A Greedy Approach to the Canny-Emiris Formula

Carles Checa
National & Kapodistrian University of Athens and Athena Research Center
Athens, Greece

Ioannis Emiris
National & Kapodistrian University of Athens and Athena Research Center
Athens, Greece

ABSTRACT
The Canny-Emiris formula [3] gives the sparse resultant as a ratio between the determinant of a Sylvester-type matrix and a minor of it, by a subdivision algorithm. The most complete proof of the formula was given by D’Andrea et al. in [9] under general conditions on the underlying mixed subdivision. Before the proof, Canny and Pedersen had proposed [5] a greedy algorithm which provides smaller matrices, in general. The goal of this paper is to give an explicit class of mixed subdivisions for the greedy approach such that the formula holds, and the dimensions of the matrices are reduced compared to the subdivision algorithm. We measure this reduction for the case when the Newton polytopes are zonotopes generated by \( n \) line segments (where \( n \) is the rank of the underlying lattice), and for the case of multihomogeneous systems. This article comes with a JULA implementation of the treated cases.

CCS CONCEPTS
• Mathematics of computing → Solvers; Combinatorial algorithms.

KEYWORDS
Combinatorics, resultant theory, mixed subdivision, computer algebra, zonotopes

ACM Reference Format:
Carles Checa and Ioannis Emiris. 2022. A Greedy Approach to the Canny-Emiris Formula. In Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation (ISSAC ’22), July 4–7, 2022, Villeneuve-d’Asq, France. ACM, New York, NY, USA, 9 pages. https://doi.org/10.1145/3476446.3536180

1 INTRODUCTION
Sparse resultants offer a standard and efficient way of studying algebraic systems while exploiting their structure. We examine matrix-based methods for expressing and computing this resultant.

The Canny-Emiris formula was conjectured in [3] as a rational formula for the sparse resultant that generalizes Macaulay’s classic formula in [15]. It gives a combinatorial construction of a Sylvester-type matrix \( \mathcal{H}_{\mathcal{A}, \rho} \) depending on the family of supports \( \mathcal{A} = (\mathcal{A}_0, \ldots, \mathcal{A}_n) \) in a lattice \( M \) of rank \( n \), and a mixed subdivision \( \rho \) defined from a lifting function \( \rho \) on the Minkowski sum \( \Delta \) of the Newton polytopes \( \Delta_i = \text{conv}(\mathcal{A}_i) \).

Each row of this matrix corresponds to a lattice point \( b \in M \) contained in a translation \( \delta \) of the polytope \( \Delta \). Moreover, to each lattice point we can associate a type vector \( t_b = (t_{b,0}, \ldots, t_{b,n}) \) corresponding to the dimensions of the components \( \Delta_i \subset \Delta \) of the cell \( D \in S(\rho) \) in which \( b \) is lying. Imposing that
\[
\sum_{i=0}^{n} t_{b,i} = n \quad \forall b \in (\Delta + \delta) \cap M,
\]
it is possible to build such matrix \( \mathcal{H}_{\mathcal{A}, \rho} \) and a principal submatrix \( E_{\mathcal{A}, \rho} \) of \( \mathcal{H}_{\mathcal{A}, \rho} \) so that the sparse resultant is:
\[
\text{Res}_{\mathcal{A}} \equiv \frac{\det(\mathcal{H}_{\mathcal{A}, \rho})}{\det(E_{\mathcal{A}, \rho})}.
\]

D’Andrea, Jerónimo and Sombra proved this formula in [9] under the assumption that \( S(\rho) \) admits an incremental chain of mixed subdivisions:
\[
S(\rho_0) \leq \cdots \leq S(\rho_n) \leq S(\rho)
\]
satisfying some combinatorial properties (where \( \leq \) denotes refinement). This proof extended the first proof given in [7] for generalized unmixed systems.

On the other hand, Canny and Pedersen gave in [5] another approach for the construction of the matrix \( \mathcal{H}_{\mathcal{A}, \rho} \). Starting at a lattice point \( b \in M \), one can construct the matrix by only adding the rows corresponding to the columns that have a nonzero entry in a previously considered row. This is a greedy way of understanding these matrices: their construction only considers the strictly necessary rows and columns given the mixed subdivision \( S(\rho) \).

The goal of this paper is to give a family of lifting functions \( \rho \) that correspond to mixed subdivisions for which: i) the proof of the formula in [9] holds and ii) the size of the matrices is reduced. This family is associated with a vector \( v \in \text{Hom}(M_\mathbb{Z}, \mathbb{R}) \) outside the hyperplane arrangement associated with the polytope \( \Delta \).

We expect this family of lifting functions to reduce the size of the Canny-Emiris matrices obtained by the greedy algorithm for a general sparse system. In this paper, we measure this reduction only for the case where the Newton polytopes are zonotopes generated by \( n \) independent line segments. Namely, we simplify the computations to the case where the supports \( \mathcal{A}_0, \ldots, \mathcal{A}_n \) are:
\[
\mathcal{A}_i = \{(b_j)_{j=1, \ldots, n} \in \mathbb{Z}^n \mid 0 \leq b_j \leq a_{ij} \} \quad i = 0, \ldots, n
\]
assuming that \( 0 < a_{ij} \leq \cdots \leq a_{n-1,j} \) for all \( j = 1, \ldots, n \). The main results Thm. 3.1 and Thm. 3.2 show that the greedy algorithm will
where \([2, 1, 12, 19, 4]\) for some of these cases, we expect our approach to have an easier generalization to general sparse systems through the use of the type functions and the underlying combinatorics.

In order to find all the lattice points in cells with a given type vector, we introduce the type functions:

\[\varphi_b : \{1, \ldots, n\} \rightarrow \{0, \ldots, n\} \quad t_{b,i} = |\varphi_b^{-1}(i)|.\]

These combinatorial objects contain all the information of the cells of the mixed subdivision and can help us construct the matrices. In Cor. 3.3, we give a combinatorial measure of the number of rows of the matrix \(H\), given by the greedy algorithm as:

\[\sum_{\varphi_b : \{1, \ldots, n\} \rightarrow \{0, \ldots, n\}} \prod_{j=1}^n a_{\varphi(b)(j)}\]

where \(\varphi_b\) satisfies \(|\varphi_b^{-1}(0, \ldots, I)| \leq I + 1\).

We show that some multihomogeneous resultant matrices can be seen as an instance of the previous case by embedding their Newton polytopes and the mixed subdivisions of their Minkowski sum into an \(n\)-zonotope. Moreover, we add restrictions to the type functions so that they also count the size of these matrices (Cor. 4.1). Despite the existence of many exact determinantal formulas ([2, 1, 12, 19, 4]) for some of these cases, we expect our approach to have a simpler generalization to general sparse systems through the use of the type functions and the underlying combinatorics.

The paper is organized as follows. In the Sec. 1, we summarize the proof of the Canny-Emiris formula in [9] and we explain the greedy approach in [5]. In Sec. 2, a concrete family of mixed subdivisions is given by considering a hyperplane arrangement associated with the Newton polytopes and it is proved that the Canny-Emiris matrices hold in this case. Sec. 3 is devoted to combinatorially finding the size of the Canny-Emiris matrices when the Newton polytopes are zonotopes generated by \(n\) line segments. In Sec. 4, the multihomogeneous resultant is seen as an instance of the previous case. Moreover, we count the number of lattice points that the greedy approach gives on these mixed subdivisions in a combinatorial way. In the conclusions, we present a list of possible ways to generalize the Canny-Emiris matrices to other sparse systems.

This article comes with a JULIA implementation of the treated cases (zonotopes and multihomogeneous systems) that can be found in (https://github.com/carleschecanualart/CannyEmiris). More than improving the existing formulas (which, in general, smaller Sylvester matrices), the goal of this implementation is to introduce the type functions in the construction of the matrices.

