Some Symmetric Properties on \((LCS)_n\)-manifolds

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ABSTRACT. We analyze the \((LCS)_n\)-manifolds endowed with some symmetric properties, focusing on Ricci tensor and the 1-form \(\gamma\). We study some properties of special Weakly Ricci-Symmetric \((LCS)_n\)-manifolds and also shown that Weakly \(\phi\)-Ricci Symmetric \((LCS)_n\)-manifold is an \(\eta\)-Einstein manifold.

1. Introduction

The study of Riemannian symmetric manifolds began with the work of E. Cartan [2]. A Riemannian manifold \((M^n, g)\) is said to be locally symmetric due to E. Cartan [2] if its curvature tensor \(R\) satisfies the relation \(\nabla R = 0\), where \(\nabla\) denotes the operator of covariant differentiation with respect to the metric tensor \(g\). A non-flat Riemannian manifold \((M^n, g)\) is called a weakly symmetric manifold if its curvature tensor \(R\) of type \((0,4)\) satisfies the condition

\[
(\nabla_W R)(X, Y, Z, U) = A(W)R(X, Y, Z, U) + B(X)R(W, Y, Z, U) + H(Y)R(X, W, Z, U) + D(Z)R(X, Y, W, U) + E(U)R(X, Y, Z, W),
\]

for all vector fields \(W, X, Y, Z, U \in \chi(M^n)\), where \(A, B, H, D\) and \(E\) are 1-forms (not simultaneously zero) and \(\nabla\) denotes the operator of covariant differentiation with respect to the Riemannian metric \(g\). The 1-forms are called the associated 1-forms of the manifold and an \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\).

In 1989, Tamassy and Binh [13, 14] introduced the notion of weakly symmetric and weakly Ricci-symmetric Riemannian manifolds and studied such structures on
Sasakian manifolds and proved that such a structure does not exist always. Weakly symmetric and weakly Ricci-symmetric structures are also studied by Shaikh and Jana [10, 11]. A Riemannian manifold \((M^n, g)\) is called weakly Ricci symmetric if its Ricci tensor \(S\) of type \((0,2)\) is not identically zero and satisfies the condition:

\[
(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(Z, X) + D(Z)S(Y, X),
\]

where \(A, B\) and \(D\) are 1-forms (not simultaneously zero). Such an \(n\)-dimensional manifold is denoted by \((WRS)_n\).

The above relation can be written as

\[
(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + S(Y, X)\sigma,
\]

where \(\sigma\) is the vector field associated to the 1-form \(D\) such that \(D(Z) = g(Z, \sigma)\) and \(Q\) is the Ricci operator, i.e., \(g(QX, Y) = S(X, Y)\) for all \(X, Y\).

In this paper, we study some symmetric properties of \((LCS)_n\) manifolds by presenting two new sections, preceded by a preliminaries section containing some background on \((LCS)_n\) manifolds.

The first one is devoted to the study of special weakly Ricci-symmetric \((LCS)_n\) manifolds. We begin by studying special Weakly Ricci-Symmetric \((LCS)_n\)-manifolds with a cyclic parallel Ricci tensor, in this case the 1-form \(\gamma\) must vanish and also we show that if a special weakly Ricci-symmetric \((LCS)_n\)-manifolds does not satisfy the condition of Einstein manifold, then the 1-form \(\gamma\) is non zero. Further it is proved that the Ricci tensor is parallel in a special weakly Ricci-symmetric \((LCS)_n\)-manifold.

Finally, in the last section we study weakly \(\phi\)-Ricci symmetric \((LCS)_n\) manifold and proved that it is an \(\eta\)-Einstein manifold. Moreover we have seen that \(\phi\)-Ricci symmetric \((LCS)_n\)-manifold is an Einstein manifold.

