Sharp Weyl Law for Signed Counting Function of Positive Interior Transmission Eigenvalues

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Abstract

We consider the interior transmission eigenvalue (ITE) problem, which arises when scattering by inhomogeneous media is studied. The ITE problem is not self-adjoint. We show that positive ITEs are observable together with plus or minus signs that are defined by the direction of motion of the corresponding eigenvalues of the scattering matrix (when the latter approach $z = 1$). We obtain a Weyl type formula for the counting function of positive ITEs, which are taken together with ascribed signs.

1 Main results

Let $\mathcal{O} \in \mathbb{R}^d$ be an open bounded domain with $C^2$ boundary $\partial \mathcal{O}$ and the outward normal $\nu$. Interior transmission eigenvalues (ITEs) are defined as values of $\lambda \in \mathbb{C}$ for which the problem

$$
- \Delta u - \lambda u = 0, \quad x \in \mathcal{O}, \quad u \in H^2(\mathcal{O}), \\
- \Delta v - \lambda n(x)v = 0, \quad x \in \mathcal{O}, \quad v \in H^2(\mathcal{O}), \\
u - v = 0, \quad x \in \partial \mathcal{O}, \\
\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \mathcal{O}
$$

(1) (2) (3)

has a non-trivial solution. Here $n(x) > 0, x \in \overline{\mathcal{O}}$, is a smooth positive function, $H^2(\mathcal{O})$ is the Sobolev space.

This spectral problem for a system of two equations in a bounded domain $\mathcal{O} \in \mathbb{R}^d$ appears naturally when the scattering transmission problem (scattering of plane waves by

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an inhomogeneous medium) is studied. The scattering problem is stated as
\[
-\Delta u - \lambda u = 0, \quad x \in \mathbb{R}^d \setminus \mathcal{O},
\]
\[
-\Delta v - \lambda n(x)v = 0, \quad x \in \mathcal{O},
\]
\[
u \quad u - v = 0, \quad x \in \partial \mathcal{O},
\]
\[
\frac{\partial u}{\partial n} - \frac{\partial v}{\partial n} = 0, \quad x \in \partial \mathcal{O},
\]
where \( u \) is the sum of the incident plane wave and the scattered wave, i.e., \( u = e^{ik(\omega,x)} + \psi_{sc} \), \( \lambda = k^2 \), \( \psi_{sc} \) satisfies the radiation conditions:
\[
\psi_{sc} = f(k,\theta,\omega) e^{ikr} \frac{r^{d+1}}{r^2} + O \left( r^{-d+1} \right), \quad \theta = \frac{x}{r}, \quad r = |x| \to \infty.
\]

The main relation between the scattering and ITE problems is due to the following fact: if the far-field operator:
\[
F = F(k) : L^2(S^{d-1}) \to L^2(S^{d-1}), \quad F\phi = \int_{S^{d-1}} f(k,\theta,\omega)\phi(\omega) dS_\omega
\]
has zero eigenvalue at the frequency \( k = k_0 > 0 \), then \( \lambda = k_0^2 \) is an ITE [11]. Indeed, if \( \mu = \mu(\omega) \) is an eigenfunction of \( F \) with zero eigenvalue, then the interior transmission eigenfunction \((u,v)\) can be obtained from the corresponding (with the same \( k = k_0 \)) solution of the exterior problem by integration over the unit sphere with the weight \( \mu(\omega) \). In order to justify the latter statement, we need only to show that the integral \( \int_{S^{d-1}} u_{ext}\mu(\omega) dS_\omega \), where \( u = u_{ext} \) is the exterior part of the scattering problem, can be extended smoothly into \( \mathcal{O} \) as a solution of the Helmholtz equation. We have
\[
\int_{S^{d-1}} u_{ext}\mu(\omega) dS_\omega = \int_{S^{d-1}} e^{ik(\omega,x)} \mu(\omega) dS_\omega + \int_{S^{d-1}} \psi_{sc}\mu(\omega) dS_\omega.
\]
Since \( F\mu = 0 \), the outgoing wave \( \int_{S^{d-1}} \psi_{sc}\mu(\omega) dS_\omega \) has zero amplitude, and therefore it is equal to zero identically. Thus (7) provides the needed extension.

This relation between ITEs and operator \( F \) is very important in the study of scattering by inhomogeneous media. In particular, it is known that positive ITEs are observable. They have been extensively used in the study of the inverse problem, starting from papers [11], [5], [22].

The transmission problem considered above has a simple analogue, which is scattering by a soft or rigid obstacle \( \mathcal{O} \). It is the exterior problem with the Dirichlet or Neumann boundary condition. The corresponding interior problem in the latter case is the eigenvalue problem for the Dirichlet or Neumann (negative) Laplacian in \( \mathcal{O} \). Unlike the Dirichlet or Neumann eigenvalues, the ITEs are defined by a much more complicated spectral problem, which is neither symmetric nor elliptic.

One of the important properties of the eigenvalues for the Dirichlet or Neumann negative Laplacian is the Weyl law for the counting function of the eigenvalues.
The goal of this paper is to obtain an analogue of the Weyl law for the signed counting function of positive ITEs and establish an important connection between positive ITEs and the scattering matrix.

Due to the lack of symmetry (and ellipticity), the discreteness of the spectrum of the ITE problem, the existence of real eigenvalues, and their asymptotics can not be obtained by soft arguments. Moreover, the existence of non-real ITEs was shown in [18], and an example of an elliptic ITE problem where the set of ITEs is not discrete can be found in [13, Examples 1,2].

There is extensive literature (see the review [4]) on the properties of ITEs and corresponding eigenfunctions. The following results are most closely related to our study. It was shown in [24] that the set of ITEs is discrete if $n(x) \neq 1$ everywhere at the boundary of the domain $\partial O$. The latter condition (which means that the inhomogeneity has a sharp boundary) will be assumed to hold in our study. It was shown in [1],[6],[9],[14],[21] that the standard Weyl estimate holds for the complex ITEs located in an arbitrary cone containing the real positive semi-axis:

$$\# \{ i : |\lambda^T_i| \leq \lambda \} = \lambda^\frac{d}{2} \frac{\omega_d}{(2\pi)^d} \left[ \text{Vol}(O) + \int_O n^\frac{d}{2}(x)dx \right] + o(\lambda^\frac{d}{2}), \quad \lambda \to \infty, \quad (8)$$

where $\omega_d$ is the volume of the unit ball in $\mathbb{R}^d$.

Earlier, in a series of articles [15],[16],[17], we have shown that if

$$\gamma := \text{Vol}(O) - \int_O n^\frac{d}{2}(x)dx \neq 0, \quad (9)$$

then the set of positive ITEs (which are the most important for applications) is infinite, and moreover,

$$\# \{ 0 < \lambda^T_i < \lambda \} \geq \frac{\omega_d}{(2\pi)^d} |\gamma| \lambda^\frac{d}{2} + O(\lambda^{\frac{d}{2}-\delta}), \quad \lambda \to \infty. \quad (10)$$

Obviously, the coefficient $|\gamma|$ is always smaller than the corresponding coefficient in (8).

We plan to show (and this is the main result of this paper) that there are signs that can be naturally ascribed to positive ITEs $\lambda^T_i$ in such a way that the Weyl law for the signed counting function (that counts the eigenvalues with the ascribed signs) for positive ITEs is valid with the coefficient $\gamma$ in the first term.

We will ascribe a value $\sigma_i = \pm 1$ to each simple positive ITE. Moreover, these values are observable and related to the scattering matrix. They will be defined below. (In the case of an ITE of geometric multiplicity $n > 1$, we ascribe a coefficient $\sigma_i$, $|\sigma_i| \leq n$, to the whole group, not to each of ITEs separately). The main result of this paper is formulated as follows.

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1Paper [14] concerns the anisotropic ITE problem, papers [1],[9] concern the case of $n(x) > 1, x \in \overline{O}$, and papers [6],[21] concern the case of $n(x) \neq 1, x \in \partial O$. 

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3
Theorem 1.1. Let \( n(x) \neq 1, x \in \partial \mathcal{O} \). The Weyl law holds for the signed counting function of the transmission problem:

\[
\sum_{i : 0 < \lambda_i^\gamma < \lambda} \sigma_i = \frac{\omega_d}{(2\pi)^d} \gamma \lambda^\gamma + O(\lambda^{\gamma - \delta}), \quad \lambda \to \infty, \quad \delta = \frac{1}{2d},
\]

(11)

We expect that a similar result is valid for other scattering problems related to Maxwell or Dirac equations, scattering on graphs, etc.

Let us specify \( \sigma_i \). Consider the scattering matrix of the transmission problem (4), (5) in \( \mathbb{R}^d \). It is a unitary operator of the form

\[
S(k) = I + 2ik\pi F : L_2(S^{d-1}) \to L_2(S^{d-1}), \quad \alpha = \frac{1}{4\pi} \left( \frac{k}{2\pi i} \right)^{\frac{d-3}{2}}, \quad (12)
\]

where the far-field operator \( F \) is defined by (6). The eigenvalues \( z_j(k) \) of the scattering matrix belong to the unit circle \( C = \{ z : |z| = 1 \} \). The operator \( F \) is compact and its eigenvalues converge to zero, and therefore for each \( k > 0 \), the eigenvalues \( \{ z_j(k) \} \) converge to \( z = 1 \) as \( i \to \infty \). Functions \( z_j(k) \) are not always analytic at points where \( z_j = 1 \) (at the essential point of the spectrum of \( S(k) \)). For example, \( z = 1 \) is never an eigenvalue of \( S(k) \) if \( \mathcal{O} \) has a corner, see [2].

It turns out that the eigenvalues \( z_j(k) \) are distributed over the circle \( C \) very non-uniformly. One of the half circles \( C_{\pm} = C \cap \{ \pm 3z > 0 \} \) always contains at most a finite number of the eigenvalues, while the opposite half circle contains infinitely many of them (with the limiting point at \( z = 1 \)). This fact was discovered first in [7, 8] in the case of scattering by a soft or rigid obstacle when the exterior Dirichlet or Neumann problem is considered instead of the transmission problem. It was shown in [7, 8] that \( C_+ \) contains a finite number of eigenvalues in the case of the Dirichlet boundary conditions, and \( C_- \) has this property when the Neumann boundary condition is imposed. A similar result was obtained recently [12] for the transmission problem under the assumption that \( n(x) \neq 1 \) in the whole domain \( \overline{\mathcal{O}} \). The choice of the half circle depends in this case of the sign of \( n(x) - 1, x \in \overline{\mathcal{O}} \).

The first result of the present paper is an extension of the latter statement to the case when it is only assumed that \( n(x) \neq 1 \) at the boundary. We also will show that if \( n \) changes sign on the boundary, then both half circles contain infinitely many eigenvalues of the scattering matrix. Namely, the following theorem will be proved.

