Plasmon mode as a detection of the chiral anomaly in Weyl semimetals

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Weyl semimetals (SMs) are a new class of gapless topological phase, which can be seen as three-dimensional (3D) analogs of graphene. Weyl fermions emerge from the band degenerate points—the Weyl nodes—in the momentum space, which are characterized by their chirality. Due to the fermion doubling theorem, Weyl nodes with opposite chirality always appear in pairs. Each Weyl node behaves like a magnetic monopole in the momentum space, which acts as the source/drain of the chiral anomaly: a nontrivial momentum-space topology of Weyl nodes gives rise to a number of novel electromagnetic responses. On the material side, Weyl SMs have been proposed for strongly correlated iridates, semiconductors, and other materials. In addition, 3D Dirac materials have recently been realized in both Cd$_3$As$_2$ and Na$_3$Bi, which could greatly facilitate the search for Weyl SMs.

A remarkable phenomenon associated with Weyl nodes is the so-called chiral anomaly, in which the application of a pair of parallel electric field $E$ and magnetic field $B$ induces a charge imbalance between the two Weyl nodes with opposite chirality. This chiral anomaly can be utilized to detect 3D Weyl SMs in experiments. For example, a large longitudinal magnetococonductivity was proposed as a consequence of the chiral anomaly, which, however, is difficult to identify unambiguously in magnetotransport data. Recently, nonlocal transport, optical conductivity, and optical absorption measurements have also been proposed to probe the chiral anomaly in 3D Weyl SMs.

In this Rapid Communication, we propose an alternative detection method of the chiral anomaly by employing the plasmon mode in 3D Weyl SMs. We show that the chiral anomaly would lead to a new plasmon mode in intrinsic Weyl SMs. The chiral anomaly causes a redshift of the frequency of plasmon mode in doped Weyl SMs. Once the small Fermi surface crosses the Weyl node that corresponds to the Lifshitz transition (LT) point, the frequency turns out to be a violetshift. Therefore, the plasmon mode can be regarded as a signature of the chiral anomaly in 3D Weyl SMs. We also show how to extract the information of the LT point from the plasmon dispersion.

We begin with a low-energy effective Hamiltonian for Weyl fermions in the vicinity of the Weyl node of chirality $\chi = \pm$:

$$\mathcal{H} = \chi \hbar v_F k \cdot \sigma - \mu_{\chi},$$

where $v_F$ is the Fermi velocity, $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ refers to the three Pauli matrices, and $\mu_{\chi}$ stands for the chirality-dependent chemical potential given by a superposition of the equilibrium carrier density and the pumped carrier density originating from the chiral anomaly. The latter grows linearly with time but the large momentum internode scattering would counteract this imbalance of carriers between two Weyl nodes. Eventually the system reaches a non-equilibrium steady state characterized by an internode relaxation time $\tau_v$. Consequently, the density of electrons pumped into or out of the neighborhood of Weyl node $\chi$ can be expressed as

$$\Delta \rho_{\chi} \equiv \chi \frac{e^2}{4\pi^2 \hbar^2} E \cdot B \tau_v.$$  

We also define several chirality-dependent quantities: the fermi wave vector $k_{F,\chi}^3 = 6\pi^2 n_{\chi}$, the chemical potential $\mu_{\chi} = \hbar v_F k_{F,\chi}$, and the charge density $n_{\chi} = n + \Delta \rho_{\chi}$. When the two Weyl nodes are equally populated, the corresponding Fermi wave vector and the chemical potential become $k_{F}^3 = 6\pi^2 n$, $\mu = \hbar v_F k_F$. For convenience, we restrict our discussion to $E \cdot B > 0$. For the undoped case with vanishing equilibrium chemical potential, $\mu_{\chi}$ depends only on the pumped charge associated with the chiral anomaly:

$$\mu_{\chi} = \chi \left( \frac{3e^2 \hbar v_F^3}{2} E \cdot B \tau_v \right)^{1/3}.$$  

Meanwhile for the doped case with a finite chemical potential $\mu$, we obtain the corresponding chirality-
dependent chemical potential as
\[ \mu_\chi = (1 + \chi^3)^{1/3} \mu, \]
where we have introduced a dimensionless ratio between the pumped charge and the equilibrium charge
\[ \gamma = \left( \frac{3e^2 \hbar v_F E \cdot B}{2 \mu^3} \right)^{1/3}. \]

It follows from Eq. (5) that by tuning the external fields the system undergoes a chirality-dependent LT at \( \gamma = \pm 1 \), i.e., the change of the topology of the chirality-dependent Fermi surface. In the following we shall work in the weak magnetic field limit, thus neglect the Landau level structure of Weyl nodes. In addition, we will focus on the \( n \)-doped case with a finite positive equilibrium chemical potential \( \mu > 0 \) throughout this paper (the discussion of the \( p \)-doped case is similar).