2. Preliminaries

In 2003, A.A. Shaikh [9] introduced the notion of Lorentzian concircular structure manifolds (briefly \((LCS)_n\)-manifolds). An \(n\)-dimensional Lorentzian manifold \(M^n\) is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric \(g\), that is, \(M^n\) admits a smooth symmetric tensor field \(g\) of type \((0,2)\) such that for each point \(p \in M\), the tensor \(g_p : T_p M^n \times T_p M^n \to R\) is a non-degenerate inner product of signature \((-+, \ldots, +)\), where \(T_p M^n\) denotes the tangent vector space of \(M^n\) at \(p\) and \(R\) is the real number space. A non-zero vector \(v \in T_p M^n\) is said to be timelike if it satisfies \(g_p(v, v) < 0\).

**Definition 2.1.** In a Lorentzian manifold \((M^n, g)\) a vector field \(P\) defined by

\[
g(X, P) = A(X),
\]

for any \(X \in \chi(M^n)\) is said to be a concircular vector field

\[
(\nabla_X A)(Y) = \alpha(g(X, Y) + \omega(X)A(Y)),
\]
where $\alpha$ is a non-zero scalar and $\omega$ is a closed 1-form.

Let $M^n$ be an $n$-dimensional Lorentzian manifold admitting a unit timelike concircular vector field $\xi$, called the characteristic vector field of the manifold. Then we have

$$\begin{align*}
(2.1) \quad g(\xi, \xi) &= -1.
\end{align*}$$

Since $\xi$ is a unit concircular vector field, there exists a non-zero 1-form $\eta$ such that

$$\begin{align*}
(2.2) \quad g(X, \xi) &= \eta(X).
\end{align*}$$

The equation of the following form holds

$$\begin{align*}
(2.3) \quad (\nabla_X \eta)(Y) &= \alpha [g(X, Y) + \eta(X)\eta(Y)], \quad (\alpha \neq 0)
\end{align*}$$

for all vector fields $X, Y$, where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ and $\alpha$ is a non-zero scalar function satisfying

$$\begin{align*}
(2.4) \quad \nabla_X \alpha &= (X\alpha) = d\alpha(X) = \rho \eta(X),
\end{align*}$$

$\rho$ being a certain scalar function given by $\rho = -\langle \xi, \alpha \rangle$. If we put

$$\begin{align*}
(2.5) \quad \phi X &= \frac{1}{\alpha} \nabla_X \xi,
\end{align*}$$

then from (2.3) and (2.4), we have

$$\begin{align*}
(2.6) \quad \phi X &= X + \eta(X)\xi,
\end{align*}$$

from which it follows that $\phi$ is a symmetric $(1,1)$ tensor. Thus the Lorentzian manifold $M$ together with the unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and $(1,1)$ tensor field $\phi$ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold)[9]. Especially, if we take $\alpha = 1$, then we can obtain the Lorentzian para-Sasakian structure of Matsumoto [6]. In a $(LCS)_n$-manifold, the following relations hold:

$$\begin{align*}
(2.7) \quad \eta(\xi) &= -1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \\
(2.8) \quad g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\
(2.9) \quad R(X, Y)Z &= (\alpha^2 - \rho)[g(Y, Z)X - g(X, Z)Y], \\
(2.10) \quad \eta(R(X, Y)Z) &= (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \\
(2.11) \quad (\nabla_X \phi)(Y) &= \alpha [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \\
(2.12) \quad S(X, \xi) &= (n-1)(\alpha^2 - \rho)\eta(X), \\
(2.13) \quad S(\phi X, \phi Y) &= S(X, Y) + (n-1)(\alpha^2 - \rho)\eta(X)\eta(Y), \\
(2.14) \quad Q \xi &= (n-1)(\alpha^2 - \rho)\xi.
\end{align*}$$
for any vector fields \(X, Y, Z\), where \(R, S\) denotes respectively the curvature tensor and the Ricci tensor of the manifold.

