Estimation and Simulation of a COGARCH(p,q) model in the YUIMA project

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Abstract

In this paper we show how to simulate and estimate a COGARCH(p,q) model in the R package yuima. Several routines for simulation and estimation are available. Indeed for the generation of a COGARCH(p,q) trajectory, the user can choose between two alternative schemes. The first is based on the Euler discretization of the stochastic differential equations that identifies a COGARCH(p,q) model while the second one considers the explicit solution of the variance process.

Estimation is based on the matching of the empirical with the theoretical autocorrelation function. In this case three different approaches are implemented: minimization of the mean square error, minimization of the absolute mean error and the generalized method of moments where the weighting matrix is continuously updated.

Numerical examples are given in order to explain methods and classes used in the yuima package.

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1 Introduction

The Continuous-Time GARCH(1,1) process has been introduced in [18] as a continuous counterpart of the discrete-time GARCH(1,1) model proposed by [3]. The idea is to develop in continuous time a model that is able to capture some stylized facts observed in financial time series [10] exploiting only one source of randomness for returns and for variance dynamics. Indeed, in the Continuous-Time GARCH (COGARCH hereafter), the stochastic differential equation for variance is driven by the discrete part of the quadratic variation of the same Lévy process used for modeling returns. The continuous nature of the COGARCH makes it particularly appealing for describing the behavior of high frequency data [see [13] for an application of method of moments using intraday returns].

The generalization to higher order COGARCH(p,q) processes has been proposed in [5, 9]. Starting from the observation that the variance of a GARCH(p,q) is an ARMA(q, p-1), the variance is modeled with a CARMA(q,p-1) process [see [7, 27, 6] and many others] driven by the discrete part of the quadratic variation of the Lévy process in the returns. Although this representation is different from the one used by [18] for the COGARCH(1,1) process, this last can be again retrieved as a special case.

Many authors recently have investigated the COGARCH(1,1) model from a theoretical and an empirical point of view [see [22, 10, 23, 12] and many others]. Some R codes for estimation and simulation of a COGARCH(1,1) driven by a Compound Poisson and Variance Gamma are available in [20]. For the general COGARCH(p,q), the main contribution remain the seminal works [5] and [9]. The aim of this paper is to describe the simulation and the estimation schemes in the yuima package [26] for a COGARCH(p,q) model driven by a general Lévy process. Based on our knowledge yuima is the first R package available on CRAN that allows the user to manage a higher order COGARCH(p,q) model. Moreover, the estimation algorithm gives the option to recover the increments of the underlying noise process and estimates the Lévy measure parameters. We recall that a similar procedure is available in yuima also for the CARMA model [?], see for a complete discussion IacusMercur2015. The yuima package is developed within the YUIMA project [8] whose aim is to provide an environment for simulation and estimation of stochastic differential equations.

The outline of the paper is as following. In Sect. 2 we discuss the main properties of the COGARCH(p,q) process. In particular we review the condition for existence of a strictly stationary variance process, its higher moments and the behavior of the autocorrelation of the square increments of the COGARCH(p,q) model. In Sect. 3 we analyze two different simulation schemes. The first is based on the Euler discretization while the second one uses the solution of the state process in the CARMA(q,p-1) model. Sect. 4 is devoted to the estimation algorithm. In Sect. 5 we show the main classes and corresponding methods in yuima package and in the Sect. 6 we present some numerical examples about the simulation and the estimation of a COGARCH(p,q) model.
2 COGARCH Models driven by a Lévy process

In this section we review the mathematical definition of a COGARCH(p,q) process and its
properties. In particular we focus on the conditions for the existence of a strictly stationary
COGARCH(p,q) process and compute the first four unconditional moments. The existence of
higher order moments plays a central role for the computation of the autocorrelation function
of the squared increments of the COGARCH(p,q) model and consequently the estimation
procedure implemented in the yuima package.

The COGARCH(p,q) process, introduced in [5] as a generalization of the COGARCH(1,1)
model, is defined through the following system of stochastic differential equations:

\[
\begin{align*}
\frac{dG_t}{\sqrt{V_t}} &= \frac{dL_t}{\sqrt{V_t}} \\
V_t &= a_0 + a^\top Y_t \\
\frac{dY_t}{a_0 + a^\top Y_t} &= AY_t - dt + e(a_0 + a^\top Y_t) [L, L]_t^d
\end{align*}
\]

where \(q \geq p \geq 1\). The state space process \(Y_t\) is a vector with \(q\) components:

\[Y_t = [Y_{1,t}, \ldots, Y_{q,t}]^\top.\]

The vector \(a \in \mathbb{R}^q\) is defined as:

\[a = [a_1, \ldots, a_p, a_{p+1}, \ldots, a_q]^\top\]

with \(a_{p+1} = \cdots = a_q = 0\). The companion \(q \times q\) matrix \(A\) is

\[
A = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-b_q & -b_{q-1} & \cdots & -b_1
\end{bmatrix}.
\]

The vector \(e \in \mathbb{R}^q\) contains zero entries except the last component that is equal to one.

\([L, L]^d_t\) is the discrete part of the quadratic variation of the underlying Lévy process \(L_t\) and is
defined as:

\[\int_0^t (\Delta L_s)^2.\]

Remark 1 A COGARCH(p,q) model is constructed starting from the observation that in the
GARCH(p,q) process, its discrete counterpart, the dynamics of the variance is a predictable
ARMA(q, p-1) process driven by the squares of the past increments. In the COGARCH(p,q)
case, the ARMA process leaves the place to a CARMA(q,p-1) model [see [7] for details about
the CARMA(p,q) driven by a Lévy process] and the infinitesimal increments of the COGA-
RCH(p,q) are defined through the differential of the driven Lévy \(L_t\) as done in [1].

As observed above the COGARCH(p,q) model generalizes the COGARCH(1,1) process that
has been introduced following different arguments from those for the (p, q) case. However
choosing \(q = 1\) and \(p = 1\) in [1] the COGARCH(1,1) process developed in [18, 13] can be
retrieved through straightforward manipulations and, for obtaining the same parametrization
in Proposition 3.2 of [18], the following equalities are necessary:

\[\omega_0 = a_0b_1, \quad \omega_1 = a_1e^{-b_1}\] and \(\eta = b_1.\)

Before introducing the conditions for strict stationarity and the existence of unconditional
higher moments, it is worth noting that the state space process \(Y_t\) can be seen as a Multi-
variate Stochastic Recurrence Equation and the corresponding theory can be applied to the
COGARCH(p,q) process [see [5] [9] more details] in order to derive its main features. In the case of the Compound Poisson driven noise, the representation through the stochastic difference equations is direct in the sense that the random coefficients of the state process $Y_t$ can be written explicitly while in the general case, it is always possible to identify a sequence of Compound Poisson processes that as limit to the chosen driven Lévy process.

