Sidorenko’s conjecture for a class of graphs: an exposition

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A famous conjecture of Sidorenko [2] and Erdős-Simonovits [3] states that if \( H \) is a bipartite graph then the random graph with edge density \( p \) has in expectation asymptotically the minimum number of copies of \( H \) over all graphs of the same order and edge density. The goal of this expository note is to give a short self-contained proof (suitable for teaching in class) of the conjecture if \( H \) has a vertex complete to all vertices in the other part. This was originally proved in [1].

**Theorem 1** Sidorenko’s conjecture holds for every bipartite graph \( H \) which has a vertex complete to the other part.

The original formulation of the conjecture by Sidorenko is in terms of graph homomorphisms. A homomorphism from a graph \( H \) to a graph \( G \) is a mapping \( f : V(H) \rightarrow V(G) \) such that for each edge \( (u, v) \) of \( H \), \( (f(u), f(v)) \) is an edge of \( G \). Let \( h_{H}(G) \) denote the number of homomorphisms from \( H \) to \( G \). We also consider the normalized function \( t_{H}(G) = h_{H}(G) / |G|^{|H|} \), which is the fraction of mappings \( f : V(H) \rightarrow V(G) \) which are homomorphisms. Sidorenko’s conjecture states that for every bipartite graph \( H \) with \( m \) edges and every graph \( G \), \( t_{H}(G) \geq t_{K_{2}}(G)^{m} \). We will prove that this is the case for \( H \) as in Theorem 1.

We use a probabilistic technique known as dependent random choice. The idea is that most small subsets of the neighborhood of a random vertex have large common neighborhood. Our first lemma gives a counting version of this technique. We will then combine this with a simple embedding lemma to give a lower bound for \( t_{H}(G) \) in terms of \( t_{K_{2}}(G) \). For a vertex \( v \) in a graph \( G \), the neighborhood \( N(v) \) is the set of vertices adjacent to \( v \). For a sequence \( S \) of vertices of a graph \( G \), the common neighborhood \( N(S) \) is the set of vertices adjacent to every vertex in \( S \).

**Lemma 1** Let \( G \) be a graph with \( N \) vertices and \( pN^{2}/2 \) edges. Call a vertex \( v \) bad with respect to \( k \) if the number of sequences of \( k \) vertices in \( N(v) \) with at most \( (2n)^{-n-1} p^{k} N \) common neighbors is at least \( \frac{1}{2n}|N(v)|^{k} \). Call \( v \) good if it is not bad with respect to \( k \) for all \( 1 \leq k \leq n \). Then the sum of the degrees of the good vertices is at least \( pN^{2}/2 \).

**Proof:** We write \( v \sim k \) to denote that \( v \) is bad with respect to \( k \). Let \( X_{k} \) denote the number of pairs \( (v, S) \) with \( S \) a sequence of \( k \) vertices, \( v \) a vertex adjacent to every vertex in \( S \), and \( |N(S)| \leq (2n)^{-n-1} p^{k} N \). We have

\[
(2n)^{-n-1} p^{k} N \cdot N^{k} \geq X_{k} \geq \sum_{v, v \sim k} \frac{1}{2n}|N(v)|^{k} \geq \frac{1}{2n} N \left( \sum_{v, v \sim k} |N(v)|/N \right)^{k} = \frac{1}{2n} N^{1-k} \left( \sum_{v, v \sim k} |N(v)| \right)^{k}.
\]

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The first inequality is by summing over $S$, the second inequality is by summing over vertices $v$ which are bad with respect to $k$, and the third inequality is by convexity of the function $f(x) = x^k$. We therefore get

$$\sum_{v, v \sim k} |N(v)| \leq (2n)^{-n/k}pN^2 \leq \frac{1}{2n}pN^2.$$ 

Hence,

$$\sum_{\text{good}} |N(v)| \geq \sum_v |N(v)| - \sum_{k=1}^n \sum_{v, v \sim k} |N(v)| \geq pN^2 - n \cdot \frac{1}{2n}pN^2 = pN^2/2,$$

as required. \qed

**Lemma 2** Suppose $\mathcal{H}$ is a hypergraph with $v$ vertices and at most $e$ edges and $\mathcal{G}$ is a hypergraph on $N$ vertices with the property that for each $k$, $1 \leq k \leq v$, the number of sequences of $k$ vertices of $\mathcal{G}$ that do not form an edge of $\mathcal{G}$ is at most $\frac{1}{2e}N^k$. Then the number of homomorphisms from $\mathcal{H}$ to $\mathcal{G}$ is at least $\frac{1}{2}N^v$.