Theorem 1.2. 1. If \( n(x) < 1, x \in \partial \mathcal{O} \), then \( S(k) \) has at most a finite number of eigenvalues \( z_j(k) \) in \( C_+ \) for each fixed \( k > 0 \) (as in the case of the Dirichlet boundary condition).

2. If \( n(x) > 1, x \in \partial \mathcal{O} \), then \( S(k) \) has at most a finite number of eigenvalues \( z_j(k) \) in \( C_- \) for each fixed \( k > 0 \) (as in the case of the Neumann boundary condition).

3. If \( n(x) - 1 \) takes both positive and negative values on \( \partial \mathcal{O} \), then \( S(k), k > 0, \) has infinitely many eigenvalues in both \( C_+ \) and \( C_- \).
After we prove the theorem, we’ll study the ITE problem under the first two assumptions of the theorem. We’ll concentrate our attention on the half-circle with a finite number of points $z_j(k)$. Our next goal is to study the motion of the points $z_j(k)$ along the half-circle when $k$ is increasing. We distinguish clock- and counter clock-wise motion of these points where $k$ plays the role of time. We are primarily interested in what happens when a point $z_j(k)$ reaches $z = 1$ at some moment $k = k_0$, for example when $k \to k_0 - 0$. When $k > k_0$, the point may stay on the same half circle, move to another half-circle or disappear at all. It is difficult to find a proper description of all possibilities and justify them due to the lack of smoothness of the eigenvalues at the points where $z_j(k) = 1$. Thus we will split the motion of each point $z_j(k)$ into two parts: before $z_j(k)$ reaches $z = 1$ at $k = k_0$ and after that event, without any attempt to relate the eigenvalues before $k_0$ and after $k_0$.

Denote by $m^+ = m^+(k_0)$ ($m^- = m^-(k_0)$) the number of eigenvalues $z_j(k)$ of the scattering matrix $S(k)$ that reach the point $z = 1$ at the moment $k = k_0$ while moving in the chosen half-circle clock-wise (counter clock-wise, respectively). For example, if $n(x) < 1$ on $\partial \mathcal{O}$, then the chosen half-circle is $C_+$, and $m^±$ is the number of eigenvalues such that

$$\lim_{k \to k_0^±} z_j(k) = 1 + i0.$$

We will prove

**Theorem 1.3.** 1) Let $n(x) < 1$, $x \in \partial \mathcal{O}$. For each ITE $\lambda_i^\ell = k_i^2$ of multiplicity $m_i$, there are $m_i^+ \leq m_i$ eigenvalues $z_j(k)$ of the scattering matrix in the upper half circle $C_+$ that approach $z = 1$ when $k \to k_i - 0$ (they are moving clock-wise) and $m_i^- \leq m_i$ eigenvalues that approach $z = 1$ when $k \to k_i + 0$ (they are moving counter clock-wise). There is an arc of the unit circle defined by $0 < \arg z < \delta$ that is free of all other points $z_j(k)$, $|k - k_i| \ll 1$.

2) Let $n(x) > 1$, $x \in \partial \mathcal{O}$. For each ITE $\lambda_i^\ell = k_i^2$ of multiplicity $m_i$, there are $m_i^+ \leq m_i$ eigenvalues $z_j(k)$ of the scattering matrix in the lower half circle $C_-$ that approach $z = 1$ when $k \to k_i + 0$ and $m_i^- \leq m_i$ eigenvalues that approach $z = 1$ when $k \to k_i - 0$. There is an arc of the unit circle defined by $-\delta < \arg z < 0$ that is free of all other points $z_j(k)$, $|k - k_i| \ll 1$.

3) The statements above remain valid if $\lambda = k_0^2 > 0$ is not an ITE (i.e., $m_i = 0$). In this case, the corresponding arc of the unit circle is free of the eigenvalues of the scattering matrix when $|k - k_0| \ll 1$.

Note that $m_i^\pm$ can take any integer values in the segment $[0, m_i]$. There are no relations between them such as $m_i^+ + m_i^- = m_i$. However, one can derive from the arguments in the proof that their sum is $m_i$ modulus two. In particular, if $m_i = 1$, i.e., an ITE is simple, then one of the numbers $m_i^\pm$ is one and another is zero. A more specific result is given by the following theorem.
Theorem 1.4. If $\lambda_i^T = k_i^2$ is an ITE whose geometric and algebraic multiplicities coincide (there are no adjoint eigenfunctions) and $(u_j, v_j)$, $1 \leq j \leq i$, is a basis in the eigenspace, then $m_i^+ - m_i^-$ is the signature of the following matrix $A = \{a_{j,l}\}$

$$m_i^+ - m_i^- = \text{sgn}\{a_{j,l}\}, \quad a_{j,l} = \int_{\mathcal{O}} (u_j u_l - n v_j v_l) \, dx, \quad 1 \leq j, l \leq m_i.$$

An analogue of Theorem 1.3 was proved in [7], [8] in the case of scattering by a soft or rigid obstacle $\mathcal{O}$. In the latter case, the interior Dirichlet or Neumann problem is considered instead of the interior transmission problem, and the eigenvalues of the Dirichlet or Neumann Laplacian in $\mathcal{O}$ are used instead of ITEs. The symmetry and simplicity of the corresponding interior problem implies that all the scattering eigenvalues in both cases (Dirichlet/Neumann condition) move in the same clock-wise direction. Moreover, in both cases, $m_i^+ = m_i$, $m_i^- = 0$ for all $i \geq 1$. Perhaps this explains why the signed Weyl formulas have not appeared in the literature earlier.

Some results related to Theorem 1.4 can be found in [12]. If $n(x) \neq 1$, $x \in \mathcal{O}$, and certain conditions on an ITE $\lambda_i^T = k_i^2$ hold, it was proved there that the eigenvalue $z_j(k)$ closest to $z = 1$ and located in the appropriate half circle converges to $z = 1$ when $k$ approaches $k_i$ from the appropriate side.

Our last and main result is as follows: Theorem 1.3 holds with $\sigma_i$ defined by the formula

$$\sigma_i = m_i^+ - m_i^-.$$  \hspace{1cm} (13)

Remark 1. There are a couple of cases when formula (11) can be obtained by direct calculations. The asymptotics of the counting function for positive ITE was found in [19] when $\mathcal{O}$ is a ball and $n(x)$ is a constant. It was shown in [19] that the counting function of positive ITEs coincides with the right-hand side in (11). Since the Weyl law and sign Weyl law have the same main term, it implies that the majority of $\sigma_i$ has the same sign, which is equal to sign$(1 - n)$. It also follows from calculations in that paper (section 3.3) that $\sigma_i \text{sign}(1 - n) = 1$ for the problem under consideration if ITE is simple. Another example concerns the anisotropic model. This model can be obtained from (1)-(3) by substituting $n$ by $n/a$ in (2) and replacing the second boundary condition by $\frac{\partial u}{\partial \nu} = a \frac{\partial v}{\partial \nu}$. Note that the results of the present article also hold for anisotropic problem, and the proofs require only minor changes. In a very trivial situation of $a = n = \text{const} \neq 1$, the spectrum of the ITE problem is a union of Dirichlet and Neumann spectrums for the negative Laplacian. So, all the ITEs are real and [8] provides the asymptotic for their counting function. But, obviously, $\gamma = 0$ in this case. So we see that the asymptotics for counting function of positive ITEs and signed counting function for positive ITEs are different.

Remark 2. Note that (11) provides information on the obstacle and coefficient $n(x)$ in the same way as the standard Weyl formula for the interior Dirichlet or Neumann problem allows one to recover the volume of the domain.

Conjecture. We believe that Theorem 1.4 remains valid without the assumption on the absence of the adjoint eigenvectors. One only needs to construct matrix $A$ using a
basis \((u_i, v_i)\) in the root subspace that corresponds to the ITE \(\lambda_i^T\). Supporting arguments will be provided after the proof of Theorem 1.4.

The rest of the paper is organized as follows. Section II concerns the relationships between Dirichlet-to-Neumann maps, ITEs, and the far-field operator. It starts (part (A)) with certain properties of the Dirichlet-to-Neumann maps and their relation to ITEs, followed (part (B)) by a representation of the far-field operator \(F\) through a combination of the Dirichlet-to-Neumann operators. While one of the factors in this representation of \(F\) is related to ITEs, this connection between \(F\) and ITEs will be deepened. Additional properties of the Dirichlet-to-Neumann maps are obtained in part (C), followed by an alternative way to define ITEs (part (D)) and an alternative representation of the far-field operator \(F\) (part (E)). While one of the factors in this representation of \(F\) is related to ITEs, this connection between \(F\) and ITEs will be deepened. Additional properties of the Dirichlet-to-Neumann maps are obtained in part (C), followed by an alternative way to define ITEs (part (D)) and an alternative representation of the far-field operator \(F\) (part (E)). In particular, a Weyl type formula for a signed sum similar to (but different from) the one in Theorem 1.1 is proved in Theorem 2.9 of part (D).

Section III completes the proofs of the main theorems. Theorem 1.2 is proved first. The proof of Theorem 1.3 starts with general facts on quadratic forms defined by unitary operators. Then it is shown that Theorem 1.3 holds with \(m_+ - m_- = \alpha^+ - \alpha^-\). The latter relation together with Theorem 2.9 implies Theorem 1.1. Theorem 1.4 is proved at the end of the section.

2 Relations between the scattering matrix and ITEs; supplementary lemmas.

(A). Dirichlet-to-Neumann operators and their relationship with ITEs. The following operators

\[
N_0^{in}, N_n^{in}, N_n^{out} : H^s(\partial O) \to H^{s-1}(\partial O) \quad \text{and} \quad N_0^{in} - N_n^{in} : H^s(\partial O) \to H^{s+1}(\partial O)
\]  

(14)

will be used heavily to prove the main result. Here \(H^s(\partial O)\) is the Sobolev space, \(N_n^{in}(\lambda)\) is the Dirichlet-to-Neumann map for equation (2) in \(O\), which is defined as follows:

\[
N_n^{in}(\lambda) : v \mid_{\partial O} \to \frac{\partial v}{\partial \nu} \mid_{\partial O},
\]  

(15)

\(N_0^{in}(\lambda)\) is the same operator for equation (1), and \(N_n^{out}(\lambda)\) is the Dirichlet-to-Neumann map for equation (11) outside \(O\), which maps the Dirichlet data to the Neumann data of the solution that satisfies the radiation condition at infinity:

\[
u_r - i k u = o(r^{(1-d)/2}), \quad r \to \infty, \quad k^2 = \lambda.
\]

We use the same normal vector \(\nu\) for the exterior problem that was chosen (see (3)) for the interior problem.

