Distributed Computation of Exact Average Degree and Network Size in Finite Number of Steps under Quantized Communication

Apostolos I. Rikos, Themistoklis Charalambous, Christoforos N. Hadjicostis, and Karl H. Johansson

Abstract—We consider the problems of computing the average degree and the size of a given network in a distributed fashion under quantized communication. We present two distributed algorithms which rely on quantized operation (i.e., nodes process and transmit quantized messages), and are able to calculate the exact solutions in a finite number of steps. Furthermore, during the operation of our algorithms, each node is able to determine in a distributed manner whether convergence has been achieved and thus terminate its operation. To the best of the authors’ knowledge, these algorithms are the first to find the exact solutions under quantized communication (i.e., there is no error in the final calculation). Additionally, note that our network size calculation algorithm is the first in the literature which calculates the exact size of a network in a finite number of steps without introducing a final error. This error in other algorithms can be either due to quantization or asymptotic convergence. In our case, no error is introduced since the desired result is calculated in the form of a fraction involving a quantized numerator and a quantized denominator. Finally, we demonstrate the operation of our algorithms and their potential advantages.

I. INTRODUCTION

Many distributed algorithms which operate over multi-agent systems rely on knowledge of the network’s parameters, such as the average degree and/or the size of the network. Knowledge of the average node degree or the size of the network plays an important role in various applications, including consensus based distributed optimization [1], infection propagation strategies [2], antidote distribution to control epidemics [3], and the networked prisoner’s dilemma game [4]. Furthermore, knowing the network’s parameters is important for detecting topological changes [5], estimating the maximum and the minimum of the initial measurements in the presence of noise via a soft-max operation [6], and controlling renewable energy resources, while maintaining an average degree that fulfills structural properties [7].

Various methods have been proposed in the literature for calculating the size of a given network. Current approaches rely on statistical methods that require the exchange of excessive information between nodes, random walk strategies, random sampling, and capture-recapture strategies [8]–[14]. Furthermore, the problem of calculating the average degree of a given network has been analyzed in [15]–[17]; however, finding its solution in a distributed fashion has received limited attention. Specifically, only [18] presents an asymptotic distributed algorithm for computing the average degree of a network. To the best of the authors’ knowledge, every algorithm in the literature which calculates the size and the average degree of a given network operates with real values and exhibits asymptotic convergence. However, operation with real values in practice requires channels with infinite bandwidth (since an irrational number requires an infinite number of bits for representation). Therefore, nodes need to approximate such numbers, but still require high bandwidth. This is undesired in practical applications, since the communication overhead of each node becomes the major operational bottleneck over large scale networks. Furthermore, asymptotic convergence introduces a final error on the calculated result because most algorithms need to be terminated after a pre-defined finite number of iterations. This error may lead to imprecise calculation of the desired quantity, which may be of significant magnitude for the case of large scale networks. In summary, calculating network parameters in a finite number of steps with quantized communication remains largely unexplored.

Main Contributions. Our paper is a major departure from the current literature and aims to bridge this gap. Compared to existing approaches, the operation of our algorithms allows nodes to process and transmit quantized values. It is important to note that quantized operation allows more efficient usage of network resources compared to real valued operation, as nodes require less bits to store and transmit information. Furthermore, compared to algorithms that exhibit asymptotic convergence, our algorithms converge in a finite number of steps without introducing any final error, as nodes compute the final result in the form of a quantized fraction. The main contributions are the following.

• We present a novel distributed algorithm for computing the average degree of a given network. Our algorithm operates with quantized values and is able to calculate the exact result without any error; see Algorithm 1.
• We present a novel distributed algorithm for computing the size of a given network. Our algorithm operates with quantized values and calculates the exact result without introducing any final error; see Algorithm 2. Furthermore, the algorithm’s operation relies on the election of a leader node (if a leader is not already assigned/decided). For this reason, we present a novel strategy for leader election with quantized processing and communication. Note that this is the first leader election strategy which relies on quantized operation; see Algorithm 2A. We show that the leader election strategy converges to the exact result (i.e., the election of one leader node) after a small number of time steps, with high probability; see Theorem 3. Note that when a leader is not already assigned, the algorithm converges to the exact result with high probability, since the leader election process is successful with high probability.
• We show that both our algorithms converge in finite time. For both algorithms, we provide upper bounds on the number of time steps needed for convergence. Our provided bounds rely on a known upper bound on the diameter rather than the size of the network.

• Both algorithms utilize a distributed stopping strategy with which nodes are able to determine whether convergence has been achieved.

II. NOTATION AND PRELIMINARIES

The sets of real, rational, and integer numbers are denoted by \( \mathbb{R} \), \( \mathbb{Q} \), and \( \mathbb{Z} \), respectively. The symbol \( \mathbb{Z}_{\geq 0} \) ( \( \mathbb{Z}_{>0} \) ) denotes the set of nonnegative (positive) integer numbers (similarly \( \mathbb{Z}_{\leq 0} \), \( \mathbb{Z}_{<0} \)). Vectors are denoted by small letters, matrices are denoted by capital letters, and the transpose of a matrix \( A \) is denoted by \( A^T \). For a matrix \( A \in \mathbb{R}^{n \times n} \), the entry at row \( i \) and column \( j \) is denoted by \( A_{ij} \).

