In this article we propose the calculation of the unconditional Wiener measure functional integral with a term of the fourth order in the exponent by an alternative method as in the conventional perturbative approach. In contrast to the conventional perturbation theory, we expand into power series the term linear in the integration variable in the exponent. In such a case we can profit from the representation of the integral in question by the parabolic cylinder functions. We show that in such a case the series expansions are uniformly convergent and we find recurrence relations for the Wiener functional integral in the $N$-dimensional approximation. In continuum limit we find that the generalized Gelfand - Yaglom differential equation with solution yields the desired functional integral (similarly as the standard Gelfand - Yaglom differential equation yields the functional integral for linear harmonic oscillator).

I. INTRODUCTION

We will define the continuum functional integral as the limit of a finite dimensional integral. This finite dimensional integral is derived from the continuum one by the time-slicing method. We avoid so problems with continuum integral measure because we consider continuum limit of the result of a finite dimensional integral. In order to evaluate finite dimensional integrals we must first solve the problem of the calculation of one dimensional integral:

$$ I_1 = \int_{-\infty}^{+\infty} dx \exp\{-Ax^4 + Bx^2 + Cx\}, $$

where $Re\, A > 0$. The exact analytical result for this integral is not known yet, therefore one uses approximative methods of calculation.

The usual perturbative approach is based on Taylor’s decomposition of the fourth order term with consecutive replacements of the order of integration and summation:

$$ I_1 = \sum_{n=0}^{\infty} \frac{(-A)^n}{n!} \int_{-\infty}^{+\infty} dx \, x^{4n} \exp\{-Bx^2 + Cx\} $$

The above integrals can be calculated, but their sum is divergent.

However, $I_1 = I_1(A, B, C)$ is an entire function for any complex values of $B$ and $C$, since there exist all integrals

$$ \partial_C^n \partial_B^m I_1(A, B, C) = (-1)^{n+m} \int_{-\infty}^{+\infty} dx \, x^{2m+n} \exp\{-Ax^4 + Bx^2 + Cx\}, \, Re\, A > 0. $$

Consequently, the power series expansions of $I_1 = I_1(A, B, C)$ in $C$ and/or $B$ have an infinite radius of convergence (and in particular they are uniformly convergent on any compact set of values of $C$ and/or $B$). We shall frequently
use the power series expansion in $C$:

$$I_1 = \sum_{n=0}^{\infty} \left( \frac{-C}{n!} \right)^n \int_{-\infty}^{+\infty} dx \, x^n \exp\{-(Ax^4 + Bx^2)\}$$

(3)

The similar idea of the first order term expansions in the fourth order action was used by Tuszynski et al.\[1\] for an evaluation of a non-Gaussian models for critical fluctuations of the Landau-Ginsburg model of phase transitions.

The integral in (3) can be expressed in terms of the parabolic cylinder function $D_{\nu}(z)$, (see, for instance, [2]). For $n$ odd, due to symmetry of the integrand the integrals are zero, for $n$ even, $n = 2m$ we have:

$$\frac{e^{z^2/4}}{(\sqrt{2A})^{m+1/2}} \Gamma(m + 1/2) \, D_{-m-1/2}(z) = \int_{0}^{\infty} dy \, y^{m-1/2} \exp\{-Ay^2 - By\},$$

where

$$z = \frac{B}{\sqrt{2A}}.$$

Explicitly, for the Eq. (3) we have:

$$I_1 = e^{z^2/4} \, \frac{\Gamma(1/2)}{(2A)^{1/4}} \sum_{m=0}^{\infty} \frac{(\xi)^m}{m!} D_{-m-1/2}(z) , \quad \xi = \frac{C^2}{4\sqrt{2A}}.$$

(4)

This sum is convergent for any values of $C$, $B$ and $A$ positive.

The convergence of the infinite series in Eq. (4) can be shown as follows. For $|z|$ finite, $|z| < \sqrt{|\nu|}$ and $|\arg(-\nu)| \leq \pi/2$ and if $|\nu| \to \infty$, the following asymptotic relation is valid [3]:

$$D_{\nu}(z) = \frac{1}{\sqrt{2}} \exp\left[\frac{\nu}{2} (\ln(-\nu) - 1) - \sqrt{-\nu} \, z\right] \left[1 + O\left(\frac{1}{\sqrt{|\nu|}}\right)\right].$$

(5)

The $m$ term of the sum in Eq. (4) possesses the asymptotic:

$$\frac{1}{m!} \exp\left[\frac{-(m+1/2)}{2} (\ln(m+1/2) - 1) - \sqrt{(m+1/2)} \, z + m \ln \xi\right].$$

(6)

This means, applying the Bolzano-Cauchy criterium that the sum in Eq. (4) is not only absolutely, but uniformly convergent for the finite values of the constants of the integral [1].

II. EVALUATION OF THE FUNCTIONAL INTEGRAL BY TIME SLICING METHOD

We suppose that Gaussian integration over momenta is done. Our aim is to evaluate the continuum unconditional Wiener measure functional integral:

$$Z = \int [D\varphi(x)] \exp(-S) ,$$

where the continuum action contains the fourth-order term:

$$S = \int_0^{\beta} d\tau \left[ c/2 \left( \frac{\partial \varphi(\tau)}{\partial \tau} \right)^2 + b\varphi(\tau)^2 + a\varphi(\tau)^4 \right].$$

(7)

The functional integral $Z$ is defined by a limiting procedure from the finite dimensional integral $Z_N$, obtained from the continuum integral, when the infinite measure $[D\varphi(x)]$ is replaced by the finite dimensional measure $\prod_{i=1}^{N} d\varphi_i(x)$ [4]:

$$Z_N = \prod_{i=1}^{N} \left( \frac{d\varphi_i}{2\pi \triangle c} \right) \exp\left\{ -\sum_{i=1}^{N} \Delta \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\triangle} \right)^2 + b\varphi_i^2 + a\varphi_i^4 \right] \right\},$$

(8)
where $\Delta = \beta/N$. The *unconditional measure* integration is characterized by integration over variable $\varphi_N$. This is the only difference from conditional measure integration, when the $\varphi_N$ variable is fixed. The detailed discussion of the conditional Wiener measure case is technically more involved and was not in the program of this article. We evaluated this case in a rather simplified form in the Appendix 4 of our article I [3]. We included a brief discussion of this case in conclusions. The continuum unconditional Wiener measure functional integral is defined by the formal limit:

$$ Z = \lim_{N \to \infty} Z_N. $$

To evaluate this limit we follow the idea of the Gelfand-Yaglom proof of the functional integral for the harmonic oscillator [6], based on the iterative procedure for the finite dimensional representation of the functional integral. The idea of $N$ dimensional integration of (8) is explained in Appendix A, we quote here the result:

$$ Z_N = \left[2\pi(1+b\Delta^2/c)\right]^{-\frac{N}{2}} \left[2\pi(1/2+b\Delta^2/c)\right]^{-1/2} \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \prod_{i=1}^{N} \left[\frac{(\xi_i)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1}+k_i+1/2) \Delta_{-k_{i-1}-k_i-1/2} (z_i)\right], $$

where $k_0 \equiv k_N \equiv 0, \xi_1 = \xi_2 = \cdots = \xi_{N-2} = \xi = (1+b\Delta^2/c)^{-1}, \xi_{N-1} = (1+b\Delta^2/c)^{-1} \omega_0 (1/2+b\Delta^2/c)^{-1/2}$, $\xi_N = 0$, $z_1 = z_2 = \cdots = z_{N-1} = z = c(1+b\Delta^2/c)/\sqrt{2a\Delta^3}$, $z_N = c(1/2+b\Delta^2/c)/\sqrt{2a\Delta^3}$.

### III. GENERALIZED GELFAND–YAGLOM EQUATION (GGYE)

Let us rewrite the result for the $N$ dimensional integral (9) in the form:

$$ Z_N = \left[ \prod_{i=0}^{N} 2(1+b\Delta^2/c)\omega_i \right]^{-\frac{1}{2}} S_N $$

(10)

with

$$ S_N = \sum_{k_1,\ldots,k_{N-1}=0}^{\infty} \prod_{i=1}^{N} \left[\frac{(\xi_i)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1}+k_i+1/2) \Delta_{-k_{i-1}-k_i-1/2} (z_i)\right], $$

(11)

where the constants and symbols in the above relation are connected to the constants of the model by the relations: $\omega_i = 1-A^2/\omega_{i-1}, \omega_0 = (1/2+b\Delta^2/c)/(1+b\Delta^2/c), A = \sqrt{1+b\Delta^2/c}$. We prove the above form (11) of the $N$ dimensional integral (9) later, now we use Eq. (11) for explanation of the Gelfand–Yaglom procedure of the construction of the difference equation. This difference equation is converted to differential equation in the continuum limit $N \to \infty$.

Let us define functions $F_k$ by:

$$ F_k = \frac{\prod_{i=0}^{k} 2(1+b\Delta^2/c)\omega_i}{S_k^2} $$

(12)

The function $F_k$ is defined from the relation of the $N$ dimensional integral (10) with $\Delta$ fixed and the variable $N \to k$. The quantity $Z_N$ given in Eq. (9) is related to $F_N$ as follows:

$$ Z_N = \frac{1}{\sqrt{F_N}} $$

The aim of the Gelfand-Yaglom construction is to find the continuum limit of the difference equation for the function $F_k$. Solution of this differential equation is connected to the continuum functional integral by:

$$ Z(\beta) = \frac{1}{\sqrt{F(\beta)}}, $$

where $\beta$ is the upper bound of the time interval in the action (7).
The idea of the GGYE construction is based on the recurrence form for the factor $\omega_i$. We replace $\omega_i$ by the functions $F_{k,\pm1}$ and $S_{k,\pm1}$. Pedagogical descriptions of this procedure can be found in Appendix B, there is a proof of the lemma:

**Lemma.** Let $F_k$ be the function defined by:

$$F_k = \prod_{i=0}^{k} \frac{2(1 + b\triangle^2/c)\omega_i}{S_k^2}$$  \hspace{1cm} (13)

with $\omega_i$ defined by recurrence relation:

$$\omega_i = 1 - A^2/\omega_{i-1},$$

and $\omega_0 = (1/2 + b\triangle^2/c)/(1 + b\triangle^2/c)$, $A = \frac{1}{2(1+b\triangle^2/c)}$. The constants $b, c, \triangle$ are parameters of the model.

