Rigidity of Asymptotically Hyperbolic Manifolds

Yuguang Shi ∗
Key Laboratory of Pure and Applied Mathematics
School of Mathematics Science,
Peking University
Beijing, 100871, China
email:ygshi@math.pku.edu.cn

Gang Tian †
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA02139
email:tian@math.mit.edu

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Abstract
In this paper, we prove a rigidity theorem of asymptotically hyperbolic manifolds only under the assumptions on curvature. Its proof is based on analyzing asymptotic structures of such manifolds at infinity and a volume comparison theorem.

1 Introduction
In this paper, we study the rigidity problem for asymptotically hyperbolic manifolds. Much progress has been made on this problem. In [7], using the Dirac operator, Min-oo proved that a spin manifold of dimension $n$ must be a hyperbolic space if it is asymptotic to hyperbolic space in a strong sense and its scalar curvature is not less than $-n(n-1)$. His argument was refined and new exciting results were obtained by Andersson and Dahl [2] and X.Wang [11]. For even dimensional manifolds, Leung proved in [6] that any conformally compact Einstein manifold $(\mathbb{B}^n,g)$ which is asymptotically hyperbolic of order greater than $2$ must be hyperbolic. By exploring properties of positive eigenfunctions,

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J. Qing proved that a conformally compact Einstein manifold with round sphere as its conformal infinity has to be a hyperbolic space when the dimension is not greater than 7 (cf. [8]). He did not need to assume that the manifold considered is spin. However, his approach relies on the positive mass theorem for asymptotically flat manifolds. In all above results, one needs to assume that there are nice coordinates at infinity and in such coordinates, the metrics tensor behaves well. In view of geometry, it would be natural to ask whether such an assumption can be replaced by an intrinsic geometric condition. In this paper, we will show a rigidity theorem of this type only under the assumption on curvature.

Let \((X^{n+1}, g)\) be a complete noncompact Riemannian manifold, we call it an asymptotically locally hyperbolic manifold, which we abbreviate as ALH in the following, of order \(\alpha\) if \(|K(x) + 1| = O(e^{-\alpha \rho(x)})\), where \(K(x)\) is the sectional curvature of \(g\) at point \(x\) in any direction and \(\rho(x) = dist_g(x, o)\).

Recall that a Riemannian manifold \(X\) has a pole \(o\) if the exponential map \(\exp_o : T_oX \to X\) is a diffeomorphism. Without loss of generality, in our case, we may assume that the sectional curvature is negative outside a unit ball of \((X, g)\). We have:

**Theorem 1.1.** Suppose that \((X^{n+1}, g)\) \(n \geq 2\) and \(n \neq 3\) is an ALH manifold of order \(\alpha\) with a pole and there is a \(\rho > 1\) such that the geodesic sphere with radius \(\rho\) and center at the pole is convex. If we further have \(\alpha > 2\) and \(\text{Ric}(g) \geq -ng\), then \((X^{n+1}, g)\) is isometric to \(H^{n+1}\).

As a corollary, we have:

**Corollary 1.2.** Suppose that \((X^{n+1}, g)\) \(n \geq 2\) and \(n \neq 3\) is an ALH manifold of order \(\alpha\) \((\alpha > 2)\), \(K \leq 0\) and \(\text{Ric}(g) \geq -ng\), then \((X^{n+1}, g)\) is isometric to \(H^{n+1}\).

Let \(Rm^0\) denotes the traceless part of the curvature tensor\(^1\), \(\|Rm^0\|\) denote the norm of the tensor for \((X, g)\), then for \(n = 3\), we have:

**Theorem 1.3.** Suppose that \((X^4, g)\) is an ALH manifold of order \(\alpha > 2\) with a pole and there is a \(\rho > 1\) such that the geodesic sphere with radius \(\rho\) and center at the pole is convex. If we further have \(\|Rm^0\| \in L^1(X)\) and \(\text{Ric}(g) \geq -3g\), then \((X^4, g)\) is isometric to \(H^4\).

We will use the volume comparison theorem to prove above theorem. In order to use the volume comparison, we need to estimate the volume growth of geodesic spheres at infinity. We will carry this out in several steps. First, we show that by changing the metric conformally, we can compactify \((X, g)\) in an appropriate way. Next, we will show that the boundary of compactified Riemannian manifold is isometric to standard sphere, in this step, we first verify that the boundary is conformal to the standard sphere. It follows from the assumption on curvature that the boundary is diffeomorphic to the standard sphere, hence, it suffices to show that the boundary is locally conformally flat.

\(^1\)The metric \(g\) is of constant sectional curvature iff \(Rm^0\) vanishes. This property determines \(Rm^0\) uniquely.
By a direct computation, we can show that the Weyl tensor of the boundary vanishes, if the induced metric on the boundary is sufficiently smooth we know that it is locally conformally flat. However, since the metric on the compactified boundary is not necessarily smooth enough, we have to check what the locally conformal flatness of the boundary means in our current case. Under the assumption on Ricci curvatures in above theorem, we observe that the scalar curvature and volume of the boundary of compactified manifolds is less or equal to those of the standard sphere. It follows that the scalar curvature of the boundary is actually equal to that of the standard sphere, hence, if \( n = 2 \), we see that the boundary is isometric to the standard sphere; if \( n \geq 3 \), then by Obata’s theorem, we know that the boundary is also isometric to the standard sphere; Finally, we can show that the volume of geodesic spheres of \((X,g)\) is equal to that of corresponding geodesic spheres in \( \mathbb{H}^{n+1} \) with the same radius, then, by the volume comparison theorem, we prove the main theorem.

This assumption \( \alpha > 2 \) should be optimal, since there are many asymptotically hyperbolic Einstein metrics on \( \mathbb{B}^4 \) with \( \alpha = 2 \), we refer the readers to Theorem C and Appendix in \([1]\) for details. In the case of \( n = 3 \), in order to show locally conformal flatness of the boundary, one has to check that certain linear combination of covariant derivatives of Schouten tensor vanishes, for time being, we do not know how to deduce this one from the assumption \( \alpha > 2 \), this is the reason why we need the extra assumption \( \|Rm^0\|_{g} \in L^1(X) \), we doubt its necessity. We also think that the assumption on existence of a pole is unnecessary. In order to remove the assumption on pole, one may study asymptotics of certain eigenfunctions at infinity and use appropriate power of them to scale metrics as we do in the next section. We will discuss this in a future paper. also one can generalize arguments to study rigidity of asymptotic symmetric spaces, one particularly interesting case is for asymptotic complex hyperbolic Kähler manifolds. We expect that a similar result can be proved for Kähler manifolds by assuming that bisectional curvature tends to \(-1\) at a sufficiently fast rate.