Let \((y_1, \ldots, y_{d-1})\) be local coordinates on \(\partial O\) with the dual variables \((\xi_1, \ldots, \xi_{d-1})\) and let \(\sum g_{i,j}(y)dy_idy_j\) be the first fundamental form on \(\partial O\). Then \(|\xi^*| = (\sum g^{i,j}(y)\xi_i\xi_j)^{1/2}\) is the length of the covector in the cotangent bundle \(T^*(\partial O)\).
Lemma 2.1. 1) The first two operators in (14) are self-adjoint elliptic pseudo-differential operators of the first order. They are meromorphic in $\lambda$ when $\lambda > 0$ with poles at eigenvalues of the Dirichlet problem for equations (1) and (2), respectively. Their residues have finite ranges. The principal symbols of these two operators are equal to $|\xi^*|$. 

2) Operator $N^\text{out}$ is an analytic in $\lambda, \lambda > 0$, elliptic pseudo-differential operator of the first order with the principal symbol $-|\xi^*|$. For every $\varphi \neq 0$,

$$\Im(N^\text{out}\varphi, \varphi) = \sqrt{\lambda} \int_{|\theta|=1} |f(\lambda, \theta)|^2 dS > 0,$$

where $f$ is the far-field amplitude of the solution of the exterior problem with the Dirichlet data $\varphi$.

3) The last operator in (14) is a meromorphic in $\lambda$ elliptic pseudo-differential operator of order $-1$ with the principal symbol $\frac{\lambda(n(x)-1)}{2|\xi^*|}$.

Proof. The first statement and the symbol of $N^\text{out}$ are well known, see more details in [16]. Formula (39) is a direct consequence of the Green formula. The positivity of (39) and analyticity of $N^\text{out}$ are due to the absence of eigenvalues of the exterior Dirichlet problem imbedded into the continuous spectrum. The last statement can be found in [16, Lemma 1.1]. It is justified by calculating the first three terms of the full symbol of operators $N^\text{in}_0$ and $N^\text{in}_n$ (the first two terms of the symbols are canceled when the difference is taken).

Definition. The kernel of a meromorphic operator function is the set of elements that are mapped to zero by both the analytical and the principal part of the operator.

The following lemma is a direct consequence of the definition of ITEs.

Lemma 2.2. [15, 16] A point $\lambda = \lambda_0$ is an ITE if and only if the operator $N^\text{in}_0(\lambda) - N^\text{in}_n(\lambda)$ has a non-trivial kernel at $\lambda = \lambda_0$ or the following two conditions hold

1) $\lambda = \lambda_0$ is an eigenvalue of the Dirichlet problem for $-\Delta$ and for equation (2), i.e., $\lambda = \lambda_0$ is a pole for both $N^\text{in}_0(\lambda)$ and $N^\text{in}_n(\lambda)$.

2) The ranges of the residues of operators $N^\text{in}_0(\lambda)$ and $N^\text{in}_n(\lambda)$ at the pole $\lambda = \lambda_0$ have a non-trivial intersection.

Moreover, the multiplicity of the interior transmission eigenvalue $\lambda = \lambda_0$ in all the cases is equal to $m_1 + m_2$, where $m_1$ is the dimension of the kernel of the operator $N^\text{in}_0(\lambda) - N^\text{in}_n(\lambda)$, and $m_2$ is the dimension of the intersection of the ranges of the residues at the pole $\lambda = \lambda_0$ ($m_2 = 0$ if $\lambda = \lambda_0$ is not a pole).

If $\lambda = \lambda_0$ satisfies the latter two conditions, we will call it a singular ITE. Thus, singular ITEs belong to the intersection of three spectral sets: $\{\lambda^T\}$ and sets of the eigenvalues of the Dirichlet problems for $-\Delta$ and for equation (2).
A relation between D-to-N operators (14) and the far-field operator. We will provide below some properties of operators (14), but first let us formulate some relationships of these operators with the scattering matrix and with ITEs that will allow us to connect the latter objects. The relation to $S(k)$ is given by the following statement.

**Theorem 2.3.** Let operator $L : L_2(S^{d-1}) \to L_2(\partial O)$ be defined by

$$(L \varphi)(r) = \int_{S^{d-1}} e^{ikr \cdot \omega} \varphi(\omega) dS_\omega, \quad (L^* u)(\theta) = \int_{\partial O} e^{-ik\theta \cdot r} u(r) dS.$$ (17)

Then the following factorization formula (where $\lambda = k^2$) is valid for the far-field operator:

$$F = \alpha L^*(N_{in}^0 - N_{out})(N_{in}^n - N_{out})^{-1}(N_{in}^0 - N_{in}^n)\mathcal{L}, \quad \alpha = \frac{1}{4\pi} \left( \frac{k}{2\pi i} \right)^{\frac{d-3}{2}}.$$ (18)

**Proof.** Let $E = E(x)$ be the solution of the equation $(\Delta + k^2)E = -\delta(x), \; x \in \mathbb{R}^d$, that satisfies the radiation condition at infinity: $E_x - ikE = o(r^{(1-d)/2}), \; r \to \infty$. The scattered wave $\psi_{sc}$ defined in (4), (5) can be written as

$$\psi_{sc}(x) = \int_{\partial O} \left( \frac{\partial E(x-y)}{\partial \nu} \psi_{sc} - E(x-y) \frac{\partial \psi_{sc}}{\partial \nu} \right) dS_y, \; x \in \mathbb{R}^d \setminus O.$$ The Green formula for functions $E$ and the solution of the Helmholtz equation in $O$ with the Dirichlet data $\psi_{sc}$ at the boundary implies that

$$\int_{\partial O} \frac{\partial E(x-y)}{\partial \nu} \psi_{sc} dS_y = \int_{\partial O} E(x-y)(N_{in}^0 \psi_{sc}) dS_y, \; x \in \mathbb{R}^d \setminus O.$$ Thus

$$\psi_{sc} = \int_{\partial O} E(x-y)(N_{in}^0 - N_{out})\psi_{sc} dS_y, \; x \in \mathbb{R}^d \setminus O,$$

which leads to the following formula for the scattering amplitude (i.e., for the kernel of operator (6)):

$$f(k, \theta, \omega) = \alpha L^*(N_{in}^0 - N_{out})\psi_{sc} dS_y, \quad \alpha = \frac{1}{4\pi} \left( \frac{k}{2\pi i} \right)^{\frac{d-3}{2}}.$$ (19)

It remains only to find $\psi$ on $\partial O$ from (5). Let us denote function $e^{ik(\omega,x)}$ by $h$. Since $u_\nu = (\psi_{sc})_\nu + h_\nu = N_{out} \psi_{sc} + N_{in}^0 h$, relations (5) can be rewritten as

$$\psi_{sc} + h = v, \quad N_{out} \psi_{sc} + N_{in}^0 h = N_{in} v, \; x \in \partial O.$$ We apply operator $N_{in}^n$ to the first equality and then subtract the second one. This leads to

$$\psi_{sc} = (N_{in}^n - N_{out})^{-1}(N_{in}^0 - N_{in}^n)h, \; x \in \partial O,$$

which, together with (19), completes the proof.

The following properties of operators (17) will be essential for us.
Lemma 2.4. If $-k^2$ is not an eigenvalue of the Dirichlet Laplacian in $\mathcal{O}$, then the kernels of operators $L^*$ and $L$ are trivial, and therefore their ranges are dense.

Proof. The statement about the kernel of $L^*$ is proved in [7, Lemma 5.2]. One can find the statement about the kernel of $L$ in the same paper, but it is difficult to refer to a specific place. Thus we will prove it. Obviously, function $L\varphi$ satisfies the equation $\Delta v + k^2 v = 0$, $x \in \mathbb{R}^d$. One can show that for each fixed $k_0 > 0$ and an arbitrary non-zero $\varphi$, function $L\varphi$ can not vanish identically in $\mathbb{R}^d$, and therefore it can not be equal to zero identically in $\mathcal{O}$. Hence, if $L\varphi = 0$ on $\partial \mathcal{O}$, then $v = L\varphi$ is an eigenfunction of the Dirichlet problem.

(C). Additional properties of the D-to-N operators (14). Recall that an operator function $A(\lambda): H_1 \rightarrow H_2$, $k \in D$, in Hilbert spaces $H_1, H_2$ is called Fredholm finitely meromorphic if 1) it is meromorphic in $\lambda \in D$, 2) it is Fredholm for each $\lambda$ that is not a pole of $A(\lambda)$, 3) if $\lambda = \lambda_0$ is a pole, then the principal part at the pole has a finite range and the analytic part is Fredholm at $\lambda = \lambda_0$.

Lemma 2.5. [3] Let $A(\lambda)$ be Fredholm finitely meromorphic in a connected set $D$. If there is a point $\lambda = \lambda_0$ where the operator $A(\lambda)$ is one-to-one and onto, then the operator function $A^{-1}(\lambda)$ is also Fredholm finitely meromorphic.

Lemma 2.6. Operators (14), their inverses, and $(N_{in}^{in} - t)^{-1}$, $(N_{in}^{in} - t)^{-1}$, $(N_{out}^{out} - t)^{-1}$ where $t$ is a constant are Fredholm finitely meromorphic in $\lambda > 0$.

Moreover, operators $(N_{in}^{in} - N_{out}^{out})^{-1}$ and $(N_{out}^{out} - t)^{-1}$ are analytic in $\lambda$, $\lambda > 0$.

Proof. Let us call operators (14), their shifts by $t$, and operator $N_{in}^{in} - N_{out}^{out}$ the direct operators, and their inverses the inverse operators. The direct operators are meromorphic in $\lambda$ with finite ranges of residues due to Lemma 2.1. They are Fredholm since they are elliptic. For each direct operator, one can easily find a point $\lambda = \lambda_0$ where the kernel of the operator is trivial. Indeed, for the last operator in (14), one can take any $\lambda_0$ that is not an ITE (see Lemma 2.6). Such a $\lambda_0$ exists since the set of ITEs is discrete (see [16]). For other direct operators, except the last one, one can choose any $\lambda_0 > 0$ that is not an eigenvalue of the corresponding Dirichlet or impedance (with the boundary condition $u_\nu - tu = 0$ on $\partial \mathcal{O}$) problem. Below we show that any $\lambda_0 > 0$ can be chosen for the last direct operator. Since any elliptic operator on a compact manifold has index zero, all the direct operators are also onto when $\lambda = \lambda_0$, and Lemma 2.5 is applicable to these operators. Hence the inverse operators are also Fredholm and finitely meromorphic in $\lambda > 0$.

