A fundamental new type of birational modification which first occurs in dimension three is the \textit{simple flip}. This is a birational map $Y \dasharrow Y^+$ which induces an isomorphism $(Y - C) \cong (Y^+ - C^+)$, where $C$ and $C^+$ are smooth rational curves such that $K_Y \cdot C < 0$ and $K_{Y^+} \cdot C^+ > 0$. ($Y$ and $Y^+$ should be allowed to have “terminal” singularities.) Mori’s celebrated theorem \cite{22} shows that these flips exist when numerically expected.

A closely related type of modification is the \textit{simple flop}. This has a similar definition, except that $Y$ and $Y^+$ should be Gorenstein, with $K_Y \cdot C = K_{Y^+} \cdot C^+ = 0$. (This is more than an analogy: every flip has a branched double cover which is a flop, and this construction was used in Mori’s proof.) For both flips and flops, the curves $C$ and $C^+$ can be contracted to points (in $Y$ and $Y^+$, respectively), yielding the same normal variety $X$. The birational map $Y \dasharrow Y^+$ can thus be described in terms of the two contraction morphisms $\pi: Y \rightarrow X$ and $\pi^+: Y^+ \rightarrow X$.

In this paper, we study the case of simple flops with $Y$ smooth, so that $\pi: Y \rightarrow X$ is an irreducible small resolution of a Gorenstein threefold singularity $P \in X$. (It is called “small” because the exceptional set is a curve rather than a divisor, and “irreducible” because the curve has only one component.) In fact, the study of flops with $Y$ smooth can be reduced to a study of Gorenstein threefold singularities with small resolutions, thanks to a theorem of Reid \cite{30} (cf. also Kollár \cite{11}) which produces a second small resolution $\pi^+: Y^+ \rightarrow X$ out of the original $\pi: Y \rightarrow X$. Pinkham \cite{27} showed that in this situation, $X$ is Gorenstein if it is merely assumed to be Cohen-Macaulay.

Early examples of small resolutions were constructed in an \textit{ad hoc} manner. The first class of examples is given by the singularities $x^2 + y^2 + z^2 + t^{2k} = 0$: a particularly nice description of the associated simple flops (with pictures!) can be found in a paper of Reid \cite[§5]{30}. 

\[ \text{GORENSTEIN THREEFOLD SINGULARITIES WITH SMALL RESOLUTIONS VIA INVARIANT THEORY FOR WEYL GROUPS} \]

\text{SHELDON KATZ AND DAVID R. MORRISON}
These flops are exactly those for which the normal bundle of \( C \) in \( Y \) is \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) or \( \mathcal{O} \oplus \mathcal{O}(-2) \). A second example of simple flops, in which the normal bundle is \( \mathcal{O}(1) \oplus \mathcal{O}(-3) \), was found by Laufer \[19\]; variants of this example were also investigated by the second author, Pinkham \[27\], and Reid \[30\]. Some other flops were studied in previous work of the authors \([23]\) and \([16]\).

In general, if \( Y \to X \) is a small resolution of the isolated Gorenstein threefold singularity \( P \in X \), then by a lemma of Reid \[30\], the general hyperplane section of \( X \) through \( P \) has a rational double point at \( P \), and the proper transform of that surface on \( Y \) gives a “partial resolution” of the rational double point. Pinkham \[27\] used this observation to give a construction which includes all possible Gorenstein threefold singularities with small resolutions. In the irreducible case, each such singularity can be described by a map from the disk to a space \( \text{PRes}(S, v) \) which parametrizes deformations which partially resolve. The natural map \( \text{PRes}(S, v) \to \text{Def}(S) \) to the deformation space then gives a map from the disk to \( \text{Def}(S) \) which describes the space \( X \) as the pullback of a (semi-)universal family.

At first glance, Pinkham’s construction appears to give a countable number of families of Gorenstein threefold singularities with irreducible small resolutions. The discrete data which appear in the construction are the type of the rational double point, together with a choice of component in the exceptional divisor of the minimal resolution of that point. However, the construction uses a particular hyperplane section through \( P \), and there is no guarantee that this hyperplane section is “general”. (Examples for which it is not general were known to Pinkham at the time he gave the construction.)

We have discovered that there are in fact only six families of Gorenstein threefold singularities with irreducible small resolutions. They can be distinguished by a very simple invariant (the “length”) which was introduced by Kollár \[10, pp. 95, 96\] a few years ago. The precise statement of our main theorem can be found in section 1.

Our methods do much more than simply characterize the six families: our techniques can be used to calculate the map \( \text{PRes}(S, v) \to \text{Def}(S) \) quite explicitly in each of the six cases. Composing this map with a general map from the disk to \( \text{PRes}(S, v) \) describes the most general Gorenstein threefold with irreducible small resolution of each type. For example, when the length is 1, the result is precisely the class of examples \( x^2 + y^2 + z^2 + t^{2k} = 0 \) mentioned above.

In order to prove the main theorem, we need to solve a fundamental problem: compute the singularity type of the generic hyperplane
section for irreducible small resolutions which have been produced by Pinkham’s construction. The main tools we use to solve this problem are derived from the theory of simultaneous resolution of rational double points, as developed by Brieskorn [5], [6], [7] and Tyurina [34]. In this theory, the deformation space $\text{Def}(S)$ is identified with a quotient space $V/\mathfrak{W}$, where $V$ is the complex root space and $\mathfrak{W}$ is the Weyl group of a certain root system $R$. (The Dynkin diagram of this root system coincides with the dual graph of the minimal resolution of the rational double point.) A simultaneous resolution of the semi-universal deformation over $V/\mathfrak{W}$ is possible after making the base change by $V \to V/\mathfrak{W}$. Pinkham’s construction of the space $\text{PRes}(S, v)$ included an identification of it with the quotient space $V/\mathfrak{W}_0$, where $\mathfrak{W}_0$ is the Weyl group of a certain subsystem of $R$. Thus, an understanding of the map $\text{PRes}(S, v) \to \text{Def}(S)$ can be obtained if one has an adequate understanding of the invariant theory of the Weyl groups $\mathfrak{W}$ and $\mathfrak{W}_0$.

Our first theorem is a slight extension of the fundamental theorem on simultaneous resolution. We need some additional information about the loci in which certain curves in the exceptional set deform, but most importantly, we need a version of the proof which provides the simultaneous resolution in an explicitly computable form. With a small amount of modification, the approach of Tyurina [34] as amplified by Pinkham [25] provides us with what we need.

The theory of simultaneous resolution implies that the coefficients of the defining polynomial of a semi-universal deformation serve as generators of the algebra of $\mathfrak{W}$-invariant polynomials on $V$. (By a theorem of Coxeter [11] and Chevalley [9], this is a free polynomial algebra.) Our next theorems show that these generators are in an appropriate sense unique up to $\mathbb{C}^*$-action, and can be computed from the invariant theory alone. This is an important step, since it allows us to recover the defining polynomials of various semi-universal deformations directly from the invariant theory. It has the effect of reducing our fundamental problem to a problem in the invariant theory of Weyl groups.

The problem in invariant theory can be solved by hand when the Weyl groups in question are those associated to the root systems $A_{n-1}$ or $D_n$, but in the cases of $E_6$, $E_7$, and $E_8$, the invariant theory is not so well understood. We computed the invariants in these cases with the aid of the symbolic computing languages Maple and Reduce, running on a Macintosh II and on a Sun 3/60 workstation. We have also developed a number of tools for manipulating these invariants, which enabled us to extract enough information relating $\mathfrak{W}$-invariants to $\mathfrak{W}_0$-invariants to solve our fundamental problem.
We describe our algorithm in sufficient detail that it could be implemented in any symbolic manipulation language. We have, however, been somewhat selective in displaying the calculated results in the paper. The implementations in MAPLE and REDUCE were carried out independently by the second and first authors, respectively, and run on different machines. We are happy to report that the two sets of calculated results agree in every particular.

One of the by-products of the work described here is a new explicit set of generators for the invariants of the Weyl groups of $E_6$, $E_7$, and $E_8$. Following ideas of Tyurina, these generators are obtained by computing the anti-pluricanonical mappings for an appropriate family of del Pezzo surfaces, and putting the defining polynomials of the images into a semi-universal form. This calculation was attempted in 1918 by C. C. Bramble \cite{Bramble} in the case of $E_7$, but our machine-aided calculation reveals that Bramble made a few errors. A corrected version of Bramble’s calculation appears in Appendix 2; the corresponding calculation for $E_6$ is in Appendix 1.

As the final draft of this paper was being prepared, we learned of some recent work of Shioda \cite{Shioda} who has also done an explicit calculation of simultaneous resolutions using a quite different approach.

János Kollár has communicated to us another possible approach to proving our main theorem \cite{Kollar}, using techniques of Clemens and Jiménez \cite{ClemensJimenez} and an analysis of higher order neighborhoods similar to that used by Mori for extremal rays in \cite{Mori}.

The paper is organized as follows. Section 1 contains the statement of the main theorem, and a discussion of all of the ingredients needed to state it, including Pinkham’s construction and Kollár’s “length” invariant. In section 2, we set up the notation for root systems, and we introduce the notion of distinguished polynomials, which provide an efficient means for comparing invariants. In section 3, we establish notation for the rational double points, and state the simultaneous resolution theorem in the form we will need it. We prove that theorem in sections 4 and 5, giving explicit constructions of simultaneous resolutions. In section 6, we show that the simultaneous resolutions we have constructed are essentially unique, and we use this uniqueness to analyze the defining polynomials of simultaneous partial resolutions. In section 7, we use the distinguished polynomials to compare invariants in different coordinate systems, and section 8 contains the proof of the main theorem, modulo some computer calculations. The computer-dependent portions of the paper have been isolated in sections 9 and 10, which describe the algorithm for explicitly putting the defining polynomials of the simultaneous resolutions for $E_6$, $E_7$, and $E_8$ into
semi-universal form, and for manipulating the invariant polynomials to finish the proof of the main theorem.

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1. Reid’s lemma, Pinkham’s construction, Kollár’s invariant, and the statement of the main theorem.

The analysis of Gorenstein threefold singularities with small resolutions begins with the following lemma of Reid [30, (1.1),(1.14)].

Reid’s Lemma. Let $\pi: Y \to X$ be a resolution of an isolated Gorenstein threefold singularity $P \in X$. Suppose that the exceptional set of $\pi$ has pure dimension 1. Let $X_0$ be a generic hyperplane section of $X$ which passes through $P$. Then $X_0$ has a rational double point at $P$.

Moreover, if $X_0$ is any hyperplane section through $P$ with a rational double point, and $Y_0$ is its proper transform, then $Y_0$ is normal, and the minimal resolution $Z_0 \to X_0$ factors through the induced map $\pi|_{Y_0}: Y_0 \to X_0$.

Following Wahl [37], a map $Y_0 \to X_0$ through which the minimal resolution $Z_0 \to X_0$ factors is called a partial resolution of $X_0$ (provided that $Y_0$ is normal). There is a natural graph associated to such a map. Start with the dual graph $\Gamma$ of the components of the exceptional divisor of the minimal resolution $Z_0 \to X_0$. The curves contracted by $Y_0 \to X_0$ correspond to vertices in the graph which span a subgraph $\Gamma_0$; we call $\Gamma_0 \subset \Gamma$ the partial resolution graph of $\pi$. Figure 1 below and figure 2 in section 7 show some partial resolution graphs. In both figures, the vertices corresponding to $\Gamma_0$ are shown with an open circle (○), while those corresponding to $\Gamma - \Gamma_0$ are shown with a closed circle (●).

Based on Reid’s lemma, Pinkham [27] gave a construction which in principle describes all Gorenstein threefold singularities with small resolutions. Let $Y \to \text{Def}(Y_0)$ and $X \to \text{Def}(X_0)$ be semi-universal deformations of $Y_0$ and $X_0$, respectively. By a theorem of Wahl [35, Theorem 1.4], all deformations of $Y_0$ blow down to give deformations of $X_0$, and there is a natural map $\tau: \text{Def}(Y_0) \to \text{Def}(X_0)$. If we choose a local equation $\{f = 0\}$ for $X_0$ in a neighborhood $U$ of $P$ in $X$, then $f$ can be regarded as a map $f: U \to \Delta$, where $\Delta \subset \mathbb{C}$ is a small disk, such that $X_0 \cap U = f^{-1}(0)$. The fibers of $f$ give a deformation of $X_0$,
so there is a classifying map $\mu_f : \Delta \to \text{Def}(X_0)$ which enables us to recover $U$. Similarly, the fibers of $f \circ \pi$ give a deformation of $Y_0$, and the classifying map $\mu_{f \circ \pi} : \Delta \to \text{Def}(Y_0)$ satisfies $\tau \circ \mu_{f \circ \pi} = \mu_f$.

The only data which are really necessary to describe the map $\tau : \text{Def}(Y_0) \to \text{Def}(X_0)$ are the type $S$ of the rational double point ($S$ is one of $A_{n-1}$, $D_n$, $E_6$, $E_7$, or $E_8$) and the subgraph $\Gamma_0 \subset \Gamma$. Having fixed such data, we denote the map by $\tau : \text{Pres}(S, \Gamma_0) \to \text{Def}(S)$. The space $\text{Pres}(S, \Gamma_0)$ represents the functor of deformations of a singularity of type $S$ which partially resolve, with partial resolution type specified by $\Gamma_0$. When $\Gamma_0$ consists of a single vertex $v$, we abbreviate the notation to $\text{Pres}(S, v)$.

Pinkham’s general construction then goes as follows. Fixing $S$ and $\Gamma_0$ determines the map $\tau : \text{Pres}(S, \Gamma_0) \to \text{Def}(S)$. For any map $\nu : \Delta \to \text{Pres}(S, \Gamma_0)$ there is an induced map $\mu := \tau \circ \nu$. Pulling back the semi-universal families by $\nu$ and $\mu$ gives threefolds $Y \to X$. Pinkham shows that if $\nu$ is sufficiently general, then $Y$ is smooth, $X$ is Gorenstein with an isolated singular point, and $Y \to X$ is a small resolution. Reid’s lemma implies that all Gorenstein threefold singularities with small resolutions arise in this way.

However, as Pinkham observed [27, p. 367], “Although we have a construction for the singularities that arise, we do not really have a classification. For example, the construction does not explain how to get the generic hyperplane . . . ”. In fact, the generic hyperplane section of $X$ can have a much simpler singularity type than $S$ (the type of the hyperplane section which was used to construct $Y \to X$). The main theorem of this paper will compute the singularity type of the generic hyperplane section in the case that the exceptional set of $\pi : Y \to X$ is irreducible.

There is a fundamental invariant of singularities $P \in X$ with an irreducible small resolution which was introduced by Kollár [10, pp. 95, 96]. Let $m_{P,X}$ denote the maximal ideal sheaf of $P$ in $X$.

**Definition.** Let $\pi : Y \to X$ be an irreducible small resolution of an isolated threefold singularity $P$. (That is, the exceptional set $\pi^{-1}(P)$ is an irreducible curve $C$.) The length of $P$ is the length at the generic point of the scheme supported on $C$ with structure sheaf $\mathcal{O}_Y/\pi^{-1}(m_{P,X})$.

**Remark.** A different invariant, the normal bundle sequence of the exceptional curve, was introduced by Pinkham in [27]. This is the collection of normal bundles of the sequence of curves $C_1, \ldots$, beginning with the exceptional curve $C$, with $C_{i+1}$ being the negative section of
the exceptional divisor of the blowup along $C_i$. The sequence terminates when the normal bundle becomes $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. There are five possible types of normal bundle sequences, listed in [27, p. 367]. The length and the normal bundle sequence measure a kind of “thickness” of the singularity and exceptional curve, respectively. It can be shown that these invariants are related as follows. Length 1 corresponds to Pinkham’s cases (1) or (2), length 2 corresponds to case (3), lengths 3, 4, 6 each correspond to case (4), and length 5 corresponds to case (5).

In the Gorenstein case, if $X_0$ is any hyperplane section of $X$ with rational double points, then the length of $X$ can also be computed from the sheaf $\mathcal{O}_{Y_0}/\pi^{-1}(m_{\mathfrak{q}, X_0})$. The length therefore coincides with the multiplicity of the curve $C$ in the maximal ideal cycle of the rational double point. (The maximal ideal cycle is the same as Artin’s fundamental cycle [1].)

This property of the length puts a constraint on the possible deformations of the hyperplane section. If we have a family $(X_0)_t$ of hyperplane sections through $P$, then we can get different partial resolution graphs for different values of $t$. However, the multiplicity of the curve $C$ in the maximal ideal cycle must be independent of $t$, since it always coincides with the length. This leads to the following definition.

**Definition.** A partial resolution $Y_0 \to X_0$ with irreducible exceptional set $C$ is primitive if for every nontrivial 1-parameter deformation of $Y_0 \to X_0$ for which $C$ deforms, the multiplicity of $C$ in the maximal ideal cycle at the generic point of the family is strictly less than the multiplicity at the special point.

The point of the definition is that if $Y \to X$ is an irreducible small resolution with a hyperplane section $X_0 \subset X$ yielding a primitive $Y_0 \to X_0$, then a 1-parameter deformation from $X_0$ to a general hyperplane section through $P$ must be trivial. Thus, $X_0$ has the same singularity type as a general hyperplane section of $X$ through $P$.

The primitive partial resolution graphs can be computed from the deformation theory of rational double points. There are exactly six of them: the six graphs shown in figure 1. The numbers labeling the vertices in the figure are the multiplicities of the corresponding curves in the maximal ideal cycle. We omit the proof that these six graphs are the only primitive ones; the proof is a fairly straightforward computation, but the result is not logically necessary for the purposes of this paper. In fact, it can be obtained as a corollary of our main theorem.

Note that the primitive partial resolution graphs shown are uniquely determined by the multiplicity of $C$ in the maximal ideal cycle. It follows from our proof of the main theorem that every partial resolution
Figure 1.

graph admits a deformation to a primitive graph through graphs for which \( C \) has constant multiplicity in the maximal ideal cycle.

We can now state our main theorem.
Main Theorem. The generic hyperplane section of an isolated Gorenstein threefold singularity which has an irreducible small resolution defines one of the primitive partial resolution graphs in figure 1. Conversely, given any such primitive partial resolution graph, there exists an irreducible small resolution $Y \to X$ whose general hyperplane section is described by that partial resolution graph.

It follows that there are exactly six basic types of simple flops on smooth threefolds.

Corollary. The general hyperplane section of $X$ is uniquely determined by the length of the singular point $P$.

Two special cases of this theorem were known previously. If $X$ has some hyperplane section of type $A_n$, then the theorem asserts that the general hyperplane section has type $A_1$. This was known to several people (including Mori, Pinkham, Shepherd-Barron and the second author) about 10 years ago, but has apparently never been published. And in [16], the first author did the case in which $X$ has some hyperplane section of type $D_4$.

The converse statement in the theorem follows from the discussion above: Pinkham showed that examples exist for any partial resolution graph, and as we have observed, a hyperplane section leading to a primitive partial resolution graph must be the general hyperplane section. The first statement in the main theorem will be proved in section 8.

2. Root systems, Weyl groups, and distinguished polynomials.

In this section, we establish some notation for certain root systems and their Weyl groups. As is customary in algebraic geometry, we take root systems to have negative definite inner product; aside from this, we follow the notation of Bourbaki [2] fairly closely for $A_n-1$ and $D_n$, making some departures in the case $E_n$.

We begin with an inner product space $H^{n+1}$ ($n \geq 1$) over a field $k$ of characteristic 0, with orthogonal basis $e_0, e_1, \ldots, e_n$ such that $(e_0 \mid e_0) = 1$ and $(e_i \mid e_i) = -1$ for $i \geq 1$. We let $e_0^*, e_1^*, \ldots, e_n^*$ be the dual basis of the dual space $(H^{n+1})^*$.

$H^{n+1}$ contains the lattice

$$H^{n+1}_Z := \{ x = \sum \xi_i e_i \in H^{n+1} \mid \xi_i \in \mathbb{Z} \text{ and } \sum \xi_i \in 2\mathbb{Z} \}.$$ 

We define three subspaces of $H^{n+1}$. $V_{E_n}$ is the orthogonal complement of the special vector $k := -3e_0 + \sum_{i=1}^n e_i$, $V_{D_n}$ is the orthogonal complement of the first basis vector $e_0$, and $V_{A_{n-1}} = V_{E_n} \cap V_{D_n}$ is the
orthogonal complement of the subspace spanned by $k$ and $e_0$. The lattice $H^{n+1}_k$ induces (by intersection with these subspaces) lattices $L_{E_n}$, $L_{D_n}$, and $L_{A_{n-1}}$. A root in one of these lattices is an element of norm $-2$.

The set of roots in the lattice $L_{E_n}$ ($n \geq 3$), $L_{D_n}$ ($n \geq 2$), or $L_{A_{n-1}}$ ($n \geq 1$) is called a root system of type $E_n$, $D_n$, or $A_{n-1}$, respectively. We denote this set of roots by $R_{E_n}$ (resp. $R_{D_n}$ or $R_{A_{n-1}}$). The lattice itself is called the root lattice, and the vector space $V_{E_n}$ (resp. $V_{D_n}$ or $V_{A_{n-1}}$) is called the root space over $k$. It is customary in Lie theory to take $k = \mathbb{R}$; here, we are primarily interested in the case $k = \mathbb{C}$, and in that case we refer to $V$ as the complex root space.

We have included a degenerate case $A_0$, as well as two cases in which the root system is reducible: $R_{D_2} \cong R_{A_1} \cup R_{A_1}$, and $R_{E_3} \cong R_{A_2} \cup R_{A_1}$. In addition, there are some isomorphisms among the irreducible ones: $R_{D_3} \cong R_{A_3}$, $R_{E_6} \cong R_{A_4}$, and $R_{E_8} \cong R_{D_5}$.

When $n \geq 1$, the lattice $L_{A_{n-1}}$ can be generated by the root basis $v_1, ..., v_{n-1}$, where $v_i := e_i - e_{i+1}$. This root basis has as its Dynkin diagram $\Gamma_{A_{n-1}}$:

\begin{center}
\begin{tikzpicture}
    \draw (0,0) -- (1,0) -- (2,0) -- (3,0) -- (4,0);
    \draw (0,0) -- (0,-1);
    \node at (0.5,0) {$v_1$}; \node at (2.5,0) {$v_{n-1}$}; \node at (1.5,-1) {$v_2$};
\end{tikzpicture}
\end{center}

Notice that in the degenerate case $A_0$, we have the empty root basis, which forms a basis for the zero vector space $V_{A_0}$.

When $n \geq 2$, adding the root $v_n := e_{n-1} + e_n$ to the set $v_1, ..., v_{n-1}$ produces a root basis $v_1, ..., v_{n-1}, v_n$ of $L_{D_n}$, which has Dynkin diagram $\Gamma_{D_n}$:

\begin{center}
\begin{tikzpicture}
    \node at (0,0) {$v_1$}; \node at (1,0) {$v_2$}; \node at (2,0) {$v_{n-2}$}; \node at (3,0) {$v_{n-1}$}; \node at (1,1) {$v_n$}; \draw (0,0) -- (1,0) -- (2,0) -- (3,0) -- (2,1);
\end{tikzpicture}
\end{center}

Only the two end vertices $v_{n-1}$ and $v_n$ appear in the reducible case $D_2$.

Finally, if $3 \leq n \leq 8$, adding the root $v_0 := e_0 - e_1 - e_2 - e_3$ to the set $v_1, ..., v_{n-1}$ produces a root basis $v_0, v_1, ..., v_{n-1}$ of $L_{E_n}$. This has Dynkin diagram $\Gamma_{E_n}$:

\begin{center}
\begin{tikzpicture}
    \node at (0,0) {$v_0$}; \node at (1,0) {$v_1$}; \node at (2,0) {$v_2$}; \node at (3,0) {$v_3$}; \node at (4,0) {$v_4$}; \node at (5,0) {$v_{n-1}$}; \node at (1,1) {$v_4$}; \node at (1,2) {$v_5$}; \draw (0,0) -- (1,0) -- (2,0) -- (3,0) -- (4,0) -- (5,0) -- (4,1) -- (3,2);
\end{tikzpicture}
\end{center}
In the reducible case $E_3$, this diagram consists of $v_1$ joined to $v_2$ on the left, and $v_0$ below.

Many of our constructions for a root system $R$ will depend on its type $S$, which is one of $A_{n-1}$, $D_n$, or $E_n$ in all cases we consider. Constructions which depend on the type may fail to be invariant under the isomorphisms of root systems $R_{D_3} \cong R_{A_3}$, $R_{E_4} \cong R_{A_4}$, and $R_{E_5} \cong R_{D_5}$.

We single out certain linear functions on our root spaces. Let $V$ be one of our root spaces $V_{A_{n-1}}$, $V_{D_n}$, or $V_{E_n}$. We define the distinguished functionals on the root space $V$ to be the $n$ functions $t_1, \ldots, t_n$ given by

$$t_i := \left( \frac{1}{3} e_i^* + e_i^* \right)|_V.$$

It is not difficult to express the distinguished functionals $t_1, \ldots, t_n$ in terms of the dual basis $v_\alpha^*, v_{\alpha+1}^*, \ldots, v_\beta^*$ of the root basis $v_\alpha, v_{\alpha+1}, \ldots, v_\beta$; the result is shown in the third column of table [1]. (In order to simplify the notation, we have introduced a few extra $v_i^*$'s into the formulas, which should be set equal to zero, as indicated in the second column of the table.) In the fourth column of table [1], we have solved for the dual basis in terms of the distinguished functionals. A key step in doing this is to let $s_1 := t_1 + \cdots + t_n$ be the sum of the distinguished functionals, and to note that $s_1 = 0$ in the case of $A_{n-1}$.

The formulas in table [1] imply that the ring $\mathbb{C}[V]$ of polynomial functions on $V$ is generated by $t_1, \ldots, t_n$, and is in fact isomorphic to $\mathbb{C}[t_1, \ldots, t_n]/I$, where $I = (t_1 + \cdots + t_n)$ in case $A_{n-1}$, and $I = (0)$ otherwise.

We collect the distinguished functionals into the distinguished polynomial

$$f_S(U; t) := \prod_{i=1}^n (U + t_i),$$

where $S$ denotes the type of the root system (one of $A_{n-1}$, $D_n$, or $E_n$) and $t$ denotes $(t_1, \ldots, t_n)$. If we let the symmetric group $\mathfrak{S}_n$ act by permuting $(t_1, \ldots, t_n)$ in the usual way, then the distinguished polynomial $f_S(U; t)$ lies in the subring $(\mathbb{C}[V][\mathfrak{S}_n])[U]$ of polynomials whose coefficients are invariant under $\mathfrak{S}_n$. It can thus be written in the form

$$f_S(U; t) = U^n + \sum_{i=1}^n s_i U^{n-i},$$

where the coefficients $s_1, \ldots, s_n$ are the elementary symmetric functions of the distinguished functionals $t_1, \ldots, t_n$. (Note that $s_1$ is the sum of the distinguished functionals, agreeing with the definition given...
\[ v_0^* = 0, \quad v_n^* = 0 \]

\[ t_i = -v_{i-1}^* + v_i^* \quad (1 \leq i \leq n) \]

\[ v_i^* = t_1 + \cdots + t_i \quad (1 \leq i \leq n - 1) \]

\[ D_n \]

\[ v_0^* = 0 \]

\[ t_{n-1} = -v_{n-2}^* + v_{n-1}^* + v_n^* \]

\[ t_n = -v_{n-1}^* + v_n^* \]

\[ v_{n-1}^* = \frac{1}{2}s_1 - t_n \]

\[ v_n^* = \frac{1}{2}s_1 \]

\[ E_n \]

\[ v_n^* = 0 \]

\[ t_1 = -\frac{2}{3}v_0^* + v_1^* \]

\[ t_2 = -\frac{2}{3}v_0^* - v_1^* + v_2^* \]

\[ t_3 = -\frac{2}{3}v_0^* - v_2^* + v_3^* \]

\[ t_i = \frac{1}{3}v_0^* - v_{i-1}^* + v_i^* \quad (4 \leq i \leq n) \]

\[ v_0^* = \frac{3}{n-9}s_1 \]

\[ v_1^* = \frac{2}{n-9}s_1 + t_1 \]

\[ v_2^* = \frac{4}{n-9}s_1 + t_1 + t_2 \]

\[ v_i^* = \frac{9-i}{n-9}s_1 + t_1 + \cdots + t_i \quad (3 \leq i \leq n - 1) \]

**Table 1.**

above.) However, the product expansion for \( f_S(U; t) \) given in equation (1) is only valid in the larger ring \( (\mathbb{C}[V])[U] \).

It is important to observe that the construction of both the distinguished functionals and the distinguished polynomial depend on the type of the root system having been identified as one of \( A_{n-1}, D_n, \) or \( E_n \). For example, although \( E_4 \) and \( A_4 \) are isomorphic as root systems, they have different distinguished functionals and polynomials. (Even the degrees of the distinguished polynomials are different.)

Notice also that when our definitions are applied to the degenerate case \( A_0 \), the single distinguished functional \( t_1 \) is identically zero, and the distinguished polynomial is simply \( f_{A_0}(U; t) = U \in \mathbb{C}[U] \).

We next describe the action of the Weyl group \( \mathfrak{W} \) of our root system. \( \mathfrak{W} \) is the subgroup of \( \text{Aut}(V^*) \) generated by the reflections \( r_v \) in the roots \( v \) which belong to the root basis. We are particularly interested in the invariant polynomials for the action of \( \mathfrak{W} \) on \( V^* \). A theorem of Coxeter [1] and Chevalley [3] guarantees that the ring \( \mathbb{C}[V]^{\mathfrak{W}} \) of
invariant polynomials is a free polynomial algebra over $C$. This implies that the quotient space $V/\mathfrak{W}$ is smooth.

The reflections $r_v \in \text{Aut}(V^*)$ which generate $\mathfrak{W}$ are restrictions of reflections in $\text{Aut}((H^{n+1})^*)$, which we also denote by $r_v$. The action of each $r_v$ on $(H^{n+1})^*$ is easily described. For example, the reflections $r_v$ for $1 \leq i \leq n-1$ act on $(H^{n+1})^*$ by fixing $e_i^*$ and mapping $e_j^*$ to $e_{\sigma(j)}^*$, where $\sigma$ is the simple transposition $(i, i+1)$ of the set $\{1, \ldots, n\}$. It follows that the action of these $r_v$ on $V^*$ maps $t_j^*$ to $t_{\sigma(j)}^*$ in each of the three cases $V = V_{A_{n-1}}$, $V = V_{D_n}$, and $V = V_{E_n}$.

The Weyl group $\mathfrak{W}_{A_{n-1}}$ is the group generated by the reflections $r_v, \ldots, r_{v_{n-1}}$, and therefore coincides with the symmetric group $S_n$. Since $t_1 + \cdots + t_n = 0$, the invariant polynomials for the action of $\mathfrak{W}_{A_{n-1}}$ are generated by the nonzero coefficients $s_2, \ldots, s_n$ of the distinguished polynomial $f_{A_{n-1}}(U; t)$.