The organization of this paper is as follows: In Section 2, we discuss the compactification and conformal structure of \((X,g)\) at infinity; In Section 3, we show that the boundary of the compactified manifold is isometric to the standard sphere and then use it to deduce the main theorem.

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## 2 Compactification and conformal structure at infinity

In this section, we give a compactification of \((X,g)\) at infinity and study the induced conformal structure at infinity. This compactification is crucial in the proof of our main theorem.
Let $\Sigma_\rho$ be the geodesic sphere in $(X, g)$ with radius $\rho$ and a fixed center $o$. Define $\bar{g}$ to be $\sinh^{-2} \rho g$, then we have:

**Theorem 2.1.** There is a subsequence of $(\Sigma_\rho, \bar{g}_\rho)$ which converges to a $W^{2,p} \cap C^{1,\alpha}$ Riemannian manifold $(\Sigma_\infty, \bar{g}_\infty)$ in the weakly $W^{2,p}$-topology, where $p \in (1, \infty)$ and $\alpha \in (0, 1)$ are arbitrary. Here by a $W^{2,p} \cap C^{1,\alpha}$ structure on $(\Sigma_\infty, \bar{g}_\rho)$, we mean that there is a covering $\{U_i\}$ of $\Sigma_\infty$ by coordinates $\phi_i : U_i \mapsto \mathbb{R}^n$ such that the transition functions $\phi_i \circ \phi_j^{-1}$ and the metric tensors $\phi_i^{*}\bar{g}$ are in $W^{2,p} \cap C^{1,\alpha}$. Furthermore, $(\Sigma_\infty, \bar{g}_\infty)$ is conformally equivalent to the standard sphere. Here $\bar{g}_\rho$ denotes the restriction of $\bar{g}$ to $\Sigma_\rho$.

By the compactness theorem proved in [4], in order to have the convergence property of $(\Sigma_\rho, \bar{g}_\rho)$, we only need to show the following

**Lemma 2.2.** There exists a constant $C$ such that $|\text{Rm}(\bar{g}_\rho)| \leq C$, $\text{Vol}(\Sigma_\rho, \bar{g}_\rho) \geq C^{-1}$, $\text{diam}(\Sigma_\rho, \bar{g}_\rho) \leq C$, where $\text{Rm}(\bar{g})$ denotes the curvature tensor of $\bar{g}$.

Let us first recall some basic formulae. For time being, we assume $\bar{g}=u^2 g$. Let $\{\omega_i\}_{1 \leq i \leq n+1}$ be a local orthonormal coframe of $g$ such that $\omega_{n+1} = d\rho$ and $\{\omega_i\}_{1 \leq i \leq n}$ is tangent to $\Sigma_\rho$. For convenience, we also denote $g = d\rho^2 + g_{ij}(\rho, \theta) d\theta_i d\theta_j$. Then we have structure equations

\[
\begin{aligned}
\begin{cases}
    d\omega_a = \sum_{b=1}^{n+1} \omega_{ab} \wedge \omega_b, & \omega_{ab} + \omega_{ba} = 0 \\
    d\omega_{ab} = \sum_{c=1}^{n+1} \omega_{ac} \wedge \omega_{cb} - \frac{1}{2} \sum_{c,d=1}^{n+1} R_{abcd} \omega_c \wedge \omega_d,
\end{cases}
\end{aligned}
\]  (2.1)

where $R_{abcd}$ denote components of the curvature tensor. The second fundamental form, denoted by $A = (h_{ij})_{1 \leq i,j \leq n}$, of $\Sigma_\rho$ with respect to $g$ is given by

\[
\omega_{n+1}|\Sigma = \sum_{j=1}^{n} h_{ij} \omega_j, \quad h_{ij} = h_{ji},
\]

where $\cdot|\Sigma$ denotes the restriction of an 1-form to $\Sigma_\rho$. The corresponding mean curvature is given by $H = \sum_{i=1}^{n} h_{ii}$.

Let $\eta_a = u \omega_a$ ($1 \leq a \leq n+1$), then $\{\eta_a\}_{1 \leq a \leq n+1}$ is an orthonormal coframe for the metric $\bar{g}$, and

\[
\begin{aligned}
\begin{cases}
    d\eta_a = \sum_{b=1}^{n+1} \eta_{ab} \wedge \eta_b, & \eta_{ab} + \eta_{ba} = 0 \\
    d\eta_{ab} = \sum_{c=1}^{n+1} \eta_{ac} \wedge \eta_{cb} - \frac{1}{2} \sum_{c,d=1}^{n+1} \tilde{R}_{abcd} \eta_c \wedge \eta_d
\end{cases}
\end{aligned}
\]  (2.2)

where $\tilde{R}_{abcd}$ are components of the curvature tensor of $(X, \bar{g})$ in the coframe $\{\eta_a\}_{1 \leq a \leq n+1}$. By a direct computation, we see that $\eta_{ab} = \omega_{ab} - (\log u) \omega_a + (\log u) \omega_b$.

Here for any smooth function $f$ on $X$, $f_\alpha$ is defined by $df = \sum_{a=1}^{n+1} f_a \omega_a$. Thus, we get

\[
\eta_{n+1,i}|\Sigma_\rho = (h_{ij} + \frac{\partial}{\partial \rho} (\log u) \delta_{ij}) u^{-1} \eta_j.
\]
It follows that the second fundamental form of $\Sigma_\rho$ with respect to $\bar{g}$ and $\{\eta_i\}_{1 \leq i \leq n}$ is given by

$$h_{ij} = (h_{ij} + \frac{\partial}{\partial \rho}(\log u)\delta_{ij})u^{-1}. \quad (2.3)$$

On the other hand, we can deduce from the structure equations for curvatures

$$\frac{\partial h_{ij}}{\partial \rho} + \sum_{k=1}^{n} h_{ik} h_{kj} = -R_{n+1 n+1 j}, \quad (2.4)$$

