A new approximation method for geodesics on the space of Kähler metrics using complexified symplectomorphisms and Gröbner Lie series

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Abstract

It has been shown that the Cauchy problem for geodesics in the space of Kähler metrics with a fixed cohomology class on a compact complex manifold $M$ can be effectively reduced to the problem of finding the flow of a related Hamiltonian vector field $X_{H}$, followed by analytic continuation of the time to complex time.

This opens the possibility of expressing the geodesic $\omega_{t}$ in terms of Gröbner Lie series of the form $\exp(\sqrt{-1} t X_{H})(f)$, for local holomorphic functions $f$. The main goal of this paper is to use truncated Lie series as a new way of constructing approximate solutions to the geodesic equation. For the case of an elliptic curve and $H$ a certain Morse function squared, we approximate the relevant Lie series by their first twelve terms, calculated with the help of Mathematica. This leads to approximate geodesics which hit the boundary of the space of Kähler metrics in finite geodesic time. For quantum mechanical applications, one is interested also on the non-Kähler polarizations that one obtains by crossing the boundary of the space of Kähler structures. Properties of the approximate geodesics and its extensions are also studied using Mathematica.

Keywords: Kähler geometry; complex homogeneous Monge-Ampère equation; Lie series; imaginary time Hamiltonian symplectomorphisms.

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1 Introduction

If a manifold \( M \) has a Kähler metric than it has an infinite-dimensional space of such metrics. Indeed, let \((M,J_0,\omega)\) be a compact Kähler manifold with Kähler form \( \omega \) and complex structure \( J_0 \). Then the space of Kähler potentials on \((M,J_0)\) with fixed cohomology class \([\omega]\) is naturally identified with an open subset in the space of functions on \( M \)

\[
H = \{ \phi \in C^\infty(M) : \omega_\phi := \omega + i\partial_0\bar{\partial}_0\phi > 0 \},
\]
where \( \partial_0, \bar{\partial}_0 \) denote the \( J_0 \)-Dolbeaut operators. A natural metric on \( H \) is the Mabuchi metric \( [M, \text{Sem}, \text{Do1}, \text{Do2}] \)

\[
g_\phi(h_1, h_2) = \int_M h_1 h_2 \frac{\omega_\phi^n}{n!}, \quad h_1, h_2 \in T_\phi H = C^\infty(M).
\]

The expression for the curvature of this metric \( [\text{Sem}, \text{Do1}] \) suggests that the space of Kähler metrics with cohomology class \([\omega]\),

\[
H_0 = \{ \omega_\phi, \phi \in H \} \cong H/\mathbb{R},
\]
is a realization of the infinite-dimensional symmetric space that would correspond to

\[
H_0 \cong G_\mathbb{C}/G,
\]
where \( G \) denotes the group of Hamiltonian symplectomorphisms of \((M,\omega)\) and \( G_\mathbb{C} \) denotes its (non-existent) complexification.

A path of Kähler potentials in \( H \), \( \phi_t \) for \( t \) in some open interval in \( \mathbb{R} \), is a geodesic if it satisfies

\[
\ddot{\phi}_t = \frac{1}{2}||\nabla \dot{\phi}_t||^2_{\phi_t}.
\]

It is well-known that this is equivalent to the homogeneous complex Monge–Ampère equation

\[
(\Omega + i\partial\bar{\partial}\Phi)^{(n+1)} = 0
\]
where \( \Omega \) is the pull-back of \( \omega \) to \( A \times M \), \( \Phi(w,z,\bar{z}) = \phi_t(z,\bar{z}) \), \( \partial \) is the Dolbeaut operator for \((w,z), \) \( w \) is an auxiliary complex variable on an annulus \( A \subset \mathbb{C} \), with \( t = \log|w| \) and \((z,\bar{z})\) are local coordinates on \( M \). As described by Donaldson \( [\text{Do1}] \), these geodesic paths would correspond to one-parameter subgroups in \( G_\mathbb{C} \), generated by “complexified” Hamiltonian flows on \( M \), with the Hamiltonian \( H \) given by the initial velocity \( \dot{\phi}_0 \). Complex time Hamiltonian evolution has been studied both in Kähler geometry \( [\text{Sem}, \text{Do1}, \text{Do2}, \text{BLU}, \text{MN}] \) and in quantum physics \( [\text{Th}, \text{HK}, \text{GS}, \text{KMN1}, \text{KMN2}] \). The geodesic path \( \phi_t \) corresponds to a family of diffeomorphisms of \( M \), \( \varphi_{it} \in Diff(M) \), which are Moser maps satisfying

\[
\varphi_{it}^*(\omega + i\partial_0\bar{\partial}_0\phi_t) = \omega.
\]