It has been demonstrated that no plasmon exists in undoped Weyl or Dirac SMs within the random phase approximation (RPA). However, when the chiral anomaly occurs, the anomalous charge transfer between the two Weyl nodes forces the Fermi surfaces to move away from their equilibrium position in opposite directions as shown in Fig.1. Thus the chemical potentials of the two Weyl nodes are \( \mu_+ \) and \( \mu_- \) satisfying the relation \( \mu_+ = -\mu_- = \mu > 0 \). In principle, the metallic nature of intrinsic Weyl SMs with chiral anomaly would support plasmon modes. In the following, we present an exact and general proof of the existence of the novel plasmon due to the chiral anomaly in undoped Weyl SMs.

The general form of the wave vector \( q \)- and frequency \( \omega \)-dependent dielectric function within the RPA is given by
\[ \varepsilon(q, \omega) = 1 - V(q) \Pi(q, \omega), \]
where \( V(q) = 4 \pi e^2 / \kappa q^2 \) is the Fourier transform of the 3D Coulomb interaction with \( \kappa \) being the effective dielectric constant. Let us consider one of the Weyl nodes. The noninteracting polarization function \( \Pi(q, \omega) \) reads
\[ \Pi(q, \omega) = \frac{g}{L^3} \sum_{kss'} \frac{f(\epsilon_{k}) - f(\epsilon_{k'})}{\hbar \omega + \epsilon_{k} - \epsilon_{k'} + i \eta} F_{ss'}(k, k'), \]
where \( g \) is the number of pairs of Weyl nodes, \( \eta \) is a positive infinitesimal, and \( s, s' = \pm \) are the band indices. The overlap of eigenstates \( F_{ss'}(k, k') \) is given by
\[ F_{ss'}(k, k') = \frac{1 + ss' \cos \theta_{kk'}}{2}, \]
where \( \theta_{kk'} \) is the angle between the 3D wave vectors \( k' = k + q \). \( f(x) = [1 + \exp \{ \beta(x - \mu) \}]^{-1} \) is the Fermi distribution function with \( \beta = 1/k_B T \).

To proceed with the theoretical details, we assume zero temperature \( T = 0 \) K. The Fermi distribution function \( f(x) \) turns into a simple step function \( \theta(\mu - x) \). Because of the general relation of the polarization function
\[ \Pi(q, \omega) = \Pi(q, \omega), \]

![FIG. 1: (Color online) The distribution of electrons in the two Weyl nodes \( \chi = \pm \) induces by the chiral anomaly \( (E \cdot B) \neq 0 \) in undoped Weyl SMs.](image)
that of doped Weyl SMs with chemical potential $\mu = |\mu_{\pm}|$.

Next we set out to find the plasmon dispersion, which can be obtained within the RPA by finding zeros of the dielectric function\textsuperscript{14},

$$
\varepsilon(q, \omega - i\Gamma) = 0,
$$

where $\Gamma$ is the decay rate of the plasmon. For weak damping, Eq. (11) reduces to the following approximate equation

$$
\text{Re} \ \varepsilon(q, \omega) = 0.
$$

For the long wavelength approximation $q \ll \omega \ll \mu$, due to $\text{Im} \ \varepsilon(q \rightarrow 0, \omega) = 0$, Eq. (11) reduces to

$$
\text{Re} \ \varepsilon(q \rightarrow 0, \omega) = 0.
$$

To order $q^0$, the real part of the dielectric function has the form

$$
\text{Re} \ \varepsilon(q \rightarrow 0, \omega) = \kappa^*(\omega) - \frac{4\alpha_\kappa g \mu^2}{3\pi \omega^2},
$$

where the function $\kappa^*(\omega)$ is defined as

$$
\kappa^*(\omega) = 1 + \frac{\alpha_\kappa g}{3\pi} \log \left| \frac{4\Lambda^2}{4\mu^2 - \omega^2} \right|.
$$