In the following, we require that matrix $A$ can be diagonalized, that means:

$$A = SDS^{-1}$$

with

$$S = \begin{bmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_q \\ \vdots & \ddots & \vdots \\ \lambda_1^{-1} & \ldots & \lambda_q^{-1} \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_q \end{bmatrix}$$

(3)

where the $\lambda_1, \lambda_2, \ldots, \lambda_q$ are the eigenvalues of matrix $A$ and are ordered as follows:

$$\Re \{ \lambda_1 \} \geq \Re \{ \lambda_2 \} \geq \ldots \geq \Re \{ \lambda_q \}.$$  

Applying the theory of stochastic recurrence equations, [5] provide a sufficient condition for the strict stationarity of a COGARCH(p,q) model. We review the result for the stationarity of the state process $Y_t$. Two fundamental assumptions are the fact that the eigenvalues $\lambda_1, \ldots, \lambda_q$ are distinct and the underlying process $L$ must have a non-trivial $\nu_l(l)$ measure. Then, the process $Y_t$ converges in distribution to the random variable $Y_\infty$ if exist some $r \in [1, +\infty]$ such that:

$$\int_{-\infty}^{+\infty} \ln (1 + \|S^{-1}e^{a_1^T S}l\|^2) \, d\nu_l(l) \leq \Re \{ \lambda_1 \}$$

(4)

for some matrix $S$ such that the matrix $A$ is diagonalizable. If we choose as a starting condition $Y_0 \overset{d}{=} Y_\infty$ than the process $Y_t$ is strictly stationary and consequently the variance $V_t$ is also a strictly stationary process. Unlike for the general case, the inequality in (4) gives only a sufficient condition about the strict stationarity, and it is difficult to verify in practice. As shown in [18] and remarked in [5], the condition in (4) is also necessary for the COGARCH(1,1) case and can be simplified as:

$$\int_{-\infty}^{+\infty} \ln (1 + a_1^2) \, d\nu_l(l) \leq b_1.$$  

(5)

Using again the SRE theory, it is also possible to determine the condition for existence of higher moments of the state process $Y_t$. In this way it is possible to determine the autocorrelations of the squared COGARCH increments that are used in the estimation procedure illustrated in Section 4. As reported in [5], the $\kappa$-th order moment of the process $Y_t$ exists finite if, for some $r \geq 1$ and $S$ such that the matrix $A$ is diagonalizable, the following conditions hold:

$$E (L_t^{\kappa}) < +\infty, \quad \int_{-\infty}^{+\infty} \left[ (1 + \|S^{-1}e^{a_1^T S}l\|^2)^\kappa - 1 \right] \, d\nu_l(l) < \Re \{ \lambda_1 \} \kappa.$$  

(6)

1The Stochastic Recurrence Equations theory [4] [17] has been also used to prove the strictly and weakly stationarity for the GARCH(p,q) model [1]

2In the yuima package a diagnostic for condition (4) is available choosing matrix $S$ as done in (3) and $r = 2$. We remark that the process $Y_t$ is stationary if the diagnostic gives a positive answer otherwise we can conclude nothing about the stationarity of the process.
As a special case of (6), the unconditional stationary mean \( E(\mathbf{Y}_\infty) = -a_0 \mu \mathbf{y}(A + \mu \mathbf{e}^\top)^{-1} \mathbf{e} \) of the vector process \( \mathbf{Y}_t \) exists if

\[
E \left( L^2 \right) < +\infty, \| S^{-1} \mathbf{e} \mathbf{a}^\top S \| r < \mathbb{R} \{ \lambda_1 \}
\]

where

\[
\mu := \int_{-\infty}^{+\infty} t^2 d\nu_L(l)
\]

is the second moment of the Lévy measure \( \nu_L(l) \). It is worth noting that the condition (2) ensures also the strict stationarity since, using \( \ln(1 + x) \leq x \), we have the following sequences of inequalities:

\[
\int_{-\infty}^{+\infty} \ln \left( 1 + \| S^{-1} \mathbf{e} \mathbf{a}^\top S \| r t^2 \right) d\nu_L(l) \leq \| S^{-1} \mathbf{e} \mathbf{a}^\top S \| r \mu \leq \mathbb{R} \{ \lambda_1 \}.
\]

For the stationary covariance matrix [9]

\[
\text{cov}(\mathbf{Y}_\infty) = \frac{a_0^2 b^2 \rho}{(b_0 - \mu a_1)^2 (1 - m)} \int_{0}^{+\infty} e^{\mathbf{A}^\top} \mathbf{e} \mathbf{e}^\top e^{\mathbf{A} t} d\mathbf{t},
\]

the existence of the second moment of \( \mathbf{Y}_\infty \) in (6) becomes:

\[
E \left( L^4 \right) < +\infty, \| S^{-1} \mathbf{e} \mathbf{a}^\top S \| r \rho < 2 \left( -\mathbb{R} \{ \lambda_1 \} - \| S^{-1} \mathbf{e} \mathbf{a}^\top S \| r \mu \right)
\]

where

\[
\rho := \int_{-\infty}^{+\infty} t^4 d\nu_L(l)
\]

is the fourth moment of the Lévy measure \( \nu_L(l) \) and

\[
m := \int_{0}^{+\infty} \mathbf{a}^\top e^{\mathbf{A}^\top} \mathbf{e} \mathbf{e}^\top e^{\mathbf{A} t} d\mathbf{t}.
\]

Before introducing the higher moments and the autocorrelations, we recall the conditions for the nonnegativity for a strictly stationary variance process in (1). Indeed, under the assumption that all the eigenvalues of matrix \( A \) are distinct and the relation in (4) holds, the variance process \( V_t \geq a_0 > 0 \) a.s. if:

\[
a^\top e^{A t} \mathbf{e} \geq 0, \forall t \geq 0.
\]

The condition in (9) is costly since we need to check it each time. Nevertheless some useful controls are available [28].

1. A necessary and sufficient condition to guarantee that \( a^\top e^{A t} \mathbf{e} \geq 0 \) in the COGARCH(2,2) case is that the eigenvalues of \( A \) are real and \( a_2 \geq 0 \) and \( a_1 \geq -a_2 \lambda(A) \) where \( \lambda(A) \) is the biggest eigenvalue.

