FINITENESS OF FPPF COHOMOLOGY

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Abstract. Let $R$ be a Henselian DVR with finite residue field. Let $G$ be a finite type, flat $R$-group scheme (not necessarily commutative) with smooth generic fiber. We show that $H^1_{fppf}(\text{Spec } R, G)$ is finite. We then give an application of the global analogue of this finiteness result to PEL type integral models of Shimura varieties.

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1. Introduction

We prove the finiteness of $H^1$ for the fppf cohomology in greater generality than the known results in existing literature.

Theorem 1.1. (Theorem 3.4) Let $K$ be a number field, $S$ a finite set of places of $K$, and $G$ a flat group scheme of finite type over $\mathcal{O}_{K, S}$ such that the connected component of identity in the generic fibre of $G$ is reductive. Then the set $H^1(\text{Spec } \mathcal{O}_{K, S}, G)$ is finite.

We also give an application of Theorem 1.1 to the Hodge morphism

$\mathcal{S}_K(G, X) \to \mathcal{S}_{K'}(\text{GSp}, S^\pm)$

where $\mathcal{S}_K(G, X)$ is a PEL type integral model for the Shimura variety $\text{Sh}_K(G, X)$ and $\mathcal{S}_{K'}(\text{GSp}, S^\pm)$ is a Siegel integral model (see section 4). The proof of the above theorem 1.1 reduces to proving the following local analogue.

Theorem 1.2. (Corollary 2.7) Let $R$ be a Henselian DVR with finite residue field. For $n$ large enough, the natural morphism,

$\delta_n : H^1_{fppf}(\text{Spec } R, G) \to H^1_{fppf}(\text{Spec } R/\pi^n R, G)$

is injective. Consequently, $H^1_{fppf}(\text{Spec } R, G)$ is finite.

The proof of Theorem 1.2 amounts to applying results of [GR03] on the vanishing of $\mathbb{E}xt^1$ of certain cotangent complexes, which measures obstruction to deformation of torsors. In fact, we prove a slightly stronger result than Theorem 1.2 (see Proposition 2.6).
2. The local case

2.1. Let $(R, \pi)$ be a Henselian pair. Let $G$ be a finite type, flat $R$-group scheme (not necessarily commutative) that is smooth over $R[1/\pi]$. We place ourselves in the situation of [GR03, 5.4.7.], where $F = R[X_1, \ldots, X_N]$ is a free $R$-algebra of finite type, and $J \subset F$ is a finitely generated ideal such that $S = F/J$ and $G \cong \text{Spec } S$. Let $P, P'$ be two arbitrary $G$-torsors and $\text{Isom}(P, P')$ be the Isom scheme. To differentiate the notations, we will write $G \cong \text{Spec } F_G/J_G$ and $\text{Isom}(P, P') \cong \text{Spec } F_{\text{isom}(P, P')}/J_{\text{isom}(P, P')}$. For any $a := (a_1, \ldots, a_N) \in R^N$, let $p_a \subset F$ be the ideal generated by $(X_1 - a_1, \ldots, X_N - a_N)$. Let $I \subset R$ be an ideal. We set the following ideal $H_R(F, J)$ of $F$ as in [GR03, Definition 5.4.1],

\begin{equation}
H_R(F, J) := \text{Ann}_F \text{Ext}_S^1(\mathbb{L}_{S/R}, J/J^2),
\end{equation}

where $\mathbb{L}_{S/R}$ is the cotangent complex.

**Lemma 2.3.** [GR03, Lemma 5.4.8] Let $n, h \in \mathbb{N}$ and $a \in R^N$ such that

\begin{equation}
t^h \in H_R(F, J) + p_a \quad J \subset p_a + t^nIF \quad n > 2h.
\end{equation}

Then there exists $b \in R^N$ such that

\begin{equation}
b - a \in t^{n-h}IR^N \quad \text{and} \quad J \subset p_b + (t^{n-h}I)^2F.
\end{equation}

**Definition 2.5.** Let $h_G$ denote the smallest integer satisfying the condition of 2.4 where $F = F_G$ and $J = J_G$.

One can likewise define the integer $h_{\text{isom}(P, P')}$. 

**Proposition 2.6.** Let $(R, \pi)$ be a general Henselian pair with $\pi$ a non-zero-divisor. Let $G$ be a finite type, flat $R$-group scheme (not necessarily commutative) that is smooth over $R[1/\pi]$. There exists an $N$, which only depends on $G$, such that $H^1(R, G) \rightarrow H^1(R/\pi^N, G)$ is injective.

**Proof.** Suppose $[P], [P'] \in H^1(\text{Spec } R, G)$ map to the same class $[P_n] \in H^1(\text{Spec } R/\pi^n R, G)$ under $\delta_n$. Thus we have $[\text{Isom}(P, P')] \in \delta_n^{-1}([\text{Isom}(P_n, P_n)])$. Now $\text{Isom}(P_n, P_n)$ is the trivial $\text{Aut}_{G_n}(P_n)$-torsor on $\text{Spec } R/\pi^n$, i.e. we have a section

\[\sigma_n: \text{Spec } R/\pi^n \rightarrow \text{Isom}(P_n, P_n).\]

