We present modeling and analysis of a hysteretic deformable mirror where the facesheet interacts with a continuous layer of piezoelectric material that can be actuated distributively by a matrix of electrodes through multiplexing. Moreover, a method for calculating the actuator influence functions is described considering the particular arrangement of electrodes. The results are presented in a semi-analytical model to describe the facesheet’s deformation caused by a high density array of actuators, and validated in a simulation. The proposed modeling of an interconnection layout of electrodes is used to determine the optimal pressures the actuators have to exert for achieving a desired surface deformation.

1. INTRODUCTION

Deformable mirrors (DMs) are instruments used for the correction of light wavefront aberrations in many imaging and non-imaging applications such as 3D imaging to increase the realism of depth perception, microscopes to correct static lenses, or laser material processes for controlling the laser beam shape and size to increase accuracy. In general, DMs are distinguished in segmented and continuous facesheet mirrors, and can be further classified by means of their actuator type that is mounted below the reflective top layer to deform the mirror surface. Depending on the application, various actuator technologies are used which include, for example, piezoelectric, electrostatic, magnetostrictive and shape memory alloy actuators as well as voice coil / reluctance actuators [1]. Furthermore, DMs are applied in adaptive optical systems and key instruments for space telescopes. When a distorted incoming wavefront arrives at the telescope, a wavefront sensor is used to measure the wavefront distortion and subsequently used to adjust the shape of DM in order to correct the distorted wavefront. Future large space telescopes like LUVOIR [2] use corona-graphic instruments for high-contrast imaging of exoplanets. Although thousands of exoplanets have been identified, the current state of technology limits our capability in measuring and understanding these exoplanets beyond their mass, radius, orbital period and distance to the host star. To overcome these challenges and provide the required capabilities for a direct exoplanet imaging space mission, DMs strive among others after high actuator density, meaning that the number of actuators must be increased to the maximum that can still guarantee practical operability including for the wire bonding, harness and electronics. Current developments can reach 100 to 6000 actuators but rarely higher[1, 3]. One of the major limitations for employing DMs with a large number of actuators on a space mission is the reliability of the associated cable harness and electronics. If every actuator has to be driven continuously to hold a specific position, a dedicated channel consisting of a digital-to-analog converter and a high voltage amplifier is required per actuator, resulting in bulk electronics. The recently presented concept of a high pixel number deformable mirror utilizing piezoelectric hysteresis for stable shape configurations [4], abbreviated hysteretic deformable mirror (HDM), demonstrates a new concept of DM whose actuation mechanism consists of multilayered piezoelectric actuators with high hysteresis. A schematic illustration of the HDM can be found in Figure 1. Due to its design and working principle it is possible to employ a large number of actuators and reach a high-resolution accuracy in correcting wavefront aberrations. In addition, it benefits from time-division multiplexing which reduces the number of wires needed to connect and address the actuators. Subsequently, the HDM provides a very simple electrode layout, as illustrated in Figure 2. The top and bottom electrodes are rotated by 90° to form intersecting areas of the electrodes presenting the actuators. The actuation is bundled by sharing the same electrodes for actuators along a line. The voltage is transmitted over a shared
Motivated by this novel concept, we present the modeling and analysis of a mirror’s facesheet that is subjected to the key characteristics of the HDM including a high actuator density and an interconnection layout. The mirror is described with a mechanical model to show the relation between the facesheet deflection and the pressures applied by the actuators. We follow the approach presented by Claffin and Bareket [5] in assuming that the deflection is governed by Poisson’s equation. To guarantee a high accuracy in modeling, we incorporate the particular arrangement of the electrodes in the HDM into the solution to Poisson’s equation and compute the required pressures to fit several Zernike polynomials [6], which are the preferred representation for light wavefront aberrations in adaptive optical systems. The simulation is performed for low actuator numbers to demonstrate the calculation method with the given conditions, and high actuator numbers which will allow a high spatial frequency wavefront correction. The results including the method’s accuracy and limits of applicability are discussed.

The paper is organized as follows: in Section 2 we present the semi-analytical plate model to calculate the facesheet deflection caused by a high density array with square pressure planes of the actuators interacting with the facesheet. Section 3 describes the least-square fitting to determine optimal actuator pressures for representing wavefront aberrations and presents simulation results for a 5 × 5 as well as a 129 × 129 actuator array. Results are discussed and the conclusions are given in Section 4.

2. SEMI-ANALYTICAL PLATE MODEL

An influence function defines the characteristic shape of the mirror surface corresponding to the deformation caused by one actuator. Several methods currently exist for modeling these influence functions of continuous facesheet mirrors. Besides the usage of Gauss functions and splines [7–9], or biharmonic plate equation [10], influence functions can be modeled by application of Kirchhoff or van Kármán theory [11–13] for plate deformations smaller than the plate thickness. Methods using the thin plate theory to calculate influence functions for real time computation for specific mirror geometries are given in [14, 15]. Furthermore, models based on the Kirchhoff plate model, for example, include assumptions for actuator forces that either presuppose the exerted force as point load or approximated electrode areas with constantly distributed loads as well as boundary conditions presenting circularly clamped DMs [5, 16] or a free outer edge [17]. Next to these modeling approaches which mainly consider the static characteristics, detailed review and analysis of DM’s dynamic properties for control purposes can be found in [18].

