EIGENFUNCTION ESTIMATES AND EMBEDDING THEOREMS
FOR THE DUNKL HARMONIC OSCILLATOR

JESÚS A. ÁLVAREZ LÓPEZ AND MANUEL CALAZA

Abstract. Eigenfunction estimates and embedding results are proved for the Dunkl harmonic oscillator on the line. These kind of results are generalized to operators on \( \mathbb{R}_+ \) of the form

\[
P = -\frac{d^2}{dx^2} + sx^2 - 2f_1 \frac{dx}{x} + f_2,
\]

where \( s > 0 \), and \( f_1 \) and \( f_2 \) are functions satisfying

\[
f_2 = \sigma (\sigma - 1)x^{-2} - f_2' \quad \text{for some } \sigma > -1/2.
\]

Contents

1. Introduction 1
2. Preliminaries 5
  2.1. Dunkl operator 5
  2.2. Dunkl harmonic oscillator 6
  2.3. Generalized Hermite polynomials 7
3. Estimates of the generalized Hermite functions 9
  3.1. Second perturbation of \( H \) 9
  3.2. Description of \( q_k \) 11
  3.3. Location of the zeros of \( \xi_k \) and \( \xi_k' \) 12
  3.4. Estimates of \( \xi_k \) 16
4. Perturbed Schwartz space 22
5. Perturbed Sobolev spaces 27
6. Perturbation of \( H \) on \( \mathbb{R}_+ \) 34
7. Examples 35
  7.1. Case where \( f_1 \) is a multiple of \( x^{-1} \) 35
  7.2. Case where \( f_1 \) is a multiple of other potential functions 37
  7.3. Case where \( f_1 \) is a multiple of \( g'/g \) for some function \( g \) 37
  7.4. Transformation of \( P \) by changes of variables 38
References 38

1. Introduction

The Dunkl operator \( T_\sigma \) on \( C^\infty(\mathbb{R}) \), depending on some \( \sigma > -1/2 \), is the perturbation of the usual derivative that can be defined by setting

\[
T_\sigma = \frac{d}{dx} \quad \text{on even}
\]

1991 Mathematics Subject Classification. 34L05; 33C45; 41A10.

Key words and phrases. Dunkl harmonic oscillator; eigenfunction estimates; Sobolev embedding.

The first author is partially supported by MICINN, Grants MTM2008-02640 and MTM2011-25656, and by MEC, Grant PR2009-0409.
functions and \( T_\sigma = \frac{d}{dx} + 2\sigma \frac{1}{x} \) on odd functions. This kind of operator, more generally on \( \mathbb{R}^n \), was introduced by C.F. Dunkl [13, 14, 15, 16, 17]. It gave rise to what is now called Dunkl theory (see the survey article [35]). This area had a big development in the last years, mainly due to its applications in Quantum Calogero-Moser-Sutherland models (see e.g. 6, 9, 11, 24, 25, 33, 2). In particular, the Dunkl harmonic oscillator [33, 18, 30, 29] is \( L_\sigma = -T_\sigma^2 + sx^2 \), depending on \( s > 0 \); i.e., it is given by using \( T_\sigma \) instead of \( d/dx \) in the expression of the usual harmonic oscillator \( H = -\frac{d^2}{dx^2} + x^2 \).

On the other hand, let \( p_k \) is the sequence of orthogonal polynomials for the measure \( e^{-sx^2}|x|^{2\sigma} \ dx \), taken with norm one and positive leading coefficient. Up to normalization, these are the generalized Hermite polynomials [35, p. 380, Problem 25]; see also [10, 12, 19, 11, 33, 34]. Let \( x_{k,k} < x_{k,k-1} < \cdots < x_{k,1} \) denote the roots of each \( p_k; \) in particular, \( x_{k,k}/2 \) is the smallest positive root if \( k \) is even. The corresponding generalized Hermite functions are \( \phi_k = p_k e^{-sx^2/2} \).

It is known that \( L_\sigma \), with domain the Schwartz space \( \mathcal{S}(\mathbb{R}) \), is essentially self-adjoint in \( L^2(\mathbb{R}, |x|^{2\sigma} \ dx) \). Moreover the spectrum of its self-adjoint extension, denoted by \( \mathcal{L}_\sigma \), consists of the eigenvalues \( (2k+1+2\sigma)s \) \((k \in \mathbb{N})\), with corresponding eigenfunctions \( \phi_k \).

We show asymptotic estimates of the functions \( \phi_k \) as \( k \to \infty \), which are used to prove embedding theorems, and these results are extended to other related perturbations of \( H \). Even though we consider only the Dunkl harmonic oscillator on the line to begin with, this work is more difficult than in the case of \( H \), and has some new features. It may also give a hint of how to proceed for higher dimension.

To get uniform estimates, we consider the functions \( \xi_k = |x|^\sigma \phi_k \) instead of \( \phi_k \). They satisfy the equation \( \xi_k'' + q_k \xi_k = 0 \), where \( q_k = (2k+1+2\sigma)s - s^2x^2 - \bar{\sigma}_k x^{-2} \) with \( \bar{\sigma}_k = \sigma (\sigma - (-1)^k) \). Let \( \widehat{I}_k = q^{-1}(\mathbb{R}_+) \) (the oscillation region), which is of the form: \((-b_k,-a_k) \cup (a_k,b_k)\) if \( \bar{\sigma}_k > 0 \) (for \( k > 0 \)), \((-b_k,b_k)\) if \( \bar{\sigma}_k = 0 \), or \((-b_k,0) \cup (0,b_k)\) if \( \bar{\sigma}_k < 0 \), where \( b_k > a_k > 0 \) with \( b_k \in O(k^{1/2}) \) and \( a_k \in O(k^{-1/2}) \) as \( k \to \infty \).

If \( \bar{\sigma}_k \geq 0 \), then \( \widehat{I}_k = \widehat{I}_k \). When \( \bar{\sigma}_k < 0 \) and \( k \) is large enough, the equation \( q_k(b) = 4\pi/b^2 \) has two positive solutions, \( b_{k,+} < b_{k,-} \), with \( b_{k,+} \in O(k^{-1/2}) \). Then set \( \widehat{J}_k = (-b_{k,-} \cup [b_{k,-},b_{k,+}] \).

**Theorem 1.1.** There exist \( C, C', C'' > 0 \), depending on \( \sigma \) and \( s \), such that, for \( k \geq 1 \):

1. \( \xi_k^2(x) \leq C/s_k(x) \) for all \( x \in \widehat{J}_k \);
2. if \( k \) is odd or \( \sigma \geq 0 \), then \( \xi_k^2(x) \leq C'k^{-1/6} \) for all \( x \in \mathbb{R} \); and
3. if \( k \) is even and \( \sigma < 0 \), then \( \xi_k^2(x) \leq C''k^{-1/6} \) if \( |x| \geq x_{k,k}/2 \).

In the case of Theorem 1.1(iii), the estimate of \( \xi_k \) cannot be extended to \( \mathbb{R} \setminus \{0\} \) because these functions are unbounded near zero. Therefore some condition of the type \( |x| \geq x_{k,k}/2 \) must be assumed; the meaning of this condition is clarified by pointing out that \( x_{k,k}/2 \in O(k^{-1/2}) \) as \( k \to \infty \). This weakness is complemented by the following result.

**Theorem 1.2.** Suppose that \( \sigma < 0 \). There exist \( C'' > 0 \), depending on \( \sigma \) and \( s \), such that \( \phi_k^2(x) \leq C'' \) for all \( k \) even and all \( x \in \mathbb{R} \).

The following theorem asserts that the type of asymptotic estimates of Theorem 1.1(ii),(iii) are optimal.
Theorem 1.3. There exist $C^{(IV)}, C^{(V)} > 0$, depending on $\sigma$ and $s$, such that, for $k \geq 1$:

(i) $\max_{x \in \mathbb{R}} \xi_k^2(x) \geq C^{(IV)k^{-1/6}}$; and,

(ii) if $k$ is even and $\sigma < 0$, then $\max_{|x| \geq x_{k,i}} \xi_k^2(x) \geq C^{(V)k^{-1/6}}$.

To prove Theorems 1.1–1.3 we apply the method that Bonan-Clark have used with $H$. The estimates are satisfied by the functions $\xi_k$ instead of $\phi_k$ because the method can be applied to the conjugation $K_{\sigma} = |x|^{\sigma}L_{\sigma} |x|^{-\sigma}$. This method has two steps: first, it estimates the distance from any point $x$ in an oscillation region to some root $x_{k,i}$, and, second, the value of $\xi_k^2(x)$ is estimated by using $|x-x_{k,i}|$. These computations for $K_{\sigma}$ become much more involved than in $H$; indeed, several cases are considered separately, some of them with significant differences; for instance, some roots $x_{k,i}$ may not be in the oscillation region $I_k$, and the functions $\xi_k$ may not be bounded, as we said.

The asymptotic distribution of the roots $x_{k,i}$ as $k \to \infty$ also has a well known measure theoretic interpretation $[20, 39, 40]$; specially, the generalized Hermite polynomials are considered in $[39]$ Section 4. However the weak convergence of measures considered in those publications does not seem to give the asymptotic approximation of the roots needed in the first step.

For each $m \in \mathbb{N}$, let $S^m$ be the Banach space of functions $\phi \in C^m(\mathbb{R})$ with $\sup_{x} |x^i \phi^{(j)}(x)| < \infty$ for $i + j \leq m$; thus $S = \bigcap_m S^m$ with the corresponding Fréchet topology. On the other hand, for each real $m \geq 0$, let $W^m_\sigma$ be the version of the Sobolev space obtained as Hilbert space completion of $S$ with respect to the scalar product defined by $\langle \phi, \psi \rangle_{W^m_\sigma} = (1 + L_{\sigma})^m \langle \phi, \psi \rangle_\sigma$, where $\langle \cdot, \cdot \rangle_\sigma$ denotes the scalar product of $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$. Let also $W^{\infty}_\sigma = \bigcap_m W^m_\sigma$ with the corresponding Fréchet topology. The subindex ev/odd is added to any space of functions on $\mathbb{R}$ to indicate its subspace of even/odd functions. The following embedding theorems are proved; the second one is a version of the Sobolev embedding theorem.

Theorem 1.4. For each $m \geq 0$, $S^m_{\sigma, ev/odd} \subset W^m_{\sigma, ev/odd}$ continuously if $m' \in \mathbb{N}$, $m' - m > 1/2$, and

\[
M^m_{m', ev/odd} = \begin{cases} 
\frac{3m'}{2} + \frac{m'}{2} [\sigma]([\sigma] + 3) + [\sigma] & \text{if } \sigma \geq 0 \text{ and } m' \text{ is even} \\
\frac{3m'}{2} + \frac{m'}{2} [\sigma]([\sigma] + 3) + [\sigma] & \text{if } \sigma < 0 \text{ and } m' \text{ is even}, \\
\frac{3m'}{2} + \frac{m'}{2} [\sigma]([\sigma] + 3) + [\sigma] & \text{if } \sigma \geq 0 \text{ and } m' \text{ is odd} \\
\frac{3m'}{2} + \frac{m'}{2} [\sigma]([\sigma] + 3) + [\sigma] & \text{if } \sigma < 0 \text{ and } m' \text{ is odd}, \\
\frac{3m'}{2} + \frac{m'}{2} [\sigma]([\sigma] + 3) + [\sigma] & \text{if } \sigma \geq 0 \text{ and } m' \text{ is odd} \\
\frac{3m'}{2} + \frac{m'}{2} [\sigma]([\sigma] + 3) + [\sigma] & \text{if } \sigma < 0 \text{ and } m' \text{ is odd}.
\end{cases}
\]

Theorem 1.5. For all $m \in \mathbb{N}$, $W^{m'}_{\sigma} \subset S^m$ continuously if

\[
m' - m > \begin{cases} 
4 + \frac{1}{2} [\sigma]([\sigma] + 1) & \text{if } \sigma \geq 0 \\
4 & \text{if } \sigma < 0.
\end{cases}
\]

Moreover $W^{m'}_{\sigma, ev} \subset S^0_{ev}$ continuously if $\sigma < 0$ and $m' > 2$.

Corollary 1.6. $S = W^{\infty}_{\sigma}$ as Fréchet spaces.

In other words, Corollary 1.6 states that an element $\phi \in L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$ is in $S$ if and only if the “Fourier coefficients” $\langle \phi, \phi_k \rangle_{\sigma}$ are rapidly decreasing on $k$. This also
means that \( S = \bigcap_m D(L_m^n) \) (\( S \) is the smooth core of \( L_{\sigma} \) [7]) because the sequence of eigenvalues of \( L_{\sigma} \) is in \( O(k) \) as \( k \to \infty \).

We introduce a perturbed version \( S_{\sigma}^m \) of every \( S^m \), whose definition involves \( T_{\sigma} \) instead of \( \frac{d}{dx} \) and is inspired by the estimates of Theorems [1.1] and [1.2]. They satisfy much simpler embedding results: \( S_{\sigma}^{m'} \subset W_{\sigma}^m \) if \( m' - m > 1/2 \), and \( W_{\sigma}^{m'} \subset S_{\sigma}^m \) if \( m' - m > 1 \). The proof of the second embedding uses the estimates of Theorems [1.1] and [1.2]. Even though \( S = \bigcap_m S_{\sigma}^m \), the inclusion relations between the spaces \( S_{\sigma}^m \) and \( S_{\sigma}^{m'} \) are complicated, which motivates the complexity of Theorems [1.4] and [1.5].

Next, we consider other perturbations of \( H \) on \( \mathbb{R}_+ \). Let \( S_{\text{ev}, U} \) denote the space of restrictions of even Schwartz functions to some open set \( U \), and set \( \phi_{k, U} = \phi_k|_U \). The notation \( S_{\text{ev}, +} \) and \( \phi_{k, +} \) is used if \( U = \mathbb{R}_+ \).

**Theorem 1.7.** Let

\[
P = H - 2f_1 \frac{d}{dx} + f_2
\]

where \( f_1 \in C^1(U) \) and \( f_2 \in C(U) \) for some open subset \( U \subset \mathbb{R}_+ \) of full Lebesgue measure. Assume that

\[
f_2 = \sigma(\sigma - 1)x^{-2} - f_1^2 - f_1'
\]

for some \( \sigma > -1/2 \). Let

\[
h = x^\sigma e^{-f_1},
\]

where \( F_1 \in C^2(U) \) is a primitive of \( f_1 \). Then the following properties hold:

(i) \( P \), with domain \( hS_{\text{ev}, U} \), is essentially self-adjoint in \( L^2(\mathbb{R}_+, e^{2F_1} dx) \);

(ii) the spectrum of its self-adjoint extension, denoted by \( \mathcal{P} \), consists of the eigenvalues \((4k + 1 + 2\alpha)s\) (\( k \in \mathbb{N} \)) with multiplicity one and normalized eigenfunctions \( \sqrt{2}x^\sigma \phi_{2k, U} \); and

(iii) the smooth core of \( \mathcal{P} \) is \( hS_{\text{ev}, U} \).

This theorem follows by showing that the stated condition on \( f_1 \) and \( f_2 \) characterizes the cases where \( P \) can be obtained by the following process: first, restricting \( L_{\sigma} \) to even functions, then restricting to \( U \), and finally conjugating by \( h \). The term of \( P \) with \( \frac{d}{dx} \) can be removed by conjugation with the product of a positive function, obtaining the operator \( H + \sigma(\sigma - 1)x^{-2} \).

Several examples of such type of operator \( P \) are given. For instance, we get the following.

**Corollary 1.8.** Let \( P = H - 2c_1x^{-1}\frac{d}{dx} + c_2x^{-2} \) for some \( c_1, c_2 \in \mathbb{R} \). If there is some \( a \in \mathbb{R} \) such that

\[
a^2 + (2c_1 - 1)a - c_2 = 0,
\]

\[
\sigma := a + c_1 > -1/2,
\]

then:

(i) \( P \), with domain \( x^a S_{\text{ev}, +} \), is essentially self-adjoint in \( L^2(\mathbb{R}_+, x^{2c_1} dx) \);

(ii) the spectrum of its self-adjoint extension, denoted by \( \mathcal{P} \), consists of the eigenvalues \((4k + 1 + 2\alpha)s\) (\( k \in \mathbb{N} \)) with multiplicity one and normalized eigenfunctions \( \sqrt{2}x^\sigma \phi_{2k, +} \); and

(iii) the smooth core of \( \mathcal{P} \) is \( x^a S_{\text{ev}, +} \).

In Corollary 1.8 for some \( c_1, c_2 \in \mathbb{R} \), there are two values of \( a \) satisfying the stated condition, obtaining two different self-adjoint operators defined by \( P \) in
different Hilbert spaces. For instance, the Dunkl harmonic oscillator $L_\sigma$ may define self-adjoint operators even when $\sigma \leq -1/2$.

Corollary 1.8 will be applied in [1] to prove a new type of Morse inequalities on strata of compact stratifications [37, 26, 41] with adapted metrics [8, 9, 27, 28, 5], where Witten’s perturbation [42] is used for the minimal/maximal ideal boundary conditions (i.b.c.) [7] of de Rham complex. It will be a continuation of the type of analysis on non-complete Riemannian manifolds began by J. Cheeger [8, 9]. The type of Morse functions on strata considered in [1] may not extend to the ambient stratification; thus they are different from the functions considered by M. Goresky and R. MacPherson [21].

More precisely, consider a cone $c(L)$ whose link is a compact stratification $L$. For any stratum $N$ of $L$, we have a corresponding stratum $M = N \times \mathbb{R}^+$ of $c(L)$, and let $\rho$ be the function on $M$ defined by the second factor projection. The study of our version of Morse functions boils down to the case of this stratum $M$ with the function $\pm \frac{1}{2} \rho^2$ and an adapted metric of the form $g = \rho^2 \tilde{g} + d\rho^2$, where $\tilde{g}$ is an adapted metric on $N$. The corresponding Witten’s perturbation of the de Rham differential operator $d$ is $d_s = e^{\pm \rho^2/2} d e^{\pm \rho^2/2} (s > 0)$. Let $\Delta_s$ denote the corresponding Witten’s perturbed Laplacian on $M$, and let $\tilde{\Delta}$ be the Laplacian on $N$. According to [8, 9], $\tilde{\Delta}$ defines a self-adjoint operator with discrete spectrum in the Hilbert space of $L^2$ differential forms on $N$. By using its spectral decomposition, $\Delta_s$ splits into sum differential operators whose Laplacians can be reduced to operators of the type considered in Corollary 1.8. The two possible choices of $a$ in Corollary 1.8 will give rise to the minimal/maximal i.b.c. of $d_s$. In this way, Corollary 1.8 provides the local analysis needed to show the desired Morse inequalities.

Acknowledgment. Part of this research was made during the visit of the first author to the Centre de Recerca Matemàtica (CRM) during the research program “Foliations”, in April–July, 2010.

2. Preliminaries

Most of the contents of this section are taken or adapted from [33].

2.1. Dunkl operator. Recall that, for any $\phi \in C^\infty = C^\infty(\mathbb{R})$, there is some $\psi \in C^\infty$ such that $\phi(x) - \phi(0) = x\psi(x)$, which also satisfies

$$
\psi^{(m)}(x) = \int_0^1 t^m \phi^{(m+1)}(tx) \, dt
$$

for all $m \in \mathbb{N}$ (see e.g. [22, Theorem 1.1.9]). The notation $\psi = x^{-1}\phi$ is used.

The Dunkl operator, in the case of dimension one, is the differential-difference operator $T_{\sigma}$ on $C^\infty$, depending on a parameter $\sigma \in \mathbb{R}$, defined by

$$(T_{\sigma} \phi)(x) = \phi'(x) + 2\sigma \frac{\phi(x) - \phi(-x)}{x}.$$  

It can be considered as a perturbation of the derivative operator $\frac{d}{dx}$.