2. Under condition \( 2 \leq p \leq q \), that all eigenvalues of \( A \) are negative and ordered in an increasing way \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{p-1} \) and \( \gamma_j \) are the roots of \( a(z) = 0 \) ordered as \( 0 > \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_{p-1} \). Then sufficient condition for (9) is

\[
\sum_{i=1}^{k} \gamma_i \leq \sum_{i=1}^{k} \lambda_i, \forall k \in \{1, \ldots, p-1\}.
\]

3. For a COGARCH(1,q) model a sufficient condition that ensures (9) is that all eigenvalues must be real and negative.
Combining the requirement in (4) with that in (9) it is possible to derive the higher moments and the autocorrelations for a COGARCH(p,q) model. As a first step, we define the returns of a COGARCH(p,q) process on the interval \((t, t + r] \), \(\forall t \geq 0\) as:

\[ G_t^{(r)} := \int_t^{t+r} \sqrt{r} dL_s. \]  

(10)

Let \(L_t\) be a symmetric and centered Lévy process such that the fourth moment of the associated Lévy measure is finite, we define the matrix \(\tilde{A}\) as:

\[ \tilde{A} := A + \mu e a^\top. \]

It is worth noting that the structure of \(\tilde{A}\) is the same of \(A\) except for the last row \(q\) where \(\tilde{A}_{q} = (-b_1 + \mu a_1, \ldots, -b_1 + \mu a_q)\). For any \(t \geq 0\) and for any \(h \geq r > 0\), the first two moments of (10) are

\[ E \left[ \left( G_t^{(r)} \right) \right] = 0 \]

and

\[ E \left[ \left( G_t^{(r)} \right)^2 \right] = \frac{\alpha_0 b_r}{b_q - \mu a_1} E \left[ L_t^2 \right]. \]  

(12)

The computation of the autocovariances and variance for squared returns (10) require the following quantities defined as:

\[ P_0 = 2\mu^2 \left[ 3\tilde{A}^{-1} \left( \tilde{A}^{-1} \left( e^{\tilde{A}r} - r I \right) - r I \right) \right] \text{cov} \left( \epsilon_{\infty} \right) \]

\[ P_{bh} = \mu^2 e^{Ah} \tilde{A}^{-1} \left( I - e^{\tilde{A}r} \right) \tilde{A}^{-1} \left( e^{\tilde{A}r} - r I \right) \text{cov} \left( \epsilon_{\infty} \right) \tilde{A}^\top \]  

(13)

\[ Q_0 = 6\mu \left[ \left( r I - \tilde{A}^{-1} \left( e^{\tilde{A}r} - r I \right) \right) \text{cov} \left( \epsilon_{\infty} \right) - \tilde{A}^{-1} \left( e^{\tilde{A}r} - r I \right) \text{cov} \left( \epsilon_{\infty} \right) \tilde{A}^\top \right] e \]

\[ Q_{bh} = \mu e^{Ah} \tilde{A}^{-1} \left( I - e^{-\tilde{A}r} \right) \left[ \tilde{A}^{-1} \left( e^{\tilde{A}r} - r I \right) \text{cov} \left( \epsilon_{\infty} \right) \tilde{A}^\top \right] e \]  

(14)

and

\[ R = 2r^2 \mu^2 + \rho. \]  

(15)

The terms \(P_{bh}\) and \(P_0\) are \(q \times q\) matrices. \(Q_{bh}\) and \(Q_0\) are \(q \times 1\) vectors and the term \(R\) is a scalar. The \(q \times q\) matrix \(\text{cov} \left( \epsilon_{\infty} \right)\) is defined as:

\[ \text{cov} \left( \epsilon_{\infty} \right) = \rho \int_0^{\infty} e^{\tilde{A}t} e e^\top e^{\tilde{A}^\top t} dt \]  

(16)

where \(\mu, \rho\) and \(m\) have been defined before.

Using the \(q \times q\) matrix \(P_{bh}\) and the \(q \times 1\) vector \(Q_{bh}\) the autocovariances of the squared returns (10) are defined as:

\[ \gamma_r \left( h \right) := \text{cov} \left[ \left( G_t^{(r)} \right)^2, \left( G_{t+h}^{(r)} \right)^2 \right] = \frac{a_0 b_r^2 (a^\top P_h a + a Q_h)}{(1 - m) (b_q - \mu a_1)^2}, \quad h \geq r \]

(17)

while the variance of the process \(\left( G_t^{(r)} \right)^2\) is

\[ \gamma_r \left( 0 \right) := \text{var} \left( \left( G_t^{(r)} \right)^2 \right) = \frac{a_0 b_r^2 (a^\top P_0 a + a Q_0 + R)}{(1 - m) (b_q - \mu a_1)^2}. \]  

(18)
Combining the autocovariances in (17) with (12) and (18) we obtain the autocorrelations:

\[ \rho_r(h) := \frac{\gamma_r(h)}{\gamma_r(0)} = \frac{(a^T P_h a + a Q_h)}{(a^T P_0 a + a Q_0 + R)}. \]  

(19)

We conclude this Section with the following remark. As done for the GARCH(p,q) model the autocovariances of the returns \( G_t^{(r)} \) are all zeros.

### 3 Simulation of a COGARCH(P,Q) model

In this Section we illustrate the theory behind the simulation routines available in the yuima package when a COGARCH(p,q) model is considered. The corresponding sample paths are generated according two different schemes.

The first method, in the Yuima project [8], is based on the Euler-Maruyama [14] discretization of the system in (1). In this case the algorithm follows these steps:

1. We determine an equally spaced grid from 0 to \( T \) composed by \( N \) intervals with the same length \( \Delta t \).

\[ 0, \ldots, (n-1) \times \Delta t, n \times \Delta t, \ldots, N \times \Delta t. \]

2. We generate the trajectory of the underlying Lévy process \( L_t \) sampled on the grid defined in step 1.

3. Choosing a starting point for the state process \( Y_0 = y_0 \) and \( G_0 = 0 \), we have

\[
Y_n = (I + A \Delta t) Y_{n-1} + e (a_0 + a^T Y_{n-1}) \Delta [L L]_n^d. \]

(20)

The discrete quadratic variation of the \( L_t \) process is approximated by

\[
\Delta [L L]_n^d = (L_n - L_{n-1})^2. \]

(21)

4. Once the approximated state process \( Y_n \) is obtained we can generate the trajectory of Variance and the process \( G_n \) according the following equations:

\[
V_n = a_0 + a^T Y_{n-1} \]

and

\[
G_n = G_{n-1} + \sqrt{V_n} (L_n - L_{n-1}). \]

(22)

(23)

It is worth noting that, although the discretized version of the state process \( Y_n \) in (20) can be seen as a stochastic recurrence equation, the conditions for stationarity and non-negativity for the variance \( V_n \) process are not the same of ones analyzed in the Section 2. In particular it is possible to determine an example where the discretized variance process \( V_n \) assumes negative values while the true process is non negative with probability one.

