Abstract

We study kernel least-squares estimation under a norm constraint. This form of regularisation is known as Ivanov regularisation and it provides better control of the norm of the estimator than the well-established Tikhonov regularisation. This choice of regularisation allows us to dispose of the standard assumption that the reproducing kernel Hilbert space (RKHS) has a Mercer kernel, which is restrictive as it usually requires compactness of the covariate set. Instead, we assume only that the RKHS is separable with a bounded and measurable kernel. We provide rates of convergence for the expected squared $L^2$ error of our estimator under the weak assumption that the variance of the response variables is bounded and the unknown regression function lies in an interpolation space between $L^2$ and the RKHS. We then obtain faster rates of convergence when the regression function is bounded by clipping the estimator. In fact, we attain the optimal rate of convergence. Furthermore, we provide a high-probability bound under the stronger assumption that the response variables have subgaussian errors and that the regression function lies in an interpolation space between $L^\infty$ and the RKHS. Finally, we derive adaptive results for the settings in which the regression function is bounded.

Keywords: Ivanov Regularisation, RKHSs, Mercer Kernels, Interpolation Spaces, Training and Validation

1. Introduction

One of the key problems to overcome in nonparametric regression is overfitting, due to estimators coming from large hypothesis classes. To avoid this phenomenon, it is common to ensure that both the empirical risk and some regularisation function are small when defining an estimator. There are three natural ways to achieve this goal. We can minimise the empirical risk subject to a constraint on the regularisation function, minimise the regularisation function subject to a constraint on the empirical risk or minimise a linear combination of the two. These techniques are known as Ivanov regularisation, Morozov regularisation
Ivanov and Morozov regularisation can be viewed as dual problems, while Tikhonov regularisation can be viewed as the Lagrangian relaxation of either.

Tikhonov regularisation has gained popularity as it provides a closed-form estimator in many situations. In particular, Tikhonov regularisation in which the estimator is selected from a reproducing kernel Hilbert space (RKHS) has been extensively studied (Steinwart and Christmann, 2008; Caponnetto and de Vito, 2007; Mendelson and Neeman, 2010; Steinwart, Hush, and Scovel, 2009). Although Tikhonov regularisation produces an estimator in closed form, it is Ivanov regularisation which provides the greatest control over the hypothesis class, and hence over the estimator it produces. For example, if the regularisation function is the norm of the RKHS, then the bound forces the estimator to lie in a ball of predefined radius inside the function space. An RKHS norm measures the smoothness of a function, so the norm constraint bounds the smoothness of the estimator. By contrast, Tikhonov regularisation provides no direct bound on the smoothness of the estimator.

The control we have over the Ivanov-regularised estimator is useful in many settings. The most obvious use of Ivanov regularisation is when the regression function lies in a ball of known radius in the RKHS. In this case, Ivanov regularisation can be used to constrain the estimator to lie in the same ball. Ivanov regularisation can also be used within larger inference methods. It is compatible with validation, allowing us to control an estimator selected from an uncountable collection. This is because the Ivanov-regularised estimator is continuous in the size of the ball containing it, so the estimators parametrised by an interval of ball sizes can be controlled simultaneously using chaining.

The norm bound provided by Ivanov regularisation can also be useful in addressing other problems. These include the use of optimal transport for the covariate shift problem, in which we are interested in bounding the squared $L^2(Q)$ error of our estimator where $Q$ is a probability measure other than the distribution of the covariates $P$. Having control over the norm of the estimator means that we can use the reproducing property and the Cauchy–Schwarz inequality to bound the difference in the value of the estimator at two different points. When the kernel of the RKHS is Lipschitz continuous, this bound can be expressed in terms of the distance between the two points. Integrating an expression of this form with respect to a joint distribution over the two points which has different fixed marginal distributions gives the transport cost for the induced optimal transport problem. The squared $L^2(Q)$ error can be bounded by the sum of the squared $L^2(P)$ error and the transport cost for the corresponding optimal transport problem.

In addition to the useful properties of the Ivanov-regularised estimator, Ivanov regularisation can be performed almost as quickly as Tikhonov regularisation. The Ivanov-regularised estimator is a support vector machine (SVM) with regularisation parameter selected to match the norm constraint, as discussed in Appendix B. This parameter can be selected to within a tolerance $\varepsilon$ using interval bisection with order $\log(1/\varepsilon)$ iterations. In general, Ivanov regularisation requires the calculation of $\log(1/\varepsilon)$ SVMs.

In this paper, we study the behaviour of the Ivanov-regularised least-squares estimator with regularisation function equal to the norm of the RKHS. We derive a number of novel results...
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concerning the rate of convergence of the estimator in various settings and under various assumptions. Ivanov regularisation allows us to significantly weaken some of the fundamental assumptions made when studying Tikhonov-regularised estimators. For example, Ivanov regularisation allows us to dispose of the restrictive Mercer kernel assumption.

The current theory of RKHS regression for Tikhonov-regularised estimators only applies when the RKHS has a Mercer kernel. If the RKHS $H$ has a Mercer kernel $k$ with respect to the covariate distribution $P$, then there is a simple decomposition of $k$ and succinct representation of $H$ as a subspace of $L^2(P)$. These descriptions are in terms of the eigenfunctions and eigenvalues of the kernel operator $T$ on $L^2(P)$. Many results have assumed a fixed rate of decay of these eigenvalues in order to produce estimators whose squared $L^2(P)$ error decreases quickly with the number of data points (Mendelson and Neeman, 2010; Steinwart et al., 2009). However, the assumptions necessary for $H$ to have a Mercer kernel are in general quite restrictive. The usual set of assumptions is that the covariate set $S$ is compact, the kernel $k$ of $H$ is continuous on $S \times S$ and the covariate distribution satisfies $\text{supp } P = S$ (see Section 4.5 of Steinwart and Christmann, 2008). In particular, the assumption that the covariate set $S$ is compact is inconvenient and there has been some research into how to relax this condition by Steinwart and Scovel (2012).

By contrast, we provide results that hold under the significantly weaker assumption that the RKHS is separable with a bounded and measurable kernel $k$. We can remove the Mercer kernel assumption because we control empirical processes over balls in the RKHS instead of relying on the representation of the RKHS given by Mercer’s theorem. We first prove an expectation bound on the squared $L^2(P)$ error of our estimator of order $n^{-\beta/2}$ under the weak assumption that the response variables have bounded variance. Here, $n$ is the number of data points and $\beta$ parametrises the interpolation space between $L^2(P)$ and $H$ containing the regression function. The definition of an interpolation space is given in Subsection 1.1. The expected squared $L^2(P)$ error can be viewed as the expected squared error of our estimator at a new independent covariate with the same distribution $P$. If we also assume that the regression function is bounded, then it makes sense to clip our estimator so that it takes values in the same interval as the regression function. This further assumption allows us to achieve an expectation bound on the squared $L^2(P)$ error of the clipped estimator of order $n^{-\beta/(1+\beta)}$.

We then move away from the average behaviour of the error towards its behaviour in the worst case. We obtain high-probability bounds of the same order, under the stronger assumption that the response variables have subgaussian errors and the interpolation space is between $L^\infty$ and $H$. The second assumption is quite natural as we already assume that the regression function is bounded, and $H$ can be continuously embedded in $L^\infty$ since it has a bounded kernel $k$. This assumption also means that the set of possible regression functions is independent of the covariate distribution, which may be advantageous in some scenarios. For example, it makes sense when considering the covariate shift problem, as discussed above.

When the regression function is bounded, we also analyse an adaptive version of our estimator, which does not require us to know which interpolation space contains the regression function. This adaptive estimator obtains bounds of the same order as the non-adaptive
one. Furthermore, our results match the order $n^{-\beta/(\beta+1)}$ of Steinwart et al. (2009) when $p = 1$. In particular, this shows that our expectation results are of optimal order for our setting.

1.1 RKHSs and Their Interpolation Spaces

A Hilbert space $H$ of real-valued functions on $S$ is an RKHS if the evaluation functional $L_x : H \rightarrow \mathbb{R}$ by $L_x h = h(x)$ is bounded for all $x \in S$. In this case, $L_x \in H^*$ the dual of $H$ and the Riesz representation theorem tells us that there is some $k_x \in H$ such that $h(x) = \langle h, k_x \rangle_H$ for all $h \in H$. The kernel is then given by $k(x_1, x_2) = \langle k_{x_1}, k_{x_2} \rangle_H$ for $x_1, x_2 \in S$, and is symmetric and positive-definite.

Now suppose that $(S, S)$ is measurable space on which $P$ is a probability measure. We can define a range of spaces between $L^2(P)$ and $H$. Let $(Z, \|\cdot\|_Z)$ be a Banach space and $(V, \|\cdot\|_V)$ be a subspace of $Z$. Then the $K$-functional of $(Z, V)$ is

$$K(z, t) = \inf_{v \in V} (\|z - v\|_Z + t\|v\|_V)$$

for $z \in Z$ and $t > 0$. For $\beta \in (0, 1)$ and $1 \leq q < \infty$, we define

$$\|z\|_{\beta,q} = \left( \int_0^\infty (t^{-\beta} K(z, t))^{q-1} dt \right)^{1/q}$$

and

$$\|z\|_{\beta,\infty} = \sup_{t > 0} (t^{-\beta} K(z, t))$$

for $z \in Z$. The interpolation space $[Z, V]_{\beta,q}$ is defined to be the set of $z \in Z$ such that $\|z\|_{\beta,q} < \infty$ for $\beta \in (0, 1)$ and $1 \leq q \leq \infty$. Smaller values of $\beta$ give larger spaces. The space is not much larger than $V$ when $\beta$ is close to 1, but we obtain spaces which get closer to $Z$ as $\beta$ decreases. Hence, we can define the interpolation spaces $[L^2(P), H]_{\beta,q}$, where $L^2(P)$ is the space of $P$-almost-sure equivalence classes of measurable functions $f$ on $(S, S)$ such that $f^2$ is integrable with respect to $P$. We will work with $q = \infty$, which gives the largest space of functions for a fixed $\beta \in (0, 1)$. Note that although $H$ is not a subspace of $L^2(P)$, the above definitions are still valid as there is a natural embedding of $H$ into $L^2(P)$ as long as the functions in $H$ are measurable on $(S, S)$. We will also require $[L^\infty, H]_{\beta,q}$, where $L^\infty$ is the space of bounded measurable functions on $(S, S)$.

1.2 Literature Review

The current literature assumes that the RKHS $H$ has a Mercer kernel $k$, as discussed above. Earlier research in this area, such as that of Caponnetto and de Vito (2007), assumes that the regression function is at least as smooth as an element of $H$. However, their paper still allows for regression functions of varying smoothness by letting the regression function be of the form $g = \mathcal{T}^{(\beta-1)/2} h$ for $\beta \in [1, 2]$ and $h \in H$. Here, $\mathcal{T} : L^2(P) \rightarrow L^2(P)$ is the kernel operator and $P$ is the covariate distribution. By assuming that the $i$th eigenvalue of $\mathcal{T}$ is
of order $i^{-1/p}$ for $p \in (0, 1]$, the authors achieve a squared $L^2(P)$ error of order $n^{-\beta/(\beta+p)}$ with high probability by using SMVs. This squared $L^2(P)$ error is shown to be of optimal order for $\beta \in (1, 2]$.

Later research focuses on the case in which the regression function is at most as smooth as an element of $H$. Often, this research demands that the response variables are bounded. For example, Mendelson and Neeman (2010) assume that $g \in T^{\beta/2}(L^2(P))$ for $\beta \in (0, 1)$ to obtain a squared $L^2(P)$ error of order $n^{-\beta/(1+p)}$ with high probability by using Tikhonov-regularised least-squares estimators. The authors also show that if the eigenfunctions of the kernel operator $T$ are uniformly bounded in $L^\infty$, then the order can be improved to $n^{-\beta/(\beta+p)}$. Steinwart et al. (2009) relax the condition on the eigenfunctions to the condition

$$||h||_\infty \leq C_p ||h||_H^p ||h||_{L^2(P)}^{1-p}$$

for all $h \in H$ and some constant $C_p > 0$. The same rate is attained by using clipped Tikhonov-regularised least-squares estimators, including clipped SMVs, and is shown to be optimal. The authors assume that $g$ is in an interpolation space between $L^2(P)$ and $H$, which is slightly more general than the assumption of Mendelson and Neeman (2010). A detailed discussion about the image of $L^2(P)$ under powers of $T$ and interpolation spaces between $L^2(P)$ and $H$ is given by Steinwart and Scovel (2012).

Lately, the assumption that the response variables must be bounded has been relaxed to allow for subexponential errors. However, the assumption that the regression function is bounded has been maintained. For example, Fischer and Steinwart (2017) assume that $g \in T^{\beta/2}(L^2(P))$ for $\beta \in (0, 2]$ and that $g$ is bounded. The authors also assume that $T^{\alpha/2}(L^2(P))$ is continuously embedded in $L^\infty$ with respect to an appropriate norm on $T^{\alpha/2}(L^2(P))$ for some $\alpha < \beta$. This gives the same squared $L^2(P)$ error of order $n^{-\beta/(\beta+p)}$ with high probability by using SVMs.

### 1.3 Contribution

In this paper, we provide bounds on the squared $L^2(P)$ error of our Ivanov-regularised least-squares estimator when the regression function comes from an interpolation space between $L^2(P)$ and an RKHS $H$, which is separable with a bounded and measurable kernel $k$. We use the norm of the RKHS as our regularisation function. Under the weak assumption that the response variables have bounded variance, we prove a bound on the expected squared $L^2(P)$ error of order $n^{-\beta/2}$ (Theorem 5 on page 12). If we assume that the regression function is bounded then we can clip the estimator and achieve an expected squared $L^2(P)$ error of order $n^{-\beta/(1+\beta)}$ (Theorem 8 on page 14).

Under the stronger assumption that the response variables have subgaussian errors and the regression function comes from an interpolation space between $L^\infty$ and $H$, we show that the squared $L^2(P)$ error is of order $n^{-\beta/(1+\beta)}$ with high probability (Theorem 16 on page 23). For the settings in which the regression function is bounded, we use training and validation on the data in order to select the size of the constraint on the norm of our estimator. This gives us an adaptive estimation result which does not require us to know which interpolation space contains the regression function. We obtain a squared $L^2(P)$ error of order $n^{-\beta/(1+\beta)}$.
in expectation and with high probability, depending on the setting (Theorems 13 and 19 on pages 19 and 27). In order to perform training and validation, the response variables in the validation set must have subgaussian errors. The expectation results are of optimal order.