Consider the decomposition $C^\infty = C^\infty_{ev} \oplus C^\infty_{od}$, as direct sum of subspaces of even and odd functions. The matrix expressions of operators on $C^\infty$ will be considered with respect to this decomposition. The operator of multiplication by a function $h$
will be denoted also by $h$. We can write
\[ \frac{d}{dx} = \begin{pmatrix} 0 & \frac{d}{dx} \\ \frac{d}{dx} & 0 \end{pmatrix}, \quad x = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \]
\[ T_\sigma = \begin{pmatrix} 0 & \frac{d}{dx} + 2\sigma x^{-1} \\ \frac{d}{dx} + 2\sigma & 0 \end{pmatrix} \]
on $C^\infty$. With
\[ \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}, \]
we have
\[ [T_\sigma, x] = 1 + 2\Sigma, \]
\[ T_\sigma \Sigma + \Sigma T_\sigma = x \Sigma + \Sigma x = 0. \] (7)

Consider the perturbed factorial $m!_\sigma$ of each $m \in \mathbb{N}$, which is inductively defined by setting $0!_\sigma = 1$, and
\[ m!_\sigma = \begin{cases} (m-1)!_\sigma m & \text{if } m \text{ is even} \\ (m-1)!_\sigma (m+2\sigma) & \text{if } m \text{ is odd} \end{cases} \]
for $m > 0$. Observe that $m!_\sigma > 0$ if $\sigma > -1/2$, which will be the case of our interest; otherwise, $m!_\sigma$ may be $\leq 0$. For $k \leq m$, even when $k!_\sigma = 0$, the quotient $m!_\sigma/k!_\sigma$ can be understood as the product of the factors from the definition of $m!_\sigma$ which are not included in the definition of $k!_\sigma$. For any $\phi \in C^\infty$ and $m \in \mathbb{N}$, we have
\[ (T_\sigma^m \phi)(0) = \frac{m!_\sigma}{m!} \phi^{(m)}(0). \] (9)
This equality follows by (6) and induction on $m$.

2.2. Dunkl harmonic oscillator. Recall that, for dimension one, the harmonic oscillator, and the annihilation and creation operators are
\[ H = -\frac{d^2}{dx^2} + s^2 x^2, \quad A = sx + \frac{d}{dx}, \quad A' = sx - \frac{d}{dx} \]
on $C^\infty$. By using $T_\sigma$ instead of $d/dx$, we get a perturbations of $H$, $A$ and $A'$ called Dunkl harmonic oscillator, and Dunkl annihilation and creation operators:
\[ L = -T_\sigma^2 + s^2 x^2 = H - 2\sigma \begin{pmatrix} x^{-1} & \frac{d}{dx} \\ 0 & 0 \end{pmatrix}, \]
\[ B = sx + T_\sigma = A + 2\sigma \begin{pmatrix} 0 & x^{-1} \\ 0 & 0 \end{pmatrix}, \]
\[ B' = sx - T_\sigma = A' - 2\sigma \begin{pmatrix} 0 & 0 \\ 0 & x^{-1} \end{pmatrix}. \]

By (7) and (8),
\[ L = BB' - (1 + 2\Sigma)s = B'B + (1 + 2\Sigma)s = \frac{1}{2} (BB' + B'B), \]
\[ [L, B] = -2sB, \quad [L, B'] = 2sB', \]
\[ [B, B'] = 2s(1 + 2\Sigma), \]
\[ [L, \Sigma] = B\Sigma + \Sigma B = B'\Sigma + \Sigma B' = 0. \] (10) (11) (12) (13)
Recall also that the Schwartz space \( \mathcal{S} = \mathcal{S}(\mathbb{R}) \) is the space of functions \( \phi \in C^\infty \) such that

\[
\| \phi \|_{S^m} = \sum_{i+j \leq m} \sup_x |x^i \phi^{(j)}(x)|
\]

is finite for all \( m \in \mathbb{N} \) (including zero). This defines a sequence of norms \( \| \|_{S^m} \) on \( \mathcal{S} \), which is endowed with the corresponding Fréchet topology. The Banach space completion of \( \mathcal{S} \) with respect to each norm \( \| \|_{S^m} \) will be denoted by \( S^m \). We have \( S^{m+1} \subset S^m \) continuously, and \( S = \bigcap_m S^m \). Let us remark that \( \| \phi' \|_{S^m} \leq \| \phi \|_{S^{m+1}} \) for all \( m \).

The above decomposition of \( C^\infty \) can be restricted to each \( S^m \) and \( \mathcal{S} \), giving \( S^m = S^m_\text{ev} \oplus S^m_\text{odd} \) and \( \mathcal{S} = \mathcal{S}_\text{ev} \oplus \mathcal{S}_\text{odd} \). The matrix expressions of operators on \( \mathcal{S} \) will be considered with respect to this decomposition. For \( \phi \in C^\infty_\text{ev} \), \( \psi = x^{-1} \psi \) and \( i, j \in \mathbb{N} \), it follows from (10) that

\[
|x^i \psi^{(j)}(x)| \leq \int_0^1 t^{j-1} |(tx)^i \phi^{(j+1)}(tx)| \, dt \leq \sup_{y \in \mathbb{R}} |y^i \phi^{(j+1)}(y)|
\]

for all \( x \in \mathbb{R} \). Thus \( \| \psi \|_{S^m} \leq \| \phi \|_{S^{m+1}} \) for all \( m \in \mathbb{N} \), obtaining that \( S_\text{odd} = x \mathcal{S}_\text{ev} \) and \( x^{-1} : C^\infty_\text{odd} \to C^\infty_\text{ev} \) restricts to a continuous operator \( x^{-1} : S_\text{odd} \to \mathcal{S}_\text{ev} \).

Therefore \( x : \mathcal{S}_\text{ev} \to S_\text{odd} \) is an isomorphism of Fréchet spaces, and \( T_\sigma, B, B' \) and \( L \) define continuous operators on \( \mathcal{S} \).

Let \( \langle \ , \, \rangle_\sigma \) and \( \| \|_\sigma \) denote the scalar product and norm of \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \). Assume from now on that \( \sigma > -1/2 \), and therefore \( \mathcal{S} \) is a dense subset of \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \). In \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \), with domain \( \mathcal{S} \), \( -T_\sigma \) is adjoint of \( T_\sigma \), \( B' \) is adjoint of \( B \), and \( L \) is essentially self-adjoint. The self-adjoint extension of \( L \), with domain \( \mathcal{S} \), will be denoted by \( \mathcal{L} \), or \( \mathcal{L}_\sigma \). Its spectrum consists of the eigenvalues \( (2k+1+2\sigma)s \) (\( k \in \mathbb{N} \)). The corresponding normalized eigenfunctions \( \phi_k \) are inductively defined by

\[
\phi_0 = s^{(2\sigma+1)/4} \Gamma(\sigma+1/2)^{-1/2} e^{-sx^2/2} ,
\]

\[
\phi_k = \begin{cases} 
(2ks)^{-1/2} B' \phi_{k-1} & \text{if } k \text{ is even} \\
(2(k+2\sigma)s)^{-1/2} B' \phi_{k-1} & \text{if } k \text{ is odd}
\end{cases}
\]

for \( k \geq 1 \). We also have

\[
B \phi_0 = 0 ,
\]

\[
B \phi_k = \begin{cases} 
(2ks)^{1/2} \phi_{k-1} & \text{if } k \text{ is even} \\
(2(k+2\sigma)s)^{1/2} \phi_{k-1} & \text{if } k \text{ is odd}
\end{cases}
\]

for \( k \geq 1 \). These assertions follow from (10)–(13) like in the case of \( H \).

2.3. **Generalized Hermite polynomials.** From (14), (15) and the definition of \( B' \), it follows that the functions \( \phi_k \) are the generalized Hermite functions \( \phi_k =
\( p_k e^{-sx^2/2} \), where \( p_k \) is the sequence of polynomials inductively defined by

\[
p_0 = s^{(2\sigma+1)/4} \Gamma(\sigma + 1/2)^{-1/2},
\]

\[
p_k = \begin{cases} (2ks)^{-1/2}(2sx_k p_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{-1/2}(2sx_k p_{k-1} - T_\sigma p_{k-1}) & \text{if } k \text{ is odd} \end{cases}
\]

for \( k \geq 1 \). Up to normalization, these are the generalized Hermite polynomials; i.e., the orthogonal polynomials associated with the measure \(|x|^{2\sigma} e^{-sx^2} \, dx \) \[36\] p. 380, Problem 25]. Each \( p_k \) is of precise degree \( k \), even/odd if \( k \) is even/odd, and with positive leading coefficient, denoted by \( \gamma_k \). By \[19\],

\[
\gamma_k = \begin{cases} (k-1/2)^{1/2}x_k \gamma_{k-1} & \text{if } k \text{ is even} \\ (k+2\sigma)^{-1/2}(2s)^{1/2} \gamma_{k-1} & \text{if } k \text{ is odd} \end{cases}
\]

We also have

\[
T_\sigma p_0 = 0,
\]

\[
T_\sigma p_k = \begin{cases} (2ks)^{1/2}p_{k-1} & \text{if } k \text{ is even} \\ (2(k+2\sigma)s)^{1/2}p_{k-1} & \text{if } k \text{ is odd} \end{cases}
\]

The following recursion formula follows directly from \[19\] and \[22\]:

\[
p_k = \begin{cases} k^{-1/2}(2s)^{1/2}xp_{k-1} - (k - 1 + 2\sigma)^{1/2}p_{k-2} & \text{if } k \text{ is even} \\ (k+2\sigma)^{-1/2}(2s)^{1/2}xp_{k-1} - (k - 1)^{1/2}p_{k-2} & \text{if } k \text{ is odd} \end{cases}
\]

We have \( p_k(0) = 0 \) if and only if \( k \) is odd, and \( p'_k(0) = 0 \) if and only if \( k \) is even. By \[21\] and induction on \( k \),

\[
p_k(0) = (-1)^{k/2} \sqrt{\frac{(k-1 + 2\sigma)(k-3 + 2\sigma) \cdots (1 + 2\sigma)}{k(k-2) \cdots 2}} p_0
\]

if \( k \) is even. When \( k \) is odd, by \[22\] and \[23\],

\[
(T_\sigma p_k)(0) = (-1)^{(k-1)/2} \sqrt{\frac{(k+2\sigma)(k - 2 + 2\sigma) \cdots (1 + 2\sigma)2s}{(k-1)(k-3) \cdots 2}} p_0
\]

obtaining

\[
p'_k(0) = \frac{(-1)^{(k-1)/2}}{1 + 2\sigma} \sqrt{\frac{(k+2\sigma)(k - 2 + 2\sigma) \cdots (1 + 2\sigma)2s}{(k-1)(k-3) \cdots 2}} p_0
\]

by \[19\]. From \[23\] and by induction on \( k \), we also get

\[
x^{-1} p_k = \sum_{\ell \in \{0,2,\ldots,k-1\}} (-1)^{\frac{k-\ell}{2}} \sqrt{\frac{(k-1)(k-3) \cdots (\ell + 2\sigma)2s}{(k+2\sigma)(k - 2 + 2\sigma) \cdots (\ell + 1 + 2\sigma)2s}} p_\ell
\]

if \( k \) is odd.\(^3\)

The following assertions come from the general theory of orthogonal polynomials \[36\] Chapter III. All zeros of each polynomial \( p_k \) are real and of multiplicity one. Each open interval between consecutive zeros of \( p_k \) contains exactly one zero of \( p_{k+1} \), and at least one zero of every \( p_\ell \) with \( \ell > k \). Moreover \( p_k \) has exactly \( \lfloor k/2 \rfloor \)

\(^3\)As a convention, the product of an empty set of factors is 1. Thus \((k-1)(k-3) \cdots (\ell + 2) = 1 \) for \( \ell = k - 1 \) in \[26\]. Similarly, \[23\] and \[24\] also hold for \( k = 0 \) and \( k = 1 \), respectively.
positive zeros and \( \lfloor k/2 \rfloor \) negative zeros. The zeros of each \( p_k \) will be denoted \( x_{k,1} > x_{k,2} > \cdots > x_{k,k} \). On each interval \( (x_{k,i}, x_{k,i}) \), the function \( p_{k+1}/p_k \) is strictly increasing, and satisfies

\[
\lim_{x \to x_{k,i}^\pm} \frac{p_{k+1}(x)}{p_k(x)} = \mp \infty.
\]

For every polynomial \( p \) of degree \( \leq k - 1 \), we have

\[
p^2(x) \leq \int_{-\infty}^{\infty} p^2(t) |t|^{2\sigma} e^{-sx^2} \, dt \cdot \sum_{\ell=0}^k p^2_k(x) \quad (27)
\]

for all \( x \in \mathbb{R} \). The Gauss-Jacobi formula states that there are \( \lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,k} \in \mathbb{R} \) such that, for any polynomial \( p \) of degree \( \leq 2k - 1 \),

\[
\int_{-\infty}^{\infty} p(x) |x|^{2\sigma} e^{-sx^2} \, dx = \sum_{i=1}^k p(x_{k,i}) \lambda_{k,i} \quad (28)
\]

Lemma 2.1. We have

\[
p_k(x_{k,i}) \lambda_{k,i} = \begin{cases} 2s & \text{if } k \text{ is even} \\ 2s/(1 + 2\sigma) & \text{if } k \text{ is odd} \end{cases}
\]

Proof. This is a direct adaptation of the proof of [4, Corollary 3]. With

\[
p = \frac{p_k p_{k-1}}{x - x_{k,i}},
\]

the formula (28) becomes

\[
\frac{\gamma_k}{\gamma_{k-1}} = p_k(x_{k,i}) p_{k-1}(x_{k,i}) \lambda_{k,i},
\]

and the result follows from (20)–(22). \( \Box \)

3. Estimates of the generalized Hermite functions

To get uniform estimates of the functions \( \phi_k \), they are multiplied by \( |x|^{-\sigma} \), obtaining eigenfunctions of another perturbation of \( H \).

3.1. Second perturbation of \( H \). Now, consider the perturbed derivative,

\[
E_\sigma = |x|^\sigma T_\sigma |x|^{-\sigma} = \begin{pmatrix} 0 & \frac{d}{dx} + \sigma x^{-1} \\ \frac{d}{dx} - \sigma x^{-1} & 0 \end{pmatrix},
\]

and the perturbed harmonic oscillator,

\[
K = |x|^\sigma L |x|^{-\sigma} = -E_\sigma^2 + s^2 x^2 = \begin{pmatrix} H + \sigma(x - 1)x^{-2} & 0 \\ 0 & H + \sigma(x + 1)x^{-2} \end{pmatrix},
\]

defined on

\[
|x|^\sigma \mathcal{S} = |x|^\sigma \mathcal{S}_{\text{ev}} \oplus |x|^\sigma \mathcal{S}_{\text{odd}}.
\]

According to Sections 2.2 and 2.3 and since \( |x|^\sigma : L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \to L^2(\mathbb{R}, dx) \) is a unitary isomorphism, \( K \) is essentially self-adjoint in \( L^2(\mathbb{R}, dx) \), and the spectrum of its self-adjoint extension, denoted by \( \mathcal{K} \), or \( \mathcal{K}_\sigma \), consists of the eigenvalues \( (2k + 1 + 2\sigma)s \) \( (k \in \mathbb{N}) \) of multiplicity one, and corresponding normalized eigenfunctions

\[
\xi_k = |x|^\sigma \phi_k = p_k |x|^\sigma e^{-sx^2/2}.
\]
Each $\xi_k$ is $C^\infty$ on $\mathbb{R} \setminus \{0\}$, and it is $C^\infty$ on $\mathbb{R}$ if and only if $\sigma \in \mathbb{N}$. If $\sigma > 0$ or $k$ is odd, then $\xi_k$ is defined and continuous on $\mathbb{R}$, and $\xi_k(0) = 0$. If $\sigma < 0$ and $k$ is even, then $\xi_k$ is only defined on $\mathbb{R} \setminus \{0\}$; in fact, by (24),

$$\lim_{x \to 0} \xi_k(x) = (-1)^{k/2} \infty .$$

By (22) and (23),

$$\xi_k' = \left(p_k' + \frac{\sigma}{x} - sx\right) p_k \left|x\right|^\sigma e^{-sx^2/2} \quad (29)$$

$$= \begin{cases} 
(\sqrt{2k}p_{k-1} + (\frac{\sigma}{x} - sx)p_k) \left|x\right|^\sigma e^{-sx^2/2} & \text{if } k \text{ is even} \\
(\sqrt{2(k+2\sigma)}p_{k-1} - (\frac{\sigma}{x} + sx)p_k) \left|x\right|^\sigma e^{-sx^2/2} & \text{if } k \text{ is odd}
\end{cases}$$

$$= \begin{cases} 
((sx + \frac{\sigma}{x})p_k - \sqrt{2(k+1+2\sigma)}p_{k+1}) \left|x\right|^\sigma e^{-sx^2/2} & \text{if } k \text{ is even} \\
((sx - \frac{\sigma}{x})p_k - \sqrt{2(k+1)}p_{k+1}) \left|x\right|^\sigma e^{-sx^2/2} & \text{if } k \text{ is odd}.
\end{cases} \quad (30)$$

By (21), (24) and (25),

$$\lim_{x \to 0^\pm} \xi_k'(x) = \begin{cases} 
0 & \text{if } \sigma > 0 \text{ or } \sigma = 0 \\
\pm p_k(0) & \text{if } \sigma = 1 \\
\pm (-1)^{k/2} \infty & \text{if } 0 < \sigma < 1 \\
\mp (-1)^{k/2} \infty & \text{if } -1/2 < \sigma < 0
\end{cases}$$

if $k$ is even,

$$\lim_{x \to 0^\pm} \xi_k'(x) = \begin{cases} 
0 & \text{if } \sigma > 0 \\
p_k'(0) & \text{if } \sigma = 0 \\
(-1)^{(k-1)/2} \infty & \text{if } -1/2 < \sigma < 0
\end{cases}$$

if $k$ is odd, and

$$\lim_{x \to 0^\pm} (\xi_k \xi_k')(x) = \begin{cases} 
0 & \text{if } k \text{ is odd or } \sigma = 0 \text{ or } \sigma > 1/2 \\
\pm p_k'(0)/2 & \text{if } k \text{ is even and } \sigma = 1/2 \\
\pm \infty & \text{if } k \text{ is even and } 0 < \sigma < 1/2 \\
\mp \infty & \text{if } k \text{ is even and } -1/2 < \sigma < 0
\end{cases} \quad (31)$$

By (30),

$$\frac{\xi_k'}{\xi_k} = \begin{cases} 
(sx + \frac{\sigma}{x} - \sqrt{2(k+1+2\sigma)}p_{k+1}/p_k) & \text{if } k \text{ is even} \\
(sx - \frac{\sigma}{x} - \sqrt{2(k+1)}p_{k+1}/p_k) & \text{if } k \text{ is odd}
\end{cases} \quad (32)$$

which generalizes a formula of (22) for the Hermite functions.