In order to estimate $h_{ij}$, we need the following

**Lemma 2.3.** Suppose that $f$ is a smooth function and for any $\rho > 0$, we have $|f(\rho) - 1| \leq K e^{-\alpha \rho}$ for some $\alpha > 2$ and $\frac{1}{4} \leq f(\rho)$. If $y$ is a solution of the equation

$$y' + y^2 = f(\rho) \quad \text{and} \quad y(0) > 0.$$ 

Then there is a constant $C > 0$, which depends only on $K$ and $y(0)$, such that

$$|y - 1| \leq C e^{-2\rho}.$$

**Proof.** We will prove this lemma in the following steps.

**Claim 1:** $0 < y(\rho) \leq \rho + C_1$ for any $\rho > 0$.

Here and in the sequel, $C_i$ always denotes a constant which depends only on $y(0)$ and $K$. Clearly, $y(\rho) \leq \rho + C_1$. To see that $y(\rho) > 0$, we first observe that $f \geq \frac{1}{14}$, hence, by using the equation, $y'(\rho) > 0$ whenever $y(\rho) < \frac{1}{12}$. It follows that $y$ increases in the region where $y < \frac{1}{2}$. Then the claim follows from $y(0) > 0$.

**Claim 2:** $|y - 1| \leq C_2 e^{-\rho}$.

Set $v = y - 1$, we have $\rho + C_1 - 1 \geq v \geq -1$ and $-1 \leq v(0) \leq y(0)$. Choose $\beta = 1 + \frac{\alpha - 2}{\alpha} < \alpha$. Then $|v| \leq C_3 e^{(\alpha - \beta)\rho}$, consequently, using the equation for $y$, we can deduce

$$(v^2)' + 2v^2 \leq (v^2)' + (4 + 2\beta)v^2 \leq C_4 e^{-\beta \rho}, \quad 2 < \beta < \alpha.$$ 

It follows

$$(v^2 e^{2\rho})' \leq C_4 e^{(2 - \beta)\rho},$$

Integrating this inequality, we get

$$v^2 \leq (v^2(0)) + \frac{2C_4}{\alpha - 2} e^{-2\rho} \leq C_5 e^{-2\rho}.$$ 

Claim 2 follows.

Now we can finish the proof of this lemma. By Claim 2, we have

$$|v| \leq C_5 e^{-\rho}.$$ 

Using this and the equation for $y$, we have

$$(v^2)' + (4 - 2|v|)v^2 \leq 2K e^{-\alpha \rho} |v|.$$
Suppose that we have proved $v^2 \leq e^{-2\beta_k}$ for some $\beta_k \geq 1$, then it follows from the above

$$(v^2 e^{4\rho})' \leq C_6 (e^{(4-\alpha-\beta_k)\rho} + e^{(4-3\beta_k)\rho}).$$

Integrating this, we get

$$v^2 \leq C_7 (e^{-4\rho} + e^{-\min\{3\beta_k, \alpha + \beta_k\}\rho}).$$

If $\min\{3\beta_k, \alpha + \beta_k\} \geq 4$, we are done, otherwise, then we take $\beta_{k+1} = \frac{1}{2} \min\{3\beta_k, \alpha + \beta_k\} \geq \beta_k + \frac{3\beta_k}{2} - 1$ and repeat the above process. Then the lemma follows after finitely many iterations.

Lemma 2.4. Let $A = \sum_{ij} h_{ij} \omega_i \otimes \omega_j$ be the second fundamental form of $\Sigma_\rho$ in $(X, g)$ and write

$$h_{ij} = \delta_{ij} + T_{ij} e^{-2\rho},$$

then $\|T\|_g \leq C$.\phantom{<}\phantom{\infty}$, where $T = \sum_{ij} T_{ij} \omega_i \otimes \omega_j$.

Remark 2.5. If $\hat{h}_{ij}$ denotes components of the second fundamental form of $\Sigma_\rho$ in $(X, g)$ in the coordinate frame $\{\frac{\partial}{\partial \theta}\}$, then we have

$$\hat{h}_{ij} = g_{ij} + p_{ij} e^{-2\rho}.\phantom{<}\phantom{\infty}$$

Write $\omega_i = \sum_j b_{ij} d\theta^j$, we have $(g_{ij}) = (b_{ij})^T \cdot (b_{ij})$ and $(p_{ij}) = (b_{ij})^T \cdot (T_{ij}) \cdot (b_{ij})$.

Proof. Let $\lambda_{\max}$ and $\lambda_{\min}$ be the largest and smallest eigenvalue of matrix $(h_{ij})$, then they are Lipschitz, and we claim that

$$\frac{d}{d\rho} \lambda_{\max} + 2 \lambda_{\max} = 1 + O(e^{-\alpha\rho}).\phantom{<}\phantom{\infty}$$

$$\frac{d}{d\rho} \lambda_{\min} + 2 \lambda_{\min} = 1 + O(e^{-\alpha\rho}).\phantom{<}\phantom{\infty}$$

In fact, for any $\rho = \rho_0$, let $V$ be the unit eigenvector of $\lambda_{\max}$, then $V^T (h_{ij}) V|_{\rho=\rho_0} = \lambda_{\max}(\rho_0)$, and $V^T (h_{ij}) V \leq \lambda_{\max}(\rho)$ for any $\rho$, thus,

$$\frac{d}{d\rho} \lambda_{\max}|_{\rho=\rho_0} = V^T (h_{ij}) V|_{\rho=\rho_0},$$

hence,

$$\frac{d}{d\rho} \lambda_{\max}|_{\rho=\rho_0} + 2 \lambda_{\max}|_{\rho=\rho_0} = 1 + O(e^{-\alpha\rho}),$$

which implies (2.2) is true, by the same reason, (2.3) is true too. On the other hand, when $\rho$ is sufficiently large the eigenvalue of matrix $(R_{n+1 n+1})$ is less than $-\frac{1}{4}$, and note that there is a convex geodesic sphere with sufficiently large radius, hence, we may assume the initial data of equation (2.3),(2.4) is positive, then by Lemma 2.3, we see Lemma 2.4 is true.\phantom{\infty}
Due to Lemma 2.4, we have \( \sup_{\Sigma} |T_{ij}| \leq C < +\infty \) for any \( \rho \geq 1 \).