In order to clarify deeper this issue we consider a COGARCH(1,1) model driven by a Variance Gamma Lévy process [see [21] [19] for more details about the VG model]. In this case, the condition for the non-negativity of the Variance in (9) is ensured if \( a_0 > 0 \) and \( a_1 > 0 \) while the strict stationarity condition in (5) for the COGARCH(1,1) is \( E[L^2] = 1 \) and \( a_1 - b_1 < 0 \). The last two requirements guarantee also the existence of the stationary unconditional mean for the process \( V_t \). We define the model using the yuima function setCogarch. Its usage is completely explained in the Section 5. The following command line instructs yuima to build the COGARCH(1,1) model with VG noise:
model1 <- setCogarch(p = 1, q = 1, work = FALSE, 
+ measure=list("rngamma(z, 1, sqrt(2), 0, 0)"), measure.type = "code", 
+ Cogarch.var = "G", V.var = "v", Latent.var="x", XinExpr=TRUE)

Choosing the following values for the parameters
param1 <- list(a1 = 0.038, b1 = 301, a0 =0.01, x01 = 0)
the COGARCH(1,1) is stationary and the variance is strictly positive. Nevertheless, if we 
simulate the trajectory using the Euler discretization, the value of $\Delta t$ can lead to negative 
values for the process as shown in the example:
Terminal1=5
n1=750
samp1 <- setSampling(Terminal=Terminal1, n=n1)
set.seed(123)
sim1 <- simulate(model1, sampling = samp1, true.parameter = param1, 
+ method="euler")
plot(sim1, main="Sample Path of a VG COGARCH(1,1) model with Euler scheme")

Looking to the Figure, we observe a divergent and oscillatory behaviour for the simulated 
state $Y_n$ and the variance $V_n$ processes while, from theoretical point of view, the conditions 
for nonnegativity and stationarity for the variance of a COGARCH(1,1) model are satisfied 
by the values used for the parameters.
To overcome this problem, we provide an alternative simulation scheme based on the solution 
of the process $Y_n$ given the starting point $Y_{n-1}$.

Applying the Ito’s Lemma for semimartingales [25] to the transformation $e^{-At}Y_t$, we have:

$$e^{-At}Y_t = Y_{t-At} - \int_{t-At}^{t} e^{-Au}Y_u \, du + \int_{t-At}^{t} e^{-Au} \, dY_u + \sum_{s \leq t} \left[ e^{-As} (Y_s - Y_{s-}) - e^{-As} (Y_s - Y_{s-}) \right] .$$

We substitute the definition of $Y_t$ in [1] and get

$$e^{-At}Y_t = Y_{t-At} - \int_{t-At}^{t} e^{-Au}Y_u \, du + \int_{t-At}^{t} e^{-Au} \, dY_u + \int_{t-At}^{t} e^{-Au} \, (a_0 + a'Y_u) \, d \{LL\}_u.$$
Using the following property for an exponential matrix

\[ Ae^{At} = A \left( I + At + \frac{1}{2} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \ldots \right) = \left( A + tA^2 + \frac{1}{2} t^2 A^3 + \frac{1}{3!} t^3 A^4 + \ldots \right) = \left( I + tA + \frac{1}{2} t^2 A^2 + \frac{1}{3!} t^3 A^3 + \ldots \right) = e^{At} A, \]

we get

\[ Y_t = e^{At} Y_{t-\Delta t} + \int_{t-\Delta t}^t e^{A(t-u)} (a_0 + a' Y_{u-}) \, d[LL]_u^d \quad (24) \]

Except for the case where the noise is a Compound Poisson, the simulation scheme follows the same steps of the Euler-Maruyama discretization where the state space process \( Y_n \) on the sample grid is generated according the approximation of the relation in (24):

\[ Y_n = e^{A\Delta t} Y_{n-1} + e^{A(\Delta t)} (a_0 + a' Y_{n-1}) \left( [LL]_n^d - [LL]_{n-1}^d \right) \quad (25) \]

or equivalently:

\[ Y_n = a_0 e^{A(\Delta t)} e^{\Delta [LL]_n^d} + e^{A\Delta t} \left( I + e a' \Delta [LL]_n^d \right) Y_{n-1}. \quad (26) \]

where \( \Delta [LL]_{n}^d := [LL]_{n}^d - [LL]_{n-1}^d \) is the increment of the discrete part of quadratic variation.

In the previous example, the sample path is simulated according the recursion in (26) choosing method = mixed in the simulate function as done below:

```r
> set.seed(123)
> sim2 <- simulate(model1, sampling = samp1, true.parameter = param1, +    method="mixed")
> plot(sim2, main="Sample Path of a VG COGARCH(1,1) model with mixed scheme")
```

In the case of the COGARCH(p,q) driven by a Compound Poisson Lévy process, a trajectory can be simulated without any approximation of the solution in (24). Indeed it is possible to determine the time where the jumps occur and then evaluate the corresponding quadratic variation in an exact way. Once the trajectory of a random time is obtained, the piecewise constant interpolation is used on the fixed grid, in order to maintain the càdlàg property of the sample path.

### 4 Estimation of a COGARCH(P,Q) model in the yuima package

In this Section we explain the estimation procedure that we propose in the yuima package for the COGARCH(p,q) model. As done for the CARMA(p,q) model driven by Lévy process [15], even in this case a three step estimation procedure is proposed that allows the user to obtain estimated values for the COGARCH(p,q) and the parameters of the Lévy measure.

This procedure is structured as follows:

3In a general Compound Poisson the jump time follow an exponential r.v. with rate \( \lambda \).
1. Using the moment matching procedure explained below, we estimate the COGARCH(p,q) parameters \( a := [a_1, \ldots, a_p] \), \( b := [b_1, \ldots, b_q] \) and the constant term \( a_0 \) in the Variance process \( V_t \). In this phase, the estimation is obtained by minimizing some distances between the empirical and theoretical autocorrelation function.

2. Once the COGARCH(p,q) parameters are available, we recover the increments of the underlying Lévy process using the methodology describe below.

3. In the third step, we use the increments obtained in the second step and estimate the Lévy measure parameters by means of maximum likelihood estimation procedure.

Let \( G_0, G_1, \ldots, G_n, \ldots, G_T \) be the observed values of the process subsampled at equally spaced instants 0, \( h, 2h, \ldots, Nh \) where the length of each interval is \( h = \frac{T}{N} \). The time of the last observation is \( T \) and \( N \) is the number of observations of the process \( G_n \).

In the following we assume that the underlying Lévy process is symmetric and centered in zero. Starting from the sample \( \{G_n\}_{n=0}^{N} \) we define the COGARCH(p,q) increments of lag one as:

\[
G_n^{(1)} := G_n - G_{n-1}
\]

and the increments of lag \( r \) as:

\[
G_n^{(r)} := G_n - G_{n-r}
\]

where \( r \geq 1 \) is a natural number and for \( r = 1 \) the definition in (4) coincides with (27).

It is worth mentioning that the increments \( G_n^{(r)} \) can be obtained as a time aggregation of increments of lag one as follows:

\[
G_n^{(r)} = \sum_{h=n-r}^{n} G_h^{(1)}
\]

The time aggregation in (28) can be useful during the optimization routine when the values of increments \( G_h^{(1)} \) are very small in absolute value.
Using the sample \( \{G_n^{(r)}\}_{n \geq r} \) we compute the empirical second moment
\[
\hat{\mu}_r := \frac{1}{N - d - r + 1} \sum_{n=r}^{N-d} (G_n^{(r)})^2
\]
and empirical autocovariance function \( \hat{\gamma}(h) \) is defined as:
\[
\hat{\gamma}_r(h) := \frac{1}{N - d - r + 1} \sum_{n=r}^{N-d} \left( (G_{n+h}^{(r)})^2 - \hat{\mu}_r \right) \left( (G_n^{(r)})^2 - \hat{\mu}_r \right) \quad h = 0, 1, \ldots, d
\]
where \( d \) is the maximum lag considered.