Neglecting the logarithmic corrections in Eq. (15), we can obtain the lowest plasmon frequency $\omega_0 \approx \sqrt{\frac{4\alpha_\kappa g \mu^2}{3\pi}}$. Two remarks are in order here. First, the linear dependence of $\omega_0$ in $\mu$ also holds for the Dirac SMs\textsuperscript{12} and Weyl SMs in the absence of the chiral anomaly\textsuperscript{3}. Second, recalling the chirality-dependent chemical potential in Eq. (2), one immediately finds that $\omega_0 \propto |\mathbf{B}|^{1/3}$. This is different from the results of $\omega_0 \propto |\mathbf{B}|^{1/2}$ in previous works\textsuperscript{11,25}, where the authors have focused on the contribution of the zeroth Landau level in a large magnetic field.

Taking into account the leading order contribution of $q$, we get

$$
\text{Re} \ \varepsilon(q \rightarrow 0, \omega) = \kappa^*(\omega) - \frac{4\alpha_\kappa g \mu^2}{3\pi \omega^2} \left[ 1 - \frac{q^2}{(2\mu)^2} \left( 1 + \mathcal{F}(2\mu, \omega) \right) \right], \tag{17}
$$

with $\mathcal{F}(x, y) = \frac{x^2 + \frac{y^2}{2} - \frac{x^2 y^2}{4}}{y^2 (x^2 - y^2)^2}$. To gain some insight into the long wavelength plasmon dispersion, we write down an approximate expression from Eqs. (13) and (17) as

$$
\omega \approx \omega_0 \left[ 1 - \frac{q^2}{8\mu^2} \left( 1 + \mathcal{F}(2\mu, \omega_0) \right) \right]. \tag{18}
$$

For comparison, we compute the exact solution, the long wavelength solution, and the approximate solution, respectively, which are plotted in Fig. 2. In the long wavelength regime, all the three solutions are in good agreement with each other. It should be noted that the lower branch of the approximate solution is fully in the intraband single particle excitation (SPE) region, which is merely an artifact due to the weak damping approximation of Eq. (11).

We assign the chirality-dependent chemical potential $\mu_{\pm}$ for the large and small Fermi surfaces with $\mu_+ > |\mu_-|$. In the long wavelength approximation $q \ll \omega \ll |\mu_-| \leq \mu_+$, to order $q^2$, the real part of the polarization function takes the form

$$
\text{Re} \ \varepsilon(q \rightarrow 0, \omega) = \kappa^*(\omega) - \frac{2\alpha_\kappa g (\mu_+^2 + \mu_-^2)}{3\pi \omega^2} \times \left[ 1 - \frac{q^2}{4(\mu_+^2 + \mu_-^2)} \sum_{\lambda = \pm} \left( 1 + \mathcal{F}(2\mu_\lambda, \omega_0) \right) \right]. \tag{19}
$$

To obtain an approximate behavior of the long wavelength plasmon dispersion, one can arrive at an expression from Eqs. (13) and (19) as

$$
\omega \approx \omega_0 \left[ 1 - \frac{q^2}{8(\mu_+^2 + \mu_-^2)} \sum_{\lambda = \pm} \left( 1 + \mathcal{F}(2\mu_\lambda, \omega_0) \right) \right]. \tag{20}
$$

FIG. 2: (Color online) Plasmon modes in undoped 3D Weyl SMs with chiral anomaly, i.e., $\mathbf{E} \cdot \mathbf{B} \neq 0$ are calculated within the RPA. The red dotdashed line shows the long wavelength plasmon mode, the blue dashed line corresponds to the approximate solution obtained from Eq. (12), and the green solid line represents the exact solution of Eq. (11). The shaded area indicates the intraband and interband SPE regions.
where the notations $\kappa^* (\omega)$ and $\omega_0$ are given by

$$\omega_0 = \sqrt{\frac{\alpha_K}{\kappa^*(\omega_0)}} \sqrt{\frac{2\gamma}{3\pi} (\mu_+^2 + \mu_-^2)}, \quad (21)$$

$$\kappa^*(\omega) = 1 + \frac{\alpha_K g}{6\pi} \left( \sum_{\lambda, \pm} \log \frac{4\lambda^2}{4\mu_\lambda^2 - \omega^2} \right), \quad (22)$$