In the case of the Weyl group $\mathfrak{W}_{D_n}$, the reflections $r_v$ for $1 \leq i \leq n-1$ again act as permutations of $t_1, \ldots, t_n$, although this time the sum $s_1$ is not necessarily zero. $\mathfrak{W}_{D_n}$ is generated by those reflections together with $r_{v_n}$. The action of $r_{v_n}$ on $(H^{n+1})^*$ is not hard to compute: we have

$$r_{v_n}(e_i^*) = \begin{cases} 
  e_i^* & \text{if } 0 \leq i \leq n-2, \\
  -e_n^* & \text{if } i = n-1, \\
  -e_{n-1}^* & \text{if } i = n.
\end{cases}$$

It follows that the action on $V_{D_n}^*$ is given by

$$r_{v_n}(t_i) = \begin{cases} 
  t_i & \text{if } 1 \leq i \leq n-2, \\
  -t_n & \text{if } i = n-1, \\
  -t_{n-1} & \text{if } i = n.
\end{cases}$$

The invariant polynomials for this action are generated by the product $t_1 \cdots t_n$, together with the elementary symmetric functions of the squares $t_1^2, \ldots, t_n^2$. These invariants are captured by the constant term $s_n = f_{D_n}(0; t)$ of the distinguished polynomial, together with the coefficients of a polynomial $g_{D_n}(Z; t)$ whose defining property is

$$g_{D_n}(Z; t) := \prod_{i=1}^{n}(Z + t_i^2). \quad (2)$$

(Note that $g_{D_n}(0; t) = f_{D_n}(0; t)^2 = s_n^2$, so its constant term is not needed to generate the $\mathfrak{W}$-invariants.) We call $g_{D_n}(Z; t)$ the second distinguished polynomial; it is only defined for root systems of type $D_n$. 
The second distinguished polynomial $g_{D_n}(Z; t)$ lies in the ring $(\mathbb{C}[V]^{S_n})[Z]$ of polynomials whose coefficients are $\mathfrak{W}$-invariants, although the defining property (2) only holds in the larger ring $(\mathbb{C}[V])[Z]$. The second distinguished polynomial could also have been defined by the property

$$g_{D_n}(-U^2; t) = f_{D_n}(U; t) \cdot f_{D_n}(-U; t)$$

which holds in the auxiliary ring $(\mathbb{C}[V]^{S_n})[U]$. Following Tyurina [34], we can express this property directly in the subring $(\mathbb{C}[V]^{S_n})[Z]$ of $(\mathbb{C}[V])[Z]$ in the following way. Collect terms of even and odd degree in the distinguished polynomial:

$$f_{D_n}(U; t) = U \cdot P_{D_n}(-U^2; t) + Q_{D_n}(-U^2; t).$$

This defines two polynomials

$$P_{D_n}(Z; t), Q_{D_n}(Z; t) \in (\mathbb{C}[V]^{S_n})[Z].$$

Then equation (3) can be rewritten as

$$g_{D_n}(Z; t) = Z \cdot P_{D_n}(Z; t)^2 + Q_{D_n}(Z; t)^2.$$   

We also record for later use the existence of a polynomial $G_{D_n}(Z, U; t) \in (\mathbb{C}[V]^{S_n})[Z, U]$ such that

$$U \cdot P_{D_n}(Z; t) + Q_{D_n}(Z; t) = (Z + U^2) \cdot G_{D_n}(Z, U; t) + f_{D_n}(U; t).$$

Since $G_{D_n}(Z, U; t)$ has degree $n - 2$ in $U$, if we define

$$\tilde{G}_{D_n}(Z, u, v; t) := v^{n-2} \cdot G_{D_n}(Z, u/v; t),$$

then $\tilde{G}_{D_n}(Z, u, v; t)$ is a polynomial which is homogeneous in $u, v$.

Finally, in the case of the Weyl group $\mathfrak{W}_{E_n}$, the reflections $r_i$ for $1 \leq i \leq n - 1$ still act as permutations of $t_1, \ldots, t_n$, and again the sum $s_1$ is not necessarily zero. $\mathfrak{W}_{E_n}$ is generated by those reflections together with $r_{v_0}$. The action of $r_{v_0}$ on $(H^{n+1})^*$ is not hard to compute: we have

$$r_{v_0}(e_i^*) = \begin{cases} 
    e_0^* + (e_0^* + e_1^* + e_2^* + e_3^*) & \text{if } i = 0, \\
    e_i^* - (e_0^* + e_1^* + e_2^* + e_3^*) & \text{if } 1 \leq i \leq 3, \\
    e_i^* & \text{if } 4 \leq i \leq n.
\end{cases}$$

It follows that the action on $V_{E_n}^*$ is given by

$$r_{v_0}(t_i) = \begin{cases} 
    t_i - \frac{2}{3}(t_1 + t_2 + t_3) & \text{if } 1 \leq i \leq 3, \\
    t_i + \frac{2}{3}(t_1 + t_2 + t_3) & \text{if } 4 \leq i \leq n.
\end{cases}$$

(8)
The invariant polynomials for the action of \( \mathfrak{W}_{E_n} \) are not as simple to describe as those for the actions of \( \mathfrak{W}_{A_{n-1}} \) and \( \mathfrak{W}_{D_n} \). In the reducible case \( E_3 \), it is not hard to verify (using equation (8)) that the polynomials

\[
\begin{align*}
\varepsilon_2^{(1)} & := s_1^2 \\
\varepsilon_2^{(2)} & := s_2 \\
\varepsilon_3 & := s_3 - \frac{1}{3} s_1 s_2 + \frac{2}{27} s_1^3
\end{align*}
\]

are invariant under \( \mathfrak{W}_{E_3} \), and generate the ring of invariants \( \mathbb{C}[V_{E_3}]^{\mathfrak{W}_{E_3}} \). In the cases of \( E_4 \) and \( E_5 \), bases for the rings of invariants can be calculated by using the isomorphisms \( R_{E_4} \cong R_{A_4} \) and \( R_{E_5} \cong R_{D_5} \). We carry this out in the corollary to lemma \( \mathfrak{W} \) in section 7.

The invariant polynomials are extremely complicated in the classical cases \( E_6, E_7, \) and \( E_8 \). Calculations of some of these were made in the early 1950s by Coxeter [11], Frame [13], and Racah [29], although published explicit formulas are limited to the case of \( E_6 \). One of the by-products of the work discussed here is an explicit description of a basis for the invariant polynomials in these three cases. (These invariants are of course polynomials in the symmetric functions \( s_1, \ldots, s_n \), since they are invariant under the subgroup \( \mathfrak{W}_{A_{n-1}} \).) The basis we compute appears in Appendix 1 for \( E_6 \) and in Appendix 2 for \( E_7 \). The polynomials in the basis are too large to be worth writing down in the case of \( E_8 \), but an algorithm for computing with those polynomials is given in section 9.

3. RATIONAL DOUBLE POINTS AND SIMULTANEOUS RESOLUTION.

Associated to each irreducible root system of type \( S = A_{n-1}, D_n, \) or \( E_n \) is a rational double point with the same name. The minimal resolution of this complex surface singularity has a dual graph which is isomorphic to the corresponding Dynkin diagram, and the singularity is determined up to isomorphism by the diagram. For each singularity type, we fix a representative for the isomorphism class: the hypersurface in \( \mathbb{C}^3 \) whose defining polynomial is given in the middle column of table [2]. Notice that as in section 2, our choices depend on the type \( S \) rather than just on the isomorphism class.

Each of our chosen representatives admits a \( \mathbb{C}^* \)-action, as indicated in the last column of table [3] if \( \mathbb{C}^3 \) is given the specified \( \mathbb{C}^* \)-action then the defining polynomial is \textit{weighted homogeneous}, i.e., is an eigenfunction for the \( \mathbb{C}^* \)-action. In the cases of \( A_{n-1} \) and \( E_4 \), there are several possible \( \mathbb{C}^* \)-actions, any one of which will suit our purposes.
Table 2.

By a theorem of Pinkham [24], a singularity with a $\mathbb{C}^*$-action admits a $\mathbb{C}^*$-semi-universal deformation. (Such a deformation is semi-universal for deformations with a $\mathbb{C}^*$-action, and is also semi-universal for arbitrary deformations.) For a surface in $\mathbb{C}^3$ defined by a weighted homogeneous polynomial $F$, this can be obtained as follows. Choose weighted homogeneous polynomials $G_1, \ldots, G_r$ which descend to a basis of the vector space $\mathbb{C}[X,Y,Z]/(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z})$. The polynomial $\Phi := F + \mu_1 G_1 + \cdots + \mu_r G_r$ will then define a $\mathbb{C}^*$-semi-universal deformation as a hypersurface in $\mathbb{C}^{3+r}$. The variable $\mu_i$ is given the difference of the weights of $F$ and $G_i$ as its weight.

In the case of rational double points, even after fixing the defining polynomial as in table 2, there are still some choices to be made, for there may be more than one choice of weighted homogeneous polynomials which give a basis of $\mathbb{C}[X,Y,Z]/(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \frac{\partial F}{\partial Z})$. We implicitly fix one such choice in table 3.

**Definition.** The defining polynomial $\Phi_S$ of a $\mathbb{C}^*$-semi-universal deformation of a rational double point is said to be in preferred versal form if it has the form given in table 3.

| $S$ | Defining Polynomial | Action of $\lambda \in \mathbb{C}^*$ |
|-----|---------------------|-------------------------------------|
| $A_{n-1}$  
$(n \geq 2)$ | $-XY + Z^n$ | $(X, Y, Z) \mapsto (\lambda^k X, \lambda^{n-k} Y, \lambda Z)$ |
| $D_n$  
$(n \geq 3)$ | $-X^2 - Y^2 Z + Z^{n-1}$ | $(X, Y, Z) \mapsto (\lambda^{n-1} X, \lambda^{n-2} Y, \lambda^2 Z)$ |
| $E_4$ | $-XY + Z^5$ | $(X, Y, Z) \mapsto (\lambda^k X, \lambda^5 - k Y, \lambda Z)$ |
| $E_5$ | $-X^2 - Y^2 Z + Z^4$ | $(X, Y, Z) \mapsto (\lambda^4 X, \lambda^3 Y, \lambda^2 Z)$ |
| $E_6$ | $-X^2 - XZ^2 + Y^3$ | $(X, Y, Z) \mapsto (\lambda^6 X, \lambda^4 Y, \lambda^3 Z)$ |
| $E_7$ | $-X^2 - Y^3 + 16YZ^3$ | $(X, Y, Z) \mapsto (\lambda^9 X, \lambda^6 Y, \lambda^4 Z)$ |
| $E_8$ | $-X^2 + Y^3 - Z^5$ | $(X, Y, Z) \mapsto (\lambda^{15} X, \lambda^{10} Y, \lambda^6 Z)$ |
We have chosen the notation in table 3 to make the $\mathbb{C}^*$-action explicit: the $\mathbb{C}^*$-action on a $\mathbb{C}^*$-semi-universal deformation whose defining polynomial is in preferred versal form is determined by giving $X$, $Y$, $Z$ the same eigenvalues as in table 2, and giving each coefficient with subscript $i$ the eigenvalue $\lambda_i$.

| $S$ | Preferred Versal Form |
|-----|------------------------|
| $A_{n-1}$ $(n \geq 2)$ | $-XY + Z^n + \sum_{i=2}^{n} \alpha_i Z^{n-i}$ |
| $D_n$ $(n \geq 3)$ | $X^2 + Y^2 Z - Z^{n-1} - \sum_{i=1}^{n-1} \delta_{2i} Z^{n-i-1} + 2\gamma_n Y$ |
| $E_4$ | $-XY + Z^5 + \varepsilon_2 Z^3 + \varepsilon_4 Z + \varepsilon_5$ |
| $E_5$ | $X^2 + Y^2 Z - Z^4 - \varepsilon_2 Z^3 - \varepsilon_4 Z^2 + 2\varepsilon_5 Y - \varepsilon_6 Z - \varepsilon_8$ |
| $E_6$ | $-X^2 - XZ^2 + Y^3 + \varepsilon_2 Y Z^2 + \varepsilon_5 Y Z + \varepsilon_6 Z^2 + \varepsilon_8 Y + \varepsilon_9 Z + \varepsilon_{12}$ |
| $E_7$ | $-X^2 - Y^3 + 16 Y Z^3 + \varepsilon_2 Y^2 Z + \varepsilon_6 Y^2 + \varepsilon_8 Y Z + \varepsilon_{10} Z^2 + \varepsilon_{12} Y + \varepsilon_{14} Z + \varepsilon_{18}$ |
| $E_8$ | $-X^2 + Y^3 - Z^5 + \varepsilon_2 Y Z^3 + \varepsilon_8 Y Z^2 + \varepsilon_{12} Z^3 + \varepsilon_{14} Y Z + \varepsilon_{18} Z^2 + \varepsilon_{20} Y + \varepsilon_{24} Z + \varepsilon_{30}$ |

Table 3.
be calculated later). Substituting $8X$ and $4Y$ for $X$ and $Y$ respectively, then dividing by 64 gives the desired defining polynomial.

Our first theorem is a slight refinement of the famous theorem on simultaneous resolution of rational double points, due to Brieskorn [5], [6], [7] and Tyurina [34]. (We follow Tyurina’s approach, as amplified by Pinkham [25].) Recall that a simultaneous resolution of a family $X \to \mathcal{D}$ is a resolution of singularities of $X$ which also resolves the singularities of each fiber of the map $X \to \mathcal{D}$. In general, one can only hope to find a simultaneous resolution after making a base change $\mathcal{R} \to \mathcal{D}$ and pulling back the original family to a family $X \times_{\mathcal{D}} \mathcal{R}$.

Let $R$ be an irreducible root system of type $S = A_{n-1}, D_n$, or $E_n$, in which a basis of simple roots $\{v_i\}$ has been chosen. Let $V$ be the complex root space, let $W$ be the Weyl group acting on $V$ by reflections, and let $\rho: V \to V/W$ be the quotient by $W$. Let $X_0$ be the corresponding rational double point with $\mathbb{C}^*$-action, let $Z_0 \to X_0$ be the minimal resolution, and fix an identification between the basis of simple roots $\{v_i\}$ of $R$ and the irreducible exceptional curves $\{C_i\}$ on $Z_0$ which preserves the associated graphs. For each positive root $v = \sum a_i v_i$, the curve $C_v = \sum a_i C_i$ is an effective rational curve of self-intersection $-2$ on $Z_0$. The deformations of $Z_0$ to which $C_v$ lifts forms a natural subset of all deformations of $Z_0$ (cf. [36, §2]).

We fix coordinates $X, Y, Z$ on $\mathbb{C}^3$.

**Theorem 1.** Let $R$ be an irreducible root system of type $S$, and let $\rho: V \to V/W$ and $Z_0 \to X_0$ be as above. Let $\mathbb{C}^*$ act on the complex root space $V$ via $v \mapsto \lambda \cdot v$ for $\lambda \in \mathbb{C}^*$, and give $V/W$ the induced $\mathbb{C}^*$-action. Then there is a $\mathbb{C}^*$-semi-universal deformation $X \to V/W$ of $X_0$ and a $\mathbb{C}^*$-equivariant simultaneous resolution $Z \to X \times_{\rho V}$ inducing $Z_0 \to X_0$ with the following properties:

1. $X$ can be embedded as a hypersurface in $\mathbb{C}^3 \times V/W$ whose defining polynomial $\Phi_S$ with respect to the coordinates $X, Y, Z$ is in preferred versal form.
2. The coefficients of the defining polynomial $\Phi_S$ are explicitly computable functions on $V/W$, which give a basis of $\mathbb{C}[V]^W$.
3. For each positive root $v \in R$, the set of deformations of $Z_0$ in the family $Z$ to which the curve $C_v$ lifts is parametrized by the hyperplane $v^\perp$ (the orthogonal complement of $v$ in $V$).

The two novelties in this statement are the phrase “explicitly computable”, and the third part of the theorem (although the latter is implicit in Slodowy’s exposition [33], and in some of Looijenga’s work [20]). We will give a proof of this theorem, based on two constructions.
of Tyurina \[34\], in the next two sections. We have explicitly computed these coefficients as functions on \(V/W\), and we give our results in equation (10) for \(A_{n-1}\), equation (11) for \(D_n\), equations (16) and (17) for \(E_4\) and \(E_5\) respectively, and Appendices 1 and 2 for \(E_6\) and \(E_7\) respectively. (In the last two cases, we “clear denominators” before displaying the result.) We have not attempted to write down the formulas for \(E_8\), although we give an algorithmic method of computing with them in section 9.

4. Simultaneous resolutions for \(A_{n-1}\) and \(D_n\).

In this section, we will prove theorem 1 in the cases of \(A_{n-1}\) and \(D_n\). It is easy to see that the theorem then follows in cases \(E_4\) and \(E_5\), since the \(\mathbb{C}^*\)-semi-universal deformation of \(E_4\) (resp. \(E_5\)) in table 3 coincides with the one for \(A_4\) (resp. \(D_5\)). We postpone the cases of \(E_6\), \(E_7\), and \(E_8\) to the next section.

We will construct the required deformations \(\mathcal{X}\) of rational double points and their simultaneous resolutions \(Z\), using the distinguished functionals \(t_1, \ldots, t_n\) (and the elementary symmetric functions \(s_1, \ldots, s_n\) in those distinguished functionals) as the key ingredients in the construction. The construction is due to Kas \[15\] (for type \(A_{n-1}\) only) and Tyurina \[34\], both heavily influenced by work of Brieskorn \[5\].

We begin with the case of \(A_{n-1}\). Define functions \(\alpha_i \in \mathbb{C}[V]^W\) by

\[
\alpha_i := \begin{cases} 
0 & \text{for } -1 \leq i \leq 1, \\
s_i & \text{for } 2 \leq i \leq n.
\end{cases}
\]

Since the Weyl group \(W\) coincides with the symmetric group \(S_n\) and \(s_1 = 0\) in \(\mathbb{C}[V]\), it follows that \(\{\alpha_i\}_{2 \leq i \leq n}\) is a basis of \(\mathbb{C}[V]^W\).

Define a hypersurface \(\mathcal{X} \subset \mathbb{C}^3 \times V/W\) by means of the polynomial

\[
\Phi_{A_{n-1}} := -XY + Z^n + \sum_{i=2}^{n} \alpha_i Z^{n-i}
\]

which is in preferred versal form. Notice that this can also be written in the form

\[
\Phi_{A_{n-1}} = -XY + f_{A_{n-1}}(Z; t),
\]

and the functions \(\alpha_i\) can be interpreted as the coefficients of \(f_{A_{n-1}}(Z; t)\).

Let \(\rho: V \rightarrow V/W\) be the quotient map. To construct a simultaneous resolution of \(\mathcal{X} \times_{\rho} V\), we recall that in the ring \((\mathbb{C}[V])[U]\), the distinguished polynomial can be written in the factored form given in
equation (1). Therefore,

$$\rho^*(\Phi_{A_{n-1}}) = -XY + \prod_{i=1}^{n}(Z + t_i).$$

We construct a simultaneous resolution $Z \to \mathcal{X} \times_{\rho} V$ by taking the closure of the graph of the morphism

$$\mathcal{X} \times_{\rho} V \to (\mathbb{P}^1)^{n-1},$$

$$(X, Y, Z, t_1, \ldots, t_n) \mapsto [X, \prod_{\nu=1}^{i}(Z + t_{\nu})],$$

Assertions 1 and 2 of theorem 1 are clear, so we need only verify assertion 3.

Let $(u_k, v_k)$ be homogeneous coordinates on the $k$th $\mathbb{P}^1$ arising from the resolution described above. Then

$$-XY + \prod_{i=1}^{n}(Z + t_i) = 0$$

$$Xv_j = u_j \prod_{i=1}^{j}(Z + t_i) \quad (1 \leq j \leq n - 1),$$

$$\prod_{i=k+1}^{j}(Z + t_i)u_jv_k = u_kv_j \quad (1 \leq k < j \leq n - 1)$$

are equations for $Z$. Thinking of $V_{A_{n-1}}$ as a hyperplane in $\mathbb{C}^n$, these equations show that the $\pi$-exceptional part of the fiber of $Z$ over $(t_1, \ldots, t_n) = (0, \ldots, 0)$ is given as the union of the curves $C_i$, $1 \leq i \leq n - 1$, where $C_i$ is defined by $X = Y = Z = 0$, $u_j = 0$ for $j < i$, $v_k = 0$ for $k > i$.

All positive roots are of the form $v = v_i + \cdots + v_{j-1}$. From this, it is easy to see that the locus where $C_v = C_i + \cdots + C_{j-1}$ deforms is given by $t_i = t_j$, which is indeed the orthogonal complement of $v$. To see this, note that if $t_i = t_j$, and are distinct from the other $t_k$, the $\pi$-exceptional part of the fiber of $Z$ over $(t_1, \ldots, t_n)$ is given by

$$Z = -t_i = -t_j, \quad u_k = 0 \quad (k < i), \quad v_k = 0 \quad (k \geq j), \quad u_nv_k \prod_{n=k+1}^{m}(t_n - t_i) = u_sv_m, \quad (i \leq k < m < j).$$

For general parameter values, this is easily seen to be a $\mathbb{P}^1$, specializing to $C_i + \cdots + C_{j-1}$ as the parameters approach 0. Since the genus is constant, this is a flat family. Since the general fiber of this family is irreducible, this is the only possible flat family over the generic point of $t_i = t_j$ which is contained in the exceptional locus.

Since $Z$ may be obtained from $\mathcal{X} \times_{\rho} V$ by successively blowing up Weil divisors, the exceptional set lies entirely over the discriminant $\prod_{i<j}(t_i - t_j)$. Hence the locus over which any (union of) exceptional
curves deform is contained in the locus where some $t_i$ equals some $t_j$. This proves the assertion.

Next we turn to the case of $D_n$. Define functions $\gamma_n, \delta_{2i} \in \mathbb{C}[V]^W$ by

$$
\gamma_n := t_1 \cdots t_n = s_n
$$

$$
\delta_{2i} := \text{the } i^{th} \text{ elementary symmetric function of } t_1^2, \ldots, t_n^2
$$

(11)

(Note that $\delta_{2n} = (\gamma_n)^2$.) The functions $\gamma_n, \delta_2, \ldots, \delta_{2n-2}$ generate $\mathbb{C}[V]^W$. Moreover, comparing (11) with the definition of the second distinguished polynomial in (2), we see that

$$
g_{D_n}(Z; t) = Z^n + \sum_{i=1}^{n-1} \delta_{2i} Z^{n-i} + (\gamma_n)^2
$$

$$
f_{D_n}(0; t) = \gamma_n.
$$

Define a hypersurface $X \subset \mathbb{C}^3 \times V/W$ by means of the polynomial

$$
\Phi_{D_n} := X^2 + Y^2Z - Z^{n-1} - \sum_{i=1}^{n-1} \delta_{2i} Z^{n-i-1} + 2\gamma_n Y
$$

which is in preferred versal form. Notice that this can also be written in the form

$$
\Phi_{D_n} = X^2 + Y^2Z - \frac{1}{Z}(g_{D_n}(Z; t) - f_{D_n}(0; t))^2 + 2f_{D_n}(0; t)Y.
$$

Let $\mathfrak{S}_n \subset \mathfrak{W}$ be the Weyl group of the subsystem of the root system $R$ generated by $v_1, \ldots, v_{n-1}$, which acts on $V$ by permutations of the distinguished functionals $t_1, \ldots, t_n$. The ring of invariant functions $\mathbb{C}[V]^{\mathfrak{S}_n}$ is generated by $s_1, \ldots, s_n$.

In the ring $(\mathbb{C}[V]^{\mathfrak{S}_n})(Z)$, we can use the properties of $g_{D_n}(Z; t)$ given in equations (4) and (5), which are expressed in terms of the polynomials $P_{D_n}(Z; t)$ and $Q_{D_n}(Z; t)$. Now (3) implies that $Q_{D_n}(0; t) = s_n$, so we can also define

$$
S_{D_n}(Z; t) := \frac{Q_{D_n}(Z; t) - s_n}{Z}.
$$

(12)

Let $\tau: V/\mathfrak{S}_n \rightarrow \mathfrak{W}/\mathfrak{W}$ be the natural map. The first step in Tyurina’s construction of the simultaneous resolution is done most naturally on $X \times_\tau V/\mathfrak{S}_n$. We abbreviate $P_{D_n}(Z; t)$, $Q_{D_n}(Z; t)$, and $S_{D_n}(Z; t)$ by $P$, $Q$, and $S$, respectively. Using (4), (5), (12) and some algebraic tricks as in Tyurina [34], we find that the defining polynomial of $X \times_\tau V/\mathfrak{S}_n$
can be written as
\[
\tau^*(\Phi_{D_n}) = X^2 + Y^2Z - \frac{1}{Z}(ZP^2 + (ZS + s_n)^2 - s_n^2) + 2s_nY
\]
\[
= (X - P)(X + P) + (YZ + ZS + 2s_n)(Y - S).
\]
Let \( \mathcal{Y} \rightarrow \mathcal{X} \times_\tau V/G_n \) be the blowup of the ideal \((X - P, Y - S)\). It is easy to see that the entire singular locus of \( \mathcal{Y} \) is contained in the coordinate chart \( \mathcal{Y}^0 \) defined by the substitution \( X - P = (Y - S)U \). The defining polynomial in this chart becomes
\[
\Phi_{Y^0} = U \cdot (2P + (Y - S)U) + (YZ + ZS + 2s_n)
\]
\[
= (Y - S)(Z + U^2) + 2UP + 2Q
\]
\[
= (Y - S + 2G)(Z + U^2) + 2f_{D_n}(U; t)
\]
(using the polynomial \( G = G_{D_n}(Z, U; t) \) defined in equation (6)).

Now let \( \sigma: V \rightarrow V/G_n \) be the natural quotient map. Since \( f_{D_n}(U; t) \) factors in \((\mathbb{C}[V])[U]\), we have
\[
\sigma^*(\Phi_{Y^0}) = (Y - S + 2G)(Z + U^2) + 2 \prod_{i=1}^{n}(U + t_i),
\]
which after a change of coordinates is the same family used in the case of \( A_{n-1} \). We can thus construct a resolution \( Z \rightarrow \mathcal{Y} \times_\sigma V \) (which also resolves \( \mathcal{X} \times_\rho V \)) by means of the construction used for \( A_{n-1} \).

Again, we only need to verify assertion 3 from theorem 1.

Let \([u, v]\) be homogeneous coordinates on the \( \mathbb{P}^1 \) of the first blow up for \( D_n \) as described above, with \( U = u/v \). The homogeneous form of the equation \( \sigma^*(\Phi_{Y^0}) \) is
\[
(v^{n-2}(Y - S) + 2\tilde{G})(v^2Z + u^2) + 2 \prod_{i=1}^{n}(u + t_i v) = 0
\]
(using the homogeneous counterpart \( \tilde{G} = \tilde{G}_{D_n}(Z, u, v; t) \) of \( G \) which was defined in equation (7)). The \( A_{n-1} \) singularity is then contained in the affine piece \( v = 1 \). Let \([u_k, v_k]\) be homogeneous coordinates on the \( k \)th \( \mathbb{P}^1 \) used in resolving the \( A_{n-1} \) singularity. Then we get as equations for \( Z_{D_n} \):
\[
(v^2Z + u^2)u_{n-1} = vu_{n-1}(u + t_nv)
\]
\[
v^{n-2-l}(v^2Z + u^2)v_l = u_l \prod_{i=l+1}^{n}(u + t_i v) \quad (1 \leq l \leq n - 2)
\]
\[
v^{k-j}v_ju_k = v_ku_j \prod_{i=j+1}^{k}(u + t_i v) \quad (1 \leq j < k \leq n - 1).
\]
These equations show that the $\pi$-exceptional part of the fiber of $\mathcal{Z}$ over $(t_1, \ldots, t_n) = (0, \ldots, 0)$ is given as the union of the curves $C_i$, $1 \leq i \leq n$, where for $i < n$, $C_i$ is defined by $X = Y = Z = 0$, $u = 0$, $u_j = 0$ for $j > i$, $v_j = 0$ for $j < i$ and $C_n$ is defined by $X = Y = Z = 0$, $uv_{n-1} = vu_{n-1}$, $v^{n-2}u_l = u^{n-2}u_l$ for $1 \leq l \leq n - 2$, $v^{k-j}v_ju_k = v_ku_j \prod_{i=j+1}^k (u + t_i v)$ for $1 \leq j < k \leq n - 1$.

The positive roots are of five types: $r = v_i + \cdots + v_{j-1}$ ($i \leq j \leq n$), $s = v_n$, $t = v_j + \cdots + v_{n-2} + v_n$ ($j \leq n-2$), $u = v_j + \cdots + v_n$ ($j \leq n-2$), $v = v_j + \cdots + v_{k-1} + 2v_k + \cdots + 2v_{n-2} + v_{n-1} + v_n$ ($j < k \leq n-2$).

Exactly as in the $A_{n-1}$ case, it is easy to see from the above construction that the locus where $C_i$ deforms is given by $t_i = t_j$. For each of the remaining cases, we exhibit the equations of a divisor in $\mathcal{Z}$ which defines a flat family of generically irreducible curves over a hyperplane in the parameter space; the hyperplane will be given first.

