On flavor symmetries of phenomenologically viable string compactifications

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Abstract. Heterotic orbifolds can explain the origin of flavor symmetries and the flavor representations of matter fields in particle physics as a result of the geometric properties of the associated string states in the compact space. After a review of the method to obtain flavor symmetries in these models, we determine the most frequent non-Abelian flavor symmetries appearing in promising Abelian heterotic orbifolds. Interestingly, these symmetries correspond only to $D_4$, $\Delta(54)$ and products of these symmetries and Abelian factors. A large set of promising models exhibits purely Abelian flavor symmetries. We finally explore the phenomenological potential of a sample model endowed with $\Delta(54)$ assuming certain ad hoc flavon expectation values.

1. Introduction

One of the goals of flavor phenomenology is to discover the underlying structure in particle physics that may solve some questions left unanswered in the SM, such as the origin of the family replication, the patterns of quark and lepton mixing matrices, the origin of CP violation and the absence of flavor-changing neutral currents (FCNC). The field theoretic approach consists in first freely choosing a non–Abelian discrete symmetry within $SU(3)_F^5$, the largest global symmetry of the SM in the absence of Yukawa couplings, and then introducing a number of ad hoc matter fields, some of them with unjustified expectation values (VEVs), to fulfill different basic phenomenological constraints, such as the quark and lepton masses and mixings. Once these restrictions are met, this bottom-up approach delivers a number of consequences, which frequently include interesting new physics. There are plenty of useful symmetries which have been thoroughly studied (see e.g. [1, 2] for a review), and it is hard to learn which of them corresponds to the actual description of our Universe.

Looking for the origin of such symmetries might at least reduce the number of possibilities. In particular, given the constraining environment of string theory, one may wonder what kind of flavor symmetries can emerge in string compactifications that reproduce many properties of the standard model (SM) or its minimal supersymmetric extension (MSSM). The first general studies of this question were [3, 4], in the context of orbifold compactifications of the heterotic string perhaps due to their geometric simplicity. Around those studies, there has been some progress in understanding the qualities of flavor symmetries arising in some phenomenologically viable heterotic orbifolds [5, 6, 7], their enhancements [8] and their generalizations in models endowed with magnetic fluxes [9]. Recently, there has also been progress in the study of flavor symmetries from promising orientifold D-brane models [10, 11].
In this paper, we focus on the $E_8 \times E_8$ heterotic string compactified on symmetric, toroidal $Z_N$ and $Z_N \times Z_M$ orbifolds, which are the simplest compactifications since, among other features, the resulting space corresponds to a special point in the Calabi-Yau moduli space, where the underlying CFT of string theory is valid. Thus, obtaining the structure of the couplings in the effective theory can be done by computing correlation functions of asymptotic string states and vertex operators [12, 13, 14, 15]. These computations lead to a set of selection rules that determine which couplings among the effective fields are non-vanishing [16, 17, 18, 19, 20, 21, 22], and that can be expressed in terms of discrete symmetries. As we shall review in section 2, based on [3, 4], these symmetries are the core of the flavor symmetries from heterotic orbifolds.

One question we address in this paper is what symmetries actually emerge from orbifold compactifications that fulfill a minimal set of necessary conditions that guarantee their phenomenological viability. Here, to be considered phenomenologically promising, an orbifold model must yield the SM gauge group, such that the hypercharge generator be non-anomalous and (with normalization) compatible with grand unification, three generations of quarks and leptons, at least a couple of Higgs superfields, $H_u$ and $H_d$, and only vectorlike exotics w.r.t. the SM gauge group. These models are identified from a set of several millions of consistent orbifold compactifications, what renders the task very time-consuming. Fortunately, this search becomes accessible thanks to tools such as the orbifolder [23], which automatizes the computation of matter spectra and the selection of the promising models.

The orbifolder allows one to perform a search of viable orbifold models by scanning randomly the parameter space and comparing the spectra of the generated models. In section 3, we apply this technique to obtain a sample of randomly generated viable orbifold models, similar to those presented in [24, tab A.1], in order to identify the most favored flavor symmetries. Our statistical results, that coincide with those of [24] except for few cases ($Z_7$ and $Z_2 \times Z_4$), help us remark that the most common non-Abelian symmetries arising in phenomenologically promising models are the dihedral group $D_4$, $\Delta(54)$ and products of these symmetries with Abelian discrete symmetries.

