CONSERVATION LAWS AND SYMMETRIES OF
TIME-DEPENDENT GENERALIZED KDV EQUATIONS

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ABSTRACT. A complete classification of low-order conservation laws is obtained for time-dependent generalized Korteweg-de Vries equations. Through the Hamiltonian structure of these equations, a corresponding classification of Hamiltonian symmetries is derived. The physical meaning of the conservation laws and the symmetries is discussed.

1. Introduction. This paper is devoted to a study of conservation laws and symmetries for a class of time-dependent generalized Korteweg-de Vries equations

$$ u_t + f(t, u)u_x + u_{xxx} = 0 $$

with

$$ f|_{u=0} = 0, \quad f_u \neq 0. $$

This class is preserved under the equivalence transformations

$$ t \rightarrow \tilde{t} = t + t_0, \quad u \rightarrow \tilde{u} = u + u_0, \quad t_0, u_0 = \text{const}. $$

Many interesting evolution equations are included in this class (1): the KdV equation ($f = u$) models the dynamics of shallow water waves; the modified KdV equation ($f = u^2$) is a model for acoustic waves in anharmonic lattices [20] and Alfvén waves in collision-free plasmas [11]; a combined KdV-mKdV equation ($f = au + bu^2$, with $a$ and $b$ being arbitrary nonzero constants) arises in plasma physics and solid-state physics, modelling wave propagation in nonlinear lattices [19] and thermal pulses in solids [15, 18]. KdV-type equations having time-dependent coefficients arise in several applications [8, 9, 6, 12] and can be mapped into this class (1) by a point transformation [4].

All of these equations (1) have a Hamiltonian structure, on any fixed spatial domain $\Omega \subseteq \mathbb{R}$, as given by

$$ u_t = \mathcal{H}(\delta H/\delta u) $$

with the local Hamiltonian functional

$$ H = \int_{\Omega} \left( \frac{1}{2} u_x^2 - F(t, u) \right) dx, \quad F(t, u) = u \int f(t, u) \, du - \int uf(t, u) \, du, $$

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The Hamiltonian operator \( \mathcal{H} \) is a total \( x \)-derivative
\[
\mathcal{H} = D_x. \tag{6}
\]
The Hamiltonian \( H \) will be a conserved integral when and only when the nonlinearity satisfies \( f_t = 0 \), which corresponds to a generalized KdV equation \( u_t + f(u)u_x + u_{xxx} = 0 \).

Previous work on special families of equations in the class (1) can be found in Refs.\([5, 10, 17, 14]\).

In section 2, all low-order conservation laws of this class of generalized KdV equations (1)–(2) will be classified. In section 3, a corresponding classification of Hamiltonian symmetries will be derived by using the Hamiltonian structure (4) of the generalized KdV equations. The physical meaning of the symmetries and the conservation laws is discussed. Finally, some concluding remarks will be made in section 4.

2. Conservation laws. Conservation laws are of basic importance in the study of evolution equations because they provide physical, conserved quantities for all solutions \( u(x, t) \), and they can be used to check the accuracy of numerical solution methods [16, 7].

A local conservation law for a time-dependent generalized KdV equation (1) is a continuity equation
\[
D_t T + D_x X = 0 \tag{7}
\]
holding for all solutions \( u(x, t) \) of equation (1), where the conserved density \( T \) and the spatial flux \( X \) are functions of \( t, x, u, \) and \( x \)-derivatives of \( u \). If \( T = D_x \Theta \) and \( X = -D_t \Theta \) hold for all solutions, then the continuity equation (7) becomes an identity. Conservation laws of this form are called locally trivial, and two conservation laws are considered to be locally equivalent if they differ by a locally trivial conservation law. The global form of a non-trivial conservation law is given by
\[
\frac{d}{dt} \int_\Omega T \, dx = -\left. X \right|_{\partial \Omega} \tag{8}
\]
where \( \Omega \subseteq \mathbb{R} \) is any fixed spatial domain.

