REDUCTION AND CHAOTIC BEHAVIOR OF POINT VORTICES ON A PLANE AND A SPHERE

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Abstract. We offer a new method of reduction for a system of point vortices on a plane and a sphere. This method is similar to the classical node elimination procedure. However, as applied to the vortex dynamics, it requires substantial modification. Reduction of four vortices on a sphere is given in more detail. We also use the Poincaré surface-of-section technique to perform the reduction a four-vortex system on a sphere.

1. Equations of motion and first integrals of a vortex system on a plane.

Let us dwell briefly on the basic forms of the equations and first integrals of the point vortices’ motion on a plane. A more complete account can be found in [1, 2], where the hydrodynamic assumptions are also given, under which these equations are valid.

The equations of motion of $n$ point vortices with Cartesian coordinates $(x_i, y_i)$ and intensities $\Gamma_i$ can be written in the Hamiltonian form:

$$\Gamma_i x_i = \frac{\partial H}{\partial y_i}, \quad \Gamma_i y_i = -\frac{\partial H}{\partial x_i}, \quad 1 \leq i \leq n, \quad (1)$$

where the Hamiltonian is

$$H = -\frac{1}{4\pi} \sum_{i<j} \Gamma_i \Gamma_j \ln M_{ij}, \quad M_{ij} = |r_i - r_j|^2, \quad r_i = (x_i, y_i). \quad (2)$$

Here, the Poisson bracket is

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right). \quad (3)$$

Since equations (1) are invariant under the group of motions of plane $E(2)$, they have, beside the Hamiltonian, three integrals of motion:

$$Q = \sum_{i=1}^{n} \Gamma_i x_i, \quad P = \sum_{i=1}^{n} \Gamma_i y_i, \quad I = \sum_{i=1}^{n} \Gamma_i (x_i^2 + y_i^2), \quad (4)$$

which, however, are not involutive:

$$\{Q, P\} = \sum_{i=1}^{N} \Gamma_i, \quad \{P, I\} = -2Q, \quad \{Q, I\} = 2P. \quad (5)$$
Hereinafter, it will be more convenient to use instead of $I$ an integral of the form

$$D = \sum_{i<j}^n \Gamma_i \Gamma_j |r_i - r_j|^2 = (\sum_{i=1}^n \Gamma_i) I - P^2 - Q^2.$$  \hfill (6)

From these integrals, one can make two involutive integrals, $Q^2 + P^2$ and $I$; then, according to the general theory [5], the system’s order can be reduced by two degrees of freedom. Thus, the three-vortex problem is reduced to a system with one degree of freedom and, therefore, is integrable (Gröbli, Kirchhoff, Poincaré) [2], while the four-vortex problem is reduced to a system with two degrees of freedom. The latter problem, generally, is not integrable [6].

Effective reduction in the system of four vortices with intensities of the same sign was done by K. M. Khanin in [4]. He considered two pairs of vortices, for each of which corresponding action-angle variables were selected, while the general system was obtained as perturbation of the two unperturbed problems. Applying this procedure for construction of the perturbation, he proved (using the methods of KAM-theory) the existence of quasiperiodic solutions. As a small parameter, he took the inverse of the distance between the two pairs of vortices.

Reduction in the four-vortex problem in the case when all the four vortices have equal intensities and in the case when there are two identical pairs of vortices was done in the papers [4] and [12], respectively. A generalization of the latter reduction to the case of $N$ vortex pairs is offered in [13]. In [21], the KAM-theory is applied to the reduced equations of the four-vortex problem.

Reduction by one degree of freedom using the translational invariants $P$ and $Q$ was done by Lim in [7]. He introduced Jacobi (barycentric) coordinates (centered, in this case, at the center of vorticity), which have well-known analogs in the classical $n$-body problem in celestial mechanics [3]. Note that even this (partial) reduction made it possible to apply some methods of KAM-theory to investigation of the motion of point vortices [7].

In the paper [8], a formal Lie algebraic construction was applied to the case of $N$ arbitrary vortices to reduce the system’s order by two degrees of freedom.

2. Reduction on a plane. Let us reduce the system’s order by two degrees of freedom in the problem of $N$-vortices of arbitrary intensities. To this end, we will use an analogy with the planar $n$-body problem from celestial mechanics.