For each $1 \leq l \leq n - 2$,

\[
v^{n-2}u_l = u_l(u + t_{l+1}v) \cdots (u + t_{n-2}v) \quad (1 \leq l \leq n - 2)
\]

For each $1 \leq j < k \leq n - 1$,

\[
v^{k-j}v_ju_k = v_ku_j \prod_{i=j+1}^k (u + t_i v)
\]

For each $1 \leq j \leq n - 2$,

\[
v^{n-j-2}v_j = v_j(u + t_{j+1}v) \cdots (u + t_{n-1}v)
\]

For each $1 \leq l \leq n - 2$,

\[
v^{n-2}u_l = u_l(u + t_{l+1}v) \cdots (u + t_{j-1}v)(u + t_{j+1}v) \cdots (u + t_{n-1}v) \quad (1 \leq l \leq j)
\]

For each $1 \leq j < k \leq n - 1$,

\[
v^{k-j}v_ju_k = v_ku_j \prod_{i=j+1}^k (u + t_i v)
\]
\[ t_j + t_{n-1} = 0 \]
\[ X = 0, \ Y = -t_1 \cdots t_{j-1}t_{j+1} \cdots t_{n-2}t_n, \ Z = -t^2_{n-1} \]
\[ (u^2 - t^2_{n-1}u^2)v_{n-1} = vu_{n-1}(u + t_nv) \]
\[ (u - t_{n-1}v)v^{n-l-2}_{l} = u_l(u + t_{l+1}v) \cdots (u + t_{n-2}v)(u + t_nv) \quad (j + 1 \leq l \leq n - 2) \]
\[ v^{n-j-2}v_j = u_j(u + t_{j+1}v) \cdots (u + t_{n-2}v)(u + t_nv) \]
\[ v^{n-l-2}v_l = u_l(u + t_{l+1}v) \cdots (u + t_{j-1}v)(u + t_{j+1}v) \cdots (u + t_{n-2}v)(u + t_nv) \quad (1 \leq l \leq j) \]
\[ v^{k-j}v_ju_k = v_ku_j \prod_{i=j+1}^{k} (u + t_iv) \quad (1 \leq j < k \leq n - 1). \]

\[ t_j + t_k = 0 \]
\[ X = 0, \ Y = -t_1 \cdots t_{j-1}t_{j+1} \cdots t_{k-1}t_{k+1} \cdots t_n, \ Z = -t^2_k \]
\[ (u^2 - t^2_kv^2)v_{n-1} = vu_{n-1}(u + t_nv) \]
\[ (u - t_kv)v^{n-l-2}v_l = u_l(u + t_{l+1}v) \cdots (u + t_{n-2}v)(u + t_nv) \quad (k \leq l \leq n - 2) \]
\[ (u - t_kv)v^{n-j-1}v_{j-1} = u_{j-1}(u + t_{j+1}v) \cdots (u + t_nv) \]
\[ v^{n-j-1}v_{j-1} = u_{j-1}(u + t_{j+1}) \cdots (u + t_{k-1})(u + t_{k+1}) \cdots (u + t_n) \]
\[ v^{n-l-2}v_l = u_l(u + t_{l+1}v) \cdots (u + t_{j-1}v)(u + t_{j+1}v) \cdots \]
\[ \cdots (u + t_{k-1}v)(u + t_{k+1}v) \cdots (u + t_nv) \quad (1 \leq l \leq j - 2) \]
\[ v^{k-j}v_ju_k = v_ku_j \prod_{i=j+1}^{k} (u + t_iv) \quad (1 \leq j < k \leq n - 1). \]

For each of the families described above in the four cases \( w = s, t, u, v \), the following statement holds. The general fiber is a \( \mathbb{P}^1 \), specializing to \( C_w \) as the parameters approach 0. Since the genus is constant, this is a flat family. Since the general fiber of this family is irreducible, this is the only possible flat family over the generic point of the corresponding hyperplane in the parameter space which is contained in the exceptional locus.

Since \( Z \) may be obtained from \( \mathcal{X}_{D_0} \) by successively blowing up Weil divisors, the exceptional set lies entirely over the discriminant \( \prod_{i<j}(t^2_i - t^2_j) \). Hence the locus over which any (union of) exceptional curves deform is contained in the locus where some \( t_i \) equals plus or minus some \( t_j \). In each case, this is the desired orthogonal complement.
5. Simultaneous resolutions for \( E_6, E_7, \) and \( E_8 \).

In this section, we complete the proof of theorem 1, treating the cases of \( E_6, E_7, \) and \( E_8 \). We use another construction of Tyurina [34], one which had been anticipated by Bramble [4] in the case of \( E_7 \) in 1918. This construction is discussed in considerable detail by Pinkham [25], whose approach we follow, and also by Mérindol [21]. The strategy in this case is to first build \( Z \), and then recover \( X \). In fact, \( Z \) is constructed as an open subset of a relative projective model \( \bar{Z} \to V \).

We begin with \( \mathbb{P}^2 \) with homogeneous coordinates \([x, y, z]\), and let \( C \) be the cuspidal cubic with equation \( x^3 = yz^2 \). The smooth points of this rational curve form an open set \( C_0 \subset C \) isomorphic to the affine line. The map \( \eta: \mathbb{A}^1 \to \mathbb{P}^2 \) defined by \( \eta(U) := [U, U^3, 1] \) gives such an isomorphism.

Let \( V \) be the root space of \( E_n \), and let the distinguished functionals \( t_1, \ldots, t_n \) serve as coordinates on \( V \). Let \( \bar{Z}_0 = \mathbb{P}^2 \times V \), and \( C_0 = C \times V \). Define \( n \) sections \( \sigma_j: V \to \bar{Z}_0 \), \( 1 \leq j \leq n \) as follows:

\[
\sigma_j(t_1, \ldots, t_n) := (\eta(t_j), (t_1, \ldots, t_n)).
\]

Now for \( j = 1, \ldots, n \), let \( \bar{Z}_j \) be the blowup of \( \bar{Z}_{j-1} \) along the proper transform of the section \( \sigma_j \), and let \( C_j \) be the proper transform of \( C_{j-1} \) on \( \bar{Z}_j \). We use \( \bar{Z} \) and \( C \) to denote \( \bar{Z}_n \) and \( C_n \), respectively, and let \( p: \bar{Z} \to V \) be the natural map. The fibers of \( p \) are all smooth surfaces.

As Pinkham points out, each fiber \( \bar{Z}_x := p^{-1}(x) \) of \( p \) is the blowup of \( \mathbb{P}^2 \) in a collection of points \( \eta(t_1), \ldots, \eta(t_n) \) in “almost general position” in the sense of Demazure [12]. In particular, \( \omega_{\bar{Z}_x}^{-1} \) is nef. We use the notation \( \omega_{\bar{Z}/V}^{-k} \) to stand for \( (\omega_{\bar{Z}/V}^{-1})^{\otimes k} \). Let

\[
\bar{\mathcal{P}} := \text{Proj}_V \left( \bigoplus_{k \geq 0} p_*(\omega_{\bar{Z}/V}^{-k}) \right)
\]

be the relative anti-canonical model. The fibers of \( \bar{\mathcal{P}} \to V \) are “generalized del Pezzo surfaces”, that is, del Pezzo surfaces with rational double points allowed.

Let \( \mathcal{Z} := \bar{Z} - C \) and \( \mathcal{P} := \bar{\mathcal{P}} - C \). (We have abused notation, and denoted the image of \( C \) in \( \bar{\mathcal{P}} \) again by \( C \).) As Pinkham shows, the Weyl group \( \mathfrak{W} = \mathfrak{W}_{E_n} \) acts on \( \mathcal{P} \) by Cremona transformations (and permutations of \( \{\sigma_1, \ldots, \sigma_n\} \)). In Pinkham’s version of the construction, the parameter space is \((C_0)^n\) rather than \( V \); Pinkham computes the induced action of \( \mathfrak{W} \) on \((C_0)^n\) obtaining the formulas on p. 196 of [25]. Since those formulas agree with our equation (8) which describes the action of \( \mathfrak{W} \) on \( V \) when the distinguished functionals are used as...
Equation | Root
---|---
\(t_i - t_j = 0\) | \(e_i - e_j\)
\(t_i + t_j + t_k = 0\) | \(e_0 - e_i - e_j - e_k\)
\(\sum_{j=1}^{6} t_{ij} = 0\) | \(2e_0 - \sum_{j=1}^{6} e_{ij}\)
\(2t_{i1} + \sum_{j=2}^{7} t_{ij} = 0\) | \(3e_0 - 2e_{i1} - \sum_{j=2}^{7} e_{ij}\)

Table 4.

coordinates\(^1\) our identification of \(V\) with \((C_0)^n\) makes the map \(\mathcal{P} \to V\) into a \(\mathcal{W}\)-equivariant map. We define \(\mathcal{X} := \mathcal{P}/\mathcal{W}\).

If we give \(\mathbb{P}^2\) and \(V\) the linear \(\mathbb{C}^*\)-actions defined by

\[
[x, y, z] \mapsto [\lambda x, \lambda^3 y, z]
\]

\[(t_1, \ldots, t_n) \mapsto (\lambda t_1, \ldots, \lambda t_n)
\]

for \(\lambda \in \mathbb{C}^*\), then the entire construction becomes \(\mathbb{C}^*\)-equivariant. Pinkham shows that \(\mathcal{X} \to V/\mathcal{W}\) is a \(\mathbb{C}^*\)-semi-universal deformation of the central fiber \(X_0\), and that \(\mathcal{Z} \to \mathcal{P} = \mathcal{X} \times_{V/\mathcal{W}} V\) is a \(\mathbb{C}^*\)-equivariant simultaneous resolution.

We need to verify the properties stated in theorem 1. The third property is the easiest this time, since it was essentially checked by Pinkham. For each \(x \in V\), the singularities of the fiber \(\tilde{P}_x\) correspond to the “effective roots” of \(\tilde{Z}_x\). These can be seen from the geometry of the set of points blown up: they correspond to 2 points being infinitely near, 3 points being collinear, 6 points being conconic, or (in the case of \(E_8\)) 8 points lying on a nodal cubic with one of the points being the node. For each possible “effective root”, Pinkham computes the equation of the locus in \(V\) in which the root is effective, giving the results in a table on p. 193 of [25]. We display those equations in table 4, using our identification of \((C_0)^n\) with \(V\). The other column of the table gives the root \(v\) in \(V\) such that the equation is proportional to the equation \(v^+ = 0\).

To finish the proof of theorem 1, we must explain how to embed \(\mathcal{X}\) into \(\mathbb{C}^3 \times V/\mathcal{W}\). We will actually embed \(\mathcal{P}\) into \(\mathbb{C}^3 \times V\), and then note the \(\mathcal{W}\)-invariance of our construction. And that embedding in turn will be a restriction of a projective embedding of \(\mathcal{P}\).

\(^1\)This is why we chose the distinguished functionals as we did.
To describe a projective embedding of $\bar{\mathcal{P}}$, we extend Demazure’s analysis [12] of anti-pluricanonical mappings to the case of families of surfaces. (A similar extension in another context has been made by Mérindol [21].) For a single generalized del Pezzo surface $\bar{\mathcal{P}}_x$, Demazure found the following.

In the case of $E_6$, the anti-canonical map embeds $\bar{\mathcal{P}}_x$ into $\mathbb{P}^3$, and the image is a cubic surface with rational double points. In the case of $E_7$, the anti-canonical map $\bar{\mathcal{P}}_x \to \mathbb{P}^2$ is a finite map of degree 2, and the anti-bicanonical map embeds $\bar{\mathcal{P}}_x$ into the weighted projective space $\mathbb{P}^{(1,1,1,2)}$. $\bar{\mathcal{P}}_x$ can be described as a double cover of $\mathbb{P}^2$ branched along a quartic curve. In the case of $E_8$, the anti-canonical system is a pencil with a base point, and the anti-bicanonical map is again of degree 2, this time mapping to the weighted projective space $\mathbb{P}^{1,1,2}$ (which can be embedded as a quadric cone in $\mathbb{P}^3$). $\bar{\mathcal{P}}_x$ is a double cover of $\mathbb{P}^{1,1,2}$, branched on a curve of graded degree 6; the anti-tricanonical map embeds $\bar{\mathcal{P}}_x$ into $\mathbb{P}^{(1,1,2,3)}$.

Our immediate goal is to show that these projective embeddings can be described globally over $V$ as embeddings into $\mathbb{P} \times V$ for the appropriate weighted projective space $\mathbb{P}$. Then we will show that the image in the case of $E_6$, and branch loci in the cases of $E_7$ and $E_8$, can be similarly globally defined over $V$ by a single polynomial. We will explain how to explicitly compute these polynomials in section 9; here we simply show that they exist. A related approach to constructing projective embeddings of $\bar{\mathcal{P}}$ appears in [21].

Lemma 1.

1. The sheaf $p_*(\omega^{-k}_{\bar{Z}/V})$ is locally free of rank $h^0(\bar{Z}_x, \omega^{-k}_{\bar{Z}_x}) = 1 + \frac{(k^2+k)(9-n)}{2}$ in case $E_n$.

2. In cases $E_7$ and $E_8$, the map $\text{Sym}^2(p_*\omega^{-1}_{\bar{Z}/V}) \to p_*\omega^{-2}_{\bar{Z}/V}$ is injective as a morphism of vector bundles, and its cokernel is locally free of rank 1.

3. In the case of $E_8$, the natural map $p_*\omega^{-1}_{\bar{Z}/V} \otimes p_*\omega^{-2}_{\bar{Z}/V} \to p_*\omega^{-3}_{\bar{Z}/V}$ has a cokernel which is locally free of rank 1. Its kernel coincides with $\text{Ker}(p_*\omega^{-1}_{\bar{Z}/V} \otimes \text{Sym}^2(p_*\omega^{-1}_{\bar{Z}/V}) \to \text{Sym}^3(p_*\omega^{-1}_{\bar{Z}/V}))$, which is locally free of rank 2.

---

2 The weighted projective spaces occurring in the description are only implicit in Demazure's paper.

3 We use the term graded degree (rather than weighted degree) for a polynomial in a weighted projective space, to avoid confusion with the weights under the background $\mathbb{C}^*$-action.
Proof. (1) Since $\omega_{Zx}^{-1}$ is nef, $H^1(\mathcal{O}_x, \omega_{Zx}^{-k}) = 0$ for all $k \geq 0$. Thus, each of the sheaves $p_*(\omega_{Zx}^{-k})$ is locally free. The rank follows from Riemann-Roch, since $c^2_1 = 9 - n$ for $E_n$.

(2) According to Demazure [12, V, Proposition 1b], the anticanonical mapping in the case of $E_7$ is 2-1. In particular, its image is not contained in any quadric, hence the desired injectivity statement. In the case of $E_8$, the anticanonical map maps to $\mathbb{P}^1$, hence its image is not contained in a quadric either. The rank 1 assertion follows from the ranks listed in the table of the lemma.

(3) By Demazure [op. cit.], in the case of $E_8$ the antibicanonical mapping is 2-1; in particular, its image is not contained in a graded cubic hypersurface (thinking of the mapping as factoring through the weighted projective space $\mathbb{P}(1,1,2)$). This gives the equality of the two kernels, where $\text{Sym}^2 p_*(\omega_{Zx}^{-1})$ is identified with its image in $p_*(\omega_{Zx}^{-2})$ by part 2 already proven. The second mentioned kernel is easily computed to have rank 2 (the mapping is surjective); from which it follows that the cokernel of the first mapping has rank 1. Q.E.D.

Lemma 2. There exist $\mathbb{C}^*$-invariant subspaces

$L_1 \subset H^0(V, p_*(\omega_{Zx}^{-1}))$ of dimension $10 - n$, in case $E_n$,
$L'_2 \subset H^0(V, p_*(\omega_{Zx}^{-2}))$ of dimension 1, in cases $E_7$ and $E_8$, and
$L'_3 \subset H^0(V, p_*(\omega_{Zx}^{-3}))$ of dimension 1 in case $E_8$,

such that the natural maps

$L_1 \otimes \mathcal{O}_V \to p_*(\omega_{Zx}^{-1})$
$L'_2 \otimes \mathcal{O}_V \to \text{Coker}(\text{Sym}^2(p_*(\omega_{Zx}^{-1})) \to p_*(\omega_{Zx}^{-2}))$
$L'_3 \otimes \mathcal{O}_V \to \text{Coker}(p_*(\omega_{Zx}^{-1} \otimes p_*(\omega_{Zx}^{-2}) \to p_*(\omega_{Zx}^{-3}))$

are $\mathbb{C}^*$-equivariant isomorphisms of sheaves. In other words, the targets of these maps are trivial locally free sheaves.

Proof. The key ingredient is Quillen’s affirmative answer [28] to Serre’s conjecture that a finitely generated projective module over a polynomial ring must be free. Since $p_*(\omega_{Zx}^{-1})$ is coherent and $V$ is affine, $M = H^0(V, p_*(\omega_{Zx}^{-1}))$ is a finitely generated module over the polynomial algebra $\mathbb{C}[V]$. Since $M$ is also locally free, it is in fact projective. (Cf. [4, Chapter II, §5.2, Theorem 1].) Quillen’s theorem then implies that $M$ is free. This proves the triviality of the target of the first map. The other two cases are similar; just observe that the sheaves on the right hand side of the arrows are coherent and locally free of rank 1 by
lemma \[\PageIndex{4}\]. Since all maps in question are $\mathbb{C}^*$-equivariant, we may choose $\mathbb{C}^*$-eigensections as generators of these trivial bundles. \hfill Q.E.D.

We let $L = \bigoplus L_k$ be the graded $\mathbb{C}[V]$-algebra generated by $L_1$ in the case of $E_6$, by $L_1$ and $L'_2$ in the case of $E_7$, and by $L_1, L'_2, L'_3$ in the case of $E_8$. (We have $L_2 = \text{Sym}^2 L_1 \oplus L'_2$ in cases $E_7$ and $E_8$, and $L_3 = \text{Sym}^3 L_1 \oplus (L_1 \otimes L'_2) \oplus L'_3$ in case $E_8$.) The algebra $L$ has 4 generators, and $\text{Proj}_V(L)$ is a relative weighted projective space of dimension 3 over $V$, isomorphic to $\mathbb{P}^{(1,1,1,1)} \times V$, $\mathbb{P}^{(1,1,1,2)} \times V$, or $\mathbb{P}^{(1,1,2,3)} \times V$, respectively. Moreover, there is a natural embedding of $\mathcal{P}$ into $\text{Proj}_V(L)$, where it forms a hypersurface.

We now show that $\mathcal{P}$ can be defined by a weighted homogeneous polynomial globally over $V$.

**Lemma 3.** There is a weighted homogeneous polynomial $\Phi_{E_n} \in L_k$ which generates the ideal of $\mathcal{P}$ in $\text{Proj}_V(L)$, where $k = 3, 4, 6$ in cases $E_6, E_7, E_8$, respectively. In the $E_7$ case, by an appropriate choice of $L'_2$ and a generator $X_2$ of $L'_2$, the polynomial takes the form $X_2^2 - f_4$, for some $f_4 \in \text{Sym}^4(L_1) \subset L_4$. In the $E_8$ case, by an appropriate choice of $L'_3$ and a generator $X_3$ of $L'_3$, the polynomial takes the form $X_3^2 - f_6$, for some $f_6 \in \text{Sym}^6(L_1) \oplus (\text{Sym}^4(L_1) \otimes L'_2) \oplus (\text{Sym}^2(L_1) \otimes \text{Sym}^2(L'_2)) \oplus \text{Sym}^3(L'_2) \subset L_6$.

**Proof.** By \[\PageIndex{2}, \text{V.3}\], it follows in the case of $E_6$ that each $\tilde{P}_x$ is a cubic. Hence the sheaf $\ker(\text{Sym}^3 p_*(\omega^{-1}_{Z/V}) \to p_*(\omega^{-3}_{Z/V}))$ is invertible. Since it is also coherent, being the kernel of a morphism of coherent sheaves, its triviality follows from Quillen’s theorem. This yields a cubic with coefficients in $\mathbb{C}[V]$ which serves as a global defining polynomial for $\mathcal{P}$.

In the case of $E_7$, pick a generator $X_2$ for $L'_2$, and define a map

$$p_*(\text{Sym}^2 \omega^{-1}_{Z/V}) \oplus p_*(\text{Sym}^4 \omega^{-1}_{Z/V}) \to p_*(\omega^{-4}_{Z/V})$$

by $(a, b) \mapsto aX_2 + b$. It follows from \[\PageIndex{12}, \text{V.4}\] that this map is surjective on each fiber; by the triviality of the bundles, it must be surjective on global sections as well. Thus, since $(X_2)^2 \in H^0(V, p_*(\omega^{-4}_{Z/V}))$, there exist global sections $a \in H^0(V, p_*(\text{Sym}^2 \omega^{-1}_{Z/V}))$, $b \in H^0(V, p_*(\text{Sym}^4 \omega^{-1}_{Z/V}))$ such that $(X_2)^2 = aX_2 + b$. Since $p_*(\omega^{-1}_{Z/V})$ is trivial, it follows that $H^0(V, p_*(\text{Sym}^k \omega^{-1}_{Z/V})) = \text{Sym}^k H^0(V, p_*(\omega^{-1}_{Z/V}))$, hence $a$ and $b$ can be described as linear combinations of polynomials on $V$ times monomials in a basis for $L_1$. We thus get $\mathcal{P}$ as defined by the graded quartic polynomial $\Phi_{E_7} := -(X_2)^2 + aX_2 + b$ with coefficients in $\mathbb{C}[V]$. To put the defining polynomial into the form claimed, we need only complete the square as in \[\PageIndex{12}, \text{V.4}\]. This is tantamount to making a different choice for $L'_2$. 


The case of $E_8$ is similar.

Since $\mathcal{P}$ is invariant under the $C^*$-action, the defining polynomial $\Phi_{E_n}$ must be weighted homogeneous. Q.E.D.

In order to calculate explicit generators for the algebra $L$, we interpret global sections of $p_*(\omega_{Z/V}^{-k})$ as being the defining polynomials of curves of degree $3k$ in $\mathbb{P}^2 \times V$ with base conditions. The base conditions state that the curve should pass through the zero-cycle $\eta(t_1) + \cdots + \eta(t_n)$ with multiplicity $k$. Choosing a basis for the space of such polynomials determines a rational map $\pi: \mathbb{P}^2 \times V \to \mathbb{P}^N$, which can be interpreted as the blowup of $\sigma_1, \ldots, \sigma_n$ followed by the anti-$k$-canonical map of $Z$.

To guarantee the $\mathfrak{W}$-invariance of our defining polynomial $\Phi_{E_n}$ for $\mathcal{P}$ (and hence obtain a defining polynomial for $X$) we must choose our generators of $L$ quite carefully.

**Definition.** Let $m = (t_1, \ldots, t_n) \subset \mathbb{C}[V]$ be the maximal ideal of the origin $0 \in V$. We say that the weighted homogeneous polynomials $X, Y, Z, W \in \mathbb{C}[V][x, y, z]$ form a good generating set for $L$ if they generate $L$ and satisfy the defining conditions given in table 5.

The defining conditions given in table 5 were obtained as follows. Each element of $L_k$ when restricted to the central fiber $\overline{Z_0}$ becomes an element of $H^0(\overline{Z_0}, \omega_{\overline{Z_0}}^{-k})$. In the case of $E_n$, the polynomials of degree $3k$ which belong to that space are exactly the ones which have a zero of order at least $nk$ at $U = 0$ when they are pulled back via $\eta: C_0 \to \mathbb{P}^2$, and whose partial derivatives with respect to $x, y$, and $z$ up through order $k - 1$ have a zero of order at least $k$ when they are pulled back via $\eta$. It is easy to see that these conditions on the partial derivatives are superfluous when applied to a monomial, since $n \geq 6$, which is more than the largest order of vanishing at $U = 0$ among $x, y$, and $z$ after pulling back via $\eta$. A set of generators for the anti-pluricanonical ring of $\overline{Z_0}$ is easy to find using this description. We have implicitly listed one such set in table 5, as the right-hand sides of congruences for $X, Y, Z, W$. (The last column of the table includes congruences for $X$ derived from the defining conditions in the cases of $E_7$ and $E_8$.) The $C^*$-action preserves the central fiber $\overline{Z_0}$, so we chose a generating set of polynomials which are weighted homogeneous on the central fiber.

We can see the weights of the generators from the information given in table 5: they are $(9, 7, 6, 3)$ for $(X, Y, Z, W)$ in the case of $E_6$, $(15, 9, 7, 3)$ for $(X, Y, Z, W)$ in the case of $E_7$, and $(24, 16, 9, 3)$ for $(X, Y, Z, W)$ in the case of $E_8$. Moreover, the last column of the table contains the coefficients in the case of $E_7$ were chosen to match the notation of Bramble [4].
Defining Conditions | Other Properties
--- | ---
\( W = x^3 - yz^2 \) | \( W = x^3 - yz^2 \)
\( Z \equiv y^2 z \mod m \) | \( Y^3 - XZ^2 - X^2 W \equiv 0 \mod m \)
\( Y \equiv xy^2 \mod m \) | \( Z \equiv y^3 \mod m \)
\( X \equiv y^3 \mod m \) | \( Y \equiv xy^2 \mod m \)
\( W = x^3 - yz^2 \) | \( X \equiv 8y^5z \mod m \)
\( Z \equiv xy^2 \mod m \) | \( 16YZ^3 - X^2 - Y^3 W \equiv 0 \mod m \)
\( Y \equiv 4y^3 \mod m \) | \( X = \frac{1}{3} \frac{\partial(Y,Z,W)}{\partial(x,y,z)} \)
\( X = \frac{1}{3} \frac{\partial(Y,Z,W)}{\partial(x,y,z)} \) | \( X \equiv y^8z \mod m \)
\( W = x^3 - yz^2 \) | \( Y^3 - X^2 - Z^5 W \equiv 0 \mod m \)
\( Z \equiv y^3 \mod m \) | \( Y \equiv xy^5 \mod m \)
\( Y \equiv xy^5 \mod m \) | \( X = \frac{-1}{6} \frac{\partial(Y,Z,W)}{\partial(x,y,z)} \)
\( X = \frac{-1}{6} \frac{\partial(Y,Z,W)}{\partial(x,y,z)} \)

Table 5.

shows the leading order terms of the polynomial \( \Phi_{E_n} \) (determined by elimination from the defining conditions), and determines its weight as 21, 30, or 48, respectively.

**Proposition 1.** There exists a good generating set \( X, Y, Z, W \) for \( L \) such that when the defining polynomial \( \Phi_{E_n} \in \mathbb{C}[V][X, Y, Z, W] \) of \( \mathcal{P} \subset \text{Proj}_V(L) \) is restricted to the affine chart \( W = 1 \), it gives a \( \mathbb{C}^* \)-semi-universal deformation of the rational double point which is in preferred versal form.

**Proof.** We first claim that there exist good generating sets for \( L \). We can start with the polynomial \( x^3 - yz^2 \) as one of the generators, since it belongs to \( L \) in all three cases. On the central fiber \( Z_0 \), the generator \( x^3 - yz^2 \) can be extended to a generating set for the entire algebra \( \bigoplus H^0(Z_0, \omega_{Z_0}^{-k}) \) as indicated in table 5. The additional generators are \( (y^2z, xy^2, y^3) \) in case \( E_6 \), \( (xy^2, 4y^3, 8y^5z) \) in case \( E_7 \), and \( (y^3, xy^5, y^8z) \) in case \( E_8 \).
Since each of the bundles involved is trivial, these generators on the central fiber can be lifted to generators of \( L \) itself. The only thing left to show is that in the cases of \( E_7 \) and \( E_8 \), we may use the Jacobian determinant as the generator of top degree. To see this, we only need to note that this Jacobian determinant does indeed belong to the algebra (since it satisfies the base conditions), and its restriction mod \( \mathfrak{m} \) is \( 8y^5z \), resp. \( y^6z \), as required.

Now let \( \bar{X}, \bar{Y}, \bar{Z}, \bar{W} \) be a good generating set for \( L \), and let \( \bar{\Phi}_{E_n} \) be the defining polynomial of \( \mathcal{P} \) with respect to this generating set. In cases \( E_7 \) and \( E_8 \), the map determined by \( \bar{Y}, \bar{Z}, \bar{W} \) expresses \( \mathcal{P} \) as a double cover of a weighted projective space. Since \( \bar{X} \) is the Jacobian determinant of this map (up to a constant), it vanishes exactly on the ramification locus of this double cover. It follows that \( \bar{\Phi}_{E_n} \) takes the form \( -\bar{X}^2 + \bar{\Phi}_{E_n}(\bar{Y}, \bar{Z}, \bar{W}) \) in these two cases.

In all three cases, we know the weight \( w \) of \( \bar{\Phi}_{E_n} \), and also its graded degree \( d \) in the algebra \( L \). Any monomial \( m \) in \( \bar{X}, \bar{Y}, \bar{Z}, \bar{W} \) which appears in \( \bar{\Phi}_{E_n} \) must have graded degree \( d \), and weight \( w_m \leq w \). We know the weight of the coefficient of the monomial will then be \( w - w_m \). Using table \( \mathbf{[1]} \) to determine the leading order terms, we can then write \( \bar{\Phi}_{E_n} \) as follows, with undetermined coefficients\( \mathbf{[2]} \). (We adopt the convention that equation numbers which are followed by \( a \), \( b \), or \( c \) refer to the cases of \( E_6 \), \( E_7 \), or \( E_8 \), respectively.) The notation is chosen so that the subscript on a coefficient shows its weight.

\[
\begin{align*}
\bar{\Phi}_{E_6} &= -\bar{X}^2 \bar{W} - \bar{X} \bar{Z}^2 + \bar{Y}^3 + \bar{\varphi}_1 \bar{Y}^2 \bar{Z} + \bar{\varphi}_2 \bar{X} \bar{Y} \bar{W} + \bar{\varepsilon}_2 \bar{Y} \bar{Z}^2 \\
&{} + \bar{\varphi}_3 \bar{X} \bar{Z} \bar{W} + \bar{\varphi}_4 \bar{Y}^2 \bar{W} + \bar{\varphi}_5 \bar{Y} \bar{Z} \bar{W} + \bar{\varphi}_6 \bar{X} \bar{W}^2 \\
&{} + \bar{\varepsilon}_6 \bar{Z}^2 \bar{W} + \bar{\varepsilon}_8 \bar{Y} \bar{W}^2 + \bar{\varepsilon}_9 \bar{Z} \bar{W}^2 + \bar{\varepsilon}_{12} \bar{W}^3 \tag{13a}
\end{align*}
\]

\[
\begin{align*}
\bar{\Phi}_{E_7} &= -\bar{X}^2 - \bar{Y}^3 \bar{W} + 16 \bar{Y} \bar{Z}^3 + \bar{\varepsilon}_2 \bar{Y}^2 \bar{Z} \bar{W} + \bar{\varphi}_2 (16 \bar{Z}^4 - \bar{Y}^2 \bar{Z} \bar{W}) \\
&{} + \bar{\varphi}_4 \bar{Y} \bar{Z}^2 \bar{W} + \bar{\varepsilon}_6 \bar{Y} \bar{Z} \bar{W}^2 + \bar{\varphi}_6 (16 \bar{Z}^3 \bar{W} - \bar{Y}^2 \bar{W}^2) \\
&{} + \bar{\varepsilon}_8 \bar{Y} \bar{Z} \bar{W}^2 + \bar{\varepsilon}_{10} \bar{Z}^2 \bar{W}^2 + \bar{\varepsilon}_{12} \bar{Y} \bar{W}^3 + \bar{\varepsilon}_{14} \bar{Z} \bar{W}^3 + \bar{\varepsilon}_{18} \bar{W}^4 \tag{13b}
\end{align*}
\]

\[
\begin{align*}
\bar{\Phi}_{E_8} &= -\bar{X}^2 + \bar{Y}^3 - \bar{Z}^5 \bar{W} + \bar{\varepsilon}_2 \bar{Y} \bar{Z}^3 \bar{W} + \bar{\varphi}_4 \bar{Y}^2 \bar{Z} \bar{W} + \bar{\varphi}_6 \bar{Z}^4 \bar{W} \\
&{} + \bar{\varepsilon}_8 \bar{Y} \bar{Z} \bar{W}^2 + \bar{\varphi}_{10} \bar{Y}^2 \bar{W}^2 + \bar{\varepsilon}_{12} \bar{Z}^3 \bar{W}^3 + \bar{\varepsilon}_{14} \bar{Y} \bar{Z} \bar{W}^3 \\
&{} + \bar{\varepsilon}_{18} \bar{Z}^2 \bar{W}^4 + \bar{\varepsilon}_{20} \bar{Y} \bar{W}^4 + \bar{\varepsilon}_{24} \bar{Z} \bar{W}^5 + \bar{\varepsilon}_{30} \bar{W}^6 \tag{13c}
\end{align*}
\]

Notice that this polynomial restricts to one in preferred versal form in the affine \( W = 1 \) exactly when all of the \( \bar{\varphi}_i \) terms vanish. We

\[\text{The two strange terms in the expression for } \bar{\Phi}_{E_7} \text{ are yet another artifact of making our notation match that of Bramble } \mathbf{[3]}.\]
therefore wish to make a change of generating set to ensure that this occurs.

