Scalable Reconstruction of Unitary Processes and Hamiltonians

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Based on recently introduced efficient quantum state tomography schemes, we propose a scalable method for the tomography of unitary processes and the reconstruction of Hamiltonians. As opposed to the exponential scaling with the number of subsystems of standard quantum process tomography, the method relies only on measurements of linearly many local observables and either (a) the ability to prepare eigenstates of locally informationally complete operators or (b) access to an ancilla of the same size as the to-be-characterized system and the ability to prepare a maximally entangled state on the combined system. As such, the method requires at most linearly many states to be prepared and linearly many observables to be measured. The quality of the reconstruction can be quantified with the same experimental resources that are required to obtain the reconstruction in the first place. Our numerical simulations of several quantum circuits and local Hamiltonians suggest a polynomial scaling of the total number of measurements and post-processing resources.

I. INTRODUCTION

Quantum process tomography [1–6] is the standard for the verification and characterization of quantum operations on well-controlled quantum systems. Among others, recent experimental demonstrations of quantum simulators of multi-partite quantum systems [7, 8] have demonstrated that, by now, the number of well controllable qubits is in a regime for which conventional tomography techniques fail as the required experimental and numerical post-processing resources scale exponentially with the number of qubits. While there has been a considerable effort to introduce scalable techniques that allow for an efficient reconstruction [9–17] and verification [18–20] of quantum states, quantum process tomography still leaves much to be desired.

The most straightforward approach to process tomography is based on the idea of probing the quantum channel with an informationally complete set of states. After sending each of these input states through the channel, the process underlying the dynamics is characterized by performing full state tomography on all of the output states. The so-obtained map fully characterizes the channel [1–3]. This strategy is referred to as standard quantum process tomography (SQPT) and works well as long as the underlying system is of low dimension. Considering multi-partite quantum systems, however, reveals the disadvantages of this technique: Not only the number of states to be sent through the channel and to be indentified grows exponentially with the number of subsystems $n$, but also the number of parameters that need to be determined for each of the output states. There is, however, a strategy to overcome the exponential scaling of the number of input and output states: Ancilla-assisted process tomography (AAPT) [4–6]. AAPT requires an ancillary system of the same size $n$ as the system itself, the preparation of a maximally entangled input state $|\Phi\rangle \propto \sum_i |i\rangle |i\rangle$ on the combined system, and full tomography of the resulting state after application of the channel, $\hat{\varrho}_C = (I \otimes E) (|\Phi\rangle \langle \Phi|)$, see Fig. 1. This method is based on the correspondence between quantum states and quantum processes known as the Choi-Jamiolkowski Isomorphism [21, 22]. The incorporation of an ancillary system considerably reduces the complexity of the preparation stage (as opposed to exponentially many input states in SQPT, only one state has to be prepared and only one output state has to be characterized) but comes at the cost of needing access to an independent ancilla system of the same size as the system that is to be analyzed. Still, for a complete characterization of the output state $\hat{\varrho}_C$, a number...
of measurements exponentially large in $n$ is required [23].

In this work, we combine recent results from efficient and scalable state tomography [9, 11] with the idea of AAPT and briefly discuss how to avoid the need for an ancilla system, see Fig. 1. This allows us to formulate a scalable process tomography scheme where the number of input and output states, together with the experimental measurement settings to characterize the latter, grows only linearly with the number of qubits. We discuss the application of our scalable process tomography technique with respect to unitary processes (in particular, for circuits preparing a Greenberg-Horne-Zeilinger (GHZ) state and performing the quantum Fourier transform) in Section III A (see Fig. 2) and the reconstruction of time-independent local Hamiltonians in Section III B (see Fig. 3 and Fig. 4). We note that the reconstruction is applicable in principle for arbitrarily long times as long as the conditions for efficient state reconstruction are met. Furthermore, the quality of the reconstruction may be quantified using the same experimental resources that are also required to obtain the reconstruction. For unitary processes, $\hat{\varrho}_E$ is guaranteed to be pure such that certifiability of the reconstruction of $E$ is inherited by the certifiability of $\hat{\varrho}_E$ [9]; see Ref. [20] for an analogous mixed-state certificate.