For the sake of simplicity, let

$$\bar{\sigma}_k = \sigma(\sigma - (-1)^k) .$$

Each $\xi_k$ satisfies

$$\xi_k'' + q_k \xi_k = 0 , \quad (33)$$

where

$$q_k = (2k + 1 + 2\sigma)s - s^2 x^2 - \bar{\sigma}_k x^{-2} .$$
3.2. Description of \( q_k \). The following elementary analysis of the functions \( q_k \) will be used in Sections 3.3 and 3.4. If \( k \) is even, then \( \bar{\sigma}_k = 0 \) if \( \sigma \in \{0, 1\} \), and \( \bar{\sigma}_k < 0 \) if \( 0 < \sigma < 1 \), and \( \bar{\sigma}_k > 0 \) otherwise. When \( k \) is odd, we have \( \bar{\sigma}_k = 0 \) if \( \sigma = 0 \), and \( \sigma \bar{\sigma}_k > 0 \) if \( \sigma \neq 0 \). Each \( q_k \) is defined and smooth on \( \mathbb{R} \) just when \( \bar{\sigma}_k = 0 \), otherwise it is defined and smooth only on \( \mathbb{R} \setminus \{0\} \). Moreover \( q_k \) is even and

\[
q_k' = -2s^2 x + 2\bar{\sigma}_k x^{-3}.
\]

Observe that

\[
\begin{align*}
\lim_{x \to \pm\infty} q_k(x) &= -\infty, \\
\lim_{x \to 0} q_k(x) &= \begin{cases} -\infty & \text{if } \bar{\sigma}_k > 0 \\
\infty & \text{if } \bar{\sigma}_k < 0 \end{cases},
\end{align*}
\]

\[
\begin{align*}
\lim_{x \to \pm\infty} q_k'(x) &= \mp\infty, \\
\lim_{x \to 0^\pm} q_k'(x) &= \begin{cases} \pm\infty & \text{if } \bar{\sigma}_k > 0 \\
\mp\infty & \text{if } \bar{\sigma}_k < 0 \end{cases}.
\end{align*}
\]

We have the following cases for the zeros of \( q_k' \):

- If \( \bar{\sigma}_k > 0 \), then \( q_k' \) has two zeros, which are

\[
\pm x_{\text{max}} = \pm \sqrt{\bar{\sigma}_k/s}.
\]

At these points, \( q_k \) reaches its maximum, which equals \( c_{\text{max}} \) for

\[
c_{\text{max}} = 2k + 1 + 2\sigma - 2\sqrt{\bar{\sigma}_k}.
\]

Notice that, in this case, \( c_{\text{max}} = 0 \) if \( k = 0 \) and \( \sigma = -1/8 \), \( c_{\text{max}} < 0 \) if \( k = 0 \) and \(-1/2 < \sigma < -1/8\), and \( c_{\text{max}} > 0 \) otherwise.

- If \( \bar{\sigma}_k = 0 \), then \( q_k' \) has one zero, which is 0, where \( q_k \) reaches its maximum \( c_{\text{max}} \) as above with \( c_{\text{max}} = 2k + 1 + 2\sigma > 0 \).

- If \( \bar{\sigma}_k < 0 \), then \( q_k' > 0 \) on \( \mathbb{R}^- \) and \( q_k' < 0 \) on \( \mathbb{R}^+ \).

We have the following possibilities for the zeros of \( q_k \):

- If \( \bar{\sigma}_k > 0 \) and \( c_{\text{max}} > 0 \), then \( q_k \) has four zeros, which are

\[
\begin{align*}
\pm a_k &= \pm \sqrt{\frac{2k + 1 + 2\sigma - \sqrt{(2k + 1 + 2\sigma)^2 - 4\bar{\sigma}_k}}{2s}}, \\
\pm b_k &= \pm \sqrt{\frac{2k + 1 + 2\sigma + \sqrt{(2k + 1 + 2\sigma)^2 - 4\bar{\sigma}_k}}{2s}}.
\end{align*}
\]

(34)

- If \( \bar{\sigma}_k > 0 \) and \( c_{\text{max}} = 0 \), or \( \bar{\sigma}_k \leq 0 \), then \( q_k \) has two zeros, \( \pm b_k \), defined by (34).

- If \( \bar{\sigma}_k > 0 \) and \( c_{\text{max}} < 0 \), then \( q_k < 0 \).

If \( q_k \) has four zeros, \( \pm a_k \) and \( \pm b_k \), then

\[
s(b_k - a_k)^2 = c_{\text{max}},
\]

(35)

and

\[
2s a_k^2 = \frac{4\bar{\sigma}_k}{2k + 1 + 2\sigma + \sqrt{(2k + 1 + 2\sigma)^2 - 4\bar{\sigma}_k}},
\]

obtaining

\[
a_k \in O(k^{-1/2})
\]

(36)

as \( k \to \infty \).
If \( q_k \) has at least two zeros, \( \pm b_k \), then
\[
2s(b_k^2 - b_{k+1}^2) = 2 + 4 \frac{k^2 - \ell^2 + (1 + 2\sigma)(k - \ell) + \sigma}{\sqrt{(2k + 1 + 2\sigma)^2 - 4\sigma} + \sqrt{(2\ell + 1 + 2\sigma)^2 - 4\sigma}}
\]
for \( \ell \leq k \), obtaining
\[
b_{k+1} - b_k \in O(k^{-1/2})
\]
as \( k \to \infty \), and
\[
b_k - b_\ell \geq C(k - \ell)k^{-1/2}
\]
for some \( C > 0 \) if \( k \) and \( \ell \) are large enough. If \( \sigma_k = 0 \), then \( s\bar{b}_k^2 = c_{\text{max}} \).

Like in [4], the maximal open intervals where \( q_k \) is defined and \( > 0 \) (respectively, \( < 0 \)) will be called oscillation (respectively, non-oscillation) intervals of \( \xi_k \); this terminology is justified by Lemma 3.1 below. We have the following possibilities for the oscillation intervals:

- If \( \sigma_k > 0 \) and \( c_{\text{max}} > 0 \), then \( \xi_k \) has two oscillation intervals, \((a_k, b_k)\) and \((-b_k, -a_k)\), containing \( x_{\text{max}} \) and \(-x_{\text{max}}\), respectively.
- If \( \sigma_k > 0 \) and \( c_{\text{max}} \leq 0 \), then \( \xi_k \) has no oscillation intervals.
- If \( \sigma_k < 0 \), then \( \xi_k \) has two oscillation intervals, \((-b_k, 0)\) and \((0, b_k)\).
- If \( \sigma_k = 0 \), then \( \xi_k \) has one oscillation interval, \((-b_k, b_k)\).

### 3.3. Location of the zeros of \( \xi_k \) and \( \xi'_k \)

In \( \mathbb{R} \setminus \{0\} \), the functions \( \xi_k \) and \( p_k \) have the same zeros. Then \( \xi_k \) and \( \xi'_k \) have no common zeros by (29). The functions \( \xi_k \) and \( \xi_1 \) have no zeros in \( \mathbb{R} \setminus \{0\} \), and the two zeros \( \pm x_{2,1} \) of \( \xi_2 \) are in \( \mathbb{R} \setminus \{0\} \).

**Lemma 3.1.** On \( \mathbb{R} \setminus \{0\} \):

(i) the zeros of \( \xi'_k \) belong to the oscillation intervals of \( \xi_k \);
(ii) if \( k \) is odd or \( \sigma \geq 0 \), the zeros of \( \xi_k \) belong to the oscillation intervals of \( \xi_k \); and
(iii) if \( k \) is even and \( \sigma < 0 \), the zeros of \( \xi_k \), possibly except \( \pm x_{k,2,1} \), belong to the oscillation intervals of \( \xi_k \).

**Proof.** It is enough to consider the zeros in \( \mathbb{R}_+ \) because \( \xi_k \) is either even or odd. We can also assume that \( \xi_k \xi'_k \) has zeros on \( \mathbb{R}_+ \), otherwise there is nothing to prove.

Let \( x_+ \) and \( x^* \) denote the minimum and maximum of the zeros of \( \xi_k \xi'_k \) in \( \mathbb{R}_+ \).

By (33),
\[
(\xi_k \xi'_k)' = \xi_k^2 - q_k \xi'_k^2 > 0
\]
on the non-oscillation intervals, and therefore \( \xi_k \xi'_k \) is strictly increasing on those intervals. In particular, since \( \xi_k \xi'_k \) is strictly increasing on \((b_k, \infty)\) and \((\xi_k \xi'_k)(x) \to 0 \) as \( x \to \infty \), it follows that \( x^* < b_k \). This shows the statement when there is one oscillation interval of the form \((-b_k, b_k)\). So it remains to consider the case where there is an oscillation interval of \( \xi_k \) in \( \mathbb{R}_+ \) of the form \((a_k, b_k)\). This holds when \( k \) is odd and \( \sigma > 0 \), \( k = 0 \) and \( \sigma \in (-1/8, 0) \cup (1, \infty) \), or \( k \in 2\mathbb{Z}_+ \) and \( \sigma \in (-1/2, 0) \cup (1, \infty) \).

If \( k \) is odd and \( \sigma > 0 \), or \( k \) is even and \( \sigma \in (1, \infty) \), then \( x_+ > a_k \) because \( \xi_k \xi'_k \) is strictly increasing on \((0, a_k)\) and \((\xi_k \xi'_k)(x) \to 0 \) as \( x \to 0^+ \) by (31).

Finally, assume that \( k \in 2\mathbb{Z}_+ \) and \( \sigma \in (-1/2, 0) \). Then the above arguments do not work because \( \xi_k \xi'_k \) is not positive on \((0, a_k)\). Let \( f \) be the function on \( \mathbb{R}_+ \) defined by \( f(x) = s2 + \sqrt{-1/s} \). We have \( f(x) \to -\infty \) as \( x \to 0^+ \), and \( f' = s - \frac{2}{x^2} > 0 \) on \( \mathbb{R}_+ \). Moreover \( \sqrt{-1/s} \) is the unique zero of \( f \) in \( \mathbb{R}_+ \).
If \(x_*\) is a zero of \(\xi_k'\), then \(\xi_k\) (and \(p_k\) as well) has no zeros in \([-x_*, x_*]\). Therefore 0 is the unique zero of \(p_{k+1}\) in this interval. So \(p_{k+1}/p_k > 0\) on \((0, x_*)\). Since

\[
0 = f(x_*) - \sqrt{2(k + 1 + 2\sigma)s} \frac{p_{k+1}(x_*)}{p_k(x_*)}
\]

by (32), it follows that \(f(x_*) > 0\), obtaining \(x_*>\sqrt{-\sigma/s}\). But

\[
a_k^2 = \frac{2k + 1 + 2\sigma - \sqrt{2(k + 1)^2 + 8(k + 1)\sigma}}{2s} < -\frac{\sigma}{s}
\]

because \(k > 1\), obtaining \(x_* > a_k\).

If \(x_*\) is a zero of \(\xi_k\) (i.e., \(x_* = x_{k,k/2}\)), then the other positive zeros of \(\xi_k\xi_k'\) are > \(a_k\) because this function is strictly increasing on \((0, a_k)\).

In the case of Lemma 3.1(iii), the zero \(\pm x_{k,k/2}\) of \(\xi_k\) may be in an oscillation interval, in a non-oscillation intervals or in their common boundary point. For instance, for \(k = 2\),

\[
p_2 = \left(\sqrt{\frac{2}{1 + 2\sigma}} - \frac{1 + 2\sigma}{2}\right) p_0
\]

by (19), obtaining

\[
x_{2,1}^2 = \frac{1 + 2\sigma}{2s}.
\]

Moreover

\[
a_2^2 = \frac{5 + 2\sigma - \sqrt{25 + 24\sigma}}{2s}.
\]

So

\[
x_{2,1} - a_2 = \frac{-4 + \sqrt{25 + 24\sigma}}{2s},
\]

and therefore \(\sigma > -3/8\) if and only if \(x_{2,1} > a_2\). So \((a_2, b_2)\) contains no zero of \(\xi_2\) when \(\sigma \in (-1/2, -3/8]\). For \(k > 2\), every oscillation interval of \(\xi_k\) contains some zero of \(\xi_k\) by Lemma 3.1.

**Lemma 3.2.** There exist \(C_0, C_1, C_2 > 0\), depending on \(\sigma\), such that, if \(k \geq C_0\) and \(I\) is any oscillation interval of \(\xi_k\), then there is some subinterval \(J \subset I\) so that:

(i) for every \(x \in J\), there exists some zero \(x_{k,i}\) of \(\xi_k\) in \(I\) such that

\[
|x - x_{k,i}| \leq \frac{C_1}{\sqrt{q_k(x)}};
\]

(ii) each connected component of \(I \setminus J\) is of length \(\leq C_2k^{-1/2}\).

**Proof.** According to Section 3.2, for any \(c > 0\) with \(cs \in q_k(I)\), the set \(I_c = I \cap q_k^{-1}(cs, \infty)\) is a subinterval of \(I\), whose boundary in \(I\) is \(I \cap q_k^{-1}(cs)\).

**Claim 1.** If \(\text{length}(I_c) \geq 2\pi/\sqrt{cs}\), then each boundary point of \(I_c\) in \(I\) satisfies the condition of (i) with \(x_{k,i} \in I_c\) and \(C_1 = 2\pi\).

Let \(f_c\) be the function on \(\mathbb{R}\) defined by \(f_c(x) = \sin(\sqrt{cs}x)\), whose zeros are \(\ell\pi/\sqrt{cs}\) for \(\ell \in \mathbb{Z}\). Since \(f_c' + csf_c = 0\) and \(cs \leq q_k\) on \(I_c\), the zeros of \(\xi_k\) in \(I_c\) separate the zeros of \(f_c\) in \(I_c\) by Sturm’s comparison theorem. If \(\text{length}(I_c) \geq 2\pi/\sqrt{cs}\), then each boundary point \(x\) of \(I_c\) is at a distance \(\leq 2\pi/\sqrt{cs}\) of two consecutive zeros of \(f_c\) in \(I_c\), and there is some zero of \(\xi_k\) between them, which shows Claim 1, because \(q_k(x) = cs\).
Now we have to analyze each type of oscillation interval separately, corresponding to the possibilities for $\bar{\sigma}_k$ and $c_{\text{max}}$. When there are two oscillation intervals of $\xi_k$, it is enough to consider only the oscillation interval contained in $\mathbb{R}_+$ because the function $\xi_k$ is either even or odd.

The first type of oscillation interval is of the form $I = (a_k, b_k)$, which corresponds to the conditions $\bar{\sigma}_k > 0$ and $c_{\text{max}} > 0$. We have $cs \in q_k(I)$ when $0 < c \leq c_{\text{max}}$. Then $q_k^{-1}(cs)$ consists of the points

$$
\pm a_{k,c} = \pm \sqrt{\frac{2k + 1 + 2\sigma - c - \sqrt{(2k + 1 + 2\sigma - c)^2 - 4\bar{\sigma}_k}}{2s}},
$$

$$
\pm b_{k,c} = \pm \sqrt{\frac{2k + 1 + 2\sigma - c + \sqrt{(2k + 1 + 2\sigma - c)^2 - 4\bar{\sigma}_k}}{2s}},
$$

and we get $I_c = [a_{k,c}, b_{k,c}]$. Since

$$
s(b_{k,c} - a_{k,c})^2 = c_{\text{max}} - c,
$$

we have $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$ if and only if $c(c_{\text{max}} - c) \geq 4\pi^2$, which means that $c_{\text{max}} \geq 4\pi$ and $c_- \leq c \leq c_+$ for

$$
c_\pm = \frac{c_{\text{max}} \pm \sqrt{c_{\text{max}}^2 - 16\pi^2}}{2}.
$$

Since $c_{\text{max}} \in O(k)$ as $k \to \infty$, there is some $C_0 > 0$, depending on $\sigma$, such that $c_{\text{max}} \geq 4\pi$ for all $k \geq C_0$. Assuming $k \geq C_0$, let $a_{k,\pm} = a_{k,c\pm}$ and $b_{k,\pm} = b_{k,c\pm}$, which satisfy

$$
a_k < a_{k,-} < a_{k,+} < b_{k,+} < b_{k,-} < b_k.
$$

Fix any $x \in I$ and let $q_k(x) = cs$. First, $x \in [a_{k,-}, a_{k,+}] \cup [b_{k,+}, b_{k,-}]$ if and only if $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$, and in this case $x$ satisfies the condition of (i) with $x_{k,i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. Second, if $x \in (a_k, a_{k,-}) \cup (b_{k,-}, b_k)$, then $\text{length}(I_c) < 2\pi/\sqrt{cs}$, $I_c \supset I_{c-}$, and we already know that $I_{c-}$ contains some zero of $\xi_k$. Hence $x$ also satisfies the condition of (i) with $C_1 = 2\pi$. And third, if $x \in (a_{k,+}, b_{k,+})$, then

$$
s(b_{k,+} - a_{k,+})^2 = c_{\text{max}} - c_+ = c_- \leq \frac{16\pi^2}{c_+} \leq \frac{32\pi^2}{c_{\text{max}}} \leq \frac{32\pi^2}{c}
$$

by (40), obtaining

$$
\text{length}(I_{c-}) \leq \frac{4\sqrt{2}\pi}{\sqrt{cs}}.
$$

Since $I_c \subset I_{c+}$ and it is already proved that $I_{c+}$ contains some zero of $\xi_k$, it follows that $x$ also satisfies the condition of (i) with $C_1 = 4\sqrt{2}\pi$. Summarizing, (i) holds in this case with $J = I$ and $C_1 = 4\sqrt{2}\pi$ if $c_{\text{max}} \geq 4\pi$. In this case, (ii) is obvious because $J = I$.

The second type of oscillation interval is of the form $I = (0, b_k)$, which corresponds to the condition $\bar{\sigma}_k < 0$. Now, $cs \in q_k(I)$ for any $c > 0$, the set $q_k^{-1}(cs)$ consists of the points $\pm b_{k,c}$, defined like in (39), and we have $I_c = (0, b_{k,c}]$. The equality $cs = q_k(2\pi/\sqrt{cs})$ holds when

$$
(2k + 1 + 2\sigma)^2 - 4\bar{\sigma}_k - 16\pi^2 > 0
$$

(41)
and $c$ is

$$c_{\pm} = 2\pi \frac{2k + 1 + 2\sigma \pm \sqrt{(2k + 1 + 2\sigma)^2 - 4\sigma k - 16\pi^2}}{\sigma k - 4\pi^2}.$$  

Assuming (41), we have $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$ if and only if $c_- \leq c \leq c_+$. Let $b_{k, \pm} = b_{k, c, \pm}$, satisfying $0 < b_{k, +} < b_{k, -} < b_k$.

Fix any $x \in I$ and let $q_k(x) = cs$. First, $x \in [b_{k, +}, b_{k, -}]$ if and only if $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$; in this case, $x$ satisfies the condition of (i) with $x_{k, i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. And second, if $x \in (b_{k, -}, b_k)$, then $\text{length}(I_c) < 2\pi/\sqrt{cs}$. $I_c \supset I_{c_-}$, and we already know that $I_{c_-}$ contains some zero of $\xi_k$. Hence $x$ also satisfies the condition of (i) with $C_1 = 2\pi$. So, when (41) is true, (i) holds with $J = [b_{k, +}, b_k]$ and $C_1 = 2\pi$.

Notice that $c_+ \in O(k)$ as $k \to \infty$. Then there are some $C_0, C_2 > 0$, depending on $\sigma$, such that, if $k \geq C_0$, then (41) holds and $sb_{k, +}^2 = 4\pi^2/c_+ \leq C_2 k^{-1}$, showing (ii) in this case.

The third and final type of oscillation interval is $I = (-b_k, b_k)$, which corresponds to the condition $\sigma_k = 0$. We have $cs \in q_k(I)$ when $0 < c \leq c_{\text{max}}$. Then $q_k^{-1}(cs)$ consists of the points $\pm b_{k, c, \pm}$, defined like in (39), and we get $I_c = [-b_{k, c, \pm}, b_{k, c, \pm}]$. Since

$$sb_{k, c, \pm}^2 = c_{\text{max}} - c,$$

we have $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$ if and only if $c(c_{\text{max}} - c) \geq \pi^2$, which means that $c_{\text{max}} \geq \pi$ and $c_\pm \leq c \leq c_+$ for

$$c_{\pm} = \frac{c_{\text{max}} \pm \sqrt{c_{\text{max}}^2 - 4\pi^2}}{2}.$$  

Since $c_{\text{max}} \in O(k)$ as $k \to \infty$, there is some $C_0 > 0$, depending on $\sigma$, such that $c_{\text{max}} \geq 4\pi$ for all $k \geq C_0$. Assuming $k \geq C_0$, let $b_{k, \pm} = b_{k, c, \pm}$, which satisfy $0 < b_{k, +} < b_{k, -} < b_k$.