Suppose that \(X,Y,Z,W\) is another good generating set of \(L\). Since the mod \(m\) restriction of a good generating set is determined by table 3, the change of generators must restrict to the identity mod \(m\). Moreover (as follows from properties of the Jacobian determinant), in cases \(E_7\) and \(E_8\) we have \(\bar{X} = X\). Considering as before the graded degrees of elements of the algebra \(L\), and the fact that each term in the equation expressing the change of generators must have a coefficient with nonnegative weight, we find that the change of generators must take the form shown below, with undetermined coefficients \(\psi_i\).

\[
\begin{align*}
\bar{X} &= X + \psi_2 Y + \psi_3' Z + \psi_6 W \\
\bar{Y} &= Y + \psi_1 Z + \psi_3 W \\
\bar{Z} &= Z + \psi_3'' W \\
\bar{W} &= W \\
\end{align*}
\]

\[
\begin{align*}
\bar{X} &= X \\
\bar{Y} &= Y + \psi_2 Z + \psi_6 W \\
\bar{Z} &= Z + \psi_4 W \\
\bar{W} &= W \\
\end{align*}
\]

\[
\begin{align*}
\bar{X} &= X \\
\bar{Y} &= Y + \psi_4 Z W + \psi_{10} W^2 \\
\bar{Z} &= Z + \psi_6 W \\
\bar{W} &= W \\
\end{align*}
\]

To finish the proof, we substitute equation (14) into equation (13) and collect the coefficients of the monomials \(\{Y^2Z, XYW, Z^3, XZW, Y^2W, XW^2\}\) in case \(E_6\), \(\{Z^4, YZ^2W, Z^3W\}\) in case \(E_7\), and \(\{Y^2ZW, Z^4W^2, Y^2W^2\}\) in case \(E_8\). (These are the monomials which we cannot allow if we wish to achieve preferred versal form.) Equating all such coefficients to zero gives the following set of equations.

\[
\begin{align*}
0 &= 3\psi_1 + \bar{\psi}_1 \\
0 &= -2\psi_2 + \bar{\psi}_2 \\
0 &= -\psi_3' + \bar{\psi}_3' + \bar{\varepsilon}_2 \psi_1 + \psi_1^3 + \bar{\psi}_1 \psi_1^2 \\
0 &= -2\psi_3' - 2\psi_3'' + \bar{\psi}_3'' + \bar{\psi}_2 \psi_1 \\
0 &= 3\psi_4 + \bar{\psi}_1 \psi_3' - \psi_2^2 + \bar{\psi}_2 \psi_2 + \bar{\psi}_4 \\
0 &= -2\psi_6 + \bar{\psi}_2 \psi_4 + \bar{\phi}_3' \psi_3' - \psi_3'^2 + \bar{\psi}_6 \\
\end{align*}
\]
0 = 16\psi_2 + 16\bar{\phi}_2
0 = 48\psi_4 - 3\psi_2^2 + \bar{\phi}_4 + 2\bar{\epsilon}_2\psi_2 - 2\bar{\phi}_2\psi_2
0 = 16\psi_6 + 16\bar{\phi}_6 - \psi_2^3 + 48\psi_2\psi_4 + \bar{\epsilon}_2\psi_2^2
+ 64\bar{\phi}_2\psi_4 - \bar{\phi}_2\psi_2^2 + \bar{\phi}_4\psi_2
(15b)

0 = 3\psi_4 + \bar{\phi}_4
0 = -5\psi_6 + \bar{\phi}_6 + \bar{\epsilon}_2\psi_4
0 = 3\psi_{10} + \bar{\phi}_{10} + \bar{\phi}_4\psi_6
(15c)

These equations are in a kind of triangular form, and it is clear they can be solved for the unknown coefficients \psi_i in terms of the coefficients \bar{\epsilon}_i and \bar{\phi}_i. Making the corresponding change of generator produces the desired generators for L. Q.E.D.

Remark. For the purposes of practical computation of these coefficients \psi_i (which we carry out in section 9) it is important to observe that these equations only depend on a subset of the \bar{\epsilon}_i and \bar{\phi}_i. Explicitly, in case E_6, they depend on \{\bar{\phi}_1, \bar{\phi}_2, \bar{\epsilon}_2, \bar{\phi}_3, \bar{\phi}_4, \bar{\phi}_6\}; in case E_7, on \{\bar{\epsilon}_2, \bar{\phi}_2, \bar{\phi}_4, \bar{\phi}_6\}; and in case E_8, on \{\bar{\epsilon}_2, \bar{\phi}_4, \bar{\phi}_6, \bar{\phi}_{10}\}.

To complete the proof of theorem 1, we first observe that the equation \(W = 0\) defines \(C \subset \bar{\mathcal{P}}\) in the generators we are considering. What we still need to check is that the defining polynomial \(\Phi_{E_n}\) which we have found is invariant under the Weyl group, and so can be used to define \(\mathcal{X}\) as well as \(\bar{\mathcal{P}}\). This essentially follows from an argument of Pinkham [25, pp. 196-198], in the following way.

Let \(\tilde{\mathcal{X}} \to \text{Def}(X_0)\) be the \(\mathbb{C}^*\)-semi-universal deformation given by the hypersurface in preferred versal form. (The defining polynomial of that hypersurface is the polynomial \(\Phi_{E_n}\), with \(W\) set equal to 1.) Since \(\mathcal{P}\) is a deformation of \(X_0\) with \(\mathbb{C}^*\)-action, there is a natural \(\mathbb{C}^*\)-equivariant map \(\rho: V \to \text{Def}(X_0)\) such that \(\mathcal{P} \cong \tilde{\mathcal{X}} \times_\rho V\). (The effect of proposition \([1]\) is to compute this map explicitly.) The isomorphism \(\mathcal{P} \cong \tilde{\mathcal{X}} \times_\rho V\) is in fact \(\mathfrak{W}\)-equivariant, where \(\mathfrak{W}\) acts by Cremona transformations on the left, and as the Weyl group on the right. Thus, \(\rho\) factors through a map \(V/\mathfrak{W} \to \text{Def}(X_0)\). But the weights of the \(\mathbb{C}^*\)-actions are the same for these two spaces, so the map is an isomorphism. It follows that \(\mathcal{P}/\mathfrak{W}\) gives a semi-universal deformation \(\mathcal{X}\), and that \(\Phi_{E_n}\) is \(\mathfrak{W}\)-invariant.
This argument also implies that the \( W \)-invariant functions \( \varepsilon_i \) serve as generators of the ring \( \mathbb{C}[V]^{W} \) of Weyl group invariants. Those generators (multiplied by appropriate integers) are shown explicitly in cases \( E_6 \) and \( E_7 \) in Appendices 1 and 2, respectively.

6. Singularities of the simultaneous partial resolutions.

In this section, we show that the semi-universal deformations constructed in theorem 1 are essentially unique (up to the \( \mathbb{C}^* \)-action). We then use this uniqueness result to study the singularities of simultaneous partial resolutions.

Lemma 4. Let \( V \) be the root space of an irreducible root system \( R \). Give \( V \) the linear \( \mathbb{C}^* \)-action \( x \mapsto \lambda \cdot x \) for all \( x \in V \), \( \lambda \in \mathbb{C}^* \). Let \( \gamma: V \to V \) be a \( \mathbb{C}^* \)-equivariant map such that \( \gamma(v^\perp) \subset v^\perp \) for all \( v \in R \). Then \( \gamma = \lambda \cdot 1_V \) for some constant \( \lambda \in \mathbb{C} \).

Proof. We first note that \( \gamma \) must be a linear map. For if we expand \( \gamma \) in a Taylor series at 0 and compare coefficients in the equation \( \gamma(\lambda x) = \lambda \gamma(x) \), we see that all higher order terms must vanish.

Let \( \{v_i\} \) be a basis of \( V \) consisting of roots \( v_i \in R \). Define vectors \( v_j^\vee \in V \) by the property \( (v_j^\vee | v_i) = \delta_{ij} \). (The vectors \( v_j^\vee \) correspond to the elements of the dual basis \( v_j^* \) of \( V^* \) under the isomorphism \( V \cong V^* \) determined by the inner product.)

Now \( \mathbb{C} \cdot v_j^\vee = \bigcap_{i \neq j} v_i^\perp \). Thus \( v_j^\vee \in \bigcap_{i \neq j} v_i^\perp \), which implies that \( \gamma(v_j^\vee) \in \bigcap_{i \neq j} v_i^\perp \). Hence \( \gamma(v_j^\vee) = \lambda_j v_j^\vee \) for some constant \( \lambda_j \).

If the vertex \( v_j \) is connected to the vertex \( v_k \) in the Dynkin diagram, then \( v_j + v_k \) must be a root as well. Let \( w_{jk} = v_j^\vee - v_k^\vee \). Then \( w_{jk} \) lies in the space \( (v_j + v_k)^\perp \cap \bigcap_{i \neq j,k} v_i^\perp \), and in fact it spans that space. Thus, since that space is preserved by \( \gamma \), we must have \( \gamma(w_{jk}) = \lambda_{jk} w_{jk} \) for some constant \( \lambda_{jk} \). But then

\[
\lambda_{jk}(v_j^\vee - v_k^\vee) = \gamma(w_{jk}) = \gamma(v_j^\vee) - \gamma(v_k^\vee) = \lambda_j v_j^\vee - \lambda_k v_k^\vee.
\]

It follows that \( \lambda_j = \lambda_k = \lambda_{jk} \).

Thus, since \( R \) has a connected Dynkin diagram, all the \( \lambda_j \) must be equal to the same constant \( \lambda \). Q.E.D.

Let \( Z_0 \to X_0 \) be the minimal resolution of a rational double point of type \( S \), and let \( \mathcal{X} \to V/W \) and \( Z \to \mathcal{X} \times_{V/W} V \) be the deformation and simultaneous resolution constructed in theorem 1 (which we fix once and for all). We call \( \mathcal{X} \to V/W \) the standard deformation and \( Z \to \mathcal{X} \times_{V/W} V \) the standard simultaneous resolution of type \( S \). The coefficients of the defining polynomial \( \Phi_S \) of \( \mathcal{X} \) are specific functions on the deformation space: we call them the standard coordinate functions.
on $\text{Def}(S) = V/W$. We call the standard coordinate function of highest weight the “constant term”, since it occurs as the constant term in the defining polynomial of the hypersurface. Notice that all of these definitions depend on identifying the type as $S$.

**Theorem 2.** Let $X' \to \mathcal{D}$ be a nontrivial $\mathbb{C}^*$-equivariant deformation of $X_0$, and let $\mathcal{R}$ be a vector space with a linear $\mathbb{C}^*$-action of pure weight one. Suppose that $\mathcal{R} \to \mathcal{D}$ is a $\mathbb{C}^*$-equivariant map, and that there is a simultaneous resolution $Z' \to X' \times_{\mathcal{D}} \mathcal{R}$ inducing $Z_0 \to X_0$. For each root $v \in \mathcal{R}$, let $H_v \subset \mathcal{R}$ be the locus in $\mathcal{R}$ parametrizing deformations of $Z_0$ to which $C_v$ lifts.

Suppose that $\alpha: \mathcal{R} \to V$ is a $\mathbb{C}^*$-equivariant surjective map such that $\alpha(H_v) \subset v^\perp$ for all $v \in \mathcal{R}$. Then $\alpha$ descends to a map $\tilde{\alpha}: \mathcal{D} \to V/W$, and $X'$ is isomorphic to $X \times_{\tilde{\alpha}} \mathcal{D}$, the pullback of the standard deformation via $\tilde{\alpha}$.

**Proof.** Let $X \to V/W$ be the standard deformation, which is $\mathbb{C}^*$-semi-universal by construction. By a theorem of Pinkham [24], since $X' \to \mathcal{D}$ is a $\mathbb{C}^*$-equivariant deformation of $X_0$, there is a $\mathbb{C}^*$-equivariant map $\tilde{\beta}: \mathcal{D} \to V/W$ such that $X'$ is isomorphic to the pullback $X \times_{\tilde{\beta}} D$. Since $X'$ admits a simultaneous resolution after base change to $\mathcal{R}$, there is a map $\beta: \mathcal{R} \to V$ which induces $\tilde{\beta}$. Since the locus in $V$ parametrizing deformations of $Z_0$ to which $C_v$ lifts is $v^\perp$ by construction, the functoriality of that property of deformations implies that $\beta(H_v) \subset v^\perp$ for all $v \in R$.

It will suffice to show that $\beta = \lambda \cdot \alpha$ for some $\lambda \in \mathbb{C}$. For if that is the case, then $\lambda$ cannot be zero since $X' \to \mathcal{D}$ is a nontrivial deformation. The desired map $\tilde{\alpha}$ will then simply be $\lambda^{-1} \cdot \tilde{\beta}$, and by using the action of $\lambda \in \mathbb{C}^*$ on $X$, $X \times_{\tilde{\alpha}} \mathcal{D}$ will be isomorphic to $X \times_{\beta} \mathcal{D}$, and therefore to $X'$.

To show that $\beta = \lambda \cdot \alpha$, let $\{v_1, \ldots, v_r\}$ be a root basis of $V$, and define

$$K = \bigcap_{i=1}^r H_{v_i}.$$ 

Since $\bigcap_{i=1}^r v_i^\perp = (0)$, we have $K \subset \ker \alpha$ and $K \subset \ker \beta$. There are thus induced maps $\tilde{\alpha}: \mathcal{R}/K \to V$ and $\tilde{\beta}: \mathcal{R}/K \to V$, and it will suffice to show that $\tilde{\beta} = \lambda \cdot \tilde{\alpha}$.

Since $\alpha$ is surjective, if we define $m := \dim \mathcal{R}$ then we have

$$m - r \leq \dim K \leq \dim \ker \alpha = m - r,$$

which implies that $\ker \alpha = K$. Thus, $\tilde{\alpha}$ is an isomorphism. Let $\gamma := \tilde{\beta} \circ \tilde{\alpha}^{-1} \in \text{Aut} V$. $\gamma$ is clearly $\mathbb{C}^*$-equivariant. And for each $v \in R$ we
have
\[ \gamma(v^\perp) = \beta(\mathcal{H}_v \mod \mathcal{K}) \subset v^\perp. \]
Therefore, \( \gamma \) satisfies the hypotheses of the lemma, so \( \gamma = \lambda \cdot 1_V \) for some \( \lambda \). It follows that \( \beta = \lambda \cdot \alpha \), as required. Q.E.D.

**Corollary.** If \( X' \to V/\mathcal{W} \) is any \( \mathbb{C}^* \)-semi-universal deformation parametrized by \( V/\mathcal{W} \) such that the spaces \( \mathcal{H}_v \) coincide with the linear subspaces \( v^\perp \subset V \), then \( X' \) is isomorphic to the standard deformation \( X \). In particular, \( X' \) can be embedded as a hypersurface in \( \mathbb{C}^3 \times V/\mathcal{W} \) with defining polynomial given in table 3, where the coefficients in the defining polynomial are the standard coordinate functions on \( V/\mathcal{W} \).

We define **standard coordinate functions** on \( V/\mathcal{W} \) in two other cases. The formulas for the standard coordinate functions on \( V_{D_n}/\mathcal{W}_{D_n} \) given in equation (11) make sense even when \( n = 2 \), and generate \( \mathbb{C}[V_{D_2}^{\mathcal{W}_{D_2}}] \); we call them the **standard coordinate functions** on \( V_{D_2}/\mathcal{W}_{D_2} \). And in the case of \( E_3 \), we define the **standard coordinate functions** on \( V_{E_3}/\mathcal{W}_{E_3} \) to be the generators \( \varepsilon_2^{(1)} \), \( \varepsilon_2^{(2)} \), and \( \varepsilon_3 \) of \( \mathbb{C}[V_{E_3}^{\mathcal{W}_{E_3}}] \), which are given in equation (9).

In these cases, the functions do not directly have an interpretation as coefficients of a semi-universal deformation. We relate the standard coordinate functions on \( V_{D_2}/\mathcal{W}_{D_2} \) to the deformation theory in the next lemma. A similar calculation could be done for \( E_3 \), but we omit it since we do not need the result.

**Lemma 5.** Let \( f_{D_2}(U; t) \) be the distinguished polynomial for \( D_2 \), let \( \gamma_2 = f_{D_2}(0; t) \), and let \( g_{D_2}(Z; t) = Z^2 + \delta_2 Z + (\gamma_2)^2 \) be the second distinguished polynomial, so that \( \gamma_2 \) and \( \delta_2 \) are the standard coordinate functions on \( V_{D_2}/\mathcal{W}_{D_2} \). Then the “constant terms” for the two root systems of type \( A_1 \) which are the irreducible constituents of \( R_{D_2} \) are \(-\frac{1}{4}(\delta_2 - 2\gamma_2)\) and \(-\frac{1}{4}(\delta_2 + 2\gamma_2)\) respectively.

**Proof.** The irreducible constituents of \( R_{D_2} \) are spanned by the root vectors \( v_1 \) and \( v_2 \), respectively. Let \( t'_1, t'_2 \) be the distinguished functionals for the \( A_1 \) spanned by \( v_1 \), and \( t''_1, t''_2 \) be those for \( v_2 \). The “constant terms” are then \( t'_1 t'_2 \) and \( t''_1 t''_2 \), respectively. Moreover, the definition of distinguished functionals implies

\[
\begin{align*}
  t'_1 &= v'_1, & t'_2 &= -v'_1, \\
  t''_1 &= v''_2, & t''_2 &= -v''_2.
\end{align*}
\]

Now according to table [1],
\[ v_1^* = \frac{1}{2} s_1 - t_2 = \frac{1}{2} (t_1 - t_2) \]
\[ v_2^* = \frac{1}{2} s_1 = \frac{1}{2} (t_1 + t_2), \]
where \( t_1 \) and \( t_2 \) are the distinguished functionals for \( D_2 \). Thus,
\[ t'_1 t'_2 = -(v_1^*)^2 = -\frac{1}{4} (t_1 - t_2)^2 = -\frac{1}{4} (s_1^2 - 4s_2) \]
\[ t''_1 t''_2 = -(v_2^*)^2 = -\frac{1}{4} (t_1 + t_2)^2 = -\frac{1}{4} s_1^2. \]
If we write \( f_{D_2}(U; t) = U^2 + s_1 U + s_2, \) then \( \gamma_2 = s_2 \). Furthermore, \( g_{D_2}(-U^2; t) = U^4 + (2s_2 - s_1^2)U^2 + s_2^2, \) which implies that \( \delta_2 = s_1^2 - 2s_2 \). It follows that \( t'_1 t'_2 = -\frac{1}{4} (\delta_2 - 2\gamma_2) \) and \( t''_1 t''_2 = -\frac{1}{4} (\delta_2 + 2\gamma_2). \) Q.E.D.

We now turn to the study of simultaneous partial resolutions. Let \( Y_0 \to X_0 \) be a partial resolution of a singularity of type \( S \), and let \( Z_0 \to Y_0 \) be the minimal resolution. The \( \mathbb{C}^* \)-action on \( X_0 \) lifts to a \( \mathbb{C}^* \)-action on \( Y_0 \). As in section 1, there is an associated partial resolution graph \( \Gamma_0 \subset \Gamma \). We write \( \Gamma - \Gamma_0 = \bigcup \Gamma_i \) as a union of its connected components, and let \( \mathcal{W}_0 = \prod \mathcal{W}_i \) be the subgroup of \( \mathcal{W} \) generated by reflections corresponding to vertices of \( \Gamma - \Gamma_0 \). (Such vertices are illustrated with closed circles (●) in figures [1] and [2].) The components \( \Gamma_i \) correspond to the singular points \( Q_i \) of \( Y_0 \).

Let \( \mathcal{Z} \to \mathcal{X} \) be the standard simultaneous resolution of type \( S \). By the techniques of [35], deformations of \( Z_0 \) can be partially blown down to deformations of \( Y_0 \), and \( \mathbb{C}^* \)-actions are preserved when this is done. Doing this universally gives a family \( \hat{\mathcal{Z}} \to V \). Moreover, as Pinkham argues in [27], the map \( \mathcal{Z} \to V \) is \( \mathcal{W}_i \)-equivariant for each \( i \), since it provides a model for simultaneous resolution of \( Q_i \). (The argument uses a result of Burns and Wahl [8], as refined by Pinkham [26].) Thus, if we let \( \mathcal{Y} = \hat{\mathcal{Z}}/\mathcal{W}_0 \), then \( \mathcal{Y} \to V/\mathcal{W}_0 \) is a \( \mathbb{C}^* \)-semi-universal deformation of \( Y_0 \). We call this the **standard simultaneous partial resolution** of type \( (S, \Gamma_0) \).

Let \( Y^{(i)}_0 \) resp. \( \mathcal{Y}^{(i)} \) be the union of all \( \mathbb{C}^* \)-orbits in \( Y_0 \) resp. \( \mathcal{Y} \) whose closure contains \( Q_i \). Then \( Y^{(i)}_0 \) is a \( \mathbb{C}^* \)-neighborhood of \( Q_i \in Y_0 \) and \( \mathcal{Y}^{(i)} \to V/\mathcal{W}_0 \) is a deformation of \( Y^{(i)}_0 \).

Now \( Q_i \in Y^{(i)}_0 \) itself a rational double point, whose associated root system \( R_i \) is isomorphic to the subsystem of \( R \) spanned by the roots from \( \Gamma_i \). If we specify the type of this rational double point as one of \( A_{n-1}, D_n, \) or \( E_n \), and identify the complex root space of the root system \( R_i \) with the subspace \( V_i \subset V \) spanned by the roots from \( \Gamma_i \),
then there is an associated standard deformation $X_i \to V_i/W_i$ of type $R_i$.

**Theorem 3.** Let $pr_i: V/W_0 \to V_i/W_i$ be the map induced by the orthogonal projection $V \to V_i$. Then $Y^{(i)}$ is isomorphic to $X_i \times_{pr_i} V/W_0$. In other words, there is a neighborhood of $Q_i \in Y$ which can be embedded in $\mathbb{C}^3 \times V/W_0$ in such a way that the coefficients of the defining polynomial are pullbacks via $pr_i$ of the standard coordinate functions on $V_i/W_i$.

**Proof.** Let $Z \to X \times V$ be the standard simultaneous resolution. Denoting the quotient map $V \to V/W_0$ by $\sigma$, there is a simultaneous resolution $Z^{(i)} \to Y^{(i)} \times \sigma V$, where $Z^{(i)}$ is the union of all $\mathbb{C}^*$-orbits in $Z$ whose closure intersects some exceptional curve lying over $Q_i$. For each root $v \in R_i$, the locus in $V$ parametrizing deformations of $Z_0$ to which $C_v$ lifts is precisely the orthogonal complement of $v$ in $V$. This space is clearly mapped into the orthogonal complement of $v$ in $V_i$ by the orthogonal projection $V \to V_i$. The theorem follows by applying theorem 2 to this situation, using the root system $R_i$ in place of $R$, and the orthogonal projection in place of $\alpha$. Q.E.D.

Consider now the case of an irreducible small resolution, so that $\Gamma_0$ consists of a single vertex $v$. If we write $\Gamma - \{v\} = \bigcup \Gamma_i$ as a union of its irreducible components, then theorem 3 provides us with a natural isomorphism $\text{PRes}(S, v) \cong V/W_0$, compatible with the maps $pr_i: V/W_0 \to V_i/W_i$ which are induced by the orthogonal projections. In fact, $W_0$ fixes the hyperplane $\ker(v^*) \subset V$, and the projections $pr_i$ induce an isomorphism $\ker(v^*)/W_0 \cong \bigoplus V_i/W_i$. Here, $v^* \in V^*$ is the element dual to $v$ in the basis dual to the root basis.

We define standard coordinate functions on the partial resolution spaces $\text{PRes}(S, v)$ in the following way. The definition depends on choosing a decomposition $\Gamma - \{v\} = \bigcup \Gamma^{(j)}$, where this time each $\Gamma^{(j)}$ is a union of connected components of $\Gamma - \{v\}$. We assume that each corresponding root system $R^{(j)}$ is either irreducible or of type $D_2$ or $E_3$; the definition also depends on identifying the type of each $R^{(j)}$ as one of $A_{k-1}$, $D_k$, or $E_k$. We define the **standard coordinate functions on** $\text{PRes}(S, v) \cong V/W_0$ to be the linear functional $v^*$ together with the pullbacks of the standard coordinate functions on the spaces $V^{(j)}/W_0$ via the mappings $V/W_0 \to V^{(j)}/W_0$ which are induced by orthogonal projection. These will be the coefficients of the defining polynomials of various open sets on the partial resolution, except in the cases where one of the constituents is a reducible root system of type $D_2$ or $E_3$. 
7. Relations among distinguished polynomials.

Let $R$ be a root system of type $S = A_{n-1}, D_n,$ or $E_n$. The complex root space $V_R$ has a root basis which can be written in the form $v_\alpha, v_{\alpha+1}, \ldots, v_\beta$, where $\alpha, \alpha + 1, \ldots, \beta$ is a sequence of consecutive integers. There is a natural dual basis $v_\alpha^*, v_{\alpha+1}^*, \ldots, v_\beta^*$ of the dual space $V^*$.

The basis element $v_k$ can be regarded as a vertex of the Dynkin diagram $\Gamma_R$. We fix $v_k$, and regard the complement $\Gamma_R - \{v_k\}$ as forming two root systems, either of which may be empty or reducible: $R'$ is the part spanned by vertices to the left of $v_k$ in $\Gamma_R - \{v_k\}$, and $R''$ is the part spanned by vertices to the right. (We use the orientation of the Dynkin diagrams displayed in section 2.) This is admittedly ambiguous in a few cases, so to make everything completely clear, we specify that when $(S,v_k) = (D_n,v_{n-1})$ or $(E_n,v_0)$, we take $R'$ to be spanned by $\Gamma_R - \{v_k\}$, and $R''$ to be empty.

We fix types $S'$ and $S''$ of the root systems $R'$ and $R''$, as indicated in table 6. (We have omitted the case $(S,v_k) = (D_n,v_n)$, since we will not need it later.) Having identified the type $S'$, there are distinguished functionals $t'_1, \ldots, t'_n$, and a distinguished polynomial $f_{S'}(U; t')$. The coefficients of this polynomial are denoted by $s'_i$, and the standard coordinate functions on $V'/W'$ are denoted by $\alpha'_i$, $\gamma'_i$, $\delta'_i$, or $\varepsilon'_i$, as appropriate. We use analogous notation for $R''$, replacing "'" by "" throughout.

By composing with the orthogonal projection maps $V_R \to V'_R$ and $V_R \to V''_R$, we can regard the distinguished functionals on $V'_R$ and $V''_R$ as linear functionals on $V_R$. We denote them again by $t'_i$ and $t''_i$, suppressing mention of the orthogonal projection maps for simplicity of notation.

**Proposition 2.** Let $v_k$ be a vertex of the Dynkin diagram $\Gamma_R$, and let $\tilde{v}_k \in V_R$ be the vector specified in table [3]. Then there is an orthogonal direct sum decomposition

$$V_R = (C \cdot \tilde{v}_k) \oplus V'_R \oplus V''_R,$$

and the dual space $V^*_R$ can be generated by the distinguished functionals of $R'$ and $R''$ together with the functional $\mu_1 := v_k^*.$

The distinguished functionals of $R$ are therefore linear combinations of these generators, and those linear combinations can be expressed by means of a relation among distinguished polynomials. The relation involves $f_{S'}(U; t), f_{S''}(U; t'), f_{S'''}(U; t''),$ and $\mu_1$, with some linear changes of variable and possible extra linear factors, and is given explicitly in table [4].
| $S$ | $k$ | $S'$ | $S''$ | $\bar{v}_k$ |
|-----|-----|------|-------|-------------|
| $A_{n-1}$ | any | $A_{k-1}$ | $A_{n-k-1}$ | $v_k + \sum_{i=1}^{k-1} \frac{i}{k} v'_i + \sum_{i=1}^{n-k} \frac{n-k-i}{n-k} v''_i$ |
| $D_n$ | $\leq n-2$ | $A_{k-1}$ | $D_{n-k}$ | $v_k + \sum_{i=1}^{k-1} \frac{i}{k} v'_i + \sum_{i=1}^{n-k} \frac{n-k-i}{n-k} v''_i + \frac{1}{2} v''_{n-k-1} + \frac{1}{2} v''_{n-k}$ |
| $D_n$ | $n$ | $A_{n-1}$ | | $v_n + \sum_{i=1}^{n-2} \frac{2i}{n} v'_i + \frac{n-2}{n} v''_{n-1}$ |
| $E_n$ | 0 | $A_{n-1}$ | | $v_0 + \frac{n-3}{n} v'_1 + \frac{2n-6}{n} v'_2 + \sum_{i=3}^{n-1} \frac{3n-3i}{n} v'_i$ |
| $E_n$ | 1 | $D_{n-1}$ | | $v_1 + \sum_{i=1}^{n-2} \frac{i}{2} v'_i + \frac{n-1}{4} v''_{n-2} + \frac{n-3}{4} v''_{n-1}$ |
| $E_n$ | 2 | $A_1$ | $A_{n-2}$ | $v_2 + \frac{1}{2} v'_1 + \frac{n-3}{n-1} v''_1 + \sum_{i=2}^{n-2} \frac{2n-2i-2}{n-1} v''_i$ |
| $E_n$ | $\geq 3$ | $E_k$ | $A_{n-k-1}$ | $v_k + \frac{3v'_i + 2v''_i + 4v''_i}{9-k} + \sum_{i=3}^{k-1} \frac{9-i}{9-k} v'_i + \sum_{i=1}^{n-k-1} \frac{n-k-i}{n-k} v''_i$ |

**Table 6.**

Proof. We identify the root bases of $V_{R'}$ and $V_{R''}$ with subsets of the root basis of $V_R$ as follows. If $S \neq E_n$ or $k \neq 1, 2$ we identify $v'_i$ with $v_i$ for all basis vectors of $V_{R'}$, and $v''_i$ with $v_{k+i}$ for all basis vectors of $V_{R''}$. If $S = E_n$ and $k = 1$ or 2, we use the identifications indicated in figure 2. With this notation established, it is easy to check that the vector $\bar{v}_k$ defined in table 3 is orthogonal to both $V_{R'}$ and $V_{R''}$. Moreover, since $\Gamma_{R'}$ is disjoint from $\Gamma_{R''}$, the spaces $V_{R'}$ and $V_{R''}$ are themselves mutually perpendicular. The claimed orthogonal direct sum decomposition follows.

