SUMS OF COMMUTATORS IN PURE C*-ALGEBRAS

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Abstract. In a pure C*-algebra (i.e., one having suitable regularity properties in its Cuntz semigroup), any element on which all bounded traces vanish is a sum of 7 commutators.

1. Introduction

This paper is concerned with the problem of representing trace zero elements in a C*-algebra as sums of commutators. This problem has a long history, going back to the result by Shoda that a matrix of zero trace is expressible as a single commutator (i.e., has the form $xy - yx$). In a general C*-algebra, one can deduce from the Hahn-Banach theorem that the elements that vanish on every bounded trace belong to the norm closure of the linear span of the commutators. One can even arrange, by a result of Cuntz and Pedersen [CP79], for a series of commutators converging in norm to any given trace zero element. A problem that has occupied numerous authors ([Fac82,Mar06,Tho95,Pop02,Ng12,Rob15]) is that of turning this infinite sum of commutators into a finite one. Examples in [PP70] and more recently [Rob15] show that this is not always possible; not even for simple nuclear C*-algebras with a unique tracial state. Marcoux [Mar06], continuing work of Fack [Fac82] and Thomsen [Tho95], was the first to show that C*-algebraic regularity properties, such as Blackadars’s strict comparison of projections, could be used to obtain a positive answer. This idea has proved fruitful, and the present paper extends further the work in this direction. We prove our results in the setting of pure C*-algebras; i.e., C*-algebras whose Cuntz semigroups have certain algebraic regularity properties. The class of pure C*-algebras includes all Z-stable C*-algebras (i.e., those tensorially absorbing the Jiang-Su algebra) and tensorially prime examples such as the reduced C*-algebra of the free group in infinitely many generators.

Let us fix some notation: Let $A$ be a C*-algebra. By a commutator in $A$ we understand an element of the form $xy - yx$; we denote it by $[x, y]$. We denote by $[A, A]$ the linear span of the commutators of $A$.

Following Winter [Win12], we say that $A$ is pure if its Cuntz semigroup has the properties of almost divisibility and almost unperforation. The latter property is equivalent to the strict comparison of positive elements by lower semicontinuous 2-quasitraces (see Section 3). We prove the following theorem:

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Theorem 1.1. Let $A$ be a pure $C^*$-algebra whose lower semicontinuous $[0, \infty]$-valued 2-quasitraces are traces. Then for every $h \in \overline{A, A}$ we have $h = \sum_{i=1}^{7} [x_i, y_i]$, with $x_i, y_i \in A$ such that $\|x_i\| \cdot \|y_i\| \leq C\|h\|$ for all $i$, and where $C$ is a universal constant.

A significant departure in this theorem from past results is that the existence of a unit in the $C^*$-algebra is not assumed. This brings new technical difficulties that can nevertheless be overcome. The assumption of simplicity for the $C^*$-algebra, typically present in previous results on this question, has also been dropped.

Part of the motivation for this paper has been to investigate pure $C^*$-algebras for their own sake. Indeed, toward the proof of Theorem 1.1 we establish a number of results on pure $C^*$-algebras of intrinsic interest. Pure $C^*$-algebras arise naturally in the classification program for simple nuclear $C^*$-algebras and in various $C^*$-algebra constructions. For example, the tensor product of any $C^*$-algebra with the Jiang-Su algebra $\mathcal{Z}$ is pure. However, while it is reasonable to expect that many naturally occurring simple $C^*$-algebras are pure, there is no evidence that $\mathcal{Z}$-stability is a prevalent property beyond the realm of nuclear $C^*$-algebras. Theorem 1.1 applies to infinite reduced free products which can be both nonexact and tensorially prime (see Example 4.11).

On the way to proving Theorem 1.1, we investigate traces of products and ultraproducts of $C^*$-algebras; a topic also of independent interest. Traces of ultraproducts show up in the recent work on the Toms-Winter conjectures: [KR13, TWW12, BBS+15]. Given a $C^*$-algebra $A$ let us denote by $T_1(A)$ the traces on $A$ of norm at most one (endowed with the weak$^*$-topology). We prove the following theorem:

Theorem 1.2. Let $A_1, A_2, \ldots$ be $C^*$-algebras with strict comparison of positive elements by traces. The following are true:

(i) The convex hull of the sets $T_1(A_1), T_1(A_2), \ldots$ is weak$^*$ dense in $T_1(\prod_{n=1}^{\infty} A_n)$.

(ii) For any free ultrafilter $U$ in $\mathbb{N}$ we have $T_1(\prod_U A_i) = \prod_U T_1(A_i)$.

Both (i) and (ii) also hold if we instead assume that the $C^*$-algebras $A_1, A_2, \ldots$ all have strict comparison of full positive elements by bounded traces and that their primitive spectra are compact.

A special case of the theorem above is [Oza13, Theorem 8], where the $C^*$-algebras are unital, $\mathcal{Z}$-stable, and exact. Here, $\mathcal{Z}$-stability and exactness have been replaced by strict comparison by traces (which we show implies that “2-quasitraces are traces”).

Here is a brief overview of the paper: In Section 2 we introduce the notion of “commutator bounds” for a $C^*$-algebra and discuss its basic properties. We then go over a number of techniques, particularly a method originally due to Fack, for proving that a $C^*$-algebra has finite commutator bounds. We take special care to adapt these techniques to the non-unital case. In Section 3 we investigate the property of strict comparison by traces and some variations on it. We show that strict comparison by lower semicontinuous traces implies that (lower semicontinuous) 2-quasitraces are traces. In Section 4 we prove Theorems 1.1 and Theorem 1.2 stated above. In Section 5 we use nilpotents of order 2, rather than commutators, to represent trace zero elements of a pure $C^*$-algebra. In the last section of the
paper we look at multiplicative commutators of unitaries and the kernel of the de la Harpe-Skandalis determinant in a pure C*-algebra.

2. Commutator bounds

Let us start by fixing some notation. Let $A$ be a C*-algebra. Let $A_{sa}$ and $A_+$ denote the sets of selfadjoint and positive elements of $A$ respectively. Let $A^\sim$ denote the unitization of $A$ and $M(A)$ the multiplier C*-algebra of $A$.

By a commutator in $A$ we understand an element of the form $[x, y] := xy - yx$, with $x, y \in A$. We denote the linear span of the commutators by $[A, A]$. We regard $A/([A, A])$ as a Banach space under the quotient norm and let $\text{Tr}: A \to A/([A, A])$ denote the quotient map (called the universal trace on $A$). We regard $\text{Tr}$ as also defined on $M_n(A)$ for all $n \in \mathbb{N}$ by $\text{Tr}((a_{i,j})_{i,j=1}^n) = \text{Tr}(\sum_{i=1}^n a_{ii})$.

We denote by $T_1(A)$ the traces on $A$ of norm at most 1; i.e., the positive linear functionals on $A$ that vanish on $[A, A]$ and have norm at most 1. It follows from Hahn-Banach’s theorem and the Jordan decomposition of bounded traces that

$$\|\text{Tr}(a)\| = \sup\{|\tau(a) | \tau \in T_1(A)|\}.$$  

(See [CP79, Theorem 2.9] and [Tho95, Proof of Lemma 3.1]). In particular,

$$[A, A] = \ker \text{Tr} = \bigcap\{\ker \tau | \tau \in T_1(A)\}.$$

We will often write $a \sim_{\text{Tr}} b$ meaning that $\text{Tr}(a - b) = 0$; i.e., $a - b \in [A, A]$.

In [Mar06], Marcoux calls commutator index of a C*-algebra the least $m \in \mathbb{N}$ such that every element $h \sim_{\text{Tr}} 0$ is expressible as a sum of $m$ commutators. We introduce here a variation on this concept where only approximation by sums of commutators is required. Furthermore, we keep track of the norms of the elements appearing in the commutators.

**Definition 2.1.** Let us say that a C*-algebra $A$ has commutator bounds $(m, C)$ if for all $h \in [A, A]$ and $\varepsilon > 0$, there exist $x_1, y_1, \ldots, x_m, y_m \in A$ such that

$$\left\| h - \sum_{i=1}^m [x_i, y_i] \right\| < \varepsilon \quad (2.1)$$

and

$$\sum_{i=1}^m \|x_i\| \cdot \|y_i\| \leq C\|h\|. \quad (2.2)$$

If (2.1) and (2.2) hold with $\varepsilon = 0$ for some $x_i, y_i \in A$ then we say that $A$ has commutator bounds $(m, C)$ with no approximations.

**Remark 2.2.** We can alternatively define commutator bounds without assuming $h \in [A, A]$ as follows: for each $h \in A$ and $\varepsilon > 0$ there exist $x_i, y_i \in A$, with $i = 1, \ldots, m$ such that

$$\| h - \sum_{i=1}^m [x_i, y_i] \| \leq \|\text{Tr}(h)\| + \varepsilon.$$
and \( \sum_{i=1}^{m} \|x_i\| \cdot \|y_i\| \leq C\|h\| \).

Many classes of C*-algebras can be shown to have finite commutator bounds: unital C*-algebras with no bounded traces have finite commutator bounds with no approximations ([Pop02]); C*-algebras of nuclear dimension \( m \in \mathbb{N} \) have commutator bounds \( (m+1, m+1) \) ([Rob15, Remark 3.2]); by Theorem 1.4 from the introduction (proven below), pure C*-algebras whose 2-quasitraces are traces have commutator bounds \( (7, C^\infty) \) with no approximations. On the other hand, even among simple unital nuclear C*-algebras there are some that have no finite commutator bounds ([Rob15, Theorem 1.4]).

Before going over a number of results on the computation of commutator bounds, let’s discuss an application of this concept to traces of products and ultraproducts. Let \( A_i, i = 1, 2, \ldots \) be C*-algebras. Recall that the product C*-algebra \( \prod_{i=1}^{\infty} A_i \) is the C*-algebra of bounded sequences \( (a_i)_{i=1}^{\infty} \), with \( a_i \in A_i \) for all \( i \). For a given free ultrafilter \( U \) of the positive integers, the ultraproduct \( \prod_{i=1}^{\infty} A_i \) is the quotient of \( \prod_{i=1}^{\infty} A_i \) by the ideal of sequences \( (a_i)_{i=1}^{\infty} \) such that \( \lim_{U} a_i = 0 \). For each \( n \in \mathbb{N} \), let us view \( \tau \in T_1(A_n) \) as an element of \( T_1(\prod_{i=1}^{\infty} A_i) \) by \( \tau((a_i)_i) = \tau(a_i) \). For each sequence of traces \( (\tau_i)_i \), with \( \tau_i \in T_1(A_i) \) for all \( i \), there is a trace on \( \prod_{i=1}^{\infty} A_i \) given by \( T_1(\prod_{i=1}^{\infty} A_i) \ni (a_i)_i \mapsto \lim_{U} \tau_i(a_i) \). Let us denote by \( \prod_{i=1}^{\infty} T_1(A_i) \) the weak* closure (in \( T_1(\prod_{i=1}^{\infty} A_i) \)) of the set of traces that arise in this way.

**Proposition 2.3.** Let \( A_i, i = 1, 2, \ldots, \) be C*-algebras, all with commutator bounds \( (m, C) \) for some \( m \in \mathbb{N} \) and \( C > 0 \). Then the convex span of the sets \( T_1(A_i) \), with \( i = 1, 2, \ldots \), is weakly dense in \( T_1(\prod_{i=1}^{\infty} A_i) \). Also, \( \prod_{i=1}^{\infty} T_1(A_i) = T_1(\prod_{i=1}^{\infty} A_i) \) for any free ultrafilter \( U \).

**Proof.** The proof of is exactly the same as that of Ozawa’s [Oza13, Theorem 8], except that [Oza13, Theorem 6] is replaced with this paper’s Remark 2.2 (Notice that Ozawa denotes by \( \prod_{i=1}^{\infty} T_1(A_i) \) the set of tracial states obtained as limits along the ultrafilter \( U \) rather than its weak* closure.)

Let’s now look into permanence properties for the commutator bounds:

**Proposition 2.4.** Let \( A \) be a C*-algebra and \( I \) a closed two-sided ideal of \( A \).

(i) If \( A \) has commutator bounds \( (m, C) \) then so do \( I \) and \( A/I \).

(ii) If \( I \) and \( A/I \) have commutator bounds \( (m, C) \) and \( (n, D) \), respectively, then \( A \) has commutator bounds \( (m+n, C+D) \).

(iii) Let \( A \) be the inductive limit of C*-algebras \( (A_\lambda)_{\lambda \in \Lambda} \), each with commutator bounds \( (m, C) \). Then \( A \) has commutator bounds \( (m, C) \) too.

**Proof.** (i) Let \( h \in [I, I] \). Since \( A \) has commutator bounds \( (m, C) \), we can find \( x_1, y_1, \ldots, x_m, y_m \in A \) that satisfy (2.1) and (2.2). Let \( (e_\lambda)_{\lambda} \) be an approximately central approximate unit of \( I \). Then for \( \lambda \) large enough \( x'_i = x_i e_\lambda \) and \( y'_i = y_i e_\lambda \) satisfy (2.1) and (2.2) and belong to \( I \).

Let us suppose now that \( h \in A/I \) and \( h \sim_{T_1} 0 \). It suffices to assume that \( \|h\| = 1 \).