Thus $\text{Isom}(P, P')(R/\pi^n) \neq \emptyset$. By [GR03, Lemma 5.4.13] whose notations we also adopt, as long as $n > 2h_{\text{isom}(P, P')}$, there exists a lifting

\[b_{P, P'} = \sigma_{P, P'}: \text{Spec } R \rightarrow \text{Isom}(P, P'),\]

such that $\sigma_{P, P'}|_{\text{Spec } R/\pi^{-n-h_{\text{isom}(P, P')}}} = \sigma_{P, P'}|_{\text{Spec } R/\pi^{-n-h_{\text{isom}(P, P')}}}$. In other words, as long as $n > 2h_{\text{isom}(P, P')}$, we have $\text{Isom}(P, P')(R) \neq \emptyset$, which implies that $\text{Isom}(P, P')$ is a trivial $\text{Aut}_G(P)$-torsor on $\text{Spec } R$, thus $P \cong P'$ as $G$-torsors on $\text{Spec } R$, and thus $[P] = [P'] \in H^1(\text{Spec } R, G)$. Thus the map $\delta_n$ is injective.

Thus it only remains to show that one can indeed find such a uniform integer $n$ that works for all $G$-torsors $P, P'$ on $\text{Spec } R$, i.e. we want to find an integer $n$ such that $n > 2h_{\text{isom}(P, P')}$ for all $G$-torsors $P, P'$ on $\text{Spec } R$. By the following Lemma 2.12, this is indeed the case; therefore we have our desired result. \[\square\]
Corollary 2.7. Let \( R \) be a Henselian DVR with finite residue field. Let \( G \) be a finite type, flat \( R \)-group scheme (not necessarily commutative) with smooth generic fiber\(^1\). For \( n \) large enough, the natural morphism,

\[
\delta_n: H^1_{\text{fppf}}(\text{Spec } R, G) \to H^1_{\text{fppf}}(\text{Spec } R/\pi^n R, G)
\]

is injective. Consequently, \( H^1_{\text{fppf}}(\text{Spec } R, G) \) is finite.

Proof. This follows directly from Proposition 2.6. \( \square \)

2.8. Recall the ideal \( H_R(F, J) \) from 2.2

\[
H_R(F, J) := \text{Ann}_F \mathbb{E}xt^1_S(L_{S/R}, J/J^2) \tag{2.9}
\]

By [GR03, Lemma 5.4.2(iii)], we have

\[
H_R(F, J) = \bigcap_{\text{all } S\text{-modules } N} \text{Ann}_F \mathbb{E}xt^1_S(L_{S/R}, N)
\]

Therefore, we can rewrite

\[
H_R(F, J) := H_R(F/J) := H_R(S), \text{ i.e. the ideal } H_R(F, J) \text{ only depends on } S = F/J, \text{ and is therefore independent of the presentation of } S.
\]

In order to prove the desired Lemma 2.12, our goal is to show that

\[
H_R(F_G/J_G) = H_R(F_{\text{Isom}}/J_{\text{Isom}}) \tag{2.10}
\]

To this end, we prove the following series of lemmas.

Lemma 2.11. The formation of \( H_R(F, J) \) commutes with flat base change. i.e.

\[
H_R(S) \otimes_R R' \cong H_R(S \otimes_R R')
\]

Proof. First note that the formation of the cotangent complexes and the formation of \( \mathbb{E}xt^1_S \) both commute with flat base change, i.e. we have

\[
\mathbb{E}xt^1_S(L_{S/R}, J/J^2) \otimes_R R' \cong \mathbb{E}xt^1_{S \otimes_R R'}(L_{S \otimes_R R'}, J'/J'^2)
\]

where \( J' = J \otimes_R R' \).

Since \( G \) and \( \text{Isom} \) are complete local intersections, their cotangent complexes \( L_{S/R} \) are perfect complexes concentrated in degrees 0 and 1. Therefore \( \mathbb{E}xt^1_S(L_{S/R}, J/J^2) \) is a finite-type \( S \)-module. Therefore, the formation of the annihilator \( \text{Ann} \mathbb{E}xt^1_S(L_{S/R}, J/J^2) \) commutes with flat base change. Thus we have

\[
(\text{Ann}_F \mathbb{E}xt^1_S(L_{S/R}, J/J^2)) \otimes_R R' \cong \text{Ann}_{F'} \mathbb{E}xt^1_{S \otimes_R R'}(L_{S \otimes_R R'}, J'/J'^2)
\]

where \( F' = F \otimes_R R' \). Since the ideal \( H_R(F, J) \) is independent of the presentation in terms of \( S = F/J \), we have

\[
H_R(S) \otimes_R R' \cong H_R(S \otimes_R R')
\]

i.e. the formation of the ideal \( H_R(F, J) \) commutes with flat base change. \( \square \)

Lemma 2.12. For any \( G \)-torsors \( P, P' \) on \( \text{Spec } R \),

\[
h_{\text{Isom}}(P, P') = h_{\text{Aut}_G}(P) = h_P = h_{P'} = h_G
\]

\(^1\)Technically speaking, we also need to assume that \( G \) is affine, but the result holds in the stated generality by essentially the same proof, see 2.15 and 2.18.
Proof. It suffices to show that \( h_{\text{Isom}(P,P')} = h_G \), the other equalities follow by exactly the same argument. To check whether \( N \geq h \), by [GR03, Lemma 5.4.8], we want to check whether \( S/H_R(S) \) is killed by \( \pi^N \). We have the following diagram

\[
\begin{array}{ccc}
S/H_R(S) & \xrightarrow{\pi^N} & S/H_R(S) \\
\downarrow & & \downarrow \\
(S/H_R(S)) \otimes_R R' & \xrightarrow{\pi^N} & (S/H_R(S)) \otimes_R R'
\end{array}
\]

