KRIGING PREDICTION WITH ISOTROPIC MATÉRN CORRELATIONS: ROBUSTNESS AND EXPERIMENTAL DESIGNS

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We investigate the prediction performance of the kriging predictors. We derive some non-asymptotic error bounds for the prediction error under the uniform metric and $L_p$ metrics when the spectral densities of both the true and the imposed correlation functions decay algebraically. The Matérn family is a prominent class of correlation functions of this kind. We show that, when the smoothness of the imposed correlation function exceeds that of the true correlation function, the prediction error becomes more sensitive to the space-filling property of the design points. In particular, we prove that, the above kriging predictor can still reach the optimal rate of convergence, if the experimental design scheme is quasi-uniform. We also derive a lower bound of the kriging prediction error under the uniform metric and $L_p$ metrics. An accurate characterization of this error is obtained, when an oversmoothed correlation function and a space-filling design is used.

1. Introduction. In contemporary mathematical modeling and data analysis, we often face the challenge of reconstructing smooth functions from scattered observations. Gaussian process regression, also known as kriging, is a widely used approach. Unlike the usual interpolation methods, kriging can provide uncertainty quantification of the underlying function in terms of its posterior distribution given the data. We refer to [9, 35, 42] for the theoretical foundation and the practical implementation of kriging, as well as its applications to spatial statistics, computer experiments, and machine learning.

The main idea of kriging is to model the underlying function as a realization of a Gaussian process. Under a Gaussian process model, the conditional distribution of the function value at an untried point given the data is normal, and can be expressed explicitly. In practice, we usually use the curve of

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conditional expectation as a surrogate model of the underlying function. Despite the known pointwise distributions, many basic properties of the kriging predictive curves remain as open problems. In this work, we focus on three fundamental aspects of kriging: 1) convergence of kriging predictive curves in function spaces; 2) robustness of kriging prediction against misspecification of the correlation functions; 3) effects of the design of experiments. Understanding the above properties of kriging can provide guidelines for choosing suitable correlation functions and experimental designs, which would potentially help the practical use of the method.

In this article, we focus on the isotropic Matérn correlation family. We suppose the underlying function is a random realization of a Gaussian process with an isotropic Matérn correlation function, and we reconstruct this function using kriging with a misspecified isotropic Matérn correlation function. We summarize our main results in Section 1.1. In Section 1.2, we make some remarks on related areas and research problems, and discuss the differences between the existing and the present results. In Section 2, we state our problem formulation and discuss the required technical conditions. Our main results are presented in Section 3. A simulation study is reported in Section 4, which assesses our theoretical findings regarding the effects of the experimental designs. Technical proofs are given in Section 6.

1.1. Summary of our results. We consider the reconstruction of a sample path of a Gaussian process over a compact set $\Omega \subset \mathbb{R}^d$. The shape of $\Omega$ can be rather general, subject to a few regularity conditions presented in Section 2.2. Table 1.1 shows a list of results on the rate of convergence of Gaussian process regression in the $L^p(\Omega)$ norm, with $1 \leq p \leq \infty$ under different designs and misspecified correlation functions. Table 1.1 covers results on both the upper bounds and the lower bounds. The lower bounds are given in terms of the sample size $n$ and the true smoothness $\nu_0$; and the upper bounds depend also on the imposed smoothness $\nu$, and two space-filling metrics of the design: the fill distance $h_{X,\Omega}$ and the mesh ratio $\rho_{X,\Omega}$. Details of the above notation are described in Section 2.2.

All results in Table 1.1 are obtained by the present work, except the shaded row which was obtained by our previous work [55]. Compared to [55], this work makes significant advances. First, this work establishes the convergence results when an oversmoothed correlation function is used, i.e., $\nu > \nu_0$. Specifically, the results in [55] depends only on $h_{X,\Omega}$, and cannot be extended to oversmoothed correlations. In this work, we prove some new approximation results for radial basis functions (see Section 3.4), and establish the theoretical framework for oversmoothed correlations. In the present
theory, the upper bounds in oversmoothed cases depend on both $h_{X,\Omega}$ and $\rho_{X,\Omega}$. We also present the bounds under the $L_p(\Omega)$ norms with $1 \leq p < \infty$ as well as the lower-bound-type results in this article.

Our findings in Table 1.1 lead to a remarkable result for the so-called quasi-uniform designs (see Section 3.3). We show that under quasi-uniform designs and oversmoothed correlation functions, the lower and upper rates coincide, which means that the optimal rates are achievable. This result also implies that the prediction performance does not deteriorate largely as an oversmoothed correlation function is imposed, provided that the experimental design is quasi-uniform.

| Case       | Design                               |
|------------|--------------------------------------|
|            | General design                        |
| $\nu \leq \nu_0$, $1 \leq p < \infty$ | Upper rate $\sigma h_{X,\Omega}^\nu$ |
|            | Lower rate $\sigma h_{X,\Omega}^\nu$ |
| $\nu \leq \nu_0$, $p = \infty$       | Upper rate $\sigma h_{X,\Omega}^\nu \log^{1/2}(1/h_{X,\Omega})$ |
|            | Lower rate $\sigma h_{X,\Omega}^\nu \sqrt{\log n}$ |
| $\nu > \nu_0$, $1 \leq p < \infty$   | Upper rate $\sigma h_{X,\Omega}^\nu \rho_{X,\Omega}^{\nu - \nu_0}$ |
|            | Lower rate $\sigma h_{X,\Omega}^\nu \rho_{X,\Omega}^{\nu - \nu_0} \sqrt{\log n}$ |
| $\nu > \nu_0$, $p = \infty$          | Upper rate $\sigma h_{X,\Omega}^\nu \rho_{X,\Omega}^{\nu - \nu_0} \log^{1/2}(1/h_{X,\Omega})$ |
|            | Lower rate $\sigma h_{X,\Omega}^\nu \rho_{X,\Omega}^{\nu - \nu_0} \sqrt{\log n}$ |

Table 1

Summary of the $L_p$ convergence rates for kriging prediction error with isotropic Matérn correlation functions. In addition to the rates of convergence, all random errors in Table 1.1 decay at sub-Gaussian rates. The rates on the shaded row were presented in our previous work [55]. The results for all other cases are obtained in the current work.

1.2. Comparison with related areas. Although the general context of function reconstruction is of interest in a broad range of areas, the particular settings of this work include: 1) Random underlying function: the underlying function is random and follows the law of a Gaussian process; 2) Interpolation: besides the Gaussian process, no random error is present, and therefore an interpolation scheme should be adopted; 3) Misspecification: Gaussian process regression is used to reconstruct the underlying true function, and the imposed Gaussian process may have a misspecified correlation function; 4) Scattered inputs: the input points are fixed, with no particular structure. These features differentiate our objective from the existing areas of function reconstruction. In this section, we summarize the distinctions between the current work and four existing areas: average-case analysis of numerical problems, nonparametric regression, posterior contraction of Gaussian process priors, and scattered data approximation. Despite the differences in the scope, some of the mathematical tools in these ar-
eas are used in the present work, including a lower-bound result from the average-case analysis (Lemma 6.8), and some results from the scattered data approximation (see Section 3.4).

1.2.1. Average-case analysis of numerical problems. Among the existing areas, the average-case analysis of numerical problems has the closest model settings compared with ours, where the reconstruction of Gaussian process sample paths is considered. The primary distinction between this area and our work is the objective of the study: the average-case analysis aims at finding the optimal algorithms (which are generally not the Gaussian process regression), while we are interested in the robustness of the Gaussian process regression. Besides, the average-case analysis focuses on the optimal designs, while our study also covers general scattered designs.

One specific topic in the average-case analysis focuses on the following quantity,

\[
 e_{p}^{\text{avg}}(\phi, N) = \left( \int_{F_1} \| f - \phi(Nf) \|_{L_p(\Omega)}^p \mu(df) \right)^{1/p},
\]

(1.1)

where \( \phi : N(F_1) \rightarrow L_p(\Omega) \) is an algorithm, \( Nf = [f(x_1), \ldots, f(x_n)] \) with \( x_i \in \Omega \), and \( F_1 \) is a function space equipped with Gaussian measure \( \mu \). It is worth noting that the results in the present work also imply some results in the form (1.1), where \( \phi \) has to be a kriging algorithm.

Results on the lower bounds of (1.1). For \( p = 2 \), the lower bound was provided by [32]; also see Lemma 6.8. If one further assumes that \( \Omega = [0, 1]^d \), Proposition VI.8 of [36] shows that the error (1.1) has a lower bound with the rate \( n^{-\nu_0/d} \). One dimensional problems with correlation functions satisfying the Sacks-Ylvisaker conditions are extensively studied; see [23, 36, 37, 39, 40, 41].

Results on the upper bound of (1.1). Upper-bound-type results are pursued in average-case analysis under the optimal algorithm \( \phi \) and optimal designs of \( \{x_1, \ldots, x_n\} \). If \( \Omega = [0, 1]^d \), [36] shows that the rate \( n^{-\nu_0/d} \) can be achieved by piecewise polynomial interpolation and specifically chosen designs; see Remark VI.3 of [36], also see page 34 of [31] and [16].

For \( 1 \leq p < \infty \) and the Matérn correlation function in one dimension, the error in average case \( e_{p}^{\text{avg}}(\phi, N) \) can achieve the rate \( n^{-\nu_0} \) by using piecewise polynomial interpolation; See Proposition IV.36 of [36]. For the Matérn correlation function in one dimension, the quantity

\[
 e_{L_\infty,p}^{\text{avg}}(\phi, N) = \left( \int_{F_1} \| f - \phi(Nf) \|_{L_\infty(\Omega)}^p \mu(df) \right)^{1/p},
\]

(1.2)
can achieve the rate $n^{-\nu_0} \sqrt{\log n}$ by using Hermite interpolating splines [5] for $1 \leq p < \infty$.

Other definitions of the error are also studied in average-case analysis. See [8, 12, 18, 19, 21, 22] for examples.

1.2.2. Nonparametric regression. The problem of interest in nonparametric regression is to recover a deterministic function $f$ under $n$ noisy observations $(x_i, y_i), i = 1, \ldots, n$, under the model

\begin{equation}
    y_i = f(x_i) + \epsilon_i, \quad i = 1, \ldots, n,
\end{equation}

where $\epsilon_i$'s are the measurement error. Assuming that the function $f$ has smoothness $\nu_0$, the optimal (minimax) rate of convergence is $n^{-\nu_0/(2\nu_0 + d)}$ [44]. A vast literature proposes and discusses methodologies regarding the nonparametric regression model (1.3), such as smoothing splines [15], kernel ridge regression [47], local polynomials [45], etc. Because of the random noise, the rates for nonparametric regression are slower than those of the present work, as well as those in other interpolation problems.

1.2.3. Posterior contraction of Gaussian process priors. In this area, the model setting is similar to nonparametric regression, i.e., the underlying function is assumed deterministic and the observations are subject to random noise. The problem of interest is the posterior contraction of the Gaussian process prior. An incomplete list of papers in this area includes [6, 7, 14, 29, 33, 48, 49, 53]. Despite the use of Gaussian process priors, the mathematical tools involved in the above papers are not applicable in the present context because of the differences in the model settings.

1.2.4. Scattered data approximation. In the field of scattered data approximation, the goal is to approximate, or often, interpolate a deterministic function $f$ with its exact observations $f(x_i), i = 1, \ldots, n$, where $x_i$'s are data sites. For function $f$ with smoothness $\nu_0$, the $L_\infty$ convergence rate is $n^{-\nu_0/d}$ [56, 57]. Although this area focus on a purely deterministic problem, some of the results in this field will serve as the key mathematical tool in this work.

2. Problem formulation. In this section we discuss the interpolation method considered in this work, and the required technical conditions.

