SOLVING THE SELESNICK-BURRUS FILTER DESIGN EQUATIONS USING COMPUTATIONAL ALGEBRA AND ALGEBRAIC GEOMETRY

JOHN B. LITTLE
Department of Mathematics and Computer Science
College of the Holy Cross

September 19, 2002

Abstract. In a recent paper, I. Selesnick and C.S. Burrus developed a design method for maximally flat FIR low-pass digital filters with reduced group delay. Their approach leads to a system of polynomial equations depending on three integer design parameters \(K, L, M\). In certain cases (their “Region I”), Selesnick and Burrus were able to derive solutions using only linear algebra; for the remaining cases (“Region II”), they proposed using Gröbner bases. This paper introduces a different method, based on multipolynomial resultants, for analyzing and solving the Selesnick-Burrus design equations. The results of calculations are presented, and some patterns concerning the number of solutions as a function of the design parameters are proved.

§1. Introduction

In this paper we will present an application of techniques from computational commutative algebra and algebraic geometry to a problem in signal processing. We will see that recent developments in the theory of multipolynomial resultants give a powerful method for solving an interesting family of problems in digital filter design.

We begin by recalling some basic concepts about digital filters. (A good general reference for this material is [PM].) A digital signal is a quantized function of a discrete variable, (e.g. time). If we ignore quantization effects, therefore, a signal can be represented mathematically by a sequence of complex numbers \(x[n]\) indexed by \(n \in \mathbb{Z}\). For many purposes, an appropriate class of signals is the sequence space \(\ell_2\), since the finiteness of the \(\ell_2\) norm corresponds to a finite energy condition on signals. Signal processing operations can be described mathematically by means of operators \(\Gamma : \ell_2 \to \ell_2\). In the signal processing context, these are called digital filters. Here, we only consider filters that are linear and shift-invariant: If \(k\) is fixed and \(y[n] = x[n + k]\) for all \(n\), then \(\Gamma(y)[n] = \Gamma(x)[n + k]\).

A linear, shift-invariant filter is characterized completely by its transfer function \(H(z)\), the \(z\)-transform of its impulse response (see §1 below). Design methods for filters to perform specified operations on signals can often be formulated as finding...
solutions of systems of polynomial equations on the coefficients in transfer functions \( H(z) \) of some specified form. For this reason, techniques from computational commutative algebra have begun to find uses in this area.

In this article we will focus on one particular filter design method introduced by Selesnick and Burrus in [SB]. Their idea was to specify \( H(z) \) for a low-pass, finite impulse response (FIR) filter (see §I) by imposing three types of conditions:

1. A given number \( M \) of flatness conditions at \( \omega = 0 \) on the square magnitude response
   \[
   F(\omega) = |H(e^{i\omega})|^2
   \]
   (that is, the vanishing of the derivatives of all orders up to \( 2M \) of \( F(\omega) \) at \( \omega = 0 \) – note that \( F \) is an even function so the derivatives of odd orders at \( \omega = 0 \) are zero automatically),

2. A second number \( L \) of flatness conditions at \( \omega = 0 \) on the group delay
   \[
   G(\omega) = \frac{d}{d\omega} \arg H(e^{i\omega})
   \]
   (that is, the vanishing of the derivatives of all orders up to \( 2L \) of \( G(\omega) \) at \( \omega = 0 \) – note that \( G \) is also an even function of \( \omega \)), and

3. A third number \( K \) of zeroes of \( H(e^{i\omega}) \) at \( \omega = \pi \).

The parameters \( K, L, M \) can be specified independently and this approach can be seen as a generalization of earlier work on maximally flat filters by Hermann, Baher, and others in certain special cases. Each of these types conditions leads to polynomial equations of degree \( \leq 2 \) on the coefficients \( h[n] \) in \( H(z) = \sum_{n=0}^{N-1} h[n] z^{-n} \), and solutions exist provided \( N - 1 \geq K + L + M \). The equations have a particularly simple form if the filter moments

\[
m_k = \sum_{n=0}^{N-1} n^k h[n]
\]

are used as the variables. Following Selesnick and Burrus, we express everything in terms of the \( m_k \).

Selesnick and Burrus establish a subdivision of these problems into two classes. The easier cases (Region I) occur for \( L \) relatively large compared to \( M \):

\[
\left\lfloor \frac{M - 1}{2} \right\rfloor \leq L \leq M.
\]

In these cases, Selesnick and Burrus give an analytic solution procedure depending only on linear algebra. The more difficult cases (Region II) occur when \( L \) is relatively small compared to \( M \):

\[
0 < L < \left\lfloor \frac{M - 1}{2} \right\rfloor - 1
\]

In Region II, Selesnick and Burrus used \textit{lex} Gröbner basis computations to solve the resulting filter design equations in a few cases. However, the complexity of this approach severely limited the range of cases they were able to handle.
Some remaining problems left unsolved by Selesnick and Burrus’s work are

(1) to develop an efficient method to solve the filter design equations in the Region II cases, and
(2) to understand the structure of the solutions of the equations for Region II in more detail— in particular to determine for given $K, L, M$, how many solutions there are, how many are real, how many yield monotone decreasing square magnitude response $|H(e^{i\omega})|^2$, and so forth.

While we cannot claim a complete solution to these problems, in this article we first introduce a different solution strategy for the Selesnick-Burrus equations in the Region II cases which has allowed us to compute solutions in cases with much larger values of $L, M$ than those reported in [SB]. Our approach is based on a careful study of the form of the equations, combined with an application of multipolynomial resultants to eliminate variables and obtain a univariate polynomial in the 1st filter moment $m_1$. This strategy is laid out in more detail in (3.10) below. (For general background on multipolynomial resultants, see [CLO] Chapters 3 and 7, [EM], [S] and [CE] for more details on the sparse version, and [KSY] for Dixon resultants. [M] contains a number of practical recipes for applying these ideas to solve systems of equations.)

Second, we attempt to explain some of the intriguing patterns we have noticed in the solutions, in particular in the number of distinct complex solutions of the Selesnick-Burrus equations along the “diagonals” $M = 2L + q$ for various values of $q$. For a given $q$ and $L$ sufficiently large these systems have a similar shape, and for the first few values of $q$ giving cases in Region II, we have been able to analyze the form of the resultant and determine the degree of the univariate polynomial in $m_1$ obtained by elimination in all cases.

The organization of the paper is as follows. §2 contains some additional concepts and terminology on digital filters, a presentation of the exact form of the Selesnick-Burrus equations from [SB], and a small example (the case $K = 1, L = 1, M = 5$), which illustrates some key features of these problems. In §3, we lay out a successful solution strategy for the Region II problems based on resultants. The first step consists of two reductions that permit the direct elimination of variables in the full Selesnick-Burrus system of $K + L + M$ equations in $K + L + M$ unknowns to yield a much more manageable system of $M - L - 1$ equations in $M - L - 1$ variables that we call the reduced Selesnick-Burrus system. The general strategy is presented, followed by some experimental results.

First, we present an outline of a calculation determining the real solutions of the Selesnick-Burrus system with $K = 2, L = 2, M = 10$, and the square magnitude response curves of the corresponding filters. For this calculation we use a method based on the Dixon resultant, combined with numerical rootfinding. All the calculations were carried out in the Maple 8 computer algebra system.

Second, we give a table showing the number of distinct solutions of the Selesnick-Burrus systems for most of the cases with $M \leq 14$ in Region II (see Figure 2 below). A number of the entries in this table were computed by Robert Lewis of Fordham University using his Fermat system and code for Dixon resultants. The resultant strategy would allow the computation of many additional cases with $M \geq 15$ as well. By way of comparison, we note that Selesnick and Burrus were only able to handle cases with $M \leq 7$ in their paper.
In the remainder of the paper we study some of the patterns that are apparent in Figure 2. §4 is devoted to a study of the properties of the coefficient matrices of the linear parts of the reduced Selesnick-Burrus systems, matrices whose coefficients are polynomials in the variable \( t = m_1 \). By some fairly intricate algebraic maneuvering, we are able to express these matrices in a very useful form using some notions from the calculus of finite differences. In particular, the entries can be expressed in terms of polynomials of the form \( D_K^j(i - t)^i \big|_{i=0} \), where \( D_K^j \) are certain finite difference operators. This allows us to determine the Smith normal form of these matrices, hence to completely understand the dependence of the ranks of various submatrices on \( t \).

The cases with \( M = 2L + 3 \) are studied intensively in §5, and the following main theorem is established (compare with the data in the table in Figure 2 below).

(5.1) Theorem. In the cases \( M = 2L + 3, \ L \geq 0 \), (the “corners” in Region II boundary), for all \( K \geq 1 \), the univariate polynomial in \( t \) in the elimination ideal of the Selesnick-Burrus equations obtained via Strategy (3.10) has degree \( 8L + 8 \).

In §6, we undertake a similar study of the cases with \( M = 2L + 4 \) and establish our second main theorem.

(6.1) Theorem. In the cases \( M = 2L + 4, \ L \geq 1 \), for all \( K \geq 1 \), the univariate polynomial in \( t \) in the elimination ideal of the Selesnick-Burrus equations obtained via Strategy (3.10) has degree \( 12L + 14 \).

The proofs of Theorems (5.1) and (6.1) show in essence how to construct the appropriate resultant matrices, so they give a general, extremely efficient, way to solve all cases with \( M = 2L + 3, 2L + 4 \). Similar results are possible in principle for the lower diagonals \( M = 2L + q, \ q \geq 5 \) as well. But we will not attempt to prove formulas for the number of solutions in those cases here because the resultants necessary to handle them become progressively more complicated to analyze.

In a companion article, [LL], we will discuss the properties of the Selesnick-Burrus filters from Region II in more detail.

The author would like to thank Ivan Selesnick for several valuable conversations, and Robert Lewis for permission to present his computational results here.

§2. Preliminaries on Filter Design and the Selesnick-Burrus Equations

Let \( \delta \) be the signal
\[
\cdots, 0, 0, 1, 0, 0, \cdots
\]
(1 at \( n = 0 \)). \( \delta \) is called the unit impulse at \( n = 0 \). Let \( \Gamma \) be a linear, shift-invariant filter as in §1. The output \( \Gamma(\delta) \) from the filter on input \( \delta \) is called the impulse response of \( \Gamma \). A beautiful consequence of the linearity and shift invariance hypotheses is that the impulse response of a filter determines the output on any other input signal. For, we can write
\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k].
\]
If $h[n]$ are the coefficients of the impulse response and $y = \Gamma(x)$ is the output, then by linearity and shift-invariance,

$$
y[n] = \sum_{k=-\infty}^{\infty} x[k] \Gamma(\delta)[n - k]
$$

(2.1)

$$
= \sum_{k=-\infty}^{\infty} x[k] h[n - k]
$$

In other words, the output is the (discrete) convolution of the input and the impulse response.

It is standard in signal processing to package the signals $x[n], y[n], h[n]$ by their "$z$-transforms" $X, Y, Z$. For instance, the definition of the $z$-transform of the signal $x[n]$ is

$$
X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.
$$

The $z$-transform of the impulse response, $H(z)$, is called the transfer function of the filter. In our cases, $h[n]$ will be nonzero for only finitely many $n$. Such filters are called finite impulse response, or FIR filters. For an FIR filter, the transfer function is a rational function, hence has a well-defined value at all $z$ in the complex plane, except for a pole at $z = 0$.

Note that the coefficient of $z^{-n}$ in the product $H(z)X(z)$ is the discrete convolution from (2.1)

$$
\sum_{k=-\infty}^{\infty} h[k] x[n - k],
$$

which is the same as $y[n]$. In other words, the $z$-transform of the output is the product of the transfer function and the $z$-transform of the input: $Y(z) = H(z)X(z)$.

Note that the restriction of $H(z)$ to the unit circle in the complex plane,

$$
H(e^{i\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-i n \omega},
$$

is the (discrete-time) Fourier transform of $h$, so $H(z)$ also determines the frequency response characteristics of the filter on input signals.

Filter design problems, such as the one studied in [SB], ask for constructions of filters adapted to perform some specified operation on input signals. An important approach is to obtain the desired behavior by designing the form of the transfer function $H(z)$. For instance, we might seek to construct:

1. "Low-pass" filters in order to remove high-frequency components of signals. These typically smooth out or blur signals and can be used to remove high-frequency "noise".
2. "High-pass" filters to remove low-frequency components of input signals. These typically pick out fine details, or rapid changes in the input and can be used to detect features.
The paper of Selesnick and Burrus proposes a way to design maximally flat low-pass FIR filters with reduced group delay. These filters are specified by three positive integer parameters denoted $K, L, M$. For an FIR low-pass filter with transfer function

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n},$$

let $F(\omega)$ be the square magnitude response and $G(\omega)$ be the group delay response as in §1. Selesnick and Burrus show that if $K, L, M \in \mathbb{N}$, and $K + L + M + 1 = N$, $L \leq M$, then the filter coefficients $h[n]$ can be determined to make:

\begin{align*}
F^{(2i)}(0) &= 0, \quad i = 1, \ldots, M, \\
G^{(2j)}(0) &= 0, \quad j = 1, \ldots, L, \\
(1 + z^{-1})^K | H(z). \tag{2.2}
\end{align*}

The meaning of the first condition is that $F(\omega)$ is flat to order $2M$ at $\omega = 0$. Similarly the second equation says $G(\omega)$ is flat to order $2L$ at $\omega = 0$. The final equation can also be interpreted as a flatness condition, since it implies that $|H(\omega)|^2$ has a zero of order $2K$ at the normalized frequency $\omega = \pi$, which corresponds to $z = -1$ under $z = e^{i\omega}$.

