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An analytical study of the dynamic behavior of Lotka-Volterra based models of COVID-19

Wael W. Mohammed \textsuperscript{a,e,}\textsuperscript{*}, E.S. Aly \textsuperscript{b}, A.E. Matouk \textsuperscript{c,d}, S. Albosaily \textsuperscript{a}, E.M. Elabbasy \textsuperscript{e}

\textsuperscript{a} Department of Mathematics, Faculty of Science, University of Ha'il, Ha'il 2440, Saudi Arabia
\textsuperscript{b} Mathematics Department, Faculty of Science, Jazan University, P.O. Box 218, Jasan, Saudi Arabia
\textsuperscript{c} Department of Mathematics, College of Science Al-Zulfi, Majmaah University, Al-Majmaah 11952, Saudi Arabia
\textsuperscript{d} College of Engineering, Majmaah University, Al-Majmaah 11952, Saudi Arabia
\textsuperscript{e} Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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ABSTRACT

COVID-19 has become a world wide pandemic since its first appearance at the end of the year 2019. Although some vaccines have already been announced, a new mutant version has been reported in UK. We certainly should be more careful and make further investigations to the virus spread and dynamics. This work investigates dynamics in Lotka-Volterra based Models of COVID-19. The proposed models involve fractional derivatives which provide more adequacy and realistic description of the natural phenomena arising from such models. Existence and boundedness of non-negative solution of the fractional model is proved. Local stability is also discussed based on Matignon’s stability conditions. Numerical results show that the fractional parameter has effect on flattening the curves of the coexistence steady state. This interesting foundation might be used among the public health strategies to control the spread of COVID-19 and its mutated versions.

Introduction

In the past months, many scientists have presented papers on the spread of the newly developed COVID-19 virus, which discuss how the virus has spread and proposals or expectations on how to treat it medically. Temperature and humidity are from the factors that affect the spread of corona virus. In our work, we will present an analytical study of the dynamic behavior of a new mathematical model of COVID-19 with which we will be interested in studying how the virus infection spreads and transmits from one person to another based on Lotka-Volterra modeling. We will be interested in studying the equilibrium of the dynamic system and finding numerical and existence of algebraic solutions to the system. This research will contribute to positive results in developing accurate, rapid and economical detection and monitoring mechanisms to reduce the risk of new corona virus [1–9]. Indeed, mathematical models can mimic the effects of the disease on many different factors and levels including high or low temperatures and high or low humidity effect. Data of susceptible, infected and recovered patients will be collected and analyzed from cold, temperate and hot countries. If we assume that we have a single patient and we begin to study how the disease affects the interactions between cells in that single patient who has become a host of the virus (models inside the host) then we discussed all the methods and factors that help and work to spread infection and virus from this person to the surrounding.

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Indeed, mathematical models can mimic the effects of the disease on many different factors and levels including high or low temperatures and high or low humidity effect. Data of susceptible, infected and recovered patients will be collected and analyzed from cold, temperate and hot countries. If we assume that we have a single patient and we begin to study how the disease affects the interactions between cells in that single patient who has become a host of the virus (models inside the host) then we discussed all the methods and factors that help and work to spread infection and virus from this person to the surrounding.

\textsuperscript{*} Corresponding author.

E-mail addresses: wael.mohammed@mans.edu.eg (W.W. Mohammed), elkhateeb@jazanu.edu.sa (E.S. Aly), aematouk@hotmail.com (A.E. Matouk).

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environment through several population groups supposed to be present in it through his daily life that may be separate (geographically). Recently, there exist models that simulate the spread of disease within cities and among the population, such as those used to predict a COVID-19 outbreak.

Fractional calculus has become chief tool to model physical, engineering and biological models [11-33]. Fractional calculus has also effective tools to describe epidemiological models because it provides more adequacies in estimating the natural behaviors of the model and it also provides higher degrees of freedom. Furthermore, the fractional models involve memory and hereditary properties which are essential to describe the behaviors of ecological models. On the other hand, Lotka-Volterra (LV) family of systems can be used to describe the dynamics of some real world models since the earliest models published by Alfred Lotka in 1925 [25] and V. Volterra in 1926 [26]; For example, LV modeling was used to explore interaction between demand forecasting and competing groups in industry [34]; LV modeling was also used to describe the business cycles in macroeconomics [35]. In addition, the fractional-order LV models showed better memory and hereditary properties in some ecological models [11-14],

In this work, we study some dynamical behaviors in Lotka-Volterra based models of COVID-19 with fractional order. We also discuss existence and boundedness of non-negative solution of the fractional model. We study the stability conditions of the model’s equilibrium points according to Matignon’s inequalities. We show the effect of fractional parameter on flattening the curves of the coexistence steady state of this mode that could be used among the public health strategies to mitigate the spread of COVID-19 and its mutated versions.

