Contact Processes on Random Regular Graphs

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Abstract

We show that the contact process on a random $d$-regular graph initiated by a single infected vertex
obeys the “cutoff phenomenon” in its supercritical phase. In particular, we prove that when the infection
rate is larger than the critical value of the contact process on the infinite $d$-regular tree there are positive
constants $C, p$ depending on the infection rate such that for any $\varepsilon > 0$, when the number $n$ of vertices is
large then (a) at times $t < (C - \varepsilon) \log n$ the fraction of infected vertices is vanishingly small, but (b) at
time $(C + \varepsilon) \log n$ the fraction of infected vertices is within $\varepsilon$ of $p$, with probability $p$.

1 Introduction

The contact process with infection rate $\lambda > 0$ on a connected, locally finite graph $G = (V_G, E_G)$ is a
continuous-time Markov chain $(\xi_t)_{t \geq 0}$ with state space $\{\text{subsets of } V_G\}$ that evolves as follows: (1) each
infected site (that is, each vertex in $\xi_t$) recovers at rate 1, and upon recovery is removed from $\xi_t$; and (2)
each healthy site (that is, each vertex not in $\xi_t$) becomes infected at rate $\lambda$ times the number of currently
infected neighbors, and upon infection is added to $\xi_t$. See [12] for a formal construction, or alternatively [9]
for the standard graphical representation using independent Poisson processes.

The behavior of the contact process on the infinite $d$-regular tree $G = T_d$ is reasonably well understood.
When $d = 2$ (where $T_2 = \mathbb{Z}$) there is a single survival phase [12]. When $d \geq 3$, there are two survival phases:
in particular, there are critical values $0 < \lambda_1(T_d) < \lambda_2(T_d) < \infty$ such that if $\lambda \leq \lambda_1$, then the contact
process dies out almost surely; if $\lambda_1 < \lambda \leq \lambda_2$, then the contact process survives globally with positive
probability but dies out locally almost surely; and if $\lambda > \lambda_2$ then the contact process survives locally with
positive probability. (See [17] for the case $d \geq 4$ and [11, 18] for $d = 3$.) The parameter range $\lambda \in (\lambda_1, \lambda_2)$
is called the weak survival phase, and $\lambda > \lambda_2$ is the strong survival phase.

When $G$ is finite there is no survival phase, since the absorbing state $\emptyset$ is accessible from every state $\xi \in \{\text{subsets of } V_G\}$. Nevertheless, when the graph is large it will contain long linear chains, and so if
the infection rate is above the critical value $\lambda_1(\mathbb{Z})$ the contact process will, with non-negligible probability,
survive for a long time in a quasi-stationary state before ultimately dying out. This suggests several problems
of natural interest:

(A) How does the survival time scale with the size of the graph?
(B) What is the nature of the quasi-stationary state?
(C) How does the process behave in its initial “explosive” stage?

These questions have been studied for several important families of graphs, notably the finite tori [16],
finit regular trees of large depth [19], and versions of the “small worlds” networks of Watts and Strogatz [8]. Stacey has shown that when $G_L$ is a finite $d$-homogeneous rooted tree of depth $L$, the extinction time
of a contact process started from full occupancy in $G_L$ grows linearly in $L$ when $\lambda < \lambda_2(T_d)$; but when $\lambda > \lambda_2(T_d)$ it grows doubly exponentially in $L$, and almost exponentially in the number of vertices. In a more recent paper [10], it has been proved that the extinction time grows exponentially in the number of vertices.

In this paper we consider a different class of graphs, the \textit{random $d$-regular graphs}. These are of interest for a number of reasons: they are expanders, they are locally tree-like, and they are (unlike the finite trees) \textit{statistically homogeneous} in a certain sense. See [20] for a survey. The behavior of several common stochastic models on random $d$–regular graphs has been studied. Lubetsky and Sly [13] have shown that the simple random walk on a large random $d$-regular graph undergoes \textit{cutoff}, that is, the transition to stationarity occurs in a narrow time window. Bhamidi, Hofstad, and Hooghiemstra [2] have shown that distance between two randomly chosen vertices in first passage percolation on a random $d$–regular graph is concentrated around $C\log n$. Chatterjee and Durrett [5] have shown that the \textit{threshold} contact process on a random $d$–regular graph exhibits a phase transition in the infectivity parameter. More recently, Ding, Sly, and Sun [7] have shown that the independence number of a random $d$-regular graph is sharply concentrated about its median.

The main result of this paper is that the contact process on a random regular graph exhibits a \textit{cutoff} phenomenon analogous to that for the simple random walk. We shall assume throughout that $nd$ is even and $d \geq 3$. Let $G \sim \mathcal{G}(n,d)$ be a random graph uniformly distributed over the set of all $d$-regular graphs on the vertex set $V = [n]$. Given $G$, for any subset $A \subset [n]$, let $\xi^A_0 = A$ (when $A = \{u\}$ is a singleton, we will write $\xi^u_0$ for the contact process with initial state $\xi^u_0 = \{u\}$). We shall be primarily interested in the “meta-stable” phase, where the infection rate $\lambda$ exceeds $\lambda_1(T_d)$, and our main focus will be the following question: for a typical pair of vertices, what is the time needed for a contact process started from one vertex to infect the other? Since the diameter of a typical random regular graph is on the order of $\log n$, we expect the infection time to be on the same order. The main result of this paper, Theorem 1.1, implies that this conjecture is true.

We say that a property holds \textit{asymptotically almost surely} if the set of graphs in $\mathcal{G}(n,d)$ satisfying this property has probability approaching 1 as $n$ goes to infinity. To denote conditional probabilities and expectations given the graph $G$, we will use a subscript $P_G$ and $E_G$. Our main results are the following two theorems.

**Theorem 1.1.** Fix arbitrary $u \neq v \in [n]$, and let $G \sim \mathcal{G}(n,d)$ be a random $d$-regular graph on the vertex set $V = [n]$. For each $\lambda \geq \lambda_1(T_d)$ let $p_\lambda > 0$ and $c_\lambda > 0$ be the survival probability and growth rate (cf. Proposition 2.7) for the contact process started from the root on the infinite tree $T_d$ with infection rate $\lambda$. Let $\varepsilon > 0$. Then for any $0 < \varepsilon < 1/8$ there exist constants $g_n(\varepsilon) \to 0$ as $n \to \infty$ such that for asymptotically almost every $G$,

\[
P_G\{v \in \xi^u_s \text{ for some } s \leq (1 - \varepsilon)c_\lambda^{-1}\log n\} \leq g_n(\varepsilon)
\]

and

\[
P_G\{v \in \xi^u_{s(1+\varepsilon)c_\lambda^{-1}\log n}\} \geq (1 - g_n(\varepsilon))p_\lambda^2.
\]

**Theorem 1.2.** Let $\xi^G$ be a contact process with initial configuration $[n]$. Fix $\varepsilon > 0$; then for each $\delta > 0$ there exist constants $f_n(\delta) \to 0$ as $n \to \infty$ such that for asymptotically almost every $G$,

\[
P_G\{(1 - \delta)n\lambda \leq |\xi^G_{s(1+\varepsilon)c_\lambda^{-1}\log n}| \leq (1 + \delta)n\lambda\} \geq 1 - f_n(\delta),
\]

Assertion [1] will be proved in section 3, and assertion [2] in section 4. Theorem 1.2 will be proved in section 5. Throughout this paper we fix $\lambda > \lambda_1(T_d)$.

## 2 Preliminaries: Contact Process on the Infinite Regular Tree

In this section, $\xi_t$ will denote the contact process started from a single vertex $O$ (the \textit{root}) on the $d$-regular tree $T_d$. The $d$–regular tree is a non-amenable graph, in the sense that its Cheeger constant is positive. This
can be expressed as follows. For a finite subset $S$ of vertices of $\mathbb{T}_d$, call $v \in S$ a border point if among the $d$ connected components obtained by removing $v$ from $\mathbb{T}_d$, at least one of them contains no other vertices in $S$. Let $B(S)$ be the set of border points in $S$; then

$$|B(S)| \geq \left(1 - \frac{1}{d-1}\right)|S|.$$ 

See, for instance, Lemma 6.2 of [17] for a proof. We will denote by $h(\mathbb{T}_d)$ the constant $1 - 1/(d-1)$.

The nonamenability of $\mathbb{T}_d$ implies that the supercritical contact process on $\mathbb{T}_d$ grows exponentially. Here is a precise formulation, proved in [14] and [15].

**Proposition 2.1.** There exist constants $c_\lambda > 0$ and $C_d > 0$ such that

$$\exp(c_\lambda t) \leq \mathbb{E}(|\xi_t|) \leq C_d \exp(c_\lambda t).$$

We will make frequent use of an auxiliary process, the severed contact process. We follow the terminology and notation of [15] and [17]. Define a branch $\mathcal{B}$ to be the connected component of the root in the subgraph obtained by removing a distinguished subset of $d-1$ edges, each having an endpoint at the root $O$. The severed contact process is the contact process restricted to $\mathcal{B}$, i.e., infection is not allowed to travel across any of the $d-1$ removed edges. We will use the letter $\eta$ to denote the severed contact process; in particular, $\eta_t^S$ is the severed contact process with initial configuration $S \subset \mathcal{B}$, and $\eta_t^O = \eta_t^{(O)}$ the severed contact process started with $O$ infected at time $0$. Clearly, the severed contact process $\eta_t$ is stochastically dominated by the contact process $\xi_t$. In the standard graphical representation [9] contact process $\xi_t$ and the severed contact process are naturally coupled in such a way that $\eta_t^O \subset \xi_t^S$ for any initial configuration $S$ and all $t \geq 0$. Hence the expected cardinality of infected sites in the severed contact process is no larger than that of the original one. However, the following proposition states that the severed process has comparable cardinality in expectation.