### 1.1 The Canny-Emiris formula

Let \(M\) be a lattice of rank \(n\) and \(M_\mathbb{R} = M \otimes \mathbb{R}\) the corresponding real vector space. Let \(N = \text{Hom}(M, \mathbb{Z})\) be its dual and \(\mathbb{T}_N = N \otimes \mathbb{C}^\times\) the underlying torus. Let \(\mathcal{A}_0, \ldots, \mathcal{A}_n \subset M\) be a family of supports corresponding to the polynomials:

\[F_i = \sum_{a \in \mathcal{A}_i} u_{i,a} x^a \in \mathbb{Z}[u_{i,a}] \quad a \in \mathcal{A}_i \quad i = 0, \ldots, n\]

where \(x^a\) are the characters in \(\mathbb{T}_N\) of the lattice points \(a \in \mathcal{A}_i\). Let \(\Delta_i = \text{conv} (\mathcal{A}_i) \subset M_\mathbb{R}\) for \(i = 0, \ldots, n\) be the convex hulls of the supports, also known as Newton polytopes, and \(\Delta\) their Minkowski sum in \(M_\mathbb{R}\).

The incidence variety \(Z(F)\) is defined as the zero set in \(\mathbb{T}_F \times \prod_{i=0}^n \mathbb{P}^\Delta_i\) of the polynomials \(F = (F_0, \ldots, F_n)\). Denote by \(\pi : \mathbb{T}_F \times \prod_{i=0}^n \mathbb{P}^\Delta_i \rightarrow \prod_{i=0}^n \mathbb{P}^\Delta_i\) the projection onto the second factor and let \(\pi_c(Z(F))\) be the direct image of the zero set of \(F_0, \ldots, F_n\).

**Definition 1.1.** The sparse resultant, denoted as \(\text{Res}_{\mathcal{A}}\), is any primitive polynomial in \(\mathbb{Z}[u_{i,a}]\) defining the direct image \(\pi_c(Z(F))\).

There are some lattice operations that can help us simplify the computation of these objects.

**Lemma 1.1.** [9, Prop. 3.2] Let \(\phi : M \rightarrow M'\) be a monomorphism of lattices of rank \(n\). Then, \(\text{Res}_{\phi(\mathcal{A})} = \text{Res}_{\mathcal{A}}^{\phi(M)}\).

**Remark 1.1.** Moreover, the sparse resultant is invariant under translations. Therefore, we can always assume \(0 \in \mathcal{A}_i\) for all \(i = 0, \ldots, n\).

**Definition 1.2.** A mixed subdivision of \(\Delta\) is a decomposition of this polytope in a union of cells \(\Delta = \sqcup \Delta_i\) such that: i) the intersection of two cells is either a cell or empty, ii) every face of a cell is also a cell of the subdivision, iii) every cell \(D\) has a component structure \(D = D_0 + \cdots + D_n\) where \(D_i\) is a cell of the subdivision in \(\Delta_i\).

The usual way to construct mixed subdivisions is by considering piecewise affine convex lifting functions \(\rho_i : \Delta_i \rightarrow \mathbb{R}\) as explained in [14]. A global lifting function \(\rho : \Delta \rightarrow \mathbb{R}\) is obtained after taking the inf-convolution of the previous functions, as explained in [9, Sec. 2].

**Definition 1.3.** A mixed subdivision of \(\Delta\) is tight if, for every \(n\)-cell \(D\), its components satisfy:

\[\sum_{i=0}^n \dim D_i = n.\]

In the case of \(n+1\) polynomials and \(n\) variables, this property guarantees that every \(n\)-cell has a component that is 0-dimensional. The cells that have a single 0-dimensional component are called mixed (i-mixed if it is the \(i\)-th component). The rest of the cells are called non-mixed.

Let \(\delta\) be a generic vector such that the lattice points in \(\Delta + \delta\) lie in \(n\)-cells. Then, consider:

\[\mathcal{B} = (\Delta + \delta) \cap M.\]

Each element \(b\) in \(\mathcal{B}\) lies in one of these translated cells \(D + \delta\) and let \(D_i\) be the components of this cell. As the subdivision is tight, there is at least one \(i\) such that \(\dim D_i = 0\).

Following the language of [16], we call \(t_b = (t_{b,0}, \ldots, t_{b,n})\) the type vector associated with \(b\) defined as \(t_{b,i} = \dim D_i\) for \(b \in D + \delta\).
Definition 1.4. The row content is a function
\[ rc : B \to \bigcup_{i=1}^n \{ i \} \times A_i \]
where for \( b \in B \) lying in an \( n \)-cell \( D, \) \( rc(b) \) is a pair \((i(b), a(b))\)
with \( i(b) = \max\{ i \mid \theta_{b,i} = 0 \} \) and \( a(b) = D_{i(b)}. \)

This provides a partition of \( B \) into subsets:
\[ B_i = \{ b \in B \mid i(b) = i \}. \]

Finally, we construct the Canny-Emiris matrices \( \mathcal{H}_{A,\rho} \) whose rows correspond to the coefficients of the polynomials \( \chi^{b-a(b)} F_{i(b)} \) for each of the \( b \in B. \) In particular, the entry corresponding to a pair \( b, b' \in B \) is:
\[ \mathcal{H}_{A,\rho}[b, b'] = \begin{cases} u_{i(b)} b' - b + a(b) \in A_i \\ 0 \end{cases} \]

Remark 1.2. Each entry contains, at most, a single coefficient \( u_{i,a} \). In particular, the row content allows us to choose a maximal submatrix of \( \mathcal{H}_{A,\rho} \) from the matrix of a map sending a tuple of polynomials \( (G_0, \ldots, G_n) \) to \( G_0 F_0 + \cdots + G_n F_n. \) These class of matrices are called Sylvester-type matrices.

Let \( C \subset B \) be a subset of the supports in translated cells. The matrix \( \mathcal{H}_{A,\rho,C} \) is defined by considering the submatrix of the corresponding rows and columns associated with elements in \( C. \) In particular, we look at the set of lattice points lying in translated non-mixed cells and consider:
\[ B^0 = \{ b \in B \mid b \text{ lies in a translated non-mixed cell} \} \]

With this, we form the principal submatrix:
\[ E_{A,\rho} = \mathcal{H}_{A,\rho,B^0} \]

The Canny-Emiris conjecture states that the sparse resultant is the quotient of the determinants of these two matrices:
\[ \text{Res}_A = \frac{\det(\mathcal{H}_{A,\rho})}{\det(E_{A,\rho})}. \]

This result was conjectured by Canny and Emiris [3] and proved by D’Andrea, Jerónimo, and Sombra [9] under the restriction that the mixed subdivision \( S(\rho) \) given by the lifting \( \rho \) satisfies a certain condition. This condition is given on a chain of mixed subdivisions.

Definition 1.5. Let \( S(\phi), S(\psi) \) be two mixed subdivisions of \( \Delta = (\Delta_0, \ldots, \Delta_n). \) We say that \( S(\psi) \) refines \( S(\phi) \) and write \( S(\phi) \leq S(\psi) \) if for every cell \( C \in S(\psi) \) there is a cell \( D \in S(\phi) \) such that \( C \subset D. \) An incremental chain of mixed subdivisions \( S(\theta_0) \leq \cdots \leq S(\theta_n) \) is a chain of mixed subdivisions of \( (\Delta_0, \ldots, \Delta_n) \) refining each other.

Remark 1.3. In [9, Def. 2.4], a common lifting function \( \omega \in \prod_{i=1}^n \mathbb{R}^{A_i} \) is considered and the \( S(\theta_i) \) are given by the lifting functions \( \omega^{S(\theta_i)} = (\omega_{\theta_0}, \ldots, \omega_{\theta_i-1}, 0) \) as long as \( S(\theta_i) \leq S(\theta_{i+1}) \). The last zero represents the lifting on \( (\Delta_i, \ldots, \Delta_n) \). The resulting mixed subdivision is the same as if we considered the zero lifting in \( \Sigma_{i=1}^n \Delta_i \).