It remains to show that operators $N_{in}^{in} - N_{out}^{out}$ and $N_{out}^{out} - t$ do not have kernels when $\lambda > 0$ (which implies that their inverse operators are analytic). Since operator $N_{in}^{in}$ is symmetric, (39) implies that

$$\text{Im}((N_{in}^{in} - N_{out}^{out})f, f) = -\text{Im}(N_{out}^{out}f, f) < 0, \quad \text{if} \quad f \neq 0,$$
i.e., operator $N_n^{\text{in}} - N_n^{\text{out}}$, $\lambda > 0$, does not have a kernel. These arguments remain valid if $N_n^{\text{in}}$ is replaced by $t$.

(\textit{D). An alternative way to define ITEs.}) Due to Lemma 2.2 the factor before $\mathcal{L}$ in formula (18) for $F$ establishes a connection between the scattering matrix (related to $F$ by (12)) and ITEs. However, the existence of singular ITEs makes it difficult to work with this factorization of $F$ and to use the factor $(N_n^{\text{in}} - t)$, which may have a pole and a kernel at the same ITE. To avoid this difficulty, $F$ will be represented in a different form, where the following operator will be used to relate $F$ and ITEs:

$$R(\lambda) = (N_n^{\text{in}} - t)^{-1} - (N_0^{\text{in}} - t)^{-1} : H^s(\partial \mathcal{O}) \to H^{s+3}(\partial \mathcal{O}),$$

(20)

where $t$ is a constant. To be rigorous, one needs to write $tI$ here and in other similar formulas, but we will omit the identity operator $I$. We are going to study the operator (20) now, and will discuss the relation between $F$ and $R$ later.

Obviously, boundary conditions (3) are equivalent to the following ones

$$u = v, \quad x \in \partial \mathcal{O},$$

$$\frac{\partial u}{\partial \nu} - tv = \frac{\partial v}{\partial \nu} - tv, \quad x \in \partial \mathcal{O},$$

(21)

where $t > 0$ is arbitrary. Consider the set $\{t_s(\lambda)\}$, $\lambda > 0$, of values of $t$ for which the impedance problem

$$-\Delta v - \lambda n(x)v = 0, \quad x \in \mathcal{O}, \quad v \in H^2(\mathcal{O}); \quad \frac{\partial v}{\partial \nu} - tv = o, \quad x \in \partial \mathcal{O},$$

(22)

has a non-trivial solution. This set is discrete and countable. Equivalently, one can define this set as the set of values of $t$ for which operator $N_n^{\text{in}}(\lambda) - t$ has a non-trivial kernel (see the definition of the kernel in Section 2 (B) if $N_n^{\text{in}}(\lambda) - t$ has a pole). One could also describe it as the set of eigenvalues of operator $N_n^{\text{in}}(\lambda)$, but it would be a little vague at points $\lambda$ where $N_n^{\text{in}}(\lambda)$ has a pole.

The equivalence of two definitions of the set $\{t_s\}$ (through the impedance problem and through the kernel of $N_n^{\text{in}}(\lambda) - t$) is obvious when $\lambda$ is not a pole of $N_n^{\text{in}}$ (i.e., $\lambda$ is not an eigenvalue of the Dirichlet problem for equation (2)). If $\lambda = \lambda_0$ is a pole of $N_n^{\text{in}}(\lambda)$, then it is obvious that the existence of the kernel of $N_n^{\text{in}}(\lambda) - t$ implies the existence of the solution of the impedance problem. The inverse statement needs a short justification. We will not provide it, since we will not need this inverse statement.

We will choose and fix an arbitrary value of $t$ in (21) such that for all $s, i$,

$$t \notin \{t_s(\lambda_i^T)\} \cup \{t_s(\lambda_i^D)\} \cup \{t_s(\lambda_i^{n,D})\}.$$  

(23)

Since we are not going to vary $t$ (except in one insignificant place that will not affect any previous arguments), we usually will not mark explicitly the dependence of any operators or functions on $t$. Thus the value of $t$ in (20) is fixed.
Lemma 2.7. Operator (20) is meromorphic in \( \lambda \), \( \lambda > 0 \). It is an elliptic pseudodifferential operator of order \(-3\) with the principal symbol \( \frac{N_{n}(x)}{2|\xi|^{3}} \). It has a non-trivial kernel only if \( \lambda \) is an ITE and the dimension of the kernel coincides with the multiplicity of the ITE. It does not have poles at ITEs.

Proof. Operator (20) can be written as

\[
R(\lambda) = (N_{0}^{in} - t)^{-1}(N_{0}^{in} - N_{n}^{in})(N_{n}^{in} - t)^{-1}.
\]

This formula together with Lemmas 2.1, 2.6 immediately imply the first two statements of Lemma 2.7. Further, \( R(\lambda)\varphi = 0 \) is equivalent to

\[
(N_{n}^{in} - t)^{-1}\varphi = (N_{0}^{in} - t)^{-1}\varphi.
\]

Both sides here are zeroes if and only if \( \varphi \) is the Dirichlet data of a singular interior transmission eigenfunction. If they are not zeroes, then \( \psi = (N_{0}^{in} - t)^{-1}\varphi \) satisfies \( N_{n}^{in}\psi = N_{0}^{in}\psi \) and defines non-singular ITEs. Finally, from (23) it follows that operator (20) does not have poles at ITEs.

We will use the approach to count ITEs that was developed in [16], but now we will use operator (20) instead of \( N_{n}^{in} - N_{0}^{in} \). We fix an arbitrary invertible symmetric elliptic operator \( D \) on \( \partial \mathcal{O} \) of the second order. Let

\[
\hat{R}(\lambda) = \sigma DRD, \quad \sigma = \text{sign}_{x \in \partial \mathcal{O}}(n(x) - 1),
\]

(\( \sigma \) is not to be confused with \( \sigma_{i} \) defined in [13]), and let \( \{\mu_{j}(\lambda)\} \) be the set of real eigenvalues of \( \hat{R}(\lambda) \), where \( \lambda \) is not a pole of \( \hat{R}(\lambda) \). Let \( n^{-}(\lambda) \geq 0 \) be the number of negative eigenvalues \( \mu_{j}(\lambda) \). From Lemma 2.7 it follows that \( \hat{R}(\lambda) \) is an elliptic operator of the first order with a positive principal symbol. Thus \( \mu_{j}(\lambda) \to \infty \) as \( j \to \infty \) and \( n^{-}(\lambda) \) is well defined if \( \lambda \) is not a pole of \( \hat{R}(\lambda) \). We will work with operator \( \hat{R} \) instead of \( R \) in order to deal with an operator whose eigenvalues converge to infinity, not to zero. On the other hand, operator \( D \) establishes a one-to-one correspondence between the kernels of \( R \) and the kernels of \( \hat{R} \). Thus ITEs can be defined as values of \( \lambda = \lambda_{i}^{T} \) where \( \hat{R}(\lambda) \) has a kernel, and \( m_{i} \) eigenvalues of \( \hat{R}(\lambda) \) vanish at each ITE \( \lambda = \lambda_{i}^{T} \) of multiplicity \( m_{i} > 0 \).

Since operator (20) is self-adjoint and analytic in \( \lambda \) in a neighborhood of each ITE, its eigenvalues and eigenfunctions can be chosen to be analytic in these neighborhoods (see [20, Example 3, XIII.12]). The following lemma follows from here, Lemma 2.7 and the theorem on the spectral decomposition of self-adjoint operators (after an appropriate enumeration of the eigenvalues \( \mu_{j} \)):

Lemma 2.8. Let \( \lambda = \lambda_{0} = \lambda_{i}^{T} \) be an ITE of order \( m_{i} \). Then there exists \( \delta > 0 \) such that

\[
\hat{R}(\lambda) = \sum_{j=0}^{m_{i}} \mu_{j}(\lambda)P_{\varphi_{j}(\lambda)} + K(\lambda), \quad |\lambda - \lambda_{0}| < \delta,
\]

(25)
where $\mu_j(\lambda)$ are analytic (when $|\lambda - \lambda_0| < \delta$) eigenvalues of $\hat{R}(\lambda)$ such that $\mu(\lambda_0) = 0$ and the corresponding eigenfunctions $\varphi_j(\lambda)$ are analytic and orthogonal, $P_{\varphi_j(\lambda)}$ is the projection on $\varphi_j$, and the kernel of $K(\lambda)$ coincides with $\text{span}\{\varphi_j(\lambda)\}$.

The inverse operator has the form:

$$\hat{R}^{-1}(\lambda) = \sum_{j=0}^{m} \mu_j^{-1}(\lambda)P_{\varphi_j(\lambda)} + K_1(\lambda), \quad |\lambda - \lambda_0| < \delta,$$

(26)

where $K_1$ is analytic in $\lambda$, $|\lambda - \lambda_0| < \delta$ (it is inverse to $K$ on the subspace orthogonal to $\text{span}\{\varphi_j(\lambda)\}$).

Let us denote by $\alpha_i^+$, $(\alpha_i^-)$ the number of eigenvalues $\mu_j(\lambda)$ whose Taylor expansion at $\lambda = \lambda_i^+$ starts with an odd power of $\lambda - \lambda_i^+$ and the coefficient for this power has the same sign as $-\sigma$ (respectively, $\sigma$), where $\sigma$ is defined by (24).

**Theorem 2.9.** Let $n(x) \neq 1$, $x \in \partial \Omega$. Then

$$\sum_{i : 0 < \lambda_i^+ < \lambda} (\alpha_i^+ - \alpha_i^-) = \frac{\omega_d}{(2\pi)^d} \gamma \lambda^\frac{d}{2} + O(\lambda^{\frac{d}{2} - \delta}), \quad \lambda \to \infty,$$

(27)

where $\delta = \frac{1}{2d}$.

