AN ALGORITHM THAT DECIDES TRANSLATION EQUIVALENCE IN A FREE GROUP OF RANK TWO

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Abstract. Let $F_2$ be a free group of rank 2. We prove that there is an algorithm that decides whether or not, for given two elements $u,v$ of $F_2$, $u$ and $v$ are translation equivalent in $F_2$, that is, whether or not $u$ and $v$ have the property that the cyclic length of $\phi(u)$ equals the cyclic length of $\phi(v)$ for every automorphism $\phi$ of $F_2$. This gives an affirmative solution to problem F38a in the online version (http://www.grouptheory.info) of [1] for the case of $F_2$.

1. Introduction

Let $F_n$ be the free group of rank $n \geq 2$ on the set $\Sigma$. As usual, for a word $v$ in $F_n$, $|v|$ denotes the length of the reduced word over $\Sigma$ representing $v$. A word $v$ is called cyclically reduced if all its cyclic permutations are reduced. A cyclic word is defined to be the set of all cyclic permutations of a cyclically reduced word. By $[v]$ we denote the cyclic word associated with a word $v$. Also by $\|v\|$ we mean the length of the cyclic word $[v]$ associated with $v$, that is, the number of cyclic permutations of a cyclically reduced word which is conjugate to $v$. The length $\|v\|$ is called the cyclic length of $v$.

In [2], Kapovich-Levitt-Schupp-Shpilrain introduced and studied in detail the notion of translation equivalence in free groups. The following definition is a combinatorial version of translation equivalence:

Definition 1.1 [2, Corollary 1.4]. Two words $u,v \in F_n$ are called translation equivalent in $F_n$ if the cyclic length of $\phi(u)$ equals the cyclic length of $\phi(v)$ for every automorphism $\phi$ of $F_n$.

Several different sources of translation equivalence in free groups were provided by Kapovich-Levitt-Schupp-Shpilrain [2] and Lee [3]. Pointing out in [2] that hyperbolic equivalence in surface...
groups (cf. [5]) and character equivalence in free groups are algorithmically decidable, Kapovich-Levitt-Schupp-Shpilrain raised the question about the existence of an algorithm which decides translation equivalence in free groups.

The purpose of the present paper is to prove that translation equivalence is algorithmically decidable in $F_2$.

**Theorem 1.2.** There exists an algorithm that decides whether or not, for given two elements $u$, $v$ of $F_2$, $u$ and $v$ are translation equivalent in $F_2$.

In conclusion as will be shown in Section 3, the algorithm in the statement of Theorem 1.2 is as follows.

**Algorithm.** Let $F_2 = \langle x, y \rangle$, and let $\Omega$ be the set of all chains of Whitehead automorphisms of $F_2$ of the form either

$$(\{y\}, x)^{m_k}(\{x\}, y)^{l_k} \cdots (\{y\}, x)^{m_1}(\{x\}, y)^{l_1},$$

or

$$(\{y\}, x^{-1})^{m_k}(\{x\}, y^{-1})^{l_k} \cdots (\{y\}, x^{-1})^{m_1}(\{x\}, y^{-1})^{l_1},$$

where $k \in \mathbb{N}$, each $l_i$, $m_i \geq 0$ and $\sum_{i=1}^k (l_i + m_i) \leq 2\|u\| + 3$. Then $\Omega$ is clearly a finite set. Check if $\|\psi(u)\| = \|\psi(v)\|$ for every $\psi \in \Omega$. If so, conclude that $u$ and $v$ are translation equivalent in $F_2$; otherwise conclude that $u$ and $v$ are not translation equivalent in $F_2$.

Here, as in [4], a Whitehead automorphism $\sigma$ of $F_n$ is defined to be an automorphism of one of the following two types (cf. [6]):

(W1) $\sigma$ permutes elements in $\Sigma^{\pm 1}$.

(W2) $\sigma$ is defined by a set $S \subset \Sigma^{\pm 1}$ and a letter $a \in \Sigma^{\pm 1}$ with both $a$, $a^{-1} \notin S$ in such a way that if $c \in \Sigma^{\pm 1}$ then (a) $\sigma(c) = ca$ provided $c \in S$ and $c^{-1} \notin S$; (b) $\sigma(c) = a^{-1}ca$ provided both $c$, $c^{-1} \in S$; (c) $\sigma(c) = c$ provided both $c$, $c^{-1} \notin S$. 
If $\sigma$ is of type (W2), we write $\sigma = (S, a)$. By $(\bar{S}, a^{-1})$, we mean the Whitehead automorphism $(\Sigma^{\pm 1} - S - a^{\pm 1}, a^{-1})$. It is then easy to check that

\[(1.1) \quad (S, a)(w) = (\bar{S}, a^{-1})(w)\]

for every cyclic word $w$ in $F_n$.

