GENERALIZED MITTAG-LEFFLER STABILITY OF FRACTIONAL
IMPULSIVE DIFFERENTIAL SYSTEM

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Abstract. This paper establishes sufficient conditions for Generalized Mittag-Leffler stability of a class of impulsive fractional differential system with Hilfer order. The analysis extends through both, instantaneous and non-instantaneous impulsive conditions. The theory utilizes continuous Lyapunov functions, to ascertain the stability conditions. An example is given discussing for various ranges.

1. Introduction

Many biological happenings sustain perturbations for a period of time. Some of such perturbations may persist for a very short span or it may stretch to a finite time interval. In accordance with the duration of perturbation, systems are branched as instantaneous impulsive system and non-instantaneous impulsive system respectively. The former impulsive system finds its way in models where the system changes its constraint suddenly. For example, physical system where there is sudden change of speed, direction or sudden change in the body condition after a drug is injected are modeled as instantaneous impulsive system. For applications of instantaneous impulsive system, refer to [17] by Stamova and Stamov, where, various models involving impulsive conditions are discussed in detail. On the other hand, the latter gets involved when the perturbations are not negligible. Such a situation was first analyzed by Hernández and O’Regan in [9], where they proposed a new impulsive conditions where the perturbations prolong for a finite interval of time and not just at some fixed moments. Precisely, instead of impulsive points $t_k$, they considered the finite time interval $[t_k, p_k]$ where the perturbation occurs. For instance, a differential equation model that reveals the impact of the drug by a body for a certain period of time admits an non-instantaneous impulse. Besides the field of medicine, most of the real-life problems including the study of geographical conditions, to estimate the impact of global warming and in varied field of physics, involve non-instantaneous impulses. The detailed theory and application regarding non-instantaneous impulses is available in whole lot in the literature, see for instance, the book by Agarwal et al. [1]. Numerous research articles constantly emerge that deal with the above two impulse conditions.

In both the impulsive systems, in particular, the state of the system keeps on varying. Thus stability is one property which has to be addressed as it determines the stable region of the state of the system. As an example, a power system uses stability analysis to prove if it has ability to withstand the impact of reasonable fluctuation (or impulses). While discussing the practical-oriented systems, their corresponding models with non-integer order is more productive and it enhances the accuracy that a system needs. For instance, a proportional-integral-derivative controller (PID controller or three-term controller) is a system with control loop employing feedback that is broadly used in industrial control.
systems and in wide range of other applications that demand continuous modulated control. Instead of classical controller, Podlubny [14] considered $PI^\lambda D^\mu$ controller combining fractional order integrand ($I^\lambda$) and fractional order derivative ($D^\mu$). An illustration is also provided in his work that proves that $PI^\lambda D^\mu$ controller works better than the classical PID controller. While embracing fractional model, choosing an effective fractional order is vital.

Generalized Riemann-Liouville fractional derivative arose as a theoretical model of dielectric relaxation in glass-forming materials in a work by Hilfer [11]. This fractional derivative named as Hilfer fractional derivative finds its place in the recent literature. This paper adapts the Hilfer fractional derivative as it unifies the two classical fractional derivatives- Caputo and Riemann-Liouville. There are a hand full of papers that proves the existence and uniqueness of solution of impulsive system with Hilfer fractional derivative with different constraints such as nonlocal conditions by Gou and Li [7], approximate controllability by Jiang and Niazi [5], with delay conditions by Ahmed et al. [4], etc.

The study of stability analysis for non-integer by Podlubny et al. [12] leads to numerous research articles that have arisen in this direction in the recent past. The work of Stamova [16] on impulsive fractional order is a classical result. Even though stability analysis has been done for Hilfer fractional system by Rezazadeh et al. [15], Wang et al. [18], in order to step further, stability analysis of impulsive differential system with Hilfer fractional derivative is a much needed topic to understand the Hilfer derivative in a much deeper sense. Present work studies the Generalized Mittag-Leffler stability of a Hilfer fractional differential system involving both, instantaneous and non-instantaneous impulsive conditions using Lyapunov approach.

The structure of the paper is in the following sequence. Section 2 covers the essential notions that are used in the rest of the paper. In Section 3, both the impulsive system are outlined and the stability conditions for both the system are established. Section 4 elaborates the stability in the case of non-instantaneous perturbation, while Section 5 renders the same when the perturbation is instantaneous along with an example. The paper culminates with a concluding remark.

2. Essential notions

Let $\Delta$ be the open set of the n-dimensional Euclidean space $\mathbb{R}^n$ with norm $\| \cdot \|$ and let $\mathbb{R}_+ := [0, \infty)$. The fractional integral of order $\mu$ and for an integrable function $g$ is given as [13, Sec 2.3.2],

$$I_0^\mu g(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} g(s) ds, \quad 0 < \mu < 1.$$

Here $\Gamma(\cdot)$ is the well known gamma function. Also, the respective fractional derivative of the two classical derivatives, Caputo and Riemann-Liouville of order $\mu$ are given by [13, Sec 2.4.1],

$$C D_0^\mu g(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{g'(s)}{(t-s)^\mu} ds, \quad t > 0, \quad 0 < \mu < 1,$$

and

$$L D_0^\mu g(t) = \frac{1}{\Gamma(1-\mu)} \left( \frac{d}{dt} \right) \int_0^t \frac{g(s)}{(t-s)^\mu} ds, \quad t > 0, \quad 0 < \mu < 1,$$

where $n = [\mu] + 1$ and $[\mu]$ denotes the integral part of number $\mu$. The Hilfer fractional derivative is of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ of function $g(t)$ is defined by Hilfer
Criteria for Generalized Mittag-Leffler stability of impulsive differential system with Hilfer fractional order

\[ D_{0^+}^{\mu,\nu} g(t) = I_{0^+}^{\nu(1-\mu)} D I_{0^+}^{(1-\nu)(1-\mu)} \]

where \( D := \frac{d}{dt} \). Results regarding the existence of solution for Hilfer fractional derivative by Furati et al. in [6] and Gu and Trujillo in [8] unlocked the stream of research with fractional Hilfer derivative. Caputo and Riemann-Liouville can be considered as a special case of Hilfer fractional derivative as

\[
D_{0^+}^{\mu,\nu} = \begin{cases} 
D I_{0^+}^{1-\mu}, & \nu = 0 \\
I_{0^+}^{1-\mu} D = C D_{0^+}^{\mu}, & \nu = 1
\end{cases}
\]