(2.13)

The map \( \pi^N \) in the second row being zero implies that in the first row being zero. By Lemma 2.11, we have

\[
\left( S_G/H_R(S_G) \right) \otimes_R R' \cong S_G \otimes_R R' \cong H_R(S_G) / H_R(S_G) \otimes_R R'
\]

In particular, if \( N > 0 \) such that \( \pi^N \) kills \( S_G/H_R(S_G) \), then \( \pi^N \) kills

\[
S_G \otimes_R R' / H_R(S_G \otimes_R R')
\]

by the above diagram. Since \( G \otimes_R R' \cong \text{Isom}(P, P') \otimes_R R' \), we have that \( \pi^N \) also kills

\[
S_{\text{Isom}(P,P')} \otimes_R R' / H_R(S_{\text{Isom}(P,P')}) \cong S_G \otimes_R R' / H_R(S_G \otimes_R R')
\]

By the analogous diagram for \( \text{Isom}(P, P') \) as 2.13, \( \pi^N \) also kills \( S_{\text{Isom}}/H_R(S_{\text{Isom}}) \). Thus \( N \geq h_{\text{Isom}} \). We simply take the minimal such \( N \) to obtain \( h_G = h_{\text{Isom}} \). Likewise, the same \( h \) holds for all inner forms of \( G \). \( \square \)

Along the way we have also proven the following:

**Corollary 2.14.** \( H_R(F_G/J_G) = H_R(F_{\text{Isom}}/J_{\text{Isom}}) \).

**2.15.** In the case where \( G \) is quasi-affine instead of affine, we can proceed similarly. We write

\[
G = \bigcup_{i=1}^{m} (\text{Spec } S_i)
\]

Let \( \mathcal{O}_G \) be the structure sheaf of \( G \). Let \( H_R(G) \) be the ideal sheaf given by \( H_R(S_i) \) on each affine open \( \text{Spec } S_i \). Using the same argument as in Lemma 2.11 on affines, one can see that the formation of the ideal sheaf \( H_R(\cdot) \) commutes with flat base change as well. Consider the relative Spec

\[
\text{Spec}_{\mathcal{O}_G}(\mathcal{O}_G/H_R(G))
\]

which is a closed subscheme of \( G \). In particular, we also have

**Lemma 2.16.** Let \( G, G' \) be fppf locally isomorphic. Then:

\( \pi^N \) is zero on \( \text{Spec}_{\mathcal{O}_G}(\mathcal{O}_G/H_R(G)) \) \( \iff \) \( \pi^N \) is zero on \( \text{Spec}_{\mathcal{O}_{G'}}(\mathcal{O}_{G'}/H_R(G')) \)

**Proof.** One can work Zariski locally to reduce to the affine case, which is proven in Lemma 2.12. \( \square \)

**Remark 2.17.** When \( G \) is quasi-affine, \( \text{Isom}(P, P') \) is representable by a quasi-affine scheme as well. This isn’t necessarily true for groups \( G \) that are not quasi-affine.
Remark 2.18. We can relax the conditions on $G$ even further, and let $G$ be any flat, finite type, generically smooth group scheme. Now $\text{Isom}(P, P')$ is an algebraic space. Cotangent complexes for algebraic spaces are defined in [LMB00]. One can then check, via a similar argument as 2.11, that the formation of $\mathcal{H}_K(-)$ in the category of algebraic spaces commutes with étale localization.

3. The global case

Proposition 3.1. Let $K$ be a number field, $S$ a finite set of places of $K$, and $G$ a connected reductive group over $\mathcal{O}_{K,S}$. Then the fibres of the map

$$H^1(\text{Spec} \mathcal{O}_{K,S}, G) \to H^1(K, G)$$

are finite.

Proof. Suppose $P, P'$ are $G$-torsors over $\mathcal{O}_{K,S}$ such that $P|_K \sim P'|_K$. For any prime $v \notin S$ of $K$, we denote $P_v := P|_{\mathcal{O}_{K_v}}$ and $P'_v := P'|_{\mathcal{O}_{K_v}}$. By Lang’s Lemma, such $P_v$ and $P'_v$ are trivial, and in particular they’re isomorphic. The isomorphism $P|_K \sim P'|_K$ extends over $\text{Spec} \mathcal{O}_{K,T}$ for some finite set of primes $T \supset S$.

For $v \notin S$ there are tautological isomorphisms $P_v|_{K_v} \sim P'|_{K_v}$, and $P'$ can be constructed by modifying these isomorphisms by an element of $T_v \in G(K_v)$ for $v \in T - S$ and gluing the $P_v$ to $P|_{\mathcal{O}_{K,T}}$ along the resulting isomorphisms. If $g_v \in G(\mathcal{O}_{K_v})$ for $v \in T - S$, or all the $g_v$ are equal to a single element $g \in G(\mathcal{O}_{K,T})$, then $P$ and $P'$ are isomorphic.