Every local conservation law can be expressed in an equivalent, characteristic form (analogous to the characteristic form for symmetries) [16] which is given by a divergence identity
\[
D_t \hat{T} + D_x \hat{X} = (u_t + f(t, u)u_x + u_{xxx})Q \tag{9}
\]
holding off of the set of solutions of the evolution equation (1), where \( \hat{T} = T + D_x \Theta \) and \( \hat{X} = X - D_t \Theta \) are a conserved density and spatial flux that are locally equivalent to \( T \) and \( X \), and where [2]
\[
Q = E_u(\hat{T}) \tag{10}
\]
is a function of \( t, x, u, \) and \( x \)-derivatives of \( u \). This function is a called a multiplier [16, 1, 7]. Here \( E_u \) denotes the Euler operator with respect to \( u \) [16].

For any evolution equation, there is a one-to-one correspondence between non-zero multipliers and non-trivial conservation laws up to local equivalence [16, 2], and the conservation laws of basic physical interest arise from multipliers of low order [3]. These multipliers for the evolution equation (1) take the form
\[
Q(t, x, u, u_x, u_{xx}) \tag{11}
\]
which correspond to conserved densities of the form
\[ T(t, x, u, u_x) \] (12)
modulo a trivial conserved density.

A function \( E_u \) will be a multiplier iff \( E_u((u_t + f(t, u)u_x + u_{xxx})Q) = 0 \) holds identically, since the kernel of the Euler operator consists of total divergences [16, 7]. This condition splits with respects to any \( x \)-derivatives of \( u \) that do not appear in \( Q \). The resulting overdetermined system consists of
\[ 0 = D_t Q + f D_x Q + D_x^3 Q \] (13)
and
\[ Q_u = E_u(Q), \quad Q_{ux} = -E_u^{(1)}(Q), \quad Q_{uxx} = E_u^{(2)}(Q) \] (14)
holding for all solutions \( u(x, t) \) of the evolution equation (1). The first equation (13) turns out to be the adjoint of the determining equation for symmetries (cf. (44)). The remaining equations (14) constitute the Helmholtz equations [2, 3] which are necessary and sufficient for \( Q \) to have the variational form (10). Here \( E_u^{(1)} \) and \( E_u^{(2)} \) denote higher Euler operators [16, 3].

It is straightforward to set up and solve this determining system (13)–(14) subject to the classification conditions (2). The computation is simplest when we separate it into two main cases: \( f_{tu} = 0 \), and \( f_{tu} \neq 0 \). We merge the resulting subcases after first having solved the determining system in each of these two cases and then having used the equivalence transformations (3). (For solving the determining system, we use the Maple package “rifsimp” which provides a complete classification of all solutions.)

The multipliers (11) for general \( f(t, u) \) are linear combinations of
\[ Q_1 = 1; \] (15)
\[ Q_2 = u. \] (16)

All special forms of \( f(t, u) \) for which additional multipliers (11) are admitted consist of:
\[ f(t, u) = a(u), \quad a(u) \text{ arbitrary}, \] (17a)
\[ Q_3 = -u_{xx} - \int a(u) du; \] (17b)
\[ f(t, u) = t^{-2/3}a(t^{1/3}u), \quad a(v) \text{ arbitrary}, \] (18a)
\[ Q_4 = -tu_{xx} + \frac{1}{3}xu - t^{1/3} \int a(t^{1/3}u) du; \] (18b)
\[ f(t, u) = a(t)u, \quad a(t) \text{ arbitrary}, \] (19a)
\[ Q_5 = (\int a(t) dt) u - x; \] (19b)
\[ f(u) = a(t)u, \quad a(t) \text{ satisfies } a^2a''' - 13a''a'a + 24a'^3 = 0, \] (20a)
\[ Q_6 = -2a(t)^{-3}u_{xx} - a(t)^{-2}u^2 - 2xa'(t)a(t)^{-4}u \]
\[ -x^2(4a'(t)^2 - a(t)a''(t))a(t)^{-6}, \] (20b)
\[ f(u) = at^{-1/3}u + bu + cu^2, \quad a, b, c \text{ constant}, \] (21a)
A first-order separable ODE
\[ Q_7 = -b t u_{xx} - \frac{1}{6} t (2 c^2 u^3 + 3 c b u^2 + b^2 u) \]
\[ - \frac{1}{4} a t^{2/3} (2 c u^2 + b u) + \frac{1}{6} x (2 c u + b); \]  \hfill (21b)