**Proposition 2.2.** There exists a constant $A_1 = A_1(\lambda, d) > 0$ such that

$$\mathbb{E}|\eta_t| \geq A_1 \exp(c_\lambda t), \forall t \geq 0.$$

**Proof.** Inequality (5) of [15] states, in our notation, that

$$\int_0^t \mathbb{E}|\eta_s| ds \geq \frac{1}{\lambda} \left(\frac{1}{d} \mathbb{E}|\xi_t| - 1 \right).$$

On the other hand, it is easy to see that

$$\mathbb{E}|\eta_t| \leq \mathbb{E}|\xi_t| \leq \frac{C_d}{d} \exp(c_\lambda s).$$

Now fix a constant $T > 0$ (to be determined later); we have

$$\frac{1}{\lambda} \left(\frac{1}{d} \mathbb{E}|\xi_{nT}| - 1 \right) \leq \int_0^{nT} \mathbb{E}|\eta_s| ds$$

$$\leq \int_0^{(n-1)T} \frac{C_d}{d} \exp(c_\lambda s) ds + \int_{(n-1)T}^{nT} \mathbb{E}|\eta_s| ds$$

$$= \frac{C_d}{dc_\lambda} (\exp(c_\lambda (n-1)T) - 1) + \int_{(n-1)T}^{nT} \mathbb{E}|\eta_s| ds,$$

and therefore

$$\int_{(n-1)T}^{nT} \mathbb{E}|\eta_s| ds \geq \frac{1}{d\lambda} \mathbb{E}|\xi_{nT}| - \frac{1}{\lambda} - \frac{C_d}{dc_\lambda} (\exp(c_\lambda (n-1)T) - 1)$$

$$= \exp(c_\lambda nT) \left(\frac{1}{d\lambda} - \frac{C_d}{dc_\lambda} \frac{1}{\exp(c_\lambda T)}\right) + \left(\frac{C_d}{dc_\lambda} - \frac{1}{\lambda}\right),$$

3
This inequality holds for any \( T > 0 \), so we can choose \( T \) large enough that \( \frac{1}{\lambda} - \frac{C_d}{d \lambda} \exp (c \lambda T) > 0 \). Fix such \( T \); then there exists \( t_\star \in [(n - 1)T, nT] \) such that

\[
\mathbb{E} |\eta_{t_\star}| \geq \frac{1}{T} \left( \exp (c \lambda nT) \left( \frac{1}{d \lambda} - \frac{C_d}{d \lambda} \exp (c \lambda T) \right) + \left( \frac{C_d}{d} - \frac{1}{\lambda} \right) \right).
\]

Now we use the fact

\[
\mathbb{E} |\eta_{s - t}| \geq (\mathbb{E} (\exp (\lambda T)) - 1) \mathbb{E} |\eta_t|.
\]

This holds because if we run the severed contact process up to time \( s \), and then keep only those infected vertices that are border points of \( \eta_t \), and run severed contact processes from each of these border points for another \( t \) inside the unoccupied branch at time \( s \), the resulting infection set is dominated by the original severed contact process at time \( t + s \). (The extra \(-1\) is because the origin \( O \) has some edges removed and we might not be able to run a severed contact process from it) Therefore, for any time \( t \in [nT, (n + 1)T] \), we have

\[
\mathbb{E} |\eta_t| \geq (\mathbb{E} (\exp (\lambda T)) - 1) \mathbb{E} |\eta - t| \geq (\mathbb{E} (\exp (\lambda T)) - 1) \times \inf_{s \in [0, nT]} \mathbb{E} |\eta_s|,
\]

and so we conclude that

\[
\inf_{t \geq 0} \frac{\mathbb{E} |\eta_t|}{\exp (c \lambda t)} > 0.
\]

Next we will show that, conditioned on survival up to a large time \( t \), the cardinality of the contact process is concentrated around its expectation, at least in exponential rate.

**Proposition 2.3.** For any fixed \( \delta > 0 \), we have

\[
\mathbb{P} \{ \exp (c \lambda (1 - \delta) t) \leq |\xi_t| \leq \exp (c \lambda (1 + \delta) t) \mid |\xi_t| > 0 \} \to 1, \quad \text{as } t \to \infty.
\]

The proof relies on another useful estimate, Theorem 4 of Athreya [1]. Athreya’s result is proved for the case where \( p_0 = 0 \) (where \((p_i)_{i \in \mathbb{N}}\) is the offspring distribution of the branching process), but it can be generalized painlessly to the case \( p_0 > 0 \). We record this extension of Athreya’s theorem as Lemma 2.4.

**Lemma 2.4.** Suppose \( (Z_n)_{n \in \mathbb{N}} \) is a branching process with mean offspring number \( \mu > 1 \), and such that \( \mathbb{E} (\exp (\theta_0 Z_1) \mid Z_0 = 1) < \infty \) for some \( \theta_0 > 0 \). Then there exists \( \theta_1 > 0 \) such that

\[
\sup_{n \geq 1} \mathbb{E} \exp \left( \frac{\theta_1 Z_n}{\mu^n} \right) < \infty.
\]

**Proof of Proposition 2.3.** We will deal with the two inequalities in the event in (3) separately.

(A) First, we will show that

\[
\mathbb{P} \{ |\xi_t| \leq \exp (c \lambda (1 + \delta) t) \} \to 1, \quad \text{as } t \to \infty.
\]

Since the event \( \{ |\xi_t| > 0 \} \) has probability bounded away from 0, it will follow that the conditional probability given \( \{ |\xi_t| > 0 \} \) also converges to 1.

To prove (4), we will build a discrete-time branching random walk on the tree which stochastically dominates the contact process. Specifically, we define \((BRW)_{n \in \mathbb{N}}\) as follows:

1. \( BRW_0 = \xi_0 = \{ O \} \) and \( BRW_T = \xi_T \) (the state of the contact process at time \( T \)).

2. Given \( BRW_n \), for each particle in \( BRW_n \), run independent contact processes for time \( T \) starting at the locations of these particles. Notice that we allow multiple particles to occupy the same vertex. The set of all particles (and their locations) is defined to be \( BRW_n(T) \).
The process $BRW_{nT}$ is a discrete-time branching random walk, and so its cardinality $(|BRW_{nT}|)_n$ is a Galton-Watson branching process. The mean offspring number is $\mathbb{E}[\xi_T] \leq C_d \exp(c_d T)$. Moreover, the distribution of $|BRW_T|$ is dominated by a geometric distribution, because the contact process is dominated by a Yule process, which has a geometric distribution at any specific time, so the finite moment generating function assumption in Lemma 2.4 holds here. Consequently, Lemma 2.4 implies that if $T$ is sufficiently large, then
\[
P\{|\xi_{nT}| \geq \exp(c_\lambda (1+ \delta) nT)\} \to 0, \quad \text{as } n \to \infty.
\]
Since the branching random walk stochastically dominates the contact process, the result (1) follows, at least for $t$ in the arithmetic progression $\{nT\}_{n \geq 0}$. To extend (1) to all $t$, we use a simple “filling” argument, as follows.

Suppose that there exists a sequence of time points $t_n \to \infty$ such that $\mathbb{P} \{|\xi_{t_n}| \geq \exp(c_\lambda (1+ \delta) t_n)\} \geq \varepsilon > 0$ for some $\varepsilon$, and without loss of generality $t_n \in [k_n T, (k_n + 1)T)$. Assume that $T$ is large enough so that not only
\[
P\{|\xi_{nT}| \geq \exp(c_\lambda (1+ \delta) nT)\} \to 0, \quad \text{as } n \to 0,
\]
but also
\[
P\{|\xi_{nT}| \geq \exp(c_\lambda (1+ \delta/2) nT)\} \to 0, \quad \text{as } n \to 0.
\]
Given that $|\xi_{t_n}| \geq \exp(c_\lambda (1+ \delta) t_n)$, for each infection at $t_n$, there is a positive (and fixed) probability $p_T$ that it remains alive for at least time $T$, and therefore there is a constant order lower bound for the probability that at time $(k_n + 1)T$, there are at least $0.99 p_T \exp(c_\lambda (1+ \delta) k_n T)$ infections. This would contradict the fact that
\[
P\{|\xi_{nT}| \geq \exp(c_\lambda (1+ \delta/2) nT)\} \to 0, \quad \text{as } n \to 0,
\]
because when $n$ is large enough, $0.99 p_T \exp(c_\lambda (1+ \delta) nT) > \exp(c_\lambda (1+ \delta/2) (n+1)T)$.

(B) Next, we show that
\[
P\{|\xi_t| \geq \exp(c_\lambda (1- \delta) t) \mid |\xi_t| > 0\} \to 1, \quad \text{as } t \to \infty \tag{5}
\]
by showing that for each $\delta > 0$ there exists $T > 0$ such that
\[
P\{\liminf_{n \to \infty} \log(|\xi_{nT}|)/(nT) \geq C_\lambda (1- \delta) \mid |\xi_t| > 0, \forall t > 0\} = 1. \tag{6}
\]
Together with another “filling” argument, this will prove that (6) holds along the entire time axis, and (4) will follow easily. We will prove (6) by establishing the following two assertions:

Assertion 1: On the event that a severed contact process survives, the exponential growth rate is as desired.

Assertion 2: On the event that a contact process survives, we can find a surviving severed contact process embedded in it.

For Assertion 1, recall that for a severed contact process $\eta$, we have $\mathbb{E}[|\eta_t|] \geq A_1 \exp(c_\lambda t)$, by Proposition 2.2. Now construct a branching process as follows:

(1) At time 0 there is 1 particle at the root.

(2) Run the severed contact process up to time $T$, then keep only the infections on the “border”, i.e., infections which have at least one uninfected branch (at time $T$) of the tree connected to it.

(3) For all remaining infections, specify one uninfected branch for each infection, and run a severed contact process within this branch with the infection serving as the new “root”. Repeat (2) and (3).

Denote the cardinality of infected vertices at time $nT$ as $\{X_n\}$. The above branching process has mean offspring number
\[
\mathbb{E}X_1 \geq h(\mathbb{T}_d)\mathbb{E}|\eta_T| - 1 \geq h(\mathbb{T}_d)A_1 \exp(c_\lambda T) - 1 \geq \exp(c_\lambda (1- \delta) T),
\]
provided \( T \) is large enough. It is clear that \( X_n \leq |\eta_n T| \leq |\xi_n T| \). By the Kesten-Stigum theorem, if a branching process \( (X_n)_{n \in \mathbb{N}} \) with mean offspring number \( \mu \) satisfies \( \mathbb{E} X_1 \log_+ X_1 < \infty \), then \( \lim X_n/ \mu^n > 0 \) almost surely on the event of survival. This proves Assertion 1.

Assertion 2 is even easier. From Assertion 1, we know that the chance that the severed contact process survives is \( p > 0 \), so on the event \( |\xi_t| \geq M \), there will be at least \( |h(T_d)M| \) border points at time \( t \), and the chance there is at least one surviving severed contact process is at least \( 1 - (1 - p)^{|h(T_d)M|} \). On the event that the contact process survives, we know \( |\xi_t| \to \infty \) as \( t \to \infty \), so the chance that we can find a surviving severed contact process is 1.

As we will see in later sections, we will explore the random regular graph with the growth of the contact process, and it is important to know up to time \( t \) how big is the explored set compared to the cardinality of the infected set at time \( t \). Therefore we would like to investigate the growth rate of the quantity \( |\cup_{s \leq t} \xi_s| \).