Graph-Theoretic Concepts. Consider a network of \( n \) (\( n \geq 2 \)) nodes communicating only with their immediate neighbors. The communication topology is captured by a directed graph (digraph) defined as \( G_d = (\mathcal{V}, \mathcal{E}) \). In digraph \( G_d \), \( \mathcal{V} = \{ v_1, v_2, \ldots, v_n \} \) is the set of nodes, whose cardinality is denoted as \( n = |\mathcal{V}| \geq 2 \), and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} - \{(v_j, v_j) \mid v_j \in \mathcal{V} \} \) is the set of edges (self-edges excluded) whose cardinality is denoted as \( m = |\mathcal{E}| \). A directed edge from node \( v_i \) to node \( v_j \) is denoted by \( m_{ij} \triangleq (v_j, v_i) \in \mathcal{E} \), and captures the fact that node \( v_j \) can receive information from node \( v_i \) (but not the other way around). We assume that the given digraph \( G_d = (\mathcal{V}, \mathcal{E}) \) is strongly connected. This means that for each pair of nodes \( v_j, v_i \in \mathcal{V}, v_j \neq v_i \), there exists a directed path\(^1\) from \( v_i \) to \( v_j \). Furthermore, the diameter \( D \) of a digraph is the longest shortest path between any two nodes \( v_j, v_i \in \mathcal{V} \) in the network. The subset of nodes that can directly transmit information to node \( v_j \) is called the set of in-neighbors of \( v_j \) and is represented by \( \mathcal{N}_j^- = \{ v_i \in \mathcal{V} \mid (v_j, v_i) \in \mathcal{E} \} \). The cardinality of \( \mathcal{N}_j^- \) is called the in-degree of \( v_j \) and is denoted by \( D_j^- \). The subset of nodes that can directly receive information from node \( v_j \) is called the set of out-neighbors of \( v_j \) and is represented by \( \mathcal{N}_j^+ = \{ v_i \in \mathcal{V} \mid (v_i, v_j) \in \mathcal{E} \} \). The cardinality of \( \mathcal{N}_j^+ \) is called the out-degree of \( v_j \) and is denoted by \( D_j^+ \).

Node Operation. The operation of each node \( v_j \in \mathcal{V} \) respects the quantization of information flow. At time step \( k \in \mathbb{Z}_{\geq 0} \), each node \( v_j \) maintains the mass variables \( y_j[k] \in \mathbb{Z} \) and \( z_j[k] \in \mathbb{Z}_{>0} \), which are used to communicate with other nodes. The state variables \( y_j^i[k] \in \mathbb{Z} \) and \( z_j^i[k] \in \mathbb{Z}_{>0} \) are used to store the received messages. The voting variables \( m_j \) and \( M_j \) are used to determine whether convergence has been achieved and the nodes can stop their operation. Furthermore, we assume that each node \( v_j \) is aware of its out-neighbors and can directly transmit messages to each out-neighbor separately. In order to randomly determine which out-neighbor to transmit to, each node \( v_j \) assigns a nonzero probability \( b_{ij} \) to each of its outgoing edges \( v_j \in \mathcal{N}_j^+ \). For every node, this probability assignment can be captured by an \( n \times n \) column stochastic matrix \( B \equiv [b_{ij}] \). A simple choice is to set these probabilities to be equal, i.e.,
\[
  b_{ij} = \begin{cases} 
    \frac{1}{1 + D_j^+}, & \text{if } i = j \text{ or } v_i \in \mathcal{N}_j, \\
    0, & \text{otherwise}.
  \end{cases}
\]
Each nonzero entry \( b_{ij} \) of matrix \( B \) represents the probability of node \( v_j \) transmitting towards out-neighbor \( v_i \in \mathcal{N}_j \).

A. Synchronous max/min - Consensus

The max-consensus algorithm computes the maximum value of the network in a finite number of time steps in a distributed fashion [19]. For every node \( v_j \in \mathcal{V} \), if the updates of the node’s state are synchronous, then the update rule is:
\[
  x_j[k+1] = \max_{v_i \in \mathcal{N}_j^- \cup \{v_j\}} \{x_i[k]\}.
\]
It has been shown (see, e.g., [20, Theorem 5.4]) that the max-consensus algorithm converges to the maximum value among all nodes in a finite number of steps \( s \), where \( s \leq D \). Similar results hold for the min-consensus algorithm.

B. Quantized Average Consensus

The objective of quantized average consensus problems is the development of distributed algorithms which allow nodes to process and transmit quantized information. During their operation, each node utilizes short communication packages and eventually obtains after a finite number of time steps a state \( q^* \) in the form of a quantized fraction, which is equal to the \textit{exact} real average \( q \) of the initial states. Note that in this paper we consider the case where quantized values are represented by integer\(^2\) numbers.

Since each node processes and transmits quantized information, we adopt the algorithm in [22]. This algorithm is preliminary for our results in this paper and during its operation, each node is able to achieve quantized average consensus after a finite number of time steps. The operation of the algorithm presented in [22], assumes that each node \( v_j \) in the network has an integer initial state \( y_j[1] \in \mathbb{Z} \). Initially, each node \( v_j \) assigns a nonzero probability to each outgoing edge and to a virtual self-edge as \( (1) \). At each time step \( k \), each node \( v_j \in \mathcal{V} \) maintains its mass variables \( y_j[k], z_j[k] \), and its state variables \( y_j^i[k], z_j^i[k] \). It updates the values of the mass variables as
\[
  y_j[k+1] = y_j[k] + \sum_{v_i \in \mathcal{N}_j^-} 1_{j,j}[k] y_i[k], \quad (2a)
\]
\[
  z_j[k+1] = z_j[k] + \sum_{v_i \in \mathcal{N}_j^-} 1_{j,j}[k] z_i[k], \quad (2b)
\]
where \( 1_{j,j}[k] = 1 \) if a message is received at \( v_j \) from \( v_i \) at \( k \) (0 otherwise). If the following condition holds:

\(^1\)A directed path from \( v_i \) to \( v_j \) exists if we can find a sequence of nodes \( v_{i_1} \equiv v_i, v_{i_2}, \ldots, v_{i_t} \equiv v_j \) such that \( (v_{i_{\tau+1}}, v_{i_\tau}) \in \mathcal{E} \) for \( \tau = 0, 1, \ldots, t - 1 \).