2. Preliminary Lemmas

We begin this section by setting some notation. As in [2], if $w$ is a cyclic word in $F_n$ and $a, b \in \Sigma^{\pm 1}$, we use $n(w; a, b)$ to denote the total number of occurrences of the subwords $ab$ and $b^{-1}a^{-1}$ in $w$. Then clearly $n(w; a, b) = n(w; b^{-1}, a^{-1})$. Similarly we denote by $n(w; a)$ the total number of occurrences of $a$ and $a^{-1}$ in $w$. Then again clearly $n(w; a) = n(w; a^{-1})$. As in [4], for two automorphisms $\phi$ and $\psi$ of $F_n$, by writing $\phi \equiv \psi$ we mean the equality of $\phi$ and $\psi$ over all cyclic words in $F_n$, that is, $\phi(w) = \psi(w)$ for every cyclic word $w$ in $F_n$.

From now on, let $F_2$ be the free group of rank 2 on the set $\{x, y\}$.

**Lemma 2.1.** Let $\alpha$ be a Whitehead automorphism of $F_2$ of type (W2). Then exactly one of $\alpha \equiv 1$, $\alpha \equiv (\{x\}, y)$, $\alpha \equiv (\{x\}, y^{-1})$, $\alpha \equiv (\{y\}, x)$ and $\alpha \equiv (\{y\}, x^{-1})$ is necessarily satisfied.

**Proof.** Let $\alpha$ be a Whitehead automorphism of $F_2$ of type (W2). By the definition of (W2), $\alpha$ is one of $(\{x\}, y)$, $(\{x^{-1}\}, y)$, $(\{x^{\pm 1}\}, y)$, $(\{x\}, y^{-1})$, $(\{x^{-1}\}, y^{-1})$, $(\{x^{\pm 1}\}, y^{-1})$, $(\{x\}, x)$, $(\{y^{-1}\}, x)$, $(\{y^{\pm 1}\}, x)$, $(\{y\}, x^{-1})$, $(\{y^{-1}\}, x^{-1})$ and $(\{y^{\pm 1}\}, x^{-1})$. Among these, $(\{x^{\pm 1}\}, y)$, $(\{x^{\pm 1}\}, y^{-1})$, $(\{y^{\pm 1}\}, x)$ and $(\{y^{\pm 1}\}, x^{-1})$ play the same role as the identity over every cyclic word in $F_2$. Moreover, by (1.1), $(\{x^{-1}\}, y) \equiv (\{x\}, y^{-1})$, $(\{x^{-1}\}, y^{-1}) \equiv (\{x\}, y)$, $(\{y^{-1}\}, x) \equiv (\{y\}, x^{-1})$ and $(\{y^{-1}\}, x^{-1}) \equiv (\{y\}, x)$ in $F_2$, thus proving the lemma. \(\square\)

Now for the rest of the paper, let $\sigma = (\{x\}, y)$ and $\tau = (\{y\}, x)$ be Whitehead automorphisms of $F_2$. Then obviously $\sigma^{-1} = (\{x\}, y^{-1})$ and $\tau^{-1} = (\{y\}, x^{-1})$. 
Lemma 2.2. In $F_2$, we have

\[
\begin{align*}
\tau^{-1}\sigma &\equiv \pi\tau, & \tau^{-1}\pi &\equiv \pi\sigma, & \sigma^{-1}\pi &\equiv \pi\tau, \\
\sigma\tau^{-1} &\equiv \pi\sigma^{-1}, & \tau\pi &\equiv \pi\sigma^{-1}, & \sigma\pi &\equiv \pi\tau^{-1}, \\
\tau\sigma^{-1} &\equiv \pi^{-1}\tau^{-1}, & \tau\pi^{-1} &\equiv \pi^{-1}\sigma^{-1}, & \sigma\pi^{-1} &\equiv \pi^{-1}\tau^{-1}, \\
\sigma^{-1}\tau &\equiv \pi^{-1}\sigma, & \tau^{-1}\pi &\equiv \pi^{-1}\sigma^{-1}, & \sigma^{-1}\pi^{-1} &\equiv \pi^{-1}\tau,
\end{align*}
\]

where $\pi$ is a Whitehead automorphism of $F_2$ of type (W1) that sends $x$ to $y$ and $y$ to $x^{-1}$.