**Proof of Lemma 2.2:** By a direct computation, we have

\[
\bar{R}_{abcd} = u^{-2} R_{abcd} - u^{-2}(\log u)_{bf} - (\log u)_b (\log u_a) (\delta_{ac} \delta_{db} - \delta_{ad} \delta_{bc}) - u^{-2} \nabla \log u |^2 (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}),
\]

(2.7)

where \( \bar{R}_{abcd} \) denote the components of the curvature tensor of \((X, \bar{g})\) in the coframe \(\{\eta_a\}\). By our assumption on asymptotic hyperbolicity, we may write

\[
R_{abcd} = (\delta_{bc} \delta_{ad} - \delta_{ac} \delta_{bd}) + E_{abcd},
\]

where \( |E_{abcd}| = O(e^{-\alpha \rho}) \).

Now let \( u = \sinh^{-1} \rho \). Noticing that for any \( 1 \leq a, b \leq n \)

\[
(\log u)_{ab} = (\log u)_{\rho}(\delta_{ab} + T_{ab} e^{-2\rho}),
\]

we can deduce from the above and Lemma 2.4

\[
\bar{R}_{abcd} = \frac{1}{4}(T_{bd} \delta_{ac} - T_{bc} \delta_{ad} - T_{ad} \delta_{bc} + T_{ac} \delta_{bd}) + E_{abcd}, \quad 1 \leq a, b, c, d \leq n \quad (2.8)
\]

here, \( |E_{abcd}| = O(e^{(2-\alpha)\rho}) \) as \( \rho \) tends to infinity.

\[
\bar{R}_{n+1b} = O(e^{(2-\alpha)\rho}),
\]

\[
\bar{R}_{n+1b} = T_{bd} - \frac{1}{2} \delta_{bd} + O(e^{(2-\alpha)\rho}).
\]

Let \( \hat{h}_{ij} \) be the components of the second fundamental form of \( \Sigma_\rho \subset X \) with respect to the metric \( \bar{g} \). It follows from (2.3) that:

\[
\hat{h}_{ij} = O(e^{-\rho}). \quad (2.9)
\]

Now let us estimate the volume and diameter of \( (\Sigma_\rho, \bar{g}) \). We can write \( g \) in the form \( d\rho^2 + g_{ij}(\rho, \theta)d\theta^i d\theta^j \), then we have

\[
\frac{\partial}{\partial \rho} g_{ij} = 2 \hat{h}_{ij}.
\]

Using the facts that \( \hat{h}_{ij} = g_{ij} + p_{ij} e^{-2\rho} \) and \( -c(g_{ij}) \leq (\hat{h}_{ij}) \leq c(g_{ij}) \) for some constant \( c \), we can show that there exists a constant \( \Lambda \) independent of \( \rho \) such that

\[
\Lambda^{-1} e^{2\rho}(\delta_{ij}) \leq (g_{ij}) \leq \Lambda e^{2\rho}(\delta_{ij}). \quad (2.10)
\]

It follows that \( \text{diam}(\Sigma_\rho, \bar{g}) \leq C_2 \) and \( \text{Vol}(\Sigma_\rho, \bar{g}) \geq \delta_0 > 0 \). The proof of Lemma 2.2 is completed.

\[\text{Without loss of generality, we may assume that } \alpha \leq 4.\]
By using (2.8), (2.9) and the Gauss equations, we see that the sectional curvature of \((\Sigma_\rho, \bar{g}_\rho)\) is uniformly bounded. Then it follows from Lemma 2.2 and [4] that there exists a sequence of \((\Sigma_\rho, \bar{g}_\rho_i)\), which will be denoted by \((\Sigma, \bar{g}_i)\), converges to \((\Sigma_\infty, \bar{g}_\infty)\) in the sense of weak topology of \(W^{2,p}\) for any \(p < \infty\), and for any \(q \in \Sigma_\infty\), there is a coordinate charts \((B_q, \theta^i)\) in which the components of \(\bar{g}_\infty\) is \(C^{1,\alpha} \cap W^{2,p}\), \(\forall p < +\infty\) and the curvature of \((\Sigma_\infty, \bar{g}_\infty)\) is bounded.

Let \(\hat{R}_{ijkl}\) be the components of the curvature tensor of \((\Sigma_\rho, \bar{g}_\rho)\) under the orthonomal frame \(\eta^i (1 \leq i \leq n)\), then the Weyl tensor is:

\[
\hat{W}_{ijkl} = \hat{R}_{ijkl} - \frac{1}{n-2}(\delta_{il} \delta_{jk} + \delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + \delta_{ij} \delta_{kl} - \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + \frac{\hat{\mathcal{R}}}{(n-1)(n-2)}(\delta_{ik} \delta_{jl} - \delta_{jk} \delta_{il})
\]

Combined with (2.8) and (2.9), we see that \(\|\hat{W}\| = o(1)\) as \(\rho\) tends to infinity, and the Ricci tensor is of the form

\[
\hat{R}_{ij} = \frac{1}{4}[(n-2)T_{ij} + tr_g T \delta_{ij}] + E_{ij},
\]

where \(E_{ij} = \bar{g}^{kl} E_{ikjl}\) and \(|E_{ij}| = o(1)\) as \(\rho\) tends to infinity.

Recall that the Schouten tensor of \(g\) is

\[
\hat{S}_{ij} = \frac{1}{n-2}(2\hat{R}_{ij} - \frac{\hat{R}}{n-1}\delta_{ij}).
\]  

Therefore, the Weyl tensor of \((\Sigma_\infty, \bar{g}_\infty)\) vanishes in the \(L^p\)-sense. Together with Gauss equations and Codazzi equations and (2.11), we deduce

\[
\bar{\nabla}_k \hat{S}_{ij} - \bar{\nabla}_j \hat{S}_{ik} = \frac{1}{2}\bar{e}^{2\rho} \sinh \rho R_{n+1,ij} + \frac{2}{n-2} \bar{\nabla}_k E_{ij} - \bar{\nabla}_j E_{ik}
\]

\[
- \frac{1}{(n-1)(n-2)}(\bar{\nabla}_k E \delta_{ij} - \bar{\nabla}_j E \delta_{ik})
\]

If \(n = 3\), by the assumption that \(\|Rm^0\|_g \in L^1(X)\), we see that there are \(\rho_i\) which tend to infinity such that

\[
\int_{\Sigma_{\rho_i}} \|Rm^0\|_g \sinh^3 \rho_i \to 0,
\]

in particular, we have

\[
\int_{\Sigma_{\rho_i}} |R_{4ijkl}| e^{3\rho_i} \to 0.
\]