**Proof.** Let us fix an arbitrary point $\alpha > 0$ that is not a pole of $\hat{R}(\lambda)$, and is smaller than the smallest eigenvalues for the Dirichlet problems for equations (1) and (2). Let us evaluate the difference $n^-(\lambda') - n^-(\alpha)$ by moving $\lambda$ from $\lambda = \alpha$ to a value $\lambda = \lambda'$ if $\lambda' > \alpha$. The eigenvalues $\mu_j(\lambda)$ are meromorphic in $\lambda$ and may enter/exit the negative semi-axis $R_{\mu}^- = \{\mu : \mu < 0\}$ only through the end points of the semi-axis. Thus we can split $n^-(\lambda') - n^-(\alpha)$ as

$$n^-(\lambda') - n^-(\alpha) = n_1(\lambda') + n_2(\lambda'),$$

(28)

where $n_1(\lambda')$ is the number of eigenvalues $\mu_j(\lambda)$ that enter/exit the negative semi-axis $R_{\mu}^-$ through the point $\mu = -\infty$ (when $\lambda$ changes from $\alpha$ to $\lambda' > \alpha$) and $n_2(\lambda')$ is the number of eigenvalues $\mu_j(\lambda)$ that enter/exit the negative semi-axis $R_{\mu}^-$ through the point $\mu = 0$. Obviously,

$$n_2(\lambda) = \sum_{i : \alpha < \lambda_i^+ < \lambda} \sigma(\alpha_i^+ - \alpha_i^-).$$

(29)

Next, one can show that

$$n_1(\lambda') = \sigma(N_{\mu}^l(\lambda') - N_l^t(\lambda')),$$

(30)

where $N_n^l(\lambda), N_l^t(\lambda)$ are counting functions for operators $\frac{1}{n(x)}\Delta$ and $-\Delta$, respectively, with the impedance boundary condition $\frac{\partial u}{\partial v} - tu = 0$. In order to obtain (30), one needs to note that $\mu_j(\lambda) \to -\infty$ only when $\lambda$ passes through the poles of operator (20). These poles occur exactly at eigenvalues of the corresponding impedance boundary problem. A
The statement of the theorem follows immediately from (29), (31), (32). The latter estimate can be justified absolutely similarly to an analogous estimate (14) in Lemma 2.10. For each positive number \( n \), An important part of the proof of Theorem 2.9 is the following estimate from above for the number \( n^-(\lambda) \) of negative eigenvalues of the operator \( R(\lambda) \):

\[
n^-(\lambda) = O(\lambda^{d/2-\delta}), \quad \lambda \to \infty.
\]

The latter estimate can be justified absolutely similarly to an analogous estimate (14) in ([17]). The statement of the theorem follows immediately from (29), (31), (32). \( \square \)

Our next goal is to show that \( \alpha_i^+ - \alpha_i^- = \sigma_i \). Then Theorem 1.3 will follow from Theorem 2.9.

(E). An alternative representation of the far-field operator.

Lemma 2.10. For each positive \( \lambda = k^2 > 0 \), operator \( \frac{1}{\alpha}F \) can be written as follows

\[
\frac{1}{\alpha}F = Q^*[R_1(\lambda) + R(\lambda)^{-1} + iI(\lambda)]Q, \quad Q = (N^{in} - t)^{-1}(N^{in} - N_0^{in})L,
\]

where 1) operators \( R_1, R, I \) are symmetric; 2) operator \( R \) is defined in (29); 3) operators \( R_1, I \) are analytic in \( \lambda \); 4) \( R_1 \) is an elliptic operator of order one; 5) operator \( I \) is infinitely smoothing (has order \(-\infty\)) and non-negative. It is strictly positive for all \( \lambda > 0 \) except possibly at most countable set \( \{\lambda_i\} \) that does not contain any of ITEs \( \lambda_i^2 \) or eigenvalues of the Dirichlet problem for equation (2) and does not have any finite limit points.

**Proof.** We write \( F \) (given in Theorem 2.3) in the form

\[
\alpha^{-1}F = \mathcal{L}^*[(N^{in} - N_0^{in})(N^{in} - N^{out})^{-1}(N^{in} - N_0^{in}) + (N_0^{in} - N_0^{in})]L
\]

\[
= \mathcal{L}^*(N_0^{in} - N_0^{in})(N^{in} - N^{out})^{-1} + (N^{in} - N_0^{in})^{-1}(N_0^{in} - N_0^{in})L = Q^*\hat{F}Q,
\]

where

\[
\hat{F} = (N^{in} - t)(N^{in} - N^{out})^{-1} + (N_0^{in} - N_0^{in})^{-1}(N^{in} - t).
\]

We need to show that \( \hat{F} = R_1 + (\hat{R})^{-1} + iI \). Let us split the right-hand side in (34) in two terms and rearrange the second one. We have

\[
(N^{in} - t)(N_0^{in} - N^{in})^{-1}(N^{in} - t) = (N^{in} - N_0^{in} + N_0^{in} - t)(N_0^{in} - N_0^{in})^{-1}(N_0^{in} - t)
\]

\[
= -N_n^{in} + t + (N_0^{in} - t)(N_0^{in} - t + t - N_0^{in})^{-1}(N_0^{in} - t) = -N_n^{in} + t + (R)^{-1}.
\]

Hence it remains to show that

\[
F_1 := (N^{in} - t)(N^{in} - N^{out})^{-1}(N^{in} - t) - N_n^{in} + t
\]

(35)
has the form $R_1 + iI$ where $R_1, I$ have the properties listed in Lemma 2.10.

One can easily single out the imaginary part of the operator $F_1$:

$$I(\lambda) = \Re F_1 = (N_{in}^n - t)(N_{in}^n - N_{out}^n)^{-1}\Re N_{out}^n(N_{in}^n - N_{out}^n)^{-1*}(N_{in}^n - t).$$

In order to obtain this formula, one can add the factor $(N_{in}^n - N_{out}^n)^*(N_{in}^n - N_{out}^n)^{-1*}$ after the negative power in expression (35) for $F_1$, and then use the symmetry of $N_{in}^n$ and the relation $\Re(N_{out}^n)^* = -\Re N_{out}^n$. Let us justify all the properties of $I(\lambda)$. Operator $(N_{in}^n - N_{out}^n)^{-1}$ is analytic in $\lambda$ due to Lemma 2.10. Operator $N_{in}^n$ has poles, but the product $P := (N_{in}^n - t)(N_{in}^n - N_{out}^n)^{-1}$ is analytic. The easiest way to see the latter property is to replace the first factor in the product $P$ by $(N_{in}^n - N_{out}^n) + (N_{out}^n - t)$. Thus $I(\lambda)$ is analytic in $\lambda$. The product $P$ is an operator of order zero, and $\Re N_{out}^n$ has order $-\infty$ (see (39)). Thus the order of $I(\lambda)$ is $-\infty$. Finally,

$$(I(\lambda)\varphi, \varphi) = \Re(N_{out}^n \psi, \psi), \quad \psi = (N_{in}^n - N_{out}^n)^{-1*}(N_{in}^n - t)\varphi.$$  

The latter expression is positive if $\psi \neq 0$ due to (39). Thus, in order to obtain the last property of $I(\lambda)$, it remains to find points $\lambda$ where operator $(N_{in}^n - N_{out}^n)^{-1*}(N_{in}^n - t)$ has a non-trivial kernel, i.e., the inverse operator

$$(N_{in}^n - t)^{-1}(N_{in}^n - N_{out}^n)^* = (N_{in}^n - t)^{-1}(N_{in}^n - t + (N_{out}^n)^*) = I - (N_{in}^n - t)^{-1}((N_{out}^n)^* - t)$$  

(where $I$ is the identity operator) has a pole. The latter may occur only when $N_{in}^n - t$ has a non-trivial kernel. The corresponding set $\{\lambda_n\}$ is the set of eigenvalues of the impedance problem (22) (where $t$ is fixed), and it does not include the ITEs and eigenvalues of the Dirichlet problem due to (23). Hence all the properties of operator $I(\lambda)$ are justified.

To obtain the properties of operator $R_1$, we rewrite (35) in the form

$$F_1 = (N_{in}^n - N_{out}^n + N_{out}^n - t)(N_{in}^n - N_{out}^n)^{-1}(N_{in}^n - t) - N_{in}^n + t$$

$$= (N_{out}^n - t)(N_{in}^n - N_{out}^n)^{-1}(N_{in}^n - t) = (N_{out}^n - t)(N_{in}^n - N_{out}^n)^{-1}(N_{in}^n - N_{out}^n + N_{out}^n - t)$$

$$= (N_{out}^n - t) + (N_{out}^n - t)(N_{in}^n - N_{out}^n)^{-1}(N_{out}^n - t).$$

Then Lemmas 2.1 and 2.6 imply the analyticity of $F_1$. Since $F_1(\lambda) = R_1(\lambda) + iI(\lambda)$ and $I$ is analytic, operator $R_1$ is analytic. From Lemma 2.1 and formula (35), it follows that the principal symbol of $F_1$ is equal to $-|\xi^*|/2$. Thus operator $R_1$ has order one since $I(\lambda)$ is an infinitely smoothing operator. The proof of Lemma 2.10 is complete.

As we mentioned earlier, it is more convenient for us to work with operator $\hat{R}$ instead of $R$, and therefore we will use the following version of (33):

$$\frac{1}{\alpha}F = Q^*D[\hat{R}_1(\lambda) + \sigma\hat{R}(\lambda)]DQ, \quad Q = (N_{in}^n - t)^{-1}(N_{in}^n - N_{in}^n_0)\mathcal{L}, \quad (36)$$

where operators $\hat{R}_1 = D^{-1}R_1D^{-1}$, $\hat{I} = D^{-1}ID^{-1}$ have the same properties as operators $R_1, I$, respectively, with the only difference that $\hat{R}_1$ has order $-3$. 

15
3 Proof of the main theorems

Proof of Theorem 1.2. Let \( n(x) < 1 \) on \( \partial \Omega \) (i.e., \( \sigma < 0 \)). Let \( T^+ = \text{Span}\{\varphi_i^+\} \), where \( \varphi_i^+ \) are the eigenfunctions of the scattering matrix \( S(k) \) with the eigenvalues \( z_i \) in the upper half complex plane \( \Im z \geq 0 \). In order to prove the first statement of the theorem, we need to show that the space \( T^+ \) is finite-dimensional.

From (12) it follows that \( \Re(\alpha - 1)F\varphi, \varphi \geq 0 \). This and the orthogonality of functions \( \varphi_i^+ \) imply that
\[
\Re(\alpha - 1)F\varphi, \varphi \geq 0, \quad \varphi \in T^+.
\]
(37)

On the other hand, from (18) and Lemma 2.1 it follows that
\[
(\alpha - 1)F = L^* \hat{F} L, \quad \hat{F} \text{ is a pseudo-differential operator with the principal symbol } \lambda (n(x) - 1)\xi^*/2.
\]

For every \( \varphi \in L^2(\partial \Omega) \), we have
\[
(\alpha - 1)F\varphi, \varphi = (\hat{F}\psi, \psi), \quad \psi = L\varphi.
\]
Since \( \hat{F} \) is an elliptic operator of order one with a negative principal symbol, there exists \( a > 0 \) such that
\[
\Re(\hat{F}\psi, \psi) \leq -a\|\psi\|_{H^{1/2}} + C\|\psi\|_{L^2(\partial \Omega)}, \quad \psi \in \mathcal{L}T^+.
\]
(38)

From here, (37), and the Sobolev imbedding theorem it follows that the set
\[
\mathcal{L}T^+ \cap \{\|\psi\|_{L^2(\partial \Omega)} = 1\}
\]
is compact in \( L^2(\partial \Omega) \). Thus, the linear space \( \mathcal{L}T^+ \) is finite-dimensional. Now Lemma 2.4 implies that the space \( T^+ \) is finite-dimensional.