Fix any $x \in I$ and let $q_k(x) = cs$. First, $b_{k, +} \leq |x| \leq b_{k, -}$ if and only if $\text{length}(I_c) \geq 2\pi/\sqrt{cs}$; in this case, $x$ satisfies the condition of (i) with $x_{k, i} \in I_c$ and $C_1 = 2\pi$ by Claim 1. Second, if $|x| > b_{k, -}$, then $\text{length}(I_c) < 2\pi/\sqrt{cs}$. $I_c \supset I_{c_-}$, and we already know that $I_{c_-}$ contains some zero of $\xi_k$. Hence $x$ also satisfies the condition of (i) with $C_1 = 2\pi$. And third, if $|x| < b_{k, +}$, then

$$sb_{k, +}^2 = c_{\text{max}} - c_+ = c_- = \frac{4\pi^2}{c_+} \leq \frac{8\pi^2}{c_{\text{max}}} \leq \frac{8\pi^2}{c}$$

by (42), obtaining

$$\text{length}(I_{c_+}) \leq \frac{\sqrt{2} \pi}{\sqrt{cs}}.$$  

Since $I_c \subset I_{c_+}$ and it is already proved that $I_{c_+}$ contains some zero of $\xi_k$, it follows that $x$ also satisfies the condition of (i) with $C_1 = \sqrt{2}\pi$. Summarizing, (i) holds in this case with $J = I$ and $C_1 = 2\pi$. In this case, (ii) is also obvious because $J = I$.

\textbf{Lemma 3.3.} There exist $C_0', C_1', C_2' > 0$, depending on $\sigma$ and $s$, such that, if $k \geq C_0'$ and $I$ is any oscillation interval of $\xi_k$, then there is some subinterval $J' \subset I$ so that:

(i) $q_k \geq C_1' k^{1/3}$ on $J'$; and

(ii) each connected component of $I \setminus J'$ is of length $\leq C_2' k^{-1/6}$. 
Proof. We use the notation of the proof of Lemma 3.2. The same type of argument can be used for all types of oscillation intervals. Thus, e.g., suppose that \( I \) is of the type \((0, b_k)\). Since \( b_k \in O(k^{1/2}) \) as \( k \to \infty \), we have \( b'_k = b_k - k^{-1/6} \in I \) for \( k \) large enough, and

\[
g_k(b'_k) = -s^2(k^{-1/3} - 2b_kk^{-1/6} - 4s_k((b_k - k^{-1/6})^2 - b_k'^2) \in O(k^{1/3})
\]
as \( k \to \infty \). So there are \( C'_0, C'_1 \) > 0, depending on \( \sigma \) and \( s \), such that \( b'_k \in I \) and \( c' = q_k(b'_k) \geq C'_1k^{1/3} \) for \( k \geq C'_0 \). Then (i) and (ii) hold with \( J' = I_c = (0, b_k) \). □

Corollary 3.4. There exist \( C''_0, C''_1 > 0 \), depending on \( \sigma \) and \( s \), such that, if \( k \geq C''_0 \) and \( I \) is any oscillation interval of \( \xi_k \), then, for each \( x \in I \), there exists some zero \( x_{k,i} \) of \( \xi_k \) in \( I \) so that

\[
|x - x_{k,i}| \leq C''_0k^{-1/6} .
\]

Proof. With the notation of Lemmas 3.2 and 3.3 let \( C''_0 = \max\{C_0, C'_0\} \) and \( C''_1 = \max\{C_2, C'_2\} \). Assume \( k \geq C''_0 \) and consider the subinterval \( J'' = J \cap J' \subset I \). By Lemmas 3.2(ii) and 3.3(iii), each connected component of \( \mathbb{I} \setminus J'' \) is of length \( \leq C''_2k^{-1/6} \). Then, for each \( x \in I \), there is some \( x'' \in J'' \) such that \( |x - x''| \leq C''_2k^{-1/6} \). By Lemmas 3.2(i) and 3.3(i), there is some zero \( x_{k,i} \) of \( \xi_k \) in \( I \) such that

\[
|x'' - x_{k,i}| = \frac{C_1}{\sqrt{q_k(x''')}} \leq \frac{C_1}{C''_1}k^{-1/6} .
\]

Hence

\[
|x - x_{k,i}| \leq (C''_0 + C_1/\sqrt{C''_1})k^{-1/6} . \quad □
\]

3.4. Estimates of \( \xi_k \).

Lemma 3.5. Let \( I \) be an oscillation interval of \( \xi_k \), let \( x \in I \) and let \( x_{k,i} \) be a zero of \( \xi_k \) in \( I \). Then

\[
\xi^2_k(x) \begin{cases} \frac{8s}{3} |x - x_{k,i}| & \text{if } k \text{ is even} \\ \frac{8s}{3(1+2\sigma)} |x - x_{k,i}| & \text{if } k \text{ is odd} \end{cases}.
\]

Proof. We can assume that there are no zeros of \( \xi_k \) between \( x \) and \( x_{k,i} \). For the sake of simplicity, suppose also that \( x_{k,i} < x \) and \( \xi_k > 0 \) on \( (x_{k,i}, x) \); the other cases are analogous. The key observation of [4] is that then the graph of \( \xi_k \) on \( [x_{k,i}, x] \) is concave down, and therefore

\[
\frac{1}{2}\xi_k(x) - x_{k,i} \leq \int_{x_{k,i}}^{x} \xi_k(t) \, dt .
\]

By Schwartz’s inequality and (28), it follows that

\[
\left(\frac{1}{2}\xi_k(x) - x_{k,i}\right)^2 \leq \left( \int_{-\infty}^{\infty} p_k^2(t) |t|^{2\sigma} e^{-st^2} \, dt \right) \left( \int_{x_{k,i}}^{x} (t - x_{k,i})^2 \, dt \right) = p_k^2(x_{k,i}) \lambda_{k,i} \frac{(x - x_{k,i})^3}{3} ,
\]

and the result follows by Lemma 2.1. □

With the notation of Lemma 3.2 for each \( k \geq C_0 \), let \( \tilde{I}_k \) denote the union of the oscillation intervals of \( \xi_k \), and let \( \tilde{J}_k \subset \tilde{I}_k \) denote the union of the corresponding subintervals \( J \) defined in the proof of Lemma 3.2. More precisely:

- if \( \tilde{\sigma}_k > 0 \) and \( c_{\max} > 0 \), then \( \tilde{J}_k = \tilde{I}_k = (-a_k, -b_k) \cup (a_k, b_k) \);
EIGENFUNCTION ESTIMATES AND EMBEDDING THEOREMS

...for $x > b$

...$\xi$ and

...Proof of Theorem 1.1. Which may be empty if there are no oscillation intervals.

...Proof. Let $k < C$

...If $x$ that $\bar{x} \in \hat{I}_k$. Then (ii) follows by Corollary 3.4 and Lemma 3.5.

...Consider the case $\sigma < 0$ and $k$ even, when Theorem 1.1 does not provide any estimate of $\xi_k^2$ around zero. According to Section 2.3, the function $p_k^2(x)$ on the region $|x| \leq x_{k,k/2}$ reaches its maximum at $x = 0$, and moreover $p_k^2(0) < p_k^2$ by (24). Hence $\phi_k^2(x) < p_k^2$ for $|x| \leq x_{k,k/2}$, which complements Theorem 1.1(iii). On the other hand, $\phi_k^2(x) \leq \xi_k^2(x)$ for $|x| \leq 1$. Moreover $x_{k,k/2} \leq 1$ for $k$ large enough by Corollary 3.4 since $a_k \to 0$ as $k \to \infty$. So Theorem 1.2 follows from Theorem 1.1(iii).

...The following lemmas will be used in the proof of Theorem 1.3.

**Lemma 3.6.** There is some $F > 0$ such that, for $k \geq 1$ and $x \geq b_{k+1}$,

...$\xi_k(x) \leq \frac{Fk^{-5/12}}{(x - b_k)^2}$.

**Proof.** Let $x_0 \in (x_{k,1}, b_k)$ such that $\xi'_k(x_0) = 0$. Since

...$\xi'_k(x) = \int_{x_0}^x \xi''_k(t) \, dt$

...and $\xi'_k(x) < 0$ for $x > b_k$, we get

...$\int_{x_0}^x q_k(t)\xi_k(t) \, dt > 0$

...for $x > b_k$. Because $\xi_k(x) > 0$ for $x > x_0$, $q_k(x) > 0$ for $x_0 < x < b_k$ and $q_k(x) < 0$ for $x > b_k$, it follows that

...$\int_{x_0}^{b_k} q_k(t)\xi_k(t) \, dt > -\int_{b_k}^x q_k(t)\xi_k(t) \, dt$. (43)
According to Corollary 3.4 and Theorem 1.1(ii),(iii), for \( k \geq C'_0 \) and with \( \tilde{C} = \max\{C', C''\} \), we get
\[
\int_{x_0}^{b_k} q_k(t) \xi_k(t) \, dt \leq \tilde{C}^{1/2} k^{-1/12} \int_{x_0}^{b_k} q_k(t) \, dt
\]
\[
= \tilde{C}^{1/2} k^{-1/12} \left( (2k + 1 + 2\sigma) s(b_k - x_0) - \frac{s^2}{3} (b_k^3 - x_0^3) + \sigma_k (b_k^{-1} - x_0^{-1}) \right)
\]
\[
\leq \tilde{C}^{1/2} k^{-1/12} \left( (2k + 1 + 2\sigma) sC''_1 k^{-1/6} - \frac{s^2}{3} (b_k^3 - b_k - C''_1 k^{-1/6})^3 + \frac{|\sigma_k| C''_1 k^{-1/6}}{b_k (b_k - C''_1 k^{-1/6})} \right)
\]
\[
\leq \tilde{C}^{1/2} k^{-1/12} \left( (2k + 1 + 2\sigma) sC''_1 k^{-1/6} - s^2 \left( C''_1 b_k^2 k^{-1/6} - C''_1 b_k^2 k^{-1/3} - \frac{C''_1^3}{3} k^{-1/2} \right) + \frac{|\sigma_k| C''_1 k^{-1/6}}{b_k (b_k - C''_1 k^{-1/6})} \right).
\]
Since
\[
2k + 1 + 2\sigma - sb_k^2 = \frac{\sigma_k}{sb_k^2},
\]
there is some \( F_0 > 0 \) such that
\[
\int_{x_0}^{b_k} q_k(t) \xi_k(t) \, dt \leq F_0 k^{1/12} \quad (44)
\]
for all \( k \in \mathbb{N} \).

On the other hand,
\[
- \int_{b_k}^{x} q_k(t) \xi_k(t) \, dt \geq -\xi_k(x) \int_{b_k}^{x} q_k(t) \, dt.
\]
With the substitution \( u = t - b_k \), we get
\[
q_k(t) = -s^2 u (u + 2b_k) + \frac{\sigma_k}{b_k^2} - \sigma_k (u + b_k)^{-2},
\]
giving
\[
-\xi_k(x) \int_{b_k}^{x} q_k(t) \, dt = \xi_k(x) \left( s^2 \left( \frac{1}{3} (x - b_k)^3 + b_k (x - b_k)^2 \right) - \frac{\sigma_k}{b_k^2} (x - b_k) - \sigma_k (x^{-1} - b_k^{-1}) \right)
\]
\[
\geq \xi_k(x) \left( s^2 b_k (x - b_k)^2 - \frac{|\sigma_k|}{b_k^2} (x - b_k) - |\sigma_k| b_k^{-1} \right)
\]
\[
\geq \xi_k(x) \left( s^2 b_k - \frac{|\sigma_k|}{b_k^2 (b_{k+1} - b_k)} \right) (x - b_k)^2 - |\sigma_k| b_k^{-1}
\]
for \( x \geq b_{k+1} \). By (37), it follows that there is some \( F_1 > 0 \) such that
\[
- \int_{b_k}^x q_k(t) \xi_k(t) \, dt \geq F_1 \xi_k(x) k^{1/2} (x - b_k)^2
\]
for all \( k \) and \( x \geq b_{k+1} \). Now the result follows from (38)–(39). \( \square \)

**Lemma 3.7.** For each \( \epsilon > 0 \), there is some \( G > 0 \) such that, for all \( k \in \mathbb{N} \),
\[
\max_{|x - x_{k,1}| \leq \epsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \xi_{k}^2(x) \leq G k^{1/6}.
\]

**Proof.** Take any \( x \in \mathbb{R} \) such that \( |x - x_{k,1}| \leq \epsilon k^{-1/6} \). By Corollary 3.4,
\[
|x - b_k| < |x - x_{k,1}| + |x_{k,1} - b_k| \leq (\epsilon + C_1') k^{-1/6}
\]
for \( k \geq C_0' \). In particular, \( b_k < x \) if \( k \) is large enough. With this assumption, let \( \ell_0, \ell_1, \ell_2 \in \mathbb{N} \) satisfying \( 0 < \ell_0 < \ell_1 < \ell_2 - 1 \), where \( \ell_0 \) and \( \ell_1 \) will be determined later, and \( \ell_2 \) is the maximum of the naturals \( \ell < k \) with \( b_\ell \leq x \) for all \( \ell' \leq \ell \). Let
\[
f_\pm(t) = \sqrt{2t + 1 + 2\sigma \pm 1}
\]
for \( t \geq 1 \). We have
\[
f_\pm(\ell) - \sqrt{\delta b_\ell} = 2 \pm (2\ell + 1 + 2\sigma + 1 + \delta t \sqrt{2(2\ell + 1 + 2\sigma)^2 - 4 \sigma t}) (f_\pm(\ell) + \sqrt{\delta b_\ell})
\]
for \( \ell \in \mathbb{Z}_+ \). So, assuming that \( k \) is large enough, we can fix \( \ell_0 \), independently of \( k \) and \( x \), so that
\[
f_\mp(\ell) < \sqrt{\delta b_\ell} < f_\pm(\ell)
\]
for all \( \ell \geq \ell_0 \). We have \( f_+(\ell_1) < f_-(\ell_2) \) because \( \ell_1 < \ell_2 - 1 \). Moreover observe that
\[
f_+(t) = (2(t + 1 + \sigma))^{-1/2} > 0,
\]
\[
f_+(t) = -(2(t + 1 + \sigma))^{-3/2} < 0
\]
for all \( t \geq 1 \). Then, by Lemma 3.5,
\[
\sum_{\ell = \ell_0}^{\ell_1 - 1} \xi_{k}^2(x) \leq \sum_{\ell = \ell_0}^{\ell_1 - 1} F^2 \sum_{\ell = \ell_0}^{\ell - 1} \frac{\ell^{-5/6}}{(x - b_{\ell})^4}
\]
\[
\leq F^2 \frac{\ell_1^{-1/6}}{\ell_0} \sum_{\ell = \ell_0}^{\ell_1 - 1} \frac{\ell^{-5/6}}{(f_-(\ell_2) - f_+(\ell_1))^4} \leq F^2 \frac{\ell_1^{-5/6}}{\ell_0} \int_{\ell_0}^{\ell_1} \frac{t^{-5/6} \, dt}{(f_-(\ell_2) - f_+(t))^4}.
\]
After integrating by parts four times, we get
\[
\int_{\ell_0}^{\ell_1} \frac{t^{-5/6} \, dt}{(f_-(\ell_2) - f_+(\ell_1))^4} \leq \frac{\ell_1^{-5/6} f_+^{-1}(\ell_1)}{3 (f_-(\ell_2) - f_+(\ell_1))^3} + \frac{5 \ell_1^{-11/6} f_+^{-2}(\ell_1)}{36 (f_-(\ell_2) - f_+(\ell_1))^2}
\]
\[
+ \frac{55 \ell_1^{-17/6} f_+^{-3}(\ell_1)}{216 (f_-(\ell_2) - f_+(\ell_1))} + \frac{365 \ell_1^{-23/6} f_+^{-4}(\ell_1) \ln(f_-(\ell_2))}{1296 \ell_1^{-23/6} f_+^{-3}(\ell_1) \ln(f_-(\ell_2))}
\]
\[
+ \frac{21505}{7776} \int_{\ell_0}^{\ell_1} t^{-29/6} f_+^{-4}(t) \, dt.
\]
Therefore, since \( f'_x(t) \in O(t^{-1/2}) \) as \( t \to \infty \), there exists some \( G_1 > 0 \), independent of \( k \) and \( x \), such that

\[
\sum_{\ell=\ell_0}^{\ell_1-1} \xi^2_\ell(x) \leq G_1 \left( \frac{\ell_1^{-1/3}}{(f_-(\ell_2) - f_+(\ell_1))^3} + \frac{\ell_1^{-5/6}}{(f_-(\ell_2) - f_+(\ell_1))^2} \right.
\]

\[
\left. + \frac{\ell_1^{-4/3}}{f_-(\ell_2) - f_+(\ell_1)} + \ell_1^{-11/6} \ln(f_-(\ell_2)) + \ln(f_-(\ell_2)) \right).
\]

We have

\[
\ell_1^{-11/6} \ln(f_-(\ell_2)) + \ln(f_-(\ell_2)) \leq \ell_2^{1/6}
\]

for \( k \) large enough. Then \( \sum_{\ell=1}^{\ell_1-1} \xi^2_\ell(x) \) has an upper bound of the type of the statement if \( \ell_1 \) satisfies

\[
\max \left\{ \frac{\ell_1^{-1/3}}{(f_-(\ell_2) - f_+(\ell_1))^3}, \frac{\ell_1^{-5/6}}{(f_-(\ell_2) - f_+(\ell_1))^2}, \frac{\ell_1^{-4/3}}{f_-(\ell_2) - f_+(\ell_1)} \right\} \leq \ell_2^{1/6}.
\]

(47)

On the other hand, according to Theorem (ii),(iii),

\[
\sum_{\ell=1}^{\ell_2} \xi^2_\ell(x) \leq C \sum_{\ell=1}^{\ell_2} \ell_1^{-1/6} \leq C \int_{\ell_1}^{\ell_2} y^{-1/6} dy = \frac{6C}{5} (\ell_2^{5/6} - \ell_1^{5/6}),
\]

where \( C = \max\{C', C''\} \). Then \( \sum_{\ell=1}^{\ell_2} \xi^2_\ell(x) \) has an upper bound of the type of the statement if

\[
\ell_2^{5/6} - \ell_1^{5/6} \leq G_2 \ell_2^{1/6}
\]

for some \( G_2 > 0 \), independent of \( k \) and \( x \), which is equivalent to

\[
\ell_1 \geq \ell_2 \left(1 - G_2 \ell_2^{-2/3}\right)^{6/5}.
\]

(48)

Thus we must check the compatibility of (47) with (48) for some \( \ell_1 \) and \( G_2 \). By (48) and since, for each \( G_2, \delta > 0 \), we have \( G_2 \ell_2^{-2/3} \leq \ell_2^{-\frac{4}{3} + \delta} \) for \( k \) large enough, we can replace (47) with

\[
\max \left\{ \frac{\ell_2^{-1/3}(1 - \ell_2^{-\frac{4}{3} + \delta})^{-2/5}}{(f_-(\ell_2) - f_+(\ell_1))^3}, \frac{\ell_2^{-5/6}(1 - \ell_2^{-\frac{4}{3} + \delta})^{-1}}{(f_-(\ell_2) - f_+(\ell_1))^2}, \frac{\ell_2^{-4/3}(1 - \ell_2^{-\frac{4}{3} + \delta})^{-8/5}}{f_-(\ell_2) - f_+(\ell_1)} \right\} \leq \ell_2^{1/6}
\]

for some \( \delta > 0 \), which is equivalent to

\[
\ell_1 \leq \frac{1}{2} \left( \sqrt{2(\ell_2 + \sigma)} - \ell_2 \left(1 - \ell_2^{-\frac{4}{3} + \delta}\right)^{b} \right)^2 - 1 - \sigma
\]

for

\[
(a, b) \in \{(-1/6, -2/15), (-1/2, -1/2), (-3/2, -8/5)\}.
\]

Thus the compatibility of (47) with (48) holds if there is some \( G_2, \delta > 0 \) such that

\[
\ell_2 \left(1 - G_2 \ell_2^{-2/3}\right)^{6/5} \leq \frac{1}{2} \left( \sqrt{2(\ell_2 + \sigma)} - \ell_2 \left(1 - \ell_2^{-\frac{4}{3} + \delta}\right)^{b} \right)^2 - 2 - \sigma,
\]

which is equivalent to

\[
G_2 \geq \ell_2^{2/3} \left(1 - \left(\frac{1}{2} \left( \sqrt{2(1 + \sigma \ell_2^{-1})} - \ell_2^{-\frac{4}{3} \delta} \left(1 - \ell_2^{-\frac{4}{3} + \delta}\right)^{b} \right)^2 - (2 + \sigma) \ell_2^{-1}\right)^{5/6}\right).
\]
There is some $G_2 > 0$ satisfying this condition because the l'Hôpital rule shows that, for $\delta$ small enough, each function
\[ t^{2/3} \left( 1 - \frac{1}{2} \left( \sqrt{2(1 + \sigma t^{-1})} - t^\alpha (1 - t^{-\frac{\sigma}{2} + \delta})^\frac{1}{6} - (2 + \sigma) t^{-1} \right) \right)^{5/6} \]
is convergent in $\mathbb{R}$ as $t \to \infty$.