The following proposition states that up to a constant factor, it is comparable to \( |\xi_t| \).

**Proposition 2.5.** There exists \( B_1 = B_1(\lambda, d) < \infty \) such that

\[
\exp(c_3 t) \leq \mathbb{E} |\cup_{s \leq t} \xi_s| \leq B_1 \exp(c_3 t), \forall t \geq 0.
\]

**Proof.** The first inequality \( \exp(c_3 t) \leq \mathbb{E} |\cup_{s \leq t} \xi_s| \) follows directly from Proposition 2.1. For the second inequality, we claim that

\[
\mathbb{E} \left| \cup_{u \leq s \leq t} \xi_u \right| \leq \mathbb{E} \left| \cup_{u \leq t} \xi_u \right| + \mathbb{E} |\xi_t| \times \mathbb{E} \left| \cup_{u \leq t} \xi_u \right|.
\]

This holds because if a site has been infected by time \( t + s \), there are two possibilities: either it was infected by time \( t \), or it is infected during the time interval \( [t, t + s] \) by an infection alive at time \( t \). These account for the two terms on the right hand side of (7).

Thus, for any fixed \( T > 0 \), we have

\[
\mathbb{E} \left| \cup_{u \leq (n+1)T} \xi_u \right| \leq \mathbb{E} \left| \cup_{u \leq nT} \xi_u \right| + \mathbb{E} |\xi_{nT}| \times \mathbb{E} \left| \cup_{u \leq T} \xi_u \right|,
\]

and

\[
\frac{\mathbb{E} \left| \cup_{u \leq (n+1)T} \xi_u \right|}{\mathbb{E} |\xi_{(n+1)T}|} \leq \frac{\mathbb{E} \left| \cup_{u \leq nT} \xi_u \right| + \mathbb{E} |\xi_{nT}| \times \mathbb{E} \left| \cup_{u \leq T} \xi_u \right|}{\mathbb{E} |\xi_{(n+1)T}|} \leq \frac{\mathbb{E} \left| \cup_{u \leq nT} \xi_u \right| + \mathbb{E} |\xi_{nT}| \times \mathbb{E} \left| \cup_{u \leq T} \xi_u \right|}{1/C_3^2 \mathbb{E} |\xi_{nT}| \mathbb{E} |\xi_T|} \leq \frac{C_3^2 \mathbb{E} \left| \cup_{u \leq nT} \xi_u \right|}{\mathbb{E} |\xi_T|} + \frac{C_3^2 \mathbb{E} \left| \cup_{u \leq T} \xi_u \right|}{\mathbb{E} |\xi_T|},
\]

so if we let \( r_n = \mathbb{E} \left| \cup_{u \leq nT} \xi_u \right|/\mathbb{E} |\xi_{nT}| \), then

\[
\sup_n r_n < \infty.
\]

As long as \( C_3^2/\mathbb{E} |\xi_T| < 1 \), which holds if we take \( T \) large enough, we conclude that \( \sup_n r_n < \infty \), that is,

\[
\mathbb{E} \left| \cup_{s \leq t} \xi_s \right|/\mathbb{E} |\xi_{nt}| < \infty.
\]

Using an argument similar to that in the proof of Proposition 2.2 we can extend this to

\[
\sup_{t \geq 0} \frac{\mathbb{E} \left| \cup_{s \leq t} \xi_s \right|}{\mathbb{E} |\xi_t|} < \infty,
\]

which translates to the desired inequality, by Proposition 2.1. \( \square \)
Proposition 2.6. For any fixed $\delta > 0$, we have
\[
\mathbb{P}\{\exp(c_\lambda(1-\delta)t) \leq \cup_{s \leq t} |\xi_s| \leq \exp(c_\lambda(1+\delta)t) \mid |\xi_t| > 0\} \to 1, \text{ as } t \to \infty.
\]

Proof. This follows by the same argument as in proof of part (i) of Proposition 2.3. \hfill \Box

Proposition 2.7. There exists $B_1 = B_1(\lambda, d) < \infty$ such that
\[
A_1 \exp(c_\lambda t) \leq \mathbb{E}\{\cup_{s \leq t} \eta_s\} \leq B_1 \exp(c_\lambda t), \forall t \geq 0,
\]
where $A_1$ is from Proposition 2.3.

Proof. This is an immediate corollary of Proposition 2.3. \hfill \Box

Next we will introduce an important concept that will figure prominently in the arguments of sections 4 and 5. For any vertex $x \in \xi_t$, say that $x$ is a pioneer point if $x \in B(\cup_{s \leq t} \xi_s)$, in other words, there exists a branch of the tree connected to $x$ which has been completely uninfected up to time $t$. We call such a branch a free branch. Notice that a pioneer point is automatically a border point of $\xi_t$. We will use $\zeta_t$ to denote the collection of pioneer points at time $t$. The next proposition describes the approximate size of $\zeta_t$ conditional on survival.

Proposition 2.8. For any $\delta > 0$,
\[
\mathbb{P}\{|\zeta_t| \geq \exp(c_\lambda(1-\delta)t) \mid |\xi_t| > 0\} \to 1, \text{ as } t \to \infty.
\]

Proof. From Proposition 2.3 and 2.6 we may assume that conditional on the event $\{ |\xi_t| > 0\}$, both of the following events occur:

1. $|\zeta_t| \geq \exp(c_\lambda(1-\delta)t)$;
2. $|\cup_{s \leq t} \xi_s| \leq \exp(c_\lambda(1+\delta)t)$.

Assuming these, it is easy to deduce that there are at least $(1 - o(1)) \exp(c_\lambda(1-\delta)t)$ vertices in $\zeta_t$ such that for each such vertex there is a branch connected to it which contains no more than $\exp(3c_\lambda \delta t)$ vertices in $\cup_{s \leq t} \xi_s$. Fix such a vertex $x \in \zeta_t$ and such a branch that has no more than $\exp(3c_\lambda \delta t)$ vertices in $\cup_{s \leq t} \xi_s$. Notice that the vertices that belong to $\cup_{s \leq t} \xi_s$ in this branch are connected. Therefore, there exists a path $y_0y_1 \ldots y_L$ in this branch, where $L \leq \log_{d-1}(\exp(3c_\lambda \delta t)) \leq 6d c_\lambda t$, such that

1. $y_0 = x$;
2. $y_i$ is connected to $y_{i+1}$ in this branch, for $0 \leq i \leq L - 1$;
3. $y_L$ is connected to a branch that has no vertices in $\cup_{s \leq t} \xi_s$.

Now run the contact process for another time $6d c_\lambda t$, and hope that at the end there is some chance of creating a pioneer point at $y_L$ by infections along the path $y_0y_1 \ldots y_L$. We will say that we have a successful infection event if all of the following events happen:

1. between time $[i, i + 1]$, the infection at $y_i$ infects $y_{i+1}$ before itself dies, and after getting infected, the vertex $y_{i+1}$ neither dies nor infects other vertices in the interval $[i, i + 1]$;
2. after $y_L$ first becomes infected between time $[L - 1, L]$, this infection neither dies, nor infects other vertices before time $6d c_\lambda t$.

The (conditional) probabilities of (1) and (2) are bounded away from 0. Given that a successful infection event occurs, the vertex $y_L$ becomes a pioneer point in $(\zeta_{(1+6d c_\lambda)t})$. On the other hand, a successful infection event will happen with chance at least $q^{6d c_\lambda t+1}$, where $q = \min(q_1, q_2) > 0$.

For different $y_0$'s, their corresponding successful infection events are mutually independent, and so the number of successful infection events stochastically dominates the Binomial distribution

\[
W \sim \text{Binomial}((1 - o(1)) \exp(c_\lambda(1-\delta)t), q^{6d c_\lambda t+1}),
\]
and as long as $\delta$ is sufficiently small, with probability approaching 1,

$$W > (1 - o(1)) \exp(c_\lambda(1 - \delta)t)q^{6\delta c_\lambda t + 1} > \exp(c_\lambda(1 - D\delta)t),$$

for some $D > 0$. Therefore, conditional on the event $|\xi_t| > 0$, with probability approaching 1,

$$|\zeta_{(1+6\delta c_\lambda)t}| > \exp(c_\lambda(1 - D\delta)t).$$

This is essentially the desired conclusion if $\delta$ is small enough.

Similarly, for the severed contact process $\eta_t$, we can define pioneer points to be those vertices in $\eta_t$ that are also border points of $\bigcup_{s \leq t} \eta_s$. Denote the set of such pioneer points by $\psi_t$. By the same proof as for Proposition 2.8, we obtain the following proposition.

**Proposition 2.9.** For any $\delta > 0$,

$$\mathbb{P}\{|\psi_t| \geq \exp(c_\lambda(1 - \delta)t) \ | \ |\eta_t| > 0\} \to 1, \text{ as } t \to \infty.$$ 

Finally, we show that the event that the severed contact process grows exponentially faster than it is supposed to is exponentially unlikely.

**Proposition 2.10.** For all $\delta > 0$, there exists a constant $K > 0$ and $\gamma > 1$ such that

$$\mathbb{P}\{|\bigcup_{s \leq t} \eta_s| \geq \exp((1 + \delta)c_\lambda t)\} \leq \exp(-K\gamma^t), \text{ as } t \to \infty.$$ 

**Proof.** This is an immediate result from Proposition 2.3 and 2.7 once we observe that $(|\bigcup_{s \leq nT} \eta_s|)_n$ is dominated by a branching process with mean offspring number $\mathbb{E}|\bigcup_{s \leq T} \eta_s|$. □

### 3 Contact Process on a Random Regular Graph

#### 3.1 The configuration model for random regular graphs

A **random $d$-regular graph** is a graph chosen uniformly from the collection $G(n, d)$ of all $d$-regular graphs on the vertex set $[n]$. We assume that $dn$ is even. A useful way to construct a random $d$-regular graph is the **configuration model** introduced by Bollobás [3] (also see [4] and [20]). This works as follows. To each of the $n$ vertices $u$, associate $d$ distinct half-edges $(u, i)$, and perform a uniform perfect matching on these $dn$ half-edges. Using this matching, construct a (multi-)graph by placing an edge between vertices $u$ and $v$ for every pair of half-edges $(u, i)$ and $(v, j)$ that are matched. The resulting graph need not be connected, and it might have multiple edges and self-loops; however, the probability that the configuration model produces a simple, connected graph is bounded away from 0 as $n \to \infty$ (cf. [20]). Moreover, given that the resulting graph is simple (that is, has no self-loops or multiple edges), it is uniformly distributed over $G(n, d)$. Thus, whenever an event holds w.h.p. for the (multi-)graph obtained from the configuration model, it also holds w.h.p. under the uniform distribution on $G(n, d)$.