Orbifold compactifications do not only constrain the symmetry groups that can be used as flavor symmetries, but also the number of effective matter fields appearing in the models and their flavor representations. Therefore, in contrast to field-theoretic models where flavor representations can be chosen at convenience, in string models one is restricted to use the field representations given by the theory. It is possible to go even beyond this statement: if all moduli fields determining the size and shape of the compactification space are fixed, even the dynamics of possible flavon fields and hence their VEVs are fully set by the theory. Unfortunately, despite recent progress in this direction [25, 26], it is still too early to define whether moduli can be stabilized in this context. Consequently, the details of the phenomenology that can be extracted from these constructions must rely on admissible (compatible with the small-field limit of SUGRA), yet ad hoc, VEVs of moduli and would-be flavon fields. This is the approach we follow in section 4, where we review the main phenomenological results of our previous work [7], based on the $\Delta(54)$ flavor symmetry, which is the favored symmetry in $Z_3 \times Z_3$ orbifolds.

2. Geometry of flavor in heterotic orbifolds

We follow the discussion of [3, 4], that leads to the building blocks of the flavor symmetries of heterotic orbifolds.

2.1. Elements of heterotic orbifolds

Here, we introduce the basic formalism of Abelian orbifolds, by using a $Z_N \times Z_M$ as the standard. $Z_N$ orbifolds follow readily by ignoring all the elements related to the second symmetry. $Z_N \times Z_M$ heterotic orbifolds, with $N, M \in \mathbb{Z}$ are characterized by the quotient of a six-dimensional torus $T^6$ divided by the joint action of two Abelian isometries of $T^6$. Since $Z_N \times Z_M$ must be an
these must be embedded into the gauge degrees of freedom of the heterotic strings. We consider coordinates of a semi-simple Lie algebra. It is convenient to express this geometry in terms of the complex origin is always a fixed point in the orbifold. Besides this trivial fixed point, there are non-moded out. If the generators of isometry of \( T^6 \), the geometry of} the torus defines which symmetries, up to deformations, can be moded out. If the generators of \( Z_N \) and \( Z_M \), \( \vartheta \) and \( \omega \) respectively, are considered as rotations in six dimensions (i.e. ignoring roto-translations), all admissible choices of \( N, M \) and the number of inequivalent torus geometries are displayed in table 1.

The geometry of \( T^6 \) is encoded in its six-dimensional lattice \( \Gamma \), whose basis vectors are \( \{e_1, \ldots, e_6\} \), which build, in the simplest cases, a space with metric given by the Cartan matrix of a semi-simple Lie algebra. It is convenient to express this geometry in terms of the complex coordinates \( z_1, z_2, z_3 \), on which the orbifold generators act as

\[
\begin{align*}
\vartheta : & \quad z_i \rightarrow z_i e^{2\pi i v_i}, & 0 \leq |v_i| < 1, \\
\omega : & \quad z_i \rightarrow z_i e^{2\pi i w_i}, & 0 \leq |w_i| < 1,
\end{align*}
\]

where the so-called twist vectors \( v = (v_1, v_2, v_3) \) and \( w = (w_1, w_2, w_3) \) are subject to the \( \mathcal{N} = 1 \) conditions \( \pm v_1 \pm v_2 \pm v_3 = 0 \) and \( \pm w_1 \pm w_2 \pm w_3 = 0 \). In table 1 we list our choice of the twist vectors for all admissible \( Z_N \) and \( Z_N \times Z_M \) orbifolds.

The action of the orbifold on \( T^6 \) is not free; that is, some points are left invariant or fixed (up to translations \( n_a e_\alpha, n_a \in \mathbb{Z} \), in the torus). For example, since \( \vartheta \) and \( \omega \) are only rotations, the origin is always a fixed point in the orbifold. Besides this trivial fixed point, there are non-trivial fixed points away from the origin. The fixed points turn out to be curvature singularities of the compact space, but do not lead to undesirable gravitational effects in the four-dimensional uncompactified space, which is flat at first approximation.

Together with torus translations, the orbifold rotations build the space group \( S = (\mathbb{Z}_N \times \mathbb{Z}_M) \rtimes \Gamma \), whose elements are \( g = (\vartheta^n \omega^m, n_a e_\alpha) \), with \( 0 \leq n < N \), \( 0 \leq m < M \) and \( n_a \in \mathbb{Z} \). The action of the space group on the complex coordinates is given by

\[
S : \quad z \rightarrow gz = \vartheta^n \omega^m z + n_a e_\alpha.
\]

It is possible to associate each fixed point \( z_f \) with a space group element, called constructing element \( g_f \), such that \( z_f = g_f z_f \). We notice that for each choice of \( (n, m) \) there are different choices of \( \{n_a\} \) leading to different fixed points. The fixed points are truly inequivalent if their corresponding constructing elements belong to different conjugacy classes within \( S \). So, at the end, for each sector \( (n, m) \) there is a finite number of fixed points.