which can be traced back to the counterpart of Eq. (16) in the undoped case by taking $\mu_+ = \mu_-$. The plasmons can also be revealed as sharp peaks in the energy loss function (ELF), defined as the imaginary part of the inverse dielectric function, i.e., $\text{Im} [1/\varepsilon(q, \omega)]$ that can be probed in various spectroscopy experiments, such as the electron energy-loss spectroscopy. As shown in Fig.4(a), in the presence of the chiral anomaly the plasmon exhibits some exotic features in the ELF spectrum. As long as the ratio $\gamma$ gradually increases from 0 to 1, the plasmon frequency with chiral anomaly $\omega_{ch}$ has a redshift with respect to the frequency without chiral anomaly $\omega_{eq}$ and finally reaches a minimum $\omega_{\text{min}}$. On the other hand, once the small Fermi surface crosses the Weyl node, i.e., $\gamma > 1$, the plasmon frequency becomes larger than $\omega_{\text{min}}$ and then has a continuous violetshift.

The behavior of plasmons of doped Weyl SMs under the influence of the chiral anomaly can be captured by our long wavelength expressions in Eq. (21) and summarized as follows

$$\begin{align*}
\omega_{\text{min}} < \omega_{ch} < \omega_{eq}, & \quad 0 < \gamma < 1; \\
\omega_{min} < \omega_{ch}, & \quad 1 < \gamma.
\end{align*} \quad (23)$$

Therefore, the unique features of undamped plasmon mode can clearly characterize the chiral anomaly in doped Weyl SMs. It should be emphasized that compared with other methods, our method possesses the advantage that we can directly determine the position of the chirality-dependent LT point from the plasmon dispersion, as shown in Fig.4(b), which coincides with the minimal frequency of plasmon mode. Actually, when $|\gamma - 1| \to 0$, the small Fermi level is close to the Weyl node point, such that small energy or momentum could induce interband transition and lead to a large number of electron-hole excitations. The plasmon mode will be damped by these electron-hole excitations. Increasing $q$ will broaden the damping region (see Fig.4(b)). Hence, the fate of the plasmon mode indeed connects with the chirality-dependent LT.

In summary, we investigated the chiral anomaly effect on the plasmon mode in 3D Weyl SMs within the RPA. We proved that a new plasmon mode would emerge in undoped Weyl SMs due to the chiral anomaly. We also demonstrated the unusual properties of the plasmon mode in doped Weyl SMs and further pointed out that the plasmon can be taken as a fingerprint of the chiral anomaly. Finally, we showed how to identify the chirality-dependent LT point from the plasmon dispersion. Our work sheds light on the probing of the chiral anomaly also makes Weyl SMs promising candidates for plasmonics.

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The detailed calculation of the polarization function is given in the Supplemental Material. It should be emphasized that unlike the 2D Dirac fermion systems, the imaginary part \( \text{Im} \Pi(q, \omega) \) does not approach zero as \( \omega \to \infty \), so that one needs to apply the generalized Kramers-Kronig relation to evaluate the real part of polarization function of the intrinsic case from its imaginary part.

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In our numerical calculation, we make use of the parameters: \( g = 12 \) (for the materials predicted in Ref. \textsuperscript{28} and the cutoff of wave vector \( \Lambda = 10^2 \mu \), Fermi velocity \( v_F = 6.85 \times 10^5 \text{m} \cdot \text{s}^{-1} \), effective dielectric constant \( \kappa = 20 \). It should be noted that the different values of these parameters just quantitatively change the plasmon frequency.

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Supplementary material for “Plasmon mode as a detection of the chiral anomaly in Weyl semimetals”

In this supplementary material we present the major steps of calculating the polarization function of a Weyl node in the 3D Weyl semimetal. Those of the other Weyl nodes can be obtained in a similar manner. The polarization function can be decomposed into two parts

\[
\Pi(q, \omega) = \Pi^-(q, \omega) + \Pi^+(q, \omega),
\]

where \( \Pi^\pm(q, \omega) \) are defined by

\[
\Pi^-(q, \omega) = \frac{g}{L^3} \sum_k \left( \frac{[f(\epsilon_{k^-}) - f(\epsilon_{k'^-})](1 + \cos \theta_{kk')}/2}{\hbar \omega + \epsilon_{k^-} - \epsilon_{k'^-} + i\eta} + \frac{f(\epsilon_{k^-})(1 - \cos \theta_{kk')}/2}{\hbar \omega + \epsilon_{k^-} - \epsilon_{k'^-} + i\eta} - \frac{f(\epsilon_{k'^-})(1 - \cos \theta_{kk'})/2}{\hbar \omega + \epsilon_{k'^-} - \epsilon_{k^-} + i\eta} \right),
\]

