Minkowski gauges and deviation measures

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Abstract

We propose to derive deviation measures through the Minkowski gauge of a given set of acceptable positions. We show that given a suitable acceptance set, any positive homogeneous deviation measure can be accommodated in our framework. In doing so, we provide a new interpretation for such measures, namely, that they quantify how much one must shrink a position for it to become acceptable. In particular, the Minkowski gauge of a set which is convex, stable under scalar addition, and radially bounded at non-constants, is a generalized deviation measure. Furthermore, we explore the relations existing between mathematical and financial properties attributable to an acceptance set on the one hand, and the corresponding properties of the induced measure on the other. In addition, we show that any positive homogeneous monetary risk measure can be represented through Minkowski gauges. Dual characterizations in terms of polar sets and support functionals are provided.

Keywords: Risk measures, Deviation measures, Acceptance sets, Convex analysis, Minkowski gauges.

1 Introduction

In modern financial theory — since the iconic paper of Markowitz (1952) — the standard deviation has been the measure most used to quantify the risk of a financial position. More recently, due to the increasing necessity of paying attention to tail risks, monetary risk measures — which respect monotonicity and translation invariance (cash additivity) — came to light. Following the seminal paper of Artzner et al. (1999), theoretical properties that are desirable for a risk measure have been widely studied. Nonetheless, there is no consensus so far about which are the best properties (axioms) such a risk measure ought to satisfy, and even less regarding the best way to measure financial risk. The axiomatic approach of Rockafellar et al. (2006a) represents a landmark in the literature, setting the tone for recent developments with the introduction of generalized deviation measures — generalizations of the standard deviation and similar measures. A deviation measure is a functional $D$ defined on a space $\mathcal{X}$ comprised of a suitable class of random variables, that captures the degree of “non-constancy” (dispersion) of a financial position. Such measures have been proved useful in financial problems as can be seen in Rockafellar et al. (2006b), Pflug (2006), Grechuk et al. (2009), Rockafellar and Uryasev (2013) among others.

Owing to the aforementioned lack of consensus regarding an appropriate way to measure risk, a handful of coherent and convex risk measures have been proposed and, as a dénouement, many generalized and convex deviation measures as well. Furthermore, due to the importance of variability, Righi and Ceretta (2016), Berkhouch et al. (2018) and Righi (2019) bring forward some novel convex risk measures, in the sense of Föllmer and Schied (2002), which explicitly take variability into account. Empirically, convex risk measures in the former class displayed consistently better performance for optimal portfolio strategies, as seen in the work of Righi and Borenstein (2018).

In the present paper we propose a novel way to obtain deviation measures, by using the Minkowski gauge as a means to ascribe a deviation to an arbitrary set of acceptable positions — and, importantly, we show that every deviation measure can be thought of in this framework. One common interpretation for deviation measures is that they quantify the distance between a random variable and constancy; our approach offers an alternative understanding: that they capture the amount that an agent must shrink a given position for it to be considered acceptable. Thus, in our quest to define a new class of deviation measures, we first have a glance into the framework of acceptance sets related to risk measures, whose
purpose is to define the range of positions which have an acceptable risk. Artzner et al. (1999) were the first to propose the concept, after which Delbaen (2002), Frittelli and Scandolo (2006), Artzner et al. (2009) among others, have deepened the literature.

In the preceding framework, we consider a vector space $\mathcal{X}$ comprised of a suitable class of random variables (feasible outcomes). An element $X \in \mathcal{X}$ is to be interpreted as a real-valued, random result of a given asset, corresponding to a certain position whose realized value depends on the outcome $\omega$ of the market; in this context, $X(\omega) > 0$ is a gain, whereas the reverse inequality corresponds to a loss. We would like to highlight the generality of our framework: we impose little restrictions on the space $\mathcal{X}$, namely, we ask for it to be a topological vector space, which include the most used spaces in the literature, such as the $L^p$ and Orlicz spaces. Given any functional $f: \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, one can interpret the value $f(X)$ as representing the financial risk of a position $X$ — however, this interpretation is too general and so it is customary, in the literature, to restrict attention to two broad classes of functionals, namely the class of monetary risk measures and the class of deviation measures\footnote{The tenured reader is probably familiar with the fact that the terminology monetary risk and deviation “measure” is misleading as the objects under study are not bona fide measures (as in “$\sigma$-finite measure” for instance) but rather functionals (possibly non-linear) on a topological vector space.}. Regarding the former it is well known that, under some weak assumptions on a set $A \subseteq \mathcal{X}$ of acceptable positions, a monetary risk measure can always be expressed in the form $\rho(X) = \inf\{m \in \mathbb{R} : X + m \in A\}$. The main message of this paper is that, under possibly different assumptions on $A$, a deviation measure takes the form $D(X) = \inf\{m > 0 : m^{-1}X \in A\}$. Thus, there is a straightforward connection between deviation measures and the concept of a Minkowski functional (or gauge) which is of utmost importance in the theory of topological vector spaces. This has the important consequence that such deviation measures are, in a sense, a type of generalized seminorm. For precise definitions, see Sections 4 and 6.

The underlying acceptance set $A$ that we have in mind is, at least in principle, quite arbitrary — a convenient choice is to take $A$ as an acceptance set (a sub-level set, that is) corresponding to some pre-specified deviation measure. Nonetheless, there is no all-encompassing approach so far by which one can introduce such acceptance sets for deviation measures. The aforesaid lack in the literature stems, mainly, from the fact that the classical notion of an acceptance set requires, indispensably, that the involved measures respect the axioms of translation invariance and monotonicity, whereas deviation measures are translation insensitive and, in general, not monotone. Besides, deviation measures are, by definition, non-negative, while the classical approach for monetary risk measures requires the position to have a risk lower than 0. Therefore, simply replacing a monetary risk measure by a deviation risk measure $D$ is of no use, as in this case a set of the form $A = \{X : D(X) \leq 0\}$, is too restrictive — after all, it would consider only constants to be acceptable! Secondly, we are specially interested in recovering the deviation measure corresponding to a given acceptance set, and the set $\{m : X + m \in A\}$, where $A$ is induced by a deviation measure, would be empty for any $X \notin A$ and equal to the whole real line if $X \in A$, even if 0 is replaced by a positive constant in the definition of $A$. Hence, for the best of our knowledge, there is no direct adaptation, from the preexisting notion of an acceptance set associated to a risk measure, that would allow one to encompass deviation measures.

We are set out to fill this gap, by proposing an acceptance set for deviation measures, in the form of a sub-level set $A_m = \{X : D(X) \leq k\}$. At any rate, the highlight of our study is to provide a comprehensive approach for recovering the underlying deviation measure from any given set of acceptable positions. Drawing inspiration from the well known representation of a monetary risk measure as an infimum over the set of acceptable cash additions on a given position, our approach consists in obtaining a deviation functional by taking the infimum over the set of acceptable positive expansions and contractions of such a position. This change in perspective is required because deviation measures are commonly characterized by at least two axioms, namely, non-negativity and translation insensitivity, but unfortunately these cannot be used to reduce the dispersion of a financial position. Therefore, in order to extract a deviation measure from a given acceptance set, other axioms are likely going to be required, for example convexity, quasi-convexity, etc.

Under convexity, one possibility is to adapt the approaches put forth by Frittelli and Scandolo (2006) and Artzner et al. (2009), where there are multiple eligible assets. These aim to recover the underlying measure of a set $A$ via $p_A(X) = \inf\{\pi(Y) : X + Y \in A\}$, where $\pi : \mathcal{C} \to \mathbb{R}$ is the cost to execute $Y$, and $\mathcal{C}$ is a set of feasible strategies. However, the preceding infimum yields a measure which is neither translation insensitive nor non-negative — not a problem if one has risk measures in mind, but an impassable hurdle if the aim is to obtain measures of deviation. An alternative within reach is to assume that there exists some (constant) risk-free asset $c$, in which case — for a given position $X$ and an acceptance set $A$ — we can use convexity to reduce the position’s risk, up to the point where it becomes acceptable; in other
words, by recovering the underlying measure of $A$ via $D_A(X) = \inf\{\lambda \in [0,1): (1 - \lambda)X + \lambda c \in A\}$. In this setting, if $A_D$ is the sub-level set $\{X \in \mathcal{X}: D(X) \leq 1\}$ corresponding to some previously given convex deviation measure $D$, then we have the equivalences

$$D_{A \downarrow D}(X) = \inf\{\lambda \in [0,1): D((1 - \lambda)X + \lambda c) \leq 1\} = \inf\{\lambda \in [0,1): D((1 - \lambda)X) \leq 1\}.$$

The quantity $D_{A \downarrow D}(X)$ can be understood as the amount by which we must shrink the position $X$ until it becomes acceptable. There is an important drawback in this approach, however — namely, that any two acceptable positions will always have the same deviation, whereas in general we want the “better” position to have a smaller deviation.

In view of the above, we see that the idea of shrinking and expanding a position is closely related to positive homogeneity. Indeed, under positive homogeneity we can see the mapping $\lambda \mapsto D(\lambda X)$, where $\lambda > 0$, as controlling simultaneously the size and the deviation of the position $X$. It appears only natural, then, to stipulate that a measure of ‘non-constancy’ be positive homogeneous, even more so considering that most of the prominent deviation measures found in the literature do satisfy this requirement — and, besides, the relevant deviation measures that are not positive homogeneous, such as the variance and the entropic deviation [Föllmer and Knispel, 2011], are only one transformation away from positive homogeneity (for instance, the standard variation in relation to the variance, etc.). In summary, positive homogeneity should translate into the following two properties for the corresponding acceptance set: in case the position $X$ does not lie in $A_D$, we should be able to shrink the position until it “fits” in the set, and if $X \in A_D$, then we should be able enlarge the position up to a limit where it still “fits” in the set. Last but not least, we assume throughout that it is possible to invest the excess capital resulting from shrinkage (similarly, to borrow the demanding capital for the enlargement) into a (constant) risk-free asset, i.e. we require our acceptance set to be stable with respect to translation by a constant. This can be interpreted as follows: adding a constant to a given position has no effect on whether the latter is acceptable or not. This property is true, in particular, whenever our acceptance set $A$ is generated by a deviation measure, in which case (owing to translation insensitivity) allocation of capital in a risk-free manner leads to no change in the deviation of the position: in other terms, we have that $D(\lambda X + (1 - \lambda)c) = D(\lambda X)$, where $\lambda$ is the amount to be shrunk and $c$ is the risk-free asset.

Monetary risk measures are representable as the minimum translation factor (corresponding to cash addition/subtraction) which makes a given position acceptable. With deviation measures, on the other hand, we propose to consider the least scaling factor (corresponding to expansion/shrinkage) which makes said position acceptable. The function which describes the latter concept is the so called Minkowski gauge, which — for a given star-shaped acceptance set $A$ — assigns a non-negative real number

$$f_A(X) = \inf\{m \in \mathbb{R}_+^\ast: m^{-1}X \in A\}.$$ (1)

to each $X \in \mathcal{X}$. See Figure 1.
Figure 1: Representation of the Minkowski gauge $f_A$ of a set $A$. 

[Diagram of a set $A$ with a point $X$ outside the set, connected by a line to the origin 0.]
Acceptance sets generated by a deviation measure $D$ at a certain level $k$ (that is, sub-level sets of the form $\{X: D(X) \leq k\}$), may have the financial intuition that $k$ represents an agent’s coefficient of aversion with respect to the deviation. Obviously, an agent with greater $k$ has higher compliance regarding exposure to dispersion. Therefore, to compare positions of agents with different degrees of aversion, we must bring the deviation measure to the same level for all agents — this is the case even if the distinct agents agree about which deviation measure should be used. Since each set is uniquely determined by a positive number $k$, it is then possible, because of positive homogeneity, to normalize each set by multiplying it by the constant $k^{-1}$. In line with the preceding heuristics, we propose to recover the normalized deviation measure through the identity $D(X) = k \cdot \inf \{m \in \mathbb{R}_+: m^{-1}X \in A^k_D\}$.

Pflug and Romisch (2007) previously studied deviation measures generated by Minkowski gauges as in equation (1). The authors focused on sets of the form $A = \{X \in \mathcal{F}: \mathbb{E}(h \circ X) \leq h(1)\}$ for a convex, symmetric, non-negative real function $h$ with $h(0) = 0$ and $0 < h(x) < \infty$ for $x \neq 0$, thus establishing a relation between financial risk and Orlicz norms. In particular, if $h$ is invertible on $[0, +\infty)$, then the set $A$ is a sub-level set of the form $A^h_f$, with the functional $f$ constrained to be of the form $f(X) = h^{-1}(\mathbb{E}(h \circ X))$. They in any event propose deviations of the form $f_A(X - \mathbb{E}X)$ and $f_A((X - \mathbb{E}X)^{-})$, and explore to exhaustion the different representations of this kind of functional. A homologous approach was studied in Bellini et al (2018), who consider return risk measures $\tilde{\rho}$, which are analogous to monetary risk measures but applied to the return of a position, not its profit/loss. Such a functional is defined on the strictly positive returns $(\{X \in \mathcal{F}^+ : \mathbb{E}(h \circ X) \leq h(1)\})$. Indeed, given a suitable acceptance set $A = A^h_{\tilde{\rho}}$, the return risk measure can be precisely recovered through the Minkowski gauge of $A$, i.e. $\tilde{\rho} = f_{A^h_{\tilde{\rho}}}$.

When it comes to the interplay between Minkowski functionals and acceptance sets, two important questions arise: (i) given an arbitrary functional $f: \mathcal{F} \to \mathbb{R}_+ \cup \{+\infty\}$, under what conditions can we find a set $A \subseteq \mathcal{F}$ such that $f = f_A$? (ii) if $f$ is of the form $f = f_A$ for some $A \subseteq \mathcal{F}$, what is the relation between $A$ and the sub-level set $A^h_f$? In particular, answering the preceding questions will tell us when it is the case that $f = f_A^h$. Fortunately, answers to both questions are readily available: $f$ is of the form $f = f_A$ for some $A$ if and only if it is positive homogeneous, and any $A$ satisfying $\{X: f(X) < 1\} \subseteq A \subseteq \{X: f(X) \leq 1\}$ will do. Additionally, if $A$ is closed and star-shaped, then one has necessarily $A = A^h_f$ — see Lemma 3.8. More important, then, is to establish relations between properties of a positive homogeneous functional $f$ and properties of the set $A^h_f$ and, reciprocally, between properties of a set $A$ and properties of the corresponding Minkowski gauge $f_A$. Of particular interest to us — as we focus our attention on $f$’s that are risk functionals — are questions like: what does $A$ have to be like to ensure that $f_A$ is a deviation measure? A convex deviation measure? (And so on). Similarly, if $f$ is a deviation measure, is it the gauge of some set? Of $A^h_f$? What can we say about $A^h_f$? What if $f$ is a convex deviation measure? (And so on). See Theorem 5.3.

Our main goal, then, is to attain a generalized deviation measure $f_A \equiv D_A$ that is generated by a given acceptance set $A$, from which we could know how much we ought to shrink a position in order to make it acceptable (i.e. to make it “enter” $A$). Here, by generated we mean that it is the Minkowski gauge of the acceptance set. In this context, it is of crucial importance to better understand the manner whereby each of the desired properties to be satisfied by the underlying acceptance set impacts the associated deviation measure. We may be willing to impose, for instance, that our acceptance set be star-shaped, because scaled down positions should have lower dispersion, and less dispersion should be “more acceptable”. Or we could ask that $A$ be stable under scalar addition, understanding in this case that we do not care about the location of a random variable, being interested only in its dispersion, its asymmetry, etc. In other words, under stability for scalar addition we treat the positions $X$ and $X + c$ equally: one is acceptable if and only if the other one is as well, and, as a consequence, the Minkowski gauge of such an $A$ will be a translation insensitive functional. Last but not least, if we are inclined towards the requirement that $A$ be radially bounded at non-constants, then our inclination actually means that the constant positions are the only ones we allow to be indefinitely scaled up. All in all, an acceptance set $A$ which is star-shaped, stable under scalar addition, and radially bounded at non-constants turns out to generate a deviation measure $D_A \equiv f_A$. This is the content of our Proposition 4.2. If, in addition, we are interested in considering the effects of diversification, then we should be willing to assume that $A$ is a convex set; in this case $D_A$ is a convex functional, which tells us that “diversification is good”. We see, then, that convexity, together with the previous conditions, hand us a generalized deviation measure — This is the message of Proposition 4.3. Furthermore, if we understand that the most conservative quantification of risk is provided by the Lower Range deviation measure $LR(X) := \mathbb{E}[X] - \text{ess inf} X$,
then our acceptance set should include the sub-level sets \( A^1_{LR} \) (Proposition 4.13). Or, if we are only concerned with distributional/statistical attributes of financial outcomes, then our acceptance set \( A \) should be required to be law invariant — in this case, the Minkowski functional does its magic again and delivers a law invariant deviation measure \( f_A \) (Proposition 4.10). Last but not least, if we want an acceptance set that does not reward (nor punishes) diversification with comonotone pairs, we should impose an acceptance set which is comonotone, in which case our deviation would be comonotone additive (Corollary 4.7).

The remainder of this paper is structured as follows: Section 2 introduces our notation and framework, and also provides the underlying financial intuition backing set properties that shall be used throughout this paper. We also state some relevant results concerning relations between attributes of sets in a (topological vector spaces). In Section 3 we focus on the Minkowski gauge as a general functional, recalling some definitions from the literature, as well as introducing new ones; we restate some important results from existing work, and, finally, we also provide new results. In section 4 we explore the Minkowski gauge as a deviation measure, developing the role of specific properties for the set and its impact on the properties for the generated functional. In section 5 we develop the idea of an acceptance set generated by a deviation measure by exploring the reverse implications from section 4. Section 6 is the icing on the cake: we establish links existing between monetary risk measures and Minkowski gauges through our proposed framework, and obtain a characterization of a class of positive homogeneous monetary risk measures, in terms of the notion of a risk system, which we introduce.

2 Preliminaries and some set properties

In all that follows, \((\Omega, \mathcal{F}, P)\) is a fixed, underlying probability space. Every equality and inequality is to be understood as holding \(\mathcal{P}\)-almost surely. As usual, we write, for \(p \in (0, \infty)\), \(L^p \equiv L^p(\Omega, \mathcal{F}, P) := \{\text{the set of all } (\mathcal{P}\text{-equivalence classes of}) \text{ random variables } X \text{ such that } \mathbb{E}|X|^p < \infty\}\), whereas \(L^0 \equiv L^0(\Omega, \mathcal{F}, P) := \{\text{the set of all } (\mathcal{P}\text{-equivalence classes}) \text{ of random variables on } (\Omega, \mathcal{F}, P)\}\), and \(L^\infty \equiv L^\infty(\Omega, \mathcal{F}, P) := \{\text{the set of all } (\mathcal{P}\text{-equivalence classes of}) \text{ random variables which are } \mathcal{P}\text{-essentially bounded}\}\). We work with a Hausdorff topological vector space \(\mathcal{X}\), and assume beforehand that the inclusions \(L^0 \supseteq \mathcal{X} \supseteq L^\infty \) hold.

The generic elements of \(\mathcal{X}\) are denoted by \(X, Y, Z\), etc, and are to be interpreted as the random result of a \emph{financial position}, which we assume throughout to be perfectly liquid and discounted by a risk-free rate. \(\mathcal{X}'\) denotes the topological dual of \(\mathcal{X}\), and we shall write \(\langle X, Y \rangle := X'(Y)\) whenever \(X \in \mathcal{X}\) and \(X' \in \mathcal{X}'\): notice that this notation gives \(\langle X, Y \rangle = \mathbb{E}XY\) if \(X \in L^\infty\) and \(Y \in L^p\), with \(1 \leq p < \infty\) and \(p^{-1} + q^{-1} = 1\), via the identification \(L^q \equiv (L^p)'\). Furthermore, we write \(\langle \mathcal{X}, \mathcal{X}' \rangle = \mathcal{X} \times \mathcal{X}'\), and call this construct the \emph{dual pair}. With this notation and terminology, the mapping \(\langle X, X' \rangle \mapsto \langle X', X \rangle\) gives a bilinear functional defined on the dual pair, one that separates points of both \(\mathcal{X}\) and \(\mathcal{X}'\). The positive and negative parts of an element \(X \in \mathcal{X}\) are denoted by \(X^+ := \max(X, 0)\) and \(X^- := \min(-X, 0)\), respectively. We define the cone \(\mathcal{X}_+\) of non-negative positions as \(\mathcal{X}_+ := \{X \in \mathcal{X}: X \geq 0\}\) (this is the range of \(X \mapsto X^+\)), and similarly \(\mathcal{X}_- := \{X \in \mathcal{X}: X \leq 0\}\). With a slight abuse of notation, we consider the inclusion \(\mathbb{R} \subseteq \mathcal{X}\) by identifying each \(x \in \mathbb{R}\) with the equivalence class of random variables equal to \(x\) almost surely. A pair of random variables is said to be \emph{comonotone} if the inequality

\[
(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0, \quad \omega, \omega' \in \Omega
\]

holds \(\mathbb{P} \otimes \mathbb{P}\)-almost surely. As usual, \(F_X\) represents the cumulative distribution function of a random variable \(X\), while \(F_X^{-1}\) denotes its left quantile function, that is to say, \(F_X^{-1}(\alpha) := \inf\{q \in \mathbb{R}: F_X(q) \geq \alpha\}\). We write \(X =_D Y\) whenever \(X\) and \(Y\) are equal in distribution, a fact which we also express by writing \(Y \in \mathcal{L}_X\) (and this already defines \(\mathcal{L}_X\) implicitly). As mentioned, we denote the property of \(X\) being almost surely greater than \(Y\) by \(X \geq Y\), while for a generic partial order \(\succeq\) we write \(X \succeq Y\), also adopting the obvious convention that the notation \(X \succeq Y\) means precisely that \(X \succeq Y\). If not clear from context, we shall mention explicitly the partial order under consideration. We say that \(X\) is \emph{greater than} \(Y\) in the \emph{dispersive order of distributions}, written \(Y \preceq_D X\), if the inequality \(F_X^{-1}(u) - F_Y^{-1}(v) \geq F_Y^{-1}(u) - F_X^{-1}(v)\) holds for every \(0 < v < u < 1\). In all that follows, \(\mathbb{R}_+\) denotes the set \([0, +\infty)\), whereas \(\mathbb{R}^*_+ := (0, +\infty)\).

Given \(A, B \subseteq \mathcal{X}\) we define the set \(A + B\) by saying that \(Z \in A + B\) if and only if \(Z = X + Y\) for some \(X \in A\) and some \(Y \in B\). Similarly, for a \(\lambda \in \mathbb{R}\), we write \(Z \in \lambda A\) if and only if \(Z = \lambda X\) for some \(\lambda \in \Lambda\) and some \(X \in A\). For simplicity, we write \(\lambda A := \{\lambda x: x \in A\}\) and \(\lambda A := \{\lambda x: x \in A\}\) when one of the

\footnote{These inclusions are assumed to hold algebraically — no \textit{a priori} assumption is made on the relation between the topologies involved.}
involved sets is a singleton: in particular, we define the ray of $X \in \mathcal{X}$ as $R_X := \mathbb{R}_+^* X$. In the same manner, $X + A := \{X\} + A$, etc. We also denote by $\text{bd}(A)$, $\text{int}(A)$, $\text{cl}(A)$, $\text{conv}(A)$, $\text{cl-conv}(A)$, $\text{cone}(A)$, and $\mathcal{A}^c$ respectively the boundary, interior, closure, convex hull, closed convex hull, conic hull, closed conic hull and the complement of $A$. Any $A \subseteq \mathcal{X}$ is called an \textit{acceptance set}, and we say that a given position $X$ is \textbf{acceptable} (w.r.t. $A$) if and only if is an element of $A$.

We now focus on properties for sets that are considered alongside the text. We make an effort to clarify the financial intuition behind each of these attributes. Since not every attribute appearing in our axiom scheme is fundamental in functional and convex analysis — and thus it is likely that some of these attributes are unknown to the reader —, we shall resort to figures as a means to illustrate them and help to develop the intuition. In these figures, we are considering $\Omega$ as the \textit{binary market}, i.e., $\Omega = \{0, 1\}$; in this setting, one can take $\mathcal{X} = L^0(\Omega) \subseteq \mathbb{R}^2$, where the latter equivalence is given via the identification of a random variable $X$ with the ordered pair $(X(0), X(1))$ in the Cartesian plane. Importantly, notice that in this context the inclusion $\mathbb{R} \subseteq \mathbb{R}^2$ corresponds to the diagonal $\{(u, v) \colon v = u, u \in \mathbb{R}\}$, which may be be different from what the reader has in mind at first thought.

\textbf{Definition 2.1.} Let $A \subseteq \mathcal{X}$ and $\{A(k) \colon k \in \mathbb{R}\} \subseteq 2^\mathcal{X}$. We say that

\begin{enumerate}[(i)]
\item \textbf{(Law invariance)} $A$ is \textbf{law invariant} if $X \in A$ and $X =_d Y$ implies $Y \in A$.

This means that a financial position having the same distribution of a given, acceptable position is also acceptable; that is, when deciding whether a position is to be deemed acceptable, we only care about its statistical properties.

\item \textbf{(Monotonicity)} $A$ is \textbf{monotone} with respect to a given partial order $\preceq$ if the conditions $X \in A$ and $X \preceq Y$ imply $Y \in A$. $A$ is said to be \textbf{anti-monotone} (w.r.t $\preceq$) if the conditions $Y \in A$ and $X \preceq Y$ imply $X \in A$. For convenience, we say that $A$ is $\preceq$-\textbf{monotone} whenever $A$ is monotone with respect to $\preceq$, and similarly for anti-monotonicity.

Under monotonicity, a position is deemed acceptable whenever a “worse” (smaller) one is also acceptable (from a financial perspective, this is not very interesting). Anti-monotonicity, on the other hand, captures the notion that being “bigger” according to some partial order is actually worse, e.g. the dispersive order of distribution. Under anti-monotonicity, then, a position is regarded as acceptable whenever a “better” position is also acceptable. Note that if $A$ is monotone then $A^c$ is anti-monotone: indeed, letting $A$ be monotone and $X \preceq Y$, then $X \in A$ implies that $Y \in A$, which is equivalent to say that $Y \not\in A$ implies that $X \not\in A$, thus yielding anti-monotonicity of $A^c$.

\item \textbf{(Conicity)} $A$ is a \textbf{cone with vertex at the origin}, or simply a \textbf{cone}, if $\lambda X \in A$ for every $\lambda \geq 0$ and every $X \in A$. $A$ is said to be a \textbf{cone with vertex at} $V \in \mathcal{X}$ if $A$ is of the form $A = V + C$ for some cone $C$. A cone with vertex at $V$ is \textbf{degenerate} if it is a singleton; otherwise, it is said to be a \textbf{proper cone with vertex at} $V$.

Conicity means that if a position is acceptable, then every non-negative multiple of the position is deemed acceptable as well. This is a reasonable assumption when we are concerned with losses, but not so much for dispersion, as it allows scaling any acceptable position up in an unbounded fashion.