\(^2\)Following [21] we assume that the state of each node is integer valued. This abstraction subsumes a class of quantization effects (e.g., uniform quantization).
(C1): \( z_j[k+1] \geq 1 \), then, node \( v_j \) updates its state variables as
\[
\begin{align*}
z_j^*[k+1] &= z_j[k+1], \\
y_j^*[k+1] &= y_j[k+1], \\
q_j^*[k+1] &= \frac{y_j^*[k+1]}{z_j^*[k+1]}.
\end{align*}
\]

Then, it transmits its mass variables \( z_j[k+1], y_j[k+1] \) to one randomly selected out-neighbor or to itself according to \( z_j \). If it transmits its mass variables, it sets them equal to zero \( z_j[k+1] = 0, y_j[k+1] = 0 \). Finally, it receives the values \( y_i[k] \) and \( z_i[k] \) from its in-neighbors \( v_i \in \mathcal{N}_j^{-} \), it executes the operations in (2a), (2b), and repeats the operation.

**Definition 1.** The system achieves exact quantized average consensus if, for every \( v_j \in \mathcal{V} \), there exists \( k_0 \in \mathbb{Z}_{\geq 0} \) so that for every \( v_j \in \mathcal{V} \) we have \( y_j^*[k] = \sum_{i=1}^{n} y_i^*[1] / n \) and \( z_j^*[k] = n \), which means that \( q_j^*[k] = \frac{\sum_{i=1}^{n} y_i^*[1]}{n} = q \), for \( k \geq k_0 \), where \( q \) is the desirable (real) average of the initial states.

The following result from [22] analyzes the convergence of the quantized average consensus algorithm.

**Theorem 1 ([22]).** Consider a strongly connected digraph \( \mathcal{G}_d = (\mathcal{V}, \mathcal{E}) \) with \( n = |\mathcal{V}| \) nodes and \( m = |\mathcal{E}| \) edges and \( z_j[1] = 1 \) and \( y_j[1] \in \mathbb{Z} \) for every node \( v_j \in \mathcal{V} \) at time step \( k = 1 \). Suppose that each node \( v_j \in \mathcal{V} \) follows the Initialization and Iteration steps as described in the algorithm in [22]. With probability one, we can find \( k_0 \in \mathbb{N} \), so that for every \( k \geq k_0 \) we have \( y_j^*[k] = \sum_{i=1}^{n} y_i^*[1] / n \) and \( z_j^*[k] = n \), which means that \( q_j^*[k] = \frac{\sum_{i=1}^{n} y_i^*[1]}{n} \), for every \( v_j \in \mathcal{V} \) (i.e., for \( k \geq k_0 \) every node \( v_j \) has calculated \( q \) as the ratio of two integer values).

**III. Problem Formulation**

Consider a network modelled as a directed graph \( \mathcal{G}_d = (\mathcal{V}, \mathcal{E}) \). In this paper, we develop a distributed algorithm that allows nodes to address the problems P1 and P2 presented below, while processing and transmitting quantized information via available communication links.

**P1.** After a finite number of time steps, each node \( v_j \) obtains a fraction \( q_j^* \), which is equal to
\[
q_j^* = \frac{\sum_{i=1}^{n} D_{ji}^+}{n},
\]
which means that \( q_j^* \) is equal to the average degree in the network. Each node \( v_j \) processes and transmits quantized values, and ceases transmissions once (4) holds for every node.

**P2.** After a finite number of time steps, each node \( v_j \) obtains a value \( z_j^* \), which is equal to
\[
z_j^* = n,
\]
which means that \( z_j^* \) is equal to the number of nodes in the network. Each node \( v_j \) processes and transmits quantized values, and ceases transmissions once (5) holds for every node.

**IV. Distributed Average Degree Computation**

In this section we present a distributed algorithm which solves problem P1. Our algorithm is detailed below as Algorithm 1. For solving the problem in a distributed way we make the following assumption.

**Assumption 1.** An upper bound \( D' \) of the diameter of the network \( D \) is known to all nodes \( v_j \in \mathcal{V} \).

Assumption 1 is necessary for coordinating min- and max-consensus algorithms, as it will be described later.

We now describe the main operations of Algorithm 1.

**Initialization.** Each node \( v_j \in \mathcal{V} \) assigns a nonzero probability to each outgoing edge and a virtual self edge, so that the sum of the nonzero probabilities is equal to one. It sets its mass variable \( y_j[1] \) to be equal to the node’s out-degree and its mass variable \( z_j[1] \) to be equal to one. Also it sets its initial state variables \( y_j^*[1], z_j^*[1] \) to be equal to the initial mass variables \( y_j[1], z_j[1] \), respectively, and the state variable \( q_j^*[1] \) to be equal to the fraction \( y_j[1]/z_j[1] \). Then, it transmits its mass variables to a randomly chosen out-neighbor and sets them equal to zero.

**Iteration - Step 1. Calculating the Average Network Degree:** Every node \( v_j \) receives the transmitted mass variables of its in-neighbors, and sums them with the stored mass variables. Then, if its mass variable \( z_j[k+1] \) is nonzero, (i) it updates its state variables to be equal to the mass variables, and (ii) it chooses randomly an out-neighbor (or itself) and transmits the mass variables \( y_j[k+1] \) and \( z_j[k+1] \). Eventually, after a finite number of time steps, the ratio of state variables \( y_j^*[k+1]/z_j^*[k+1] \) of each node \( v_j \) is equal to the average degree in the network.