Proof. For the first equality, check that $(\pi\tau)^{-1}\tau^{-1}\sigma = (\{y^{\pm 1}\},x^{-1})(\{x^{\pm 1}\},y) \equiv 1$ in $F_2$. In a similar way, the rest of the equalities can be checked. $\square$

Lemma 2.3. For every automorphism $\phi$ of $F_2$, $\phi$ can be represented as $\phi \equiv \beta\phi'$, where $\beta$ is a Whitehead automorphism of $F_2$ of type (W1) and $\phi'$ is a chain of one of the forms

\begin{align*}
(C1) \phi' &\equiv \tau^{m_k}\sigma^{l_k}\cdots\tau^{m_1}\sigma^{l_1} \\
(C2) \phi' &\equiv \tau^{-m_k}\sigma^{-l_k}\cdots\tau^{-m_1}\sigma^{-l_1}
\end{align*}

with $k \in \mathbb{N}$ and both $l_i, m_i \geq 0$ for every $i = 1, \ldots, k$.

Proof. By Whitehead’s Theorem (cf. [6]) together with Lemma 2.1, an automorphism $\phi$ of $F_2$ can be expressed as

\begin{equation}
\phi \equiv \beta'\tau^{q_t}\sigma^{p_t}\cdots\tau^{q_1}\sigma^{p_1},
\end{equation}

where $\beta'$ is a Whitehead automorphism of $F_2$ of type (W1), $t \in \mathbb{N}$ and both $p_i, q_i$ are (not necessarily positive) integers for every $i = 1, \ldots, t$. If not every $p_i$ and $q_i$ has the same sign (including 0), apply repeatedly Lemma 2.2 to the chain on the right-hand side of (2.1) to obtain that either $\phi \equiv \beta'\pi^r\tau^{m_k}\sigma^{l_k}\cdots\tau^{m_1}\sigma^{l_1}$ or $\phi \equiv \beta'\pi^r\tau^{-m_k}\sigma^{-l_k}\cdots\tau^{-m_1}\sigma^{-l_1}$, where $\pi$ is as in Lemma 2.2, $r \in \mathbb{Z}$, $k \in \mathbb{N}$, and both $l_i, m_i \geq 0$ for every $i = 1, \ldots, k$. Putting $\beta = \beta'\pi^r$, we get the required result. $\square$

Under the same notation as in the statement of Lemma 2.3, we define the length of an automorphism $\phi$ of $F_2$ as $\sum_{i=1}^k (m_i + l_i)$, which is denoted by $|\phi|$. Then obviously $|\phi| = |\phi'|$. 


Lemma 2.4. Let \( u, v \) be elements in \( F_2 \). Also let \( m \) be an arbitrary positive integer, and let \( \Lambda \) be the set of all chains of the form (C1) or (C2) of length less than or equal to \( m \). Suppose that \( \| \psi(u) \| = \| \psi(v) \| \) for every \( \psi \in \Lambda \). Then we have both \( n([\psi(u)];x) = n([\psi(v)];x) \) and \( n([\psi(u)];y) = n([\psi(v)];y) \) for every \( \psi \in \Lambda \).

Proof. Under the given hypothesis of the lemma, [2, Lemma 2.2] yields that \( n([u];x) = n([v];x) \) and \( n([u];y) = n([v];y) \), thus proving the lemma when \( \psi = 1 \). Now assuming that the assertion of the lemma is true for every \( \psi_1 \in \Lambda \) with \( |\psi_1| = m' < m \), we shall prove that \( n([\psi_2(u)];x) = n([\psi_2(v)];x) \) and \( n([\psi_2(u)];y) = n([\psi_2(v)];y) \) for every \( \psi_2 \in \Lambda \) with \( |\psi_2| = m' + 1 \). Such \( \psi_2 \) can be expressed as \( \sigma^{\pm 1} \psi_1 \) or \( \tau^{\pm 1} \psi_1 \) for some \( \psi_1 \in \Lambda \) with \( |\psi_1| = m' \).