To determine the influence functions as precisely as possible with static characteristics, it is necessary to define the electrode areas according to their actual shapes. Given the concept of the HDM, the electrodes have an interconnection layout creating square pressure planes lying under a thin circular facesheet. The actuators are separated by a specified distance. To describe the surface displacement, it is necessary to integrate over the area of each pressure plane. Therefore, each plane is separated into several areas which can be described by a coordinate transformation using Cartesian coordinates as well as the radial and angular limits. It is assumed that the thickness of electrodes can be neglected and the piezoelectric actuators modeled as springs in parallel to a force source over an area which creates pressure on the facesheet.

A. Determination of influence matrix

We consider the Poisson equation [19]

$$\nabla^2 z = -\frac{q}{\mathcal{T}} \tag{1}$$

governing the relation between small surface displacements $z$ of a thin facesheet with surface tension $\mathcal{T}$ generated by an exerted pressure $q$. The solution to Poisson’s equation in polar coordinates $(r, \phi)$ can be given by

$$z(r, \phi, \hat{r}, \hat{\phi}) = C \iint_A F(r, \phi, \hat{r}, \hat{\phi}(r)) \cdot q(\hat{r}, \hat{\phi}) \ d\hat{r} \ d\hat{\phi} \tag{2}$$

with

$$A = \{(r, \phi) | \phi_1(r) \leq \hat{\phi} \leq \phi_2(r), 0 \leq r \leq 1\} \tag{3}$$

where $z(r, \phi)$ is the out-of-plane displacement of the thin facesheet, $(\hat{r}, \hat{\phi})$ are the integration variables, $q(\hat{r}, \hat{\phi})$ are the distributed forces over the particular electrode area, and constant
\[ C = \frac{a^2}{T} \] contains the relation between the facesheet radius \( a \) and the surface tension for normalization of the function \( F \) to unity. Edge deflection and slopes are both equal to zero. Furthermore, \( F \) is defined as

\[
F(r, \phi, \hat{r}, \hat{\phi}(\hat{r})) = \begin{cases} 
  f_1(r, \phi, \hat{r}, \hat{\phi}(\hat{r})) & \text{if } 0 < \hat{r} < r \\
  f_2(r, \phi, \hat{r}, \hat{\phi}(\hat{r})) & \text{if } r < \hat{r} < 1.
\end{cases}
\]

(4)

The resulting deflection will be the integral of \( z(r, \phi, \hat{r}, \hat{\phi}) \) over the area \( A \) of the facesheet

\[
z(r, \phi) = \frac{q(r, \phi)}{2\pi} \int_0^1 \int_0^{2\pi} F(r, \phi, \hat{r}, \hat{\phi}(\hat{r})) \, d\phi \, d\hat{r}
\]

(5)

assuming that \( q(\hat{r}, \hat{\phi}) \) is a piecewise constant function on \( R_1 < \hat{r} < R_2 \) which gives \( q(r, \phi) \). Note that \( R_1 \) and \( R_2 \) designate the smallest and greatest radius for describing the electrodes, respectively.

Following the approach of Claflin and Bareket [5], the equation for calculation of the surface deflection on a specific point on the clamped facesheet can be formulated as

\[
z(r, \phi) = \sum_{n=1}^{N_e} M(r, \phi) q(r, \phi)
\]

(6)

where \( M \) represents the coefficients derived from the solutions of the Poisson equation, \( q(r, \phi) \), are piecewise constant pressures exerted on the respective \( j \)-th electrode, and \( N_e \) is the total number of electrodes.

The exact shape of an electrode is defined via a coordinate transformation. This allows us to implement the information later to the Poisson’s equation (Eq. 2) and find a solution. The electrode is split into parts based on areas of radial limits. These radial limits are used to implement the transformation from Cartesian to polar coordinates. Thus, \( \phi \) depends on \( r \). For convenience, the integration with respect to \( \phi \) is performed first, and results in

\[
z(r, \phi) = \frac{q(r, \phi)}{2\pi} C \left\{ -\ln(r) \int_0^r \left( \phi_2(\hat{r}) - \phi_1(\hat{r}) \right) \right.
\]

\[
\left. -r \sum_{n=1}^\infty \frac{1}{n^n} \left( \left( \frac{r}{\hat{r}} \right)^n - \left( \frac{\hat{r}}{r} \right)^n \right) \times [\sin(n(\phi_2(\hat{r}) - \phi)) - \sin(n(\phi_1(\hat{r}) - \phi))] d\hat{r} \right.
\]

\[
+ \int_0^r \frac{r}{\hat{r}} \ln \left( \frac{1}{\hat{r}} \right) \left( \phi_2(\hat{r}) - \phi_1(\hat{r}) \right) \right.
\]

\[
\left. -r \sum_{n=1}^\infty \frac{1}{n^n} \left( \left( \frac{r}{\hat{r}} \right)^n - \left( \frac{\hat{r}}{r} \right)^n \right) \times [\sin(n(\phi_2(\hat{r}) - \phi)) - \sin(n(\phi_1(\hat{r}) - \phi))] d\hat{r} \right\}.
\]

(7)

The introduced coordinate transformation is inserted and the integration with respect to \( r \) is solved as a function of the position of the electrodes. We define five cases according to the actuator position (Figure 3), as follows: Case 1, central actuator; Case 2, diagonal actuators; Case 3, midline actuators; Case 4, actuators above the diagonals; and Case 5, actuators below the diagonal. The definition of each radial limit can be found in Table 1, and is visualized in Figure 4-8 together with a respective pressure plane.