Let in continuum limit the following condition is valid:

$$\lim_{\triangle \to 0} (\triangle O_1 + \triangle^2 O_2) = 0,$$  \hspace{1cm} (14)

where

$$O_1 = -\frac{4b}{c} F_k \left( \frac{S_{k+1} - S_k}{\triangle S_{k+1}} \right) + 2 \frac{F_k - F_{k-1}}{\triangle} \left[ \frac{b}{c} + \left( \frac{S_{k+1} - S_k}{\triangle S_{k+1}} \right)^2 \right] + 4 F_{k-1} \left( \frac{S_{k+1} - S_k}{\triangle S_{k+1}} \right) \left( \frac{S_{k+1} - 2S_k + S_{k-1}}{\triangle^2 S_{k+1}} \right)$$

$$O_2 = \frac{2b}{c} F_k \left( \frac{S_{k+1} - S_k}{\triangle S_{k+1}} \right)^2 - F_{k-1} \left( \frac{S_{k+1} - 2S_k + S_{k-1}}{\triangle^2 S_{k+1}} \right)^2.$$  \hspace{1cm} (15)

Let in the limit $\triangle \to 0$, the functions $F(\tau)$ and $S(\tau)$ are the continuum limit of the function $F_k$ and $S_k$, when we define: $\lim_{\triangle \to 0} k.\triangle = \tau$.

Then $F(\tau)$ is the solution of the differential equation:

$$\frac{\partial^2}{\partial \tau^2} F(\tau) + 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right) \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right) = F(\tau) \left( \frac{2b}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \frac{\partial}{\partial \tau} \ln S(\tau))^2 \right), \tau \in (0, \beta)$$  \hspace{1cm} (16)

with initial conditions

$$F(0) = \frac{1}{S^2(0)},$$

$$\frac{\partial}{\partial \tau} F(\tau) \big|_{\tau=0} = -\frac{\partial}{\partial \tau} \left( \frac{1}{S^2(\tau)} \right) \big|_{\tau=0}.$$

The nontrivial dynamics is hidden in the function $S(\tau)$.

**Note:** When $S(\tau)$ is known exactly the above equation can be simplified by the substitution:

$$F(\tau) = \frac{y(\tau)}{S^2(\tau)}.$$

For the new variable $y(\tau)$ we find a simple equation:

$$\frac{\partial^2}{\partial \tau^2} y(\tau) = y(\tau) \left( \frac{2b}{c} \right),$$  \hspace{1cm} (18)

accompanied by initial conditions:

$$y(0) = 1, \frac{\partial y(\tau)}{\partial \tau} \big|_{\tau=0} = 0.$$

Thus, in the case when function $S(\tau)$ is known exactly, the problem of the functional integral calculation is trivial.

Problems arise in situations, when $S(\tau)$ is known approximately, as a result of a perturbative approach. Below we define a reasonable approximation of $S(\tau)$ valid in the proximity of $\tau = 0$. But for finite (large) $\tau = \beta$ this asymptotic expansion of $S(\tau)$ does not have to be valid. However, in our case, the development of the function $F(\tau)$ from $\tau = 0$ to $\tau = \beta$ is controlled by a differential equation. Approximative knowledge of the function $S(\tau)$ leads to a more reliable result for $F(\beta)$ as the solution of Eq. (18). This philosophy of the calculation corresponds to ideas of evaluation of physical quantities be the renormalization group approach.
IV. EVALUATION OF THE FUNCTION \( S_N \)

The exact result of \( N \)–dimensional integration \(^5\) contains summations of products of parabolic cylinder functions. We were not able to find a simple formula for such a summation (although, parabolic cylinder functions belong to the representation of the group of the upper-triangle matrices, therefore, due to the group theoretical background we believe to the simplification of the product of two such functions). In what follows, we explain our approach to provide the summation over indexes \( k_i \). For a single index \( k_i \) we are dealing with the sum of the series:

\[
\sum_{k_i=0}^{\infty} a_{k_i} \cdot a_{k_i} = \frac{(\xi e^{2\lambda})^{k_i}}{(2k_i)!} \Gamma(k_i-1+k_i+1/2)D_{-k_i-1-k_i-1/2}(z) \Gamma(k_i+k_i+1+1/2)D_{-k_i-k_i+1+1/2}(z) \tag{19}
\]

The series \(^{19}\) is uniformly convergent. For asymptotic values of \( k_i \) following the Stirling formula for logarithm of gamma functions and the asymptotic relation \(^5\) for parabolic cylinder functions we find \(^5\):

\[
\ln a_{k_i} = -(k_i - \frac{k_i-1+k_i+1}{2}) \ln k_i + 2k_i(\ln(\xi z) - \ln 2 + 1/2) - 2\sqrt{k_i} z + (k_i-1+k_i+1) \ln z + 1/2 \ln \pi + z^2/2 + o(k_i^{-1/2})
\tag{20}
\]

The leading term of the above relation is

\[
\ln(a_{k_i}) \sim -k_i \ln(k_i),
\]

and

\[
a_{k_i} \sim (\xi e^{2\lambda})^{k_i} \frac{(\sqrt{k_i} z)^{k_i-1+k_i+1}}{k_i! \exp(2\sqrt{k_i} z)}.
\]

This asymptotic behavior of \( a_{k_i} \) is sufficient for a proof of the uniform convergence of the series. The convergence criteria are fulfilled for an arbitrary group of sums over \( k_i \) index in the result \(^{19}\) for \( N \)–dimensional integral.

In fact in \(^{19}\) we have an \( N \)-tuple sum of two index quantities (closely related to \( a_{k_i} \)). Let us for simplicity discuss the \( N \) tuple sum of the two index quantities:

\[
\sum_{k_1,\ldots,k_N=0}^{\infty} \alpha_{k_0,k_1}\alpha_{k_1,k_2}\alpha_{k_2,k_3}\cdots\alpha_{k_N-1,k_N}\alpha_{k_N,k_{N+1}},
\]

\[
\alpha_{k_i-1,k_i} = \frac{(\xi e^{2\lambda})^{k_i}}{(2k_i)!} \Gamma(k_i-1+k_i+1/2)D_{-k_i-1-k_i-1/2}(z) \tag{21}
\]

If each individual sum over \( k_i \) exists and is finite, we divide the sum over the index \( k_i \) to a finite principal sum and a remainder \( \varepsilon_i(K_0) \), which can be made as small as possible by suitable selection of the upper summation limit \( K_0 \), comparing it to the principal part of the sum:

\[
\sum_{k_i=0}^{\infty} \alpha_{k_i-1,k_i}\alpha_{k_i,k_i+1} = \sum_{k_i=0}^{K_0} \alpha_{k_i-1,k_i}\alpha_{k_i,k_i+1} + \sum_{k_i=K_0+1}^{\infty} \alpha_{k_i-1,k_i}\alpha_{k_i,k_i+1} = (1 + \varepsilon_i(K_0)) \sum_{k_i=0}^{K_0} \alpha_{k_i-1,k_i}\alpha_{k_i,k_i+1} \tag{22}
\]

Let us define the ” principal sum” \( \Sigma(m,N-m) \) of \(^{21}\) as follows: the first \( m \) summations run to infinity and the last \( N-m \) summations run to \( K_0 \):

\[
\Sigma(m,N-m) = \sum_{k_1,\ldots,k_m=0}^{\infty} a_{k_1,k_2,k_3}\cdots a_{k_{m-1},k_m} b_{k_m}, \tag{23}
\]

where

\[
b_{k_m} = \sum_{k_{m+1},\ldots,k_N=0}^{K_0} a_{k_m,k_{m+1}} a_{k_{m+1},k_{m+2}}\cdots a_{k_N,k_{N+1}} \tag{24}
\]
In this notation, the sum \( 21 \) is \( \Sigma(N,0) \), our aim is to estimate this sum by its principal sum \( \Sigma(0,N) \), when all summations run over finite range. Performing in \( \Sigma(m,N - m) \) the sum over \( k_m \), we obtain:

\[
\Sigma(m,N - m) = (1 + \varepsilon_m)\Sigma(m-1,N - m + 1).
\]

We have the inequality:

\[
\Sigma(0,N) \leq \Sigma(N,0) \leq (1+\varepsilon)^N \Sigma(0,N).
\]

For any fixed \( N \) we choose \( K_0 \) so that \( \varepsilon = \max_i \varepsilon_i(K_0) \).

Following this discussion, the finite dimensional integral will converge if the upper bound \( \varepsilon \) to remainders approaches to zero as

\[
\varepsilon \sim N^{-\theta}, \quad \theta > 0.
\]

This will guarantee the convergence of \( \Sigma(0,N) \) to the continuum integral.

In what follows, we describe the idea of the evaluation of ”principal sum” for Eq. \( 9 \) and how to estimate the remainder to this leading term. The procedure of the summation consists of the following steps. At first, we represent one of the parabolic cylinder functions \( D_{m-1/2}(z) \) by Poincaré-type expansion \( 3 \), valid for real index and positive argument of the function, under the assumption that the index of the function is finite and the argument is going to infinity. For the dimension of the integral sufficiently great this is consistent, because \( z \sim N^{3/2} \). We have:

\[
D_{m-1/2}(z) = e^{z^2/4} z^{m+1/2} D_{m-1/2}(z) = \sum_{j=0}^{\infty} (-1)^j \frac{(m + 1/2)_j}{j! (2z^2)^j} + \epsilon_{\mathcal{J}}(m, z), \quad (25)
\]

where \( \epsilon_{\mathcal{J}}(m, z) \) is the remainder of the Poincaré-type expansion of the \( D \) function. For Poincaré-type expansion the upper bound of remainder was calculated by Olver \( 7 \). We use the improved upper bound evaluated by Temme \( 8 \). The upper bound for remainder in definition \( 25 \) reads:

\[
|\epsilon_{\mathcal{J}}(m, z)| \leq \frac{2z^2}{z^2 - 2m} \frac{(m + 1/2)_2 \mathcal{J}}{\mathcal{J} - 1)! (2z^2)^{\mathcal{J}} 1 \mathcal{F}_2 \left( \frac{\mathcal{J}}{2}, \frac{1}{2}, \frac{\mathcal{J}}{2} + 1; 1 - \frac{m^2}{z^2} \right) \exp \left( \frac{4\theta}{z^2 - 2m} 1 \mathcal{F}_2 \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1 - m^2}{z^2} \right) \right), \quad (26)
\]

where

\[
\theta = \frac{|m^2/4 + \frac{3}{16} + \frac{2m}{z^2} (1 + \frac{m}{2z^2})}{(z^2 - 2m)^2}.
\]