The results not involving validation are summarised in Table 1. The columns for which there is an $L^\infty$ bound on the regression function also make the $L^2(P)$ interpolation assumption. Orders of bounds marked with $(\ast)$ are known to be optimal.

| Regression Function | $L^2(P)$ Interpolation | $L^\infty$ Bound | $L^\infty$ Interpolation |
|---------------------|------------------------|------------------|------------------------|
| Response Variables  | Bounded Variance       | Bounded Variance | Subgaussian Errors      |
| Bound Type          | Expectation            | Expectation      | High Probability        |
| Bound Order         | $n^{-\beta/2}$         | $n^{-\beta/(1+\beta)}$ $(\ast)$ | $n^{-\beta/(1+\beta)}$ |

Table 1: Orders of bounds on squared $L^2(P)$ error

The validation results are summarised in Table 2. Again, the columns for which there is an $L^\infty$ bound on the regression function also make the $L^2(P)$ interpolation assumption. The assumptions on the response variables relate to those in the validation set, which has $\tilde{n}$ data points. We assume that $\tilde{n}$ is at least some multiple of $n$. Again, orders of bounds marked with $(\ast)$ are known to be optimal.

| Regression Function | $L^\infty$ Bound | $L^\infty$ Interpolation |
|---------------------|------------------|------------------------|
| Response Variables  | Subgaussian Errors | Subgaussian Errors      |
| Bound Type          | Expectation      | High Probability        |
| Bound Order         | $n^{-\beta/(1+\beta)}$ $(\ast)$ | $n^{-\beta/(1+\beta)}$ |

Table 2: Orders of validation bounds on squared $L^2(P)$ error

2. Problem Definition and Assumptions

We now formally define our regression problem and the assumptions that we make in this paper. For a topological space $T$, let $\mathcal{B}(T)$ be the Borel $\sigma$-algebra of $T$. Let $(S, \mathcal{S})$ be a measurable space. Assume that $(X_i, Y_i)$ are $(S \times \mathbb{R}, \mathcal{S} \otimes \mathcal{B}(\mathbb{R}))$-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for $1 \leq i \leq n$, which are i.i.d. with $X_i \sim P$ and $Y_i^2$ integrable. We denote integration with respect to $\mathbb{P}$ by $\mathbb{E}$. Since $\mathbb{E}(Y_i|X_i)$ is $\sigma(X_i)$-measurable, where $\sigma(X_i)$ is the $\sigma$-algebra generated by $X_i$, we have that $\mathbb{E}(Y_i|X_i) = g(X_i)$ for some function $g$ which is measurable on $(S, \mathcal{S})$ (Section A3.2 of Williams, 1991). From the definition of conditional expectation and the identical distribution of the $(X_i, Y_i)$, it is clear that we can choose $g$ to be the same for all $1 \leq i \leq n$. The conditional expectation used is that of Kolmogorov, defined using the Radon–Nikodym derivative. Its definition is unique almost surely. Since $Y_i^2$ is integrable, $g \in L^2(P)$ by Jensen’s inequality. In addition to

$$\mathbb{E}(Y_i|X_i) = g(X_i),$$

we assume

$$\text{var}(Y_i|X_i) \leq \sigma^2.$$
We also assume that \( g \in [L^2(P), H]_{\beta, \infty} \) with norm at most \( B \), where \( H \) is a separable RKHS of measurable functions on \((S, S)\) and \( \beta \in (0, 1) \). Finally, we assume that \( H \) has kernel \( k \) which is bounded and measurable on \((S \times S, S \otimes S)\), with

\[
\|k\|_{\infty} = \sup_{x \in S} k(x, x)^{1/2} < \infty.
\]

We can guarantee that \( H \) is separable by, for example, assuming that \( k \) is continuous and \( S \) is a separable topological space (Lemma 4.33 of Steinwart and Christmann, 2008). The fact that \( H \) has a kernel \( k \) which is measurable on \((S \times S, S \otimes S)\) guarantees that all functions in \( H \) are measurable on \((S, S)\) (Lemma 4.24 of Steinwart and Christmann, 2008).

Let \( B_H \) be the unit ball of \( H \) and \( r > 0 \). Our assumption that \( g \in [L^2(P), H]_{\beta, \infty} \) with norm at most \( B \) for \( \beta \in (0, 1) \) provides us with the following approximation result. Theorem 3.1 of Smale and Zhou (2003) shows that

\[
\inf \{ \|h - g\|_{L^2(P)}^2 : h \in rB_H \} \leq \frac{B^2(1-\beta)}{r^{2\beta/(1-\beta)}}
\]

when \( H \) is dense in \( L^2(P) \). This additional condition is present because these are the only interpolation spaces considered by the authors, and in fact the result holds by the same proof even when \( H \) is not dense in \( L^2(P) \). Hence, for all \( \alpha > 0 \) there is some \( h_{r,\alpha} \in rB_H \) such that

\[
\|h_{r,\alpha} - g\|_{L^2(P)}^2 \leq \frac{B^2(1-\beta)}{r^{2\beta/(1-\beta)}} + \alpha. \tag{3}
\]

We will estimate \( h_{r,\alpha} \) for small \( \alpha > 0 \) by

\[
\hat{h}_r = \arg\min_{f \in rB_H} \frac{1}{n} \sum_{i=1}^n (f(X_i) - Y_i)^2.
\]

We also define \( \hat{h}_0 = 0 \). The estimator \( \hat{h}_r \) is calculated in Appendix B and shown to be a \((H, B(H))\)-valued measurable function on \((\Omega \times [0, \infty), F \otimes B([0, \infty]))\), where \( r \) varies in \([0, \infty)\). Lemma 21 summarises these results in this appendix.

### 2.1 Clipping

In the majority of this paper, we will assume that \( \|g\|_{\infty} \leq C \). Since \( g \) is bounded in \([-C, C]\), we can make \( h_{r,\alpha} \) and \( \hat{h}_r \) closer to \( g \) by constraining them to lie in the same interval. Similarly to Chapter 7 of Steinwart and Christmann (2008) and Steinwart et al. (2009), we define the projection \( V : \mathbb{R} \to [-C, C] \) by

\[
V(t) = \begin{cases}  
-C & \text{if } t < -C \\
t & \text{if } |t| \leq C \\
C & \text{if } t > C 
\end{cases}
\]

for \( t \in \mathbb{R} \).
2.2 Validation

Let us assume that we have an independent second data set \((\tilde{X}_i, \tilde{Y}_i)\) which are \((\mathcal{S} \times \mathbb{R}, \mathcal{S} \otimes \mathcal{B}(\mathbb{R}))\)-valued random variables on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\) for \(1 \leq i \leq \tilde{n}\). Let the \((\tilde{X}_i, \tilde{Y}_i)\) be i.i.d. with \(\tilde{X}_i \sim P \) and

\[
\mathbb{E}(\tilde{Y}_i | \tilde{X}_i) = g(\tilde{X}_i).
\]

Furthermore, we assume that the moment generating function of \(\tilde{Y}_i - g(\tilde{X}_i)\) conditional on \(\tilde{X}_i\) satisfies

\[
\mathbb{E} (\exp(t(\tilde{Y}_i - g(\tilde{X}_i)))) | \tilde{X}_i) \leq \exp(\tilde{\sigma}^2 t^2/2)
\]

for all \(t \in \mathbb{R}\) and \(1 \leq i \leq \tilde{n}\). The random variable \(\tilde{Y}_i - g(\tilde{X}_i)\) is said to be \(\tilde{\sigma}^2\)-subgaussian conditional on \(\tilde{X}_i\). This assumption is made so that we can use chaining in the proofs of Lemmas 10 and 17. Let \(\rho \geq 0\) and \(R \subseteq [0, \rho]\) be compact. Furthermore, let

\[
F = \{ \hat{V}h_r : r \in R \}
\]

and

\[
r_0 = \arg\min_{r \in R} \mathbb{E}\left( \| \hat{V}h_r - g \|_{L^2(P)}^2 \right).
\]

The minimum is attained as it is the minimum of a continuous function over a compact set. In the event of ties, we may take \(r_0\) to be the infimum of all points attaining the minimum. We will estimate \(r_0\) by

\[
\hat{r} = \arg\min_{r \in R} \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} (\hat{V}h_r(\tilde{X}_i) - \tilde{Y}_i)^2.
\]

We can uniquely define \(\hat{r}\) in the same way as \(r_0\). Lemma 23 in Appendix B shows that the estimator \(\hat{r}\) is a random variable on \((\Omega, \mathcal{F})\). Hence, \(\hat{h}_r\) is a \((H, \mathcal{B}(H))\)-valued random variable on \((\Omega, \mathcal{F})\).

2.3 Suprema of Stochastic Processes

We require the measurability of certain suprema over subsets of \(H\). Since \(H\) is a separable metric space, its subsets are also separable. For example, \(rB_H\) has a countable dense subset \(A_r\). Furthermore, \(H\) is continuously embedded in \(L^2(P_n)\) and \(L^2(P)\) because its kernel \(k\) is bounded. Here, \(P_n\) is the empirical distribution of the \(X_i\). Hence, for example,

\[
\sup_{f \in rB_H} \left\| f \right\|_{L^2(P_n)}^2 - \left\| f \right\|_{L^2(P)}^2 = \sup_{f \in A_r} \left\| f \right\|_{L^2(P_n)}^2 - \left\| f \right\|_{L^2(P)}^2.
\]

This is a random variable on \((\Omega, \mathcal{F})\) since the right-hand side is a supremum of countably many random variables on \((\Omega, \mathcal{F})\). Therefore, this quantity has a well-defined expectation and we can also apply Talagrand’s inequality to it.
3. Expectation Bound for Unbounded Regression Function

First, we will bound the difference between \( \hat{h}_r \) and \( h_{r,\alpha} \) in the \( L^2(P_n) \) norm.

**Lemma 1** The definition of \( \hat{h}_r \) shows

\[
\| \hat{h}_r - h_{r,\alpha} \|^2_{L^2(P_n)} \leq \frac{4}{n} \sum_{i=1}^{n} (Y_i - g(X_i))(\hat{h}_r(X_i) - h_{r,\alpha}(X_i)) + 4\| h_{r,\alpha} - g \|^2_{L^2(P_n)}, \tag{4}
\]

**Proof** Since \( h_{r,\alpha} \in rB_H \), the definition of \( \hat{h}_r \) gives

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{h}_r(X_i) - Y_i)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (h_{r,\alpha}(X_i) - Y_i)^2.
\]

Expanding

\[
(\hat{h}_r(X_i) - Y_i)^2 = \left( (\hat{h}_r(X_i) - h_{r,\alpha}(X_i)) + (h_{r,\alpha}(X_i) - Y_i) \right)^2,
\]

substituting into the above and rearranging gives

\[
\frac{1}{n} \sum_{i=1}^{n} (\hat{h}_r(X_i) - h_{r,\alpha}(X_i))^2 \leq \frac{2}{n} \sum_{i=1}^{n} (Y_i - h_{r,\alpha}(X_i))(\hat{h}_r(X_i) - h_{r,\alpha}(X_i)).
\]

Substituting

\[
Y_i - h_{r,\alpha}(X_i) = (Y_i - g(X_i)) + (g(X_i) - h_{r,\alpha}(X_i))
\]

into the above and applying the Cauchy–Schwarz inequality to the second term gives

\[
\| \hat{h}_r - h_{r,\alpha} \|^2_{L^2(P_n)} \leq \frac{2}{n} \sum_{i=1}^{n} (Y_i - g(X_i))(\hat{h}_r(X_i) - h_{r,\alpha}(X_i))
\]

\[
+ 2\| g - h_{r,\alpha} \|_{L^2(P_n)} \| \hat{h}_r - h_{r,\alpha} \|_{L^2(P_n)}.
\]

For constants \( a, b \geq 0 \) we have

\[
x^2 \leq a + 2bx \implies x^2 \leq 2a + 4b^2
\]

for \( x \in \mathbb{R} \) by completing the square and rearranging. Applying this result to the above inequality proves the lemma.

We now take the expectation of (4).

**Lemma 2** By bounding the expectation of the right-hand side of (4), we have

\[
\mathbb{E} \left( \| \hat{h}_r - h_{r,\alpha} \|^2_{L^2(P_n)} \right) \leq \frac{4\| \kappa \|_{\infty} \sigma r}{n^{1/2}} + \frac{4B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + 4\alpha.
\]

**Proof** We have

\[
\mathbb{E} \left( \| h_{r,\alpha} - g \|^2_{L^2(P_n)} \right) = \| h_{r,\alpha} - g \|^2_{L^2(P)} \leq \frac{B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + \alpha
\]
by (3) and
\[ E \left( \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) \hat{h}_{r,\alpha}(X_i) \right) = E \left( \frac{1}{n} \sum_{i=1}^{n} E(Y_i - g(X_i)|X_i) \hat{h}_{r,\alpha}(X_i) \right) = 0. \]

The remainder of this proof method is due to Remark 6.1 of Sriperumbudur (2016). Since \( \hat{h}_r \in rB_H \), we have
\[
\frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) \hat{h}_r(X_i) \leq \sup_{f \in rB_H} \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) f(X_i)
\]
\[
= \sup_{f \in rB_H} \left( \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) k(X_i, f) \right)
\]
\[
= r \left\| \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) k(X_i) \right\|_H
\]
\[
= r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} (Y_i - g(X_i))(Y_j - g(X_j)) k(X_i, X_j) \right)^{1/2}
\]
by the Cauchy–Schwarz inequality. By Jensen’s inequality and the independence of the \((X_i, Y_i)\), we have
\[
E \left( \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) \hat{h}_r(X_i) \right \rvert X) \leq r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \text{cov}(Y_i, Y_j \rvert X) k(X_i, X_j) \right)^{1/2}
\]
\[
\leq r \left( \frac{\sigma^2 \|k\|^2_\infty}{n^2} \right)^{1/2}
\]
and again, by Jensen’s inequality, we have
\[
E \left( \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) \hat{h}_r(X_i) \right) \leq r \left( \frac{\sigma^2 \|k\|^2_\infty}{n} \right)^{1/2}
\]
The result follows from Lemma 1.

The next step is to move our bound on the expectation of the squared \( L^2(P_n) \) norm of \( \hat{h}_r - h_{r,\alpha} \) to the expectation of the squared \( L^2(P) \) norm of \( \hat{h}_r - h_{r,\alpha} \).

**Lemma 3** By using Rademacher processes, we can show
\[
E \left( \sup_{f \in rB_H} \left\| f \|_{L^2(P_n)}^2 - \| f \|_{L^2(P)}^2 \right\| \right) \leq \frac{8 \|k\|^2_\infty r^2}{n^{1/2}}.
\]

**Proof** Let the \( \varepsilon_i \) be i.i.d. Rademacher random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) for \( 1 \leq i \leq n \), independent of the \( X_i \). Lemma 2.3.1 of van der Vaart and Wellner (2013) shows
\[
E \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i)^2 - \int f^2 \, dP \right| \leq 2 E \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i)^2 \right|
\]
by symmetrisation. Since
\[ |f(X_i)| \leq \|k\|_\infty r \]
for all \( f \in rB_H \), we find
\[
\frac{f(X_i)^2}{2\|k\|_\infty r}
\]
is a contraction vanishing at 0 as a function of \( f(X_i) \) for all \( 1 \leq i \leq n \). By Theorem 3.2.1 of Giné and Nickl (2016), we have
\[
\mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \right) \leq 2 \mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \right).
\]
We now follow a similar argument to the proof of Lemma 2. We have
\[
\sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| = \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i kX_i, f \right|_H = r \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i kX_i \right\|_H = r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j k(X_i, X_j) \right)^{1/2}.
\]
By Jensen’s inequality, we have
\[
\mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \right) \leq r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \text{cov}(\varepsilon_i, \varepsilon_j | X) k(X_i, X_j) \right)^{1/2} = r \left( \frac{1}{n^2} \sum_{i=1}^{n} k(X_i, X_i) \right)^{1/2}
\]
and again, by Jensen’s inequality, we have
\[
\mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \right) \leq r \left( k(X_i, X_i) \right)^{1/2}.
\]
The result follows.

The next result follows as a simple consequence of Lemmas 2 and 3

**Corollary 4** Since \( \hat{h}_r - h_{r,\alpha} \in 2rB_H \), we have
\[
\mathbb{E} \left( \left\| \hat{h}_r - h_{r,\alpha} \right\|_{L^2(P)}^2 \right) \leq \frac{4\|k\|_\infty \sigma r}{n^{1/2}} + \frac{4B^2(1-\beta)}{r^{2\beta/(1-\beta)}} + \frac{32\|k\|_\infty^2 r^2}{n^{1/2}} + 4\alpha.
\]
All that remains to is to combine Corollary 7 with (3) by using
\[ \| \hat{h}_r - g \|_{L^2(P)}^2 \leq \left( \| \hat{h}_r - h_{r,\alpha} \|_{L^2(P)} + \| h_{r,\alpha} - g \|_{L^2(P)} \right)^2 \]
\[ \leq 2 \| \hat{h}_r - h_{r,\alpha} \|_{L^2(P)}^2 + 2 \| h_{r,\alpha} - g \|_{L^2(P)}^2 \]
and to let \( \alpha \to 0 \).

