Dynamics of quantum-classical hybrid system: effect of matter-wave pressure

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Radiation pressure affects the kinetics of a system exposed to the radiation and it constitutes the basis of laser cooling. In this paper, we study matter-wave pressure through examining the dynamics of a quantum-classical hybrid system. The quantum and classical subsystem have no explicit coupling to each other, but affect mutually via a changing boundary condition. Two systems, i.e., an atom and a Bose-Einstein condensate (BEC), are considered as the quantum subsystems, while an oscillating wall is taken as the classical subsystem. We show that the classical subsystem would experience a force proportional to \( Q^{-3} \) from the quantum atom, whereas it acquires an additional force proportional to \( Q^{-2} \) from the BEC due to the atom-atom interaction in the BEC. These forces can be understood as the matter-wave pressure.

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I. INTRODUCTION

It is well known that electromagnetic radiation exerts a pressure upon any surface exposed to it, the pressure was deduced theoretically by James Clerk Maxwell and Adolfo Bartoli more than a century ago, and proven experimentally by Lebedev, Ernest Fox Nichols and Gordon Ferrie Hullin in the last century. Recently, this research field becomes active again due to the breakthrough in nanofabrication and in ultracold science, this together with the coupling of coherent optical system to micromechanical devices, has opened up the exciting new field of research, cavity optomechanics.

Quantum mechanics tells us that all matter exhibits both wave-like and particle-like properties, called wave-particle duality, the wavelength is inversely proportional to the momentum of a particle and the frequency is directly proportional to the particle energy. These facts give rise to a question: Can the matter wave exert a pressure \( \text{(matter-wave pressure)} \) upon a surface exposed to it, like the electromagnetic radiation does? To answer this question, it is good to study a coupled quantum-classical hybrid system, and examine how the quantum subsystem affects the kinetics of the classical subsystem.

On the other hand, classical-quantum hybrid system on its own is an interesting topic to study. By classical-quantum hybrid system we mean a composite system consisting of a quantum and a classical subsystem. For a closed hybrid system, the quantum subsystem may be treated classically for specific issues addressed. However, this approach in general is inadequate owing to the loss of quantum features. In Ref. [15], the authors present a general framework for exact treatment of such a hybrid system. When the quantum subsystem is dynamically fast and the classical subsystem is slow, a vector potential arises. This vector potential, on one hand, gives rise to the familiar Berry phase in the fast quantum dynamics, on the other hand, it yields a Lorentz-like force in the slow classical dynamics. In the formalism, the Hamiltonian of the fast quantum subsystem depends explicitly on the freedom of the classical subsystem. In contrast, here we study the dynamics of quantum-classical hybrid system without any explicit interaction, instead an boundary condition will be considered that connects the quantum and the classical subsystem. An atom and a BEC are taken as the quantum subsystem, both of them move in an one-dimensional square well with an oscillating wall as their boundary. The difference between these two quantum subsystems is that the BEC has atom-atom interaction, whereas the atom would not have. This setting is exactly a model we need to discuss the problem of matter-wave pressure.

The paper is organized as follows. After a brief introduction to the formalism for classical-quantum hybrid system in Sec.II, we study the kinetics of the classical subsystem (i.e., the oscillating wall) under the effect of a quantum atom in Sec.III. Treating the atom as a classical particle, the kinetics of the wall is also examined in this section. In Sec.IV, we investigate the kinetics of the wall exposed to a Bose-Einstein condensate. The atom-atom coupling in the BEC will make the force acting on the wall different from the case in Sec. III. Finally we conclude our results in Sec.V.

