PARTITIONS OF FROBENIUS RINGS
INDUCED BY THE HOMOGENEOUS WEIGHT

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Abstract. The values of the homogeneous weight are determined for finite Frobenius rings that are a direct product of local Frobenius rings. This is used to investigate the partition induced by this weight and its dual partition under character-theoretic dualization. A characterization is given of those rings for which the induced partition is reflexive or even self-dual.

1. Introduction

The homogeneous weight has been studied extensively in the literature of codes over rings. It has been introduced by Constantinescu and Heise [7] as a generalization of both the Hamming weight on finite fields and the Lee weight on \( \mathbb{Z}_4 \). Its main feature is that the average weight of the elements in a nonzero principal ideal is the same constant for all such ideals. The weight has been further generalized to arbitrary non-commutative finite rings by Greferath and Schmidt [18] as well as Honold and Nechaev [23]. The homogeneous weight has proven to be an important tool in ring-linear coding. For instance, in [11] Duursma et al. construct non-linear codes with the best parameters so far using certain ring-linear codes and where the ring is endowed with the homogenous weight.

These and other properties of the homogeneous weight have led to a detailed study of this weight. Among other things, it has been shown that the MacWilliams extension theorem remains true for isomorphisms preserving the homogeneous weight, see Constantinescu et al. [8] for codes over the integer residue ring \( \mathbb{Z}_N \), Wood [28] and Greferath and Schmidt [18] for codes over general finite Frobenius rings, and Greferath et al. [16] for codes over the Frobenius module of a finite ring.

On the other hand, so far no explicit MacWilliams identity for the homogeneous-weight enumerators of codes has been established in any general form. Such identities relate a suitably defined weight enumerator of a code to the weight enumerator (or a dual version thereof) of its dual code. MacWilliams identities are well known for many weight functions, e.g., the Hamming weight, the complete weight, the symmetrized Lee weight, and more.

The non-existence of an explicit MacWilliams identity for the homogeneous weight is due to the fact that the partition induced by this weight does not behave as well under dualization as those for the Hamming weight or the other weights just

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mentioned. More precisely, the induced partition is in general not self-dual with respect to a certain character-theoretic dualization. In this paper we will study the homogeneous weight partition on a certain class of finite Frobenius rings, which includes all commutative Frobenius rings, and provide a characterization of those rings for which the partition is reflexive (that is, coincides with its bidual) or even self-dual. Reflexivity, which is weaker than self-duality, guarantees a MacWilliams identity because in this case the partition and its dual have the same number of partition sets. In this case we will also provide the associated Krawtchouk coefficients. With these data, an explicit MacWilliams identity relating the corresponding partition enumerators is simply an instance of the general theory about reflexive partitions, see for instance [14] or Camion [5]. A precursor of these ideas is the paper [3] by Byrne et al.

We will study the homogeneous weight on Frobenius rings that are a direct product of local Frobenius rings. For the latter the homogeneous weight is well-known [3] and takes a very simple form. Making use of an explicit formula for the values of the homogeneous weight provided by Honold [20], we will be able to compute the values of the homogeneous weight on the specified Frobenius rings. This is carried out in Section 3.

In Section 4 we then go on and study the partition induced by the homogeneous weight. We will see that all elements outside the socle have the same weight, thus form one partition set, whereas in the socle the homogeneous partition is closely related to the product of Hamming partitions that are induced by a suitable direct product representation of the ring. In fact, we will show that the homogeneous partition is reflexive if and only if its restriction to the socle coincides with the just described product of the Hamming partitions. We also give a characterization of the rings for which the homogeneous partition is reflexive. It is given in terms of the orders of the residue fields of the local component rings. Finally, we will prove that the homogeneous weight partition is self-dual if and only if it is reflexive and the ring is semisimple.

2. Frobenius rings and partitions

Throughout this section, let $G = (G, +)$ be a finite abelian group, and let $R$ be a finite ring with unity. Denote its group of units by $R^*$. Our main subject is partitions and their duals of $R$ or $R^n$, but occasionally we need to consider partitions of a group; mainly for the additive group of an ideal of $R$. For this reason we present the main notions of this section for groups with special emphasis on rings.

Denote by $\hat{G}$ the complex character group $\text{Hom}(G, \mathbb{C}^*)$ of $G$ with addition $(\chi_1 + \chi_2)(a) := \chi_1(a)\chi_2(a)$. The principal character is denoted by $\varepsilon$, thus $\varepsilon(a) = 1$ for all $a \in G$. It is well known that $G$ and $\hat{G}$ are isomorphic. The most fundamental property of characters on $G$ is the orthogonality relation
\begin{equation}
\sum_{a \in G} \chi(a) = |G|\delta_{\chi, \varepsilon},
\end{equation}
where $\delta$ denotes the Kronecker symbol. For a subgroup $H \leq G$ the dual subgroup is $H^\perp := \{ \chi \in \hat{G} \mid \chi(h) = 1 \text{ for all } h \in H \}$.

For a ring $R$ and $n \in \mathbb{N}$ we denote by $\hat{R}^n$ the character group of the additive group $(R^n, +)$. This group can be endowed with an $R$-$R$-bimodule structure via the left and right scalar multiplications
\begin{equation}
(r \cdot \chi)(v) = \chi(vr) \quad \text{and} \quad (\chi \cdot r)(v) = \chi(rv) \quad \text{for all } r \in R \text{ and } v \in R^n.
\end{equation}
Recall that $R$ is a Frobenius ring if $R_{\text{soc}}(R) \cong R/(\text{rad}(R))$, where $\text{soc}(R)$ denotes the socle of the left $R$-module $R$ and $\text{rad}(R)$ is the Jacobson radical of $R$ (it is well-known that the existence of a left isomorphism implies the right analogue). Since $\text{rad}(R)$ is a two-sided ideal, $R/\text{rad}(R)$ is even a ring. Furthermore, $\text{soc}(R) = \text{soc}(R_R)$, and we will simply write $\text{soc}(R)$ for the socle. If $R$ is local, that is, $\text{rad}(R)$ is the unique maximal left (resp. right) ideal of $R$, then $\text{soc}(R)$ is the unique minimal ideal [25, Ex. (3.14)]. In this case $R/\text{rad}(R)$ is called the residue field of $R$.

It follows from Lamprecht [26] (see also Hirano [19, Th. 1], Honold [20, p. 409], and Wood [29, Th. 3.10]) that $R$ is Frobenius if and only if $\hat{R}$ and $R$ are isomorphic left $R$-modules, or, equivalently, isomorphic right $R$-modules. More precisely, there exists a character $\chi \in \hat{R}$ such that $R \to \hat{R}$, $r \mapsto r \cdot \chi$, (resp. $R \to \hat{R}$, $r \mapsto \chi \cdot r$) is an isomorphism of left (resp. right) $R$-modules. Any such $\chi$ is called a generating character of $R$. By virtue of [29, Prop. 5.1] that any two generating characters $\chi, \chi'$ differ by a unit, i.e., $\chi' = w \cdot \chi$ and $\chi' = \chi \cdot u'$ for some $u, u' \in R^*$.

As a consequence, for each $n$, the maps
\begin{align}
\alpha_l : R^n &\to \hat{R}^n, \quad v \mapsto \chi(\langle -, v \rangle), \\
\alpha_r : R^n &\to \hat{R}^n, \quad v \mapsto \chi(\langle v, - \rangle),
\end{align}
are left (resp. right) $R$-module isomorphisms (here $\langle v, w \rangle$ denotes the standard inner product on $R^n$).

Many standard examples of rings are Frobenius, for instance, the integer residue rings $\mathbb{Z}_N$, finite fields, finite chain rings, finite group rings as well as matrix ring over Frobenius rings, and direct products of Frobenius rings. For details we refer to Wood [29, Ex. 4.4] and Lam [25, Sec. 16.B]. The ring $\mathbb{F}_2[x, y]/(x^2, y^2, xy)$ is a local, non-Frobenius ring; see [6, Ex. 3.2].

For a subgroup $C$ of $R^n$ (called an additive code), the dual code is, as before, $C^\perp = \{ \chi \in \hat{R}^n \mid \chi(v) = 1 \text{ for all } v \in C \}$. It is obvious that if $C$ is a left (resp. right) submodule, then $C^\perp$ is a right (resp. left) submodule of $\hat{R}$. Moreover, $C^\perp \cap C = [C^\perp : C] = [R^n : C]$. Applying the isomorphisms in (2.3) to the additive group of $C^\perp$ results in the left and right hand side dual (see also [29, Th. 7.7])
\begin{align}
C^\perp &= \alpha_l^{-1}(C^\perp) = \{ v \in R^n \mid \langle w, v \rangle = 0 \text{ f. a. } w \in C \} \quad \text{for } C \leq_R (R^n), \\
C &= \alpha_r^{-1}(C^\perp) = \{ v \in R^n \mid \langle v, w \rangle = 0 \text{ f. a. } w \in C \} \quad \text{for } C \leq (R^n)_R.
\end{align}

Since $C^\perp$ is a right $R$-module and $C$ a left one, the maps $\alpha_l$ and $\alpha_r$ act only as group isomorphisms. It follows $(C^\perp)_R \overset{\perp}{=} C$ for any $C \leq_R (R^n)$ (and $(C^\perp)_L \overset{\perp}{=} C$ for right submodules), which forms a generalization of the double annihilator property of Frobenius rings [25, Th. 15.1].