The detailed summary of the calculation of the coefficients resulting from the solution to Poisson’s equation can be found in Appendices A to G.
The actuators which lie below the diagonals (visualized in Figure 6) were described by use of four radial limits, \( r_1, r_3, r_2 \) and \( r_4 \). The numbering of the radii is systematically distributed according to the corner position. The calculation of coefficients for this case can be found in Appendix F.

**B. Actuator model**

The actuators become coupled through the stiffness of the facesheet. Usually, DMs profit by low inter-actuator coupling, denoting the mechanical coupling between neighboring actuators, which improves the surface accuracy. If significant inter-actuator coupling is present, this needs to be considered in the modeling and control processes [20].

![Fig. 9. Simplified actuator model, modeled by a stiffness \( k \) in parallel to a force source over an area \( \Phi \) acting on the mirror facesheet.](image)

Here, we introduce the model of actuators based on two components, which correspond to a spring in parallel with a force source over an area. Consequently, the relation from Equation (6) may be described by

\[
z(r, \phi) = \sum_{j=1}^{N_L} M(r, \phi) \left( \Phi_{P_j}(V) - k \bar{z}_j \right) \tag{8}
\]

with

\[
\Phi_{P_j}(V) = Y_j \Phi_{T_j}(V) \tag{9}
\]

and

\[
\bar{z}_j := \frac{\sum_{i \in E_j} z_i / n_e}{A_e} \tag{10}
\]

where \( \Phi_{P_j}(V) \) denotes the Preisach operator capturing the highly nonlinear hysteresis of the actuators in regard to the total deformation in relation of the initial thickness dimension, the diagonal matrix containing the Young's modulus \( Y_j \), the longitudinal elongations of the actuators \( \Phi_{T_j}(V) \), the diagonal stiffness matrix containing the actuators' stiffness \( k \), and the mean surface deflection above the respective electrode with area \( A_e \). \( \bar{z}_j \) calculated by means of \( n_e \) surface displacement points \( z_i \) on a specific position within the electrode area. It is assumed that all the actuators are identical and can exert an asymmetric butterfly loop as exemplary presented in Figure 10. A framework to model the electric-field dependence on the strain in piezoelectric materials purposely designed to exhibit loops with

| Case 1 | Case 2 | Case 3 | Case 4 | Case 5 |
|--------|--------|--------|--------|--------|
| \( p = 0 \) | \( p \leq r_1 \) | \( p \leq r_{1e} \) | \( p \leq r_1 \) | \( p \leq r_1 \) |
| \( p \geq r_1 \) | \( p \geq r_4 \) | \( p \geq r_4 \) | \( p \geq r_4 \) | \( p \geq r_4 \) |
| \( 0 < p \leq r_{1e} \) | \( r_1 < p \leq r_2 \) | \( r_2 < p \leq r_{1e} \) | \( r_2 < p \leq r_{1e} \) | \( r_2 < p \leq r_{1e} \) |
| \( r_{1e} < p < r_1 \) | \( r_2 < p < r_4 \) | \( r_4 < p < r_4 \) | \( r_4 < p < r_4 \) | \( r_4 < p < r_4 \) |

**Table 1. Definition of radial limits for splitting the electrode areas.**
remnant deformation was presented by Jayawardhana et al. [21] based on the use of the Preisach operator. The complete formal definition of the Preisach operator is given in [22].

![Graph showing asymmetric butterfly hysteresis loop with remnant deformation.](image)

**Fig. 10.** Asymmetric butterfly hysteresis loop with remnant deformation, the measured data of which was collected from previous material tests. The axial displacement was measured while a certain voltage was applied.

3. RESULTS AND DISCUSSION

A. Least-square fitting

The preferred representation for light wavefront aberrations in adaptive optical systems is via Zernike polynomials. They are defined on a unit circle using polar coordinates $(r, \theta)$ as functions of azimuthal frequency $m$ and radial degrees $n$, where $m \leq n$. The set of polynomials [6] can be given by

\[
\begin{align*}
Z_m^n(r, \theta) &= R_m^n(r) \cos(m\theta) & \text{for } m \geq 0 \\
Z_m^n(r, \theta) &= R_m^n(r) \sin(m\theta) & \text{for } m < 0
\end{align*}
\]  

(11)

where

\[
R_m^n(r) = \sum_{S=0}^{(n-m)/2} \frac{(-1)^S (n-S)!r^{n-2S}}{S!(n+m)/2 - S![(n-m)/2 - S]!}
\]  

(12)

To calculate the required pressure terms to fit several Zernike polynomials, each displacement of a respective point on the facesheet which is defined by $(r, \phi)$ is fit to the corresponding point on Zernike polynomials. An over-determined set of equations is solved in the least-square sense resulting in

\[
\Phi = (\mathbf{M}^T \mathbf{M})^{-1} \mathbf{M} \mathbf{z}_d + \mathbf{M} \mathbf{k}_d.
\]  

(13)

which aims at minimizing the root-mean-square deviation (RMSD) between the two quantities.

B. Simulation results

Using MATLAB R2019a, a low density array with 5×5 actuators and a high density array with 129×129 actuators were generated. To decrease the computational effort in the latter case, the coefficient calculations were executed in parallel per 5 actuators and run in a compute cluster (Peregrine HPC cluster). For all experiments, we used a partition of two Intel Xeon E5 2680 v3 or v4 (2.50GHz or 2.40GHz respectively) CPUs with 5GB of memory. Thereby, the computational time was decreased to about 2h when all jobs ran in parallel.