\item \textbf{(Radial boundedness)} $A$ is \textbf{radially bounded} if, for every non-zero $X \in A$, there is some $\delta_X \in (0, \infty)$, such that $\delta X \not\in A$ whenever $\delta \in [\delta_X, \infty)$. The set $A$ is said to be \textbf{radially bounded at non-constants} if $A \setminus \mathbb{R}$ is radially bounded.

Radial boundedness is, in a sense, the opposite of conicity: it says that there is always a bound on how much it is possible to scale up a position while keeping it acceptable. It means precisely that $A$ contains no cone (except for the trivial cone $\{0\}$) — see Figure 2 for an example. As constants have no dispersion, financially it makes sense to always consider them acceptable; that is to say, when we are mainly concerned with positions that are acceptable with respect to their dispersion, it is fruitful to limit the scaling up of all positions except for constants. In this case we should require that $A$ be radially bounded at non-constants. Figure 3 shows a set which is radially bounded at non-constants but it is not radially bounded.
(v) (Stability under scalar addition) $A$ is **stable under scalar addition** if $A + \mathbb{R} = A$, that is, if $X + c \in A$, for all $X \in A$ and $c \in \mathbb{R}$.

In our framework, as scalar addition does not affect the dispersion of a financial position, it is a reasonable property to be imposed on acceptance sets — see Figure 3 for an example.

It is important to note that stability under scalar addition is incompatible (from a financial perspective) with monotonicity (or anti-monotonicity) with respect to some partial orders of interest, such as the “almost surely $\geq$” order. To illustrate, assume $A$ is $\leq$-monotone, stable under scalar addition and that $0 \in A$. Then $L^\infty \subseteq A$: indeed, since $0 \in A$, stability under scalar addition immediately entails $\mathbb{R} \subseteq A$. Then, for any $Y \in L^\infty$ it follows that $Y \geq \text{ess inf} Y \in \mathbb{R} \subseteq A$, so monotonicity gives us $Y \in A$. Clearly such an $A$ is way too large to be of any practical interest from a financial perspective. Also, stability under scalar addition is clearly incompatible with radial boundedness, as a non-empty acceptance set that respects stability under scalar addition contains at least the whole real line, and hence it cannot be radially bounded. However, a set which is radially bounded at non-constants, such as the one in Figure 3 undoubtedly can accommodate stability under scalar addition.
Figure 3: A set $A$ which is absorbing, radially bounded at non-constants, stable under scalar addition and star-shaped. The subspace $\mathbb{R}$ of constant random variables is represented by the thick black diagonal.

(vi) (Absorbency) $A$ is absorbing if, for every $X \in \mathcal{X}$, there is some $\delta_X > 0$ such that $[0, \delta_X]X \subseteq A$, that is, if $0 \leq \lambda \leq \delta_X$, then $\lambda X \in A$.

$A$ being absorbing means that, for any random variable $X \in \mathcal{X}$ (not necessarily in $A$), the line segment joining 0 to a suitable rescaling of $X$ lies entirely in $A$. Absorbing sets are of interest in part because any positive homogeneous function is completely determined by its values on any absorbing set. Furthermore, when $A$ is absorbing, it is possible to shrink any position until it “fits” in the set. In other words, any position may be scaled to a point where it becomes acceptable. Importantly, in a topological vector space, every neighborhood of zero is an absorbing set. Figure 4 shows an example of an absorbing set.

Figure 4: A set $A$ which is absorbing and radially bounded. Notice that $\delta_X$ is not uniquely defined.
(vii) **(Convexity)** A is **convex** if $\lambda X + (1 - \lambda Y) \in A$, for every pair $X, Y \in A$ and every $\lambda \in [0, 1]$. Convexity is a fundamental property in the theory of vector spaces. In our context, it is closely related to the concept of diversification, in the following sense: if an acceptance set $A$ is convex, then one cannot obtain an unacceptable position via a convex combination of acceptable positions, i.e. we cannot get worse off when we diversify. Analogously, if the complement of an acceptance set $A$ is convex, then we cannot get better off by taking convex combinations of non-acceptable positions.

(viii) **(Star-shapedness)** A is **star-shaped** if $\lambda X \in A$, for every $X \in A$ and $\lambda \in [0, 1]$. A is said to be **costar-shaped** if $A^c$ is star-shaped. A being star-shaped means that the line segment joining 0 to $X$ lies entirely in $A$, for every $X$ already lying in $A$ (thus, star-shapedness does not imply absorbency). For a star-shaped set $A$, given any $X \in \mathcal{X}$, there exists some non-negative number $\lambda_X$ (possibly with $\lambda^* = \infty$) such that that $\mathbb{R}_+ X \cap A \supseteq (0, \lambda_X)X$ and $\mathbb{R}_+ X \cap A^c \supseteq (\lambda_X, \infty)X$; note that if $A$ is absorbing then we can take $\lambda_X > 0$, and if $A$ is radially bounded then we can take $\lambda_X < \infty$. For sets containing zero, star-shapedness is a slightly weaker requirement than convexity: if $0 \in A$ and $A$ is convex, then $A$ is star-shaped. Figure 5 displays a star-shaped set which is not absorbing nor convex, while Figure 4 shows a set that is not star-shaped, although absorbing. Notice that $A \neq \emptyset$ being costar-shaped implies $\lambda X \in A$, for every $X \in A$ and $\lambda \in (1, \infty)$. Star-shapedness captures the financial notion that any scaled down version of an acceptable position should also be deemed acceptable. This is clearly a desirable property, as it intuitively means that if an agent accepts to invest a certain amount in a stock, then she also finds it acceptable to invest a lesser amount in the same stock.

![Figure 5: A set A which is star-shaped set and radially bounded.](image)

(ix) **(Strong star-shapedness)** A is **strongly star-shaped** if $A$ is star-shaped and, for each $X \in \mathcal{X}$, the ray $R_X \equiv (0, \infty)X$ intersects the boundary of $A$ at most once, i.e. the set $R_X \cap \text{bd} A$ is either empty or a singleton. For a similar concept, see [Rubinov and Gasimov (2004)](https://example.com). Figure 6 provides an example of a strongly star-shaped set having the origin as a boundary point.
Moreover, notice that for any $c \in \text{Lemma 2.2.}$

and $0 \preceq \preceq \text{Additionally, anti-monotonicity w.r.t. given } X.$

Proof. \text{Lemma 2.3. Let } A \subseteq \mathcal{X}$ is symmetric if $X \in A$ implies $-X \in A.$

While symmetry is useful, especially as — whenever $\mathcal{X}$ is a normed space — open balls centered at the origin are symmetric, this attribute is not desirable from a financial perspective, when $A$ represents a collection of acceptable positions. Indeed, there is no reason to require nor to expect that, for a given portfolio $X$ which is deemed acceptable, the corresponding short position $-X$ should be considered acceptable as well.

We now move to stating some results — involving the concepts introduced above — which will be used throughout the text.

Lemma 2.2. Let $A \subseteq \mathcal{X}$ be non-empty. If $A$ is $\preceq_{\mathcal{D}}$-anti-monotone, then it is stable under scalar addition, star-shaped and law invariant.

Proof. Let $A$ be non-empty and assume it is $\preceq_{\mathcal{D}}$-anti-monotone. Let $X \in A.$ Clearly $F^{-1}_{c}(u) - F^{-1}_{c}(v) = 0$ for any $c \in \mathbb{R}$ and $0 < v < u < 1.$ Hence, it is clear that $c \preceq_{\mathcal{D}} X$ for any $c \in \mathbb{R},$ which entails $\mathbb{R} \subseteq A.$ Moreover, notice that $F^{-1}_{X+c}(u) - F^{-1}_{X+c}(v) = F^{-1}_{c}(v) + c - F^{-1}_{c}(c) - c = F^{-1}_{c}(u) - F^{-1}_{c}(v)$ for any $c \in \mathbb{R}$ and $0 < v < u < 1.$ Therefore, $X + c \preceq_{\mathcal{D}} X$ for all $c \in \mathbb{R},$ and due to anti-monotonicity of $A,$ we get $X + c \in A$ (this holds for all $X \in A$ and $c \in \mathbb{R}$). Furthermore, $A$ is star-shaped: indeed, given $X \in A$ we have $F^{-1}_{X}(u) - F^{-1}_{X}(v) \geq \lambda(F^{-1}_{X}(u) - F^{-1}_{X}(v)) = F^{-1}_{\lambda X}(u) - F^{-1}_{\lambda X}(v),$ for any $\lambda \in [0, 1]$ and $0 < v < u < 1.$ Hence, $\lambda X \preceq_{\mathcal{D}} X$ for any $\lambda \in [0, 1],$ from which star-shapedness of $A$ follows.

Additionally, anti-monotonicity w.r.t. $\preceq_{\mathcal{D}}$ clearly implies that $A$ is law invariant, as if $Y$ and $X$ follow the same distribution it is obvious that $Y \preceq_{\mathcal{D}} X$.

Lemma 2.3. Let $B \subseteq \mathcal{X}.$ Then its law invariant hull $\mathcal{L}_B := \{X \in \mathcal{X} : X =_d Y, \text{ for some } Y \in B\}$ inherits from $B$ the attributes of stability under scalar addition, star-shapedness, absorbency, conicity, symmetry and $\preceq_{\mathcal{D}}$-monotonicity.

Proof. If $B$ is stable under scalar addition, then taking any $Y \in \mathcal{L}_B$ and $c \in \mathbb{R}$ we see — as, per definition, it holds that $Y =_d X$ for some $X \in B$ — that $Y + c =_d X + c \in B,$ that is $Y + c \in \mathcal{L}_B.$

Figure 6: A set $A$ which is strongly star-shaped, with $0 \in \text{bd}(A).$
Assume now that $B$ is a cone, and let $Y \in \mathcal{L}_B$ and $\lambda > 0$. We have $Y \equiv Y$ for some $X \in B$, and, since $B$ is a cone, $\lambda X \in B$. But $\lambda Y \equiv \lambda X$, and this is all we need to conclude that $\mathcal{L}_B$ is also a cone. A similar argument yields that $\mathcal{L}_B$ is star-shaped (resp., absorbing) whenever $B$ is.

For symmetry, just note that $X \equiv Y$ if and only if $-X \equiv -Y$. Finally, $\mathcal{L}_B$-monotonicity is clear as the dispersive order of distributions is defined in terms of distributions alone.

**Remark 2.4.** Not every property that seems plausibly heritable turns out to be so: take, for instance, radial boundedness of $B$. It seems reasonable — since no random variable in $B$ can be scaled up indeterminately while remaining acceptable — that the same should be true of $\mathcal{L}_B$. However, the following counterexample shows that this is false: let $\Omega = \{0, 1\}$ be the Bernoulli space comprised of all sequences of $0$'s and $1$'s, that is, the generic element $\omega \in \Omega$ is of the form $\omega = (\omega_1, \omega_2, \ldots)$ with $\omega_n \in \{0, 1\}$ for all $n$. The probability measure $\mathbb{P}$ is defined, for each $n$ and each ntuple $x_1, \ldots, x_n \in \{0, 1\}$, via

$$\mathbb{P}(\omega \in \Omega : \omega_1 = x_1, \ldots, \omega_n = x_n, \omega_{n+1} = R, \omega_{n+1} = R, \ldots) = 1/2^n.$$

Now define $X_n(\omega) = n \times I(\omega_n = 1)$, and put $B = \{X_1, X_2, \ldots\}$. Such $B$ is radially bounded, since for any fixed element $X_n \in B$, there is only one element of $B$ in the direction $0X_n$. However, $\mathcal{L}_B$ is not radially bounded: indeed, since $nX_1 \equiv X_n$, we have that $nX_1 \in \mathcal{L}_B$ for all $n$, and thus $\mathcal{L}_B$ is not radially bounded in the direction of $X_1$. Similarly, $X_2/2 \equiv X_{2n}/2n$ and thus we have $nX_2 \in \mathcal{L}_B$ for all $n$, and so on.

**Lemma 2.5.** Let $A \subseteq \mathcal{X}$. If $A$ is closed, star-shaped, and contains a proper cone with vertex at some constant $x \in \mathbb{R}$, then $A$ is not radially bounded. Hence, if $A$ is closed, star-shaped, and radially bounded, then every proper cone with vertex at a constant intersects $A^c$.

**Proof.** As $A$ contains a proper cone with vertex at some constant $x \in \mathbb{R}$, there exists a non-zero $X \in \mathcal{X}$ such that $\{x + \lambda X : \lambda \geq 0\} \subseteq A$. As $A$ is star-shaped, we have that $k(x + \lambda X) \in A$ for all $k \in [0, 1]$ and all $\lambda \geq 0$; in particular, taking $\lambda = 1/k$, we have $kx + X \in A$ for all $k \in [0, 1]$ and, as $A$ is closed, $X = \lim_{k \to 0} kx + X \in A$. To conclude that $A$ is not radially bounded, it is sufficient to show that there is no $\delta_X > 0$ such that $\delta_X X \notin A$ for $\delta \geq \delta_X$. So, let us fix an arbitrary $\delta_X > 0$ and put $k_n = 1/n$ and let $\lambda_n = \delta_X / k_n$. As $A$ is closed, we have $\lim_{n \to \infty} (k_n x + k_n \lambda_n X) \in A$. Now, clearly the preceding limit equals $\delta_X X$ and so, as $\delta_X$ was chosen arbitrarily, we can conclude that $A$ is not radially bounded.

**Remark 2.6.** A quick inspection of the proof of Lemma 2.5 tells us that it remains true even when the vertex $x$ is not assumed to be a constant. In any case, we opt to state it for constant vertices since this is the case which will be used later on in the text.

**Lemma 2.7.** Let $X \in \mathcal{X}$. Then the family $C_X \equiv \{Y \in \mathcal{X} : Y$ is comonotone to $X\}$ is a convex cone which is closed with respect to the topology of convergence in probability. Furthermore, if $(X, Y)$ is a comonotone pair, then any two elements of the convex cone $C_{X,Y} \equiv \text{conv}(\text{cone}(\{X\} \cup \{Y\}))$ are comonotone to one other.

**Proof.** In what follows all equalities and inequalities are in the $\mathbb{P} \otimes \mathbb{P}$-almost sure sense, that is, they hold for any pair $(\omega, \omega')$ lying in an event $\Omega_1 \subseteq \Omega \times \Omega$ having total $\mathbb{P} \otimes \mathbb{P}$ measure. To see that $C_X$ is a cone, note that for any $Y \in C_X$ we have, by definition, $(X(\omega) - X(\omega')) \times (Y(\omega) - Y(\omega')) \geq 0$, for any $(\omega, \omega') \in \Omega_1$. Hence, for any $\lambda \geq 0$ and $(\omega, \omega') \in \Omega_1$,

$$(X(\omega) - X(\omega'))(\lambda Y(\omega) - \lambda Y(\omega')) = \lambda(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0,$$

yielding $\lambda Y \in C_X$. For convexity, let $Y, Z \in C_X$. Then, for $\lambda \in [0, 1]$ we have that

$$\lambda X(\omega) + (1 - \lambda) Z(\omega) = \lambda \{X(\omega) - X(\omega')\}[Y(\omega) - Y(\omega')] + (1 - \lambda) [X(\omega) - X(\omega')]\{Z(\omega) - Z(\omega')\} \geq 0$$

whenever $(\omega, \omega') \in \Omega_1$. To see that $C_X$ is closed in the asserted sense, consider a convergent sequence $\{Y_n\} \subseteq C_X$ with $Y_n \to Y$ in probability. By standard facts of measure theory, there is a subsequence $\{Y_{n(k)}\}$ such that $Y_{n(k)} \to Y$ almost surely. Clearly this yields that $Y$ is comonotone to $X$.\footnote{$\Omega_1$ can be taken as the countable intersection of the events where the required inequalities (for any pairing of $X, Y, Y_n, Z$ and $W$) hold.}

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For the second claim, let \( Z, W \in C_{X,Y} \). By definition we have \( Z = \gamma_1(\lambda_1 X) + (1 - \gamma_1)(\delta Y) \) for some triplet \((\gamma_1, \lambda_1, \delta_1)\) with \(0 \leq \gamma_1 \leq 1\) and \(0 \leq \lambda_1, \delta_1\), and similarly \( W = \gamma_2(\lambda_2 X) + (1 - \gamma_2)(\delta Y) \) for some triplet \((\gamma_2, \lambda_2, \delta_2)\) with \(0 \leq \gamma_2 \leq 1\) and \(0 \leq \lambda_2, \delta_2\). Then, for \((\omega, \omega') \in \Omega_1\), expanding the product

\[
(Z(\omega) - Z(\omega'))(W(\omega) - W(\omega'))
\]
yields a weighted sum whose terms are all non-negative. This completes the proof. \( \blacksquare \)

**Remark 2.8.** Note that the set \( C := \bigcap_{Y \in C_X} C_Y \), where \( C_X \) and \( C_Y \) are defined as in the proposition above, is a non-empty, closed, and convex set, such that all its elements are comonotone to one another. In particular, \( R \subseteq C \).

We now define and explore a very important concept regarding duality in convex analysis, namely the **polar** of a set.

**Definition 2.9.** For a dual pair \((\mathcal{X}, \mathcal{X}')\), the **polar** \( A^\circ \) of a non-empty set \( A \subseteq \mathcal{X} \) is defined through

\[
A^\circ := \{ X' \in \mathcal{X}' : \sup_{X \in A} \langle X, X' \rangle \leq 1 \},
\]
and the **bipolar** of \( A \) is the set given by

\[
A^{\circ\circ} := \{ X \in \mathcal{X} : \sup_{X' \in A^\circ} \langle X, X' \rangle \leq 1 \}.
\]

**Remark 2.10.** Notice that the bipolar is always defined with the dual pair \((\mathcal{X}, \mathcal{X}')\) in mind, which forces the inclusion \( A^{\circ\circ} \subseteq \mathcal{X} \). If instead one had the bidual \( \mathcal{X}'' \) in mind (or, which is the same, the dual pair \((\mathcal{X}', \mathcal{X}'')\)), it would then be natural to define \( (A^\circ)^\circ := \{ X'' \in \mathcal{X}'' : \sup_{X' \in A^\circ} \langle X', X'' \rangle \leq 1 \} \). In this case, however, unfortunately one may have \( A^{\circ\circ} \neq (A^\circ)^\circ \). This is a detail that is frequently overlooked in the literature, although it has important consequences: for instance, see the Bipolar Theorem (item \( \text{[vi]} \) in Lemma 2.11, and also example \( \text{[vii]} \) below.

**Lemma 2.11.** Given a dual pair \((\mathcal{X}, \mathcal{X}')\), let \( A, B, \{ A_i \}_{i \in I} \) be subsets of \( \mathcal{X} \):

(i) If \( A \subseteq B \), then \( B^\circ \subseteq A^\circ \).

(ii) \( (\lambda A)^\circ = \lambda^{-1} A^\circ \) for each \( \lambda \neq 0 \).

(iii) \( \cap A_i^\circ = (\cup A_i)^\circ \).

(iv) \( A^\circ \) is nonempty, convex, weakly*-closed and contains 0.

(v) If \( A \) is absorbing, then \( A^\circ \) is weakly*-bounded, i.e. the set \( \{ \langle XX' \rangle : X \in A \} \) is bounded in \( \mathbb{R} \), for every \( X' \in \mathcal{X}' \).

(vi) The bipolar \( A^{\circ\circ} \) is the convex, weak-closed hull of \( A \cup \{0\} \).

(vii) If \( A \) is a cone, then \( A^\circ = \{ X' \in \mathcal{X}' : \langle X, X' \rangle \leq 0, \forall X \in A \} \).

(viii) If \( A \) is star-shaped and stable under scalar addition, then \( \langle 1, X' \rangle = 0 \) for all \( X' \in A^\circ \).

**Proof.** For items (i) to (vi), see Lemma 5.102 and Theorem 5.103 of Aliprantis and Border (2006). Item \( \text{[vii]} \) follows from an argument similar to the proof that \( B_0 = B_0^\circ \) in Proposition 3.19 below. For item \( \text{[viii]} \), let \( X' \in A^\circ \). Then — as \( R \subseteq A \) and \( A + R \subseteq A \) by assumption — we have, for any \( X \in A \) and \( c \in \mathbb{R} \),

\[
\langle X, X' \rangle + c(1, X') = \langle X + c, X' \rangle \leq 1
\]
and, as \( c \) is arbitrary, it is necessarily true that \( \langle 1, X' \rangle = 0 \). \( \blacksquare \)

**Remark 2.12.** Item \( \text{[vi]} \) above is the famous Bipolar Theorem, which states, in other words, that if \( A \) is closed, convex and contains zero, then \( A = A^{\circ\circ} \). It is important to have in mind that \( A^{\circ\circ} \subseteq \mathcal{X} \) by definition. The following (counter)example provides a reasoning for the bipolar to be defined in \( \mathcal{X} \) and not in \( \mathcal{X}'' \).
Example 2.13. Let $\mathcal{X} = L^1$, so that $\mathcal{X}' = L^\infty$ and $\mathcal{X}'' = b_{a}$. With the dual pair $(L^1, L^\infty)$ in mind, if $A$ is the unit ball in $\mathcal{X}'$, then clearly $A^\odot \supseteq \text{ball}(L^\infty)$. To see that the converse inclusion $A^\odot \subseteq \text{ball}(L^\infty)$ also holds, notice that if $X' \in \mathcal{X}'$ is such that $\|X'\|_\infty > 1$ then, since the random variable $X = \mathbf{1}_{[X' > \lambda]}/\mathbb{P}[X' > \lambda]$ belongs to $\text{ball}(L^1)$ for any conformable $1 < \lambda < \|X'\|_\infty$, we have for such an $X$

\[
\langle X, X' \rangle = \frac{1}{\mathbb{P}[X' > \lambda]} \int_{[X' > \lambda]} X' \, d\mathbb{P} \geq \frac{1}{\mathbb{P}[X' > \lambda]} \int_{[X' > \lambda]} \lambda \, d\mathbb{P} > 1,
\]

hence $X' \notin A^\odot$. Fix $B := A^\odot$ and, now with the dual pair $(L^\infty, b_{a})$ in mind, notice that given any $X'' \in b_{a}$ with total variation less than 1, clearly one has $\langle X'', X' \rangle \leq 1$ for all $X' \in B$. That is, $X'' \in B^\odot$. However, since $L^1$ is not reflexive, not every such $X''$ is the image of an $X \in L^1$ via the canonical embedding. Therefore, $(A^\odot)^\odot \supseteq A^{\odot\odot}$.

3 Minkowski gauge

There is plethora of results concerning the Minkowski gauge to be found in the realms of functional and convex analysis. In this section we recall and introduce important concepts, state some known results from the literature, and rediscover others that are of special interest from a financial perspective. Before defining the Minkowski gauge, we turn our focus to relevant properties — which regard functionals in general, not only the Minkowski gauge — that are considered alongside the text.

Definition 3.1. Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be an arbitrary, extended real-valued functional on $\mathcal{X}$. A sub-level set of a functional $f$ (defined on $\mathcal{X}$) at level $k \in \mathbb{R}$ is denoted by $A^k_f := \{ X \in \mathcal{X} : f(X) \leq k \}$. Moreover, we say that

(i) (Non-negativity): $f$ is non-negative if $f(X) > 0$ for any non-constant $X$ and $f(X) = 0$ for any constant $X$.

If $f$ is a deviation measure, non-negativity tells us that that the deviation can only assume strictly positive values, except when evaluated at constants — which have no deviation.

(ii) (Translation insensitivity) $f$ is translation insensitive if $f(X + c) = f(X)$ for any $X \in \mathcal{X}$ and $c \in \mathbb{R}$.

Whenever $f$ is a deviation measure, translation insensitivity ensures that the deviation does not change if a constant amount is added to a given position.

(iii) (Translation invariance) $f$ is translation invariant if $f(X + c) = f(X) - c$ for any $X \in \mathcal{X}$ and $c \in \mathbb{R}$.

Unlike translation insensitivity — which is typically a property imposed on deviation measures — translation invariance is one of the requirements defining a monetary risk measure; it says that the (monetary) risk of a position is reduced by the exact same amount of an invested sure gain on that position.

(iv) (Monotonicity) $f$ is monotone (w.r.t. a given partial order $\preceq$) whenever $Y \preceq X$ implies $f(Y) \leq f(X)$. If $f$ is monotone, than $f$ is said to be anti-monotone (w.r.t. $\preceq$). For simplicity, whenever the partial order is not explicitly mentioned, we are assuming that it is the “almost surely $\leq$” partial order.

From a financial perspective, imposing anti-monotonicity on a risk functional $f$ corresponds to the requirement that, if a position yields better results than another in every possible state of the world, then the former necessarily has lower risk than the latter.

(v) (Positive homogeneity) $f$ is positive homogeneous if $f(\lambda X) = \lambda f(X)$ for all $X \in \mathcal{X}$ and $\lambda \geq 0$.

For a risk measure $f$, positive homogeneity has the financial interpretation that the risk of a position increases proportionally to its magnitude.

(vi) (Convexity) $f$ is convex if $f(\lambda X + (1 - \lambda) Y) \leq \lambda f(X) + (1 - \lambda) f(Y)$, for every pair $X, Y \in \mathcal{X}$ and all $\lambda \in [0, 1]$. 

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From the financial viewpoint, convexity is a property which ensures that diversification reduces risk. A mapping \( f: \mathcal{X} \to \mathbb{R} \cup \{+\infty\} \) with \( f(0) = 0 \) is said to be a sub-linear functional whenever it satisfies any two of the following properties: (a) positive homogeneity; (b) convexity; (c) sub-additivity (the latter means that \( f(X + Y) \leq f(X) + f(Y) \) for any \( X, Y \in \mathcal{X} \)).

(vii) (Lower Range Dominance) \( f \) is lower-range dominated if domain(\( f \)) \( \subseteq L^1 \) and \( f(X) \leq \mathrm{EX} - \operatorname{ess \inf} X =: \mathrm{LR}(X) \) for all \( X \).

Lower range dominance is an essential property, as it reveals the interplay between coherent risk measures and generalized deviation measures — see in [Rockafellar et al. (2000a)] for instance.

(viii) (Law Invariance) \( f \) is law invariant if \( F_X = F_Y \) implies \( f(Y) = f(X) \).

If \( f \) is a risk functional, law invariance encapsulates the notion that, in appraising the risk of a position, we should only care about its statistical properties — as these properties embody the uncertainty (w.r.t. the market outcome) faced by a given agent. Law invariance is also important in empirical implementations, as it allows the theoretical risk measure to be estimated from historical data.

(ix) (Lower-Semicontinuity) \( f \) is lower-semicontinuous if the set \( A^f \) is closed, for all real \( k \).

In the case when \( \mathcal{X} \) is a metric space, lower-semicontinuity is equivalent to the following property: given any convergent sequence \( \{X_n\} \subseteq \mathcal{X} \), it holds that \( f(\lim X_n) \leq \lim \inf f(X_n) \). The convex envelop of a mapping \( f: \mathcal{X} \to \mathbb{R} \) is defined to be the extended real valued function \( \operatorname{conv} f \) given by \( \operatorname{conv} f(X) := \sup g(X), X \in \mathcal{X} \), where the supremum runs through all affine, continuous \( g: \mathcal{X} \to \mathbb{R} \) satisfying \( g \leq f \). Note that \( \operatorname{conv} f \) is convex and lower-semicontinuous.