**Theorem 5** We have
\[ E \left( \| \hat{h}_r - g \|_{L^2(P)}^2 \right) \leq \frac{8 \| k \|_\infty \sigma r}{n^{1/2}} + \frac{10B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + \frac{64 \| k \|_\infty^2 r^2}{n^{1/2}}. \]

Based on this bound, the asymptotically optimal choice of \( r \) is
\[ \left( \frac{5\beta}{32(1-\beta)} \right)^{(1-\beta)/2} \| k \|_\infty^2 r \] which gives a bound of
\[ 2 \left( \frac{32}{32(1-\beta)} \right)^{1-\beta} \left( \frac{32(1-\beta)}{5\beta} \right)^{\beta} \| k \|_\infty^2 B^2 \sigma n^{-\beta/2} + 8 \left( \frac{32}{32(1-\beta)} \right)^{(1-\beta)/2} \| k \|_\infty^2 \sigma B n^{-(1+\beta)/4}. \]

4. Expectation Bound and Validation for Bounded Regression Function

We will now also assume that \( \| g \|_\infty \leq C \) and clip our estimator. It follows from (3) that
\[ \| \hat{h}_{r,\alpha} - Vh_{r,\alpha} \|_{L^2(P)}^2 \leq \frac{B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + \alpha \]
and from Lemma 2 that
\[ E \left( \| \hat{h}_r - Vh_{r,\alpha} \|_{L^2(P)}^2 \right) \leq \frac{4 \| k \|_\infty \sigma r}{n^{1/2}} + \frac{4B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + 4\alpha. \]

4.1 Expectation Bound

As in Section 3, the next step is to move our bound on the expectation of the squared \( L^2(P_n) \) norm of \( V\hat{h}_r - Vh_{r,\alpha} \) to the expectation of the squared \( L^2(P) \) norm of \( V\hat{h}_r - Vh_{r,\alpha} \). Working with clipped functions allows us to achieve a difference between the two squared norms of order \( r/n^{1/2} \) instead of order \( r^2/n^{1/2} \). This is substantially smaller.

**Lemma 6** By using Rademacher processes, we can show
\[ E \left( \sup_{f \in L_B^H} \left( \| Vf - Vh_{r,\alpha} \|_{L^2(P_n)} - \| Vf - Vh_{r,\alpha} \|_{L^2(P)} \right) \right) \leq \frac{64 \| k \|_\infty C r}{n^{1/2}}. \]
Proof Let the $\varepsilon_i$ be i.i.d. Rademacher random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ for $1 \leq i \leq n$, independent of the $X_i$. Lemma 2.3.1 of van der Vaart and Wellner (2013) shows

$$
\mathbb{E} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \int (Vf - Vh_{r,\alpha})^2 d\mathbb{P} \right|
$$

is at most

$$
2 \mathbb{E} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 \right|
$$

by symmetrisation. Since

$$
|Vf(X_i) - Vh_{r,\alpha}(X_i)| \leq 2C
$$

for all $f \in rB_H$, we find

$$
\frac{(Vf(X_i) - Vh_{r,\alpha}(X_i))^2}{4C}
$$

is a contraction vanishing at 0 as a function of $Vf(X_i) - Vh_{r,\alpha}(X_i)$ for all $1 \leq i \leq n$. By Theorem 3.2.1 of Giné and Nickl (2016), we have

$$
\mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 \right| \right) \leq 2 \mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i Vf(X_i) \right| \right).
$$

Therefore,

$$
\mathbb{E} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \int (Vf - Vh_{r,\alpha})^2 d\mathbb{P} \right|
$$

is at most

$$
16C \mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i (Vf(X_i) - Vh_{r,\alpha}(X_i)) \right| \right) \leq 32C \mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i Vf(X_i) \right| \right)
$$

by the triangle inequality, since $h_{r,\alpha} \in rB_H$. Again, by Theorem 3.2.1 of Giné and Nickl (2016), we have

$$
\mathbb{E} \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \int (Vf - Vh_{r,\alpha})^2 d\mathbb{P} \right|
$$

is at most

$$
64C \mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i Vf(X_i) \right| \right)
$$

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since $V$ is a contraction vanishing at 0. We now follow a similar argument to the proof of Lemma 2. We have

$$\sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| = \sup_{f \in rB_H} \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i k_{X_i}, f \right\|_H = r \left\| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i k_{X_i} \right\|_H = r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \varepsilon_i \varepsilon_j k(X_i, X_j) \right)^{1/2}.$$ 

By Jensen’s inequality, we have

$$\mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \right) \leq r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \text{cov}(\varepsilon_i, \varepsilon_j|X)k(X_i, X_j) \right)^{1/2} = r \left( \frac{1}{n^2} \sum_{i=1}^{n} k(X_i, X_i) \right)^{1/2},$$

and again, by Jensen’s inequality, we have

$$\mathbb{E} \left( \sup_{f \in rB_H} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right| \right) \leq r \left( \frac{\|k\|_\infty^2}{n} \right)^{1/2}.$$ 

The result follows.

The next result follows as a simple consequence of Lemmas 2 and 6

**Corollary 7** We have

$$\mathbb{E} \left( \|V \hat{h}_r - Vh_{r,\alpha}\|_{L^2(P)}^2 \right) \leq \frac{4\|k\|_\infty^2 (\sigma + 16C)r}{n^{1/2}} + \frac{4B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + 4\alpha.$$ 

All that remains to is to combine Corollary 7 with (3) by using

$$\|V \hat{h}_r - g\|_{L^2(P)}^2 \leq \left( \|V \hat{h}_r - Vh_{r,\alpha}\|_{L^2(P)} + \|Vh_{r,\alpha} - g\|_{L^2(P)} \right)^2 \leq 2\|V \hat{h}_r - Vh_{r,\alpha}\|_{L^2(P)}^2 + 2\|Vh_{r,\alpha} - g\|_{L^2(P)}^2$$

and to let $\alpha \to 0$.

**Theorem 8** We have

$$\mathbb{E} \left( \|V \hat{h}_r - g\|_{L^2(P)}^2 \right) \leq \frac{8\|k\|_\infty^2 (\sigma + 16C)r}{n^{1/2}} + \frac{10B^2/(1-\beta)}{r^{2\beta/(1-\beta)}}.$$ 

Based on this bound, the optimal choice of $r$ is

$$\left( \frac{5\beta}{2(1-\beta)} \right)^{(1-\beta)/(1+\beta)} \|k\|_\infty^{(1-\beta)/(1+\beta)} (\sigma + 16C)^{-(1-\beta)/(1+\beta)} B^{2/(1+\beta)} n^{(1-\beta)/(2(1+\beta))},$$

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which gives a bound of
\[
2 \cdot 5^{(1-\beta)/(1+\beta)} \cdot 4^{2\beta/(1+\beta)} \left( \frac{2\beta}{1-\beta} \right)^{(1-\beta)/(1+\beta)} + \left( \frac{1-\beta}{2\beta} \right)^{2\beta/(1+\beta)} \right) \times \|k\|^{2\beta/(1+\beta)}(\sigma + 16C)^{2\beta/(1+\beta)}B^{2/(1+\beta)}n^{-\beta/(1+\beta)}.
\]

4.2 Validation Bound

The proof method for the validation bound is similar to that of the expectation bound. First, we will bound the difference between \(V\hat{h}_0\) and \(V\hat{h}_{r_0}\) in the \(L^2(\hat{P}_n)\) norm, where \(\hat{P}_n\) is the empirical distribution of the \(\hat{X}_i\).

**Lemma 9** The definition of \(\hat{r}\) shows

\[
\|V\hat{h}_\hat{r} - V\hat{h}_{r_0}\|_{L^2(\hat{P}_n)}^2 \leq \frac{4}{n} \sum_{i=1}^{\hat{n}} (\hat{Y}_i - g(\hat{X}_i))(V\hat{h}_\hat{r}(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i)) + 4\|V\hat{h}_{r_0} - g\|_{L^2(\hat{P}_n)}^2.
\]  

**Proof** Since \(r_0 \in R\), the definition of \(\hat{r}\) gives

\[
\frac{1}{n} \sum_{i=1}^{\hat{n}} (V\hat{h}_\hat{r}(\hat{X}_i) - \hat{Y}_i)^2 \leq \frac{1}{n} \sum_{i=1}^{\hat{n}} (V\hat{h}_{r_0}(\hat{X}_i) - \hat{Y}_i)^2.
\]

Expanding

\[
(V\hat{h}_\hat{r}(\hat{X}_i) - \hat{Y}_i)^2 = \left((V\hat{h}_\hat{r}(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i)) + (V\hat{h}_{r_0}(\hat{X}_i) - \hat{Y}_i)\right)^2,
\]

substituting into the above and rearranging gives

\[
\frac{1}{n} \sum_{i=1}^{\hat{n}} (V\hat{h}_\hat{r}(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i))^2 \leq \frac{2}{n} \sum_{i=1}^{\hat{n}} (\hat{Y}_i - V\hat{h}_{r_0}(\hat{X}_i))(V\hat{h}_\hat{r}(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i)).
\]

Substituting

\[
\hat{Y}_i - V\hat{h}_{r_0}(\hat{X}_i) = (\hat{Y}_i - g(\hat{X}_i)) + (g(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i))
\]

into the above and applying the Cauchy–Schwarz inequality to the second term gives

\[
\|V\hat{h}_\hat{r} - V\hat{h}_{r_0}\|_{L^2(\hat{P}_n)}^2 \leq \frac{2}{n} \sum_{i=1}^{\hat{n}} (\hat{Y}_i - g(\hat{X}_i))(V\hat{h}_\hat{r}(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i)) + 2\|g - V\hat{h}_{r_0}\|_{L^2(\hat{P}_n)}\|V\hat{h}_\hat{r} - V\hat{h}_{r_0}\|_{L^2(\hat{P}_n)}.
\]

For constants \(a, b \geq 0\) we have

\[
x^2 \leq a + 2bx \implies x^2 \leq 2a + 4b^2
\]

for \(x \in \mathbb{R}\) by completing the square and rearranging. Applying this result to the above inequality proves the lemma.
A stochastic process $W$ on $(\Omega, \mathcal{F})$ indexed by a metric space $(M, d)$ is $d^2$-subgaussian if it is centred and $W(s) - W(t)$ is $d(s,t)^2$-subgaussian for all $s, t \in M$. $W$ is separable if there exists a countable set $M_0 \subseteq M$ such that the following holds for all $\omega \in \Omega_0$, where $\mathbb{P}(\Omega_0) = 1$. For all $s \in M$ and $\varepsilon > 0$, $W(s)$ is in the closure of

$$\{W(t) : t \in M_0, d(s, t) \leq \varepsilon\}.$$  

We also need to introduce the concept of covering numbers for the next result. The covering number $N(M, d, \varepsilon)$ is the minimum number of $d$-balls of size $\varepsilon > 0$ needed to cover $M$. Whenever we require the covering number to be measurable in $\varepsilon$, such as when integrating over $\varepsilon$, we will replace the covering number with a measurable upper bound. We now take the expectation of (5).

**Lemma 10** By bounding the expectation of the right-hand side of (5), we have

$$\mathbb{E}\left(\|V \hat{h}_f - V \hat{h}_{r_0}\|^2_{L^2(\mathbb{P}_n)}\right)$$

is at most

$$\frac{16\bar{\sigma}C}{\bar{n}^{1/2}} \left(2 \log \left(2 + \frac{\|k\|_\infty^2 \rho^2}{C^2}\right)^{1/2} + \pi^{1/2}\right) + 4 \mathbb{E}\left(\|V \hat{h}_{r_0} - g\|^2_{L^2(\mathbb{P})}\right).$$

**Proof** We have

$$\mathbb{E}\left(\|V \hat{h}_{r_0} - g\|^2_{L^2(\mathbb{P}_n)}\right) = \mathbb{E}\left(\|V \hat{h}_{r_0} - g\|^2_{L^2(\mathbb{P})}\right).$$

Let $f_0 = V \hat{h}_{r_0}$ and

$$W(f) = \frac{1}{\bar{n}} \sum_{i=1}^{\bar{n}} W_i(f),$$

where

$$W_i(f) = (\tilde{Y}_i - g(\tilde{X}_i))(f(\tilde{X}_i) - f_0(\tilde{X}_i))$$

for $1 \leq i \leq \bar{n}$ and $f \in F$. Note that the $W_i$ are independent and centred. It is clear that $W_i(f_1) - W_i(f_2)$ is $\bar{\sigma}^2\|f_1 - f_2\|^2_{L^2}$-subgaussian given $\tilde{X}_i$ for $1 \leq i \leq \bar{n}$ and $f_1, f_2 \in F$, so the process $W$ is $\bar{\sigma}^2\|\cdot\|_{L^2}$-subgaussian given $\tilde{X}$. By Lemma 22, we have that $(F, \bar{\sigma}\|\cdot\|_{\infty}/\bar{n}^{1/2})$ is separable. Hence, $W$ is separable on $(F, \bar{\sigma}\|\cdot\|_{\infty}/\bar{n}^{1/2})$ since it is continuous. The diameter of $(F, \bar{\sigma}\|\cdot\|_{\infty}/\bar{n}^{1/2})$ is

$$D = \sup_{f_1, f_2 \in F} \bar{\sigma}\|f_1 - f_2\|_{\infty}/\bar{n}^{1/2} \leq 2\bar{\sigma}C/\bar{n}^{1/2}.$$  

From Lemma 28 in Appendix D, we have

$$\int_0^\infty \left(\log(N(F, \bar{\sigma}\|\cdot\|_{\infty}/\bar{n}^{1/2}, \varepsilon))\right)^{1/2} d\varepsilon = \int_0^\infty \left(\log(N(F, \|\cdot\|_{\infty}, \bar{n}^{1/2} \varepsilon/\bar{\sigma}))\right)^{1/2} d\varepsilon$$

$$= \frac{\bar{\sigma}}{\bar{n}^{1/2}} \int_0^\infty \left(\log(N(F, \|\cdot\|_{\infty}, u))\right)^{1/2} du$$

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By using Rademacher processes, we can show Lemma 11 shows the result follows.

The next step is to move our bound on the expectation of the squared $L^2(\hat{P}_n)$ norm of $V\hat{h}_r - V\hat{h}_{r_0}$ to the expectation of the squared $L^2(P)$ norm of $V\hat{h}_r - V\hat{h}_{r_0}$.