\[
\Pi^+(q, \omega) = \frac{g}{L^3} \sum_k \left( \frac{[f(\epsilon_{k'^+}) - f(\epsilon_{k'^+})(1 + \cos \theta_{kk'})/2}{\hbar \omega + \epsilon_{k'^+} - \epsilon_{k'^-} + i\eta} + \frac{f(\epsilon_{k'^+})(1 - \cos \theta_{kk'})/2}{\hbar \omega + \epsilon_{k'^+} - \epsilon_{k'^-} + i\eta} - \frac{f(\epsilon_{k'^+})(1 - \cos \theta_{kk'})/2}{\hbar \omega + \epsilon_{k'^-} - \epsilon_{k'^-} + i\eta} \right).
\]

Due to the causality \( \text{Re} \Pi^-(q, -\omega) = \text{Re} \Pi^+(q, \omega) \), in the following we focus only on the case for \( \omega > 0 \). We firstly evaluate the polarization function of the intrinsic case with \( \mu = 0 \) that implies \( \Pi^+(q, \omega) \) vanishes. After some simple algebra, we can obtain

\[
\Pi(q, \omega) = -\frac{g}{16\pi^2q} \int_0^\Lambda dk \int_{|k-q|}^{k+q} dk' \left( (k' - k)^2 - q^2 \right) \left( \frac{1}{\omega - k - k' + i\eta} - \frac{1}{\omega + k + k' + i\eta} \right),
\]

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34 The detailed calculation of the polarization function is given in the Supplemental Material. It should be emphasized that unlike the 2D Dirac fermion systems, the imaginary part \( \text{Im} \Pi(q, \omega) \) does not approach zero as \( \omega \to \infty \), so that one needs to apply the generalized Kramers-Kronig relation to evaluate the real part of polarization function of the intrinsic case from its imaginary part.

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37 S. A. Maier, \textit{Plasmonics: Fundamentals and Applications} (Springer, 2007).
where \( \Lambda \) is the cutoff. Using the Dirac identity \( \frac{1}{i \pi \eta} = \mathcal{P} \frac{1}{x} + i \pi \delta(x) \) one can get

\[
\text{Im } \Pi^-(q, \omega) = \frac{g}{16 \pi^2 q} \int_0^\Lambda dk \int_{|k-q|}^{k+q} dk' \left[ (k' - k)^2 - q^2 \right] \delta(\omega - k - k'),
\]

\[
\text{Re } \Pi^-(q, \omega) = - \frac{g}{16 \pi^2 q} \mathcal{P} \int_0^\Lambda dk \int_{|k-q|}^{k+q} dk' \left[ (k' - k)^2 - q^2 \right] \left( \frac{1}{\omega - k - k'} - \frac{1}{\omega + k + k'} \right),
\]

where the notation \( \mathcal{P} \) means the principal value of the integral. It is straightforward to calculate the imaginary part of the intrinsic polarization function

\[
\text{Im } \Pi^-(q, \omega) = - \frac{g q^2 \theta(\omega - q)}{24 \pi}.
\]

In fact, there are two different methods to calculate the real part of the polarization function. One is to directly carry out the integral. The other is to apply the Kramers-Kröning relation.

We at first perform the integration Eq. (29) to get the real part of the intrinsic polarization function. It is convenient to decompose this real part into two terms

\[
\text{Re } \Pi^-(q, \omega) = \text{Re } \Pi_1^-(q, \omega) + \text{Re } \Pi_2^-(q, \omega),
\]

where

\[
\text{Re } \Pi_1^-(q, \omega) = \frac{g}{16 \pi^2 q} \mathcal{P} \int_0^\Lambda dk \int_{|k-q|}^{k+q} dk' \left( (-3k + k') + \frac{(2k + \omega)^2 - q^2}{k' + k + \omega} \right),
\]

\[
\text{Re } \Pi_2^-(q, \omega) = \frac{g}{16 \pi^2 q} \mathcal{P} \int_0^\Lambda dk \int_{|k-q|}^{k+q} dk' \left( (-3k + k') + \frac{(2k - \omega)^2 - q^2}{k' + k - \omega} \right).
\]