First let \( \psi_2 = \sigma^{\pm 1} \psi_1 \). Then clearly \( n([\psi_2(u)];x) = n([\psi_1(u)];x) \) and \( n([\psi_2(v)];x) = n([\psi_1(v)];x) \). Since \( n([\psi_1(u)];x) = n([\psi_1(v)];x) \) by the induction hypothesis, we have \( n([\psi_2(u)];x) = n([\psi_2(v)];x) \). Moreover it is clear that \( n([\psi_2(u)];y) = \| \psi_2(u) \| - n([\psi_2(u)];x) \) and \( n([\psi_2(v)];y) = \| \psi_2(v) \| - n([\psi_2(v)];x) \). Since \( \| \psi_2(u) \| = \| \psi_2(v) \| \) by the hypothesis of the lemma, we finally have \( n([\psi_2(u)];y) = n([\psi_2(v)];y) \).

The other case where \( \psi_2 = \tau^{\pm 1} \psi_1 \) is similar. \( \square \)

For a cyclic word \( w \) in \( F_2 \) and a Whitehead automorphism, say \( \sigma \), of \( F_2 \), a subword of the form \( x y^r x^{-1} \) \((r \neq 0)\), if any, in \( w \) is invariant in passing from \( w \) to \( \sigma(w) \), although there occurs cancellation in \( \sigma(x y^r x^{-1}) \) (note that \( \sigma(x y^r x^{-1}) = x y \cdot y^r \cdot y^{-1} x^{-1} = x y^r x^{-1} \)). Such cancellation is called trivial cancellation. And cancellation which is not trivial cancellation is called proper cancellation. For example, a subword \( x y^{-r} x \) \((r \geq 1)\), if any, in \( w \) is transformed to \( x y^{-r+1} x y \) by applying \( \sigma \), and the cancellation occurring in \( \sigma(x y^{-r} x) \) is proper cancellation.

Lemma 2.5. Let \( w \) be a cyclic word in \( F_2 \), and let \( \psi \) be a chain of the form (C1) (or (C2)). If \( \psi \) contains at least \( \|w\| \) factors of \( \sigma \) (or \( \sigma^{-1} \)), then there cannot occur proper cancellation in passing from \( \psi(w) \) to \( \sigma \psi(w) \) (or \( \psi(w) \) to \( \sigma^{-1} \psi(w) \)); if \( \psi \) contains at least \( \|w\| \) factors of \( \tau \) (or \( \tau^{-1} \)), then
there cannot occur proper cancellation in passing from $\psi(w)$ to $\tau \psi(w)$ (or $\psi(w)$ to $\tau^{-1} \psi(w)$).

Proof. We shall show that if $\psi$ is a chain of the form (C1) such that $\psi$ contains at least $\|w\|$ factors of $\sigma$, then no proper cancellation occurs in passing from $\psi(w)$ to $\sigma \psi(w)$ (the other cases are similar). Supposing that there is a chain $\psi'$ of type (C1) such that no proper cancellation occurs in passing from $\psi'(w)$ to $\sigma \psi'(w)$, we see that proper cancellation cannot occur in passing from $\sigma \psi'(w)$ to $\sigma^2 \psi'(w)$ or in passing from $\tau^t \sigma \psi'(w)$ to $\sigma \tau^t \sigma \psi'(w)$ for any $t \geq 1$. Hence if there was proper cancellation in passing from $\psi(w)$ to $\sigma \psi(w)$, then proper cancellation would also occur at every step of applying $\sigma$ in $\psi$. However since cancelled $y^\pm 1$ in proper cancellation at every step of applying $\sigma$ in the chain $\sigma \psi$ must originally exist in $w$ and since the chain $\sigma \psi$ contains more than $\|w\|$ factors of $\sigma$, we reach a contradiction. \qed

3. Proof of Theorem 1.2

We shall prove the following.

(*) Let $\Omega$ be the set of all chains of the form (C1) or (C2) of length less than or equal to $2\|u\| + 3$. Suppose that $\|\psi(u)\| = \|\psi(v)\|$ for every $\psi \in \Omega$. Then $u$ and $v$ are translation equivalent in $F_2$.

Once (*) is proved, the translation equivalence of $u$, $v$ in $F_2$ is algorithmically decidable as follows.