Now, if $\ell_2 < k - 1$, let $\ell_3$ denote the minimum integer $\ell < k$ such that $b_{\ell'} > x$ for all $\ell' \geq \ell$. Also, let $\bar{\sigma}_{\min/\max}$ denote the minimum/maximum values of $\bar{\sigma}_\ell$ for $\ell \in \mathbb{N}$. Then
\[
\sqrt{\frac{2(\ell_3 - 1) + 1 + 2\sigma + \sqrt{(2(\ell_3 - 1) + 1 + 2\sigma)^2 + 4\bar{\sigma}_{\min}}}{2s}} \leq x
\]
\[
< \sqrt{\frac{2(\ell_2 + 1) + 1 + 2\sigma + \sqrt{(2(\ell_2 + 1) + 1 + 2\sigma)^2 + 4\bar{\sigma}_{\max}}}{2s}} ,
\]
resulting in
\[ 2(\ell_3 - \ell_2) - 4
\]
\[ < \sqrt{2(\ell_2 + 1) + 1 + 2\sigma)^2 + 4\bar{\sigma}_{\max} - \sqrt{2(\ell_3 - 1) + 1 + 2\sigma)^2 + 4\bar{\sigma}_{\min}} .
\]
If $\ell_3 > \ell_2 + 1$, it follows that
\[ (2(\ell_2 + 1) + 1 + 2\sigma)^2 + 4\bar{\sigma}_{\max} > (2(\ell_3 - 1) + 1 + 2\sigma)^2 + 4\bar{\sigma}_{\min} ,
\]
giving
\[ 2\sqrt{\bar{\sigma}_{\max} - \bar{\sigma}_{\min}} > \sqrt{2(\ell_2 + 1) + 1 + 2\sigma)^2 - (2(\ell_3 - 1) + 1 + 2\sigma)^2}
\[
\geq 2(\ell_3 - \ell_2) - 4 .
\]
Therefore $\sum_{\ell=\ell_3+1}^{\ell_0} \xi(x)$ has an upper bound of the type of the statement by Theorem 1.1(ii),(iii).

Let
\[ h(t) = (2t + 1 + 2\sigma)s - s^2x^2 - \bar{\sigma}_{\max}x^{-2} \]
for $t \geq 0$. According to Theorem 1.1(i), if $\ell_3 < k - 1$, then
\[
\sum_{\ell=\ell_3+1}^{k-1} \xi^2(x) \leq C \sum_{\ell=\ell_3+1}^{k-1} \frac{1}{\sqrt{q_\ell(x)}} \leq C \sum_{\ell=\ell_3+1}^{k-1} \frac{1}{\sqrt{h(\ell)}} \leq C \int_{\ell_3}^{k-1} \frac{dt}{\sqrt{h(t)}}
\[
= C \frac{1}{2s} \left( \sqrt{h(k-1)} - \sqrt{h(\ell_3)} \right) \leq C \frac{1}{2s} \sqrt{2(k-1 - \ell_3)} .
\]
Hence $\sum_{\ell=\ell_3+1}^{k-1} \xi^2(x)$ also has an upper bound like in the statement because, by [38], [37] and [40], there is some $G_3, G_4 > 0$ such that
\[ G_3(k-1 - \ell_3)k^{-1/2} \leq b_{k-1} - b_{\ell_3} \leq b_{k-1} - x \leq G_4k^{-1/6} . \]

Proof of Theorem 1.3 By [38],
\[
1 = \int_{-\infty}^{\infty} \left( \frac{p_k(x)}{x - x_{k,1}} \right)^2 \frac{|x|^{2\sigma}e^{-sx^2}}{p_k^2(x_{k,1}) \lambda_{k,1}} dx .
\]
Thus, by (27) and Lemma 3.7

\[
\int_{|x-x_k,1| \leq \varepsilon k^{-1/6}} \left( \frac{p_k(x)}{x-x_k,1} \right)^2 \frac{|x|^{2\sigma} e^{-sx^2}}{p_k^2(x_k,1) \lambda_{k,1}} \, dx
\]

\[
\leq \int_{|x-x_k,1| \leq \varepsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \varepsilon^2 \xi_{\ell}^2(x) \, dx \leq 2\varepsilon k^{-1/6} \max_{|x-x_k,1| \leq \varepsilon k^{-1/6}} \sum_{\ell=0}^{k-1} \xi_{\ell}^2(x) \leq 2\varepsilon G
\]

for any \( \varepsilon > 0 \). It follows that

\[
\int_{|x-x_k,1| \geq \varepsilon k^{-1/6}} \left( \frac{p_k(x)}{x-x_k,1} \right)^2 \frac{|x|^{2\sigma} e^{-sx^2}}{p_k^2(x_k,1) \lambda_{k,1}} \, dx \geq \frac{1}{2}
\]

(49)

when \( \varepsilon \leq \frac{1}{4G} \), which implies part (i) by Lemma 2.1

When \( k \) is even and \( \sigma < 0 \), either \( 0 < x_{k,k/2} < a_k \) or \( |x_{k,k/2} - a_k| \leq C_1^\sigma k^{-1/6} \) for \( k \) large enough according to Corollary 3.3. Moreover \( |x_{k,1} - b_k| \leq C_1^\sigma k^{-1/6} \) for \( k \) large enough by Corollary 3.4 as well. So, by (30) and (55), there are some \( C_0, C_1 > 0 \), independent of \( k \), such that

\[
x_{k,k/2} \leq a_k + C_1^\sigma k^{-1/6} \leq C_0 k^{-1/2},
\]

\[
x_{k,1} - x_{k,k/2} \geq b_k - a_k - 2C_1^\sigma k^{-1/6} = \sqrt{\frac{C_{\text{max}}}{s}} - 2C_1^\sigma k^{-1/6} \geq C_1 k^{1/2}
\]

On the other hand, by (27), there is some \( C_2 > 0 \), independent of \( k \), such that \( \xi_k^2(x) \leq C_2 |x|^{2\sigma} \) for \( |x| \leq x_{k,k/2} \). Therefore

\[
\int_{|x| \leq x_{k,k/2}} \frac{\xi_k^2(x)}{(x-x_{k,1})^2} \, dx \leq \frac{C_2}{(x_{k,1} - x_{k,k/2})^2} \int_{|x| \leq x_{k,k/2}} |x|^{2\sigma} \, dx
\]

\[
= \frac{2C_2 x_{k,k/2}^{2\sigma+1}}{(2\sigma+1)(x_{k,1} - x_{k,k/2})^2} \leq \frac{2C_2 C_0^{2\sigma+1}}{(2\sigma+1)C_1^{2\sigma+1}} \lambda_{k,1}^{-\sigma-1} \leq \frac{2C_2 C_0^{2\sigma+1}}{(2\sigma+1)C_1^{2\sigma+1}} \lambda_{k,1}^{-\sigma-1}
\]

This inequality and (49) imply part (ii). \( \Box \)

4. Perturbed Schwartz space

We introduce a perturbed version \( \mathcal{S}_\sigma \) of \( \mathcal{S} \). It will be shown that \( \mathcal{S}_\sigma = \mathcal{S} \) after all, but the relevance of this new definition to study \( L \) will become clear in the next section; in particular, the norms used to define \( \mathcal{S}_\sigma \) will be appropriate to show embedding results, like a version of the Sobolev embedding theorem. Since \( \mathcal{S}_\sigma \) must contain the functions \( \phi_k \), Theorems 1.1 and 1.2 indicate that different definitions must be given for \( \sigma \geq 0 \) and \( \sigma < 0 \).

When \( \sigma \geq 0 \), for any \( \phi \in C^\infty \) and \( m \in \mathbb{N} \), let

\[
\|\phi\|_{\mathcal{S}_\sigma^m} = \sum_{i+j \leq m} \sup_{x \in [0,1]^i} |x^{i}T_j^m \phi(x)|.
\]

This defines a norm \( \| \cdot \|_{\mathcal{S}_\sigma^m} \) on the linear space of functions \( \phi \in C^\infty \) with \( \|\phi\|_{\mathcal{S}_\sigma^m} < \infty \), and let \( \mathcal{S}_\sigma^m \) denote the corresponding Banach space completion. There is a canonical inclusion \( \mathcal{S}_\sigma^m \subset \mathcal{S}_\sigma^m \), and the perturbed Schwartz space is defined as \( \mathcal{S}_\sigma = \bigcap_m \mathcal{S}_\sigma^m \), endowed with the corresponding Fréchet topology. In particular, \( \mathcal{S}_0 \) is the usual Schwartz space \( \mathcal{S} \). Like in the case of \( \mathcal{S} \), there are direct sum decompositions into subspaces of even and odd functions, \( \mathcal{S}_\sigma^m = \mathcal{S}_{\sigma,\text{ev}}^m \oplus \mathcal{S}_{\sigma,\text{odd}}^m \) for each \( m \in \mathbb{N} \), and \( \mathcal{S}_\sigma = \mathcal{S}_{\sigma,\text{ev}} \oplus \mathcal{S}_{\sigma,\text{odd}} \).
When \( \sigma < 0 \), the spaces of even and odd functions are considered separately. Let
\[
\| \phi \|_{S^m_{\sigma}} = \sum_{i+j \leq m, \ i+j \text{ even}} \sup_x |x^i(T^j_\sigma \phi)(x)| + \sum_{i+j \leq m, \ i+j \text{ odd} \neq 0} \sup_x |x^i(T^j_\sigma \phi)(x)|
\] (51)
for \( \phi \in C^\infty \), and let
\[
\| \phi \|_{S^m_{\sigma}} = \sum_{i+j \leq m, \ i+j \text{ even}} \sup_{x \neq 0} |x^i(T^j_\sigma \phi)(x)| + \sum_{i+j \leq m, \ i+j \text{ odd} \neq 0} \sup_x |x^i(T^j_\sigma \phi)(x)|
\] (52)
for \( \phi \in C^\infty_{\sigma} \). These expressions define a norm \( \| \cdot \|_{S^m_{\sigma}} \) on the linear spaces of functions \( \phi \) in \( C^\infty_{\sigma} \) and \( C^\infty_{\sigma} \) with \( \| \phi \|_{S^m_{\sigma}} < \infty \). The corresponding Banach space completions will be denoted by \( S^m_{\sigma, \text{odd}} \) and \( S^m_{\sigma, \text{ev}} \). Let \( S^m_{\sigma} = S^m_{\sigma, \text{ev}} \oplus S^m_{\sigma, \text{odd}} \), which is also a Banach space by considering e.g. the norm, also denoted by \( \| \cdot \|_{S^m_{\sigma}} \), defined by the maximum of the norms on both components. There are canonical inclusions \( S^m_{\sigma+1} \subset S^m_{\sigma} \), and let \( S_{\sigma} = \bigcap_m S^m_{\sigma} \), endowed with the corresponding Fréchet topology. We have \( S_{\sigma} = S_{\sigma, \text{ev}} \oplus S_{\sigma, \text{odd}} \), and \( S_{\sigma, \text{odd}} \) is dense in \( S_{\sigma} \) for all \( m \); thus \( S_{\sigma} \cap C^\infty \) is dense in \( S_{\sigma} \).

Obviously, \( \Sigma \) defines a bounded operator on each \( S^m_{\sigma} \). It is also easy to see that \( T^j_\sigma \) defines a bounded operator \( S^{m+1}_{\sigma} \to S^m_{\sigma} \) for any \( m \); notice that, when \( \sigma < 0 \), the role played by the parity of \( i+j \) fits well to prove this property. Similarly, \( x \) defines a bounded operator \( S^{m+1}_{\sigma} \to S^m_{\sigma} \) for any \( m \) because
\[
[T^j_\sigma, x] = \begin{cases} jT^{j-1}_\sigma & \text{if } j \text{ is even} \\ (j + 2\Sigma)T^{j-1}_\sigma & \text{if } j \text{ is odd} \end{cases}
\]
by (1) and (2). So \( B \) and \( B' \) define bounded operators \( S^{m+1}_{\sigma} \to S^m_{\sigma} \) too, and \( L \) defines a bounded operator \( S^{m+2}_{\sigma} \to S^m_{\sigma} \). Therefore \( T^j_\sigma, x, \Sigma, B, B' \) and \( L \) define continuous operators on \( S_{\sigma} \).

In order to prove Theorems 1.4 and 1.5 we introduce an intermediate weakly perturbed Schwartz space \( S_{w, \sigma} \). Like \( S_{\sigma} \), it is defined as a Fréchet space of the form \( S_{w, \sigma} = \bigcap_m S^m_{w, \sigma} \), where each \( S^m_{w, \sigma} \) is the Banach space defined like \( S^m_{\sigma} \) by using \( \frac{d}{dx} \) instead of \( T_\sigma \) in the right hand sides of (20)–(22); in particular, \( S^0_{w, \sigma} = S^0_{\sigma} \) as Banach spaces. The notation \( \| \cdot \|_{S^m_{w, \sigma}} \) will be used for the norm of \( S^m_{w, \sigma} \). As before, \( S_{w, \sigma} \) consists of functions which are \( C^\infty \) on \( \mathbb{R} \setminus \{0\} \) but a priori possibly not even defined at zero, \( S_{w, \sigma} \cap C^\infty \) is dense in \( S_{w, \sigma} \), and there is a canonical decomposition \( S_{w, \sigma} = S_{w, \sigma, \text{ev}} \oplus S_{w, \sigma, \text{odd}} \) given by the subspaces of even and odd functions, and \( \frac{d}{dx} \) and \( x \) define bounded operators on \( S^{m+1}_{w, \sigma} \to S^m_{w, \sigma} \). Thus \( \frac{d}{dx} \) and \( x \) define continuous operators on \( S_{w, \sigma} \).

**Lemma 4.1.** If \( \sigma \geq 0 \), then \( S^{m+\lceil \sigma \rceil}_{w, \sigma} \subset S^m_{w, \sigma} \) continuously for all \( m \).

**Proof.** Let \( \phi \in S \). For all \( i \) and \( j \), we have
\[
|x|^\sigma |x^i \phi(j)(x)| \leq |x^{i+\lceil \sigma \rceil} \phi(j)(x)|
\]
for $|x| \geq 1$, and
\[ |x|^\sigma x^i \phi^{(j)}(x) \leq |x^i \phi^{(j)}(x)| \]
for $|x| \leq 1$. So
\[ \|\phi\|_{\mathcal{S}_{w,\sigma}^n} \leq \|\phi\|_{S_{m+|\sigma|}} \]
for all $m$.

**Lemma 4.2.** If $\sigma \geq 0$, then $\mathcal{S}_{w,\sigma}^m \subset S^m$ continuously for all $m$, where
\[ m' = m + 1 + \frac{1}{2} \lfloor \sigma \rfloor (\lfloor \sigma \rfloor + 1) \]

**Proof.** Let $\phi \in \mathcal{S}_{w,\sigma}$. For all $i$ and $j$,
\[ |x^i \phi^{(j)}(x)| \leq |x|^\sigma |x^i \phi^{(j)}(x)| \quad (53) \]
for $|x| \geq 1$. It remains to prove an inequality of this type for $|x| \leq 1$, which is the only difficult part of the proof. It will be a consequence of the following assertion.

**Claim 2.** For each $n \in \mathbb{N}$, there are finite families of real numbers, $c^n_{a,b}$, $d^n_{k,\ell}$ and $e^n_{u,v}$, where the indices $a, b, k, \ell, u$ and $v$ run in finite subsets of $\mathbb{N}$ with $b, \ell, v \leq M_n = 1 + \frac{n(n+1)}{2}$ and $k \geq n$, such that
\[ \phi(x) = \sum_{a,b} c^n_{a,b} x^a \phi^{(b)}(1) + \sum_{k,\ell} d^n_{k,\ell} x^k \phi^{(\ell)}(x) + \sum_{u,v} e^n_{u,v} x^u \int_x^1 t^n \phi^{(v)}(t) \, dt \]
for all $\phi \in C^\infty$.

Assuming that Claim 2 is true, the proof can be completed as follows. Let $\phi \in \mathcal{S}_{w,\sigma}$ and set $n = \lfloor \sigma \rfloor$. For $|x| \leq 1$, according to Claim 2
\[ |\phi(x)| \leq \sum_{a,b} |c^n_{a,b}| |\phi^{(b)}(1)| + \sum_{k,\ell} |d^n_{k,\ell}| |x^k \phi^{(\ell)}(x)| \]
\[ + \sum_{u,v} |e^n_{u,v}| 2 \max_{|t| \leq 1} |t^n \phi^{(v)}(t)| \]
\[ \leq \sum_{a,b} |c^n_{a,b}| |\phi^{(b)}(1)| + \sum_{k,\ell} |d^n_{k,\ell}| |x|^\sigma |\phi^{(\ell)}(x)| \]
\[ + \sum_{u,v} |e^n_{u,v}| 2 \max_{|t| \leq 1} |t|^\sigma |\phi^{(v)}(t)| . \]
Let $m, i, j \in \mathbb{N}$ with $i + j \leq m$. By applying the above inequality to the function $x^i \phi^{(j)}$, and expressing each derivative $(x^i \phi^{(j)})^{(r)}$ as a linear combination of functions of the form $x^p \phi^{(q)}$ with $p + q \leq i + j + r$, it follows that there is some $C \geq 1$, depending only on $\sigma$ and $m$, such that
\[ |x^i \phi^{(j)}(x)| \leq C \|\phi\|_{\mathcal{S}_{w,\sigma}^{i+j+m}} \quad (54) \]
for $|x| \leq 1$. By (53) and (54),
\[ \|\phi\|_{S^m} \leq C \|\phi\|_{\mathcal{S}_{w,\sigma}^m} \]
with $m' = m + M_n$. 