It is easy to see that the Selesnick-Burrus conditions (2.2) can be expressed as polynomial equations in the filter coefficients. However, the form of these equations becomes significantly simpler if they are expressed in terms of the filter moments,

\begin{equation}
m_k = \sum_{n=0}^{N-1} n^k h[n]. \tag{2.3}
\end{equation}

The explicit form of the equations is derived in [SB] as follows:

1. The flatness conditions on $F$ at $\omega = 0$ are quadratic in the $m_i$:

\begin{equation}
0 = \binom{2i}{i} m_i^2 + 2 \sum_{\ell=0}^{i-1} \binom{2i}{\ell} (-1)^{i+\ell} m_\ell m_{2i-\ell}, \quad i = 1, \ldots, M. \tag{2.4a}
\end{equation}

2. The flatness conditions on $G$ at $\omega = 0$ are also quadratic in the $m_i$:

\begin{equation}
0 = \sum_{\ell=0}^{j} \left(1 - \frac{2\ell}{2j+1}\right) \binom{2j+1}{\ell} (-1)^\ell m_\ell m_{2j+1-\ell}, \quad j = 1, \ldots, L. \tag{2.4b}
\end{equation}

(These are derived from $G^{(2j)}(0) = 0$, using the conditions $F^{(2i)}(0) = 0$, $i = 1, \ldots, M$.)

3. Finally, the zero of order $K$ at $z = -1$ is equivalent to saying that the remainder of $H(z)$ on division by $(1 + z^{-1})^K$ is zero. This yields $K$ linear equations on $m_i$.

At first glance this looks like an underdetermined system with $2M + 1$ variables $m_i$, $i = 0, \ldots, 2M$, and $K + L + M = N - 1$ equations. However, the moments
$m_k, k \geq N$ are not independent variables. They can all be expressed in terms of $m_0, \ldots, m_{N-1}$ by solving systems of linear equations. We will normalize our filters by requiring that $m_0 = 1$. This accomplishes a first reduction to a system of $N - 1$ equations in $N - 1$ variables. We expect only finitely many solutions and the real solutions are of the greatest interest.

(2.5) Example. We study the Selesnick-Burrus equations in the relatively simple case $L = 1, M = 5, K = 1$. There are 6 quadratic equations, from setting

\[
\begin{align*}
2m_1^2 &- 2m_2, \\
6m_2^2 &+ 2m_4 - 8m_1m_3, \\
20m_3^2 &- 2m_6 + 12m_1m_5 - 30m_2m_4, \\
70m_4^2 &+ 2m_8 - 16m_1m_7 + 56m_2m_6 - 112m_3m_5, \\
252m_5^2 &- 2m_{10} + 20m_1m_9 - 90m_2m_8 + 240m_3m_7 - 420m_4m_6 \\
- m_3 &+ m_1m_2
\end{align*}
\]
equal to zero, and similarly 4 additional linear equations:

\[
\begin{align*}
- 315 + 14496m_1 + 23912m_3 - 9310m_4 + 8m_7 - 196m_6 + 1904m_5 - 30184m_2, \\
2m_8 &- 728m_6 + 9408m_5 - 51632m_4 + 141120m_3 - 185152m_2 + 91392m_1 - 2205, \\
4m_9 &- 17052m_6 + 247380m_5 - 1445010m_4 + 4105160m_3 - 5529048m_2 \\
&+ 2784096m_1 - 72765, \\
m_{10} &- 43407m_6 + 670320m_5 - 4070200m_4 + 11869200m_3 - 16288944m_2 \\
&+ 8326080m_1 - 231525.
\end{align*}
\]

In this small example, we can apply a “brute force” method to derive a solution. This is also essentially the method used by Selesnick and Burrus to handle the more difficult problems in their Region II. The lex Gröbner basis for the whole system with $m_1 > m_2 > \cdots > m_5$ is in generic “Shape Lemma” ([CLO], Chapter 2, §4) form. The last element is a univariate polynomial of degree 16 in $m_1$. Using numerical root-finding, we find 6 approximate real roots: $m_1 \approx 0.4470426799, 1.233505559, 2.558981682, 4.441018318, 5.766494441, \text{ and } 6.955295732$. Then the other moments $m_j$ and filter coefficients $h[i]$ can be determined by backsolving in the Gröbner basis and using the equations (2.3).

We can see a general feature of the Selesnick-Burrus equations here. Note that the 6 real roots form three pairs of the form $r, 7 - r$. In fact, for all $K, L, M$, the mapping

\[
m_1 \mapsto (L + M + K) - m_1
\]
gives the effect of time reversal (that is, taking the original transfer function $H(z) = \sum_{n=0}^{N-1} h[n]z^{-n}$ to the reversed $\hat{H}(z) = \sum_{n=0}^{N-1} h[N-1-n]z^{-n}$). It is not difficult to see that the whole Selesnick-Burrus system – (2.4a), (2.4b), and the linear equations expressing the higher moments in terms of the lower ones – is invariant under time reversal. Up to time reversal, there are 3 distinct real filters satisfying the Selesnick-Burrus conditions in this case. The plot in Figure 1. shows the square
magnitude response curves for the three filters. Note that two are apparently monotone decreasing, while one has a pronounced “ripple” in the “passband”. The filters with monotone square magnitude responses would be much more useful for actual low-pass filtering applications.

![Figure 1.](image)

The case we treated above: $L = 1, M = 5, K = 1$ is just within Selesnick and Burrus’s Region II (see §1). However, “brute-force” methods only work in very small cases in Region II! For instance, when $L = 0$ it can be seen in several different ways that there are $2^M$ complex solutions of the Selesnick-Burrus equations. Thus solving the systems with $L = 0$ becomes exponentially more complex as $M$ increases.

§3. A Solution Strategy in Region II

In this section, we will present a strategy for solving the Selesnick-Burrus equations in Region II that is much more efficient than “brute force” elimination as in Example (2.5). The idea is to exploit the special structure of the Selesnick-Burrus equations as much as possible. We will also report some results obtained by this strategy.

First, following Selesnick and Burrus, we show how to reduce the number of variables from $N − 1 = K + L + M$ to $M − L − 1$ and obtain an equivalent system of equations that we will call the reduced Selesnick-Burrus system for a given collection of parameters $K, L, M$. The computations involved in these steps are minimal.

The first part of this reduction is to use the simple observation that the last condition $(1 + z^{-1})^K H(z)$ in Selesnick and Burrus’ formulation implies that the moments $m_0, \ldots, m_{N−1}$ already satisfy certain linear equations, and hence all of the equations can be expressed in terms of the moments in the column vector $\vec{m} = (m_0, \ldots, m_{L+M})^T$. (As noted before, we will also normalize $m_0 = 1$.)

To see how this works in detail, write $H(z) = (1 + z^{-1})^K P(z)$, let $\vec{h}$ be the column vector $(h[0], h[1], \ldots, h[N−1])^T$, and let $\vec{p} = (p[0], p[1], \ldots, p[N−1−K])^T$ be the column vector of coefficients in $P$. Then we have an equation

\[
\vec{h} = T\vec{p},
\]

where $T$ is an $N \times (N − K) = (K + L + M + 1) \times (L + M + 1)$ matrix whose rows and columns are shifted copies of the vector of binomial coefficients $\binom{K}{j}$, $j = 0, \ldots, K$. 
By the definition (2.3) of the moments, we have

(3.2) \( \vec{m} = Q\vec{h} = QT\vec{p} \),

where \( Q \) is an \( (L + M + 1) \times (K + L + M + 1) \) “Vandermonde-type” matrix, whose \( i \)th row is the vector of \( i \)th powers of the integers \( 0, 1, \ldots, K + L + M \).

Combining (3.1) and (3.2), we have the equality

\( \vec{h} = T(QT)^{-1}\vec{m} \)

Hence, we can express \( m_k \) for \( k > L + M \) as

(3.3) \[ m_k = (0, 1^k, 2^k, \ldots, (L + M)^k)T(QT)^{-1}\vec{m} \]

The second part of this reduction is to use some observations about the Selesnick-Burrus quadratic equations (2.4a) and (2.4b), and the affine variety they define over the field \( \mathbb{C} \). It is well-known that there is nothing special about varieties defined by quadrics, but the Selesnick-Burrus equations have a very particular form. First note that the quadratic Selesnick-Burrus polynomials do not depend on the parameter \( K \). Let \( J_{L,M} \) be the ideal they generate in \( \mathbb{C}[m_1, \ldots, m_{2M}] \). In addition, we have the following observations.

(3.4) Lemma. Let \( V_{L,M} = V(J_{L,M}) \) be the affine variety defined by the Selesnick-Burrus quadrics for a given pair of parameters \( L, M \).

a. The Selesnick-Burrus quadrics are homogeneous if we assign

weight\((m_i) = i\).

b. \( V_{L,M} \) contains a rational normal curve passing through each of its points.

c. \( V_{L,M} \) is a smooth variety in \( \mathbb{C}^{2M} \) of dimension \( M - L \).

Proof. All of these claims are easy consequences of the form of the quadrics. \( \square \)

In fact, we can see much more about the variety \( V_{L,M} \) if look at another generating set for the ideal that defines it. Before giving the general statement, we again take up the case \( K = 1, L = 1, M = 5 \) considered in Example (2.5).

(3.5) Example. Recall the Selesnick-Burrus quadrics given in Example (2.5). If we compute a \( \text{lex} \) Gröbner basis for \( J_{1,5} \) with \( m_{10} > m_9 > \cdots > m_1 \) we find:

\[ G = \{m_2 - m_1^2, \ m_3-m_1^3, \ m_4-m_1^4, \]
\[ m_6 - 6m_1m_5 + 5m_5^2, \ m_8 - 8m_1m_7 + 112m_3^2m_5 - 105m_1^8, \]
\[ m_{10} - 10m_1m_9 + 240m_3^2m_7 - 126m_5^2 + 3780m_3^2m_5 + 3675m_1^{10}\}, \]

the Gröbner basis \( G \) shows a very nice parametrization for \( V_{1,5} \). If we let

\[ \varphi : \mathbb{C}^4 \rightarrow \mathbb{C}^{10} \]
\[ (t, a, b, c) \mapsto (t, t^2, t^3, t^4, a, 6at - 5t^6, b, 8bt - 112at^3 + 105t^8, c, \]
\[ 10ct - 240bt^3 + 126a^2 - 3780at^5 - 3675t^{10}) \]
The image of $\varphi$ is precisely $V_{1,5}$.

We next indicate a connection between the Selesnick-Burrus systems and some classical topics in algebraic geometry. These observations are needed here only to verify that the hypotheses of [BEM] are satisfied for these systems and can be omitted if the reader is not familiar with these concepts. However, they motivated a large portion of our work on this problem.

The related ideal

$$J' = \langle m_2^2 - m_1^2, m_3 - m_1^3, m_4 - m_1^4, m_6 - 6m_1m_5, m_8 - 8m_1m_7, m_{10} - 10m_1m_9 \rangle$$

is equal to the ideal generated by the $2 \times 2$ minors of

$$\begin{pmatrix}
    m_1 & m_2 & m_3 & m_4 & m_6 & m_8 & m_{10} \\
    1 & m_1 & m_2 & m_3 & 6m_5 & 8m_7 & 10m_9
\end{pmatrix}$$

Hence, $S = V(J')$ is an affine 4-fold rational normal scroll (see, e.g., [H]) – the union of $\mathbb{C}^3$'s spanned by related points on a rational normal curve of degree 4 and 3 lines. Moreover, $V_{1,5}$ is the image of the scroll $S$ under a certain upper-triangular automorphism $\alpha$ of $\mathbb{C}^{10}$. We also see that the projection of $V_{1,5}$ into the coordinate subspace $\mathbb{C}^8$ with coordinates $m_1, \ldots, m_8$ is itself a rational scroll of dimension 3. (It is only the quadratic term $a^2$ in the last coordinate that keeps $V_{1,5}$ from being a rational scroll itself.)

Similar results hold for all the $V_{L,M}$. These observations imply that $V_{L,M}$ is a unirational variety for all $L, M$. The additional linear equations define the affine part of a 0-dimensional linear section of $V_{L,M}$. Because of this, the Selesnick-Burrus systems fall into the general context discussed in the paper [BEM], and we can use the main theorem there to eliminate variables using resultants (without using Gröbner bases). We will use this approach in the following.