**Lemma 11** By using Rademacher processes, we can show

$$\mathbb{E} \left( \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} (f(\tilde{X}_i) - V\hat{h}_{r_0}(\tilde{X}_i))^2 - \int (f - V\hat{h}_{r_0})^2 dP \right| \right)$$

is at most

$$\frac{64C^2}{\tilde{n}^{1/2}} \left( 2 \log \left( 2 + \frac{\|k\|_{L^\infty}^2}{C^2} \right) \right)^{1/2} + \pi^{1/2}.$$ 

**Proof** Let $f_0 = V\hat{h}_{r_0}$ and the $\varepsilon_i$ be i.i.d. Rademacher random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ for $1 \leq i \leq \tilde{n}$, independent of $\tilde{X}$, $X$ and $Y$. Lemma 2.3.1 of van der Vaart and Wellner (2013) shows

$$\mathbb{E} \left( \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} (f(\tilde{X}_i) - f_0(\tilde{X}_i))^2 - \int (f - f_0)^2 dP \right| \right)$$

is at most

$$2 \mathbb{E} \left( \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} \varepsilon_i (f(\tilde{X}_i) - f_0(\tilde{X}_i))^2 \right| \right)$$
by symmetrisation. Since
\[ |f(\tilde{X}_i) - f_0(\tilde{X}_i)| \leq 2C \]
for all \( f \in F \), we find
\[ \frac{(f(\tilde{X}_i) - f_0(\tilde{X}_i))^2}{4C} \]
is a contraction vanishing at 0 as a function of \( f(\tilde{X}_i) - f_0(\tilde{X}_i) \) for all \( 1 \leq i \leq \tilde{n} \). By Theorem 3.2.1 of Giné and Nickl (2016), we have
\[ W \]
for all \( f \in F \), we find
\[ W \]
is at most
\[ 2E \left( \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} \varepsilon_i(f(\tilde{X}_i) - f_0(\tilde{X}_i)) \right| \right) . \]
Therefore,
\[ E \left( \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} (f(\tilde{X}_i) - f_0(\tilde{X}_i))^2 - \int (f - f_0)^2 dP \right| \right) \]
is at most
\[ 16C \left( \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} \varepsilon_i(f(\tilde{X}_i) - f_0(\tilde{X}_i)) \right| \right) . \]
We now follow a similar argument to the proof of Lemma 10. Let
\[ W(f) = \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} W_i(f) \]
where
\[ W_i(f) = \varepsilon_i(f(\tilde{X}_i) - f_0(\tilde{X}_i)) \]
for \( 1 \leq i \leq \tilde{n} \) and \( f \in F \). Note that the \( W_i \) are independent and centred. We see that
\[ W_i(f_1) - W_i(f_2) \]
is \( \|f_1 - f_2\|_{\infty}^2 \)-subgaussian given \( \tilde{X}_i \) for \( 1 \leq i \leq \tilde{n} \) and \( f_1, f_2 \in F \) by Hoeffding’s lemma, so the process \( W \) is \( \|\cdot\|_{\infty}/\tilde{n}^{1/2} \)-subgaussian given \( \tilde{X} \). By Lemma 22, we have that \( (F, \|\cdot\|_{\infty}/\tilde{n}^{1/2}) \) is separable. Hence, \( W \) is separable on \( (F, \|\cdot\|_{\infty}/\tilde{n}^{1/2}) \) since it is continuous. The diameter of \( (F, \|\cdot\|_{\infty}/\tilde{n}^{1/2}) \) is
\[ D = \sup_{f_1, f_2 \in F} \|f_1 - f_2\|_{\infty}/\tilde{n}^{1/2} \leq 2C/\tilde{n}^{1/2} . \]
From Lemma 28 in Appendix D, we have
\[ \int_0^\infty (\log(N(F, \|\cdot\|_{\infty}/\tilde{n}^{1/2}, \varepsilon)))^{1/2} d\varepsilon = \int_0^\infty (\log(N(F, \|\cdot\|_{\infty}, \tilde{n}^{1/2} \varepsilon)))^{1/2} d\varepsilon \]
\[ = \frac{1}{\tilde{n}^{1/2}} \int_0^\infty (\log(N(F, \|\cdot\|_{\infty}, u)))^{1/2} du \]
is finite. Hence, by Theorem 2.3.7 of Giné and Nickl (2016) and Lemma 28 in Appendix D, we have
\[
\mathbb{E} \left( \sup_{f \in F} |W(f)| \bigg| \tilde{X}, X, Y \right)
\]
is at most
\[
\mathbb{E}(|W(f_0)| \bigg| \tilde{X}, X, Y) + 2^{5/2} \int_{0}^{C/\tilde{n}^{1/2}} (\log(2N(F, \|\cdot\|_\infty/\tilde{n}^{1/2}, \varepsilon)))^{1/2} d\varepsilon
\]
\[
= 2^{5/2} \int_{0}^{C/\tilde{n}^{1/2}} (\log(2N(F, \|\cdot\|_\infty, \tilde{n}^{1/2})u))^{1/2} du
\]
\[
= 2^{5/2} \tilde{n}^{1/2} \left( \left( \log \left( 2 + \frac{\|k\|_\infty^2 \rho^2}{C^2} \right) \right)^{1/2} + \frac{\pi^{1/2} C}{2} \right)
\]
\[
= 4C \tilde{n}^{1/2} \left( \left( 2 \log \left( 2 + \frac{\|k\|_\infty^2 \rho^2}{C^2} \right) \right)^{1/2} + \pi^{1/2} \right).
\]
The result follows.

The next result follows as a simple consequence of Lemmas 10 and 11.

**Corollary 12** We have
\[
\mathbb{E} \left( \|\hat{V} \hat{h} - V \hat{h}_{r_0}\|_{L^2(P)}^2 \right)
\]
is at most
\[
\frac{16(\tilde{\sigma} + 4C)C}{\tilde{n}^{1/2}} \left( 2 \log \left( 2 + \frac{\|k\|_\infty^2 \rho^2}{C^2} \right) \right)^{1/2} + 4 \mathbb{E} \left( \|\hat{V} \hat{h}_{r_0} - g\|_{L^2(P)}^2 \right).
\]

All that remains is to combine Corollary 12 with a bound on
\[
\mathbb{E} \left( \|\hat{V} \hat{h}_{r_0} - g\|_{L^2(P)}^2 \right)
\]
from Theorem 8 and to let \( \alpha \to 0 \).

**Theorem 13** We have
\[
\mathbb{E} \left( \|\hat{V} \hat{h} - g\|_{L^2(P)}^2 \right)
\]
is at most
\[
\frac{32(\tilde{\sigma} + 4C)C}{\tilde{n}^{1/2}} \left( 2 \log \left( 2 + \frac{\|k\|_\infty^2 \rho^2}{C^2} \right) \right)^{1/2} + 10 \mathbb{E} \left( \|\hat{V} \hat{h}_{r_0} - g\|_{L^2(P)}^2 \right).
\]

If \( R = [0, \rho] \) for \( \rho = an^{1/2} \) and \( a > 0 \), then \( r = an^{(1-\beta)/(2(1+\beta))} \in R \). Therefore, by the definition of \( r_0 \), we have
\[
\mathbb{E} \left( \|\hat{V} \hat{h}_{r_0} - g\|_{L^2(P)}^2 \right) \leq \mathbb{E} \left( \|\hat{V} \hat{h} - g\|_{L^2(P)}^2 \right).
\]
By Theorem 8, this gives a bound of on
\[ E \left( \| \hat{V}r - g \|_{L^2(P)}^2 \right) \]
of order
\[ \log(n)^{1/2} / \tilde{n}^{1/2} + n^{-\beta/(1+\beta)}, \]
which is of order
\[ n^{-\beta/(1+\beta)} \]
when \( \tilde{n} \) is at least a multiple of \( n \).

If \( R = \{ s_i = bi : 0 \leq i \leq I - 1 \} \cup \{ s_I = an^{1/2} \} \) and \( \rho = an^{1/2} \) for \( a, b > 0 \) and \( I = \lceil an^{1/2} / b \rceil \), then there is a unique \( 0 \leq i \leq I \) such that \( an^{(1-\beta)/(2(1+\beta))} \leq s_i < an^{(1-\beta)/(2(1+\beta))} + b \). By the definition of \( r_0 \), we have
\[ E \left( \| \hat{V}r_0 - g \|_{L^2(P)}^2 \right) \leq E \left( \| \hat{V}s_i - g \|_{L^2(P)}^2 \right). \]

By Theorem 8, this gives a bound of on
\[ E \left( \| \hat{V}r - g \|_{L^2(P)}^2 \right) \]
of order
\[ \log(n)^{1/2} / \tilde{n}^{1/2} + n^{-\beta/(1+\beta)}, \]
which is of order
\[ n^{-\beta/(1+\beta)} \]
when \( \tilde{n} \) is at least a multiple of \( n \).

5. High-Probability Bound and Validation for \( L^\infty \) Interpolation Regression Function

We will now assume that \( g \in [L^\infty, H]_{\beta,\infty} \) with norm at most \( B \) instead of \( g \in [L^2(P), H]_{\beta,\infty} \) with norm at most \( B \). This assumption is stronger as it implies that the norm of \( g \in [L^2(P), H]_{\beta,\infty} \) is
\[ \sup_{t > 0} \left( t^{-\beta} \inf_{h \in H} (\| g - h \|_{L^2(P)} + t \| h \|_H) \right) \leq \sup_{t > 0} \left( t^{-\beta} \inf_{h \in H} (\| g - h \|_{L^\infty} + t \| h \|_H) \right) \leq B. \]

We will still assume that \( \| g \|_\infty \leq C \). Hence, the results of Section 4 continue to apply. Again, by Theorem 3.1 of Smale and Zhou (2003), we have
\[ \inf \left\{ \| h - g \|_{L^\infty}^2 : h \in rB_H \right\} \leq \frac{B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}}. \]

Hence, for all \( \alpha > 0 \) there is some \( h_{r,\alpha} \in rB_H \) such that
\[ \| h_{r,\alpha} - g \|_{L^\infty}^2 \leq \frac{B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}} + \alpha. \]
This implies
\[ \|h_{r,\alpha} - g\|_{L^2(P)}^2 \leq \frac{B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}} + \alpha \]
and so in this setting we can allow \( h_{r,\alpha} \) from Section 4 to be the same as the one defined here. This assumption means that we do not have to consider the variability of
\[ \|h_{r,\alpha} - g\|_{L^2(P_n)}^2 \]
around its mean in the proof of Lemma 14. This would add a term of order \( r^2/n^{1/2} \) to the bound in that lemma, substantially increasing its size. We also now assume that the moment generating function of \( Y_i - g(X_i) \) conditional on \( X_i \) satisfies
\[ \mathbb{E}(\exp(t(Y_i - g(X_i)))|X_i) \leq \exp(\sigma^2 t^2/2) \]
for all \( t \in \mathbb{R} \) and \( 1 \leq i \leq n \). The random variable \( Y_i - g(X_i) \) is said to be \( \sigma^2 \)-subgaussian conditional on \( X_i \). This condition implies (1) and our weaker assumption (2). It allows us to analyse a quadratic form evaluated at the \( Y_i - g(X_i) \) in the proof of Lemma 14 by using Lemma 26 in Appendix C. All probabilities mentioned in this section will be taken with respect to \( P \) on \((\Omega, \mathcal{F})\) unless otherwise stated.

5.1 High-Probability Bound

We first bound the squared \( L^2(P_n) \) norm of \( \hat{h}_r - h_{r,\alpha} \).

**Lemma 14** Let \( t > 0 \). With probability at least \( 1 - 3e^{-t} \), we have
\[ \|\hat{h}_r - h_{r,\alpha}\|_{L^2(P_n)}^2 \leq \frac{4\|k\|_{\infty}\sigma r \left( (1 + t + 2(t^2 + t)^{1/2})^{1/2} + (2t)^{1/2} \right)}{n^{1/2}} + \frac{4B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}} + 4\alpha. \]

**Proof** From the proof of Lemma 2, we have
\[ \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i))\hat{h}_r(X_i) \leq r \left( \frac{1}{n^2} \sum_{i,j=1}^{n} (Y_i - g(X_i))(Y_j - g(X_j))k(X_i, X_j) \right)^{1/2}. \]

Let \( K \) be the \( n \times n \) matrix with \( K_{i,j} = k(X_i, X_j) \) and let \( \varepsilon \) be the vector of the \( Y_i - g(X_i) \). Then
\[ \frac{1}{n^2} \sum_{i,j=1}^{n} (Y_i - g(X_i))(Y_j - g(X_j))k(X_i, X_j) = \varepsilon^T(n^{-2}K)\varepsilon. \]

Furthermore, since \( k \) is a measurable function on \((S \times S, S \otimes S)\), we have that \( n^{-2}K \) is measurable on \((\Omega, \mathcal{F})\) and non-negative-definite. Let \( a_i \) be the eigenvalues of \( n^{-2}K \) for \( 1 \leq i \leq n \). Then
\[ \max_i a_i \leq \text{tr}(n^{-2}K) \leq n^{-1}\|k\|_{\infty}^2 \]
and
\[ \text{tr}((n^{-2}K)^2) = \|a\|_2^2 \leq \|a\|_1^2 \leq n^{-2}\|k\|_{\infty}^4. \]
Therefore, by Lemma 26 in Appendix C, we have

\[ \epsilon^T (n^{-2} K) \epsilon \leq \| k \|_\infty^2 \sigma^2 n^{-1} (1 + t + 2(t^2 + t)^{1/2}) \]

and

\[ \frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) \hat{h}_r(X_i) \leq \frac{\| k \|_\infty \sigma r (1 + t + 2(t^2 + t)^{1/2})^{1/2}}{n^{1/2}} \]

with probability at least \( 1 - e^{-t} \). Furthermore,

\[ -\frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) h_{r,\alpha}(X_i) \]

is subgaussian given \( X \) with parameter

\[ \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 h_{r,\alpha}(X_i)^2 \leq \frac{\| k \|_\infty^2 \sigma^2 r^2}{n}. \]

So we have

\[ -\frac{1}{n} \sum_{i=1}^{n} (Y_i - g(X_i)) h_{r,\alpha}(X_i) \leq \frac{\| k \|_\infty \sigma r (2t)^{1/2}}{n^{1/2}} \]

with probability at least \( 1 - e^{-t} \) by Chernoff bounding. Finally,

\[ \| h_{r,\alpha} - g \|^2_{L^2(P_n)} \leq \frac{B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}} + \alpha \]

by (6). The result follows.

We will now apply \( V \) again. It follows from (6) that

\[ \| V h_{r,\alpha} - g \|^2_{L^2(P)} \leq \frac{B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}} + \alpha \]

and from Lemma 14 that for \( t > 0 \) we have

\[ \| V \hat{h}_r - V h_{r,\alpha} \|^2_{L^2(P_n)} \leq \frac{4 \| k \|_\infty \sigma r ((1 + t + 2(t^2 + t)^{1/2})^{1/2} + (2t)^{1/2})}{n^{1/2}} + \frac{4B^{2/(1-\beta)}}{r^{2\beta/(1-\beta)}} + 4\alpha \]

with probability at least \( 1 - 3e^{-t} \). We now move our probability bound on the squared \( L^2(P_n) \) norm of \( V \hat{h}_r - V h_{r,\alpha} \) to the squared \( L^2(P) \) norm of \( V \hat{h}_r - V h_{r,\alpha} \).

**Lemma 15**  Let \( t > 0 \). With probability at least \( 1 - e^{-t} \) we have

\[ \sup_{f \in r B_H} \left( \| V f - V h_{r,\alpha} \|^2_{L^2(P_n)} - \| V f - V h_{r,\alpha} \|^2_{L^2(P)} \right) \]

is at most

\[ \frac{64 \| k \|_\infty C r}{n^{1/2}} + \left( \frac{32 C^4}{n} + \frac{1024 \| k \|_\infty C^3 r}{n^{3/2}} \right)^{1/2} t^{1/2} + \frac{8 C^2 t}{3n}. \]
Proof Let

\[ Z = \sup_{f \in rB_H} \left\| Vf - Vh_{r,\alpha} \right\|_{L^2(P)}^2 - \left\| Vf - Vh_{r,\alpha} \right\|_{L^2(P)}^2 \]

\[ = \sup_{f \in rB_H} \left| \sum_{i=1}^{n} n^{-1} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \left\| Vf - Vh_{r,\alpha} \right\|_{L^2(P)}^2 \right| . \]

We have

\[ \mathbb{E} \left( n^{-1} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \left\| Vf - Vh_{r,\alpha} \right\|_{L^2(P)}^2 \right) = 0, \]

\[ n^{-1} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \left\| Vf - Vh_{r,\alpha} \right\|_{L^2(P)}^2 \leq \frac{4C^2}{n}, \]

\[ \mathbb{E} \left( n^{-2} (Vf(X_i) - Vh_{r,\alpha}(X_i))^2 - \left\| Vf - Vh_{r,\alpha} \right\|_{L^2(P)}^2 \right)^2 \leq \frac{16C^4}{n^2} \]

for all \( 1 \leq i \leq n \) and \( f \in rB_H \). Furthermore, \( rB_H \) is separable since \( H \) is, so \( Z \) is a random variable on \((\Omega, \mathcal{F})\) and we can use Talagrand’s inequality (Theorem A.9.1 of Steinwart and Christmann, 2008) to show

\[ Z > \mathbb{E}(Z) + \left( 2t \left( \frac{16C^4}{n} + \frac{8C^2 \mathbb{E}(Z)}{n} \right) \right)^{1/2} + \frac{8tC^2}{3n} \]

with probability at most \( e^{-t} \). From Lemma 6, we have

\[ \mathbb{E}(Z) \leq \frac{64\|k\|_{\infty}Cr}{n^{1/2}}. \]

The result follows. \( \square \)

We obtain a bound on the \( L^2(P) \) norm of \( \hat{V}h_r - Vh_{r,\alpha} \) as a simple consequence of Lemmas 14 and 15. All that remains is to combine this bound with (6) in the same way as the proof of Theorem 8 and to let \( \alpha \to 0 \).