Let \( \varepsilon > 0 \) be given. By [NR14, Lemma 2.1 (i)], there exists a lift \( h' \in A \) of \( h \) such that \( h' \sim_{T_1} 0 \) and \( \|h'\| \leq 1 + \frac{\varepsilon}{2} \). Since \( A \) has commutator bounds \( (m, C) \), there exist
Since $\varepsilon > 0$ be given. Since the quotient $C^*$-algebra $A/I$ has commutator bounds $(n, D)$, there exist $x_1', y_1', \ldots, x_n', y_n' \in A/I$ such that $\|h' - \sum_{i=1}^{n} [x_i', y_i']\| < \frac{\varepsilon}{3}$ and $\sum_{i=1}^{n} \|x_i'\| \cdot \|y_i'\| \leq D\|h'\|$. For $i = 1, \ldots, n$, let us find lifts $x_i''$ and $y_i''$ in $A$ of $x_i'$ and $y_i'$, respectively, such that $\|x_i''\| = \|x_i'\|$ and $\|y_i''\| = \|y_i'\|$. Let $(e_\lambda)_\lambda$ be an approximately central approximate unit of the ideal $I$. We choose an index $\lambda$ such that

$$\|h - e_\lambda h - \sum_{i=1}^{n} [x_i, y_i]\| < \frac{\varepsilon}{3},$$

and

$$\|h - \sum_{i=1}^{n} [x_i, y_i]\| < \frac{\varepsilon}{3},$$

where $x_i = (1 - e_\lambda)^{1/2} x_i'' (1 - e_\lambda)^{1/2}$ and $y_i = (1 - e_\lambda)^{1/2} y_i'' (1 - e_\lambda)^{1/2}$ for $i = 1, \ldots, n$. Note that

$$\sum_{i=1}^{n} \|x_i\| \cdot \|y_i\| \leq \sum_{i=1}^{n} \|x_i''\| \cdot \|y_i''\| \leq D\|h'\| \leq D\|h\|.$$ 

Since $e_\lambda h e_\lambda \in I$ and the ideal $I$ has commutator index $(m, C)$, we can find elements $x_{n+1}, y_{n+1}, \ldots, x_{n+m}, y_{n+m} \in I$ such that

$$\|e_\lambda h e_\lambda - \sum_{i=n+1}^{n+m} [x_i, y_i]\| < \frac{\varepsilon}{3},$$

$$\sum_{i=n+1}^{n+m} \|x_i\| \cdot \|y_i\| \leq C\|e_\lambda h e_\lambda\| \leq C\|h\|.$$
Hence, \( \sum_{i=1}^{m+n} \|x_i\| \cdot \|y_i\| \leq (C + D)\|h\| \) and

\[
\left\| h - \sum_{i=1}^{m+n} [x_i, y_i] \right\| \leq \|h - e_\lambda h e_\lambda - (1 - e_\lambda)h(1 - e_\lambda)\| + \left\| (1 - e_\lambda)h(1 - e_\lambda) - \sum_{i=1}^{n} [x_i, y_i] \right\| < \varepsilon.
\]

(iii) Since we have already shown that the commutator bounds pass to quotients, we may assume that \( A_\lambda \subseteq A \) for all \( \lambda \) and that \( \bigcup_{\lambda \in \Lambda} A_\lambda = A \).

Let \( h \in A \) be such that \( h \sim_{TV} 0 \). Let \( \varepsilon > 0 \) be given. Let us prove the existence of \( x_1, y_1, \ldots, x_m, y_m \in A \) satisfying (2.1) and (2.2). It is clear that we may reduce ourselves to the case \( \|h\| = 1 \). We claim there exists \( \lambda \in \Lambda \) and a contraction \( h' \in A_\lambda \) such that \( \|h - h'\| < \varepsilon/2 \) and \( h' \sim_{TV} 0 \in A_\lambda \). To prove this, we first approximate \( h \) sufficiently by a finite sum of commutators: \( \|h - \sum_{j=1}^{k} [v_j, w_j]\| < \varepsilon/4 \). Next, we choose \( \lambda \in \Lambda \) and \( v'_1, w'_1, \ldots, v'_k, w'_k \in A_\lambda \) such that \( \|v_j - v'_j\| \) and \( \|w_j - w'_j\| \) are sufficiently small for all \( j \), so that \( \|h - \sum_{j=1}^{k} [v'_j, w'_j]\| < \varepsilon/4 \). Finally, we set

\[
h' = \left(1 + \frac{\varepsilon}{4}\right)^{-1} \sum_{j=1}^{k} [v'_j, w'_j].
\]

Notice then that

\[
\|h - h'\| \leq \left\| h - \left(1 + \frac{\varepsilon}{4}\right) h' \right\| + \frac{\varepsilon}{4} \|h'\| < \frac{\varepsilon}{2},
\]

and that \( h' \sim_{TV} 0 \) in \( A_\lambda \), as desired.

Since \( A_\lambda \) has commutator bounds \((m, C)\), there exist \( x_1, y_1, \ldots, x_m, y_m \in A_\lambda \) such that \( \|h' - \sum_{j=1}^{m} [x_i, y_i]\| < \varepsilon/2 \) and \( \sum_{i=1}^{m} \|x_i\| \cdot \|y_i\| \leq C \|h'\| \leq C \). These are the desired elements.

It is possible to reduce the number of commutators by passing from a C*-algebra \( A \) with commutator bounds \((m, C)\) to a matrix algebra \( M_n(A) \). This, however, is achieved at the expense of increasing the constant \( C \). Marcoux obtains such a reduction for unital C*-algebras in [Mar10] Lemma 4.1 and [Mar06] Lemma 2.2. Here, we cover the non-unital case and give explicit bounds for the norms of the commutators.

**Lemma 2.5.** Let \( d_1, \ldots, d_n \) in \( A \) be such that \( \sum_{i=1}^{n} d_i = 0 \). Then there exist \( X, Y \in M_n(A) \), with \( \|X\| \cdot \|Y\| \leq 4n \max_i \|d_i\| \), such that the main diagonal of \( [X, Y] \) equals \((d_1, \ldots, d_n)\).

**Proof.** For \( k = 1, \ldots, n - 1 \), let \( s_k = \sum_{i=1}^{k} d_i \). Let us write \( s_k = q_k r_k \), for some \( q_k, r_k \in A \) such that \( \|q_k\| = \|r_k\| = \|s_k\|^\frac{1}{2} \) (e.g., \( q_k = v |s_k|^{\frac{1}{2}} \) and \( r_k = |s_k|^{\frac{1}{2}} \), where
Let us denote $s_k = v|s_k|$ as the polar decomposition of $s_k$ in $A^{**}$. Let

\[
X = \begin{pmatrix}
0 & q_1 & r_1 \\
r_1 & q_2 & \ddots \\
0 & r_2 & \ddots & q_{n-1} \\
r_3 & \ddots & \ddots & \ddots & q_{n-1}
\end{pmatrix}, \quad Y = \begin{pmatrix}
0 & q_1 & r_1 \\
r_1 & q_2 & \ddots \\
0 & r_2 & \ddots & q_{n-1} \\
r_3 & \ddots & \ddots & \ddots & q_{n-1}
\end{pmatrix}.
\]

A straightforward computation shows that $X$ and $Y$ are as required.

**Lemma 2.6.** Let $h \in M_n(A)$, with $h = (h_{j,k})_j,k$. Suppose that $h_{j,j} = [x_j, y_j]$ for some $x_j, y_j \in A$, for $j = 1, \ldots, n$. Then $h = [X, Y]$ for some $X, Y \in M_n(A)$ such that $\|X\| : \|Y\| \leq 36n^2(n-1)\|h\| + 3n \sum_{k=1}^{n} \|x_k\| : \|y_k\|$.

**Proof.** Let $h \in M_n(A)$ and $x_j, y_j \in A$, with $j = 1, \ldots, n$, be as in the statement of the lemma. Let us also assume that $x_j$ is a contraction for all $j$ (replacing $y_j$ with $\|x_j\|y_j$ if necessary). Let $\lambda_j = 3(j-1)$ and $d_j = x_j + \lambda_j 1 \in A^\sim$, for $j = 1, \ldots, n$. Notice that the spectrum of $d_j$ is contained in $\{ z \in \mathbb{C} \mid |z - \lambda_j| < 1 \}$ for all $j$. In particular, the spectra of the $d_j$'s are pairwise disjoint. Notice also that $[d_j, b] = [x_j, b]$ for all $b \in A^\sim$ and $j = 1, \ldots, n$. Let us fix $k, j$ with $1 \leq k, j \leq n$ and $k \neq j$. By [Her89, Corollary 3.2], the Rosenblum operator $T_{k,j} : A^\sim \to A^\sim$, defined by $T_{k,j}(b) = d_k b - b d_j$, is invertible. Let $b_{k,j} \in A^\sim$ be such that $T_{k,j}(b_{k,j}) = d_k b_{k,j} - b_{k,j} d_j = h_{k,j}$. Since $h_{k,j} \in A$, and by our choice of $\lambda_k$ and $\lambda_j$, we must have that $b_{k,j} \in A$.

Let us define $b_{k,k} := y_k$ for $k = 1, \ldots, n$. Let $X, Y \in M_n(A)$ be given by

\[
X = \text{diag}(d_1, \ldots, d_n), \quad Y = (b_{k,j})_{k,j=1}^n.
\]

It is a straightforward computation show that $[X, Y] = a$ (see [Mar06, Lemma 2.2]).

Let us find bounds for the norms of $X$ and $Y$. The bound $\|X\| = \max_j \|d_j\| \leq 3n$ is straightforward. In order to bound $\|Y\|$, we first estimate $\|b_{k,j}\|$. Fix $k, j$ such that $1 \leq k, j \leq n$ and $k \neq j$. By [Her89, Corollary 3.20],

\[
b_{k,j} = \frac{1}{2\pi i} \int_{\Gamma_k} (d_k - \alpha 1)^{-1} h_{k,j} (d_j - \alpha 1)^{-1} d\alpha,
\]

where $\Gamma_k$ is the positively oriented simple closed contour given by $\Gamma_k(t) = \lambda_k + \frac{3}{2} e^{it}$, for $t \in [0, 2\pi]$. We have

\[
(d_k - \alpha 1)^{-1} = \frac{1}{\lambda_k - \alpha} \left( 1 - \frac{d_k - \lambda_k 1}{\alpha - \lambda_k} \right)^{-1}.
\]

But $\|d_k - \lambda_k 1\| = \|x_k\| \leq 1$ and $|\alpha - \lambda_k| = 3/2$ for all $\alpha \in \Gamma_k$. So, $\|d_k - \lambda_k 1\| \leq 1/(3/2) = 2/3$, from which we deduce that

\[
\|(d_k - \alpha 1)^{-1}\| \leq \frac{2}{3} \sum_{l=0}^{\infty} (2/3)^l = 2
\]
for all $\alpha \in \Gamma_k$. Next, since $k \neq j$, $|\lambda_j - \alpha| \geq 3/2$ for all $\alpha \in \Gamma_k$. Also, $\|d_j - \lambda_j 1\| = \|x_j\| \leq 1$. Hence,

$$
\|(d_j - \alpha 1)^{-1}\| = \frac{1}{\|\lambda_j - \alpha\|} \left\| \left(1 - \frac{d_j - \lambda_j}{\alpha - \lambda_j}\right)^{-1}\right\| \leq \frac{2}{3} \sum_{l=0}^{\infty} (2/3)^l = 2,
$$

for all $\alpha \in \Gamma_k$. Thus, $\|(d_k - \alpha 1)^{-1}\| \leq 2$ and $\|(d_j - \alpha 1)^{-1}\| \leq 2$ for all $\alpha \in \Gamma_k$. From this and (2.4), we get

$$
\alpha \Gamma_k = \left(\frac{\alpha}{\pi}\right) \cdot \text{length}(\Gamma_k) = \frac{1}{\pi} \|h_{k,j}\| (6\pi) = 12\|h_{k,j}\| \leq 12\|h\|,
$$

for all $k \neq j$. Recall that, by our conventions, $\|b_{k,k}\| = \|y_k\| = \|x_k\| \cdot \|y_k\|$. Then $\|Y\| \leq n(n-1)12\|h\| + \sum_{k=1}^{n} \|x_k\| \cdot \|y_k\|$. This, together with $\|X\| \leq 3n$, proves the lemma.

**Theorem 2.7.** Let $A$ be a $C^*$-algebra and $n \in \mathbb{N}$.

(i) If $M_n(A)$ has commutator bounds $(m, C)$ (with no approximations) then $A$ has commutator bounds $(mn^2, Cn)$ (with no approximations).

(ii) If $A$ has commutator bounds $(m, C)$ (with no approximations), then $M_n(A)$ has commutator bounds $(2, C')$ (with no approximations) for all $n \geq m$, where $C' \leq 36n^3 + (2C - 36)n^2 + n$.

**Proof.** (i) Let $h \in A$ with $h \sim_{\text{Tr}} 0$. Then $a \otimes 1_n \in M_n(A)$ and $a \otimes 1_n \sim_{\text{Tr}} 0$ in $M_n(A)$. Let $\varepsilon > 0$ be given. Since $M_n(A)$ has commutator bounds $(m, C)$, there exist $X_j, Y_j \in M_n(A)$ ($1 \leq j \leq m$) such that

$$
\|h \otimes 1_n - \sum_{j=1}^{m} [X_j, Y_j]\| < \varepsilon,
$$

(2.5)

$$
\sum_{j=1}^{m} \|X_j\| \cdot \|Y_j\| \leq C\|a \otimes 1_n\| = C\|h\|.
$$

(2.6)

Averaging along the main diagonal in (2.5) we get

$$
\|h - \frac{1}{n} \sum_{j=1}^{m} \sum_{k,l=1}^{n} [x_{j,k}, y_{j,l,k}]\| < \varepsilon,
$$

where $X_j = (x_{j,k,l})_{k,l=1}^{n}$ and $Y_j = (y_{j,k,l})_{k,l=1}^{n}$. On the other hand, using (2.6) we get

$$
\frac{1}{n} \sum_{j=1}^{m} \sum_{k,l=1}^{n} \|x_{j,k,l}\| \cdot \|y_{j,l,k}\| \leq \frac{1}{n} \sum_{j=1}^{m} \sum_{k,l=1}^{n} \|X_j\| \cdot \|Y_j\| = n \sum_{j=1}^{m} \|X_j\| \cdot \|Y_j\| \leq nC\|h\|,
$$

as required. The same arguments above, but with $\varepsilon = 0$, prove the result for commutator bounds with no approximations.

(ii) Let us deal with the case of commutator bounds with no approximations. Let $h \in M_n(A)$ be such that $h \sim_{\text{Tr}} 0$. Then $\sum_{i=1}^{n} h_{i,i} \sim_{\text{Tr}} 0$ in $A$. But $A$ has commutator bounds $(m, C)$ with no approximations. Hence, $\sum_{i=1}^{n} h_{i,i} = \sum_{i=1}^{n} [x_i, y_i]$ for some
Suppose that the following are true:

Let Theorem 2.9.

Proof. Let us regard The assumption that the decomposition of $x$ exists $v \in z H$. Hence, the elements

stating the theorem, we introduce some definitions and prove a lemma.

Despite its technical statement, Theorem 2.9 constitutes our main tool in proving that a $C^*$-algebra has finite commutator bounds with no approximations. Before stating the theorem, we introduce some definitions and prove a lemma.

Let us define the direct sum of positive elements in $A \otimes K$. Fix isometries $v_1, v_2 \in B(\ell_2)$ generating the Cuntz algebra $O_2$. Let us regard them as elements of the multiplier algebra $M(A \otimes K)$ via the natural embeddings $1 \otimes B(\ell_2) \subseteq M(A) \otimes B(\ell_2) \subseteq M(A \otimes K)$. Then, given $a, b \in (A \otimes K)_+$, let us define $a \oplus b \in (A \otimes K)_+$ by $a \oplus b = v_1 a v_1^* + v_2 b v_2^*$.

