A characterization of $Q$-polynomial distance-regular graphs using the intersection numbers

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Abstract
We consider a primitive distance-regular graph $\Gamma$ with diameter at least 3. We use the intersection numbers of $\Gamma$ to find a positive semidefinite matrix $G$ with integer entries. We show that $G$ has determinant zero if and only if $\Gamma$ is $Q$-polynomial.

1 Introduction
Let $\Gamma$ denote a distance-regular graph with diameter $D \geq 3$. In the literature there are a number of characterizations for the $Q$-polynomial condition on $\Gamma$. There is the balanced set characterization [9, Theorem 1.1], [10, Theorem 3.3]. There is a characterization involving the dual distance matrices [10, Theorem 3.3]. There is a characterization involving the intersection numbers $a_i$ [8, Theorem 3.8]; cf. [3, Theorem 5.1]. There is a characterization involving a tail in a representation diagram [5, Theorem 1.1]. There is a characterization involving a pair of primitive idempotents [6, Theorem 1.1]; cf. [7, Theorem 1.1].

In this paper we obtain the following characterization of the $Q$-polynomial property. Assume $\Gamma$ is primitive. We use the intersection numbers of $\Gamma$ to find a positive semidefinite matrix $G$ with integer entries. We show that $G$ has determinant zero if and only if $\Gamma$ is $Q$-polynomial. Our main result is Theorem 18.

2 Preliminaries
Let $X$ denote a nonempty finite set. Let $\text{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of the matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. For $B \in \text{Mat}_X(\mathbb{C})$ let $\overline{B}$ and $B^t$ denote the complex conjugate and the transpose of $B$, respectively. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors with coordinates indexed by $X$ and entries in $\mathbb{C}$. Observe that $\text{Mat}_X(\mathbb{C})$ acts...
on $V$ by left multiplication. We endow $V$ with the Hermitean inner product $(\cdot, \cdot)$ such that $(u, v) = u^*\mathcal{F}v$ for all $u, v \in V$. The inner product $(\cdot, \cdot)$ is positive definite. For $B \in \text{Mat}_X(\mathbb{C})$ we obtain $(u, Bv) = (B^*u, v)$ for all $u, v \in V$. We endow $\text{Mat}_X(\mathbb{C})$ with the Hermitean inner product $(\cdot, \cdot)$ such that $(R, S) = \text{tr}(R^*S)$ for all $R, S \in \text{Mat}_X(\mathbb{C})$. The inner product $(\cdot, \cdot)$ is positive definite.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$ and edge set $R$. Let $\partial$ denote the shortest path-length distance function for $\Gamma$. Define the diameter $D := \max\{\partial(x, y) | x, y \in X\}$. For a vertex $x \in X$ and an integer $i \geq 0$ define $\Gamma_i(x) = \{y \in X | \partial(x, y) = i\}$. For notational convenience abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, we call the graph $\Gamma$ regular with valency $k$ whenever $|\Gamma(x)| = k$ for all $x \in X$. We say that $\Gamma$ is distance-regular whenever for all integers $h, i, j$ ($0 \leq h, i, j \leq D$) and for all $x, y \in X$ with $\partial(x, y) = h$, the number

$$p_{ij}^h := |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of $x$ and $y$. The integers $p_{ij}^h$ are called the intersection numbers of $\Gamma$. From now on we assume $\Gamma$ is distance-regular with diameter $D \geq 3$. We abbreviate $c_i := p_{1,i-1}^i$ ($1 \leq i \leq D$), $a_i := p_{1i}^i$ ($0 \leq i \leq D$), $b_i := p_{i,i+1}^i$ ($0 \leq i \leq D - 1$), $k_i := p_{ii}^0$ ($0 \leq i \leq D$), and $c_0 = 0$, $b_D = 0$. Observe that $\Gamma$ is regular with valency $k = b_0$ and $c_i + a_i + b_i = k$ ($0 \leq i \leq D$). By [2] p. 127] the following holds for $0 \leq h, i, j \leq D$: (i) $p_{ij}^h = 0$ if one of $h, i, j$ is greater than the sum of the other two; and (ii) $p_{ij}^h \neq 0$ if one of $h, i, j$ equals the sum of the other two. For $0 \leq i \leq D$, let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $(x, y)$-entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x, y) = i, \\
0 & \text{if } \partial(x, y) \neq i, \end{cases} \quad x, y \in X.$$ 

We call $A_i$ the $i$-th distance matrix of $\Gamma$. We call $A = A_1$ the adjacency matrix of $\Gamma$. Observe that $A_i$ is real and symmetric for $0 \leq i \leq D$. Note that $A_0 = I$, where $I$ is the identity matrix. Observe that $\sum_{i=0}^D A_i = J$, where $J$ is the all-ones matrix in $\text{Mat}_X(\mathbb{C})$. Observe that for $0 \leq i, j \leq D$, $A_iA_j = \sum_{h=0}^D p_{ij}^h A_h$. For integers $h, i, j$ ($0 \leq h, i, j \leq D$) we have

$$p_{0j}^h = \delta_{hj} \quad (1)$$

$$p_{ij}^0 = \delta_{ij}k_i \quad (2)$$

$$p_{ij}^h = p_{ji}^h \quad (3)$$

$$k_hp_{ij}^h = k_ip_{ij}^h = k_jp_{ij}^h. \quad (4)$$

For $0 \leq i, j \leq D$ we have $A(A_iA_j) = (AA_i)A_j$. This gives a recursion

$$c_{i+1}p_{i+1,j}^h + a_i p_{ij}^h + b_{i-1}p_{i-1,j}^h = c_h p_{ij}^{h-1} + a_h p_{ij}^h + b_h p_{ij}^{h+1} \quad (5)$$
that can be used to compute the intersection numbers.