Our next Lemma establishes an important common feature of all of the Selesnick-Burrus systems which we will exploit to define the reduced Selesnick-Burrus system for a given set of parameters $K, L, M$ with $M > L$.

\textbf{(3.6) Lemma.} Assume $M > L$. The Selesnick-Burrus quadrics (2.4a) and (2.4b) imply that $m_k = m_1^k$ for all $k, 1 \leq k \leq 2L + 2$.

\textbf{Proof.} The proof is by induction on $L$, the base case being $L = 1$. In that case, we have from (2.4a) with $M = 1, 2$, and $m_0 = 1$:

\begin{equation}
(3.7) \quad 2m_1^2 - 2m_2, \quad 6m_2^3 + 2m_4 - 8m_1m_3
\end{equation}

From (2.4b) with $L = 1$:

\begin{equation}
(3.8) \quad -m_3 + m_1m_2
\end{equation}

The equation $m_2 = m_1^2$ follows directly from the first equation in (3.7). Substituting in (3.8), we have $m_3 = m_1^3$. Then substituting in the second equation in (3.8), we have $m_4 = m_1^4$.

The induction step is similar. Assume we have shown that the quadrics (2.4a) with $1 \leq i \leq M$ and (2.4b) $1 \leq j \leq L$ imply $m_k = m_1^k$ for all $1 \leq k \leq 2L + 2$. 

\[\]
Consider these quadrics, plus (2.4a) with \( i = M + 1 \) and (2.4b) with \( j = L + 1 \). By the induction hypothesis, we substitute \( m_k = m_1^k \) for \( 1 \leq k \leq 2L + 2 \). Then substituting into (2.4b) with \( j = L + 1 \), we have

\[
0 = \left( \sum_{\ell=0}^{L} \left( 1 - \frac{2\ell}{2L + 3} \right) \left( \frac{2L + 3}{\ell} \right) (-1)^\ell \right) m_1^{2L+3} + (-1)^{L+1} \left( 1 - \frac{2L + 2}{2L + 3} \right) \left( \frac{2L + 3}{L + 1} \right) m_2^{2L+3}.
\]

This implies \( m_2^{2L+3} = m_1^{2L+3} \) because, applying some standard binomial coefficient identities,

\[
\sum_{\ell=0}^{L+1} \left( 1 - \frac{2\ell}{2L + 3} \right) \left( \frac{2L + 3}{\ell} \right) (-1)^\ell = \sum_{\ell=0}^{L+1} (-1)^\ell \left( \frac{2L + 2}{\ell} \right) - \left( \frac{2L + 2}{\ell - 1} \right) = 0.
\]

Then, we substitute \( m_k = m_1^k \) for \( k = 1, \ldots, 2L + 3 \) into (2.4a) with \( i = 2L + 4 \) to deduce \( m_2^{2L+4} = m_1^{2L+4} \). □

(3.9) Definition. The reduced Selesnick-Burrus system for given parameters \( K, L, M \) is the system of equations obtained from the full Selesnick-Burrus system of \( N - 1 = K + L + M \) equations in \( N - 1 \) variables \( m_1, \ldots, m_{N-1} \) as follows.

1. First, substitute in the equations (2.4a) for \( i \geq L + 2 \), for all the moments \( m_k \) for \( k > L + M \) from equations (3.3) above.
2. Write \( m_1 = t \) and substitute \( m_k = t^k \) for all \( 1 \leq k \leq 2L + 2 \) in these equations. Also set \( m_0 = 1 \).

The result is a system of \( M - L - 1 \) equations in the \( M - L - 1 \) variables

\[
t, m_{2L+3}, \ldots, m_{L+M}.
\]

The quadrics (2.4a) with \( i \leq L + 1 \) and all of the quadrics (2.4b) are discarded since they have been used to derive the equations \( m_k = t^k \).

The Gröbner basis computation we used in Example (2.5) and substitution of the parametrization of the variety defined by the Selesnick-Burrus quadrics does the same sort of elimination of variables as given in part 2 of the reduction described here (and more). Note that the linear equations we discussed above, for instance in Example (2.5), have been subsumed in the equations (3.3). We have eliminated the higher moments \( m_k, k > L + M \) using them, so they do not appear explicitly in the reduced system. The parameter \( K \) enters only in the form of the \( T \) matrix in (3.3). Changing \( K \) changes the coefficients of the equations but not their Newton polytopes or the number of solutions (provided \( K \geq 1 \)).

It will be most useful to view the polynomials in the reduced system as polynomials in the moments \( m_{2L+3}, \ldots, m_{L+M} \), whose coefficients are polynomials in \( t \). For the Region I cases considered by Selesnick and Burrus, these polynomials are linear in \( m_{2L+3}, \ldots, m_{L+M} \), and this is what allows the use of purely linear algebra techniques to eliminate and obtain a univariate polynomial in \( t \).
In fact the Region II cases are characterized by the fact that the reduced system
still has non-linear terms in $m_{2L+3}, \ldots, m_{L+M}$. The precise form of the reduced
system is determined by “how far down into” Region II we are from the boundary.
That is, for $L$ sufficiently large, all the cases along the “diagonals” defined by
$M = 2L + q$, for fixed $q \geq 3$ will have a similar shape. (There are also “special
cases” along early portions of lower diagonals $M = 2L + q$ with $q \geq 5$. These are
different from the stable form because the nonlinear terms are different.)

(3.10) Strategy. To study the Selesnick-Burrus equations for cases in Region II,
we propose the following strategy.

(1) Form the reduced system as in (3.9), and view it as a system of $M - L - 1$
linear and quadratic equations in the $M - L - 2$ variables $m_{2L+3}, \ldots, m_{L+M}$,
with the variable $t$ “hidden in the coefficients.”

(2) Use the linear equations in the reduced system to solve for a subset of the
remaining higher moments in terms of the lower moments, and substitute
into the quadratic equations.

(3) Use an appropriate formulation for multipolynomial resultants to eliminate
the remaining undetermined moments and produce a univariate polynomial
in $t$.

In order to compute examples, we have used several different resultant formula-
tions. For instance, in §5 below, we will see that the cases with $M = 2L + 3$ can be
handled by using the multipolynomial resultant of a general system of $L + 1$ ho-
ogeneous linear equations and 1 homogeneous quadratic equation in $L + 2$ variables.
This resultant is denoted by $Res_{81, \ldots, 1, 2}$ in [CLO], Chapter 3.

Mixed sparse resultants (see [CE], [S]), Dixon (or Bézout) resultants (see [KSY]),
and even the naive approach of iterated pairwise Sylvester resultants all work rea-
sonably well on the smaller examples. Dixon resultants seem to be far superior
for the larger cases. In almost all cases, some care is needed to eliminate extraneous
factors in the computed polynomial in $t$. One useful criterion here is the
fact mentioned above in (2.6) that the Selesnick-Burrus system is invariant under
time-reversal. Thus correct univariate polynomial in $t$ must be invariant under
$t \mapsto (K + L + M) - t$. This strategy is particularly well adapted for the problem
of determining the number of complex solutions of the design equations as a func-
tion of the design parameters $K, L, M$. In combination with numerical rootfinding
methods, it can also serve as a template for a general solution method for the
Selesnick-Burrus systems. We illustrate this below.

To indicate the scale of the problems that this strategy allows us to solve, we
provide the following table giving the degree of the univariate polynomial in $t$
generating the elimination ideal of the Selesnick-Burrus system for given $L, M$. In
most cases the computation was done with $K = 1$ for simplicity, but the degree
will be the same for all $K \geq 1$.

In this table, the entries along the diagonal $M = 2L + 3$ are the first within
Region II; the Region I cases with $M < 2L + 3$ are not shown. For purposes of
comparison, the entries for $M \leq 7$ were also reported by Selesnick and Burrus in
[SB]. The entries with $M \geq 8$ and $L > 0$ are new. Starred entries were computed
by Robert Lewis of Fordham University, using his Fermat system and his routines
for Dixon resultants. The blank entries are somewhat beyond the scope of current
software. On the other hand, many cases with $M \geq 15$ would also be tractable by these methods.

| M/L | 0 | 1 | 2 | 3 | 4 | 5 |
|-----|---|---|---|---|---|---|
| 5   | 32| 16|   |   |   |   |
| 6   | 64| 26|   |   |   |   |
| 7   | 128| 48| 24|   |   |   |
| 8   | 256| 78| 38|   |   |   |
| 9   | 512| 152| 66| 32|   |   |
| 10  | 1024| 278| 112| 50|   |   |
| 11  | 2048| 512*| 192*| 86| 40|   |
| 12  | 4096| 944*| 358*| 142| 62|   |
| 13  | 8192| 572*| 240*| 106| 48|   |
| 14  | 16384| 1020*| 402*| 174*| 74|   |

**Figure 2.**

We will now present an outline of the resultant computation for the case $K = 2, L = 2, M = 10$ and show how the methods described in [BEM] and [M] can be used to derive all the real solutions. The reduced Selesnick-Burris system in this case is a system of $M - L - 1 = 7$ equations in the 7 variables $t = m_1$, and the $m_j$, $j = 7, \ldots, 12$. We will begin by using the resultant to eliminate $m_j$, $j = 7, \ldots, 12$ and yield a univariate polynomial in $t$ satisfied by all the solutions. This is done by “hiding the variable $t$ in the coefficients” of the system as described, for instance, in [M].

For simplicity, we will write $m_7 = x$, $m_8 = y$, $m_9 = z$, $m_{10} = u$, $m_{11} = v$, $m_{12} = w$, and denote the $j$th equation by $a_j(x, y, z, u, v, w) = 0$. The first three equations are

\[
\begin{align*}
0 &= a_1(x, y, z, u, v, w) = 7t^8 + y - 8tx \\
0 &= a_2(x, y, z, u, v, w) = -84t^{10} - u + 10tx - 45yt^2 + 120xt^3 \\
0 &= a_3(x, y, z, u, v, w) = 462t^{12} + w - 12tv + 66t^2u - 220xt^3 + 495yt^4 - 792t^5x
\end{align*}
\]

The remaining four equations are significantly more complicated and will be omitted here. (The complete computation is available as a Maple 8 worksheet from the author’s homepage by downloading

```
mathcs.holycross.edu/~little/SB2210.mws
```

To run this and other examples, the procedures in the file

```
mathcs.holycross.edu/~little/CompFileLatest.map
```

should also be downloaded.)

The Dixon resultant computation proceeds as follows. We introduce a second set of variables $X, Y, Z, U, V, W$ and compute the $7 \times 7$ determinant $\Delta$ whose $j$th
row is the transpose of
\[
\begin{pmatrix}
\alpha_j(x, y, z, u, v, w) \\
\alpha_j(x, y, z, u, v, w) - \alpha_j(x, y, z, u, v, w) \\
\alpha_j(x, y, z, u, v, w) - \alpha_j(x, y, z, u, v, w) \\
\alpha_j(x, y, z, u, v, w) - \alpha_j(x, y, z, u, v, w) \\
\alpha_j(x, y, z, u, v, w) - \alpha_j(x, y, z, u, v, w) \\
\alpha_j(x, y, z, u, v, w) - \alpha_j(x, y, z, u, v, w) \\
\end{pmatrix}
\]

The expanded form of the determinant can be written as a matrix product \( \Delta = RM \cdot C \), where \( R \) is a 44-component row vector containing monomials in \( x, y, z, u, v, w \), \( M \) is a 44 \times 36 matrix whose entries are polynomials in \( t \), and \( C \) is 36-component column vector whose entries are monomials in \( X, Y, Z, U, V, W \). The rank of the matrix \( M \) in this case is 24.

By the main result of [BEM], any 24 \times 24 submatrix \( M' \) of \( M \) of rank 24 has determinant equal to a multiple of the resultant of the system. For a particular choice of maximal rank submatrix, we computed and factored the determinant yielding a reducible polynomial with one factor of degree 112 in \( t \) and other factors of smaller degrees. The factor of degree 112 is the resultant; the others are extraneous factors that depend on the choice of the submatrix \( M' \).

Using Maple’s \texttt{fsolve} routine, 12 approximate real roots were determined, \( t = 0.021826159039817 \cdot \ldots \), \( 1.14111245031295 \cdot \ldots \), \( 2.46849175059426 \cdot \ldots \), \( 4.77577862421111 \cdot \ldots \), \( 5.42248255383217 \cdot \ldots \), \( 6.63285847397435 \cdot \ldots \) and six additional roots obtained from these by time reversal – \( t \mapsto 14 - t \) (note that \( K + L + M = 2 + 2 + 10 = 14 \)). In this computation, a 170 decimal digit floating-point number system was necessary to obtain accurate results. The use of the moment variables in the Selesnick-Burrus formulation simplifies the form of the equations immensely and makes the symbolic approach we have used feasible. But it also imposes a severe numerical conditioning penalty in return.