Theorem 16 Let \( t > 0 \). With probability at least \( 1 - 4e^{-t} \), we have

\[ \| \hat{V}h_r - g \|_{L^2(P)}^2 \leq \frac{8\|k\|_{\infty} \sigma r ((1 + t + 2t^2 + t^{1/2})^{1/2} + (2t)^{1/2})}{n^{1/2}} + \frac{10B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} \]

\[ + \frac{128\|k\|_{\infty} Cr}{n^{1/2}} + \frac{8\left( \frac{2C^4}{n} + \frac{64\|k\|_{\infty}C^2 r}{n^{3/2}} \right)^{1/2}}{t^{1/2}} + \frac{16C^2}{3n}. \]

For \( t \geq 1 \), this bound can be simplified to

\[ \frac{8(2C^4 + 8\|k\|_{\infty}^{1/2} C^3 r^{1/2} + \|k\|_{\infty}(5\sigma + 16C)r)^{1/2}}{n^{1/2}} + \frac{10B^2/(1-\beta)}{r^{2\beta/(1-\beta)}} + \frac{16C^2}{3n}. \]

Based on this bound, the asymptotically optimal choice of \( r \) is

\[ \left( \frac{5\beta}{2(1-\beta)} \right)^{(1-\beta)/(1+\beta)} \|k\|_{\infty}^{-1/(1-\beta)/(1+\beta)} \]

\[ \times (5\sigma + 16C)^{-(1-\beta)/(1+\beta)} B^{2/(1+\beta)} t^{-(1-\beta)/(2(1+\beta))} n^{(1-\beta)/(2(1+\beta))}, \]

which, ignoring factors involving only \( \beta \), gives a bound of asymptotic order

\[ \|k\|_{\infty}^{2\beta/(1+\beta)} (5\sigma + 16C)^{2\beta/(1+\beta)} B^{2/(1+\beta)} t^{\beta/(1+\beta)} n^{-\beta/(1+\beta)}. \]
5.2 Validation Bound

Before continuing, we will redefine \( r_0 \) to be the minimiser over \( R \) of the \( 1 - 4e^{-t} \) quantile of

\[
\|\hat{V} h_r - g\|^2_{L^2(P)}.
\]

The minimum is attained as it is the minimum of a continuous function over a compact set. In the event of ties, we may take \( r_0 \) to be the infimum of all points attaining the minimum. We will also introduce some new notation. Let \( U \) and \( V \) be random variables on \((\Omega, F)\). Then

\[
\|U\|_{\psi_2} = \inf\{c > 0 : \mathbb{E} \psi_2(|U|/c) \leq 1\},
\]

\[
\|U|V\|_{\psi_2} = \inf\{c > 0 : \mathbb{E}(\psi_2(|U|/c)|V) \leq 1\},
\]

where

\[
\psi_2(x) = \exp(x^2) - 1
\]

for \( x \in \mathbb{R} \). We first bound the squared \( L^2(\tilde{\mathcal{P}}_n) \) norm of \( V \hat{h}_r - V \hat{h}_{r_0} \).

**Lemma 17** Let \( t > 0 \). With probability at least \( 1 - 2e^{-t} \), we have

\[
\|V \hat{h}_r - V \hat{h}_{r_0}\|^2_{L^2(\tilde{\mathcal{P}}_n)}
\]

is at most

\[
\frac{2^8 \cdot 3\tilde{\sigma}C}{\tilde{n}^{1/2}} \left( \left( 2 \log \left( 1 + \frac{k\|\rho\|^2}{8C^2} \right) \right)^{1/2} + \pi^{1/2} \right) t^{1/2} + \frac{2^9/2C^2t^{1/2}}{\tilde{n}^{1/2}} + 4\|V \hat{h}_{r_0} - g\|^2_{L^2(P)}.
\]

**Proof** We first reintroduce the process from the proof of Lemma 10. Let \( f_0 = V \hat{h}_{r_0} \) and

\[
W(f) = \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} W_i(f),
\]

where

\[
W_i(f) = (\tilde{Y}_i - g(\tilde{X}_i))(f(\tilde{X}_i) - f_0(\tilde{X}_i))
\]

for \( 1 \leq i \leq \tilde{n} \) and \( f \in F \). Note that the \( W_i \) are independent and centred. It is clear that \( W_i(f_1) - W_i(f_2) \) is \( \tilde{\sigma}^2 \|f_1 - f_2\|^2_{\infty,\tilde{n}} \)-subgaussian given \( \tilde{X}_i \) for \( 1 \leq i \leq \tilde{n} \) and \( f_1, f_2 \in F \), so the process \( W \) is \( \tilde{\sigma}^2 \|\cdot\|^2_{\infty,\tilde{n}} \)-subgaussian given \( \tilde{X} \). By Lemma 22, we have that \( (F, \tilde{\sigma} \|\cdot\|_{\infty,\tilde{n}^{1/2}}) \) is separable. Hence, \( W \) is separable on \( (F, \tilde{\sigma} \|\cdot\|_{\infty,\tilde{n}^{1/2}}) \) since it is continuous. The diameter of \((F, \tilde{\sigma} \|\cdot\|_{\infty,\tilde{n}^{1/2}})\) is

\[
D = \sup_{f_1, f_2 \in F} \tilde{\sigma} \|f_1 - f_2\|_{\infty,\tilde{n}^{1/2}} \leq 2\tilde{\sigma}C/\tilde{n}^{1/2}.
\]

From Lemma 28 in Appendix D, we have

\[
\int_{0}^{\infty} (\log(N(F, \tilde{\sigma} \|\cdot\|_{\infty,\tilde{n}^{1/2}}, \varepsilon)))^{1/2} d\varepsilon = \int_{0}^{\infty} (\log(N(F, \|\cdot\|_{\infty}, \tilde{n}^{1/2} \varepsilon/\tilde{\sigma})))^{1/2} d\varepsilon
\]

\[
= \frac{\tilde{\sigma}}{\tilde{n}^{1/2}} \int_{0}^{\tilde{n}^{1/2}} (\log(N(F, \|\cdot\|_{\infty}, u)))^{1/2} du
\]

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is finite. Hence, by Exercise 1 of Section 2.3 of Giné and Nickl (2016) and Lemma 28, we have
\[ \left\| \sup_{f \in F} |W(f)| \right\|_{\psi_2} \]
is at most
\[ \left\| W(f_0) \right\|_{\psi_2} + 2^{9/2} \cdot 3^{1/2} \int_0^{2\sigma C/n^{1/2}} (\log N(F, \sigma, n^{1/2} \varepsilon))^{1/2} d\varepsilon \]
\[ = 2^{9/2} \cdot 3^{1/2} \int_0^{2\sigma C/n^{1/2}} (\log N(F, \sigma, n^{1/2} \varepsilon))^{1/2} d\varepsilon \]
\[ = 2^{9/2} \cdot 3^{1/2} \sigma C \int_0^{2C} (\log N(F, \sigma, u))^{1/2} du \]
\[ \leq 2^{9/2} \cdot 3^{1/2} \sigma C \left( 2 \left( \log \left( 1 + \frac{\|k\|_\infty^2 \rho^2}{8C^2} \right) \right)^{1/2} + \pi^{1/2} \right). \]
By Lemma 24 in Appendix C, we have
\[ \sup_{f \in F} |W(f)| \]
is subgaussian with parameter
\[ \frac{2^{11} \cdot 3^{2} \sigma C^2}{n} \left( 2 \log \left( 1 + \frac{\|k\|_\infty^2 \rho^2}{8C^2} \right) \right)^{1/2} + \pi^{1/2} \]
and so is at most
\[ \frac{2^6 \cdot 3 \sigma C}{n^{1/2}} \left( 2 \log \left( 1 + \frac{\|k\|_\infty^2 \rho^2}{8C^2} \right) \right)^{1/2} + \pi^{1/2} \]
with probability at least \( 1 - e^{-t} \) by Chernoff bounding. In particular,
\[ \frac{1}{n} \sum_{i=1}^{\hat{n}} (\hat{Y}_i - g(\hat{X}_i))(V\hat{h}_r(\hat{X}_i) - V\hat{h}_{r_0}(\hat{X}_i)) \]
is at most
\[ \frac{2^6 \cdot 3 \sigma C}{n^{1/2}} \left( 2 \log \left( 1 + \frac{\|k\|_\infty^2 \rho^2}{8C^2} \right) \right)^{1/2} + \pi^{1/2} \]
with probability at least \( 1 - e^{-t} \). Furthermore,
\[ \|V\hat{h}_{r_0} - g\|_{L^2(P)}^2 = \frac{1}{n} \sum_{i=1}^{\hat{n}} (V\hat{h}_{r_0}(\hat{X}_i)^2 - g(\hat{X}_i))^2 - \|V\hat{h}_{r_0} - g\|_{L^2(P)}^2 + \|V\hat{h}_{r_0} - g\|_{L^2(P)}^2. \]
Since
\[ \left\| (V \hat{h}_{r_0}(\tilde{X}_i)^2 - g(\tilde{X}_i)) - V \hat{h}_{r_0} - g\right\|_{L^2(P)}^2 \leq 4C^2 \]
for all \( 1 \leq i \leq \tilde{n} \), we find
\[ \|V \hat{h}_{r_0} - g\|_{L^2(P)}^2 - \|V \hat{h}_{r_0} - g\|_{L^2(P)}^2 > t \]
with probability at most
\[ \exp \left( -\frac{\tilde{n}t^2}{32C^4} \right). \]
by Hoeffding’s inequality. Therefore, we have
\[ \|V \hat{h}_{r_0} - g\|_{L^2(P)}^2 - \|V \hat{h}_{r_0} - g\|_{L^2(P)}^2 \leq \frac{2^{5/2}C^2t^{1/2}}{\tilde{n}^{1/2}} \]
with probability at least \( 1 - e^{-t} \). The result follows.

We now move our probability bound on the squared \( L^2(\tilde{P}_n) \) norm of \( V \hat{h}_{r_0} - V \hat{h}_{r_0} \) to the squared \( L^2(P) \) norm of \( V \hat{h}_{r_0} - V \hat{h}_{r_0} \).

**Lemma 18** Let \( t > 0 \). With probability at least \( 1 - e^{-t} \), we have
\[ \sup_{f \in F} \left| \frac{1}{\tilde{n}} \sum_{i=1}^{\tilde{n}} (f(\tilde{X}_i) - V \hat{h}_{r_0}(\tilde{X}_i))^2 - \int (f - V \hat{h}_{r_0})^2 dP \right| \]
is at most
\[ \frac{64C^2}{\tilde{n}^{1/2}} \left( 2 \log \left( 2 + \frac{\|k\|_{\infty}^2\rho^2}{C^2} \right) \right)^{1/2} + \pi^{1/2} \]
\[ + \frac{4C^2}{\tilde{n}^{1/2}} \left( 2 + \frac{64}{\tilde{n}^{1/2}} \left( \left( 2 \log \left( 2 + \frac{\|k\|_{\infty}^2\rho^2}{C^2} \right) \right)^{1/2} + \pi^{1/2} \right) \right)^{1/2} t^{1/2} + \frac{8C^2t}{3\tilde{n}}. \]

**Proof** Let
\[ Z = \sup_{f \in F} \left| \left\| f - V \hat{h}_{r_0} \right\|_{L^2(\tilde{P}_n)}^2 - \left\| f - V \hat{h}_{r_0} \right\|_{L^2(P)}^2 \right| \]
\[ = \sup_{f \in F} \left| \sum_{i=1}^{\tilde{n}} \tilde{n}^{-1} \left( (f(\tilde{X}_i) - V \hat{h}_{r_0}(\tilde{X}_i))^2 - \left\| f - V \hat{h}_{r_0} \right\|_{L^2(P)}^2 \right) \right|. \]
We have
\[ \mathbb{E} \left( \tilde{n}^{-1} \left( (f(\tilde{X}_i) - V \hat{h}_{r_0}(\tilde{X}_i))^2 - \left\| f - V \hat{h}_{r_0} \right\|_{L^2(P)}^2 \right) \right) = 0, \]
\[ \tilde{n}^{-1} \left| (f(\tilde{X}_i) - V \hat{h}_{r_0}(\tilde{X}_i))^2 - \left\| f - V \hat{h}_{r_0} \right\|_{L^2(P)}^2 \right| \leq \frac{4C^2}{\tilde{n}}, \]
\[ \mathbb{E} \left( \tilde{n}^{-2} \left( (f(\tilde{X}_i) - V \hat{h}_{r_0}(\tilde{X}_i))^2 - \left\| f - V \hat{h}_{r_0} \right\|_{L^2(P)}^2 \right)^2 \right) \leq \frac{16C^4}{\tilde{n}^2}. \]
for all $1 \leq i \leq \tilde{n}$ and $f \in F$. Furthermore, $F$ is separable by Lemma 22, so $Z$ is a random variable on $(\Omega, \mathcal{F})$ and we can use Talagrand’s inequality (Theorem A.9.1 of Steinwart and Christmann, 2008) to show

$$Z > \mathbb{E}(Z) + \left(2t \left(\frac{16C^4}{n} + \frac{8C^2 \mathbb{E}(Z)}{\tilde{n}}\right)\right)^{1/2} + \frac{8tC^2}{3\tilde{n}}$$

with probability at most $e^{-t}$. From Lemma 11, we have

$$\mathbb{E}(Z) \leq \frac{64C^2}{\tilde{n}^{1/2}} \left(2\log \left(2 + \frac{\|k\|_{\infty}^2 \rho^2}{C^2}\right)\right)^{1/2} + \pi^{1/2}.$$ 

The result follows.

We obtain a bound on the $L^2(P)$ norm of $V\hat{h}_r - V\hat{h}_{r_0}$ as a simple consequence of Lemmas 17 and 18. All that remains to is to combine this bound with a bound on the $1 - 4e^{-t}$ quantile of

$$\|V\hat{h}_{r_0} - g\|_{L^2(P)}^2$$

from Theorem 16 and to let $\alpha \to 0$.