After some cumbersome but straightforward calculation, we can get

\[
\text{Re } \Pi_1^-(q, \omega) = - \frac{2g q^2}{96 \pi^2} \log \left( \frac{2(\Lambda + \omega)^2 - q^2}{(q + \omega)^2} \right) + \frac{g}{96 \pi^2} \left[ +6 \omega(q + 2\Lambda) \right]
\]

\[
\text{Re } \Pi_2^-(q, \omega) = \frac{2g q^2}{96 \pi^2} \log \left( \frac{2(\Lambda - \omega)^2 - q^2}{(q - \omega)^2} \right) + \frac{g}{96 \pi^2} \left[ -6 \omega(q + 2\Lambda) \right]
\]

and

\[
\text{Re } \Pi^-(q, \omega) = - \frac{2g q^2}{96 \pi^2} \left( \log \left( \frac{(2\Lambda + \omega)^2 - q^2}{(q + \omega)^2} \right) + \log \left( \frac{(2\Lambda - \omega)^2 - q^2}{(q - \omega)^2} \right) \right)
\]

\[
\text{Re } \Pi_1^-(q, \omega) = \frac{g}{96 \pi^2} \left[ \frac{(2\Lambda + \omega)^2}{2\Lambda + \omega - q} - 2(2\Lambda + \omega)^2 - 3q(2\Lambda + \omega) \log \frac{2\Lambda + \omega + q}{2\Lambda + \omega - q} + \frac{16}{3} q^2 \right]
\]

\[
\text{Re } \Pi_2^-(q, \omega) = \frac{g}{96 \pi^2} \left[ \frac{(2\Lambda - \omega)^2}{2\Lambda - \omega - q} - 2(2\Lambda - \omega)^2 - 3q(2\Lambda - \omega) \log \frac{2\Lambda - \omega + q}{2\Lambda - \omega - q} + \frac{16}{3} q^2 \right].
\]

Two remarks about the real part of the intrinsic polarization function are in order here. First, \( \text{Re } \Pi_2^-(q, \omega) \) can also be obtained by replacing \( \omega \) with \( -\omega \) in \( \text{Re } \Pi_1^-(q, \omega) \). Second, it can be seen that \( \text{Re } \Pi^-(q, \omega) \) is an even function in \( \omega \) and will be valid for an arbitrary frequency.

Since the cutoff \( \Lambda \) is much larger than both \( q \) and \( \omega \), it is instructive to take a look at the expression of \( \text{Re } \Pi^-(q, \omega) \) in large \( \Lambda \) limit. Making use of the limits

\[
\lim_{\frac{t}{q} \to \infty} \left( \frac{t^3}{q} \log \frac{t + q}{t - q} - 2t^2 \right) = \frac{2q^2}{3}, \quad \lim_{\frac{t}{q} \to \infty} t \log \frac{t + q}{t - q} = 2q,
\]

(37)
we immediately verify that these underlined terms in Eq. (36) vanish with $t = 2\Lambda \pm \omega$ and then get a simple expression of $\text{Re} \, \Pi^{-}(q, \omega)$

$$\text{Re} \, \Pi^{-}(q, \omega) = -\frac{gq^{2}}{48\pi^{2}} \log \left( \frac{(2\Lambda + \omega)^{2} - q^{2}}{(2\Lambda - \omega)^{2} - q^{2}} \right).$$  

(38)

We can further simplify the terms of $q$ and $\omega$ in the numerator of the logarithmic function and have

$$\text{Re} \, \Pi^{-}(q, \omega) = -\frac{gq^{2}}{24\pi^{2}} \log \left| \frac{4\Lambda^{2}}{q^{2} - \omega^{2}} \right|.$$  

(39)

Now we turn to calculate the real part of the intrinsic polarization function $\Pi^{-}(q, \omega)$ using the Kramers-Krönig relations. Since the imaginary part does not approach zero as $\omega \rightarrow \infty$ one need to utilize the generalized Kramers-Krönig relation with one subtractions.

$$\text{Re} \, \Pi^{-}(q, \omega) = \text{Re} \, \Pi^{-}(q, 0) + \frac{\omega}{\pi} \int_{-\infty}^{\infty} d\xi \frac{\text{Im} \, \Pi^{-}(q, \xi)}{\xi(\xi - \omega)}.$$  