To determine the other components of the solution, we use the form of the row monomial vector \( R \) in the equation \( \Delta = RM \cdot C \) above. The entries of \( R \) corresponding to the rows of the maximal rank 24 \times 24 submatrix \( M' \) contain the six monomials \( x, y, z, u, v, w \). Substituting each of the \( t \) values above in turn, the vector in the kernel of \( (M')^{tr} \) with first component equal to 1 has 6 components equal to the \( x, y, z, u, v, w \) values in the corresponding solution of the system. We then determine the values of the filter coefficients from the moments from (3.2) and (3.3) above.

The square magnitude responses of the 6 real filters found above are shown in Figure 3. Of these, four are apparently monotone decreasing, one has a maximum,
and one has a minimum and a maximum. The four monotone filters come from the 
t-values closest to the center value \( t = \frac{K + L + M}{2} = 7 \).

![Figure 3](image)

Timings for this computation are as follows (all done in Maple 8 on a SunBlade 100 workstation with a 500 MHz UltraSPARC processor and 256MB of RAM, running Solaris). The symbolic part of the computation (the computation of the Dixon resultant and factoring the univariate polynomial) takes approximately 320 seconds. (There is a certain amount of randomness built into the choice of the maximal rank submatrix \( M' \), however, and the time can vary depending on which submatrix is used.) The numerical part (the rootfinding steps) can be done quickly (i.e. in less CPU time than the symbolic computation, even with the high-precision arithmetic) with an ad hoc “by-hand” search for the real roots in the interval \([0, 7]\) and a fast iterative method like Newton-Raphson. (With the “brute-force” application of Maple’s \texttt{fsolve} command described above, and illustrated in the worksheet mentioned before, the numerical part of the computation takes much longer, of course – about 8200 seconds, including the plotting of the square magnitude response curves.)

We have used similar numerical computations to solve the reduced system and determine the filter coefficients of the real solutions in many of the cases reported above in Figure 2. As is indicated by this example, we note that the degrees give only one measure of the complexity of these computations.

In a companion paper [LL], we discuss some properties of the filters obtained by these computations in more detail. In the next sections here we will focus instead on some of the patterns that seem to appear when the table in Figure 2 is examined carefully.

\section*{§4. A Technical Interlude}

In this section we will prove a number of technical lemmas on the Smith normal form of certain matrices that appear when the linear equations in the reduced
Selesnick-Burrus systems (3.9) are reformulated in a particularly useful way. For simplicity, we will describe the general form of these matrices in this section in the abstract, so to speak; we will delay showing how the Selesnick-Burrus equations fit these patterns until §5 and §6.

We will need the following notation.

**Notation.** Let \( j, K, \ell \) denote nonnegative integers, and \( t \) an indeterminate. All vectors are infinite, indexed by the nonnegative integers, \( \mathbb{Z}_{\geq 0} \).

a. We will write \( \Delta^j \) for the vector of coefficients in the \( j \)th forward difference operator, each entry divided by \( j! \), “padded” with additional zero entries on the right:

\[
\Delta^j = \frac{1}{j!} \left( (-1)^j \binom{j}{0}, (-1)^{j-1} \binom{j}{1}, (-1)^{j-2} \binom{j}{2}, \ldots, \binom{j}{j}, 0, \ldots \right).
\]

The indices of the nonzero entries shown run from 0 to \( j \).

b. Similarly, we will write \( \Delta^j_\ell \) for right shift by \( \ell \) of the vector above, so the \( (-1)^j \binom{j}{j} \) occurs in position \( \ell \), and zeroes appear in locations 0 through \( \ell - 1 \).

c. We will write

\[
D^j_K = \frac{1}{2^K} \sum_{\ell=0}^{K} \binom{K}{\ell} \Delta^j_\ell.
\]

The vector \( D^j_K \) can also be viewed as the padded vector of coefficients of a difference operator.

d. We will write \( (i-t)^\ell \) for the vector with entries

\[
((0-t)^\ell, (1-t)^\ell, (2-t)^\ell, \ldots)
\]

e. We will use the shorthand

\[
[j, \ell; K] = \langle D^j_K, (i-t)^\ell \rangle,
\]

where \( \langle, \rangle \) is the formal dot product on vectors indexed by \( \mathbb{Z}_{\geq 0} \). Note that all of the vectors \( D^j_K \) we consider have only a finite number of nonzero terms, so convergence is automatic. The sum is the value at \( i = 0 \) of the result of applying the operator \( D^j_K \) to the function of the discrete variable \( i \) given by \( (i-t)^\ell \). This is a polynomial of degree \( \ell - j \) in \( t \) if \( \ell \geq j \), and equals zero otherwise because all \( j \)th differences of a polynomial of degree \( < j \) in \( i \) vanish.

f. An expression of the form \( [j, \ell; K](a) \) will denote the value obtained by substituting \( t = a \) in the polynomial \( [j, \ell; K] \).

(4.1) **Lemma.** The \( [j, \ell; K] \) polynomials have the following properties.

a. ("reflection identity") Up to a sign, \( [j, \ell; K] \) is symmetric about \( t = \frac{j+K}{2} \):

\[
[j, \ell; K](j + K - t) = (-1)^{j+\ell}[j, \ell; K](t)
\]

We call \( t = \frac{j+K}{2} \) the **center value** of \( [j, \ell; K] \).
b. (“center value zero”) If \( \ell \) and \( j \) have opposite parity, then
\[
[j, \ell; K] \left( \frac{j + K}{2} \right) = 0.
\]

c. (“boost identity”) \([j, \ell; K]\) satisfies
\[
[j, \ell; K](t - 1) = [j, \ell; K](t) + (j + 1)[j + 1, \ell; K](t).
\]

**Proof.** Part a follows from a direct computation. Because of the symmetry of the binomial coefficients in the \( \Delta_k^j \), \( D_K^j \) is symmetric about \( \frac{j + K}{2} \), up to the sign \((-1)^j\). Therefore we have
\[
[j, \ell; K]((j + K) - t) = \langle D_K^j, (i - ((j + K) - t))^\ell \rangle
\]
\[= (-1)^\ell \langle D_K^j, ((j + K) - i)^\ell \rangle = (-1)^{j + \ell} \langle D_K^j, (i - t)^\ell \rangle = (-1)^{j + \ell} [j, \ell; K](t).
\]

Part b follows immediately from part a.

Part c is shown by another calculation. In terms of the shift operator \( E \), we have
\[
D_K^{j+1} = \frac{1}{2^{j+1}}(E + 1)^K(E - 1)^{j+1},\quad \text{so}
\]
\[
(j + 1)[j + 1, \ell; K](t) = (j + 1)\langle D_K^{j+1}, (i - t)^\ell \rangle
\]
\[
= \frac{1}{2^j j!}(E + 1)^K(E - 1)^{j+1},(i - t)^\ell \rangle
\]
\[= \langle D_K^j E, (i - t)^\ell \rangle - \langle D_K^j, (i - t)^\ell \rangle
= [j, \ell; K](t - 1) - [j, \ell; K](t). \square
\]

The specific matrices that will appear in the analysis of the linear equations in the reduced Selesnick-Burrus filter design systems have the following forms \( A(s, m; K) \) and \( \hat{A}(s, m; K) \), for certain positive integers \( s, m \) depending on the flatness parameters \( L, M \) from the filter design problem. First we introduce the matrix \( A(s, m; K) = (4.2a) \)
\[
\begin{pmatrix}
[2s - 1, 2s; K] & [2s, 2s; K] & \ldots & [2s + m - 1, 2s; K] \\
[2s - 1, 2s + 2; K] & [2s, 2s + 2; K] & \ldots & [2s + m - 1, 2s + 2; K] \\
\vdots & \vdots & \ddots & \vdots \\
[2s - 1, 2s + 2m; K] & [2s + 1, 2s + 2m; K] & \ldots & [2s + m - 1, 2s + 2m; K]
\end{pmatrix}
\]

We will write \( \delta(s, m; K) = \det A(s, m; K) \).

Similarly, \( \hat{A}(s, m; K) = (4.2b) \)
\[
\begin{pmatrix}
[2s, 2s; K] & [2s + 1, 2s; K] & \ldots & [2s + m, 2s; K] \\
[2s, 2s + 2; K] & [2s + 1, 2s + 2; K] & \ldots & [2s + m, 2s + 2; K] \\
\vdots & \vdots & \ddots & \vdots \\
[2s, 2s + 2m; K] & [2s + 1, 2s + 2m; K] & \ldots & [2s + m, 2s + 2m; K]
\end{pmatrix}
\]
We write \( \tilde{\delta}(s, m; K) = \det \tilde{A}(s, m; K) \).

For example, with \( s = 3, m = 1, \) and \( K = 2 \), the matrix \( A(3, 1; 2) \) is

\[
A(3, 1; 2) = \begin{pmatrix}
21 - 6t & 1 \\
-56t^3 + 588t^2 - 2212t + 2940 & 476 - 224t + 28t^2
\end{pmatrix}.
\]

The entry in the second row and second column is \([2s, 2s + 2; K] = [6, 8; 2]\).

The following observation will simplify our work considerably.

**Observation.** Since \([j, \ell; K]\) is zero if \( j > \ell \), note that all the entries on the first row of the matrix \( \tilde{A}(s, m; K) \) except the first are zero. Expanding along the first row, we have

\[
\tilde{\delta}(s, m; K) = \delta(s + 1, m - 1; K).
\]

Therefore, for our purposes it will suffice to study the \( \delta(s, m; K) \).

Our main goal in the remainder of this section is to determine the Smith normal form of the matrices \( A(s, m; K) \) above, and hence to determine \( \delta(s, m; K) \). Recall that the Smith normal form of a square matrix \( A \) with entries in \( \mathbb{C}[t] \) is the diagonal matrix obtained by doing elementary row and column operations. The diagonal entries satisfy the following property for all \( n \leq \text{rank}(A) \): the product of the first \( n \) diagonal entries is equal to the monic greatest common divisor of all the \( n \times n \) minors of \( A \). The properties of the Smith normal form follow from the standard theory of homomorphisms between modules over a PID such as \( \mathbb{C}[t] \) (see for instance [Ja]).

We introduce the following additional notation to facilitate working column by column in \( A(s, m; K) \). Note that the entries in \( A(s, m; K) \) all have the form \([j, \ell; K]\) with \( 2s \leq \ell \leq 2s + 2m, \ell \) even. The entries in the first column have \( j = 2s - 1 \). The entries in the second have \( j = 2s \), and so forth. We will write \( A^j \) for the column in \( A = A(s, m; K) \) in which the entries are \([j, \ell; K]\) for \( 2s \leq \ell \leq 2s + 2m, \ell \) even.

Our first result shows that \( \delta(s, m; K) \) is symmetric about \( t = \frac{2s + m - 1 + K}{2} \), up to a sign.

**Lemma.** Let \( \delta(s, m; K) \) be as above, and let \( c_A = 2s + m - 1 + K \) \((c_A^2 \text{ is the center value of the entries of the last column in } A(s, m; K)) \). Then

\[
\delta(s, m; K)(c - t) = \pm \delta(s, m; K)(t).
\]

**Proof.** Consider the column \( A^{2s + m - 1 - p} \) for each \( 0 \leq p \leq m \). The center value for the entries in this column is \( t = \frac{c_A - 1}{2} \). By Lemma (4.1), part a, we have the corresponding column in \( A(s, m; K)(c - t) \) equals

\[
A^{2s + m - 1 - p}(c - t) = A^{2s + m - 1 - p}[(c - p) - (t - p)]
= (-1)^{2s + m - 1 - p} A^{2s + m - 1 - p}(t - p)
\]

Then we apply the “boost identity” (Lemma (4.1), part c) repeatedly to deduce that \( A^{2s + m - 1 - p}(t - p) \) equals \( A^{2s + m - 1 - p}(t) \), plus a linear combination of the terms \( A^{2s + m - 1 - p+q}(t) \), for \( 1 \leq q \leq p \). It follows that the the column in \( A^{2s + m - 1 - p}(c - t) \)
is in the span of the columns in $A^j(t)$ with $2s + m - 1 - p \leq j \leq 2s + m - 1$, and hence $\delta(s, m; K)(c - t) = \pm \delta(s, m; K)(t)$.

Some factors in $\delta(s, m; K)$ are immediately clear from Lemma (4.1), part b. If $j$ is odd, then the center value root $t = \frac{j + K}{2}$ of the entries in the column $A^j$ is also a root of $\delta(s, m; K)$. Moreover, it will will follow from the next lemma that $(t - \frac{j + K}{2})^e$ with $e > 1$ divides $\delta(s, m; K)$ in some cases.