The parameter \( \lambda \) satisfies \( \lambda = \mu + \nu - \mu \nu, \ 0 < \lambda \leq 1 \). The Laplace transform with respect to the Hilfer fractional derivative, is given as [15]

\[
L[0 D_{0^+}^{\mu,\nu} x(t)] = s^\mu L[x(t)] - s^{\nu(\mu-1)} I_{0^+}^{(1-\nu)(1-\mu)} x(0^+), \ \Re(s) > 0.
\]

The Mittag-Leffler function with one parameter say \( \mu \) and two parameters \( \mu \) and \( \lambda \) are given respectively by, (see [13] for details)

\[
E_\mu(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + 1)} \quad \text{and} \quad E_{\mu,\lambda}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \lambda)}, \ \mu > 0, \ \lambda > 0, \ z \in \mathbb{C}.
\]

and for \( \lambda = 1 \), \( E_\mu(z) = E_{\mu,1}(z) \).

The Laplace transforms with respect to one and two parameter of Mittag-Leffler function are given respectively by

\[
L\{E_\mu(-\gamma t^\mu)\} = \frac{s^{\mu-1}}{s^\mu + \gamma} \quad \text{and} \quad L\{t^{\lambda-1} E_{\mu,\lambda}(-\gamma t^\mu)\} = \frac{s^{\mu-\lambda}}{s^\mu + \gamma}, \ \gamma \in \mathbb{R}.
\]

In the study of stability analysis, Lyapunov’s direct method which is a sufficient condition is predominantly used in the literature. By determining a suitable Lyapunov function the stability is validated. Stability for Riemann-Liouville fractional system

\[
_0 D_t^\mu x(t) = g(t, x), \ \mu \in (0, 1]
\]

in terms of Mittag-Leffler was first given by Podlubny et al. [12].

**Definition 2.1.** [12] The solution of (2.1) is said to be Mittag-Leffler stable if

\[
\|x(t)\| \leq \left[ n[x(t_0)](t - t_0)^{1-\lambda} E_{\mu,\lambda}(-\gamma(t - t_0)^{\mu}) \right]^b,
\]

where \( t_0 \) is the initial time, \( \mu \in (0, 1) \), \( \lambda \in [0, 1] \), \( \gamma \geq 0 \), \( b > 0 \). Also \( n(0) = 0 \), \( n(x) \geq 0 \) and \( n(x) \) is locally Lipschitz on \( x \in \mathbb{R}^n \) with Lipschitz constant \( n_0 \).

Another definition which emphasis on the existence of Hilfer fractional derivative is given by the following:

**Definition 2.2.** [6] The function \( u(t) \in C^{\mu,\nu}([0, \infty), \mathbb{R}) \) if \( u(t) \) is differentiable on \([0, \infty)\) and the Hilfer fractional \( _0 D_t^{\mu,\nu} u(t) \) exists for \( t \in [0, \infty) \).
3. FORMULATION OF STABILITY CONDITIONS FOR A NON-LINEAR SYSTEM

In this section we propose the Generalized Mittag-Leffler stability condition for nonlinear impulsive differential system with Hilfer fractional derivative.

To commence with, an initial value problem for the nonlinear Hilfer fractional non-instantaneous impulsive differential system defined as below is considered.

\[
\begin{aligned}
& \begin{cases}
\upsilon_l D_t^{\mu_l} x(t) = g(t, x(t)), & t \notin \{t_k, t_{k+1}\}, \\
x(t) = \phi_k(t, x(t), x(t_k-0)), & t \in (t_k, t_{k+1}), \\
I_{t_0}^{\lambda_1} |x(t)|_{t=0} = x_0.
\end{cases}
\end{aligned}
\]  

(3.1)

Here, \(D_t^{\mu_l}\) denotes the Hilfer fractional derivative of order \(0 < \mu < 1\), type \(0 \leq \nu \leq 1\) and \(\lambda = \mu + \nu - \mu \nu\). \(I_{t_0}^{\lambda_1} (\cdot)\) is the Riemann-Liouville fractional integral. The sequence \(\{t_k\}_{k=1}^{\infty}\) and \(\{p_k\}_{k=0}^{\infty}\) are assumed such that \(p_0 = 0 < t_k < p_k < t_{k+1}\), for \(k = 1, 2, \ldots\), with \(\lim_{k \to \infty} t_k = \infty\). Let the given initial time be \(t_0 \in (0, t_1) \cup \cup_{k=1}^{\infty} (p_k, t_{k+1})\). The nonlinear function \(g\) is defined as \(g : \cup_{k=0}^{\infty} (p_k, t_{k+1}) \times \mathbb{R}^n \to \mathbb{R}^n\) and \(x_0 \in \mathbb{R}^n\). The non-instantaneous impulsive function \(\phi_k(t, x, z)\) for \(k = 1, 2, \ldots\), for the Hilfer fractional system is defined as \(\phi_k : \cup_{k=1}^{\infty} [t_k, p_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\) and the interval \((t_k, p_k)\) for \(k = 1, 2, \ldots\), are the intervals of non-instantaneous impulses system. In order to study the Generalized Mittag-Leffler stability of the zero solution of the given impulsive system, first some conditions that guarantee the existence of a zero solution is assumed. Then for such a zero solution, the below defined Generalized Mittag-Leffler stability condition is studied with the help of continuously differentiable Lyapunov functions.

**Definition 3.1.** The zero solution of Hilfer fractional non-instantaneous impulsive differential system (3.1) is called Generalized Mittag-Leffler stable if

\[
\|x(t)\| \leq N \prod_{i=1}^{k} \left( \frac{\lambda_1}{p_i} \left\| x_0 \right\|_{\mathbb{R}^n} \right)^\lambda \left( t_i - p_{i-1} \right)^{\lambda-1} \left( t - p_k \right)^{\lambda-1} E_{\mu,\lambda} \left( \frac{\lambda_1}{p_i} \right) \left( -\gamma (t_i - p_{i-1})^\mu \right) E_{\mu,\lambda} \left( \frac{\lambda_1}{p_i} \right) \left( -\gamma (t - p_k)^\mu \right) \right]^\frac{1}{\mu}, \quad t \in (p_k, t_{k+1}], \ k = 1, 2, \ldots.
\]