II. GENERAL FORMALISM

In this section, we present a brief introduction to the formalism for quantum-classical hybrid system. For more detail, we refer the reader to Ref.[15]. Consider a

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The quantum-classical hybrid system described by

\[ H = (\Psi|\hat{H}_1(Q)|\Psi) + H_2(P,Q), \]  

(1)

where \( \hat{H}_1 \) is the Hamiltonian for the quantum subsystem and \( |\Psi\rangle = (\psi_1,\psi_2,\ldots,\psi_N)^T \) denotes its quantum state. \( H_2(P,Q) \) represents the classical subsystem and \( P, Q \) stand for its coordinates and momenta, respectively. It has been shown\([19, 20]\) that a quantum system possesses a vector potential, which we can obtain a vector potential that the classical subsystem feels, as

\[ A = \sum_{n=1}^N I_n A_n, \quad A_n = i\langle \phi_n \rangle \frac{\partial}{\partial Q} |\phi_n\rangle, \]  

(4)

where \( I_n = \hbar |c_n|^2 \). This leads to a dynamical equation for the classical subsystem,

\[ M\ddot{Q} = -\frac{\partial \mathcal{H}_I}{\partial \dot{Q}} - \frac{\partial V_2}{\partial Q} + \dot{Q} \times \mathcal{B}, \]  

(5)

with \( \mathcal{H}_I = \mathcal{H}_I(I_1,Q) = \sum_n E_n(Q) I_{1n}/\hbar \), \( V_2 \) is a potential and \( \mathcal{B} = \nabla \times A = \sum_n I_{1n} \nabla \times A_n \) is a gauge field like magnetic field.

III. ATOM AS THE QUANTUM SUBSYSTEM

Consider a bipartite hybrid system which consists of a single atom in an infinitely deep well and a moving wall, the moving wall acts as a boundary for the atom (see Fig. 1), and the whole system is restricted to move in one-dimension. The Hamiltonian of such a system can be described by

\[ \hat{H} = \hat{H}_1 + H_2, \]  

(6)

where \( \hat{H}_1 \) denotes the Hamiltonian of the atom trapped in the infinitely deep well, \( H_2 \) stands for the Hamiltonian of the moving wall which is considered as a classical harmonic oscillator. Here \( \hat{H}_1 = p^2/2m \), and \( H_2 = \int \frac{1}{2} M \omega^2(Q - Q_0)^2 \). \( p \) and \( P \) denote the momentum along the \( x \) axis for the quantum atom and classical wall respectively; Note \( p \) is an operator while \( P \) is a c-number here; \( m \) and \( M \) stand for the mass of the atom and the oscillator respectively; \( V_2 = \frac{1}{2} M \omega^2(Q - Q_0)^2 \) is the potential energy of the harmonic oscillator; \( Q \) is the coordinate of the moving wall and \( Q_0 \) is its equilibrium position; \( \omega \) denotes the vibration frequency of the classical subsystem. In this hybrid system, we consider a situation that there is no explicit interaction between the quantum atom and the moving wall. The moving wall affects the dynamics of the quantum atom by changing its boundary condition. Interesting features arise in this case as we will show below.

In our model the classical subsystem only changes the boundary condition of the quantum subsystem, which would reflect in the instantaneous eigenstate and the corresponding eigenvalue of the quantum subsystem given below

\[ |\phi_n(Q)\rangle = \int_0^Q \phi_n(q,Q)|q\rangle dq, \]

\[ \phi_n(Q) = \sqrt{\frac{2}{Q}} \sin \frac{n\pi q}{Q}, \]

\[ E_n(Q) = \frac{\hbar^2 \pi^2 n^2}{2mq^2}, \]  

(7)

where \( |q\rangle \) denotes the eigenstate of coordinate \( q \). Then the Hamiltonian of the quantum subsystem can be rewritten as

\[ \mathcal{H}_I(I_1,Q) = \sum_n E_n(Q) I_{1n}/\hbar = \sum_n \frac{n^2 \pi^2 \hbar^2}{2mQ^2} |c_n|^2. \]  

(8)

With these results, the vector potential \( A \) is then

\[ A = \sum_{n=1}^N I_{1n} A_n \]
FIG. 2: The coordinate of the classical subsystem $Q$ (in units of nm) as a function of time (in units of $10^{-7}$ s) with $B = 0.01$, $\omega = 1$ (in units of $2\pi \times 10^7$ Hz), $Q_0 = 1$, the initial condition is $Q(0) = 1.1$ and $\dot{Q}(0) = 0$. In the red solid curve, we consider the effect of the quantum subsystem, while in the blue dash curve we plot the free evolution of the oscillator.

\[
\sum_{n=1}^{\infty} i\hbar |c_n|^2 \int_0^Q \phi_n^*(q, Q) \frac{\partial}{\partial Q} \phi_n(q, Q) dq = 0,
\]

leading to an observation that only scalar potential (from the quantum subsystem) affects the dynamics of the classical subsystem. It is worth noting that this conclusion depends on the model, i.e., the magnetic-like gauge fields are not zero in general.