Note, in the case (20a), the third-order ODE possesses two first integrals \(-2a^p a'' + (p + 13)a^{p-1}a'' = c = \text{const.} \) for \( p = -5 \) and \( p = -7 \). This yields a reduction to a first-order separable ODE
\[ a' = a^3 (c_1 + c_2 a^2)^{1/2}, \quad c_1, c_2 \text{ constant} \]  \hfill (22)
which has the quadrature
\[ \frac{2c_1^{1/2} + (c_1 + c_2 a^2)^{1/2}}{a} \exp \left( - \frac{c_1^{1/2} (c_1 + c_2 a^2)^{1/2}}{c_2 a^2} \right) \]
\[ = \exp \left( \frac{2c_1^{3/2}}{c_2} (t + c_3) \right), \quad c_1, c_2, c_3 \text{ constant.} \]  \hfill (23)

For each multiplier admitted by a time-dependent generalized KdV equation (1), a corresponding conserved density and flux can be derived (up to local equivalence) by integration of the divergence identity (9) \cite{7, 3}. We obtain the following results.

**Theorem 2.1.** (i) All conservation laws given by low-order conserved densities (12) admitted by the class of time-dependent generalized KdV equations (1) for arbitrary \( f(t,u) \) (satisfying conditions (2)) are linear combinations of:
\[ T_1 = u, \quad X_1 = u_{xx} + \int f(t, u) \, du; \]  \hfill (24)
\[ T_2 = \frac{1}{2} u^2, \quad X_2 = u u_{xx} - \frac{1}{2} u^2 + \int u f(t, u) \, du. \]  \hfill (25)

(ii) The class of time-dependent generalized KdV equations (1) admits additional conservation laws given by low-order conserved densities (12) only for \( f(t,u) \) of the form (17a), (18a), (19a), (20a) (satisfying conditions (2)). The admitted conservation laws in each case are given by:
\[ T_3 = \frac{1}{2} u_{xx}^2 - \int A(u) \, du, \]  \hfill (26a)
\[ X_3 = -\frac{1}{2} u_{xx}^2 - A(u) u_{xx} - u_x u_x - \frac{1}{2} A(u)^2, \]  \hfill (26b)
\[ A(u) = \int a(u) \, du; \]  \hfill (26c)
\[ T_4 = \frac{1}{2} u_{xx}^2 + \frac{1}{6} x u^2 - \int A(t^{1/3}u) \, du, \]  \hfill (27a)
\[ X_4 = -\frac{1}{6} t u_{xx}^2 - A(t^{1/3}u) u_{xx} + \frac{1}{6} x (2u u_{xx} - u_x^2) \]
\[ + \frac{1}{6} x t^{-1} (u A(t^{1/3}u) - \int A(t^{1/3}u) \, du) - \frac{1}{2} t^{-1} A(t^{1/3}u)^2, \]
\[ A(v) = \int a(v) \, dv; \]  \hfill (27c)
\[ T_5 = \frac{1}{2} A(t) u^2 - x u, \]  \hfill (28a)
\[ X_5 = -\frac{1}{2} x (2u u_{xx} + a(t) u^2) + u_x + \frac{1}{2} A(t) (2u u_{xx} - u_x^2) - \frac{1}{3} a(t) A(t) u^3, \]  \hfill (28b)
\[ A(t) = \int a(t) \, dt; \]  \hfill (28c)
\[ T_6 = a(t)^{-3} u_{xx}^2 - \frac{1}{3} a(t)^{-2} u^3 - c_1 x^2 u - (c_2 + c_1 a(t)^{-2})^{1/2} x u^2, \]  \hfill (29a)
\[ X_6 = -a(t)^{-3}(u_{xx}^2 + 2u_t u_x) - a(t)^{-2}u_x^2 + 2(c_2 + c_1 a(t)^{-2})^{1/2} uu_x \\
-\frac{1}{4}a(t)^{-1} u^4 + 2c_1(u - xu_x) - \frac{1}{2}c_1 x^2(2u_{xx} + a(t)u^2) \\
-x(c_2 + c_1 a(t)^{-2})^{1/2}(2uu_{xx} - u_x^2) - \frac{4}{5}x(c_1 + c_2 a(t)^{2})^{1/2} u^3; \]