As a special case for \(k = 0\), the above condition is for \(t \in [t_0, t_1]\) and

\[
\|x(t)\| \leq N \prod_{i=1}^{k} \left( \frac{\lambda_1}{p_i} \left\| x_0 \right\|_{\mathbb{R}^n} \right)^\lambda \left( t_i - p_{i-1} \right)^{\lambda-1} E_{\mu,\lambda} \left( \frac{\lambda_1}{p_i} \right) \left( -\gamma (t_i - p_{i-1})^\mu \right) E_{\mu,\lambda} \left( \frac{\lambda_1}{p_i} \right) \left( -\gamma (t_i - p_{i-1})^\mu \right) \right]^\frac{1}{\mu}, \quad t \in (t_k, p_k],
\]

where \(k = 1, 2, \ldots\). Without loss of generality \(p_0 = t_0\) can be assumed. While \(x(t) = x(t; t_0, x_0), t_0 \in (0, t_1) \cup \cup_{k=1}^{\infty} (p_k, t_{k+1})\) is the initial time, \(x_0 \in \mathbb{R}^n, \mu \in (0, 1), \lambda \in [0, 1], \alpha, \beta, \gamma\) and \(N\) are positive constants.

Next, an initial value problem for the nonlinear Hilfer fractional instantaneous impulsive differential equation illustrated below is considered. This problem can be considered as a special case of non-instantaneous equation, wherein, if \(p_l = t_l\), for \(l = 0, 1, \ldots\) in Definition (3.1), the corresponding solution gives the solution of the Hilfer fractional instantaneous differential equation:

\[
\begin{aligned}
& \begin{cases}
\upsilon_l D_t^{\mu_l} x(t) = g(t, x(t)), & t \neq t_l, \ l = 0, 1, \ldots \\
x(t_l) = \psi_l (x(t_l-0)), & l = 1, 2, \ldots \\
I_{t_0}^{\lambda_1} |x(t)|_{t=0} = x_0.
\end{cases}
\end{aligned}
\]  

(3.2)

As specified earlier, \(D_t^{\mu_l} x(t)\) is the Hilfer fractional derivative of order \(0 < \mu < 1\), \(0 \leq \nu \leq 1\) and \(0 \leq \lambda \leq 1\). Further, \(x_0 \in \mathbb{R}^n, t_0 \in \mathbb{R}_+/\{t_l\}\) and the nonlinear function \(g\) is
defined by $g: [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n; \psi: \mathbb{R}^n \to \mathbb{R}^n$. The increasing sequence $\{t_l\}_{l=1}^{\infty}$ be such that $0 < t_l < t_{l+1}$, $l = 1, 2, \ldots$, with $\lim_{l \to \infty} t_l = \infty$.

With the same approach, first certain conditions that guarantee the existence of solution is outlined. Then with the existence of zero solution, the Generalized Mittag-Leffler stability condition defined below is verified.

**Definition 3.2.** The zero solution of Hilfer fractional instantaneous impulsive differential system (3.2) is called Generalized Mittag-Leffler stable if

$$ ||x(t)|| \leq N \left(t_0 I^{1-\lambda}_t \|x_0\|^a b \right) \sum_{l=1}^{l} \left( \prod_{i=1}^{l}(t_i - t_{i-1})^{\lambda-1}(t - t_i)^{\lambda-1} \right) \frac{1}{\gamma} \left( - \gamma(t - t_i)^{\mu} \right) E_{\mu, \lambda} \left( - \gamma(t - t_i)^{\mu} \right), \quad t \in (t_i, t_{i+1}), \quad l = 1, 2, \ldots.$$  

As a special case, for $l = 0$, $t \in (t_0, t_1)$, the condition is given by

$$ ||x(t)|| = \left( \frac{\beta}{\alpha} (t - t_0)^{\lambda-1} \|x_0\|^a b \right) E_{\mu, \lambda} \left( - \gamma(t - t_0)^{\mu} \right), \quad t \in (t_0, t_1).$$

For the impulsive points,

$$ ||x(t)|| \leq N \left( \prod_{i=1}^{l}(t_i - t_{i-1})^{\lambda-1}(t_i - t_{i-1})^{\lambda-1} \right) \frac{1}{\gamma} \left( - \gamma(t_i - t_{i-1})^{\mu} \right) E_{\mu, \lambda} \left( - \gamma(t_i - t_{i-1})^{\mu} \right), \quad t = t_l, \quad l = 1, 2, \ldots,$$

where $x(t) = x(t; t_0, x_0)$, $t_0 \in [0, \infty) \setminus \{t_l\}$ is the initial time for $l = 1, 2, \ldots$, $\mu \in (0, 1)$, $\lambda \in (0, 1)$, $\alpha$, $\beta$, $\gamma$ and $N$ are positive constants.

**Remark 3.1.**

1. For non-impulsive conditions the Definition (4) of [18] is a particular case of the Definition (3.2).

2. If the zero solution of Hilfer fractional differential equation is Generalized Mittag-Leffler stable with respect to both non-instantaneous and instantaneous impulses, then for $t_0 \leq \xi_1 < \xi_2 < \xi$ and for $t \geq t_0$,

$$ ||x(t)|| \leq N \left( t_0 I^{1-\lambda}_t \|x_0\|^a b \right) E_{\mu, \lambda} \left( - \gamma(t - t_0)^{\mu} \right) \frac{1}{\gamma} \left( - \gamma(t - t_0)^{\mu} \right) \leq E_{\mu, \lambda} \left( - \gamma(t - t_0)^{\mu} \right)$$

as the inequality satisfies the condition,

$$ E_{\mu, \lambda} \left( - \gamma(\xi_2 - \xi_1)^{\mu} \right) E_{\mu, \lambda} \left( - \gamma(\xi - \xi_2)^{\mu} \right) \leq E_{\mu, \lambda} \left( - \gamma(\xi - \xi_1)^{\mu} \right)$$

3. The Mittag-Leffler stability analysis for the Caputo fractional differential equation has been discussed extensively by Agarwal, Hristova and O’Regan. In particular, the complete analysis of stability conditions of Impulsive Caputo fractional system is given in [2, 3]. As Hilfer fractional order system finds itself strong in many research articles, similar analysis for Hilfer fractional order is a prospective one.

4. As stated in [12], Mittag-Leffler stability and generalized Mittag-Leffler stability imply asymptotic stability.

The following Lemma will be used in the study of Generalized Mittag-Leffler stability for both the impulsive systems.

**Lemma 3.1.** Assume that

1. $g(t, 0) = 0$ for $t \geq 0$.