Thus one sees that there is a surjection from the set

$$(3.2) \lim_{T} G(\mathcal{O}_{K,S}) \setminus \prod_{v \notin T - S} G(K_v)/G(\mathcal{O}_{K_v}) = G(\mathcal{O}_{K,S}) \setminus G(\mathcal{A}^S_1)/\prod_{v \notin S} G(K_v)$$

to any fibre of the map $H^1(\text{Spec} \mathcal{O}_{K,S}, G) \to H^1(K, G)$. Here $\mathcal{A}^S_1$ denotes the adeles of $K$ with trivial components at primes in $S$. (In fact it is not hard to see that this map is a bijection.)

By a result of Borel and Harish-Chandra [BHC62], the set in 3.2 is finite, and thus the map $H^1(\text{Spec} \mathcal{O}_{K,S}, G) \to H^1(K, G)$ has finite fibres. \hfill \box

Now we can combine all the above results and prove the following.

Proposition 3.3. Let $K$ be a number field, $S$ a finite set of places of $K$, and $G$ a smooth group over $\mathcal{O}_{K,S}$ such that the connected component of the identity $G^0 \subset G$ is reductive, and $G/G^0$ is étale. Then $H^1(\text{Spec} \mathcal{O}_{K,S}, G)$ is finite.

Proof. If $G$ is finite étale, by the global Poitou-Tate duality, we have $H^i(\text{Spec} \mathcal{O}_{K,S}, G)$ is finite for $i = 0, 1, 2$. Thus by dévissage it suffices to consider the case when $G$ is connected reductive, which we now assume. Note that by Lemma 3.1, we may replace $S$ by a larger finite set $S'$. Thus we may assume that the simply connected cover $G^{sc}$ of the derived group $G^{der}$ of $G$ is étale over $G^{sc}$ and hence of order invertible on $\text{Spec} \mathcal{O}_{K,S}$. We may further assume that the abelianization $G^{ab}$ splits over a field which is unramified at $v \notin S$.

We now consider a number of cases.

(I) Suppose $G$ is a torus. Let $L/K$ be a finite extension over which $G$ splits. Our assumptions imply that we may take $L/K$ to be unramified at primes $v \notin S$. By Hilbert Theorem 90 and our assumption on $G^{ab}$, we have

$$H^1(\text{Spec} \mathcal{O}_{K,S}, G) = H^1(\text{Gal}(L/K), G(\mathcal{O}_{L,S})).$$
As an abelian group $G(O_{L,S})$ is a product of copies of $O_{L,S}$. In particular, this is a finitely generated abelian group. Hence $H^1(\text{Gal}(L/K), G(O_{L,S}))$ is finite; it is a finitely generated abelian group killed by the order of $\text{Gal}(L/K)$.

(II) Suppose $G = G^{\text{sc}}$. Then by Lemma 3.1, it suffices to prove that $H^1(K, G)$ is finite, which follows from the Hasse principle.

This, together with the remark above when $G$ is finite, implies the lemma when $G$ is simply connected.

Finally, combining this with the case of a torus implies the Lemma in general. □

**Theorem 3.4.** Let $K$ be a number field, $S$ a finite set of places of $K$, and $G$ a flat group scheme of finite type over $O_{K,S}$ such that the connected component of identity in the generic fibre of $G$ is reductive. Then the set $H^1(\text{Spec} O_{K,S}, G)$ is finite.

**Proof.** Let $T \supset S$ be a finite set of places of $K$ such that $G|_{O_{K,T}}$ satisfies the conditions of Proposition 3.3. We consider the map

$$H^1(\text{Spec} O_{K,S}, G) \rightarrow H^1(\text{Spec} O_{K,T}, G) \times \prod_{v \in T - S} H^1(O_{K_v}, G).$$

By Proposition 3.3, $H^1(\text{Spec} O_{K,T}, G)$ is finite. And by Proposition 2.7, each $H^1(O_{K_v}, G)$ is finite, and thus $\prod_{v \in T - S} H^1(O_{K_v}, G)$ as a finite product of finite sets is clearly finite. Thus the RHS of the above map (3.5) is finite. Thus in order to show that $H^1(\text{Spec} O_{K,S}, G)$ is finite, it suffices to show that the above map (3.5) has finite fibres. This can be shown using exactly the same argument as in Lemma 3.1—the only difference is that $P$ and $P'$ are isomorphic over $O_{K_v}$ for $v \in T - S$, hence trivial. □

4. An application to integral models of Shimura varieties of PEL type

From now on, we fix a PEL datum, i.e. we fix a semisimple $\mathbb{Q}$-algebra $B$ with a maximal order $O_B$. The algebra $B$ is endowed with a positive involution $*$ and $O_B$ is stable under the involution $*$. Let $(\mathcal{A}, \lambda, \iota)$ be an abelian scheme with $O_B$-action $\iota$ and a symmetric $O_B$-linear polarization $\lambda$.

Let $\text{Aut}(\mathcal{A})$ be the $\mathbb{Z}$-group whose points in a $\mathbb{Z}$-algebra $R$ are given by $\text{Aut}(\mathcal{A})(R) = (\text{End}_S \mathcal{A} \otimes \mathbb{Z} R)^*$. Let $L$ be the $\mathbb{Z}$-scheme of linear maps $O_B \rightarrow \text{End}_S \mathcal{A}$, i.e. we have $L := \text{Hom}_\mathbb{Z}(O_B, \text{End}_S \mathcal{A})$, whose $R$-points (for any $\mathbb{Z}$-algebra $R$) are given by

$L(R) := \text{Hom}_\mathbb{Z}(O_B \otimes \mathbb{Z} R, \text{End}_S \mathcal{A} \otimes \mathbb{Z} R) \cong \text{Hom}_\mathbb{Z}(O_B, \text{End}_S \mathcal{A}) \otimes \mathbb{Z} R$

Now, for a fixed endomorphism structure $\iota : O_B \rightarrow \text{End}_S(\mathcal{A})$, we have $\iota \in L(\mathbb{Z})$.