Algorithm. Let $\Omega$ be the set of all chains of the form (C1) or (C2) of length less than or equal to $2\|u\| + 3$ (note that $\Omega$ is a finite set). Check if $\|\psi(u)\| = \|\psi(v)\|$ for every $\psi \in \Omega$. If so, conclude that $u$ and $v$ are translation equivalent in $F_2$; otherwise conclude that $u$ and $v$ are not translation equivalent in $F_2$.

Let $\phi$ be an automorphism of $F_2$. By Lemma 2.3, $\phi$ can be represented as $\phi \equiv \beta \phi'$, where $\beta$ is a Whitehead automorphism of $F_2$ of type (W1) and $\phi'$ is of the form either (C1) or (C2). We
proceed with the proof of (\ast) by induction on $|\phi'|$. Suppose that $\phi'$ is a chain of the form (C1) with $|\phi'| > 2\|u\| + 3$ (the case for (C2) is similar). Assuming that $\|\psi(u)\| = \|\psi(v)\|$ for every chain $\psi$ of the form (C1) or (C2) with $|\psi| < |\phi'|$, we shall show that $\|\phi'(u)\| = \|\phi'(v)\|$, which is equivalent to showing that $\|\phi(u)\| = \|\phi(v)\|$. Suppose that $\phi'$ ends with $\tau$ (the case where $\phi'$ ends with $\sigma$ is similar), that is,

$$
\phi' = \tau^{m_k} \sigma^{l_k} \cdots \tau^{m_1} \sigma^{l_1},
$$

where both $l_i, m_i \geq 0$ for every $i = 1, \ldots, k$ and $m_k > 0$. Put

$$
\phi_1 = \tau^{m_k-1} \sigma^{l_k} \cdots \tau^{m_1} \sigma^{l_1}.
$$

Also put

$$
u_1 = \phi_1(u) \quad \text{and} \quad v_1 = \phi_1(v).
$$

It then follows from $\tau(u_1) = \phi'(u)$ and $\tau(v_1) = \phi'(v)$ that

$$
\|\phi'(u)\| = \|u_1\| + n([u_1]; y) - 2n([u_1]; y, x^{-1})
$$

(3.1)

$$
\|\phi'(v)\| = \|v_1\| + n([v_1]; y) - 2n([v_1]; y, x^{-1}).
$$

By the induction hypothesis, we have $\|u_1\| = \|v_1\|$. Moreover, by Lemma 2.4, we have $n([u_1]; y) = n([v_1]; y)$. So it suffices to show $n([u_1]; y, x^{-1}) = n([v_1]; y, x^{-1})$ to get the equality $\|\phi'(u)\| = \|\phi'(v)\|$. Clearly the chain $\phi_1$ has length $|\phi_1| = |\phi'| - 1 \geq 2\|u\| + 3$. Hence either $\sigma$ or $\tau$ occurs at least $\|u\| + 2$ times in $\phi_1$. We consider two cases accordingly.

**Case 1.** $\sigma$ occurs at least $\|u\| + 2$ times in $\phi_1$.

In this case, clearly $l_k > 0$. Put

$$
u_2 = \tau^{m_k-1} \sigma^{l_k-1} \cdots \tau^{m_1} \sigma^{l_1}(u) \quad \text{and} \quad \nu_2' = \sigma^{l_k-1} \cdots \tau^{m_1} \sigma^{l_1}(u).
$$

Then $u_2 = \tau^{m_k-1}(\nu_2')$. In the following claims, we shall make some observations about the cyclic word $[u_2']$. 
Claim 1. (i) If $l_k - 1 > 0$, then $[u'_2]$ does not have $x^2$ or $x^{-2}$ as a subword.

(ii) Let $l_k - 1 = 0$. Then the cyclic word $[\sigma^{l_{k-1}} \sigma^{m_1 \sigma^{l_1}}(u)]$ does not have $x^2$ or $x^{-2}$ as a subword. If there is a subword $x^2$ or $x^{-2}$ in $[u'_2]$, then it is actually part of the subword $yx^2$ or $x^{-2}y^{-1}$, respectively.