\textbf{(4.4) Lemma.} Let $R = [2s - 1, 2s + m - 1] \cap \mathbb{N}$. If $j \in R$ is odd and $j + 2p$ is also in $R$, then

$$A^j \left( \frac{j + K + 2p}{2} \right) \in \text{Span} \left\{ A^{j'} \left( \frac{j + K + 2p}{2} \right) : j' \in R, j' > j, j' \text{ even} \right\}.$$ 

\textbf{Proof.} Note that $\frac{j + K + 2p}{2}$ is the center value of the entries in the column $A^{j+2p}$. The proof is a kind of double induction argument – descending induction on the odd $j \in R$, and ascending induction on $p \geq 0$ such that $j + 2p \in R$. In the base case for the outer induction, $j$ is the largest odd integer in $R$. In this case, necessarily, $p = 0$. But then $t = \frac{j + K}{2}$ is the center value root of the column $A^j$, so the conclusion of the Lemma follows. Similarly, if $j$ is any odd integer in $R$ and $p = 0$ we see that

$$A^j \left( \frac{j + K}{2} \right) = 0 \in \text{Span} \left\{ A^{j'} \left( \frac{j + K + 2p}{2} \right) : j' \in R, j' > j, j' \text{ even} \right\}.$$ 

For the inductive step, assume that the conclusion of the lemma holds for a given $j, p$, and also for all odd $j > j$ and all $q$ such that $j + 2q \in R$. If $j + 2(p + 1) \in R$, then we consider $A^j \left( \frac{j + K + 2(p + 1)}{2} \right)$. By the “boost identity” from Lemma (4.1), part c, for all even $\ell$, $2s \leq \ell \leq 2s + 2m$, we have $[j, \ell; K] \left( \frac{j + K + 2(p + 1)}{2} \right) = [j, \ell; K] \left( \frac{j + K + 2p}{2} \right) - (j + 1)[j + 1, \ell; K] \left( \frac{j + K + 2(p + 1)}{2} \right).$

Hence

$$A^j \left( \frac{j + K + 2(p + 1)}{2} \right) = A^j \left( \frac{j + K + 2p}{2} \right) - (j + 1)A^{j+1} \left( \frac{j + K + 2(p + 1)}{2} \right).$$

In the second term, $j + 1$ is even and $> j$, so we do not need to do anything further with that. We apply the inductive hypothesis to the first term:

$$A^j \left( \frac{j + K + 2p}{2} \right) \in \text{Span} \left\{ A^{j'} \left( \frac{j + K + 2p}{2} \right) : j' \in R, j' > j, j' \text{ even} \right\}.$$ 

The entries in the $A^{j'} \left( \frac{j + K + 2p}{2} \right)$ appearing in the linear combination are the $[j', \ell; K] \left( \frac{j + K + 2p}{2} \right)$. By the “boost identity” from Lemma (4.1) part c again, we have $[j', \ell; K] \left( \frac{j + K + 2p}{2} \right) = [j', \ell; K] \left( \frac{j + K + 2(p + 1)}{2} \right) + (j' + 1)[j' + 1, \ell; K] \left( \frac{j + K + 2(p + 1)}{2} \right).$
Hence $A^j \left( \frac{j + K + 2(p + 1)}{2} \right) \in \text{Span} \left\{ A^j \left( \frac{j + K + 2(p + 1)}{2} \right), A^j+1 \left( \frac{j + K + 2(p + 1)}{2} \right) \right\}$.

In the first vector in this set, $j' > j$ is even and this term matches the conclusion. In the second vector, $j' + 1 > j$ is odd. Moreover, since for suitable $q$, $j + 2(p + 1) = (j' + 1) + 2q = \tilde{j} + 2q \in R$, we may apply the induction hypothesis to conclude that the second vector is also in $\text{Span} \{ A^j \left( \frac{j + K + 2(p + 1)}{2} \right) : j' \in R, j' > j, j' \text{ even} \}$. □

The main consequence we will draw from this lemma is the following corollary giving information about the Smith normal form of $A(s, m; K)$ and $\delta(s, m; K)$.

(4.5) Corollary. Let $p \geq 0$, let $2s - 1 + 2p \in R$, and let $t = \frac{2s - 1 + 2p + K}{2}$, the center value for the column $A^{2s - 1 + 2p}$. Then the rank of $A(s, m; K)$ at this $t$ is at most $m - p$ (i.e., the rank drops by at least $p + 1$ at this $t$). Hence $(2t - (2s - 1 + 2p + K))$ divides the last $p + 1$ entries on the diagonal of the Smith normal form of $A(s, m; K)$, and $(2t - (2s - 1 + 2p + K))^{p+1}$ divides $\delta(s, m; K)$.

Proof. By standard properties of the Smith normal form, all the claims here follow from the statement about the rank of $A(s, m; K)$ at $t = \frac{2s - 1 + 2p + K}{2}$. That statement follows directly from Lemma (4.4): At this $t$, the $p + 1$ columns $A^{2s - 1 + 2q}$, $0 \leq q \leq p$ are all in the span of the remaining columns of $A(s, m; K)$. □

For future reference, we note that Lemma (4.3) (the symmetry of $\delta(s, m; K)$ about $t = c_A = \frac{2s - 1 + m + K}{2}$ up to sign) implies the existence of additional roots of $\delta(s, m; K)$ greater than $c_A$.

The foregoing establishes lower bounds on the multiplicities of the roots of $\delta(s, m; K)$ at the center value roots of the columns $A^j$ for odd $j$. We next show that there are also roots of $\delta(s, m; K)$ at the center value $t$-values of the columns $A^j$ for even $j$.

(4.6) Lemma. Let $2s + 2p \in R$ and consider the center value $t = \frac{2s + 2p + K}{2}$ for the column $A^{2s + 2p}$. If $j$ is odd and $j < 2s + 2p$, then

$$A^j \left( \frac{2s + 2p + K}{2} \right) \in \text{Span} \left\{ A^j \left( \frac{2s + 2p + K}{2} \right) : j' < 2s + 2p, j' \text{ even} \right\}.$$

Proof. The proof is similar to the proof of Lemma (4.4) except that now we will proceed by ascending induction on $p$, and descending induction on $j$ such that $j < 2s + 2p$. The base cases are $p = 0$, $j = 2s - 1$, and more generally, $p$ arbitrary and $j = 2s + 2p - 1$. By the “boost identity” ((4.1) part c), $[2s + 2p - 1, \ell; K] \left( \frac{2s + 2p + K}{2} \right) = (4.7)$ $[2s + 2p - 1, \ell; K] \left( \frac{2s + 2p + K}{2} - 1 \right) - (2s + 2p)[2s + 2p, \ell; K] \left( \frac{2s + 2p + K}{2} \right)$
Next, apply the “reflection identity” ((4.1) part \(a\)) to the first term on the right. The center value for \(A^{2s+2p-1}\) is \(\frac{2s+2p-1+K}{2}\), so
\[
2s + 2p - 1 + K - \left(\frac{2s + 2p + K}{2}\right) = \frac{2s + 2p + K}{2}.
\]
Hence, since \(\ell\) is even and \(2s + 2p - 1\) is odd,
\[
(4.8) \quad [2s + 2p - 1, \ell; K] \left(\frac{2s + 2p + K}{2}\right) = -[2s + 2p - 1, \ell; K] \left(\frac{2s + 2p + K}{2}\right)
\]
Combining (4.7) and (4.8) for all even \(\ell\), \(2s + 2p - 1\) is odd,
\[
A^{2s+2p-1} \left(\frac{2s + 2p + K}{2}\right) \in \text{Span} \left\{ A^{2s+2p} \left(\frac{2s + 2p + K}{2}\right) \right\}.
\]
So the conclusion of the Lemma holds in these cases.

For the inductive step, assume that the conclusion of the Lemma holds for all odd integers \(j\) between \(j + 2\) and \(2s + 2p\) with the current \(p\), and for all \(\hat{p} < p\). Consider the entries \([j, \ell, K]\) in \(A^j\). By the “boost identity” ((4.1) part \(c\)), \([j, \ell, K]\) \(\left(\frac{2s+2p+K}{2}\right) = \]
\[
[j, \ell, K] \left(\frac{2s + 2p + K}{2} - 1\right) - (j + 1)[j + 1, \ell, K] \left(\frac{2s + 2p + K}{2}\right).
\]
Hence
\[
A^j \left(\frac{2s + 2p + K}{2}\right) \in \text{Span} \left\{ A^j \left(\frac{2s + 2p + K}{2} - 1\right), A^{j+1} \left(\frac{2s + 2p + K}{2}\right) \right\}.
\]
In the second vector on the right, \(j + 1 > j\) is even so this term matches the conclusion of the Lemma. In the first vector, \(\frac{2s+2p+K}{2} - 1 = \frac{2s+2(p-1)+K}{2}\) is the center value for the column \(A^{2s+2(p-1)}\), and \(j < 2s + 2(p - 1)\). By the induction hypothesis, \(A^j \left(\frac{2s+2(p-1)+K}{2}\right) \in \]
\[
\text{Span} \left\{ A^{j'} \left(\frac{2s + 2(p - 1) + K}{2}\right) : j' < j' \leq 2s + 2(p - 1), j' \text{ even} \right\}.
\]
But for each entry in one of these \(A^{j'}\), we can apply the “boost identity” again: \([j', \ell; K] \left(\frac{2s+2(p-1)+K}{2}\right) = \]
\[
[j', \ell; K] \left(\frac{2s + 2p + K}{2}\right) + (j' + 1)[j' + 1, \ell; K] \left(\frac{2s + 2p + K}{2}\right).
\]
Hence
\[
A^{j'} \left(\frac{2s + 2(p - 1) + K}{2}\right) \in \text{Span} \left\{ A^{j'} \left(\frac{2s + 2p + K}{2}\right), A^{j'+1} \left(\frac{2s + 2p + K}{2}\right) \right\}.
\]
The first vector in the spanning set matches the conclusion of the Lemma since \(j'\) is even. In the second term, \(j' + 1 > j\) is odd. Hence that vector can be written as a linear combination as in the conclusion of the Lemma by the induction hypothesis.

\(\square\)

Here too, the main consequence we will draw from this lemma is a corollary giving information about the Smith normal form of \(A(s, m; K)\) and \(\delta(s, m; K)\).
(4.9) Corollary. Let \( p \geq 0 \), let \( 2s + 2p \in R \), and let \( t = \frac{2s + 2p + K}{2} \), the center value for the column \( A^{2s+2p} \). Then the rank of \( A(s; m; K) \) at this \( t \) is at most \( m - p \) (i.e., the rank drops by at least \( p + 1 \) at this \( t \)). Hence \( (2t - (2s + 2p + K)) \) divides the last \( p + 1 \) entries on the diagonal of the Smith normal form of \( A(s; m; K) \), and \( (2t - (2s + 2p + K))^{p+1} \) divides \( \delta(s; m; K) \).

Proof. As in the proof of Corollary (4.5), all the claims here follow from the statement about the rank of \( A(s; m; K) \) at \( t = \frac{2s + 2p + K}{2} \). That statement follows directly from Lemma (4.6): At this \( t \), the \( p + 1 \) columns \( A^{2s-1+2q} \), \( 0 \leq q \leq p \) are all in the span of the remaining columns of \( A(s; m; K) \). \( \square \)

As in the case of the center value zeroes from Corollary (4.5), Lemma (4.3) (the symmetry, up to a sign, of \( \delta(s; m; K) \) under \( t \mapsto c_A - t \), where \( c_A = 2s - 1 + m + K \)) implies the existence of a second, symmetrically located collection of roots of \( \delta(s; m; K) \) greater than \( c_A \). We are now ready for the major result of this section.

(4.10) Theorem. Let \( c_A = 2s - 1 + m + K \) as above. The determinant \( \delta(s; m; K) \) can be written in the form:

\[
\delta(s; m; K) = a \prod_{i=2s-1+K}^{2s-1+K+2m} (2t - i)^{\frac{m-|c_A - i|}{2} + 1}
\]

for some constant \( a \). In the Smith normal form of \( A(s; m; K) \), the \( (m + 1, m + 1) \) entry is (a constant times) the product \( \prod_{i=2s-1+K}^{2s-1+K+2m} (2t - i) \) (one factor for each root), the \( (m, m) \) entry is a divisor of this polynomial whose roots are the roots of \( \delta(s; m; K) \) of multiplicity \( \geq 2 \), and so forth.

The \( |c_A - i| \) in the exponent ensures the symmetry of the exponents in this expansion about \( c_A \). To make this somewhat intricate statement more intelligible, before proceeding to the proof, we give a small example. Consider the \( 4 \times 4 \) matrix \( A(2,3;2) \), which has the shape

\[
A(2,3;2) = \begin{pmatrix}
[1] & [0] & 0 & 0 \\
[3] & [2] & [1] & [0] \\
[5] & [4] & [3] & [2] \\
[7] & [6] & [5] & [4]
\end{pmatrix}.
\]

Here and in the rest of this article we will use the standing notational convention:

Notation. \([d]\) is shorthand for a polynomial of degree exactly \( d \) in \( t \).