The estimate \( 26 \) is valid for \( 2\sqrt{m} \leq z \) \( 8 \). Before insertion of the Poincaré-type expansion \( 25 \) into \( 19 \) for one of the \( D \), we divide the sum over \( k_i \) into two parts. One, over finite \( k_i \), where Poincaré-type expansion is correct and the second, the remainder, small compared to the first part due to uniform convergence:

\[
\sum_{j=0}^{\infty} \frac{(-1)^j}{j! (2z^2)^j} \sum_{k=0}^{K_0} \left( \frac{\varepsilon_{2k_i}}{2k_i)! \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 2j + 1/2) D_{-k_i,k_i,1/2}(z) + (27)
\]

\[
+ \sum_{k_i=0}^{K_0} \left( \frac{\varepsilon_{2k_i}}{2k_i)! \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) D_{-k_i,k_i,1/2}(z) + (28)
\]

\[
+ \sum_{K_0+1}^{\infty} \left( \frac{\varepsilon_{2k_i}}{2k_i)! \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) D_{-k_i,k_i,1/2}(z) D_{-k_i,k_i,1/2}(z),
\]

where \( 2\sqrt{K_0} < z \). In the next step we extend the summation in the first, (“leading”) term up to infinity by adding and subtracting the terms, allowing to sum over the index \( k_i \) according to the relation \( 3 \):

\[
e^{z^2/4} \sum_{k=0}^{\infty} \frac{(\nu)_k}{k!} t^k D_{-\nu-k}(x) = \varepsilon(x-t)^2/4 D_{-\nu} (x-t).
\]

We summed up the product of the two functions \( D \), the result is the function \( D \) with a new argument. The pedagogical description of this procedure can be found in article \( 1 \) \( 8 \). We show the idea of the evaluation of the leading term in Appendix C.
We have shown, that the leading term for the $N$ dimensional integral is of the form:

$$Z^{\text{cut}}_{N-1} = \left( \prod_{i=0}^{N-1} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_i}} \right) \sum_{\mu=0}^{\omega} (-1)^{\mu} \frac{1}{\mu!(2\pi)^{2\mu}} (N)^{2\mu}$$

(29)

where the symbol $(N)^{2\mu}$ is defined by the recurrence relation:

$$(AQ\Lambda Q\Lambda^{-1})^{2\mu - p} (\Lambda)^{2\mu - p} =$$

$$= \sum_{j=0}^{p} \frac{a^{2\mu - j}}{(AQ\Lambda Q\Lambda^{-1})^{p - j}} \sum_{\lambda=\left(\mu + 1\right)}^{\mu} (AQ\Lambda^{-2}Q\Lambda^{-1})^{2\lambda - j} (\Lambda - 1)^{2\lambda} \left( \frac{\mu}{\lambda} \right) (Q\Lambda^{-1})^{\mu - \lambda}, \ p \in \mathbb{R}, 0 < 2\mu >$$

(30)

where the recurrence in $(\Lambda)^{2\mu - p}$ starts from:

$$(1)^2 = \frac{1}{\omega_0^2} a_1^2$$

Putting $\Lambda = N$, and $p = 2\mu$ in Eq. (30) we obtain the relation for the leading part of the $N-$ dimensional integral, which in the continuum limit coincides with the functional integral searched, if the remainder of the leading part in the continuum limit disappears. The sum over index $\mu$ in Eq. (29) is an asymptotic sum, therefore the upper summation limit must be taken symbolically. The estimate of the remainder to $Z^{\text{cut}}_{N-1}$ is discussed in Appendix D. We have shown, that this remainder can be made smaller than $N^{-1-\theta}$, $\theta > 0$, which is the necessary condition for the vanishing remainder in the continuum limit $N \to \infty$.

V. SOLUTION OF THE RECURRENCE RELATION FOR $(\Lambda)^{2\mu - p}$

To evaluate the $N$ dimensional integral, we solve the recurrence relation (30) for arbitrary value $\mu = d$:

$$(AQ\Lambda Q\Lambda^{-1})^{2d - p} (\Lambda)^{2d} =$$

$$= \sum_{j=0}^{p} \frac{a^{2d - j}}{(AQ\Lambda Q\Lambda^{-1})^{p - j}} \sum_{\lambda=\left(\mu + 1\right)}^{\mu} (AQ\Lambda^{-2}Q\Lambda^{-1})^{2\lambda - j} (\Lambda - 1)^{2\lambda} \left( \frac{d}{\lambda} \right) (Q\Lambda^{-1})^{d - \lambda}, \ p \in \mathbb{R}, 2d >$$

The right hand side of the equation is $(2d, p)$-th matrix element of the product of three matrices. For fixed $d$ on the left hand side of the equation, we read only the $d$-th column of a matrix, which is recurrently tied to the matrix in the center of the product on the left hand side. We define an auxiliary matrix $X^d(\Lambda)$ by the following matrix equation:

$$X^d_{p,\mu}(\Lambda) = \sum_{j=0}^{p} \sum_{\lambda=\left(\mu + 1\right)}^{\mu} A^d_{p,j}(\Lambda - 1) C^d_{j,\lambda}(\Lambda - 1) M^d_{\lambda,\mu}(\Lambda - 1).$$

To evaluate the matrix $C^d(\Lambda)$, we must calculate $X^d(\Lambda)$ for all dimensions $d$ up to $\mu$, for each dimension to extract $d$-th column of matrix $X^d(\Lambda)$ and to compose from these columns the matrix $C^d(\Lambda)$. We define such linear operation as follows:

1. Let $A^d$ and $M^d$ are the matrices of the dimensions $(2\mu + 1)(2\mu + 1)$ and $(\mu + 1)(\mu + 1)$ respectively, $C^d$ is the matrix of dimensions $(2\mu + 1)(\mu + 1)$. These matrices possess nonzero main minors of the dimensions $(2d + 1)(2d + 1)$, $(d + 1)(d + 1)$, and $(2d + 1)(d + 1)$, respectively. The definition of the matrices $A^d(\Lambda)$ and $M^d(\Lambda)$ is in Appendix E.

2. Matrix $X^d$ is the one column matrix defined by the relation:

$$\tilde{X}^d(\Lambda) = X^d(\Lambda) \ast \Phi^d,$$

where $\Phi^d$ is the projector of the $d$-th column of the matrix $X^d(\Lambda)$ into $d$-th column of the matrix $\tilde{X}^d(\Lambda)$. $\Phi^d$ is a matrix with a single nonzero term

$$\{\Phi^d\}_{d,k} = \delta_{d,k}.$$
3. The matrix $X^d(\Lambda)$ is defined by relation:

$$X^d(\Lambda) = A^d(\Lambda - 1) \ast C^d(\Lambda - 1) \ast M^d(\Lambda - 1).$$

4. Then, for $C^d(\Lambda)$ we have the result:

$$C^d(\Lambda) = \sum_{i=0}^{d} A^{d-i} \ast C^{d-i} \ast M^{d-i}(\Lambda - 1).$$

5. After evaluation of the full recurrence we find:

$$C^d(\Lambda) = \sum_{i=0}^{d} \sum_{i_1=0}^{d-i} \cdots \sum_{i_d=0}^{d-i} \left\{ A^{d-i} \ast A^{d-i_1-i_2-\cdots-i_d} \right\} \ast \left\{ C^{d-i} \ast C^{d-i_1} \ast M^{d-i_1} \right\} \ast \left\{ M^{d-i-i_1-i_2-\cdots-i_d} \ast M^{d-i-i_1-i_2-\cdots-i_d} \right\}$$

The evaluations of multiple products of the matrices is given in Appendix E. Remember that for the function $S_\Lambda$ defined in Eq. (29) the only important matrix element is $\{ C(\Lambda)^{2\mu} \}_{2\mu,2\mu}$ then we find the result:

$$\left\{ C(\Lambda)^{2\mu} \right\}_{2\mu,2\mu} = \sum_{i_1=0}^{\mu} \sum_{i_2=0}^{\mu-i_1} \cdots \sum_{i_d=0}^{\mu-i_1-i_2-\cdots-i_d} \left( I_1 \cdots I_{d-1} \right) Q^{1}_{\Lambda-i_1} \cdots Q^{d}_{\Lambda-i_1-i_2} \times$$

$$\times \left( \frac{1}{A_{Q_0}Q_1} \right)^j \left\{ \left( I_2 \right) a_{\Lambda-j}^{2\lambda} Q_0^{4j} Q_1^{4j} \right\}$$

We introduced the abbreviation

$$I_j = \mu - (i_j + i_{j+1} + \cdots + i_\Lambda).$$

$x_\Lambda$ are independent variables and $D_\xi$ is the differential operator given as:

$$D_\xi = 3/4 \partial^2_\xi + 3\xi \partial^3_\xi + \xi^2 \partial^4_\xi$$

The asymptotic decomposition of the function $S_\Lambda$ reads:

$$S_\Lambda = \sum_{\mu=0}^{3\mu} \left( -1 \right)^{\mu} \frac{\mu!}{(2\mu)^3 \Delta^3} \Delta^{3\mu} \left\{ C(\Lambda)^{2\mu} \right\}_{2\mu,2\mu} \triangleq 3\mu$$

since $\{ C(\Lambda)^{2\mu} \}_{2\mu,2\mu} = (\Lambda)^{2\mu}$. The quantity $z^2 \Delta^3$ is finite in the continuum limit $\Delta \to \infty$ and the factor $\Delta^{3\mu}$ ensures that only the leading term of $\{ C(\Lambda)^{2\mu} \}_{2\mu,2\mu}$ survives the continuum limit. Due to the analytic form for $\{ C(\Lambda)^{2\mu} \}_{2\mu,2\mu}$ we can express $S_\Lambda$ in the continuum $\Delta \to 0$ limit as well as in asymptotic $\mu \to \infty$ limit. The analytical evaluations of the lower $\mu$ terms and the relation for the asymptotic terms is done in details in our article. We evaluated the analytical result for the first three terms (i.e. $\mu = 1, 2, 3$). As an illustration we give the first of them:

$$\left\{ C(\Lambda)^{2} \right\}_{2,2} = \frac{\mu!}{(2\mu)!} \sum_{k=2}^{\Lambda} Q_k^1 b_k^2 J(0, 0; 2, 1; b_k/b_{k-1})$$
Fig. 1: \(b\) dependence of the continuum function \(S(a, b, c, \tau)\) for fixed values \(a = 0.1, c = 0.5, \tau = 1\). The first three nontrivial terms of the asymptotic series \((33)\) were used.

where

\[
J(l, M;l; n, i; b_j/b_{j-1}) = \sum_{p=l}^{M1N} \left( \frac{2i_j - l}{p - l} \right) \left( \frac{2i_j - 1/2}{2i_j - p} \right) \frac{n!(2i_j - p)!}{(n - 2i_j - p)!} \left( \frac{b_j}{b_j - 1} \right)^p,
\]

and

\[
b_j = \frac{1}{Q_{j+1}Q_j} + \cdots + \frac{1}{Q_{\Lambda}Q_{\Lambda - 1}}.
\]

The continuum limit corresponds to the prescription:

\[
k, \Delta \rightarrow x, \quad \Delta, \Lambda \rightarrow \tau,
\]

\[
\sum_{k=2}^{\Lambda} \frac{1}{\Delta} \int_0^\tau dx,
\]

When \(\Lambda \rightarrow N\), then \(\tau \rightarrow \beta\), where \(\beta\) is the constant of the model, and \(N = \frac{\beta}{\Delta}\).