Once the generic geometrical aspects of the compactification in six dimensions have been set, these must be embedded into the gauge degrees of freedom of the heterotic strings. We consider
here the $\mathcal{N} = 1$ $E_8 \times E_8$ heterotic string. Modular invariance of the partition function demands that each orbifold twist be embedded either as a rotation or as a shift vector in the 16 dimensions of $E_8 \times E_8$, and that the $T^6$ translations be connected to a so-called Wilson line. We denote the shift vectors as $V, W$ for the embedding of $\vartheta, \omega$, and the Wilson lines as $A_\alpha$, $\alpha = 1, \ldots, 6$, as the embedding of $e_\alpha$. Constructing elements in $S$ are then embedded into the gauge degrees of freedom as

$$g = (\vartheta^n \omega^m, n_\alpha e_\alpha) \leftrightarrow V_g \equiv nV + mW + n_\alpha A_\alpha.$$  \hspace{1cm} (3)

The gauge embedding is subject to some constraints. First, both $V$ and $W$ must be consistent with a $\mathbb{Z}_N \times \mathbb{Z}_M$ action. This amounts to requiring e.g. that $NV$ must lie in the root lattice of $E_8 \times E_8$, $\Lambda$. Analogous conditions must be imposed to $W$. Secondly, Wilson lines must be consistent with the torus geometry and the orbifold action on it. The fact that the $\Gamma$ basis vectors $e_\alpha$ are in general related by the action of $\vartheta$ and $\omega$ translates to relations among different $A_\alpha$. Furthermore, from these considerations, just as shift vectors, Wilson lines $A_\alpha$ have an order $N_\alpha$, such that $N_\alpha A_\alpha \in \Lambda$ (without summation over $\alpha$). Finally, modular invariance additionally imposes in $\mathbb{Z}_N \times \mathbb{Z}_M$ heterotic orbifolds that [28]

$$N (V^2 - v^2) \equiv 0 \mod 2,$$

$$M (W^2 - w^2) \equiv 0 \mod 2,$$

$$M (V \cdot W - v \cdot w) \equiv 0 \mod 2,$$

$$N_\alpha (V \cdot A_\alpha) \equiv 0 \mod 2, \quad \alpha = 1, \ldots, 6,$$

$$N_\alpha (W \cdot A_\alpha) \equiv 0 \mod 2,$$

$$A_\alpha^2 \equiv 0 \mod 2,$$

$$\gcd(N_\alpha, N_\beta) (A_\alpha \cdot A_\beta) \equiv 0 \mod 2, \quad \alpha \neq \beta.$$

If all the previous requirements are fulfilled, all ingredients can be used to compactify a heterotic string. Note that $A_\alpha = 0$ for all states have identical gauge numbers.

We now turn to the matter fields $\Phi$ in orbifold compactifications. These correspond to closed string states $|\Phi\rangle$ that are invariant under the orbifold action and that must be massless because the mass of massive strings is some factor of the Planck scale, $M_{pl}$, and thus too high to appear in the effective theory at low energies. Closed strings comprise left-movers, that equip physical states with gauge quantum numbers, and right-movers providing the so-called $H$-momentum [20], which is basically the momentum of the state in the compact dimensions.

In heterotic orbifolds, bulk or *untwisted fields* correspond to the orbifold-invariant states arising directly from the ten-dimensional closed strings of the uncompactified heterotic string, whose field limit is ten-dimensional $\mathcal{N} = 1$ supergravity endowed with an $E_8 \times E_8$ Yang-Mills theory. Thus, the four-dimensional gauge superfields, generating the unbroken gauge group $G_{4D} \subset E_8 \times E_8$, and some four-dimensional matter states with non-trivial gauge quantum numbers under $G_{4D}$ live in the bulk of a heterotic orbifold.

Additionally, there are the so-called *twisted fields*, which arise from strings that are closed only due to the action of the orbifold. Twisted fields are always localized at singularities of the orbifold and are thus related to a constructing element $g$ and its embedding into the gauge degrees of freedom $V_g$ defined in (3). Since it is $V_g$ what defines the gauge quantum numbers of the localized strings, we notice that for each sector $(n, m)$ all states have identical gauge numbers unless $A_\alpha \neq 0$ for some $\alpha$, i.e. unless there are non-trivial Wilson lines.

2.2. Coupling selection rules

Couplings among string states are subject to a set of constraints called *string selection rules* [16, 17, 18, 19, 20, 21, 22], due to symmetries of the underlying CFT of the compactified string theory. The main restrictions are

1. *Gauge invariance*: the sum of all gauge quantum numbers must be trivial.