**Iteration - Step 2. Distributed Stopping:** Every \( k = tD' + 1 \) time steps, where \( t \in \mathbb{N} \), each node \( v_j \) sets its voting variables \( m_j \) and \( M_j \) to be equal to the fraction \( y_j^*[k]/z_j^*[k] \) of the state variables. It broadcasts its voting variables to its out-neighbors and receives the corresponding \( m_i \) and \( M_i \) from its in-neighbors \( v_i \in \mathcal{N}_j^{-} \). It stores the min and max among all received and its own voting values to the variables \( m_j \) and \( M_j \), respectively. Then, the min – and max – consensus algorithms are performed for \( D' - 1 \) steps. When \( k = (t + 1)D' \) time steps, each node \( v_j \) checks whether \( M_j, m_j \) have equal values; if this holds, then every node terminates the operation of the algorithm. If not, the process continues, with step \( k = (t + 1)D' + 1 \), in which each node \( v_j \) sets its voting variables \( m_j \) and \( M_j \) to be equal to the fraction \( y_j^*[k]/z_j^*[k] \) of the state variables, and the min – and max – consensus algorithms are restarted. Note that the \( m_j \) and \( M_j \) are fractions of integers. Therefore, for both the min – and max – consensus, each node \( v_j \) at time step \( k \) sends a pair of integer values \((y_j^*[k], z_j^*[k])\) whose ratio \( y_j^*[k]/z_j^*[k] \) is used during the consensus operation.

We now analyze the convergence of Algorithm 1.

**Theorem 2.** Consider a strongly connected digraph \( \mathcal{G}_d = (\mathcal{V}, \mathcal{E}) \) with \( n = |\mathcal{V}| \) nodes and \( m = |\mathcal{E}| \) edges. At time step \( k = 1 \), each node \( v_j \) follows the Initialization and Iteration steps as described in Algorithm 1. For any probability \( p_0 \),...
Algorithm 1 Average Degree Computation Algorithm

**Input:** A strongly connected digraph $G_d = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges. Each node $v_j \in V$ has knowledge of an upper bound $D'$ of the network diameter.

**Initialization:** Each node $v_j \in V$:
1) assigns a nonzero probability $b_{lj}$ to each of its outgoing edges $m_{lj}$, where $v_l \in N^+_j \cup \{v_j\}$, as follows

$$b_{lj} = \begin{cases} \frac{1}{1 + D'}, & \text{if } l = j \text{ or } v_l \in N^+_j, \\ 0, & \text{if } l \neq j \text{ and } v_l \notin N^+_j, \end{cases}$$

2) sets $y_j[1] := D^+_j, z_j[1] = 1$.
3) sets $y_j'[1] := y_j[1], z_j'[1] = 1, q_j'[1] := y_j'[1]/z_j'[1]$.
4) chooses $v_l \in N^+_j \cup \{v_j\}$ randomly according to $b_{lj}$, and transmits $y_j[1]$ and $z_j[1]$ towards $v_l$.

**Iteration:** For $k = 1, 2, \ldots$, each node $v_j \in V$, does the following:

- **while true** then
  1) if $k \mod D' = 1$ then sets $M_j = m_j = y_j'[k]/z_j'[k]$;
  2) broadcasts $M_j, m_j$ to every $v_l \in N^+_j$;
  3) receives $M_i, m_i$ from every $v_l \in N^-_j$;
  4) sets $M_j = \max_{v_l \in N^+_j \cup \{v_j\}} m_j, m_j = \min_{v_l \in N^-_j \cup \{v_j\}} m_i$;
  5) receives $y_i[k]$ and $z_i[k]$ from $v_i \in N^-_j$ and updates according to (2a)-(2b);
  6) if $z_j[k + 1] > 1$, then
    6.1) sets $z_j'[k + 1] = z_j[k + 1], y_j'[k + 1] = y_j[k + 1], q_j'[k] = y_j'[k]/z_j'[k]$;
    6.2) chooses $v_l \in N^+_j \cup \{v_j\}$ randomly according to $b_{lj}$, and transmits $y_l[k + 1]$ and $z_l[k + 1]$ towards $v_l$.
  7) if $k \mod D' = 0$ then, if $M_j = m_j$ then stops operation.

**Output:** (4) holds for every $v_j \in V$.

where $0 < p_0 < 1$, there exists $k_0 \in \mathbb{N}$, so that with probability at least $p_0$, every node $v_j$ addresses problem $P1$ in Section III.

**Proof.** See Appendix A.

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V. DISTRIBUTED NETWORK SIZE COMPUTATION

In this section we present a distributed algorithm which solves problem $P2$. Our algorithm is detailed below as Algorithm 2. Note here that Assumption 1 also holds during the operation of the proposed algorithm and we additionally make the following assumption.

**Assumption 2.** All nodes $v_j \in V$ have knowledge of a constant $U_v \in \mathbb{N}$.

Assumption 2 is necessary for executing the max-consensus algorithm in order to elect a leader node with high probability. This can be a preset value for running the protocol, irrespective of the network.

We now describe the main operations of Algorithm 2.

**Initialization - Step 1. Probability Assignment:** Each node $v_j \in V$ assigns a nonzero probability to each outgoing edge and a virtual self edge, so that the sum of nonzero probabilities is equal to one.

**Initialization - Step 2. Leader Election:** Each node executes Algorithm $2A$. During its operation, each node in the network executes a max-consensus for $U_vD'$ time steps (i.e., executes a max-consensus algorithm $U_v$ times). More specifically, each node randomly picks a quantized value and executes the first max-consensus for $D'$ time steps. Once the first max-consensus converges, we have that (i) the node (or nodes) that picked the maximum value pick again randomly a quantized value, and (ii) the node (or nodes) that did not pick the maximum value choose a value equal to $-1$. Then, the max-consensus is executed again for $D'$ time steps. This process is repeated $U_v$ times (i.e., for $U_vD'$ time steps). After the execution of Algorithm $2A$, we have that one node $v_j \in V$ is the leader (i.e., flag$_d'^j = 1$) and every node $v_i \in V \setminus \{v_j\}$ is a follower (i.e., flag$_d'^j = 0$), with high probability.