**Lotka-Volterra based models of COVID-19**

The mathematical model, we will discuss in this paper, is inspired form the classic Lotka-Volterra (LV) model [34-35] for simulation predator–prey dynamics. Here, we can assume that the virus acts as a predator that preys on individuals. Then, the continuous mutation processes of the virus, within the host or the infected individual population, have similar role as the feeding processes of the predator in the LV model. We will get the healthy individual population by \( y_1(t) \) at time \( t \). the infected individual population is given by \( y_2(t) \) at time \( t \). Let \( b > 0 \) represents the infection rate (1-protection rate), the immigration rate of healthy individuals is given by \( a > 0 \), and \( c > 0 \) will introduce the immigration rate of infected individuals. Finally, the death rate is given by \( d > 0 \) and the cure rate is given by \( e > 0 \).

\[
\frac{dy_1}{dt} = ay_1(t) - by_1(t)y_2(t) + ey_2(t),
\]

\[
\frac{dy_2}{dt} = by_1(t)y_2(t) + (c - d)e y_2(t).
\] (1)

Obviously, system (1) has two different equilibrium points; the trivial and coexistence points, respectively:

\[ E_1 = (0, 0), \quad E_2 = \left( \frac{e + d - c}{b} \frac{a(e + d - c)}{b(d - c)}, d \neq c, b > 0 \right) \] (2)

On the other hand, if model (1) involves fractional derivative, higher accuracy and degrees of freedom will be obtained. Therefore, we introduce the Caputo fractional differential operator as follows [36,37]:

\[
D^\alpha e(t) = \frac{1}{\Gamma(\sigma - \alpha)} \int_0^t (t-\eta)^{\sigma-1-\alpha} e^{\sigma(\eta)} d\eta, \quad \alpha > 0, \quad \sigma \in Z^+, \quad \alpha \in (\sigma - 1, \sigma).
\] (3)

Hence, replacing the integer-order derivatives with the fractional operator (3) into model (1), yields

\[ D^\alpha y_1(t) = ay_1(t) - by_1(t)y_2(t) + ey_2(t), \]

\[ D^\alpha y_2(t) = by_1(t)y_2(t) + (c - d)e y_2(t), \quad 0 < \alpha \leq 1. \] (4)

Remark 1. We can add some stochastic term for system (4) but for simplicity here we ignored it.

To prove existence and boundedness of non-negative solution of system (4), we introduce the following theorem:

**Theorem 1.** Any solution starts in the set

\[ \Psi = \{(y_1, y_2) \in \mathbb{R}^2_+ : y_i \geq 0, i = 1, 2, \sum_{i=1}^2 ny_i > 0, \eta = \max(a, c, d)\}, \]

is bounded.

then \( P = y_1(t) + y_2(t) \) Proof. Let

\[ D^\alpha P(t) \leq (P(t)), \] (5)

Applying the Laplace transform, the last inequality (5) implies that:

\[ \lim_{t \to \infty} E_1(-\eta^\alpha) = 0. \]

Obviously

\[ P(t) \leq P(0) E_1(-\eta^\alpha), \quad E_1(t) = \sum_{i=0}^{\infty} \left( \frac{t}{(1 + i\eta)} \right)^i \]

Therefore, any solution starts in this set is bounded. Our objectives are study the dynamics of model (4) as shown by following sections.