2. *$H$-momentum conservation*: all $H$-momenta of the states must add up to zero.
Space-group invariance: the product of constructing elements must be trivial, which for \( r \) interacting strings means

\[
\prod_{f=1}^{r} \left( \theta_{(f)}, n_{\alpha}^{(f)} e_{\alpha} \right) \equiv \left( 1, \bigcup_{f} (1 - \theta_{(f)}) \Gamma \right),
\]

where \( g_{f} = (\theta_{(f)}, n_{\alpha}^{(f)} e_{\alpha}), \theta_{(f)} = \vartheta^{n_{(f)}^{(f)}} \omega^{m_{(f)}}, \) corresponds to the constructing element of a twisted state localized at \( z_{f} \), and \( \bigcup_{f} (1 - \theta_{(f)}) \Gamma \) is known as the invariant sublattice of fixed points. The space group selection rule (5) indicates if \( r \) closed strings can interact or not taking as criterion their localizations in the compact space.

These selection rules establish for which combination of string states there is a non-zero correlation function, and thus a non-vanishing coupling for the associated effective fields. We can easily see that the only selection rule that is relevant for flavor symmetries is the space-group invariance. Notice that the rotational part of that selection rule implies

\[
\sum_{f=1}^{r} n_{(f)} = 0 \mod N, \quad \sum_{f=1}^{r} m_{(f)} = 0 \mod M,
\]

which is a way to express invariance under a \( \mathbb{Z}_{N} \times \mathbb{Z}_{M} \) symmetry for fields with charges \((n_{(f)}, m_{(f)})\). In explicit examples, it can be shown, as we do in the next section, that the translational part also implies the existence of additional orbifold and geometry dependent Abelian symmetries with charges \( n_{\alpha}^{(f)} \). That is, in the four-dimensional model emerging from an Abelian heterotic orbifold, space-group invariance amounts to including additional Abelian symmetries and assign thus appropriate discrete charges to each field in the model.

2.3. Non-Abelian flavor symmetries

As guiding examples of the origin of flavor symmetries, let us consider the cases illustrated in figure 1. The first figure corresponds to a \( \mathbb{Z}_{2} \) symmetry that, instead of being moded out of the whole \( \mathbb{T}^{2} \), divides only one compact dimension, i.e. \( S^{1} \). This scenario is realized in heterotic

![Figure 1](image_url)

Figure 1: Geometrical origin of the \( D_{4} \) and \( \Delta(54) \) flavor symmetries in orbifolds. When unaffected by Wilson lines, the fixed points of an \( S^{1}/\mathbb{Z}_{2} (\mathbb{T}^{2}/\mathbb{Z}_{3}) \) orbifold realize an \( S_{2} (S_{3}) \) permutation symmetry. String selection rules impose an additional \( \mathbb{Z}_{2} \times \mathbb{Z}_{2} (\mathbb{Z}_{3} \times \mathbb{Z}_{3}) \) symmetry based on the localization charges of twisted states. The resulting flavor symmetry is \( S_{2} \times \mathbb{Z}_{2}^{2} = D_{4} (S_{3} \times \mathbb{Z}_{3}^{2} = \Delta(54)) \).
or orbifolds when $T^6$ can be factorized as $T^2 \times T^4$ and there is a Wilson line associated to one of the compact directions ($e_2$ in the depicted case). The $\mathbb{Z}_2$ acts as a reflection on the points of $S^1$, so that the elements of $\mathbb{Z}_2$ that act on $z_1$ are expressed as $\{\vartheta^0 = 1, \vartheta^1 = -1\}$. Thus, the action of $\vartheta$ on the $\Gamma$ basis vector reads $e_1 \rightarrow -e_1$. With this at hand, we observe that two fixed points occur in the sector $\vartheta^n$ with $n = 1$ and are given by the constructing elements $g_0 = (\vartheta, 0)$ and $g_1 = (\vartheta, e_1)$, corresponding to $z_0 = 0$ and $z_1 = \frac{1}{2}e_1$.

From the rotational part of space-group invariance, we realize that the only non-vanishing couplings are those satisfying $\vartheta^n = 1$, i.e. when an even number of strings from the $\vartheta$ sector interact. This implies that states are charged with a $\mathbb{Z}_2$ charge that can be either $n = 0$ for untwisted sector states or $n = 1$ for states in the $\vartheta$ sector. On the other hand, the translational component of space-group invariance requires verifying first what the invariant sublattice is. Since $(1 - \vartheta)e_1 = 2e_1$, then the invariant sublattice is given by any integer multiple of $2e_1$. Consider now a coupling between, say, two twisted states, related to the constructing elements $(\vartheta, n_1^{(1)}e_1)$ and $(\vartheta, n_1^{(2)}e_1)$. The product of these states yields $(1, (n_1^{(1)} - n_1^{(2)})e_1)$, which is equivalent to $(1, (n_1^{(1)} + n_1^{(2)})e_1)$ because they differ by a contribution of the invariant sublattice. Hence, the selection rule implies that $n_1^{(1)} + n_1^{(2)}$ must be even, or $\sum f n_1^{(f)} = 0 \mod 2$ in general, i.e. another $\mathbb{Z}_2$ that associates the charge $n_1^{(f)}$ to the twisted states. We find thus that couplings between two twisted states are allowed only if both states lie at the same fixed point. We have found a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry whose generators can be expressed in the space of the $\vartheta$ fixed points as the matrices $R = \text{diag}(-1, -1)$ and $T = \text{diag}(1, -1)$.