**Theorem 19** Let $t > 0$. With probability at least $1 - 3e^{-t}$, we have

$$\|V\hat{h}_r - V\hat{h}_{r_0}\|_{L^2(P)}^2$$

is at most

$$\frac{2^9 \cdot 3\tilde{C}C}{\tilde{n}^{1/2}} \left(2\log \left(1 + \frac{\|k\|_{\infty}^2 \rho^2}{8C^2}\right)\right)^{1/2} + \frac{1}{\tilde{n}^{1/2}} + \frac{2^{11/2}C^2 t^{1/2}}{3\tilde{n}}$$

$$+ \frac{2^7 C^2}{\tilde{n}^{1/2}} \left(2\log \left(2 + \frac{\|k\|_{\infty}^2 \rho^2}{C^2}\right)\right)^{1/2} + \frac{1}{\tilde{n}^{1/2}}$$

$$+ \frac{2^{7/2} C^2}{\tilde{n}^{1/2}} \left(1 + \frac{2^5}{\tilde{n}^{1/2}} \left(2\log \left(2 + \frac{\|k\|_{\infty}^2 \rho^2}{C^2}\right)\right)^{1/2} + \frac{1}{\tilde{n}^{1/2}}\right)^{1/2} - 10\|V\hat{h}_{r_0} - g\|_{L^2(P)}^2.$$ 

If $R = [0, \rho]$ for $\rho = an^{1/2}$ and $a > 0$, then $r = an^{(1-\beta)/(2(1+\beta))} \in R$. Therefore, by the definition of $r_0$, we have that the $1 - 4e^{-t}$ quantile of

$$\|V\hat{h}_{r_0} - g\|_{L^2(P)}^2$$

is at most that of

$$\|V\hat{h}_r - g\|_{L^2(P)}^2,$$

which is bounded in Theorem 16. For $t \geq 1$, we obtain a bound on

$$\|V\hat{h}_r - g\|_{L^2(P)}^2.$$
of order
\[ t^{1/2} \log(n)^{1/2}/\tilde{n}^{1/2} + t^{1/2}n^{-\beta/(1+\beta)} \]
with probability at least \(1 - 7e^{-t}\), which is of order
\[ t^{1/2}n^{-\beta/(1+\beta)} \]
when \(\tilde{n}\) is at least a multiple of \(n\).

If \(R = \{s_i = bi : 0 \leq i \leq I - 1\} \cup \{s_I = an^{1/2}\}\) and \(\rho = an^{1/2}\) for \(a, b > 0\) and \(I = [an^{1/2}/b]\), then there is a unique \(0 \leq i \leq I\) such that \(an^{(1-\beta)/(2(1+\beta))} \leq s_i < an^{(1-\beta)/(2(1+\beta))} + b\). By the definition of \(r_0\), we have that the \(1 - 4e^{-t}\) quantile of
\[ \|V\hat{h}_{r_0} - g\|_{L^2(P)}^2 \]
is at most that of
\[ \|V\hat{h}_{s_i} - g\|_{L^2(P)}^2, \]
which is bounded in Theorem 16. For \(t \geq 1\), we obtain a bound on
\[ \|V\hat{h}_{\hat{r}} - g\|_{L^2(P)}^2 \]
of order
\[ t^{1/2} \log(n)^{1/2}/\tilde{n}^{1/2} + t^{1/2}n^{-\beta/(1+\beta)} \]
with probability at least \(1 - 7e^{-t}\), which is of order
\[ t^{1/2}n^{-\beta/(1+\beta)} \]
when \(\tilde{n}\) is at least a multiple of \(n\).

6. Discussion

In this paper, we show how Ivanov regularisation can be used to produce smooth estimators which have a small squared \(L^2(P)\) error. We consider the case in which the regression function lies in an interpolation space between \(L^2(P)\) and an RKHS. We achieve these bounds without the standard assumption that the RKHS has a Mercer kernel, which means that our results apply even when the covariate set is not compact. In fact, our only assumption on the RKHS is that it is separable, with a bounded and measurable kernel. Under the weak assumption that the response variables have bounded variance, we prove an expectation bound on the squared \(L^2(P)\) error of our estimator of order \(n^{-\beta/2}\). If we assume that the regression function is bounded, then we can clip the estimator and show that this has an expected squared \(L^2(P)\) error of order \(n^{-\beta/(1+\beta)}\). Under the stronger assumption that the response variables have subgaussian errors and that the regression function comes from an interpolation space between \(L^\infty\) and \(H\), we show that the squared \(L^2(P)\) error is of order \(n^{-\beta/(1+\beta)}\) with high probability. For the settings in which the regression function is bounded, we can use training and validation on the data set. This allows us to select the size of the norm constraint for our Ivanov regularisation without knowing which interpolation
space contains the regression function. The response variables in the validation set must have subgaussian errors.

The order \( n^{-\beta/(\beta+1)} \) matches that of Steinwart et al. (2009) when \( p = 1 \). In particular, this shows that our expectation results are of optimal order. However, we do not make the restrictive assumption that \( k \) is a Mercer kernel of \( H \), as discussed above. This is because we use Ivanov regularisation instead of Tikhonov regularisation and control empirical processes over balls in the RKHS. By contrast, the current theory uses the embedding of \( H \) in \( L^2(P) \) from Mercer’s theorem to achieve bounds on Tikhonov-regularised estimators (Mendelson and Neeman, 2010; Steinwart et al., 2009). For this reason, it seems unlikely that Tikhonov-regularised estimators such as SVMs would perform well in the absence of a Mercer kernel, although it would be interesting to investigate whether or not this is the case.

It would be useful to extend the lower bound of order \( n^{-\beta/(\beta+1)} \) to the case in which the regression function lies in an interpolation space between \( L^\infty \) and the RKHS. This would show that our high-probability bound is also of optimal order. However, it is possible that estimation can be performed with a high-probability bound on the squared \( L^2(P) \) error of smaller order.

Appendix A. RKHSs and the Empirical Distribution

In this appendix, we will provide a decomposition of \( H \) based on \( P_n \). Recall that the evaluation functional \( L_x : H \rightarrow \mathbb{R} \) by \( L_x h = h(x) \) is bounded for all \( x \in S \). Since \( L_x \) is a bounded linear functional, it is also continuous. Therefore,

\[
L_x^{-1}\{0\} = \{ f \in H : f(x) = 0 \}
\]

is a closed subspace of \( H \). Let

\[
Z = \{ f \in H : \| f \|_{L^2(P_n)} = 0 \} = \bigcap_{i=1}^{n} L_{X_i}^{-1}\{0\},
\]

which is a closed subspace of \( H \). There exists a projection \( Q : H \rightarrow Z \) which is orthogonal with respect to the inner product on \( H \).

We now consider \( f - Qf \) for \( f \in H \). Let us define the integral operator \( \hat{T} : H \rightarrow H \) by

\[
\hat{T}f = \frac{1}{n} \sum_{i=1}^{n} f(X_i)k_{X_i}.
\]

The operator \( \hat{T} \) is clearly linear and is bounded because \( H \) is an RKHS. Furthermore, \( \text{im} \hat{T} \) is finite-dimensional, so \( \hat{T} \) is compact. Finally,

\[
\langle \hat{T}f_1, f_2 \rangle_H = \langle f_1, f_2 \rangle_{L^2(P_n)}
\]
is symmetric in $f_1, f_2 \in H$, so $\hat{T}$ is self-adjoint. Hence, the spectral theorem (Corollary 4 in Chapter 14 of Bollobás, 1999) tells us that

$$f - \sum_{i=1}^{m} \langle f, \hat{a}_i \rangle_H \hat{a}_i \in \ker \hat{T},$$

where the $\hat{a}_i \in (\ker \hat{T})^\perp$ for $1 \leq i \leq m$ are orthonormal in $H$ and eigenvectors of $\hat{T}$ with corresponding eigenvalues $\hat{\lambda}_i \neq 0$ and $m = \text{rk} \hat{T} \leq n$. In fact,

$$\hat{\lambda}_i = \langle \hat{\lambda}_i \hat{a}_i, \hat{a}_i \rangle_H = \langle \hat{T} \hat{a}_i, \hat{a}_i \rangle_H = \langle \hat{a}_i, \hat{a}_i \rangle_{L^2(P_n)} \geq 0$$

and $\hat{\lambda}_i \neq 0$, so $\hat{\lambda}_i > 0$ for $1 \leq i \leq m$. Order the $\hat{\lambda}_i$ so that they are non-increasing. More generally, the $\hat{a}_i$ have the useful property that

$$\langle \hat{a}_i, f \rangle_{L^2(P_n)} = \langle \hat{T} \hat{a}_i, f \rangle_H = \langle \hat{\lambda}_i \hat{a}_i, f \rangle_H = \hat{\lambda}_i \langle \hat{a}_i, f \rangle_H$$

for any $f \in H$. So $\hat{e}_i = \hat{\lambda}_i^{-1/2} \hat{a}_i \in (\ker \hat{T})^\perp$ for $1 \leq i \leq m$ are orthonormal in $L^2(P_n)$ and eigenvectors of $\hat{T}$ with corresponding eigenvalues $\hat{\lambda}_i$. Furthermore,

$$f - \sum_{i=1}^{m} \langle f, \hat{e}_i \rangle_{L^2(P_n)} \hat{e}_i \in \ker \hat{T}$$

for all $f \in H$.

Let us examine $\ker \hat{T}$. Recall that

$$Z = \{ f \in H : \|f\|_{L^2(P_n)} = 0 \}.$$ 

Clearly $Z \subseteq \ker \hat{T}$, but we also have that if $f \in \ker \hat{T}$, then

$$\|f\|_{L^2(P_n)}^2 = \langle f, f \rangle_{L^2(P_n)} = \langle \hat{T} f, f \rangle_H = 0.$$ 

Hence, $\ker \hat{T} \subseteq Z$ and therefore $\ker \hat{T} = Z$. This means $\hat{e}_i \in Z^\perp$ for $1 \leq i \leq m$ and

$$f - \sum_{i=1}^{m} \langle f, \hat{e}_i \rangle_{L^2(P_n)} \hat{e}_i \in Z$$

for all $f \in H$. Recall that $Q$ is the orthogonal projection from $H$ onto $Z$, so $Qf$ is the unique element in $Z$ such that $f - Qf \in Z^\perp$ for $f \in H$. Since

$$f - \sum_{i=1}^{m} \langle f, \hat{e}_i \rangle_{L^2(P_n)} \hat{e}_i \in Z$$

and

$$\sum_{i=1}^{m} \langle f, \hat{e}_i \rangle_{L^2(P_n)} \hat{e}_i \in Z^\perp,$$
we find

\[ Qf = f - \sum_{i=1}^{m} \langle f, \hat{e}_i \rangle_{L^2(P_n)} \hat{e}_i \]

or

\[ f = \sum_{i=1}^{m} \langle f, \hat{e}_i \rangle_{L^2(P_n)} \hat{e}_i + Qf. \]

Finally, since the \( \hat{e}_i \in \text{sp}\{k_{X_j} : 1 \leq j \leq n\} \), we have

\[ Z^\perp \subseteq \text{sp}\{k_{X_j} : 1 \leq j \leq n\}. \]

On the other hand, if \( f \in \text{sp}\{k_{X_j} : 1 \leq j \leq n\} \) and \( z \in Z \), then

\[ \langle f, z \rangle_H = 0. \]

So \( \text{sp}\{k_{X_j} : 1 \leq j \leq n\} \subseteq Z^\perp \).

Therefore, \( Z^\perp = \text{sp}\{k_{X_j} : 1 \leq j \leq n\} \) and

\[ H = \text{sp}\{k_{X_j} : 1 \leq j \leq n\} \oplus Z. \]

### A.1 Eigenvectors and Eigenvalues of \( \hat{T} \) and \( n^{-1}K \)

We will now investigate the \( \hat{e}_j \) and \( \hat{\lambda}_j \). We can write

\[ \frac{1}{n} \sum_{i=1}^{n} \hat{e}_j(X_i)k_{X_i} = \hat{\lambda}_j \hat{e}_j. \]

Since \( \hat{\lambda}_j > 0 \), we see that the values of \( \hat{e}_j(X_i) \) for \( 1 \leq i \leq n \) uniquely define \( \hat{e}_j \). By evaluating this expression at \( X_l \) for \( 1 \leq l \leq n \), we obtain

\[ \frac{1}{n} \sum_{i=1}^{n} \hat{e}_j(X_i)k(X_i, X_l) = \hat{\lambda}_j \hat{e}_j(X_l), \]

or

\[ \frac{1}{n} \sum_{i=1}^{n} k(X_i, X_l)\hat{e}_j(X_l) = \hat{\lambda}_j \hat{e}_j(X_l) \]

since \( k \) is symmetric. If we let \( K \) be the \( n \times n \) symmetric matrix with \( K_{l,j} = k(X_l, X_j) \), \( E \) be the \( n \times m \) matrix with \( E_{i,j} = n^{-1/2}\hat{e}_j(X_i) \) and \( \Lambda \) be the \( m \times m \) diagonal matrix with \( \Lambda_{j,j} = \hat{\lambda}_j \), then this can be written as

\[ n^{-1}KE = EL. \]

So the columns of \( E \) are eigenvectors of \( n^{-1}K \), with eigenvalues found in \( \Lambda \). Furthermore, since the \( \hat{e}_j \) are orthonormal in \( L^2(P_n) \), we find \( E^TE = I \). We also find

\[ m = \text{rk} \hat{T} \leq \text{rk}(n^{-1}K). \]

Now let us consider a diagonalisation of the symmetric matrix

\[ n^{-1}K = ADA^T, \]

where \( A \) is an orthogonal matrix and \( D \) is a diagonal matrix with non-increasing entries. The columns of \( A \) are eigenvectors of \( n^{-1}K \) with eigenvalues found in \( D \). Since the kernel
is non-negative definite, \( n^{-1}K \) is non-negative definite and \( D_{j,j} \geq 0 \) for \( 1 \leq j \leq n \). For \( 1 \leq j \leq \text{rk}(n^{-1}K) \), let

\[ f_j = D_{j,j}^{-1}n^{-1/2} \sum_{i=1}^{n} A_{i,j} k X_i. \]

Then

\[ \hat{T} f_j = D_{j,j}^{-1}n^{-1/2} \sum_{i=1}^{n} A_{i,j} \hat{T} k X_i, \]

\[ = D_{j,j}^{-1}n^{-1/2} \sum_{j=1}^{n} A_{i,j} n^{-1} \sum_{l=1}^{n} k(X_l, X_l) k X_i \]

\[ = n^{-1/2} \sum_{l=1}^{n} \left( n^{-1} \sum_{i=1}^{n} k(X_l, X_i) A_{i,j} D_{j,j}^{-1} \right) k X_i \]

\[ = n^{-1/2} \sum_{l=1}^{n} A_{l,j} k X_l \]

\[ = D_{j,j} f_j, \]

so \( f_j \) is an eigenvector of \( \hat{T} \) with eigenvalue \( D_{j,j} > 0 \). Furthermore,

\[ f_j(X_i) = D_{j,j}^{-1}n^{-1/2} \sum_{l=1}^{n} A_{l,j} k(X_l, X_i) \]

\[ = n^{-1/2} \sum_{l=1}^{n} k(X_l, X_i) A_{l,j} D_{j,j}^{-1} \]

\[ = n^{1/2} A_{i,j}. \]

Since \( A \) is orthogonal, we find that the \( f_j \) for \( 1 \leq j \leq \text{rk}(n^{-1}K) \) are orthonormal in \( L^2(P_n) \). Hence,

\[ \text{rk}(n^{-1}K) \leq \text{rk} \hat{T} = m. \]

Combining this with \( m \leq \text{rk}(n^{-1}K) \), we find that \( \text{rk} \hat{T} = m = \text{rk}(n^{-1}K) \). Furthermore, we have

\[ f = \sum_{j=1}^{m} \langle f, f_j \rangle_{L^2(P_n)} f_j + Qf \]

for all \( f \in H \) because the \( f_j \) form a basis of \( \text{sp}\{k X_i : 1 \leq i \leq n\} \) which is orthonormal in \( L^2(P_n) \).