As is well known, in the $n$-body problem, one starts with elimination of the center of mass [3]. To do this, one can introduce the Jacobi variables

$$\xi_2 = r_2 - r_1, \quad \xi_3 = r_3 - \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}, \quad \ldots, \quad \xi_n = r_n - \frac{m_1 r_1 + \ldots + m_{n-1} r_{n-1}}{m_1 + \ldots + m_{n-1}},$$

which define the positions of the bodies in the frame of reference fixed to the center of mass ($\sum m_i r_i = 0$). Clearly, a similar procedure in the $n$-vortex problem (masses are replaced with vorticities $m_i \rightarrow \Gamma_i$) allows us to eliminate the center of vorticity. In other words, we use here the group of translations from [7] [10]. The obtained reduced system is invariant under rotations (group $SO(2)$) about the center of vorticity. In celestial mechanics, the reduction with the rotational symmetry is known as Jacobi’s node elimination and can be performed in various ways (see [3] [20]).

In vortex dynamics, the above procedure differs substantially (for the equations of motion are the first order differential equations in time) and can be performed by using polar coordinates for the Jacobi variables ($\xi_{ix} = \sqrt{2p_i} \cos \varphi_i; \xi_{iy} = \sqrt{2p_i} \sin \varphi_i$) with subsequent elimination of the angle of total rotation of the vortex system as a whole.

**Proposition 2.1.** A planar system of $N$ arbitrary vortices of nonzero total intensity ($\sum_{i=1}^N \Gamma_i \neq 0$) allows reduction by two degrees of freedom. The canonical
variables of the reduced system are

\[ q_i = \frac{\Gamma_{i+2} \sum_{k=1}^{i+1} \Gamma_k}{\sum_{k=1}^{i+2} \Gamma_k}, \quad \psi_i = \varphi_{i+2} - \varphi_2, \quad i = 1, \ldots, N - 2, \tag{7} \]

where

\[ \rho_i = \frac{1}{2} |r_i - R^{(i-1)}|^2, \quad \varphi_i = \arctg \left( \frac{y_i - y^{(i-1)}}{x_i - x^{(i-1)}} \right), \quad i = 2 \ldots N, \tag{8} \]

Here, \( R^{(i)} = \frac{\sum_{j=1}^{i} \Gamma_j r_j}{\sum_{j=1}^{i} \Gamma_j} \) defines the center of vorticity of the \( i \) vortices, while \( r_i \) is the radius-vector of the \( i \)th vortex (see Fig. 1).

**Proof.** Let us first calculate the Poisson bracket of \( \rho_i \) and \( \varphi_i \):

\[ \{ \rho_i, \varphi_j \} = \delta_{ij} \frac{\sum_{k=1}^{i} \Gamma_k}{\sum_{i=1}^{N} \Gamma_i \sum_{k=1}^{i-1} \Gamma_k}, \quad i = 2, \ldots, N. \tag{9} \]

If the numerator and the denominator do not vanish, we normalize the variables \( \rho_i \) so that the \( q_i, \psi_i \) found from (7) are canonical. It is easy to show that if \( \sum_{i=1}^{N} \Gamma_i = 0 \), then the vortices can be divided into a few groups, such that for each group \( \sum_{k=1}^{i} \Gamma_k = 0 \), \( i = 1, \ldots, N - 1 \).

Now we show that on a common level surface of the integrals \( Q, P, \) and \( D \), the Hamiltonian (4) can be expressed in terms of the variables (7). Indeed, the squared mutual distances \( M_{ij} \) can be expressed in terms of the vectors \( s_i, i = 1 \ldots N \), from the \( i \)th vortex to the center of vorticity of the \( (i - 1) \)th vortex' subsystem:

\[ M_{ij} = |s_i - s_j + \sum_{k=j}^{i-1} \frac{\Gamma_k s_k}{\sum_{l=1}^{i-1} \Gamma_l}|^2, \quad i > j, \quad s_i = r_i - \frac{\sum_{j=1}^{i-1} \Gamma_j r_j}{\sum_{j=1}^{i-1} \Gamma_j}. \tag{10} \]