We now turn to partitions of a group $G$. A partition $\mathcal{P} = (P_m)_{m=1}^M$ will mostly be written as $\mathcal{P} = P_1 | P_2 | \ldots | P_M$. The sets of the partition are called its blocks, and we write $|\mathcal{P}|$ for the number of blocks in $\mathcal{P}$. Two partitions $\mathcal{P}$ and $\mathcal{Q}$ are called identical if $|\mathcal{P}| = |\mathcal{Q}|$ and the blocks coincide after suitable indexing. Moreover, $\mathcal{P}$ is called finer than $\mathcal{Q}$ (or $\mathcal{Q}$ is coarser than $\mathcal{P}$), written as $\mathcal{P} \leq \mathcal{Q}$, if every block of $\mathcal{P}$ is contained in a block of $\mathcal{Q}$. Note that if $\mathcal{P} \leq \mathcal{Q}$ then $|\mathcal{P}| \geq |\mathcal{Q}|$. Denote by $\sim_\mathcal{P}$ the equivalence relation induced by $\mathcal{P}$, thus, $v \sim_\mathcal{P} v'$ if $v, v'$ are in the same block of $\mathcal{P}$.

The following notion of a dual partition will be crucial for us. The left-sided version has been introduced for Frobenius rings by Byrne et al. [3, p. 291] and goes back to the notion of $F$-partitions as introduced by Zinoviev and Ericson in [30].
also [31]. Reflexive partitions, defined below, are exactly the partitions that induce abelian association schemes as studied in a more general context by Delsarte [9], Camion [5], and others, see also [10]. For an overview of these various approaches and their relations in the language of partitions, see also [14].

Throughout, we use the notation \([n] := \{1, \ldots, n\}\) and \([n]_0 := \{0, \ldots, n\}\).

**Definition 2.1.** (a) Let \(\mathcal{P} = P_1 | P_2 | \ldots | P_M\) be a partition of \(G\). The dual partition, denoted by \(\hat{\mathcal{P}}\), is the partition of \(\hat{G}\) defined via the equivalence relation \(\chi \sim \hat{\chi}' : \iff \sum_{g \in P_m} \chi(g) = \sum_{g \in \hat{P}_m} \chi'(g)\) for all \(m \in [M]\). The partition \(\mathcal{P}\) is called reflexive if \(\hat{\mathcal{P}} = \mathcal{P}\).

(b) Let \(R\) be a Frobenius ring with generating character \(\chi\), and let \(\mathcal{P} = P_1 | P_2 | \ldots | P_M\) be a partition of the group \((R^n, +)\). The left and right \(\chi\)-dual partition of \(\mathcal{P}\), denoted by \(\hat{\mathcal{P}}^{[x,l]}\) and \(\hat{\mathcal{P}}^{[x,r]}\) are defined as the preimage of \(\hat{\mathcal{P}}\) under the isomorphisms \(\alpha_l\) and \(\alpha_r\) in (2.3). Thus, \(\hat{\mathcal{P}}^{[x,l]}\) and \(\hat{\mathcal{P}}^{[x,r]}\) are the partitions of \(R^n\) given by the equivalence relations

\[
v \sim_{\hat{\mathcal{P}}^{[x,l]}} v' : \iff \sum_{w \in P_m} \chi(\langle w, v \rangle) = \sum_{w \in P_m} \chi(\langle w, v' \rangle) \quad \text{for all } m \in [M]
\]

and

\[
v \sim_{\hat{\mathcal{P}}^{[x,r]}} v' : \iff \sum_{w \in P_m} \chi(\langle v, w \rangle) = \sum_{w \in P_m} \chi(\langle v', w \rangle) \quad \text{for all } m \in [M].
\]

It is not hard to find examples of rings and partitions for which the left and right \(\chi\)-dual partitions do not coincide for any generating character \(\chi\) (take, e.g., the ring in [29, Ex. 1.4(iii)]).

The dual partitions are related via \(\hat{\mathcal{P}}^{[x,l]} = \hat{\mathcal{P}}^{[x,r]} = \hat{\mathcal{P}}\), see [1, Prop. 4.4].

Thus, \(\mathcal{P} = \hat{\mathcal{P}}^{[x,l]}\) if and only if \(\mathcal{P} = \hat{\mathcal{P}}^{[x,r]}\), and either identity characterizes reflexivity. Moreover,

\[(2.6) \quad \mathcal{P} = \hat{\mathcal{P}}^{[x,l]} \iff \mathcal{P} = \hat{\mathcal{P}}^{[x,r]}.
\]

If \(\mathcal{P} = \hat{\mathcal{P}}^{[x,l]}\) the partition \(\mathcal{P}\) is called \(\chi\)-self-dual.

The (generalized) Kravchouk coefficients \(K_{\ell,m}\), \(\ell \in [L], m \in [M]\), of a partition pair \((\mathcal{P} = (P_m)_{m=1}^M, \hat{\mathcal{P}} = (Q_{\ell})_{\ell=1}^L)\), are defined as \(K_{\ell,m} = \sum_{g \in P_m} \chi(g)\), where \(\chi\) is any element in \(Q_{\ell}\). In the ring setting this read as follows. Let \(\hat{\mathcal{P}}^{[x,l]} = Q_{l1} | \ldots | Q_{lL}\) and \(\hat{\mathcal{P}}^{[x,r]} = Q'_{l1} | \ldots | Q'_{lL}\). Hence \(Q_{l\ell}' = \alpha_{l-1}(Q_{l\ell})\) and \(Q_{l\ell}' = \alpha_{r-1}(Q_{l\ell})\) for \(\ell \in [L]\). Then (2.3) yields

\[(2.7) \quad K_{\ell,m} = \sum_{w \in P_m} \chi(\langle w, v' \rangle) = \sum_{w \in P_m} \chi(\langle v'', w \rangle),
\]

where \(v'\) is any element in \(Q_{l\ell}'\) and \(v''\) is any element in \(Q_{l\ell}'\). In particular, \(K_{\ell,m}\) does not depend on the sidedness of the dual partition.

We briefly illustrate that the dual partition depends in general on the choice of the generating character \(\chi\), even in the commutative case.

**Example 2.2.** On the field \(\mathbb{F}_4 = \{0, 1, a, a^2\}\), where \(a^2 = a + 1\), consider the characters \(\chi, \hat{\chi}\) given by \(\chi(0) = \chi(1) = 1, \chi(a) = \chi(a^2) = -1\) and \(\hat{\chi}(0) = \hat{\chi}(a) = 1, \hat{\chi}(1) = \hat{\chi}(a^2) = -1\). Thus we have isomorphisms \(\alpha : \mathbb{F}_4 \rightarrow \mathbb{F}_4, r \mapsto r \cdot \chi\) and
β : \mathbb{F}_4 \to \mathbb{F}_4, r \mapsto r \cdot \bar{\chi}. Consider the partition \( P = 0 \mid 1 \mid a, a^2 \). Writing \( \hat{P}^{[x]} \) for \( \hat{P}^{[x]} = \mathbb{P}^{[x]} \), one easily verifies that \( \hat{P}^{[x]} = P \), and thus \( P \) is \( \chi \)-self-dual. On the other hand, \( \hat{P}^{[x]} = 0 \mid 1 \mid a^2 \mid a \). Hence \( P \) is not \( \bar{\chi} \)-self-dual.

For many standard partitions on Frobenius rings, e.g., the Hamming partition, the dual does not depend on the choice of the generating character. In the next section we will see that this is also the case for the main topic of this paper, partitions induced by the homogeneous weight.

Let us return to the general situation of partitions of groups. We have the following simple observations.

**Remark 2.3.**

(a) The singleton \( \{ \varepsilon \} \) is always a block of \( \hat{P} \), which follows from the orthogonality relations (2.1). Using (2.3) we also conclude that \( \{ 0 \} \) is a block of \( \hat{P}^{[x]} \) and \( \hat{P}^{[x]} \).

(b) If \( P \leq Q \), then \( \hat{P}^{[x]} \leq \hat{Q}^{[x]} \) and \( \hat{P}^{[x,r]} \leq \hat{Q}^{[x,r]} \). This follows directly from the definition of the dual partitions since each block of \( Q \) is the union of blocks of \( P \).