In the continuum limit we obtain:

\[
Q_k \rightarrow 2 \cosh(\gamma x)
\]

\[
b_k \rightarrow \frac{1}{\Delta \gamma} (\tanh(\gamma \tau) - \tanh(\gamma x))
\]

where \(\gamma = \sqrt{2b/c}\), \(b\) and \(c\) are parameters of the model. The continuum limit of the relation \((33)\) will be called \(S(a, b, c, \tau)\). We show the first nontrivial term \((\mu = 1)\) of the three evaluated now:

\[
\{ C^2(a, b, c, \tau) \}_{2,2} = \frac{3}{8\gamma^2} [3\gamma \tau \tanh^2(\gamma \tau) + \tanh(\gamma \tau) - \gamma \tau]
\]  

\((34)\)

For the calculated higher terms we have analytical formulas also as results of algebraic evaluation by Mathematica\(^{[10]}\). The continuum function \(S(a, b, c, \tau)\) for the first three nontrivial contributions is shown in Fig. 1.

The corresponding term for the Gelfand-Yaglom equation, \(-2\partial_\mu^2 \ln(S(a, b, c, \tau)) - 4(\partial_\mu \ln(S(a, b, c, \tau)))^2\), is shown in Fig. 2.
VI. CONCLUSIONS

We presented an analytical method of evaluation of the unconditional Wiener measure functional integral with a fourth order term in the action. No simple analytical form of such an integral is known, perturbative methods of evaluation are needed. Instead of a standard perturbative procedure we expand the linear term of the action. Such integral is an entire function of all remaining parameters with infinite radii of convergence. We find an analytical result for the functional integral in the form of the solution of the “generalized Gelfand-Yaglom” (GGY) equation for the case of an anharmonic oscillator with positive coupling and no positivity requirement for the quadratic term. We calculated the asymptotic solution of the GGY equation up to third order in the coupling constant.

The same method of evaluation could be applied for the case of the functional integral with the conditional Wiener measure for anharmonic oscillator, with more interesting physical results. The approach based on the GGY equation can be applied too. However in this case the method is technically rather involved. An evaluation for a simplified case when the endpoints are fixed to zero is presented in our article I [5]. In this case the functional integral is a propagator with coinciding endpoints well known as Moeler’s formula [4] in the case of harmonic oscillator. We evaluated the correction to Moeler’s formula $S(\beta)$ up to the first nontrivial term and we found:

$$S(\beta) = 1 - \frac{3a}{32c^2 \gamma^3} \left\{ -3 \coth(\gamma \beta) + 2 \gamma \beta \left[ \coth^2(\gamma \beta) + \frac{1}{2} \sinh^2(\gamma \beta) \right] \right\}$$

(36)

A more complete evaluation of the functional integral with conditional Wiener measure and a more systematic study of the anharmonic oscillator is in progress.

When the frequency $b(\tau)$ and the coupling constant $a(\tau)$ are time dependent we obtain for the anharmonic oscillator by the time slicing method described in the article an equation which in the continuum limit reads:

$$-2 \partial^2_{\tau} \ln(S(a, b, c, \tau)) - 4(\partial_{\mu} \ln(S(a, b, c, \tau)))^2$$
\[
\frac{\partial^2}{\partial \tau^2} F(\tau) + 4 \frac{\partial}{\partial \tau} F(\tau) \frac{\partial}{\partial \tau} \ln S(\tau) = F(\tau) \left[ \frac{2b(\tau)}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right)^2 \right],
\]  
(37)

where \( S(\tau) \) can be evaluated by a method similar to that of Section IV. The time dependence of the functions \( a(\tau), b(\tau) \), instead of constants \( a, b \), does not complicate summations over the index in individual time-slice intervals. The nonlocal character of the result for \( N \)-dimensional integral, represented by the dependence of parabolic cylinder functions on two summation indexes is represented by the function \( \xi_i \), introduced in the evaluation procedure:

\[
(1 + b_{N-m-1} \Delta^2 / c) (1 + b_{N-m} \Delta^2 / c) \xi_{N-m-1}^2 = 1
\]

After the full recurrence procedure of the evaluation of the \( N \)-dimensional integral, we obtain the result:

\[
Z_{cut}^{N-1} = \left\{ \prod_{i=0}^{N-1} \frac{1}{\sqrt{2(1 + b_{N-i} \Delta^2 / c) \omega_i}} \right\} S_{N-1},
\]  
(38)

where \( S_{N-1} \) is a function with structure unimportant for this moment. The function \( \omega_i \) is defined by the recurrence relation:

\[
\omega_i = 1 - \frac{\xi_{N-i}^2}{4 \omega_{i-1}}.
\]

We obtained Eq. (37) without detailed evaluation of the function \( S(\tau) \), by the same method as in Section III. By substitution

\[
F(\tau) = \frac{y(\tau)}{S^2(\tau)}
\]

we convert (37) to selfadjoint [11] equation:

\[
y''(\tau) = \left( \frac{2b(\tau)}{c} \right) y(\tau).
\]  
(39)

In the case of a linear harmonic oscillator with time dependent frequency, we obtain for the function \( F(\tau) \), leading to the inverse square root of the functional integral, the simple second order selfadjoint equation discussed for the first time by Lewis [12]. The time evolution of QM systems of a broader class of time dependent Hamiltonians was discussed by Šamaj [13]. Equation (37) belongs to the class of general linear second order differential equations discussed exhaustively by Kamke [11]. Equation (39) corresponds to its reduced normal form with invariant

\[
I = -\frac{2b(\tau)}{c}.
\]

In Kamke [11], the variable substitution and function replacement leading to the differential equation of the second order with constant invariant are proven. Therefore, as in the case of the harmonic oscillator, in the case of anharmonic oscillator there exists a possibility to convert the problem with time dependent frequency and coupling constant to the constant coefficient linear equation of the second order problem.

Functional integral methods play an important role in quantum mechanics. One has the tools to evaluate the mean values of observables without a necessity to solve equations of motion. For example, the conditional measure functional integral in quantum mechanics represents particle propagation. Its integrated form over final positions describes unconditional functional integral in quantum mechanics. In statistical physics, the same quantity, after Wick rotation \( i t \to -\tau \), represents the partition function. In fact, all such functional integrals offer important information about physical quantities, e.g. spectrum of Hamiltonian, mean values of observables, etc.

The anharmonic oscillator can be considered as \( \phi^4 \) theory in \((1 + 0)\) dimensions and can be regarded as a toy model for understanding of QCD, as the anharmonic oscillator [14] studied in the perturbative approach was. In our description we have a possibility to study the case with positive mass, corresponding to the anharmonic oscillator and, what is more interesting, the case with negative frequency. It will be interesting to analyze the analog of the GGY equation in field theory also.

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APPENDIX A: EVALUATION OF THE $N$–DIMENSIONAL INTEGRAL

A pedagogical evaluation of the integral is done in our article I [3], here we recall the most important steps. We are going to evaluate the $N$–dimensional integral defined by the relation:

$$Z_N = \int_{-\infty}^{+\infty} \prod_{i=1}^{N} \left( \frac{d\varphi_i}{2\pi\Delta_i} \right) \exp \left\{ - \sum_{i=1}^{N} \Delta_i \left[ c/2 \left( \frac{\varphi_i - \varphi_{i-1}}{\Delta_i} \right)^2 + b_\varphi^2_i + a_\varphi^4_i \right] \right\} . \quad (A1)$$

First, we rewrite the sum in exponential function in a form convenient for consecutive integrations:

$$L_N = \Delta a_\varphi^4_1 + \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c})_\varphi^2_1 - \frac{c}{\Delta} \varphi_2 - \cdots + \Delta a_\varphi^4_i + \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c})_\varphi^2_i - \frac{c}{\Delta} \varphi_{i+1} - \cdots + \Delta a_\varphi^4_N + \frac{c}{\Delta} (1/2 + \frac{b\Delta^2}{c})_\varphi^2_N . \quad (A2)$$

We expand the exponential factor containing terms linear in the integration variable into Taylor’s series. Using the integration formula [2]:

$$\int_{0}^{\infty} x^{\alpha-1} \exp(-px^2 - qx) \, dx = \Gamma(\alpha)(2p)^{-\alpha/2} \exp \left( \frac{q^2}{8p} \right) D_{-\alpha} \left( \frac{q}{\sqrt{2p}} \right) , \quad (A3)$$

we find for integration over the variable $\varphi_1$:

$$Z_1 = \frac{1}{\sqrt{2\pi(1 + b\Delta^2/c)}} \sum_{k_1=0}^{\infty} \left( \frac{c}{\Delta(1 + b\Delta^2/c)} \right)^{k_1} (2k_1)! \varphi_2^{2k_1} \Gamma(k_1 + 1/2)D_{-k_1-1/2}(z) , \quad (A4)$$

where we used the notation:

$$D_{-k_1-1/2}(z) = z^{k_1+1/2} e^{\frac{z^2}{2}} D_{-k_1-1/2}(z) , \quad (A5)$$

The term $\varphi_2^{2k_1}$ in Eq. (A4) will play active rôle in the integration over the variable $\varphi_2$:

$$Z^\text{loc}_2 = \int_{-\infty}^{+\infty} \frac{d\varphi_2}{2\pi\Delta_i} \varphi_2^{2k_1} \exp \left\{ - \Delta a_\varphi^4_2 - \frac{c}{\Delta} (1 + \frac{b\Delta^2}{c})_\varphi^2_2 + \frac{c}{\Delta} \varphi_3 \right\} . \quad (A6)$$