(x) (Upper-Semicontinuity) \( f \) is upper-semicontinuous if the set \( \{X \in \mathcal{X} : f(X) \geq k\} \) is closed for all real \( k \).

In the case when \( \mathcal{X} \) is a metric space, upper-semicontinuity is equivalent to the following property: given any convergent sequence \( \{X_n\} \subseteq \mathcal{X} \), it holds that \( f(\lim X_n) \geq \lim \sup f(X_n) \). Note that a functional \( f \) is continuous if and only if it is both upper- and lower-semicontinuous.

(xi) (Symmetry) \( f \) is symmetric if \( f(X) = f(-X) \) for all \( X \in \mathcal{X} \).

In the theory of topological vector spaces, symmetry is one of the sine qua non conditions in defining a seminorm; indeed, a seminorm is precisely the Minkowski gauge of a symmetric, convex, and absorbing set. Although some deviation measures — the standard deviation, for example — do enjoy this attribute, symmetry is not really something desirable in our framework. Indeed, we are interested in quantifying the downside risk or deviation of a position, and thus dispersion above the mean can even be considered as “good”.

(xii) (Comonotone Additivity) \( f \) is comonotone additive if \( f(X + Y) = f(X) + f(Y) \) for every pair \( X, Y \in \mathcal{X} \) such that \( X \) and \( Y \) are comonotone.

Comonotone additivity implies that a comonotone pair does not yield a gain, nor a loss, in diversification. This property sums up the notion that, for such a pair, an agent should be indifferent about how the two positions are kept, whether they are held in the same portfolio or separately.

We are now in place to introduce the main tool used in this paper — the Minkowski gauge — as well as some related functionals:

**Definition 3.2.** Let \( A \subseteq \mathcal{X} \). The Minkowski gauge of \( A \) is the functional \( f_A: \mathcal{X} \to \mathbb{R}_+ \cup \{\infty\} \) defined, for \( X \in \mathcal{X} \), by

\[
  f_A(X) := \inf \{ m \in \mathbb{R}_+^* : m^{-1}X \in A \},
\]

where \( \inf \emptyset = \infty \). The cogauge of \( A \) is the functional \( \varphi_A: \mathcal{X} \to \mathbb{R}_+ \cup \{\infty\} \) defined, for \( X \in \mathcal{X} \), by

\[
  \varphi_A(X) := \sup \{ m \in \mathbb{R}_+^* : m^{-1}X \in A \},
\]

where \( \sup \emptyset = 0 \). Additionally, the support function \( h_{A^\circ}: \mathcal{X} \to \mathbb{R}_+ \cup \{\infty\} \) on the polar \( A^\circ \) is defined, for \( X \in \mathcal{X} \), as

\[
  h_{A^\circ}(X) := \sup \{(X, X') : X' \in A^\circ \}.
\]

4It is well known that, for such an \( f \), any two of this three axioms imply the remaining one — see [Aliprantis and Border (2006)].

5In the present setting, the convention \( \sup \emptyset = 0 \) is a sensible one, as we are taking the supremum over some subset of \( (0, \infty) \).
In words, the Minkowski gauge answers the following question: given a set \( A \) of acceptable positions, how much should we shrink (or "gauge") a certain position \( X \) for it to become acceptable? The value \( f_A(X) \) is the required amount of shrinkage. Notice that the following inclusions always hold:

\[
\{ X \in \mathcal{X} : f_A(X) < 1 \} \subseteq A \subseteq A_f^A.
\]

The cogauge, in turn, is a useful concept that is closely linked to the Minkowski gauge: if we take a set \( A \) comprised of non-acceptable positions, then the cogauge gives the most that we can shrink a position while keeping it non-acceptable. Importantly, for a star-shaped set \( A \), gauge and cogauge are linked — see Corollary 3.3 below. For more details on cogauges, we refer the reader to [Rubinov and Yagubov 1986; Rubinov 2000; Zaffaroni 2003, 2013] and references therein.

Before continuing with the Minkowski gauge, let us state a useful result for positive homogeneous functionals on topological vector spaces.

**Lemma 3.3.** Let \( f : \mathcal{X} \to \mathbb{R} \cup \{ \infty \} \). If \( f \) is positive homogeneous, then the set \( E := \{ X \in \mathcal{X} : f(X) = 1 \} \) has empty interior.

**Proof.** Let us proceed by contraposition by showing that if \( E \) has non-empty interior, then \( f \) is not positive homogeneous. Assume, then, that \( X \in \text{int} E \), and let \( V \) denote an open neighborhood of \( X \) with \( V \subseteq E \). By continuity of scalar multiplication, for small enough \( u > 0 \) we have \((1+u)X \in V \subseteq E \). But then \( f((1+u)X) = 1 < (1+u)f(X) \), so \( f \) is not positive homogeneous.

We begin with a result which we use many times in the remainder of the paper. It relates star-shapedness with the fact that the infimum in the definition of the Minkowski gauge is taken over an interval.

**Lemma 3.4.** Let \( A \subseteq \mathcal{X} \) and \( X \in \mathcal{X} \). Then

(i) \( f_A(X) = \infty \) if and only if \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = \emptyset \) if and only if \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = \mathbb{R}_+^* \). Moreover, if \( A \) is star-shaped, then

(ii) \( f_A(X) = 0 \) if and only if \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = \mathbb{R}_+^* \) if and only if \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = \emptyset \).

If in addition \( 0 < f_A(X) < \infty \), then one of the following holds:

(iii) \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = \{ f_A(X), \infty \} \) and \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = (0, f_A(X)) \) (this is true in particular when \( A \) is closed).

(iv) \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = \{ f_A(X), \infty \} \) and \( \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} = (0, f_A(X)) \) (this is true in particular when \( A \) is open).

**Proof.** The first item is immediate. For the remaining assertions, let \( T_X(m) := -m^{-1}X \) for \( m \in \mathbb{R}_+^* \). Clearly \( T_X \) is a continuous mapping from \( \mathbb{R}_+^* \) to \( \mathcal{X} \). We have \( T_X^{-1}(A) = \{ m \in \mathbb{R}_+^* : -m^{-1}X \in A \} \) and similarly \( T_X^{-1}(A') = \{ m \in \mathbb{R}_+^* : -m^{-1}X \notin A \} \). Assume now that \( A \) is star-shaped and \( m \in T_X^{-1}(A) \). Then, if \( m' > m \), we have \( m' \in T_X^{-1}(A) \) as well. This establishes that \( T_X^{-1}(A) \) is always an interval with \( \infty \) as its right endpoint, and by definition the left endpoint is \( f_A(X) \), thus establishing (ii), (iii) and (iv), where the topological assertions follow by continuity of \( T_X \).

We then have the following direct corollary on the relation between gauge and cogauge.

**Corollary 3.5.** Let \( A \subseteq \mathcal{X} \) be star-shaped. Then the equality

\[
f_A(X) = \varphi_{A'}(X)
\]

holds for all \( X \in \mathcal{X} \).

**Remark 3.6.** Let \( A \subseteq \mathcal{X} \) and \( A' \subseteq \mathcal{X}' \). Then the Minkowski gauge \( f_{A \times A'} : \mathcal{X} \times \mathcal{X}' \to \mathbb{R}_+ \) is, by definition, given by \( f_{A \times A'}(X, X') = \inf \{ m \in \mathbb{R}_+^* : (X, X')/m \in A \times A' \} \), for all \( X \in \mathcal{X} \) and \( X' \in \mathcal{X}' \). Therefore, if we both \( A \subseteq \mathcal{X} \) and \( A' \subseteq \mathcal{X}' \) are star-shaped sets, we get that \( f_{A \times A'}(X, X') = \max(f_A(X), f_A(X')) \), for all \( X \in \mathcal{X} \) and \( X' \in \mathcal{X}' \). To see this, notice that the following chain holds:

\[
f_{A \times A'}(X, X') = \inf \{ m \in \mathbb{R}_+^* : (X, X')/m \in A \times A' \}
\]

\[
= \inf \{ m \in \mathbb{R}_+^*: X/m \in A \text{ and } X'/m \in A' \}
\]

\[
= \max \{ \inf \{ m \in \mathbb{R}_+^* : X/m \in A \}, \inf \{ m \in \mathbb{R}_+^* : X'/m \in A' \} \}
\]

\[
= \max(f_A(X), f_A(X')).
\]
A special case occurs when $A$ is closed and convex, and $A' = A^\circ$: in this scenario one has $f_{A \times A^\circ}(X, X') = \max(f_A(X), f_{A^\circ}(X'))$.

Importantly, the Minkowski gauge of a set $A$ is positive homogeneous whenever $0 \in A$:

**Lemma 3.7.** Let $A \subseteq \mathcal{X}$. Then the following holds, for each $X \in \mathcal{X}$:

(i) $f_{\lambda A}(X) = \lambda^{-1} f_A(X)$, for every $\lambda \in \mathbb{R}^*_+$.

(ii) If $0 \in A$, then $f_A$ is positive homogeneous.

**Proof.** Let $X \in \mathcal{X}$. For the first item, given $\lambda \in \mathbb{R}^*_+$, we have

$$f_{\lambda A}(X) = \inf \{ m \in \mathbb{R}^*_+ : m^{-1} X \in \lambda A \} = \inf \{ m \in \mathbb{R}^*_+ : (\lambda m)^{-1} X \in A \} = \inf \{ m \lambda^{-1} \in \mathbb{R}^*_+ : m^{-1} X \in A \} = \lambda^{-1} f_A(X)$$

as claimed.

For the second item, clearly $f_A(0X) = f_A(0) = \inf \{ m \in \mathbb{R}^*_+ : m^{-1} 0 \in A \} = \inf \mathbb{R}^*_+ = 0 = 0 f_A(X)$. Moreover, given $\lambda > 0$, we have

$$f_A(\lambda X) = \inf \{ m \in \mathbb{R}^*_+ : \lambda m^{-1} X \in A \} = \inf \{ m \in \mathbb{R}^*_+ : m^{-1} X \in \lambda^{-1} A \} = f_{\lambda^{-1} A}(X).$$

Then, by item (i) we have $f_{\lambda^{-1} A}(X) = \lambda f_A(X)$, so the claim holds. \(\blacksquare\)

Item (i) in the above proposition tells us how a certain set operation on $A$ (in this case, rescaling) modifies the corresponding gauge. This is further explored in Lemmata 3.8 and 3.11 below, which also establish connections between properties of acceptance sets and properties of the associated Minkowski gauges. See Figure 7 to build up the intuition backing these results.

**Lemma 3.8.** (Lemma 5.49 of Aliprantis and Border (2006)) Let $A, B \subseteq \mathcal{X}$ be non-empty. Then the following holds:

(i) If $A \subseteq B$, then $f_A(X) \geq f_B(X)$, for all $X \in \mathcal{X}$.

(ii) $f_A(-X) = f_{-A}(X)$ for all $X \in \mathcal{X}$; in particular, if $A$ is symmetric, so is $f_A$.

(iii) If $A$ contains a cone $M$, then $f_A(X) = 0$, for all $X \in M$; in particular as $\{0\}$ is a cone, if $0 \in A$ then $f_A(0) = 0$.

(iv) If $A$ is closed and star-shaped, then $A = A^+_1$.

(v) If $A$ and $B$ are star-shaped, then $f_{A \cup B}(X) = \max(f_A(X), f_B(X))$

**Remark 3.9.** Note that $A \cap (-A)$ can be interpreted as a *symmetrization* of $A$, and whenever $A$ is closed, star-shaped, convex, stable under scalar addition and radially bounded at non constants, one has $f_{A \cap (-A)}(X) = \max(f_A(X), f_{-A}(X))$, yielding a symmetric generalized deviation measure (i.e., a seminorm). The spaces generated by symmetrized sets as the ones just described were studied in Right (2017).

**Remark 3.10.** Let $0 \in A$, and let $st(A)$ be defined by the condition that $Z \in st(A)$ if and only if $Z = \lambda X$ for some $\lambda \in [0, 1]$ and some $X \in A$ (that is, $st(A) = [0, 1] A$ in our preceding notation). It is clear that $st(A)$ is the smallest star-shaped set that contains $A$. Also, as an arbitrary intersection of star-shaped sets is still star-shaped, we see that $st(A)$ is equal to the intersection of all star-shaped sets that contain $A$. Therefore, we have that $B := \{ X \in \mathcal{X} : f_A(X) < 1 \} \subseteq st(A) \subseteq A^+_1$. In order to see this, note that $A^+_1 \equiv \{ X \in \mathcal{X} : f_A(X) \leq 1 \}$ is clearly star-shaped, as $f_A$ is positive homogeneous, and that $A \subseteq A^+_1$ because $X \in A$ implies $f_A(X) \leq 1$. Thus, we know that $st(A) = [0, 1] A \subseteq [0, 1] A^+_1 = A^+_1$. Hence, we only need to show that $B \subseteq st(A)$ which is also clear as $f_A(X) < 1$ implies that there is some $m \in (0, 1]$ such that $mX \in A$. We also have the identities $f_B = f_A = f_{A^+_1} = st(A)$, by item (i) in Lemma 3.8.

**Lemma 3.11.** Let $A, B \subseteq \mathcal{X}$. Then, for each $X \in \mathcal{X}$, the following holds:

(i) $f_{A \cup B}(X) = \min \{ f_A(X), f_B(X) \}$.

(ii) If $B$ is a cone, then $f_{A + B}(X) = \inf_{Z \in B} f_A(X - Z)$.

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Proof. Write $y = \min \{f_A(X), f_B(X)\}$.

For item (i) if $m < y$ then $m^{-1}X \notin A$ and $m^{-1}X \notin B$. By contraposition, $y$ is a lower bound for the set $\{m \in \mathbb{R}_+: m^{-1}X \in A \cup B\}$. Thus $f_{A \cup B}(X) \geq y$. The reversed inequality is immediate: if $m$ is such that $m^{-1}X \in A$, then obviously $m^{-1}X \in A \cup B$ and $f_{A \cup B}(X) \leq f_A(X)$. Similarly, we have $f_{A \cup B}(X) \leq f_B(X)$ and, a fortiori, $f_{A \cup B}(X) \leq y$.

For item (ii), let $m > 0$. Then one has $m^{-1}X \in A + B$ if and only if $m^{-1}X = a + b$ for some $a \in A$ and some $b \in B$, if and only if $m^{-1}X - b = a$, for some $b \in B$ and some $a \in \mathcal{X}$ such that $f_A(a) \leq 1$, if and only if $f_A(m^{-1}X - b) \leq 1$ for some $b \in B$. By positive homogeneity, the latter sentence is equivalent to the following: there exists a $b \in B$ such that $f_A(X - mb) \leq m$. Additionally — as $B$ is a cone — if there is an element $b \in B$ that respects $f_A(X - mb) \leq m$, then by letting $d = mb$ we see that there is an element $d \in B$ such that $f_A(X - d) \leq m$, and the reciprocal of the previous sentence is obviously also true: that is, it holds that $f_A(X - mb)$ for some $b \in B$ if and only if $f_A(X - d) \leq m$ for some $d \in B$. In view of the above equivalences, by writing $M_b := \{m \in \mathbb{R}_+: f_A(X - b) \leq m\}$ and noticing that $f_A(X - b) = \inf M_b$, we finally have that

$$f_{A+B}(X) = \inf_{b \in B} M_b$$

as asserted.

Remark 3.12. In the context of item (ii) from last lemma, we have from Lemma 3.8 that $f_B(X) = 0$ for any $X \in B$ since $B$ is a cone. Thus, $f_{A+B} = \inf_{Z \in B} \{f_A(X - Z) + f_B(Z)\} = \inf_{X \in \mathcal{X}} \{f_A(X - Z) + f_B(Z)\}$. The last equality holds because, for any $Z \notin B$, as $B$ is a cone, it follows by Lemma 3.4 that $f_B(Z) = +\infty$. This concept is closely related to inf-convolution and optimal risk sharing. Inf-convolution is a well known operation for functionals in convex analysis — for details of the use of inf-convolution in risk sharing we refer the reader to [Barrieu and El Karoui 2005, Jouini et al. 2008] and [Righi 2020a].

Figure 7: This figure illustrates items (i) and (v) in Lemma 3.8 as well as item (i) in Lemma 3.11 and item (ii) in Proposition 3.13. Here, we have $A \cap B = \{(x,y) \in \mathbb{R}^2_+: \max(0.8x, y) \leq 1\}$ and $A \cup B = \{(1.25x, y) \in \mathbb{R}^2: \min(x,y) \in [0,1] \text{ or } \max(x,y) \in [0,1]\}$.

The next result deals with the solution of the minimization problem appearing in the definition of the Minkowski gauge.

Proposition 3.13. Let $A \subseteq \mathcal{X}$ be non-empty. Then, we have the following:

(i) If $A$ is absorbing, then $f_A$ is finite-valued.

(ii) If $A \cup B$ is absorbing, then $f_{A \cup B}$ is finite-valued.

(iii) If $A \cap B$ is absorbing, then $f_{A \cap B}$ is finite-valued.

Proof. Write $y = \min \{f_A(X), f_B(X)\}$.

For item (i) if $m < y$ then $m^{-1}X \notin A$ and $m^{-1}X \notin B$. By contraposition, $y$ is a lower bound for the set $\{m \in \mathbb{R}_+: m^{-1}X \in A \cup B\}$. Thus $f_{A \cup B}(X) \geq y$. The reversed inequality is immediate: if $m$ is such that $m^{-1}X \in A$, then obviously $m^{-1}X \in A \cup B$ and $f_{A \cup B}(X) \leq f_A(X)$. Similarly, we have $f_{A \cup B}(X) \leq f_B(X)$ and, a fortiori, $f_{A \cup B}(X) \leq y$.

For item (ii), let $m > 0$. Then one has $m^{-1}X \in A + B$ if and only if $m^{-1}X = a + b$ for some $a \in A$ and some $b \in B$, if and only if $m^{-1}X - b = a$, for some $b \in B$ and some $a \in \mathcal{X}$ such that $f_A(a) \leq 1$, if and only if $f_A(m^{-1}X - b) \leq 1$ for some $b \in B$. By positive homogeneity, the latter sentence is equivalent to the following: there exists a $b \in B$ such that $f_A(X - mb) \leq m$. Additionally — as $B$ is a cone — if there is an element $b \in B$ that respects $f_A(X - mb) \leq m$, then by letting $d = mb$ we see that there is an element $d \in B$ such that $f_A(X - d) \leq m$, and the reciprocal of the previous sentence is obviously also true: that is, it holds that $f_A(X - mb)$ for some $b \in B$ if and only if $f_A(X - d) \leq m$ for some $d \in B$. In view of the above equivalences, by writing $M_b := \{m \in \mathbb{R}_+: f_A(X - b) \leq m\}$ and noticing that $f_A(X - b) = \inf M_b$, we finally have that

$$f_{A+B}(X) = \inf_{b \in B} M_b$$

as asserted.

Remark 3.12. In the context of item (ii) from last lemma, we have from Lemma 3.8 that $f_B(X) = 0$ for any $X \in B$ since $B$ is a cone. Thus, $f_{A+B} = \inf_{Z \in B} \{f_A(X - Z) + f_B(Z)\} = \inf_{X \in \mathcal{X}} \{f_A(X - Z) + f_B(Z)\}$. The last equality holds because, for any $Z \notin B$, as $B$ is a cone, it follows by Lemma 3.4 that $f_B(Z) = +\infty$. This concept is closely related to inf-convolution and optimal risk sharing. Inf-convolution is a well known operation for functionals in convex analysis — for details of the use of inf-convolution in risk sharing we refer the reader to [Barrieu and El Karoui 2005, Jouini et al. 2008] and [Righi 2020a].

Figure 7: This figure illustrates items (i) and (v) in Lemma 3.8 as well as item (i) in Lemma 3.11 and item (ii) in Proposition 3.13. Here, we have $A \cap B = \{(x,y) \in \mathbb{R}^2_+: \max(0.8x, y) \leq 1\}$ and $A \cup B = \{(1.25x, y) \in \mathbb{R}^2: \min(x,y) \in [0,1] \text{ or } \max(x,y) \in [0,1]\}$.

The next result deals with the solution of the minimization problem appearing in the definition of the Minkowski gauge.

Proposition 3.13. Let $A \subseteq \mathcal{X}$ be non-empty. Then, we have the following:

(i) If $A$ is absorbing, then $f_A$ is finite-valued.
(ii) If $A$ is star-shaped and $y := f_A(X) \in \mathbb{R}^*_+$, then $y^{-1}X \in \text{bd}(A)$, i.e., $f_A(X) = \varphi_A^{-}(X) = 1$ implies $X$ lies in the boundary of $A$.

(iii) If $A$ is strongly star-shaped, $X \in \text{bd}(A)$ and $X \neq 0$ then, $f_A(X) = 1$.

(iv) If $f_A(X) \in \mathbb{R}^*_+$, then

$$f_A(X) = \inf\{m \in \mathbb{R}^*_+ : m^{-1}X \in A\} = (\sup\{m \in \mathbb{R}^+_+ : mX \in A\})^{-1}.$$  

(v) If $R_X \cap A = \emptyset$ then $f_A(X) = \infty$. In particular if $0 \notin A$ then $f_A(0) = \infty$.

(vi) If $A$ is closed, absorbing and radially bounded, then the infimum in equation (2) is attained for any $X \in \mathcal{X}$ \setminus \{0\}, that is, $X \in f_A(X)A$ for any $X \in \mathcal{X}$.

(vii) If $A$ is closed, then the infimum in equation (2) is attained for any $X$ such that $f_A(X) \in \mathbb{R}^*_+$.

**Proof.** Let $X \in \mathcal{X}$ and write $y := f_A(X)$.

For the first item, there exists — by the absorbing property — some $\delta_X \in \mathbb{R}^*_+$ such that the inclusion $[0, \delta_X]X \subseteq A$ holds. It is straightforward to see that in this case the set $\{m \in \mathbb{R}^*_+ : m^{-1}X \in A\}$ is never empty. Therefore, $f_A(X) < \infty$.

For the second item, it suffices to consider the case $f_A(X) = 1$, as the general case then easily follows from positive homogeneity. In order to verify that $X \in \text{bd}(A)$, we only have to exhibit sequences $\{Y_n\} \subseteq A$ and $\{Z_n\} \subseteq A^c$ such that $\lim Y_n = \lim Z_n = X$. Let, then, $Y_n$ be defined through $Y_n := (1+1/2^n)-X$ and, similarly, $Z_n := (1-1/2^n)-X$. Continuity of scalar multiplication immediately yields the desired equality of limits, so it only remains to show that $Y_n \rightarrow A$ and $Z_n \rightarrow A^c$ for all $n$. For such, just notice that due to star-shapedness through Lemma 3.3 — if $m > 1$, then $m^{-1}X \in A$, so $Y_n \rightarrow A$, and if $0 < m < 1$, then $m^{-1}X \notin A$, so $Z_n \notin A$.

For item (iii) as $X \neq 0$ by assumption, we see that whenever the ray $R_X \equiv \{\lambda X : \lambda > 0\}$ has a non-empty intersection with $\text{bd}(A)$, it necessarily also holds that $f_A(X) \in \mathbb{R}^*_+$. Therefore, as $X \in \text{bd}(A)$, we also have that $f_A(X)^{-1}X \in \text{bd}(A)$, by item (iii). Thus, we have $X \in R_X \cap \text{bd} A$ and $f_A(X)^{-1}X \in R_X \cap \text{bd} A$, and hence strong star-shapedness of $A$ tells us that $f_A(X) = 1$.

For item (iv) notice that

$$y = \inf\{m \in \mathbb{R}^*_+ : m^{-1}X \in A\} = \inf\{m^{-1} \in \mathbb{R}^*_+ : mX \in A\}.$$  

Now, if $x^{-1} > 0$ is a lower bound for the set $\{m^{-1} \in \mathbb{R}^*_+ : mX \in A\}$, then $x$ is an upper bound for the set $\{m \in \mathbb{R}^+_+ : mX \in A\}$; if $x^{-1}$ is the largest such lower bound, then $x$ is the smallest such upper bound. That is to say, one has $y^{-1} = \sup\{m \in \mathbb{R}^+_+ : mX \in A\}$.

Item (v) is clear as if $\{\lambda X : \lambda > 0\} \cap A = \emptyset$ then the set $\{m \in \mathbb{R}^*_+ : m^{-1}X \in A\}$ is empty and the infimum of such set is $\infty$.

For item (vi) notice that if $A$ is radially bounded, then $f_A(X) > 0$, for every non-zero $X \in \mathcal{X}$. Indeed, if $A$ is radially bounded then — by definition — for each $X$ there is a $mX > 0$ such that $m^{-1}X \notin A$, for all $m < mX$. Therefore, it holds that $\inf\{m \in \mathbb{R}^*_+ : m^{-1}X \in A\} > 0$. Now, let $T_X : \mathbb{R}^*_+ \rightarrow \mathcal{X}$ be defined by $T_X(m) = m^{-1}X$. Clearly, $T_X$ is continuous. Thus, if $A$ is a closed subset of $\mathcal{X}$ so is $T_X^{-1}(A)$ a closed subset of $\mathbb{R}^*_+$. Also, if $A$ is absorbing, then $T_X^{-1}(A)$ is non-empty. Finally, since radial boundedness ensures $f_A(X) > 0$, it follows that $\inf\{T_X^{-1}(A) \cap T_X^{-1}(A)\}$ as stated.

The proof of item the last item is identical to the previous one.

We have already seen above that if $0 \in A$, then its Minkowski gauge $f_A$ is positive homogeneous and that, if the stronger requirement $0 \in \text{int} A$ (i.e., $A$ is absorbing) holds, then $f_A$ is also finite-valued. The next lemma shows that positive homogeneity is also a sufficient condition ensuring that an arbitrary functional $f$ (which does not assume negative values) is the Minkowski gauge of some subset of $\mathcal{X}$.

We opt to state the result as it appears in [Aliprantis and Border (2006)] where it is assumed at the outset that $\text{range}(f) \subseteq \mathbb{R}^+_+$. This assumption can be easily dropped; if so, the set $V$ appearing in Proposition 3.14 is no longer (necessarily) absorbing. Instead, in this case the condition $0 \in V$ must hold.

**Proposition 3.14.** (Lemma 5.50 and Theorem 5.52 of Aliprantis and Border (2006)) Let $A, B \subseteq \mathcal{X}$ be non-empty, and let $f : \mathcal{X} \rightarrow \mathbb{R}^+_+$ be an arbitrary function. Then the following holds:
(i) \( f \) is positive homogeneous if and only if it is the Minkowski gauge of an absorbing set, in which case for every \( V \subseteq \mathcal{X} \) satisfying 
\[ \{ X \in \mathcal{X} : f(X) < 1 \} \subseteq V \subseteq A^1_f, \]
we have \( f_V = f \).

(ii) \( f \) is sub-linear (positive homogeneous and convex) if and only if it is the Minkowski gauge of a convex absorbing set \( V \), in which case we may take \( V = A^1_f \).

(iii) \( f \) is sub-linear and symmetric if and only if it is the Minkowski gauge of a symmetric, convex absorbing set \( V \), in which case we may take \( V = A^1_f \).

(iv) \( f \) is sub-linear and lower-semicontinuous if and only if it is the Minkowski gauge of an absorbing, closed convex set \( V \), in which case we may take \( V = A^1_f \).