For any $b \in \text{Aut}(\mathcal{A})(R)$, we can define an element $\iota_b \in L(R)$ given by

$$\iota_b(a) = b(a)b^{-1} \quad \text{for any } a \in O_B \otimes \mathbb{Z} R.$$

(Once that this construction is inspired by explicit computations of matrices.)

This gives us a map of $\mathbb{Z}$-schemes $\psi : \text{Aut}(\mathcal{A}) \rightarrow L$, which on the level of $R$-points is given by

$$\psi(R) : \text{Aut}(\mathcal{A})(R) \rightarrow L(R)
\quad b \mapsto \iota_b.$$
Next we construct a subgroup scheme $H$ of $\text{Aut}(A)$. We start with our fixed $\iota : \mathcal{O}_B \to \text{End}_S(A)$. Denote by $B'$ the commutant of $(\iota \otimes \mathbb{Q})(B)$ in $\text{End}_S(A \otimes \mathbb{Q})$. We also denote $\mathcal{O}_{B'} := B' \cap \text{End}_S(A)$. We define an algebraic group $H$ over $\mathbb{Z}$, whose $R$-points (for $\mathbb{Z}$-algebra $R$) are given by $H(R) := (\mathcal{O}_{B'} \otimes_{\mathbb{Z}} R)^\times = ((B' \cap \text{End}_S(A) \otimes_{\mathbb{Z}} R)^\times$. Note that the group $H$ is smooth, with connected geometric fibres. To see this, denote $\text{rank}_\mathbb{Z} \mathcal{O}_{B'} = n$. Consider the affine space $\mathbb{A}^n$ over $\mathbb{Z}$ corresponding to $\mathcal{O}_{B'}$. We can embed $H \subset \mathbb{A}^n$ (this can be seen explicitly by writing down the equations of $H$). Thus $H$ is open in the affine space $\mathbb{A}^n$. Thus $H$ is smooth with connected geometric fibres. The generic fibre $H_Q$ is reductive, and so $H_{\mathbb{Z}[1/N]}$ is reductive for some integer $N$.

Note that the semisimple algebra $\text{End}_S(A)$ is equipped with the Rosatti involution $\dagger$, and $\mathcal{O}_{B'}$ inherits an involution from $\text{End}_S(A)$. For any involution $* \circ \mathcal{O}_{B'}$, we denote by $\mathcal{O}_{B'}^* \subset \mathcal{O}_{B'}$ the subgroup fixed by $*$. Note that $\mathcal{O}_{B'}^*$ is a saturated subgroup of $\mathcal{O}_{B'}$. We define $H^* := H \cap \mathcal{O}_{B'}^*$, where the intersection is the scheme-theoretic intersection. In particular, when we take the specific involution $* = \dagger$ (the Rosati involution), we obtain the $\mathbb{Z}$-scheme $H^\dagger := H \cap \mathcal{O}_{B'}^\dagger$. We also denote by $\text{Aut}(A)^\dagger$ the part of $\text{Aut}(A)$ fixed by $\dagger$. (Aut$(A)^\dagger$ is also viewed as a $\mathbb{Z}$-scheme.) Note that $H^\dagger$ is a sub-scheme of Aut$(A)^\dagger$. Since $H \subset \text{Aut}(A)$ commutes with the image of $\iota$, this induces a map $\text{Aut}(A)/H \to L$, identifying $\text{Aut}(A)/H$ with a locally closed subscheme of $L$.

Let $I \subset \text{Aut}(A)$ be the algebraic group over $\mathbb{Z}$ consisting of elements $m$ with $mm^\dagger = 1$. By positivity of the Rosatti involution, $I$ is compact, i.e. $I(\mathbb{R})$ is compact. We define a map of $\mathbb{Z}$-schemes $\wp : \text{Aut}(A) \to \text{Aut}(A)^\dagger$ given by $m \mapsto mm^\dagger$ and we denote by $H \subset \text{Aut}(A)$ the preimage of $H^\dagger$ under this map. By Definition, $I = \ker \wp \subset \text{Aut}(A)$. In fact, $I \subset H$.

Now we recall some facts about torsors. For a scheme $T$ and a group scheme $G$ over $T$, we denote by $H^1(G,T)$ the set of isomorphism classes of $G$-torsors on $T$.

**Lemma 4.1.** Let $G$ be a flat group scheme of finite type over a scheme $T$ and $G' \subset G$ a closed, flat subgroup. Then there is a sequence of maps

$$G(T) \to G/G'(T) \overset{\delta}{\to} H^1(T,G')$$

where if $x,x' \in G/G'(T)$ then $\delta(x) = \delta(x')$ if and only if there exists a $g \in G(T)$ with $g \cdot x = x'$.

**Proof.** For $x \in G/G'(T)$ the element $\delta(x)$ is the class of the $G$-torsor given by $F_x$, the fibre over $x$ of the map $G \to G/G'$.