**Initialization - Step 3. Initialization of Mass and State Variables:** Each node initializes its mass variables and its state variables according to the result of Algorithm $2A$ (i.e., the leader node initializes its mass variables different than the follower nodes). Specifically, the leader node initializes both $y$ and $z$ to be equal to 1. Every follower node initializes $y$ to be equal to 0 and $z$ to be equal to 1. Then, every node sets its state variables to be equal to the mass variables.

**Iteration - Step 1. Calculating the Network Size:** Every node $v_j$ receives the transmitted mass variables of its neighbors, and sums them with the stored mass variables. Then, if its mass variable $z_j[k + 1]$ is nonzero, (i) it updates its state variables to be equal to the mass variables, and (ii) it chooses randomly an out-neighbor (or itself) and transmits the mass variables $y_j[k + 1]$ and $z_j[k + 1]$. Eventually, after a finite number of time steps, the state variable $z_j[k + 1]$ of each node $v_j$ is equal to the number of nodes in the network.

**Iteration - Step 2. Distributed Stopping:** The operation of this step is identical to “Iteration - Step 1. Distributed Stopping” of Algorithm 1. It is omitted due to space considerations.

We now analyze the convergence of Algorithm 2.

**Theorem 3.** Consider a strongly connected digraph $G_d = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges. At time step $k = 1$, each node $v_j$ follows the Initialization and Iteration steps as described in Algorithm 2. For any probability $p_0$, where $0 < p_0 < 1$, there exists $k_0 \in \mathbb{N}$, so that with probability at least $p_0$, every node $v_j$ addresses problem $P2$ in Section III.

**Proof.** See Appendix B.

**Remark 1.** Apart from guaranteeing that the exact number of nodes is computed, Algorithm 2 establishes also that the process is terminated (so that other algorithms can be initiated after the convergence of Algorithm 2). If operation termination was not required, then a variation of Algorithm 2 could be used to calculate the network size in a finite number of steps without electing a leader node. Specifically, during this variation each node $v_j$ initializes $z_j[1] = 1$ and executes iteration steps 2–6 of Algorithm 2. During this execution, there exists $k_0$, for which $z_j[k] = n$ for every $v_j \in V$, for $k \geq k_0$. 

\[ \square \]
Algorithm 2 Distributed Network Size Computation Algorithm

Input: A strongly connected digraph $G_d = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges. Each node $v_j \in V$ has knowledge of an upper bound $D'$ of the network diameter.

Initialization: Each node $v_j \in V$ does the following:
1) Assigns a nonzero probability $b_{ij}$ to each of its outgoing edges $m_{ij}$, where $v_i \in N_j^+ \cup \{v_j\}$, as follows
   
   $$b_{ij} = \begin{cases} \frac{1}{1+D'}, & \text{if } l = j \text{ or } v_l \in N_j^+, \\ 0, & \text{if } l \neq j \text{ and } v_l \notin N_j^+. \end{cases}$$
2) Calls Algorithm 2A;
   1) if flag$^{ld}$ = 1, sets $y_j[1] := 1$, $z_j[1] := 1$;
   2) if flag$^{ld}$ = 0, sets $y_j[1] := 0$, $z_j[1] := 1$;
3) Sets $y_j'[1] := y_j[1], z_j'[1] := 1$.

Iteration: For $k = 1, 2, \ldots$, each node $v_j \in V$, does the following:
- while true then
  1) if $k \mod D' = 1$ then sets $M_j = m_j = y_j[k];$
  2) steps 2 - 7 are same as Algorithm 1;

Output: (5) holds for every $v_j \in V$.

Algorithm 2A Leader Election with Quantized Information

Input: A strongly connected digraph $G_d = (V, E)$ with $n = |V|$ nodes and $m = |E|$ edges. Each node $v_j \in V$ has knowledge of an upper bound $D'$ of the network diameter.

Initialization: Each node $v_j \in V$ sets flag$^{ld} = 1$.

Iteration: For $k = 1, 2, \ldots$, each node $v_j \in V$, does the following:
1) if $k \mod D' = 1$ then
   1.1) if flag$^{ld} = 1$ then chooses randomly $\eta_j' \in \{0, 1, \ldots, \#U_p\}$, and sets $M_j' = \eta_j'$;
   1.2) if flag$^{ld} = 0$ then sets $\eta_j' := -1$ and $M_j' = \eta_j'$;
2) broadcasts $M_j'$ to every $v_i \in N_j^+$;
3) receives $M_i'$ from every $v_i \in N_j^+$;
4) sets $M_j' = \max_{v_i \in N_j^+ \cup \{v_j\}} M_i'$;
5) if $k \mod D' = 0$ then if $M_j' \neq \eta_j'$ then sets flag$^{ld} = 0$;

Output: flag$^{ld}$ for every $v_j \in V$.

Remark 2. Algorithm 2A is executed as an initialization step of Algorithm 2. However, both can be executed in parallel. The strategy of parallel execution significantly decreases the required number of time steps for convergence of Algorithm 2 as demonstrated in Section VI (see Fig. 4).

VI. SIMULATION RESULTS

In this section, we present simulation results in order to demonstrate the operation of our proposed algorithms and their potential advantages. For both algorithms we focus on a random digraph of 20 nodes and show how the nodes’ states converge to the desired value in finite time. We also demonstrate the distributed stopping capabilities of our algorithms. Finally, we analyze numerically the convergence time of our algorithms, and we present the distributions of the required number of time steps for convergence. Our simulations emphasize the novelty of our algorithms which, to the best of our knowledge, are the first that use quantized values to calculate the exact values of the average degree and size of a network while also providing strong theoretical guarantees. Note here that comparing against other algorithms from current literature is not feasible due to the fact that all other methods in the literature operate with real-values and exhibit asymptotic convergence.