(v) \( f \) is sub-linear and continuous if and only if it is the Minkowski gauge of a convex neighborhood \( V \) of zero, in which case we may take \( V = A^1_f \).

(vi) \( f \) is sub-linear, symmetric and continuous if and only if it is the Minkowski gauge of a unique closed, symmetric and convex neighborhood \( V \) of zero, namely \( V = A^1_f \).

Remark 3.15. A locally convex topology is a topology generated by a family of seminorms. In particular, the neighborhood base at zero is given by the collection of all \( A^k_p \), with \( k > 0 \) and \( p \) belonging to some collection of seminorms. Now, Lemma 3.8 item (vi) actually tells us that each \( p \) is the Minkowski gauge of some unique closed, symmetric, convex neighborhood \( A \) of zero, namely \( A = A^1_p \), with \( p = f_A \). Distinctively, Theorem 5.73 of Aliprantis and Border (2006) tells us that any locally convex topology is generated by the family of gauges of the convex symmetric closed neighborhoods of zero.

Next, we show that one can establish a relation between the convex envelope of a positive homogeneous functional \( f \) defined on \( \mathcal{X} \) and the Minkowski gauge of the closed convex hull of \( A^1_f \).

**Proposition 3.16.** Let \( A \subseteq \mathcal{X} \). If \( 0 \in A \), then the Minkowski gauge of the closed convex hull of \( A \) is equal to the convex envelope of the Minkowski gauge of \( A \), i.e. one has 
\[ f_{\text{cl-conv}A}(X) = \text{conv} \, f_A(X) \]
for all \( X \in \mathcal{X} \).

**Proof.** First, notice that any lower-semicontinuous sub-linear function \( g \geq 0 \) that is dominated by \( f_A \) can be written as \( g = f_C \), with \( C \) a closed convex set given by \( C = A \subseteq A^1_A \subseteq A \) (see items (ii) and (iii) in Lemma 3.8 item (iv) in Proposition 3.14) and also items (viii) and (xiii) in Theorem 5.3 below, where the absorbing condition can be dropped by letting \( g \) assume \( +\infty \). Reciprocally, if \( C \) is any closed convex set such that \( A \subseteq C \), then the sub-linear function \( g := f_C \geq 0 \) is dominated by \( f_A \). In summary, there is a one-to-one correspondence between the class \( S_A(f_A) \) comprised of all lower-semicontinuous sub-linear mappings \( g : \mathcal{X} \to \mathbb{R}_+ \cup \{ +\infty \} \) dominated by \( f_A \) and the class \( C \) comprised of all closed convex sets \( C \supseteq A \). Therefore, since by definition \( \text{cl-conv}A = \bigcap_{C \in C} C \), an easy generalization of item (v) in Lemma 3.8 entails 
\[ f_{\text{cl-conv}A}(X) = \sup_{C \subseteq C} f_C(X) = \sup_{g \in S_A(f_A)} g(X). \]

Now, let \( S(f) \) be the set of all lower-semicontinuous sub-linear functions dominated by a mapping \( f \), and \( \mathcal{A}(f) \) the set of all continuous affine functions dominated by \( f \). The supremum over \( S_A(f_A) \) in the above expression corresponds to the supremum over all lower-semicontinuous sub-linear functions with values in \( \mathbb{R}_+ \cup \{ +\infty \} \) that are dominated by \( f_A \) and it clearly coincides with the supremum over all (not necessarily positive) lower-semicontinuous sub-linear functions that are dominated by \( f_A \). That is, we have 
\[ \sup_{g \in S_A(f_A)} g(X) = \sup_{g \in S(f_A)} g(X). \]

As any lower-semicontinuous sub-linear function can be written as the supremum of the continuous affine functions that it dominates (by taking its convex envelope), we have that 
\[ \sup_{f \in S(f_A)} \sup_{g \in \mathcal{A}(f)} g(X) = \sup \left\{ g(X) : g \in \bigcup_{f \in S(f_A)} \mathcal{A}(f) \right\} \]
\[ = \sup \left\{ g(X) : g \in \mathcal{A}(f) \right\} \]
\[ = \text{conv} \, f_A(X). \]
and this completes the proof. ■

Remark 3.17. If the convex envelope of a function $f$ is defined as the supremum over the (not necessarily continuous) affine functions that it dominates, then $\text{conv} f$ is not necessarily lower-semicontinuous. Nevertheless, the proposition above can easily be adapted to yield the equality $\text{conv} f = f_{\text{conv} A}$ by changing convex, closed sets for convex sets and dropping all the requirements of continuity over $g, f$ and the affine functions appearing in the proof.

Under strong star-shapedness we have the following result regarding to continuity.

Proposition 3.18. Let $A$ be strongly star-shaped and closed. If $0 \in \text{bd}(A)$, then $f_A$ is continuous except at 0. If $0 \in \text{int}(A)$, then $f_A$ is continuous everywhere.

Proof. First, note that if $A$ is closed and star-shaped, then due to Lemma 5.8 item (v), we have $A = A_A^1$, from which it follows that $f_A$ is lower-semicontinuous — see Theorem 5.3 item (i). Now, by definition of strong star-shapedness it is always true that $0 \in A$. We will show first the case $0 \in \text{bd}(A)$, and then consider the case $0 \in \text{int}(A)$.

Assume then that $0 \in \text{bd}(A)$. Note that if we let $B := A \setminus \{0\}$, then it is an easy check to see that $f_B(X) = f_A(X)$ for all $X \in \mathcal{X} \setminus \{0\}$ and $f_B(0) = \infty$. Indeed, for $X \neq 0$ the conditions $m^{-1}X \in A$ and $m^{-1}X \in B$ are clearly equivalent, whereas for $X = 0$ the condition $X \in mA$ is always true whereas $X \in mB$ is vacuous. Hence, we have that $f_B$ is lower-semicontinuous everywhere, except at 0. Furthermore, we have $B = A_{fn}^1$. To see it, note that $A_{fn}^1 = \{X \in \mathcal{X} : f_B(X) \leq 1\}$, and obviously $0 \notin A_{fn}^1$ as $f_B(0) = \infty$. Therefore,

$$A_{fn}^1 = A_{fn}^1 \setminus \{0\}$$

$$= \{X \in \mathcal{X} : f_B(X) \leq 1\} \setminus \{0\}$$

$$= \{X \in \mathcal{X} \setminus \{0\} : f_B(X) \leq 1\}$$

$$= \{X \in \mathcal{X} \setminus \{0\} : f_A(X) \leq 1\}$$

$$= \{X \in \mathcal{X} : f_A(X) \leq 1\} \setminus \{0\}$$

$$= B.$$

It remains to show that the set $V := \{X \in \mathcal{X} : f_B(X) < 1\}$ is open, from which we will know (again from Theorem 5.4) item (iii) that $f_B$ is upper-semicontinuous. This will give us then that $f_B$ is continuous everywhere except at 0, which in turn entails continuity of $f_A$ everywhere except at 0. To see that $V$ is indeed an open set, note that — due to Proposition 3.18 items (iii), (iii) and (v) — if $X \neq 0$ then $X \in \text{bd}(A)$ if and only if $f_A(X) = 1$, hence $\text{bd}(A) = \{X \in \mathcal{X} : f_A(X) = 1\} \setminus \{0\}$. Now, the reader should realize that, again since $f_B(0) = \infty$,

$$V = \{X \in \mathcal{X} : f_B(X) < 1 \text{ and } X \neq 0\}$$

$$= \{X \in \mathcal{X} : f_A(X) < 1 \text{ and } X \neq 0\}$$

$$= (A \setminus \text{bd}(A)) \setminus \{0\}$$

$$= A \setminus \text{bd}(A)$$

$$= \text{int}(A),$$

and as $\text{int}(A)$ is by definition an open set, the claim that $V$ is open holds.

Finally, if $0 \in \text{int}(A)$, as we already have that $f_A$ is lower-semicontinuous, it is enough to show that it is also upper-semicontinuous. It suffices to show that the set $U := \{X \in \mathcal{X} : f_A(X) < 1\}$ is open. Clearly, again due to Proposition 3.18 items (iii), (iii) and (v), we have $\text{bd}(A) = \{X \in \mathcal{X} : f_A(X) = 1\}$. Hence, $\text{int}(A) = A \setminus \text{bd}(A) = U$ and the claim follows. ■

The following result characterizes polar sets through Minkowski gauges. Recall that, by definition, the polar of a set $A \subseteq \mathcal{X}$ is given by $A^\circ = \{X' \in \mathcal{X}' : \langle X, X' \rangle \leq 1 \text{ for all } X \in A\}$.

Proposition 3.19. Let $A$ be star-shaped. Then it holds that

$$A^\circ = \{X' \in \mathcal{X}' : \langle X, X' \rangle \leq 1 \text{ for all } X \in A\} = B_0 \cap B \cap B_{\infty}.$$

Proof. Notice that we can write

$$A^\circ = \{X' \in \mathcal{X}' : \langle X, X' \rangle \leq 1 \text{ for all } X \in A\} = B_0 \cap B \cap B_{\infty},$$

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Similarly, we can write the right hand side in (5) as
\[ \{X' \in X' : \langle X, X' \rangle \leq f_A(X) \text{ for all } X \in X' \} = B_0^* \cap B^* \cap B_\infty^*, \]
where
\[ B_0^* = \{X' \in X' : \langle X, X' \rangle \leq 0 \text{ for all } X \in X' \text{ such that } f_A(X) = 0\}, \]
\[ B^* = \{X' \in X' : \langle X, X' \rangle \leq f_A(X) \text{ for all } X \in X' \text{ such that } 0 < f_A(X) < \infty\}, \]
\[ B_\infty^* = \{X' \in X' : \langle X, X' \rangle \leq \infty \text{ for all } X \in X' \text{ such that } f_A(X) = \infty\}. \]

Clearly \( B_\infty = B_\infty^* = X' \) since \( B_\infty \) is defined by a vacuous sentence and the upper bound \( f_A(X) = \infty \) in \( B_\infty^* \) is non-binding. Thus, to establish the proposition it suffices to show that \( B_0 = B_0^* \) and \( B = B^* \).

For the equality \( B_0 = B_0^* \), suppose \( X' \in B_0 \) and let \( X \in X' \) be such that \( f_A(X) = 0 \). If \( \langle X, X' \rangle \leq 0 \) then there is nothing to show as in this case \( X' \in B_0^* \). If \( \langle X, X' \rangle \geq 0 \), then — as \( f_A \) is positive homogeneous — we have \( f_A(\lambda X) = 0 \) for all \( \lambda > 0 \) and, by assumption, \( \langle \lambda X, X' \rangle \leq 1 \) for all \( \lambda > 0 \), which necessarily entails \( \langle X, X' \rangle = 0 \). Thus, \( B_0 \subseteq B_0^* \). That \( B_0 = B_0^* \) is obvious. Hence, \( B_0 = B_0^* \).

For the equality \( B = B^* \), suppose \( X' \in B \) and let \( X \in X' \) be such that \( 0 < f_A(X) < \infty \). Writing \( Y = X/f_A(X) \), we have \( Y \in A \) by item (iii) in Proposition 3.14 and \( 0 < f_A(Y) < \infty \) by positive homogeneity. Thus \( \langle Y, X' \rangle \leq 1 \) or, which is the same, \( \langle X, X' \rangle \leq f_A(X) \). The preceding argument shows that, \( B \subseteq B^* \). Reciprocally, suppose \( X' \in B^* \) and let \( X \in A \) be such that \( 0 < f_A(X) < \infty \). Writing \( Y = f_A(X)X \in X' \), then again positive homogeneity entails \( 0 < f_A(Y) = f_A(X)^2 < \infty \). Thus, \( \langle Y, X' \rangle \leq f_A(Y) \) or, equivalently, \( \langle f_A(X)X, X' \rangle \leq f_A(X)^2 \), from which we deduce that \( \langle X, X' \rangle \leq 1 \) since \( f_A(X) \leq 1 \). Therefore, \( B^* \subseteq B \), which establishes the equality in \( B = B^* \).

A well know result in convex analysis is the duality associating the Minkowski gauge of a set \( A \) with the support function of its polar \( h_A^\circ(X) = \sup_{X' \in A} \langle X, X' \rangle, X \in X' \). This is related to the convex biconjugate of the Fenchel-Moreau Theorem (when \( X' \) is a locally convex topological space) via the conjugate and biconjugate functions (the latter is also called the penalty function in the jargon of convex risk measures). If the gauge is a proper, convex and weakly lower-semicontinuous functional, then the penalty is precisely the characteristic function of the polar. Below we present this duality result for topological vector spaces, without relying on the Fenchel-Moreau Theorem.

**Proposition 3.20** (Dual representation). Let \( A \) be a closed, convex set such that \( 0 \in A \). Then we have the identity
\[ f_A(X) = h_A^\circ(X) \]
for all \( X \in X' \).

**Proof.** For simplicity, let us write \( h := h_A^\circ \). First of all, notice that \( h: X' \rightarrow \mathbb{R}_+ \cup \{\infty\} \) is a lower-semicontinuous, sub-linear function. Then, by item (iv) in Proposition 3.14 (we drop the absorbing condition as \( h \) may assume infinity), we have
\[ h = f_A_1^\circ. \]

Therefore, it is enough to show that \( A_1^\circ = A \). Note that
\[ A_1^\circ = \{X \in X' : h(X) \leq 1\} = \{X \in X' : \sup_{X' \in A} \langle X, X' \rangle \leq 1\} = \{X \in X' : \langle X, X' \rangle \leq 1 \text{ for all } X' \in A^\circ\} = A^\circ. \]

\[ \text{The characteristic function of the polar assumes 0 if } X \in A^\circ \text{ and } \infty \text{ otherwise.} \]
Lastly, the Bipolar Theorem (item $[\text{vi}]$ in Lemma 2.11) entails $A = A^\circ\circ$. Hence,

$$h = f_{A^\circ} = f_{A^\circ\circ} = f_A$$

as claimed. \[\Box\]

**Remark 3.21.** The equality in the proposition above holds even if $A$ is empty, as in this case $f_A \equiv \infty$ and $A^\circ = \emptyset$, so $h_{A^\circ} \equiv \infty$. It is also interesting to remember that if $A = \emptyset$, then $f_A \equiv 0$, $A^\circ = \{0\}$ and $h_{A^\circ} \equiv 0$. Furthermore, note that $A^\circ = \partial f_A(0) := "the set of sub-gradients of f_A at 0"$.

**Remark 3.22.** By the Bipolar Theorem, for a closed, star-shaped set $A$, we have that $A^\circ\circ = \text{conv}A$. Additionally, Proposition 3.19 above tells us that $h_{A^\circ} = f_{A^\circ\circ} = f_{\text{conv}A}$. However, by Proposition 3.16 as $A$ is closed, $f_{\text{conv}A} = \text{conv}f_A$. Therefore, we have the following representation for the support function $h_{A^\circ}$ in terms of the convex envelope of $f_A$:

$$h_{A^\circ}(X) = \text{conv}f_A(X), \quad X \in \mathcal{X}.$$ (7)

4 Minkowski gauges as deviation measures

In this section we explore deviation measures induced by a Minkowski gauge, interpreting this functional as measuring the amount of shrinkage on a financial position required to accommodate it in the base set $A$ of acceptable positions. Specifically, we present results that link properties of $A$ to financial properties of $f_A$. Whenever possible — and due to its importance for duality —, we also establish a connection with the support function $h_{A^\circ}$.

Before proceeding, let us introduce some further terminology. A non-negative and translation insensitive functional $D: \mathcal{X} \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is called a deviation measure; if moreover $D$ is convex, then it is said to be a convex deviation measure. Non-negativity and translation insensitivity are taken as axioms in defining deviation measures because they capture, respectively, the intuitions that (i) a position whose payoff does not depend on the market outcome should display zero dispersion, and; (ii) adding a fixed amount of cash to a given position should not alter its “degree of non-constancy”. A positive homogeneous, convex deviation measure is said to be a generalized deviation measure. Notice that the sub-level set $A^\circ_B$ of a deviation measure $D$, for $B > 0$, is never empty — indeed, it contains at least the set of all constant positions. Of course, we say that $D$ is law invariant, $\preceq$-monotone, comonotone additive, lower-range dominated; etc, if it fulfills the corresponding properties.

**Remark 4.1.** Rockafellar and Uryasev 2013 proposed measures of error to quantify the “non-zeroness” of a random variable. By definition, a functional $\varepsilon: L^p \rightarrow \mathbb{R}_+ \cup \{\infty\}$ is called a measure of error if it is lower-semicontinuous, sub-linear, positive homogeneous and satisfies (i) $\varepsilon(X) = 0$ if and only if $X = 0$ almost surely; and (ii) if $\lim \varepsilon(X_n) = 0$ then $\lim \mathbb{E}X_n = 0$. By the authors’ Quadrangle Theorem, if $\varepsilon$ is a measure of error, then the functional $D$ defined, for $X \in \mathcal{X}$, by $D(X) := \min_{c \in \mathbb{R}} \varepsilon(X - c)$, is a convex deviation measure. Furthermore, such $D$ is a generalized deviation measure whenever the inequality $\varepsilon(X) \leq \|X\|$ holds for every $X \leq 0$. From this we can conclude that, given a functional $\varepsilon$ satisfying all those conditions, the identity $D(X) = f_{(A^\circ + B)}(X)$ holds — see Lemma 4.11. Indeed, if the minimum is attained, it holds that $f_{(A^\circ + B)}(X) = \inf_{c \in \mathbb{R}} f_A(X - c) = \min_{c \in \mathbb{R}} \varepsilon(X - c) = D(X)$.

We begin, then, by characterizing deviation measures in general. It is specially compelling that the support functional inherits the defining attributes of deviation measures:

**Proposition 4.2.** Let $A \subseteq \mathcal{X}$ be star-shaped. Then:

(i) If $A$ is radially bounded, then $f_{A \cup R}$ and $h_{(A \cup R)^\circ}$ are both non-negative. Thus, if $A$ is radially bounded at non-constants, then $f_A$ and $h_{A^\circ}$ are both non-negative.

(ii) If $A$ is stable under scalar addition, then $f_A$ and $h_{A^\circ}$ are both translation insensitive.

In particular, if $A$ is star-shaped, radially bounded at non-constants and stable under scalar addition, then $f_A$ is a deviation measure.

**Proof.** For the first item, recall from item $[\text{vi}]$ in Proposition 3.13 that $f_A(X) > 0$ for every non-zero $X \in \mathcal{X}$, whenever $A$ is radially bounded. Observe that, as $R$ is a subspace of $\mathcal{X}$, one has $f_R(X) = 0$ for every $X \in R$, whereas $f_R(X) = \infty$, for each (a.s.) non-constant $X$. Then, as $A$ is radially bounded, we have $f_{A \cup R}(X) = \min(f_A(X), f_R(X)) = f_A(X) > 0$, for every $X \notin R$, and for $c \in R$ we have that $f_{A \cup R}(c) = \min(f_A(X), f_R(c)) = \min(f_A(X), 0) = 0$. Now, let $B := A \cup R$. Note that, by item $[\text{iii}]$ of
By definition and positive homogeneity we have

\[ f(A) = \min(h_{A^c}, h_{A^c}). \]

Additionally, by Proposition 2.20, \( h_{A^c}(X) = \emptyset(X) = 0 \) for all \( X \in \mathbb{R} \) and \( h_{A^c}(X) = \infty \) otherwise (i.e. if \( X \) is non-constant). Hence, it is clear that \( h_{B^c}(X) = 0 \) for constant \( X \).

For the case of non-constant \( X \), in order to demonstrate that \( h_{B^c}(X) > 0 \), it is enough to show that in this case \( h_{A^c}(X) > 0 \), since \( h_{B^c}(X) = \infty \) and \( h_{A^c} = \min(h_{A^c}, h_{A^c}). \) We shall do so by showing that \( \{ X \in \mathscr{A} : h_{A^c}(X) = 0 \} = 0 \). Letting \( C \) denote the set in the lefthandside in the latter equation, first note that \( C \) is clearly a cone. Furthermore it is an easy check to show that \( A^c \subseteq C^c \). By definition, then, it holds that \( C^c \subseteq A^c \). This, together with item (vi) in Theorem 2.11 yields that \( C \subseteq C^c \subseteq A^c = \text{cl-conv}A \), and as the closed convex hull of a radially bounded set is again radially bounded, it follows that \( \text{cl-conv}A \) is radially bounded. Now, by definition, the only non-empty cone that is contained in a radially bounded set is the trivial cone \( \{0\} \), it follows that \( C = \{0\} \). Therefore, for \( X \neq 0 \), it holds that \( h_{A^c}(X) > 0 \) and the claim holds. The proof of item (ii) is complete once we recall that \( A \) is radially bounded at non-constants if and only if \( A \setminus \mathbb{R} \) is radially bounded.

For item (ii), since \( \mathbb{R} \subseteq A \) clearly we have \( f_A(c) = 0 \), for every \( c \in \mathbb{R} \). It is also clear that for such a \( c \) one has \( X + c \in A \) if and only if \( X \in A \). In particular the condition \( (X + c)/m \in A \) is equivalent to \( X/m \in A \), hence

\[ f_A(X + c) = \inf \{ m > 0 : (X + c)/m \in A \} = \inf \{ m > 0 : m^{-1}X \in A \} = f_A(X), \]

for any \( c \in \mathbb{R} \). Lastly, recall from item (viii) in Lemma 2.11 that \( (1, X') = 0 \) for all \( X' \in A^c \). By the definition of \( h_{A^c} \), together with linearity we have that \( (1, X') = 0 \) and

\[ h_{A^c}(X + c) = \sup_{X' \in A^c} \langle X, X' \rangle + c(1, X') = h_{A^c}(X) \]

as claimed.

**Remark 4.3.** For a given, non-empty \( A \subseteq \mathscr{A} \), it is always true that \( A + \mathbb{R} \) is stable under scalar addition. Assuming further that \( A \) is star-shaped, closed and radially bounded yields that \( A + \mathbb{R} \) is radially bounded at non-constants. Indeed, by Lemma 2.25 in this case \( A \) contains no proper cone with vertex at some \( x \in \mathbb{R} \). Hence, \( A + \mathbb{R} \) contains no cones other than \( \mathbb{R} \) and \( \{0\} \), that is, \( A + \mathbb{R} \) is radially bounded at non-constants. In this case, \( f_{A + \mathbb{R}} \) is a deviation measure. An (apparent) sensible choice for the acceptance set \( \mathbb{A} \) would be a sub-level set \( A^c \) corresponding to some pre-specified coherent risk measure \( \rho \) (see Section 3 for the definition). However, such a set is never radially bounded. Nevertheless, if we insist on taking \( B := A^c + \mathbb{R} \) in order to force translation insensitivity, then we would have that \( B \equiv \{ X \in \mathscr{A} : \rho < \infty \} \), which again is of no interest as it is clearly a cone, with \( f_B(X) = 0 \) for \( X \in B \) and \( f_B(X) = \infty \) otherwise. Said another way, in this case \( f_B \) is the characteristic function of \( \rho \).

Diversification (or lack thereof) is a very important feature from the financial point of view, a concept which mathematically is expressed in terms of convexity. The next result conveys sufficient conditions to be imposed on the acceptance set in order to ensure that the corresponding Minkowski gauge be a generalized deviation measure. As the polar and its support are convex, irrespective of the chosen acceptance set, we refrain from presenting the analogous result for the support of the polar in the proposition below.

**Proposition 4.4.** Let \( A \subseteq \mathscr{A} \). The following assertions hold:

(i) If \( A \) is convex and \( 0 \in A \), then \( f_A \) is sub-linear.

(ii) If \( A \) is star-shaped and \( A^c \) is convex, then \( f_A \) is super-linear (concave and positive homogeneous) on \( \text{cone}(A^c) \), i.e. \( f_A(X + Y) \geq f_A(X) + f_A(Y) \) for any \( X, Y \in \text{cone}(A^c) \).

In particular if \( A \) is convex, radially bounded, stable under scalar addition and contains the origin, then \( f_A \) is a generalized deviation measure.

**Proof.** We already have positive homogeneity for both items, by Lemma 3.17 as \( 0 \in A \) in each case.

For the first item, it remains to show that \( f_A \) is convex, so fix \( \lambda \in [0, 1] \) and \( X, Y \in \mathscr{A} \). Define

\[ A := \{ \alpha \in \mathbb{R}^*_+ : \lambda X \in A \} \quad \text{and} \quad B := \{ \beta \in \mathbb{R}^*_+ : (1 - \lambda)Y \in \beta A \} \].

By definition and positive homogeneity we have \( \inf A = f_A(\lambda X) = \lambda f_A(X) \) and \( \inf B = f_A((1 - \lambda)Y) = (1 - \lambda)f_A(Y) \). We only need to consider the case where both \( A \) and \( B \) are non-empty, as otherwise the upper bound \( f_A(X + (1 - \lambda)Y) \leq \infty \) holds trivially. Take \( \alpha \in A \) and \( \beta \in B \). Then, convexity of \( A \)
yields \( \lambda X + (1 - \lambda)Y \in (\alpha + \beta)A \), and hence \( f_A(\lambda X + (1 - \lambda)Y) \leq \alpha + \beta \). Therefore, \( f_A(\lambda X + (1 - \lambda)Y) \leq \inf A + \inf B = \lambda f_A(X) + (1 - \lambda)f_A(Y) \).