Definition 1.6. The mixed volume of \( n \) polytopes \( P_1, \ldots, P_n \subset \mathbb{R}^d \), denoted as \( \text{MV}(P_1, \ldots, P_n) \), is the coefficient of \( \prod_{i=1}^n \lambda_i \) in:
\[ \text{Vol}_n(a_1 P_1 + \cdots + a_n P_n), \]
which is a polynomial in \( \lambda_1, \ldots, \lambda_n \) [6, Theorem 6.7].

Proposition 1.1. [13, Thm. 3.4] Let \( S(\rho) \) be a tight mixed subdivision of \( \Delta = (\Delta_0, \ldots, \Delta_n). \) For \( i = 0, \ldots, n, \) the mixed volume of all the polytopes except \( \Delta_i \) equals the volume of the i-mixed cells.
\[ \text{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) = \sum_{D \text{ i-mixed}} \text{Vol}_n D \]

In particular, \( \text{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) \) equals the degree of the sparse resultant in the coefficients of \( F_i \) (6, Chap. 7, Thm. 6.3)). Each of the rows of \( \mathcal{H}_{A,\rho} \) will correspond to a lattice point \( b \) and each entry on that row will have degree 1 with respect to the coefficients of \( F_{i(b)} \) and zero with respect to the coefficients of the rest of polynomials. Therefore, if we add the lattice points in i-mixed cells, the degree of \( \mathcal{H}_{A,\rho} \) with respect to the coefficients of \( F_i \) will be at least the degree of the resultant with respect to the same coefficients.

Definition 1.7. [9, Def. 3.4] The fundamental subfamily of \( A \) is the family of supports \( A_i = (A_i)_{i \in \mathbb{I}} \) such that the resultant has positive degree with respect to the coefficients of \( F_i \) for \( i \in \mathbb{I}. \) This definition can be given in other equivalent terms as shown in [18, Cor. 1.1].

Remark 1.4. Using Prop. 1.1, we can see that if the fundamental subfamily is empty, then the resultant is equal to 1 while if the fundamental subfamily is \( \{ i \} \) then \( A_i \) is given by a single point \( (a) \) and the resultant is \( u_{i,a}^m \) for \( m_i = \text{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n). \) The Canny-Emiris formula holds naturally [9, Prop. 4.26] in both two cases.

Definition 1.8. An incremental chain \( S(\theta_0) \leq \cdots \leq S(\theta_n) \) is admissible if for each \( i = 0, \ldots, n, \) each \( n \)-cell \( D \) of the subdivision \( S(\theta_i) \) satisfies either (i) the fundamental subfamily of \( A_D \) contains at most one support, or (ii) \( B_{D,i} \) is contained in the union of the translated i-mixed cells of \( S(\rho_D). \) A mixed subdivision \( S(\rho) \) is called admissible if it admits an admissible incremental chain \( S(\theta_0) \leq \cdots \leq S(\theta_n) \leq S(\rho) \) refining it.

With all these properties, together with the use of the product formulas ([18],[10],[9]), one can reproduce the proof of the Canny-Emiris formula given in [9, Thm. 4.27] under the conditions of admissibility in \( S(\rho). \)

1.2 The greedy algorithm

Using the previous notation, we state the greedy algorithm in [5] for the construction of the matrix. Let \( b \in B \) be a lattice point in a translated cell. The first step of the algorithm is to add the row of the matrix corresponding to \( b, \) and then continue by considering the lattice points corresponding to the columns that have a nonzero entry in this row. These lattice points are:
\[ b - a(b) + A_i(b). \]

All these lattice points will have to be added as rows of the matrix. If we add the lattice point \( b' \) after another lattice point \( b, \) we say that we reach \( b' \) from \( b. \) The algorithm terminates when there are no more lattice points to add and it might give a matrix \( \mathcal{H}_G \) which has smaller dimensions than the matrix \( \mathcal{H}_{A,\rho} \) constructed by using all the lattice points in \( B. \) This algorithm also provides a minor \( E_{A,\rho} \) considering the rows and columns in \( \mathcal{H}_G \) corresponding to lattice
points in non-mixed cells.

It was not proved by Canny and Pedersen whether this approach would always include all the lattice points in mixed cells as rows of the matrix, independently of the starting point. As these points are necessary to achieve the degree of the resultant (Prop. 1.1), we consider them to be the starting points of the algorithm.

**Remark 1.5.** We know that the entry corresponding to the diagonal of the matrix $\mathcal{H}_{\mathcal{A},\rho,C}$ will be $\prod_{b \in C} \mathcal{u}(b, a(b))$ for any subset $C \subset B$. This term can be used in order to deduce that these matrices have non-zero determinant [9, Prop. 4.13].

**Theorem 1.1.** If the Canny-Emiris formula holds for a mixed subdivision $S(\rho)$ and the greedy approach gives matrices $\mathcal{H}_G$ and $\mathcal{E}_G$, then:

$$\text{Res}(\mathcal{A}) = \frac{\det(\mathcal{H}_G)}{\det(\mathcal{E}_G)}$$

**Proof.** In general, there is a subset $G \subset B$ corresponding to the rows and columns of $\mathcal{H}_G$. We assume that $G$ contains all the lattice points in translated mixed cells.

Let $\mathcal{H}_{\mathcal{A},\rho}$ be the matrix containing all lattice points in translated cells of $S(\rho)$. Without loss of generality, we can assume that the matrix takes the following form:

$$\mathcal{H}_{\mathcal{A},\rho} = \begin{pmatrix} \mathcal{H}_G & 0 \\ \mathcal{H}_{B-G} \end{pmatrix}$$

where $\mathcal{H}_G$ is the minor corresponding to the lattice points in $G$ and $\mathcal{H}_{B-G}$ is the minor corresponding to the lattice points not in $G$. The zeros appear due to the fact that there is no pair $b \not\in G$, $b' \in G$ such that $b = b' - a(b') + a(b)$. The same block-triangular structure also appears in the principal submatrix $\mathcal{E}_{\mathcal{A},\rho}$ and all the lattice points that are not in $G$ must be non-mixed, implying that $\mathcal{E}_{B-G} = \mathcal{H}_{B-G}$.

Finally, using the fact that the determinant of a block-triangular matrix is the product of the determinants of the diagonal blocks, we can prove the resultant formula:

$$\text{Res}(\mathcal{A}) = \frac{\det(\mathcal{H}_{\mathcal{A},\rho})}{\det(\mathcal{E}_{\mathcal{A},\rho})} = \frac{\det(\mathcal{H}_G)}{\det(\mathcal{E}_G)} \frac{\det(\mathcal{H}_{B-G})}{\det(\mathcal{E}_{B-G})} = \frac{\det(\mathcal{H}_G)}{\det(\mathcal{E}_G)}$$

**Example 1.1.** Let $f_0, f_1, f_2$ be three bilinear equations corresponding to the supports $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$. A possible mixed subdivision $S(\rho)$ is the following:

```
[ ][ ][ ]
[ ][ ][ ]
[ ] [ ] [ ]
```

where the dots indicate the lattice points in translated mixed cells. The number of lattice points in translated cells is 9. However, if we construct the matrix greedily starting from the lattice points in translated mixed cells, we have an $8 \times 8$ matrix.

2 A FAMILY OF LIFTING FUNCTIONS

In this section, we give a family of lifting functions associated to the polytopes $\Delta_0, \ldots, \Delta_8$ and we prove the Canny-Emiris formula holds for the corresponding mixed subdivisions.

**Definition 2.1.** We can define a hyperplane arrangement $\mathcal{H} \subset \mathbb{N}^n_k$ by considering the span of the $(n - 1)$-dimensional cones of the normal fan of $\Delta$ ([20] for more on polytopes and hyperplane arrangements).