According to (8), \( s_i = (\sqrt{2\rho_i} \cos \varphi_i, \sqrt{2\rho_i} \sin \varphi_i) \); here, \( \rho_i \) and \( \varphi_i \) are expressed in terms of \( q_i, \psi_i \) and the angle \( \psi_0 = \varphi_2 \) of total rotation of the vortex system as a whole about the common center of vorticity:

\[ \rho_2 = \left( \frac{D}{2 \sum_{i=1}^{N} \Gamma_i} - \sum_{k=1}^{N-2} q_k \right) \frac{\Gamma_1 + \Gamma_2}{\Gamma_1 \Gamma_2}, \quad \rho_i = \sum_{k=1}^{i} \frac{\Gamma_k q_{i-2}}{\sum_{k=1}^{i} \Gamma_k}, \quad \psi_i = \psi_{i-2} + \psi_0, \quad i = 3, \ldots, N. \tag{11} \]
Since the squared mutual distances are expressed in terms of all possible scalar products, \((s_i, s_j) = 2\sqrt{\rho_i \rho_j \cos(\varphi_i - \varphi_j)}\), they do not depend on \(\psi_0\). Thus, on the common level surface of the system’s integrals, the squared mutual distances and, consequently, the system’s Hamiltonian are expressed in terms of the variables (7).

Let us discuss in more detail the special case \(\sum_{i=1}^{N} \Gamma_i = 0\), which has no analog in celestial mechanics (since bodies’ masses are always positive, \(m_i > 0\)). In this case we say that the center of vorticity is at infinity, and the variables (7) are undefined (since one of the denominators vanishes). A corresponding reduction is described in the following way:

**Proposition 2.2.** When \(\sum_{j=1}^{N} \Gamma_j = 0\), the canonical variables of the reduced system (those obtained from reduction by two degrees of freedom), \(\tilde{\rho}_i\), \(\tilde{\varphi}_i\), \(i = 1, \ldots, N - 2\), are given by

\[
\tilde{\rho}_i = \frac{\Gamma_i \sum_{k=1}^{i-1} \Gamma_k}{\sum_{k=1}^{i} \Gamma_k} \rho_{i+1}, \quad \tilde{\varphi}_i = \varphi_{i+1}, \quad i = 1, \ldots, N - 2,
\]

where \(\rho_i\) and \(\varphi_i\) are given by (8).

**Proof.** As we showed earlier, the Poisson bracket of \(\rho_i\) and \(\varphi_i\) is defined by (9). Now we have \(\{\rho_N, \varphi_N\} = 0\), while \(\rho_N\) and \(\varphi_N\) are the integrals of motion. They are related to the standard integrals of motion (4) and (6) in the following way:

\[
P = \Gamma_N \sqrt{2\rho_N \cos \varphi_N}, \quad Q = \Gamma_N \sqrt{2\rho_N \sin \varphi_N}, \quad D = -\Gamma_N^2 \rho_N.
\]

As in the previous Proposition, the squared mutual distances \(M_{ij}\) are expressed in terms of \(s_i = (\sqrt{2\rho_i \cos \varphi_i}, \sqrt{2\rho_i \sin \varphi_i})\), according to (10). Hence, the system’s Hamiltonian on a common level surface of the system’s integrals depends on \(2(N - 2)\) variables, \((\rho_i, \varphi_i, i = 2 \ldots N - 1)\), and on the values of the two integrals, \(\rho_N\) and \(\varphi_N\).

There is an even more special (but no less interesting) case: \(\sum_{j=1}^{N} \Gamma_j = 0\) and \(D = 0\). Since in this case we have three involutive integrals (1), we can reduce the system’s order by three degrees of freedom. For example, the four-vortex problem yields under these conditions a special case of integrability [17, 18], while a system of five point vortices is reduced to a system with two degrees of freedom. Indeed,

**Proposition 2.3.** When \(\sum_{j=1}^{N} \Gamma_j = 0\) and \(D = 0\), a system allows reduction by three degrees of freedom. The canonical variables of the reduced system are defined by (7), \(i = 1 \ldots N - 3\).