Finally, in the absence of Wilson lines associated with the direction $e_1$, both singularities are indistinguishable from the point of view of a four-dimensional observer because the gauge quantum numbers of states living at both singularities are equal. Therefore, there is a permutation symmetry $S_2$ of the states at the singularities, whose generator can be expressed as

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7}$$

Noting that conjugating with $P$ any element of $\mathbb{Z}_2 \times \mathbb{Z}_2$ generated by $R$ and $T$ yields another element of this group, we see that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is a normal subgroup of the full discrete symmetry, which can thus be written as $S_2 \times (\mathbb{Z}_2 \times \mathbb{Z}_2)$, isomorphic to $D_4$. This is the flavor symmetry of the fields arising from an $S^1 / \mathbb{Z}_2$ sector of a heterotic orbifold.

As we discuss in full detail in [7], similar considerations yield the flavor symmetry $\Delta(54)$ in the $T^2 / Z_3$ orbifold, depicted in figure 1(b). In this case, $\vartheta e_1 \rightarrow e_2$ and $\vartheta e_2 \rightarrow -e_1 - e_2$. Further, the constructing elements associated with the three fixed points of the $\vartheta$ sector are $(\vartheta, n_1 e_1)$, $n_1 = 0, 1, 2$. With the invariant sublattice given by the basis $\{3e_1, 3e_2\}$, one can show that space-group invariance implies that only couplings of a multiple of three states are allowed and that $\sum f n_1^{(f)} = 0 \mod 3$ must be satisfied. The related Abelian symmetry is then $\mathbb{Z}_3 \times \mathbb{Z}_3$, generated by $R = \text{diag}(\rho, \rho, \rho)$ and $T = \text{diag}(1, \rho, \rho^2)$, with $\rho = e^{2\pi i/3}$. Furthermore, in the absence of Wilson lines related to $T^2$, there is a permutation symmetry $S_3$, whose generators are

$$\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The multiplicative closure of all elements reveals that the emerging flavor symmetry is $S_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)$, isomorphic to $\Delta(54)$.

Another interesting aspect is that the flavor representations of matter fields are also determined by this structure. States associated to the fixed points of $S^1 / \mathbb{Z}_2$ can only build doublets 2 of $D_4$. In $T^2 / \mathbb{Z}_3$, the only matter representations in the $\vartheta$ sector are triplets 3.
of $\Delta(54)$ (in the notation of [29]); in the $\vartheta^2 = \vartheta^{-1}$ sector the representations are conjugate, i.e. $3i_2$. Bulk states are in all cases trivial singlets of the corresponding flavor symmetries. No further representations appear unless states form condensates.

The permutation symmetries are broken completely if there are non-vanishing Wilson lines associated to the compact directions. Thus, the flavor symmetries suffer an explicit breakdown to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ in $S^1/\mathbb{Z}_2$ or $T^2/\mathbb{Z}_3$, respectively, when affected by non-trivial Wilson lines. Flavor symmetries can also be broken spontaneously to non-Abelian subgroups if some fields localized at the singularities develop VEVs. The details depend on the VEV structure and the number of flavon fields.

This discussion has been explicitly developed for all possible sub-orbifolds (in less than six dimensions) appearing in Abelian toroidal heterotic orbifolds [3], resulting in a reduced number of family symmetries. The findings include, besides $D_4$ and $\Delta(54)$, only the symmetries $(D_4 \times D_4)/\mathbb{Z}_2$, $(D_4 \times \mathbb{Z}_4)/\mathbb{Z}_2$, $(D_4 \times \mathbb{Z}_8)/\mathbb{Z}_2$ and $S_7 \times \mathbb{Z}_2^3$. As we shall shortly see, not all of these symmetries are realized in phenomenologically viable models.

3. Favored flavor symmetries in $\mathbb{Z}_N$ and $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds

The tools explained in the previous section can be applied to all Abelian orbifolds. Therefore, given a set of consistent shift vectors and Wilson lines, it is possible first to obtain the spectrum and then to determine the flavor symmetry by inspecting which Wilson lines are trivial and what is the orbifold action on the $\Gamma$ basis vectors associated to them.