Substituting these equations together with the potential energy $V_2$ into Eq. (4), we get the dynamical equation for the classical subsystem of the linear system (one-dimensional)

\[
\dot{Q} = \frac{B}{Q^3} - \omega^2 (Q - Q_0),
\]

where $B = \sum_n \frac{a^2 n^2 \hbar^2}{2mM}$. This equation includes the effect of the quantum subsystem and describes the kinetics of the classical subsystem. The first term in Eq. (10) represents a force acting on the wall from the atom. We explain this force as a consequence of matter-wave pressure for the following reasons. (1) Suppose that the population of the atom in level $n$ remains unchanged, the total energy of the atom is $E_1 = \sum_n \frac{a^2 n^2 \hbar^2}{2mM} |c_n|^2$, where $|c_n|^2$ is the probability of the atom in level $n$. Consider a small vibration $\Delta Q$ of the wall, the work done by the atom is $\dot{F} \cdot \Delta Q$, and we have $\dot{F} = -\frac{\partial E_1}{\partial Q}$, this exactly leads to the first term in Eq. (10). (2) The wave-function of the atom is $|\Phi\rangle = \sum_n c_n |\phi_n(Q)\rangle$, with the energy $E_1$ and momentum $\vec{p} = \sum_n |c_n|^2 \vec{p}_n = \sum_n |c_n|^2 \frac{a^2 n^2 \hbar}{2mM}$, the probability current is $J = \sum_n J_n = \sum_n |c_n|^2 \frac{a^2 n^2 \hbar}{2mM}$. This means that the atom exerts a pressure force given by $\vec{F} = \sum_n 2\hbar n J_n = \sum_n |c_n|^2 \frac{a^2 n^2 \hbar^2}{mM}$ on the wall.

In Fig. 2 we plot the coordinate of the classical subsystem $Q$ as a function of time. It is clear that $Q$ is a oscillating function of time. The blue-dash line denotes a free harmonic oscillator and the red-solid line is for the case with the effect of the quantum subsystem. Here the parameters are chosen as $B = 0.01$, $\omega = 1$, $Q_0 = 1$, $Q(0) = 1.1$ and $\dot{Q}(0) = 0$, where $Q(0)$ and $\dot{Q}(0)$ denote the initial maximal displacement from the equilibrium and the initial velocity of the moving wall, respectively.

Comparing with the free harmonic oscillator (the blue dash line in Fig. 2), we find that when we consider the effect of quantum subsystem the equilibrium point moves to the right side. This is a consequence of the matter-wave pressure. In Fig. 3, we present the time evolution of the coordinate of the classical subsystem $Q$ under different initial amplitudes ($Q(0) - Q_0$). For the red solid curve, we choose $Q(0) = 1.1$, while for the blue dash curve, $Q(0) = 1.8$. The other parameters are $B = 0.1$, $\omega = 1$, and $Q_0 = 1$. We find that the effect of the quantum subsystem increases with $Q(0) - Q_0$. The curve becomes sharper when the coordinate moves toward the origin. This feature becomes evident for the case of large initial amplitude. Next we show the coordinate $Q$ as a function of time with different initial equilibrium position $Q_0$ in Fig. 4. We find that for the same initial amplitude, the larger initial equilibrium position $Q_0$ is, the smaller the effect of quantum subsystem on the classical subsystem.