The orthogonal direct sum decomposition can be regarded as a change of basis from $\{v_i\}$ to $\{v'_i\} \cup \{v''_i\} \cup \{\bar{v}_k\}$ in the space $V$. If we use maps $\sigma'$ and $\sigma''$ to describe the identification of root bases, then this change of basis can be written

$$v'_i = v_{\sigma'(i)}$$
$$v''_i = v_{\sigma''(i)}$$
$$\bar{v}_k = \sum a'_i v'_i + \sum a''_i v''_i,$$

where the coefficients $a'_i$ and $a''_i$ are found in table 3. (In all but two cases, $\sigma'(i) = i$ and $\sigma''(i) = k + i$.) The corresponding change of dual
basis takes the form
\[
\begin{align*}
v_{\sigma'(i)}^* &= a'_i \bar{v}_k^* + v_i'^* \\
v_{\sigma''(i)}^* &= a''_i \bar{v}_k^* + v_i''^* \\
v_k^* &= \bar{v}_k^*.
\end{align*}
\]

It follows that \( \mu_1 = v_k^* = \bar{v}_k^* \) can be used along with the distinguished functionals on \( V_{R'} \) and \( V_{R''} \) to generate \( V_R^* \).

To finish the proof, we must carry out the calculation which leads to table 7. We will do this in a few cases, and leave the remaining ones to the reader.

We first treat an easy case: the case \( S = A_{n-1} \). Using the fourth column of table 1 applied to \( R' \) and \( R'' \), we can write the change of

| \( S \) | \( k \) |
|-------|-------|
| \( A_{n-1} \) | any | \( f_{A_{n-1}}(U; t) = f_{A_{k-1}}(U + \frac{1}{k} \mu_1; t') \cdot f_{A_{n-k-1}}(U - \frac{1}{n-k} \mu_1; t'') \) |
| \( D_n \leq n-2 \) | \( D_n \) | \( f_{D_n}(U; t) = f_{A_{k-1}}(U + \frac{1}{k} \mu_1; t') \cdot f_{D_{n-k}}(U; t'') \) |
| \( D_n \) | \( n \) | \( f_{D_n}(U; t) = f_{A_{n-1}}(U + \frac{2}{k} \mu_1; t') \) |
| \( E_n \) | \( 0 \) | \( f_{E_n}(U; t) = f_{A_{n-1}}(U - \frac{9-n}{3n} \mu_1; t') \) |
| \( E_n \) | \( 1 \) | \( (-1)^n \cdot f_{E_n}(U; t) = (-U + \frac{1}{3} \rho_1 - \frac{9-n}{6} \mu_1) \cdot f_{D_{n-1}}(-U - \frac{1}{6} \rho_1 + \frac{9-n}{12} \mu_1; t') \) |
| \( E_n \) | \( 2 \) | \( f_{E_n}(U; t) = f_{A_1}(U + \frac{2}{3} \sigma_1 + \frac{9-n}{6n-6} \mu_1; t') \cdot \frac{f_{A_{n-2}}(U - \frac{1}{4} \mu_1; t'')}{(U - \frac{1}{4} \sigma_1 - \frac{9-n}{4n-3} \mu_1)} \cdot f_{A_{n-2}}(U - \frac{1}{4} \sigma_1 - \frac{9-n}{4n-3} \mu_1; t'') \cdot f_{A_{n-k-1}}(U - \frac{1}{9-k} \tau_1 - \frac{9-n}{(9-k)(n-k)} \mu_1; t'') \) |
| \( E_n \geq 3 \) | \( E_n \) | \( f_{E_n}(U; t) = f_{E_k}(U; t') \cdot f_{A_{n-k-1}}(U - \frac{1}{9-k} \tau_1 - \frac{9-n}{(9-k)(n-k)} \mu_1; t'') \) |

**Notation:**
- \( \mu_1 \) denotes the coordinate function \( v_k^* \)
- \( \rho_1 \) denotes the coefficient of \( U^{n-2} \) in \( f_{D_{n-1}}(U; t') \)
- \( \sigma_1 \) denotes a root of \( f_{A_{n-2}}(U; t'') \)
- \( \tau_1 \) denotes the coefficient of \( U^{k-1} \) in \( f_{E_k}(U; t') \)

**Table 7.**
basis as
\[
\begin{align*}
v_i^* &= \frac{i}{k} \bar{v}_k^* + v_i'^* = \frac{i}{k} \mu_1 + t'_1 + \cdots + t'_i, \quad 1 \leq i \leq k - 1 \\
v_k^* &= \bar{v}_k^* = \mu_1 \\
v_{k+i}^* &= \frac{n-k-i}{n-k} \bar{v}_k^* + v_i''^* = \frac{n-k-i}{n-k} \mu_1 + t''_1 + \cdots + t''_i, \quad 1 \leq i \leq n - k - 1
\end{align*}
\]

Then using the third column of Table I applied to \(R\), we get
\[
\begin{align*}
t_i &= \frac{i}{k} \mu_1 + t'_i, \quad 1 \leq i \leq k \\
t_{k+i} &= -\frac{i}{n-k} \mu_1 + t''_i, \quad k + 1 \leq k + i \leq n.
\end{align*}
\]

It follows that
\[
f_{A_{n-1}}(U; t) = f_{A_{k-1}}(U + \frac{1}{k} \mu_1; t') \cdot f_{A_{n-k-1}}(U - \frac{1}{n-k} \mu_1; t'').
\]

We next treat the case of \(R = E_n\), with \(k = 1\), which is displayed in the left half of Figure 2. In this case,
\[
\begin{align*}
v_0^* &= \frac{n-3}{4} \bar{v}_1^* + v_{n-1}'^* = \frac{n-3}{4} \mu_1 + \frac{1}{2} s'_1 \\
v_1^* &= \bar{v}_1^* = \mu_1 \\
v_2^* &= \frac{n-1}{4} \bar{v}_1^* + v_{n-2}'^* = \frac{n-1}{4} \mu_1 + \frac{1}{2} s'_1 - t'_{n-1} \\
v_{n-i}^* &= \frac{i}{2} \bar{v}_1^* + v_i'^* = \frac{i}{2} \mu_1 + s'_1 - t'_{i+1} - \cdots - t'_{n-1}, \quad 1 \leq i \leq n - 3
\end{align*}
\]

which implies
\[
\begin{align*}
t_1 &= \frac{n-6}{6} \mu_1 - \frac{1}{3} s'_1 \\
t_{i+1} &= \frac{n-9}{12} \mu_1 + \frac{1}{6} s'_1 - t'_{n-i}, \quad 2 \leq i + 1 \leq n.
\end{align*}
\]

It follows that
\[
f_{E_n}(U; t) = (-1)^{n-1} \cdot (U + \frac{9-n}{6} \mu_1 - \frac{1}{3} s'_1) \cdot f_{D_{n-1}}(-U - \frac{n-9}{12} \mu_1 - \frac{1}{6} s'_1; t'),
\]
since the right-hand side is equal to
\[
(-1)^{n-1} \cdot (U + \frac{n-9}{6} \mu_1 - \frac{1}{3} s'_1) \cdot \Pi(-U - \frac{n-9}{12} \mu_1 - \frac{1}{6} s'_1 + t'_j)
\]
\[
= (U + \frac{n-9}{6} \mu_1 - \frac{1}{3} s'_1) \cdot \Pi(U + \frac{n-9}{12} \mu_1 + \frac{1}{6} s'_1 - t'_j)
\]
\[
= \Pi(U + t_i).
\]

Finally, we treat the case \( R = E_n, k = 2 \), which is displayed in the right half of figure 2. In this case,
\[
v_0^* = \frac{n-3}{n-1} v_2^* + v_1^{**} = \frac{n-3}{n-1} \mu_1 + t_1''
\]
\[
v_1^* = \frac{1}{2} v_2^* + v_1^* = \frac{3}{2} \mu_1 + t'_1
\]
\[
v_2^* = \tau^* = \mu_1
\]
\[
v_{i+1} = \frac{2n-2i}{n-1} v_2^* + v_i'' = \frac{2n-2i}{n-1} \mu_1 + t_1'' + \cdots + t_i'', \quad 2 \leq i \leq n-2
\]
which implies
\[
t_i = \frac{n-9}{6n-6} \mu_1 - \frac{2}{3} t''_1 + t'_i, \quad 1 \leq i \leq 2
\]
\[
t_{i+1} = \frac{n-9}{3n-3} \mu_1 + \frac{1}{3} t''_1 + t''_i, \quad 2 \leq i \leq n-1.
\]
(Notice that the functional \( \frac{n-9}{3n-3} \mu_1 + \frac{1}{3} t''_1 + t''_1 \) is “missing” here.) It follows that
\[
f_{E_n}(U; t) = f_{A_1}(U + \frac{n-9}{6n-6} \mu_1 - \frac{2}{3} t''_1; t') \cdot \frac{f_{A_{n-2}}(U + \frac{n-9}{3n-3} \mu_1 + \frac{1}{3} t''_1; t'')}{(U + \frac{n-9}{3n-3} \mu_1 + \frac{1}{3} t''_1)}.
\]
Since \(-t''_1\) is a root of \( f_{A_{n-1}}(U; t'') \), if we define \( \sigma_1 := -t''_1 \), the formula in the table follows.

All remaining cases are left to the reader. \( \text{Q.E.D.} \)

Proposition 4 provides a method for explicitly calculating the map
\( \text{PRes}(S, v_k) \to \text{Def}(S) \) (which can also be written as \( V/\mathfrak{m} \to \mathfrak{m}/\mathfrak{m} \)).
What we wish to calculate explicitly is the map on coordinate rings
\( \mathbb{C}[V]\mathfrak{m} \subset \mathbb{C}[V]^{\mathfrak{m}} \). In other words, we want to express the standard coordinate functions on \( V/\mathfrak{m} \) as polynomials in the standard coordinate functions on \( V/\mathfrak{m} \).

Now each standard coordinate function \( \varphi_j \) on \( V/\mathfrak{m} \) is a function of \( s_1, \ldots, s_n \) (the coefficients of the distinguished polynomial \( f_S(U; t) \)).
Using proposition 4 these in turn are expressed as functions of the coefficients \( s'_i \) and \( s'_j \) of the distinguished polynomials of \( f_{S'}(U; t') \) and \( f_{S''}(U; t'') \), together with \( \mu_1 \) (and possibly an auxiliary variable \( \rho_1, \sigma_1, \) or \( \tau_1 \) which will eliminate itself in the end). The expression for \( \varphi_j \) in terms of these variables is invariant under \( \mathfrak{m}/\mathfrak{m} \), and so can be expressed as a polynomial in \( \mu_1 \) together with the pullbacks of the standard coordinate functions on \( V'/\mathfrak{m}' \) and \( V''/\mathfrak{m}'' \). One approach to finding this polynomial expression is the method of undetermined coefficients.
We carry this out for the cases of $A_{n-1}$ and $D_n$, collecting the information we require in the form of certain congruences. The part of the computation which we need in the $E_n$ cases will be stated in table 9 in section 8, and verified in section 10.

**Proposition 3.**

1. If $R = A_{n-1}$, then $\alpha_{n-1} \equiv \alpha'_{k-1} \alpha''_{n-k} + \alpha'_{k} \alpha''_{n-k-1} \mod \mu_1$, and $\alpha_n \equiv \alpha'_{k} \alpha''_{n-k} \mod \mu_1$.
2. If $R = D_n$ and $k = 1$, then $\delta_{2n-4} \equiv \delta''_{2n-4} \mod \mu_1$.
3. If $R = D_n$ and $k \leq n-2$, let $J_4$ denote the ideal generated by all monomials of degree 4 in the standard coordinate functions on $\text{PRes}(D_n, v_k)$. Then $\gamma_n \equiv \alpha'_{k} \gamma''_{n-k} \mod \mu_1$, and $\delta_{2n-2} \equiv (\alpha'_{k})^2 \delta''_{2n-2k-2} \mod J_4$.
4. If $R = D_n$ and $k = n$, then $\gamma_n \equiv \alpha'_{n} \mod \mu_1$.

**Proof.** We will prove the third statement, and leave the others (which are easier) to the reader. When $R = D_n$ and $k \leq n-2$ we have

$$f_{D_n}(U; t) = f_{A_{k-1}}(U + \frac{1}{k} \mu_1; t') \cdot f_{D_{n-k}}(U; t'')$$

$$\equiv f_{A_{k-1}}(U; t') \cdot f_{D_{n-k}}(U; t'') \mod \mu_1$$

$$\equiv \alpha'_{k} \gamma''_{n-k} \mod (\mu_1, U)$$

It follows that $\gamma_n \equiv \alpha'_{k} \gamma''_{n-k} \mod \mu_1$. Moreover, if we define $\tilde{g}_{A_{k-1}}(Z; t')$ by

$$\tilde{g}_{A_{k-1}}(-U^2; t') = f_{A_{k-1}}(U + \frac{1}{k} \mu_1; t') \cdot f_{A_{k-1}}(-U + \frac{1}{k} \mu_1; t')$$

then

$$\tilde{g}_{A_{k-1}}(Z; t') \equiv (\alpha'_{k-1}^2 - 2\alpha'_{k-2} \alpha'_{k} + \frac{2}{k} \mu_1 \alpha'_{k-1} \alpha'_{k-2})Z$$

$$+ (\alpha'_{k}^2 + \frac{2}{k} \mu_1 \alpha'_{k-1} \alpha'_{k}) \mod (J_4, Z^2)$$

while

$$g_{D_{n-k}}(Z; t'') \equiv \delta''_{2n-2k-2}Z + (\gamma''_{n-k})^2 \mod Z^2.$$

Hence, if we multiply these congruences and retain only terms of degree at most three in the standard coordinate functions on $\text{PRes}(D_n, v_k)$, we get

$$g_{D_n}(Z; t) \equiv (\alpha'_{k})^2 \delta''_{2n-2k-2}Z \mod (J_4, Z^2).$$

The congruence for $\delta_{2n-2}$ follows.

Q.E.D.

In order to effectively apply the last line of table 9 when $k = 4$ or 5, we need to compute the distinguished polynomials $f_{E_4}(U; t)$ and $f_{E_5}(U; t)$.
Lemma 6. Let \( t_1, \ldots, t_n \) be the distinguished functionals for \( E_4 \), resp. \( E_5 \), and let \( \tilde{t}_1, \ldots, \tilde{t}_5 \) be the distinguished functionals for \( A_4 \), resp. \( D_5 \). If we identify these root systems in such a way that \( \tilde{v}_i = v_i \) for \( 1 \leq i \leq n - 1 \) and \( \tilde{v}_n = v_0 \) then

\[
f_{A_4}(U; \tilde{t}) = (U + \frac{3}{5}s_1) \cdot f_{E_4}(U - \frac{2}{5}s_1; t)
\]

and

\[
f_{D_5}(U; \tilde{t}) = f_{E_5}(U - \frac{1}{2}s_1; t).
\]

Proof. In the case of \( E_4 \), we calculate with table I as follows.

\[
\begin{align*}
\tilde{t}_1 &= \tilde{v}_1^* = v_1^* = -\frac{2}{5}s_1 + t_1 \\
\tilde{t}_2 &= -\tilde{v}_1^* + \tilde{v}_2^* = -v_1^* + v_2^* = -\frac{2}{5}s_1 + t_2 \\
\tilde{t}_3 &= -\tilde{v}_2^* + \tilde{v}_3^* = -v_2^* + v_3^* = -\frac{2}{5}s_1 + t_3 \\
\tilde{t}_4 &= -\tilde{v}_3^* + \tilde{v}_4^* = -v_3^* + v_4^* = \frac{3}{5}s_1 - t_1 - t_2 - t_3 \\
\tilde{t}_5 &= -\tilde{v}_4^* = -v_4^* = \frac{3}{5}s_1 
\end{align*}
\]

Now since \( \frac{3}{5}s_1 - t_1 - t_2 - t_3 = -\frac{2}{5}s_1 + t_4 \) we get

\[
(U + \tilde{t}_1) \cdots (U + \tilde{t}_5) = (U - \frac{2}{5}s_1 + t_1) \cdots (U - \frac{2}{5}s_1 + t_4) \cdot (U + \frac{3}{5}s_1),
\]

and the first equation follows. The case of \( E_5 \) is similar (and easier), and will be left to the reader. Q.E.D.

Corollary. The standard coordinate functions on \( \text{Def}(E_4) \) and \( \text{Def}(E_5) \), expressed in terms of the elementary symmetric functions \( s_i \) of their respective distinguished functionals \( t_i \), are given by

\[
\begin{align*}
\varepsilon_2 &= s_2 - \frac{3}{5}s_1^2 \\
\varepsilon_3 &= s_3 - \frac{1}{5}s_2s_1 + \frac{2}{25}s_1^3 \\
\varepsilon_4 &= s_4 + \frac{1}{5}s_3s_1 - \frac{8}{25}s_2s_1^2 + \frac{12}{125}s_1^4 \\
\varepsilon_5 &= \frac{3}{5}s_1s_4 - \frac{6}{25}s_3s_1^2 + \frac{12}{125}s_2s_1^3 - \frac{72}{3125}s_1^5
\end{align*}
\]
in the case of $E_4$, and by

\[
\begin{align*}
\varepsilon_2 &= -2s_2 + \frac{5}{4}s_1^2 \\
\varepsilon_4 &= s_2^2 - 2s_2s_1^2 + \frac{5}{8}s_1^4 + s_1s_3 + 2s_4 \\
\varepsilon_5 &= -\frac{1}{8}s_2s_1^3 + \frac{1}{32}s_1^5 + \frac{1}{4}s_1^2s_3 - \frac{1}{2}s_1s_4 + s_5 \\
\varepsilon_6 &= \frac{3}{4}s_2s_1^4 - \frac{3}{4}s_2s_1^4 - s_2s_1s_3 - 2s_2s_4 + \frac{5}{32}s_1^6 + \frac{3}{4}s_1^3s_3 + \frac{1}{2}s_1^2s_4 \\
&\quad - 3s_1s_5 - s_3^2 \\
\varepsilon_8 &= \frac{3}{16}s_2s_1^4 - \frac{1}{8}s_2s_1^4 - \frac{1}{2}s_2s_1s_3 + 3s_2s_1s_5 + \frac{5}{256}s_1^8 + \frac{3}{16}s_1^5s_3 \\
&\quad - \frac{1}{8}s_1^4s_4 - \frac{1}{2}s_1^3s_5 + \frac{1}{2}s_1^2s_3^2 - s_1s_4s_3 - 2s_5s_3 + s_4^2
\end{align*}
\]

(17)

in the case of $E_5$.

Proof. Write $f_{E_n}(U; t) = U^n + \sum_{i=0}^{n-1} s_iU^{n-i}$ for $n = 4$ or $5$. Use the formulas of the lemma to calculate $f_{A_4}(U; \tilde{t})$ and $f_{D_5}(U; \tilde{t})$. In the case of $E_4$, the standard coordinate functions $\varepsilon_i$ can be read off as the coefficients of $U^{5-i}$ in $f_{A_4}$. In the case of $E_5$, form the second distinguished polynomial $g_{D_5}$ via $g_{D_5}(-U^2; \tilde{t}) = f_{D_5}(U; \tilde{t})f_{D_5}(-U; \tilde{t})$. The standard coordinate function $\varepsilon_2i$ can be read off as the coefficient of $U^{5-i}$ in $g_{D_5}(U; \tilde{t})$, while $\varepsilon_5$ is simply the coefficient of $U^0$ in $f_{D_5}(U; \tilde{t})$. Q.E.D.

8. Proof of the main theorem.

In this section, we prove the main theorem, assuming the validity of some results to be stated in table [4]. The proof will be complete when we verify that table in section 10.

The partial resolution graphs shown in figure [5] determine a singularity type $S_\ell$ for each length $\ell$ between 1 and 6, which we call the associated type of the length. Explicitly, this type is

| $\ell$ | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|
| $S_\ell$ | $A_1$ | $D_4$ | $E_6$ | $E_7$ | $E_8$ | $E_8$ |

Figure [6] (in section 1) illustrates the partial resolution graphs \{v\} $\subset \Gamma_{S_\ell}$, where $v$ is the vertex corresponding to the unique component in the maximal ideal cycle of length $\ell$.

Our aim is to show that for $\pi: Y \to X$ of length $\ell$, the singularity type of the general hyperplane section is $S_\ell$. We say that the singularity type of a rational double point is at worst $S$ if its dual graph is isomorphic to a (proper or improper) subgraph of $\Gamma_S$.

Lemma 7.

1. The partial resolution graphs shown in figure [5] are primitive.
2. Let \( \pi: Y \to X \) be an irreducible small resolution of an isolated Gorenstein threefold singularity, and let \( \ell \) be the length. Suppose that \( X \) has a hyperplane section whose singularity type is at worst \( S_\ell \). Then the generic hyperplane section defines the primitive partial resolution graph corresponding to \( S_\ell \) given in figure [\text{\ref{fig}}] .

Proof.

(1) For each \( \ell \) between 1 and 6, let \( n(\ell) \) be the minimum \( n \) such that there is a rational double point whose dual graph has \( n \) vertices, and at least one component has multiplicity exactly \( \ell \) in the maximal ideal cycle. Examining the maximal ideal cycles of the rational double points, it is easy to see that for each \( \ell \), there is a unique such rational double point with \( n(\ell) \) vertices, namely the one shown in figure [\text{\ref{fig}}] . Now for any nontrivial 1-parameter deformation of a rational double point, the dual graph of the minimal resolution of the general fiber is isomorphic to a proper subgraph of the dual graph of the special fiber. It follows that each graph shown in figure [\text{\ref{fig}}] is primitive: any proper subgraph will have fewer vertices, and so cannot have a component in its maximal ideal cycle of multiplicity \( \ell \).

(2) Fix \( \ell \). For the singularity of type \( S_\ell \), there is a unique component of multiplicity exactly \( \ell \) in the maximal ideal cycle. Moreover, no proper subgraph of \( \Gamma_{S_\ell} \) has any component with multiplicity exactly \( \ell \) in its maximal ideal cycle. Thus, since the length is \( \ell \), the partial resolution graph determined by the given hyperplane section must be of the type shown in figure [\text{\ref{fig}}] (which indicates the unique component of multiplicity \( \ell \)). On the other hand, since the graphs in figure [\text{\ref{fig}}] are primitive, it follows that this is also the type of the generic hyperplane section.

Q.E.D.

Consider now \( \pi: Y \to X \), an irreducible small resolution of an isolated Gorenstein threefold singularity \( P \in X \), and a hyperplane section \( \{ f = 0 \} \) which has a rational double point. This determines a partial resolution graph \( \{ v \} \subset \Gamma_{S_\ell} \), and the length \( \ell \) coincides with the multiplicity of \( v \) in the maximal ideal cycle. There is a natural classifying map \( \mu_f: \Delta \to \text{Def}(S) = V/W \) which allows us to recover a neighborhood of \( P \in X \) as the pullback of the standard deformation \( X \to V/W \). The map \( \mu_f \) determines a discrete valuation \( \nu_f: \mathbb{C}[V]^W \to \mathbb{Z} \) which is defined by

\[
\nu_f(\varphi) = \text{order of vanishing at 0 of } \mu_f^*(\varphi).
\]

Thanks to lemma [\text{\ref{lem}}], in order to prove the main theorem it suffices to show that \( X \) has some hyperplane section whose singularity type is at worst \( S_\ell \). The following proposition shows how to use the discrete
valuation $\nu_f$ applied to the standard coordinate functions on $\text{Def}(S)$ (or in the $E_7$ case, to certain simple polynomial expressions in these functions) to bound the singularity type of the general hyperplane section.

**Proposition 4.** Let $X$ be a threefold with an isolated rational Gorenstein singular point $P$, and let $\{ f = 0 \}$ be a hyperplane section through $P$ with a rational double point of type $S$. Let $\mu_f : \Delta \to \text{Def}(S) = V/W$ be the classifying map, and let $\nu_f : \mathbb{C}[V]^W \to \mathbb{Z}$ be the associated discrete valuation.

1. Suppose that $S = A_{n-1}$, and let $\{ \alpha_i \}$ be the standard coordinate functions on $\text{Def}(S)$. If $\nu_f(\alpha_{n-1}) = 1$ or $\nu_f(\alpha_n) = 2$ then the general hyperplane section of $X$ has singularity type at worst $A_1$.

2. Suppose that $S = D_n$, and let $\{ \gamma_n, \delta_i \}$ be the standard coordinate functions on $\text{Def}(S)$. If $\nu_f(\gamma_n) = 1$ or $\nu_f(\delta_{2n-4}) = 1$ then the general hyperplane section of $X$ has singularity type at worst $A_1$, while if $\nu_f(\gamma_n) = 2$ or $\nu_f(\delta_{2n-2}) = 3$ then the general hyperplane section of $X$ has singularity type at worst $D_4$.

3. Suppose that $S = E_6, E_7,$ or $E_8$, and let $\{ \varepsilon_i \}$ be the standard coordinate functions on $\text{Def}(S)$. Define $\tilde{\varepsilon}_i = \varepsilon_i$ if $S \neq E_7$ or $i \neq 12, 18$, and in the case of $E_7$, define

$$\tilde{\varepsilon}_{12} = \varepsilon_{12} + \frac{1}{3} \varepsilon_6^2 \quad \text{and} \quad \tilde{\varepsilon}_{18} = \varepsilon_{18} + \frac{1}{3} \varepsilon_6 \varepsilon_{12} + \frac{2}{27} \varepsilon_6^3.$$

Let $M_f$ be the set of monomials $T^d Y^k Z^\ell$ such that $\varepsilon_i Y^k Z^\ell$ is one of the terms in the polynomial in preferred versal form, and $\nu_f(\tilde{\varepsilon}_i) = d < \infty$. If any of the monomials in $M_f$ are listed in the right half of table 8, then the general hyperplane section of $X$ has singularity type at worst $S$, where $S$ is the label on the leftmost column in the right half of the table which contains some monomial from $M_f$.

Moreover, if $X$ has an irreducible small resolution of length $\ell$, then $M_f$ contains no monomials to the left of the column whose label is the associated type $S_\ell$.

The left half of table 8 has been included to make it easier to find which monomials in $M_f$ come from which standard coordinate functions $\varepsilon_i$. It is not actually necessary for the description of the link between $M_f$ and the singularity type of the general hyperplane section.

**Proof.** The proof is based on the classification of rational double points by means of their Newton polygons. A convenient reference for this is [31, (4.9)(3)]. In brief, suppose that $\{ f = 0 \} \subset \mathbb{C}^3$ has a rational
double point at the origin. Then the type $S$ is determined by the defining polynomial $f$ as follows.

1. $S = A_0$ (i.e. the surface is smooth at the origin) if and only if $f$ contains a linear term. (Notice that $f$ contains no constant term, since the origin lies on the surface.)

2. $S = A_1$ if and only if the quadratic part $f_2$ of $f$ has rank 3.

3. $S = A_{n-1}, n > 2$ if and only if the quadratic part $f_2$ of $f$ has rank 2. The value of $n$ is determined by the higher order terms. In particular, if the cubic part $f_3$ of $f$ is nonzero and involves none of the variables appearing in $f_2$, then $S = A_2$.

4. If $f_2$ has rank 1, choose coordinates so that $f = x^2 + g(y, z)$, and $g$ has no quadratic part. Note that $g_3$ is a homogeneous cubic in 2 variables.
   (a) $S = D_4$ if and only if the cubic part $g_3$ of $g$ has three distinct linear factors.
   (b) $S = D_n, n > 4$ if and only if the cubic part $g_3$ of $g$ has two distinct linear factors. (The value of $n$ is determined by the higher order terms.)
   (c) If $g_3$ has a unique linear factor, write $g_3 = h^3$.
      (i) $S = E_6$ if and only if $h$ does not divide the quartic part $g_4$ of $g$.
      (ii) $S = E_7$ if and only if the quartic part $g_4$ of $g$ is divisible by $h$ but not by $h^2$.
      (iii) $S = E_8$ otherwise.

| $E_6$ | $E_7$ | $E_8$ | $A_0$ | $A_1$ | $A_2$ | $D_4$ | $D_k$ | $E_6$ | $E_7$ |
|------|------|------|------|------|------|------|------|------|------|
| $\varepsilon_8$ | $\varepsilon_6$ | $\varepsilon_{12}$ | $\varepsilon_{10}$ | $\varepsilon_{14}$ | $\varepsilon_{18}$ | $\varepsilon_{20}$ | $\varepsilon_{24}$ | $\varepsilon_{30}$ |
| $TYZ^2$ | $TY^2$ | $TZ^3$ | $T^2YZ$ | $T^2T^2$ | $T^3Y$ | $T^3Z$ | $T^4$ |

Table 8.
The threefold $X$ has defining polynomial $\mu_f^*(\Phi_S)$, where $\Phi_S$ is the polynomial in preferred versal form of type $S$. If $\varphi_i Y^k Z^{d-\infty}$ is a term in $\Phi_S$ and if $\nu_f(\varphi_i) = d < \infty$, then the monomial $T^d Y^k Z^{d-\infty}$ appears in the defining polynomial of $X$ with a nonzero coefficient. (Here, $T$ is the coordinate on the disk $\Delta$.) In this way, we can analyze the low-degree terms appearing in the defining polynomial of $X$ by using the set $M_f$.

We define the leading terms of $\Phi_S$ to be those which have constant coefficients; there are 2 or 3 such terms. For all other coefficients $\varphi_i$, we have $\nu_f(\varphi_i) \geq 1$. Table 8 has been constructed so that all potential low-degree terms (other than leading terms) are shown there.

Suppose first that $S = A_{n-1}$. The only possible monomial of degree 1 in the defining polynomial is $T$, and this occurs if and only if $\nu_f(\alpha_n) = 1$. This is the condition for $X$ (and its general hyperplane section) to be smooth at the origin; in this case, the singularity type is certainly "at worst" $A_1$.

If $\nu_f(\alpha_n) > 1$, we consider quadratic terms. The leading term of degree 2 is $-XY$, while other potential quadratic terms must be chosen from $\{TZ, T^2\}$. If at least one of those potential terms occurs with a nonzero coefficient, then the rank of the quadratic part of the defining polynomial is at least 3. And this implies that the quadratic part of the general hyperplane section will have rank 3, and so will have a singularity of type $A_1$. But to guarantee that at least one of the potential terms occurs, we simply need $\nu_f(\alpha_{n-1}) = 1$ or $\nu_f(\alpha_n) = 2$.

Suppose next that $S = D_n$. Again, the only possible monomial of degree 1 in the defining polynomial is $T$, and this occurs if and only if $\nu_f(\delta_{2n-2}) = 1$. This is the condition for $X$ (and its general hyperplane section) to be smooth at the origin; in this case, the singularity type is certainly "at worst" $A_1$ or even $D_4$.

If $\nu_f(\delta_{2n-2}) > 1$, we consider quadratic terms. The leading term of degree 2 is $X^2$, while other potential quadratic terms must be chosen from $\{TY, TZ, T^2\}$. If at least one of the terms $TY, TZ$ occurs with a nonzero coefficient, then the rank of the quadratic part of the defining polynomial is at least 3. And this implies that the quadratic part of the general hyperplane section will have rank 3, and so will have a singularity of type $A_1$. But to guarantee that at least one of those terms occurs, we simply need $\nu_f(\gamma_n) = 1$ or $\nu_f(\delta_{2n-4}) = 1$.