In order to assess the mechanical model, a second simulation in Matlab was generated fitting the mirror surface to selected Zernike polynomials as in Figure 11 and 12. The procedure of this approach included three steps. The first step consisted of reducing the mirror surface to an active area due to the boundary conditions to circumvent an increasing fitting error caused by zero deflection at the clamped edge. Secondly, a mask was generated to match selected points of the Zernike polynomial disc plot to the surface points of the mirror. This mask was created with a partition in radial and angular coordinates according to $r_0 < r_1 < \cdots < r_{(n-1)} < r_n$ with $r_0 = 0$ and $r_n = 1$, and $\phi_0 < \phi_1 < \cdots < \phi_{(n-1)} < \phi_n$ with $\phi_0 = 0$ and $\phi_n = 2\pi$ respectively. In the third step, the RMSD of the estimator $z_d$ with respect to the actual surface deflection $z$ was calculated (Eq. (14)) to evaluate the mirror accuracy.

\[
\text{RMSD}(z) = \sqrt{\frac{\sum_{x=1}^{X} (z_x - \hat{z}_x)^2}{X}}
\]  

(14)

There were 4961 surface points selected based on the described partition. The active area was restricted to 40% of the mirror diameter. Figures 13 and 14, and Figures 15 and 16 show the results for a 5×5 actuator array of fitting the mirror surface to selected Zernike polynomial examples. Figures 17 and 18, and Figures 19 and 20 show the results for a 129×129 actuator array of fitting the mirror surface to selected examples. Table 2 summarizes the RMSDs for the first 15 Zernike polynomials with low and high density arrays.

![Graph showing results.](image)

**Fig. 11.** Graphic representation with a vertical colorbar giving the normalized surface displacement of the Zernike polynomial $Z_3^3$ with radial degree 3 and azimuthal degree 1.

Considering the fitting results for a low density array with 5×5 actuators, the intersection layout became clear and the positions of the few actuators play a major role for the final results. The fitting errors are between 3.8% and 25.9%. Comparing these results to a high density array with 129×129 actuators, we observe that the RMSDs decrease drastically. With 129×129 actuators, we have deviations between 0.000298% and 0.524411%. The fitting errors for low and high density arrays behave in a similar manner with increasing degree of the Zernike polynomial.

4. CONCLUSIONS

This study investigated the fundamental characteristics of actuator positions of high density arrays and presented a generalization of cases to calculate every actuator position of deformable mirrors for application in a novel hysteretic deformable mirror. Based on the introduced coordinate transformation while
Fig. 12. Graphic representation with a vertical colorbar giving the normalized surface displacement of the Zernike polynomial $Z_{04}^4$ with radial degree 4 and azimuthal degree 0.

Fig. 13. $5 \times 5$ actuator array fitted to Zernike polynomial $Z_{13}^1$ with an amplitude of $1 \mu m$ in a graphic representation showing the active area of the mirror as unit disc with vertical colorbar giving the surface displacement.

Fig. 14. Surface displacement along the radial line $\phi = 0$ for a $5 \times 5$ actuator array fitted to Zernike polynomial $Z_{13}^1$ with an amplitude of $1 \mu m$.

Fig. 15. $5 \times 5$ actuator array fitted to Zernike polynomial $Z_{04}^0$ with an amplitude of $1 \mu m$ in a graphic representation showing the active area of the mirror as unit disc with vertical colorbar giving the surface displacement.

Fig. 16. Surface displacement along the radial line $\phi = 0$ for a $5 \times 5$ actuator array fitted to Zernike polynomial $Z_{04}^4$ with an amplitude of $1 \mu m$.

Fig. 17. $129 \times 129$ actuator array fitted to Zernike polynomial $Z_{13}^1$ with an amplitude of $1 \mu m$ in a graphic representation showing the active area of the mirror as unit disc with vertical colorbar giving the surface displacement.
Fig. 18. Surface displacement along the radial line $\phi = 0$ for a $129 \times 129$ actuator array fitted to Zernike polynomial $Z_3^1$ with an amplitude of $1 \mu m$.

Fig. 19. $129 \times 129$ actuator array fitted to Zernike polynomial $Z_3^4$ with an amplitude of $1 \mu m$ in a graphic representation showing the active area of the mirror as unit disc with vertical colorbar giving the surface displacement.

Fig. 20. Surface displacement along the radial line $\phi = 0$ for a $129 \times 129$ actuator array fitted to Zernike polynomial $Z_3^4$ with an amplitude of $1 \mu m$.