2.1. Background. Let $Z(x)$ be an underlying Gaussian process, with $x \in \mathbb{R}^d$. We suppose $Z(\cdot)$ is a stationary Gaussian process with mean zero. The
covariance function of $Z$ is denoted as 

$$\text{Cov}(Z(x), Z(x')) = \sigma^2 \Psi(x - x'),$$

where $\sigma^2$ is the variance, and $\Psi$ is the correlation function satisfying $\Psi(0) = 1$. The correlation function $\Psi$ is a symmetric positive semi-definite function on $\mathbb{R}^d$. Since we are interested in interpolation, we require that $Z(\cdot)$ is mean square continuous, or equivalently, $\Psi$ is continuous on $\mathbb{R}^d$. Then it follows from the Bochner’s theorem ([13] page 208; Theorem 6.6 of [56]) that, there exists a finite nonnegative Borel measure $F_{\Psi}$ on $\mathbb{R}^d$, such that

$$\Psi(x) = \int_{\mathbb{R}^d} e^{i\omega^T x} F_{\Psi}(d\omega). \quad (2.1)$$

In particular, we are interested in the case where $\Psi$ is also positive definite and integrable on $\mathbb{R}^d$. In this case, it can be proven that $F_{\Psi}$ has a density with respect to the Lebesgue measure. See Theorem 6.11 of [56]. The density of $F_{\Psi}$, denoted as $f_{\Psi}$, is known as the spectral density of $Z$ or $\Psi$.

In this work, we suppose that $f_{\Psi}$ decays algebraically. A prominent class of correlation functions of this type is the isotropic Matérn correlation family [42, 43], given by

$$\Psi(x; \nu, \phi) = \frac{1}{\Gamma(\nu)2^{\nu-1}} (2\sqrt{\nu\phi}\|x\|)^\nu K_\nu(2\sqrt{\nu\phi}\|x\|), \quad (2.2)$$

with the spectral density [46]

$$f_{\Psi}(\omega; \nu, \phi) = \pi^{-d/2} \frac{\Gamma(\nu + d/2)}{\Gamma(\nu)} (4\nu\phi^2)^\nu (4\nu\phi^2 + \|\omega\|^2)^{-\nu-(d/2)} , \quad (2.3)$$

where $\phi, \nu > 0$, $K_\nu$ is the modified Bessel function of the second kind and $\| \cdot \|$ denotes the Euclidean metric. It is worth noting that (2.3) is bounded above and below by $(1 + \|\omega\|^2)^{-(\nu+d/2)}$ multiplying two constants, respectively.

Now we consider the interpolation problem. Suppose we have a scattered set of points $X = \{x_1, \ldots, x_n\} \subset \Omega$. Here the set $\Omega$ is the region of interest, which is a subset of $\mathbb{R}^d$. The goal of kriging is to recover $Z(x)$ given the observed data $Z(x_1), \ldots, Z(x_n)$. A standard predictor is the best linear predictor [42, 43], given by the conditional expectation of $Z(x)$ on $Z(x_1), \ldots, Z(x_n)$, as

$$E[Z(x)|Z(x_1), \ldots, Z(x_n)] = r^T(x)K^{-1}Y, \quad a.s., \quad (2.4)$$

where $r(x) = (\Psi(x - x_1), \ldots, \Psi(x - x_n))^T$, $K = (\Psi(x_j - x_k))_{jk}$ and $Y = (Z(x_1), \ldots, Z(x_n))^T$. 


The best linear predictor in (2.4) depends on the correlation function \( \Psi \). However, in practice \( \Psi \) is commonly unknown. Thus, we may inevitably use a misspecified correlation function, denoted by \( \Phi \). Suppose that \( \Phi \) has a spectral density \( f_{\Phi} \). We also suppose that \( f_{\Phi} \) decays algebraically, but the decay rate of \( f_{\Phi} \) can differ from that of \( f_{\Psi} \).

We consider the predictor given by the right-hand side of (2.4), in which the true correlation function \( \Psi \) is replaced by the misspecified correlation function \( \Phi \). Clearly, such a predictor is no longer the best linear predictor. Nevertheless, it still defines an interpolant, denoted by 

\[
\mathcal{I}_{\Phi,X} Z(x) = r^T_1(x) K_1^{-1} Y,
\]

where \( r_1(x) = (\Phi(x - x_1), \ldots, \Phi(x - x_n))^T \), \( K_1 = (\Phi(x_j - x_k))_{jk} \) and \( Y = (Z(x_1), \ldots, Z(x_n))^T \). In (2.5), \( \mathcal{I}_{\Phi,X} \) denotes the interpolation operator given by the kriging predictor, which can be applied not only to a Gaussian process, but also to a deterministic function in the same vein.

2.2. Notation and conditions. We do not assume any particular structure of the design points \( X = \{x_1, \ldots, x_n\} \). Our error estimate for the kriging predictor will depend on two dispersion indices of the design points.

The first one is the fill distance, defined as 

\[
h_{X,\Omega} := \sup_{x \in \Omega} \inf_{x_j \in X} \|x - x_j\|.
\]

The second is the separation radius, given by 

\[
q_X := \min_{1 \leq j \neq k \leq n} \|x_j - x_k\|/2.
\]

It is easy to check that \( h_{X,\Omega} \geq q_X \). Define the mesh ratio \( \rho_{X,\Omega} := h_{X,\Omega}/q_X \geq 1 \). Because we are only interested in the prediction error when the design points are sufficiently dense, for notational simplicity, we assume that \( h_{X,\Omega} < 1 \).

We assume the following conditions throughout this article. First, we need a geometric condition on the experimental region \( \Omega \), given by Condition 2.1.

**Definition 2.1.** A set \( \Omega \subset \mathbb{R}^d \) is said to satisfy an interior cone condition if there exists an angle \( \alpha \in (0, \pi/2) \) and a radius \( R > 0 \) such that for every \( x \in \Omega \), a unit vector \( \xi(x) \) exists such that the cone

\[
C(x, \xi(x), \alpha, R) := \left\{ x + \lambda y : y \in \mathbb{R}^d, \|y\| = 1, y^T \xi(x) \geq \cos \alpha, \lambda \in [0, R] \right\}
\]

is contained in \( \Omega \).
**Condition 2.1.** The experimental region $\Omega \subset \mathbb{R}^d$ is bounded and satisfies an interior cone condition.

Condition 2.1 holds in most practical situations, because the commonly encountered experimental regions, like the rectangles or balls, satisfy interior cone conditions. Figure 1 (page 258 of [38]) is an illustration of the $\alpha$-interior cone condition.

![Illustration of an interior cone condition](image)

**Fig 1. An illustration of an interior cone condition (page 258 of [38]).**

Conditions 2.2-2.3 require that the spectral densities decay in an algebraic order. Condition 2.4 is a regularity condition about the differentiability of $\Psi$ and $\Phi$.

**Condition 2.2.** There exist constants $c_2 \geq c_1 > 0$ and $\nu_0 > 0$ such that, for all $\omega \in \mathbb{R}^d$,

$$c_1(1 + \|\omega\|^2)^{-(\nu_0 + d/2)} \leq f_\Psi(\omega) \leq c_2(1 + \|\omega\|^2)^{-(\nu_0 + d/2)}.$$

**Condition 2.3.** There exist constants $c_3 \geq c_3 > 0$ and $\nu > 0$ such that, for all $\omega \in \mathbb{R}^d$,

$$c_3(1 + \|\omega\|^2)^{-(\nu + d/2)} \leq f_\Phi(\omega) \leq c_4(1 + \|\omega\|^2)^{-(\nu + d/2)}.$$

**Condition 2.4.** The correlation functions $\Psi$ and $\Phi$ are differentiable, i.e., $f_\Psi(\omega)$ and $f_\Phi(\omega)$ satisfy

$$\int_{\mathbb{R}^d} \|\omega\| f_\Psi(\omega) d\omega < +\infty, \text{ and } \int_{\mathbb{R}^d} \|\omega\| f_\Phi(\omega) d\omega < +\infty.$$
In the rest of this paper, we use the following conventions. For two positive sequences $a_n$ and $b_n$, we write $a_n \asymp b_n$ if, for some constants $C, C' > 0$, $C a_n / b_n \leq C'$ for all $n$. Similarly, we write $a_n \gtrsim b_n$ if $a_n \geq C b_n$ for some constant $C > 0$. We denote the cardinality of set $X$ by $\text{card}(X)$.

3. Main results. In this section, we present our main theoretical results on the prediction error of kriging.

3.1. Upper and lower bounds of the uniform kriging prediction error. This work aims at studying the prediction error of the kriging algorithm (2.5), i.e., $|Z(x) - \mathcal{I}_{\Phi,X} Z(x)|$. In this subsection, we consider the prediction error of the kriging algorithm (2.5) under a uniform metric, given by

$$
\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)|.
$$

(3.1)

The quantity (3.1) was first considered by Wang, Tuo and Wu in [55]. Under Conditions 2.1-2.4, they derived an upper bound of (3.1) under the case $\nu \leq \nu_0$. We show this result in Theorem 3.1 for the completeness of work. Here we are interested in the case $\nu > \nu_0$, that is, the imposed correlation function is smoother than the true correlation function. In Theorem 3.2, we provide an upper bound of the prediction error for $\nu > \nu_0$. In addition to the upper bounds, we obtain a lower bound of the uniform kriging prediction error in Theorem 3.3.

**Theorem 3.1.** Suppose Conditions 2.1-2.4 hold and $\nu \leq \nu_0$. Then there exist constants $C_1, C_2 > 0, C_3 > e$ and $h_0 \in (0, 1]$, such that for any design $X$ with $h_{X,\Omega} \leq h_0$ and any $t > 0$, with probability at least $1 - 2 \exp\{-t^2/(C_1 \sigma^2 h_{X,\Omega}^{2\nu_0})\}$, the kriging prediction error has the upper bound

$$
\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)| \leq C_2 \sigma h_{X,\Omega}^{\nu_0} \log^{1/2}(C_3/h_{X,\Omega}) + t.
$$

Here the constants $C_1, C_2, C_3$ depend only on $\Omega, \Phi,$ and $\Psi$.

**Theorem 3.2.** Suppose Conditions 2.1-2.4 hold and $\nu > \nu_0$. Then there exist constants $C_1, C_2 > 0, C_3 > e$ and $h_0 \in (0, 1]$, such that for any design $X$ with $h_{X,\Omega} \leq h_0$ and any $t > 0$, with probability at least $1 - 2 \exp\{-t^2/(C_1 \sigma^2 h_{X,\Omega}^{2\nu_0})\}$, the kriging prediction error has the upper bound

$$
\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)| \leq C_2 \sigma h_{X,\Omega}^{\nu_0} \log^{1/2}(C_3/h_{X,\Omega}) + t.
$$

Here the constants $C_1, C_2, C_3$ depend only on $\Omega, \Phi,$ and $\Psi$. 
Theorem 3.3. Suppose Conditions 2.1-2.4 hold. Then there exist constants \( C_1, C_2 > 0 \), such that for any design \( X \) satisfying \( \text{card}(X) = n \) and any \( t > 0 \), with probability at least \( 1 - 2 \exp\{-t^2/(2C_1\sigma^2A)\} \), the kriging prediction error has the lower bound

\[
\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)| \geq C_2 \sigma n^{-\frac{\nu_0}{4}} \sqrt{\log n} - t,
\]

where \( A = h_{X,\Omega}^\nu \) if \( \nu \leq \nu_0 \), and \( A = h_{X,\Omega}^{2(\nu-\nu_0)} \rho_{X,\Omega} \) if \( \nu > \nu_0 \). Here the constants \( C_1, C_2 > 0 \) depend only on \( \Omega, \Phi, \) and \( \Psi \).

3.2. Bounds for the \( L_p \) norms of the kriging prediction error. Now we consider the \( L_p \) norm of the kriging prediction error, given by

\[
\|Z - \mathcal{I}_{\Phi,X} Z\|_{L_p(\Omega)} := \left( \int_{\Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)|^p dx \right)^{1/p},
\]

with \( 1 \leq p < \infty \). The upper bounds of the \( L_p \) norms of the kriging prediction error with undersmoothed and oversmoothed correlation functions are provided in Theorems 3.4 and 3.5, respectively.

Theorem 3.4. Suppose Conditions 2.1-2.4 hold and \( \nu \leq \nu_0 \). Then there exist constants \( C_1, C_2 > 0 \) and \( h_0 \in (0,1] \), such that for any design \( X \) with \( h_{X,\Omega} \leq h_0 \) and any \( t > 0 \), with probability at least \( 1 - \exp\{-t^2/(C_1\sigma^2 h_{X,\Omega}^\nu)\} \), the kriging prediction error has the upper bound

\[
\|Z - \mathcal{I}_{\Phi,X} Z\|_{L_p(\Omega)} \leq C_2 \sigma h_{X,\Omega}^\nu + t.
\]

The constants \( C_1, C_2 \) depend only on \( \Omega, \Phi, \Psi \) and \( p \).