It follows that for any \(\phi \in C^\infty(\Sigma_{\rho_i})\),

\[
\int_{\Sigma_{\rho_i}} \phi R_{4ijkl} e^{2\rho_i} \sinh \rho_i \to 0.
\]

Without loss of generality, we may assume \((\Sigma_{\rho_i}, \bar{g}_{\rho_i})\) converges to \((\Sigma_\infty, \bar{g}_\infty)\), for simplicity, in the sequel, \((\Sigma_\infty, \bar{g}_\infty)\) will be denoted by \((\Sigma, \bar{g})\) and the components of its curvature tensor will be simply denoted by \(\hat{R}_{ijkl}\). Then if \(n = 3\),
we see that the Schouten tensor of \((\Sigma, g)\) satisfies the following equations in the sense of distribution,

\[
\bar{\nabla}_k \bar{S}_{ij} - \bar{\nabla}_j \bar{S}_{ik} = 0,
\]

(2.14)

that is, for any \(\phi \in C^\infty(\Sigma)\), we have

\[
\int_{\Sigma} \bar{S}_{ij} \bar{\nabla}_k \phi - \bar{S}_{ik} \bar{\nabla}_j \phi = 0,
\]

where \(\bar{\nabla}_k\) are covariant derivatives of \((\Sigma, \bar{g})\) with respect to an orthonormal basis \(\{e_i\}_{1 \leq i \leq n}\).

Now, we are in the position to show the following:

**Theorem 2.6.** Suppose that \((\Sigma, g)\) is an \(n\)-dimensional Riemannian manifold and the metric \(g\) is in \(W^{2,p}\) for any \(1 < p < \infty\). If its curvature tensor is bounded and Weyl tensor \(W = 0\) if \(n > 3\); and (2.14) is true if \(n = 3\), then \(g\) is locally conformally flat, i.e., for any point \(q \in V\), there is a neighborhood \(U\), such that in \(U\), we have a positive \(W^{2,p}\) function \(f\) with \(g = fg_{\text{euc}}\), where \(g_{\text{euc}}\) denotes a flat metric on \(U\).

Clearly, it is a local result, hence, we need to consider only the problem in a local coordinate chart, i.e., we assume that \(\Sigma\) is a ball \(B^n \subset \mathbb{R}^n\) and \(g = g_{ij} dx_i dx_j\), where \(x_1, \cdots, x_n\) are Euclidean coordinates of \(\mathbb{R}^n\). By our assumption, \(g_{ij}\) are \(W^{2,p}\) functions on \(B^n\) for any \(1 < p < \infty\). It follows the curvature tensor \(R_{ijkl}\) and the Christoffel symbol \(\Gamma^i_{jk}\) are in \(L^p\) and \(W^{1,p}\) respectively. Hence, we can define covariant derivatives of \(R_{ijkl,h}\) in the sense of distribution, that is, for any \(\phi \in C^\infty_0(B^n)\), we have:

\[
\int_{B^n} R_{ijkl,h} \phi \sqrt{\det(g)} dx = - \int_{B^n} R_{ijkl} \frac{\partial}{\partial x^h} (\phi \sqrt{\det(g)}) dx + \int_{B^n} R_m \Gamma \phi \sqrt{\det(g)} dx,
\]

where \(\det(g) = \det(g_{ij})\) and \(R_m \Gamma\) refers to a bilinear form of \(R_{ijkl}\) and \(\Gamma_{ijk}\).

Since \(R_{ijkl}\) are in \(L^p\) and \(\frac{\partial}{\partial x^h} (\phi \sqrt{\det(g)})\) is in \(C^\alpha\) for some \(\alpha > 0\), the right hand side of the above equation is well defined. Similarly, we can define \(R_{ijkl,hm}\) in the sense of distribution, that is, for any \(\phi \in C^\infty_0(B^n)\), we have:

\[
\int_{B^n} R_{ijkl,hm} \phi \sqrt{\det(g)} dx = \int_{B^n} R_{ijkl} \frac{\partial^2}{\partial x^h \partial x^m} (\phi \sqrt{\det(g)}) dx - \int_{B^n} R_m \frac{\partial}{\partial x^m} (\Gamma \phi \sqrt{\det(g)}) dx + \int_{B^n} R_m \frac{\partial}{\partial x^m} (\Gamma \phi \sqrt{\det(g)}) dx.
\]

Now we have:

**Lemma 2.7.** Suppose that \(g \in W^{2,p}\) for some \(p > 1\), then in the distributional sense, we have the second Bianchi identity

\[
R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l} = 0
\]

(2.15)

and

\[
R_{ik,tt} = R_{ik,tl} + Ric * Rm.
\]

(2.16)
That is, for any $\phi \in C_0^\infty(B^n)$, we have
\[
\int_{B^n} (R_{ijkl,h} + R_{ijlh,k} + R_{ijhk,l}) \phi \sqrt{\det(g)} \, dx = 0
\]
and
\[
\int_{B^n} (R_{ik,lt} - R_{ik,tl} - \text{Ric} \ast \text{Rm}) \phi \sqrt{\det(g)} \, dx = 0.
\]
Here $R_{ij}$ is the Ricci tensor of $g$ and $\text{Ric} \ast \text{Rm}$ denotes a bilinear form of Ricci tensor and curvature tensor.

**Proof.** By the assumption, we may take a sequence of smooth metrics $g_i$ which converges to $g$ in $W^{2,p}$. Since (2.15) and (2.16) hold for the curvature tensor of $g_i$ and curvature tensors of $g_i$ converge to that of $g$ in $L^p$, we see that (2.15) and (2.16) hold for $g$, too.

Next we construct harmonic coordinates around any point of manifold. Without loss of generality, we only need to show

**Lemma 2.8.** Suppose that $g_{ij}$ are in $W^{2,p}$ on $B^n$ for any $1 < p < \infty$, then there are harmonic coordinates $(z^1, \cdots, z^n)$ around $o \in B^n$ with $z^i$ in $W^{3,p}$.