II. SCALABLE PROCESS TOMOGRAPHY

Let us first consider the AAPT scheme and restrict, without loss of generality, to two-level systems, so qubits. Further, we arrange the ancilla-$r$-system as depicted in Fig. 1 with odd sites representing the ancilla and even sites denoting the system itself. In this enumeration, the maximally-entangled input state $|\Phi\rangle$ takes the form of a product of Bell states $|\phi^+\rangle_{1,2} \otimes \cdots \otimes |\phi^+\rangle_{2n-1,2n}$, $|\phi^+\rangle \propto |00\rangle + |11\rangle$, and hence only requires local two-qubit manipulations for its experimental generation. The system (i.e., the even sites) is sent through the channel and state tomography has to be performed on the resulting state $\hat{\varrho}_c$. Without any prior knowledge about the underlying quantum channel, full state tomography is inevitable and one seemingly faces the notorious curse of dimensionality. A large class of quantum states, however, may be reconstructed from a number of measurements and with post-processing resources that both scale only polynomially in $n$ [9–11]. If the output state $\hat{\varrho}_c$ happens to fall within this class of states, scalable reconstruction of the channel may be achieved. The input to the scalable state tomography schemes [9–11] are given by the measurement data of the following observables (or, alternatively, any other local operator basis)

$$\hat{P}_{k;\alpha_1,\ldots,\alpha_r} = 1_{1,\ldots,k} \otimes \hat{\sigma}_{k+1}^{\alpha_1} \otimes \cdots \otimes \hat{\sigma}_{k+r}^{\alpha_r} \otimes 1_{k+r+1,\ldots,2n},$$

(1)

for all $\alpha_i \in \{x,y,z\}$, $k = 0, \ldots, 2n-r$, and a fixed $r$ independent of the size of the system. Here, $\hat{\sigma}_x^i$, $\hat{\sigma}_y^i$, $\hat{\sigma}_z^i$ are the Pauli matrices for qubit $i$. There are $(2n-r+1) \times 3^r$ such operators, i.e., the number of observables that are required for these reconstruction schemes scales linearly in $n$. Note that the restriction to local information is not mandatory, any output state that is uniquely characterized by a number of measurements that scales moderately with $n$ can be reconstructed by these techniques [11] and, hence, belongs to the class of states for which our process tomography procedure is applicable.

Interestingly, the necessary information may also be obtained without the need of an ancilla, yet then increasing the demand at the preparation stage: By virtue of the identity

$$\text{tr}[(\hat{A} \otimes \hat{S}) \hat{\varrho}_c] = \frac{\text{tr}[\hat{E}(\hat{A}^\dagger)\hat{S}]}{2^n},$$

(2)

which holds for any operator $\hat{S}$ (acting on the system (ancilla), and the fact that each $\hat{P}_{k;\alpha_1,\ldots,\alpha_r}$ is of the form $\hat{P}_A \otimes \hat{P}_S$ ($A$: ancilla, $S$: system), one may obtain the necessary expectation values by preparing the eigenstates of the Pauli matrices in $\hat{P}_A$ on the system, sending them through the channel and measuring $\hat{P}_S$ on the resulting state (see Fig. 1 and Appendix A for details) – a scheme that requires no ancilla, the preparation of linearly many states and the measurement of linearly many observables.

While the preparation/measurement strategy we just outlined may be favourable from an experimental perspective, we will present our scheme in the framework of AAPT as certain intuitions are particularly transparent in this setting. In the remainder of this section, we dis-

![Figure 2. Reconstruction of the quantum circuits in Eq. (10).](image-url)
cuss how unitary operators and the Hamiltonians governing time evolution can be constructed from \( \hat{g} \). In principle, the scheme is applicable to non-unitary channels as well, as long as the corresponding state \( \hat{g} \) permits reconstruction.