Taking both integration steps together we have:

$$Z_2 = \left( \frac{1}{\sqrt{2\pi(1 + b\Delta^2/c)}} \right)^2 \sum_{k_1,k_2=0}^{\infty} \xi^{2k_1} \left( \frac{c}{\Delta(1 + b\Delta^2/c)} \right)^{k_2} \Gamma(k_1 + 1/2)D_{-k_1-1/2}(z) \Gamma(k_1 + k_2 + 1/2)D_{-k_1-k_2-1/2}(z) , \quad (A7)$$

where we used a new symbol:

$$\xi = \frac{1}{(1 + b\Delta^2/c)} .$$

Eq. (A7) is the result of both $\varphi_1$ and $\varphi_2$ integrations. We can repeat this procedure for other integration variables $\varphi_3, \cdots, \varphi_{N-1}$. For the integration over variable $\varphi_N$ one has no linear term in the exponent, therefore we don’t expand anything and this last integration will not add summation over the index $k_N$ to the final formula. We have:

$$Z^\text{loc}_N = \int_{-\infty}^{+\infty} \frac{d\varphi_N}{2\pi\Delta_i} \varphi_N^{2k_N-1} \exp \left\{ - \Delta a_\varphi^4_N - \frac{c}{\Delta} (1/2 + \frac{b\Delta^2}{c})_\varphi^2_N \right\} . \quad (A8)$$
and the result is:

\[ Z_N^{loc} = \left( \frac{c(1/2 + b\Delta^2/c)}{\sqrt{2\pi(1/2 + b\Delta^2/c)}} \right)^{k_{N-1}} \Gamma(k_{N-1} + 1/2)D_{-k_{N-1}-1/2}(z_N). \]  

(A9)

Remember the difference in definitions of \( z_N \) and \( z_i, \ i = 1, 2, \cdots, N - 1 \):

\[ z_N = \frac{c(1/2 + b\Delta^2/c)}{\sqrt{2a\Delta^3}} \]

and also the term

\[ \left( \frac{c(1/2 + b\Delta^2/c)}{\Delta} \right)^{k_{N-1}}, \]

which modify the definition of:

\[ \xi_{N-1} = \sqrt{\frac{1}{1 + b\Delta^2/c}} \sqrt{\frac{1}{(1/2 + b\Delta^2/c)}}. \]

For the \( N \)-dimensional integral we obtain finally the exact result:

\[ Z_N = \left[ 2\pi(1 + b\Delta^2/c) \right]^{N-1} \left[ 2\pi(1/2 + b\Delta^2/c) \right]^{-1/2} \sum_{k_1,\cdots,k_{N-1}=0}^{\infty} \prod_{i=1}^{N} \left[ \frac{(\xi_i)^{2k_i}}{(2k_i)!} \Gamma(k_{i-1} + k_i + 1/2)D_{-k_{i-1}-k_i-1/2}(z_i) \right], \]

where \( k_0 \equiv k_N \equiv 0 \), and \( \xi_1 = \xi_2 = \cdots = \xi_{N-2} = \xi \), also \( \xi_N = 1 \), and \( z_1 = z_2 = \cdots = z_{N-1} = z \).

**APPENDIX B: PROOF OF THE GENERALIZED GELFAND-YAGLOM DIFFERENTIAL EQUATION**

We rewrite the \( N \)-dimensional integral given in Eqs. (10), (11) as follows:

\[ Z_N = \frac{S_{N-1}(\Delta)}{\sqrt{\prod_{i=0}^{N-1} 2(1 + b\Delta^2/c)\omega_i}}, \]

where

\[ \omega_i = 1 - \frac{A^2}{\omega_{i-1}}, \quad \omega_0 = \frac{1/2 + b\Delta^2/c}{1 + b\Delta^2/c}, \quad A = \frac{1}{2(1 + b\Delta^2/c)}. \]

The value of the functional integral in the continuum limit is formally defined by

\[ Z = \lim_{N \to \infty} Z_N \]

Let us define the function

\[ F_k = \prod_{i=0}^{k} \frac{S_i^2}{S_k} \]

(B1)

Here \( S_k \) is the function \( S_N(\Delta) \), with \( N \) replaced by \( k \) and \( \Delta \) is fixed. Let us stress the relation between the \( N \)-dimensional integral and the function \( F_N \):

\[ Z_N = \frac{1}{\sqrt{F_N}} \]

The aim of the Gelfand-Yaglom construction is to find in the continuum limit such a differential equation in variable \( \tau \sim k.\Delta \) that its solution is connected to the continuum functional integral by relation:

\[ Z(\beta) = \frac{1}{\sqrt{F(\beta)}}, \quad \beta \sim N.\Delta. \]
In the spirit of the Gelfand-Yaglom construction we are going to express the relation for \( F_{k+1} \) by help of \( F_k \) and \( F_{k-1} \). We have:

\[
F_{k+1} = \frac{k+1}{\Delta S_{k+1}} \prod_{i=0}^{k+1} 2(1 + b \Delta^2/c) \omega_i = \frac{2(1 + b \Delta^2/c) \omega_{k+1}}{S_{k+1}^2} \prod_{i=0}^{k+1} 2(1 + b \Delta^2/c) \omega_i = (B2)
\]

\[
\frac{2(1 + b \Delta^2/c)}{S_{k+1}^2} \prod_{i=0}^{k+1} 2(1 + b \Delta^2/c) \omega_i = \frac{k}{S_{k+1}^2} 2(1 + b \Delta^2/c) \omega_i
\]

Regarding to the definition of the function \( F_k \), in Eq. (B1), we have:

\[
F_{k+1} = 2(1 + b \Delta^2/c) F_k \frac{S_k^2}{S_{k+1}^2} - F_{k-1} \frac{S_{k-1}^2}{S_{k+1}^2} = (B3)
\]

After some algebra we find:

\[
F_{k+1} - 2F_k + F_{k-1} - 2(F_k - F_{k-1}) \left( \frac{S_k^2 - S_{k+1}^2}{S_{k+1}^2} \right) = \frac{2b \Delta^2}{c} F_k - F_{k-1} \left( \frac{S_{k-1}^2 - 2S_k^2 + S_{k+1}^2}{S_{k+1}^2} \right)
\]

We need not to know the structure of functions \( S_k \) to derive the identities:

\[
S_k^2 - S_{k+1}^2 = -2S_{k+1}(S_{k+1} - S_k) + (S_{k+1} - S_k)^2 = (B5)
\]

and

\[
S_{k-1}^2 - 2S_k^2 + S_{k+1}^2 = 2(S_{k+1} - S_k)^2 + 2S_{k+1}(S_{k+1} - 2S_k + S_{k-1}) - 4(S_{k+1} - S_k)(S_{k+1} - 2S_k + S_{k-1}) + (S_{k+1} - 2S_k + S_{k-1})^2.
\]

Inserting these identities into Eq. (B4) we find a difference equation which, divided by \( \Delta^2 \), takes the form:

\[
\frac{F_{k+1} - 2F_k + F_{k-1}}{\Delta^2} + 4 \frac{F_k - F_{k-1}}{\Delta} \frac{S_{k+1} - S_{k}}{\Delta S_k} = F_k \left[ \frac{2b/c - 2S_{k+1} - 2S_k + S_{k-1}}{\Delta^2 S_k} - 2 \left( \frac{S_{k+1} - S_k}{\Delta S_k} \right)^2 \right] (B7)
\]

\[
+ \Delta O_1 + \Delta^2 O_2,
\]

where

\[
O_1 = -\frac{4b}{c} F_k \left( \frac{S_{k+1} - S_k}{\Delta S_{k+1}} \right) + 2 \frac{F_k - F_{k-1}}{\Delta} \left[ \frac{b}{c} + \left( \frac{S_{k+1} - S_k}{\Delta S_{k+1}} \right)^2 \right] + 4F_{k-1} \left( \frac{S_{k+1} - S_k}{\Delta S_{k-1}} \right) \left( \frac{S_{k+1} - 2S_k + S_{k-1}}{\Delta^2 S_{k+1}} \right)
\]

\[
O_2 = \frac{2b}{c} F_k \left( \frac{S_{k+1} - S_k}{\Delta S_{k+1}} \right)^2 - F_{k-1} \left( \frac{S_{k+1} - 2S_k + S_{k-1}}{\Delta^2 S_{k+1}} \right)^2 (B8)
\]

In the continuum limit \( \Delta \to 0 \) we replace \( k, \Delta \) by \( \tau \). Under the condition

\[
\lim_{\Delta \to 0} (\Delta O_1 + \Delta^2 O_2) = 0 , (B9)
\]

we obtain the differential equation:

\[
\frac{\partial^2}{\partial \tau^2} F(\tau) + 4 \frac{\partial}{\partial \tau} F(\tau) \frac{\partial}{\partial \tau} \ln S(\tau) = F(\tau) \left[ \frac{2b}{c} - 2 \frac{\partial^2}{\partial \tau^2} \ln S(\tau) - 4 \left( \frac{\partial}{\partial \tau} \ln S(\tau) \right)^2 \right] (B10)
\]
The initial conditions for $S(\tau)$ continuous and finite in $\tau = 0$ are:

$$F(0) = \frac{1}{S^2(0)}, \quad (B11)$$

$$\frac{\partial}{\partial \tau} F(0) = \lim_{\Delta \to 0} \frac{F_1 - F_0}{\Delta} = -\left( \frac{1}{S^2(0)} \right)'.$$

### APPENDIX C: EVALUATION OF THE LEADING PART OF $S_N$

In this appendix we evaluate the finite range $k$, summations by the recurrence method. Let us start with summation over the index $k_{N-1}$ of the Eq. (9). The finite sum to be done is:

$$Z_1^{cut} = \sum_{k_{N-1}=0}^{K_0} \left[ \frac{1}{\sqrt{2\pi(1 + b\Delta^2/c)\omega_0}} \frac{(\xi^2/\omega_0)^{k_{N-1}}}{(2k_{N-1})!} \Gamma(k_{N-1} + k_{N-1} + 1/2)D_{-k_{N-2} - k_{N-1} - 1/2} (z) \right]$$

$$\left[ \frac{1}{\sqrt{2\pi(1 + b\Delta^2/c)\omega_0}} \Gamma(k_{N-1} + 1/2)D_{-k_{N-1} - 1/2} (z_0) \right] \quad (C1)$$

Let us recall the dependence of the function $D$ on the parabolic cylinder function $D$:

$$D_{-m-1/2}(z) = z^{m+1/2} e^{\frac{z^2}{4}} D_{-m-1/2}(z)$$

and also the definitions of the variables $\xi, \omega_0, z$ and $z_0$:

$$z = \frac{c(1 + b\Delta^2/c)}{2a\Delta^3}, \quad z_0 = \frac{c(1/2 + b\Delta^2/c)}{2a\Delta^3},$$

$$\xi = \frac{1}{1 + b\Delta^2/c}, \quad \omega_0 = \frac{1/2 + b\Delta^2/c}{1 + b\Delta^2/c}.$$