An important feature of the configuration model is that, at any stage, the first half-edge in the next random pair can be selected using any rule, as long as the second half-edge is chosen uniformly at random from the remaining half-edges (see [20]).

#### 3.2 Growth and exploration constructions

##### 3.2.1 Vanilla version

There are two layers of randomness in our model: first, the graph $G$ is chosen from the uniform distribution on the set $G(n, d)$ of $d$-regular graphs with vertex set $[n]$, and then a contact process is run on $G$. We would
like to construct a probability space where we use the configuration model to grow the contact process and the random graph $G$ in tandem, building edges of $G$ only at those times $t$ when the contact process attempts a new infection from a vertex whose neighborhood structure is not yet completely determined.

Suppose we want to run a contact process $(\xi_t^*)_t$ with initial configuration $\{u\}$. Let $U_t$ be the set of unmatched half-edges up to time $t$. At time 0, only vertex $u$ is infected, and no edges are yet determined, so $U_0$ is full, that is, it contains all $nd$ half-edges. The recovery and infection times of the contact process are determined by a system of independent Poisson processes attached to the vertices of the graph, two to each vertex (one for recoveries, the other for outgoing infections).

At any time $t$ when an infected vertex $v$ attempts an infection, one of the $d$ half-edges incident to $v$ is selected uniformly at random. If this half-edge is already matched to a half-edge $(w,j)$ then vertex $w$ is infected, if it is currently healthy, or left infected if already infected. If, on the other hand, one of the unmatched half-edges $(v,i)$ incident to $v$ is selected then one of the other remaining unmatched half-edges $(w,j)$ is chosen at random from $U_t\backslash\{(v,i)\}$ and matched with $(v,i)$, and vertex $w$ is infected. After this we remove $(v,i)$ and $(w,j)$ from $U_t$.

We will refer to this construction as the vanilla version of the grow and explore process, and denote by $P$ and $E$ the probability and expectation operator for the probability space on which the underlying Poisson processes and other random variables used in the matchings and infection attempts are defined. If we run the grow and explore process up to time $T$, it is possible that $U_T$ will not be not empty; in this case we match the remaining half-edges in $U_T$ to complete the graph $G$. We condition on the event that the obtained graph is simple.

**Proposition 3.1.** If the grow and explore process is run up to time $T$, then, conditional on the event that the resulting graph $G$ is simple, the pair $(G, (\xi_t^*)_{0\leq t\leq T})$ will have the same joint distribution as for the contact process on a random regular graph.

**Proof.** First of all, $G$ is uniform over $\mathcal{G}(n,d)$, because whenever we pair two unmatched half-edges, the second half-edge is always chosen uniformly at random from the unmatched pool.

Secondly, the interoccurrence times between infection attempts are i.i.d. exponentials, and in each attempt the active infection chooses one of its neighbors at random and independent of everything else. Meanwhile each vertex allows at most one infection at a time. Therefore $(\xi_t^*)_{0\leq t\leq T}$ is a version of the contact process on $G$.

The above construction of the grow and explore process assumes a singleton initial configuration and that the graph is initially completely unexplored. It is clear that the construction can be trivially modified so as to work with an arbitrary initial configuration and with part of the graph initially explored.

### 3.2.2 Cover tree version with singleton initial configuration

Next we describe a variation of the grow and explore process, in which we grow a contact process $\tilde{\xi}_t$ on the infinite cover tree and in tandem assign labels $v \in [n]$ to the vertices of $T = T_d$ in such a way that $\tilde{\xi}$ partially projects, via the labeling $\phi$ of vertices, to a contact process $\xi_t$ on $G$. The assignment of labels to vertices of $T_d$ will result in a (random) labeling function
\[
\phi : \mathcal{V}_{T_d} \rightarrow [n]
\]
that will determine the covering map from $T_d$ to $G$ and the edge structure of $G$, as well as the projection mapping from the vertex set of $T_d$ to that of $G$. The construction will require that some – but not all – of the vertices in $\tilde{\xi}_t$ be selected for projection to $\xi_t$; thus, at any time $t$ the set $\tilde{\xi}_t$ will be partitioned as $\tilde{\xi}_t = \tilde{\xi}_t,\text{BLUE} \cup \tilde{\xi}_t,\text{RED}$, and
\[
\xi_t = \phi(\tilde{\xi}_t,\text{BLUE}).
\]
Where appropriate, we will denote vertices of $T_d$ via a tilde, e.g., $\tilde{x}$, and use $x$ to denote the corresponding vertex $x \in [n]$, so that $\phi(\tilde{x}) = x$.

Fix a vertex $u \in [n]$; the singleton $\{u\}$ will be the initial configuration of the (projected) contact process on $G$. Denote the root vertex of the infinite tree $T_d$ by $\tilde{u}$, and declare $\phi(\tilde{u}) = u$. Let $\tilde{\xi}_t$ be a contact process on $T_d$ with initial configuration $\tilde{\xi}_0 = \tilde{u}$; assume that this is constructed in the usual way, using independent Poisson processes attached to the vertices $\tilde{v}$ of $T_d$ to determine the times at which recoveries and attempted infections occur, and independent uniform random variables to determine which neighbor of a vertex $\tilde{v}$ will be selected when vertex $\tilde{v}$ attempts an infection. At time $t = 0$ only the label $\phi(\tilde{u}) = u$ is determined; the function $\phi$ is augmented only at those times when a blue vertex of $\tilde{\xi}_t$ attempts an infection. The rules by which these augmentations occur are as follows.

Suppose at time $t$, an infected vertex $\tilde{x} \in \tilde{\xi}_t,\text{BLUE}$ attempts an infection. At this time $t$ some of the neighbors of $\tilde{x}$ might have been labeled, and others might not; denote by $\tilde{y}_1, \ldots, \tilde{y}_k$ the neighbors that have been labeled, with $\phi(\tilde{y}_1) = y_1$, and by $\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_{d-f}$ the neighbors that have not yet been labeled. Moreover, at this time some of the neighbors of $x$ in $G$ will have been determined, including $y_1, y_2, \ldots, y_k$, but possibly also some others, which we denote by $y_{t+1}, y_{t+2}, \ldots, y_{t+k}$, where $t + k \leq d$. Because the infection attempt entails choosing one of the $d$ neighbors of $\tilde{x}$ at random to serve as the target of the attempt, there are 3 possibilities:

1. With probability $l/d$, one of the vertices $\tilde{y}_1, \ldots, \tilde{y}_k$ is chosen. In this case $\phi$ is not augmented.
2. With probability $k/d$, one of the vertices $\tilde{z}_1, \ldots, \tilde{z}_{d-f}$ is chosen randomly, say $\tilde{w}$, and one of the labels $y_{t+1}, \ldots, y_{t+k}$ is chosen uniformly at random to serve as the label $\phi(\tilde{w})$ for the vertex $\tilde{w}$.
3. With probability $1 - l/d - k/d$, an unused half-edge $(x, i)$ incident to $x$ is chosen randomly, and another unused half-edge $(w, j)$ then chosen randomly from among all remaining unused half-edges and matched with $(x, i)$. When this happens we add an edge connecting $x$ and $w$ to $G$ and remove the two half-edges $(x, i)$ and $(w, j)$ from the set of unused half-edges.

To complete the construction, we must specify how the vertices of $\tilde{\xi}_t$ are to be colored (red or blue). This is done as follows. First, a vertex $\tilde{v}$ is assigned a color only at a time $t$ when it enters (or re-enters, if it was previously infected but subsequently recovered) $\tilde{\xi}_t$. Second, if $\tilde{v}$ is infected by a red vertex $\tilde{x}$ then it is colored red. Third, if $\tilde{v}$ is infected by a blue vertex $\tilde{x}$ then it is colored blue unless the label $v$ assigned to $\tilde{v}$ has also been assigned to one of the other vertices of $\tilde{\xi}_t,\text{BLUE}$; in this case $\tilde{v}$ is colored red.

Notice that in this construction, when $\phi(\tilde{\xi}_t,\text{BLUE})$ attempts an infection, any of its $d$ neighbors is equally likely to be the target of the infection attempt, and when a new edge is added to $G$ it follows the configuration model. Therefore, the projection obeys the same rules as in the vanilla version of grow and explore described above. This proves the following.

**Proposition 3.2.** The pair $(G, (\phi(\tilde{\xi}_t,\text{BLUE}))_{0 \leq t \leq T})$ obtained by running the cover tree version of grow and explore has the same law as in the vanilla version of the grow and explore process.

The cover tree version of grow and explore has two constituent processes: the contact process $\tilde{\xi}_t$ on the cover tree and the labeling process. We will call these the grow process and the explore process, respectively.

To emphasize the initial configuration $\{u\}$, its corresponding contact process on the cover tree is denoted as $\tilde{\xi}_t^u$. Later we will run several contact processes on multiple vertices, and adding the superscript will help us distinguish them.

The following figures illustrate concrete examples of how the construction works.
Figure 1: This graph records the order that vertices on a 3-regular cover tree are first infected. 0 indicates the vertex is in the initial configuration.

Figure 2: Given Figure 1, we try to project it onto the finite 3-regular graph. Suppose the top vertex has already been labelled as a, we sequentially label vertex 1,2,3,4 according to the law we describe before. When we label vertex 3 we accidentally use label c again, so the finite graph immediately becomes the one on the right. Remember that we only allow one infection per vertex, so whenever we observe multiple infections at the same time on a vertex we will remove the whole infection trail except the chronically first one.

Figure 3: Given Figure 1 and 2, suppose the infection trail coming from vertex 3 to vertex 4 is not removed, then when we label vertex 4, according to our law it has chance 1/2 to be b and chance 1/2 to be drawn from R. The left graph is on the cover tree and the right one is on the finite graph.