For example, $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold models in their simplest geometry $(1,1)$ in the nomenclature [27] admit three independent Wilson lines, each associated to one of the $z^i$ complex planes with basis vectors $e_{2i-1}$ and $e_{2i}$. The global flavor symmetry in the absence of Wilson lines is $(S_3 \times S_3 \times S_3) \times \mathbb{Z}_3^5$, somewhat similar to the naive expectation of $\Delta(54)^3$. Now, if one of the Wilson lines is non-zero, then the flavor symmetry is $(S_3 \times S_3) \times \mathbb{Z}_3^5$ due to the explicit breaking of one $S_3$. Two non-vanishing Wilson lines yield $S_3 \times \mathbb{Z}_3^5$, which can be rewritten as $\Delta(54) \times \mathbb{Z}_3^3$. (This symmetry shall play an important role in the next section.) And three non-trivial Wilson lines lead to a purely Abelian $\mathbb{Z}_3^3$ flavor symmetry.

There are other more complicated cases, such as $\mathbb{Z}_6$-II orbifolds, whose structure leads to the possibility of three non-trivial Wilson lines: one related to the $T^2$ of the $z^2$ plane, and two associated to the $e_5$ and $e_6$ directions. From the orbifold action determined by the twist $v = (1,2,-3)/6$, we see that the orbifold corresponds to the factorization $T^2/\mathbb{Z}_6 \times T^2/\mathbb{Z}_3 \times S^1/\mathbb{Z}_2 \times S^1/\mathbb{Z}_2$, where the Wilson lines can affect the last three factors. In this case, it is not enough to know the number of non-zero Wilson lines of a given model to figure out the flavor symmetry. One needs also the direction with which it is associated. If a model is furnished with one non-trivial Wilson line, then the original flavor symmetry $(S_3 \times S_2 \times S_2) \times (\mathbb{Z}_2^3 \times \mathbb{Z}_2^3)$ is broken down to one of two possibilities, $\Delta(54) \times D_4 \times \mathbb{Z}_2$ or $(S_2 \times S_2) \times (\mathbb{Z}_2^3 \times \mathbb{Z}_2^3)$, isomorphic to $(D_4 \times D_4)/\mathbb{Z}_2 \times \mathbb{Z}_2^3$. Two non-vanishing Wilson lines render the flavor symmetries $\Delta(54) \times \mathbb{Z}_2^3$ or $D_4 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_2$. And three non-zero Wilson lines lead to a $\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$ flavor symmetry.

This exercise can be performed for phenomenologically promising heterotic orbifolds. The use of the orbifolder [23] leads to large sets of models with a priori defined properties. Particularly, one can obtain models with the following features: 1) SM gauge group with non-anomalous and correctly normalized hypercharge, 2) three SM generations, 3) at least a couple of Higgses $H_u$ and $H_d$, and 4) no chiral exotics. These models will be considered phenomenologically viable. Then, identifying their flavor symmetries can help envisage the phenomenological potential of these constructions.

We have performed non-exhaustive scans of all orbifolds listed in tables 2, with the geometries specified in the second column of each table in the notation of [27]. That is, we have used all possible geometries (ignoring roto-translations) of $\mathbb{Z}_N$ orbifolds and only the simplest ones for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds. Our results, that shall further detailed elsewhere, are statistically equivalent.
to those of [24, tab A.1], except in the $Z_7$ orbifold where we do find one promising model with one Wilson line and in the $Z_2 \times Z_4$ orbifold, where we find that most of the models are endowed with two non-vanishing Wilson lines, instead of three.

Out of nearly 8,000 models, about 30% of them arise from $Z_2 \times Z_4$ models and around 10+10% from $Z_4 \times Z_4$ and $Z_6 \times Z_6$ orbifolds. About 35% of the models saturate the number of non-vanishing Wilson lines. In these cases, the flavor symmetries are purely Abelian, rendering the models less promising. These appear mostly in $Z_{12}$ and $Z_6 \times Z_6$ orbifolds, which may suggest that some efforts in the literature about $Z_{12}$-I orbifolds should be redirected.

We have examined the most common flavor symmetries in each orbifold geometry. Our results are listed in table 2. There, we display only the non-Abelian flavor symmetries identified, omitting smaller Abelian factors, for the sake of simplicity. In the cases where only Abelian flavor symmetries appear, we present only the largest Abelian group. The symmetries in squared brackets correspond to less favored non-Abelian flavor symmetries that arise in our models.

In many cases where most of the models exhibit Abelian flavor symmetries, there are also some viable models with non-Abelian flavor symmetries. For example, in $Z_8$-I with torus geometry $(2, 1)$, besides the flavor group $Z_8$, there are about 25% of the models endowed with a $(D_4 \times Z_8)/Z_2$ flavor symmetry.