**Proof.** Let $\Gamma^i_{jk}$ denote the Christoffel symbols of $g$ in euclidean coordinates $x^1, \cdots, x^n$. Define $y_i$ by
\[
x^i = y^i - \Gamma^i_{jk}(o) y^j y^k \quad (1 \leq i \leq n),
\]
by the Inverse Theorem, we see that $y_i$ are smooth functions of $(x^1, \cdots, x^n)$ around $o$ and form coordinates. Let $\bar{\Gamma}^i_{jk}$ be the Christoffel symbols of $g$ in coordinates $(y^1, \cdots, y^n)$, then by direct computations, we see
\[
\bar{\Gamma}^k_{ij} \frac{\partial x^s}{\partial y^k} = \frac{\partial^2 x^s}{\partial y^j \partial y^i} + \frac{\partial x^l}{\partial y^j} \frac{\partial x^m}{\partial y^i} \Gamma^s_{lm}.
\]
It follows that $\bar{\Gamma}^k_{ij}(o) = 0$, consequently, $\Delta y^i = 0$ at $o$. This implies that $\|\Delta y^i\|_{L^\infty(B_\epsilon(o))}$ tends to 0 as $\epsilon$ goes to zero. Consider the following boundary value problem on $B_\epsilon(o)$
\[
\begin{cases}
\Delta z^i = 0 \\
z^i|_{\partial B_\epsilon} = y^i|_{\partial B_\epsilon}.
\end{cases}
\]
then, using standard estimates for elliptic equations, we get
\[
\|z^i - y^i\|_{C^{1,\alpha}} \leq C \|\Delta y^i\|_{L^\infty(B_\epsilon)}.
\]
Therefore, $z^1, \cdots, z^n$ form local coordinates on $B_\epsilon(0)$ when $\epsilon$ is sufficiently small. Clearly, $z^i$ are in $W^{3,p}(B_\epsilon)$ and harmonic with respect to $g$. The lemma is proved.

By a direct computation and Lemma 2.8, we see that metric tensor of $g$ in coordinates $z^1, \cdots, z^n$ is also in $W^{2,p}$. In the following, we will consider the problem in these harmonic coordinates, and the metric components will be still denoted by $g_{ij}$. 
Lemma 2.9. Let $R$ be the scalar curvature of $g$ and bounded, then when $\epsilon$ is sufficiently small, the following equation

$$\begin{cases}
\Delta u - \frac{n-2}{4(n-1)} Ru = 0 \\
u|_{\partial B} = 1|_{\partial B},
\end{cases}$$

has a positive solution in $W^{2,p}(B_{\epsilon})$.

Proof. We note that when $\epsilon$ is sufficiently small, the first Dirichlet eigenvalue can be arbitrarily large, and $R$ is bounded, hence, the corresponding homogeneous equation has only trivial solution, and this implies the above equation has nonnegative solution, then by Lemma 3.4 in [10] (p34), we see that the solution has to be positive. This finishes the proof of the lemma. \qed

In order to show Theorem 2.6, we need the following lemma (see Theorem 17.2.7, [5], p18 for its proof).

Lemma 2.10. Let $a_{ij}(x)$ be Lipschitz continuous in an open set $\Omega \subset \mathbb{R}^n$, and assume that the matrix $(a_{ij})$ is positive definite and $u \in L^2_{\text{loc}}(\Omega)$. Then

$$\sum \frac{\partial}{\partial x^j}(a_{jk} \frac{\partial u}{\partial x^k}) = f,$$

implies $u \in W^{1,2}_{\text{loc}}(\Omega)$ if $f \in H^{-1}_{\text{loc}}(\Omega)$, moreover, if $f \in L^2_{\text{loc}}(\Omega)$, then $u \in W^{2,2}_{\text{loc}}(\Omega)$. Here $H^{-1}_{\text{loc}}(\Omega)$ is the dual space of $W^{1,2}_{\text{loc}}(\Omega)$.

Now we can finish the proof of Theorem 2.6. Since the scalar curvature of $(\Sigma, g)$ is bounded, by Lemma 2.9, we may choose a sufficiently small neighborhood of $q$ such that there is a positive $W^{2,p}$ function $u$ on this neighborhood such that the scalar curvature of $\bar{g} = u^{\frac{4}{n-2}}g$ vanishes. It is easy to show that $\bar{g}$ is also in $W^{2,p}$ for any $1 < p < \infty$, moreover, its Weyl tensor also vanishes if $n \geq 4$ and (2.14) still holds if $n = 3$. By Lemma 2.8, we choose harmonic coordinates of the metric $\bar{g}$ with metric tensor $\bar{g}_{ij}$ in $W^{2,p}$. It suffices to show that the corresponding Ricci tensor is smooth in these harmonic coordinates. In the sequel, we will do everything in these coordinates.

Since the Weyl tensor and the scalar curvature vanish, we have

$$\bar{R}_{ijkl} = \frac{1}{n-2}(\bar{R}_{ikl} \bar{g}_{jl} - \bar{R}_{jkl} \bar{g}_{il} + \bar{R}_{jl} \bar{g}_{ik} - \bar{R}_{il} \bar{g}_{jk}) \quad (2.17)$$

On the other hand, by Lemma 2.7, we have the second Bianchi identity for $\bar{g}$, hence, by a direct computation, we deduce