2. $V(t, x(t))$ be a continuously differentiable function defined by

$$ V(t, x(t)) : \mathbb{R}_+ \times \Lambda \to \mathbb{R}_+, \quad \Lambda \in \mathbb{R}^n, \quad 0 \in \Lambda.$$
(3) \( V(t, x(t)) \) is locally Lipschitz with respect to the second variable \( x \).

(4) \( V(t, 0) = 0 \) for \( t \in \mathbb{R}_+ \).

(5) \( \alpha \|x\|^{a} \leq V(t, x(t)) \leq \beta \|x\|^{ab} \), for \( t \geq \tau \), \( x \in \Lambda \).

\( \tau D^{\mu,\nu}_t V(t, x(t)) \leq -\gamma \|x\|^{ab} \), for \( t \in [\tau, t_{m+1}] \).

holds for \( \tau \in [p_m, t_{m+1}] \), \( m \geq 0 \), \( m \in \mathbb{N} \), where \( \mu \in (0, 1) \), \( \nu \in [0, 1] \), \( \alpha \), \( \beta \), \( \gamma \), \( a \), \( b \), are arbitrary positive constants, \( \tilde{x}_0 \in \Lambda \) and \( x(t) = x(t; \tau, \tilde{x}_0) \in C^{\mu,\nu}([\tau, t_{m+1}], \Lambda) \) is a solution of Hilfer fractional impulsive differential system \((4.1)\).

Then

\[
V(t, x(t)) \leq \tau I_t^{1-\lambda} V(\tau, x(\tau)) (t - \tau)^{\lambda-1} E_{\mu,\lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^{\mu} \right), \quad t \in [\tau, t_{m+1}]
\]

and

\[
\|x(t; \tau, \tilde{x}_0)\| \leq \left[ \frac{\beta}{\alpha} (t - \tau)^{\lambda-1} \|\tilde{x}_0\|^{ab} E_{\mu,\lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^{\mu} \right) \right]^{\frac{1}{\alpha}}, \quad t \in [\tau, t_{m+1}].
\]

Proof. From the conditions 5-(i) and 5-(ii) given, it follows that

\[
\tau D^{\mu,\nu}_t V(t, x(t)) \leq -\gamma \|x\|^{ab}. \quad t \in [\tau, t_{m+1}].
\]

Combining both the above inequality results in the inequality

\[
\tau D^{\mu,\nu}_t V(t, x(t)) \leq -\frac{\gamma}{\beta} V(t, x(t)), \quad t \in [\tau, t_{m+1}].
\]

There exists a function \( W(t) \in C([\tau, p_m], [0, \infty)) \) such that

\[
\tau D^{\mu,\nu}_t V(t, x(t)) + W(t) = -\frac{\gamma}{\beta} V(t, x(t)), \quad t \in [\tau, t_{m+1}]. \tag{3.3}
\]

Taking the Laplace transform of the above equation for \( t \in [\tau, t_{m+1}] \) gives

\[
s^{\mu} V(s) - s^{\nu(\mu-1)} [\tau I_t^{1-\lambda} V(\tau, x(\tau))] + W(s) = -\frac{\gamma}{\beta} V(s),
\]

where \( V(s) = L[V(t, x(t)], W(s) = L[W(t)] \). Further simplification leads to

\[
V(s) = \frac{s^{\nu(\mu-1)} [\tau I_t^{1-\lambda} V(\tau, x(\tau))] - W(s)}{s^{\mu} + \frac{\gamma}{\beta}}.
\]

If \( x(\tau) = 0 \), then \( \tau I_t^{1-\lambda} V(\tau, x(\tau)) = 0 \) and the solution to \((4.1)\) becomes zero. If \( x(\tau) = 0 \), then \( \tau I_t^{1-\lambda} V(\tau, x(\tau)) = 0 \). As \( V(t, x(t)) \) is locally Lipschitz with respect to the second term, from the existence and uniqueness theorem [12, Theorem 3.4] and inverse Laplace transform, a unique solution of \((3.3)\) exists and is given as

\[
V(t, x(t)) = \left[ \tau I_t^{1-\lambda} V(\tau, x(\tau)) \right] (t - \tau)^{\lambda-1} E_{\mu,\lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^{\mu} \right)
- W(t) \ast \left[ (t - \tau)^{\lambda-1} E_{\mu,\mu} \left( \frac{-\gamma}{\beta} (t - \tau)^{\mu} \right) \right].
\]

Since \( (t - \tau)^{\lambda-1} \geq 0 \) and \( E_{\mu,\mu} \left( \frac{\gamma}{\beta} (t - \tau)^{\mu} \right) \geq 0 \), it follows that

\[
V(t, x(t)) \leq \left[ \tau I_t^{1-\lambda} V(\tau, x(\tau)) \right] (t - \tau)^{\lambda-1} E_{\mu,\lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^{\mu} \right).
\]
As, \( V(\tau, x(\tau)) \leq \beta \|x(\tau)\|^a \) it implies that
\[
V(t, x(t)) \leq \left[ xI^{1-\lambda}_t \|x(\tau)\|^a \beta(t - \tau)^{\lambda - 1} E_{\mu, \lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^\mu \right) \right].
\]
But from condition 5-(i), it can be concluded that
\[
\alpha \|x\|^a \leq \left[ xI^{1-\lambda}_t \|x(\tau)\|^a \beta(t - \tau)^{\lambda - 1} E_{\mu, \lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^\mu \right) \right]
\implies \|x(t; \tau, \hat{x}_0)\| \leq \left[ xI^{1-\lambda}_t \|x(\tau)\|^a \beta(t - \tau)^{\lambda - 1} E_{\mu, \lambda} \left( \frac{-\gamma}{\beta} (t - \tau)^\mu \right) \right]^\frac{1}{\alpha}.
\]
This completes the proof of the Lemma.

\[ \square \]

4. Stability results for non-instantaneous impulsive system

For an arbitrary initial value \( \tau \in [p_k, t_{k+1}) \), \( k = 0, 1, \ldots \) a general Hilfer fractional initial value problem can be described as
\[ \tau D^{\mu, \nu}_t x(t) = g(t, x(t)), \quad t \in [\tau, t_{k+1}], \]
\[ \tau I^{(1-\lambda)}_t [x(t)]_{t=\tau} = \hat{x}_0. \]
(4.1)