As has been shown in various forms in the literature [9, 31, 3, 14], a partition \( P \) of \( R^n \) and its dual partition \( \hat{P} \) of \( \hat{R}^n \) allow a MacWilliams identity: applying a certain MacWilliams transformation to the \( \hat{P} \)-partition enumerator of a code \( C \subseteq \hat{R}^n \) results in the \( P \)-enumerator of its dual \( C^⊥ \subseteq R^n \). For an overview in the language of this paper see [14, Sec. 2]. The most symmetric situation arises for reflexive (or even self-dual) partitions, in which case the transformation can be carried out in both directions, and thus the two enumerators determine each other uniquely. Most, if not all, classical examples of MacWilliams identities are instances of this general MacWilliams identity based on a self-dual partition (for instance, for the Hamming weight, symmetrized Lee weight, complete weight, and the rank metric).

In the next section we will study the partition on a Frobenius ring induced by the homogeneous weight and investigate for which rings the partition is reflexive or even self-dual. Our main tool for characterizing reflexivity of a partition is the following convenient criterion from [14, Th. 3.1]. In the framework of association schemes the result also appears in [15, Th. 10.1] (if the blocks of the partition are invariant under taking inverses) and in [21, Fact V.2].

**Theorem 2.4.** For any partition \( P \) on \( G \) we have \(|P| \leq |\hat{P}| \) and \( \hat{P} \leq P \). Moreover, \( P \) is reflexive if and only if \(|P| = |\hat{P}| \).

We close this section with several specific instances of reflexive partitions that will be needed in the next section.

**Example 2.5.** Let \( R \) be a Frobenius ring with generating character \( \chi \), and denote by \( P^{[x]} \) (resp. \( P^{[x,r]} \)) the partition given by the orbits of the left (resp. right) action of \( R^* \) on \( R \). Thus, the blocks of \( P^{[x]} \) are given by the distinct orbits \( O_{x,l} = \{ ux \mid u \in R^* \} \), \( x \in R \), whereas the blocks of \( P^{[x,r]} \) are given by the orbits \( O_{x,r} = \{ xu \mid u \in R^* \} \).

It follows from [1, Prop. 4.6] that \( \hat{P}^{[x]} = P^{[x,r]} \) and \( \hat{P}^{[x,r]} = P^{[x,l]} \).

The Hamming partition and its Krawtchouk coefficients will be needed in the following form in the next section.

**Example 2.6** (see [24, Lem. 2.6.2]). Let \( G = A_1 \times \ldots \times A_n \), where each \( A_i \) is a finite abelian group of order \( q \). Then \( \hat{G} = \hat{A}_1 \times \ldots \times \hat{A}_n \). Let \( P \) be the partition
of $G$ induced by the Hamming weight. It is well known that the dual partition $\hat{P}$ is the partition induced by the Hamming weight on $A_1 \times \ldots \times A_n$ and the Krawtchouk coefficients are $K_{\ell,m} = K_m^{(n,q)}(\ell)$, where
\begin{equation}
K_m^{(n,q)}(x) = \sum_{j=0}^{m} (-1)^j (q-1)^{m-j} \binom{x}{j} \binom{n-x}{m-j}
\end{equation}
is the Krawtchouk polynomial. All of this shows that the Hamming partition on $G$ is reflexive. Finally, the Hamming partition on the module $R^n$ is self-dual with respect to any generating character $\chi$ by virtue of (2.3).

When studying the partition induced by the homogeneous weight, we will need to consider product partitions. Let $G = A_1 \times \ldots \times A_n$, where $A_1, \ldots, A_n$ are finite abelian groups, and let $P_i$ be a partition of $A_i$ for $i \in [n]$. Write $P = P_{1,1} | P_{1,2} | \ldots | P_{n,M}$. Then the product partition on $G$ is defined as
\begin{equation}
Q := P_1 \times \ldots \times P_n := (P_{1,m_1} \times \ldots \times P_{n,m_n})_{(m_1,\ldots,m_n) \in [M_1] \times \ldots \times [M_n]}.
\end{equation}
Parts of the next result also appeared in [31, Th. 4] and [5, Th. 4.86].

**Theorem 2.7** ([14, Th. 4.3 and proof]). The dual partition of $Q$ is $\hat{Q} = \hat{P}_1 \times \ldots \times \hat{P}_n$. As a consequence, if each $P_i$ is reflexive then $Q$ is reflexive. The Krawtchouk coefficients of $(Q, \hat{Q})$ are
\begin{align*}
K_{(\ell_1,\ldots,\ell_n),(m_1,\ldots,m_n)} &= \prod_{i=1}^{n} K_{\ell_i,m_i}^{(i)} \\
\text{for } (\ell_1,\ldots,\ell_n), (m_1,\ldots,m_n) &\in [M_1] \times \ldots \times [M_n], \text{ and where } K_{\ell_i,m_i}^{(i)} \text{ are the Krawtchouk coefficients of } (P_i, \hat{P}_i).
\end{align*}

The following result shows that the trivial extension of a partition behaves well under dualization. The notation $\chi|_H$ stands for the restriction of $\chi$ to $H$, whereas $\varepsilon_G$ and $\varepsilon_H$ denote the principal characters on $G$ and $H$, respectively.

**Proposition 2.8.** Let $H \leq G$ be a subgroup of $G$, and let $P = P_0 | \ldots | P_M$ be a partition of $H$. Let $P = Q_0 | Q_1 | \ldots | Q_L$, where $Q_0 = \{\varepsilon_H\}$ (see Remark 2.3(a)). Thus $P$ is a partition of $H$. Define $P_{-1} := G \setminus H$. Then $P' = P_{-1} | P_0 | \ldots | P_M$ is a partition of $G$. The dual partition of $P'$ is given by $\hat{P}' = Q'_{-1} | Q'_0 | Q'_1 | \ldots | Q'_L$, where $Q'_{-1} = H^\perp \setminus \{\varepsilon_G\}$, $Q'_0 = \{\varepsilon_G\}$ and $Q'_\ell = \{\chi \in G | H^\perp \mid \chi|_H \in Q_\ell\}$ for $\ell \in [L]$. The Krawtchouk coefficients of $(P', \hat{P}')$ are
\begin{equation}
(K'_{\ell,m})_{\ell=-1,\ldots,L, m=-1,\ldots,M} = \begin{pmatrix}
-|H| & |P_0| & \ldots & |P_M| \\
|P_{-1}| & K_{0,0} & \ldots & K_{0,M} \\
0 & K_{1,0} & \ldots & K_{1,M} \\
& \vdots & \ddots & \vdots \\
0 & K_{L,0} & \ldots & K_{L,M}
\end{pmatrix},
\end{equation}
where $K_{\ell,m}, \ell \in [L]_0, m \in [M]_0$, are the Krawtchouk coefficients of $(P, \hat{P})$. As a consequence, if $P$ is reflexive then so is $P'$.

**Proof.** We have to consider various cases.

1) Let $\chi \in Q'_\ell$ for $\ell \in [L]$. Then $\sum_{a \in P_m} \chi(a) = \sum_{a \in P_m} \chi|_H(a) = K_{\ell,m}$ for each $m \in [M]_0$. Furthermore, (2.1) yields $\sum_{a \in G \setminus H} \chi(a) = -\sum_{a \in H} \chi(a) = \ldots$
2) Next, let \( \chi \in Q'_{-1} = H^\perp \setminus \{ \varepsilon_G \} \). Then \( \chi(a) = 1 \) for all \( a \in P_m, m \in [M]_0 \). Thus \( \sum_{a \in P_m} \chi(a) = |P_m| \) for all \( m \in [M]_0 \). Moreover, \( \sum_{a \in G \backslash H} \chi(a) = -\sum_{a \in H} \chi(a) = -|H| \).

3) For \( \chi = \varepsilon_G \) we have \( \sum_{a \in P_m} \chi(a) = |P_m| \) for all \( m \in \{-1,0,\ldots,M \} \). For the same reason, \( |P_m| = K_{a,m} \) for all \( m \in [M]_0 \).

In all of these cases the sums do not depend on the choice of \( \chi \) within the specified set, and thus the partition \( Q := Q'_{-1} | Q'_0 | Q'_1 | \ldots | Q'_L \) is finer than or equal to \( \overrightarrow{\mathcal{P}} \).

The above also establishes the Krawtchouk coefficients stated in (2.10). Since \( \hat{\mathcal{P}} \) is the dual partition of \( \mathcal{P} \), Definition 2.1(a) implies that no two rows of the matrix in (2.10) coincide. This means that if \( \chi \in Q'_\ell \) and \( \chi' \in Q'_{\ell'} \), where \( \ell \neq \ell' \), then \( \chi \not\sim_{\mathcal{P}'} \chi' \). Thus \( \overrightarrow{\mathcal{P}} = Q \), as desired. The statement concerning reflexivity follows from Theorem 2.4. 