The empirical autocorrelations are:
\[
\hat{\rho}_r(h) = \frac{\hat{\gamma}_r(h)}{\gamma_r(0)} \quad (29)
\]

We use the theoretical and empirical autocorrelations in order to define the moment conditions for the estimation procedure. By the introduction of the \( q+p \) vector \( \theta := (a, b) \) we define the vector function
\[
g(\theta) := E[f(G_n^{(r)}, \theta)] \quad (30)
\]
where \( f(G_n^{(r)}, a, b) \) is a \( d \) dimensional real function where the components are defined as:
\[
f_h(\Delta G_n^{(r)}, \theta) = \rho_r(h) - \frac{\left( (G_{n+h}^{(r)})^2 - \hat{\mu}_r \right) \left( (G_n^{(r)})^2 - \hat{\mu}_r \right)}{\gamma_r(0)} \quad h = 1, \ldots, d. \quad (31)
\]

In the estimation algorithm, we replace the expectation in (30) with the sample counterpart. The components of vector \( \hat{g}(\theta) = [\hat{g}_1(\theta), \ldots, \hat{g}_d(\theta)]^T \) are defined as:
\[
\hat{g}_h(\theta) = \frac{1}{N - d - r + 1} \sum_{n=r}^{N-d} \left[ \rho_r(h) - \frac{\left( (G_{n+h}^{(r)})^2 - \hat{\mu}_r \right) \left( (G_n^{(r)})^2 - \hat{\mu}_r \right)}{\gamma_r(0)} \right] \\
= \rho_r(h) - \hat{\rho}_r(h), \quad h = 1, \ldots, d. \quad (32)
\]

The vector \( \theta = (a, b) \) containing the COGARCH(p,q) parameters are obtained by minimizing some distances between empirical and theoretical autocorrelations. The optimization problem is:
\[
\min_{\theta \in \mathbb{R}^{q+p}} d(\rho_r, \hat{\rho}_r)
\]
where the components of vectors \( \rho_r \) and \( \hat{\rho}_r \) are the theoretical and empirical autocorrelations defined in (19) and (29) respectively. Function \( d(x, y) \) measures the distance between vectors \( x \) and \( y \). In the yuima environment, three distances are available and listed below:

1. the \( L_1 \) norm
\[
\|\hat{g}(\theta)\|_1 = \frac{d}{h=1} |\hat{g}_h(\theta)|. \quad (33)
\]
2. the squared of $L_2$ norm

$$\|\hat{g}(\theta)\|_2^2 = \sum_{h=1}^{d} (\hat{g}_h(\theta))^2.$$  \hfill (34)

3. The quadratic form

$$\|\hat{g}(\theta)\|^2_W = \hat{g}(\theta)^T W \hat{g}(\theta)$$  \hfill (35)

where the positive definite weighting matrix $W$ is chosen to obtain efficient estimators between those that belong to the class of asymptotically normal estimators.

It is worth noting that the objective function $\|\hat{g}(\theta)\|^2_W$ is a special case of the function $\|\hat{g}(\theta)\|^2_W$ where the weighting matrix $W$ coincides with the identity matrix. Both distances are related with the Generalized Method of Moments (GMM) introduced by [11]. Under some regularity conditions [24], the GMM estimators are consistent and, for any general positive definite matrix $W$, their asymptotic variance-covariance matrix $V$ are:

$$V = \frac{1}{N - d - r + 1} (D^TWD)^{-1} D^TWSWD (D^TWD)^{-1},$$

where $D$ is defined as:

$$D = E \left[ \frac{\partial f(G_n(\cdot), \theta)}{\partial \theta} \right].$$  \hfill (36)

While

$$S = E \left[ f(G_n(\cdot), \theta) f(G_n(\cdot), \theta)^T \right].$$  \hfill (37)

For the squared $L_2$ norm in (34) matrix $V$ becomes:

$$V = \frac{1}{N - d - r + 1} (D^T D)^{-1} D^T S D (D^T D)^{-1}$$

while for (35), as observed above, the choice of the matrix $W$ is done in order to obtain efficient estimators in the class of all asymptotically normal estimators. To obtain this result we prefer to use the Continuously Updated GMM estimator [12]. In this case the matrix $W$ is determined simultaneously with the estimation of the vector parameters $\theta$. Introducing the function $\|\hat{g}(\theta)\|^2_W$ as the sample counterpart of the quadratic form in (35), the minimization problem becomes:

$$\min_{\theta \in \mathbb{R}^{p+q}} \|g(\theta)\|^2_W = \hat{g}(\theta)^T W(\theta) \hat{g}(\theta)$$

where function $W(\theta)$ maps from $\mathbb{R}^{p+q}$ to $\mathbb{R}^{d \times d}$ and is defined as:

$$W(\theta) = \left( \frac{1}{N - r - d + 1} \sum_{n=r}^{N-r-d} f(c_n^{(r)}, \theta) f(c_n^{(r)}, \theta)^T \right)^{-1}.$$  \hfill (39)

Observe that $W(\theta)$ is a consistent estimator of matrix $S^{-1}$ that means:

$$\hat{W}(\theta) \xrightarrow{P} S^{-1}$$  \hfill (40)

consequently the asymptotic variance-covariance matrix $V$ in (35) becomes:

$$V = \frac{1}{N - r - d + 1} (D^T S^{-1} D)^{-1}.$$  \hfill (41)

Once the estimates of vector $\theta$ are obtained, the user is allowed to retrieve the increments of the underlying Lévy process according the following procedure. This stage is independent
on the nature of the Lévy measure but it is only based on the solution of the state process $Y_t$ and on the approximation of the quadratic variation with the squared increments of the Lévy driven process.

Starting from the discrete time equally spaced observations $G_1, G_2, \ldots, G_{N\Delta t}$, we remark that the increment $\Delta G_t := G_t - G_{t-}$ can be approximated using the observations $\{G_{n\Delta t}\}_{n=0}^N$ as follows:

$$\Delta G_t \approx \Delta G_{n\Delta t} = G_{n\Delta t} - G_{(n-1)\Delta t}$$

Recalling that $G_t = \sum_{0 \leq s \leq t} \sqrt{V_s} (\Delta L_s)$, the approximation in (42) becomes:

$$\Delta G_t \approx \sqrt{V_{n\Delta t}} (\Delta L_{n\Delta t})$$

where $V_{n\Delta t}$ is the value of the variance process at the observation time $t = n\frac{T}{N}$ and $\Delta L_{n\Delta t} = L_{n\Delta t} - L_{(n-1)\Delta t}$ is the discretized Lévy increment from $t - \Delta t$ to $t$.