Next, let us introduce a preorder relation on the positive element of a $C^*$-algebra. Let $a, b \in A_+$. Let us write $a \preceq b$ if $a = x^* x$ and $x x^* \in \text{her}(b)$ for some $x \in A$. This relation is called Blackadar’s relation in $[\text{ORT11}]$. It can be alternatively described as saying that the right ideal $aA$ embeds into $bA$ as a Hilbert module. (Thus, it is clearly transitive.) We will make repeated use of the following fact (see $[\text{ORT11}]$ Proposition 4.6): Say $a = x^* x$ and $x x^* \in \text{her}(b)$. Let $x = w|x|$ be the polar decomposition of $x$ in $A^{**}$. Then $vy \in \text{her}(b)$ for any $y \in \text{her}(a)$.

**Lemma 2.8.** Let $a, b \in A_+$ be such that $a \preceq b \otimes n$ (in $A \otimes K$). Then for all $h \in \text{her}(a)$ there exist $z_1, w_1, \ldots, z_n, w_n \in A$ and $h' \in \text{her}(b)$ such that $h = \sum_{i=1}^n [z_i, w_j] + h'$, $\|z_j\| \cdot \|w_j\| \leq \|h\|$ for all $j$, and $\|h'\| \leq n\|h\|$.

**Proof.** Let us regard $A$ as a subalgebra of $M_n(A)$ embedded in the top left corner. The assumption $a \preceq b \otimes n$ can be rephrased as $a \preceq b \otimes 1_n$ in $M_n(A)$. That is, there exists $x \in M_n(A)$ such that $a = xx^*$ and $x x^* \in \text{her}(b \otimes 1_n)$. Let $x = v|x|$, with $v \in M_n(A)'$, be the polar decomposition of $x$. Let $(v_1, \ldots, v_n)$ denote the first row of $v$ (the rest of the rows are 0). Finally, let us write $h = h_1 h_2$, with $h_1, h_2 \in \text{her}(a)$ such that $\|h_1\| \cdot \|h_2\| = \|h\|$ (e.g., as in the proof of Lemma 2.5). Then

$$h = h_1 \left( \sum_{j=1}^n h_1 v_j v_j^* \right) h_2 = \sum_{j=1}^n [h_1 v_j^*, v_j h_2] + \sum_{j=1}^n v_j h_2 h_1 v_j^*.$$ 

Hence, the elements $z_j = h_1 v_j^*, w_j = v_j h_2$, for $j = 1, \ldots, n$, and $h' = \sum_{j=1}^n v_j h_2 h_1 v_j^*$ are as required.

**Theorem 2.9.** Let $A$ be a $C^*$-algebra and $e_0 \in A_+$ a strictly positive element. Suppose that the following are true:
(i) There exist an integer \( L \geq 1 \) and pairwise orthogonal positive elements \( e_1, e_2, \ldots \in A_+ \) such that \( e_j \leq \Theta^L e_{j+1} \) for \( j = 0, 1, \ldots. \)

(ii) There exist constants \( C > 0, M \in \mathbb{N}, \) and \( 0 < \lambda < 1, \) such that for all \( j \in \{0, 1, \ldots\} \) and \( h \in \text{her}(e_j) \) such that \( h \sim_{T_1} 0 \) (in \( \text{her}(e_j) \)), there exist \( x_1, y_1, \ldots, x_M, y_M \in \text{her}(e_j) \) such that

\[
\|h - \sum_{i=1}^{M} [x_i, y_i]\| \leq \lambda\|h\|,
\]

and \( \|x_i\| \cdot \|y_i\| \leq C\|h\| \) for all \( i. \)

Then \( A \) has finite commutator bounds \((\overline{M}, \overline{C})\) with no approximations, where \( \overline{M} \) and \( \overline{C} \) depend only on \( L, M, \lambda, C. \)

Proof. Let us choose \( L_1 \in \mathbb{N} \) such that \( \lambda^{L_1} < 1/(2L) \). From hypothesis (ii) we deduce the following:

(ii') for all \( j \in \{0, 1, \ldots\} \) and \( h \in \text{her}(e_j) \) such that \( h \sim_{T_1} 0 \) (in \( \text{her}(e_j) \)), there exist \( x_1, y_1, \ldots, x_{L_1}, y_{L_1} \in \text{her}(e_j) \) such that

\[
\|h - \sum_{i=1}^{L_1} [x_i, y_i]\| < \lambda^{L_1}\|h\| < \frac{1}{2L}\|h\|,
\]

and \( \|x_i\| \cdot \|y_i\| \leq C^{1/2}\|h\|^{1/2} \) for all \( i. \)

Let \( h \in A \) be such that \( h \sim_{T_1} 0. \) By hypothesis (i) and Lemma 2.8 we have

\[
h = \sum_{j=1}^{L} [z_j, w_j] + h_1,
\]

where \( h_1 \in \text{her}(e_1) \) and \( \|h_1\| \leq L\|h\|. \) By (ii') above, applied in the hereditary algebra \( \text{her}(e_1) \), there exist \( x_1^{(1)}, y_1^{(1)}, \ldots, x_{L_1}^{(1)}, y_{L_1}^{(1)} \in \text{her}(e_1) \) and \( h_1' \in \text{her}(e_1) \), such that

\[
h_1 = \sum_{i=1}^{L_1} [x_i^{(1)}, y_i^{(1)}] + h_1',
\]

and \( \|h_1'\| \leq \frac{1}{2L}\|h_1\| \leq \frac{\|h\|}{2}. \) Again by hypothesis (i) and Lemma 2.8 we have

\[
h_1' = \sum_{j=1}^{L} [z_j^{(1)}, w_j^{(1)}] + h_2,
\]

where \( h_2 \in \text{her}(e_2), \) \( z_j^{(1)}, w_j^{(1)} \in \text{her}(e_1 + e_2) \) for all \( j, \) and \( \|h_2\| \leq L\|h_1'\|. \) Applying (ii') in \( \text{her}(e_2) \), we get \( h_2 = \sum_{i=1}^{L_2} [x_i^{(2)}, y_i^{(2)}] + h_2', \) with \( \|h_2'\| \leq \frac{\|h_1'\|}{2} \leq \frac{\|h\|}{4}. \)

Continuing in this way, we construct, for each \( n \in \mathbb{N}, \) elements

1. \( h_n, h_n' \in \text{her}(e_n) \) such that \( \|h_n\| \leq \frac{1}{2^n}\|h\|, \) \( \|h_n\| \leq \frac{1}{2^n}\|h\|, \)
In this case we say that \( a \) are pairwise orthogonal. Also, the norm estimates on the elements \( A \) guarantee that these series converge. Furthermore, it is clear that norm estimates

\[
\|a\|, \|b\| \leq \|h\|^{\frac{1}{2}}
\]

for all \( j \), and

\[
A_j = 1 + 2^{\frac{1}{2}} + 3^{\frac{1}{2}} + \cdots + n^{\frac{1}{2}}.
\]

It follows that

\[
h_1 = \sum_{k=1}^{L_1} \sum_{i=1}^{M} [x_i^{(k)}, y_i^{(k)}] + \sum_{k=1}^{L_1} \sum_{j=1}^{L} [z_j^{(k)}, w_j^{(k)}] + h_{n+1}.
\]

We can gather terms belonging to orthogonal hereditary subalgebras and define

\[
X_i = \sum_{n=1}^{\infty} x_i^{(n)} , \quad Y_i = \sum_{n=1}^{\infty} y_i^{(n)} ,
\]

for \( i = 1, \ldots, L_1 M \), and

\[
Z_j,0 = \sum_{n \text{ even}} z_j^{(n)} , \quad Z_j,1 = \sum_{n \text{ even}} z_j^{(n)} ,
\]

\[
W_j,0 = \sum_{n \text{ odd}} w_j^{(n)} , \quad W_j,1 = \sum_{n \text{ even}} w_j^{(n)} ,
\]

for \( j = 1, \ldots, L \). Note that the terms in the series defining the elements \( X_i, Y, Z_j, k, W_j, k \) are pairwise orthogonal. Also, the norm estimates on the elements \( x_i^{(n)}, y_i^{(n)}, z_j^{(n)}, w_j^{(n)} \) guarantee that these series converge. Furthermore, it is clear that norm estimates on \( X_i, Y, Z_j, k, W_j, k \) can be obtained from those on \( x_i^{(n)}, y_i^{(n)}, z_j^{(n)}, w_j^{(n)} \). We have

\[
h = \sum_{j=1}^{L} [z_j, w_j] + \sum_{i=1}^{L_1 M} [X_i, Y_i] + \sum_{j=1}^{L} [Z_j,0, W_j,0] + \sum_{j=1}^{L} [Z_j,1, W_j,1].
\]

This shows that \( A \) has commutator bounds \((3L + L_1 M, C')\) with no approximations, for some \( C' \). \( \square \)

3. Strict comparison of positive elements

Here we review and explore the strict comparison of positive elements by traces and 2-quasitraces. Some of these results will be used in the proof of Theorem \[\square\] in the next section.

Let us start by recalling the definition of the Cuntz semigroup. Let \( A \) be a C*-algebra. Let \( a, b \in (A \otimes \mathcal{K})_+ \). Let us write \( a \prec_{Cu} b \) if \( d_n b d_n \to a \) for some \( d_n \in A \otimes \mathcal{K} \).

In this case we say that \( a \) is Cuntz smaller than \( b \). Let us write \( a \sim_{Cu} b \) if \( a \prec_{Cu} b \)
and \( b \preceq_{Cu} a \), in which case we say that \( a \) and \( b \) are Cuntz equivalent. Let \([a]\) denote the Cuntz class of \( a \in (A \otimes K)_+\).

The Cuntz semigroup of \( A \), denoted by \( \text{Cu}(A) \), is defined as the quotient set \((A \otimes K)/\sim_{Cu}\), endowed the following order and addition: \([a] \leq [b] \) if \( a \preceq_{Cu} b \) and \([a] + [b] = [a \oplus b] \), with the direct sum \( a \oplus b \in (A \otimes K)_+ \) as defined in the previous section. The reader is referred to \([\text{APT}11]\) and \([\text{APT}14]\) for the basic theory of the Cuntz semigroup (some of which will be used below).

Let us denote by \( T(A) \) the cone of lower semicontinuous \([0, \infty]\)-valued traces on \( A \); i.e., the lower semicontinuous maps \( \tau: A_+ \to [0, \infty] \) that are additive, homogeneous, map 0 to 0, and satisfy that \( \tau(x^*x) = \tau(xx^*) \) for all \( x \in A \). Let us denote by \( \text{QT}(A) \) the lower semicontinuous \([0, \infty]\)-valued 2-quasitraces on \( A_+ \). Traces and 2-quasitraces extend uniquely to traces and quasitraces on \((A \otimes K)_+\), and we shall regard them as defined on this domain (see \([\text{BK}04\text{, Remark 2.27 (viii)}]\)). Recall (from the previous section) that we denote by \( T_1(A) \) the convex set of traces on \( A \) of norm at most 1.

A topology on \( \text{QT}(A) \) can be defined as follows: Let \((\tau_\lambda)_\lambda \) be a net in \( \text{QT}(A) \) and \( \tau \in \text{QT}(A) \). Let us say that \( \tau_\lambda \to \tau \) if for any \( a \in (A \otimes K)_+ \) and \( \varepsilon > 0 \) we have

\[
\limsup_{\lambda} \tau_\lambda((a - \varepsilon)_+) \leq \tau(a) \leq \liminf_{\lambda} \tau_\lambda(a).
\]

In this way \( \text{QT}(A) \) is a compact Hausdorff space and \( T(A) \) and \( T_1(A) \) are closed subsets of \( \text{QT}(A) \) (see \([\text{ERS}11\text{, Subsections 3.2 and 4.1}]\)).

The dimension function associated to \( \tau \in \text{QT}(A) \) is defined as

\[
d_\tau(a) = \lim_n \tau(a^{1/n}) = \|\tau|_{\text{her}(a)}\|
\]

for all \( a \in (A \otimes K)_+ \). The value \( d_\tau(a) \) depends only on the Cuntz class of the positive element \( a \). Thus, by a slight abuse of notation, we also write \( d_\tau([a]) \).

The ordered semigroup \( \text{Cu}(A) \) is called \emph{almost unperforated} if

\[
(k + 1)x \leq ky \implies x \leq y
\]

for all \( k \in \mathbb{N} \) and \( x, y \in \text{Cu}(A) \). The ordered semigroup \( \text{Cu}(A) \) is called \emph{almost divisible} if for all \( n \in \mathbb{N} \), \( x \in \text{Cu}(A) \) and \( x' \ll x \) (i.e., \( x' \) compactly contained in \( x \)), there exists \( y \in \text{Cu}(A) \) such that

\[
y \leq x \text{ and } x' \leq (n + 1)y.
\]

The \( \text{C}^* \)-algebra \( A \) is called \emph{pure} if \( \text{Cu}(A) \) is both almost unperforated and almost divisible. By \([\text{Ror04}\text{, C}^*\text{-algebras that absorb tensorially the Jiang-Su algebra are pure. There are, however, tensorially prime C}^*\text{-algebras.}]

It is shown in \([\text{ERS}11\text{, Proposition 6.2}]\) (and in \([\text{Ror04}\text{, Corollary 4.6}]\) for simple \( \text{C}^* \)-algebras) that almost unperforation in \( \text{Cu}(A) \) is equivalent to the property of strict comparison of positive elements by 2-quasitraces. We consider here the following generalization of the latter property:

**Definition 3.1.** Let \( A \) be a \text{C}^*\text{-algebra and } K \subseteq \text{QT}(A) \text{ a compact subset. Let us say that } A \text{ has strict comparison of positive elements by } 2\text{-quasitraces in } K \text{ if
Proof. (i) Let \( \tau \in K \) and some \( \gamma > 0 \) implies that \( [a] \leq [b] \) for all \( a, b, d, K \). Thus, one of these open sets covers \( K \), as shown in the proof of [ERS11, Lemma 5.11], we have:

\[
\tau((a - \varepsilon)_+) \leq \left(1 - \frac{\gamma}{2}\right)d_\tau((b - \delta)_+),
\]

for all \( \tau \in K \).

\[\begin{align*}
\{\tau \in K \mid d_\tau((a - \varepsilon)_+) > 1\} & \subseteq \{\tau \in K \mid \left(1 - \frac{\gamma}{2}\right)d_\tau(a) > 1\}. \\
The left side is compact while the right side is covered by the open sets & \{\tau \in K \mid \left(1 - \frac{\gamma}{2}\right)d_\tau((b - \frac{1}{n})_+) > 1\}, \quad n = 1, 2, \ldots.
\end{align*}\]

Thus, one of these open sets covers \( \{\tau \in K \mid d_\tau((a - \varepsilon)_+) > 1\} \). This proves the lemma.

\[\square\]

Lemma 3.3. Let \( K \subseteq QT(A) \) be compact and \( a, b, d, K \). Suppose that \( d_\tau(a) \leq (1 - \gamma)d_\tau(b) \) for all \( \tau \in K \) and some \( \gamma > 0 \). Then for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
d_\tau((a - \varepsilon)_+) \leq \left(1 - \frac{\gamma}{2}\right)d_\tau((b - \delta)_+),
\]

for all \( \tau \in K \).