### A. Reconstruction of Unitary Channels

We aim at reconstructing unitary channels, so channels of the form \( \mathcal{E}(\hat{\rho}) = \hat{U} \rho \hat{U}^\dagger \). For those, it is guaranteed that the resulting state is pure, i.e., \( \hat{g} = |\psi_{\mathcal{E}}\rangle \langle \psi_{\mathcal{E}}| \). The unitary may then be obtained from the identity \( \langle j | \hat{U} | i \rangle = 2^{n/2} \langle j | \psi_{\mathcal{E}} \rangle \). The output of the state-reconstruction algorithm of Refs. [9, 11] provide us with a pure estimate \( |\psi_{\mathcal{E}}^{\text{rec}}\rangle \) of \( \hat{g} \) given in the form of a matrix product state (MPS) [24, 25],

\[
|\psi_{\mathcal{E}}^{\text{rec}}\rangle = \sum_{i_1, \ldots, i_{2n}} A_{i_1}^{(1)} \cdots A_{i_{2n}}^{(2n)} |i_1, \ldots, i_{2n}\rangle,
\]

where \( A_{i_k}^{(k)} \in \mathbb{C}^{D_{k-1} \times D_k} \) with \( D_0 = 1 = D_{2n} \) and summation is over all \( i_k = 1, 2, k = 1, \ldots, 2n \). Given this form, the matrix product operator (MPO) representation of the estimate \( \hat{U}_{\text{rec}} \) to \( \hat{U} \), \( \langle j | \hat{U}_{\text{rec}} | i \rangle = 2^{n/2} \langle j | \psi_{\mathcal{E}}^{\text{rec}} \rangle \), may straightforwardly be obtained by grouping the \( n \) pairs of ancilla and system sites:

\[
\hat{U}_{\text{rec}} = 2^{n/2} \sum_{i_1, \ldots, i_{2n}} \left[ A_{i_1}^{(1)} A_{i_2}^{(2)} \cdots \left( A_{i_{2n-1}}^{(2n-1)} A_{i_{2n}}^{(2n)} \right) \right] |i_1\rangle \langle i_1| \otimes \cdots \otimes |i_{2n}\rangle \langle i_{2n}-1|.
\]

As the input state \( |\Phi\rangle \) can efficiently be represented as an MPS (given the sites are labeled as depicted in Fig. 1), we expect our efficient process tomography scheme to work for quantum circuits that consist of algebraically many local gates, i.e., for gates having an efficient MPO representation. The latter holds (approximately), e.g., for the quantum circuit that prepares the GHZ state (implements the quantum Fourier transform), which we consider below.

### B. Hamiltonian Reconstruction

Assume that the quantum process is in fact the time evolution under a local Hamiltonian, i.e., \( \hat{U} = e^{-i \hat{H} t} \), with \( \hat{H} = \sum_i \hat{h}_i \), where \( \hat{h}_i \) only acts on a fixed number of neighbouring sites (we will consider nearest-neighbour Hamiltonians throughout). With the tools to reconstruct unitary processes at hand, it remains to address the question of how to find a valid estimate of the Hamiltonian \( \hat{H} \) governing the time evolution. To obtain \( \hat{H} \) from this unitary, we will use the identity

\[
x = \sin(x) \frac{\arccos(\cos(x))}{\sqrt{1 - \cos^2(x)}}, \quad x \in (-\pi, \pi),
\]

which converges for \(|z - 1| < 2\) [26]. The basic idea is that from \( \hat{U} = e^{-i \hat{H} t} \), we know that \( 2i \sin(\hat{H} t) = \hat{U}^\dagger - \hat{U} \),

\[
\arccos(z) = \sum_{k=0}^{\infty} c_k (z - 1)^k, \quad c_k = \frac{(-1)^k}{2^k} \prod_{j=1}^{k} \frac{j}{j + \frac{1}{2}}
\]

for a given \( N \) and \( \sin(\hat{H}_{\text{rec}} t) \) and \( \cos(\hat{H}_{\text{rec}} t) \) as above. This approach has two advantages over using a power series expansion of the logarithm: It is valid for larger values of \( \|\hat{H}t\| \) and the series converges much faster. Note that even if the reconstruction of \( |\psi_{\mathcal{E}}\rangle \) is perfect, \( \hat{U}_{\text{rec}} \) may differ from \( \hat{U} \) by a global phase \( \phi \), such that \( \hat{U}^\dagger e^{i\phi} \hat{U} = I \) and Eq. (5) impose \( \|\hat{H} - \phi I\| < \pi \). To remedy this problem, we use \( \hat{U}_{\text{rec}} \text{tr}(\hat{U}_{\text{rec}}^*) / \text{tr}(\hat{U}_{\text{rec}}) \) as our actual estimate.