These observations can be understood by examining the dynamical equation Eq. (10). In this equation $M \dot{Q}$ denotes the resultant force of the moving wall and $M\omega^2 (Q - Q_0)$ is the elastic spring force which keeps the moving wall in harmonic oscillating. Quantum subsystem brings in a rightward force (acting on $Q$) with $B_0 = BM$ which is inversely proportional to the cube of the coordinate $Q$ of the moving wall. This is the reason why the equilibrium position moves more to the right side than the free harmonic oscillator. The quantum subsystem induced force results from the matter-wave pressure, hence by examining the kinetics of the classical subsystem, we may recognize some features of the matter-wave pressure. It is easy to see that this force decreases rapidly with the increasing of $Q$. This is the reason why the effect of the
plotted for $Q$ coordinate was treated as a quantum system, in this figure we show the $Q$ objects. This plot is for different initial equilibrium positions are 25 and 0, respectively. The atom was assumed initially with the wall. We assume that the collision is elastic such that both energy and momentum are conserved. The numerical results are presented in Fig. 5 where we plot the coordinates of the atom ($q$) and wall ($Q$) as a function of time. Similar to the quantum case, the smaller the well is (or, the smaller the $Q_0$), the larger the effect of the atom on the wall. The difference is that a quantum atom exerts a force proportional to $1/Q^3$ on the wall, while a classical atom provides a force on average proportional to $1/Q$ according to the law of conservation of momentum. As a consequence, the kinetics of the wall behaves differently for $Q \to 0$.

IV. BOSE-EINSTEIN CONDENSATE AS THE QUANTUM SUBSYSTEM

In this section we will take a BEC as the quantum subsystem to study the kinetics of the classical subsystem. The difference between the atom and the BEC that will manifest in our study is the atom-atom interaction in the BEC. We will show in the following that this atom-atom interaction results in an additional force proportional to $Q^{-2}$ to the classical subsystem. For simplicity, we consider a BEC in one-dimensional square well with an moving wall as its boundary. The setting is the same as in Fig. 1 but the atom is replaced with a BEC. The Hamiltonian which describes such a system can be written as,

$$H = \int \psi^*(x) \hat{H}_1(x, Q) \psi(x) d^3x + H_2(P, Q),$$

where $\hat{H}_1$ denotes the Hamiltonian for the BEC, and $\psi(x)$ is the wave function of the BEC. $H_2(P, Q)$ is the Hamiltonian of the classical moving wall and $P, Q$ are its coordinate and momentum, respectively. The effective Hamiltonian of a BEC in a potential $V(x)$ takes,

$$\hat{H}_1 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + g|\psi(x, t)|^2,$$

where the potential $V(x)$ in our model is,

$$V(x) = \begin{cases} 0 & 0 \leq x \leq Q \\ +\infty & x < 0 \text{ or } x > Q \end{cases}.$$

Here $x$ stands for the coordinate of the BEC and the atom-atom coupling constant in BEC is denoted by $g$. The stationary GP equation can be written as

$$\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + g|\psi(x, t)|^2\right) \psi(x, t) = u\psi(x, t).$$

Here $u$ denotes the chemical potential. In terms of Jacobi elliptic functions, the eigenfunction for repulsive interaction $g > 0$ and attractive interaction $g < 0$ can be written as,[21, 22]

$$\varphi_{+, j}(x) = b_{+, j} \text{sn}(a_{+, j} \cdot x + \delta_{+, j}, k_{+, j}), \quad \text{for } g > 0,$$

$$\varphi_{-, j}(x) = b_{-, j} \text{cn}(a_{-, j} \cdot x + \delta_{-, j}, k_{-, j}), \quad \text{for } g < 0,$$

with

$$k_{\pm, j} = \frac{\hbar^2}{2m} |g|,$$

\[ a_{\pm, j} = \frac{\hbar^2}{2m} \sqrt{2g}. \]
where $\text{sn}$, $\text{cn}$ are the Jacobi elliptic functions and $k_{\pm,j}$ ($j = 1, 2, 3, ...$) are the modular number of the Jacobi elliptic function, $\delta_{\pm,j}$ ($j = 1, 2, 3, ...$) are constants which will be given below. We use $+$ and $-$ to denote the case of $g > 0$ and $g < 0$ respectively, and $j$ labels the eigenfunctions.