Theorem 3.5. Suppose Conditions 2.1-2.4 hold and \( \nu > \nu_0 \). Then there exist constants \( C_1, C_2 > 0 \) and \( h_0 \in (0,1] \), such that for any design \( X \) with \( h_{X,\Omega} \leq h_0 \) and any \( t > 0 \), with probability at least \( 1 - \exp\{-t^2/(C_1\sigma^2 h_{X,\Omega}^{2(\nu-\nu_0)} \rho_{X,\Omega}^{2(\nu-\nu_0)})\} \), the kriging prediction error has the upper bound

\[
\|Z - \mathcal{I}_{\Phi,X} Z\|_{L_p(\Omega)} \leq C_2 \sigma h_{X,\Omega}^{\nu_0} \rho_{X,\Omega}^{(\nu-\nu_0)} + t.
\]

The constants \( C_1, C_2 \) depend only on \( \Omega, \Phi, \Psi \) and \( p \).

Regarding the lower prediction error bounds under the \( L_p \) norm, we obtain a result analogous to Theorem 3.3. Theorem 3.6 suggests a lower bound under the \( L_p \) norm, which differs from that in Theorem 3.3 only by a \( \sqrt{\log n} \) factor.
Theorem 3.6. Suppose Conditions 2.1-2.4 hold. There exist constants
\( C_1, C_2 > 0 \), such that for any design \( X \) satisfying \( \text{card}(X) = n \) and any \( t > 0 \), with probability at least \( 1 - 2 \exp\{-t^2/(2C_1\sigma^2A)\} \), the kriging prediction error has the lower bound
\[
\|Z - I_{\Phi,X}Z\|_{L_p(\Omega)} \geq C_2\sigma n^{-\frac{\nu}{2}} - t
\]
for \( 1 \leq p < \infty \), where \( A = h_{X,\Omega}^{2\nu} \) if \( \nu \leq \nu_0 \), and \( A = h_{X,\Omega}^{2\nu}\rho_{X,\Omega}^{2(\nu - \nu_0)} \) if \( \nu > \nu_0 \). Here the constants \( C_1, C_2 > 0 \) depend only on \( \Omega, \Phi, \Psi \), and \( p \).

3.3. Asymptotic results and quasi-uniform designs. The results in Theorems 3.1, 3.2, 3.4 and 3.5 are presented in a non-asymptotic manner, i.e., the design \( X \) is fixed. The asymptotic results, which are traditionally of interest in spatial statistics, can be inferred from these non-asymptotic results. Here we consider the so-called fixed-domain asymptotics [43, 20], in which the domain \( \Omega \) is kept unchanged and the design points become dense over \( \Omega \).

To state the asymptotic results, suppose we have a sequence of designs with increasing number of points, denoted by \( X = \{X_1, X_2, \ldots\} \). We regard \( X \) as a sampling scheme which generates a sequence of designs. For instance, we may consider the design sequence generated by random sampling or maximin Latin hypercube designs.

Without loss of generality, we assume that \( \text{card}(X_n) = n \), where \( n \) takes its value in an infinite subset of \( \mathbb{N} \). This assumption enables direct comparison between our upper and lower bounds. Given the sampling scheme \( X \), we denote \( h_n := h_{X_n,\Omega}, q_n := q_{X_n} \) and \( \rho_n = h_n/q_n \).

We collect the asymptotic rates analogous to the upper bounds in Corollaries 3.1 and 3.2. Their proofs are straightforward.

Corollary 3.1. Suppose Conditions 2.1-2.4 hold. In addition, we suppose the sampling scheme \( X \) is asymptotically dense over \( \Omega \), that is, \( h_n \to 0 \) as \( n \to \infty \). We further assume \( h_{X_n,\Omega}^{\nu_0}\rho_n^{\nu - \nu_0} \to 0 \) if \( \nu > \nu_0 \). Then the uniform kriging prediction error has the order of magnitude
\[
\sup_{x \in \Omega} |Z(x) - I_{\Phi,X_n}Z(x)| = \begin{cases} 
O_p \left( h_n^{\nu} \log^{1/2}(1/h_n) \right) & \text{if } \nu \leq \nu_0, \\
O_p \left( h_{X_n,\Omega}^{\nu_0}\rho_n^{\nu - \nu_0} \log^{1/2}(1/h_n) \right) & \text{if } \nu > \nu_0.
\end{cases}
\]

Corollary 3.2. Under the conditions of Corollary 3.1, for \( 1 \leq p < \infty \), the kriging prediction error has the order of magnitude in \( L_p(\Omega) \)
\[
\|Z(x) - I_{\Phi,X_n}Z(x)\|_{L_p(\Omega)} = \begin{cases} 
O_p \left( h_n^{\nu} \right) & \text{if } \nu \leq \nu_0, \\
O_p \left( h_{X_n,\Omega}^{\nu_0}\rho_n^{\nu - \nu_0} \right) & \text{if } \nu > \nu_0.
\end{cases}
\]
From Corollaries 3.1 and 3.2, we find that the upper bounds of kriging prediction error strongly depend on the sampling scheme $\mathcal{X}$. For any sampling scheme, it can be shown that $h_n \gtrsim n^{-1/d}$ and $q_n \lesssim n^{-1/d}$ [3, 17]. In fact, it is possible to have $h_n \asymp q_n \asymp n^{-1/d}$. In this situation, $\rho_n$ is uniformly bounded above by a constant.

**Definition 3.1.** A sampling scheme $\mathcal{X}$ is said quasi-uniform if there exists a constant $C > 0$ such that $\rho_n \leq C$ for all $n$.

If a sampling scheme is quasi-uniform and $\nu \geq \nu_0$, then the order of magnitude in Corollaries 3.1 and 3.2 agree with the lower bounds in Theorems 3.3 and 3.6, respectively, implying that these bounds are sharp. We summarize the results in Corollary 3.3.

**Corollary 3.3.** Suppose Conditions 2.1-2.4 hold and $\nu \geq \nu_0$. In addition, we suppose the sampling scheme $\mathcal{X}$ is quasi-uniform. Then the kriging prediction error has the exact order of magnitude

$$\sup_{x \in \Omega} |Z(x) - I_{\Phi,X_n} Z(x)| \asymp n^{-\nu_0/d} \log^{1/2} n,$$

$$\|Z(x) - I_{\Phi,X_n} Z(x)\|_{L^p(\Omega)} \asymp n^{-\nu_0/d}, \quad 1 \leq p < \infty.$$  

It is not hard to find a quasi-uniform sampling scheme. For example, a hypercube grid sampling in $\Omega = [0,1]^d$ is quasi-uniform because $\rho_n = \sqrt{d}$ is a constant. However, random samplings do not belong to the quasi-uniform class. We illustrate the impact of the experimental designs in Example 3.1.

**Example 3.1.** The random sampling in $[0,1]$ is not quasi-uniform. To see this, let $x_1, \ldots, x_n$ be mutually independent random variables following the uniform distribution on $[0,1]$. Denote their order statistics as

$$0 = x(0) \leq x(1) \leq \cdots \leq x(n) \leq x(n+1) = 1.$$  

Clearly, we have

$$\rho_n = \frac{\max_{0 \leq j \leq n} |x(j+1) - x(j)|}{\min_{0 \leq j \leq n} |x(j+1) - x(j)|}.$$  

Let $y_1, \ldots, y_n$ be mutually independent random variables following the exponential distribution with mean one. It is well known that $(x(1), \ldots, x(n+1))$ has the same distribution as

$$\left(\frac{y_1}{\sum_{j=1}^n y_j}, \ldots, \frac{\sum_{j=1}^n y_j}{\sum_{j=1}^n y_j}\right).$$
Thus \( \rho_n \) has the same distribution as \( \max y_j / \min y_j \). Clearly, \( \max y_j \asymp \log n \) and \( \min y_j \asymp 1/n \). This implies \( \rho_n \asymp n \log n \). Similarly, we can see that \( h_n \) has the same distribution as \( \max y_j / \sum_{k=1}^{n} y_k \), which is of the order \( O_P(n^{-1} \log n) \).

Now consider the kriging predictive curve under \( \Omega = [0, 1] \) and random sampled design points and an oversmoothed correlation, i.e., \( \nu > \nu_0 \). According to Corollary 3.1, its uniform error has the order of magnitude \( O_P(n^{-\nu} \log^{\nu+1/2} n) \), which decays to zero if \( \nu < 2 \nu_0 \).

In Section 4, we will conduct simulation studies to verify our theoretical assertions on the rates of convergence in this example. It can be seen from Table 2 in Section 4 that the numerical results agree with our theory.

3.4. Discussion on a major mathematical tool and a byproduct on radial basis function approximation. The theory of radial basis function approximation is an essential mathematical tool for developing the upper bounds in this work, as well as those in our previous work [55]. We refer to [56] for an introduction of the radial basis function approximation theory.

The classic framework on the error analysis for radial basis function approximation employs the reproducing kernel Hilbert spaces as a necessary mathematical tool. The development of Wang, Tuo and Wu [55] relies on these classic results. These results, however, are not applicable in the current context when \( f_\Psi / f_\Phi \) is not uniformly bounded.

The current research is partially inspired by the “escape theorems” for radial basis function approximation established by [4, 24, 25, 26, 27, 28]. These works show that, some radial basis functions interpolants still provide effective approximation, even if the underlying functions are too rough to lie in the corresponding reproducing kernel Hilbert spaces.

In light of these existing results, we obtain an escape theorem for the kriging predictive variance, given in Lemma 6.2 in Section 6. Lemma 6.2 serves as a key component in the proofs of Theorems 3.2 and 3.5. It is worth noting that Lemma 6.2 is not a direct consequence from any of the above literature. Besides, Lemma 6.2 can render a partly new escape theorem for radial basis function approximation as a byproduct, given in Theorem 3.7.

A primary objective of the radial basis function approximation theory is to study the approximation error

\[
g - \mathcal{I}_\Phi x g,
\]

for a deterministic function \( g \). As in a typical escape theorem, we assume that \( g \) lies in a (fractional) Sobolev space which is a superset of the reproducing kernel Hilbert space generated by \( \Phi \).
Our convention of the Fourier transform is \( \hat{g}(\omega) = \int_{\mathbb{R}^d} g(x) e^{-i\omega^T x} dx \). Regarding the Fourier transform as a mapping \( \hat{g} : L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d) \) [56], the norm of the (fractional) Sobolev space \( W_2^\beta(\mathbb{R}^d) \) for a real number \( \beta > 0 \) (also known as the Bessel potential space) is

\[
\|g\|^2_{W_2^\beta(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{g}(\omega)|^2 (1 + |\omega|^2)^\beta d\omega,
\]

for \( g \in L_2(\mathbb{R}^d) \). For \( \alpha = (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{N}_0^d \), we shall use the notation \( |\alpha| = \sum_{j=1}^d \alpha_j \). For \( x = (x_1, \ldots, x_d)^T \), we denote

\[
D^\alpha g = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} g \quad \text{and} \quad x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}.
\]

**Theorem 3.7.** Let \( \Phi \) be a continuous kernel function satisfying

\[
c_1(1 + \|\omega\|^2)^{-\tau} \leq \hat{\Phi}(\omega) \leq c_2(1 + \|\omega\|^2)^{-\tau},
\]

and \( g \) be a function in \( W_2^\beta(\mathbb{R}^d) \) with \( d/2 < \beta < \tau \). Let \( \alpha \in \mathbb{N}_0^d \) satisfies \( |\alpha| \leq \beta - d/2 \). Suppose \( \Omega \subset \mathbb{R}^d \) is bounded and satisfies an interior cone condition. Then there exist constants \( C > 0 \) and \( h_0 \in (0, 1] \) such that for any design \( X \) with \( h_X, \Omega \leq h_0 \), we have

\[
\sup_{x \in \Omega} |D^\alpha g(x) - D^\alpha I_{\Phi,X} g(x)| \leq Ch_{X,\Omega}^{\beta-d/2-|\alpha|} \rho_{X,\Omega}^{\tau-\beta} \|g\|_{W_2^\beta(\mathbb{R}^d)}.
\]

Given the condition (3.3), it is known that the reproducing kernel Hilbert space generated by \( \Phi \) is equivalent to \( W_2^\tau(\mathbb{R}^d) \), which is a proper subset of \( W_2^\beta(\mathbb{R}^d) \) since \( \beta < \tau \). So Theorem 3.7 shows that the radial basis function approximation may still give reasonable error bounds even if the underlying function does not lie in the reproducing kernel Hilbert space.