The initial state \( |\Phi\rangle \) is an MPS of low bond dimension and, as under local Hamiltonians quantum correlations build up in a light-cone-like picture, \( \hat{g} \) will still have a small bond dimension for small times [27, 28]. We thus expect the reconstruction to succeed for small times.

### III. NUMERICAL SIMULATIONS

For our numerical simulations we proceed as follows: For several exemplary channels \( \mathcal{E} \), we numerically obtain \( |\psi_{\mathcal{E}}\rangle \) [29] and simulate measurements of the local observables \( \hat{P}_{k_{\alpha_1}, \ldots, \alpha_r} \) by drawing \( M \) times per observable according to \( \hat{g} \). In this way, we take statistical errors into
account, with statistical errors for measurements without ancilla being very similar [30]. The resulting empirical mean values (i.e., the simulated estimates of the weights \( \tilde{\hat{H}}_{k,\alpha_1,...,\alpha_r} \) of all eigenprojectors \( \tilde{\hat{H}}_{k,\alpha_1,...,\alpha_r} \), \( s_i = \pm 1 \), of \( \hat{P}_{k,\alpha_1,...,\alpha_r} \) are then used to obtain a reconstruction \( |\psi_E^{\text{rec}}\rangle \) of \( \psi_E \). This is done by exploiting the algorithm described in [9] to obtain an initial state for the scalable maximum-likelihood algorithm of Ref. [11]. The result is an MPS representation of \( |\psi_E^{\text{rec}}\rangle \) which can be converted into an MPO representation of \( \hat{U}_{\text{rec}} \) as described above in Eq. (4).

With the estimate \( |\psi_E^{\text{rec}}\rangle \) and thus the corresponding operator \( \hat{U}_{\text{rec}} \) at hand, we then quantify the quality of the reconstruction scheme by

\[
F = F(\hat{U}, \hat{U}_{\text{rec}}) = |\langle \psi_E | \psi_E^{\text{rec}} \rangle|^2,
\]

and note that this is in one-to-one correspondence to other distance measures for unitary channels used in the literature [31–33] [34].

In the case of Hamiltonian reconstruction, we assess our reconstructed estimates as follows: First, note that two Hamiltonians \( \hat{H} \) and \( \hat{H} + \lambda \hat{1} \), \( \lambda \in \mathbb{R} \), are physically indistinguishable. Therefore, we measure relative distances between Hamiltonians according to [35]

\[
D(\hat{H}, \hat{H}') = \min_{\lambda \in \mathbb{R}} \| \hat{H} - \hat{H}' - \lambda \hat{1} \|,
\]

which is independent of energy offsets in both \( \hat{H}' \) and \( \hat{H} \). We choose the operator norm \( \| \cdot \| \) motivated by its property

\[
|\langle \hat{A}(t) \rangle_{\hat{H}} - \langle \hat{A}'(t) \rangle_{\hat{H}'}| \leq 2\| \hat{H} - \hat{H}' \| \| \hat{A} \|,
\]

where \( \hat{A}(t) \) and \( \hat{A}'(t) \) are the Heisenberg picture time evolutions of \( \hat{A} \) according to \( \hat{H} \) and \( \hat{H}' \), respectively. In other words, the operator norm distance defines a timescale on which two Hamiltonians may be considered equivalent.

For all results below, we repeat the whole procedure of sampling from the simulated state, reconstructing it, and assessing the quality of the reconstruction several times. All results shown are mean values of \( F(\hat{U}, \hat{U}_{\text{rec}}) \) and \( D(\hat{H}, \hat{H}_{\text{rec}}) \) over several runs, with deviations that are, for the number of measurements per observable considered, smaller than the size of the markers. Next, we present numerical results for the reconstruction of quantum circuits and Hamiltonians and study the performance as a function of the number of qubits \( n \), the number \( M \) of measurements per observable, and the block size \( r \) of the subsystems on which measurements are performed. We simulate circuits and Hamiltonians on up to 32 qubits. Hence, reconstructing the unitary uses pure state reconstruction on up to 64 qubits.