**Definition 2.2.** An \((LCS)_n\)-manifold \(M^n\) is said to be \(\eta\)-Einstein if its Ricci tensor \(S\) is of the form

\[
S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),
\]

for any vector fields \(X\) and \(Y\), where \(a\) and \(b\) are some functions.

### 3. On Special Weakly Ricci-Symmetric \((LCS)_n\)-manifolds

An \(n\)-dimensional Riemannian manifold \((M^n, g)\) is called a special Weakly Ricci-Symmetric manifold \((SWRS)_n\) if

\[
(\nabla_X S)(Y, Z) = 2\gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(Y, X),
\]

for any vector fields \(X, Y\) on \(M^n\), where \(\gamma\) is a 1-form defined by

\[
\gamma(X) = g(X, W),
\]

where \(W\) is the associated vector field.

**Theorem 3.1. In a special weakly Ricci-symmetric \((LCS)_n\)-manifold, the Ricci tensor is parallel.**

**Proof.** Putting \(Z = \xi\) in (3.1), we get

\[
(\nabla_X S)(Y, \xi) = 2\gamma(X)S(Y, \xi) + \gamma(Y)S(X, \xi) + \gamma(\xi)S(Y, X),
\]

The left hand side of above equation can be written as

\[
(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi).
\]

By using (2.13), (3.2) and (3.4), the equation (3.3) can be written as

\[
\nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi) = 2(n-1)(\alpha^2 - \rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2 - \rho)\gamma(Y)\eta(X) + \eta(W)S(Y, X).
\]

Taking \(Y = \xi\) in (3.5) and using (2.10), (2.12), (2.13) and (3.2), we get

\[
-2(n-1)(\alpha^2 - \rho)\gamma(X) + (n-1)(\alpha^2 - \rho)\eta(W)\eta(X) + (n-1)(\alpha^2 - \rho)\eta(W)\eta(X) = 0.
\]

Putting \(X = \xi\) in (3.6), we obtain

\[
\eta(W) = 0.
\]

Using (3.7) in (3.6) gives

\[
\gamma(X) = 0,
\]
Some Symmetric Properties on $(LCS)_n$-manifolds

for any vector fields $X$ on $M^n$.

Hence in view of (3.8), we obtain from (3.1) that

$$\nabla_X S = 0.$$

\[\blackqed\]

**Theorem 3.2.** Let $(M^n, g)$ be a special Weakly Ricci-Symmetric $(LCS)_n$-manifold with a cyclic parallel Ricci tensor. Then the 1-form $\gamma$ must vanish.

**Proof.** Taking the cyclic sum of (3.1), we get

$$\nabla_X Y + \nabla_Y Z + \nabla_Z X = 0. \tag{3.9}$$

Let $M^n$ admit a cyclic parallel Ricci tensor. Then (3.9) reduces to

$$\gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(X, Y) = 0. \tag{3.10}$$

Taking $Z = \xi$ in (3.10) and using (2.7) and (3.2), we get

$$\gamma(X)S(Y, Z) + (n-1)(\alpha^2 - \rho)\gamma(Y)\eta(X) + \eta(W)S(X, Y) = 0. \tag{3.11}$$

Now putting $Y = \xi$ in (3.11) and using (2.7), (2.12) and (3.2), we get

$$-\gamma(X)\eta(W) + \eta(W)\gamma(Y) = 0. \tag{3.12}$$

Further, taking $X = \xi$ in (3.12) and using (2.7) and (3.2), we obtain

$$\eta(W) = 0. \tag{3.13}$$

So by the use of (3.13) in (3.12), we have $\gamma(X) = 0$, for any vector field $X$ on $M^n$.

This completes the theorem. \[\blackqed\]

**Theorem 3.3.** If a special Weakly Ricci-symmetric $(LCS)_n$-manifold is not an Einstein manifold, then 1-form $\gamma \neq 0$.