\[ T_7 = \frac{1}{8}c t u_x^2 - \frac{1}{12} t(u^2 + bu)^2 + \frac{1}{6} x(u^2 + bu) - \frac{1}{3} at^{2/3}(\frac{1}{2}cu^3 + \frac{1}{4}bu^2), \]

\[ X_7 = -\frac{1}{4}c t(u_{xx}^2 + 2u_t u_x) - \frac{1}{6} t(2c^2 u^3 + 3bcu^2 + b^2 u)u_{xx} + \frac{1}{12} t b^2 u_x^2 \\
- \frac{1}{18} t u^3 + \frac{1}{12} x(u^2 + bu)^2 + \frac{1}{6} x((2cu + b)u_{xx} - bu_x^2) \\
+ \frac{1}{2} axt^{-1/3}(\frac{1}{2}cu^3 + \frac{1}{4}bu^2) - \frac{1}{12} a^{2/3}(\frac{1}{2}cu^4 + bu^3) \\
- \frac{1}{6} at^{2/3}((2cu^2 + bu)u_{xx} - \frac{1}{2} bu_x^2 + 2c^2 u^5 + \frac{5}{2} bcu^4 + \frac{5}{2} b^2 u^3) \\
- \frac{1}{6}(2cu + b)u_x. \]

Note that \( u_t \) can be eliminated in the spatial flux expressions by use of the evolution equation (1).

The physical meaning of these conservation laws (24)–(30) can be seen by considering their global form (8).

For general \( f(t, u) \), the two admitted conservation laws (24) and (25) yield the conserved integrals

\[ C_1 = \int_{\Omega} u \, dx, \]

\[ C_2 = \int_{\Omega} \frac{1}{2} u^2 \, dx. \]

These represent the total mass and the \( L^2 \)-norm for solutions \( u(x, t) \).

In the time-independent case (17a), where \( f(t, u) = a(u) = A'(u) \), the conservation law (26) yields the conserved integral

\[ C_3 = \int_{\Omega} \left( \frac{1}{2} u_x^2 - \int A(u) \, du \right) \, dx \]

which represents the Hamiltonian or the total energy for solutions \( u(x, t) \).

In the time-dependent nonlinear case (18a), where \( f(t, u) = t^{-2/3}a(t^{1/3}u) = t^{-2/3}A'(t^{1/3}u) \), the conservation law (27) yields the conserved integral

\[ C_4 = \int_{\Omega} \left( \frac{1}{2} t u_x^2 + \frac{1}{6} xu^2 - \int A(t^{1/3}u) \, du \right) \, dx \]

which represents a dilational energy for solutions \( u(x, t) \).