For the second item, star-shapedness of \( A \) and equation (4.4) tell us that \( f_A = \varphi_{A^c} \). Hence, as positive homogeneity already holds, it suffices to show that \( \varphi_{A^c} \) is a concave functional on \( \text{cone}(A^c) \) whenever \( A^c \) is convex. To see that this is the case, let \( B = A^c \) and fix \( \lambda \in [0,1] \) and \( X, Y \in \text{cone}(A^c) \). Let us first consider the case where \( 0 < \lambda < 1 \) and where both \( X \) and \( Y \) are non-zero. In this scenario the sets \( A_1 := \{ \alpha \in \mathbb{R}_+^* : \lambda X \in \alpha B \} \) and \( B_1 := \{ \beta \in \mathbb{R}_+^* : (1 - \lambda)Y \in \beta B \} \) are both non-empty (for instance, \( X \in \text{cone}(B) \) means precisely that \( X = aZ \) for some \( a > 0 \) and some non-zero \( Z \in B \), and in this case we have \( \lambda a \in A_1 \)). By definition and using positive homogeneity of \( f_A \) together with the equality \( f_A = \varphi_B \), we have \( \sup A_1 = \varphi_B(\lambda X) = \lambda \varphi_B(X) \) and \( \sup B_1 = \varphi_B((1 - \lambda)Y) = (1 - \lambda)\varphi_B(Y) \). Taking \( \alpha \in A_1 \) and \( \beta \in B_1 \), convexity of \( B \) yields \( \lambda X + (1 - \lambda)Y \in (\alpha + \beta)B \), so \( \varphi_B(\lambda X + (1 - \lambda)Y) \geq \alpha + \beta \). Therefore, \( \varphi_B((1 - \lambda)Y) \geq \sup A_1 + \sup B_1 = \lambda \varphi_B(X) + (1 - \lambda)\varphi_B(Y) \). The remaining cases are just a matter of adapting the following argument: if, say, \( \lambda X = 0 \), then \( A = \emptyset \) and \( \varphi_B(\lambda X + (1 - \lambda)Y) = \varphi_B((1 - \lambda)Y) = (1 - \lambda)\varphi_B(Y) = \lambda \varphi_B(X) + (1 - \lambda)\varphi_B(Y) \). This completes the proof.

\[ \blacksquare \]

Remark 4.5. Unfortunately, item (ii) in Proposition 4.4 cannot be relaxed as to accommodate super-linearity of \( f_A \) on the whole \( \mathcal{F} \). However, if we are willing to let go from the identity \( f_A = \varphi_{A^c} \), it is possible to define the cogauge in a slightly different manner, by assigning the value \( \varphi_B(X) := -\infty \) whenever \( \{ n \in \mathbb{R} : n^{-1}X \in B \} = \emptyset \); in this case, an easy adaptation of the proof of item (i) in Proposition 4.4 yields concavity of \( \varphi_B \) for convex \( B \). This alternative definition of the cogauge was studied in [Barbara and Crouzeix (1994)]. To see that the assumptions in item (ii) of Proposition 4.4 do not, in general, yield super-linearity of \( f_A \) on the whole \( \mathcal{F} \), consider the following counterexample, illustrated in Figure 8: let \( \Omega = \{0,1\} \) be the binary market and identify \( L^0 \equiv \mathbb{R}^2 \) as usual. Let \( A := \{ (x, y) \in \mathbb{R}^2 : y - |x| \leq 1 \} \). In this case, the set \( C := A \setminus \text{cone}(A^c) \) is a cone and hence, for any \( X \in C \), we have that \( f_A(X) = 0 \), whereas \( f_A(X) > 0 \) if \( X \notin C \). Now let \( Y = (1, 1/2) \in \text{int} B \), \( Z = (1, 1) \in \text{bd} B \) and \( W = (1, 2) \in \text{bd} A \). We have \( f_A(Z) = 0 < f_A(W) \), but \( Z \) is a convex combination of \( W \) and \( Y \), so \( f_A \) is not concave on the whole domain.

Figure 8: A star-shaped set \( A \) (in gray) with convex complement for which \( f_A \) is not concave.

Consider a cone \( C \) comprised of positions that do not provide any benefit or detriment from diversification. By a \textit{benefit} from diversifying a position \( X \) with an asset \( Y \) we mean that the risk, or dispersion, of the overall portfolio will not increase if we take a convex combination of \( X \) and \( Y \) when compared to any one of the individual positions. In the acceptance set, such reasoning is reflected by noting that if both \( X \) and \( Y \) are acceptable, then their convex combinations cannot be worse — that is to say, convex combinations of acceptable positions are acceptable as well. This rationale says that the acceptance set \( A \), or at least its positions also lying in \( C \), should be a convex set, i.e we should require that \( C \cap A \) be convex. On the other hand, by a \textit{detriment} from diversifying \( X \) with a position \( Y \in C \), we mean the exact opposite: that the risk or dispersion of any convex combination of \( X \) and \( Y \) should not be less then the individual positions. With respect to an acceptance set \( A \), this means that if both \( X \) and \( Y \) are not deemed acceptable \((X, Y \notin A)\), then combining them in a convex fashion yields a unacceptable position as well. Hence, the complement of \( A \) should be convex, at least when restricted to \( C \): we should
also require that $A^c \cap C$ be a convex set. Importantly, when restricted to such a cone, the Minkowski gauge of a star-shaped set $A$ is linear:

**Proposition 4.6.** Let $A$ be a star-shaped set, and let $C \subseteq \text{cone}(A^c)$ be a cone for which both $A \cap C$ and $A^c \cap C$ are convex sets. Then $f_A$ respects $f_A(X + Y) = f_A(X) + f_A(Y)$ for every $X, Y \in C$.

**Proof.** Let $g$ be the restriction of $f_A$ to the cone $C$, i.e. $g: C \to \mathbb{R}_+ \cup \{\infty\}$ is such that $g(X) = f_A(X) = \max(f_A(X), f_C(X)) = f_{A \cap C}(X)$ for all $X \in C$. It suffices to show that $g$ is additive; we shall proceed by showing that this function is concave and sub-linear. Sub-linearity of $g$ is yielded by item (ii) of Proposition 3.11 as $A \cap C$ is a convex set containing the origin by assumption, and thus $f_{A \cap C}$ is sub-linear on the whole $\mathcal{C}$, in particular when restricted to $C$. For concavity, we shall summon the coauge to help us: as $A$ is a star-shaped set, the gauge coincides with the coauge of its complement, i.e. $f_A = f_{A^c}$. Similarly, if we only impose those conditions on $A^c$, then $f_A$ is translation insensitive, i.e. it is a deviation measure.

The next result concerns law invariance.

**Corollary 4.7.** Let $A \subseteq \mathcal{X}$ be radially bounded with $0 \in A$. Suppose both $A$ and $A^c$ are convex for comonotone pairs, i.e. $\lambda X + (1 - \lambda)Y \in A$ for all $\lambda \in [0, 1]$ whenever $X, Y \in A$ are comonotone, and similarly for $A^c$. Then $A$ is star-shaped and $f_A$ is comonotone additive.

**Proof.** First, notice that $A$ is star-shaped. Indeed, any $X \in A$ is comonotone to 0, and by assumption $A$ is convex for this pair, i.e. $\lambda X \equiv \lambda X + (1 - \lambda)0 \in A$ for any $0 \leq \lambda \leq 1$. Furthermore, as $A$ is radially bounded, it follows that $\text{cone}(A^c) = \mathcal{X}$ and so any cone that we may take is contained in $\text{cone}(A^c)$.

Now let $X$ and $Y$ be any comonotone pair. Note that any two members of the set $C_{X,Y} = \text{conv}(\text{cone}(\{X\} \cup \{Y\}))$ are comonotone to one another (see Lemma 2.7). Now, if we take any $Z, W \in C_{X,Y} \cap A$, as they are a comonotone pair, by assumption we have that $\lambda Z + (1 - \lambda)W \in C_{X,Y} \cap A$. Hence, $C_{X,Y} \cap A$ is a convex set. The same argument tells us that $C_{X,Y} \cap A^c$ is also convex. Thus, by Proposition 4.6, we have that $f_A(X + Y) = f_A(X) + f_A(Y)$.

**Remark 4.8.** If the conditions in the corollary above and in Proposition 4.6 are imposed only on $A$ (and not necessarily on $A^c$), then we have in the proposition that $f_A$ is convex on $C$, and in the corollary that $f_A$ is convex for comonotone pairs. Similarly, if we only impose those conditions on $A^c$, then the resulting $f_A$ is concave.

**Remark 4.9.** Note that the assumptions on Corollary 4.7 above — particularly radial boundedness — imply that $f_A(X) > 0$ for any $X \in \mathcal{X} \setminus \{0\}$. Hence, such a set $A$ cannot yield a deviation measure, as it cannot fulfill the axiom of non-negativity. Notwithstanding, we can take $A + R$ as the acceptance set — which, due to item (ii) of Lemma 3.11, yields a Minkowski gauge that satisfies $f_{A + R}(X) = \inf_{\varepsilon \in \mathbb{R}} f_{A}(X - \varepsilon)$. Now let $X$ be non-constant, and notice that any constant is comonotone to $X$. From the Corollary 4.7, we have that $\inf_{\varepsilon \in \mathbb{R}} f_{A}(X - \varepsilon) = f_{A}(X) + \inf_{\varepsilon \in \mathbb{R}} f_{A}(-\varepsilon) = f_{A}(X) > 0$, whereas for constant $X$ it is clear that $\inf_{\varepsilon \in \mathbb{R}} f_{A}(X - \varepsilon) = \inf_{\varepsilon \in \mathbb{R}} f_{A}(0) = 0$. Therefore, $f_{A + R}$ is non-negative, and $A + R$ is clearly stable under scalar addition. Consequently, in view of Proposition 3.12, item (ii) it holds that $f_{A + R}$ is translation insensitive, i.e. it is a deviation measure.

The next result concerns law invariance.

**Proposition 4.10.** If $A \subseteq \mathcal{X}$ is law invariant then $f_A$ is law invariant. Furthermore, if $(\Omega, \mathcal{F}, \mathbb{P})$ is an atomless probability space, then for $\mathcal{X} = L^p$, $p \in [1, \infty)$, it holds that $A^\circ$ and $h_{A^\circ}$ are law invariant. If additionally $A^\circ \subseteq L^1$, then the preceding is also true for $\mathcal{X} = L^\infty$.  

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Proof. First, let $X =_{d} Y \in \mathcal{X}$ and $m \in \mathbb{R}^{+}_{\ast}$. Clearly one has $m^{-1}X =_{d} m^{-1}Y$ and thus, as $A$ is law invariant by assumption, the condition $m^{-1}X \in A$ holds if and only if it holds that $m^{-1}Y \in A$. This leads to

$$f_A(X) = \inf \{ m \in \mathbb{R}^{+}_{\ast} : m^{-1}X \in A \} = \inf \{ m \in \mathbb{R}^{+}_{\ast} : m^{-1}Y \in A \} = f_A(Y).$$

We will establish the second claim for $X \in L^1$. The case $1 < p < \infty$ is analogous. If $X \in A$ then by assumption the set $\{ Y \in L^p : Y =_{d} X \}$ is contained in $A$. Two applications of Lemma 4.60 in Föllmer and Schied (2016) then give us

$$\sup_{Y \in \mathcal{L}_X} \mathbb{E}[XY'] = \int_0^1 f_{X^{-1}}(t)F_{X^{-1}}(t)dt = \sup_{Y' \in \mathcal{L}_{X'}} \mathbb{E}[XY']$$

for $X \in L^1$ and $X' \in L^{\infty}$. Furthermore, it is well known that this fact can be generalized to $X \in L^p$ and $X' \in L^q, p \in [1, \infty)$ — see Filipović and Svindland (2012) for instance. Now, for any $X' \in A^\circ$, the above yields

$$1 \geq \sup_{Y \in \mathcal{L}_X} \mathbb{E}[XY'] = \sup_{Y' \in \mathcal{L}_{X'}} \mathbb{E}[XY'],$$

for all $X \in A$. Therefore, if $X' \in A^\circ$ and $Y' =_{d} X'$, the above gives $\mathbb{E}[XY'] \leq \sup_{Z' \in \mathcal{L}_{X'}} \mathbb{E}[XZ'] \leq 1$ for all $X \in A$, that is to say, $Y' \in A^\circ$. In summary, the polar $A^\circ$ is law invariant. For the support function, since $X \in \mathcal{L}_X = \mathcal{L}_{Y}$, we have

$$h_{A^\circ}(X) \equiv \sup_{X' \in A^\circ} \mathbb{E}[XX'] \leq \sup_{X' \in A^\circ} \sup_{Z' \in \mathcal{L}_{Y'}} \mathbb{E}[ZZ'] = \sup_{X' \in A^\circ} \sup_{Z' \in \mathcal{L}_{X'}} \mathbb{E}[YY'] = \sup_{X' \in A^\circ} \mathbb{E}[YY'] = h_{A^\circ}(Y),$$

where equality (8) follows from the fact that the collection $\{ \mathcal{L}_{X'} : X' \in A^\circ \}$ is a partition of $A^\circ$. Finally, by symmetry we also have the reversed inequality $h_{A^\circ}(Y) \leq h_{A^\circ}(X)$, which establishes the stated result.

**Remark 4.11.** It is a well known result in measure theory that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ has no atoms if and only if there is a random variable $U$ defined on $\Omega$ having a uniform distribution on the unit interval — see for instance Proposition 6.9 in Delbaen (2002). At first sight this seems to exclude important examples such as the binary market, which is usually modelled via the sample space $\Omega = \{0,1\}$ equipped with the discrete $\sigma$-field $\mathcal{F} = 2^{\Omega}$ and a probability measure characterized by a real number $0 \leq p \leq 1$ for which $\mathbb{P}\{1\} = p$. Of course, there is no continuous uniform random variable defined on this space. A possible workaround is as follows: extend the underlying probability space via $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) = (\Omega \times (0,1], 2^\Omega \otimes \text{Borel}(\mathbb{R}), \mathbb{P} \otimes \text{Lebesgue})$ and set $U(\omega,t) := t$ which clearly has a uniform distribution. Now, for each random variable $X$ on the original sample space there corresponds a random variable $\hat{X}$ on the extended space which ‘only depends on the first coordinate’, $\hat{X}(\omega,t) := X(\omega)$. In this framework, by letting $\hat{L}^0$ denote the collection of all such random variables, it is clear that we have the identification $\hat{L}^0 \equiv \mathbb{R}$ and that $\hat{L}^0 \subseteq \hat{L}^\infty(\mathbb{P})$. Thus, we can still apply Proposition 4.10 in the context of the binary market whenever $A \subseteq \hat{L}^0$.

Remember that the “law invariant hull” $\mathcal{L}_B := \{ X \in \mathcal{X} : X =_{d} Y, \text{ for some } Y \in B \}$ of a set $B$, inherits some interesting properties from $B$ — see remark 2.3. We then have the following connection.

**Proposition 4.12.** Let $B \subseteq \mathcal{X}$. Then the equality

$$f_{\mathcal{L}_B}(X) = \inf_{Y \in \mathcal{L}_X} f_B(Y)$$

holds for all $X \in \mathcal{X}$.

**Proof.** Let $X \in \mathcal{X}$ and $m \in \mathbb{R}^{+}_{\ast}$. Now, we have $X/m \in \mathcal{L}_B$ if and only if there exists an $Y \in \mathcal{X}$ such that $X/m =_{d} Y/m$ and $Y/m \in B$, if and only if there exists an $Y \in \mathcal{L}_X$ such that $Y \in mB$. Therefore, $\{ m \in \mathbb{R}^{+}_{\ast} : X/m \in \mathcal{L}_B \} = \bigcup_{Y \in \mathcal{L}_X} \{ m \in \mathbb{R}^{+}_{\ast} : Y \in mB \}$ and then

$$f_{\mathcal{L}_B}(X) = \inf_{Y \in \mathcal{L}_X} \{ m \in \mathbb{R}^{+}_{\ast} : Y \in mB \} = \inf_{Y \in \mathcal{L}_X} \inf \{ m \in \mathbb{R}^{+}_{\ast} : Y \in mB \} = \inf_{Y \in \mathcal{L}_X} f_B(Y),$$

as stated.
We now explore lower-range dominance of $f_A$ and $h_{A\odot}$. It is not a surprise that lower-range dominance of these functionals has a connection with the relation between $A$ and $A_{LR}^1$.

**Proposition 4.13.** Let $\mathcal{X} = L^p$, where $p \in [1, \infty)$. If $A_{LR}^1 \subseteq A$, then $f_A$ and $h_{A\odot}$ are lower-range dominated.

**Proof.** We shall prove the case $\mathcal{X} = L^1$ and $\mathcal{X}' = L^\infty$, since if we show the claim for all $X \in L^1$ then it also holds for the $L^p$ spaces due to the inclusion $L^1 \supseteq L^p$, for $p \in (1, \infty]$. Write $B := A_{LR}^1$. As $B \subseteq A$ and since $LR$ is sub-linear, we have that $f_A(X) \leq f_B(X) = LR(X)$, for all $X \in \mathcal{X}$, by Lemma 3.8 and Proposition 3.12. Therefore, $f_A$ is lower-range dominated. For $h_{A\odot}$, Lemma 2.11 tells us that $A\odot \subseteq B\odot$ and since $B$ is a closed, convex set containing the origin, the dual representation holds. Hence, for $X \in \mathcal{X}$, we have $h_{A\odot}(X) \leq h_{B\odot}(X) = f_B(X) = LR(X)$ as stated. 

**Remark 4.14.** A natural way to force lower-range dominance is by taking an acceptance set of the form $A = B \cup A_{LR}^1$, where $B \subseteq \mathcal{X}$ is a given set of acceptable positions. This yields $f_A = \min(f_B, f_{A_{LR}^1})$. However, while the union operation preserves properties like stability under scalar multiplication, star-shapedness, law invariance and radial boundedness at non-constants, it is possible that convexity may be lost.

**Remark 4.15.** Under the conditions of Proposition 4.13 we have that $X' \leq 1$ for all $X \in A\odot$. To see it, first note that as $B := A_{LR}^1$ is closed and star-shaped (by Theorem 3.8 since LR is lower-semicontinuous and positive homogeneous), Proposition 3.13 then tells us that the polar can be written as $B\odot = \{X' \in \mathcal{X}' : \mathbb{E}[X X'] \leq LR(X) \text{ for all } X \in \mathcal{X}\}$. Therefore we have

$$B\odot = \{X' \in \mathcal{X}' : \mathbb{E}[X X'] \leq LR(X) \forall X \in \mathcal{X}\} = \{X' \in \mathcal{X}' : \mathbb{E}[X (X' - 1)] \leq - \text{ess inf } X \forall X \in \mathcal{X}\}$$

$$= \{X' \in \mathcal{X}' : \mathbb{E}[X (1 - X')] \geq \text{ess inf } X \forall X \in \mathcal{X}\},$$

hence, $\inf_{X' \in B\odot} \mathbb{E}[X (1 - X')] \geq \text{ess inf } X$ for every $X \in \mathcal{X}$. Now, letting $X' \in \mathcal{X}'$ be such that $\text{ess sup } X' > 1$, and, for $\omega \in \Omega$, defining

$$X(\omega) := \begin{cases} 0, & 1 - X'(\omega) \geq 0 \\ 1, & 1 - X'(\omega) < 0, \end{cases}$$

we clearly have that $X \in L^\infty \subseteq L^p$ and $\mathbb{E}[X (1 - X')] < 0$ whereas $\text{ess inf } X = 0$. Therefore $X' \notin B\odot$.

We now explore monotonicity with respect to a given partial order $\preceq$. Despite the fact that this kind of property is not studied much in the literature (both for gauges and deviations), it becomes crucial for decision making.

**Proposition 4.16.** Let $\preceq$ be a partial order that is stable under positive scalar multiplication$\text{7}$ and let $A \subseteq \mathcal{X}$. Then, we have the following:

(i) If $A$ is monotone with respect to $\preceq$, then $f_A$ is anti-monotone with respect to $\preceq$.

(ii) If $A$ is anti-monotone with respect to $\preceq$, then $f_A$ is monotone with respect to $\preceq$.

**Proof.** For the first item, let $X \preceq Y$. If $m \in \mathbb{R}_+$ is such that $m^{-1}X \in A$, then $m^{-1}Y \in A$, as $A$ is monotone. Thus, $\{m \in \mathbb{R}_+ : m^{-1}X \in A\} \subseteq \{m \in \mathbb{R}_+ : m^{-1}Y \in A\}$ and hence $f_A(X) \geq f_A(Y)$. Similarly, for item (ii) let $Y \preceq X$. If $m \in \mathbb{R}_+$ is such that $m^{-1}X \in A$, then $m^{-1}Y \in A$, as $A$ is anti-monotone. Thus, $\{m \in \mathbb{R}_+ : m^{-1}X \in A\} \subseteq \{m \in \mathbb{R}_+ : m^{-1}Y \in A\}$ and hence $f_A(X) \geq f_A(Y)$. 

For the next proposition, recall that $\succeq_D$ denotes the dispersive order of distributions, according to which one has $Y \succeq_D X$ if and only if the inequality $F_Y^{-1}(u) - F_Y^{-1}(v) \leq F_X^{-1}(u) - F_X^{-1}(v)$ holds for every $0 < v < u < 1$.

**Proposition 4.17.** Let $\neq A \subseteq \mathcal{X}$ be $\preceq_D$-anti-monotone. Then

(i) If $(A, \succeq_D)$ has a maximal element $X$, then $A$ is stable under convex combinations of comonotone pairs and radially bounded at non-constants. Furthermore, $f_A$ admits the following representations:

$$f_A(Y) = \inf\{m \in \mathbb{R}_+ : F_Y^{-1}(u) - F_Y^{-1}(v) \leq m(F_X^{-1}(u) - F_X^{-1}(v)), \forall 0 < v < u < 1\}$$

$$= \sup\{m \in \mathbb{R}_+ : F_Y^{-1}(u) - F_Y^{-1}(v) \geq m(F_X^{-1}(u) - F_X^{-1}(v)), \forall 0 < v < u < 1\}$$

$$= \inf\{m \in \mathbb{R}_+ : Y \preceq_D mX\}.$$ 

7That is to say, it holds that $Y \preceq X$ if and only if $\lambda Y \preceq \lambda X$ for all $\lambda \in \mathbb{R}_+$. 

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(ii) If \((A^c, \preceq_D)\) has a minimal element \(X\), then \(A^c\) is stable under convex combinations of comonotone pairs. Furthermore, \(f_A\) admits the following representation:

\[
\begin{align*}
\quad f_A(Y) &= \inf \{m \in \mathbb{R}^+_\times : F_Y^{-1}(u) - F_Y^{-1}(v) < m(F_X^{-1}(u) - F_X^{-1}(v)), \text{ for some } 0 < v < u < 1 \} \\
&= \sup \{m \in \mathbb{R}^+_\times : F_Y^{-1}(u) - F_Y^{-1}(v) \geq m(F_X^{-1}(u) - F_X^{-1}(v)), \forall 0 < v < u < 1 \} \\
&= \sup \{m \in \mathbb{R}^+_\times : mX \preceq_D Y \}.
\end{align*}
\]

**Proof.** Before proceeding, notice that \(A\) is necessarily star-shaped.

For item (i), let \(X\) be a maximal element of \(A\). First, note that the quantile function is comonotone additive, in the sense that \(F_{X+Z} = F_X + F_Z\) whenever \((Y, Z)\) is a comonotone pair — see Lemma 4.90 in [Föllmer and Schied, 2002]. Hence, for any \(W\) that is a convex combination of some comonotone pair \(Y, Z \in A\), it follows that

\[
\begin{align*}
F_W^{-1}(u) - F_W^{-1}(v) &= F_{\lambda Z + (1-\lambda)Y}^{-1}(u) - F_{\lambda Z + (1-\lambda)Y}^{-1}(v) \\
&= \lambda F_Z^{-1} + (1-\lambda)F_Y^{-1}(u) - \lambda F_Z^{-1} + (1-\lambda)F_Y^{-1}(v) \\
&= \lambda (F_Z^{-1}(u) - F_Z^{-1}(v)) + (1-\lambda)(F_Y^{-1}(u) - F_Y^{-1}(v)) \\
&\leq \max (F_Z^{-1}(u) - F_Z^{-1}(v), F_Y^{-1}(u) - F_Y^{-1}(v)) \\
&\leq F_X^{-1}(u) - F_X^{-1}(v),
\end{align*}
\]

for all \(0 < v < u < 1\), which shows that \(W \in A\).

To see that \(A\) is radially bounded at non-constants, note that one has \(F_Y^{-1}(u) - F_Y^{-1}(v) = 0\) for all \(0 < v < u < 1\) if and only if \(Y\) is constant. Hence, for a non-constant \(Y\), there is some \(u\) and \(v\) with \(u > v\) such that \(c := F_Y^{-1}(u) - F_Y^{-1}(v) > 0\). Also, we have that \(\lambda c = F_Y^{-1}(u) - F_Y^{-1}(v)\) for any \(\lambda > 0\). Therefore, as \(k := F_X^{-1}(u) - F_X^{-1}(v) \geq F_Y^{-1}(u) - F_Y^{-1}(v)\), it is obvious that one can find a \(\gamma\) such that for any \(\lambda \geq \gamma\) the inequality \(\lambda c > k\) holds. This implies that \(\lambda Y \preceq X\) never holds, and hence — as \(X\) is a maximal element of \(A\) — we must have \(\lambda Y \notin A\). As \(Y \in A\) was arbitrary, it follows that \(A\) is radially bounded at non-constants.

For the stated representations, note that \(Y \in A\) if and only if \(Y \preceq X\). Therefore, the following holds,

\[
\begin{align*}
\quad f_A(Y) &= \inf \{m \in \mathbb{R}^+_\times : m^{-1}Y \in A \} \\
&= \inf \{m \in \mathbb{R}^+_\times : Y \preceq mX \} \\
&= \inf \{m \in \mathbb{R}^+_\times : F_Y^{-1}(u) - F_Y^{-1}(v) \leq m(F_X^{-1}(u) - F_X^{-1}(v)), \forall 0 < v < u < 1 \}.
\end{align*}
\]

Furthermore, remember that \(0 \in A\) and that, if \(A\) is stable under convex combinations of comonotone pairs, then \(A\) is star-shaped (see Corollary 4.7). Hence we have, by equation (4), that \(f_A = \varphi_A\) and so

\[
\begin{align*}
\quad f_A(Y) &= \varphi_A(Y) \\
&= \sup \{m \in \mathbb{R}^+_\times : m^{-1}Y \in A^c \} \\
&= \sup \{m \in \mathbb{R}^+_\times : m^{-1}Y \notin A \} \\
&= \sup \{m \in \mathbb{R}^+_\times : Y \preceq_D mX \text{ does not hold} \} \\
&= \sup \{m \in \mathbb{R}^+_\times : F_Y^{-1}(u) - F_Y^{-1}(v) > m(F_X^{-1}(u) - F_X^{-1}(v)), \text{ for some } 0 < v < u < 1 \}.
\end{align*}
\]

For the second item, let \(X\) be a minimal element of \(A^c\). First, we shall show that \(A^c\) is monotone with respect to the dispersive order of distributions: let \(Y \in A^c\) and \(Y \preceq Z\). Suppose, by contradiction, that \(Z \in A\). Then, as \(A\) is anti-monotone, we should have \(Y \in A\), an absurd. Hence, \(Z \in A^c\). Now, notice that for any \(W\) that is a convex combination of some comonotone pair \(Y, Z \in A^c\), i.e. \(W = \lambda Z + (1-\lambda)Y\) for some \(\lambda\) in the unit interval, the following holds for all \(0 < v < u < 1\):

\[
\begin{align*}
F_W^{-1}(u) - F_W^{-1}(v) &= F_{\lambda Z + (1-\lambda)Y}^{-1}(u) - F_{\lambda Z + (1-\lambda)Y}^{-1}(v) \\
&= \lambda F_Z^{-1} + (1-\lambda)F_Y^{-1}(u) - \lambda F_Z^{-1} + (1-\lambda)F_Y^{-1}(v) \\
&= \lambda (F_Z^{-1}(u) - F_Z^{-1}(v)) + (1-\lambda)(F_Y^{-1}(u) - F_Y^{-1}(v)) \\
&\geq \min (F_Z^{-1}(u) - F_Z^{-1}(v), F_Y^{-1}(u) - F_Y^{-1}(v)) \\
&\geq F_X^{-1}(u) - F_X^{-1}(v).
\end{align*}
\]

Therefore, as \(A^c\) is monotone w.r.t. \(\preceq_D\), we have \(W \in A^c\).
Finally, for the stated representations note that $Y \in A^c$ if and only if $X \leq Y$. Therefore, the following holds,

$$f_A(Y) = \inf\{m \in \mathbb{R}^+_1: m^{-1}Y \in A\}$$

$$= \inf\{m \in \mathbb{R}^+_1: m^{-1}Y \notin A^c\}$$

$$= \inf\{m \in \mathbb{R}^+_1: mX \preceq Y \text{ does not hold}\}$$

$$= \inf\{m \in \mathbb{R}^+_1: F_Y^{-1}(u) - F_Y^{-1}(v) < m(F_X^{-1}(u) - F_X^{-1}(v)), \text{ for some } 0 < v < u < 1\}.$$

For the second representation, as $A$ is star-shaped, we are once again allowed to summon the cogauge in order to obtain

$$f_A(Y) = \varphi_{A^c}(Y)$$

$$= \sup\{m \in \mathbb{R}^+_1: m^{-1}Y \in A^c\}$$

$$= \sup\{m \in \mathbb{R}^+_1: mX \preceq Y\}$$

$$= \sup\{m \in \mathbb{R}^+_1: F_Y^{-1}(u) - F_Y^{-1}(v) \geq m(F_X^{-1}(u) - F_X^{-1}(v)), \forall 0 < v < u < 1\}.$$

This completes the proof.