Let $M$ denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by $A$. By [2, p. 127] the matrices $A_0, A_1, ..., A_D$ form a basis for $M$. We call $M$ the Bose-Mesner algebra of $\Gamma$. By [2, p. 45], $M$ has a basis $E_0, E_1, ..., E_D$ such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $E_i^2 = E_i$ ($0 \leq i \leq D$); (iv) $E_i E_j = \delta_{ij} E_i$ ($0 \leq i, j \leq D$). The matrices $E_0, E_1, ..., E_D$ are called the primitive idempotents of $\Gamma$, and $E_0$ is called the trivial idempotent. For $0 \leq i \leq D$ let $m_i$ denote the rank of $E_i$. Let $\lambda$ denote an indeterminate. Define polynomials $\{v_i\}_{i=0}^{D+1}$ in $\mathbb{C}[\lambda]$ by $v_0 = 1$, $v_1 = \lambda$, and

$$\lambda v_i = c_{i+1} v_{i+1} + a_i v_i + b_{i-1} v_{i-1} \quad (1 \leq i \leq D),$$

where $c_{D+1} := 1$. By [2, p. 128] and [11, Lemma 3.8], the following hold: (i) deg $v_i = i$ ($0 \leq i \leq D + 1$); (ii) the coefficient of $\lambda^i$ in $v_i$ is $(c_1 c_2 ... c_i)^{-1}$ ($0 \leq i \leq D + 1$); (iii) $v_i(A) = A_i$ ($0 \leq i \leq D$); (iv) $v_{D+1}(A) = 0$; (v) the distinct eigenvalues of $\Gamma$ are precisely the zeros of $v_{D+1}$; call these $\theta_0, \theta_1, ..., \theta_D$. Define a matrix $B \in \text{Mat}_{D+1}(\mathbb{C})$ as follows:

$$B = \begin{bmatrix} a_0 & b_0 & 0 \\ c_1 & a_1 & b_1 \\ & c_2 & a_2 & \cdots \\ & & \cdots & \cdots & b_{D-1} \\ 0 & & & c_D & a_D \end{bmatrix}.$$  

We call $B$ the intersection matrix of $\Gamma$. Note that $A$ has the same minimal polynomial as $B$. Moreover the minimal polynomial of $B$ is the characteristic polynomial of $B$. For an eigenvalue $\theta$ of $B$ we have $vB = \theta v$ where $v$ is a row vector $v = (v_0(\theta), v_1(\theta), ..., v_D(\theta))$. Define polynomials $\{u_i\}_{i=0}^D$ in $\mathbb{C}[\lambda]$ by $u_0 = 1$, $u_1 = \lambda/k$, and

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \quad (1 \leq i \leq D - 1).$$

Observe that $u_i = v_i/k_i$ ($0 \leq i \leq D$). For an eigenvalue $\theta$ of $B$ we have $Bu = \theta u$ where $u$ is a column vector $u = (u_0(\theta), u_1(\theta), ..., u_D(\theta))^t$. By [2, p. 131, 132],

$$A_j = \sum_{i=0}^D v_j(\theta_i) E_i \quad (0 \leq j \leq D), \quad (6)$$

$$E_j = |X|^{-1} m_j \sum_{i=0}^D u_i(\theta_j) A_i \quad (0 \leq j \leq D). \quad (7)$$

Since $E_i E_j = \delta_{ij} E_i$ and by (6), (7) we have $A_j E_i = E_i A_j$ ($0 \leq i, j \leq D$).

For $1 \leq i \leq D$ let $\Gamma_i$ denote the graph with vertex set $X$ where vertices are adjacent in $\Gamma_i$ whenever they are at distance $i$ in $\Gamma$. We observe that $\Gamma_1 = \Gamma$. The graph $\Gamma$ is said to be primitive whenever $\Gamma_i$ is connected for $1 \leq i \leq D$. 


Lemma 1. (See [2 Proposition 4.4.7].) Assume \( \Gamma \) is primitive. Then \( u_i(\theta_j) \neq 1 \) for \( 1 \leq i, j \leq D \).

We now define a matrix \( S \in \text{Mat}_{D+1}(\mathbb{C}) \).

**Definition 2.** Let \( S \in \text{Mat}_{D+1}(\mathbb{C}) \) denote the transition matrix from the basis \( \{A_i\}_{i=0}^D \) of \( M \) to the basis \( \{E_i\}_{i=0}^D \) of \( M \). Thus

\[
E_j = \sum_{i=0}^D S_{ij} A_i \quad (0 \leq j \leq D),
\]

\[
A_j = \sum_{i=0}^D (S^{-1})_{ij} E_i \quad (0 \leq j \leq D).
\]

**Lemma 3.** The entries of \( S \) and \( S^{-1} \) are given below. For \( 0 \leq i, j \leq D \),

\[
S_{ij} = |X|^{-1} m_j u_i(\theta_j), \quad (S^{-1})_{ij} = v_j(\theta_i).
\]

**Proof.** Immediate from Definition 2 and (6), (7). \( \square \)

We recall the \( Q \)-polynomial property. Let \( \circ \) denote the entry-wise multiplication in \( \text{Mat}_X(\mathbb{C}) \). Note that \( A_i \circ A_j = \delta_{ij} A_i \) for \( 0 \leq i, j \leq D \). So \( M \) is closed under \( \circ \). By [11, p. 377], there exist scalars \( q_{ij}^h \in \mathbb{C} \) such that

\[
E_i \circ E_j = |X|^{-1} \sum_{h=0}^D q_{ij}^h E_h \quad (0 \leq i, j \leq D). \tag{8}
\]

We call the \( q_{ij}^h \) the Krein parameters of \( \Gamma \). By [2 p. 48, 49], these parameters are real and nonnegative for \( 0 \leq h, i, j \leq D \). The graph \( \Gamma \) is said to be \( Q \)-polynomial with respect to the ordering \( E_0, E_1, ..., E_D \) whenever the following hold for \( 0 \leq h, i, j \leq D \):

(i) \( q_{ij}^h = 0 \) if one of \( h, i, j \) is greater than the sum of the other two; and (ii) \( q_{ij}^h \neq 0 \) if one of \( h, i, j \) equals the sum of the other two. Let \( E \) denote a primitive idempotent of \( \Gamma \). We say that \( \Gamma \) is \( Q \)-polynomial with respect to \( E \) whenever there exists a \( Q \)-polynomial ordering \( E_0, E_1, ..., E_D \) of the primitive idempotents such that \( E = E_1 \).