### A. Quantum Circuits

We demonstrate the feasibility of our scalable tomography scheme by considering the GHZ circuit, which prepares an \( n \)-qubit GHZ state from \( |0\ldots0\rangle \) and the quan-
tum Fourier transform [1],
\[
\hat{\text{GHZ}} = \text{CN}_{n-1,n} \text{CN}_{n-2,n-1} \cdots \text{CN}_{1,2} \hat{H}_1,
\]
\[
\hat{\text{QFT}} = \prod_{k=1}^{n} \left( \prod_{j=1}^{n-k} \text{CR}_{k,k+j} \left( \pi/2^j \right) \right) \hat{H}_k,
\]
where we use the convention \( \prod_{j=1}^{k} \hat{U}_i = \hat{U}_k \cdots \hat{U}_1 \) for products of non-commuting operators. Here, \( \hat{H}_k \) denotes the Hadamard gate acting on qubit \( k \), and \( \text{CN}_{i,t+1} \) \( (\text{CR}_{i,j}(\phi)) \) denotes the two-qubit conditional not (conditional rotation) gate [1]. The reconstruction results are summarized in Fig. 2.

**B. Hamiltonian reconstruction**

We determine an estimate of the Hamiltonian that governs the time evolution by the series given in Eq. (7) with \( N = 3 \). With this, and the assumption that \( \min_{\lambda \in \mathbb{R}} \| \hat{H} - \lambda \| = \| \hat{H} \| [36] \), one has
\[
D(\hat{H}, \hat{H}_{\text{rec}}) \leq \frac{\| \hat{H} - \hat{H}_{\text{rec}} \|}{\| \hat{H} \|} \leq \frac{1}{10^3} \| \hat{H} t \|^6 + \mathcal{O}(\| \hat{H} t \|^8). \tag{11}
\]

Fig. 3 shows results for an isotropic Heisenberg Hamiltonian. We have also studied the critical Ising model and a Hamiltonian with random nearest-neighbour interaction, which show very similar behaviour and the corresponding results may be found in Appendix B.

The distance \( D(\hat{H}, \hat{H}_{\text{rec}}) \) between the reconstructed and the exact Hamiltonian shown in Fig. 3 displays the following features: First, the reconstruction is expected to fail for \( \| \hat{H} t \| \geq \pi \) (see Eq. (5)) and, indeed, \( D(\hat{H}, \hat{H}_{\text{rec}}) \) is large in this area (indicated by the grey background). Secondly, Eq. (11) suggests that close to \( \| \hat{H} t \| = \pi \), \( D(\hat{H}, \hat{H}_{\text{rec}}) \) should scale as \( \| \hat{H} t \|^6 / 140 = t^6 / (140 t_n^6) \) (thick grey line in Fig. 3). Thirdly, we observe that for infinitely many measurements per observ-able, \( M = \infty \), and fixed \( t/t_n = \| \hat{H} t \| \sim nt \), the distance \( D(\hat{H}, \hat{H}_{\text{rec}}) \) decreases with system size, a behaviour inherited from the quality of the reconstruction \( |\psi_{\text{rec}}^T\rangle \) (see left of Fig. 3): The fidelity \( F(\hat{U}, \hat{U}_{\text{rec}}) \) is limited by the amount of block entanglement in \( |\psi_{\text{rec}}\rangle \). At a fixed time \( t \), an area law [37] holds for this entanglement such that it is bounded even for arbitrarily large systems [27, 28].