**Remark 3.1.** Theorem 4.2 of [28] states that under the conditions of Theorem 3.7 in addition to \( |\beta| > d/2 \), we have

\[
\|g - I_{\Phi,X} g\|_{W_2^\beta(\Omega)} \leq Ch_{X,\Omega}^{\beta-\mu} \rho_{X,\Omega}^{\tau-\beta} \|g\|_{W_2^\beta(\mathbb{R}^d)},
\]

for \( 0 \leq \mu \leq \beta \). This result implies a weaker version of Theorem 3.7. Combining (3.5) with the real interpolation theory for Sobolev spaces (See, e.g., Theorem 5.8 and Chapter 7 of [1]), yields (3.4). Doing this requires that \( |\beta| > d/2 \), and this unnecessary condition is removed in our Theorem 3.7.
4. Simulation studies. The objective of this section is to verify whether the rate of convergence given by Corollary 3.1 is accurate. We consider the settings in Example 3.1. We have shown that under a random sampling over the experimental region $\Omega = [0, 1]$, the kriging prediction error has the rate $O_p(n^{\nu - 2\nu_0} \log^{\nu+1/2} n)$ for $\nu > \nu_0$. If a grid sampling is used, Corollaries 3.1 and 3.3 show that the error has the order of magnitude $n^{-\nu_0} \log^{1/2} n$ for $\nu > \nu_0$.

We denote the expectation of (3.1) with random sampling and grid sampling by $E_{\text{rand}}$ and $E_{\text{grid}}$, respectively. Our idea of assessing the rate of convergence is described as follows. If the error rates are sharp, we have the approximations

$$
\log E_{\text{rand}} \approx (\nu - 2\nu_0) \log n + (\nu + \frac{1}{2}) \log \log n + \log c_1,
$$

$$
\log E_{\text{grid}} \approx -\nu_0 \log n + \frac{1}{2} \log \log n + \log c_2,
$$

for random samplings and grid samplings, respectively, where $c_1, c_2$ are constants. Since $\log \log n$ grows much slower than $\log n$, we can regard the $\log \log n$ term as a constant and get the second approximations

$$
(4.1) \quad \log E_{\text{rand}} \approx (2\nu_0 - \nu) \log(1/n) + C_1,
$$

$$
(4.2) \quad \log E_{\text{grid}} \approx \nu_0 \log(1/n) + C_2.
$$

To verify the above formulas via numerical simulations, we can regress $\log E_{\text{rand}}$ and $\log E_{\text{grid}}$ on $\log(1/n)$ and examine the estimated slopes. If the bounds are sharp, the estimated slopes should be close to the theoretical assertions $2\nu_0 - \nu$ and $\nu_0$, respectively.

In our simulation studies, we consider the sample sizes $n = 10^k$, for $k = 2, 3, ..., 15$. For each $k$, we simulate 100 realizations of a Gaussian process. For a specific realization of a Gaussian process, we generate $10^k$ independent and uniformly distributed random points as $X$, and use $\sup_{x \in \Omega_1} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)|$ to approximate the uniform error $\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)|$, where $\Omega_1$ is the first 200 points of the Halton sequence [30]. We believe that the points are dense enough so that the approximation can be accurate. Then the regression coefficient is estimated using the least squares method. For grid sampling, the training sample sizes and the testing points are the same. The results are presented in Table 2. The first two columns of Table 2 show the true and imposed smoothness parameters of the Matérn correlation functions. The fourth and the fifth columns show the convergence rates obtained from the simulation studies and the theoretical analysis, respectively. The sixth column shows the relative difference between the fourth and the fifth
columns, given by $|\text{estimated slope-theoretical slope}|/(\text{theoretical slope})$. The last column gives the $R$-squared values of the linear regression of the simulated data.

| $\nu_0$ | $\nu$ | Design | ES   | TS   | RD    | $R^2$  |
|--------|-------|--------|------|------|-------|-------|
| 1.1    | 1.3   | RS     | 0.9011 | 0.9  | 0.0012 | 0.8579 |
|        |       | GS     | 1.0670 | 1.1  | 0.0300 | 0.9992 |
| 1.1    | 2.8   | RS     | 0.1653 | No convergence | NA | 0.0308 |
|        |       | GS     | 1.0968 | 1.1  | 0.0030 | 0.9995 |
| 2.1    | 2.8   | RS     | 1.523  | 1.4  | 0.088  | 0.9834 |
|        |       | GS     | 2.0953 | 2.1  | 0.0022 | 0.9992 |
| 1.5    | 3.5   | RS     | 0.1083 | No convergence | NA | 0.0991 |
|        |       | GS     | 1.4982 | 1.5  | 0.0012 | 0.9989 |

Table 2

Numerical studies on the convergence rates of kriging prediction with oversmoothed correlation functions. The following abbreviations are used: RS=Random sampling, GS=Grid sampling, ES=Estimated slope, TS=Theoretical slope, RD=relative difference.

In the setting of Rows 2, 3, 5-7 and 9 of Table 2, our theory suggests the prediction consistency, i.e., $h_n^{\nu_0} \rho_n^{\nu-\nu_0}$ tends to zero. It can be seen that the estimated slopes coincide with our theoretical assertions for these cases. Also, the $R$-squared values for these rows are high, which implies a good model fitting of (4.1)-(4.2). When $h_n^{\nu_0} \rho_n^{\nu-\nu_0}$ goes to infinity, our simulation results suggest a very slow rate of convergence. Specifically, under the random sampling scheme and $(\nu_0, \nu) = (1.1, 2.8)$ and $(\nu_0, \nu) = (1.5, 3.5)$, the estimated rates of convergence are 0.1653 and 0.1083, respectively. Also, the $R$-squared values are very low. These slow rates and poor model fitting imply that the kriging predictor could be inconsistent. Figure 2 shows the scattered plots of the raw data and the regression lines under the four combinations of $(\nu_0, \nu)$ in Table 2.

5. Concluding remarks. The error bounds presented in this work are not only valuable in mathematics. They can also provide guidelines for practitioners of kriging. Especially, our work confirms the importance of the design of experiments: if the design is quasi-uniform, the use of an oversmoothed correlation would not be an issue.

As a final remark, we compare the rates in this work with the ones in radial basis function approximation [11, 56]. For the radial basis function approximation problems, we adopt the standard framework so that the underlying function lies in the reproducing kernel Hilbert space generated by the correlation function. For the $L_\infty$ norm, the obtained optimal rate of convergence for kriging is $O_P(n^{-\nu_0/d} \sqrt{\log n})$; while that for the radial basis function approximation is $O(n^{-\nu_0/d})$. So there is a difference in the $\sqrt{\log n}$...
Fig 2. The regression line of $\log \mathcal{E}_{\text{unif}}$ and $\log \mathcal{E}_{\text{grid}}$ on $\log(1/n)$, under the four combinations of $(\nu_0, \nu)$ in Table 2. Each point denotes one average prediction error for each $n$.

For $L_p$ norms with $1 \leq p < \infty$, the difference is more dramatic. While the optimal rate of convergence for kriging is $O_p(n^{-\nu_0/d})$, that for radial basis function approximation is $O(n^{-\nu_0/d-\min(1/2,1/p)})$. This gap between the optimal rates can be explained, as the support of a Gaussian process is essentially larger than the corresponding reproducing kernel Hilbert space [50].

6. Proofs. This section consists of our technical proofs.

6.1. Generic theorems on the upper bounds. Here we extend the framework suggested by [55] and prove a new maximum inequality for Gaussian
process regression, given in Lemma 6.1. First we introduce the necessary notation. For \( \alpha = (\alpha_1, \ldots, \alpha_d)^T \in \mathbb{N}_0^d \) and \( x = (x_1, \ldots, x_d)^T \), we denote \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \). Define
\[
K_1^{-1} r_1(x) =: a(x) =: (a_1(x), \ldots, a_n(x))^T,
\]
where \( K_1 = (\Phi(x_j - x_k))_{jk}, r_1(x) = (\Phi(x - x_1), \ldots, \Phi(x - x_n))^T \) are given in (2.5). Given \( \alpha \in \mathbb{N}_0^d \), we define the quasi-power function \( Q(\alpha)(x) \) as
\[
Q(\alpha)(x) := \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} D^\alpha a_j(x) e^{i\omega^T x_j} - (i\omega)^\alpha e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega.
\]
(6.1)

The maximum inequality in Lemma 6.1 depends on two quantities \( Q_0 \) and \( Q_1 \), defined as
\[
Q_0 := \sup_{x \in \Omega} Q^{(0)}(x) \quad \text{and} \quad Q_1 := \sup_{x \in \Omega, |\alpha| = 1} Q^{(1)}(x).
\]

The proof of Lemma 6.1 is in Section 6.7.

**Lemma 6.1.** Suppose Condition 2.4 is met. There exist constants \( C_1, C_2 \) depending only on \( \Omega \), such that for any \( t > 0 \), with probability no less than \( 1 - 2 \exp\{-t^2/(2\sigma^2 Q_0^2)\} \),
\[
\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\nu, X} Z(x)| \leq C_1 \sigma \int_0^{Q_0} \sqrt{\log\left(1 + \frac{C_2 Q_1}{\epsilon}\right)} d\epsilon + t.
\]

To use Lemma 6.1, we need to obtain upper bounds of \( Q^{(\alpha)}(x) \) for \( |\alpha| = 0, 1 \). Here we pursue a more general result, which allows \( |\alpha| \) to be a general non-negative integer. To do this, we need to further assume that \( v \) and \( v_0 \) in Conditions 2.2-2.3 satisfy \( v \geq |\alpha| \) and \( v_0 \geq |\alpha| \) to ensure the \( |\alpha| \)-th order differentiability of \( a(x) \). This requirement reduces to Condition 2.4 when \( |\alpha| = 1 \).

In the following Lemma 6.2, we show that (6.1) may have a reasonable upper bound even if \( f_\Psi / f_\Phi \) is unbounded. We called it an “escape theorem” in Section 3.4. We present its proof in Section 6.8.

**Lemma 6.2.** Suppose Conditions 2.1-2.4 are met. Besides, \( v \geq |\alpha| \) and \( v_0 \geq |\alpha| \). If \( v > v_0 \), then there exist constants \( C > 0 \) and \( h_0 \in (0, 1] \) independent of \( X \) and \( x \) such that
\[
Q^{(\alpha)}(x) \leq C h_{X, \Omega}^{v_0 - |\alpha|} h_{X, \Omega}^{\nu - v_0} \frac{\nu - v_0}{\nu - v_0} \frac{\nu - v_0}{\nu - v_0}
\]
holds for all \( x \in \Omega \) and all \( X \) satisfying \( h_{X, \Omega} \leq h_0 \).
In the rest of this section, we provide technical proofs of the theoretical results in Sections 3 as well as related lemmas.