On the other hand, if neither of those terms occurs, yet $T^2$ occurs, then the rank of the quadratic part is 2. (This happens when $\nu_f(\gamma_n) > 1$, $\nu_f(\delta_{2n-4}) > 1$, and $\nu_f(\delta_{2n-2}) = 2$.) Now the leading term $Y^2 Z$ also appears in our defining polynomial. Since this is a term of degree 3 which involves neither of the variables $X, T$ which appear in the quadratic part, there will be hyperplane sections of type $A_2$. (For
example, the hyperplane section defined by \( Y = Z \) will be of type \( A_2 \).

It follows that the singularity type of the general hyperplane section is at worst \( A_2 \), and hence is certainly at worst \( D_4 \).

So we may assume that \( \nu_f(\gamma_n) > 1 \), \( \nu_f(\delta_{2n-4}) > 1 \), and \( \nu_f(\delta_{2n-2}) > 2 \). The defining polynomial can then be written in the form \( X^2 + G(Y, Z, T) \), and the cubic part of \( G \) takes the form \( G_3(Y, Z, T) = Y^2Z + T \cdot H(Y, Z, T) \). Moreover, the only monomials which could appear in \( T \cdot H(Y, Z, T) \) are \( T^2Z, T^2Y, T^2Z, T^3 \). In order for the general hyperplane section of \( G_3 \) to fail to have 3 distinct linear factors, \( G_3 \) must be nonreduced. Since \( G_3 = 0 \) defines a plane cubic, this implies that \( G_3 \) itself factors in the form \( H_1^2H_2 \), where \( H_1, H_2 \) are two (possibly equal) linear polynomials.

Considering this factorization mod \( T \), we see that it must take the form

\[
G_3(Y, Z, T) = (Y + \alpha T)^2 \cdot (Z + \beta T).
\]

Moreover, since \( TY^2 \) is not one of the monomials which can occur in \( G_3 \), \( \beta \) must in fact be 0. Thus, if the general hyperplane section fails to have type \( D_4 \), \( Z \) must divide \( G_3 \). The presence of either of the monomials \( T^2Y \) or \( T^3 \) will prevent this, and their presence is guaranteed by the conditions \( \nu_f(\gamma_n) = 2 \) and \( \nu_f(\delta_{2n-2}) = 3 \), respectively. So when either of these conditions holds, the general hyperplane section must be of type \( D_4 \).

Suppose finally that \( S = E_n \). We define \( \tilde{X} = X + \frac{1}{2}Z^2 \) in case \( E_6 \) and \( \tilde{X} = X \) in cases \( E_7 \) and \( E_8 \). Then the leading terms take the form

\[
-\tilde{X}^2 + \frac{1}{4}Z^4 + Y^3, \quad -\tilde{X}^2 - Y^3 + 16YZ^3, \quad -\tilde{X}^2 + Y^3 - Z^5,
\]

The monomials in these leading terms together with the monomials in \( M_f \) will include all monomials of low degree in the defining polynomial \( \mu_f^*(\Phi_S) \).

The analysis of the cases with a linear part or with a quadratic part of rank bigger than 1 proceeds almost exactly as in the case of \( D_n \), using \( -\tilde{X}^2 \) in place of \( X^2 \) for the leading term of degree 2, and \( \pm Y^3 \) for the leading term of degree 3. It yields the criteria for having hyperplane sections of types \( A_0 \), \( A_1 \), and \( A_2 \) which are stated in table 8.

(The only remarks that need to be added to the argument given in the \( D_n \) case concern the \( E_7 \) case, since we use two modified coefficients \( \varepsilon_{12} \) and \( \varepsilon_{18} \) in that case. The remarks (which follow from the defining formulas for the modified coefficients) are that \( \nu_f(\varepsilon_{12}) = 1 \) if and only if \( \nu_f(\varepsilon_{12}) = 1 \), and that when \( \nu_f(\varepsilon_{12}) > 1 \), we have \( \nu_f(\varepsilon_{18}) = 2 \) if and only if \( \nu_f(\varepsilon_{18}) = 2 \). Thus, the orders of vanishing which predict the presence of the monomials \( TY \) and \( T^2 \) are being calculated properly.)
So we may assume that none of the monomials $T^2$, $TY$, $TZ$, or $T$ appear in our defining polynomial. The defining polynomial can be written in the form $-X^2 + G(Y,Z,T)$, and this time the cubic part of $G$ takes the form $G_3(Y,Z,T) = \pm Y^3 + T \cdot H(Y,Z,T)$. As before, the general hyperplane section will be of type $D_4$ unless $G_3$ can be factored in the form $\pm H_1^2 H_2$, where $H_1$, $H_2$ are two (possibly equal) linear polynomials.

Considering this factorization mod $T$, we see that it takes the form

$$\pm G_3(Y,Z,T) = (Y + \alpha T)^2 \cdot (Y + \beta T).$$

In particular, for such a factorization to exist, $G_3$ must be a function of $Y$ and $T$ alone. The presence of any of the monomials $TYZ$, $TZ^2$, or $T^2Z$ in $M_f$ prevents this, and forces the general hyperplane section to have type $D_4$.

If none of those monomials is present in $M_f$, then $G_3$ is a homogeneous binary cubic, and the general hyperplane section has type $D_k$ unless $G_3$ is the cube of a linear polynomial. (The type will be $D_4$ if there are three distinct linear factors of $G_3$, and will be $D_k$, $k > 4$ if there are only two.) Now for $S \neq E_7$, the monomial $TY^2$ cannot occur in $G_3$ (as is clear from table [8]). In this case, if $G_3$ is a cube it must be $Y^3$, and the presence of either of the monomials $T^2 Y$ or $T^3$ in $M_f$ will prevent this from happening, and lead to the general hyperplane section having type $D_k$.

The argument is more complicated in the case of $E_7$. We define $\tilde{Y} = Y - \frac{1}{3} \varepsilon_6$, and note that

$$-\tilde{Y}^3 + \varepsilon_{12} \tilde{Y} + \varepsilon_{18} = -Y^3 + \varepsilon_6 Y^2 + \varepsilon_{12} Y + \varepsilon_{18}.$$

In this case, if $G_3$ is a cube, then its cube root $H$ must be the linear part (with respect to $Y$, $T$) of $-\tilde{Y}$. Thus, if either $\nu_f(\varepsilon_{12}) = 2$ or $\nu_f(\varepsilon_{18}) = 3$, then $G_3$ cannot be a cube, and the singularity type must be $D_k$.

We now assume that $G_3$ is in fact a cube, and let $H$ be its cube root. If $S = E_6$, then the general hyperplane section is at worst $E_6$ (which is certainly at worst $E_7$), and we are finished.

Suppose instead that $S = E_7$ and $\nu_f(\varepsilon_6) = 1$. If $\varepsilon_6 \equiv \alpha T \mod T^2$, then $H = -Y + \frac{1}{3} \alpha T$. The quartic part of $G$ includes the leading term $16YZ^3$. But since the monomial $TZ^3$ cannot occur in $G$ and $\alpha \neq 0$, it follows that the quartic part of $G$ cannot be divisible by $H$. Thus, the general hyperplane section has type $E_6$.

---

6The argument could have been simplified, eliminating the use of $\tilde{\varepsilon}_{12}$ and $\tilde{\varepsilon}_{18}$, were it not for our desire to match notation with Bramble [4].
We may therefore assume that either $S = E_7$ and $\nu_f(\varepsilon_6) > 1$, or that $S = E_8$. In either case, the cube root $H$ is exactly $\pm Y$. This divides the quartic part of the leading term (which is $16YZ^3$ or $0$, respectively). We can therefore identify which cases have general hyperplane section $E_6$ or $E_7$ by finding in $M_f$ a monomial not divisible by $Y$, or one not divisible by $Y^2$, respectively. This is exactly what is done in the final two columns of table 8.

To prove the last statement in the proposition, note that all labels $L$ to the left of $S_\ell$ in table 8 correspond to singularities with the property that the maximum multiplicity which occurs in the maximal ideal cycle for the singularity is strictly less than $\ell$. In addition, any singularity whose type is at worst $L$ has this same property. But if $X$ has an irreducible small resolution of length $\ell$, no such singularity can be the general hyperplane section. Thus, there can be no monomials in columns to the left of that labeled by $S_\ell$. Q.E.D.

Lemma 8. Let $\pi : Y \to X$ be an irreducible small resolution of an isolated Gorenstein threefold singularity, and let $\{f = 0\}$ define a hyperplane section with a rational double point. Let $\mu_{f\pi} : \Delta \to \text{PRes}(S, v) = V/\mathfrak{m}_v$ be the classifying map determined by $f$, and define a discrete valuation $\nu_{f\pi} : C[V]^{\mathfrak{m}_v} \to \mathbb{Z}$ by

$$
\nu_{f\pi}(\varphi) = \text{order of vanishing at 0 of } \mu_{f\pi}^*(\varphi).
$$

If $\varphi_i \in C[V]^{\mathfrak{m}_v}$ is any standard coordinate function on $\text{PRes}(S, v)$, then $\nu_{f\pi}(\varphi_i) \geq 1$. If $\varphi_i$ is in fact a “constant term”, then $\nu_{f\pi}(\varphi_i) = 1$.

Proof. The first assertion holds since $\mu_{f\pi}(0) = 0$. The second holds since $Y$ is smooth at the singular point associated with $\varphi_i$. Q.E.D.

Consider again the inclusion of rings $C[V]^{\mathfrak{m}_v} \subset C[V]^{\mathfrak{m}_0}$, which corresponds to the natural projection $\sigma : \text{PRes}(S, v) \to \text{Def}(S)$. The larger ring $C[V]^{\mathfrak{m}_0}$ is a free polynomial ring generated by the standard coordinate functions on $\text{PRes}(S, v)$. Thus, each element of the smaller ring $C[V]^{\mathfrak{m}_v}$ can be written as a polynomial in those standard coordinate functions. Each monomial in such an expression has a degree (in the standard coordinate functions which are generating the ring) as well as a weight under the background $\mathbb{C}^*$-action. For a fixed weight $i$ and degree $d$, we denote by $P_{i,d}$ the subspace of polynomials in $C[V]^{\mathfrak{m}_v}$ whose weight is $i$ and whose degree is less than or equal to $d$.

Lemma 9. Fix a partial resolution type $(S, v)$. Let $\pi : Y \to X$ be an irreducible small resolution of an isolated Gorenstein threefold singular point, and let $\{f = 0\}$ be a hyperplane section with partial resolution type $(S, v)$. Let $\nu_f : C[V]^{\mathfrak{m}_v} \to \mathbb{Z}$ be the associated discrete valuation.
Suppose that \( \varphi_i \in \mathbb{C}[V]^{\mathfrak{m}_i} \) is a function of weight \( i \), and \( m_i \in P_{t,d} \) is a monomial of degree exactly \( V/\nu \). Suppose further that there is an ideal \( I \subset \mathbb{C}[V]^{\mathfrak{m}_i} \) whose intersection with \( P_{t,d} \) is \( \{0\} \) such that

\[
\varphi_i \equiv c \cdot m_i \mod I
\]

for some nonzero constant \( c \). If \( \nu_{f_{\pi}}(m_i) = d \), then \( \nu_f(\varphi_i) = d \).

Proof. Consider the classifying map \( \mu_f : \Delta \to \text{Def}(X_0) = \text{Def}(S) = V/\mathfrak{m} \) determining our threefold \( X \), and its associated discrete valuation \( \nu_f : \mathbb{C}[V]^{\mathfrak{m}_i} \to \mathbb{Z} \), as well as the related map and valuation \( \mu_{f_{\pi}} \) and \( \nu_{f_{\pi}} \). Note that \( \nu_{f_{\pi}} \) extends \( \nu_f \). Suppose that \( \varphi_i - c \cdot m_i \) has degree at most \( d \). Then it lies in \( I \cap P_{t,d} \) and so must be 0. Thus, we may assume that \( \varphi_i - c \cdot m_i \) has degree strictly greater than \( d \). It follows by lemma 8 that \( \nu_{f_{\pi}}(\varphi_i - c \cdot m_i) > d \). Since \( \nu_{f_{\pi}}(c \cdot m_i) = d \), it follows that \( \nu_f(\varphi_i) = d \). Q.E.D.

Proof of the Main Theorem. If \( \pi : Y \to X \) is as above, we set about showing that the orders of the standard coordinate functions on \( \text{Def}(S) \) (or the expressions \( \bar{e}_i \) in the case of \( E_7 \)) satisfy the relevant hypothesis from proposition 4. We show that a particular monomial occurs in the defining polynomial of \( X \) with nonzero coefficient, then we use lemmas 8 and 9 together with proposition 4 to conclude that the general hyperplane section of \( X \) is as claimed. We assume henceforth that \( (S,v_k) \) is not one of the pairs illustrated in figure 1. This puts certain restrictions on \( n \) and \( k \) which we will exploit without comment.

The first case to consider is \( S = A_{n-1} \). The standard coordinate functions on \( \text{PRes}(A_{n-1},v_k) \) are \( \mu_1, \alpha'_1, \ldots, \alpha'_k, \alpha''_2, \ldots, \alpha''_{n-k} \). By proposition 3, \( \alpha_{n-1} = (\alpha'_{k-1} \alpha''_{n-k} + \alpha'_{k} \alpha''_{n-k-1}) \mod \mu_1 \) and \( \alpha_n = \alpha'_{k} \alpha''_{n-k} \mod \mu_1 \).

Suppose that \( k = 1 \) and \( n \geq 2 \). The length is 1, and there is only one “constant term” in this case: \( \alpha''_{n-1} \). (Note that there would be no constant terms whatsoever in case \( n = 1 \).) Since \( \alpha'_0 = 1 \) and \( \alpha'_1 = 0 \) by definition, we have

\[
\alpha_{n-1} \equiv \alpha''_{n-1} \mod (\mu_1).
\]

Furthermore \( P_{n-1,1} = \mathbb{C} \cdot \alpha''_{n-1} \), so its intersection with the ideal \( I = (\mu_1) \) is \( \{0\} \). (Again we have used \( n \geq 2 \).) By lemma 3, \( \nu_f(\alpha''_{n-1}) = 1 \). By lemma 3 we conclude that \( \nu_f(\alpha_{n-1}) = 1 \). By proposition 4 the singularity type of the general hyperplane section is at worst \( A_1 \).

Suppose instead that \( 1 < k < n-1 \). (We can omit the case \( k = n-1 \) by symmetry.) In this case, the length is 1, there are two “constant terms” \( \alpha'_k \) and \( \alpha''_{n-k} \), and we have

\[
\alpha_n \equiv \alpha'_{k} \alpha''_{n-k} \mod \mu_1.
\]
We have additional congruence in this case: $$\nu \equiv 1 \pmod{k, n-k} \leq n-2$$, there can be no affine linear function in these variables of weight $n-1$. Thus $\psi = 0$, so $\varphi = 0$ and the intersection of $P_{n,2}$ with $(\mu_1)$ must be $\{0\}$. By lemma 8, we conclude that $\nu_f(\alpha_n) = 2$. By proposition 4, it follows that the singularity type of the general hyperplane section is at worst $A_1$.

The next case to consider is $S = D_n$, $k \leq n-2$, with $n \geq 4$. The standard coordinate functions on $P\mathbb{R}(D_n, v_k)$ are $\mu_1, \alpha_1', \ldots, \alpha_k', \delta_2', \ldots, \delta_{2n-2k-2}', \gamma_{n-k}$.

By proposition 3, we have $\nu_f(\alpha_k') = 2$. By lemma 9, we conclude that $\nu_f(\beta_{n-k}) = 2$. By lemma 9, we conclude that $\nu_f(\gamma_{n-k}) = 2$. Moreover, proposition 4 provides us with an additional congruence in this case:

$$\delta_{2n-2} = \delta_{2n-2} \pmod{\mu_1}.$$

We have $P_{n-2,2} = \mathbb{C} \cdot \delta_{2n-4}$, whose intersection with the ideal $I = (\mu_1)$ is $\{0\}$. By lemma 8, $\nu_f(\delta_{2n-2}) = 1$. By lemma 9, we conclude that $\nu_f(\delta_{2n-2}) = 1$. By proposition 4, the singularity type of the general hyperplane section is at worst $A_1$.

If $1 < k < n-1$, the length is 2. Moreover, proposition 3 gives

$$\delta_{2n-2} = \delta_{2n-2} \pmod{J_4}.$$ 

It is clear from considering degrees that $P_{n-2,2}$ intersects $J_4$ trivially.

If $k < n-2$, then there are two constant terms $\alpha_k'$ and $\delta_{2n-2k-2}'$. By lemma 8, $\nu_f((\alpha_k')^2 \delta_{2n-2k-2}'') = 3$. By lemma 9, we conclude that $\nu_f(\delta_{2n-2}) = 3$. By proposition 4, it follows that the singularity type of the general hyperplane section is at worst $D_4$.

If $k = n-2$, then by lemma 8 the constant terms are $\alpha_{n-2}', -\frac{1}{4}(\delta_{2n-2k-2}' + 2\gamma_{n-k}')$. According to lemma 8, each of these has order 1 with respect to the discrete valuation $\nu_f$; hence, at least one of $\nu_f(\gamma_{n-k}')$ and $\nu_f(\delta_{2n-2}'')$ is also equal to 1.

If $\nu_f(\delta_{2n-2}') = 1$, then $\nu_f((\alpha_k')^2 \delta_{2n-2k-2}'') = 3$ and we can use the same argument as in the case $k < n-2$ to conclude that the general hyperplane section has type at worst $D_4$. On the other hand, if $\nu_f(\gamma_{n-k}') = 1$, we use the additional congruence

$$\gamma_n \equiv \alpha_{n-2}' \gamma_{n-k}' \pmod{\mu_1}$$

provided by proposition 3. We have $P_{n,2} \cap (\mu_1) = \{0\}$ since the maximum weight among standard coordinate functions on $P\mathbb{R}(D_n, v_{n-1})$...
is $n - 2$, and we have $\nu_f(\alpha_{n-2}^{''}) = 2$. By lemma \[3\], we conclude that $\nu_f(\gamma_n) = 2$. By proposition \[4\], it follows that the singularity type of the general hyperplane section is at worst $D_4$.

The third case to consider is $S = D_n$, $k = n$, in which the length is 1. (By symmetry, we can omit the case $k = n - 1$.) The standard coordinate functions on $\text{PRes}(D_n, v_n)$ are $\mu_1, \alpha_2', \ldots, \alpha_n'$, and $\alpha_n'$ is the unique “constant term”. By proposition \[4\], we have
\[
\gamma_n \equiv \alpha_n' \mod \mu_1,
\]
and lemma \[8\] implies that $\nu_f(\alpha_n') = 1$. Moreover, since $P_{n,1} = \mathbb{C} \cdot \alpha_n'$, its intersection with $(\mu_1)$ is $\{0\}$. By lemma \[4\], we conclude that $\nu_f(\gamma_n) = 1$. By proposition \[4\], it follows that the singularity type of the general hyperplane section is at worst $A_1$.

Finally, we consider the cases with $S = E_n$. Among the standard coordinate functions on $\text{PRes}(S, v_k)$, let $\bar{\varphi}_N$ be the “constant term” of highest weight, say weight $N$. (This is unique, since we are avoiding the case $(S, v_k) = (E_6, v_3)$.) Let $I$ be the ideal in $\mathbb{C}[V]^W$ which is generated by all the standard coordinate functions on $\text{PRes}(S, v_k)$ other than $\bar{\varphi}_N$. We select a standard coordinate function $\varepsilon_i$ as indicated in table \[8\], and calculate it mod $I$ using the relations given in proposition \[4\]. Table \[8\] shows the results of this calculation: we will describe the calculation itself in section 10.

The calculated result takes the form
\[
\varepsilon_i \equiv c \cdot (\bar{\varphi}_N)^d \mod I
\]
for some nonzero constant $c$. (The key point of the calculation is showing that this constant is not 0.) Moreover, since $i = Nd$ and $N$ is the highest weight among standard coordinate functions on $\text{PRes}(S, v_k)$, any monomial of weight $i$ other than $(\bar{\varphi}_N)^d$ must have degree strictly greater than $d$. It follows that $I \cap P_{i,d} = \{0\}$ and thus by lemma \[8\], $\nu_f(\varepsilon_i) = d$.

Let $M_f$ be the set of monomials from proposition \[4\]. Since $\nu_f(\varepsilon_i) = d$, we conclude that $M_f$ contains the monomial shown in the next-to-last column of table \[8\]. The label $L$ of the column in table \[8\] in which that monomial appears has been reproduced in the last column of table \[8\]. In each case, the label $L$ coincides with the type $S_\ell$ which is associated with the length $\ell$. (The length itself is shown in the second column.) Thus, by proposition \[4\], the set $M_f$ can contain no monomials to the left of the column labeled by $L$ in table \[8\], and the singularity type of the general hyperplane section is at worst $L = S_\ell$. The main theorem then follows from lemma \[4\]. Q.E.D.
9. The computation of preferred versal form in the $E_n$ cases.

In this section, we explain how to explicitly compute a defining polynomial in preferred versal form for the standard deformation in the $E_n$ cases. The result will be a formula for the standard coordinate functions $\varepsilon_i$ in terms of the elementary symmetric functions $s_1, \ldots, s_n$ of the distinguished functionals $t_1, \ldots, t_n$. One’s natural inclination is to expand everything completely as polynomials in $s_1, \ldots, s_n$, and simply work with things in expanded form. For this computation, however, that would not be a wise strategy: the “constant term” in the case of $E_8$ is a polynomial with 2462 terms.\footnote{It is possible that the coefficients of a few of these terms may be zero.}

To keep things explicit, and yet in a compact format, we introduce two notions. By a set of substitution rules, we mean a set of expressions of the form $v_i = f_i(x_1, \ldots, x_k)$ which express certain variables $v_i$ as polynomial functions of other variables $x_j$. (This notion is more flexible than the notion of a ring homomorphism, since the rings to which $v_i$ and $x_j$ belong do not need to be specified until the substitution rules are used.) We say that a set of substitution rules $R$ is given in solve-list format when it is specified by three objects $R'$, $P$, and $L$ which satisfy a certain condition, as follows. $R'$ is another set of substitution rules, $P$ is a polynomial, and $L$ is an ordered list, the solve-list, consisting of pairs $(m_i, v_i)$ where $m_i$ is a monomial and $v_i$ is a variable. (The substitution rules $R'$ take the form $w_\alpha = f_\alpha(y_1, \ldots, y_\ell)$, where the $w_\alpha$ are distinct from the $v_i$ but the $y_\beta$ may include some $v_i$'s.) The condition which must be satisfied is this: if $c_i$ is the coefficient of the monomial $m_i$ in the expression obtained by substituting the rules $R'$ into the polynomial $P$, then the variable $v_i$ appears linearly in $c_i$ with a nonzero constant coefficient, and does not appear in any $c_j$ with $j < i$. (It is allowed, however, that $v_i$ appear in $c_j$ with $j > i$, and it may even appear in a nonlinear fashion.)

The algorithm for producing the rules $R$ from the triple $R'$, $P$, $L$ is simple: compute the coefficients $c_i$, and for $i = 1, 2, \ldots$, successively solve the equations $c_i = 0$ for the variables $v_i$. In solving the $i^{th}$ equation, one uses previously found values of $v_j$ for $j < i$ to eliminate the variables $v_j$ from the expression $c_i$. We refer to this process as expanding the solve-list.

Specifying the rules $R$ by means of $R'$, $P$, and $L$ (without actually expanding the solve-list) can give a relatively compact representation of a complicated set of rules. Moreover, our later application of these explicit calculations will be of the following form: calculate what happens
when the rules $\mathcal{R}$ are restricted to a subspace which is parametrized in a simple way. In carrying out those applications, it will be much to our advantage to begin by pulling back the ingredients of the solve-list format (i.e. $\mathcal{R}', P$, and $L$) to the parameter space, and then computing the pullback of the rules $\mathcal{R}$ by directly expanding the pulled-back solve-list. This part of the computation is explained in section 10.

We have already encountered (during the proof of proposition 1) a set of substitution rules which is best expressed in solve-list format. Let $\mathcal{R}_\mu$ denote the set of substitution rules given in equation (14), which describes a change of generators in the algebra $L$. Then the substitution $R_\psi$ (which describes the coefficients to use in $\mathcal{R}_\mu$ which will produce the preferred versal form) can be given in solve-list format by means of the substitution rules $\mathcal{R}_\mu$, the polynomial $\Phi_{E_n}$, and the solve-list

\[ Y^2Z \quad XYW \quad Z^4 \quad XZW \quad Y^2W \quad XW^2 \]

\[ \psi_1 \quad \psi_2 \quad \psi'_3 \quad \psi''_3 \quad \psi_4 \quad \psi_6 \]

\[ Z^4 \quad YZ^2W \quad Z^3W \]

\[ \psi_2 \quad \psi_4 \quad \psi_6 \]

\[ Y^2ZW \quad Z^4W^2 \quad Y^2W^2 \]

\[ \psi_4 \quad \psi_6 \quad \psi_{10} \]

(We continue to use the convention that equation numbers which are followed by $a$, $b$, or $c$ refer to the cases of $E_6$, $E_7$, or $E_8$, respectively.)

We computed the corresponding expressions $c_i$ explicitly in equation (15), and verified that \{ $c_i = 0$ \} has the appropriate triangular form.

Throughout this section, the subscript on a variable (when present) indicates its weight with respect to the $\mathbb{C}^*$-action. We retain the notation introduced in section 5.

The first step in our computation is to express the anti-pluricanonical mappings explicitly in coordinates, and thereby obtain a good generating set $X, Y, Z, W$ for $L$. We identify these generators with polynomials in $\mathbb{C}[V][x, y, z]$ which satisfy certain base conditions. The anti-canonical mapping (which corresponds to $L_1$) is given by the cubics passing through the zero-cycle $\eta(t_1) + \cdots + \eta(t_n)$. In other words, we want cubics $F \in \mathbb{C}[V][x, y, z]$ such that $U^n$ divides $\Psi_n(U)$, where $\Psi_n(U) := U^n - s_1U^{n-1} + \cdots + (-1)^ns_n$ is the monic polynomial of degree $n$ whose roots are $t_1, \ldots, t_n$. It is not difficult to find a basis for these in each case (by hand), and to make the basis match the first part of the normalizations established in table 3. In the case of $E_6$, we
get
\[ \tilde{W} := x^3 - yz^2 \]
\[ \tilde{Z} := y^2z - s_1x^2y + s_2xyz - s_3x^3 + s_4x^2z - s_5xz^2 + s_6z^3 \]
\[ \tilde{Y} := xy^2 - s_1y^2z + s_2x^2y - s_3xyz + s_4x^3 - s_5x^2z + s_6xz^2 \]
\[ \tilde{X} := y^3 + (s_2 - s_1^2)xy^2 - (s_3 - s_1s_2)y^2z + (s_4 - s_1s_3)x^2y \]
\[ - (s_5 - s_1s_4)xyz + (s_6 - s_1s_5)x^3 + s_1s_6x^2z \]
which gives a good generating set for \( L \), since \( L \) is generated by \( L_1 \) in this case.

In the case of \( E_7 \), a basis for the cubics is given by the first three lines of
equation (19b) below.\footnote{This basis is chosen to match the one used by Bramble \cite{B}; a simpler choice would have used \( \tilde{Y} - s_1^2\tilde{Z} \).} This is completed to a good generating set
for \( L \) by using \( \frac{1}{3} \) of the Jacobian determinant as the fourth generator.

\[ \tilde{W} := x^3 - yz^2 \]
\[ \tilde{Z} := xy^2 - s_1y^2z + s_2x^2y - s_3xyz + s_4x^3 - s_5x^2z + s_6xz^2 - s_7z^3 \]
\[ \tilde{Y} := 4y^3 + (4s_2 - 4s_1^2 + s_1^2)y^2z - (4s_3 - 4s_1s_2 + s_1^2)y^2z + (4s_4 \]
\[ - 4s_1s_3 + s_1^2s_2)xy^2 - (4s_5 - 4s_1s_4 + s_1^2s_3)xyz + (4s_6 \]
\[ - 4s_1s_5 + s_1^2s_4)x^3 - (4s_7 - 4s_1s_6 + s_1^2s_5)x^2z + (-4s_1s_4) \]
\[ + s_1^2s_6)xz^2 - s_1^2s_7z^3 \]
\[ \tilde{X} := \frac{1}{3} \frac{\partial(\tilde{Y}, \tilde{Z}, \tilde{W})}{\partial(x,y,z)}. \]

In the case of \( E_8 \), a basis for the cubics in \( L_1 \) is given by

\[ \tilde{W} := x^3 - yz^2 \]
\[ \tilde{Z} := y^3 + (s_2 - s_1^2)xy^2 - (s_3 - s_1s_2)y^2z + (s_4 - s_1s_3)x^2y \]
\[ - (s_5 - s_1s_4)xyz + (s_6 - s_1s_5)x^3 - (s_7 - s_1s_6)x^2z \]
\[ + (s_8 - s_1s_7)xz^2 + s_1s_8z^3. \]

A basis for the sextics which determine the anti-bicanonical map is then
given by quadratic expressions in these cubics (i.e. \( \text{Sym}^2 L_1 \))
together with a new sextic \( F \). To match the normalizations from table 4, we
assume that \( F \) has weight 16 and satisfies \( F \equiv xy^5 \mod m \). We then
need to ensure that \( F \) has multiplicity 2 along the zero-cycle \( \eta(t_1) + \ldots + \eta(t_8) \). We do this by imposing two conditions on \( F \): (i) \( \eta^*(F) = \Psi_8(U)^2 \)
and (ii) \( \Psi_8(U) \) divides \( \eta^*(\partial F/\partial x) \). The first condition guarantees that
\( F \) meets \( C \) with multiplicity at least 2 at each point in the zero-cycle.
(Note that \( F \) and \( \Psi_8(U)^2 \) are monic of the same degree). The second
guarantees that *one* of the partial derivatives of $F$ vanishes at those points. But now by the chain rule,
\[ \eta^*(\frac{\partial F}{\partial x}) + 3U^2\eta^*(\frac{\partial F}{\partial y}) = 2\Psi_8(U)'\Psi_8(U). \]

It follows that $\Psi_8(U)$ divides $\eta^*(\partial F/\partial y)$ as well, which implies that the other partial derivative vanishes at those points, and therefore that $F$ has multiplicity 2 along the zero-cycle $\eta(t_1) + \cdots + \eta(t_n)$.

Finding $F$ explicitly is now a matter of solving the equations in the coefficients of $F$ implied by these conditions. The answer is not unique, but still depends on 2 free parameters. We used MAPLE and REDUCE to solve the equations, and made a choice for the free parameters which makes the coefficients of $x^3y^3$ and $x^6$ both 0. We use this polynomial $F$ as the third element $\bar{Y}$ of a good generating set for $L$, and complete the good generating set by using $-\frac{1}{6}$ of the Jacobian determinant as $\bar{X}$, that is,
\[ \bar{X} := -\frac{1}{6} \frac{\partial (Y, Z, W)}{\partial (x, y, z)}. \]

This good generating set is shown explicitly in Appendix 0; in particular, the polynomial used as $\bar{Y}$ is displayed there explicitly.