$$\bar{g}^{jh} \bar{R}_{jk,h} = 0. \quad (2.18)$$

If $n \geq 4$, using (2.17), (2.18) and the Bianchi identity, we can also derive

$$\bar{R}_{dl,k} - \bar{R}_{ik,l} = 0. \quad (2.19)$$
When \( n = 3 \), since the scalar curvature vanishes, the above equation is nothing but (2.14). It follows
\[
\bar{g}^{kt} \bar{R}_{it,kt} - \bar{g}^{kt} \bar{R}_{ik,lt} = 0,
\]
and because of (2.16) in Lemma 2.7, we have
\[
\bar{g}^{kt} \bar{R}_{ik,lt} = \bar{g}^{kt} \bar{R}_{ik,tl} + \bar{g} * \bar{R}ic * \bar{R}m.
\]
Note that
\[
\bar{g}^{kt} \bar{R}_{il,kt} = (\bar{g}^{kt} \bar{R}_{ik,t})_l = 0,
\]
so we have
\[
\bar{g}^{kt} \bar{R}_{il,kt} = \bar{g} * \bar{R}ic * \bar{R}m.
\]
Since \( \bar{g} \) is in \( W^{2,p} \), the above equation can be written as
\[
\frac{\partial}{\partial x}(\bar{g}^{kt} \frac{\partial \bar{R}_{il}}{\partial x^k}) = \partial \bar{g} * \partial \bar{R}ic + \bar{g} * \bar{R}ic * \bar{R}m.
\] (2.20)
Noticing that \( \bar{g} \in W^{2,p} \) and \( \bar{R}m \in L^p \) for any \( p > 1 \), we see that the right hand side of (2.20) is in \( H^{-1} \), which is dual to \( W^{1,2}_0 \). Then it follows from Lemma 2.10 that \( \bar{R}_{ij} \) are actually in \( W^{1,2}_0 \), in turns, this implies that the right side of (2.20) is in \( L^2_{loc} \), then again by Lemma 2.10, we see that \( \bar{R}_{ij} \) are in \( W^{2,2}_{loc} \), then it follows from the standard theory for elliptic equations that \( \bar{R}_{ij} \) are actually \( C^{2,\alpha}_{loc} \), therefore, \( \bar{g} \) is smooth, and consequently, by the classical Weyl Theorem, it is locally conformal flat. Theorem 2.6 is proved.

Now, we can prove Theorem 2.1.

**Proof of Theorem 2.1**: It only remains to show that \( (\Sigma, \bar{g}) \) is conformally equivalent to the standard sphere. By the assumption of Theorem 2.1, we see that \( \Sigma \) is diffeomorphic to \( S^n \). On the other hand, by Theorem 2.6, we know that \( (\Sigma, \bar{g}) \) is a locally conformally flat manifold, so is conformally equivalent to \( S^n \). Theorem 2.1 is proved.

### 3 Proof of Main Theorems

To prove Theorem 1.1 and Theorem 1.3, we need to compare both the volume and the scalar curvature of \( (\Sigma, \bar{g}) \) which is the boundary of Riemannian manifold \( (X, \bar{g}) \) with those corresponding quantities of the standard sphere. Using this, we are able to show that \( (\Sigma, \bar{g}) \) is actually isometric to \( S^n \). Then by the Volume Comparison theorem, we can conclude that the original manifold \( (X, g) \) is isometric to \( H^{n+1} \).

**Lemma 3.1.** Let \( \omega_n \) denote the volume of \( S^n \) and \( \bar{R} \) be the scalar curvature of \( (\Sigma, \bar{g}) \), then we have \( \text{Vol}(\Sigma, \bar{g}) \leq \omega_n \) and \( \bar{R} \leq n(n - 1) \).

**Proof.** Recall that \( \bar{R} \) is the scalar curvature of \( (\Sigma, \bar{g}) \), then by the computations in last section, we have
\[
\bar{R} = \frac{n-1}{2} \sum_{i,j=1}^{n} g^{ij} p_{ij} + o(1), \text{ as } \rho \to \infty
\]
and
\[ H = n + e^{-2\rho} \sum_{i,j=1}^{n} g^{ij} p_{ij}, \]
where \( H \) denotes the mean curvature of \( \Sigma_{\rho} \) in \( (X, g) \). On the other hand, because of \( \text{Ric}(g) \geq -ng \), we can use the Laplacian Comparison Theorem to get
\[ H|_{\Sigma_{\rho}} = \triangle_g p_{\Sigma_{\rho}} \leq n \coth \rho \]
(3.1)
It follows
\[ \sum_{i,j=1}^{n} g^{ij} p_{ij} \leq \frac{2ne^{2\rho}}{e^{2\rho} - 1}, \]
and consequently,
\[ \hat{R} \leq n(n - 1) + o(1), \]
letting \( \rho \) go to \( \infty \), we get \( \bar{R} \leq n(n - 1) \).

To show \( \text{Vol}(\Sigma, \bar{g}) \leq \omega_n \), we only need to prove for any \( \rho > 0 \),
\[ \text{Vol}(\Sigma_{\rho}, g) \leq (\sinh \rho)^n \omega_n. \]
(3.2)
For any \( \delta > 0 \), integrating (3.1), we obtain
\[ \int_{B_{\tau+\delta} \setminus B_{\tau}} \triangle_g \rho dV_g \leq \int_{B_{\tau+\delta} \setminus B_{\tau}} n \coth \rho dV_g, \]
which is equivalent to
\[ \frac{\text{Vol}(\Sigma_{\tau+\delta} - \text{Vol}(\Sigma_{\tau}))}{\delta} \leq \frac{n}{\delta} \int_{\tau}^{\tau+\delta} \coth \rho \text{Vol}(\Sigma_{\rho}) d\rho. \]
Let \( \delta \to 0 \), we have
\[ (\log ( (\sinh \tau)^{-n} \text{Vol}(\Sigma_{\tau}) ))' \leq 0. \]
Hence \( (\sinh \tau)^{-n} \text{Vol}(\Sigma_{\tau}) \) is non-increasing with \( \tau \). Since \( \lim_{\tau \to 0} (\sinh \tau)^{-n} \text{Vol}(\Sigma_{\rho}) = \omega_n \), we see from the above that (3.2) is true. This implies that \( \text{Vol}(\Sigma, \bar{g}) \leq \omega_n \).
Thus Lemma 3.1 is proved.

Our next goal is to establish

**Lemma 3.2.** The limit space \( (\Sigma, \bar{g}) \) is isometric to the standard sphere \( (S^n, g_0) \)

**Proof.** If \( n = 2 \), we only need to show \( \bar{R} = 2 \), suppose not, we have \( \bar{R} < 2 \) and \( \text{Vol}(\Sigma, \bar{g}) \leq 4\pi \), this is in contradiction with Gauss-Bonnet formula.

If \( n \geq 3 \), it suffices to prove that \( \bar{R} = n(n - 1) \). In fact, we can write
\[ \bar{g} = u^{\frac{4}{n-2}} g_0 \]
(3.3)
for some \( u > 0 \) which belongs to \( W^{2,p} \) for any \( p < \infty \). If \( \tilde{\mathcal{R}} = n(n-1) \), \( u \) satisfies a semi-linear elliptic equation and the standard regularity theory implies that \( u \) is smooth. Then Lemma 3.2 follows from the Obata theorem.