Average Degree Computation of a Random Network of 20 Nodes. In this section we demonstrate the operation of Algorithm 1 over a random digraph of 20 nodes. Each edge of the digraph was generated with probability 0.5, and the diameter is equal to $D = 3$. The average degree of the digraph is equal to $206/20 = 10.3$. During the execution of Algorithm 1 each node has knowledge of an upper bound on the network’s diameter equal to $D' = 4$.

In Fig. 1, we plot the evolution of the state variable $q_j[k]$ of every node $v_j$. We can see that Algorithm 1 converges after 74 time steps to the exact solution. Specifically, each node calculates the quantized fraction $106/20$ which is equal to the average degree in the network. Furthermore, we can see that after 78 iterations each node terminates its operation since it has knowledge of $D' = 4$ which is an upper bound on the network diameter.

In Fig. 2, we present 10000 executions of Algorithm 1 over a random digraph of 20 nodes. Each edge of the digraph was generated with probability 0.5, and the diameter is equal to $D = 3$ for each of the 10000 executions. The average number of time steps for convergence of Algorithm 1 is equal to 120.96. In Fig. 2 we can see that in most cases Algorithm 1 requires 70–160 iterations for convergence.
Size Computation of a Random Network of 20 Nodes.
In this section we demonstrate the operation of Algorithm 2 over a random digraph of 20 nodes. The parameters of the digraph are the same as in the previous example where we demonstrated Algorithm 1, and each node also has knowledge of \( D' = 4 \). Furthermore, we have that \( U_v = 20 \). This means that Algorithm 2A is executed for 80 time steps.

In Fig. 3, we plot the evolution of the state variable \( z_{sj}[k] \) of every node \( v_j \). We can see that Algorithm 2 calculates after 160 time steps the quantized fraction \( 1/20 \), where the denominator is equal to \( D = 3 \) for each of the 10000 executions, and each node also has knowledge of \( D' = 4 \). Note that in Fig. 3, we plot the evolution of \( z_{sj}[k] \) for time steps \( k \geq 80 \), since Algorithm 2A is executed for 80 time steps.

In Fig. 4, we present 10000 executions of Algorithm 2 over a random digraph of 20 nodes. Each edge of the digraph was generated with probability 0.5, the diameter is equal to \( D = 3 \) for each of the 10000 executions, and each node also has knowledge of \( D' = 4 \). In Fig. 4 (A) we execute Algorithm 2A before Algorithm 2 for \( U_v = 20 \) and \( D' = 4 \) (i.e., we execute Algorithm 2A for 80 time steps). In Fig. 4 (B) we execute Algorithm 2A in parallel with Algorithm 2. The average number of time steps for convergence of Algorithm 2 is equal to 198.64 for (A), and 119.60 for (B). In Fig. 4 (A) we can see that in most cases Algorithm 2 requires 150–240 iterations for convergence. However, in Fig. 4 (B) we can see that in most cases it requires 80–140. More specifically, in Fig. 4 (B) we can see that 1400 executions of Algorithm 2 require 80 – 90 time steps to converge. This is mainly due to the fact that the Iteration Steps of Algorithm 2 have reached convergence, but nodes need to implement Algorithm 2A for 80 time steps. As a result, parallel execution of Algorithm 2A with Algorithm 2 is better, since a significantly smaller number of time steps is required for convergence.

Leader Election via Algorithm 2A. We now analyze the probability of having one leader after the execution of Algorithm 2A, over a random digraph of 20 nodes. In Fig. 5 we present 10000 executions of Algorithm 2A for \( U_v = 10 \). We plot the instances for which we have more than one leader nodes during the execution of Algorithm 2A, for the cases where (i) \( \eta_j \in \{0, 1, \ldots, 15\} \), (ii) \( \eta_j \in \{0, 1, \ldots, 31\} \), and (iii) \( \eta_j \in \{0, 1, \ldots, 255\} \) for every node \( v_j \). We can see that if we increase the range of \( \eta_j \) for every node \( v_j \), the convergence rate of Algorithm 2A improves greatly. Also, after 5 executions of Algorithm 2A (i.e., \( U_v = 5 \)), the number of instances where we have multiple leader nodes is equal to zero. This means that during Algorithm 2A for \( U_v \geq 5 \), only one node is leader node.

VII. CONCLUSIONS AND FUTURE DIRECTIONS
We showed that our algorithms are able to compute the exact values of the average degree and the size of a network after a finite number of time steps. Additionally, our algorithms allow each node to determine in a distributed fashion whether convergence has been achieved, and thus terminate its operation. Furthermore, in order to implement our algorithms, we present the first leader election strategy which relies on quantized operation (i.e., nodes process and transmit quantized information). Finally, we have demonstrated the operation of our algorithm over random directed networks and illustrated its finite time convergence.

Extending our algorithms to achieve fully asynchronous operation which is desirable in large-scale networks, and operating over unreliable networks (e.g., packet dropping links) are two of our main future directions.

APPENDIX A
PROOF OF THEOREM 2
We first consider Lemma 1, which is necessary for our subsequent development.