3. Explicit values of the homogeneous weight

In this section we consider the homogeneous weight and determine its values for those finite Frobenius rings that are isomorphic to a product of local rings. This includes all finite commutative Frobenius rings. In the subsequent section, the results will be used to study the partition induced by the homogeneous weight.

Throughout, let \( R \) be a finite Frobenius ring with group of units \( R^* \), and fix a generating character \( \chi \). The following definition is taken from Greferath and Schmidt [18].

**Definition 3.1.** The (left) homogeneous weight on \( R \) with average value \( \gamma \) is a function \( \omega : R \to \mathbb{Q} \) such that \( \omega(0) = 0 \) and

(i) \( \omega(x) = \omega(y) \) for all \( x, y \in R \) such that \( Rx = Ry \),

(ii) \( \sum_{y \in Rx} \omega(y) = \gamma |Rx| \) for all \( x \in R \setminus \{0\} \); in other words, the average weight over each nonzero principal ideal is \( \gamma \).

In [18, Th. 1.3] Greferath/Schmidt proved the existence and uniqueness of the homogeneous weight with given average value for any finite ring, and in [18, Cor. 1.6] the same authors show that (ii) is satisfied by the homogeneous weight for all nonzero ideals of \( R \) (for this result the Frobenius property is essential).

It is easy to see that the Hamming weight on \( R \) is homogeneous if and only if \( R \) is a field, in which case it has average value \( \frac{q-1}{q} \), where \( q = |R| \).

Without loss of generality we restrict ourselves to the homogeneous weight with average value \( \gamma = 1 \), which we call the normalized homogeneous weight.

Thanks to Honold [20, p. 412], an explicit formula for the normalized homogeneous weight on a Frobenius ring is known and reads as

\[
\omega(r) = 1 - \frac{1}{|R^*|} \sum_{u \in R^*} \chi(ru) = 1 - \frac{1}{|R^*|} \sum_{u \in R^*} \chi(ur) \quad \text{for } r \in R; \tag{3.1}
\]

see [17, Prop.1.3] for a short proof verifying that the function satisfies Definition 3.1. A specific instance of this formula, tailored to Galois rings, appears also in [27]. One should note that (3.1) does not depend on the choice of \( \chi \), which is also a consequence of the fact that the set of generating characters of \( R \) is given by \( \{ w \chi \mid u \in R^* \} = \{ \chi \cdot u \mid u \in R^* \} \). It follows from (3.1) that the left homogeneous weight is also right homogeneous, i.e., it satisfies the right counterparts of Definition 3.1.
and (ii); see [20, Th. 2]. Furthermore, the identities in (3.1) yield
\[
(3.2) \quad \sum_{u \in R^*} \chi(ru) = \sum_{u \in R^*} \chi(ur) = |R^*|(1 - \omega(r)) \text{ for } r \in R,
\]
which will be useful later for determining the values of the homogeneous weight on arbitrary Frobenius rings.

**Definition 3.2.** Denote by \( \mathcal{P}_{\text{hom}} \) the partition of \( R \) induced by homogeneous weight. It is thus given by the equivalence relation \( x \sim_{\mathcal{P}_{\text{hom}}} x' \iff \omega(x) = \omega(x') \) for \( x, x' \in R \).

Recall from Example 2.5 the partitions \( \mathcal{P}^{\ast, l} \) and \( \mathcal{P}^{\ast, r} \) induced by the left and right action of \( R^* \) on \( R \). The identities in (3.1) imply \( \mathcal{P}^{\ast, l} \leq \mathcal{P}_{\text{hom}} \) and \( \mathcal{P}^{\ast, r} \leq \mathcal{P}_{\text{hom}} \).

Thus, \( \mathcal{P}^{\ast, r} \leq \mathcal{P}_{\text{hom}}^{[x, r]} \) and \( \mathcal{P}^{\ast, l} \leq \mathcal{P}_{\text{hom}}^{[x, l]} \) due to Example 2.5 and Remark 2.3(b).

Using the fact that the orbit \( O_{x, r} \) of \( x \in R \) under the group \( R^* \) is given by \( O_{x, r} = \{xu_1, \ldots, xu_m\} \), where \( u_1, \ldots, u_m \) are representatives of the distinct right cosets of the stabilizer subgroup of \( x \in R \), one obtains with (3.2)
\[
(3.3) \quad \sum_{b \in O_{x, r}} \chi(ab) = \frac{|O_{x, r}|}{|R^*|} \sum_{u \in R^*} \chi(axu) = |O_{x, r}|(1 - \omega(ax)) \quad \text{for all } a, x \in R.
\]

Consider now a block \( P \) of \( \mathcal{P}_{\text{hom}} \). Then \( P = \bigcup_{i=1}^M O_{x_i, r} \) for a certain number \( M \) of distinct orbits \( O_{x_i, r} \), and for any generating character \( \chi \) we have \( \sum_{b \in P} \chi(ab) = \sum_{i=1}^M |O_{x_i, r}|(1 - \omega(ax_i)) \). Thus we obtain

**Remark 3.3.** The dual partition \( \mathcal{P}_{\text{hom}}^{[x, r]} \) does not depend on \( \chi \) and will simply be denoted by \( \mathcal{P}_{\text{hom}}^{[x, r]} \). Hence
\[
a \sim_{\mathcal{P}_{\text{hom}}^{[x, r]}} a' \iff \sum_{b \in P} \chi(ab) = \sum_{b \in P} \chi(a'b) \quad \text{for all blocks } P \text{ of } \mathcal{P}_{\text{hom}}.
\]

The above computation also shows that the Krawtchouk coefficients of the pair \( (\mathcal{P}_{\text{hom}}, \mathcal{P}_{\text{hom}}^{[x, r]}) \) are rational numbers. Analogous statements are true for the left dual of \( \mathcal{P}_{\text{hom}} \).

The following examples illustrate that the homogeneous partition on a Frobenius ring may exhibit a variety of different properties. Since all examples are commutative we will simply write \( \mathcal{P}_{\text{hom}} \) instead of \( \mathcal{P}_{\text{hom}}^{[x, r]} \).

**Example 3.4.** (a) On \( \mathbb{Z}_8 \) we have \( \mathcal{P}_{\text{hom}} = 0 \mid 1, 2, 3, 5, 6, 7 \mid 4 \). Its dual is \( \mathcal{P}_{\text{hom}} = 0 \mid 1, 3, 5, 7 \mid 2, 4, 6 \), which has been observed already in [3, Ex. 2.9]. Thus \( \mathcal{P}_{\text{hom}} \) is not self-dual, but reflexive, due to Theorem 2.4.

(b) On \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) the homogeneous partition is given by \( 00, 11 \mid 01, 10 \). The fact that \( \{00\} \) is not a block of \( \mathcal{P}_{\text{hom}} \) shows that \( \mathcal{P}_{\text{hom}} \) is not the dual of any partition; see Remark 2.3(a). In particular, \( \mathcal{P}_{\text{hom}} \) is not reflexive and thus not self-dual. In Remark 3.10(b) a characterization will be presented for a Frobenius ring to contain a nonzero element with homogeneous weight zero.

(c) Consider \( \mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\} \) and the ring \( \mathbb{Z}_2 \times \mathbb{F}_4 \). Using for instance the character \( \chi \) on \( \mathbb{F}_4 \) given in Example 2.2 one derives \( \mathcal{P}_{\text{hom}} = 00 \mid 10 \mid 01, 0\alpha, 0\alpha^2 \mid 11, 1\alpha, 1\alpha^2 \). It will follow from Theorem 4.4 (or can be checked directly) that \( \mathcal{P}_{\text{hom}} \) is self-dual and thus reflexive.

In order to determine the values of the homogeneous weight explicitly, we start with the following well-known case.
Example 3.5 ([3, Ex. 2.8]). Let $R$ be a local Frobenius ring with residue field $R/\text{rad}(R)$ of order $q$. Then the normalized homogeneous weight is given by $\omega(0) = 0$ and $\omega(a) = \frac{q}{q-1}$ for $a \in \text{soc}(R) \backslash \{0\}$ and $\omega(a) = 1$ otherwise. Hence $|\mathcal{P}_{\text{hom}}| = 3$. In Theorem 4.4 (see also Example 4.5(a)) we will see that $\mathcal{P}_{\text{hom}}$ is reflexive, and we will also determine the dual partition.

We now proceed to determine the values of the homogeneous weight for finite Frobenius rings that can be written as the direct product of local Frobenius rings. Thanks to [25, Th. 15.27], this covers all finite commutative Frobenius rings. We first summarize some basic properties for direct products of Frobenius rings.