### A.2 Correspondence Between \( \hat{T} \) and \( n^{-1}K \)

We have shown a bijection between sets of eigenvalues and eigenvectors of \( \hat{T} \) and \( n^{-1}K \). Let \( \hat{e}_j \) be eigenvectors of \( \hat{T} \) for \( 1 \leq j \leq m \) which are orthonormal in \( L^2(P_n) \) with non-increasing
eigenvalues $\hat{\lambda}_j$ and
\[ f = \sum_{j=1}^{m} \langle f, \hat{e}_j \rangle_{L^2(P_n)} \hat{e}_j + Qf \]
for all $f \in H$. Also, let
\[ n^{-1}K = ADA^T \]
where $A$ is an orthogonal matrix and $D$ is a diagonal matrix with non-increasing non-negative entries, $\text{rk}(n^{-1}K)$ of which are non-zero. Then the bijection is given by $\text{rk}(n^{-1}K) = m$, $D_{j,j} = \hat{\lambda}_j$ for $1 \leq j \leq m$ and
\[ A_{i,j} = n^{-1/2} \hat{e}_j(X_i) \]
for $1 \leq i \leq n$ and
\[ \hat{e}_j = \hat{\lambda}_j^{-1/2} \sum_{i=1}^{n} A_{i,j} k_{X_i}. \]
By Lemma 4.25 of Steinwart and Christmann (2008), the function $\Phi : S \to H$ by $\Phi(x) = k_x$ is a $(H, B(H))$-valued measurable function on $(S, S)$. Hence, $k_{X_i}$ is a $(H, B(H))$-valued random variable on $(\Omega, \mathcal{F})$ for $1 \leq i \leq n$. This means that the $\hat{e}_j$ being $(H, B(H))$-valued random variables and the $\hat{\lambda}_j$ and $m$ being random variables on $(\Omega, \mathcal{F})$ is equivalent to $Q$, $D$ and $\text{rk}(n^{-1}K)$ being measurable on $(\Omega, \mathcal{F})$. Quintana and Rodríguez (2014) show that a strictly-positive-definite matrix which is measurable on $(\Omega, \mathcal{F})$ can be diagonalised by an orthogonal matrix and a diagonal matrix which are both measurable on $(\Omega, \mathcal{F})$. The result holds for non-negative-definite matrices by adding the identity matrix before diagonalisation. Hence, the non-negative definite matrix $n^{-1}K$, which is measurable on $(\Omega, \mathcal{F})$, can be diagonalised in such a way that $A$, $D$ and $\text{rk}(n^{-1}K)$ are measurable on $(\Omega, \mathcal{F})$. It follows that for this diagonalisation of $n^{-1}K$, the $\hat{e}_j$ are $(H, B(H))$-valued random variables and the $\hat{\lambda}_j$ and $m$ are random variables on $(\Omega, \mathcal{F})$. We choose the above quantities so that they satisfy these measurability conditions. We collect the key results from this appendix in one lemma.

**Lemma 20** There exist eigenvectors $\hat{e}_j$ of $\hat{T}$ for $1 \leq j \leq m$ which are orthonormal in $L^2(P_n)$ and $(H, B(H))$-valued random variables on $(\Omega, \mathcal{F})$ with non-increasing eigenvalues $\hat{\lambda}_j$ which are random variables on $(\Omega, \mathcal{F})$, as is $m$, such that
\[ f = \sum_{j=1}^{m} \langle f, \hat{e}_j \rangle_{L^2(P_n)} \hat{e}_j + Qf \]
for all $f \in H$. Here, $Q$ is the orthogonal projection of $H$ onto
\[ Z = \{ f \in H : \|f\|_{L^2(P_n)} = 0 \} \]
with $Z^\perp = \text{sp}\{k_{X_i} : 1 \leq i \leq n\}$ and $\hat{T} : H \to H$ is the integral operator
\[ \hat{T}f = \frac{1}{n} \sum_{i=1}^{n} f(X_i) k_{X_i}. \]
Let
\[ n^{-1}K = ADA^T \]
where \( A \) is an orthogonal matrix and \( D \) is a diagonal matrix with non-increasing entries which are both measurable on \((Ω, F)\). Such a diagonalisation exists. Then \( m = \text{rk}(n^{-1}K) \) and \( \hat{\lambda}_j = D_{j,j} \) for \( 1 \leq j \leq m \) and we can choose
\[ \hat{e}_j = \hat{\lambda}_j^{-1/2} \sum_{i=1}^{n} A_{i,j} k_{X_i}. \]

### Appendix B. Estimator Calculation and Measurability

We will now look at how to calculate \( \hat{h}_r \). Lemma 20 shows that any \( f \in H \) can be represented as
\[ f = \sum_{j=1}^{m} \langle f, \hat{e}_j \rangle_{L^2(P_n)} \hat{e}_j + Qf, \]
where the projection \( Q : H \rightarrow Z \) is orthogonal with respect to the inner product on \( H \) and
\[ Z = \{ f \in H : \| f \|_{L^2(P_n)} = 0 \}, \]
with \( Z^\perp = \text{sp}\{ k_{X_i} : 1 \leq i \leq n \} \). The \( \hat{e}_j \) in \( H \) for \( 1 \leq j \leq m \) are orthonormal in \( L^2(P_n) \) and orthogonal in \( H \) with \( \| \hat{e}_j \|_H^2 = \hat{\lambda}_j^{-1} \) for \( \hat{\lambda}_j > 0 \) non-increasing. They are also eigenvectors of the integral operator \( \hat{T} : H \rightarrow H \) by
\[ \hat{T}f = \frac{1}{n} \sum_{i=1}^{n} f(X_i)k_{X_i}, \]
with corresponding eigenvalues \( \hat{\lambda}_j \). Let \( K \) be the \( n \times n \) matrix with \( K_{i,j} = k(X_i, X_j) \), \( E \) be the \( n \times m \) matrix with \( E_{i,j} = n^{-1/2} \hat{e}_j(X_i) \) and \( \Lambda \) be the \( m \times m \) diagonal matrix with \( \Lambda_{j,j} = \hat{\lambda}_j \). The columns of \( E \) are eigenvectors of \( n^{-1}K \) with eigenvalues found in \( \Lambda \) and \( E^T E = I \).

We have
\[ \hat{h}_r = \arg \min_{f \in rB_H} \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \langle f, \hat{e}_j \rangle_{L^2(P_n)} \hat{e}_j(X_i) - Y_i \right)^2. \]
Let \( a_j = \langle f, \hat{e}_j \rangle_{L^2(P_n)} \) for \( 1 \leq j \leq m \). Then
\[ \| f \|_H^2 = a^T \Lambda^{-1} a + \| Qf \|_H^2 \]
and
\[ \hat{h}_r = \arg \min_{f \in rB_H} (Ea - n^{-1/2}Y)^T (Ea - n^{-1/2}Y), \]
so we will let \( Qf = 0 \). We can then write the norm constraint as
\[ a^T \Lambda^{-1} a + s = r^2 \]
where \( s \geq 0 \) is a slack variable. The Lagrangian of this problem can be written as

\[
L(a, s; \mu) = (Ea - n^{-1/2} Y)^T (Ea - n^{-1/2} Y) + \mu (a^T \Lambda^{-1} a + s - r^2)
\]

\[
= a^T (I + \mu \Lambda^{-1}) a - 2n^{-1/2} Y^T Ea + \mu s + n^{-1} Y^T Y - \mu r^2
\]

where \( \mu \) is the Lagrangian multiplier for the norm constraint. We seek to minimise the Lagrangian for a fixed value of \( \mu \). Note that we require \( \mu \geq 0 \) for the Lagrangian to have a finite minimum due to the term in \( s \). We have

\[
\frac{\partial L}{\partial a} = 2(I + \mu \Lambda^{-1}) a - 2n^{-1/2} E^T Y,
\]

so the minimising value of \( a \) is given by

\[
a = n^{-1/2} (I + \mu \Lambda^{-1})^{-1} E^T Y.
\]

We now search for a value of \( \mu \) such that \( a \) and \( s \) satisfy the norm constraint. We will call this value \( \mu(r) \). There are two cases. If

\[
r^2 \leq n^{-1} Y^T E \Lambda^{-1} E^T Y,
\]

then

\[
a = n^{-1/2} (I + \mu(r) \Lambda^{-1})^{-1} E^T Y, \quad s = 0
\]

minimises \( L \) for \( \mu = \mu(r) \geq 0 \) and satisfies the norm constraint, where \( \mu(r) \) satisfies

\[
n^{-1} Y^T E(I + \mu(r) \Lambda^{-1})^{-1} \Lambda^{-1}(I + \mu(r) \Lambda^{-1})^{-1} E^T Y = r^2.
\]

Otherwise,

\[
a = n^{-1/2} E^T Y, \quad s = r^2 - n^{-1} Y^T E \Lambda^{-1} E^T Y
\]

minimises \( L \) for \( \mu = \mu(r) = 0 \) and satisfies the norm constraint. Hence, the Lagrangian sufficiency theorem shows

\[
\hat{h}_r = n^{-1/2} \sum_{j=1}^m \frac{\hat{\lambda}_j}{\hat{\lambda}_j + \mu(r)} (E^T Y)_{j\cdot} \hat{e}_j
\]

for \( r > 0 \). We also have \( \hat{h}_0 = 0 \).

Since \( \mu(r) \) is strictly decreasing for

\[
r^2 \leq n^{-1} Y^T E \Lambda^{-1} E^T Y,
\]

we find

\[
\{ \mu(r) \leq \mu \} = \{ n^{-1} Y^T E(I + \mu \Lambda^{-1})^{-1} \Lambda^{-1}(I + \mu \Lambda^{-1})^{-1} E^T Y \leq r^2 \}
\]

for \( \mu \in [0, \infty) \). So \( \mu(r) \) is measurable on \((\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}([0, \infty)))\), where \( r \) varies in \([0, \infty)\). Hence, \( \hat{h}_r \) is a \((H, \mathcal{B}(H))\)-valued measurable function on \((\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}([0, \infty)))\), where \( r \) varies in \([0, \infty)\).
To calculate $\mu(r)$, we first use that $\mu(r) = 0$ if
\[
r \geq (n^{-1}Y^TE\Lambda^{-1}E^TY)^{1/2}.
\]
Otherwise, $\mu(r) > 0$ and
\[
r^2 = n^{-1}Y^TE(I + \mu(r)\Lambda^{-1})^{-1}\Lambda^{-1}(I + \mu(r)\Lambda^{-1})^{-1}E^TY
\leq n^{-1}Y^TE\Lambda E^TY \mu(r)^{-2}.
\]
Hence,
\[
\mu(r) \leq (n^{-1}Y^TE\Lambda E^TY)^{1/2}r^{-1}.
\]
Since $\mu(r)$ is strictly decreasing when $\mu(r) > 0$, we can use interval bisection to calculate $\mu(r)$ for a fixed $r > 0$ to within a given tolerance.

**B.1 Large $n$**

When $n$ is large, it may be quicker to avoid diagonalising $n^{-1}K$ and to instead solve a system of $n$ linear equations at each iteration of interval bisection. As noted above, we restrict our search for the estimator

\[
\hat{h}_r = \arg\min_{f \in rB_H} \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - Y_i)^2
\]
to $Z^\perp = \text{sp}\{kX_i : 1 \leq i \leq n\}$. Let
\[
f = \sum_{j=1}^{n} b_j k_j
\]
for $b \in \mathbb{R}^n$. The Lagrangian can be written as
\[
L(b, s; \mu) = n^{-1}(Kb - Y)^T(Kb - Y) + \mu(b^TKb + s - r^2)
= b^T(n^{-1}K^2 + \mu K)b - 2n^{-1}Y^TKb + \mu s + n^{-1}Y^TY - \mu r^2.
\]
We have
\[
\frac{\partial L}{\partial b} = 2(n^{-1}K^2 + \mu K)b - 2n^{-1}KY.
\]
This being 0 is equivalent to
\[
b = (K + n\mu I)^{-1}Y + w
\]
for some $w \in \text{ker} K$. However, the same $f$ is produced for all $w \in \text{ker} K$ because the squared $H$ norm of
\[
\sum_{j=1}^{n} w_j k_j,
\]
is $w^TKw = 0$, so we are free to take $w = 0$. The solution to the problem is given by
\[
b = (K + n\mu(r)I)^{-1}Y
\]
when $\mu(r) > 0$ or $\mu(r) = 0$ and $K$ is invertible, where $\mu(r)$ is the same as above.

To calculate $\mu(r)$, note that if $\mu(r) > 0$ then

$$r^2 = Y^T(K + n\mu(r)I)^{-1}K(K + n\mu(r)I)^{-1}Y$$

$$\leq n^{-2}Y^TKY\mu(r)^{-2}.$$  

Hence,

$$\mu(r) \leq n^{-1}(Y^TKY)^{1/2}r^{-1}.$$  

Since $\mu(r)$ is strictly decreasing when $\mu(r) > 0$, we can use interval bisection to calculate $\mu(r)$ for a fixed $r > 0$ to within a given tolerance. When $K$ is invertible, we can also use that $\mu(r) = 0$ if

$$r \geq (Y^T K^{-1} Y)^{1/2}.$$  

We summarise the calculation of $\hat{h}_r$ in the following lemma.

**Lemma 21** We have

$$\hat{h}_r = n^{-1/2} \sum_{j=1}^m \frac{\hat{\lambda}_j}{\hat{\lambda}_j + \mu(r)}(E^TY)_{\hat{e}_j}$$

for $r > 0$ and $\hat{h}_0 = 0$. If

$$r^2 \leq n^{-1}Y^T E \Lambda^{-1} E^T Y,$$

then $\mu(r) \geq 0$ satisfies

$$n^{-1}Y^T E(I + \mu(r) \Lambda^{-1})^{-1} \Lambda^{-1} (I + \mu(r) \Lambda^{-1})^{-1} E^T Y = r^2.$$  

Otherwise, $\mu(r) = 0$. We have that $\mu(r)$ is strictly decreasing when $\mu(r) > 0$, and $\mu(r)$ is measurable and $\hat{h}_r$ is a $(H, B(H))$-valued measurable function on $(\Omega \times [0, \infty), \mathcal{F} \otimes B([0, \infty)))$, where $r$ varies in $[0, \infty)$.

When $\mu(r) > 0$ or $\mu(r) = 0$ and $K$ is invertible, we have

$$\hat{h}_r = \sum_{j=1}^n ((K + n\mu(r)I)^{-1}Y)_{\hat{e}_j}k_{X_j}.$$  

If $\mu(r) > 0$, then

$$Y^T(K + n\mu(r)I)^{-1}K(K + n\mu(r)I)^{-1}Y = r^2.$$  

When $K$ is invertible, $\mu(r) = 0$ if

$$r^2 \geq Y^T K^{-1} Y.$$
B.2 Continuity

We will prove a continuity result about our estimator.

**Lemma 22** Let $r, s \in [0, \infty)$. We have

$$
\|\hat{h}_r - \hat{h}_s\|_H^2 \leq |r^2 - s^2|.
$$

**Proof** Let $s > r$. If $r > 0$ then

$$
\langle \hat{h}_r, \hat{h}_s \rangle_H = n^{-1} \sum_{j=1}^{m} \frac{\hat{\lambda}_j}{(\hat{\lambda}_j + \mu(r))(\hat{\lambda}_j + \mu(s))}(E^T Y)^2_j
\geq n^{-1} \sum_{j=1}^{m} \frac{\hat{\lambda}_j}{(\hat{\lambda}_j + \mu(r))^2}(E^T Y)^2_j
= \|\hat{h}_r\|_H^2.
$$

Furthermore, if $\mu(r) > 0$ then $\|\hat{h}_r\|_H^2 = r^2$ and

$$
\|\hat{h}_r - \hat{h}_s\|_H^2 = \|\hat{h}_r\|_H^2 + \|\hat{h}_s\|_H^2 - 2\langle \hat{h}_r, \hat{h}_s \rangle_H
\leq \|\hat{h}_s\|_H^2 - \|\hat{h}_r\|_H^2
= \|\hat{h}_s\|_H^2 - r^2
\leq s^2 - r^2.
$$

Otherwise, $\mu(r) = 0$ and so $\mu(s) = 0$, which means $\hat{h}_r = \hat{h}_s$. If $r = 0$ then $\hat{h}_r = 0$ and

$$
\|\hat{h}_r - \hat{h}_s\|_H^2 = \|\hat{h}_s\|_H^2 \leq s^2.
$$

Hence, whenever $r < s$ we have

$$
\|\hat{h}_r - \hat{h}_s\|_H^2 \leq s^2 - r^2.
$$

The result follows.

