ON THE DEGREE OF ALGEBRAIC CYCLES ON HYPERSURFACES

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Abstract. Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$. Griffiths and Harris conjectured in 1985 that the degree of every curve $C \subset X$ is divisible by $d$. Despite substantial progress by Kollár in 1991, this conjecture is not known for a single value of $d$. Building on Kollár’s method, we prove this conjecture for infinitely many $d$, the smallest one being $d = 5005$. The set of these degrees $d$ has positive density. We also prove a higher-dimensional analogue of this result and construct smooth hypersurfaces defined over $\mathbb{Q}$ that satisfy the conjecture.

1. Introduction

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

In their famous work [GH85], Griffiths and Harris made five conjectures about curves on a very general hypersurface $X \subset \mathbb{P}^4$. The weakest one is:

Conjecture 1 (Griffiths–Harris). Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree $d \geq 6$. Then the degree of every curve $C \subset X$ is divisible by $d$.

This would follow from the stronger conjectures that $C$ is algebraically or even rationally equivalent to a multiple of a plane section. Their strongest conjecture, stating that $C$ is a complete intersection with a surface in $\mathbb{P}^4$, was disproven by Voisin [Voi89]. In contrast, Wu proved in [Wu90] that every curve $C \subset X$ of degree at most $2d - 2$ is a complete intersection with a surface in $\mathbb{P}^4$. In particular, the degree of every curve $C \subset X$ is at least $d$. Nevertheless, Conjecture 1 is still open in any degree $d$.

More generally, one might conjecture:

Conjecture 2. Let $n \geq 1$ be an integer, and let $X \subset \mathbb{P}^{n+1}$ be a very general hypersurface of degree $d \geq 2n$. Then the degree of every positive-dimensional closed subvariety $Z \subset X$ is divisible by $d$.

Date: 1st June 2022.

2020 Mathematics Subject Classification. 14C25 (primary); 14C30, 14D06, 14J70.
Note that this conjecture is wrong if $1 < d < 2n$ because a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d < 2n$ contains a line. The case $n = 1$ of this conjecture is trivial, and the case $n = 2$ follows from the Noether–Lefschetz theorem [Lef24]. For $n \geq 3$, however, Conjecture 2 is not known for a single $d$ yet.

1.1. Kollár’s method. For a very general hypersurface $X \subset \mathbb{P}^4$ of degree $d$, let us write

$$f_3(d) = \gcd\{\deg C \mid C \subset X \text{ curve}\}.$$ 

Conjecture 1 states that $f_3(d) = d$ for all $d \geq 6$.

The Trento examples [K+91], mostly due to Kollár, achieve substantial progress towards Conjecture 1 via specialization arguments. By degenerating a very general hypersurface $X \subset \mathbb{P}^4$ into a singular projection of a smooth projective threefold $Y$, the following results are obtained:

(i) $d \mid 6 \cdot f_3(d^3)$ for all $d \geq 1$ (Kollár, see also [SV05, section 2])
(ii) $d \mid 6 \cdot f_3(3d^2)$ for all $d \geq 4$ (Kollár)
(iii) $d \mid 2 \cdot f_3(6d)$ for all $d \geq 9$ (van Geemen, improved by [DHS94])

These results naturally generalize to arbitrary dimension $n \geq 3$ (see section 2), but they do not prove Conjecture 2 for any $d \geq 2n$.

1.2. Main results. The main purpose of this article is to show the following:

**Theorem 3.** Let $n \geq 3$ be an integer. Then there exists a set of degrees $d$ with positive density such that Conjecture 2 is true in degree $d$.

In particular, the case $n = 3$ proves the conjecture of Griffiths and Harris for infinitely many degrees $d$. The smallest of them is $d = 5005$. Hence, $d = 5005$ is the first degree where Conjecture 1 is currently known.

For a projective variety $X$, let us introduce the group

$$Z^{2c}(X) = \frac{H^{c,c}(X,\mathbb{Z})}{\langle \text{alg. classes} \rangle},$$

which measures the failure of the integral Hodge conjecture on $X$ in codimension $c$. As a consequence of Theorem 3, we get:

**Corollary 4.** Let $n \geq 3$ be an integer. Then there exists a set of degrees $d$ with positive density such that

$$Z^{2c}(X) \cong \mathbb{Z}/d$$

for a very general hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d$ and for $\frac{n}{2} < c < n$. 
In particular, the integral Hodge conjecture fails for very general hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of these degrees \( d \).