Remark 3.6. Let $R = R_1 \times \ldots \times R_t$, where each $R_i$ is a finite (not necessarily local) Frobenius ring. Let $\chi_i$ be a generating character of $R_i$ for all $i \in [t]$. Then $R$ is a Frobenius ring with character module $\hat{R} = \hat{R}_1 \times \ldots \times \hat{R}_t$. A generating character $\chi$ of $R$ is given by $\chi := (\chi_1, \ldots, \chi_t)$ defined as

$$\chi(a_1, \ldots, a_t) = \prod_{i=1}^{t} \chi_i(a_i) \text{ for } (a_1, \ldots, a_t) \in R.$$ 

Moreover, $\text{rad}(R) = \text{rad}(R_1) \times \ldots \times \text{rad}(R_t)$ and $\text{soc}(R) = \text{soc}(R_1) \times \ldots \times \text{soc}(R_t)$. Finally, $R = \text{soc}(R) \iff \text{rad}(R) = \{0\} \iff \text{rad}(R_i) = \{0\}$ for all $i \iff R$ is semisimple. This last case will be of particular interest to us.

The above leads to the following identity for the values of the homogenous weight. The formula has also recently been derived with the aid of the Möbius function by Byrne et al. [4, Lem. 7], and for the case where all $R_i$ are commutative principal ideal rings, by Fan et al. [13, Th. 4.1], see [12, p. 4]. We provide an alternative proof using (3.2).

Proposition 3.7. Let $R = R_1 \times \ldots \times R_t$ be a direct product of (not necessarily local) Frobenius rings. For $i \in [t]$ let $\omega_i$ be the normalized homogeneous weight on $R_i$. Then the normalized homogeneous weight on $R$ is given by

$$\omega(a_1, \ldots, a_t) = 1 - \prod_{i=1}^{t} (1 - \omega_i(a_i)) \text{ for } (a_1, \ldots, a_t) \in R.$$ 

Proof. Let $\chi_i$ be a generating character of $R_i$, and let $\chi$ be as in Remark 3.6. Using (3.2) and $|R^*| = \prod_{j=1}^{t} |R_j^*|$, we compute

$$\sum_{(u_1, \ldots, u_t) \in R^*} \chi(a_1 u_1, \ldots, a_t u_t) = \sum_{u_1 \in R_1^*} \ldots \sum_{u_t \in R_t^*} \prod_{i=1}^{t} \chi_i(a_i u_i)$$

$$= \prod_{i=1}^{t} \sum_{u_i \in R_i^*} \chi_i(a_i u_i) = \prod_{i=1}^{t} |R_i^*| \left(1 - \omega_i(a_i)\right)$$

$$= |R^*| \prod_{i=1}^{t} \left(1 - \omega_i(a_i)\right).$$

Now (3.1) leads to the desired identity. \qed

The previous results provide us immediately with an explicit formula for the homogeneous weight in the following case.
Proposition 3.8. Let \( R = R_1 \times \ldots \times R_t \), where each \( R_i \) is a finite local Frobenius ring with residue field \( R_i/\text{rad}(R_i) \) of order \( q \). Then

\[
\omega(a) = \begin{cases} 
1 - \left(\frac{-1}{q-1}\right)^{\text{wt}(a)} & \text{if } a \in \text{soc}(R), \\
1 & \text{otherwise},
\end{cases}
\]

where \( \text{wt}(a) := |\{i \mid a_i \neq 0\}| \) denotes the Hamming weight of \( a = (a_1, \ldots, a_t) \).

In the same way we can compute the homogeneous weight on any finite Frobenius ring that is given as as a direct product of local rings. We will need to keep track of the orders of the residue fields of the component rings and thus fix the following notation. For the rest of this paper, let \( R \) be a finite Frobenius ring of the form

\[
R = R_1 \times \ldots \times R_t, \quad \text{where } R_i = R_{i,1} \times \ldots \times R_{i,n_i} \text{ and } R_{i,j} \text{ local},
\]

where \( |R_{i,j}/\text{rad}(R_{i,j})| = q_i \) for all \( j \in [n_i] \) and \( q_1, \ldots, q_t \) distinct.

Recall that \( \text{soc}(R_{i,j}) \cong R_{i,j}/\text{rad}(R_{i,j}) \).

Propositions 3.7 and 3.8 yield the following generalization of Example 3.5.

Theorem 3.9. Let \( R \) be as in (3.4) and write its elements as \( a = (a_1, \ldots, a_t) \), where \( a_i \in R_i \). Using the Hamming weight \( \text{wt} \) on each \( R_i \), the homogeneous weight on \( R \) is given by

\[
\omega(a_1, \ldots, a_t) = \begin{cases} 
1 - \prod_{i=1}^t \left(\frac{-1}{q_i-1}\right)^{\text{wt}(a_i)} & \text{if } a \in \text{soc}(R), \\
1 & \text{otherwise}.
\end{cases}
\]

We close this section with the following immediate insight about the induced partition \( \mathcal{P}_{\text{hom}} \).

Remark 3.10. (a) \( \omega(a) \neq 1 \) for all \( a \in \text{soc}(R) \). Thus \( R \setminus \text{soc}(R) \) is a block of \( \mathcal{P}_{\text{hom}} \). This generalizes the situation for local Frobenius rings in Example 3.5.

(b) There exists a nonzero \( a \in R \) such that \( \omega(a) = 0 \) if and only if there exists an \( i \in [t] \) such that \( q_i = 2 \) and \( n_i \geq 2 \). In this case, \( \mathcal{P}_{\text{hom}} \) is not reflexive due to Remark 2.3(a). Example 3.4(b) is the smallest such ring.

A detailed study of the homogeneous partition and its dual is presented in the next section.

4. The partition induced by the homogeneous weight

In this section we focus on rings as in (3.4) and characterize those rings for which the partition \( \mathcal{P}_{\text{hom}} \) induced by the homogeneous weight is reflexive or even self-dual. Thanks to (2.6) we may and will restrict ourselves to the right dual of \( \mathcal{P}_{\text{hom}} \) in order to study self-duality (and reflexivity). In the reflexive case we also determine the right dual partition and the Krawtchouk coefficients explicitly.

Recall from Remark 3.3 that the right dual partition of \( \mathcal{P}_{\text{hom}} \) does not depend on the choice of the generating character and is denoted by \( \overline{\mathcal{P}}_{\text{hom}} \).

Some basic properties of \( \mathcal{P}_{\text{hom}} \) were presented already in Remark 3.10. We now focus on a simple consequence of Theorem 3.9 that will turn out to be crucial for characterizing reflexivity.

Recall that \( R \) is as in (3.4). Theorem 3.9 shows that the homogeneous weight of \( a = (a_1, \ldots, a_t) \in R \) depends on the Hamming weights \( \text{wt}(a_i) \). In particular, if \( a, b \in \text{soc}(R) \) are such that \( \text{wt}(a_i) = \text{wt}(b_i) \) for all \( i \in [t] \), then \( \omega(a) = \omega(b) \).
This shows that the homogeneous partition is closely related to the product of the Hamming partitions on \( \text{soc}(R_i) \), \( i \in [t] \).

We will therefore study this product partition first and come back to the homogeneous partition thereafter. As we will show later, the homogeneous partition is reflexive if and only if it coincides on the socle with the product of the Hamming partitions.

Let \( \mathcal{H}_i \) be the Hamming partition on \( \text{soc}(R_i) = \text{soc}(R_{i,1}) \times \ldots \times \text{soc}(R_{i,n_i}) \), thus \( \mathcal{H}_i = P_{t,0} | \ldots | P_{t,n_i} \) with blocks

\[
P_{i,j} = \{(a_{i,1}, \ldots, a_{i,n_i}) \in \text{soc}(R_i) \mid \text{wt}(a_{i,1}, \ldots, a_{i,n_i}) = j\}.
\]

Denote the induced product partition on \( \text{soc}(R_1) \times \ldots \times \text{soc}(R_t) = \text{soc}(R) \) by \( \mathcal{H} := \mathcal{H}_1 \times \ldots \times \mathcal{H}_t \). By (2.9) it consists of the blocks

\[
P_m := P_{1,m_1} \times \ldots \times P_{t,m_t},
\]

(4.1)

where \( m := (m_1, \ldots, m_t) \in \mathcal{M} := [n_1]_0 \times \ldots \times [n_t]_0 \).