An important result in the literature of risk and deviation measures is the following dual representation for convex deviation measures:

**Theorem 4.18** ([Rockafellar et al. (2006a), Theorem 1]). A given functional $D: L^2 \to \mathbb{R}_+ \cup \{+\infty\}$ is a lower-semicontinuous generalized deviation measure if and only if it has a representation of the form

$$D(X) = \mathbb{E}X - \inf_{Q \in \mathcal{Q}} \mathbb{E}[XQ], \text{ for all } X \in L^2$$

(9)

in terms of a convex envelope $\mathcal{Q} \subseteq L^2$ satisfying the following:

(Q1) $\mathcal{Q}$ is non-empty, closed and convex;

(Q2) for each non-constant $X$ there is a $Q \in \mathcal{Q}$ for which $\mathbb{E}(XQ) < \mathbb{E}X$;

(Q3) $\mathbb{E}Q = 1$ for all $Q \in \mathcal{Q}$.

Additionally, the set $\mathcal{Q}$ above is uniquely determined by $D$ through

$$\mathcal{Q} = \{Q \in L^2: D(X) \geq \mathbb{E}X - \mathbb{E}[XQ], \text{ for all } X\},$$

and the finiteness of $D$ is equivalent to boundedness of $\mathcal{Q}$. Furthermore, $D$ is lower-ranging dominated if and only if $\mathcal{Q}$ has the additional property that

(Q4) $Q \geq 0$ for all $Q \in \mathcal{Q}$.

With regard to our framework, we have the following correspondences for the dual representation in the generalized and law invariant cases.

**Corollary 4.19.** Let $A \subseteq \mathcal{X}$. Suppose $A$ is convex, radially bounded, stable under scalar addition and contains the origin. Then $D_A \equiv f_A$ is a generalized deviation measure, and admits the dual representation

$$D_A(X) = \mathbb{E}X - \inf_{Q \in \mathcal{Q}} \mathbb{E}[XQ] = \sup_{X' \in A^\circ} \langle X, X' \rangle = h_{A^\circ}(X), \quad X \in \mathcal{X},$$

where $\mathcal{Q} = 1 - A^\circ$. Furthermore, if $A^+_1 \subseteq A$, then $X' \leq 1$ for any $X' \in A^\circ$.

**Proof.** For the Minkowski gauge of a closed convex set with $0 \in A$, the dual coincides with the support function of the polar — see Proposition 3.20. Furthermore, we have a one-to-one correspondence of the properties of the well known theorem of dual representation for deviation measures of [Rockafellar et al. (2006a)] (see Theorem 4.18 above) with the acceptance set $A$ and its polar $A^\circ \equiv 1 - \mathcal{Q}$. (Q1) is direct from Lemma 2.11.

A closed convex set $A$ is a cone if and only if its polar can be written as

$$A^\circ = \{X' \in \mathcal{X}': \sup_{X \in A} \langle X, X' \rangle \leq 0\},$$

where $f_A := \infty \times I_{A^c}$. Therefore, if $A$ is radially bounded (i.e. $A$ contains no cone with vertex at the origin), there is some $X' \in A^\circ$ such that $\langle X, X' \rangle > 0$, for all $X \in \mathcal{X}$. This yields that, if $A$ is radially
bounded at non-constants, the previous sentence holds for all non-constant \( X \in \mathcal{X} \), which, again under \( Q = 1 - A^\circ \) is equivalent to (Q2).

Stability under scalar addition of \( A \) implies that, for all \( X' \in A^\circ \), it holds that \( \langle 1, X' \rangle = 0 \), and it makes \( h_{A^\circ} \) a translation insensitive functional (Proposition 4.2). This is equivalent to (Q3).

Clearly, the boundedness of \( Q \) is equivalent to the boundedness of \( A^\circ \), which in turn is equivalent to \( A \) being absorbing (see Lemma 2.11) and \( f_A \) finite (see Proposition 3.13). If \( A_{1R}^\circ \subseteq A \), then by Proposition 4.13 one has \( X' < 1 \), for every \( X' \in A^\circ \) and, consequently, (Q4).

**Proposition 4.20.** Assume \((\Omega, \mathcal{F}, \mathbb{P})\) is an atomless probability space, and put \( \mathcal{X} := L^p (p \in [1, \infty]) \). Let moreover \( B \) denote a law invariant, closed, radially bounded, convex subset containing the origin, and define \( A := B + \mathbb{R} \). Then \( f_A \) is a law invariant, lower semicontinuous generalized deviation measure, and the following representation holds, for all \( X \in \mathcal{X} \):

\[
f_A(X) = \sup_{X' \in A^\circ} \int_0^1 F_{X'}^{-1}(t)F_X^{-1}(t)dt = \sup_{\psi \in \Lambda} \int_0^1 \psi(t)F_X^{-1}(t)dt = \sup_{g \in G} \int_0^1 g(t)F_X^{-1}(dt),
\]

where \( \Lambda \) is a collection of nondecreasing functions \( \psi \in L^q[0,1] \) such that \( \int_0^1 \psi(t) dt = 0 \), and where \( G \) is a collection of positive concave functions \( g: [0,1] \to \mathbb{R} \). If in addition \( B^c \) is convex for comonotone pairs, then \( f_A \) is also comonotone additive, then the supremum in the above representations is attained for some, respectively, \( X' \in A^\circ, \psi \in \Lambda, g \in G \), for any \( X \in \mathcal{X} \).

**Proof.** First of all, notice that if \( B \) is a law invariant, closed, radially bounded, convex subset containing the origin, then, by the reasoning in Remark 4.3, the set \( A := B + \mathbb{R} \) is radially bounded at non constants and stable under scalar addition. Furthermore, the operation of set addition preserves convexity, law invariance and closedness; hence, \( A \) is convex, law invariant and closed. This yields that \( f_A \) is a law invariant, lower semi continuous generalized deviation measure. The stated representations follow from Propositions 2.1 and 2.2 of Grechuk et al. (2009). Also, Proposition 2.4 of the same paper yields, under comonotonic additivity — which is given by the convexity for comonotonic pairs of \( B^c \) and remark 4.9 — that \( f_A(X) = \int_0^1 g(t)F_X^{-1}(dt), \) for some positive concave function \( g: [0,1] \to \mathbb{R} \).

**Remark 4.21.** By taking \( B = A^\circ_1 \) for a law invariant measure of error \( \varepsilon \), one obtains a set \( B \) that fulfills the requirements from the above proposition. To ensure comonotonicity, one can take \( B \) of the form \( B = A^\circ_1 \) as above, with the additional requirement that the error measure \( \varepsilon \) be comonotone additive.

### 5 Acceptance sets for deviation measures

So far we have (mostly) focused on the scenario where an acceptance set \( A \) is given, and studied the relations existing between properties of this set and associated properties of its Minkowski gauge, especially how the former manifest on the latter. Now, a special case occurs when the acceptance set itself is already induced by a given, specified \textit{a priori} deviation measure. Remember that under the mild requirement that \( A \) is closed and star-shaped we have \( A = A_{1A}^\circ \) (as ensured by item (iv) in Lemma 3.8). Additionally, item (i) in Proposition 3.14 tells us that a positive homogeneous function \( f \) coincides with the Minkowski gauge of \( A_{1f}^\circ \) (where the requirement that the underlying set be absorbing may be dropped when \( +\infty \in range f \)).

The crucial fact explored in this section is that we actually have a \textit{two-way correspondence} between properties of the functional and the properties of the associated acceptance set. In particular, a lower-semicontinuous, convex deviation measure yields an acceptance set which is stable under scalar addition, convex, closed and radially bounded at non-constants.

**Proposition 5.1.** Let \( \{A(k) : \ k \in \mathbb{R}^*_+\} \) be a positive homogeneous family of subsets of \( \mathcal{X} \). Then, for each real positive \( k \) and \( \lambda \), we have the following:

\[
k f_{A(k)} = \lambda f_{A(\lambda)} = \inf\{m \in \mathbb{R}^*_+ : X \in A(m)\}. \tag{10}
\]

**Proof.** For the first equality, let \( k, \lambda \in \mathbb{R}^*_+ \). Due to positive homogeneity, we have \( k^{-1}\lambda A(k) = A(\lambda) \). Therefore,

\[
k f_{A(k)}(X) = \lambda \inf\{mk\lambda^{-1} \in \mathbb{R}^*_+: X \in mA(k)\} = \lambda \inf\{m \in \mathbb{R}^*_+: X \in mk^{-1}\lambda A(k)\} = \lambda \inf\{m \in \mathbb{R}^*_+: X \in mA(\lambda)\} = \lambda f_{A(\lambda)}(X).
\]
The second equality follows trivially by setting $\lambda = 1$ above and noticing that $mA(1) = A(m)$. ■

**Remark 5.2.** The representation $\inf\{m \in \mathbb{R}_+: X \in A(m)\}$ appearing in equation (10) was studied in the context of risk measures in [Drapeau and Kupper (2013)] under some extra conditions on the family $\{A(k)\}$.

For the theorem below, recall that $A^k_f = \{X \in \mathcal{X}: f(X) \leq k\}$. The following theorem provides a characterization for acceptance sets generated by deviation measures, i.e. sub-level sets corresponding to non-negative, translation insensitive functionals on $\mathcal{X}$. These results are new in the literature, and can be seen as reciprocals for the results studied in the previous sections.

**Theorem 5.3.** Let $f,f': \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be positive homogeneous functionals. Then we have the following, for all positive real $k$,

(i) The collection $\{A^k_f: \lambda \in \mathbb{R}_+\}$ is a positive homogeneous family with each of its members being a star-shaped set. Moreover, if $f$ does not assume negative values, then the following string of equalities holds, for all $X \in \mathcal{X}$:

$$f(X) = f_{A^k_f}(X) = kf_{A^k_f}(X) = \inf \{\lambda \in \mathbb{R}_+: X \in A^k_f\}.$$

(ii) If $f$ is finite, then $A^k_f$ is absorbing.

(iii) If $f$ is translation insensitive, then $A^k_f + \mathbb{R} = A^k_f$.

(iv) If $f$ is non-negative, then $A^k_f$ is radially bounded at non-constants and $\mathbb{R} \subseteq A^k_f$.

(v) If $f$ is a convex functional, then $A^k_f$ is a convex set.

(vi) If $f$ is a concave functional, then $(A^k_f)^\circ$ is a convex set.

(vii) If $f$ is law invariant, then so is $A^k_f$.

(viii) If $f \leq f'$, then $A^k_{f'} \subseteq A^k_f$. In particular, if $f$ is lower-range dominated then $A^k_{LR} \subseteq A^k_f$.

(ix) If $f$ is symmetric, then so is $A^k_f$.

(x) If $f(X) > 0$ for all $X \in \mathcal{X}$, then $A^k_f$ is radially bounded.

(xi) If $f$ respects $f(X + Y) = f(X) + f(Y)$ for $X,Y$ in some convex cone $C$, then $A^k_f \cap C$ and $(A^k_f)^\circ \cap C$ are convex sets. In particular, if $f$ is comonotone additive, then both $A^k_f$ and its complement are stable under convex combinations of comonotone pairs in $\mathcal{X}$.

(xii) If $f$ is lower-semicontinuous, then $A^k_f$ is closed.

(xiii) If $f$ is continuous, then $A^k_f$ is strongly star-shaped.

(xiv) If $f$ is monotone, then $A^k_f$ is anti-monotone and $A^k_{C,f}$ is monotone.

**Proof.** For item (i), star-shapedness of each $A^k_f$ is clear as if $f(X) \leq k$, then $\lambda f(X) \leq k$, for any $\lambda \in [0,1]$. Also, note that (by positive homogeneity of $f$)

$$\lambda A^k_f = \{\lambda X \in \mathcal{X}: f(X) \leq k\} = \{X \in \mathcal{X}: f(X) \leq \lambda k\} = A^{\lambda k}_f,$$

yielding the positive homogeneity for the generated family. It remains to prove that $f = f_{A^k_f}$, as the remaining equalities will follow from Proposition 5.1. Now, with $A = A^1_f$, we have (again by positive homogeneity of $f$)

$$f_A(X) = \inf \left\{m \in \mathbb{R}_+: m^{-1}X \in \{Z: f(Z) \leq 1\} \right\}$$

$$= \inf \left\{m \in \mathbb{R}_+: f(X) \leq m \right\} = f(X).$$

Item (ii) is clear, as if $f$ is a positive homogeneous finite function and $k > 0$ then, for any $X \in \mathcal{X}$ such that $f(X) > 0$ one has $f(\frac{kX}{f(X)}) = k$. Therefore, we have $tX \in A^k_f$ for any $0 \leq t \leq \frac{k}{f(X)}$. Of course, if $f(X) \leq 0$ then there is nothing to prove, as in this case we have $f(X) \leq k$, that is, $X \in A^k_f$. 32
For item (iii), let $Y \in A^k_f + \mathbb{R}$, that is, $Y = X + c$ for some $X \in A^k_f$ (meaning $f(X) \leq k$) and some $c \in \mathbb{R}$. Then $f(Y) = f(X + c) = f(X) \leq k$ as $f$ is translation insensitive. This yields $A^k_f + \mathbb{R} \subseteq A^k_f$. The reverse inclusion holds by definition.

Item (iv) follows from the fact that, for any non-constant $X$, we have $f(X) > 0$ (by assumption). Hence, by positive homogeneity of $f$, there is some $\delta_X := k/f(X) > 0$ such that $f(mX) > k$ for all $m > \delta_X$. Furthermore, as $f(c) = 0 < k$ for any $c \in \mathbb{R}$ it follows that $\mathbb{R} \subseteq A^k_f$.

For item (v), let $X, Y \in A_f^k$ and let $Z$ be any convex combination of $X$ and $Y$. It follows from the convexity of $f$ that $f(Z) \leq \max(f(X), f(Y)) \leq k$, hence the claim holds.

For item (vi), let $B = (A^k_f)^c$, $X, Y \in B$ and assume $Z$ is any convex combination of $X$ and $Y$. It follows from the concavity of $f$ that $f(Z) \geq \min(f(X), f(Y)) > k$, hence the claim holds.

Regarding item (vii), let $X \in A^k_f$ and assume $Y = dX$. Then, due to law invariance of $f$, we have $f(Y) = f(X) \leq k$, that is $Y \in A^k_f$.

For item (viii), let $X \in A^k_f$. Clearly the claim holds, as $f(X) \leq f'(X) \leq k$. The particular case for when $f$ is lower-range dominated is obvious from the definition.

To prove item (ix) simply note that if $X \in A^k_f$, then — due to the symmetry of $f$ — it holds that $f(-X) = f(X) \leq k$, that is $-X \in A^k_f$.

Item (x) follows the same reasoning as item (iv).

For item (xi), note that the restriction of $f$ to $C$ is both convex and concave, hence the convexity of $A^k_f \cap C$ follows the same reasoning that item (v) and the convexity of $(A^k_f)^c \cap C$ from item (vi). For the case when $f$ is additive comonotone, let $X, Y$ be a comonotone pair. Due to Lemma 2.7, the set $C_{X,Y}$ is a convex cone whose members are all comonotone to one another, and $f$ is additive on $C_{X,Y}$. The particular case for $X, Y \in A^k_f$, then $Z \in A^k_f \cap C_{X,Y} \subseteq A^k_f$ whenever $X, Y \in A^k_f$, and similarly $Z \in (A^k_f)^c$ whenever $X, Y \in (A^k_f)^c$.

Item (xii) is just the definition of lower-semicontinuity.

For item (xiii) we shall show only for the case $A := A^1_f$. It holds for general $A^k_f$ due to item (i). By continuity of $f$, we have that $A$ is closed whereas the set $B := \{X \in \mathcal{X} : f(X) < 1\}$ is open. Evidently, $A^c$ is open and $B^c$ is closed, and the inclusions $B \subseteq \text{int} A$ and $A^c \subseteq \text{int}(B^c)$ hold; in particular this gives $0 \in \text{int} A$ as $f$ is positive homogeneous, so $A$ is absorbing and $f(X) = f(A) < \infty$ for all $X$. Therefore, $B^c \cap A = \{X \in \mathcal{X} : f(X) = 1\} = \text{bd}(A)$, where the second equality is yielded by Lemma 3.3. We must show that, for each $X$, the ray $R_X := \{\lambda X : \lambda \in \mathbb{R}_+\}$ intersects $\text{bd}(A)$ at most once. For all $X$ such that $f(X) \leq 0$ it is clear that $R_X \subseteq B$ (so $R_X \cap \text{bd}(A) = \emptyset$). It remains to consider the case $0 < f(X) < \infty$. Clearly, $f(\lambda X) = 1$ for $\lambda^{-1} := f(X)$, so $R_X \cap \text{bd}(A)$ is nonempty. Moreover, if $\gamma > \lambda$ then clearly $f(\gamma X) > 1$ by positive homogeneity, and if $0 < \gamma < \lambda$ then $\gamma X \in B$; in any case $\gamma X \notin \text{bd} A$.

Lastly, for item (xiv) again we shall show only for the case $A := A^1_f$ and $B := A^1_f$, as it holds for general $A^k_f$ and $A^k_f$ due to item (i). Let $Y \in A$ and $X \preceq Y$. Now, remember that for any $Z \in \mathcal{X}$, $Z \in A$ if and only if $f(Z) \leq 1$. Then we have, by monotonicity of $f$, that $f(X) \leq f(Y) \leq 1$, hence $X \in A$, establishing the anti-monotonicity of $A$. By the same token, let $X \in B$ and $X \preceq Y$. Again, we have that for any $Z \in \mathcal{X}$, $Z \in B$ if and only if $-f(Z) \leq 1$, and by anti-monotonicity of $-f$ it follows that $1 \geq -f(X) \geq -f(Y)$. This completes the proof.

Now, we analyze how some operations on a deviation measure are reflected on its corresponding acceptance set. For a comprehensive theory on combinations of monetary risk measures, see |Right (2020)|.

**Proposition 5.4.** Let $f, f' : \mathcal{X} \to \mathbb{R}_+ \cup \{\infty\}$ be positive homogeneous functionals and $k, \lambda \in \mathbb{R}_+$. Then:

(i) $A^k_{\min(f,f')} = A^k_f \cup A^k_{f'}$ and $A^k_{\max(f,f')} = A^k_f \cap A^k_{f'}$.

(ii) $X \in A^k_f$ if and only if there are non-negative constants $c$ and $d$, and positive homogeneous functions $g$ and $h$ such that $k = c + d$, $f = g + h$ and $X \in A^g_c \cap A^h_d$. In particular, one has $A^k_{f + f'} \supseteq A^k_f \cap A^k_{f'}$.

(iii) $A^k_{f'} = \lambda^{-1} A^k_f$.

**Proof.** For the first item, if $X \in A^k_{\min(f,f')}$, then $f(X) \leq k$ or $f'(X) \leq k$. That is, $X \in A^k_f \cup A^k_{f'}$. Reciprocally, if $X \in A^k_f \cup A^k_{f'}$, then we must have $f(X) \leq k$ or $f'(X) \leq k$, so $\min(f(X), f'(X)) \leq k$ which is the same as $X \in A^k_{\min(f,f')}$. The equality $A^k_{\max(f,f')} = A^k_f \cap A^k_{f'}$ follows from a similar argument.
Item (ii) is established as follows: assume \( X \in A_{\sigma}^{k} \cap A_{h}^{d} \), where \( k = c + d \) and \( f = g + h \). Then, by definition, it holds that \( g(X) \leq c \) and \( h(X) \leq d \). Hence, \( f(X) \equiv g(X) + h(X) \leq c + d = k \), which is the same as \( X \in A_{\sigma}^{k} \). For the reverse inclusion, assume \( X \in A_{\sigma}^{k} \). Then, trivially, there are non-negative constants \( c := k \) and \( d := 0 \), and positive homogeneous functions \( g := f \) and \( h := 0 \) such that \( X \in A_{\sigma}^{k} \cap A_{h}^{d} \equiv A_{\sigma}^{k} \). The last equivalence follows from the fact that \( A_{\sigma}^{k} = \{ X \in \mathcal{X} : 0(X) \leq 0 \} \equiv \mathcal{X} \).

Finally, for the last item we have \( X \in A_{\sigma}^{k} \) if and only if \( f(X) \leq k/\lambda \) if and only if \( X \in A_{\sigma}^{k/\lambda} \). The latter set is equal to \( \lambda^{-1}A_{\sigma}^{k} \) by item [iii] in Theorem 5.3.

5.1 Deviation measures: some examples

In this section we discuss a few examples of well-known deviation measures and their respective acceptance sets.

Example 5.5. Variance (\( \sigma^2 \)): One of the most widely used measures to quantify dispersion. It is defined, for \( X \in \mathcal{X} \subseteq L^1 \) (recall that we allow for deviations measures to assume \(+\infty\)), as

\[
\sigma^2(X) = \mathbb{E}[(X - \mathbb{E}X)^2],
\]

and the associated acceptance sets are given by

\[
A_{\sigma^2}^k = \{ X \in \mathcal{X} : \sigma^2(X) \leq k \}, \quad k > 0.
\]

As the variance is not positive homogeneous, it does not coincide with the Minkowski gauge of \( A_{\sigma^2}^k \); indeed, for \( A = A_{\sigma^2}^k \), we have

\[
f_A(X) = \frac{\sigma(X)}{\sqrt{k}}.
\]

Also, notice that \( \sigma^2(X) < \infty \) if and only if \( X \in L^2 \).

Example 5.6. Standard deviation (\( \sigma \)): The measure used to quantify risk in the seminal paper of Markowitz (1952). It has served as inspiration for the class of generalized deviation measures. It is defined, for \( X \in \mathcal{X} \subseteq L^1 \), as

\[
\sigma(X) = \sqrt{\sigma^2(X)} = \|X - \mathbb{E}X\|_2,
\]

and the associated acceptance sets are given by

\[
A_{\sigma}^k = \{ X \in \mathcal{X} : \sigma(X) \leq k \}, \quad k > 0.
\]

(Note that \( A_{\sigma}^k = A_{\sigma^2}^{k^2} \).) If \( X \preceq Y \) then \( \|X - \mathbb{E}X\|_2 \geq \|Y - \mathbb{E}Y\|_2 \), for a detailed proof and more details see Shaked (1982). Furthermore, writing \( A_{\sigma}^k \) := \( A_{\| \cdot \|_2}^k \), we have that

\[
\sigma(X) = k f_{A_{\sigma}^k}(X) = k f_{A_{\| \cdot \|_2}^k}(X),
\]

where the first equality above follows from Theorem 5.5, item [i], and the second one comes from item [ii] in Lemma 5.10, together with the identity \( k f_{A_{\sigma}^k} = \| \cdot \|_2 \) yielded by item [iii] in Theorem 5.3 and the well known fact that \( \inf_{z \in \mathbb{R}} \|X - z\|_2 = \|X - \mathbb{E}X\|_2 = \sigma(X) \) (indeed, \( \| \cdot \|_2 \) is the measure of error associated with the standard deviation). Notice that \( \sigma(X) \) is finite if and only if \( X \in L^2 \). In Figure 5 below, we can see the acceptance set \( A_{\sigma}^1 \) in blue, (note that \( A_{\sigma}^1 = A_{\sigma^2}^{1^2} \)) and the closed unit ball (on the norm \( \| \cdot \|_2 \)) in red. The figure also illustrates the relation \( A_{\sigma}^2 + \mathbb{R} = A_{\sigma}^1 \).

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8 Importantly, here \( \| \cdot \|_2 \) does not represent the Euclidian norm.
Figure 9: The sub-level sets $A^1_{\sigma}$ (in blue) and $A^1_{||\cdot||_2}$ (in red) in the binary market $\Omega = \{0, 1\}$ with $P\{0\} = 1/4$ and $P\{1\} = 3/4$.

Example 5.7. Standard lower-semi-deviation ($\sigma_-$): It is a generalized deviation measure that considers only the negative part of the deviation $X - EX$. This one is defined, for $X \in \mathcal{X} \subseteq L^1$, as

$$\sigma_-(X) = ||(X - EX)^-||_2.$$

The corresponding acceptance sets are given by

$$A^k_{\sigma_-} = \{X \in \mathcal{X} : ||(X - EX)^-||_2 \leq k \} = \{X \in \mathcal{X} : \sigma^2(X | EX \geq X) \leq k^2/P(EX \geq X) \}, \; k > 0,$$

where $\sigma^2(X | EX \geq X) := E \{(X - EX)^2 | EX \geq X \}$ is the conditional variance of $X$ given that $X$ lies in the lower tail of its distribution. Importantly, the set $A^k_{\sigma_-}$ contains every random variable whose standard deviation is bounded above by $k$, as $||(X - EX)^-||_2 \leq ||X - EX||_2$ clearly yields $A^k_{\sigma} \subseteq A^k_{\sigma_-}$. Such fact, can be seen in Figure 10, where the acceptance set $A^1_{\sigma}$ of the standard deviation is depicted in blue, and $A^1_{\sigma_-}$ is represented in red. In particular, $\sigma_-$ is finite on a subspace which is larger than $\{X \in \mathcal{X} : \sigma(X) < \infty \}$. 


Figure 10: The sub-level sets $A_1^\sigma$ (in red) and $A_1^{-\sigma}$ (in blue) in the binary market $\Omega = \{0, 1\}$ with $\mathbb{P}\{0\} = 1/4$ and $\mathbb{P}\{1\} = 3/4$.