6.2. **Proof of Theorem 3.2.** We use Lemma 6.1 to prove Theorem 3.2. First, note that there exists a constant $C$ so that

$$Q_0 \leq C \sup_{x \in \Omega} Q^{(0)}(x)$$

and

$$Q_1 \leq C \sup_{x \in \Omega, |\alpha|=1} Q^{(\alpha)}(x),$$

which, together with Lemma 6.2, yields

$$Q_0 \leq C' \frac{\nu_0}{h_{X,\Omega}} \rho_{X,\Omega}^{\nu-\nu_0}$$

and

$$Q_1 \leq C' \frac{\nu_0-1}{h_{X,\Omega}} \rho_{X,\Omega}^{\nu-\nu_0},$$

for some constant $C'$ independent of $X$. Now a direct application of Lemma 6.1 yields that, for any $t > 0$, with probability no less than $1 - 2 \exp\{-t^2/(2C'^2 \sigma^2 h_{X,\Omega}^2 \rho_{X,\Omega}^{2(\nu-\nu_0)})\}$, the kriging prediction error can be bounded above as

$$\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X} Z(x)| \leq C_1 \sigma \sqrt{\int_0^{h_{X,\Omega}} \rho_{X,\Omega}^{\nu-\nu_0} \log \left(1 + \frac{C_2 C' \rho_{X,\Omega}^\nu}{\epsilon} \right) d\epsilon + t}$$

$$= C_3 C' \sigma \rho_{X,\Omega}^{\nu-\nu_0} \int_0^{h_{X,\Omega}} \sqrt{\log \left(1 + \frac{C_2}{\epsilon} \right)} d\epsilon + t. \tag{6.2}$$

To estimate the integral in (6.2), we apply the Cauchy-Schwarz inequality to get

$$\int_0^{h_{X,\Omega}} \sqrt{\log \left(1 + \frac{C_2}{\epsilon} \right)} d\epsilon \leq h_{X,\Omega}^{1/2} \left( \int_0^{h_{X,\Omega}} \log \left(1 + \frac{C_2}{\epsilon} \right) d\epsilon \right)^{1/2}$$

$$= h_{X,\Omega}^{1/2} \left( C_2 \log \left( \frac{C_2 + h_{X,\Omega}}{C_2} \right) + h_{X,\Omega} \log \left( \frac{C_2 + h_{X,\Omega}}{h_{X,\Omega}} \right) \right)^{1/2}. \tag{6.3}$$

Finally, for $x \geq 0$, we have the basic inequality $\log(1+x) \leq x$, which implies

$$\log \left( \frac{C_2 + h_{X,\Omega}}{C_2} \right) \leq \frac{h_{X,\Omega}}{C_2}.$$

Consequently, by incorporating the condition $h_{X,\Omega} \leq 1$, we get

$$C_2 \log \left( \frac{C_2 + h_{X,\Omega}}{C_2} \right) + h_{X,\Omega} \log \left( \frac{C_2 + h_{X,\Omega}}{h_{X,\Omega}} \right) \leq h_{X,\Omega} \log \left( \frac{e(C_2 + 1)}{h_{X,\Omega}} \right) \tag{6.4}.$$

Then we complete the proof by combining (6.2), (6.3) and (6.4).
6.3. Proof of Theorem 3.3. We first present some lemmas used in this proof.

Lemma 6.3 (Theorem 2.1.1 of [2]). Let $G(x)$ be a separable zero-mean Gaussian process with $x$ lying in a $d$-compact set $\Omega$, where $d$ is a metric defined by $d(x,x') = \mathbb{E}(G(x) - G(x'))^2$. Let $\sigma_\Omega^2 = \sup_{x \in \Omega} \mathbb{E}G(x)^2$. Then, we have
\[
\mathbb{P} \left( \mathbb{E} \sup_{x \in \Omega} G(x) - \sup_{x \in \Omega} G(x) \geq t \right) \leq e^{-t^2/\sigma_\Omega^2},
\]
and $\mathbb{E} \sup_{x \in \Omega} G(x) < \infty$.

Lemma 6.4 (Theorem 6.5 of [52]). Let $G(x)$ be as in Lemma 6.3. For some universal constant $C$, we have
\[
\mathbb{E} \sup_{x \in \Omega} G(x) \geq C \sup_{\eta > 0} \eta \sqrt{\log N(\eta, \Omega, d)},
\]
where $N(\epsilon, \Omega, d)$ is the $\epsilon$-covering number of the metric space $(\Omega, d)$.

Lemma 6.5 (Theorem of [11], page 119). The $n$th approximation number of the embedding $\text{id} : W_{2+\nu/2} \rightarrow L_\infty$ $A_n$ satisfies that for all $n \in \mathbb{N}$,
\[
A_n \geq c_1 n^{-\frac{\nu}{\nu_0}},
\]
where $c_1$ is a positive constant depending on $\Omega$ and $\nu_0$.

Lemma 6.6 (Theorem 5.14 of [57]). Suppose Condition 2.3 is met. Then there exist constants $c > 0$ and $h_0 \in (0,1]$ depending only on $\Omega$ and $\nu$, such that for any $X$ satisfying $h_{X,\Omega} \leq h_0$, $\sup_{x \in \Omega} (\Phi(x - x) - r^T_1(x)K_1^{-1}r_1(x)) \leq \epsilon h_{X,\Omega}^2$, where $r_1(x) = (\Phi(x - x_1), ..., \Phi(x - x_n))^T$, and $K_1 = (\Phi(x_j - x_k))_{jk}$.

Without loss of generality, assume $\sigma = 1$, because otherwise we can consider the lower bound of $\sup_{x \in \Omega} |Z(x) - \mathcal{I}\Phi,x Z(x)|/\sigma$ instead. Let $g(x) = Z(x) - \mathcal{I}\Phi,x Z(x)$, which is still a Gaussian process since $\mathcal{I}\Phi,x$ is linear.

The rest of our proof consists of the following steps. In steps 1 and 2, we bound $\sigma_\Omega^2 = \sup_{x \in \Omega} \mathbb{E}g(x)^2$. In step 3, we provide a lower bound of $\mathbb{E} \sup_{x \in \Omega} g(x)$. In the last step, we invoke Lemma 6.3 to obtain the desired results.

**Step 1: Obtaining an upper bound of $\sigma_\Omega^2$**

Direct calculation shows that
\[
\sigma_\Omega^2 = \sup_{x \in \Omega} (\Phi(x - x) - 2r^T_1(x)K_1^{-1}r_1(x) + r^T_1(x)K_1^{-1}K_1^{-1}K_1^{-1}r_1(x)),
\]
where $r(\cdot), r_1(\cdot), K,$ and $K_1$ are the same as in (2.4) and (2.5). It follows from (2.1) that

$$
\Psi(x) = \int_{\mathbb{R}^d} e^{i\omega^T x} f_\Psi(\omega) d\omega.
$$

Then for any $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$,

$$
u^T Ku = \sum_{j,k=1}^n u_j \bar{u}_k \Psi(x_j - x_k) = \sum_{j,k=1}^n u_j \bar{u}_k \int_{\mathbb{R}^d} e^{i\omega^T (x_j - x_k)} f_\Psi(\omega) d\omega
$$

(6.7)

By (6.6) and (6.7), and letting $u = K_1^{-1} r_1(x) =: a(x) =: (a_1(x), \ldots, a_n(x))^T$, we obtain

$$
\sigma_\Omega^2 = \sup_{x \in \Omega} \int_{\mathbb{R}^d} \left| \sum_{j=1}^n a_j(x) e^{i\omega^T x_j} - e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega.
$$

(6.8)

Note $Q_0^2 = \sigma_\Omega^2$. If $\nu > \nu_0$, by Lemma 6.2, we have

$$
\sigma_\Omega^2 \leq C_1 h_\Omega^{2\nu_0} \rho_X(\Omega)^{2(\nu - \nu_0)}.
$$

(6.9)

If $\nu \leq \nu_0$, then $f_\Psi / f_\Phi$ is uniformly bounded by some constant $C_2$. By (6.8),

$$
\sigma_\Omega^2 \leq C_2 \sup_{x \in \Omega} \int_{\mathbb{R}^d} \left| \sum_{j=1}^n a_j(x) e^{i\omega^T x_j} - e^{i\omega^T x} \right|^2 f_\Phi(\omega) d\omega
$$

$$
\leq C_2 \sup_{x \in \Omega} \int_{\mathbb{R}^d} \left| \sum_{j=1}^n a_j(x) e^{i\omega^T x_j} - e^{i\omega^T x} \right|^2 f_\Phi(\omega) d\omega
$$

$$
= C_2 \sup_{x \in \Omega} (\Phi(x - x) - r_1^T(x) K_1^{-1} r_1(x)).
$$

Applying Lemma 6.6 yields

$$
\sigma_\Omega^2 \leq C_3 h_\Omega^{2\nu}.
$$

(6.10)

**Step 2: Obtaining a lower bound of $\sigma_\Omega^2$**
Because the best linear predictor of $Z(x)$ conditioning on $Z(x_1), \ldots, Z(x_n)$ is $T_{\Psi,X} Z(x)$, we have

$$\sigma_n^2 = \sup_{x \in \Omega} \mathbb{E} g(x)^2 \geq \sup_{x \in \Omega} \mathbb{E} (Z(x) - T_{\Psi,X} Z(x))^2$$

$$(6.11)\quad = \sup_{x \in \Omega} (\Psi(x - x) - r^T(x) K^{-1} r(x)).$$

To show a lower bound of $\sigma_n^2$, we define the operator $T_a : W_2^{\nu_0 + \frac{d}{2}} \to L_\infty$ for $a(x) = (a_1(x), \ldots, a_n(x))^T$, by

$$T_a f = \sum_{j=1}^n a_j(x) \delta_{x_j}(f),$$

where $\delta_x$ for $x$ denotes the point evaluation functional, i.e., $\delta_x(f) = f(x)$. It can be seen that $T_n$ is a rank $n$ operator. By the proof of Lemma 6.5, there exists a function $\phi \in W_2^{\nu_0 + \frac{d}{2}}$ such that $\|\phi\|_{W_2^{\nu_0 + \frac{d}{2}}} = 1$, and

$$(6.12)\quad \|\phi - T_a \phi\|_{L_\infty} \geq A_n,$$

where $A_n$ denotes the $n$th approximation number of the identity operator $id : W_2^{\nu_0 + \frac{d}{2}} \to L_\infty$. By the Sobolev embedding theorem [1], $id$ is an operator with finite norm. By (6.5) and (6.12), we have

$$C_4 n^{-\frac{2\nu_0}{d}} \leq \sup_{x \in \Omega} |\phi(x) - T_a \phi(x)|^2$$

$$= \sup_{x \in \Omega} \left| \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix^T \omega} \phi(\omega) - \sum_{j=1}^n a_j(x) e^{ix_j^T \omega} \hat{\phi}(\omega) d\omega \right|^2$$

$$\leq \sup_{x \in \Omega} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| e^{ix^T \omega} - \sum_{j=1}^n a_j(x) e^{ix_j^T \omega} \right| \hat{\phi}(\omega) d\omega \right)^2$$

$$\leq \sup_{x \in \Omega} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| e^{ix^T \omega} - \sum_{j=1}^n a_j(x) e^{ix_j^T \omega} \right|^2 f_\Psi(\omega) d\omega \right) \left( \int_{\mathbb{R}^d} \left| \hat{\phi}(\omega) \right|^2 f_\Psi(\omega) d\omega \right)$$

$$\leq C_5 \sup_{x \in \Omega} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| e^{ix^T \omega} - \sum_{j=1}^n a_j(x) e^{ix_j^T \omega} \right|^2 f_\Psi(\omega) d\omega \right) \times$$

$$\left( \int_{\mathbb{R}^d} \left| \hat{\phi}(\omega) \right|^2 (1 + \|\omega\|^2)^{(\nu_0 + d/2)} d\omega \right)$$

$$\leq C_6 \sup_{x \in \Omega} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| e^{ix^T \omega} - \sum_{j=1}^n a_j(x) e^{ix_j^T \omega} \right|^2 f_\Psi(\omega) d\omega \right).$$
The first equality is true because the construction of \( \phi \) allows us to use Fourier inversion theorem, the third inequality is true because of the Cauchy-Schwarz inequality, and the fourth inequality is true because of Condition 2.2. Therefore, there exists a constant \( C_7 \) such that for any \( a(x) = (a_1(x), \ldots, a_n(x))^T \),

\[
\sup_{x \in \Omega} \left( \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix^T\omega} - \sum_{j=1}^{n} a_j(x)e^{ix_j^T\omega} \right)^2 f_\Psi(\omega)d\omega \geq C_7 n^{-\frac{2\nu_0}{d}}.
\]

In particular, by (6.11), we set \( a(x) = K^{-1}r(x) \) to get

\[
\sigma^2 \geq C_7 n^{-\frac{2\nu_0}{d}}.
\]

**Step 3: Obtaining a lower bound of** \( \mathbb{E} \sup_{x \in \Omega} g(x) \)

Suppose we add a set of another \( n \) points \( X_1 = \{x'_1, \ldots, x'_n\} \). We have for all \( x'_j \),

\[
d(x, x'_j)^2 = \mathbb{E}(Z(x) - \mathcal{I}_{\Phi,X}Z(x) - (Z(x'_j) - \mathcal{I}_{\Phi,X}Z(x'_j)))^2
\]

\[
\geq \mathbb{E}(Z(x) - \mathcal{I}_{\Phi,X\cup X_1}Z(x))^2,
\]

because \( \mathcal{I}_{\Phi,X\cup X_1}Z(x) \) is the best linear predictor and \( \mathcal{I}_{\Phi,X}Z(x) + Z(x'_j) - \mathcal{I}_{\Phi,X}Z(x'_j) \) is a linear predictor. In Step 2 we showed that given these 2n points \( X \cup X_1 \), there exists at least one point \( x \) such that \( \mathbb{E}(Z(x) - \mathcal{I}_{\Phi,X\cup X_1}Z(x))^2 \geq C_7(2n)^{-\frac{2\nu_0}{d}} \). Therefore, there exists at least one point \( x \) such that \( d(x, x'_j)^2 \geq C_7(2n)^{-\frac{2\nu_0}{d}} \) for all \( x'_j \), which implies \( N(C_7(2n)^{-\frac{\nu_0}{d}} / 2, \Omega, d) \geq n \). Let \( \eta = C_7(2n)^{-\frac{\nu_0}{d}} / 2 \). By Lemma 6.4, we have

\[
(6.13) \quad \mathbb{E} \sup_{x \in \Omega} g(x) \geq C_8 n^{-\frac{\nu_0}{d}} \sqrt{\log n}.
\]

**Step 4: Bounding** \( \mathbb{P}(\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X}Z(x)| \leq C_2 n^{-\frac{\nu_0}{d}} \sqrt{\log n} - t) \).