For the dual partition the following identifications are useful. Note that \( \text{rad}(R) \cong \text{rad}(R_{1,1}) \times \ldots \times \text{rad}(R_{t,1})/ \text{rad}(R_{t,1}) \) for \( R \) as in (3.4), and similarly

\[
R_i/\text{rad}(R_i) \cong R_{i,1}/\text{rad}(R_{i,1}) \times \ldots \times R_{i,n_i}/\text{rad}(R_{i,n_i}).
\]

This allows us to consider the Hamming weight on \( R_i/\text{rad}(R_i) \). For \( a_i = (a_{i,1}, \ldots, a_{i,n_i}) \in R_i \) put

\[
\text{wt}(a_i + \text{rad}(R_i)) := |\{j \mid a_{i,j} \not\in \text{rad}(R_{i,j})\}|.
\]

Now we can formulate the following duality.

**Theorem 4.1.** Consider \( \mathcal{H} = (P_m)_{m \in \mathcal{M}} \) as above. Define the partition \( \mathcal{H}' \) of \( R \) as

\[
\mathcal{H}' = (P_m)_{m \in \mathcal{M} \cup \{0\}}, \text{ where } P_0 := R/\text{soc}(R).
\]

The right dual partition \( \widehat{\mathcal{H}}^{[\chi,\cdot]} \) does not depend on \( \chi \) and coincides with the left dual partition \( \widehat{\mathcal{H}}^{[\cdot,\chi]} \). It is given by

\[
\widehat{\mathcal{H}}^{[\chi,\cdot]} = (Q_m)_{m \in \mathcal{M} \cup \{0\}},
\]

where \( Q_0 = \text{rad}(R) \setminus \{0\} \), \( Q_0 = \{0\} \) and for \( m = (m_1, \ldots, m_t) \in \mathcal{M} \setminus \{0\} \)

\[
Q_m = \{(a_1, \ldots, a_t) \in R \setminus \text{rad}(R) \mid \text{wt}(a_i + \text{rad}(R_i)) = m_i \text{ for all } i \in [t]\}.
\]

In particular, \( \mathcal{H}' \) is reflexive. If \( R = \text{soc}(R) \) then \( |\mathcal{H}'| = s := \prod_{i=1}^{t} (n_i + 1) \), and \( \mathcal{H}' \) is simply the product of the Hamming partitions and thus self-dual. If \( R \neq \text{soc}(R) \) then \( |\mathcal{H}'| = s + 1 \).

**Proof.** We make use of Theorem 2.7 and Proposition 2.8. For this we consider \( \text{soc}(R) \) as a subgroup of \( R \). Then \( \mathcal{H} \) is a partition of this subgroup, and it is given as the product of the Hamming partitions \( \mathcal{H}_i \) of \( \text{soc}(R_i) \). By Example 2.6, the dual partitions \( \widehat{\mathcal{H}}_i \) are the Hamming partitions on the character groups \( \text{soc}(R_i) \), and Theorem 2.7 implies that \( \widehat{\mathcal{H}} \) is the partition \( \widehat{\mathcal{H}}_1 \times \ldots \times \widehat{\mathcal{H}}_t \) of \( \text{soc}(R) = \text{soc}(R_1) \times \ldots \times \text{soc}(R_t) \). Proposition 2.8 yields that the partition \( \widehat{\mathcal{H}}' \) of the group \( \widehat{R} \) consists of the blocks \( \text{soc}(R)^{\perp} \setminus \{\varepsilon\} \), \( \{\varepsilon\} \), and the blocks

(4.2) \[
\{(\chi_{i,1} a_1, \ldots, \chi_{i,t} a_t) \in \widehat{R} \setminus \text{soc}(R)^{\perp} \mid \text{wt}(\chi_{i,1} a_1)_{|\text{soc}(R_i)} = m_i \text{ f. a. } i \in [t]\}
\]

for all \( m \in \mathcal{M} \setminus \{0\} \), and where \( \chi_i \) is a fixed generating character of \( R_i \) (see also Remark 3.6). Now the isomorphism \( \alpha_r \) from (2.3) turns the partition \( \widehat{\mathcal{H}}' \) of \( \widehat{R} \) into
the partition $\tilde{\mathcal{H}}^{[x,r]}$ of $R$. Since soc($R$) is a right ideal, (2.5) yields $\alpha_{-1}(\text{soc}(R)^{+}) = \text{ann}_{Y}(\text{soc}(R)) = \text{rad}(R)$. As a consequence, the partition $\tilde{\mathcal{H}}^{[x,r]}$ consists of the blocks rad($R$) \{0\}, \{0\}, and the sets $Q_m$ given in the theorem. All of this proves the desired duality. It is clear that the left-sided version of (4.2) is true as well, and the analogous proof establishes $\tilde{\mathcal{H}}^{[x,r]} = \tilde{\mathcal{H}}^{[x,t]}$. The cardinality of $\mathcal{H}$' is clear from $|\mathcal{M}| = s$, and reflexivity follows from Theorem 2.4. Finally, if $R = \text{soc}(R)$, then rad($R$) = \{0\}, so that in this case the blocks $P$ and $Q$ are missing, and the sets $Q_m$ are the blocks of the product partition $\mathcal{H}$ on soc($R$) = $R$. Therefore $\mathcal{H}'$ is self-dual.

From now on we will simply write $\tilde{\mathcal{H}}$ for the dual partition.

**Corollary 4.2.** The Krawtchouk coefficients of the pair $(\mathcal{H}', \tilde{\mathcal{H}}')$ from Theorem 4.1 are given by

$$K_{\ell,m} = \begin{cases} -|\text{soc}(R)| & \text{if } m = \circ = \ell \\ \frac{1}{|R \setminus \text{soc}(R)|} & \text{if } m = \circ, \ell = 0 \\ & \text{if } m = \circ, \ell \notin \{0,\circ\} \end{cases}$$

$$\left|P_{1,m_1} \times \ldots \times P_{t,m_t}\right| = \prod_{i=1}^{t} (m_i(q_i - 1)^{m_i})$$

for all $\ell, m \in \mathcal{M} \cup \{\circ\}$, and where $K_{\ell,m}(q_i(q_i - 1)^{m_i})$ are the Krawtchouk coefficients of the Hamming partition of $\text{soc}(R_{i,1}) \times \ldots \times \text{soc}(R_{i,n_i})$. In the special case where $R = \text{soc}(R)$, and thus each $R_{i,j}$ is a field, only the last case occurs.

**Proof.** This is a consequence of Proposition 2.8 along with Theorem 2.7 and the classical Krawtchouk coefficients for the Hamming partition given in Example 2.6.

Now we can return to the homogeneous weight. Note first that the equivalence relation corresponding to the partition $\mathcal{H}'$ is given by

$$a \sim_{\mathcal{H}'} b \iff \begin{cases} a, b \in R \setminus \text{soc}(R) \text{ or} \\ a, b \in \text{soc}(R) \text{ and } \text{wt}(a_i) = \text{wt}(b_i) \text{ for all } i \in [t]. \end{cases}$$

A comparison to the homogeneous weight in Theorem 3.9 shows that $\mathcal{H}' \leq \mathcal{P}_{\text{hom}}$. In other words, $a \sim_{\mathcal{H}'} b \Rightarrow \omega(a) = \omega(b)$ for all $a, b \in R$. The converse of this implication is not true in general. In other words, $\mathcal{P}_{\text{hom}}$ may be strictly coarser than $\mathcal{H}'$. Theorem 3.9 indicates that the particular values of $q_1, \ldots, q_t$ decide on the difference between these two partitions. We cast the following definition. It is simply made to reflect the case where $\mathcal{H}' = \mathcal{P}_{\text{hom}}$, as we will show in Theorems 4.4 and 4.7.

**Definition 4.3.** Given the list $\mathcal{L} := [(q_1, n_1), \ldots, (q_t, n_t)]$ of distinct prime powers $q_i$ and multiplicities $n_i \in \mathbb{N}$. Then $\mathcal{L}$ is called separating if the following holds: whenever $m := (m_1, \ldots, m_t), \ell := (\ell_1, \ldots, \ell_t) \in \mathcal{M} = [n_1]_0 \times \ldots \times [n_t]_0$ and $m \neq \ell$, then

$$\prod_{i=1}^{t} (q_i - 1)^{m_i} \neq \prod_{i=1}^{t} (q_i - 1)^{\ell_i} \text{ or } \sum_{i=1}^{t} m_i \neq \sum_{i=1}^{t} \ell_i \mod 2.$$ 

We call a list $[q_1, \ldots, q_t]$ of distinct prime powers separating, if the list $[(q_1, 1), \ldots, (q_t, 1)]$ is separating. An integer $N$ is separating if its list of distinct prime factors is separating.
As we will see below, for a separating list $\mathcal{L}$ the partition $\mathcal{H}'$ separates the elements of $R$ according to their homogeneous weight.