The first statement of the theorem is proved. To prove the second statement, one needs only to replace \( T^+ \) by \( T^- = \text{Span}\{\varphi_i^-\} \), where \( \varphi_i^- \) are the eigenfunctions with the eigenvalues \( z_i \) in the lower half complex plane, and use the positivity of the principal symbol of \( \hat{F} \). Let us prove the last statement.

Assume that the space \( T^- = \text{Span}\{\varphi_i^-\} \) is finite-dimensional, where \( \varphi_i^- \) are the eigenfunctions of the scattering matrix \( S(k) \) with the eigenvalues \( z_i \) in the lower half complex plane \( \Im z < 0 \). Then, similarly to (37), we have
\[
\Re(\alpha - 1)F\varphi, \varphi \geq 0, \quad \varphi \in (T^-)^\perp,
\]
and therefore,
\[
\Re(\hat{F}\psi, \psi) \geq 0, \quad \psi \in \mathcal{L}(T^-)^\perp.
\]
(40)

We fix an \( \varepsilon > 0 \) so small that the set \( \Gamma^- = \partial \Omega \cap \{x : n(x) < 1 - \varepsilon\} \) is not empty. Let \( n' \) be an infinitely smooth function in \( \Omega \) such that \( 0 < n' < 1 \) and \( n' \) coincides with \( n(x) \) in a \( d \)-dimensional neighborhood of \( \Gamma^- \). From standard local a priori estimates for
the solutions of elliptic equations it follows that the operator \( G = N_{n}^{in} - N_{n}'^{in} \) is infinitely smoothing on functions \( \psi \in L_2 \). The latter space consists of functions from \( L_2(\partial \Omega) \) with the support in \( \overline{\Gamma} \). Denote by \( \hat{F}' \) operator (18) with \( n \) replaced by \( n' \). Since (38) holds for \( \hat{F}' \), it is valid for \( \hat{F} \) when \( \psi \in L_2 \). This and (40) imply that

\[
0 \leq -a \| \psi \|_{H^{1/2}} + C \| \psi \|_{L_2(\partial \Omega)}, \quad \psi \in L_2(\Omega) \cap L_2(\partial \Omega) \subset L_{2}^{-} \cap L_{2}^{-} \cap \{ \| \psi \|_{L_2^{-}} = 1 \}.
\]

The inequality above and the Sobolev imbedding theorem lead to the compactness of the set \( L_2(\Omega) \cap L_{2}^{-} \cap \{ \| \psi \|_{L_2^{-}} = 1 \} \). The compactness is possible only if the linear space \( L_2(\Omega) \cap L_{2}^{-} \) (which is equal to \( (L^*T)^{-} \cap L_2^{-} \)) is finite-dimensional. The latter contradicts the assumption that \( T^{-} \) is finite-dimensional. Similarly, one can prove that \( T^{+} \) can not be finite-dimensional.

**Proof of Theorems 1.1 and 1.3.** Step 1. Quadratic forms related to \( S(k) \). The following general statement plays an important role in the proof of the main results. Let \( 0 < \alpha_1 < \alpha_2 < \pi \). Denote by \( S_{\alpha_1, \alpha_2} \) the closed domain in the upper half complex plane bounded by the arc and the chord of the unit circle with the end points at \( e^{i\alpha_1}, e^{i\alpha_2} \).

**Lemma 3.1.** Let \( U \) be a unitary operator in a Hilbert space \( \mathcal{H} \). Let \( \mathcal{H}_0 \subset \mathcal{H} \) be an \( m \) dimensional subspace, and let \( \mathcal{H}_1 \subset \mathcal{H} \) be a subspace of co-dimension \( m \). Then

1. The range (the set of values) of the quadratic form \( \langle U\varphi, \varphi \rangle, \varphi \in \mathcal{H}, \| \varphi \| = 1 \), coincides with the polygon with the vertices (there may be infinitely many of them) at the eigenvalues of \( U \).

2. If \( \langle U\varphi, \varphi \rangle \in S_{\alpha_1, \alpha_2} \) for each \( \varphi \in \mathcal{H}_0, \| \varphi \| = 1 \), then \( U \) has at least \( m \) eigenvalues \( z \) (with the multiplicities taken into account) with \( \text{arg} z \in (\alpha_1, \alpha_2) \).

3. If \( \langle U\varphi, \varphi \rangle, \varphi \in \mathcal{H}_1, \| \varphi \| = 1 \), does not have values in \( S_{\alpha_1, \alpha_2} \), then \( U \) has at most \( m \) eigenvalues \( z \) (with the multiplicities taken into account) with \( \text{arg} z \in (\alpha_1, \alpha_2) \).

**Proof.** The form has values

\[
\sum_i t_i^2 e^{i\gamma_i}, \quad \text{where} \quad \sum_i t_i^2 = 1.
\]

Here \( e^{i\gamma_i} \) are the eigenvalues of \( U \). This implies the first statement. If the second assumption holds, then the existence of at least one eigenvalue follows immediately from the first statement. If there are only \( m_1 < m \) linearly independent normalized eigenfunctions \( \varphi_j, 1 \leq j \leq m_1 \), with eigenvalues on the arc that bounds \( S_{\alpha_1, \alpha_2} \), then one can apply the first statement of the lemma to the same quadratic form on the space orthogonal to \( \text{span} \varphi_j \) and prove the existence of one more eigenvalue on the same arc.

Let us prove the last statement. Assume that \( U \) has more than \( m \) eigenvalues with \( \text{arg} z \in (\alpha_1, \alpha_2) \). Then there exists an eigenfunction that belongs to \( \mathcal{H}_1 \) and has an eigenvalue \( e^{i\alpha}, \alpha_1 < \alpha < \alpha_2 \). The form on this eigenfunction is equal to \( e^{i\alpha} \).
The next lemma contains some statements on relations between the far-field operator $F$ (see [2]) and the eigenvalues $z_j(k)$ of the scattering matrix $S(k)$.

**Lemma 3.2.** (A) Let $n(x) < 1$ on $\partial \mathcal{O}$. Then

1) If there exist $m^\pm$-dimensional subspaces $\Phi^\pm = \Phi^\pm(k)$ in $L_2(S^{d-1})$ such that the following relations hold for the far field operator $F$ when $k \to k_0 \mp 0$:

$$0 < \text{arg}(\alpha^{-1}F\varphi, \varphi) < \delta(k), \quad 0 \neq \varphi \in \Phi^\pm, \quad \lim_{k \to k_0 \mp 0} \delta(k) = 0, \quad (41)$$

then the scattering matrix $S(k)$, $\pm(k_0 - k) > 0$, has at least $m^\pm$ eigenvalues $z_j(k) = z_j^\pm(k), 1 \leq j \leq m^\pm$, on $C_+$ that approach $z = 1$ moving clockwise (counter clockwise, respectively), i.e.,

$$\lim_{k \to k_0 \mp 0} z_j(k) = 1 + i0.$$  

2) If there exist subspaces $\Phi'_\pm = \Phi'_\pm(k)$ in $L_2(S^{d-1})$ of co-dimensions $m^\pm$ such that

$$\arg(\alpha^{-1}F\psi, \psi) \notin (0, \delta), \quad 0 \neq \psi \in \Phi'_\pm, \quad \varepsilon > \pm(k_0 - k) > 0, \quad (42)$$

then the scattering matrix $S(k)$ has at most $m^\pm$ eigenvalues on the arc $0 < \arg z < \delta$ of the unit circle when $\varepsilon > \pm(k_0 - k) > 0$.

3) Thus if both assumptions 1) and 2) hold, then $S(k)$ has $m^+$ ($m^-$) eigenvalues on $C_+$ that approach $z = 1$ moving clockwise (counter clockwise, respectively) when $k \to k_0 \mp 0$, and all other eigenvalues on $C_+$ are separated from $z = 1$ when $k$ is close enough to $k_0$.

(B) Let $n(x) > 1$ on $\partial \mathcal{O}$. Then

1') If there exist $m^\pm$-dimensional subspaces $\Phi^\pm = \Phi^\pm(k) \subset L_2(S^{d-1})$ such that

$$\pi - \delta(k) < \text{arg}(\alpha^{-1}F\varphi, \varphi) < \pi, \quad 0 \neq \varphi \in \Phi^\pm, \quad \lim_{k \to k_0 \pm 0} \delta(k) = 0, \quad (43)$$

then the scattering matrix $S(k)$ has at least $m^\pm$ eigenvalues on $C_+$ that approach $z = 1$ moving clockwise (counter clockwise, respectively).

2') If there exist subspaces $\Phi'_\pm = \Phi'_\pm(k) \subset L_2(S^{d-1})$ of co-dimension $m^\pm$ such that

$$\arg(\alpha^{-1}F\psi, \psi) \notin (\pi - \delta, \pi), \quad 0 \neq \psi \in \Phi'_\pm, \quad \varepsilon > \pm(k - k_0) > 0,$$

then the scattering matrix $S(k)$ has at most $m^\pm$ eigenvalues on the arc $\pi - \delta < \arg z < \pi$ of the unit circle when $\varepsilon > \pm(k - k_0) > 0$.

**Proof.** The eigenvalues of the unitary operator $S(k)$ belong to the unit circle. Therefore, from [12] it follows that the eigenvalues of the operator $\alpha^{-1}F(k)$ belong to the circle of radius $k|\alpha|^2/2\pi$ centered at $ik|\alpha|^2/2\pi$, and moreover, if $\sigma = -1$, then the values of the quadratic form $(S(k)\varphi, \varphi)$ with $\|\varphi\| = 1$ belong to the set $S_{\sigma, \gamma}$ if and only if (41) holds with $\delta(k) = \gamma$. Similarly, if $\sigma = 1$, then the values of the quadratic form $(S(k)\varphi, \varphi), \|\varphi\| = 1,$
belong to the set $S_{-\gamma,0}$ if and only if \( (13) \) holds with $\delta(k) = \gamma$. Thus Lemma 3.2 is the direct consequence of Lemma 3.1.