Taking the boundary conditions $\varphi_{\pm,j}(0) = \varphi_{\pm,j}(Q) = 0$ and the normalization condition $\int_0^Q |\varphi_{\pm,j}(x)|^2 dx = 1$ into account, we obtain,

$$
\begin{align*}
\delta_{+,j} &= 0, \\
a_{+,j} &= \frac{2jk_{+,j}K(k_{+,j})}{Q}, \quad j = 1, 2, 3, ... \\
b_{+,j} &= \sqrt{\frac{k_{+,j}K(k_{+,j})}{[K(k_{+,j}) - E(k_{+,j})]^2}} \\
\delta_{-,j} &= -K(k_{-,j}), \\
a_{-,j} &= \frac{2jk_{-,j}K(k_{-,j})}{Q}, \quad j = 1, 2, 3, ...
\end{align*}
$$

Specifically, for $g > 0$

$$
\mathcal{H}_{+,j} = \frac{\hbar^2}{2m \frac{Q^2}{Q^2}} \int_0^{(2j-1)K(k_{+,j})} [(k_{+,j} + 1)\text{sn}^2 z - 2k_{+,j}\text{sn}^4 z] dz
$$

and for $g < 0$

$$
\mathcal{H}_{-,j} = \frac{\hbar^2}{2m \frac{Q^2}{Q^2}} \int_{-K(k_{-,j})}^{-(2j-1)K(k_{-,j})} [1 - (2k_{-,j} + 1)\text{sn}^2 y + 2k_{-,j}\text{sn}^4 y] dy
$$

Here $z = \frac{2jk_{+,j}x}{Q}$ and $y = K(k_{-,j})(\frac{2j-1}{Q} - 1)$. Substituting these equations together with potential $V = \frac{1}{2}M\omega^2(Q - Q_0)^2$ into Eq. (3), we obtain a kinetical equation for the classical subsystem. For repulsive interaction, i.e., $g > 0$, it is

$$
\dot{Q} = \frac{C_1}{Q^3} + \frac{D_1}{Q^2} - \omega^2(Q - Q_0),
$$

with

$$
C_1 = \frac{2jh^2 k_{+,j}K(k_{+,j})^2}{Mm[K(k_{+,j}) - E(k_{+,j})]} \int_0^{(2j-1)K(k_{+,j})} [(k_{+,j} + 1)\text{sn}^2 z - 2k_{+,j}\text{sn}^4 z] dz > 0,
$$

and

$$
D_1 = \frac{gk_{+,j}K(k_{+,j})}{2jM[K(k_{+,j}) - E(k_{+,j})]^2} \int_0^{(2j-1)K(k_{+,j})} \text{sn}^4 z dz > 0.
$$

For $g < 0$,

$$
\dot{Q} = \frac{C_2}{Q^3} + \frac{D_2}{Q^2} - \omega^2(Q - Q_0),
$$

where

$$
C_2 = \frac{2jh^2 k_{-,j}K(k_{-,j})^2}{Mm[(k_{-,j}-1)K(k_{-,j}) + E(k_{-,j})]} \int_{-(k_{-,j})}^{-(2j-1)K(k_{-,j})} [1 - (2k_{-,j} + 1)\text{sn}^2 y + 2k_{-,j}\text{sn}^4 y] dy > 0,
$$

$$
b_{-,j} = \sqrt{\frac{k_{-,j}K(k_{-,j})}{[E(k_{-,j}) + (k_{-,j}) - 1]K(k_{-,j})]} Q
$$

where $K(k_{\pm,j})$ and $E(k_{\pm,j})$ are the first and the second elliptic integrals, respectively.