The list $[2, 3, 7, 13]$ is not separating, and the condition is violated in two ways: $(2 - 1)(3 - 1)(7 - 1) = (13 - 1)$ and $(3 - 1)(7 - 1) = (2 - 1)(13 - 1)$. The list $[3, 7, 13]$ is separating. The list $[(2, 1), (3, 2), (5, 1)]$ is not separating because $(3 - 1)^2 = (2 - 1)(5 - 1)$. The list $[(3, 2), (5, 1)]$ is separating, but $[(3, 4), (5, 2)]$ is not. The case $m = 0$ shows that if $[(q_1, n_1), \ldots, (q_t, n_t)]$ is separating and $q_i = 2$ for some $i$, then $n_i = 1$. Finally, if $[q_1, \ldots, q_t]$ is not separating, then $t \geq 4$.

Starting with three distinct primes and using sufficiently large primes one easily shows that there exist infinitely many separating integers.

**Theorem 4.4.** Let $R$ be as in (3.4) and let $\mathcal{L} = [(q_1, n_1), \ldots, (q_t, n_t)]$ be separating. Then $\mathcal{P}_{\text{hom}} = \mathcal{H}'$, where $\mathcal{H}'$ is as in Theorem 4.1. As a consequence, $\mathcal{P}_{\text{hom}}$ is reflexive and $\widehat{\mathcal{P}_{\text{hom}}} = \widehat{\mathcal{P}}_{\text{hom}} = \mathcal{H}'$. Moreover, if $R = \text{soc}(R)$, i.e., $R$ is semisimple, then the homogeneous partition coincides with the product of the Hamming partitions $\mathcal{H}_i$ on each $R_i$, and thus is self-dual.

**Proof.** First, the separating property guarantees that if $q_i = 2$ then $n_i = 1$. With Remark 3.10(b) we conclude that $\omega(a) \neq 0$ for all $a \neq 0$. Thus $\{0\}$ is a block of $\mathcal{P}_{\text{hom}}$. Moreover, $R \setminus \text{soc}(R)$ is a block of $\mathcal{P}_{\text{hom}}$ due to Remark 3.10(a). Since both sets are also blocks of $\mathcal{H}'$ and $\mathcal{H}' \leq \mathcal{P}_{\text{hom}}$, it remains to show that for all $a, b \in \text{soc}(R)$ such that $\omega(a) = \omega(b)$ we have $a \sim_{\mathcal{H}'} b$. By Theorem 3.9 $\omega(a) = \omega(b)$ yields $(-1)^{\text{wt}(b)} \prod_{i=1}^{t}(q_i - 1)^{\text{wt}(a_i)} = (-1)^{\text{wt}(a)} \prod_{i=1}^{t}(q_i - 1)^{\text{wt}(b_i)}$, where $\text{wt}(a) = \sum_{i=1}^{t} \text{wt}(a_i)$ and similarly for $b$. Since $q_i - 1 > 0$, this leads to $\text{wt}(a) \equiv \text{wt}(b) \mod 2$ and $\prod_{i=1}^{t}(q_i - 1)^{\text{wt}(a_i)} = \prod_{i=1}^{t}(q_i - 1)^{\text{wt}(b_i)}$. Since $\mathcal{L}$ is separating, this yields $\text{wt}(a_i) = \text{wt}(b_i)$ for all $i \in [t]$, and therefore $a \sim_{\mathcal{H}'} b$. This concludes the proof. \[\square\]

Before turning to the non-separating case, let us present some examples of the homogeneous weight on (products of) integer residue rings. We use the notation from Theorem 4.1. The last result allows us to simply write $\widehat{\mathcal{P}_{\text{hom}}}$ for the dual partitions.

**Example 4.5.** (a) Let $t = 1$, $n_1 = 1$, and $q_1 = q$, thus $R$ is a local Frobenius ring with $|R/\text{rad}(R)| = q$. Then $\mathcal{M} = \{0, 1\}$ and the partitions $\mathcal{P}_{\text{hom}} = \mathcal{H}' = P_0 \mid P_1 \mid P_0$ and $\widehat{\mathcal{P}_{\text{hom}}} = \widehat{\mathcal{H}'} = Q_0 \mid Q_0 \mid Q_1$ read as

\[\mathcal{P}_{\text{hom}} = \{0\} \mid \text{soc}(R) \setminus \{0\} \mid R \setminus \text{soc}(R) \text{ and } \widehat{\mathcal{P}_{\text{hom}}} = \{0\} \mid \text{rad}(R) \setminus \{0\} \mid R^*.\]

This generalizes the dual partition $\widehat{\mathcal{P}_{\text{hom}}}$ of $\mathbb{Z}_8$ presented in Example 3.4(a). The Krawtchouk matrix, indexed row- and columnwise by the blocks of $\widehat{\mathcal{P}_{\text{hom}}}$ and $\mathcal{P}_{\text{hom}}$ in the given order, is given by

\[
\begin{pmatrix}
1 & q - 1 & |R| - q \\
1 & q - 1 & -q \\
1 & -1 & 0
\end{pmatrix}
\]

For $R = \mathbb{Z}_q$, this matrix also appears in [5, p. 1553] by Camion.

(b) Consider the ring $R = \mathbb{Z}_{pq} \times \mathbb{Z}_{pq}$, where $p, q$ are distinct primes and $r, s \geq 1$. The list $[p, q]$ is separating. Note that $R \cong \mathbb{Z}_N$, where $N = p^r q^s$, and hence $N$ is separating. For the component rings, socle and radical satisfy $\text{soc}(\mathbb{Z}_{pq}) \setminus \{0\} = \mathcal{O}_{p^{-1}}$ (the $(\mathbb{Z}_{pq})^*$-orbit of the element $p^{-1}$) and $\text{rad}(\mathbb{Z}_{pq}) = (p)$ and analogously for $\mathbb{Z}_{pq}$. 

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\[\text{Advances in Mathematics of Communications} \quad \text{Volume 8, No. 2 (2014), 191–207}\]
The homogeneous partition $\mathcal{P}_{\text{hom}}$ and the values of the homogeneous weight are given by (in the order $P_{(0,0)} | P_{(1,0)} | P_{(0,1)} | P_{(1,1)} | P_\circ$

\[
\begin{array}{|c|c|c|c|c|}
\hline
\mathcal{P}_{\text{hom}} & \{0\} & O_{p^{-1}} \times \{0\} & O_{q^{-1}} \times O_{q^{-1}} & R \setminus (p^{q-1} \times (q^{q-1}) \\
\omega & 0 & p - 1 & q - 1 & 1 - p - q \\
\hline
\end{array}
\]

the dual partition is $\mathcal{P}_{\text{hom}}^* = 0 | (p) \times (q) \setminus \{0\} | (p) \times Z_{q^*} | Z_{p^*} \times (q) | Z_{p^*} \times Z_{q^*},$ and the Krawtchouk matrix has the form

\[
K = \begin{pmatrix}
    1 & p - 1 & q - 1 & (p - 1)(q - 1) & p^* q^* - pq \\
    1 & p - 1 & q - 1 & (p - 1)(q - 1) & -pq \\
    1 & p - 1 & -1 & 1 - p & 0 \\
    1 & -1 & q - 1 & 1 - q & 0 \\
    1 & -1 & -1 & 1 & 0
\end{pmatrix}.
\]

If $N = pq,$ the last block of $\mathcal{P}_{\text{hom}}$ and the second block of $\mathcal{P}_{\text{hom}}^*$ are missing, and so are the last column and second row of $K.$ For this case, the values of the homogeneous weight also appears in $[2, \text{Ex. 3}]$ by Byrne.

Here is the smallest non-separating integer $N.$ As one may expect, the homogeneous partition is not reflexive.

**Example 4.6.** Consider $\mathbb{Z}_N,$ where $N$ is the non-separating integer $N = 2 \cdot 3 \cdot 7 \cdot 13 = 546.$ Note that $\mathbb{Z}_N \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_{13}$ and since $n_i = 1$ for all $i,$ the partition $\mathcal{H}'$ is simply the orbit partition under the multiplicative action of $\mathbb{Z}_N.$ Hence $|\mathcal{H}'| = 2^4 = 16.$ But the elements in the blocks $P_{(0,1,1,0)}$ and $P_{(1,0,0,1)}$ (for the notation see (4.1)) have the same homogeneous weight, $11/12,$ and similarly the elements in $P_{(0,0,0,1)} \cup P_{(1,1,1,0)}$ have weight $13/12.$ This leads to $|\mathcal{P}_{\text{hom}}| = 14.$ On the other hand, using a computer algebra program one computes that the dual partition $\mathcal{P}_{\text{hom}}^*$ coincides with the orbit partition $\mathcal{H}'$ and thus $|\mathcal{P}_{\text{hom}}^*| = 16.$ Hence $\mathcal{P}_{\text{hom}}$ is not reflexive.