For sufficiently large systems, we hence expect \( F(\hat{U}, \hat{U}_{\text{rec}}) \) at fixed \( t/t_n \) to be independent of the system size \( n \) and therefore increase with \( n \) if \( t/t_n = t \cdot n \) is kept fixed. Finally, let us discuss the dependence of the distance between the exact and the reconstructed Hamiltonian in Fig. 3 on the number \( M \) of measurements per observable. First of all, with a finite number of measurements no reconstruction will be possible at small times, because the signal of the Hamiltonian in \( U \approx \mathbb{1} - i \hat{H} t \) will be smaller than the noise. Further, the data suggests that, at small times, \( D(\hat{H}, \hat{H}_{\text{rec}}) \propto \frac{1}{t/t_n} \frac{n}{\sqrt{M}} \), a behaviour one would expect if one assumes that the relative error \( D(\hat{H}, \hat{H}_{\text{rec}}) \) is proportional to the ratio \( R/S \) of the noise amplitude \( R = n/\sqrt{M} \) (motivated by the fact that we have measured \( \mathcal{O}(n) \) observables, each of which has been estimated to within a standard deviation given, for sufficiently large \( M \), by \( 1/\sqrt{M} \)) and the strength of the signal \( S = \| \hat{H} t \| = t/t_n \).

To summarize, measuring at larger times gives a larger signal and a smaller error, but we are limited by the condition \( t/t_n < \pi \) imposed by Eq. (5). To keep the relative error constant at fixed \( t/t_n \), we require \( M \propto n^2 \), resulting in a total number of measurements \( \propto n^3 \).
1. Reconstruction for long times and enforcing a local reconstruction

In its present formulation, the reconstruction scheme is limited to \( t/t_n < \pi \), a restriction that may be overcome by measuring at two different times \( t, t' \). The times up to which the fidelity \( F(\hat{U}, \hat{U}_{\text{rec}}) \) is sufficiently high is only limited by \( r - \) increasing \( r \) will increase the time up to which full information about \( \hat{U} \) may be obtained by measuring on \( r \) consecutive qubits. In fact, as can be seen on the left of Fig. 3, for \( n = 32 \) and the relatively small \( r = 3 \), the fidelity \( F(\hat{U}, \hat{U}_{\text{rec}}) \) is still quite high at \( t/t_n = \pi \) while the reconstruction of \( \hat{H} \) fails for these times. Measuring at \( t, t' \) and obtaining \( \hat{U} = e^{-i\hat{H}t}, \hat{U}' = e^{-i\hat{H}t'} \) by reconstruction, we are only limited by \( |t' - t| < t_n\pi \) when reconstructing \( \hat{H} \) from \( \hat{U}^\dagger \hat{U}' = e^{i\hat{H}(t-t')}. \) Fig. 4 shows results of this reconstruction scheme with \( t/t_n = 3.51 \) and \( t' > t \). Reconstructing the Hamiltonian from \( \hat{U}(t' - t) = \hat{U}^\dagger \hat{U}' \), the time difference \( t' - t \) clearly assumes the role of the time \( t \) when reconstructing the Hamiltonian from \( \hat{U} \). Therefore, all scaling properties carry over as long as \( \hat{U} \) and \( \hat{U}' \) can be obtained with sufficiently high fidelity. We simulated measurements on blocks of \( r = 5 \) consecutive sites to satisfy this requirement. Reconstructing from \( \hat{U}(t' - t) \) for \( M = \infty \), the relative error \( \text{D}(\hat{H}, \hat{H}_{\text{rec}}) \) is finite for \( t' - t = 0 \) and decreases up to its minimum around \( t' - t \approx t_n \). The reason is that the error in \( \hat{U}(t' - t) \) remains finite as \( t' - t \to 0 \) because \( t \) and, as a consequence, the error in \( \hat{U} \) are fixed. This finite error may also become larger than the signal amplitude \( ||\hat{H}t|| \), explaining the increase towards \( t' - t = 0 \).

Note that, from \( \hat{U}(t' - t) \), we can also reconstruct Hamiltonians that are time-dependent for times before \( t \) and nearly constant between \( t \) and \( t' \). In this way, stroboscopic reconstructions of a time-dependent Hamiltonian may be obtained after large propagation times. Furthermore, \( t/t_n < \pi \) becomes more restrictive as \( n \) increases, thus the usefulness of taking measurements at two times increases for larger systems.

Of course, making use of additional information can only improve the scheme. As an example, suppose that we know that the Hamiltonian is nearest-neighbour only. One may then project the reconstructed \( \hat{H}_{\text{rec}} \) onto a nearest-neighbour Hamiltonian. As can be seen in Fig. 4, this reduces the error dramatically.