For \( n = 3 \), the previous result from [DHS94] (item (iii) above) allows to disprove the integral Hodge conjecture for a set of degrees with density \( \frac{1}{6} \).

With our approach, we can actually show the failure of the integral Hodge conjecture for a set of degrees with density 1:

**Theorem 5.** Let \( n \geq 3 \) be an integer. Then there exists a set of degrees \( d \) with density 1 such that the integral Hodge conjecture for very general hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of degree \( d \) is false in every codimension \( c \) with \( \frac{n}{2} < c < n \).

Theorems 3 and 5 as well as the known results (i), (ii), and (iii) concern very general hypersurfaces. This might possibly exclude all hypersurfaces defined over number fields. However, Totaro proved in [Tot13] that (i), (ii), and (iii) are in most cases also valid for certain smooth hypersurfaces \( X \subset \mathbb{P}^4 \) defined over \( \mathbb{Q} \). Using Totaro’s results, we show:

**Theorem 6.** There exists a smooth hypersurface \( X \subset \mathbb{P}^4 \) of degree \( d \geq 6 \) defined over \( \mathbb{Q} \) such that the degree of every curve \( C \subset X_{\mathbb{Q}} \) is divisible by \( d \).

### 1.3. General observations and notation

Let \( X \subset \mathbb{P}^{n+1} \) be a very general hypersurface of degree \( d \) and let \( \alpha = c_1(O_X(1)) \in H^{1,1}(X, \mathbb{Z}) \) denote the hyperplane class.

By the Lefschetz hyperplane theorem, the restriction map

\[ H^i(\mathbb{P}^{n+1}, \mathbb{Z}) \to H^i(X, \mathbb{Z}) \]

is an isomorphism for \( i < n \). Therefore,

\[ H^{2c}(X, \mathbb{Z}) = H^{c,c}(X, \mathbb{Z}) = \mathbb{Z} \cdot \alpha^c \quad \text{for } c < \frac{n}{2}. \]

Hence, Conjecture 2 is true for subvarieties \( Z \subset X \) of codimension \( c < \frac{n}{2} \).

Moreover, \( Z^{2c}(X) \) is trivial for \( c < \frac{n}{7} \).

For \( \frac{n}{2} < c \leq n \), Poincaré duality implies that

\[ H^{2c}(X, \mathbb{Z}) = H^{c,c}(X, \mathbb{Z}) = \mathbb{Z} \cdot \frac{1}{d} \alpha^c \quad \text{for } \frac{n}{2} < c \leq n. \]

Therefore, no topological obstructions on the degree of subvarieties \( Z \subset X \) of codimension \( c \) with \( \frac{n}{2} < c \leq n \) exist. In this case, we can rephrase Conjecture 2 in terms of the group \( Z^{2c}(X) \). Since \( \alpha^c \) is clearly algebraic, \( Z^{2c}(X) \) is a quotient of \( \mathbb{Z}/d \). The order of \( Z^{2c}(X) \) is given by the greatest common divisor of the degrees of all subvarieties \( Z \subset X \) of codimension \( c \). Hence,
Conjecture 2 in codimension $c$ for $\frac{n}{2} < c < n$ is equivalent to $Z^{2c}(X) \cong \mathbb{Z}/d$. In particular, Theorem 3 implies Corollary 4.

Note that it suffices to prove Conjecture 2 for curves $C \subset X$ because every positive-dimensional subvariety $Z \subset X$ gives rise to a curve $C \subset X$ of the same degree after intersecting $Z$ with a suitable linear subspace.

If $X \subset \mathbb{P}^{n+1}$ is very general, the order of the group $Z^{2n-2}(X)$ only depends on $n$ and $d$. We set $f_n(d) := |Z^{2n-2}(X)|$ for $n \geq 3$ and $d \geq 1$. In other words,

$$f_n(d) = \gcd\{\deg C \mid C \subset X \text{ curve}\}.$$

For $n = 3$, this agrees with the definition of $f_3(d)$ given earlier.