The second step of our computation is to compute the defining polynomial $\Phi_{E_n}$ of $\mathcal{P} \subset \text{Proj}_V(L)$ with respect to the generating set $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$. Such a polynomial was shown to exist in lemma 3, and its general form was given in equation (13); we must compute the unknown coefficients which appear in that equation. Thanks to the remark following proposition 1, we only need the coefficients of low weight at this stage of the computation.

Let $\pi: \mathbb{P}^2 \times V \to \mathcal{P} \subset \text{Proj}_V(L)$ be the rational map determined by the generating set $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$. Now $\Phi_{E_n}$ vanishes on the image of $\pi$. Thus, if we write $\tilde{\Phi}_{E_n}$ with undetermined coefficients and compute $\pi^*(\tilde{\Phi}_{E_n})$, every coefficient in the expression for $\pi^*(\tilde{\Phi}_{E_n})$ must vanish. This produces some equations for the undetermined coefficients.

**Proposition 5.** Let $\mathcal{R}_\pi$ be the set of substitution rules which describes the map $\pi$ with respect to the coordinates $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$, as given in equations (13a), (13b) and Appendix 0. Then the coefficients of low weight in the equation $\tilde{\Phi}_{E_n}$ can be described by a set of substitution rules $\mathcal{R}_\nu$, which are given in solve-list format by means of the substitution
rules $\mathcal{R}_\pi$, the polynomial $\Phi_{E_6}$, and the solve-list

$$
\begin{array}{cccccccc}
x^2y^6z & x^4y^6z & x^2y^6z^2 & x^4y^6z^2 & y^6z^2 & x^5y^4z & x^4y^4z & x^6y^5 \\
\xi_1 & \xi_2 & \xi_2 & \xi_3 & \xi_3 & \xi_4 & \xi_5 & \xi_6
\end{array}
$$

(20a)

$$
\begin{array}{cccc}
x^4y^5 & x^8y^5 & x^8y^6 & x^8y^8 \\
\xi_2 & \xi_4 & \xi_6 & \xi_10
\end{array}
$$

(20b)

$$
\begin{array}{cccc}
x^4y^4 & x^5y^4 & x^8y^4 & x^7y^4 & x^8y^10 \\
\xi_2 & \xi_4 & \xi_6 & \xi_8 & \xi_10
\end{array}
$$

(20c)

Proof. We carry out the expansion specified by the solve-list format, but work mod $m$. The congruences in table 2 give substitutions for $\tilde{X}$, $\tilde{Y}$, $\tilde{Z}$, $\tilde{W}$ which are valid mod $m$. If we make those substitutions into equation (13) and collect terms of low degree in $z$ we obtain:

$$
\pi^*(\Phi_{E_6}) \equiv \psi_1x^2y^6z + \psi_2x^4y^5 + (\xi_2 - \xi_2)xy^6z^2 + \psi_3y^3x^y + (\psi_3' - \psi_3)y^6z^3 + \psi_4(x^5y^4 - x^2y^5z^2) + \xi_5(x^4y^4z - xy^5z^3) + \psi_6x^6y^3 + (\xi_6 - 2\psi_6)x^3y^4z^2 + \xi_8(x^7y^2 - 2x^4y^2z^3) + \xi_9(x^6y^2z - 2x^3y^2z^3) + \xi_{12}(x^9 - 3x^6yz^2)
\mod (m, 3^4)
$$

$$
\pi^*(\Phi_{E_7}) \equiv 16\xi_2x^4y^8 + (16\psi_2 - 16\xi_2)xy^9z^2 + \psi_4(4x^5y^7 - 4x^2y^8z^2) + 16\xi_6x^6y^6 + (16\psi_6 - 32\xi_6)x^3y^7z^2 + \xi_8(4x^7y^5 - 8x^4y^6z^2) + \xi_{10}(x^8y^4 - 2x^5y^5z^2) + \xi_{12}(x^9y^3 - 12x^6y^4z^2) + \xi_{14}(x^{10}y^2 - 3x^7y^3z^2) + \xi_{18}(x^{12} - 4x^9yz^2)
\mod (m, 3^3)
$$

$$
\pi^*(\Phi_{E_8}) \equiv 8\xi_2x^4y^{14} + \psi_4x^5y^{13} + \psi_6x^6y^{12} + \xi_8x^7y^{11} + \psi_{10}x^8y^{10} + \xi_{12}x^9y^9 + \xi_{14}x^8y^{10} + \xi_{18}x^9y^{12} + \xi_{20}x^{13}y^5 + \xi_{21}x^{15}y^3 + \xi_{30}x^{18}
\mod (m, 3).
$$

Since each equation which is to be solved is a coefficient of a monomial in $x, y, z$, it is homogeneous (with respect to the background $\mathbb{C}^*$-action). For each such equation, the $\xi_i$'s and $\psi_i$'s involved are either the ones given above, or ones of strictly lower weight (since they are multiplied by nontrivial functions of the $t_i$'s). Thus, if we proceed from lower weight to higher weight in solving these equations, and use the leading order terms above as a guide to the order in which equations of the same weight should be solved, we arrive at the solve-lists stated in the proposition.

Q.E.D.

The solve-lists given in proposition 3 can be extended to determine the entire defining polynomial in these coordinates. We did this, and
expanded the extended solve-lists using MAPLE and REDUCE in the cases of $E_6$ and $E_7$. In the case of $E_6$, we obtained the expanded defining polynomial

$$
\Phi_{E_6} = -X^2 \bar{W} - X \bar{Z}^2 + \bar{Y}^3 + s_1 \bar{Y}^2 \bar{Z} - s_2 X \bar{Y} \bar{W} + 0 \bar{Y} \bar{Z}^2 - s_3 X \bar{Z} \bar{W} + 0 \bar{Z}^2 \bar{W} + (s_5 - s_1 s_4) \bar{Y} \bar{Z} \bar{W} + (2 s_6 - s_1 s_5) \bar{X} \bar{W}^2 + 0 \bar{Z} \bar{W}^2 \\
+ (s_2 s_6 + s_1^2 s_6 - s_2 s_1 s_5 + s_5 s_3) \bar{Y} \bar{W}^2 + (s_2 s_1 s_6 - s_3 s_6) \bar{Z} \bar{W}^2 \\
+ (s_1 s_6 s_2 s_3 - s_6^2 + s_5 s_1 s_6 - s_1^2 s_6 - s_3^2 s_6) \bar{W}^3.
$$

In the case of $E_7$, the defining polynomial which we found agrees with the one found by Bramble [4, p. 357], and we have not reproduced it here. The defining polynomial for $E_8$ is very large, and we have not attempted to write it down.

We can now describe a set of substitution rules $R_\pi$ which describes the map $\pi$ with respect to the coordinates $X, Y, Z, W$ as being a composition

$$
R_\pi = R_{\mu}^{-1} \circ \bar{R_\pi} \circ R_\psi \circ R_\nu,
$$

where we have used the fact that the $R_\mu$ as given in equation (14) can be solved for $X, Y, Z, W$ as functions of the other variables, yielding $R_{\mu}^{-1}$.

**Proposition 6.** Let $R_\pi = R_{\mu}^{-1} \circ \bar{R_\pi} \circ R_\psi \circ R_\nu$ be the set of substitution rules which describes the map $\pi$ with respect to the coordinates $X, Y, Z, W$. Then the coefficients in the defining polynomial $\Phi_{E_n}$ can be described by a set of substitution rules which are given in solve-list format by means of the polynomial $\Phi_{E_n}$ (with undetermined coefficients), the substitution rules $R_\pi$, and the solve-list

$$(21)$$

| $x y^0 z^2$ | $x^4 y^4 z^2$ | $x^4 y^4 z^2$ | $x^4 y^4 z^2$ | $x^4 y^4 z^2$ | $x^4 y^4 z^2$ |
|-----------|----------------|----------------|----------------|----------------|----------------|
| $\varepsilon_2$ | $\varepsilon_5$ | $\varepsilon_6$ | $\varepsilon_8$ | $\varepsilon_9$ | $\varepsilon_{12}$ |

$$(21b)$$

| $x^3 y^3$ | $x^6 y^6$ | $x^9 y^9$ | $x^3 y^3$ | $x^6 y^6$ | $x^9 y^9$ | $x^{10} y^{12}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|----------------|
| $\varepsilon_2$ | $\varepsilon_6$ | $\varepsilon_8$ | $\varepsilon_{10}$ | $\varepsilon_{12}$ | $\varepsilon_{14}$ | $\varepsilon_{18}$ |

$$(21c)$$

| $x^3 y^{13}$ | $x^6 y^{16}$ | $x^9 y^{19}$ | $x^{12} y^{22}$ | $x^{15} y^{25}$ | $x^{18} y^{28}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| $\varepsilon_2$ | $\varepsilon_6$ | $\varepsilon_8$ | $\varepsilon_{12}$ | $\varepsilon_{14}$ | $\varepsilon_{18}$ |

$$(21d)$$

| $x^3 y^{16}$ | $x^6 y^{19}$ | $x^9 y^{22}$ | $x^{12} y^{25}$ | $x^{15} y^{28}$ | $x^{18} y^{31}$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| $\varepsilon_2$ | $\varepsilon_6$ | $\varepsilon_8$ | $\varepsilon_{12}$ | $\varepsilon_{14}$ | $\varepsilon_{18}$ | $\varepsilon_{24}$ | $\varepsilon_{30}$ |
Proof. The defining polynomial in preferred versal form with undetermined coefficients is:

\[ \Phi_{E_6} = -X^2 W - X Z^2 + Y^3 + \varepsilon_2 Y Z^2 + \varepsilon_5 Y Z W + \varepsilon_6 Z^2 W + \varepsilon_8 Y W^2 + \varepsilon_9 Z W^2 + \varepsilon_{12} W^3 \]  \hfill (22a)

\[ \Phi_{E_7} = -X^2 - Y^3 W + 16 Y Z^3 + \varepsilon_2 Y^2 Z W + \varepsilon_6 Y^2 W^2 + \varepsilon_8 Y Z W^2 + \varepsilon_{10} Z^2 W^2 + \varepsilon_{12} Y W^3 + \varepsilon_{14} Z W^3 + \varepsilon_{18} W^4 \]  \hfill (22b)

\[ \Phi_{E_8} = -X^2 + Y^3 - Z^5 W + \varepsilon_2 Y Z^3 W + \varepsilon_8 Y Z^2 W^2 + \varepsilon_{12} Z^3 W^3 + \varepsilon_{14} Y Z W^3 + \varepsilon_{18} Z^2 W^4 + \varepsilon_{20} Y W^4 + \varepsilon_{24} Z W^5 + \varepsilon_{30} W^6 \]  \hfill (22c)

As before, we carry out the procedure specified by the solve-list format, working mod \( m \). The congruences in table 5 give substitutions for \( X, Y, Z, W \) which are valid mod \( m \). If we make those substitutions into the equation for \( \Phi_{E_n} \) above and collect terms of low degree in \( z \) we obtain:

\[ \pi^*(\Phi_{E_6}) \equiv \varepsilon_2 x y^6 z^2 + \varepsilon_5 x^4 y^4 z + \varepsilon_6 x^3 y^4 z^2 + \varepsilon_8 (x^7 y^2 - 2x^4 y^3 z^2) + \varepsilon_9 x^6 y^2 z + \varepsilon_{12} (x^9 - 3x^6 y z^2) \mod (m, z^3) \]

\[ \pi^*(\Phi_{E_7}) \equiv 16 \varepsilon_2 x^4 y^8 + 16 \varepsilon_6 x^6 y^6 + \varepsilon_8 4x^7 y^5 + \varepsilon_{10} x^8 y^4 + \varepsilon_{12} 4x^9 y^3 + \varepsilon_{14} x^{10} y^2 + \varepsilon_{18} x^{12} \mod (m, z) \]

\[ \pi^*(\Phi_{E_8}) \equiv \varepsilon_2 x^4 y^{14} + \varepsilon_8 x^7 y^{11} + \varepsilon_{12} x^9 y^9 + \varepsilon_{14} y^8 x^{10} + \varepsilon_{18} x^{12} y^6 + \varepsilon_{20} x^{13} y^5 + \varepsilon_{24} x^{15} y^3 + \varepsilon_{30} x^{18} \mod (m, z^3) \]

The same argument used in the proof of proposition 5 shows that if we proceed from lower weight to higher weight in solving these equations, we arrive at the solve-lists stated in the proposition. Q.E.D.

It is important to notice that the solve-lists for \( E_7 \) and \( E_8 \) given in proposition 6 do not involve \( z \); it is therefore possible (and desirable) to set \( z = 0 \) at the beginning of any computation involving these solve-lists.

We have expanded these solve-lists using MAPLE and REDUCE in the cases of \( E_6 \) and \( E_7 \); the results of this computation are displayed in Appendices 1 and 2, respectively. (The results for \( E_7 \) can be found, with
some errors, in Bramble [4]. Appendix 2 gives the corrected results."
The results for $E_8$ are too large to contemplate writing down. We stress
however that even in the cases of $E_6$ and $E_7$, further calculations with
these formulas are best done by leaving them in solve-list format as
long as possible, and only expanding the solve-lists at the very end,
after restricting to a suitable subspace.

10. Restricted polynomials and the main computation.

In this section, we will justify the congruences stated in table 9.
These congruences all take the form $\varepsilon_i \equiv c \cdot (\tilde{\varphi}_N)^d \mod I$. We will
in fact compute $\varepsilon_i$ modulo another ideal $J \subset I$ which is generated
by a certain subset of the set of standard coordinate functions, and
note that our desired congruence is an immediate consequence of this
computation. In order to describe the ideal $J$ and our computational
method, we must first describe in parametric form some subspaces of
the deformation spaces $\text{Def}(S)$.

Let $R$ be a root system of type $S$, let $V_R$ be the complex root space,
and let $t_1, \ldots, t_n$ be the distinguished functionals, with $s_1, \ldots, s_n$ their
elementary symmetric functions. Suppose that we are given a vector
space $W$ and a monic polynomial $r_S(U)$ of degree $n$ in $U$, whose co-
efficients lie in the ring $\mathbb{C}[W]$ of polynomial functions on $W$. This
determines a map $\psi: W \to V_R/\mathfrak{S}_n$ by means of the action on polyno-
mials $\psi^*: \mathbb{C}[V_R]^{\mathfrak{S}_n} \to \mathbb{C}[W]$ defined by

$$\psi^*(s_i) = \text{the coefficient of } U^{n-i} \text{ in } r_S(U).$$

In particular, the pullback of the distinguished polynomial is $\psi^*(f_S(U; t)) = r_S(U)$. We call $r_S(U)$ the restricted polynomial associated to $\psi$.

We wish to describe a particular case of this construction in which
the image of $\psi$ is defined by an ideal $J_S$ which is generated by a subset
of the standard coordinate functions on $\text{Def}(S)$. When we have done
so, we will call the generators of $J_S$ the vanishing coordinates, and
refer to coordinates on $W$ as parameters. We want $J_S$ to be as large
as possible, yet not to contain the “constant term”. Equivalently, $W$
parametrizes a subspace of $\text{Def}(S)$ on which many of the standard
coordinate functions (but not the “constant term”) vanish.

If $S = A_{n-1}$, let $J_{A_{n-1}}$ be the ideal generated by all the standard
coordinate functions on $\text{Def}(A_{n-1})$ other than the “constant term”.
Since the standard coordinate functions are the elementary symmetric
functions $s_i$ themselves, it is easy to construct a restricted polynomial
for the ideal $J_{A_{n-1}}$. There are two natural choices: we use either
$r_{A_{n-1}}(U) = U^n + \lambda_1^n$, or $r_{A_{n-1}}(U) = U^n - \lambda_1^n$.
(We will use the second form when we need an explicit root $\lambda_1$ of the restricted polynomial.)
If \( S = D_n \) and \( n \) is even, we again let \( J_{D_n} \) be the ideal generated by all the standard coordinate functions on \( \text{Def}(D_n) \) other than the “constant term”. The construction of a restricted polynomial in this case is based on a special factorization property: if we define

\[
F(U) = U^n - \lambda_{n-1} U,
\]
and

\[
G(-U^2) = F(U) \cdot F(-U)
\]
then

\[
G(Z) = Z^n + \lambda_{n-1}^2 Z.
\]

Thus, if we let \( r_{D_n}(U) = F(U) \), then the pullbacks via \( \psi^* \) of the standard coordinate functions will be the coefficients of \( G(Z) \), together with one coefficient of \( F(U) \). It follows that \( \psi^* \) of all standard coordinate functions except for the “constant term” vanish.

In the remaining cases, we can find an ideal \( J_{S} \) and a restricted polynomial as follows. Begin with parameters \( \lambda_1, \ldots, \lambda_n \), the initial restricted polynomial \( U^n + \sum \lambda_i U^{n-i} \), and the map given by \( \psi^*(s_i) = \lambda_i \). The pullbacks of the standard coordinate functions \( \varphi_j \) via \( \psi^* \) can be computed as functions of the \( \lambda_i \). If the weight \( j \) of a standard coordinate function \( \varphi_j \) is at most \( n \), then \( \lambda_j \) appears in the formula for that coordinate function with a nonzero constant coefficient. (This must be checked case by case.) It follows that if we set all such standard coordinate functions equal to zero, we get a triangular system of linear equations in a subset of the set \( \{ \lambda_i \} \) of parameters. These equations can be derived from equations (11), (16), and (17) in the cases of \( D_7 \), \( E_4 \), and \( E_5 \) respectively, and from Appendices 1 and 2 in the case of \( E_6 \) and \( E_7 \). We used MAPLE and REDUCE to solve the equations in those 5 cases; the resulting restricted polynomials are shown in table 10.

If \( \varphi_N \) is the “constant term” (or for that matter any of the standard coordinate functions on \( \text{Def}(S) \)) it is a straightforward matter to compute \( \psi^*(\varphi_N) \), based on the description of the mapping \( \psi \) which is given by the coefficients of the restricted polynomial as shown in table 10. For this purpose, one again uses equations (11), (16), and (17) in the cases of \( D_7 \), \( E_4 \), and \( E_5 \), respectively. We have carried out this computation using MAPLE and REDUCE, and displayed the answers in the last column of table 10.

In the case of \( E_6 \) and \( E_7 \), it is more efficient to perform this computation directly from the solve-list description of the standard coordinate functions on \( \text{Def}(S) \) which was given in section 9. (That is, we use the explicit description of \( \psi^*(s_i) \) to substitute for \( s_i \) in the ingredients of the solve-list, and then solve the resulting equations.) When this is done, formulas \( e_{12}(\lambda_1, \lambda_3, \lambda_4) \) and \( e_{18}(\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_7) \) are obtained.
which express the pullbacks of the respective “constant terms” $\varepsilon_{12}$ and $\varepsilon_{18}$ in terms of the parameters. We calculated these formulas using MAPLE and REDUCE, but as they are a bit long, we have not displayed them in the table.

We are now ready to explain how the congruences in table 3 are derived. Let $(E_n, v_k)$ be one of the pairs considered in table 3. The “constant term” of highest weight $\varphi_N$ is induced from the projection onto a subspace $\tilde{V}/\mathfrak{m}$ corresponding to an irreducible subsystem $\tilde{R}$ of the root system $R_{E_n}$. (In all cases except $(E_7, v_2)$ and $(E_8, v_2)$, $R$ is the “left part” $R'$; in those two cases, $R$ is the “right part” $R''$.) As in section 8, let $I$ be the ideal in $\mathbb{C}[V]$ which is generated by all the standard coordinate functions on $\text{PRes}(E_n, v_k)$ which come from $R - \{v_k\} - \tilde{R}$. Since the “constant term” of highest weight is associated to $\tilde{R}$ but does not belong to $J_{R'}$, we have $J \subset I$. We extend the map $\tilde{\psi}: \tilde{W} \to \tilde{V}/\mathfrak{g}_{\tilde{n}}$ to a map $\psi: \tilde{W} \to V/(\mathfrak{g}_{\tilde{n}}' \times \mathfrak{g}_{\tilde{n}}'')$ by simply composing it with the natural inclusion $\tilde{V}/\mathfrak{g}_{\tilde{n}} \subset V/(\mathfrak{g}_{\tilde{n}}' \times \mathfrak{g}_{\tilde{n}}'')$.

We need to compute $\varepsilon_i$ modulo $J$. Since $J$ vanishes on the image of $\psi$, it suffices to compute $\psi^*(\varepsilon_i)$ in terms of the pullbacks of the standard coordinate functions on $\text{PRes}(E_n, v_k)$ via $\psi^*$. The first step is to use proposition 4 and table 5, which relates the distinguished polynomial for $R$ to those for $R'$ and $R''$. Now $\psi^*(f_{S'}(U; t'))$ and $\psi^*(f_{S''}(U; t''))$ can be computed immediately: one of them is the restricted polynomial for $\tilde{R}$, and the other one is just a power of $U$ (since all of the corresponding standard coordinate functions vanish when pulled back via $\psi$). Table 5 can then be used to compute $\psi^*(f_S(U; t))$; we carry out this computation below.

We first consider the case $(E_n, v_0)$, in which the complementary root system has type $A_{n-1}$. We have $\psi^*(\mu_1) = 0$ and

$$\psi^*(f_{A_{n-1}}(U; t')) = r_{A_{n-1}}(U) = U^n + \lambda_n.$$  

(We use the first form of the restricted polynomial since we do not need a root.) Thus, by table 5, $\psi^*(f_{E_n}(U; t)) = U^n + \lambda_n$.

We next consider the case $(E_n, v_1)$, in which the complementary root system has type $D_{n-1}$. Since $\psi^*(f_{D_{n-1}}(U; t')) = r_{D_{n-1}}(U)$, the coefficient $\psi^*(\mu_1)$ of $U^{n-2}$ in this polynomial is 0 when $n - 1$ is even, and $\lambda_1$ when $n - 1 = 7$. We also have $\psi^*(\mu_1) = 0$. Thus by table 7, in the
case \((E_7, v_1)\) we get
\[
\psi^*(f_{E_7}(U; t)) = (-1)^7 \cdot (-U) \cdot \tilde{\psi}^*(f_{D_6}(-U; t')) = U^7 + \lambda_5 U^2
\]
while in the case \((E_8, v_1)\) we get
\[
\psi^*(f_{E_8}(U; t)) = (-1)^8 \cdot (-U + \frac{1}{3} \lambda_1) \cdot \tilde{\psi}^*(f_{D_7}(-U - \frac{1}{6} \lambda_1; t')) = (-U + \frac{1}{3} \lambda_1) \cdot r_{D_7}(-U - \frac{1}{6} \lambda_1).
\]

We next consider the case \((E_n, v_2)\), in which the complementary root system has components of type \(A_1\) and \(A_{n-2}\). In this case, \(\psi^*(f_{A_{n-2}}(U; t'')) = r_{A_{n-2}}(U)\), which we write this time in the form \(U^{n-1} - \lambda_1^{n-1}\) so that \(\psi^*(\sigma_1) = \lambda_1\) is a root of this polynomial. Now
\[
r_{A_{n-2}}(U) = \frac{U^{n-2} - \sum_{i=0}^{n-2} \lambda_1^i}{U - \lambda_1}.
\]
Moreover, \(\psi^*(\mu_1) = 0\), and \(\psi^*(f_{A_1}(U + \frac{2}{3} \lambda_1; t')) = (U + \frac{2}{3} \lambda_1)^2\). Table 4 then implies
\[
\psi^*(f_{E_n}(U; t)) = (U + \frac{2}{3} \lambda_1)^2 \cdot \sum_{i=0}^{n-2} \lambda_1^i (U - \frac{1}{3} \lambda_1)^{n-2-i}
\]
Finally, we consider the case \((E_n, v_k)\) with \(k \geq 4\), in which the complementary root system has components of type \(E_k\) and \(A_{n-k-1}\). In this case, we have \(\psi^*(f_{E_k}(U; t')) = r_{E_k}(U) = U^k + \lambda_1 U^{k-1} + \cdots\), which implies that the coefficient \(\psi^*(\tau_1)\) of \(U^{k-1}\) in this polynomial is \(\lambda_1\). Since \(\psi^*(\mu_1) = 0\) and \(\psi^*(f_{A_{n-k-1}}(U; t'')) = U^{n-k}\), table 7 implies that
\[
\psi^*(f_{E_n}(U; t)) = (U - \frac{1}{9 - k} \lambda_1)^{n-k} \cdot r_{E_k}(U).
\]
The second step in the computation of \(\varepsilon_i\) modulo \(J\) is to use the coefficients of the pulled-back distinguished polynomial \(\psi^*(f_{E_n}(U; t))\) as ingredients for the solve-lists in section 9, and obtain (using MAPLE and REDUCE) a formula for \(\psi^*(\varepsilon_i)\) in terms of the parameters. Now we have already computed \(\psi^*(\tilde{\varphi}_N)\) in terms of the parameters, as indicated in table 10. So we simply need to compare the formulas for \(\psi^*(\varepsilon_i)\) and \(\psi^*(\tilde{\varphi}_N)\).

We illustrate this comparison with an example which can be carried out by hand. The case we consider is \((E_7, v_1)\), in which the complementary root system has type \(D_6\). We have \(\psi^*(f_{E_7}(U; t)) = U^7 + \lambda_5 U^2\), which implies that \(\psi^*(\varepsilon_{10})\) is computed by setting \(s_5 = \lambda_5\) and all other \(s_j = 0\) in the formula for \(\varepsilon_{10}\). The only term that then remains is the
term coming from $s^2_5$ in the original formula for $\varepsilon_{10}$. Now inspection of Appendix 2 shows that the coefficient of $s^2_5$ in the formula for $16\varepsilon_{10}$ is 256. Thus,

$$\psi^*(\varepsilon_{10}) = 16\lambda_5^2 = 16\psi^*(s_5)^2$$

which implies that

$$\varepsilon_{10} \equiv 16s^2_5 \mod J,$$

as required.

To return to the general argument: in all cases from table 9 except $(E_8, v_1)$ and $(E_8, v_7)$, the only monomial in the standard coordinate functions on $\text{PRes}(E_n, v_k)$ which has weight $i$ and which does not pull back to zero under $\psi^*$ is $(\tilde{\varphi}_N)^d$. Thus, in those cases it follows that $\psi^*(\varepsilon_i)/\psi^*(\tilde{\varphi}_N)^d$ is a constant. We calculated these constants using MAPLE and REDUCE, obtaining the values indicated in table 9. This verifies the congruences $\varepsilon_i \equiv c \cdot (\tilde{\varphi}_N)^d \mod J$, which suffices since $J \subset I$ in each case.

In the two remaining cases $(E_8, v_1)$ and $(E_8, v_7)$, the congruences which hold modulo $J$ are

$$\varepsilon_{24} \equiv 0 \cdot (\delta'_8)^3 + \frac{1}{16} \cdot (\delta'_8)^2 \mod J, \text{ and}$$

$$\varepsilon_{18} \equiv -\frac{1}{3072} \cdot (\varepsilon'_8\varepsilon'_{10}) + \frac{1}{64} \cdot (\varepsilon'_{10}) \mod J,$$

respectively, and these imply the desired congruences modulo $I$. To verify these congruences, we also need to calculate

$$\psi^*(\delta'_8) = \lambda_1^3\lambda_5 - \frac{3}{4}\lambda_1^5\lambda_3 + \frac{3}{2}\lambda_1^2\lambda_3^2 + \frac{5}{64}\lambda_1^8 - 2\lambda_3\lambda_5,$$

$$\psi^*(\varepsilon'_8) = e_8(\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_7),$$

$$\psi^*(\varepsilon'_{10}) = e_{10}(\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_7).$$

The first of these formulas is obtained from equation (11), while the second and third lines refer to formulas which we have calculated explicitly with MAPLE and REDUCE using the solve-list method, but do not display here. (Notice that the calculations of $\delta'_8$ and $\varepsilon'_8$ are indicated in table 10.) Now the coefficients in equations (23) and (24) can be calculated with the method of undetermined coefficients. That is, there will be some relation of the form

$$\psi^*(\varepsilon_{24}) = c_1 \cdot \psi^*(\delta'_8)^3 + c_2 \cdot \psi^*(\delta'_8)^2 \mod J, \text{ or}$$

$$\psi^*(\varepsilon_{18}) = c_1 \cdot \psi^*(\varepsilon'_8) \cdot \psi^*(\varepsilon'_{10}) + c_2 \cdot \psi^*(\varepsilon'_{18}) \mod J,$$

respectively. Substituting the calculated values of $\psi^*$ allowed us to solve (using MAPLE and REDUCE) for the undetermined coefficients $c_1$, $c_2$.

This completes the verification of table 9, and the proof of the main theorem.
We would like to offer two pieces of advice to the ambitious reader who wishes to duplicate our symbolic calculations. First, it is essential when computing with solve-lists to keep them unexpanded as long as possible. Even when a solve-list must be expanded, it may be that all relevant information can be extracted by only partially expanding the solve list, solving for a proper subset of the variables.

Second, the absence of $z$ from the monomials in the solve-lists (20c), (21b), and (21c) means that $z$ can be set equal to 0 before the expansion of these solve-lists begins. This cuts down the size of the computation tremendously.