Proof. As shown in the proof of [ERS11, Lemma 5.11], we have

\[
\{\tau \in K \mid d_\tau((a - \varepsilon)_+) > 1\} \subseteq \{\tau \in K \mid \left(1 - \frac{\gamma}{2}\right)d_\tau(a) > 1\}.
\]

The left side is compact while the right side is covered by the open sets

\[
\{\tau \in K \mid \left(1 - \frac{\gamma}{2}\right)d_\tau((b - \frac{1}{n})_+) > 1\}, \quad n = 1, 2, \ldots.
\]

Thus, one of these open sets covers \( \{\tau \in K \mid d_\tau((a - \varepsilon)_+) > 1\} \). This proves the lemma.

Lemma 3.4. Let \( A \) be a \( C^* \)-algebra with strict comparison of positive elements by \( K \), where \( K \subseteq QT(A) \) is compact. Let \( a, b, d, K \).

(i) If \( d_\tau(a) \leq d_\tau(b) \) for all \( \tau \in K \) then \( d_\tau(a) \leq d_\tau(b) \) for all \( \tau \in QT(A) \).

(ii) If \( \tau(a) \leq \tau(b) \) for all \( \tau \in K \) then \( \tau(a) \leq \tau(b) \) for all \( \tau \in QT(A) \).

Proof. (i) Let \( k \in \mathbb{N} \). Then \( d_\tau(a \otimes 1_k) \leq (1 - \frac{1}{k+1})d_\tau(b \otimes 1_{k+1}) \) for all \( \tau \in K \). Since \( A \) has strict comparison by 2-quasitraces in \( K \), \( a \otimes 1_k \not\leq b \otimes 1_{k+1} \), which in turn implies that \( d_\tau(a) \leq (1 + \frac{1}{k+1})d_\tau(b) \) for all \( \tau \in QT(A) \). Letting \( k \to \infty \) we get that \( d_\tau(a) \leq d_\tau(b) \) for all \( \tau \in QT(A) \), as desired.

(ii) Let \( \varepsilon > 0 \). By a compactness argument as in the proof of Lemma 3.3, we find that there exists \( \delta > 0 \) such that \( \tau((a - \varepsilon)_+) \leq \tau((b - \delta)_+) \) for all \( \tau \in K \) (see [ERS11, Propositions 5.1 and 5.3]). Let \( f_\varepsilon, g, g_\delta : QT(A) \to [0, \infty] \) be as follows:

\[
f_\varepsilon(\tau) = \tau((a - \varepsilon)_+), \quad g(\tau) = \tau(b), \quad \text{and} \quad g_\delta(\tau) = \tau((b - \delta)_+) \text{ for all } \tau.
\]

The functions \( f_\varepsilon \) and \( g \) (and also \( g_\delta \)) belong to the realification of \( Cu(A) \), as defined in [Rob13]. That is, \( f_\varepsilon = f_n \to g = g_n \to \), where \( f_n(\tau) = \frac{1}{n}d_\tau([a_n]) \) and \( g_n(\tau) = \frac{1}{n}d_\tau([b_n]) \),
for some \([a_n], [b_n] \in \text{Cu}(A)\) and \(r_n, s_n \in \mathbb{N}\) for \(n = 1, \ldots\). (This follows from the fact that
\[
\tau(c) = \int_0^{||c||} d_r([(c - t)_+]) dt
\]
for all \(c \in (A \otimes \mathcal{K})_+\) and \(\tau \in \text{QT}(A)_+\).) By Propositions 5.1 and 5.3, the function \(g_\delta\) is way below \(g\), so that \(g_\delta \leq g_n\) for some \(n\). Hence, \(f_m(\tau) \leq g_n(\tau)\) for all \(\tau \in K\) and all \(m \in \mathbb{N}\); i.e., \(\frac{1}{r_m} d_r(a_m) \leq \frac{1}{s_n} d_r(b_n)\) for all \(\tau \in K\) and \(m\). By (i), this same inequality holds for all \(\tau \in \text{QT}(A)_+\); whence, \(f_m \leq g\) for all \(m\). Passing to the supremum over \(m\) we get \(\tau(a - \varepsilon)_+ \leq \tau(b)\) for all \(\tau \in \text{QT}(A)_+\) and \(\varepsilon > 0\). Now passing to the supremum over all \(\varepsilon > 0\) we get \(\tau(a) \leq \tau(b)\) for all \(\tau \in \text{QT}(A)_+\), as desired.

There is a version of the previous lemma for strict comparison of full positive elements by traces:

**Lemma 3.5.** Let \(A\) be a C*-algebra with \(\text{Prim}A\) compact and with strict comparison of full positive elements by traces. Let \(a, b \in (A \otimes \mathcal{K})_+\) be full positive elements.

(i) If \(d_r(a) \leq d_r(b)\) for all \(\tau \in T(A)\) then \(d_r(a) \leq d_r(b)\) for all \(\tau \in \text{QT}(A)_+\).

(ii) If \(\tau(a) \leq \tau(b)\) for all \(\tau \in T(A)\) then \(\tau(a) \leq \tau(b)\) for all \(\tau \in \text{QT}(A)_+\).

**Proof.** The same proof as Lemma 3.4 with the obvious modifications, works here. When taking functional calculus cut-downs \((a - \varepsilon)_+\) and \((b - \delta)_+\), we must take care to choose them so that they are still full (which is possible by the compactness of \(\text{Prim}(A)_+\)).

**Theorem 3.6.** Let \(A\) be a C*-algebra.

(i) If \(A\) has strict comparison of positive elements by traces then every lower semicontinuous 2-quasitrace on \(A\) is a trace.

(ii) If \(\text{Prim}A\) is compact and \(A\) has strict comparison of full positive elements by traces, then every densely finite lower semicontinuous 2-quasitrace on \(A\) is a trace.

**Proof.** (i) Let \(a, b \in A_+\). Let \(c, d \in M_2(A)\) be defined as
\[
c = \begin{pmatrix} a + b & 0 \\ 0 & a \end{pmatrix}, \quad d = \begin{pmatrix} a \\ b \end{pmatrix}.
\]
Then \(\tau(c) = \tau(d)\) for all \(\tau \in T(A)\). By Lemma 3.4 (ii), we get that \(\tau(c) = \tau(d)\) for all \(\tau \in \text{QT}(A)_+\). But \(\tau(c) = \tau(a + b)\) and \(\tau(d) = \tau(a) + \tau(b)\). So all \(\tau \in \text{QT}(A)_+\) are additive, as desired.

(ii) The same proof as in (i), but relying now on Lemma 3.5 (ii), shows in this case that the lower semicontinuous 2-quasitraces on \(A\) are additive on pairs of full positive elements. Let us now prove that the densely finite ones are additive on any pair of positive elements. Let \(\tau\) be one such 2-quasitrace and let \(a, b \in A_+\). Say \(w \in A_+\) is full (whose existence is guaranteed by the compactness of \(\text{Prim}(A)_+\)) and
let $e_1, e_2, \ldots$ be an approximate unit of $C^*(a, b, w)$ such that $e_{n+1}e_n = e_n$ for all $n$. Notice that $e_n$ is full for large enough $n$ by the compactness of $\text{Prim}(A)$. So

\[
\tau(e_nae_n + e_nb e_n) + 2\tau(e_{n+1}) = \tau(e_nae_n + e_{n+1} + e_nb e_n + e_{n+1})
\]

\[
= \tau(e_nae_n + e_{n+1}) + \tau(e_nb e_n + e_{n+1})
\]

\[
= \tau(e_nae_n) + \tau(e_nb e_n) + 2\tau(e_{n+1}).
\]

In the first and third equalities we have used the additivity of $\tau$ on commutative $C^*$-algebras and in the middle equality the additivity of $\tau$ on pairs of full elements. Since $\tau(e_{n+1}) < \infty$, we get

\[
\tau(e_nae_n + e_nb e_n) = \tau(e_nae_n) + \tau(e_nb e_n),
\]

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using the lower semicontinuity of $\tau$ we obtain that $\tau(a + b) = \tau(a) + \tau(b)$, as desired. \qed

Remark 3.7. In view of part (i) of the previous theorem, the property of strict comparison of positive elements by traces is equivalent to “strict comparison by 2-quasitraces” and “2-quasitraces are traces” (all traces and 2-quasitraces are assumed to be lower semicontinuous). Observe also that if $A$ has strict comparison of positive elements by 2-quasitraces and its densely finite 2-quasitraces are traces, then $A$ has strict comparison of full positive elements by traces. Indeed, if $b \in (A \otimes K)_+$ is a full element the inequality $d_\tau(a) \leq (1 - \varepsilon) d_\tau(b)$ need only be verified on all densely finite traces (otherwise $\tau(b) = \infty$).

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 from the introduction. The proof is preceded by a number of preparatory results.

**Theorem 4.1.** Let $A_1, A_2, \ldots$ be $C^*$-algebras with strict comparison by traces. Let $h \in \prod_{n=1}^\infty A_n$ be such that $h_n \sim_{T^1} 0$ for all $n \in \mathbb{N}$. Then $h \sim_{T^1} 0$.

**Proof.** Let $h \in \prod_{n=1}^\infty A_n$ be such that $h_n \sim_{T^1} 0$ for all $n$. First, let us show how to reduce ourselves to the case that the $C^*$-algebras $A_n$ are $\sigma$-unital for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $h_n \in [A_n, A_n]$, there exist finite sums of commutators $\sum_{i=1}^{N_n} [x^{(k)}_i, y^{(k)}_i]$, with $x^{(k)}_i, y^{(k)}_i \in A$ for all $i, k$, converging to $h_n$ as $k \to \infty$. Consider the $C^*$-subalgebra $B_n = \text{her}(\{x^{(k)}_i, y^{(k)}_i \mid i = 1, \ldots, N_n, k \in \mathbb{N}\})$. Then $B_n$ has strict comparison by traces and is $\sigma$-unital. Furthermore, $h_n \in [B_n, B_n]$. To prove the theorem, it suffices to show that $h \sim_{T^1} 0$ in $\prod_{n=1}^\infty B_n$. Thus, from this point on, we assume that the $C^*$-algebras $A_1, A_2, \ldots$ are all $\sigma$-unital.

Let us set $\prod_{n=1}^\infty A_n = A$. As before, let $h \in A$ be such that $h_n \sim_{T^1} 0$ for all $n$. We may assume that $\|h\| \leq 1$. Let us suppose, for the sake of contradiction, that $\mu(h) \neq 0$ for some trace $\mu$ on $A$ of norm 1. Notice then that $(h_n)_+ \sim_{T^1} (h_n)_-$ for all $n \in \mathbb{N}$, but $\mu(h_+) \neq \mu(h_-)$. We wish, however, to find positive elements $a_n, b_n \in A_n$ agreeing on all traces in $T(A_n)$ (i.e., l.s.c. and $[0, \infty]$-valued) while at the same time $\mu((a_n)_n) \neq \mu((b_n)_n)$. Let us show how to achieve this: Assume, without loss of generality, that $\mu(h) > 0$. Set $\mu(h) = \delta$. Fix $n \in \mathbb{N}$. Let $(e^{(k)}_n)_i$ be an approximate
unit of $A_n$ such that $e^{(i+1)}_n e^{(i)}_n = e^{(i)}_n$ for all $i$. We can find $h^{(i)}_n \in \text{her}(e^{(i)}_n)$ for $i$ large enough such that $\|h^{(i)}_n - h_n\| < \frac{\varepsilon}{2}$ and $h^{(i)}_n \sim_{T} 0$ in her$(e^{(i)}_n)$. (This is achieved as follows: first, sufficiently approximate $h_n$ by a finite sum of commutators; next, multiply the elements in these commutators by $e^{(i)}_n$ and let $i \to \infty$.) Let

$$a_n := (h^{(i)}_n)_+ + e^{(i+1)}_n,$$

$$b_n := (h^{(i)}_n)_- + e^{(i+1)}_n.$$ 

Let $\tau \in T(A_n)$. Suppose first that $\tau(e^{(i+1)}_n) < \infty$. Then $\tau$ is bounded on her$(e^{(i)}_n)$, and so $\tau(h^{(i)}_n) = 0$, because $h^{(i)}_n \sim_{T} 0$ in her$(e^{(i)}_n)$. Hence,

$$\tau(a_n) = \tau((h^{(i)}_n)_+) + \tau(e^{(i+1)}_n)$$
$$= \tau(h^{(i)}_n)_- + \tau(e^{(i+1)}_n)$$
$$= \tau(b_n).$$

On the other hand, if $\tau(e^{(i+1)}_n) = \infty$ then again we find that $\tau(a_n) = \infty = \tau(b_n)$. Furthermore,

$$\tau((a_n - t)_+) = \tau((h^{(i)}_n)_+ + (e^{(i+1)}_n - t)_+)$$
$$= \tau((h^{(i)}_n)_- + (e^{(i+1)}_n - t)_+)$$
$$= \tau((b_n - t)_+)$$

for all $0 \leq t < 1$ and all $\tau \in T(A_n)$. (This equality is again verified both if $\tau((e^{(i+1)}_n - t)_+) < \infty$ and if $\tau((e^{(i+1)}_n - t)_+) = \infty$.)

Let $a = (a_n)_n$ and $b = (b_n)_n$. Notice that $\mu(a) - \mu(b) = \mu((h^{(i)}_n)_n) > \delta/2$. Let $K = \bigcup_{n=1}^{\infty} T(A_n)$. Let us show that $\tau(a) = \tau(b)$ for all $\tau \in K$. Consider the set

$$Q = \{ \tau \in T(A) \mid \tau((a - t)_+) = \tau((b - t)_+) \text{ for all } 0 \leq t < 1 \}.$$ 

Clearly, $\tau(a) = \tau(b)$ for all $\tau \in Q$. So it suffices to show that $K \subseteq Q$. By our construction of $a$ and $b$, we have $\bigcup_{n=1}^{\infty} T(A_n) \subseteq Q$. We will be done once we have shown that $Q$ is closed in $T(A)$. Suppose that $\lambda \tau \to \tau$ in $T(A)$, with $\tau_{\lambda} \in Q$ for all $\lambda$. Let $0 \leq t < 1$ and choose $t < t' < 1$. Then

$$\tau((a - t')_+) \leq \liminf_{t' \to t} \tau_{\lambda}((a - t')_+) \leq \limsup_{t' \to t} \tau_{\lambda}((b - t')_+) \leq \tau((b - t)_+).$$

Passing to the supremum over all $t' > t$ on the left, we get that $\tau((a - t)_+) \leq \tau((b - t)_+)$. By symmetry, we also have $\tau((b - t)_+) \leq \tau((a - t)_+)$. Thus, $\tau \in Q$ as desired.