**Example 5.8.** Lower range deviation (LR): It is the ‘most conservative’ among the class of lower-range dominated generalized deviation measures, defined for $X \in L^1$ as

$$\text{LR}(X) = \mathbb{E}[X - \text{ess inf } X],$$

with acceptance set

$$A_{LR}^k = \{X \in L^1 : \mathbb{E}X - \text{ess inf } X \leq k\} = \{X \in L^1 : \text{ess sup}(-X) \leq \mathbb{E}[-X] + k\}.$$

Thus, $A_{LR}^k$ is comprised of all positions $X$ whose penalized expected loss $\mathbb{E}(-X) + k$ is bounded below by the maximum loss $\text{ess sup}(-X)$. Furthermore writing $A = \text{ball}_{\|\cdot\|_1}(0; k) \cap L^1_+$, we have that,

$$\text{LR}(X) = k f_{A_{LR}^k}(X) = k f_{A+R}(X).$$

The second equality follows from the fact that $k f_{A}(X)$ assumes $\infty$ for all $X \leq 0$, and equals $\mathbb{E}|X|$ otherwise; thus it coincides with the error function associated to the lower-range deviation — see Lemma 3.11. In Figure 11, we can see the acceptance set $A_{LR}^1$ in blue, and the closed unit ball (on the norm $\|\cdot\|_1$) restricted to $R^+_2$ in red. The fact that $A + R = A_{LR}^1$ is clear from this figure.
Example 5.9. Upper range deviation (UR): Defined, for $X \in \mathcal{X} \subseteq L^1$, as

$$UR(X) = \text{ess sup} X - \mathbb{E}X = LR(-X),$$

this measure is the symmetric opposite of LR. Its acceptance set is given by

$$A^{k}_{UR} = \{X \in \mathcal{X} : \text{ess sup} X - \mathbb{E}X \leq k\} = \{X \in \mathcal{X} : \text{ess sup} X \leq \mathbb{E}X + k\}.$$ 

Furthermore, writing $A = \text{ball}_{\| \cdot \|_1}(0; k) \cap \mathcal{X}_{-}$ we have that

$$UR(X) = k f_{A^{k}_{UR}}(X) = k f_{A^{k}_{LR}}(X),$$

where the second equality follows from the same reasoning as the one for LR.

Example 5.10. Full range deviation (FRD): Can be considered the most extreme generalized deviation measure, defined for $X \in \mathcal{X} = \{X \in L^0 : \text{ess inf} X < \infty \text{ or } \text{ess sup} X > -\infty\}$ as

$$FRD(X) = \text{ess sup} X - \text{ess inf} X,$$

with acceptance set

$$A^{k}_{FRD} = \{X \in \mathcal{X} : \text{ess sup} X \leq k + \text{ess inf} X\}.$$ 

Furthermore, writing $A = A^{k}_{\| \cdot \|_\infty}$ we have that

$$FRD(X) = k f_{A^{k}_{FRD}}(X) = 2 k f_{A^{k}_{LR}}(X),$$

where the second equality is due to Lemma 3.11 and the fact that $2 k f_{A}(X) = 2 \|X\|_\infty$, which is the error function associated to the full range deviation. Note that $FRD(X) < \infty$ if and only if $X \in L^\infty$. In Figure 11, we can see the acceptance set $A^{1}_{LR}$ in blue, and the closed unit ball (on the norm $\| \cdot \|_\infty$), scaled down in half, in red. Clearly, $A^{0.5}_{\| \cdot \|_\infty} + \mathbb{R} = A^{1}_{LR}$. 

Figure 11: The sub-level sets $A^{1}_{LR}$ (in blue) and $A = \text{ball}_{\| \cdot \|_1}(0; 1) \cap \mathcal{X}_{+}$ (in red) in the binary market $\Omega = \{0, 1\}$ with $\mathbb{P}\{0\} = 1/4$ and $\mathbb{P}\{1\} = 3/4$. 

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Figure 12: The sub-level sets $A_{RD}^{k}$ (in blue) and $A = A_{\|\cdot\|_{\infty}}^{0.5}$ (in red) in the binary market $\Omega = \{0,1\}$ with $P\{0\} = 1/4$ and $P\{1\} = 3/4$.

**Example 5.11.** Expected shortfall deviation (ESD): A generalized deviation measure derived from the (standard) expected shortfall. It is defined, for $X \in \mathcal{X} \subseteq L^1$ and $0 < \alpha \leq 1$, by $\text{ESD}_{\alpha}(X) = \text{ES}_{\alpha}(X - \mathbb{E}X)$ with,

$$\text{ES}_{\alpha}(X) = - \int_{0}^{\alpha} \frac{1}{\alpha} F_{X}^{-1}(t) \, dt,$$

and $\text{ESD}_{\alpha}(X) = \mathbb{E}X - \text{ess inf} \, X = \text{LR}(X)$ for $\alpha = 0$. Note that if we take $\gamma = 1 - \alpha$ we have that $\text{ES}_{\alpha}(X) = \int_{\gamma}^{1} \frac{1}{1 - \gamma} F_{X}^{-1}(t) \, dt$. Furthermore, if $F_{X}$ is continuous, then the following representation also holds.

$$\text{ESD}_{\alpha}(X) = \text{ES}_{\alpha}(X - \mathbb{E}X) \equiv -\mathbb{E}(X - \mathbb{E}X \mid X \leq F_{X}^{-1}(\alpha)) = \mathbb{E}(X) - \mathbb{E}(X \mid X \leq F_{X}^{-1}(\alpha))$$

with acceptance set

$$A_{\text{ESD}_{\alpha}}^{k} = \{X \in \mathcal{X} : k - \text{ES}_{\alpha}(X) \geq \mathbb{E}X\}$$

If we let the Koenker-Bassett error be defined as $\text{KB}_{\alpha}(X) = \mathbb{E} \left[ \alpha^{-1}(1 - \alpha)X^{-} + X^{+} \right]$, which is the error function associated with the ESD, then we have $\text{KB}_{\alpha} = k f_A$, with $A = A_{\text{KB}_{\alpha}}^{k}$. Hence — by Lemma 3.11 — it holds that

$$\text{ESD}_{\alpha}(X) = k f_{A_{\text{ESD}_{\alpha}}}^{\text{k}}(X) = k f_{A + \mathbb{R}}(X).$$

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Figure 13: The sublevel sets $A^k_{\text{ESD}}$ (in blue) and $A = A^k_{\text{KB,} \alpha}$ (in red), with $\alpha = 0.1$, in the binary market $\Omega = \{0, 1\}$ with $P\{0\} = \frac{1}{4}$ and $P\{1\} = \frac{3}{4}$.

6 Minkowski gauge and monetary risk measures

In this section we study the relation between Minkowski gauges and (positive homogeneous) monetary risk measures. First, we recall some terminology. A translation invariant and anti-monotone functional that does not attain $-\infty$ is said to be a monetary risk measure, which we denote generically by $\rho: X \rightarrow \mathbb{R} \cup \{\infty\}$. If such a $\rho$ is moreover convex, then we say that $\rho$ is a convex risk measure. A positive homogeneous, convex risk measure is said to be a coherent risk measure. A sub-level set of a monetary risk measure is never empty; yielding well defined acceptance sets $A^k_{\rho} = \{X \in X: \rho(X) \leq k\}$ for $k \geq 0$.

There is a compelling interplay between coherent risk measures and generalized deviation measures: Rockafellar et al. (2006a) show that any coherent risk measure $\rho$ gives rise to a generalized deviation measure via $D_\rho(X) := \rho(X - \mathbb{E}X)$, whenever $\rho$ is strict, i.e. $\rho(X) > \mathbb{E}X$ for non-constant $X$. Conversely, a generalized deviation measure which is lower-range dominated turns out to deliver a coherent risk measure through $\rho(X) := -\mathbb{E}X + D(X)$.

Any monetary risk measure — say, $\rho$ — maps onto the real line; thus, such $\rho$ clearly cannot be expressed as a gauge, although it does admit a representation of the form

$$\rho(X) = \inf \{m \in \mathbb{R}: X + m \in A^0_{\rho}\}, \ X \in \mathcal{X}. \tag{11}$$

Notwithstanding, if $\rho$ is positive homogeneous, then its positive and negative parts (denoted respectively by $\rho^+$ and $\rho^-$) can easily be written as gauges. It is evident that $A^0_{\rho} \subseteq A^1_{\rho}$: the acceptance set $A := A^1_{\rho}$ contains the cone of all “riskless positions”, i.e. those $X$ such that $\rho(X) \leq 0$. Therefore, $f_A(X)$ can appraise the risk of a position $X$. By the same token, $A^1_{-\rho}$ contains the cone of the non-acceptable positions, i.e. those $X$ such that $\rho(X) > 0$, so $f_{A^1_{-\rho}}(X)$ assesses the “risklessness” of a given position $X$. The next results formalizes the preceding line of thought, and also provides a general representation for positive homogeneous monetary risk measures — and does so without resorting to convexity assumptions, thus characterizing a departure from the standard approach found in the literature, where convexity plays a major role.

**Theorem 6.1.** Let $\rho: \mathcal{X} \rightarrow \mathbb{R} \cup \{\infty\}$ be a monetary risk measure. If $\rho$ is positive homogeneous, then it holds that $\rho^+ = f_{A^1_{\rho}}$ and $\rho^- = f_{A^1_{-\rho}}$. In particular, one has

$$\rho(X) = f_{A^1_{\rho}}(X) - f_{A^1_{-\rho}}(X)$$

9Recall that in this paper we use the term acceptance set to refer to sub-level sets of the form $A^k_{\rho} := \{X \in \mathcal{X}: \rho(X) \leq k\}$ whereas in the literature on monetary risk measures an acceptance set takes the form $\{X \in \mathcal{X}: \rho(X) \leq 0\}$.
for all $X \in \mathcal{X}$.

Proof. As $\rho$ does not attain $-\infty$ and is positive homogeneous, we have that $-\rho$ is also positive homogeneous and does not attain $+\infty$, so $A_{-}\rho$ is an absorbing set and $f_{A_{-}\rho}$ is finite. This avoids the indeterminacy $-\infty - \infty$. Now let $B_{-} := \{X \in \mathcal{X}: \rho(X) \leq 0\}$ and $B_{+} := \{X \in \mathcal{X}: \rho(X) \geq 0\}$. Clearly, for $X \in B_{+}$ we have $f_{A_{+}\rho}(X) = 0 = \rho^{+}(X)$, and similarly for $X \in B_{-}$ we have $f_{A_{-}\rho}(X) = 0 = \rho^{-}(X)$.

Moreover, whenever $X \in B_{-}$ it holds that

$$f_{A_{-}\rho}(X) = \inf \{ m \in R_{+}^{*}: -\rho(X) \leq m \} = -\rho(X) = \rho^{-}(X),$$

as $-\rho(X) \geq 0$, and for $X \in B_{+}$ we have

$$f_{A_{+}\rho}(X) = \inf \{ m \in R_{+}^{*}: \rho(X) \leq m \} = \rho(X) = \rho^{+}(X),$$

as $\rho(X) \geq 0$. This establishes the equalities $\rho^{+} = f_{A_{+}\rho}$ and $\rho^{-} = f_{A_{-}\rho}$. □

Remark 6.2. Note that the Theorem 6.1 holds even if $\rho$ is not monetary. In fact any positive homogeneous functional $f: \mathcal{X} \rightarrow R \cup \{\infty\}$ can be represented by $f(X) = f_{A_{+}\rho}(X) - f_{A_{-}\rho}(X)$.

Now, assume we are given four sets (let us call them $C_{+}$, $C_{-}$, $A_{+}$ and $A_{-}$) that compose a “system of acceptable and non-acceptable positions”, or simply a risk system. We shall interpret $C_{+}$ as a cone of positions deemed “riskless”, i.e. with no positive risk (this set must contain the cone of non-negative random variables), and $C_{-}$ as a cone of risky assets, i.e. those with non-negative risk (a set which contains, at least, the set of non-positive random variables). Clearly, we want any position to lie in at least one of those sets, so $C_{+} \cup C_{-} = \mathcal{X}$. These two sets give us information on whether a position has positive or negative risk. However, such information is “not enough” if our objective is to differentiate — through gauges — how risky (or riskless) a position is. Therefore, we require two additional sets in our system: one (call it $A_{+}$) to gauge the riskiness of a position and the other (call it $A_{-}$) to gauge its risklessness. Fortunately, we have so far introduced and developed the tools that allow us to formalize this reasoning.

We shall make the requirement that the set $A_{+}$ contain the cone $C_{+}$ of riskless positions, but we are flexible in allowing that it also comprises some risky, but acceptable, positions. As convex combinations should not increase risk, $A_{+}$ should also be convex. Additionally, as $X \leq Y$ clearly implies that $X$ has greater risk than $Y$, we demand that $A_{+}$ be monotone as well. Furthermore, to asses the “risklessness” of the position, we make the requirement that the complement of $A_{-}$ only contain positions that yield at least as much as a risk free position. In other words, we should have $1 \in bd(A_{+}^{c}) = bd(A_{-})$, and $A_{-}^{c}$ should be monotone and convex (for the same reason that we want $A_{+}$ to be monotone and convex), yielding anti-monotonicity of $A_{-}$. We require that $A_{-}$ contain $C_{-}$. As an example of such a system of sets, suppose we have already been handed a given continuous, coherent risk measure, say $\rho$. Taking then $C_{+} := A^{\rho}\rho$, $C_{-} := A_{+}^{\rho}\rho$, $A_{+} := A_{+}\rho$, and $A_{-} := A_{-}\rho$ provides a system with the required properties (see Theorem 6.3). Figure 14 illustrates another possible choice for $A_{+}$ and $A_{-}$ in the context of the binary market, i.e. with $\mathcal{X} = R^{2}$. Recall that $\mathcal{X}$ is the set comprised of positions $X \geq 0$ whereas $\mathcal{X}$ contains all positions $X \leq 0$. For the next definition, recall that $R_{X} = (0, +\infty)X$.

Definition 6.3. Let $A := (A_{+}, A_{-}, C_{+}, C_{-})$ be a quadruple of non-empty subsets of $\mathcal{X}$. We say that $A$ is a risk system on $\mathcal{X}$ if the following conditions are satisfied:

(A1) $A_{+}$ is strongly star-shaped, closed, and $\leq$-monotone, with $-1 \in bd(A_{+})$ and $0 \in int(A_{+})$.

(A2) $C_{+}$ is a conic, $\leq$-monotone subset of $\mathcal{X}$. Moreover, the inclusion $C \subseteq C_{+} \subseteq A_{+}$ holds for any cone $C$ contained in $A_{+}$. 

(B1) $A_{-}$ is strongly star-shaped, closed, and $\leq$-anti-monotone, with $1 \in bd(A_{-})$ and $0 \in int(A_{-})$.

(B2) $C_{-}$ is a conic, $\leq$-anti-monotone subset of $\mathcal{X}$. Moreover, the inclusion $C \subseteq C_{-} \subseteq A_{-}$ holds for any cone $C$ contained in $A_{-}$.

(C) $C_{+} \cup C_{-} = \mathcal{X}$, and $C_{+} \cap C_{-}$ has an empty interior.

(D) For every $X \notin C_{+}$, the sets $A_{+} \cap (R_{+}X + R_{-})$ and $A_{+}^{c} \cap (R_{+}X + R_{-})$ are convex. Similarly, for every $X \notin C_{-}$, the sets $A_{-} \cap (R_{+}X + R_{+})$ and $A_{-}^{c} \cap (R_{+}X + R_{+})$ are convex.
If the risk system further respects the requirement that

(E) \( A_+ \) is convex and \( A_- \) has convex complement,

then \( A \) is said to be a \textbf{coherent risk system on} \( \mathcal{X} \). We say a risk measure is \textbf{generated by the risk system} \( A \) if it takes the form \( \rho_A := f_{A_+} - f_{A_-} \). Similarly, we say that the quadruple \( A_{\rho} = (A^1_{\rho}, A^0_{\rho}, A^0_{\rho}, A^0_{\rho}) \) is the \textbf{risk system generated by} \( \rho \) whenever \( \rho \) is a continuous, positive homogeneous monetary risk measure.

\textbf{Remark 6.4.} The functional \( \rho_A \) appearing in the above definition, which we have boldly referred to as a risk measure, is indeed a monetary risk measure, as ensured by Theorem 6.3 below. In fact, this Theorem tells us that every risk system on \( \mathcal{X} \) gives rise to a continuous, positive homogeneous, monetary risk measure. The latter result is complemented by Corollary 6.10 which tells us that if the generating risk system is moreover required to be coherent, then the corresponding function is in fact a continuous coherent risk measure. Theorem 6.8 provides the reciprocal to these assertions.

\textbf{Remark 6.5.} It will be convenient, in order to make the proofs of the results below clearer, to introduce the set \( K_X \), defined for \( x \in (C_- \cap C_+)^c \), as

\[
K_X := \begin{cases} 
\mathbb{R}_+ X + \mathbb{R}_+, & x \notin C_- \\
\mathbb{R}_+ X + \mathbb{R}_-, & x \notin C_+ 
\end{cases}
\]

(12)

In this case, assumption \( \text{[D]} \) can be restated as

\( \text{(D')} \) If \( x \notin C_+ \), then \( A_+ \cap K_X \) and \( A^- \cap K_X \) are convex. If \( x \notin C_- \), then \( A_- \cap K_X \) and \( A^+ \cap K_X \) are convex.

\textbf{Remark 6.6.} The requirements \( 0 \in \text{int}(A_+) \) and \( 0 \in \text{int}(A_-) \) in assumptions \( \text{[A1]} \) and \( \text{[B1]} \) can be relaxed if we consider in \( \mathcal{X} \) a topology not coarser than the (metric) topology of convergence in probability. Indeed, if that requirements are dropped from those assumptions, it still holds that \( 0 \in \text{int} A_+ \) and \( 0 \in \text{int} A_- \). To see that this is the case, notice that, since \( A_+ \) is a strongly star-shaped set with \( -1 \in \text{bd} A_+ \), one has that \( \lambda \in \text{int} A_+ \) for any \( \lambda \in (-1, 0) \). In particular, \( -1/2 \in \text{int} A_+ \). Now proceed by contradiction: assume \( 0 \in \text{bd} A_+ \), so that there is a sequence \( Z_n \notin A_+ \) converging to the origin in probability. For such a sequence, let \( X_n := Z_n \mathbb{I}_{|Z_n| \leq 0} \). Note that monotonicity of \( A_+ \) forces \( X_n \notin A_+, \) since \( X_n \leq Z_n \). Moreover, as it also holds that \( |X_n| \leq |Z_n| \), we have for any \( \varepsilon > 0 \) that

\[
P(\omega: |X_n(\omega)| \geq \varepsilon) \leq P(\omega: |Z_n(\omega)| \geq \varepsilon) \to 0 \quad \text{as} \quad n \to \infty.
\]

Therefore, \( X_n \to 0 \) in probability. Let now \( W_n := X_n - 1/2 \). We have \( W_n \to -1/2 \) in probability, but monotonicity of \( A_+ \) and the fact that \( X_n \notin A_+ \) oblige \( W_n \notin A_+ \) for all \( n \). This contradicts the fact that \( -1/2 \in \text{int} A_+ \). A similar argument yields \( 0 \in \text{int}(A_-) \).

We begin by obtaining some properties of the sets in a risk system.

\textbf{Lemma 6.7.} Let \( A = (A_+, A_-, C_+, C_-) \) be a risk system on \( \mathcal{X} \). Then the following holds:

(i) \( C_+ \) and \( C_- \) are closed cones, with \( \mathcal{X}_+ \subseteq C_+ \) and \( \mathcal{X}_- \subseteq C_- \).

(ii) \( 1 \in \text{int} C_+ \) and \( -1 \in \text{int} C_- \).

(iii) \( \rho_A \) is a continuous function.

(iv) \( \text{bd} C_+ = \text{bd} C_- = C_+ \cap C_- \) is a cone and \( \rho_A(X) = 0 \) if and only if \( X \in C_+ \cap C_- \).

(v) \( \mathbb{R}_+ C_+ + \mathbb{R}_+ \subseteq C_+ \) and \( \mathbb{R}_+ C_- + \mathbb{R}_- \subseteq C_- \).

\textbf{Proof.} We shall prove the statements concerning \( C_+ \) and \( A_+ \), as the ones for \( C_- \) and \( A_- \) are analogous.

For item (i), assumption \( \text{[A2]} \) requires that \( C_+ \) be the largest conic set contained in \( A_+ \). This is well defined since an arbitrary union of cones is again a cone, and we know by assumption \( \text{[A1]} \) that \( A_+ \) contains at least one cone, namely \( \mathcal{X}_+ \). Assumption \( \text{[A2]} \) then tells us that \( \text{cl}(C_+) \subseteq C_+ \subseteq A_+ \) since \( A_+ \) is closed and the closure of a cone is again a cone.

For the second item, the assumption that \( 1 \in \text{bd} A_- \) tells us that it must be the case that \( 1 \notin C_- \) — otherwise, we would have \( \lambda \in \text{bd} A_- \) for every \( \lambda > 0 \), violating the requirement that \( A_- \) be also strongly star-shaped. Since \( C_- \) is closed by item (i), we have that \( C_- \) is open, and assumption \( \text{[C]} \) entails \( C_- \subseteq C_+ \). Therefore, \( 1 \in \text{int} C_+ \).
For item (iii), we have that the functional \( \rho = \rho_A \) is continuous, as both \( f_{A+} \) and \( f_{A-} \) are continuous due to assumptions [A1] and [A2] and Proposition 5.13.

For (iv), the fact that \( C_+ \cap C_- \) in item (v) is a (closed) cone is immediate. Furthermore, note that \( \rho(X) \leq 0 \) if and only if \( X \in C_+ \) (see item [iii] in Lemma 5.3), i.e. \( \{ X \in \mathcal{X} : \rho(X) \leq 0 \} = C_+ \), additionally, continuity of \( \rho \) yields that \( B := \{ X \in \mathcal{X} : \rho(X) < 0 \} \) is an open set, and clearly \( \text{cl} B = C_+ \), hence, \( \text{bd} C_+ = \{ X \in \mathcal{X} : \rho(X) = 0 \} = \text{bd} C_- \). For the last equality, is enough to show that if \( \rho(X) < 0 \) then \( X \notin C_- \), however it is clear as \( \rho(X) \geq 0 \) if and only if \( X \in C_- \). This also yields that \( \rho(X) = 0 \) if and only if \( X \in C_+ \cap C_- \).

The fifth item is easily seen to hold: indeed, if \( X \in C_+ \) and \( y, z \geq 0 \), then concity of \( C_+ \) entails \( yX \in C_+ \) and monotonicity then yields \( yX + z \in C_+ \) since \( yX + z \geq yX \).

First, we provide some sufficient conditions granting that a risk measure results in a risk system.

**Theorem 6.8.** If \( \rho \) is a continuous, positive homogeneous monetary risk measure, then the quadruple \( \mathcal{A}_\rho = (A_+ , A_- , C_+ , C_- ) := (A_+^\rho , A_-^\rho , A_0^\rho , A_0^{0}\rho) \) is a risk system. Additionally, if \( \rho \) is coherent, then \( \mathcal{A}_\rho \) is a coherent risk system.

**Proof.** For monotonicity in the assumptions, first note that due to the \( \leq \)-anti-monotonicity of \( \rho \), \( A_+^\rho \) and \( A_0^\rho \) are clearly \( \leq \)-monotone, while due to the monotonicity of \( -\rho \), \( A_-^\rho \) and \( A_0^{0}\rho \) are \( \leq \)-anti-monotone, see Theorem 5.3 item [xiv]. For assumption [A1] and [B2] the continuity of \( \rho \) and \( -\rho \) entail strong star-shapedness of \( A_+^\rho \) and \( A_-^\rho \) — see Theorem 5.3 item [xiii]. Continuity also yields that both sets are closed, and that they contain \( 0 \) in its interior. To see that the latter is true, take the pre-image \( B \) of the open unit ball on the real line, that is, \( B = \{ X : |f(X)| < 1 \} \); by definition of continuity, \( B \) is an open set in \( \mathcal{X} \), contained in \( A_+^\rho \) and containing the origin. To see that \( 1 \in \text{bd} A_+^\rho \), since we already have that \( A_+^\rho \) is strongly star-shaped and closed, and since \( \rho(0 + 1) = \rho(0) - 1 \) by translation invariance and positive homogeneity of \( \rho \), clearly \( -\rho(1) = 1 \) and thus \( 1 \in A_+^\rho \). Finally, for any \( \delta > 0 \) we have \( -\rho((1 + \delta) - (\rho(1))) = 1 + \delta > 1 \), so the constant position \( 1 + \delta \) does not lie in \( A_+^\rho \), and similarly \( -\rho((1 - \delta)) < 1 \) so \( 1 - \delta \) in \( A_+^\rho \). This establishes \( 1 \in \text{bd} A_+^\rho \). The argument for \( -1 \in \text{bd} A_-^\rho \) is similar. To see that \( \mathcal{X}_+ \subseteq A_+^\rho \), just notice that we have — by anti-monotonicity of \( \rho \) — that \( \rho(X) \leq \rho(0) = 0 \leq 1 \) whenever \( 0 \leq X \) (\( X \in \mathcal{X}_+ \)). Similarly, monotonicity of \( -\rho \) tells us that \( -\rho(X) \leq -\rho(0) = 0 \leq 1 \) whenever \( X \leq 0 \) (\( X \in \mathcal{X}_- \)). To demonstrate that \( A_0^\rho \) is the largest cone (clearly it is a cone) contained in \( A_+^\rho \), it suffices to show that the set \( \{ 0 \} \cup (A_+^\rho \setminus A_0^\rho) \) contains no proper cone (otherwise, if \( C \) were such a cone, then \( C \cap A_0^\rho \) would be a cone contained in \( A_+^\rho \) larger than \( A_0^\rho \)). But \( \{ 0 \} \cup (A_+^\rho \setminus A_0^\rho) \equiv \{ 0 \} \cup \{ X \in \mathcal{X} : \rho(X) \in [0,1] \} \), and positive homogeneity of \( \rho \) forbids the latter set from containing any proper cone. The same reasoning yields that \( A_0^{0}\rho \) is the largest cone contained in \( A_+^\rho \).

For assumption [C1], it is obvious that \( A_0^\rho \cup A_0^{0}\rho = \mathcal{X} \), and since \( A_+^\rho \cap A_0^{0}\rho = \text{bd}(A_+^\rho) \cap \text{bd}(A_0^{0}\rho) \) clearly this set has an empty interior.

For assumption [D1] we have to show that, for \( X \) as above, the sets \( A_+ \cap K_X \) and \( (A_+^\rho)^c \cap K_X \) are convex, where \( K_X \) is defined as in equation (12). Let us start with \( X \notin C_+ \equiv A_+^\rho \). First of all, notice that \( K_X \) is a convex cone: indeed, let \( Z, Y \in K_X \). By definition, \( Z = \lambda X + z \) for some \( \lambda \geq 0 \) and some \( z \leq 0 \), and similarly \( Y = \delta X + y \) for some \( \delta \geq 0 \) and some \( y \leq 0 \). Then, if \( \alpha \geq 0 \), clearly \( \alpha Z = (\alpha \lambda)X + \alpha z \in K_X \), and if \( \alpha \in [0,1] \), we have \( \alpha Z + (1 - \alpha)Y = (\alpha \lambda + (1 - \alpha)\delta)X + \alpha z + (1 - \alpha)y \in K_X \), thus establishing that \( K_X \) is a convex cone. Moreover, for \( Z, Y \) as above, we have \( \rho(Z + Y) = \rho((\lambda + \delta)X + (z + y)) = (\lambda + \delta)\rho(X) - (z + y) = \rho(\lambda X + z) + \rho(\delta X + y) = \rho(Z) + \rho(Y) \). Now it is just a matter of evoking item [xii] of Theorem 5.3 to conclude that both \( K_X \cap A_+^\rho \) and \( K_X \cap (A_+^\rho)^c \) are convex sets. The same line of thought will show that assumption [D2] holds for \( X \notin C_+ \).

Lastly, if \( \rho \) is coherent (in particular, convex) then \( A_+^\rho \) is convex by item [vi] in Theorem 5.3 and since \( -\rho \) is concave, item [vi] of the same theorem tells us that \( (A_+^\rho)^c \) is a convex set.
The generating functional is not translation invariant.