For the less ambitious reader, MAPLE source files for all calculations described in the paper are available upon request (directed to the second author).
\[ W = x^3 - yz^2 \]
\[ Z = y^3 + (s_2 - s_1^2)xy^2 - (s_3 - s_1 s_2)y^2z + (s_4 - s_1 s_3)x^2y - (s_5 - s_1 s_4)xyz \]
\[ + (s_6 - s_1 s_5)x^3 + (s_7 - s_1 s_6)x^2z + (s_8 - s_1 s_7)xz^2 + s_1 s_8 z^3 \]
\[ \bar{Y} = xy^5 - 2s_1 y^5z + (s_1^2 + 2s_2) y^4 x^2 + (-2s_3 - 2s_1 s_2) y^4 z x + (s_2^2 + 2s_1 s_3 + 2s_4) z^2 y^4 \]
\[ + (2s_1 s_5 + s_6 + 2s_3 s_1 + s_3^2 + s_3^2 - 3s_7^2 s_4 + 2s_2 s_4 + s_5^2 - s_3 s_4^2) z^2 y^3 x \]
\[ + (2s_1 s_2 s_4 - 2s_3 s_4 + s_5 s_1^2 - 2s_1 s_2 s_3 - s_3^4 s_4 - s_2 s_5^2 - s_7 + s_3 s_1 - s_1 s_6 + 2s_3^2 s_1 \]
\[ + s_3 s_1^3 - s_3 s_2^3) z^3 y^3 + (-2s_5 s_2 - s_7 + 2s_1^2 s_2 s_3 + s_3 s_2^2 - s_1 s_6 - s_3 s_1 - 2s_6 s_1 \]
\[ + s_3 s_1^4 - s_3 s_2^4) z^3 y^3 + (s_4 s_2^2 - s_1 s_7 + 2s_3 s_1 s_4 - s_4^2 \]
\[ - s_3 s_1^2 - 2 s_3 s_1^2 - s_1 s_6 + s_6 s_4 - s_3 s_5^2 + s_3 s_1^5 + s_4 s_1^4 + s_3 s_1 s_4^2 \]
\[ + (3s_1 s_7 + s_3 s_2 + 2s_8 + 2s_3 s_5 + 2s_3 s_1^2 + 2s_4 + s_3 s_5^2 - s_4 s_1^4 \]
\[ - s_3 s_1^2 + s_3 s_1^2 + s_3 s_1 - s_3 s_1 - 2s_3 s_1 s_4 \]z^2 y^2 x^2 + (s_5 s_2 s_1 - s_4 s_1^2 + s_1^2 s_4 s_3 \]
\[ - 2s_1 s_4^2 + s_4 s_1^4 + s_3 s_1^3 - s_6 s_3 + s_1 s_4 - s_2 s_4^2 - s_2 s_7 - 2s_4 s_2^2 - s_5 s_1^4 \]z y x^4
\[ + (s_4 s_1^2 - s_6 s_1^3 - s_5 s_2 s_1^2 - 3s_1 s_8 - s_4 s_1^5 + 2s_1 s_4^2 + s_1 s_7 - 2s_5 s_4 - s_2 s_7 \]
\[ + 2s_4 s_2 s_1^4 - s_6 s_3 - s_5 s_2^2 - s_1 s_4 s_3 + s_6 s_1^4 \]z^3 y^2 x + (s_2 s_8 - 2s_5 s_3^2 s_2 - s_1 s_5 s_2^2 \]
\[ + s_5 s_1^5 + s_7 s_3 + s_3 s_6 s_1^2 - s_6 s_1^4 + s_3 s_7 + s_2 s_6 + s_6 s_4 + s_3 s_5 s_1^2 - 2s_1 s_4 s_5 \]
\[ - s_8 s_1^2) z^4 y^2 + (s_7 s_3 - s_2 s_6 + s_6 s_4 - s_1 s_7 - s_5 s_1^5 + s_5^2 + s_2 s_8 + s_6 s_1^4 - s_3 s_5 s_1^2 \]
\[ + 2s_1 s_4 s_5 + 2s_5 s_1^3 s_2 + s_1 s_5 s_2^2 - s_2 s_6 s_1^2 + s_8 s_1^2) z^2 y^2 x^3 + (s_6 s_1^2 s_3 + s_4 s_7 - 2s_6 s_3^2 s_2 \]
\[ - s_7 s_1^4 - s_5 s_6 + s_6 s_1^5 - s_6 s_1 s_2^2 + s_8 s_1^3 + s_7 s_2^2 - 2s_6 s_1 s_4 + s_3 s_8 + s_2 s_7 s_1^2) z x^5 \]
\[ + (-s_5 s_6 - 3s_4 s_7 + s_6 s_1 s_2^2 + 2s_6 s_1^2 s_2 - s_8 s_1^3 - 3s_3 s_8 - s_6 s_1^2 s_3 - s_6 s_1^5 + 2s_6 s_1 s_4 \]
\[ + s_7 s_1^4 - s_2 s_7 s_1^2 - s_7 s_2^2) z^3 y x^2 + (2s_7 s_5 s_1^2 - s_2 s_7 s_3 + s_8 s_1^4 + s_2^2 + s_5 s_7 - s_8 s_2 s_1^2 \]
\[ + 2s_8 s_1^2 + s_8 s_1 + s_4 s_8 + 2s_7 s_1 s_4 - s_7 s_2^2) z^2 x^4 + (s_1 s_7 s_3 - s_2 s_7 s_1 + s_8 s_2 s_1^2 \]
\[ + 3s_4 s_8 + s_5 s_7 - 2s_7 s_1 s_4 - 2s_7 s_2 s_1^2 - s_8 s_4^2 + s_8 s_2^2 + s_7 s_1^2) z^4 y x + (s_8 s_1 s_2 \]
\[ + 2s_1 s_8 s_4 + 2s_8 s_1^2 s_2 - s_8 s_1^5 - s_3 s_8 s_2 - s_5 s_8) z^5 y + (s_8 s_1^5 - 2s_6 s_7 - s_8 s_1 s_2 \]
\[ - 2s_1 s_8 s_4 - 2s_8 s_1^2 s_2 - s_5 s_8 + s_3 s_8 s_1^2) z^3 x^3 + (s_7^2 + 2s_6 s_8) z^4 x^2 - 2s_7 s_8 x z^5 + s_8^2 z^6 \]
\[ \bar{X} = \frac{1}{6} \frac{\partial (Y, Z, W)}{\partial (x, y, z)} \]
6 \varepsilon_2 = -2 s_1^2 + 3 s_2

81 \varepsilon_5 = 4 s_1^5 - 15 s_1^3 s_2 + 27 s_1^2 s_3 - 27 s_1 s_4 + 81 s_5

1944 \varepsilon_6 = -16 s_1^6 + 72 s_1^4 s_2 - 216 s_1^3 s_3 + 27 s_1^2 s_2^2 + 216 s_1^2 s_4
+ 162 s_1 s_2 s_3 + 324 s_1 s_5 - 81 s_2^3 + 324 s_2 s_4 - 243 s_3^2 - 1944 s_6

34992 \varepsilon_8 = -64 s_1^8 + 384 s_1^6 s_2 - 864 s_1^5 s_3 - 324 s_1^4 s_2^2 + 864 s_1^4 s_4
+ 1944 s_1^3 s_2 s_3 - 2592 s_1^3 s_5 - 486 s_1^2 s_2^3 - 1944 s_1^2 s_2 s_4 - 2916 s_1^2 s_3^2
+ 34992 s_1 s_2^2 s_6 + 2916 s_1 s_2 s_3^2 - 11664 s_1 s_2 s_5 + 5832 s_1 s_3 s_4 - 729 s_2^4
+ 5832 s_2^2 s_4 - 4374 s_2 s_3^2 + 17496 s_3 s_5 - 11664 s_4^2

78732 \varepsilon_9 = 64 s_1^9 - 432 s_1^7 s_2 + 1296 s_1^6 s_3 + 324 s_1^5 s_2^2 - 1296 s_1^5 s_4
- 3888 s_1^4 s_2 s_3 - 1944 s_1^4 s_5 + 1215 s_1^3 s_2^3 + 972 s_1^3 s_2 s_4 + 5832 s_1^3 s_3^2
- 14580 s_1^3 s_6 - 2187 s_1^2 s_2^2 s_3 + 17496 s_1^2 s_2 s_5 - 8748 s_1^2 s_3 s_4
+ 2187 s_1 s_2^2 s_4 + 52488 s_1 s_2 s_6 - 8748 s_1 s_4^2 - 6561 s_2^3 s_5 - 78732 s_3 s_6
+ 26244 s_4 s_5

11337408 \varepsilon_{12} = -256 s_1^{12} + 2304 s_1^{10} s_2 - 6912 s_1^9 s_3 - 4320 s_1^8 s_2^2
+ 6912 s_1^8 s_4 + 36288 s_1^7 s_2 s_3 + 10368 s_1^7 s_5 - 6480 s_1^6 s_2^3
- 20736 s_1^6 s_2 s_4 - 54432 s_1^6 s_3^2 + 217728 s_1^6 s_6 - 11664 s_1^5 s_2^2 s_3
- 186624 s_1^5 s_2 s_5 + 93312 s_1^5 s_3 s_4 + 10935 s_1^4 s_2^2 + 11664 s_1^4 s_2^2 s_4
+ 104976 s_1^4 s_2 s_3^2 - 1189728 s_1^4 s_2 s_6 + 93312 s_1^4 s_4^2 - 78732 s_1^3 s_2^3 s_3
+ 437400 s_1^3 s_2^2 s_5 - 209952 s_1^3 s_2 s_3 s_4 - 104976 s_1^3 s_3^2
+ 2729376 s_1^3 s_3 s_6 - 279936 s_1^3 s_4 s_5 + 13122 s_1^2 s_2^5 + 196830 s_1^2 s_2^2 s_3^2
+ 629856 s_1^2 s_2^2 s_6 - 944784 s_1^2 s_2 s_3 s_5 - 209952 s_1^2 s_2 s_4^2
+ 314928 s_1^2 s_3^2 s_4 - 7558272 s_1^2 s_4 s_6 + 2834352 s_1 s_2^5 s_5^2 - 78732 s_1 s_2^4 s_3
+ 314928 s_1 s_2^3 s_5 + 157464 s_1 s_2^2 s_3 s_4 - 236196 s_1 s_2 s_3^3
+ 3779136 s_1 s_2 s_3 s_6 - 1259712 s_1 s_2 s_4 s_5 + 472392 s_1 s_3^2 s_5
+ 629856 s_1 s_3 s_4^2 + 13122 s_1^2 s_6 - 157464 s_2^4 s_4 + 118098 s_2^3 s_3^2
- 472392 s_2^2 s_3 s_5 + 629856 s_2^2 s_4^2 - 472392 s_2 s_3^2 s_4 + 177147 s_3^4
- 2834352 s_3^2 s_6 + 1889568 s_3 s_4 s_5 - 839808 s_4^3
Appendix 2. Standard coordinates for $E_7$.

This Appendix gives the standard coordinate functions $\varepsilon_i$ for $E_7$, and can also serve as a correction to the formulas of Bramble [1, pp. 358-360]. The $A_i$ which we calculate here are integer multiples of the $\varepsilon_i$ which clear denominators; Bramble’s paper contains the same multiples. (Note that our $\varepsilon_i$ correspond to his $\alpha_{ijk}$.) It is very impressive to observe that Bramble, calculating by hand, was correct in the calculation of $A_2$, $A_6$, $A_8$, and $A_{10}$, and had only two incorrect coefficients for $A_{14}$. However, the formulas for $A_{12}$ and $A_{18}$ from [1] are mostly wrong.

\[ A_2 = \varepsilon_2 = 3 s_1^2 - 4 s_2 \]
\[ A_6 = 48 \varepsilon_6 = 18 s_1^6 - 72 s_1^4 s_2 + 96 s_1^3 s_3 + 32 s_1^2 s_2^2 - 96 s_1^2 s_4 - 32 s_1 s_2 s_3 + 96 s_1 s_5 - 64 s_2 s_4 + 48 s_3^2 + 384 s_6 \]
\[ A_8 = 48 \varepsilon_8 = -27 s_1^8 + 144 s_1^6 s_2 - 192 s_1^5 s_3 - 160 s_1^4 s_2^2 + 192 s_1^4 s_4 + 320 s_1^3 s_2 s_3 - 192 s_1^3 s_5 - 128 s_1^2 s_2 s_4 - 160 s_1^2 s_3^2 + 128 s_1 s_3 s_4 - 2304 s_1 s_7 + 768 s_2 s_6 + 384 s_3 s_5 - 256 s_4^2 \]
\[ A_{10} = 16 \varepsilon_{10} = 3 s_1^{10} - 20 s_1^8 s_2 + 32 s_1^7 s_3 + 32 s_1^6 s_2^2 - 32 s_1^6 s_4 - 96 s_1^5 s_2 s_3 + 32 s_1^5 s_5 + 64 s_1^4 s_2 s_4 + 80 s_1^4 s_3^2 - 128 s_1^4 s_6 - 128 s_1^3 s_3 s_4 - 256 s_1^3 s_7 + 256 s_1^2 s_2 s_6 + 128 s_1^2 s_3 s_5 + 512 s_1 s_2 s_7 - 512 s_1 s_3 s_6 - 1024 s_3 s_7 + 256 s_5^2 \]
\[ A_{12} = 6912 \varepsilon_{12} = -297 s_1^{12} + 2376 s_1^{10} s_2 - 3456 s_1^9 s_3 - 5616 s_1^8 s_2^2 + 3456 s_1^8 s_4 + 14976 s_1^7 s_2 s_3 - 3456 s_1^7 s_5 + 3328 s_1^6 s_2^3 - 11520 s_1^6 s_2 s_4 - 11520 s_1^5 s_2 s_5 + 19584 s_1^5 s_3 s_4 - 20736 s_1^5 s_7 - 1536 s_1^4 s_2 s_4 + 13440 s_1^4 s_3 s_5 + 34560 s_1^4 s_2 s_6 - 14976 s_1^4 s_3 s_5 - 11520 s_1^4 s_4 s_2 + 3072 s_1^3 s_2 s_3 s_4 + 55296 s_1^3 s_2 s_7 - 10240 s_1^3 s_3^3 - 55296 s_1^3 s_3 s_5 + 18432 s_1^3 s_4 s_5 - 18432 s_1^2 s_2^2 s_6 - 9216 s_1^2 s_2 s_3 s_5 - 6144 s_1^2 s_2 s_4^2 + 12288 s_1^2 s_3^2 s_4 - 55296 s_1^2 s_3 s_7 + 27648 s_1^2 s_5^2 - 110592 s_1 s_2^2 s_7 + 73728 s_1 s_2 s_3 s_6 - 18432 s_1 s_3^2 s_5 + 6144 s_1 s_3 s_4^2 + 221184 s_1 s_4 s_7 - 110592 s_1 s_5 s_6 + 55296 s_2 s_3 s_7 + 36864 s_2 s_4 s_6 - 55296 s_3^2 s_6 + 18432 s_3 s_4 s_5 - 8192 s_4^3 - 110592 s_5 s_7 - 110592 s_6^2 \]
\[ A_{14} = 768 \varepsilon_{14} = 27 s_1^{14} - 252 s_1^{12} s_2 + 384 s_1^{11} s_3 + 752 s_1^{10} s_2^2 - 384 s_1^{10} s_4 - 2176 s_1^9 s_2 s_3 + 384 s_1^9 s_5 - 704 s_1^8 s_2^3 + 1792 s_1^8 s_2 s_4 + 1664 s_1^8 s_3^2 \]
\[ A_{18} = -768 s_{1}^{8} s_{6} + 2816 s_{1}^{7} s_{2}^{2} s_{3} - 1280 s_{1}^{7} s_{2} s_{5} - 2944 s_{1}^{7} s_{3} s_{4} + 768 s_{1}^{7} s_{7} - 1536 s_{1}^{6} s_{2}^{2} s_{4} - 3968 s_{1}^{6} s_{2} s_{3}^{2} + 2816 s_{1}^{6} s_{2} s_{6} + 2432 s_{1}^{6} s_{3} s_{5} + 1280 s_{1}^{6} s_{4}^{2} + 4608 s_{1}^{5} s_{2} s_{3} s_{4} - 2048 s_{1}^{5} s_{2} s_{7} + 2048 s_{1}^{5} s_{3}^{3} - 5120 s_{1}^{5} s_{3} s_{6} - 2048 s_{1}^{5} s_{4} s_{5} - 1024 s_{1}^{4} s_{2}^{2} s_{6} - 512 s_{1}^{4} s_{2} s_{3} s_{5} - 1024 s_{1}^{4} s_{2} s_{4}^{2} - 4096 s_{1}^{4} s_{3}^{2} s_{4} + 8192 s_{1}^{4} s_{3} s_{7} + 8192 s_{1}^{4} s_{4} s_{6} - 1536 s_{1}^{4} s_{5}^{2} + 4096 s_{1}^{3} s_{2}^{2} s_{7} + 2048 s_{1}^{3} s_{3}^{2} s_{5} + 2048 s_{1}^{3} s_{3} s_{4}^{2} + 16384 s_{1}^{3} s_{4} s_{7} - 12288 s_{1}^{3} s_{5} s_{6} - 30720 s_{1}^{2} s_{2} s_{3} s_{7} - 4096 s_{1}^{2} s_{4} s_{6} + 8192 s_{1}^{2} s_{2} s_{5}^{2} - 2048 s_{1}^{2} s_{3}^{3} s_{6} - 2048 s_{1}^{2} s_{3} s_{4} s_{5} - 12288 s_{1}^{2} s_{5} s_{7} + 12288 s_{1}^{2} s_{6}^{2} - 8192 s_{1}^{2} s_{4} s_{7} + 32768 s_{1}^{2} s_{3}^{2} s_{7} + 8192 s_{1}^{3} s_{3} s_{6} - 8192 s_{1}^{3} s_{5}^{2} + 49152 s_{1}^{4} s_{6} s_{7} - 24576 s_{2} s_{5} s_{7} + 16384 s_{3} s_{4} s_{7} - 24576 s_{3} s_{5} s_{6} + 8192 s_{4} s_{5}^{2} + 49152 s_{7}^{2} s_{6}^{2} - 756 s_{1}^{16} s_{2} + 1152 s_{1}^{15} s_{3} + 3264 s_{1}^{14} s_{2}^{2} - 1152 s_{1}^{14} s_{4} - 9600 s_{1}^{13} s_{2} s_{3} + 1152 s_{1}^{13} s_{5} - 5888 s_{1}^{12} s_{2}^{3} + 8448 s_{1}^{12} s_{2} s_{4} + 7488 s_{1}^{12} s_{3}^{2} + 24576 s_{1}^{11} s_{2}^{2} s_{3} - 7680 s_{1}^{11} s_{2} s_{5} - 13824 s_{1}^{11} s_{3} s_{4} + 4608 s_{1}^{11} s_{7} + 3584 s_{1}^{10} s_{2}^{4} - 16896 s_{1}^{10} s_{2} s_{4} + 36864 s_{1}^{10} s_{2} s_{3}^{2} - 3072 s_{1}^{10} s_{2} s_{6} + 12288 s_{1}^{10} s_{3} s_{5} + 6912 s_{1}^{10} s_{4} s_{2} - 17920 s_{1}^{9} s_{2}^{3} s_{3} + 12800 s_{1}^{9} s_{2}^{2} s_{5} + 53760 s_{1}^{9} s_{2} s_{3} s_{4} - 29184 s_{1}^{9} s_{2} s_{7} + 20480 s_{1}^{9} s_{3}^{3} + 7680 s_{1}^{9} s_{3} s_{6} - 12288 s_{1}^{9} s_{4} s_{5} + 5120 s_{1}^{8} s_{2}^{3} s_{4} + 37120 s_{1}^{8} s_{2}^{2} s_{3}^{2} + 13312 s_{1}^{8} s_{2}^{2} s_{6} - 39424 s_{1}^{8} s_{2} s_{3} s_{5} - 21504 s_{1}^{8} s_{2} s_{4} - 49152 s_{1}^{8} s_{3}^{2} s_{4} + 52224 s_{1}^{8} s_{3} s_{7} + 24576 s_{1}^{8} s_{4} s_{6} - 16128 s_{1}^{8} s_{5}^{2} - 20480 s_{1}^{7} s_{2}^{2} s_{3} + 106496 s_{1}^{7} s_{2}^{2} s_{7} - 40960 s_{1}^{7} s_{2} s_{3}^{3} - 81920 s_{1}^{7} s_{2} s_{3} s_{6} + 40960 s_{1}^{7} s_{2} s_{4} s_{5} + 45056 s_{1}^{7} s_{2}^{3} s_{5} + 36864 s_{1}^{7} s_{3} s_{4}^{2} - 24576 s_{1}^{7} s_{4} s_{7} - 8192 s_{1}^{6} s_{2}^{3} s_{6} - 4096 s_{1}^{6} s_{2}^{2} s_{3} s_{5} - 8192 s_{1}^{6} s_{2}^{2} s_{4}^{2} + 49152 s_{1}^{6} s_{2}^{2} s_{5}^{2} + 299008 s_{1}^{6} s_{2} s_{3} s_{7} - 90112 s_{1}^{6} s_{2} s_{4} s_{6} + 86016 s_{1}^{6} s_{2} s_{5}^{2} + 20480 s_{1}^{6} s_{3}^{4} + 77824 s_{1}^{6} s_{3}^{2} s_{6} - 77824 s_{1}^{6} s_{3} s_{4} s_{5} - 8192 s_{1}^{6} s_{5}^{3} + 73728 s_{1}^{6} s_{6}^{2} - 212992 s_{1}^{5} s_{2}^{3} s_{7} + 131072 s_{1}^{5} s_{2}^{2} s_{3} s_{6} - 40960 s_{1}^{5} s_{2}^{2} s_{5}^{2} + 24576 s_{1}^{5} s_{2} s_{3} s_{4}^{2} + 139264 s_{1}^{5} s_{2} s_{4} s_{7} - 49152 s_{1}^{5} s_{2} s_{5} s_{6} - 49152 s_{1}^{5} s_{3}^{3} s_{4} + 229376 s_{1}^{5} s_{3}^{2} s_{7} + 90112 s_{1}^{5} s_{3} s_{4} s_{6} - 128880 s_{1}^{5} s_{3} s_{5}^{2} + 32768 s_{1}^{5} s_{4}^{2} s_{5} \]
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\[ + 147456 s_1^5 s_6 s_7 + 778240 s_1^4 s_2^2 s_3 s_7 + 81920 s_1^4 s_2^2 s_4 s_6 - 81920 s_1^4 s_2^2 s_5^2 - 286720 s_1^4 s_2 s_3^2 s_6 + 40960 s_1^4 s_2 s_3 s_4 s_5 - 32768 s_1^4 s_2 s_4^3 - 122880 s_1^4 s_2 s_5 s_7 - 245760 s_1^4 s_2 s_6^2 + 81920 s_1^4 s_3^2 s_5 - 409600 s_1^4 s_3 s_4 s_7 + 122880 s_1^4 s_3 s_5 s_6 - 131072 s_1^4 s_4^2 s_6 + 122880 s_1^4 s_4 s_5^2 + 147456 s_1^4 s_7^2 - 327680 s_1^3 s_2^2 s_4 s_7 - 819200 s_1^3 s_2 s_3^2 s_7 + 163840 s_1^3 s_2 s_3 s_5^2 - 393216 s_1^3 s_2 s_4 s_7 + 163840 s_1^3 s_3^3 s_6 - 163840 s_1^3 s_3^2 s_4 s_5 + 65536 s_1^3 s_3 s_4^3 + 196608 s_1^3 s_3 s_5 s_7 + 393216 s_1^3 s_3 s_6^2 - 262144 s_1^3 s_4^2 s_7 + 393216 s_1^3 s_4 s_5 s_6 - 294912 s_1^3 s_5^3 - 983040 s_1^2 s_2^2 s_5 s_7 + 589824 s_1^2 s_2^2 s_6^2 + 1572864 s_1^2 s_2 s_3 s_4 s_7 - 393216 s_1^2 s_2 s_3 s_5 s_6 - 131072 s_1^2 s_2 s_4^2 s_6 + 131072 s_1^2 s_2 s_4 s_5^2 - 393216 s_1^2 s_2 s_5^2 + 327680 s_1^2 s_3^2 s_7 - 131072 s_1^2 s_3 s_4 s_6 + 65536 s_1^2 s_3^2 s_5^2 - 65536 s_1^2 s_3 s_4^2 s_5 + 393216 s_1^2 s_3 s_6 s_7 + 393216 s_1^2 s_4 s_5 s_7 - 393216 s_1^2 s_4 s_6^2 + 1179648 s_1 s_2^2 s_6 s_7 + 1572864 s_1 s_2 s_3 s_5 s_7 - 1179648 s_1 s_2 s_3 s_6^2 - 262144 s_1 s_2 s_4^2 s_7 - 1441792 s_1 s_3^2 s_4 s_7 + 393216 s_1 s_3^2 s_5 s_6 + 262144 s_1 s_3 s_4^2 s_6 - 131072 s_1 s_3 s_4 s_5^2 + 393216 s_1 s_3 s_7^2 - 1572864 s_1 s_4 s_6 s_7 - 1179648 s_1 s_5^2 s_7 + 1179648 s_1 s_5 s_6^2 + 589824 s_2 s_7^2 - 1179648 s_2 s_3 s_6 s_7 - 393216 s_2 s_4 s_5 s_7 + 589824 s_3^2 s_6^2 + 524288 s_3 s_4^2 s_7 - 393216 s_3 s_4 s_5 s_6 + 65536 s_4^2 s_5^2 - 1572864 s_4 s_7^2 + 2359296 s_5 s_6 s_7 \]
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| \((S, v_k)\) | length | \(S'\) | \(S''\) | congruence | monomial | type |
|----------------|--------|--------|--------|------------|----------|------|
| \((E_6, v_0)\) | 2      | \(A_5\) | -      | \(\varepsilon_6 \equiv -\alpha_6' \mod I\) | \(TZ^2\) | \(D_4\) |
| \((E_6, v_4)\) | 2      | \(E_4\) | \(A_1\) | \(\varepsilon_5 \equiv -\varepsilon_5' \mod I\) | \(TYZ\) | \(D_4\) |
| \((E_6, v_5)\) | 1      | \(E_5\) | \(A_0\) | \(\varepsilon_8 \equiv -\frac{1}{4}\varepsilon_8' \mod I\) | \(TY\) | \(A_1\) |
| \((E_7, v_0)\) | 2      | \(A_6\) | -      | \(\varepsilon_{14} \equiv 64(\alpha_7')^2 \mod I\) | \(T^2Z\) | \(D_4\) |
| \((E_7, v_1)\) | 2      | \(D_6\) | -      | \(\varepsilon_{10} \equiv 16\delta_{10}' \mod I\) | \(TZ^2\) | \(D_4\) |
| \((E_7, v_2)\) | 3      | \(A_1\) | \(A_5\) | \(\varepsilon_6 \equiv -12\alpha_6'' \mod I\) | \(TY^2\) | \(E_6\) |
| \((E_7, v_4)\) | 3      | \(E_4\) | \(A_2\) | \(\varepsilon_{10} \equiv 16(\varepsilon_5')^2 \mod I\) | \(T^2Z^2\) | \(E_6\) |
| \((E_7, v_5)\) | 2      | \(E_5\) | \(A_1\) | \(\varepsilon_8 \equiv -4\varepsilon_8' \mod I\) | \(TYZ\) | \(D_4\) |
| \((E_7, v_6)\) | 1      | \(E_6\) | \(A_0\) | \(\varepsilon_{12} \equiv 16\varepsilon_{12}' \mod I\) | \(TY\) | \(A_1\) |
| \((E_8, v_0)\) | 3      | \(A_7\) | -      | \(\varepsilon_{24} \equiv (\alpha_8')^3 \mod I\) | \(T^3Z\) | \(E_6\) |
| \((E_8, v_1)\) | 2      | \(D_7\) | -      | \(\varepsilon_{24} \equiv -\frac{1}{16}(\delta_{12}')^2 \mod I\) | \(T^2Z\) | \(D_4\) |
| \((E_8, v_2)\) | 4      | \(A_1\) | \(A_6\) | \(\varepsilon_{14} \equiv (\alpha_7'')^2 \mod I\) | \(T^2YZ\) | \(E_7\) |
| \((E_8, v_5)\) | 4      | \(E_5\) | \(A_2\) | \(\varepsilon_8 \equiv -\frac{1}{4}\varepsilon_8' \mod I\) | \(TYZ^2\) | \(E_7\) |
| \((E_8, v_6)\) | 3      | \(E_6\) | \(A_1\) | \(\varepsilon_{12} \equiv \varepsilon_{12}' \mod I\) | \(TZ^2\) | \(E_6\) |
| \((E_8, v_7)\) | 2      | \(E_7\) | \(A_0\) | \(\varepsilon_{18} \equiv \frac{1}{64}\varepsilon_{18}' \mod I\) | \(TZ^2\) | \(D_4\) |

**Table 9.** Key computations
|\( S \) | Vanishing Coordinates | Restricted Polynomial \( r_S(U) \) | “Constant Term” \( \psi^*(\varphi_N) \) |
|---|---|---|---|
| \( A_{n-1} \) | \( \alpha_j, 2 \leq j \leq n-1 \) | \( U^n + \lambda_n \) or \( U^n - \lambda_n \) | \( \lambda_n \) or \( -\lambda_n \) |
| \( D_n \) | \( \gamma_n, \delta_{2j}, 1 \leq j \leq n-2 \) | \( U^n - \lambda_{n-1}U \) | \( \lambda_{n-1}^2 \) |
| \( D_7 \) | \( \delta_2, \delta_4, \delta_6, \gamma_7 \) | \( U^7 + \lambda_1 U^6 + \frac{1}{2} \lambda_1^2 U^5 + \lambda_3 U^4 + (\lambda_1 \lambda_3 - \frac{1}{8} \lambda_1^4)U^3 + \lambda_5 U^2 + (\lambda_1 \lambda_5 - \frac{1}{2} \lambda_3 \lambda_1^2 + \frac{1}{16} \lambda_6 + \frac{1}{2} \lambda_3^2)U \) | \( (\lambda_1 \lambda_5 - \frac{1}{2} \lambda_3 \lambda_1^3 + \frac{1}{16} \lambda_6^2 + \frac{1}{2} \lambda_3^2)^2 \) |
| \( E_4 \) | \( \varepsilon_2, \varepsilon_3, \varepsilon_4 \) | \( U^4 + \lambda_1 U^3 + \frac{3}{5} \lambda_1^2 U^2 + \frac{1}{25} \lambda_1^3 U + \frac{11}{125} \lambda_1^4 \) | \( \frac{243}{3125} \lambda_1^5 \) |
| \( E_5 \) | \( \varepsilon_2, \varepsilon_4, \varepsilon_5 \) | \( U^5 + \lambda_1 U^4 + \frac{5}{8} \lambda_1^2 U^3 + \lambda_3 U^2 + (\frac{15}{128} \lambda_1^4 + \frac{1}{2} \lambda_1 \lambda_3)U + \left(\frac{27}{250} \lambda_1^5 - \frac{1}{2} \lambda_1^2 \lambda_3\right) \) | \( \frac{2601}{16384} \lambda_1^5 + \frac{9}{4} \lambda_3^2 \lambda_1^2 - \frac{153}{125} \lambda_1^5 \lambda_3 \) |
| \( E_6 \) | \( \varepsilon_2, \varepsilon_5, \varepsilon_6 \) | \( U^6 + \lambda_1 U^5 + \frac{2}{3} \lambda_1^2 U^4 + \lambda_3 U^3 + \lambda_4 U^2 + (\frac{1}{3} \lambda_1 \lambda_4 - \frac{1}{3} \lambda_3 \lambda_1^2 + \frac{2}{27} \lambda_1^5)U + (\frac{5}{18} \lambda_1^2 \lambda_4 - \frac{1}{9} \lambda_3 \lambda_1^3 + \frac{11}{486} \lambda_1^6 - \frac{1}{9} \lambda_3^2) \) | \( e_{12}(\lambda_1, \lambda_3, \lambda_4) \) |
| \( E_7 \) | \( \varepsilon_2, \varepsilon_6 \) | \( U^7 + \lambda_1 U^6 + \frac{3}{4} \lambda_1^2 U^5 + \lambda_3 U^4 + \lambda_4 U^3 + \lambda_5 U^2 + (\frac{1}{8} \lambda_1^2 + \frac{3}{64} \lambda_1^6 - \frac{3}{16} \lambda_3 \lambda_1^2 \) + \lambda_5 U^2 + (\frac{1}{4} \lambda_1 \lambda_5 + \frac{3}{8} \lambda_1^2 \lambda_4)U + \lambda_7 \) | \( e_{18}(\lambda_1, \lambda_3, \lambda_4, \lambda_5, \lambda_7) \) |

Table 10.