We recall the dual Bose-Mesner algebra of \( \Gamma \). Fix a vertex \( x \in X \). For \( 0 \leq i \leq D \) let \( E_i^* = E_i^*(x) \) denote the diagonal matrix in \( \text{Mat}_X(\mathbb{C}) \) with \((y, y)\)-entry

\[
(E_i^*)_{yy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{if } \partial(x, y) \neq i,
\end{cases} \quad y \in X.
\]

We call \( E_i^* \) the \( i \)-th dual idempotent of \( \Gamma \) with respect to \( x \). Observe that (i) \( \sum_{i=0}^D E_i^* = I \); (ii) \( E_i^* = E_i^* (0 \leq i \leq D) \); (iii) \( E_i^* = E_i^* (0 \leq i \leq D) \); (iv) \( E_i^* E_j^* = \delta_{ij} E_i^* (0 \leq i, j \leq D) \). By construction \( E_0^*, E_1^*, ..., E_D^* \) are linearly independent. Let \( M^* = M^*(x) \)
denote the subalgebra of $\text{Mat}_X(\mathbb{C})$ with basis $E_0^*, E_1^*, ..., E_D^*$. We call $M^*$ the dual Bose-Mesner algebra of $\Gamma$ with respect to $x$.

We now recall the dual distance matrices of $\Gamma$. For $0 \leq i \leq D$ let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $(y, y)$-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy} \quad y \in X. \quad (9)$$

We call $A_i^*$ the dual distance matrix of $\Gamma$ with respect to $x$ and $E_i$. By [11], p. 379, the matrices $A_0^*, A_1^*, ..., A_D^*$ form a basis for $M^*$. Observe that (i) $A_0^* = I$; (ii) $\sum_{i=0}^D A_i^* = |X|E^*_0$; (iii) $A_i^* = A_i^*_E^* (0 \leq i \leq D)$; (iv) $A_i^* = A_i^* (0 \leq i \leq D)$; (v) $A_i^*A_j^* = \sum_{h=0}^D q_{ij}^h A_h^* (0 \leq i, j \leq D)$. From (3), (7) we have

$$A_j^* = m_j \sum_{i=0}^D u_i(\theta_j)E_i^* \quad (0 \leq j \leq D), \quad (10)$$

$$E_j^* = |X|^{-1} \sum_{i=0}^D v_j(\theta_i)A_i^* \quad (0 \leq j \leq D). \quad (11)$$

**Lemma 4.** The matrix $|X|S$ is the transition matrix from the basis $\{E_i^*\}_{i=0}^D$ of $M^*$ to the basis $\{A_i^*\}_{i=0}^D$ of $M^*$. Thus

$$A_j^* = |X| \sum_{i=0}^D S_{ij}E_i^* \quad (0 \leq j \leq D),$$

$$E_j^* = |X|^{-1} \sum_{i=0}^D (S^{-1})_{ij}A_i^* \quad (0 \leq j \leq D).$$

**Proof.** Immediate from Lemma 3 and (10), (11). \hfill \Box

### 3 The matrices $S^{alt}$, $(S^{-1})^{alt}$, $S'$

Recall the matrix $S$ from Definition 2. We now modify the matrices $S, S^{-1}$ to obtain $D \times D$ matrices $S^{alt}, (S^{-1})^{alt}$ as follows:

$$(S^{alt})_{ij} = S_{ij} - S_{0j} \quad (1 \leq i, j \leq D), \quad (12)$$

$$(S^{-1})^{alt}_{ij} = (S^{-1})_{ij} \quad (1 \leq i, j \leq D). \quad (13)$$

**Lemma 5.** The following (i)–(iv) hold.

(i) $S^{alt}$ is the transition matrix from $\{A_2E_i^* - AE_i^*A_2\}_{i=1}^D$ to $\{A_2A_i^* - AA_i^*A_2\}_{i=1}^D$.

(ii) $S^{alt}$ is the transition matrix from $\{A_3E_i^* - E_i^*A_3\}_{i=1}^D$ to $\{A_3A_i^* - A_i^*A_3\}_{i=1}^D$.

(iii) $S^{alt}$ is the transition matrix from $\{A_2E_i^* - E_i^*A_2\}_{i=1}^D$ to $\{A_2A_i^* - A_i^*A_2\}_{i=1}^D$. 
(iv) $S^\text{alt}$ is the transition matrix from $\{AE^*_i - E^*_i A\}_{i=1}^D$ to $\{AA^*_i - A^*_i A\}_{i=1}^D$.

(v) $(S^{-1})^\text{alt}$ and $S^\text{alt}$ are inverses.