Lemma 1 ([23]). Consider a strongly connected digraph \( G_d = (\mathcal{V}, \mathcal{E}) \) with \( n = |\mathcal{V}| \) nodes and \( m = |\mathcal{E}| \) edges. Suppose that each node \( v_j \) assigns a nonzero probability \( b_{lj} \) to each of its outgoing edges \( m_{lj} \), where \( v_l \in N_j^+ \cup \{v_j\} \), as follows

\[
b_{lj} = \begin{cases} \frac{1}{1+D_j^+}, & \text{if } l = j \text{ or } v_l \in N_j^+, \\ 0, & \text{if } l \neq j \text{ and } v_l \notin N_j^+. \end{cases}
\]

At time step \( k = 0 \), node \( v_j \) holds a “token” while the other nodes \( v_l \in \mathcal{V} - \{v_j\} \) do not. Each node \( v_j \) transmits the “token” (if it has it, otherwise it performs no transmission)
according to the nonzero probability \( b_{ij} \) it assigned to its outgoing edges \( m_{ij} \). The probability \( P_{T_i}^D \) that the token is at node \( v_j \) after \( D \) time steps satisfies \( P_{T_i}^D \geq (1 + D_{\max}^+)^{-D} > 0 \), where \( D_{\max} = \max_{v_j \in V} D_j^+ \).

We now consider Lemma 2, which analyzes the probability according to which two tokens performing a random walk visit a specific node at the same time step. The proof is similar to Lemma 1, mutatis mutandis, and is omitted due to space limitations.

**Lemma 2.** Consider a strongly connected digraph \( G_d = (V, \mathcal{E}) \) with \( n = |V| \) nodes and \( m = |\mathcal{E}| \) edges. Suppose that each node \( v_j \) assigns a nonzero probability \( b_{ij} \) to each of its outgoing edges \( m_{ij} \), where \( v_i \in N_j^+ \cup \{v_j\} \), as follows:

\[
b_{ij} = \begin{cases} 
\frac{1}{1 + D_j^+}, & \text{if } l = j \text{ or } v_l \in N_j^+, \\
0, & \text{if } l \neq j \text{ and } v_l \notin N_j^+.
\end{cases}
\]

At time step \( k = 0 \), nodes \( v_i, v_j \) hold a “token” while the other nodes \( v_l \in V \setminus \{v_i, v_j\} \) do not (i.e., there are two tokens in the network). Each node \( v_j \) transmits the “token” (if it has it, otherwise it performs no transmission) according to the nonzero probability \( b_{ij} \) assigned to its outgoing edges \( m_{ij} \). After \( D \) time steps, the probability \( P_{T_i}^D \) that the two tokens visit a specific node at the same time step is \( P_{T_i}^D \geq \sum_{l=1}^{n} (1 + D_{\max}^+)^{-2D} > 0 \), where \( D_{\max} = \max_{v_j \in V} D_j^+ \).

Now we present the proof of Theorem 2. The operation of Algorithm 1 can be interpreted as the “random walk” of \( n \) “tokens” in a Markov chain. Each token has a pair of values \( y, z \). If two (or more) tokens visit the same node at the same time step \( k \), they “merge” to a new single token (i.e., if two (or more) tokens merge, their \( y \) values sum to the new \( y \) value, and their \( z \) values sum to the new \( z \) value). Then, the new token performs a random walk in a Markov chain. Once all \( n \) tokens merge to a final single token, this final single token has a pair of values \( y, z \) whose ratio \( y/z \) is equal to the average degree in the network. Thus, executing Algorithm 1 for an additional finite number of time steps, this final single token will visit every node in the network.

The structure of the proof comprises of three parts. In the first part (Part I), we calculate the number of time steps \( k_0 \) after which all tokens have merged to one single final token with probability at least \( \sqrt{p_0} \). In the second part (Part II), we calculate the number of time steps \( k_d' \) after which the final single token has visited every node in the network with probability at least \( \sqrt{p_0} \). In the third part (Part III), we calculate the number of time steps \( k_0 \) after which (4) holds for every node, and each node ceases transmissions.

**Part I.** During the operation of Algorithm 1, from Lemma 2 we have that, after \( D \) time steps, the probability \( P_{T_i}^D \) that two (or more) tokens visit a specific node at the same time step is \( P_{T_i}^D \geq \sum_{l=1}^{n} (1 + D_{\max}^+)^{-2D} > 0 \), where \( D_{\max} = \max_{v_j \in V} D_j^+ \). This means that, after \( D \) time steps, the probability \( P_{T_i}^D \) that two (or more) tokens do not visit a specific node at the same time step is

\[
P_{N_T^D \setminus i}^D = 1 - \sum_{l=1}^{n} (1 + D_{\max}^+)^{-2D}.
\]

By extending this analysis, we choose \( \epsilon' \) (where \( 0 < \epsilon' < 1 \)) for which it holds that

\[
\epsilon' \leq 1 - 2^{\frac{\log_2 \sqrt{p_0}}{n}}.
\]

After \( \tau' \) time steps where

\[
\tau' \geq \left\lceil \frac{\log \epsilon'}{\log (1 - (1 + D_{\max}^+)^{-2D})} \right\rceil,
\]

and \( \epsilon' \) fulfills (7), we have that the probability \( P_{T_i}^{\tau_D^D} \) that two (or more) tokens do not visit a specific node at the same time step is \( P_{T_i}^{\tau_D^D} \leq \left[ P_{N_T^D}^D \right]^{\tau} \leq \epsilon' \). This means that after \( \tau' \) time steps, the probability \( P_{T_i}^{\tau_D^D} \) that two (or more) tokens visit a specific node at the same time step is \( P_{T_i}^{\tau_D^D} \geq 1 - \epsilon' \).

