt{\Phi(D_1^* \cdots D_m^* D_{m;1}^*, I)}, \quad \text{(B28)}
$$

From the definition, we have that $\sqrt{\Upsilon(\mathcal{A}_i^*)} - \Phi(D_i^*, I) \leq \gamma_D \left( 1 - \sqrt{\Phi(D_i^*, I)} \right)^2$. We can use a telescopic expansion to get

$$
\prod_i \sqrt{\Upsilon(\mathcal{A}_i^*)} - \prod_i \Phi(D_i^*, I) \leq \gamma_D \sum_{k=1}^m \left( \prod_{i=k+1}^m \Phi(D_i^*, I) \right) \left( 1 - \sqrt{\Phi(D_k^*, I)} \right)^2 \left( \prod_{i=1}^{k-1} \sqrt{\Upsilon(\mathcal{A}_i^*)} \right) \quad \text{(B29)}
$$

$$
\leq \gamma_D \sum_k \left( 1 - \sqrt{\Phi(D_k^*, I)} \right)^2 \quad \text{(B30)}
$$
Using this and going through the same telescoping expansion as featured in the proof of theorem[7] we get
\[
\left| \sqrt{\Upsilon(A_{m:1})} - \prod_i \sqrt{\Upsilon(A_i^*)} \right| \leq 2\gamma_D \sum_k \left( 1 - \sqrt{\Phi(D_k^*, I)} \right)^2 + 4\gamma_D \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \left( 1 - \sqrt{\Phi(D_k^*, I)}(1 - \sqrt{\Phi(D_k^*, I)}) \right)
\]
\[
= 2\gamma_D \left( \sum_{k=1}^{m} \left( 1 - \sqrt{\Phi(D_k^*, I)} \right)^2 \right)^2 \quad \text{(Complete the square)}
\]
\[
\leq 2\gamma_D \left( \sum_{k=1}^{m} \left( 1 - \sqrt{\Upsilon(A_k^*)} \right)^2 \right)^2 \quad \text{(Complete the square)}
\]
\[
= \frac{2\gamma_D}{\prod_{i=1}^{m} \sqrt{\Upsilon(A_i^*)}} \left( \prod_{i=1}^{m} \Upsilon(A_i^*) \right)^2 \left( \prod_{k=1}^{m} \Upsilon(A_k^*) \right)^2 \quad \text{(B31)}
\]
where on the last line we used
\[
\left( \prod_{i=1}^{m} \Upsilon(A_i^*) \right) \left( 1 - \sqrt{\Upsilon(A_i^*)} \right) \leq \prod_{i=1}^{k-1} \sqrt{\Upsilon(A_i^*)} - \prod_{i=1}^{k} \sqrt{\Upsilon(A_i^*)} \quad \text{(B32)}
\]
A few straightforward algebraic manipulations on eq. (B31) yield
\[
\left| \Upsilon(A_{m:1}) - \prod_i \Upsilon(A_i^*) \right| \leq \frac{4\gamma_D}{\prod_i \sqrt{\Upsilon(A_i^*)}} \left( \prod_i \Upsilon(A_i^*) \right)^2 + \frac{4\gamma_D^2}{\prod_i \Upsilon(A_i^*)^2} \left( \prod_i \Upsilon(A_i^*) \right)^4 \quad \text{(B33)}
\]
To remove the LK approximations on the LHS, we first use
\[
\prod_i \Upsilon(A_i^*) - \prod_i \Upsilon(A_i) = \sum_{k=1}^{m} \Upsilon(A_i)(\Upsilon(A_k^*) - \Upsilon(A_k)) \prod_{i=1}^{k-1} \Upsilon(A_i) \quad \text{(B34)}
\]
\[
\geq \sum_{k=1}^{m} (\Upsilon(A_k^*) - \Upsilon(A_k)) \quad \text{(B35)}
\]
\[
\geq \sum_{k=1}^{m} (1 - \Upsilon(A_k))^2 \quad \text{(B36)}
\]
followed by a usage of theorem[1] to translate \(\Upsilon(A_{m:1}^*)\) into \(\Upsilon(A_{m:1})\).

\[\square\]