Let us now show that $A$ has strict comparison of positive elements by $K$ (as defined in Definition [3.3]). Let $c, d \in (A \otimes K)_+$ be such that $d_{\tau}(c) \leq (1 - \gamma) d_{\tau}(d)$ for all $\tau \in K$ and some $\gamma > 0$. In order to show that $[c] \leq [d]$, it suffices to show that $[(c - \varepsilon)_+] \leq [d]$ for all $\varepsilon > 0$. But, for each $\varepsilon > 0$ and $\delta > 0$, we have $(c - \delta)_+ \sim_{C_u} c' \in M_N(A)$ and $(d - \delta)_+ \sim_{C_u} d' \in M_N(A)$ for some $N > 0$. Thus, applying Lemma [3.3], we may reduce the proof to the case that $c, d \in M_N(A)$ for
some $N \in \mathbb{N}$. Let us assume this. Let us fix $\varepsilon > 0$. Again by Lemma 3.3 there exists $\delta > 0$ such that
\[
d_\tau((c - \varepsilon)_+) \leq \left(1 - \frac{\gamma}{2}\right)d_\tau((d - \delta)_+), \text{ for all } \tau \in K.
\]
Since $M_N(A) \cong \prod_n M_N(A_n)$, we can write $c = (c_n)_n$ and $d = (d_n)_n$, with $c_n, d_n \in M_N(A_n)$ for all $n$. Projecting onto $A_n$, we get
\[
d_\tau((c_n - \varepsilon)_+) \leq \left(1 - \frac{\gamma}{2}\right)d_\tau((d_n - \delta)_+), \text{ for all } \tau \in T(A_n).
\]
Since the $C^*$-algebra $A_n$ has strict comparison of positive elements by traces, we get that $[(c_n - \varepsilon)_+] \leq [(d_n - \delta)_+]$. Thus, $(c_n - 2\varepsilon)_+ = x_n^*x_n$ and $x_nx_n^* \in \text{her}((d_n - \delta)_+)$ for some $x_n \in A_n$. Then $(c - 2\varepsilon)_+ = x^*x$ and $xx^* \leq Md$ for some $M > 0$, where $x = (x_n)_n$. Hence, $[(c - 2\varepsilon)_+] \leq [d]$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we get $[c] \leq [d]$, as desired.

We now know that $\tau(a) = \tau(b)$ for all $\tau \in K$ and that $A$ has strict comparison by $K$. By Lemma 3.3 (ii), we conclude that $\tau(a) = \tau(b)$ for all $\tau \in T(A)$. But this contradicts that $\mu(a) \neq \mu(b)$, which completes the proof.

Essentially the same proof, with some modifications, yields the following theorem:

**Theorem 4.2.** Let $A_1, A_2, \ldots$ be $C^*$-algebras with $\text{Prim}(A_n)$ compact and with strict comparison of full positive elements by traces. Let $h \in \prod_{n=1}^\infty A_n$ be such that $h_n \sim_{T^*} 0$ for all $n \in \mathbb{N}$. Then $h \sim_{T^*} 0$.

**Proof.** Let us sketch the necessary modifications to the proof of Theorem 4.1 that yield a proof of the present theorem: As before, we can reduce to the case that the $C^*$-algebras $A_1, A_2, \ldots$ are $\sigma$-unital. To this end, we use that $\text{Prim}(A_n)$ is if and only if there exist $c_n \in (A_n)_+$ and $\varepsilon > 0$ such that $(c_n - \varepsilon)_+$ is full. Now defining $B_n = \text{her}((c_n, x_i^{(k)}, y_i^{(k)} | i, k))$, we guarantee that $B_n$ has compact primitive spectrum for all $n$ and is $\sigma$-unital. Next, the elements $a_n$ and $b_n$ in the first part of the proof are constructed as before, except that we take care that they be full elements. This is possible since the elements of the approximate unit $(e_n^{(i)})_i$ are full for large enough $i$. The definitions of the sets $K$ and $Q$ remain unchanged, and again we find that $K \subseteq Q$ and that $Q$ is closed. In the next segment of the proof of Theorem 4.1 it is shown that $A$ has strict comparison of positive elements by $K$. The same arguments can be used to show that, in the present case, the strict comparison by $K$ holds for full positive elements; i.e., assuming that $c$ and $d$ are full. We finish the proof as before, now relying on Lemma 3.4 rather than Lemma 3.3.$\Box$

We deduce from the previous theorems the following corollaries:

**Corollary 4.3.** There exists $N \in \mathbb{N}$ such that if $A$ is a $C^*$-algebra with strict comparison of positive elements by traces and $h \in A$ is such that $h \sim_{T^*} 0$, then
\[
\|h - \sum_{i=1}^N [x_i, y_i]\| \leq \frac{1}{2}\|h\|
\]
for some $x_i, y_i \in A$ such that $\|x_i\| \cdot \|y_i\| \leq \|h\|$ for all $i$.

**Proof.** Let us suppose for the sake of contradiction that no such $N$ exists. Then there exist $C^*$-algebras $A_1, A_2, A_3, \ldots$ with strict comparison by traces and contractions $h_n \in A_\infty$ such that $h_n \sim_{Tr} 0$ and the distance from $h_n$ to elements of the form $\sum_{i=1}^n [u_i, v_i]$, with $\|u_i\|, \|v_i\| \leq 1$ for all $i$, is at least $1/2$ for all $n \in \mathbb{N}$. Let $h = (h_n) \in \prod_{n=1}^{\infty} A_n$. By Theorem 4.4, $h \sim_{Tr} 0$. Hence, $\|h - \sum_{j=1}^N [x_j, y_j]\| < 1/2$ for some $x_j, y_j \in \prod_{n=1}^{\infty} A_n$. Increasing $N$ if necessary, we may assume that $\|x_j\|, \|y_j\| \leq 1$ for all $j = 1, \ldots, N$. We get a contradiction projecting onto the $N$-th coordinate. \hfill \Box

The same proof, now relying on Theorem 4.2, yields the following corollary:

**Corollary 4.4.** There exists $N' \in \mathbb{N}$ such that if $A$ is a $C^*$-algebra with $\text{Prim}(A)$ compact and with strict comparison of full positive elements by traces, and $h \in A$ is such that $h \sim_{Tr} 0$, then

$$\|h - \sum_{i=1}^{N'} [x_i, y_i]\| \leq \frac{1}{2}\|h\|$$

for some $x_i, y_i \in A$ such that $\|x_i\| \cdot \|y_i\| \leq \|h\|$ for all $i$.

From these corollaries we deduce the theorem on traces of products and ultraproducts stated in the introduction:

**Proof of Theorem 4.3.** Imitate the proof of [Oza13, Theorem 8], now relying on Corollary 4.3 or Corollary 4.4 instead of on [Oza13, Theorem 6]. \hfill \Box

In order to obtain finite commutator bounds for a pure $C^*$-algebra whose 2-quasitraces are traces, we intend to apply Theorem 4.2. We have already shown that condition (ii) of that theorem is met by this class of $C^*$-algebras (in Corollary 4.3). In the next lemmas we establish the existence of a sequence of pairwise orthogonal positive elements as in Theorem 4.2 (i).

Recall that, given positive elements $a$ and $b$, by $b \preceq a$ we mean that $b = x^*x$ and $xx^* \in \text{her}(a)$ for some $x \in A$.

**Lemma 4.5.** Let $A$ be a pure $C^*$-algebra and $a, b \in (A \otimes K)_+$. If $d_\tau(b) \leq \gamma d_\tau(a)$ for all $\tau \in \text{QT}(A)$ and some $\gamma < 1/2$ then $b \preceq a$.

**Proof.** The proof follows closely that of [BRT+12, Theorem 4.4.1], but we take care to remove the assumption on finite quotients needed there. First, using functional calculus, let us find $b_i, b'_i, b''_i \in C^*(b)_+$, with $i = 1, 2, \ldots$, such that

1. $b'_i b_i = b_i$ and $b''_i b'_i = b'_i$ for all $i$,
2. $b''_i \perp b''_j$ for all $i, j$ such that $i \neq j$ and $i - j$ is even,
3. $b = \sum_{i=1}^{\infty} b_i$.

Since $b \preceq \bigoplus_{i=1}^{\infty} \frac{1}{i} b_i$, it suffices to show that $\bigoplus_{i=1}^{\infty} \frac{1}{i} b_i \preceq a$. Let us prove this.

We have

$$\sum_{i=1}^{\infty} d_\tau(b''_i) \leq \sum_{i=1}^{\infty} d_\tau(b'_i) + \sum_{i=1}^{\infty} d_\tau(b''_{2i-1}) \leq 2d_\tau(b) \leq 2\gamma d_\tau(a).$$
Since $A$ is pure, we can choose $d \in (A \otimes \mathbb{K})_+$ such that $d_\tau(b''_1 \oplus d) \leq \gamma_1 d_\tau(a)$ for some $2\gamma < \gamma_1 < 1$, and $[b''_1] \leq [k[d]]$ for some $k \in \mathbb{N}$. (E.g., choose $d$ such that $N[d] \leq [a] \leq (N + 1)[d]$ with $N$ large enough.) It follows by the strict comparison property of $A$ that $[b''_1] \leq [a]$. Since $[b''_1] \ll [b''_1]$ (where $\ll$ is the relation of compact containment in $\Cu(A)$), there exists $\varepsilon > 0$ such that $[b''_1] \leq [(d - \varepsilon)_+].$ Let $d'' = (d - \varepsilon)_+.$ We then have that $b''_1 \oplus d'' \preceq (a - \delta)_+$ for some $\delta > 0.$ Let $v \in (A \otimes \mathbb{K})^{**}$ be a partial isometry implementing this subequivalence. Let $c'_1, e \in \her((a - \delta)_+)$ be given by $c'_1 = vb''_1 v^*$ and $e = vd'' v^*.$ Let $g_\delta(a) \in C^*(a)_+$ be strictly positive and such that $g_\delta(a)(a - \delta)_+ = (a - \delta)_+.$ Finally, set $a_1 = g_\delta(a) - c'_1.$ Then
\[\sum_{i=2}^{\infty} d_\tau(b''_1) + d_\tau(c'_1) \leq 2\gamma d_\tau(a) \leq 2\gamma d_\tau(a_1) + d_\tau(c'_1),\]
for all $\tau \in \QT(A).$ If $d_\tau(c'_1) < \infty,$ we get
\[\sum_{i=2}^{\infty} d_\tau(b''_1) \leq 2\gamma d_\tau(a_1),\]
Suppose that $d_\tau(c'_1) = \infty.$ Then $d_\tau(b''_1) = d_\tau(c'_1) = \infty.$ Since $[b''_1] \leq [k[d''].$ we also have $d_\tau(e) = d_\tau(d') = \infty.$ But $a_1 e = (g_\delta(a) - c'_1)e = e.$ Hence, $d_\tau(a_1) = \infty.$ So again we have \((4.1)\). Let $c_1 = vb''_1 v^*$ and notice that $a_1 \perp c_1.$ We can repeat the same arguments, now finding positive elements $c_2, a_2 \in \her(a_1)$ such that $b_2 \sim c_2, a_2 \perp c_2$ and $\sum_{i=2}^{\infty} d_\tau(b''_1) < 2\gamma d_\tau(a_2).$ Continuing this process ad infinitum, we obtain $c_1, c_2, \ldots \in \her(a)$ such that $b_i \sim c_i$ for all $i$ and $c_1 \perp c_j$ for all $i \neq j.$ Hence, \[\bigoplus_{i=1}^{\infty} b_i \sim \sum_{i=1}^{\infty} c_i \in \her(a),\] which proves the lemma. \hfill \Box

The previous lemma implies that in a pure $C^*$-algebra the ordered semigroup $W(A)$ is hereditary in $\Cu(A).$ This will not be needed later on but has independent interest. Recall that $W(A)$ is defined as
\[W(A) = \{[a] \in \Cu(A) \mid a \in \bigcup_{n=1}^{\infty} M_n(A)\}.
\]

**Corollary 4.6.** Let $A$ be a pure $C^*$-algebra. The ordered semigroup $W(A)$ is a hereditary (in the order-theoretic sense) subsemigroup of $\Cu(A).$

**Proof.** Let $a \in A \otimes \mathbb{K}$ and $b \in M_n(A)$ be positive elements such that $[a] \leq [b].$ By Lemma 4.5 $a \preceq b^{\oplus 3} \in M_{3n}(B).$ Hence $a \sim a' \in \her(b^{\oplus 3}) \subseteq M_{3n}(B),$ which in turn implies that $[a] = [a'] \in W(A).$ \hfill \Box

In the following lemma we make use of the abundance of soft elements in the Cuntz semigroup of a pure $C^*$-algebra (see [APT14]). It will not be necessary here to recall their definition and multiple properties. We will merely need the following fact: *Let $A$ be a pure $C^*$-algebra and $[a] \in Cu(A).$ Then there exists $[a]_s \in Cu(A)$ such that $[a]_s \leq [a], d_\tau([a]_s) = d_\tau([a])$ for all $\tau \in QT(A),$ and $[a]_s$ is exactly divisible by all $n \in \mathbb{N};$ i.e., for each $n \in \mathbb{N}$ there exists $[b] \in Cu(A)$ such that $n[b] = [a]_s.$* The proof of this fact can be extracted from [APT14] as follows: By [APT14] Theorem 7.3.11, the Cuntz semigroup of a pure $C^*$-algebra has $Z$-multiplication (in the sense
of [APT14 Definition 7.1.3]). Here $Z$ denotes the Cuntz semigroup of the Jiang-Su C*-algebra $Z$. Then $[a] = 1 \cdot [a]$, with $1 \in Z$ denoting the “soft” 1, has the desired properties. See [APT14 Proposition 7.3.16].

**Lemma 4.7.** Let $A$ be a $\sigma$-unital pure C*-algebra and $e_0 \in A_+$ a strictly positive element. Then there exist pairwise orthogonal positive elements $e_1, e_2, \ldots \in A_+$ such that $e_i \precsim e_0^5$ for all $i \geq 1$ and $e_0 \precsim e_1^{11}$.