\( \square \)

**Step 2. Plan to complete the proofs of Theorems 1.1 and 1.3.** Let $\lambda_0 = \lambda^T$ be an ITE of multiplicity $m_i$ and let $\beta^\pm_i$ be the number of eigenvalues $\mu_j$, $j \leq m_i$, in formulas (25), (26) that are negative when $1 \gg \varepsilon \geq \pm \sigma (\lambda - \lambda_0) > 0$. Let us stress that we consider only those $\mu_j(\lambda)$ that vanish at $\lambda = \lambda_0$. Obviously, $\beta^\pm_i = \alpha^\pm_i + r$, where $\alpha^\pm_i$ are defined in (27) and $r$ is the number of eigenvalues whose Taylor expansion starts with an even positive power of $\lambda - \lambda_0$ and has a negative coefficient for this power. In particular,

\[
\beta^+_i - \beta^-_i = \alpha^+_i - \alpha^-_i.
\] (44)

In Step 3, we are going to show that there exists a $\beta^+_i$-dimensional subspace $\Phi = \Phi^+(\lambda)$ in $L_2(S^{d-1})$ on which (41) holds when $k \to k_0$ and there is a $\beta^-_i$-dimensional subspace $\Phi = \Phi^-(\lambda)$ in $L_2(S^{d-1})$ on which (41) holds when $k \to k_0 + 0$. We refer to relation (41) below, but in fact we are going to justify simultaneously (41) when $\sigma < 0$ and its analogue for $\sigma > 0$ stated in part (B) of Lemma 3.2. In Step 4, we will prove (42) with $\Phi^i = \Phi^i_0$ of co-dimension $\beta^+_i$ when $k \to k_0$ (and its analogue from part (B) of the same lemma). Then Lemma 3.2 will justify all the statements of Theorem 1.3 with $m^+_i = \beta^+_i$. In particular, the last statement of the theorem will be justified because the arguments below are valid when $m_i = 0$, i.e., $\lambda = \lambda_0$ is not an ITE (more details will be given in Step 4). Since the relation $m^+_i = \beta^+_i$ will be established, (41) will imply that $\alpha^+_i - \alpha^-_i = m^+_i - m^-_i$. Thus Theorem 1.1 will be a consequence of Theorem 2.9. Hence the proofs of Theorems 1.1 and 1.3 will be completed as soon as (41) and (42) are established.

Let us make one more remark concerning the next steps. The ITEs are values of $\lambda$, and the Dirichlet-to-Neumann maps are functions of $\lambda$, while it is customary to consider the far-field operator and the scattering matrix as functions of $k$. Many formulas below will contain simultaneously $k$ and $\lambda$. It is always will be assumed (without reminders) that $\lambda = k^2$.

**Step 3. Establishing (47).** We will consider only the case of $\sigma (\lambda - \lambda_0) \to +0$, since the arguments in the case of $\sigma (\lambda - \lambda_0) \to -0$ are no different (only the pluses in the indices must be replaced by minuses in the latter case).

Denote by $\hat{F}$ the operator in the square brackets in the right-hand side of formula (36). Let

\[
\hat{F}^+ = \hat{F}^+(\lambda):= \text{span}\{\varphi_j, 1 \leq j \leq \beta^-_j\},
\] (45)

where $\varphi_j = \varphi_j(\lambda)$ are functions defined in (25). The enumeration is such that the functions in (25) with $\mu_j(\lambda) < 0$ when $\sigma (\lambda - \lambda_0) \to +0$ are listed first. Then from (36) it follows that

\[
(\hat{F} \varphi, \varphi) = \sigma \sum_{j=1}^{\beta^+_i} c^2_j \mu_j^{-1}(\lambda) + O(1), \quad \varphi = \sum_{j=1}^{\beta^+_i} c_j \varphi_j \in \hat{F}^+, \quad \sigma (\lambda - \lambda_0) \to +0.
\] (46)
This implies that \(|\Im(\hat{F}\varphi,\varphi)| = O(1)| and

\[
\frac{|\Im(\hat{F}\varphi,\varphi)|}{\Re(\hat{F}\varphi,\varphi)} \to 0, \quad \sigma(\lambda - \lambda_0) \to +0, \quad 0 \neq \varphi \in \Phi^-.
\] (47)

The imaginary part of the form (36) is positive (due to Lemma 2.10) and the real part has the same sign as \(-\sigma\). Thus (47) justifies (41) for \(\sigma < 0\) and its analogue for \(\sigma > 0\) from part (B) of Lemma 3.2 but both relations are justified for the operator \(\hat{F}\) (on the space \(\Phi^-\)) instead of the operator \(\alpha^{-1}F\). Then the same relations for \(\hat{F}\) hold for an arbitrary \(\beta_1^+\)-dimensional subspace \(\Phi^+_\lambda\) in \(L_2(S^{d-1})\) if it is close enough to \(\hat{F}^+\lambda\), where the distance between these subspaces may depend on \(\lambda\). Since operator \(DQ\) for each \(\lambda \in (\lambda_0 - \varepsilon, \lambda_0)\) and small enough \(\varepsilon\) has a dense range (see Lemma 2.4), one can find functions \(\psi_j\) such that \(DQ\psi_j\) are so close to \(\varphi_j\) that (41) holds for operator \(\hat{F}\) on the subspace \(\Phi^+\lambda = \text{span}\{DQ\psi_j\}\). Then (41) and its analogue for \(\sigma > 0\) hold for \(\alpha^{-1}F\) with \(\Phi^+ = \text{span}\psi_j\).

Step 4. Establishing (42). As in the previous step, we could prove simultaneously (42) and its analogue for \(\sigma > 0\). However, we will assume that \(\sigma < 0\) to make the text more transparent. Also, we are going to consider only the case of \(\lambda < \lambda_0 = \lambda_0^T\), since the case of \(\lambda > \lambda_0\) is no different.

Let us show that (42) holds with \(\Phi' = (Q^*D\hat{F}^+)^{1}\), where \(\hat{F}^+\) is defined in (45). Due to Lemma 2.4, it is possible to choose \(\varepsilon > 0\) small enough so that the kernel of the operator \(Q^*D\) is trivial when \(\lambda_0 - \varepsilon < \lambda < \lambda_0\), and therefore the dimension of \(Q^*D\hat{F}^+\) is \(\beta_1^+\).

If \(\psi \in \Phi'\), then \(\varphi := Q\psi\) is smooth enough since \(Q\) contains the factor \(\mathcal{L}\), which is an infinitely smoothing operator. In particular, \(\varphi \in H^1(\partial\mathcal{O})\). Furthermore, \(D\varphi \perp \hat{F}^+\), and (due to (33))

\[
(\alpha^{-1}F\psi,\psi) = ([R_1(\lambda) + R(\lambda)^{-1} + iI(\lambda)]\varphi,\varphi).
\]

Hence it is enough to show that

\[
\arg([R_1 + R^{-1} + iI]\varphi,\varphi) \notin (0, \delta) \quad \text{when} \quad \lambda_0 - \varepsilon < \lambda < \lambda_0
\] (48)

for smooth functions \(\varphi \neq 0\) such that \(D\varphi \perp \hat{F}^+\). Obviously, it is enough to consider smooth functions \(\varphi \neq 0\) from the space

\[
\Phi_1 = (D\hat{F}^+)^{1} \cap \{\|\varphi\|_{H^{1/2}(\partial\mathcal{O})} = 1\} \cap \{\Re(\hat{F}\varphi,\varphi) > 0\},
\]

and (48) will be proved if we show the existence of constants \(\gamma_1, \gamma_2 > 0\) such that the following estimates are valid for the real and imaginary parts of the form (48):

\[
([R_1 + R^{-1}]\varphi,\varphi) < \gamma_1, \quad (I\varphi,\varphi) > \gamma_2 > 0 \quad \text{for} \quad \varphi \in \Phi_1.
\] (49)

From (24) and (26) it follows that

\[
(R^{-1}\varphi,\varphi) = \sigma(\hat{R}^{-1}D\varphi, D\varphi) = \sum_{j=\beta_1^+}^{m_1} \frac{\sigma}{\mu_j(\lambda)} \|P_{\varphi_j}D\varphi\|^2 + \sigma(K_1 D\varphi, D\varphi)
\]
\( \leq (\sigma K_1(\lambda) D\varphi, D\varphi). \)

We omitted the terms under the summation sign here, since \( \mu_j(\lambda) > 0 \) (and \( \sigma \mu_j(\lambda) < 0 \)) when \( j > \beta^+_1, \lambda_0 - \varepsilon < \lambda < \lambda_0 \). For each \( \lambda \neq \lambda_0 \), operator \( \hat{R}(\lambda) \) is an elliptic pseudodifferential operator of the first order with a negative principal symbol (see Lemma 2.7 and formula (24)), and therefore \( \hat{R}^{-1}(\lambda) \) is an elliptic pseudodifferential operator of order \(-1\) with a negative principal symbol. Operator \( K_1(\lambda) \) differs from \( \hat{R}^{-1}(\lambda) \) by a projection on a finite-dimensional space spanned by \( C^\infty \) functions. Thus it has the same properties as \( \hat{R}^{-1}(\lambda) \), but additionally it is analytic in \( \lambda \) (see Lemma 2.5). Hence there is a constant \( a_1 > 0 \) such that

\[
(R^{-1}(\lambda) \varphi, \varphi) \leq (\sigma K_1(\lambda) D\varphi, D\varphi) \leq -a_1 \|D\varphi\|_{H^{-1/2} + O(\|D\varphi\|_{H^{-1}})} \\
\leq -a_1 \|\varphi\|_{H^{3/2}} + O(\|\varphi\|_{H^1(\partial\Omega)}), \quad \lambda_0 - \varepsilon < \lambda < \lambda_0.
\]

Operator \( R_1 \) is an elliptic operator of the first order, and it is analytic in \( \lambda \) in a neighborhood of \( \lambda_0 \) (see Lemma 2.10). Thus

\[
([R_1 + R^{-1}] \varphi, \varphi) \leq -a_1 \|\varphi\|_{H^{3/2}} + a_2(\|\varphi\|_{H^1}), \quad \lambda_0 - \varepsilon < \lambda < \lambda_0.
\]

This implies the first estimate in (49) and also the compactness of the set \( \Phi_1 \) in \( H^1(\partial\Omega) \). Indeed, since \( \|\varphi\|_{H^1} = 1 \) in \( \Phi_1 \), from the line above it follows that \( \Re(\hat{F}\varphi, \varphi) > 0 \) on \( \Phi_1 \) only if \( \|\varphi\|_{H^{3/2}} \) is bounded. Thus the set \( \Phi_1 \) is compact in \( H^1(\partial\Omega) \) due to the Sobolev imbedding theorem.

Further, due to (39), \( \Im(\hat{F}\varphi, \varphi) > 0 \) on each element \( 0 \neq \varphi \in H^1(\partial\Omega) \) for \( \lambda_0 - \varepsilon/2 \leq \lambda \leq \lambda_0 \) (the end points are included). Then the compactness of \( \Phi_1 \) in \( H^1(\partial\Omega) \) implies that \( \Im(\hat{F}\varphi, \varphi) \) has a positive lower bound on \( \Phi_1 \), i.e., the second estimate in (49) holds. Thus (42) is justified.