For instance, the entry \([3]\) in the first column is the polynomial \([3, 6; 2] = 425 - 420t + 150t^2 - 20t^3\). The entries marked 0 are actual zeroes.

According to the formula in the statement of the Theorem, the center value of the 4th column is \( t = \frac{c_A}{2} \), where \( c_A = 2s - 1 + m + K = 2 \cdot 2 - 1 + 3 + 2 = 8 \). The set of roots is symmetric about \( t = 4 \). The “predicted” value for \( \delta(2,3;2) \) is

\[
\delta(2,3;2) = a(2t - 5)(2t - 6)(2t - 7)^2(2t - 8)^2(2t - 9)^2(2t - 10)(2t - 11)
\]
for some constant \( a \). Using the computer algebra system Maple, we find
\[
\delta(2, 3; 2) = 672 \left(2 t - 11\right) \left(t - 3\right) \left(t - 5\right) \left(2 t - 5\right) \left(2 t - 7\right)^2 \left(2 t - 9\right)^2 (t - 4)^2,
\]
and the Smith normal form of \( A(2, 3; 2) \) is:
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & p_3(t) & 0 \\
0 & 0 & 0 & p_7(t)
\end{pmatrix},
\]
where
\[
p_3(t) = t^3 - 12 t^2 + \frac{191}{4} t - 63 = \frac{1}{8} \left(2 t - 7\right) (2t - 8) (2t - 9)
\]
and
\[
p_7(t) = t^7 - 28 t^6 + \frac{665}{2} t^5 - 2170 t^4 + \frac{134449}{16} t^3 - \frac{77203}{4} t^2 + \frac{389415}{16} t - \frac{51975}{4}.
\]
\( p_7(t) \) is the monic polynomial with roots \( t = 5/2, 7/2, 4/2, 9/2, 11/2 \) (all multiplicity 1). We now proceed to the proof of Theorem (4.10).

**Proof.** It follows from Corollaries (4.5) and (4.9) that the product in equation (4.11) divides \( \delta(s, m; K) \). If we knew that \( \delta(s, m; K) \) had the form given in (4.11), then the claims about the Smith normal form of \( A(s, m; K) \) would also follow from these Corollaries. Hence, to prove the Theorem it suffices to prove that the degree of \( \delta(s, m; K) \) equals the degree of the product in (4.11) in \( t \). To compute the degree of \( \delta(s, m; K) \), recall the form of the matrix \( A(s, m; K) \) given in (4.2a). We have
\[
A(s, m; K) = \begin{pmatrix}
[1] & [0] & 0 & 0 & \cdots & 0 \\
[3] & [2] & [1] & [0] & \cdots & 0 \\
[5] & [4] & [3] & [2] & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
[2m+1] & [2m] & [2m-1] & [2m-2] & \cdots & [m+1]
\end{pmatrix}
\]
where, as earlier, \([d]\) denotes a polynomial of degree \( d \) in \( t \). The 0 entries are actual zeroes. By examining the form of this matrix, it is not difficult to see that because of the zeroes above the main diagonal, every nonzero product of entries, one from each row and one from each column, has the same total degree as the product of the entries on the main diagonal:
\[
1 + 2 + \cdots + (m + 1) = \frac{(m + 1)(m + 2)}{2}.
\]
Hence the degree of \( \delta(s, m; K) \) is no larger than \( \frac{(m + 1)(m + 2)}{2} \).

But on the other hand, we will see that the product in (4.11) also has degree \( \frac{(m + 1)(m + 2)}{2} \). Hence it follows that \( \delta(s, m; K) \) equals the product in (4.11). To compute the degree of (4.11), we consider the cases \( m \) even and \( m \) odd separately. If \( m = 2q \) is even, then the central value of the last column of the matrix gives one of the central value roots. The sum of the multiplicities in (4.11) gives
\[
2(1 + 1 + 2 + 2 + \cdots + q + q) + q + 1 = (q + 1)(2q + 1) = \frac{(m + 1)(m + 2)}{2}.
\]
Similarly with \( m = 2q + 1 \) an odd number, the total degree is
\[
2(1 + 1 + 2 + 2 + \cdots + q + q + (q + 1)) + q + 1 = (q + 1)(2q + 3) = \frac{(m + 1)(m + 2)}{2},
\]
which concludes the proof. \( \square \)

By the Observation above concerning the \( \tilde{A}(s, m; K) \) matrices, we have a parallel formula for \( \tilde{\delta}(s, m; K) \).

**Corollary.** Let \( \tilde{c} = 2s + m + K \). The determinant \( \tilde{\delta}(s, m; K) \) can be written in the form:
\[
(4.13) \quad \tilde{\delta}(s, m; K) = a \prod_{i=2s+1+K}^{2s-1+K+2m} (2t - i)^{\frac{m-1}{2}[i+1]} + 1
\]
for some constant \( a \). In the Smith normal form of \( \tilde{A}(s, m; K) \), the \((m, m)\) entry is a constant times the product \( \prod_{i=2s+1+K}^{2s-1+K+2m} (2t - i) \) (one factor for each root), the \((m - 1, m - 1)\) entry is the divisor of this whose roots are the roots of \( \tilde{\delta}(s, m; K) \) of multiplicity \( \geq 2 \), and so forth.

**Proof.** This follows directly from Theorem (4.10), using the relation \( \tilde{\delta}(s, m; K) = \delta(s + 1, m - 1; K) \). \( \square \)

Here is an example, showing \( \tilde{\delta}(2, 3; 2) \) for comparison with \( \delta(2, 3; 2) \) computed earlier. Using Maple, we have
\[
\tilde{\delta}(2, 3; 2) = 36 \ (t - 5) \ (2t - 7) \ (2t - 11) \ (t - 4) \ (-9 + 2t)^2,
\]
which agrees with (4.13) for this \( s, m, K \).

§5. **The \( M = 2L + 3 \) Diagonal**

In this section we will discuss the Selesnick-Burrus systems for parameters \( L, M \) satisfying \( M = 2L + 3 \). In particular, we will prove the following theorem which explains one pattern that can be seen in the table given in Figure 2.

**Theorem.** In the cases \( M = 2L + 3, L \geq 0 \), (the “corners” in Region II boundary), for all \( K \geq 1 \), the univariate polynomial in \( t \) in the elimination ideal of the Selesnick-Burrus equations obtained via Strategy (3.10) has degree \( 8L + 8 \).

Before giving the details of the proof, we outline the method we will use. Along the first diagonal in Region II, of the \( M - L - 1 = L + 2 \) equations in the reduced Selesnick-Burrus system, \( L + 1 \) are inhomogeneous linear equations in \( m_{2L+3}, \ldots, m_{3L+3} \) whose coefficients are polynomials in \( t \). We will begin by showing how the coefficients matrix of this linear part of the system can be rewritten as the matrix \( A(L + 2, L; K) \) as defined in §4, times a suitable invertible lower-triangular matrix. The last equation (from the flatness condition \( F^{(2M)}(0) = F^{(4L+6)}(0) = 0 \)) contains the non-linear term \( m_{2L+3}^2 \), plus linear terms in \( m_{2L+3}, \ldots, m_{3L+3} \). To
eliminate to a univariate polynomial in \( t \), we will use a formula for the multivariable resultant \( \text{Res}_{1, \ldots, d} \) from Proposition 5.4.4 of [Jo] (see also Exercise 10 of Chapter 3, §3 in [CLO]). (This formula may be proved by the basic approach of solving the linear equations for \( m_{2L+3}, \ldots, m_{3L+3} \) in terms of \( t \) by Cramer’s Rule, then substituting into the last equation to obtain a univariate polynomial in \( t \).) We will need to keep careful track of the factorizations of \( \delta(L + 2, L; K) = \det A(L + 2, L; K) \) from Theorem (4.10). The steps in this outline will be accomplished in a series of Lemmas.

**Lemma 5.2.** The \( L + 1 \) linear equations in the reduced Selesnick-Burrus system with \( M = 2L + 3 \) can be rewritten in the form

\[
A(L + 2, L; K) \cdot \mathbf{m}_r = \mathbf{b},
\]

where \( A(L + 2, L; K) \) is the matrix defined in (4.2a), \( \mathbf{L} \) is a constant lower-triangular matrix with diagonal entries equal to 1, \( \mathbf{m}_r = (m_{2L+3}, \ldots, m_{3L+3})^T \), and \( \mathbf{b} = ([2L + 4], [2L + 6], \ldots, [4L + 4])^T \).

**Proof.** Recall the form of the Selesnick-Burrus quadrics from (2.4a):

\[
0 = \binom{2j}{j} m_j^2 + 2 \sum_{\ell=0}^{j-1} \binom{2j}{\ell} (-1)^{j-\ell} m_\ell m_{2j-\ell}
\]

The linear equations in the reduced Selesnick-Burrus system come from these for \( j = L + 2, \ldots, 2L + 2 \), via the reduction process described in (3.9). We begin by rearranging these equations to the following form by separating the terms involving the variables \( m_{2L+3}, \ldots, m_{3L+3} \) from those depending on the higher moments \( m_{3L+4}, \ldots, m_{4L+4} \). We have

\[
W_1 V_1 T(QT)^{-1} \mathbf{m} + W_2 V_2 T(QT)^{-1} \mathbf{m} = \mathbf{b}^T,
\]

where

1. the matrix \( W_1 \) comes from the coefficients of the \( m_k, 2L + 3 \leq k \leq 3L + 3 \) in (5.3):

\[
W_1 = \begin{pmatrix}
-\binom{2L+4}{1} t & \binom{2L+4}{2} t^2 & 0 & 0 & \cdots \\
-\binom{2L+6}{3} t^3 & \binom{2L+6}{2} t^2 & -\binom{2L+6}{1} t & \binom{2L+6}{0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
-\binom{2L+4}{L+1} t^{L+1} & \binom{2L+4}{2L} t^{2L} & -\binom{2L+4}{2L-1} t^{2L-1} & \binom{2L+4}{2L-2} t^{2L-2} & \cdots \\
\end{pmatrix}
\]

2. the matrix \( W_2 \) comes from the coefficients of the \( m_k, 3L + 4 \leq k \leq 4L + 4 \) in (5.3):

\[
W_2 = (-1)^L \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
(\frac{4L+4}{L}) t^L & (\frac{4L+4}{L-1}) t^{L-1} & \cdots & (\frac{4L+4}{0}) \\
\end{pmatrix}
\]
(3) the matrices \( V_1, V_2 \) are Vandermonde-type matrices:

\[
V_1 = \begin{pmatrix}
0 & 1^{2L+3} & \cdots & (3L + 3 + K)^{2L+3} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1^{3L+3} & \cdots & (3L + 3 + K)^{3L+3}
\end{pmatrix},
\]

and

\[
V_2 = \begin{pmatrix}
0 & 1^{3L+4} & \cdots & (3L + 3 + K)^{3L+4} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1^{4L+4} & \cdots & (3L + 3 + K)^{4L+4}
\end{pmatrix}
\]

(4) \( \vec{m} = (1, t, \ldots, t^{2L+2}, m_{2L+3}, \ldots, m_{3L+3})^T \)

(5) \( \vec{b} \) has the same form as \( \vec{b} \) in the statement of the Lemma but is not the entire vector of \( t \) terms. (There are also terms depending only on \( t \) that come from the matrix product \( (W_1V_1 + W_2V_2)T(QT)^{-1}\vec{m} \).

(6) the matrices \( Q \) and \( T \) are as in the discussion leading up to (3.3).

Since the first \( 2L + 3 \) entries of \( \vec{m} \) depend only on \( t \), the coefficients of \( m_{2L+3} \) through \( m_{3L+3} \) in our equations come from the product

\[
(W_1V_1 + W_2V_2) \cdot T \cdot \vec{m}_r,
\]

where \( T \) is the submatrix of \( T(QT)^{-1} \) containing all the entries from the last \( L + 1 \) columns. The other terms in the product \( (W_1V_1 + W_2V_2) \cdot T(QT)^{-1} \cdot \vec{m} \) containing only powers of \( t \) go into the vector \( \vec{b} \), and (5.4) becomes

\[
(W_1V_1 + W_2V_2) \cdot T \cdot \vec{m}_r = \vec{b}.
\]

The fact that establishes the connection between these equations and the matrices \( A(s, m; K) \) considered in §4 is the following observation. In the matrix \( T \), the final column is the vector \( D_K^{3L+3} \) as in the Notation at the start of §4, written as a column. This follows if we think of the columns of \( (QT)^{-1} \) as operators acting on the rows of \( QT \), thought of as power functions of a discrete variable. Similarly, the next-to-last column of \( T(QT)^{-1} \) is a linear combination \( D_K^{3L+2} + \alpha D_K^{3L+3} \) for some constant \( \alpha \), and so on. In general, we have

\[
T = (D_K^{2L+3}|D_K^{2L+4}| \cdots |D_K^{3L+3}) \cdot \mathcal{L}
\]

for a lower-triangular square \((L + 1) \times (L + 1)\) matrix \( \mathcal{L} \) with diagonal entries 1.