If $x,x' \in G/G'(T)$, we consider the subscheme of $G$ whose points in a $T$-scheme $S$ are given by the set of $y \in G(S)$ such that $y \cdot x = x'$. This subscheme is a $G$-torsor, isomorphic to the scheme of isomorphisms $\text{Isom}(F_x,F_x')$. In particular, $F_x$ and $F_x'$ are isomorphic if and only if this $G$-torsor is trivial; that is, if and only if $x$ and $x'$ differ by a point of $G(T)$.

Here we recall the following corollary of the Noether-Skolem Theorem

**Lemma 4.2.** For any two $k$-algebra homomorphisms $f,g$ from a simple $k$-algebra $A$ to a central simple $k$-algebra $B$, there exists $b \in B^\times$ such that $f(x) = bg(x)b^{-1}$ for all $x \in A$. 
We will also need the following easy observation: For any \( \iota : \mathcal{O}_B \to \End(\mathcal{A}) \) given in the PEL Shimura data, the image of \( \iota \) lies in \( \End(\mathcal{A})^1 \), where \( \dagger \) is the Rosati involution induced from the \( \mathcal{O}_B \)-linear polarization \( \lambda \) in our Shimura datum. (i.e. the image of \( \iota \) is stable under \( \dagger \), by which we mean
\[
i(\alpha)^\dagger = \iota(\alpha) \quad \text{for all } \alpha \in \mathcal{O}_B.
\]
Moreover, \( \dagger \) induces an anti-involution on \( \mathcal{O}_B \) via \( \iota \).

**Theorem 4.3.** The morphism on the level of integral models
\[
\Phi : \mathcal{S}_K(G, X) \to \mathcal{S}_K(G\Sp, S^\perp)
\]
induced (on the level of points) by \( (\mathcal{A}, \lambda, \iota) \to (\mathcal{A}, \lambda) \) has finite fibres.

**Proof.** We do it in steps.

**Step 1:** The Theorem reduces to showing that \( I(\mathbb{Z})/\tilde{H}(\mathbb{Z})/H(\mathbb{Z}) \) is finite.

**Proof.** Suppose \( (\mathcal{A}, \lambda, \iota) \) and \( (\mathcal{A}, \lambda, \iota') \) are two triples (on the left-hand side of \( \Phi \)) mapping to the same principally polarized abelian scheme \( (\mathcal{A}, \lambda) \) on the right-hand side. By the Noether-Skolem Theorem, \( \iota'(a) = b\iota(a)b^{-1} \) for some \( b \in \Aut(\mathcal{A})(\mathbb{Q}) \) and for all \( a \in \mathcal{O}_B \).

Since the images of both \( \iota' \) and \( \iota \) are stable under \( \dagger \), we have for any \( a \in \mathcal{O}_B \),
\[
\iota'(a) = (\iota(a))^\dagger = (b\iota(a)b^{-1})^\dagger = (b^{-1})^\dagger\iota(a)^\dagger = (b^{-1})^\dagger \iota(a)b^\dagger.
\]
Combined with the original identity \( \iota'(a) = b\iota(a)b^{-1} \), we obtain for any \( a \in \mathcal{O}_B \),
\[
bc(a)b^{-1} = (b^{-1})^\dagger \iota(a)b^\dagger.
\]
Since the polarization \( \lambda \) is required to be \( \mathcal{O}_L \)-linear and symmetric, \( (b^{-1})^\dagger = (b^\dagger)^{-1} \), and thus for any \( a \in \mathcal{O}_B \),
\[
(b^\dagger b)\iota(a)(b^\dagger b)^{-1} = \iota(a). \quad (\ast)
\]

Now, \( (\ast) \) implies that, for any \( a \in \mathcal{O}_B \)
\[
(b^\dagger b)\iota(a)(b^\dagger b)^{-1} = \iota(a) = b\iota(a)(b^\dagger b)(b^\dagger b)^{-1}
\]
Multiplying both sides by \( (b^\dagger b) \) on the right, we have, for any \( a \in \mathcal{O}_B \),
\[
(b^\dagger b)\iota(a) = (b^\dagger b)(b^\dagger b)^{-1}
\]
Thus \( b^\dagger b \in B' \), and since \( b^\dagger b \in \End_\mathbb{Q} \mathcal{A} \otimes \mathbb{Q} \), we have \( b^\dagger b \in H(\mathbb{Q}) \). Since \( (b^\dagger)^\dagger = b \), we have \( (b^\dagger b)^\dagger = b^\dagger b \), and thus we also have \( b^\dagger b \in H^1(\mathbb{Q}) \subset \Aut(\mathcal{A})^1(\mathbb{Q}) \). Recall the map we defined earlier
\[
(4.4) \quad \varphi : \Aut(\mathcal{A}) \to \Aut(\mathcal{A})^\dagger
\]
\[
(4.5) \quad m \mapsto mm^\dagger
\]
and recall that we also defined \( \tilde{H} := \varphi^{-1}(H^\dagger) \subset \Aut(\mathcal{A}) \). Thus we have
\[
b \in \tilde{H}(\mathbb{Q}) \subset \Aut(\mathcal{A})(\mathbb{Q}).
\]
On the other hand, since \( \iota' = b\iota b^{-1} \) sends \( \mathcal{O}_B \) to \( \End(\mathcal{A}) \), we also have \( b \in \Aut(\mathcal{A})(\mathbb{Z}) \). Moreover, we can consider \( b \mod \tilde{H} \in (\Aut(\mathcal{A})/H)(\mathbb{Z}) \subset L(\mathbb{Z}) \). By Proposition 3.4, \( H^1(\Spec \mathbb{Z}, H) \) is a finite set.