Proof. (i), (v) For $1 \leq j \leq D$ we write $A_2 A^*_j A - A A^*_j A_2$ in terms of $\{A_2 E^*_i A - AE^*_i A_2\}_{i=1}^D$. By Lemma 4 and (12) and since $\sum_{i=0}^D E^*_i = I$, we have

$$A_2 A^*_j A - A A^*_j A_2 = |X| \sum_{i=0}^D (A_2 E^*_i A - AE^*_i A_2)S_{ij}$$

$$= |X|(A_2 E_0^* A - AE_0^* A_2)S_{0j} + |X| \sum_{i=1}^D (A_2 E^*_i A - AE^*_i A_2)S_{ij}$$

$$= |X|(A_2(I - (E_1^* + \cdots + E_D^*)) A - A(I - (E_1^* + \cdots + E_D^*)) A_2)S_{0j}$$

$$+ |X| \sum_{i=1}^D (A_2 E^*_i A - AE^*_i A_2)S_{ij}$$

$$= |X| \sum_{i=1}^D (A_2 E^*_i A - AE^*_i A_2)(S_{ij} - S_{0j})$$

$$= |X| \sum_{i=1}^D (A_2 E^*_i A - AE^*_i A_2)(S^\text{alt})_{ij}.$$

Next, for $1 \leq j \leq D$ we write $A_2 E^*_j A - AE^*_j A_2$ in terms of $\{A_2 A^*_i A - AA^*_i A_2\}_{i=1}^D$. By Lemma 4 and (13) and since $A_0^* = I$, we find

$$A_2 E^*_j A - AE^*_j A_2 = |X|^{-1} \sum_{i=0}^D (A_2 A^*_i A - AA^*_i A_2)(S^{-1})_{ij}$$

$$= |X|^{-1}(A_2 A_0^* A - AA_0^* A_2)(S^{-1})_{0j}$$

$$+ |X|^{-1} \sum_{i=1}^D (A_2 A^*_i A - AA^*_i A_2)(S^{-1})_{ij}$$

$$= |X|^{-1} \sum_{i=1}^D (A_2 A^*_i A - AA^*_i A_2)(S^{-1})_{ij}$$

$$= |X|^{-1} \sum_{i=1}^D (A_2 A^*_i A - AA^*_i A_2)(S^{-1})_{ij}^\text{alt}.$$

The result follows.

(ii) – (iv) Similar to the proof of (i).
Define a matrix

\[ S' = \begin{bmatrix} S^\text{alt} & 0 \\ S^\text{alt} & S^\text{alt} \\ 0 & S^\text{alt} \end{bmatrix}, \]

where \( S^\text{alt} \) is from (12). Observe that \( S' \) is \( 4D \times 4D \).

**Lemma 6.** \( \det(S') = (\det(S^\text{alt}))^4 \). Moreover \( S' \) is invertible.

**Proof.** Immediate from the construction of \( S' \). \( \square \)

**Corollary 7.** The matrix \( S' \) is the transition matrix from

\[ \{ A_2 E_i^* A - AE_i^* A_2 \}_{i=1}^D, \{ A_3 E_i^* - E_i^* A_3 \}_{i=1}^D, \{ A_2 E_i^* - E_i^* A_2 \}_{i=1}^D, \{ AE_i^* - E_i^* A \}_{i=1}^D \]

to

\[ \{ A_2 A_i^* A - AA_i^* A_2 \}_{i=1}^D, \{ A_3 A_i^* - A_i^* A_3 \}_{i=1}^D, \{ A_2 A_i^* - A_i^* A_2 \}_{i=1}^D, \{ AA_i^* - A_i^* A \}_{i=1}^D. \]

**Proof.** Immediate from Lemma 5. \( \square \)

## 4 The bilinear form \( \langle \,, \rangle \)

Recall the positive definite Hermitean bilinear form \( \langle \,, \rangle \) on \( \text{Mat}_X(\mathbb{C}) \).

**Lemma 8.** (See [11 Lemma 3.2].) For \( 0 \leq h, i, j, r, s, t \leq D \),

(i) \( \langle E_i^* A_j E_h^*, E_r^* A_s E_t^* \rangle = \delta_{ir} \delta_{js} \delta_{ht} k_{ij} p_{ht}^h \),

(ii) \( \langle E_i^* A_j E_h^*, E_r A_s^* E_t \rangle = \delta_{ir} \delta_{js} \delta_{ht} m_{ij} q_{ht}^h \).

**Corollary 9.** (See [11 Lemma 3.2].) For \( 0 \leq h, i, j \leq D \),

(i) \( E_i^* A_j E_h^* = 0 \) if and only if \( p_{ij}^h = 0 \),

(ii) \( E_i^* A_j E_h^* = 0 \) if and only if \( q_{ij}^h = 0 \).

**Lemma 10.** For \( 0 \leq h, i, j, r, s, t \leq D \) we have

\[ \langle A_i E_j^* A_h, A_r E_s^* A_t \rangle = \sum_{\ell=0}^D k_{\ell} p_{ij}^h p_{rs}^\ell p_{ht}^\ell. \]
Proof. Since $A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h$ and $E_i E_j = \delta_{ij} E_i$ (0 ≤ $h, i, j \leq D$) and by Lemma 8 and (4), we obtain

$$\langle A_i E_j^* A_h, A_r E_s^* A_t \rangle = tr((A_i E_j^* A_h)^t (A_r E_s^* A_t))$$

$$= tr(A_h E_j^* A_i A_r E_s^* A_t)$$

$$= \sum_{\ell=0}^{D} p_{ir}^\ell tr(A_h E_j^* A_i A_r E_s^* A_t)$$

$$= \sum_{\ell=0}^{D} p_{ir}^\ell tr(E_j^* A_i E_s^* A_t A_h)$$

$$= \sum_{\ell=0}^{D} \sum_{w=0}^{D} p_{ir}^\ell P_{ht}^w tr(E_j^* A_i E_s^* A_w A_t A_h)$$

$$= \sum_{\ell=0}^{D} \sum_{w=0}^{D} p_{ir}^\ell P_{ht}^w tr(E_j^* E_s^* A_i E_s^* E_t^* A_w A_t A_h)$$