Table 2. Summary of root-mean square deviations (RMSDs) for the first 15 Zernike polynomials (ZPs) with a $5 \times 5$ actuator array (low density (LD)) and $129 \times 129$ actuator array (high density (HD)).

| ZPs     | LD: RMSDs in [%] | HD: RMSDs in [%] |
|---------|------------------|------------------|
| $Z_{-1}^1$ | 4.426           | 0.001433         |
| $Z_1^1$   | 4.352           | 0.001373         |
| $Z_{-2}^2$ | 3.806           | 0.000298         |
| $Z_2^2$   | 12.840          | 0.065971         |
| $Z_2^2$   | 9.498           | 0.004775         |
| $Z_{-3}^3$ | 10.138          | 0.007609         |
| $Z_3^3$   | 13.937          | 0.131942         |
| $Z_{-4}^4$ | 13.756          | 0.134622         |
| $Z_4^4$   | 9.974           | 0.007343         |
| $Z_{-4}^4$ | 11.376          | 0.005252         |
| $Z_{-2}^4$ | 12.662          | 0.082268         |
| $Z_4^0$   | 25.917          | 0.524411         |
| $Z_4^2$   | 20.755          | 0.409452         |
| $Z_4^4$   | 13.069          | 0.011954         |

Solving the Poisson equation, it was possible to model exactly the shape of the pressure planes and guarantee a more realistic description of the actuator influence functions. By calculating the coefficient matrix in a cluster, the computational time was decreased and presents a usable method for computations on deformable mirrors with high actuator densities. Furthermore, the mirror model includes the mechanical coupling between the actuators and the facesheet. The presented results contribute to achieve a higher accuracy in modeling the actuator influence functions according to the actual properties of the DM, and therefore decrease fitting errors. It provides a framework on how to consider high actuator densities and calculate them in a reasonable way regarding actuator position case classification and computation time.

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**DISCLOSURE**

The authors declare no conflicts of interest.
A. RESPECTIVE FORMULAS

For actuators of the right side of the plate, the left corner of a pressure plane is denoted with \(x_1\), the right corner with \(x_2\), the lower corner with \(y_1\) and the upper corner with \(y_2\). The designation is mirrored with actuators on the left side of the plate. In general, it can be said that \(|x_1| \leq |x_2|\) and \(|y_1| \leq |y_2|\).

Table 3 summarizes the definition of all radial limits with coordinate transformations for splitting the electrode areas. Symbols which are assigned to reoccurring formulas are listed in Table 4.

Table 3. Definition of radial limits in interval \(J\) with coordinate transformation for splitting the electrode areas.

| Case 1 | Boundaries | Coordinate transformation |
|--------|------------|--------------------------|
| \(I_1\) | \(0 \leq r \leq r_{1e}\) | \(0 < \phi < 2\pi\) |
| \(I_{11}\) | \(r_{1e} < r \leq r_{1}\) | \(\arccos(x_1/\bar{r}) < \phi \leq \arcsin(y_2/\bar{r})\) |
| \(I_{12}\) | \(r_{1} < r \leq r_{4}\) | \(\arcsin(y_1/\bar{r}) < \phi \leq \arcsin(y_2/\bar{r})\) |
| \(I_{13}\) | \(r_{4} < r \leq r_{4e}\) | \(\arccos(x_1/\bar{r}) < \phi \leq \arcsin(y_1/\bar{r})\) |
| \(I_{14}\) | \(r_{4e} < r \leq r_{4}\) | \(\arcsin(y_1/\bar{r}) < \phi \leq \arcsin(x_2/\bar{r})\) |

Table 4. Assignment of symbols to reoccurring formulas.

| Symbol | Formula |
|--------|---------|
| \(\kappa_{x_1}\) | \(x_1\sqrt{1 - x_2^2/\bar{r}^2}\) |
| \(\kappa_{x_2}\) | \(x_2\sqrt{1 - x_2^2/\bar{r}^2}\) |
| \(\epsilon_{x_1}\) | \(y_1\sqrt{1 - y_1^2/\bar{r}^2}\) |
| \(\epsilon_{x_2}\) | \(y_2\sqrt{1 - y_2^2/\bar{r}^2}\) |

B. NUMERICAL INTEGRATION

Appendix B contains the two sub-integrals which are solved numerically. 174 is the maximum number of \(n\) terms required for convergence [5].

\[
f_{2n} := \sum_{n=1}^{\infty} \int_{R_1}^{R_2} \frac{p}{n^2} \left( (\bar{r})^n - \left(\frac{\bar{r}}{\bar{r}}\right)^n \right) \times \sin(n(\phi_2(r) - \phi)) - \sin(n(\phi_1(r) - \phi)) \right) \, dr
\]

C. COEFFICIENT CALCULATION IN CASE 1

Appendix C contains the formulas for calculating the coefficients \(M\) for actuators that can be categorized in Case 1.

\(1. \ r_i = 0\)

\[
M = (1/(2\pi)) \times ((f_{1(i_1)}(r_{1e})) \times (f_{2(i_1)}(r_{1})) - f_{2(i_1)}(r_{1e}))
+ (f_{2(i_2)}(r_{1}) - f_{2(i_2)}(r_{1e})) + (f_{2(i_2)}(r_{1}) - f_{2(i_2)}(r_{1e}))
+ (f_{2(i_4)}(r_{1}) - f_{2(i_4)}(r_{1e}))
\]

\(2. \ 0 < r_i \leq r_{1e}\)

\[
M = (1/(2\pi)) \times ((f_{1(i_1)}(r_{1})) - f_{2(i_1)}(0)) - f_{1n(i_1)}
+ (f_{2(i_1)}(r_{1}) - f_{2(i_1)}(r_{1})) - f_{2n(i_1)}
+ (f_{2(i_2)}(r_{1}) - f_{2(i_2)}(r_{1})) - f_{2n(i_2)}
+ (f_{2(i_4)}(r_{1}) - f_{2(i_4)}(r_{1})) - f_{2n(i_4)}
\]