We know that $f_n(d) \mid d$ for all degrees $d$, and Conjecture 2 in degree $d$ is equivalent to $f_n(d) = d$.

1.4. Proof idea and overview. Looking at the existing results towards Conjecture 1, statement (iii) seems to be more powerful than (i) and (ii). However, the main idea for proving Theorem 3 is to combine (i), (ii), and (iii) in order to show $d \mid f_3(d)$ for certain degrees $d$. This is based on the observation that $d \mid f_n(d_1)$ and $d \mid f_n(d_2)$ together imply $d \mid f_n(d_1 + d_2)$, as can be seen by another degeneration argument.

In section 2, we develop higher-dimensional analogues of the Trento examples [K+91]. These allow to carry out our approach in arbitrary dimension $n \geq 3$.

In section 3, we will prove the following statement behind Theorem 3:

**Proposition 7.** Let $n \geq 3$ be an integer. Then we have $f_n(d) = d$ if $d$ is coprime to $n!$ and the largest prime power $q$ dividing $d$ satisfies

$$((\binom{n}{2} - 1) \cdot q^n + (n! - \binom{n}{2}) \cdot q^{n-1} + (2^n + 1) \cdot n!) \leq d.$$

For $n = 3$, the smallest degree $d$ with this property is

$$d = 5 \cdot 7 \cdot 11 \cdot 13 = 5005.$$

In section 4, we will see that the positive integers $d$ fulfilling the condition in Proposition 7 have positive density for all $n \geq 3$, thus completing the proof of Theorem 3.

Finally, we prove Theorem 5 in section 5 and Theorem 6 in section 6.
Acknowledgements

I am grateful to my supervisor Stefan Schreieder for many helpful suggestions and comments concerning this work. Further, I would like to thank the anonymous referee for useful remarks that improved the exposition.

This project received partial funding by the DFG project “Topological properties of algebraic varieties” (grant no. 416054549) and by the ERC grant “RationAlgic” (grant no. 948066). The final part of this project was carried out while the author was in residence at Institut Mittag-Leffler, supported by the Swedish Research Council under grant no. 2016-06596.

2. The Trento examples

In this section, we take a closer look at the Trento examples from [K+91] by generalizing them to arbitrary dimension $n \geq 3$.

All examples rely on the following lemma:

**Lemma 8** (Kollár). Let $n \geq 3$ be an integer. Suppose that there exists a smooth projective variety $Y$ of dimension $n$ with a very ample line bundle $L$ such that $L^n = d$ and $k \mid B \cdot L$ for every curve $B \subset Y$. Then we have

$$k \mid n! \cdot f_n(d).$$

**Proof.** We consider the embedding $Y \subset \mathbb{P}^N$ given by the very ample line bundle $L$, and take a general linear projection

$$\pi: Y \to \mathbb{P}^{n+1}.$$ 

Then $\pi(Y) \subset \mathbb{P}^{n+1}$ is a hypersurface of degree $L^n = d$.

By [Mat73] (see also [BE10]), each fiber of $\pi$ has at most $n + 1$ distinct points (note that the fibers of $\pi$ might have a much larger degree than $n + 1$ due to their non-reduced scheme structure, but for our argument we only need that they consist of at most $n + 1$ points topologically). Moreover, only finitely many fibers have exactly $n + 1$ distinct points. Hence, for every curve $C \subset \pi(Y)$, the curve $B = \pi^{-1}(C)_{\text{red}} \subset Y$ admits a finite surjective map $\pi|_B: B \to C$ of degree at most $n$. Therefore, we have $B \cdot L \mid n! \cdot \deg C$ and thus $k \mid n! \cdot \deg C$.

Now if $X \subset \mathbb{P}^{n+1}$ is a very general hypersurface of degree $d$, every curve on $X$ specializes to a curve $C \subset \pi(Y)$ of the same degree. For more details on this degeneration argument, see [SV05, section 2]. Since $k \mid n! \cdot \deg C$, it follows that $k \mid n! \cdot f_n(d)$. □
Corollary 9 (Kollár). For all \( n \geq 3 \) and \( d \geq 1 \), we have
\[
d \mid n! \cdot f_n(d^n)
\]
In particular, we have \( d \mid f_n(d^n) \) if \( d \) is coprime to \( n! \).