Using the discretization scheme introduced in (25), the process $Y_{n\delta t}$ is written as:

$$Y_{n\delta t} \approx e^{A\Delta t} Y_{(n-1)\delta t} + e^{A(\Delta t)} e^{(a_0 + a'Y_{t-\Delta t})} \left([LL]_m^d - [LL]_{(n-1)}^d\right)$$

since the difference $[LL]_n^d - [LL]_{(n-1)}^d \approx (\Delta L_{n\Delta t})^2$ and using the result in (43), the difference equation (44) can be approximated in terms of the squared increments of COGARCH(p,q) process and we have:

$$Y_t \approx e^{A\Delta t} Y_{t-\Delta t} + e^{A(\Delta t)} e^{(a_0 + a'Y_{t-\Delta t})} (\Delta L_t)^2$$

Choosing $Y_0$ equal to the unconditional mean of the process $Y_t$, we are able to retrieve its sample path according to the recursive equation in (45). The only quantities that we need to compute are the squared increments of the COGARCH(p,q) process on the grid $\{0, \Delta t, 2\Delta t, \ldots, n\Delta t, \ldots, N\Delta t\}$. The estimated state process in (45) is also useful for getting the estimated trajectory of the variance process. Finally note the Lévy increment at a general time $t = n\Delta t$ is obtained as:

$$\Delta L_{n\delta t} = \frac{\Delta G_{n\delta t}}{\sqrt{V_{n\delta t}}}$$

Once the increments of the underlying Lévy are obtained, it is possible to obtain the estimates for the Lévy measure parameters through the Maximum Likelihood Estimation procedure [we refer to the yuima documentation [8, 26] for the available Lévy processes].

5 Package R

In this Section we illustrate the main classes and methods in yuima that allow the user to deal with a COGARCH(p,q) model. The routines implemented are based on the results considered in Section 3 for simulation and Section 4 for the estimation procedures. In particular we classify these classes and methods in three groups. The first group contains the classes and functions that allow the user to define a specific COGARCH(p,q) model in the yuima framework. The second group is used for the simulation of the sample paths for the COGARCH(p,q) model and the third is related to the possibility of estimation using simulated or real data.
5.1 Classes and Methods for the definition of a COGARCH(P,Q) model

The main object for a COGARCH(p,q) process in the yuima environment is an object of yuima.cogarch-class that contains all the information about a specific COGARCH(p,q) process. This class extends the yuima.model-class and it has only one additional slot, called @info, that contains an object of cogarch.info-class. We refer to the yuima documentation for a complete description of the slots that constitute the objects of class yuima.model. In this paper we focus only on the object of class cogarch.info. In particular its structure is composed by the slots listed below:

- @p is an integer that is the number of moving average coefficients in the Variance process \( V_t \).
- @q is an integer number that corresponds to the dimension of autoregressive coefficients in the variance process \( V_t \).
- @ar.par contains a string that is the label for the autoregressive coefficients.
- @ma.par is the Label for the moving average coefficients.
- @loc.par indicates the name of the location coefficient in the process \( V_t \).
- @Cogarch.var string that contains the name of the observed process \( G_t \).
- @V.var is the Label of the \( V_t \) process.
- @Latent.var indicates the label of the state process \( Y_t \).
- @XinExpr is a logical variable. If the value is FALSE, default value, the starting condition for the state process \( Y_t \) is a zero vector. Otherwise the user has to fix the starting condition as argument true.parameter in the method simulate.
- @measure identifies the Lévy measure of the underlying noise and consequently the discrete part of the quadratic variation that drives the state process.
- @measure.type says if the Lévy measure belongs to the family of Compound Poisson or is another type of Lévy.

The user builds an object of class yuima.cogarch through to the constructor setCogarch:

```r
setCogarch(p, q, ar.par = "b", ma.par = "a", loc.par = "a0", Cogarch.var = "g", V.var = "v", Latent.var = "y", jump.variable = "z", time.variable = "t", measure = NULL, measure.type = NULL, XinExpr = FALSE, startCogarch = 0, work = FALSE, ...)
```

The arguments used in a call of the function setCogarch are illustrated in the following list:

- p: A non negative integer that is the number of the moving average coefficients in the variance process.
- q: A non negative integer that indicates the number of the autoregressive coefficients in the variance process.
- ar.par: A character-string that is the label of the autoregressive coefficients.
- ma.par: A character-string that is the label of the autoregressive coefficients.
- loc.par: A string that indicates the label of the location coefficient in the variance process.
- Cogarch.var: A character-string that is the label of the observed COGARCH process.
• **V.var**: A character-string that is the label of the latent variance process.

• **Latent.var**: A character-string that is the label of the latent process in the state space representation for the variance process.

• **jump.variable**: Label of the underlying Lévy process.

• **time.variable**: Label of the time variable.

• **measure**: Lévy measure of the underlying Lévy process.

• **measure.type**: Label that indicates whether the underlying noise is a Compound Poisson process or another Lévy without the diffusion component.

• **XinExpr**: A logical variable that identifies the starting condition. In particular, the default value **XinExpr = FALSE** implies that the starting condition for the state process is zero. Alternatively **XinExpr = TRUE** means that the user is allowed to specify as parameters the starting values for each component of the state variable.

• **startCogarch**: Initial value for the COGARCH process.

• ...: Arguments to be passed to **setCogarch** such as the slots of the **yuima.model-class**.