Proof of Claim 1. (i) Let $l_k - 1 > 0$. Since the chain $\sigma^{l_{k-2}} \sigma^{m_1 \sigma^{l_1}}$ contains at least $\|u\|$ factors of $\sigma$, by Lemma 2.5 no proper cancellation occurs in passing from $[\sigma^{l_{k-2}} \sigma^{m_1 \sigma^{l_1}}(u)]$ to $[\sigma^{l_{k-1}} \sigma^{m_1 \sigma^{l_1}}(u)] = [u'_2]$. This yields that $x^2$ or $x^{-2}$ cannot occur in $[u'_2]$ as a subword.

(ii) Let $l_k - 1 = 0$. Then $l_{k-1} > 0$ and the chain $\sigma^{l_{k-1}-1} \sigma^{m_1 \sigma^{l_1}}$ contains at least $\|u\|$ factors of $\sigma$. Again by Lemma 2.5, no proper cancellation occurs in passing from $[\sigma^{l_{k-1}-1} \sigma^{m_1 \sigma^{l_1}}(u)]$ to $[\sigma^{l_{k-1}} \sigma^{m_1 \sigma^{l_1}}(u)]$. This yields that $x^2$ or $x^{-2}$ cannot occur in $[\sigma^{l_{k-1}} \sigma^{m_1 \sigma^{l_1}}(u)]$ as a subword.

Thus if there exists $x^2$ or $x^{-2}$ in $[u'_2]$ as a subword, it must have newly occurred in passing from $[\sigma^{l_{k-1}} \sigma^{m_1 \sigma^{l_1}}(u)]$ to $[\sigma^{l_{k-1}} \sigma^{m_1 \sigma^{l_1}}(u)] = [u'_2]$. This implies that if there is a subword $x^2$ or $x^{-2}$ in $[u'_2]$, it is actually part of the subword $yx^2$ or $x^{-2}y^{-1}$, respectively.

Claim 2. The cyclic word $[u'_2]$ can be written as $[w_1 z_1 \cdots w_t z_t]$, where $z_i$ is either $x y^t x^{-1}$ or $x y^{-t} x^{-1}$ $(t \geq 1)$, and $w_i$ contains no $y x^{-1}$ or $x y^{-1}$ as a subword and neither begins with nor ends with $x^\pm 1$.

Proof of Claim 2. Since the chain $\sigma^{l_{k-1}} \sigma^{l_1}$ contains at least $\|u\|$ factors of $\sigma$, by Lemma 2.5 no proper cancellation occurs in passing from $[u'_2]$ to $[\sigma(u'_2)]$. This implies that any subword $yx^{-1}$ or $xy^{-1}$, if any, in $[u'_2]$ must be part of a subword of the form $x y^t x^{-1}$ or $x y^{-t} x^{-1}$ $(t \geq 1)$, respectively, in $[u'_2]$.

Suppose that $x y^t x^{-2}$ or $x^2 y^{-t} x^{-1}$ $(t \geq 1)$ occurs in $[u'_2]$ as a subword. By Claim 1 (i), this happens only when $l_k - 1 = 0$. Also by the second part of Claim 1 (ii), any subword of the form $x y^t x^{-2}$ or $x^2 y^{-t} x^{-1}$ $(t \geq 1)$ in $[u'_2]$ is part of a subword of the form $x y^t x^{-s} y^{-1}$ or $y x^s y^{-t} x^{-1}$
(s ≥ 2), respectively, in \([u'_2]\). But then a subword of the form \(yx^{-s}y^{-1}\) or \(yx^{s}y^{-1}\) (s ≥ 2) must exist in \([\sigma^{l_{k-1}} \cdots \tau^{m_1}\sigma^{l_1}(u)]\), a contradiction to the first part of Claim 1 (ii).

□

Now put \(u'_1 = \sigma(u'_2)\).

By Claim 2, we have \([u'_1] = [\sigma(w_1 z_1 \cdots w_t z_t)] = [w'_1 z_1 \cdots w'_t z_t]\), where \(w'_i = (\{x\}, y)(w_i)\). Then \(w'_i\) contains no \(yx^{-1}\) or \(xy^{-1}\) as a subword and has the same initial and terminal letters as \(w_i\) does for each \(i\). Since \(u_1\) and \(u_2\) are obtained by applying \(\tau^{m_k-1}\) to \(u'_1\) and \(u'_2\), respectively, we see that

\[ n([u_1]; y, x^{-1}) = n([u_2]; y, x^{-1})\]