Let $\rho : \mathcal{F} \to \mathbb{R}_+$ be a risk measure, and assume that $\rho$ satisfies the assumptions of Lemma 3.8 and item (i) of Lemma 3.11. The quadruple $(A_+, A_-, C_+, C_-)$ in this example is not a risk system, since assumption (D) in the definition is not satisfied. For example, taking $X \notin C_-$ as depicted below, the set $K_X \cap A_+^*$ is not convex. The problem here arises from the fact that the generating functional is not translation invariant.

![Figure 14: Sets $A_+ = \{(x,y) \in \mathbb{R}^2 : y \geq e^{-(1+x^2)} - 2\}$ (in blue) and $A_- = \{(x,y) \in \mathbb{R}^2 : \min(xy,x|y|) \leq 1\}$ (in red). Here, one has $C_+ = \mathcal{F}_+$ and $C_- = \mathcal{F}(\mathcal{F}^c \setminus \mathcal{F}_+)$. The quadruple $(A_+, A_-, C_+, C_-)$ in this example is not a risk system, since assumption (D) in the definition is not satisfied. For example, taking $X \notin C_-$ as depicted below, the set $K_X \cap A_+^*$ is not convex. The problem here arises from the fact that the generating functional is not translation invariant.]

**Theorem 6.9.** Let $\mathcal{A}$ be a risk system. Then the functional $\rho_\mathcal{A}$ defined, for $X \in \mathcal{F}$, via $\rho_\mathcal{A}(X) = f_{A_+}(X) - f_{A_-}(X) + \varphi_{C_-}(X)$ is a continuous, positive homogeneous monetary risk measure. Furthermore, $\rho_\mathcal{A}^+ = f_{A_+}$ and $\rho_\mathcal{A}^- = f_{A_-}$.

**Proof.** To ease notation, let $\rho := \rho_\mathcal{A}$. Clearly, $\rho$ does not attain $\infty (-\infty)$ as $A_+ (A_-)$ contains an open neighborhood of 0, which makes it an absorbing set and, therefore, we have $f_{A_+} < \infty (f_{A_-} < \infty)$. Thus, $\rho = f_{A_+} - f_{A_-} \in \mathbb{R}$. Furthermore, as assumption (C) gives us $C_+ \cup C_- = \mathcal{F}$, we have then by item (ii) of Lemma 3.8 and item (ii) of Lemma 3.11 that $f_{A_+} = f_{A_-} = f_\mathcal{A}$ for all $X \in \mathcal{F}$, i.e. at most one of $f_{A_+}(X)$ and $f_{A_-}(X)$ is non-zero. This yields $\rho^+ = f_{A_+}$ and $\rho^- = f_{A_-}$. To see that $\rho$ is anti-monotone, we have by Proposition 4.16 together with assumptions (A1) and (B1) that $f_{C_+}$ and $-f_{C_-}$ are anti-monotone functions, so their sum is as well.

Positive homogeneity of $\rho$ is inherited, directly from the definition, from $f_{A_+}$ and $f_{A_-}$.

Continuity of $f_{A_+}$ and $f_{A_-}$ follows from the fact that $A_-$ and $A_+$ are closed, strongly star-shaped and $0 \in \text{int } A_-$ (see Proposition 3.18). Hence, $\rho$ as the sum of continuous functions, is continuous.

We now proceed to establish translation invariance in each one of the following scenarios, for $\alpha \geq 0$:

1. $X + \alpha$ with $X \notin C_-$;
2. $X - \alpha$ with $X \notin C_+$;
3. $X \pm \alpha$ with $X \in C_- \cap C_+$;
4. $X - \alpha$ with $X \notin C_-$, broken in the respective subcases:
   (a) $X - \alpha \notin C_-$;
   (b) $X - \alpha \in C_- \cap C_+$;
   (c) $X - \alpha \notin C_+$;
5. $X + \alpha$ with $X \notin C_+$.

For 1 and 2, note that if $X \notin C_-$ then $X + \alpha \notin C_-$ for all $\alpha \in \mathbb{R}_+$, and similarly, if $X \notin C_+$ then $X - \alpha \notin C_+$ for all $\alpha \in \mathbb{R}_+$. (by $\leq$-anti-monotonicity of $C_-$ and $\leq$-monotonicity of $C_+$, respectively.)
Moreover, as $C_-$ is the largest conic set contained in $A_-$, we have that $A_- \setminus C_-$ is radially bounded. This yields that cone $((A_- \setminus C_-)^c)$ is empty. Moreover, we have that $\text{cone}((A_- \setminus C_-)^c) = \text{cone}(X \setminus A_-) \cup C_- = \text{cone}(A_+^c) \cup C_-$. Then, since cone $((A_+^c) \cap C_- = \{0\}$, we obtain $\text{cone}((A_- \setminus C_-)^c) \cup C_- = \text{cone}(A_+^c) \setminus \{0\}$. Hence, from assumption [C] we have that the condition $X \notin C_-$ implies $X \in \text{cone}(A_+^c)$. From this we have, by $\leq$-monotonicity of $C_+$ in assumption [A2] that for $X \notin C_-$ it holds that $K_X \subseteq \text{cone}(A_+^c)$ (where $K_X$ is defined as in equation (12)).

By Proposition [4.2] and assumption [D] — and recalling that $f_{A_-}(1) = 1$ because $A_-$ is a strongly star-shaped set with $1 \in \text{bd}(A_-)$ — we have, for $X \notin C_-$ and $\alpha \in \mathbb{R}_+$,

$$f_{A_-}(X + \alpha) = f_{A_-}(X) + \alpha f_{A_-}(1) = f_{A_-}(X) + \alpha$$

and $f_{A_+}(X) = f_{A_+}(X + \alpha) = 0$. Hence,

$$\rho(X + \alpha) = f_{A_+}(X + \alpha) - f_{A_-}(X + \alpha) = -f_{A_-}(X - \alpha) = \rho(X) - \alpha,$$

fulfilling item 1. On the other hand, for item 2, letting $X \notin C_+$ and $\alpha \in \mathbb{R}_+$ we have $f_{A_+}(X) = f_{A_+}(X - \alpha) = 0$ and, by the same reasoning as above (item 1), now using the fact that $f_{A_+}(-1) = 1$,

$$f_{A_+}(X - \alpha) = f_{A_+}(X + \alpha \cdot (-1)) = f_{A_+}(X) + z f_{A_+}(-1) = f_{A_+}(X) + \alpha,$$

and then

$$\rho(X - \alpha) = f_{A_+}(X - \alpha) - f_{A_-}(X - \alpha) = f_{A_+}(X) + \alpha = \rho(X) + \alpha.$$

Now, for item 3, let $X \in C_+ \cap C_-$, and let $X_n := X + 1/n$, so that $\{X_n\} \subseteq (C_+ \cap C_-) = C_-^c$ is a sequence converging to $X$. To see that $X + 1/n \notin C_-$ whenever $X \in C_+ \cap C_-$ notice that $X/2 \in \text{bd}(C_+)$ and $1/(2n) \in \text{int}(C_+)$, the latter being true since otherwise we would have $\mathbb{R}_+ \subseteq \text{bd}(C_-)$; therefore, we have $2(X/2 + 1/(2n)) \in \text{int}(C_+)$. Note that, as $A_-$ is strongly star-shaped with 0 in its interior, $f_{A_-}$ is continuous, and so is $\rho^\ast$. Therefore, using item 1 and given $\alpha \in \mathbb{R}_+$,

$$\rho(X + \alpha) = -f_{A_-}(X + \alpha) = -\lim_{n \to \infty} f_{A_-}(X_n + \alpha) = -(\lim_{n \to \infty} f_{A_-}(X_n + \alpha)) = -f_{A_-}(X) - c,$$

and so $\rho(X + \alpha) = \rho(X) - c$. Similarly,

$$0 = \rho(X) = \rho(X + \alpha - \alpha) = \rho(X - \alpha) - \alpha,$$

and then $\rho(X - \alpha) = \alpha = \alpha + \rho(X)$.

We now focus on the remaining cases, items 4 and 5, and in fact we shall consider item 4 only, since the other is quite similar. In what follows, then, we are taking $X \notin C_-$. For such $X$, we know that $R_X$ is not entirely contained in $A_-$. (otherwise, the ray $\mathbb{R}_+ X$ is a conic set contained in $A_-$, and, a fortiori, in $C_-$). Therefore, by star-shapedness of $A_-$ and item [11] in Lemma 3.4 we get that $f_{A_+}(X) > 0$, which in turn implies $\rho(X) < 0$. Now, let $\alpha \in \mathbb{R}_+$ and define $Y = X - \alpha$. From assumption [C] we then have three possibilities: either $Y \notin C_-$ (item 4.a), or $Y \in C_+ \cap C_-$ (item 4.b), or else $Y \notin C_+^c$ (item 4.c).

Items 4.a and 4.b follow from the identities

$$\rho(X) = \rho(Y + \alpha) = \rho(Y) = \rho(X - \alpha),$$

which yield $\rho(X - \alpha) = \rho(X) + \alpha$, (the equality * comes from case 1 for item 4.a, and from case 3 for item 4.b).

Lastly, for 4.c (that is, $X \notin C_-^c$ and $Y = X - \alpha \notin C_+$), we will first show that there exists some scalar $m$ such that $X - m \in C_+ \cap C_-$. Let $s$ be a real, positive constant. We know that $X \in \text{int}(C_+)$ since $C_-$ is open, and also $s \in \text{int}(C_+)$. Therefore, $X + s \in \text{int}(C_+)$, as $C_-$ is monotone — being the complement of an anti-monotone set —, and so $X + s \notin C_-$. By the same token we have $X - s \notin C_-$ for small enough $s$. Furthermore, we know that there is some real $k > 0$ such that $X - k \notin C_+$ (take $k = 1$), so let us define the sets

$$S := \{s \in \mathbb{R}: X - s \notin C_-\} \quad \text{and} \quad K := \{k \in \mathbb{R}: X - k \notin C_+\},$$

and put moreover $s^* := \sup S$ and $k^* := \inf K$. Henceforth, all $k \in K$ and all $s \in S$. Due to monotonicity (and anti-monotonicity) of $C_-$ and $C_+$, we have that $s < k$. Furthermore, we know that $X - s^* \in C_-$
and \( X - k \ast \in C_+ \). Now let \( m \) be such that \( s \leq s^* \leq m \leq k^* \leq k \), then due to monotonicity of \( C_+ \) it follows that \( X - m \in C_+ \) and due to anti-monotonicity of \( C_- \) we have that \( X - m \in C_- \). Hence, \( X - m \in C_+ \cap C_- \), which yields \( \rho(X - m) = 0 \), and by case 3, \( \rho(X - m) = \rho(X) + m \), i.e. \( \rho(X) = -m \). Thus, \( X + \rho(X) \in C_+ \cap C_- \). Indeed, we have \( Z \subset C_+ \cap C_- \) if and only if \( \rho(Z) = 0 \). Therefore, \( \rho(X + \rho(X) - \beta) = \rho(X + \rho(X)) + \beta \) for \( \beta \in \mathbb{R} \), by case 3. Furthermore, as \( X + \rho(X) \in C_+ \) and \( Y = X - \alpha \not\in C_+ \), monotonicity of \( C_+ \) tells us that \( \mathbb{P}(X - \alpha \geq X + \rho(X)) < 1 \) and then \( \rho(X) \geq -\alpha \). By letting \( \gamma := \rho(X) + \alpha \geq 0 \), we finally have that

\[
\rho(X - \alpha) = \rho(X + \rho(X) - \gamma) = \rho(X + \rho(X)) + \gamma = \rho(X) + \gamma - \rho(X) = \rho(X) + \alpha.
\]

This completes the proof.

**Corollary 6.10.** If \( A \) is a coherent risk system, then the functional \( \rho_A := f_{A_+} - f_{A_-} \) is a continuous, coherent risk measure.

**Proof.** Let \( \rho := \rho_A \). From Theorem 6.5, the only thing remaining to be established is convexity of \( \rho \). We shall to separate it in four instances: 1) \( X, Y \in C_- \); 2) \( X, Y \not\in C_- \); 3) \( X \in C_-, Y \not\in C_- \); 4) \( Y \in C_-, X \not\in C_- \). Note that, as we are under translation invariance it is enough to show only quasi-convexity, i.e. \( \rho(\lambda X + (1 - \lambda)Y) \leq \max(\rho(X), \rho(Y)) \) for any \( X, Y \in \mathcal{X} \) and any \( \lambda \in [0, 1] \), from which convexity follows. In order to see it, note that

\[
\rho(\lambda X + (1 - \lambda)Y) - \lambda \rho(X) - (1 - \lambda)\rho(Y) = \rho(\lambda(X + \rho(X)) + (1 - \lambda)(Y + \rho(Y))) \\
\leq \max(\rho(X + \rho(X)), \rho(Y + \rho(Y))) \\
= 0.
\]

Hence, \( \rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y) \).

For the first instance note that, due to Proposition 4.4 and assumption \( \{E\} \) \( f_{A_+} \) is convex. From this, we have that \( \rho \) is convex for pairs \( X, Y \in C_- \), (as in such case \( f_{A_+}(X) = f_{A_+}(Y) = 0 \)).

For the second instance, due to Proposition 4.4 and assumption \( \{E\} \) \( f_{A_+} \) is concave when restricted to \( \text{cone}(A_-^e) \). Thus, we have that \( \rho \) is convex for \( X, Y \in \text{cone}(A_-^e) \) (as \( f_{A_+}(X) = f_{A_+}(Y) = 0 \) and \( f_{A_-} \) is convex). We shall show that \( C_- \subset \text{cone}(A_-^e) \). Remember that the complement of a cone (union with 0) is also a cone. By assumptions \( \{B2\} \) and \( \{C\} \) we have the implications \( C_- \subset A_- \Rightarrow [0] \cup C_- \supseteq \text{cone}(A_-^e) \Rightarrow \text{cone}(A_-^e) \supseteq [0] \cup \text{cone}(A_-^e)^c \). Now, by assumption \( \{B2\} \) \( C_- \) is the largest cone contained in \( A_- \). Thus, \( C_- = (\text{cone}(A_-^e))^c \cup [0] \supseteq (\text{cone}(A_-^e))^c \), and taking the complement one last time implies \( C_- \subseteq \text{cone}(A_-^e) \), as desired.

Lastly, note that the third and fourth cases are actually equivalents. Thus, let \( X \in C_- \) and \( Y \not\in C_- \); this yields that \( f_{A_+}(X) = f_{A_+}(Y) = 0 \). Hence, by assumption \( \{E\} \) we have for any \( \lambda \in [0, 1] \),

\[
\rho(\lambda X + (1 - \lambda)Y) \leq \lambda f_{A_+}(X) + (1 - \lambda)f_{A_+}(Y) = f_{A_+}(\lambda X - (1 - \lambda)Y) \\
\leq f_{A_+}(X) - f_{A_+}(X) \\
= \rho(X) \leq \max(\rho(X), \rho(Y)).
\]

Hence, \( \rho \) is quasi-convex, and, by Theorem 6.5 is also translation invariant. Therefore, \( \rho \) is convex, and, the result holds.

**Corollary 6.11.** Let \( A = (A_+, A_-, C_+, C_-) \) be a risk system on \( \mathcal{X}^e \). Then the risk system generated by \( \rho_A \) coincides with \( A \), that is, \( A \rho_A = A \). Moreover, if \( \rho \) is a continuous, positive homogeneous monetary risk measure, then \( \rho = \rho_A \).

**Proof.** Write \( \rho := \rho_A \) for simplicity. Note that, we have to show the following identities:

\[
A_+ = A_+^e, \quad A_- = A_-^e, \quad C_+ = A_0^e, \quad \text{and} \quad C_- = A_0^e.
\]

First note that by Theorems 6.1 and 6.9, we have that \( \rho^+ = f_{A_+} = f_{A_+^e} \) and \( \rho^- = f_{A_-} = f_{A_-^e} \). Now, \( A_0^e = A_0^e \), hence, as \( A_+ \) is closed and star-shaped, we get \( A_0^e = A_+ \), by item [iv] in Lemma 3.8. For the equality \( A_- = A_-^e \), note that \( A_0^e = A_0^e = A_0^e \), and by the same reasoning as above, \( A_- = A_-^e \). That \( C_+ = A_0^e \) (respectively, \( C_- = A_0^e \)) easily follows from noticing that \( A_0^e \) (respectively, \( A_0^e \)) is the largest cone contained in \( A_0^e \) (respectively \( A_0^e \)).

For the remaining statement, let \( \rho \) be a continuous positive homogeneous monetary risk measure. That \( \rho = \rho_A \) follows directly from Theorem 6.1 and the preceding items, by noting that \( \rho_{A_+} = f_{A_+} - f_{A_-} \).

\[\text{45}\]
Remark 6.12. If one seeks additional protection against risk, it is possible to further penalize a given monetary acceptance set. For example, if \( \rho \) is a coherent risk measure and \( c > 0 \) is the “amount of additional protection required”, one can obtain an acceptance set of the form

\[
A_\rho - c = \{ X \in \mathcal{X} : \rho(X) \leq 0 \} - c = \{ X : \rho(X) \leq -c \}.
\]

However, this approach does not take variability in account; if the aim is to appraise variability, then one can take the intersection \( A_\rho^c \cap A_D^k \), where \( D \) is some generalized deviation measure. A possible way to extract a risk measure from this set is to take

\[
\inf \{ m \in \mathbb{R} : X + m \in \lambda (A_\rho^c \cap A_D^k) \} = \rho(X) + \frac{c}{k} D(X).
\]

Moreover, by Proposition 4.7 of Righi (2019), if

\[
\inf_{D(X)>0} \left( \frac{\text{ess sup}(-X) - \rho(X)}{D(X)} \right) \geq \frac{c}{k},
\]

then \( \rho(X) + \frac{c}{k} D(X) \) is coherent as well.

Remark 6.13. Given a convex lower-semicontinuous function \( f : \mathcal{X} \to \mathbb{R} \cup \{ \infty \} \), the space \( L^f := \{ X \in \mathcal{X} : f_A^1(|X|) < \infty \} \) generated by the Minkowski gauge of its sub-level set, inspired on Orlicz spaces, is a Banach space. This approach was studied, for instance, in Kupper and Svindland (2011), Owari (2014), Svindland (2009) and Liebrich and Svindland (2017) on the context of risk measures and the so-called economic index of riskiness.

6.1 Monetary measures: some examples

In this section we discuss a few examples of well-known monetary risk measures and their respective acceptance sets in our framework.

Example 6.14. Expected Loss (E): One of the simplest monetary measures. It is defined, for \( X \in \mathcal{X} \subseteq L^1 \), as

\[ E(X) = \mathbb{E}[-X] \]

and the associated acceptance sets are given by

\[
A_E^1 = \{ X \in \mathcal{X} : \mathbb{E}(X) \geq -1 \} \supseteq A_E^0 = \{ X \in \mathcal{X} : \mathbb{E}(X) \geq 0 \}
\]

\[
A_{-E}^1 = \{ X \in \mathcal{X} : \mathbb{E}(X) \leq 1 \} \supseteq A_{-E}^0 = \{ X \in \mathcal{X} : \mathbb{E}(X) \leq 0 \}.
\]

Clearly, taking \( A_+ = A_E^1 \), \( C_+ = A_E^0 \), \( A_- = A_{-E}^1 \) and \( C_- = A_{-E}^0 \) yields a risk system — see Figure 15 where the black line represents \( C_+ \cap C_- \). Therefore, due to Theorem 6.1, the Expected Loss has the following representation.

\[ E(X) = f_{A_E^1}(X) - f_{A_{-E}^1}(X) = f_{A_E^0}(X) - f_{A_{-E}^0}(X) \]
Figure 15: Risk system of Expected Loss in the binary market $\Omega = \{0, 1\}$, with $\mathbb{P}\{0\} = \frac{1}{4}$ and $\mathbb{P}\{1\} = \frac{3}{4}$. The thick diagonal through the origin is $C_+ \cap C_-$. 

Example 6.15. Mean plus standard deviation ($E\sigma_{\beta}$): It is a well known measure of risk, defined as 

$$E\sigma_{\beta}(X) = E(X) + \beta \sigma(X),$$

where $\beta \in \mathbb{R}$ is a prescribed constant. This risk measure may fail to be anti-monotone if 

$$\beta > \inf_{\sigma(X) > 0} \left( \frac{\text{ess sup}(-X) - E(X)}{\sigma(X)} \right).$$

(See remark 6.12). The associated acceptance sets are given by 

$$A^1_{E\sigma_{\beta}} = \{X \in \mathcal{X}: E(1 + X) \geq \beta \sigma(X)\} \supseteq A^0_{E\sigma_{\beta}} = \{X \in \mathcal{X}: E(X) \geq \beta \sigma(X)\}$$

$$A^1_{-E\sigma_{\beta}} = \{X \in \mathcal{X}: E(X - 1) \leq \beta \sigma(X)\} \supseteq A^0_{-E\sigma_{\beta}} = \{X \in \mathcal{X}: E(X) \leq \beta \sigma(X)\},$$

Again, by taking a suitable $\beta$, we have that $A_+ = A^1_{E\sigma_{\beta}}$, $C_+ = A^0_{E\sigma_{\beta}}$, $A_- = A^1_{-E\sigma_{\beta}}$ and $C_- = A^0_{-E\sigma_{\beta}}$ yield a risk system in $\mathcal{X}$. Besides, the representation given by Theorem 6.1 due to Example 5.6 is 

$$E\sigma_{\beta}(X) = E(X) + f_{A^2_{-1}}(X).$$

Unfortunately, in $\mathbb{R}^2$, the only suitable choice of $\beta$ is $\beta = 0$; hence, in this setting the sets $A^1_{E\sigma_{\beta}}$ are always of the form represented in Figure 15 and there is no risk system that generates $E\sigma_{\beta}$.
Figure 16: $A_{E\sigma_1}$ and $A_{E\sigma_2}$ in the binary market $\Omega = \{0, 1\}$, with $P\{0\} = 1/4$ and $P\{1\} = 3/4$.

Example 6.16. Maximum loss (ML): Defined for $X \in \mathcal{X}$ as

$$ML(X) = -\text{ess inf } X = \text{ess sup}(-X),$$

with acceptance sets

$$A_{ML}^1 = \{X \in \mathcal{X} : P(X < -1) = 0\} \supseteq A_{ML}^0 = \{X \in \mathcal{X} : P(X < 0) = 0\} = \mathcal{X}^-$$

$$A_{ML}^{-1} = \{X \in \mathcal{X} : P(X \leq 1) > 0\} \supseteq A_{ML}^0 = \{X \in \mathcal{X} : P(X \leq 0) > 0\}.$$

Figure 17 illustrates the maximum loss on $\mathcal{X} = \mathbb{R}^2$; note that the sets appearing here are the same for all probability measure absolutely continuous with respect to $P$. Furthermore notice that $C_+ = \mathbb{R}_+^2$.

Figure 17: Risk system of Maximum Loss.
Remark 6.17. The above example does give rise to a coherent risk system, as ML is a continuous coherent risk measure on $\mathbb{R}^2$. However, if $\mathcal{F} = L^p$ is infinite dimensional, where $p \in [0, \infty)$, then ML may fail to be lower semi-continuous, in which case neither $A_- \equiv A^-_{\text{ML}}$ nor $C_- \equiv A^0_{\text{ML}}$ are closed. In fact, the following counterexample shows that, if $\mathcal{F} = L^0(0,1]$, then we cannot have a risk system with $C_+ = \mathcal{F}_+$: let $(\Omega, \mathcal{F}, \mathbb{P}) = ((0,1], \text{Leb})$ be the unit interval equipped with Lebesgue measure. Let $K_+ = \mathcal{F}_+ = \{X \in L^0: X \geq 0\}$ and $K_- = \text{cl}(K_+^\circ)$. Clearly $K_+$ is monotone and so $K_-$ is anti-monotone. We now show that there $1 \notin \text{int}(K_+)$ in the topology of convergence in probability. Let

$$X_n := \mathbb{I}(1/n, 1] - \mathbb{I}(0, 1/n], \quad n \in \mathbb{N}.$$ 

Clearly $X_n \notin K_+$, and $X_n \to 1$ in probability, so $1 \in \text{bd}(K_+)$ = $\text{bd}(K_-)$. In fact, an easy adaptation of the argument above shows that $K_+$ has empty interior! In particular, $K_+ \subseteq K_-$. Thus, the sets $K_+$ and $K_-$ above never coincide with the maximal cones $C_+$ and $C_-$ from a risk system $(A_+, A_-, C_+, C_-)$, since Lemma 6.7 ensures that $1 \notin \text{int}(K_+)$. 

Example 6.18. Value at Risk (VaR$_\alpha$): A (non-convex) monetary risk measure defined, for $X \in \mathcal{F} \subseteq L^0$ and $0 < \alpha < 1$, as

$$\text{VaR}_\alpha(X) := -\inf\{x \in \mathbb{R}: F_X(x) \geq \alpha\} = -F^{-1}_X(\alpha)$$

with acceptance sets

$$A^1_{\text{VaR}_\alpha} = \{X \in \mathcal{F}: \mathbb{P}(X < -1) \leq \alpha\} \supseteq A^0_{\text{VaR}_\alpha} = \{X \in \mathcal{F}: \mathbb{P}(X < 0) \leq \alpha\} \supseteq A^{-1}_{\text{VaR}_\alpha} = \{X \in \mathcal{F}: \mathbb{P}(X \leq 0) \geq \alpha\}.$$

The representation of the risk system for the Value at Risk on the plane is actually equal to the one for Maximum Loss, whenever $\alpha \leq \min(\mathbb{P}\{0\}, \mathbb{P}\{1\})$ — see Figure 17. If $\alpha \geq \max(\mathbb{P}\{0\}, \mathbb{P}\{1\})$ then the graph of $\text{VaR}_\alpha(−X)$ coincides with the one in Figure 17.

Example 6.19. Expected Shortfall (ES$_\alpha$): A coherent risk measure defined for $X \in \mathcal{F} \subseteq L^1$ and $0 \leq \alpha \leq 1$ as

$$\text{ES}_\alpha(X) := -\int_0^\alpha \frac{1}{\alpha} F^{-1}_X(t) \, dt \equiv \text{ESD}_\alpha(X) - \mathbb{E}[X],$$

with acceptance sets

$$A^1_{\text{ES}_\alpha} = \left\{X \in \mathcal{F}: \int_0^\alpha F^{-1}_X(t) \, dt \geq -\alpha\right\} \supseteq A^0_{\text{ES}_\alpha} = \left\{X \in \mathcal{F}: \int_0^\alpha F^{-1}_X(t) \, dt \geq 0\right\} \supseteq A^{-1}_{\text{ES}_\alpha} = \left\{X \in \mathcal{F}: \int_0^\alpha F^{-1}_X(t) \, dt \leq 0\right\}.$$ 

Figure 18 illustrates the risk system of ES$_\alpha$ in the context of the binary market.

Figure 18: Risk system of Expected Shortfall in the binary market $\Omega = \{0, 1\}$, with $\mathbb{P}\{0\} = 0.05$ and $\mathbb{P}\{1\} = 0.95$. 

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