The solution of the (IVP) initial value problem of Hilfer fractional non-instantaneous differential system (3.1) is given by
\[
x(t) = x(t; t_0, x_0) = \left\{ \begin{array}{ll}
X_k(t), & t \in (p_k, t_{k+1}], \quad k = 0, 1, \ldots \\
\phi_k(t, x(t; t_0, x_0), X_k(t_k - 0)), & t \in (t_k, p_k], \quad k = 1, 2, \ldots 
\end{array} \right.
\]
(4.2)
On considering interval by interval, it can be noted that

**Remark 4.1.** (1) For \( t \in [t_0, t_1] \), the solution \( X_0(t) \) of the system (3.1) coincides with the solution of IVP (4.1) for \( \tau = t_0 \), \( k = 0 \) and \( \hat{x}_0 = x_0 \);
(2) For \( t \in (t_1, p_1] \), the solution of the system (3.1) satisfies the equation
\[
x(t; t_0, x_0) = \phi_1(t, x(t; t_0, x_0), X_1(t_1 - 0));
\]
(3) For \( t \in (p_1, t_2] \), the solution \( X_1(t) \) of the system (3.1) coincides with the solution of IVP (4.1) for \( \tau = p_1 \), \( k = 1 \) and \( \hat{x}_0 = \phi_1(p_1, x(p_1; t_0, x_0), X_1(t_1 - 0)) \);
(4) For \( t \in (t_2, p_2] \), the solution of the system (3.1) satisfies the equation
\[
x(t; t_0, x_0) = \phi_2(t, x(t; t_0, x_0), X_2(t_2 - 0));
\]
(5) For \( t \in (p_2, t_3] \) the solution \( X_2(t) \) of the system (3.1) coincides with the solution of IVP (4.1) for \( \tau = p_2 \), \( k = 2 \) and \( \hat{x}_0 = \phi_2(p_2, x(p_2; t_0, x_0), X_2(t_2 - 0)) \);
and so on.

In general, the solution \( x(t; t_0, x_0) \), \( t \geq t_0 \) satisfies the integral equation.
\[
x(t) = \left\{ \begin{array}{ll}
t^{\lambda - 1} x_0 + \frac{1}{\Gamma(\lambda)} \int_{t_0}^{t} (t - s)^{\mu - 1} f(s, x(s))ds, & t \in [t_0, t_1], \\
\phi_k(t, x(t; t_0, x_0), x(t_k - 0; t_0, x_0)), & t \in (t_k, p_k], \quad k = 1, 2, \ldots, \\
\phi_k(p_k, x(p_k; t_0, x_0), x(t_k - 0; t_0, x_0)) + \frac{1}{\Gamma(\mu)} \int_{t_k}^{t} (t - s)^{\mu - 1} f(s, x(s))ds, & t \in (p_k, t_{k+1}], \quad k = 1, 2, \ldots.
\end{array} \right.
\]

The following conditions are assumed to guarantee the existence of solution \( x(t; t_0, x_0) \) of Hilfer fractional non-instantaneous differential equation:
**Condition 1.** The function $g \in C \left( \left[ 0, t_1 \right] \cup \bigcup_{k=1}^{\infty} [p_k, t_{k+1}] \times \mathbb{R}^n, \mathbb{R}^n \right)$ for $t \in (0, t_1) \cup \bigcup_{k=1}^{\infty} (p_k, t_{k+1}]$ with $g(t, 0) = 0$ is such that for any initial point $(\hat{t}_0, \hat{x}_0) \in (0, t_1) \cup \bigcup_{k=1}^{\infty} (p_k, t_{k+1}] \times \mathbb{R}^n$, the IVP for general Hilfer fractional differential system (4.1) with $\tau = t_0$ has a solution $x(t; \hat{t}_0, \hat{x}_0) \in C^{n,\nu} \left( [\hat{t}_0, \hat{t}_{m+1}], \mathbb{R}^n \right)$, where $m = \min \{ k : \hat{t}_0 < t_{k+1} \}$.

**Condition 2.** The function $\phi_k : [t_k, p_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ for any $k = 1, 2, \ldots$ are such that the equation $x = \phi_k(t, x, z)$ has a solution $x = \zeta_k(t, z)$, $t \in [t_k, p_k]$. The function $\zeta_k$ is defined as $\zeta_k \in C ([t_k, p_k] \times \mathbb{R}^n, \mathbb{R}^n)$, with $\zeta_k(t, 0) = 0$ for $t \in [t_k, p_k]$, $k = 1, 2, \ldots$.

The following theorem provides the condition for the zero solution of Hilfer fractional system with non-instantaneous impulses to satisfy the Mittag-Leffler stable condition:

**Theorem 4.1.** Let the assumed conditions [1] and [2] hold. $\Delta \in \mathbb{R}^n$; $0 \in \Delta$. Further, let the Lyapunov function $V(t, x(t))$ be a continuously differentiable function defined by

$$V(t, x(t)) : \mathbb{R}_+ \times \Delta \to \mathbb{R}_+$$

and locally Lipschitz with respect to the second variable along with $V(t, 0) = 0$ for $t \geq 0$, such that

1. For $t \geq 0$, $x \in \mathbb{R}^n$,

$$\alpha \|x\|_a \leq V(t, x(t)) \leq \beta \|x\|^{ab},$$

where $\alpha$, $\beta$, $a$, $b$ are positive constants, with $\beta \leq 1$.

2. For any $\tau \in (t_0, t_1) \cup \bigcup_{k=0}^{\infty} [p_k, t_k]$ and any solution $x(t) \in C^{n,\nu} ([\tau, t_{m+1}], \mathbb{R}^n)$ of fractional system (4.1), the inequality

$$\tau D^{\mu,\nu}_t V(t, x(t)) \leq \gamma \|x\|^{ab}, \quad t \in (\tau, t_{m+1})$$

holds, where $m = \min \{ k : \tau < p_k \}$, $\mu \in (0, 1)$, $\nu \in [0, 1]$, $\lambda \in [0, 1]$, and $\gamma > 0$.

3. For any $k = 0, 1, 2, \ldots$, the inequality

$$V(t, \zeta_k(t, x)) \leq \sigma \|x\|^a,$$

for, $t \in (t_k, p_k)$, $x \in \mathbb{R}^n$

holds, where, $\sigma$ is a positive constant such that $\sigma \leq \alpha$.

Then the zero solution of Hilfer fractional differential equation (3.1) is Generalized Mittag-Leffler stable with respect to non-instantaneous impulses.