### 5.2 Classes and Methods for the simulation of a COGARCH(P,Q) model

The **simulate** is a method for an object of class **yuima.model**. It is also available for an object of class **yuima.cogarch**. The function requires the following inputs:

```r
simulate(object, nsim=1, seed=NULL, xinit, true.parameter, space.discretized = FALSE, increment.W = NULL, increment.L = NULL, method = "euler", hurst, methodfGn = "WoodChan", sampling=sampling, subsampling=subsampling, ...)```

In this work we focus on the argument **method** that identifies the type of discretization scheme for the time when the object belongs to the class of **yuima.cogarch**. The default value **euler** means that the simulation of a sample path is based on the Euler-Maruyama discretization of the stochastic differential equations. This approach is available for all objects of class **yuima.model**. For the COGARCH(p,q) an alternative simulation scheme is available choosing **method = mixed**. In this case the generation of trajectories is based on the solution (24) for the state process. In particular if the underlying noise is a Compound Poisson Lévy process, the trajectory is built using a two step algorithm. First the jump time is simulated internally using the Exponential distribution with parameter \( \lambda \) and then the size of jump is simulated using the random variable specified in the slot **yuima.cogarch@model@measure**. For the other Lévy processes, the simulation scheme is based on the discretization of the state process solution (25) in Section 5.

### 5.3 Classes and Methods for the estimation of a COGARCH(P,Q) model

The **cogarch.gmm** class is a class of the **yuima** package that contains estimated parameters obtained by the **gmm** function.

- **@model** is an object of of **yuima.cogarch-class**.

- **@objFun** is an object of class **character** that indicates the objective function used in the minimization problem. L2 refers to the squared of L2 norm in (34), L2CUE for the quadratic form (35) and the L1 for the L1 norm in (33).

- **@call** is an object of class **language**.
• @coef is an object of class numeric that contains the estimated parameters.
• @fullcoef is an object of class numeric that contains the estimated and fixed parameters from the user.
• @vcov is an object of class matrix.
• @min is an object of class numeric.
• @minuslogl is an object of class function.
• @method is an object of class character.

The cogarch.gmm.incr is a class of the yuima package that extends the cogarch.gmm-class and is filled from the function gmm.

• Incr.lev is an object of class zoo that contains the estimated increments of the noise obtained using cogarchNoise.
• model is an object of yuima.cogarch-class.
• logl.incr is an object of class numeric that contains the value of the log-likelihood for the estimated Levy increments.
• objFun is an object of class character that indicates the objective function used in the minimization problem. The values are the same for the slot @objFun in an object of class cogarch.gmm.
• call is an object of class language.
• coef is an object of class numeric that contains the estimated parameters.
• fullcoef is an object of class numeric that contains estimated and fixed parameters.
• vcov is an object of class matrix.
• min is an object of class numeric.
• minuslogl is an object of class function.
• method is an object of class character.

The function gmm returns the estimated parameters of a COGARCH(p,q) model. The parameters are obtained by matching the theoretical with the empirical autocorrelation function.

\[
gmm(yuima, data = NULL, start, method="BFGS", fixed = list(), lower, upper, lag.max = NULL, equally.spaced = TRUE, Est.Incr = "NoIncr", objFun = "L2")
\]

• yuima is a yuima object or an object of yuima.cogarch-class
• data is an object of class yuima.data-class contains the observations available at uniformly spaced instants of time. If data=NULL, the default, the function uses the data in an object of yuima-class.
• start is a list that contains the starting values for the optimization routine.
• method is a string that indicates one of the methods available in the function optim.
• fixed a list of fixed parameters in the optimization routine.
• lower a named list for specifying lower bounds for parameters.
• upper a named list for specifying upper bounds for parameters.
• lag.max maximum lag for which we calculate the theoretical and empirical acf. Default is \(\sqrt{N}\) where \(N\) is the number of observations.
• `equally.spaced` Logical variable. If `equally.spaced = TRUE`, in each unitary interval we have the same number of observations. If `equally.spaced = FALSE`, each unitary interval is composed by different number of observations.

• `Est.Incr` a string variable. If `Est.Incr = "NoIncr"`, default value, `gmm` returns an object of class `cogarch.gmm-class` that contains the COGARCH parameters. If `Est.Incr = "Incr"` or `Est.Incr = "IncrPar"` the output is an object of class `cogarch.gmm.incr-class`. In the first case the object contains the increments of the underlying noise while in the second case it contains also the estimated parameters of the Lévy measure.

• `objFun` a string variable that identifies the objective function in the optimization step. `objFun = "L2"`, default value, the objective function is a quadratic form where the weighting matrix is the identity one. `objFun = "L2CUE"` the weighting matrix is estimated using Continuously Updating GMM (L2CUE). `objFun = "L1"`, the objective function is the mean absolute error. In the last case standard errors for estimators are not available.

Function `gmm` uses function `cogarchNoise` for the estimation of the underlying Lévy in a COGARCH(p,q) model. This function assumes that the underlying Lévy process is symmetric and centered in zero.

\[ \text{cogarchNoise(yuima.cogarch, data=NULL, param, mu=1)} \]

The arguments of the `cogarchNoise` are listed below

• `yuima.cogarch` is an object of `yuima-class` or an object of `yuima.cogarch-class` that contains all the information about the COGARCH(p,q) model.

• `data` is an object of class `yuima.data-class` that contains the observations available at uniformly spaced instants of time. If `data=NULL`, the default, the `cogarchNoise` uses the data in an object of `yuima.data-class`.

• `param` is a list of parameters for the COGARCH(p,q) model.

• `mu` is a numeric object that contains the value of the second moments of the Lévy measure.

6 Numerical results

In this section we show how to use the `yuima` package for the simulation and the estimation of a COGARCH(p,q) model driven by different symmetric Lévy processes. As a first step we focus on a COGARCH(1,1) model driven by different Lévy processes available on the package. In particular we consider the cases in which the driven noise are a Compound Poisson with jump size normally distributed and a Variance Gamma process. In the last part of this section, we show also that the estimation procedure implemented seems to be adequate even for higher order COGARCH models. In particular we simulate and then estimate different kind of COGARCH(p,q) models driven by a Compound Poisson process where the distribution of the jump size is a normal.

6.1 Simulation and Estimation of a COGARCH(1,1)

The first example is a COGARCH(1,1) model driven by a Compound Poisson process. As a first step, we choose the set of the model parameters:

```
> numSeed <- 200
> param.cp <- list(a1 = 0.038, b1 = 0.053, a0 = 0.04/0.053, 
+ lambda = 1, età=0, sig2=1, x01 = 50.33)
```

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\(a_1, b_1\) and \(a_0\) are the parameters of the state process \(Y_t\). \(\lambda\) is the intensity of the Compound Poisson process while \(\eta\) and \(\sigma^2\) are the mean and the variance of the jump size. \(x_{0,1}\) is the starting point of the process \(X_t\), the chosen value is the stationary mean of the state process and it is used in the simulation algorithm.

In the following command line we define the model using the `setCogarch` function.

```r
> mod.cp <- setCogarch(p = 1, q = 1, work = FALSE,
+ measure=list(intensity="lambda", df=list("dnorm(z,eta,sig2")
+ measure.type = "CP",
+ Cogarch.var = "g", V.var = "v", Latent.var="x",
+ XInExpr=TRUE)
```

We simulate a sample path of the model using the Euler discretization. We fix \(\Delta t = \frac{1}{15}\) and the command lines below are used to instruct `yuima` for the choice of the simulation scheme:

```r
> Term <- 1600
> num <- 24000
> set.seed(numSeed)
> samp.cp <- setSampling(Terminal=Term, n=num)
> sim.cp <- simulate(mod.cp, true.parameter=param.cp,
+ sampling=samp.cp, method="euler")
```

In the following figure we show the behaviour of the simulated trajectories for the COGARCH(1,1) model \(G_t\), the variance \(V_t\) and the state space \(X_t\):

```r
> plot(sim.cp, main = "simulated COGARCH(1,1) model driven by a Compound Poisson process")
```

We use the two step algorithm developed in Section 4 for the estimation of the COGARCH\((p,q)\) and the Lévy measure parameters. In the `yuima` function `gmm`, we fix `objFun` = L2 meaning that the objective function used in the minimization is the mean squared error. Setting also `Est.Incr=IncrPar`, the function `gmm` returns the estimated increments of the underlying noise.