$$= \sum_{\ell=0}^{D} \sum_{w=0}^{D} p_{ir}^\ell P_{ht}^w tr(E_j^* E_s^* A_i E_s^* E_t^* E_j^* A_w A_t A_h)$$

$$= \sum_{\ell=0}^{D} \sum_{w=0}^{D} p_{ir}^\ell P_{ht}^w tr((E_s^* A_i E_j^*)^t (E_s^* A_w E_j^*))$$

$$= \sum_{\ell=0}^{D} \sum_{w=0}^{D} p_{ir}^\ell P_{ht}^w tr(E_s^* A_i E_j^*, E_s^* A_w E_j^*)$$

$$= \sum_{\ell=0}^{D} \sum_{w=0}^{D} \delta_{tw} p_{ir}^\ell p_{ht}^{w\ell} k_{\ell j}$$

$$= \sum_{\ell=0}^{D} k_{\ell} p_{ir}^\ell p_{ht}^{\ell} p_{ht}^\ell. \quad \square$$

Definition 11. Let $G$ denote the matrix of inner products for

$$A_2 E_i^* A - AE_i^* A_2, A_3 E_i^* - E_i^* A_3, A_2 E_i^* - E_i^* A_2, AE_i^* - E_i^* A,$$

where 1 ≤ $i \leq D$. Thus the matrix $G$ is $4D \times 4D$.

Theorem 12. The entries of $G$ are as follows: For 1 ≤ $i, j \leq D$, where $\phi/2$ is a weighted sum involving the following terms and coefficients:
Proof. By Lemma [10] and using (11)–(15), we obtain

\[
\langle A_2 E_i^* A - AE_i^* A_2, A_2 E_j^* A - AE_j^* A_2 \rangle
\]

\[
= \langle A_2 E_i^* A, A_2 E_j^* A \rangle - \langle A_2 E_i^* A, AE_j^* A_2 \rangle - \langle AE_i^* A_2, A_2 E_j^* A \rangle + \langle AE_i^* A_2, AE_j^* A_2 \rangle
\]

\[
= \sum_{\alpha=0}^{D} k_{\alpha} p_{22}^\alpha p_{ij}^\alpha p_{11}^\alpha - \sum_{\beta=0}^{D} k_{\beta} p_{21}^\beta p_{ij}^\beta p_{12}^\beta - \sum_{\gamma=0}^{D} k_{\gamma} p_{12}^\gamma p_{ij}^\gamma p_{21}^\gamma + \sum_{\eta=0}^{D} k_{\eta} p_{11}^\eta p_{ij}^\eta p_{22}^\eta
\]

\[
= 2 \left( \sum_{\alpha=0}^{2} k_{\alpha} p_{22}^\alpha p_{ij}^\alpha p_{11}^\alpha - \sum_{\beta=1}^{3} k_{\beta} (p_{ij}^{2})^2 p_{ij}^\beta \right)
\]

\[
= 2(k_{0} p_{22}^0 p_{ij}^0 p_{11}^0 + k_{1} p_{22}^1 p_{ij}^1 p_{11}^1 + k_{2} p_{22}^2 p_{ij}^2 p_{11}^1 - k_{1}(p_{ij}^{2})^2 p_{ij}^1 - k_{2}(p_{ij}^{2})^2 p_{ij}^2 - k_{3}(p_{ij}^{2})^2 p_{ij}^3)
\]

\[
= 2(k_{2} p_{ij}^1 + (k_{0} a_{1} a_{2} - k_{b}^2) p_{ij}^1 + k_{2}(c_{2}(b_{1} - 1) - a_{2}(a_{1} + 1) + b_{2}(c_{3} - 1))) p_{ij}^2
\]

\[
= k_{3} c_{3} p_{ij}^3.
\]

Similarly, for 1 \leq h \leq 3,

\[
\langle A_h E_i^* - E_i^* A_h, A_2 E_j^* A - AE_j^* A_2 \rangle
\]

\[
= \langle A_h E_i^* A, A_2 E_j^* A \rangle - \langle A_h E_i^* A, AE_j^* A_2 \rangle - \langle E_i^* A_h, A_2 E_j^* A \rangle + \langle E_i^* A_h, AE_j^* A_2 \rangle
\]

\[
= \sum_{\alpha=0}^{D} k_{\alpha} p_{22}^\alpha p_{ij}^0 p_{01}^\alpha - \sum_{\beta=0}^{D} k_{\beta} p_{12}^\beta p_{ij}^0 p_{02}^\beta - \sum_{\gamma=0}^{D} k_{\gamma} p_{02}^\gamma p_{ij}^0 p_{11}^\gamma + \sum_{\eta=0}^{D} k_{\eta} p_{01}^\eta p_{ij}^0 p_{22}^\eta
\]

\[
= 2(k_{1} p_{h2}^1 p_{ij}^1 - k_{2} p_{h1}^2 p_{ij}^1)
\]

\[
= 2(k_{2} p_{h1}^2 p_{ij}^1 - k_{2} p_{h2}^2 p_{ij}^1)
\]

\[
= 2 k_{2} p_{h1}^2 (p_{ij}^1 - p_{ij}^2).
\]
Similarly, for $1 \leq h, \ell \leq 3$,