**Example 2.1.** A polytope $\Delta$, together with its normal fan (blue) and the hyperplane arrangement $\mathcal{H}_\Delta$ (red).

```
[ ] [ ] [ ]
[ ] [ ] [ ]
[ ] [ ] [ ]
```

**Definition 2.2.** Let $\mathcal{H}$ be the hyperplane arrangement associated to $\Delta$ and take a vector $v \in \mathbb{N}^n_k$ which does not lie in $\mathcal{H}$. We consider lifting functions $\omega_i : \mathcal{A}_i \rightarrow \mathbb{R}$ defined as:

$$\omega_i(x) = \lambda_i(x, x) \quad i = 0, \ldots, n \quad x \in \Delta_i$$

for $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ satisfying $\lambda_0 > \cdots > \lambda_n \geq 0$ and small enough. Let $\rho = (\omega_0, \ldots, \omega_n)$ be a lifting giving a mixed subdivision $S(\rho)$.

**Theorem 2.1.** $S(\rho)$ is an admissible mixed subdivision.

**Proof.** Let $S(\theta_i)$ be the mixed subdivision obtained from $\theta_i = (\omega_0, \ldots, \omega_{i-1}, 0)$. Using [9, Prop. 2.11], for each $i = 0, \ldots, n$, there is an open neighborhood of $0 \in U \subset \mathbb{R}$ such that for $\omega_i \in U$ we have $S(\theta_i) \subset S(\theta_{i+1})$. For $\lambda_i > \lambda_{i+1} > 0$ small enough, $\omega_i$ lies in $U$. Therefore, the $(\theta_i)$ form an incremental chain.

All the lattice points with row content 0 are 0-mixed. Therefore, $S(\theta_i)$ satisfies ii) in Definition 1.8. Let $D$ be an n-cell of $S(\theta_i)$. If $\dim D_t = 0$, then the fundamental subfamily of $\mathcal{A}_D$ is at most $\{i\}$ as shown in Rem. 1.4. We show that, for our choice of the lifting function, the rest of cells $D$ satisfy ii) in Definition 1.8.

Let $D \in S(\theta_i)$ such that $\dim D_t > 0$. Suppose that this cell contains a lattice point $b \in B$ that has row content $i$ but is not $i$-mixed. Therefore, this lattice point $b$ will be in a cell of $S(\rho)$ with a 0-dimensional $j$-th component for some $j < i$. Take $C \supset D$ in $S(\theta_j)$ containing the previous lattice point $b$. If $\dim C_j > 0$, then the lifting function $\omega_j = \lambda_j(x, x)$ takes the same value in all the points of $C_j$. Therefore, the vector $e$ is normal to $C_j$ and has to be contained in the hyperplane arrangement associated to $\Delta$. As this is not the case, $\dim C_j = 0$ and consequently $\dim D_j = 0$, contradicting the initial hypothesis.

This result can be extended to any chain $\lambda_0 > \lambda_1 > \cdots > \lambda_n \geq 0$. The details of such proof were added in an appendix in the arxiv version of this text.

**Remark 2.1.** This choice of the lifting function can also be seen as a case of the approach of [7], in a first proof of the rational formula for generalized unmixed systems.

3 THE CASE OF n-ZONOTOPES

For simplicity, we suppose that our lattice is $\mathbb{Z}^n$. 

286
Definition 3.1. A zonotope is a polytope given as a sum of line segments. An n-zonotope is generated by n line segments, which span a lattice of dimension n.

Consider linearly independent \( v_1, \ldots, v_n \in \mathbb{Z}^n \) and the line segments \( \overline{v_i v_j} \subset \mathbb{R}^n \) forming an n-zonotope \( Z \subset \mathbb{R}^n \). If the Newton polytopes are n-zonotopes whose defining line segments are integer multiples of the \( \overline{v_i v_j} \), we can write the supports of the system as:

\[
\mathcal{A}_j^i = \left\{ \sum_{j=1}^{n} \lambda_j v_j \in \mathbb{Z}^n \mid \lambda_j \in \mathbb{Z}, \ 0 \leq \lambda_j \leq a_{ij} \right\},
\]

for some \( a_{ij} \in \mathbb{Z}_{>0} \). Let V be the nonsingular matrix whose columns are the coordinates of the \( v_i \) in the canonical basis of \( \mathbb{Z}^n \) for \( j = 1, \ldots, n \) and consider it as a monomorphism of lattices \( V: \mathbb{Z}^n \to \mathbb{Z}^n \) of rank n. Let \( e_1, \ldots, e_n \) be the canonical basis of \( \mathbb{Z}^n \).

Corollary 3.1. Let \( \mathcal{A}_1', \ldots, \mathcal{A}_n' \) be the previous family of supports, then \( \text{Res}_{\mathcal{A}} = \text{Res}_{\mathcal{A}'} \), where:

\[
\mathcal{A}_i = \left\{ (b_j)_{j=1, \ldots, n} \in \mathbb{Z}^n \mid 0 \leq b_j \leq a_{ij} \right\}, \quad i = 0, \ldots, n.
\]

Proof. Using Lemma 1.1, we can view the map \( V: \mathbb{Z}^n \to \mathbb{Z}^n \) as a monomorphism of lattices sending the canonical basis \( e_j \) to \( v_i \) for \( i = 1, \ldots, n \). The absolute value of the determinant \(|\text{det}(V)|\) is the index of the image. This last result happens after the reduction of V to its Smith normal form [17, Thm. 2.3]. □

Remark 3.1. The normal vectors of the n-zonotope are given by n pairs \((\eta_j, -\eta_j)_{j=1, \ldots, n} \in \mathcal{N} \). The results that follow in this section can be proved without using Cor. 3.1, after changing bj by \((b, \eta_j)\) for \( j = 1, \ldots, n \), choosing \( \eta_j \) to be the element in the pair such that \( 0 \leq (b, \eta_j) \leq a_{ij} \).

In order to prove our results, we assume that the \( a_{ij} \) are ordered, meaning that \( 0 < a_{ij} \leq a_{ij} \leq \cdots \leq a_{n-j} \) for \( j = 1, \ldots, n \), where we exclude \( \mathcal{A}_0 \) from this assumption. Consider a translation \( \delta \in \mathbb{R}^n \) which is negative in each component and small enough. Then, the lattice points in translated cells of a mixed subdivision of the previous system are:

\[
\mathcal{B} = \left\{ (b_j)_{j=1, \ldots, n} \in \mathbb{Z}^n \mid 0 \leq b_j < \sum_{k=0}^{n} a_{ij} \right\}.
\]

Let \( v \not\in \cup_{i=1}^{n} \{ x_j = 0 \} \) be the mixed subdivision \( S(p) \) as in the previous section. We define \( v \) as the row content. For \( j = 1, \ldots, n \), and get the following result.

Proposition 3.1. Let \( b \in \mathcal{B} \) and \( i \in \{0, \ldots, n\} \). Then:

\[
t_{b,i} = \left\{ j \in \{1, \ldots, n\} \mid \sum_{k=0}^{i-1} a_k b_k \leq b_j < \sum_{k=0}^{i} a_k b_k \right\}
\]

and the row content \( i(b) \) is the maximum index in \( \{0, \ldots, n\} \) such that:

\[
D_j \in \mathcal{N} = \sum_{k=0}^{i(b)-1} a_k b_k \leq b_j < \sum_{k=0}^{i(b)} a_k b_k.
\]

with the support \( a(b) \in \mathcal{A}_{i(b)} \) satisfying:

\[
a(b)_j = \begin{cases} 0 & b_j < \sum_{k=0}^{i(b)-1} a_k b_k, \\ a_{i(b)} b_j & b_j \geq \sum_{k=0}^{i(b)} a_k b_k. \end{cases}
\]

Proof. It can be deduced from Def. 2.2 that the mixed subdivision \( S(\theta_i) \) is given by the lifting function \( \theta_i = \sum_{j=1}^{n} \theta_{ij} \) where the lifting in the component \( x_j \) only depends on \( \theta_{ij} \). As we said in Def. 1.2, each of the \( \theta_{ij} \) are given as the inf-convolution of the lifting function in each of the polytopes. The first non-trivial lifting function is \( S(\theta_0) \): it is piecewise linear and by Rem. 1.3 depends only on the lifting given in \( \Delta_0 \). Therefore, after taking the inf-convolution, we get:

\[
\theta_{0,j} = \begin{cases} 0 & \{ x_j = 0 \}, \\ \lambda_0 v_j & \{ x_j = a_{0,j} \}, \\ \lambda_0 v_j & \{ x_j = \sum_{k=0}^{a_{ii}} a_{0,k} \}. \end{cases}
\]

This structure is guaranteed by the fact that \( \lambda_0 v_j < 0 \). One can see that \( S(\theta_i) \) has two types of cells \( D \) with respect to the j-th coordinates: those in which \( \theta_{ij} v_j \in D_0 \) which have to be in \( x_j < a_{ij} \) for \( x \in D \) and those with \( \theta_{ij} v_j \not\in D_0 \) in \( x_j \geq a_{ij} \). In terms of the lattice points, as we are considering \( \delta_i < 0 \), those \( b \in \mathcal{B} \) lying in a cell \( D \) with \( \theta_{ij} v_j \in D_0 \) satisfy that \( 0 \leq b_j < a_{ij} \) and the rest satisfy \( b_j \geq a_{ij} \). As a consequence, \( t_{b,0} \) must be the number of \( j \in \{1, \ldots, n\} \) such that \( 0 \leq b_j < a_{ij} \).