Proof. We apply Lemma 8 to \( Y = \mathbb{P}^n \) and \( L = \mathcal{O}_{\mathbb{P}^n}(d) \). \( \square \)

Corollary 10 (Kollár). For all \( n \geq 3 \) and \( d \geq 4 \), we have
\[
d \mid n! \cdot f_n\left(\binom{n}{2}d^{n-1}\right)
\]
In particular, we have \( d \mid f_n\left(\binom{n}{2}d^{n-1}\right) \) if \( d \) is coprime to \( n! \).

Proof. We take \( Y = S \times \mathbb{P}^{n-2} \) where \( S \subset \mathbb{P}^3 \) is a very general surface of degree \( d \geq 4 \). On \( Y \), we consider the very ample line bundle
\[
L = \text{pr}_1^*\mathcal{O}_S(1) \otimes \text{pr}_2^*\mathcal{O}_{\mathbb{P}^{n-2}}(d)
\]
Then we have \( L^n = \binom{n}{2}d^{n-1} \) (note that the factor \( \binom{n}{2} \) is accidentally missing in [K+91]). If \( B \subset Y \) is a curve, we obtain
\[
B \cdot L = (\text{pr}_1)_*B \cdot \mathcal{O}_S(1) + (\text{pr}_2)_*B \cdot \mathcal{O}_{\mathbb{P}^{n-2}}(d) \equiv 0 \pmod{d},
\]
because \( d \) divides the degree of the curve \( \text{pr}_1(B) \subset S \) by the Noether–Lefschetz theorem [Lef24]. Therefore, Lemma 8 implies the result. \( \square \)

Corollary 11 (van Geemen, Debarre–Hulek–Spandaw). For all \( n \geq 3 \) and \( d \geq 2^n + 1 \), we have
\[
d \mid (n-1)! \cdot f_n(n! \cdot d)
\]
In particular, we have \( d \mid f_n(n! \cdot d) \) if \( d \) is coprime to \( (n-1)! \).

Proof. Let \( (Y, L) \) be a very general polarized Abelian variety of dimension \( n \) and type \((1, \ldots, 1, d)\). Then we have \( L^n = n! \cdot d \). It was shown in [DHS94] that the line bundle \( L \) is very ample.

Since \( L \) is of type \((1, \ldots, 1, d)\), we have
\[
c_1(L) = dx_1 \wedge dx_2 + \cdots + dx_{2n-3} \wedge dx_{2n-2} + d \cdot dx_{2n-1} \wedge dx_{2n} \in H^2(Y, \mathbb{Z})
\]
for a suitable basis \( dx_1, \ldots, dx_{2n} \) of \( H^1(Y, \mathbb{Z}) \). From this we see that
\[
\frac{c_1(L)^{n-1}}{(n-1)!} = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{2n-3} \wedge dx_{2n-2} + d \cdot \ldots \in H^{2n-2}(Y, \mathbb{Z})
\]
is not divisible by any integer larger than 1. If \( Y \) is very general, the algebraic classes in \( H^{2n-2}(Y, \mathbb{Z}) \) are rational multiples of \( c_1(L)^{n-1} \), and thus integral multiples of \( c_1(L)^{n-1}/(n-1)! \). Therefore, the degree of every curve \( B \subset Y \) is divisible by \( L^n/(n-1)! = nd \). Hence, Lemma 8 gives \( nd \mid n! \cdot f_n(n! \cdot d) \). \( \square \)
3. Proof of Proposition 7

We combine Corollaries 9, 10, and 11 via the following simple observation:

**Lemma 12.** If \( d \mid f_n(d_1) \) and \( d \mid f_n(d_2) \), then \( d \mid f_n(d_1 + d_2) \).

**Proof.** Let \( C \subset X \) be a curve on a very general hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( d_1 + d_2 \). By the same degeneration argument which we used in the proof of Lemma 8, every hypersurface \( X \subset \mathbb{P}^{n+1} \) of degree \( d_1 + d_2 \) contains a curve of the same degree as \( C \). In particular, we can choose \( X = X_1 \cup X_2 \) to be the union of very general hypersurfaces \( X_1, X_2 \subset \mathbb{P}^{n+1} \) of degrees \( d_1 \) and \( d_2 \), respectively. Then every irreducible component of a curve \( C \subset X_1 \cup X_2 \) lies on \( X_1 \) or \( X_2 \). By assumption, the degrees of these components are divisible by \( d \). We conclude that \( d \mid \deg C \). □

Now we can prove Proposition 7.