\(3. \ r_{1e} < r_i < r_1\)

\[
M = (1/(2\pi)) \times ((f_{1(i_1)}(r_{1})) - f_{2(i_1)(r_{1})) - f_{1n(i_1)}
+ (f_{2(i_1)}(r_{1}) - f_{2(i_1)}(r_{1})) - f_{2n(i_2)}
+ (f_{2(i_4)}(r_{1}) - f_{2(i_4)}(r_{1})) - f_{2n(i_4)}
\]

\(4. \ r_i \geq r_1\)

\[
M = (1/(2\pi)) \times ((f_{1(i_1)}(r_{1})) - f_{2(i_1)}(0)) - f_{1n(i_1)}
+ (f_{2(i_1)}(r_{1}) - f_{2(i_1)}(r_{1})) - f_{2n(i_1)}
+ (f_{2(i_2)}(r_{1}) - f_{2(i_2)}(r_{1})) - f_{2n(i_2)}
+ (f_{2(i_4)}(r_{1}) - f_{2(i_4)}(r_{1})) - f_{2n(i_4)}
\]
A. Sub-functions of \( f_1 \)
\[
f_1(t_1) = -(\pi r^2 \log(r_1))
\]
\[
f_1(t_{1b}) = (r(-(-k_{x2} - k_{y2} + r_{a2} - r_{b2}) \log(r_1))/2
\]
\[
f_1(t_{1c}) = (r(k_{x2} + k_{y2} - r_{a1} + r_{b2}) \log(r_1))/2
\]
\[
f_1(t_{1d}) = (r(-k_{x1} - k_{y1} + r_{a1} - r_{b1}) \log(r_1))/2
\]
\[
f_1(t_{1e}) = (r(k_{x2} + k_{y2} - r_{a2} + r_{b1}) \log(r_1))/2
\]

B. Sub-functions of \( f_2 \)
\[
f_2(t_{1a}) = 2\pi(r^2/4 + (r^2 \log(r_1))/2
\]
\[
f_2(t_{1b}) = (3\epsilon k_{x2}r^2 - 3\epsilon k_{y2}/r^2 - (r^2 \epsilon_{x2}) + (x^2 r_2)/2
\]
\[
+ (r^2 \epsilon_{b2}/r^2 - y^2 \epsilon_{b2}/r^2 - r_1 \log(r_1))/2
\]
\[
+ (r \epsilon_{y1} + r \epsilon_{b2}) \log(r_1))/2
\]
\[
f_2(t_{1c}) = (3\epsilon k_{x2}/r^2 - 3\epsilon k_{y2}/r^2 - (r^2 \epsilon_{x2}) + (x^2 r_2)/2
\]
\[
+ (r^2 \epsilon_{b2}/r^2 - y^2 \epsilon_{b2}/r^2 - r_1 \log(r_1))/2
\]
\[
+ (r \epsilon_{y1} + r \epsilon_{b2}) \log(r_1))/2
\]
\[
f_2(t_{1d}) = (3\epsilon k_{x2}/r^2 - 3\epsilon k_{y2}/r^2 - (r^2 \epsilon_{x2}) + (x^2 r_2)/2
\]
\[
+ (r^2 \epsilon_{b2}/r^2 - y^2 \epsilon_{b2}/r^2 - r_1 \log(r_1))/2
\]
\[
+ (r \epsilon_{y1} + r \epsilon_{b2}) \log(r_1))/2
\]

D. COEFFICIENT CALCULATION IN CASE 2
Appendix D contains the formulas for calculating the coefficients \( \mathcal{M} \) for actuators that can be categorized in Case 2.

1. \( r_1 \leq r_1 \)
\[
\mathcal{M} = (1/2\pi)((f_2(t_{1b})(r_2) - f_2(t_{1c})(r_1)) - f_2n(t_{1i}))
\]
\[
+ ((f_2(t_{1c})(r_2) - f_2(t_{1d})(r_2)) - f_2n(t_{1i}))
\]

2. \( r_1 < r_2 \)
\[
\mathcal{M} = (1/2\pi)((f_1(t_{1})(r_1) - f_1(t_{1d})(r_1)) - f_1n(t_{1i}))
\]
\[
+ ((f_2(t_{1b})(r_2) - f_2(t_{1c})(r_1)) - f_2n(t_{1i}))
\]
\[
+ ((f_2(t_{1c})(r_4) - f_2(t_{1d})(r_2)) - f_2n(t_{1i}))
\]

3. \( r_2 < r_3 < r_4 \)
\[
\mathcal{M} = (1/2\pi)((f_1(t_{1})(r_1) - f_1(t_{1d})(r_2)) - f_1n(t_{1i}))
\]
\[
+ ((f_2(t_{1b})(r_2) - f_2(t_{1c})(r_1)) - f_2n(t_{1i}))
\]
\[
+ ((f_2(t_{1c})(r_4) - f_2(t_{1d})(r_2)) - f_2n(t_{1i}))
\]

4. \( r_3 \geq r_4 \)
\[
\mathcal{M} = (1/2\pi)((f_1(t_{1})(r_1) - f_1(t_{1d})(r_1)) - f_1n(t_{1i}))
\]
\[
+ ((f_2(t_{1b})(r_4) - f_2(t_{1d})(r_4)) - f_2n(t_{1i}))
\]