Consider now the structure of the mixed subdivision \( S(\theta_{i+1}) \) with respect to the j-th coordinate. Using the inf-convolution, we get:

\[
\theta_{ij} = \begin{cases} 0 & \{ x_j = 0 \}, \\ \lambda_0 v_j & \{ x_j = a_{0,j} \}, \\ (\lambda_0 + \lambda_i) v_j & \{ x_j = a_{0,j} + a_{1,j} \}, \\ \cdots \\ (\lambda_0 + \cdots + \lambda_{i-1}) v_j & \{ x_j = \sum_{k=0}^{i-1} a_{ij} \}, \\ (\lambda_0 + \cdots + \lambda_{i-1}) v_j & \{ x_j = \sum_{k=0}^{i} a_{ij} \}. \end{cases}
\]

Again, this structure is guaranteed by \( \lambda_0 v_j < 0 \) and by \( \lambda_i > \lambda_i \).

With the same argument as before, the cells in which \( \theta_{ij} v_j \not\in D_i \) are precisely those that satisfy \( \sum_{k=0}^{i-1} a_k b_k \leq b_j < \sum_{k=0}^{i} a_k b_k \) for \( b \in D \). As before, this also proves that \( t_{b,j} \) is the number of \( j \in \{1, \ldots, n\} \) such that \( \sum_{k=0}^{i-1} a_k b_k \leq b_j < \sum_{k=0}^{i} a_k b_k \). We refer to Fig. 1 in A for an explicit example of this construction.

The second claim follows from the definition of row content with respect to the type vector \( t_b \). Let \( b \in \mathcal{B} \) and let \( i(b) \) be its row content. For \( j = 1, \ldots, n \), we either have \( b_j \leq \sum_{k=0}^{i(b)-1} a_k b_k \) or \( b_j \geq \sum_{k=0}^{i(b)} a_k b_k \). In the first case, the vertex associated to the row content, will be in the face of \( D_i(b) \) defined by the equality \( \{ x_j = 0 \} \) and in the second case, the one defined by the equality \( \{ x_j = a_{i(b)} \} \). □

Remark 3.2. If \( v_j > 0 \), we would change the inequalities by \( \sum_{k=0}^{i(b)} a_k b_k \leq b_j < \sum_{k=0}^{i(b)} a_k b_k \) but the results that follow would not change. Any other mixed subdivisions of this particular system can also be formed this way.

Definition 3.2. The type function \( \varphi_b : \{1, \ldots, n\} \to \{0, \ldots, n\} \) associated to each lattice point \( b \in \mathcal{B} \) is defined as the vector of indices satisfying:

\[
\sum_{k=0}^{i(b)-1} a_k b_k \leq b_j < \sum_{k=0}^{i(b)} a_k b_k.
\]
Following Prop. 3.1, it satisfies that $t_{b,i} = \left\lfloor \frac{1}{a_{b,i}} \right\rfloor$.

From the components of $a(b)$ in Prop. 3.1, we deduce that the range of values for $(b - a(b) + \mathcal{A}_{i(b)})$ is:

$$
\begin{cases}
[k, k + a_i(b)] & b > \sum_{i=0}^{k} a_i(b) \\
[k - a_i(b), k] & b < \sum_{i=0}^{k} a_i(b)
\end{cases}
$$

**Corollary 3.2.** The range of possible type functions for $b' \in b - a(b) + \mathcal{A}_{i(b)}$ are:

$$
\varphi_{b'}(j) \in \begin{cases}
\varphi_{b}(j) - 1, \varphi_{b}(j) & i(b) < \varphi_{b}(j) \\
\varphi_{b}(j), \ldots, 1, (b) & i(b) > \varphi_{b}(j)
\end{cases}
$$

**Proof.** Take $I$ to be the index such that $\sum_{i=0}^{I-1} a_k < b - \sum_{i=0}^{I} a_k$ and we get the inequalities:

$$
\begin{cases}
b_j - a_i(b) \geq \sum_{i=0}^{I-1} a_k & i(b) < I \\
b_j + a_i(b) < \sum_{i=0}^{I} a_k & i(b) > I
\end{cases}
$$

In the first row, we used that $a_i(b) < a_{I-I}$.

**Definition 3.3.** We define the greedy subset $G \subset B$ to be formed by all the lattice points $b \in B$ such that:

$$
\sum_{i=0}^{I} t_{b,i} \leq I + 1 \quad \forall I < n.
$$

**Theorem 3.1.** Let $b \in G$ and $b' \notin G$. Then, $b' \notin b - a(b) + \mathcal{A}_{i(b)}$.

**Proof.** Let $I$ be the greatest index such that $\sum_{i=0}^{I} t_{b,i} > I + 1$. As it is the greatest, we must have $t_{b,i} = 0$ and $\sum_{i=0}^{I} t_{b,i} < n - I - 1$.

On the other hand, $\sum_{i=0}^{I} t_{b,i} \leq I + 1$. Using Cor. 3.2, the previous sum cannot grow in $b - a(b) + \mathcal{A}_{i(b)}$ when $i(b) > I$. If $\sum_{i=0}^{I} t_{b,i} = I + 1$, then $\sum_{i=0}^{I} t_{b,i} < n - I$ which implies that there is $i > I$ with $t_{b,i} = 0$ and $i(b) > I$.

Suppose $\sum_{i=0}^{I} t_{b,i} < I + 1$ and $i(b) < I$. Using Cor. 3.2, we have:

$$
\sum_{i=I+1}^{n} t_{b,i} \geq n - I \quad \text{and} \quad \sum_{i=I+1}^{n} t_{b,i} \geq n - I - 1
$$

for $b' b - a(b) + \mathcal{A}_{i(b)}$. Therefore,

$$
\sum_{i=I+2}^{n} t_{b,i} < n - I - 1 \leq \sum_{i=I+1}^{n} t_{b,i}
$$

meaning that it is not possible that $b'$ has a type function on the range of $b - a(b) + \mathcal{A}_{i(b)}$.

**Definition 3.4.** Let $I_b \in \{0, \ldots, n\}$ be the index satisfying:

$$
I_b = \max \{i \in \{0, \ldots, n\} \mid t_{b,i} \geq 2\}
$$

$b$ lies in a non-mixed cell

$b$ lies in a mixed cell

Let $g_b = \{i < I_b \mid t_{b,i} = 0\}$ be the number of zeros that $t_{b,i}$ has before $I_b$.

**Lemma 3.1.** Let $b \in G$ and suppose that $g_b = 0$. Then, $b$ lies in a mixed cell.

**Proof.** Suppose that $b$ lies in a non-mixed cell. This would mean that there is no zero before $I_b$ implying that $\sum_{i=0}^{I_b} t_{b,i} = \sum_{i=0}^{I_b} t_{b,i} + t_{b,i} \geq I_b + 2$.