**Proof.** Let the arbitrary initial time be $t_0$, such that $t_0 \in (0, t_1) \cup \bigcup_{k=0}^{\infty} [t_k, p_k]$. With no loss of generality, let the initial time be assumed as $t_0 \in (0, t_1)$. For the arbitrary initial point $x_0 \in \mathbb{R}^n$, the solution of Hilfer fractional impulsive system (3.1) is considered as $x(t; t_0, x_0)$. The stability is proved at first for few initial intervals and then extended to the entire interval under consideration.

**Step 1:** For the interval $t \in [t_0, t_1]$:

The solution $X_0(t)$ coincides with the solution of the general Hilfer impulsive problem (1.1). Here $\tau = t_0$; $k = 0$, $\tilde{x}_0 = x_0$.

According to Lemma 3.1, the solution can be written as,

$$\|x(t; t_0, x_0)\| \leq \left[ \int_{t_0}^t (t - s)^{1-\lambda} \|x_0\|^{ab} \frac{\beta}{\alpha} (t - s)^{\lambda-1} E_{\mu,\lambda} \left( -\frac{\gamma}{\beta} (t - s)^{\mu} \right) \right]^\frac{1}{2}.$$  

Since $\beta \leq 1$,

$$\|x(t; t_0, x_0)\| \leq \left[ \int_{t_0}^t (t - s)^{1-\lambda} \|x_0\|^{ab} \frac{\beta}{\alpha} (t - s)^{\lambda-1} E_{\mu,\lambda} (-\gamma (t - s)^{\mu}) \right]^\frac{1}{2}.$$  

(4.6)
Hence, it can be concluded that
\[ \alpha \| x(t; t_0, x_0) \|^a \leq V(t, x(t; t_0, x_0)). \]

From the Remark 4.1, the solution takes the form \( x(t; t_0, x_0) = \phi_1(t, x(t_0, x_0), X_1(t_1 - 0)). \)

But, according to the condition 2, the equation \( x(t) = \phi_1(t, x, x(t)) \) has a solution \( x = \zeta_1(t, x(t_1)) \) for the interval \( (t_1, p_1) \).

Employing (4.6), results in
\[ \alpha \| x(t; t_0, x_0) \|^a \leq V(t, x(t, t_0, x_0)) = V(t, \zeta_1(t, x(t_1)), t_0, x_0). \]

Based on condition (4.5), it follows that
\[ V(t, \zeta_1(t, x(t_1)); t_0, x_0) \leq \sigma \| x(t_1 - 0; t_0, x_0) \|^a. \]

As a result, it is deduced as
\[ \alpha \| x(t; t_0, x_0) \|^a \leq \sigma \| x(t_1 - 0; t_0, x_0) \|^a. \]

Employing (4.6), results in
\[ \alpha \| x(t; t_0, x_0) \|^a \leq \sigma \mu \frac{1}{\alpha} (1 - t_0)^{\mu - 1} E_{\mu, \lambda} \left( -\frac{\gamma}{\beta} (t_1 - t_0)^{\mu} \right). \]

Hence, it can be concluded that
\[ \| x(t; t_0, x_0) \| \leq \left[ \mu \frac{1}{\alpha} (1 - t_0)^{\mu - 1} E_{\mu, \lambda} \left( -\frac{\gamma}{\beta} (t_1 - t_0)^{\mu} \right) \right]^\frac{1}{\mu} . \]

**Step 2:** For the interval \( t \in (t_1, p_1) \):

In accordance with the solution (4.2) of IVP for the system (3.1), \( t = t_1 - 0, k = 1 \). From the given condition (4.3), it follows that,
\[ a \| x(t; t_0, x_0) \|^a \leq V(t, x(t_0, x_0)). \]

Hence, it can be concluded that
\[ \alpha \| x(t; t_0, x_0) \|^a \leq V(t, x(t; t_0, x_0)) = V(t, \zeta_1(t, x(t_1)); t_0, x_0). \]

As a result, it is deduced as
\[ \alpha \| x(t; t_0, x_0) \|^a \leq \sigma \| x(t_1 - 0; t_0, x_0) \|^a. \]

Employing (4.6), results in
\[ \alpha \| x(t; t_0, x_0) \|^a \leq \sigma \mu \frac{1}{\alpha} (1 - t_0)^{\mu - 1} E_{\mu, \lambda} \left( -\frac{\gamma}{\beta} (t_1 - t_0)^{\mu} \right). \]

**Step 3:** For the interval \( t \in (p_1, t_2) \):

The solution for \( t \in (p_1, t_2) \) according to (4.1) is given by \( x(t; t_0, x_0) = X_1(t) \). From conditions (4.3) and (4.4), it can be deduced that
\[ \tau D_t^{\alpha\nu} V(t, x(t)) \leq \gamma \| X_1(t) \|^a \leq \frac{-\gamma}{\beta} V(t, X_1(t)). \]

Here \( \tau = p_1, k = 1, \gamma = \frac{\gamma}{\beta} \) and \( x_0 = x(t_1; t_0, x_0) = \phi_1(p_1, x(p_1; t_0, x_0), X_1(t_1 - 0)) = X_1(t_1). \)

From Lemma 3.1, it can be observed that,
\[ V(t, X_1(t)) \leq [p_1 \mu^{\lambda - 1} E_{\mu, \lambda} \left( -\frac{\gamma}{\beta} (t - p_1)^{\mu} \right)]. \]

But,
\[ V(p_1, X_1(p_1)) = V(p_1, x(t_1; t_0, x_0)) \]
\[ = V(p_1, \phi_1(p_1, x(p_1; t_0, x_0), X_1(t_1 - 0))) \]
\[ = V(t_1, \zeta_1(p_1, x(t_1 - 0); t_0, x_0)). \]

Utilizing condition (4.5) and \( \beta \leq 1 \), gives
\[ V(p_1, X_1(p_1)) \leq \sigma \| x(t_1 - 0; t_0, x_0) \|^a \]
\[ \leq \sigma \mu \frac{1}{\alpha} (1 - t_0)^{\mu - 1} E_{\mu, \lambda} \left( -\frac{\gamma}{\beta} (t_1 - t_0)^{\mu} \right). \]
and hence it can be concluded that
\[
V(t, X_1(t)) \leq \beta \left(t_0 I_{t_1}^{1-\lambda} \left[p_1 I_{t_1}^{1-\lambda} V(t_1, X_1(t_1))\right] \right)(t_1 - t_0)^{\lambda-1}(t - p_1)^{\lambda-1}
\]