By Lemma 6.3, for any \( t > 0 \), we have

\[
2e^{-t^2/\sigma^2} \geq 2\mathbb{P} \left( \sup_{x \in \Omega} g(x) \leq \mathbb{E} \sup_{x \in \Omega} g(x) - t \right)
\]

\[
\geq 2\mathbb{P} \left( \sup_{x \in \Omega} g(x) \leq C_9 n^{-\frac{\nu_0}{d}} \sqrt{\log n} - t \right)
\]

\[
(6.14) \quad \geq \mathbb{P} \left( \sup_{x \in \Omega} |g(x)| \leq C_9 n^{-\frac{\nu_0}{d}} \sqrt{\log n} - t \right),
\]

where the second inequality is true because of (6.13) and Lemma 6.4, and the last inequality is true because \( g(x) \) is symmetric.

Combining (6.9), (6.10), and (6.14) finishes the proof.
6.4. Proof of Theorem 3.4. The proofs of Theorems 3.4, 3.5 and 3.6 rely on the following concentration inequality of Gaussian processes. The proof of Lemma 6.7 is provided in Section 6.10.

**Lemma 6.7.** Let $G$ be a separable Gaussian process on a $d$-compact $\Omega$ with mean zero, then for all $u > 0$ and $1 \leq p < \infty$,

$$
P \left( \|G\|_{L^p(\Omega)} - \mathbb{E}\|G\|_{L^p(\Omega)} > u \right) \leq e^{-u^2/(2C\sigma^2_\Omega)},$$

$$
\text{and } P \left( \|G\|_{L^p(\Omega)} - \mathbb{E}\|G\|_{L^p(\Omega)} < -u \right) \leq e^{-u^2/(2C\sigma^2_\Omega)},
$$

where $\sigma^2_\Omega = \sup_{x \in \Omega} \mathbb{E}G(x)^2$, and $C$ is a constant only depending on $\Omega$.

Without loss of generality, assume $\sigma = 1$, because otherwise we can consider the upper bound of $\|Z - \mathcal{I}_{\Phi, X}Z\|_{L^p(\Omega)}/\sigma$ instead. Let $g(x) = Z(x) - \mathcal{I}_{\Phi, X}Z(x)$, which is still a Gaussian process since $\mathcal{I}_{\Phi, X}$ is linear. Let $\sigma^2_\Omega = \sup_{x \in \Omega} \mathbb{E}g(x)^2$.

By Fubini’s theorem,

$$
\mathbb{E}\|Z - \mathcal{I}_{\Phi, X}Z\|_{L^p(\Omega)}^p = \int_{x \in \Omega} \mathbb{E}|Z(x) - \mathcal{I}_{\Phi, X}Z(x)|^p \, dx
$$

$$
= \int_{x \in \Omega} \frac{2^{p/2} \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}} \left(\mathbb{E}(Z(x) - \mathcal{I}_{\Phi, X}Z(x))^2\right)^{p/2} \, dx
$$

$$
\leq C_1 \sigma^p_\Omega.
$$

(6.15)

The second equality of (6.15) is true because $Z(x) - \mathcal{I}_{\Phi, X}Z(x)$ follows a normal distribution with mean zero, and the absolute moments of a normal random variable $X_\sigma \sim N(0, \sigma^2)$ can be expressed by its variance as

$$
\mathbb{E}|X_\sigma|^p = \sigma^p \cdot \frac{2^{p/2} \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}};
$$

see [54]. By combining Lemma 6.7, (6.10) and (6.15), we have

$$
e^{-u^2/2\sigma^2_\Omega} \geq P \left( \|g\|_{L^p(\Omega)} > \mathbb{E}\|g\|_{L^p(\Omega)} + u \right)
$$

$$
\geq P \left( \|g\|_{L^p(\Omega)} > \mathbb{E}\|g\|_{L^p(\Omega)}^p + u^p \right)^{1/p}
$$

$$
\geq P \left( \|g\|_{L^p(\Omega)} > \mathbb{E}\|g\|_{L^p(\Omega)}^p + u^p \right)^{1/p} \sigma_\Omega
$$

$$
\geq P \left( \|g\|_{L^p(\Omega)} > 2^{1-1/p}(C_1/\sigma_\Omega + u) \right),
$$

(6.16)

where the second inequality follows from the Jensen’s inequality and the $c_r$-inequality. Combining (6.10) and (6.16) completes the proof.
6.5. Proof of Theorem 3.5. The proof of Theorem 3.5 is similar to the proof of Theorem 3.4. The only difference here is that at the last step we combine (6.9) and (6.16) to get the desired results.

6.6. Proof of Theorem 3.6. The proof of Theorem 3.6 depends on the eigenvalues of the true correlation function \( \Psi \). Because \( \Psi \) is a positive definite function, by Mercer’s theorem (see [34] for example), there exists a countable set of positive eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots > 0 \) and an orthonormal basis for \( L_2(\Omega) \) \( \{ \varphi_k \}_{k \in \mathbb{N}} \) such that

\[
\Psi(x - y) = \sum_{k=1}^{\infty} \lambda_k \varphi_k(x) \varphi_k(y),
\]

where the summation is uniformly and absolutely convergent. The following lemma, which states lower bounds of the expectation of the kriging prediction error, is a direct consequence of Theorem 1.2 of [32].

**Lemma 6.8.** Suppose Conditions 2.1 holds. Let \( \lambda_k \)'s be eigenvalues of \( \Psi \). Then the expectation of the integrated squared prediction error has the lower bound

\[
\mathbb{E} \| Z - \mathcal{I}_{\Phi,X} Z \|^2_{L_2(\Omega)} \geq \sigma \sum_{k=n+1}^{\infty} \lambda_k.
\]

**Lemma 6.9.** Suppose Conditions 2.1, 2.2 and 2.4 hold. Let \( \lambda_k \) be as in (6.17). Then, \( \lambda_k \propto k^{-2\alpha_0/d-1} \).

**Proof of Lemma 6.9.** Let \( T \) be the embedding operator of \( \mathcal{N}_\Psi(\Omega) \) into \( L_2(\Omega) \), and \( T^* \) be the adjoint of \( T \). By Proposition 10.28 in [56],

\[
T^*v(x) = \int_{\Omega} \Psi(x, y)v(y)dy, \quad v \in L_2(\Omega), \quad x \in \Omega.
\]

By Corollary 10.13 of [56], \( W_2^{\alpha_0+\frac{d}{2}} \) coincide with \( \mathcal{N}_\Psi(\Omega) \). By Theorem 5.7 in [10], \( T \) and \( T^* \) have the same singular values. By Theorem 5.10 in [10], for all \( k \in \mathbb{N} \), \( a_k(T) = \mu_k(T) \), where \( a_k(T) \) denotes the approximation number for the embedding operator (as well as the integral operator), and \( \mu_k \) denotes the singular value of \( T \). By Theorem in Section 3.3.4 in [11], the embedding operator \( T \) has approximation numbers satisfying

\[
C_3 k^{-\alpha_0/d-1/2} \leq a_k \leq C_4 k^{-\alpha_0/d-1/2}, \forall k \in \mathbb{N},
\]
Applying (6.10) to \(\|Z(x) - I_{\Psi,X}Z(x)\|_{L_p(\Omega)/\sigma}\) instead. Let \(g(x) = Z(x) - I_{\Psi,X}Z(x)\), which is still a Gaussian process since \(I_{\Psi,X}\) is linear. Let \(\sigma_{\Omega}^2 = \sup_{x \in \Omega} \mathbb{E} g(x)^2\). The upper bounds of \(\sigma_{\Omega}^2\) under the cases of \(\nu > \nu_0\) and \(\nu \leq \nu_0\) are given in (6.9) and (6.10), respectively.

Consider a quasi-uniform design \(X'\) with \(\text{card}(X') = n\). Obviously \(h_{X',\Omega} \leq h_{X',\Omega}\). By Proposition 14.1 of [56], \(h_{X',\Omega} \leq Cn^{-1/d}\). By Hölder’s inequality, we have \(\|f\|_{L_2(\Omega)} \leq \|f\|_{L_1(\Omega)} \|f\|_{L_3(\Omega)}^{1/4}\) for any \(f \in L_1(\Omega)\), which implies

\[
\int_{x \in \Omega} \mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2 dx \leq \int_{x \in \Omega} (\mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2)^{1/2} dx \left( \int_{x \in \Omega} (\mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2)^{3/2} dx \right)^{1/4}.
\]

Applying (6.10) to \(\sup_{x \in \Omega} \mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2\) with \(\nu = \nu_0\) yields

\[
\int_{x \in \Omega} (\mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2)^{3/2} dx \leq C_1 \left( \sup_{x \in \Omega} (\mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2)^{3/2} \right)^{1/4} \leq C_2 h_{X',\Omega}^{3\mu_0} \leq C_3 n^{-\frac{3\mu_0}{d}}.
\]

The left hand side of (6.19) can be bounded from below by using Lemma 6.8, which yields

\[
\int_{x \in \Omega} \mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2 dx \geq \left( \sum_{k=2^n+1}^{\infty} \lambda_k \right)^{1/2}. \tag{6.21}
\]