Now, let us prove Claim 2. By induction on \( n \) and using integration by parts, it is easy to prove that

\[
\int_x^1 t^n \phi^{(n+1)}(t) dt = \sum_{r=0}^n (-1)^{n-r} \frac{n!}{r!} (\phi^{(r)}(1) - x^r \phi^{(r)}(x)).
\]  

(55)

This shows directly Claim 2 for \( n \in \{0, 1\} \). Proceeding by induction, let \( n > 1 \) and assume that Claim 2 holds for \( n - 1 \). By (55), it is enough to find appropriate expressions of \( x^r \phi^{(r)}(x) \) for \( 0 < r < n \). For that purpose, apply Claim 2 for \( n - 1 \) to each function \( \phi^{(r)} \), and multiply the resulting equality by \( x^r \) to get

\[
x^r \phi^{(r)}(x) = \sum_{a,b} e^{a,b} \int x^{r+a} \phi^{(r+b)}(1) + \sum_{k,\ell} \phi^{n-1} \phi^{k+\ell}(x)
\]

\[
+ \sum_{u,v} e^{n-1} x^{r+u} \int t^{n-1} \phi^{(r+v)}(t) dt,
\]

where \( a, b, k, \ell, u \) and \( v \) run in finite subsets of \( \mathbb{N} \) with \( b, \ell, v \leq M_{n-1} \) and \( k \geq n-1 \); thus \( r + k \geq n \) and

\[
r + b, r + \ell, r + v \leq n - 1 + M_{n-1} = M_n - 1.
\]

Therefore it only remains to rise the exponent of \( t \) by a unit in the integrals of the last sum. Once more, integration by parts makes the job:

\[
\int_x^1 t^n \phi^{(r+v+1)}(t) dt = \phi^{(r+v)}(1) - x^n \phi^{(r+v)}(x) - n \int_x^1 t^{n-1} \phi^{(r+v)}(t) dt. \quad \square
\]

**Lemma 4.3.** If \( \sigma < 0 \), then \( S_{w, \sigma}^{m+1} \subset S^m \) continuously for all \( m \).

**Proof.** Let \( i, j \in \mathbb{N} \) such that \( i + j \leq m \). Since

\[
|x^i \phi^{(j)}(x)| \leq \begin{cases} |x|^\sigma |x^i \phi^{(j)}(x)| & \text{if } 0 < |x| \leq 1 \\ |x|^\sigma |x^{i+1} \phi^{(j)}(x)| & \text{if } |x| \geq 1 \end{cases}
\]

for any \( \phi \in C^\infty \), we get \( \|\phi\|_{S^m} \leq \|\phi\|_{S_{w, \sigma}^{m+1}}. \quad \square
\]

**Lemma 4.4.** If \( \sigma < 0 \), then \( S_{w, \sigma}^{m+2} \subset S_{w, \sigma}^m \) continuously for all \( m \).

**Proof.** This is proved by induction on \( m \). We have \( \|\phi\|_{S_{w, \sigma}^m} = \|\phi\|_{S^0} \) on \( C^\infty_{ev} \). On the other hand, for \( \phi \in C^\infty_{od} \) and \( \psi = x^{-1} \phi \in C^\infty_{ev} \), we get

\[
|x|^\sigma |\psi(x)| \leq \begin{cases} |\psi(x)| & \text{if } 0 < |x| \leq 1 \\ |\phi(x)| & \text{if } |x| \geq 1 \end{cases}
\]

So, by (6)

\[
\|\phi\|_{S_{w, \sigma}^m} \leq \max\{\|\psi\|_{S^0}, \|\psi\|_{S^0}\} \leq \|\phi\|_{S^0}. \quad \square
\]

Now, assume that \( m > 0 \) and the result holds for \( m - 1 \). Let \( i, j \in \mathbb{N} \) such that \( i + j \leq m \), and let \( \phi \in S_{w, \sigma} \). If \( i = 0 \) and \( j \) is odd, then \( \phi^{(j)} \in S_{w, \sigma} \). Thus there is some \( \psi \in S_{w, \sigma} \) such that \( \phi^{(j)} = x^{j-1} \psi \), obtaining

\[
|x|^\sigma |\phi^{(j)}(x)| \leq \begin{cases} |\psi(x)| & \text{if } 0 < |x| \leq 1 \\ |\phi^{(j)}(x)| & \text{if } |x| \geq 1 \end{cases}
\]
Corollary 4.7. is bounded by Lemmas 4.1 and 4.2. □

Lemma 4.8. \( S_{w,\sigma} \) is a Fréchet space.

Corollary 4.9. \( S = S_{w,\sigma} \) as Fréchet spaces.

Corollary 4.10. \( x^{-1} \) defines a bounded operator \( S_{w,\sigma}^{m'} \to S_{w,\sigma,ce}^{m} \), where

\[
m' = \begin{cases} 
    m + 2 + \frac{1}{2} \lfloor \sigma \rfloor (\lfloor \sigma \rfloor + 3) & \text{if } \sigma \geq 0 \\
    m + 4 & \text{if } \sigma < 0 
\end{cases}
\]

Proof. If \( \sigma \geq 0 \), the composite

\[
S_{w,\sigma,od}^{m+2+\frac{1}{2} \lfloor \sigma \rfloor (\lfloor \sigma \rfloor + 3)} \to S_{od}^{m+\lfloor \sigma \rfloor + 1} \xrightarrow{x^{-1}} S_{ev}^{m+\lfloor \sigma \rfloor} \to S_{w,\sigma,ce}^{m}
\]

is bounded by Lemmas 4.1 and 4.2. If \( \sigma < 0 \), the composite

\[
S_{w,\sigma,od}^{m+4} \to S_{od}^{m+3} \xrightarrow{x^{-1}} S_{ev}^{m+2} \to S_{w,\sigma,ce}^{m}
\]

is bounded by Lemmas 4.1 and 4.2. □

Corollary 4.11. \( x^{-1} \) defines a continuous operator \( S_{w,\sigma} \to S_{w,\sigma,ce} \).

Lemma 4.12. \( S_{w,\sigma,ce}^{M_{m,ce}/odd} \subset S_{\sigma,ce/odd}^{m} \) continuously for all \( m \), where

\[
M_{m,ce/odd} = \begin{cases} 
    \frac{3 m + 2 m}{2} + \frac{m}{4} \lfloor \sigma \rfloor (\lfloor \sigma \rfloor + 3) & \text{if } \sigma \geq 0 \text{ and } m \text{ is even} \\
    \frac{3 m + 1}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is even} 
\end{cases}
\]

\[
M_{m,ce} = \begin{cases} 
    \frac{3 m - 1}{2} + \frac{m - 1}{4} \lfloor \sigma \rfloor (\lfloor \sigma \rfloor + 3) & \text{if } \sigma \geq 0 \text{ and } m \text{ is odd} \\
    \frac{3 m - 1}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} 
\end{cases}
\]

\[
M_{m,odd} = \begin{cases} 
    \frac{3 m + 1}{2} + \frac{m + 1}{4} \lfloor \sigma \rfloor (\lfloor \sigma \rfloor + 3) & \text{if } \sigma \geq 0 \text{ and } m \text{ is odd} \\
    \frac{3 m + 3}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} 
\end{cases}
\]
Proof. The result follows by induction on \( m \). The statement is true for \( m = 0 \) because \( S^0_{w,\sigma} = S^0_\sigma \) as Banach spaces. Now, take any \( m > 0 \), and assume that the result holds for \( m - 1 \).

For \( \phi \in C^\infty_\sigma \), \( i + j \leq m \) with \( j > 0 \) and \( x \in \mathbb{R} \), we have
\[
|x^i T^j_\sigma \phi(x)| = |x^i T^j_\sigma^{-1} \phi'(x)| ,
\]
obtaining
\[
\|\phi\|_{S^m_\sigma} \leq \|\phi'\|_{S^{m-1}_\sigma} + \|\phi\|_{S^0_{w,\sigma}} .
\]
But, by the induction hypothesis and since \( M_{m,\text{ev}} = M_{m-1,\text{odd}} + 1 \), there are some \( C, C' > 0 \), independent of \( \phi \), such that
\[
\|\phi'\|_{S^{m-1}_\sigma} \leq C \|\phi'\|_{S_{w,\sigma}^{m-1,\text{odd}}} \leq C' \|\phi\|_{S_{w,\sigma}^{M_{m,\text{ev}}}} .
\]
For \( \phi \in C^\infty_{\text{odd}} \), and \( i, j \) and \( x \) as above, we have
\[
|x^i T^j_\sigma \phi(x)| \leq |x^i T^j_\sigma^{-1} \phi'(x)| + 2 |\sigma| |x^i T^j_\sigma^{-1} x^{-1} \phi(x)| ,
\]
obtaining
\[
\|\phi\|_{S^m_\sigma} \leq \|\phi'\|_{S^{m-1}_\sigma} + 2 |\sigma| \|x^{-1}\phi\|_{S^{m-1}_\sigma} + \|\phi\|_{S^0_{w,\sigma}} .
\]
But, by the induction hypothesis, Corollary 4.6 and since
\[
M_{m,\text{odd}} = \begin{cases} M_{m-1,\text{ev}} + 2 + \frac{1}{2}[\sigma]([\sigma] + 3) & \text{if } \sigma \geq 0 \\ M_{m-1,\text{ev}} + 4 & \text{if } \sigma < 0 , \end{cases}
\]
there are some \( C, C > 0 \), independent of \( \phi \), such that
\[
\|\phi'\|_{S^{m-1}_\sigma} + 2 |\sigma| \|x^{-1}\phi\|_{S^{m-1}_\sigma} \leq C \left( \|\phi'\|_{S_{w,\sigma}^{m-1,\text{ev}}} + \|x^{-1}\phi\|_{S_{w,\sigma}^{M_{m-1,\text{ev}}}} \right) \leq C' \|\phi\|_{S_{w,\sigma}^{M_{m,\text{odd}}}} . \quad \Box
\]

Corollary 4.9. \( S_{w,\sigma} \subset S_{\sigma} \) continuously.

5. Perturbed Sobolev spaces

Observe that \( S_{\sigma} \subset L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \). Like in the case where \( \mathcal{S} \) is considered as domain, it is easy to check that, in \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \), with domain \( S_{\sigma} \), \( B \) is adjoint of \( B' \) and \( L \) is symmetric.

Lemma 5.1. \( S_{\sigma} \) is a core\(^4\) of \( L \).

Proof. Let \( R \) denote the restriction of \( L \) to \( S_{\sigma} \). Then \( \mathcal{L} \subset \mathcal{R} \subset \mathcal{R}^{*} \subset \mathcal{L}^{*} = \mathcal{L} \) in \( L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \) because \( \mathcal{S} \subset S_{\sigma} \) by Corollaries 4.5 and 4.9 \( \Box \)

For each \( m \in \mathbb{N} \), let \( W^m_{\sigma} \) be the Hilbert space completion of \( \mathcal{S} \) with respect to the scalar product \( \langle \cdot, \cdot \rangle_{W^m_{\sigma}} \) defined by
\[
\langle \phi, \psi \rangle_{W^m_{\sigma}} = \langle (1 + L)^m \phi, \psi \rangle_{\sigma} .
\]
The corresponding norm will be denoted by \( \| \cdot \|_{W^m_{\sigma}} \), whose equivalence class is independent of the parameter \( s \) used to define \( L \). In particular, \( W^0_{\sigma} = L^2(\mathbb{R}, |x|^{2\sigma} \, dx) \). As usual, \( W^m_{\sigma} \subset W^m_{\sigma'} \) when \( m' > m \), and let \( W^\infty_{\sigma} = \bigcap_m W^m_{\sigma} \), which is endowed with the induced Fréchet topology. Once more, there are direct sum decompositions into subspaces of even and odd (generalized) functions, \( W^m_{\sigma} = W^m_{\sigma,\text{ev}} \oplus W^m_{\sigma,\text{odd}} \) and \( W^\infty_{\sigma} = W^\infty_{\sigma,\text{ev}} \oplus W^\infty_{\sigma,\text{odd}} \).

\(^4\)Recall that a core of a closed densely defined operator \( T \) between Hilbert spaces is any subspace of its domain \( \mathcal{D}(T) \) which is dense with the graph norm.
According to Lemma 5.1 the space $W^m_\sigma$ can be defined for any real number $m$ by using $(1 + \mathcal{L})^m$, and moreover $S_\sigma$ can be used instead of $S$ in its definition.

Obviously, $L$ defines a bounded operator $W^m_{\sigma+2} \to W^m_{\sigma}$ for each $m \geq 0$, and therefore a continuous operator on $W^\infty_{\sigma}$. Moreover, by (13), $\Sigma$ defines a bounded operator on each $W^m_{\sigma}$, and therefore a continuous operators on $W^\infty_{\sigma}$.

**Lemma 5.2.** $B$ and $B'$ define bounded operators $W^{m+1}_{\sigma} \to W^m_{\sigma}$ for each $m$.

**Proof.** This follows by induction on $m$. For $m = 0$, by (10), for each $\phi \in S$,

$$
\|B\phi\|^2_\sigma = \|B'\phi\|^2_\sigma = \langle (L - (1 + 2\Sigma)s)\phi, \phi \rangle_\sigma \leq C_0 \|\phi\|^2_{W^1_\sigma}
$$

for some $C_0 > 0$ independent of $\phi$. It follows that $B$ and $B'$ define bounded operators $W^1_\sigma \to L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$.

Now take $m > 0$ and assume that there are some $C_{m-1}, C'_{m-1} > 0$ so that

$$
\|B\phi\|^2_{W^m_\sigma} \leq C_{m-1} \|\phi\|^2_{W^m_\sigma}, \quad \|B'\phi\|^2_{W^{m-1}_\sigma} \leq C'_{m-1} \|\phi\|^2_{W^m_\sigma}
$$

for all $\phi \in S$. Then, by (11),

$$
\|B\phi\|^2_{W^m_\sigma} = \langle (1 + L)B\phi, B\phi \rangle_{W^m_{\sigma-1}} = \|B\phi\|^2_{W^m_{\sigma-1}} + \langle LB\phi, B\phi \rangle_{W^m_{\sigma-1}}
$$

$$
= (1 - 2s) \|B\phi\|^2_{W^m_{\sigma-1}} + \langle BL\phi, B\phi \rangle_{W^m_{\sigma-1}} \leq C_{m-1} \|\phi\|^2_{W^m_{\sigma-1}} + \|BL\phi\|_{W^m_{\sigma-1}} \|B\phi\|_{W^m_{\sigma-1}}
$$

$$
\leq C_m \|\phi\|^2_{W^m_{\sigma+1}}
$$

for some $C_m > 0$ independent of $\phi$. Similarly

$$
\|B'\phi\|^2_{W^{m+1}_\sigma} \leq C'_{m} \|\phi\|^2_{W^m_{\sigma+1}}
$$

for some $C'_m > 0$ independent of $\phi$.

**Remark 1.** $B'$ is not adjoint of $B$ in $W^m_{\sigma}$ for $m > 0$.

$L$ and $\Sigma$ preserve $W^m_{\sigma, ev}$ and $W^m_{\sigma, odd}$ for each $m$, whilst $B$ and $B'$ interchange these subspaces.

The motivation of our tour through perturbed Schwartz spaces is the following embedding results; the second one is a version of the Sobolev embedding theorem.

**Proposition 5.3.** $S'^{m'}_\sigma \subset W^m_{\sigma}$ continuously if $m' - m > 1/2$.

**Proposition 5.4.** $W^m_{\sigma} \subset S^m_\sigma$ continuously if $m' - m > 1$.

**Corollary 5.5.** $S_\sigma = W^\infty_{\sigma}$ as Fréchet spaces.

For each non-commutative polynomial $p$ (of two variables, $X$ and $Y$), let $p'$ denote the non-commutative polynomial obtained by reversing the order of the variables in $p$; e.g., if $p(X,Y) = X^2 Y^3 X$, then $p'(X,Y) = XY^3 X^2$. It will be said that $p$ is symmetric if $p(X,Y) = p'(Y,X)$. Notice that any non-commutative polynomial of the form $p'(Y,X)p(X,Y)$ is symmetric. Given any non-commutative polynomial $p$, the continuous operators $p(B, B')$ and $p'(B', B)$ on $S_\sigma$ are adjoint from each other in $L^2(\mathbb{R}, |x|^{2\sigma} \, dx)$; thus $p(B, B')$ is a symmetric operator if $p$ is symmetric. The following lemma will be used in the proof of Proposition 5.3.
Lemma 5.6. For each non-negative integer \( m \), we have
\[
(1 + L)^m = \sum_a q'_a(B', B) q_a(B, B')
\]
for some finite family of homogeneous non-commutative polynomials \( q_a \) of degree \( \leq m \).

Proof. The result follows easily from the following assertions.

Claim 3. If \( m \) is even, then \( L^m = g_m(B, B')^2 \) for some symmetric homogeneous non-commutative polynomial \( g_m \) of degree \( m \).

Claim 4. If \( m \) is odd, then
\[
L^m = g'_{m,1}(B', B) g_{m,1}(B, B') + g'_{m,2}(B', B) g_{m,2}(B, B')
\]
for some homogeneous non-commutative polynomials \( g_{m,1} \) and \( g_{m,2} \) of degree \( m \).

If \( m \) is even, then \( L^{m/2} = g_m(B, B') \) for some symmetric homogeneous non-commutative polynomial \( g_m \) of degree \( \leq m \) by (14). So \( L^m = g_m(B, B')^2 \), showing Claim 3.

If \( m \) is odd, then write \( L^{m/2} = f_m(B, B') \) as above for some symmetric homogeneous non-commutative polynomial \( f_m \) of degree \( \leq m - 1 \). Then, by (14),
\[
L^m = \frac{1}{2} f_m(B, B')(BB' + B'B) f_m(B, B')
\]
Thus Claim 4 follows with
\[
g_{m,1}(B, B') = \frac{1}{\sqrt{2}} B' f_m(B, B') \quad \text{and} \quad g_{m,2}(B, B') = \frac{1}{\sqrt{2}} B f_m(B, B'). \quad \square
\]

Proof of Proposition 5.3 when \( \sigma \geq 0 \). By the definitions of \( B \) and \( B' \), for each non-commutative polynomial \( p \) of degree \( \leq m' \) (of three variables), there exists some \( C_p > 0 \) such that \( |x|^{\sigma} |p(x, B, B')\phi| \) is uniformly bounded by \( C_p \|\phi\|_{S_{m'}} \) for all \( \phi \in S_{\sigma} \). Write
\[
(1 + L)^m = \sum_a q'_a(B', B) q_a(B, B')
\]
according to Lemma 5.6, and let
\[
\bar{q}_a(x, B, B') = x^{m'-m} q_a(B, B')
\]
Then, for each \( \phi \in S_{\sigma} \),
\[
\|\phi\|_{W^m}^2 = \sum_a \| q_a(B, B') \phi \|_{S_{\sigma}}^2
\]
\[
= \sum_a \int_{-\infty}^{\infty} |(q_a(B, B')\phi)(x)|^2 |x|^{2\sigma} \, dx
\]
\[
\leq 2 \sum_a \left( C^2_{q_a} + C^2_{q_a} \int_{1}^{\infty} x^{-2(m'-m)} \, dx \right) \|\phi\|_{S_{m'}}^2,
\]
where the integral is finite because \( -2(m'-m) < -1 \). \( \square \)

Proof of Proposition 5.3 when \( \sigma < 0 \). Now, for each homogeneous non-commutative polynomial \( p \) of degree \( d \leq m' \), there is some \( C_p > 0 \) such that:

- \( |p(x, B, B')\phi| \) is uniformly bounded by \( C_p \|\phi\|_{S_{m'}} \) for all \( \phi \in S_{\sigma, ev} \) if \( d \) is even, and by \( C_p \|\phi\|_{S_{m'}} \) for all \( \phi \in S_{\sigma, odd} \) if \( d \) is odd; and
• \(|x|^\sigma |p(x, B, B')\phi|\) is uniformly bounded by \(C_p \|\phi\|_{S_{\sigma,\text{odd}}}^m\) for all \(\phi \in S_{\sigma,\text{odd}}\)

if \(d\) is even, and by \(C_p \|\phi\|_{S_{\sigma,\text{ev}}}^m\) for all \(\phi \in S_{\sigma,\text{ev}}\) if \(d\) is odd.