A. Sub-functions of \( f_1 \)
\[
f_1(t_{1a}) = (r(k_{x1} + k_{y1} - r_{a1} + r_{b1}) \log(r_1))/2
\]
\[
f_1(t_{1b}) = (r(-k_{x1} - k_{y1} + r_{a1} - r_{b2}) \log(r_1))/2
\]

B. Sub-functions of \( f_2 \)
\[
f_2(t_{1a}) = (-3\epsilon k_{x2})/r^2 - (3\epsilon k_{y2})/r^2 - (r^2 \epsilon_{x2}) + (x^2 r_2)/2
\]
\[
+ (r^2 \epsilon_{b2}/r^2 - y^2 \epsilon_{b2}/r^2 - r_1 \log(r_1))/2
\]
\[
+ (r \epsilon_{y1} + r \epsilon_{b2}) \log(r_1))/2
\]
\[
f_2(t_{1b}) = (-3\epsilon k_{x2})/r^2 - (3\epsilon k_{y2})/r^2 - (r^2 \epsilon_{x2}) + (x^2 r_2)/2
\]
\[
+ (r^2 \epsilon_{b2}/r^2 - y^2 \epsilon_{b2}/r^2 - r_1 \log(r_1))/2
\]
\[
+ (r \epsilon_{y1} + r \epsilon_{b2}) \log(r_1))/2
\]

E. COEFFICIENT CALCULATION IN CASE 3
Appendix E contains the formulas for calculating the coefficients \( \mathcal{M} \) for actuators that can be categorized in Case 3.

1. \( r_1 \leq r_1 \)
\[
\mathcal{M} = (1/2\pi)((f_2(t_{1d})(r_1) - f_2(t_{1c})(r_1)) - f_{2n}(t_{1i}))
\]
\[
+ (f_2(t_{1c})(r_1) - f_2(t_{1b})(r_1)) - f_{2n}(t_{1i}))
\]
\[
+ (f_2(t_{1b})(r_1) - f_2(t_{1a})(r_1)) - f_{2n}(t_{1i}))
\]
\[
+ (f_2(t_{1a})(r_1) - f_2(t_{1d})(r_1)) - f_{2n}(t_{1i}))
\]

2. \( r_1 < r_1 \)
\[
\mathcal{M} = (1/2\pi)((f_1(t_{1})(r_1) - f_1(t_{1d})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1d})(r_1) - f_1(t_{1c})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1c})(r_1) - f_1(t_{1b})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1b})(r_1) - f_1(t_{1a})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1a})(r_1) - f_1(t_{1d})(r_1)) - f_{1n}(t_{1i}))
\]

3. \( r_1 < r_1 \)
\[
\mathcal{M} = (1/2\pi)((f_1(t_{1})(r_1) - f_1(t_{1d})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1d})(r_1) - f_1(t_{1c})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1c})(r_1) - f_1(t_{1b})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1b})(r_1) - f_1(t_{1a})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1a})(r_1) - f_1(t_{1d})(r_1)) - f_{1n}(t_{1i}))
\]

4. \( r_4 < r_4 \)
\[
\mathcal{M} = (1/2\pi)((f_1(t_{1})(r_1) - f_1(t_{1d})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1d})(r_1) - f_1(t_{1c})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1c})(r_1) - f_1(t_{1b})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1b})(r_1) - f_1(t_{1a})(r_1)) - f_{1n}(t_{1i}))
\]
\[
+ (f_1(t_{1a})(r_1) - f_1(t_{1d})(r_1)) - f_{1n}(t_{1i}))
\]
Appendix F contains the formulas for calculating the coefficients $\mathcal{M}$ for actuators that can be categorized in Case 4.

1. $r_1 \leq r_1$

$$M = \frac{1}{(2\pi)} \left( \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

2. $r_1 < r_1 \leq r_3$

$$M = \frac{1}{(2\pi)} \left( \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

3. $r_3 < r_1 \leq r_2$

$$M = \frac{1}{(2\pi)} \left( \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

4. $r_2 < r_1 \leq r_4$

$$M = \frac{1}{(2\pi)} \left( \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

$$+ \left( f_{2i}(t_3) - f_{1i}(t_1) \right) \left( f_{2i}(t_3) - f_{1i}(t_1) \right) - f_{2n}(t_1)$$

5. $r_4 \leq r_1$
\[ r_2 < r_1 \leq r_3 \]
\[ M = (1/(2\pi))((f_1(t_1)(r_1) - f_1(t_1)(r_2)) - f_{1n}(t_1)) + ((f_2(t_1)(r_3) - f_2(t_1)(r_1)) - f_{2n}(t_1)) + ((f_1(t_1)(r_2) - f_1(t_1)(r_1)) - f_{1n}(t_1)) + ((f_2(t_1)(r_4) - f_2(t_1)(r_3)) - f_{2n}(t_1)) \]

\[ r_3 < r_1 < r_4 \]
\[ M = (1/(2\pi))((f_1(t_1)(r_1) - f_1(t_1)(r_3)) - f_{1n}(t_1)) + ((f_2(t_1)(r_4) - f_2(t_1)(r_1)) - f_{2n}(t_1)) + ((f_1(t_1)(r_2) - f_1(t_1)(r_1)) - f_{1n}(t_1)) + ((f_1(t_1)(r_3) - f_1(t_1)(r_2)) - f_{1n}(t_1)) \]

\[ r_1 \geq r_4 \]
\[ M = (1/(2\pi))((f_1(t_1)(r_2) - f_1(t_1)(r_1)) - f_{1n}(t_1)) + ((f_1(t_1)(r_3) - f_1(t_1)(r_2)) - f_{1n}(t_1)) + ((f_1(t_1)(r_4) - f_1(t_1)(r_3)) - f_{1n}(t_1)) \]