In the time-dependent linear cases (19a) and (20a), the two conservation laws (28) and (29) respectively yield the conserved integrals

\[ C_5 = \int_{\Omega} \left( \frac{1}{2} A(t)u^2 - xu \right) \, dx \]

where \( f(t, u) = a(t)u = A'(t)u \), and

\[ C_6 = \int_{\Omega} \left( a(t)^{-3} u_x^2 - \frac{1}{3} a(t)^{-2} u^3 - (c_2 + c_1 a(t)^{-2})^{1/2} xu^2 - c_1 x^2 u \right) \, dx \]

where \( f(t, u) = a(t)u \) with \( a(t) \) given by expression (23). Since \( f(t, u) \) is linear in \( u \) in these two cases, the evolution equation (1) has the form

\[ u_t + a(t)uu_x + u_{xxx} = 0 \]
which is a KdV equation with a time-dependent coefficient, where \( a(t)u \) physically represents an advective velocity. Then the first conserved integral \((35)\) describes a generalized Galilean momentum, and the second conserved integral \((36)\) describes a generalized dilational energy. In particular, when \( a = \text{const.} \), these conserved integrals reduce to the ordinary Galilean momentum \( \int_{\Omega} \left( \frac{1}{2}atu^2 - xu \right) dx \) and the ordinary energy \( a^{-3} \int_{\Omega} (u_x^2 - au^3) \) for the KdV equation.

In the quadratic case \((21a)\), where \( f(u) = at^{-1/3}u + bu + cu^2 \), the conservation law \((30)\) yields the conserved integral

\[
C_7 = \int_{\Omega} \left( \frac{1}{2} ctu_x^2 - \frac{1}{12}t(cu^2 + bu)^2 + \frac{1}{6}x(cu^2 + bu) - \frac{1}{4}at^{2/3}(4cu^3 + 3bu^2) \right) dx
\]

which represents a combined Galilean energy-momentum for solutions \( u(x, t) \). In particular, when \( a = b = 0 \), this conserved integral reduces to the Galilean energy \( \int_{\Omega} \left( \frac{1}{2}tu_x^2 - \frac{1}{12}t(4u^2) + \frac{1}{6}xu^2 \right) dx \) for the mKdV equation, while when \( a = c = 0 \), the Galilean momentum for the KdV equation is obtained.

3. **Symmetries.** Symmetries are a basic structure of evolution equations as they can be used to find invariant solutions and yield transformations that map the set of solutions \( u(x, t) \) into itself [16, 7].

An infinitesimal symmetry for a time-dependent generalized KdV equation \((1)\) is a generator

\[
X = \xi \partial_x + \tau \partial_t + \eta \partial_u
\]

whose prolongation leaves invariant the equation \((1)\), where \( \xi, \tau, \) and \( \eta \) are functions of \( t, x, u, \) and \( x \)-derivatives of \( u \). The symmetry is trivial if it leaves invariant every solution \( u(x, t) \) of the equation \((1)\). This occurs when (and only when) \( \xi, \tau, \) and \( \eta \) satisfy the relation

\[
\eta = u_x \xi + u_t \tau
\]

for all solutions \( u(x, t) \). The corresponding generator \((39)\) of a trivial symmetry is given by

\[
X_{\text{triv}} = \xi \partial_x + \tau \partial_t + (\xi u_x + \tau u_t) \partial_u
\]

which has the prolongation \( \text{pr}X_{\text{triv}} = \xi D_x + \tau D_t \).

Any symmetry generator is equivalent [16, 7] to a generator

\[
\hat{X} = X - X_{\text{triv}} = P \partial_u, \quad P = \eta - \xi u_x - \tau u_t
\]

under which \( u \) is infinitesimally transformed while \( x \) and \( t \) are invariant, due to the relation

\[
\text{pr}X - \text{pr}\hat{X} = \xi D_x + \tau D_t.
\]

This generator \((42)\) defines the **characteristic form** for the infinitesimal symmetry. Invariance of the evolution equation \((1)\) is then given by the condition [16, 3]

\[
0 = D_t P + D_x (fP) + D_x^2 P
\]

holding for all solutions \( u(x, t) \) of the equation \((1)\).