We then apply Lemma 4.1 above to the following sequence
\[
\Aut(\mathcal{A})(\mathbb{Z}) \to (\Aut(\mathcal{A})/H)(\mathbb{Z}) \to H^1(\Spec \mathbb{Z}, H)
\]
and conclude that \( \Aut(\mathcal{A})(\mathbb{Z}) \) has only finitely many orbits on \( (\Aut(\mathcal{A})/H)(\mathbb{Z}) \).
Since any such \( \iota' \) is determined by its conjugating element \( b \in \tilde{H}(\mathbb{Q}) \) which in fact comes from \( b \in \tilde{H}(\mathbb{Z}) \) (because we fixed an \( \iota \) at the beginning), to show that there are only finitely many possibilities for \( \iota' \), it suffices to show that there are finitely many

\[
b \in \tilde{H}(\mathbb{Z}) = \text{Aut}(\mathcal{A})(\mathbb{Z}) \cap \tilde{H}(\mathbb{Q}).
\]

And since \( \text{Aut}(\mathcal{A})(\mathbb{Z}) \) has finitely many orbits on \( (\text{Aut}(\mathcal{A})/H)(\mathbb{Z}) \), it suffices to show that \( \tilde{H}(\mathbb{Z})/H(\mathbb{Z}) \) is finite.

On the other hand, \( I \) is compact, thus \( I(\mathbb{Z}) \) is compact and discrete, so \( I(\mathbb{Z}) \) must be finite. Thus to show that \( \tilde{H}(\mathbb{Z})/H(\mathbb{Z}) \) is finite, it suffices to show that \( I(\mathbb{Z}) \backslash \tilde{H}(\mathbb{Z})/H(\mathbb{Z}) \) is finite.

**Proof.** Recall the map \( \varphi \) we defined earlier. Now we restrict the map \( \varphi \) to the preimage of \( H^\dagger \) (which we called \( \tilde{H} \)), and consider

\[
\varphi : \tilde{H} \rightarrow H^\dagger
\]

\[
m \mapsto m^\dagger m
\]

Since \( I = \ker \varphi \), the above map factors through the projection \( \tilde{H} \rightarrow I \backslash \tilde{H} \), and thus the map \( I \backslash \tilde{H} \hookrightarrow H^\dagger \) identifies \( I \backslash \tilde{H} \) with a subspace of \( H^\dagger \). This identification is \( H \)-equivariant when we let \( H \) act on \( H^\dagger \) via \( m \mapsto h^\dagger mh \) for \( h \in H \) and \( m \in H^\dagger \).

Thus in order to show that \( I(\mathbb{Z}) \backslash \tilde{H}(\mathbb{Z})/H(\mathbb{Z}) \) is finite, it suffices to show that \( H^\dagger(\mathbb{Z})/H(\mathbb{Z}) \) is finite.

**Step II:** The theorem further reduces to showing that \( H^\dagger(\mathbb{Z})/H(\mathbb{Z}) \) is finite.

**Proof.** Consider the fibre of \( H^\dagger \) (which we called \( \tilde{H} \)), and denote the scheme

\[
\text{Lie}(\tilde{H}) \twoheadrightarrow \tilde{H}
\]

where we denote the scheme \( \text{Lie}(\tilde{H}) \) smooth and surjective over \( \mathbb{Z}(\frac{1}{2}) \).

**Step III:** In order to show that \( H^\dagger(\mathbb{Z})/H(\mathbb{Z}) \) is finite, we first show the map \( \vartheta \) (to be defined below) is smooth and surjective over \( \mathbb{Z}(\frac{1}{2}) \).

**Proof.** Consider the map

\[
\vartheta : H \times H^\dagger \rightarrow H^\dagger \times H^\dagger
\]

\[
(h, m) \mapsto (h^\dagger mh, m)
\]

Consider the fibre of \( \vartheta \) over a point \( (m, m') \) and we obtain a scheme

\[
\vartheta^{-1}(\{(m, m')\}) = \{(h, m') \in H \times \{m'\} : h^\dagger m'h = m\} := H_{m,m'} \times \{m'\}
\]

where we denote the scheme \( H_{m,m'} := \{h \in H : h^\dagger m'h = m\} \). Consider the involution on \( \mathcal{O}_{B'} \) given by

\[
\dagger(m) : \mathcal{O}_{B'} \rightarrow \mathcal{O}_{B'}
\]

\[
h \mapsto m^{-1}h^\dagger m
\]

Define the scheme \( H_{\dagger(m)} := \{h \in H : h^\dagger mh = m\} = \{h \in H : h^{\dagger(m)}h = 1\} \). Now, the scheme \( H_{m,m'} \) is either empty, or a torsor under \( H_{\dagger(m)} \). The generic fibre of \( H_{\dagger(m)} \) has a connected component of identity which is reductive. To compute the dimension of the fibres of \( H_{\dagger(m)} \) over \( \mathbb{Z}(\frac{1}{2}) \), we study its Lie algebra \( \text{Lie}(H_{\dagger(m)}) \).

Note that \( \text{Lie}(H_{\dagger(m)}) = (\text{Lie}(H))_{\dagger(m)} = \text{Lie}(H)_{\dagger(m)} = \text{Lie}(H)_{\dagger(m)}^{-1} \).