Let \( d\tilde{V} \) and \( dV_0 \) be the volume elements of \((\Sigma, \tilde{g})\) and \((S^n, g_0)\), respectively, then by (3.3), \( d\tilde{V} = u^{\frac{2n}{n-2}}dV_0 \). The following equation is well-known

\[
\tilde{\mathcal{R}} = u^{\frac{n+2}{n}}(n-1)u - \frac{4(n-1)}{n-2}\Delta_{S^n}u,
\]

it follows

\[
\int_{S^n} \tilde{\mathcal{R}} u^{\frac{2n}{n-2}}dV_0 = \int_{S^n} ((n-1)nu^2 + \frac{4(n-1)}{n-2}|\nabla_{S^n}u|^2)dV_0,
\]

since \( \tilde{\mathcal{R}} \leq n(n-1) \), we get

\[
n(n-1)(\int_{S^n} u^{\frac{2n}{n-2}}dV_0)\frac{n}{n-2} \geq \frac{\int_{S^n} ((n-1)nu^2 + \frac{4(n-1)}{n-2}|\nabla_{S^n}u|^2)dV_0}{(\int_{S^n} u^{\frac{2n}{n-2}}dV_0)\frac{n}{n-2}}.
\]

Using the fact that \( d\tilde{V} = u^{\frac{2n}{n-2}}dV_0 \) and \( Vol(\Sigma, \tilde{g}) \leq \omega_n \), we see that:

\[
n(n-1)\omega_n \frac{n}{n-2} \geq \frac{\int_{S^n} ((n-1)nu^2 + \frac{4(n-1)}{n-2}|\nabla_{S^n}u|^2)dV_0}{(\int_{S^n} u^{\frac{2n}{n-2}}dV_0)\frac{n}{n-2}}.
\] (3.4)

Write \( g_0 = \psi \frac{1}{n-2}ds^2_{\mathbb{R}^n} \), where \( \psi(x) = (1+|x|^2)^{\frac{2-n}{2}} \), then \( dV_0 = \psi^{\frac{2n}{n-2}}dx \), where \( dx \) is the volume element of \( \mathbb{R}^n \), we have:

The RHS of (3.4) = \[ \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} |\nabla_{\mathbb{R}^n}(u\psi)|^2dx \] (3.5)

On the other hand, by a direct computation, we have:

\[
n(n-1)\omega_n \frac{n}{n-2} = \frac{4(n-1)}{n-2} \int_{\mathbb{R}^n} |\nabla_{\mathbb{R}^n}\psi|^2dx \] (3.6)

Putting (3.4), (3.5) and (3.6) together, we obtain

\[
\frac{\int_{\mathbb{R}^n} |\nabla_{\mathbb{R}^n}\psi|^2dx}{(\int_{\mathbb{R}^n} \psi^{\frac{2n}{n-2}}dx)\frac{n}{n-2}} \geq \frac{\int_{\mathbb{R}^n} |\nabla_{\mathbb{R}^n}u\psi|^2dx}{(\int_{\mathbb{R}^n} (u\psi)^{\frac{2n}{n-2}}dx)\frac{n}{n-2}}.
\] (3.7)

Note that \( \psi = (1+|x|^2)^{\frac{2-n}{2}} \), we know that the LHS of (3.7) is the best Sobolev constant for \( \mathbb{R}^n \), hence, the equality in (3.7) holds, so \( \tilde{\mathcal{R}} = n(n-1) \). Thus we see that \((\Sigma, \tilde{g})\) is nothing but \((S^n, g_0)\). Lemma 3.1 is proved. \[ \square \]
Proof of Theorem 1.1: In the proof of Lemma 3.1, we have shown that 
\((\sinh \rho)^n \text{Vol}(\Sigma_{\rho}, g)\) is non-increasing. By Lemma 3.2 and the fact that \((\Sigma_{\rho}, \bar{g}_{\rho})\) subconverges to \((\Sigma, \bar{g})\) in the Cheeger-Gromov topology, we get

\[
\lim_{\rho \to \infty} (\sinh \rho)^{-n} \text{Vol}(\Sigma_{\rho}, g) = \omega_n.
\]

Hence, by the Volume Comparison Theorem, we have that for any \(\rho > 0\), 
\((\sinh \rho)^{-n} \text{Vol}(\Sigma_{\rho}, g) = \omega_n\). Now we claim that

\[
\Delta_{g_{\rho}} = H|_{\Sigma_{\rho}} = n \coth \rho, \quad \forall \rho > 0.
\]

If it is false, there is a point \(p \in \Sigma_{\rho}\) such that \(\Delta_{g_{\rho}}|_p < n \coth \rho|_p\), so

\[
\int_{B_{\rho+\delta}(o)\setminus B_{\rho}(o)} \Delta_{g_{\rho}} \tau dV_g < \int_{B_{\rho+\delta}(o)\setminus B_{\rho}(o)} n \coth \tau dV_g,
\]

or equivalently

\[
\text{Vol}(\Sigma_{\rho+\delta}) - \text{Vol}(\Sigma_{\rho}) < n \int_{\rho}^{\rho+\delta} \coth \tau Area(\Sigma_{\tau}) d\tau.
\]

This contradicts to that \(\text{Vol}(\Sigma_{\tau}) = (\sinh \tau)^n \omega_n\). Hence for any \(\rho > 0\), \(H|_{\Sigma_{\rho}} = n \coth \rho\) and consequently

\[
\frac{\partial H}{\partial \rho} + \frac{H^2}{n} = n.
\]

However, from (2.2), we see that

\[
\frac{\partial H}{\partial \rho} + \frac{H^2}{n} \leq n,
\]

moreover, the equality holds if and only if \(h_{\rho ij} = \coth \rho g_{ij}\). On the other hand, a direct computation shows that

\[
\frac{\partial g_{ij}}{\partial \rho} = 2h_{\rho ij} = 2 \coth \rho g_{ij},
\]

and

\[
\lim_{\rho \to 0} \rho^{-2} g_{ij} = (g_0)_{ij}.
\]

Hence, \(g_{ij} = (\sinh \rho)^2 (g_0)_{ij}\), where \(g_0\) is the standard metric on \(\mathbb{S}^n\). Therefore, we see \(g = d\rho^2 + (\sinh \rho)^2 (g_0)_{ij} d\theta^i d\theta^j\), that is, \((X, g)\) is isometric to \(H^{n+1}\). Theorem 1.1 is proved.

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