3.3 Proof of Assertion (1) of Theorem 1.1

Let $\xi_t^u = \phi(\tilde{\xi}_t, \text{BLUE})$ be the contact process constructed from the cover tree version of the grow and explore process. Let $t_1 = (1 - \varepsilon) \log n / c_\lambda$. To prove (1), it suffices to show that

$$P\{\exists s \leq t_1 \text{ such that } v \in \xi^u_s \} \to 0, \text{ as } n \to \infty.,$$

(8)

because if $P\{\exists s \leq t_1 \text{ s.t. } \xi^u_s(v) = 1\} \leq l_n$ and $l_n \to 0$ as $n \to \infty$, then by Markov’s inequality,

$$P\{G : P_G(\exists s \leq t_1 \text{ such that } v \in \xi^u_s) \geq \sqrt{l_n}\} \leq \sqrt{l_n}.$$

Now the event that there exists $s \leq t_1$ such that $v \in \xi^u_s$ coincides with the event that at least one vertex in $\cup_{s \leq t_1} \xi^u_s$ is labelled $v$. Consequently, to bound the probability of this event, it suffices to show that

(1) as $n \to \infty$, $P\{| \cup_{s \leq t_1} \tilde{\xi}_s^u | \leq n^{1 - \varepsilon/2}\} \to 1$; and
(2) Given $|\cup_{s \leq t_1} \tilde{\xi}_s^v| \leq n^{1-\varepsilon/2}$, the conditional probability that label $v$ is not used in the labeling process before time $t_1$ approaches 1.

Assertion (1) follows directly from Proposition 2.8. To prove (2), observe that, because all labels other than $u$ are equally likely to be used in the labeling process, and because these probabilities add up to at most $n^{1-\varepsilon/2}$, the chance that label $v$ appears in $\cup_{s \leq t_1} \tilde{\xi}_s^v$ is at most $n^{1-\varepsilon/2}/(n-1) \to 0$ as $n \to \infty$.

### 3.4 Cover tree version with general initial configuration

In the next section we will build the contact process with generic initial configuration, therefore we need to generalize our grow and explore process. Suppose the initial configuration of the contact process is a set $\xi_0 = \{u_1, u_2, \ldots, u_k\} \subset \{1, 2, \ldots, n\}$, and some of the half-edges are already paired up. We will construct a cover tree for each of the $u_i$’s, and assign a labeling function at each of these cover trees, i.e., we have

$$\phi_1 \times \phi_2 \times \cdots \times \phi_k : \mathcal{V}_{\mathcal{T}_d}^k \to [n]^k,$$

where $\mathcal{V}_{\mathcal{T}_d}^k$ is the cartesian product of $k$ copies of $\mathcal{V}_{\mathcal{T}_d}$ and $[n]^k$ is the cartesian product of $k$ copies of $[n]$.

At each $u_i$, we associate an independent contact process on $\mathcal{T}_d$. We call it $(\tilde{\xi}_s^{u_i})_{s \geq 0}$, with $\tilde{\xi}_0^{u_i} = \tilde{u}_i$ and $\phi_i(\tilde{u}_i) = u_i$. Similar to the definition in section 3.2, each $\tilde{\xi}_t^i$ is partitioned as $\tilde{\xi}_t^{i, \text{BLUE}} \cup \tilde{\xi}_t^{i, \text{RED}}$, and

$$\xi_t = \bigcup_{i=1}^k \phi_i(\tilde{\xi}_t^{i, \text{BLUE}}).$$

When there is an infection attempt in $\tilde{\xi}_t^{i, \text{BLUE}}$, the law of updating the labeling function $\phi_i$ is exactly the same as in section 3.2, such that whenever $\xi_t$ attempts an infection, any of its $d$ neighbors is equally likely to be the target. Notice now all $\phi_i$’s share the same unused pool of half-edges. The rule for coloring $\tilde{\xi}_t^i$ is essentially the same as in section 3.2, i.e., descendants of red vertices are red; descendants of blue vertices are blue unless there is another vertex in $\tilde{\xi}_t^{j, \text{BLUE}}$ for some $j$ that has the same label assigned to it, in which case the newborn infection is colored red. It is not hard to obtain the following analogy of Proposition 3.2.

**Proposition 3.3.** The pair $(G, (\xi_t)_{0 \leq t \leq T})$ obtained by running the above version of grow and explore has the same law as in the vanilla version of the grow and explore process.

In following sections we slightly abuse notations by ignoring the index of $\phi_i$ and use $\phi$ as the only labeling function.

Similarly we can substitute the process on the tree with the severed contact process and use the same labeling law. This will be useful in the next section.

### 3.5 Building independent contact processes

Suppose now we would like to build several independent contact processes, for example two independent contact processes with different initial configuration, $(\xi_t^0)_{0 \leq t \leq T_1}$ and $(\xi_t^0)_{0 \leq t \leq T_2}$ on the same random regular graph. To do so, we can first run the grow and explore process to obtain $(\tilde{\xi}_t^0)_{0 \leq t \leq T_1}$ and the labeling function $\phi$. After that, we run an independent contact process on the cover tree, $(\tilde{\xi}_t^0)_{0 \leq t \leq T_2}$. In order to construct the labeling function for $(\tilde{\xi}_t^0)_{0 \leq t \leq T_2}$, we have to be consistent with what has been explored on the finite graph so far. That is, besides obeying the rule of constructing the labeling function in previous sections, the pool of unused half-edges is whatever remained after labeling $(\tilde{\xi}_t^0)_{0 \leq t \leq T_1}$. In other words the processes on the cover trees are independent, while the labeling process have to be mutually consistent.

It is not hard to see that the processes obtained in this manner have the desired distribution. Moreover, the order of labeling the processes on the tree does not matter: we can instead label $(\tilde{\xi}_t^0)_{0 \leq t \leq T_2}$ and then $(\tilde{\xi}_t^0)_{0 \leq t \leq T_1}$ without changing the distribution of the processes on the finite graph.
4 A Second Moment Argument

4.1 Heuristics and Strategy

In this section we shall prove assertion (2) of Theorem 1.1. This states that for any two vertices $u, v \in [n]$ the conditional probability, given the graph $G$, that $v \in \xi^u_t$ converges to $p^2_\lambda$ as $n \to \infty$, where $t_+ := (1 + \varepsilon) c_1^{-1} \log n$. The rationale, in brief, is as follows. For most vertices $u$ and most $d$-regular graphs $G$ the contact process $\xi^u_t$ on $G$ looks – at least locally – like a contact process on the infinite regular tree $\mathbb{T}_d$ initiated by a single infected vertex $\hat{u}$ at the root. The chance that such a contact process survives is $p_\lambda$, so the chance that the contact process $\xi^u_t$ on $G$ survives for a significant amount of time (call this event quasi-survival) should also be about $p_\lambda$.

The contact process is self-dual, in particular, the Poisson processes used in the standard graphical construction can be reversed without change of distribution. Thus, the event that $v \in \xi^u_t$ has the same $\mathbb{P}_G$-probability as the event that $u \in \xi^v_t$, and these events have the same $\mathbb{P}_G$-probability that two independent contact processes $\xi^u_t$ and $\xi^v_t$ started at $u$ and $v$ will intersect at time $t/2$. But this will only happen if both contact processes survive for time $t/2$, and for large $t$ the probability of this will be about $p^2_\lambda$. Hence, for large $t$, with high probability,

$$
\mathbb{P}_G\{v \in \xi^u_t\} = \mathbb{P}_G\{\xi^u_{t/2} \cap \xi^v_{t/2} \neq \emptyset\} \leq p^2_\lambda(1 + o(1)).
$$

(9)

This argument shows that $p^2_\lambda$ is the largest possible asymptotic value for the probability in relation (2). To show that this value is actually attained, we will show that conditional on the event of quasi-survival for two independent contact processes $\xi^u_t$ and $\xi^v_t$, the random sets $\xi^u_{t+2}$ and $\xi^v_{t+2}$ will almost certainly overlap. For this, we will argue that on the event of quasi-survival, the cardinality of $\xi^u_{t+2}$ will be at least $n^{1/2+\delta}$ for some $\delta > 0$ depending on $\varepsilon$, and that $\xi^v_{t+2}$ is approximately distributed as a random subset of $[n]$ of cardinality $n^{1/2-\delta}$. Since two such independent random subsets will intersect with high probability, this suggests that

$$
\mathbb{P}_G\{\xi^u_{t+2} \cap \xi^v_{t+2} \neq \emptyset\} \approx p^2_\lambda.
$$

Unfortunately, because the labeling processes used in constructing the two contact processes $\xi^u_t$ and $\xi^v_t$ will interfere, it will turn out that the random sets $\xi^u_{t+2}$ and $\xi^v_{t+2}$ are not independent, and so a more circuitous argument will be needed.

Henceforth, let $t_1 = (1 - \varepsilon) \log n/c_\lambda$, $t_2 = (1 + 3\varepsilon) \log n/c_\lambda$ and $\Delta t = t_2 - t_1 = 2\varepsilon \log n/c_\lambda$. Notice that $t_1 + t_2 = t_+$. We will run $\xi^u_t$ for $t_2$ and $\xi^v_t$ for $t_1$. In order to show $\mathbb{P}\{\xi^u_{t_2} \cap \xi^v_{t_1} \neq \emptyset\} \geq (1 - o(1))p^2_\lambda$, we will apply the second moment method. Let us briefly describe the idea.

Fix $0 < \varepsilon < 1/8$ and let $0 < \delta < \varepsilon$. We will first grow $\xi^u_t$ up to time $t_1$, and $\xi^v_t$ up to time $t_1 - 1$. We call the processes to be in stage 1. Conditional on the event that both contact processes survive (which has probability about $p^2_\lambda$), with high probability, the sets of pioneer points, $|\xi^u_{t_1-1}|$ and $|\xi^v_{t_1-1}|$ will be moderately large (at least $n^{(1-\varepsilon)(1-\delta)/2}$), while $|\cup_{s \leq t_1} \xi^u_s|$ and $|\cup_{s \leq t_1} \xi^v_s|$ will not be too large (no more than $n^{(1-\varepsilon)(1-\delta)/2}$). Based on these, with high probability, in the exploration process, we observe (1) the labels assigned to $\cup_{s \leq t_1} \xi^u_s$ have no overlap with the labels assigned to $\cup_{s \leq t_1} \xi^v_s$, and (2) the labeling in $\cup_{s \leq t_1} \xi^u_s$ does not lead to coexisting infections on the same site in the finite graph, and thus $|\xi^u_{t_1}| = |\xi^v_{t_1}|$, for all $s \leq t_1$, and the same holds for $|\xi^v_{t_1}|$ up to time $t_1 - 1$. If all above events occur (which has probability $(1 - o(1))p^2_\lambda$), we call it a good event.