**Proof.** As we showed in the previous Proposition, in the case of \(\sum_{j=1}^{N} \Gamma_j = 0\), the change of variables (8) reduces the system’s dimensionality by two. Upon the reduction procedure, we obtain the variables \(\rho_k\), \(\varphi_k\), \(k = 2 \ldots N - 1\). Now, to reduce the system’s order by one more degree of freedom, we should the variables (7) defined now for \(i = 1 \ldots N - 3\). Using the methods of Proposition 2.1, we can show that, on the common level surface of the first integrals, the Hamiltonian is expressed in terms of the variables \(q_i, \psi_i, i = 1 \ldots N - 3\), and depends on the parameters \(\rho_N\) and \(\varphi_N\).
3. The equations of motion and first integrals of a vortex system on $S^2$.
For $n$ point vortices on $S^2$, the Hamiltonian equations of motion in terms of spherical coordinates $(\theta_i, \varphi_i)$ can be written as follows [14]:

$$\dot{\theta}_i = \{\theta_i, \mathcal{H}\}, \quad \dot{\varphi}_i = \{\varphi_i, \mathcal{H}\}, \quad i = 1, \ldots, n, \quad (14)$$

with the Poisson bracket

$$\{\varphi_i, \cos \theta_k\} = \frac{\delta_{ik}}{R^2 \Gamma_i}, \quad (15)$$

where $\Gamma_i$ are the vortices’ intensities, and the Hamiltonian is

$$\mathcal{H} = -\frac{1}{4\pi} \sum_{i<k}^{n} \Gamma_i \Gamma_k \ln(M_{ik}) = -\frac{1}{4\pi} \sum_{i<k}^{n} \Gamma_i \Gamma_k \ln\left(4R^2 \sin^2 \frac{\gamma_{ik}}{2}\right). \quad (16)$$

Here, $R$ is the radius of the sphere, $M_{ij}$ is the squared distance between the $i$th and the $j$th vortices, measured along the chord, and $\gamma_{ik}$ is the angle between the vectors from the sphere’s center to the point vortices $i$ and $k$.

Beside the Hamiltonian, the equations (14) admit three additional independent involutive integrals:

$$F_1 = R \sum_{i=1}^{n} \Gamma_i \sin \theta_i \cos \varphi_i, \quad F_2 = R \sum_{i=1}^{n} \Gamma_i \sin \theta_i \sin \varphi_i, \quad F_3 = R \sum_{i=1}^{n} \Gamma_i \cos \theta_i. \quad (17)$$

The vector $F$, with components $(F_1, F_2, F_3) = F = \sum \Gamma_i r_i$ (where $r_i$ are the radius-vectors of the vortices), is called the moment of vorticity, its components commuting in the following way:

$$\{F_i, F_j\} = \frac{1}{R} \varepsilon_{ijk} F_k. \quad (18)$$

As in the planar case, the system can be reduced by two degrees of freedom, using involutive integrals, eg., $F_3$ and $F^2$.

Thus, for the case of three vortices, we obtain a completely integrable system (this system was independently found and studied in [14][15][19]). The four-vortex problem is reduced to a system with two degrees of freedom and, generally, is not integrable [11].

4. Reduction on a sphere. On a sphere, as distinct from a plane, it is impossible to regard symmetry transformations as translations and rotations (there are only rotations); nevertheless, the above reduction algorithm (which generalizes the Jacobi reduction) can itself be generalized. As above, we will consider in succession the moment of vorticity of two, three, ..., and $n$ vortices:

$$F_2 = \Gamma_1 r_1 + \Gamma_2 r_2, \quad \ldots, \quad F_n = \Gamma_1 r_1 + \ldots + \Gamma_n r_n = F,$$

where $r_i \in \mathbb{R}^3$ are the Cartesian coordinates of the vortices on a sphere embedded in $\mathbb{R}^3$. The squared moments $F^2_k$, $k = 2, \ldots, n$, have the following (obvious) properties:

1°. all $F^2_k$ commute with each other

$$\{F^2_k, F^2_m\} = 0;$$

2°. a squared moment $F^2_k$ commutes with the coordinates of all the vortices numbered $1, 2, \ldots, k$

$$\{F^2_k, x_i\} = \{F^2_k, y_i\} = \{F^2_k, z_i\} = 0, \quad i = 1, \ldots, k.$$
Thus, the squared moments \( F_2^2, \ldots, F_{n-1}^2 \) are invariant under the action of the group \( SO(3) \), commute with each other, and their number is half the dimension of the reduced system. Using \( 2^\alpha \), it is easy to add some relative angular variables of the reduced system to this set.