**Lemma 3.2.** If $t_{b,i} = 0$ and $b \in G$, $\sum_{i=0}^{I_b} t_{b,i} < I + 1$.

**Proof.** Otherwise, $\sum_{i=0}^{I_b} t_{b,i} \geq I + 1$.

**Theorem 3.2.** Let $G$ be the greedy subset and $b \in G$ such that $g_b = K$ for $K > 0$. Then, there is $b' \in G$ with $g_{b'} = K - 1$ such that for some $b' \in b' - a(b') + \mathcal{A}_{i(b')}$. $\varphi_{b'} = \varphi_b$. As a consequence, we reach $b$ from $b'$.

**Proof.** Consider $t_b$ to be the type vector of $b$ and suppose that $t_b$ has two or more zeros after $I_b$. Then,

$$
\sum_{i=I_b+1}^{n} t_{b,i} \leq n - I_b - 2
$$

implying that $\sum_{i=0}^{I_b} t_{b,i} \geq I_b + 2$, and $b \notin G$.

If $i_b$ has one zero after $I_b$, it implies that $i(b) > I_b$. If $g_{b} > 0$, it needs to have at least one zero before $I_b$. Therefore, the type vector contains a sequence of the form

$$
(\ldots, 0, 1, \ldots, 1, b_h, \ldots)
$$

for some $I' < I \leq I_b$ with $t_{b,i} \geq 2$. Consider the type function:

$$
\varphi_{b'}(j) = \begin{cases}
\varphi_{b}(j) - 1 & I' < \varphi_{b}(j) \leq I \\
\varphi_{b}(j) & \text{otherwise}
\end{cases}
$$

The corresponding type vector $t_{b'}$ contains a sequence:

$$
(\ldots, 1, \ldots, 1, I_{b_b}, 1, \ldots, 1, b_h, \ldots)
$$

Using Lemma 3.2, $\sum_{i=0}^{I'} t_{b,i} < I' + 1$, therefore we will have:

$$
\sum_{i=0}^{I'} t_{b',i} \leq I' + 1.
$$

The same will hold for all the partial sums from $I' \leq I_b$ implying there is $b' \in G$ with type function $\varphi_{b'}$. Using Cor. 3.2, $\varphi_b$ is in the range of type functions in $b' - a(b') + \mathcal{A}_{i(b')}$. As long as $i(b') < n$, we can find $b' \in G$ such that:

$$
(b - b') + a(b') < a_{i(b')} \quad \text{for} \quad b \in b' - a(b') + \mathcal{A}_{i(b')}
$$

so $b \in b' - a(b') + \mathcal{A}_{i(b')}$. If $i(b) = i(b') \equiv n$, we must have $a(b) = a(b') = 0$, so we reach a point $b' \in b' - a(b') + \mathcal{A}_{i(b')}$ in the same cell as $b$ such that:

$$
(b - b') < (b - b') \quad \forall I' < \varphi_{b}(j) \leq I
$$

As $i(b)$ is always the same, after a finite number of steps, we have $b \in b' - a(b') + \mathcal{A}_{i(b')}$. 

298
If \( t_b \) does not have any zero after \( I_b \), then \( i(b) < I_b \). The vector contains a sequence that looks like

\[
i(b) \quad (\ldots, 0, t_b, i(b) + 1, \ldots, t_b I_b, \ldots)\]

for \( t_b I_b \geq 1 \) with \( i(b) < I_b \). In this case, consider the type function:

\[
\varphi_b(j) = \begin{cases} 
\varphi_b(j-1) - i(b) < \varphi_b(j) \leq I_b & \text{otherwise} \\
\varphi_b(j) - i(b) & \text{otherwise} 
\end{cases}
\]

with type vector \((\ldots, t_b, i(b) + 1, \ldots, t_b I_b, 0, \ldots, \ldots)\).

\[
\sum_{i=0}^{n} t_{p,i} \geq n - i(b) + \sum_{i=1}^{t_b} (t_{p,i} - 1) \implies \sum_{i=0}^{n} t_{p,i} \leq i(b) - \sum_{i=1}^{t_b} (t_{p,i} - 1) \leq i(b) + 1
\]

which implies that

\[
\sum_{i=0}^{n} t_{p,i} - t_{p,i} + t_{p,i} b_i + 1 \leq i(b) - \sum_{i=1}^{t_b} (t_{p,i} - 1) + t_{p,i} b_i + 1 \leq i(b) + 1
\]

This argument holds for bounding the partial sums for \( I > i(b) \) so there is \( b' \in G \) with type function \( \varphi_{b'} \) and \( \varphi_b \) is in the range of type functions in \( b' - a(b') + A(i(b)) \). In this case, it is not possible that \( i(b') = n \) so the same argument on the previous case holds in order to say that \( b \in b' - a(b') + A(i(b)) \).

Thm. 3.1 and Thm. 3.2 imply that if we start the greedy algorithm from the lattice points in mixed cells, we will reach exactly the lattice points in \( G \). This actually reduces the size of the Canny-Emiris matrices.

**Corollary 3.3.** The size of the matrix \( H_{G} \) is:

\[
\sum_{\varphi_{b}: \{1, \ldots, n\} \to \{0, \ldots, n\}} \prod_{j=1}^{n} a_{\varphi_{b}(j)}
\]

where the sum is over the functions that satisfy \( \varphi_{b}^{-1}(\{0, \ldots, I\}) \subseteq I + 1 \) \( \forall l < n \).

**Proof.** Each type function \( \varphi_{b} \) corresponds to a cell \( D \in S(\rho) \). The lattice points \( b \in D \) satisfy Definition 3.2. Therefore, for each \( j \), there are \( a_{\varphi_{b}(j)} \) possible values of \( b_{j} \). The product over all of them gives the desired count.

We could not yet prove whether this is minimal with respect to the application of the greedy approach to any other admissible mixed subdivision, but the many of the examples that we have verify this conjecture. We should remark that this combinatorial formula (and the one in Cor. 3.3) should be compared with the same sum over all the type functions without the restriction. For practical purposes, we show an example of our computations of the matrix dimensions in Fig. 2 in \( A \).

### 4 MULTIHOMOGENEOUS FORMULAS

**Example 4.1.** Let \( f_0, f_1, f_2 \) be three homogeneous polynomials of degrees 2, 2, 1 respectively. We choose \( \nu = (-1, -2) \) and \( \delta = (-3/4, -3/4) \) and define an admissible mixed subdivision \( S(\rho) \) in the Minkowski sum \( \Delta \) of their Newton polytopes \( \Delta_i \). Let \( B \) be the set of lattice points in \( \Delta + \delta \). Consider a system of polynomials whose Newton polytopes are \( n \)-zonotopes generated by the vectors \( w_1 = (1, 0) \) and \( w_2 = (-1, 1) \) and let \( a_{0,1} = a_{0,2} = a_{1,1} = a_{1,2} = 2 \) and \( a_{2,1} = a_{2,2} = 1 \) be the bounds of the supports as in Section 3. Let \( S(\overline{p}) \) be the mixed subdivision in the Minkowski sum \( \Delta \) of \( \Delta_i \) of this system given by the same \( \nu, \delta \) as the previous, and let \( \overline{B} \) be the set of lattice points in \( \Delta + \delta \).

It turns out that the mixed subdivision \( S(\rho) \) embeds, into \( S(\overline{p}) \) implying that \( B = \overline{B} \cap \Delta \). As the greedy reduction applies to the second system, it must apply to the first as well. We get a \( 9 \times 9 \) matrix \( H_{G} \) for the homogeneous system, excluding the black lattice point in the figure.