With the notation of Lemma 5.6, let \(d_q\) denote the degree of each homogenous non-commutative polynomial \(q_a\), and let \(q_a(x, B, B')\) be defined like in the previous

case. Then, as above,

\[
\|\phi\|_{W_{\sigma}^m}^2 \leq 2 \sum_{a \text{ even}} \left( C_{q_a}^2 \int_0^1 x^{2\sigma} dx + C_{q_a}^2 \int_1^\infty x^{-2(m'-m) + 2\sigma} dx \right) \|\phi\|_{S_{\sigma,\text{ev}}}^2
\]

\[
+ 2 \sum_{a \text{ odd}} \left( C_{q_a}^2 \int_0^1 x^{2\sigma} dx + C_{q_a}^2 \int_1^\infty x^{-2(m'-m) + 2\sigma} dx \right) \|\phi\|_{S_{\sigma,\text{odd}}}^2
\]

for \(\phi \in S_{\sigma,\text{ev}}\), and

\[
\|\phi\|_{W_{\sigma}^m}^2 \leq 2 \sum_{a \text{ even}} \left( C_{q_a}^2 + C_{q_a}^2 \int_1^\infty x^{-2(m'-m)} dx \right) \|\phi\|_{S_{\sigma,\text{ev}}}^2
\]

\[
+ 2 \sum_{a \text{ odd}} \left( C_{q_a}^2 + C_{q_a}^2 \int_1^\infty x^{-2(m'-m) + 2\sigma} dx \right) \|\phi\|_{S_{\sigma,\text{odd}}}^2
\]

for \(\phi \in S_{\sigma,\text{ev}}\), where the integrals are finite because \(-1/2 < \sigma < 0\) and \(-2(m'-m) < -1\). □

Let \(\mathcal{C}\) denote the space of rapidly decreasing sequences of real numbers. Recall that a sequence \(c = (c_k) \in \mathbb{R}^N\) is rapidly decreasing if

\[
\|c\|_{c_m} = \sup_k |c_k|(1 + k)^m
\]

is finite for all \(m \geq 0\). These expressions define norms \(\|c_m\|\) on \(\mathcal{C}\). Let \(\mathcal{C}_m\) denote the completion of \(\mathcal{C}\) with respect to \(\|\cdot\|_{c_m}\), which consists of the sequences \(c \in \mathbb{R}^N\) with \(\|c\|_{c_m} < \infty\). So \(\mathcal{C} = \bigcap_{m} \mathcal{C}_m\) with the induced Fréchet topology. Also, for each \(m \geq 0\), let \(\ell^2_m\) denote the Hilbert space completion of \(\mathcal{C}\) with respect to the scalar product \(\langle , \rangle_{\ell^2_m}\) defined by

\[
\langle c, c' \rangle_{\ell^2_m} = \sum_k c_k c'_k (1 + k)^m
\]

for \(c = (c_k)\) and \(c' = (c'_k)\). The corresponding norm will be denoted by \(\|\cdot\|_{\ell^2_m}\). Thus \(\ell^2_m\) is a weighted version of \(\ell^2\); in particular, \(\ell^2_0 = \ell^2\). Let \(\ell^2_\infty = \bigcap_{m} \ell^2_m\) with the corresponding Fréchet topology.

A sequence \(c = (c_k)\) will be called even/odd if \(c_k = 0\) for all odd/even \(k\). We get the following direct sum decompositions into subspaces of even and odd sequences:

\[
\mathcal{C}_m = \mathcal{C}_{m,\text{ev}} \oplus \mathcal{C}_{m,\text{odd}}, \quad \mathcal{C} = \mathcal{C}_{\text{ev}} \oplus \mathcal{C}_{\text{odd}},
\]

\[
\ell^2_m = \ell^2_{m,\text{ev}} \oplus \ell^2_{m,\text{odd}}, \quad \ell^2_\infty = \ell^2_{\infty,\text{ev}} \oplus \ell^2_{\infty,\text{odd}}.
\]

Lemma 5.7. \(\ell^2_{2m} \subset \mathcal{C}_m\) and \(\mathcal{C}_{m'} \subset \ell^2_m\) continuously for all \(m\) if \(2m' - m > 1\).

Proof. It is easy to see that

\[
\|c\|_{c_m} \leq \|c\|_{\ell^2_m}, \quad \|c\|_{\ell^2_m} \leq \|c\|_{c_{m'\text{ev}}} \left( \sum_k (1 + k)^{m-2m'} \right)^{1/2}
\]

for any \(c \in \mathcal{C}\), where the last series is convergent because \(m - 2m' < -1\). □
Corollary 5.8. \( \ell_2^\infty = C \) as Fréchet spaces.

According to Section 2.2 the “Fourier coefficients” mapping \( \phi \mapsto (\langle \phi_k, \phi \rangle_\sigma) \) defines a quasi-isometry \( W^{m'}_\sigma \to \ell^2_m \) for all \( m \), and therefore an isomorphism \( W^{m'}_\sigma \to C \) of Fréchet spaces. Notice that the “Fourier coefficients” mapping can be restricted to the even and odd subspaces.

Corollary 5.9. Any \( \phi \in L^2(\mathbb{R}, |x|^{2s} \, dx) \) is in \( S_\sigma \) if and only if its “Fourier coefficients” \( \langle \phi_k, \phi \rangle_\sigma \) are rapidly decreasing on \( k \).

Proof. By Corollary 5.5 the “Fourier coefficients” mapping defines an isomorphism \( S_\sigma \to C \) of Fréchet spaces.

There is also a version of the Rellich theorem stated as follows.

Proposition 5.10. The operator \( W^{m'}_\sigma \to W^m_\sigma \) is compact for \( m' > m \).

By using the “Fourier coefficients” mapping, Proposition 5.10 follows from the following lemma (see e.g. [32, Theorem 5.8]).

Lemma 5.11. The operator \( \ell^2_{m'} \to \ell^2_m \) is compact for \( m' > m \).

Proof of Proposition 5.10. For \( \phi \in S_\sigma \), its “Fourier coefficients” \( c_k = \langle \phi_k, \phi \rangle_\sigma \) form a sequence \( c = (c_k) \) in \( C \), and

\[
\sum_k |c_k|(1 + k)^{m/2} \leq \|c\|_{m'} \left( \sum_k (1 + k)^{m-m'} \right)^{1/2}
\]

by Cauchy-Schwartz inequality, where the last series is convergent since \( m - m' < -1 \). Therefore

\[
\sum_k |c_k|(1 + k)^{m/2} \leq C \|\phi\|_{W^{m'}}
\]

for some \( C > 0 \) independent of \( \phi \).

On the other hand, for all \( i, j \in \mathbb{N} \) with \( i + j \leq m \), there is some homogeneous noncommutative polynomial \( p_{i,j} \) of degree \( i + j \) such that \( x^i T^j_\sigma = p_{i,j}(B, B') \). Then, by (15)–(17),

\[
|\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| \leq C_{i,j}(1 + k)^{m/2} \sum_{|\ell-k| \leq m} |c_\ell|
\]

for some \( C_{i,j} > 0 \) independent of \( \phi \).

Now suppose that \( \sigma \geq 0 \). By (56), (57) and Theorem 1.3(ii), there is some \( C_{i,j} > 0 \) independent of \( \phi \) and \( x \) so that

\[
|x|^s |x^i T^j_\sigma \phi(x)| \leq |x|^s \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| |\phi_k(x)|
\]

\[
= \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| |\phi_k(x)| \leq C'_{i,j} \|\phi\|_{W^{m'}}
\]

for all \( x \). Hence \( \|\phi\|_{S_\sigma^m} \leq C'\|\phi\|_{W^{m'}} \), for some \( C' > 0 \) independent of \( \phi \).

Finally assume that \( \sigma < 0 \). By (56), (57) and Theorem 1.3 there is some \( C'_{i,j} > 0 \), independent of \( \phi \) and \( x \), so that

\[
|x^i T^j_\sigma \phi(x)| \leq \sum_k |\langle \phi_k, x^i T^j_\sigma \phi \rangle_\sigma| |\phi_k(x)| \leq C'_{i,j} \|\phi\|_{W^{m'}}
\]
Lemma 5.16. By (26), this composite is an extension of the map \( x \), and Corollary 5.15. it without using (26) and the perturbed Sobolev spaces? Question 5.14. The proof of Corollary 5.13 is very indirect. Is it possible to prove it without using (26) and the perturbed Sobolev spaces?

Proof. By the Cauchy-Schwartz inequality,

\[
\begin{align*}
\|d\|_{\mathcal{C}_m} & = \sup_{\ell} \sum_{k \in \{\ell+1,\ell+3,\ldots\}} \sqrt{(k-1)(k-3)\cdots(\ell+2)2s} |c_k| (1 + \ell)^m \\
& \leq \sqrt{2s} \sup_{\ell} \sum_{k \in \{\ell+1,\ell+3,\ldots\}} |c_k| (1 + \ell)^m \\
& \leq \sqrt{2s} \|c\|_{\ell^2_{m'}}^2 \sup_{\ell} \left( \sum_{k \in \{\ell+1,\ell+3,\ldots\}} (1 + k)^{-m'-(1 + \ell)^m} \right)^{1/2} \\
& \leq \sqrt{2s} \|c\|_{\ell^2_{m'}} \left( \sum_{k} (1 + k)^{m-m'} \right)^{1/2},
\end{align*}
\]

where the last series is convergent since \( m - m' < -1 \). \( \square \)

Corollary 5.13. \( x^{-1} \) defines a bounded operator \( \mathcal{S}_{m'}^{\mathcal{m}}_{\sigma,\text{odd}} \to \mathcal{S}_{m,\text{ev}}^{\mathcal{m}} \) if \( m' - m > 1 \).

Proof. Since \( 2m' > m + 5 \), there are \( m_1, m_2, m_3 \geq 0 \) such that

\[
m' - m_3 > 1/2, \quad m_3 - m_2 > 1, \quad 2m_2 - m_1 > 1, \quad m_1 - m > 1.
\]

Then, by Propositions 5.3 and 5.5, Lemmas 5.7 and 5.12 and using the “Fourier coefficients” mapping, we get the following composition of bounded maps:

\[
\mathcal{S}_{m,\text{odd}}^{\mathcal{m}} \to \mathcal{W}_{\sigma,\text{odd}}^{m_3} \to \ell^2_{m_3,\text{odd}} \overset{\Xi}{\to} \mathcal{C}_{m,\text{ev}} \to \ell^2_{m_1,\text{ev}} \to \mathcal{W}_{\sigma,\text{ev}}^{m_2} \to \mathcal{S}_{m,\text{ev}}^{m}.
\]

By (26), this composite is an extension of the map \( x^{-1} : \mathcal{S}_{\text{odd}} \to \mathcal{S}_{\text{ev}} \). \( \square \)

Question 5.14. The proof of Corollary 5.13 is very indirect. Is it possible to prove it without using (26) and the perturbed Sobolev spaces?

Corollary 5.15. \( x^{-1} \) defines a continuous operator \( \mathcal{S}_{\sigma,\text{odd}} \to \mathcal{S}_{\sigma,\text{ev}}^{m} \).

Lemma 5.16. \( \mathcal{S}_{\sigma,\text{ev}} \subset \mathcal{S}_{w,\sigma,\text{ev}}^{m} \) and \( \mathcal{S}_{\sigma}^{m+2} \subset \mathcal{S}_{w,\sigma}^{m} \) continuously for \( m \geq 1 \).
Corollary 5.18. Let us construct a sequence of naturals $M_{m, ev/odd}$ such that $S_{\sigma, ev/odd}^{M_{m, ev/odd}} \subset S_{w, \sigma, ev/odd}^m$ continuously for all $m$. Like in the proof of Lemma 4.8, we proceed by induction on $m$, with $M_{0, ev/odd} = 0$. For $m > 0$, assume that the terms $M_{m-1, ev/odd}$ are constructed.

For $\phi \in C_{\infty}^w$, $i + j \leq m$ with $j > 0$ and $x \in \mathbb{R}$, we have

$$|x^i \phi^{(j)}(x)| = |x^i (T_0 \phi)^{(j-1)}(x)|,$$

obtaining

$$\|\phi\| S_{w, \sigma}^m \leq \|T_0 \phi\| S_{w, \sigma}^{m-1} + \|\phi\| S_{w, \sigma}^m.$$

But there are some $C, C' > 0$, independent of $\phi$, such that

$$\|T_0 \phi\| S_{w, \sigma}^{m-1} \leq C \|T_0 \phi\| S_{\sigma}^{M_{m-1, odd}} \leq C' \|\phi\| S_{w, \sigma}^{M_{m, ev}}$$

with

$$M_{m, ev} = M_{m-1, odd} + 1. \quad (59)$$

For $\phi \in C_{\infty}^w$, and $i, j$ and $x$ as above, we have

$$|x^i \phi^{(j)}(x)| \leq |x^i (T_0 \phi)^{(j-1)}(x)| + 2 \sigma |x^i (x^{-1} \phi)^{(j-1)}(x)|,$$

obtaining

$$\|\phi\| S_{w, \sigma}^m \leq \|T_0 \phi\| S_{w, \sigma}^{m-1} + 2 |\sigma| \|x^{-1} \phi\| S_{w, \sigma}^{m-1} + \|\phi\| S_{w, \sigma}^m.$$ But, by Corollary 5.13, there are some $C, C' > 0$, independent of $\phi$, such that

$$\|T_0 \phi\| S_{w, \sigma}^{m-1} + 2 |\sigma| \|x^{-1} \phi\| S_{w, \sigma}^{m-1} \leq C \left( \|\phi\| S_{\sigma}^{M_{m-1, ev}} + \|x^{-1} \phi\| S_{\sigma}^{M_{m-1, ev}} \right)$$

$$\leq C' \|\phi\| S_{\sigma}^{M_{m, odd}}$$

if

$$M_{m, odd} \geq M_{m-1, ev} + 1, \quad 2M_{m, odd} > M_{m-1, ev} + 5. \quad (60)$$

The conditions (59) and (60) are satisfied with $M_{1, ev} = 1$, $M_{1, odd} = 3$ and $M_{m, ev/odd} = m + 2$ for $m \geq 2$. 

\[ \square \]

Corollary 5.17. $S_{\sigma, ev/odd}^{M_{m, ev/odd}} \subset S_{\sigma, ev/odd}^m$ continuously for all $m$, where

$$M_{m, ev/odd} = \begin{cases} \frac{3m}{2} + \frac{m}{2} \lfloor \sigma \rfloor \lfloor \sigma \rfloor + 3 + \lfloor \sigma \rfloor & \text{if } \sigma \geq 0 \text{ and } m \text{ is even} \\ \frac{3m}{2} + 2 & \text{if } \sigma < 0 \text{ and } m \text{ is even} \end{cases}$$

$$M_{m, ev} = \begin{cases} \frac{3m-1}{2} + \frac{m-1}{4} \lfloor \sigma \rfloor \lfloor \sigma \rfloor + 3 + \lfloor \sigma \rfloor & \text{if } \sigma \geq 0 \text{ and } m \text{ is odd} \\ \frac{3m+1}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} \end{cases}$$

$$M_{m, odd} = \begin{cases} \frac{3m+1}{2} + \frac{m+1}{4} \lfloor \sigma \rfloor \lfloor \sigma \rfloor + 3 + \lfloor \sigma \rfloor & \text{if } \sigma \geq 0 \text{ and } m \text{ is odd} \\ \frac{3m+1}{2} & \text{if } \sigma < 0 \text{ and } m \text{ is odd} \end{cases}$$

\[ \square \]

Corollary 5.18. $S_{\sigma, ev}^{M'} \subset S_{\sigma, ev}^m$ continuously for all $m$, where

$$m' = m + 3 + \frac{\lfloor \sigma \rfloor \lfloor \sigma \rfloor + 1}{2}.$$

Moreover $S_{\sigma, ev}^1 \subset S_{\sigma, ev}^0$ continuously.

\[ \square \]

Proof. This follows from Lemmas 4.1, 4.3 and 4.8.
Corollary 5.19. $S_\sigma = S$ as Fréchet spaces.

Proof. This is a consequence of Corollaries 5.17 and 5.18. □

Now, Theorems 1.4 and 1.5 follow from Corollaries 5.17 and 5.18 and Propositions 5.3 and 5.4.

6. Perturbation of $H$ on $\mathbb{R}_+$

More general perturbations of $H$ can be obtained with conjugation of $L$ by the operator of multiplication by functions which are defined and positive almost everywhere (with respect to the Lebesgue measure), like we did in Section 5.1 with the function $|x|^\sigma$. We will only consider conjugations of the even and odd components of $L$ separately, and acting on spaces of functions on $\mathbb{R}_+$. This will be also enough for the application indicated in Section 1.

Let $L_{ev/odd}$, or $L_{\sigma, ev/odd}$, denote the restriction of $L$ to $S_{ev/odd}$. Since the function $|x|^{2\sigma}$ is even, there is an orthogonal decomposition $L^2(\mathbb{R}, |x|^{2\sigma} \, dx) = L^2_0(\mathbb{R}, |x|^{2\sigma} \, dx) \oplus L^2_{odd}(\mathbb{R}, |x|^{2\sigma} \, dx)$ as direct sum of subspaces of even and odd functions. Then $L_{ev/odd}$ is essentially self-adjoint in $L^2_{ev/odd}(\mathbb{R}, |x|^{2\sigma} \, dx)$, and its self-adjoint extension $L_{ev/odd}$, or $L_{\sigma, ev/odd}$, is obtained by restriction of $L$. We also get an obvious version of Corollary 1.6 for $L_{ev/odd}$.

Fix open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure. Let $S_{ev/odd, U}$ denote the linear subspace of $C^\infty(\mathbb{R}_+)$ consisting of the restrictions to $U$ of the functions in $S_{ev/odd}$. The restriction to $U$ defines a linear isomorphism

$$S_{ev/odd} \cong S_{ev/odd, U},$$

and a unitary isomorphism

$$L^2_{ev/odd}(\mathbb{R}, |x|^{2\sigma} \, dx) \cong L^2(\mathbb{R}_+, 2x^{2\sigma} \, dx).$$

Let $L_{ev/odd, U}$, or $L_{\sigma, ev/odd, U}$, denote the operator defined by $L_{ev/odd}$ on $S_{ev/odd, U}$ via (61). Let also $\phi_k|U = \phi_k|U$, whose norm in $L^2(\mathbb{R}_+, x^{2\sigma} \, dx)$ is $1/\sqrt{2}$ since (62) is unitary. When $U = \mathbb{R}_+$, the notation $S_{ev/odd, +}$, $L_{ev/odd, +}$, or $L_{\sigma, ev/odd, +}$, and $\phi_k, +$ will be used. Moreover let $L_{ev/odd, +}$, or $L_{\sigma, ev/odd, +}$, be the self-adjoint operator in $L^2(\mathbb{R}_+, x^{2\sigma} \, dx)$ that corresponds to $L_{ev/odd}$ via (62).

Going one step further, for any positive function $h \in C^2(U)$, the operator (of multiplication by) $h$ defines a unitary isomorphism

$$h : L^2(\mathbb{R}_+, x^{2\sigma} \, dx) \cong L^2(\mathbb{R}_+, x^{2\sigma} h^{-2} \, dx).$$

We get that $hL_{ev/odd, U}h^{-1}$, with domain $hS_{ev/odd, U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, x^{2\sigma} h^{-2} \, dx)$, and its self-adjoint extension is $hL_{ev/odd, +}h^{-1}$. Via (62) and (63), we obtain an obvious version of Corollary 1.6 for $hL_{ev/odd, +}h^{-1}$. By using

$$\left[ \frac{d}{dx}, h \right] = h', \quad \left[ \frac{d^2}{dx^2}, h \right] = 2h' \frac{d}{dx} + h'',$$

it easily follows that $hL_{ev/odd, U}h^{-1}$ is of the form (1) with $f_1 \in C^1(U)$ and $f_2 \in C(U)$. Then Theorem 1.7 is a consequence of the following.