**Proposition 4.1.** A system of \( N \) vortices on a sphere, when \( F_N = \sum_{i=1}^{N} \Gamma_i r_i \neq 0 \), allows reduction by two degrees of freedom, using the canonical variables \( \rho_i, \psi_i \) given by

\[
\rho_i = |F_{i+1}|, \quad \tan \psi_i = \frac{\rho_i(F_{i+1}, r_{i+1} \times r_{i+2})}{(r_{i+1} \times F_{i+1}, r_{i+2} \times F_{i+1})}, \quad i = 1, \ldots, N-2, \quad (19)
\]

where \( \psi_i \) is the angle between the planes \((F_{i+1}, r_{i+2})\) and \((F_{i+1}, r_{i+1})\) (see Fig. 2).

**Proof.**

1. It can be shown straightforwardly from (19) that the variables \( \rho_i, \psi_i \) commute in the following way:

\[
\{\rho_i, \rho_j\} = \{\psi_i, \psi_j\} = 0, \quad \{\rho_i, \psi_j\} = \delta_{ij} \frac{\rho_i}{F_i}. \quad (20)
\]

2. Then we show that on the common level surface of the integrals \( F_N \), the Hamiltonian (16) can be expressed in terms of the variables (19). Since \( r_i \) and \( F_i \), \( i = 2, \ldots, N \), are linearly related:

\[
r_i = \frac{1}{r_i^2}(F_i - F_{i-1}), \quad i = 2, \ldots, N,
\]

the squared mutual intervortical distances \( M_{ij} \) can be expressed in terms of scalar products of \( F_1 = \Gamma_1 r_1, F_2, \ldots, F_N \). So, to prove the Proposition, it is sufficient to show that these scalar products on the common level surface of the integral \( F_N \) are completely defined by the variables (19).

Consider an algorithm for construction of the vectors \( F_i, i = 1 \ldots N - 1 \), using the known variables (19) and the values of \( F_N \). By construction, the angle \( \psi_{i-2} \) is the angle between the planes \((F_{i-1}, r_i)\) and \((F_{i-1}, r_{i-1})\), or, what is the same, between the planes \((F_{i-1}, F_i)\) and \((F_{i-1}, F_{i-2})\). Hence,

\[
F_{i-2} = \frac{F_{i-1}(F_{i-1}, F_{i-2})}{F_{i-1}^2} + \left( n_1 \sin \psi_{i-2} + n_2 \cos \psi_{i-2} \right) \sqrt{F_{i-2}^2 - \frac{(F_{i-1}, F_{i-2})^2}{F_{i-1}^2}},
\]

\[
n_1 = \frac{F_i \times F_{i-1}}{|F_i \times F_{i-1}|}, \quad n_2 = \frac{F_{i-1} \times (F_i \times F_{i-1})}{|F_{i-1} \times (F_i \times F_{i-1})|}. \quad (21)
\]
Using the definitions of $F_i$ and $\rho_i$ [19], we find that

$$\langle F_i, F_{i-1} \rangle = \frac{1}{2}(\rho_{i-1}^2 + \rho_{i-2}^2 - \Gamma_{i-1}^2 R^2), \quad F_i^2 = \rho_{i-1}^2. \tag{22}$$

Substituting these relations into (21), we obtain the recurrent expression for $F_{i-2}$ in terms of $F_{i-1}$, $F_i$, $\rho_i$, and $\psi_i$:

$$F_{i-2} = F_{i-1} \frac{\rho_{i-2}^2 + \rho_{i-3}^2 - \Gamma_{i-1}^2 R^2}{2\rho_{i-2}^2} + \left(n_1 \sin \psi_{i-2} + n_2 \cos \psi_{i-2}\right) \times$$

$$\times \sqrt{\rho_{i-3}^2 - \frac{(\rho_{i-2}^2 + \rho_{i-3}^2 - \Gamma_{i-1}^2 R^2)^2}{4\rho_{i-2}^2}}, \quad i = 3, \ldots, N,$$ \tag{23}

where $\rho_0 = |F_1| = \Gamma_1 R$.