**Proof of Proposition 7.** Since \( d \) is a product of pairwise coprime powers of primes, it suffices to show \( q \mid f_n(d) \) for every prime power \( q \mid d \). By assumption, we have

\[
\left( \binom{n}{2} - 1 \right) \cdot q^n + (n! - \binom{n}{2}) \cdot q^{n-1} + (2^n + 1) \cdot n! \leq d.
\]

(*)

We choose \( i \in \{0, \ldots, \binom{n}{2} - 1\} \) such that

\[
d \equiv i \cdot q^n \pmod{\binom{n}{2}}.
\]

This is possible because \( \binom{n}{2} \) divides \( n! \) and \( q \) is coprime to \( n! \).

Then we choose \( j \in \{0, \ldots, n! - 1\} \) such that

\[
d \equiv i \cdot q^n + j \cdot q^{n-1} \pmod{n!}.
\]

Our choice of \( i \) implies that \( j \) is divisible by \( \binom{n}{2} \), so we have \( j \leq n! - \binom{n}{2} \).

By our choice of \( j \), there exists an integer \( k \) such that

\[
d = i \cdot q^n + j \cdot q^{n-1} + k \cdot n!.
\]

Since \( q \mid d \), we have \( q \mid k \). And from (*) we get \( k \geq 2^n + 1 \).

Now we have:

- \( q \mid f_n(q^n) \) by Corollary 9
- \( q \mid f_n(\binom{n}{2}q^{n-1}) \) by Corollary 10
- \( k \mid (n-1)! \cdot f_n(k \cdot n!) \) by Corollary 11 and thus \( q \mid f_n(k \cdot n!) \)

Combining these results via repeated usage of Lemma 12, we obtain

\[
q \mid f_n \left( i \cdot q^n + \frac{j}{\binom{n}{2}} \cdot \binom{n}{2} q^{n-1} + k \cdot n! \right) = f_n(d) .
\]

□
Remark 13. Using only Corollaries 9 and 11, one can show a weaker statement where (⋆) is replaced by the assumption
\[(n! - 1) \cdot q^n + (2^n + 1) \cdot n! \leq d .\]
It turns out that this stronger condition results in a set of degrees \(d\) with the same density. Therefore, Corollary 10 is not strictly necessary to obtain Theorem 3. However, only together with Corollary 10 we can prove Conjecture 1 for \(d = 5005\).

4. Some analytic number theory

A set \(A\) of positive integers has positive density if
\[
\liminf_{m \to \infty} \frac{|A \cap \{1, \ldots, m\}|}{m} > 0 .
\]

In this section, we want to prove the following:

Proposition 14. Let \(n \geq 1\) be an integer and \(\lambda > 0\) a real number. Then the positive integers \(d\) coprime to \(n!\) such that the largest prime power dividing \(d\) is not larger than \(\lambda \cdot d^{1/n}\) have positive density.

Together with Proposition 7, this will complete the proof of Theorem 3, since for \(d \gg 0\) the condition \(q \leq \left(\binom{n}{2}\right)^{-1/n} d^{1/n}\) implies
\[
\left(\left(\binom{n}{2}\right) - 1\right) \cdot q^n + (n! - \binom{n}{2}) \cdot q^{n-1} + (2^n + 1) \cdot n! \leq d ,
\]
so Proposition 7 applies to a set of degrees \(d\) with positive density.