We use the asymptotic Poincaré-type expansion of the parabolic cylinder function, which for $D$ means:

$$D_{-k_{N-1} - 1/2}(z_0) \equiv \frac{k_{N-1}^{+1/2}}{z_0} e^{\frac{z^2}{4}} D_{-k_{N-1} - 1/2}(z_0) = \sum_{j=0}^{J} (-1)^j \frac{(k_{N-1} + 1/2)_{2j}}{j! (2z_0^2)^j} + \varepsilon J(k_{N-1}, z_0) \quad (C2)$$

In the last relation, $J$ denotes the number of terms of the asymptotic expansions convenient to take into account. We apply asymptotic expansion for the function $D_{-k_{N-1} - 1/2}(z_0)$ in Eq. (C1). In the truncated sum we interchange the order of the finite summations over indices $k_{N-1}$ and $j$. We replace $D$ by $D$, therefore a corresponding power of the variable $z$ will play an important role. In this way we obtain the relation:

$$Z_1^{cut} = \frac{\Gamma(1/2)}{\sqrt{2\pi(1 + b\Delta^2/c)\omega_0}} \sum_{j=0}^{J} \frac{(-1)^j}{j! (2z_0^2)^j} \exp(\frac{z^2}{4}) z^{k_{N-2} + 1/2}$$

$$\sum_{k_{N-1}=0}^{K_0} \frac{(\frac{\xi^2}{\omega_0})^{k_{N-1}}}{(k_{N-1})!} \Gamma(k_{N-1} + k_{N-2} + 1/2) (k_{N-1} + 1/2)_{2j} D_{-k_{N-1} - k_{N-2} - 1/2} (z), \quad (C3)$$

where we simplified the calculations by identities:

$$(2k)! = 2^{2k} k! (1/2)_k, \quad \Gamma(k + 1/2) = \Gamma(1/2)(1/2)_k.$$  

Let us study in detail the sum

$$\sum_{k_{N-1}=0}^{K_0} \frac{(\frac{\xi^2}{\omega_0})^{k_{N-1}}}{(k_{N-1})!} \Gamma(k_{N-1} + k_{N-2} + 1/2) (k_{N-1} + 1/2)_{2j} D_{-k_{N-1} - k_{N-2} - 1/2} (z) \quad (C4)$$
This sum is uniformly convergent, therefore we can extend the summation up to infinity by adding the corresponding terms, which appear also in the remainder with opposite sign. To be able to provide the sum over the index \( k_{N-1} \), we must modify the Pochhammer symbol

\[
(k_{N-1} + 1/2)_{2j} = (k_{N-1} + 1/2) \cdots (k_{N-1} + 1/2 + 2j - 1) .
\]

We see, that this object is a polynomial in the variable \( k_{N-1} \) of the \( 2j \)-th order. We rewrite the polynomial in another form:

\[
(k_{N-1} + 1/2)_{2j} = \sum_{i=0}^{\min(2j,k_{N-1})} a_i^{2j} \frac{(k_{N-1})!}{(k_{N-1} - i)!}
\]

The coefficients \( a_i^{2j} \) are given by recurrence procedure from the relation:

\[
\sum_{k_{N-1}=0}^{K_0} \frac{(k_{N-1} + 1/2)_{2j}}{(k_{N-1})!} f(k_{N-1}) = \sum_{i=0}^{2j} a_i^{2j} \sum_{k_{N-1}=i}^{K_0} \frac{1}{(k_{N-1} - i)!} f(k_{N-1}) \quad \text{(C5)}
\]

From the above definition, we find the recurrence equation:

\[
a_i^k = (k - 1/2 + i) a_i^{k-1} + a_i^{k-1}
\]

while the initial conditions are:

\[
a_0^j = 1, \quad a_0^j = (1/2)_j, \quad a_{j+1}^j = 0
\]

The solution of this recurrence equation is:

\[
a_i^j = \binom{j}{i} \frac{(1/2)_j}{(1/2)_i}
\]

Inserting all these replacements into Eq. (C4), with help of the identity

\[
\Gamma(k_{N-2} + k_{N-1} + 1/2) = \Gamma(k_{N-2} + i + 1/2) (k_{N-2} + i + 1/2)_{k_{N-1} - i}
\]

after some algebra, introducing a new summation index \( k = k_{N-1} - i \), we obtain the formula:

\[
Z_1^{\text{cut}} = \frac{z^{k_{N-2} + 1/2}}{\sqrt{2(1 + b\Delta^2/c)\omega_0}} \sum_{j=0}^J \frac{(-1)^j}{j!(2z_0^2)^j} \frac{e^{z^2/4}}{\sqrt{2\pi(1 + b\Delta^2/c)}} \sum_{i=0}^{2j} a_i^{2j} \times \Gamma(k_{N-2} + i + 1/2) \left( \frac{z \xi^2}{4\omega_0} \right)^i \sum_{k=0}^{K_0} \frac{(-z \xi^2)^k}{k!} (k_{N-2} + i + 1/2)_k D_{-k_{N-2} - i - 1/2 - k} (z)
\]

In the above relation we extend summation over the index \( k \) up to infinity. The sum over \( k \) is now prepared for application of the identity:

\[
e^{z^2/4} \sum_{k=0}^\infty \frac{(v)}{k!} t^k D_{-\nu - k} (x) = e^{(x-t)^2/4} D_{-\nu} (x-t)
\]

The result of the first recurrence step, replacing \( D \) by \( D \), reads:

\[
Z_1^{\text{cut}} = \frac{1}{\sqrt{2(1 + b\Delta^2/c)\omega_0}} \frac{(\omega_1)^{-k_{N-2}}}{\sqrt{2\pi(1 + b\Delta^2/c)\omega_1}} \sum_{j=0}^J \frac{(-1)^j}{j!(2z_0^2)^j} \sum_{i=0}^{2j} a_i^{2j} \left( \frac{\xi^2}{4\omega_0} \right)^i \Gamma(k_{N-2} + i + 1/2) D_{-k_{N-2} - i - 1/2} (z_1)
\]

where

\[
z_1 = z \left( 1 - \xi^2/(4\omega_0) \right), \quad \omega_1 = \frac{z_1}{z} = 1 - \xi^2/(4\omega_0).
\]
For the following summation over the index \( k_{N-2} \) we have:

\[
Z_2 = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_1}} \frac{1}{\sqrt{2\pi(1+b\Delta^2/c)}} \sum_{k_{N-2}=0}^{\infty} \left[ \frac{1}{\sqrt{2\pi(1+b\Delta^2/c)(2k_{N-2})!}} \right] \left( \frac{\xi}{\omega_1} \right)^{k_{N-2}} \Gamma(k_{N-3}+k_{N-2}+1/2)D_{-k_{N-3}-k_{N-2}-1/2}(z) \\
\times \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(2z^2)^j} \sum_{i=0}^{2j} (1)^{2j}_i \left( \frac{\xi^2}{4\omega_0^2\omega_1} \right)^i \Gamma(k_{N-2}+i+1/2)D_{-k_{N-2}-i-1/2}(z_1),
\]

where

\[
(1)^{2j}_i = \frac{1}{\omega_0^{2j}} a_i^{2j}
\]

will define the first step of the new recurrence relation.

Now, due to the uniform convergence of the sum over the index \( k_{N-2} \) we will evaluate the leading part of \( Z_2 \) as a finite sum over index \( k_{N-2} \). In the finite sum, we use the asymptotic Poincaré-type expansion of the parabolic cylinder function \( D_{-k_{N-3}-i-1/2}(z_1) \). We have then in the relation for \( Z_2^{\text{cut}} \) finite summations only and we change the order of the sums. We have:

\[
Z_2^{\text{cut}} = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_1}} \frac{1}{\sqrt{2\pi(1+b\Delta^2/c)}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(2z^2)^j} \sum_{i=0}^{2j} (1)^{2j}_i \left( \frac{A^2}{\omega_0^2\omega_1} \right)^i \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(2z^2)^l} \frac{1}{\omega_1^l} z^{k_{N-3}+1/2}
\]

\[
\times \sum_{k_{N-2}=0}^{N_0} \left( \frac{\xi^2}{k_{N-2}} \right)^{k_{N-2}} \Gamma(k_{N-3}+k_{N-2}+1/2) D_{-k_{N-3}-k_{N-2}-1/2}(z) \Gamma(k_{N-2}+1/2)\frac{1}{2l+i},
\]

where \( A = \xi/2 \). Summing over the index \( k_{N-2} \) as in the first recurrence step, we have:

\[
Z_2^{\text{cut}} = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_1}} \frac{1}{\sqrt{2\pi(1+b\Delta^2/c)}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(2z^2)^j} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(2z^2)^l} \frac{1}{\omega_1^l}
\]

\[
\times \sum_{i=0}^{2j} (1)^{2j}_i \left( \frac{A^2}{\omega_0^2\omega_1} \right)^i \sum_{p=0}^{2l+i} a_p^{2l+i} \left( \frac{A^2}{\omega_1^2\omega_2} \right)^p \Gamma(k_{N-3}+p+1/2)D_{-k_{N-3}-p-1/2}(z_2),
\]

where one defines new variables:

\[
z_2 = z\omega_2, \quad \omega_2 = 1 - \frac{A^2}{\omega_1}.
\]

The summations over indices \( j, l \) is done as follows:

\[
\sum_{j=0}^{J} \sum_{l=0}^{L} \frac{(-1)^{j+l}}{j!!(2z^2)^j+l} f(l+j)g(j)h(l) =
\]

\[
= \sum_{\mu=0}^{J} \frac{(-1)^\mu f(\mu)}{\mu!(2z^2)^\mu} \sum_{j=0}^{\mu} \binom{\mu}{j} g(j)h(\mu-j) + \sum_{\mu=J+1}^{2J} \frac{(-1)^\mu f(\mu)}{(2z^2)^\mu} \sum_{j=J+1}^{\mu} \frac{g(j)h(\mu-j)}{j!(\mu-j)!}.
\]
The first term on the right hand side of the above relation will contribute to the leading part of Eq. (C14), while the second term, where index \( \mu > J \), will contribute to the remainder, due to the term \( z^{-2\mu} \sim N^{-3\mu} \) which may be made as small as possible by choosing \( J \) properly. Interchanging the order of summations:

\[
\sum_{i=0}^{2j} \sum_{p=0}^{2\mu - 2j + i} \rightarrow \sum_{p=0}^{\mu} \sum_{i=\max[0, p-2\mu+2j]}^{2j}
\]

we find the result of the second recurrence step:

\[
Z_{\text{cut}}^2 = \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_0}} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_1}} \frac{(z/z_2)^{k_{N-3}}}{\sqrt{2(1+b\Delta^2/c)\omega_2}}
\]

\[
\times \sum_{\mu=0}^{\mathcal{J}+\mathcal{C}} (-1)^\mu \sum_{p=0}^{2\mu} (2)_p^\mu \left( \frac{\Lambda^2}{\omega_1\omega_2} \right)^p \Gamma(k_{N-3} + p + 1/2) D_{-k_{N-3}-p-1/2}(z_2),
\]

where the second recurrence step of the function \((2)_p^\mu\) is defined by:

\[
(2)_p^\mu = \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{1}{\omega_1^{2\mu-2j}} \sum_{i=\max[0, p-2\mu+2j]}^{2j} \left( \frac{A^2}{\omega_0\omega_1} \right)^i \left( \frac{1}{2} \right)_i \ a_{p,i} 2^{\mu - 2j + i}
\]