**Proof.** Let $[e_0] \preceq [e_0]$ be the soft element associated to the Cuntz semigroup element $[e_0]$. Let us find $[f_1], [f_2], \ldots \in Cu(A)$ such that $5[f_1] = [e_0]_s$ and $2[f_{i+1}] = [f_i]$ for all $i \geq 1$. Let $f \in (A \otimes K)_+$ be given by $f = f_1 \oplus f_2 \oplus f_3 \oplus \cdots$. Then

$$d_\tau(f_i) = \frac{2}{5} d_\tau(f_{i+1}^5),$$

$$d_\tau(a) = \frac{5}{11} d_\tau(f_1^{11}),$$

$$d_\tau(f) = \frac{2}{5} d_\tau(e_0),$$

for all $\tau \in QT(A)$. Hence, by Lemma 4.5, $f_i \precsim f_{i+1}^5$, $e_0 \precsim f_1^{11}$, and $f \precsim e_0$. Let $v \in (A \otimes K)^*$ be the partial isometry implementing the comparison $f \precsim a$. Then, the positive elements $e_i = v f_i v^*$, with $i = 1, 2, \ldots$, have the desired properties. \hfill $\Box$

Let us now prove Theorem 1.1 from the introduction.

**Proof of Theorem 1.1.** We first prove that every C*-algebra $A$ as in the theorem has finite commutator bounds with no approximations. We then reduce the number of commutators to 7.

Let $h \in [A, A]$. Then $\sum_{i=1}^n x_i^{(n)} y_i^{(n)} \to h$ for some $x_i^{(n)}, y_i^{(n)} \in A$. Passing to the hereditary C*-subalgebra $\text{her}(\{x_i^{(n)}, y_i^{(n)} \mid i = 1, \ldots, k_n, n = 1, \ldots \})$ if necessary, we may assume that $A$ is $\sigma$-unital (since hereditary subalgebras of pure C*-algebras are again pure). Let $e_0 \in A_+$ be a strictly positive element. By Lemma 4.7, $A$ contains a sequence of pairwise orthogonal positive elements $(e_i)_{i=1}^\infty$ such that $e_i \precsim e_{i+1}^{11}$ for all $i$ and $e_0 \precsim e_1^{11}$. Furthermore, Corollary 4.3 is applicable to every hereditary subalgebra of $A$. Thus, Theorem 2.7 is applicable to $A$. That is, $A$ has finite commutator bounds $(n, C)$ with no approximations, for some $n \in \mathbb{N}$ and $C > 0$.

Let us now reduce the number of commutators to 7. Since $A$ is pure, we can find $b \in (A \otimes K)_+$ such that $(2n+1)[b] \leq [e_0] \leq (2n+2)[b]$. Then

$$d_\tau(b^{\otimes n}) \leq \frac{n}{2n+1} d_\tau(e_0) \text{ and } d_\tau(e_0) \leq \frac{2n+2}{5n} d_\tau(b^{\otimes 5n}).$$

Hence, by Lemma 4.5, there exists $f \in A_+$ such that $b^{\otimes n} \sim f$ and $e_0 \precsim f^{\otimes 5}$. Now let $h \in A$ be such that $h \sim_{TB} 0$. From Lemma 2.8 we obtain that $h = \sum_{i=1}^n \|x_i \cdot y_i\| + h'$, for some $h' \in \text{her}(f)$ such that $h' \sim_{TB} 0$ in $\text{her}(f)$. But $\text{her}(f) \cong M_0(\text{her}(b))$. Thus, by Theorem 2.7 (ii), $h' = [x_0, y_0] + [x_7, y_7]$. Furthermore, the sum $\sum_{i=1}^7 \|x_i\| \cdot \|y_i\|$ is bounded by $C'\|h\|$ for some $C' > 0$, as can be seen from the statements of Lemma 2.8 and Theorem 2.7. \hfill $\Box$
In Theorem [10] it is possible to reduce further the number of commutators under a variety of additional assumptions. We show in Theorem [4.10] below that if the C*-algebra is assumed to be unital, then three commutators suffice. We need a couple of lemmas.

**Lemma 4.8.** Let $d \in M(A)$ be a multiplier positive contraction satisfying that $d \oplus d \oplus d \preceq 1 - d$ in $M(A)$. Then for each $h \in A$ there exist $x,y \in A$ and $h' \in \text{her}(1 - d)$ such that $h = [x,y] + h'$ and $\|x\| \cdot \|y\| \leq 2\|h\|$.

**Proof.** We have $h = dh + (1 - d)hd + (1 - d)h(1 - d)$. Let

$$f = dh + (1 - d)hd,$$

$$e = d + hdh^* + (1 - d)hdh^*(1 - d).$$

Then $f \in \text{her}(e)$ and $e \preceq d \oplus d \oplus d \preceq 1 - d$. So we can apply Lemma 2.8 with $n = 1$ to $f$. We get $f = [x,y] + f'$, with $f' \in \text{her}(1 - d)$ and $\|x\| \cdot \|y\| \leq \|f\| \leq 2\|h\|$. Hence, $h = [x,y] + (1 - d)h(1 - d) + f'$, which proves the lemma. \qed

Let $Z_{n-1,n}$ denote the dimension drop C*-algebra:

$$Z_{n-1,n} = \{f \in M_{(n-1)n}(C[0,1]) \mid f(0) \in M_{n-1} \otimes 1_n, f(1) \in M_n \otimes 1_{n-1}\}.$$

**Proposition 4.9.** Let $A$ be a pure unital C*-algebra. Then for all $n \in \mathbb{N}$ there exists a unital homomorphism $\phi: Z_{n-1,n} \to A$.

**Proof.** Let $n \in \mathbb{N}$. Since $\text{Cu}(A)$ is almost divisible, we can find $[a] \in \text{Cu}(A)$ such that $n[a] \leq [1]$ and $[1] \leq (n+1)[(a - \varepsilon)_+]$ for some $\varepsilon > 0$. In turn, this implies that there exist pairwise orthogonal positive elements $b_1, b_2, \ldots, b_n \in A_+$ such that $(a - \frac{\varepsilon}{n})_+ \sim b_1 \sim \cdots \sim b_n$. Using functional calculus, let us modify $b_i$ so that there exists $b'_i \in \text{her}(b_i)_+$, with $b_ib'_i = b'_i$ and $b'_i \sim (a - \varepsilon)_+$ for all $i$. Set $\sum_{i=1}^n b_i = b$. Then $(1 - b)b'_i = (1 - b_i)b'_i = 0$ for all $i$. That is, $1 - b \perp b'_i$ for all $i$. We have

$$d_\tau(1 - b) + nd_\tau(b'_i) = d_\tau(1 - b) + d_\tau\left(\sum_{i=1}^n b'_i\right) \leq d_\tau(1 - b) + (n+1)d_\tau(b'_i),$$

for all $\tau \in \text{QT}(A)$. Hence, $d_\tau(1 - b) \leq d_\tau(b'_i) = \frac{1}{3}d_\tau(b'_1 \oplus b'_2 \oplus b'_3)$. By Lemma 4.5, this implies that $1 - b \preceq b'_1 \oplus b'_2 \oplus b'_3$. Let us assume now that $n = 3k$, for some $k \in \mathbb{N}$. By [RW10] Proposition 5.1, there exists a unital homomorphism from the dimension drop C*-algebra $Z_{k,k+1}$ into $A$. \qed

**Theorem 4.10.** Let $A$ be a pure C*-algebra with compact Prim(A) and with strict comparison of full positive elements by traces. Then $A$ has commutator bounds $(7, C_1)$, with no approximations, for some universal constant $C_1$. If $A$ is unital then it has commutator bounds $(3, C_2)$, with no approximations, for some universal constant $C_2$. 

Lemma 5.1. Let \( A \) be a C*-algebra. Let \( N_2 = \{ x \in A \mid x^2 = 0 \} \); i.e., \( N_2 \) denotes the set of nilpotent elements of order 2 in \( A \) (a.k.a, square zero elements).

**Proof.** Let us first show that \( A \) has finite commutator bounds. To this end, we proceed as in the proof of Theorem 1.1 but with a few small modifications. (The main difference with Theorem 1.1 being that now we only require strict comparison by traces on full positive elements.) Lemma 4.7 is applicable to \( A \), yielding a sequence of pairwise orthogonal positive elements \((e_i)_{i=1}^\infty\) such that \( e_0 \precsim e_1^{\otimes 11} \) and \( e_i \precsim e_i^{\otimes 7} \) for all \( i \). Here \( e_0 \in A_+ \) is strictly positive. The hereditary subalgebras \( \text{her}(e_i) \) have compact primitive spectrum for all \( i \) (since they are full in \( B \)). Hence, Corollary 4.4 is applicable in each of them. Now Theorem 2.9 implies that \( A \) has finite commutator bounds \((n, C)\) with no approximations.

The arguments for reducing the number of commutators to 7 in the proof of Theorem 1.1 apply here as well.

Let us now show that if \( A \) is unital then the number of commutators can be reduced to 3. By Proposition 4.9, the dimension drop C*-algebra \( Z_{n,n+1} \) maps unitaly into \( A \). By [RW10] Proposition 5.1, there exist \( f_1, \ldots, f_n \in A_+ \) such that \( f_1 \sim f_i \) for all \( i \) and \( 1 - \sum_{i=1}^n f_i \precsim (f_1 - \varepsilon)_+ \) for some \( \varepsilon > 0 \). Let us assume without loss of generality that \( n \geq 3 \). By Lemma 4.8 for each \( h \in [A, A] \) we have \( h = [x_1, y_1] + h' \), with \( h' \in \text{her}(f_1, \ldots, f_n) \). But \( \text{her}(f_1, \ldots, f_n) \cong M_n(\text{her}(f_1)) \), and \( \text{her}(f_1) \) has commutator bounds \((n, C)\) with no approximations. It follows by Theorem 2.7 that \( h' = [x_2, y_2] + [x_3, y_3] \). □

**Example 4.11.** Let \((A_i, \tau_i)\), with \( i = 1, 2, \ldots \), be unital C*-algebras with faithful tracial states. Assume that for infinitely many indices \( i_1, i_2, \ldots \) there exist unitaries \( u_{i_n} \in A_{i_n} \) such that \( \tau_{i_n}(u_{i_n}) = 0 \) for all \( n \). Let \( A = A_1 \ast A_2 \ast \cdots \) and \( \tau = \tau_1 \ast \tau_2 \ast \cdots \) be the reduced free product C*-algebra and tracial state. It is known that \( A \) is simple and \( \tau \) is its unique tracial state (by [Avi82]). Furthermore, by [Rob12] Proposition 6.3.2, \( A \) has strict comparison of positive elements by the trace \( \tau \). It follows that \( A \) is pure and, by Theorem 3.6, that the only bounded 2-quasitraces on \( A \) are the scalar multiples of \( \tau \). By Theorem 4.10 if \( h \in A \) is such that \( \tau(h) = 0 \) then \( h \) is a sum of 3 commutators.

5. **SUMS OF NILPOTENTS OF ORDER 2**

Let \( A \) be a C*-algebra. Let \( N_2 = \{ x \in A \mid x^2 = 0 \} \); i.e., \( N_2 \) denotes the set of nilpotent elements of order 2 in \( A \) (a.k.a, square zero elements).

**Lemma 5.1.** Let \( z \in N_2 \). Then \( z = [u, v] \) and \( z + z^* = [w, w] \) for some \( u, v, w \in A \).

**Proof.** We may assume that \( \|z\| = 1 \). The universal C*-algebra generated by a square zero contraction is \( M_2(C_0(0, 1)) \). Thus, there exists a homomorphism \( \phi: M_2(C_0(0, 1)) \to A \) such that \( \bar{z} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\phi} z \). So it suffices to express \( \bar{z} \) and \( \bar{z} + \bar{z}^* \) as commutators. Indeed,

\[
\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \left[ \begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix} \right],
\]

\[
\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \frac{1}{4} \left[ \begin{pmatrix} t^{\frac{1}{2}} & -t^{\frac{1}{2}} \\ t^{\frac{1}{2}} & -t^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} t^{\frac{1}{2}} & t^{\frac{1}{2}} \\ -t^{\frac{1}{2}} & -t^{\frac{1}{2}} \end{pmatrix} \right].
\]

□
The preceding lemma implies that linear span of $N_2$ is contained in $[A, A]$. In Theorem 5.3 below we show, conversely, that if $A$ is a pure unital $C^*$-algebra then every commutator is expressible as a sum of at most $14 \times 256$ order 2 nilpotents.

**Lemma 5.2.** Let $A$ be a $C^*$-algebra and $a, b \in A$.

(i) If $a, b \in N_2$ then $[a, b]$ is a sum of 3 order 2 nilpotents.

(ii) If $A = M_2(B)$ or $A = M_3(B)$ then $[a, b]$ is at most a sum of 14 order 2 nilpotents.

In both cases, the norm of the nilpotents is bounded by $C||a|| \cdot ||b||$, where $C$ is a universal constant.

**Proof.** (i) Let $a, b \in N_2$. Normalizing $a$ and $b$ if necessary we may assume that they are contractions. Then, as pointed out in [Mar02, Lemma 3.2],

$$[a, b] = (ab + aba - b - ba) + (-aba) + (b),$$

and each term on the right hand side is an order 2 nilpotent of norm at most 4.

(ii) This is proven by Marcoux in [Mar02, Theorem 5.6 (ii)], for $n = 2$, and in [Mar02, Theorem 3.5(ii)] for $n = 3$ (see also remarks after [Mar06, Theorem 5.1]). Although in the statements of these theorems Marcoux assumes that $B$ is unital, a quick inspection of the proofs reveals that this is not used. □

**Theorem 5.3.** Let $A$ be a pure unital $C^*$-algebra and $a, b \in A$.

(i) Then $[a, b]$ is a sum of at most $14 \times 256$ nilpotents of order 2. The norm of the order 2 nilpotents is bounded by $C||a|| \cdot ||b||$, where $C$ is a universal constant.

(ii) If $[a, b]$ is selfadjoint then it is a sum of $14 \times 256$ commutators of the form $[x^*, x]$ with $x \in N_2$, with $||x||^2 \leq C' ||a|| \cdot ||b||$ and $C'$ is a universal constant.

**Proof.** (i) Let us choose $s_1$ and $s_2$ such that $0 < s_1 < s_2 < 1$. Let $f_1, f_2, f_3, f_4 \in C([0, 1])_+$ be functions such that $f_1$ is supported on $[0, s_1)$, $f_2$ is supported on $(0, s_2]$, $f_3$ is supported on $(s_1, 1)$, $f_4$ is supported on $(s_2, 1]$, and $f_1 + f_2 + f_3 + f_4 = 1$. Let us regard $C([0, 1])$ embedded in $\mathbb{Z}_{2,3}$ via the map $f \mapsto f \cdot 1_6$. Further, by Proposition 4.9 $\mathbb{Z}_{2,3}$ maps unitally into $A$. In this way, we can view $f_1, f_2, f_3, f_4$ as elements of $A$. Then

$$[a, b] = \sum_{1 \leq i,j,k,l \leq 4} [f_iaf_j, f_kbfi].$$

We will show that each of the 256 terms on the right hand side is expressible as a sum of at most 14 nilpotents of order 2.