Plugging (6.20) and (6.21) into (6.19), we have

\[
\int_{x \in \Omega} (\mathbb{E}(Z(x) - I_{\Psi,X \cup X'}Z(x))^2)^{1/2} dx \geq C_3^{-1} n^{3\mu_0/d} \left( \sum_{k=2^n+1}^{\infty} \lambda_k \right)^2. \tag{6.22}
\]
By Fubini’s theorem and (6.22), it can be seen that
\[
\mathbb{E}\|Z - \mathcal{I}_{\Phi,X}Z\|_{L^1(\Omega)} = \int_{x \in \Omega} \mathbb{E}|Z(x) - \mathcal{I}_{\Phi,X}Z(x)| \, dx \\
= \int_{x \in \Omega} \frac{2^{1/2}}{\sqrt{\pi}} (\mathbb{E}(Z(x) - \mathcal{I}_{\Phi,X}Z(x))^2)^{1/2} \, dx \\
\geq C_4 \int_{x \in \Omega} (\mathbb{E}(Z(x) - \mathcal{I}_{\Psi,X \cup X'}Z(x))^2)^{1/2} \, dx \\
\geq C_5 n^{\frac{3\nu_0}{4}} \left( \sum_{k=2n+1}^{\infty} \lambda_k \right)^2,
\]
where the first inequality is because \(\mathcal{I}_{\Psi,X \cup X'}Z(x)\) is the best linear predictor of \(Z(x)\). By Lemma 6.9, it can be checked that \(\sum_{k=2n+1}^{\infty} \lambda_k \simeq n^{-\nu_0/d}\). Then by (6.23), we have
\[
\mathbb{E}\|Z - \mathcal{I}_{\Phi,X}Z\|_{L^1(\Omega)} \geq C_5 n^{\frac{3\nu_0}{4}} \left( \sum_{k=2n+1}^{\infty} \lambda_k \right)^2 \geq C_6 n^{-\nu_0/d},
\]
For \(1 \leq p < \infty\) and any \(u > 0\), applying Lemma 6.7 yields
\[
e^{-u^2/(2C_7 \sigma_1^2)} \geq \mathbb{P} \left( \|g\|_{L_p(\Omega)} < \mathbb{E}\|g\|_{L_p(\Omega)} - u \right) \\
\geq \mathbb{P} \left( \|g\|_{L_p(\Omega)} < 2^{1-p} (\mathbb{E}\|g\|_{L_p(\Omega)})^p - u^p \right) \\
\geq \mathbb{P} \left( \|g\|_{L_p(\Omega)} < 2^{1-p} (C_8 \mathbb{E}\|g\|_{L_1(\Omega)})^p - u^p \right) \\
\geq \mathbb{P} \left( \|g\|_{L_p(\Omega)} < C_9 n^{-\nu_0p/d} - u^p \right) \\
\geq \mathbb{P} \left( \|g\|_{L_p(\Omega)} < C_{10} n^{-\nu_0/d} - u \right).
\]
In (6.25), the second inequality is because of Jensen’s inequality; the third inequality is because of the fact \(\|g\|_{L_p(\Omega)} \geq C_8 \|g\|_{L_1(\Omega)}\) for some constant \(C_8 > 0\) depending on \(p\) and \(\Omega\); the fourth inequality is by (6.24); and the last inequality is true because of the elementary inequality \((a+b)^p \geq a^p + b^p\) for \(a, b > 0\). Thus, we finish the proof of Theorem 3.6.

6.7. Proof of Lemma 6.1. Define
\[
d^2(x, x') = \mathbb{E}(Z(x) - \mathcal{I}_{\Phi,X}Z(x) - Z(x') + \mathcal{I}_{\Phi,X}Z(x'))^2.
\]
We first present the following lemma used in the proof of Lemma 6.1. Lemma 6.10 is a direct result of Corollary 2.2.8 of [51], by applying it to the Gaussian process of interest \(Z(x) - \mathcal{I}_{\Phi,X}Z(x)\).
Lemma 6.10. There exists a universal constant $C > 0$, such that for any $t > 0$, with probability no less than $1 - 2\exp\{-2t^2/D^2\}$,

$$\sup_{x \in \Omega} |Z(x) - \mathcal{I}_{\Phi,X}Z(x)| \leq C \int_0^{D/2} \sqrt{\log N(\epsilon, \Omega, d)} d\epsilon + t,$$

where $N(\epsilon, \Omega, d)$ is the $\epsilon$-covering number of the metric space $(\Omega, d)$, and $D$ is the diameter of $\Omega$.

Now we are ready to prove Lemma 6.1. Using the definition of $\mathcal{I}_{\Phi,X}$ in (2.5), one can expand (6.26) and find the explicit formula

\begin{align*}
(6.27) & \quad d^2(x, x')/\sigma^2 = \Psi(x - x) - 2r_1^T(x)K_1^{-1}r(x) + r_1^T(x)K_1^{-1}KK_1^{-1}r_1(x) \\
& \quad + \Psi(x' - x') - 2r_1^T(x')K_1^{-1}r(x') + r_1^T(x')K_1^{-1}KK_1^{-1}r_1(x') \\
& \quad - 2[\Psi(x - x') - r_1^T(x')K_1^{-1}r(x) - r_1^T(x')K_1^{-1}r_1(x) + r_1^T(x)K_1^{-1}KK_1^{-1}r_1(x')],
\end{align*}

where $r, K, r_1, K_1$ are the same as in (2.4) and (2.5).

Recall that for any $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$,

\begin{equation}
(6.28) \quad u^T K u = \sum_{j,k=1}^n u_j \bar{u}_k \Psi(x_j - x_k) = \sum_{j,k=1}^n u_j \bar{u}_k \int_{\mathbb{R}^d} e^{i \omega^T (x_j - x_k)} f_\Psi(\omega) d\omega
\end{equation}

Replace $u$ in (6.28) by $K_1^{-1}r_1(x) =: a(x) =: (a_1(x), \ldots, a_n(x))^T$. Then we obtain

\begin{equation}
(6.29) \quad r_1^T(x)K_1^{-1}KK_1^{-1}r_1(x) = \int_{\mathbb{R}^d} \left| \sum_{j=1}^n a_j(x) e^{i \omega^T x_j} \right|^2 f_\Psi(\omega) d\omega.
\end{equation}

Note that (6.29) is the third term of (6.27). Similarly, we can represent each term of (6.27) using $f_\Psi$. After putting them all together, we can verify via elementary algebraic calculations that,

\begin{equation}
(6.30) \quad d^2(x, x') = \sigma^2 \int_{\mathbb{R}^d} \left| \sum_{j=1}^n (a_j(x) - a_j(x')) e^{i \omega^T x_j} - (e^{i \omega^T x} - e^{i \omega^T x'}) \right|^2 f_\Psi(\omega) d\omega.
\end{equation}
The representation (6.30) admits the first upper bounds of $D$ and $N(\epsilon, \Omega, d)$. First,

$$D^2 = \sup_{x, x' \in \Omega} d^2(x, x')$$

(6.31) \[ \leq 4\sigma^2 \sup_{x \in \Omega} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} a_j(x) e^{i\omega^T x_j} - e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega = 4\sigma^2 Q_0^2. \]

Condition 2.4 implies that $a_j(\cdot)$’s are differentiable. Then it follows from the mean value theorem that

$$d^2(x, x') \leq \sigma^2 \sum_{|\alpha|=1} \sup_{y \in \Omega} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} D^\alpha a_j(y) e^{i\omega^T x_j} - (i\omega)^\alpha e^{i\omega^T y} \right|^2 f_\Psi(\omega) d\omega \cdot \|x - x'\|^2$$

$$= \sigma^2 Q^2 \|x - x'\|^2.$$

Therefore, by the definition of the covering number, we have

(6.32) \[ N(\epsilon, \Omega, d) \leq N(\epsilon/(\sigma Q), \Omega, \|\cdot\|). \]

The right side of (6.32) involves the covering number of a Euclidean ball, which is studied in the literature; see Lemma 2.5 of [47]. This result leads to the bound

(6.33) \[ \log N(\epsilon, \Omega, d) \leq C_1 \log \left(1 + \frac{C_2 \sigma Q_1}{\epsilon}\right). \]

for some constants $C_1, C_2 > 0$. By (6.31), (6.33), and using Lemma 6.10, we finish the proof.

6.8. Proof of Lemma 6.2. We first present the following lemma used in this proof. Let $\pi_l(\mathbb{R}^d)$ be the set of $d$-variate polynomials with absolute degree no more than $l$. Lemma 6.12 is implied in the proof of Lemma 3.3 of [28].

**Lemma 6.11 (Theorem 11.8 of [56]).** Suppose Condition 2.1 holds. Let $l \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq l$. Then there exist constants $h_0, c_1^{(\alpha)}, c_2^{(\alpha)} > 0$ such that for all $X = \{x_1, \ldots, x_n\} \subset \Omega$ with $h_{X,\Omega} \leq h_0$ and every $x \in \Omega$ there exist numbers $\tilde{a}_1^{(\alpha)}(x), \ldots, \tilde{a}_n^{(\alpha)}(x)$ with

1. $\sum_{j=1}^{n} \tilde{a}_j^{(\alpha)}(x) p(x_j) = D^\alpha p(x)$ for all $p \in \pi_l(\mathbb{R}^d)$,
2. $\sum_{j=1}^{n} |\tilde{a}_j^{(\alpha)}(x)| \leq c_1^{(\alpha)} h_{X,\Omega}$,
3. $\tilde{a}_j^{(\alpha)}(x) = 0$, if $\|x - x_j\| > c_2^{(\alpha)} h_{X,\Omega}$. 


LEMMA 6.12. Let \( X = \{ x_1, \ldots, x_n \} \) be a set of scattered points with the separation radius \( q_X \). Let \( \beta > d/2 \) and \( u_1, \ldots, u_n \in \mathbb{R} \) be arbitrary. Then, there exists a constant \( \kappa \) depending only on \( d \) and \( \beta \), such that

\[
\int_{||\omega|| \geq \kappa/q_X} \left| \sum_{j=1}^{n} u_j e^{i\omega^T x_j} \right|^2 (1 + ||\omega||^2)^{-\beta} d\omega \leq 2 \int_{||\omega|| \leq \kappa/q_X} \left| \sum_{j=1}^{n} u_j e^{i\omega^T x_j} \right|^2 (1 + ||\omega||^2)^{-\beta} d\omega.
\]

Now we prove Lemma 6.2. Define

\[
\left[ P^{(\alpha)}(x) \right]^2 := \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} D^{\alpha} a_j(x) e^{i\omega^T x_j} - (i\omega)^{\alpha} e^{i\omega^T x_j} \right|^2 f_\Phi(\omega) d\omega,
\]

which is obtained by replacing the factor \( f_\Psi(\omega) \) in (6.1) by \( f_\Phi(\omega) \). Fix \( x \) and \( \alpha \), define the quadratic function

\[
Q(u) := \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} u_j e^{i\omega^T x_j} - (i\omega)^{\alpha} e^{i\omega^T x_j} \right|^2 f_\Phi(\omega) d\omega,
\]

for \( u = (u_1, \ldots, u_n)^T \). Clearly \( \left[ P^{(\alpha)}(x) \right]^2 = Q(D^{\alpha} a(x)) \). Elementary calculations show that \( Q(u) \) is minimized at \( u = D^{\alpha} a(x) \), i.e.,

\[
\left[ P^{(\alpha)}(x) \right]^2 = Q(D^{\alpha} a(x)) \leq Q(u)
\]

for all \( u \in \mathbb{R}^n \).

Consider the \( l \)th order Taylor polynomial of \( e^x \), which is \( p(x) = \sum_{j=0}^{l} x^j/j! \), with \( l \geq \nu + d/2 \). Let \( h_{X,\Omega} \leq h_0 \), where \( h_0 \) is as in Lemma 6.11. Then all conditions of Lemma 6.11 are fulfilled since we impose Condition 2.1. Let \( \tilde{a}^{(\alpha)} = (\tilde{a}_1^{(\alpha)}(x), \ldots, \tilde{a}_n^{(\alpha)}(x))^T \) be the vector given by Lemma 6.11 for the polynomial space \( \pi_{l}(\mathbb{R}^d) \) and the points \( X = \{ x_1, \ldots, x_n \} \subset \Omega \). Then it follows from the same arguments in establishing Theorem 5.14 of [57] that, under Condition 2.3,

\[
Q(\tilde{a}^{(\alpha)}) \leq C h_{X,\Omega}^{2(\nu-|\alpha|)}
\]

which, together with (6.34), yields

\[
\left[ P^{(\alpha)}(x) \right]^2 \leq C h_{X,\Omega}^{2(\nu-|\alpha|)}.
\]
Let $\xi \geq 1$ to be determined later. Write

$$\left[ Q^{(\alpha)}(x) \right]^2 = \int_{||\omega|| \leq \xi} \left| \sum_{j=1}^{n} D^{\alpha} a_j(x) e^{i\omega^T x_j} - (i\omega)^{\alpha} e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega$$

$$+ \int_{||\omega|| \geq \xi} \left| \sum_{j=1}^{n} D^{\alpha} a_j(x) e^{i\omega^T x_j} - (i\omega)^{\alpha} e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega$$

$$=: I_1 + I_2.$$ 

First we consider $I_2$. We invoke Lemma 6.11 to find $\tilde{a}^{(\alpha)}(x) = (\tilde{a}^{(\alpha)}_1(x), \ldots, \tilde{a}^{(\alpha)}_n(x))$ so that (6.35) holds. Using Conditions 2.2-2.3 and the basic inequality $|w + z|^2 \leq 2|w|^2 + 2|z|^2$, we find that