We use the following easy lemma on the distribution of prime powers:

Lemma 15. Let \(\Pi(m)\) denote the number of prime powers \(\leq m\). Then
\[
\frac{\Pi(m)}{m} \to 0 .
\]

Proof. By the prime number theorem, we have
\[
\frac{\pi(m)}{m} \to 0 ,
\]
where \(\pi(m)\) counts the prime numbers \(\leq m\). Now if \(p^e \leq m\) is a prime power with \(e \geq 2\), we have \(e \leq \log_2 m\) and \(p \leq \sqrt[2]{m}\), so we conclude by noting that
\[
\frac{\log_2 m \cdot \sqrt[2]{m}}{m} \to 0 .
\]

We also need the following consequence of Mertens’ theorem:
Lemma 16. We have
\[ \sum_{x^{1/n} < p \leq x} \frac{1}{p} \xrightarrow{x \to \infty} \log n, \]
where the sum runs only over prime numbers \( p \).

Proof. By [Mer74], there exists a constant \( C \) such that
\[ \sum_{p \leq x} \frac{1}{p} - \log \log x \xrightarrow{x \to \infty} C. \]
Since \( \log x - \log \log x^{1/n} = \log n \), we conclude that
\[ \sum_{x^{1/n} < p \leq x} \frac{1}{p} = \sum_{p \leq x} \frac{1}{p} - \sum_{p \leq x^{1/n}} \frac{1}{p} \xrightarrow{x \to \infty} \log n. \]
\[ \square \]

Now we can prove Proposition 14.

Proof of Proposition 14. Let \( \alpha > 1 \) be a real number. Dickman proved in [Dic30] that the positive integers \( d \) whose largest prime divisor is not larger than \( d^{1/\alpha} \) have density \( \rho(\alpha) \), where \( \rho \) denotes the Dickman function. More generally, this result was proven for arithmetic progressions in [Buc49]. Therefore, the positive integers \( d \) coprime to \( n! \) whose largest prime divisor is not larger than \( d^{1/\alpha} \) have density
\[ \frac{\varphi(n!)}{n!} \cdot \rho(\alpha), \]
where \( \varphi \) denotes Euler’s totient function. Since \( \rho \) is continuous, it follows that the positive integers \( d \) coprime to \( n! \) whose largest prime divisor is not larger than \( \lambda \cdot d^{1/n} \) have density
\[ \frac{\varphi(n!)}{n} \cdot \rho(n) > 0. \]

In other words, Proposition 14 holds if we replace ‘prime power’ by ‘prime number’. Hence, it suffices to show that the positive integers \( d \) divisible by a prime power \( q = p^e > \lambda \cdot d^{1/n} \) with \( e \geq 2 \) have density 0.

For a given \( x \), let us consider the number \( N(x) \) of positive integers \( d \leq x \) with this property. Any such \( d \) can be written as
\[ d = q \cdot r, \]
where \( q = p^e \geq \lambda \cdot d^{1/n} \) is a prime power with \( e \geq 2 \). For fixed \( q \leq x^{1/n} \), there are at most \( \lambda^{-n} q^{n-1} \) possibilities for \( d \) because
\[ r = \frac{d}{q} \leq \frac{\lambda^{-n} q^n}{q} = \lambda^{-n} q^{n-1}. \]
For fixed $q > x^{1/n}$, there are at most $\frac{x}{q}$ possibilities for $d$ because
\[ r = \frac{d}{q} \leq \frac{x}{q}. \]
Together we obtain the upper bound
\[ N(x) \leq \lambda^{-n} \cdot \sum_{q \leq x^{1/n}} q^{n-1} + x \cdot \sum_{x^{1/n} < q \leq x} \frac{1}{q}, \]
where both sums run only over prime powers $q = p^e$ with $e \geq 2$.