Our goal is to show that conditional on the good event happening in stage 1, if we run $\xi^u_t$ for another time $\Delta t$, and run $\xi^v_t$ for another time $1$, then with high probability, there will be at least one common label between them at the end. We call this period to be stage 2. It suffices to consider certain subsets of both contact processes which are much easier to deal with. At the end of stage 1, we only keep the pioneer points of $\xi^u_{t_1}$. For each such pioneer point (call it $i$), together with its free branch, we run an independent severed contact process $(\tilde{\eta}^u_i)_{s \geq 0}$ inside this branch for another duration of $\Delta t$, and label all of them. We run a similar
process for each pioneer point of $\tilde{\eta}_{\Delta}^{v}$ (call it $j$, and the severed contact process $\hat{\eta}_{\Delta}^{v}$) for time 1, and label all of them. Such constructed $\phi(\cup \hat{\eta}_{\Delta}^{v})$ should be regarded as a subset of $\xi_{t_{1}}^{u}$, and $\phi(\cup j \hat{\eta}_{\Delta}^{v})$ should be regarded as a subset of $\xi_{t_{1}}^{u}$.

For each pair of $(i, j)$, let $I_{i,j}$ be the event that at the end of stage 2 we observe a common label assigned between $\hat{\eta}_{\Delta}^{v}$ and $\hat{\eta}_{\Delta}^{w}$ in a very specific way which we will state later. We will use a second moment argument to show that $\sum_{i} \sum_{j} I_{i,j} \rightarrow \infty$ high with probability, which implies that $\xi_{t_{1}}^{u} \cap \xi_{t_{1}}^{w} \neq \emptyset$ with high probability.

Let us formulate the terminologies above in more details. We define $F$ (the good event mentioned above) to be the event that all of the following events happen in stage 1:

1. $|\tilde{\xi}_{s_{1}}^{u}| \geq n^{(1-\varepsilon)(1-\delta)/2}$.
2. $|\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}| \leq n^{(1-\varepsilon)(1+\delta)/2}$.
3. In the exploration process of $|\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}|$, there is not a pair of distinct vertices in $|\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}|$ on the cover tree that are assigned the same label on the finite graph. The same holds for $|\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{w}|$.

Notice that on $F$, all infections in $\tilde{\xi}_{s_{1}}^{u}$ and $\tilde{\xi}_{s_{1}}^{w}$ are colored BLUE. Without loss of generality we will label $\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}$ first, then $\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{w}$.

**Proposition 4.1.** $\mathbb{P}\{F\} \geq (1 - o(1))p_{\lambda}^{2}$, as $n \rightarrow \infty$.

**Proof.** For the two contact processes on the cover tree, with probability at least $p_{\lambda}^{2}$, one survive up to time $t_{1}$ and the other up to time $t_{1} - 1$. Then it follows from Proposition 2.6 and 2.8 that (1) and (2) in the definition of $F$ hold simultaneously with probability at least $(1 - o(1))p_{\lambda}^{2}$.

Given (1) and (2), (3) is a statement on the exploration process, and we will use the configuration model to estimate its probability. When we first label the vertices in $\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}$, (3) requires that whenever there is a vertex to be labeled, it cannot use any of those labels already used. For example, $u$ is automatically the first label; for the second vertex, (3) requires it not to be labeled $u$. Therefore, the chance that the labeling in $\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}$ is consistent with (3) is

$$\prod_{m=1}^{A-1} \left(1 - \frac{(d-2)m + d + 1}{dn - 2m + 1}\right)^{B-1},$$

where $A = |\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}| \leq n^{(1-\varepsilon)(1+\delta)/2}$.

Similarly, given the labeling in $|\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{v}|$ is consistent with (3), the chance that the labeling in $|\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{w}|$ comply with (3), is given by

$$\prod_{m=1}^{B-1} \left(1 - \frac{(d-2)(A-1) + (d-2)(m-1) + 1}{dn - (2A - 2) - 2m + 1}\right)^{A-1},$$

where $B = |\cup_{s \leq t_{1}} \tilde{\xi}_{s}^{w}| \leq n^{(1-\varepsilon)(1+\delta)/2}$.

So the chance that (3) holds given (1) and (2) is the product of (10) and (11), which is no less than

$$\left(1 - \frac{(d-2)(A + B - 2) + 1}{dn - o(n)}\right)^{A+B} = \exp\left(-O\left(\frac{(A + B)^{2}}{n}\right)\right) \geq \exp\left(-O(n^{-\varepsilon-\delta+\delta})\right) = 1 - o(1),$$

as $n \rightarrow \infty$,}
and therefore the $\mathbb{P}$-probability of observing $F$ is at least $(1 - o(1))p_\lambda^2$. \hfill \Box

From now on, we assume that in stage 1, $F$ happens. Notice on $F$, there is a one-to-one correspondence between infections on the cover tree and infections on the finite graph up to time $t_1$ ($t_1 - 1$) for the contact processes started from $u$ ($v$). For each pioneer point $i \in \zeta_u t_1$ together with its free branch, let $(\tilde{\eta}_s^i)_s \geq 0$ denote the severed contact process in this branch with $\{i\}$ being the initial configuration. Similarly, for each pioneer point $j \in \zeta_{v t_1 - 1}$ let $(\tilde{\eta}_s^j)_s \geq 0$ denote the corresponding severed contact process. See Figure 4 for a graphical illustration.

![Figure 4](image)

**Figure 4:** $i_1, \ldots, i_A$ are all pioneer points of $\tilde{\xi}_u^v$; $j_1, \ldots, j_B$ are all pioneer points of $\tilde{\xi}_{v t_1 - 1}^v$. We will run independent severed contact processes inside these branches associated with the pioneer points.

Next we will define the event $I_{ij}$ for $i \in \tilde{\xi}_{v t_1}^u$ and $j \in \tilde{\xi}_{v t_1 - 1}^v$. We say $I_{i,j}$ happens if in stage 2 all of the following events happen:

1. for $(\tilde{\eta}_s^j)_0 \leq s \leq 1$, $j$ infects the neighbor in its free branch, call it $y$, before time 1, and the infection at $y$ stays alive but does not give rise to an infection till the end of time 1;
2. for $(\tilde{\eta}_s^i)_0 \leq s \leq \Delta t$, up to time $\Delta t$, labels assigned to vertices in $\cup_{s \leq t_1} \tilde{\xi}_s^u$ do not appear in the label set assigned to $\cup_{s \leq \Delta t} \tilde{\eta}_s^i$; or any $\cup_{s \leq \Delta t} \tilde{\eta}_s^i$ for $i' \in \tilde{\xi}_{v t_1}^u$, $i' \neq i$; also, distinct vertices in $\cup_{s \leq \Delta t} \tilde{\eta}_s^i$ are assigned distinct labels;
3. $x_0$, a pioneer point of $\tilde{\eta}_{\Delta t - 1}^i$, infects the neighbor in its free branch (call this neighbor $x_1$) in time interval $[\Delta t - 1, \Delta t]$, and the infection at $x_1$ stays alive till the end of $\Delta t$ without infecting other vertices; furthermore, $x_1$ is assigned the same label as $y$.

See Figure 5 for a graphical illustration.
Now in order to show with probability approaching 1, there exists some $i$ and $j$ such that $I_{i,j}$ happens, we will show the following assertion on the first moment

$$
\sum_{i \in \tilde{\zeta}_u^t} \sum_{j \in \tilde{\zeta}_v^{t-1}} \mathbb{P}\{I_{ij} | F\} \to \infty, \text{ as } n \to \infty,
$$

(FM)

and the following assertion on the second moment

$$
\sum_{i \in \tilde{\zeta}_u^t} \sum_{j \in \tilde{\zeta}_v^{t-1}} \sum_{i' \in \tilde{\zeta}_u^t} \sum_{j' \in \tilde{\zeta}_v^{t-1}} \mathbb{P}\{I_{ij} \cap I_{i'j'} | F\} = (1 + o(1)) \left( \sum_{i \in \tilde{\zeta}_u^t} \sum_{j \in \tilde{\zeta}_v^{t-1}} \mathbb{P}\{I_{ij} | F\} \right)^2, \text{ as } n \to \infty.
$$

(SM)

The next two subsections are devoted to proving (FM) and (SM).

### 4.2 1st Moment Calculation

**Proof of (FM).** Since $\mathbb{P}\{I_{ij} | F\}$ is constant among all pairs $(i, j) \in \tilde{\zeta}_u^t \times \tilde{\zeta}_v^{t-1}$, it suffices to estimate a single term. Let us first estimate the size of $| \cup_{s \leq \Delta t} \tilde{\eta}_s^i |$. From Proposition 2.10

$$
\mathbb{P}\{| \cup_{s \leq \Delta t} \tilde{\eta}_s^i | > n^{2(1+\delta)} \gamma \} \leq \exp(-K\gamma^{\Delta t}) = \exp(-K\gamma^{2\epsilon \log n/c_\lambda}),
$$

which is decaying faster than any polynomial of $1/n$. On the other hand, on $F$, there are no more than $n^{1-\epsilon}(1+\delta)/2$ different $i$'s. The same argument works for $j$'s. Therefore by a union bound, with probability $1 - o(1)$, we have

$$
| \cup_{s \leq \Delta t} \tilde{\eta}_s^i | \leq n^{2(1+\delta)} \epsilon, \text{ for all } i \in \tilde{\zeta}_u^t; \ | \cup_{s \leq \Delta t} \tilde{\eta}_s^j | \leq n^{2(1+\delta)} \epsilon, \text{ for all } j \in \tilde{\zeta}_v^{t-1}.
$$

(12)
Let us estimate how many half-edges will be used at the end of stage 2. If (12) holds,

1. labeling $\cup_{s \leq t_2} \tilde{x}_s$ and $\cup_{s \leq t_1} \tilde{x}_s$ have used no more than $O(n(1-\varepsilon)(1+\delta)/2)$ half-edges;
2. labeling $\cup_{s \leq \Delta t} \tilde{\eta}_s$ and $\cup_{s \leq \Delta t} \tilde{\eta}_s$ for all $i' \neq i$ and $j' \neq j$ will use no more than $O(n(1-\varepsilon)(1+\delta)/2 \times n^{2(1+\delta)\varepsilon})$, which is $O(n(1+\delta)(1+3\varepsilon)/2)$ half-edges;
3. labeling $\cup_{s \leq \Delta t} \tilde{\eta}_s$ and $\cup_{s \leq 1} \tilde{\eta}_s$ will use no more than $O(n^{2(1+\delta)\varepsilon})$ half-edges.