Similar to Example 4.1, let’s now consider multihomogeneous polynomial systems and embed them into \( n \)-zonotopes. Let \( n_1, \ldots, n_r \in \mathbb{N}_{>0} \) be natural numbers and let \( M = \oplus_{i=1}^{n} \mathbb{Z}^{n_i} \) be our lattice. Each multihomogeneous polynomial system can be written as:

\[
F_i = \sum_{a \in A_i} u_{a} x^{a} \quad i = 0, \ldots, n
\]

where the supports are:

\[
A_i = \{(b_{j})_{j=1}^{n_{i}} \in \oplus_{j=1}^{n} \mathbb{Z}^{n_{i}} | b_{j} \geq 0, \sum_{j=0}^{n_{i}} b_{j} \leq d_{i,j}\}
\]

where \( d_{i} = (d_{i,1}, \ldots, d_{i,s}) \) is the multidegree of \( F_i \). Each of these supports can be embedded into the following sets of supports:

\[
A_i = \{(b_{j}) \in \oplus_{j=1}^{n} \mathbb{Z}^{n_{i}} | 0 \leq \sum_{j=0}^{n_{i}} b_{j} \leq d_{i,j}\}
\]

Let \( \Delta_{i}, \Delta \) be the Newton polytopes of each of the systems and \( \Delta, \Delta \) be their respective Minkowski sums.

**Lemma 4.1.** The Newton polytopes \( \Delta_{i} \) of the system of polynomials with supports in \( \overline{A}_{i} \) are \( n \)-zonotopes whose line segments \((w_{j,l})_{l=1}^{n} \) are given by the columns of the matrix:

\[
W = \begin{bmatrix}
W_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & W_s
\end{bmatrix}, \quad W_l = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & 0 & 0 & 1
\end{bmatrix}
\]

where the square blocks \( W_l \) are of size \( n_l \) for \( l = 1, \ldots, s \). Moreover, \( \overline{A} = \bigcup_{i=1}^{r} \bigcup_{j=1}^{n} \{(x, w_{j,l}) : 0 \leq \sum_{j=0}^{n_{i}} b_{j} \leq d_{i,j}\} \subseteq \prod_{i=1}^{r} \mathbb{R}^{n_{i}} \) is the hyperplane arrangement associated to \( \Delta \).
Proof. Let \( b \in S^n \cap \mathbb{Z}^n \) be a lattice point. As the columns of \( W \) form a basis of the lattice, we can write \( b = \sum_{i=1}^{s} \sum_{j=1}^{n_i} \lambda_{ij} w_{ij} \) and these coefficients are precisely \( \lambda_{ij} = \sum_{j=1}^{n_i} b_{ij} \). Then,

\( b \in \mathcal{N} \iff 0 \leq \lambda_{ij} \leq d_{ij} \) \( l = 1, \ldots, s \), \( j = 1, \ldots, n_l \).

The normal vectors to the faces of \( \mathcal{N} \) are given by the columns \( \langle \eta_{ij} \rangle_{i=1, \ldots, s} \) of the matrix:

\[
H = \begin{bmatrix}
H_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & H_s
\end{bmatrix}, \quad H_l = \begin{bmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 0 & \ldots & 0 \\
1 & 1 & 1 & 0 & \ldots & 1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
1 & 1 & 1 & 1 & \ldots & 1
\end{bmatrix}
\]

One can check that \( \langle w_{ij}, \eta_{ij} \rangle \neq 0 \), if and only if, \( l = l' \) and \( j = j' \). Therefore, \( v \in \mathcal{H} \), if and only if, it belongs to the span of \( \sum_{j=1}^{s} n_j - 1 \) columns of \( H \), and this will only happen if \( \langle v, w_{ij} \rangle = 0 \) for some pair \( j, l \).

Remark 4.1. As a consequence of Lemma 4.1, we can apply the results of Sec. 3 to the system with supports \( \mathcal{N} \). The matrix \( H \) gives the normals to our polytopes, so we can use it in the sense of Rem. 3.1.

Let \( v \notin \mathcal{H} \) and suppose that we take \( \langle v, w_{ij} \rangle < 0 \), for \( l = 1, \ldots, s \), and \( j = 1, \ldots, n_l \). Consider \( S(\mathcal{P}) \) to be the admissible mixed subdivision of \( \mathcal{N} \) given by \( v \) as in Sec. 2. Let \( S(\rho) \) be the mixed subdivision given by the same vector in \( \Delta \). Using \( \langle v, w_{ij} \rangle < 0 \), one can check that this mixed subdivision is also admissible as \( v \) does not belong to the hyperplane arrangement \( \mathcal{H} \) associated to \( \Delta \). Let \( \mathcal{B}, \mathcal{B} \) be the sets of lattice points in translated cells of \( \Delta \) and \( \mathcal{N} \) respectively. We can see the polytopes \( \Lambda_l \) as a product of simplices \( \Delta_{l,1} \times \cdots \times \Delta_{l,s} \) in each of the factors of \( M_\rho = \prod_{l=1}^{s} \mathbb{R}^{n_l} \).

Theorem 4.1. The mixed subdivision \( S(\rho) \) coincides with \( S(\mathcal{P}) \cap \Delta \).

Proof. The vector \( v \in \prod_{l=1}^{s} \mathbb{R}^{n_l} \) has to satisfy:

\[ v_{il} < 0 \iff v_{il} < v_{il-1} < \cdots < v_{il} < 0 \] for \( l = 1, \ldots, s \), \( 1 \leq j \leq n_l \). In other words \( v_{il} < v_{il-1} < \cdots < v_{il} < 0 \). This means that the mixed subdivision \( S(\mathcal{B}) \) lifts the vertices of \( \Lambda_0 \) in the order \( 0, d_{01} \), \( d_{02} \) \( \cdots \), \( d_{0s} \) from higher to lower in the same lines of Prop. 3.1. This means that the cells \( D \in S(\rho) \) such that \( d_{01} \), \( d_{02} \) \( \cdots \), \( d_{0s} \) are contained between the hyperplanes \( \langle x, w_{ij} \rangle = \sum_{j=0}^{l-1} d_{ij} \rangle \) and \( \langle x, w_{ij} \rangle = \sum_{j=0}^{l} d_{ij} \rangle \) for \( l = 1, \ldots, s \), \( j = 1, \ldots, n_l \). \( S(\rho) \) has the same structure. Therefore, \( S(\rho) \) coincides with \( S(\mathcal{P}) \cap \Delta \).

Therefore, if we apply the greedy algorithm to the multihomogeneous system with supports in the \( \mathcal{N} \), we will obtain the same greedy subset \( G \subset \mathcal{N} \), with the restriction on the type vectors given in Definition 3.3. In particular, the domain of the type functions will now be a multiset in each group of variables:

\[ \varphi_{\rho} : \{1, \ldots, n_1\}, \ldots, \{1, \ldots, n_s\} \rightarrow \{0, \ldots, n\} \]

The following proposition gives conditions to guarantee that the type function \( \varphi_{\rho} \) corresponds to a lattice point \( b \in \mathcal{B} \).

Proposition 4.1. A lattice point \( b \in \mathcal{B} \) belongs to \( \mathcal{B} \) iff its type function satisfies:

\[ \varphi_{\rho}(j) \leq \varphi_{\rho}(j^*), \quad \forall j < j^* \] \( j, j^* = 1, \ldots, n_l, \ l = 1, \ldots, s \).

Corollary 4.1. The size of the matrix \( H_{\rho} \) for multihomogeneous systems is:

\[
\sum_{s=1}^{n} \prod_{l=1}^{n} d_{ij} (\mathcal{N}_{k,l}) = \prod_{k=0}^{n} \mathcal{N}_{k,l} \rho(j, k) \]

where \( \mathcal{N}_{k,l} \rho(j, k) = \{(j \in \{1, \ldots, n_l\} : \varphi_{\rho}(j) = k\} \) and \( \mathcal{N}_{k,l} \rho \) satisfies the restrictions of Cor. 3.3 and Prop. 4.1.

We refer to the arxiv version for the proofs of Props. 4.1 and Cor. 4.1. There exist exact determinant resultant formulas for some multihomogeneous cases, obtained by using the Weyman complex and other techniques [2, 1, 11, 12, 19]. Our approach does not improve those cases, but the use of type functions might be easier to generalize to a general case. Let’s give an example of the size of these matrices with respect to some of the existing formulas.

Example 4.2. For the polynomial system of Ex. 1.1, there are exact formulas of Sylvester type [11] which give a matrix of size 6, smaller than that of size 8.