**Proof:** Consider a random mapping of the vertices of $\mathcal{H}$ to the vertices of $\mathcal{G}$. The probability that a given edge of $\mathcal{H}$ does not map to an edge of $\mathcal{G}$ is at most $\frac{1}{2}$. By the union bound, the probability that there is an edge of $\mathcal{H}$ that does not map to an edge of $\mathcal{G}$ is at most $e \cdot \frac{1}{2e} = 1/2$. Hence, with probability at least 1/2, a random mapping gives a homomorphism, so there are at least $\frac{1}{2}N^v$ homomorphisms from $\mathcal{H}$ to $\mathcal{G}$. \qed

**Lemma 3** Let $H = (V_1, V_2, E)$ be a bipartite graph with $n$ vertices and $m$ edges such that there is a vertex $u \in V_1$ which is adjacent to all vertices in $V_2$. Let $G$ be a graph with $N$ vertices and $pN^2/2$ edges, so $t_{K_2}(G) = p$. Then the number of homomorphisms from $H$ to $G$ is at least $(2n)^{-n^2}p^mN^n$.

**Proof:** Let $n_i = |V_i|$ for $i \in \{1, 2\}$. We will give a lower bound on the number of homomorphisms $f : V(H) \to V(G)$ that map $u$ to a good vertex $v$ of $G$. Suppose we have already picked $f(u) = v$. Let $\mathcal{H}$ be the hypergraph with vertex set $V_2$, where $S \subset V_2$ is an edge of $\mathcal{H}$ if there is a vertex $w \in V_1 \setminus \{v\}$ such that $N(w) = S$. The number of vertices of $\mathcal{H}$ is $n_2$, which is at most $n$, and the number of edges of $\mathcal{H}$ is $n_1 - 1$, which is also at most $n$. Let $\mathcal{G}$ be the hypergraph on $N(v)$, where a sequence $R$ of $k$ vertices of $N(v)$ is an edge of $\mathcal{G}$ if $|N(R)| \geq (2n)^{-(n-1)}p^kN$. Since $v$ is good, for each $k$, $1 \leq k \leq v$, the number of sequences of $k$ vertices of $\mathcal{G}$ that are not the vertices of an edge of $\mathcal{G}$ is at most $\frac{1}{2v}N^k$. Hence, by Lemma 2 there are at least $\sum_{v, v \text{ good}} \frac{1}{2} |N(v)|^{n_2}$ homomorphisms $g$ from $\mathcal{H}$ to $\mathcal{G}$. Pick one such homomorphism $g$, and let $f(x) = g(x)$ for $x \in V_2$. By construction, once we have picked $f(u)$ and $f(V_2)$, there are at least $(2n)^{-n^2}p^{N(w)/N}$ possible choices for $f(w)$ for each vertex $w \in V_1$. Hence, the number of homomorphisms from $H$ to $G$ is at least

$$\sum_{\text{good}} \frac{1}{2} |N(v)|^{n_2} \prod_{w \in V_1 \setminus \{v\}} (2n)^{-(n-1)p^{N(w)/N}} = \sum_{\text{good}} \frac{1}{2} (2n)^{-n(n-1)}p^{m-n_2}N^{n_1-1} \sum_{\text{good}} |N(v)|^{n_2} \geq \frac{1}{2} (2n)^{-n(n-1)}p^{m-n_2}N^{n_1-1} \left( \sum_{\text{good}} |N(v)/N|^{n_2} \right) \geq \frac{1}{2} (2n)^{-n(n-1)}p^{m-n_2}N^{n_1}(pN/2)^{n_2} \geq (2n)^{-n^2}p^mN^n.$$
The first inequality is by convexity of the function $q(x) = x^k$ and the second inequality is by the lower bound on the sum of the degrees of good vertices given by Lemma 1.

We next complete the proof of Theorem 1 by improving the bound in the previous lemma on the number of homomorphisms from $H$ to $G$ using a tensor power trick. The tensor product $F \times G$ of two graphs $F$ and $G$ has vertex set $V(F) \times V(G)$ and any two vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in $F \times G$ if and only if $u_i$ is adjacent with $v_i$ for $i \in \{1, 2\}$. Let $G^1 = G$ and $G^r = G^{r-1} \times G$. Note that $t_H(F \times G) = t_H(F) \times t_H(G)$ for all $H, F, G$.

**Proof of Theorem 1:** Suppose for contradiction that there is a graph $G$ such that $t_H(G) < t_{K_2}(G)^m$. Denote the number of edges of $G$ as $pN^2/2$, so $t_{K_2}(G) = p$. Let $c = t_H(G) / t_{K_2}(G)^m < 1$. Let $r$ be such that $c^r < (2n)^{-m^2}$. Then

$$t_H(G^r) = t_H(G)^r = c^r t_{K_2}(G)^{rm} = c^r t_{K_2}(G^r)^m < (2n)^{-m^2} t_{K_2}(G^r)^m.$$ 

However, this contradicts Lemma 3 applied to $H$ and $G^r$. This completes the proof.

**References**

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