With [22] it is easy to show that

$$F_i = \alpha_i F_N + \beta_i F_{N-1} + \gamma_i F_N \times F_{N-1}, \quad i = 1, \ldots, N - 2, \tag{24}$$

where the coefficients $\alpha_i$, $\beta_i$, $\gamma_i$ are expressed in terms of the coordinates [19]. Hence, all the scalar products ($F_i, F_j$) are expressed in terms of the coordinates [19] and the values of ($F_N, F_N$), ($F_{N-1}, F_{N-1}$), and ($F_N, F_{N-1}$). Using [22], we can express these values in terms of the coordinates [19] and the constants of the integrals $F_N$. In this way, the scalar products ($F_i, F_j$) (and, consequently, the mutual intervortical distances) would depend only on the variables [19]. Hence, the variables [19] allow the reduction by two degrees of freedom.

Let us now discuss the special case of further reduction by one more degree of freedom. This reduction is quite similar to the case of planar motion of the vortices with $\sum_{j=1}^{N} \Gamma_j = 0$, $D_N = 0$. We have

**Proposition 4.2.** When $F_N = \sum_{i=1}^{N} \Gamma_i r_i = 0$, the system [14] allows reduction by three degrees of freedom. The canonical variables of the reduced system are given by [19] with $i = 1, \ldots, N - 3$.

**Proof.** To prove the Proposition, we show, as in the previous case, that the scalar products ($F_i, F_j$) depend only on the variables $\rho_i, \psi_i$, $i = 1, \ldots, N - 3$.

Using [22], the vectors $F_i$ can be expressed in terms of $F_{N-1}$ and $F_{N-2}$:

$$F_i = \tilde{\alpha}_i F_{N-1} + \tilde{\beta}_i F_{N-2} + \tilde{\gamma}_i F_{N-1} \times F_{N-2}, \quad i = 1, \ldots, N - 3, \tag{25}$$

where $\tilde{\alpha}_i$, $\tilde{\beta}_i$, and $\tilde{\gamma}_i$ depend on the coordinates $\rho_i, \psi_i$, $i = 1, \ldots, N - 3$. Thus, the scalar products ($F_i, F_j$) are expressed in terms of the coordinates $\rho_i, \psi_i$, $i = 1, \ldots, N - 3$, and ($F_{N-1}, F_{N-1}$), ($F_{N-2}, F_{N-2}$), and ($F_{N-1}, F_{N-2}$). Here, $F_{N-1} = -\Gamma_{N-1} r_N$, therefore, $\rho_{N-2} = |F_{N-1}| = R|\Gamma_N|$ is an integral of motion. Hence, using [22], we can express the scalar products ($F_{N-1}, F_{N-1}$), ($F_{N-2}, F_{N-2}$), and ($F_{N-1}, F_{N-2}$) in terms of $\rho_i, \psi_i$, $i = 1, \ldots, N - 3$. In this way, the mutual intervortical distances depend only on $\rho_i, \psi_i$, $i = 1, \ldots, N - 3$, and the transformation [19] allows the reduction by three degrees of freedom.

Note that under the specified conditions, a four-vortex system on a sphere is integrable [2, 9].

5. **Explicit reduction of a four-vortex system on a sphere.** Now let us present a reduced system of four vortices on a sphere in explicit form. The system’s Hamiltonian is expressed in terms of the mutual distances by [10]. The mutual distances between four vortices on a sphere, given in terms of the canonical
variables of the reduced system \( \{ (\rho_i, \psi_j) \} = R^{-1} \delta_{ij}, i, j = 1, 2 \) and the squared integral \( F^2 = F_4^2 = c^2 = \text{const} \) read:

\[
M_{12} = \frac{(\Gamma_1 + \Gamma_2)^2 R^2 - \rho_1^2}{\Gamma_1 \Gamma_2}, \quad M_{23} = \frac{(\Gamma_2 + \Gamma_3)^2 R^2 - \rho_2^2 + 2(F_1 F_3)}{\Gamma_2 \Gamma_3},
\]

\[
M_{13} = \frac{(\Gamma_1 + \Gamma_3)^2 - \Gamma_2^2 - \Gamma_3^2}{\Gamma_1 \Gamma_3} R^2 + \rho_1^2 - 2(F_1, F_3), \quad M_{14} = 2R^2 + 2 \frac{(F_1, F_3) - (F_1, F)}{\Gamma_1 \Gamma_4},
\]

\[
M_{34} = \frac{(\Gamma_3 + \Gamma_4)^2 R^2 - \rho_1^2 + c^2 + 2(F_2, F)}{\Gamma_3 \Gamma_4},
\]

\[
M_{24} = \frac{(2\Gamma_2 \Gamma_4 - \Gamma_3^2) R^2 + \rho_2^2 + \rho_2^2 - 2(F_2, F) + 2(F_1, F) - 2(F_1, F_3)}{\Gamma_2 \Gamma_4}.
\]