Globally, we also find that about 30% of the models yield a $D_4$ symmetry multiplied by purely Abelian factors. Around 21% of the models have non-trivial combinations of $D_4$ with itself and

| $Z_N$ | flavor symmetry | # WL | $Z_N \times Z_M$ | flavor symmetry | # WL |
|-------|-----------------|------|-----------------|-----------------|------|
| $Z_3$ | (1, 1) no viable model | - | $Z_2 \times Z_2$ | (1, 1) $(D_4 \times D_4)/Z_2 [D_4]$ | 4/6 |
| $Z_4$ | (1, 1) no viable model | - | $Z_2 \times Z_4$ | (1, 1) $[D_4 \times Z_4]/Z_2$ | 2/4 |
| $Z_5$ | (2, 1) $Z_4$ | 3/3 | $Z_2 \times Z_6$ | (1, 1) $[D_4 \times Z_4]/Z_2$ | 3/4 |
| $Z_6$ | (3, 1) $Z_4$ | 2/2 | $Z_2 \times Z_6$ | (2, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |
| $Z_7$ | (2, 1) $Z_4$ | 3/3 | $Z_2 \times Z_7$ | (4, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |
| $Z_8$ | (2, 1) $Z_4$ | 2/2 | $Z_2 \times Z_8$ | (4, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |
| $Z_9$ | (2, 1) $Z_4$ | 2/2 | $Z_2 \times Z_9$ | (4, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |
| $Z_{10}$ | (2, 1) $Z_4$ | 2/2 | $Z_2 \times Z_{10}$ | (4, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |
| $Z_{11}$ | (2, 1) $Z_4$ | 2/2 | $Z_2 \times Z_{11}$ | (4, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |
| $Z_{12}$ | (2, 1) $Z_4$ | 2/2 | $Z_2 \times Z_{12}$ | (4, 1) $[D_4 \times Z_4]/Z_2$ | 2/3 |

Table 2: Statistically preferred flavor symmetries in phenomenologically viable $Z_N$ and $Z_N \times Z_M$ heterotic orbifolds. The first two columns of each table correspond to the label of Abelian orbifolds, following the notation of [27]. The third column displays the most common flavor symmetry appearing in promising models; the symmetries in squared brackets are non-Abelian flavor symmetries that appear less frequently in these models. Additional Abelian factors are not displayed. The last column counts the number of independent non-trivial Wilson lines needed to build most frequent promising models over the maximal number of Wilson lines allowed by the geometry.

About 21% of the models have non-trivial combinations of $D_4$ with itself and
other Abelian factors or quotients, and close to 5% of the models are furnished with a $\Delta(54)$ flavor symmetry. That is, considering the models that exhibit these flavor symmetries together with those endowed with purely Abelian flavor groups, we obtain around 90% of all our models.

4. Phenomenological consequences of flavor symmetries in string models

Our results of the previous section show that most promising heterotic orbifolds enjoy a $D_4$ non-Abelian flavor symmetry. The phenomenological potential of this symmetry in this context has been studied in detail in [6, 30], where admissible Yukawa textures for quarks and (charged and neutral) leptons as well as promising supersymmetric soft terms (that help avoid FCNC in supersymmetric models) are achieved. It is interesting that the same flavor symmetry appears frequently in appealing D-brane models [11].

We have also found that $\Delta(54)$ is the second most favored non-Abelian flavor symmetry appearing in viable heterotic orbifolds and it has been largely ignored even from a bottom-up approach. Thus, it is necessary to study its phenomenological consequences. To do so, we recover here the main properties of the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold model studied in [7]. In that work, one of the about 800 promising models was chosen due to its simplicity. The model is defined by the shift vectors

\[
3V = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -2, 0, 0, 1, 1, 1, 1, 4\right), \quad (9a)
\]
\[
3W = \left(0, 1, 1, 4, 0, 0, 1, 1; 1, -1, 4, -4, -1, 0, 0, 1\right), \quad (9b)
\]

and the Wilson lines

\[
3A_1 = 3A_2 = \left(-\frac{7}{2}, -\frac{3}{2}, \frac{9}{2}, \frac{7}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, -3, 0, -2, 0, -2, -4, 3, -2\right), \quad (10a)
\]
\[
3A_3 = 3A_4 = \left(3, 3, -3, -2, -1, 2, 4, -4; -3, 1, -1, -4, 1, 1, 4, 1\right). \quad (10b)
\]

These parameters yield the unbroken gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y \times [SU(2) \times U(1)]^1$, where the additional $SU(2)$ factor is considered hidden because no SM–field carries a charge under that group. All fields in the spectrum are charged under the additional $U(1)$ factors, but these shall be broken by flavon VEVs. The flavor symmetry is $\Delta(54) \times \mathbb{Z}_3^3$.