7. Proof of theorem[8]

Proof. Let’s compose a decoherent sequence with a unitary operation:
\[
\sqrt{\Phi(U \circ A_{m:1}^*)} = \left| \frac{\text{Tr} U |A_1^m\rangle \cdots |A_1^1\rangle}{d} \right| . \quad \text{(B37)}
\]
WOLOG, we pick the global phase of \(U\) such that \(\sqrt{\Phi(U, I)} = \text{Tr} U/d \in \mathbb{R}_+\). Our goal is to bound
\[
\left| \frac{\text{Tr} U |A_1^m\rangle \cdots |A_1^1\rangle}{d} \right|^2 = \left| \text{Re} \left( \frac{\text{Tr} U |A_1^m\rangle \cdots |A_1^1\rangle}{d} \right) \right|^2 + \left| \text{Im} \left( \frac{\text{Tr} U |A_1^m\rangle \cdots |A_1^1\rangle}{d} \right) \right|^2 . \quad \text{(B38)}
\]
Let’s first bound the amplitude of the imaginary term:

$$\left| \text{Im} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i|}{d} \right\} \right| = \left| \text{Im} \left\{ \frac{\text{Tr} \left( (U - \text{Tr}(U)/d) I \right) |A_i^n| \cdots |A_i|}{d} \right\} \right| $$

(Adding real terms.)

$$\leq \frac{\text{Tr} \left( (U - \text{Tr}(U)/d) I \right) |A_i^n| \cdots |A_i|}{d} \leq \frac{\|U - \text{Tr}(U)/d\|_2}{\sqrt{d}} \frac{\|A_i^n| \cdots |A_i|\|_2}{\sqrt{d}}$$

(Cauchy-Schwarz ineq.)

$$= \sqrt{1 - \Phi(U, I)} \sqrt{\sum_{i=m}^{1} A_i^n} - \Phi(A,m+1, I) \tag{B39}$$

By combining eqs. (B25), (B30) and (B31), we get

$$\left| \text{Im} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i|}{d} \right\} \right|^2 \leq \left[ \left( 1 - \frac{\gamma_D}{\prod_i \sqrt{\Phi(A_i^n, I)}} \right)^2 + \frac{\gamma_D^2}{4 \prod_i \Phi(A_i^n, I)^2} \right] \left( 1 - \frac{\gamma_D}{\prod_i \sqrt{\Phi(A_i^n, I)}} \right)^4$$

meaning that the imaginary term is absolutely insignificant. To bound the real part of the trace, we mimic most of the proof technique used to prove theorem $[7]$. Let’s express the LK operators as $|A_i^n| = \sqrt{\Phi(A_i^n, I)} I_d + \Delta_i$ and $U = \Phi(U, I) I_d + \Delta_{m+1}$, and apply a telescopic expansion twice:

$$\text{Re} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i|}{d} \right\} = \sqrt{\Phi(U, I) I_d \prod_i \sqrt{\Phi(A_i^n, I)}} + \sum_{k=1}^{m} \text{Re} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i| \Delta_k}{d} \right\} \prod_{i=1}^{k} \sqrt{\Phi(A_i^n, I)}$$

(Telescopic sum)

$$= \sqrt{\Phi(U, I) I_d \prod_i \sqrt{\Phi(A_i^n, I)}} + \sum_{k=1}^{m} \sqrt{\Phi(U, I) I_d \prod_i \sqrt{\Phi(A_i^n, I)}} \text{Re} \left\{ \frac{\text{Tr} \Delta_k}{d} \right\} / \sqrt{\Phi(A_i^n, I)}$$

$$+ \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \text{Re} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i| \Delta_i \Delta_k}{d} \right\} \prod_{i=1}^{\ell-1} \sqrt{\Phi(A_i^n, I)}$$

(Telescopic sum, again)

By construction, $\text{Tr} \Delta_i = 0$, which leaves us, after a simple application of the triangle inequality, with

$$\left| \text{Re} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i|}{d} \right\} \right| - \sqrt{\Phi(U, I) I_d \prod_i \sqrt{\Phi(A_i^n, I)}} \leq \sum_{k=1}^{m} \sum_{\ell=k+1}^{m} \text{Re} \left\{ \frac{\text{Tr} U |A_i^n| \cdots |A_i| \Delta_i \Delta_k}{d} \right\} \prod_{i=1}^{\ell-1} \sqrt{\Phi(A_i^n, I)}$$

$$+ \sum_{k=1}^{m} \text{Re} \left\{ \frac{\text{Tr} \Delta_{m+1} \Delta_k}{d} \right\} \prod_{i=1}^{m} \sqrt{\Phi(A_i^n, I)} \tag{B41}$$
The first term on the LHS is upper-bounded by \( \frac{\gamma_D}{D} \prod_{i=1}^{D} \frac{\Phi(A_i^*, I)}{\Phi(A_i^*, I)} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right)^2 \) by using the same derivation as for eq. (B23). The second term is bounded as follows:

\[
\left| \sum_{k=1}^{m} \Re \left\{ \frac{\text{Tr} \Delta_{m+1} \Delta_k}{d} \right\} \prod_{i \neq k}^{m} \sqrt{\Phi(A_i^*, I)} \right| \leq \sum_{k=1}^{m} \Re \left\{ \frac{\text{Tr} \Delta_{m+1} \Delta_k}{d} \right\} \prod_{i \neq 1}^{m} \sqrt{\Phi(A_i^*, I)} \quad \text{(Triangle ineq.)}
\]

\[
\leq \sum_{k=1}^{m} \Re \left\{ \frac{\text{Tr} \Delta_{m+1} \Delta_k}{d} \right\} \quad \text{(\( \sqrt{\Phi(A_i^*, I) \leq 1 \))}
\]

\[
\leq \sum_{k=1}^{m} \frac{\Delta_k (\Delta_{m+1} + \Delta_{m+1}^\dagger)/2}{d} \quad (\Delta_k = \Delta_{m+1}^\dagger \text{ for } k \leq m.)
\]

\[
\leq \sum_{k=1}^{m} \frac{||\Delta_k||_2}{\sqrt{d}} \frac{||\text{Re}(U) - \text{Tr}(U)||_2}{\sqrt{d}} \quad \text{(Cauchy-Schwarz ineq.)}
\]

\[
\leq \sum_{k=1}^{m} \sqrt{\gamma_D} \sqrt{\gamma_U} \left( 1 - \sqrt{\Phi(A_k^*, I)} \right) \left( 1 - \sqrt{\Phi(U, I)} \right) \quad \text{(Equability.)}
\]

\[
\leq \sqrt{\frac{\gamma_D \gamma_U}{\prod_{i=1}^{m} \Phi(A_i^*, I)}} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right) \left( 1 - \sqrt{\Phi(U, I)} \right) \quad \text{(B42)}
\]

Using \(|a^2 - b^2| \leq |a - b||a + b|\), and reuniting the pieces, we get

\[
\left| \Phi U \circ A_{m:1} - \Phi(U, I) \prod_{i=1}^{m} \Phi(A_i^*, I) \right| \leq \sqrt{\frac{\gamma_D \gamma_U}{\prod_{i=1}^{m} \Phi(A_i^*, I)}} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right) \left( 1 - \sqrt{\Phi(U, I)} \right)
\]

\[
+ \frac{\gamma_D}{\prod_{i=1}^{m} \Phi(A_i^*, I)} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right)^2
\]

\[
+ (1 - \sqrt{\Phi(U, I)}) \left[ \frac{\gamma_D}{\prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)}} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right)^2 \right]
\]

\[
+ \frac{\gamma_D}{4 \prod_{i=1}^{m} \Phi(A_i^*, I)} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*, I)} \right)^4
\]

\[
+ \gamma_D \sum \left( 1 - \sqrt{\Phi(A_i^*, I)} \right)^2 + \frac{2\gamma_D}{\prod_{i=1}^{m} \Phi(A_i^*)} \left( 1 - \prod_{i=1}^{m} \sqrt{\Phi(A_i^*)} \right)^2 \left( \text{B43} \right)
\]

A straightforward application of eq. (B26) and theorem 2 on the LHS (to get rid of the *) yields eq. (63).

\[ \square \]

**Appendix C: Proof of theorem 5**

**Proof.** Let \( \sigma_i \) be the singular values of the LK operator of \( A \). The first part of the proof revolves around

\[
E(\sigma_i^2) \leq E(\sigma_i) \leq \sqrt{E(\sigma_i^2)} \quad \text{, (C1)}
\]

which implies that

\[
\Upsilon(A^*) \leq (E(\sigma_i))^2 \leq \sqrt{\Upsilon(A^*)} \quad \text{. (C2)}
\]
First, let’s demonstrate the lower bound eq. \((55b)\):

\[
\max_{V \in SU(d)} \Phi(V \circ A, U) \geq \max_{V \in SU(d)} \Phi(V \circ A^*, U)
\]

(\text{Theorem 2})

\[
= (\mathbb{E}(\sigma_i))^2
\]

(\text{C3})

\[
\geq \Upsilon(A^*)
\]

(\text{Equation (C2)})

\[
\geq \Upsilon(A) - (1 - \Upsilon(A))^2.
\]