\[
\langle A_h E_i^* - E_i^* A_h, A_\ell E_j^* - E_j^* A_\ell \rangle \\
= \langle A_h E_i^*, A_\ell E_j^* \rangle - \langle A_h E_i^*, E_j^* A_\ell \rangle - \langle E_i^* A_h, A_\ell E_j^* \rangle + \langle E_i^* A_h, E_j^* A_\ell \rangle \\
= \langle A_h E_i^* A_0, A_\ell E_j^* A_0 \rangle - \langle A_h E_i^* A_0, A_0 E_j^* A_\ell \rangle - \langle A_0 E_i^* A_h, A_\ell E_j^* A_0 \rangle \\
+ \langle A_0 E_i^* A_h, A_0 E_j^* A_\ell \rangle \\
= \sum_{\alpha=0}^{D} k_\alpha p_{h\alpha}^0 p_{ij}^0 p_{00}^0 - \sum_{\beta=0}^{D} k_\beta p_{h0}^\beta p_{ij}^\beta p_{00}^\beta - \sum_{\gamma=0}^{D} k_\gamma p_{0\alpha}^\gamma p_{ij}^\gamma p_{h0}^\gamma + \sum_{\eta=0}^{D} k_\eta p_{00}^\eta p_{ij}^\eta p_{h\ell}^\eta \\
= 2(k_0 p_{h\alpha}^0 - \delta_{h\alpha} p_{ij}^h) \\
= 2(\delta_{h\ell} k_{ij}^h - \delta_{h\ell} k_{ij}^h) \\
= 2\delta_{h\ell} k_{ij}^h (\delta_{ij}^h k_{ij}^h). 
\]

(16)

The result follows.

In Appendix 2, we give the matrix $G$ for $D = 3$.

**Definition 13.** For $1 \leq i \leq D$ let $B_i$ denote the matrix of inner products for $A_2 A_i^* A - A A_i^* A_2, A_3 A_i^* - A_i^* A_3, A_2 A_i^* - A_i^* A_2, A A_i^* - A_i^* A$.

So the matrix $B_i$ is $4 \times 4$.

**Definition 14.** Let $\tilde{G}$ denote the matrix of inner products for $A_2 A_i^* A - A A_i^* A_2, A_3 A_i^* - A_i^* A_3, A_2 A_i^* - A_i^* A_2, A A_i^* - A_i^* A$,

where $1 \leq i \leq D$. Thus the matrix $\tilde{G}$ is $4D \times 4D$.

**Lemma 15.** The matrix $\tilde{G}$ has the form

\[
\tilde{G} = \begin{bmatrix} B_1 & 0 \\ B_2 & \ddots \\ 0 & \ddots & B_D \end{bmatrix},
\]

where $B_1, B_2, ..., B_D$ are from Definition 13.

**Proof.** Recall that $A_0^*, A_1^*, ..., A_D^*$ form a basis for $M^*$. Therefore the sum $M M^* M = \sum_{i=0}^{D} M A_i^* M$ is direct. The summands are mutually orthogonal by Lemma $D(ii)$. The result follows.

**Lemma 16.** $\det(\tilde{G}) = \prod_{i=1}^{D} \det(B_i)$.

**Proof.** Immediate from Lemma 15.
5 The main result

In this section we obtain our main result, which is Theorem 18.

Lemma 17. The following (i)–(iii) hold.

(i) \( \tilde{G} = (S')^t GS' \).

(ii) \( \det(G) = (\det(S'))^{-2} \det(\tilde{G}) \).

(iii) \( \det(G) = (\det(S^{alt}))^{-8} \prod_{i=1}^{D} \det(B_i) \).

Proof. (i) Immediate from Definition 11, Definition 14, and Corollary 7.

(ii) Follows from (i).

(iii) Follows from (ii) and Lemmas 6, 16.

Theorem 18. Let \( \Gamma \) denote a primitive distance-regular graph with diameter \( D \geq 3 \). Then \( \Gamma \) is \( Q \)-polynomial if and only if \( \det(G) = 0 \).

Proof. To prove the theorem in one direction, assume that \( \Gamma \) is \( Q \)-polynomial with respect to the ordering \( E_0, E_1, ..., E_D \). Write \( A^* = A^*_s \). By Theorem 3.3 and Lemma 11, we obtain \( A^* A_3 - A_3 A^* \in \text{Span} \{ A^* A_2 - A_2 A^*, A^* A_2 - A_2 A^*, A^* A - AA^* \} \). Thus \( A^* A_2 - A_2 A^*, A^* A_3 - A_3 A^* \) are linearly dependent. Now the matrix \( B_1 \) from Definition 13 has determinant zero. Now \( \det(G) = 0 \) by Lemma 17(iii).

For the other direction, assume \( \det(G) = 0 \). By Lemma 17(iii) and since \( S^{alt} \) is invertible, there exists an integer \( t \) (1 \( \leq t \leq D \)) such that \( \det(B_t) = 0 \). Now \( A^*_t A_2 - A_2 A^*_t, A^*_t A_3 - A_3 A^*_t \) are linearly dependent. We will show that \( A^*_t A_3 - A_3 A^*_t \in \text{Span} \{ A^*_s A_2 - A_2 A^*_s, A^*_s A_2 - A_2 A^*_s, A^*_s A - AA^*_s \} \). To do this we show that \( A^*_s A_2 - A_2 A^*_s \) is linearly independent. Suppose not. Then there exist scalars \( a, b, c \), not all zero, such that

\[
ax(A^*_t A_2 - A_2 A^*_t) + bx(A^*_t A_2 - A_2 A^*_t) + c(A^*_t A - AA^*_t) = 0. \tag{17}
\]

Abbreviate \( \theta^*_i = m_x u_i(\theta_i) \) \((0 \leq i \leq D)\). So \( A^*_t = \sum_{i=0}^{D} \theta^*_i E^*_i \). By Lemma 11

\[
\theta^*_i \neq \theta^*_0 \quad (1 \leq i \leq D). \tag{18}
\]

For \( 1 \leq h \leq 3 \) pick \( z \in X \) such that \( \partial(x, z) = h \). Compute the \((x, z)\)-entry in (17).