We now focus on the kinetic equation of the classical subsystem. In our model, it is easy to show that $\mathcal{B} = \nabla \times \mathbf{A} = 0$, and the classical Hamiltonian for the BEC takes

$$
\mathcal{H} = \int_0^Q \psi^*(x,t)\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + \frac{1}{2}g\psi^*(x,t)\psi(x,t)\right] \psi(x,t) dx.
$$
and

\[ D_2 = \frac{g k^2 \omega^2 K(k_{-j})}{2j M [(k_{-j} - 1) K(k_{-j}) + E(k_{-j})]^{3/2}} \int_{-\infty}^{(2j-1)K(k_{-j})} \text{cn}^4 y dy < 0. \]

It is easy to show that if there is no interaction between the atoms in BEC (namely, \( g = 0 \)), \( C_1 \) and \( D_i \) (\( i = 1, 2 \)) reduce to \( C_1 = C_2 = \frac{2 \pi^2 k^2}{M m} \), and \( D_1 = D_2 = 0 \). This is exactly the result given in Sec.III, where the atom was taken as the quantum subsystem. Observing Eqs. (21) and (22), we find that the classical wall experiences a force proportional to \( Q^{-3} \), which is the same as that we discussed in Sec.III. In addition to this force, a force \((\sim Q^{-2})\) inversely proportional to the square of the coordinate of the classical subsystem appears. This force is due to the interaction between atoms in the BEC, which is different from the result we discussed in Sec.III.

In Fig. 6 we plot the coordinate of the classical subsystem \( Q \) as a function of time for different \( D_1 \) and \( D_2 \). Here we choose the parameters as \( C_1 = C_2 = 0.01 \), \( \omega = 1 \), \( Q_0 = 3 \), \( Q(0) = 3.1 \) and \( \dot{Q}(0) = 0 \). In contrast, we plot the situation without atom-atom interaction as the green solid line. From the figure, we find that the equilibrium point of the classical subsystem moves right for the case of \( D_1 = 2 \) and \( D_2 = 0.5 \), whereas for the case of \( D_2 = -2 \) and \( D_2 = -0.5 \), the equilibrium position moves left. These can be understood by analyzing the dynamical equation Eqs. (21) and (22). In Eq. (21) \( M \dot{Q} \) is the resultant force of the moving wall and \( M \omega^2 (Q - Q_0) \) denotes the elastic spring force which keeps the moving wall in harmonic oscillation. Quantum subsystem brings in a rightward force \((\bar{C}_1/k)\) similar to the case in Sec.III. Another term that is inversely proportional to the square of the coordinate \((\bar{D}_2/k)\) comes from the repulsive interaction between atoms in the BEC. It results in a repulsive force for the moving wall, too. There together can explain why the equilibrium position of the wall moves right with respect to the case without atom-atom coupling. For similar reasons the equilibrium position of the wall moves left for attractive atom-atom interaction.

**V. CONCLUSION AND DISCUSSIONS**

The dynamics of quantum-classical hybrid system has been studied in this paper. Two quantum subsystems are taken to discuss the kinetics of the classical subsystem. When the quantum subsystem is an atom, the classical subsystem experiences a force \((\bar{C}_1/k)\) proportional to its distance \( Q \) to the fixed wall, meanwhile the energy of the atom has been changed because the classical subsystem provides a moving boundary, even if the atom remains in the same level in time evolution. When the quantum subsystem is a BEC, the BEC would exert an additional force proportional to \( Q^{-2} \) to the moving wall. This force comes from the atom-atom interaction in the BEC, hence it can act as a witness of the nonlinearity in BEC. With current technology, a SiN membrane of effective mass \( M = 4 \times 10^{-12} \text{Kg} \) is possible in laboratory, a vibration frequency \( 2\pi \times 1.3 \text{ MHz} \) sets the time scale of the dynamics to \( \sim 10^{-7} \text{s} \), \( m = 10^{-27} \text{Kg} \), \( M = 10^{-13} \text{Kg} \), and the eigenstate index \( n \sim 500 \) may lead to \( B \sim 0.1 \) (in units of \( (\text{Hz})^2 m^2 \)) in the first example, with which the kinetics of the classical system has been changed sharply. This estimation is conservative. In fact, the frequency of the membrane can be \( \sim \text{GHz} \), leading to a time scale \( \sim 10^{-9} \text{s} \). To our best knowledge, this is the first time to show theoretically the effect of matter-wave pressure, though the experimental observation is a challenge task.

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