(\text{Lemma 1})

Demonstrating the upper bound eq. \((55b)\) follows the same reasoning:

\[
\max_{V \in SU(d)} \Phi(V \circ A, U) \leq \max_{V \in SU(d)} \Phi(V \circ A^*, U) + (1 - \Upsilon(A))(1 - \max_{V \in SU(d)} \Phi(V \circ A, U))
\]

(\text{Theorem 2})

\[
\leq (\mathbb{E}(\sigma_i))^2 + (1 - \Upsilon(A))^2 + (1 - \Upsilon(A))^3
\]

(\text{Lower bound eq. } (55b))

\[
\leq \sqrt{\Upsilon(A^*)} + (1 - \Upsilon(A))^2 + (1 - \Upsilon(A))^3.
\]

(\text{C4})

To tighten the lower bound at line \((C3)\) we may use the WSE decoherence constant:

\[
(\mathbb{E}(\sigma_i))^2 = \sqrt{\Upsilon(A^*)} - \gamma_D (1 - \mathbb{E}(\sigma_i))^2
\]

(\text{Equiability})

\[
\geq \sqrt{\Upsilon(A^*)} - \gamma_D (1 - \sqrt{\Upsilon(A^*)})^2
\]

(\text{Equation (C2)})

\[
\geq \sqrt{\Upsilon(A)} - (1 - \Upsilon(A))^2 - \gamma_D (1 - \sqrt{\Upsilon(A)})^2
\]

(\text{Lemma 1})

\[
\geq \sqrt{\Upsilon(A)} - (1 - \Upsilon(A))^2 - \gamma_D \left(1 - \sqrt{\Upsilon(A)}\right)^2,
\]

(\text{C5})

which completes the proof.

\[\square\]

1. \textbf{Proof of theorem 9}

\textit{Proof}. Let’s factor the decoherent channels into their (left) polar decomposition \(A_i = D_i \circ V_i\). By relabeling \((V_{\text{c}i})^{-1} \circ D_i \circ V_{\text{c}i} = D'_{\text{c}i}\) (notice that \(D'_{\text{c}i}\) are decoherent), we have

\[
A_{m:1} = V_{m:1} \circ D'_{m:1}.
\]

(\text{C6})

First, let’s find a lower bound on \(\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1})\). A way to do this is to pick a wisely chosen argument for \(W\). Let’s pick \(W = U_{m:1} \circ (V_{m:1})^\dagger\):

\[
\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) \geq \Phi(U_{m:1} \circ (V_{m:1})^\dagger \circ A_{m:1}, U_{m:1})
\]

(\text{Theorem 1})

\[
\geq \Phi(D''_{m:1}, \mathcal{I})
\]

(\text{C7})

\[
\geq \prod_{i=1}^m \Phi(D''_{i}, \mathcal{I}) - \prod_{i=1}^m \frac{\gamma_D}{\Phi(D''_{i}, \mathcal{I})} \left(1 - \prod_{i=1}^m \sqrt{\Phi(D''_{i}, \mathcal{I})}\right)^2
\]

(\text{Equation (B37)})

\[
- \frac{\gamma_D^2}{4 \prod_{i=1}^m \Phi(D''_{i}, \mathcal{I})^2} \left(1 - \prod_{i=1}^m \sqrt{\Phi(D''_{i}, \mathcal{I})}\right)^4
\]

(\text{Equation (C3)})

\[
\geq \prod_{i=1}^m \Phi(D''_{i}, \mathcal{I})
\]

(\text{C8})

\[
- \frac{\gamma_D}{\prod_{i=1}^m \sqrt{\Upsilon(A_i)}} \left(1 - \prod_{i=1}^m \sqrt{\Upsilon(A_i)}\right) - \frac{\gamma_D^2}{4 \prod_{i=1}^m \Upsilon(A_i)^2} \left(1 - \prod_{i=1}^m \sqrt{\Upsilon(A_i)}\right)^4.
\]

(\text{Equation (C2)})

To bound \(\prod_{i=1}^m \Phi(D''_{i}, \mathcal{I})\), we express it as a sum of three terms:

\[
\prod_{i=1}^m \sqrt{\Upsilon(A_i)} + \left(\prod_{i=1}^m \sqrt{\Upsilon(A_i)} - \prod_{i=1}^m \sqrt{\Upsilon(A_i)}\right) + \left(\prod_{i=1}^m \Phi(D''_{i}, \mathcal{I}) - \prod_{i=1}^m \sqrt{\Upsilon(A_i)}\right).
\]