(40)

$$\text{Im} \, \Pi^{-}(q, \omega) = \text{Im} \, \Pi^{-}(q, 0) - \frac{\omega}{\pi} \int_{-\infty}^{\infty} d\xi \frac{\text{Re} \, \Pi^{-}(q, \xi)}{\xi(\xi - \omega)}.$$  

(41)

The zero frequency term $\text{Re} \, \Pi^{-}(q, 0)$ in Eq. (10) can be obtained from Eq. (29)

$$\text{Re} \, \Pi^{-}(q, 0) = -\frac{4gq^{2}}{96\pi^{2}} \log \left| \frac{(2\Lambda)^{2} - q^{2}}{q^{2}} \right| + \frac{2g}{96\pi^{2}} \left\{ \frac{(2\Lambda)^{3}}{q} \log \frac{2\Lambda + q}{2\Lambda - q} - \frac{2(2\Lambda)^{2} - 3q(2\Lambda) \log \frac{2\Lambda + q}{2\Lambda - q}}{3} \right\}.$$  

(42)

In the limit of large cutoff $\Lambda$, we find that the above underlined term vanishes. Neglecting the $q^{2}$ term in the numerator of the logarithmic function yields

$$\text{Re} \, \Pi^{-}(q, 0) = -\frac{gq^{2}}{24\pi^{2}} \log \frac{4\Lambda^{2}}{q^{2}}.$$  

(43)

The second term in Eq. (10) is calculated by carrying out the integration

$$\text{Re} \, \Pi^{-}(q, \omega) - \text{Re} \, \Pi^{-}(q, 0) = \frac{\omega}{\pi} \int_{-\infty}^{\infty} d\xi \frac{\text{Im} \, \Pi^{-}(q, \xi)}{\xi(\xi - \omega)} = -\frac{gq^{2}}{24\pi^{2}} \log \left| \frac{q^{2}}{q^{2} - \omega^{2}} \right|.$$  

(44)

Substituting Eq. (13) into Eq. (14) leads to the same result as Eq. (39), which differs slightly from the counterpart in Ref. 5. It should be pointed out that $\text{Re} \, \Pi^{-}(q, \omega)$ satisfies Eq. (11) by considering $\text{Im} \, \Pi^{-}(q, 0) = 0$. Therefore, the polarization function of the intrinsic case turns out to be

$$\Pi^{-}(q, \omega) = -\frac{gq^{2}}{24\pi^{2}} \left[ \log \left| \frac{4\Lambda^{2}}{q^{2} - \omega^{2}} \right| + i\pi \theta(\omega - q) \right].$$  

(45)

Following the similar procedure, one can reach the polarization function $\Pi^{+}(q, \omega)$ of the extrinsic case with $\mu > 0$

$$\text{Im} \, \Pi^{+}(q, \omega) = -\frac{gq^{2}}{8\pi^{2}} \left[ \theta(q - \omega) \left( \frac{\pi G(q, \omega)}{q^{2}} \theta(2\mu + \omega - q) - \frac{\pi G(q, -\omega)}{q^{2}} \theta(2\mu - \omega - q) \right) \right. $$

$$+ \theta(\omega - q) \left( -\frac{\pi}{3} \frac{G(-q, -\omega)}{q^{2}} \theta(q + \omega - 2\mu) \theta(2\mu + q - \omega) \right) \right],$$  

(46)

$$\text{Re} \, \Pi^{+}(q, \omega) = -\frac{gq^{2}}{8\pi^{2}} \left[ \frac{8q^{2} - G(q, \omega)H(q, \omega)}{3q^{2}} - \frac{G(-q, \omega)H(-q, \omega)}{q^{2}} - \frac{G(q, -\omega)H(q, -\omega)}{q^{2}} \right.$$

$$\left. - \frac{G(-q, -\omega)H(-q, -\omega)}{q^{2}} \right],$$  

(47)

where the functions $G(q, \omega)$ and $H(q, \omega)$ are defined by

$$G(q, \omega) = \frac{1}{12q} \left[ (2\mu + \omega)^{3} - 3q^{2}(2\mu + \omega) + 2q^{3} \right],$$  

(48)

$$H(q, \omega) = \log \left| \frac{2\mu + \omega - q}{q - \omega} \right|.$$  

(49)
Combining $\Pi^-(q, \omega)$ with $\Pi^+(q, \omega)$, we finally obtain the total polarization function of the 3D Weyl SM in Eq. (24).

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