A symmetry will generate a point transformation on \( (x, t, u) \) iff the coefficients \( \xi, \tau, \) and \( \eta \) in its characteristic function \( P \) depend only on \( x, t, u \) [16, 7], yielding a generator with the form

\[
X = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \eta(x, t, u) \partial_u
\]
For any Hamiltonian evolution equation, there is a correspondence that produces a symmetry from each admitted conservation law. This correspondence is a Hamiltonian analog of Noether’s theorem. It can be formulated for an evolution equation (1), with the Hamiltonian structure (6), through the explicit relation [16, 13]

\[ P = \mathcal{H}(\delta C/\delta u) = D_x Q \]

(46)

involving the characteristic function \( P \) of the symmetry (42) and the multiplier \( Q \) associated to the conserved integral \( C = \int_\Omega T \, dx \) given by a conservation law (7).

This correspondence is one way: every conservation law yields a symmetry. The converse holds iff the symmetry has the form (46), which requires that \( E_u(P) = 0 \).

The classification of low-order conservation laws stated in Theorem 2.1 for the class of time-dependent generalized KdV equations (1) yields the following symmetries produced from the conservation law multipliers.

The multipliers (15) and (16) given by the two conservation laws (24) and (25) admitted for general \( f(t, u) \) produce the symmetry characteristic functions

\[ P_1 = 0; \quad P_2 = u_x. \]

(47)

(48)

The additional multipliers (17a)–(21a) given by the conservation laws (26)–(30) which are admitted for forms of \( f(t, u) \) respectively yield the symmetry characteristic functions

\[ P_3 = -u_{xxx} - a(u)u_x; \]

(49)

\[ P_4 = -tu_{xxx} + \frac{1}{3} x u_x + \frac{1}{6} u - t^{2/3} a(t^{1/3} u)u_x; \]

(50)

\[ P_5 = (\int a(t) \, dt) u_x - 1; \]

(51)

\[ P_6 = -2a(t)^{-3} u_{xxx} - 2a(t)^{-2} u u_x - 2x a'(t) a(t)^{-4} u_x - 2a'(t) a(t)^{-4} u - 2x(4a'(t)^2 - a(t)a''(t)) a(t)^{-6}; \]

(52)

\[ P_7 = -btu_{xxx} - t(c^2 u^2 + cbu + \frac{1}{6} b^2)u_x - at^{2/3}(cu + \frac{1}{2} b)u_x + \frac{1}{6} c x u_x + \frac{1}{6} (2cu + b). \]

(53)

By evaluating these characteristic functions (48)–(53) on solutions \( u(x, t) \), we can eliminate all \( u_{xxx} \) terms to obtain \( P(t, x, u, u_x, u_t) = \eta(t, x, u) - \xi(t, x, u)u_x - \tau(t, x, u)u_t \) in each case. This leads to the following symmetry classification.

**Theorem 3.1.** (i) The symmetries corresponding to the two low-order conserved integrals (31)–(32) admitted by the class of time-dependent generalized KdV equations (1) for arbitrary \( f(t, u) \) (satisfying conditions (2)) are generated by:

\[ X_1 = 0, \quad X_2 = \partial_x. \]

(54)

(ii) The additional symmetries corresponding to the low-order conserved integrals (33)–(38) admitted for special forms of \( f(t, u) \) (satisfying conditions (2)) are generated in each case by:

\[ X_3 = \partial_t, \quad f(t, u) = a(u), \quad a(u) \text{ arbitrary}; \]

(55a)

\[ X_4 = \frac{1}{3} x \partial_x + t \partial_t + \frac{1}{3} u \partial_u, \quad f(t, u) = t^{-2/3} a(t^{1/3} u), \quad a(v) \text{ arbitrary}; \]