IV. CONCLUSION AND OUTLOOK

We studied in detail the application of recent scalable state tomography models to quantum process tomography. At the hand of unitary channels—quantum circuits such as the quantum Fourier transform and unitary time-evolution governed by local Hamiltonians—favourable scaling with the number of qubits was numerically demonstrated. The scheme, as presented, relies on an ancilla system, the preparation of a maximally entangled state on the combined system, and local measurements after application of the channel. We also discussed an alternative scalable scheme without the need for an ancilla, which displays the same scaling properties as the ancilla-assisted scheme. The quality of the reconstructed unitary channel may be quantified using the certificate introduced in Ref. [9].

We have also shown how local Hamiltonians may be reconstructed from their corresponding unitary after short evolution time. We have discussed and numerically demonstrated how the restriction of small evolution times may be relaxed by taking measurements at two different times. This enables the reconstruction and verification of a quantum device at arbitrarily large times for as long as the conditions for efficient state tomography are met, even if, intermittently, the device has passed through highly entangled states. Furthermore, the knowledge of the Hamiltonian being local may be incorporated and has, for the Hamiltonians that we studied, improved fidelities considerably (by roughly an order of magnitude).

Using the mixed-state tomography methods introduced in Refs. [10, 11], we expect non-unitary channels to be similarly amenable to the scheme studied here. Assessing the quality of such a reconstruction, however, will rely on the ability to quantify the quality of mixed-state reconstructions – a goal that is, in particular for many qubits and sufficient generality, still to be met.

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is obtained by evaluating the product using MPO representations and compressing the result [38] to bond dimension 16. The total compression error $[2(1 - \sqrt{\mathcal{F}})]^{1/2}$ was smaller than $2 \times 10^{-3}$, three orders of magnitude smaller than all the reconstruction fidelities considered. The time evolution $\hat{U} = e^{-i\hat{H}t}$ of a local Hamiltonian can be obtained up to a certain time scale with DMRG/MPO methods [38–40]. We obtained $\hat{U}$ up to $t_{\text{max}} = \frac{1}{\eta}$ from a second-order Trotter expansion of $e^{-i\hat{H}t}$ with 2000 Trotter steps and MPO bond dimension 128; with $\eta = \frac{1}{n-1} \sum_{k=1}^{n-1} \|h_{k,k+1}\|$, $\hat{H} = \sum_{k=1}^{n-1} \hat{h}_{k,k+1}$.

[30] These statistical errors directly translate to statistical errors for the scheme without ancilla: With ancilla and for each observable $\hat{F}_{k_1,\ldots,k_n}$, we simulate $M$ measurements. This translates to preparing each of the $2^n$ eigenstates of $\hat{\sigma}_{k_1}^z \otimes \cdots \otimes \hat{\sigma}_{k_n}^z$ an average of $M/2^n$ times and measuring corresponding local observables on $s'$ sites after application of the channel. Here, $s \leq \lceil \frac{n}{2} \rceil$ and $s + s' = r$.

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Appendix A: Technical Details on Ancilla-Assisted Tomography and the Reduction of Experimental Effort

Let

$$|\Phi\rangle = \frac{1}{2^{n/2}} \sum_{i=1}^{2^n} |i\rangle|i\rangle$$

be the maximally entangled state on the combined system. Then

$$\hat{\varrho}_E = (1_A \otimes \mathcal{E})(|\Phi\rangle\langle\Phi|) = \frac{1}{2^n} \sum_{i,j} |i\rangle \langle j| \otimes \mathcal{E}(|i\rangle \langle j|),$$

e.g., \langle i|\langle j|\hat{\varrho}_E|i'\rangle|j'\rangle = \langle j|\mathcal{E}(|i\rangle \langle i'|)|j'\rangle/2^n and \hat{\varrho}_E thus completely characterizes the channel \(\mathcal{E}\). Now consider the observables

$$\hat{P}_{k:a_1,...,a_r} = 1_{1,...,k} \otimes \hat{\sigma}_{k+1}^{a_1} \otimes \cdots \otimes \hat{\sigma}_{k+r}^{a_r} \otimes 1_{k+r+1,...,2n}$$