$$I_2 = \int_{||\omega|| \geq \xi} \left| \sum_{j=1}^{n} \left\{ D^{\alpha} a_j(x) - \tilde{a}^{(\alpha)}(x) \right\} e^{i\omega^T x_j} + \sum_{j=1}^{n} \tilde{a}^{(\alpha)}(x) e^{i\omega^T x_j} - (i\omega)^{\alpha} e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega$$

$$\leq 2c_2 \int_{||\omega|| \geq \xi} \left| \sum_{j=1}^{n} \left\{ D^{\alpha} a_j(x) - \tilde{a}^{(\alpha)}(x) \right\} e^{i\omega^T x_j} \right|^2 (1 + ||\omega||^2)^{-(\nu_0 + d/2)} d\omega$$

$$+ 2c_2 \int_{||\omega|| \geq \xi} \left| \sum_{j=1}^{n} \tilde{a}^{(\alpha)}(x) e^{i\omega^T x_j} - (i\omega)^{\alpha} e^{i\omega^T x} \right|^2 (1 + ||\omega||^2)^{-(\nu_0 + d/2)} d\omega$$

$$=: 2c_2 I_3 + 2c_2 I_4.$$ 

Now the idea is to use Lemma 6.12 to bound $I_3$. To this end, we set $\xi = \kappa/q_X$, where $\kappa$ is the constant given by Lemma 6.12. For simplicity, we require that $\kappa \geq 1$. Then we have

$$I_3 \leq 2 \int_{||\omega|| \leq \kappa/q_X} \left| \sum_{j=1}^{n} \left\{ D^{\alpha} a_j(x) - \tilde{a}^{(\alpha)}(x) \right\} e^{i\omega^T x_j} \right|^2 (1 + ||\omega||^2)^{-(\nu_0 + d/2)} d\omega$$

$$\leq 2^{\nu - \nu_0 + 1} c_3 \kappa^{2(\nu - \nu_0)} q_X^{2(\nu - \nu_0)} \int_{||\omega|| \leq \kappa/q_X} \left| \sum_{j=1}^{n} \left\{ D^{\alpha} a_j(x) - \tilde{a}^{(\alpha)}(x) \right\} e^{i\omega^T x_j} \right|^2 f_\Psi(\omega) d\omega$$

$$\leq 2^{\nu - \nu_0 + 2} c_3^{-1} \kappa^{2(\nu - \nu_0)} q_X^{-2(\nu - \nu_0)} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} D^{\alpha} a_j(x) e^{i\omega^T x_j} - e^{i\omega^T x} \right|^2 f_\Psi(\omega) d\omega$$
where the last inequality follows from (6.35) and (6.36). Using the same idea, we can also bound $I_1$ as

$$ I_1 \leq 2^{\nu - \nu_0} \kappa^2(\nu - \nu_0) \sum_{j=1}^n \tilde{a}^{(\alpha)}(x) e^{i \omega^T x_j} - e^{i \omega^T x} \bigg| f_\Phi(\omega) d\omega $$

where $\kappa = \frac{\kappa}{X, \Omega}$.

Finally we bound $I_4$. The idea is to use the property (2) of Lemma 6.11. Since $\sum_{i=1}^n |\tilde{a}^{(\alpha)}(x)| \leq c_1^{(\alpha)} h_{X, \Omega}^{-|\alpha|}$, we have

$$ \sum_{i=1}^n \tilde{a}^{(\alpha)}(x) e^{i \omega^T x_j} - (i \omega)^\alpha e^{i \omega^T x} \bigg| \leq 2 \left( c_1^{(\alpha)} \right)^2 h_{X, \Omega}^{-2|\alpha|} \|\omega\|^2|\alpha|. $$

Substituting the above inequality into the definition of $I_4$ yields

$$ I_4 \leq 2 \int_{\|\omega\|^2 \geq \kappa / q_x} \left( \left( c_1^{(\alpha)} \right)^2 h_{X, \Omega}^{-2|\alpha|} \|\omega\|^2|\alpha| \right) \left( 1 + \|\omega\|^2 \right)^{-\nu_0 + d/2} d\omega. $$

For $\beta > d/2$, let

$$ C_{\beta-d/2} := \int_{\|\omega\|^2 \geq 1} \left( 1 + \|\omega\|^2 \right)^{-\beta} d\omega < \infty. $$

Then a change-of-variable argument yields

$$ \int_{\|\omega\|^2 \geq \xi} \left( 1 + \|\omega\|^2 \right)^{-\beta} d\omega = \xi^{d-2\beta} \int_{\|\omega\|^2 \geq 1} \left( \xi^{-2} + \|\omega\|^2 \right)^{-\beta} d\omega \leq C_{\beta-d/2} \xi^{d-2\beta}, $$

where the inequality follows from the fact that $\xi = \kappa / q_x \geq \kappa / h_{X, \Omega} \geq 1$. Combining (6.39) and (6.40), we obtain

$$ I_4 \leq 2 \left( c_1^{(\alpha)} \right)^2 C_{\nu_0} \kappa^{-2\nu_0} q_x^{2 \nu_0} h_{X, \Omega}^{-2|\alpha|} + 2 C_{\nu_0} \kappa^{-2(\nu_0 - |\alpha|)} h_{X, \Omega}^{2(\nu_0 - |\alpha|)} $$

$$ \leq 2 C_{\nu_0} \left( \left( c_1^{(\alpha)} \right)^2 C_{\beta} \kappa^{-2\nu_0 + \kappa - 2(\nu_0 - |\alpha|)} h_{X, \Omega}^{2(\nu_0 - |\alpha|)} \right), $$

where $\kappa = \frac{\kappa}{X, \Omega}$. 

Combining (6.39) and (6.40), we obtain
where the last inequality follows from the fact that $q_X \leq h_{X,\Omega}$.

By combining (6.37), (6.38) and (6.41), we conclude that $[Q^{(\alpha)}(x)]^2$ is bounded by a multiple of $h^{2(\nu_0 - |\alpha|)}h_{X,\Omega}^{2(\nu - \nu_0)}$, which implies the desired result.

6.9. Proof of Theorem 3.7. The condition $g \in W^{\beta}_2(\mathbb{R}^d)$ with $\beta > d/2$ implies that $g$ can be recovered from its Fourier transform via

$$g(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{g}(\omega)e^{i\omega^T x} dx.$$ 

This implies that

$$|D^\alpha g(x) - D^\alpha \mathcal{I}_{\Phi,X} g(x)| = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left| \sum_{j=1}^{n} D^\alpha a_j(x)e^{i\omega^T x_j} - (i\omega)^\alpha e^{i\omega^T x} \right| \hat{g}(\omega) d\omega.$$ 

Then via the Cauchy-Schwarz inequality we arrive at

$$|D^\alpha g(x) - D^\alpha \mathcal{I}_{\Phi,X} g(x)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \sum_{j=1}^{n} D^\alpha a_j(x)e^{i\omega^T x_j} - (i\omega)^\alpha e^{i\omega^T x} \right)^2 d\omega \left( \int_{\mathbb{R}^d} |\hat{g}(\omega)|^2 (1 + \|\omega\|^2)^\beta d\omega \right)^{1/2}$$

$$= \frac{1}{(2\pi)^d} Q^{(\alpha)}(x) \|g\|_{W^{\beta}_2(\mathbb{R}^d)},$$

where $Q^{(\alpha)}$ is defined in (6.1) with $f_\psi(\omega) = (1 + \|\omega\|^2)^{-\beta}$. Now the desired result follows from Lemma 6.2.

6.10. Proof of Lemma 6.7. We will use the following lemma, which provides a concentration inequality of a Lipschitz continuous function.

**Lemma 6.13 (Lemma 2.1.6 of [2]).** Let $G$ be a $k$-dimensional vector of centered, unit-variance, independent Gaussian variables. If $h : \mathbb{R}^k \to \mathbb{R}$ has Lipschitz constant $L$, then for all $u > 0$.

$$\mathbb{P}(h(G) - \mathbb{E}h(G) > u) \leq e^{-u^2/(2L^2)}.$$ 

We start with the finite case. Suppose $\Omega$ is finite, so that we can write it as $\{1, \ldots, k\}$. Let $w = (w_1, \ldots, w_k)^T$ be the weights of $G$ with $\sum_{j=1}^k w_j = \text{Vol}(\Omega)$. Then $\|G\|_{L^p(\Omega)}$ becomes $\|G\|_w$, where $G = (G_1, \ldots, G_k)^T$ and

$$\|G\|_w = \left( \sum_{j=1}^k w_j |G_j|^p \right)^{1/p}.$$
Let $K$ be the $k \times k$ covariance matrix of $G$ on $\Omega$, with components $K_{ij} = \mathbb{E}(G_iG_j)$. Let $W$ be a vector of independent, standard Gaussian variables, and $A$ is a matrix such that $A^TA = K$. Thus $G$ has the same law of $AW$.

Consider the function $h(x) = \|Ax\|_w$. Let $\|a\|_p$ be the $l_p$ norm of a vector $a$, and $e_j$ be the vector with $j$th element one and other elements zero. Then

$$|h(x) - h(y)| = \|Ax\|_w - \|Ay\|_w \leq \|A(x - y)\|_w$$

$$= \left( \sum_{j=1}^{k} w_j |(A(x - y))_j|^p \right)^{1/p} = \left( \sum_{j=1}^{k} w_j |e_j^T A(x - y)|^p \right)^{1/p}$$

$$\leq \left( \sum_{j=1}^{k} w_j \|e_j^T A\|^p \|x - y\|^p \right)^{1/p}.$$ 

The first inequality is by the triangle inequality, and the last inequality is by the Cauchy-Schwarz inequality. Note

$$\|e_j^T A\|^2 = e_j^T A^TAe_j = \mathbb{E}(G_j^2) \leq \sigma^2_0.$$ 

Therefore, we have

$$|h(x) - h(y)| \leq \text{Vol}(\Omega)^{1/p} \sigma_\Omega \|x - y\| \leq \max\{\text{Vol}(\Omega), 1\} \sigma_\Omega \|x - y\|,$$

which implies $h$ is a Lipschitz continuous function with Lipschitz constant $\max\{\text{Vol}(\Omega), 1\} \sigma_\Omega$. By the equivalence in law of $G$ and $AW$, and Lemma 6.13, we finish the proof for the finite case for the first inequality in Lemma 6.7. The second inequality in Lemma 6.7 can be proved similarly by considering $h(x) = -\|Ax\|_w$.

For the general $\Omega$, we use approximation. Let $\Omega_{nj}, j = 1, ..., n$, be a partition of $\Omega$ such that $\max\{\text{Vol}(\Omega_{nj}) \to 0$ as $n$ goes to infinity. Thus, $\Omega = \bigcup_{j=1}^{n_\Omega} \Omega_{nj}$ and $\sum_{j=1}^{n_\Omega} \text{Vol}(\Omega_{nj}) = \text{Vol}(\Omega).$ By Lemma 6.3, $\sup_{x \in \Omega} G(x)$ is finite a.s., which implies $G$ is $L_p$-integrable since $G$ is continuous a.s. (see Theorem 2.1.3 of [2]). Let

$$G_n = \left( \sum_{j=1}^{n_\Omega} \text{Vol}(\Omega_{nj})|G(x_{nj})|^p \right)^{1/p}$$

with $x_{nj} \in \Omega_{nj}$. By the separability of $G$, we have $G_n \to \|G\|_{L_p(\Omega)}$ a.s.. By the monotone convergence, $\sigma^2_\Omega_n \to \sigma^2_\Omega,$ where $\sigma^2_\Omega_n = \sup_{x \in \{x_{nj}\}_{j=1}^{n_\Omega}} \mathbb{E}G(x_{nj})^2.$

Since $G_n \leq \sup_{x \in \Omega} G(x)$ and $\mathbb{E}\sup_{x \in \Omega} G(x) < \infty$, by Lebesgue’s dominated convergence theorem, we have

$$\mathbb{P}(G_n > u) \to \mathbb{P}(\|G\|_{L_p(\Omega)} > u), \mathbb{P}(G_n < u) \to \mathbb{P}(\|G\|_{L_p(\Omega)} < u), \mathbb{E}(G_n) \to \mathbb{E}(\|G\|_{L_p(\Omega)}).$$

Thus, from the finite version of the proof, we finish the proof for the general case.
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