**Step 5.** All the arguments in the previous step, used to prove (42), are valid when \( m_i = 0 \) (i.e., for \( \lambda = \lambda_0 \), which is not an ITE) if \( \lambda_0 \) does not belong to the exceptional set \( \{\hat{\lambda}_s\} \) defined in Lemma 2.10. This set is discrete and consists of eigenvalues of the impedance problem (22) where \( t \) was fixed at an earlier stage (see the proof of Lemma 2.10). After (42) is proved for all \( \lambda \) except a fixed exceptional set \( \{\hat{\lambda}_s\} \), we can change the value of \( t \) to another value \( t = t_s \) for which \( \lambda = \hat{\lambda}_s \) is not an eigenvalue of the impedance problem (22) with \( t = t_s \). Then (42) will be justified for \( \lambda = \hat{\lambda}_s \). In fact, we can find a value of \( t = \hat{t} \) that can be used simultaneously for all points \( \hat{\lambda}_s \), but we do not need to do it.

The proof of Theorems 1.1 and 1.3 is complete.

**Proof of Theorem 1.4.** First, let us show that the eigenvalues of the operator \( R(\lambda) \), defined in (24), can only have simple zeroes. Indeed, if \( \lambda > 0 \) is not a pole of the operator \( R^{-1}(\lambda) \), then \( R^{-1}(\lambda) \) maps an arbitrary function \( f \in H^{3/2}(\partial\Omega) \) into

\[
R^{-1}(\lambda)f = (\frac{\partial u}{\partial v} - tu)|_{x \in \partial\Omega} = (\frac{\partial u}{\partial v} - tv)|_{x \in \partial\Omega}.
\]

(50)
where \((u, v)\) is the solution of the problem

\[
\Delta u + \lambda u = 0, \quad u \in H^2(\mathcal{O}),
\]

\[
\Delta v + \lambda n(x)v = 0, \quad v \in H^2(\mathcal{O}),
\]

\[
u - v = f, \quad x \in \partial \mathcal{O},
\]

\[
\frac{\partial u}{\partial \nu} - tu = \frac{\partial v}{\partial \nu} - tv, \quad x \in \partial \mathcal{O}.
\]

One can express solution \((u, v)\) of (51)-(53) through the resolvent of the ITE problem by looking for \((u, v)\) as a sum of two terms, where the first term \((u_1, v_1)\) satisfies only the boundary conditions, and the second term is the solution of the problem (51)-(53) with homogeneous boundary conditions and the right-hand side in the equations defined by the first term. Hence the operator \(R(\lambda)\) implies that the eigenvalues of the operator \(\Delta\) have poles at most of first order. Therefore, (50) implies that the eigenvalues of the operator \(R(\lambda)\) may have zeroes only of the first order at \(\lambda = \lambda_0\).

Now let \(\lambda = \lambda_0\) be an ITE, and let \(\{\varphi_j(\lambda)\}\) be a system of eigenfunctions of the operator \(\hat{R}(\lambda)\) with the eigenvalues \(\mu_j(\lambda), \mu_j(\lambda_0) = 0\), defined in Lemma 2.8. We can assume that functions \(\{\varphi_j\}\) are normalized. Then

\[
\mu_j(\lambda_0) = \sigma \nu_j(\lambda_0) \neq 0, \quad 1 \leq j \leq m_i.
\]

Further, due to (50)- (53), the functions \(\psi_j = \psi_j(\lambda_0)\) in the kernel of \(R(\lambda_0)\) that correspond to \(\mu_j(\lambda)\) are the impedance values (53) of the components of the eigenfunctions \((u_j, v_j)\) of the ITE problem with the eigenvalue \(\lambda = \lambda_0\), i.e., (for transparency, we omit index \(j\) below)

\[
\psi = (\frac{\partial u}{\partial \nu} - tu)|_{x \in \partial \mathcal{O}} = (\frac{\partial v}{\partial \nu} - tv)|_{x \in \partial \mathcal{O}}, \quad \lambda = \lambda_0.
\]

Let \(u(\lambda)\) and \(v(\lambda)\) be the solutions of (51) and (52), respectively, with the boundary conditions (56). Using the Green formula and facts that \(\frac{\partial u}{\partial \nu} + tv' = 0\) at the boundary and \(\Delta v' + \lambda n v' = -nv\) in the domain, we obtain that

\[
\frac{d}{d\lambda}((\lambda_n^m(\lambda) - t)^{-1} \lambda^{i-j}, \psi) = \int_{\partial \mathcal{O}} v' \overline{\psi} dS = \int_{\partial \mathcal{O}} v' \overline{\psi} dS - \int_{\partial \mathcal{O}} (\frac{\partial v'}{\partial \nu} - tv') \overline{\psi} dS
\]

\[
= \int_{\mathcal{O}} v' (\Delta v + \lambda n(x) v) - \int_{\mathcal{O}} (\Delta v' + \lambda n(x) v') \overline{v} = \int_{\mathcal{O}} n(x)|v|^2 dx.
\]
A similar relation (with $v$ replaced by $u$) is valid when $n(x) \equiv 1$. Thus
\[
\nu'(\lambda_0) = \left( \frac{d}{d\lambda} ((N^\text{in}_n - t)^{-1} - (N^\text{in}_0 - t)^{-1})(\lambda) \psi, \psi) \right)_{\lambda = \lambda_0} = \int_O (n|v|^2 - |u|^2) dx,
\]
and therefore
\[
\mu_j'(\lambda_0) = \sigma \int_O (n|v_j|^2 - |u_j|^2) dx \neq 0, \quad 1 \leq j \leq m_i.
\] (57)

This non-degeneracy is an essential part of the proof of the theorem, it was established in (55) using the simplicity of ITEs.

Functions $(u_j, v_j)$ form a basis in the eigenspace of the ITE problem (due to Lemma 2.7). Due to the non-degeneracy (57), one can apply the Gramm orthogonalization procedure with respect to the indefinite metric $J(u, v) = \int_O (n|v|^2 - |u|^2) dx$ and reduce the matrix $A$ defined in Theorem 1.4 to a diagonal form. From here it follows that
\[
\text{sgn} A = -\sigma \sum_{j=1}^{m_i} \text{sign} \mu_j'(\lambda_0).
\]

Since $\mu_j'(\lambda_0) \neq 0$, from the definition of $\beta^\pm_i$ given in Step 2 of the proof of Theorem 1.3 it follows that the right hand side in the formula above is equal to $\beta^+_i - \beta^-_i$. It was also shown in the proof of Theorem 1.3 that $m^\pm_i = \beta^\pm_i$. This complete the proof of Theorem 1.4.

Let us provide some arguments supporting the conjecture stated at the end of the introduction. For simplicity let us assume that there is a unique Jordan block of size $m > 1$ corresponding to an ITE $\lambda^T_i$. The relation between the operator $R^{-1}(\lambda)$ and the resolvent of the ITE problem that was established at the beginning of the proof of Theorem 1.4 implies that there is a single eigenvalue $\mu(\lambda)$ of the operator $\hat{R}(\lambda)$ that vanishes at $\lambda = \lambda^T_i$, and this eigenvalue has zero of order $m$ at $\lambda = \lambda^T_i$. Then from the proof of Theorem 1.3 it follows that $\sigma_i = \pm 1$ if $m$ is odd and $\sigma_i = 0$ if $m$ is even. Since the ITE problem is symmetric with respect to the indefinite metric $J(u, v) = \int_O (|u|^2 - n|v|^2) dx$, the same relation is valid for the signature of the matrix $A$, see [10].

References

[1] E.Blasten, L.Päivärinta, Completeness of generalized transmission eigenstates, Inverse Problems, 29 (2013) 104002.

[2] E.Blasten, L.Päivärinta, J.Sylvester, Do corners always scatter?, arXiv:1211.1848 [math.AP], 2012 To appear in Comm.Math.Phys.

[3] P.Bleher, Operators that depend meromorphically on a parameter (Russian), Vestnik Moskov. Univ., Ser. I, Mat. Meh., 24, no. 5 (1969), 30–36.
[4] F. Cakoni, H. Haddar, Transmission Eigenvalues in Inverse Scattering Theory, Inside Out II, MSRI Publications, Volume 60 (2012).

[5] D. Colton, P. Monk, Quarterly Journal of Mechanics and Applied Mathematics, 41 (1988), 97-125.

[6] M. Dimassi, V. Petkov, Upper bound for the counting function of interior transmission eigenvalues, arXiv:1308.2594 [math.SP] (2013).

[7] J. P. Eckmann, C.-A. Pillet, Spectral Duality for planar billiards. Commun. Math. Phys. 170 (1995), 283-313.

[8] J. P. Eckmann, C.-A. Pillet, Zeta functions with Dirichlet and Neumann boundary conditions for exterior domains, Helv. Phys. Acta, 70 (1997), 44-65.

[9] Faierman Melvin, Transmission eigenvalues for parame ter-elliptic boundary problems, preprint.

[10] Israel Gohberg, Peter Lancaster, L. Rodman, Indefinite Linear Algebra and Applications, Birkhauser, 2000.

[11] A. Kirsch, The denseness of the far field patterns for the transmission problem IMA J. Appl. Math., 37 (1986), 213–223

[12] A. Kirsch, A. Lechleiter, The inside–outside duality for scattering problems by inhomogeneous media, Inverse Problems, 29 (2013), 104011

[13] E. Lakshtanov, B. Vainberg, Ellipticity in the interior transmission problem in anisotropic media, SIAM J. Math. Anal., 44 (2012), 1165-1174.

[14] E. Lakshtanov, B. Vainberg, Remarks on interior transmission eigenvalues, Weyl formula and branching billiards, J. Phys. A: Math. Theor., 45 (2012), 125202.

[15] E. Lakshtanov, B. Vainberg, Bounds on positive interior transmission eigenvalues, Inverse Problems, 28 (2012), 105005.

[16] E. Lakshtanov, B. Vainberg, Applications of elliptic operator theory to the isotropic interior transmission eigenvalue problem, Inverse Problems, 29 (2013), 104003.

[17] E. Lakshtanov, B. Vainberg, Weyl type bound on positive Interior Transmission Eigenvalues, Communications in PDE, accepted, (2013).

[18] Y. J. Leung, D. Colton, Complex transmission eigenvalues for spherically stratified media, Inverse Problems, 28 (2012), 075005.

[19] H. Pham, P. Stefanov, Weyl asymptotics of the transmission eigenvalues for a constant index of refraction, arXiv:1309.3616 [math.SP].
[20] M. Reed, B. Simon, Methods of Modern Mathematical Physics, IV, Academic Press, 1978.

[21] Luc Robbiano, Spectral analysis on interior transmission eigenvalues, arXiv:1310.6273 [math.AP], 2013.

[22] B.Rynne, B.Sleeman, The interior transmission problem and inverse scattering from inhomogeneous media, SIAM J. Math. Anal., 22 (1991), 61755–62.

[23] Yu. Safarov and D. Vassiliev, The Asymptotic Distribution of Eigenvalues of Partial Differential Operators, American Mathematical Society, (1997, 1998).

[24] J.Sylvester, Discreteness of Transmission Eigenvalues via Upper Triangular Compact Operators, SIAM J. Math. Anal., 44(1) (2011), 341-354.