Therefore we see that during the labeling procedure, at any time there are at least $\lambda$ labels in the prescribed manner is at least

Therefore, from the above analysis, a counting argument similar to the proof of Proposition 4.1 will conclude that with probability $1 - o(1)$, the labels assigned to $\cup_{s \leq \Delta t} \tilde{\eta}_s$ and $\cup_{s \leq 1} \tilde{\eta}_s$ do not overlap with those assigned to $\cup_{s \leq t_2} \tilde{x}_s$, $\cup_{s \leq t_1} \tilde{x}_s$, $\cup_{s \leq \Delta t} \tilde{\eta}_s$ and $\cup_{s \leq 1} \tilde{\eta}_s$ for all $i' \neq i$ and $j' \neq j$; moreover, different vertices in $\cup_{s \leq \Delta t} \tilde{\eta}_s$ are assigned different labels; the same holds for $\cup_{s \leq 1} \tilde{\eta}_s$.

Now let us discuss how to produce a common label in the prescribed way in the definition of $I_{ij}$. Let $p_{\lambda, \text{severed}}$ be the chance that the severed contact process with rate $\lambda$ survives. It is not hard to deduce that $p_{\lambda, \text{severed}} > 0$ using proposition 2.2 and thus

$$\mathbb{P}\{\tilde{\eta}_{\Delta t-1} \neq \emptyset\} \geq p_{\lambda, \text{severed}} > 0.$$ From Proposition 2.2 with probability $(1 - o(1))p_{\lambda, \text{severed}}$, $|\tilde{\psi}_{\Delta t-1}| \geq e^{-c_1 n^{2(1-\delta)\varepsilon}}$.

With some positive probability $q_\lambda > 0$, for $(\tilde{\eta}_s)_{0 \leq s \leq 1}$, $j$ infects the neighbor in its free branch (call this neighbor $y$) before time 1, and the infection at $y$ stays alive but does not give rise to an infection till the end of time 1. Fix a vertex $x_0 \in \tilde{\eta}_{\Delta t-1}$ which is a pioneer point, then $q_\lambda$ is also the probability that the infection at $x_0$ infects the neighbor in its free branch (call this neighbor $x_1$) before time 1, and the infection at $x_1$ stays alive but does not give rise to an infection till the end of time 1. See figure 5.

As long as when we label $y$, we choose a label which has been chosen for the first time (which is of probability $1 - o(1)$, because the number of used half-edges is of order $O(n(1+\delta)(1+3\varepsilon)/2)$), then when we label $x_1$, the chance of choosing the same label as $y$ is at least $(d-1)/dn$. Now we can apply a Binomial type of argument to show that conditional on $|\tilde{\psi}_{\Delta t-1}| \geq e^{-c_1 n^{2(1-\delta)\varepsilon}}$, the chance of observing a common label in the prescribed manner is at least

$$\frac{1}{2}(1 - o(1))e^{-c_1 n^{2(1-\delta)\varepsilon}} \frac{d - 1}{dn} q_\lambda^2 = C_1 n^{2(1-\delta)\varepsilon - 1},$$

for some $C_1 > 0$.

Combining all the calculation above, we conclude that

$$\mathbb{P}\{I_{ij} \mid F\} \geq (1 - o(1))p_{\lambda, \text{severed}} C_1 n^{2(1-\delta)\varepsilon - 1} \geq C_2 n^{2(1-\delta)\varepsilon - 1},$$

for some $C_2 > 0$. Since there are at least $n^{(1-\varepsilon)(1-\delta)/2}$ distinct $i$’s and $n^{(1-\varepsilon)(1-\delta)/2}$ distinct $j$’s on $F$, we conclude

$$\sum_{i \in \mathcal{C}^t_i} \sum_{j \in \mathcal{C}^t_j} \sum_{i' \in \mathcal{C}^t_{i-1}} \sum_{j' \in \mathcal{C}^t_{j-1}} \mathbb{P}\{I_{ij} \cap I_{i'j'} \mid F\} \geq n^{(1-\varepsilon)(1-\delta)/2} n^{(1-\varepsilon)(1-\delta)/2} C_2 n^{2(1-\delta)\varepsilon - 1} = C_2 n^{\varepsilon - (1+\varepsilon) \varepsilon - 1} \rightarrow \infty,$$

as $n \to \infty$, which is (EM).

\[\square\]

### 4.3 2nd Moment Calculation

This subsection is dedicated to showing (SM). We expand the second moment as follows,

$$\sum_{i \in \mathcal{C}^t_i} \sum_{j \in \mathcal{C}^t_j} \sum_{i' \in \mathcal{C}^t_{i-1}} \sum_{j' \in \mathcal{C}^t_{j-1}} \mathbb{P}\{I_{ij} \cap I_{i'j'} \mid F\} = I + II + III + IV,$$

17
where

\[
I = \sum_{i \in \xi^e_1} \sum_{j \in \xi^e_{1-1}} \mathbb{P}\{I_{ij} | F\},
\]

\[
II = \sum_{i \in \xi^e_1} \sum_{j \not\in \xi^e_{1-1}} \mathbb{P}\{I_{ij} \cap I_{ij'} | F\},
\]

\[
III = \sum_{i \not\in \xi^e_1} \sum_{j \in \xi^e_{1-1}} \mathbb{P}\{I_{ij} \cap I_{ij'} | F\},
\]

\[
IV = \sum_{i \not\in \xi^e_1} \sum_{j \not\in \xi^e_{1-1}} \mathbb{P}\{I_{ij} \cap I_{ij'} | F\}.
\]

Now (SM) becomes

\[
I + II + III + IV = (1 + o(1)) I^2, \quad \text{as } n \to \infty.
\]

We have already shown that \( I \to \infty \) as \( n \to \infty \), so

\[
I = o(I^2).
\]

We will show the following assertions in the next subsections,

\[
II = o(I^2), \quad \text{(13)}
\]

\[
III = o(I^2), \quad \text{(14)}
\]

\[
IV = (1 + o(1)) I^2. \quad \text{(15)}
\]

### 4.3.1 Proof of (13) and (14)

**Proof.** We will only prove (13) because the proof of (14) is essentially the same. We will find an upper bound for

\[
\mathbb{P}\{I_{ij} \cap I_{ij'} | F\}.
\]

From the proof of (12) we know that with probability at least \( 1 - n^{-4} \), (12) holds. We condition on (12) happening. Suppose \( y \) is the neighbor of \( j \) in \( j \)'s free branch, and \( y' \) is the neighbor of \( j' \) in \( j' \)'s free branch. In order that \( I_{ij} \cap I_{ij'} \) happens, at least we should observe that when we label \( y \) we assign some label that has already been used in \( \cup_{s \leq \Delta t} \tilde{\eta}^e_s \), and the same is true for \( y' \). \( y \) has chance no more than

\[
\frac{(d - 1)n^{2(1+\delta)}\varepsilon}{dn - O(n^{(1+\delta)(1+3\varepsilon)/2})}
\]

of using a label already used in \( \cup_{s \leq \Delta t} \tilde{\eta}^e_s \), because the numerator is an upperbound of qualified half-edges in the pool, and the denominator is an lower bound of the number of unused half-edges at the end of stage 2 if (12) holds. The same holds for \( y' \).

Therefore we obtain that

\[
\mathbb{P}\{I_{ij} \cap I_{ij'} \} \leq \frac{1}{n^2} + \left( \frac{(d - 1)n^{2(1+\delta)}\varepsilon}{dn - O(n^{(1+\delta)(1+3\varepsilon)/2})} \right)^2 = O(n^{4(1+\delta)\varepsilon - 2}).
\]

On \( F \), there are no more than \( n^{(1-\varepsilon)(1+\delta)/2} \) different \( i \)'s, and no more than \( n^{(1-\varepsilon)(1+\delta)/2} \) different \( j \)'s and hence no more than \( n^{(1-\varepsilon)(1+\delta)} \) different pairs of \((j,i)\). Therefore we sum over all \( i,j,j' \) and obtain

\[
\sum_{i \in \xi^e_1} \sum_{j \not\in \xi^e_{1-1}} \sum_{j' \not\in \xi^e_{1-1}} \mathbb{P}\{I_{ij} \cap I_{ij'} | F\} \leq n^{(1-\varepsilon)(1+\delta)/2} n^{(1-\varepsilon)(1+\delta)} O(n^{4(1+\delta)\varepsilon - 2}) = o(I^2).
\]

Therefore we’ve proved (12). \( \square \)
4.3.2 Proof of (15)

For \( i \neq i' \in \mathcal{C}_{i_1} \) and \( j \neq j' \in \mathcal{C}_{i_{t-1}} \) we will estimate

\[
P \{ I_{ij} \cap I_{ij'} | F \}.
\]

Our goal is to show the above quantity is almost the same as

\[
P \{ I_{ij} | F \} \cdot P \{ I_{ij'} | F \}.
\]

From the proof of (12) we know that with probability at least \( 1 - n^{-4}, (12) \) holds.

First of all, the severed contact processes on the cover trees, namely \( \bar{\eta}'_{\Delta t}, \bar{\eta}'_{\Delta t}, \eta'_{\Delta t} \) and \( \bar{\eta}'_{\Delta t} \) are independent. Given a realization of the above 4 processes, now we consider the labeling process on them. Let \( A \) be a specific pattern of labeling \( \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t} \) and \( \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \), and let \( B \) be a specific pattern of labeling \( \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t} \) and \( \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \).

Notice that

\[
P \{ I_{ij} | F \} = \sum_{A \text{ compatible with } I_{ij}} P \{ \text{pattern } A | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \cdot P \{ \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \},
\]

\[
P \{ I_{ij'} | F \} = \sum_{B \text{ compatible with } I_{ij'}} P \{ \text{pattern } B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \cdot P \{ \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \}, \tag{16}
\]

where in expressions such as \( P \{ \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \) we only care about information that is relevant to the labeling process, such as which vertices are ever infected and the relative order of appearances of infections. Irrelevant information such as when exactly an infection appears is not included in this probability.

From now on let us use pattern \( A \& B \) to denote pattern \( A \) on \( \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \) and pattern \( B \) on \( \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \). Also notice that if pattern \( A \) is compatible with \( I_{ij} \) and pattern \( B \) is compatible with \( I_{ij'} \) then the pattern \( A \& B \) is compatible with \( I_{ij} \cap I_{ij'} \). Therefore we can also express

\[
P \{ I_{ij} \cap I_{ij'} | F \} = \sum_{A \text{ compatible with } I_{ij}} \sum_{B \text{ compatible with } I_{ij'}} P \{ \text{pattern } A \& B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \cdot P \{ \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \}. \tag{17}
\]

From the above expressions, it suffices to compare

\[
P \{ \text{pattern } A | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \times P \{ \text{pattern } B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \}. \tag{18}
\]

and

\[
P \{ \text{pattern } A \& B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \}. \tag{19}
\]

We can rewrite (19) as

\[
P \{ \text{pattern } A | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \times P \{ \text{pattern } B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \text{ pattern } A \}. \tag{20}
\]

The following proposition compares (18) and (20).