\footnote{The fact that these maps are labelled by \( it \) instead of simply \( t \) is explained below.}
These can be described by a system of non-linear PDEs and also in terms of a lifting of the Hamiltonian flow to a complexification of \( M \). In [MN], in the real-analytic setting and for \( M \) compact, it was shown how the path of Moser maps can be explicitly described by Hamiltonian evolution of local holomorphic coordinates analytically continued to complex time. For sufficiently small \(|t|\), one defines a new global complex structure \( J_t \) on \( M \) (which is biholomorphic to \( J_0 \)) via new local \((J_t\text{-holomorphic})\) coordinates defined by the Lie series

\[
z_{it} = e^{itX_H}z,
\]

where \( X_H \) is the Hamiltonian vector field of \( H \) with respect to \( \omega \). One then obtains a geodesic path of Kähler structures \((M,\omega_{\phi_t},J_0)\). In the symplectic description, which we will use below, one fixes the symplectic form rather than the complex structure (see Theorem 4.1 and Proposition 9.1 in [MN]), so that the geodesic path becomes

\[
(\omega,J_t) = \phi_t^* (\omega_{\phi_t},J_0).
\]

In this paper, we use the method proposed in [MN] to reduce the Cauchy problem for geodesics in \( \mathcal{H}_\omega \) to finding the associated \( \omega \)-Hamiltonian flow followed by an appropriate complexification, in the setting of the Gröbner theory of Lie series of vector fields [Gro]. The main goal of the present paper consists then in applying [MN] to constructing approximate solutions to the geodesic equations by taking only the first \( N \) terms in the relevant Lie series.

We consider the Cauchy problem with initial flat Kähler metric on the two-dimensional torus \( \mathbb{T}^2 \) and \( \frac{\partial \phi_t}{\partial t}|_{t=0} = H \), with \( H \) the square of a particular Morse function on \( \mathbb{T}^2 \). With the help of Mathematica and of a supercomputer, we calculate approximately \((N = 12)\), for different values of \( t \), the conformal factor of the metrics along the geodesics. The solution remains Kähler for geodesic time \( t \) inside the interval \( t \in (-T_1, T_2) \) for certain finite positive values \( T_1, T_2 \). It hits the boundary of the space of Kähler metrics both at (negative) time \( -T_1 \) and (positive) time \( T_2 \). For \( t : t > T_2 \) and \( t < -T_1 \) the solution corresponds to a mixed polarization (in the sense of geometric quantization) with open regions with both positive and negative definite metric. Since these polarizations are of interest for quantum physics (see [KMN2] for a discussion) we study the behaviour of the metric also in this region.

We remark that Lie series have been also successfully applied to the approximate integration of ordinary differential equations, in particular in celestial mechanics (see eg [BHT, ED, HLW]).

2 Geodesic equation on the space of Kähler metrics and imaginary time Hamiltonian symplectomorphisms

For the compact Kähler manifold \((M,J_0,\omega)\) and in the context described above, consider the following Cauchy problem for geodesics

\[
\begin{align*}
\ddot{\phi}_t &= \frac{1}{2} ||\nabla \dot{\phi}_t||^2_{\phi_t}, \\
\dot{\phi}_0 &= 0 \\
\dot{\phi}_t|_{t=0} &= H
\end{align*}
\] (2.1)
with real analytic initial data.

In [MN], the following algorithm has been proposed for reducing the highly nonlinear partial differential equation (2.1) (which is equivalent to the HCMA equation [Sem, Do1, Do2]) to finding the flow of the Hamiltonian vector field $X_H$ of $H$, for the initial symplectic form $\omega$, followed by a particular complexification.