Here, the vortices' intensities are arbitrary. The scalar products are:

\[
(F_1, F_2) = \frac{1}{2}(\rho_1^2 + \Gamma_1^2 R^2 - \Gamma_2^2 R^2), \quad (F_2, F_3) = \frac{1}{2}(\rho_2^2 + \rho_1^2 - \Gamma_3^2 R^2),
\]

\[
(F_3, F) = \frac{1}{2}(c^2 + \rho_2^2 - \Gamma_2^2 R^2),
\]

\[
(F_2, F) = \frac{(F_3, F)(F_2, F_3)}{\rho_2^2} + \left( c^2 - \frac{(F_3, F)^2}{\rho_2^2} \right) \cos \psi_2 \sqrt{\frac{\rho_1^2 \rho_2^2 - (F_2, F_3)^2}{\rho_2^2 c^2 - (F_3, F)^2}},
\]

\[
(F_1, F_3) = \frac{(F_2, F_3)(F_1, F_2)}{\rho_1^2} + \left( \rho_2^2 - \frac{(F_2, F_3)^2}{\rho_1^2} \right) \cos \psi_1 \sqrt{\frac{\rho_1^2 \Gamma_1^2 R^2 - (F_1, F)^2}{\rho_1^2 \rho_2^2 - (F_1, F_3)^2}}.
\]

\[
(F_1, F) = \frac{(F_2, F)(F_1, F_2)}{\rho_1^2} + \left( \frac{(F_3, F) - (F_2, F)(F_2, F_3)}{\rho_1^2} \right) \cos \psi_1 \times
\]

\[
\times \sqrt{\frac{\rho_1^2 \Gamma_1^2 R^2 - (F_1, F)^2}{\rho_1^2 \rho_2^2 - (F_1, F_3)^2}} + \frac{\sin \psi_1 \sin \psi_2}{\rho_1 \rho_2} \sqrt{(\rho_1^2 \Gamma_1^2 R^2 - (F_1, F)^2)(\rho_2^2 c^2 - (F_3, F)^2)}.
\]

6. **The Poincaré section of a four-vortex system on a sphere.** The obtained systems of reduced canonical variables can be applied to various analytical and numerical studies. Consider, for example, chaos in the system of four vortices on a sphere.

Below, we present Poincaré surface-of-section plots for a reduced system of four vortices on a sphere. As far as we know, such maps have not been plotted before. As a plane of section, we choose \( \psi_1 = \text{const} \). The intersection between this plane and the isoenergetic surface \( \mathcal{H}(\rho_1, \psi_1, \rho_2, \psi_2) = E = \text{const} \) is some two-dimensional surface (generally, disconnected) in the space \( \rho_1, \rho_2, \psi_2 \). The system’s phase flow on this surface generates a Poincaré map. As a rule, the surface is complicated, therefore, we give here a 3D view of the Poincaré map without projecting it on a plane.

**Remark 1.** Construction of a Poincaré section in three-dimensional space allows us to ignore various singularities of the projection and does not result in emergence of fictitious objects, mentioned in [12] (namely, crescent-shaped tori, which are due to the projection’s singularities).
Figure 3. The Poincaré maps for the problem of four vortices of equal intensities on a sphere, $D = 3.55$. The plane of section is $\psi_1 = \frac{3\pi}{2}$. The energy values are: $E = 0.8$ (Figs. a and b), $E = 0.67$ (Figs. c and d), $E = 0.64$ (Fig. e).
Figure 3 shows the phase portraits in the space $\rho_1, \rho_2, \psi_2$ for the case of four equal vortices and $D = 3.55$. The plane of section is $\psi_1 = \frac{\pi}{2}$. The energy values are: $E = 0.8$ (Figs. a and b), $E = 0.67$ (Figs. c and d), $E = 0.64$ (Fig. e).

Note that for small (close to the Thompson configuration), as well as for sufficiently large, energies, the system’s phase portrait is almost regular (Fig. 3 a, b, e). While for intermediate energy values, the system’s phase flow becomes almost completely chaotic (Fig. 3 c, d).

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