Now employing condition (4.3), results in
\[
\|x(t; t_0, x_0)\| \leq \left[\frac{\beta}{\alpha} \left(t_0 I_{t_1}^{1-\lambda} \left[p_1 I_{t_1}^{1-\lambda} \|x_0\|^a b\right] \right)(t_1 - t_0)^{\lambda-1}(t - p_1)^{\lambda-1}
\]
\[
E_{\mu,\lambda} (-\gamma(t_1 - t_0)^\mu) E_{\mu,\lambda} (-\gamma(t - p_1)^\mu) \right]^{\frac{1}{\lambda}}.
\]

Step 4: For the interval \( t \in (t_2, p_2) \):
With the similar procedure it can be deduced that
\[
\|x(t; t_0, x_0)\| \leq \left[\frac{\beta}{\alpha} \left(t_0 I_{t_1}^{1-\lambda} \left[p_1 I_{t_1}^{1-\lambda} \|x_0\|^a b\right] \right)(t_1 - t_0)^{\lambda-1}(t - p_1)^{\lambda-1}
\]
\[
E_{\mu,\lambda} (-\gamma(t_1 - t_0)^\mu) E_{\mu,\lambda} (-\gamma(t - p_1)^\mu) \right]^{\frac{1}{\lambda}}.
\]

Extending this procedure for further intervals confirms that the zero solution of the given system (3.1) is Mittag-Leffler stable. \( \square \)

5. Stability results for Hilfer fractional instantaneous impulsive system

The solution of the instantaneous impulsive Hilfer fractional system (3.2) is given by
\[
x(t) = x(t; t_0, x_0)
\]
\[
= \begin{cases}
X_l(t), & t \in (t_l, t_{l+1}), \ l = 0, 1, \ldots \\
\psi_l(x_l(t_l - 0)), & t = t_l, \ l = 1, 2, \ldots
\end{cases}
\]

The above solution \( X_l(t) \) is the solution of the IVP (4.1), for \( \tau = t_l \) and
\[
t_l I_{t_l}^{(1-\lambda)}[x(t)]_{t=t_l} = \hat{x}_0 = \psi_l(x(t_l - 0)).
\]

In general, the solution \( x(t; t_0, x_0), t > t_0 \) satisfies the integral equation
\[
x(t) = \begin{cases}
\psi_l(x_l(t_l - 0); t_0, x_0) \\
+ \frac{1}{\Gamma(\mu)} \int_{t_0}^t (t - s)^{\mu-1} g(s, x(s; t_0, x_0)) ds, \ \text{for} \ l = 0, 1, \ldots
\end{cases}
\]

Analogous to the conditions for non-instantaneous impulsive system, the following conditions are assumed for the existence of the zero solution for the instantaneous system (3.2).

**Condition 3.** The function \( g \in C \left([0, \infty) / \{t_l\} \times \mathbb{R}^n, \mathbb{R}^n\right) \) for \( t \neq t_l \) with \( g(t, 0) = 0 \) is such that for any initial point \((\hat{t}_0, \hat{x}_0) \in [0, \infty) / \{t_l\} \times \mathbb{R}^n\), the IVP for general Hilfer fractional differential system (4.1) with \( \tau = \hat{t}_0 \) has a solution \( x(t; \hat{t}_0, \hat{x}_0) \in C^{\mu,\nu} \left([\hat{t}_0, t_{m+1}], \mathbb{R}^n\right) \), where \( m = \min \{l : \hat{t}_0 < t_{l+1}\} \).

**Condition 4.** The function \( \psi_l : \mathbb{R}^n \to \mathbb{R}^n \) for any \( l = 0, 1, \ldots \) with \( \psi_l(0) \equiv 0 \).
The following theorem provides the condition for the zero solution of Hilfer fractional system with instantaneous impulses to satisfy the Mittag-Leffler stable condition:

**Theorem 5.1.** Let the assumed conditions \(3\) and \(4\) hold. \(\Delta \in \mathbb{R}^n; 0 \in \Delta\). Further let the Lyapunov function \(V(t, x(t))\) be continuously differentiable function defined by

\[
V(t, x(t)) : \mathbb{R}_+ \times \Delta \rightarrow \mathbb{R}_+ 
\]

and locally Lipschitz with respect to the second variable along with \(V(t, 0) = 0\) for \(t \geq 0\), such that

1. For \(t \geq 0\), \(x \in \Delta\),
   \[
   \alpha \|x\|^a \leq V(t, x(t)) \leq \beta \|x\|^b, \quad (5.2)
   \]
   where \(\alpha, \beta, a, b\) are positive constants, with \(\beta \leq 1\).
2. For any \(\tau \neq t_i\), \(l = 0, 1, \ldots\) and any solution \(x(t) \in C^{\mu,\nu}([\tau, t_m], \mathbb{R}^n)\) of fractional system \((1.1)\), the inequality
   \[
   \beta D_t^{\mu,\nu} V(t, x(t)) \leq \gamma \|x\|^b, \quad t \in (\tau, t_m) \quad (5.3)
   \]
   holds, where \(m = \min\{l : \tau < t_l\}, \mu \in (0, 1), \nu \in [0, 1], \lambda \in [0, 1] \) and \(\gamma > 0\).
3. For any \(l = 1, 2, \ldots\), the inequality
   \[
   V(t_l, \psi_l(x(t))) \leq \sigma \|x\|^a, \quad x \in \mathbb{R}^n \quad (5.4)
   \]
   holds, where, \(\sigma\) is a positive constant such that \(\sigma \leq \alpha\).

Then the zero solution of Hilfer fractional differential equation \((3.2)\) is Generalized Mittag-Leffler stable with respect to instantaneous impulses.

**Proof.** Let the arbitrary initial time with no loss of generality be assumed as \(t_0 \in [0, t_1)\). For the arbitrary initial point \(x_0 \in \mathbb{R}^n\), with the initial time \(t_0\), the solution is given by \(x(t; t_0, x_0)\). As in the previous theorem for non-instantaneous impulsive system, the proof of the theorem is carried out interval by interval and using induction it is extended to a general interval.

**Step 1:** For the interval \(t \in (t_0, t_1)\):

The solution \(X_0(t)\) of \((3.2)\) coincides with the solution of the general Hilfer impulsive problem \((4.1)\). Here \(\tau = t_0, k = 0, \bar{x}_0 = x_0\).