To finish the proof of the Lemma, we substitute (5.6) into (5.5) and rearrange the terms again:

\[
(W_1V_1 + W_2V_2) \cdot (D_K^{2L+3}|D_K^{2L+4}| \cdots |D_K^{3L+3}) \cdot \mathcal{L} \cdot \vec{m}_r + \vec{b}.
\]

We have

\[
U_1 := V_1 (D_K^{2L+3}|D_K^{2L+4}| \cdots |D_K^{3L+3}) = (D_K^j t^j),
\]

for \( 2L + 3 \leq j \leq 3L + 3 \) and \( 2L + 3 \leq \ell \leq 3L + 3 \), and

\[
U_2 := V_2 (D_K^{2L+3}|D_K^{2L+4}| \cdots |D_K^{3L+3}) = (D_K^j t^j),
\]
for $2L + 3 \leq j \leq 3L + 3$ and $3L + 4 \leq \ell \leq 4L + 4$.

Consider the $(I, J)$ entry of the product

$$(W_1V_1 + W_2V_2) \left( D_{K}^{2L+3} | D_{K}^{2L+4} | \cdots | D_{K}^{3L+3} \right),$$

which is the dot product of the $I$th row of $W_1$ with the $J$th column of $U_1$, plus the dot product of the $I$th row of $W_2$ with the $J$th column of $U_2$. The form of the entries in $W_1$ and $W_2$ on the $I$th row is $(-1)^q \left( 2L + 2(I+1) \right)^t q$ for $q$ from $2I-1$ down to 0. Hence this sum of dot products equals

$$\sum_{q=0}^{2I-1} (-1)^q \left( 2L + 2(I+1) \right)^t q D_{K}^{J(I+1)-q} = D_{K}^{J(i-t)} 2L+2(i+1)$$

$$\quad = [J, 2L + 2(I + 1); K],$$

using the notation introduced in §4. As $I$ runs from 1 to $L + 1$ and $J$ runs from $2L+3$ to $3L+3$, we see that these entries form the matrix $A(L+2, L; K)$ as claimed.

For a general system of $L+1$ linear homogeneous equations and one homogeneous quadratic equation in $L+2$ variables, if the linear equations are written as $A\vec{x} = 0$, and the quadratic equation is $Q(\vec{x}) = 0$, then by the result from Proposition 5.4.4 of [Jo] mentioned before, the multivariable resultant $\text{Res}_{1,\ldots,1,2}$ equals

$$Q(\delta_1, -\delta_2, \delta_3, \ldots, (-1)^{L+1}\delta_{L+2})$$

where $\delta_I = \det A_I$, and $A_I$ is the $(L+1) \times (L+1)$ submatrix of $A$ obtained by deleting column $I$.

We apply this to our reduced Selesnick-Burrus system. Write the augmented matrix of the linear equations as

$$A = (A(L+2, L, K) \cdot \mathcal{L} | -\vec{b})$$

where $\mathcal{L}$ is the lower triangular matrix and $\vec{b}$ is the column vector $([2L+4], [2L+6], \ldots, [4L+4])^{tr}$ from Lemma (5.2). Our next Lemma shows that the determinants of the minors of $A$ have a common factor of degree $L(L-1)/2$. To prepare for this statement, we introduce the following notation. Let $\delta$ be the product of the first $L$ diagonal entries (elementary divisors) in the Smith normal form of $A(L+2, L; K)$:

$$\delta = \prod_{i=2L+5+K}^{4L+1+K} (2L-i)^{L-3L+3+K-i-1}$$

(There is one factor in this product for each of the roots of multiplicity $\geq 2$ of $\delta(L+2, L; K)$, and the exponents are each 1 less than the corresponding exponents in $\delta(L+2, L; K)$.).
Lemma. Let $A$ be as in (5.6) and $\delta_i$ be the $i$th minor of $A$ as above. If $1 \leq i \leq L+1$, then

$$\delta_i = [4L + 3 + i] \cdot \delta$$

where $[4L + 3 + i]$ is some polynomial in $t$ of degree $4L + 3 + i$, and $\delta$ is the product from (5.8). If $i = L+2$, then

$$\delta_{L+2} = \prod_{i=2L+3}^{4L+3+K} (2t - i)^{2L+3+K-1} = [2L + 1] \cdot \delta$$

Proof. We begin by computing the minor $A_{L+2}$. Since $L$ is a constant lower-triangular matrix with diagonal entries equal to 1, $A_{L+2} = \delta(L + 2; L; K) = \det(A(L + 2; L; K))$. We use Theorem (4.10) to compute this. We have $c_A = 3L + 3 + K$ and

$$\delta_{L+2} = \delta(L + 2; L; K) = a \prod_{i=2L+3}^{4L+3+K} (2t - i)^{2L+3+K-1}$$

for some constant $a$. By the properties of the Smith normal form, we know that at a root $t = t_0$ of multiplicity $r$, the rank of $A(L + 2; L; K)$ is $L + 1 - r$, so every $(L + 2 - r) \times (L + 2 - r)$ submatrix of $A(L + 2; L; K)$ will have zero determinant at $t = t_0$.

Now consider the other minors $A_i$, for $1 \leq i \leq L+1$, and expand the determinant along the column containing the entries from the vector $\vec{b}$. Each term in this expansion is the product of an entry from $\vec{b}$ times the determinant of an $L \times L$ submatrix of $A(L + 2; L; K) \cdot L$. Hence by the statement at the end of the last paragraph, $\delta_i$ is divisible by $\delta$. The remaining factor in $A_i$ comes by examining the degrees of the entries of the matrix as in the proof of Theorem (4.10). Note that if $L = 1$, the starting value of the index $i$ is greater than the final value. In that case $\delta = 1$. In all other cases, the degree of $\delta$ is $\left(\frac{L}{2}\right)$ (see the proof of Theorem (4.10)).

We now consider the quadratic polynomial in the reduced Selesnick-Burrus system. Let $z$ be a homogenizing variable. Then the homogenized version of $Q$, the equation from $F^{(4L+6)}(0) = 0$, has the form

$$(5.10) \quad [0]m_{2L+3}^2 + [2L + 3]m_{2L+3}z + \cdots + [L + 3]m_{3L+3}z + [4L + 6]z^2$$

We analyze the result of substituting the $(-1)^{i+1}\delta_i$ into this polynomial as in (5.7).

Lemma. The resultant of our system has the form

$$[8L + 8] \delta^2,$$

where $[8L + 8]$ denotes a polynomial of degree $8L + 8$ in $t$, and $\delta$ is the product from (5.8).
SOLVING THE SELESNICK-BURRUS FILTER DESIGN EQUATIONS

Proof. To obtain the resultant of our equations to eliminate \( m_{2L+3}, \ldots, m_{3L+3} \), we substitute

\[
\begin{align*}
m_{2L+3} &= \delta_1 \\
& \vdots \\
m_{3L+3} &= \delta_{L+1} \\
z &= \delta_{L+2}
\end{align*}
\]

into (5.10) (following equation (5.7) above), and use Lemma (5.9). We obtain the following expression for the resultant:

\[
|0|(4L + 4\delta^2) + [2L + 3][4L + 4\delta)(2L + 1\delta) + \cdots + \\
[L + 3][(5L + 4\delta)(2L + 1\delta) + [4L + 6][(2L + 2\delta)^2 \\
= [8L + 8]\delta^2.
\]

□

The factor of degree \( 8L+8 \) is the univariate polynomial in \( t \) that we want, and this concludes the proof of Theorem (5.1). The other factor in (5.12) is extraneous in the sense that the \( t \) with \( \delta(t) = 0 \) do not give solutions of the whole Selesnick-Burrus system. In fact, it can be seen that the linear equations in the reduced system are inconsistent for those \( t \). In algebraic geometric terms, the resultant of the homogenized system contains information about all the solutions of the equations in projective space, including solutions “at infinity”. The common factor \( \delta^2 \) gives solutions at infinity, and the degree in \( t \) of the full polynomial in (5.11) is the degree of the projective closure of the affine variety defined by the Selesnick-Burrus quadrics – the deformed rational scroll as in the discussion given in Example (3.5) in the case \( L = 1, M = 5 \). In that case there are no solutions at infinity (since \( \delta = 1 \)). However for \( L \geq 2 \), there are always such solutions. For example with \( L = 2 \), there are 24 solutions of the Selesnick-Burrus system for all \( K \geq 1 \), but the degree of the variety defined by the quadrics is 26. The factor \( \delta^2 = \left( \binom{2}{2} \right)^2 = [2] \) accounts for the difference. Similarly, with \( L = 3 \), there are 32 solutions of the Selesnick-Burrus equations for all \( K \geq 1 \), but the degree of the variety defined by the quadrics is 38. Again, the factor \( \delta^2 = \left( \binom{3}{2} \right)^2 = [6] \) accounts for the difference.

In the companion article [LL], we will give more details on the structure of the filters corresponding to the \( 8L + 8 \) solutions of the Selesnick-Burrus equations for small \( L \). For example, rather extensive calculations suggest the following conjectures.

(5.12) Conjectures. Consider the Selesnick-Burrus equations with \( M = 2L + 3 \), and \( K \geq 1 \).

1. The polynomial of degree \( 8L + 8 \) is irreducible over \( \mathbb{Q} \), hence has \( 8L+8 \) distinct solutions in \( \mathbb{C} \).
2. Of the \( 8L + 8 \) solutions, \( 2(L + 2) \) are real (yielding \( L + 2 \) different filters because of the invariance under time reversal).
3. Exactly four of these (2 different filters), those with \( t = m_1 \) closest to the center value \( \frac{K+L+M}{2} \), yield monotone decreasing square magnitude response.
The other solutions correspond to filters with progressively greater oscillation and greater maximum “passband ripple” as the distance from $t = m_1$ to $K + \frac{t}{L}$ increases.

The beginnings of this pattern can be seen in Example (3.5), which gives the case $L = 1, M = 5$.

§6. The $M = 2L + 4$ Diagonal

In this section we will discuss the Selesnick-Burrus systems with $M = 2L + 4$, the second diagonal in Region II in the table given in Figure 2. Our goal is to prove a result parallel to Theorem (5.1) giving the degree of the univariate polynomial in $t$ whose roots give the different solutions.

(6.1) Theorem. In the cases $M = 2L + 4$, $L \geq 1$, for all $K \geq 1$, the univariate polynomial in $t$ in the elimination ideal of the Selesnick-Burrus equations obtained via Strategy (3.10) has degree $12L + 14$.

Our proof will follow the same pattern as the proof of Theorem (5.1). First, we analyze the form of the equations in these cases. We rewrite the linear equations in a suitable form making use of the results of §4. Then the univariate polynomial is obtained via an elimination of variables tailored to the form of these equations.

We begin by noting that the reduced Selesnick-Burrus system in these cases has the following form. The first $L + 1$ equations (from the flatness conditions $F^{(2L+4)}(0) = \cdots = F^{(4L+4)}(0)$ are linear in the $L + 2$ variables $m_{2L+3}, \ldots, m_{3L+4}$. The remaining two equations have nonlinear terms. The condition $F^{(4L+6)}(0) = 0$ gives a reduced equation containing $m^2_{2L+3}$, plus linear terms in all the variables. (This is the same as the last equation in the $M = 2L + 3$ cases.) In addition, the condition $F^{(4L+8)}(0) = 0$ gives a reduced equation containing $m^2_{2L+4}, m_{2L+3}m_{2L+5}$, terms, plus linear terms. Following the strategy (3.10), we solve the linear equations for $L + 1$ of the variables in terms of the others, substitute into the quadrics, then compute the Sylvester resultant of the 2 quadrics. (Our approach here is closely related to one way to derive the multipolynomial resultant for a system of $L + 1$ homogeneous linear and 2 homogeneous quadratic equations in $L + 3$ variables: $Res_{1,1,2,2}$, but it seems to be easier in this case to use an ad hoc approach.)

We begin with the following Lemma describing the linear equations. Since the precise statement involves some new quantities, we will sketch the derivation first, then give the formulation of the Lemma we will use. First, an argument exactly like the proof of Lemma (5.2) shows that the linear equations can be rewritten in the form

$$A \cdot \mathcal{L} \cdot \vec{m}_r = \vec{b}$$

where $A$ is the $(L + 1) \times (L + 2)$ matrix:

$$
\begin{pmatrix}
[2L + 3, 2L + 4; K] & [2L + 4, 2L + 4; K] & \cdots & [3L + 4, 2L + 4; K] \\
[2L + 3, 2L + 6; K] & [2L + 4, 2L + 6; K] & \cdots & [3L + 4, 2L + 6; K] \\
\vdots & \vdots & \ddots & \vdots \\
[2L + 3, 4L + 4; K] & [2L + 4, 4L + 4; K] & \cdots & [3L + 4, 4L + 4; K]
\end{pmatrix},
$$

$\mathcal{L}$ is a certain lower-triangular constant matrix with 1’s on the main diagonal, $\vec{m}_r = (m_{2L+4}, \ldots, m_{3L+4})^T$, and $\vec{b} = ([2L + 4], [2L + 6], \ldots, [4L + 4])^T$. We will
Cramer’s Rule. For $1 \leq L$ the reduced Selesnick-Burrus system with $\text{det} A \cdot \mathcal{L}$ (a certain linear combination of the entries on row $i$ of the matrix $A$). After we subtract all terms involving $m_{2L+3}$ to the right-hand sides of the equations, we obtain the following result, because the submatrix of $A$ consisting of all entries in the last $L+1$ columns is precisely the matrix $\tilde{A}(L+2, L; K)$ from (4.2b).