- **Step 1: Lie series of $J_0$-holomorphic coordinate functions** – Let $U_\alpha$ be an open cover of $M$ and $(U_\alpha, z_1^\alpha, \cdots, z_n^\alpha)$, denote $J_0$-holomorphic coordinate charts. For $\tau \in \mathbb{C}$, and for sufficiently small $|\tau|$, find the Lie series of $X_H$ for every $J_0$-holomorphic coordinate function and

  $$z_j^\alpha(\tau) = e^{\tau X_H}(z_j^\alpha) = \sum_{k=1}^{\infty} \frac{\tau^k}{k!} (X_H)^k(z_j^\alpha). \quad (2.2)$$

- **Step 2: Define the Moser isotopy** – In this step one uses the constructive proof of Theorem 2.6 of [MN] to turn the complex symplectomorphisms (2.2) into Moser diffeomorphisms $\varphi_\tau$ such that

  $$z_j^\alpha(\tau) = \varphi_\tau^*(z_j^\alpha) = e^{\tau X_H}(z_j^\alpha), \quad |\tau| < T. \quad (2.3)$$

  The functions $z_j^\alpha(\tau)$ are $J_\tau$-holomorphic for the complex structure

  $$J_\tau = \varphi_\tau^*(J_0) := (\varphi_\tau^*)^{-1} \circ J_0 \circ \varphi_\tau$$

- **Step 3: Restricting to imaginary time and geodesics** – As shown in Proposition 9.1 of [MN], by restricting the Moser isotopy of step 2, $\varphi_\tau$, to imaginary $\tau = it, t \in \mathbb{R}$, the path of symplectic forms

  $$\omega_t = (\varphi_{it}^{-1})^*(\omega), \quad |t| < T, \quad (2.4)$$

  is a geodesic path in $\mathcal{H}_\omega$ and its Kähler potential is a solution of the Cauchy problem (2.1). The expression for the Kähler potential in terms of the imaginary time symplectomorphisms is given by (4.1)–(4.3) and (6.7) of [MN]. Below, however, we will focus on finding approximate expressions for the geodesics in terms of the Kähler forms as in (2.4).

3 New approximate method for finding geodesics and description of the computational method

Let the geodesic $\omega_t$ in (2.4) be written in the form

$$\omega_t = \frac{i}{2} \sum_{j,j} h_{jj}(z, \bar{z}; t) dz_j \wedge d\bar{z}_j.$$

Then (2.4) is equivalent to,

$$(\varphi_{it})^*(\omega_t) = \omega, \quad |t| < T, \quad (3.1)$$
or, in local coordinates,
\[
\sum_{j,j'} h_{jj'}(z(it), \bar{z}(it); t) \, dz_j(it) \wedge d\bar{z}_{j'}(it) = \sum_{j,j'} h_{jj'}(z, \bar{z}; 0) \, dz_j \wedge d\bar{z}_{j'}.
\]

For complex one-dimensional manifolds one obtains
\[
h_{11}(z, \bar{z}, t) = \frac{i}{2} \frac{h_{11}(z, \bar{z}, 0)}{\frac{\partial \bar{z}(it)}{\partial z} - \frac{\partial z(it)}{\partial \bar{z}}}. \tag{3.2}
\]

We define the N-th order approximation to the Lie series in (2.2) as
\[
z_j(\tau; N) = \sum_{k=0}^{N} \frac{\tau^k}{k!} (X_H)^k (z_j) \tag{3.3}
\]

\textbf{Definition 3.1} The Lie series N-th order approximation to the geodesics is (for n = 1) defined to be the path of metrics given by the following conformal factors
\[
h_{11}^{(N)}(t) = \frac{i}{2} \frac{h_{11}(z, \bar{z}, 0)}{\frac{\partial \bar{z}(it; N)}{\partial z} - \frac{\partial z(it; N)}{\partial \bar{z}} = \sum_{k=0}^{N} a_k(x, y)t^k. \tag{3.4}
\]

The computational implementation is then straightforward. The chosen software was Mathematica, as it provides the necessary tools for heavy symbolical manipulations. Nevertheless, some implementation challenges became quickly evident. Given a reasonably non-trivial Hamiltonian, the successive application of the chain and Leibniz’s rules leads to an exponential growth of the number of additive terms in each of the terms of the series. This means that for a reasonable approximation of N > 4 the final formula is quite long. For N = 12, the .txt file with the result obtained by Mathematica is around 30 Mbytes. Consequently, computing a single value with this formula was a very slow process and the computation of all the values necessary to generate the graphs required heavy parallelization. We thus resorted to the Baltasar supercomputer, run by CENTRA, and used a node of 48 cores to run the code.