**Local Stability of model (4)**

The Jacobian of model (4) is

\[ J = \begin{pmatrix} a - by_2 & -by_1 + e \\ by_2 & by_1 + (c - d - e) \end{pmatrix}. \] (6)

Henceforth, we refer by LAS to the locally asymptotically stable equilibria. Based on Matignon’s stability theory [35], the equilibrium point of the linearized model (4) is LAS if all the eigenvalues \( \lambda_i, i = 1, 2 \) of the matrix \( J \) fulfill the following inequalities:

\[ |\arg(\lambda_i)| > \pi/2, \quad i = 1, 2. \] (7)

If at least one \( \lambda \) does not achieve inequalities (7), the equilibrium state is not LAS. Hence, we introduce the following stability results:

**Theorem 2.** The trivial equilibrium point of system (4) is not LAS. Proof. For the trivial equilibrium point \( E_1 = (0, 0) \), the related eigenvalues are \( \lambda_1 = a > 0, \lambda_2 = c - (d + e) \) which imply that \( |\arg(\lambda_1)| = 0 \). i.e., \( E_1 \) is not LAS. Moreover, if \( c < d + e \) then \( |\arg(\lambda_2)| = \pi \) that also implies that trivial point is saddle.

**Lemma 1.** If the immigration rate of infected individuals is more than the summation of death rate and cure rate, then the trivial equilibrium point is unstable node in the integer-order case.

**Theorem 3.** The coexistence equilibrium point of system (4) is LAS if the death rate is greater than the immigration rate of infected individuals. However, if the immigration rate of infected individuals is more than the summation of the death rate and the cure rate then the coexistence point is saddle.

Proof. For the coexistence point \( E_2 = ((-c + d + e)/b, -a(-c + d + e)/(bc - bd)) \), the matrix \( J \) has the eigenvalues \( \lambda_{1,2} = \frac{-c + d + e}{-c + d + e} \). When \( d > c, \) then \( \lambda_i, i = 1, 2 \) have negative real parts which imply that the coexistence point is LAS. Furthermore, if \( c > d + e \) then \( |\arg(\lambda_1)| = \pi \) and \( |\arg(\lambda_2)| = 0 \) which implies that the coexistence point is saddle.

**Lemma 2.** If the death rate is greater than the immigration rate of infected individuals, the coexistence point \( E_2 = ((-c + d + e)/b, -a(-c + d + e)/(bc - bd)) \) is stable node in the integer-order case provided that \( a > \frac{4(c-d)^2}{e^2(c+d+e)}. \)
Numerical results

Here, we discuss some COVID-19 data obtained for Saudi Arabia and collected during the period from March 18, 2020 until November 18, 2020. The death rate resulting from COVID-19 is approximately $d = 0.02$ and the cure rate is approximately $e = 0.98$. The infection rate is approximately $b = 0.01$. The immigration rate of healthy individuals is approximately $a = 0.012$ and the immigration rate of infected individuals is approximately $c = 0.0001$.

The model (4) is numerically integrated using the fore-mentioned values of model’s parameters and with orders $\alpha = 1$, $\alpha = 0.9$, $\alpha = 0.8$, $\alpha = 0.7$ and $\alpha = 0.5$. The simulation results are summarized in Figs. 1 and 2. The initial conditions used in Figs. 1, 2 are respectively, $(y_1, y_2) = (0.01, 0.01)$ and $(y_1, y_2) = (99.99, 60.3)$.

The results show that model (4) converges to the coexistence equilibrium point better than the integer-order counterpart (see Fig. 2). To explain this interesting foundation, we recall that the memory effect in the fractional case eliminates the oscillations in its integer-order case after some times. Therefore, model (4), with fractional case, is more adequate to describe the dynamics of the proposed Lotka-Volterra model of COVID-19. The simulation results also illustrate that the fractional parameters can be used as controllers to flatten the curves of infected individuals. So, such models may suggest better public health strategies to mitigate or slow down the fast spread of the deadly pandemic.

Conclusion

This work has investigated some dynamical behaviors in Lotka-Volterra based models of COVID-19. The proposed models involve the fractional-order case and its corresponding integer-order counterpart. The existence and boundedness of non-negative solution of the fractional model has been proved. Conditions for local stability have been discussed for both fractional and integer-order cases. The influence of the fractional parameter on mitigating the spread of the pandemic has been shown via numerical simulation results.

CRediT authorship contribution statement

W.W. Mohammed: Data curation, Formal analysis, Validation.
E.S. Aly: Data curation, Writing - original draft.
A.E. Matouk: Data curation, Software, Writing - original draft.
S. Albosaily: Data curation, Software.
E.M. Elabbasy: Formal analysis, Validation.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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