**6.2 Lemma.** Using the notation introduced above, the $L + 1$ linear equations in the reduced Selesnick-Burrus system with $M = 2L + 4$ can be rewritten in the form

$$\tilde{A}(L+2, L; K) \cdot \mathcal{L} \cdot \vec{m}_r = \vec{b},$$

where $\vec{b} = ([2L + 4] - \{2L + 3, 2L + 4; K\}m_{2L+3}, \ldots, [4L + 4] - \{2L + 3, 4L + 4; K\}m_{2L+3})^{tr}$.

We can solve the system $\tilde{A}(L+2, L; K) \cdot \mathcal{L} \cdot \vec{m}_r = \vec{b}$ for the moments in $\vec{m}_r$ using Cramer’s Rule. For $1 \leq i \leq L+1$, this gives

$$m_{2L+3+i} = \frac{\det A_i}{\delta(L + 2, L; K)}$$

where $A_i$ is the matrix obtained from $\tilde{A}(L+2, L; K) \cdot \mathcal{L}$ by replacing column $i$ with the vector $\vec{b}$. Next, we consider what happens when we substitute from (6.3) into the first nonlinear equation (from $F^{(4L+6)}(0) = 0$). We will show that the result is an equation of the form

$$[0]m_{2L+3}^2 + [2L + 3]m_{2L+3} + [4L + 6] = 0$$

(in other words, the denominators from (6.3) cancel with terms in the numerators in this equation). The situation that produces this cancellation is described in the following general lemma.

**6.5 Lemma.** Consider a system of equations of the form

$$a_{11}(t) x_1 + a_{12}(t) x_2 + \cdots + a_{1n} x_n = r_1(t)$$
$$a_{21}(t) x_1 + a_{22}(t) x_2 + \cdots + a_{2n} x_n = r_2(t)$$
$$\vdots$$
$$a_{n-1,1}(t) x_1 + a_{n-1,2}(t) x_2 + \cdots + a_{n-1,n} x_n = r_{n-1}(t)$$
$$a_{n1}(t) x_1 + a_{n2}(t) x_2 + \cdots + a_{nn} x_n = r_n(t) + cx_1^2,$$

where $a_{ij}(t)$ and $r_i(t)$ are in $\mathbb{C}[t]$. Let $A = (a_{ij}(t))$ be the full $n \times n$ matrix of coefficients of the linear terms, and let $A' = (a_{ij}(t))$, $1 \leq i \leq n - 1$, $2 \leq j \leq n$ be the matrix of coefficients of $x_2, \ldots, x_n$ in the first $n - 1$ equations. Assume, up to a constant factor, $\det A'$ is the product of the of the first $n - 1$ diagonal entries of the Smith normal form of $A$. Then solving for $x_2, \ldots, x_n$ from the first $n - 1$ equations by Cramer’s Rule and substituting into the last equation produces an equation of the form

$$cx_1^2 + B(t)x_1 + C(t) = 0,$$
where \( B(t), C(t) \in \mathbb{C}[t] \).

**Proof.** To make the connection between \( A \) and \( A' \) clearer, we note that \( A' = A_{n1} \) (submatrix obtained by deleting row \( n \) and column 1). We will number the rows in \( A' \) by indices 1 through \( n-1 \) and the columns by indices 2 through \( n \) in the following. As described above for the Selesnick-Burrus equations, take the first \( n-1 \) equations, subtract the \( x_1 \) terms to the right sides, and apply Cramer’s Rule to solve for \( x_2, \ldots, x_n \) in terms of \( x_1 \), yielding:

\[
x_j = \frac{\det A'_j}{\det A'},
\]

for \( 2 \leq j \leq n \), where \( A'_j \) is the matrix obtained from \( A' \) by replacing the \( j \)th column (recall, this means the column containing the \( a_{ij}(t), \) \( 1 \leq i \leq n-1 \) for this \( j \)) with the vector

\[
(r_1(t) - a_{1j}(t)x_1, \ldots, r_{n-1}(t) - a_{n-1,j}(t)x_1)^t.
\]

If we expand \( \det A'_j \) along the column (6.6) in each case we obtain an expression

\[
\det A'_j = (-1)^{j+1} \det A_{nj}x_1 + \sum_{i=1}^{n-1} (-1)^{i+j} r_i(t) \det A'_{ij},
\]

where \( A_{nj} \) is the submatrix of \( A \) obtained by deleting row \( n \) and column \( j \), and \( A'_{ij} \) is a minor of \( A' \) (which is also a submatrix of \( A \) obtained by deleting two rows and two columns). Substitute for \( x_1 \) in the last equation in the system and rearrange, taking all the \( r_j(t) \) terms to the right hand side. The coefficient of \( r_j(t) \) is \( 1/\det A' \) times \( (-1)^{n+j} \det A_{j1} \). Hence we obtain

\[
\frac{1}{\det A'} \left( \sum_{j=1}^{n} a_{nj} \cdot (-1)^{j+1} \det A_{n,j} \right) x_1 = cx_1^2 + \frac{1}{\det A'} \left( \sum_{j=1}^{n} (-1)^{n+j} \det A_{j1} r_j(t) \right)
\]

Up to a sign, the coefficient of \( x_1 \) is \( 1/\det A' \) times the determinant of \( A \), expanded along the \( n \)th row. So this can be rewritten as

\[
(-1)^{n-1} \frac{\det A}{\det A'} = cx_1^2 + \frac{1}{\det A'} \left( \sum_{j=1}^{n} (-1)^{n+j} \det A_{j1} r_j(t) \right)
\]

By hypothesis, \( \det A' \) divides \( \det A \) and all of the \( \det A_{j1} \), which finishes the proof.

\( \square \)

Now, we must show that the linear equations in the Selesnick-Burrus system satisfy the hypotheses of the Lemma. But this follows from the determinant formulas from §4. In our case, \( x_1 = m_{2L+1} \), and \( x_2, \ldots, x_n \) are \( m_{2L+1}, \ldots, m_{4L+4} \). Continuing from Lemma (6.2), \( A \) is the matrix \( A(L+2, L+1; K) \) (times a lower triangular factor of determinant 1), and \( A' \) is \( A'(L+2, L; K) \) (times another lower triangular factor of determinant 1). Hence by Theorem (4.10) we have \( c_A = 2s + m - 1 + K = 3L + 4 + K \), and

\[
\det A = \delta(L+2, L+1; K) = a \prod_{i=2L+3+K}^{4L+5+K} (2t-i)^{\left\lfloor \frac{L+1-|3L+4+K-1|+1}{2} \right\rfloor} + 1
\]
for some constant $a$. Similarly by Corollary (4.12), we have $c = 3L + 4 + K$ and
\[
\det A' = \delta(L + 2, L; K) = a' \prod_{i=2L+5+K}^{4L+3+K} (2t - i)^{\frac{L-1-(3L+i+K-1)}{2} + 1},
\]
for some constant $a'$. Because of the $L - 1$ in the exponent, each factor in $\det A'$ occurs with multiplicity one less than in $\det A$. Hence $\det A'$ is precisely the product of the first $L$ diagonal entries of the Smith normal form of the $(L + 1) \times (L + 1)$ matrix $A$, or equivalently, $\frac{\det A'}{\det A}$ is a polynomial whose roots are all the roots of $\det A = 0$, but all with multiplicity 1. Hence the conclusion of Lemma (6.5) holds, and we obtain an equation of the form (6.4).

Because of similar cancellations, the final nonlinear equation (from $F^{(4L+8)}(0) = 0$) has the form
\[
(6.6) \quad 2m_{2L+3}^2 + \frac{4L + 4}{2L - 1} m_{2L+3} + \frac{6L + 7}{2L - 1} = 0
\]
after we substitute for $m_{2L+4}, \ldots, m_{3L+4}$ from (6.3). The polynomial of degree $2L - 1$ in the denominators is the same in both terms after the first, and equals the last diagonal entry in the Smith normal form of $A(L + 2, L; K)$ (the reduced polynomial of the determinant $\tilde{\delta}(L + 2, L; K)$).

The final step is to eliminate $t$ from the two equations (6.4) and (6.6). For this, we use the determinant form of the Sylvester resultant (see [CLO]) of two quadratic polynomials, after clearing the denominators in (6.6). We have
\[
Res = \det \left( \begin{array}{cccc}
0 & [2L + 3] & [4L + 6] & 0 \\
[2L + 1] & [2L + 3] & [4L + 6] & 0 \\
0 & [4L + 4] & [6L + 7] & 0 \\
0 & [4L + 4] & [6L + 7] & \end{array} \right) \\
= [12L + 14]
\]
This concludes the proof of Theorem (6.1).

Computation of these polynomials for a number of $L$ and $K$ suggests that the polynomial of degree $[12L + 14]$ is always irreducible over $\mathbb{Q}$, hence has distinct roots in $\mathbb{C}$. But we do not have a proof of this fact. The filters obtained in this case are considered in [LL].

The strategy from (3.10) that we have used here and in §4 can also be used to analyze the lower diagonals $M = 2L + q, q \geq 5$ in Region II. For instance, for $q = 5$, solving the linear part of the reduced Selesnick-Burrus system and substituting into the remaining equations leads to a system of 3 quadrics in 2 variables (or 3 homogeneous variables). Explicit determinantal formulas for the multipolynomial resultant $Res_{2,2,2}$ (see [CLO], Chapter 3, §2) can be applied, and it can be seen that for $L \geq 2$, the degree of the univariate polynomial in $t$ is $20L + 26$. We will not present the details of that case here.

However, the resultants needed to eliminate variables in the final, nonlinear system get progressively harder to analyze as $q$ increases. Unfortunately, the Dixon resultants leading to the most efficient computations tend to have many extraneous factors that must be accounted for. As a result, they are less convenient for the type of analysis done here.
REFERENCES

[BEM] L. Busé, M. Elkadi, and B. Mourrain, Generalized Resultants over Unirational Algebraic Varieties, J. Symb. Comp. 29 (2000), 515-526.

[CE] J. Canny and I. Emiris, An Efficient Algorithm for the Mixed Sparse Resultant, Applied Algebra, Algebraic Algorithms, and Error Correcting Codes (AAECC-10) (G. Cohen, T. Mora, and O. Moreno, eds.), Lecture Notes in Computer Science 673, Springer Verlag, New York, NY, 1993, pp. 89-104.

[CLO] D. Cox, J. Little, and D. O'Shea, Using Algebraic Geometry, Graduate Texts in Mathematics 185, Springer Verlag, New York, NY, 1998.

[EM] I. Emiris and B. Mourrain, Matrices in Elimination Theory, J. Symb. Comp. 28 (1999), 3-44.

[H] J. Harris, Algebraic Geometry, A First Course, Graduate Texts in Mathematics 133, Springer Verlag, New York, NY, 1992.

[Ja] N. Jacobson, Basic Algebra I, W. H. Freeman, San Francisco, CA, 1974.

[Jo] J. Jouanolou, Le formalisme du résultant, Advances in Mathematics 90 (1991), 117-263.

[KSY] D. Kapur, T. Saxena, and L. Yang, Algebraic and Geometric Reasoning using Dixon Resultants, Proceedings of ACM International Symposium on Symbolic and Algebraic Computation, Montreal, 1995.

[LL] J. Little, and R. Lewis, On Selesnick and Burrus’s Maximally Flat Filters with Reduced Group Delay, in preparation.

[M] B. Mourrain, An introduction to algebraic methods for solving polynomial equations, preprint, February 2001.

[PM] J. Proakis and D. Manolakis, Digital Signal Processing, Prentice Hall, Upper Saddle River, NJ, 1996.

[SB] I. Selesnick, and C. S. Burrus, Maximally Flat Lowpass FIR Filters with Reduced Delay, IEEE Trans. Circuits and Systems II 45(1) (1998), 53-68.

[S] B. Sturmfels, Sparse elimination theory, Computational Algebraic Geometry and Commutative Algebra (D. Eisenbud and L. Robbiano, eds.), Cambridge University Press, Cambridge, UK, 1993, pp. 264-298.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE COLLEGE OF THE HOLY CROSS
WORCESTER, MA 01610

E-mail address: little@mathcs.holycross.edu