For \( h = 3 \) we get \( ac_3(\theta^*_1 - \theta^*_0) = 0 \). For \( h = 2 \) we get \( ab_2(\theta^*_1 - \theta^*_0) + c(\theta^*_0 - \theta^*_1) = 0 \). For \( h = 1 \) we get \( ab_1(\theta^*_1 - \theta^*_0) + c(\theta^*_0 - \theta^*_1) = 0 \). Solving these equations we obtain \( a(\theta^*_1 - \theta^*_0) = 0 \) and \( b = 0, c = 0 \). Recall that \( a, b, c \) are not all zero, so \( a \neq 0 \) and \( \theta^*_1 = \theta^*_0 \). Now (17) becomes \( AA^*_s A_2 - A_2 A^*_s A = 0 \). Recall \( c_2 A_2 = A^2 - A_1 A - kI \). We have \( AA^*_s A^2 + kA^*_s A = A^2 A^*_s A + kAA^*_s \). Thus \( [A, AA^*_s A + kA^*_s] = 0 \). For \( 0 \leq i, j \leq D \)
such that \( i \neq j \) we have \( E_i A_i^* E_j(\theta_i \theta_j + k) = 0 \). By Corollary\[^9\] \( E_i A_i^* E_j \neq 0 \) if and only if \( q_{ij}^i \neq 0 \), and in this case \( \theta_i \theta_j + k = 0 \). Since \( q_{ii}^i = 1 \) and \( \theta_i = k \), we have \( k \theta_i + k = 0 \) and hence \( \theta_i = -1 \). Define a diagram with nodes \( 0, 1, \ldots, D \). There exists an arc between nodes \( i, j \) if and only if \( i \neq j \) and \( q_{ij}^i \neq 0 \). Observe that node 0 is connected to node \( t \) and no other nodes. By \[^2\] Proposition 2.11.1] and Lemma\[^1\] the diagram is connected. Thus there exists a node \( s \) with \( s \neq 0 \) and \( s \neq t \) that is connected to node \( t \) by an arc. In other words \( q_{st}^i \neq 0 \). So \( \theta_s \theta_t + k = 0 \) and hence \( \theta_s = k \), a contradiction. Therefore \( AA_i^* A_2 - A_2 A_i^* A, A_i^* A_2 - A_2 A_i^* A, A_i A - A A_i^* \) are linearly independent. So \( A_i^* A_3 - A_3 A_i^* \in \text{Span} \{ AA_i^* A_2 - A_2 A_i^* A, A_i^* A_2 - A_2 A_i^* A, A_i A - A A_i^* \} \). Now by \[^10\] Theorem 3.3] and \[^18\], \( \Gamma \) is a \( Q \)-polynomial with respect to \( E = E_t \). \[ \square \]

6 Appendix 1

Recall the distance-regular graph \( \Gamma \) with diameter \( D \). Recall for \( 0 \leq h \leq D \)

\[
\begin{align*}
    p_{1,h-1}^h &= c_h, & p_{1h}^h &= a_h, & p_{1,h+1}^h &= b_h, \\
    p_{h,h-1}^1 &= \frac{k_1 c_h}{k}, & p_{h1}^1 &= \frac{k_h a_h}{k}, & p_{h,h+1}^1 &= \frac{k_h b_h}{k}.
\end{align*}
\]

We now give \( p_{2j}^h \) for \( h - 2 \leq j \leq h + 2 \).

\[
\begin{align*}
    p_{2,h-2}^h &= \frac{c_{h-1} c_h}{c_2}, \\
    p_{2,h-1}^h &= \frac{c_h (a_{h-1} + a_h - a_1)}{c_2}, \\
    p_{2h}^h &= \frac{c_h (b_h - 1) + a_h (a_h - a_1 - 1) + b_h (c_h + 1)}{c_2}, \\
    p_{2,h+1}^h &= \frac{b_h (a_h + 1 + a_h - a_1)}{c_2}, \\
    p_{2,h+2}^h &= \frac{b_h b_{h+1}}{c_2}.
\end{align*}
\]

We now give \( p_{3j}^h \) for \( h - 3 \leq j \leq h + 3 \).

\[
\begin{align*}
    p_{3,h-3}^h &= \frac{c_{h-2} c_{h-1} c_h}{c_2 c_3}, \\
    p_{3,h-2}^h &= \frac{(a_h - a_2) c_{h-1} c_h + c_{h-1} c_h (a_h - a_2 + a_h - a_1)}{c_2 c_3}, \\
    p_{3,h-1}^h &= \frac{c_{h-1} c_h (b_h - 1) + c_h a_{h-1} (a_h - a_1 - 1) + b_h b_{h-1} (c_h - 1)}{c_2 c_3} \\
    &\quad + \frac{c_h (a_h - a_2) (a_h - a_1 - 1) + b_h b_{h-1} (c_h - 1)}{c_2 c_3} - \frac{b_1 c_h}{c_3}, \\
    p_{3,h+1}^h &= \frac{b_h c_h c_{h+1}}{c_2 c_3}, \\
    p_{3,h+2}^h &= \frac{b_h b_{h+2}}{c_2 c_3}, \\
    p_{3,h+3}^h &= \frac{b_h b_{h+3}}{c_2 c_3}.
\end{align*}
\]
Recall the matrix $G$ from Theorem 12. In this appendix we give $G$ for $D = 3$.