We can see that after \( \Lambda \) recurrence steps the result of the \( \Lambda \) summations over the indices \( k_i \) can be read as:

\[
Z_{\Lambda} = \left\{ \prod_{i=0}^{\Lambda} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_i}} \right\} (\omega_\Lambda)^{-k_{N-1-\Lambda}}
\]

\[
\times \sum_{\mu=0}^{\mathcal{J}} (-1)^\mu \sum_{p=0}^{2\mu} (\Lambda)_p^\mu \left( \frac{A^2}{\omega_{\Lambda-1}\omega_\Lambda} \right)^p \Gamma(k_{N-\Lambda-1} + p + 1/2) D_{-k_{N-\Lambda-1}-p-1/2}(z_\Lambda).
\]

We have evaluated the recurrence relations:

\[
z_\Lambda = z\omega_\Lambda, \quad \omega_\Lambda = 1 - \frac{A^2}{\omega_{\Lambda-1}}, \quad \omega_0 = \frac{1/2 + b\Delta^2/c}{1 + b\Delta^2/c}, \quad A = \frac{\xi}{2},
\]

and introduced the recurrence definition for the function \((\Lambda)_p^\mu\):

\[
(\Lambda)_p^\mu = \sum_{j=0}^{\mu} \binom{\mu}{j} \frac{1}{\omega_{\Lambda-1}^{2\mu-2j}} \sum_{i=\max[0, p-2\mu+2j]}^{2j} \left( \frac{A^2}{\omega_{\Lambda-2\omega_{\Lambda-1}} \omega_{\Lambda-1}} \right)^i \left( \Lambda - 1 \right)_i \ a_{p,i} 2^{\mu - 2j + i}
\]

where the recurrence procedure begins from \((1)_i\) given in Eq. (C12).

After the last recurrence step, for \( \Lambda = N - 1 \), we are left with the relation of the form \((C18)\) where \( k_{N-\Lambda-1} \equiv 0 \) and the index of the \( D \) function is only \(-p - 1/2\). We expand the \( D_{-p-1/2}(z_{N-1}) \) as in all previous recurrence steps and find the relation:

\[
Z_{N-1}^\text{cut} = \left\{ \prod_{i=0}^{N-1} \frac{1}{\sqrt{2(1+b\Delta^2/c)\omega_i}} \right\} \sum_{\mu=0}^{\mathcal{J}} (-1)^\mu \sum_{p=0}^{2\mu} (2)_p^\mu \left( \frac{A^2}{\omega_{N-2\omega_{N-1}} \omega_{N-1}} \right)^i \left( N - 1 \right)_i \ a_{0,i} 2^{i+1}
\]

Following the definition of the \( a_i^\mu \) symbols, we have:

\[
(1/2)_{2l+i} = a_{0}^{2l+i}
\]
Then in the last part of the preceding equation we read:

$$\sum_{l=0}^{\mu} \binom{\mu}{l} \frac{1}{\omega_{N-1}^{2l}} \sum_{i=0}^{2l} \frac{A_{2i}^2}{(N-2\omega_{N-1})^i} (N-1)^{2j} a_0^{2l+i} = (N)^{2\mu}$$

(C21)

Following the calculations done in this Appendix, we conclude, that it is possible to realize summations in the exact formula for the $N-$ dimensional integral at least by help of asymptotic expansions of the parabolic cylinder functions. It is possible to provide the continuum limit of our result and there are no additional terms contributing to the result in the continuum limit. The result reads

$$Z_{N-1}^{cut} = \left\{ \prod_{i=0}^{N-1} \frac{1}{\sqrt{2(1 + b\Delta^2/c)\omega_i}} \right\} \sum_{\mu=0}^{J} \frac{(-1)^\mu}{\mu!(2z^2)^\mu} (N)^{2\mu}$$

(C22)

This expression is sufficient for the calculation of the continuum unconditional Wiener measure functional integral by the Gelfand-Yaglom procedure leading to the differential equation of the second order. The second part of the relation (C22) represents the expansion of an unknown function.

In the following calculation the key role play objects $\omega_i$ defined by the recurrence relation

$$\omega_i = 1 - \frac{A^2}{\omega_{i-1}}$$

with the first term

$$\omega_0 = 1/2 + B,$$

where

$$B = \frac{b\Delta^2/c}{2(1 + b\Delta^2/c)},$$

$\omega_i$ defined in this way are represented by continued fractions. The continued fraction can be represented by a simpler relation as the solution of its $n$–th convergent problem of continued fractions [15]. Let us shortly explain this procedure.

Let us have a continued fraction of the form:

$$\omega = a_1 + \frac{b_1}{a_2 + \frac{b_2}{\ldots}}$$

The $n$–th convergent is defined as

$$\omega_n = \frac{p_n}{q_n}$$

where $p_n$ and $q_n$ are given by equations:

$$p_n = a_n p_{n-1} + b_n p_{n-2}$$
$$q_n = a_n q_{n-1} + b_n q_{n-2}$$

(C23)

Solutions of these recurrence equations have the form:

$$p_n = \tilde{w}_1 \rho_1^n + \tilde{w}_2 \rho_2^n$$
$$q_n = w_1 \rho_1^n + w_2 \rho_2^n$$

(C24)

where $\rho_{1,2}$ are solutions of the characteristic equation, in our case a homogenous one:

$$\rho^2 - a_n \rho - b_n = 0$$

For the continued fraction in question we have:

$$a_n = 1, b_n = -A^2$$
and the solution of the characteristic equation is:

\[ \rho_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4A^2} \right) \]

The constants \( \tilde{w}_1 \) and \( w_1 \) are fixed by \( \omega_0 \) and \( \omega_1 \) terms, that adjust the initial conditions:

\[ p_0 = 1 + 2B \]
\[ p_1 = 1 + 2B - A^2 \]
\[ q_0 = 2 \]
\[ q_1 = 1 + 2B \]

The \( n - \text{th} \) convergent method solution is completed by the relations:

\[ \tilde{w}_{1,2} = \frac{1}{2} \left( (1 + 2B) \pm \left( \sqrt{1 - 4A^2} + \frac{2B}{\sqrt{1 - 4A^2}} \right) \right), \]
\[ w_{1,2} = 1 \pm \frac{2B}{\sqrt{1 - 4A^2}}, \]

the particular characteristic follows from the above solution:

\[ p_n = q_{n+1} \]

which simplifies our calculation significantly. In forthcoming calculations we have introduced more convenient variables:

\[ Q_i = \frac{q_i}{A^i} = w_1 \left( \frac{\rho_1}{A} \right)^i + w_2 \left( \frac{\rho_2}{A} \right)^i \]

and also performed the replacement:

\[ \frac{A^2}{\omega_{k-1}^2} = \frac{Q_{k-1}^2}{Q_k^2}. \]

In what follows, we:
- replace the summation index \( i \) by the summation index \( j \) defined by \( i = 2\lambda - j \).
- interchange the order of summations over indexes \( j \) and \( \lambda \).
- rewrite the recurrence relation Eq. (C19) as follows:

\[ (AQ^\lambda \Lambda^{-1})^{2\mu - p} (\Lambda^{2\mu - p} = \]

\[ = \sum_{j=0}^{p} (AQ^\lambda \Lambda^{-1})^{2\lambda - j} (\Lambda^{-1})^{2\lambda - j} \left( \frac{\mu}{\lambda} \right) (\Lambda^{2\mu - \lambda}, p \in \mathbb{N}, 0, 2\mu > \right), \quad (C25) \]

the recurrence starts from the term (C12)

**APPENDIX D: EVALUATION OF THE REMAINDER TO THE LEADING PART OF \( S_N \)**

Let us describe the evaluation of the remainder to the leading part of the sum calculated in Appendix C. The remainder consists of the infinite sum of the original series, the finite sum over the remainder of the Poincaré expansion of the parabolic cylinder function, the second term of the finite sum Eq.(C15) and the relation added to the leading part for possibility to perform an infinite summation over the parabolic cylinder function:

\[ R(S, K_0) = -\sum_{j=0}^{\mathcal{J}} \frac{(-1)^j}{j! (2z)^j} \sum_{k_i=K_0}^{\infty} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 2j + 1/2) \mathcal{D}_{-k_i-1-k_i-1/2} (z) + \]
\[ + \sum_{k_i=K_0}^{K_0} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) \mathcal{D}_{-k_i-k_i-1/2} (z) \mathcal{D}_{-k_i-1-k_i-1/2} (z) \]
\[ + \sum_{k_i=K_0+1}^{\infty} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i + k_i + 1/2) \Gamma(k_i + k_i + 1/2) \mathcal{D}_{-k_i-k_i-1/2} (z) \mathcal{D}_{-k_i-1-k_i-1/2} (z). \]
The finite sum in remainder:

\[
\sum_{k_i=0}^{K_i} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i-1 + k_i + 1/2) \Gamma(k_i + k_i+1 + 1/2) D_{-k_i-1-k_i-1/2} (z) \epsilon_f (k_i + k_i+1, z)
\]  

is bounded by the relation (26):

\[
\frac{M}{(J - 1)! (2\pi^2)^{J}} \sum_{k_i=0}^{K_0} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i-1 + k_i + 1/2) \Gamma(k_i + k_i+1 + 1/2) D_{-k_i-1-k_i-1/2} (z)
\]

where

\[
M = \max \left( \left( \frac{2z^2}{\pi^2 - 2a} \right) - 1 F_2(J/2, 1/2; J/2 + 1; 1 - \frac{a^2}{z^2}) \exp \left( \frac{4\delta}{z^2 - 2a} \right) \right),
\]

and \( a = k_i + k_i+1, k_i \geq 1 = 1, 2, \ldots, K_0 \). Let us remember that \( z^2 \sim N^3 \). Since we have the freedom to choose the parameter \( J \), the upper bound on this contribution to the remainder can be made as small as necessary power of \( N \).