**A. Sub-functions of** \( f_1 \)
\[ f_{11}(t_1) = (r(\epsilon_{x_1} + \epsilon_{y_1}) + r\alpha_1 + r\beta_1 \log(r_1))/2 \]
\[ f_{12}(t_1) = (r(\epsilon_{x_1} - \epsilon_{y_2}) + r\beta_1 \log(r_1))/2 \]
\[ f_{13}(t_1) = (r(-\epsilon_{x_2} - \epsilon_{y_2} + r\alpha_2 - r\beta_2) \log(r_1))/2 \]

**B. Sub-functions of** \( f_2 \)
\[ f_{21}(t_1) = (-3r\epsilon_{x_1})/4 - (3r\epsilon_{y_1})/4 + (r^2\alpha_1)/4 - (r^2\gamma_1)/2 \]
\[ - (r^2\beta_1)/4 - (y_1^3\beta_1)/2 - (r(\epsilon_{x_1} - \epsilon_{y_2}) \log(r_1))/2 \]
\[ - (r(\epsilon_{y_1} + r\beta_1) \log(r_1))/2 \]

\[ f_{22}(t_1) = (-3r\epsilon_{x_2})/4 + (3r\epsilon_{y_2})/4 - (r^2\alpha_1)/4 - (r^2\gamma_2)/2 \]
\[ + (r^2\beta_2)/4 + (y_2^3\beta_2)/2 + (r(\epsilon_{x_2} - \epsilon_{y_2}) \log(r_1))/2 \]
\[ + (r(\epsilon_{y_2} + r\beta_2) \log(r_1))/2 \]

**REFERENCES**

1. P.Y. Madec, “Overview of deformable mirror technologies for adaptive optics and astronomy,” in *Adaptive Optics Systems III*, vol. 8447 B. L. Ellerbroek, E. Marchetti, and J.-P. Véran, eds., International Society for Optics and Photonics (SPIE, 2012), pp. 22 – 39.
2. The LUVOR Team, “The LUVOIR Mission Concept Study Final Report,” arXiv preprint, arXiv: 1912.06219 (2019).
3. R. H. Freeman and J. E. Pearson, “Deformable mirrors for all seasons and reasons,” Appl. Opt. 21, 580–588 (1982).
4. R. Huisman et al., “Deformable mirror concept utilizing piezoelectric hysteresis for stable shape configurations,” in preparation. (2020).
5. E. Scott Clafflin and Noah Bareket, “Configuring an electrostatic membrane mirror by least-squares fitting with analytically derived influence functions,” J. Opt. Soc. Am. A 3, pp. 1833–1839 (1986).
6. V. Lakshminarayanan and A. Fleck, “Zernike polynomials: A guide,” Journal of Modern Optics - J MOD OPTIC. 58, 1678–1678 (2011).
7. L. Huang, C. Rao, and W. Jiang, “Modified gaussian influence function of deformable mirror actuators,” Opt. Express 16, 108–114 (2008).
8. R. K. Tyson, *Adaptive Optics Engineering Handbook* (CRC Press, 1999).
9. R. K. Tyson and B. W. Frazier, *Field Guide to Adaptive Optics* (SPIE Press, 2004).
10. R.F.M.M. Hamelinck, “Adaptive deformable mirror : based on electromagnetic actuators,” Eindhoven : Technische Universiteit Eindhoven (2010).
11. R. P. Grosso and M. Yellin, “The membrane mirror as an adaptive optical element,” J. Opt. Soc. Am. 67 pp. 399–406 (1977).
12. L. Arnold, “Influence functions of a thin shallow meniscus-shaped mirror,” Appl. Opt. 36, 2019–2028 (1997).
13. K. Bush, D. German, B. Klemme, A. Marrs and M. Schoen, “Electrostatic membrane deformable mirror wavefront control systems: design and analysis,” SPIE Optics & Photonics; Advanced Wavefront Control: Methods, Devices, and Applications VII 7466, 74660G (2009).
14. A. Menikoff, “Actuator influence functions of active mirrors,” Appl. Opt. 30 pp. 833–838 (1991).
15. T. Rupper, “Modeling and control of deformable membrane mirrors,” Adapt. Opt. Prog. (2012).
16. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics, Part II* (McGraw-Hill, New York, 1953).
17. John W. Hardy, *Adaptive Optics for Astronomical Telescopes* (Oxford Series in Optical and Imaging Sciences, Oxford University Press, 1998).
18. B. Jayawardhana, M. A. Vasquez-Beltran, W. J. van de Beek, C. de Jonge, M. Acuautla, S. Damero, R. Peletier, B. Noheda, and R. Huisman, “Modeling and Analysis of Butterfly Loops via Preisach Operators and its Application in a Piezoelectric Material,” in *2018 IEEE Conference on Decision and Control (CDC)*, (2018), pp. 6894–6899.
19. M. A. Vasquez-Beltran, B. Jayawardhana, and R. Peletier, “Asymptotic Stability Analysis of Lur’e Systems With Butterfly Hysteresis Nonlinearities,” IEEE Control. Syst. Lett. 4, 349–354 (2020).