We could also exploit the incremental algorithm for constructing the Canny-Emiris formula [4], but we would be losing the combinatorial properties. Therefore, we would not have a proof of the formula for such matrices or we wouldn’t be able to guarantee that they have a non-zero determinant as in Rem. 1.5. Moreover, such implementation requires the precomputing of mixed volumes.

5 CONCLUSIONS

The main contribution of this paper is a first approach to estimating the possible reduction of the size of the Canny-Emiris matrices produced by the greedy algorithm in [5]. Apart from the treated cases, we could consider other systems for which the mixed subdivision can be embedded in an \( n \)-zonotope and impose restrictions on the type functions accordingly. We could also try to drop the hypothesis that \( a_{0j} \leq \cdots \leq a_{n-1,j} \), the examples show that, for that case, the reduction in the cells that are not in \( G \) is lower. We also expect to measure when the Newton polytopes are \( m \)-zonotopes for \( m > n \). In such cases, the examples show that there will still be some reduction.

Example 5.1. Here an example for \( \mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0, 0), (1, 0) \} \) \( (1, 1), (1, -1), (1, -2), (2, 0) \) \( \subset \mathbb{Z} \). Our choice of the mixed subdivision would give the drawing in Fig. 3. There is still a reduction on the lattice points of the cells not in \( G \) (in black), but not all the lattice points can be excluded.

This article includes an implementation of the Canny-Emiris formula in JULLIA for the case of \( n \)-zonotopes and multihomogeneous systems which does not depend on the choice of a lifting function whose code and explanation can be found at this link. The practical use of the construction of symbolic resultant matrices can be found in [8] and others.

ACKNOWLEDGMENTS

The project has received funding from EU’s H2020 research & innovation programme under the Marie Skłodowska-Curie grant agreement No 860843 (GRAPES). We thank Elias Tsigaridas and Christos Konaxis for their support and Carlos D’Andrea for answering our questions. We thank the reviewers for relevant comments.
REFERENCES

[1] M. R. Bender, J.C. Faugere, A. Mantzaflaris, and E. Tsigaridas. 2021. Koszul-type determinantal formulas for families of mixed multilinear systems. SIAM Jnl. on App. AG, 5, 4, 589–619. DOI: 10.1137/20M1332190.

[2] M.R. Bender, J.-C. Faugère, and E. Tsigaridas. 2018. Towards mixed gröbner basis algorithms: the multihomogeneous and sparse case. ISSAC - 43rd Intern. Symp. on Symbolic & Algebraic Computation. DOI: 10.1145/3208976.3209018. https://hal.inria.fr/hal-01787423.

[3] J.F. Canny and I.Z. Emiris. 1993. An efficient algorithm for the sparse mixed resultant. Appl. algebra, algebraic algo. & error-correcting codes, 673, 89–104.

[4] J.F. Canny and I.Z. Emiris. 1995. Efficient incremental algorithms for the sparse resultant and the mixed volume. Jour. Symb. Comput., 20, 117–149. DOI: 10.1006/jsco.1995.1041.

[5] J.F. Canny and P. Pedersen. 1993. An Algorithm for the Newton Resultant. Technical report. Cornell University, NY, USA.

[6] D. Cox, J. Little, and D. O’Shea. 2015. Using Algebraic Geometry. Volume 185. ISBN: 0-87144-103-6. DOI: 10.1007/b138611.

[7] C. D’Andrea. 2001. Macaulay style formulas for sparse resultants. Trans. AMS, 354, (August 2001). DOI: 10.2307/3073009.

[8] C. D’Andrea and I.Z. Emiris. 2001. Computing sparse projection operators. Contemporary Mathematics, 286, 121–140.

[9] C. D’Andrea, G. Jeronimo, and M. Sombra. 2020. The canny-emiris conjecture for the sparse resultant. CoRR, abs/2004.14622. arXiv: 2004.14622. https://arxiv.org/abs/2004.14622.

[10] C. D’Andrea and M. Sombra. 2015. A poisson formula for the sparse resultant. Proc. London Math. Society, 110, 4, 932–964. ISSN: 0024-6115. DOI: 10.1112/plms/pdu069. http://dx.doi.org/10.1112/plms/pdu069.

[11] A. Dickenstein and I.Z. Emiris. 2003. Multihomogeneous resultant formulae by means of complexes. Journal of Symbolic Computation, 36, 317–342. DOI: 10.1016/S0747-7171(03)00086-5.

[12] I. Emiris and A. Mantzaflaris. 2012. Multihomogeneous resultant formulae for systems with scaled support. J. Symbolic Computation, 820–842. DOI: 10.1016/j.jsc.2011.12.010. https://hal.inria.fr/inria-00355881.

[13] I.Z. Emiris and A. Rege. 1994. Monomial bases and polynomial system solving. Proc. Intern. Symp. Symbolic & Algebraic Computation, 114–122. M.A.H. MacCallum, editor. DOI: 10.1145/190347.190374. https://doi.org/10.1145/190347.190374.

[14] M. Gelfand, M. Kapranov, and A. Zelevinsky. 1994. Discriminants, resultants, and multidimensional determinants. Birkhauser.

[15] F.S. Macaulay. 1903. Some formulae in elimination. Proc. Lond. Math. Soc. 35, 3–27.

[16] T. Michiels and R. Cools. 2000. Decomposing the secondary cayley polytope. Discrete and Computational Geometry, 23, 3, 367–380. ISSN: 1432-0444. DOI: https://doi.org/10.1007/PL00009506.

[17] R. P. Stanley. 2016. Smith normal form in combinatorics. Journal of Combinatorial Theory, Series A, 144, 476–495. Fifty Years of the Journal of Combinatorial Theory. ISSN: 0097-3165. DOI: https://doi.org/10.1016/j.jcta.2016.06.013.

[18] B. Sturmfels. 1994. On the Newton polytope of the resultant. J. Algebraic Combin., 3, 207–236.

[19] B. Sturmfels and A. Zelevinsky. 1994. Multigraded resultants of sylyester type. Journal of Algebra, 163, 115–127.

[20] G.M. Ziegler. 1995. Lectures on Polytopes. (1st edition). Number 152 in Grad Texts in Math. Springer, New York.

A APPENDIX: FIGURES

Figure 1: This table explains how the process of passing from the proposed lifting on $A_0, A_1, A_2$ to the mixed subdivision works in the $j$-th coordinate for $v_j < 0$ for any of the two components of Ex. 1.1. One clearly sees that, for instance, $0a_0, 0a_2, 0a_0 < D_0$, if and only if, $x_0 < a_0$ for $x D_0$. The product of two such subdivisions gives the mixed subdivision in Ex. 1.1.

| Step | Lifting | Subdivision |
|------|---------|-------------|
| $S(\theta_0)$ | $0$ | $0$ | $0$, $a_0$ + $a_1$, $a_2$, $a_0$, $a_1$, $a_2$ |
| $S(\theta_1)$ | $\lambda_0$ | $\lambda_2$ | $0$, $a_0$, $a_1$, $a_2$ |
| $S(\theta_2)$ | $\lambda_2$ | $(\lambda_0 + \lambda_1)x_j$ | $0$, $a_0$, $a_1$, $a_2$, $a_0$, $a_1$, $a_2$ |
| $S(\rho)$ | $\lambda_2$ | $(\lambda_0 + \lambda_1)x_j$ | $0$, $a_0$, $a_1$, $a_2$, $a_0$, $a_1$, $a_2$ |

Figure 2: This table represents the size of the matrices we achieve for zonotopes of dimensions from 2 to 5 with $a_{ij} = 1$ using the greedy approach versus the original Canny-Emiris formula. We also compare to the degree of the resultant.

| Dimension | Canny-Emiris | Greedy | Resultant degree |
|-----------|--------------|--------|-----------------|
| 2         | 9            | 8      | 6               |
| 3         | 64           | 50     | 24              |
| 4         | 625          | 432    | 360             |
| 5         | 7776         | 4802   | 3720            |

Figure 3: Mixed subdivision corresponding to Example 5.1.