Let us examine the commutator $[f_iaf_j, f_kbfi]$. Since $\text{her}_A (f_1 + f_2 + f_3) \cong M_2(B)$ and $\text{her}_A (f_2 + f_3 + f_4) \cong M_3(B')$ for some $B, B' \subseteq A$, if not all four functions appear in $[f_iaf_j, f_kbfi]$ then by Lemma 5.2 this commutator is expressible as a sum of at most 14 order 2 nilpotents.

Let us examine the commutators $[f_iaf_j, f_kbfi]$ where all four functions appear; i.e., such that $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Let us assume that $i = 1$. If $l = 3$ or $l = 4$ then $[f_iaf_j, f_kbfi]$ is itself an order 2 nilpotent and we are done. Suppose that $l = 2$. Since we are assuming that all four indices must appear, either $k = 4$ and $j = 3$, or
k = 3 and j = 4. Suppose that k = 4 and j = 3. Since f_1af_3 and f_3bf_2 are both order 2 nilpotents, [f_1af_3, f_3bf_2] is a sum of three order 2 nilpotents by Lemma \[5.2\] (i). Suppose now that k = 3 and j = 4. The commutator is then [f_1af_4, f_3bf_2], which can be dealt with as follows:

\[
[f_1af_4, f_3bf_2] = [f_1a(f_4f_3)^{\frac{1}{2}}, (f_4f_3)^{\frac{1}{2}}bf_2] + [(f_4f_3)^{\frac{1}{2}}b(f_2f_1)^{\frac{1}{2}}, (f_2f_1)^{\frac{1}{2}}a(f_4f_3)^{\frac{1}{2}}] + [(f_2f_1)^{\frac{1}{2}}a(f_4f_3)^{\frac{1}{2}}, f_3b(f_2f_1)^{\frac{1}{2}}].
\]

Each of the commutators on the right side is a commutator of order 2 nilpotents and is thus expressible as a sum of three order 2 nilpotents. So, [f_1af_4, f_3bf_2] is expressible as a sum of 9 nilpotents of order 2.

Let us assume now that i \neq 1. As argued above, we may reduce ourselves to the case that one of the indices j, k, l is 1. On the grounds of the symmetry of our set-up, any of these cases can be dealt with just as we did above for i = 1. (Notice that only the orthogonality relations between the functions were used in our analysis; the asymmetry of the dimension drop C*-algebra played not role.) We are thus done.

(ii) Suppose that [a, b] is selfadjoint. By (i), it can be written as a sum with 14 \times 256 terms, each of the form \(z + z^\ast\), with \(z \in N_2\). In turn, each of these terms is expressible as a commutator of the form \([x^\ast, x]\), with \(x \in N_2\), by Lemma \[5.1\] □

**Theorem 5.4.** Let A be a pure unital C*-algebra whose bounded 2-quasitraces are traces. Then the sets \([A, A]\), \([A, A]\), and the linear span of \(N_2\), are equal. Each \(h \in A\) such that \(h \sim_{T_0} 0\) is expressible as a sum of 3 \times 14 \times 256 square zero elements. If \(h\) is selfadjoint then it is also expressible as sum of 3 \times 14 \times 256 commutators of the form \([x^\ast, x]\), with \(x \in N_2\).

**Proof.** The assumptions on A imply that it has strict comparison of full positive elements by traces (see Remark \[3.7\]). By Theorem \[1.10\] each \(h \in [A, A]\) is expressible as a sum of 3 commutators. Each of these commutators, in turn, is expressible as a sum of 14 \times 256 square zero elements. If \(h\) is selfadjoint, then it also expressible as a sum of 14 \times 256 terms of the form \(z + z^\ast\), with \(z \in N_2\), and each of these is a commutator \([x^\ast, x]\) with \(x \in N_2\). □

6. The kernel of the determinant map

Let us briefly recall the definition of the de la Harpe-Skandalis determinant, as defined in \[dlHS84a\]. Let A be a C*-algebra. Let GL_\infty(A) denote the infinite general linear group of A and GL_0^\infty(A) the connected component of the identity. (If A is non-unital, the general linear groups GL_n(A) are defined as the subgroup of GL_n(A^\sim) of elements of the form 1_n + x, with \(x \in M_n(A)\), and GL_\infty(A) is the direct limit.) Let \(x \in GL_\infty(A)\). Let \(\eta: [0, 1] \to GL_0^\infty(A)\) be a path such that \(\eta(0) = 1\) and \(\eta(1) = x\). Let

\[
\Delta_\eta(x) = \frac{1}{2\pi i} \text{Tr}
\left( \int_0^1 \eta'(t) \eta(t)^{-1} \right) \in A/[A, A].
\]
If $\zeta$ is another path connecting $1$ to $x$, then $\tilde{\Delta}_\eta(x) - \tilde{\Delta}_\zeta(x)$ belongs to the (additive) subgroup $\{\text{Tr}(p) - \text{Tr}(q) \mid p, q$ projections in $M_\infty(A)\}$, which we denote by $\text{Tr}(K_0(A))$. The de la Harpe-Skandalis determinant $\Delta_{\text{Tr}}(x)$ is defined as the image of $\tilde{\Delta}_\eta(x)$ in the quotient of $A/[A, A]$ by $\text{Tr}(K_0(A))$. If $x = \prod_{k=1}^n e^{ih_k}$ then $\Delta_{\text{Tr}}(x)$ can be computed to be the image of $\sum_{k=1}^n h_k$ in this quotient.

Let $U(A)$ denote the unitary group of $A$ and $U^0(A)$ the connected component of the identity. (If $A$ is non-unital, $U(A)$ is defined as the unitaries in $U(A^n)$ of the form $1 + x$, with $x \in A$.) Given unitaries $u, v \in U^0(A)$, let us denote by $(u, v)$ the multiplicative commutator $uvu^{-1}v^{-1}$. Let $DU^0(A)$ denote the derived or commutator subgroup of $U^0(A)$. It is clear that $DU^0(A)$ is contained in the kernel of $\Delta_{\text{Tr}}$ (since $\Delta_{\text{Tr}}$ is a group homomorphism with abelian codomain). In this section we prove the following theorem:

**Theorem 6.1.** Let $A = M_3(B)$, where $B$ is a unital pure C*-algebra whose bounded 2-quasitraces are traces. Then $\ker \Delta_{\text{Tr}} \cap U^0(A) = DU^0(A)$.

The proof is preceded by a number of lemmas.

**Lemma 6.2.** Let $A$ be a pure unital C*-algebra and $p \in M_m(A)$ a projection. Then there exists $h \in A_{sa}$ such that $h \sim_{\text{Tr}} p$ and $e^{ih} = (u, v)$ for some unitaries $u, v \in U^0(A)$.

**Proof.** Let $B = pM_m(A)p$. Choose $n \in \mathbb{N}$. Since $B$ is pure and unital, there exists a unital homomorphism $\phi : Z_{n^{-1}} \to B$, by Proposition 4.1. Let $e \in Z_{n^{-1}}$ be a positive element such that $\text{rank}(e(t)) = n$ for all $t \in (0, 1]$ and $\text{rank}(e(0)) = n - 1$ (so that $(n - 1)|e| \leq |1| \leq n|e|$) in the Cuntz semigroup of $Z_{n^{-1}}$. In the proof of Lemma 5.4 of [NR14] a selfadjoint element $h \in \text{her}(e)$ is constructed such that $h \sim_{\text{Tr}} 1$ (in $Z_{n^{-1}}$) and $e^{ih} = (u, v)$ for some unitaries $u, v \in U^0(\text{her}(e))$. Moving these elements with the homomorphism $\phi$, we get $e' \in B_+, h' \in \text{her}(e')_{sa}$, and $u', v' \in U^0(\text{her}(e'))$, such that $(n - 1)|e'| \leq |p| \leq n|e'|$, $h' \sim_{\text{Tr}} p$, and $e'^{ih'} = (u', v')$. Now choose $n > 2m$. Then $(2m + 1)|e'| \leq |p| \leq m|1|$, where $1$ is the unit of $A$. By Lemma 4.5, this implies that $e' \preceq 1$; i.e., there exists $x \in M_m(A)$ such that $x^*x = e'$ and $xx^* \in A$. Let $x = w|x|$ be the polar decomposition of $x$ (in $M_m(A)^{**}$). Then the selfadjoint $h'' = wh'u^*$, and the unitaries $u'' = wwu^*$ and $v'' = wvv^*$ have the desired properties. \hfill $\square$

**Lemma 6.3.** Let $A$ be a pure unital C*-algebra. Let $u \in U^0(A)$ be such that $\Delta_{\text{Tr}}(u) = 0$. Then $u = \prod_{j=1}^M (u_j, v_j) \cdot e^{ih}$ for some $u_1, v_1, \ldots, u_M, v_M \in U^0(A)$ and some $h \in A_{sa}$ such that $h \sim_{\text{Tr}} 0$.

**Proof.** Since $u \in U^0(A)$ we have $u = \prod_{j=1}^n e^{ih_j}$, where $h_1, \ldots, h_n \in A_{sa}$, and since $\Delta_{\text{Tr}}(u) = 0$ we also have $\sum_{j=1}^n h_j \sim_{\text{Tr}} p - q$ for some projections $p, q \in M_m(A)$. Applying the previous lemma, we can write

$$u = (u', v')(u'', v'') \prod_{j=1}^{n+2} e^{ih_j},$$

where now $\sum_{j=1}^{n+2} h_j \sim_{\text{Tr}} 0$. It thus suffices to prove the lemma for the unitary $\prod_{j=1}^{n+2} e^{ih_j}$. Let $N \in \mathbb{N}$. Then
\[
\prod_{j=1}^{n+2} e^{ih_j} = \prod_{j=1}^{n+2} (e^{ih_j/N})^N = \prod_{j=1}^{M} (u_j, v_j) \cdot \left( \prod_{j=1}^{n+2} e^{ih_j/N} \right)^N.
\]

Here the commutators $(u_j, v_j)$ result simply from rearranging the factors of the first product. In particular, $u_j, v_j \in U^0(A)$ for all $j$. We can choose $N$ large enough so that $\prod_{j=1}^{n+2} e^{ih_j/N} = e^{ih}$, for some $h \in A_{sa}$. By [dlHS84a, Lemma 3(b)], the trace of the logarithm of a product of $n+2$ unitaries belonging to a sufficiently small neighborhood of the identity is equal to the sum of the trace of the logarithm of each of the unitaries. Thus, for $N$ large enough, we have with $h \sim_{\text{Tr}} \sum_{j=1}^{n+2} h_j/N \sim_{\text{Tr}} 0$.

Therefore,
\[
u = \prod_{j=1}^{M} (u_j, v_j) \cdot e^{iNh},
\]

with $h \in A_{sa}$ such that $h \sim_{\text{Tr}} 0$. This proves the lemma

\[\square\]

**Lemma 6.4.** Let $m \in \mathbb{N}$, $R > 0$, and $\varepsilon > 0$. Then there exists $M \in \mathbb{N}$ with the following property: If $A$ is a C*-algebra and $a_1, \ldots, a_m \in A_{sa}$ are such that $\|a_i\| \leq R$ for all $i$, then there exist $x_1, y_1, \ldots, x_M, y_M \in A_{sa}$ and $c \in A_{sa}$ such that
\[
e^{i(a_1 + \cdots + a_m)} = \prod_{k=1}^{M} (e^{ix_k}, e^{iy_k}) \cdot e^{ia_1} \cdots e^{ia_m} \cdot e^{ic},
\]

\[
\|x_k\|, \|y_k\| \leq \varepsilon \cdot \sum_{j=1}^{m} \|a_j\|, \text{ for } k = 1, \ldots, M,
\]

\[
\|c\| \leq \varepsilon \cdot \sum_{j=1}^{m} \|a_j\|, \text{ and } c \sim_{\text{Tr}} 0.
\]

**Proof.** By [Suz77, Theorem 2], for $\lambda \in \mathbb{R}$ such that $|\lambda| < \frac{1}{mR} \cdot (\ln 2 - \frac{1}{2})$ we have
\[
e^{i\lambda a_1 + \cdots + i\lambda a_m} = e^{i\lambda a_1} e^{i\lambda a_2} \cdots e^{i\lambda a_m} \cdot e^{ic(\lambda)},
\]

where $\|c(\lambda)\| \leq L \max_j \|a_j\|^2$ and the constant $L > 0$ is dependent on $m$ and $R$ only. Furthermore, by [dlHS84a, Lemma 3(b)], for $|\lambda|$ small enough (depending only on $m$ and $R$), the trace of the logarithm of the right side is equal to $\sum_{j=1}^{m} i\lambda \text{Tr}(a_j) + \text{Tr}(c(\lambda))$; whence $c(\lambda) \sim_{\text{Tr}} 0$. Now let us choose $\lambda = \frac{1}{N}$, with $N \in \mathbb{N}$. Then for $N$ large enough (depending only on $m$ and $R$) we have that
\[
e^{ia_1/N + \cdots + ia_m/N} = e^{ia_1/N} e^{ia_2/N} \cdots e^{ia_m/N} \cdot e^{cN},
\]

(6.1)
where $c_N \in A_{sa}$ satisfies that $c_N \sim_{Tr} 0$ and

$$\|c_N\| \leq \frac{L}{N^2} \cdot \max_i \|a_i\|^2 \leq \frac{LR}{N^2} \cdot \max_i \|a_i\|.$$  

Raising to the $N$ on both sides of (6.1) we get

$$e^{ia_1 + \ldots + i a_m} = (e^{a_1/N} e^{a_2/N} \ldots e^{a_m/N} e^{c_N})^N$$

$$= \prod_{k=1}^M (e^{ix_k}, e^{iy_k}) \cdot e^{a_1} e^{a_2} \ldots e^{a_m} e^{Nc_N}.$$  

The commutators $(e^{ix_k}, e^{iy_k})$ in the last expression result from rearranging the terms in $(e^{a_1/N} e^{a_2/N} \ldots e^{a_m/N} \cdot e^c)^N$. Notice that $M$ depends only on $N$ and $m$. Choosing $N > \frac{1}{\varepsilon}$ we arrange for $\|x_k\|, \|y_k\| \leq \varepsilon \max_j \|a_j\|$. Choosing $N > \frac{LR}{\varepsilon}$ we also get that $\|Nc_N\| \leq \frac{LR}{N} \cdot \max_i \|a_i\| \leq \varepsilon \max_j \|a_j\|$.