**Proof.** For an Einstein manifold $(\nabla_X Y) = 0$ and $S(Y, Z) = kg(Y, Z)$.

By (3.1), we have

$$2\gamma(X)S(Y, Z) + \gamma(Y)S(X, Z) + \gamma(Z)S(Y, X) = 0. \tag{3.14}$$

Taking $Z = \xi$ in (3.14) and using (2.13) and (3.2), we get

$$2(n-1)(\alpha^2 - \rho)\gamma(X)\eta(Y) + (n-1)(\alpha^2 - \rho)\gamma(Y) \eta(X) + \eta(W)S(Y, X) = 0. \tag{3.15}$$
Taking $X = \xi$ in (3.15) and using (2.12) (2.13) and (3.2), we get

\[2(n-1)(\alpha^2 - \rho)\eta(W)\eta(Y) - (n-1)(\alpha^2 - \rho)\gamma(Y) + (n-1)(\alpha^2 - \rho)\eta(W)\eta(Y) = 0.\]

Again taking $Y = \xi$ in (3.16) and by virtue of (2.12) and (3.2), we get

\[\eta(W) = 0.\]

Using (3.17) in (3.16), we get \[\gamma(Y) = 0,\]

for any vector field $Y$ on $M^n$. \hfill \Box

4. Weakly $\phi$-Ricci Symmetric $(LCS)_n$-manifolds

**Definition 4.3.** A $(LCS)_n$-manifold is said to be weakly $\phi$-Ricci Symmetric if the Ricci operator satisfies

\[\phi^2((\nabla_X Q)(Y)) = A(X)\phi^2(QY) + B(Y)\phi^2(QX) + S(Y, X)\phi^2(\sigma).\]

Especially, if the 1-forms $A = B = D = 0$, then (4.1) turns into the notion of $\phi$-Ricci symmetric introduced by Shukla and Shukla [12].

Let us take $(LCS)_n$-manifold, which is weakly $\phi$-Ricci symmetric. Then from (2.8), equation (4.1) becomes

\[\nabla_X Q(Y) + \eta((\nabla_X Q)(Y))\xi = A(X)[QY + \eta(QY)]\xi + B(Y)[QX + \eta(QX)]\xi + S(Y, X)[\sigma + \eta(\sigma)]\xi,\]

from which it follows that

\[g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z) = A(X)[S(Y, Z) + \eta(QY)\eta(Z)] + B(Y)[S(X, Z) + \eta(QX)\eta(Z)] + S(Y, X)[\sigma + \eta(\sigma)]\xi.\]

Putting $Y = \xi$ in (4.2) and using (2.9), (2.12) and (2.14), we get

\[\alpha + B(\xi)S(X, Z) = -(n-1)(\alpha^2 - \rho)[-\alpha g(X, Z) + \eta(X)D(Z) + (B(\xi) + \eta(\sigma))\eta(X)\eta(Z)].\]

Replacing $X$ by $\phi X$ and $Z$ by $\phi Z$ in (4.3), we have

\[[\alpha + B(\xi)]S(\phi X, \phi Z) = \alpha(n-1)(\alpha^2 - \rho)g(\phi X, \phi Z).\]

By virtue of (2.8) and (2.13), equation (4.4) becomes

\[S(X, Z) = kg(X, Z) + l\eta(X)\eta(Z),\]
Some Symmetric Properties on \((LCS)_n\)-manifolds

where

\[
  k = \frac{\alpha(n - 1)(\alpha^2 - \rho)}{\alpha + B(\xi)} \quad \text{and} \quad l = \frac{(n - 1)(\alpha^2 - \rho)B(\xi)}{\alpha + B(\xi)},
\]

provided \(\alpha + B(\xi) \neq 0\).

Hence we can state,

**Theorem 4.4.** A weakly \(\phi\)-Ricci symmetric \((LCS)_n\)-manifolds is an \(\eta\)-Einstein manifold.

From Theorem 4.4., it can be easily seen that

**Corollary 4.1.** A \(\phi\)-Ricci symmetric \((LCS)_n\)-manifold is an Einstein manifold.

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