Lemma 6.1. For $\sigma > -1/2$, a positive function $h \in C^2(U)$, and an operator $P$ of the form (1) with $f_1 \in C^1(U)$ and $f_2 \in C(U)$, we have $P = hL_{\sigma, ev, U}h^{-1}$ on $hS_{ev, U}$ if and only if (2) and (3) are satisfied with some primitive $F_1 \in C^2(U)$ of $f_1$. 
Proof. By (64),

\[ h^{-1}Ph = -h^{-1} \frac{d^2}{dx^2} h + sx^2 - 2h^{-1}f_1 \frac{d}{dx} h + f_2 \]

\[ = -\frac{d^2}{dx^2} h^{-1} \left( 2h \frac{d}{dx} h'' + s \right) \]

\[ - 2f_1 \frac{d}{dx} - 2h^{-1}f_1h' + f_2 \]

\[ = H - 2(h^{-1}h' + f_1) \frac{d}{dx} - h^{-1}h'' - 2h^{-1}f_1h' + f_2. \]

So \( P = hL_{\sigma, ev, U}h^{-1} \) if and only if

\[ h^{-1}h' = \sigma x^{-1} - f_1, \]

(65)

\[ f_2 = h^{-1}h'' + 2h^{-1}h'f_1. \]

(66)

The equality (65) is equivalent to (3), and gives

\[ h^{-1}h'' = (\sigma x^{-1} - f_1)^2 - \sigma x^{-2} + f_1'. \]

So, by (66),

\[ f_2 = (\sigma x^{-1} - f_1)^2 - \sigma x^{-2} + f_1' + 2(\sigma x^{-1} - f_1)f_1 \]

\[ = \sigma(\sigma - 1)x^{-2} - f_1^2 - f_1'. \]

It follows that (65) and (66) are equivalent to (3) and (2). \( \square \)

Remark 2. By (64), we get an operator of the same type if \( h \) and \( \frac{d}{dx} \) is interchanged in (1).

Remark 3. By using (64) with \( h = x^{-1} \) on \( \mathbb{R}^+ \), it is easy to check that \( L_{\sigma, odd, +} = xL_{1+\sigma, ev, +}x^{-1} \) on \( S_{odd, +} = xS_{ev, +} \) for all \( \sigma > -1/2 \). So no new operators are obtained with the conjugation \( L_{\sigma, odd, U} \) by \( h \).

Remark 4. If \( f_1 \) is a rational function, then the function \( f_2 \), given by (2), is also rational.

Remark 5. The term of \( P \) with \( \frac{d}{dx} \) can be removed by conjugation, obtaining the operator \( H + \sigma(\sigma - 1)x^{-2} \), given by restricting \( K_{\sigma} \), first to even functions and second to \( \mathbb{R}^+ \). In this way, we get all operators of the form \( H + cx^{-2} \) with \( c > -1/4 \).

7. Examples

7.1. Case where \( f_1 \) is a multiple of \( x^{-1} \). A particular class of (1) is given by the operators of the form

\[ P = H - 2c_1x^{-1} \frac{d}{dx} + c_2x^{-2} \]

(67)

for \( c_1, c_2 \in \mathbb{R} \). In this case, we can take \( F_1 = c_1 \log x \). Then \( e^{F_1} = x^{c_1} \), (3) gives \( h = x^a \) with \( a = \sigma - c_1 \), and (2) becomes \( c_2x^{-2} = (a^2 + a(2c_1 - 1))x^{-2} \). Therefore Corollary 1.8 follows from Theorem 1.7.

Remark 6. According to Remark 2, we get an operator of the same type if \( x^{-1} \) and \( \frac{d}{dx} \) is interchanged in (67).
The existence of \( a \in \mathbb{R} \) satisfying (6) is characterized by the condition
\[
(2c_1 - 1)^2 + 4c_2 \geq 0.
\]
Observe that (6S) is satisfied if \( c_2 \geq \min\{0, 2c_1\} \). In particular, we have the following special cases.

**Example 7.1.** Suppose that \( c_2 = 0 \); i.e., \( P = H - 2c_1x^{-1}\frac{d}{dx} \). Thus \( P = L_{c_1, ev,+} \) if \( c_1 > -1/2 \); however, this inequality is not required \( a \) priori. Then (4) means that \( a \in \{0, 1 - 2c_1\} \), and (6) gives

\[
\sigma = \begin{cases} 
    c_1 & \text{if } a = 0 \\
    1 - c_1 & \text{if } a = 1 - 2c_1.
\end{cases}
\]

In the case \( a = 0 \) and \( \sigma = c_1 \), the condition \( c_1 > -1/2 \) is needed to apply Corollary 1.8. In this case, Corollary 1.8 holds for \( P = L_{c_1, ev,+} \) on \( S_{ev,+} \), which is a direct consequence of the known properties of \( L_{c_1} \) (Section 2.2 and Corollary 1.6).

Nevertheless, Corollary 1.8 gives new information in the case \( a = 1 - 2c_1 \) and \( \sigma = 1 - c_1 \); we have \( \sigma > -1/2 \) just when \( c_1 < 3/2 \) (\( c_1 \leq -1/2 \) is allowed!). When this inequality is satisfied, Corollary 1.8 states that \( P \), with domain \( x^{1-2c_1} S_{ev,+} \), is also essentially self-adjoint in \( L^2(\mathbb{R}_+, x^{2c_1} dx) \); the spectrum of its self-adjoint extension \( P \) consists of the eigenvalues \( (4k + 3 - 2c_1)s \) \( k \in \mathbb{N} \) with multiplicity one; the corresponding normalized eigenfunctions are \( \sqrt{2} x^{1+2c_1} \phi_{2k,+} \); and \( D^\infty(P) = x^{1-2c_1} S_{ev,+} \).

Thus, when \(-1/2 < c_1 < 3/2 \), we have got two essentially self-adjoint operators in \( L^2(\mathbb{R}_+, x^{2c_1} dx) \) defined by \( P \), with domains \( S_{ev,+} \) and \( x^{1-2c_1} S_{ev,+} \), which are equal just when \( c_1 = 1/2 \). In particular, if \( c_1 = 0 \), these operators are defined by \( H \) with domains \( S_{ev,+} \) and \( x S_{ev,+} = S_{odd,+} \).

**Example 7.2.** Suppose that \( c_2 = 2c_1 \); i.e., \( P = H - 2c_1x^{-1}\frac{d}{dx} + 2c_1x^{-2} \). Then (4) means that \( a \in \{1, -2c_1\} \), and (6) gives

\[
\sigma = \begin{cases} 
    1 + c_1 & \text{if } a = 1 \\
    -c_1 & \text{if } a = -2c_1.
\end{cases}
\]

In the case \( a = 1 \) and \( \sigma = 1 + c_1 \), we have \( \sigma > -1/2 \) if and only if \( c_1 > -3/2 \). When this inequality is satisfied, Corollary 1.8 states that \( P \), with domain \( x S_{ev,+} \), is essentially self-adjoint in \( L^2(\mathbb{R}_+, x^{2c_1} dx) \); the spectrum of its self-adjoint extension \( P \) consists of the eigenvalues \( (4k + 3 + 2c_1)s \) \( k \in \mathbb{N} \) with multiplicity one; the corresponding normalized eigenfunctions are \( \sqrt{2} x\phi_{2k,+} \); and \( D^\infty(P) = x S_{ev,+} \).

In the case \( a = -2c_1 \) and \( \sigma = -c_1 \), we have \( \sigma > -1/2 \) just when \( c_1 < 1/2 \). When this inequality is satisfied, Corollary 1.8 states that \( P \), with domain \( x^{-2c_1} S_{ev,+} \), is essentially self-adjoint in \( L^2(\mathbb{R}_+, x^{2c_1} dx) \); the spectrum of its self-adjoint extension \( P \) consists of the eigenvalues \( (4k + 1 - 2c_1)s \) \( k \in \mathbb{N} \) with multiplicity one; the corresponding normalized eigenfunctions are \( \sqrt{2} x^{-2c_1} \phi_{2k,+} \); and \( D^\infty(P) = x^{-2c_1} S_{ev,+} \).

Thus, when \(-3/2 < c_1 < 1/2 \), we have got two essentially self-adjoint operators in \( L^2(\mathbb{R}_+, x^{2c_1} dx) \) defined by \( P \), with domains \( x S_{ev,+} \) and \( x^{-2c_1} S_{ev,+} \), which are equal just when \( c_1 = -1/2 \). In particular, if \( c_1 = 0 \), we get again that these operators are defined by \( H \) with domains \( x S_{ev,+} = S_{odd,+} \) and \( S_{ev,+} \).
7.2. Case where $f_1$ is a multiple of other potential functions. Suppose that $f_1 = cx^r$ for $c, r \in \mathbb{R}$ with $r \neq -1$. Given any $\sigma > -1/2$, now (2) becomes
\[ f_2 = \sigma(\sigma - 1)x^{-2} + c^2 x^{2r} - crx^{r-1}. \]
Moreover we can take $F_1 = \frac{cx^{r+1}}{r+1}$, obtaining
\[ h = x^\sigma \exp \left( -\frac{cx^{r+1}}{r+1} \right) \]
according to (3). Then Theorem 1.7 asserts that the operator
\[ P = H - 2c \frac{dx}{dx} + \sigma(\sigma - 1)x^{-2} - c^2 x^{2r} - crx^{r-1}, \]
with domain $h \mathcal{S}_{ev,+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2F_1} dx)$; the spectrum of its self-adjoint extension $P$ consists of the eigenvalues $(4k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2} h \phi_{2k,+}$; and the smooth core of $P$ is $h \mathcal{S}_{ev,+}$.

7.3. Case where $f_1$ is a multiple of $g'/g$ for some function $g$. The operators of Section 7.1 can be generalized as follows. For an open subset $U \subset \mathbb{R}_+$ of full Lebesgue measure, take $f_1 = cg'/g$ for $c \in \mathbb{R}$ and some non-vanishing function $g \in C^2(U)$. Given any $\sigma > -1/2$, the equality (2) gives
\[ f_2 = \sigma(\sigma - 1)x^{-2} - c(c - 1) \frac{g''}{g} - c \frac{g'''}{g}. \]
In this case, we can take $F_1 = c \log |g|$, obtaining $h = x^\sigma |g|^{-c}$ by (3). Then Theorem 1.7 states that the operator
\[ P = H - 2c \frac{g'}{g} \frac{dx}{dx} + \sigma(\sigma - 1)x^{-2} + c(c - 1) \frac{g''}{g} - c \frac{g'''}{g}, \]
with domain $x^\sigma |g|^{-c} \mathcal{S}_{ev,U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, |g|^{2c} dx)$; the spectrum of its self-adjoint extension $P$ consists of the eigenvalues $(4k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2} x^\sigma |g|^{-c} \phi_{2k,U}$; and the smooth core of $P$ is $x^\sigma |g|^{-c} \mathcal{S}_{ev,U}$. This agrees with Corollary 1.8 when $g = x$.

Example 7.3. If we take $g = \cos x$, which does not vanish on $U = \mathbb{R}_+ \setminus (2N + 1)\frac{i\pi}{2}$, we get that, for any $\sigma > -1/2$, the operator
\[ P = H - 2c \tan x \frac{dx}{dx} + \sigma(\sigma - 1)x^{-2} + c(c - 1) \tan^2 x - c, \]
with domain $x^\sigma |\cos x|^{-c/2} \mathcal{S}_{ev,U}$, is essentially self-adjoint in $L^2(\mathbb{R}_+ \setminus |\cos x|^{2c} dx)$; the spectrum of its self-adjoint extension $P$ consists of the eigenvalues $(4k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2} x^\sigma |\cos x|^{-c/2} \phi_{2k,U}$; and the smooth core of $P$ is $x^\sigma |\cos x|^{-c/2} \mathcal{S}_{ev,U}$.

Similar examples can be given with other trigonometric and hyperbolic functions.

Example 7.4. For $g = e^x$, it follows that, for any $\sigma > -1/2$, the operator
\[ P = H - 2c \frac{dx}{dx} + \sigma(\sigma - 1)x^{-2} - c^2, \]
with domain $x^\sigma e^{-cx} \mathcal{S}_{ev,+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2cx} dx)$; the spectrum of its self-adjoint extension $P$ consists of the eigenvalues $(4k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$)
with multiplicity one and normalized eigenfunctions $\sqrt{2} x^\sigma e^{-cx/2} \phi_{2k,+}$; and the smooth core of $\mathcal{P}$ is $x^\sigma e^{-cx} \mathcal{S}_{ev,+}$.

**Example 7.5.** With more generality, for $g = e^x$ ($0 \neq n \in \mathbb{Z}$) and any $\sigma > -1/2$, the operator

$$
P = H - 2cnx^{n-1} \frac{d}{dx} + \sigma((\sigma - 1)x^{-2} - c(c - 1)n^2x^{2(n-1)} - c(n(n-1)x^{n-2} + n^2x^{2(n-1)}) ,
$$

with domain $x^\sigma e^{-cx} \mathcal{S}_{ev,+}$, is essentially self-adjoint in $L^2(\mathbb{R}_+, e^{2cx} \, dx)$; the spectrum of its self-adjoint extension $\mathcal{P}$ consists of the eigenvalues $(4k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2} x^\sigma e^{-cx} \phi_{2k,+}$; and the smooth core of $\mathcal{P}$ is $x^\sigma e^{-cx} \mathcal{S}_{ev,+}$.

7.4. **Transformation of $P$ by changes of variables.** We can use arbitrary changes of variables to provide a larger family of essentially self-adjoint operators whose spectrum can be described. For instance, the operator $P$ on $\mathbb{R}_+$, given in Section 7.3, can be transformed into a differential operator on $\mathbb{R}$ with the change of variable $x = \log y$, where now $y$ denotes the standard coordinate of $\mathbb{R}_+$. Since $dx/dy = 1/y = e^{-x}$, we get

$$
\frac{d}{dy} = e^{-y} \frac{d}{dx}, \quad \frac{d^2}{dy^2} = e^{-2y} \left( \frac{d^2}{dx^2} - \frac{d}{dx} \right).
$$

So this change of variables transforms the operator $P$ of (1) (on functions of $y$) into the operator

$$
P_1 = -e^{-2x} \frac{d^2}{dx^2} + s^2 e^{2x} - 2 \left( f_1(e^x)e^{-x} - e^{-2x} \right) \frac{d}{dx} + f_2(e^x)
$$

(on functions of $x$), and transforms $L^2(\mathbb{R}_+, e^{2\phi_1(y)} \, dy)$ into $L^2(\mathbb{R}_+, e^{2\phi_1(e^y)} \, e^x \, dx)$. Suppose that $f_1$ and $f_2$ satisfy (2) for some $\sigma > -1/2$, and let $h \in C^2(U)$ be defined by (3) for some primitive $F_1 \in C^2(U)$ of $f_1$. Let also $V = \{ x \in \mathbb{R} \mid e^x \in U \}$. Then $P_1$, with domain

$$
\left\{ h(e^x) \phi(e^x) \mid \phi \in \mathcal{S}_{ev,U} \right\} \subset C^2(V) ,
$$

is essentially self-adjoint in the eigenvalues $(4k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$) with multiplicity one and normalized eigenfunctions $\sqrt{2} h(e^x) \phi_{2k,U}(e^x)$; and the smooth core of $\mathcal{P}_1$ is $\mathcal{S}_{ev,U}$.

**References**

[1] J.A. Álvarez López and M. Calaza Cabanas, *Witten’s perturbation on strata*, in preparation.

[2] T.H. Baker and P.J. Forrester, *The Calogero-Sutherland model and generalized classical polynomials*, Comm. Math. Phys. 188 (1997), 175–216.

[3] ———, *The Calogero-Sutherland model and polynomials with prescribed symmetry*, Nucl. Phys. B 492 (1997), 682–716.

[4] S.S. Bonan and D.S. Clark, *Estimates of the Hermite and the Freud polynomials*, J. of Approx. Theory 63 (1990), 210–224.

[5] J.P. Brasselot, G. Hector, and M. Saralegi, *$L^2$-cohomologie des espaces stratifiés*, Manuscripta Math. 76 (1992), 21–32.

[6] L. Brink, T.H. Hansson, S. Konstein, and M.A. Vasiliev, *The Calogero model—anyonic representation, fermionic extension and supersymmetry*, Nuclear Phys. B 401 (1993), 591–612.

[7] J. Brüning and M. Lesch, *Hilbert complexes*, J. Funct. Anal. 108 (1992), 88–132.
[8] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace Operator (Univ. Hawaii, Honolulu, Hawaii, 1979) (Providence, R.I., 1980), Proc. Sympos. Pure Math., vol. XXXVI, Amer. Math. Soc., 1980, pp. 91–146.

[9] , *Spectral geometry of singular Riemannian spaces*, J. Differential Geom. 18 (1983), 575–657. MR 85d:58083

[10] T.S. Chihara, *Generalized Hermite polynomials*, Ph.D. thesis, Purdue University, 1955.

[11] , *An introduction to orthogonal polynomials*, Mathematics and its Applications, vol. 13, Gordon and Breach Science Publishers, New York-London-Paris, 1978.

[12] D. Dickinson and S.A. Warsi, *On a generalized Hermite polynomial and a problem of Carlitz*, Bull. Un. Mat. Ital. 18 (1963), 256–259.

[13] C.F. Dunkl, *Reflection groups and orthogonal polynomials on the sphere*, Math. Z. 197 (1988), 33–60.

[14] , *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. 311 (1989), 167–183.

[15] , *Integral kernels with reflection group invariants*, Canad. J. Math. 43 (1991), 1213–1227.

[16] , *Hankel transforms associated to finite reflection groups*, Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991) (Providence, RI), Contemp. Math., vol. 138, Amer. Math. Soc., 1992, pp. 123–138.

[17] , *Symmetric functions and $B_N$-invariant spherical harmonics*, J. Phys. A 35 (2002), 10391–10408.

[18] M. Dutta, S.K. Chatterjea, and K.L. More, *On a class of generalized Hermite polynomials*, Bull. Inst. Math. Acad. Sinica 3 (1975), 377–381.

[19] P. Erdös and P. Turán, *On interpolation. III*, Ann. Math. 41 (1940), 510–555.

[20] M. Goresky and R. MacPherson, *Morse theory and intersection homology theory*, Analysis and Topology on Singular Spaces, II, III, Luminy, 1981, Astérisque, vol. 135-192, 1983, pp. 135–192.

[21] N. Nagase, *Sheaf theoretic $L^2$-cohomology*, Adv. Stud. Pure Math. 8 (1986), 273–279.

[22] A. Nowak and K. Stempak, *Imaginary powers of the Dunkl harmonic oscillator*, SIGMA Symmetry Integrability Geom. Meth. Methods Appl. 5 (2009), Paper 016, 12 pp.

[23] A. Pasquier, *A lecture on the Calogero-Sutherland models*, Integrable models and strings (Espoo, 1993) (Berlin), Lecture Notes in Phys., vol. 436, Springer, 1994, pp. 36–48.

[24] J. Roe, *Elliptic operators, topology and asymptotic methods*, second ed., Pitman Research Notes in Mathematics, vol. 395, Addison Wesley Longman Limited, Edinburgh Gate, Harlow, Essex CM20 2JE, England, 1998.

[25] M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus*, Nonselfadjoint operators and related topics (Beersheva, 1992) (Basel) (A. Flett and I. Gohberg, eds.), Oper. Theory Adv. Appl., vol. 73, Birkhäuser, 1994, pp. 309–396.

[26] M. Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. 192 (1998), 519–542.

[27] , *Dunkl operators: theory and applications*, Orthogonal polynomials and special functions (Leuven, 2002) (Berlin) (E. Koelink and W. Van Assche, eds.), Lecture Notes in Math., vol. 1817, Springer, 2003, pp. 93–135.
[36] G. Szegő, *Orthogonal polynomials*, fourth ed., Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, RI, 1975.

[37] R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. 75 (1969), 240—284.

[38] H. Ujino and M. Wadati, *Rodrigues formula for Hi-Jack symmetric polynomials associated with the quantum Colegero model*, J. Phys. Soc. Japan 65 (1996), 2423–2439.

[39] W. Van Assche, *Some results on the distribution of the zeros of orthogonal polynomials*, Journal of Computational and Applied Mathematics 12-13 (1985), 615–623.

[40] W. Van Assche and J.L. Teugels, *Second order asymptotic behaviour of the zeros of orthogonal polynomials*, Rev. Roumaine Math. Pures Appl. 32 (1987), 15–26.

[41] A. Verona, *Stratified mappings—structure and triangulability*, Lecture Notes in Math., vol. 1102, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[42] E. Witten, *Supersymmetry and Morse theory*, J. Differential Geom. 17 (1982), 661–692.