**Proposition 6.5.** There exists $N \in \mathbb{N}$ such that the following holds: If $A = M_2(B)$, where $B$ is a pure $C^*$-algebra with compact Prim$(B)$ and whose bounded 2-quasitraces are traces, and $h \in A_{sa}$ is such that $h \sim_{Tr} 0$ and $\|h\| \leq 1$, then

$$e^{ih} = \prod_{j=1}^N (u_j, v_j) \cdot e^{ic}$$

for some $c \in A_{sa}$ and some $u_1, v_1, \ldots, u_N, v_N \in U^0(A)$ such that $c \sim_{Tr} 0$, $\|c\| \leq \frac{1}{2} \|h\|$, and $\|u_j - 1\|, \|v_j - 1\| \leq \|e^{ih} - 1\|^{\frac{1}{2}}$, for all $j = 1, \ldots, N$.

**Proof.** Let $h \in A_{sa}$ be such that $h \sim_{Tr} 0$ and $\|h\| \leq 1$. By Theorem 4.10, $h$ is a sum of 7 commutators, and by Lemma 5.2 (ii), each of these commutators is a sum of at most 14 nilpotents of order 2. Furthermore, since $h$ is selfadjoint we can assume that these nilpotent elements have the form $[z^*, z]$, with $z^2 = 0$. We thus have $h = \sum_{k=1}^m [z_k^*, z_k]$, for some $z_1, \ldots, z_m \in A$ such that $z_k^2 = 0$. Here $m = 7 \times 14$ and $\|z_k\|^2 \leq C' \|h\|$ for all $k$, where $C'$ is a universal constant. However, dividing the elements $z_k$ by a large natural number ($> C'^2$) and enlarging $m$, we can assume instead that $\|z_k\|^2 \leq \|h\|$ for all $k = 1, \ldots, m$. Let us assume this. Notice that now $\|[z_k^*, z_k^*]\| \leq \|h\| \leq 1$. By the previous lemma applied with $\varepsilon = \frac{1}{2}$, $m$, and $R = \frac{1}{2}$, we have

$$e^{ih} = e^{[z_1, z_1^*] + \cdots + [z_m, z_m^*]}$$

$$= \prod_{k=1}^M (e^{ix_k}, e^{iy_k}) \cdot \prod_{k=1}^m e^{[z_k^*, z_k]} \cdot e^{ic},$$

where $x_1, y_1, \ldots, x_M, y_M \in A$ and $c \in A_{sa}$ are such that $c \sim_{Tr} 0$, $\|c\| \leq \frac{1}{2} \|h\|$, and $\|x_k\|, \|y_k\| \leq \frac{1}{2} \|h\|$ for all $k$. Notice that $\|e^{ix_k} - 1\|, \|e^{iy_k} - 1\| \leq \|e^{ih} - 1\|^{\frac{1}{2}}$ for all $k = 1, \ldots, M$. It remains to show that the terms $e^{[z_k, z_k^*]}$ are also expressible.
as commutators. By [NR14] Lemma 2.4 (ii), for all \(z \in A\) such that \(z^2 = 0\) and \(\|z\|^2 \leq \frac{\pi}{2}\) we have \(e^{i[z^* z]} = (u, v)\) for some unitaries \(u, v \in U^0(A)\) such that \(\|u - 1\|, \|v - 1\| \leq \|e^{i[z^* z]} - 1\|^\frac{1}{2}\). Applying this to each \(z_k\), we get \(e^{i[z_k z_k]} = (u_k, v_k)\), where \(u_k, v_k \in U^0(A)\) are such that

\[
\|u_k - 1\|, \|v_k - 1\| \leq \|e^{i[z_k z_k]} - 1\|^\frac{1}{2} \leq \|e^{ih} - 1\|^\frac{1}{2}
\]

for all \(k = 1, \ldots, m\). \(\square\)

**Lemma 6.6.** Let \(A\) be a pure \(C^*\)-algebra with strictly positive element \(d \in A_+\). Let \(\varepsilon > 0\). Then there exist pairwise orthogonal positive elements such that \(a_i \sim b_i\) and \(b_i \preceq a_{i+1} + b_{i+1}\) for all \(i\), and \([(d - \varepsilon)_+] \leq 11[a_1]\).

Proof. Let \([d]_s \in Cu(A)\) denote the “soft” element associated to \([d]\). Using that this element is infinitely divisible, we can find \([c_1], [c_2], [c_3], \ldots \in Cu(A)\) such that \([c_i] = 2[c_{i+1}]\) for all \(i\) and \(10[c_1] = [d_0]\). Let us pick the representatives \(c_i\) such that \(\|c_i\| \rightarrow 0\). Since \([d] \leq 11[c_1]\), there exists \(\delta_1 > 0\) such that \([(d - \varepsilon)_+] \leq 11[(c_1 - \delta_1)_+]\).

Let us continue choosing \(\delta_2, \delta_3\) such that \((c_i - \delta_i)_+ \preceq (c_{i+1} - \delta_{i+1})_+ + (c_{i+1} - \delta_{i+1})_+\) for all \(i\). Consider the element

\[c = \bigoplus_{i=1}^{\infty} (c_i + c_i) \in (A \otimes K)_+\].

Notice that \(d_\tau(c) \leq \frac{2}{5} d_\tau(d)\) for all \(\tau \in QT(A)\). By Lemma 4.5 \(c \preceq d\). Let \(v \in \text{her}(d)^{**}\) denote the partial isometry implementing this subequivalence. Let us define

\[a_i = v((c_i - \delta_i)_+ \oplus 0)v^*, \]
\[b_i = v(0 \oplus (c_i - \delta_i)_+)v^*, \]

for all \(i\). The elements \(a_1, b_1, a_2, b_2, \ldots\) are pairwise orthogonal. They satisfy \(a_i \sim b_i\) and \(b_i \preceq a_{i+1} + b_{i+1}\) for all \(i\) and \([(d - \delta\varepsilon)_+] \leq 11[(c_1 - \delta_1)_+] = 11[a_1]\), as desired. \(\square\)

**Lemma 6.7.** Let \(A\) be a pure \(C^*\)-algebra with \(\text{Prim}(A)\) compact and whose bounded 2-quasitraces are traces. Let \(a_1, b_1, a_2, b_2, \ldots \in A_+\) be pairwise orthogonal positive elements as in the previous lemma. Let \(h \in \text{her}(a_1)\) be a selfadjoint element such that \(h \sim_{\mathbb{T}} 0\). Then \(e^{ih} \in DU^0(A)\).

Proof. The proof uses the the multiplicative version of “Fack’s technique”, as applied in [NR14] Lemma 6.5.

Since \(e^{ih} = (e^{ih/N})^N\) for all \(N \in \mathbb{N}\) we can assume that \(\|h\| < \delta\) for any prescribed \(\delta\). Let us choose \(\delta\) such that [WSS84] Proposition 5.18 is applicable to any unitary within a distance of at most \(\delta\) of 1.

By Proposition 6.5 applied in \(\text{her}(a_1 + b_1)\), there exist unitaries \(u_i(1), v_i(1) \in U^0(\text{her}(a_1 + b_1))\), for \(i = 1, \ldots, N\), such that

\[e^{ih} = \prod_{i=1}^{N} (u_i(1), v_i(1)) e^{ih_i'},\]
where $h'_1 \in \text{her}(a_1 + b_1)_{sa}$, $h'_1 \sim_{Tr} 0$, and $\|h_1\|' < \frac{\|h\|}{\sqrt{2}}$. Next, by [dlHS84b, Proposition 5.18] (see also [NR14, Lemma 6.4]), there exist unitaries $w_1^{(1)}, x_1^{(1)}, w_2^{(1)}, x_2^{(1)}$ in $U^0(\text{her}(a_1 + b_1))$ such that
\[
edih = (w_1^{(1)}, x_1^{(1)})(w_2^{(1)}, x_2^{(1)})e^{ih'_1},
\]
and $h'_1 \in \text{her}(b_2)_{sa}$. Finally, by [dlHS84b, Lemma 5.17] applied in $\text{her}(b_1 + a_2 + b_2)$, we have $e^{ih''_1} = (y^{(1)}, z^{(1)})e^{ih_1}$, with $y^{(1)}, z^{(1)} \in U^0(\text{her}(b_1 + a_2 + b_2))$ and $h_2 \in \text{her}(a_2 + b_2)_{sa}$. Next, we apply again Proposition 6.5 in $\text{her}(a_2 + b_2)$:
\[
edih = \prod_{i=1}^N (u_i^{(2)}, v_i^{(2)}) e^{ih'_2},
\]
with $h'_2 \sim_{Tr} 0$ and $\|h'_2\| < \frac{\|h\|}{\sqrt{2}}$, followed by applications of [dlHS84b, Proposition 5.18] and [dlHS84b, Lemma 5.17]:
\[
edih = (w_1^{(2)}, x_1^{(2)})(w_2^{(2)}, x_2^{(2)})e^{ih''_2},
\]
where $h''_2 \in \text{her}(b_2)_{sa}$ and $h'_2 \sim_{Tr} 0$, and $h_3 \in \text{her}(a_3 + b_3)_{sa}$ and $h_3 \sim_{Tr} 0$. Continuing this strategy we construct, for each $n \in \mathbb{N}$,
1. unitaries $u_1^{(n)}, v_1^{(n)}, \ldots, u_N^{(n)}, v_N^{(n)}$ and $w_1^{(n)}, x_1^{(n)}, w_2^{(n)}, x_2^{(n)}$ in $U^0(\text{her}(b_n))$,
2. unitaries $y^{(n)}, z^{(n)}$ in $U^0(\text{her}(b_n + a_{n+1} + b_{n+1}))$, and
3. a self-adjoint $h_n \in \text{her}(e_n)_{sa}$

such that $h_n \sim_{Tr} 0$ and
\[
edih = \prod_{k=1}^{n-1} \left( \prod_{i=1}^N (u_i^{(k)}, v_i^{(k)})(w_1^{(k)}, x_1^{(k)})(y^{(k)}, z^{(k)}) \right) e^{ih_n}.
\]
Notice that $h_n \to 0$. Thus, the above formula yields and expression of $e^{ih}$ as an infinite product of commutators. By the pairwise orthogonality of the $a_n$s and $b_n$s, we can gather the terms of this infinite product into subsequences, each of them equal to a finite product of commutators. This is done in the same manner as in the proof of [dlHS84b, Proposition 6.1]. First, we group together the commutators $(y^{(k)}, z^{(k)})$ in the product above:
\[
edih = \prod_{k=1}^{n-1} \left( \prod_{i=1}^N (u_i^{(k)}, v_i^{(k)})(w_1^{(k)}, x_1^{(k)})(y^{(k)}, z^{(k)}) \right) e^{ih_n},
\]
where
\[
\begin{align*}
\hat{u}_i^{(k)} &= (y^{(k-1)}, z^{(k-1)})u_i^{(k)}(y^{(k-1)}, z^{(k-1)})^{-1}, \\
\hat{v}_i^{(k)} &= (y^{(k-1)}, z^{(k-1)})v_i^{(k)}(y^{(k-1)}, z^{(k-1)})^{-1}, \\
\hat{w}_j^{(k)} &= (y^{(k-1)}, z^{(k-1)})w_j^{(k)}(y^{(k-1)}, z^{(k-1)})^{-1}, \\
\hat{x}_j^{(k)} &= (y^{(k-1)}, z^{(k-1)})x_j^{(k)}(y^{(k-1)}, z^{(k-1)})^{-1}, 
\end{align*}
\]
for all $i = 1, \ldots, N, j = 1, 2,$ and $k = 2, \ldots, n$. Since $(y^{(k)}, z^{(k)}) - 1$ belongs to
$\text{her}(b_k) \cap \text{her}(a_k + b_k)$, the modified unitaries $\tilde{u}_i^{(k)}, \tilde{v}_i^{(k)}, \tilde{w}_j^{(k)}, \tilde{x}_j^{(k)}$ continue to
belong to $U^0(\text{her}(a_k + b_k))$. Therefore,

$$
(1) \quad \prod_{k=1}^{N}(\prod_{i=1}^{N}(\tilde{u}_i^{(k)}, \tilde{v}_i^{(k)})) \cdot (w_2^{(k)}, x_2^{(k)}) \cdot (w_2^{(k)}, x_2^{(k)})
$$

is a product of $N + 2$ commutators,

$$
(2) \quad \prod_{k=1}^{N}(y^{(2k-1)}), z^{(2k-1)} \text{ is a single commutator},
$$

$$
(3) \quad \prod_{k=1}^{N}(y^{(2k)}), z^{(2k)} \text{ is a single commutator}.
$$

We thus arrive at an expression of $e^{ih}$ as a product of $N + 4$ commutators. \hfill \Box

Proof of Theorem 6.1. It is clear that every unitary in $U^0(A)$ is in $\Delta_{T_1}$. To prove the converse, it suffices, by Lemma 6.3, to show that $e^{ih}$, with $h \sim_{T_1} 0$ is a finite product of commutators. Writing $e^{ih} = (e^{ih/N})^N$, we can assume that $\|h\| < \delta$ for any prescribed $\delta$. We will specify how small should $\delta$ be soon. By [dlHS84a], Proposition 5.18, $e^{ih}$ is a product of commutators times $e^{ih'}$, with $h' \in \text{her}(e_{1,1})$ and $h' \sim_{T_1} 0$. Let us choose $d \in \text{her}(e_{1,2} + e_{3,3})$ and $\varepsilon > 0$ such that $N[d] \leq [e_{1,2} + e_{3,3}] \leq (N + 1)[(d - 2\varepsilon)_+]$, for $N$ large enough (how large to be specified soon). Let us find pairwise orthogonal elements $d_1, \ldots, d_N \in \text{her}(e_{2,2} + e_{3,3})$ such that $d_i \sim (d - \varepsilon)_+$ for all $i$. Since $[e_{1,1}] \leq N[(d - 2\varepsilon)_+]$, we have $e_{1,1} \preceq (\sum_i d_i - \varepsilon)_+$. By repeated application of [dlHS84a], we can express $e^{ih'}$ as a finite product of commutators times $e^{ih''}$, with $h'' \in \text{her}((d_1 - \varepsilon)_+)$ selfadjoint such that $h'' \sim_{T_1} 0$ (in her$((d_1 - \varepsilon)_+)$). Let us choose a sequence $a_1, b_1, a_2, b_2, \ldots \in \text{her}(d_2 + \cdots + d_1)$ as in Lemma 6.3. Then $12[d_1] = [d_2 + \cdots + d_1] \leq 11[a_1]$. Thus, $(d_1 - \varepsilon)_+ \preceq a_1$. We can therefore express $e^{ih''}$ as a commutator times $e^{ih'''}$, with $h''' \in \text{her}(a_1)$ and $h''' \sim_{T_1} 0$. By Lemma 6.7, $e^{ih'''}$ is a finite product of commutators. \hfill \Box

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