The relevant matter spectrum is displayed in table 3, where the first eight columns refer to SM matter states, including the Higgs fields and right-handed neutrinos, and the last five columns display the properties of flavon fields. In this model, SM fermion fields transform as triplets of the $\Delta(54)$ flavor symmetry while the Higgs fields do not transform because they are bulk fields.

Since the SM matter fields are charged under flavor symmetries, only the presence of the properly charged $s$ and $\phi$ flavon fields allows for Yukawa couplings in the (non–renormalizable)
superpotential, given by

\begin{align*}
W_Y &= y^u_{ijk} Q_i H_u \bar{u}_j \phi^u_k s_u + y^d_{ijk} Q_i H_d \bar{d}_j \phi^{(d,e)}_k s^{(d,e)} + y^e_{ijk} L_i H_d \bar{e}_j \phi^{(d,e)}_k s^{(d,e)} \\
&+ y^\nu_{ijkl} L_i H_u \bar{\nu}_j \phi^\nu_k, \\
&i, j, k = 1, 2, 3,
\end{align*}

(11)

where the summation over repeated indices must follow the rules of the product of $\Delta(54)$ representations that lead to invariant singlets. We see from this superpotential, that quarks and charged leptons acquire masses through dimension–6 operators, and the Dirac neutrino masses as well as the right-handed Majorana neutrino masses are generated at renormalizable level. This is optimal because guarantees that the largest masses are those of right-handed neutrinos.

Assuming that most properties are conserved even after supersymmetry breakdown, we observe that choosing some ad hoc hierarchical flavon–VEV alignments results in the following flavor phenomenology features:

• correct masses for quarks and charged leptons;
• proper Gatto-Sartori-Tonin relation in the quark sector (although the other two mixing angles are very small);
• a mass relation between the down–quark sector and the charged leptonic sector

\[ m_s - m_d \approx m_\mu - m_e \approx m_\tau . \]

(12)

• compatibility (only) with normal hierarchy of neutrino masses;
• smallest neutrino mass of order 6–7 meV;
• total neutrino masses of order 65–70 meV; and
• PMNS matrix compatible with current constraints (atmospheric and reactor mixing angles are in the 3$\sigma$ region of the global best fit), with the atmospheric mixing angle greater than 45 degrees.

Interestingly, an inverted hierarchy being disfavored as well as the atmospheric mixing angle lying in the second octant, are features compatible with recent findings of the T2K collaboration [31]. These results render the neutrino sector of $\Delta(54)$ heterotic orbifolds much more promising than the other sectors and let us assert that $Z_3 \times Z_3$ heterotic orbifolds and $\Delta(54)$ as a flavor symmetry provide a fertile playground for useful phenomenology.

5. Concluding remarks

In this paper, we firstly reviewed the geometric origin of flavor symmetries. They appear as a result of the effective properties of matter fields due to their localization in the compact space. The selection rules that define the non-zero couplings among fields are the key string ingredient in the construction of flavor symmetries. Non-Abelian symmetries turn out to be semi-direct products of a permutation symmetry, arising from the degeneracy of localized states in the absence of non-trivial Wilson lines, multiplied with some Abelian discrete symmetries inherited from the selection rules.

We have then performed a large scan of promising heterotic orbifolds and identified the statistically favored flavor symmetries in those models. We find that in about 50% of the explored models the $D_4$ flavor symmetry is preferred. The second most frequent non-Abelian flavor symmetry in phenomenologically viable models is $\Delta(54)$. It is somewhat surprising that about 35% of the models carry purely Abelian groups as flavor symmetries.

Since $D_4$ has been explored in the past, here we reviewed the main features of a $Z_3 \times Z_3$ orbifold model giving rise to the SM particle spectrum that forms $\Delta(54)$ flavor representations.
Exploiting the couplings structure of the model and choosing some ad hoc flavon VEVs, the model reproduces all particle masses, but exhibits issues to yield the right quark and charged-lepton mixing angles. Yet the neutrino sector, that allows for see-saw neutrino masses, can meet the experimental constraints for all mixing angles, favors a normal hierarchy in the neutrino sector, and provides promising values for the left-handed neutrino masses. One is thus encouraged to explore other models with this symmetry to exhaust the phenomenological potential of $\Delta(54)$.

Two reasonable questions arise from this study. Given the most favored symmetries, can we provide a generic recipe for the flavor phenomenology arising from string models? Is it possible to falsify some of these promising heterotic orbifolds on the basis of their flavor symmetries? In clarifying these general questions, it would also be convenient to perform a more exhaustive scan, including all possible toroidal geometries of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, roto-translations, and the enhancement(s) of these symmetries at special modular values. These open challenges shall be the aim of future works.

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