Based on Lemma \(3.1\), the solution can be written as,

\[
\|x(t; t_0, x_0)\| \leq \left[ t_0 \int_{t_0}^{t} \|x_0\|^{ab} \frac{\beta}{\alpha} (t - t_0)^{\lambda - 1} \mathcal{E}_{\mu,\lambda} (-\gamma (t - t_0)^{\mu}) \right]^{\frac{1}{a}}. \quad (5.5)
\]

**Step 2:** For the interval \(t = t_1\):

For \(t = t_1\), from the solution given in \((5.1)\), \(x(t; t_0, x_0) = \psi_1(x_1(t_1 - 0))\). Using the condition \((5.2)\) and \((5.4)\), it follows that,

\[
\alpha \|x(t_1; t_0, x_0)\|^a \leq V(t_1, x(t_1; t_0, x_0)) = V(t_1, \psi_1(x_1(t_1 - 0))) \leq \sigma \|x(t_1; t_0, x_0)\|^a.
\]

As a result, it is deduced that

\[
\alpha \|x(t; t_0, x_0)\|^a \leq \sigma \|x(t_1 - 0; t_0, x_0)\|^a.
\]
Applying (5.5) results in
\[ \| x(t; t_0, x_0) \| \leq \left[ \int_{t_0}^{t} \| x_0 \|^{\beta \mu} \right]^{\frac{1}{\alpha}} (t_1 - t_0)^{\lambda-1} E_{\mu, \lambda} (-\gamma(t_1 - t_0)\mu) \].

**Step 3:** For the interval \( t \in (t_1, t_2) \):
The solution for \( t \in (t_1, t_2) \) according to (5.1) is given by \( x(t; t_0, x_0) = X_1(t) \). From conditions (5.2) and (5.3), it can be deduced that
\[ \tau \int_{t_0}^{t} V(t, x(t)) \leq \gamma \| X_1(t) \|^{\beta \mu} \leq -\frac{\gamma}{\beta} V(t, X_1(t)). \]

Here \( \tau = t_1 \), \( \beta = \frac{\gamma}{\beta} \), and \( \tilde{x}_0 = x(t_1; t_0, x_0) = \psi_1(x_1(t_1 - 0)) = X_1(t_1) \).
From Lemma 3.1 it can be observed that,
\[ V(t, X_1(t)) \leq [\mu, I_{t_1}^{1-\lambda} V(t_1, X_1(t_1))](t_1 - t_1)^{\lambda-1} E_{\mu, \lambda} \left( -\frac{\gamma}{(\beta)^2} (t_1 - t_1)^{\mu} \right). \]
But,
\[ V(t_1, X_1(t_1)) = V(t_1, x(t_1; t_0, x_0)) = V(t_1, \phi_1(x_1(t_1 - 0))). \]
By means of condition (5.4) and as \( \beta \leq 1 \), gives
\[ V(t_1, X_1(p_1)) \leq \sigma \| x(t_1 - 0; t_0, x_0) \|^{\mu} \leq \sigma \left[ \| x(0; t_0, x_0) \|^{\beta \mu} \right]^{\frac{1}{\alpha}} (t_1 - t_0)^{\lambda-1} E_{\mu, \lambda} (-\gamma(t_1 - t_0)\mu) \]
and hence it can be concluded that,
\[ V(t, X_1(t)) \leq \mu \int_{t_0}^{t} \left[ \mu, I_{t_1}^{1-\lambda} V(t_1, X_1(t_1)) \right](t_1 - t_1)^{\lambda-1} E_{\mu, \lambda} (-\gamma(t_1 - t_0)\mu) E_{\mu, \lambda} (-\gamma(t_1 - t_1)\mu). \]

Now employing condition (5.2) results in
\[ \| x(t; t_0, x_0) \| \leq \left[ \frac{\beta}{\alpha} \left( t_0 \mu, I_{t_1}^{1-\lambda} \| x_0 \|^{\beta \mu} \right) (t_1 - t_0)^{\lambda-1} (t_1 - t_1)^{\lambda-1} E_{\mu, \lambda} (-\gamma(t_1 - t_0)\mu) E_{\mu, \lambda} (-\gamma(t_1 - t_1)\mu) \right]^{\frac{1}{\alpha}}. \]

**Step 4:** For the interval \( t = t_2 \):
With the similar procedure it can be deduced that
\[ \| x(t; t_0, x_0) \| \leq \left[ \frac{\beta}{\alpha} \left( t_0 \mu, I_{t_1}^{1-\lambda} \| x_0 \|^{\beta \mu} \right) (t_1 - t_0)^{\lambda-1} (t_1 - t_1)^{\lambda-1} E_{\mu, \lambda} (-\gamma(t_1 - t_0)\mu) E_{\mu, \lambda} (-\gamma(t_1 - t_1)\mu) \right]^{\frac{1}{\alpha}}. \]
Extending the procedure inductively confirms that the zero solution of the given system (3.2) is Mittag-Leffler stable. □
Example 1. Consider the initial value problem for Hilfer fractional instantaneous impulsive system with
\[
\begin{align*}
\partial_t^{\mu,\nu} x(t) &= -x, & t \neq t_l, & l = 0, 1, \ldots \\
x(t_l) &= ax, & a \neq 0, & t = t_l, l = 1, 2, \ldots \\
I_{t_0^+}^{(1-\lambda)}[x(t)]|_{t=0} &= 0.5.
\end{align*}
\]
(5.6)
The solution of the above problem are calculated using the Definition (3.2). For values with \(a = 2, x_0 = 0.5, \mu = 0.3\), the stability graph is plotted for different values of \(\nu\). The resulting graph illustrates the jumps in the impulsive points \(t_l = 0, 1, \ldots\) Further, when \(\nu\) approaches zero, the solution is approaching zero swiftly can be observed.

![Graphs showing stability with different \(\nu\) values](image)

**Figure 1**

![Graphs showing stability with different \(\nu\) values](image)

**Figure 2**

Note that, if \(\nu = 1\), the graph coincides with the Figure 2 by Agarwal et al [2].

6. CONCLUDING REMARKS

Mittag-Leffler stability condition for systems with both instantaneous impulses and non-instantaneous impulses having Hilfer fractional order is discussed in detail. By varying the value of \(\nu\), we can interpolate the results on the stability of the solution of the system 3.1 and 3.2 between Caputo and Riemann-Liouville fractional operators.

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