with \(\hat{\sigma}_i^x, \hat{\sigma}_i^y, \hat{\sigma}_i^z\) are the Pauli matrices on site \(i\). Measuring these observables, we obtain the expectation values of the eigenprojectors

$$\hat{\Pi}_{k:a_1,...,a_r}^{s_1,...,s_r} = 1_{1,...,k} \otimes \hat{\Pi}_{k+1:a_1}^{s_1} \otimes \cdots \otimes \hat{\Pi}_{k+r:a_r}^{s_r} \otimes 1_{k+r+1,...,2n},$$

where \(\hat{\Pi}_{k+i:a_i}^{s_i} = |a_i, s_i\rangle\langle a_i, s_i|\), \(s_i \in \{-1,1\}\).

For given \(\hat{\Pi}_{k:a_1,...,a_r}^{s_1,...,s_r}\), write \(\hat{\Pi}_{k:a_1,...,a_r}^{s_1,...,s_r} = \hat{P}_A \otimes \hat{P}_S\) with the direct product referring to ancilla \((A)\) vs. system \((S)\). One finds

$$\langle \hat{P}_A \otimes \hat{P}_S \rangle_{\varrho_E} = \text{tr}[\hat{\varrho}_E (\hat{P}_A \otimes \hat{P}_S)] = \frac{1}{2^n} \sum_{i,j,k,l} \langle i| \langle j| \hat{\Pi}_{k:a_1,...,a_r}^{s_1,...,s_r} \hat{P}_A \hat{P}_S \rangle |i\rangle |j\rangle
= \frac{1}{2^n} \sum_{i,j,k} \langle i| \langle j| \hat{P}_A \hat{P}_S \rangle |i\rangle |j\rangle = \frac{1}{2^n} \sum_{i,j,l} \langle l| \hat{P}_A \hat{P}_S \rangle |i\rangle |j\rangle
= \frac{1}{2^n} \sum_{i,j} \langle l| \hat{P}_A \hat{P}_S \rangle |i\rangle |j\rangle
= \frac{1}{2^n} \text{tr}[\mathcal{E}(\hat{P}_A^t) \hat{P}_S],$$

where \(\hat{P}_A^t = \sum_{i,l} \langle l| \hat{P}_A |i\rangle \langle i| \hat{P}_S \rangle \) is the transpose of \(\hat{P}_A\) in the basis in which \(|\Phi\rangle\) is entangled. Note that \(\hat{P}_A\) is, up to a prefactor, a mixed product state. The expectation value \(\langle \hat{P}_A \otimes \hat{P}_S \rangle_{\varrho_E}\) may thus be obtained by preparing the state \(\hat{P}_A\), sending it through the channel and measuring an appropriate product of Pauli matrices on the output state.

Appendix B: Hamiltonian Reconstruction for Ising and Random Hamiltonians

Fig. 3 in the main text shows the performance of our reconstruction scheme for local Hamiltonians using the example of the isotropic Heisenberg Hamiltonian. Fig. 5 shows data for the critical Ising model and for a Hamiltonian with random nearest-neighbour interaction, with matrix elements chosen uniformly from \([-1,1]\). Reconstruction works equally well for the critical Ising model and the randomly chosen nearest-neighbour interaction.
Figure 5.  **Left:** Fidelity between exact state $|\psi_E\rangle$ and tomographic estimate $|\psi_{\text{rec}}^E\rangle$ for a unitary time evolution $\hat{U} = e^{-i\hat{H}t}$ with $\hat{H}$ the Hamiltonian of the critical Ising model (top row) and a Hamiltonian with random nearest-neighbour interaction (bottom row). The Hamiltonians act on $n$ qubits and the tomographic estimate is based on complete measurements on blocks of $r = 3$ consecutive qubits with $M$ measurements per observable.  **Right:** Inverse relative distance between $\hat{H}$ and $\hat{H}_{\text{rec}}$ reconstructed from $|\psi_{\text{rec}}^E\rangle$. The unit of time is $t_n = 1/\|\hat{H}\| \sim 1/n$. Results are very similar to the isotropic Heisenberg data shown in Fig. 3 in the main text.