Now, since \( \dagger(m) \) and \( \dagger \) are conjugate to each other, they have the same number of eigenvalues equal to \(-1\) when acting on \( B' \). Thus the dimension of this Lie
algebra $\text{Lie}(H)^{\dagger(m)} = -1$ over any point of $\text{Spec } \mathbb{Z}[1/2]$ is the same as the dimension of $(\text{Lie} H)^{\dagger} = -1$, which is equal to $\dim H - \dim H^\dagger$. On the other hand, since

$$\dim(\text{Lie} H)^{\dagger} = -1 \geq \dim H^{\dagger(m)} \geq \dim H - \dim H^\dagger$$

Thus all inequalities in the above achieve equality:

$$\dim H^{\dagger(m)} = \dim(\text{Lie} H)^{\dagger} = -1 = \dim \text{Lie}(H^{\dagger(m)}).$$

Thus $H^{\dagger(m)}$ is smooth over $\mathbb{Z}[1/2]$ and $\dim H^{\dagger(m)} = \dim H - \dim H^\dagger$. Now, since $H_{m,m'}$ is a torsor under $H^{\dagger(m)}$ which is smooth of dimension independent of $m$, we have that $H_{m,m'}$ is also smooth of dimension independent of $m$ or $m'$. Therefore, the map $\vartheta$ has smooth equidimensional fibres $H_{m,m'}$ over $\mathbb{Z}[1/2]$, in particular $\vartheta$ is flat over $\mathbb{Z}[1/2]$, as it is a map of smooth $\mathbb{Z}$-schemes.

To show that $\vartheta$ is surjective over $\mathbb{Z}[1/2]$: we first note that the above argument implies that $\vartheta$ is an open map. In particular, we consider for any $m \in H^\dagger$, the restriction $\vartheta|_{H^\dagger \times m} : H^\dagger \times m \rightarrow H^\dagger \times m$. Therefore, in any fibre of $H^\dagger$ over a point of $\text{Spec } \mathbb{Z}[1/2]$, the orbits of $H$ acting on $H^\dagger$ are open. Since $H^\dagger$ has connected fibres, this implies that there is only one orbit, and thus $H$ acts transitively on the fibres of $H^\dagger$. Hence $\vartheta$ is surjective over $\mathbb{Z}[1/2]$.

**Step IV: The situation over $\mathbb{Z}$: reduces to showing that $F_m(\mathbb{Z})/H(\mathbb{Z})$ is finite.**

**Proof.** Let $(H^\dagger)_I$ denote the blow up of $H^\dagger$ in an ideal $I$ supported over the prime 2. We consider the inverse limit $(H^\dagger)^\sim = \lim_I (H^\dagger)_I$. The map $(H^\dagger)^\sim \rightarrow H^\dagger$ induces a bijection on $\mathbb{Z}$-points. Let $m \in H^\dagger$, and consider the orbit map $H \rightarrow H^\dagger$ obtained by taking the fibre of $\vartheta$ over $m$. We denote by $H^\sim_m$ the proper transform of $H$ with respect to $(H^\dagger)^\sim \rightarrow H^\dagger$. That is, $H^\sim_m$ is the closure of the generic fibre of $H$ in $H \times (H^\dagger)^\sim$. By [RG71, Theorem 5.2.2.], the map $H^\sim_m \rightarrow (H^\dagger)^\sim$ is flat, hence open. We denote by $F_m$ its image. This is an open set containing the point corresponding to $m$. Let $F = \bigcup_m F_m$, an open subset of $(H^\dagger)^\sim$ containing $H^{\dagger(m)}(\mathbb{Z})$. Since $(H^\dagger)^\sim$ is quasi-compact (it is an inverse limit of quasi-compact schemes), $F$ is a union of finitely many $F_m$. Thus it suffices to show that $F_m(\mathbb{Z})/H(\mathbb{Z})$ is finite. 

**Step V: The finiteness of $F_m(\mathbb{Z})/H(\mathbb{Z})$ reduces to the finiteness of $H/H^{\dagger(m)}(\mathbb{Z})$.**

**Proof.** Note that $H^\sim_m \rightarrow F_m$ is of finite type, thus by a theorem of Skolem ([MB89], if $m' \in F_m(\mathbb{Z})$, there is an $h \in H^\sim_m(\mathcal{O}_K) = H(\mathcal{O}_K)$ with $h \cdot m = m'$ for some number field $K$.

We denote by $H^{\dagger(m)}$ the closure of $H^{\dagger(m)}$ in $H$, so that this is a finite type flat group scheme over $\mathbb{Z}$. Then the image of $h$ in $H^\dagger_m := H/H^{\dagger(m)}$ is in $H/H^{\dagger(m)}(\mathbb{Z}) := H^\dagger_m(\mathbb{Z})$ since $m, m' \in H^\dagger(\mathbb{Z})$. Thus in order to show that $F_m(\mathbb{Z})/H(\mathbb{Z})$ is finite, it suffices to show that $H/H^{\dagger(m)}(\mathbb{Z})$ is finite.

Finally, to show that $H/H^{\dagger(m)}(\mathbb{Z})$ is finite, by Lemma 4.1, it suffices to show that the set of torsors $H^\dagger(\text{Spec } \mathbb{Z}, H^{\dagger(m)})$ is finite, and this follows from Proposition 3.4. Thus we have finished the proof of Theorem 4.3. 

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