4 Description of the results.

For the numerical analysis we took the Hamiltonian:
\[
H = \frac{1}{8} \left( \sin^2(\pi x) + \sin^2(\pi y) \right)^2 \tag{4.1}
\]
in the unit square $[0,1] \times [0,1]$ with initial Kähler structure given by the flat metric and the standard complex structure with local holomorphic coordinate, $z = x + iy$. An approximate expression for the conformal factor was calculated using 12 terms of the Lie series. Except for Figure 6, all the graphs where obtained by sampling points on uniform $50 \times 50$ lattice (for each of the subfigures in Figure 6 a $200 \times 200$ lattice was used instead).

Figure 1: Plot of $\log \left( |a_{12}(t)/h_{11}^{(11)}(t)| \right)$ along the diagonal $x = y \in [0, 0.5]$, where $a_j$ is $j$th term in the series and $t$ a value in (imaginary) time. (See equation (3.4).) The valley in the figure corresponds to $t = 0$ and the time axis runs in the transverse direction for $t \in [-1, 1]$.

We begin with a brief comment on error analysis and on the chosen intervals. It is known that below a certain value of geodesic time the Lie series is absolutely convergent [Gro]. In our analysis, to estimate the error of the truncation of the series, we used the ratio of the absolute value of the last term considered in (3.4) to the absolute value of the sum of all the lower order terms (see the related discussion around equation (66) of [ED]). This gives us some indication of convergence and of the magnitude of the error.

In Figure 1 we plot the logarithm of the absolute value of the ratio between the last term considered in (3.4) and the sum of the lower order terms, for points in the first half of the diagonal of the unit square (this is $(x,y)$ such that $x = y < 0.5$) and for values of imaginary time $t$ in $[-1, 1]$. We can observe that for $t < 0.5$ we have negative values of the error indicator for all sampled points, while for larger $t$ the same is not true.

Figure 2: Positive small imaginary time evolution of $h_{11}$
and inclusively we find positive values of the logarithm that suggest a significant error. As such we restricted our analysis to $t$ smaller than 0.5.

Let us begin the analysis of the conformal factor with $t > 0$ in Figure 2. In Figure 2c there are regions where the metric is no longer positive definite. (For example, in the regions around the saddle points and the maximum point of $H$. See also Figure 6.) Although the metric is no longer everywhere positive beyond this value of time, it is still interesting for applications in geometric quantization. The critical time, that is the earliest time at which the conformal factor is not strictly positive, is higher than but close to $t = 0.118$.

In Figure 5 we have similar plots but for $t < 0$ where the critical time is lower than but close to $t = 0.121$, just after Figure 3c. Due to the nature of the computation of the conformal factor, the transition between signs always occurs at points where the conformal factor blows up (due to zeros of the denominator in (3.4)).

We also focus the analysis around the minimum and the maximum. Around the minimum, $H$ is close to the square of the Hamiltonian for the harmonic oscillator, though the region where such analogy is valid narrows with time. This phenomenon of the shrinking of the region of validity of the harmonic oscillator approximation is more evident for negative time evolution, where the singularity line approaches the origin rather quickly, as it is clear in Figure 4d.

The evolution around the maximum is more complicated. Nonetheless, one also has elliptic behaviour around the maximum point in a (very small) region that also reduces in time. For the maximum this reduction is much faster and more evident since it is limited by singularity lines for both positive and negative imaginary time evolution.

Finally, we present the evolution of sign of the conformal factor for a value of $t$ high
Figure 5: Evolution of $h_{11}$ around the maximum

(a) $t = 0.05$  
(b) $t = 0.5$  
(c) $t = -0.05$  
(d) $t = -0.5$

Figure 6: Evolution of the sign of $h_{11}$. Regions with positive sign (+) of the conformal factor are in tan and the blue is on regions with negative sign (−).

enough to give regions with negative conformal factor but such that we can still trust the approximation. This images, in Figure 6, present a very clear geometrical picture of the evolution. We observed in the approximation scheme that taking more and more terms in the Lie series produces more and more detail in the patterns in the figures, but the overall structure remains similar.

Also, one can very clearly identify some similarities in the patterns of evolution for positive and negative time. One interesting fact is the similarity of the band structure of the regions of positive versus negative conformal factor around the saddle points.

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