Arguing similarly, we have

\[ n([v_1]; y, x^{-1}) = n([v_2]; y, x^{-1})\]

where \(v_2 = \tau^{m_k-1}\sigma^{l_{k-1}} \cdots \tau^{m_1}\sigma^{l_1}(v)\). Furthermore, since

\[ -2n([u_2]; y, x^{-1}) = \|\tau^{m_k}\sigma^{l_{k-1}} \cdots \tau^{m_1}\sigma^{l_1}(u)\| - \|u_2\| - n([u_2]; y)\]

\[ -2n([v_2]; y, x^{-1}) = \|\tau^{m_k}\sigma^{l_{k-1}} \cdots \tau^{m_1}\sigma^{l_1}(v)\| - \|v_2\| - n([v_2]; y)\]

by the induction hypothesis applied to both \(\|\tau^{m_k}\sigma^{l_{k-1}} \cdots \tau^{m_1}\sigma^{l_1}(u)\| = \|\tau^{m_k}\sigma^{l_{k-1}} \cdots \tau^{m_1}\sigma^{l_1}(v)\|\)

and \(\|u_2\| = \|v_2\|\) together with Lemma 2.4 applied to \(n([u_2]; y) = n([v_2]; y)\), we finally have

\[ n([u_1]; y, x^{-1}) = n([u_2]; y, x^{-1}) = n([v_2]; y, x^{-1}) = n([v_1]; y, x^{-1})\]

that is, \(n([u_1]; y, x^{-1}) = n([v_1]; y, x^{-1})\), as required.

**Case 2.** \(\tau\) occurs at least \(\|u\| + 2\) times in \(\phi_1\).

We divide this case into two subcases.

**Case 2.1.** \(m_k ≥ 2\).

Put

\[ u_3 = \tau^{m_k-2}\sigma^{l_{k}} \cdots \tau^{m_1}\sigma^{l_1}(u)\] \[ v_3 = \tau^{m_k-2}\sigma^{l_{k}} \cdots \tau^{m_1}\sigma^{l_1}(v)\]
Here since the chain $\tau^{m_k-2}\sigma^l \ldots \tau^m \sigma^1$ contains at least $\|u\| + 1$ factors of $\tau$, by Lemma 2.5 no proper cancellation occurs in passing from $[u_3]$ to $[\tau(u_3)] = [u_1]$. Hence we have $n([u_1]; y, x^{-1}) = n([u_3]; y, x^{-1})$. Similarly $n([v_1]; y, x^{-1}) = n([v_3]; y, x^{-1})$. Since

$$-2n([u_3]; y, x^{-1}) = \|\tau^{m_k-1}\sigma^l \ldots \tau^m \sigma^1(u)\| - \|u_3\| - n([u_3]; y)$$

$$-2n([v_3]; y, x^{-1}) = \|\tau^{m_k-1}\sigma^l \ldots \tau^m \sigma^1(u)\| - \|v_3\| - n([v_3]; y),$$

the desired equality $n([u_1]; y, x^{-1}) = n([v_1]; y, x^{-1})$ follows from the induction hypothesis and Lemma 2.4.

**Case 2.2.** $m_k = 1$.

In this case clearly $m_{(k-1)} > 0$. Put

$$u_4 = \sigma^l \tau^{m_{(k-1)}-1} \ldots \tau^m \sigma^1(u) \quad \text{and} \quad v_4 = \sigma^l \tau^{m_{(k-1)}-1} \ldots \tau^m \sigma^1(v).$$

As in Case 1, we can see that

$$n([u_1]; y, x^{-1}) = n([u_4]; y, x^{-1})$$

$$n([v_1]; y, x^{-1}) = n([v_4]; y, x^{-1}).$$

Then since

$$-2n([u_4]; y, x^{-1}) = \|\tau \sigma^l \tau^{m_{(k-1)}-1} \ldots \tau^m \sigma^1(u)\| - \|u_4\| - n([u_4]; y)$$

$$-2n([v_4]; y, x^{-1}) = \|\tau \sigma^l \tau^{m_{(k-1)}-1} \ldots \tau^m \sigma^1(u)\| - \|v_4\| - n([v_4]; y),$$

the required equality $n([u_1]; y, x^{-1}) = n([v_1]; y, x^{-1})$ follows from the induction hypothesis and Lemma 2.4.

The proof of ($\ast$), and hence the proof of Theorem 1.2, is now completed. \qed

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