In the asymptotic region of \( k_i > K_0 \) we expand one of the function \( D \) to double asymptotic expansions proposed by Temme [16]:

\[
D_{-a-1/2}(z) = \frac{\exp (-A z^2)}{(1 + 4\lambda)^{1/4}} \left[ \sum_{k=0}^{n-1} \frac{f_k(\lambda)}{z^{2k}} + \frac{1}{2^n} R_n(a, z) \right]
\]

where the following quantities were introduced:

\[
\lambda = \frac{a}{z^2}, \quad w_0 = \frac{1}{2} [\sqrt{1 + 4a} - 1], \quad A = \frac{1}{2} w_0^2 + w_0 - \lambda - \lambda \ln w_0 + \lambda \ln \lambda,
\]

the functions \( f_k(\lambda) \) are calculated in [16]. We find for the infinite part of the sum decompositions:

\[
\sum_{k_i=K_0+1}^{n} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i-1 + k_i + 1/2) \Gamma(k_i + k_i+1 + 1/2) D_{-k_i-1-k_i-1/2} (z)
\]

It was shown in [3] that the last part of this contribution can be made as small as we need due to the freedom in the choice of the parameter \( \alpha \) representing the number of the functions \( f_k(\lambda) \) taken into account in the Temme double asymptotic decomposition [16] of the parabolic cylinder function.

Now we are going to estimate the last part of the remainder, corresponding to the difference of the series:

\[
\sum_{k_i=K_0+1}^{n} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i-1 + k_i + 1/2) \Gamma(k_i + k_i+1 + 1/2) D_{-k_i-1-k_i-1/2} (z)
\]

Since both series are uniformly convergent, we exchange the order of summations, to obtain:

\[
\sum_{k_i=K_0+1}^{n} \frac{(\xi)^{2k_i}}{(2k_i)!} \Gamma(k_i-1 + k_i + 1/2) \Gamma(k_i + k_i+1 + 1/2) D_{-k_i-1-k_i-1/2} (z)
\]

\[
\left\{ \exp (-A z^2) \left[ \sum_{k=0}^{n-1} \frac{f_k(\lambda)}{z^{2k}} \right] - \sum_{j=0}^{J} \frac{(-1)^j (k_i + k_i+1 + 1/2)_{2j}}{j! (2z^2)^j} \right\}
\]
The double asymptotic expansion reduces to Poincaré - type expansion \[8\] when \( a \) is fixed after expanding the quantities in \( \lambda = a/z^2 \) for small values of this parameter. In the difference

\[
\left\{ \frac{\exp (-A z^2)}{(1 + 4 \lambda)^{1/4}} \sum_{k=0}^{n-1} f_k(\lambda) \frac{z^{2k}}{\lambda^{2k}} \right\} - \sum_{j=0}^{\mathcal{J}} (-1)^j (k_i + k_{i+1} + 1/2) j_{2j} \left( \frac{z}{2} \right)^j
\]

all terms where \( k, j \leq \min(n, \mathcal{J}) \) cancel one another and the rest of terms is proportional or smaller a \( z^{-2} \min(n, \mathcal{J}) \). As in the case of previous contributions to the remainder, this means that this part of the remainder can be made as small as we need in the power of \( 1/N \).

APPENDIX E: DEFINITION OF THE MATRICES

The definition of the matrices \( A^d(\Lambda - 1) \), \( C^d(\Lambda - 1) \), \( M^d(\Lambda - 1) \) is the following:

1. The \( A^d \) is the lower triangular matrix with zeros over the main diagonal of the dimension \((2\mu + 1)(2\mu + 1)\). The principal minor of the dimension \((2d + 1)(2d + 1)\) non-zero only with elements:

\[
\{A^d(\Lambda - 1)\}_{p,j} = \frac{\alpha_{2d-p}^{2d-j}}{(AQ_{\Lambda} Q_{\Lambda-1})^{p-j}}.
\]

2. The \( C^d_{p,\lambda}(\Lambda) \) is the upper triangular matrix with zeros under the main diagonal of the dimension \((2\mu + 1)(\mu + 1)\). The nonzero elements form the main minor of the dimension \((2d + 1)(d + 1)\) with \( \lambda \)th column:

\[
C^d_{p,\lambda}(\Lambda) = (AQ_{\Lambda} Q_{\Lambda-1})^{2\lambda - p} (A^{4\mu})^{2\lambda - p},
\]

where \( p = 0, 1, \ldots, 2\lambda \) and \( \lambda = 0, 1, \ldots, d \).

3. The \( M^d_{\lambda,k} \) is the upper triangular matrix with zeros under the main diagonal of the dimension \((\mu + 1)(\mu + 1)\). The nonzero elements form the main minor of the dimension \((d + 1)(d + 1)\):

\[
M^d_{\lambda,k}(\Lambda - 1) = \binom{k}{\lambda} (Q_{\Lambda-1}^4)^{k-\lambda}, \quad d \geq k \geq \lambda \geq 0.
\]

For a product of two lower-triangular matrices \( A^{I_3}(2) \) and \( A^{I_2}(1) \) we have \[9\]:

\[
\sum_{j=\lambda}^{p} \left\{ A^{I_3}(2) \right\}_{p,j} \left\{ A^{I_2}(1) \right\}_{j,\lambda} = \sum_{j=\lambda}^{p} \frac{\alpha_{2I_3-p}^{2I_3-j}}{(AQ_{3} Q_{2})^{p-j}} \frac{\alpha_{2I_2-j}^{2I_2-\lambda}}{(AQ_{2} Q_{1})^{j-\lambda}} =
\]

\[
= 2^{2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!((p-\lambda)!)!} \partial^2 \epsilon \left( \epsilon^{4I_3 - 2p} \sum_{j=\lambda}^{p} \left( p - \lambda \right)^{j - \lambda} \left( \frac{\epsilon}{AQ_{3} Q_{2}} \right)^{p-j} \left( \frac{1}{AQ_{2} Q_{1}} \right)^{j-\lambda} \right)_{\epsilon = 1},
\]

where \( \epsilon \) is an auxiliary variable.

We have used two identities for the summation over the index \( j \) and the definition for index \( I_j \):

\[
\alpha_{2I_3-p}^{2I_3-j} \alpha_{2I_2-j}^{2I_2-\lambda} = 2^{2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!((p-\lambda)!)!} \left( \frac{4I_3 - 2j)!}{(4I_2 - 2j)!} \right)
\]

and

\[
\frac{(4I_3 - 2j)!}{(4I_2 - 2j)!} = \partial^2 \epsilon^{4I_3 - 4I_2 + (\epsilon^{4I_3 - 2j})}_{|\epsilon = 1}
\]

\[
I_j = d - (i_j + i_{j+1} + \cdots + i_{\lambda})
\]
In the above relation the summation over the index \( j \) can be performed explicitly. Introducing a new auxiliary variable:

\[ \xi = \epsilon^2, \]

we find:

\[ \sum_{j=\lambda}^{p} \left\{ K_j^{(2)}(\lambda) \right\}_{p,j} \left\{ K_j^{(1)}(\lambda) \right\}_{j,\lambda} = 2^{-2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!(p-\lambda)!} 2^{4I_2} D_\xi^{I_j} \left\{ \frac{1}{\xi^{p-2I_j}} \left( \frac{\xi}{AQ_3Q_1} + \frac{1}{AQ_2Q_1} \right)^{p-\lambda} \right\}_{\xi=1}, \]

where \( D_\xi = 3/4\partial_\xi^2 + 3\xi\partial_\xi + \xi^2 \partial_\xi^2 \) is calculated from \( \partial_\xi^2 \).

For the resulting product of all matrices \( K_j^{(k)} \) we find:

\[ \left\{ K_{d-i\lambda}(\Lambda - 1) * K_{d-i\lambda-i\lambda-1}(\Lambda - 2) * \cdots * K_{d-i\lambda-i\lambda-1-\cdots-i_2}(1) \right\}_{p,\lambda} = 2^{-2(p-\lambda)} \frac{(4I_2 - 2\lambda)!}{(4I_3 - 2p)!(p-\lambda)!} 2^{4I_2 + i_3 + \cdots + i_\lambda} \times \left( \prod_{m=2}^{\Lambda} D_{\xi_m}^{I_m} \left[ \frac{1}{\xi^{p-2I_m}} \left( \frac{1}{AQ_2Q_1} + \frac{\xi_2}{AQ_3Q_2} + \cdots + \frac{\xi_2 \cdots \xi_{\lambda-1}}{AQ_\lambda Q_{\lambda-1}} \right)^{p-\lambda} \right] \right)_{(all\xi_m=1)} \]

By symbol * we indicate the product of matrices. Evaluating the product of matrices:

\[ \left\{ M_{d-i\lambda-i\lambda-1-\cdots-i_2}(1) * \cdots * M_{d-i\lambda-i\lambda-1}(\Lambda - 2) * M_{d-i\lambda}(\Lambda - 1) \right\} \]

we use that \( M_{d-i\lambda}(j) \) is a one-column matrix with \( I_{j+1} \) non-zero elements in the \( (j+1) \) column:

\[ \left\{ M_{d-i\lambda}(j) \right\}_{\lambda,j+1} = \left( \begin{array}{c} I_{j+1} \\ \lambda \end{array} \right) Q_j^{4(I_{j+1} - \lambda)}, \]

where \( \lambda = 0, 1, \ldots, I_{j+1} \). Product of such matrices is a one-column matrix with elements:

\[ \left\{ M_{d-i\lambda-i\lambda-1-\cdots-i_2}(1) * \cdots * M_{d-i\lambda-i\lambda-1}(\Lambda - 2) * M_{d-i\lambda}(\Lambda - 1) \right\}_{\lambda,I_\lambda} = \left( \begin{array}{c} I_2 \\ \lambda \\ I_2 \\ \cdots \\ I_{\lambda-1} \\ I_{\lambda} \end{array} \right) Q_1^{4(I_2 - \lambda)} Q_2^{4(I_3 - I_2)} \cdots Q_{\lambda-1}^{4(I_{\lambda-1} - I_{\lambda-2})} \]

From the definition (30) of recurrence steps we have for the matrix \( C_{d-i\lambda}(1) \) nonzero elements:

\[ \left\{ C_{d-i\lambda}(1) \right\}_{j,\lambda} = \frac{Q_{d\lambda}}{(AQ_1Q_0)^{2\lambda}} Q_{2\lambda-j}^{2\lambda}, \]

with conditions for indexes:

\[ 0 \leq j \leq 2\lambda \leq 2I_\lambda \leq 2\mu. \]

Collecting all partial results together, inserting them into Eq. (31) and remembering that for function \( S_\lambda \) defined in Eq. (28) only matrix elements \( \left\{ C(\lambda)^{2\mu} \right\}_{2\mu,2\mu} \) are important, we find the result (32).

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