**Example 19.** Assume $D = 3$. The rows and columns of $G$ are indexed by the following matrices, in the specified order:

- block 1: $A_3E_1^* - E_1^*A_3$, $A_3E_2^* - E_2^*A_3$, $A_3E_3^* - E_3^*A_3$
- block 2: $A_2E_1^* - E_1^*A_2$, $A_2E_2^* - E_2^*A_2$, $A_2E_3^* - E_3^*A_2$
- block 3: $AE_1^* - E_1^*A$, $AE_2^* - E_2^*A$, $AE_3^* - E_3^*A$
- block 4: $A_2E_1^* - AE_1^*A_2$, $A_2E_2^* - AE_2^*A_2$, $A_2E_3^* - AE_3^*A_2$

So the matrix $G$ is $12 \times 12$. $G$ has the form

$$G = \begin{bmatrix}
X & 0 & 0 & S \\
0 & Y & 0 & T \\
0 & 0 & Z & U \\
S & T & U & W
\end{bmatrix},$$

where each block is a $3 \times 3$ symmetric matrix as shown below.

$$X = \begin{bmatrix}
2k_3k & -2k_3c_3 & -2k_3a_3 \\
-2k_3c_3 & 2k_3(k_2 - p_{22}^3) & -2k_3p_{23}^3 \\
-2k_3a_3 & -2k_3p_{23}^3 & 2k_3(k_3 - p_{33}^3)
\end{bmatrix},$$

$$Y = \begin{bmatrix}
2k_2(k - c_2) & -2k_2a_2 & -2k_2b_2 \\
-2k_2a_2 & 2k_2(k_2 - p_{22}^2) & -2k_2p_{23}^2 \\
-2k_2b_2 & -2k_2p_{23}^2 & 2k_2(k_3 - p_{33}^2)
\end{bmatrix}.$$
From Appendix 1, we find
\[
Z = \begin{bmatrix}
2k(k - a_1) & -2kb_1 & 0 \\
-2kb_1 & 2k(k_2 - p_{22}^1) & -2kp_{23}^1 \\
0 & -2kp_{23}^1 & 2k(k_3 - p_{33}^1)
\end{bmatrix},
\]
\[
S = \begin{bmatrix}
2k_2b_2(a_1 - c_2) & 2k_2b_2(b_1 - a_2) & -2k_2b_2^2 \\
2k_2b_2(b_1 - a_2) & 2k_2b_2(p_{22}^1 - p_{22}^2) & 2k_2b_2(p_{23}^1 - p_{23}^2) \\
-2k_2b_2^2 & 2k_2b_2(p_{23}^1 - p_{23}^2) & 2k_2b_2(p_{33}^1 - p_{33}^2)
\end{bmatrix},
\]
\[
T = \begin{bmatrix}
2k_2a_2(a_1 - c_2) & 2k_2a_2(b_1 - a_2) & -2k_2a_2b_2 \\
2k_2a_2(b_1 - a_2) & 2k_2a_2(p_{22}^1 - p_{22}^2) & 2k_2a_2(p_{23}^1 - p_{23}^2) \\
-2k_2a_2b_2 & 2k_2a_2(p_{23}^1 - p_{23}^2) & 2k_2a_2(p_{33}^1 - p_{33}^2)
\end{bmatrix},
\]
\[
U = \begin{bmatrix}
2k_2c_2^1(a_1 - c_2) & 2k_2c_2(b_1 - a_2) & -2k_2c_2b_2 \\
2k_2c_2(b_1 - a_2) & 2k_2c_2(p_{22}^1 - p_{22}^2) & 2k_2c_2(p_{23}^1 - p_{23}^2) \\
-2k_2c_2b_2 & 2k_2c_2(p_{23}^1 - p_{23}^2) & 2k_2c_2(p_{33}^1 - p_{33}^2)
\end{bmatrix}.
\]

The matrix $\mathbb{W}$ is symmetric with entries
\[
\mathbb{W}_{11} = 2(k^2k_2 + (k_2a_1a_2 - kb_1^2)a_1 + (k_2(b_1 - 1) + a_2(a_2 - a_1 - 1) \\
+ b_2(c_3 - 1)) - k_2a_2^2c_2),
\]
\[
\mathbb{W}_{12} = 2((k_2a_1a_2 - kb_1^2)b_1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)) \\
- k_2a_2^2a_2 - k_3c_3^2),
\]
\[
\mathbb{W}_{13} = 2((k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)) - k_2a_2^2b_2 - k_3c_3^2a_3),
\]
\[
\mathbb{W}_{22} = 2(kk_2^2 + (k_2a_1a_2 - kb_1^2)p_{22}^1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) \\
+ b_2(c_3 - 1)) - k_2a_2^2p_{22}^1 - k_3c_3^2p_{22}^1),
\]
\[
\mathbb{W}_{23} = 2((k_2a_1a_2 - kb_1^2)p_{23}^1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)) \\
- k_2a_2^2p_{23}^1 - k_3c_3^2p_{23}^1),
\]
\[
\mathbb{W}_{33} = 2(kk_2^2k_3 + (k_2a_1a_2 - kb_1^2)p_{33}^1 + (k_2(c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) \\
+ b_2(c_3 - 1)) - k_2a_2^2p_{33}^1 - k_3c_3^2p_{33}^1).
\]

From Appendix 1, we find
\[
p_{22}^1 = \frac{k_2a_2}{k}, \quad p_{22}^2 = \frac{c_2(b_1 - 1) + a_2(a_2 - a_1 - 1) + b_2(c_3 - 1)}{c_2}, \quad p_{22}^3 = \frac{c_3(a_2 + a_3 - a_1)}{c_2},
\]
\[
p_{23}^1 = \frac{k_2b_2}{k}, \quad p_{23}^2 = \frac{b_2(a_3 + a_2 - a_1)}{c_2}, \quad p_{23}^3 = \frac{c_3(b_2 - 1) + a_3(a_3 - a_1 - 1) - b_3}{c_2},
\]
\[
p_{33}^1 = \frac{k_3a_3}{k}, \quad p_{33}^2 = \frac{b_2(c_3(b_2 - 1) + a_3(a_3 - a_1 - 1) - b_3)}{c_2c_3}, \quad p_{33}^3 = \frac{c_3(b_3 - 1) + a_3(a_3 - a_1 - 1) - b_3}{c_2c_3} - \frac{b_1a_3}{c_3}.
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