As we show next, the result of this example is true for all non-separating cases. The proof, however, is surprisingly technical, and we are not aware of any simpler argument. The difficulty is due to the fact that a non-separating list allows for many unions of sets $P_m$ to form blocks of $\mathcal{P}_{\text{hom}},$ and a precise statement about the dual partition must therefore involve the detailed Krawtchouk coefficients. One should note that the most technical part, Case 2 below, does indeed occur, as shown by the previous example.

**Theorem 4.7.** Let $R$ be as in (3.4) and let $\mathcal{L} = [(q_1, n_1), \ldots, (q_t, n_t)]$ be non-separating. Then the partition $\mathcal{P}_{\text{hom}}$ is not reflexive. More precisely, $\mathcal{P}_{\text{hom}} > \mathcal{H}'$ whereas $\mathcal{P}_{\text{hom}}^* \geq \mathcal{H}'.$ Thus $|\mathcal{P}_{\text{hom}}| > |\mathcal{P}_{\text{hom}}^*|.$

**Proof.** According to Theorem 4.1 we have $\mathcal{H}' = (P_m)_{m \in \mathcal{M} \cup \{\circ\}}$ and $\mathcal{H}' = (Q_{\ell})_{\ell \in \mathcal{M} \cup \{\circ\}},$ where $P_m$ is as in (4.1) and $Q_{\ell}$ as in Theorem 4.1. Let $K_{\ell,m}$ be the Krawtchouk coefficients of the pair $(\mathcal{H}', \mathcal{H}').$ It is clear from (4.3) that $\mathcal{P}_{\text{hom}} \geq \mathcal{H}'$ and thus $\mathcal{P}_{\text{hom}}^* \geq \mathcal{H}'$ due to Remark 2.3(b). Theorem 3.9 shows that for $m \in \mathcal{M}$

\[
a \in P_m \implies \omega(a) = 1 - \prod_{i=1}^t \left( -\frac{1}{q_i} \right)^{m_i}.
\]
From the fact that \( \mathcal{L} \) is non-separating it follows that there exist two distinct blocks \( P_m, P_{m'} \) such that all elements of \( P_m \cup P_{m'} \) have the same homogeneous weight. Thus \( \mathcal{P}_{\text{hom}} > \mathcal{H}' \).

We have to show that \( \mathcal{P}_{\text{hom}}^{-r} = \mathcal{H}' \). First of all, the block \( Q_0 = \{0\} \) of \( \mathcal{H}' \) is also a block of \( \mathcal{P}_{\text{hom}}^{-r} \), as this is a general property of dual partitions, see Remark 2.3(a).

Next, recall that the index \( \diamond \) occurs only if \( R \neq \text{soc}(R) \) and that in this case \( P_0 \) is a block of \( \mathcal{P}_{\text{hom}}^{-r} \). Corollary 4.2 shows that \( K_{\ell,0} = 0 \) for all \( \ell \notin \{0, \diamond\} \) whereas \( K_{0,0} < 0 \) and \( K_{0,\diamond} > 0 \). This implies that the block \( Q_0 \) of \( \mathcal{H}' \) is also a block of \( \mathcal{P}_{\text{hom}}^{-r} \).

Now we come to the main part of the proof. Assume that \( \ell, \ell' \in \mathcal{M}\setminus\{0\} \) are such that \( Q_\ell \) and \( Q_{\ell'} \) are contained in the same block of \( \mathcal{P}_{\text{hom}}^{-r} \). We have to show that \( \ell = \ell' \).

Clearly, any block \( P \) of \( \mathcal{P}_{\text{hom}}^{-r} \), where \( P \neq P_0 \), is of the form \( P = \bigcup_{m \in \mathcal{N}} P_m \) for a subset \( \mathcal{N} \subseteq \mathcal{M} \). Therefore

\[
\sum_{b \in P} \chi(ab) = \sum_{m \in \mathcal{N}} \sum_{b \in P_m} \chi(ab) = \sum_{m \in \mathcal{N}} K_{\ell,m} \quad \text{for any } a \in Q_\ell.
\]

Hence our assumption on \( \ell, \ell' \) may be written as

\[
(4.5) \quad \sum_{m \in \mathcal{N}} K_{\ell,m} = \sum_{m \in \mathcal{N}} K_{\ell',m} \quad \text{for all blocks } \bigcup_{m \in \mathcal{N}} P_m \text{ of } \mathcal{P}_{\text{hom}}^{-r}.
\]

We will make use of the Krawtchouk coefficients for the case where \( m = e_i = (0, \ldots, 1, \ldots, 0) \) (with 1 in the \( i \)th position). From Corollary 4.2 we have \( K_{\ell,ei} = K_{\ell_1,q_i}(\ell_i) = (n_i - \ell_i)q_i - n_i \), which along with (4.5) results in the implication

\[
(4.6) \quad P_{ei} \text{ is a block of } \mathcal{P}_{\text{hom}}^{-r} \implies \ell_i = \ell_i'.
\]

In order to proceed, we assume without loss of generality that

\[
(4.7) \quad q_1 < \ldots < q_t.
\]

Case 1: Suppose that for all \( i \in [t] \) we have \( q_i - 1 \neq \prod_{j=1}^i (q_j - 1)^{m_j} \) whenever \( \sum_{j=1}^i m_j \) is odd. Then (4.4) shows that each \( P_{ei} \) is a block of \( \mathcal{P}_{\text{hom}}^{-r} \), and hence (4.6) implies \( \ell = \ell' \).

Case 2: Let \( i \in [t] \) be minimal such that \( q_i - 1 = \prod_{j=1}^i (q_j - 1)^{m_j} \) for some \( m \neq e_i \) and \( \sum_{j=1}^i m_j \) odd. Then the same argument as in Case 1 along with (4.7) shows that the sets \( P_{ei} \), where \( a < i \), are blocks of \( \mathcal{P}_{\text{hom}}^{-r} \). Hence by (4.6)

\[
(4.8) \quad \ell_a = \ell_a' \quad \text{for } a = 1, \ldots, i - 1.
\]

Let \( m^{(1)}, \ldots, m^{(s)} \) be all indices with odd sum and such that

\[
q_i - 1 = \prod_{j=1}^i (q_j - 1)^{m^{(r)}} \quad \text{for } r \in [s].
\]

Then the set \( P' := \bigcup_{m \in \mathcal{N}} P_m \), where \( \mathcal{N} = \{e_i, m^{(1)}, \ldots, m^{(s)}\} \), is a block of \( \mathcal{P}_{\text{hom}}^{-r} \). (4.7) yields \( m^{(r)}_j = 0 \) for \( j \geq i \). Thus the case \( m \neq \diamond \neq \ell \) in Corollary 4.2 along with (2.8) shows that

\[
K_{\ell,m^{(r)}} = K_{\ell_1,\ldots,\ell_{i-1},0,\ldots,0,m^{(r)}} \quad \text{for all } r \in [s]
\]

and analogously for \( \ell' \). Hence \( K_{\ell,m^{(r)}} = K_{\ell',m^{(r)}} \) by (4.8) and

\[
\sum_{m \in \mathcal{N}} K_{\ell,m} = \sum_{r=1}^s K_{\ell,m^{(r)}} + K_{\ell,ei} = \sum_{r=1}^s K_{\ell',m^{(r)}} + K_{\ell,ei}.
\]
Now (4.5) yields $K_{ℓ,e_i} = K_{ℓ',e_i}$ and hence $ℓ_i = ℓ'_i$.

Now we may continue in the same fashion for the index set $\{i+1, \ldots, t\}$. If there is no index $i' > i$ such that $q_j - 1 = \prod_{j=1}^{t}(q_j - 1)^{m_j}$ and $\sum_{j=1}^{t} m_j$ is odd, then we may argue as in Case 1. Otherwise, we choose the smallest $i' > i$ and argue as in Case 2. Proceeding in this manner we finally arrive at $ℓ = ℓ'$, as desired.

We close the paper with the following summary.

**Corollary 4.8.** Let $R$ be as in (3.4).

(a) $\mathcal{P}_{\text{hom}}$ is reflexive if and only if it coincides on the socle with the product of the Hamming partitions on $R_i$, $i = 1, \ldots, t$, and this is the case if and only if $[(q_1, n_1), \ldots, (q_t, n_t)]$ is separating. In this case $\mathcal{P}_{\text{hom}} = \mathcal{P}_{\text{hom}}$. Moreover, $\mathcal{P}_{\text{hom}}$ is self-dual if and only if $R$ is semisimple and $[(q_1, n_1), \ldots, (q_t, n_t)]$ is separating.

(b) The homogeneous partition $\mathcal{P}_{\text{hom}}$ on $\mathbb{Z}_N$ is reflexive if and only if $N$ is separating. The partition is self-dual if and only if $N$ is square-free and separating.

We leave it to future research to determine the values and the corresponding partition of the homogeneous weight for general Frobenius rings.

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