(56a)

\[ X_5 = \frac{1}{3} x \partial_x + t \partial_t + \frac{1}{3} u \partial_u, \quad f(t, u) = t^{-2/3} a(t^{1/3} u), \quad a(v) \text{ arbitrary}; \]

(56b)
\[ X_5 = a(t)\partial_x - \partial_u, \quad (57a) \]

\[ f(t, u) = a(t)u, \quad a(t) \text{ arbitrary}; \quad (57b) \]

\[ X_6 = -2a'(t)a(t)^{-4}x\partial_x + 2a(t)^{-3}\partial_t \]
\[ + (2x(a(t)a''(t) - 4a'(t)^2)a(t)^{-6} - 2a'(t)a(t)^{-4})\partial_u, \quad (58a) \]

\[ f(u) = a(t)u, \quad a(t) \text{ satisfies } a^2a''' - 13a'a'' + 24a' = 0; \quad (58b) \]

\[ X_7 = (\frac{1}{5}cx + \frac{1}{6}b^2t - \frac{1}{4}ab t^{2/3})\partial_x + ct\partial_t + (\frac{1}{6}a + \frac{1}{6}b)\partial_u, \quad (59a) \]

\[ f(u) = at^{-1/3}u + bu + cu^2, \quad a, b, c \text{ constant.} \quad (59b) \]

Note, from the quadrature (23) for the ODE for \(a(t)\) in case (58), we can express

\[ X_6 = -2(c_2 + c_1a(t)^2)^{1/2}x\partial_x + 2a(t)^{-3}\partial_t - 2(c_1x + (c_2 + c_1a(t)^2)^{1/2})\partial_u. \quad (60) \]

All of the symmetries (54)–(59) are point symmetries. Their physical meaning will now be discussed.

For general \(f(t, u)\), the symmetry \(X_1\) obtained from the conservation law (24) is trivial. Consequently, the conservation law (24) represents a Casimir of the Hamiltonian structure [16]. The other symmetry \(X_2\) is a space translation.

In the time-independent case (55b), where the conserved integral (33) represents the Hamiltonian or the total energy for solutions \(u(x, t)\), the symmetry \(X_3\) is a time translation.

In the time-dependent nonlinear case (56b), where the conserved integral (34) represents a dilational energy for solutions \(u(x, t)\), the symmetry \(X_4\) is a scaling.

In the time-dependent linear cases (57b) and (58b), where the two conserved integrals (35) and (36) respectively represent a generalized Galilean momentum and a generalized dilational energy, the first symmetry \(X_5\) is a generalized Galilean boost and the second symmetry \(X_6\) is a generalized dilation. Note the evolution equation (1) in these cases has the form of a time-dependent KdV equation (37) in which \(a(t)u\) physically represents an advective velocity. In particular, when \(a = \text{const.}\), these two symmetries reduce to an ordinary Galilean boost and a time translation.

In the quadratic case (59b), where the conserved integral (38) represents a combined Galilean energy-momentum for solutions \(u(x, t)\), the symmetry \(X_7\) is a scaling combined with a Galilean boost. In particular, when \(a = b = 0\), this symmetry reduces to the scaling symmetry for the mKdV equation, while when \(a = c = 0\), the Galilean boost symmetry for the KdV equation is obtained.

4. Concluding remarks. The classifications of low-order conservation laws and associated Hamiltonian symmetries obtained in this paper can be extended to a wider class of evolution equations

\[ u_t + f(t, u)u_x + b(t)u + c(t)u_{xxx} = s(t) \quad (61) \]

by use of a mapping

\[ x \rightarrow \tilde{x} = x - \zeta(t), \quad t \rightarrow \tilde{t} = \tau(t), \quad u \rightarrow \tilde{u} = \lambda(t)u + \nu(t) \quad (62) \]

with \(\tau'(t) \neq 0\) and \(\lambda(t) \neq 0\). This will be carried out in a subsequent paper [4].
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