Therefore, after \( k_0' \geq (n - 1)\tau' \) time steps, the probability \( P_{T_i}^{(n-1)\tau_D^D} \) that two (or more) tokens visit a specific node for \( n - 1 \) instances at the same time step is

\[
P_{T_i}^{(n-1)\tau_D^D} \geq (1 - \epsilon')^{(n-1)} \geq \sqrt{p_0}.
\]

**Part II.** During the operation of Algorithm 1, from Lemma 1 we have that the probability \( P_{T_i}^D \) that a token visits a specific node after \( D \) time steps is \( P_{T_i}^D \geq (1 + D_{\max}^+)^{-D} > 0 \), where \( D_{\max} = \max_{v_j \in V} D_j^+ \). This means that the probability \( P_{NT}^D \) that a token does not visit a specific node after \( D \) steps is

\[
P_{NT}^D \leq 1 - (1 + D_{\max}^+)^{-D}.
\]

We choose \( \epsilon'' \) (where \( 0 < \epsilon'' < 1 \)) for which it holds that

\[
\epsilon'' \leq 1 - 2^{\frac{\log_2 \sqrt{p_0}}{n}}.
\]

After \( \tau'' D \) time steps where

\[
\tau'' \geq \left\lceil \frac{\log \epsilon''}{\log (1 - (1 + D_{\max}^+)^{-D})} \right\rceil,
\]

and \( \epsilon'' \) fulfills (11), we have that the probability \( P_{NT}^D \) that one token does not visit a specific node is \( P_{NT}^D \leq \left[ P_{NT}^D \right]^{\tau''} \leq \epsilon'' \). This means that after \( \tau'' D \) time steps, the probability \( P_{NT}^D \) that one token visits a specific node is \( P_{NT}^D \geq 1 - \epsilon'' \).

Therefore, after \( k_0'' \geq (n - 1)\tau'' D \) time steps, the probability \( P_{NT}^{(n-1)\tau'' D} \) that one token visits a specific node for \( n - 1 \) instances (i.e., it visits every node in the network) is

\[
P_{NT}^{(n-1)\tau'' D} \geq (1 - \epsilon'')^{(n-1)} \geq \sqrt{p_0}.
\]

**Part III.** During the operation of Algorithm 1, from Lemmas 1 and 2, we can state that after \( (n - 1)\tau'' D + (n - 1)\tau'' D \) time steps, where \( \tau'' \) fulfills (8) and \( \tau'' \) fulfills (12), we have that (4) holds for every \( v_j \in V \) with probability at least \( p_0 \). Then, after an additional number of \( D' \) time steps, each node will determine whether convergence has been achieved, and thus it will cease transmissions. As a result, during the operation of Algorithm 1, after \( k_0 \geq (n - 1)\tau'' D + (n - 1)\tau'' D + D' \) time steps, we have that each node addresses problem \( P_1 \) in Section III with probability at least \( p_0 \).

**APPENDIX B**

**PROOF OF THEOREM 3**

We first consider Lemma 3, which is necessary for our subsequent development.
Lemma 3. Consider a strongly connected digraph $\mathcal{G}_d = (\mathcal{V}, \mathcal{E})$ with $n = |\mathcal{V}|$ nodes and $m = |\mathcal{E}|$ edges. Each node executes Algorithm 2A. After $U_d$ time steps, a single node $v_j$ is the leader and every other node $v_i \in \mathcal{V}\setminus\{v_j\}$ is a follower, with probability that goes to one, as $U_d$ goes to infinity.

Proof. When node $v_j$ selects a value randomly depending on the number of bits allocated for communication, it basically samples from a random variable $X_j$ with probability mass function a discrete uniform distribution on the integers $0, 1, 2, \ldots, M - 1$, where $M := 2^{\text{bits}}$. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed (i.i.d) random variables (representing the random variables of the $n$ nodes in the network).

Let $Y = \max\{X_1, \ldots, X_n\}$. The probability of event $A_\ell$ being that $\ell < n$ nodes have the maximum value is given by

$$P[A_\ell] = \binom{n}{\ell} \left( \frac{1}{M} \right) ^\ell \left( \frac{Y - 1}{M} \right) ^{n - \ell}.$$  

Let $n(j)$ denote the number of nodes participating in the max-consensus algorithm in round $j$, $j \in \{1, 2, \ldots, U_d\}$. In the worst case (when $n(j) = 2$), a node is eliminated at round $j$ with probability at least $1 - 1/M$. Now, we proceed with a very conservative analysis to prove Lemma 3 (an extended and less conservative analysis will be given in the journal version of the paper). Consider a sequence of rounds $\{1, 2, \ldots, U_d\}$: at each round a node is eliminated with probability (at least) $p = 1 - 1/M$. Otherwise, with probability (less than) $1 - p = 1/M$, no node is eliminated. Thus, the probability that we have a leader after $U_d$ rounds is the probability that we have $n-1$ eliminations: this is bounded from below by

$$\sum_{k=n-1}^{U_d} \binom{U_d}{k} p^k (1-p)^{U_d-k} = 1 - \sum_{k=0}^{n-2} \binom{U_d}{k} p^k (1-p)^{U_d-k} > 1 - (n-1) \binom{U_d}{n-1} p^{n-1} (1-p)^{U_d-(n-1)},$$

where $U_d$ is assumed to be larger than $2(n-1)$ for the last inequality to hold (but $U_d$ is not required to be even relevant with the network size, as shown in Fig 5).

The major advantage of this method is that the number of nodes participating in the next max-consensus is limited to the number of nodes that had the same maximum value in the preceding max-consensus round. Therefore, especially for a considerably large value of $M$, the number of nodes participating in the next max-consensus is much smaller. The procedure continues until there is a round with no two nodes that select the same maximum integer value (cf. Fig 5).

Once Algorithm 2 finishes the Initialization steps and elects a leader node (see Algorithm 2A) with high probability (see Lemma 3), it executes its Iteration steps. However, the Iteration steps of Algorithm 2 are identical to Algorithm 1. Thus the proof of Theorem 3 is similar to Theorem 2 and is omitted.

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