Using Lemma 15, we get
\[ \frac{1}{x} \cdot \sum_{q \leq x^{1/n}} q^{n-1} \leq \frac{1}{x} \cdot \Pi(x^{1/n}) \cdot \left( x^{1/n} \right)^{n-1} \leq \frac{\Pi(x^{1/n})}{x^{1/n}} \xrightarrow{x \to \infty} 0, \]
so in order to prove $\frac{N(x)}{x} \to 0$ for $x \to \infty$, it remains to show that
\[ \sum_{x^{1/n} < q \leq x} \frac{1}{q} \xrightarrow{x \to \infty} 0. \]

For $q = p^e \leq x$, we have $e \leq \log_2 x$ and thus
\[
\limsup_{x \to \infty} \sum_{x^{1/n} < q \leq x} \frac{1}{q} = \limsup_{x \to \infty} \sum_{e=2}^{\left\lfloor \log_2 x \right\rfloor} \sum_{x^{1/n} < p^e \leq x} \frac{1}{p^e} \\
\leq \limsup_{x \to \infty} \left( \sum_{e=2}^{\left\lfloor \log_2 x \right\rfloor} \sum_{x^{1/n} < p^e \leq x^{1/e}} \frac{x^{1/e} - x^{1/e} \log_2 x}{p} + \sum_{e=n+1}^{\left\lfloor \log_2 x \right\rfloor} \frac{x^{1/e}}{x^{1/n}} \right). 
\]

Here we used $\frac{1}{p^e} = p^{-(e-1)}$ for $p > x^{1/e}$ if $2 \leq e \leq n$, and $\frac{1}{p^e} \leq \frac{1}{x^{1/n}}$ for $p > x^{1/en}$ if $e \geq n + 1$. Applying Lemma 16 for each $2 \leq e \leq n$, and using $x^{1/e} \leq x^{1/(n+1)}$ for $e \geq n + 1$, we obtain
\[ \cdots \leq \limsup_{x \to \infty} \left( \sum_{e=2}^{n} \frac{x^{1/e} \log n}{x^{1/n}} + \frac{\log_2 x}{x^{1/(n+1)}} \right) = 0. \]

**Remark 17.** The proof shows that the density in Theorem 3 amounts to
\[ \frac{\varphi(n!)}{n!} \cdot \rho(n). \]
For example, the density for $n = 3$ is $\frac{1}{3} \cdot \rho(3) \approx 1.6\%$. 

5. Failure of the integral Hodge conjecture

By the work of Kollár [K+91], hypersurfaces provide an example for varieties where the integral Hodge conjecture fails due to a non-torsion cohomology class. Theorem 5 says that this counterexample works for almost all degrees \( d \) (in the sense of density).

**Proof of Theorem 5.** The failure of the integral Hodge conjecture in degree \( d \) is equivalent to \( f_n(d) \neq 1 \). Hence, we need to show that the positive integers \( d \) with \( f_n(d) \neq 1 \) have density 1. If \( d \) has a prime divisor \( p \) coprime to \( n! \) such that
\[
\left( \frac{n}{2} - 1 \right) \cdot p^n + \left( n! - \frac{n}{2} \right) \cdot p^{n-1} + (2^n + 1) \cdot n! \leq d ,
\]
then the proof of Proposition 7 shows that \( p \mid f_n(d) \). Therefore, for every prime \( p > n \) all sufficiently large multiples \( d \) of \( p \) satisfy \( f_n(d) \neq 1 \).

For a given \( \varepsilon > 0 \), we can find distinct primes \( p_1, \ldots, p_N > n \) such that
\[
\frac{1}{p_1} + \cdots + \frac{1}{p_N} > \frac{1}{\varepsilon}
\]
since the sum of the reciprocals of all primes diverges. We know from the previous paragraph that for \( d \gg 0 \), we might have \( f_n(d) = 1 \) only if \( d \) is not divisible by any of the primes \( p_1, \ldots, p_N \). Hence, the density of these \( d \) is at most
\[
\left( 1 - \frac{1}{p_1} \right) \cdots \left( 1 - \frac{1}{p_N} \right) < \frac{1}{(1 + \frac{1}{p_1}) \cdots (1 + \frac{1}{p_N})} < \frac{1}{\frac{1}{p_1} + \cdots + \frac{1}{p_N}} < \varepsilon .
\]
This concludes the proof. \( \square \)

**Remark 18.** For any degree \( d \geq 1 \), there exist special smooth hypersurfaces \( X \subset \mathbb{P}^{n+1} \) of degree \( d \) which do satisfy the integral Hodge conjecture in every codimension \( c \) with \( \frac{n}{2} < c < n \). For example, we can take the Fermat hypersurface
\[
\{ x_0^d + \cdots + x_{n+1}^d = 0 \} \subset \mathbb{P}^{n+1},
\]
since it contains an \( (n - c) \)-dimensional linear subspace for any \( \frac{n}{2} < c < n \).