```r
> res.cp <- gmm(sim.cp, start = param.cp, objFun = "L2", Est.Incr = "IncrPar")
```

The results can be displayed using the method `summary` and in the following figure we report the recovered increments of the underlying noise process.

```r
> summary(res.cp)
```

Two Stages GMM estimation

Call:
gmm(yuima = sim.cp, start = param.cp, Est.Incr = "IncrPar", objFun = "L2")
Coefficients:

|          | Estimate | Std. Error |
|----------|----------|------------|
| b1       | 6.783324e-02 | 0.06862392 |
| a1       | 3.403071e-02 | 0.02897625 |
| a0       | 1.032014e+00 | NA         |
| lambda   | 1.073912e+00 | NA         |
| eta      | 6.818470e-09 | NA         |
| sig2     | 7.837838e-01 | NA         |

Log.objFun L2: -3.491179

Number of increments: 24000

Average of increments: -0.002114

Standard Dev. of increments: 0.256610

-2 log L of increments: 2851.529874

Summary statistics for increments:

|        | Min.  | 1st Qu. | Median | Mean  | 3rd Qu. | Max.  |
|--------|-------|---------|--------|-------|---------|-------|
| -2 Incr.L | -2.840000 | 0.000000 | 0.000000 | -0.002114 | 0.000000 | 3.686000 |

> plot(res.cp, main = "Compound Poisson Increment of a COGARCH(1,1) model")

We are able also to generate the original process using the increments stored into the object `res.cp` using the `simulate` function.

> traj.cp<- simulate(res.cp)

> plot(traj.cp, main = "estimated COGARCH(1,1) driven by compound poisson process")

In the next example, we simulate and estimate a COGARCH(1,1) model driven by a Variance Gamma process. We set the values for the parameters and define the model using the following command lines:

> param.VG <- list(a1 = 0.038, b1 = 0.053, a0 = 0.04/0.053,
+ lambda = 1, alpha = sqrt(2), beta = 0, mu = 0, x01 = 50.33)

> cog.VG <- setCogarch(p = 1, q = 1, work = FALSE,
+ measure=list("rngamma(z, lambda, alpha, beta, mu")),
We obtain a trajectory for COGARCH(1,1) with Variance Gamma noise.

```r
> set.seed(numSeed)
> samp.VG <- setSampling(Terminal = Term, n = num)
> sim.VG <- simulate(cog.VG, true.parameter = param.VG,
> + sampling = samp.VG, method = "euler")
> plot(sim.VG, main = "simulated COGARCH(1,1) model driven by a Variance Gamma process")
```

and then we estimate the model parameters:

```r
> res.VG <- gmm(sim.VG, start = param.VG, Est.Incr = "IncrPar")
> summary(res.VG)
```

Two Stages GMM estimation

Call:
gmm(yuima = sim.VG, start = param.VG, Est.Incr = "IncrPar")

Coefficients:

| Estimate | Std. Error | b1    | 0.051449188 | 0.04168365 |
|----------|------------|-------|-------------|-------------|
| a1       | 0.028791052| 0.01810412 |
| a0       | 1.248576654| NA    |
| lambda   | 1.049274382| 0.09432438 |
| alpha    | 1.466220182| 0.08789087 |
| beta     | 0.051526860| 0.03929050 |
| mu       | 0.003357025| 0.02936151 |
Log.objFun L2: -3.755496

Number of increments: 24000

Average of increments: 0.003635

Standard Dev. of increments: 0.258127

-2 log L of increments: 4291.499302

Summary statistics for increments:

|       | Min.       | 1st Qu.  | Median | Mean   | 3rd Qu. | Max.    |
|-------|------------|----------|--------|--------|---------|---------|
|       | -5.548000  | -0.001729| 0.00000| 0.003635| 0.002092| 4.005000|

> plot(res.VG, main = "Variance Gamma Increment of a COGARCH(2,1) model")

Even in this case we can obtain the COGARCH(1,1) trajectory using the estimated increments as follows:

> traj.VG <- simulate(res.VG)
> plot(traj.VG, main="estimated COGARCH(1,1) model driven by Variance Gamma process")
6.2 COGARCH(p,q) model driven by a Compound Poisson process

We conclude this Section by illustrating an example using COGARCH(p,q) process. In this way, we show the ability of the *yuima* package in managing in a complete way these models. For this reason in the following we consider a COGARCH(2,1) driven by Compound Poisson Processes where the jump size is normally distributed.

We define the COGARCH(2,1) model in the *yuima* using the command lines:

```r
> param.cp2 <- list(a0 = 0.5, a1 = 0.1, b1 = 1.5, b2 = 0.5,
+ lambda = 1, eta = 0, sig2 = 1, x01 = 2.5, x02 = 0)
> mod.cp2 <- setCogarch(p = 1, q = 2, work = FALSE,
+ measure = list(intensity = "lambda", df = list("dnorm(z,eta,sig2)")),
+ measure.type = "CP", Cogarch.var = "y", V.var = "v",
+ Latent.var = "x", XinExpr = TRUE)
```

We simulate a trajectory.

```r
> samp.cp2 <- setSampling(Terminal = Term, n = num)
> set.seed(numSeed)
> sim.cp2 <- simulate(mod.cp2, sampling = samp.cp2,
+ true.parameter = param.cp2, method="euler")
> plot(sim.cp2, main = "simulated COGARCH(2,1) model driven by a Compound Poisson process")
```

We estimate the model parameters and recover the underlying Lévy noise increments:

```r
> res.cp2 <- gmm(yuima = sim.cp2, start = param.cp2, Est.Incr = "IncrPar")
> summary(res.cp2)
```

### Two Stages GMM estimation

Call:

```r
gmm(yuima = sim.cp2, start = param.cp2, Est.Incr = "IncrPar")
```

Coefficients:

|     | Estimate | Std. Error |     |
|-----|----------|------------|-----|
| b2  | 0.0569630413 | 0.20054247 |     |
| b1  | 0.9520642366 | 3.54500502 |     |
| a1  | 0.0281299955 | 0.09775311 |     |
| a0  | 0.2956658497 | NA         |     |
| lambda | 1.0423762156 | NA         |     |
```

---

22
eta  0.0002425553  NA  
sig2  0.8154399532  NA

Log.objFun L2:  -3.323979

Number of increments: 24000

Average of increments: -0.001929

Standard Dev. of increments: 0.258830

-2 log L of increments: 2861.417140

Summary statistics for increments:

| Min.   | 1st Qu. | Median | Mean   | 3rd Qu. | Max.    |
|--------|---------|--------|--------|---------|---------|
| -3.054000 | 0.000000 | 0.000000 | -0.001929 | 0.000000 | 3.483000 |

> plot(res.cp2, main = "Compound Poisson Increment of a COGARCH(2,1) model")

The path of the COGARCH(2,1) driven by the estimated increments are reported below:

> traj.cp2 <- simulate(res.cp2)
> plot(traj.cp2, main = "estimated COGARCH(2,1) model driven by a Compound Poisson process")

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