(\text{C9})
To bound the second term, we used a telescopic expansion:

\[
\prod_{i=1}^{m} \sqrt{Y(A_i^*)} - \prod_{i=1}^{m} \sqrt{Y(A_i)} = \sum_{k=1}^{m} \left( \prod_{i=k+1}^{m} \sqrt{Y(A_i)} \right) \left( \sqrt{Y(A_k^*)} - \sqrt{Y(A_k)} \right) \left( \prod_{i=1}^{k-1} \sqrt{Y(A_i)} \right)
\]

\[
\geq - \sum_{k=1}^{m} \left( \sqrt{Y(A_k^*)} - \sqrt{Y(A_k)} \right)
\]

\[
\geq - \sum_{k=1}^{m} \left( Y(A_k^*) - \sqrt{Y(A_k^*)} - (1 - Y(A_k))^2 \right) \quad \text{(Lemma 1)}
\]

\[
\geq - \sum_{k=1}^{m} (1 - Y(A_k))^2,
\]

where we used \( \sqrt{x} - \sqrt{x - (1 - x)^2} \leq (1 - x)^2 \) for \( x \in [1/2, 1] \) on the last line. The third term of eq. (C9) is bounded as follows:

\[
\prod_{i=1}^{m} \Phi(D_i^{*, a}, I) - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \geq - \gamma_D \sum_{k=1}^{m} \left( 1 - \sqrt{Y(A_k^*)} \right)^2 \quad \text{(Equation (B30))}
\]

\[
\geq - \gamma_D \sum_{k=1}^{m} (1 - \sqrt{Y(A_k^*)})^2. \quad \text{(Equation (C2))}
\]

Reuniting the pieces together, we get

\[
\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) \geq \prod_{i=1}^{m} \sqrt{Y(A_i^*)} - \gamma_D \sum_{k=1}^{m} (1 - \sqrt{Y(A_k^*)})^2 - \sum_{k=1}^{m} (1 - Y(A_k))^2
\]

\[
- \frac{\gamma_D}{\prod_{i=1}^{m} \sqrt{Y(A_i^*)}} \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right)^2 - \frac{\gamma_D^2}{4 \prod_{i=1}^{m} \sqrt{Y(A_i^*)}^2} \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right)^4. \quad \text{(C11)}
\]

With regards to the upper bound, we can use theorem 2 and eq. (C11) to get

\[
\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) \leq \max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) + \frac{5}{4} \left( \sum_{k=1}^{m} (1 - Y(A_k)) \right)^2 + \frac{1}{2} \left( \sum_{k=1}^{m} (1 - Y(A_k)) \right)^3
\]

\[
+ \sum_{k=1}^{m} (1 - Y(A_k)) \left\{ \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right) + \gamma_D \sum_{k=1}^{m} (1 - \sqrt{Y(A_k^*)}) + \prod_{k=1}^{m} (1 - Y(A_k))^2
\]

\[
+ \frac{\gamma_D}{\prod_{i=1}^{m} \sqrt{Y(A_i^*)}} \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right)^2 + \frac{\gamma_D^2}{4 \prod_{i=1}^{m} \sqrt{Y(A_i^*)}^2} \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right)^4 \right\}, \quad \text{(C12)}
\]

which simplifies the maximization. Indeed, by using the flavored Von-Neumann inequality (lemma 3), followed by eq. (C2), we get

\[
\max_{W \in SU(d)} \Phi(W \circ A_{m:1}, U_{m:1}) \leq \left| \frac{\text{Tr} |D'_{m:1}|}{d} \right|^2 \quad \text{(Lemma 4)}
\]

\[
\leq \sqrt{Y(D'_{m:1})} \quad \text{(Equation (C2))}
\]

\[
\leq \prod_{i} \sqrt{Y(A_i^*)} + \frac{2 \gamma_D}{\prod_{i=1}^{m} \sqrt{Y(A_i^*)}} \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right)^2 \quad \text{(Equation (B31))}
\]

\[
\leq \prod_{i} \sqrt{Y(A_i)} + \sum_{k=1}^{m} (1 - Y(A_k))^2 + \frac{2 \gamma_D}{\prod_{i=1}^{m} \sqrt{Y(A_i^*)}} \left( 1 - \prod_{i=1}^{m} \sqrt{Y(A_i^*)} \right)^2 \quad \text{(Equation (C10))}
\]