**Remark 19.** There are infinitely many degrees \( d \) for which we are not able to disprove the integral Hodge conjecture. In particular, this problem remains open when \( d \) is a prime number, in which case the failure of the integral Hodge conjecture is equivalent to Conjecture 2.
6. Example over $\mathbb{Q}$

The basic idea in [Tot13] is to replace the original degeneration arguments with degenerations to positive characteristic. To prove Theorem 6, we apply this idea to the proof of Proposition 7 and use some of Totaro’s results.

**Proposition 20.** There exists a smooth hypersurface $X \subset \mathbb{P}^4$ of degree $d = 7 \cdot 13 \cdot 19 \cdot 31 = 53599$ defined over $\mathbb{Q}$ such that the degree of every curve $C \subset X_{\mathbb{Q}}$ is divisible by $d$.

**Proof.** We will show the following lemma:

**Lemma 21.** Let $q$ be any of the four prime divisors of $d$. Then there exists a prime $p$ and a hypersurface $Y \subset \mathbb{P}^4_{\mathbb{F}_p}$ of degree $d$ such that the degree of every curve $C \subset Y_{\mathbb{F}_p}$ is divisible by $q$. Moreover, $p$ can be chosen such that finitely many given primes are avoided.

Once this lemma is proven, we proceed as follows: We ensure that the primes $p$ for each prime divisor $q \mid d$ are pairwise different. Then we use the Chinese remainder theorem to construct a smooth hypersurface $X \subset \mathbb{P}^4$ defined over $\mathbb{Q}$ that simultaneously specializes to all four hypersurfaces $Y \subset \mathbb{P}^4_{\mathbb{F}_p}$ from Lemma 21. This $X$ satisfies our claim. □

**Proof of Lemma 21.** Since $q \equiv 1 \pmod{6}$, we can write $d = q^3 + 6k$ for some integer $k$. Note that $k \geq 38$. As in the proof of Lemma 12, we want to take $Y = Y_1 \cup Y_2$, where $Y_1, Y_2 \subset \mathbb{P}^4_{\mathbb{F}_p}$ are two hypersurfaces of degrees $q^3$ and $6k$, respectively, such that every curve $C \subset (Y_i)_{\mathbb{F}_p}$ has degree divisible by $q$.

We first construct $Y_1$. A priori, [Tot13, Corollary 4.2] only gives hypersurfaces $Y_1 \subset \mathbb{P}^4$ over $\overline{\mathbb{F}}_p$ with this property for every $p > q^3$. However, as in the proofs of [Tot13, Lemma 5.1] and [Tot13, Theorem 6.1], we can apply [Tot13, Lemma 4.3] to $\mathbb{P}^3_{\mathbb{Q}}$ (polarized by $\mathcal{O}_{\mathbb{P}^3}(q)$) to get a rational map to $\mathbb{P}^4_{\mathbb{Z}}$ and obtain hypersurfaces $Y_1 \subset \mathbb{P}^4$ over $\mathbb{F}_p$ after excluding finitely many primes $p$.

Now we construct $Y_2$. The proof of [Tot13, Theorem 6.1] yields a prime $p$ and a hypersurface $Y_2 \subset \mathbb{P}^4_{\mathbb{F}_p}$ of degree $6k$ such that $k \mid 6 \cdot \deg C$ for every curve $C \subset (Y_2)_{\mathbb{F}_p}$. Since $k$ is a multiple of $q$ and $q$ is coprime to $6$, it follows that $q \mid \deg C$. Furthermore, we can guarantee that $p$ is different from finitely many given primes (including also the primes where the construction of $Y_1$ does not work) by doing the argument of [Tot13, Theorem 6.1] over $\mathbb{Z}[1/P]$ instead of $\mathbb{Z}$, where $P$ is the product of these finitely many primes. □
Remark 22. For simplicity, we gave only one specific example over $\mathbb{Q}$. The above argument obviously works for other values than $d = 53599$ as well.

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