B.3 Validation

We also have the estimator $\hat{r}$ of $r_0$ when performing validation.

**Lemma 23** We have that $\hat{r}$ is a random variable on $(\Omega, \mathcal{F})$.

**Proof** Let

$$
W(s) = \frac{1}{n} \sum_{i=1}^{\tilde{n}} (V\hat{h}_s(\tilde{X}_i) - \tilde{Y}_i)^2
$$
for $s \in R$. Note that $W(s)$ is a random variable on $(\Omega, \mathcal{F})$ and continuous in $s$ by Lemma 22. Since $R \subseteq \mathbb{R}$, it is separable. Let $R_0$ be a countable dense subset of $R$. Then

$$\inf_{s \in R} W(s) = \inf_{s \in R_0} W(s)$$

is a random variable on $(\Omega, \mathcal{F})$ as the right-hand side is the infimum of countably many random variables on $(\Omega, \mathcal{F})$. Let $r \in [0, \rho]$. By the definition of $\hat{r}$, we have

$$\{\hat{r} \leq r\} = \bigcup_{s \in R \cap [0, r]} \{W(s) \leq \inf_{t \in R} W(t)\}.$$

Since $R \cap [0, r] \subseteq \mathbb{R}$, it is separable. Let $A_r$ be a countable dense subset of $R \cap [0, r]$. By the sequential compactness of $R \cap [0, r]$ and continuity of $W(s)$, we have

$$\{\hat{r} \leq r\} = \bigcap_{a \in \mathbb{N}} \bigcup_{s \in A_r} \{W(s) \leq \inf_{t \in R} W(t) + a^{-1}\}.$$

This set is an element of $\mathcal{F}$.

Appendix C. Subgaussian Random Variables

The following result is Lemma 2.3.1 from Giné and Nickl (2016).

**Lemma 24** Let $U$ and $V$ be random variables. If $\|U\|_{\psi_2} < \infty$ and $E U = 0$, then $U$ is $6\|U\|_{\psi_2}$-subgaussian. If $U$ is $\sigma^2$-subgaussian, then $\|U\|_{\psi_2}^2 \leq 6\sigma^2$. If $\|U|V\|_{\psi_2}^2 < \infty$ and $E(U|V) = 0$, then $U$ is $6\|U|V\|_{\psi_2}^2$-subgaussian given $V$. If $U$ is $\sigma^2$-subgaussian given $V$, then $\|U|V\|_{\psi_2}^2 \leq 6\sigma^2$.

The following result relates a quadratic form of subgaussians to that of centred normal random variables.

**Lemma 25** Let $\varepsilon_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are independent conditional on some sub-$\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$ and let

$$E(\exp(t\varepsilon_i)|\mathcal{G}) \leq \exp(\sigma^2t^2/2)$$

for all $t \in \mathbb{R}$. Also, let $\delta_i$ for $1 \leq i \leq n$ be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ which are independent of each other and $\mathcal{G}$ with $\delta_i \sim N(0, \sigma^2)$. If $A$ is an $n \times n$ non-negative-definite matrix which is measurable on $(\Omega, \mathcal{G})$, then

$$E(\exp(t\varepsilon^T A\varepsilon)|\mathcal{G}) \leq E(\exp(t\delta^T A\delta)|\mathcal{G})$$

for all $t \geq 0$.

**Proof** This proof method uses techniques from the proof of Lemma 9 of Abbasi-Yadkori, Pál, and Szepesvári (2011). We have

$$E(\exp(t\varepsilon_i/\sigma)|\mathcal{G}) \leq \exp(t_i^2/2)$$
for all \(1 \leq i \leq n\) and \(t_i \in \mathbb{R}\). Furthermore, the \(\varepsilon_i\) are independent conditional on \(\mathcal{G}\), so
\[
\mathbb{E}(\exp(t^T\varepsilon/\sigma) | \mathcal{G}) \leq \exp(\|t\|^2/2).
\]
Quintana and Rodríguez (2014) show that a strictly-positive-definite matrix which is measurable on \((\Omega, \mathcal{G})\) can be diagonalised by an orthogonal matrix and a diagonal matrix which are both measurable on \((\Omega, \mathcal{G})\). The result holds for non-negative-definite matrices by adding the identity matrix before diagonalisation, so let \(A\) have the square root \(A^{1/2}\) which is measurable on \((\Omega, \mathcal{G})\). We can then replace \(t\) with \(sA^{1/2}u\) for \(s \in \mathbb{R}\) and \(u \in \mathbb{R}^n\) to get
\[
\mathbb{E}(\exp(su^T A^{1/2}\varepsilon/\sigma) | \mathcal{G}) \leq \exp(s^2\|A^{1/2}u\|^2/2).
\]
Integrating over \(u\) with respect to the distribution of \(\delta\) gives
\[
\mathbb{E}(\exp(s^2 \varepsilon^T A\varepsilon/2) | \mathcal{G}) \leq \mathbb{E}(\exp(s^2 \delta^T A\delta/2) | \mathcal{G}).
\]
The result follows.

Having established this relationship, we can now obtain a probability bound on a quadratic form of subgaussians by using Chernoff bounding. The following is a conditional subgaussian version of the Hanson–Wright inequality. It is similar to Theorem 2.1 of Hsu, Kakade, and Zhang (2012). All probabilities mentioned will be taken with respect to \(\mathbb{P}\) on \((\Omega, \mathcal{F})\) unless otherwise stated.

**Lemma 26** Let \(\varepsilon_i\) for \(1 \leq i \leq n\) be random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) which are independent conditional on some sub-\(\sigma\)-algebra \(\mathcal{G} \subseteq \mathcal{F}\) and let
\[
\mathbb{E}(\exp(t\varepsilon_i) | \mathcal{G}) \leq \exp(\sigma^2 t^2/2)
\]
for all \(t \in \mathbb{R}\). If \(A\) is an \(n \times n\) non-negative-definite matrix which is measurable on \((\Omega, \mathcal{G})\) and \(t \geq 0\), then
\[
\varepsilon^T A\varepsilon \leq \sigma^2 \text{tr}(A) + 2\sigma^2\|A\|t + 2\sigma^2(\|A\|^2t^2 + \text{tr}(A^2)t)^{1/2}
\]
with probability at least \(1 - e^{-t}\) conditional on \(\mathcal{G}\). Here, \(\|A\|\) is the operator norm of \(A\), which is measurable on \((\Omega, \mathcal{G})\).

**Proof** This proof method follows that of Theorem 3.1.9 of Giné and Nickl (2016). Quintana and Rodríguez (2014) show that a strictly-positive-definite matrix which is measurable on \((\Omega, \mathcal{G})\) can be diagonalised by an orthogonal matrix and a diagonal matrix which are both measurable on \((\Omega, \mathcal{G})\). The result holds for non-negative-definite matrices by adding the identity matrix before diagonalisation, so let
\[
A = QDQ^T,
\]
where \(Q\) is an \(n \times n\) orthogonal matrix and \(D\) is an \(n \times n\) diagonal matrix with non-negative entries which are both measurable on \((\Omega, \mathcal{G})\). Also, let \(\delta_i\) for \(1 \leq i \leq n\) be random variables on \((\Omega, \mathcal{F}, \mathbb{P})\) which are independent of each other and \(\mathcal{G}\), with \(\delta_i \sim \mathcal{N}(0, \sigma^2)\). Then, by Lemma 25 and the fact that \(Q^T\delta\) has the same distribution as \(\delta\), we have
\[
\mathbb{E}(\exp(t\varepsilon^T A\varepsilon) | \mathcal{G}) \leq \mathbb{E}(\exp(t\delta^T A\delta) | \mathcal{G}) = \mathbb{E}(\exp(t\delta^T D\delta) | \mathcal{G})
\]

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for all $t \geq 0$. Furthermore,
\[
\mathbb{E}(\exp(t\delta^2_i/\sigma^2)) = \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{1/2}} \exp(tx^2 - x^2/2)dx = \frac{1}{(1 - 2t)^{1/2}},
\]
so
\[
\mathbb{E}(\exp(t(\delta^2_i/\sigma^2 - 1))) = \exp(-(\log(1 - 2t) + 2t)/2).
\]
We have
\[
-2(\log(1 - 2t) + 2t) \leq \sum_{i=2}^{\infty} (2t)^i(2/i) \leq 4t^2/(1 - 2t)
\]
for $0 \leq t \leq 1/2$. Therefore,
\[
\mathbb{E}(\exp(tD_{i,i}(\delta^2_i - \sigma^2))|\mathcal{G}) \leq \exp\left(\frac{\sigma^4D^2_{i,i}t^2}{1 - 2\sigma^2D_{i,i}t}\right)
\]
for $0 \leq t \leq 1/(2\sigma^2D_{i,i})$ and $1 \leq i \leq n$, and
\[
\mathbb{E}(\exp(t(\delta^T D\delta - \sigma^2\operatorname{tr}(D)))|\mathcal{G}) \leq \exp\left(\frac{\sigma^4\operatorname{tr}(D^2)t^2}{1 - 2\sigma^2(\max_i D_{i,i})t}\right)
\]
for $0 \leq t \leq 1/(2\sigma^2(\max_i D_{i,i}))$. By Chernoff bounding, we have
\[
\varepsilon^T A\varepsilon - \sigma^2 \operatorname{tr}(A) > s
\]
for $s \geq 0$ with probability at most
\[
\exp\left(\frac{\sigma^4\operatorname{tr}(A^2)t^2}{1 - 2\sigma^2\|A\|t} - ts\right)
\]
conditional on $\mathcal{G}$ for $0 \leq t \leq 1/(2\sigma^2\|A\|)$. Letting
\[
t = \frac{s}{2\sigma^4\operatorname{tr}(A^2) + 2\sigma^2\|A\|s}
\]
gives the bound
\[
\exp\left(-\frac{s^2}{4\sigma^4\operatorname{tr}(A^2) + 4\sigma^2\|A\|s}\right).
\]
Rearranging gives the result.

Appendix D. Covering Numbers

The following result gives a bound on the covering numbers of $F$.

Lemma 27 From Lemma 22, we have
\[
N(F, \|\cdot\|_\infty, \varepsilon) \leq 1 + \frac{\|k\|^2_\infty \rho^2}{2\varepsilon^2}.
\]
Proof Let \( a \in \mathbb{N} \) and \( r_i \in R \) and \( f_i = V \hat{h}_{r_i} \in F \) for \( 1 \leq i \leq a \). Also, let \( f = V \hat{h}_r \in F \) for \( r \in R \). Since \( V \) is a contraction,
\[
\| \hat{h}_r - \hat{h}_{r_i} \|_\infty \leq \varepsilon \implies \| f - f_i \|_\infty \leq \varepsilon.
\]
Lemma 22 shows
\[
|r^2 - r_i^2| \leq \varepsilon^2 / \| k \|_\infty^2 \implies \| \hat{h}_r - \hat{h}_{r_i} \|_\infty \leq \varepsilon.
\]
Hence, if we let
\[
r_i^2 = \varepsilon^2 (2i - 1) / \| k \|_\infty^2
\]
and
\[
\rho^2 - \varepsilon^2 (2a - 1) / \| k \|_\infty^2 \leq \varepsilon^2 / \| k \|_\infty^2,
\]
then we find \( N(F, \| \cdot \|_\infty, \varepsilon) \leq a \). Rearranging the above shows that we can choose
\[
a = \left\lceil \frac{\| k \|_\infty^2 \rho^2}{2 \varepsilon^2} \right\rceil
\]
and the result follows.

We will also calculate integrals of these covering numbers.

Lemma 28 Let \( a \geq 1 \). From Lemma 27, we have
\[
\int_0^L \left( \frac{\log(aN(F, \| \cdot \|_\infty, \varepsilon))}{\varepsilon} \right)^{1/2} d\varepsilon \leq \left( \log \left( \frac{1 + \| k \|_\infty^2 \rho^2}{2L^2} \right) a \right)^{1/2} \frac{L}{2} + \left( \frac{\pi}{2} \right)^{1/2} L
\]
for \( L \in (0, \infty) \). When \( a = 1 \), we have
\[
\int_0^L \left( \frac{\log(N(F, \| \cdot \|_\infty, \varepsilon))}{\varepsilon} \right)^{1/2} d\varepsilon \leq 2 \left( \log \left( 1 + \frac{\| k \|_\infty^2 \rho^2}{8C^2} \right) \right)^{1/2} C + (2\pi)^{1/2} C
\]
for \( L \in (0, \infty] \).

Proof Let \( L \in (0, \infty) \). Then
\[
\int_0^L \left( \frac{\log(aN(F, \| \cdot \|_\infty, \varepsilon))}{\varepsilon} \right)^{1/2} d\varepsilon \leq \int_0^L \left( \log \left( a \left( 1 + \frac{\| k \|_\infty^2 \rho^2}{2L^2u^2} \right) \right) \right)^{1/2} d\varepsilon
\]
by Lemma 27. Changing variables to \( u = \varepsilon / L \) gives
\[
L \int_0^1 \left( \log \left( a \left( 1 + \frac{\| k \|_\infty^2 \rho^2}{2L^2u^2} \right) \right) \right)^{1/2} du
\leq L \int_0^1 \left( \log \left( a \left( 1 + \frac{\| k \|_\infty^2 \rho^2}{2L^2} \right) \frac{1}{u^2} \right) \right)^{1/2} du
= L \int_0^1 \left( \log \left( a \left( 1 + \frac{\| k \|_\infty^2 \rho^2}{2L^2} \right) \right) + \log \left( \frac{1}{u^2} \right) \right)^{1/2} du.
\]
For \(a, b \geq 0\) we have \((a + b)^{1/2} \leq a^{1/2} + b^{1/2}\), so the above is at most

\[
L \int_0^1 \left( \log \left( a \left( 1 + \frac{\|k\|^2\rho^2}{2L^2} \right) \right) \right)^{1/2} du + L \int_0^1 \left( \log \left( \frac{1}{u^2} \right) \right)^{1/2} du
\]

\[
= L \left( \log \left( a \left( 1 + \frac{\|k\|^2\rho^2}{2L^2} \right) \right) \right)^{1/2} + L \int_0^1 \left( \log \left( \frac{1}{u^2} \right) \right)^{1/2} du.
\]

Changing variables to

\[
s = \left( \log \left( \frac{1}{u^2} \right) \right)^{1/2}
\]

shows

\[
\int_0^1 \left( \log \left( \frac{1}{u^2} \right) \right)^{1/2} du = \int_0^\infty s^2 \exp(-s^2/2) ds
\]

\[
= \frac{1}{2} \int_{-\infty}^\infty s^2 \exp(-s^2/2) ds
\]

\[
= \frac{(2\pi)^{1/2}}{2}
\]

\[
= \left( \frac{\pi}{2} \right)^{1/2},
\]

since the last integral is a multiple of the variance of a \(N(0, 1)\) random variable. The first result follows. Note that \(N(F, \|\cdot\|_\infty, \varepsilon) = 1\) whenever \(\varepsilon \geq 2C\), as the ball of radius \(2C\) about any point in \(F\) is the whole of \(F\). Hence, when \(a = 1\), we have

\[
\int_0^L \left( \log(N(F, \|\cdot\|_\infty, \varepsilon)) \right)^{1/2} d\varepsilon \leq \int_0^\infty \left( \log(N(F, \|\cdot\|_\infty, \varepsilon)) \right)^{1/2} d\varepsilon
\]

\[
= \int_0^{2C} \left( \log(N(F, \|\cdot\|_\infty, \varepsilon)) \right)^{1/2} d\varepsilon
\]

\[
\leq 2 \left( \log \left( 1 + \frac{\|k\|^2\rho^2}{8C^2} \right) \right)^{1/2} + (2\pi)^{1/2} C
\]

for \(L \in (0, \infty]\).

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