**Proposition 4.2.** If (12) happens, then

\[
P \{ \text{pattern } A | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \} \leq (1 + o(1)) P \{ \text{pattern } A | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \}, \tag{21}
\]

and

\[
P \{ \text{pattern } B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t}, \text{ pattern } A \} \leq (1 + o(1)) P \{ \text{pattern } B | F, \cup_{s \leq \Delta t} \bar{\eta}'_{\Delta t}, \cup_{s \leq 1} \bar{\eta}'_{\Delta t} \}. \tag{22}
\]
We say that a vertex which is (15).

Therefore throughout this section we let \( t \in (1 \pm \varepsilon) \log n / c \lambda \).

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Proof. Let us only show (22) since the proof of (21) is the same. The difference between

\[
P\{\text{pattern } B | F, \cup s \leq \Delta \eta_s', \cup s \leq \Delta \eta_s'' \}, \text{ pattern } A \}
\]

and

\[
P\{\text{pattern } B | F, \cup s \leq \Delta \eta_s', \cup s \leq \Delta \eta_s'' \}
\]
is that whenever we draw an unused half-edge from the pool, the number of unused half-edge is reduced because some are already used in pattern A. However if (12) happens, then

1. labeling pattern A uses no more than \( O(n^{2(1+\delta)\varepsilon}) \) half-edges;
2. the total number of half-edges to be drawn in stage 2 is no more than \( O(n^{(1+\delta)(1+3\varepsilon)/2}) \);
3. at any time in stage 2, the number of unused half-edges is at least \( K(n) \), where

\[
K(n) = dn - O(n^{(1+\delta)(1+3\varepsilon)/2}).
\]

Therefore

\[
P\{\text{pattern } B | F, \cup s \leq \Delta \eta_s', \cup s \leq \Delta \eta_s'' \}, \text{ pattern } A \}
\]
is no more than

\[
\left( \frac{K(n)}{K(n) - O(n^{2(1+\delta)\varepsilon})} \right)^{O(n^{(1+\delta)(1+3\varepsilon)/2})} P\{\text{pattern } B | F, \cup s \leq \Delta \eta_s', \cup s \leq \Delta \eta_s'' \},
\]

which is

\[
(1 + o(1)) P\{\text{pattern } B | F, \cup s \leq \Delta \eta_s', \cup s \leq \Delta \eta_s'' \}.
\]

\[\square\]

Proof of (12). Based on whether (12) happen or not, by (10), (17) and Proposition 4.2,

\[
P\{I_{ij} \cap I_{ij'} | F \} = \frac{1}{n^4} + (1 + o(1)) P\{I_{ij} | F\} P\{I_{ij'} | F\},
\]

therefore when we sum over all \( i, i', j, j' \), we obtain

\[
\sum_{i \neq i' \in \zeta_1} \sum_{j \neq j' \in \zeta_1} P\{I_{ij} \cap I_{ij'} | F\} \leq o(1) + (1 + o(1)) \left( \sum_{i \in \zeta_1} \sum_{j \in \zeta_1} P\{I_{ij} | F\} \right)^2,
\]

which is (13).

\[\square\]

5 Asymptotic infection density

Throughout this section we let \( t_+ = (1 + \varepsilon) \log n / c \lambda \).

Fix \( 0 < \varepsilon < 1/8 \) and let \( g_n(\varepsilon) \) be in Theorem 1.1. We say that a pair of vertices \( (u, v) \in [n] \times [n] \) is good if

\[
P_G \{ v \in \zeta_*^u \} \geq (1 - g_n(\varepsilon)) p_X^2.
\]

We say that a vertex \( u \in [n] \) is good if the set \{ \( v \in [n] : v \neq u, (u, v) \) is a good pair \} has cardinality at least \( (1 - \sqrt[4]{g_n(\varepsilon)}) (n - 1) \). Using Markov inequality it is easy to deduce from Theorem 1.1 that

Proposition 5.1. For asymptotically almost every \( G \), the number of good pairs is at least \( (1 - \sqrt[4]{g_n(\varepsilon)}) n (n - 1) \), and the number of good vertices is at least \( (1 - \sqrt[4]{g_n(\varepsilon)}) n \).
The choice of $\sqrt{\frac{d}{n}}$ and $\sqrt{\frac{n}{d}}$ in the definition of good pair/vertex and in the above proposition is not crucial; we only need them to be $o(1)$ terms. The above proposition shows that such defined good pairs/vertices are indeed typical.

The next proposition states that a contact process started from a good vertex has decent chance to survive time $t_+$.

**Proposition 5.2.** Suppose $u \in [n]$ is a good vertex, and $\xi^u$ is a contact process with initial state $\{u\}$ on $G$. Then there exist constants $h_n(\varepsilon) \to 0$ as $n \to \infty$ such that for asymptotically almost every $G \sim G(n,d)$,

$$(1 + h_n(\varepsilon))p_{\lambda} \geq P_G\{\xi^u_{t_+} \neq \emptyset\} \geq (1 - h_n(\varepsilon))p_{\lambda}.$$

**Proof.** First of all $P_G\{\xi^u_{t_+} \neq \emptyset\} \leq (1 + o(1))p_{\lambda}$. This is because

$$P_G\{\xi^u_{t_+} \neq \emptyset\} \leq P\{\tilde{\xi}_{t_+} \neq \emptyset\} = (1 + o(1))p_{\lambda},$$

where $(\tilde{\xi}_t)_{t \geq 0}$ is a contact process on $T_d$ with the root as the initial configuration.

It remains to show the lower bound. Denote

$$S_u = \sum_{v \in V(G), v \neq u} 1_{\{v \in \xi^u_{t_+}\}}.$$

In the remains of this section, $o(1)$ terms only depend on $n$ but not $u$ or $G$. Since $u$ is a good vertex,

$$E_G S_u \geq (1 - o(1))p^2_{\lambda} n.$$

On the other hand,

$$S_u^2 = \sum_{v \in [n] \setminus \{u\}} \sum_{w \in [n] \setminus \{u\}} 1_{\{v \in \xi^u_{t_+}, w \in \xi^u_{t_+}\}},$$

and

$$E_G S_u^2 = \sum_{v \in [n] \setminus \{u\}} \sum_{w \in [n] \setminus \{u\}} E_G\{v \in \xi^u_{t_+}, w \in \xi^u_{t_+}\}.$$

Let $t_+ = t_{+1} + t_{+2}$, where $t_{n,1} = t_{n,2} = t_+/2$. Using the graphical representation, the event $\{v \in \xi^u_{t_+}, u \in \xi^u_{t_+}\}$ happens if and only if there exits open paths starting from $u$ and reach both $v$ and $w$ in time $t_+$. That requires both of the following events to happen:

1. $u$ infects some (random) subset $Z \subset [n]$ at time $t_{+1}$;
2. $Z$ infect both $v$ and $w$ in time interval $[t_{+1}, t_{+2}]$.

Let $\xi^v$, $\xi^w$ be 3 mutually independent contact processes, with initial configurations being $\{u\}$, $\{v\}$ and $\{w\}$ respectively. By duality we can reverse the time axis of the second event, and observing the above two events is no easier than observing the following two events:

3. $\xi^u$ survives to time $t_{+1}$;
4. $\xi^v$ and $\xi^w$ both survive to time $t_{+2}$.

It is easy to see that the $P_G$-probability of observing (3) and (4), is no bigger than $(1 + o(1))p^3_{\lambda}$, because of (23).

Therefore, for each pair $(v, w)$, we have

$$E_G\{v \in \xi^u_{t_+}, w \in \xi^u_{t_+}\} \leq (1 + o(1))p^3_{\lambda},$$

which implies

$$E_G S_u^2 \leq (1 + o(1))p^3_{\lambda} n^2.$$
Now since $u$ is a good vertex,
\[ \mathbb{E}_G\{S_u | S_u > 0\} = \frac{\mathbb{E}_G S_u}{\mathbb{P}_G\{S_u > 0\}} \geq \frac{(1 - o(1))p^2_n}{\mathbb{P}_G\{S_u > 0\}}, \]
while
\[ \mathbb{E}_G\{S_u^2 | S_u > 0\} = \mathbb{E}_G \frac{S_u^2}{\mathbb{P}_G\{S_u > 0\}} \leq \frac{(1 + o(1))\text{survival}^2 n^2}{\mathbb{P}_G\{S_u > 0\}}. \]
However by Jensen’s inequality, $\mathbb{E}_G\{S_u | S_u > 0\}^2 \leq \mathbb{E}_G\{S_u^2 | S_u > 0\}$, and we must have
\[ \mathbb{P}_G\{S_u > 0\} \geq (1 - o(1))p_\lambda. \]

From the proof of the above proposition we can also obtain a good estimate of the size of $\xi^u_t$.

**Proposition 5.3.** Suppose $u \in [n]$ is a good vertex, and $\xi^u_t$ is a contact process with initial state $\{u\}$. Fix $\varepsilon > 0$. Then there exist constants $k_u(\varepsilon) \to 0$ as $n \to \infty$ such that for asymptotically almost every $G$,
\[ \mathbb{P}_G\{(1 - \delta)np_\lambda \leq S_u \leq (1 + \delta)np_\lambda | S_u > 0\} \geq 1 - k_u(\varepsilon). \]

**Proof.** From Proposition 5.2, $\mathbb{P}_G\{S_u > 0\} \geq (1 - o(1))p_\lambda$. On the other hand $\mathbb{P}_G\{S_u > 0\} \leq (1 + o(1))p_\lambda$. These two bounds combined with the calculation in the proof of Proposition 5.2 we obtain
\[ \mathbb{E}_G\{S_u^2 | S_u > 0\} \leq (1 + o(1))\mathbb{E}_G\{S_u | S_u > 0\}^2. \]
Then we apply Chebyshev’s inequality. \hfill \Box

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let $S = \sum_{v \in [n]} 1_{\{v \in \xi^u_t\}}$. By duality of the contact process, Proposition 5.1 and Proposition 5.2,
\[ \mathbb{E}_G S \geq (1 - o(1))np_\lambda. \]
On the other hand, using a similar argument as in the proof of Proposition 5.2 we have
\[ \mathbb{E}_G S^2 \leq (1 + o(1))n^2 p_\lambda^2. \]
Therefore we obtain
\[ \mathbb{E}_G S^2 \leq (1 + o(1))(\mathbb{E}_G S)^2. \]
Then we apply Chebyshev’s inequality. \hfill \Box

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